A foliated higher dimensional space-time: Implications for BSM physics and unification of forces

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Abstract

We present some interesting physical consequences from investigating the $D$-dimensional space-time manifold (bulk) which is foliated by the $(D - 4)$-dimensional space-like submanifolds, realized as the internal spaces, being smooth copies of the compact connected Lie group. The principal bundle is used to describe the bulk leading to the natural present of the conventional gauge fields. These fields point precisely out the local directions of the usual 4-dimensional world, realized as the external spaces to be transversal to the internal spaces, being full independent to those along the internal spaces in the bulk. Consequently, any particle propagating along the external directions may be coupled to these gauge fields. The gauge interaction, the 4-dimensional tensor gravity and that mediated by modulus fields of the internal spaces are unified in the same geometric framework of the bulk. The remarkable extension of usual gauge symmetry by putting it into structure of the bulk leads to interesting results, for example, the quantum geometrodynamics of the space-time. The physical size of the internal spaces is fixed dynamically by the moduli stabilization potential which completely arise from the intrinsic geometry of the bulk. The low energy bulk gravity in the weak field limit is treated around the classical ground state of the bulk. Additionally, the dynamical description of the fundamentally 4-dimensional Weyl spinor fields in the bulk is given in a detail study. An important result is to explain the natural smallness origin of the neutrino masses. Finally, we study the fields of carrying out the non-trivial representations of the above Lie group whose vacuum expectation values would generate the masses for the above gauge fields. We then suggest a dark matter candidate coming from such fields.

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I. INTRODUCTION

The present observed universe appears to have three spatial dimensions and one dimension of time. However, the possibility that there may in fact be more has been suggested by many reasons, even if experimental observations have not confirmed any evidence to the existence of additional dimensions so far. The first attempt was proposed by Kaluza and Klein to unify the electromagnetic interaction and the gravity described by general relativity in a 5-dimensional space-time. The gauge potential, or the photon, is realized as extra components of the generic metric. In modern Kaluza-Klein scenarios [1, 2] tried to unify the forces of the Standard Model (SM) to the gravity, the gauge group is realistically associated with the isometry group of the internal manifold. The string/M theories defined consistently suppose the existence of additional dimensions. Their low energy limit is well known as the supergravity theories involving the noncommutative extra dimensions beside the original ones.

It is very interesting in recent years that higher dimensions theories with additional dimensions being much larger than the Planck length have devoted enormously to solve problems in particle physics as well as cosmology. These have thus opened a new window to seek Beyond the Standard Model (BSM) physics. The large extra dimensions \[ \text{was first suggested by Arkani-Hamed, Dimopoulos and Dvali (also known as the ADD model) as solution for hierarchy problem. According to this scenario, our 4-dimensional world could be realized as a 3-brane embedded in a higher dimensional space-time. This brane has the natural origin in the superstring theory. The particles of the SM are confined on this brane, except for the gravity that can propagate in the whole bulk space-time. The coupling strength of the bulk gravity would thus be diluted over the large volume of extra dimensions. As a result, the effective 4-dimensional Planck mass } M_4 \text{ is no longer the fundamental scale, but is related to the fundamentally higher dimensional Planck mass } M_D \text{ and the volume of } (D - 4)\text{-dimensional extra space } V_{D-4} \text{ as, } M_4 = M_D V_{D-4}. \text{ The hierarchy problem is overcome with } M_D \sim \text{TeV and } V_{D-4} \text{ to be large enough. On the other hand, this model could help to explain the weakness of the usual 4-dimensional gravity compared other interactions. In addition, Randall and Sundrum proposed the warped extra dimensions [4] which provides an alternative solution for the problem above. The hierarchy is addressed due to the exponential redshift factor presented in the metric of the nonfactorizable geometry on the (visible) IR 3-brane located at one fixed point of the orbifold } S^1/Z_2. \text{ The phenomenological implications of this model are very distinctive to those of the ADD model. On the contrary, all of particles are free to propagate in the whole bulk known by the name of Universal Extra Dimensions (UED)[5]. In order for extra dimensions that are not escaped by the observation, they must be curled up sufficiently small of the size, } R^{-1} \gtrsim \text{TeV.}

The studying of the black-hole solutions in the higher dimensional space-time has led to the interesting and important results. It has been aware growingly that the physics of higher dimensional black-holes can be markedly different and much richer than that in four dimensions [6]. The supergravity theories also include variously of the solitonic solutions of black \( p \)-brane [7], which are the extended objects of the 4-dimensional classical black-hole solutions. In certain limit, the black \( p \)-brane picture is dual to the \( Dp \)-brane picture obtained in the weak string coupling limit of the corresponding superstring theory, meaning that they describe the same physics. This was to motivate the conjecture of the AdS/CFT correspondence [8, 10] which relates the weak coupled classical gravity to the strong coupled field theory without the gravity in one lower dimension.
It has been well known that the Lie group is especially important with wide range of applications in other fields of modern theoretical physics.\textsuperscript{1} This is due to the richness of the Lie group which comes at least in the following three interesting varieties. The first one is related to the representation concept of the Lie group which has become thoroughly familiar with the development of modern elementary particle theory over much of the past century. In this way, the elementary particles have internal symmetry space corresponding to a certain representation space of a given Lie group. The internal degrees of freedom will become dynamical if the internal space is given locally. The gauge symmetry principle, one of the fundamental principles of subatomic particle physics, thus requires introducing the gauge vector bosons to remain the symmetry represented by that Lie group. These bosons play the role of mediating the interaction among the elementary particles. The second one is related the smooth action of the Lie group on the manifolds which often appears in the physics as the symmetric transformation group of the space-time. For example, it is already quite familiar with the Poincaré group acting on the 4-dimensional Minkowski space-time. It should be noted that this is the linear action. As we will see in main parts of the present work, the action of the Lie group is given in the more abstract way. The two varieties indicated above show that the Lie group provides a natural framework to relate between the continuous symmetries of the quantum field theory to the conserved quantities. The third one is that Lie group is also a differentiable manifold itself beside the group structure in the usual sense. More precisely, every Lie group is a parallelizable manifold since its tangent bundle is trivial.

The interesting aspects of the Lie group mentioned above are of significant motivations for using it in the higher dimensional extension of the space-time performed in this paper. The purpose of this paper is natural of the braneless higher dimensional extension. The higher dimensional space-time which we will consider in the whole work is described by a $D$-dimensional differentiable manifold denoted $B^D$ with $(D - 4)$ space-like extra dimensions. The bulk space-time is supposed to hold an internal structure which is similar to that the elementary particles do as discussed above. It is constructed by attaching a smooth copy of the Lie group $G$ at each point of a 4-dimensional pseudo-Riemannian manifold denoted $M^4$ with the Lorentz metric. For much of what follows we are going to be interesting in the special unitary and orthogonal Lie groups. From this point of view, an underline geometry of $B^D$ is always given as the manifold foliated by disjoint submanifolds of $(D - 4)$ dimensions which are diffeomorphic to the Lie group $G$. They are also called the leaves of the foliation. Thus the internal spaces are connected and compact. It is important to note that at starting point we consider the internal spaces to be compact in the sense of the mathematics. The most useful mathematical description for geometry of the foliated bulk space-time under consideration is known as the $G$-principal bundle. In the literature, $B^D$ and $M^4$ are commonly called the total and base spaces, respectively, while each internal space is called the fibre. This object is presented in very systematic and detailed way in textbooks to see for example in \cite{12,13}, or is described in connection with application for the gauge theory \cite{14}. It should be noted that in the usual gauge theory the base manifold is the space-time while the total manifold is the configuration space of field given by the associated vector bundle.

The geometry of the $G$-principal bundle is very dramatic that can offer a new way of looking for

\textsuperscript{1} Here, we are only interesting in the application for high energy physics. Others such as in condensed matter theory or statistical physics lead also to the important significance.
deeper structure of the space-time. In this case, the novel and important physical consequences of the \( D \)-dimensional theory will be derived in the natural way. Another interesting point is that effects of the extra dimensions are elegantly hidden from the present experiments on particle accelerators, astrophysics and cosmology.

This paper is organized as follows. In Sec. II, we present relevant geometry ingredients which are base for the study of this framework, and some interesting physical implications inferred from these in great detail. We study the dynamics of pure bulk gravity in Sec. III in which the stabilization potential of the internal volume modulus fields is given by the novel features of the bulk space-time. Consequently, the size of the internal spaces is determined by the dynamics of the theory corresponding with the modulus fields to get the vacuum expectation value (VEV). In Sec. IV, the classical ground state of the bulk space-time is determined, and the perturbative description of the bulk gravity in the low energy limit is investigated around this background. In Sec. V, we show how the 4-dimensional Weyl spinor fields occur naturally on \( B^D \), and their dynamical Lagrangian is constructed consistently. In particular, these results are applied in the realistic model to explain the small mass origin of the observed neutrinos. In Sec. VI, the dynamics of the fields of carrying out the local non-trivial representations of the Lie group \( G \) is considered. We then suggest a candidate for the dark matter coming from such fields. Finally, we devote to conclusions and comments in the last section, Sec. VII.

II. GEOMETRICAL SETTINGS OF HIGHER DIMENSIONAL SPACE-TIME

A. Local coordinate systems

To begin a determination of local coordinate systems on the bulk \( B^D \), we should first analyze its local structure. As has already been proposed, the higher dimensional space-time \( B^D \) is in general factorized locally as the topology product of two spaces, \( U \times G \). Here \( U \) is an open subset of the manifold \( M^4 \) which is smoothly equivalent to an open subset \( V \) of \( \mathbb{R}^4 \) due to \( M^4 \) being the differential manifold.\(^2\) On the other hand, \( B^D \) looks like \( U \times G \) locally but be different from \( M^4 \times G \) globally. Therefore, there exists a diffeomorphism map \( f \) smoothly deforming \( U \times G \) to a local neighbourhood \( \pi^{-1}(U) \) on the bulk which is constructed as an inverse image of the surjective projection

\[
\pi : B^D \longrightarrow M^4, \tag{1}
\]

as follows

\[
f : U \times G \longrightarrow \pi^{-1}(U), \tag{2}
\]

which is called a local trivialization. Through such differentiable maps, it is quite clear to see that the internal spaces are smooth copies of the Lie group \( G \) as already mentioned in the introduction. As a result, there is exactly one-to-one correspondence between a point on \( B^D \) and a pair of two elements \( (x, g) \), with \( x \in U \subset M^4 \) and \( g \in G \), locally. Particularly, with respect the connected Lie group interested in this framework every element \( g \in G \) is expressed uniquely in the exponential map as, \( g = \exp\{i\theta^a T_a\} \), which is not implicitly truth for the non-connected Lie group. The generators

\(^2\) A set of these open subsets \( \{V\} \) provides local coordinate systems on the 4-dimensional manifold \( M^4 \).
$T_a, a = 1, \ldots, D - 4,$ satisfy the following commutation relation of the Lie algebra $\mathfrak{g}$ of the Lie group $G$

$$[T_a, T_b] = i f_{ab}^c T_c,$$  \hspace{1cm} (3)

where $f_{ab}^c$ are the structure constants. A set of $(D - 4)$ real numbers $\{\theta^a\}$ parameterizing each element of the Lie group $G$ defines a smooth manifold. By this, the geometric aspect of the Lie group $G$ is emerged by identifying an element of $G$ with a point in that manifold.

Using the local description of $B^D$ above, coordinates of a point on $B^D$ are thus given as

$$X^M = x^\mu, \quad M = 0, 1, 2, 3,$$

$$X^M = \frac{\theta^a}{\Lambda} \equiv \hat{\theta}^a, \quad M = 4, \ldots, D - 1,$$  \hspace{1cm} (4)

where $\{x^\mu\} \in \mathbb{R}^4$ defines the coordinates for a point $x \in U \subset \mathcal{M}^4$. In the following, the indices $M, N, \ldots$ running from 0 to $D - 1$ are used to denote the higher dimensional indices, the 4-dimensional external indices are associated with $\mu, \nu, \ldots = 0, 1, 2, 3$, and the internal indices are labeled by $a, b, \ldots = 1, \ldots, D - 4$ or $i, j, \ldots = 1, \ldots, D - 4$ as explained below. The coordinates $x^\mu$ are realized as the 4-dimensional external coordinates while $\theta^a$ (or $\hat{\theta}^a$) are realized as the internal coordinates, or fibre ones, which offer obviously the global coordinate system on each internal space. It should be noted that due to $\theta^a$ being dimensionless a new energy scale $\Lambda$ characterizing physically to the internal spaces is taken in a natural way. This energy scale will be responsible for exciting the physical states coming from the extra dimensions. Thus, it should be high enough.

Unlike the rule of the coordinate transformation corresponding with the global coordinate system, that of the local one is produced by the non-empty overlap of any two local neighbourhoods, $\pi^{-1}(U)$ and $\pi^{-1}(U')$, defined by

$$x'^\mu = \Lambda(x)^\mu_\nu x^\nu, \quad e^{i\theta^a T_a} = h(x)e^{i\theta^a T_a}.$$  \hspace{1cm} (5)

Here $\Lambda(x)$ and $h(x) = \exp \{i \alpha^a(x) T_a\}$ are elements of the linear transformation group $GL(4, \mathbb{R})$ and the Lie group $G$, respectively. All of the local coordinate systems on the bulk along with (5) contain the global information about $B^D$. It can be seen in (5) that new internal coordinates are related to old internal coordinates via the left multiplication by $h(x)$. Then $\theta'^a$ are explicitly expressed in terms of $\theta^a$ and $\alpha^a(x)$ as

$$\theta' = -i \ln \left( e^{i\alpha(x)} e^{i\theta} \right)$$

$$= -i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left( e^{i\alpha(x)} e^{i\theta} - 1 \right)^n$$

$$= \theta + \alpha(x) + \frac{i}{2} [\alpha(x), \theta] - \frac{1}{12} [\alpha(x), [\alpha(x), \theta]] + \ldots,$$  \hspace{1cm} (6)

where the internal coordinates and the internal transformation parameters have been written in the compact form as, $\beta = \beta^a T_a$, with $\beta$ referred to $\theta'$, $\theta$ and $\alpha(x)$. The invariant principle of the physical laws requires that higher dimensional physical quantities have to be covariant under the local coordinate transformations above. On the other hand, they are entirely independent of choosing a local coordinate system to describe the physical system leading to the same result for all local observers.
The functions $\Lambda(x)$ and $h(x)$ are usually called transition ones which give us a different perspective to describe geometry of the given bulk space-time. $B^D$ can be constructed by smoothly gluing the local pieces, which each of them is given of the form $V \times G$, with an open subset $V$ belonging $\mathbb{R}^4$, together under an equivalence relation. This identifies any two points of two different local pieces in a region of overlap in the rule. On the other hand, these functions will determine the way of how to twist the local pieces as pasted together. The homotopy groups on $GL(4, \mathbb{R})$ and $G$ which classify respectively the smooth maps of $\Lambda(x)$ and $h(x)$ will establish the topological properties of the bulk. The fact how to paste these local pieces together is completely determined by the sources of matter in the bulk. This means that the present of the sources will tell us about the curvature of the space-time bulk according to the idea of general relativity. However, in order to study systematically the non-trivial geometrical properties of the bulk as well as be convenient in the practical computation, appropriately geometric quantities assigned to the physical significances should be introduced given in later on, such as connection and its curvature, or metric tensor. The homotopy groups mentioned above are then evaluated by integrating the characteristic classes, which are constructed in term of the curvature of the connection, over the base space.

B. The action of the Lie group $G$ on $B^D$

One specially essential property of $B^D$ is that the Lie group $G$ acts freely on it in sense of the active diffeomorphism transformation. This action occurs in the extremely natural way of which does not have to be imposed by hand as well as does not depend of deforming of the bulk under the present of matter sources. It should be emphasized that for many applications in physics, the Lie group often appears as the set of the passive diffeomorphism transformations. For example, the local coordinate transformations are given above. The definition of this action is determined via the right multiplication by an element of the Lie group $G$ on the second factor of the diffeomorphism map given in (2). Since this is often known as the right action. The coordinates of two points on $B^D$ related together under this transformation are thus taken as

$$X^M = (x^\mu, \hat{\theta}^a) \xrightarrow{R_g} X'^M = (x^\mu, \hat{\theta}'^a), \quad (7)$$

where $R_g$ is denoted to the action of an element $g \in G$ on $B^D$, and $\exp\{i\theta^a T_a\} = \exp\{i\theta^a T_a\}g$. It can be easily checked that the definition (7) is independent of choosing of the local coordinate system. This is because the left and right multiplications on the Lie group are completely independent with each other. So it is clearly a good definition. Furthermore, from the definition (7), we can easily see that the orbit of any point of $B^D$ under the action of $G$ is an internal space containing that point. Since this action is transitive on each internal space.

In particular, it can make sense of interesting physics as we will see directly in what follows next. A physical field is called to be invariant under the action of $G$ if its values on the same fibre are be related together by active diffeomorphism transformations acting on its field configuration space which induce the corresponding right action on $B^D$. It is important to note that the action of $G$ is not necessary to remain the dynamical action functional due to that it is not of the symmetric transformation in the usual sense. This means that the Lie group $G$ becomes the symmetry group

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3 In the case that we impose an action functional to be invariant under the action of $G$, as a result, the physics at
with respect to the field function (or field operator for the corresponding quantum version), but being not of that of the corresponding action functional. Therefore, the phenomenology coming from such field appears at different locations on the same fibre to almost be distinct physically. This active action of $G$ will be shown to play an extremely important role for the dynamics of the fields, which are symmetric under $G$, in the internal spaces. Strictly speaking, their internal dynamics is governed by the *intrinsic symmetry nature* under $G$ rather than by the sources. As indicated in the paper [15], the active diffeomorphism invariance is a property of the dynamical theory itself whereas the passive diffeomorphism invariance is a property of the formulation of a dynamical theory.

C. The connection one-form

Each local coordinate system on the bulk given in the previous subsection induces naturally a local basis for the tangent spaces on it presented by

$$\{\partial_M\} = \left(\{\partial_\mu\}, \{\Lambda \partial_a \equiv \hat{\partial}_a\}\right),$$

which is also called the holonomic frame fields. Under the local coordinate transformations (5), these basic vectors rotate as follows

$$\partial_\mu \longrightarrow \partial'_\mu = \partial_{x'^\mu} \partial_\mu + \frac{\partial \hat{\theta}^a}{\partial x'^\mu} \hat{\partial}_a,$$

$$\hat{\partial}_a \longrightarrow \hat{\partial}'_a = \frac{\partial \hat{\theta}^b}{\partial \hat{\theta}^a} \hat{\partial}_b.$$  

(9)

It is direct to see that vectors $\hat{\partial}_a$ transform in the covariant way under the rule of the fibre coordinate transformation. This means that the total of $\{\hat{\partial}_a\}$ under the compatibility condition corresponding to the second transformation of (9) are equally good. Since they provide the well-defined local bases for all of the tangent spaces of the internal spaces on $B^D$. Each of these spaces is also called the vertical tangent subspace defined intrinsically as the kernel of differential map $\pi_*$ induced by the projection map $\Pi$ in the following

$$V_X B^D = \{V \in T_X B^D | \pi_*[V] = 0\},$$

(10)

where $T_X B^D$ and $V_X B^D$ are denoted to the tangent space on $B^D$ and its vertical tangent subspace, respectively, at a point of $B^D$ with the coordinates $X^M = (x^\mu, \hat{\theta}^a)$. On the other hand, the directions along the internal spaces are defined independently of choosing a local basis. A complement part of a vertical vector, called the horizontal vector which is to belong the imagine set of $\pi_*$, however, is not specified with the present information. This is because it is not defined canonically, or depends of a local basis chosen as seen more clearly from the transformation in the first line of (9). An explicitly points related by $G$ is identical. So the physical space-time now becomes $B^D / G = M^4$. This has been considered, for instance, with the 5-dimensional space-time $M^4 \times S^1 / Z_2$ in which the orbifold $S^1 / Z_2$ is derived by identifying physically at points on the circle $S^1$ under the group $Z_2$. 

The geometric description of such vector is taken as a tangent vector of a 4-dimensional submanifold of $B^D$ which is transversal to the fibres. Such submanifold is in general smoothly equivalent to the base manifold $M^4$ locally, and be in fact identified as the usual 4-dimensional world. The horizontal vectors would thus point out the local directions of this world in the bulk space-time $B^D$.\footnote{In the context of the gauge theory, the parallel transport of a physical field of the value in the associated vector bundle along a curve $\gamma(t)$ in the space-time (base manifold) is defined uniquely by moving it along a curve $\pi(t)$ in the total manifold which is derived by the horizontal lift of $\gamma(t)$.}

The only way to determine space of the horizontal vectors at each point on $B^D$ is by introducing an additional structure of connection $\omega(X)$ \cite{12,13}. It is a $\mathfrak{g}$-valued one-form defined globally on $B^D$. This connection is used to geometrically define an unique decomposition of each tangent space on $B^D$ into the following form

$$T_x B^D = H_x B^D \oplus V_x B^D,$$

where the horizontal tangent subspace $H_x B^D$ is determined by the kernel of $\omega$

$$H_x B^D = \{ V \in T_x B^D | \omega(X)[V] = 0 \}. \quad (12)$$

It is important to note that the separation into the direct sum above is not based on the familiar orthogonal approach by the metric tensor not being introduced yet to define a scalar product for the tangent vectors. But, it is done in terms of the kernel and image of the differential operator which exists independently of the metric endowed on $B^D$. In this way the connection $\omega$ plays the role of the projective operator mapping the vertical component of a tangent vector on $B^D$ into an element of the Lie algebra $\mathfrak{g}$. Furthermore, this projection is an isomorphism between each vertical tangent subspace and $\mathfrak{g}$ because of that the internal spaces are smooth copies of the Lie group $G$. Notice that, due to the fact that the tangent spaces on the same fibre are related with each other by a set of the pushforward maps $\{ R_g \}$ induced by the right action, an arbitrary tangent space on $B^D$ will generate all of the remaining ones on the same fibre. The connection $\omega(X)$ is thus required to satisfy the following $G$-invariant condition

$$R^*_g \omega(x, \theta') = g^{-1} \omega(x, \hat{\theta}) g, \quad (13)$$

where $R^*_g$ is the pullback map induced by the right action $R_g$ with $g \in G$, and $\exp \{ i \theta^\alpha T_a \} = \exp \{ i \theta^\alpha T_a \} g$.

It is convenient to take the connection explicitly in a local neighbourhood $\pi^{-1}(U)$ as follows

$$\omega(X) = g^{-1} i \pi^* A(x) g + g^{-1} d g$$

$$= i g^{-1} T_a g (d \theta^\alpha + g_\alpha A^\mu_\alpha(x) dx^\mu), \quad (14)$$

where $g = e^{i \theta^\alpha T_a}$, $d$ is the exterior differential operator, the pullback map $\pi^*$ which is induced by the projection operator $\pi$ acts on $A(x) = g_\alpha A^\mu_\alpha(x) T_a dx^\mu$ which is a $\mathfrak{g}$-valued one-form defined on a local neighbourhood $U$ of the base manifold $M^4$ in which $A^\mu_\alpha$ with the dimension $[A^\mu_\alpha] = 1$ are called the gauge fields in the terminology of the physical literature, and $g_\alpha$ is the dimensionless gauge coupling constant characterizing the strength of the interaction mediated by the $A^\mu_\alpha$. This will be seen more clearly below. It is interesting that a set of one-forms, $\{ d \theta^\alpha + g_\alpha A^\mu_\alpha dx^\mu / \Lambda \}$, specify a local basis for
the dual spaces of the vertical tangent subspaces. The connection \( \omega \) taken in two different local
neighbourhoods coincides together in an intersection region leading the gauge transformation for
\( A^a_\mu \) as

\[
A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} \left[ h(x)A_\nu(x)h(x)^{-1} + \frac{1}{i g_i} h(x)\partial_\nu h(x)^{-1} \right],
\]

where \( h(x) \) is given in (5), and \( A_\mu = A^a_\mu T_a \). Eq. (15) leads to that the one-forms,
\( d\hat{\theta}^a + g_i A^a_\mu dx^\mu/\Lambda \), are covariant under the rule of the fibre coordinate transformation. Since the total of the set of these
one-forms form the local vertical coframes well-defined. We can see that this transformation rule is to
combine of both the 4-dimensional external coordinate and the usual gauge transformations.\(^5\) This
is due to that it has an origin from the local coordinate transformations of the higher dimensional
space-time. It is also interesting to note that as analyzed in references \([16, 18]\) the coordinate
transformation can be combined with the gauge transformation to form a gauge-covariant coordinate
transformation. It can be verified that an infinitesimal variation of \( A^a_\mu \) is taken as

\[
\delta A^a_\mu(x) = -\partial_\nu \epsilon^\nu(x) A^a_\mu(x) - \epsilon^\nu(x) \partial_\nu A^a_\mu(x) + f^{ab}_c \alpha^b(x) A^c_\nu(x) - \partial_\mu \alpha^a(x),
\]

where infinitesimal parameters \( \epsilon^\nu(x) \) and \( \alpha^a(x) \) are defined as, \( x'^\mu = x^\mu + \epsilon^\mu(x) \) and \( \exp\{i\alpha^a(x)T_a\} \cong 1 + i\alpha^a(x)T_a \), respectively. The first two terms in Eq. (16) come from the infinitesimal transformation
of the 4-dimensional external coordinates while the remaining ones present the usually infinitesimal
gauge transformation.

In order to see much more clearly about the geometrical significance of (12), let us consider a curve
in \( B^D \) which is expressed in a local neighbourhood as, \( \gamma(t) = (x^\mu(t), \hat{\theta}^a(t)) \), flowing transversely to
the fibres. This means that all of its tangent vectors denoted by \( V(t) \) are horizontal. They are thus
annihilated by the connection, \( \omega[V(t)] = 0 \), leading to the following relation

\[
\frac{dg(t)}{dt} = -i g_i A^a_\mu T_a \frac{dx^\mu(t)}{dt} g(t),
\]

where \( g(t) = e^{i\theta^a(t)T_a} \). Then, we find

\[
\frac{d\theta^a(t)}{dt} = -g_i A^a_\mu \frac{dx^\mu(t)}{dt},
\]

that results in the formal expression for \( V(t) \) as

\[
V(t) = \frac{dx^\mu(t)}{dt} \partial_\mu + \frac{d\theta^a(t)}{dt} \hat{\theta}_a
= \frac{dx^\mu(t)}{dt} \left( \partial_\mu - \frac{g_i A^a_\mu}{\Lambda} \hat{\theta}_a \right) \equiv \frac{dx^\mu(t)}{dt} \hat{\mu}.
\]

\(^5\) In the usual gauge theory, all of the fields are to live on the base manifold \( M^4 \) which is the 4-dimensional Minkowski-
flat space-time. In this case, there exists of course the global coordinate systems on \( M^4 \) in which the group of the
coordinate transformation is the conventional Poincaré group. Since we will have the familiar gauge transformation
rule for \( A^a_\mu \). It is noticed that by the pullback map \( \pi^* \) the gauge fields \( A^a_\mu \) in our framework are to lay on the
total manifold \( B^D \). This leads to that the transformation (13) will in general not agree with the usual gauge
transformation even if \( M^4 \) being Minkowski-flat.
The vectors $\hat{\partial}_\mu$ are covariant under the 4-dimensional external coordinate transformation resulting from Eq. (15). Thus all of $\{\hat{\partial}_\mu\}$ will serve the equally good local frames for whole horizontal tangent subspaces on the bulk. A dual basis for $\{\hat{\partial}_\mu\}$ is easily found as $\{dx^\mu\}$. On the other hand, the local directions of the usual 4-dimensional space-time in $B^D$ are now completely defined with the present of the gauge fields $A^a_\mu$.

It is worth remarking an important point in the analysis presented above that the usual 4-dimensional directions to be transversal to fibres found by the gauge fields $A^a_\mu$ do not break explicitly the higher dimensional covariance. In much previous scenarios of the higher dimensional space-time extension, these directions are often defined by the present of 3–brane. However, this breaks explicitly the higher dimensional Poincaré symmetry

$$\mathbb{R}^{1,D-1} \times SO(D-1,1) \rightarrow \mathbb{R}^{1,3} \times SO(3,1) \times SO(D-4),$$

where the first factor corresponds to translational symmetries and the Lorentz group along the 3-brane, and the second one does to the rotation group along $(D-4)$ extra dimensions which are transverse to 3-brane.\(^6\) Indeed, this result is very important to point out that the external and internal directions in $B^D$ are exactly independent together in the non-trivial geometry structure rather than in the trivial sense of the global Cartesian product. It thus leads to a very interesting consequence of that unlike three usual space-directions the space-like internal directions are completely split up the time-directions. This implies that the speed of the dynamical propagating in the internal spaces is in principle not unlimited by the speed of light $c$ which is the largest speed for that in the 4-dimensional external spaces. Another one is that any vector field $V(X)$ on $B^D$ is always written in horizontal and vertical components in the following form

$$V(X) = V^\mu_H(X)\hat{\partial}_\mu + V^a_V(X)\hat{\partial}_a.$$

It should be noted that $V^\mu_H(X)$ and $V^a_V(X)$ do not mix with each other, so they are thought of as the independently physical fields at the fundamentally higher dimensional level. It is also straightforward to generalize that for a general tensor field on the bulk in which there will have additionally hybrid fields including both the horizontal and vertical indices.

We have seen that the gauge fields $A^a_\mu$ gives the principled way of how to move from an internal space to another. Since an arbitrary particle moving along the 4-dimensional external directions of the bulk may be coupled with $A^a_\mu$ which does not exist clearly in the usual gauge theory. This is

\(^6\) In this case, 3–brane is to correspond with the rigid object. Consequently, the amplitude for a scattering process between two particles which live on 3-brane exchanged by gauge bosons propagating into the bulk will be divergent even at tree level by the contribution of the KK gauge boson modes. If, however, 3-brane is flexible, then braons describing the fluctuations of 3-brane in the extra dimensions will be included in the computation of the amplitude. So contribution from the KK gauge boson modes is automatically suppressed by an exponential factor \(^{19}\). In fact, the braons are to correspond with the Goldstone bosons of translational invariance in extra dimensions broken spontaneously \(^{20,22}\). The description of the effective theory for the flexible 3-brane universe is given in more detail by the work \(^{23}\). Some works also considered that this translational invariance is possible an approximate symmetry, so some braons can have non-zero masses. They always couple by pairs, since the lightest braon is stable. It would also be difficult to detect them in observations. Thus the braon can provide natural candidates to the dark matter \(^{24}\). Braon phenomenology at the LHC is given in Ref. \(^{25}\).
easily seen by the appearance of the gauge fields $A^a_\mu$ in the derivatives $\partial_\mu$ implying that the field propagating in the bulk having non-zero derivatives in the internal variables would interact to $A^a_\mu$. It is interesting to interpret that corresponding gauge charges are generated dynamically through the physical kinetics in the internal spaces. This means that such interaction is of the manifestation of properties of the internal dynamics. In addition to such coupling perspective, there will be known the coupling occurring in the context of the usual gauge theory in which the gauge fields $A^a_\mu$ are coupled to the fields of carrying out non-trivial representations of the group $G$, as we will discuss in Sec. [VI]. Eventually, the connection one-form is realized as the elegant geometrical object in the mathematical point of view while the gauge fields $A^a_\mu$ carry out the interesting physical significance as force-carrying particles.

**D. Curvature of connection one-form**

What we have already seen in the last subsection is the natural existence of the connection one-form expressed locally in the gauge fields $A^a_\mu$ lying $B^D$. Their physical interpretation is specially interesting to allow the determination of the 4-dimensional external directions as well as lead to the interaction. Next, we are going to introduce a curvature two-form $\Omega$ for this given connection. It is a totally antisymmetric rank-2 covariant tensor with values of $g$. The present of the curvature would provide much more important information to better understand the non-triviality of the bundle $B^D$. Notice that, the curvature two-form is also defined globally over the whole bulk, and in the same internal space it must satisfy the $G$-invariant condition in analogy with that of the connection one-form $\omega$.

Then $\Omega$ is defined by the Cartan’s structure equation as

$$\Omega = d\omega + \omega \wedge \omega,$$  \hspace{1cm} (22)

where $\wedge$ is the wedge product on the space of the differential forms living the bulk. From this definition, one can show that $\Omega$ acts on any pair of horizontal vectors to yield an element of $g$ but vanishing for the remaining cases. In other works, only the horizontal components of the curvature are not zero in which they are defined explicitly as follows

$$\Omega_{\mu\nu}(X) = ig_i e^{-i\theta}F_{\mu\nu}(x)e^{i\theta},$$ \hspace{1cm} (23)

where

$$F_{\mu\nu}(x) = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_i f^{a}_{bc}A^b_\mu A^c_\nu)T_a,$$ \hspace{1cm} (24)

is well-known as the usual Yang-Mills field strength tensor. It is important to emphasize that above explicit expression will be very useful to compute in practice as seen later in which kinetic term for the gauge fields $A^a_\mu$ is constructed in term of the curvature two-form. In an overlap region between any two neighbourhoods, the curvature $\Omega$ has to obey, $\Omega = \Omega'$, leading to the rule of the gauge transformation for $F_{\mu\nu}$ as follows

$$F'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} h(x)F_{\rho\lambda}(x)h^{-1}(x),$$ \hspace{1cm} (25)

where $h(x)$ is given in (5).
The geometrical interpretation of the curvature $\Omega$ can be taken in several different ways. However, it is almost given to be similarly to that of the Riemann curvature tensor which measures the non-commutativity of the parallel transport around a loop on the base space. Let us consider the parallel displacement of a field with values of the total space along a small loop of an infinitesimal parallelogram on the base space. This is performed via a horizontal lifting curve in the total space which is in general not closed. The process failure to close is measured by the curvature to be proportional to $F_{\mu\nu}$. The curvature thus would tell us how to distinguish a given connections on $B^D$ that is sufficiently different to the flat connection, in which its curvature vanishes $F_{\mu\nu} = 0$, corresponding with the trivial bundle $B^D$.

**E. Global frame of the vertical tangent subspaces**

An interesting result derived from the geometry of the bulk is that the group Lie $G$ acting smoothly and freely on $B^D$ induces naturally so-called fundamental vector fields defined globally on it. By the right action of $G$, we can define a flow on the bulk given locally as, $f(x, e^{i\theta})e^{iAt}$, with the local trivialization $f$ and $A$ to be an element of $\mathfrak{g}$, which lies clearly within an internal space. It is also of the one-parameter group of transformations on $B^D$. Value of such a vector field is determined at a point with the coordinates, $X^M = (x^\mu, \hat{\theta}^a)$, by

$$V(X) = \frac{d}{dt} \left[ f(x, e^{i\theta})e^{iAt} \right] \bigg|_{t=0},$$

which is obviously a vertical vector generated by $A$. In this way, we can construct any fundamental vector field corresponding with an element of $\mathfrak{g}$ by assigning $V(X)$ at each point on $B^D$ via a set of pushforward differential maps $\{R_g\}$ acting on the vector given in Eq. (26). This construction leads to such vector fields to be invariant under $G$. All of them form a vector space preserving the structure of the Lie algebra $\mathfrak{g}$, thus, it is isomorphic to $\mathfrak{g}$. It can be argued that this space is thought of as an extension of the space of the left-invariant vector fields on the Lie group $G$ in the case of the non-trivial bundle $B^D$.

The basic vector fields of the space of the fundamental vector fields denoted by $K_i(X)$ ($\equiv \hat{\partial}_i$), with $i = 1, 2, ..., D - 4$, satisfy the following Lie bracket

$$[K_i(X), K_j(X)] = f^k_{ij}K_k(X),$$

where $f^k_{ij} = \Lambda f^k_{ij}$ are the structure constants of this space with $f^k_{ij}$ being those of $\mathfrak{g}$. These structure constants can also realized as a tensor of type $(1, 2)$ transforming under the global rotation of the frame fields $K_i$ read

$$K_i(X) \longrightarrow K'_i(X) = A^j_iK_j(X),$$

where $A^j_i$ is in general a constant matrix of the general linear group $GL(D - 4, \mathbb{R})$. We should note that each vertical tangent subspace is also isomorphic to $\mathfrak{g}$ due to that the internal spaces are to deform smoothly of the Lie group $G$. It is thus very important that $\{K_i\}$ constitutes a global frame for the whole vertical tangent subspaces on $B^D$.

These frame fields are written explicitly in a local neighbourhood as

$$K_i(X) = K_i^a(X)\hat{\theta}_a.$$
Here $K_i^a(X)$ is dimensionless, and it transforms as a covector under the rule of the fibre coordinate transformation and as a vector under the transformation. The fact that a horizontal vector field is transported along an flow generated by a fundamental vector field remains again another one. On the other hand, we must have the condition on the following Lie derivatives, $[V^H, K_i(X)] \in H_X B^D$, for $V^H$ to be a horizontal vector field and all of $i = 1, ..., D - 4$. Thus, this results in the equations for $K_i^a(X)$ that are derived from the Lie derivatives of $\hat{\partial}_\mu$ along each $K_i(X)$ to vanish identically as

$$\left(\partial_\mu - \frac{g_i A^b_\mu}{\Lambda} \hat{\partial}_b\right) K_i^a(X) = 0. \quad (30)$$

These first-order partial differential equations may be considered as the equations describing the evolution of the frame fields $K_i(X)$ under the dynamics of the bulk space-time. It is important to be emphasized that these equations are properly defined in a pure geometric way rather than from the equations of motion obtained by varying the corresponding action. This means that the fields $K_i(X)$ would be not realized as the physical degrees of freedom in the usual sense, but are the geometric axial fields. Nevertheless, their effects on the physical phenomenology would be shown in indirect ways.

The fundamental vector fields are used to describe the infinitesimal displacement behaviour of the action of the Lie group $G$ on $B^D$. It is expressed in terms of parameters of the infinitesimal transformation $\varepsilon^i$ as

$$x'^\mu = x^\mu, \quad \theta'^a = \theta^a + \varepsilon^i K_i^a. \quad (31)$$

The flows of $K_i(X)$ running along the internal spaces lead to the infinitesimal displacements obviously only performed along these spaces. Since the 4-dimensional external coordinates in the same local neighbourhood do not change. Notice also that, $\varepsilon^i$ are independent on the bulk coordinates because the action of the Lie group $G$ is defined globally. In the modern Kaluza-Klein theories, the local isometry on the internal space requires $\varepsilon^i = \varepsilon^i(x)$. Such transformation thus provides a geometrical origin for the local gauge transformation in the 4-dimensional point of view. The second equation of Eq. (31) suggests that $K_i(X)$ play the role of the infinitesimal generators corresponding with the active diffeomorphism transformations resulting from the action of $G$ on the bulk. With respect a physical field $\psi$ which is diffeomorphism invariance under the action of $G$, then all Lie derivatives of $\psi$ along the fields $K_i$ vanish

$$\mathcal{L}_{K_i} \psi = 0. \quad (32)$$

Here the Lie derivatives are defined for general geometrical objects rather than only the usual tensor fields on $B^D$. The vector fields are characterized by Eq. (32) to point out the locally symmetric directions for the corresponding field in the internal spaces. A specially interesting case in which the right-hand of Eq. (32) is not vanished but is proportion to $\psi$ will lead to the conformal diffeomorphism invariance for $\psi$ under $G$.

Let us introduce dual fields of $K_i(X)$, denoted by $K^i(X)$, satisfying the following conditions

$$\langle K^i, K_j \rangle = \delta^i_j. \quad (33)$$

which form a global basis of the internal one-form fields on $B^D$. Their local form is taken as

$$K^i(X) = K^i_a(X) \left( d\hat{\theta}^a + \frac{g_i A^a_\mu d\chi^\mu}{\Lambda} \right), \quad (34)$$
where $K^i_a(X)$ transform as a vector under the rule of the fibre coordinate transformation and as covector under the rotation of $K_i$ given in (28). With the help of both (27) and (30) equations, it can be easily verified that $K^i(X)$ obey the equations extended from the usual Maurer-Cartan’s structure equations as follows

$$dK^i = -\frac{1}{2} f_{jk}^i K^j \wedge K^k + \frac{g_i}{2\Lambda} F_{\mu\nu} dx^\mu \wedge dx^\nu,$$

(35)

where $F_{\mu\nu}^i$ is defined in terms of $A^i_\mu = K^i_a A^a_\mu$ of the following form

$$F_{\mu\nu}^i = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g_i f_{jk}^i A^j_\mu A^k_\nu.$$  

(36)

$F_{\mu\nu}^i$ can be interpreted in similar way as the components of the Yang-Mills field strength tensor of the new gauge fields $A^i_\mu$. The geometrical interpretation of $F_{\mu\nu}^i$ is to measure the failure of integrability of the horizontal distribution because the Lie bracket of any two horizontal frame fields is given as

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = -\frac{g_i}{\Lambda} F_{\mu\nu}^i K_i,$$

(37)

which leads to a vertical vector. In the case of all the components $F_{\mu\nu}^i$ vanishing, the horizontal distribution is integrable since by the Frobenius theorem the higher dimensional space-time $B^D$ is also foliated by the 4-dimensional external submanifolds.

III. BULK GRAVITY

In the preceding section the fundamentally geometrical ingredients underlying the higher dimensional space-time has been established. In what follows of this section we are going to investigate systematically the aspects of the higher dimensional gravity.

The fundamental variables describing the dynamics of the gravity are as usual related to the geometrical quantities of the space-time manifold. In this way the space-time is a geometrically dynamical object and its curvature should be changed under the present of sources. In the minimal formalism the gravitational degrees of freedom for our purpose consist of the gauge fields $A^a_\mu$ together with the metric tensor equipped on the bulk manifold. An ansatz for the bulk metric is given locally in the most general form as follows

$$ds^2_B = g_{\mu\nu} dx^\mu dx^\nu - \gamma_{ij} K^i K^j,$$

(38)

which describes separately two infinitesimally invariant intervals corresponding with the horizontal and vertical ones. The metrics $g_{\mu\nu}$ and $\gamma_{ij}$ are thought of as the external and internal gravity fields, respectively, that in general depend on the bulk coordinates. Both of them are non-degenerate and symmetric rank-2 covariant tensor fields. The signature of the metric $g_{\mu\nu}$ is chosen as, $(+ - - -)$, since the sign “−” in the second term of Eq. (38) refers the space-like nature of the internal spaces. This means that the metric $\gamma_{ij}$ provides exactly a scalar product of positive definition on each vertical tangent subspace. The internal metric $\gamma_{ij}$ derived in this way can be interpreted in sense of the volume modulus fields which set dynamically the size of the internal spaces. They are obviously global degrees of freedom. This is of special interest in our purpose in which the dynamic treatment
of the internal gravity would just be much simpler than that with local degrees of freedom as seen later on. As well-known in the description of the gravity interaction by local variables, the curvature of the space-time manifold at each point is not determined completely by the energy and momentum tensor flowing through each point for dimensions being larger than three. The reason for this is that there are not enough constraint equations to determine all of local degrees of freedom. All horizontal indices are lowered and raised with the external metric $g_{\mu \nu}$ and its dual $g^{\mu \nu}$, respectively, whereas the internal metric $\gamma_{ij}$ and its dual $\gamma^{ij}$ do also too for all vertical indices. In addition to the fields created the bulk gravity, we also have the geometric axial field $K^i_a$, or its dual $K_{ia}$. We can make an interesting observation that only the gauge fields $A^a_{\mu}$ are the gravitational degrees of freedom being independent of the smooth structure on $B^D$. Note that the description of the gravitational dynamics being independent of the background of the space-time in sense of the gravity as the gauge theory has been developed by many works, the interesting readers can see in [26] where the usual metric is generated effectively.

Under this description, all of them are manifestly independent with each other. Thus they are realized as the fundamental fields unified in the same geometrical structure of the higher dimensional space-time. This suggests another viable approach to the treatment of the force unification. This is perfectly different to the usual higher dimensional gravity whose fundamentally dynamical degrees of freedom are described by the higher dimensional metric tensor $G_{MN}$. In such case, the bulk metric $G_{MN}$ is possible to be parameterized in a convenient way in terms of a 4-dimensional metric tensor, $n$ 4-dimensional vectors, and $n(n+1)/2$ scalars, with $n$ extra-dimensions. However, they would not be realized as independent fields at the higher dimensional level. These are only derived effectively by the dimensional reduction through a compactification mechanism of extra dimensions. The 4-dimensional gravity action in the effective theory then includes the physical degrees of freedom associated with the KK towers of such 4-dimensional fields. We should note that both of two scenarios in general are of equal independent components in which depending on symmetry presenting in theory some unphysical degrees of freedom are particularly removed by fixing gauge.

It should be emphasized that beside the above indicated fields, in particular, new degrees of freedom may occur in a remarkable way in the higher dimensional extensions of the gravity. For example, the contorsion field which is considered even in four dimensions will be discussed below. An alternative possible is of the $(p+1)$-form gauge fields $(p > 0)$ [27], the generalization of the conventional 1-form gauge fields, which occur almost in supergravity theories. Their sources are the charged objects in the higher dimensional space-time, $p$-branes. One of these, the rank-2 antisymmetric Kalb-Ramond (KR) field [28] effectively arising from a massless excitation in the heterotic string theories whose source is the fundamental string, was considered in the ADD [29] and Randall-Sundrum [30] compactification scenarios. Its geometric interpretation is specially interestingly known as the space-time torsion.

We have seen from the previous analysis that the internal coframe fields $K^i$ are defined globally on the bulk. Since the internal metric on $B^D$ enjoys naturally an interesting one where $\gamma_{ij}$ is a constant matrix with respect these fields. Note that one may choose others so that $\gamma_{ij}$ would be diagonal. In principle, this process is completely taken because $\gamma_{ij}$ is symmetric and the group rotating the coframe fields $K^i$ is in general the linear transformation group $GL(D-4, \mathbb{R})$. It thus
leads to the metric $\gamma_{ij}$ that may be expressed as follows

$$\gamma_{ij} = v_i \delta_{ij},$$

where $v_i$ are all positive constants, and $\delta_{ij}$ is the Kronecker delta function. Furthermore, the internal metric just derived above may also be re-scaled by an appropriately conformal factor in order for one of the components $v_i$ to be equal of the constant value, 1. Thus, without loss of generality we can choose $v_i = 1$ for $i = D - 4$. It is very important to note that the above given conclusion would be incorrect if a manifold is impossible to define the global coframe fields. This is because of the metric in such case carrying out local degrees of freedom which would be changed under the local coordinate transformation. An interesting interpretation of $(D - 5)$ undetermined values $v_i$, $i = 1,...D - 5$, is that they should be fixed dynamically corresponding with VEVs of the modulus fields. This can be thought of as the physical compactification mechanism stabilizing the size of the internal spaces. Correspondingly, in the present situation we would like to study the dynamics of the internal gravity with the scheme to be as economical as possible in which the relevant dynamic fields are given in the following form

$$\gamma_{ij}(X) = e^{\phi_i(X)} \delta_{ij},$$

due to the fact that $\gamma_{ij}$ is positive definition as mentioned above, here $\phi_{D - 4} = 0$. The transformations given in Eq. (28) preserving the form (40) correspond with taking a dilation transformation on $\phi_i(X)$ with $i = 1,...,D - 5$, as

$$\phi_i(X) \rightarrow \phi_i(X) + \lambda_i,$$

where $\lambda_i$ are constants. For $i = 1,...,D - 5$, $v_i$ are expected to relate with VEVs of the modulus fields $\langle \phi_i \rangle$, as, $v_i = e^{\langle \phi_i \rangle}$. Furthermore, we consider the modulus fields being independent of the internal coordinates, $\phi_i = \phi_i(x)$, which correspond with the metric on each internal space to be invariant under the action of $G$. As we will see later on, the internal metric under consideration is consistent with that at the ground state of the bulk.

To evaluate a change of any tensor field on the bulk space-time along a vector field, we need to introduce a linear connection to construct an operator of the covariant derivative. It is interesting that there exist many possible connections on the foliated manifolds as given in [11]. However, the fact that connection will be considered in the present work is torsion-free (or symmetric) and compatible with metric, also well-known as the Levi-Civita connection. It is determined uniquely on a (pseudo-) Riemannian manifold. Dealing with non-symmetric metric connection has received some interesting results in four-dimensions as well as higher-dimensions in which new degrees of freedom coming from contorsion field would be included into. In general, the coefficients of the given connection and components of the Riemann curvature tensor are taken exactly in the terms of the bulk metric $G_{MN}$ and non-holonomic functions $C^P_{MN}$ as

$$\Gamma^P_{MN} = \frac{G^{PQ}}{2}(\partial_M G_{NQ} + \partial_N G_{MQ} - \partial_Q G_{MN}) + \frac{C^{PQ}}{2}(C_{QMN}^O G_{ON} + C_{QNM}^O G_{OM}) + \frac{C^P_{MN}}{2},$$

$$R^O_{MPN} = \partial_N [\Gamma^O_{PM}] - \partial_P [\Gamma^O_{NM}] + \Gamma^Q_{PM} \Gamma^O_{NQ} - \Gamma^Q_{NM} \Gamma^O_{PQ} + C^Q_{PN} \Gamma^O_{QM}.$$  

Note that, in the non-holonomic basis the torsion-free condition of the metric connection does not result in the usual symmetric condition, $\Gamma^P_{MN} = \Gamma^P_{NM}$. However, in the natural or holonomic basis,
\((\{\partial_\mu\}, \{\hat{\partial}_i\})\), in which all of the factors \(C^P_{MN}\) vanish identically, then we will get again the familiar expressions for \(\Gamma^P_{MN}\) as well as \(R^R_{MPN}\). It is convenient to adapt the basic, \((\{\hat{\partial}_\mu\}, \{\hat{\partial}_i\})\), where explicit expressions for \(C^P_{MN}\) and \(\Gamma^P_{MN}\) are given in the appendix. It is important that with the Levi-Civita connection on \(B^D\) the covariant derivative of a horizontal vector in another one would in general not result in a vector of the horizontal tangent subspace, and similarly neither do vertical vectors.

For our setup, the dynamics of the pure bulk gravity is governed by the classical action to be invariant under both the local coordinate transformations (5) and the global transformation (28) taking the form

\[
S_G = M_*^{D-2} \int_{B^D} d^D X \sqrt{|G|} \left[ \frac{1}{2g^2_M} \text{Tr} (\Omega_{\mu\nu} \Omega^{\mu\nu}) + R - V(\phi_i) \right].
\]

(43)

The fundamental Planck scale \(M_*\) determines the scale of the higher dimensional quantum gravity. \(G = (-1)^{D-4}gK^2\gamma\), with \(K = \det(K^{\alpha}_{\alpha})\), is the determinant of the bulk metric expressed through that of the external and internal metrics, denoted by \(g\) and \(\gamma\), respectively. The trace operator, \(\text{Tr}\), refers to the non-degenerate ad-invariant inner product considered in our present paper on the Lie algebra \(g\). \(\Omega^{\mu\nu} = g^{\mu\rho}g^{\nu\lambda}\Omega_{\rho\lambda}\), are the contravariant components of the curvature \(\Omega\). The action \(S_G\) above possesses the first term with quadratic curvature realized as the term of the Yang-Mills type for the gauge field \(A^a_\mu\) in a natural way. The scalar curvature \(R\) constructed as follows

\[
R = G^{MN}R_{MN} = g^{\mu\nu}R^M_{\mu\nu\rho} - \gamma^{ij}R^M_{ijM},
\]

(44)

presents the higher dimensional Einstein-Hilbert action whose explicit expansion is taken in the appendix. In the above given action we have introduced the potential for the modulus fields \(V(\phi_i)\) that can also be interpreted as an extension of \(\phi_i\)-dependence of bulk cosmological constant. The present of this term and the first one is thought of as the crucial nature of the present framework.

The structure of the potential \(V(\phi_i)\) constructed in such a way that the indices of \(\bar{f}_{jk}\), the internal metric and its dual are contracted with each other has a remarkable form as follows

\[
V(\phi_i) = \Lambda_B + \left( a_1 f^{k}_{ij}f^{l}_{k}i_j \gamma^{ij} + a_2 f^{i}_{ik}f^{j}_{k}j_{il} \gamma^{ij} \gamma^{kl} \gamma^{pq} \right)
\]

\[
= \Lambda_B + \Lambda^2 \left( a_1 \sum_i f^{k}_{iij}f^{f}_{ikl}e^{-\phi_i} + a_2 \sum_{i,k,j} (f^{k}_{ij})^2 e^{\phi_k - \phi_i - \phi_j} \right),
\]

(45)

where \(\Lambda_B\) is a bulk cosmological constant, and \(a_{1,2}\) are dimensionless coupling constants. It is very important to see that an essential point to give rise \(V(\phi_i)\) is the structure constants of Lie algebra \(g\). In the usual way to generate a potential for the metric field we may only contract the indices of the metric to those of its dual without derivatives leading to cosmological constant contribution. Therefore, the above given potential would be a trivial constant with respect internal manifolds that do not correspond to smooth deforming of the non-Abelian Lie groups. Such manifolds have been most commonly studied in the previously higher dimensional theories such as \(n\)-torus \(T^n\) or two-sphere \(S^2\). As will be shown below, \(V(\phi_i)\) is also added on other contributions coming from the bulk determinant and the scalar curvature \(R\) to generate a complete potential of the moduli stabilization.
Using the explicit expression for $\Omega_{\mu\nu}$ derived in the previous section and choosing a basis of $g$ so that, $\text{Tr}(T_aT_b) = \delta_{ab}/2$, the first term in the action (43) is then taken as

$$S_G^1 = -\left(\frac{M_s}{\Lambda}\right)^{D-4} \int_{B^D} d^4x d^{D-4}\theta \sqrt{|G|} \frac{F_{\mu\nu}F^{\mu\nu}}{4}. \quad (46)$$

The expression under the integral is just the conventional Yang-Mills Lagrangian but for the gauge fields $A^a_\mu$ lying on the total space $B^D$ rather than on the base space. To get the standard form, the factor $(M_s/\Lambda)^{D-4}$ should be absorbed by normalizing the gauge fields $A^a_\mu$ which will lead to new gauge coupling

$$g = \left(\frac{\Lambda}{M_s}\right)^{D-2}, \quad (47)$$

that remains dimensionless. It is thus clear to say that the higher dimensional gravity includes the 4-dimensional gauge interaction of the symmetric group $G$ mediated by the gauge fields $A^a_\mu$. In calculation of physical processes we have to add on the gauge fixing and ghost terms for $A^a_\mu$.

Now let us be straightforward to get explicit terms governing the dynamics of the external gravitational field and the modulus fields. The corresponding action is obtained by expanding the last two terms in the action (43) given up to total derivatives as

$$S_G^2 = \int d^4x d^{D-4}\theta \sqrt{|g|} K \left[ \frac{M^2 g}{M_P} \frac{e^{\phi/\sqrt{M_6}}} {\bar{M}_P} \mathcal{L}_g + \mathcal{L}_{\bar{\phi}} = \frac{g_2^2}{4} \left( \frac{M_P}{\Lambda} \right)^2 \sum_i e^{(\phi+\bar{\phi})/\sqrt{M_P}} F_{\mu\nu}^i F^i_{\mu\nu} \right], \quad (48)$$

$$\mathcal{L}_g = \hat{R} + \frac{1}{4} \left( \hat{\partial}_\mu g^{\mu\nu} \hat{\partial}^\nu g_{\mu\rho} - g^{\mu\lambda} g^{\rho\nu} \hat{\partial}_\mu g_{\nu\rho} \right), \quad (49)$$

$$\mathcal{L}_{\bar{\phi}} = -\frac{\sqrt{M_P}}{2} \sum_i \hat{\partial}_\mu \bar{\phi}_i \hat{\partial}^\mu \bar{\phi}_i - \bar{V}(\bar{\phi}_i). \quad (50)$$

The contravariant partial derivatives are defined by

$$\hat{\partial}^\mu = g^{\mu\nu} \hat{\partial}_\nu, \quad \hat{\partial}^i = \gamma^{ij} \hat{\partial}_j = e^{-\phi/\sqrt{M_P}} \hat{\partial}_i, \quad (51)$$

with $\bar{M}_P^2 = M_6^{D-2}/\Lambda^{D-4}$. We have normalized the modulus fields as

$$\bar{\phi}_i = \bar{M}_P \phi_i, \quad (52)$$

which have the dimension of mass, and $\bar{\phi} = \sum_i \bar{\phi}_i/2$. The dynamics of $g_{\mu\nu}$ and $\bar{\phi}_i$ in $B^D$ is determined by $\mathcal{L}_g$ and $\mathcal{L}_{\bar{\phi}}$, respectively. The last term in the action (43) plays the coupling role in the high energy, but will contribute an induced kinetic energy term for the gauge fields $A^a_\mu$ in the low energy limit after integrating over the fixed volume of the internal spaces. The 4-dimensional standard scalar curvature $\hat{R}$ is given to associate with the 4-dimensional metric $g_{\mu\nu}$ as

$$\hat{R} = g^{\mu\nu} \left( \hat{\partial}_\nu \Gamma^\lambda_{\mu\rho} - \hat{\partial}_\lambda \Gamma^\nu_{\mu\rho} + \Gamma^\rho_{\lambda\mu} \Gamma^\mu_{\nu\rho} - \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\rho\mu} \right), \quad (53)$$

where $\Gamma^\lambda_{\mu\nu}$ are the horizontal components of the Christoffel symbols $\Gamma^P_{MN}$. Therefore, it consists of two parts on which one presents the external kinetic energy term for $g_{\mu\nu}$ or the conventional 4-dimensional Einstein-Hilbert term, and another presents the interaction between $g_{\mu\nu}$ and the gauge.
The potential \( \tilde{V}(\tilde{\phi}_i) \) is taken as

\[
\tilde{V} = (\Lambda M_P)^2 e^{\tilde{\phi}/M_P} \left[ \frac{\Lambda B}{\Lambda^2} + \left( a_1 - \frac{1}{2} \right) \sum_i f_{ik} f_{kl} e^{-\tilde{\phi}_i/M_P} + \left( a_2 + \frac{1}{4} \right) \sum_{i,k,j} (f_{ij})^2 e^{\tilde{\phi}_k - \tilde{\phi}_i - \tilde{\phi}_j}/M_P \right].
\]

(54)

Note that the terms corresponding to the factors, \(-1/2\) and \(1/4\), in parentheses are of the scalar curvature of the internal spaces.

In the action (48), the present of the conformal factor \( e^{\tilde{\phi}/M_P} \) defining volume of the internal spaces leads to the non-minimally coupled modulus fields to the 4-dimensional tensor gravity field. It is well known as working on the so-called Jordan frame for \( g_{\mu\nu} \). This factor also results in the kinetic energy term of the fields \( \tilde{\phi}_i \) having the non-canonical form. Thus, it is often referred to the coupling parameter. The effective four-dimensional Plank scale will be fixed by this factor after the modulus fields get VEVs. It is interesting to observe in the case of without the gauge fields \( A^a_\mu \) that the action \( S^2_{\tilde{G}} \) above can be recognized as a more general variant of the Brans-Dicke theory. By performing an appropriate Weyl rescaling on the metric \( g_{\mu\nu} \) one can obtain scalar fields coupling minimally to new external metric field which also obeys the Einstein equation. Although such Einstein frame may be useful for practical computation such as with the cosmological equations, the physical significance is often not more direct and easily applicable. These is mainly due to that the matters moving in the space-time with the new external metric, excepting the radiation fields having the null geodesics, do not follow the geodesics leading to non-universal couplings. It is almost thus thought of that the physical metric is given in which all matter fields follow the geodesics.

It is quite clearly that the potential \( \tilde{V}(\tilde{\phi}_i) \) will be a promise of moduli stabilization at fundamental level. This achieves only if \( \tilde{V}(\tilde{\phi}_i) \) holds a stable minimum associated with that VEVs of the modulus fields are no longer chosen in an arbitrary way but are completely determined. On the other hand, the size of the internal spaces is dynamically fixed. In contrast, there will occur corresponding massless scalar fields in the effective low energy theory which cause dangerous phenomenological problems. In this case an effective potential for the modulus fields has to be generated in various ways. It has been shown that such potential can arise via quantum effects of pure geometrical and non-geometrical fields [31–33]. The dynamics of a bulk scalar field as proposed in [34] also provides a mechanism for stabilizing the size of the extra dimension in the Randall-Sundrum model. An alternative possibility is to come from the gaugino condensation [35, 36] which is also considered to break dynamically the supersymmetry [37]. Even though the modulus fields are stabilized by \( \tilde{V}(\tilde{\phi}_i) \), KK spectrum of an arbitrary particle propagating in the vacuum background of the modulus fields would still be unspecified by corresponding Laplace operator unknown. In fact, we have to finish a last step by solving Eq. (30) or Eq. (35) to determine the internal partial derivatives \( \hat{\partial}_i \). However, in the low energy regime in which the twisting of the bundle \( B^D \) is essentially trivial, since this is done in a much more simple way. It is due to that the internal frame fields \( \hat{\partial}_i \) and their dual \( K^i(x, \theta) \) may be approximately taken as those of the left-invariant vector fields and one-forms, respectively, on the Lie group \( G \).

In order to verify the existence of the stable minimum with respect the moduli stabilization potential \( \tilde{V}(\tilde{\phi}_i) \) above, let us consider a simple illustration example of the internal spaces which are smoothly equivalent to the group manifold \( SU(2) \). It is well known that this manifold is diffeomorphic to the 3-sphere. For this case, it is not difficult to find the corresponding potential for two
FIG. 1: The shape of the potential $\tilde{V}(\tilde{\phi})$, defined in unit of $(\Lambda M_p)^2$, is plotted as the function, for convenience, of $m_i \equiv e^{\tilde{\phi}_i}/M_P$ ($i = 1, 2$) in the case of the internal manifolds being diffeomorphic to the Lie group SU(2). The parameters of the potential are valued as $\Lambda_1 = 5$, $a_1 = 0$, and $a_2 = \frac{3}{4}$. Left panel: The three-dimensional plot of the given potential has an absolute minimum seen quite clearly. Right panel: The contour plot of the given potential with “+” referring the absolute minimum.

modulus fields in which some conditions on the parameters of the potential are required to guarantee it being bounded from below. Its form is shown, for example, in FIG. 1 with de Sitter vacuum. Note that, in the realistic theory the minimum value of the moduli stabilization potential has to be fine tuned in the way of total vacuum energy density being consistent with the cosmological constant $[38, 39]$. We would like to emphasize that this illustration may also be true for more complex cases of the special unitary group as well as the special orthogonal group.

It is important that $\tilde{V}(\tilde{\phi})$ is emerged from the intrinsic properties of the higher dimensional space-time. Therefore, the present work may provide the natural mechanism to understand the dynamics of the extra dimensional compactification. Notice that the classical potential of the volume modulus fields can also get effective contributions as indicated above. It is interesting to think of that the inflation can be explained in the framework in connection with the moduli stabilization attracted in much of the previous works, for example, see in [40].

Let us end this section with a few important comments on the construction of the higher dimensional gravitational interaction above. According to this description we can see that the curvature of the 4-dimensional external and internal spaces and the non-triviality of the bundle $B^D$ are to depend intimately together due to that they are laid in the same geometrical structure. On the other hand, the external and internal gravitational interactions together with the gauge interaction most widely known as of the quantum field theory mediated by $A^a_{\mu}$ are unified in the same mathematical structure. The particular importance of this unification is basic to argue that in the very strong field of the external gravity, such as inside a black-hole, the quantum effects coming from this gauge interaction and the modulus fields become very important contribution creating the quantum geometrodynamics of the space-time, and vice versa. Clearly, within the present framework the geometry of usual 4-dimensional universe is quantum-mechanically dynamic. This is due to the horizontal frame $\{\hat{\partial}_\mu\}$ (as we have pointed out above that these vectors determine locally the
4-dimensional external directions in $B^D$) spanning the 4-dimensional external spaces to present the quantum effects rather than quantization of the components $g_{\mu\nu}$ of the external metric tensor as almost expected. In the low energy where the gauge fields $A^a_{\mu}$ are hidden by, for instance, the Higgs mechanism which will be discussed in the later section, the external gravity would be replaced by the classical description of general relativity. In this way, the quantum behavior of the gravitational interaction can be understood manifestly by the quantum dynamics of the gauge interaction expected to become the strong coupling in short distance scales, or the high energy scales, and that of the modulus fields. This is clearly a good way toward a quantum gravity description near the Planck scale. These would also generate the corrections to the usual 4-dimensional Einstein-Hilbert action including the higher order terms in the curvature tensor or scalar. In summary, in the large distances, the gravity is well described by general relativity in which gravitational effects originating from extra dimensions are strongly suppressed. However, as the energy increasing highly, the usual 4-dimensional classical gravity will get modification from such quantum effects.

IV. GROUND STATE OF BULK AND PERTURBATIVE GRAVITY

A. Vacuum solution of $B^D$

Let us look for what means actually by classical vacuum state of the higher dimensional space-time proposed in the present work. In the classical ground state, the matter sources are absent since the twisting of the higher dimensional bulk is obviously trivial. This means that there is a canonical splitting of $B^D$ into the 4-dimensional space-time and the $(D-4)$-dimensional extra space without the gauge fields $A^a_{\mu}$. On other the hand, we may consider a solution describing the ground state geometry of the bulk with the topology given as

$$B^D = M^4 \times G,$$

with the corresponding metric read

$$G^{0}_{MN} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & e^{\langle \tilde{\phi}_i \rangle / M_P} \delta_{ij} \end{pmatrix}.$$  \hspace{1cm} (56)

Notice that, the ground state of $B^D$ must admit the internal metric associated with the stable minimum of the potential $\tilde{V}(\tilde{\phi}_i)$ in which the modulus fields get VEVs denoted by $\langle \tilde{\phi}_i \rangle$ (with $i = 1, ..., D-5$), and $\langle \tilde{\phi}_{D-4} \rangle = 0$. Indeed, this is applicable for the geometry of the ground state because $K_i$ (or $\tilde{\partial}_i$) and $K^i$ now correspond to the basic fields of the left-invariant vector fields and one-forms on the Lie group $G$. In this case, it is interesting to result in the left-invariant metric on $G$. The above topological product implies that the vacuum geometry of the bulk is foliated by either $G$ or $M^4$.

By the perfect feature of the ground state geometry, we are ready to write the Einstein field equation for the external metric $g_{\mu\nu}(x)$ as follows

$$\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(\hat{R} - \Lambda_{eff}) = 0.$$  \hspace{1cm} (57)
Here $\hat{R}_{\mu\nu}$ and $\hat{R}$ are the Ricci and scalar curvatures, respectively, of the manifold $M^4$. The effective cosmological constant $\Lambda_{\text{eff}}$ is given by

$$\Lambda_{\text{eff}} = e^{-\langle \tilde{\phi} \rangle / M_P} \frac{\tilde{V}_{\text{min}}}{M_P^2}, \quad (58)$$

where $\tilde{V}_{\text{min}}$ is the value of the potential $\tilde{V}(\tilde{\phi}_i)$ at its absolute minimum, and $\langle \tilde{\phi} \rangle = \sum_i \langle \tilde{\phi}_i \rangle / 2$. The solution for Eq. (57) can be derived by the Ricci tensor satisfying in the following relation

$$\hat{R}_{\mu\nu} = {\Lambda_{\text{eff}} \over 2} g_{\mu\nu}. \quad (59)$$

This solution results in that the manifold $M^4$ is the 4-dimensional Einstein space which has maximal symmetry. If $\Lambda_{\text{eff}}$ is non-zero, depending on its sign to be negative or positive, then $M^4$ is of the Anti de Sitter or the de Sitter space, respectively. With respect the remaining case that $\tilde{V}_{\text{min}}$ vanishes where the parameters of the potential must be finely-tuned to the contributions in the potential exactly canceled with each other, we have the solution of the Ricci-flat 4-dimensional manifold. One of such ones is the 4-dimensional Minkowski-flat space-time which has also to fulfil the stronger condition that all the components of the Riemannian tensor vanish identically. At classical level, the Minkowski space-time which is stable by the positive energy theorem [41] has the lowest energy. Therefore, the classical ground state for $B^D$ is determined by the geometry $\mathbb{R}^{1,3} \times G$ whose isometry group is the product of the 4-dimensional Poincaré group and the Lie group $G$.

We want to emphasize that it was not in fact easy to obtain a ground state solution of the topology (55) in the modern Kaluza-Klein unification in which $M^4$ is the 4-dimensional Minkowski-flat space-time. This is only satisfied if the internal manifold is Ricci-flat. Such manifold could be, for example, the $n$-torus, or the Calabi-Yau manifold which have been considered for the compactification of the extra dimensions in the superstring theories. It is easily verified that with respect to the present work the internal manifolds in the ground state is not Ricci-flat but has non-zero constant curvature. However, it is important to note that the Ricci-flat condition is just a too strong restriction on the internal manifold due to that such manifold only admits the Abelian isometry groups. This difficult can be solved by including additionally the matter fields (the non-geometrical fields) to guarantee the present of the non-Abelian isometry groups [17]. Thus it is clearly lost the original Kaluza-Klein idea. An interesting consideration to overcome in more natural way is the present of a totaly antisymmetric third-rank tensor [42] which arises naturally in the eleven-dimensional Kaluza-Klein supergravity [43]. Another possible mechanism which does not have to introduce the matter fields is to add the higher curvature terms [44].

So far we have concentrated mostly to discuss the ground state of $B^D$ in the classical level. However, a question can be asked whether the vacuum state found above is stable or modified under the quantum effects. In the following we would like to comment briefly about a true vacuum solution for $B^D$ in point of view of the quantum field theory in which the non-perturbative effects would reveal the structure of this vacuum. However, a more detailed study for such is beyond the scope of the present work but will be done in our feature work. It is important to recall that the true vacuum in the Yang-Mills theory has the extremely complicated structure due to the non-trivial topology in the space of the gauge fields. It is defined as a linear combination of topologically
inequivalent energy-degenerate vacua

\[ |\theta_{va}\rangle = \sum_{n=-\infty}^{+\infty} e^{in\theta_{va}} |n\rangle, \quad (60) \]

which is also well known as the \( \theta \)-vacuum where \( \theta_{va} \) is the vacuum angle satisfying \( 0 \leq \theta_{va} \leq 2\pi \). The \( n \)th-topological vacuum \( |n\rangle \) is corresponded to the winding number \( n \), an integer number, which is a geometric invariant characterizing the non-trivial topology of the bundle \( B^D \). This quantum number is derived by integrating the second Chern character over the 4-dimensional Minkowski base manifold. This suggests us that the classical ground state has been looked for above only corresponding with the case of the trivial winding number, \( n = 0 \). On the other hand, it is inconsistent with the \( \theta \)-vacuum. In this case there will not be simple to take the gauge fields \( A_a^\mu \) to be zero as performed above, but \( \langle A_\mu \rangle = 0 \). We can now consider the metric for the true vacuum geometry of \( B^D \) in term of the warped-product ansatz as follows

\[ G^{(n)}_{MN}(X) = \begin{pmatrix} f^{(n)}(\hat{\theta})g^{(n)}_{\mu\nu}(x) & 0 \\ 0 & e^{(\tilde{\phi}_i)/M_P \delta_{ij}} \end{pmatrix}, \quad (61) \]

which is associated with the topological vacuum \( |n\rangle \). For a reason that will be explained in the next subsection. Note that in the case of the non-trivial bundle \( B^D, n \neq 0, g^{(n)}_{\mu\nu}(x) \) may not be of the Ricci-flat solution. Classical, the system obviously lies at the rest at the ground state of the topological quantum number being zero. Quantum-mechanically, zero-point fluctuations would play the important role which leads to the tunneling among different topological vacuum solutions, also well known as the instanton effects. Therefore, the geometry of the ground state for \( B^D \) is precisely a superposition of these solutions presenting the quantum nature of the space-time. The instanton effects are expected to make some interestingly physical implications as well as modify the classical properties of theory. Notice that, we have not been interesting in the quantum effects on the classical potential \( \tilde{V}(\tilde{\phi}_i) \) in the above given analysis.

\section*{B. Effective dynamics of the bulk gravitation below the compactification scale}

The aim of this subsection is to study the physical particle spectrum of the bulk gravity in the effective low energy theory beneath the compactification scale in which we will mainly focus the 4-dimensional external gravitational field and the modulus fields. As usual, the low energy degrees of freedom of the higher dimensional gravity are obtained by the decomposition of the higher dimensional massless graviton via the KK dimensional reduction. The result includes, in the 4-dimensional viewpoint, the massless zero modes of graviton, vectors, radion (trace of the internal metric), scalars (traceless of the internal metric), and their KK excitations \[45, 46\]. In the case of the internal space to be an orbifold, the zero mode of vectors will be eliminated due to the orbifold projection. For the present work, the low energy investigation of the bulk gravity may be done in a simple way. The effective degrees of freedom consist of the massive gauge bosons \( A_\mu^a \) whose masses are generated by the Higgs mechanism that will be discussed in the later section and slightly perturbative fields around the classical background solution given above.
In the fact, the present observed universe is essentially flat meaning that the external spaces may be approximately considered as the 4-dimensional Minkowski space-time. Since in the low energy limit we can investigate the physical progress on the bulk geometry to correspond with the above classical background. Notice that, because of our present context which is a braneless scenario, this background is not broken by the present of branes as their surface tension exceeds the fourth power of the fundamental Plank scale, $M^4_{P}$.

In the weak field limit, the effective dynamics of the 4-dimensional external gravitational field and the modulus fields may be studied by expanding them perturbatively around the ground state metric at the leading order as

$$g_{\mu\nu} = \eta_{\mu\nu} + M_{Pl}^{-1} h_{\mu\nu}, \quad \tilde{\phi}_i = \langle \tilde{\phi}_i \rangle + \langle V_G \rangle^{-\frac{2}{D}} \tilde{h}_i, \quad i \neq D - 4,$$

where $\eta_{\mu\nu}$ is the 4-dimensional Minkowski background. The effective 4-dimensional Plank scale $M_{Pl}$ and the vacuum volume of the internal spaces $\langle V_G \rangle$ are determined immediately via the fundamental parameters as follows

$$M^2_{Pl} = \frac{M^2_{*} \langle V_G \rangle}{\Lambda^{D-4}}, \quad \langle V_G \rangle = \int_G d^{D-4}\theta K(\theta)e^{(\bar{\phi})/M_P}.$$

It is very important to note that the first equality of (63) is the usual relation to solve the hierarchy problem in the ADD model with the large extra-dimensions, a $n$-torus of equal radii $R$, in which $\langle V_G \rangle/\Lambda^{D-4} \sim R^{D-4}$ and $M_* \sim 1\text{TeV}$. Notice that, the specific energy scale $\Lambda$ of the compact internal spaces in this framework should be so much larger than the inverse radii of the extra dimensions in the ADD model. This is because all of particles are free to propagate in the whole bulk being similar to the UED [5]. Since such extra dimensions would not just play much the role of understanding the hierarchy problem between the weak and the Plank scales in the observed world. The $h_{\mu\nu}$ and $\tilde{h}_i (i \neq D - 4)$ express the small quantum fluctuations of the 4-dimensional tensor gravitational and modulus fields, respectively. This means that both of them have to satisfy the condition $|h_{\mu\nu}|, |\tilde{h}_i| \ll 1$. In the above given definition, these fields have been normalized so that they have the canonical kinetic term and the mass dimension. Notice that, $h_{\mu\nu}$ is the symmetric rank-2 tensor of ten independent components which is of the reducible representation of the 4-dimensional Lorentz group. Thus it can be split into irreducible ones in such a way as, $2 \oplus 1 \oplus 0 \oplus 0$. However, the fact that some degrees of freedom are consistently eliminated by constraint equations leads to the physical degrees of freedom to be less than original ones.

An effective Lagrangian determining the free evolution of the field describing the 4-dimensional curvature perturbation in the bulk can be obtained by inserting the first expansion of (62) into the Lagrangian (49). Then we find

$$\mathcal{L}_h = \frac{1}{2} \partial^\lambda h_{\mu\nu} \partial_\lambda h_{\mu\nu} - \partial^\lambda h_{\mu\nu} \partial_\mu h_{\lambda\nu} + \partial^\mu h \partial^\nu h_{\mu\nu} - \frac{1}{2} \partial^\mu h \partial_\mu h - \frac{1}{2} (h_{\mu\nu} \Box_G h_{\mu\nu} + h \Box_G h),$$

where $h_{\mu\nu} = \eta^{\mu\lambda} \eta^{\nu\rho} h_{\lambda\rho}$, $h = \eta^{\mu\nu} h_{\mu\nu}$, $\partial^\mu = \eta^{\mu\nu} \partial_\nu$, and $\Box_G$ is the Laplace operator on the fixed internal spaces given as

$$\Box_G = -\sum_i e^{-(\tilde{\phi}_i)/M_P} K_i^a(\theta) K_i^b(\theta) \partial_a \partial_b.$$
By the compact topology of the internal spaces, the eigenvalues of \( \Box_G \) are nonnegative and discrete. The equations of motion derived by varying \( \mathcal{L}_h \) with respect to the quantum field \( h_{\mu\nu} \) is of the form

\[
\Box h_{\mu\nu} - \partial_\lambda \partial_\mu h_{\nu}^\lambda - \partial_\lambda \partial_\nu h_{\mu}^\lambda + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h + \Box_G (h_{\mu\nu} + \eta_{\mu\nu} h) = 0. \tag{66}
\]

We can show that both the Lagrangian (64) and Eq. (66) are invariant under the local gauge transformation

\[
h_{\mu\nu}(x, \hat{\theta}) \longrightarrow h_{\mu\nu}(x, \hat{\theta}) - \partial_\mu \varepsilon_\nu(x) - \partial_\nu \varepsilon_\mu(x), \tag{67}
\]

which is induced from the local external coordinate transformation. However, it is very important to see that this gauge transformation is not able to fix a gauge to eliminate some unphysical degrees of freedom. This is due to the fact that the field \( h_{\mu\nu} \) is in general to depend on both the external and the internal coordinates while the gauge transformation parameters only depend on the external coordinates. It is interestingly equivalent to the interpretation that under the above gauge transformation \( g_{\mu\nu} \) transforms locally on the external spaces but globally on the internal ones. The global transformation is then impossible to fix gauge hence in usual sense the transformation (67) does not generate literally the local gauge transformation. On the contrary, it would remove eight degrees of freedom to result in two physical ones for \( h_{\mu\nu} \). This is of the case that the field \( h_{\mu\nu} \) is independent on the internal coordinates.

The quantum fluctuations of the 4-dimensional external gravity can be KK expanded as

\[
h_{\mu\nu}(X) = \sum_n h^{(n)}_{\mu\nu}(x) Y^{(n)}(\hat{\theta}), \tag{68}
\]

where \( Y^{(n)}(\hat{\theta}) \) are the eigenvector of the operator \( \Box_G \) corresponding to the eigenvalue \( \lambda^2_n \), with \( n \) denoted to a set of quantum numbers characterizing to each eigenvector. Each KK mode \( h^{(n)}_{\mu\nu}(x) \) satisfies the equations of motion in analogy with the Fierz-Pauli ones for the massive graviton but the mass term in this equation replaced by

\[
\lambda^2_n (h^{(n)}_{\mu\nu} + \eta_{\mu\nu} h^{(n)}) \tag{69}. 
\]

An important observation is that such mass terms clearly violate the Fierz-Pauli tuning excepting \( \lambda^2_0 = 0 \) corresponding to the massless zero mode which carries out two physical degrees of freedom.\(^7\)

With respect to every KK excitation, we only find the constraint equations, \( \partial_\mu h^{(n)}_{\mu\nu} + \partial_\nu h^{(n)} = 0 \), which would remove the four degrees of freedom. The constraint on the trace, \( h^{(n)} = 0 \), is not able to be derived unless the sign of the last term in (69) is changed oppositely. Thus, each KK excited mode propagating in the effective 4-dimensional Minkowski space-time carries exactly out six physical degrees of freedom. It is very important that this gives rise a 4-dimensional massive graviton of five physical degrees of freedom and a real scalar ghost of instability (a scalar field with negative kinetic energy) which breaks the unitarity of theory.

According to the analysis above, the linearization of the 4-dimensional tensor gravity field around the classical background of \( B^D \) yields to the unstable massive scalar excitations in the effective 4-dimensional theory. However, a possible way to naturally overcome the problem of the scalar ghosts

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\({}^7\) Notice that the coefficient, \(-1\), between \( h_{\mu\nu} \) and \( \eta_{\mu\nu} h \) in the mass term of the Fierz-Pauli equations is tuned by hand to have the consistence theory. It not enforced by any symmetric property of theory [48]. For the present the coefficient, \(+1\), in the mass term given in (69) is enforced by the space-like feature of the internal kinetic term.
is that the 4-dimensional external gravitational field is invariant under the action of $G$ in general up to a conformal factor as

$$g_{\mu\nu}(x, \hat{\theta}) = e^{\sigma_g(h)}g_{\mu\nu}(x, \hat{\theta}),$$

(70)

where $\exp\{i\theta^a T_a\} = \exp\{i\theta^a T_a\} h$, and $\sigma_g(h)$ is a smooth real scalar function on $G$. With the help of the local canonical section taken as

$$s : U \longrightarrow \pi^{-1}(U)$$

$$x \longmapsto s(x) = f(x, e),$$

(71)

we can particularly get an interesting and useful expression for $g_{\mu\nu}(x, \hat{\theta})$ as follows

$$g_{\mu\nu}(x, \hat{\theta}) = e^{\sigma_g(\hat{\theta})}g_{\mu\nu}(x).$$

(72)

In this way, $g_{\mu\nu}(x) = g_{\mu\nu}[s(x)]$ plays the role of usual 4-dimensional gravity field in the effective 4-dimensional theory. The warped scalar function, $\sigma_g(\hat{\theta}) = \sigma_g[\exp\{i\theta^a T_a\}]$, determines conformal diffeomorphism invariance of the 4-dimensional external gravitational field under the Lie group $G$. This warped function must clearly satisfy the equation in analogy with Eq. (32) written in the case of the conformal diffeomorphism invariance. It is also thought of as an intrinsic characteristic of the 4-dimensional external gravitational field playing the internal dynamical progress itself. This can be manifested in an indirect way through the coupling to the gauge fields $A^a_\mu$. On the other hand, by determining such coupling strength in the effective low energy theory we are in principle possible to deduce certain information about the conformal diffeomorphism invariance of $g_{\mu\nu}(x, \hat{\theta})$. In the low energy limit, this warped factor will create a vacuum energy density, or cosmological constant contribution, which is obtained by integrating the last two terms in Eq. (64) over the vacuum volume of the internal spaces. Such density is clearly very large because of the fact that it is proportion to $\Lambda^2$, thus need to be exactly cancelled to other contributions, for example, the minimum energy of the moduli stabilization potential $\tilde{V}(\tilde{\phi}_i)$. It is also important to notice that an interesting consequence of this invariance is that there will not occur the KK partners of the usual graviton in the effective low energy theory. This means that the important phenomenological constraints coming from the KK excitations of the usual graviton [45, 46, 49, 50] would not transparently happen in the present scenario.

Next let us specialize in the physical quantum fluctuations of the modulus fields by plugging the second expansion of (62) back into (50). Then relevant Lagrangian can be obtained as

$$\mathcal{L}_{\bar{h}} = \frac{1}{2} \left( \partial_\mu \tilde{h}^T \partial^\mu \tilde{h} - \tilde{h}^T M_{\tilde{h}} \tilde{h} \right) + \mathcal{L}_{\text{self-int}},$$

(73)

where $\tilde{h}$ is a $(D - 5)$-component column vector defined as, $\tilde{h} = (\tilde{h}_1...\tilde{h}_{D-5})^T$, $M_{\tilde{h}}$ is a mass mixing matrix among them, and $\partial^\mu = \eta^{\mu\nu} \partial_\nu$. The physical excitations describing the curvature perturbation of the internal spaces correspond with the eigenvectors of $M_{\tilde{h}}$ whose masses squared, the eigenvalues of this matrix, are order of $\Lambda^2$. These massive scalar particles are hidden in the present world due to the compactification scale, $\sim \Lambda$, assumed to be very high compared the electroweak scale. They would be often coupled to the effective 4-dimensional matter fields with the coupling strength, $\sim E/M_{Pl}$, in analogy with the usual graviton doing. Since these couplings are strongly suppressed in the energy $E$ to be so much smaller than $M_{Pl}$. It is noticed that there would also be mixing
between $\tilde{h}_i$ and other scalars such as the Higgs boson of the SM. This mixing results in modification in both their mass spectrum and coupling, thus may affect significantly, for instance, on the ordinary Higgs phenomenology.

In general, probing geometry of extra dimensions as well as their working energy scale can be mainly taken by detecting the KK modes. In the energy region to be lower than the typical scale of the compact internal spaces, only the fundamental state and some following light KK excitations are taken into account of the phenomenologically significant contributions. The higher KK excitations are completely integrated out thus not relevant. In particular, for producing the KK excitations on colliders, such as at LHC or ILC, only some first KK modes may be produced whereas the heavy massive KK states could not appear even the increase in colliding energy. This is due to the quantum fluctuations of the modulus fields that are created at a consistent energy scale and enlarged in increasing of colliding energy. Strictly speaking, the quantum-mechanical effects of the modulus fluctuations become now very important leading to the vacuum volume of the internal spaces being no long stable. Thus the structure of the KK tower is broken down at such scale. Additionally, another contribution can modify importantly the KK spectrum arising from the gauge fields $A_\mu$ which would also deform smoothly the background geometry of the internal spaces.

V. 4-DIMENSIONAL WEYL FERMIONS ON BULK

The fermions play actually the central role in understanding of the present world. They have been thought of as the fundamental building blocks of the matter in the universe. Their nature is well defined to be chiral. In the previously higher dimensional extensions, the fundamental fermions, being often vector-like, are given as the spinorial representations of the higher dimensional Lorentz group. A problem arising from these is that beside the SM fermions there would have the presence of very many extra fermions, which may transform the non-triviality under the gauge group of the SM, in the low energy particle spectrum. Many consistent mechanisms have been proposed to hide these extra fermions such as the orbifold compactification \[51–54\] or using the domain wall \[55–57\]. In the following we expect to show in our framework that the introduction of the 4-dimensional Weyl fermions on the bulk $B^D$ at fundamental level is a natural result of the intrinsical geometry of the higher dimensional space-time. The dynamics of these fermions in the bulk will then be provided in detailed investigation.

Let us recall that the horizontal tangent subspace at each point on $B^D$ spanned by $\{\hat{\partial}_\mu\}$ is endowed with the given metric $g_{\mu\nu}(x, \hat{\theta})$ defined in the corresponding dual basis. However, one may take the vielbein field $e_\alpha^\mu(x, \hat{\theta})$ to change from the basis $\{\hat{\partial}_\mu\}$ to another basis $\{\hat{e}_\alpha\}$ as

$$\hat{e}_\alpha = e_\alpha^\mu(x, \hat{\theta})\hat{\partial}_\mu,$$

so that in this basis the 4-dimensional external metric is Minkowski-flat. This therefore implies that the vierbein field must satisfy the following relation

$$\eta_{\alpha\beta} = e_\alpha^\mu(x, \hat{\theta})e_\beta^\nu(x, \hat{\theta})g_{\mu\nu}(x, \hat{\theta}),$$

which defines the local 4-dimensional Minkowski space-time at each point on $B^D$ in which the gravity is absent. The indices $\alpha, \beta,...$ will be used to denote the local 4-dimensional Lorentz ones.
Note that the metric $\eta_{\alpha\beta}$ here would not be realized as the perturbative background metric, but is exactly the metric of the local 4-dimensional Minkowski space-time, often called the axial metric. In the formulation of the vierbein field, the 4-dimensional external gravity would be described by $e_\alpha^\mu$ instead of the metric $g_{\mu\nu}$. It is important to note that there are many local 4-dimensional Lorentz frames that also yield the same Eq. (75). Consequently, these will lead to the local Lorentz transformation rotating among these frames as

\[ e_\alpha^\mu(x, \hat{\theta}) \rightarrow e_\alpha'^\mu(x, \hat{\theta}) = \Lambda(x, \hat{\theta})_\beta^\alpha e_\beta^\mu(x, \hat{\theta}), \]

\[ \eta_{\alpha\beta} = \Lambda(x, \hat{\theta})_\gamma^\alpha \Lambda(x, \hat{\theta})^\sigma_\beta \eta_{\gamma\sigma}, \]

(76)

where $\Lambda(x, \hat{\theta})$ is obviously an element of the Lorentz group $SO(3,1)$. The properly physical significance of this group in the case of the curved space-time is taken as the symmetry group of the local 4-dimensional inertial frames.

As pointed out in the preceding section, values of the 4-dimensional external metric $g_{\mu\nu}(x, \hat{\theta})$ on the same internal space are related together which have resulted in the expression, $g_{\mu\nu}(x, \hat{\theta}) = \exp\{\sigma_g(\hat{\theta})\} g_{\mu\nu}(x)$. By using the local canonical section in the similar fashion as what has been performed with respect $g_{\mu\nu}(x, \hat{\theta})$ and combining with Eq. (75), the expression for the vierbein field can thus be defined as

\[ e_\alpha^\mu(x, \hat{\theta}) = e^{-\frac{\sigma_g(\hat{\theta})}{2}} e_\alpha^\mu(x). \]

(77)

Here $e_\alpha^\mu(x)$ satisfies the following relation

\[ \eta_{\alpha\beta} = e_\alpha^\mu(x) e_\beta^\nu(x) g_{\mu\nu}(x), \]

(78)

which defines the local 4-dimensional inertial frame on an effective 4-dimensional space-time manifold. This result also leads to, $\Lambda(x, \theta) = \Lambda(x)$, meaning that each element of the local Lorentz group $SO(3,1)$ only depends on the external coordinates. This is clearly consistent with the local 4-dimensional external coordinate transformation.

Next let us proceed to discuss how the 4-dimensional Weyl spinors occur on $B^D$. With the novel geometrical structure of the bulk, we may easily see that they are the simplest non-trivial irreducible representations of the local Lorentz group $SO(3,1)$ just determined above. This results in that the chiral 4-dimensional fermions will occur naturally at the fundamental level in the present framework. Under the rotation of the local Lorentz frames, they transform as follows

\[ \Psi_L \rightarrow \Psi'_L = e^{\frac{\xi_m}{2} \sigma_m} \Psi_L, \]

(79)

\[ \Psi_R \rightarrow \Psi'_R = e^{\frac{\eta_m}{2} \sigma_m} \Psi_R, \]

(80)

where $\Psi_{L,R}$ are denoted to the left-handed and right-handed Weyl spinors written in term of the two-component spinor, respectively. They are both complex objects depending in general on the bulk coordinates. Both $\xi^m$ and $\eta^m$, with the index $m$ running from 1 to 3, are complex functions in only the external coordinates. These function are exactly expressed in the terms of local transformation parameters, a $4 \times 4$ antisymmetric real matrix, of the group $SO(3,1)$. The matrices $\sigma_m$ are the usual Pauli ones. However, it is almost to work the Weyl spinors obtained from the projection of the Dirac spinor, $\Psi_D = (\Psi_L \quad \Psi_R)^T$, by the operators $P_{L,R} = (1 \mp \gamma_5)/2$. This spinor transforms under the local Lorentz group $SO(3,1)$ as

\[ \Psi_D \rightarrow \Psi'_D = e^{\frac{\xi^m}{2} \sigma_m} \Psi_D, \]

(81)
where $\Sigma_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta]$ are the generators of the Lorentz group $SO(3,1)$ determined in terms of the usual Dirac matrices $\gamma^\alpha$ obeying the Clifford algebra, $\{\gamma_\alpha, \gamma_\beta\} = \eta_{\alpha\beta}$. Their counterparts in the curved space-time given by $\gamma_\mu = e^\alpha_\mu \gamma^\alpha$ satisfying the analogous one, $\{\gamma_\mu, \gamma_\nu\} = \eta_{\mu\nu}$. The parameters $e^{\alpha\beta}$ of the above local transformation are a $4 \times 4$ antisymmetric matrix with elements which are real functions in the external coordinates. In what follows we will work the Weyl spinors expressed via the Dirac spinor.

We would now like to find a suitable Lagrangian describing the propagation of the 4-dimensional Weyl spinor fields given above in the higher dimensional space-time $B^D$. We first have to note carefully that with respect to the present case the usual kinetic term of these fields is impossible to determine the dynamics in all of directions of the bulk $B^D$. An obstruction is crucially due to the fact that these fields would behave the same as a spin-$1/2$ field when moving along the 4-dimensional external directions but as a scalar field when moving in the internal directions. The scalar-like behavior is realized in sense of that the usual spin concept is lost under the viewpoint of the internal evolution in which instead of this they carry out the conventionally internal charges inherited from their non-trivial representations under the group $SO(3,1)$. We will bring out the meaning in more detail below in which the corresponding Lagrangian will been found. This means that the physical manifestation is distinctly different to correspond with the dynamics taken in the external and internal spaces. This analysis thus give a better understanding of their dynamic properties in $B^D$ which will be reasonable to construct a consistent Lagrangian. Notice that it can also be straightforward for the more generic case in which fields have an arbitrary spin. A full Lagrangian thus includes two independent separating parts which determine the evolution along the external and internal directions, respectively. The evolution of a Weyl spinor field with specifically given chirality along the 4-dimensional external directions are characterized by the usual Lagrangian

$$L_{4D}^\Psi = \bar{\Psi}i\gamma^\alpha e^{\alpha}_\mu D_\mu \Psi,$$

(82)

where $\Psi$ is denoted to the (right-handed) left-handed Weyl spinor field with the dimension $[\Psi] = \frac{D-1}{2}$. In particular, it may be dimensionally renormalized in the natural way as

$$\Psi_{\text{nor}} = \frac{\Psi}{\Lambda^{\frac{D-1}{2}}},$$

(83)

where the factor $1/\Lambda^{\frac{D-1}{2}}$ is taken from the bulk volume element. The redefinition above leads to $[\Psi_{\text{nor}}] = \frac{3}{2}$ which is equal to the dimension of the fermion in the 4-dimensional space-time. It is important to note that this redefined field $\Psi_{\text{nor}}$ is not treated as the infraredly effective field, but it is still the fundamental field. This is because the redefinition is formally independent on a cutoff scale at which the corresponding low energy description will break down. Moreover, other fundamental fields living on $B^D$ can also be taken similarly by such way to have the same dimension as the corresponding field in the 4-dimensional theory. By this, it is interesting to observe that the dimension of coupling constants in the higher dimensional theory, e.g. as that for the gauge, Yukawa interactions or the self-couplings of the scalar fields, would be respectively same as in the usual 4-dimensional theory. In the following discussion we will be interesting in the renormalized fields as above in which “nor” will be dropped. On the other hand, the factor $1/\Lambda^{D-4}$ in the original matter action is completely eliminated through the redefinition of the field. The covariant derivatives $D_\mu$ acting $\Psi$ are given by

$$\hat{D}_\mu = \hat{\partial}_\mu + \frac{i}{2} \omega^{\alpha\beta}_\mu \Sigma_{\alpha\beta},$$

(84)
where $\omega_{\mu}^{\alpha\beta}$ is the torsion-free spin connection which is expressed in terms of the vierbein field as

$$\omega_{\mu}^{\alpha\beta} = e_{\alpha}^{\nu} \left( \hat{\partial}_\mu e_{\beta\nu} + e_{\beta\lambda} \Gamma_{\mu\lambda}^{\nu} \right),$$

(85)

with $e_{\alpha}^{\nu}$ to be inverse of the vierbein field $e_{\alpha}^{\nu}$, and $e_{\beta\nu} = \eta_{\beta\gamma} e_{\gamma\nu}$. It is important to notice that the 4-dimensional Weyl spinor fields moving along the 4-dimensional external directions with the specific chirality are massless due to their mass term vanishing. Next, let us attempt to construct an internal Lagrangian for $\Psi$. As just discussed above, in order to do this we must first find out the bilinear terms of $\Psi$ to be a Lorentz scalar which satisfying only consist of, $\bar{\Psi}^c \Psi$. Here $\bar{\Psi}^c = C \Psi^*$ is the charge conjugate field of $\Psi$, with $*$ denoted to the complex conjugate and $C$ to be a 4 $\times$ 4 matrix satisfying the following conditions, $C^T C = 1$ and $C^T \gamma^\mu C = -\gamma^\mu$. It should be noted that another term mixing between the left-handed and right-handed Weyl spinors is not included by the property of the chirality in which they transform differently under the local gauge group. This term is thus forbidden obviously, and only generated via the coupling to the Higgs field after this field gets non-zero VEV. From the above argument, we can find out the following expectant Lagrangian

$$\mathcal{L}_G^\Psi = \frac{1}{2\Lambda} \left( -\hat{\partial}_i \bar{\Psi}^c \hat{\partial}^i \Psi - M_\Psi^2 \bar{\Psi}^c \Psi \right) + \text{H.c.},$$

(86)

where real number $M_\Psi^2$ appearing in the above Lagrangian plays the role of mass squared characterizing to the internal motion. It thus is naturally taken to the same order as the scale $\Lambda$. We can see that this Lagrangian being analogous to that of the scalar field is quadratic in the internal derivatives $\hat{\partial}_i$. At first important sight, all of the terms in the Lagrangian (86) vanish identically. This is due to the present of the totally antisymmetric tensor $\epsilon_{ab}$ (with $a, b = 1, 2$ and $\epsilon_{12} = 1$) in components of the bilinear combinations of $\Psi$ where it is often realized classically as the ordinary commuting field, or $c$-number. This means that the internal dynamics of the 4-dimensional Weyl spinor fields does not really make sense in the classical theory, but is of the quantum-mechanical concept in which they should be anticommuting fields. On the other hand, there has no an equivalence in the classical limit, thus, the Lagrangian (86) can be accepted without deriving from a classical Lagrangian. At second important one, $\Psi$ in the Lagrangian (86) is not able to play the role of the physical field under the internal point of view. This is because each term of (86) is not hermitian leading to that the corresponding Hamiltonian is unbounded from below. A combination of $\Psi$ and $\Psi^c$, however, defined as

$$\Psi_M = \Psi + \Psi^c,$$

(87)

provides the well-defined physical field. Therefore, $\mathcal{L}_G^\Psi$ is physically rewritten as

$$\mathcal{L}_G^\Psi = \frac{1}{2\Lambda} \left( -\hat{\partial}_i \Psi_M^T C \hat{\partial}^i \Psi_M - M_\Psi^2 \Psi_M^T C \Psi_M \right).$$

(88)

The field $\Psi_M$ is known as the Majorana spinor field (the real Dirac spinor) having the degrees of freedom to be equal to those of $\Psi$. It satisfies the reality condition, $\Psi_M^c = \Psi_M^*$, being invariant under the Lorentz transformation.\(^8\) The Lagrangian (86) describes the physical progress of a scalar-like

\(^8\) Notice that, the Dirac matrices $\gamma^\mu$ transform under a group of the unitary matrices as, $\gamma^\mu \rightarrow U \gamma^\mu U^\dagger$, corresponding to the rotation of the basis of the Clifford algebra. Using this transformation, one can look at a novel basis in which $C = 1$ and all $\gamma^\mu$ are pure imaginary known as the Majorana basis. Hence the reality condition of the field $\Psi$ is now simple as, $\Psi^c = \Psi^* = \Psi$, which is precisely same as that for a scalar field. The corresponding internal Lagrangian is thus of that of the real scalar field.
neutral Grassman field in the internal spaces in which it carries out the globally SO(3, 1) internal symmetry in the usual sense. These tell us that a quantum particle of carrying out the (right) left-handed Weyl spinor representation propagating in $B^D$ is completely different from its antiparticle of carrying out the (left) right-handed one under the 4-dimensional external viewpoint whereas it is its own antiparticle under the internal viewpoint. The structure of the KK tower for $\Psi_M$ is determined through the operator, $(\Box_G - M^2_\Psi)/\Lambda$. Since the lowest level has non-zero energy of the value $M^2_\psi/\Lambda$.

In the fact, there exists a significant coupling between $\Psi$ and the modulus fields given as

$$\sum_{i}^{D-5} \lambda_{\tilde{\phi}_i \Psi} \tilde{\phi}_i \bar{\Psi} \gamma^0 \Psi + H.c,$$

(89)

where $\lambda_{\tilde{\phi}_i \Psi}$ are dimensionless coupling constants. Notice that, the higher-order terms in $\tilde{\phi}_i$ in the coupling above have been skipped due to be suppressed by the positive powers of $M_p$, while the zero-order term leading to the mass term for $\Psi$ must obviously be zero. The present of these terms are to come from the expansion of the invariant terms in analogy in the non-trivial ones given in Eq. (45) coupled to $\bar{\Psi} \gamma^0 \Psi$.

It is worth stressing here one remarkable property of the internal Lagrangian given above. It may be easily verified that the Lagrangian (86) is only allowed with respect the fields which are neutral under all the local gauge charges, such as the right-handed neutrinos occurring in extensions of the SM. In other words, this is not of the case of the fields of carrying out such charges, for example, the fermions of the SM, due to that their Lagrangian would break the local gauge symmetry explicitly.

If a 4-dimensional Weyl spinor field $\Psi$ (for what is analyzed in the following we will still use $\Psi$ to refer the field under consideration) has the exactly conserved quantum charges, then its internal Lagrangian is precisely forbidden. Consequently, this results in that such field is by itself active (conformal) diffeomorphism invariance under the action of $G$. The values of $\Psi$ on the same internal space are thus related together as

$$\Psi(x, \hat{\theta}_2) = D(g) \Psi(x, \hat{\theta}_1),$$

(90)

where $D(g)$, $g \in G$, is in general an element of the linear transformation group on the representation space of $\Psi$, and $\exp\{i \hat{\theta}_a^2 T_a\} = \exp\{i \hat{\theta}_1^a T_a\} g$. To see Eq. (90) more concretely, for simplicity, let us consider the field $\Psi$ to carry additionally out the local gauge symmetry $SU(N)$ in which it is of the fundamental representation of this group. The following procedure can be similarly generalized to the more complicated case arising in the realistic model. In this case, $D(g)$ can be split into a direct product of the form

$$D(g) = D_{su}(g) \otimes D_{sp}(g) \equiv D_{su}(g) D_{sp}(g),$$

(91)

where $D_{su}(g)$ and $D_{sp}(g)$ are matrices rotating in the space of the fundamental representation of the group $SU(N)$ and in the space of the spinorial representation of the Lorentz group $SO(3, 1)$, respectively. Let us note that $D_{su}(g)$ is not necessary to be an element of the group $SU(N)$ (similar to $D_{sp}(g)$). This is because the transformation in Eq. (90) does not play the gauge transformation role, but acts as active diffeomorphism transformation. In order for the transformation rule in Eq. (90) well-defined, it must indeed be independent of the gauge transformation of the group $SU(N)$ (and also the local Lorentz transformation group) read

$$\Psi'(x', \hat{\theta}_1') = U(x', \hat{\theta}_1) \Psi(x, \hat{\theta}_1),$$

(92)

$$\Psi'(x', \hat{\theta}_2') = U(x', \hat{\theta}_2) \Psi(x, \hat{\theta}_2).$$

(93)
Here \( U(x, \hat{\theta}_{12}) = \exp\{i\alpha^m(x, \hat{\theta}_{12})I_m \} \) are elements of the group \( SU(N) \) which have to be related with each other due to the condition in Eq. (33), with \( I_m \) \((m = 1, \ldots, N^2 - 1)\) to be generators of the group \( SU(N) \) given in the fundamental representation. This requirement suggests the relation, 
\[
\Psi'(x', \hat{\theta}'_2) = D(g)\Psi'(x', \hat{\theta}'_1),
\]
which then leads to the following condition for \( D(g) \)
\[
D(g) = U(x, \hat{\theta}_2)D(g)U^{-1}(x, \hat{\theta}_1).
\]

We can easily see that Eq. (94) is only satisfied if \( D(g) \) commutes with \( U \), or \([D(g), I_m] = 0\), meaning that \( D(g) \) is necessary a matrix to be proportional to the identity matrix, and each element \( U(x, \hat{\theta}) \) of the gauge group \( SU(N) \) does not depend on the internal coordinates, \( U(x, \hat{\theta}) = U(x) \).

The last condition will be particularly important to give an interesting result of which the gauge fields corresponding with the gauge group \( SU(N) \) have only the 4-dimensional external components denoted by \( B_\mu \). Moreover, their internal Lagrangian built uniquely as
\[
\sim g^{\mu\nu}\text{Tr}[\hat{\partial}_i B_\mu \hat{\partial}_i B_\nu],
\]
is forbidden by violating the gauge invariance. Thus, \( B_\mu \) are by themselves the \( G \)-invariant gauge fields in general up to a conformal factor. Notice that, their 4-dimensional external Lagrangian is given by the usual Yang-Mills one. So by analogy with what has been done for the 4-dimensional external metric field, we can obtain an useful expression for \( \Psi \) as
\[
\Psi(x, \hat{\theta}) = D(\hat{\theta})\psi(x).
\]

Here \( D(\hat{\theta}) \equiv D[\exp\{i\theta^a T_a\}] \) presents the intrinsic diffeomorphism invariance of \( \Psi \) under the action of \( G \), and obviously satisfies Eq. (32). The field \( \psi(x) \) encodes the 4-dimensional behavior of the fundamental field \( \Psi \), and occurs in the 4-dimensional effective description in the low energy limit. Similarly, there have no the KK excitation states of such fields in the effective low energy theory, for instance, the fermions and the gauge bosons in the SM.

**The smallness of the neutrino masses.**– Let us close this section by providing a phenomenologically significant consequence of the dynamical description for the 4-dimensional Weyl spinor fields above. It is well established by the experimental observations that the masses of the neutrinos are so much smaller than those of the charged leptons and quarks in the SM. In the next analysis, we desire to provide a possibility to realize the natural origin of this problem. We consider the minimal extension of the SM given by including three right-handed neutrinos \( \nu_R \) associated with three lepton families. They are all obviously singlet under the gauge group of the SM. As seen above, all of the charged leptons, the quarks and the gauge bosons of the SM are diffeomorphism invariance by themselves under the action of the Lie group \( G \). The right-handed neutrinos and the SM Higgs doublet are only possible to have the non-trivial dynamics in the internal spaces. The electroweak symmetry is spontaneously broken by the VEV of this scalar field whose breaking scale is possible to be much lower than the compactified one. Hence a mass Lagrangian for the neutrinos in the flavor basis is taken in the effective low energy description as
\[
L^\text{mass}_\nu = -\sum_n \left( m_n \bar{\nu}_L \nu_{nR} + \frac{1}{2} M_n \bar{\nu}^c_{nR} \nu_{nR} \right) + \text{H.c.},
\]
where \( \nu_{nR} \) is the \( n \)th KK mode of the right-handed neutrinos, and all of the fermions have been normalized canonically. Notice that the flavor indices have been suppressed for convenience.

Here \( (x, \hat{\theta}_{12}) \) are elements of the group \( SU(N) \) which have to be related with each other due to the condition in Eq. (33), with \( I_m \) \((m = 1, \ldots, N^2 - 1)\) to be generators of the group \( SU(N) \) given in the fundamental representation. This requirement suggests the relation,
Dirac masses are defined by

\[ m_n = \kappa_n v_w, \]

(98)

where \( v_w \) is the electroweak symmetry breaking scale corresponding to the VEV of the SM Higgs doublet. The dimensionless coefficient \( \kappa_n \) contains the information about the corresponding bulk Yukawa coupling constant, the functions determining the diffeomorphism invariance of the left-handed neutrinos under \( G \) and the profile wave function of the \( n \)th KK mode of \( \nu_R \). Such constants are effectively matched as 4-dimensional Yukawa coupling constants of the neutrinos. The Majorana masses occurring in a natural way in the 4-dimensional effective theory are taken as

\[ M_n = \frac{M_{2R}^2}{\Lambda} + \frac{\lambda_{n}^2}{\Lambda} + \sum_i^{D-5} \lambda_{\phi_i} \langle \phi_i \rangle. \]

(99)

The mass \( M_{2R}^2/\Lambda \) comes from the Lagrangian describing the internal dynamics for the right-handed neutrinos. The second term is mass of the \( n \)th KK right-handed neutrinos. The contribution by the last term is produced by the coupling between the right-handed neutrinos and the modulus fields whose form is given by (89). Therefore, the active neutrinos acquire the masses through the 5-dimensional effective operator [58] corresponding to the KK modes of the right-handed neutrinos integrated out as

\[ m_{\nu}^{ac} = m_D M^{-1} m_D^T. \]

(100)

Here \( m_D \) and \( M \) are \((1 \times \infty)\) Dirac and \((\infty \times \infty)\) Majorana mass matrices, respectively, constructed from the above mass expressions. Notice that, a next order contribution for the active neutrino masses is induced by the novel interaction [89]. This is completely analogous to the generation of the small neutrino masses arising from the six dimensional effective operator studied within the NMSSM framework. We can approximately ignore the mixing between the left-handed neutrinos and the KK excitations of the right-handed neutrinos, thus they decouple to the mass spectrum of the active neutrinos given at the lowest order

\[ m_{\nu} \cong \frac{m_{0D}}{M_0}. \]

(101)

Eventually, the smallness of the neutrino masses at the sub-eV scale to be consistent with the neutrino oscillation data is realized via the type I seesaw mechanism which requires the scale \( M_0 \sim 10^6 \) GeV if the Dirac masses \( m_{0D} \) of the neutrinos are of order of the electron mass. Therefore, we can say that the small mass appearance of the observed neutrinos is just of consequences of the higher dimensional space-time geometry.

VI. FIELDS WITH INTERNAL GAUGE CHARGES

We have largely seen above how to find the successfully dynamical description for the 4-dimensional Weyl spinor fields in \( B^D \) which are of the representations of the symmetry group SO(3,1). In this section, we will proceed to be mainly interesting in fields on \( B^D \) carrying out remarkable representations of the symmetry group \( G \) of the internal spaces. These fields will correspond to the conventional source for the gauge fields \( A_{\mu}^a \). Let us consider a field \( \Phi(x, \tilde{\theta}) \) which
is scalar with respect of both the local external coordinate transformation group \( GL(4, \mathbb{R}) \) and the local Lorentz group \( SO(3,1) \), but takes a \( d \)-dimensional representation of the Lie group \( \mathcal{G} \). This multiple is expressed in term of a column vector with \( d \) scalar fields. Under the local coordinate transformations, the field \( \Phi \) rotates as follows

\[
\Phi(x, \hat{\theta}) \rightarrow \Phi'(x', \hat{\theta}') = U \Phi(x, \hat{\theta}),
\]

where,

\[
U = D[h(x)] = \exp\{i\alpha^a(x)M_a\},
\]

is a \( d \times d \) matrix corresponding to a representation of the element \( h(x) \in \mathcal{G} \) given in \([5]\) at which \( M_a \) are also \( d \times d \) matrices associated with the representation of the generators \( T_a \in \mathfrak{g} \). We can make an observation that the transformation \([\text{102}]\) is nearly analogy to that in the usual gauge theory, but here it has a deep connection with the internal structure of the higher dimensional space-time. Thus, the present of such fields in Nature would be specially useful for better intuitive understanding of symmetric aspects that the internal structure of the higher dimensional space-time holds. Moreover, they would play the crucial role as the important source for the physical kinetics of \( A^a_{\mu} \) to generate the geometrical twisting of the bundle \( B^D \). A geometrical interpretation of such field is taken as a local section on the associated vector bundle with the base space \( B^D \) rather than \( M^4 \) as in the original gauge theory.

The physical progress of the field \( \Phi \) in the bulk are determined by the following Lagrangian

\[
\mathcal{L}_\Phi = (D^\mu \Phi)\dagger(D_\mu \Phi) - (\hat{\partial}^i \Phi)\dagger(\hat{\partial}_i \Phi) - V(\Phi),
\]

where the horizontal covariant derivatives \( D_\mu \) are defined in a natural way by the gauge fields \( A^a_\mu \) on \( B^D \) as

\[
D_\mu = \hat{\partial}_\mu + ig_iA^a_\mu M_a.
\]

The above given Lagrangian to be invariant under \([\text{102}]\) gives rise the gauge transformation rule as

\[
A^a_\mu(x')M_a = \frac{\partial x'^\nu}{\partial x^\mu} \left[ UA^a_\mu(x)M_aU^{-1} + \frac{1}{ig_i}U\partial_i U^{-1}\right].
\]

This is to correspond with the transformation \([\text{15}]\), but written in the form of the given representation. The infinitesimal form of \([\text{105}]\) can be obtained with the same result as in Eq. \([\text{16}]\).

The scalar potential \( V(\Phi) \) is introduced in the following form

\[
V(\Phi) = \mu^2\Phi\dagger\Phi + \lambda(\Phi\dagger\Phi)^2.
\]

The mass squared \( \mu^2 \), as analyzed above, involves two independent masses squared, meaning that \( \mu^2 = \mu_{\text{ex}}^2 + \mu_{\text{in}}^2 \), in which \( \mu_{\text{ex}} \) and \( \mu_{\text{in}} \) are the characteristic masses for the dynamics in the external and internal spaces, respectively. The quartic-order coupling constant \( \lambda \) is real and dimensionless. We have also assumed that trilinear couplings of \( \Phi \) violates the gauge invariance \([\text{102}]\) explicitly, since these are absent in the above potential. In addition, the field \( \Phi \) of course couples to the modulus fields (in such a way to be similar as the field \( \Psi \) discussed in the previous section coupled) and the SM Higgs doublet \( H \) taken as

\[
\left(\sum_{i}^{D-5}\lambda_i\tilde{\phi}_i + \sum_{i,j}^{D-5}\lambda_{ij}\tilde{\phi}_i\tilde{\phi}_j\right)\Phi\dagger\Phi, \ \lambda_{\Phi H^\dagger}\Phi H^\dagger H.
\]
where the coupling constants $\lambda_i$ have the mass dimension whereas $\lambda_{ij}$ and $\lambda_{\Phi H}$ are dimensionless ones.

Due to the absence of the Yukawa couplings between $\Phi$ with the fermions given in the preceding section which are not allowed by the transformation [102], so the total Lagrangian possesses an accidentally exact $Z_2$-discrete symmetry

$$\Phi \rightarrow -\Phi.$$  \hspace{1cm} (108)

Without loss of generality, other fields can be taken to transform the trivial way (or being even) under this symmetry. Clearly, this is to come of both the symmetries of the higher dimensional space-time and the renormalizable coupling terms.

In analogy with the standard procedure to construct the kinetic term for the gauge fields in the conventional gauge theory, we need first to define the covariant field strength tensor. The easiest way to derive this tensor is through the commutator of covariant derivatives

$$\hat{F}_{\mu\nu} = \frac{1}{g_i} [D_\mu, D_\nu] = -\frac{1}{\Lambda} F^i_{\mu\nu} \hat{\partial}_i + i F^a_{\mu\nu} M_a.$$  \hspace{1cm} (109)

The Lagrangian for $A^a_{\mu}$ can then be given as

$$\mathcal{L}_A = \frac{1}{2} \left( \frac{M_*}{\Lambda} \right)^{D-4} \text{Tr}[\hat{F}_{\mu\nu} \hat{F}^{\mu\nu}] = \left( \frac{M_*}{\Lambda} \right)^{D-4} \left( -\frac{g_i^2 M_*^2}{4 \Lambda^2} \sum_i e^{\tilde{\phi}_i/M_P} F^i_{\mu\nu} F^{i\mu\nu} - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \right),$$  \hspace{1cm} (110)

where $\text{Tr}$ refers a symmetric inner product determined by

$$\text{Tr}[\hat{\partial}_i \hat{\partial}_j] = -\frac{g_i^2 M_*^2}{2} \delta_{ij} e^{\tilde{\phi}_i/M_P}, \hspace{0.5cm} \text{Tr}[M_a M_b] = \delta_{ab}, \hspace{0.5cm} \text{Tr}[\hat{\partial}_i M_a] = 0.$$  \hspace{1cm} (111)

Comparing with (46) and the last term in (48), we can easily see that the last expression of (110) agrees with the dynamical Lagrangian for the gauge fields $A^a_{\mu}$.

Before proceeding to come the last section, we would like to suggest an extremely natural candidate for the dark matter (DM) coming from the scalar multiple $\Phi$ above if it is neutral under the gauge symmetry of the SM. We consider the case which the $G$ gauge sector is at broken phase to be consistent with the above analysis whose scale would be expected to be higher than the electroweak scale. To maintain the multiple $\Phi$ which plays the role in the dark matter, in addition to this we should introduce a different multiple $\Phi'$ to break spontaneously the gauge symmetry and generate masses for the gauge bosons $A^a_{\mu}$. Since the massive gauge bosons $A^a_{\mu}$ are hidden in the present observed world. For example, with $G$ to be the Lie group SU(2) as we already discussed previously, one can easily check a possibility that $\Phi$ and $\Phi'$ are taken as the double and triplet of this group, respectively. This means that the vacuum would be chosen spontaneously as, $\langle \Phi \rangle = 0$ (thus $\Phi$

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9 The quantum anomaly often caused by the fermions transforming non-triviality under the local gauge symmetry is automatically cancelled in this framework.
would be realized as the inert multiple) and \( \langle \Phi' \rangle \neq 0 \), which is assumed in the existence. It is of the vacuum configuration described by the following surface

\[
F_{\text{vac}}(\langle \Phi \rangle, \langle \Phi' \rangle) = 0. \tag{112}
\]

This associates with much infinitely degenerate ground states defined classically via the vanishing of the first-order derivatives of the potential \( V(\Phi, \Phi') \). Note that due to the potential and its space of minima having the same symmetry since this minimal surface must of course have the isometric group \( G \). As a consequence, the \( Z_2 \) symmetry given in (108) remains to be conserved by \( \Phi \) which guarantees the stability with respect the lightest particle of the old charge under this parity. There has in general the mass splitting in \( \Phi \) by coupling to \( \Phi' \). Hence the lightest component in the zero-mode of the multiple \( \Phi \) is responsible to the dark matter. In this case, the SM Higgs field and the massive gauge bosons \( A_\mu^a \) provides predominantly the portal which links the visible sector (the SM particles) and the dark sector whose origin is to come from the extension of the space-time structure. Notice that, these two sectors can also communicate together via higher dimension operators. Since the DM pair annihilation into the light SM particles and the DM scattering on nuclei are transmitted via both the Higgs field and \( A_\mu^a \). It is important that if mass of the DM candidate is smaller than the half of the mass of Higgs boson determined by ATLAS and CMS collaborations [59, 60], it would contribute in the invisible decay of the Higgs bosons with the current experimental data at LHC given in [61] leading to constrains on its mass and couplings. It is interesting to see that the stability of the dark matter is fully guaranteed by the intrinsically dynamical symmetry of theory rather than the symmetry imposed by the hand. To show the above given analysis more clearly, a detail study of dark matter phenomenology and constraints will be taken in [62].

**VII. CONCLUSIONS AND COMMENTS**

This work has presented the importantly physical implications coming from a remarkably geometrical form of the higher dimensional space-time manifold of \( D \)-dimensions denoted by \( B^D \), also called bulk. We assume that the bulk manifold under the dynamics motivated by the sources is always foliated by \((D - 4)\)-dimensional submanifolds deforming smoothly of the compact connected Lie group \( G \) mainly considered in this paper as \( SU(N) \) and \( SO(N) \) ones. The mathematical description of the bulk is thus given by the principal bundle context. Using the local smooth equivalence, we have appropriately built the local coordinate systems to define each point on \( B^D \) in which an energy scale \( \Lambda \) occurs in the natural way to be physically specific for the internal spaces which are dimensionless in the mathematical description. It is very important to be followed by this geometrical structure that the internal directions and the external directions identified as those of the usual 4-dimensional world exist full independently together in \( B^D \). This is due to the fact that these directions are determined through the vertical and horizontal tangent vectors which are tangent to the fibres and the 4-dimensional submanifolds being transversal to the fibres, respectively. The local bases for the vertical vectors are defined with the given data of \( B^D \). However, with respect to the horizontal vectors these only achieve if the \( g \) valued connection one-form expressed in terms of the gauge fields \( A_\mu^a \) is introduced additionally. These gauge fields will now serve as the physical objects to point precisely out the local directions of the 4-dimensional external spaces. Consequently, an arbitrary particle moving along the external directions would be possible to interact with \( A_\mu^a \). The
corresponding gauge charges are generated dynamically by the evolution in the internal spaces. In this way, our framework is an extension of the conventional gauge interaction in connecting closely to the structure of the space-time.

The inclusion of the pure bulk gravity dynamics is investigated corresponding with the fundamental degrees of freedom in the most minimal scheme consisting of the gauge fields $A_\mu^a$, the 4-dimensional external metric field $g_{\mu\nu}$ and the modulus fields $\phi_i$, $i = 1, \ldots, D - 5$. These fields are unified in the same geometrical framework of the higher dimensional space-time. Remarkably, the potential of the moduli stabilization is constructed in the novel way that gives rise the mechanism to fix dynamically the size of the internal spaces. This is well done due to that the Lie group $G$ acts freely on $B^D$ and transitively on each internal space. It has been explicitly shown, for instance, the Lie group $SU(2)$. The perturbative investigation of the higher dimensional gravity is then given around the vacuum background of the bulk which is determined classically. However, we find that the KK spectrum of the 4-dimensional graviton consists of the unstable massive scalar fields which can be removed if the 4-dimensional external metric is in general conformal diffeomorphism invariance under the action of $G$. This leads to that there have no the KK partners for the usual graviton in the effective low energy theory. Thus, the phenomenological consequences related to the KK gravitons are missed in this framework. The gauge fields $A_\mu^a$ get masses to become massive via the Higgs mechanism corresponding with the gauge symmetry group $G$ spontaneously broken. By the above moduli stabilization potential, the modulus excitations have also heavy masses. The effective bulk gravity in the low energy region is thus described by the usual 4-dimensional Einstein gravity, the non-Abelian gauge theory mediated by the massive gauge bosons, and a set of $(D - 5)$ massive scalar fields which determine the quantum excitations of the modulus fields. In this way, the mechanism to hidden the effects of the extra dimensions in the experimental seeking on accelerators is very natural. Therefore, we are completely not worry the existence of massless particles in the effective low energy theory which would lead to the unwanted violations of the equivalence principle because of contributing to Newton’s law.

We have further studied the dynamics of the 4-dimensional Weyl spinor fields in $B^D$ which are the simplest non-trivial representations of the Lorentz group $SO(3, 1)$. This group is determined as the local symmetry group of the local 4-dimensional inertial frames corresponding with the Minkowski-flat form of the horizontal tangent subspaces. The natural existence of these fields on $B^D$ would thus yield the very favorable light to overcome the chiral fermion problem occurring almost in the 4-dimensional effective theory matched from the higher dimensional theory. An interesting result inherited from this description is that their physical behaviour is manifested in a distinctly different way under the dynamics in the 4-dimensional external and internal spaces. Strictly speaking, observers will see them to behave as the spin-$1/2$ particles under the usual 4-dimensional external point of view but as the scalar-like particles with respect to the internal point of view, i.e. their spin would be lost, and instead of this they carry out the conventional internal symmetry of the Lorentz group $SO(3, 1)$. Based on this analysis, the external Lagrangian for these fields has been found with the usual Dirac one but written in the curved space-time. Whereas the internal Lagrangian has the form in analogy to that of the real scalar field, but only making quantum sense without the classical equivalence. There is an important consequence related to this construction in which the internal dynamic Lagrangian is forbidden with respect to the 4-dimensional Weyl spinor fields of carrying out the usually local gauge charges. Such fields are thus imposed by themselves the active
(conformal) diffeomorphism invariance under the action of the Lie group \( G \). In particular, another interesting result derived provides a possible natural interpretation for the tiny mass origin of the neutrinos observed in the present experiments via the type I see-saw mechanism.

It is of interest to investigate the fields transforming the non-triviality under the local symmetry group of the internal spaces. They are constructed in a similar way as in the context of the usual gauge theory. A geometrical description for these fields is given as the local sections of the associated vector bundle on the \( B^D \) rather than on the base space of \( B^D \). The 4-dimensional Weyl fermions discussed above are obviously neutral under this group. We also suggest a natural candidate for the dark matter coming from these novel fields to be neutral under the gauge group of the SM. An accidentally exact \( Z_2 \)-discrete symmetry conserved in the broken phase of the \( G \) gauge symmetry protects its stability.

Beyond those, a further issue for next work is to consider the supersymmetric extension in which the bulk space-time is included additionally the anticommuting extra dimensions \([63]\). This will be specially interesting by the fact that \( B^D \) contains both 4D Poincaré and gauge symmetries. Therefore, the rules of building the supersymmetry in the previously higher dimensional space-time is clearly impossible to apply in such case.

Finally, before ending this discussion we wish to take the following useful comment. Although we have been only interesting in the internal spaces which are equivalent smoothly to the compact connected Lie group, the above given analysis will be able to be extended to some the manifolds which do not carry out the structure of the Lie group. As well known, quite many manifolds are constructed as the coset space \( G/H \) that is the quotient of Lie group \( G \) and its Lie subgroup \( H \). If we identify physically at the points on the bulk \( B^D \) which belong the same orbit under the action of \( H \), then this result in the internal spaces which are the smooth copies of \( G/H \). Note that, because the action of \( H \) is free so the resulting internal spaces are also smooth manifolds unless the above identification is taken by rather the discrete subgroups than the continuous ones leading to the internal spaces with singularities, or well-known as orbifolds. These have been used in the scenarios of particle physics with extra dimensions to get the low energy realistic particle spectrum as well as the gauge symmetry or supersymmetry breaking.

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Appendix: Non-holonomic functions, Christoffel symbols and bulk scalar curvature

The frame fields on the bulk are given by, \( \{ E_M(X) \} = (\{ \hat{\partial}_\mu \}, \{ \hat{\partial}_i \}) \). The non-holonomic functions \( C^p_{MN} \) are then determined through the Lie bracket of any two frame fields as

\[
[E_M(X), E_N(X)] = C^p_{MN} E_P(X),
\]  
(A.1)
whose explicit expression is defined as follows

\[ C^\lambda_{\mu\nu} = 0, \quad C^i_{\mu\nu} = -\frac{g_i}{\Lambda} F^i_{\mu\nu}, \]  
\[ C^\nu_{\mu i} = -C^\nu_{i\mu} = 0, \quad C^j_{\mu i} = -C^j_{i\mu} = 0, \]  
\[ C^k_{ij} = T^k_{ij}, \]  
(A.2)

With the bulk metric given generally in Eq. (38), the explicit expression for the Christoffel symbols as

\[ \Gamma^\rho_{\mu\nu} = \frac{\partial_x}{2} \left( \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right), \]  
\[ \Gamma^i_{\mu\nu} = \frac{1}{2} \left( \gamma^{ij} \partial_j g_{\mu\nu} - \frac{g_i}{\Lambda} F^i_{\mu\nu} \right), \]  
\[ \Gamma^\nu_{\mu i} = \Gamma^\nu_{i\mu} = \frac{1}{2} \frac{g^\nu_{ij}}{\gamma_{ij}} \partial_\mu \gamma_{ik}, \]  
\[ \Gamma^\mu_{ij} = \frac{1}{2} g^{\mu\nu} \partial_\nu \gamma_{ij}, \]  
\[ \Gamma^k_{ij} = \frac{1}{2} \left[ \gamma_m^{kl} \left( T^m_{ij} \gamma_{mi} + T^m_{ij} \gamma_{mi} \right) + T^k_{ij} \right]. \]  
(A.5)

The scalar curvature of the bulk is defined by

\[ R = \hat{R} + \frac{1}{4} \partial_\mu g^{\mu\nu} \partial^\nu g_{\mu\nu} - \frac{1}{4} \left( g^{\mu\nu} \partial_\rho g_{\mu\nu} \right) \left( g^\rho_{\lambda\nu} \partial_\lambda g^\nu_{\rho\lambda} \right) - \frac{1}{4} \partial_\mu \gamma_{ij} \partial_\nu \gamma_{ij} + \frac{1}{4} \left( \gamma_{ij} \partial_\mu \gamma_{ij} \right) - V(\gamma_{ij}) - \frac{g_i^2}{4\Lambda^2} \gamma_{ij} F^i_{\mu\nu} F^{j\mu\nu} + \sum_{i=1}^{4} \nabla_M X^M_i, \]

where \( \hat{R} \) is the 4-dimensional standard scalar curvature given in Eq. (53), the potential \( V(\gamma_{ij}) \) for the internal metric corresponding to the scalar curvature of the internal spaces is given as

\[ V(\gamma_{ij}) = \Lambda^2 \left( -\frac{1}{2} f^k_{ij} f^l_{kj} \gamma_{ij} + \frac{1}{4} f^p_{ik} f^q_{jk} \gamma_{ij} \gamma_{kl} \gamma_{pq} \right), \]  
(A.11)

and

\[ X^M_1 = \begin{pmatrix} -g^{\mu\nu} \gamma_{ij} \partial_\nu \gamma_{ij} \\ 0 \end{pmatrix}, \quad X^M_2 = \begin{pmatrix} 0, \Lambda \gamma_{ij} g^{\mu\nu} \partial_\nu g_{\mu\nu} \end{pmatrix}, \]  
\[ X^M_3 = \begin{pmatrix} g^{\mu\nu} \gamma_{ij} \partial_\nu \gamma_{ij} \\ 0 \end{pmatrix}, \quad X^M_4 = \begin{pmatrix} 0, -\Lambda \gamma_{ij} g^{\mu\nu} \partial_\nu g_{\mu\nu} \end{pmatrix}. \]  
(A.12)

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