Generalized gauge transformations in phase space picture of quantum mechanics: Kirkwood representation

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Abstract. Here, the ordinary gauge transformations in configuration space are generalized to the phase space. The corresponding gauge functions are separable in terms of those of the configuration and the momentum spaces. Furthermore, we consider the gauge transformations in phase space for which their gauge functions are not, in general, separable. It is shown that, the assumption of the general global gauge symmetry in phase space immediately yields the quantization of the electric charge. We obtain this result without assuming the concept of the magnetic monopole that Dirac did in his approach to the problem of electric charge quantization and which is not detected yet.

1. Introduction
In 1932 the phase space approach to quantum mechanics is proposed by Wigner [1]. Since then, the phase space picture of quantum mechanics is extended and formulated as different representations [2]. Recent years have seen a surge of interest in applying the representations known as the Husimi [1], the Glauber-Sudarshan [1, 5], Q- [6], generalized P- [7], and Kirkwood representations [8], to the problems encountered in quantum mechanics, quantum statistical mechanics, quantum optics, condensed matter physics, etc. [9, 10, 11]. Different classes of symmetry transformations in phase space picture of quantum mechanics including the rotation, translation, squeezing and boost are extensively investigated by Kim et al [12]. As an important symmetry, the gauge symmetry is widely studied in configuration space by different authors [13, 14, 15]. Here we study the gauge symmetry using the Kirkwood representation of the phase space. We introduce the gauge transformations in phase space corresponding to the ordinary gauges in the configuration space. We show that the gauge functions in the phase space are separable as an additive form in terms of those in configuration and momentum spaces.

We further generalized the gauge transformations in phase space to include the cases that are not, in general, separable. We show that, the Kirkwood equation giving the temporal evolution of Kirkwood distribution functions is invariant under the general gauge symmetry. It can be easily shown that the general gauge transformations are a subset of the canonical transformations in the extended phase space [16]. We show that an immediate result of the general global gauge symmetry, as a special case of the general gauge symmetry in phase space, is the quantization of the electric charge.
In 1932 P.A.M. Dirac formulated the quantization of the electric charge by assuming the dual symmetry in Maxwell’s equation as well as the existence of the magnetic monopole [17]. Despite the elaborate and mathematically sophisticated approach of Dirac to the problem of the charge quantization, unfortunately, the magnetic monopole has not already been detected. However, in our approach, the existence of the magnetic monopole is no longer necessary to obtain the quantization of electric charge. In the next section a review of the ordinary gauge transformations in quantum mechanics is given. In Section 3 the generalization of the gauge symmetry in phase space quantum mechanics is presented. This is followed by introducing the relations between gauge functions in configuration, momentum and phase space. In section 4 the quantization of electric charge is obtained by assuming the global gauge symmetry in phase space. Section 5 is devoted to the conclusions and remarks.

2. Ordinary gauge transformations

In quantum mechanics, the gauge transformations in configuration representation are defined as unitary transformations on the wave functions as follows

$$\psi'(q,t) = e^{-i\alpha \Lambda} \psi(q,t),$$

where $\alpha = e/\hbar c$. Here one can distinguish between three kinds of gauges: 1) the global gauge defined as $\Lambda = \text{Const.}$; 2) the local defined as $\Lambda = \Lambda(q,t)$ and finally 3) the nonlinear gauges defined as $\Lambda = \Lambda(\psi, \psi^*, q,t)$ [18]. Total electric charge is denoted by "$e$". Similarly, in momentum representation, one may assume that the wave function $\phi(p,t)$ evolves by a gauge transformation as

$$\phi'(p,t) = e^{-i\alpha \Gamma} \phi(p,t),$$

where $\Gamma = \Gamma(p,t)$ is the gauge function in momentum space. In momentum representation one still has three kinds of gauges corresponding to those of configuration representation mentioned above. One may emphasize that the Schrödinger equation

$$H(-i\hbar \frac{\partial}{\partial q}, q)\psi(q,t) = i\hbar \frac{\partial \psi(q,t)}{\partial t},$$

is invariant under the gauge transformations, i.e.,

$$H'(-i\hbar \frac{\partial}{\partial q}, q)\psi'(q,t) = i\hbar \frac{\partial \psi'(q,t)}{\partial t}.$$ 

Therefore, Hamiltonian transforms as

$$H \rightarrow H' = e^{i\alpha \Lambda} H e^{-i\alpha \Lambda} - i\hbar e^{i\alpha \Lambda} \partial (e^{-i\alpha \Lambda}) / \partial t.$$

The minimal coupling Hamiltonian of a system of charged particles interacting with an electromagnetic field is given by

$$H = \frac{(P - \frac{e}{c} A(q,t))^2}{2m} + V(q) - \frac{e}{c} \phi(q,t).$$

One may look to the gauge transformations of minimally coupled Hamiltonian

$$H' = \frac{(P - \frac{e}{c} A'(q,t))^2}{2m} + V(q) - \frac{e}{c} \phi'(q,t),$$

from two equivalent points of view: 1) In the conventional approach, the effect of a gauge transformation on the vector and scalar potentials are as $A' = A + \nabla \Lambda$ and $\phi' = \phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t}$. One
can show that the Maxwell’s equations are invariant using the new vector and scalar potentials. In other words, the electromagnetic fields are invariant under a gauge transformation. However, except in a special gauge, the Hamiltonian and conjugate momentum, are not, in general, the conventional energy and kinetic momentum, respectively.

2) An alternative look to the gauge transformations is to consider them as canonical transformations on the phase space coordinates, i.e.,

\[
\begin{align*}
Q &= q, \\
P &= p + \nabla \Lambda, \\
H' &= H + \frac{1}{c} \frac{\partial \Lambda}{\partial t}.
\end{align*}
\]

For an arbitrary \( \Lambda(q, t) \) this transformation is canonical, where the corresponding transformation matrix is

\[
M = \begin{pmatrix}
1 & 0 \\
\nabla^2 \Lambda & 1
\end{pmatrix},
\]

and satisfies \( MJ\tilde{M} = J \). The gauge symmetry of quantum electrodynamics in configuration space is responsible for conservation of the electric charge.

3. Gauge Transformations in Phase Space
Among infinitely many representations of Phase space quantum mechanics, the Kirkwood representation is a possible one [8]. In this representation the distribution function is defined as

\[
\chi = \psi\phi^* e^{-ipq/\hbar},
\]

and satisfies a Schrödinger like equation

\[
h\chi = i\hbar \frac{\partial \chi}{\partial t}.
\]

In Eq. (11), the extended Hamiltonian \( h \) is defined as [8]

\[
h = H(p + \pi_q, q) - H(p, q + \pi_p).
\]

One can use Eqs. (1), (2) and (10), to obtain the gauge transformed distribution function

\[
\chi' = \psi'\phi'^* e^{-ipq/\hbar} = e^{-i\varepsilon/\hbar} \psi e^{i\Gamma/\hbar} \phi e^{-ipq/\hbar} = e^{-i\varepsilon(p, q, t)/\hbar} \chi.
\]

Therefore, the distribution functions as well as the wave functions evolve by unitary transformations and the gauge function in phase space, \( g(p, q, t) \), is separable in terms of gauge functions in configuration, \( \Lambda \), and momentum, \( \Gamma \), spaces, respectively

\[
g(p, q, t) = \Lambda(q, t) - \Gamma(p, t).
\]

The gauge functions \( \Lambda \) and \( \Gamma \), in general, are not independent. To understand the relation between gauge functions in configuration and momentum space, let’s have a look to the integral form of the Kirkwood distribution function of Eq. (10), i.e., [19]

\[
\chi = \frac{1}{2\pi\hbar} \int \psi(q)\psi^*(q + \lambda)e^{ip\lambda/\hbar} d\lambda.
\]
Applying the same gauge transformation on Eq. (15), one gets

\[ \chi' = e^{-ie(\Lambda - \Gamma)/\hbar c} \chi \]

\[ = \frac{1}{2\pi\hbar} \int e^{-ie\Lambda(q)/\hbar c} \psi(q) e^{ie\Lambda(q+\lambda)/\hbar c} \psi^*(q + \lambda) e^{ip\lambda/\hbar} d\lambda, \]

\[ \rightarrow e^{ie\Lambda/\hbar c} = \frac{\int e^{ie\Lambda(q')/\hbar c} \psi^*(q') e^{ipq'/\hbar} dq'}{\int \psi^*(q') e^{ipq'/\hbar} dq'}. \] (16)

Equation (16) shows the relation between the gauge functions in configuration and momentum space and could be used to construct the general gauge function in phase space, for a given ordinary gauge in the configuration space. One may release the separable condition on Eq. (14), and look after a larger class of gauge functions that may include that of Eq. (14) as a special case. Then, a general gauge transformation on the extended Hamiltonian for a particle in a potential \( V(q) \), is given by

\[ h' = \frac{(p + \pi_q - \kappa \nabla_q g)^2}{2m} - \frac{p^2}{2m} + V(q + \pi_p - \kappa \nabla_p g) - \hbar \kappa \frac{\partial g}{\partial t}. \] (17)

Applying the separation condition, Eqs. (14), one gets

\[ h' = \left\{ \frac{(p + \pi_q - \kappa \nabla_q g)^2}{2m} + V(q) \right\} - \left\{ \frac{p^2}{2m} + V(q + \pi_p - \kappa \nabla_p \Gamma) \right\} - \hbar \kappa \frac{\partial (\Lambda - \Gamma)}{\partial t}. \] (18)

It can be easily shown that for arbitrary gauge functions \( \Lambda(q, t) \) and \( \Gamma(p, t) \), the Hamiltonian (17) or (18) could be obtained by the following extended canonical transformation on \( h \)

\[ q \rightarrow q, \quad \pi_q \rightarrow \pi_q - \kappa \nabla g, \]

\[ p \rightarrow p, \quad \pi_p \rightarrow \pi_p - \kappa \nabla p g. \] (19)

Therefore, the dynamical equation for the distribution function, Eq. (11), is invariant under the canonical transformations (19) in phase space. Due to the arbitrariness of general gauge function \( g(p, q, t) \), the general gauge transformation is the gauge symmetry transformation of our system in phase space. The set of ordinary gauge operators, \( G' \), is a subset of general gauge operators \( G \), i.e.,

\[ \forall g = \Lambda - \Gamma \in G' | G' \subset G. \] (20)

By extending the Hamiltonian (6) by the extension rule given in [16] one finds

\[ h' = \left\{ \frac{(p + \pi_q - \xi A(q, t))^2}{2m} + v(q) + e\phi(q, t) \right\} \]

\[ - \left\{ \frac{(p - \xi A(q + \pi_p, t))^2}{2m} + v(q + \pi_p) + e\phi(q + \pi_p, t) \right\}, \] (21)

and applying a general gauge symmetry (19) one gets

\[ h' = \left\{ \frac{(p + \pi_q - \xi A(q, t) - \alpha \nabla_q g(p, q, t))^2}{2m} + v(q) + e\phi(q, t) \right\} \]

\[ - \left\{ \frac{(p - \xi A(q + \pi_p - \alpha \nabla p g(p, q, t), t))^2}{2m} + v(q + \pi_p - \alpha \nabla p g(p, q, t)) \right\} \]

\[ + e\phi(q + \pi_p - \alpha \nabla p g(p, q, t), t) - \hbar \alpha \frac{\partial g(p, q, t)}{\partial t}. \] (22)
Clearly the operations of extension and general gauge transformations do not commute. If one applies a general gauge transformation before extension of Hamiltonian, the gauge functions in configuration space will contain the conjugate momenta as well as the coordinates, while, the theory of gauge symmetry emphasize that the gauge functions should be in terms of the coordinates and not the conjugate momenta. Therefore, the extended Hamiltonian (22), which follows a gauge transformation after extension, is conventionally preferred with respect to one made by the extension of a gauge transformed Hamiltonian. Note that for the global gauges in phase space, where the gauge functions are constant, or at most are a function of time, \( g = g(t) \), the operation of extension and gauge transformations do commute. In contrast to the configuration space, one cannot interpret a general gauge transformation of the extended Hamiltonian (22), as a transformation on vector and scalar potentials (6). As mentioned after Eq. (7), the ordinary gauge transformations could be interpreted either as the change in the vector and the scalar electromagnetic potentials or as a canonical transformations in the configuration space. However, in the case of the general gauge transformation it is clear from Hamiltonian (22) that it can only be interpreted as a canonical transformation in the extended phase space.

In the general gauge transformed Hamiltonian (22), the changes in the vector potential appears in two different ways: i) in the first half of Eq. (22) the vector potential changes as \( A'(q, t) = A(q, t) + \alpha \nabla p g(p, q, t) \), ii) where in the second half one has \( A'(q, t) = A(q + \pi_p - \alpha \nabla p g(p, q, t), t) \). The same is true for scalar potential. Therefore, the change in the vector and scalar potentials has no unique definition in the general gauge transformed extended Hamiltonian (23), although, both form of general gauge transformed vector and scalar potentials satisfy the Maxwell equations.

4. Charge Quantization

From Eq. (16) it can be easily shown that a gauge function in momentum space corresponding to a global gauge in configuration space, is itself a global gauge, too. If one substitute \( \Lambda = \text{Const.} \) in Eq. (16) then

\[
e^{\frac{ie\Gamma}{\hbar c}} = e^{\frac{ie\Lambda}{\hbar c}} \Lambda = \frac{e}{\hbar c} \Gamma + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots
\]

(23)

All ordinary global gauges in configuration space are mapped to the gauges in phase space with discrete values, where

\[
g = \Lambda - \Gamma = \left( \frac{2\pi c}{e} \right)n\hbar.
\]

(24)

Equation (24) is a additional quantization condition in phase space representation of quantum mechanics that gives the quantization of electric charge. In Eq. (24) one may set \( n = 1 \) to obtain the quanta of electric charge \( e_1 \), then

\[
g = 2\pi\hbar/e_1,
\]

(25)

where quanta of charge \( e_1 \) is the charge of electron. Substituting Eq. (25) in Eq. (24) gives the quantization of electric charge

\[
e = ne_1,
\]

(26)

as a special property of the general gauge invariance in phase space quantum mechanics. The charge quantization first proposed in 1931 by P.A.M. Dirac was as follows

\[
e e_m = 2\pi c n\hbar.
\]

(27)

He obtained it by assuming the dual symmetry of Maxwells equations and the concept of magnetic monopole with the "magnetic charge" \( e_m \). The assumption of existence of magnetic
monopole is essential in the derivation of charge quantization in the Dirac approach. Nevertheless a great effort has been done during the past 76 years, no experimental evidence has been observed for detecting the magnetic monopoles. In our method, however, the assumption of the magnetic monopole is not necessary.

5. Conclusions
Symmetries and the corresponding conservation laws, in general, play an important role in the study of physical problems. Here, the Kirkwood representation of phase space quantum mechanics is considered and its gauge symmetry is investigated. The relation between the gauge functions in configuration and momentum space and their corresponding general gauge function in the phase space are obtained by extending the ordinary gauge transformations in configuration space into phase space. An immediate result of the general global gauge symmetry is the phase space is shown to be the quantization of the electric charge. In contrast to the Dirac’s method, which is based on the existence of the magnetic monopole that has never been identified, we obtain the quantization of electric charge on the basis of the global gauge symmetry property inherent in the phase space.

References
[1] Wigner E P 1932 Phys. Rev. 40 749.
[2] Lee H W 1995 Physics Reports 259 147.
[3] Husimi K 1940 Proc. Phys. Math. Soc. Jpn. 22 264.
[4] Glauber R J 1963 Physical Review 130 2529.
[5] Sudarshan E C G 1963 Physical Review Letter 10 277.
[6] Mehta C L 1964 J. Math. Phys. 5 677.
[7] Drummond P D and Gardiner C W 1980 J. Phys. A: Math. Gen. 13 2353.
[8] Nasiri S, Sobouti Y and Taati F 2006 Journal of Math. Physics 47 092106.
[9] Sirera M and Prez A 1999 Physical Review D 52 125011.
[10] Nouri S 1998 Physical Review A 57 1526.
[11] Khademi S and Nasiri S 2002 Iranian Journal of Science and Technology 20 1.
[12] Kim Y S and Noz M E 1991 Phase Space Picture of Quantum Mechanics (New York: World Scientific).
[13] Kobe D H and Wen E C T 1980 Phys. Letter 80A 121.
[14] Kobe D H 1982 Am. J. Phys. 50 128.
[15] Baxter C 1991 Phys. Review A 44 3178.
[16] Sobouti Y and Nasiri S 1993 Int. J. Mod. Phys. B 7 3255.
[17] Dirac P A M 1932 Proc. R. Soc. A 133 60.
[18] Doebner H D and Goldin G A 1994 J. Phys. A: Math. Gen. 27 1771.
[19] Hillery M, OConnell R F, Scully M O and Wigner E P 1984 Phys. Rep. 106 121.