A Note on the Gessel Numbers

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Abstract

The Gessel number $P(n, r)$ represents the number of lattice paths in a plane with unit horizontal and vertical steps from $(0, 0)$ to $(n + r, n + r - 1)$ that never touch any of the points from the set $\{(x, x) \in \mathbb{Z}^2 : x \geq r\}$. In this paper, we use combinatorial arguments to derive a recurrence relation between $P(n, r)$ and $P(n - 1, r + 1)$. Also, we give a new proof for a well-known closed formula for $P(n, r)$. Moreover, a new combinatorial interpretation for the Gessel numbers is presented.

Keywords: Gessel numbers, Catalan numbers, central binomial coefficient, lattice paths.

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1 Introduction

Let $n$ be a non-negative integer and let $r$ be a positive integer. The Gessel number $P(n, r)$ counts all lattice paths [1, p. 191] in plane with $(1, 0)$ and $(0, 1)$ steps from $(0, 0)$ to $(n + r, n + r - 1)$ that never touch any of the points from the set $\{(x, x) \in \mathbb{Z}^2 : x \geq r\}$.

By using a combinatorial argument and an instance of the Pfaff-Saalschütz theorem, Gessel proved that (see [1, p. 191])

$$P(n, r) = \frac{r}{2(n + r)} \binom{2n}{n} \binom{2r}{r}.$$  (1)

Let $C_n = \frac{1}{n+1} \binom{2n}{n}$ denote the $n$th Catalan number. The Catalan numbers are well studied in the literature (e.g. see [3, 7]) and there are many counting problems in combinatorics whose solutions are given by the Catalan numbers. For example, the Catalan number $C_n$ is the number of all paths in a plane from $(0, 0)$ to $(n, n)$ with $(1, 0)$ and $(0, 1)$ steps such that they never rise above the line $y = x$ (see [3, Example 9.1], [6, Problem 158], and [4, Eq. (10.11)]).

By setting $r = 1$ in Equation (1), it follows that $P(n, 1) = C_n$. Recently, it is shown [5, Cor. 4] that the sum $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m C_k C_{2n-k}$ is divisible by $\binom{2n}{n}$ for all non-negative integers $n$ and for all positive integers $m$. The Gessel numbers play an interesting role in this proof [5, Eq. (68)].
It is known [1, p. 191] that, for a fixed positive integer $r$, the smallest positive integer $K$, such that $\frac{K}{n+r} \binom{2n}{n}$ is an integer for every $n$ is $\frac{r}{2} \binom{2r}{r}$. Guo gave a generalization of the Gessel numbers [2, Eq. (1.10)].

In this paper, we study the Gessel numbers by establishing a close relationship between the Gessel and the Catalan numbers. After we establish such a relationship, we present a new combinatorial interpretation for the Gessel Numbers. This combinatorial interpretation is contained in our main result, Theorem 6.

2 Main results

Our first result expresses $P(n, r)$ in terms of the Catalan numbers.

**Proposition 1.** Let $n$ be a non-negative integer and let $r$ be a positive integer. Then

$$P(n, r) = \sum_{k=0}^{r-1} \binom{2k}{k} C_{n+r-k-1}.$$  \hspace{1cm} (2)

**Proof.** Let $k$ be a non-negative integer, and let $p$ be a path from $(0, 0)$ to $(n+r, n+r-1)$, with $(1, 0)$ and $(0, 1)$ steps, that never touches any of the points from the set $\{(x, x) \in \mathbb{Z}^2 : x \geq r\}$. Assume that the last point point of intersection of $p$ and the line $y = x$, when taking the path $p$ from left to right, is the point $(k, k)$, where $0 \leq k \leq r-1$.

The number of “permitted” paths from $(0, 0)$ to $(k, k)$ is $\binom{2k}{k}$, since every such path does not contain “forbidden” points. Every such path passes through the point $(k+1, k)$, as $(k, k)$ is the last intersection point between this path and the line $y = x$. Note that the segment that connects points $(k+1, k)$ and $(n+r, n+r-1)$ is parallel with the line $y = x$.

The number of “permitted” paths from $(k+1, k)$ to $(n+r, n+r-1)$ is the same as the number of all paths from $(k+1, k)$ to $(n+r, n+r-1)$ with $(1, 0)$ and $(0, 1)$ steps that never rise above the line $y = x - 1$. It follows that there are $C_{n+r-k-1}$ such paths.

Therefore, the number of all paths whose last intersection with the line $y = x$ is the point $(k, k)$ is $\binom{2k}{k} C_{n+r-k-1}$. Since $k$ takes values from 0 to $r-1$, we obtain Equation (2). \hfill \Box

We use the previous proposition to obtain a recurrence relation between $P(n-1, r+1)$ and $P(n, r)$.

**Theorem 2.** Let $n$ and $r$ be positive integers. Then

$$P(n-1, r+1) - P(n, r) = \binom{2r}{r} C_{n-1}.$$  \hspace{1cm} (3)

**Proof.** The number $P(n-1, r+1)$ counts all paths from $(0, 0)$ to $(n+r, n+r-1)$, with $(1, 0)$ and $(0, 1)$ steps, that never touch any of the points from the set $\{(x, x) \in \mathbb{Z}^2 : x \geq r+1\}$. 2
Therefore, the number $P(n-1, r+1) - P(n, r)$ counts all paths from $(0,0)$ to $(n+r, n+r-1)$, with $(1,0)$ and $(0,1)$ steps, whose last point of intersection with the line $y = x$ is the point $(r, r)$.

The number of “permitted” paths from $(0,0)$ to $(r, r)$ is $\binom{2r}{r}$, since every such path does not contain “forbidden” points.

After $(r, r)$, every such path must pass through $(r+1, r)$. Note that the segment that connects points $(r+1, r)$ and $(n+r, n+r-1)$ is parallel with the line $y = x$. The number of “permitted” paths from $(r+1, r)$ to $(n+r, n+r-1)$ is the same as the number of all paths from $(r+1, r)$ to $(n+r, n+r-1)$, with $(1,0)$ and $(0,1)$ steps, that never rise above the line $y = x - 1$. By a well-known [4, Eq. (10.11)] property of the Catalan numbers, it follows that there are $C_{n-1}$ such paths.

Therefore, the number of all paths whose last point of intersection with the line $y = x$ is the point $(r, r)$ is $(\binom{2r}{r}) C_{n-1}$.

By using recurrence relation (3) and induction on $n$, we give a proof of Equation (1).

Let $S(n, r)$ denote $\frac{r}{2(n+r)} \binom{2n}{n} \binom{2r}{r}$. We will show that $P(n, r) = S(n, r)$ for all non-negative integers $n$ and for all positive integers $r$. We use induction on $n$.

For $n = 0$, since the “final” point $(r, r - 1)$ is below the first “forbidden” point $(r, r)$, the Gessel number $P(0, r)$ counts all paths in a plane from $(0,0)$ to $(r, r - 1)$ with $(1,0)$ and $(0,1)$ steps without any restrictions. Hence, $P(0, r) = \binom{2r-1}{r}$ or $P(0, r) = \frac{1}{2} \binom{2r}{r}$.

Therefore, it follows that $P(0, r) = S(0, r)$ for all positive integers $r$.

Let us assume that $P(n-1, r) = S(n-1, r)$ for some positive integer $n$ and for all positive integers $r$.

We use a well-known [3, p. 26] recurrence relation for the central binomial coefficient:

$$\binom{2(r+1)}{r+1} = \frac{2(2r+1)}{r+1} \binom{2r}{r}.$$ (4)

Then we have that the following equalities hold:

$$P(n, r) = P(n-1, r+1) - \binom{2r}{r} C_{n-1} \quad \text{(by Equation (3))}$$

$$= S(n-1, r+1) - \binom{2r}{r} C_{n-1} \quad \text{(by the induction hypothesis)}$$

$$= \frac{r+1}{2(n+r)} \binom{2(n-1)}{n-1} \binom{2(r+1)}{r+1} - \binom{2r}{r} C_{n-1}$$

$$= \frac{r+1}{2(n+r)} \binom{2(n-1)}{n-1} 2(2r+1) \binom{2r}{r} - \binom{2r}{r} C_{n-1} \quad \text{(by Equation (4))}$$
This completes our proof by induction.

**Definition 3.** Let $n$ be a non-negative integer and let $r$ be a positive integer. The number $Q(n, r)$ counts all lattice paths in a plane with $(1, 0)$ and $(0, 1)$ steps from $(0, 0)$ to $(n+r, n+r-1)$ that never touch any of the points from the set $\{(x, x) \in \mathbb{Z}^2 : 1 \leq x \leq r\}$.

**Proposition 4.** Let $n$ and $r$ be positive integers. Then

$$Q(n, r) = \sum_{k=1}^{n} C_{r+k-1} \binom{2(n-k)}{n-k}. \quad (5)$$

**Proof.** Let $k$ be a non-negative integer, and let $p$ be a path from $(0, 0)$ to $(n+r, n+r-1)$, with $(1, 0)$ and $(0, 1)$ steps, that never touches any of the points from the set $\{(x, x) \in \mathbb{Z}^2 : 1 \leq x \leq r\}$. There are two cases to consider.

**The first case:** A path $p$ intersects the line $y = x$ only at the point $(0, 0)$. In this case, a path $p$ must begin with a $(1, 0)$ step. Note that the segment that connects the points $(1, 0)$ and $(n+r, n+r-1)$ is parallel with the line $y = x$. The number of “permitted” paths from $(1, 0)$ to $(n+r, n+r-1)$ is the same as the number of all paths from $(1, 0)$ to $(n+r, n+r-1)$, with $(1, 0)$ and $(0, 1)$ steps, that never rise above the line $y = x - 1$. It follows that there are $C_{n+r-1}$ such paths.

**The second case:** A path $p$ intersects the line $y = x$ in at least two points. Let $(r+k, r+k)$ be the first point of intersection between $p$ and the line $y = x$ after the point $(0, 0)$. Here, $1 \leq k \leq n-1$. Note that, in this case, $n \geq 2$.

Let $m$ be a positive integer. It is readily verified that the number of all paths in a plane from $(0, 0)$ to $(m, m)$, with $(1, 0)$ and $(0, 1)$ steps, that intersect the line $y = x$ only at points $(0, 0)$ and $(m, m)$ is $2C_{m-1}$. Therefore, the number of “permitted” paths from $(0, 0)$ to $(r+k, r+k)$ is $2C_{r+k-1}$. 

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The number of “permitted” paths from \((r+k, r+k)\) to \((n+r, n+r-1)\) is the same as the number of all paths from \((r+k, r+k)\) to \((n+r, n+r-1)\) with \((1,0)\) and \((0,1)\) steps, since every such path does not contain “forbidden” points. It follows that there are 
\[
\frac{1}{2} \binom{2(n-k)}{n-k}
\] such paths (see [4, Equation (10.3)]).

Therefore, the number of all paths that intersect the line \(y = x\) at the point \((r+k, r+k)\) for the first time after the point \((0,0)\) is \(C_{r+k-1}(2(n-k))\). Since \(k\) can take values from 1 to \(n-1\), it follows that there are 
\[
\sum_{k=1}^{n-1} C_{r+k-1}\left(\frac{2(n-k)}{n-k}\right)
\] such paths.

Putting ever together, it holds:
\[
Q(n,r) = C_{n+r-1} + \sum_{k=1}^{n-1} C_{r+k-1}\left(\frac{2(n-k)}{n-k}\right)
\]
\[
= \sum_{k=1}^{n} C_{r+k-1}\left(\frac{2(n-k)}{n-k}\right).
\]

\(\Box\)

Remark 5. Note that, for \(n = 0\), the number \(Q(0,r)\) is equal to \(C_{r-1}\).

We now use Proposition 1 and Proposition 4 to prove our main result that gives us a new combinatorial interpretation for the Gessel numbers.

**Theorem 6.** Let \(n\) and \(r\) be positive integers. Then
\[
P(n,r) = Q(r,n).
\] (6)

**Proof.** By setting \(n := r\) and \(r := n\) in Proposition 4, it follows that
\[
Q(r, n) = \sum_{k=1}^{r} C_{n+k-1}\left(\frac{2(r-k)}{r-k}\right).
\] (7)

By substituting \(t\) for \(r-k\), it follows that Equation (7) becomes
\[
Q(r, n) = \sum_{t=0}^{r-1} C_{n+r-t-1}\left(\frac{2t}{t}\right).
\] (8)

By using Proposition 1 and Equation (8), it follows that \(Q(r, n) = P(n,r)\). \(\Box\)

Theorem 6 gives a new combinatorial interpretation for Gessel numbers. By Theorem 6, the Gessel number \(P(n,r)\) is the number of all lattice paths in a plane with \((1,0)\) and \((0,1)\) steps from \((0,0)\) to \((n+r, n+r-1)\) that never touch any of the points from the set \(\{(x,x) \in \mathbb{Z}^2 : 1 \leq x \leq n\}\).
3 Concluding remarks

We end this paper with some formulas for the Gessel numbers. Let \( n \) be a non-negative integer, and let \( r \) be a positive integer. By Equation (1), Theorem 6, and Remark 5, it follows that

\[
Q(n, r) = \begin{cases} 
C_{r-1}, & \text{if } n = 0 \\
\frac{n}{2(n+r)} \binom{2n}{n} \binom{2r}{r}, & \text{if } n > 0.
\end{cases}
\]  

(9)

Let \( n \) and \( r \) be positive integers. Then the following formulas are true:

\[
\frac{1}{2} \binom{2n+2r}{n+r} - P(n, r) = \sum_{k=1}^{n} \binom{2(r+k-1)}{r+k-1} C_{n-k},
\]  

(10)

\[
\frac{1}{2} \binom{2n+2r}{n+r} - Q(n, r) = \sum_{l=1}^{r} \binom{2(n+r-l)}{n+r-l} C_{l-1}.
\]  

(11)

The left side of Equation (10) represents the number of all lattice paths in a plane with \((1,0)\) and \((0,1)\) steps from \((0,0)\) to \((n+r, n+r-1)\) whose intersection with the set \(\{(x,x) \in \mathbb{Z}^2 : r \leq x \leq n+r-1\}\) is non-empty. It is readily verified that there are \(\binom{2(r+k-1)}{r+k-1} C_{n-k}\) lattice paths in a plane with \((1,0)\) and \((0,1)\) steps from \((0,0)\) to \((n+r, n+r-1)\) whose last point of intersection with the set \(\{(x,x) \in \mathbb{Z}^2 : r \leq x \leq n+r-1\}\) is the point \((r+k-1, r+k-1)\). Here, \(1 \leq k \leq n\).

Similarly, the left side of Equation (11) represents the number of all lattice paths in a plane with \((1,0)\) and \((0,1)\) steps from \((0,0)\) to \((n+r, n+r-1)\) whose intersection with the set \(\{(x,x) \in \mathbb{Z}^2 : 1 \leq x \leq r\}\) is non-empty. It is readily verified that there are \(\binom{2(n+r-l)}{n+r-l} C_{l-1}\) lattice paths in a plane with \((1,0)\) and \((0,1)\) steps from \((0,0)\) to \((n+r, n+r-1)\) whose first point of intersection with the set \(\{(x,x) \in \mathbb{Z}^2 : 1 \leq x \leq r\}\), after the point \((0,0)\), is the point \((l,l)\). Here, \(1 \leq l \leq r\).

Note that, by using Equations (10) and (11), one can give another proof of Theorem 6.

Remark 7. By using a combinatorial argument, Gessel proved [1, Equation (39)] the following formula:

\[
\sum_{k=0}^{n} P(k, r) \binom{2n-2k}{n-k} = \frac{1}{2} \binom{2n+2r}{n+r},
\]  

(12)

where \( n \) is a non-negative integer and \( r \) is a positive integer. It is known that Equation (12) uniquely determines the numbers \( P(n, r) \). Gessel used Equation (12) in order to prove Equation (1).

Let \( n \) and \( r \) be positive integers. By using Equation (12) and Theorem 6, it can be proved that

\[
\sum_{k=0}^{r-1} \binom{2k}{k} Q(n, r-k) = \frac{1}{2} \binom{2n+2r}{n+r} - \frac{1}{2} \binom{2n}{n} \binom{2r}{r}.
\]  

(13)

Note that, for positive integers \( n \), Equation (13) uniquely determines the numbers \( Q(n, r) \).
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