The Beta Power Muth Distribution: Regression Modeling, Properties and Data Analysis

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Abstract

In this paper, we propose a new flexible lifetime distribution. The proposed distribution will be referred to as beta power Muth distribution. It can be used to model increasing, decreasing, bathtub shaped or upside-down bathtub hazard rates. Some properties of the new model are obtained including moments, quantile function and moments of the order statistics. The unknown model parameters are estimated by the maximum likelihood method of estimation. A Monte Carlo simulation study is carried out to assess the performance of the maximum likelihood estimates. Two reliability data sets are applied to illustrate the usefulness and flexibility of the proposed model. In addition, we introduce a new location-scale regression model based on the logarithm of the proposed distribution and provide a real data application.

Key Words: Beta distribution, Muth distribution, data analysis, maximum likelihood estimation, regression modeling.

Mathematical Subject Classification: 60E05, 62E15, 62F10.

1. Introduction

In reliability and lifetime analysis, the hazard function, also called hazard rate, failure rate or instantaneous failure rate, has a crucial role to characterize real lifetime data. Several real-life data sets have a non-monotone hazard rates such as the bathtub shapes and upside-down bathtub (unimodal) hazard rates. The most popular traditional distributions do not provide a good fit for modeling this kind of the data sets. Hence, many parametric probability distributions have been introduced to analyze real data sets with non-monotone hazard rates.

One of the most recent models is that of Jodrá et al. (2017) who have proposed a new two-parameter lifetime distribution with bathtub-shaped and increasing failure rate called the power Muth (PM) distribution, where the Muth distribution was first proposed by Muth (1977) in the context of reliability theory. A random variable $Y$ is said to follow the Muth distribution, with shape parameter $\alpha \in [0,1]$, if its probability density function (pdf) has the form:

$$g_Y(y) = (e^{\alpha y} - \alpha) e^{\left[\frac{\alpha y - \frac{1}{\alpha} (e^{\alpha y} - \alpha)}{\gamma (y - \alpha)}\right]}, \quad y > 0.$$ 

Since the Muth distribution was introduced, it has been largely overlooked in the literature until the paper of Leemis and McQueston (2008), where they referred its relation with the exponential distribution. After seven years, some mathematical properties of the Muth distribution are derived, for the first time, by Jodrá et al. (2015). Then, Jodrá et al. (2017) extend the Muth distribution, by using the transformation $X = \beta Y^{1/\gamma}$, where $\beta > 0$, $\gamma > 0$ and $Y$ have the Muth distribution with parameter $\alpha = 1$. The pdf of $X$ is
\[ g(x) = \frac{y}{\beta^\gamma} x^{y-1} \left( e^{(x/\beta)^\gamma} - 1 \right) e^{\left( e^{(x/\beta)^\gamma} - 1 \right)}, \]

and the cumulative distribution function (cdf) of \( X \) is

\[ G(x) = 1 - e^{\left( (x/a)^\gamma - (e^{(x/a)^\gamma} - 1) \right)}, \]

where \( x > 0, \beta \) is a scale parameter and \( \gamma \) is a shape parameter. Also, Jodrá et al. (2017) studied various statistical properties of this distribution and showed that it gives the best fit for two real data sets than many other distributions.

The PM distribution does not provide enough flexibility for analyzing different types of lifetime data. To increase the flexibility for modelling purposes, it will be useful to consider further alternatives to this distribution. Therefore, the goal of this paper is to introduce a new four-parameter generalization that can capture decreasing, increasing, unimodal (upside-down bathtub) and bathtub-shaped hazard rate functions. The new distribution will be called the beta PM (BPM) distribution. We study some properties of the new model, give maximum likelihood estimation of the parameters, derive the elements of the observed information matrix and apply this model to real-life data sets. Also, based on this distribution, we present a new regression model.

The rest of the paper is organized as follows. In Section 2, we define the BPM model. In Sections 3, 4 and 5 we explicit expressions for the quantile function, moments and the moments of the order statistics respectively. In Section 6, we discuss maximum likelihood estimation of the model parameters. In Section 7, we conduct a simulation study to check the performance of the maximum likelihood estimates. A new regression model and residual analysis are presented in Section 8. Three applications are given in Section 9. Conclusions are given in Section 10.

2. The model definition

Eugene et al. (2002) proposed the beta-generated family of distributions by using the beta random variable. For an arbitrary baseline cdf \( G(x) \), the cdf of the beta generalized family is defined by

\[ F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1} dt, \]

where \( a > 0 \) and \( b > 0 \) are two shape parameters, \( I_p(a,b) = \frac{B(a+b)}{B(a,b)} \) is the incomplete beta function ratio, \( B_\gamma(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt \) is the incomplete beta function, \( B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \) is the complete beta function and \( \Gamma(\cdot) \) is the gamma function. The pdf corresponding to (3) is

\[ f(x) = \frac{g(x)}{B(a,b)} [G(x)]^{a-1}[1-G(x)]^{b-1}, \]

where \( g(x) = dG(x)/dx \) is the baseline pdf. By using Equation (3), several authors have defined and studied many new distributions. For example, Eugene et al. (2002) introduced the beta-normal distribution. Nadarajah and Kotz (2005) defined and studied the beta exponential. Famoye et al. (2005) introduced the beta-Weibull distribution. Singla et al. (2012) defined the beta generalized Weibull distribution. Shkhatreh et al. (2016) introduced the beta generalized linear exponential distribution. Benkhelifa (2017) proposed the beta generalized Gompertz distribution. Awodutire et al. (2020) introduced the beta type-I generalized half logistic distribution. Benkhelifa (2021) proposed the beta reduced modified Weibull distribution.

By substituting (2) in (3), the cdf of the BPM distribution with four parameters \( (\beta > 0, \gamma > 0, a > 0 \text{ and } b > 0) \) can be defined by

\[ F(x) = I_{1-e^{(x/\beta)^\gamma} - (e^{(x/\beta)^\gamma} - 1)}(a, b) = \frac{1}{B(a,b)} \int_0^{1-e^{(x/\beta)^\gamma} - (e^{(x/\beta)^\gamma} - 1)} t^{a-1}(1-t)^{b-1} dt. \]
The pdf of the BPM distribution is given by

\[ f(x) = \frac{\gamma x^{\gamma - 1} e^{\left(x/\beta\right)^{\gamma}} e^{\left(\frac{x}{\beta}\right)^{\gamma} \left(1 - e^{\left(x/\beta\right)^{\gamma} \left(1 - e^{\left(x/\beta\right)^{\gamma}}\right)}\right)^{a-1}}}{\beta^{\gamma} B(a,b)} \quad (6) \]

It is clear that the PM distribution with parameters \( \beta \) and \( \gamma \) is a special sub-model for \( a=b=1 \). For \( b=1 \), we obtain the exponentiated PM (EPM) distribution, which is proposed by Irshad et al. (2021). Hereafter, a random variable \( X \) with pdf (6) will be denoted by \( X \sim BPM(\beta, \gamma, a, b) \).

Figure 1 shows some possible shapes of the pdf (6) of the BPM distribution for some parameter values of \( \beta, \gamma, a \) and \( b \). Then, we observe that the density function (6) can take various forms depending on the parameter values. It is evident that the BPM distribution is much more flexible than the PM distribution.

The hazard function of the BPM distribution is given by

\[ h(x) = \frac{\gamma x^{\gamma - 1} e^{\left(x/\beta\right)^{\gamma}} e^{\left(\frac{x}{\beta}\right)^{\gamma} \left(1 - e^{\left(x/\beta\right)^{\gamma} \left(1 - e^{\left(x/\beta\right)^{\gamma}}\right)}\right)^{a-1}}}{\beta^{\gamma} B(a,b) \left(1-e^{\left(\frac{x}{\beta}\right)^{\gamma} \left(1 - e^{\left(x/\beta\right)^{\gamma}}\right)}\right)^{a-1}} \]

In Figure 2, we plot the hazard rate function of the BPM distribution for selected values of \( \beta, \gamma, a \) and \( b \). We observe that the hazard rate function of the BPM distribution can be increasing, decreasing, and bathtub shaped or unimodal shaped depending on the values of the parameters. Therefore, the BPM distribution is quite flexible and can be used to fit various types of data sets in different domains.
3. Quantile function

From Proposition 3 of Jodrá et al. (2017), we have the quantile function of the PM distribution

\[ Q_{PM}(u) = \beta \left[ \ln \left( \frac{u - 1}{e} \right) \right]^{1/\gamma}, \quad 0 < u < 1, \]

where \( W_{-1} \) is the negative branch of the Lambert \( W \) function. Then, by inverting BPM cdf (5), we obtain the quantile function of the BPM distribution as follows

\[ Q(u) = \beta \left[ \ln \left( \frac{Q_{a,b}(u) - 1}{e} \right) \right]^{1/\gamma}, \quad 0 < u < 1, \quad (7) \]

where \( Q_{a,b}(u) \) is the \( u \)th quantile of a beta distribution with parameters \( a \) and \( b \). Therefore, it is easy to simulate the BPM distribution. Let \( V \) be a beta random variable with parameters \( a > 0 \) and \( b > 0 \). Then, the random variable

\[ X = \beta \left[ \ln \left( \frac{V - 1}{e} \right) \right]^{1/\gamma}, \quad (8) \]

follows the BPM distribution. From Equation (8), we can generate a random variable \( X \) having the BPM distribution when the parameters are known. The median can be derived from (7) by setting \( u = 1/2 \).

4. Moments

Here, we give the moments of the BPM distribution. We can determine the skewness, kurtosis and the expected lifetime of a device in lifetime data. The following theorem gives the \( r \)th moment of the BPM distribution in terms of the generalized integro-exponential function, which is defined by (see Milgram, 1985):

\[ E_{x}^{r} = \frac{1}{\Gamma(k + 1)}. \]
where $z \in \mathbb{R}$, $s \in \mathbb{R}$ and $k > -1$.

**Theorem 1.** If $X \sim \text{BPM}(\beta, \gamma, a, b)$, then the $r$th moment of $X$ is given by

$$E(X^r) = \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} \frac{\beta^r e^{(j+b)}}{(j+b)B(a,b)} E_{r-(j+b)+1}^{r-1}(j+b).$$

**Proof.** The $r$th moment of $X$ is

$$E(X^r) = \int_0^\infty x^r f(x) dx.$$  

From (6) we have

$$E(X^r) = \frac{\gamma}{\beta^r B(a,b)} \int_0^\infty x^{r+\gamma-1} \left( e^{(x/\beta)^\gamma} - 1 \right) e^{(j+b)} \left( e^{(x/\beta)^\gamma} - \left( e^{(x/\beta)^\gamma} \right)^{a-1} \right) dx.$$  

By making use of the binomial series expansion, if $\eta > 0$ is real non-integer and $|z| < 1$,

$$(1 - z)^{\eta-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\eta-1}{j} z,$$

we obtain

$$E(X^r) = \frac{\gamma}{\beta^r B(a,b)} \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} \int_0^\infty x^{r+\gamma-1} \left( e^{(x/\beta)^\gamma} - 1 \right) e^{(j+b)} \left( e^{(x/\beta)^\gamma} - \left( e^{(x/\beta)^\gamma} \right)^{a-1} \right) dx.$$  

By setting $u = e^{(x/\beta)^\gamma}$ in the above integral, we get

$$E(X^r) = \frac{\beta^r}{B(a,b)} \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} e^{(j+b)} \int_1^\infty (\ln u)^{r} (u - 1) u^{(j+b)-1} e^{-(j+b)u} du.$$  

Then

$$E(X^r) = \frac{\beta^r}{B(a,b)} \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} e^{(j+b)} \left[ \int_1^\infty (\ln u)^{r} u^{(j+b)-1} e^{-(j+b)u} du - \int_1^\infty (\ln u)^{r} u^{(j+b)-1} e^{-(j+b)u} du \right].$$  

From the generalized integro-exponential function we have

$$E(X^r) = \frac{\beta^r}{B(a,b)} \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} e^{(j+b)} \Gamma \left( \frac{r}{\gamma} + 1 \right) \left[ E_{r-(j+b)+1}^{r/\gamma}(j+b) - E_{r-(j+b)+1}^{r/\gamma}(j+b) \right].$$  

From Equation (2.4) of Milgram (1985), we have

$$E_k^s(z) = \frac{zE_k^s(z) - E_k^{s-1}(z)}{1 - s}, \quad z > 0, \ s \neq 1, \ k \geq 0.$$  

Therefore, we obtain the desired result.

**5. Moments of order statistics**

In this section, the $p$th moment of the $i$th order statistic for the BPM distribution is given which have some applications in reliability and lifetime analysis. Let $X_{1:n}, \ldots, X_{n}$ be a simple random sample from BPM distribution and let $X_{1:n}, \ldots, X_{n:n}$ denote the order statistics obtained from this sample. The following theorem gives the $p$th moment of the $i$th order statistic $X_{i:n}$.
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By using (9), we get

where

Proof. By the definition, the \( \rho \)th moment of \( X_{\infty} \) is

where \( f_{i,n}(x) \) is the pdf of \( X_{\infty} \) given by

where \( F(x) \) and \( f(x) \) are given by (5) and (6), respectively. By using (9), the BPM cdf function (5) for \( b > 0 \) real non-integer can be rewritten as

From Gradshteyn and Ryzhik (2000), we have

where \( d_0 = a_0^\gamma \) and \( d_m = (ma_0)^{-1} \sum_{q=1}^m q(s + 1) - m \) \( a_q \) \( d_{m-q} \), \( m \geq 1 \). Therefore

Then

By using (9), we get

Therefore

\[ E(X_{\infty}^p) = \frac{\gamma}{\beta^p B(i, n - i + 1)} \sum_{l=0}^{n-i} \sum_{j=0}^{\infty} (-1)^{k+l} \binom{n-i}{l} \frac{2a+j-1}{k} d_j x^{\rho-1} \left( e^{\frac{x}{\beta Y}} - 1 \right) e^{\frac{x}{\beta Y} - (e^{\frac{x}{\beta Y} - 1})^a} \]
Similarly, as a proof of Theorem 1, we obtain \(E(X_{i:n})\).

6. Estimation

Here, the maximum likelihood estimates (MLEs) of the parameters of the BPM distribution are presented. Let \(x_1, \ldots, x_n\) be \(n\) random observations from the BPM distribution with unknown parameter vector \(\xi = (\beta, \gamma, a, b)^T\). Then, the log-likelihood function, denoted by \(\ell(\xi)\), for \(\xi\) is given by

\[
\ell(\xi) = n \ln y + ny \ln \beta - n \ln B(a, b) + (y - 1) \sum_{i=1}^{n} \ln(x_i) + \sum_{i=1}^{n} \ln \left( \frac{e^{x_i / \beta} - 1}{1 - e^{x_i / \beta}} \right) - b \sum_{i=1}^{n} (e^{x_i / \beta} - 1) + (a - 1) \sum_{i=1}^{n} \ln \left( 1 - \frac{e^{x_i / \beta}}{\left( e^{x_i / \beta} - 1 \right)} \right).
\]

Therefore, the log-likelihood equations are obtained, by taking the partial derivatives of \(\ell(\xi)\) with respect to \(\beta, \gamma, a\) and \(b\), as follows

\[
U_\beta(\xi) = \frac{ny}{\beta} - \frac{y}{\beta^{y+1}} \sum_{i=1}^{n} x_i^y e^{x_i / \beta} - 1 - \frac{by}{\beta^{y+1}} \sum_{i=1}^{n} x_i^y + \frac{by}{\beta^{y+1}} \sum_{i=1}^{n} x_i^y \frac{e^{x_i / \beta} - 1}{1 - e^{x_i / \beta}},
\]

\[
U_\gamma(\xi) = \frac{ny}{\gamma} - n \ln \beta + \sum_{i=1}^{n} \ln(x_i) + \frac{1}{\beta^y} \sum_{i=1}^{n} x_i^y \ln \left( \frac{e^{x_i / \beta} - 1}{1 - e^{x_i / \beta}} \right) + \frac{b}{\beta^y} \sum_{i=1}^{n} x_i^y \ln \left( e^{x_i / \beta} \right) \left( 1 - e^{x_i / \beta} \right),
\]

\[
U_a(\xi) = -n \psi(a) + \psi(a + b) + \sum_{i=1}^{n} \ln \left( 1 - \frac{e^{x_i / \beta}}{\left( e^{x_i / \beta} - 1 \right)} \right),
\]

and

\[
U_b(\xi) = -n \psi(b) + \psi(a + b) + \frac{1}{\beta^y} \sum_{i=1}^{n} x_i^y - \sum_{i=1}^{n} \frac{e^{x_i / \beta} - 1}{\beta^y},
\]

where \(\psi(\cdot)\) is the digamma function.

Solving the non-linear likelihood equations \(U_\beta(\xi) = 0, U_\gamma(\xi) = 0, U_a(\xi) = 0\) and \(U_b(\xi) = 0\) simultaneously, we obtain the MLE \(\hat{\xi} = (\hat{\beta}, \hat{\gamma}, \hat{a}, \hat{b})^T\). To construct approximate confidence intervals, we use the multivariate normal \(N_4(0, J^{-1}(\xi))\) distribution, where \(J^{-1}(\xi)\) is the inverse of the expected information matrix evaluated at \(\xi\). The information matrix given by

\[
J(\xi) = -\begin{pmatrix}
U_{\beta \beta} & U_{\beta \gamma} & U_{\beta a} & U_{\beta b} \\
U_{\beta \gamma} & U_{\gamma \gamma} & U_{\gamma a} & U_{\gamma b} \\
U_{\beta a} & U_{\gamma a} & U_{a a} & U_{a b} \\
U_{\beta b} & U_{\gamma b} & U_{a b} & U_{b b}
\end{pmatrix}
\]

whose elements are given in Appendix B. So, the approximate confidence intervals for \(\beta, \gamma, a\) and \(b\) are given, respectively, by

\[
\hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{\text{var}(\hat{\beta})}, \quad \hat{\gamma} \pm Z_{\frac{\alpha}{2}} \sqrt{\text{var}(\hat{\gamma})}, \quad \hat{a} \pm Z_{\frac{\alpha}{2}} \sqrt{\text{var}(\hat{a})} \quad \text{and} \quad \hat{b} \pm Z_{\frac{\alpha}{2}} \sqrt{\text{var}(\hat{b})}.
\]
where $\text{var}(\cdot)$ is the diagonal element of $J^{-1}(\xi)$ corresponding to each parameter and $Z_{\omega/2}$ is the quantile $100(1-\omega/2)$% of the standard normal distribution.

### 7. Monte Carlo simulation study

In this section, we conduct a simulation study to check the performance of the MLEs of the parameters of the BPM distribution. From Equation (8), we generate random samples of sizes $n=20, 50, 100, 200$ and $500$ from the BPM distribution using the following sets of parameters:

- Set I: $\beta = 0.2, \gamma = 0.5, a = 2.5, b = 5$,
- Set II: $\beta = 2.5, \gamma = 1, a = 2, b = 1$,
- Set III: $\beta = 5, \gamma = 0.2, a = 2, b = 2$.

The simulation is performed via the statistical software $R$ through the command $mle$. The number of Monte Carlo replications made was $N=1000$. The evaluation of the performance is based on the bias and the mean squared errors (MSE) defined as follows:

$$
\text{Bias} = \frac{1}{N} \sum_{i=1}^{N} (\hat{\xi}_i - \xi) \quad \text{and} \quad \text{MSE} = \frac{1}{N} \sum_{i=1}^{N} (\hat{\xi}_i - \xi)^2,
$$

where $\xi = \beta, \gamma, a$ and $b$. The results of our simulation study are summarized in Table 1. We can see that the bias and MSE of the MLEs decrease when the sample size increases, as expected. This verifies the consistency properties of the MLEs, i.e., we can conclude that the maximum likelihood method performs well for estimating the parameters of the BPM distribution.

#### Table 1: Monte Carlo simulation results for the BPM bias and MSEs.

| Sample size | Parameter | Set I | Set II | Set III |
|-------------|-----------|-------|--------|---------|
| $n=20$      | $\beta$  | 0.5622| 0.3469 | 0.0982  | 0.0268  | 0.0658  | 0.0195  |
|             | $\gamma$ | 0.5418| 0.3348 | 0.1808  | 1.3121  | 0.9343  | 1.2206  |
|             | $a$      | -0.8340| 0.7457 | 0.1658  | 0.1787  | 0.6632  | 2.9043  |
|             | $b$      | 0.4960| 0.2925 | 0.3241  | 0.7857  | 0.8042  | 1.6549  |
| $n=50$      | $\beta$  | 0.4996| 0.2496 | 0.8433  | 0.0159  | 0.0226  | 0.0065  |
|             | $\gamma$ | 0.3911| 0.2842 | 0.1434  | 0.6799  | 0.1474  | 0.3497  |
|             | $a$      | -0.6893| 0.5903 | 0.0498  | 0.0566  | 0.3276  | 2.1792  |
|             | $b$      | 0.3811| 0.2254 | 0.1752  | 0.1827  | 0.2269  | 0.3907  |
| $n=100$     | $\beta$  | 0.4261| 0.2201 | 0.2230  | 0.0104  | 0.0158  | 0.0062  |
|             | $\gamma$ | -0.1893| 0.0573 | 0.1353  | 0.4343  | 0.0338  | 0.0654  |
|             | $a$      | 0.0243| 0.0010 | 0.0019  | 0.0231  | 0.2480  | 1.2747  |
|             | $b$      | -0.0764| 0.0106 | 0.1373  | 0.1063  | 0.0611  | 0.1206  |
| $n=200$     | $\beta$  | 0.2330| 0.0462 | 0.0073  | 0.0100  | 0.0133  | 0.0057  |
|             | $\gamma$ | -0.1448| 0.0373 | 0.0424  | 0.1470  | 0.0134  | 0.0255  |
|             | $a$      | 0.0161| 0.0006 | -0.0073 | 0.0118  | 0.2289  | 0.6935  |
|             | $b$      | -0.0584| 0.0073 | 0.1074  | 0.0630  | 0.0106  | 0.0553  |
| $n=500$     | $\beta$  | 0.1172| 0.0228 | 0.0012  | 0.0007  | -0.0209 | 0.0015  |
|             | $\gamma$ | 0.0011| 0.00042| 0.0279  | 0.0576  | -0.0066 | 0.0094  |
|             | $a$      | -0.0030| 0.003 | -0.0138 | 0.0058  | 0.0238  | 0.3746  |
|             | $b$      | 0.0011| 0.0004 | 0.0756  | 0.0369  | -0.0207 | 0.0246  |

### 8. The LBPM Regression Model

In some practical applications, the lifetimes are affected by many explanatory variables and for this reason the regression models are widely used to estimate univariate survival functions for censored data. Among them, the location-scale regression model is distinguished since it is frequently used in clinical trials. In this section, we introduce a new location-scale regression model based on the logarithm of the BPM distribution. If $X \sim \text{BPM}(\beta, \gamma, a, b)$ then the random variable $Y = \ln X$ has the log-BPM (LBPM) distribution. The density function of $Y$, replacing $\sigma = 1/\gamma$ and $\mu = \ln \beta$, is given by
The linear location-scale regression model linking the response variable $y_i$ and the explanatory variable vector $v_i^T = (v_{i1}, ..., v_{ip})$ is given by

$$y_i = v_i^T \theta + \sigma z_i, \quad i = 1, ..., n,$$

where the random error $z_i$ has density function (12) and $\theta = (\theta_1, ..., \theta_p)^T$ is the unknown vector of regression coefficients. The parameter $\mu_i = v_i^T \theta$ is the location of $y_i$. The location parameter vector $\mu = (\mu_1, ..., \mu_p)^T$ is represented by a linear model $\mu = V \theta$, where $V = (v_1, ..., v_p)^T$ is a known model matrix. The LBPM regression model contains LEPM regression for $(b = 1)$ and LPM for $(a = b = 1)$ regression models as special sub-models.

Suppose $(y_1, v_1), ..., (y_n, v_n)$ is sample of $n$ independent observations, where the random response is defined by: $y_i = \min \{\ln(x_i), \ln(c_i)\}$. Let $F$ and $C$ be the sets of individuals for which $y_i$ is the log-lifetime or log-censoring, respectively. The log-likelihood function for the vector of parameters $\tau = (a, b, \sigma, \theta^T)^T$ is given by:

$$\ell(\tau) = \sum_{i \in F} \ln f(y_i) + \sum_{i \in C} \ln S(y_i),$$

where $f(y_i)$ is the density function (10) and $S(y_i)$ is the survival function (11) of $Y_i$. The log-likelihood function for $\tau$ is:

$$\ell(\tau) = -r [\ln \sigma + \ln B(a, b)] + \sum_{i \in F} \ln (t_i) + \sum_{i \in F} \ln (e^{t_i} - 1) + b \sum_{i \in F} (e^{t_i} - e^{e^{t_i}} + 1) + (a - 1) \sum_{i \in F} \ln \left(1 - e^{e^{t_i} - e^{e^{t_i}} + 1}\right) + \sum_{i \in C} \ln \left(1 - l \exp(e^{t_i} - e^{e^{t_i}} + 1)(a, b)\right),$$

where $r$ is the number of uncensored observations (failures) and $t_i = e^{z_i}$. We can obtain the MLE $\hat{\tau}$ of $\tau$ by maximizing the loglikelihood function $\ell(\tau)$. To compute this estimate, we use the procedure `optim` in R software.

8.1. Residual Analysis

Residual analysis has an important role in judging the adequacy of the fitted model. To study departures from error assumptions and the presence of outliers, we consider here residual analysis based on the martingale and modified deviance residuals.

8.1.1. Martingale residual

The martingale residual is defined in counting processes, for more details see Fleming and Harrington (1994). The martingale residuals for LBPM model is
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\[ r_{M_i} = \begin{cases} 
1 + \ln \left( \frac{1 - I_{\hat{a}, \hat{b}}^{\text{exp}}(e_{t_i}^* - e_{t_i}^* + 1)}{1 - \exp(e_{t_i}^* - e_{t_i}^* + 1)} \left( \hat{a}, \hat{b} \right) \right) & \text{if } i \in F, \\
\ln \left( \frac{1 - I_{\hat{a}, \hat{b}}^{\text{exp}}(e_{t_i}^* - e_{t_i}^* + 1)}{1 - \exp(e_{t_i}^* - e_{t_i}^* + 1)} \left( \hat{a}, \hat{b} \right) \right) & \text{if } i \in C,
\end{cases} \]

where \( t_i^* = \frac{y_i - \hat{\mu}^T \hat{\theta}}{\hat{\sigma}} \) with \( \hat{\mu} = v_i^T \hat{\theta} \).

8.1.2. Modified Deviance Residual

The main drawback of the martingale residual is that when the fitted model is correct, it is not symmetrically distributed about zero. To overcome this problem, modified deviance residual was proposed by Therneau et al. (1990). The modified deviance residual for LBPM model is

\[ r_{D_i} = \begin{cases} 
\text{sign}(r_{M_i}) \left( -2 \ln \left( \frac{1 - I_{\hat{a}, \hat{b}}^{\text{exp}}(e_{t_i}^* - e_{t_i}^* + 1)}{1 - \exp(e_{t_i}^* - e_{t_i}^* + 1)} \left( \hat{a}, \hat{b} \right) \right) \right)^{1/2} & \text{if } i \in F, \\
\text{sign}(r_{M_i}) \left( -2r_{M_i} \right)^{1/2} & \text{if } i \in C,
\end{cases} \]

where \( r_{M_i} \) is the martingale residual.

9. Data Analysis

In order to show the flexibility of the BPM distribution we use two reliability real data sets with different shapes. For these data sets, we compare the fit of the BPM distribution with the PM, EPM (Irshad et al., 2021), beta generalized Weibull (BGW) (Singla et al., 2012), beta Weibull (BW) (Famoye et al., 2005), Kumaraswamy Weibull (KW) (Corderio et al., 2010), McDonald Weibull (Mc-W) (Corderio et al. 2014), transmuted Weibull (TW) (Aryall and Tsokos, 2011), beta generalized Gompertz (BGG) (Benkhelifa, 2017), alpha power Weibull (APW) (Nassar et al., 2017), new generalized odd log-logistic flexible Weibull (GOLLFW) (Prataviera et al., 2018), exponentiated additive Weibull distribution (EAW) (Ahmad and Ghazal, 2020). The pdf's of these distributions are given in Appendix A.

In order to verify which distribution fits better to the data sets, we compute the values of the log-likelihood functions \((-2\hat{\ell})\), Akaike information criterion (AIC), consistent Akaike information criteria (CAIC), Bayesian information criterion (BIC) and the Kolmogorov-Smirnov (K-S) statistic with corresponding p-value. The better model corresponds to smaller values of these measures and high p-value.

9.1. First data set: devices failure time data

The first data set is given by Aarset (1987) and represents the time to first failure of 50 devices (in weeks). This data set is: 0.1, 0.2, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 85, 85, 85, 85, 85, 85, 86, 86. The TTT-plot of this data set, in Figure 3(a), shows a convex shape followed by a concave shape. This corresponds to a bathtub shaped hazard rate function. So, the BPM distribution is appropriate for modeling the first data.

Table 2 presents the values of the MLEs of the parameters for all fitted distributions. We see from Table 3 that the BPM distribution has the smallest values of \(-2\hat{\ell}\), AIC, BIC, CAIC and K-S and largest p-value. Hence, the BPM distribution gives an excellent fit than the others models for the first data set. In addition, we plot the histogram of this data set and the fitted pdfs in Figure 4(a). This Figure shows that the BPM pdf provides a closer fit to the histogram than other distribution. The plots of the estimated cdfs and empirical cumulative function are displayed in Figure 4(b). These plots reveal that the BPM cdf is the closest curve to the empirical cumulative function. So, we conclude that the BPM model is the best.

The variance-covariance matrix for the estimated parameters of the BPM distribution is given by
\[ J^{-1}(\xi) = \begin{pmatrix} 2.31703 & 0.02339 & 0.00624 & 0.02355 \\ 0.02339 & 0.00567 & -0.00024 & -0.00034 \\ 0.00624 & -0.00024 & 0.00025 & 0.00017 \\ 0.02355 & -0.00034 & 0.00017 & 0.00053 \end{pmatrix} \]

So, the approximate 95% confidence intervals for the parameters \( \beta, \gamma, a \) and \( b \), are [51.5405, 57.5074], [2.4914, 2.7866], [0.0539, 0.1161], [0.0409, 0.1307] respectively.

### Table 2: MLEs of the model parameters and the corresponding standard errors given in parentheses.

| Model | \( \hat{\beta} \) | \( \gamma \) | \( \lambda \) | \( \hat{a} \) | \( \hat{b} \) |
|-------|-----------------|------------|------------|--------------|--------------|
| BPM   | 54.5240(1.522)  | 2.6390(0.0753) | -          | 0.0850(0.01587) | 0.0858(0.0229) |
| PM    | 33.5676(5.3343) | 0.4254(0.0557) | -          | -             | -             |
| EPM   | 82.809(2.6231)  | 2.4668(0.0776) | -          | 0.1377(0.0203) | -             |
| BW    | 5.1699(0.0064)  | -           | 0.0217(0.0239) | 0.0899(0.1123) | 0.0726(0.1214) |
| KW    | 1.8942(0.5097)  | -           | 0.0028(0.0071) | 0.1395(0.0724) | 0.1369(0.0698) |
| Mc-W  | 1.7122(0.3520)  | 1.8145(0.1028) | 0.0169(0.0016) | 0.2786(0.1772) | 0.0525(0.1186) |
| TW    | 0.8958(0.1286)  | -           | 0.0386(0.0230) | -0.3279(0.2592) | -             |
| BGW   | 0.0037(0.0071)  | 2.5748(2.9461) | 0.0688(0.0211) | 0.1172(0.1231) | 0.0983(0.1161) |
| APW   | 0.8355(0.1372)  | -           | 0.0586(0.1372) | 4.5265(4.067)  | -             |
| GOLLFW | 29.757(7.32\times 10^{-5}) | 0.1058(2.76 \times 10^{-2}) | -0.0383(3.25 \times 10^{-3}) | 0.1870(8.02 \times 10^{-3}) |
| EAW   | 1.5554(0.4801)  | 1.2450(0.1315) | 0.1051(0.0312) | 0.0019(0.0041) | 7.8701(0.01369) |

### Table 3: The statistics: \(-2\hat{\ell}, \text{AIC}, \text{BIC}, \text{CAIC}, \text{K-S} \) and p-value.

| Model | \(-2\hat{\ell}\) | AIC    | BIC    | CAIC   | K-S    | p-value |
|-------|-----------------|--------|--------|--------|--------|---------|
| BPM   | 429.3273        | 437.3273 | 444.9754 | 438.2162 | 0.1200 | 0.4675  |
| PM    | 476.6327        | 480.6327 | 484.4567 | 480.888 | 0.19217 | 0.0498  |
| EPM   | 452.7223        | 458.7223 | 464.4584 | 459.2441 | 0.2053 | 0.0295  |
| BW    | 461.6844        | 471.6844 | 481.2445 | 473.048 | 0.1628 | 0.1414  |
| KW    | 457.7916        | 465.7916 | 473.4397 | 466.6805 | 0.1397 | 0.2826  |
| Mc-W  | 461.7876        | 469.7876 | 477.4357 | 470.6765 | 0.1526 | 0.1941  |
| TW    | 460.388         | 470.388 | 479.9481 | 471.7516 | 0.1739 | 0.0971  |
| BGG   | 480.7334        | 486.7334 | 492.4695 | 487.2551 | 0.1841 | 0.0673  |
| APW   | 439.0648        | 449.0648 | 458.6249 | 450.4284 | 0.1276 | 0.3893  |
| GOLLFW| 440.1259        | 448.1259 | 455.7740 | 449.0148 | 0.1493 | 0.2149  |
| EAW   | 461.1976        | 471.1976 | 480.7577 | 472.5613 | 0.1456 | 0.2392  |

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Figure 3: TTT-plot for (a) Aarset data (b) number of successive failures data.

Figure 4: (a) Plots of the histogram and the fitted densities (b) Plots of the empirical cdf and estimated cdfs of Aarest data.
9.2. Second data set: Number of successive failures data

The data have been presented by Proshkan (1963). The data set is: 194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 14 12, 239, 14, 18, 39, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71. This data set has an upside-down bathtub shaped failure rate function as shown by the scaled TTT-plot, which has a concave shape followed by a convex shape; see Figure 3(b). The BPM distribution is appropriate for modeling these data.

Table 4 gives the MLEs of the parameters of all models used here for the second data set. The values of -2\(\hat{\xi}\), AIC, BIC, CAIC, K-S and its p-value are listed in Table 5. From this Table, we can see that the BPM distribution as the best fit for the second data set. Figures 5(a), 5(b) illustrate the pdfs and empirical cdfs, respectively, of the comparative models to show the over fitting of the BPM distribution.

The estimated variance-covariance matrix of the BPM distribution for the this data set is

\[
J^{-1}(\hat{\xi}) = \begin{pmatrix}
0.79321 & 0.00764 & -1.98644 & 0.08687 \\
0.00764 & 0.00045 & -0.03481 & -0.00061 \\
-1.98644 & -0.03481 & 6.48624 & -0.15314 \\
0.08687 & -0.00061 & -0.15314 & 0.01519
\end{pmatrix}.
\]

Then, the approximate 95% confidence intervals for the parameters \(\beta\), \(\gamma\), \(a\) and \(b\), are, respectively, [0, 0.6518], [0.1558, 0.2395], [0.2233, 0.2068], [0, 0.4112].

| Model | \(\hat{\beta}\)         | \(\hat{\gamma}\)     | \(\hat{\lambda}\) | \(\hat{a}\)    | \(\hat{b}\)    |
|-------|-------------------------|-----------------------|-------------------|----------------|----------------|
| BPM   | 0.9062(0.8906)          | 0.1976(0.0213)        | -                 | 5.2150(2.5468) | 0.1697(0.1323) |
| PM    | 63.6606(6.1905)         | 0.3648(0.0204)        | -                 | -              | -              |
| EPM   | 3.8633(4.7341)          | 0.1616(0.0381)        | -                 | 5.1627(2.7911) | -              |
| BGW   | 0.4288(0.2270)          | 0.6925(1.4808)        | 0.1005(0.3520)    | 6.3630(13.5748)| 5.2195(16.5406)|
| BW    | 0.4010(0.1856)          | -                     | 0.1279(0.2949)    | 4.6125(3.8441) | 5.2263(17.7615)|
| KW    | 0.7284(0.0911)          | -                     | 0.2923(0.8001)    | 2.9535(1.8066) | 0.1656(0.2089) |
| Mc-W  | 0.6004(0.1272)          | 19.9958(3.0166)       | 0.1445(0.1258)    | 2.6628(1.3279) | 0.7559(0.1167) |
| TW    | 0.9668(0.0565)          | -                     | 0.0099(0.0037)    | 0.4663(0.2808) | -              |
| BGG   | 0.0336(1.82\times 10^{-2}) | 10.2667(7.35\times 10^{-4}) | 2.9\times 10^{-2} (1\times 10^{-3}) | 0.144(3.69\times 10^{-2}) | 0.248(1.06\times 10^{-1}) |
| APW   | 0.4765(4.76\times 10^{-2}) | -                     | 0.3054(4.02\times 10^{-2}) | 197.44(2.12\times 10^{-5}) | -              |
| GOLFW | 1.6681(0.0718)          | 2.9655(0.7605)        | -                 | 0.0019(0.0004) | 2.4440(0.7314) |
| EAW   | 0.3394(0.1710)          | 0.0030(0.0253)        | 1.0060(0.2716)    | 0.5481(0.5842847) | 6.4281(6.4167) |

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Table 5: The statistics: $-2\hat{\ell}$, AIC, BIC, CAIC, K-S and p-value.

| Model  | $-2\hat{\ell}$ | AIC       | BIC       | CAIC      | K-S       | p-value |
|--------|----------------|-----------|-----------|-----------|-----------|---------|
| BPM    | 2064.106       | 2072.106  | 2085.052  | 2072.325  | 0.0361    | 0.9669  |
| PM     | 2080.665       | 2084.665  | 2091.138  | 2084.73   | 0.0706    | 0.3065  |
| EPM    | 2066.035       | 2072.035  | 2081.745  | 2072.166  | 0.0447    | 0.8475  |
| BGW    | 2066.094       | 2076.094  | 2092.276  | 2072.623  | 0.0442    | 0.8556  |
| BW     | 2066.151       | 2074.151  | 2087.097  | 2074.37   | 0.0443    | 0.8529  |
| KW     | 2064.732       | 2072.732  | 2085.677  | 2072.950  | 0.0395    | 0.9307  |
| Mc-W   | 2066.23        | 2076.23   | 2092.412  | 2076.560  | 0.0455    | 0.8298  |
| TW     | 2070.853       | 2076.853  | 2086.563  | 2076.984  | 0.0465    | 0.8095  |
| BGW    | 2064.774       | 2074.774  | 2089.956  | 2074.103  | 0.0381    | 0.9587  |
| APW    | 2071.108       | 2077.108  | 2086.817  | 2077.238  | 0.0504    | 0.7259  |
| GOLLFW | 2106.803       | 2114.803  | 2127.749  | 2115.022  | 0.0969    | 0.05816 |
| EAW    | 2065.315       | 2075.315  | 2091.497  | 2075.644  | 0.04231   | 0.8947  |

Figure 5: (a) Plots of the histogram and the fitted densities (b) Plots of the empirical cdf and estimated cdfs of successive failures data.

9.3. Third data set: **HIV survival data**

This data set is reported in Hosmer and Lemeshow (1999) and also it is available in Bolstad2 package of R software. The sample size is n=100 on HIV+ subjects belonging to an health maintenance organization, where the goal is to evaluate the survival time of these subjects. Alizadeh et al. (2017) adopted the log-odd power Cauchy-Weibull (LOPCW) regression model to analyse this data set. We use the same data set to prove the flexibility of LBPM regression model, where the aim of this study is to relate the survival time (y) with the history of drug use (v). The variables are: $y_i$: observed survival time (in months), $cens_i$: censoring indicator (0= alive at study end or lost to follow up, 1= death due to AIDS or AIDS related factors) and $v_{1i}$ (1= yes, 0= no) represents the history of the drug use. The regression model fitted to the data set is given by

$$y_i = \theta_0 + \theta_1 v_i + \sigma z_i,$$

where $y_i$ has the LBPM density (10) for $i=1,\ldots,100$. We compare the LBPM regression model with the LPM, LEPM, LOPCW and log-Weibull (LW) regression models. Table 6 gives the MLEs, their approximate standard errors and p-values obtained from the fitted these models, $-2\hat{\ell}$, AIC, BIC and CAIC statistics. These results indicate...
that the LBPM regression model has the lowest values of $-\hat{\ell}$, AIC, BIC and CAIC. Therefore, it is clear that the LBPM model provides an adequate fit to HIV survival data. We can observe that the explanatory variable $v_{11}$ is significant at the level of 1%.

Figure 6(a) displays the modified deviance residuals (see Section 8) against the index of the observations. Figure 6(b) gives the normal probability plot with generated envelope. We conclude that none of observed values appear as possible outliers. Therefore, the fitted model is very suitable for this data set.

Table 6: MLEs of the model parameters standard errors in (.) and p-values in [.] and the values of $-\hat{\ell}$, AIC, BIC and CAIC.

| Model   | $a$      | b   | $\sigma$ | $\theta_0$ | $\theta_1$ | $-\hat{\ell}$ | AIC    | BIC    | CAIC    |
|---------|----------|-----|----------|------------|------------|----------------|--------|--------|---------|
| LBPM    | 53.559   | 0.500 | 9.624    | -4.7662    | -0.826     | 241.853        | 251.853| 264.878| 252.491 |
|         | (77.175) | (0.079) | (3.613) | (3.674)    | (0.270)    |                |        |        |         |
| LEPM    | 46.161   | 12.737 | -6.316   | -0.851     | 281.759    | 289.759        | 300.180| 290.180|         |
|         | 96.609   | 7.370  | 7.273    | 0.270      |            |                |        |        |         |
| LPM     |          | 2.703 | 2.666    | -1.000     | 298.915    | 304.915        | 312.730| 305.165|         |
|         |          | (0.221)| (0.175) | (0.235)    |            |                |        |        |         |
| LOPCW   | 3.287    | 3.784 | 3.843    | -1.001     | 286.660    | 294.661        | 305.080| 295.081|         |
|         | (2.921)  | (3.348)| (1.206) | (0.252)    |            |                |        |        |         |
| LW      | 1.071    | 3.003 | -1.052   | 292.875    | 298.875    | 306.690        | 299.125|        |         |
|         | (0.888)  | (0.166)| (0.239) |            |            |                |        |        |         |

Figure 6: (a) Index plot of the modified deviance residuals and (b) Normal probability plot for the modified deviance residuals with envelope from the fitted LBPM regression model.
10. Conclusions

We have introduced a new four-parameter model called the beta power Muth distribution. This distribution has as sub-models the power Muth and exponentiated power Muth distributions. We have derived explicit expressions for the moments, quantile function and moments of the order statistics associated with the proposed model. We have used the maximum likelihood method to estimate the model parameters. We have presented two examples involving reliability data sets. One of the data sets has a bathtub shaped failure rate and the other has an upside-down bathtub shaped failure rate function. For these data sets, our model provides the best fit than other competitive models. Further, we have defined the LBPM regression model and shown that this model gives an adequate fit to HIV survival data.

Appendix A.

In this appendix, we give the pdf of each distribution used in the application section.

- **EPM distribution**
  \[ f(x) = \frac{a\gamma x^{\gamma-1}(e^{(x/\beta)^\gamma} - 1)e^{\frac{(x/\beta)^\gamma}{(e^{(x/\beta)^\gamma} - 1)}}}{\beta^\gamma} \left(1 - e^{\frac{(x/\beta)^\gamma}{(e^{(x/\beta)^\gamma} - 1)}}\right)^{\alpha-1}, \]
  where \( x, \beta, \gamma, \alpha > 0. \)

- **BGW distribution**
  \[ f(x) = \frac{\beta\gamma x^{\beta-1}e^{-\lambda x^\beta}}{B(a, b)} \left(1 - e^{-\lambda x^\beta}\right)^{\gamma a-1} \left(1 - \left(1 - e^{-\lambda x^\beta}\right)^{\gamma b}\right)^{-1}, \]
  where \( x, \beta, \gamma, \lambda, a, b > 0. \)

- **BW distribution**
  \[ f(x) = \frac{\beta\lambda x^{\beta-1}}{B(a, b)} \left(1 - e^{-\lambda x^\beta}\right)^{\alpha-1} e^{-b\lambda x^\beta}, \]
  where \( x, \beta, \lambda, a, b > 0. \)

- **Me-W distribution**
  \[ f(x) = \frac{\beta\gamma x^{\beta-1}e^{-\lambda x^\beta}}{B(a/\gamma, b)} \left(1 - e^{-\lambda x^\beta}\right)^{\alpha a-1} \left(1 - \left(1 - e^{-\lambda x^\beta}\right)^{\gamma b}\right)^{-1}, \]
  where \( x, \beta, \gamma, \lambda, a, b > 0. \)

- **TW distribution**
  \[ f(x) = \lambda \beta x^{\beta-1}e^{-\lambda x^\beta} \left(1 + a - 2ae^{-\lambda x^\beta}\right), \]
  where \( x, \beta, \lambda, a > 0. \)

- **BGG distribution**
  \[ f(x) = \frac{\beta\gamma e^{\lambda x^\beta} e^{-\frac{\beta}{\lambda} e^{\lambda x^\beta}}}{B(a, b)} \left(1 - e^{-\frac{\beta}{\lambda} e^{\lambda x^\beta}}\right)^{\gamma a-1} \left(1 - \left(1 - e^{-\frac{\beta}{\lambda} e^{\lambda x^\beta}}\right)^{\gamma b}\right)^{-1}, \]
  where \( x, \beta, \gamma, \lambda, a, b > 0. \)

- **APW distribution**
  \[ f(x) = \frac{\ln a}{a - 1} \lambda \beta x^{\beta-1}e^{-\lambda x^\beta} a^{1-e^{-\lambda x^\beta}}, \]
  where \( x, a > 0, a \neq 1, \lambda, \beta > 0. \)
• GOLLFW distribution
\[
f(x) = \beta \gamma \left( a + \frac{b}{x^2} \right) \exp \left[ \left( ax - \frac{b}{x} \right) - \kappa_{ab}(x) \right] (1 - \exp[-\kappa_{ab}(x)])^{\gamma - 1} \left[ 1 - \left( 1 - \exp[-\kappa_{ab}(x)] \right)^{\gamma \beta - 1} \right] \{1 - \exp[-\kappa_{ab}(x)]\}^{\gamma \beta} + \left[ 1 - \{1 - \exp[-\kappa_{ab}(x)]\}^{\gamma \beta} \right]^{-2},
\]
where \( x, \beta, \gamma, a, b > 0 \) and \( \kappa_{ab}(x) = \exp \left( ax - \frac{b}{x} \right) \).

• EAW distribution
\[
f(x) = b (a \beta x^{\beta - 1} + \lambda y x^{\lambda - 1}) [1 - \exp(-ax^{\beta} - y x^{\lambda})]^{\theta - 1} \exp(-ax^{\beta} - y x^{\lambda}),
\]
where \( x, \beta, \gamma, \lambda, a, b > 0 \).

**Appendix B.**

Let \( t_i = \frac{x_i}{\beta} \). The elements of the observed information matrix are:

\[
U_{\beta \beta} = \frac{ny}{\beta^2} + \frac{y(y + 1)}{\beta^{y+2}} \sum_{i=1}^{n} x_i^{\gamma} e^{t_i^{\gamma}} - \frac{y^2}{\beta^{2y+2}} \sum_{i=1}^{n} x_i^{2\gamma} e^{t_i^{\gamma}} + \frac{by(y + 1)}{\beta^{y+2}} \sum_{i=1}^{n} x_i^{\gamma} - \frac{by(y + 1)}{\beta^{y+2}} \sum_{i=1}^{n} x_i^{\gamma} e^{t_i^{\gamma}}
\]

\[
- \frac{by^2}{\beta^{2y+2}} \sum_{i=1}^{n} x_i^{2\gamma} e^{t_i^{\gamma}} + \frac{(a - 1)y(y + 1)}{\beta^{y+2}} \sum_{i=1}^{n} x_i^{\gamma} \left( e^{t_i^{\gamma}} - 1 \right) e^{\left( t_i^{\gamma} - t_i^{\gamma+1} \right)} - \frac{y}{\beta^{y+2}} \sum_{i=1}^{n} x_i^{2\gamma} \left( e^{2t_i^{\gamma}} - e^{t_i^{\gamma} + 1} \right) + \frac{(a - 1)y^2}{\beta^{2y+2}} \sum_{i=1}^{n} x_i^{2\gamma} \left( 1 - e^{t_i^{\gamma}} \right)^2 e^{\left( t_i^{\gamma} - e^{t_i^{\gamma+1}} \right)}
\]

\[
U_{\beta \alpha} = \frac{y}{\beta^{y+1}} \sum_{i=1}^{n} x_i^{\gamma} \left( e^{t_i^{\gamma}} - 1 \right) e^{\left( t_i^{\gamma} - e^{t_i^{\gamma+1}} \right)}
\]

\[
U_{\beta \beta} = \frac{y}{\beta^{y+1}} \sum_{i=1}^{n} x_i^{\gamma} + \frac{y}{\beta^{y+1}} \sum_{i=1}^{n} x_i^{\gamma} e^{t_i^{\gamma}},
\]

\[
U_{\gamma \alpha} = \frac{1}{\beta y} \sum_{i=1}^{n} x_i^{\gamma} \ln(t_i) \left( 1 - e^{t_i^{\gamma}} \right) e^{\left( t_i^{\gamma} - e^{t_i^{\gamma+1}} \right)},
\]

\[
U_{\gamma \beta} = \frac{1}{\beta y} \sum_{i=1}^{n} x_i^{\gamma} \left( 1 - e^{t_i^{\gamma}} \right) e^{\left( t_i^{\gamma} - e^{t_i^{\gamma+1}} \right)},
\]
\[ U_{yb} = -\frac{1}{b^\gamma} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) \left( 1 - e^{t_i^\gamma} \right), \]

\[ U_{by} = -\frac{n}{b} - \frac{\gamma}{b^\gamma+1} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) e^{t_i^\gamma} + \frac{b}{b^\gamma+1} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) \left( 1 + e^{t_i^\gamma} \right) - \frac{b}{b^\gamma+1} \sum_{i=1}^{n} x_i^\gamma \]

\[ + \frac{1}{b^{2\gamma+1}} \sum_{i=1}^{n} \beta \gamma x_i^\gamma e^{t_i^\gamma} + \gamma x_i^\gamma e^{t_i^\gamma} \ln(t_i) \left( e^{t_i^\gamma} - 1 \right)^2 \]

\[ + \frac{b}{b^{2\gamma+1}} \sum_{i=1}^{n} \beta \gamma x_i^\gamma e^{t_i^\gamma} + \gamma x_i^\gamma e^{t_i^\gamma} \ln(t_i) \left( e^{t_i^\gamma} - 1 \right)^2 \]

\[ + \frac{\gamma(a - 1)}{b^{2\gamma+1}} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) \left( 1 + e^{t_i^\gamma} \right) - \frac{b}{b^{\gamma+1}} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) \left( 1 - e^{t_i^\gamma} \right) e^{(t_i^\gamma - e^{t_i^\gamma+1})} \]

\[ + \frac{\gamma(a - 1)}{b^{2\gamma+1}} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) \left( e^{t_i^\gamma} + (1 - e^{t_i^\gamma})^2 \right) e^{(t_i^\gamma - e^{t_i^\gamma+1})} \]

\[ + \frac{\gamma(a - 1)}{b^{2\gamma+1}} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) (1 - e^{t_i^\gamma})^2 e^{(t_i^\gamma - e^{t_i^\gamma+1})} \]

\[ U_{yy} = -\frac{n}{b^\gamma} - \frac{\ln b}{b^\gamma} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) \frac{e^{t_i^\gamma}}{e^{t_i^\gamma} - 1} + \frac{1}{b^\gamma} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) e^{t_i^\gamma} \left[ \ln(x_i) \left( e^{t_i^\gamma} - 1 \right) - t_i^\gamma \ln(t_i) \right] \]

\[ - \frac{b}{b^\gamma} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) \ln(x_i) + \frac{\beta \gamma}{b^\gamma} \sum_{i=1}^{n} x_i^\gamma e^{t_i^\gamma} \ln(t_i) \]

\[ - \frac{b}{b^\gamma} \sum_{i=1}^{n} x_i^\gamma e^{t_i^\gamma} \ln(t_i) + t_i^\gamma \ln(t_i) \]

\[ - \frac{(\alpha - 1) \ln b}{b^\gamma} \sum_{i=1}^{n} x_i^\gamma \ln(t_i) \left( 1 - e^{t_i^\gamma} \right) e^{(t_i^\gamma - e^{t_i^\gamma+1})} \]

\[ + \frac{a - 1}{b^\gamma} \sum_{i=1}^{n} x_i^\gamma \ln(t_i)^2 \left( 1 - e^{t_i^\gamma} \right) e^{(t_i^\gamma - e^{t_i^\gamma+1})} \]

\[ U_{aa} = -n[\psi'(a) - \psi'(a + b)], U_{ab} = n[\psi'(a + b)] \text{ and } U_{bb} = -n[\psi'(b) - \psi'(a + b)], \]

where \( \psi'(\cdot) \) is the trigamma function.
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