Large deviations of the free energy in the O’Connell-Yor polymer

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Abstract

We investigate large deviations of the free energy in the O’Connell-Yor polymer through a variational representation of the positive real moment Lyapunov exponents of the associated parabolic Anderson model. Our methods yield an exact formula for all real moment Lyapunov exponents of the parabolic Anderson model and a dual representation of the large deviation rate function with normalization $n$ for the free energy.

1 Introduction

The O’Connell-Yor semi-discrete polymer was introduced in [17] in connection with a generalization of the Brownian queueing model. For $n \in \mathbb{N}$ and $\beta > 0$, the authors define a polymer partition function by

$$Z_n(\beta) = \int_{0 < s_1 < \cdots < s_{n-1} < n} \exp \left[ \beta (B_1(0, s_1) + \cdots + B_n(s_{n-1}, n)) \right] ds_1 \cdots ds_{n-1},$$

where $\{B_i\}_{i=1}^\infty$ is a family of i.i.d. standard Brownian motions. Based on the work of Matsumoto and Yor [13], the authors showed the existence of a stationary version of this model satisfying an analogue of Burke’s theorem. This property makes (1) one of the three polymer models considered exactly solvable, the others being the log-gamma polymer introduced by Seppäläinen in [20] and the strict-weak gamma polymer studied in [7] and [16]. Subsequent work on the representation theoretic underpinnings of the exact solvability of these models

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can be found in the work of Borodin and Corwin on Macdonald processes [4] and the work of O’Connell connecting (1) to the quantum Toda lattice [15].

This model has now been extensively studied, so we may only provide a brief and incomplete summary of the known results. Moriarty and O’Connell [14] rigorously computed the free energy; Seppäläinen and Valkó [21] identified the scaling exponents; Borodin, Corwin, and Ferrari [6] showed that the model lies in the KPZ universality class by proving the Tracy-Widom limit; and Borodin and Corwin [5] proved a contour integral representation for the integer moments and computed their large $n$ asymptotics.

Georgiou and Seppäläinen [9] computed the large deviation rate function with normalization $n$ for the free energy in the log-gamma polymer using a method introduced by Seppäläinen in [19]. Structurally the log-gamma polymer and the O’Connell-Yor polymer are quite similar, so it is natural to expect that the techniques of [9] should also apply in this setting. This is our starting point.

The main result of this paper can be thought of as an extension of the asymptotics studied in [5]. There, the authors use a contour integral representation for the integer moments of $Z_n(\beta)$ to compute

$$\lim_{n \to \infty} \frac{1}{n} \log E[Z_n(\beta)^p]$$

for any $p \in \mathbb{N}$. In [5, Appendix A.1], the authors give a non-rigorous computation of the free energy of this model using the replica heuristic, which agrees with the exact solution in [14]. A key step in this argument is an unjustified analytic continuation of the integer moment Lyapunov exponents. We are able to compute the limit above for all $p \in \mathbb{R}$, which confirms this analytic continuation.

Throughout the paper, we take advantage of the relationship between (1) and the zero temperature version of (1), the Brownian queue. A connection between this queueing model and the largest eigenvalue of a standard GUE matrix was discovered independently by Baryshnikov [1, Theorem 0.7] and Gravner, Tracy, and Widom [10], both in 2001. The known large deviations [12] for top eigenvalues give useful estimates in several of the proofs that follow. The precise results we use are collected in subsection A.3 of the appendix.

This polymer and the log-gamma polymer are the only positive temperature polymer models for which precise large deviations have been studied. The left tail large deviations, which have non-universal scalings (see [2]), remain open for both models.

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2 Preliminaries and statement of results

2.1 Definition of the polymer model and statement of results

Let $B$ and $\{B_i\}_{i=0}^\infty$ be independent two-sided standard Brownian motions. We adopt the notational convention $B_i(s, t) = B_i(t) - B_i(s)$ and similarly for $B$. Define a family of partition functions for $j, n \in \mathbb{Z}_+$ with $j < n$ and $s, t \in (0, \infty)$ with $s < t$ by

$$Z_{j,n}(u, t) = \int_{u_i < \cdots < u_{n-1} < t} e^{B_j(u_i) + \sum_{i=j+1}^{n-1} B_i(u_{i-1}, u_i) + B_n(u_{n-1}, t)} ds_j \cdots u_{n-1}. \quad (2)$$

For the case $j = n$, we define

$$Z_{j,j}(u, t) = e^{B_j(u,t)}. \quad (3)$$

We will refer to the $j, n$ variables as space and the $u, t$ variables as time. Translation invariance of Brownian motion and our assumption that the environment is i.i.d. immediately imply that the distribution of the partition function is shift invariant.

It follows from Brownian scaling that for $\beta > 0$ and $n > 1$ we have

$$Z_n(\beta) \overset{d}{=} \beta^{-2(n-1)} Z_{1,n}(0, \beta^2 n).$$

For the remainder of the paper, we will only consider partition functions of the form $Z_{j,n}(u, t)$ as results for these partition functions can easily be translated into results for $Z_n(\beta)$ using this distributional identity.

We now argue that the partition function as defined above is supermultiplicative for shifts with a positive time component. Clearly, whenever $t, u > 0$

$$Z_{j,j}(0, t + u) = Z_{j,j}(0, t) Z_{j,j}(t, t + u).$$

For $m, n \in \mathbb{N}$ and $t, u > 0$, we have

$$Z_{0,n+m}(0, t + u) = \int \cdots \int e^{B_0(0, u_0) + \sum_{i=1}^{n+m-1} B_i(u_{i-1}, u_i) + B_{n+m}(u_{n+m-1}, t + u)} du_0 \cdots u_{n+m-1}$$

$$\geq \int \cdots \int e^{B_0(0, u_0) + \sum_{i=1}^{n+m-1} B_i(u_{i-1}, u_i) + B_{n+m}(u_{n+m-1}, t + u)} du_0 \cdots u_{n+m-1}$$

$$= Z_{0,n}(0, t) Z_{n,n+m}(t, t + u). \quad (4)$$

When $j < n$ and $t, u > 0$, we recall the semi-martingale decomposition of $\log Z_{j,n}(u, t)$ in filtration of the Brownian environment.

$$\log Z_{j,n}(u, t) = B_n(t) - B_j(u) + \log C_{j,n}(u, t) \quad (5)$$
where
\[ C_{j,n}(u,t) = \int_u^t \int_{u_j}^t \cdots \int_{u_{n-2}}^t e^{B_j(u) + \sum_{i=j}^{n-1} B_i(u_{i-1}, u_i) - B_n(u_{n-1})} du_{n-1} \cdots du_{j+1} du_j \]
is strictly increasing in \( t \) and strictly decreasing in \( u \). It follows that
\[ Z_{j,n}(0, t+u) \geq Z_{j,n}(0, t) Z_{n,n}(t, t+u), \]
\[ Z_{j,n}(0, t+u) \geq Z_{j,j}(0, t) Z_{j,n}(t, t+u). \]

Therefore, for \( n, m \in \mathbb{Z}_+ \) and \( u, t > 0 \), we find
\[ Z_{0,n+m}(0, t+u) \geq Z_{0,n}(0, t) Z_{n,n+m}(t, t+u). \]

Note that the two partition functions on the right are independent by the independence of the environment.

The free energy for [13] was computed in [14]. We mention that, as in [9, Lemma 4.1], once one knows the existence and continuity of the free energy, a variational problem similar to the one we study for the rate function in this paper can be used to compute the value of the free energy. We have

**Lemma 2.1.** [14] Fix \( s, t \in (0, \infty) \). Then the almost sure limit
\[ \rho(s, t) = \lim_{n \to \infty} \frac{1}{n} \log Z_{1,\lfloor ns \rfloor}(0, nt) \]
exists and is given by
\[ \rho(s, t) = \inf_{\theta > 0} \{ \theta t - s \Psi_0(\theta) \} = t \Psi^{-1}_1 \left( \frac{t}{s} \right) - s \Psi_0 \left( \Psi^{-1}_1 \left( \frac{t}{s} \right) \right) \]
where \( \Psi_0(x) = \frac{d}{dx} \log \Gamma(x) \) is the digamma function, \( \Psi_1(x) = \frac{d}{dx} \Psi_0(x) \) is the trigamma function, and \( \Psi^{-1}_1(\Psi_1(x)) = \Psi_1(\Psi^{-1}_1(x)) = x \).

The main result of this paper is a computation of the real moment Lyapunov exponents of the parabolic Anderson model associated to [2] and, through an application of the Gärtner-Ellis theorem, the large deviation rate function with normalization \( n \) for the free energy of the polymer. Specifically, we have

**Theorem 2.2.** Let \( s, t \in (0, \infty) \) and \( \xi \in \mathbb{R} \). Then
\[ \Lambda_{s,t}(\xi) = \lim_{n \to \infty} \frac{1}{n} \log E \left[ e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt)} \right] \]
\[ = \begin{cases} \xi \rho(s, t) & \xi \leq 0 \\ \min_{\mu > 0} t \left( \frac{\xi^2}{2} + \xi \mu \right) - s \log \frac{\Gamma(\mu + \xi)}{\Gamma(\mu)} & \xi > 0 \end{cases} \]
and \( \Lambda_{s,t}(\xi) \) is a differentiable function of \( \xi \in \mathbb{R} \).
Theorem 2.3. Fix $s, t \in (0, \infty)$. The distributions of $n^{-1} \log Z_{1,|ns|}(0, nt)$ satisfy a large deviation principle with normalization $n$ and convex good rate function

$$I_{s,t}(x) = \begin{cases} 
\infty & x < \rho(s, t) \\
\Lambda_{s,t}^*(x) & x \geq \rho(s, t)
\end{cases}$$

where $\Lambda_{s,t}^*(x) = \sup_{\xi \in \mathbb{R}} \{x\xi - \Lambda_{s,t}(\xi)\}$ is the Legendre-Fenchel transform of $\Lambda_{s,t}(\xi)$.

2.2 Definition of the stationary model and proof outline

For $\theta > 0$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$ we define another partition function by

$$Z_\theta^n(t) = \int_{-\infty < u_0 < u_1 < \cdots < u_n - 1 < t} e^{\theta u_0 - B(u_0) + B_1(u_0, u_1) + \cdots + B_n(u_{n-1}, t)} du_0 \ldots u_{n-1}.$$ 

It is notationally convenient to further define

$$Z_\theta^0(t) = e^{\theta t - B(t)}.$$ 

Set $Y_\theta^0(t) = B(t)$ and for $k \geq 1$ recursively define

$$r_\theta^k(t) = \log \int_{-\infty}^{t} e^{Y_\theta^{k-1}(u, t) - \theta(t-u) + B_k(u, t)} du,$$

$$Y_\theta^k(t) = Y_\theta^{k-1}(t) + r_\theta^0(0) - r_\theta^k(t),$$

where we again use the notational convention $Y_\theta^k(u, t) = Y_\theta^k(t) - Y_\theta^k(u)$. Induction shows

$$\sum_{k=1}^{n} r_\theta^k(t) = B(t) - \theta t + \log Z_\theta^n(t).$$

\cite{[17]} Theorem 5] gives the Burke property: for each $\theta > 0$ and $t \in \mathbb{R}$, the random variables $r_\theta^k(t)$ are i.i.d.. Dufresne’s identity \cite{[17]} Theorem 7] implies that $r_\theta^k(t)$ has the distribution of the negative logarithm of a random variable with density $\Gamma(\theta)^{-1}x^{\theta-1}e^{-x}$.

By considering where paths leave the potential of the Brownian motion $B$, we obtain a decomposition of $Z_{1,|ns|}(nt)$ into terms that involve the partition function we are studying:

$$Z_{1,|ns|}(nt) = \int_{0}^{nt} Z_0^\theta(u)Z_{1,|ns|}(u, nt)du + \sum_{j=1}^{|ns|} Z_j^\theta(0)Z_{j,|ns|}(0, nt)$$

$$= n \int_{0}^{t} Z_0^\theta(nu)Z_{1,|ns|}(nu, nt)du + \sum_{j=1}^{|ns|} Z_j^\theta(0)Z_{j,|ns|}(0, nt). \quad (7)$$
It is convenient to rewrite the previous decomposition in a form that is better suited to analysis of large deviations:

\[ e^{\sum_{k=1}^{\lfloor ns \rfloor} r_k^r(nt)} = n \int_0^t \frac{Z^\theta_0(nu)}{Z^\theta_0(nt)} Z_{1, \lfloor ns \rfloor}^\theta(nu, nt) du + \sum_{j=1}^{\lfloor ns \rfloor} \frac{Z^\theta_j(0)}{Z^\theta_0(nt)} Z_j, \lfloor ns \rfloor(0, nt). \] (8)

We now briefly outline the proof of Theorem 2.2.

In order to compute the positive moment Lyapunov exponents, we consider the dual problem of establishing right tail large deviations for the free energy. We begin by showing that a sufficiently regular right tail large deviation rate function exists using subadditivity arguments. It then follows from (8) that this right tail rate function solves a variational problem in terms of computable rate functions coming from the stationary model. Taking Legendre-Fenchel transforms brings us back to the study of moment Lyapunov exponents and gives the variational problem a linear structure which makes it tractable.

For non-positive exponents, we use crude estimates on the partition function to identify the limit. We are able to do this because the left tail large deviations for the free energy are are strictly subexponential while the moment Lyapunov exponents are only sensitive to exponential scale large deviation.

3 Right tail large deviations

3.1 Existence and structure of the right tail rate function

We now turn to the problem of showing the existence and regularity of the right tail rate function for the polymer free energy. As is typical for right tail large deviations, these properties follow from (almost) subadditivity arguments. Because the partition function degenerates for steps with no time component and we do not restrict attention to integer \( s \), it is necessary to tilt time slightly in this argument. It is not hard to see that this change does not affect the free energy.

**Theorem 3.1.** For all \( s \geq 0, \ t > 0 \) and \( r \in \mathbb{R} \), the limit

\[ J_{s,t}(r) = \lim_{x \to \infty} -\frac{1}{x} \log P \left( \log Z_{0, \lfloor xs \rfloor}(0, xt - 1) \geq xr \right) \]

exists and is \( \mathbb{R}_+ \) valued. Moreover, \( J_{s,t}(r) \) is continuous, convex, subadditive, and positively homogeneous of degree one as a function of \( (s, t, r) \in [0, \infty) \times (0, \infty) \times \mathbb{R} \). For fixed \( s \) and \( t \), \( J_{s,t}(r) \) is increasing in \( r \) and \( J_{s,t}(r) = 0 \) if \( r \leq \rho(s, t) \).
Proof. Define the function $T: [0, \infty) \times (1, \infty) \times \mathbb{R} \to \mathbb{R}_+$ by

$$T(x, y, z) = -\log P \left( \log Z_{0,[x]}(0, y - 1) \geq z \right).$$

Lemma A.1 in the appendix implies that $P \left( \log Z_{0,[x]}(0, y - 1) \geq z \right) \neq 0$ and therefore that this function is well-defined.

Take $(x_1, y_1, z_1), (x_2, y_2, z_2) \in [0, \infty) \times (1, \infty) \times \mathbb{R}$ and call $x_{1,2} = |x_1 + x_2| - |x_1| - |x_2| \in \{0, 1\}$. By (6) we have

$$Z_{0,[x_1+x_2]}(y_1+1,y_2-1) \geq Z_{0,[x_1]}(y_1-1)Z_{[x_1],[x_1+x_2]}(y_1-1,y_1+y_2-1).$$

Independence and translation invariance then imply

$$P \left( \log Z_{0,[x_1+x_2]}(y_1+1,y_2-1) \geq z_1 + z_2 \right) \geq P \left( \log Z_{0,[x_1]}(y_1-1) \geq z_1 \right) P \left( \log Z_{0,[x_2]}(y_2) \geq z_2 \right).$$

If $x_{1,2} = 0$ then, recalling that $\log Z_{[x_2],[x_2]}(u,t) = B_{[x_2]}(u,t)$, we find

$$P \left( \log Z_{0,[x_2]}(y_2) \geq z_2 \right) \geq P \left( \log Z_{0,[x_2]}(y_2-1) \geq z_2 \right) P \left( \log Z_{[x_2],[x_2]}(y_2-1,y_2) \geq 0 \right) \geq \frac{1}{2} P \left( \log Z_{0,[x_2]}(y_2-1) \geq z_2 \right).$$

Similarly, when $x_{1,2} = 1$ we have

$$P \left( \log Z_{0,[x_2]+1}(y_2) \geq z_2 \right) \geq P \left( \log Z_{0,[x_2]}(y_2-1) \geq z_2 \right) P \left( \log Z_{0,1}(y_2,1) \geq 0 \right).$$

Setting $C = \max\{\log(2), -\log P \left( \log Z_{0,1}(y_2,1) \geq 0 \right)\} < \infty$, we find that

$$T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \leq T(x_1, y_1, z_1) + T(x_2, y_2, z_2) + C.$$
Fix \((t, r) \in (0, \infty) \times \mathbb{R}\) and a sequence \((s_k, t_k, r_k) \in [0, \infty) \times (0, \infty) \times \mathbb{R}\) with \((s_k, t_k, r_k) \to (0, t, r)\). Recall that \(\log Z_{0,0}(0, t) = B_0(t)\), so we may compute with the normal distribution to find \(J_{0,t}(r) = \frac{r^2}{2t} 1_{\{r \geq 0\}}\). If \(s_k = 0\) for all sufficiently large \(k\) then clearly \(J_{s_k,t_k}(r_k) \to J_{0,t}(r)\). We may therefore assume without loss of generality that \(s_k > 0\) for all \(k\). First observe that if \(r \leq 0\), then \(J_{0,t}(r) = 0\) and the lower bound is trivial.

If \(r > 0\), we may assume without loss of generality that there exists \(c > 0\) with \(r_k > c\) for all \(k\). By Lemma A.3 in the appendix, for all sufficiently large \(k\) we have

\[
J_{s_k,t_k}(r_k) \geq s_k J_{\text{GUE}} \left( \frac{r_k - s_k \log t_k - s_k + s_k \log s_k}{2 \sqrt{t_k s_k}} - 1 \right),
\]

where \(J_{\text{GUE}}(r) = 4 \int_0^r \sqrt{x(x+2)} dx\). Using this formula and calculus, we find that

\[
\lim_{k \to \infty} s_k J_{\text{GUE}} \left( \frac{r_k - s_k \log t_k - s_k + s_k \log s_k}{2 \sqrt{t_k s_k}} - 1 \right) = \frac{r^2}{2t}
\]

and therefore continuity follows. Lemma 2.1 implies that \(J_{s,t}(r) = 0\) for \(r \leq \rho(s, t)\).

**Remark 3.2.** Note that we only address the spatial boundary in the previous result. The reason for this is that the right tail rate function is not continuous at \(t = 0\) for any \(s > 0\) and \(x \in \mathbb{R}\). To see this, we can use the lower bound for \(J_{s,t}(r)\) coming from Lemma A.3 which will be non-trivial for \(t\) sufficiently small. As \(t \downarrow 0\), this lower bound tends to infinity.

**Lemma 3.3.** Fix \((s, t, r) \in (0, \infty)^2 \times \mathbb{R}\). For any sequences \(s_n, t_n \in \mathbb{N} \times (0, \infty)\) with \(\frac{1}{n}(s_n, t_n) \to (s, t)\) we have

\[
J_{s,t}(r) = \lim_{n \to \infty} -\frac{1}{n} \log P \left( \log Z_{0,s_n}(0, t_n) \geq nr \right).
\]

**Proof.** Fix \(\epsilon < \min(s, t)\) and positive. We will assume that \(n\) is large enough that the following conditions hold:

\[
\left\lfloor \left( \frac{s - \epsilon}{2} \right) n \right\rfloor \leq s_n < \left\lfloor \left( \frac{s + \epsilon}{2} \right) n \right\rfloor, \quad \left( \frac{t - \epsilon}{2} \right) n \leq t_n < \left( \frac{t + \epsilon}{2} \right) n - 2.
\]

We have

\[
Z_{0,s_n}(0, t_n) \geq Z_{0,\left\lfloor (s-\epsilon)n \right\rfloor}(0, (t-\epsilon)n - 1) Z_{\left\lfloor (s-\epsilon)n \right\rfloor, \left\lfloor (t-\epsilon)n \right\rfloor}(t-\epsilon)n - 1, t_n).
\]

It follows that

\[
P \left( \log Z_{0,s_n}(0, t_n) \geq nr \right) \\
\geq P \left( \log Z_{0,\left\lfloor (s-\epsilon)n \right\rfloor}(0, (t-\epsilon)n - 1) \geq nr \right) P \left( \log Z_{0,\left\lfloor (s-\epsilon)n \right\rfloor}(0, t_n - (t-\epsilon)n + 1) \geq 0 \right).
\]
Call \( s(n) = s_n - [(s - \epsilon)n] \) and \( t(n) = t_n - (t - \epsilon)n + 1 \) and divide the interval \((0, t(n))\) into \( s(n) \) uniform subintervals. We may bound \( Z_{0, s(n)}(0, t(n)) \) below by a product of i.i.d. random variables:

\[
Z_{0, s(n)}(0, t(n)) \geq \prod_{i=1}^{s(n)} Z_{i-1, i} \left( \left( i - 1 \right) \frac{t(n)}{s(n)}, i \frac{t(n)}{s(n)} \right).
\]

Therefore,

\[
P \left( \log Z_{0, s(n)}(0, t(n)) \geq 0 \right) \geq P \left( \log Z_{0, 1} \left( 0, \frac{t(n)}{s(n)} \right) \geq 0 \right)^{s(n)}.
\]

Notice that \( \lim \frac{s(n)}{n} = \epsilon \) and \( \lim \frac{t(n)}{n} = \epsilon \), so we may further assume without loss of generality that \( \frac{1}{2} < \frac{t(n)}{s(n)} < 2 \) for all \( n \). We have

\[
Z_{0, 1} \left( 0, \frac{t(n)}{s(n)} \right) \geq Z_{0, 1} \left( 0, \frac{1}{2} \right) Z_{1, 1} \left( \frac{1}{2}, \frac{t(n)}{s(n)} \right) = Z_{0, 1} \left( 0, \frac{1}{2} \right) e^{B_1 \left( \frac{1}{2}, \frac{t(n)}{s(n)} \right)}
\]

so that

\[
P \left( \log Z_{0, 1} \left( 0, \frac{t(n)}{s(n)} \right) \geq 0 \right) \geq \frac{1}{2} P \left( \log Z_{0, 1} \left( 0, \frac{1}{2} \right) \geq 0 \right).
\]

Therefore for \( C = \log(2) - \log P \left( \log Z_{0, 1} \left( 0, \frac{1}{2} \right) \geq 0 \right) \) and all \( \epsilon < \min(s, t) \) we have

\[
\limsup \frac{1}{n} \log P \left( \log Z_{0, s(n)}(0, t_n) \geq nr \right) \leq J_{s, t} - \epsilon + \epsilon C
\]

sending \( \epsilon \downarrow 0 \) and applying continuity of the rate function gives one inequality. A similar argument gives the lim inf inequality.

\[\square\]

**Corollary 3.4.** Fix \((s, t, r) \in (0, \infty)^2 \times \mathbb{R}\). Then we have

\[
J_{s, t}(r) = \lim_{n \to \infty} - \frac{1}{n} \log P \left( \log Z_{1, s(n)}(0, nt) \geq nr \right).
\]

The Legendre-Fenchel transform of a function on \( \mathbb{R} \) is defined by \( f^*(\xi) = \sup_{x \in \mathbb{R}} \{x \xi - f(x)\} \). The next corollary is easy to see from convexity of \( J_{s, t}(x) \) in \((s, t, x) \in (0, \infty)^2 \times \mathbb{R}\). Details can be found in the first few lines of the proof of [9, Lemma 4.6].

**Lemma 3.5.** For all \( \xi > 0 \), \( J^*_{s, t}(\xi) \) is convex as a function of \((s, t) \in (0, \infty)^2\).

We denote by \( f \Box g(x) = \inf_{y \in \mathbb{R}} \{ f(x - y) + g(y) \} \) the infimal convolution of \( f \) and \( g \). Versions of the next lemma appear in several other papers, so we elect not to re-prove it. The exact statement we need appears in [9].
Lemma 3.6. ([4, Lemma 3.6]) Suppose that for each \( n \), \( X_n \) and \( Y_n \) are independent, that the limits

\[
\lambda(s) = \lim_{n \to \infty} -\frac{1}{n} \log P(X_n \geq ns), \quad \phi(s) = \lim_{n \to \infty} -\frac{1}{n} \log P(Y_n \geq ns)
\]

exist, and that \( \lambda \) is continuous. If there exists \( a_\lambda \) and \( a_\phi \) so that \( \lambda(a_\lambda) = \phi(a_\phi) = 0 \), then

\[
\lim_{n \to \infty} -\frac{1}{n} \log P(X_n + Y_n \geq nr) = \begin{cases} 
\inf_{a_\lambda \leq s \leq r-a_\phi} \{ \phi(r-s) + \lambda(s) \} & r \geq a_\phi + a_\lambda \\
0 & r \leq a_\phi + a_\lambda 
\end{cases}
\]

\[ = \lambda \square \phi(r) \]

3.2 Boundary rate functions for the stationary model

To organize the exposition, we introduce notation for the rate functions which appear in the variational problem for \( J_{s,t}(x) \). The first limit below can be computed as the right branch of a Cramér rate function, while the second follows from standard estimates for the normal distribution. For \( s, t > 0 \), set

\[
U^\theta_s(x) = -\lim \frac{1}{n} \log P \left( \sum_{k=1}^{\lfloor ns \rfloor} r^\theta_k(0) \geq nx \right) = \begin{cases} 
0 & x \leq -s\Psi_0(\theta) \\
x(\theta - \Psi_0^{-1}(-\frac{x}{s})) + s \log \frac{\Gamma(\theta)}{\Gamma(\Psi_0^{-1}(-\frac{x}{s}))} & x > -s\Psi_0(\theta)
\end{cases}
\]

\[
R^\theta_t(x) = -\lim \frac{1}{n} \log P(B(nt) - \theta nt \geq nx) = \begin{cases} 
0 & x \leq -\theta t \\
\frac{1}{2} \left( \frac{x+\theta t}{\sqrt{t}} \right)^2 & x > -\theta t
\end{cases}
\]

We may continuously (jointly in \( (s, x) \)) extend \( U^\theta_s(x) \) to

\[
U^\theta_0(x) = \begin{cases} 
0 & x \leq 0 \\
x\theta & x > 0
\end{cases}
\]

We record the Legendre-Fenchel transforms of these functions below:

\[
(U^\theta_s)^*(\xi) = \begin{cases} 
\infty & \xi < 0 \text{ or } \xi \geq \theta \\
s \log \frac{\Gamma(\theta - \xi)}{\Gamma(\theta)} & 0 \leq \xi < \theta 
\end{cases}, \quad (R^\theta_t)^*(\xi) = \begin{cases} 
\infty & \xi < 0 \\
t(\frac{\xi}{2} - \theta \xi) & \xi \geq 0
\end{cases}
\]

Similarly, we can introduce the right tail rate functions which we will show correspond to the two parts of the decomposition of the stationary model in [8]
For \(a \in [0,t)\), \(u \in (0,s)\), \(v \in [0,s]\), and \(x \in \mathbb{R}\) define
\[
G^\theta_{a,s,t}(x) = -\lim_{n\to\infty} \frac{1}{n} \log P\left( B(na,nt) - \theta n(t-a) + \log Z_{1|ns}(na,nt) \geq nx \right) \\
H^\theta_{a,v,s,t}(x) = -\lim_{n\to\infty} \frac{1}{n} \log P\left( -\log Z^\theta_u(nt) + \log Z^\theta_{[n\theta]}(0) + \log Z^\theta_{[n\theta],|ns]}(0,nt) \geq nx \right).
\]
Recall that \(\log Z^\theta_j(0) = \sum_{k=1}^j r^\theta_k(0)\) is measurable with respect to the sigma algebra \(\sigma(B(s), B_k(s) : 1 \leq k \leq j; s \leq 0)\) and that for \(0 \leq u < nt\), \(\log Z_{j|ns}(u,t)\) is measurable with respect to the sigma algebra \(\sigma(B_k(s_k) : j \leq k \leq [ns], u \leq s_j \leq nt)\). Combining the independence of the environment with the computations above, Corollary 3.4 and Lemma 3.6 imply that \(G^\theta_{a,s,t}(x)\) and \(H^\theta_{a,v,s,t}(x)\) are well-defined. In particular, we immediately obtain

**Corollary 3.7.** For \(a \in [0,t)\) and \(u \in (0,s)\), and \(v \in [0,s]\)
\[
G^\theta_{a,s,t}(x) = R^\theta_{t-a} \Box J_{s,t-a}(x), \quad H^\theta_{a,v,s,t}(x) = R^\theta_{t} \Box U^\theta_u \Box J_{s-v,t}(x).
\]

As all of the rate functions above are non-negative, continuous, and convex functions it follows from general theory that both of these infimal convolutions are convex in \(x\) [18 Theorem 5.4] and it is not hard to see that they are also proper. For such functions, infimal convolution is Legendre-Fenchel dual to addition [18 Theorem 16.4], so it follows that

**Corollary 3.8.**
\[
(G^\theta_{a,s,t})^*(\xi) = (R^\theta_{t-a})^*(\xi) + (J_{s,t-a})^*(\xi),

(H^\theta_{a,v,s,t})^*(\xi) = (R^\theta_{t})^*(\xi) + (U^\theta_u)^*(\xi) + (J_{s-v,t})^*(\xi).
\]

We will need to arrange the error estimates in such a way that we will need a strong kind of local uniform continuity of \(H^\theta_{a,v,s,t}(x)\). This result is the only point in the paper where we directly use the continuity up to the boundary in Theorem 3.1.

**Lemma 3.9.** Fix \(\theta, s, t > 0\) and a compact set \(K \subseteq \mathbb{R}\). Then
\[
\lim_{\delta, \gamma \to 0} \sup_{a,b \in [0,s], |b-a| < \delta} \left| H^\theta_{a,b,s,t+\gamma}(r_1) - H^\theta_{a,b',s,t}(r_2) \right| = 0
\]

*Proof.* Notice that \(\theta t, a\Psi_0(\theta),\) and \(\rho(s-b,t+\gamma)\) are bounded for \(a, b \in [0,s]\) and \(\gamma \in [0,t]\).

Using this fact and the formula for \(H^\theta_{a,b,s,t+\gamma}(r)\) coming from Corollary 3.7 and Lemma 3.6 there exists a compact set \(K'\) containing \(K\) so that for all \(r \in K, a, b \in [0,s],\) and \(\gamma \in [0,t]\)
\[
H^\theta_{a,b,s,t+\gamma}(r) = \inf_{x \in K'} \{ R^\theta_{t} \Box U^\theta_u(x) + J_{s-b,t+\gamma}(r-x) \}
\]

It is not hard to see that \((a, x) \mapsto R^\theta_{t} \Box U^\theta_u(x)\) is continuous on \([0,s] \times \mathbb{R}\). By Theorem 3.1 for any compact set \(K'\) we have joint uniform continuity of \((a, b, \gamma, r, x) \mapsto R^\theta_{t} \Box U^\theta_u(x) + J_{b,t+\gamma}(r-x)\) on the compact set \([0,s]^2 \times [0,t] \times K' \times K'\) and so the result follows. \(\Box\)
We also need uniform control on \( G_{a,s,t}^\theta(x) \). The proof of the next result is the same as the proof of Lemma 3.9 but the statement is weaker. The difference between these two results comes from the fact that \( R_{t}^\theta(x) \) does not extend to \( t = 0 \) continuously for \( x > 0 \).

**Lemma 3.10.** Fix \( \theta, s, t > 0 \) and \( 0 < \delta \leq t \) and a compact set \( K \subseteq \mathbb{R} \). Then

\[
\lim_{\epsilon \downarrow 0} \sup_{\gamma \downarrow 0, a_1 \in [0, t-\delta] : |a_1-a_2| < \gamma, r_1, r_2 \in K : |r_1-r_2| < \epsilon} |G_{a_1,s,t}^\theta(r_1) - G_{a_2,s,t}^\theta(r_2)| = 0
\]

Next, we show that \( G_{a,s,t}^\theta(x) \) degenerates to \( \infty \) locally uniformly near \( a = t \).

**Lemma 3.11.** Fix \( \theta, s, t > 0 \) and \( K \subset \mathbb{R} \) compact. Then

\[
\liminf_{a \uparrow t} \inf_{x \in K} G_{a,s,t}^\theta(x) = \infty
\]

**Proof.** We have

\[
G_{a,s,t}^\theta(x) = \begin{cases} 
0 & x \leq -\theta(t-a) + \rho(s, t-a) \\
\inf_{-\theta(t-a) \leq y \leq x - \rho(s, t-a)} \{ J_{s,t-a}(x-y) + R_{t-a}^\theta(y) \} & x > -\theta(t-a) + \rho(s, t-a)
\end{cases}
\]

Fix \( \epsilon > 0 \). The formula in Lemma 2.1 shows that \( \rho(s, t-a) \to -\infty \) as \( a \uparrow t \), so that for all \( x \in \mathbb{R} \) and \( a \) sufficiently close to \( t, x > -\theta(t-a) + \rho(s, t-a) \). For \( a \) sufficiently large that this holds for all \( x \in K \), we have

\[
J_{s,t-a}(x-y) + R_{t-a}^\theta(y) \geq \begin{cases} 
J_{s,t-a}(x + \theta(t-a) - \epsilon) & y \in [-\theta(t-a), -\theta(t-a) + \epsilon] \\
R_{t-a}^\theta(-\theta(t-a) + \epsilon) & y \in [-\theta(t-a) + \epsilon, x - \rho(s, t-a) - \epsilon] \\
R_{t-a}^\theta(x - \rho(s, t-a) - \epsilon) & y \in [x - \rho(s, t-a) - \epsilon, x - \rho(s, t)]
\end{cases}
\]

By Lemma A.3 for all \( x \in K \) and \( a \) sufficiently large, we have

\[
J_{s,t-a}(x) \geq sJ_{\text{GUE}} \left( \frac{x - s \log(t-a) - s(1 - \log(s))}{2\sqrt{(t-a)s}} - 1 \right)
\]

Combining this with the exact formula for \( R_{t-a}^\theta(x) \) and optimizing the lower bounds over \( x \in K \) shows that the infimum over \( x \in K \) of the minimum of these three lower bounds tends to infinity, giving the result. \( \square \)
3.3 A variational problem for the right tail rate function

Fix $a \in [0, t)$ and $0 < \delta \leq t - a$. Then \[ \text{in A.8} \] immediately implies the following lower bounds.

\[
\log \left( n \int_a^{a+\delta} \frac{Z^\theta_0(nu)}{Z^\theta_0(nt)} Z_{1,[ns]}(nu, nt) du \right) \leq \sum_{k=1}^{[ns]} r_k^\theta(nt) \tag{9}
\]

\[
- \log Z_0^\theta(nt) + \log Z_j^\theta(0) + \log Z_{j,[ns]}(0, nt) \leq \sum_{k=1}^{[ns]} r_k^\theta(nt) \tag{10}
\]

We introduce the notation $a \land b = \max(a, b)$ and $a \land b = \min(a, b)$. For any partition $\{a_i\}_{i=0}^N$ of $[0, t]$, we have

\[
\sum_{k=1}^{[ns]} r_k^\theta(nt) \leq \left\{ \max_{0 \leq t \leq N-1} \log \left( n \int_{a_i}^{a_{i+1}} \frac{Z^\theta_0(nu)}{Z^\theta_0(nt)} Z_{1,[ns]}(nu, nt) du \right) \right\}
\]

\[
\vee \left\{ \max_{1\leq j \leq [ns]} - \log Z_0^\theta(nt) + \log Z_j^\theta(0) + \log Z_{j,[ns]}(0, nt) \right\} + \log(N + 1 + ns)
\tag{11}
\]

Our goal in this section is to show that estimates \[ \text{in A.9, A.10, and A.11} \] above lead to a variational characterization of the right tail rate function $J_{s,t}(x)$:

\[
U^\theta_s(x) = \min\left\{ \inf_{0 \leq a < t} G_{a,s,t}^\theta(x), \inf_{0 \leq a < s} H_{a,a,s,t}^\theta(x) \right\}
\]

\[
= \min\left\{ \inf_{0 \leq a < t} R_{t-a}^\theta \square J_{s,t-a}(x), \inf_{0 \leq a < s} R_{t}^\theta \square U^\theta_a \square J_{s-t,a}(x) \right\}. \tag{12}
\]

We break the estimates into two inequalities.

**Lemma 3.12.** Fix $\theta > 0$, $(s,t) \in (0, \infty)^2$ and $x \in \mathbb{R}$. Then

\[
U^\theta_s(x) \leq \min\left\{ \inf_{0 \leq a < t} G_{a,s,t}^\theta(x), \inf_{0 \leq a < s} H_{a,a,s,t}^\theta(x) \right\}.
\]

**Proof.** For $a \in [0, s)$, taking $j = \lfloor an \rfloor$ in inequality \[ \text{in A.10} \] above immediately implies

\[
U^\theta_s(x) \leq H_{a,a,s,t}^\theta(x). \tag{13}
\]

Fix $\delta \in (0, t)$; then for all $a \in [0, t - \delta)$ and all $u \in [0, a + \delta]$, we have

\[
Z_{1,1}(nu, n(a + \delta)) Z_{1,[ns]}(n(a + \delta), nt) \leq Z_{1,[ns]}(nu, nt). \tag{14}
\]

It then follows that

\[
P\left( \log \left( n \int_a^{a+\delta} \frac{Z^\theta_0(nu)}{Z^\theta_0(nt)} Z_{1,[ns]}(nu, nt) du \right) \geq nx \right)
\]

\[
\geq P\left( \log Z_{1,[ns]}(n(a + \delta), nt) + \log \frac{Z^\theta_0(n(a + \delta))}{Z^\theta_0(nt)} \right.
\]

\[
+ \log \left( n \int_a^{a+\delta} \frac{Z^\theta_0(nu)}{Z^\theta_0(n(a + \delta))} Z_{1,1}(nu, n(a + \delta) du \right) \geq nx \right).
\]

13
Fix $\epsilon > 0$. By independence of the Brownian environment, we find that
\[
-\frac{1}{n} \log P \left( \log \left( n \int_{a}^{a+\delta} \frac{Z_{0}^{\theta}(nu)}{Z_{0}^{\theta}(nt)} Z_{1,|ns|}(nu, nt) du \right) \geq nx \right)
\leq -\frac{1}{n} \log P \left( \log Z_{1,|ns|}(n(a+\delta), nt) + \log \frac{Z_{0}^{\theta}(n(a+\delta))}{Z_{0}^{\theta}(nt)} \geq n(x+\epsilon) \right)
+ -\frac{1}{n} \log P \left( \log \left( n \int_{a}^{a+\delta} \frac{Z_{0}^{\theta}(nu)}{Z_{0}^{\theta}(n(a+\delta))} Z_{1,1}(nu, n(a+\delta)) du \right) \geq -n\epsilon \right).
\]

It is not hard to see that as $n \to \infty$ the second term tends to zero, since the term in the integral is the exponential of a Brownian motion with positive drift. Indeed using the lower bound obtained by considering the minimum of the Brownian increments on the interval $[a, a+\delta]$, we can show that the probability of being larger than $-n\epsilon$ tends to one. Taking lim sup and recalling inequality (9) we obtain
\[
U_{s}^{\theta}(x) \leq G_{a,\delta,s,t}^{\theta}(x+\epsilon).
\]

By Lemma 3.10 we may take $\delta, \epsilon \downarrow 0$ in (15). Optimizing over $a$ in the resulting equation and in (13) gives the result.

Lemma 3.13. Fix $\theta > 0$, $(s, t) \in (0, \infty)^2$ and $x \in \mathbb{R}$. Then
\[
U_{s}^{\theta}(x) \geq \min \{ \inf_{0 \leq a < t} G_{a,s,t}^{\theta}(x), \inf_{0 \leq a < s} H_{a,a,s,t}^{\theta}(x) \}.
\]

Proof. Fix a large $p > 1$ and small $\epsilon, \gamma > 0$. Consider uniform partitions $\{a_{i}\}_{i=0}^{M}$ of $[0, t]$ and $\{b_{i}\}_{i=0}^{N}$ of $[0, s]$ of mesh $\nu = \frac{t}{M+1}$ and $\delta = \frac{s}{N+1}$ respectively. We will add restrictions on these parameters later in the proof. Take $n$ sufficiently large that $[b_{i}n] < [b_{i+1}n]$ for all $i$.

Fix $j < |ns|$ not equal to any of the partition points $[b_{i}n]$ and consider $i$ so that $[b_{i}n] < j < [b_{i+1}n]$. Notice that $Z_{0}^{\theta}(nt)$ is $\sigma(B(nt))$ measurable and $Z_{j}^{\theta}(0)$ is measurable with respect to $\sigma(B(s), B_{1}(s), \ldots, B_{j}(s) : s \leq 0)$, so these random variables and $Z_{j,|ns|}(u, v)$ are mutually independent if $0 \leq u < v$. It follows from translation invariance and this independence that
\[
P \left( - \log Z_{0}^{\theta}(nt) + \log Z_{j}^{\theta}(0) + \log Z_{j,|ns|}(0, nt) \geq nx \right)
= P \left( - \log Z_{0}^{\theta}(nt) + \log Z_{j}^{\theta}(0) + \log Z_{j,|ns|}(n\gamma, n(t + \gamma)) \geq nx \right)
\]
We have
\[
Z_{[b_{i}n],|ns|}(0, n(t + \gamma)) \geq Z_{[b_{i}n],j}(0, n\gamma)Z_{j,|ns|}(n\gamma, n(t + \gamma)).
\]
It then follows that
\[
P \left( \log Z_0^\theta(nt) + \log Z_j^\theta(0) + \log Z_{j,n}\nu(0, nt) \geq nx \right)
\]
\[
\leq P \left( \log Z_0^\theta(nt) + \log Z_{[b_{i+1}]}^\theta(0) + \log Z_{[b_{i+1}]}(0, n(t + \gamma)) \geq n(x - 2\epsilon) \right)
\]
\[
+ P \left( \log Z_{[b_{i+1}]}(0, n) \geq -ne + P \left( \sum_{k=j}^{[b_{i+1}]} r_k^\theta(0) \leq -ne \right). \right.
\]

Using the moment bound in Lemma A.2 with \( \xi = -p \) for \( p > 1 \) and the exponential Markov inequality gives the bound
\[
P \left( \log Z_{[b_{i+1}]}(0, n) \leq -ne \right) \leq e^{-np(\epsilon - p\gamma - \delta \log \frac{x}{\epsilon}) - 4p + o(n)}.
\]

For the last inequality, we first require \( \gamma < \frac{x}{4p} \) and then take \( \delta \) small enough that \( \delta \log \frac{\delta}{\gamma} < \frac{x}{4} \). The exponential Markov inequality and the known moment generating function of the i.i.d. sum give the bound
\[
P \left( \sum_{k=j}^{[b_{i+1}]} r_k^\theta(0) \leq -ne \right) \leq e^{-np(\epsilon - \delta p^{-1} \log(\Gamma(\theta + p) \Gamma(\theta)^{-1}))} \leq e^{-n p + o(n)}
\]
where in the last step we additionally require \( \delta < \frac{p}{4} \log(\Gamma(\theta + p) \Gamma(\theta)^{-1}) \). For the case that \( j \) is a partition point, we have
\[
P \left( \log Z_0^\theta(nt) + \log Z_{[b_{i+1}]}(0) + \log Z_{[b_{i+1}]}(0, nt) \geq nx \right)
\]
\[
\leq P \left( \log Z_0^\theta(nt) + \log Z_{[b_{i+1}]}(0) + Z_{[b_{i+1}]}(0, n(t + \gamma)) \geq n(x - 2\epsilon) \right)
\]
\[
+ P \left( \sum_{k=j}^{[b_{i+1}]} r_k^\theta(0) \leq -2ne \right).
\]

and the same error bound as above applies. We now turn to the problem of estimating the integral
\[
P \left( \log \left( n \int_{a_i}^{a_{i+1}} \frac{Z_0^\theta(nu)}{Z_0^\theta(nt)} Z_{1,\nu}(nu, nt) du \right) \geq nx \right)
\]
\[
\leq P \left( \log \left( \frac{Z_0^\theta(na_i)}{Z_0^\theta(nt)} Z_{1,\nu}(na_i, nt) \right) \geq n(x - \epsilon) \right)
\]
\[
+ P \left( \log \left( n \int_{a_i}^{a_{i+1}} \frac{Z_0^\theta(nu)}{Z_0^\theta(na_i)} \frac{Z_{1,\nu}(nu, nt)}{Z_{1,\nu}(na_i, nt)} du \right) \geq ne \right).
\]

By Lemma A.5, we have
\[
P \left( \log \left( n \int_{a_i}^{a_{i+1}} \frac{Z_0^\theta(nu)}{Z_0^\theta(na_i)} \frac{Z_{1,\nu}(nu, nt)}{Z_{1,\nu}(na_i, nt)} du \right) \geq ne \right) \leq \exp \left\{ -n \left( \frac{\epsilon - \theta \nu}{2 \sqrt{\nu}} \right)^2 + o(n) \right\}
\]
where we require \( \nu < \frac{\pi}{2} \).

Take \( n \) sufficiently large that \( \log(ns + N) \leq n\epsilon \). It follows from (11) and union bounds that

\[
\frac{1}{n} \log P \left( \sum_{k=1}^{\lfloor ns \rfloor} \nu_k^\theta(nt) \geq nx \right) \leq \frac{1}{n} \log(ns + N)
\]

\[
+ \max_{0 \leq i \leq M-1} \left\{ \frac{1}{n} \log P \left( \log \left( n \int_{a_i}^{a_{i+1}} Z_0^\theta(nu)Z_0^\theta(nt)^{-1}Z_{1,\lfloor ns \rfloor}(nu,nt)du \right) \geq n(x - \epsilon) \right) \right\}
\]

\[
\vee \max_{1 \leq j \leq \lfloor ns \rfloor} \left\{ \frac{1}{n} \log P \left( \log Z_0^\theta(nt) + \log Z_j^\theta(0) + \log Z_{j,\lfloor ns \rfloor}(0,nt) \geq n(x - \epsilon) \right) \right\}.
\]

Combining this with the previous estimates, multiplying by \(-1\) and sending \( n \to \infty \) gives

\[
U_s^\theta(x) \geq \min_{0 \leq i \leq M-1} G_{a_i,s,t}^\theta(x - 2\epsilon) \wedge \left( \frac{\epsilon - \theta \nu}{2\sqrt{\nu}} \right)^2 \wedge \frac{p\epsilon}{2} \wedge \min_{0 \leq i \leq N-1} H_{b_i+1,b_i,s,t+\gamma}(x - 3\epsilon)
\]

\[
\geq \inf_{a \in [0,t]} G_{a,s,t}^\theta(x - 2\epsilon) \wedge \left( \frac{\epsilon - \theta \nu}{2\sqrt{\nu}} \right)^2 \wedge \frac{p\epsilon}{2}
\]

\[
\wedge \left\{ \inf_{a \in [0,s]} H_{a,a,s,t}(x) - \sup_{a,b \in [0,s]:|b-a|<\delta} |H_{a,b,s,t+\gamma}(x - 3\epsilon) - H_{a,b,s,t}(x)| \right\}.
\]

We first send \( \delta \downarrow 0 \), then \( \gamma \downarrow 0 \), then \( \nu \downarrow 0 \), then \( p \uparrow \infty \). By Lemma 3.11, there is \( \eta > 0 \) so that for all \( \epsilon \in [0,1] \), we have

\[
\inf_{a \in [0,t]} G_{a,s,t}^\theta(x - 2\epsilon) = \inf_{a \in [0,t-\eta]} G_{a,s,t}^\theta(x - 2\epsilon).
\]

Now, take \( \epsilon \downarrow 0 \) and use Lemmas 3.9 and 3.10. This gives the desired bound

\[
U_s^\theta(x) \geq \min \left\{ \inf_{a \in [0,t]} G_{a,s,t}^\theta(x), \inf_{a \in [0,t]} H_{a,a,s,t}(x) \right\}.
\]

We now turn the variational problem for the right tail rate functions into a variational problem involving Legendre-Fenchel transforms.

**Lemma 3.14.** For any \( \theta > 0 \) let \( \xi \in (0,\theta) \). Then \( J_{s,t}^*(\xi) \) satisfies the variational problem

\[
0 = \max \left\{ \sup_{0 \leq a < t} (t - a) \left( \frac{1}{2} \xi^2 - \theta \xi \right) - s \log \frac{\Gamma(\theta - \xi)}{\Gamma(\theta)} + J_{s,t-a}^*(\xi), \right. \]

\[
\left. \sup_{0 \leq a < s} t \left( \frac{1}{2} \xi^2 - \theta \xi \right) - (s - a) \log \frac{\Gamma(\theta - \xi)}{\Gamma(\theta)} + (J_{s-a,t})^*(\xi) \right\}.
\]
Proof. [Lemma 3.12 and Lemma 3.13 imply (12)] Infimal convolution is Legendre-Fenchel dual to addition for proper convex functions [18, Theorem 16.4] so we find

\[
(U^\theta_s)^*(\xi) = \sup_{x \in \mathbb{R}} \{ \xi x - \min_{0 \leq a < t} R^\theta_{t-a} J_{s,t-a}(x), \inf_{0 \leq a < s} R^\theta_a \square J_{s-a,t}(x) \} \\
= \sup_{x \in \mathbb{R}} \max \{ \sup_{0 \leq a < t} \xi x - R^\theta_{t-a} J_{s,t-a}(x), \sup_{0 \leq a < s} \xi x - R^\theta_a \square J_{s-a,t}(x) \} \\
= \max \{ \sup_{0 \leq a < t} (R^\theta_{t-a})^*(\xi) + J^*_{s,t-a}(\xi), \sup_{0 \leq a < s} (R^\theta_a)^*(\xi) + (U^\theta_s)^*(\xi) + (J_{s-a,t})^*(\xi) \}.
\]

If \( \xi \in (0, \theta) \), then \((U^\theta_s)^*(\xi) < \infty\), so we may subtract \((U^\theta_s)^*(\xi)\) from both sides. Substituting in the known Legendre-Fenchel transforms gives the result. \(\square\)

### 3.4 Solving the variational problem

Next, we show that the variational problem in [Lemma 3.14] identifies \(J^*_{s,t}(\xi)\) for \(\xi > 0\). To show the analogous result in [9], the authors followed the usual approach of rephrasing the variational problem as a Legendre-Fenchel transform in the space-time variables and appealing to convex analysis. The elementary approach presented in the next proposition has the advantage of allowing us to avoid some of the technicalities in that argument and is the main reason we are able to appeal to the Gärtner-Ellis theorem to prove the large deviation principle.

**Proposition 3.15.** Let \(I \subseteq \mathbb{R}\) be open and connected and let \(h, g : I \to \mathbb{R}\) be twice continuously differentiable functions with \(h'(\theta) > 0\) and \(g'(\theta) < 0\) for all \(\theta \in I\). For \((x, y) \in (0, \infty)^2\), define

\[
f_{x,y}(\theta) = x h(\theta) + y g(\theta)
\]

and suppose that \(\frac{d^2}{d\theta^2} f_{x,y}(\theta) > 0\) for all \((x, y) \in (0, \infty)^2\) and that \(f_{x,y}(\theta) \to \infty\) as \(\theta \to \partial I\) (which may be a limit as \(\theta \to \pm \infty\)). If \(\Lambda(x, y)\) is a continuous function on \((0, \infty)^2\) with the property that for all \((x, y) \in (0, \infty)^2\) and \(\theta \in I\) the identity

\[
0 = \left\{ \sup_{0 \leq a < x} \Lambda(x - a, y) - f_{x-a,y}(\theta) \right\} \lor \left\{ \sup_{0 \leq b < y} \Lambda(x, y - b) - f_{x,y-b}(\theta) \right\}
\]

(16)

holds, then

\[
\Lambda(x, y) = \min_{\theta \in I} f_{x,y}(\theta).
\]
Proof. Fix \((x, y) \in (0, \infty)^2\) and call \(\nu = \frac{y}{x}\). Under these hypotheses, there exists a unique \(\theta_{x,y}^* = \arg\min_{\theta \in I} f_{x,y}(\theta) = \theta_{1,\nu}^*\). Identity (16) implies that for all \(a \in [0, x)\) and \(b \in [0, y)\) we have

\[
\Lambda(x - a, y) \leq f_{x-a,y}(\theta_{x-a,y}^*), \quad \Lambda(x, y - b) \leq f_{x,y-b}(\theta_{x,y-b}^*),
\]

and therefore for any \(\theta \in I\), \(a \in [0, x)\) and \(b \in [0, y)\),

\[
\begin{align*}
\Lambda(x - a, y) - f_{x-a,y}(\theta) &\leq f_{x-a,y}(\theta_{x-a,y}^*) - f_{x-a,y}(\theta), \\
\Lambda(x, y - b) - f_{x,y-b}(\theta) &\leq f_{x,y-b}(\theta_{x,y-b}^*) - f_{x,y-b}(\theta).
\end{align*}
\tag{17, 18}
\]

Uniqueness of minimizers implies that \(f_{x-a,y}(\theta_{x-a,y}^*) - f_{x-a,y}(\theta) < 0\) unless \(\theta = \theta_{x-a,y}^*\) and similarly \(f_{x,y-b}(\theta_{x,y-b}^*) - f_{x,y-b}(\theta) < 0\) unless \(\theta = \theta_{x,y-b}^*\). Notice that \(\theta_{1,\nu}^*\) solves

\[
0 = h'(\theta_{1,\nu}^*) + \nu g'(\theta_{1,\nu}^*).
\tag{19}
\]

By the implicit function theorem, we may differentiate the previous expression with respect to \(\nu\) to obtain

\[
\frac{d\theta_{1,\nu}^*}{d\nu} = -\frac{g'(\theta_{1,\nu}^*)}{h''(\theta_{1,\nu}^*) + \nu g''(\theta_{1,\nu}^*)} > 0.
\tag{20}
\]

Now, set \(\theta = \theta_{x,y}^*\) in (16). Equality (20) implies that for \(a \in (0, x)\) and \(b \in (0, y)\), \(\theta_{x,y-b}^* < \theta_{(x,y)}^* < \theta_{(x-a,y)}^*\). Then (17) and (18) give us the inequalities

\[
\begin{align*}
\Lambda(x - a, y) - f_{x-a,y}(\theta_{x,y}^*) &\leq f_{x-a,y}(\theta_{x-a,y}^*) - f_{x-a,y}(\theta_{x,y}^*), \\
\Lambda(x, y - b) - f_{x,y-b}(\theta_{x,y}^*) &\leq f_{x,y-b}(\theta_{x,y-b}^*) - f_{x,y-b}(\theta_{x,y}^*).
\end{align*}
\tag{21, 22}
\]

Notice that (16) implies either there exists \(a_n \to a \in [0, x]\) or \(b_n \to b \in [0, y]\) so that one of the following hold:

\[\Lambda(x - a_n, y) - f_{x-a_n,y}(\theta_{x,y}^*) \to 0, \quad \Lambda(x, y - b_n) - f_{x,y-b_n}(\theta_{x,y}^*) \to 0.\]

Our goal is to show that the only possible limits are \(a_n \to 0\) or \(b_n \to 0\), from which the result follows from continuity. Continuity and inequalities (21) and (22) rule out the possibilities \(a \in (0, x)\) and \(b \in (0, y)\) respectively. It therefore suffices to show that

\[
\begin{align*}
\limsup_{a \to x^-} f_{x-a,y}(\theta_{x-a,y}^*) - f_{x-a,y}(\theta_{x,y}^*) &< 0, \\
\limsup_{b \to y^-} f_{x,y-b}(\theta_{x,y-b}^*) - f_{x,y-b}(\theta_{x,y}^*) &< 0.
\end{align*}
\tag{23, 24}
\]
We will only write out the proof of (23), since the proof of (24) is similar. For any fixed 
\( a \in (0, x) \), we have

\[
f_{x-a,y}(\theta^*_{x-a,y}) - f_{x-a,y}(\theta^*_x) < 0.
\]

It suffices to show that the previous expression is decreasing in \( a \). Differentiating the previous
expression and using (19) and the fact that \( \theta^*_x < \theta^*_x \), we find

\[
\frac{d}{da} \left( (x-a)h(\theta^*_{x-a,y}) + yg(\theta^*_{x-a,y}) - [(x-a)h(\theta^*_x) + yg(\theta^*_x)] \right)
= h(\theta^*_x) - h(\theta^*_x) < 0.
\]

Corollary 3.16. For all \( \xi > 0 \),

\[
J^*_{s,t}(\xi) = \min_{\theta > \xi} \left( -\frac{\xi^2}{2} + \theta \xi \right) + s \log \frac{\Gamma(\theta - \xi)}{\Gamma(\theta)}
= \min_{\mu > 0} \left( \frac{\xi^2}{2} + \xi \mu \right) - s \log \frac{\Gamma(\mu + \xi)}{\Gamma(\mu)}.
\]

Proof. It is evident from the variational representation [Lemma 3.14] that \( J^*_{s,t}(\xi) \) is not infinite
for any choice of the parameters \( \xi, s, t > 0 \). It then follows from [Lemma 3.5] and [18] Theorem
10.1 that \( J^*_{s,t}(\xi) \) is continuous in \((s, t) \in (0, \infty)^2 \).

Fix \( \xi \) and set \( I = \{ \theta : \theta > \xi \} \). Define \( h(\theta) = -\frac{\xi^2}{2} + \theta \xi \) and \( g(\theta) = \log \frac{\Gamma(\theta - \xi)}{\Gamma(\theta)} \) for \( \theta \in I \). It is classical that \( \Psi_1(x) > 0 \) and \( \Psi_2(x) < 0 \), so that \( g'(\theta) = \Psi_0(\theta - \xi) - \Psi_0(\theta) < 0 \) and \( g''(\theta) = \Psi_1(\theta - \xi) - \Psi_1(\theta) > 0 \). The rest of the conditions in the statement of [Proposition 3.15]
are clear.

The second equality is the substitution \( \mu = \theta - \xi \).

4 Moment Lyapunov exponents and the LDP

The next result would be Varadhan’s theorem if \( J_{s,t}(x) \) were a full rate function, rather than
a right tail rate function. The proof is the same as the standard proof of that result (see
the proof of [8 Theorem 4.3.1] or [9 Lemma 5.1]) so we omit it. The exponential moment
bound needed for the proof follows from [Lemma A.2].

Lemma 4.1. For \( \xi > 0 \),

\[
J^*_{s,t}(\xi) = \lim_{n \to \infty} \frac{1}{n} \log E \left[ e^{\xi \log Z_{1,\lfloor nt \rfloor}(0,nt)} \right]
\]

and in particular the limit exists.
Lemma 4.1 shows that $J_{s,t}^*(\xi)$ is the $\xi$ moment Lyapunov exponent for the parabolic Anderson model associated to this polymer. With this in mind, the second formula in the statement of Corollary 3.16 above agrees with the conjecture in [5, Appendix A1].

Our next goal is to show that the left tail large deviations are not relevant at the scale we consider. This proof is based on the proof of [9, Lemma 4.2], which contains an error: the upper bound in the first line of [9, (4.10)] and the lower bound in the second are inconsistent if, for example, $s \in \mathbb{N}$ and $t$ is irrational. We avoid this problem by repeating a similar argument for $s \in \mathbb{Q}$, but the proof does not give sufficiently strong uniform control to recover the statement in that paper when we pass to irrational $s$.

Proposition 4.2. Fix $s,t > 0$. For all $\epsilon > 0$

$$\liminf_{n \to \infty} -\frac{1}{n} \log P \left( \log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s, t) - \epsilon) \right) = \infty.$$  

Proof. First we consider the case $s \in \mathbb{Q}$. There exists $M \in \mathbb{N}$ large enough that $M(s \wedge t) \geq 1$ and for all $m \geq M$ we have

$$\frac{1}{m} E \log Z_{1,\lfloor ms \rfloor}(0, mt) \geq \rho(s, t) - \epsilon.$$  

Fix $m \geq M$ so that $ms \in \mathbb{N}$. We will denote coordinates in $\mathbb{R}^{\lfloor ns \rfloor - 1}$ by $(u_1, \ldots, u_{\lfloor ns \rfloor - 1})$. For $a, b, s, t \in (0, \infty)$ and $n, k, l \in \mathbb{N}$, define a family of sets $A_{k,a}^{l,b} \subset \mathbb{R}^{\lfloor ns \rfloor - 1}$ by

$$A_{k,a}^{l,b} = \{0 < u_1 < \cdots < u_{k-1} < a < u_k < \cdots < u_{k+l-1} < a + b < u_{k+l} < \cdots < u_{\lfloor ns \rfloor - 1} < nt\}.$$  

For $j, k, \in \mathbb{Z}^+$, set

$$A_k^j \equiv A_{j,\lfloor ms \rfloor}^{\lfloor ms \rfloor, mt}.$$  

For each $n$ sufficiently large that the expression below is greater than one, define

$$N = \left\lfloor \frac{n}{m} - \left\lfloor \sqrt{n} \right\rfloor - 2 \right\rfloor,$$  

so that we have

$$(n - 2m)t \leq (N + \left\lfloor \sqrt{n} \right\rfloor + 1)mt \leq (n - m)t,$$  

$$\left\lfloor \sqrt{n} \right\rfloor + 1 \leq \left\lfloor ns - N[ms] \right\rfloor \leq \left(\left\lfloor \sqrt{n} \right\rfloor + 2\right) ms.$$  

With this choice of $N$, for $0 \leq k \leq \left\lfloor \sqrt{n} \right\rfloor$ and $0 \leq j \leq N - 1$, $A_k^j$ is nonempty. Then for $0 \leq k \leq \left\lfloor \sqrt{n} \right\rfloor$, define

$$D_k = \bigcap_{j=0}^{N-1} A_k^j \cap \left\{u : 0 < u_1 < \cdots < u_{(N+1)ms-1} < t \left(\frac{n-m}{2}\right) < u_{(N+1)ms} < \cdots < u_{\lfloor ns \rfloor - 1} < nt\right\}.$$  

20
To simplify the formulas that follow, we introduce the notation $s_j = jms$ and $t_j^k = (j+k)mt$. In words, we can think of $D_k$ as the collection of paths from 0 to $nt$ which traverse the bottom line until $t_0^k$, then for $0 \leq j \leq N-1$ move from $t_j^k$ to $t_{j+1}^k$ along the next $ms$ lines. The path then moves from $t_N^k$ to $t \left(n - \frac{m}{2}\right)$ along the next $ms$ lines and finally proceeds to $nt$ along the remaining lines. Observe that $\{D_k\}_{k=0}^{\lfloor \sqrt{n]\rfloor}$ is a pairwise disjoint, non-empty family of sets. With the convention $u_0 = 0$ and $u_{[ns]} = nt$, we have the bound

$$Z_{1,[ns]}(0, nt) \geq \sum_{k=0}^{\lfloor \sqrt{n}\rfloor} \int_{D_k} e^{\sum_{i=1}^{[ns]} B_i(u_{i-1}, u_i)} du_1 \ldots u_{[ns]} - 1.$$ 

In the integral over $D_k$, for each $0 \leq j \leq N$ we add and subtract $B_{s_j}(t_j^k)$ in the exponent. Similarly, add and subtract $B_{s_{N+1}}(t (n - \frac{m}{2}))$. The reason for this step is that this will make the product of integrals coming from the definition of $D_k$ into a product of partition functions, as when we showed submultiplicativity of the partition function in [4]. Introduce

$$H_k^n = \inf_{t_k^n = u_0 < u_1 < \cdots < u_{ms_1} < u_{ms} = n (t - \frac{m}{2})} \sum_{i=0}^{ms-1} B_{s_{N+i}}(u_{i-1}, u_i)$$

and observe that $H_0^n \leq B_{s_N}(t_N^0; t_N^k) + H_k^n$. Let $C > 0$ be a uniform lower bound in $n$ (recall that $m$ is fixed) on the Lebesgue measure of the Weyl chamber in the definition of $H_{\lfloor \sqrt{n}\rfloor}^n$. Such a bound exists by [25]. Set $I_n = \max_{t_N^0 \leq u \leq t_{N-1}} B_{s_N}(t_{N-1}, u)$. We have the lower bound

$$Z_{1,[ns]}(0, nt) \geq C Z_{s_{N+1},[ns]} \left(t \left(n - \frac{m}{2}\right), nt\right) e^{H_0^n - I_n} \left(\sum_{k=0}^{\lfloor \sqrt{n}\rfloor} e^{B_0(t_0^k)} \prod_{j=0}^{N-1} Z_{s_j, s_{j+1}}(t_j^k, t_{j+1}^k)\right).$$

We therefore have the upper bound

$$P \left(\log Z_{1,[ns]}(0, nt) \leq -n(\rho(s, t) - 6\epsilon)\right) \leq P \left(\log Z_{(N+1)ms,[ns]} \left(t \left(n - \frac{m}{2}\right), nt\right) \leq -n\epsilon - \log C\right) + P \left(\sum_{0 \leq k \leq \lfloor \sqrt{n}\rfloor} \log Z_{s_j+1, s_{j+1}}(t_j^k, t_{j+1}^k) \leq -n(\rho(s, t) - 2\epsilon)\right) + P \left(H_0 \leq -n\epsilon\right) + P \left(\min_k B_0(t_0^k) \leq -n\epsilon\right) + P \left(I_n \geq n\epsilon\right).$$

It follows from translation invariance, Lemma A.6 and (26) that

$$P \left(\log Z_{(N+1)ms,[ns]} \left(t \left(n - \frac{m}{2}\right), nt\right) \leq -n\epsilon - \log \frac{mt}{12}\right) = O \left(e^{-n^3}\right).$$
We have
\[
\begin{align*}
P \left( \max_{0 \leq k \leq \lfloor \sqrt{n} \rfloor} \sum_{j=0}^{N-1} \log Z_{s_j+1,s_{j+1}}(t_j^k, t_{j+1}^k) \leq -n(\rho(s,t) - 2\epsilon) \right) \\
= P \left( \sum_{j=0}^{N-1} \log Z_{s_j+1,s_{j+1}}(t_j^1, t_{j+1}^1) \leq -n(\rho(s,t) - 2\epsilon) \right) \leq O(e^{-cn^\frac{3}{2}})
\end{align*}
\]
for some $c_1 > 0$. The first equality comes from the fact that the terms in the maximum are i.i.d. and the second comes from standard large deviation estimates for an i.i.d. sum once we recall that $N = \frac{n}{m} + o(n)$.

Recall that by (25) $n(t - \frac{m}{2}) - t_0^N = O(\sqrt{n})$. It follows from Lemma A.4 that there exist $c_2, C_2 > 0$ so that
\[
P(H_0 \leq -n\epsilon) \leq C_2 e^{-c_2 n^\frac{3}{2}}.
\]
The remaining two terms can be controlled with the reflection principle and are $O\left( e^{-\frac{1}{4} n^{\frac{3}{2}}} \right)$.

Now let $s$ be irrational. For each $k$, fix $\tilde{s}_k < s$ rational with $e^{-k} < |\tilde{s}_k - s| < 2e^{-k}$ and set $\tilde{t}_k = t - \frac{1}{k}$. Call $\alpha_k = s - \tilde{s}_k$ and $\beta_k = t - \tilde{t}_k = \frac{1}{k}$. Subadditivity gives
\[
P\left( \log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s,t) - \epsilon) \right) \leq P\left( \log Z_{1,\lfloor n\tilde{s}_k \rfloor}(0, n\tilde{t}_k) \leq n\left( \rho(\tilde{s}_k, \tilde{t}_k) - \frac{\epsilon}{2} \right) \right)
+ P\left( \log Z_{\lfloor ns \rfloor,\lfloor ns \rfloor}(n\tilde{t}_k, nt) \leq n\left( \rho(s,t) - \rho(\tilde{s}_k, \tilde{t}_k) - \frac{\epsilon}{2} \right) \right).
\]

Since $\tilde{s}_k$ is rational, we have already shown that the first term is negligible. Take $k$ sufficiently large that $\rho(s,t) - \rho(\tilde{s}_k, \tilde{t}_k) - \frac{\epsilon}{2} < -\frac{\epsilon}{4}$. By Lemma A.3 we find
\[
\liminf_{n \to \infty} \frac{1}{n} P\left( \log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s,t) - \epsilon) \right) \geq \alpha_k J_{GUE} \left( \frac{\epsilon - \alpha_k \log \beta_k - \alpha_k + \alpha_k \log \alpha_k}{2\sqrt{\alpha_k \beta_k}} \right).
\]
Using formula (31), $J_{GUE}(r) = 4 \int_0^r \sqrt{x(x+2)} dx$, it is not hard to see that as $k \to \infty$, this lower bound tends to infinity.

\begin{lemma}
Fix $s, t > 0$ and $\xi < 0$. Then
\[
\lim_{n \to \infty} \frac{1}{n} \log E\left[ e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt)} \right] = \xi \rho(s,t).
\]
\end{lemma}

\begin{proof}
Fix $\epsilon > 0$ and small and recall that Lemma A.2 and Jensen’s inequality imply that for any $\xi < 0$, $\sup_n \frac{1}{n} \log E\left[ e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt)} \right] < \infty$. The lower bound is immediate from
\[
\frac{1}{n} \log E\left[ e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt)} \right] \geq \frac{1}{n} \log E\left[ e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt)} 1_{\{ \log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s,t)+\epsilon) \}} \right]
\geq \xi (\rho(s,t) + \epsilon) + \frac{1}{n} \log P(\log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s,t) + \epsilon))
\]

\end{proof}
once we recall that \( P(\log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s, t) + \epsilon)) \to 1 \).

For the upper bound, we decompose the expectation as follows

\[
E [e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt)}] = E \left[ e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt) 1_{\{\log Z_{1,\lfloor ns \rfloor}(0, nt) > n(\rho(s, t) - \epsilon)\}}} \right] \\
+ E \left[ e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt) 1_{\{\log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s, t) - \epsilon)\}}} \right].
\]

Recalling that \( P(\log Z_{1,\lfloor ns \rfloor}(0, nt) > n(\rho(s, t) - \epsilon)) \to 1 \), this leads to

\[
\limsup_{n \to \infty} \frac{1}{n} \log E [e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt)}] \leq \max \left\{ \xi (\rho(s, t) - \epsilon), \limsup_{n \to \infty} \frac{1}{n} \log E \left[ e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt) 1_{\{\log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s, t) - \epsilon)\}}} \right] \right\}.
\]

By Cauchy-Schwarz and Proposition 4.2

\[
\limsup_{n \to \infty} \frac{1}{n} \log E [e^{\xi \log Z_{1,\lfloor ns \rfloor}(0, nt) 1_{\{\log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s, t) - \epsilon)\}}}] \leq \frac{1}{2} \sup_{n} \frac{1}{n} \log E \left[ e^{2\xi \log Z_{1,\lfloor ns \rfloor}(0, nt)} \right] + \limsup_{n \to \infty} \frac{1}{2n} \log P(\log Z_{1,\lfloor ns \rfloor}(0, nt) \leq n(\rho(s, t) - \epsilon)) = -\infty. \quad \Box
\]

Combining the previous results, we are led to the proof of Theorem 2.2, from which we immediately deduce Theorem 2.3.

**Proof of Theorem 2.2.** Lemmas 4.1 and 4.3 give the limit for \( \xi \neq 0 \) and the limit for \( \xi = 0 \) is clear.

Differentiability is clear for \( \xi < 0 \) and the left derivative at zero is \( \rho(s, t) \). For \( \xi > 0 \), there is a unique \( \mu(\xi) \) solving

\[
\Lambda_{s,t}(\xi) = t \left( \frac{\xi^2}{2} + \xi \mu(\xi) \right) - s \log \frac{\Gamma(\mu(\xi) + \xi)}{\Gamma(\mu(\xi))}. \quad (27)
\]

This \( \mu(\xi) \) is given by the unique solution to

\[
0 = t\xi + s \left( \Psi_0(\mu(\xi)) - \Psi_0(\mu(\xi) + \xi) \right), \quad (28)
\]

which can be rewritten as

\[
\frac{1}{\xi} \left( \Psi_0(\mu(\xi) + \xi) - \Psi_0(\mu(\xi)) \right) = \frac{t}{s}.
\]

By the mean value theorem, there exists \( x \in [0, \xi) \) so that

\[
\Psi_1^{-1} \left( \frac{t}{s} \right) - x = \mu(\xi).
\]
Using this, it is not hard to see that $\Lambda_{s,t}(\xi)$ is continuous at $\xi = 0$. The implicit function theorem implies that $\mu(\xi)$ is smooth for $\xi > 0$. Differentiating (27) with respect to $\xi$ and applying (28) we have

$$d d\xi \Lambda_{s,t}(\xi) = t(\xi + \mu(\xi)) - s\Psi_0(\mu(\xi) + \xi).$$

Substituting in for $\mu(\xi)$, appealing to continuity, and taking $\xi \downarrow 0$ gives

$$\lim_{\xi \downarrow 0} d d\xi \Lambda_{s,t}(\xi) = t\Psi - s\Psi_0(\Psi^{-1}(t) - 1) - s\Psi_0(\Psi^{-1}(t) - 1).$$

which implies differentiability at zero and hence at all $\xi$.

Proof of Theorem 2.3. The large deviation principle holds by Theorem 2.2 and the Gärtner-Ellis theorem [8, Theorem 2.3.6].
A.2 Moment estimate for the partition function

Lemma A.2. Fix $t > 0$, $n \in \mathbb{N}$ and $\xi \in \mathbb{R}$ with $|\xi| > 1$. Then there is a constant $C > 0$ depending only on $\xi$ so that

$$E[Z_{0,n}(0,t)^\xi] \leq C \left( \frac{\sqrt{n}}{t} \left( \frac{te^n}{n} \right)^n \right)^{\xi} e^{\frac{1}{2} \xi^2 t}.$$ 

Proof. By Jensen’s inequality with respect to the uniform measure on $A_{n,t} = \{ s \in \mathbb{R}^n : 0 < s_1 < \cdots < s_n < t \}$ and Tonelli’s theorem we find

$$E[Z_{0,n}(0,t)^\xi] = E \left[ \left( \int_{0<s_1<\cdots<s_n<t} e^{\sum_i B(s_i,s_{i+1})} ds_1 \ldots ds_n \right)^\xi \right]$$

$$\leq |A_{n,t}|^{-1} |A_n|^{\xi} \int_{0<s_1<\cdots<s_n<t} E[e^{\xi \sum_i B(s_i,s_{i+1})}]ds_1 \ldots ds_n$$

$$= |A_{n,t}|^{\xi} e^{\frac{1}{2} \xi^2 t},$$

where we have used independence of the Brownian increments and the moment generating function of the normal distribution to compute the last line. The remainder of the statement of the lemma comes from the identity $|A_{n,t}| = t^{n-1} (n-1)!$ and Stirling’s approximation to $n!$. \(\Box\)

A.3 Bounds from the GUE connection

Let $\lambda_{GUE,n}$ be the top eigenvalue of an $n \times n$ GUE random matrix with entries that have variance $\sigma^2 = \frac{1}{4n}$. Then [1, Theorem 0.7] and [10] give

$$\lambda_{GUE,n} = \frac{d}{2\sqrt{n}} \max_{0=u_0<u_1<\cdots<u_{n-1}<u_n=1} \sum_{i=1}^n B_i(u_{i-1},u_i).$$

The right tail rate function of $\lambda_{GUE,n}$ can be computed ([12], (1.25)), [3]) for $r > 0$ to be

$$J_{GUE}(r) = \lim_{n \to \infty} -\frac{1}{n} \log P(\lambda_{GUE,n} \geq 1 + r) = 4 \int_0^r \sqrt{x(x+2)}dx$$

(31)

Lemma A.3. Suppose that $r, s, t > 0$ and $(s_n, t_n, r_n) \in \mathbb{N} \times (0, \infty) \times \mathbb{R}$ satisfy $n^{-1}(s_n, t_n, r_n) \to (s, t, r)$. If $r - s \log t - s + s \log s > 2\sqrt{ts}$, then

$$\liminf_{n \to \infty} -\frac{1}{n} \log P(\log Z_{0,s_n}(0,t_n) \geq r_n) \geq sJ_{GUE} \left( \frac{r - s \log t - s + s \log s}{2\sqrt{ts}} - 1 \right)$$

and if $r + s \log t + s - s \log s > 2\sqrt{ts}$, then

$$\liminf_{n \to \infty} -\frac{1}{n} \log P(\log Z_{0,s_n}(0,t_n) \leq -r_n) \geq sJ_{GUE} \left( \frac{r + s \log t + s - s \log s}{2\sqrt{ts}} - 1 \right).$$
Proof. Define $A_{n,t} = \{ u \in \mathbb{R}^n : 0 < u_1 < \cdots < u_n < t \}$ and observe that

$$|A_{n,t}| = \frac{t^{n-1}}{(n-1)!} \leq \frac{\sqrt{n}}{t \sqrt{2\pi}} \left( \frac{te}{n} \right)^n.$$ 

Using this fact and bounding $Z_{0,n}(0,t)$ as defined in (2) above with the maximum value of the Brownian increments, we obtain

$$\log Z_{0,sn}(0,t_n) \leq \log \left( \frac{\sqrt{sn}}{t_n \sqrt{2\pi}} \left( \frac{tn^e}{sn} \right)^{sn} \right) + \max_{0=u_0<u_1<\cdots<u_{sn}=t_n} \sum_{i=0}^{s_n-1} B_i(u_i, u_{i+1})$$

$$d \leq \log \left( \frac{\sqrt{sn}}{t_n \sqrt{2\pi}} \left( \frac{tn^e}{sn} \right)^{sn} \right) + 2\sqrt{tnsn} \lambda_{\text{GUE,sn}}$$

The result then follows from the inequality

$$P(\log Z_{0,sn}(0,t_n) \geq r_n) \leq P \left( \lambda_{\text{GUE,sn}} \geq \frac{r_n - sn \log tn - sn + sn \log sn}{2\sqrt{tnsn}} \right) \leq Ce^{-cn^{2-\alpha}}$$

The proof of the second bound follows a similar argument: we bound the partition function below with the minimum of the Brownian increments, apply the upper bound from Stirling’s approximation to $n!$, and appeal to Brownian reflection symmetry.

Lemma A.4. Fix $\epsilon > 0$ and let $s \in \mathbb{N}$ and $t_n = O(n^\alpha)$ for some $\alpha < 1$. Then there exist $c, C > 0$ so that

$$P \left( \max_{0=u_0<u_1<\cdots<u_{sn}=t_n} \sum_{i=0}^{s_n-1} B_i(u_i, u_{i+1}) \geq n\epsilon \right) \leq Ce^{-cn^{2-\alpha}}$$

Proof. Standard large deviation estimates for largest eigenvalues [12, (2.7)] give the result: there exist $c, C > 0$ such that

$$P \left( \max_{0=u_0<u_1<\cdots<u_{sn}=t_n} \sum_{i=0}^{s_n-1} B_i(u_i, u_{i+1}) \geq n\epsilon \right) \leq P \left( \lambda_{\text{GUE,sn}} \geq \frac{n \sqrt{\epsilon}}{\sqrt{s} } \right) \leq Ce^{-cn^{2-\alpha}}.$$ 

A.4 Upper tail coarse graining estimate

Lemma A.5. Fix $\alpha \in [0, t)$ and $\epsilon > 0$. Then for $\nu < \min(\epsilon, t - \alpha)$

$$P \left( \log n \int_{a}^{a+\nu} \frac{Z_{0}^\theta(nu)}{Z_{0}^\theta(na)} \cdot \frac{Z_{1,[ns]}(nu, nt)}{Z_{1,[ns]}(na, nt)} du \geq n\epsilon \right) \leq \exp \left\{ -n \frac{1}{4} \left( \frac{\epsilon - \theta \nu}{\sqrt{\nu}} \right)^2 + o(n) \right\}.$$
Proof. By (6), we have for all $u \in (a, a + \nu)$

$$Z_{1,1}(na, nu)^{-1}Z_{1,|n\nu|}(nu, nt)^{-1} \geq Z_{1,|n\nu|}(na, nt)^{-1}$$

so it follows that

$$P \left( \log n \int_a^{a+\nu} \frac{Z_0^\theta(nu)Z_{1,|n\nu|}(nu, nt)}{Z_0^\theta(na)Z_{1,|n\nu|}(na, nt)} \, du \geq ne \right) \leq P \left( \log n \int_a^{a+\nu} \frac{Z_0^\theta(nu)Z_{1,1}(na, nu)}{Z_0^\theta(na)} \, du \geq ne \right)$$

$$= P \left( \log n \int_a^{a+\nu} e^{\theta_n(u-a)-B(na,nu)-B_1(na,nu)} \, du \geq ne \right)$$

$$\leq P \left( \max_{0 \leq u \leq 1} B(u) + B_1(u) \geq \sqrt{n} \left( \frac{\epsilon - \theta \nu}{\sqrt{\nu}} - \frac{\log(n\nu)}{\sqrt{n\nu}} \right) \right),$$

where the last inequality comes from Brownian translation invariance, symmetry, and scaling. Recall that $B + B_1$ has the same process level distribution as $\sqrt{2B}$. The result follows from the reflection principle. \qed

A.5 Left tail error bound

Lemma A.6. Take sequences $t_n, s_n, r_n$ such that there exist $a, b > 0$ with $a < t_n < b$, $r_n \to r > 0$ and $s_n \in \mathbb{N}$ satisfies $s_n \log(s_n) = o(n)$. Then there exist constants $c, C > 0$ such that

$$P \left( \log Z_{0,s_n}(0, t_n) \leq -nr_n \right) \leq Ce^{-cn^2}.$$

Proof. We have $Z_{0,s_n}(0, t_n) \geq \prod_{i=0}^{s_n-1} Z_{i,i+1} \left( \frac{i}{s_n}, \frac{i+1}{s_n} \right)$ where the $Z_{i,i+1} \left( \frac{i}{s_n}, \frac{i+1}{s_n} \right)$ are i.i.d.. As above in (30), there exist i.i.d. random variables $X_i \sim N \left( \log \left( \frac{i}{s_n} \right), \frac{2}{3s_n} \right)$ with $Z_{i,i+1} \left( \frac{i}{s_n}, \frac{i+1}{s_n} \right) \geq X_i$. It follows that

$$P \left( \log Z_{0,s_n}(0, t_n) \leq -nr_n \right) \leq P \left( \sum_{i=0}^{s_n-1} X_i \leq -nr_n \right) = P \left( N(0, 1) \geq n \frac{r_n}{\sqrt{3t_n}} + \frac{s_n}{n \sqrt{3t_n}} \log \left( \frac{t_n}{s_n} \right) \right).$$

Recall that $\frac{r_n}{\sqrt{3t_n}} + \frac{s_n}{n \sqrt{3t_n}} \log \left( \frac{t_n}{s_n} \right)$ is a bounded sequence and without loss of generality is bounded away from zero. The result follows from normal tail estimates. \qed

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