Perturbative partition function for a squashed $S^5$

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We compute the index of 6d $\mathcal{N} = (1, 0)$ theories on $S^5 \times \mathbb{R}$ containing vector and hypermultiplets. We only consider the perturbative sector without instantons. By compactifying $\mathbb{R}$ to $S^1$ with a twisted boundary condition and taking the small radius limit, we derive the perturbative partition function on a squashed $S^5$. The 1-loop partition function is represented in a simple form with the triple sine function.

Subject Index B10, B16

1. Introduction

The sphere partition functions play important roles in the recent progress in supersymmetric gauge theories and string/M-theory. The $S^4$ partition function [1] provides data on the 4d side of the Alday–Gaiotto–Tachikawa (AGT) relation [2], which connects 4d $\mathcal{N} = 2$ supersymmetric gauge theories and 2d conformal field theories. The $S^3$ partition function [3–5] is used to investigate the dynamics of 3d supersymmetric gauge theories. In particular, it reproduces the $N^{3/2}$ behavior of the free energy of 3d gauge theories realized on M2-branes [6–8]. The $S^5$ partition function [9–11] has also attracted great interest recently. The motivation comes from interests in 5d conformal field theories [12] and the conjecture [13,14] concerning the relation between 5d supersymmetric Yang–Mills theories and mysterious 6d $(2, 0)$ supersymmetric theories. See also Refs. [15–20] for recent studies of 5d supersymmetric Yang–Mills theories on various backgrounds.

These partition functions depend on continuous parameters. This fact is important because we can extract physical information from the dependence on the continuous parameters. Mass parameters, such as real mass parameters in 3d $\mathcal{N} = 2$ gauge theories and complex ones in 4d $\mathcal{N} = 2$ gauge theories can be easily turned on as expectation values of background vector multiplets. We also have a similar mass parameter in the 5d $\mathcal{N} = 2$ case [11]. The introduction of deformation parameters of spheres is more involved. Two kinds of deformed spheres have been studied in the literature: ellipsoids and squashed spheres. The partition function for 3d ellipsoids is computed in [21], and the same form of the partition function is reproduced for squashed $S^3$ in [22]. The partition function for 4d ellipsoids is worked out in [23] and its consistency with the AGT relation is found. The purpose of this paper is to determine the partition function for squashed $S^5$.

A convenient way to construct a squashed $S^n$ is dimensional reduction from $S^n \times \mathbb{R}$. If we compactify $\mathbb{R}$ with a twisted boundary condition we obtain a squashed $S^n$. Supersymmetric theories on

\footnote{The term “squashed spheres” often refers to both kinds of deformation in the literature. In this paper, for distinction, we use this term only for the specific deformation of the sphere which is presented in Sect. 2.1.}
squashed $S^3$ are constructed in [22] by using this dimensional reduction, and their one-loop partition function can be obtained by taking a certain limit of the 4d index [24–26]. In this paper we apply the same method to 5d $\mathcal{N} = 1$ supersymmetric gauge theories and derive the partition function in the perturbative sector for squashed $S^5$ with general squashing parameters $\phi_i (i = 1, 2, 3)$. The final result is

$$Z_{\text{pert}} = C(\omega) \left( \frac{\text{rank } G}{\prod_{i=1}^{r} \int_{-\infty}^{\infty} d\sigma_i} \right) \exp \left[ -\frac{(2\pi)^3}{\omega_1 \omega_2 \omega_3} \mathcal{F}(\sigma) \right] \prod_{\alpha \in \text{Root}} S_3(-i\alpha(\sigma), \omega) \prod_{\rho \in D} S_3(-i\rho(\sigma) + \frac{\omega_1 + \omega_2 + \omega_3}{2}, \omega), (1)$$

where $\mathcal{F}$ is the prepotential of the theory and $S_3(z, \omega)$ is the triple sine function with periods $\omega = (\omega_1, \omega_2, \omega_3)$. The squashing parameters appear in this partition function through the periods $\omega_i = 1 + i\phi_i$. We assume that the hypermultiplets belong to the representation $R + \bar{R}$ of the gauge group. $\alpha$ runs over all (positive and negative) roots of the gauge group $G$, and $\rho$ runs over the weights in the representation $R$. We can introduce mass parameters for hypermultiplets straightforwardly by shifting the weights $\rho$. The prepotential $\mathcal{F}$ is normalized so that the Chern–Simons term is given by

$$S = \frac{i}{6} \frac{\partial^3 \mathcal{F}(\sigma)}{\partial \sigma^a \partial \sigma^b \partial \sigma^c} \int A^a \wedge F^b \wedge F^c + \ldots. (2)$$

The overall factor $C(\omega)$ depends only on $\omega$, and its explicit form is given in Eq. (85).

This paper is organized as follows. In the next section, we summarize the background manifold that we consider and the supersymmetry on it. In Sect. 3 we compute the index of vectors and hypermultiplets in 6d $\mathcal{N} = (1, 0)$ theories by localization. In Sect. 4, we take the small radius limit of the index and obtain the $S^5$ partition function. The last section is devoted to discussion.

**Note added**

While this work was being completed, a paper by G. Lockhart and C. Vafa appeared [38] which also studies the partition function for a deformed $S^5$.

### 2. Background manifold and supersymmetry

#### 2.1. Squashed sphere

A squashed $S^{2r-1}$ is easily obtained by dimensional reduction from the $2r$-dimensional manifold $S^{2r-1} \times \mathbb{R}$ with the metric

$$ds^2 = dt^2 + \sum_{i=1}^{r} |dz_i|^2. (3)$$

In this paper we only consider the $r = 3$ case. The complex coordinates $z_i (i = 1, \ldots, r)$ are constrained by

$$\sum_{i=1}^{r} |z_i|^2 = R^2. (4)$$

In the following we set $R = 1$. If we compactified the “time” direction by the identification $(t, z_i) \sim (t + \beta, z_i)$ and took the small radius limit $\beta \to 0$, we would obtain the round $S^{2r-1}$. Instead, we
consider the twisted identification

\[(t, z_i) \sim (t + \beta, e^{i\phi_i / \beta} z_i). \tag{5}\]

The parameters \(\phi_i\) are called squashing parameters. If we introduce real coordinates \(\rho_i\) and \(\theta_i\) by

\[\rho_i e^{i\theta_i} = e^{-i\phi_i / \beta} z_i, \tag{6}\]

the identification becomes \((t, \rho_i, \theta_i) \sim (t + \beta, \rho_i, \theta_i)\). In the new coordinate system the metric becomes

\[ds_6^2 = v^2(dt + W)^2 + ds_5^2, \tag{7}\]

where \(v\) and \(W\) are the scalar function and the differential defined by

\[v^2 = 1 + \sum_{i=1}^{r} \phi_i^2 \rho_i^2, \quad W = \frac{1}{v^2} \sum_{i=1}^{r} \phi_i \rho_i^2 d\theta_i. \tag{8}\]

\(ds_5^2\) in (7) is the metric of the squashed sphere. Its explicit form is

\[ds_5^2 = \sum_{i=1}^{r} (d\rho_i^2 + \rho_i^2 d\theta_i^2) - v^2 W^2. \tag{9}\]

### 2.2. \(\mathcal{N} = (1, 0)\) supersymmetry on \(S^5 \times \mathbb{R}\)

The 6d \(\mathcal{N} = (1, 0)\) superconformal symmetry on an arbitrary conformally flat background is parameterized by symplectic Majorana–Weyl spinors \(\epsilon\) and \(\kappa\) satisfying

\[D_M \epsilon = \Gamma_M \kappa. \tag{10}\]

The spinors \(\epsilon\) and \(\kappa\) have positive and negative chirality, respectively. We use \(M, N, \ldots\) for 6d vector indices and \(\mu, \nu, \ldots\) for 5d ones. 6d and 5d Dirac matrices are denoted by \(\Gamma_M\) and \(\gamma_\mu\), respectively. Our choices for the representation of Dirac matrices are given in Appendix A. The R-symmetry is \(SU(2)_R\), and both \(\epsilon\) and \(\kappa\) are \(SU(2)_R\) doublets. We use \(I, J, \ldots = 1, 2\) for \(SU(2)_R\) indices when we show them explicitly.

Let us consider supersymmetry on \(S^5 \times \mathbb{R}\). We use vector indices \(\mu, \nu, \ldots = 1, \ldots, 5\) for \(S^5\) and 6 for \(\mathbb{R}\). The \(\mathcal{N} = (1, 0)\) supersymmetry algebra contains 16 supercharges. Because \(S^5 \times \mathbb{R}\) is conformally flat, we can realize all 16 supersymmetries on \(S^5 \times \mathbb{R}\) as well as on \(\mathbb{R}^6\), and the Killing equation (10) has 16 linearly independent solutions. The general solution is

\[\epsilon_I = \begin{pmatrix} e^{-\frac{i}{2} \epsilon_i(4)} \eta^I_I + e^{\frac{i}{2} \epsilon^I(4)} \bar{\eta}^I_I \\ 0 \end{pmatrix}, \tag{11}\]

where \(\eta^I_I\) and \(\bar{\eta}^I_I\) \((i = 1, 2, 3, 4)\) are Grassmann-odd constant parameters, and \(\epsilon_i(4)\) and \(\epsilon^I(4)\) are bases of Killing spinors on \(S^5\) satisfying

\[D_\mu \epsilon_i(4) = -\frac{i}{2r} \gamma_\mu \epsilon_i(4), \quad D_\mu \epsilon^I(4) = +\frac{i}{2r} \gamma_\mu \epsilon^I(4). \tag{12}\]

Note that \(\epsilon_I\) are eight-component 6d spinors while \(\epsilon_i(4)\) and \(\epsilon^I(4)\) are defined as four-component 5d spinors. Let \(SO(6)_{iso}\) be the isometry of \(S^5\). \(\epsilon_i(4)\) and \(\epsilon^I(4)\) belong to 4 and \(\bar{4}\) of \(SO(6)_{iso}\).
respectively. The 16 parameters $\eta^i_j$ and $\bar{\eta}^i_{ii}$ correspond to the 16 supercharges in the $\mathcal{N} = (1, 0)$ superconformal algebra. The spinors $\epsilon$ and $\kappa$ are related by

$$\kappa = -\frac{1}{2r} \Gamma_{\text{iso}} \Gamma^0 \epsilon,$$

(13)

where $\Gamma_{\text{iso}}$ is the $SO(6)_{\text{iso}}$ chirality operator that acts on $\epsilon_i(4)$ and $\epsilon^i(\bar{4})$ as +1 and −1, respectively.

We consider a gauge theory whose action contains the Yang–Mills term. Because $1/g^2_{\text{YM}}$ has mass dimension one in 5d and the Yang–Mills term is a kind of mass deformation, the superconformal symmetry in 5d is broken to the rigid supersymmetry. Correspondingly, we impose the condition

$$\Gamma_{\text{iso}} \epsilon = \tau_3 \epsilon,$$

(14)

on the parameter $\epsilon$. This condition admits eight supersymmetries with parameters

$$\eta^i_1 \equiv \eta^i, \quad \eta^i_2 = 0, \quad \bar{\eta}^i_{ii} = 0, \quad \bar{\eta}^i_{2i} = \bar{\eta}^i.$$

(15)

The general Killing spinor for rigid supersymmetry has components

$$\epsilon_1 = \begin{pmatrix} e^{-\frac{i}{2} \epsilon_1(4)} \eta^i \\ 0 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} e^{\frac{i}{2} \epsilon^i(\bar{4}) \bar{\eta}^i} \\ 0 \end{pmatrix}.$$

(16)

For the computation of the index with the help of the localization technique, we have to choose one supersymmetry $Q$. Before giving our choice of $Q$ we first choose a complex structure (Kähler form) $I_{\mathbb{C}^3}$ on $\mathbb{C}^3$ spanned by $z_i$ in (4),

$$I_{\mathbb{C}^3} = -\frac{i}{2} dz^*_i \wedge dz_i.$$

(17)

Let $SU(3)_V \times U(1)_V$ be the subgroup of $SO(6)_{\text{iso}}$ which keeps (17) invariant. $U(1)_V$ is generated by $I_{\mathbb{C}^3}$, and we parameterize $U(1)_V$-orbits by a coordinate $\psi$ so that

$$I_{\mathbb{C}^3} = -\mathcal{L}_\psi,$$

(18)

where $\mathcal{L}_\psi$ is the Lie derivative along the vector $\partial_\psi$. With this $U(1)_V$ symmetry we naturally represent $S^5$ as the Hopf fibration over $\mathbb{C}P^2$. $I_{\mathbb{C}^3}$ naturally induces the complex structure (Kähler form) of $\mathbb{C}P^2$, and its pullback to $S^5 \times \mathbb{R}$ is denoted by $I$.

The metric on $S^5 \times \mathbb{R}$ is

$$ds^2 = e^M e^M = e^m e^m + e^5 e^5 + e^6 e^6, \quad (M = 1, \ldots, 6, \quad m = 1, 2, 3, 4),$$

(19)

where $e^5$ and $e^6$ are

$$e^5 = d\psi + V, \quad e^6 = dt.$$

(20)

$V$ is a differential on $\mathbb{C}P^2$ satisfying $dV = -2i$, and $t \equiv x^6$ is the coordinate along $\mathbb{R}$. We choose the vielbein on $\mathbb{C}P^2$, $e^m$ ($m = 1, 2, 3, 4$), so that

$$I = e^2 \wedge e^1 + e^4 \wedge e^3.$$

(21)

The choice of the complex structure $I_{\mathbb{C}^3}$ breaks $SO(6)_{\text{iso}}$ to $SU(3)_V \times U(1)_V$, and the representations $4$ and $\bar{4}$ branch into $SU(3)_V \times U(1)_V$ irreducible representations as

$$\begin{align*}
4 & = 3^+_{+2} + 1^-_{-2}, \quad \bar{4} = 3^-_{-2} + 1^+_{+2}.
\end{align*}$$

(22)

Correspondingly, Killing spinors split as

$$\epsilon_i(4) \rightarrow \left( \epsilon_i(3^+_{+2}), \epsilon_i(1^-_{-2}) \right), \quad \epsilon^i(\bar{4}) \rightarrow \left( \epsilon^i(3^-_{-2}), \epsilon^i(1^+_{+2}) \right).$$

(23)
We introduce the following shorthand notation for the $SU(3)_V$ singlet Killing spinors:

\[ \xi_1 = \varepsilon(1_{\frac{1}{2}}), \quad \xi_2 = \varepsilon(1_{\frac{3}{2}}). \]  

(24)

With our choice of the Dirac matrices given in Appendix A, $\xi_1$ and $\xi_2$ have the components

\[ \xi_1 = (0, 1, 0, 0)^T, \quad \xi_2 = (1, 0, 0, 0)^T. \]  

(25)

and they are related to $I$ by

\[ I_{\mu\nu} = -i\xi_1^\dagger \gamma_{\mu\nu} \xi_1 = i\xi_2^\dagger \gamma_{\mu\nu} \xi_2. \]  

(26)

Now we are ready to give our choice of $Q$. We define $Q$ by

\[ \eta Q = \delta(\eta \varepsilon), \]  

(27)

where the right-hand side is the SUSY transformation with the parameter $\varepsilon = \eta \varepsilon$. $\eta$ is a Grassmann-odd constant and $\varepsilon$ is the 6d Killing spinor with components

\[ \varepsilon_1 = \left( e^{-\frac{i}{2} \xi_1}, 0 \right), \quad \varepsilon_2 = \left( -e^{\frac{i}{2} \xi_2}, 0 \right). \]  

(28)

3. Index of $(1, 0)$ theory

3.1. Definition of the index

In this section, we derive the 5d partition function as a limit of the index of the corresponding 6d $\mathcal{N} = (1, 0)$ theory. To obtain a 5d theory with a gauge group $G$ and the matter representation $R$, we start from the 6d theory with the same $G$ and $R$. Although the 5d theory is well defined, its 6d counterpart may be anomalous. Even so, we can still use the “index” as a concise way to express the spectrum of fluctuations of fields on $S^5 \times S^1$ around saddle points. We are only interested in the modes that survive after the small radius limit $\beta \to 0$, and the “index” is useful to obtain such modes, although it cannot be regarded as a physical quantity when the theory is anomalous.

The bosonic symmetry of the 6d $\mathcal{N} = (1, 0)$ theory on $S^5 \times \mathbb{R}$ is $\mathbb{R} \times SO(6)_{iso} \times SU(2)_R$, and the choice of $Q$ breaks it to $\mathbb{R} \times U(1)_V \times SU(3)_V \times SU(2)_R$. Correspondingly, there are five Cartan generators:

\[ H = -\partial_t, \quad Q_V = -i\mathcal{L}_\psi, \quad \tau_3, \quad \lambda_3, \quad \lambda_8. \]  

(29)

$\tau_3$ is the $SU(2)_R$ Cartan generator acting on doublets as the Pauli matrix $\tau_3 = \text{diag}(1, -1)$. $\lambda_3$ and $\lambda_8$ are $SU(3)_V$ Cartan generators whose fundamental representation matrices are

\[ \lambda_3|3 = \begin{pmatrix} 1 \\
-1 \n\0 \end{pmatrix}, \quad \lambda_8|3 = \begin{pmatrix} 1 \\
1 \\
0 \n\end{pmatrix}. \]  

(30)

The quantum numbers of $\varepsilon_I$ in (28) are shown in Table 1. The general transformation generated by (29) that commutes with the supersymmetry $Q$ is

\[ O = q^H - Q_V - 2\tau_3 x Q_V + \frac{1}{2} \tau_3 \lambda_3 \lambda_8 \lambda_8. \]  

(31)

The index is defined with this operator by $[27]$:

\[ \mathcal{I}(x, y_3, y_8) = \text{tr} \left[ (-1)^F O \right]. \]  

(32)

where $F$ is the fermion number operator and the trace is taken over the Fock space of gauge-invariant states on $S^5$. This can be computed by the path integral of the theory defined on $S^5 \times S^1$. The path
Table 1. Quantum numbers of $\varepsilon_I$. 

|   | $H$ | $Q_V$ | $\tau_3$ | $\lambda_3$ | $\lambda_8$ |
|---|-----|-------|----------|-------------|-------------|
| $\varepsilon_1$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $+1$ | $0$ | $0$ |
| $\varepsilon_2$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $0$ | $0$ |

Integral in the perturbative sector, which does not contain instantons on $\mathbb{C}P^2$, reduces to the finite-dimensional integral

$$I = \oint [d\sigma] Z_0 \text{PE}_{sp}.$$  

$Z_0$ is the zero-point contribution and $I_{sp}$ is the single-particle index defined by (32) with the trace replaced by the summation over single-particle excitations, including charged particles. PE is the plethystic exponential defined by

$$\text{PE} f(x_i) = \exp \sum_{m=1}^{\infty} \frac{1}{m} f(x_i^m).$$  

In general, the single-particle index takes the form

$$I_{sp} = \sum_i n_i q^{a_i} x^{b_i} y^{c_i} y^{d_i},$$

and then the corresponding zero-point contribution $Z_0$ and the plethystic exponential $\text{PE}_{sp}$ are

$$Z_0 = \prod_i q^{n_i a_i} x^{n_i b_i} y^{n_i c_i} y^{n_i d_i},$$

$$\text{PE}_{sp} = \prod_i \left(1 - q^{a_i} x^{b_i} y^{c_i} y^{d_i}\right)^{-n_i}.$$  

We define the “modified plethystic exponential” as the product of these two factors, i.e.,

$$\text{PE}' I_{sp} = Z_0 \text{PE}_{sp} = \prod_i \left(q^{a_i} x^{b_i} y^{c_i} y^{d_i} - q^{-a_i} x^{-b_i} y^{-c_i} y^{-d_i}\right)^{-n_i}.$$  

The integration variable $\sigma$ is the $t$-component of the gauge field (see Eq. (47)) and the integration over $\sigma$ picks up gauge-invariant states. The integration measure is

$$\oint [d\sigma] = C_0 \prod_{\alpha \in \text{root}} \left(\int_0^{2\pi/\beta} d\sigma_i \right) J(\sigma), \quad J(\sigma) = \prod_{\alpha \in \text{root}} 2i \sin \frac{\beta \alpha(\sigma)}{2}.$$  

Each diagonal component of $\sigma$ takes a value in $S^1$ with period $2\pi/\beta$. The factor $C_0$ is given by

$$C_0 = \frac{1}{|W|} \left(\frac{\beta}{2\pi}\right)^{\text{rank} G},$$  

where $|W|$ is the order of the Weyl group of $G$. This factor is needed to guarantee that each gauge-invariant state contributes to the index with the correct weight.
3.2. **Vector multiplet**

The SUSY transformation laws of $\mathcal{N} = (1, 0)$ vector multiplets are

\[
\begin{align*}
\delta A_M &= -(\epsilon \Gamma_M \lambda), \\
\delta \lambda &= -\frac{1}{2} \Gamma^{MN} F_{MN} \epsilon + i D_a \tau_a \epsilon, \\
\delta D_a &= i(\epsilon \tau_a \Gamma^M D_M \lambda) - 2i(\kappa \tau_a \lambda),
\end{align*}
\]

(40)

where $\tau_a (a = 1, 2, 3)$ are $SU(2)_R$ generators. To localize the path integral, we need to deform the action by $Q$-exact terms. A standard choice of the $Q$-exact term is

\[L \propto Q[(Q\lambda)\dagger \lambda],\]

(41)

where “$\dagger$” is the Hermitian conjugate. Although the bosonic part of this Lagrangian is positive semi-definite, this action is not suitable for computation of the index because this has an unwanted explicit $t$-dependence. For example, the norm $\epsilon^\dagger \epsilon$ of the Killing spinor $\epsilon$ defined by (28) appears in (41), and depends on the coordinate $t$:

\[\epsilon^\dagger \epsilon = 2 \cosh t.\]

(42)

To avoid such $t$-dependence, we introduce the new conjugate operator “$\star$” that acts on $\epsilon$ as

\[\epsilon^\star = \epsilon^\dagger |_{t \to -t},\]

(43)

and on all other bosonic fields as “$\dagger$”. The norm of $\epsilon$ defined with $\star$ is constant:

\[\epsilon^\star \epsilon = 2.\]

(44)

We use the following $Q$-exact terms defined with $\star$, which has no explicit $t$-dependence:

\[
\frac{s}{2} Q[(Q\lambda)^\star \lambda] = s \left[ \frac{1}{2} F_{MN} F^{MN} - \frac{1}{4} \epsilon^{mnpq56} F_{mn} F_{pq} - D_a D_a \\
- \lambda \Gamma^M D_M \lambda - \frac{i}{4} I_{MN}(\lambda \Gamma^{MN} \Gamma_6 \lambda) - \frac{i}{2} (\lambda \Gamma_5 \tau_3 \lambda) + \frac{1}{2} (\lambda \Gamma_6 \tau_5 \lambda) \right].
\]

(45)

The path integral does not depend on the deformation parameter $s \in \mathbb{R}_+$, and in the weak coupling limit $s \to \infty$ it reduces to the Gaussian integral. The path integral of the auxiliary fields $D_a$ gives a constant and we neglect it.

The saddle point equations for the vector field are

\[F_{12} + F_{34} = F_{23} + F_{14} = F_{31} + F_{24} = F_{5m} = F_{6m} = F_{56} = 0.\]

(46)

In the perturbative sector with vanishing instanton number on $\mathbb{C}P^2$, (46) implies $F_{MN} = 0$, and up to gauge transformations we can set the gauge potential to be

\[A = \sigma \, dt,\]

(47)

where $\sigma$ is a constant diagonal matrix. The path integral of $A_M$ and $\lambda$ around these saddle points can be explicitly performed in the weak coupling limit.

Let us first consider the bosonic sector. The path integral of $A_M$ and $\lambda$ around these saddle points can be explicitly performed in the weak coupling limit.

To perform the path integral, we expand fluctuations of $A_M$ by harmonics on $\mathbb{C}P^2$. See Refs. [11,28] for detailed explanations for the harmonic expansion. The harmonics are classified by
SU(3)\textsubscript{V} representations labeled by two quantum numbers, \(k\) and \(m\). They are related to the SU(3) Dynkin labels \(k_1, k_2 \in \mathbb{Z}_{\geq 0}\) by

\[
k = k_1 + k_2, \quad m = k_2 - k_1.
\] (49)

For example, \((k, m) = (1, -1)\) and \((k, m) = (1, 1)\) correspond to the fundamental and anti-fundamental representations, respectively. Because both \(k_1\) and \(k_2\) are non-negative integers, \(m\) for specific \(k\) runs from \(-k\) to \(k\) with step 2, i.e.,

\[
m = -k, -k + 2, \ldots, k - 2, k.
\] (50)

Let \(\mu\) be an SU(3)\textsubscript{V} weight vector and \(\alpha\) be a weight vector in the adjoint representation of the gauge group. For specific \(\mu\) and \(\alpha\), there are generically six bosonic modes. Let \(a_i\) (\(i = 1, \ldots, 6\)) be the corresponding coefficients in the harmonic expansion of \(A_M\). By substituting the mode expansion in the action (48) we obtain the quadratic form \(a_i M_{ij} a_j\) with the matrix \(M_{ij}\) depending on the quantum numbers \(k, m\), and the Wilson line \(\sigma\). The determinant of \(M_{ij}\) is

\[
[k(k + 4)]^2 (2^2 - D_6^2) [(k + 4)^2 - D_6^2](k(k + 4) - 2m + 9 - D_6^2)(k(k + 4) + 2m + 9 - D_6^2).
\] (51)

up to an unimportant constant. The covariant derivative \(D_6\) in (51) contains the background gauge field (47):

\[
D_6 = \partial_6 - i[\sigma, \ast] = \partial_6 - i\alpha(\sigma).
\] (52)

The factor \([k(k + 4)]^2\) is canceled by the Faddeev–Popov determinant associated with the gauge-fixing term \((D_\mu A_\mu)^2\), and the other factors represent the contribution of physical modes. We can read off one-particle excitations on \(S^5\) as zeros of (51). We show the spectrum obtained in this way in Table 2.

If \(m = \pm k\), some modes in the table are absent. Furthermore, special treatment is needed when \(k = 0\). In this case, in addition to modes shown in Table 2, we have a bosonic zero mode of \(A_6\) corresponding to a residual gauge symmetry. The gauge-fixing term \(\propto (D_\mu A_\mu)\) in (48) does not fix gauge transformations with a parameter depending only on \(x^6\). We impose a further gauge-fixing condition which makes \(A_6\) a diagonal constant matrix as in (47). The integration measure \(J(\sigma)\) in (38) is the Faddeev–Popov determinant associated with this extra gauge fixing.

Next, let us consider the fermionic sector. The \(\lambda\)-bilinear part in the \(Q\)-exact action (45) is

\[
\mathcal{L}_\lambda = s \left[ -\lambda \left( \Gamma^M D_M - \frac{1}{4} I_{MN} \Gamma^{MN} + \frac{i}{2} \tau_3 \Gamma^5 - \frac{i}{2} \frac{\partial}{\partial \tau_3} \right) \lambda \right].
\] (53)

By expanding \(\lambda\) by spinor harmonics on \(\mathbb{C}P^2\), we obtain the spectrum shown in Table 3.

Let us compare the bosonic spectrum with the fermionic one. The bosonic modes [A1], [A2], [A3], and [A4] have the same quantum numbers \(H - Q_V - 2\tau_3\) and \(Q_V + \frac{3}{4} \tau_3\) as the fermionic modes \([\lambda 1],[\lambda 2],[\lambda 3]\), and \([\lambda 4]\), respectively, and almost all of them cancel to each other. Only the fermionic modes \([\lambda 1]\) and \([\lambda 3]\) with \(m = k\) are left due to the difference in the range of \(m\). (Note that \(k - m\) must be an even integer and there are no \(m = k - 1\) modes.) These surviving modes contribute to
Table 2. Single particle states of the vector field $A_M$. $\sqrt{\pm}$ represents $\sqrt{k(k+4) \pm 2m+9}$ and $\alpha$ stands for $\alpha(\sigma)$.

| ID   | $H$            | $Q_V$ | $\tau_3$ | Range of $m$                        | $H - Q_V - 2\tau_3$ | $Q + \frac{1}{2}\tau_3$ |
|------|----------------|-------|-----------|------------------------------------|----------------------|--------------------------|
| [A1] | $-i\alpha + k$ | $m$   | 0         | $-k + 2 \leq m \leq k - 2$        | $-i\alpha + k - m$  | $m$                      |
| [A2] | $-i\alpha + k + 4$ | $m$   | 0         | $-k \leq m \leq k$                | $-i\alpha + k - m + 4$ | $m$                      |
| [A3] | $-i\alpha + \sqrt{\pm}$ | $m + 3$ | 0         | $-k \leq m \leq k - 2$            | $-i\alpha + \sqrt{\pm} - m - 3$ | $m + 3$                 |
| [A4] | $-i\alpha + \sqrt{-}$ | $m - 3$ | 0         | $-k + 2 \leq m \leq k$            | $-i\alpha + \sqrt{-} - m + 3$ | $m - 3$                 |

Table 3. One-particle spectrum of $\lambda$. $\sqrt{\pm}$ and $\alpha$ stand for $\sqrt{k(k+4) \pm 2m+9}$ and $\alpha(\sigma)$, respectively.

| ID   | $H$            | $Q_V$ | $\tau_3$ | Range of $m$                        | $H - Q_V - 2\tau_3$ | $Q + \frac{1}{2}\tau_3$ |
|------|----------------|-------|-----------|------------------------------------|----------------------|--------------------------|
| $[\lambda 1]$ | $-i\alpha + k + \frac{1}{2}$ | $m - \frac{3}{2}$ | $+1$ | $-k + 2 \leq m \leq k$            | $-i\alpha + k - m$  | $m$                      |
| $[\lambda 2]$ | $-i\alpha + k + \frac{7}{2}$ | $m + \frac{3}{2}$ | $-1$ | $-k \leq m \leq k$                | $-i\alpha + k - m + 4$ | $m$                      |
| $[\lambda 3]$ | $-i\alpha + \sqrt{\pm} + \frac{1}{2}$ | $m + \frac{3}{2}$ | $+1$ | $-k \leq m \leq k$                | $-i\alpha + \sqrt{\pm} - m - 3$ | $m + 3$                 |
| $[\lambda 4]$ | $-i\alpha + \sqrt{-} - \frac{1}{2}$ | $m - \frac{3}{2}$ | $-1$ | $-k + 2 \leq m \leq k$            | $-i\alpha + \sqrt{-} - m + 3$ | $m - 3$                 |

the single-particle index by

\[
- \sum_{x \in \text{adj}} \left[ \sum_{k=1}^{\infty} q^{-i\alpha(\sigma)} x^{k} \chi(k,k)(y_3, y_8) + \sum_{k=0}^{\infty} q^{-i\alpha(\sigma)} x^{k+3} \chi(k,k)(y_3, y_8) \right],
\]  

(54)

where $\chi(k,m)(y_3, y_8)$ is the $SU(3)_V$ character

\[
\chi(k,m) = \sum_{\mu \in (k,m)} y_3^{\mu(k_3)} y_8^{\mu(k_8)}.
\]  

(55)

Note that $\alpha$ in (54) runs over all weights in the adjoint representation of the gauge group. Let us separate the Cartan part

\[
\mathcal{T}_{sp}^{\text{Cartan}} = -(\text{rank } G) \left[ \sum_{k=1}^{\infty} x^{k} \chi(k,k)(y_3, y_8) + \sum_{k=0}^{\infty} x^{k+3} \chi(k,k)(y_3, y_8) \right],
\]  

(56)

and combine the remaining part with $- \sum_{x \in \text{root}} q^{-i\alpha}$, which reproduces the measure factor

\[
J(\sigma) = PE' \left( - \sum_{x \in \text{root}} q^{-i\alpha(\sigma)} \right).
\]  

(57)

Then we have the single particle index

\[
\mathcal{T}_{sp}^{\text{vector}}(q, x, y_3, y_8) = - \sum_{x \in \text{root}} \sum_{k=0}^{\infty} q^{-i\alpha(\sigma)} x^{k} \chi(k,k)(y_3, y_8) + q^{-i\alpha(\sigma)} x^{k+3} \chi(k,k)(y_3, y_8).
\]  

(58)

With this definition of $\mathcal{T}_{sp}^{\text{vector}}$, we do not have to include $J(\sigma)$ separately in (38).

The $SU(3)_V$ representation $(k, k)$ contains weight vectors corresponding to lattice points in a triangle, as shown in Fig. 1. The multiplicity for every weight is one. We label these weights by two
Fig. 1. The \((k, k)\) representation on the weight space. This example shows \((6, 6)\).

Integers \(n_1\) and \(n_2\) satisfying

\[
n_1, n_2 \geq 0, \quad n_1 + n_2 \leq k.
\]

The eigenvalues of Cartan generators \(\lambda_3\) and \(\lambda_8\) are

\[
\mu(\lambda_3) = n_2 - n_1, \quad \mu(\lambda_8) = 2k - n_1 - n_2.
\]

and the character of this representation is

\[
\chi(k, k)(y_3, y_8) = \sum_{\mu \in (k, k)} y_3^\mu \lambda_3^\mu y_8^\mu = \sum_{n_1=0}^{k} \sum_{n_2=0}^{k-n_1} y_3^{n_2-n_1} y_8^{2k-n_1-n_2}.
\]

If we introduce \(n_3 = k - n_1 - n_2\), we can rewrite the product over representations \((k, k)\) \((k = 0, 1, 2, \ldots)\) and weights \(\mu \in (k, k)\) as the product over three independent non-negative integers \(n_1\), \(n_2\), and \(n_3\),

\[
\prod_{k=0}^{\infty} \prod_{\mu \in (k, k)} (\cdots) = \prod_{n_1=0}^{\infty} \prod_{n_2=0}^{\infty} \prod_{n_3=0}^{\infty} (\cdots).
\]

By using this, we can rewrite the infinite product appearing in the modified plethystic exponential as

\[
\text{PE'} \sum_{k=0}^{\infty} r x^k \chi(k, k)(y_3, y_8) = \prod_{k=0}^{\infty} \prod_{\mu \in (k, k)} \frac{1 - r x^k y_3^\mu \lambda_3^\mu y_8^\mu}{1 - r x^k y_3^\mu \lambda_3^\mu y_8^\mu} = \frac{1}{F(r; p)},
\]

where we defined the function

\[
F(r; p) = \prod_{n_1, n_2, n_3=0}^{\infty} \frac{1 - r p_1^{n_1} p_2^{n_2} p_3^{n_3}}{\sqrt{r p_1^{n_1} p_2^{n_2} p_3^{n_3}}},
\]

with \(p = (p_1, p_2, p_3)\) given by

\[
p_1 = \frac{x}{y_3 y_8}, \quad p_2 = \frac{x y_3}{y_8}, \quad p_3 = x y_8^2.
\]

With this function we can express the modified plethystic exponential of \(T_{\text{sp}}^\text{vector}\) as

\[
\text{PE'} T_{\text{sp}}^\text{vector} = \prod_{\alpha \in \text{root}} F(q^{-i\alpha(\sigma)}; p) F(q^{-i\alpha(\sigma)} x^3; p).
\]
3.3. Hypermultiplets

A 6d hypermultiplet contains four real scalar fields $q_i$ ($i = 1, 2, 3, 4$) and a symplectic Majorana–Weyl spinor field $\psi$ as on-shell degrees of freedom. It is difficult to give the off-shell supersymmetry transformation of hypermultiplets for full $\mathcal{N} = (1, 0)$ supersymmetry. Instead, we give the off-shell supersymmetry only for the $Q$ used in localization. This is easily done by following the prescription in [15] used for 5d hypermultiplets. In addition to the Killing spinor $\epsilon$, we introduce another spinor $\tilde{\epsilon}$ with negative chirality satisfying

$$\epsilon J \tilde{\epsilon} J = 0, \quad (\epsilon \Gamma^M \epsilon) + (\tilde{\epsilon} \Gamma^M \tilde{\epsilon}) = 0. \quad (67)$$

The transformation laws with the Grassmann-odd parameters $(\epsilon, \tilde{\epsilon}) = (\eta \epsilon, \eta \tilde{\epsilon})$ are

$$\delta q_i = i(\epsilon \rho_i \psi),$$
$$\delta \psi = i \tilde{\rho}_i \Gamma^M D_M q_i + \tilde{\rho}_i \bar{\epsilon} f_i + 4i \tilde{\rho}_i \kappa q_i,$$
$$\delta f_i = (\tilde{\epsilon} \rho_i \Gamma^M D_M \psi) + (\tilde{\epsilon} \bar{\rho}_j \tilde{\epsilon}) q_j, \quad (68)$$

where $f_i$ ($i = 1, 2, 3, 4$) are real auxiliary fields. $\rho_i$ and $\tilde{\rho}_i$ ($i = 1, 2, 3, 4$) defined by

$$\rho_i = (\sigma_1, \sigma_2, \sigma_3, -i I_2), \quad \tilde{\rho}_i = (\sigma_1, \sigma_2, \sigma_3, i I_2) \quad (69)$$

are $SU(2)_F \times SU(2)_R$ invariant tensors, where $SU(2)_F$ is the flavor symmetry acting on the scalar fields in the hypermultiplet. The conditions in (67) guarantee the closure of the algebra and the supersymmetry invariance of the action

$$\mathcal{L}_{\text{hyper}} = D_M q_i D_M q_i + \psi \Gamma^M D_M \psi + (\tau_a)_{ij} q_i [D_a, q_j] + 2 \psi \tilde{\rho}_i [\lambda, q_i] + \frac{1}{5} R q_i q_i + f_i f_i, \quad (70)$$

where $(\tau_a)_{ij} \equiv -(1/2) \text{tr}(\tau_a \rho_i \tilde{\rho}_j)$ is the 't Hooft symbol and $R$ is the scalar curvature of the background manifold.

On $S^5 \times \mathbb{R}$, we adopt the following $\tilde{\epsilon}$ and its * conjugate

$$\tilde{\epsilon}_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ ie^{-\frac{\pi}{2}} \end{pmatrix}, \quad \tilde{\epsilon}_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -ie^{\frac{\pi}{2}} \end{pmatrix}, \quad \tilde{\epsilon}^* = \tilde{\epsilon}^* \mid_{t \to -t}. \quad (71)$$

The action (70) is in fact not only $Q$-invariant but also $Q$-exact on $S^5 \times \mathbb{R}$, and can be written as

$$\mathcal{L}_{\text{hyper}} = \frac{1}{2} Q[(Q \psi)^* \psi]. \quad (72)$$

We use this action for the computation of the spectrum.

We assume that the hypermultiplet belongs to $R + \overline{R}$ of the gauge group. We denote a weight of $R$ by $\rho$, and then the weight of its anti-particle is $-\rho \in \overline{R}$. If we specify a weight vector $\rho \in R$ and an $SU(3)_V$ weight vector $\mu \in (k, m)$, we obtain the spectrum of $q_i$ excitations shown in Table 4, and $\psi$ excitations in Table 5.
By collecting all the contributions, we obtain the total index

\[ P_{\text{TEP}} \quad 2013 \]

and the modified plethystic exponential of this is

\[ \text{taking the small radius limit.} \]

we do not know the 6d lift of a general 5d theory. We will introduce the classical contribution after

points. Such a contribution exists in 5d theory, and if we could lift the action of the 5d theory to 6d,

\[ \sum_{\rho \in R} \sum_{k=0}^{\infty} \left( q^{-i\rho(\sigma)} x^{k+\frac{3}{2}} + q^{+i\rho(\sigma)} x^{k+\frac{3}{2}} \right) \chi_{(k,k)}(y_3, y_8), \]

and the modified plethystic exponential of this is

\[ \text{PE}^{\text{hyper}} = \frac{1}{\prod_{\rho \in R} F(q^{-i\rho(\sigma)} x^{\frac{1}{2}}; p) F(q^{+i\rho(\sigma)} x^{\frac{1}{2}}; p)}. \]

4. Partition function from 6d index

4.1. One-loop partition function

By collecting all the contributions, we obtain the total index

\[ \mathcal{Z}_{1\text{-loop}}^{(6d)} = \frac{\prod_{\alpha \in \text{root}} F(q^{-i\alpha(\sigma)}; p) F(q^{+i\alpha(\sigma)} x^{3}; p)}{\prod_{\rho \in R} F(q^{-i\rho(\sigma)} x^{\frac{1}{2}}; p) F(q^{+i\rho(\sigma)} x^{\frac{1}{2}}; p)}. \]

To be exact, we should include the factor e^{-S_0} that comes from the classical action \( S_0 \) at the saddle points. Such a contribution exists in 5d theory, and if we could lift the action of the 5d theory to 6d, the classical 6d action should also contribute to the index. However, we do not include it here because we do not know the 6d lift of a general 5d theory. We will introduce the classical contribution after taking the small radius limit.

To take the small radius limit, let us relate the fugacities \( q, x, y_3, \) and \( y_8 \) to the geometry of \( S^5 \times S^1. \) If we set

\[ x = q^{1+iw_0}, \quad y_3 = q^{iw_3}, \quad y_8 = q^{iw_8}, \]

and \( q = e^{-\beta}, \) the twist operator (31) becomes

\[ \mathcal{O} = q^{H+iw_0} Q_{\text{V}3+iw_3 \lambda_3+iw_8 \lambda_8} q^{-\frac{1}{2}+\frac{3}{2}w_0} \tau_i. \]
The two factors on the right-hand side correspond to geometric and internal symmetries. The geometric factor represents the shift generated by $H + iw_0 Q_V + iw_3 \lambda_3 + iw_8 \lambda_8$, and the trace in the index (32) corresponds to the identification (5) with the squashing parameters
\[ \phi_1 = w_0 - w_3 - w_8, \quad \phi_2 = w_0 + w_3 - w_8, \quad \phi_3 = w_0 + 2w_8. \] (78)

We can read off the relations
\[ p_i = q^{\rho_0}, \quad \omega_i = 1 + i \phi_i. \] (79)

from (65), (76), and (78). The partition function on the squashed sphere is given by taking the small radius limit $\beta \to 0$;
\[ Z_{\text{pert}} = \lim_{q \to 1} T(x = q^{1+iw_0}, y_3 = q^{iw_3}, y_8 = q^{iw_8}). \] (80)

In this limit the function $F$ reduces to
\[ F(q^{c}; p) \rightarrow \prod_{n_1,n_2,n_3=0}^{\infty} (c + n_1\omega_1 + n_2\omega_2 + n_3\omega_3) = \frac{1}{\Gamma_3(c, \omega)}, \] (81)

and we obtain the one-loop determinant of the 5d theory
\[ Z_{(1d)}^{(1d)} \rightarrow Z_{1-\text{loop}} = \prod_{r \in \text{root}} S_3(-i\sigma(\sigma), \omega) \prod_{r \in \bar{R}} S_3(-i\rho(\sigma) + \frac{\omega_{\text{tot}}}{2}, \omega). \] (82)

$\Gamma_3(z, \omega)$ and $S_3(z, \omega)$ are the triple gamma function and the triple sine function, respectively. In general, the multiple gamma function $\Gamma_r(z, \omega)$ [29] and the multiple sine function $S_r(z, \omega)$ [30–33] depend on the $r$-component period $\omega = \{\omega_1, \ldots, \omega_r\}$, and are defined by
\[ \Gamma_r(z, \omega) = \prod_{n \geq 0} (n \cdot \omega + z)^{-1}, \]
\[ S_r(z, \omega) = \Gamma_r(z, \omega)^{-1} \Gamma_r(\omega_{\text{tot}} - z, \omega)^{(1)} \]
\[ = \left( \prod_{n \geq 0} (n \cdot \omega + z) \right) \left( \prod_{n \geq 1} (n \cdot \omega - z) \right)^{(-1)^r-1}, \] (83)

\[ \omega_{\text{tot}} = \sum_{i=1}^{r} \omega_i. \]

The integration variables $\sigma_i$ originally take value in $S^1$ with period $2\pi/\beta$. In the small radius limit the $S^1$ is replaced by $\mathbb{R}$, and in the 5d theory $\sigma_i$ are integrated over the real axis:
\[ \prod_{i=1}^{\text{rank G}} \int_{0}^{2\pi/\beta} d\sigma_i \rightarrow \prod_{i=1}^{\text{rank G}} \int_{-\infty}^{\infty} d\sigma_i. \] (84)

We also have the constant factor independent of $\sigma$:
\[ C(\omega) = \lim_{\beta \to 0} C_{\text{PE}}^{\text{Cartan}} \frac{\text{det}_{\text{sp}}^{\text{Cartan}}}{\text{det}^{\text{Cartan}}} = \frac{1}{|W|} \left( \frac{S'_3(0; \omega)}{2\pi} \right)^{\text{rank G}}, \] (85)

where $S'_3(z; \omega) = \partial S_3(z; \omega)/\partial z$.

---

\(^2\)This factor does not affect the $N^3$ behavior of the free energy of the 5d $N = 2$ supersymmetric Yang–Mills theory studied in [37]. However, it depends on the squashing parameters $\phi_i$ and would have physical significance at finite $N$. This factor is also given in [39].
We have obtained the perturbative partition function (1) apart from the exponential factor coming from the classical action. For $SU(3) \times U(1)$ symmetric squashed $S^5$, the classical action has already been computed in [34] and shown to be proportional to the prepotential. In the following we will first confirm that for such homogeneous squashed spheres the classical action indeed gives the exponential factor in (1). For a generic squashed sphere, unfortunately, we do not know the precise form of the classical action, and we cannot prove (1). In this case we will “define” the prepotential so that it is proportional to the classical action, and determine the proportionality constant by looking at the asymptotic behavior of the one-loop factor.

4.2. $\mathcal{N} = 1/4$ case

In this and the next subsections, we consider homogeneous squashed $S^5$. Generically, a squashed $S^5$ has isometry $U(1)^3 \subset SU(4)_{\text{iso}}$, and is not homogeneous. We obtain a homogeneous squashed sphere when the operator $O$ in (31) preserves $SU(3) \times U(1)$ isometry, which may differ from $SU(3)_V \times U(1)_V$ preserved by $I_{C3}$. There are two essentially different cases. One is

$$\phi_1 = \phi_2 = \phi_3 = -u. \quad (86)$$

In this case, the compactification preserves $SU(3)_V \times U(1)_V$, and the supercharge $Q$ is $SU(3)_V$-singlet. This corresponds to the $\mathcal{N} = 1/4$ theory constructed in [34]. The periods are

$$\omega = (1 - iu)1, \quad 1 = (1, 1, 1). \quad (87)$$

The classical action at a saddle point $\sigma$ is [34] $^3$

$$S_{\mathcal{N}=\frac{1}{4}} = \frac{(2\pi)^3}{(1 - iu)^3} \mathcal{F}(\sigma), \quad (88)$$

and this gives the exponential factor in (1).

When we brought (88) from [34], we changed the convention for the prepotential. The prepotential is a cubic function of $\sigma$, and is expanded schematically as

$$\mathcal{F}(\sigma) = g_3 \sigma^2 + g_2 \sigma^2 + g_1 \sigma + g_0. \quad (89)$$

The coefficients $g_i$ ($i = 0, 1, 2, 3$) are coupling constants of the theory, and if the background is flat, they have a standard normalization. In a curved manifold, however, coefficients in the Lagrangian density may depend on the coordinates, and there is no standard choice of the normalization of $g_i$, except $g_3$, which is proportional to the Chern–Simons level and quantized. Due to this ambiguity in the definition of $g_i$ ($i = 0, 1, 2$), the prepotential depends on the convention. In particular, by the rescaling of the coefficients $g_i \rightarrow g'_i = c^{3-i} g_i$, the prepotential changes as

$$\mathcal{F}(\sigma) \rightarrow \mathcal{F}'(\sigma) = c^3 \mathcal{F}\left(\frac{\sigma}{c}\right). \quad (90)$$

If we denote the prepotential in [34] by $\mathcal{F}'$, it is related to $\mathcal{F}$ here in this way with $c = (1 + u^2)^{-1/2}$.

As is argued in [34], the perturbative partition function of $\mathcal{N} = 1/4$ theory is independent of the squashing parameter $u$. This is shown as follows. By introducing new integration variable $\sigma' = \sigma/(1 - iu)$ and using the scaling property of the multiple sine function $S_r(cz, c\omega) = S_r(z, \omega)$,
we can eliminate the $u$-dependence from the one-loop factor and prefactor $C(\omega)$ combined with the integration measure:

$$Z_{1\text{-loop}} = \prod_{\alpha \in \text{adj}} S_3(-i\alpha(\sigma'), 1) \prod_{\rho \in \text{R}} S_3(-i\rho(\sigma') + \frac{3}{2}, 1),$$

$$C(\omega) \int \frac{d \sigma}{\text{rank } G} \prod_{i=1}^{\text{rank } G} d\sigma_i = C(1) \int \frac{d \sigma_i'}{\text{rank } G} \prod_{i=1}^{\text{rank } G} d\sigma_i'.$$

(91)

We can also remove $u$ from the classical action by changing the convention of the prepotential by (90) with $c = 1 - iu$:

$$S_{\mathcal{N} = \frac{1}{4}} = (2\pi)^3 F(\sigma').$$

(92)

4.3. $\mathcal{N} = 3/4$ case

The other choice of the squashing parameter preserving an $SU(3) \times U(1)$ isometry is

$$\phi_1 = \phi_2 = -\phi_3 = u.$$

(93)

Let $SU(3)_{\text{iso}} \times U(1)_{\text{iso}}$ be the preserved isometry. In this case, $SU(3)_{\text{iso}}$ is different from $SU(3)_V$, and the supercharge $Q$ belongs to an $SU(3)_{\text{iso}}$ triplet. This gives the $\mathcal{N} = 3/4$ theory in [34]. The periods are

$$\omega_1 = 1 + iu, \quad \omega_2 = 1 + iu, \quad \omega_3 = 1 - iu.$$

(94)

The classical action at a saddle point $\sigma$ is [34]

$$S_{\mathcal{N} = \frac{3}{4}} = \frac{(2\pi)^3}{(1 + iu)^2(1 - iu)} F(\sigma).$$

(95)

(We took account of the difference in the convention for the prepotential as in Sect. 4.2.) This gives the exponential factor in (1). In this case, the partition function depends on the squashing parameter $u$ in a non-trivial way.

As is pointed out in [36], the asymptotic behavior of the one-loop partition function on the round $S^5$ can be interpreted as the quantum correction to Chern–Simons levels. This is also the case in $\mathcal{N} = 1/4$ theories because the classical action and the one-loop partition function are independent of the squashing parameter $u$. In the $\mathcal{N} = 3/4$ case, they non-trivially depend on $u$. Let us confirm that the asymptotic behavior of $Z_{1\text{-loop}}$ is still consistent with the quantization of Chern–Simons levels. For simplicity, let us consider a $U(1)$ gauge theory with the Chern–Simons term

$$\mathcal{L} = \frac{k}{6(2\pi)^2} \int A \wedge F \wedge F.$$

(96)

The prepotential contains the corresponding term

$$\mathcal{F}(a) = \frac{k}{6(2\pi)^2} a^3,$$

(97)

and this contributes to the classical action by

$$S_{\mathcal{N} = \frac{3}{4}} = \frac{\pi k}{3(1 + iu)^2(1 - iu)} \sigma^3.$$

(98)

We compare this to the asymptotic value of the one-loop partition function (82). We consider the contribution of a single hypermultiplet with charge 1. This gives the factor $1/S_3(\frac{\text{det}}{2} - i\sigma; \omega)$ in the
one-loop determinant. In the asymptotic region $|\Re \sigma| \to \infty$, the triple sine function behaves as

$$\log S_3 \left( \frac{\omega_{\text{tot}}}{2} - i\sigma, \omega \right) \sim \text{sign}(\sigma) \left( -\frac{\pi}{6\omega_1\omega_2\omega_3} \sigma^3 - \frac{\pi (\omega_1^2 + \omega_2^2 + \omega_3^2)}{24\omega_1\omega_2\omega_3} \sigma \right).$$

(We assumed $\Re \omega_i > 0$.) The coefficient of the cubic term is

$$-\text{sign}(\sigma) \frac{\pi}{6\omega_1\omega_2\omega_3} = -\text{sign}(\sigma) \frac{\pi}{6(1 + in)(1 - in)}.$$

Comparing this to (98), we find that this effectively shifts the coefficient $k$ by $\pm 1/2$. This is the same as the one-loop correction to the Chern–Simons level by the fermion in the single hypermultiplet.

4.4. General squashed $S^5$

The supersymmetric action for a general squashed $S^5$ has not been explicitly obtained, and thus we cannot directly determine the classical action $S_0(\sigma)$. However, by taking advantage of the ambiguity in the definition of the prepotential that we mentioned in Sect. 4.2, we define the prepotential so that it is proportional to the classical action at saddle points. Then we need to determine the proportionality constant. For this purpose we can use the relation between the classical action and the asymptotic form of the one-loop determinant as discussed in the previous subsection. We obtain

$$S_0 = \frac{(2\pi)^3}{\omega_1\omega_2\omega_3} F(\sigma)$$

for general squashed $S^5$, and this gives the exponential factor in (1).

5. Discussion

We computed the perturbative partition function of 5d $\mathcal{N} = 1$ supersymmetric gauge theories defined on a squashed $S^5$. The result is expressed in a simple form containing the triple sine function. This is analogous to the $S^3$ partition function written in terms of the double sine function. The $S^5$ partition function depends on the squashing parameters $(\phi_1, \phi_2, \phi_3)$ through the periods $\omega_i = 1 + i\phi_i$.

To determine the classical action at saddle points for general squashing parameters, we used the convention of the prepotential in which the classical action at saddle points is proportional to the prepotential, and the proportionality constant is determined by the consistency of the asymptotic form of the one-loop determinant factor to the quantization of Chern–Simons levels. We do not know the explicit form of the action on a general squashed $S^5$, and the precise meaning of the coefficients in the prepotential in the convention is not clear. In general, we cannot obtain Chern–Simons terms in 5d by the dimensional reduction from 6d; we need to rely on direct analysis in 5d such as the Noether procedure to construct the supersymmetric action in general squashed $S^5$.

An important application of the partition function is the confirmation of predictions of AdS/CFT (anti de Sitter/conformal field theory) correspondence. For the round $S^5$, the large $N$ partition function of $USp(2N)$ supersymmetric QCD with $N_f \leq 7$ flavors is computed in [36], and precise agreement of the large $N$ leading term to the prediction of the gravity dual is confirmed. It is interesting to extend this analysis to squashed $S^5$. Another interesting theory is the $U(N)$ supersymmetric Yang–Mills theory with a single adjoint hypermultiplet, which is related to the $(2, 0)$ theory realized on M5-branes via $S^1$ compactification. The $N^3$ behavior of this partition function is confirmed in [11,37]. It is important to establish the relation between the gauge theory and the gravity dual for an arbitrary mass parameter $\Delta$ and squashing parameters $\phi_i$.

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\(^4\)This can be obtained from some formulae of the multiple gamma function in [35].
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Appendix. Dirac matrices

We use the following representation of the 5d Dirac matrices

\[
\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & -i1_2 \\ i1_2 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}.
\] (A1)

where \(\sigma_i (i = 1, 2, 3)\) are the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (A2)

The 6d Dirac matrices are

\[
\Gamma_{\mu} = \begin{pmatrix} 0 & \gamma_{\mu} \\ \gamma_{\mu} & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & -i1_4 \\ i1_4 & 0 \end{pmatrix}, \quad \Gamma_7 = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}.
\] (A3)

The bilinear of \(SU(2)\) doublet spinors is defined by

\[
\psi \chi = \epsilon^{IJ} C^{ab} \psi_{Jb} \chi_{Ia},
\] (A4)

where \(\epsilon^{IJ}\) is an \(SU(2)\) invariant anti-symmetric tensor with component \(\epsilon^{12} = -\epsilon^{21} = 1\), and \(C^{ab}\) is the charge conjugation matrix with components

\[
C^{ab} = \begin{pmatrix} i\sigma_2 & -i\sigma_2 \\ i\sigma_2 & -i\sigma_2 \end{pmatrix}.
\] (A5)

The antisymmetric tensor is defined by

\[
\Gamma^{M_1 \cdots M_6} \Gamma^7 = \epsilon^{M_1 \cdots M_6} 1_8.
\] (A6)

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