On Visibility Graphs of Point Sets in the Plane

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Abstract

The visibility graph \( V(X) \) of a discrete point set \( X \subset \mathbb{R}^2 \) has vertex set \( X \) and an edge \( xy \) for every two points \( x, y \in X \) whenever there is no other point in \( X \) on the line segment between \( x \) and \( y \). We show that for every graph \( G \), there is a point set \( X \in \mathbb{R}^2 \), such that the subgraph of \( V(X \cup \mathbb{Z}^2) \) induced by \( X \) is isomorphic to \( G \). As a consequence, we show that there are visibility graphs of arbitrary high chromatic number with clique number six settling a question by Kára, Pór and Wood.

1 Introduction

The concept of a visibility graph is widely studied in discrete geometry. You start with a set of objects in some metric space, and the visibility graph of this configuration contains the objects as vertices, and two vertices are connected by an edge if the corresponding objects can “see” each other, i.e., there is a straight line not intersecting any other part of the configuration from one object to the other. Often, there are extra restrictions on the objects and on the direction of the lines of visibility.

Specific classes of visibility graphs which are well studied include bar visibility graphs (see [3]), rectangle visibility graphs (see [6]) and visibility graphs of polygons (see [1]). In this paper we consider visibility graphs of point sets.

Let \( X \subset \mathbb{R}^2 \) be a discrete point set in the plane. The visibility graph of \( X \) is the graph \( V(X) \) with vertex set \( X \) and edges \( xy \) for every two points \( x, y \in X \) whenever there is no other point in \( X \) on the line segment between \( x \) and \( y \), i.e., when the point \( x \) is visible from the point \( y \) and vice versa.

Kára, Pór and Wood discuss these graphs [4], and make some observations regarding the chromatic number \( \chi(V(X)) \) and the clique number \( \omega(V(X)) \), the order of the largest clique. In particular, they characterize all visibility graphs with \( \chi(V(X)) = 2 \) and \( \chi(V(X)) = 3 \), and in both cases, \( \omega(V(X)) = \chi(V(X)) \). Similarly, they show the following proposition.

Proposition 1. Let \( \mathbb{Z}^2 \) be the integer lattice in the plane, then \( \omega(V(\mathbb{Z}^2)) = \chi(V(\mathbb{Z}^2)) = 4 \).

Note that \( V(\mathbb{Z}^2) \) is not perfect as it contains induced 5-cycles. Further, it is not true in general that \( \omega(V(X)) = \chi(V(X)) \)—there are point sets with as few as nine points with \( \omega(V(X)) = 4 \) and \( \chi(V(X)) = 5 \).

For general graphs, there are examples with \( \chi(G) = k \) and \( \omega(G) = 2 \) for any \( k \), one famous example is the sequence of graphs \( M_{k-2} \) by Mycielski [5]. No similar construction is known for visibility graphs with bounded clique number. As their main result, Kára et al. construct a family of point sets with \( \chi(V(X)) \geq (c_1 \log \omega(V(X_i)))^{c_2 \log \omega(V(X_i))} \) for some constants \( c_1 \) and \( c_2 \) and with \( \omega(V(X_i)) \) getting arbitrarily large. Our main result is the following theorem.

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Theorem 2. For every graph $G$, there is a set of points $X \subset \mathbb{R}^2$ such that the subgraph of $\mathcal{V}(X \cup \mathbb{Z}^2)$ induced by $X$ is isomorphic to $G$.

Let $G_k$ be a graph with $\chi(G_k) = k$ and $\omega(G_k) = 2$, and let $X_k$ be the corresponding set given by Theorem 2. Let $Y_k \subset X_k \cup \mathbb{Z}^2$ be the subset of points contained in the convex hull of $X_k$. Then $\chi(\mathcal{V}(Y_k)) \geq \chi(G_k) = k$ and $\omega(\mathcal{V}(Y_k)) \leq \omega(G_k) + \omega(\mathcal{V}(\mathbb{Z}^2)) = 6$, so we get the following corollary settling the question from above raised by Kára et al.

Corollary 3. For every $k$, there is a finite set point $Y \subset \mathbb{R}^2$, such that $\chi(\mathcal{V}(Y)) \geq k$ and $\omega(\mathcal{V}(Y)) = 6$.

2 Proof of the Theorem

Let $G$ be a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. We will show the following lemma in the Section 3.

Lemma 4. For $M$ large enough, there is a set of prime numbers $\{p_{ij} : 1 \leq i < j \leq n\}$ with the following properties:

1. $2^M < p_{ij} < 2^{M+1}$.

2. For $1 \leq k \leq n$, let $P_k = 2^{M+k+n} \prod_{i=1}^{k-1} p_{ik} \prod_{j=k+1}^{n} p_{kj}$, with $0 \leq n_k \leq k+n-1$ chosen such that $\log_2 P_k = nM + 2k$. Then $p_{k\ell}$ is the only number in $\{p_{ij} : 1 \leq i < j \leq n\}$ which divides $P_\ell - P_k$ for $1 \leq k < \ell \leq n$.

From this, we can construct the set of points $X$ in Theorem 2:

$$X = \{x_i : 1 \leq i \leq n\} \subset \mathbb{R}^2, \text{ with } x_i = \left(2^{-nM} P_i, i \prod_{k<j} (P_j - P_k) \prod_{k<j} p_{kj} \right).$$

Before we prove the lemma, we will show that this point set has the properties stated in the theorem. For $1 \leq i < \ell \leq n$, let $m_{i\ell}$ be the slope of the line through $x_i$ and $x_\ell$. Then

$$m_{i\ell} = \frac{\ell - i}{P_\ell - P_i} \cdot \frac{2^{nM} \prod_{k<j} (P_j - P_k)}{\prod_{k<j} p_{kj}}.$$ 

There are no three colinear points in $X$, as

$$2^{nM+2i+1} \leq P_i + 1 - P_1 < 2^{nM+2i+3},$$

thus $m_{(i+1)i} > m_{(i+1)(i+2)}$, and therefore $m_{i\ell} > m_{ik}$ for $i < \ell < k$. Thus, $\mathcal{V}(X)$ is complete, and it remains to show that there is an integer point on the line segment between $x_i$ and $x_\ell$ if and only if $i\ell \notin E(G)$. To establish this goal, we look at the intersections of the line through $x_i$ and $x_\ell (i < \ell)$ with the gridline with constant $x$-coordinate $s \in \mathbb{Z}$, where $1 \leq 2^{-nM} P_i < s < 2^{-nM} P_\ell \leq 2^{2n}$. For this intersection point $z_{i\ell}^s = (s, y_{i\ell}^s)$ we have

$$y_{i\ell}^s = \frac{\prod_{k<j} (P_j - P_k)}{\prod_{k<j} p_{kj}} + (s - 2^{-nM} P_i)m_{i\ell}$$

$$= i \frac{\prod_{k<j} (P_j - P_k)}{\prod_{k<j} p_{kj}} + \frac{\ell - i}{P_\ell - P_i} \cdot \frac{2^{nM} \prod_{k<j} (P_j - P_k)}{\prod_{k<j} p_{kj}} + P_1 \frac{\ell - i}{P_\ell - P_i} \cdot \frac{\prod_{k<j} (P_j - P_k)}{\prod_{k<j} p_{kj}}.$$ 

(1) \hspace{1cm} (2) \hspace{1cm} (3)
The expression (1) is an integer since $p_{k\ell}$ divides $P_j - P_k$. By the same argument, (3) is an integer—just note further that $p_{i\ell}$ divides $P_i$. It remains the analysis of (2).

If $i\ell \notin E(G)$, then (2) is an integer. Therefore, $z_{i\ell}^s \in \mathbb{Z}^2$, and $x_i x_\ell \notin E(\mathcal{V}(X \cup \mathbb{Z}^2))$. If $i\ell \in E(G)$, observe that $p_{i\ell} > 2^M > \max\{\ell - i, s\}$, so $p_{i\ell}$ does not divide $s$ or $\ell - i$. Clearly, $p_{i\ell}$ does not divide $2^n M$, and by Lemma 4, it does not divide any of the $P_j - P_k$ other than $P_\ell - P_r$. Thus, (2) is not an integer, $z_{i\ell}^s \notin \mathbb{Z}^2$, and $x_i x_\ell \notin E(\mathcal{V}(X \cup \mathbb{Z}^2))$, proving Theorem 2.

\[\square\]

3 Proof of Lemma 4

By an inequality of Finsler [2], there are more than $2^M/(3(M + 1) \ln 2) > 2n^3$ prime numbers in the interval from $2^M$ to $2^{M+1}$.

We will pick the $p_{ij}$ sequentially in the order $p_{12}, p_{13}, \ldots, p_{1n}, p_{23}, \ldots, p_{(n-1)n}$. Assume that we have picked numbers up to but not including $p_{ij}$ according to the Lemma, and we want to pick $p_{ij}$. If $j \neq n$, we pick any prime number in the interval from $2^M$ to $2^{M+1}$ that was not selected before (condition (*)), such that $p_{ij}$ does not divide $P_k - P_\ell$ for all $1 \leq \ell < k < i$ (condition (**)). There were less than $\binom{n}{2}$ primes selected before, and each $P_k - P_\ell$ has at most $n$ prime divisors greater than $2^M$, thus at most $\binom{n}{2} + n \binom{n}{2} < n^3$ of the choices are blocked, and so this choice is possible.

If $j = n$, we have to fulfill (*) and (**), with the added condition (***) that no $p_{k\ell}$ may divide $P_i - P_r$ for $\{k, \ell\} \neq \{i, r\}$. Pick $p_{ij}$ according to (*) and (**), and assume that $p_{k\ell}$ divides $P_i - P_r$ for some $\{k, \ell\} \neq \{i, r\}$. We have $k \neq i$ as all $p_{ij}$ divide $P_i$, otherwise $p_{i\ell}$ also divides $P_r$ and thus $r = \ell$, a contradiction. Similarly, $\ell \neq i$.

Pick another number $p'_{ij}$ according to (*) and (**). If $p_{k\ell}$ divides $P_i - P_r$, then $p_{k\ell}$ divides $P'_i - P_r = (P'_i - p_{ij})/p_{ij}$, and thus $p_{k\ell}$ divides $p'_{ij} - p_{ij}$. But this is impossible since $|p'_{ij} - p_{ij}| < 2^M < p_{k\ell}$. Therefore, each $p_{k\ell}$ can block at most one choice for $p_{ij}$, so in total at most $\binom{n}{2}$ further choices are blocked by (**), and we can always find a number $p_{ij}$ with (*), (**), and (***). This concludes the proof of the lemma.

\[\square\]

4 Further Questions

We have shown that there are visibility graphs with $\chi(\mathcal{V}(X)) \geq k$ and $\omega(\mathcal{V}(X)) = 6$ for every $k$. For all visibility graphs with $\omega(\mathcal{V}(X)) \leq 3$, we know that $\chi(\mathcal{V}(X)) = \omega(\mathcal{V}(X))$. The only cases left to consider are $\omega(\mathcal{V}(X)) = 4$ and $\omega(\mathcal{V}(X)) = 5$. A similar technique of combining a visibility graph with $\omega(\mathcal{V}(X)) = 3$ with a graph $G$ with $\omega(G) = 2$ and large chromatic number will not work, since the visibility graphs with $\omega(\mathcal{V}(X)) = 3$ are too simple (all but at most two of their vertices are collinear unless $\mathcal{V}(X)$ is a special graph on six vertices). It would be no surprise to us if the chromatic number of visibility graphs with $\omega(\mathcal{V}(X)) = 5$ is bounded.

Finally, one could look for smaller point sets with $\chi(\mathcal{V}(X)) \geq k$ and $\omega(\mathcal{V}(X)) = 6$, as our sets tend to be very large.

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