Spin and mass of the nearest supermassive black hole

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Abstract. Quasi-Periodic Oscillations (QPOs) of the hot plasma spots or clumps orbiting an accreting black hole contain information on the black hole mass and spin. The promising observational signatures for the measurement of black hole mass and spin are the latitudinal oscillation frequency of the bright spots in the accretion flow and the frequency of black hole event horizon rotation. Both of these frequencies are independent of the accretion model and defined completely by the properties of the black hole gravitational field. Interpretation of the known QPO data by dint of a signal modulation from the hot spots in the accreting plasma reveals the Kerr metric rotation parameter, \(a = 0.65 \pm 0.05\), and mass, \(M = (4.2 \pm 0.2) \times 10^6 M_\odot\), of the supermassive black hole in the Galactic center. At the same time, the observed 11.5 min QPO period is identified with a period of the black hole event horizon rotation, and, respectively, the 19 min period is identified with a latitudinal oscillation period of hot spots in the accretion flow. The described approach is applicable to black holes with a low accretion rate, when accreting plasma is transparent up to the event horizon region.

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1. Introduction

Parameters of the accreting black holes are the central problem in astrophysics, because they provide the unique opportunity for the verification of General Relativity. In the Galactic center dwells the nearest supermassive black hole with a mass \(M = (4.1 \pm 0.4) \times 10^6 M_\odot\), measured by observations of the orbital parameters of two serendipitous fast moving S0 stars \([3, 27]\). It is demonstrated below the ability of the Quasi-Periodic Oscillation (QPO) observations for the measuring of both the mass and spin of the astrophysical black holes. The major problem with observations of the supermassive black hole SgrA* in the Galactic center is that it is a ‘dormant’ quasar, i.e., nearly completely inactive with a very rare splashing of activity. Nevertheless, there are serendipitous observations of QPOs from SgrA* with a sufficiently high statistical significance in the near-infrared \([25]\) and X-rays \([3]\). It must be stressed however, that the observed QPO signals from SgrA* are suffered from the red noise \([4]\) and should only be considered as upper limits with a rather low statistical evidence.
The usual interpretation relates QPOs with the resonances in the accretion disks [25, 3, 34, 13, 30, 23]. A weak point of the resonance QPO interpretation is the ambiguity caused by the dependence on the accretion model used. It seems that the resonance QPO models are applicable to black holes in the Active Galactic Nuclei (AGN) and in the stellar binaries with the high accretion rates. The others promising approaches for revealing the black hole rotation are the continuum fitting method for the relativistic accretion disk models [37, 38, 35], the modelling of the spectral lines broadening in the accretion flow [24, 10, 32] and correlation between jet power and black hole spin [36]. These approaches also depend on the used accretion models.

What is described below is the alternative QPO interpretation, related to the oscillation frequencies of the numerous hot spots in the accretion plasma [45, 1, 49, 11, 47], which are independent of the accretion model and defined completely by the properties of the black hole gravitational field. In this interpretation the QPOs instead of the pure periodic ones result from the simultaneous emission of numerous clumps of hot plasma, existing in the accretion flow at different radial distances from black hole [28, 2, 42]. The described approach is directly applicable to black holes with a low accretion rate, when accreting plasma is transparent up to the event horizon region.

The supermassive black hole in the Galactic center is the most favorable case. This supermassive black hole is activated from time to time by the episodic accretion of tidally disrupted stars [29, 20, 15, 18, 19, 26], accompanied by the formation of shocks, jets and acceleration of cosmic rays [40, 8, 44, 16]. Especially crucial for the electromagnetic extraction of energy from black holes and for the efficiency of generation and acceleration of cosmic rays is the value of Kerr metric angular momentum [9].

The nonrotating spherically symmetric Schwarzschild black holes seems to be the very exotic objects in the Universe, along with all the other nonrotating objects, like stars and galaxies. A typical black hole must rotate rather fast, obtaining angular momentum either from the collapsing progenitor massive star, or after the coalescence from another black hole, or due to the spinning up by accretion matter. We use units with $G = c = 1$ and define the spin parameter of Kerr black hole as $a = J/M^2$, where $M$ and $J$ are, respectively, the black hole mass and angular momentum. Approaching to the extreme Kerr black hole state with $a = 1$ is only possible as an infinite limiting process [5], according to the third law of the black hole (thermo)dynamics [7]. In the case of the astrophysically interesting thin disk accretion, the black hole will finally spin up to the “canonical” value of the Kerr spin parameter $a_* = 0.9982$, i.e. very near to the extremely fast rotating state $a = 1$ [46].

Equations of motion for test particles (e.g., planets or clumps of hot plasma) with a mass $\mu$ in the Kerr metric in the Boyer–Lindquist coordinates $(t, r, \theta, \varphi)$ are [14]:

$$
\rho^2 \frac{d\rho}{d\lambda} = \pm \sqrt{\mathcal{V}}, \quad \rho^2 \frac{d\theta}{d\lambda} = \pm \sqrt{\mathcal{V}_\theta},$$
(1)

$$
\rho^2 \frac{d\varphi}{d\lambda} = L \sin^{-2} \theta + a(\Delta^{-1} P - E),$$
(2)

$$
\rho^2 \frac{dt}{d\lambda} = a(L - aE \sin^2 \theta) + (r^2 + a^2) \Delta^{-1} P,$$
(3)
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where \( \lambda = \tau/\mu \), \( \tau \) — is the proper time of a particle and

\[
\begin{align*}
V_r & = P^2 - \Delta[\mu^2r^2 + (L - aE)^2 + Q], \\
V_\theta & = Q - \cos^2\theta[a^2(\mu^2 - E^2) + L^2 \sin^{-2}\theta], \\
P & = E(r^2 + a^2) - aL, \\
\rho^2 & = r^2 + a^2 \cos^2\theta, \\
\Delta & = r^2 - 2r + a^2.
\end{align*}
\]

The motion of a test particle is completely defined by three integrals of motion: the total particle energy \( E \), the azimuthal component of the angular momentum \( L \) and the Carter constant \( Q \), related to the total angular momentum of the particle and non-equatorial motion. It is useful to choose the dimensionless variables and parameters:

\[
t \Rightarrow t/M, \quad r \Rightarrow r/M, \quad a \Rightarrow a/M, \quad E \Rightarrow E/\mu, \quad L \Rightarrow L/(M\mu), \quad Q \Rightarrow Q/(M^2\mu^2). \]

The effective potentials \( V_r \) and \( V_\theta \) in (4) and (5) determine the motion of particles in the radial \( r \)-direction and latitudinal \( \theta \)-direction, respectively. The radius of the black hole event horizon is \( x_h = 1 + \sqrt{1 - a^2} \).

2. Quasi-periodic plunging trajectories

Fig. 1 shows the numerically calculated “plunging” trajectory of the planet infalling into the rotating black hole.

From equations of motion (2) and (3) follows the crucial feature of any test particle trajectory, approaching the event horizon at \( x = x_h \). Namely, by approaching to the black hole horizon, \( x \to x_h \), the trajectory is quasi-periodically “winding up” with an azimuthal angular velocity \( \Omega_\varphi = d\varphi/dt \), coming close to the angular velocity of the black hole horizon \( \Omega_h \), where

\[
\Omega_h = \frac{d\varphi}{dt} \bigg|_{x \to x_h} = \frac{a}{2(1 + \sqrt{1 - a^2})}.
\]

This general behavior of the plunging trajectories near the event horizon is valid even for the massless particles. See in Fig. 2 the corresponding plunging photon trajectory, quasi-periodically “winding up” with the angular velocity \( \Omega_\varphi \to \Omega_h \), by approaching the black hole event horizon at the southern hemisphere. As a result, the angular velocity of black hole event horizon \( \Omega_\varphi \) must be inevitably imprinted to the QPO signal from the accreting black holes.

Any source of radiation, e.g., clump of the hot plasma, approaching the event horizon of the rotating black hole will be viewed by the distant observer in a relativistic “synchrotron mode” as short splashes of radiation, beamed and boosted forward into the narrow solid angles \([6, 33, 17, 39]\), and repeated quasi-periodically with a frequency very near to \( \nu_h = \Omega_h/2\pi \).

The oscillation with \( \Omega_h \) from (9) is a first observational signature for revealing the spin of the supermassive black hole in the Galactic center. The corresponding second signature is related with the QPOs of non-equatorial bound orbits in the accretion flow.
Figure 1. The planet trajectory with a total energy $E = 0.85$, azimuthal angular momentum $L = 1.7$ and Carter constant $Q = 1$, infalling into the black hole with a spin $a = 0.9982$ and event horizon radius $x_h = 1.063$. The trajectory is “winding up” with the angular velocity $\Omega_\varphi \to \Omega_h$, by approaching the black hole horizon at the northern hemisphere. Trajectory is shown here and in the further similar Figures in the Boyer–Lindquist coordinates.

Table 1. Successive $(n = 1, 2, 3\ldots)$ radial, azimuthal, and latitudinal periods $(T_{n,r}, T_{n,\varphi}, T_{n,\theta})$ in units $MG/c^3$, measured by the distant observer, for the bound quasi-periodic orbit, shown in Figs. [3] and [4].

| n  | Azimuthal period $T_{n,\varphi}$ | Radial period $T_{n,r}$ | Latitudinal period $T_{n,\theta}$ |
|----|-------------------------------|------------------------|----------------------------------|
| 1  | 74.2                          | 181.1                  | 90.4                             |
| 2  | 18.9                          | 180.6                  | 61.9                             |
| 3  | 22.4                          |                       | 108.6                            |
| 4  | 121.3                         |                       | 40.8                             |
| 5  | 26.3                          |                       |                                  |
| 6  | 19.5                          |                       |                                  |
| 7  | 44.3                          |                       |                                  |

3. Quasi-periodic bound orbits

The specific features of the accretion disk flow near the black hole depend in part on the properties of the stable gravitationally bound orbits of test particles with $E < 1$
Figure 2. The photon trajectory with an impact parameter $b = L/E = 2$ and Carter constant $Q = 2$, infalling into the black hole with $a = 0.9982$ and $x_h = 1.063$. The trajectory is “winding up” with the angular velocity $\Omega_\varphi \to \Omega_h$, by approaching the black hole horizon at the southern hemisphere. This trajectory also corresponds to the outgoing photon for the black hole with an opposite direction of spin.

[48], which have two radial turning points, the apocenter and pericenter radii, $r_a$ and $r_p$, respectively, and also two latitudinal turning points, $\pi/2 \pm \theta_{\text{max}}$ above and below the equatorial plane, where $\theta_{\text{max}}$ is a maximal elevation angle of the orbit with respect to the equatorial plane. These turning points are defined by the zeroes of the corresponding effective potentials $V(r)$ and $V(\theta)$ from [4] and [5] respectively. The bound orbits around the rotating Kerr black hole are quasi-periodic in general, because metric depends not on the one, but on the two coordinates: the radius $r$ and the latitude $\theta$, in contrast to the Schwarzschild case. The bound orbits oscillate not in time but only in space between the two radial turning points and between the two latitudinal turning points. The pure periodic bound orbits in the Kerr metric are only the degenerate ones: they either confined in the equatorial plane of the black hole or belong to the specific case of spherical orbits [48], with a radial coordinate $r = \text{const}$.

See in Figs. 3 and 4 an example of the numerically calculated bound quasi-periodic orbit, viewed from the black hole north pole and aside, respectively. In the Table 1 are shown the successive $(n = 1, 2, 3 \ldots)$ radial $(T_{n,r}, \tau_{n,r})$, azimuthal $(T_{n,\varphi}, \tau_{n,\varphi})$ and latitudinal periods $(T_{n,\theta}, \tau_{n,\theta})$ for the bound quasi-periodic orbit, shown in Figs. 3 and 4 and measured by the distant $(T)$ and proper $(\tau)$ observers, respectively. In Fig. 5 this orbit is superimposed onto the thin and opaque accretion disk, when only the parts of
Figure 3. The bound quasi-periodic orbit with $Q = 2$, $E = 0.92$, $L = 1.9$, $x_p = 1.74$, $x_a = 9.48$ and $\theta_{\text{max}} = 36.3^\circ$, viewed from the north pole of the black hole with $a = 0.9982$ and $x_h = 1.059$. The orbit is shown thin at the beginning and thick at the ending.

Figure 4. The same orbit as in Fig 3 viewed from the north pole of the black hole.
Figure 5. The same bound quasi-periodic orbit as in Fig. 3 and 4 viewed aside, and superimposed onto the thin and opaque accretion disk. Only the parts of the orbit, which are above the accretion disk, may be viewed by the distant observer as QPOs with a frequencies near $\nu_\theta = \Omega_\theta/2\pi$.

the orbit, which are above the accretion disk, may be viewed by a distant observer. This is an illustration of the QPOs produced by one of the numerous hot spots or clumps of plasma in the accretion flow, which may be used for the determination of the black hole mass and spin [45, 11, 49, 11, 47]. The presence of bright plasma points in the turbulent accretion flow are also confirmed by numerical simulations of the thin accretion disks [28, 21, 42]. Below we calculate the corresponding oscillation periods of test particles (hot spots of plasma) in the thin accretion disk.

4. Orbital oscillations

The azimuthal and latitudinal angular velocities of the non-equatorial bound orbits, $\Omega_\varphi$ and $\Omega_\theta$, respectively, are related by the equation [48, 31]:

$$\frac{\Omega_\theta}{\Omega_\varphi} = \frac{\pi}{2} (\beta z_+) \left[ \frac{L}{a} \Pi(-z_-, k) + \frac{2 x E - a L}{\Delta} K(k) \right]^{-1},$$

where $K(k)$ and $\Pi(-z_-, k)$ are, respectively, the full elliptic integrals of the first and third kind, $k^2 = z_-/z_+$,

$$z_\pm = (2\beta)^{-1} \left[ \alpha + \beta \pm \sqrt{(\alpha + \beta)^2 - 4Q\beta} \right],$$

$\alpha = (Q + L^2)/a^2$ and $\beta = 1 - E^2$. By using the values for orbital energy $E$ and azimuthal angular momentum $L$ from [21, 22] for the test particle at the non-equatorial spherical
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Figure 6. Near the equatorial orbit of the bright clump of plasma with $Q = 0.1$, $E = 0.91$, $L = 2.715$, $x_p = 3.85$, $x_a = 5.01$ and $\theta_{\text{max}} = 6.6^\circ$, oscillating in the thin opaque accretion disk around the black hole with $a = 0.65$, $x_h = 1.76$ and $x_{\text{ms}} = 3.62$.

orbit, it can be calculated from equations of motion (2) and (3) the corresponding azimuthal angular velocity at the equatorial plane:

$$\Omega_{\varphi, \text{sph}} = \frac{x \sqrt{x^3 (3Q - Q x + x^2) + a^2 Q^2 - a (x^2 + 3Q)}}{\{x^5 - a^2 [x^2 + Q(x + 3)]\} M}.$$  \hspace{1cm} (12)

In the particular case of circular orbits in the equatorial plane ($r = \text{const}$, $Q = 0$, $\theta = \pi/2$), the angular velocity (12) is simplified to the well known form [6]:

$$\Omega_{\varphi, \text{circ}} = \frac{1}{a + x^{3/2}} \frac{1}{M}.$$  \hspace{1cm} (13)

From (10) and (12) in the limit $Q \to 0$ follows the angular velocity of the latitudinal oscillation of a near circular orbit in the thin accretion disk:

$$\Omega_\theta = \frac{2\pi}{T_\theta} = \frac{\sqrt{x^2 - 4ax^{1/2} + 3a^2}}{x(a + x^{3/2})} \frac{1}{M}.$$  \hspace{1cm} (14)

This angular velocity describes the latitudinal oscillation of the hot spot or clump of plasma in the thin accretion disk.

It is assumed that the brightest hot spots in the accretion flow are located near the inner edge of the accretion disk, corresponding to the radius of the marginally stable circular orbit [6], $x = x_{\text{ms}}$:

$$x_{\text{ms}} = 3 + Z_2 - \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)},$$  \hspace{1cm} (15)

where

$$Z_1 = 1 + (1 - a^2)^{1/3} \left[(1 + a)^{1/3} + (1 - a)^{1/3}\right]$$  \hspace{1cm} (16)

and $Z_2 = \sqrt{3a^2 + Z_1^2}$. See in Fig. 6 the example of the oscillating clump of plasma in the thin accretion disk around the moderately fast rotating black hole.

The latitudinal oscillation with an angular velocity $\Omega_\theta$ from (14), estimated at the radius $x = x_{\text{ms}}$, is the second requisite observation signature of the spinning black hole in the Galactic center.
5. Black hole spin and mass in the Galactic center

Fig. 7 shows the 1-sigma error \((M, a)\)-region for the joint resolution of equations (9) and (14) with the observed 11.5 min QPO1 period, identified with \(T_h\), and, respectively, the 19 min period QPO2, identified with \(T_\theta\). This 1-sigma \((M, a)\) region corresponds to the Kerr metric rotation parameter, \(a = 0.65 \pm 0.05\), and mass, \(M = (4.2 \pm 0.2) \times 10^6 M_\odot\), for the supermassive black hole in the Galactic center. It is clearly illustrated in Fig. 8 that a self-consistency of the observed QPO periods with \(T_h\) and \(T_\theta\) corresponds to the same value of the black hole spin, \(a = 0.65 \pm 0.05\). At the same time the values of azimuthal angular velocities \(\Omega_\phi\) of hot spots in the accretion disk are spread in a wide range. For this reason the azimuthal oscillations in the accretion disk would not produce any prominent features in the spectrum of QPOs.

Note also, that the additional three QPOs observed in the X-rays with periods around 1.7, 3.6 and 37.5 min [3]. These QPOs are less prominent than the used ones and seemingly related with the resonances in the accretion disk [25, 3, 31, 43, 30, 23] or
Figure 8. The observed QPOs with the mean periods 11.5 and 19 min (filled horizontal stripes QPO1 and QPO2) from the supermassive black hole Sgr A*, identified, respectively, with a period of the event horizon rotation $T_h$ from (9) and a period of the latitudinal oscillation of the hot plasma clump at the near-circular orbit in the thin accretion disk with $\Omega_\theta$ from (14). The filled region for $T_\theta$ corresponds to the permissible values of $Q$ and $x$ adjusted with the errors of the observed QPO periods. The observed QPO periods $T_h$ and $T_\theta$ corresponds to the same value of the black hole spin, $a = 0.65 \pm 0.05$ and mass, $M = (4.2 \pm 0.2)10^6 M_\odot$, according the joint resolution of equations (9) and (14), shown in Fig. 7

being the harmonics of the used QPOs with periods 11.5 and 19 min, respectively.

6. Conclusion

Interpretation of the known QPO data by dint of signal modulation from the hot spots in the accreting plasma reveals the Kerr metric rotation parameter, $a = 0.65 \pm 0.05$, of the supermassive black hole in the Galactic center. At the same time, the observed 11.5 min QPO period is identified with the period of the black hole horizon rotation, and, respectively, the 19 min period is identified with the latitudinal oscillation period of hot spots in the accretion flow. A major part of uncertainty in estimation of the black hole
spin related with an error in the measurements of the black hole mass in the Galactic center.

A supermassive black hole in the Galactic center acquires its angular momentum by accretion of tidally destructed stars and gas clouds with accidentally orientated individual angular momenta. For this reason a moderate spin value of the supermassive black hole Sgr A* is quite natural due to specific conditions in the Galactic center. Note also that the value of spin parameter $a = 0.65 \pm 0.05$, derived here by dint of QPOs, is in a qualitative agreement with the corresponding quite independent estimation, $a \approx 0 - 0.6$, from the millimeter VLBI observations of Sgr A* [12, 13]. At the same time, the black hole mass, $M = (4.2 \pm 0.2) \times 10^6 M_\odot$, derived here from the QPOs, is in a good agreement with a quite independent estimation, $M = (4.1 \pm 0.4) \times 10^6 M_\odot$, measured from observations of the fast moving S0 stars [3, 27].

Additionally, a moderate value of the spin indicates that the nearest to us supermassive black hole in the Galactic center is not a very efficient accelerator of cosmic rays, because the efficiency of electromagnetic extraction of energy from the rotating black hole is proportional to $a^2$ [9].

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