A new invariant for $\sigma$ models

Ioannis P. ZOIS$^1$
Mathematical Institute, 24-29 St. Giles’, Oxford OX1 3LB

$^1$izois@maths.ox.ac.uk
Abstract

We introduce a new invariant for $\sigma$ models (and foliations more generally) using the even pairing between K-homology and cyclic homology. We try to calculate it for the simplest case of foliations, namely principal bundles. We end up by discussing some possible physical applications including quantum gravity and M-Theory. In particular for M-Theory we propose an explicit topological Lagrangian and then using S-duality we conjecture on the existence of certain plane fields on $S^{11}$.

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To my brother Demetrios.
0.1 Introduction and Motivation

In [37] we proposed a Lagrangian density for the topological part of the non-supersymmetric M-theory using Polyakov’s flat bundle description of non-linear $\sigma$ models. The new key ingredient from the geometric point of view was ”characteristic classes for flat bundles”. This idea about using characteristic classes of flat bundles came from the definition of a new invariant for Haefliger structures [38].

This invariant can be defined for any foliation in general; as far as physics is concerned however (and this includes the case of M-theory treated in [37]), we are primarily concerned with a special kind of foliations, called flat foliations of bundles. This is due to the fact that as Polyakov had noticed in [18], $\sigma$ models can be thought of as flat principal bundles (see [37] for more details). Thus, hopefully, our invariant might be of some relevance whenever $\sigma$ models are met in physics.

We organise this paper as follows: in section 2 we explain the strategy of the construction; in section 3 we provide all the details; in section 4 we give the invariant formula; in section 5 we calculate this invariant for the simplest case of a principal bundle and in sections 6 and 7 we discuss some possible applications in physics. In section 3 we review some of the techniques from non-commutative topology which we shall use for defining the invariant. All other sections contain new original material.

0.2 Strategy

0.2.1 Instantons

Let us recall some facts about instantons. We would like to think of our invariant as an analogue of the instanton number for foliations.

We consider a principal bundle $(P, \pi, M, G)$, where $M$ is the base manifold assumed to be compact and 4-dim for brevity, $G$ is SU(2) for simplicity, $P$ is the total space of the bundle and $\pi$ is the projection. Assuming we have a connection $A$ on $P$ with curvature $F$, then the instanton number ignoring constants is simply $\int_M F \wedge F$, i.e. the second Chern number $c_2$ of the bundle $P$.

We would like to think of this number slightly differently: more or less by
definition, any principal bundle $P$ over $M$ defines an element (K-class) of the group $K^0(M)$ (we forget equivariant K-theory for simplicity). Using the Chern-Weil homomorphism we get the Chern classes of $P$ which belong to the cohomology groups $H^{2*}(M)$. Considering the (top dimensional) fundamental class $[M]$ of $M$ in the homology group $H_*(M)$ of $M$ and taking the pairing between homology and cohomology, which in this case is just integration over $M$, we get the instanton number. We can consider the Chern-Weil homomorphism from $K^0(M) \to H^{2*}(M)$ as a "black box" and forget all about cohomology for the moment; then the instanton number will be the result from pairings between K-theory and (singular) homology $H_*(M)$.

Our construction since we are dealing with foliations (more accurately with the space of leaves of foliations) which provide a good example of non-commutative topological spaces, imitates the above picture: to each foliation we can associate a homology class which will be the analogue of the fundamental class $[M]$ above; this class however will belong to an appropriate homology theory called **cyclic homology** and it is called **transverse fundamental class** of the foliation. Moreover one can also construct a class in K-homology, being the analogue of K-theory for our purpose. Then we use a formula for pairings between cyclic homology and K-homology to get our result.

### 0.2.2 Non-commutative Topology

In this subsection we would like to mention briefly what non-commutative topology is about. As its name suggests, this is one aspect of non-commutative geometry. Non-commutative geometry has appeared in physics literature some years ago mainly through the so called "quantised calculus". Anyway, the starting point of non-commutative topology is the fact that given any compact Hausdorff space $X$ say, the commutative ($C^*$)-algebra $C(X)$ of complex valued functions defined on $X$ can capture all the topological information of the space $X$ itself; in fact $X$ and $C(X)$ are completely equivalent, one can be uniquely constructed by the other. Conversely, given any commutative algebra $A$, say, there exists a compact Hausdorff space $X$ say, (called the spectrum of $A$) "realising" the commutative algebra $A$. Realising means that the commutative algebra $C(X)$ of complex valued functions on $X$ is essentially the algebra $A$. In mathematics terminology one says that the categories of compact Hausdorff spaces and commutative $C^*$-algebras are...
equivalent. This is the so called \textit{Gelfand’s theorem}.

We know however that there exist non-commutative $C^*$-algebras as well. The natural question then is whether one can find a "topological" realisation for them just like for the commutative ones. We are looking for a non-commutative analogue of Gelfand’s theorem. This question is not fully answered in mathematics, it is related to the famous Baum-Connes conjecture. There are some things already known in mathematics and these are related to foliations. This is what we shall be using extensively in this paper. The appropriate framework is that of K-theory and various homology theories.

During the ’70s mathematicians (Baum, Douglas, Kasparov and others) developed a K-theory for arbitrary $C^*$-algebras (commutative or not) and it is a well-known theorem due to Serre and Swan that in the commutative case this K-theory reduces to Atiyah’s original topological K-theory. Moreover in the 80’s mathematicians (Connes, Loday, Quillen and others) developed a homology theory called \textit{cyclic homology} for arbitrary algebras which again in the commutative case gives in the limit the usual simplicial homology. So non-commutative topology, in terms of K-theory and various homology theories gives a generalisation of ordinary topology through Gelfand’s theorem. A good example of a non-commutative topological space is the space of leaves of a foliation (see below for definitions). In general quotients of ordinary topological spaces by discrete groups give non-commutative (abbreviated to "nc" in the sequel) spaces. Good textbooks are [15] and [42] for an introduction on K-theory of $C^*$-algebras and cyclic homology respectively.

\textbf{0.2.3 The Invariant}

In order to construct our invariant for \textit{any} foliation [38], we use some ideas from non-commutative geometry [20], [22], [23], [27]. The strategy is as follows: given any foliation $F$ of a manifold $V$, namely an integrable subbundle $F$ of $TV$, one can associate to it another manifold $\Gamma(F)$, called the \textit{graph} (or \textit{holonomy groupoid}) of the foliation introduced in [29]. This is of dimension $\dim V + \dim F$. Using the complex line bundle $\Omega^{1/2}(\Gamma(F))$ of $1/2$-densities defined on $\Gamma(F)$, we consider the set (actually vector space) of smooth sections of this line bundle equipped with a $\ast$ product, thus obtaining an algebra. We then complete this algebra in a "minimal" manner (in standard $C^*$-algebra theory this is called the \textit{reduced} $C^*$-algebra completion), thus we obtain a
\( C^*\)-algebra denoted \( C^*(F) \) which is naturally associated to our original foliation \( F \). From now on one can forget the original foliation \( F \) of \( V \) altogether and concentrate on its corresponding \( C^*\)-algebra \( C^*(F) \). We are interested in the \( K_0 \) group of \( C^*(F) \) and in its cyclic homology groups. If we pick a metric \( g \) on the transverse bundle \( t \) of \( F \) we can construct in a natural way a \( C^*(F) \)-module \( E(F) \), thus obtaining a class \( [E(F)] \) in \( K_0(C^*(F)) \). Moreover to our foliation one can associate in a natural way a cyclic cocycle \( [F] \) in the \( q \)-th cyclic homology group of the \( C^*\)-algebra \( C^*(F) \), called the \textit{fundamental transverse cyclic cocycle} of the foliation, where \( q \) is the codimension of the foliation \( F \). Then we use the even pairing between K-homology and cyclic homology in this case, namely we consider the pairing

\[
\langle [E(F)], [F] \rangle := (m!)^{-1}(F\#Tr)(E(F), ..., E(F))
\]

as was firstly introduced in the abstract algebraic context in [23]. Hence we obtain a \textit{complex number} as a result from the above pairing and this complex number characterises our original foliation \( F \).

\section*{0.3 The constructions in detail:}

\subsection*{0.3.1 Foliations}

Let \( V \) be a smooth manifold and \( TV \) its tangent bundle. A smooth subbundle \( F \) of \( TV \) is called \textit{integrable} iff one of the following equivalent conditions is satisfied:

1. Every \( x \in V \) is contained in a submanifold \( W \) of \( V \) such that

\[
T_y(W) = F_y
\]

where \( T_y \) denotes the tangent space over \( y \).

2. Every \( x \in V \) is in the domain \( U \subset V \) of a submersion \( p : U \rightarrow \mathbb{R}^q \) (\( q=\text{codim } F \)) with \( F_y = \ker(p_*)_y \forall y \in U \).

3. \( C^\infty(V,F) = X \in C^\infty(V,TV) ; X_x \in F_x \forall x \in V \) is a Lie subalgebra of the Lie algebra of vector fields on \( V \).

4. The ideal \( J(F) \) of smooth differential forms which vanish on \( F \) is stable under differentiation: \( d(J) \subset J \)
The condition 3. is simply Frobenius’ Theorem and 4. its dual.

Example:
Any 1-dimensional subbundle $F$ of $TV$ is integrable, but for $\dim F \geq 2$ the condition is non-trivial; for instance if $V$ is the total space of a principal bundle with compact structure group, then we know that the subbundle of vertical vectors is always integrable, but the horizontal subbundle is integrable iff the connection is flat.

We shall make extensive use of this fact in this piece of work.

A foliation of $V$ is given by an integrable subbundle $F$ of $V$. The leaves of the foliation are the maximal connected submanifolds $L$ of $V$ with $T_x(L) = F_x \forall x \in L$, and the partition of $V$ into leaves $V = \cup L_a$ where $a \in A$ is characterised geometrically by its "local triviality": every point $x \in V$ has a neighborhood $U$ and a system of local coordinates $(x^i)$, $i=1,...,\dim V$ which is called foliation chart, so that the partition of $U$ into connected components of leaves, called plaques (they are the leaves of the restriction of the foliation on $U$), corresponds to the partition of $\mathbb{R}^{\dim V} = \mathbb{R}^{\dim F} \times \mathbb{R}^{\codim F}$ into the parallel affine subspaces $\mathbb{R}^{\dim F} \times pt$.

Very simple examples indicate that the leaves $L$ may not be compact even if the manifold $V$ is and that the space of leaves $X := V/F$ may not be Hausdorff for the quotient topology. The "rational torus" is such an example.

Throughout this paper we would mainly restrict our attention to two special kinds of foliations: we consider a principal bundle $P$ with structure (Lie) group $G$ (assumed compact and connected) over a compact manifold $M$. The total space $P$ has automatically a foliation induced by the fibration: the leaves are the fibers which are isomorphic to the structure group $G$ and the space of leaves is just the base space $M$ with its manifold topology. We shall be referring to this foliation as the vertical foliation of the principal bundle and it will be denoted $P_V$. Clearly, the dimension of this foliation is equal to the dimension of the group $G$, the integrable subbundle of $TP$ being in this case the vertical subbundle. The codimension is equal to the dimension of the base space $M$.

Now if in addition a flat connection is given on our principal bundle, we have another foliation of the total space which we shall be referring to as the horizontal or flat foliation and it will be denoted $P_H$. We shall study this foliation extensively in the following subsection. The dimension of this
foliation equals the dimension of the base space and the codimension equals the dimension of the group. From this one can see that the vertical and the horizontal foliations of a principal bundle are transverse to each other.

Now the vertical foliation behaves very well; everything is compact and Hausdorff, as were the spaces we started with to build our bundle. In this case the general theory of foliations gives nothing more than the well-known theory of principal bundles. However, the horizontal foliation can suffer from various "pathological" defects and for this reason it is interesting from the ncg point of view. Let us study it in greater detail.

0.3.2 Flat foliation of a principal bundle

To begin with, a flat connection on a principal bundle $P$ with structure group $G$ and base space $M$, corresponds to reduction of the structure group from $G$ to a subgroup isomorphic to a normal subgroup of the fundamental group of the base space $\pi_1(M)$. Moreover a (gauge equivalence class of a) flat connection also defines a (conjugacy class of a) representation

$$H : \pi_1(M) \to G$$

If we identify the fundamental group with the group of covering translations of the universal covering $\tilde{M}$ of $M$ we get an action $\kappa$ of $\pi_1(M)$ on $\tilde{M} \times G$ defined as follows:

$$\kappa : \pi_1(M) \times (\tilde{M} \times G) \to (\tilde{M} \times G)$$

$$(\gamma, (\tilde{m}, g)) \mapsto (\gamma(\tilde{m}), H(\gamma)(g))$$

where we use the obvious notation $\gamma \in \pi_1(M), g \in G, \tilde{m} \in M$. This action gives a commutative diagram:

$$\begin{array}{ccc}
\tilde{M} \times G & \xrightarrow{pr} & \tilde{M} \\
\downarrow \pi & & \downarrow q \\
P' = (\tilde{M} \times G)/\kappa & \xrightarrow{p} & M
\end{array}$$

(1)
where \( pr \) is the canonical projection, \( \pi \) is the quotient map by \( \varpi \), \( p \) is uniquely induced by \( pr \) and \( q \) is just the map from the universal covering space to the original space.

This construction is called suspension of the representation \( H \). One can prove that the map \( \pi \) is a covering map and that if \( F := ImH \) is endowed with the induced topology, then \( \xi_H = (P', p, M) \) is a fiber bundle with fiber \( G \), total space \( P' \), base \( M \), projection \( p \) and structure group \( F \).

To study the geometric properties of suspensions we introduce a new topology on the total space \( P' \) of \( \xi_H \). We denote by \( G^\delta \) the set \( G \) supplied with the discrete topology. Then the action \( \varpi \) of \( \pi_1(M) \) on \( \tilde{M} \times G^\delta \) remains continuous and the map \( \pi : \tilde{M} \times G^\delta \to P' \) induces on \( P' \) a new topology which is finer than its manifold topology. We denote by \( P^\delta \) the set \( P' \) supplied with this topology. The topology on \( M \times G^\delta \) and the topology \( P^\delta \) are called the leaf topologies. Then the suspension diagram below is a commutative diagram of covering maps:

\[
\begin{array}{ccc}
\tilde{M} \times G^\delta & \xrightarrow{pr} & \tilde{M} \\
 \downarrow \pi & & \downarrow q \\
P^\delta & \xrightarrow{p} & M
\end{array}
\]

(2)

The topological space \( P^\delta \) is not connected unless the fiber is contractible. A connected component of \( P^\delta \) is called a leaf of \( \xi_H \). Each point \( x = \pi(\tilde{m}, g) \in P' \) belongs to exactly one leaf which is denoted \( L_x \) and equals \( \pi(\tilde{M} \times g) \). The leaves are injectively immersed submanifolds of \( P' \) but in general not embedded. They are transverse to the fibers of \( \xi_H \). Conjugate representations \( H \) and \( H' \) give suspension bundles \( \xi_H \) and \( \xi_{H'} \) which are isomorphic.

Let now \( x = \pi(\tilde{m}, g) \). Then the representation

\[
H_x : \pi_1(L_x) \to G
\]

with image \( F_g \), is called the holonomy representation of the leaf \( L_x \) at the point \( x \). The group \( F_g \) is the holonomy group of the leaf \( L_x \) at the point \( x \). \( F_g \) is the isotropy group of \( F \) in \( g \in G \). Moreover \( \pi_1(L_x) \) is isomorphic to the isotropy group of \( \pi_1(M) \) in the point \( g \in G \), namely \( \pi_1(L_x) \cong \{ \gamma \in \pi_1(M) | H(\gamma)g = g \} \). See also [19].
There is a topological way to characterise these flat bundles which is by using *classifying spaces for flat bundles* in a fashion analogous for ordinary bundles, namely:

Let $G$ be a connected Lie group and let $G^\delta$ denote the same group with the discrete topology. The *Milnor join construction* for $G$ defines a connected space $BG$ which is the classifying space for principal $G$-bundles. The same construction applied to $G^\delta$ yields a connected topological space $BG^\delta$ which is an *Eilenberg-Maclane space* $K(G, 1)$, namely $\pi_1(BG) = G$ and $\pi_j(BG) = 0$ for $j > 1$. The inclusion $i : G^\delta \to G$ induces a continuous map $Bi : BG^\delta \to BG$. As sets these two spaces are the same with the source having finer topology than the range. The difference in these two topologies is measured by introducing the homotopy fiber $BG'$. This is defined by first replacing $Bi$ with a homotopy equivalent weak fibration over $BG$, then take for $BG'$ the (homotopy class of the) fiber. The description then is just the construction of the *Puppe Sequence* for $Bi$ (cf [30]).

Choose a base point in $BG^\delta$ and consider its image in $BG$. Then let $\Omega(BG)$ and $P(BG)$ denote the space of based loops and paths with initial point of $BG$ respectively. Let $e$ be the end point map of a path. Then one has a fibration

$$\Omega(BG) \to P(BG) \to BG$$

where the second map is $e$. Then define $BG'$ via the homotopy pull-back diagram:

$$\begin{array}{ccc}
\Omega(BG) & \longrightarrow & \Omega(BG) \\
\downarrow & & \downarrow \\
BG' & \longrightarrow & P(BG) \\
\downarrow & & \downarrow^e \\
BG^\delta & \longrightarrow & BG \\
\end{array}$$

(3)

A principal $G$-bundle $P$ over a manifold $M$ is equivalent to giving an open covering for $M$ and the transition functions. This data defines a continuous map $g_P : M \to BG$. If the transition functions are locally constant, namely if the bundle $P$ is flat, then $g_P$ can be factored through $BG^\delta$ as a continuous map.
A choice of transition functions which are locally constant is equivalent to specifying a flat G-structure on $P$. Hence $P$ has a horizontal foliation whose holonomy map $a : \pi_1(M) \to G$ defines the classifying map $Ba : M \to BG^\delta$. Conversely, given a continuous map $Ba : M \to BG^\delta$, there is induced a representation $a : \pi_1(M) \to G$ and a corresponding flat principal $G$-bundle $P_a = \tilde{M} \times_{\pi_1(M)} G$, where $\tilde{M}$ is the universal covering of $M$. The topological type of the $G$-bundle $P_a$ is determined by the composition $g_a : M \to BG^\delta \to BG$.

The principal bundle is trivial iff $g_a$ is homotopic to the constant map $M \to pt$. The choice of the homotopy is equivalent to specifying a global section on $P_a$.

0.3.3 Groupoids and $C^\ast$-algebras associated to Foliations

The next step is to associate the holonomy groupoid to any foliation. In general a groupoid is roughly speaking a small category with inverses, or more precisely

**Definition 1:**

A groupoid consists of a set $\Gamma$, a distinguished subset $\Gamma^{(0)}$ of $\Gamma$, two maps $r, s : \Gamma \to \Gamma^{(0)}$ and a law of composition $\circ : \Gamma^{(2)} := (\gamma_1, \gamma_2) \in \Gamma \times \Gamma; s(\gamma_1) = r(\gamma_2) \to \Gamma$

such that:

1. $s(\gamma_1 \circ \gamma_2) = s(\gamma_2)$, $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$ $\forall (\gamma_1, \gamma_2) \in \Gamma^{(2)}$
2. $s(x) = r(x) = x \forall x \in \Gamma^{(0)}$
3. $\gamma \circ s(\gamma) = \gamma$, $r(\gamma) \circ \gamma = \gamma \forall \gamma \in \Gamma$
4. $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$
5. Each $\gamma$ has a two sided inverse $\gamma^{-1}$, with $\gamma \gamma^{-1} = r(\gamma)$ and $\gamma^{-1} \gamma = s(\gamma)$

The maps $r, s$ are called range and source maps.

In the category theory terminology, $\Gamma^{(0)}$ is the space of objects and $\Gamma^{(2)}$ is the space of morphisms.
Definition 2:

A smooth groupoid $\Gamma$ is a groupoid together with a differentiable structure on $\Gamma$ and $\Gamma^{(0)}$ such that the maps $r, s$ are submersions and the object inclusion map $\Gamma^{(0)} \to \Gamma$ is smooth, as is the composition map $\Gamma^{(2)} \to \Gamma$.

The notion of a $\frac{1}{2}$-density on a smooth manifold allows one to define in a canonical manner the convolution algebra of a smooth groupoid $\Gamma$.

Specifically, given $\Gamma$, let $\Omega^{1/2}$ be the line bundle over $\Gamma$ whose fiber $\Omega^{1/2}_\gamma$ at $\gamma \in \Gamma, r(\gamma) = x, s(\gamma) = y$, is the linear space of maps

$$\rho : \wedge^k T_\gamma(\Gamma_x) \otimes \wedge^k T_\gamma(\Gamma_y) \to \mathbb{C}$$

such that

$$\rho(\lambda \nu) = |\lambda|^{1/2} \rho(\nu) \forall \lambda \in \mathbb{R}$$

Here $\Gamma_y = \gamma \in \Gamma; s(\gamma) = y, \Gamma^x = \gamma \in \Gamma; r(\gamma) = x$, and $k = \text{dim} T_\gamma(\Gamma^x) = \text{dim} T_\gamma(\Gamma_y)$ are the dimensions of the fibers of the submersions $r : \Gamma \to \Gamma^{(0)}$ and $s : \Gamma \to \Gamma^{(0)}$.

Then we endow the linear space $C^\infty_c(\Gamma, \Omega^{1/2})$ of smooth compactly supported sections of $\Omega^{1/2}$ with the convolution product

$$(a * b)(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} a(\gamma_1)b(\gamma_2)$$

$\forall a, b \in C^\infty_c(\Gamma, \Omega^{1/2})$ where the integral on the RHS makes sense since it is the integral of a 1-density, namely $a(\gamma_1)b(\gamma_1^{-1}\gamma)$, on the manifold $\Gamma^x$, $x = r(\gamma)$.

One then can prove that if $\Gamma$ is a smooth groupoid and $C^\infty_c(\Gamma, \Omega^{1/2})$ is the convolution algebra of smooth compactly supported $\frac{1}{2}$-densities with involution $^*$, $f^*(\gamma) = \overline{f(\gamma^{-1})}$. Then for each $x \in \Gamma^{(0)}$, the following defines an involutive representation $\pi_x$ of $C^\infty_c(\Gamma, \Omega^{1/2})$ in the Hilbert space $L^2(\Gamma_x)$:

$$(\pi_x(f)\xi)(\gamma) = \int f(\gamma_1)\xi(\gamma_1^{-1}\gamma)$$
∀γ ∈ Γ, ξ ∈ L²(Γ)

The completion of $C_c^∞(Γ, Ω^{1/2})$ for the norm $||f|| = sup_{x ∈ Γ(0)}||π_x(f)||$ is a $C^*$-algebra denoted $C^*_r(Γ)$.

Moreover one defines the $C^*$-algebra $C^*(Γ)$ as the completion of the involutive algebra $C_c^∞(Γ, Ω^{1/2})$ for the norm $||f||_{max} = sup||π(f)||$; $\{π$ involutive Hilbert space representation of $C_c^∞(Γ, Ω^{1/2})\}$

After this general introduction to groupoids and to $C^*$-algebras associated to them, now we pass to groupoids and $C^*$-algebras associated to foliations.

Let $(V, F)$ be a foliated manifold of codim $q$. Given any $x ∈ V$ and a small enough open set $W$ in $V$ containing $x$, the restriction of the foliation $F$ to $W$ has as its leaf space an open set of $R^q$ which we shall call a transverse neighborhood of $x$. In other words, this open set $W/F$ is the set of plaques around $x$. Now given a leaf $L$ of $(V, F)$ and two points $x, y ∈ L$, any simple path $γ$ from $x$ to $y$ on $L$ uniquely determines a germ $h(γ)$ of a diffeomorphism from a transverse neighborhood of $x$ to one of $y$. This depends only on the homotopy class of $γ$ and is called the holonomy of the the path $γ$. The holonomy groupoid of a leaf $L$ is the quotient of its fundamental groupoid by the equivalence relation which identifies two paths $γ_1, γ_2$ from $x$ to $y$ both in $L$ iff $h(γ_1) = h(γ_2)$. Here by the fundamental groupoid of a leaf we mean the groupoid $Γ = L × L$, $r, s$ are the two projections, $Γ^{(0)} = L$ and the composition is $(x, y) ∘ (y, z) = (x, z)$. (From this one can see that every space is a groupoid). The holonomy covering $˜L$ of a leaf $L$ is the covering of $L$ associated to the normal subgroup of its fundamental group $π_1(L)$ given by paths with trivial holonomy. The holonomy groupoid or graph of the foliation is the union $Γ$ of the holonomy groupoids of its leaves. Given an element $γ$ of $Γ$ we denote by $s(γ) = x$ the origin of the path $γ$ and by $r(γ) = y$ its end point, where $r, s$ are the range and source maps as in the general case.

An element of $Γ$ is thus given by two points $x = s(γ)$ and $y = r(γ)$ of $V$ together with an equivalence class of smooth paths : the $γ(t), t ∈ [0, 1]$ with $γ(0) = x$ and $γ(1) = y$, tangent to the bundle $F$, namely with $\frac{dγ}{dt} ∈ F_γ(t) ∀ t ∈ R$, identifying $γ_1$ and $γ_2$ as equivalent iff the holonomy of the path $γ_2γ_1^{-1}$ at the point $x$ is the identity. The graph $Γ$ has an obvious composition
law. For $\gamma_1$ and $\gamma_2$ in $\Gamma$, the composition $\gamma_1 \circ \gamma_2$ makes sense if $s(\gamma_1) = r(\gamma_2)$. The groupoid $\Gamma$ is by construction a (not necessarily Hausdorff) manifold of dimension $\text{dim}\, \Gamma = \text{dim} V + \text{dim} F$.

**Definition 3:** The $C^*$-algebra of the foliation is exactly the $C^*$-algebra of its graph, as described for arbitrary groupoids above.

For our foliations of interest, the graph $\Gamma$ is the following: for the vertical foliation is just the manifold $P \times G$ whereas for the horizontal foliation is $P \times_a \tilde{M}$, where $a$ is the representation from $\pi_1(M)$ to $G$ induced by the flat connection 1-form (via the holonomy). Moreover the distinguished subset $\Gamma^{(0)}$ in both cases is the manifold we want to foliate, namely $P$, the total space of our bundle in our case.

The $C^*$-algebras associated to our foliations are: for the vertical foliation is $C(M)$ tensored with compact operators which act as smoothing kernels along the leaves which in turn is strongly Morita equivalent to just $C(M)$, whereas for the horizontal foliation is strongly Morita equivalent (abreviated to SME) to $C(P) \rtimes \pi_1(M)$. (Note: the representation of the fundamental group of the base onto the structure Lie group induced by the flat connection 1-form used enters the definition of the crossed product). The first algebra is **commutative** (up to SME), but the second is **not**! It is for this reason that we can see now that ncg has an important role to play, in fact we are deeply in the ncg setting. Obviously if the space is simply connected, i.e. $\pi_1$ vanishes, non-commutativity is lost. We would like to emphasise that in all cases in the literature where some ”non-commutative” algebras were used, especially in connection to the well-known Connes-Lott model for electroweak theory (or even QCD), these algebras are in fact SME to commutative ones. Hence in terms of topology, this is not a real non-commutative case.

**0.3.4 K-classes associated to foliations**

We shall give the general construction for an arbitrary foliation.

Let $(V, F)$ be a foliated manifold and $t = TV/F$ the transverse bundle of the foliation. The holonomy groupoid $\Gamma$ of $(V, F)$ acts in a natural way on $t$ by the differential of the holonomy, thus for every $\gamma \in \Gamma$, $\gamma : x \to y$ determines a linear map $h(\gamma) : t_x \to t_y$. We denote this action by $h$. It is not in general
possible to find a Euclidean metric on $t$ which is invariant under the above action of $\Gamma$. Let $g$ be an arbitrary smooth Euclidean metric on the real vector bundle $t$. Thus for $\xi \in t_x$ we let $||\xi||_g = (\langle \xi, \xi \rangle_g)^{1/2}$ be the corresponding norms and inner products and drop subscript $g$ henceforth. Using $g$ we define a $C^*$-module $E$ on the $C^*$-algebra $C^*_r(V, F)$ of the foliation. Recall that $C^*_r(V, F)$ is the completion of the convolution algebra $C^*_c(\Gamma, \Omega^{1/2})$ which acts by right convolution on the linear space $C^*_c(\Gamma, \Omega^{1/2} \otimes r^*(t_C))$ denoted $\Lambda$ for simplicity and $t_C$ is the complexification of the transverse bundle $t$:

$$(\xi f)(\gamma) = \int_{\Gamma^y} \xi(\gamma_1)f(\gamma_1^{-1}\gamma)$$

where $y = r(\gamma)$. Endowing the complexified bundle $t_C$ with the inner product associated to $g$ and anti-linear in the first variable, the following formula defines a $C^*_c(\Gamma, \Omega^{1/2})$-valued inner product

$$\langle \xi, n \rangle(\gamma) = \int_{\Gamma^y} \langle \xi(\gamma_1^{-1}), n(\gamma_1^{-1}) \rangle$$

for any $\xi, n \in C^*_c(\Gamma, \Omega^{1/2} \otimes r^*(t_C))$. One then checks that the completion $E$ of the space $C^*_c(\Gamma, \Omega^{1/2} \otimes r^*(t_C))$ for the norm

$$||\xi|| = (||\xi, \xi||_{C^*_c(V, F)})^{1/2}$$

becomes a $C^*$-module over $C^*_r(V, F)$. If one takes also the action $h$ of $\Gamma$ on $t$ into account, with some extra effort one can make $E$ into a $(\Lambda, \Sigma)$-bimodule (for the definition of the algebra $\Sigma$ see below). The first construction thus gives us an element $E$ of $K_0(C^*_r(V, F))$ whereas the second gives $E$ as an element of $KK_0(\Lambda, \Sigma)$, Kasparov's bivariant $K$-Theory. (Recall that the 0th Kasparov's bivariant $K$-group in this case consists of stable isomorphism classes of $(\Lambda, \Sigma)$-bimodules). We shall use this action $h$ to define a left action of $C^*_c(\Gamma, \Omega^{1/2})$ on $E$ by:

$$(f\xi)(\gamma) = \int_{\Gamma^y} f(\gamma_1)h(\gamma_1)\xi(\gamma_1^{-1})$$

$\forall f \in C^*_c(\Gamma, \Omega^{1/2}), \xi \in E$

One then can prove that for any $f \in C^*_c(\Gamma, \Omega^{1/2})$ the above formula defines an endomorphism $\lambda(f)$ of the $C^*$-module $E$ whose adjoint $\lambda(f)^*$ is given by
\((\lambda(f)^*\xi)(\gamma) = \int_{\Gamma^w} f^#(\gamma_1)h(\gamma_1)\xi(\gamma_1^{-1}\gamma)\)

where

\(f^#(\gamma) := \tilde{f}((\gamma^{-1})\Delta(\gamma))\)

and

\(\Delta(\gamma) = (h(\gamma)^{-1})^t h(\gamma)^{-1} \in \text{End}(t_{C^r}(r(\gamma)))\)

This shows that unless the metric on \(t\) is \(\Gamma\)-invariant, the representation \(\lambda\) is not a \(*\)-representation, the subtle difference between \(\lambda(f)^*\) and \(\lambda(f^*)\) being measured by \(\Delta\). In particular \(\lambda\) is not in general bounded for the \(C^*\)-algebra norms on both \(\text{End}_{C^*_r(V,F)}E\) and \(C^*_r(V,F) \supset C^\infty_c(\Gamma, \Omega^{1/2})\). However \(\lambda\) is a closable homomorphism of \(C^*\)-algebras, namely, the closure of the graph of \(\lambda\) is the graph of a densely defined homomorphism. Then with the graph norm

\[\|x\|_\lambda = \|x\| + \|\lambda(x)\|\]

the domain \(\Sigma\) of the closure \(\tilde{\lambda}\) of \(\lambda\) is a Banach algebra which is dense in the \(C^*\)-algebra \(\Lambda = C^*_r(V,F)\). The \(C^*\)-module \(E\) is then a \((\Lambda, \Sigma)\)-bimodule. This particular module \(E\) we constructed here will be the one of the two main ingredients which define the invariant we want and we shall denote it \(E(F)\).

0.3.5 Cyclic classes associated to foliations (transverse fundamental cyclic cocycle)

We begin with some definitions from cyclic homology:

**Definition 1:**

1. A cycle of dimension \(n\) is a triple \((\Omega, d, f)\) where \(\Omega = \bigoplus_{j=0}^n \Omega^j\) is a graded algebra over \(\mathbb{C}\), \(d\) is a differential on \(\Omega\)'s and \(f : \Omega^n \to C\) is a closed graded trace on \(\Omega\).

2. Let \(A\) be an algebra over \(\mathbb{C}\). Then a cycle over \(A\) is given by a cycle \((\Omega, d, f)\) and a homomorphism \(\rho : A \to \Omega^0\).
A cycle over \( A \) of dimension \( n \) is essentially determined by its \textit{character} which is the following \((n+1)\)-linear functional on \( A \):

\[
\tau(a^0, ..., a^n) = \int \rho(a^0)d(\rho(a^1))d(\rho(a^2))...d(\rho(a^n))
\]

\( \forall a^j \in A \)

One can then prove that this is a \textit{cyclic cocycle} of \( A \), namely it defines a cohomology class in the cyclic homology of \( A \) and that the above is a necessary and sufficient statement.

We shall now describe the transverse fundamental class associated to foliations. There is a general construction for arbitrary foliations which is quite involving since one has to \textit{complete} the graded algebra. This is so because the transverse bundle of the foliation may not be integrable and in this case derivation along transverse directions will not be a differential.

We, however, are primarily interested in our two special kinds of foliations, the vertical and the horizontal foliation of a principal bundle. These foliations are transverse, both are integrable so derivatives are differentials and hence one does not have to complete the graded algebras. We refer to [27] for the general construction. Here we shall only describe the classes which are associated to our two foliations:

The vertical and the horizontal (or flat) foliations of the total space of our principal bundle \( P \) will be denoted \((P_V)\) and \((P_H)\) respectively. One then has that there is a natural cycle for the algebra of each foliation, namely:

**Vertical Foliation**, cycle denoted \([P_V]\):

The natural cycle canonically associated to the algebra \( C^\infty_c(P \times G, \Omega^{1/2}_1) \) of the vertical foliation consists of:

1. The graded algebra \( C^\infty_c(P \times G, \Omega^{1/2}_1 \otimes r^*(\wedge P^*_H)) \) where \( P_H \) is the \textit{horizontal} subbundle (i.e. the transverse bundle to the vertical foliation).

2. The differential \( d = d_V + d_H + \theta \) where

\[
d_H : C^\infty(P, \wedge^r P^*_V \otimes \wedge^s P^*_H) \rightarrow C^\infty(P, \wedge^{r+1} P^*_V \otimes \wedge^{s+1} P^*_H)
\]

\[
d_V : C^\infty(P, \wedge^r P^*_V \otimes \wedge^s P^*_H) \rightarrow C^\infty(P, \wedge^{r+1} P^*_V \otimes \wedge^s P^*_H)
\]

\( \theta \) means contraction with the section \( \theta \in C^\infty(P, P_V \otimes \wedge^2 P^*_H) \), where \( \theta \) is defined by:
\[ \theta(p_H(X), p_H(Y)) = p_V([X, Y]) \] for any pair of horizontal vector fields \( X, Y \in C^\infty(P, P_H) \) and \( (p_H, p_V) \) is the isomorphism \( TP \to P_H \oplus P_V \) given by \( P_H \).

3. The trace is defined via

\[ \tau(w) = \int_{\Gamma(0)} w \]

where \( \Gamma \) is the graph of the vertical foliation \( \Gamma = P \times G \).

Similarly one defines a fundamental class for the horizontal foliation.

One then can define the character of these cycles—essentially the trace—which is a class in the cyclic homology of the appropriate algebra for each foliation [22].

Note:
Since now we have a cyclic homology class, say \( \phi \) of the algebra of the foliation, say \( \Lambda \), we automatically have a map

\[ K_i(\Lambda) \to \mathbb{C} \]

given by pairing it with K-group elements \( (i = 0, 1 \text{ above}) \) to get index theorems for leafwise elliptic operators. Let us mention here that the analytic Index of an operator elliptic along the leaves of an arbitrary foliation say \( (V, F) \), is an element of \( K_0(C^*(V, F)) \), being in fact a generalisation of the index of families of elliptic operators considered by Atiyah and Singer. (In the Atiyah-Singer case of families of elliptic operators one is dealing with the foliation induced by the fibration, which is the commutative geometry case). The operator itself which is elliptic along the leaves of the foliation is an element of \( KK(C^*(V, F), C(V)) \).

0.4 Invariant for the nlσm

The final step then is to make use of the general formula for pairings between K-homology and cyclic homology. In more concrete terms, one has:

**Definition:**
Let \( A \) be an algebra. Then the following equality defines a bilinear pairing between K-theory and cyclic homology:
1. Even case: $K_0(A)$ and $HC^{ev}(A)$:

$\langle [e], [\phi] \rangle := (m!)^{-1}(\phi \# Tr)(e, ..., e)$

for $e \in K_0(A)$ using the idempotents’ description and $\phi \in HC^{2m}(A)$ and where $\#$ is the cup product in cyclic homology (see for instance [23] for the precise definition).

2. Odd case: $K_1(A)$ and $HC^{odd}(A)$:

$\langle [u], [\phi] \rangle = \frac{1}{\sqrt{2i}} 2^{-n} \Gamma \left( \frac{n}{2} + 1 \right)^{-1}(\phi \# Tr) (u^{-1} - 1, u - 1, u^{-1} - 1, ..., u - 1)$

This is an important point because by pairing the $C^*$-module $E$ we constructed previously naturally associated to the foliation considered with the cyclic cocycle naturally associated to the foliation, we get an invariant for arbitrary foliations. In particular if we apply this to the horizontal foliation, we get a complex number which is an invariant for the nσm. Namely one has:

$\langle [E(P_H)], [P_H] \rangle = (m!)^{-1}((P_H) \# Tr)(E(P_H), ..., E(P_H)) \in \mathbb{C}$

In more concrete terms, assuming that $E(P_H) \in M_k(C^*(P_H))$ for some $k$ (where $C^*(P_H)$ is the corresponding $C^*$-algebra to the horizontal foliation) and $[P_H] \in Z^q(C^*(P_H))$ where $Z$ denotes cyclic cocycles and $q$ is the codimension of the horizontal foliation, then $(P_H) \# Tr \in Z^q(M_k(C^*(P_H)))$ is defined by

$((P_H) \# Tr)(a^0 \otimes m^0, ..., a^q \otimes m^q) = (P_H)(a^0, ..., a^q) Tr(m^0 ... m^q)$

for any $a^i \in C^*(P_H), m^i \in M_k(\mathbb{C}), i = 1, ..., q$.

Note: Let us mention that the odd case formula is related to the $\eta$ invariant for leafwise elliptic operators, see [31], [32] which in turn is related to global anomalies and to the Freedman-Townsend invariance (cf [33], [38], [25], [41]).
0.5 An example: principal fibre bundles

In order to get some more insight to this pairing we shall try to calculate it for the case of principal bundles (vertical foliation) which is the simplest example.

We begin by describing the graph in detail: the set $\Gamma$ in this case is the manifold $P \times G$, the distinguished subset $\Gamma^{(0)} = P \times \{e\}$ and denoting the action (on the right) of $g \in G$ on $p \in P$ simply by $(p, g) \mapsto pg$, one has that the range and source maps are respectively $r(p, g) = p$ and $s(p, g) = pg$, the inverse $(p, g)^{-1} = (pg, g^{-1})$ and the law of composition is $(p_1, g_1) \circ (p_2, g_2) = (p_1, g_1 g_2)$ if $p_1 g_1 = p_2$. Obviously the set $\Gamma^{(2)} = P \times G \times G$.

Moreover we recall that the $C^*$-algebra for the vertical foliation is strongly Morita equivalent to $C(M)$,

We now make use of two important facts:

1. $K_0(C(M)) = K^0(M)$, namely Serre-Swan theorem

and

2. $H^*_{cont}(C(M)) = H_*(M)$

where on the RHS we have the ordinary homology of $M$ (with complex coefficients) and by definition for the LHS we have $H^*(C(M)) := \lim_{\to}(HC^n(C(M)), S)$ (see [23] for explanations of the notation), $HC^*$ denotes cyclic homology and ”cont” means restriction to continuous linear functionals.

The first fact says that for commutative $C^*$-algebras one gets Atiyah’s topological K-theory for the underlying space (described in terms of stable isomorphism classes of complex vector bundles over the space considered) and the second says that in the commutative case again cyclic homology is ”roughly speaking” the ordinary homology of the underlying space (and thus we see that non-commutative geometry reduces to ordinary geometry in the commutative case).

Since we have these two results in our disposal, we shall try to reduce the whole discussion in terms of bundles and ordinary homology theory because this is more comprehensible.

In order to describe the pairing then we need the transverse fundamental cyclic cocycle: we shall give a simple dimensional argument here; the exact computations are rather too technical to be presented in greater detail. The
cyclic cocycle we will get from the vertical foliation will be of dimension equal

to the codimension of the vertical foliation which is equal to the dimension

do the base space of our bundle. Moreover as we mentioned above, in this

case the $C^*$-algebra of this foliation is SME to the algebra of functions on

the base $C(M)$. This is a commutative $C^*$-algebra whose cyclic homology is

more or less the de Rham cohomology of the base space. Hence it is not too

hard to suspect that we get a top homology class, which in fact turns out to

be the fundamental class of our base space $[M]$.

For the module denoted $E$ above in this case one uses the following fact:
it is a consequence of Serre-Swan theorem mentioned above that the link

between topological K-theory and K-theory of commutative $C^*$-algebras is

that given a complex vector bundle over $M$ (thus a topological K-class),
one considers the corresponding $C(M)$-module of smooth sections of the

given complex vector bundle. Thus in this case the module $E$ we get is

$C_c^\infty(P \times G, \Omega^{1/2} \otimes r^*(t_C))$, hence we can recover the corresponding complex

vector bundle over $M$ as follows:

If we denote by $(P, \pi, G, M)$ our original principal bundle and we consider

the vertical foliation, then its normal bundle $t$ would be $\pi^*(TM)$, where $TM$
is the tangent bundle of $M$. We prefer the topological K-theory description

which in this case is rather easy to read: the bundle associated to this $C(M)$
module $E$ is:

$$
\begin{array}{ccc}
\Omega^{1/2} \otimes pr_1^*\pi^*(TM) & \longrightarrow & \pi^*(TM) \longrightarrow TM \\
\downarrow & & \downarrow \\
P \times G & \longrightarrow & P \longrightarrow M
\end{array}
$$

(4)

where the fibre of the line bundle $\Omega^{1/2}$ is the linear space of maps $\rho :$

$\wedge^{\dim G} T_\gamma(G) \otimes \wedge^{\dim G} T_\gamma(G) \rightarrow \mathbb{C}$ satisfying the well-known property for $1/2$-
densities. Hence in this case the result will be the number we get if we take
the bundle $\Omega^{1/2} \otimes pr_1^*\pi^*(TM)$ seen as a bundle over $M$, then apply
the ordinary Chern character to it and integrate over $M$. The result will be a
combination of the Pontryagin class of $TM$ and the second Chern class of $P$
(recall that we assumed $M$ to be 4-dim and $P$ is an SU(2) bundle) which is
something expected.

There are some subtleties though: we have the bundle $\Omega^{1/2} \otimes pr_1^*\pi^*(TM)$
over the graph which is $P \times G$. We want to see this as a bundle over $M$.  

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We consider firstly the factor $pr_i^*\pi^*(TM)$. This is indeed a bundle over $M$ with fibre $G \times G \times \mathbb{R}^{\dim M}$ but this is neither a vector nor a principal bundle and in order to talk about characteristic classes one actually needs the one or the other. In order not to change the topology then, which is what we are mainly interested in, we can consider the vector bundle $TM \otimes adP$ instead, where $adP$ is the adjoint bundle to $P$. To study formally the classes of $TM \otimes adP$ is an exercise in mathematics (we forget the pull-backs since they can be treated easily). The point is that we shall get a combination of the Pontryagin classes of $TM$ (or Chern if we complexify) and of Chern classes of $P$. The later is known since the bundle is given whereas the former can be computed from topological information of $M$ itself. For example for simply connected closed 4-manifolds one has from the Hirzebruch signature formula that (see [17]):

\[ p_1 = 3\tau = 3(b^+ - b^-) \]

where $\tau$ is the signature.

As about the other factor, the $1/2$-densities of the graph, seen as a rank 1 real bundle over the graph $P \times G$, is rather dull. It will be determined by $\omega_1$, the first Stiefel-Whitney class: either trivial ($\omega_1 = 0$) or it is non-orientable ($\omega_1 = 1$). \textit{(Note:} half densities over a complex manifold say $N$ with $\dim N = k$ are slightly more complicated: in this case its class will be $\frac{1}{2}c_1(\wedge^k T^*)$, where $\wedge^k T^*$ is the canonical bundle, so it will correspond to $spin^c$ structures on $N$). But we still have the same problem that over $M$ it is neither a vector nor a principal bundle. This can be overcome as before; the point is that since it is a dull bundle over $P \times G$, the projection does not change anything, so as a bundle over $M$ it will be determined by the topology of $P$, hence we also have Chern classes of $P$.

What we gave above was a qualitative description and the lesson was that the invariant will be some combination of Chern numbers of $P$ and the Pontryagin number of the tangent bundle $TM$ of $M$. The key point is that $TM$ appears because it is the transverse bundle of the vertical foliation. We expect then that characteristic classes of the transverse bundle should be important in general.

What about the flat foliation then, which is the case which is related to $n\sigma m$ in general and to M-theory in particular in physics? This is a purely nc case. Computations are much harder and a picture involving bundles is impossible since we do not have Serre-Swan theorem. Moreover cyclic
homology of nc algebras has no relation to the usual topology. We can still say something though: first of all, since we have a flat principal bundle, all its characteristic classes vanish, so we get nothing from them. Since the base space $M$ is not simply connected, gauge inequivalent classes of flat connections are characterised by their holonomy. So we expect the holonomy to play some role.

Moreover, from the vertical foliation case we saw that the normal bundle of the foliation also plays a vital role.

Note: It is not always true that in the commutative case cyclic homology identifies well with ordinary homology, in fact this is always essentially true only for $H_0$. There may be complications. However this point will not be treated in this article with greater detail. See [24].

0.6 Relation to Physics

There are three cases in physics where this invariant may play some role:

1. $Nl\sigma m$.

As it is well-known, $\sigma$ models classically describe harmonic maps between two Riemannian manifolds (target and source spaces). From the general remarks we made in the preceding section, we said that we expect the invariant to include the holonomy of the flat connection (namely $\pi_1$ of the source space) plus the topology of the space of leaves (target space). Hence this invariant should contain information about the topology of both manifolds involved in $\sigma$ models. Characteristic classes of foliations may provide the way to calculate the invariant (analogue of Chern-Weil theory, see [35]).

Moreover, in the flat foliation case the invariant describes the topological charge of the M-theory Lagrangian density suggested in [37].

Another application is the following (we thank Dr S. T. Tsou for pointing this out to us): we know from Polyakov ([18]) that Yang-Mills theories can be formulated as $n\sigma m$ on the loop space. This point of view very recently exhibited some very nice dualities of the Standard Model, see [44]. Hence the invariant, for the flat foliation case (since $n\sigma m$ can be thought of as a flat bundle with structure group the isometries of the target space), maybe of some relevance also for Yang-Mills theories, the setting however will involve loop spaces now! We do not know exactly what its physical significance would be. Moreover doing K-theory on loop spaces (infinite dimensional manifolds)
is considerably harder. There are however some path integral techniques.
(see again [44] and references therein).

2. Instantons with non simply connected boundary.

Following largely the case of instantons we suggest that this invariant is related to interpolation between gauge inequivalent vacua which exist due to the non simply connectedness of the space considered. Clearly there is extra degeneracy of the vacuum coming from the fact the our space is not simply connected. This degeneracy is of different origin than that of instantons since as it is well-known for ordinary instantons the degeneracy comes from the different topologies of the bundle considered. In more concrete terms we suppose that this invariant will be relevant in the following case: let us assume that we try to follow the discussion in the BPST famous paper on instantons [36]; if we assume that we have a space whose boundary is not just $S^3$ as in that case but a 3-manifold which has a non-trivial $\pi_1$. In this case we want the potential to become flat (pure gauge) on the boundary. However if the boundary is a 3-manifold with a non trivial fundamental group, then flat connections are not unique (up to gauge equivalence of course). We know more specifically that gauge equivalent classes of flat connections are in 1-1 correspondence with conjugate classes of representations of the fundamental group onto the structure group considered. Thus in this case the flat connection we choose will not be unique. This extra degeneracy of the vacuum comes from the different possible choices of flat connection, which is something noticed for the first time. The invariant is related to interpolation between these extra vacua.

We expect then some relation with the so called ALE gravitational instantons which are important both in quantum gravity and in gauge theory [39], [40].

3. Gravity, Non-Commutative Topological Quantum Field Theories (nc-TQFT for brevity).

In ordinary Yang-Mills theory, gauge transformations are described as automorphisms of the bundle (namely fibre preserving maps) which induce the identity on the base space (cf for example [26]). Sometimes these are called strong bundle automorphisms. If one wants to generalise this picture and attempts to include the symmetry of general relativity, namely local diffeomorphisms of the base space, then there is a problem because there are local diffeomorphisms of the base space which can not be induced by bundle automorphisms (cf [16]). In simple words: there are ”more” local dif-
feomorphisms of the base space than bundle automorphisms. The way that theoretical physicists usually try to go around this problem is - to begin with, supersymmetry and finally, - supergravity. The origins of supersymmetry are actually quantum mechanical (multiplets with same number of bosons and fermions plus symmetry between particles of different spin which should exist, if all interactions are eventually unified since gauge particles have spin-1 whereas the graviton is supposed to have spin-2. Another point of view is to examine the largest possible symmetry of the S-matrix elements in the framework of relativistic quantum field theory on Minkowski space). To make the long story short, based on the Coleman-Mandula theorem (which is responsible for introducing anti-commuting coordinates), what one actually does (following the superspace formalism for N=1 supersymmetry coming from observation that Minkowski space is actually Poincare/Lorentz) is to enlarge the base manifold (assumed to be spacetime) by adding some fermionic dimensions (non-commuting coordinates), thus obtaining another space, the so-called superspace. This superspace, in an analogous fashion, can be seen as the quotient space superPoincare/Lorentz. Supersymmetric Yang-Mills theories are principal G-bundles over superspace with G some compact and connected Lie group (usually SU(N)) whereas supergravity can be seen as a principal G-bundle over superspace where G is the superPoincare group (generalisation of Einstein-Cartan theory). However the so-called Noether technique which makes local a rigid supersymmetry and which is used mainly to construct supersymmetric interacting Lagrangians actually suggests that supersymmetric Yang-Mills theories can be equivalently be seen as principal G-bundles over ordinary spacetime with G some Super Lie Group.

Letting alone some severe criticism of supersymmetric theories (e.g. positive metric assumption), especially when the discussion comes to supergravity (the most important experimental problem of supersymmetric theories is the fact that none of the superpartners of particles has ever been observed, the way phenomenologists try to overcome this problem is to assume spontaneously breaking of supersymmetries; two of the main theoretical problems are: the aspect of supergravity as a local gauge theory which is not completely mathematically justified, for example in N=8 D=4 supergravity theory coming from D=11 N=1 supergravity which is supposed to be the best candidate for unification and according to recent progress one of the two low energy limits of M-theory, local diffeomorphisms are supposed to come from gauging the group O(8), an assumption which is based on the observation
that ordinary gravity comes from gauging the Poincare group, something which is wrong because of the existence of the "shouldering" form on arbitrary curved manifolds; the second important problem is that most of the extended supersymmetric and supergravity theories are actually up to now formulated only "on-shell", namely they are essentially classical theories - for N=1 supergravity though there is another problem, one has more than one "off-shell" formulations, this in fact has now an explanation from the recently observed string/5-brane duality in D=10 (old brane-scan); - trying to be fair, we must mention that the good features of such theories are that they offer probably the only up to now known hope for unification plus the fact that they give "less divergent" theories, something essential for perturbative quantum fields theories), – we would like to propose here another approach; our approach is more in the spirit of non-perturbative quantum field theories, in fact topological quantum field theories: instead of enlarging the base manifold by considering anti-commuting coordinates, we chose to relax the fibre preserving condition, meaning that now we allow "bundle" maps which are not fibre-preserving; in such a case, fibres may be "mixed up", for example they may be "tilted" or "broken". The resulting structure after applying these more general transformations to our original bundle may no longer be a fibre bundle, but it will still be a foliation. In this case however what we will get as the quotient space will not necessarily be the manifold we had originally in our bundle construction (supposed to be space-time) but another space of leaves with the same dimension and maybe very different topology. In this case the dimension of the leaves is kept fixed, equall to the dimension of the Lie algebra considered; had we changed that, the dimension of the space of leaves would have changed accordingly. This picture is quite close to the picture that string theorists patronise, namely that space-time is not fixed but it emerges as a ground state from some dynamical process.

In fact there is a deep result due to Thurston which for a given manifold M, say, it relates the group of local diffeomorphisms of M with the group of foliations of M [3]. In particular Thurston proves that one has an isomorphism in cohomology after some shift in the degrees between the classifying space of local diffeomorphisms and the classifying space of foliations of a closed manifold M. If we take M to be the total space of a principal bundle P over spacetime, then obviously local diffeomorphisms of the base space are included to local diffeomorphisms of the total space which in turn are very closely related to the group of foliations. Hence at least in principle looking
at the group of foliations of the total space of a principal bundle provides a framework which is rich enough in order to incorporate local diffeomorphisms of the base space, something we need in order to relate general relativity with Yang-Mills theory symmetries and this framework is mathematically rigorous.

The group then of all foliations of the total space with fixed codimension is huge. It definitely contains all foliations which are ”regular” enough in order to get manifolds diffeomorphic to our original one. Yet foliations can be really nasty: in this case the quotient space may not be a manifold at all but a ”quantum” topological space. All these cases need to be studied. For the moment we know that whenever the foliations have a corresponding $C^*$ algebra which is SME to a commutative one, then the space of leaves will be a compact Hausdorff topological space of the same dimension. If the $C^*$-algebras of two foliations can be related with a *-preserving homomorphism, then the corresponding quotient spaces will be homoeomorphic. What is the appropriate condition on the $C^*$ algebras in order to get diffeomorphic manifolds, we do not know (this point is of particular interest in 4-dim due to the existence of the so-called ”exotic structures” for 4-manifolds). The main point here is that we can ”control” how much non-commutativity we want in the $C^*$-algebra and then see what this means topologically. At this point we would like to recall that mathematically, going from classical physics to quantum is going from commuting algebras to non-commuting ones. The essence of Planck’s constant then is that it tells us ”how much” non-commutativity we want. Moreover there is the fundamental theorem for $C^*$-algebras representations, namely that for each $C^*$-algebra (commutative or not), there exists a Hilbert space whose space of bounded operators is actually ”the same” as the original $C^*$-algebra. Hence for each foliation there exists a corresponding $C^*$-algebra (commutative or not), a corresponding topological space (space of leaves which may be a manifold or a quantum topological space respectively) and finally, a Hilbert space as a representation space!

Let us now turn to something related to the above but a little more concrete: for the moment let us consider the case where the dimensions of the leaves and of the space of leaves are kept fixed. This situation has some similarities with quantum gravity seen as a TQFT (in fact we generalise that picture and we present a way to consider unified theories-namely gravity and Yang-Mills theories-as Non-Commutative Topological Quantum Field The-
ories). In [39] it was argued that TQFT may provide a framework which is rich enough for the development of a quantum theory of gravity. In that aspect, space-time was treated as an unquantized object whereas the metric was quantum mechanical. The idea in TQFT framework is to find an invariant $Z(M)$ for a topological space $M$ and then one seeks for a Lagrangian density whose partition function yields the invariant $Z(M)$, see [34]. One has to be a little more careful though in order for the Atiyah’s axioms for TQFT to be satisfied [14]. In quantum mechanics one usually has a space of quantum states associated to a given system. Often this space of states refers to a particular instant of time, which can be represented in a 4-dimensional world by a space-like hypersurface. In TQFT this vector space appears as part of the definition, when the space-time $M$ has a boundary, i.e. a space of dimension 1 less. In more concrete terms, to a $(d-1)$-dim space $\Sigma$ we associate a vector space $V(\Sigma)$ and to each $d$-dim space $M$ with $\partial M = \Sigma$ we associate a vector $Z(M)$, the partition function of the space $M$. This point can be generalised, in fact $\Sigma$ can be any embedded submanifold of $M$ with dimension 1 less. One interpretation of these conditions is that $\Sigma$ represents the ”present instant” of time and that the vectors in $V(\Sigma)$ which are determined by various choices of observables represent a memory of past facts. The primary problem nonetheless is the construction of invariants for spaces and the state spaces and partition functions for spaces with boundary are usually obtained as a by-product. Since by our proposal above one can end up with quantum topological spaces as spaces of leaves of foliations, one can call this theory non-commutative topological quantum field theory and we believe that this can provide a framework for quantum unified theories (including Yang-Mills and gravity).

The picture we have then is the following: we start with a $G$-bundle $P$ over a 4-manifold $M$. From symmetry considerations, namely we want to include local diffeomorphisms of the base space and relate them to bundle automorphisms (hence relating general relativity to gauge theory), we end up to consider all dim$G$-dim foliations of $P$. Automatically the 4-dim space which is the space of leaves is somehow ”quantized”, namely it is forced to have one of the leaf topologies. This is a difference with TQFT as explained in [39] where the metric was quantised but space-time was unquantized (needless to say, in such a case as ours, the metric is quantised too automatically). Moreover, for each foliation we have a quotient space of leaves and hence an invariant $Z(M)$ which is a complex number. The boundary of course can
be added with its vector space attached to it, one however has to examine what happens to it as foliations vary. The Lagrangian density whose partition function is this invariant for foliations is an open question. It should be related to characteristic classes for foliations. A good indication for that is the fact that the $\Gamma_q$ functor of $q$-dim Haefliger structures or $\Gamma_q$ structures as they are known in topology (and hence foliations which is an example of a $\Gamma_q$ structure, where $q$ is the codimension of the foliation) is representable, see [13]. So for the moment we do not have a "full" specific ncTQFT since we have the invariant but not the appropriate Lagrangian density. We presented though a generalisation of TQFT for non-commutative cases.

Another possible application might be the ability to construct deformed Yang-Mills theories, see [43]. In that paper, some new compactifications of the IKKT matrix theory on non-commutative tori were introduced which, in a certain sense, could be realised as deformed Yang-Mills theories. Clearly in this case our invariant will be the "instanton number" of these deformed Yang-Mills theories.

This picture also suggests that the above described non-commutative topological quantum field theories can be seen as emerging from M-theory compactified down to some non-commutative spaces (tori or other).

## 0.7 M-Theory

In this section we shall present an application to M-Theory. Since it is more extensive, we give it separately.

We know that M-Theory consists of membranes and 5-branes living on an 11-manifold ([11], [12]) and it is non perturbative. This theory has a very intriguing feature: we can only extract information about it from its limiting theories, namely either from D=11 N=1 supergravity or from superstrings in D=10. This is so because this theory is genuinely non perturbative for a reason which lies in the heart of manifold topology:

Let us recall that in string theory, the path integral involves summation over all topologically distinct diagrams (same for point particles of course). Strings are 1-branes hence in time they swep out a 2-manifold. At the tree level then we need all topologically distinct simply connected 2-manifolds (actually there is only one, as topology tells us) and for loop corrections, topology again says that topologically distinct non simply connected 2-manifolds
are classified by their genus, so we sum up over all Riemann surfaces with different genus.

It is clear then that for a perturbative quantum field theory involving p-branes we have to sum upon all topologically distinct (p+1)-dim manifolds: simply connected ones for tree level and non simply connected ones for loop corrections. Thus we must know before hand the topological classification of manifolds in the dimension of interest. That is the main problem of manifold topology in mathematics.

But now we face a deep and intractable problem: geometry tells us, essentially via a no-go theorem which is due to Whitehead from late '40's, that: "we cannot classify non simply connected manifolds with dimension greater or equal to 4"! Hence for p-branes with p greater or equal to 3, all we can do via perturbative methods is up to tree level!

What happens for 3-manifolds then (hence for membranes)? The answer from mathematics is that we do not know if all 3-manifolds can be classified! So even for 2-branes it is still unclear whether perturbative methods work (up to all levels of perturbation theory)!

The outlet from this situation that we propose here is not merely to look only at non perturbative aspects of these theories (i.e. the soliton part of the theory) and then apply S-duality, as was done up to now, but to abandon perturbative methods completely from the very beginning. There is only one way known up to now which can achieve this "radical" solution to our problem: formulate the theory as a Topological Quantum Field Theory and hence get rid of all perturbations once and for all.

Let us explain how this can be achieved.

Our approach is based on one physical "principle":

A theory containing p-branes should be formulated on an m-dim manifold which admits $\Gamma_q$-structures, where $q = m - p - 1$.

N.B.

Although we used in our physical principle $\Gamma_q$-structures which are more general than foliations, we shall use both these terms meaning essentially the same structure. The interested reader may refer to [2] for example to see the
precise definitions which are quite complicated. The key point however is that the difference between $\Gamma_q$-structures (or Haefliger structures as they are most commonly known in topology) and codim-$q$ foliations is essentially the difference between transverse and normal. This does not affect any of what we have to say, since Bott-Haefliger theory of characteristic classes is formulated for the most general case, namely $\Gamma$-structures. We would also like to mention the relation between $\Gamma$-structures and $\Omega$-spectra which is currently an active field in topology.

(For D-branes we need a variant of the above principle, namely we need what are called plane foliations but we shall not elaborate on this point here).

One way of thinking about this principle is that it is analogous to the “past histories” approach of quantum mechanics. Clearly in quantum level one should integrate over all foliations of a given codim.

A piece of warning here: this principle does not imply that all physical process between branes are described by foliations. Although the group of foliations is huge, in fact comparable in size with the group of local diffeomorphisms [3], and foliations can be really “very nasty”, we would not like to make such a strong statement. What is definitely true though is that some physical process are indeed described by foliations, hence at least this condition must be satisfied because of them.

Note:
Before going on further, we would like to make one crucial remark: this principle puts severe restrictions on the topology that the underlying manifold may have, in case of M-Theory this is an 11-manifold. It is also very important if the manifold is open or closed. This may be of some help, as we hope, for the compactification problem of string theory or even M-Theory, namely how we go from $D=10$ (or $D=11$) to $D=4$ which is our intuitive dimension of spacetime. We shall address this question in the next section. The final comment is this: this principle puts absolutely no restriction to the usual quantum field theory for point particles in $D=4$, e.g. electroweak theory or QCD. This is so because in this case spacetime is just $\mathbb{R}^4$ which is non compact and we have 0-branes (point particles) and consequently 1-dim foliations for which the integrability condition is trivially satisfied (essentially
this is due to a deep result of Gromov for foliations on open manifolds, which states that all open manifolds admit codim 1 foliations; in striking contrast, closed manifolds admit codim 1 foliations iff their Euler characteristic is zero, see for example in [2], [4] or references therein).

If we believe this principle, then the story goes on as follows: we are on an 11-manifold, call it M for brevity and we want to describe a theory containing 5-branes for example (and get membranes from S-duality). Then M should admit 6-dim foliations or equivalently codim 5 foliations. We know from Haefliger that the $\Gamma_q$-functor, namely the functor of codim q Haefliger structures and in particular codim q foliations, is representable. Practically this means that we can have an analogue of Chern-Weil theory which characterises foliations of M up to homotopy using cohomology classes of M. (One brief comment for foliations: one way of describing Haefliger structures more generally is to say that they generalise fibre bundles in exactly the same way that fibre bundles generalise Cartesian product. This observation is also important when mentioning gerbes later on).

In fact it is proved that the correct cohomology to classify Haefliger structures up to homotopy (and hence foliations which constitute a particular example of Haefliger structures) is the Gelfand-Fuchs cohomology. This is a result of Bott and Haefliger, essentially generalising an earlier result due to Godbillon and Vey which was dealing only with codim 1 foliations, [5].

Now we have a happy coincidence: the Bott-Haefliger class for a codim 5 foliation (which, recall, is what we want for 5-branes on an 11-manifold) is exactly an 11-form, something that fits well with using it as a Lagrangian density!

The construction for arbitrary codim q foliations goes as follows: let $F$ be a codim q foliation on an m-manifold M and suppose its normal bundle $\nu(F)$ is orientable. Then $F$ is defined by a global decomposable q-form $\Omega$. Let $\{(U_i, X_i)\}_{i \in I}$ be a locally finite cover of distinguished coordinate charts on M with a smooth partition of unity $\{\rho_i\}$. Then set

$$\Omega = \sum_{i \in I} \rho_i dx_i^{m-q+1} \wedge \ldots \wedge dx_i^m$$

Since $\Omega$ is integrable,
\[ d\Omega = \theta \wedge \Omega, \]  

where \( \theta \) some 1-form on M. The \((2q+1)\)-form

\[ \gamma = \theta \wedge (d\theta)^q, \]  

is closed and its de Rham cohomology class is independent of all choices involved in defining it, depending only on homotopy type of \( F \). That’s the class we want.

Clearly for our case we are on an 11-manifold dealing with 5-branes, hence 6-dim foliations, hence codim 5 and thus the class \( \gamma \) is an 11-form.

This construction can be generalised to arbitrary \( \Gamma^r_q \)-structures as a mixed de Rham-Cech cohomology class and thus gives an element in \( H^{2q+1}(B\Gamma^r_q; \mathbb{R}) \), where \( B\Gamma^r_q \) is the classifying space for \( \Gamma^r_q \)-structures. Note that in fact the BHGV class is a cobordism invariant of codim q foliations of compact \((2q+1)\)-dim manifolds. This construction gives one computable characteristic class for foliations. Optimally we would like a generalisation of the Chern-Weil construction for \( GL_q \). That is we would like an abstract GDA with the property that for any codim q foliation \( F \) on a manifold \( M \) there is a GDA homomorphism into the de Rham algebra on \( M \), defined in terms of \( F \) such that the induced map on cohomology factors through a universal map into \( H^*(B\Gamma^r_q; \mathbb{R}) \). This algebra is nothing more than the Gelfand-Fuchs Lie coalgebra of formal vector fields in one variable.

More concretely, let \( \Gamma \) be a transitive Lie-pseudogroup acting on \( \mathbb{R}^n \) and let \( a(\Gamma) \) denote the Lie algebra of formal \( \Gamma \) vector fields associated to \( \Gamma \). Here a vector field defined on on \( U \subset \mathbb{R}^n \) is called a \( \Gamma \) vector field if the local 1-parameter group which it engenders is \( \Gamma \) and \( a(\Gamma) \) is defined as the inverse limit

\[ a(\Gamma) = \lim_{\leftarrow} a^k(\Gamma) \]

of the \( k \)-jets at 0 of \( \Gamma \) vector fields. In the pseudogroup \( \Gamma \) let \( \Gamma_0 \) be the set of elements of \( \Gamma \) keeping 0 fixed and set \( \Gamma_0^k \) equal to the \( k \)-jets of elements in \( \Gamma_0 \).

Then the \( \Gamma_0^k \) form an inverse system of Lie groups and we can find a subgroup \( K \subset \lim_{\leftarrow} \Gamma_0^k \) whose projection on every \( \Gamma_0^k \) is a maximal compact
subgroup for $k > 0$. This follows from the fact that the kernel of the projection $\Gamma_{0}^{k+1} \to \Gamma_{0}^{k}$ is a vector space for $k > 0$. The subgroup $K$ is unique up to conjugation and its Lie algebra $k$ can be identified with a subalgebra of $a(\Gamma)$.

For our purposes we need the cohomology of basic elements rel $K$ in $a(\Gamma)$, namely $H_{\cdot}(a(\Gamma); K)$ which is defined as follows: Let $A\{a^{k}(\Gamma)\}$ denote the algebra of multilinear alternating forms on $a^{k}(\Gamma)$ and let $A\{a(\Gamma)\}$ be the direct limit of the $A\{a^{k}(\Gamma)\}$. The bracket in $a(\Gamma)$ induces a differential on $A\{a(\Gamma)\}$ and we write $H\{a(\Gamma)\}$ for the resulting cohomology group. The relative group $H^{\ast}(a(\Gamma); K)$ is now defined as the cohomology of the subcomplex of $A\{a(\Gamma)\}$ consisting of elements which are invariant under the natural action of $K$ and annihilated by all inner products with elements of $k$. Then the result is:

\textit{Let $F$ be a $\Gamma$-foliation on $M$. There is an algebra homomorphism}

$$
\phi : H\{a(\Gamma); K\} \to H(M; \mathbb{R})
$$

\textit{which is a natural transformation on the category $C(\Gamma)$}.

The construction of $\phi$ is as follows:

Let $P^{k}(\Gamma)$ be the differential bundle of $k$-jets at the origin of elements of $\Gamma$. It is a principal $\Gamma_{0}^{k}$-bundle. On the other hand $\Gamma$ acts transitively on the left on $P^{k}(\Gamma)$. Denote by $A(P^{\infty}(\Gamma))$ the direct limit of the algebras $A(P^{k}(\Gamma))$ of differential forms on $P^{k}(\Gamma)$. The invariant forms wrt the action of $\Gamma$ constitute a differential subalgebra denoted $A_{\Gamma}$. One can then prove that it is actually isomorphic to $A(a(\Gamma))$.

Now let $F$ be a foliation on $M$ and let $P^{k}(F)$ be the differentiable bundle over $M$ whose fibre at every point say $x \in M$ is the space of $k$-jets at this point of local projections that vanish on $x$. This is a $\Gamma_{0}^{k}$-principal bundle. Its restriction is isomorphic to the inverse image of the bundle $P^{k}(\Gamma)$, hence the differential algebra of $\Gamma$-invariant forms on $P^{k}(\Gamma)$ is mapped in the algebra $A(P^{k}(F))$ of differential forms on $P^{k}(F)$. If we denote by $A(P^{\infty}(F))$ the direct limit of $A(P^{k}(F))$ we get an injective homomorphism $\phi$ of $A(a(\Gamma))$ in $A(P^{\infty}(F))$ commuting with the differential.

This homomorphism is compatible with the action of $K$, hence induces a homomorphism on the subalgebra of K-basic elements. But the algebra $A(P^{k}(F); K)$ of K-basic elements in $A(P^{k}(F))$ is isomorphic to the algebra of differential forms on $P^{k}(F)/K$ which is a bundle over $M$ with contractible fibre $\Gamma_{0}^{k}/K$. Hence $H(A(P^{k}(F); K))$ is isomorphic via the de Rham theorem.
to $H(M; \mathbb{R})$. The homomorphism $\phi$ is therefore obtained as the composition

$$H(a(\Gamma); K) \to H(A(P^\infty(F); K)) = H(M; \mathbb{R})$$

But we think that is enough with abstract nonsense formalism. Let us make our discussion more down to earth:

Consider the GDA (over $\mathbb{R}$)

$$WO_q = \wedge(u_1, u_3, ..., u_{2(q/2) - 1}) \otimes P_q(c_1, ..., c_q)$$

with $du_i = c_i$ for odd $i$ and $dc_i = 0$ for all $i$ and

$$W_q = \wedge(u_1, u_2, ..., u_q) \otimes P_q(c_1, ..., c_q)$$

with $du_i = c_i$ and $dc_i = 0$ for $i=1,...,q$ where $deg u_i = 2i - 1$, $deg c_i = 2i$ and $\wedge$ denotes exterior algebra, $P_q$ denotes the polynomial algebra in the $c_i$’s mod elements of total degree greater than $2q$. The cohomology of $W_q$ is the Gelfand Fuchs cohomology of the Lie algebra of formal vector fields in $q$ variables. We note that the ring structure at the cohomology level is trivial, that is all cup products are zero. Then the main result is that there are homomorphisms

$$\phi : H^*(WO_q) \to H^*(B\Gamma^r_q; \mathbb{R})$$

$$\bar{\phi} : H^*(W_q) \to H^*(\bar{B}\Gamma^r_q; \mathbb{R})$$

for $r \geq 2$ with the following property ($\bar{B}\Gamma^r_q$ denotes the classifying space for framed foliations): If $F$ is a codim $q$ $C^r$ foliation of a manifold $M$, there is a GDA homomorphism

$$\phi_F : WO_q \to \wedge^*(M)$$

into the de Rham algebra on $M$, defined in terms of the differential geometry of $F$ and unique up to chain homotopy, such that on cohomology we have $\phi_F = f^* \circ \phi$, where $f : M \to B\Gamma^r_q$ classifies $F$. If the normal bundle of $F$ is trivial, there is a homomorphism

$$\bar{\phi}_F : W_q \to \wedge^*(M)$$
with analogous properties.

Combining this result with the fact that $B\Gamma_q$ is contractible, we deduce that a foliation is essentially determined by the structure of its normal bundle; the Chern classes of the normal bundle are contained in the image of the map $\phi$ above but we have additional non trivial classes in the case of foliations (which are rather difficult to find though), one of which is this BHGV class which we constructed explicitly and it is the class we use as a Lagrangian density which is purely topological since its degree fits nicely for describing 5-branes.

There is an alternative approach due to Simons [9] which avoids passing to the normal bundle using circle coefficients. What he actually does is to associate to a principal bundle with connection a family of characteristic homomorphisms from the integral cycles on a manifold to $S^1$ and then defining an extension denoted $K^{2k}_q$ of $H^{2k}(BGL_q; \mathbb{Z})$. This approach is related to gerbes. A gerbe over a manifold is a construction which locally looks like the Cartesian product of the manifold with a line bundle. Clearly it is a special case of foliations (remember our previous comment on foliations). However this approach actually suggests that they might be equivalent, if the approach of Bott-Haefliger is equivalent to that of Simons, something which is not known.

Now the conjecture is that the partition function of this Lagrangian is related to the invariant introduced in [38].

In order to establish relation with physics, we must make some identifications. The 1-form $\theta$ appearing in the Lagrangian has no direct physical meaning. In physics it is assumed that a 5-brane gives rise to a 6-form gauge field denoted $A_6$ whose field strength is simply

$$dA_6 = F_7$$  \hspace{1cm} (7)

The only way we can explain geometrically this is that this 6-form is the Poincare dual of the 6-chain that the 5-brane sweeps out as it moves in time.

We know that since we have S-duality between membranes and 5-branes, in an obvious notation one has

$$F_7 = *F_4$$  \hspace{1cm} (8)
which is the S-duality relation, where

\[ F_4 = dA_3 \]  \hspace{1cm} (9)

Observe now that the starting point for 5-brane theory is \( A_6 \) where the starting point to construct the BHGV class was the 5-form \( \Omega \). How are they related?

There are three obvious possibilities:

I. \( d\Omega = A_6 \) That would imply that \( A_6 \) is pure gauge.

II. \( dF_4 = \Omega \) This is trivial because it implies \( d\Omega = 0 \), hence \( d\Omega = \theta \wedge \Omega = 0 \).

III. The only remaining possibility is

\[ *A_6 = \Omega \]  \hspace{1cm} (10)

We call this "reality condition". So now in principle we can substitute equations (10) and (5) into (6) and get an expression for the Lagrangian which involves the gauge field \( A_6 \).

The Euler-Lagrange equations which are actually analogous to D=11 N=1 supergravity Euler-Lagrange equations (see equation (12) below) read:

\[ d * d\theta + \frac{1}{5} (d\theta)^5 = 0 \]  \hspace{1cm} (11)

The on-shell relation with D=11 N=1 supergravity is established as follows: recall that the bosonic sector of this supergravity theory is

\[ \int F_4 \wedge F_4 \wedge A_3 \]

where \( F_4 = dA_3 \) with Euler-Lagrange equations

\[ d * F_4 + \frac{1}{2} F_4 \wedge F_4 = 0 \]  \hspace{1cm} (12)

Constraining \( A_3 \) via (12), by (9), (8), (7), (10) and (5) we get a constraint for \( \theta \) which can be added to the class \( \gamma \) as a Lagrange multiplier.

In order to calculate the partition function, some additional difficulties may arise because we do not know what notion of equivalence between foliations is the appropriate one for physics in order to fix the gauge and add Faddeev-Popov terms as constraints to kill-off the gauge freedom. There
are actually four different notions of equivalence for foliations: conjugation, homotopy, integrable homotopy and foliated cobordism.

In principle, one must end up with an equivalent theory starting with membranes (that's due to S-duality), provided of course a suitable class was found. Clearly the BHGV class for a membrane would be a 17-form.

The final comment refers to [45]. In that article it was conjectured that the quantum mechanics of branes could be described as a matrix model. As it is well-known matrix models use point particle degrees of freedom. This is rather intriguing since we are talking about M-theory which contains various p-branes. In our approach though we propose a Lagrangian density which has as fundamental object a mysterious 1-form which, if seen as a gauge potential, that would imply the existence of some yet unknown underlying point particle!

0.7.1 Plane fields

We now pass on to the second question raised in this application, namely the restrictions on the topology of the underlying manifold of a theory containing p-branes via our physical principle.

It is clear from the definition that the existence of a foliation of certain dim, say d (or equivalently codim q=n-d) on an n-manifold (closed) depends:

a.) On the existence of a dim d subbundle of the tangent bundle
b.) On this d-dim subbundle being integrable.

The second question has been answered almost completely by Bott and in a more general framework by Thurston. Bott's result dictates that for a codim q subbundle of the tangent bundle to be integrable, the ring of Pontrjagin classes of the subbundle with degree > 2q must be zero. There is a secondary obstruction due to Shulman involving certain Massey triple products but we shall not elaborate on this. However Bott's result suggests nothing for question a.) above. Let us also mention that this result of Bott can be deduced by another theorem due to Thurston which states that the classifying space $B\Gamma_q^{\infty}$ of smooth codim q framed foliations is (q+1)-connected.

On the contrary, Thurston’s result reduces the existence of codim q > 1 foliations (at least up to homotopy) to the existence of q-plane fields. This is a deep question in differential topology, related to the problem of classification of closed manifolds according to their rank.
Now the problem of existence of q-plane fields has been answered only for some cases for spheres \( S^n \) for various values of \( n,q \) [6]. In particular we know everything for spheres of dimension 10 and less. We should however mention a theorem due to Winkelnkemper [7] which is quite general in nature and talks about simply connected compact manifolds of dim \( n \) greater than 5. If \( n \) is not 0 mod 4 then it admits a so-called *Alexander decomposition* which under special assumptions can give a particular kind of a codim 1 foliation with \( S^1 \) as space of leaves and a surjection from the manifold to \( S^1 \). If \( n \) is 0 mod 4 then the manifold admits an Alexander decomposition iff its signature is zero.

Let us return to string theory now: String theory works in \( D=10 \) and in this case we have the old brane-scan suggesting the string/5-brane duality. The new brane-scan contains all p-branes for \( p \leq 6 \) and some D-7 and 8-branes are thought to exist. However topology says that for a sphere in dim 10 we can have only dim 0 and dim 10 plane fields (in fact this is true for all even dim spheres), hence by Thurston only dim 0 and dim 10 foliations and then our physical principle suggests that \( S^{10} \) is ruled out as a possible underlying topological space for string theory.

What about M-Theory in \( D=11 \) then?

For the case of \( S^{11} \) then it is known that \( S^{11} \) admits a 3-plane field, hence by our physical principle a theory containing membranes can be formulated on \( S^{11} \). For \( S^{11} \) nothing is known for the existence of q-plane fields for q greater than 3. But now we apply S-duality between membranes/5-branes and conjecture that:

\[ S^{11} \text{ should admit 5-plane fields.}\]

Let us close with two final remarks:

1. There is extensive work in foliations with numerous results which actually insert many extra parameters into their study, for example metric aspects, existence of foliations with compact leaves (all or at least one or exactly one), with leaves diffeomorphic to \( \mathbb{R}^n \) for some \( n \) etc. We do not have a clear picture for the moment concerning imposing these in physics. Let us only mention one particularly strong result due to Wall generalising a result of Reeb [8]: if a closed \( n \)-manifold admits a codim 1 foliation whose
leaves are homeomorphic to $\mathbb{R}^{n-1}$, then by Thurston we know that its Euler characteristic must vanish, but in fact we have more: it has to be the $n$-torus!

The interesting point however is that although all these extended objects theories in physics are expressed as $\sigma$ models [37], hence they involve metrics on the manifold (target space) and on the worldvolumes ie on the leaves, in our approach the metric is only used in the reality condition (10) which makes connection with physical fields (that is some metric on the target space) where at the same time we do not use any metric on the source space (worldvolumes-leaves of the foliation).

2. In [37] another Lagrangian density was proposed. It is different from the one described here but they are related in an analogous way to the relation between the Polyakov and Nambu-Goto (in fact Dirac [10]) actions for the free bosonic string: extended objects basically immitate string theory and we have two formalisms: the $\sigma$ model one which is the Lagrangian exhibited in [37] using Polyakov’s picture of $\sigma$ models as flat principal bundles with structure group the isometries of the metric on the target space [18]; yet we also have the *embedded surface* picture which is the Dirac (Nambu-Goto) action and whose analogue is described in this work.

In the light of a very recent work [34], we can also make some further comments: the first is that the Moyal algebra used in order to discuss noncommutative solitons is actually Morita equivalent to the usual commutative one. This fact can be further verified from the explicit construction of an algebra homomorphism between noncommutative and ordinary Yang-Mills fields based on Gelfand’s theorem. Truly noncommutative situations appear when discussing noncommutative tori.

The next comment refers to the last section of that paper: we already know that strings in a constant magnetic field can be described from that Moyal algebra-like spacetime structure and they also discuss what may happen to 5-branes in M-theory. Our point of view coincides with theirs in the following way: we here propose that this is indeed the case for 5-branes with a C-field turned on, namely that this situation can be described by 6-dim foliations which rather correspond to a ”free” theory of branes but on a ”noncommutative” topological space, which is actually, in our case, the space of leaves of the corresponding foliation.
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