SINGULAR INTEGRALS AND COMMUTATORS IN
GENERALIZED MORREY SPACES

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Abstract. The purpose of this paper is to study singular integrals whose kernels $k(x; \xi)$ are variable, i.e. they depend on some parameter $x \in \mathbb{R}^n$ and in $\xi \in \mathbb{R}^n \setminus \{0\}$ satisfy mixed homogeneity condition of the form $k(x; \mu^{\alpha_1}\xi_1, \ldots, \mu^{\alpha_n}\xi_n) = \mu^{-\sum_{i=1}^n \alpha_i} k(x; \xi)$ with positive real numbers $\alpha_i \geq 1$ and $\mu > 0$. The continuity of these operators in $L^p(\mathbb{R}^n)$ is well studied by Fabes and Rivi`ere. Our goal is to extend their results in generalized Morrey spaces with a weight satisfying suitable dabling and integral conditions. A special attention is paid also of the commutators of the kernel with functions of bounded and vanishing mean oscillation.

1. Introduction

We consider the following integral operators

$$Kf(x) := P.V. \int_{\mathbb{R}^n} k(x; x-y)f(y)dy$$

(1.1)

and its commutators with essentially bounded functions

$$C[a,k]f(x) := P.V. \int_{\mathbb{R}^n} k(x; x-y)(a(y)-a(x))f(y)dy$$

$$= K(af)(x) - a(x)Kf(x).$$

(1.2)

The generating kernel $k(x; \xi) : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is variable, i.e. it depends on some parameter $x$ and possesses “good” properties with respect to the second variable $\xi$. This class of kernels is firstly studied by Fabes and Rivi`ere in [FR]. They generalize the classical kernels of Calderón and Zygmund $k(\xi) = \Omega(\xi)/|\xi|^n$ having homogeneity of degree $-n$ and those studied by Jones in [BJ] and satisfying homogeneity property of the form

$k(\lambda \xi, \lambda^m \tau) = \lambda^{-n-m}k(\xi, \tau), \xi \in \mathbb{R}^n, \tau \in (0, \infty), m \geq 1$. Introducing a new metric $\rho$, Fabes and Rivi`ere study (1.1) in $L^p(\mathbb{R}^n)$, where $\mathbb{R}^n$ is endowed with the topology induced by $\rho$ and defined by ellipsoids. Thus, the unite sphere with respect to $\rho$ coincides with the unite sphere $\Sigma_n$ with respect to the
Euclidean metric. This fact allows to impose on the kernel $k$ the Calderón-Zygmund conditions on the unite sphere, in spite of the lack of "symmetry" of $k$ with respect to the variables $\xi_i$, $i = 1, \ldots, n$. Let we note, that the standard parabolic metric $\tilde{\rho} = \sup\{|x|, \sqrt{t}\}$, $x \in \mathbb{R}^n$, $t \in (0, \infty)$, for instance, does not permit to define the mentioned above conditions on kernels having homogeneity of parabolic type. Using the Fourier transform in $L^2(\mathbb{R}^n)$ and the Marcinkiewicz interpolation theorem, Fabes and Rivièrè obtained that the integral operators (1.1) are continuous in $L^p(\mathbb{R}^n)$, $p \in (1, \infty)$.

In the present work we study the continuity of these operators in the generalized Morrey spaces $L^{p,\omega}(\mathbb{R}^n)$ where the function $\omega$ satisfies suitable conditions.

A special attention is paid also to the commutators $C[a, k]$ of the kernel $k$ and functions $a$ having bounded or vanishing mean oscillation. In this case we impose of the results of Coifman-Rochberg-Weiss (CRW) and Bramanti-Cerutti (BC) treating continuity in $L^p(\mathbb{R}^n)$ of commutators with constant kernels.

The technique we used is the one elaborated by Calderón and Zygmund and consisting of expansion of the kernel into spherical harmonics and restricting the considerations on integral operators with constant kernels.

2. Definitions and preliminary results

Let $\alpha_1, \ldots, \alpha_n$ be real numbers, $\alpha_i \geq 1$ and define $\alpha = \sum_{i=1}^n \alpha_i$. Following Fabes and Rivièrè ([FR]), the function $F(x, \rho) = \sum_{i=1}^n x_i^2 \rho^{-2\alpha_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x, \rho) = 1$ is unique solvable in $\rho(x)$. It is a simple matter to check that $\rho(x - y)$ defines a distance between any two points $x, y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric $\rho$ results a homogeneous metric space ([FR, Remark 1], [BC]). The balls with respect to $\rho(x)$, centered at the origin and of radius $r$ are simply the ellipsoids

$$\mathcal{E}_r(0) = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{r^{2\alpha_1}} + \ldots + \frac{x_n^2}{r^{2\alpha_n}} < 1 \right\}$$
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with Lebesgue measure \(|E_r| = C(n)r^\alpha\). It is easy to see that the unite sphere with respect to this metric coincides with the unite sphere \(\Sigma_n\) with respect to the Euclidean one.

**Definition 2.1.** The function \(k(x; \xi) : \mathbb{R}^n \times \{\mathbb{R}^n \setminus \{0\}\} \to \mathbb{R}\) is called a 
variable kernel with mixed homogeneity if:

1. for every fixed \(x\) the function \(k(x; \cdot)\) is a constant kernel satisfying
   - (i) \(k(x; \cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})\);
   - (ii) \(k(x; \mu^\alpha \xi_1, \ldots, \mu^\alpha \xi_n) = \mu^{-\alpha} k(x; \xi)\), \(\forall \mu > 0, \alpha_i \geq 1, \alpha = \sum_{i=1}^n \alpha_i\);
   - (iii) \(\int_{\Sigma_n} k(x; \xi) d\sigma_\xi = 0\) and \(\int_{\Sigma_n} |k(x; \xi)| d\sigma_\xi < \infty\);
2. for every multiindex \(\beta : \sup_{\xi \in \Sigma_n} |D_\xi^\beta k(x; \xi)| \leq C(\beta)\) independently of \(x\).

Let us note that in the special case \(\alpha_i = 1\) and thus \(\alpha = n\), Definition 2.1 gives rise to the classical Calderón-Zygmund kernels. One more example is when \(\alpha_1 = \ldots = \alpha_{n-1} = 1, \alpha_n = \bar{\alpha} \geq 1\). In this case we obtain the kernels studied by Jones in [BJ] and discussed in [FR].

For the sake of completeness we recall the definitions and some properties of the spaces we are going to use.

**Definition 2.2.** For \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\) and any ellipsoid \(E \subset \mathbb{R}^n\) centered at \(x \in \mathbb{R}^n\) and of radius \(r > 0\) set

\[
\gamma_f(R) := \sup_{r \leq R} \frac{1}{|E|} \int_E |f(y) - f_E| dy \quad \text{for every } R > 0, \tag{2.1}
\]

where \(f_E = \frac{1}{|E|} \int_E f(y) dy\) and \(|E|\) is the Lebesgue measure of \(E\), comparable to \(r^\alpha\). Then:

1. \(f \in BMO\) (bounded mean oscillation) if \(\|f\|_* := \sup_R \gamma_f(R) < \infty\).
   The quantity \(\|f\|_*\) is a norm in \(BMO\) modulo constant function under which \(BMO\) results a Banach space (see [JN]);
2. \(f \in VMO\) (vanishing mean oscillation) with \(VMO\)-modulus \(\gamma_f(R)\) if \(f\) belongs to \(BMO\) and \(\gamma_f(R) \to 0\) as \(R \to 0\) (see [S]).

For a bounded domain \(\Omega \subset \mathbb{R}^n\), we define \(BMO(\Omega)\) and \(VMO(\Omega)\) taking \(f \in L^1(\Omega)\) and \(E \cap \Omega\) instead of \(E\) in (2.1).

Let \(\omega : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+\) and for any ellipsoid \(E\) we write \(\omega(x, r) =: \omega(E)\).
Definition 2.3. A function \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \), \( p \in (1, \infty) \) belongs to the generalized Morrey space \( L^{p,\omega}(\mathbb{R}^n) \) if the following norm is finite

\[
\|f\|_{p,\omega} := \left( \sup_\mathcal{E} \frac{1}{\omega(x,|\mathcal{E}|)} \int_{\mathcal{E}} |f(y)|^p dy \right)^{1/p}.
\]  

(2.2)

The space \( L^{p,\omega}(\Omega) \) and the norm \( \|f\|_{p,\omega;\Omega} \) are defined by taking \( f \in L^p(\Omega) \) and \( E \cap \Omega \) instead of \( E \) in (2.2).

For \( \omega(x, r) = 1 \) we get the Lebesgue space \( L^p(\mathbb{R}^n) \) and for \( \omega(x, r) = r^\lambda \), \( \lambda \in (0, \alpha) \), \( L^{p,\omega}(\mathbb{R}^n) \) coincides with the Morrey space \( L^{p,\lambda}(\mathbb{R}^n) \) when \( \mathbb{R}^n \) is endowed with the metric \( \rho \). However, there exist weight functions, as \( \omega(x, r) = r^\lambda \ln(r + 2) \), \( \lambda \in (0, \alpha) \) for which \( L^{p,\omega}(\mathbb{R}^n) \) does not coincide with any Morrey space.

For a given measurable function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) define the Hardy–Littlewood maximal operator \( Mf \) and the sharp maximal operator \( f^\# \) as

\[
Mf(x) := \sup_{x \in \mathcal{E}} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |f(y)| dy, \quad f^\#(x) := \sup_{x \in \mathcal{E}} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |f(y) - f_{\mathcal{E}}| dy
\]

almost everywhere in \( \mathbb{R}^n \) and the supremum is taken over all ellipsoids \( \mathcal{E} \) centered at \( x \). Define also the operator \( M_s f(x) := (M|f|^s(x))^{1/s} \) for \( s \geq 1 \).

The next results are weighted variants of the well-known maximal and sharp inequalities obtained in Lebesgue and Morrey spaces (see [St], [CF], [DPR]).

Lemma 2.4. (Maximal inequality) ([Na]) Assume that there are constants \( C_1 \) and \( C_2 \) such that for any \( x_0 \in \mathbb{R}^n \) and for any \( r > 0 \)

\[
r \leq t \leq 2r \implies C_1 \leq \frac{\omega(x_0, t)}{\omega(x_0, r)} \leq C_2,
\]

\[
\int_r^{\infty} \frac{\omega(x_0, t)}{t^{a+1}} dt \leq C \frac{\omega(x_0, r)}{r^a}.
\]

(2.3)

For \( 1 \leq s < p < \infty \), there is a constant \( C_{p,s} > 0 \) such that for \( f \in L^{p,\omega}(\mathbb{R}^n) \)

\[
\|M_s f\|_{p,\omega} \leq C_{p,s} \|f\|_{p,\omega}.
\]

Lemma 2.5. (Sharp inequality) Let \( 0 < \sigma \leq 1 \) and \( \mathcal{E} \) be an ellipsoid centered at \( x_0 \in \mathbb{R}^n \) of radius \( r \). Suppose that \( \omega(x_0, r) \) satisfies (2.3) and

\[
\int_r^{\infty} \frac{\omega(x_0, t)}{t^{\sigma a+1}} dt \leq C \frac{\omega(x_0, r)}{r^{\sigma a}}.
\]
Then for $p \in (1, \infty)$ and $f \in L^{p,\omega}(\mathbb{R}^n)$ there exists a constant $C$ independent of $f$ such that
\[
\|f\|_{p,\omega} \leq C\|f^\#\|_{p,\omega}.
\]

**Proof.** Let $\chi_\mathcal{E}$ be the characteristic function of the ellipsoid and denote by $2\mathcal{E}$ an ellipsoid centered at $x_0$ and of radius $2r$. It is easy to verify that $M\chi_\mathcal{E}(x) \leq r^\alpha/(\rho(x-x_0) - r)^\alpha \leq 1$, for all $x \in \mathbb{R}^n$. Further, for any $x \in 2^{k+1}\mathcal{E} \setminus 2^k\mathcal{E}$, $k = 1, 2, \ldots$ the maximal function of $\chi_\mathcal{E}$ could be estimated by $r^\alpha/(2^{k+1}r - r)^\alpha \leq M\chi_\mathcal{E}(x) \leq r^\alpha/(2^kr - r)^\alpha$ which gives a reason to compare $M\chi_\mathcal{E}(x)$ with $2^{-k\alpha}$ for any $x$ as above. From the properties of the maximal function, that is $|f| \leq Mf$ and $Mf \leq f^\#$ (see [GR, p. 410]) follows

\[
J = \int_\mathcal{E} |f(y)|^p dy = \int_{\mathbb{R}^n} |f(y)|^p \chi_\mathcal{E}(y) dy
\]
\[
\leq \int_{\mathbb{R}^n} |Mf(y)|^p (M\chi_\mathcal{E}(y))^\sigma dy \leq C \int_{\mathbb{R}^n} |f^\#(y)|^p (M\chi_\mathcal{E}(y))^\sigma dy
\]
\[
\leq C \left\{ \int_{2\mathcal{E}} |f^\#(y)|^p dy
\right. \\
\left. + \sum_{k=1}^{\infty} \int_{2^{k+1}\mathcal{E} \setminus 2^k\mathcal{E}} |f^\#(y)|^p \left( \frac{r}{\rho(y-x_0) - r} \right)^{\sigma \alpha} dy \right\}
\]
\[
\leq C \left\{ \frac{\omega(2\mathcal{E})}{\omega(2\mathcal{E})} \int_{2\mathcal{E}} |f^\#(y)|^p dy
\right. \\
\left. + r^{\sigma \alpha} \sum_{k=1}^{\infty} \frac{\omega(2^{k+1}\mathcal{E})}{(2^k r)^{\sigma \alpha}} \frac{1}{\omega(2^{k+1}\mathcal{E})} \int_{2^{k+1}\mathcal{E}} |f^\#(y)|^p dy \right\}
\]
\[
\leq C r^{\sigma \alpha} \sum_{k=0}^{\infty} \frac{\omega(2^k \mathcal{E})}{(2^k r)^{\sigma \alpha}} \|f^\#\|_{p,\omega}.
\]

From the properties of the function $\omega(x_0, t)$ follows
\[
\frac{\omega(2^k \mathcal{E})}{(2^k r)^{\sigma \alpha}} \sim \int_{2^k r}^{2^{k+1} r} \frac{\omega(x_0, t)}{t^{\sigma \alpha + 1}} dt.
\]

Hence
\[
\int_{\mathcal{E}} |f(y)| dy \leq C r^{\sigma \alpha} \int_r^{\infty} \frac{\omega(x_0, t)}{t^{\sigma \alpha + 1}} dt \|f^\#\|_{p,\omega} \leq C \omega(x_0, r) \|f\|_{p,\omega}.
\]
Lemma 2.6. (John-Nirenberg type lemma) Let $a \in BMO$ and $p \in (1, \infty)$. Then for any ellipsoid $E$ holds
\[
\left( \frac{1}{|E|} \int_E |a(y) - a_E|^p \, dy \right)^{1/p} \leq C(p)\|a\|_*.
\]

One more background we need is that for spherical harmonics and their properties (see for instance [CZ], [FR], [CFL]). Recall that any homogeneous polynomial $P: \mathbb{R}^n \to \mathbb{R}$ of degree $m$ that satisfies $\Delta P(x) = 0$ is called an $n$-dimensional solid harmonic of degree $m$. Its restriction to the unit sphere $\Sigma_n$ will be called an $n$-dimensional spherical harmonic of degree $m$. Denote by $\Upsilon_m$ the space of all $n$-dimensional spherical harmonics of degree $m$. In general it results a finite-dimensional linear space with $g_m = \dim \Upsilon_m$ such that $g_0 = 1$, $g_1 = n$ and
\[
g_m = \left( \frac{m+n-1}{n-1} \right) - \left( \frac{m+n-3}{n-1} \right) \leq C(n)m^{n-2}, \quad m \geq 2. \tag{2.5}
\]
Further, let $\{Y_{sm}(x)\}_{s=1}^{g_m}$ be an orthonormal base of $\Upsilon_m$. Then $\{Y_{sm}\}_{s=1}^{g_m}$ is a complete orthonormal system in $L^2(\Sigma_n)$ and
\[
\sup_{x \in \Sigma_n} \left| D_2^{\beta} Y_{sm}(x) \right| \leq C(n)m^{|\beta|+(n-2)/2}, \quad m = 1, 2, \ldots. \tag{2.6}
\]
If, for instance, $\phi \in C^\infty(\Sigma_n)$ then $\sum_{s,m} b_{sm} Y_{sm}(x)$ is the Fourier series expansion of $\phi(x)$ with respect to $\{Y_{sm}\}_{s=1}^{g_m \infty}$ (the $\sum_{s,m}$ substitutes $\sum_{m=0}^{\infty} \sum_{s=1}^{g_m}$) and
\[
b_{sm} = \int_{\Sigma_n} \phi(y) Y_{sm}(y) \, d\sigma, \quad |b_{sm}| \leq C(n, l)m^{-2l} \sup_{y \in \Sigma_n} \left| D_2^{\beta} \phi(y) \right| \tag{2.7}
\]
for any integer $l$. In particular, the expansion of $\phi$ into spherical harmonics converges uniformly to $\phi$. For the proof of the above results see [CZ].

3. Singular integral estimates

Let $k(x; \xi)$ be a kernel in the sense of Definition 2.1. In order to ensure the existence of the operators (1.1) and (1.2) in $L^p(\mathbb{R}^n)$ we restrict our considerations to functions $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ for which the norm (2.2) is finite. For the sake of convenience we still denote these spaces by $L^{p,\omega}(\mathbb{R}^n)$. Having in mind this we define the operators $K_\varepsilon f$ and $C_\varepsilon[a, k] f$ for
We are going to prove that \((2.4)\), by

\[
\forall \varepsilon > 0,
\]

Moreover, we shall show that the last ones are also continuous in \(L^p(\mathbb{R}^n)\) and this leads also to continuity in \(L^p(\mathbb{R}^n)\)-topology the singular integrals

\[
\mathcal{K}_\varepsilon f(x) := \int_{\rho(x-y) > \varepsilon} k(x; x-y) f(y) dy,
\]

\[
\mathcal{C}_\varepsilon[a, k] f(x) := \mathcal{K}_\varepsilon(a f)(x) - a(x) \mathcal{K}_\varepsilon f(x) = \int_{\rho(x-y) > \varepsilon} k(x; x-y)[a(y) - a(x)] f(y) dy.
\]

We are going to prove that \(\mathcal{K}_\varepsilon\) and \(\mathcal{C}_\varepsilon[a, k]\) are bounded and continuous from \(L^{p,\omega}(\mathbb{R}^n)\) into itself uniformly in \(\varepsilon\). This along with the properties of the kernel \(k(x; \xi)\) will enable to let \(\varepsilon \to 0\) obtaining as limits in the \(L^{p,\omega}(\mathbb{R}^n)\)-topology the singular integrals

\[
\mathcal{K} f(x) := P.V. \int_{\mathbb{R}^n} k(x; x-y) f(y) dy = \lim_{\varepsilon \to 0} \mathcal{K}_\varepsilon f(x)
\]

\[
\mathcal{C}[a, k] f(x) := P.V. \int_{\mathbb{R}^n} k(x; x-y)[a(y) - a(x)] f(y) dy = \lim_{\varepsilon \to 0} \mathcal{C}_\varepsilon[a, k] f(x).
\]

Moreover, we shall show that the last ones are also continuous in \(L^{p,\omega}(\mathbb{R}^n)\).

Let us note assuming \(f \in L^p(\mathbb{R}^n), p \in (1, \infty)\) Fabes-Rivière ([FR]) show that \(\mathcal{K} f\) exists in \(L^p(\mathbb{R}^n)\) for \(p \in (1, \infty)\) as a limit of \(\mathcal{K}_\varepsilon f\) when \(\varepsilon \to 0\) in the \(L^p\)-norm. Moreover, the operator \(\mathcal{K} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\) is continuous and this leads also to continuity in \(L^p(\mathbb{R}^n)\) of \(\mathcal{C}[a, k] f\) if \(a(x)\) is essentially bounded. As it concerns to the commutator we are going to derive a result similar to that of Coifman-Rochberg-Weiss ([CRW, Theorem 1]), which asserts: if \(\mathcal{K}\) is Calderón-Zygmund operator in \(L^p(\mathbb{R}^n)\), \(p \in (1, \infty)\) and \(a \in BMO\) than the commutator \(\mathcal{C}[a, \cdot]\) is a well defined linear continuous operator from \(L^p(\mathbb{R}^n)\) into itself. Later, this result has been extended by Bramanti-Cerutti ([BC]) in the framework of homogeneous spaces. Based on this background about Calderón-Zygmund operators, we are going to obtain continuity in \(L^{p,\omega}(\mathbb{R}^n)\) and boundedness in terms of \(\|a\|_\ast\) for the commutators (1.2) having kernel of more general type.

**Theorem 3.1.** Let \(k(x; \xi)\) be a variable kernel of mixed homogeneity, \(f \in L^{p,\omega}(\mathbb{R}^n)\), \(p \in (1, \infty)\), \(\omega\) satisfies (2.3) and (2.4), and \(a \in BMO\). Then there exist the integrals \(\mathcal{K} f, \mathcal{C}[a, k] f \in L^{p,\omega}(\mathbb{R}^n)\) as limits of \(\mathcal{K}_\varepsilon f\) and \(\mathcal{C}_\varepsilon[a, k] f\) when \(\varepsilon \to 0\) with respect to the \(L^{p,\omega}(\mathbb{R}^n)\)-norm. The operators \(\mathcal{K}\) and \(\mathcal{C}[a, k]\)
are bounded from $L^{p,\omega}(\mathbb{R}^n)$ into itself and
\[
\|Kf\|_{p,\omega} \leq C\|f\|_{p,\omega}, \quad \|C[a, k]f\|_{p,\omega} \leq C\|a\|_\infty\|f\|_{p,\omega}
\]
where the constants depend on $n, p, \alpha$ and $k$ through the constant $C(\beta)$.

**Proof.** Let $x, y \in \mathbb{R}^n$ and $\mathcal{H} = y/\rho(y) \in \Sigma_n$. From the properties of the kernel with respect to the second variable and the completeness of $\{Y_{sm}(x)\}_{s, m}$ in $L^2(\Sigma_n)$ it follows
\[
k(x; x - y) = \rho(x - y)^{-\alpha}k(x; x - y) = \rho(x - y)^{-\alpha}\sum_{s, m} b_{sm}(x)Y_{sm}(x - y).
\]
This way, the Definition 2.1 ii) and (2.17) imply
\[
\|b_{sm}\|_\infty \leq C(n, l, k)m^{-2l}
\]
(3.1) for any integer $l > 1$. Replacing the kernel with its expansion, we get
\[
K_\varepsilon f(x) = \int_{\rho(x - y) > \varepsilon} \sum_{s, m} b_{sm}(x)\mathcal{H}_{sm}(x - y)f(y)dy,
\]
(3.2)
\[
C_\varepsilon[a, k]f(x) = \int_{\rho(x - y) > \varepsilon} \sum_{s, m} b_{sm}(x)\mathcal{H}_{sm}(x - y)[a(y) - a(x)]f(y)dy
\]
with $\mathcal{H}_{sm}(x - y)$ standing for $Y_{sm}(x - y)\rho(x - y)^{-\alpha}$. It is easy to check that $\mathcal{H}_{sm}(\cdot)$ is a constant kernel in the sense of Definition 2.1 i). Indeed, $i_a)$ and $i_b)$ are trivial while $i_c)$ follows from the fact that $Y_{sm}(x)$ is a harmonic homogeneous polynomial and the property of integral mean on sphere for harmonic functions (i.e. $Y_{sm}(0) = 0$). In order to get series expansions of $K_\varepsilon f$ and $C_\varepsilon[a, k]f$, we let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ to be such that $\rho(x - y) > \varepsilon$. Then (2.3), (2.4) and (3.1) yield
\[
\left| \sum_{m=1}^N \sum_{s=1}^{g_m} b_{sm}(x)\frac{Y_{sm}(x - y)}{\rho(x - y)^\alpha} f(y) \right| \leq C(n) \frac{|f(y)|}{\rho(x - y)^\alpha} \sum_{m=1}^\infty m^{n-2+(n-2)/2-2l}
\]
where $|f(\cdot)|\rho(\cdot)^{-\alpha} \in L^1(\mathbb{R}^n)$ for a.a. $x \in \mathbb{R}^n$ and the integer $l$ is preliminary chosen greater than $(3n - 4)/4$. Similar inequality holds also for the commutator $C_\varepsilon[a, k]f$. Thus, by the dominated convergence theorem
\[
K_\varepsilon f(x) = \sum_{s, m} b_{sm}(x)K_{sm, \varepsilon}(x),
\]
(3.3)
\[
C_\varepsilon[a, k]f(x) = \sum_{s, m} b_{sm}(x)C_{sm, \varepsilon}[a, k]f(x)
\]
with
\[ K_{sm,\varepsilon}(x) := \int_{\rho(x-y) > \varepsilon} H_{sm}(x-y)f(y)dy, \]
\[ C_{sm,\varepsilon}[a, k]f(x) := \int_{\rho(x-y) > \varepsilon} H_{sm}(x-y)[a(y) - a(x)]f(y)dy. \]

This way instead of the operators \( Kf \) and \( C[a, k]f \) we shall study the existence and boundedness in \( L^{p,\omega}(\mathbb{R}^n) \) of the singular integrals
\[ K_{sm}f(x) := P.V. \int_{\mathbb{R}^n} H_{sm}(x-y)f(y)dy, \]
\[ C_{sm}[a, k]f(x) := P.V. \int_{\mathbb{R}^n} H_{sm}(x-y)[a(y) - a(x)]f(y)dy \]
with constant kernels \( H_{sm}(\cdot) \). For what concern boundedness of \( K_{sm} \) in \( L^p(\mathbb{R}^n) \) we dispose of [FR, Theorem II.1] and this implies, through [BC, Theorem 2.5], boundedness in \( L^p(\mathbb{R}^n) \) of \( C_{sm}[a, k] \) as well. The cited results however require the kernel to have some “integral continuity”, called the Hörmander condition. It turns out that \( H_{sm}(\cdot) \) satisfies even stronger condition as shows the following lemma.

**Lemma 3.2. (Pointwise Hörmander condition)** Let \( \mathcal{E} \) and \( 2\mathcal{E} \) be ellipsoids centered at \( x_0 \) and of radius \( r \) and \( 2r \), respectively. Then
\[ |H_{sm}(x-y) - H_{sm}(x_0-y)| \leq C(n, \alpha)m^{n/2} \frac{\rho(x_0-x)}{\rho(x_0-y)^{\alpha+1}} \]
for each \( x \in \mathcal{E} \) and \( y \notin 2\mathcal{E} \).

**Proof.** We shall apply the mean value theorem to \( H_{sm} \) and therefore decay estimate for the gradient \( \nabla H_{sm} \) is needed. Let \( x \in \mathbb{R}^n \setminus \{0\} \) be an arbitrary point. The implicit function theorem applied to the equation \( F(x, \rho(x)) = 1 \) gives an expression for the gradient \( \nabla \rho(x) \) and straightforward calculations imply
\[
\frac{\partial H_{sm}}{\partial x_i}(x) = \frac{1}{\rho(x)^{\alpha+\alpha_i}} \left( -\alpha Y_{sm}(x) \frac{x_i}{\rho(x)^{\alpha_i}} \sum_{j=1}^{n} \alpha_j x_j^2 \rho(x)^{-2\alpha_j} + \frac{\partial Y_{sm}(x)}{\partial x_i}(x) \right) \\
- \sum_{k=1}^{n} \alpha_k \frac{\partial Y_{sm}(x)}{\partial x_k}(x) \frac{x_i x_k}{\rho(x)^{\alpha_i} \rho(x)^{\alpha_k} \sum_{j=1}^{n} \alpha_j x_j^2 \rho(x)^{-2\alpha_j}}. \]
Since \( \bar{x} \in \Sigma_n \) and taking into account (2.4), \( \frac{x_i}{\rho(x)^{\alpha_i}} \leq |\bar{x}| \leq 1 \) and \( \min \alpha_i \leq \sum_{j=1}^{n} \alpha_j x_j^2 / \rho(x)^{2\alpha_j} \leq \max \alpha_i \), we get
\[
\left| \frac{\partial H_{sm}}{\partial x_i}(x) \right| \leq C(n, \alpha) \frac{m^{n/2}}{\rho(x)^{\alpha+\alpha_i}} \quad \forall \, x \in \mathbb{R}^n \setminus \{0\}. \tag{3.5}
\]

Now, applying the mean value theorem to the left-hand side of (3.4) we get
\[
H_{sm}(x - y) - H_{sm}(x_0 - y) = \sum_{i=1}^{n} \frac{\partial H_{sm}}{\partial x_i}(x_0 - \xi)(x_0 - x)_i \tag{3.6}
\]
with \( \xi = y - t(x - x_0) \) and \( t \in (0, 1) \). Obviously \( \rho(y - \xi) = t \rho(x_0 - x) \leq r \) which along with \( y \notin 2\mathcal{E} \) gives that \( \xi \) does not belong to \( \mathcal{E} \) and \( \rho(x_0 - \xi) \geq \frac{1}{2} \rho(x_0 - y) \). Having in mind \( (x_0 - x)_i \leq \rho(x_0 - x)^{\alpha_i} \), (3.5) and (3.6) we obtain
\[
|H_{sm}(x - y) - H_{sm}(x_0 - y)| \leq C(n, \alpha) m^{n/2} \sum_{i=1}^{n} \frac{\rho(x_0 - x)^{\alpha_i}}{\rho(x_0 - \xi)^{\alpha+\alpha_i}}
\leq C(n, \alpha) m^{n/2} \frac{\rho(x_0 - x)}{\rho(x_0 - y)^{\alpha+1}} \sum_{i=1}^{n} \frac{\rho(x_0 - x)^{\alpha_i-1}}{\rho(x_0 - \xi)^{\alpha_i-1}}
\leq C(n, \alpha) m^{n/2} \frac{\rho(x_0 - x)}{\rho(x_0 - y)^{\alpha+1}}
\]
where we have used that \( \alpha_i \geq 1 \) and \( \rho(x_0 - x) < \frac{1}{2} \rho(x_0 - y) \leq \rho(x_0 - \xi) \) from which follows immediately the last sum is no greater than \( n \). \( \square \)

**Remark 3.3.** This result ensures the kernel \( H_{sm} \) satisfies the Hörmander integral condition (see [FR, (1.1)])
\[
\int_{\{y \in \mathbb{R}^n : \rho(y) \geq 4\rho(x)\}} |H_{sm}(y - x) - H_{sm}(y)| \, dy \leq C
\]
with a constant independent of \( x \).

In view of the cited above results there exist \( K_{sm} f, \, C_{sm}[a, k] f \in L^p(\mathbb{R}^n) \) such that
\[
\lim_{\varepsilon \to 0} \|K_{sm, \varepsilon} f - K_{sm} f\|_{L^p(\mathbb{R}^n)} = \lim_{\varepsilon \to 0} \|C_{sm, \varepsilon}[a, k] f - C_{sm}[a, k] f\|_{L^p(\mathbb{R}^n)} = 0.
\]

Our goal is to show that this convergence is fulfilled also with respect to the \( L^{p, \omega}(\mathbb{R}^n) \)-norm. The proof is broken up into several Lemmas.
Lemma 3.4. The singular integrals $K_{sm}f$ and their commutators $C_{sm}[a, k]f$ satisfy

$$(K_{sm}f)^2(x) \leq Cm^{n/2}(M(|f|^p)(x))^{1/p},$$

$$(C_{sm}[a, k]f)^2(x) \leq C\|a\|_\ast \left\{ M(|K_{sm}f|^p)(x) \right\}^{1/p} + m^{n/2}(M(|f|^p)(x))^{1/p},$$

where the constant depends on $n$, $p$ and $\alpha$ but not on $f$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $E$ for the ellipsoid $E$ centered at $x_0$ and of radius $r$. Let we consider the expression

$$I := \frac{1}{|E|} \int_E |K_{sm}f(y) - (K_{sm}f)_E| \, dy.$$

Adding and extracting $K_{sm,2r}f(x_0)$ to the function under the sign of the integral we obtain

$$I \leq \frac{2}{|E|} \int_E |K_{sm}f(y) - K_{sm,2r}f(x_0)| \, dy =: 2I(x_0,E).$$

Set $(2E)^c = \mathbb{R}^n \setminus 2E$ and write $f = f\chi_{2E} + f\chi_{(2E)^c} = f_1 + f_2$ with $\chi$ being the characteristic function of the respective set. Hence

$$I(x_0,E) \leq \frac{1}{|E|} \int_E |K_{sm}f_1(y)| \, dy + \frac{1}{|E|} \int_E |K_{sm}f_2(y) - K_{sm,2r}f(x_0)| \, dy =: I_1(x_0,E) + I_2(x_0,E).$$

From the boundedness of $K_{sm}$ in $L^p(\mathbb{R}^n)$ ([FR, Theorem II.1]) follows

$$I_1(x_0,E) \leq \frac{1}{|E|} \left( \int_E 1 \, dy \right)^{1/p'} \left( \int_E |K_{sm}f_1(y)|^p \, dy \right)^{1/p} = \frac{1}{|E|^{1/p}} \|K_{sm}f_1\|_p$$

$$ \leq \frac{C(p,\alpha)}{|E|^{1/p}} \|f_1\|_p \leq C(p,\alpha) \left( M(|f|^p)(x_0) \right)^{1/p}$$

with $1/p' + 1/p = 1$. About $I_2(x_0,E)$, we have
\[
I_2(x_0, \mathcal{E}) \leq \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \left( \int_{(2\mathcal{E}^c)} |\mathcal{H}_{sm}(y - \xi) - \mathcal{H}_{sm}(x_0 - \xi)| |f(\xi)| d\xi \right) dy \\
\leq C(n, \alpha)m^{n/2} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \left( \int_{(2\mathcal{E}^c)} \frac{\rho(x_0 - y)}{\rho(x_0 - \xi)^{\alpha + 1}} |f(\xi)| d\xi \right) dy \\
\leq C(n, \alpha)m^{n/2} \int_{\mathcal{E}} \left( \int_{(2\mathcal{E}^c)} \frac{|f(\xi)|}{\rho(x_0 - \xi)^{\alpha + 1}} d\xi \right) \\
\leq C(n, \alpha)m^{n/2} \sum_{k=1}^{\infty} \int_{2k+1\mathcal{E}^c \setminus 2k\mathcal{E}} \frac{1}{2k(\alpha + 1)^r} |\mathcal{H}_{sm}(y - \xi) - \mathcal{H}_{sm}(x_0 - \xi)| |f(\xi)| d\xi \\
\leq C(n, \alpha)m^{n/2} \left( M(|f|^p)(x_0) \right)^{1/p},
\]

after applying Lemma 3.2 for \( y \in \mathcal{E} \) and \( \xi \in (2\mathcal{E})^c \). Taking \( \sup_{\mathcal{E}} I(x_0, \mathcal{E}) \) and heaving in mind the arbitrariness of \( x_0 \), we obtain (3.7) for any \( x \in \mathbb{R}^n \).

To estimate the sharp function of the commutator we shall employ the idea of Stromberg (see [10]) which consists of expressing \( \mathcal{C}_{sm}[a, k]f \) as a sum of integral operators and estimating their sharp functions. Precisely,

\[
\mathcal{C}_{sm}[a, k]f(x) = \mathcal{K}_{sm}(a - a_\mathcal{E})f(x) - (a(x) - a_\mathcal{E})\mathcal{K}_{sm}f(x) \\
= \mathcal{K}_{sm}(a - a_\mathcal{E})f_1(x) + \mathcal{K}_{sm}(a - a_\mathcal{E})f_2(x) - (a(x) - a_\mathcal{E})\mathcal{K}_{sm}f(x) \\
=: J_1(x) + J_2(x) + J_3(x)
\]

where we have used the same truncation for the function \( f \) as in \( I(x_0, \mathcal{E}) \). Before proceed further, let us point out the obvious inequality

\[
|a_2\mathcal{E} - a_\mathcal{E}| \leq C(n, \alpha)\|a\|_* \quad \forall \ a \in BMO(\mathbb{R}^n) \tag{3.8}
\]

and its by-product

\[
|a_2k\mathcal{E} - a_\mathcal{E}| \leq C(n, \alpha)k\|a\|_* \tag{3.9}
\]

following from (3.8) by running induction. Now, for arbitrary \( p \in (1, \infty) \) and \( q \in (1, p) \), we have
Further, (3.8) and Lemma 2.6 applied to the second integral yield

$$G_1(x_0, \mathcal{E}) := \frac{1}{|\mathcal{E}|} \int_\mathcal{E} |J_1(x) - (J_1)\mathcal{E}| \, dx \leq \frac{2}{|\mathcal{E}|} \int_\mathcal{E} |\mathcal{K}_{sm}(a - a\mathcal{E})f_1(x)| \, dx$$

$$\leq \frac{2}{|\mathcal{E}|} \left( \int_\mathcal{E} |\mathcal{K}_{sm}(a - a\mathcal{E})f_1(x)|^q \, dx \right)^{1/q} \left( \int_\mathcal{E} 1 \, dx \right)^{1/q}$$

$$\leq \frac{C(q, \alpha)}{|\mathcal{E}|^{1/q}} \left( \int_{2\mathcal{E}} |(a(x) - a\mathcal{E})f_1(x)|^q \, dx \right)^{1/q}$$

$$\leq \frac{C(q, \alpha)}{|\mathcal{E}|^{1/q}} \left( \int_{2\mathcal{E}} |f(x)|^p \, dx \right)^{1/p} \left( \int_{2\mathcal{E}} |a(x) - a\mathcal{E}|^{pq/(p-q)} \, dx \right)^{(p-q)/pq}.$$  

Further, (3.8) and Lemma 2.6 applied to the second integral yield

$$\int_{2\mathcal{E}} |a(x) - a\mathcal{E}|^{pq/(p-q)} \, dx \leq C(p, q) \left( \int_{2\mathcal{E}} |a(x) - a_{2\mathcal{E}}|^{pq/(p-q)} \, dx \right. + \left. \int_{2\mathcal{E}} |a_{2\mathcal{E}} - a\mathcal{E}|^{pq/(p-q)} \, dx \right)$$

$$\leq C(p, q) \left( |2\mathcal{E}|^{1/|2\mathcal{E}|} \int_{2\mathcal{E}} |a(x) - a_{2\mathcal{E}}|^{pq/(p-q)} \, dx + |2\mathcal{E}|C(n, \alpha)\|a\|_{s}^{pq/(p-q)} \right)$$

$$\leq C(n, p, q, \alpha) |2\mathcal{E}|\|a\|_{s}^{pq/(p-q)}.$$  

Therefore,

$$G_1(x_0, \mathcal{E}) \leq C\|a\|_{s} \left( \frac{1}{|2\mathcal{E}|} \int_{2\mathcal{E}} |f(y)|^p \, dy \right)^{1/p} \leq C\|a\|_{s} \left( M(|f|^p)(x_0) \right)^{1/p}.$$

To estimate the sharp function of $J_2(x)$, we proceed analogously as we already did for $I_2(x_0, \mathcal{E})$. Precisely,

$$G_2(x_0, \mathcal{E}) := \frac{1}{|\mathcal{E}|} \int_\mathcal{E} |J_2(x) - (J_2)\mathcal{E}| \, dx \leq \frac{2}{|\mathcal{E}|} \int_\mathcal{E} |J_2(x) - J_2(x_0)| \, dx$$

and the integrand satisfies

$$|J_2(x) - J_2(x_0)| \leq \int_{(2\mathcal{E})^c} |\mathcal{H}_{sm}(x - y) - \mathcal{H}_{sm}(x_0 - y)| |a(y) - a\mathcal{E}| |f(y)| \, dy$$

$$\leq C(n, \alpha)m^{n/2} \rho(x_0 - x) \int_{(2\mathcal{E})^c} \frac{|a(y) - a\mathcal{E}| |f(y)|}{\rho(x_0 - y)^{a+1}} \, dy$$

$$\leq C(n, \alpha)m^{n/2} \left( \int_{(2\mathcal{E})^c} |f(y)|^p \rho(x_0 - y)^{a+1} \, dy \right)^{1/p} \left( \int_{(2\mathcal{E})^c} \frac{|a(y) - a\mathcal{E}|^p}{\rho(x_0 - y)^{a+1}} \, dy \right)^{1/p'}.$$
Finally, \( L \) with constants depending on \( E \) while (3.9) and Lemma 2.6 imply

\[
\int (2E)^{c} \frac{|f(y)|^p}{\rho(x_0 - y)^{\alpha + 1}} dy = \sum_{k=1}^{\infty} \int_{2^{k+1}\mathcal{E}/2^k \mathcal{E}} \frac{|f(y)|^p}{\rho(x_0 - y)^{\alpha + 1}} dy \leq \frac{2^{\alpha + 1}}{r} M(|f|^p)(x_0),
\]

while (3.3) and Lemma 2.6 imply

\[
\int (2E)^{c} \frac{|a(y) - a_\mathcal{E}|^p'}{\rho(x_0 - y)^{\alpha + 1}} dy = \sum_{k=1}^{\infty} \int_{2^{k+1}\mathcal{E}/2^k \mathcal{E}} \frac{|a(y) - a_\mathcal{E}|^p'}{\rho(x_0 - y)^{\alpha + 1}} dy 
\leq \sum_{k=1}^{\infty} \frac{1}{(2^k r)^{\alpha + 1}} \int_{2^{k+1}\mathcal{E}} |a(y) - a_\mathcal{E}|^p' dy 
\leq \sum_{k=1}^{\infty} \frac{2^{p'-1}}{(2^k r)^{\alpha + 1}} \int_{2^{k+1}\mathcal{E}} (|a(y) - a_{2^{k+1}\mathcal{E}}|^p' + |a_{2^{k+1}\mathcal{E}} - a_\mathcal{E}|^p') dy 
\leq C \sum_{k=1}^{\infty} \frac{2^{k+1}|\mathcal{E}|}{(2^k r)^{\alpha + 1}} (1 + k^{p'}) \|a\|^p' \leq C \|a\|^p' r
\]

and the constant depends on \( n, p \) and \( \alpha \). Hence

\[
G_2(x_0, \mathcal{E}) \leq C(n, p, \alpha) m^{n/2} \|a\|_* \left(M(|f|^p)(x_0)\right)^{1/p}.
\]

Finally,

\[
G_3(x_0, \mathcal{E}) := \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |J_3(x) - (J_3)_\mathcal{E}| dx \leq \frac{2}{|\mathcal{E}|} \int_{\mathcal{E}} |a(x) - a_\mathcal{E}| |\mathcal{K}_{sm} f(x)| dx 
\leq 2 \left( \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |a(x) - a_\mathcal{E}|^p' dx \right)^{1/p'} \left( \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |\mathcal{K}_{sm} f(x)|^p dx \right)^{1/p} 
\leq C(p) \|a\|_* \left(M(|\mathcal{K}_{sm} f|^p)(x_0)\right)^{1/p}.
\]

Summing up \( G_1(x_0, \mathcal{E}), G_2(x_0, \mathcal{E}) \) and \( G_3(x_0, \mathcal{E}) \) and taking the supremum with respect to \( \mathcal{E} \) and rendering in account the arbitrarity of the point \( x_0 \) we get the desired estimate for the commutator.

\begin{lemma}
The operators \( \mathcal{K}_{sm} \) and \( \mathcal{C}_{sm}[a, k] \) are continuous acting from \( L^{p, \omega}(\mathbb{R}^n) \) into itself and

\[
\|\mathcal{K}_{sm} f\|_{p, \omega} \leq C m^{n/2} \|f\|_{p, \omega}, \quad \|\mathcal{C}_{sm}[a, k] f\|_{p, \omega} \leq C m^{n/2} \|a\|_* \|f\|_{p, \omega}
\]

with constants depending on \( n, p, \) and \( \alpha \).
\end{lemma}
Proof. First of all we shall estimate the $L^{p,\omega}$-norms of the sharp functions of the considered operators. Since the expression for $(\mathcal{K}_{sm}f)^\sharp$ in (3.7) holds true for any $q \in (1, p)$ as well, the maximal inequality (Lemma 2.4) with $s = 1$ asserts
\[
\int_E |(\mathcal{K}_{sm}f)^\sharp(x)|^p dx \leq C m^{m/2} \int_E |M(|f|^q)(x)|^{p/q} dx 
\leq C m^{m/2} \omega(E) \|M(|f|^q)\|_{p/q, \omega}^{p/q} 
\leq C m^{m/2} \omega(E) \|f\|_{p/q, \omega}^p.
\]
Dividing of $\omega(E)$ and taking $\sup_E$, we arrive at
\[
\|(\mathcal{K}_{sm}f)^\sharp\|_{p, \omega} \leq C m^{n/2} \|f\|_{p, \omega}
\]
which implies the first inequality in (3.10) through Lemma 2.5. The $L^{p,\omega}$-estimate for the commutator follows in the same manner. 

Lemma 3.6. The operators $\mathcal{K}_{sm,\varepsilon}$ and $\mathcal{C}_{sm,\varepsilon}[a, k]$ are continuous acting from $L^{p,\omega}(\mathbb{R}^n)$ into itself and satisfy
\[
\|\mathcal{K}_{sm,\varepsilon}f\|_{p, \omega} \leq C m^{n/2} \|f\|_{p, \omega}, \|\mathcal{C}_{sm,\varepsilon}[a, k]f\|_{p, \omega} \leq C m^{n/2} \|a\|_* \|f\|_{p, \omega}
\]
with constants depending on $n, p$ and $\alpha$.

Proof. Let $E_\varepsilon$ and $E_{\varepsilon/2}$ be ellipsoids centered at $x \in \mathbb{R}^n$ and of radius $\varepsilon$ and $\varepsilon/2$, respectively. Writing $f = f\chi_{E_\varepsilon} + f\chi_{(E_\varepsilon)^c} = f_1 + f_2$ we obtain
\[
\mathcal{K}_{sm,\varepsilon}f(x) \leq \frac{1}{|E_{\varepsilon/2}|} \int_{E_{\varepsilon/2}} |\mathcal{K}_{sm,\varepsilon}f(x)| dy \leq \frac{1}{|E_{\varepsilon/2}|} \int_{E_{\varepsilon/2}} |\mathcal{K}_{sm}f(y)| dy 
+ \frac{1}{|E_{\varepsilon/2}|} \int_{E_{\varepsilon/2}} |\mathcal{K}_{sm,\varepsilon}f(x) - \mathcal{K}_{sm}f(y)| dy 
\leq \frac{2}{|E_{\varepsilon/2}|} \int_{E_{\varepsilon/2}} |\mathcal{K}_{sm}f_1(y)| dy 
+ \frac{1}{|E_{\varepsilon/2}|} \int_{E_{\varepsilon/2}} |\mathcal{K}_{sm,\varepsilon}f(x) - \mathcal{K}_{sm}f_2(y)| dy 
:= 2I_1(x, E_{\varepsilon/2}) + I_2(x, E_{\varepsilon/2})
\]
where $I_1$ and $I_2$ stand for the terms introduced at the proof of Lemma 3.4 and the same arguments as therein lead to
\[
|\mathcal{K}_{sm,\varepsilon}f(x)| \leq C(n, p, \alpha) m^{n/2} (M(|f|^q)(x))^{1/q}
\]
for any \( q \in (1, \infty) \). It remains to take the \( L^{p, \omega} \)-norms of the both sides above for \( 1 < q < p \) and to apply Lemma 2.4 in order to get (3.11).

The commutator estimate follows analogously.

Returning to the series expansions (3.3), we are in a position now to complete the proof of Theorem 3.1. First of all, note that

\[
\sum_{m=1}^{\infty} \sum_{s=1}^{g_m} \|b_{sm}K_{sm, \epsilon}f\|_{p, \omega} \leq C(n, p, \alpha, k) \|f\|_{p, \omega} \sum_{m=1}^{\infty} m^{-2l+n-2+n/2},
\]

\[
\sum_{m=1}^{\infty} \sum_{s=1}^{g_m} \|b_{sm}C_{sm, \epsilon}[a, k]f\|_{p, \omega} \leq C(n, p, \alpha, k) \|a\|_{*} \|f\|_{p, \omega} \sum_{m=1}^{\infty} m^{-2l+n-2+n/2}
\]

as it follows from (3.1), (2.5) and Lemma 3.6. Choosing \( l > (3n - 2)/4 \) the series in (3.3) result totally convergent in \( L^{p, \omega}(\mathbb{R}^n) \), uniformly in \( \epsilon > 0 \), whence

\[
\|K_{\epsilon}f\|_{p, \omega} \leq C \|f\|_{p, \omega}, \quad \|C_{\epsilon}[a, k]f\|_{p, \omega} \leq C \|a\|_{*} \|f\|_{p, \omega}.
\]

Setting

\[
Kf(x) := \sum_{s, m} b_{sm}(x)K_{sm}f(x), \quad C[a, k]f(x) := \sum_{s, m} b_{sm}(x)C_{sm}[a, k]f(x),
\]

we obtain as above

\[
\|Kf\|_{p, \omega} \leq C \|f\|_{p, \omega}, \quad \|C[a, k]f\|_{p, \omega} \leq C \|a\|_{*} \|f\|_{p, \omega}
\]

through (3.1), (2.5) and Lemma 3.5

Finally, the total convergence in \( L^{p, \omega}(\mathbb{R}^n) \) of the series expansions (3.3), uniformly in \( \epsilon > 0 \), gives

\[
\lim_{\epsilon \to 0} K_{\epsilon}f(x) = \sum_{s, m} b_{sm}(x) \lim_{\epsilon \to 0} K_{sm, \epsilon}f(x) = \sum_{s, m} b_{sm}(x)K_{sm}f(x) = Kf(x),
\]

\[
\lim_{\epsilon \to 0} C_{\epsilon}[a, k]f(x) = \sum_{s, m} b_{sm}(x) \lim_{\epsilon \to 0} C_{sm, \epsilon}[a, k]f(x) = C[a, k]f(x)
\]

and this completes the proof of Theorem 3.1.

It is worth noting that singular integrals like (1.1) and (1.2) appear in the representation formulas for the solutions of linear elliptic and parabolic partial differential equations. To make the obtained here results applicable to the study of regularizing properties of these operators we need of some additional local results.
Corollary 3.7. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $k(x; \xi): \Omega \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ be a variable kernel of mixed homogeneity, $a \in \text{BMO}(\Omega)$, $p \in (1, \infty)$ and $\omega$ satisfies (2.3) and (2.4). Then, for any $f \in L^{p,\omega}(\Omega)$ and almost all $x \in \Omega$, the singular integrals

$$Kf(x) = \text{P.V.} \int_{\Omega} k(x; x-y)f(y)dy$$

$$C[a, k]f(x) = \text{P.V.} \int_{\Omega} k(x; x-y)[a(y) - a(x)]f(y)dy$$

are well defined in $L^{p,\omega}(\Omega)$ and

$$\|Kf\|_{p,\omega; \Omega} \leq C\|f\|_{p,\omega; \Omega}, \quad \|C[a, k]f\|_{p,\omega; \Omega} \leq C\|a\|_\ast \|f\|_{p,\omega; \Omega}$$

with $C = C(n, p, \alpha, \Omega, k)$.

To obtain the above assertion it is sufficient to extend $k(x; \cdot)$ and $f(\cdot)$ as zero outside $\Omega$. One more necessary extension preserving the norm is that of $a$ in $\text{BMO}(\mathbb{R}^n)$ and we have it according to the results of Jones [PJ] and Acquistapace [A] (see [CFL] for details).

Another consequence of Theorem 3.1 is the “good behavior” of the commutator for $\text{VMO}$ functions $a$.

Corollary 3.8. Suppose $a \in \text{VMO}$ with $\text{VMO}$-modulus $\gamma_a$. Then, for each $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon, \gamma_a) > 0$ such that for any $\varrho \in (0, r_0)$ and any ellipsoid $\mathcal{E}_\varrho$ of radius $\varrho$ one has

$$\|C[a, k]f\|_{p,\omega; \mathcal{E}_\varrho} \leq C\varepsilon\|f\|_{p,\omega; \mathcal{E}_\varrho}$$

(3.12)

for all $f \in L^{p,\omega}(\mathcal{E}_\varrho)$.

Proof. From the properties of the $\text{VMO}$ functions [S, Theorem 1] it follows that for any $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon, \gamma_a)$ and continuous and uniformly bounded function $g$ with modulus of continuity $\omega_g(r_0) < \varepsilon/2$ such that $\|a - g\|_\ast < \varepsilon/2$. Let $\mathcal{E}_\varrho$ be an ellipsoidoid at $x_0$ and of radius $\varrho < r_0$. Following [CFT], we construct a function

$$h(x) = \begin{cases} g(x) & x \in \mathcal{E}_\varrho \\ g\left( x_0 + \varrho^a \frac{x - x_0}{\varrho(x-x_0)^{\alpha}} x_{01}, \ldots, x_{0n} + \varrho^a \frac{x - x_0}{\varrho(x-x_0)^{\alpha}} x_{0n} \right) & x \in \mathcal{E}_\varrho^c \end{cases}$$
which is uniformly continuous in $\mathbb{R}^n$. Whence the oscillation of $h$ in $\mathbb{R}^n$ is no greater than the oscillation of $g$ in $\mathcal{E}_{r_0}$. Then

$$
\|C[a, k]f\|_{p, \omega; \mathcal{E}_e} \leq \|C[a - g, k]f\|_{p, \omega; \mathcal{E}_e} + \|C[g, k]f\|_{p, \omega; \mathcal{E}_e} \\
\leq C\|a - g\|_* \|f\|_{p, \omega; \mathcal{E}_e} + C\|h\|_* \|f\|_{p, \omega; \mathcal{E}_e} \\
\leq C(\|a - g\|_* + \omega_g(r_0)) \|f\|_{p, \omega; \mathcal{E}_e} < C\varepsilon \|f\|_{p, \omega; \mathcal{E}_e}.
$$

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