ON LAWSON’S AREA-MINIMIZING HYPERCONES

YONGSHENG ZHANG

Abstract. This short note consists of two parts. In the first part, we show that the area-minimality property of all known homogeneous area-minimizing hypercones in Euclidean spaces (see [Law91] for the most up-to-date list) can be gained following Lawson’s original idea of [Law72]. The second part is devoted to proving that every area-minimizing cone $C_{m,n} \subset \mathbb{R}^{m+n+2}$ enjoys calibrations singular only at the origin. Based on this result several interesting examples in the realm of calibrated geometry are discussed in [Zha].

1. Introduction

Let $X$ be a Riemannian manifold and $G$ a compact connected group of isometries of $X$. Then Hsiang and Lawson [HL71] shows that a $G$-invariant submanifold $N$ of cohomogeneity $k$ in $X$ is minimal if and only if $N^*/G$ is minimal in $X^*/G$ with respect to certain naturally induced metric on the orbit space. Here $*$ means the union of principal orbits of action $G$. In particular they classified compact homogeneous minimal hypersurfaces in standard spheres.

Let $X$ be $\mathbb{R}^{n+1}$ endowed with the Euclidean metric. Suppose $M$ is a submanifold in the unit sphere $S^n \subset \mathbb{R}^{n+1}$. We call $C(M) = \{tx : 0 \leq t < +\infty, x \in M\}$ the cone over $M$ and $M$ the link of $C(M)$. It is well known that $M$ is minimal in $S^n$ if and only if $C(M) \sim 0$ is minimal in $\mathbb{R}^{n+1}$.

In [Law72] Lawson first showed that there always exists a $G$-invariant solution to the Plateau problem for any $G$-invariant boundary $M$, and then he studied further which homogeneous minimal hypersurfaces in their classification give rise to area-minimizing cones (in the sense of Federer, namely the truncated cone by the unit ball centered at the origin is an area-minimizer among all integral currents [FF60] with boundary $M$). He showed the following result by constructing calibration 1-forms (smooth away from boundary) on orbit spaces.

Theorem 1.1 (Lawson [Law72]). There exist link manifolds $M$ of types (I) $S^p(3)/S^p(1)^3$ in $\mathbb{R}^{14}$, $F_4/Spin(8)$ in $\mathbb{R}^{26}$, and moreover, (II) $S^{r-s+1} \times S^{s+1}$ in $\mathbb{R}^{r+s+2}$ for $r, s \geq 2$ with $r + s \geq 10$ or with $r + s = 9$ and $|r - s| \leq 5$ or with $r = s = 4$, $SO(2) \times SO(k)/\mathbb{Z}_2 \times SO(k - 2)$ in $\mathbb{R}^{2k}$ for $k \geq 10$, $SU(2) \times SU(k)/T^1 \times SO(k - 2)$ in $\mathbb{R}^{4k}$ for $k \geq 5$, $SU(5)/SU(2) \times SU(2)$ in $\mathbb{R}^{20}$, $U(1) \cdot Spin(10)/T^1 \cdot SU(4)$ in $\mathbb{R}^{32}$, and $Sp(2) \times Sp(k)/Sp(1)^2 \times Sp(k - 2)$ in $\mathbb{R}^{8k}$ for $k \geq 2$, such that $C(M)$ is area-minimizing.

Remark 1.2. Here we put overlapping types of (I) and (II) in [Law72] into class (II).

Date: January 21, 2015.
2010 Mathematics Subject Classification. Primary 53C38, Secondary 28A75.
Key words and phrases. homogeneous area-minimizing hypercone, Lawson’s calibration, comass, calibration with one singular point.
Remark 1.3. For example cones $C_{m,n} \triangleq C(S^m(\sqrt{\frac{m}{m+n}})) \times S^n(\sqrt{\frac{n}{m+n}})$ where $m = r - 1$ and $n = s - 1$ for $r, s$ in the theorem are area-minimizing.

Following Lawson’s idea, we give a proof of the affirmative part of the table on page 86 in Lawlor [Law91] by verifying the rest cases.

**Theorem 1.4.** $M$ of class (II) in the above theorem can also be of type $S^2 \times S^4$ in $\mathbb{R}^8$, $SO(2) \times SO(k)/\mathbb{Z}_2 \times SO(k - 2)$ in $\mathbb{R}^{2k}$ for $k = 9$, and $SU(2) \times SU(k)/T \times SO(k - 2)$ in $\mathbb{R}^{4k}$ for $k = 4$, such that $C(M)$ is area-minimizing. Moreover, types of $S^1 \times S^5$ and $SO(2) \times SO(8)/\mathbb{Z}_2 \times SO(6)$ correspond to stable minimal cones.

In particular for the type of $S^2 \times S^4$ in $\mathbb{R}^8$ we have the following.

**Corollary 1.5 (Simoes [Sim74, Sim73]).** $C_{2,4}$ is area-minimizing in $\mathbb{R}^8$.

According to Hardt and Simon [HS85], here comes another corollary.

**Corollary 1.6.** Area-minimizing cones $C(M)$ in Theorems 1.1 and 1.4 are strictly area-minimizing in the corresponding Euclidean spaces.

**Remark 1.7.** The strict area-minimality of $C_{2,4}$ was first proved by Lin [Lin87].

Based upon [Law72] and the last section of Federer [Fed74], we obtain our second theorem by modifying Lawson’s original calibrations.

**Theorem 1.8.** For each hypercone $C_{r-1,s-1} \subseteq \mathbb{R}^{r+s}$ in Theorems 1.1 and 1.4 there exists a calibration singular only at the origin to calibrate it.

See [Zha] for further applications on the extension of these calibrations with one singular point.

2. Preliminaries

Suppose a compact connected Lie group $G$ has an orthogonal representation on an Euclidean space with cohomogeneity 2 and $M_0$ is the principal orbit of greatest volume in $S^n$. Then a celebrated classification due to [HL71] and [TT72] is the following. We refer readers to the original papers for details about representations.
Each orbit space is isometrically the cone over an Euclidean arc $A$ in $\mathbb{R}^2$ endowed with metric $ds^2 = V^2(dx^2 + dy^2)$. Let $r$ be the projection map to orbit space. Then it can be shown that (up to a uniform constant factor) the volume of $N$ equals the length of $\pi(N)$ for any $G$-invariant connected (compactly supported and possibly singular) submanifold $N$ of codimension 2 in the Euclidean space.

In order to search for area-minimizing cone $C(M_0)$, Lawson made the following construction on orbit space for class (II), i.e., Row $1 - 6$ in the above table.

First note that $A = \frac{\pi}{4}$ for Row 2 - 6. When one considers $re^{\theta} = z^2$ with $z = x + yi$, the orbit space will be stretched to a cone $Q$ of arc $\frac{\pi}{4}$ with metrics (up to a scalar factor) $e^{2k-3}z^{4k}e^{\xi}(r^2d\theta^2 + dr^2)$, $e^{4k-6}c^4(r^2d\theta^2 + dr^2)$, $e^{8k-10}c^8(r^2d\theta^2 + dr^2)$, $e^{10r}c^{12}(r^2d\theta^2 + dr^2)$ and $e^{2r}c^2(r^2d\theta + dr^2)$ respectively. Here $c = \cos \theta$ and $s = \sin \theta$. For metric $ds^2 = r^2\cos^2\theta d\theta^2 + dr^2$ on $Q$ (including Row 1), the angle $\theta_0$ corresponding to $M_0$ now satisfies $\tan^2\theta_0 = q/p$. Set $\tau = \cos^\alpha \theta_0 \sin^q \theta_0$. Then for $d\tilde{s}^2 = \frac{1}{\tau}ds^2$ on $Q$, Lawson considered the function type

$$f = a^{-1}r^\beta\phi^\beta,$$

where $2a = l + 2$, $\beta = \frac{c^\tau}{\tau}$ and $\beta$ is some real number to be determined. Since

$$\phi'(\theta) = \phi(\theta)(q \cot \theta - p \tan \theta),$$

it follows

$$df = r^{a-1}\phi^\beta dr + (\frac{B}{a})r^\alpha\phi^\beta(q \cot \theta - p \tan \theta)d\theta.$$

Note that for a 1-form its comass (see [HL82]) and usual norm are the same. Therefore if for some $\beta$

$$r^{2a-2}\phi^\beta[1 + (\frac{B}{a})^2(q \cot \theta - p \tan \theta)^2] \leq \frac{r^\beta c^\xi}{\tau},$$

|    | $G$                | type of $M_0$ | $A$     | $V^2$            |
|----|-------------------|--------------|---------|-----------------|
| 1  | $SO(r) \times SO(r)$ | $S^{r-1} \times S^{r-1}$ | $r, s \geq 2$ | $\frac{\pi}{2}$ | $c \cdot x^{2r-2}y^{2r-2}$ |
| 2  | $SO(2) \times SO(k)$ | $\frac{SO(2) \times SO(k)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$ | $k \geq 2$ | $\frac{\pi}{4}$ | $c \cdot (xy)^{2(k-4)}(x^2 - y^2)^2$ |
| 3  | $SU(2) \times SU(k)$ | $\frac{SU(2) \times SU(k)}{T^*SU(k-2)}$ | $k \geq 2$ | $\frac{\pi}{4}$ | $c \cdot (xy)^{4k-6}(x^2 - y^2)^4$ |
| 4  | $Sp(2) \times Sp(k)$ | $\frac{Sp(2) \times Sp(k-2)}{T^{(2)}(\mathbb{Z})}$ | $k \geq 2$ | $\frac{\pi}{4}$ | $c \cdot (xy)^{8k-10}(x^2 - y^2)^8$ |
| 5  | $U(5)$ | $\frac{U(5)}{T^*SU(5)}$ | $T^*SU(5)$ | $\frac{\pi}{4}$ | $c \cdot (xy)^2 \Im(x + iy)^4$ |
| 6  | $U(1) \cdot Spin(10)$ | $\frac{U(1) \cdot Spin(10)}{T^*Spin(10)}$ | $T^*Spin(10)$ | $\frac{\pi}{4}$ | $c \cdot (xy)^2 \Im(x + iy)^4$ |
| 7  | $SO(3)$ | $\frac{SO(3)}{\mathbb{Z}_2}$ | $\mathbb{Z}_2$ | $\frac{\pi}{4}$ | $c \cdot \Im(x + iy)^4$ |
| 8  | $SU(3)$ | $\frac{SU(3)}{T^*SU(2)}$ | $T^*SU(2)$ | $\frac{\pi}{4}$ | $c \cdot \Im(x + iy)^4$ |
| 9  | $Sp(3)$ | $\frac{Sp(3)}{T^*Sp(2)}$ | $T^*Sp(2)$ | $\frac{\pi}{4}$ | $c \cdot \Im(x + iy)^4$ |
| 10 | $F_4$ | $\frac{F_4}{Spin(8)}$ | $Spin(8)$ | $\frac{\pi}{3}$ | $c \cdot \Im(x + iy)^4$ |
| 11 | $Sp(2)$ | $\frac{Sp(2)}{T^*Sp(1)}$ | $T^*Sp(1)$ | $\frac{\pi}{4}$ | $c \cdot \Im(x + iy)^4$ |
| 12 | $G_2$ | $\frac{G_2}{Spin(7)}$ | $Spin(7)$ | $\frac{\pi}{6}$ | $c \cdot \Im(x + iy)^4$ |
| 13 | $SO(4)$ | $\frac{SO(4)}{\mathbb{Z}_2 + \mathbb{Z}_2}$ | $\mathbb{Z}_2 + \mathbb{Z}_2$ | $\frac{\pi}{6}$ | $c \cdot \Im(x + iy)^4$ |
or simply

$$\phi^{2\beta-1} \left[ 1 + \frac{\beta}{\alpha}^2 (q \cot \theta - p \tan \theta)^2 \right] \leq 1,$$

then $df$ is a calibration 1-form, smooth on $Q^\circ$ and continuous to $\partial Q$, calibrating the ray $\theta = \theta_0$. Due to Lawson’s reduction of the equivariant Plateau problem, the corresponding hypercone in Euclidean space is therefore area-minimizing. Denote the left-hand side of (2.2) by $\psi$ and it is clear that $\psi$ is pointwise the square of comass of $df$ with respect to $d\tilde{s}^2$.

3. Proof of Theorem 1.4

Let us consider the derivative of $\psi$.

$$\psi' = \phi^{2\beta-1} (q \cot \theta - p \tan \theta) \left( 2\beta - 1 \right) \left[ 1 + \frac{\beta}{\alpha}^2 (q \cot \theta - p \tan \theta)^2 \right]$$

$$+ 2 \left( \frac{\beta}{\alpha} \right)^2 (q \cot \theta - p \tan \theta) \right\}$$

Denote the part in braces by $\eta_\beta$. Then

$$\eta_\beta = (2\beta - 1) \left[ 1 + \frac{\beta}{\alpha}^2 (q \cot \theta - p \tan \theta)^2 \right] + 2 \left( \frac{\beta}{\alpha} \right)^2 \left( \frac{q}{s^2} - \frac{p}{c^2} \right)$$

$$= (2\beta - 1) \left[ 1 + \frac{\beta}{\alpha}^2 (q^2 \cot^2 \theta + p^2 \tan^2 \theta - 2pq) \right] + 2 \left( \frac{\beta}{\alpha} \right)^2 \left( \frac{q}{s^2} - \frac{p}{c^2} \right)$$

$$= (2\beta - 1) - 2 \left( \frac{\beta}{\alpha} \right)^2 [(2\beta - 1)pq + p + q]$$

$$+ \left( \frac{\beta}{\alpha} \right)^2 \left[ (2\beta - 1)q^2 - 2q \right] \cot^2 \theta + [(2\beta - 1)p^2 - 2p] \tan^2 \theta$$

For a triple $(p, q, \alpha)$, if there exists some $\beta$ such that

$$\eta_\beta > 0 \text{ on } Q^\circ$$

then, based upon the sign of $\psi'$ and the fact $\psi(\theta_0) = 1$, (2.2) holds on $Q$.

A1. For simplicity, we try $\beta = 1$ for Row 1 in the table. Now

$$2a = \Sigma + 2 \text{ where } \Sigma = p + q, \ p = 2r - 2, \text{ and } q = 2s - 2.$$ 

$$4a^2 \eta_1 = 4a^2 - 8(p + q + pq) + 4(q^2 - 2q) \cot^2 \theta + 4(p^2 - 2p) \tan^2 \theta$$

$$\geq 4a^2 - 8(p + q + pq) + 8 \sqrt{(q^2 - 2q)(p^2 - 2p)} \quad ("\geq" \text{ unless } p = q = 2)$$

$$\geq 4a^2 - 8(p + q + pq) + 8(q - 2)(p - 2)$$

$$\geq (\Sigma + 2)^2 - 24\Sigma + 32$$

$$= (\Sigma - 2)(\Sigma - 18)$$

Thus, when $2(r + s) - 4 = \Sigma \geq 18$, namely $r + s \geq 11$, we have $\eta_1 > 0$. 

Actually from the first row of (3.2) one can get
\[ 4\alpha^2\eta_1 \tan^2 \theta = 4(p^2 - 2p) \tan^4 \theta + [4\alpha^2 - 8(p + q + pq)] \tan^2 \theta + 4(q^2 - 2q) \]
where \( Y = \tan^2 \theta \). Easy to check that \( \Delta \) of the quadratic is negative for \( r, s \geq 3 \) with \( 8 \leq r+s \leq 10 \). So \( \eta_1 > 0 \). In particular \( C_{2,4} \) (corresponding to \( (r, s) = (3, 5) \)) is area-minimizing.

Due to Simons [Sim68] and Simoes [Sim74, Sim73] there are no area-minimizing cones for \( r+s < 8 \) or for \( (r, s) = (2, 6) \). Therefore \( (r, s) = (2, 7) \) and \( (2, 8) \) are the only left cases to check. Although \( df \) cannot be a global calibration on \( Q \) for \( (r, s) = (2, 6) \), it can serve as a local calibration in a very thin angular neighborhood of the ray \( \theta = \theta_0 \), e.g. \( 1.145 \leq \theta \leq 1.165 \) shown in Figure 1 below. (When \( \theta = \theta_0 = \arctan \sqrt{5} \approx 1.150262, \psi = 1 \).) In fact the existence of such an angular neighborhood can be seen from \( \eta_1(\theta_0) > 0 \). As a consequence, \( C_{1,5} \) is a stable minimal hypercone in \( \mathbb{R}^8 \).

**A2.** For the rest two cases as well as for our purpose later in C2. we consider \( \beta = 1.2 > 1 \). Similarly, it follows that \( \eta_{1,2} > 0 \) for \( r, s \geq 2 \) with \( r+s \geq 9 \).

**B.** For \( SO(2) \times SO(k)/\mathbb{Z}_2 \times SO(k-2) \) in \( \mathbb{R}^{2k} \) for \( k = 9 \) (with \( \alpha = 8.5 \)) one can take \( \beta = 1.2 \). Then there are two roots \( \theta_{1,2} \) of \( \eta_{1,2} \) satisfying \( \theta_0 < \theta_1 < \theta_2 \). Since \( \psi(\theta_2) < 1 \), we know that \( df \) is a calibration on \( Q \).
For \( SO(2) \times SO(k)/\mathbb{Z}_2 \times SO(k - 2) \) in \( \mathbb{R}^{2k} \) for \( k = 8 \) (with \( \alpha = 7.5 \)), using \( \beta = 1 \), there is only one root \( \theta_1 \) of \( \eta_1 \) and \( \theta_1 > \theta_0 \). So \( df \) is a local calibration in some angular neighborhood of \( \theta = \theta_0 \), for example \(|\theta - \theta_0| < \theta_1 - \theta_0|\).

**Figure 3.** Graph of \( \psi \) with \( \beta = 1 \).

Due to technique reason we save the discussion on \( SU(2) \times SU(k)/T^1 \times SU(k - 2) \) in \( \mathbb{R}^{4k} \) for \( k = 4 \) to the end of this note.

4. Proof of Theorem 1.8

By explanations of Theorem 2 in [Law72] or the last section of [Fed74], up to a constant,

\[
\omega_f = \pi^* (df) \wedge \Omega,
\]

where \( \Omega = \Omega_0 / V \) and \( \Omega_0 \) is the oriented volume form of principal orbits, is a calibration of the cone, which is smooth away from singular orbits in the Euclidean space.

In this section, we will show that either \( \omega \) arising from our construction is smooth, or otherwise after suitable modifications one can get a smooth calibration in Euclidean space for each cone in **Row 1**.

It is easy to see that the principal orbit through \( (\overrightarrow{X}, \overrightarrow{Y}) \in \mathbb{R}^r \times \mathbb{R}^s \), where neither of \( \overrightarrow{X} \) and \( \overrightarrow{Y} \) is a null vector, is \( S^{r-1}(||\overrightarrow{X}||) \times S^{s-1}(||\overrightarrow{Y}||) \). Therefore

\[
\Omega_0^{r,s} = \Omega_r(\overrightarrow{X}) \wedge \Omega_s(\overrightarrow{Y}),
\]

where

\[
\Omega_r(\overrightarrow{X}) = \frac{1}{||\overrightarrow{X}||} \sum_{i=1}^{r} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_r,
\]

\[
\Omega_s(\overrightarrow{Y}) = \frac{1}{||\overrightarrow{Y}||} \sum_{j=1}^{s} (-1)^{j-1} y_j dy_1 \wedge \cdots \wedge \hat{dy}_j \wedge \cdots \wedge dy_s,
\]

and \( \{x_i\}, \{y_j\} \) are standard basis of \( \mathbb{R}^r \) and \( \mathbb{R}^s \) respectively.
**C1.** For an even pair \((r, s)\), i.e. \(r = 2n\) and \(s = 2m\), with \(r, s \geq 4\) and \(\beta = 1\) in A1., there is a nonzero constant \(C\) such that

\[
C \cdot \omega_f = C \cdot d (\pi^* f) \wedge \frac{\Omega^0_{r-s}}{V}
\]

\[
= d \left[ \mathbb{R}^{3-r-s} : \|X\|^{2r-2} \cdot \|Y\|^{2s-2} \right] \wedge \frac{\Omega^0_{r-s}}{V} \quad \text{(Here } R = \sqrt{\|X\|^2 + \|Y\|^2})
\]

\[
= \left\{ \mathbb{R}^{3-r-s} \left[ (r-1) \|X\|^{2r-4} \|Y\|^{2s-2} d\|X\|^2 + (s-1) \|X\|^{2r-2} \|Y\|^{2s-4} d\|Y\|^2 \right] 
\right.
\]

\[
+ (3-r-s) \mathbb{R}^{2-r-s} \|X\|^{2r-2} \cdot \|Y\|^{2s-2} dR \left. \right\} 
\]

\[
\wedge \left. \frac{1}{\|X\|^r} \left( \sum_{j=1}^{r} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge dx_l \wedge x_j \right) \right.
\]

\[
\wedge \left. \frac{1}{\|Y\|^s} \left( \sum_{j=1}^{s} (-1)^{j-1} y_j dy_1 \wedge \cdots \wedge dy_{j-1} \wedge dy_j \right) \right.
\]

\[
= \|X\|^{r-4} \|Y\|^{s-2} \cdot \Lambda_1 + \|X\|^{r-2} \|Y\|^{s-4} \cdot \Lambda_2 + \|X\|^{r-2} \|Y\|^{s-2} \cdot \Lambda_3
\]

where \(\Lambda_i\) are smooth forms away from the origin. Hence so is \(\omega_f\).

**C2.** For the remaining cases of **Row 1** in A1. and A2., each reduced ray has a calibration \(df\) where \(f = C\alpha^{-1} r^p c^k s^l\) for some positive constants \(C\) and \(k, t\) with \(2k - p > 2\) and \(2t - q > 2\). (For such \(f\) one can use \(\beta = 1\) when \(p, q > 2\) and \(\beta = 1.2\) when either of \(p\) and \(q\) equals 2.) Inspired by C1., we wish to deform \(f\) to make its behavior near \(\partial Q\) the same as in C1. and meanwhile \(d\tilde{f}\) for the deformed function \(\tilde{f}\) also a calibration on \(Q\).

Our strategy is to increase the exponential of \(f\). We will show how to remove the singularity along \(\theta = 0\) (with \(r > 0\)) for illustration. Choose an integer \(N\) such that \(N - q - 1\) is a positive even integer and \(N > t\). Then search for a suitable smooth function \(\lambda(\theta)\) such that \(\lambda(\theta) \equiv N - t\) when \(0 \leq \theta < \frac{\pi}{2}\) and \(\lambda(\theta) \equiv 0\) when \(\theta > \bar{\theta}\) for some small positive \(\bar{\theta}\) for the deformation:

\[
\tilde{f} = C\alpha^{-1} r^p c^k s^{l+\lambda}.
\]

By computation

\[
d\tilde{f} = C\alpha c^k s^{l+\lambda} dr + C\alpha^{-1} r^p c^k s^{l+\lambda} (\lambda' \ln s + (t + \lambda) \cot \theta - k \tan \theta) d\theta
\]

and similar to (2.2) the square of comass of \(d\tilde{f}\) equals

\[
\tau C^2 c^{2k-p} s^{2\lambda+2q} \left[ 1 + \left( \frac{1}{\alpha} \right)^2 (\lambda' \ln s + (t + \lambda) \cot \theta - k \tan \theta)^2 \right].
\]

In view of the square of comass of calibration \(df\),

\[
\tau C^2 c^{2k-p} s^{2\lambda-q} \left[ 1 + \left( \frac{1}{\alpha} \right)^2 (t \cot \theta - k \tan \theta)^2 \right] \leq 1 \text{ on } Q,
\]

one only needs to show (4.3) is no large than one on the support of \(\lambda\) for \(d\tilde{f}\) being a calibration.
We use the following Lipschitz function for any fixed $x$ with $0 < x < \theta_0$:

$$\lambda_x(\theta) = \begin{cases} N - t & 0 \leq \theta \leq \frac{\epsilon}{2} \\ (N - t) \left[ 1 - \frac{1}{\int_{\frac{\theta}{2}}^{x} \frac{1}{\ln \sin\gamma} \, d\gamma} \right] & \frac{\epsilon}{2} < \theta < x \\ 0 & x \leq \theta \end{cases}$$

By this choice (4.3) on $\frac{\epsilon}{2} < \theta < x$ becomes

$$(4.4) \quad \tau C^2 2^{k_p} s^{2t - 2\lambda_x - q} \left[ 1 + \left( \frac{1}{\alpha} \right)^2 \left( -\frac{N - t}{\int_{\frac{\theta}{2}}^{x} \frac{1}{\ln \sin\gamma} \, d\gamma} + (t + \lambda_x) \cot \theta - k \tan \theta \right) \right]^2.$$ 

Let us consider the behavior of (4.4) as $x$ tends to zero. By the assumption $2t + 2\lambda_x - q \geq 2\rho = 2t - q > 2$, the expression limits to

$$\lim_{t \to 0} C^2 s^{2t + 2\lambda_x - q} \left( \frac{N - t}{\int_{\frac{\theta}{2}}^{x} \frac{1}{\ln \sin\gamma} \, d\gamma} \right)^2 \leq C^2 (N - t)^2 \left( \lim_{t \to 0} \frac{\sin' x}{\int_{\frac{\theta}{2}}^{x} \frac{1}{\ln \sin\gamma} \, d\gamma} \right)^2.$$ 

Since $\int_{0}^{\alpha} \frac{1}{\ln \gamma} \, d\gamma$ is convergent for every $0 < a < \frac{\pi}{2}$, the denominator of the latter limit in the parenthesis goes to zero and one can apply L'Hôpital’s rule. Then another employment of the rule for

$$\lim_{\sin x \to 0} \frac{\rho \ln \sin x}{\sin^{-e} (x)} \quad \text{and} \quad \lim_{\frac{\theta}{2} \to 0} \frac{\rho \ln \sin \frac{\theta}{2}}{\sin^{-e} (x)}$$

leads to the answer zero.

Not hard to see there exists a smooth approximation $\lambda$ of some $\lambda_x$ such that (4.3) does not exceed one. One construction can be given as follows. Choose $x_0$ small enough so that, for some small positive $\epsilon_1 < \frac{\epsilon_0}{2}$, the following analogous expression to (4.4)

$$(4.5) \quad \tau C^2 c^{2k_p} s^{2t} \left[ 1 + \left( \frac{1}{2} \right)^2 \left( -\frac{N - t}{\int_{x_0}^{x_0 + \epsilon_0} \frac{1}{\ln \sin\gamma} \, d\gamma} + K \cot \theta - k \tan \theta \right) \right]^2$$

is less than half of one pointwise for $\frac{\pi}{2} - \epsilon_1 \leq \theta \leq x_0 + \epsilon_1$, any $\chi \in [0, 2]$ and $K \in [0, N + 1]$. Let $\Lambda_\epsilon$ be an integral convolution of $\lambda_{x_0}$ given by a compactly supported, smooth, even mollifier function over small averaging radius $\epsilon (\epsilon_1)$ at each point. Define $\sigma_\epsilon$ satisfying

$$\Lambda_\epsilon(y) = -\sigma_\epsilon(y) \frac{N - t}{\int_{\frac{\theta}{2}}^{x_0} \frac{1}{\ln \sin\gamma} \, d\gamma} \cdot \frac{1}{\sin^{-e} (x)} \quad \text{for} \quad \frac{x_0}{2} - \epsilon_1 \leq y \leq x_0 + \epsilon_1.$$ 

Then $\sigma_\epsilon$ is a smooth function with

$$\lim_{\epsilon \to 0} \sigma_\epsilon(y) = \begin{cases} 1 & \text{for} \quad \frac{x_0}{2} \leq y \leq x_0 \\ \frac{1}{2} & \text{at} \quad x_0 \quad \text{or} \quad \frac{x_0}{2} \\ 0 & \text{on} \quad \left[ \frac{x_0}{2} - \epsilon_1, \frac{x_0}{2} \right] \cup (x_0, x_0 + \epsilon_1) \end{cases}$$
So through a contradiction argument there exists some small $\varepsilon_2(< \varepsilon_1)$, such that $\sigma \varepsilon_2(y) \leq 2$ on $[\frac{x_0}{2} - \varepsilon_1, x_0 + \varepsilon_1]$. By comparing (4.3) and (4.5), it shows that $\lambda = \lambda_\varepsilon_2$ meets our needs.

According to the discussion in C1. and (4.1), $N - q - 1$ being a positive even integer asserts the smoothness of $\omega_j$ along $\{(X, Y) \in \mathbb{R}^r \times \mathbb{R}^s : \overrightarrow{X} = \overrightarrow{0} \text{ and } \overrightarrow{Y} \neq \overrightarrow{0}\}$. The same trick applies for removing the singularity along $\{(X, Y) \in \mathbb{R}^r \times \mathbb{R}^s : \overrightarrow{X} \neq \overrightarrow{0} \text{ and } \overrightarrow{Y} = \overrightarrow{0}\}$. Hence we finish the proof.

5. The Case of $SU(2) \times SU(k)/T^1 \times SU(k - 2)$ in $\mathbb{R}^{4k}$ for $k = 4$

In virtue of Mathematica it can be observed that the construction (2.1) cannot directly produce a calibration for any value $\beta$ in this case. Instead we fix $\beta = 1$ and look at the graph of $\psi = \frac{1}{7.5} \tau c^{4} s^{10} \left[ 1 + \left( \frac{1}{7.5} \right)^2 \left( 10 \cot \theta - 4 \tan \theta \right)^2 \right]$.

Since the second peak is not relatively quite high, we try to chop it off. Similarly as (4.2), we alter the order of term $c$ (or $s$ alternatively) as follows. Set

$$f_1 = \frac{1}{7.5 \tau} c^{4+2\lambda_1} s^{10},$$

where $\lambda_1$ is a function. Then the square of the comass of $df_1$ becomes

$$\left( \cdot \right) \frac{1}{\tau} c^{4+2\lambda_1} s^{10} \left[ 1 + \left( \frac{1}{7.5} \right)^2 \left( \lambda_1' \ln c + 10 \cot \theta - (4 + \lambda_1) \tan \theta \right)^2 \right].$$

Notice that $\theta_0$ (at where the first peak occurs) is less than 1.00686 and its local behavior is shown as follows:
Now we consider the ODE:

\[
\lambda_1' = 7.5 \sqrt{\frac{1}{c^{4+2\lambda_1}} s^{10} (\frac{14}{4})^2 (\frac{14}{10})^5} - 1 + 10 \cot \theta - (4 + \lambda_1) \tan \theta - \ln c,
\]

with \(\lambda_1(1.007) = 0\). Its solution tells us how to vary the exponential of \(c\) to keep the corresponding comass exactly one where it defines. Running the following codes in Mathematica

\[
\text{sl = NDSolve[}\{y'[x] == (7.5 \ast (1/(\text{Cos}[x]^{(4 + 2 y[x]) \text{Sin}[x]^{(10)} (14/4)^2 (14/10)^5}) - 1)^{(0.5)} + 10 \text{Cot}[x] - (4 + y[x]) \text{Tan}[x])/(\text{Log}[\text{Cos}[x]])], y[1.007] == 0, y, \{x, 1.007, 1.2}\}\]

we figure out the graph of \(\lambda_1\):

Let \(\theta_1\) be the zero point of \(\lambda_1\) near 1.2. Owing to the continuity of slope fields, one can construct a smooth variation \(\lambda\) of \(\lambda_1\) on an interval slightly larger than \([1.007, \theta_1]\) such that, after replacing \(\lambda_1\) by \(\lambda\), (★) is bounded from above by one. For instance, around 1.007 one can properly glue 0 and \(\lambda_1\) smoothly and on a small interval \([\theta_1, \theta_1 + \epsilon]\) (with \(\theta_1 + \epsilon < 1.2\)) a reasonable smoothly bending curve from \(\lambda_1\) down to zero function can serve for our purpose. Thus we complete the proof of Theorem 1.4.

**Acknowledgement**

The author would like to express deep gratitude to Professor H. Blaine Lawson, Jr. for his guidance and constant encouragement. He also wishes to thank Professor Hui Ma and Doctor Chao Qian for helpful comments, and the MSRI at where the main idea of this note was developed during the author’s visit there in Fall 2013.
REFERENCES

[FF60] Herbert Federer and Wendell H. Fleming, Normal and integral currents, Ann. Math. 72 (1960), 458–520.
[Fed74] Herbert Federer, Real flat chains, cochains and variational problems, Indiana Univ. Math. J. 24 (1974), 351–407.
[HS85] Robert Hardt and Leon Simon, Area minimizing hypersurfaces with isolated singularities, J. Reine. Angew. Math. 362 (1985), 102–129.
[HL82] Reese Harvey and Blaine H. Lawson, Calibrated geometries, Acta Math. 148 (1982), 47–157.
[HL71] Wu-Yi Hsiang and Blaine H. Lawson, Minimal submanifolds of low cohomogeneeity, J. Diff. Geom. 5 (1971), 1–38.
[Law91] Gary Lawlor, A Sufficient Criterion for a Cone to Be Area-Minimizing, Vol. 91, Mem. of the Amer. Math. Soc., 1991.
[Law72] Blaine H. Lawson, The Equivariant Plateau Problem and Interior Regularity, Trans. Amer. Math. Soc. 173 (1972), 231–249.
[Lin87] Fang-Hua Lin, Minimality and Stability of Minimal Hypersurfaces in \( \mathbb{R}^n \), Bull. Austral. Math. Soc. 36 (1987), 209–214.
[Sim74] Plinio Simoes, On A Class of Minimal Cones in \( \mathbb{R}^n \), Bull. Amer. Math. Soc. 3 (1974), 488–489.
[Sim73] _______. A class of minimal cones in \( \mathbb{R}^n \), \( n \geq 8 \), that minimize area, Ph.D. thesis, University of California, Berkeley, Calif., 1973.
[Sim68] James Simons, Minimal varieties in riemannian manifolds, Ann. of Math. 88 (1968), 62–105.
[TT72] Ryoichi Takagi and Tsunero Takahashi, On the principal curvatures of homogeneous hypersurfaces in a sphere, pp. 469-481, Diff. Geom., in honor of K. Yano, Kinokuniya, Tokyo, 1972.
[Zha] Yongsheng Zhang, On extending calibrations (part of “On Calibrated Geometry I: Gluing Techniques”).

Current address: SCHOOL OF MATHEMATICS AND STATISTICS, NORTEAST NORMAL UNIVERSITY, 5268 RenMin Street, NAnsQuan District, ChiangChun, Jilin 130024, P.R. China
E-mail address: yongsheng.chang@gmail.com