The Constructive Maximal Point Space

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Abstract

We argue that constructive maximality (Martin-Löf [14]) can with advantage be employed in the study of maximal point spaces, and related questions in quantitative domain theory.

1 Introduction

Maximality of points in a domain \( D \) can be difficult to treat constructively (or effectively) since its definition involves quantification over the whole of \( D \); it is not readily apparent how to reformulate the definition in a convenient way in terms of an assumed (countable) basis of \( D \). For this reason, Martin-Löf [14] worked instead with a stronger notion: constructive maximality. A point is constructively maximal if it passes “tests of fineness” involving comparison with basis elements (Section 4).

In this paper we re-examine constructive maximality, in relation to recent work on maximal point spaces, measurement (in the sense of K.Martin), partial metrics, and related topics. The maximal and the constructively maximal points of a domain \( D \) (say, \( \text{Max} \) and \( C\text{Max} \) respectively) do not coincide in general. In cases where they differ, it appears that \( C\text{Max} \) has better properties than \( \text{Max} \). A remarkable fact is that \( \text{Max} = C\text{Max} \) precisely when the so-called Lawson condition [11] holds for \( D \): see Corollary 4.4. The Lawson condition is (we can argue) a nice property for a domain to have, precisely because it forces every maximal point to be constructively maximal.

Rather than considering just the constructively maximal points of \( D \), namely those which pass all the tests of fineness, we can “measure” an arbitrary point of \( D \) by the fineness tests which it passes. In Sections 5,6 we consider weak metrics and also measurements (in a slight extension of Martin’s sense). We show in particular that every 2nd-countable locally compact space possesses a measurement.

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Domains are approached in this paper via the $R$-structures of [18] (also known as abstract bases: [1]). $R$-structures are reviewed briefly in Section 1 below. In [18] we viewed the $R$-structures, in part, as an abstraction from the spaces considered concretely by Martin-Löf in [14]. In that respect, the present paper represents a continuation of [18].

2 $R$-structures, neighbourhood systems

Definition 2.1 A set $E$ equipped with a transitive relation $<$ is said to be an $R$-structure provided that, for each $p \in E$, the set $\downarrow p = \{ x \mid x < p \}$ is $<$-directed.

A prime example of an $R$-structure is the collection of closed rational intervals, with the transitive order given by:

$$[a,b] < [c,d] \iff a < c < d < b$$

The idea is that an $R$-structure can represent, via ideal completion, a continuous domain: in the example, the interval domain (domain of closed real intervals). The main motivation (in [18]) was that computability questions concerning domains could be reduced to questions about (assumed) enumerations of the representing $R$-structures. $R$-structures have also been studied (under the name “abstract bases”), with some new applications, by A.Jung: see [1].

It will be convenient to endow each $R$-structure $(E, <)$ with a topology, namely that in which the “round” upper sets $\uparrow S$ (equivalently, sets $U$ such that $U = \uparrow U$) are taken as open. That this is a topology is contained in the following result, the straightforward proof of which is omitted:

Proposition 2.2 Let $<$ be a dense (that is, interpolative) transitive relation on the set $X$. Then the following are equivalent:

(i) The sets $\uparrow x$ $(x \in X$) provide the base of a topology on $X$;
(ii) The round upper sets constitute a topology on $X$;
(iii) $(X, <)$ is an $R$-structure.

This could be summarized by saying that the usual definition of the Alexandroff topology for a pre-ordered set “works” for an arbitrary dense order $(X, <)$ if and only if the latter is an $R$-structure.

In conformity with [14] we may read the formula “$a < b$” as “$b$ is finer than $a$”, and refer to the elements $a, b$ as “neighbourhoods”. It will be convenient to refer to the structures with which Martin-Löf works informally in [14] (and presents by means of three main examples) as “neighbourhood systems”. Besides the “finer than” relation (which we propose to axiomatize as an $R$-structure), a neighbourhood system comes equipped with two further important predicates (symmetric binary relations): overlap and lie apart. Overlap seems to be adequately captured, in an arbitrary $R$-structure, by having a
common upper bound. At any rate, we shall take this as our definition:

\[ a \mathrel{\uparrow} b \iff \exists c. a < c \land b < c. \]

“Lie apart” should be something stronger than the negation of “overlap”: for example, in the rational interval structure, two abutting intervals, say \([-1, 0]\) and \([0, 1]\), are not considered as lying apart. The following definition may be proposed:

\[ a \nparallel b \iff \exists a', b' < b. \neg (b' \mathrel{\uparrow} a') \]

This definition leaves something to be desired, from the constructive point of view, because of the embedded negation. From that point of view, it may be preferable to have \(\nparallel\) as a primitive, and try to characterize it by some axioms. For the limited purposes of this paper we shall accept the preceding definition. We return to this topic in the concluding section. In any case, for an effective treatment of the material, basic predicates should be assumed to be at least semi-decidable.

There are certain “normalizing” conditions which it may be useful to impose on an \(R\)-structure. The only one of these which we shall consider is that every neighbourhood can be refined:

\((N)\) \quad \forall a \exists b. a < b.

It is easy to see that, by removing from an arbitrary \(R\)-structure \(E\) every neighbourhood for which there is no finer neighbourhood in \(E\), we obtain an \(R\)-structure \(E'\) which satisfies (M1). Moreover, \(E'\) has the same topology as \(E\) (more precisely, its lattice of open sets is isomorphic with that of \(E\)), and the two \(R\)-structures have the same ideal completion (Section 3).

We can now formulate a useful principle, whose effect will be to enable us to construct maximal points without appealing to Zorn’s Lemma or the like:

\((M)\) \quad \forall a, b, c. b < c \Rightarrow \exists d > a. d \nparallel c \lor d > b.

This “principle” is a formal rendering of a property of neighbourhood systems which is used as an informal axiom by Martin-Löf. We may derive (M) from (N) by the following argument: Assume that \(b < c\) and \(a\) are given. Let \(d\) be any neighbourhood finer than \(a\). If \(d\) lies apart from \(c\) we are done. If not, then by the definition of \(\nparallel\) we have that \(a\) overlaps \(b\). In other words, there is a neighbourhood which refines both \(a\) and \(b\).

In this proof that (M) holds in any \(R\)-structure satisfying (N) we have (inevitably) used classical first order logic. See further the Concluding remarks (Sec. 8).

## 3 Domains

By a domain we generally understand a continuous dcpo. The domain is \(\omega\)-continuous if it has a countable basis. As general references, see [1], [10]. For most of our work here the following concrete description suffices. A \(<\)-ideal
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(or round ideal) of an \( R \)-structure \((E, <)\) is a \(<\)-directed lower set. In more detail, it is a subset \( I \) such that
\[
a < b \in I \rightarrow a \in I; \quad a \in I, b \in I \rightarrow \exists c > a, b. \ c \in I.
\]
The completion \( \bar{E} \) of \( E \) is the set of round ideals, ordered by inclusion. \( \bar{E} \) is itself an \( R \)-structure, with “finer than” given by:
\[
x <_C y \iff \exists e \in y. \ x \subseteq \downarrow e.
\]
The \( R \)-structure topology of \( \bar{E} \) coincides with the Scott topology, and \(<_C\) with the way below relation \(<_<\) of \( \bar{E} \). An \((\omega\text{-continuous})\) domain is a poset which is isomorphic with the completion of a (countable) \( R \)-structure. An \((\omega\text{-algebraic})\) domain is a poset which is isomorphic with the completion of a (countable) reflexive \( R \)-structure. Every domain (indeed, every \( R \)-structure) is locally compact.

We continue with some remarks on continuous posets and sobriety which are intended to provide some context, but are not strictly necessary for understanding the remainder of the paper. (For sober spaces, see e.g. \([8],[19]\).) First, it may be observed that the notion of an \( R \)-structure is strictly more general than that of a continuous poset. We have:

**Proposition 3.1** Let \((D, \sqsubseteq)\) be a continuous poset, with way-below relation \(<_<\). Then \((D, \sqsubseteq)\) is a \( T_0 \) \( R \)-structure whose specialization order coincides with \( \sqsubseteq \).

**Proof.** That the continuous poset \((D, \sqsubseteq)\) is an \( R \)-structure with respect to its way-below relation, is standard. Also standard is that, for all \( x, y \in D, x \subseteq y \) iff \( \forall z (z << x \rightarrow z << y) \). But this just says that \( \sqsubseteq \) is the specialization order of \((D, \sqsubseteq)\). Since this order is anti-symmetric, the \( R \)-structure is \( T_0 \).

The converse is false:

Let \( E \) be the \( R \)-structure with elements \( a_i, b_i, b (i \in N) \), and \( a_i < a_{i+k}, b_i < b_{i+k} (k \geq 0) \), \( a_i < b_i, a_i < b, b_i < b \) (all \( i \)). \( E \) is obviously \( T_0 \). But viewing \( E \) as a poset, we have: \( b_i << b_i \) is false for all \( i \), since \( \bigsqcup a_i = b \), and so no \( b_i \) is the least upper bound of elements way-below it. Thus, \( E \) is not a continuous poset.

We state without proof a characterization of the \( R \)-structures which are continuous posets:

**Proposition 3.2** A \( T_0 \) \( R \)-structure \((E, <)\) is a continuous poset if and only if each \( e \in E \) is the least upper bound of exactly one round ideal (namely \( \downarrow e \)).

For this and other reasons we regard the \( R \)-structures or neighbourhood systems as more “fundamental” than the continuous posets. (The “other reasons” include the feature, mentioned above, that the basic predicates of a neighbourhood system are semi-decidable, whereas it is not appropriate to require that \( \sqsubseteq \) be semi-decidable in general.)

It is fairly well known that the ideal completion construction, whereby a poset is completed to an algebraic dcpo, can be considered as an instance of
sobrification. For this, the poset is of course taken with Alexandroff topology, the dcpo with Scott topology. We observe that this extends readily to the completion of an $R$-structure (to a continuous dcpo). In fact, the description (of completion as sobrification) is particularly clear in the $R$-structure setting. For we view the completion $\bar{E}$ of an $R$-structure $E$ as again an $R$-structure, its topology given as for any $R$-structure (the upper sets with respect to $<$). We have in particular the following:

**Theorem 3.3** Let $(E, <)$ be an $R$-structure, and $\sqsubseteq$ its specialization order. The sobrification of $E$ is homeomorphic with $\bar{E}$ (and $E$ is sober iff $(E, \sqsubseteq)$ is a continuous dcpo).

**Proof.** (Outline) Any $<$-ideal $I$ of $E$ yields a filter-base of basic open sets as $F(I) = \{\uparrow a | a \in I\}$. This filter-base has the property
\[\uparrow a \in F(I) \Rightarrow \exists b > a. \uparrow b \in F(I)\] (1)
Indeed, it is clear that we have a (1,1)-correspondence between the round ideals and the filter-bases satisfying (1). But the filter-bases satisfying (1) are exactly the completely prime filter-bases. Thus $F(\cdot)$ gives a (1,1)-correspondence between the round ideals and the completely prime filters of (basic) open sets, and we have $e = \bigcup I$, where $I$ is a round ideal, iff $F(I)$ is a neighbourhood base of $e$.

In connection with the following theorem, we note that the statement “the space $X$ is an $R$-structure” is taken to mean that $X$ can be ordered as an $R$-structure in such a way that its topology coincides with the $R$-structure topology. If that is the case, then the ordering can be taken as that given by: $a < b \iff b \in \text{Int}(\uparrow a)$. We have:

**Theorem 3.4** For any space $X$, the following are equivalent:

(i) $X$ is an $R$-structure;
(ii) The sobrification of $X$ is a domain;
(iii) $\mathcal{O}(X)$ is a completely distributive lattice.

This result will not be used in the sequel, and we omit the proof here. The equivalence of (2) with (3) can be pieced together from [6]: see especially Chapter V, Exercise 1.10. Notice that this work provides several equivalents to the notion of a completely distributive lattice, some of which may be more attractive from a domain-theoretic point of view, in particular the following: a distributive lattice $L$ is completely distributive if and only if both $L$ and $L^{op}$ are continuous.

The significance of the fact that $\bar{E}$ is the sobrification of $E$ is that many statements about the topology of $\bar{E}$ can be translated directly into statements about $E$ (since the topologies of $E$ and $\bar{E}$ are isomorphic). For example, we have the following obvious fact:

**Proposition 3.5** An open set $U$ in an $R$-structure $(E, <)$ is dense if and only
if
\[ \forall e \in E \exists u \in U. \; e < u. \quad (2) \]

Then (2) is also a necessary and sufficient condition for the corresponding open set \( \bar{U} \) to be dense in \( \bar{E} \). (We use the notation \( U \leftrightarrow \bar{U} \) for the isomorphism between the topologies of \( E, \bar{E} \).)

Concerning \( G_\delta \) sets in \( \bar{E} \) we have:

**Proposition 3.6** Let \( X = \bigcap_n \bar{U}_n \) be a \( G_\delta \) set in \( \bar{E} \), and let \( x \) be any point of \( X \). Then
\[ x \in X \iff \forall n \exists e \in x. \; e \in U_n. \]

## 4 Maximal points

We now come to the topic of maximality of points in a domain \( \bar{E} \). This is problematic in a constructive sense, since the usual definition involves quantification over the whole of \( \bar{E} \). For this reason Martin-Löf works with a definition involving only quantification over \( E \), which we may state as:

**Definition 4.1** A point \( x \in \bar{E} \) is constructively maximal provided that
\[ \forall a, b \in E. \; a < b \Rightarrow \exists c \in x. \; (a < c \lor b \leq c) \]

The idea behind this definition may be understood by thinking of each pair \( (a, b) \), where \( a < b \), as a “test of fineness”. A point passes the test if it has \( a \) as a neighbourhood, or else has a neighbourhood lying apart from \( b \). A point is as fine as possible, or maximal, if it passes all tests.

We shall use the notation \( CMax_E \) (or just \( CMax \)) for the constructively maximal points of a domain \( \bar{E} \). As a first result about \( CMax \) we have the following (adapted from [14]):

**Proposition 4.2**

(i) Suppose that \( x, y \in CMax_E \), and that \( a \uparrow b \) for every \( a \in x, \; b \in y \). Then \( x = y \).

(ii) Suppose that \( x \in CMax_E, \; y \in \bar{E}, \) and \( x \subseteq y \). Then \( x = y \).

**Proof.**

(i) Assume that \( x \in \bar{E}, \; y \in CMax_E \), and \( a \uparrow b \) for all \( a \in x, \; b \in y \). Let \( c \) be any element of \( x \), and \( d \) an element of \( x \) that is finer than \( c \). By maximality, we have \( e \in y \) such that \( e \leq d \) or \( c < e \). but \( e \uparrow d \); so \( c < e \), and \( c \in y \). By transposing \( x, y \), the result follows.

(ii) If \( x \subseteq y \), then surely \( a \uparrow b \) for all \( a \in x, \; b \in y \). Then by the argument of (1), the hypotheses imply \( y \subseteq x \).
Thus a constructively maximal point is maximal in the ordinary sense. But the converse is not true in general, as we see from the example given in Fig.1 (in which $x \in \text{Max}$, but $x \notin \text{CMax}$).

From what has been said so far, it might seem that the set of constructively maximal points of a domain $D$ depends on the choice of basis of $D$. That this is not so will be a consequence of Theorem 4.3, giving alternative characterizations of $\text{CMax}$. We note that the Lawson topology of $D$ (= $\bar{E}$) is given by taking as subbasic open sets the Scott open sets together with the sets of the form $L_y = \{ x \mid \neg x \subseteq y \}$. We will also make use of the sets $I_x, J_x$ studied by Bukatin in [2]. Namely, $I_x = \{ y \in \bar{E} \mid \{ x, y \} \text{ has no upperbound} \}$, $J_x = \{ y \in \bar{E} \mid x \in \text{Int}(I_y) \}$. (“Int” here refers to the Scott topology.) Clearly, $J_x \subseteq I_x$ for all $X \in \bar{E}$.

**Theorem 4.3** The following are equivalent:

(i) $x \in \text{CMax}$;

(ii) every Lawson neighbourhood of $x$ contains a Scott neighbourhood of $x$;

(iii) $x \in \text{Max}$ and $I_x = J_x$.

**Proof.** (1) $\Rightarrow$ (2): Assume (1). It suffices to prove (2) for each subbasic Lawson neighbourhood $L_y$ of $x$. Thus, given $y \notin x$, choose $e, e' \in y \setminus x$ such that $e < e'$. Since $x \in \text{CMax}$, we have $c \in x$ such that $e < c$ or $e' \nless c$. This implies $e' \nless c$, and so $c$ determines a Scott neighbourhood of $x$ contained in $L_y$.

(2) $\Rightarrow$ (3): Assume (2). If $x \subset y$, then $x = y$, since otherwise $L_y$ would be a Lawson neighbourhood of $x$ not containing any Scott neighbourhood of $x$. Next, suppose $y \in I_x$. Since $x$ is maximal, this is equivalent to $y \notin x$. By (2), $y$ contains a Scott neighbourhood of $x$; thus $y \in J_x$.

(3) $\Rightarrow$ (1): Assume (3), and let $c, d \in E$ be such that $c < d$. If $c \in x$ then $x$ passes the “test” $(c, d)$. If not, choose $e$ with $c < e < d$, and let $y = \downarrow e$. Clearly, $y \in I_x$. Hence $y \in J_x$, and we have $e' \in x$ such that $e, e'$ have no
upper bound (in $E$). Hence $d \not\in d'$ for any $d' \in x$ with $e' < d'$. Again, therefore, $x$ has passed the test $(c, d)$.

The statement that the Scott and Lawson topologies agree on the maximal elements of $\bar{E}$ is known as the Lawson condition for $\bar{E}$. Notice that the Lawson condition is equivalent to the statement that condition (2) of the preceding Theorem holds for every $x \in \text{Max}$. Hence we have:

**Corollary 4.4** The Lawson condition holds for a given domain if and only if $\text{Max} = \text{CMax}$.

**Remark 4.5** Lemma 31 of Waszkiewicz [20] can be interpreted as stating that, if $D$ is an algebraic domain, then the Lawson condition implies that $\text{Max}_D = \text{CMax}_D$.

The remainder of this paper is concerned in part with showing that it is $\text{CMax}$ rather than $\text{Max}$ that has “good” properties. The role of the Lawson condition in ensuring that $\text{Max}$ behaves well is thus explained by the fact that, under this condition, $\text{Max}$ coincides with $\text{CMax}$.

As an indication of the properties of $\text{CMax}$, we have the following:

**Theorem 4.6** (i) For any $R$-structure $E$, $\text{CMax}_E$ is a $G_\delta$ regular Hausdorff subspace of $E$. If $E$ satisfies $(M)$ (or $(N)$: Sections 2,8), $\text{CMax}$ is dense.

(ii) (Baire property.) Assume $(M)$. If $(\bar{U}_i)$ is a sequence of dense open subsets of $E$, there is a dense subset $X$ (with cardinality $\leq |E|$) of $\text{CMax}$ such that $X \subseteq \bigcup_i \bar{U}_i$.

**Proof.** (1) That $\text{CMax}$ is Hausdorff is, in effect, the contrapositive of Proposition 4.2(i). Regularity is clear as well. For, we may choose for any given point $x \in \text{CMax}$ and $a \in x$ a neighbourhood $b$ such that $a < b$ and then, for each $y$ such that $a \notin y$, a neighbourhood $c_y \in y$ lying apart from $b$. Thus we get a closed neighbourhood (complement of $\bigcup_y c_y$ in $\text{CMax}$) of $x$ contained in $\bar{a}$.

To see that $\text{CMax}$ is $G_\delta$, define for each test pair $t = (a, b)$ the open set $U_t = \{e \in E | a < e \text{ or } b \not\in e\}$. Then $\text{CMax}$ is the intersection $\bigcap_t \bar{U}_t$ (cf. Prop.3.6). For density, enumerate the tests as $(t_i)_{i \in \mathbb{N}}$. For any $e \in E$ we can then, by the principle (M), successively choose $e_0, e_1, \ldots$ such that $e < e_0 < e_1 < \ldots$ and $e_i$ satisfies $t_i$ (all $i$). This sequence defines a point of $\text{CMax}$ lying in $\uparrow e$.

(2) By a slight refinement of the preceding argument. Given $e \in E$, we choose the sequence $(e_i)$ as before, but this time satisfying the extra condition that $e_i \in U_i$. To see that this is possible, suppose that $e_k$ has been chosen. Since each $U_i$ is dense, we can find $e'_k \in U_{k+1}$ such that $e_k < e'_k$, and then choose $e_{k+1} > e'_k$ such that $e_{k+1}$ satisfies $t_{k+1}$. Then take $X$ as the set of points (one for each $e$) constructed in this way.
5 Weak metrics, weights, measurements

Our aim in this and the next Section is to show that the maximality ideas considered above can be extended to quantitative domain theory. We begin by recalling some basic definitions.

**Definition 5.1** A quasi-metric on a set $X$ is a map $d : X \times X \to \mathbb{R}^0^+$ satisfying

(i) $d(x,x) = 0$

(ii) $d(x,z) \leq d(x,y) + d(y,z)$; and is $T_0$ if

(iii) $[d(x,y) = d(y,x) = 0] \Rightarrow x = y$.

A weight for $(X,d)$ is a map $w : X \to \mathbb{R}^0^+$ such that

(W) $w(x) + d(x,y) = w(y) + d(y,x)$

**Definition 5.2** A partial metric, or pmetric (Matthews), is a map $p : X \times X \to \mathbb{R}^0^+$ satisfying

(i) $p(x,y) = p(y,x)$

(ii) $[p(x,y) = p(x,x) = p(y,y)] \Rightarrow x = y$

(iii) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ ($\Delta^1$)

(iv) $p(x,x) \leq p(x,y)$

It is well-known that the notions $T_0$ weighted quasi-metric and partial metric are equivalent, via the assignment

$p(x,y) = w(x) + d(x,y)$

and its inverse (i.e. $w(x) = p(x,x), d(x,y) = p(x,y) - p(x,x)$).

The topology induced by a quasi-metric $d$ is given by the $\varepsilon$-balls $\{y | d(x,y) < \varepsilon\}$, exactly as for metrics. The topology induced by a partial metric is defined to be that induced by the associated quasi-metric.

The natural question “Which spaces are quasi-metrizable?” was studied in the first systematic paper on quasi-metrics [21]. For 2nd-countable spaces at least, the answer was extremely simple: all of them are. If $(U_n)_n$ is an enumeration of basic open sets of the space $X$, we need only put

$$d(x,y) = \sum_{x \in U_n, y \not\in U_n} 2^{-n}$$

to quasi-metrize $X$.

What if we ask for a weighted quasi-metric ($\equiv$ pmetric)? Künzi and Vajner [9] provide a subtle discussion of this question, but again the answer is very simple in the 2nd-countable case. We need only supplement the preceding definition of $d$ with:

$$w(x) = 1 - \sum_{x \in U_n} 2^{-n}$$

Since it may be argued that we need to be concerned only with the 2nd-countable case (in computer science or constructive mathematics), the problem seems to be solved almost before we have started.
But the preceding solution has a drawback. One of the main intended features of weight is that it should capture the maximal points of the space as being those of weight 0 [15]. Clearly, however (assuming $T_0$ separation), the weighting of Künzi & Vajner can assign the value 0 to at most one point of the space, namely the greatest point if it exists.

Our proposal is to use an enumeration of fineness tests, rather than a simple enumeration of basic open sets. A point is (constructively) maximal if it passes all tests. So we should be able to measure a point by how many tests it passes.

**Notation.** We shall write $x \models t$, where $t = (a, b), a < b$, for “$x$ passes the test $t$”, that is, $\exists c \in x. a < c \lor b < c$.

Given a countable $R$-structure $(E, <)$, and an enumeration $(t_i)$ of its fineness tests, we define a weight function $w$ on $\bar{E}$ by:

$$w(x) = 1 - \sum \{2^{-n} \mid x \models t_n\}$$

**Proposition 5.3**

(i) $w(x) = 0 \iff x \in CMax$

(ii) $x \subsetneq y \Rightarrow w(x) > w(y)$

(iii) $w$ is Scott-continuous as a map from $\bar{E}$ to $[0, 1]^{op}$.

**Proof.** (1),(2): obvious.

(3) Suppose that $y = \bigsqcup_i x_i$, where the join is directed. For any test $t$, we have that $y \models t$ iff $x_i \models t$ for some $i$. That is:

$$\{t \mid y \models t\} = \bigcup_i \{t \mid x_i \models t\}.$$

Hence, $w(y) = \bigsqcup_i w(x_i)$ (in the $[0, 1]^{op}$ ordering).

In this Section we have seen two kinds of tests used to define weight functions on a 2nd-countable space $X$ (where $X$ has to be of the form $\bar{E}$ for the second kind of test to be applicable). Moreover we have, in effect, seen tests on ordered pairs $(x, y) \in X \times X$ used to define a quasi-metric on $X$. Indeed, a test $t$ is in this case given by a basic open set $U$, and we stipulate:

$$(x, y) \models t \iff x \in U \rightarrow y \in U$$

We may observe some features of these tests which made the constructions “work”. For the tests on single elements (for defining weights) we have, first, that each test should be extensionally an open set. This feature by itself gives (3) of the preceding Proposition. Next, tests should be able to discriminate between points, one of which is strictly greater than the other (in the specialization order):

$$x \sqsubset y \Rightarrow \exists t. y \models t \& \neg x \models t$$

This feature lies behind (2) of the Proposition. A third, less rigid, requirement is that a point which satisfies all tests should be maximal in the space.
Turning to the tests on pairs (for defining distance functions), we observe the following. Each test $t$ determines a binary relation $R_t$, and, at each point $x$, the family of sets $R_t[x]$ is a base of neighbourhoods. (In fact, in the example above, more is true: the family $(R_t)$ of relations is a base of a quasi-uniformity, which in turn induces the topology of the space.)

In the next Section we shall present the preceding observations in a more systematic way, consider further variations on tests, and propose some applications in quantitative domain theory.

6 Measurement and distance

We recall K. Martin’s notion of measurement [12],[13], as a map from a continuous dcpo to $E = [0, \infty]^{op}$. The kernel of such a map $f$ is the set $\{x | f(x) = 0\}$.

**Definition 6.1** A (Scott) continuous map $\mu : D \to E$ on a continuous dcpo $D$ is a measurement if, for every $x \in \ker \mu$ and neighbourhood $U$ of $x$, there exists $\epsilon > 0$ such that $\mu(x) \subseteq U$, where

$$\mu(x) = \{y \in D | y \sqsubseteq x \& |\mu(x) - \mu(y)| < \epsilon\}.$$  

We note that the definition still makes sense if the dcpo $D$ is replaced by an arbitrary space $X$ (the ordering then being taken, of course, as the specialization order $\sqsubseteq_X$). It is also true that the codomain $E$ can be generalized away from $[0, \infty]^{op}$ (Martin[12]), but this extension is of little interest for us here. Also note that the condition about $\mu_{\epsilon}(x)$ is only required to hold for $x \in \ker \mu$. The definition may be parameterized by replacing $\ker \mu$ by an arbitrary subspace $Y$ of $X$ (Martin writes $\mu \to \sigma_Y$ for this, in the case that we are dealing with the Scott topology $\sigma$). In our development of measurements, the $\mu_{\epsilon}$ condition will hold for all $x \in X$, and we shall ignore the relativization of the definition to a subspace. Perhaps we may speak of an unrestricted measurement for the case that the $\mu_{\epsilon}$ condition holds over the whole space.

Given the space $(X, T)$, we may consider the topology on $X$ having as a base the collection of sets $U \cap \downarrow x$ ($U \in T, x \in X$). Martin [12] calls it the $\mu$-topology of $X$ (at least in the case that $X$ is a continuous dcpo). If $B \subseteq T$, we shall call $B$ a $\mu$-base of $X$ provided that the collection of sets $U \cap \downarrow x$ ($U \in B, x \in X$) is a base of the $\mu$-topology. Recall [19] that if $f$ is a map from a set $S$ to a space $Z$, the initial topology induced on $S$ by $f$ is $\{f^{-1}(U) | U \text{ open in } Z\}$. We then have:

**Proposition 6.2** Let $X$ be a space and $w : X \to E$ a map (not assumed to be continuous). The following are equivalent:

(i) The initial topology is a $\mu$-base for $X$.

(ii) $\{w^{-1}(\uparrow r) | r \text{ rational} \}$ is a $\mu$-base for $X$.

(iii) $w$ is a measurement.

**Proof.** That $w$ is a measurement amounts to the statement that each $w^{-1}(O)$
(O open in E) is open in X and, furthermore, for each \( x \in X \), the sets \( w^{-1}(O) \cap \downarrow x \) give a neighbourhood base at \( x \) for the \( \mu \)-topology on \( X \). In other words, the initial topology is a \( \mu \) base for \( X \). At the same time, (1) \( \iff \) (2) since each \( w^{-1}(O) \) is a join of sets \( w^{-1}(\uparrow r) \) (\( r \) rational).

The significance of (2) in the preceding Proposition is that it shows us a countable \( \mu \)-base of \( X \). Conversely, from any countable \( \mu \)-base we get a measurement:

**Proposition 6.3** Every space with a countable \( \mu \)-base possesses a measurement.

**Proof.** Assume that \( X \) has the \( \mu \)-base \( U_0, U_1, \ldots \). Define a weight function \( w \) on \( X \) by:

\[
w(x) = 1 - \sum \{2^{-n} \mid x \in U_n\}
\]

Assume that \( O \) is an open set containing a given point \( x \), and let \( k \) be such that \( x \in U_k \) and \( U_k \cap \downarrow x \subseteq O \cap \downarrow x \). Notice that, if \( y \) is any point below \( x \), the terms \( 2^{-n} \) (whose sum is deducted from 1) occurring in the expression for \( w(y) \) are a subset of those occurring in the expression for \( w(x) \). In particular, if \( y \) is below \( x \) and not in \( U_k \), then \( w(y) \geq w(x) + 2^{-k} \). Hence \( N = w^{-1}(\uparrow (w(x) + 2^{-k})) \) is a neighbourhood of \( x \) such that \( N \cap \downarrow x \subseteq O \).

**Theorem 6.4** Every 2nd-countable locally compact space \( X \) has a measurement.

**Proof.** Without loss of generality we may assume that we have a countable base \( B \) of \( X \) which is at the same time a basis of the continuous lattice \( O(X) \), and which is moreover closed under finite joins. For each pair \( t = (a, b) \), where \( a, b \in B \) and \( a \ll b \), define the open set \( U_t \) by:

\[
U_t = b \cup \bigcup \{V \mid V \text{ is open \\ & disjoint from } a\}.
\]

Now, given any \( x \in X \) and open neighbourhood \( U \) of \( x \), we may choose \( a, b \in B \) such that \( x \in a \ll b \subseteq U \). Then it is easy to check that \( U_{(a, b)} \cup \downarrow x \subseteq U \cup \downarrow x \). Thus the sets \( U_t \) give a countable \( \mu \)-base of \( X \).

As mentioned before, the measurements we construct in this way are “unrestricted”. We may also note that the preceding argument (Theorem 6.4) applies just as well if we have a compact subset \( K \) of \( X \) (rather than just a point \( x \in X \)) and open superset \( U \) of \( K \). Indeed, if \( K \) is any set relatively compact in \( U \), we can interpolate basic open sets \( a, b \) so that \( K \subseteq a \ll b \subseteq U \). Thus the measurement we have constructed is *Lebesgue*, in the sense of Martin [12].

Let us return now to the case that \( X \) is a (continuous) domain \( \bar{E} \), where \( E \) is enumerated as \( e_0, e_1, \ldots \). We note that we can achieve the same effect as with our double-element tests \( (e, e') \) using only a single element \( e \), provided that we interpret \( x \models e_i \) as:

\[
e_i \in x \lor (\forall j \leq i)(e_i < e_j \rightarrow x \text{ has a neighbourhood lying apart from } e_i).
\]
It is clear that the definitions and results of Section 4 stand, with only a
slight rewording, with tests taken in this way. Now, when \((E, <)\) is reflexive,
determining an algebraic domain \(\bar{E}\), the single element test simplifies to:
\[
x \models e \iff e \in x \lor x \text{ has a neighbourhood lying apart from } e.
\]

As has (in effect) been noted by Waszkiewicz \([20]\) we can work with tests
of this kind, in an \(\omega\)-algebraic domain, to define not only a measurement, but
a partial metric. We have only to extend the tests to pairs of points by taking
\((x, y) \models e\) to mean:
\[
e \in x \cap y \lor x, y \text{ have neighbourhoods lying apart from } e.
\]

For the following theorem, we indicate only the main steps in the proof, as a
similar result has been given by Waszkiewicz (although he takes a somewhat
indirect approach, involving an embedding into Plotkin’s universal domain \(T^\omega\)
\([17]\)).

**Theorem 6.5** Every \(\omega\)-algebraic domain \(D\) has a compatible partial metric \(p\)
such that \(p(x, x) = 0\) exactly when \(x \in \text{CMax}_D\).

**Proof.** We take \(D\) as \(\bar{E}\), with \(E\) enumerated as \(e_0, e_1, \ldots\), and \(<\) reflexive.
Define the distance function \(p\) by
\[
p(x, y) = 1 - \sum \{2^{-n} | (x, y) \models e_n\}.
\]

Consider the sharp triangle property:
\[
p(x, z) + p(y, y) \leq p(x, y) + p(y, z).
\]

For this, it suffices to show that each instance of satisfaction of a test on
the right (that is, by \((x, y)\) or by \((y, z)\)) is matched by at least one instance of
satisfaction of the same test on the left (by \((x, z)\) or \((y, y)\)). Now if \((x, y) \models e\)
or \((y, z) \models e\), we evidently have \((y, y) \models e\). So the only case we need to
consider is that in which both \((x, y) \models e\) and \((y, z) \models e\). But for this to hold,
it must occur either that all three of \(x, y, z\) have \(e\) as a neighbourhood, or
else all three have neighbourhoods apart from \(e\). Then both \((x, z) \models e\) and
\((y, y) \models e\).

The remaining properties of a \(p\)-metric are straightforward to verify. In ac-
CORDANCE with the discussion at the end of the preceding Section, we determine
the topology induced by \(p\) by considering the relations \(R_e\) where
\[
R_e(x, y) \equiv (x, x) \models e \rightarrow (x, y) \models e.
\]

(Note. The topology is that of the quasi-distance \(d(x, y) = p(x, y) - p(x, x)\).
This is \(1 - \sum \{2^{-n} | (x, y) \models e_n\}\), where \(t_n\) is the test given by
\[
(x, y) \models t_n \equiv (x, x) \models e_n \rightarrow (x, y) \models e_n.
\]
In fact it is easy to see that the sets \(R_e[x]\) give a neighbourhood base of the
Scott topology at each point \(x\).

The preceding result is a slight extension of Waszkiewicz’ \([20]\), inasmuch as
we dispense with the Lawson condition (while, of course, replacing \(\text{Max}\) with
CMax). The proof does not extend to ω-continuous domains (the argument for ∆² fails). It seems that a quite different aproach is needed to achieve such an extension.

7 Related work

In this section, we briefly consider some related works, of which we became aware only after completing the above. The first two of these were brought to our attention by the referees.

The most substantial connection is with an unpublished paper of Heckmann [7]. In that paper, Heckmann studies “domain environments”, that is, structures \((D, B, M, X)\), where \(D\) is a continuous dcpo, \(B\) a basis of \(D\), \(M\) the set of maximal elements of \(D\), and \(X\) a subset of \(M\). The idea is that a domain \(D\) may be used to represent, not only a space homeomorphic to Max\(D\), but a space homeomorphic to a subspace of Max\(D\). In his Proposition 1.10, Heckmann defines strongly maximal for an element \(x\) of \(D\) by several equivalent conditions, one of which reads:

\[(4) \text{ For every } a \gg b \text{ in } B : x \in \text{cl}(\uparrow a) \text{ implies } x \in \uparrow b.\]

It is not too difficult to see that this condition is equivalent, in classical logic, with the assertion that \(x \in CMax\). Heckmann remarks that strong maximality, as formulated in condition (4), “may be better suited for an effective treatment than maximality.” No further discussion of this point is provided, however, and it is clear that Heckmann has arrived at this view independently of Martin-Löf.

Important notions in [7] are the closed approximation and strong closed approximation properties of a domain environments. The closed approximation property is in effect a formulation of the Lawson condition. (This is not stated explicitly in the paper, but is clearly intended, as one sees from Heckmann’s Proposition 1.23 and following remark.) The reader is referred to the paper for the definition of “strong closed approximation” (SCA). From Theorem 1.22 we learn that \(M^* (= CMax)\) is the largest subset of \(M (= Max)\) enjoying the property SCA. Combining this with Corollary 1.21, which asserts that \(M\) has closed approximation iff it has SCA, it is easily deduced (though, again, not quite explicit in the paper) that the Lawson condition obtains if and only if \(Max = CMax\).

Via these identifications, there is a substantial overlap between our Section 4 and Heckmann’s results. Beyond this, the paper [7] contains much of interest for the representation of spaces by subsets of Max\(D\), and especially for the investigation of CMax itself.

The remaining cases of (possibly) related work can be treated more briefly. A referee asks whether there might be a connection between constructive maximality and the finitary subspaces of Escardó [5]. Specifically, the connection would be with Supp\(X\): the smallest finitary dense sober subspace of a sober
space $X$. It must be said that the prospects for a genuine connection look remote. The notions of constructivity involved in the two ideas seem to be quite different. Moreover, whereas $CMax(D)$ is a subset of $Max(D)$, $Supp(X)$ is always a superset of $Max(D)$. Towards reducing the gap slightly, we note (Theorem 3.3.13 of [5]) that, in the stably compact case, we have equality between $Max$ and $Supp$ provided that (actually: if and only if) $Max$ is compact. Now, in the case of a stably compact domain, we of course have equality between $Max$ and $CMax$. In ongoing work with Ralph Koppermann, we are considering the stable compactification of domains, a procedure under which the anomalous elements (those which are maximal but not constructively maximal) “disappear”. In that context, it is conceivable that some connection between the two strands of work might emerge.

Finally, we mention that Martin-Löf’s predicative definition of maximality (that is, what we call constructive maximality) has been used systematically in G. Curi’s work in formal topology [3],[4]. Curi’s work has, in its details, almost no overlap with ours, but has the advantage of being carried out in an entirely constructive fashion.

8 Concluding remarks

Mention of G. Curi and formal topology brings us back to a topic from Section 2, namely the defect in the definition of apartness given there: its reliance on classical negation. Indeed, a pair of “contrary” predicates, such as overlap and apartness, should be presented more symmetrically than was done in Section 2. The main point is that, besides being mutually exclusive, these predicates exhibit a relaxed (or perhaps one might say, rounded) exhaustiveness. That is, rather than

$$a \uparrow b \lor a \not\approx b$$

we have

$$(1) \quad a < a' \rightarrow (a \uparrow b \lor a' \not\approx b).$$

It is a feature of $R$-structures/neighbourhood systems that basic predicates occur in contrary pairs. (The predicate $<$ itself has a useful contrary, in this sense.)

As an illustration, we see that by invoking (1), we can derive the principle (M) (Section 2) from (N) by simple constructive reasoning. Indeed, suppose that we have neighbourhoods (belonging to a given $R$-structure) $a, b, c$, with $b < c$. By (1) we have that $b \uparrow a \lor c \not\approx a$. If the first alternative holds, we may choose $d$ to be any neighbourhood finer than both $a, b$; whilst if $c \not\approx a$, we choose (by (N)) $d$ to be any neighbourhood finer than $a$. So (M) is proved. We plan to investigate contrary predicate-pairs and their uses on another occasion.

A desideratum for a quasi-metric (or partial metric) over a domain $D$ is that, besides inducing the Scott topology, its symmetrization should induce the Lawson topology. For example, Lawson [11] requires, for the extension
Smyth

theorem given there, a metric which induces the Lawson topology. But this
creates a difficulty for some approaches to defining weak metrics. Indeed
partial metrics whose self-distance is 0 on $CMax$ cannot be expected to (sym-
metrically) induce the Lawson topology in all cases. To see this, let $D$ be $\mathbb{N}_{\perp}$. We require $w(n) = 0$ (all $n$), and $w(\perp) > 0$; say $w(\perp) = 1$. We calculate successively: $d(\perp, n) = 0$; $d(n, \perp) = 1$; $d*(\perp, n) = 1$. Hence $\perp$ is an isolated point, whereas in the Lawson topology $\perp$ is the limit of the sequence 0, 1, 2...

Somewhat related to this issue is the fact that, in this paper, we have only
worked with measurements and distances introduced by an “extrinsic” enu-
meration of a basis. It would be advantageous to consider quantities (metric
primitives) intrinsic to an $R$-structure $E$, and characterized by some axioms.
For example, as observed following Definition 6.1 above, $E$ can possess a mea-
surement regardless of whether it is complete. But there is much more to be
said on this.

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