The inertial Jacquet–Langlands correspondence

Andrea Dotto

Abstract
We give a parametrization of the simple Bernstein components of inner forms of a general linear group over a local field by invariants constructed from type theory, and explicitly describe its behaviour under the Jacquet–Langlands correspondence. Along the way, we prove a conjecture of Broussous, Sécherre and Stevens on preservation of endo-classes.

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1 Introduction.
The construction of types for Bernstein components of an inner form of \(\text{GL}_n(F)\), for \(F\) a local non-archimedean field, has been initiated by Bushnell and Kutzko in the split case and continued and eventually completed by Broussous, Sécherre and Stevens in general. Meanwhile, Bushnell and Henniart provided a uniform description, for varying \(n\), of the objects which enter these constructions, relying on the basic notion of endo-class of simple characters, and they started a programme aiming to use type theory to describe various instances of Langlands functoriality for general linear groups, such as the local Langlands correspondence, the Jacquet–Langlands correspondence, automorphic induction and base change of representations. This paper completes this programme for the Jacquet–Langlands correspondence at the level of inertial classes of representations.

Let \(A = M_m(D)\) be a central simple algebra over \(F\), for a central division algebra \(D\) of reduced degree \(d\) over \(F\). Then \(G = \text{GL}_m(D) = A^\times\) is an inner form of \(H = \text{GL}_n(F)\) for \(n = md\). Recall that the Jacquet–Langlands correspondence is a bijection

\[
\text{JL} : D(G) \to D(H)
\]

between the sets of essentially square-integrable representations (or discrete series representations) of these groups, characterized by the equality

\[
(-1)^m \text{tr}(\pi) = (-1)^n \text{tr(}\text{JL}\pi)
\]
on matching regular elliptic elements of \(G\) and \(H\). Here, \(\text{tr}(\pi)\) denotes the Harish-Chandra character of \(\pi\), identified with a function on regular semisimple elements.
The category of smooth representations of the groups $G$ and $H$, as for any other connected reductive group over $F$, decomposes according to the action of the Bernstein centre. A block in the Bernstein decomposition corresponds to a component of the Bernstein variety, hence to an inertial class of supercuspidal supports, and two discrete series representations are in the same block if and only if they are unramified twists of each other. Since the Jacquet–Langlands correspondence commutes with twisting by characters, it yields a bijection

$$JL : \mathcal{B}_{ds}(G) \rightarrow \mathcal{B}_{ds}(H)$$

on the sets of blocks containing discrete series representations. These are the simple blocks. The irreducible representations contained in a simple block are said to form a simple inertial class.

In order to describe this map explicitly one needs a parametrization of both sides in terms of objects that can be compared to each other, which we provide by generalizing the results of [Dol18] to inner forms of $GL_n(F)$. Precise definitions will be given later, and we just sketch the construction here. Given a simple inertial class $s$ for $GL_m(D)$, we denote by $\text{cl}(s)$ the endo-class of simple characters attached to $s$ in $[BSS12]$, which coincides with the endo-class of maximal simple characters contained in any factor of the supercuspidal support of $s$. Fixing a lift of $\text{cl}(s)$ to its unramified parameter field, and a conjugacy class $\kappa(\text{cl}(s))$ of maximal $\beta$-extensions in $G$ of endo-class $\text{cl}(s)$, we construct a second invariant of the simple inertial classes of $G$, which we denote by $s \mapsto \lambda_{\kappa(\text{cl}(s))}(s)$. It consists of a set of characters of the multiplicative group of a finite field, corresponding to a representation of a finite general linear group. Our parametrization is given by the fact that these two invariants $\text{cl}(s)$ and $\Lambda_{\kappa(\text{cl}(s))}(s)$ determine the inertial class $s$ uniquely (see theorem 3.9). To emphasize the fact that it depends on the choice of a lift $\Theta_F$, we write its inverse in a slightly different manner.

We introduce triples $(\Theta_F, \Theta_E, [\chi])$, consisting of

1. an endo-class $\Theta_F$ defined over $F$, of degree $\delta(\Theta_F)$ dividing $n$
2. a lift $\Theta_E \rightarrow \Theta_F$ of $\Theta_F$ to its unramified parameter field $E$
3. a Galois orbit of characters of $e_{n/\delta(\Theta_F)}^\times$ under the action of $\text{Gal}(e_{n/\delta(\Theta_F)}/e)$.

We will sometime refer to these as inertial triples for $GL_n(F)$. Here, $e$ denotes the residue field of $E$ and $e_{n/\delta(\Theta_F)}$ denotes an extension of degree $n/\delta(\Theta_F)$ of $e$. Our parametrization gives rise to a surjection $s_{GL_m(D)}$ from the set of inertial triples for $GL_n(F)$ to the set of simple inertial classes of irreducible representations of $GL_m(D)$, and we give an explicit description of its fibers. Furthermore, $\text{cl}(s_{GL}(\Theta_F, \Theta_E, [\chi])) = \Theta_F$ and $\Lambda_{\kappa(\text{cl}(s))}(s_{GL}(\Theta_F, \Theta_E, [\chi])) = [\chi]$ if it is computed with respect to the lift $\Theta_E \rightarrow \Theta_F$.

Using the modular version of type theory developed in [MS14], and the block decomposition of $[SS16a]$, we use a version of this parametrization which works over any algebraically closed coefficient field $R$ with characteristic different from $p$, and we study its behaviour under reduction modulo $\ell$ of an integral $\mathfrak{Q}_\ell$-representation. We also prove a compatibility result with respect to parabolic induction. To state this, let $s$ be a supercuspidal inertial class for $GL_m(D)$, and consider the inertial class $s_a$ of $GL_{am}(D)$ corresponding to the supercuspidal support $(GL_m(D)^a, \pi^{\otimes a})$ for any $\pi \in s$, where $a \geq 1$ is an integer. Then $\text{cl}(s_a) = \text{cl}(s)$ (as it only depends on the supercuspidal support) and there exists a $\beta$-extension $\kappa(\text{cl}(s))_a$ in $GL_{am}(D)$, of endo-class $\text{cl}(s)$, such that $\Lambda_{\kappa(\text{cl}(s))_a}(s_a)$ is the inflation of $\Lambda_{\kappa(\text{cl}(s))}(s)$ through the norm of the extension $e_{am/\delta(\Theta_F)}/e_{am/\delta(\Theta_F)}$.

Our main results are the following. Let $s_G$ and $s_H$ be simple inertial classes of complex representations for the groups $G$ and $H$ respectively, and assume that $s_H = JL(s_G)$.

**Theorem.** The equality $\text{cl}(s_G) = \text{cl}(s_H)$ holds.

Since $\text{cl}(s)$ coincides with the endo-class attached to a simple inertial class in $[BSS12]$ (see remark 3.12) this theorem implies conjecture 9.5 in $[BSS12]$, the “endo-class invariance conjecture”. We also study the behaviour of the second invariant in our parametrization.
Theorem. Let $\Theta_F = \text{cl}(s_G) = \text{cl}(s_H)$, and let $\epsilon^\beta_G$ and $\epsilon^\beta_H$ be the symplectic sign characters attached to any maximal simple character in $G$ and $H$ of endo-class $\Theta_F$. Let $\kappa^\text{can}_G$ and $\kappa^\text{can}_H$ be the $p$-primary conjugacy classes of maximal $\beta$-extensions in $G$ and $H$. Then

$$\Lambda_{\epsilon^\beta_G \kappa^\text{can}_G}(s_G) = \Lambda_{\epsilon^\beta_H \kappa^\text{can}_H}(s_H).$$

Recall from [Dot18] that, for a certain quadratic character $\epsilon_{\text{Gal}}$ uniquely determined by $\Theta_F$, the representation $\kappa^\text{can}_F = \epsilon_{\text{Gal}} \epsilon^\beta_H \kappa^\text{can}_H$ is the canonical $\beta$-extension of endo-class $\Theta_F$ for the group $H$: it has the property that $\Lambda_{\kappa^\text{can}_F}(\pi)$ coincides with the level zero part of the Langlands parameter $\text{rec}(\pi)$ for any simple representation $\pi$ of endo-class $\Theta_F$. If we let $\kappa^\text{can}_G = \epsilon_{\text{Gal}} \epsilon^\beta_G$, the theorem above implies an analogous property of $\kappa^\text{can}_G$. Namely, if $\pi$ is an essentially square-integrable representation of $G$, then $\Lambda_{\kappa^\text{can}_G}(\pi)$ coincides with the level zero part of $\text{rec}(\text{JL}(\pi))$.

We heavily use the techniques developed in [BH11] and [SS16b], which prove special cases of our results in the context of essentially tame endo-classes. The parametrization is constructed in section 3, applying the method of “rigidification via a lift” developed in [Dot18]. To do this, we need to generalize some well-known properties of simple characters of $\text{GL}_n(F)$ to the nonsplit case, which we do in sections 2 and 3. Section 4 develops a character formula analogous to that in [BH11], keeping track of rigidifications throughout. Then we prove the first of the theorems above, applying the method of [SS16b] and a new technique to reduce to the split case. A comparison of character formulas then implies the supercuspidal case of the second theorem, and we deduce the general case as in section 8 of [SS16b].

We end this introduction by pointing out that this paper does not accomplish a local proof of the existence of the Jacquet–Langlands correspondence. The main problem is that the type is not directly related to the character, and even less so for non-cuspidal discrete series representations. Via [MS14a], our parametrization of simple inertial classes can be made independent of the Jacquet–Langlands correspondence, and one could write down a map

$$\text{JL} : s_G(\Theta_F, \Theta_E, [\chi]) \mapsto s_H(\Theta_F, \Theta_E, [\chi])$$

(using canonical $\beta$-extensions for $G$ and $H$) and prove directly it is a bijection. The problem would then be to prove that the representations in matching inertial classes satisfy the character identity. The method we use in the paper assumes the existence of the Jacquet–Langlands transfer and manages to compute enough character values to characterize it completely, but the proof of this characterization relies upon knowing the existence of the transfer.

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Notation and conventions. Fix a local non-archimedean field $F$ of residue characteristic $p$ and an algebraic closure $\overline{F}/F$, and write $\mathfrak{f}$ for the residue field, $\mathcal{O}_F$ for the ring of integers and $\pi_F$ for a uniformizer. Similar notation will be used for other local fields and central division algebras over them (so for instance $\mathfrak{e}$ is the residue field of $E$). Write $F_d$ for the unramified extension of $F$ of degree $d$ in $\overline{F}$, and $f_d$ for the extension of $\mathfrak{f}$ of degree $d$ in the algebraic closure of $\mathfrak{f}$ given by the residue field of the maximal unramified extension of $F$ in $\overline{F}$. The group of Teichmüller roots of unity in $F$ is denoted $\mu_F$, and the absolute value on $F$ is normalized so that $|\pi_F| = |\mathfrak{f}|^{-1}$. Whenever discussing simple characters, a choice of additive character $\psi_F$ of $\mathcal{O}_F$ will be made implicitly, and whenever $E/F$ is a finite extension this will be $\psi_E = \psi_F \circ \text{tr}_{E/F}$.

Representations of a locally profinite group like $\text{GL}_m(D)$ will be tacitly assumed to be smooth. The coefficient field will change in the course of the paper, but will always be an algebraically closed field of characteristic different from $p$, and we will specify it explicitly when needed. Characters are not assumed
to be unitary, and whenever a character \( \chi \) of a group \( G \) and a representation \( \pi \) of a subgroup \( H \subseteq G \) are given, the representation \( \pi \otimes \chi |_H \) will be called a twist of \( \pi \); when \( G \) is a \( p \)-adic reductive group and \( \chi \) is an unramified character, this will be called an unramified twist.

For a central simple algebra \( A \) over \( F \) and \( E/F \) a field extension in \( A \), the commutant of \( E \) in \( A \) will be denoted \( Z_A(E) \), and the centralizer and normalizer of \( E \) in \( G = A^\times \) will be denoted \( Z_G(E) = Z_A(E)^\times \) and \( N_G(E) \) respectively. For \( x \in G \), we write \( \text{ad}(x) \) for the automorphism \( z \mapsto xzx^{-1} \) of \( A \).

For an extension \( 1/k \) of finite fields, an element \( x \in I \) is \( k \)-regular if it has \([1:k]\) different conjugates under \( \text{Gal}(1/k) \). A \( k \)-regular character of \( 1^k \) is defined similarly, via the right action \( g : \chi \mapsto g^\chi = \chi \circ g \) of \( \text{Gal}(1/k) \) on characters. In general, pullback by an automorphism \( \sigma \) will be denoted \( g^\sigma \). Notice that \( x \in 1^k \) can be \( k \)-regular and still generate a proper subgroup of \( 1^k \) (consider, for instance, an extension of prime degree).

For any character \( \alpha \) of \( 1^k \), define the stabilizer field \( k(\alpha) \) as the fixed field of \( \text{Stab}_{\text{Gal}(1/k)}(\chi) \). It only depends on the orbit of \( \alpha \) under \( \text{Gal}(1/k) \), which will be denoted \([\alpha]\). Similarly, if \( \ell \) is a prime number then the character \( \alpha \) decomposes uniquely as a product \( \alpha = \alpha_{(\ell)} \alpha^{(\ell)} \) in which \( \alpha_{(\ell)} \) has order a power of \( \ell \) and \( \alpha^{(\ell)} \) has order coprime to \( \ell \); because this decomposition is unique, the orbit \([\alpha]\) is independent of the representative \([\alpha]\), and similarly for \([\alpha_{(\ell)}]\). The orbit \([\alpha_{(\ell)}]\) is the \( \ell \)-regular part of \([\alpha]\). We’ll often apply the following lemma.

**Lemma 1.1.** If \( 1/k \) is an extension of finite fields, and \( \chi \) is a character of \( 1^k \), then there exists a unique \( k \)-regular character \( \chi^{\text{reg}} \) of \( k[\chi]^\times \) such that \( \chi = \chi^{\text{reg}} \circ N_{1/k}[\chi] \).

**Proof.** Since the norm map \( N_{1/k}[\chi] \) is surjective, for the existence part it suffices to prove that if \( N_{1/k}[\chi](x) = 1 \) then \( \chi(x) = 1 \). But by Hilbert 90, \( N_{1/k}[\chi](x) = 1 \) if and only if \( x = \frac{g}{y} \) for some \( g \in \text{Gal}(1/k[\chi]) \) and \( x \in 1^k \), and then \( \chi(x) = 1 \) as \( \chi \) is \( \text{Gal}(1/k[\chi]) \)-stable. Uniqueness holds because \( N_{1/k}[\chi] \) is surjective, and regularity holds because the stabilizer of \( \chi \) in \( \text{Gal}(1/k) \) is \( \text{Gal}(1/k[\chi]) \).

Throughout the article, the reduced degree of a central division algebra \( D \) over \( F \) (positive square root of the \( F \)-dimension) is denoted by \( \text{d} \). Usually \( \text{GL}_m(D) \) will denote an inner form of \( \text{GL}_m(F) \), so that \( n = \text{md} \). The character “absolute value of the reduced norm” is an unramified character of \( \text{GL}_m(D) \), denoted \( \nu \). An unramified twist \( \pi \otimes (\chi \circ \nu) \) will usually be written \( \chi \pi \).

## 2 Maximal simple types.

Let \( G = \text{GL}_m(D) \) for a central division algebra \( D \) of reduced degree \( d \) over \( F \) (where possibly \( D = F \)). This is the group of \( F \)-points of a connected reductive group \( G/F \), which is an inner form of \( \text{GL}_{md,F} \) splitting over \( F_d \). Fix a central simple algebra \( A \) of dimension \( n^2 \) over \( F \) and a simple left \( A \)-module \( V \) such that the opposite of the endomorphism algebra \( \text{End}_A(V) \) is isomorphic to \( D \). Then \( D \) acts to the right on \( V \), and upon a choice of basis \( A \) identifies with the matrix algebra \( \text{M}_m(D) \) (passing to the opposite algebra ensures that the multiplication is as expected). In this section we recall some basic properties of the objects which go into the definition of types for cuspidal representations of \( G \). A lot of this material is standard, but we need generalizations to the non-split case of certain well-known properties of simple characters of \( \text{GL}_n(F) \), and we couldn’t find these in the literature.

The coefficient field for representations will be an algebraically closed field \( R \) of characteristic different from \( p \). By [SS16a], an analogue of the Bernstein decomposition holds for the category of smooth \( R \)-representations of \( G \), and the blocks are in bijection with inertial equivalence classes of supercuspidal supports. We recall that a supercuspidal support consists of a pair \( (L, \sigma) \) consisting of the group \( L \) of \( F \)-points of an \( F \)-Levi factor of an \( F \)-parabolic subgroup of \( G \), together with a supercuspidal \( R \)-representation \( \sigma \) of \( L \). Two such pairs \( (L_i, \sigma_i) \) are inertially equivalent if there exists \( g \in G \) and an unramified character \( \chi \) of \( L_i \) such that \( L_2 = gL_1g^{-1} \) and \( \chi \sigma_1 = \text{ad}(g)\sigma_2 \). The block corresponding to an inertial class \([L, \sigma]\) consists of those smooth representations all of whose irreducible subquotients have supercuspidal support in \([L, \sigma]\) (see section 10.1 in [SS16a]). As stated, this definition requires the uniqueness of supercuspidal support up to conjugacy for an irreducible representation, which is proved in section 6.2 of [MS14a].

The set of irreducible representations of \( G \) contained in a block is called an inertial class. The simple inertial classes are those corresponding to inertial equivalence classes of the form \([\text{GL}_{md_{(n)}}(D), \pi_0^n] \) for a divisor \( r_0 \) of \( m \). An irreducible representation contained in a simple inertial class will be called a simple representation. Over the complex numbers, every essentially square-integrable representation is simple.
Lattice sequences. Consider lattice sequences in the space $V$, which are decreasing functions

$$\Lambda : \mathbb{Z} \to \{\sigma_D\text{-lattices in } V\}$$

where the right hand side is ordered by inclusion, such that there exists a positive integer $e$ with $\Lambda_{k+e} = \Lambda_k \sigma_D$ for all $k$. The number $e$ is called the period of the sequence; the sequence is called a chain, or a strict sequence, if it is strictly decreasing. A sequence is called uniform (see [Pro87], 1.7) if it is a chain and the dimension of $\Lambda_k/\Lambda_{k+1}$ over the residue field $d$ of $D$ is constant as $k$ varies.

Every sequence defines a hereditary $\sigma_F$-order $\mathfrak{A} = \mathfrak{P}_0(\Lambda)$ in $A$ equipped with a filtration by $\sigma_F$-lattices $\mathfrak{P}_n(\Lambda)$, via

$$\mathfrak{P}_n(\Lambda) = \{a \in A : a \Lambda_k \subseteq \Lambda_{k+n} \text{ for all } k \in \mathbb{Z}\}.$$ 

The Jacobson radical $\mathfrak{P}(\mathfrak{A})$ of $\mathfrak{A}$ then equals $\mathfrak{P}_1(\Lambda)$ (see [Sec04] 1.2), and we write $U^n(\Lambda)$ for $1 + \mathfrak{P}_n(\Lambda)$. The normalizer $\mathcal{R}$ of a sequence is defined as

$$\mathcal{R}(\Lambda) = \{g \in A^\times : \text{ there exists } n \in \mathbb{Z} \text{ such that } g(\Lambda_k) = \Lambda_{k+n} \text{ for all } k\}.$$ 

Such an integer $n$ is then unique and denoted $v_\Lambda(g)$; this defines a morphism $\mathcal{R}(\Lambda) \to \mathbb{Z}$ whose kernel $U(\mathfrak{A})$ is the unit group of $\mathfrak{A}$. The unit groups of hereditary $\sigma_F$-orders in $A$ are precisely the parahoric subgroups of $A^\times$. As in [Sec04] 1.2, this set-up defines a bijection $\Lambda \mapsto \mathfrak{P}_0(\Lambda)$ from lattice chains up to translation in $\mathbb{Z}$ to hereditary orders in $A$. It follows that the normalizer of a lattice chain coincides with the normalizer in $G$ of the corresponding hereditary order.

Let $E/F$ be a field extension in $A$. An $\sigma_D$-lattice sequence $\Lambda$ in $V$ is called $E$-pure if $E^\times \subseteq \mathcal{R}(\Lambda)$. This condition is equivalent to $\Lambda$ being an $\sigma_D$-lattice sequence in $V$ viewed as an $E$-vector space. Denote by $B = Z_A(E)$ the commutant of $E$ in $A$. This is a central simple algebra over $E$ of $E$-dimension $n'^2$, say, so $B \cong M_{n'}(D_E)$ for some central division $E$-algebra $D_E$ of $E$-dimension $d'^2$, and we have the identities

$$n' = \frac{n}{[E:F]}, \quad d' = \frac{d}{(d,[E:F])}, \quad m'd' = n'$$

as in [BH11], 2.1.1.

The need of considering general lattice sequences instead of focusing on chains, which can be done in the split case, arises from the behaviour of filtrations of hereditary orders attached to $E$-pure sequences under intersection $\mathfrak{A} \cap B$. Upon fixing a simple left $B$-module $V_E$, one has the following result.

**Theorem 2.1** (Theorem 1.4 in [SS08]). Given an $E$-pure lattice sequence $\Lambda$ in $V$, there exists a unique up to translation $\sigma_{D_E}$-lattice sequence $\Gamma$ in $V_E$ such that

$$\mathfrak{P}_k(\Lambda) \cap B = \mathfrak{P}_k(\Gamma) \text{ for all } k \in \mathbb{Z},$$

and the normalizer $\mathcal{R}(\Gamma)$ equals $\mathcal{R}(\Lambda) \cap B^\times$.

The sequence $\Gamma = \text{tr}_B \Lambda$ is called the trace of the lattice sequence $\Lambda$, and $\Lambda$ the continuation of $\Gamma$. Notice that the theorem does not say that every $\sigma_{D_E}$-lattice sequence has a continuation: this doesn’t necessarily hold (see [SS08] Exemple 1.6). Usually, $\mathfrak{B}$ will denote the hereditary order $\mathfrak{A} \cap B = \mathfrak{P}_0(\Gamma)$.

When $a, b \in \mathbb{Z}$, we can rescale a lattice sequence $\Lambda$ to

$$a\Lambda + b : k \mapsto \Lambda_{\lfloor \frac{k+b}{e} \rfloor},$$

and the set of these sequences is called the affine class of $\Lambda$. If $\Lambda = a\Lambda_0$ for a lattice chain $\Lambda_0$, the sequence $\Lambda$ will be called a multiple of $\Lambda_0$, and $\mathcal{R}(\Lambda) = \mathcal{R}(\Lambda_0)$: what changes is the filtration on this group. The map $\Lambda \mapsto \text{tr}_B(\Lambda)$ preserves affine classes. One can’t say much about the trace of an arbitrary sequence—for instance, the trace of a chain needn’t be a chain, see [BL02] section 6—but the following result on preimages holds.

**Proposition 2.2.** Assume $\Lambda$ is an $E$-pure lattice sequence in $V$ whose trace $\Gamma = a\Gamma_0$ is a multiple of a uniform chain $\Gamma_0$ of $\sigma_{D_E}$-period $r$. Then $\Lambda$ is a multiple of a uniform chain of period $\frac{re(E/F)}{(d, re(E/F))}$.

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1The notation $(d,[E:F])$ stands for the highest common factor of $d$ and $[E:F]$. 

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Proof. By [BL02] proposition II.5.4, if $\Gamma$ is a multiple of a uniform chain then so is $\Lambda$. By [SS08] théorème 1.7 and its proof, there exists a unique chain $\Lambda_0$ in $V$ whose trace is a multiple of $\Gamma_0$, and the $\sigma_D$-period of $\Lambda_0$ is $re(E/F)/(d, re(E/F))$ (see also [BSS12] lemma 4.18). The claim now follows as $\Lambda$ is a multiple of some chain, which must be $\Lambda_0$. \qed

Simple characters. We only discuss simple characters attached to simple strata of the form $[\mathfrak{A}, \beta]$, consisting of a principal $\sigma_F$-order $\mathfrak{A}$ in $A$ attached to a lattice chain $\Lambda$ in $V$, and an element $\beta \in A$ generating a field $E = F[\beta]$, such that $E^\times \subseteq \mathfrak{r}(\Lambda)$ and the condition

$$k_0(\beta, \mathfrak{A}) < 0$$

on the critical exponent holds (see for instance [Sec04] for an exposition). We follow [BH14] in shortening notation to $[\mathfrak{A}, \beta]$ for what is otherwise denoted $[\mathfrak{A}, -v_\Lambda(\beta), 0, \beta]$, as these are the only strata which will show up in what follows.

As in proposition 3.42 in [Sec04] and section 2.5 in [BH11], there exist $\sigma_F$-orders $h(\beta, \mathfrak{A}) \subseteq i(\beta, \mathfrak{A}) \subseteq \mathfrak{A}$ attached to a simple stratum $[\mathfrak{A}, \beta]$ in $A$, with a filtration by ideals $h^k(\beta, \mathfrak{A})$ and $i^k(\beta, \mathfrak{A})$. There are compact open subgroups $H(\beta, \mathfrak{A}) = h(\beta, \mathfrak{A})^\times$ and $J(\beta, \mathfrak{A}) = i(\beta, \mathfrak{A})^\times$, with filtrations by subgroups

$$J^k(\beta, \mathfrak{A}) = J(\beta, \mathfrak{A}) \cap U^k(\mathfrak{A}) = 1 + i^k(\beta, \mathfrak{A})$$

$$H^k(\beta, \mathfrak{A}) = H(\beta, \mathfrak{A}) \cap U^k(\mathfrak{A}) = 1 + h^k(\beta, \mathfrak{A}).$$

These groups are normalized by $J(\beta, \mathfrak{A})$ and by $\mathfrak{r}(\mathfrak{A}) \cap B^\times$. $H^k$ is normal in $J^k$ and the quotients $J^k/H^k$ are finite-dimensional vector spaces over $F_p$ (see [Sec04] proposition 4.3). The inclusion induces isomorphisms $\mathfrak{B}/\mathfrak{B}_1(\mathfrak{A}) \to j(\beta, \mathfrak{A})/j^1(\beta, \mathfrak{A})$ and $U(\mathfrak{B})/U^1(\mathfrak{B}) \to J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A})$.

The group $H^1(\beta, \mathfrak{A})$ carries a distinguished finite set $C(\mathfrak{A}, \beta)$ of simple characters, which is fundamental for the construction of types, and is defined and studied in [Sec04] and [SS08], section 2. These references treat the more general case of simple characters of positive level, which form a set $\{\mathfrak{A}, m, \beta\}$: one has $H^1(\mathfrak{A}, m, \beta) = C(\mathfrak{A}, 0, \beta)$. The definition of $C(\mathfrak{A}, \beta)$ also depends on the choice of an additive character $\psi$ of $F$, which is fixed throughout. Since the group $H^1(\beta, \mathfrak{A})$ is a pro-$p$ group, these characters are valued in $\mu_{\psi\Lambda}(R)$, and there is a canonical bijection from the simple characters over $\mathbb{Q}_\ell$ to those over $\mathbb{F}_\ell$, given by reduction mod $\ell$, whenever $\ell \neq p$ is a prime number.

Simple characters satisfy the “intertwining implies conjugacy” property to various degrees; in full generality, one has the following result, which can be strengthened in the split case (see [BK93] Theorem 3.5.11, and [BH11], 2.6). In order to state it, we need the notion of an embedding in $A$; this is a pair $(E, \Lambda)$, where $E$ is a field extension of $F$ in $A$, and $\Lambda$ is an $E$-pure $\sigma_D$-lattice sequence in $V$. Two embeddings are equivalent if there exists $g \in A^\times$ such that $\Lambda_1$ and $g\Lambda_2$ coincide up to translation, and $g$ conjugates the maximal unramified extensions of $F$ in $E_1, E_2$ of degree dividing $d$. Two simple strata $[\mathfrak{A}_1, \beta_1]$ have the same embedding type if the embeddings $(F[\beta_1], \Lambda_1)$ are equivalent, where $\Lambda_1$ is the chain attached to $\mathfrak{A}_1$.

Theorem 2.3 (See [BSS12] Theorem 1.12). Given two simple strata $[\mathfrak{A}, \beta_1]$ with the same embedding type, and two simple characters $\theta_i \in C(\mathfrak{A}, \beta_i)$ which intertwine in $A^\times$, let $K_i$ be the maximal unramified extension of $F$ in $F[\beta_i]$. Then there exists $u \in \mathfrak{r}(\mathfrak{A})$ such that

1. $K_2 = uK_1u^{-1}$

2. $H^1(\beta_2, \mathfrak{A}) = uH^1(\beta_1, \mathfrak{A})u^{-1}$ and $\theta_1 = ad(u)^*\theta_2$.

Endo-classes. Consider now all the groups $\text{GL}_n(F)$ and their inner forms $\text{GL}_m(D)$ for varying $n$, and the set of all simple characters of these groups. There is an equivalence relation on this set, called endo-equivalence, which is discussed in [BH96] in the split case and [BSS12] in general. An endo-class of simple characters over $F$ is an equivalence class for this equivalence relation. The endo-class of a simple character $\theta$ will be denoted $\text{cl}(\theta)$. Again, we identify endo-classes of simple $\mathbb{Q}_\ell$-characters and simple $\mathbb{F}_\ell$-characters.

It is important to notice that we might have two endo-equivalent simple characters $\theta_i$, of endo-class $\Theta_F$, defined by simple strata $[\mathfrak{A}_i, \beta_i]$, in which the extensions $F[\beta_i]$ of $F$ are not isomorphic. However, by [BH96] 8.11 and [BSS12] lemma 4.7, they will have the same ramification index and residue class degree. The degrees
is $J$-degree, are unramified, and normalize the $J$-izes $	heta$ it normalizes $A$.

Because the orders $B$ of $\mathcal{O}$, it normalizes $A$.

Proposition 2.4. Maximal simple strata are sound.

Proof. Let $[\mathfrak{A}, \beta]$ be a maximal simple stratum, corresponding to a lattice chain $\Lambda$ in $V$. By definition, $\mathfrak{A} \cap B = \mathfrak{B}$ is a maximal order in $B$. Consider the trace $\Gamma = \text{tr}_B(\Lambda)$. Then $\mathfrak{B}_0(\Gamma) = \mathfrak{B}$, and it follows that the chain $\Gamma_0$ associated to $\Gamma$ is principal of period 1. So necessarily $\Gamma = t\Gamma_0$ for some positive integer $t$. It follows that $\mathfrak{R}(\mathfrak{B}) = \mathfrak{R}(\Gamma_0)$ is actually equal to $\mathfrak{R}(\Gamma)$: what changes is the filtration on it. Since $\mathfrak{R}(\Gamma) = \mathfrak{R}(\mathfrak{A}) \cap B^\times = \mathfrak{R}(\mathfrak{B})$, that is, the stratum $[\mathfrak{A}, \beta]$ is sound.

The relation of endo-equivalence between maximal simple characters in the same group takes on a simple form: it coincides with conjugacy.

Proposition 2.5. Maximal simple strata $[\mathfrak{A}_i, \beta_i]$ in the same central simple algebra $A$ over $F$, defining endo-equivalent maximal simple characters $\theta_i$, have the same embedding type. Endo-equivalent maximal simple characters in the same group are conjugate.

Proof. Write $B_{\beta_i} = Z_A(F[\beta_i])$, and let $\Lambda_i$ be the lattice chains in $V$ corresponding to the $\mathfrak{A}_i$. By the Skolem–Noether theorem there exists $x \in A^\times$ conjugating the maximal unramified extensions of $F$ in $F[\beta_i]$, as they have the same degree $f(\Theta_F)$ over $F$, so we can assume that they both coincide with a subfield $E$ of $A$. Because the orders $\mathfrak{B}_i = \mathfrak{A}_i \cap B_{\beta_i}$ are maximal, there are extensions of $F[\beta_i]$ in $B_{\beta_i}$ which have maximal degree, are unramified, and normalize the $\mathfrak{A}_i$.

To see this, observe that $\mathfrak{B}_i^{\times}$ is a maximal compact subgroup of $B_{\beta_i}^{\times}$. Choose any maximal unramified extension $L_i$ of $F[\beta_i]$ in $B_{\beta_i}$, so that $\mathfrak{O}^{\times}_{L_i}$ is contained up to conjugacy in $\mathfrak{B}_i^{\times}$. Since $L_i^{\times} = \pi_{F[\beta]}^{Z_{\mathfrak{B}_i}} \times \mathfrak{o}_{L_i}^{\times}$, we have $L_i^{\times} \subseteq \mathfrak{R}(\mathfrak{A}_i)$. By proposition 2.4 we have $\mathfrak{R}(\mathfrak{A}_i) \cap B_{\beta_i}^{\times} = \mathfrak{R}(\mathfrak{B}_i)$ and so $L_i^{\times} \subseteq \mathfrak{R}(\mathfrak{A}_i)$.

To prove that $[\mathfrak{A}_1, \beta_1]$ have the same embedding type, it is enough to prove that $\text{tr}_{Z_A(E)(\Lambda_i)}$ are conjugate under $Z_A(E)^{\times}$ (up to translation), as then the same will hold for their continuations $\Lambda_i$ by the uniqueness statement in theorem 2.4. The sequences $\Lambda_i = \text{tr}_{B_{\beta_i}}(\Lambda_i)$ are both multiples of a chain of period 1, since $\mathfrak{B}_i$ is a maximal order. By proposition 2.2 the period of $\Lambda_i$ determines that of $\Lambda_i$, so $\Lambda_1$ and $\Lambda_2$ have the same period. Again by proposition 2.2, the sequence $\text{tr}_{Z_A(E)}(\Lambda_i) = a_i \Gamma_i$ is a multiple of a uniform chain $\Gamma_i$, as its trace to $L_i$ must be a multiple of the unique lattice chain for $L_i$. The period of $\Gamma_i$ determines that of $\Lambda_i$, hence here we deduce that $\Gamma_1$ and $\Gamma_2$ have the same period $t$. By the proof of [SS08] theorem 1.7, the integer $a_i$ then equals $\dfrac{d}{\text{trace}(E/F)} = \dfrac{d}{\text{trace}(t)}$ and is independent of $i$, and so the sequences $\text{tr}_{Z_A(E)}(\Lambda_i)$ are conjugate under $Z_A(E)^{\times}$ up to translation.

That $\theta_1$ and $\theta_2$ are conjugate now follows from theorem 2.3.

Proposition 2.6. If $[\mathfrak{A}_1, \beta_1]$ and $[\mathfrak{A}_2, \beta_2]$ are maximal simple strata in $A$ defining the simple character $\theta$, then $\mathfrak{A}_1 = \mathfrak{A}_2$ and $J^i(\beta_1, \mathfrak{A}_1) = J^i(\beta_2, \mathfrak{A}_2)$ for $i = 0, 1$.

Proof. To see that $J^i(\beta_1, \mathfrak{A}_1) = J^i(\beta_2, \mathfrak{A}_2)$ argue as in [BH14] (2.1.1). The normalizer $J(\theta)$ of $\theta$ in $G$ can be computed as follows. We know from [Sec05] proposition 2.3 that the intertwining of $\theta$ in $G$ is $J(\beta, \mathfrak{A})B_{\beta}^{\times} J(\beta, \mathfrak{A})$, for any maximal simple stratum $[\mathfrak{A}, \beta]$ defining $\theta$, and that $\mathfrak{R}(\mathfrak{B}_\beta)J(\beta, \mathfrak{A})$ normalizes $\theta$ (using that $\mathfrak{R}(\mathfrak{B}_\beta) = \mathfrak{R}(\mathfrak{A}) \cap B_\beta$ by proposition 2.4). Now assume that $g \in B_{\beta}^{\times}$ normalizes $\theta$. Then it normalizes $U^1(\beta, \mathfrak{A}) \cap B_{\beta}^{\times} = U^1(\mathfrak{B}_\beta)$ (this equality is claimed in [Sec05] after Remarque 2.4). But the normalizer of $U^1(\mathfrak{B}_\beta)$ in $B_{\beta}^{\times}$ equals $\mathfrak{R}(\mathfrak{B}_\beta)$, by the argument in [BK93] 1.1, hence $g \in \mathfrak{R}(\mathfrak{B}_\beta)$ and so the normalizer $J(\theta)$ equals $\mathfrak{R}(\mathfrak{B}_\beta)J(\beta, \mathfrak{A})$.

Then the normalizer $J(\theta)$ has a unique maximal compact subgroup $J_\theta$, which equals $J(\beta, \mathfrak{A})$ for any maximal simple stratum $[\mathfrak{A}, \beta]$ defining $\theta$, and $J_\theta$ has a unique subgroup $J^i_\theta$ that is maximal amongst its normal pro-$p$ subgroups, and $J^i_\theta$ equals then $J^i(\beta, \mathfrak{A})$. This recovers the groups $H^1, J^1$ and $J$ intrinsically to $\theta$. 
By proposition\(^2\), the strata \([\mathfrak{A}, \beta]\) have the same embedding type, hence for some \(g \in G\) we have \(g\mathfrak{A}_1 g^{-1} = \mathfrak{A}_2\). The characters \(g^* \theta\) and \(\theta\) intertwine, hence there exists \(u \in \hat{\mathfrak{A}}(\mathfrak{A}_1)\) conjugating them, by theorem\(^2\). So \((gu)^* \theta = \theta\), and then \(gu \in J(\theta)\) and conjugates \(\mathfrak{A}_1\) to \(\mathfrak{A}_2\). But \(J(\theta)\) normalizes \(\mathfrak{A}_1\), as it equals \((\hat{\mathfrak{A}}(\mathfrak{A}_1) \cap B_{\beta_1})J_\theta\), and so \(\mathfrak{A}_1 = \mathfrak{A}_2\), as \(gu\) normalizes \(\mathfrak{A}_1\) and at the same time conjugates it to \(\mathfrak{A}_2\). \(\square\)

By proposition\(^2\), the groups \(H^1(\beta, \mathfrak{A}), J^1(\beta, \mathfrak{A})\) and \(J(\beta, \mathfrak{A})\) for a simple stratum \([\mathfrak{A}, \beta]\) defining a maximal simple character \(\theta\) only depend on \(\theta\), and will be denoted \(H^1_\theta, J^1_\theta\) and \(J_\theta\).

The endo-classes of \(F\) can be lifted and restricted through tamely ramified field extensions \(E/F\). In this context there exists a restriction map

\[
\text{Res}_{E/F} : \mathcal{E}(E) \rightarrow \mathcal{E}(F)
\]

from the set of endo-classes of simple characters of \(E\) to those for \(F\). It is surjective, and its fiber over a given endo-class \(\Theta_F\) consists by definition of the set of \(E\)-lifts of \(\Theta_F\). We’ll need some details as to how the lifting can be performed in practice, in the unramified case.

**Proposition 2.7** (See [BH96] section 7 and [BSS12] sections 5 and 6). Let \(\theta\) be a maximal simple character in \(A\) defined by the simple stratum \([\mathfrak{A}, \beta]\), with endo-class \(\Theta_F\). Let \(K\) be an unramified extension of \(F\) in \(A\) such that \(\beta\) commutes with \(K\) and generates a field extension of \(K\) in \(A_K = A(\alpha)(K)\), and \(K[\beta]^\times \subseteq \hat{\mathfrak{A}}(\mathfrak{A})\). Then \(\theta_K = \theta|H^1_\theta \cap A_K\) is a simple character, with

\[
\begin{align*}
H^1_{\theta_K} &= H^1_\theta \cap A_K \\
J^1_{\theta_K} &= J^1_\theta \cap A_K \\
J_{\theta_K} &= J_\theta \cap A_K.
\end{align*}
\]

These groups will be denoted \(H^1_K, J^1_K\) and \(J_K\) respectively. The character \(\theta_K\) is called the interior \(K\)-lift of \(\theta\). Its endo-class \(\Theta_K = \text{cl}(\theta_K)\) is a \(K\)-lift of \(\Theta_F\).

If \(\alpha : F_1 \rightarrow F_2\) is a continuous isomorphism between local fields, it induces a pullback

\[
\alpha^* : \mathcal{E}(F_2) \rightarrow \mathcal{E}(F_1)
\]

on the sets of endo-classes. When a central simple algebra \(A\) over \(F_2\) is given, together with a simple character \(\theta\) in \(A^\times\), one can regard \(A\) as a central simple \(F_1\)-algebra via \(\alpha\), and then \(\text{cl}_{F_1}(\theta)\), the endo-class of \(\theta\) as a simple character over \(F_1\), is equal to \(\alpha^* \text{cl}_{F_2}(\theta)\). The functoriality property

\[
(\alpha_1 \alpha_2)^* = \alpha_2^* \alpha_1^*
\]

also holds. It follows that the group of continuous automorphisms of \(F\) acts to the right on the set \(\mathcal{E}(F)\) of endo-classes of \(F\). The action will be denoted \(g : \Theta_F \rightarrow \Theta'_F = g^* \Theta_F\).

**Moving from \(H\) to \(J\).** Let \(\theta\) be a maximal simple character in \(G\). By proposition 2.1 in [MS14b], there exists a unique irreducible representation \(\eta = \eta(\theta)\) of \(J^1_\theta\) which contains \(\theta\), called the Heisenberg representation attached to \(\theta\). The dimension of \(\theta\) is a power of \(p\) and the restriction \(\eta|H^1_\theta\) is a multiple of \(\theta\), and \(\theta\) and \(\eta(\theta)\) have the same \(G\)-intertwining. By section 2.4 of [MS14b], there exists an extension of \(\eta\) to \(J_\theta\) with the same \(G\)-intertwining as \(\theta\) and \(\eta\), called a \(\beta\)-extension or \(\beta\)-extension of \(\eta\). By [Séc05] théorème 2.28 and (2.2) in [MS14b] we know that the group of characters of \(e^\times\) is transitive on the set of \(\beta\)-extensions of \(\eta\), by the twisting action

\[
\chi : \kappa \mapsto \kappa \otimes (\chi \circ \nu_B)
\]

where \(\chi : e^\times \rightarrow R^\times\) has been inflated to \(e^\times_E\), and \(\nu_B : \mathfrak{A}^\times \rightarrow e^\times_E\) is the reduced norm.

**Proposition 2.8.** Assume that \(\mathfrak{B} = \mathfrak{A} \cap B\) is a maximal order in \(B\). Then there exists exactly one \(\beta\)-extension \(\kappa\) of \(\eta\) to \(J_\theta\) such that the determinant character of \(\kappa\) has order a power of \(p\). We will refer to \(\kappa\) as a \(p\)-primary \(\beta\)-extension.
Proof. Write $E = F[β]$. Fix an $E$-linear isomorphism

$$\Phi : B \to M_{m'}(D')$$

where $D'$ is a central division algebra of reduced degree $d'$ over $E$, such that the order $\mathcal{B}$ gets mapped to $M_{m'}(pD')$. We then get an isomorphism $Φ : J_θ/J_1^θ \to \mathcal{B}^\times/U^1(\mathcal{B}) \to GL_{m'}(d')$, for $d'$ the residue field of $D'$, via $Φ$ above and the inverse of the isomorphism $\mathcal{B}^\times/U^1(\mathcal{B}) \to J_θ/J_1^θ$ induced by the inclusion.

Let $κ$ be a $β$-extension of $η$. The determinant character $\det κ$ has prime-to-$p$ part $(\det κ)^{(p)}$ that is trivial on the pro-$p$ group $J_θ$, hence $(\det κ)^{(p)}$ is the inflation to $J_θ$ of a character $γ$ of $d^\times$ through the determinant of $GL_{m'}(d')$ and the isomorphism $Φ$. Assume that $γ$ is norm-inflated from $e^\times$. Observe that $\det(κ ⊗ (χ ⊕ ( γ) ) = \det(κ)(χ^{, \dim κ} ⊕ ν_B)$. Now since $\dim κ$ is a power of $p$ and the character group of $e^\times$ has order prime to $p$, there exists a unique $χ$ such that $χ^{, \dim κ} ⊕ ν_B|J = \det κ^{(p)}$, and the claim follows.

So it’s enough to prove that $γ$ is norm-inflated from $e^\times$, which happens if and only if $γ$ is stable under $\mathrm{Gal}(d'/e)$. If $π_{D'}$ is a uniformizer of $D'$, its conjugacy action on $B^\times$ induces under $Φ$ the Frobenius automorphism on matrix entries, so it’s enough to prove that the restriction of $(\det κ)^{(p)}$ to $B^\times$ is normalized by $π_{D'}$; and this is true because $B^\times$ intertwines $κ$, hence it intertwines $\det κ$ and $(\det κ)^{(p)}$. \[\square\]

Maximal simple types. Fix a maximal simple character $θ$ in $G$, with corresponding Heisenberg representation $η$. Let $κ$ be a $β$-extension of $η$ to $J_θ$. Let $σ$ be a cuspidal irreducible representation $J_θ/J_1^θ$, and define $λ = σ ⊕ κ$. A pair $(J_θ, λ)$ arising thus is called a maximal simple type in $G$. Over the complex numbers, these are types for the supercuspidal inertial classes of $G$. The modular case is different, due for instance to the fact that there may exist cuspidal non-supercuspidal representations, which will contain maximal simple types but won’t exhaust an inertial class. However, the following result holds, for which see the introduction to [MS14b] and the references therein.

**Theorem 2.9.** Let $ρ$ be an irreducible cuspidal representation of $G$. Then

1. $ρ$ contains a unique $G$-conjugacy class of maximal simple types.

2. If $(J, λ)$ is a maximal simple type contained in $ρ$, then $λ$ admits extensions to its normalizer $J(λ)$, and for precisely one such extension $Λ$ the compact induction $π(Λ) = \text{ind}^G_{J(λ)} Λ$ is isomorphic to $ρ$.

3. two irreducible cuspidal representations $ρ_i$ contain the same maximal simple types if and only if they are unramified twists of one another.

We will need some information on the structure of the normalizers $J(λ)$. If $(J_θ, λ)$ is any maximal simple type arising from $θ$ and $σ$, and $[\mathcal{B}, β]$ is a maximal simple stratum defining $θ$, by [MS14b] paragraph 3.4 the order $s(σ)$ of the stabilizer of $σ$ in $\mathrm{Gal}(e_{d'}/e)$ equals the index of $F[β]^\times J_θ$ in $J(λ)$. Fixing an isomorphism $B \to M_{m'}(D')$, we have $J(θ) = π(\mathcal{B}) J_θ = π_Z^\times, λ J_θ$ for any uniformizer $π_{D'}$ of $D'$, and so the index of $J(λ)$ in $J(θ)$ equals the size $b(σ)$ of the orbit of $σ$ under $\mathrm{Gal}(e_{d'}/e)$.

Symplectic signs. Let’s work over the complex numbers until the end of this section. Fix a maximal simple character $θ$ in $G$. Then one has a well-defined map

$$J_1^θ/H_1^θ × J_1^θ/H_1^θ \to μ_p(C), (x, y) \to θ[x, y],$$

where $μ_p(C)$ is the group of complex roots of unity of order $p$ and $[x, y] = xyx^{-1}y^{-1}$. By proposition 2.3 in [Sec05], this map is a symplectic form on the $\mathbb{F}_p$-vector space $J_1^θ/H_1^θ$: it is alternating, $\mathbb{F}_p$-bilinear and nondegenerate.

This is a special case of the following situation, for which we refer to [BF83], section 8, and [BH10], section 3. Consider triples $(G, N, θ)$ where $G$ is a group with a normal subgroup $N$ such that the quotient $V = G/N$ is a finite-dimensional $\mathbb{F}_p$-vector space, and $θ$ is a faithful character $θ : N \to \mathbb{C}^\times$ such that $θ$ is stable under conjugation by $G$ and $(gN, hN) \mapsto θ[g, h]$ is a symplectic form on $V$. In the above, we have $G = J_1^θ/\ker(θ)$ and $N = H_1^θ/\ker(θ)$.

**Proposition 2.10.** (See [BF83] 8.3.3) There exists a unique irreducible representation $η = η(θ)$ of $G$ which contains $θ$, called the Heisenberg representation attached to $θ$. The dimension of $θ$ is a power of $p$ and the restriction $η|N$ is a multiple of $θ$. 

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Let now $\Gamma$ be a finite cyclic subgroup of the automorphism group $\text{Aut}(G)$, of order prime to $p$, preserving $N$ and the character $\theta$, so that $V$ is a symplectic $F_p$-representation of $\Gamma$. Because $\Gamma$ is cyclic, $\eta$ extends to $\Gamma \rtimes G$. All extensions are twists of each other by characters inflated from $\Gamma$. Since the dimension of $\eta$ is a power of $p$ and the order of $\Gamma$ is prime to $p$, there exists a unique extension $\tilde{\eta}$ such that $\tilde{\eta}|\Gamma$ has trivial determinant character (as in the proof of proposition 2.8).

In section 8 of [BF83], there is defined a function $t_{\Gamma,V}$ on $\Gamma$ for each symplectic representation $V$ of $\Gamma$, with the following properties:

1. $t_{\Gamma,V}$ is valued in $\mathbb{Z}$ and nowhere vanishing.
2. if $V = V_1 \perp V_2$ is the orthogonal sum of subspaces $V_1$ and $V_2$, then $t_{\Gamma,V} = t_{\Gamma,V_1}t_{\Gamma,V_2}$
3. $t_{\Gamma,V}(x)$ only depends on the cyclic subgroup of $\Gamma$ generated by $x$.
4. if $V$ arises as $G/N$ from $G, N, \theta$ in the above situation, then $t_{\Gamma,V}(x)$ equals the trace of $x$ on $\tilde{\eta}$.

We will be interested in the sign of the function $t_{\Gamma,V}$, which will be denoted $x \mapsto \epsilon(x, V) = \frac{t_{\Gamma,V}(x)}{t_{\Gamma,V}(x)}$. Observe that $\pm \epsilon(x, V)$ is rarely a character of $\Gamma$. However, in section 3 of [BH10] there is defined a sign $\epsilon^0(V)$ and a character $\epsilon^1(x, V)$ of $\Gamma$, in such a way that if $x$ generates $\Gamma$ then $\epsilon(x, V) = \epsilon^0(V)\epsilon^1(x, V)$. By proposition 5 (where $\epsilon^1$ is denoted $t^1$) if $\Delta$ is a subgroup of $\Gamma$ with $V^\Sigma = V^\Delta$ then actually

$$\epsilon(x, V) = \epsilon^0(V)\epsilon^1(x, V)$$

holds for any generator $x$ of $\Delta$. We will apply all this in the case where $G = J_\theta / \ker(\theta)$ and $N = H^1 / \ker(\theta)$ for a maximal simple character $\theta$, and $\Gamma = \mu_K$ for certain unramified extensions $K/F$.

## 3 Invariants of simple inertial classes.

This section constructs the parametrization of simple inertial classes we shall use. We begin with a maximal simple character $\theta$ of endo-class $\Theta_F$ in $G = \text{GL}_m(D)$. The case of level zero representations can be regarded as corresponding to maximal simple characters with trivial endo-class, which (by definition) are the trivial characters of the pro-unipotent radicals $U^1(\mathfrak{B})$ of maximal compact subgroups of $G$. Choose a stratum $[\mathfrak{A}, \beta]$ defining $\theta$, and let $\mathfrak{B} = \mathfrak{A} \cap B$. Since $\theta$ is maximal, $\mathfrak{B}$ is a maximal order in the central simple algebra $B$ over the field $F[\beta]$.

Fix an $F[\beta]$-linear isomorphism

$$\Phi : B \rightarrow M_{m'}(D')$$

where $D'$ is a central division algebra of reduced degree $d'$ over $F[\beta]$, such that the order $\mathfrak{B}$ gets mapped to $M_{m'}(D_D')$. The inverse of the isomorphism $U(\mathfrak{B})/U^1(\mathfrak{B}) \rightarrow J_\theta / J_\theta^1$ induced by the inclusion, together with $\Phi$, induces an isomorphism

$$\Phi : J_\theta / J_\theta^1 \rightarrow U(\mathfrak{B})/U^1(\mathfrak{B}) \rightarrow \text{GL}_{m'}(d').$$

Notice however that there is no canonical choice of $D'$ such that $B \cong M_{m'}(D')$ (a choice of such an isomorphism amounts to a choice of a simple left module for $B$). This makes it hard to compare these quotients between endo-equivalent maximal simple characters in different groups, and other issues arise from the fact that there may be more than one simple stratum $[\mathfrak{A}, \beta]$ defining $\theta$. To deal with this issue, we introduce an analogue of the notion of tame parameter field in [BH10] 2.6. Our analogue only accounts for the unramified part of parameter fields but works uniformly across inner forms of $\text{GL}_n(F)$, and suffices to treat inertial classes of representations.

### 3.1 Lifts and rigidifications.

A parameter field for a maximal simple character $\theta$ is by a definition a subfield of $A$ of the form $F[\beta]$ for a simple stratum $[\mathfrak{A}, \beta]$ defining $\theta$. An unramified parameter field is a subfield of $A$ of the form $F[\beta]^{ur}$ for a parameter field $F[\beta]$, the maximal unramified extension of $F$ in $F[\beta]$. 

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Proposition 3.1. Let $\theta$ be a maximal simple character in $A^\times$ and let $T_1, T_2$ be unramified parameter fields for $\theta$. Then

1. there exists $j \in J^1_0$ conjugating $T_1$ to $T_2$
2. if $j \in J^1_0$ normalizes an unramified parameter field $T$ for $\theta$, then it centralizes it.

It follows that there exists exactly one isomorphism $E_1 \to E_2$ which can be realized by conjugation by elements of $J^1_0$.

Proof. This is very similar to [BH14], 2.6 Proposition. Let $[\mathfrak{A}, \beta_i]$ be strata defining $\theta$ with $T_i = F[\beta_i]^{ur}$. For the first part, given a generator $\zeta_1$ of $\mu_{T_1}$, there exists some generator $\zeta_2 \in \mu_{T_2}$ and some $j_1 \in J^1_0$ such that $\zeta_2 = \zeta_1 j_1$. This is because the inclusion yields isomorphisms $U(\mathfrak{B}_{\beta_i})/U^1(\mathfrak{B}_{\beta_i}) \to J_0/J^1_0$ embedding $\mu_{T_1}$ in the centre of $J_0/J^1_0$. The centre is given by the image of $\sigma_{D_1}$, hence might be larger than the image of $\mu_{T_1}$, but it will still be a cyclic group. Since the $\mu_{T_i}$ have the same order, as the $T_i$ have the same degree $f(\Theta_F)$ over $F$, they will have the same image under these maps.

By [BH14], 2.6 Conjugacy Lemma, $\zeta_2 = \zeta_1 j_1$ is $J^1_0$-conjugate to some $\zeta_3 = \zeta_1 j_2$, where $j_2 \in J^1_0$ commutes with $\zeta_1$. But then $j_2 = 1$ as its order has to be both a power of $p$ (as $j_2 \in J^1_0$) and prime to $p$ (as $j_2 = \zeta_1^{-1} \zeta_3$ and the factors at the right hand side commute). So the generator $\zeta_2$ of $\mu_{T_2}$ is $J^1_0$-conjugate to the generator $\zeta_1$ of $\mu_{T_1}$, and the claim follows.

The second part holds as $\mu_T$ generates $T$ over $F$ and embeds in $J_0/J^1_0$, on which the conjugation action of $J^1_0$ is trivial. ⊓⊔

The degree of an unramified parameter field of $\theta$ over $F$ equals $f(\Theta_F)$, which is independent of the choice of $[\mathfrak{A}, \beta]$, defining $\theta$, and even of the choice of a representative $\Theta$ of $\Theta_F$. Let $E = F[\Theta_F]$, the unramified extension of $F$ in $\bar{F}$ of degree $f(\Theta_F)$. By proposition 3.1 between any two unramified parameter fields $E_i$ for $\theta$ there is a distinguished isomorphism $\iota_{E_1, E_2} : E_1 \to E_2$. Choose $F$-linear isomorphisms

$$\iota_T : E \to T$$

for any parameter field $T$ for $\theta$, such that $\iota_{T_1, T_2} \iota_{T_1} = \iota_{T_2}$ throughout. Denote the system of the $\iota_T$ by $\iota$.

Now fix a parameter field $F[\beta]$ for $\theta$, and an $F[\beta]$-linear isomorphism $\Phi : B \to M_{m'}(D')$. The choice of $\iota$ yields a distinguished embedding $e \to d'$, and the extension $d'/e$ then has degree $d' = d/(d, \delta(\Theta_F))$, where $\delta(\Theta_F) = [F[\beta] : F]$, so we get a well-defined $\text{Gal}(e'/e)$-orbit of $e'$-linear isomorphisms $d' \to e'$ and $M_{m'}(d') \to M_{m'}(e')$. In all, the choice of $\iota$ specifies, for every maximal simple stratum $[\mathfrak{A}, \beta]$ defining $\theta$ and every $F[\beta]$-linear isomorphism $\Phi : B \to M_{m'}(D')$, an isomorphism

$$\Psi : J_0/J^1_0 \to \text{GL}_{m'}(e'),$$

well-defined up to the action of $\text{Gal}(e'/e)$ on matrix entries.

Proposition 3.2. The conjugacy class $[\Psi(\iota)]$ of this isomorphism under the natural action of $\text{Gal}(e'/e) \times \text{GL}_{m'}(e')$ on $\text{GL}_{m'}(e')$, by inner automorphisms and Galois action on matrix entries, is independent of the choice of $[\mathfrak{A}, \beta]$ and $\Phi$, and only depends on $\theta$ and $\iota$.

Proof. Take two maximal simple strata $[\mathfrak{A}, \beta_i]$ defining $\theta$, and fix $F[\beta_i]$-linear isomorphisms $\Phi_i : B_i \to M_{m'}(D_i')$ to central division algebras $D_i'$ over $F[\beta_i]$. We obtain isomorphisms

$$J_0/J^1_0 \to U(\mathfrak{B}_{\beta_i})/U^1(\mathfrak{B}_{\beta_i}) \to \text{GL}_{m'}(d_i') \to \text{GL}_{m'}(e_i') \quad (3.1)$$

well-defined up to Galois action on coefficients, where the first map is the inverse of the natural inclusion, the second is induced by $\Phi_i$, and the third by an arbitrary choice of an isomorphism $d_i' \\to e_i'$ that is $e'$-linear for the embedding $e \to d_i'$ induced by $\iota_{F[\beta_i]^{ur}} : E \to F[\beta_i]^{ur}$. The integers $m_i'$ and $d_i'$ coincide as they only depend on the endo-class of $\theta$.

Observe that $3.1$ arises from an analogous sequence

$$j(\beta_i, \mathfrak{A})/j^1(\beta_i, \mathfrak{A}) \to \mathfrak{B}_i/\mathfrak{P}_1(\mathfrak{B}_i) \to M_{m'}(d_i') \to M_{m'}(e_i')$$
of \( e \)-linear ring isomorphisms between \( e \)-algebras, on passing to the groups of units. The equality \( j^1(\beta, A) = j^1(\beta_2, A) \) holds since \( j^1(\beta, A) = J^1(\beta, A) - 1 \). The orders \( j(\beta, A) \) have the same group of units, since \( j(\beta, A)^{\times} = J(\beta, A) \). The quotient \( j(\beta, A)/j^1(\beta, A) \) is additively generated by its group of units (as for all matrix algebras over fields), hence \( j(\beta_1, A) = j(\beta_2, A) \).

The \( e \)-algebra structure on \( j(\beta, A)/j^1(\beta, A) \) comes from the embedding \( t_{F[\beta]^w} \) for \( i = 1, 2 \), and by construction these embeddings are conjugate by the action of \( J_0^1 \). So these two \( e \)-algebra structures coincide. The claim follows as we have two \( e \)-linear ring isomorphisms \( j(\beta, A)/j^1(\beta, A) \to M_a(\mathfrak e) \), which therefore differ by the action of \( \text{Gal}(\mathfrak e/e) \times \text{GL}_m(\mathfrak e) \) by the Skolem–Noether theorem.

We next show how the choice of a lift \( \Theta_E \to \Theta_F \) of \( \Theta_F \) to \( E \) defines such a system \( \iota \)-isomorphism. Let \( [\beta_i, \beta_i] \) for \( i = 1, 2 \) be a simple stratum in \( A \) defining \( \theta \), and consider the unramified parameter field \( T_i = F[\beta_i]^w \) of \( F \). Proposition 3.1 applies, as \( \beta_i \) commutes with \( T_i \) and \( T_i[\beta_i] = F[\beta_i] \) is a field with \( F[\beta_i]^w \subseteq \mathfrak A(\mathfrak A) \), and we get an interior lift \( \theta_{T_i} \). Fix compatible isomorphisms \( \iota_{T_i} : E \to T_i \) as in section 3.1. We get endo-classes

\[
\Theta_i^1 = \iota_{T_i}^* \text{cl}(\theta_{T_i}).
\]

**Proposition 3.3.** The endo-classes \( \Theta_E^1 \) and \( \Theta_F^1 \) are equal.

**Proof.** Because the \( \iota_{T_i} \) are compatible, we have \( \iota_{T_2} = \iota_{T_1} \circ \iota_{T_1} \), for \( \iota_{T_1, T_2} : T_1 \to T_2 \) the only isomorphism induced by conjugation by elements of \( J_0^1 \) (see proposition 3.1). The relation

\[
\Theta_E^2 = \iota_{T_2}^* \text{cl}(\theta_{T_2}) = \iota_{T_2} \iota_{T_1}^* \iota_{T_1, T_2} \text{cl}(\theta_{T_2})
\]

holds. Assume \( \iota_{T_1, T_2} \) is induced by conjugation by \( j \in J_0^1 \). Then

\[
\iota_{T_1, T_2}^* \text{cl}(\theta_{T_2}) = \text{cl}(\text{ad}(j)^* \theta_{T_2}).
\]

However, \( \text{ad}(j)^* \theta_{T_2} \) is the \( T_1 \)-lift of \( \text{ad}(j)^* \theta = \theta \), hence \( \text{ad}(j)^* \theta_{T_2} = \theta_{T_1} \), and the claim follows.

**Proposition 3.4.** The group \( \text{Gal}(E/F) \) is simply transitive on the set \( \text{Res}^{-1}_{E/F}(\Theta_F) \) of \( E \)-lifts of \( \Theta_F \).

**Proof.** The action has been defined at the end of section 2. By [BH03] 1.5.1, \( \text{Gal}(E/F) \) is transitive on \( \text{Res}^{-1}_{E/F}(\Theta_F) \), which is in bijection with the set of simple components of \( E \otimes_F F[\beta] \) for any parameter field \( F[\beta] \) for \( \theta \). But \( E \) is \( F \)-isomorphic to the maximal unramified extension of \( F \) in \( F[\beta] \), hence

\[
E \otimes_F F[\beta] \cong \prod_{\sigma : E \to F[\beta]} F[\beta]
\]

and so the fiber \( \text{Res}^{-1}_{E/F}(\Theta_F) \) has as many elements as \( \text{Gal}(E/F) \).

It follows that for any unramified parameter field \( T \) for \( \theta \) we can define \( \iota_T : E \to T \) to be the only \( F \)-linear isomorphism such that \( \iota_T^* \text{cl}(\theta_T) = \Theta_E \); by proposition 3.4, \( \iota_T \) is well-defined, and by proposition 3.3 this defines a compatible system of isomorphisms. So an inertial triple gives rise to a conjugacy class

\[
\Psi(\Theta_E) : J_0^1/J_0^1 \to \text{GL}_m(\mathfrak e)
\]

for any maximal simple character \( \theta \) in \( G \) with endo-class \( \Theta_F \), by setting \( \Psi(\Theta_E) = \Psi(i) \) for the \( i \) just constructed.

### 3.2 Level zero maps.

We begin by recalling the definition and basic properties of the \( K \)-functor associated to a \( \beta \)-extension \( \kappa \) of a maximal simple character \( \theta \) in \( G \), as in section 5 of [MS14b]. This is an exact functor from the category of smooth representations of \( G \) to the category of representations of \( J_0^1/J_0^1 \), defined by \( K_\kappa : \pi \mapsto \text{Hom}_{J_0^1}(\kappa|_{J_0^1}, \pi|_{J_0^1}) \), with the \( J_0^1 \)-action by \( f \to x \circ f \circ x^{-1} \). The behaviour of this functor on cuspidal representations of \( G \) is recorded in the following lemma.

**Lemma 3.5** (See [MS14b] leme 5.3). Let \( \rho \) be a cuspidal irreducible representation of \( G \). Then
1. if \( \rho \) does not contain \( \theta \), then \( K_\kappa(\rho) = 0 \).

2. if \( \rho \) contains the maximal simple type \( (J_0, \kappa \otimes \sigma) \) then

\[
K_\kappa(\rho) \cong \sigma \oplus \sigma^{\phi} \oplus \cdots \oplus \sigma^{\phi(b(\rho)-1)},
\]

where \( \phi \) is a generator of \( \text{Gal}(e_\ell'/e) \) and \( b(\rho) \) is the size of the orbit of \( \sigma \) under the action of \( \text{Gal}(e_\ell'/e) \).

Over \( \bar{\mathcal{O}}_\ell \) we have a bijection

\[ \sigma : (\text{orbits of } \text{Gal}(e_n/e) \text{ on } \mathbf{e}\text{-regular characters of } e_n^x) \rightarrow (\text{supercuspidal irreducible representations of } \text{GL}_n(e)) \]

characterized by a character identity on maximal elliptic tori (see [Gre55] or section 2 of [BH10]). We recall that if \( e_n^x \) is embedded in \( \text{GL}_n(e) \) via the left multiplication action on \( e_n \), and \( x \in e_n^x \) is a primitive element for the extension \( e_n/e \), then

\[
\text{tr} \sigma[\chi](x) = (-1)^{n-1} \sum_{i=0}^{n-1} \chi(F^i x)
\]

for \( F \) the Frobenius element of \( \text{Gal}(e_n/e) \).

A character \( \chi : e_n^x \rightarrow \bar{\mathcal{O}}_\ell^x \) decomposes uniquely as a product of an \( \ell \)-singular part \( \chi^{(\ell)} \) and an \( \ell \)-regular part \( \chi^{(\ell)} \), whose orbits under \( \text{Gal}(e_n/e) \) only depend on the orbit of \( \chi \). We use the mod \( \ell \) reduction map to identify the prime-to-\( \ell \) roots of unity in \( \bar{\mathcal{O}}_\ell \) and \( \bar{\mathcal{O}}_\ell' \). Then the reduction mod \( \ell \) of \( \chi \) identifies with \( \chi^{(\ell)} \).

The reduction \( r_\ell(\sigma[\chi]) \) is irreducible and cuspidal, and only depends on \( [\chi^{(\ell)}] \). We denote it by \( \sigma[\ell][\chi^{(\ell)}] \).

This defines a bijection, from the orbits of \( \text{Gal}(e_n/e) \) on the characters of \( (e_n^x)^{(\ell)} \) which have an \( \mathbf{e}\text{-regular extension to } e_n^x \), to the set of cuspidal irreducible representations of \( \text{GL}_n(e) \) over \( \bar{\mathcal{O}}_\ell' \). The representation \( \sigma[\ell][\chi^{(\ell)}] \) is supercuspidal if and only if \( [\chi^{(\ell)}] \) is itself \( \mathbf{e}\text{-regular} \). Finally, if \( \chi^{(\ell)} \) is norm-inflated from an \( \mathbf{e}\text{-regular} \) \( \mathbf{F}_\ell\text{-character } \chi^{(\ell), \text{reg}} \) of \( e_n^{x/a} \) for some positive divisor \( a \) of \( n \), then the supercuspidal support of \( r_\ell(\sigma[\chi]) \) is \( \sigma[\ell][\chi^{(\ell), \text{reg}}] \otimes a \) (see [Vig96] III.2.8 and [MS14b] théorème 2.36).

We see therefore that a \( \beta \)-extension \( \kappa \) of \( \theta \) and a lift \( \Theta_E \rightarrow \Theta_F \) attach to every cuspidal representation \( \pi \) of \( G \) containing \( \theta \) an orbit of cuspidal representations of \( \text{GL}_{m'}(e_{\ell'}) \) under \( \text{Gal}(e_{\ell'}/e) \): take the pushforward of \( K_\kappa(\pi) \) under any isomorphism in the conjugacy class \( \Psi(\Theta_E) \). Equivalently, we get an orbit of \( \text{Gal}(e_{n/\delta(\Theta_F)}/e) \) on the set of characters of \( e_{n/\delta(\Theta_F)}^x \). We introduce the notation \( X_R(\Theta_F) \) for the set of \( R^x \)-valued characters of \( e_{n/\delta(\Theta_F)}^x \), and we refer to the map just constructed

\[
\Lambda_{n,R} : (\text{cuspidal representations of endo-class } \Theta_F) \rightarrow \Gamma(\Theta_F) \backslash X_R(\Theta_F)
\]

as the level zero map attached to \( \Theta_E \rightarrow \Theta_F \) and \( \kappa \).

**Proposition 3.6.** If \( \theta_1 = \text{ad}(g)^* \theta_2 \) are conjugate maximal simple characters in \( G \), and \( \kappa_1 = \text{ad}(g)^* \kappa_2 \) are \( \beta \)-extensions of the \( \theta_i \), then \( \Lambda_{\kappa_1} = \Lambda_{\kappa_2} \). Conversely, if \( \kappa_1, \kappa_2 \) are \( \beta \)-extensions of \( \theta \) with \( \Lambda_{\kappa_1} = \Lambda_{\kappa_2} \), then \( \kappa_1 = \kappa_2 \).

**Proof.** Let \( E_1 \) be an unramified parameter field for \( \theta_1 \), and let \( \iota_{E_1} : E \rightarrow E_1 \) be the only \( F \)-linear isomorphism with \( \iota_{E_1}^* \text{cl}(\theta_1_{E_1}) = \Theta_E \). Then \( gE_1 g^{-1} \) is an unramified parameter field for \( \theta_2 \), and we have an isomorphism \( \text{ad}(g)^* \iota_{E_1} : E \rightarrow gE_1 g^{-1} \). Since \( \theta_1 = \text{ad}(g)^* \theta_2 \), the relation \( \theta_1_{E_1} = \text{ad}(g)^* \theta_2_{E_1} g_{E_1} g^{-1} \) holds on the interior lifts. Hence (\( \text{ad}(g)^* \iota_{E_1} )^* \text{cl}_{2E_1} g_{E_1} g^{-1} = \Theta_E \) and \( \text{ad}(g)^* \iota_{E_1} \) is the isomorphism specified by \( \Theta_E \). So conjugation by \( g \) preserves the classes \( \Psi(\Theta_E) \) of isomorphisms \( J_{\theta_1}/J_{\theta_1}^0 \rightarrow \text{GL}_{n/\delta(\Theta_F)}(e) \), and since \( K_{\kappa_i} = \text{ad}(g)^* K_{\kappa_2} \) the first claim follows.

Now assume that the \( \kappa_i \) are \( \beta \)-extensions of \( \theta \) and \( \Lambda_{\kappa_i} = \Lambda_{\kappa_2} \). By the proof of proposition 2.8 the \( \kappa_i \) are twists of each other by a character \( \chi^x \). Then \( \chi \) fixes all elements of \( \Gamma(\Theta_F) \backslash X_R(\Theta_F) \) giving rise to cuspidal \( R \)-representations of \( \text{GL}_{n/\delta(\Theta_F)}(e) \), because these also give rise to cuspidal representations of \( \text{GL}_{m'}(e_{\ell'}) \) (recall that \( m'd' = n/\delta(\Theta_F) \)). By proposition 2.13 in [Dot18], we have that \( \chi = 1 \).
By lemma 6.1 and 6.8 in [MS14a], we see that $\Lambda_{K,R}(\pi)$ is supercuspidal if and only if $\pi$ is supercuspidal, and we know that an inertial class of supercuspidal representations consists of unramified twists of a single representation, and is determined by the corresponding maximal simple type. So $\Lambda_{K,R}$ only depends on the inertial class, when restricted to supercuspidal representations.

To extend the level zero map to simple inertial classes, we need to study the compatibility of $K$-functors with parabolic induction, as in section 5.3 of [MS14b]. The only part of this we will need is that given a divisor $m_0$ of $m$ such that $\delta(\Theta_F)$ divides $m_0d$, there exists a unique maximal $\beta$-extension $\kappa_0$ in $GL(m)(D)$ of endo-class $\Theta_F$ with the following property. Let $\pi$ be an irreducible simple representation of $GL(m)(D)$ of endo-class $\Theta_F$, with supercuspidal support inertially equivalent to $\pi_0^{\otimes m/m_0}$. Identify $J_0/J_0^\pi$ with $GL(m')(e_{d'})$ via any isomorphism in the conjugacy class $\Psi(\Theta_F)$. Then every representation in the supercuspidal support of a Jordan–Hölder factor of $K_r(\pi)$ is contained in the orbit attached to $\Lambda_{\kappa_0,R}(\pi_0)$.

The existence of this compatible $\beta$-extension $\kappa_0$ can be proved by the same arguments as in section 2.3 of [Dat18], and the same transitivity property holds: if $m_1|m_0$ and $\delta(\Theta_F)|m_1d$, and $\kappa_1$ is compatible with $\kappa$, then $\kappa_1$ is compatible with $\kappa_0$.

We can now define the level zero map

$$\Lambda_{K,R} : (\text{simple inertial classes of endo-class } \Theta_F) \to \Gamma(\Theta_F) \setminus X_R(\Theta_F)$$

sending $[GL_{m_0}(D), \pi_0^{\otimes s}]$ to the inflation to $e_{n/\delta(\Theta_F)}^\times$ of $\Lambda_{\kappa_0,R}(\pi_0)$ through the norm.

**Example 3.7.** Let $R = \mathbb{F}_\ell$ and let $\pi$ be a cuspidal, non-supercuspidal irreducible representation of $GL_m(D)$. We have given two definitions for the level zero part of $\pi$. Choose an element of $\Psi(\Theta_F)$ and then a factor $\sigma$ of $K_r(\pi)$, viewed as a cuspidal representation of $GL_{m'}(e_{d'})$. Then $\sigma$ corresponds to an orbit $[\chi]|_{\ell}$ of characters $e_{n/\delta(\Theta_F)}^\times \to \mathbb{F}_\ell^\times$ under $\text{Gal}(e_{n/\delta(\Theta_F)}/e_{d'})$. Our first definition yields $\Lambda_{K,R}(\pi) = [\chi]$, the orbit under $\text{Gal}(e_{n/\delta(\Theta_F)}/e)$. The condition for $\sigma$ to be cuspidal is that $\chi$, viewed as a character $(e_{n/\delta(\Theta_F)})^{(\ell)} \to \overline{Q}_{\ell}^\times$, extends to an $e_{d'}$-regular character of $e_{n/\delta(\Theta_F)}^\times$. Since $\sigma$ is not supercuspidal, $\chi$ is not itself $e_{d'}$-regular, hence it is norm-inflated from an $e_{d'}$-regular character $\chi_0$ of some intermediate $e_{n/\delta(\Theta_F)r}^\times$. The supercuspidal support of $\sigma$ corresponds to the $\text{Gal}(e_{n/\delta(\Theta_F)r}/e_{d'})$-orbit of $\chi_0$.

The supercuspidal support of $\pi$ is inertially equivalent to some $[GL_{m'/r_0}(D), \pi_0^{\otimes r_0}]$. The second definition of $\Lambda_{K,R}(\pi)$ is the inflation of the character orbit $\Lambda_{\kappa_0,R}(\pi_0)$. Since $d' = d/(\ell,\delta(\Theta_F))$ and $m' = n/\delta(\Theta_F)d'$, we see that $\Lambda_{\kappa_0,R}(\pi_0)$ is an orbit of $\text{Gal}(e_{n/\delta(\Theta_F)r_0}/e)$ on the $e_{d'}$-regular characters of $e_{n/\delta(\Theta_F)r_0}^\times$, giving rise to supercuspidal representations of $GL_{m'/r_0}(e_{d'})$. By compatibility of $\kappa$ and $\kappa_0$, the supercuspidal support of $\sigma$ consists of representations from $\Lambda_{\kappa_0,R}(\pi_0)$. It follows that $r = r_0$, and the two definitions agree.

We have the following proposition concerning reduction modulo $\ell$, which uses the fact that a $\beta$-extension $\kappa$ of a maximal simple $\overline{Q}_{\ell}$-character $\theta$ is integral and the reduction $r_\ell(\kappa)$ is a $\beta$-extension of $r_\ell(\theta)$, a maximal simple $\mathbb{F}_\ell$-character (see proposition 2.37 in [MS14b]).

**Lemma 3.8.** Let $\pi$ be an integral $\overline{Q}_{\ell}$-representation of $GL_m(D)$ which is simple of endo-class $\Theta_F$. If $\tau$ is a factor of $r_\ell(\pi)$, then $\Lambda_{r_\ell(\pi)}(\mathbb{F}_{\ell}) = \Lambda_{\kappa_0}(\pi)(\ell)_{\mathbb{F}_{\ell}}$.

**Proof.** The representation $\pi$ is a subquotient of a parabolic induction $\chi_1\pi^0 \times \cdots \times \chi_n\pi^0$ for an integral supercuspidal representation $\pi^0$ of some $GL_{m/r_0}(F)$ and unramified characters $\chi_i$ valued in $\overline{Q}_{\ell}$. Then the Jordan–Hölder factors of $r_\ell(\tau)$ is a subset of those of $\bigotimes_i r_\ell(\pi^0) \times \cdots \times \bigotimes_n r_\ell(\pi^0)$.

Recall that $\Lambda_{r_\ell(\pi)}(\mathbb{F}_{\ell})$ is the inflation of the character orbit corresponding to the supercuspidal support of $K_{r_\ell(\pi)}(\tau)$, under any isomorphism $J_0/J_0^\pi \to GL_{m'}(e_{d'})$ in the conjugacy class $\Psi(\Theta_F)$. By [MS14b] lemma 5.11, the equality $r_\ell(\mathbb{K}_r(\pi)) = [K_{r_\ell(\pi)}(r_\ell(\pi))]$ holds. Every factor of $K_{r_\ell(\pi)}(\pi)$ has supercuspidal support contained in $[K_{\kappa_0}(\pi^0)^{\times n}]$, where $\kappa_0$ is compatible with $\kappa$. Hence, the supercuspidal support of the reduction of every factor of $K_{r_\ell}(\pi)$ coincides with the supercuspidal support of a factor of $r_\ell(\mathbb{K}_{r_\ell}(\pi))$. By definition, $[K_{\kappa_0}(\pi^0)]$ goes under any isomorphism in $\Psi(\Theta_F)$ to the direct sum of the representations attached to $\Lambda_{\kappa_0}(\pi^0)$. The supercuspidal support of the reduction of any of these representations is a multiple of a
representation attached to \( \Lambda_\kappa, \Omega \) \((\pi(\ell), \text{reg}) \). Hence the supercuspidal support of every factor of \( K_\kappa(\pi) \) consists of representations attached to \( \Lambda_\kappa, \Omega \) \((\pi(\ell), \text{reg}) \). Since \( K_{\kappa(i)}(\tau) \) appears in the reduction of \( K_\kappa(\pi) \), and by definition \( \Lambda_\kappa, \Omega \) \((\pi(\ell), \text{reg}) \) inflates to \( \Lambda_\kappa, \Omega \) \((\pi(\ell), \text{reg}) \), the claim follows.

To summarize the results of this section, we fix for every endo-class \( \Theta_F \) a lift \( \Theta_E \rightarrow \Theta_F \) and a conjugacy class of maximal \( \beta \)-extensions \( \kappa \) in \( GL_m(D) \) of endo-class \( \Theta_F \). In more detail, by proposition \([2.5]\) any two maximal simple characters in \( GL_m(D) \) of endo-class \( \Theta_F \) are conjugate, and it possible to choose their \( \beta \)-extensions so that, whenever \( \theta_1 = \text{ad}(g)^* \theta_2 \) for some \( g \in GL_m(D) \), one has \( \kappa_1 = \text{ad}(g)^* \kappa_2 \). For this to be well-defined we need to check that if \( g \in G \) normalizes \( \theta_1 \), then it normalizes \( \kappa_1 \); but the normalizer \( J(\theta) \) of \( \theta \) in \( G \) normalizes \( J_0 \), which is the unique maximal compact subgroup of \( J(\theta) \), and \( \theta \) and \( \kappa \) have the same \( G \)-intertwining (this is a defining property of \( \beta \)-extensions), hence the claim follows.

This allows us to define two invariants of a simple block \( s \) over \( R \), namely the endo-class \( cl(s) \) of any maximal simple character contained in the supercuspidal support of \( s \), and the level zero part \( \Lambda_{\kappa(s)}, R(s) \), where \( \kappa(s) \) is any \( \beta \)-extension in the class we have attached to \( cl(s) \), and the lift is the one we have fixed for \( cl(s) \); this is well-defined by proposition \([3.6]\).

Here is the parametrization of inertial classes we shall use.

**Theorem 3.9.** The map \( in : s \mapsto (cl(s), \Lambda_{\kappa(s)}, R(s)) \) is a bijection from the set of simple inertial classes of \( R \)-representations of \( GL_m(D) \) to the set of pairs \((\Theta_F, [\chi])\) consisting of an endo-class \( \Theta_F \) of degree dividing \( n = md \) and a character orbit \([\chi] \in \Gamma(\Theta_F) \times R(\ Theta_F) \).

**Proof.** A supercuspidal inertial class \( s \) is determined by the conjugacy class of maximal simple types it contains, which can be recovered from the image of this map. To see this, assume given \((\Theta_F, [\chi])\) such that \([\chi] \in e_{d'}\)-regular, and let \( \theta \) be a maximal simple character in \( GL_m(D) \) of endo-class \( \Theta_F \) and \( \beta \)-extension \( \kappa \). Then the maximal simple types \( (J_0, \kappa \otimes \sigma_t) \), where the \( \sigma_t \) are any two supercuspidal representations of \( GL_m(e_{d'}) \) in the orbit corresponding to \([\chi] \), inflated to \( J_0/\mathcal{J}_u \) under any element of \( \Psi(\Theta_F) \), are conjugate in \( GL_m(D) \) and correspond to a supercuspidal inertial class. Indeed, if \([A, \beta] \) is a simple stratum defining \( \theta \), and we fix an \( F[\beta]\)-linear isomorphism \( B \rightarrow M_{md}(D') \), then the normalizer \( J(\theta) = \pi_{D'}^* J_\theta \), and conjugation by \( \pi_{D'}^* \) acts as the Frobenius element of \( \text{Gal}(d'/e) \) on \( D' \). Hence our map is injective when restricted to supercuspidal inertial classes, and its image consists of those \((\Theta_F, [\chi])\) such that \([\chi] \in e_{d'}\)-regular.

The result then follows because, if \( s = [GL_m/\mathcal{r}_0(D), \pi_{\mathcal{r}_0}] \), then \( \text{inv}_{GL_m(D)}(s) = \text{inv}_{GL_m/\mathcal{r}_0(D)}[GL_m/\mathcal{r}_0(D), \pi_{\mathcal{r}_0}] \), using the result for \( \text{inv}_{GL_m/\mathcal{r}_0(D)} \), defined via the compatible \( \beta \)-extension \( \kappa(s) \), on supercuspidal inertial classes. \( \square \)

**Example 3.10.** To clarify the situation, fix \( \Theta_F \) and notice that the simple inertial classes in \( GL_m(D) \) of endo-class \( \Theta_F \) are in bijection with the union of the supercuspidal inertial classes in \( GL_U(D) \) of endo-class \( \Theta_F \), where \( t \) is a divisor of \( m \) such that \( \delta(\Theta_F) \) divides \( n/t \). We are claiming that these are also in bijection with \( \Gamma(\Theta_F) \times R(\Theta_F) \). Every element of \( \Gamma(\Theta_F) \times R(\Theta_F) \) is regular for precisely one subfield of \( e_{d'} \), and \( e_{n/\delta(\Theta_F)}/e_{d'} \) has degree \( m = \frac{n}{\delta(\Theta_F)}d' \). So the two sets are in bijection if and only if, for a divisor \( t|m \), we have that \( \delta(\Theta_F) \) divides \( n/t \) and \( m' \) are equivalent conditions.

To see this, assume first that \( \delta(\Theta_F) \) divides \( n/t \). Then \( GL(D) \) has a maximal simple character of endo-class \( \Theta_F \), with a parameter field whose commutant is isomorphic to some \( M_m(D') \) with \( m' = \frac{n}{\delta(\Theta_F)} \), whereas \( m' = \frac{n}{\delta(\Theta_F)} \). Since \( m' \) is an integer and \( m' = m't \), we have \( m' \). Conversely, assume that \( m' \) and write \( m' = m''t \) for an integer \( m'' \). Then \( n/\delta(\Theta_F) = m'd'' \) for \( n/t = \delta(\Theta_F)m''d' \) and \( \delta(\Theta_F) \) divides \( n/t \).

**Remark 3.11.** We reiterate that the construction of the map \( \Lambda_{\kappa(s)} \) depends on the choice of a lift of \( cl(s) \) to its ramified parameter field. To emphasize this, we will write the inverse to \( \text{inv} \) as a map \( (\Theta_F, \Theta_E, [\chi]) \mapsto s_G(\Theta_F, \Theta_E, [\chi]), \) with finite fibers consisting of orbits of \( \text{Gal}(e/f) \) acting diagonally by \( g \cdot (\Theta_F, \Theta_E, [\chi]) = (\Theta_F, g^* \Theta_E, (g^{-1})^* [\chi]) \).

The triple \((\Theta_F, \Theta_E, [\chi])\) corresponds to a supercuspidal inertial class of \( GL_m(F) \) if and only if \([\chi] \) consists of \( e \)-regular characters of \( e_{n/\delta(\Theta_F)} \). If this happens, then \( s_G(\Theta_F, \Theta_E, [\chi]) \) is supercuspidal for all inner forms \( G \) of \( GL_m(F) \). When the inner form is \( D^X \) for a division algebra \( D \), one has \( e_{n/\delta} = e_{d}, \) so every triple is supercuspidal for \( D^X \) — of course, this is as expected because \( D^X \) has no nontrivial rational parabolic subgroups and so every irreducible smooth representation is supercuspidal.

**Remark 3.12.** In \([BSS12]\) there is assigned an endo-class \( \Theta(s) \) to every simple inertial class \( s \) of complex representations of \( GL_m(D) \), defined to be the endo-class of any simple character contained in representations
in $s$. Since $s$ needn’t be supercuspidal, these characters needn’t be maximal simple characters, but if $s_0$ is supercuspidal then $\Theta(s_0) = \text{cl}(s_0)$ by definition. By remark 6.8 in [SS16b], $\Theta(s) = \Theta(s_0)$ if $s$ is inertially equivalent to a multiple of $s_0$ (this is implicit in the construction of compatible $\beta$-extensions). Then by construction we see that $\text{cl}(s) = \Theta(s)$ for every simple inertial class $s$.

Remark 3.13. In section 3.4 of [MS14b] there is defined a number of invariants attached to a cuspidal representation $\rho$ of $GL_n(D)$. If $(J, \kappa \otimes \sigma)$ is a maximal simple type in $\rho$, these are

1. $n(\rho)$, the torsion number, which is the number of unramified characters $\chi$ of $G$ such that $\rho \otimes \chi \cong \chi$
2. $b(\rho)$, the size of the orbit of $\sigma$ under the action of $\text{Gal}(e_d/e)$
3. $s(\rho)$, the order of the stabilizer of $\sigma$ in $\text{Gal}(e_d/e)$
4. $f(\rho) = n/e(\Theta_F)$

These only depend on the inertial class of $\rho$ and can be read off from our parametrization. We make this explicit over the complex numbers. Write $\text{inv}(\rho) = (\Theta_F, [\chi])$. We have the equality $f(\rho) = n(\rho)/s(\rho)$, by an explicit computation using [BHT] (2.6.2)(4)(b) (or see [MS14b] equation (3.6)). We also note that [BHT] section 2 defines a parametric degree for all simple representations, and for a supercuspidal $\rho$ this equals $n/s(\rho)$ (see [SS16b], section 3.1).

The stabilizer $S_1$ of any representative $\chi$ of $[\chi]$ under $\text{Gal}(e_{d/\delta(\Theta_F)}/e)$ is isomorphic to the stabilizer $S_2$ of the corresponding cuspidal representation of $GL_{m'}(e_d)$ under $\text{Gal}(e_d/e)$. Indeed, $S_1$ surjects onto $S_2$ by restriction and $S_1 \cap \text{Gal}(e_{d/\delta(\Theta_F)}/e) = 1$, because $[\chi]$ consists of $e_d$-regular characters. The quantity $s(\rho)$ therefore also equals $s[\chi]$, the order of the stabilizer of any element of $[\chi]$ under $\text{Gal}(e_{d/\delta(\Theta_F)}/e)$. We will also denote by $b[\chi]$ the size of the orbit under $\text{Gal}(e_d/e)$ of any representation of $GL_{m'}(e_d)$ of the form $\sigma(\chi)$ for $\chi \in [\chi]$.

4 The inertial Jacquet–Langlands correspondence.

We now proceed to the main theorems. As in the previous section, we fix lifts $\Theta_E \rightarrow \Theta_F$ of all endo-class of degree dividing $n$. We will work with the conjugacy class of $p$-primary maximal $\beta$-extensions in any inner form of $GL_n(F)$, which then determines a conjugacy class in any $GL_n(D)$ uniquely by compatibility; this compatible conjugacy class, however, needs not be $p$-primary. We write $A_\lambda$ and $\text{inv}$ for the corresponding level zero and invariant maps. We will work over the complex numbers unless stated otherwise.

4.1 A character formula.

Consider a supercuspidal irreducible representation $\tau$ of $G$. Let $s = s_G(\Theta_F, \Theta_E, [\chi])$ be the inertial class of $\tau$, and let $(J, \lambda)$ be a maximal simple type for $s$, so that $\tau$ is the compact induction of an extension $\Pi$ of $\lambda$ to its normalizer $J(\lambda)$. The type $(J, \lambda)$ is constructed from a maximal simple character $\theta$ with endo-class $\Theta_F$, and we fix a simple stratum $[\mathfrak{A}, \beta]$ defining $\theta$. Let $T = F[\beta]^u$, an unramified parameter field for $\theta$.

Write $B = Z_A(F[\beta]) \cong M_{m'}(D')$ for the commutant of $F[\beta]$ in $A$. Fix an extension $L/F[\beta]$ in $B$ that has maximal degree, is unramified, and normalizes the order $\mathfrak{A}$. Such an $L$ exists by the arguments in the proof of proposition 2.7. Consider the maximal unramified extension $K = L^u$ of $F$ in $L$, write $A_K$ for the commutant $Z_A(K)$, and let $G_K = A_K^\chi$. In this context, the normalizer $N_G(K)$ acts on $G_K$, and there is an isomorphism

$$N_G(K)/G_K \rightarrow \text{Gal}(K/F)$$

by the conjugation action on $K$. It follows that $\text{Gal}(K/F)$ has a right action on isomorphism classes of representations of $G_K$: if $\tau$ is a representation and $t_\alpha \in N_G(K)$ maps to $\alpha \in \text{Gal}(K/F)$, denote by $\tau^\alpha$ the representation $g \mapsto \tau(t_\alpha gt_\alpha^{-1})$. The isomorphism class of $\tau^\alpha$ is independent of the choice of preimage $t_\alpha$ of $\alpha$. If $\tau$ has endo-class $\Theta_K$, then $\tau^\alpha$ has endo-class $\Theta_K^\alpha$.

Since $\beta$ commutes with $K$ and generates a field $L = K[\beta]$ over $K$, and $L^u \subseteq \mathfrak{r}(\mathfrak{A})$, proposition 2.7 applies and $\theta$ has an interior $K$-lift $\Theta_K$. This is a character of $H^1_K = H^1_K \cap B$, and it is defined by the simple stratum $[\mathfrak{A}_K, \beta]$ for $\mathfrak{A}_K = \mathfrak{A} \cap A_K$. It is a maximal simple character, because $\beta$ generates $L$ over $K$, $L$ is self-centralizing in $A_K$, and $L$ has a unique hereditary $\mathfrak{o}_L$-order, namely $\mathfrak{o}_L$ itself, which is a maximal order.
extends to \( J \) in \([BH10]\), we can further change \( J \) induced from one of these extensions from \( \det \) anymore. We will refer to any representation obtained by raising to a suitable power of \( \theta \) of \( G \) to a suitable power of \( \beta \) \( \pi \) due to Bushnell and Henniart in the context of essentially tame endo-classes \((\text{see [BH11] section 6})\). Recall representations of \( G \) containing a maximal simple type \((\text{see [BH11] section 1.2 and the appendix to [BH96]}). \) See \([BH11]\) section 6.4 and \([BH96]\) Lemma 4.4.

Theorem 4.1. Let \( \zeta \in \mu_K \) generate the field \( K \) over \( F \), and let \( u \) be an elliptic, regular and pro-unipotent element of \( G \). Then

\[
\text{tr} \pi(\zeta u) = (-1)^{n'+1}s[\chi]^{-1}\epsilon(\zeta,V) \sum_{\alpha \in \Gal(k/F)} \chi(\zeta)\text{tr}r^\alpha(u)
\]

where \( \chi \) is evaluated at \( \zeta \) via any \( e \)-linear isomorphism \( \iota : k \to e_n/\delta(\Theta_F) \), where \( k \) is an \( e \)-algebra via \( \iota(\Theta_F)_T : e \to t \).

**Remark 4.2.** It is not immediate that the formula makes sense as written, but we will see while proving the theorem that the characters of \( \tau \) and \( \tau^\alpha \) coincide on \( u \) whenever \( \alpha \in \Gal(k/t) \), hence the right hand side is independent of the choice of representatives of \( [\chi] \) and of the choice of \( \iota \).

Recall that \( V = J^1/H^1 \) is a symplectic representation of \( \mu_K \) over \( F_p \), and that \( m' \) is defined by \( B = Z_A(F^/[\beta]) \cong M_{m'}(D') \). A pro-unipotent element \( u \) of \( G \) is one for which \( u^{m'} \to 1 \) as \( n \to +\infty \). See remark 3.13 for the definition of \( s[\chi] \).

**Proof.** The element \( \zeta u \in G \) is elliptic and regular over \( F \), since \( F[\zeta u] \) is a finite-dimensional \( F \)-subspace of \( A = M_{m'}(K) \), hence it is complete and it contains \( \zeta \); but then it contains \( u \) and \( F[z u] = K[u] \) is a maximal field extension of \( F \) in \( A \). So the Harish-Chandra character of \( \pi \) at \( \zeta u \) can be computed by the Mackey formula for an induced representation

\[
\text{tr} \pi(\zeta u) = \sum_{y \in J(\lambda) \setminus G} \text{tr} \Pi(y\zeta uy^{-1}),
\]

see \([BH11]\) section 1.2 and the appendix to \([BH96]\).

**Lemma 4.3.** If \( y \in G \) and \( y\zeta uy^{-1} \in J(\lambda) \), then \( y\zeta uy^{-1} \in J_0 \) and there exists \( \tilde{g} \) in the normalizer \( N_G(K) \) such that \( J(\lambda)\tilde{g} = J(\lambda)\tilde{g} \). For any such \( \tilde{g} \), one has \( \tilde{g}uy^{-1} \in J^1_K \).

**Proof.** Since the value of the determinant of \( \zeta u \) is zero, and \( J(\lambda)/J_0 \) is infinite cyclic generated by some power of a uniformizer of \( D' \), necessarily \( y\zeta uy^{-1} \in J_0 \) if \( y\zeta uy^{-1} \in J(\lambda) \). The quotient \( J_0/J_0^1 \) is isomorphic to a general linear group \( GL_{m'}(e_d) \), and the degree \( [K : F] = n/\delta(\Theta_F) \), so \( k^\times \) embeds in \( GL_{m'}(d') \) as a maximal elliptic torus. Now the claim follows as in the proof of \([BH10]\) Lemma 13: first prove that \( y\zeta uy^{-1} \in J_0 \) by raising to a suitable power of \( p \), and then notice that there exists some other \( \zeta' \in \mu_K \) generating \( K \) over \( F \) with \( y\zeta uy^{-1} \) conjugate in \( J_0 \) to \( \zeta' u' \) for some \( u' \in J_0^1 \). By \([BH10]\), 2.6 Conjugacy Lemma, or \( u' \in J_0^1 \). By \([BH10]\), 2.6 Conjugacy Lemma, or Lemma 14 in \([BH91]\), we can further change \( y \) in its \( J_0 \)-coset and assume that \( u' \) and \( \zeta' \) commute, and this implies that \( u' = 1 \). But then \( y\zeta uy^{-1} = \zeta' uy^{-1} \) with \( uy^{-1} \) commuting with \( \zeta' \) and contained in \( J_0 \). As the image of \( \zeta' \) in \( J_0^1/J_0^1 \) is a regular elliptic element, it commutes with no unipotent elements except the identity, so \( uy^{-1} \in J_0^1 \).

**Lemma 4.4** (Compare \([BH10]\) proposition 9). The group \( J(\lambda) \cap G_K \) equals \( J(\lambda \chi) \), and the order of the image of \( J(\lambda) \cap N_G(K) \) under the isomorphism \( N_G(K)/G_K \to \Gal(K/F) \) equals \( n/\delta(\Theta_F)b[\chi] = m'd'/b[\chi] \), where \( b[\chi] \) equals the index of \( J(\lambda) \) in \( \pi^1_{D'} \rtimes J_0 \).
Proof. We can determine an element in $\text{Gal}(K/F)$ by its action on $\mu_K$, and $\mu_K = \mu_L$. Any choice of isomorphism $\psi : J_\theta/J_\theta^1 \to \text{GL}_{m'}(e_{d'})$ in $\Psi(\Theta_E)$ induces a surjective group homomorphism

$$\bar{\psi} : \pi_{D'} \rtimes J_\theta \to \text{Gal}(e_{d'}/e) \ltimes \text{GL}_{m'}(e_{d'})$$

which sends $\pi_{D'}$ to a generator of $\text{Gal}(e_{d'}/e)$ and maps $\mu_L$ isomorphically onto its image, which is an elliptic maximal torus in $\text{GL}_{m'}(e_{d'})$, hence self-centralizing in $\text{Gal}(e_{d'}/e) \ltimes \text{GL}_{m'}(e_{d'})$ (to see this, embed this group in $\text{GL}_{m'd'}(e_{d'})$, where the image of $\mu_L$ is still an elliptic maximal torus). So, if $x \in \pi_{D'} \rtimes J_\theta$ centralizes $\mu_K$ then it is contained in $\pi_{D'} \rtimes J_\theta$, which equals $\pi_{F[\beta]} \rtimes J_\theta$ as $e_{d'} \subseteq J_\theta$. This implies that

$$\mathbf{J}(\lambda) \cap Z_G(K) = (\pi_{F[\beta]} \rtimes J_\theta) \cap Z_G(K) = \pi_{F[\beta]} \rtimes J_\theta = \mathbf{J}(\lambda_K).$$

Every automorphism of $\mu_K$ induced by a conjugation in $\text{Gal}(e_{d'}/e) \ltimes \text{GL}_{m'}(e_{d'})$ is also induced by a conjugation in $\pi_{D'} \rtimes J_\theta$; to see this, observe that if $x \in \pi_{D'} \rtimes J_\theta$ and $x \zeta x^{-1} = \zeta' u$ for some $u \in J_\theta$ then we can change $x$ in its $J_\theta$-coset and assume that $\zeta'$ and $u$ commute, by lemma 14 in [BH10]. Then since the order of $\zeta$ and $\zeta'$ is prime to $p$ and $J_\theta^1$ is a pro-$p$ group we conclude that $u = 1$, and the claim follows.

The group of automorphisms of an elliptic maximal torus $T$ in $\text{GL}_{m'}(e_{d'})$ induced by $\text{Gal}(e_{d'}/e) \ltimes \text{GL}_{m'}(e_{d'})$ is cyclic of order $m'd'$: this holds because up to conjugacy $T$ arises from restricting scalars of the $e_{m'd'}$-vector space $e_{m'd'}$ to $e_{d'}$. Restricting scalars further to $e$, we see that the normalizer of $e_{d'}$ in $\text{GL}_{m'd'}(e)$ is $\text{Gal}(e_{d'}/e) \ltimes \text{GL}_{m'}(e_{d'})$, and it contains the normalizer of $e_{m'd'}$ in $\text{GL}_{m'd'}(e)$, which induces $\text{Gal}(e_{m'd'}/e)$ on $e_{m'd'}^\times$.

The lemma now follows since the group $\mathbf{J}(\lambda)$ has index $b[\chi]$ in $\pi_{D'} \rtimes J_\theta$ and contains $J_\theta$, hence maps under $\bar{\psi}$ to $\Delta \times \text{GL}_{m'}(e_{d'})$, for $\Delta \subseteq \text{Gal}(e_{d'}/e)$ the only subgroup of index $b[\chi]$. □

The space $\mathbf{J}(\lambda) N_G(K)$ decomposes into double cosets

$$\mathbf{J}(\lambda) N_G(K) = \bigcup_{\sigma \in \text{Gal}(K/F)} \mathbf{J}(\lambda) t_\sigma G_K$$

where $t_\sigma \in N_G(K)$ induces $\sigma$ on $K$ upon conjugation, and $\mathbf{J}(\lambda) t_\sigma G_K = \mathbf{J}(\lambda) t_\tau G_K$ if and only if $\tau \sigma^{-1}$ is induced by $\mathbf{J}(\lambda)$. Then by lemma 4.3 and lemma 4.4 we may rewrite the sum as

$$\text{tr}_\pi(\zeta u) = \sum_{y \in J(\lambda) \setminus J(\lambda) N_G(K)} \text{tr}_\Pi(y \zeta uy^{-1}) = (\delta(\Theta_E) b[\chi]/n) \sum_{\alpha \in \text{Gal}(K/F) \setminus J(\lambda_K) \setminus G_K} \sum_{y \in J(\lambda_K) \setminus G_K} \text{tr}_\Pi(y \alpha(\zeta uy^{-1}).$$

Here, $\alpha(\zeta uy^{-1}) = t_\alpha \zeta uy t_\alpha^{-1}$.

These $y$ commute with all the $\alpha(\zeta)$. We are now going to fix an isomorphism $\psi : J_\theta/J_\theta^1 \to \text{GL}_{m'}(e_{d'})$ in the conjugacy class $\Psi(\Theta_E)$, and a representation $\sigma$ of $\text{GL}_{m'}(e_{d'})$ so that $\lambda = \kappa \otimes \psi^* \sigma$. Furthermore, we’ll choose a representative $\chi$ for $[\chi]$ such that $\sigma = \sigma(\chi)$ under the Green parametrization\footnote{Recall that $[\chi]$ only determines a $\text{Gal}(e_{d'}/e)$-orbit of representations of $\text{GL}_{m'}(e_{d'})$. Via the choice of $\psi$ and $\sigma$, we are fixing an element of this orbit.}. Then

$$\text{tr}_\Pi(y \alpha(\zeta uy^{-1}) = \text{tr}_\Pi(\alpha(\zeta) y \alpha(u) y^{-1}) = \text{tr}_\sigma(\zeta^\alpha) \text{tr}_{\kappa}(\zeta^\alpha y \alpha(u) y^{-1})$$

since $\Pi$ extends $\lambda$, where $\zeta^\alpha = \alpha(\zeta)$ and $\sigma$ is evaluated at $\zeta^\alpha$ via $\psi$.

**Lemma 4.5.** The equality

$$\text{tr}_{\kappa}(\zeta^\alpha y \alpha(u) y^{-1}) = \epsilon(\zeta^\alpha, V) \text{tr}_{\lambda_K}(y \alpha(u) y^{-1})$$

holds whenever $y \alpha(u) y^{-1} \in J_K$.

**Proof.** Compare [BH14] section 5.2. We use the Glauberman correspondence (see for instance [BH14] 5.1.2) for the cyclic group $\Gamma \subseteq \mu_K$ generated by $\zeta$, acting on $J_\theta^1/\ker(\theta)$ and normalizing $\eta$. This implies that there exist a unique irreducible representation $\eta^\Gamma$ of $(J_\theta^1/\ker(\theta))^\Gamma$ and sign $\epsilon = \pm 1$ such that

$$\text{tr}_{\eta}(x) = \epsilon \text{tr}(\eta\zeta x)$$

where \(\zeta\) is a generator of the cyclic subgroup generated by $J_\theta^1$. Then $\zeta$ is such a generator, and $\eta(\zeta) = \zeta^\alpha$.

Then

$$\text{tr}_{\kappa}(\zeta^\alpha y \alpha(u) y^{-1}) = \epsilon(\zeta^\alpha, V) \text{tr}_{\lambda_K}(y \alpha(u) y^{-1})$$

by the Glauberman correspondence. □
for all \( x \in (J^1/\ker(\theta))^\Gamma \) and every generator \( \zeta \) of \( \Gamma \). Recall from section 2 that \( \bar{\eta} \) is the only irreducible representation of \( \Gamma \ltimes J^1_0 \) with trivial determinant on \( \Gamma \).

By construction, \( \bar{\eta} \) is isomorphic to the restriction of the \( p \)-primary \( \beta \)-extension \( \kappa \) to \( \Gamma \ltimes J^1_0 \), since \( \det \kappa \) has order a power of \( p \) and \( \Gamma \) has order prime to \( p \). Since \( \zeta \) generates \( K \) over \( F \), the fixed point space \( (J^1_0)^\Gamma = J^1_K \), and since \( \ker(\theta) \) is a pro-\( p \) group and \( \Gamma \) has order prime to \( p \), a cohomological vanishing argument as in \( [BH10] \) proposition 6 implies that \( (J^1_0/\ker(\theta))^\Gamma = J^1_K/\ker(\theta_K) \). We claim that actually \( \bar{\eta}^\Gamma = \eta_K \), the Heisenberg representation associated to \( \theta_K \). To see this, it is enough to prove that \( \bar{\eta}^\Gamma \) contains \( \theta_K \), by the uniqueness statement in proposition 2.10. Replacing \( x \) by \( xh \) for \( h \in H^1_K \) in part 2. above, we find

\[
tr\bar{\eta}^\Gamma(xh) = etr\bar{\eta}(\xi xh) = etr\bar{\eta}(\xi x)\theta(h) = \theta_K(h)etr\bar{\eta}^\Gamma(x).
\]

Setting \( x = 1 \) and letting \( h \) vary through \( H^1_K \) yields the claim.

Finally, we compute \( \epsilon \) by letting \( x = 1 \) in \( tr\eta_k(x) = etr\bar{\eta}(\xi) \). The equality \( etr\bar{\eta}(\xi) = \dim \eta_K \) implies that \( \epsilon \) equals the sign of the trace of \( \xi \) on \( \bar{\eta} \), which by definition is \( \epsilon(\xi, V) \).

If \( y\alpha(u)y^{-1} \not\in J^1_K \), then \( y\alpha(u)y^{-1} \not\in J(\lambda) \) by lemma 4.3. So we have

\[
tr\Pi(y\alpha(u)y^{-1}) = \epsilon(\zeta, V)tr\sigma(\zeta^\alpha)tr\lambda_K(y\alpha(u)y^{-1})
\]

where the traces of \( \Pi \) and \( \lambda_K \) are extended by zero to \( G \) and \( G_K \) respectively. Since \( \tau \) is induced from an extension of \( \lambda_K \) to \( J(\lambda_K) \), we deduce that

\[
\sum_{y \in J(\lambda_K) \backslash G_K} tr\lambda_K(y\alpha(u)y^{-1}) = tr\tau(\alpha(u))
\]

and so

\[
tr\tau(\zeta u) = (\delta(\Theta_F)b[\chi]/\mu) \sum_{\alpha \in Gal(k/F)} \epsilon(\zeta, V)tr\sigma(\zeta^\alpha)tr\tau(\alpha(u)).
\]

Now the Galois twists \( \tau^\alpha = \tau \circ ad(t_\alpha) \) have character \( x \mapsto tr\tau(\alpha(x)) \), and the endo-class \( cl(\tau^\alpha) \) of \( \tau^\alpha \) satisfies \( cl(\tau^\alpha) = (cl\tau)^\alpha \). By \( [BH03] \) 1.5.1, the group \( Gal(k/F) \) is transitive over the \( K \)-lifts of \( \Theta_F \), and there's as many of these as simple components of \( K \otimes F \mathbb{F}_l \). So the stabilizer of \( cl(\tau) \) in \( Gal(k/F) \) is \( Gal(k/F) \), and for \( \alpha \in Gal(k/F) \) the supercuspidal representations \( \tau \) and \( \tau^\alpha \) of \( G_K \) have the same endo-class. By proposition 2.3 they both contain the simple character \( \theta_K \), so their restriction to \( J^1_K \) contains \( \eta_K \). Since \( J_K/J^1_K \cong \mu_L = \mu_K \), which by construction acts trivially on \( \tau \) and \( \tau^\alpha \), these representations contain the same simple type \( (J_K, \kappa_K) \). So they are inertially equivalent, and their characters therefore agree on elements as \( u \) whose reduced norm has valuation 0. We can now rearrange the sum further to

\[
tr\tau(\zeta u) = (\delta(\Theta_F)b[\chi]/\mu)\sum_{\gamma \in Gal(t/F)} \left( tr\gamma(\xi) \sum_{\delta \in Gal(k/t)} tr\sigma(\zeta^\delta) \right)
\]

since \( \epsilon(\zeta, V) \) only depends on the subgroup of \( \mu_K \) generated by \( \zeta \).

The trace \( tr\sigma(\zeta^\delta) \) can be computed as follows. We are evaluating \( \sigma \) at \( \zeta^\delta \) using a fixed choice of isomorphism \( \psi : J^1_0/\ker(\theta)^\Gamma \to GL_{m'}(e_{d'}) \) in \( \Psi(\Theta_F) \). Any such \( \psi \), comes from an isomorphism \( \psi : j_0/j_0' \to M'_{m'}(e_{d'}) \) by passing to groups of units. The elliptic maximal torus \( \psi(\mu_K) \) is conjugate to \( e_{m'/d'}^\times \), where the trace of \( \sigma = \sigma(\chi) \) is given by explicit formulas, and this isomorphism \( \mu_K \to e_{m'/d'}^\times \) (\( \psi \) followed by conjugation) comes from an \( e \)-linear isomorphism \( k \to e_{m'/d'} \) by passing to groups of units. Then one has the character formula

\[
\sum_{\delta \in Gal(k/t)} tr\sigma(\zeta^\delta) = (-1)^{m+1} \sum_{\nu \in Gal(e_{m'/d'}/e_{d'})} \sum_{\chi(\zeta^\delta u)} \chi(\zeta^\delta u)
\]

\[
= (-1)^{m+1} \sum_{\nu \in Gal(e_{m'/d'}/e_{d'})} \sum_{\chi \in [x]} \chi(\zeta^\delta)
\]

\[
= (-1)^{m+1} \sum_{\delta \in Gal(k/t)} \chi(\zeta^\delta)
\]
where \( \sigma \) is evaluated on \( \zeta^d \) via \( \psi \) and \( \chi \) is evaluated on \( \zeta^d \) via any \( e \)-linear isomorphism \( \iota : k \to m' \cdot d' \). Because the sums are taken over \( \text{Gal}(k/t) \), the answer is independent of the choice of \( \psi \) and \( \iota \), and the second line shows that the answer does not depend on the choice of \( \chi \) in \([\chi] \).

Now since \( n / \delta(\Theta_F) = m' \cdot d' \) we have

\[
(\delta(\Theta_F)b[\chi]/n)m' = b[\chi]/d' = s[\chi]^{-1},
\]

and rearranging further we obtain

\[
\text{tr}_{\pi}(\zeta u) = (-1)^{m'+1}s[\chi]^{-1}\epsilon(\zeta, V) \sum_{\gamma \in \text{Gal}(k/t)} \left(\text{tr}_{\tau}(u) \sum_{\delta \in \gamma \text{Gal}(k/t)} \chi(\zeta^d)\right)
\]

\[
= (-1)^{m'+1}s[\chi]^{-1}\epsilon(\zeta, V) \sum_{\gamma \in \text{Gal}(k/t)} \text{tr}_{\tau}(u)\chi(\zeta^d).
\]

\[\square\]

### 4.2 Results from \( \ell \)-modular representation theory.

Let \( \ell \neq p \) be a prime number, and fix an isomorphism \( \iota : C \to \overline{Q}_\ell \). In [SS16b] section 4.1 there is defined a notion of \( mod \ \ell \) inertial supercuspidal support for irreducible smooth \( \ell \)-adic representations of \( G = \text{GL}_m(D) \). It is an inertial class of supercuspidal supports for \( \text{GL}_m(D) \) over \( F_\ell \), and it only depends on the inertial class of the representation. Write \( i_\ell(s) \) for the mod \( \ell \) inertial supercuspidal support of the inertial class \( s \), and say that two classes \( s_1 \) and \( s_2 \) for the category of \( \overline{Q}_\ell \)-representations of \( G \) are in the same \( \ell \)-block if \( i_\ell(s_1) = i_\ell(s_2) \).

Given simple inertial classes \( s_i \) of complex representations of \( G \), say that they are \( \ell \)-linked if the \( \overline{Q}_\ell \)-components corresponding to them under \( \iota \) are in the same \( \ell \)-block—by [SS16b] lemma 5.2 this is independent of the choice of \( \iota \)—and that they are linked if there exist prime numbers \( \ell_1, \ldots, \ell_r \) all distinct from \( p \) and inertial classes \( s^0, \ldots, s^r \) such that \( s^0 = s_0, s^r = s_1, \) and \( s^i \) are \( \ell_i \)-linked throughout.

By lemma 4.3 in [SS16b], the mod \( \ell \) inertial supercuspidal support of an integral representation \( \pi \) coincides with the supercuspidal support of every factor of \( r_\ell(\pi) \). From this and proposition 3.8 it follows that

\[
\text{cl}(i_\ell(s)) = \text{cl}(s) \text{ and } \Lambda_{r_\ell(s)}(i_\ell(s)) = \Lambda_{r_\ell(s)}(s). \]

So simple inertial classes \( s_i \) are \( \ell \)-linked if and only if \( \text{cl}(s_1) = \text{cl}(s_2) \) and \( \Lambda_{r_\ell(s_1)} = \Lambda_{r_\ell(s_2)} \). Letting \( \ell \) vary, we see that the \( s_i \) are linked if and only if they have the same endo-class (compare [SS16b] propositions 5.5 and 5.8). The compatibility of the Jacquet–Langlands correspondence with respect to mod \( \ell \) reduction implies the following result.

**Theorem 4.6.** (Corollary 6.3 and Theorem 6.4 in [SS16b]) Let \( H = \text{GL}_n(F) \) and consider the Jacquet–Langlands transfer of simple inertial classes of complex representations

\[
\text{JL}_G : \mathcal{B}_{ds}(G) \to \mathcal{B}_{ds}(H)
\]

Let \( s_i \) be simple inertial classes for \( G \). Then \( s_1 \) and \( s_2 \) are \( \ell \)-linked if and only if \( \text{JL}_G(s_1) \) and \( \text{JL}_G(s_2) \) are \( \ell \)-linked.

### 4.3 Proof of the main theorems.

Now consider central simple algebras \( A_1 = \text{M}_n(F) \) and \( A_2 = \text{M}_m(D) \) over \( F \). Write \( \text{JL}_{G_2} : D(G_2) \to D(G_1) \) for the Jacquet–Langlands correspondence between their unit groups, as well as for the map it induces on simple inertial classes. Let \( s_i \) be simple inertial classes of the \( G_i \) with

\[
s_i = \text{JL}_{G_2}(s_2).
\]

**Theorem 4.7.** We have the equality \( \text{cl}(s_1) = \text{cl}(s_2) \).
Proof. Let \( d \) be the reduced degree of \( D \) over \( F \). For all integers \( a \geq 1 \) there exist simple inertial classes \( s_1^a \) in \( GL_{m,n}(F) \) and \( GL_{m,n}(D) \) which correspond under the Jacquet–Langlands transfer on these groups and have endo-class \( cl(s_1^a) = cl(s_1) \): it suffices to let their supercuspidal support be a multiple of the supercuspidal support of \( s_1 \). Letting \( a = d \), we can assume that \( d \) divides \( \frac{\sigma_{cl(s_i)}}{\sigma_{cl(s_j)}} \) for all \( i, j \).

We can assume that both \( s_1 \) are supercuspidal: to see this, recall that the parametric degree of a simple inertial class is preserved under the Jacquet–Langlands correspondence (see [BH11] 2.8 Corollary 1), and a simple inertial class of \( GL_{m,n}(F) \) is supercuspidal if and only if it has maximal parametric degree, we see that the transfer of a supercuspidal representation of \( GL_{m,n}(F) \) is supercuspidal. Let \( s_1^a \) be a supercuspidal inertial class for \( GL_{m,n}(F) \), with \( cl(s_1) = cl(s_1^a) \). Then \( s_1 \) and \( s_1^a \) are linked, hence by theorem 4.6 also \( s_2 \) and \( JL_{GL_{2}}^{-1}(s_1^a) \) are linked, and so they have the same endo-class. So if \( cl(JL_{GL_{2}}^{-1}(s_1^a)) = cl(s_1^a) \) then the theorem follows.

When the \( s_i \) are supercuspidal, their parametric degree is maximal and by the formulas in remark 5.13, the invariants \( [\chi_i] = \Lambda_n(s_i) \). Fix maximal simple characters \( \theta_i \) in \( G_i \) of endo-class \( cl(s_i) \), with underlying simple strata \( \{\mathfrak{X}_i, \beta_i\} \). Let \( T_i = F[\beta_i]^{ur} \). Let \( L_i \) be an extension of \( F[\beta_i] \) contained in \( Z_{A_i}(F[\theta_i]) \) which has maximal degree, is unramified, and normalizes \( \mathfrak{X}_i \), as in the proof of proposition 2.5. Let \( K_i \) be the maximal unramified extension of \( F \) contained in \( L_i \). The quantity \( t = \frac{\sigma_{cl(\theta_i)}}{\sigma_{cl(\beta_i)}} \) is preserved under the Jacquet–Langlands correspondence (it is the torsion number of the inertial class, by the formulas in remark 5.13), which is preserved since the Jacquet–Langlands correspondence commutes with twists by unramified characters, and the \( K_i \) have the same degree \( t \) over \( F \).

Because \( d \) divides \( \frac{n}{\sigma_{cl(\theta_i)}} \), it divides \( [K_i : F] \), and the commutant \( Z_{A_i}(K_2) \) is a split central simple algebra over \( K_2 \): indeed, we have \( Z_{A_i}(K_2) \cong M_{m'}(D') \) for a central division algebra \( D' \) of reduced degree \( d/(d, [K_2 : F]) = 1 \). By [BH03] 1.5.1, the group \( Gal(k/f) \) is transitive on the set of \( K \)-lifts of \( cl(s_i) \), which has \( f(cl(s_i)) \) many elements. Since \( cl(s_i) = cl(s_1) \), the representations \( s_i ^* \) as \( \gamma \) runs through \( Gal(k/f) \) are pairwise inertia inequivalent (as they have different endo-classes). They are furthermore totally ramified representations of \( \hat{G}_K \), in the sense that their unramified parameter fields all coincide with \( K \).

**Lemma 4.8** (Linear independence lemma). Let \( \pi_1, \ldots, \pi_r \) be irreducible, supercuspidal, totally ramified representations of \( GL_m(D) \) for a central division algebra \( D \) over \( F \), whose central characters agree on \( \mu_F \). Assume that they are pairwise inertia inequivalent. Then the characters \( tr \pi_i \) are linearly independent on the set of elliptic, regular, pro-unipotent elements of \( GL_m(D) \).

**Proof.** This follows from lemma 6.6 in [BH11], as we can twist the \( \pi_i \) by unramified characters of \( GL_m(D) \) until the central characters also agree on a uniformizer of \( F \). This does not change the inertial classes of the \( \pi_i \), nor the character values on elliptic, regular, pro-unipotent elements of \( GL_m(D) \) as these have reduced norms of valuation 0.

The central characters of the \( \tau_i \) are trivial on \( \mu_K \) by construction. Then by the linear independence lemma either there exists \( \gamma \in Gal(K/F) \) such that \( \tau_1^\gamma \) and \( \tau_2 \) are inertia inequivalent, or

\[ Formally, K is really an inverse limit of the diagram \( \alpha_0: K_2 \to K_1 \), and similarly for \( A_K \) and \( \alpha: Z_{A_2}(K_2) \to Z_{A_1}(K_1) \). \]
for all values of \(i, \gamma\) and \(\zeta\). That this does not happen follows when \(i = 1\) by theorem 1.1 in [SZ00], stating that there exists no character \(\chi\) of \(k^x\) such that \(\sum_{\gamma \in \text{Gal}(k/f)} \chi(\zeta^\gamma) = 0\) for all \(f\)-regular elements of \(k\).

So we have proved that \(\tau_1\) and \(\tau_2\) are inertially equivalent for some \(\gamma \in \text{Gal}(K/F)\). But then they have the same endo-class, and since their endo-classes are \(K\)-lifts of \(\text{cl}(s_1)\) and \(\text{cl}(s_2)\) respectively, the theorem follows.

We pass now to the study of the level zero part of the \(s_i\). Let us first assume that the \(s_i\) are supercuspidal. Choose maximal simple characters \(\theta_i\) contained in the \(s_i\), defined by strata \([\mathfrak{A}_i, \beta_i]\), and let \(T_i = F[\beta_i]^{w_T}\). As in the proof of proposition 2.25 we take a maximal unramified extension \(L_i = F[\beta_i]\) in \(Z_{A_i}(F[\beta_i])\) normalizing \(\mathfrak{A}_i\), and identify the maximal unramified extensions \(K_i\) of \(F\) in \(L_i\). Now we know that the \(\theta_i\) have the same endo-class, and we take the only \(F\)-linear isomorphism \(\alpha_0 : K_2 \to K_1\) such that

\[
\alpha_0^* \text{cl}(\theta_{1,K_1}) = \text{cl}(\theta_{2,K_2})
\]

for the interior lifts \(\theta_{i,K_i}\). Notice, however, that the commutants \(Z_{A_i}(K_i)\) needn’t be isomorphic. As in the proof of theorem 4.7 we write \(K\) for any of \(K_2\) and \(K_1\) and \(T\) for any of \(T_2\) and \(T_1\), using the isomorphism \(\alpha_0\).

**Theorem 4.9.** The equality \(\epsilon_{\mu K}^1(V_1)\Lambda_\kappa(s_1) = \epsilon_{\mu K}^1(V_2)\Lambda_\kappa(s_2)\) holds, where the \(\epsilon_{\mu K}^1(V_i)\) are the symplectic sign characters, and \(\kappa\) is the conjugacy class of \(p\)-primary \(\beta\)-extensions.

**Proof.** We choose supercuspidal representations \(\pi_i\) of \(G_i\) in \(s_i\), Jacquet–Langlands transfers of each other, and we let \(\tau_i\) be some \(K\)-lift of \(\pi_i\); then, because of our choice of \(\alpha_0\), \(\tau_1\) and \(\tau_2\) have the same endo-class.

Fix a root of unity \(\zeta \in \mu_K\) generating \(K\) over \(F\), and let \(u_1\) be an elliptic, regular, pro-unipotent element of \(G_{K,1} = \text{Gal}(K)\). The matching conjugacy class in \(G_{K,2}\) then consists of pro-unipotent elements, as in the case of an elliptic regular element this is a condition which can be checked on the eigenvalues of the characteristic polynomial. Let \(u_2\) be an element of this conjugacy class. We apply proposition 4.1 and obtain an equality

\[
\text{tr}\pi_i(\zeta u_i) = (-1)^{m_i+1} \epsilon(\zeta, V_i) \sum_{\gamma \in \text{Gal}(t/f)} \left( \text{tr}\tau_i^\gamma(u_i) \sum_{\delta \in \text{Gal}(k/t)} \chi_i(\zeta^\delta) \right)
\]

where \(\Lambda_\kappa(s_i) = [\chi_i]\). By the linear independence lemma, we have that \(\tau_1^\gamma\) and \(\text{JL}_{G_{K,2}}(\tau_2)\) are inertially equivalent for some \(\gamma \in \text{Gal}(k/f)\) (this is the Jacquet–Langlands correspondence for the groups \(G_{K,i}\), which are inner forms of each other). This \(\gamma\) is unique, as the \(\tau_i^\gamma\) have pairwise different endo-classes for \(\gamma \in \text{Gal}(t/f)\). By theorem 4.7 the endo-class of \(\text{JL}_{G_{K,2}}(\tau_2)\) is \(\text{cl}(\theta_{2,K_2})\). By our choice of \(\alpha_0\), this implies \(\gamma = 1\).

Fix \(u_i\) so that the characters of the \(\tau_i\) are nonzero at \(u_i\); this is possible by the linear independence lemma, because the \(\tau_i\) are totally ramified. Then the Jacquet–Langlands character relation

\[
(-1)^{n_K} \text{tr}\tau_1(u_1) = (-1)^{m_K} \text{tr}\tau_2(u_2)
\]

holds, where \(Z_{A_1}(K_1) \cong M_{n_K}(K)\) and \(Z_{A_2}(K_2) \cong M_{n_K}(D_K)\) for some central division algebra \(D_K\) over \(K\).

We now have an equality

\[
(-1)^{m+m'+m_K+1} \epsilon(\zeta, V_2) \sum_{\delta \in \text{Gal}(k/t)} \chi_2(\zeta^\delta) = (-1)^{n+n'+n_K+1} \epsilon(\zeta, V_1) \sum_{\delta \in \text{Gal}(k/t)} \chi_1(\zeta^\delta)
\]

(4.1)

on comparing \(\text{tr}\pi_1(\zeta u_1)\) and \(\text{tr}\pi_2(\zeta u_2)\) by the Jacquet–Langlands correspondences over \(F\) and over \(K\). This equality holds for all \(\zeta \in \mu_K\) generating \(K\) over \(F\)—equivalently, for all \(\zeta \in k^x\) generating \(k\) over \(f\). To be more precise\(^4\), we are evaluating \(\chi_i\) at \(\zeta^\delta \in \mu_K\) via a choice of \(e\)-linear isomorphism \(i : k_0 \to e_{n_i/\delta}(\theta_F)\), as in theorem 4.1. Since \(\alpha_0^* \text{cl}(\theta_{1,K_1}) = \text{cl}(\theta_{2,K_2})\), we have \(\alpha_0^* T_2 = \nu T_1\), hence the \(\epsilon_i\) can be chosen compatibly with \(\alpha_0 : k_2 \to k_1\), allowing us to evaluate \(\chi_i\) to \(\zeta^\delta \in \mu_K\).

\(^4\)This becomes clearer if we consider \(k\) to be an inverse limit of the diagram \(\alpha_0 : k_2 \to k_1\).
Recall that there exist a sign $\epsilon^{0}_{\mu K}(V_i)$ and a quadratic character $\epsilon^{1}_{\mu K}(z, V_i)$ of $\mu_K$ such that, whenever $z \in \mu_K$ generates a subgroup $\Delta$ of $\mu_K$ with $V_i^{\mu_K} = V_i^\Delta$, one has

$$\epsilon(z, V_i) = \epsilon^{0}_{\mu K}(V_i)\epsilon^{1}_{\mu K}(z, V_i).$$

In our case, every $\zeta$ generating $K$ over $F$ satisfies $V_i^{\zeta} = V_i^{\mu_K}$ even if $\zeta$ does not generate $\mu_K$, by a cohomological vanishing argument as in \[BH10\] proposition 6. Comparing coefficients, one gets an equality

$$(−1)^{n+n'+n_K+1}\epsilon^{0}_{\mu K}(V_1)\sum_{\delta \in \text{Gal}(k/t)} \epsilon^{1}_{\mu K}(\zeta^\delta, V_1)\chi_1(\zeta^\delta) = (−1)^{m+m'+m_K+1}\epsilon^{0}_{\mu K}(V_2)\sum_{\delta \in \text{Gal}(k/t)} \epsilon^{1}_{\mu K}(\zeta^\delta, V_2)\chi_2(\zeta^\delta)$$

which we rewrite

$$(−1)^{n'+1}\sum_{\delta \in \text{Gal}(k/t)} \epsilon^{1}_{\mu K}(\zeta^\delta, V_1)\chi_1(\zeta^\delta) = (−1)^{n+n'+n_K+m+m'+m_K+1}\epsilon^{0}_{\mu K}(V_1)\epsilon^{0}_{\mu K}(V_2)\sum_{\delta \in \text{Gal}(k/t)} \epsilon^{1}_{\mu K}(\zeta^\delta, V_2)\chi_2(\zeta^\delta).$$

This equation stays true if $\zeta$ varies over all generators of the extension $k/f$. Since we are dealing with supercuspidal inertial classes for $\text{GL}_n(F)$, the character $\chi_1$ is $e$-regular. If $\chi_1$ varies through all $e$-regular characters of $e^{x}_{n/\delta(\Theta_F)}$, and we let $\chi_2$ vary so that $s_{G_1}(\Theta_F, \Theta_E, [\chi_1]) = \text{JL}(s_{G_2}(\Theta_F, \Theta_E, [\chi_2]))$ (this is possible by theorem \[4.7\]) then equation \[4.2\] continues to stay true. At the left hand side of \[4.2\] one has the trace of a supercuspidal irreducible representation of $\text{GL}_{n/\delta(\Theta_F)}(t)$. By 2.3 Corollary in \[BH10\] we deduce that \[4.2\]

$$(\epsilon^{1}(z, V_1)\chi_1) = (\epsilon^{1}(z, V_2)\chi_2)$$

and the claim follows.

It follows from theorem \[4.9\] that, twisting the $p$-primary $B$-extension by the symplectic sign character (a quadratic character), we obtain conjugacy classes $\kappa_i$ of $\beta$-extensions in $G_i$, of endo-class $\Theta_F$, such that $\Lambda_{\kappa_1}(s_1) = \Lambda_{\kappa_2}(s_2)$ whenever the $s_i$ are supercuspidal inertial classes and Jacquet–Langlands transfers of each other. Notice also that by \[BH11\] 6.9, the sign $\epsilon^0$ and the character $\epsilon^1$ determine each other: $\epsilon^1$ is the nontrivial quadratic character if and only if $p$ is odd and $\epsilon^0 = −1$. It follows that the quadratic character $\epsilon^{1}_{\mu K}(V_1)\epsilon^{1}_{\mu K}(V_2)$ is nontrivial if and only if $p$ is odd and $n + n' + n_K + m + m' + m_K$ is odd.

**Theorem 4.10.** With the notation of the previous paragraph, the equality $\Lambda_{\kappa_1}(s_1) = \Lambda_{\kappa_2}(s_2)$ also holds for non-cuspidal $s_i$.

**Proof.** This is proved as lemma 9.11 in \[SS16\]. Write $[\chi_i]$ for $\Lambda_{\kappa_i}(s_i)$ and assert that $\chi_1$ is not $e$-regular, as the $e$-regular case has already been treated. Write $\xi(\kappa_1, \kappa_2)$ for the permutation of $\Gamma(\Theta_F)\backslash X_C(\Theta_F)$ such that

$$\xi(\kappa_1, \kappa_2)\Lambda_{\kappa_1}(x_1) = \Lambda_{\kappa_2}(x_2)$$

for all simple inertial classes $x_1 = \text{JL}_{G_2}(x_2)$ of endo-class $\Theta_F$. By the results in section \[2.2\] we see that for any prime number $\ell \neq p$ this permutation preserves the equivalence relation of having the same $\ell$-regular parts on $\Gamma(\Theta_F)\backslash X_C(\Theta_F)$. We will prove that $\xi([\chi_1]) = [\chi_2]$.

Because the parametric degree of simple inertial classes, as defined in \[BH11\], is preserved under the Jacquet–Langlands correspondence, one finds that $e([\chi_1]) = e([\chi_2])$ (since by the formulas in remark \[3.13\] the parametric degree of $s_{G_1}(\Theta_F, \Theta_E, [\chi_1])$ equals $n/s([\chi_1])$). Hence $\xi$ preserves Frobenius orbit size.

Let $a$ be some large integer ($a \geq 7$ will suffice) and write $\kappa^*_i$ for the maximal $\beta$-extension in $\text{GL}_m(D)$ compatible with $\kappa_i$. Let $s_i$ correspond to the supercuspidal support $\pi^*_{i \text{reg}}$, and let $s_{i, a}$ be the

\[\text{We couldn't apply this directly to equation \[4.11\] to deduce}

$$[\chi_1] = [\chi_2]$$

$$(-1)^{n+n'+n_K+1}\epsilon(\zeta, V_1) = (-1)^{m+m'+m_K+1}\epsilon(\zeta, V_2)$$

because the sign $\epsilon(\zeta, V_i)$ may not be constant on the $\zeta$ which generate $K$ over $F$, since these may generate proper subgroups of $\mu_K$.\]
simple inertial class with supercuspidal support \( \pi_i^{\text{sc}} \). Then the \( \mathfrak{s}^*_i \) are Jacquet–Langlands transfers of each other, and we claim that that \( \Lambda_{\kappa_i}(\mathfrak{s}^*_i) \) is the inflation \( [\chi^*_i] \) of \( \Lambda_{\kappa_i}(\mathfrak{s}_i) \). To see this, observe that \( \pi_i \) is a supercuspidal representation of some \( \text{GL}_{m_i}(F) \) or \( \text{GL}_{m_i}(D) \) and write \( \kappa_{i,*} \) for the \( \beta \)-extension in this group compatible with \( \kappa_i \). Then by construction \( \Lambda_{\kappa_i}(\mathfrak{s}_i) \) is the inflation of \( \Lambda_{\kappa_{i,*}}(\pi_i) \). By transitivity, \( \kappa_{i,*} \) and \( \kappa_i^* \) are compatible, hence \( \Lambda_{\kappa_{i,*}}(\mathfrak{s}^*_i) \) is the inflation of \( \Lambda_{\kappa_{i,*}}(\pi_i) \), and the claim follows.

So \( \xi([\kappa_1^*,\kappa_2^*])([\chi_i]^*) = [\chi_i]^* \); and since the norm is surjective in finite extensions of finite fields it suffices to prove that \( \xi([\kappa_1^*,\kappa_2^*])([\chi_i]^*) = [\chi_i]^* \). By lemma 8.5 in [SS16b] we can find a prime number \( \ell \neq p \) not dividing the order of \( e([\chi_i]^*) \), an integer \( a \geq 1 \) and an \( e \)-regular character \( \beta \) of \( e_\kappa^*(\chi^*_{m,i}(\pi_i)) \) with the same \( \ell \)-regular part of \( \chi_i^* \). Then \( \xi([\chi_i]^*) \) is \( \ell \)-regular, so it suffices to prove that \( \xi[\beta] = [\beta] \) and \( \xi[\chi_i^*] \) is \( \ell \)-regular. That \( \xi[\chi_i^*] \) is \( \ell \)-regular follows as \( \xi \) preserves parametric degrees and \( \ell \) does not divide the order of \( e([\chi_i]^*) \).

By theorem 4.4 in [Dot18] we know that there exists some \( \beta \)-extension \( \kappa \) in \( \text{GL}_{m_i}(F) \) such that \( \xi(\kappa,\kappa^*_{\beta})([\beta] = [\beta] \), hence there exists some character \( \delta \) of \( e^\times \) such that \( \xi(\kappa^*_1,\kappa^*_2)([\beta] = [\delta \beta] \) for every \( e \)-regular character \( \beta \) of \( e_\kappa^*(\chi^*_{m,i}(\pi_i)) \), because \( \kappa^*_1 \) and \( \kappa \) are unramified twists of each other. An argument analogous to the one used in the proof of theorem 4.2 in [Dot18] then proves that \( \delta = 1 \), and the theorem follows.

Finally, recall from the introduction that for a certain quadratic character \( \varepsilon \text{Gal} \) the representation \( \kappa_{\text{can}}^\text{GL}_{m_i}(F) = \varepsilon \text{Gal} \kappa_{1,i}^\text{can} \) is the canonical \( \beta \)-extension of endo-class \( \Theta_F \) for the group \( \text{GL}_{m_i}(F) \). In more detail, \( \varepsilon \text{Gal} \) is nontrivial if and only if \( p \neq 2 \) and the degree of a tame parameter field of \( \Theta_F \) over \( F \) is even. Then we define the canonical \( \beta \)-extension for \( \text{GL}_{m_i}(D) \) of endo-class \( \Theta_F \) to be the maximal \( \beta \)-extension \( \kappa_{\text{can}}^\text{GL}_{m_i}(D) = \varepsilon \text{Gal} \kappa^\text{can}_{1,i} \). It has the property that if \( p \) is an essentially square-integrable representation of \( \text{GL}_{m_i}(D) \), then \( \Lambda_{\kappa_{\text{can}}^\text{GL}_{m_i}(D)}(\pi) \) coincides with the level zero part of the Langlands parameter \( \text{rec}(\text{JL}(\pi)) \), as defined in [Dot18]. Furthermore, since the canonical \( \beta \)-extensions of endo-class \( \Theta_F \) in \( \text{GL}_{m_i}(F) \) are compatible with each other for varying \( n \), we see (by an argument similar to the proof of proposition 4.4 in [Dot18]) that the same is true for \( \text{GL}_{m_i}(D) \).

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