Half lives and Rates for Electron Capture on Drip Line Nuclei and for Beta Decay with $65 < A < 100$

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Abstract

The half lives are calculated for the process of $\beta^\pm$ decay and electron capture for nuclei in mass range $\sim 65 - 100$ relevant for the core of a massive star at late burning stage of stellar evolution that leads to supernova explosion. The rates of electron capture process are also computed for nuclei relevant during presupernova stage of massive stars. These half lives and rates are calculated by expressing $\beta^\pm$ Gamow-Teller decay strengths in terms of smoothed bivariate strength densities. These strength densities are constructed in the framework of spectral averaging theory for two body nuclear Hamiltonian in a large nuclear shell model space. The method has a natural extension to weak interaction rates for r and rp-processes.

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1 Introduction

The late evolution stages of massive stars ($> 8M_\odot$) is strongly influenced by the weak interaction processes. In the pre-collapse stage of a star the important weak processes are electron captures and beta decays. These weak processes influence the value of $Y_e$, the electron fraction. As Chandrasekhar mass is
proportional to $Y_e^2$, the value of $Y_e$ is crucial for the collapse of a supernova star and the subsequent evolution. The stability (or collapse) of the star at late stage is determined by among other things, the balancing of the gravitational collapse by electron degenerate pressure. While electron capture reduces the number of electrons for pressure support, $\beta^-$ decays increases the same. Therefore these rates are important ingredients for building a model for the core collapse supernova. With this in view, in this work we have investigated the rates and half lives for weak interaction processes for some of the nuclei relevant for presupernova and supernova environment with $65 < A < 100$. We first calculate the weak interaction strength densities using a theory based on principles of statistical nuclear spectroscopy – spectral averaging theory.

Among the earlier calculations for weak interaction rates ($\beta^{\pm}$, EC etc.), Fuller, Fowler, Newman [1] for the first time dealt with these rates at finite temperature in the mass range $21 \leq A \leq 60$. Aufderhide et al [2, 3] emphasised the need for calculating weak interaction rates for $A > 60$ nuclei as presupernova stars generate neutron rich cores with mass $A > 60$. Aufderhide et al and later Kar et al [4] have made rate calculations in the mass $A > 60$. In this work, we have explicitly constructed the bivariate weak interaction strength densities for calculation of weak interaction half lives and rates. As in an earlier work, using these methods, $\beta^-$ decay rates have been calculated [5] for nuclei in the regime $55 < A < 65$ and extended to EC for a few fp-shell nuclei, in this work we have computed EC half lives for the nuclear mass region 65 - 81 that are still relevant for supernova stars using smoothed form for EC strength densities in the framework of spectral averaging theory. We also calculate rates for some sample nuclei in this region and this would be extended to neutron rich nuclei later.

After the spontaneous nucleosynthesis ceases in late stage of steller evolution, heavier elements in the star are synthesised by r-process, s-process and rp processes. The drip line nuclei play an important role then for the understanding of stellar processes after the star runs out of fuel. With this in view we have calculated the $\beta^{\pm}$ half lives for the drip line neutron nuclei in the mass region $81 < A < 100$. In this case too the calculations are made by constructing
bivariate strength densities for $\beta^\pm$ operator under the framework of spectral averaging theory.

The spectral averaging theory which is based on statistical nuclear spectroscopy, was started with Bethe’s statistical mechanical level density formula, Wigner’s treatment of spectral fluctuations using matrix ensembles and French’s embedded ensembles and Gaussian densities. The smoothed forms of spectroscopic observables follow from the action of Central Limit Theorems (CLT) in nuclear shell model spaces. The statistical spectroscopy is in deriving and applying the smoothed forms in indefinitely large spaces with interactions by using unitary group decompositions (of Hamiltonians and the spectroscopic spaces), CLT’s locally, and convolutions - the resulting theory is the Spectral Averaging Theory in Large Shell Model Spaces (SAT-LSS) [6, 7, 8, 9, 10, 11]. Here it is seen that the essential role of interactions is to produce local spreadings of the non interacting particle (NIP) densities and the spreadings are in general Gaussian in nature. The spectral averaging theory in large shell model spaces has important nuclear physics applications like calculations of nuclear state and level densities, occupation and spin cut-off densities and calculations of occupation numbers and spin cut-off factors [10].

The theory has been extended to calculate the smoothed form for interacting particle (IP) bivariate transition strength densities ($I_H^O(E_i, E_f)$) for a one plus two body nuclear Hamiltonian $H$ and a transition operator $O$. This bivariate strength density ($I_H^O(E_i, E_f)$) takes a convolution form [8, 12, 13] with the non interacting particle (NIP) strength densities being convoluted with a spreading bivariate Gaussian due to irreducible two body (off-diagonal) part $V$ of the interaction. Mathematically, $I_H^O \Rightarrow I_h^O \otimes \rho_V^O$, where $I_h^O$ is the strength density due to effective one body part $h$ of the interacting Hamiltonian and $\rho_V^O$ is a zero centred bivariate Gaussian due to the irreducible two body (off-diagonal) part $V$ of the interaction. In spectral averaging theory the NIP part $I_h^O$ are constructed by calculating a few lower order moments of $I_h^O$ and explicit analytical formulae for NIP strength densities are worked out in [13]. This method termed as moment method is a convenient way for rapid construction of NIP strength densities. The IP bivariate strength densities thus constructed has been used to calculate
\( \beta^- \) decay rates for some fp-shell nuclei around the mass range \( 55 < A < 65 \), Giant Dipole Resonance (GDR) cross-sections etc.

The smoothed form for IP strength densities for \( \beta^\pm \) decay and EC Gamow-Teller transition operator can therefore be constructed using the formalism of spectral averaging theory in a large shell model space. With these, the half lives and rates for electron capture processes at finite temperature relevant for presupernova environment and half lives for \( \beta^\pm \) decay processes are calculated for the nuclei mentioned earlier.

The paper is organised as follows. In Section 2 we give the formalism. Section 3 deals with the calculational methods that includes choice of shell model space, the single particle energies (s.p.e.) etc. The results are discussed in Section 4. In section 5 we conclude with some discussions and future outlook.

2 Formalism

2.1 Unitary decompositon of nuclear Hamiltonian

Given \( m \) number of particles distributed in a shell model space we have an \( m \)-particle shell model space. Each shell model orbit or single particle orbit \( \alpha \) with degeneracy \( N_\alpha = 2j_\alpha + 1 \) is called a spherical orbit. A group of spherical orbits is called a unitary orbit \( \alpha \). In what follows we will use \( \alpha, \beta \) etc. (smaller in size) to denote spherical orbits and \( \alpha, \beta \) etc. (larger in size) for unitary orbits. Thus we have for \( m \) number of particles distributed in this shell model space, spherical configuration \( \mathbf{m} \equiv m_\alpha, m_\beta, \ldots (m_\alpha, m_\beta \text{ etc.} \) are number of particles in spherical orbits \( \alpha, \beta \text{ etc.} \) respectively) and unitary configuration \( [\mathbf{m}] \equiv m_\alpha, m_\beta, \ldots \). Using the convention that the spherical orbits \( \alpha \) belongs to unitary orbit \( \alpha \) the number of single particle orbits in unitary orbit \( \alpha \) is \( N_\alpha = \sum_{\alpha \in \alpha} N_\alpha \).

A further decomposition of \( m \)-particle space is possible by attaching an \( s_\alpha \) label to each spherical orbit \( \alpha \) where \( s_\alpha \) for lighter nuclei denotes \( sh\omega \) excitation value. With this the \( m \)-particle space can be decomposed into \( S \)-subspaces as
follows:

\[ m \rightarrow \sum S^\pi; S^\pi \rightarrow \sum [m]; [m] \rightarrow \sum m; S = \sum m_\alpha s_\alpha \quad (1) \]

One can recognise here the appearance of \( U(N) \) group which is generated by the \( N^2 \) operators \( a^\dagger_{j_\alpha m_\alpha}a_{j_\alpha m_\alpha}; \alpha, \beta = 1, 2, ..., N \). For \( m \) identical particles therefore there are \( \binom{N}{m} \) antisymmetric states forming an irreducible representation (irrep) for the group \( U(N) \) usually denoted by Young shape \( \{1^m\} \). The only scalar operator is the number operator \( n \) as it remains invariant under the transformation of the \( U(N) \) group. A given operator can be decomposed into tensor operators (belonging to a definite irrep of \( U(N) \)) with respect to \( U(N) \) group.

Thus the unitary group \( U(N_\alpha) \) acting on each spherical orbit \( \alpha \) generates \( m_\alpha \) of spherical configurations \( m \), i.e. \( m \) behaves as \( \{1^{m_\alpha}\} \otimes \{1^{m_\beta}\} \otimes ... \) with respect to the direct sum group \( U(N_\alpha) \oplus U(N_\beta) \oplus ... \) and the scalar operators are \( m_\alpha \)'s. Therefore for a given nuclear two body Hamiltonian \( H = h(1) + V(2) \), it should be obvious that non interacting particle part \( h(1) \) of \( H \) is a scalar with respect to spherical configuration group and

\[ h = \sum \epsilon_\alpha n_\alpha = h^{[0]} \quad (2) \]

where \( \epsilon_\alpha \) is the single particle energy (s.p.e.) of the spherical orbit \( \alpha \). For the spherical configuration scalar part \( V^{[0]} \) (must be a second order polynomial in \( n_\alpha \)) of residual interaction \( V(2) \), one has

\[ V^{[0]} = \sum_{\alpha \geq \beta} V_{\alpha \beta} \frac{n_\alpha(n_\beta - \delta_{\alpha \beta})}{(1 + \delta_{\alpha \beta})}. \quad (3) \]

In the above \( V_{\alpha \beta} \) is the average two body interaction given by

\[ V_{\alpha \beta} = \{N_{\alpha \beta}\}^{-1} \{\sum_j (2J + 1)V^{J}_{\alpha \beta}(1 + \delta_{\alpha \beta})\} \quad (4) \]

where \( N_{\alpha \beta} = N_\alpha(N_\beta - \delta_{\alpha \beta}) \) and \( J \) is the angular momentum. The remaining non-diagonal part \( V = V(2) - V^{[0]} \) of \( V(2) \) is an irreducible two body part as there cannot be an effective one body part of \( V(2) \) with respect to spherical configuration group.
The same idea can be applied for the decomposition of Hamiltonian under unitary configuration group. In this case, the unitary group $U(N_\alpha)$ acting in each unitary orbit $\alpha$ generates $m_\alpha$ of unitary configuration $[m]$; i.e. $[m]$ behaves as $\{1^{m_\alpha}\} \otimes \{1^{m_\beta}\} \otimes ...$ with respect to the direct sum group $U(N_\alpha) \oplus U(N_\beta) \oplus ...$ and the scalar operators are $m_\alpha$’s. The decomposition relevant for the present is the spherical scalar part $h[0] + V[0]$ of the Hamiltonian. They will have $[0] \oplus [1] \oplus [2]$ tensorial parts with respect to unitary configuration direct sum group.

Thus after the unitary decomposition of the nuclear Hamiltonian $H$ we have

$$H = h(1) + V(2)$$
$$\Rightarrow h[0] + V[0] + V$$
$$\Rightarrow h[0][0] + h[0][1] + V[0][0] + V[0][1] + V[0][2] + V$$ (5)

This has been demonstrated that the contribution of $V[0][2]$ part is small ($\leq 5\%$) all across the periodic table by calculating its norm for the case of ds, fp, 10-orbit and 15-orbit interactions and phenomenological interactions like surface delta interaction and pairing + Q.Q interactions[7, 10]. Therefore $V[0][2]$ can be neglected for all practical purposes.

Thus Eq. 5 reduces to

$$H = h[0][0] + h[0][1] + V[0][0] + V[0][1] + V$$
$$= h + V$$ (6)

where $h$ is the effective one body part of $H$ and $V$ is irreducible two body part.

For a unitary configuration $[m]$, the unitary decompositions of nuclear Hamiltonian $H$ are given by

$$h[0][0] = \sum_\alpha \epsilon_\alpha n_\alpha ; \hspace{0.5cm} \epsilon_\alpha = \left( \sum_{\alpha \in \alpha} \epsilon_\alpha N_\alpha \right) N_\alpha^{-1}$$

$$h[0][1] = \sum_\alpha \epsilon_\alpha^{[1]} n_\alpha ; \hspace{0.5cm} \epsilon_\alpha^{[1]} = \epsilon_\alpha - \epsilon_\alpha$$

$$V[0][0] = \sum_{\alpha \geq \beta} [V_\alpha \beta] \frac{n_\alpha (n_\beta - \delta_\alpha \beta)}{(1 + \delta_\alpha \beta)} ; \hspace{0.5cm} V_\alpha \beta = \left[ \sum_{\alpha \in \alpha , \beta \in \beta} N_\alpha \beta V_\alpha \beta \right] [N_\alpha \beta]^{-1}$$
\[ V^{[0][1]} = \sum_{\alpha} \left\{ \sum_{\beta} (m_\beta - \delta_\alpha \beta) \left[ \epsilon^{[1]_\alpha}_\beta \right] \right\} n_\alpha \]
\[ \epsilon^{[1]_\alpha}_\beta = \left\{ \left[ \sum_{\beta \in \beta} (N_\beta - \delta_\alpha \beta) V_{\alpha \beta} \right] - (N_\beta - \delta_\alpha \beta) V_{\alpha \beta} \right\} \times \{ N_\beta - 2\delta_\alpha \beta \}^{-1} \] (7)

In the above and for rest of the calculations we consider \( S \)-conserving part (see discussions above Eq. 2 for \( S \) quantum number) of the interaction. The \( S \)-mixing part \( V_{S-mix} \) of \( V(2) \) represents admixing between distant configurations (at least \( 2\hbar \omega \) away from each other) and this leads to multimodal form of densities [13] unlike the unimodal forms. Moreover \( \text{GT} \beta^\pm \) operator does not connect different \( S \)-subspaces. Hence the \( V_{S-mix} \) is not considered for the rest of the calculations.

Given \( H = h + V \) the IP strength density \( I_O^{H=h+V}(E_i, E_f) \) for a transition operator \( O \) will take a bivariate convolution form [12] with the two convoluting functions being NIP strength densities \( I_O^h \) and a normalised spreading bivariate Gaussian \( \rho_{V,BIV-G} \) due to \( V \) (interactions),
\[ I_O^{h+V}(E_i, E_f) = I_O^h \otimes \rho_{V,BIV-G}[E_i, E_f] \] (8)

For our case, \( O \equiv O(\text{GT}) \). In large spectroscopic spaces with protons and neutrons (pn), this GT bivariate strength density can be partitioned in different unitary configuration subspaces and \( S \)-subspaces (Eq. 1) and can be written as (identifying that \( O(\text{GT}) \) does not connect two different subspaces)
\[ I_O^{H=h+V}(E_i, E_f) = \sum_S \sum_{[m^{(i)},m^{(f)}],[m^{(i)},m^{(f)}] \in S} I_O^{h}[m^{(i)},m^{(f)}] \otimes \rho_{V,BIV-G}[m^{(i)},m^{(f)}][E_i, E_f] \] (9)

In general for a transition operator \( O \) and for a nuclear Hamiltonian \( H \), the strength density \( I_O^{H,m,m'} \) is given as
\[ I_O^{H,m,m'} = I^{m'}(E') \langle E'm'|O|EM \rangle^2 I^m(E) \]
\[ = \langle \langle O^\dagger \delta(H - E')O \delta(H - E) \rangle \rangle^m \] (10)
where $I^m(E')$ and $I^m(E)$ are final and initial state densities and $\langle \langle \rangle \rangle^m$ represents a trace over $m$ particle space. By the action of CLT in the spectroscopic space, the density will be a bivariate Gaussian and it is demonstrated in [10, 5] by constructing the exact NIP strength densities (and its $s$-decomposition) for $\mathcal{O}(GT)$ operator and then comparing this with the smoothed Gaussian form.

The smoothed form for strength density is constructed with marginal centroids $\epsilon_1, \epsilon_2$ and variances $\sigma_1^2, \sigma_2^2$ and calculating few lower order central bivariate moments given by

\[
\begin{align*}
\epsilon_1 &= \langle \langle O^\dagger OH \rangle \rangle^m / \langle \langle O^\dagger O \rangle \rangle^m \\
\epsilon_2 &= \langle \langle O^\dagger HO \rangle \rangle^m / \langle \langle O^\dagger O \rangle \rangle^m \\
\sigma_1^2 &= \langle \langle O^\dagger OH^2 \rangle \rangle^m / \langle \langle O^\dagger O \rangle \rangle^m \\
\sigma_2^2 &= \langle \langle O^\dagger H^2 O \rangle \rangle^m / \langle \langle O^\dagger O \rangle \rangle^m \\
\mu_{pq} &= \left( \frac{H - \epsilon_2}{\sigma_2} \right)^p \left( \frac{H - \epsilon_1}{\sigma_1} \right)^q / \langle \langle O^\dagger O \rangle \rangle^m
\end{align*}
\]

(11)

Therefore the calculation of moments are in fact calculation of $m$-particle averages or traces of the operator in question. This is done by first calculating a few basic traces and then propagating them in $m$-particle space. As here we are dealing with unitary configurations $[m]$ and unitary configuration densities (Eq. 9), we would require the unitary configuration traces of the type $\langle \rangle^m$. These moments $M_{pq}([m])$ for the construction of NIP strength densities $I^m_h$ for a one body transition operator $\mathcal{O}$ with $p + q \leq 2$ are calculated in details in Ref. [10, 4]. As the GT operator is of the type $\mathcal{O}(GT) = \epsilon_{\alpha\beta} a^\dagger_{\alpha} a_{\beta}$ ($\epsilon_{\alpha\beta}$ is the single particle matrix elements), for a given initial configuration $[m_i]$, the final configuration $[m_f]$ then is obtained uniquely as $[m_f] = [m_i] \times (1^i_{\alpha} 1^f_{\beta})$. Thus one also obtains the partial moments $M_{pq}([m_i], [m_f])$ for $I^h$. For a proton neutron (pn) configuration one writes $[m]$ as $[m_p, m_n]$.

For the construction Gaussian spreadings $\rho^V_{\mathcal{O}}([m_p, m_n]; [m_i, m_f]) (GT)(x, y)$, in Eq. 9, the following approximations are adopted. The marginal centroids,

$M_{10} = \langle \langle O^\dagger OV \rangle \rangle^{m_p, m_n} / \langle \langle O^\dagger O \rangle \rangle^{m_p, m_n} \simeq \langle V \rangle^{m_p, m_n} = 0$, as $V$ is traceless; $M_{01} = \langle \langle O^\dagger VO \rangle \rangle^{m_p, m_n} / \langle \langle O^\dagger O \rangle \rangle^{m_p, m_n} \simeq \langle V \rangle^{m_p, m_n} = 0$ and the marginal variances given by traces of the type $\langle O^\dagger OV^2 \rangle$ and $\langle O^\dagger V^2 O \rangle$ are equal to $\langle V^2 \rangle$. 

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2.2 Formalism for $\beta^{\pm}$ decay half lives and electron capture rates at finite temperature

The weak interaction ($\beta^{\pm}$ and electron capture (EC) for the present case) rate $T(E_i \rightarrow E_f)$ is the number of weak processes per second from a given initial state $|E_i\rangle$ of the parent nucleus to the final nuclear state $|E_f\rangle$ and $T(E_i \rightarrow E_f) \propto [g_V^2 B_F(E_i \rightarrow E_f) + g_A^2 B_{GT}(E_i \rightarrow E_f)]$, where $g_V$ and $g_A$ are respectively the vector and axial vector coupling constants and $B_F$ and $B_{GT}$ are the Fermi and Gamow-Teller transition strengths respectively. Including the phase space factor $f$ that incorporates the dependence of the rate on nuclear charge $Ze$ and the available energy for the weak process under consideration, $T$ takes the form $T(E_i \rightarrow E_f) = C f [g_V^2 B_F(E_i \rightarrow E_f) + g_A^2 B_{GT}(E_i \rightarrow E_f)]$ where $C$ is a constant with $Q$ the GS $Q$-value of the weak process ($\beta^{\pm}$ or EC), $Q_i = Q + E_i$. The value of $C$ is fixed by using the $\log ft$ values from pure Fermi transitions. One can write down the expressions for ground state half lives and $\beta^{\pm}$ decay and EC rates at finite temperature. For present beta decay calculations we have neglected Fermi term $B_F$ as the Fermi strength is concentrated in narrow domain and high up in energy (the centroid $\sim 1.44ZA^{-1/3}$ MeV, width $\sim 0.157ZA^{-1/3}$ MeV). For $\beta^+$ decay and electron capture processes, Fermi transitions are not possible for nuclei with $N > Z$, since for mother isospin $T_0 = (N - Z)/2$, the isospins of the states of daughter are greater by 1, i.e. $T_{\text{daughter}} = |((N+1) - (Z-1))/2| = T_0 + 1$. Writing $B_{GT}(J_iE_i \rightarrow J_fE_f) = (2J_i+1)^{-1}\sum |\langle E_fJ_fM_f|(O_{GT})^k\mu|E_iJ_iM_i\rangle|^2$ which in continuous version becomes $\{I(E_i)\}^{-1}\sum_{\alpha \in E_i, \beta \in E_f, \mu} |\langle E_f\alpha|(O_{GT})^k\mu E_i\beta\rangle|^2$, the expression for GS half life is

$$t_{1/2}(GS) = \{6250(s)\} \times \left\{ \int_0^Q \left[ \left( \frac{g_A}{g_V} \right)^2 3\mathcal{L} \right] \left[ \frac{\mathcal{H}_{O(GT)}(E_{GS}, E_f)}{\mathcal{H}(E_{GS})} \right] f(Z)dE_f \right\}^{-1}$$

In the above equation $\mathcal{L}$ is the so called quenching factor and the factor 3 comes because of the definition of $\mathcal{H}$. The usual values of $\mathcal{L}$ is 0.6 [17] for $\beta^-$ decay and 0.5 for EC/$\beta^+$ decay and $\left( \frac{g_A}{g_V} \right)^2 = 1.4$ as given in [17]. The weak interaction
rates are given by
\[
\lambda(T) = \frac{\ell n(2)(s^{-1})}{6250} \left[ \int e^{-E_i/k_B T} H(E_i) dE_i \right]^{-1} \times \\
\left[ \int dE_i e^{-E_i/k_B T} H(E_i) \left[ \int_{Q_i}^0 dE_f \left( \frac{g_A}{g_V} \right)^2 3E \right] \times \\
\frac{I_{\Omega(GT)}^H(E_i, E_f)}{I^H(E_i)} f(Z, T) \right]
\]
\[
= \frac{\ell n(2)(s^{-1})}{6250} \left[ \int e^{-E_i/k_B T} H(E_i) dE_i \right]^{-1} \times \\
\left[ \int dE_i e^{-E_i/k_B T} \left[ \int_{Q_i}^0 dE_f \left( \frac{g_A}{g_V} \right)^2 3E \right] I_{\Omega(GT)}^H(E_i, E_f) f(Z, T) \right]
\]
(13)

The phase space integral \( f \) in the above equations are given by, for the case of EC
\[
f = \int_{E_{\text{min}}}^{\infty} dE \left( E^2 - m_e^2 \right)^{1/2} \left( E - Q_i \right)^2 F_c(Z, E) \times \\
\left( \frac{1}{1 + \exp[(E - \mu_e)/k_B T]} \right) \left( \frac{1}{1 + \exp[(E - \mu_{\nu e})/k_B T]} \right)
\]
(14)

where \( E_{\text{min}} \) is greater among \(-Q_i\) and \( m_e c^2 \). In the actual calculation the integrand is expressed in terms of electron mass \( m_e \) and the integration limits are also accordingly changed. In the above \( k_B \) stands for Boltzmann constant, \( \mu_e \) is the chemical potential for electron and \( \mu_{\nu} \) is the same for neutrino. For the case of \( \beta^- \) decay the phase space factor reads as
\[
f = \int_{1}^{e_0} \frac{F_c(Z, \epsilon_e) \epsilon_e (\epsilon_e^2 - 1)^{1/2} (\epsilon_0 - \epsilon_e)^2 \epsilon_e}{\{1 + \exp[(\mu_e - \epsilon_e)/(k_B T_e)]\}} d\epsilon_e
\]
(15)

In the above equation \( \epsilon_0 = E_0/m_e; (E_0 = Q_i - E_f), \mu_e = \mu/m_e \) and \((k_B T)_e = k_B T/m_e \). \( F_c(Z, E) \) in the above equations give the Coulomb factor and for EC is given by
\[
F_c(Z, E) = F_c(Z) F_{c}^*(E)
\]
\[
F_c^* = \frac{8\pi \alpha Z(1 + S)}{|\Gamma(1 + 2s)|^2} \left( \frac{2R}{h} \right)^{2S-2} \left( \frac{1 + 0.577(\alpha Z)^2}{2 + 0.577(\alpha Z)^2} \right)
\]
\[
F_{c}^*(E) = (E^2 - m_e^2)^{S-1} \left( \frac{E}{(E^2 - m_e^2)^{1/2}} \right) \left( \frac{1}{1 - e^{-2\pi \theta}} \right)
\]
(16)
In the above, \( \alpha \) is the fine structure constant. The value of \( S \) in the present calculation is assumed to be 1 \[18\] and following \[18\] the term \( \frac{1}{1-e^{-2\pi \theta}} \) is expressed as \( \langle 2\pi \theta \rangle \) and for \( Q_i < m_e c^2 \)

\[
\langle 2\pi \theta \rangle = 2\pi \alpha Z \frac{e^{\frac{-1}{\lambda}}}{K_2(1/\lambda)} (1 + 2\lambda + 2\lambda^2) \tag{17}
\]

\[
\lambda = k_B T/m_e c^2
\]

where \( K_2 \) is second order modified Bessel function. For \( Q_i > m_e c^2 \) we use

\[
\langle 2\pi \theta \rangle = 2\pi \alpha Z \frac{Q_i}{(\epsilon_0 - m_e^2)^{1/2}} \tag{18}
\]

For \( \beta^- \) process the Coulomb factor is taken as given in Schenter and Vogel \[19\] and the expression reads as

\[
F_c(Z, \epsilon) = \sqrt{\epsilon^2 - 1} exp[\alpha(Z) + \beta(Z)\sqrt{\epsilon - 1}] ;
\]

\[
\alpha(Z) = \begin{cases} 
-0.811 + 4.46(-2)Z + 1.08(-4)Z^2 & (\epsilon - 1) < 1.2 \\
-8.46(-2) + 2.48(-2)Z + 2.37(-4)Z^2 & (\epsilon - 1) \geq 1.2 
\end{cases}
\]

\[
\beta(Z) = \begin{cases} 
0.673 - 1.82(-2)Z + 6.38(-5)Z^2 & (\epsilon - 1) < 1.2 \\
1.15(-2) + 3.58(-4)Z - 6.17(-5)Z^2 & (\epsilon - 1) \geq 1.2 
\end{cases}
\]

(19)

For the electron chemical potential we have used the expression given in \[2\],

\[
\mu_e = 1.11(\rho_7 Y_e)^{1/3} \left[ 1 + \left( \frac{\pi}{1.11} \right)^2 \frac{T^2}{(\rho_7 Y_e)^{2/3}} \right]^{-1/3} . \tag{20}
\]

In the above \( \rho_7 \) is the matter density \( \rho \) in units of \( 10^7 \) gms/cc, \( T \) is the temperature \( T \) expressed in MeV and \( Y_e \) is the electron fraction. We put \( \mu_\nu \), the neutrino chemical potential to be zero as for the densities we consider the neutrinos are free streaming.

### 3 Calculational Procedure

For the construction of strength densities we have selected a 9-orbit shell model space both for proton and neutron with \(^{56}Ni\) as core, consisting of the spherical
orbits $3p_{3/2}$, $3f_{5/2}$, $3p_{1/2}$, $4g_{9/2}$, $4d_{5/2}$, $4g_{7/2}$, $4s_{1/2}$, $4d_{3/2}$ and $5h_{11/2}$. The s.p.e. in MeV are 26.58, 26.19, 29.09, 33.91, 38.52, 42.47, 42.30, 43.15, 58.44 respectively. The initial values of s.p.e’s are taken from ref [20]. The renormalisation effects due to the closed core (in this case $s$ shell, $p$ shell, $ds$ shell and $f_{7/2}$ orbit) are then incorporated. This effect not only renormalises $fp - sdg$ and $sdg - h_{11/2}$ separation but also renormalises the single particle energies. They can be evaluated from $V_{a\beta}$ and $V_{\alpha \beta}$ discussed in Sect. 2.1. The unitary orbits are $\{3p_{3/2}, 3f_{5/2}, 3p_{1/2}\}$, $\{4g_{9/2}\}$, $\{4d_{5/2}, 4g_{7/2}, 4s_{1/2}, 4d_{3/2}\}$, $\{5h_{11/2}\}$. Thus each of the proton and neutron shell model space has been divided into four unitary orbits. For the 2-body residual interaction, we have used a phenomenological interaction namely pairing + Q.Q interaction [21] with the strength $\chi = 242/A^{-5/3}$.

The nuclei chosen for different rate and half life calculations are given in Table 1. These nuclei have been chosen from Ref. [22].

Using the formalism given in Sect. 2 one can now construct the weak interaction strength densities (bivariate Gaussian form). For the state densities $I^H(E)$, required for calculation of half lives or rates (Eq. 12, 13), we adopted the formula given in Dilg et al [23]. This has been demonstrated in [10, 11] that state densities $I^H(E)$ obtained from spectral averaging theory represent very well the results obtained from Dilg et al state density formula. Besides the assumption of marginal centroids and variances (Sect. 2.1), we also assume the bivariate correlation coefficient $\zeta$ to be independent of configuration. Moreover, for two-body EGOE (Embedded Gaussian Orthogonal Ensemble) [8], one has the result $\zeta \approx 1 - 2/m$, where $m$ is the number of active nucleons. Therefore, in the present calculation, $\zeta$ is first evaluated by treating it as a parameter with the EGOE form for $\zeta = a + b/m$ and $a$ and $b$ is evaluated by calculating $\beta^-$ decay half lives for 18 nuclei in the mass range considered and then comparing them with the experimental values of the same.

To this end, we fix $a$ and $b$ by calculating $\zeta$ for various values of $a$ and $b$ with the constraint that the value of $\zeta$ lies within 0.6 and 0.9. The value of bivariate variance (approximated as state density variance, $\langle V^2 \rangle$ (Sect. 2.1)) is taken to be 16.5 MeV$^2$. The values of $a$ and $b$ are found out by minimizing the
quantity $\sum_{i=\text{nuclei}} (\log(\tau_{1/2}^{i})_{\text{cal}} - \log(\tau_{1/2}^{i})_{\text{expt}})^2$. The values of $a$ and $b$ are obtained respectively as 0.77 and 0.3. The values of $\zeta$ and $\beta^-$ half lives for the nuclei considered are listed in the Table 2.

Having obtained the values of the parameters $a$ and $b$ for calculation of $\zeta$, we now proceed to calculate $\beta^+$ decay half lives for the nuclei given in Table 1. For this calculation, we keep $\zeta$ unchanged from the value obtained for $\beta^-$ decay half life calculations. But here we have parametrised the variance as $\sigma^2_V = A + B/m$ and obtained the parameters $A$ and $B$ by minimising $\sum_{i=\text{nuclei}} (\log(\tau_{1/2}^{i})_{\text{cal}} - \log(\tau_{1/2}^{i})_{\text{expt}})^2$ for the four nuclei considered for $\beta^+$ decay (Table 1) much the same way as determination of $\zeta$ in $\beta^-$ decay calculations. For calculating $\beta^+$ half lives we use Eqs. 12 and Eqs 16 - 18. The values of $A$ and $B$ are obtained as $A = 20.5$ and $B = 1.9$. The values of calculated half lives and the variance $\sigma^2_V$ for the four nuclei considered for $\beta^+$ decay are shown in Table 3.

Calculation of electron capture (EC) half lives are similar to that of $\beta^+$ calculations. We use the parameters $a$ and $b$ obtained from $\beta^-$ calculations to obtain correlation coefficients $\zeta$ for the five nuclei (Table 1) chosen for EC calculations. The variance $\sigma^2_V$ is parametrized as $\sigma^2_V = A + B/m$ and the values of $A$, $B$ and consequently $\sigma^2_V$ are found out by minimising $\sum_{i=\text{nuclei}} (\log(\tau_{1/2}^{i})_{\text{cal}} - \log(\tau_{1/2}^{i})_{\text{expt}})^2$. In this case the values of $A$ and $B$ are obtained as $A = 11.5$, $B = -4.6$. For these calculations we have used the Eqs. 12, 14, 16 - 18. The calculated half lives and $\sigma^2_V$ are shown in Table 3.

We have also calculated EC decay rates for typical densities and temperatures relevant for presupernova stars. The rates are evaluated for three different temperatures ($3 \times 10^9$, $4 \times 10^9$ and $5 \times 10^9$ in $^\circ K$), three different stellar densities ($10^9$, $10^8$ and $10^7$ gms/cc) and for three values for electron fraction $Y_e$ (0.50, 0.47, 0.43). The calculations use Eqs. 13,14, 15-18 and Eq. 20 are used. The calculated EC rates alongwith temperatures, $Y_e$ and stellar densities for each of the five nuclei considered, are given in Table 5.
4 Discussions and Conclusions

We have calculated the half lives and rates for weak interactions for a number of nuclei spanning a wide range \((65 < A < 100)\) in the periodic table. These calculations are performed by explicitly constructing smoothed form for bivariate strength densities for the weak interaction operator (in this case Gamow Teller operator) using convolution form within the framework of spectral averaging theory in nuclear physics. The lower mass region of the above range is relevant for presupernova and supernova stars and the higher mass region is relevant for r-process nucleosynthesis that synthesises heavier nuclei. We have used here, the principles of spectral averaging theory for calculation of EC rates for different temperatures, densities and electron fraction values in presupernova environment. Earlier such calculations with spectral averaging theory have been performed for \(\beta^-\) decay rates for presupernova stars and for a few fp-shell nuclei \([24]\). The \(\beta^+\) and \(\beta^-\) decay half lives are calculated for the nuclei in the range \(A > 80\). The calculation of the rates for typical r-process conditions for these processes will be taken up in near future.

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Table Captions

Table 1 Nuclei chosen for different weak processes.

Table 2 Calculated and experimental $\beta^-$ decay half lives and correlation coefficient $\zeta$. Method of calculating $\zeta$ is given in the text.

Table 3 Calculated and experimental $\beta^+$ decay half lives and variance $\sigma^2_V$. $Q$-values are also given.

Table 4 Calculated and experimental EC half lives and variance $\sigma^2_V$. $Q$-values are also given.

Table 5 Electron Capture rates for $^{65}$Ge, $^{69}$Se, $^{73}$Kr, $^{77}$Sr, $^{81}$Zr.
| Weak Interaction | Nuclei Chosen |
|------------------|---------------|
| Electron Capture | $^{65}_{32}\text{Ge},^{69}_{34}\text{Se},^{73}_{36}\text{Kr},^{77}_{38}\text{Sr},^{81}_{40}\text{Zr}$ |
| $\beta^+$ decay | $^{85}_{42}\text{Mo},^{89}_{44}\text{Ru},^{93}_{46}\text{Pd},^{97}_{48}\text{Cd}$ |
| $\beta^-$ decay | $^{81}_{30}\text{Zn},^{81}_{31}\text{Ga},^{81}_{33}\text{As}$ |
| | $^{82}_{32}\text{Ge},^{83}_{31}\text{Ga},^{83}_{33}\text{As}$ |
| | $^{84}_{32}\text{Ge},^{85}_{34}\text{Se},^{87}_{33}\text{As}$ |
| | $^{88}_{37}\text{Rb},^{89}_{34}\text{Se},^{89}_{35}\text{Br}$ |
| | $^{90}_{36}\text{Kr},^{91}_{33}\text{Rb},^{92}_{35}\text{Br}$ |
| | $^{92}_{38}\text{Sr},^{93}_{36}\text{Kr},^{94}_{39}\text{Y}$ |
| Sr No. | Nucl. | Z  | $Q_{val}$ (MeV) | $T_{exp.}^{1/2}$ | $T_{calc}^{1/2}$ | ζ   |
|-------|-------|----|-----------------|------------------|------------------|-----|
| 1     | $^{81}Zn$ | 30 | 11.9            | 0.29             | 0.819            | 0.7820 |
| 2     | $^{81}Ga$  | 31 | 8.32           | 1.221            | 1.483            | 0.7820 |
| 3     | $^{81}As$  | 33 | 3.86            | 33.3             | 3.662            | 0.7820 |
| 4     | $^{82}Ge$  | 32 | 4.7             | 4.6              | 53.088           | 0.7815 |
| 5     | $^{83}Ga$  | 31 | 11.5            | 0.31             | 0.127            | 0.7811 |
| 6     | $^{83}As$  | 33 | 5.46            | 13.4             | 21.134           | 0.7811 |
| 7     | $^{84}Ge$  | 32 | 7.7             | 1.2              | 2.415            | 0.7807 |
| 8     | $^{85}Se$  | 34 | 6.182           | 31.7             | 106.261          | 0.7803 |
| 9     | $^{87}As$  | 33 | 10.3            | 0.73             | 0.0632           | 0.7797 |
| 10    | $^{88}Rb$  | 37 | 5.13            | 1066.8           | 27227.014        | 0.7794 |
| 11    | $^{89}Se$  | 34 | 9.0             | 0.41             | 0.117            | 0.7791 |
| 12    | $^{89}Br$  | 35 | 8.16            | 4.4              | 1.465            | 0.7791 |
| 13    | $^{90}Kr$  | 36 | 4.39            | 32.32            | 413.661          | 0.7788 |
| 14    | $^{91}Rb$  | 37 | 5.86            | 58.4             | 290.453          | 0.7786 |
| 15    | $^{92}Br$  | 35 | 12.2            | 0.343            | 0.096            | 0.7783 |
| 16    | $^{92}Sr$  | 38 | 1.67            | 9756.            | 110696.591       | 0.7783 |
| 17    | $^{93}Kr$  | 36 | 8.6             | 1.286            | 3.706            | 0.7781 |
| 18    | $^{94}Y$   | 39 | 4.0             | 67320.           | 2590.043         | 0.7779 |
### Table 3

| Nucleus | Z | $Q_{\beta^+}$ (MeV) | $T_{1/2}^{exp}$ (sec) | $T_{1/2}^{calc}$ (sec) | $\sigma_V^2$ (MeV$^2$) |
|---------|---|---------------------|-----------------------|------------------------|------------------------|
| $^{85}$Mo | 42 | 8.6 | 0.4 | 0.076 | 20.566 |
| $^{89}$Ru | 44 | 8.92 | 0.3 | 0.22 | 20.558 |
| $^{93}$Pd | 46 | 9.5 | 0.3 | 0.14 | 20.551 |
| $^{97}$Cd | 48 | 9.6 | 0.2 | 0.32 | 20.546 |

### Table 4

| Nucleus | Z | $Q_{EC}$ (MeV) | $T_{1/2}^{exp}$ (sec) | $T_{1/2}^{calc}$ (sec) | $\sigma^2$ (MeV$^2$) |
|---------|---|----------------|-----------------------|------------------------|------------------------|
| $^{65}$Ge | 32. | 6.24 | 30.9 | 12.79 | 10.989 |
| $^{69}$Se | 34. | 6.78 | 27.4 | 10.32 | 11.146 |
| $^{73}$Kr | 36. | 6.65 | 27. | 12.58 | 11.229 |
| $^{77}$Sr | 38. | 6.85 | 9. | 9.35 | 11.281 |
| $^{81}$Zr | 40. | 7.16 | 15. | 14.04 | 11.316 |
| Nucleus | $\rho$(gms/cc) | $Y_e$ | Temperature in °K |
|---------|----------------|------|------------------|
|         |                |      | $3 \times 10^9$  | $4 \times 10^9$  | $5 \times 10^9$  |
|         |                | Y     | Rates (s$^{-1}$)  |
| $^{65}$Ge | $10^9$         | 0.50  | $2.77 \times 10^{-3}$ | $2.89 \times 10^{-3}$ | $3.07 \times 10^{-3}$ |
|         |                | 0.47  | $2.56 \times 10^{-3}$ | $2.68 \times 10^{-3}$ | $2.86 \times 10^{-3}$ |
|         |                | 0.43  | $2.28 \times 10^{-3}$ | $2.41 \times 10^{-3}$ | $2.58 \times 10^{-3}$ |
|         |                | 0.50  | $2.05 \times 10^{-4}$ | $2.47 \times 10^{-4}$ | $3.13 \times 10^{-4}$ |
|         | $10^8$         | 0.47  | $1.92 \times 10^{-4}$ | $2.34 \times 10^{-4}$ | $2.98 \times 10^{-4}$ |
|         |                | 0.43  | $1.76 \times 10^{-4}$ | $2.16 \times 10^{-4}$ | $2.79 \times 10^{-4}$ |
|         |                | 0.50  | $2.96 \times 10^{-5}$ | $5.67 \times 10^{-5}$ | $1.10 \times 10^{-4}$ |
|         | $10^7$         | 0.47  | $2.86 \times 10^{-5}$ | $5.56 \times 10^{-5}$ | $1.10 \times 10^{-4}$ |
|         |                | 0.43  | $2.71 \times 10^{-5}$ | $5.43 \times 10^{-5}$ | $1.09 \times 10^{-4}$ |
| $^{69}$Se | $10^9$         | 0.50  | $2.81 \times 10^{-3}$ | $2.92 \times 10^{-3}$ | $3.10 \times 10^{-3}$ |
|         |                | 0.47  | $2.60 \times 10^{-3}$ | $2.72 \times 10^{-3}$ | $2.89 \times 10^{-3}$ |
|         |                | 0.43  | $2.32 \times 10^{-3}$ | $2.45 \times 10^{-3}$ | $2.62 \times 10^{-3}$ |
|         |                | 0.50  | $2.20 \times 10^{-4}$ | $2.64 \times 10^{-4}$ | $3.31 \times 10^{-4}$ |
|         | $10^8$         | 0.47  | $2.07 \times 10^{-4}$ | $2.50 \times 10^{-4}$ | $3.16 \times 10^{-4}$ |
|         |                | 0.43  | $1.89 \times 10^{-4}$ | $2.31 \times 10^{-4}$ | $2.96 \times 10^{-4}$ |
|         | $10^7$         | 0.50  | $3.26 \times 10^{-5}$ | $6.16 \times 10^{-5}$ | $1.19 \times 10^{-4}$ |
|         |                | 0.47  | $3.14 \times 10^{-5}$ | $6.05 \times 10^{-5}$ | $1.18 \times 10^{-4}$ |
|         |                | 0.43  | $2.99 \times 10^{-5}$ | $5.90 \times 10^{-5}$ | $1.17 \times 10^{-4}$ |
| $^{73}$Kr | $10^9$         | 0.50  | $2.18 \times 10^{-3}$ | $2.27 \times 10^{-3}$ | $2.40 \times 10^{-3}$ |
|         |                | 0.47  | $2.01 \times 10^{-3}$ | $2.11 \times 10^{-3}$ | $2.24 \times 10^{-3}$ |
|         |                | 0.43  | $1.80 \times 10^{-3}$ | $1.90 \times 10^{-3}$ | $2.03 \times 10^{-3}$ |
|         | $10^8$         | 0.50  | $1.72 \times 10^{-4}$ | $2.05 \times 10^{-4}$ | $2.58 \times 10^{-4}$ |
|         |                | 0.47  | $1.61 \times 10^{-4}$ | $1.95 \times 10^{-4}$ | $2.46 \times 10^{-4}$ |
|         |                | 0.43  | $1.48 \times 10^{-4}$ | $1.80 \times 10^{-4}$ | $2.30 \times 10^{-4}$ |
|         | $10^7$         | 0.50  | $2.55 \times 10^{-5}$ | $4.82 \times 10^{-5}$ | $9.24 \times 10^{-5}$ |
|         |                | 0.47  | $2.46 \times 10^{-5}$ | $4.73 \times 10^{-5}$ | $9.19 \times 10^{-5}$ |
|         |                | 0.43  | $2.34 \times 10^{-5}$ | $4.61 \times 10^{-5}$ | $9.13 \times 10^{-5}$ |
Table 5. Contd.

| Nucleus | $\rho$(gms/cc) | $Y_e$ | Temperature in $^\circ$K |
|---------|----------------|-------|-------------------------|
|         |                |       | $3 \times 10^9$ | $4 \times 10^9$ | $5 \times 10^9$ |
|         |                |       | Rates (s$^{-1}$) |                 |               |
| $^{77}$Sr | $10^9$         | 0.50  | $1.86 \times 10^{-3}$ | $1.90 \times 10^{-3}$ | $1.98 \times 10^{-3}$ |
|         |                | 0.47  | $1.72 \times 10^{-3}$ | $1.77 \times 10^{-3}$ | $1.86 \times 10^{-3}$ |
|         |                | 0.43  | $1.54 \times 10^{-3}$ | $1.60 \times 10^{-3}$ | $1.68 \times 10^{-3}$ |
|         | $10^8$         | 0.50  | $1.56 \times 10^{-4}$ | $1.84 \times 10^{-4}$ | $2.26 \times 10^{-4}$ |
|         |                | 0.47  | $1.47 \times 10^{-4}$ | $1.74 \times 10^{-4}$ | $2.16 \times 10^{-4}$ |
|         |                | 0.43  | $1.35 \times 10^{-4}$ | $1.62 \times 10^{-4}$ | $2.03 \times 10^{-4}$ |
|         | $10^7$         | 0.50  | $2.39 \times 10^{-5}$ | $4.41 \times 10^{-5}$ | $8.27 \times 10^{-5}$ |
|         |                | 0.47  | $2.30 \times 10^{-5}$ | $4.33 \times 10^{-5}$ | $8.23 \times 10^{-5}$ |
|         |                | 0.43  | $2.19 \times 10^{-5}$ | $4.22 \times 10^{-5}$ | $8.18 \times 10^{-5}$ |
| $^{81}$Zr | $10^9$         | 0.50  | $1.45 \times 10^{-3}$ | $1.48 \times 10^{-3}$ | $1.55 \times 10^{-3}$ |
|         |                | 0.47  | $1.34 \times 10^{-3}$ | $1.38 \times 10^{-3}$ | $1.44 \times 10^{-3}$ |
|         |                | 0.43  | $1.20 \times 10^{-3}$ | $1.24 \times 10^{-3}$ | $1.31 \times 10^{-3}$ |
|         | $10^8$         | 0.50  | $1.15 \times 10^{-4}$ | $1.36 \times 10^{-4}$ | $1.68 \times 10^{-4}$ |
|         |                | 0.47  | $1.09 \times 10^{-4}$ | $1.29 \times 10^{-4}$ | $1.60 \times 10^{-4}$ |
|         |                | 0.43  | $9.96 \times 10^{-5}$ | $1.19 \times 10^{-4}$ | $1.50 \times 10^{-4}$ |
|         | $10^7$         | 0.50  | $1.72 \times 10^{-5}$ | $3.20 \times 10^{-5}$ | $6.04 \times 10^{-5}$ |
|         |                | 0.47  | $1.66 \times 10^{-5}$ | $3.14 \times 10^{-5}$ | $6.01 \times 10^{-5}$ |
|         |                | 0.43  | $1.58 \times 10^{-5}$ | $3.06 \times 10^{-5}$ | $5.97 \times 10^{-5}$ |