HIGH ENERGY SIGN-CHANGING SOLUTIONS FOR CORON’S PROBLEM

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Abstract: We study the existence of sign changing solutions to the following problem

\[
\begin{cases}
\Delta u + |u|^{p-1}u = 0 & \text{in } \Omega_{\varepsilon}; \\
u = 0 & \text{on } \partial\Omega_{\varepsilon},
\end{cases}
\]

where \( p = \frac{n+2}{n-2} \) is the critical Sobolev exponent and \( \Omega_{\varepsilon} \) is a bounded smooth domain in \( \mathbb{R}^n \), \( n \geq 3 \), with the form \( \Omega_{\varepsilon} = \Omega \setminus B(0, \varepsilon) \) with \( \Omega \) a smooth bounded domain containing the origin 0 and \( B(0, \varepsilon) \) the ball centered at the origin with radius \( \varepsilon > 0 \). We construct a new type of sign-changing solutions with high energy to problem (0.1), when the parameter \( \varepsilon \) is small enough.

1. Introduction

Let \( D \) be a smooth bounded domain in \( \mathbb{R}^n \), with \( n \geq 3 \), and let \( p = \frac{n+2}{n-2} \) be the critical Sobolev exponent, namely the exponent for which the Sobolev embedding \( H^1_0(D) \hookrightarrow L^{p+1}(D) \) ceases to be compact. A problem which has been widely studied in the last 30 years concerns the existence, multiplicity, and qualitative properties for positive or sign-changing solutions to

\[
\begin{cases}
\Delta u + |u|^{p-1}u = 0 & \text{in } D; \\
u = 0 & \text{on } \partial D.
\end{cases}
\]

Solvability for Problem (1.1) is not a trivial issue, since it strongly depends on the geometry of \( D \). Let us briefly summarize some classical results. A direct consequence of Pohozaev’s identity [23] is that Problem (1.1) has no positive solutions when the domain \( D \) is strictly star-shaped. On the other hand, if \( D \) is an annulus, then Kazdan and Warner [15] showed that solvability for problem (1.1) is restored: a solution is found as critical point of the energy functional associated to the problem, and the required compactness for the functional is obtained thanks to radial symmetry. A surprising result by Coron showed later on that symmetry is not really needed to have solvability: in his classical work [6] he proves the existence of a positive solution to (1.1) in the case in which \( D \) has a small (not necessarily symmetric) hole. In literature, the name Coron’s problem is referred to problem (1.1), when the domain \( D \) has a hole. The result by Coron was then generalized by Bahri and Coron in [2], where the authors showed that, under the assumption that some homology group of \( D \) with coefficients in \( \mathbb{Z}_2 \) is not trivial then problem (1.1) has at least one positive solution. Multiplicity result for positive solutions to (1.1) is obtained in [24], where the situation of a domain \( D \) with several holes is treated. On the other hand, existence and multiplicity of positive solutions has been established also in contractible domains, we refer to the works of Dancer [7], Ding [10], and Passaseo [21–22], among others.

The study of solutions for elliptic problems with critical nonlinearity which change sign has received the interest of several authors in the last years, see for instance [3, 4], [13, 14], and

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references therein. Here we focus our interest in existence and qualitative properties of sign-changing solutions to (1.1) for domains $D$ which have a hole, that is in the Coron’s setting. The first result available in literature is the one contained in [17], where a large number of sign changing solutions to (1.1) in the presence of a single small hole has been proved. To be more precise, the authors assume that the domain $D$ has the form $D = \Omega \setminus B(0,\varepsilon)$, where $\Omega$ is a smooth bounded domain containing the origin, and it is symmetric with respect to the origin, while the hole is given by $B(0,\varepsilon)$, a round ball with size $\varepsilon$. In this situation, they prove that, for any given integer $k$ there exist $\varepsilon_k > 0$ so that, for any $\varepsilon \in (0,\varepsilon_k)$ a sign-changing solution to (1.1) exists, and it has the shape of a superposition of exactly $k$ bubbles, centered at 0. A bubble is the function

\begin{equation}
U(y) = \alpha_n \left( \frac{1}{1 + |y|^2} \right)^{\frac{n-2}{2}}, \quad \alpha_n = \left[ n(n-2) \right]^{\frac{n-2}{4}},
\end{equation}

and it solves

\begin{equation}
\Delta u + u^p = 0, \quad \text{in} \quad \mathbb{R}^n.
\end{equation}

It is well known that all positive solutions to (1.3) are given by the function (1.2) and any translation, and proper dilation of it, that is by

\begin{equation}
U(x - \xi), \quad \text{for} \quad \xi \in \mathbb{R}^n, \quad \text{and} \quad \lambda^{-\frac{n-2}{2}} U\left( \frac{x}{\lambda} \right), \quad \text{for} \quad \lambda > 0,
\end{equation}

see [1, 20, 25]. The solutions found in [17] are bubble-tower, with $k$ bubbles of alternating sign, which at main order look like

\begin{equation}
u(\varepsilon) \sim \sum_{j=1}^{k} (-1)^{j+1} \lambda_j^{-\frac{n-2}{2}} U\left( \frac{x}{\lambda_j} \right), \quad \lambda_j \sim \varepsilon^{\frac{j-1}{2k}},
\end{equation}

as $\varepsilon \to 0^+$. The energy of these solutions has the asymptotic expansion

\begin{equation}e(u(\varepsilon)) = kS_n + O\left( \varepsilon^{\frac{n-2}{2}} \right), \quad \text{as} \quad \varepsilon \to 0^+, \quad \text{where} \quad S_n = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \int_{\mathbb{R}^n} |\nabla U|^2 \, dx \right).
\end{equation}

We recall the explicit definition of the energy $e$, given by

\begin{equation}
e(u) = \frac{1}{2} \int_D |\nabla u|^2 - \frac{1}{p+1} \int_D |u|^{p+1}.
\end{equation}

Substantial improvement of this result was obtained in [12] where the assumption of symmetry was removed. See also [16]. A different construction of sign-changing solutions for (1.1) on a domain with a small hole has been obtained in [5]: in this case, solutions look like the sum of a positive bubble centered inside the shrinking hole, and a number of sign-changing bubbles centered inside the domain, far from the hole. A common feature of the constructions in [5, 12, 16, 17] above described is that the building blocks that constitute the core of the shape of the solutions are given by the positive solutions to (1.3), which are completely classified by the functions given by (1.2) and (1.5).

The aim of this work is to produce a new, different construction of sign-changing solutions to problem (1.1), which are built upon a limit profile given by an explicit sign-changing solution to

\begin{equation}
\Delta u + |u|^{p-1}u = 0, \quad \text{in} \quad \mathbb{R}^n.
\end{equation}

In [8] it is proven the existence of a sequence of finite-energy, sign-changing solutions $Q_k$ to (1.6), with a crown-like shape and energy of size $(k+1)S_n(1 + o(1))$, as $k \to \infty$. For any $k$ large, $Q_k$
is given by

$$Q_k(x) = U_*(x) + \tilde{\phi}(x), \quad U_*(x) = U(x) - \sum_{j=1}^{k} U_j(x),$$

where $\tilde{\phi}$ is smaller than $U_*$, in some sense that we precise later. The function $U$ is given by (1.2) and $U_j$ are defined as

$$U_j(x) = \frac{1}{k} \sum_{j=1}^{k} U(x - \xi_j),$$

where $\xi_j = \sqrt{1 - \mu_k^2} \left( \cos \theta_j, \sin \theta_j, 0 \right) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, $\theta_j = \frac{2\pi(j - 1)}{k}$, and the parameter $\mu_k$ is defined as

$$\left[ \sum_{j=1}^{k} \frac{1}{1 - \cos \theta_j} \right] \mu_k^{-\frac{n-2}{2}} = 1 + O\left( \frac{1}{k} \right), \quad \text{for} \quad k \to \infty.$$

Observe that the previous definition gives

$$\mu_k \sim \begin{cases} k^{-2}, & \text{if} \quad n \geq 4, \\ k^{-2} |\log k|^{-2}, & \text{if} \quad n = 3. \end{cases}$$

In other words, $Q_k$ is a crown-like function, with a central positive bubble, centered at the origin in $\mathbb{R}^n$, and a large number of negative copies of bubbles, centered at the vertices of a regular $k$-polygon sit in the $(x_1, x_2, 0, \ldots, 0)$-plane, each one with a very sharp profile, as $k \to \infty$. A property of these solutions that is central for our construction to work is that they are non-degenerate, as proved in [18]. In the class of bounded functions, the linearized operator around $Q_k$

$$L(\phi) = \Delta \phi + p|Q_k|^{p-1} \phi$$

has a kernel of finite dimension, equals to $3n$. Section 2 is devoted to give a precise description of the main properties of these solutions, including their non-degeneracy.

In this paper, we construct sign changing solutions to problem (1.1) using the function $Q_k$ as main building block, thus generating a new type of sign-changing solutions, different from the ones already known in literature [5, 12, 16, 17].

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $\varepsilon > 0$ and $\Omega_\varepsilon = \Omega \setminus B(0, \varepsilon)$. We consider the following problem

$$\begin{cases} \Delta u + |u|^{p-1}u = 0 \quad \text{in} \quad \Omega_\varepsilon; \\ u = 0 \quad \text{on} \quad \partial \Omega_\varepsilon. \end{cases}$$

Our result states the following

**Theorem 1.1.** There exists an integer $k_0$ such that for any integer $k \geq k_0$, there exists $\varepsilon_k$, such that for any $\varepsilon \in (0, \varepsilon_k)$, Problem (1.10) has a sign changing solution $u_\varepsilon$, satisfying

$$u_\varepsilon(x) = d_\varepsilon^{-\frac{n-2}{2}} \varepsilon^{-\frac{n-2}{4}} Q_k \left( \frac{x}{\varepsilon d_\varepsilon} \right) (1 + o(1))$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$, and $d_\varepsilon \to d_0$ with $d_0$ is a positive constant. Moreover

$$J_\varepsilon(u_\varepsilon) = (k + 1)S_n + O(\varepsilon^{\frac{n-2}{2}}), \quad \text{as} \quad \varepsilon \to 0^+,$$
where
\[ J_\varepsilon(u_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 \, dx - \frac{1}{p+1} \int_{\Omega_\varepsilon} |u|^{p+1} \, dx. \]

This paper is organized as follows. In Section 2 we recall the main properties of the functions $Q_k$, that are used in the rest of the paper. In Section 3 we define a first approximation to problem (1.10). We give the expansion of the energy functional at the first approximation in Section 4. The proof of Theorem 1.1 is contained in Section 5. Section 6 is devoted to solve a linear problem, and Section 7 is devoted to solve a nonlinear problem. Further properties on the functions $Q_k$, which are new, are reported in the Appendix A.

2. Building blocks: Sign-Changing solutions $Q_k$ in $\mathbb{R}^n$

The solutions predicted by Theorem 1.1 are constructed as small perturbation of an initial approximation. This initial approximation is build using the entire, finite energy, sign changing solution $Q_k$ for the problem (1.6), which are mentioned in the Introduction. The existence of such solutions is proven in [8, 9]: this is a sequence of solutions, defined for any integer $k$ sufficiently large. If we define the energy by
\[ E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dy - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} \, dy, \]
we have
\[ E(Q_k) = \begin{cases} (k+1) S_n \left( 1 + O(k^{2-n}) \right) & \text{if } n \geq 4, \\ (k+1) S_3 \left( 1 + O(k^{-1} \log k) \right) & \text{if } n = 3, \end{cases} \]
as $k \to \infty$, where $S_n$ is a positive constant, depending on $n$. The solution $Q = Q_k$ decays at infinity like the fundamental solution, namely
\[ \lim_{|y| \to \infty} |y|^{n-2} Q_k(y) = \beta_n (1 + c_k) \]
where $\beta_n$ is a positive number and
\[ c_k = \begin{cases} O\left(\frac{1}{k^{n-3}}\right) & \text{if } n \geq 4, \\ O(|\log k|^{-1}) & \text{if } n = 3, \end{cases} \]
as $k \to \infty$.

Furthermore, the solution $Q = Q_k$ has a positive global non degenerate maximum at $y = 0$. To be more precisely we have
\[ Q(y) = \left[ n(n-2) \right]^{\frac{n-2}{4}} \left( 1 - \frac{n-2}{2} |y|^2 + O(|y|^3) \right) \quad \text{as } |y| \to 0. \]

Another property for the solution $Q = Q_k$ is that it is invariant under rotation of angle $\frac{2\pi}{k}$ in the $y_1, y_2$ plane, namely
\[ Q(e^{2\pi i/k} \bar{y}, y') = Q(\bar{y}, y'), \quad \bar{y} = (y_1, y_2), \quad y' = (y_3, \ldots, y_n). \]
It is even in the $y_j$-coordinates, for any $j = 2, \ldots, n$
\[ Q(y_1, \ldots, y_j, \ldots, y_n) = Q(y_1, \ldots, -y_j, \ldots, y_n), \quad j = 2, \ldots, n. \]
It respects invariance under Kelvin’s transform:
\[ Q(y) = |y|^{2-n} Q\left( \frac{y}{|y|^2} \right). \]
The function $\tilde{\phi}$ in (1.7) can be further decomposed. Let us introduce some cut-off functions $\zeta_j$ to be defined as follows. Let $\zeta(s)$ be a smooth function such that $\zeta(s) = 1$ for $s < 1$ and $\zeta(s) = 0$ for $s > 2$. We also let $\zeta^-(s) = \zeta(2s)$. Then we set
\[
\zeta_j(y) = \begin{cases} 
\zeta(k^{-1}|y| - 2|y - \xi|) & \text{if } |y| > 1, \\
\zeta(k^{-1}|y - \xi|) & \text{if } |y| \leq 1,
\end{cases}
\]
in such a way that that
\[
\zeta_j(y) = \zeta_j(y/|y|^2).
\]
The function $\tilde{\phi}$ has the form
\[
(2.7) \quad \tilde{\phi} = \sum_{j=1}^k \tilde{\phi}_j + \psi.
\]
In the decomposition (2.7) the functions $\tilde{\phi}_j$, for $j > 1$, are defined in terms of $\tilde{\phi}_1$
\[
(2.8) \quad \tilde{\phi}_j(\bar{y},y') = \tilde{\phi}_1(e^{2\pi i j k} \bar{y}, y'), \quad j = 1, \ldots, k - 1.
\]
We have that
\[
(2.9) \quad \|\psi\|_{n-2} \leq Ck^{1-\frac{n}{q}} \quad \text{if } n \geq 4, \quad \|\psi\|_{n-2} \leq \frac{C}{\log k} \quad \text{if } n = 3,
\]
where $q > \frac{n}{2}$, and
\[
(2.10) \quad \|\phi\|_{n-2} := \| (1 + |y|^{n-2}) \phi \|_{L^\infty(\mathbb{R}^n)}.
\]
On the other hand, if we rescale and translate the function $\tilde{\phi}_1$
\[
(2.11) \quad \phi_1(y) = \mu^{\frac{n-2}{2}} \tilde{\phi}_1(\xi + \mu y)
\]
we have the validity of the following estimate for $\phi_1$
\[
(2.12) \quad \|\phi_1\|_{n-2} \leq Ck^{-\frac{n}{q}} \quad \text{if } n \geq 4, \quad \|\phi_1\|_{n-2} \leq \frac{C}{k \log k} \quad \text{if } n = 3.
\]
The description of the solution $Q$ in (1.7) is thus quite accurate. For later purpose, we observe that the region where $Q$ changes sign is well understood: there exists $0 < R_1 < 1 < R_2$, positive, so that
\[
(2.13) \quad Q(y) > 0, \quad \text{for } |y| \leq R_1, \quad |y| \geq R_2.
\]
In [13], is was proved that these solutions are non degenerate. That is, fix one solution $Q = Q_k$ of problem (1.6) and define the linearized equation around $Q$ as follows
\[
(2.14) \quad L(\phi) = \Delta \phi + p|Q|^{p-1}\phi.
\]
The invariances (2.4), (2.5), (2.6), together with the natural invariance of any solution to (1.6) under translation (if $u$ solves (1.6) then also $u(y + \zeta)$ solves (1.6) for any $\zeta \in \mathbb{R}^n$) and under dilation (if $u$ solves (1.6) then $\lambda^{-\frac{n-2}{2}} u(\lambda^{-1} y)$ solves (1.6) for any $\lambda > 0$) produce some natural functions $\varphi$ in the kernel of $L$, namely
\[
L(\varphi) = 0.
\]
These are the $3n$ linearly independent functions we introduce next:
\[
(2.15) \quad z_0(y) = \frac{n-2}{2} Q(y) + \nabla Q(y) \cdot y,
\]
Observe now that no boundary. In other words, there exists a smooth injective map have that $\hat{\Theta}$.

To show (2.20), let us introduce the following operator: for any set of parameters $x$ by rotations in the $(z, y)$ plane, in the $(x, y, z)$-plane, for any choice of $y$ in $\mathbb{R}^n$. Let us be more precise. Denote by $\Theta(z)$, with $n = 1$.

One has

$$L(z_\alpha) = 0, \quad \text{for all } \alpha = 0, 1, \ldots, 3n - 1.$$  (2.20)

To show (2.21), let us introduce the following operator: for any set of parameters $A = (\lambda, \xi, a, \theta)$, for any choice of $Q$ under rotation in the $(y, z)$ plane, while the functions defined in (2.19) are related to the invariance under rotation in the $(y_1, y_2)$ plane, while the functions defined in (2.19) are related to the invariance under rotation in the $(y_1, y_2)$ plane and in the $(y_2, y_3)$ plane respectively.

Let us be more precise. Denote by $O(n)$ the orthogonal group of $n \times n$ matrices $P$ with real coefficients, so that $P^TP = I$, and by $SO(n) \subset O(n)$ the special orthogonal group of all matrices in $O(n)$ with $\det P = 1$. $SO(n)$ is the group of all rotations in $\mathbb{R}^n$, it is a compact group, which can be identified with a compact set in $\mathbb{R}^{n(n-1)/2}$. Consider the sub group $\hat{S}$ of $SO(n)$ generated by rotations in the $(x_1, x_2)$-plane, in the $(x_j, x_\alpha)$-plane, for any $j = 1, 2$ and $\alpha = 3, \ldots, n$. We have that $\hat{S}$ is compact and can be identified with a compact manifold of dimension $2n - 3$, with no boundary. In other words, there exists a smooth injective map $\chi: \hat{S} \to \mathbb{R}^{n(n-1)/2}$ so that $\chi(\hat{S})$
is a compact manifold of dimension $2n - 3$ with no boundary and $\chi^{-1} : \chi(\hat{S}) \to \hat{S}$ is a smooth parametrization of $\hat{S}$ in a neighborhood of the Identity. Thus we write

$$\theta \in \mathcal{O} = \chi(\hat{S}), \quad R_\theta = \chi^{-1}(\theta)$$

where $\mathcal{O}$ is a compact manifold of dimension $2n - 3$ with no boundary and $R_\theta$ denotes a rotation in $\hat{S}$. Let $\theta = (\theta_{12}, \theta_{13}, \ldots, \theta_{1n}, \theta_{23}, \ldots, \theta_{2n})$, and we write

$$R_\theta = P_{12}(\theta_{12})P_{13}(\theta_{13})P_{14}(\theta_{14}) \cdots P_{1n}(\theta_{1n})P_{23}(\theta_{23})P_{24}(\theta_{24}) \cdots P_{2n}(\theta_{2n}),$$

where $P_{ij}(\theta)$ is the rotation in the $(i, j)$-plane of an angle $\theta$,

$$P_{ij}(\theta) = \begin{bmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cos \theta & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cos \theta & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 
\end{bmatrix}, \quad i < j.$$

We have

$$\left( \frac{\partial}{\partial \theta_{12}} \Theta_{(1,0,\theta,0)}(x) \right)_{\theta=0} = z_{n+1}(x)$$

and, for any $l = 3, \ldots, n$,

$$\left( \frac{\partial}{\partial \theta_{1l}} \Theta_{(1,0,\theta,0)}(x) \right)_{\theta=0} = z_{n+1}(x), \quad \left( \frac{\partial}{\partial \theta_{2l}} \Theta_{(1,0,\theta,0)}(x) \right)_{\theta=0} = z_{n+l-1}(x)$$

Thus we have the validity of (2.20) as direct consequence of (2.23), (2.24), (2.25), (2.26).

In [18] it is proven that there exists a sequence of solutions $Q = Q_k$ of the form (1.7) which are non degenerate in the sense that

$$\text{Kernel}(L) = \text{Span}\{z_\alpha : \alpha = 0, 1, 2, \ldots, 3n - 1\},$$

or equivalently, any bounded (or any solution in $D^{1,2}$) of $L(\varphi) = 0$ is a linear combination of the functions $z_\alpha, \alpha = 0, \ldots, 3n - 1$. The non-degeneracy of $Q$ is a crucial property for our construction.

3. The First Approximation Solution

Let $\eta > 0$ be a fixed and small number and let us introduce a set of parameters $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ with the properties that

$$\lambda = d \sqrt{\varepsilon}, \quad \text{with} \quad \eta < d < \frac{1}{\eta}, \quad \text{for some fixed} \ \eta > 0,$$

$$\xi = \lambda \tau \in \mathbb{R}^n, \quad \text{with} \quad |\tau| < \eta$$

$$a \in \mathbb{B} := \left\{ a = (a_1, a_2, 0, \ldots, 0) \in \mathbb{R}^n : |a| < \frac{1}{2} \right\},$$

and

$$\theta = (\theta_{12}, \theta_{13}, \ldots, \theta_{1n}, \theta_{23}, \ldots, \theta_{2n}) \in \mathcal{O}$$
where $\mathcal{O}$ is a compact manifold of dimension $2n - 3$ with no boundary which was introduced in the previous section. The elements in $\mathcal{O}$ represents the invariants of any solution to (1.6) generated by rotations in the $(x_1, x_2)$-plane, in the $(x_j, x_n)$-plane, for any $j = 1, 2$ and $\alpha = 3, \ldots, n$. Here, with abuse of notation, we identify $a = (a_1, a_2) \in \mathbb{R}^2$, with $a = (a_1, a_2, 0) \in \mathbb{R}^n$.

For any set of parameters $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$, we now introduce the function,

$$Q_A(x) = \Theta_A(R^{-1}(x - \xi)),$$

where $\Theta_A$ is defined in (2.22). More explicitly,

$$Q_A(x) = \lambda^{\frac{n-2}{2}} \frac{|x - \xi|}{|x - \xi| - R_\theta a} |x - \xi|^{2-n} Q \left( \frac{x - \xi - R_\theta a}{\lambda} \right)^2 \left( \frac{x - \xi - R_\theta a}{\lambda} \right)^2 \left( \frac{|x - \xi|}{1 - 2R_\theta a \cdot \frac{x - \xi}{\lambda} + |a|^2 \frac{x - \xi}{|x - \xi|} \right)^2.$$

Observe that $Q_A$ solves the equation in (1.10), but it is far from satisfying the boundary conditions. For this reason, we correct $Q_A$, introducing its projection onto $H^1_0(\Omega_\varepsilon)$. Let us define $P_\varepsilon Q_A$ to be

$$P_\varepsilon Q_A = \begin{cases} -\Delta P_\varepsilon Q_A = |Q_A|^{p-1} Q_A & \text{in } \Omega_\varepsilon; \\ P_\varepsilon Q_A = 0 & \text{on } \partial \Omega_\varepsilon. \end{cases}$$

We next give the description of the asymptotic behavior of the projection $P_\varepsilon Q_A$ for $x \in \Omega_\varepsilon$, as $\varepsilon \to 0$. To do so, we need to introduce Green’s function $G(x, y)$ of the domain, namely $G$ satisfies

$$G(x, y) = \begin{cases} -\Delta G(x, y) = \delta(x - y) & x \in \Omega; \\ G(x, y) = 0 & x \in \partial \Omega, \end{cases}$$

where $\delta(x)$ denotes the Dirac mass at the origin, and its regular part $H(x, y) := \Gamma(x - y) - G(x, y)$, where $\Gamma$ denotes the fundamental solution of the Laplacian,

$$\Gamma(x) = \gamma_n |x|^{2-n}.$$

It is direct to see that

$$H(x, y) = \begin{cases} -\Delta H(x, y) = 0 & x \in \Omega; \\ H(x, y) = \Gamma(x - y) & x \in \partial \Omega. \end{cases}$$

Furthermore, we introduce the function

$$F(\tau, a, \theta) := \begin{cases} Q \left( \frac{\tau}{|\tau^2| - R_\theta a} \right)^2 |\tau|^{2-n} & \text{if } \tau \neq 0, \\ \lim_{z \to 0} Q(z)|z|^{n-2} & \text{if } \tau = 0. \end{cases}$$

This is a smooth function in the set of parameters $\tau, a$ and $\theta$ satisfying (3.2), (3.3), (3.4). We have the validity of the following

**Lemma 3.1.** Let $\eta > 0$ be fixed and assume that $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ satisfies (3.2)-(3.4), with the additional assumption that $\xi = \lambda \tau$, and $\tau \neq 0$. Let

$$R(x) := P_\varepsilon Q_A(x) - Q_A(x) + \gamma_n \frac{1}{\lambda^{n-2}} Q(-R_\theta a) H(x, \xi) + \frac{1}{\lambda^{n-2}} F(\tau, a, \theta) \frac{\varepsilon^{n-2}}{|x|^{n-2}},$$

where $F$ is defined in (3.10). Then there exists a positive constant $c$ such that for any $x \in \Omega \setminus B(0, \varepsilon)$

$$|R(x)| \leq c \lambda^\frac{n-2}{2} \left[ \frac{\varepsilon^{n-2}(1 + \varepsilon \lambda^{-n+1})}{|x|^{n-2}} + \lambda^2 + \frac{\varepsilon^{n-2}}{\lambda^{n-2}} \right].$$
Let us now take $\xi$ if $\lambda \tau \leq 1$. Thus $-\Delta \hat{\xi} = 0$ in $\hat{\Omega}$. Furthermore, $\hat{\Omega}_\varepsilon \rightarrow \Omega \setminus B(0,1)$ as $\varepsilon \rightarrow 0$. Observe that, if $z = \frac{e^{y-\xi}}{\lambda}$, then

$$Q_A(\varepsilon y) = \lambda^{-\frac{n-4}{2}} |z|^{2-n} Q \left( \frac{z}{|z|^2} - R_{\theta a} \right).$$

For any $y \in \partial B(0,1)$ we have that

$$\hat{R}(y) = -\lambda^{-\frac{n-4}{2}} Q_A(\varepsilon y) + \gamma \frac{1}{4} Q(-R_{\theta a}) H(\varepsilon y, \xi) + \lambda^{2-n} F(\tau, a, \theta).$$

If $\xi = \lambda \tau$, and $\tau \neq 0$, then $z = -\tau + \frac{e^{y}}{\lambda} y$, and a direct Taylor expansion gives that

$$Q_A(\varepsilon y) = \lambda^{-\frac{n-4}{2}} Q \left( -\frac{\tau}{|\tau|^2} - R_{\theta a} \right) |\tau|^{2-n} \left( 1 + O(\frac{\varepsilon}{\lambda}) \right)$$

uniformly for points $y \in \partial B(0,1)$. If $\tau = 0$, then

$$Q_A(\varepsilon y) = \lambda^{-\frac{n-4}{2}} \left( \lim_{w \rightarrow \infty} |z|^{2-n} Q(w) \right) \left( 1 + O(\frac{\varepsilon}{\lambda}) \right)$$

Thus we get the estimates

$$|\hat{R}(y)| \leq C(1 + \frac{1}{\lambda^{n-2}}) \text{ uniformly for } y \in \partial B(0,1).$$

Let us now take $y \in \partial (\frac{\Omega}{\varepsilon})$, and we have

$$\hat{R}(y) = -\lambda^{-\frac{n-4}{2}} Q_A(\varepsilon y) + \frac{\gamma}{2} |\varepsilon y|^{2-n} Q(R_{\theta a}) + \frac{1}{\lambda^{n-2}} F(\tau, a, \theta) \frac{1}{|y|^{n-2}}.$$  

Since $|z| = |\tau + \frac{e^{y}}{\lambda}| \rightarrow \infty$, as $\varepsilon \rightarrow 0$, we get

$$|\hat{R}(y)| \leq C(\lambda^2 + \left( \frac{\varepsilon}{\lambda} \right)^{n-2}) \text{ uniformly for } y \in \partial \left( \frac{\Omega}{\varepsilon} \right).$$

A comparison argument for harmonic functions implies that

$$|\hat{R}(y)| \leq C \left[ \frac{1 + \varepsilon \lambda^{1-n}}{|y|^{n-2}} + \lambda^2 + \left( \frac{\varepsilon}{\lambda} \right)^{n-2} \right].$$

This fact gives (3.15).

Let us now denote by $R_\lambda(x) = \partial_\lambda R(x)$ and define $\hat{R}_\mu(y) = \lambda^{-\frac{n-4}{2}} R(\varepsilon y)$. A direct computation shows that

$$|\hat{R}_\lambda(y)| \leq C(1 + \frac{1}{\lambda^{n-2}}) \text{ uniformly for } y \in \partial B(0,1),$$

and

$$|\hat{R}_\lambda(y)| \leq C(\lambda^2 + \left( \frac{\varepsilon}{\lambda} \right)^{n-2}) \text{ uniformly for } y \in \partial \left( \frac{\Omega}{\varepsilon} \right).$$

This fact gives (3.12).
Finally, let $R_i(x) = \partial_{x_i} R(x)$ and $\tilde{R}_i(y) = \lambda^{-\frac{d}{2}} R_i(\varepsilon y)$. We get the following estimates
\[
|\tilde{R}_i(y)| \leq C(1 + \frac{\varepsilon}{\lambda^a}) \quad \text{uniformly for } y \in \partial B(0, 1),
\]
and
\[
|\tilde{R}_i(y)| \leq C(\lambda^2 + \frac{\varepsilon^{n-2}}{\lambda^{n-1}}) \quad \text{uniformly for } y \in \partial \left( \frac{\Omega}{\varepsilon} \right).
\]
This fact gives (3.13). In a similar way, one gets estimates (3.14) and (3.15).

4. THE EXPANSION OF THE ENERGY

In this Section, we give the expansion of the energy function $J_\varepsilon(P_\varepsilon Q_A)$ which defined by
\[
J_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\Omega_\varepsilon} |u|^{p+1} dx.
\]
We have the following result.

**Proposition 4.1.** Let $\eta > 0$ be fixed, and $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ satisfies (3.1)-(3.4). Then
\[
J_\varepsilon(P_\varepsilon Q_A) = c_1 + \frac{1}{2} \left( \gamma_n^{-2} Q(-R_\eta a)^2 H(0, 0) d^n - \frac{c_2}{d^{n-2}} F(\tau, a, \theta) \right) \varepsilon^{\frac{n-2}{2}} + \Pi(d, \tau, a, \theta) \varepsilon^{\frac{n-1}{2}}
\]
and
\[
\nabla_{(d, \tau, a, \theta)} J_\varepsilon(P_\varepsilon Q_A) = \nabla_{(d, \tau, a, \theta)} \left[ \frac{1}{2} \left( \gamma_n^{-2} Q(-R_\eta a)^2 H(0, 0) d^n - \frac{c_2}{d^{n-2}} F(\tau, a, \theta) \right) \right] \varepsilon^{\frac{n-2}{2}} + \Pi(d, \tau, a, \theta) \varepsilon^{\frac{n-1}{2}}
\]
as $\varepsilon \to 0$, where $\Pi$ denote a smooth function of its variables, which is uniformly bounded as $\varepsilon \to 0$ for $(\lambda, \xi, a, \theta)$ satisfying (3.1)-(3.4). Here $F$ is the function introduced in (3.10), and $c_1$ and $c_2$ are the constants
\[
c_1 = \frac{1}{n} \int_{\mathbb{R}^n} |Q|^{p+1} dz, \quad c_2 = \frac{1}{n} \int_{\mathbb{R}^n} |Q|^p dz.
\]

**Proof.** We compute the energy
\[
J_\varepsilon(P_\varepsilon Q_A) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla P_\varepsilon Q_A|^2 dx - \frac{1}{p+1} \int_{\Omega_\varepsilon} |P_\varepsilon Q_A|^{p+1} dx.
\]
Taking into the fact that $-\Delta P_\varepsilon Q_A = |Q_A|^{p-1} Q_A$ in $\Omega_\varepsilon$, and $P_\varepsilon Q_A = 0$ on $\partial \Omega_\varepsilon$, we have
\[
\int_{\Omega_\varepsilon} |\nabla P_\varepsilon Q_A|^2 dx = \int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A P_\varepsilon Q_A dx = \int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A dx + \int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A(P_\varepsilon Q_A - Q_A) dx.
\]
Moreover, by a Taylor expansion, for some $t \in (0, 1)$,
\[
\int_{\Omega_\varepsilon} |P_\varepsilon Q_A|^{p+1} dx = \int_{\Omega_\varepsilon} |Q_A + (P_\varepsilon Q_A - Q_A)|^{p+1} dx = \int_{\Omega_\varepsilon} |Q_A|^{p+1} dx + (p+1) \int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A(P_\varepsilon Q_A - Q_A) dx + \frac{p(p+1)}{2} \int_{\Omega_\varepsilon} (t P_\varepsilon Q_A + (1-t)Q_A)^{p-1} (P_\varepsilon Q_A - Q_A)^2 dx,
\]
From (4.3), (4.4) and (4.5), we get
\[
J_\varepsilon(P_\varepsilon Q_A) = \frac{1}{n} \int_{\Omega_\varepsilon} |Q_A|^{p+1} dx - \frac{1}{2} \int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A (P_\varepsilon Q_A - Q_A) dx
\]
(4.6)
\[\hspace{1cm} - \frac{p}{2} \int_{\Omega_\varepsilon} (t P_\varepsilon Q_A + (1 - t) Q_A)^{p-1} (P_\varepsilon Q_A - Q_A)^2 dx.\]

We will estimate each term in the following, and then the result in Proposition is a consequence of the following Lemmas.

**Lemma 4.2.** Let \( \eta > 0 \) be fixed, and \( A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3} \) satisfies (3.1)-(3.4). It holds
\[
\int_{\Omega_\varepsilon} |Q_A|^{p+1} dx = \int_{\mathbb{R}^n} |Q|^{p+1} dx + \varepsilon^p \Pi(d, \tau, a, \theta),
\]
(4.7)
where \( \Pi \) is a smooth function of its variables, which is uniformly bounded as \( \varepsilon \to 0 \) for \( (\lambda, \xi, a, \theta) \) satisfying (3.1)-(3.4).

**Proof.** We decompose it as
\[
\int_{\Omega_\varepsilon} |Q_A|^{p+1} dx = \int_{\varepsilon < |x| < \delta} |Q_A|^{p+1} dx + \int_{|x| > \delta} |Q_A|^{p+1} dx
\]
for some \( \delta > 0 \) fixed and small. In the region \( \varepsilon < |x| < \delta \) we introduce the change of variables \( y = \frac{x - \xi}{\lambda} = x - \tau \), so that
\[
\int_{\varepsilon < |x| < \delta} |Q_A|^{p+1} dx = \int_{B_{1,\varepsilon}} \left[ \frac{y}{|y|} - R_\theta a |y| \right]^{2-n} |Q| \left( \frac{y}{|y|^2} - R_\theta a |y|^2 \right)^{p+1} dy.
\]
where \( B_{1,\varepsilon} := B(-\tau, \frac{\delta}{\lambda}) \setminus B(-\tau, \frac{\varepsilon}{\lambda}) \). Since \( |w|^{2-n} Q\left(\frac{w}{|w|^2}\right) = Q(w) \), and using the change of variables \( z = \frac{y}{|y|^2} \), we then have
\[
\int_{\varepsilon < |x| < \delta} |Q_A|^{p+1} dx = \int_{B_{1,\varepsilon}} \left[ |y|^{2-n} |Q| \left( \frac{y}{|y|^2} - R_\theta a \right) \right]^{p+1} dy
\]
\[
= \int_{B_{1,\varepsilon}} |y|^{2n} \left[ |Q| \left( \frac{y}{|y|^2} - R_\theta a \right) \right]^{p+1} dy
\]
\[
= \int_{B_{2,\varepsilon}} |Q| (z - R_\theta a)^{p+1} dz = \int_{\mathbb{R}^n} |Q|^{p+1} dz + \lambda^n O(1),
\]
where \( B_{2,\varepsilon} := B\left(\frac{\tau}{|\tau|^2}, 1 \frac{\delta}{\lambda}\right) \setminus B\left(\frac{\tau}{|\tau|^2}, 1 \frac{\varepsilon}{\lambda}\right) \), if \( \tau \neq 0 \), and \( B_{2,\varepsilon} := B(0, \frac{\delta}{\lambda}) \setminus B(0, \frac{\varepsilon}{\lambda}) \) if \( \tau = 0 \). Thus, the above estimate holds true for a generic function \( O(1) \) of the parameters \( (d, \tau, a, \theta) \), which is uniformly bounded as \( \varepsilon \to 0 \). On the other hand, in the set \( |x| > \delta \) we have that
\[
\int_{|x| > \delta} |Q_A|^{p+1} dx \leq C \lambda^n.
\]
This concludes the proof of the Lemma.

**Lemma 4.3.** Let \( \eta > 0 \) be fixed, and \( A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3} \) satisfies (3.1)-(3.4). It holds, as \( \varepsilon \to 0 \),
\[
\int_{\Omega_\varepsilon} |Q_A|^{p-1} Q_A (P_\varepsilon Q_A - Q_A) dx = -\gamma_n^{-2} Q(-R_\theta a)^2 H(0,0) \lambda^{n-2}
\]
(4.8)
\[\hspace{1cm} - \left( \frac{\varepsilon}{\lambda} \right)^{n-2} F(\tau, a, \theta) \int_{\mathbb{R}^n} |Q|^p dz + \Pi(d, \tau, a, \theta) \varepsilon^{\frac{n+1}{n}}.
\]
where \( \Pi \) is a smooth function of its variables, which is uniformly bounded as \( \varepsilon \to 0 \) for \((\lambda, \xi, a, \theta)\) satisfying (3.7) and (3.4). Here \( F \) is the function introduced in (3.10).

**Proof.** By Lemma 3.1, we have
\[
\int_{\Omega_{\varepsilon}} |Q_{A}|^{p-1}Q_{A}(P_{\varepsilon}Q_{A} - Q_{A})\,dx = -\gamma_{n}^{-1} \lambda^{\frac{n+2}{2}}Q(-R_{\theta a}) \int_{\Omega_{\varepsilon}} |Q_{A}|^{p-1}Q_{A}H(x, \xi)\,dx
\]
\[
-\frac{\varepsilon^{n-2}}{\lambda^{\frac{n+2}{2}}}F(\tau, a, \theta) \int_{\Omega_{\varepsilon}} |Q_{A}|^{p-1}Q_{A}\,dx + \int_{\Omega_{\varepsilon}} |Q_{A}|^{p-1}Q_{A}R(x)\,dx
\]
(4.9) \(:= I_{1} + I_{2} + I_{3}.

To estimate \( I_{1} \), we write
\[
\int_{\Omega_{\varepsilon}} |Q_{A}|^{p-1}Q_{A}H(x, \xi)\,dx = \int_{\varepsilon < |x| < \delta} |Q_{A}|^{p-1}Q_{A}H(x, \xi)\,dx + \int_{\Omega \gamma |x| > \delta} |Q_{A}|^{p-1}Q_{A}H(x, \xi)\,dx,
\]
for some positive, small and fixed \( \delta \). In the first region, the function \( H \) is smooth, and in particular it has bounded derivatives. Thus, by Taylor expansions, we get
\[
\int_{\varepsilon < |x| < \delta} |Q_{A}|^{p-1}Q_{A}H(x, \xi)\,dx = H(0, 0) \left( \int_{\varepsilon < |x| < \delta} |Q_{A}|^{p-1}Q_{A}\,dx \right)
\]
\[
+ O(\int_{\varepsilon < |x| < \delta} |Q_{A}|^{p-1}Q_{A}|x|\,dx) + \lambda O(\int_{\varepsilon < |x| < \delta} |Q_{A}|^{p-1}Q_{A}\,dx).
\]
Taking the change of variables \( y = \frac{x-\xi}{\lambda} \), using the invariance of \( Q \) under Kelvin transform, and using the change of variables \( z = \frac{y}{|y|^2} \), we get
\[
\int_{\varepsilon < |x| < \delta} |Q_{A}|^{p-1}Q_{A}\,dx = \lambda^{\frac{n+2}{2}} \int_{B_{1,\varepsilon}} \left[ |y|^{2-n} |Q| \left( \frac{y}{|y|^2} - R_{\theta a} \right) \right]^{p} \,dy
\]
\[
= \lambda^{\frac{n+2}{2}} \int_{B_{1,\varepsilon}} |y|^{-(n+2)} \left[ |Q| \left( \frac{y}{|y|^2} - R_{\theta a} \right) \right]^{p} \,dy
\]
\[
= \lambda^{\frac{n+2}{2}} \int_{B_{2,\varepsilon}} \frac{1}{|z|^{n-2}} |Q(z - R_{\theta a})|^{p} \,dz
\]
\[
= \lambda^{\frac{n+2}{2}} \int_{R^{n}} \frac{1}{|z + R_{\theta a}|^{n-2}} |Q|^{p} \,dz + O(\lambda^{\frac{n+2}{2}}),
\]
as \( \varepsilon \to 0 \). Here again \( B_{1,\varepsilon} := B(-\tau, \frac{\lambda}{\varepsilon}) \setminus B(-\tau, \frac{\lambda}{\varepsilon}) \), while \( B_{2,\varepsilon} := B(\frac{\tau}{|\tau|^2}, \frac{\lambda}{\varepsilon}) \setminus B(\frac{\tau}{|\tau|^2}, \frac{\lambda}{\varepsilon}) \), if \( \tau \neq 0 \), and \( B_{2,\varepsilon} := B(0, \frac{\lambda}{\varepsilon}) \setminus B(0, \frac{\lambda}{\varepsilon}) \) if \( \tau = 0 \). Recall now that
\[
Q(-R_{\theta a}) = \gamma_{n} \int_{R^{n}} \frac{1}{|z + R_{\theta a}|^{n-2}} |Q|^{p} \,dz
\]
Thus we get
(4.10)
\[
\int_{\varepsilon < |x| < \delta} |Q_{A}|^{p-1}Q_{A}\,dx = \gamma_{n}^{-1} \lambda^{\frac{n+2}{2}}Q(-R_{\theta a}) + O(\lambda^{\frac{n+2}{2}}).
\]
On the other hand, using again the change of variables \( y = \frac{x-\xi}{\lambda} \) one finds directly that
\[
\int_{|x| < \delta} |Q_{A}|^{p-1}Q_{A}\,dx = O(\lambda^{\frac{n+2}{2}}).
\]
On the other hand, we observe that, in the region where \( |x| > \delta \), one has
(4.11)
\[
|Q_{A}(x)| \leq C \lambda^{\frac{n+2}{2}}, \quad \text{for all} \quad x \in \Omega, \quad |x| > \delta.
\]
where the constant $C$ is independent of $\varepsilon$. Indeed, to prove (4.11), we start with the observation that, in the region under consideration, one has

$$|Q_\lambda(x)| \leq c|Q_A(x)|,$$

for some constant $c$, independent of $\varepsilon$. Now, if we set $y = \frac{x}{\lambda}$, we have, in the region under consideration

$$|Q_\lambda(x)| \leq C\lambda^{-\frac{n-2}{2}} \left| \frac{y}{|y|^2} - R\theta a \right|^{2-n} |y|^{2-n} \left( \frac{\frac{y}{|y|^2} - R\theta a}{\left( \frac{y}{|y|^2} - R\theta a \right)^2} \right).$$

Thus the validity of (4.11) follows. A direct consequence of (4.11) is

$$\int_{\Omega \cap |x| > \delta} |Q_\lambda|^{p-1}Q_\lambda H(x,0) \, dx = O(\lambda^{\frac{n+2}{2}}).$$

Then, we find

(4.12) \quad I_1 = -\gamma_n^{-2}\lambda^{n-2}Q(-R\theta a)^2H(0,0) + O(\lambda^{n-1}).

Let us now estimate $I_2$. We split the integral as follows

$$\int_{\Omega_{\varepsilon}} |Q_\lambda|^{p-1}Q_\lambda \frac{1}{|x|^{n-2}} \, dx = \int_{\varepsilon < |x| < \delta} |Q_\lambda|^{p-1}Q_\lambda \frac{1}{|x|^{n-2}} \, dx + \int_{\Omega \cap |x| > \delta} |Q_\lambda|^{p-1}Q_\lambda \frac{1}{|x|^{n-2}} \, dx.$$

Using again (4.11), we see that $\int_{\Omega \cap |x| > \delta} |Q_\lambda|^{p-1}Q_\lambda \frac{1}{|x|^{n-2}} \, dx = O(\lambda^{\frac{n+2}{2}})$. Using the invariance of $Q$ under Kelvin transform, and using the changes of variables, first $y = \frac{x}{\lambda}$ and then $z = \frac{y}{|y|^2}$, we get

$$\int_{\Omega \cap |x| < \delta} |Q_\lambda|^{p-1}Q_\lambda \frac{1}{|x|^{n-2}} \, dx = \lambda^{\frac{n-2}{2}} \int_{B_{1,\varepsilon}} \left| y \right|^{2-n} \left| Q \right| \left( \frac{y}{|y|^2} - R\theta a \right)^p \frac{1}{|y|^{n-2}} \, dy
= \lambda^{\frac{n-2}{2}} \int_{B_{1,\varepsilon}} \left| y \right|^{-2n} \left| Q \right| \left( \frac{y}{|y|^2} - R\theta a \right)^p \, dy
= \lambda^{\frac{n-2}{2}} \int_{B_{2,\varepsilon}} |Q(z - R\theta a)|^p \, dz = \lambda^{\frac{n-2}{2}} \left( \int_{\mathbb{R}^n} |Q|^p \, dz + O(\lambda^2) \right).$$

Then

(4.14) \quad I_2 = -\frac{\varepsilon^{n-2}}{\lambda^{n-2}} \left( F(\tau,\alpha,\theta) \int_{\mathbb{R}^n} |Q|^p \, dz + O(\lambda^2) \right).$$

We conclude with the estimate for $I_3$. We use the result in Lemma 3.1 and in particular estimate (3.11), to get

(4.15) \quad |I_3| = |\int_{\Omega_{\varepsilon}} |Q_\lambda|^{p-1}Q_\lambda R(x) \, dx| \leq c\varepsilon^{\frac{2}{3}}(|I_1| + |I_2|),

for some positive constant $c$. This concludes the proof of Lemma.

**Lemma 4.4.** Under the same assumptions as in Proposition 4.1, it holds

(4.16) \quad \int_{\Omega_{\varepsilon}} (tP_\varepsilon Q_A + (1-t)Q_A)^{p-1} (P_\varepsilon Q_A - Q_A)^2 \, dx = \Pi(d,\tau,\alpha,\theta) \varepsilon^{\frac{n-1}{2}}.
Proof. Using (4.6), we have

$$\left| \int_{\Omega_\varepsilon} (tP_\varepsilon Q_A + (1-t)Q_A)^{p-1} (P_\varepsilon Q_A - Q_A)^2 \, dx \right|$$

(4.17)

$$\leq c \int_{\Omega_\varepsilon} |Q_A|^{p-1} \left( \lambda^{n-2} + \frac{\varepsilon^{2(n-2)}}{\lambda^{n-2}} \right) \frac{1}{|x|^{2(n-2)}} \, dx.$$

Since

$$\lambda^{n-2} \int_{\Omega_\varepsilon} |Q_A|^{p-1} \, dx = \lambda^{n-2} \int_{|x|<\delta} |Q_A|^{p-1} \, dx + \lambda^{n-2} \int_{|x|>\delta} |Q_A|^{p-1} \, dx$$

(4.18)

$$= \lambda^{2(n-2)} \int_{|y|<\frac{\delta}{\lambda}} \left| y \right|^{2-n} |Q| \left( \frac{y}{|y|^2} - R_{\theta a} \right) \, dy + O(\lambda^{n+2})$$

$$= \lambda^{2(n-2)} \int_{|y|<\frac{\delta}{\lambda}} \left| y \right|^{-4} \left[ \left( \frac{y}{|y|^2} - R_{\theta a} \right) \right]^{p-1} \, dy + O(\lambda^{n+2})$$

$$= \lambda^{2(n-2)} \left( \int_{\mathbb{R}^n} \right) \frac{1}{|z + R_{\theta a}|^{n-2}} |Q|^{p-1} \, dz + O(1) + O(\lambda^{n+2})$$

(4.19)

$$= O(\lambda^{n+2}),$$

and

$$\frac{\varepsilon^{2(n-2)}}{\lambda^{n-2}} \int_{\Omega_\varepsilon} |Q_A|^{p-1} \frac{1}{|x|^{2(n-2)}} \, dx$$

$$= \frac{\varepsilon^{2(n-2)}}{\lambda^{n-2}} \int_{|x|<\delta} |Q_A|^{p-1} \frac{1}{|x|^{2(n-2)}} \, dx + \frac{\varepsilon^{2(n-2)}}{\lambda^{n-2}} \int_{|x|>\delta} |Q_A|^{p-1} \frac{1}{|x|^{2(n-2)}} \, dx$$

$$= \frac{\varepsilon^{2(n-2)}}{\lambda^{2(n-2)}} \int_{|y|<\frac{\delta}{\lambda}} \left| y \right|^{2-n} |Q| \left( \frac{y}{|y|^2} - R_{\theta a} \right) \, dy + O(1) \frac{\varepsilon^{2(n-2)}}{\lambda^{n-2}}$$

$$= \frac{\varepsilon^{2(n-2)}}{\lambda^{2(n-2)}} \int_{|y|<\frac{\delta}{\lambda}} \left| y \right|^{-3(n-2)} \left[ \left( \frac{y}{|y|^2} - R_{\theta a} \right) \right]^{p-1} \, dy + O(1) \frac{\varepsilon^{2(n-2)}}{\lambda^{n-2}}$$

$$= \frac{\varepsilon^{2(n-2)}}{\lambda^{2(n-2)}} \left( \int_{\mathbb{R}^n} \right) \frac{1}{|z + R_{\theta a}|^{n-6}} |Q|^{p-1} \, dz + O(1) \frac{\varepsilon^{2(n-2)}}{\lambda^{n-2}}$$

(4.19)

$$= O\left( \frac{\varepsilon^{2(n-2)}}{\lambda^{2(n-2)}} \right).$$

Then (4.16) follows from (4.17) to (4.19). \qed

We conclude this section with the proof of (4.2). More precisely, we prove

$$\partial_d J_\varepsilon(P_\varepsilon Q_A) = \partial_d \left[ \frac{1}{2} \left( \tau^{-2} Q(-R_{\theta a})^2 H(0,0) d^{n-2} + \frac{c_2}{d^{n-2}} F(\tau, a, \theta) \right) \right] \varepsilon^{\frac{n-2}{2}}$$

(4.20)

$$+ \Pi(d, \tau, a, \theta) \varepsilon^{\frac{n-1}{4}},$$
From Lemma 4.3, we have that
\begin{equation}
\partial_d J_\varepsilon(P_\varepsilon Q_A) = \partial_d \left( \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla P_\varepsilon Q_A|^2 \, dx - \frac{1}{p+1} \int_{\Omega_\varepsilon} |P_\varepsilon Q_A|^{p+1} \, dx \right)
= \int_{\Omega_\varepsilon} \nabla P_\varepsilon Q_A \nabla (\partial_d (P_\varepsilon Q_A)) \, dx - \int_{\Omega_\varepsilon} |P_\varepsilon Q_A|^p \partial_d (P_\varepsilon Q_A) \, dx
\end{equation}
Since the function \( P_\varepsilon Q_A \) satisfies (3.6), we find
\[ \partial_d J_\varepsilon(P_\varepsilon Q_A) = - \int_{\Omega_\varepsilon} \left[ |P_\varepsilon Q_A|^p - |Q_A|^{p-1} Q_A \right] \partial_d (P_\varepsilon Q_A) \, dx. \]
By a Taylor expansion, for some \( t \in (0, 1) \),
\[ |P_\varepsilon Q_A|^p = |Q_A + (P_\varepsilon Q_A - Q_A)|^p = |Q_A|^{p-1} Q_A + p|Q_A|^{p-2} Q_A (P_\varepsilon Q_A - Q_A) + \frac{p(p-1)}{2} (tP_\varepsilon Q_A + (1-t)Q_A)^{p-2} (P_\varepsilon Q_A - Q_A)^2. \]
Then
\[ \partial_d J_\varepsilon(P_\varepsilon Q_A) = - \int_{\Omega_\varepsilon} \left[ p|Q_A|^{p-2} Q_A (P_\varepsilon Q_A - Q_A) \right] \partial_d (Q_A + (P_\varepsilon Q_A - Q_A)) \, dx \]
\[ - \frac{p(p-1)}{2} \int_{\Omega_\varepsilon} (tP_\varepsilon Q_A + (1-t)Q_A)^{p-2} (P_\varepsilon Q_A - Q_A)^2 \partial_d (Q_A + (P_\varepsilon Q_A - Q_A)) \, dx \]
\[ = - \int_{\Omega_\varepsilon} \left[ p|Q_A|^{p-2} Q_A (P_\varepsilon Q_A - Q_A) \right] \partial_d Q_A \, dx \]
\[ - \int_{\Omega_\varepsilon} \left[ p|Q_A|^{p-2} Q_A (P_\varepsilon Q_A - Q_A) \right] \partial_d (P_\varepsilon Q_A - Q_A) \, dx \]
\[ - \frac{p(p-1)}{2} \int_{\Omega_\varepsilon} (tP_\varepsilon Q_A + (1-t)Q_A)^{p-2} (P_\varepsilon Q_A - Q_A)^2 \partial_d (Q_A + (P_\varepsilon Q_A - Q_A)) \, dx \]
\[ = - \int_{\Omega_\varepsilon} \left[ p|Q_A|^{p-2} Q_A (P_\varepsilon Q_A - Q_A) \right] \partial_d Q_A \, dx \]
\[ + O \left( \int_{\Omega_\varepsilon} |Q_A|^{p-2} Q_A (P_\varepsilon Q_A - Q_A)^2 \, dx \right) \]
\[ = - \int_{\Omega_\varepsilon} \partial_d \left[ |Q_A|^p (P_\varepsilon Q_A - Q_A) \right] \, dx + O \left( \int_{\Omega_\varepsilon} |Q_A|^{p-2} Q_A (P_\varepsilon Q_A - Q_A)^2 \, dx \right) \]
\[ = \partial_d \left( - \int_{\Omega_\varepsilon} |Q_A|^p (P_\varepsilon Q_A - Q_A) \, dx \right) + \int_{\Omega_\varepsilon} |Q_A|^p \partial_d (P_\varepsilon Q_A - Q_A) \, dx \]
\begin{align}
(4.21) & \quad + O \left( \int_{\Omega_\varepsilon} |Q_A|^{p-2} Q_A (P_\varepsilon Q_A - Q_A)^2 \, dx \right) \nonumber.
\end{align}
From Lemma 4.3, we have that
\[ \partial_d \left( - \int_{\Omega_\varepsilon} |Q_A|^p (P_\varepsilon Q_A - Q_A) \, dx \right) \]
\[ = \partial_d \left( \chi_n^{-2} Q(-R\theta)^2 H(0, 0) d^{n-2} + \frac{c_2}{d^{n-2}} F(\tau, a, \theta) \right) \varepsilon^{\frac{n-2}{2}} + \Pi(d, \tau, a, \theta) \varepsilon^{\frac{n-1}{2}} \]
(4.22) \(= (n - 2)d^{-1} \left( \gamma_n^{-2}Q(-R_0a)^2H(0,0)d^{n-2} - \frac{c_2}{dn-2}F(\tau, a, \theta) \right) \varepsilon^\frac{n-2}{2} + \Pi(d, \tau, a, \theta) \varepsilon^\frac{n-1}{2} \)

Moreover, for the second term in (4.21), by Lemma 3.1, we have

\[
\int_{\Omega_\varepsilon} |Q_A|^p \partial_d(P_\varepsilon Q_A - Q_A) dx \\
= \int_{\Omega_\varepsilon} |Q_A|^p \partial_d \left[ -\gamma_n^{-1} \lambda^{\frac{n-2}{2}} Q(-R_0a)H(x, \xi) - \frac{\varepsilon^{n-2}}{\lambda^{\frac{n-2}{2}}} F(\tau, a, \theta) \frac{1}{|x|^{n-2}} + R(x) \right] dx \\
= \int_{\Omega_\varepsilon} |Q_A|^p \partial_\lambda \left[ -\gamma_n^{-1} \lambda^{\frac{n-2}{2}} Q(-R_0a)H(x, \xi) - \frac{\varepsilon^{n-2}}{\lambda^{\frac{n-2}{2}}} F(\tau, a, \theta) \frac{1}{|x|^{n-2}} + R(x) \right] \frac{\partial \lambda}{\partial d} dx \\
= \frac{n - 2}{2} \varepsilon^{\frac{1}{2}} \int_{\Omega_\varepsilon} |Q_A|^p \left[ -\gamma_n^{-1} \lambda^{\frac{n-2}{2}} Q(-R_0a)H(x, \xi) + \frac{\varepsilon^{n-2}}{\lambda^{\frac{n-2}{2}+1}} F(\tau, a, \theta) \frac{1}{|x|^{n-2}} \right] dx \\
+ \varepsilon^{\frac{1}{2}} \int_{\Omega_\varepsilon} |Q_A|^p \partial_\lambda R(x) dx \\
= \frac{n - 2}{2} d^{-1} \int_{\Omega_\varepsilon} |Q_A|^p \left[ -\gamma_n^{-1} \lambda^{\frac{n-2}{2}} Q(-R_0a)H(x, \xi) + \frac{\varepsilon^{n-2}}{\lambda^{\frac{n-2}{2}+1}} F(\tau, a, \theta) \frac{1}{|x|^{n-2}} \right] dx \\
+ \varepsilon^{\frac{1}{2}} \int_{\Omega_\varepsilon} |Q_A|^p \partial_\lambda R(x) dx \\
= \frac{n - 2}{2} d^{-1} [I_1 - I_2] + \varepsilon^{\frac{1}{2}} \int_{\Omega_\varepsilon} |Q_A|^p \partial_\lambda R(x) dx.
\]

where \(I_1\) and \(I_2\) are defined in (4.19), with

\[
I_1 - I_2 = - \left( \gamma_n^{-2}Q(-R_0a)^2H(0,0)d^{n-2} - \frac{c_2}{dn-2}F(\tau, a, \theta) \right) \varepsilon^\frac{n-2}{2} + \Pi(d, \tau, a, \theta) \varepsilon^\frac{n-1}{2},
\]

and from Lemma 3.1 we have \(|\partial_\lambda R(x)| \leq c\lambda^{-1} |R(x)|\), then by (4.15), we get

\[
|\varepsilon^{\frac{1}{2}} \int_{\Omega_\varepsilon} |Q_A|^p \partial_\lambda R(x) dx| \leq c\lambda^{-1} \varepsilon^{\frac{1}{2}} \int_{\Omega_\varepsilon} |Q_A|^p R(x) dx = cd \int_{\Omega_\varepsilon} |Q_A|^p R(x) dx \\
\leq c\varepsilon^{\frac{1}{2}} (|I_1| + |I_2|).
\]

Thus

\[
\int_{\Omega_\varepsilon} |Q_A|^p \partial_d(P_\varepsilon Q_A - Q_A) dx \\
(4.23) = - \frac{n - 2}{2} d^{-1} \left( \gamma_n^{-2}Q(-R_0a)^2H(0,0)d^{n-2} - \frac{c_2}{dn-2}F(\tau, a, \theta) \right) \varepsilon^\frac{n-2}{2} + \Pi(d, \tau, a, \theta) \varepsilon^\frac{n-1}{2}.
\]

Lastly, using Lemma 3.1 as a computation in Lemma 3.3, we have

\[
(4.24) O \left( \int_{\Omega_\varepsilon} |Q_A|^{p-2}Q_A(P_\varepsilon Q_A - Q_A)^2 dx \right) = \Pi(d, \tau, a, \theta) \varepsilon^\frac{n-1}{2},
\]

as \(\varepsilon \to 0\), where \(\Pi\) is a smooth function of the variables \((d, \tau, a, \theta)\), which is uniformly bounded as \(\varepsilon \to 0\) for \((\lambda, \xi, a, \theta)\) satisfying (3.1) - (3.4).

Therefore, by (4.21), (4.22), (4.23) and (4.24), we obtain

\[
\partial_d J_\varepsilon(P_\varepsilon Q_A) = \frac{n - 2}{2} d^{-1} \left( \gamma_n^{-2}Q(-R_0a)^2H(0,0)d^{n-2} - \frac{c_2}{dn-2}F(\tau, a, \theta) \right) \varepsilon^\frac{n-2}{2} + \Pi(d, \tau, a, \theta) \varepsilon^\frac{n-1}{2} \\
= \partial_d \left[ \frac{1}{2} \left( \gamma_n^{-2}Q(-R_0a)^2H(0,0)d^{n-2} + \frac{c_2}{dn-2}F(\tau, a, \theta) \right) \varepsilon^\frac{n-2}{2} \right] + \Pi(d, \tau, a, \theta) \varepsilon^\frac{n-1}{2}.
\]
That is, \((4.20)\) holds.

5. **Scheme of the proof**

By the change of variable,

\begin{equation}
\tag{5.1}
v(y) = \varepsilon^{\frac{1}{p-1}}u(\sqrt{\varepsilon}y).
\end{equation}

Problem \((1.10)\) has a solution \(u\) if and only if \(v\) solves the following problem

\[
\begin{cases}
\Delta v + |v|^{p-1}v = 0, & \text{in } D_\varepsilon; \\
v = 0, & \text{on } \partial D_\varepsilon,
\end{cases}
\]

where \(D_\varepsilon := \frac{\Omega}{\sqrt{\varepsilon}} = \frac{\Omega}{\sqrt{\varepsilon}} \setminus B(0, \sqrt{\varepsilon}).\

In expanded variable, the solution that we are looking for looks like

\begin{equation}
\tag{5.2}
V(y) = V(y) + \phi(y), \quad \text{where } V(y) = \varepsilon^{\frac{1}{p-1}}P_\varepsilon Q_A(\sqrt{\varepsilon}y),
\end{equation}

where \(P_\varepsilon Q_A\) is defined in \((3.6)\). We observe that the function \(V\) is nothing but the projection onto \(H_0^1(D_\varepsilon)\) of the function \(\varepsilon^{\frac{1}{p-1}}Q_A(\sqrt{\varepsilon}y)\). We also observe that, if \(A = (\lambda, \xi, a, \theta)\), then

\[
\varepsilon^{\frac{1}{p-1}}Q_A(\sqrt{\varepsilon}y) = Q_{\tilde{A}}(y), \quad \text{with } \tilde{A} = (d, d\tau, a, \theta)
\]

since \(\lambda = d\sqrt{\varepsilon}\) and \(\xi = \lambda\tau\), where \(Q_A\) is given in \((3.3)\).

Rewriting the result contained in Lemma 3.1 we see that as \(\varepsilon \to 0\),

\begin{equation}
\tag{5.4}
V(y) = Q_{\tilde{A}}(y) + \varepsilon \frac{\eta}{\sqrt{\varepsilon}} \left(1 + \frac{1}{|y|^{n-2}}\right)\Xi_{\tilde{A}}(y),
\end{equation}

uniformly on compact sets of \(D_\varepsilon\). Here \(\Xi_{\tilde{A}}(y)\) is a smooth function, which is uniformly bounded for \(y \in D_\varepsilon\), as \(\varepsilon \to 0\), and for sets of parameters \(\tilde{A}\) satisfying \((3.1)-(3.4)\).

In terms of \(\phi\), problem \((5.2)\) becomes

\begin{equation}
\tag{5.5}
L(\phi) = -N(\phi) - E, \quad \text{in } D_\varepsilon, \quad \phi = 0, \quad \text{on } \partial D_\varepsilon,
\end{equation}

where

\begin{equation}
\tag{5.6}
L(\phi) = \Delta \phi + pV^{1-p} \phi, \quad N(\phi) = (V + \phi)^p - V^p - pV^{1-p} \phi,
\end{equation}

and

\begin{equation}
\tag{5.7}
E = V^p - |Q_{\tilde{A}}|^{p-1}Q_{\tilde{A}}, \quad \text{with } \tilde{A} = (d, d\tau, a, \theta).
\end{equation}

Consider the following functions, for any \(j = 0, 1, 2, \ldots, 3n - 1\),

\begin{equation}
\tag{5.8}
Z_j(y) = \varepsilon^{\frac{1}{p-1}}\bar{Z}_j(\sqrt{\varepsilon}y), \quad y \in D_\varepsilon, \quad \bar{Z}_j(x) = \Theta_{\tilde{A}}[z_j](x),
\end{equation}

where \(\Theta_{\tilde{A}}\) is the operator defined in \((2.21)\). Observe that

\begin{equation}
\tag{5.9}
Z_j(y) = \Theta_{\tilde{A}}[z_j](y).
\end{equation}

In order to solve problem \((5.5)\), we first consider the linear problem. Let \(\eta > 0\) be fixed as in \((3.1)\), and assume that the set of parameters \(A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}\) satisfies \((3.1)-(3.4)\). Given a function \(h\), we consider the problem of finding a function \(\phi\) and real numbers \(c_j, j = 0, 1, 2, \ldots, 3n - 1\) such that

\[
\begin{cases}
L(\phi) = h + \sum_{j=0,1,2,\ldots,3n-1} c_j V^{p-1} Z_j, & \text{in } D_\varepsilon; \\
\phi = 0, & \text{on } \partial D_\varepsilon; \\
\int_{D_\varepsilon} V^{p-1} Z_j \phi dy = 0, & \text{for all } j = 0, 1, 2, \ldots, 3n - 1.
\end{cases}
\]
In order to perform an invertibility theory for $L$ subject to the above orthogonality conditions, we introduce some proper weighted $L^\infty$-norms. We start with for

\begin{equation}
\|\psi\|_{\ast\ast} = \sup_{y \in I_\varepsilon} \|y\|^{n-2} \psi(y) + \sup_{y \in O_\varepsilon} |(1 + |y|^4)\psi(y)|,
\end{equation}

where

\begin{equation}
I_\varepsilon := \{ y \in D_\varepsilon, |y| < 1 \}, \quad O_\varepsilon := \{ y \in D_\varepsilon, |y| > 1 \}.
\end{equation}

This $L^\infty$ weighted norm, which allows singularity at 0, is suitable to estimate the right hand side $h$ in (5.10). The estimate of $\|E\|_{\ast\ast}$, where $E$ is the function defined in (5.7), is crucial for our argument, as it will become clear later on. We claim that there exists a positive constant $C$, independent of $\varepsilon$, so that

\begin{equation}
\|E\|_{\ast\ast} \leq C\varepsilon^{\frac{n-2}{2}}.
\end{equation}

Let us consider first $y \in D_\varepsilon$, with $|y| > 1$. Using the result in Lemma 3.1 and a Taylor expansion, in combination with (5.8) and (2.2), we immediately see that

\begin{equation}
|E(y)| \leq C\varepsilon^{\frac{n-2}{2}} \left| Q_A(y) \right|^{p-1} \leq C\varepsilon^{\frac{n-2}{2}} \frac{1}{|y|^\frac{p-2}{2}}.
\end{equation}

Let us now consider the region $y \in D_\varepsilon$, and $|y| < 1$. In this region, the function $E$ can be estimated as follows

\begin{equation}
|E(y)| \leq C \left( \frac{\varepsilon^{\frac{n-2}{2}}}{|y|^{n-2}} \right)^p \leq C \frac{\varepsilon^{\frac{n-2}{2}}}{|y|^{n-2}},
\end{equation}

since in the region we are considering one has $|y| > \sqrt{\varepsilon}$. With this, (5.13) is proven. We now introduce an appropriate norm to estimate the solutions to (5.10). This norms depends on the dimension of the space. For a function $\psi$ defined on $D_\varepsilon$, we define

\begin{equation}
\|\psi\|_{\ast} = \sup_{y \in I_\varepsilon} \left[ |y|^{\alpha} \psi(y) \right] + \sup_{y \in O_\varepsilon} \left[ (1 + |y|^\beta) \psi(y) \right],
\end{equation}

where

\begin{align*}
\alpha &= \left\{ \begin{array}{ll}
\sigma & \text{if } n \geq 5, \\
n - 4 & \text{if } n = 4,
\end{array} \right. \\
\beta &= \left\{ \begin{array}{ll}
\beta = 2 & \text{if } n \geq 5, \\
2 - \sigma & \text{if } n = 4
\end{array} \right.
\end{align*}

for some $\sigma > 0$, and

\begin{equation}
\|\psi\|_{\ast} = \sup_{y \in D_\varepsilon} \left[ (1 + |y|) \psi(y) \right] + \sup_{y \in D_\varepsilon} \left[ (1 + |y|^2) \psi(y) \right],
\end{equation}

if $n = 3$.

Equation (5.10) is solved in the following proposition, whose proof is postponed to Section 6.

**Proposition 5.1.** Let $\eta > 0$ be fixed as in (3.4), and assume that the set of parameters $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ satisfies (3.1), (3.4). Then there are numbers $\varepsilon_0 > 0$, $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C^\alpha(D_\varepsilon)$, problem (5.10) admits a unique solution $\phi := T_\varepsilon(h)$. Moreover,

\begin{equation}
\|T_\varepsilon(h)\|_{\ast} \leq C\|h\|_{\ast\ast}, \quad |c_j| \leq C\|h\|_{\ast\ast},
\end{equation}

and

\begin{equation}
\|\nabla (d, r, a, \theta) \phi\|_{\ast} \leq C\|h\|_{\ast\ast}.
\end{equation}
Based on the results in Proposition 5.1, a fixed point argument allows us to solve the nonlinear problem of finding a function $\phi$ and constants $c_j$ solutions to

\begin{equation}
\begin{aligned}
L(\phi) &= -[N(\phi) + E] + \sum_{j=0,1,\ldots,3n-1} c_j V^{p-1} Z_j, \quad \text{in } D_\varepsilon; \\
\phi &= 0, \quad \text{on } \partial D_\varepsilon; \\
\int_{D_\varepsilon} V^{p-1} Z_j \phi dy &= 0, \quad \text{for all } j = 0, 1, 2, \ldots, 3n - 1.
\end{aligned}
\end{equation}

\text{(5.19)}

The solvability of problem (5.19) is established in next Proposition, whose proof is postponed to Section 7.

**Proposition 5.2.** Assume the conditions of Proposition 5.1 are satisfied. Then there are numbers $\varepsilon_0 > 0$, $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$, there exists a unique solution $\phi = \phi(d, a, \theta)$ to problem (5.19). Moreover, the map $(d, \tau, a, \theta) \rightarrow \phi(d, \tau, a, \theta)$ is of class $C^1$ for $\| \cdot \|_*$ norm, and

\begin{equation}
\| \phi \|_* \leq C \varepsilon^{n-2},
\end{equation}

\text{(5.20)}

and

\begin{equation}
\| \nabla_{d, \tau, a, \theta} \phi \|_* \leq \varepsilon^{n-2}.
\end{equation}

\text{(5.21)}

After problem (5.10) has been solved, we find a solution to problem (5.5), if we can find a point $(d, \tau, a, \theta)$ such that coefficients $c_j$ in (5.10) satisfy

\begin{equation}
\begin{aligned}
c_j &= 0 \quad \text{for all } j = 0, 1, 2, \ldots, 3n - 1.
\end{aligned}
\end{equation}

\text{(5.22)}

For notational convenience, we introduce the set

\begin{equation}
\mathcal{A} := \{(d, \tau, a, \theta) : \text{conditions (3.1)–(3.4) are satisfied}\} \subset \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}.
\end{equation}

\text{(5.23)}

We now introduce the finite dimensional restriction $F_\varepsilon(d, \tau, a, \theta) : \mathcal{A} \rightarrow \mathbb{R}$, given by

\begin{equation}
F_\varepsilon(d, \tau, a, \theta) = I_\varepsilon(V(y) + \phi(y)),
\end{equation}

\text{(5.24)}

with $V$ defined by (5.3) and $\phi$ is the unique solution to problem (5.19) given by Proposition 5.2 and $I_\varepsilon$ is the energy functional associated to problem (5.2), given by

\begin{equation}
I_\varepsilon(v) = \frac{1}{2} \int_{D_\varepsilon} |\nabla v|^2 dy - \frac{1}{p+1} \int_{D_\varepsilon} |v|^{p+1} dy.
\end{equation}

\text{(5.25)}

**Lemma 5.3.** If $(d, \tau, a, \theta)$ is a critical point of $F_\varepsilon$, then $v(y) = V(y) + \phi(y)$ is a solution of problem (5.2).

**Proof.** We claim that if $(d, \tau, a, \theta)$ if a critical point for $F_\varepsilon$, then We first differentiate $F_\varepsilon$ with respect to $d$, then we have

\begin{equation}
\begin{aligned}
DI_\varepsilon(V + \phi)\left[\frac{\partial}{\partial d} Q_{\tilde A} + o(1)\right] &= 0, \quad DI_\varepsilon(V + \phi)\left[\frac{\partial}{\partial \tau_i} Q_{\tilde A} + o(1)\right] = 0, \quad i = 1, \ldots, n \\
DI_\varepsilon(V + \phi)\left[\frac{\partial}{\partial \theta_{12}} Q_{\tilde A} + o(1)\right] &= 0, \quad DI_\varepsilon(V + \phi)\left[\frac{\partial}{\partial a_j} Q_{\tilde A} + o(1)\right] = 0, \quad j = 1, 2 \\
DI_\varepsilon(V + \phi)\left[\frac{\partial}{\partial \theta_{1l}} Q_{\tilde A} + o(1)\right] &= 0, \quad DI_\varepsilon(V + \phi)\left[\frac{\partial}{\partial \theta_{2l}} Q_{\tilde A} + o(1)\right] = 0, \quad l = 3, \ldots, n
\end{aligned}
\end{equation}

\text{(5.26)}

Let us assume the validity of these equalities. From (5.19), we have

\begin{equation}
DI_\varepsilon(V + \phi)[Z_i + o(1)] = \sum_j c_j \int_{D_\varepsilon} V^{p-1} Z_j [Z_i + o(1)] dy
\end{equation}

\text{(5.27)}

where

\begin{equation}
\begin{aligned}
Z_0 &= \frac{\partial}{\partial d} Q_{\tilde A}, \quad Z_j = \frac{\partial}{\partial \tau_i} Q_{\tilde A}, \quad j = 1, \ldots, n, \quad Z_{n+1} = \frac{\partial}{\partial \theta_{12}} Q_{\tilde A}
\end{aligned}
\end{equation}

\text{(5.28)}
Lemma 5.4. Assume the conditions of Proposition 5.1 are satisfied. Then we have the following

\[ Z_{n+2} = \frac{\partial}{\partial a_1} Q_{\bar{A}}, \quad Z_{n+3} = \frac{\partial}{\partial a_2} Q_{\bar{A}} \]

and, for \( l = 3, \ldots, n, \)

\[ Z_{n+l+1} = \frac{\partial}{\partial \theta_{il}} Q_{\bar{A}}, \quad Z_{2n+l-1} = \frac{\partial}{\partial \theta_{2l}} Q_{\bar{A}}. \]

Using (2.23), (2.24), (2.25), (2.26), a direct computation gives

\[
\int_{D_{\varepsilon}} V^{p-1} Z_j Z_i dy = \begin{cases} 
\int_{\mathbb{R}^n} |Q|^{p-1}(y)z_i^2(y)dy + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } i = j; \\
\int_{\mathbb{R}^n} |Q|^{p-1}(y)z_1(y)z_{n+2}(y)dy + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } i = 1, j = n + 2; \\
\int_{\mathbb{R}^n} |Q|^{p-1}(y)z_2(y)z_{n+3}(y)dy + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } i = 2, j = n + 3; \\
O(\varepsilon^{\frac{n}{n-2}}) & \text{otherwise,}
\end{cases}
\]

where the functions \( z_j \) are the ones defined in (2.15), (2.16), (2.17), (2.18), (2.19). Therefore, the condition \( \nabla_{(d,\tau,a,\theta)} F_{\varepsilon}(d,\tau,a,\theta) = 0 \) give the 3n conditions

\[ DI_{\varepsilon}(V + \phi)[Z_j] = 0, \quad j = 0, \ldots, 3n - 1, \]

that give necessarily that \( c_j = 0 \) for all \( j = 0, \ldots, 3n - 1. \) This concludes the proof of the Lemma. We shall now prove (5.20). Since the arguments are similar, we prove the first formula in (5.26). Observe that

\[
\frac{\partial}{\partial d} F_{\varepsilon}(d,\tau,a,\theta) = DI_{\varepsilon}(V + \phi)[\frac{\partial}{\partial d} V + \frac{\partial}{\partial d} \phi].
\]

From Lemma 5.1 and (5.8),

\[
\frac{\partial}{\partial d} V(y) = \frac{\partial}{\partial d} Q_{\bar{A}}(y) + \varepsilon^{\frac{n}{n-2}} (1 + \frac{1}{|y|^{n-2}}) \Theta \bar{A}(y),
\]

where \( \Theta \bar{A}(y) \) is uniformly bounded as \( \varepsilon \to 0. \) Now, observe that

\[
\frac{\partial}{\partial d} Q_{\bar{A}}(y) = d^{\frac{n-4}{2}} \left| \frac{y - d\tau}{|y - d\tau|} \right| - R_{d}a \left| \frac{y - d\tau}{d} \right|^{2-n} z_0 \left( \frac{\varepsilon y - d\tau}{d} - R_{d}a \left| \frac{y - d\tau}{d} \right|^2 \right).
\]

Taking into account that \( \| \frac{\partial}{\partial d} \phi \|_* = o(1), \) as \( \varepsilon \to 0, \) we get that \( \frac{\partial}{\partial d} F_{\varepsilon} = DI_{\varepsilon}(V + \phi)[\frac{\partial}{\partial d} Q_{\bar{A}} + o(1)], \)

as \( \varepsilon \to 0. \)

\[ \square \]

Lemma 5.4. Assume the conditions of Proposition 5.1 are satisfied. Then we have the following expansion

\[ F_{\varepsilon}(d,\tau,a,\theta) - I_\varepsilon(V) = o(\varepsilon^{\frac{n-2}{2}}) \Theta, \]

where \( \Theta \) is \( C^1 \) uniformly bounded, independent of \( \varepsilon. \)

Proof. By a Taylor expansion and the fact that \( DI_{\varepsilon}(V + \phi)[\phi] = 0, \) we have

\[
F_{\varepsilon}(d,\tau,a,\theta) - I_\varepsilon(V) = I_\varepsilon(V + \phi) - I_\varepsilon(V) = \int_0^1 D^2 I(V + t\phi)[\phi,\phi]t dt
\]

\[
= \int_0^1 \int_{D_{\varepsilon}} [\|\nabla\phi\|^2 - p(V + t\phi)^{p-1}\phi^2] t dt.
\]

From (5.19), we have

\[
F_{\varepsilon}(d,\tau,a,\theta) - I_\varepsilon(V) = \int_0^1 \int_{D_{\varepsilon}} (p \left[ V^{p-1} - (V + t\phi)^{p-1} \right] \phi^2 + [N(\phi) + E] \phi) dy
\]
\[ \leq C \int_{D_\epsilon} |V^{p-1} - (V + \phi)^{p-1}| \phi^2 \, dy + \int_{D_\epsilon} |E| |\phi| \, dy + \int_{D_\epsilon} |N(\phi)| |\phi| \, dy \]

(5.28)

\[ = o(\epsilon^{-\frac{n-2}{2}}) \Theta, \]

uniformly with respect to \((d, \tau, a, \theta)\) in the considered region, where \(\Theta\) is uniformly bounded, independent of \(\epsilon\). Here we used the facts \(\|E\|_* \leq C\epsilon^{\frac{n-2}{2}}\) and \(\|\phi\|_* \leq C\epsilon^{\frac{n-2}{2}}\).

By a similarly way, using the facts \(\|\nabla_{(d, \tau, a, \theta)} E\|_* \leq C\epsilon^{\frac{n-2}{2}}\) and \(\|\partial_{(d, \tau, a, \theta)} \phi\|_* \leq C\epsilon^{\frac{n-2}{2}}\), we can obtain

\[ \nabla_{(d, \tau, a, \theta)} (F_\epsilon(d, \tau, a, \theta) - I_\epsilon(V)) = o(\epsilon^{-\frac{n-2}{2}})\Theta. \]

This ends the proof of Lemma. \(\square\)

**Proof of Theorem 1.1.** By Lemma 5.3 we know that \(u(\sqrt{\epsilon}y) = \epsilon^{-\frac{1}{p-1}} (V(y) + \phi(y))\) is a solution to problem (1.10) if and only if \((d, \tau, a, \theta)\) is a critical point of \(F_\epsilon(d, \tau, a, \theta)\). So we have to prove the existence of the critical point of \(F_\epsilon(d, \tau, a, \theta)\). We observe that, under the change of variables (5.1), we have \(I_\epsilon(v) = \hat{I}_\epsilon(u)\). From Lemma 5.4 Proposition 4.1, (4.1) and (4.2) we find

(5.29)

\[ F_\epsilon(d, \tau, a, \theta) = c_1 - \Psi(d, \tau, a, \theta) \epsilon^{\frac{n-2}{2}} + o(\epsilon^{-\frac{n-2}{2}}) \Theta(d, \tau, a, \theta), \]

where \(\Psi\) is defined as

\[ \Psi(d, \tau, a, \theta) = \frac{1}{2} \left[ \gamma_n^{-2} Q(-R_{\theta a})^2 H(0, 0) d^{n-2} + \frac{c_2}{d^{n-2}} F(\tau, a, \theta) \right], \]

with \(F\) given in (3.10) and \(\Theta\) is a smooth function of its variables, which is uniformly bounded, together with its first derivatives, as \(\epsilon \to 0\) for \((\lambda, \xi, a, \theta)\) satisfying (3.1) (3.2). Thus our result is proven provided we find a critical point, stable under \(C^1\) perturbation, of the function \(\Psi\).

Firstly, we observe that

\[ \partial_\tau \Psi(d, \tau, a, \theta) = \frac{n-2}{2} \gamma_n^{-2} Q(R_{\theta a})^2 H(0, 0) d^{n-3} - \frac{n-2}{2} \frac{c_2}{d^{n-1}} F(\tau, a, \theta). \]

We have that \(\partial_\tau \Psi_\epsilon(d_0, \tau, a, \theta) = 0\) with \(d_0 = \left( \frac{c_2 F(\tau, a, \theta)}{\gamma_n^{-2} Q(R_{\theta a})^2 H(0, 0)} \right)^{-\frac{1}{n-4}}\), and \(\partial_{dd}^2 \Psi_\epsilon(d_0, \tau, a, \theta) > 0\). Moreover, for \(d\) any and \(\theta\),

\[ (\tau, a) \mapsto \Psi(d, \tau, a, \theta) \]

has a nondegenerate maximum at \((0, 0)\).

Lastly, since \(\epsilon \in \mathcal{O}\) with \(\mathcal{O}\) is a compact manifold of dimension \(3n\) with no boundary, then, for any \(d, \tau, a, \theta\), the function \(\theta \mapsto \Psi(d, \tau, a, \cdot)\) has a stable minimum \(\bar{\theta}(d, \tau, a)\). Let \(\bar{\theta}_0 = \bar{\theta}(d_0, 0, 0)\). We conclude that \((d_0, 0, 0, \bar{\theta}_0)\) is a stable critical point for \(\Psi\). Then there exists a critical point \((d_\epsilon, \tau_\epsilon, a_\epsilon, \theta_\epsilon)\) of \(F_\epsilon\) satisfying \((d_\epsilon, \tau_\epsilon, a_\epsilon, \theta_\epsilon) \to (d_0, 0, 0, \bar{\theta}_0)\) as \(\epsilon \to 0\). This concludes the proof of Theorem 1.1.

The rest of the paper is devoted to prove in details all the facts stated until now.

6. The linear problem: proof of Proposition 5.1

**Proof of Proposition 5.1.** The first part of the proof consists in establishing the priori estimate (5.17). We do it by contradiction: assume that there exists a sequence \(\epsilon = \epsilon_l \to 0\) such that there are functions \(\phi_\epsilon\) and \(h_\epsilon\) such that

(6.1)

\[ \left\{ \begin{array}{ll}
L(\phi_\epsilon) = h_\epsilon + \sum_{j=0,1,\ldots,3n-1} c_j V^{p-1} Z_j, & \text{in } D_\epsilon; \\
\phi_\epsilon = 0, & \text{on } \partial D_\epsilon; \\
\int_{D_\epsilon} V^{p-1} Z_j \phi_\epsilon dy = 0, & \text{for all } j = 0, 1, 2, \ldots, 3n - 1,
\end{array} \right. \]
for certain constants $c_j$, depending on $\varepsilon$, with $\|h_\varepsilon\|_{**} \to 0$ while $\|\phi_\varepsilon\|_*$ remains bounded away from 0 as $\varepsilon \to 0$.

We first establish the slightly weaker assertion that

$$\|\phi_\varepsilon\|_\rho \to 0$$

with $\rho > 0$ a small fixed number, where

$$\|\psi\|_\rho = \sup_{y \in \mathcal{E}} \left[ |y|^{\alpha + \rho} \psi(y) + |y|^{\alpha + 1 + \rho} D\psi(y) \right]$$

$$+ \sup_{y \in \mathcal{O}_\varepsilon} \left[ |(1 + |y|^\beta - \rho)\psi(y)| + |(1 + |y|^\beta + 1 - \rho) D\psi(y)| \right],$$

if $n \geq 4$, and

$$\|\psi\|_\rho = \sup_{y \in \mathcal{D}_\varepsilon} \left[ |(1 + |y|^1 - \rho)\psi(y)| + |(1 + |y|^2 - \rho) D\psi(y)| \right],$$

if $n = 3$. To do this, we assume the opposite, so that with no loss of generality we may take $\|\phi_\varepsilon\|_\rho = 1$.

We claim that

$$c_j \to 0, \quad \text{as} \quad \varepsilon \to 0.$$  \tag{6.2}

Testing equation (6.4) against $Z_i$, integrating by parts twice we get that

$$\sum c_j \int_{\mathcal{D}_\varepsilon} V^{p-1} Z_j Z_i dy = \int_{\mathcal{D}_\varepsilon} [\Delta Z_i + pV^{p-1} Z_i] \phi_\varepsilon dy - \int_{\partial\mathcal{D}_\varepsilon} Z_i \partial_\nu \phi_\varepsilon - \int_{\mathcal{D}_\varepsilon} h_\varepsilon Z_i dy. \tag{6.3}$$

We claim that

$$\int_{\mathcal{D}_\varepsilon} V^{p-1} Z_j Z_i dy = \begin{cases} \int_{\mathbb{R}^n} |Q|^{p-1}(y) z_i^2(y) dy + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } i = j; \\ \int_{\mathbb{R}^n} |Q|^{p-1}(y) z_1(y) z_{n+2}(y) dy + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } i = 1, j = n + 2; \\ \int_{\mathbb{R}^n} |Q|^{p-1}(y) z_2(y) z_{n+3}(y) dy + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } i = 2, j = n + 3; \\ O(\varepsilon^{\frac{n}{n-2}}) & \text{otherwise,} \end{cases} \tag{6.4}$$

$$\int_{\mathcal{D}_\varepsilon} [\Delta Z_i + pV^{p-1} Z_i] \phi_\varepsilon = o(1)\|\phi_\varepsilon\|_\rho, \quad \text{and} \quad \int_{\partial\mathcal{D}_\varepsilon} Z_i \partial_\nu \phi_\varepsilon = o(1)\|\phi_\varepsilon\|_\rho \tag{6.5}$$

$$|\int_{\mathcal{D}_\varepsilon} h_\varepsilon Z_i| \leq C\|h_\varepsilon\|_{**}. \tag{6.6}$$

Thus, we conclude that

$$|c_j| \leq C\|h_\varepsilon\|_{**} + o(1)\|\phi_\varepsilon\|_\rho \tag{6.7}$$

from which (6.2) readily follows.

Proof of (6.4). From (5.8) and (5.9), we observe that

$$\int_{\mathcal{D}_\varepsilon} V^{p-1} Z_j Z_i dy = \int_{\mathcal{D}_\varepsilon} Q_{A_j}^{p-1} \Theta_A[z_j] \Theta_A[z_i] dy + O(\varepsilon^{\frac{n}{n-2}}),$$

where $A = (d, d\tau, a, \theta)$. Using first the change of variable $z = \frac{\nu - dr}{a}$, and then the change of variables $\eta = \frac{z}{|z|}$, we have

$$\int_{\mathcal{D}_\varepsilon} Q_{A_j}^{p-1} \Theta_A[z_j] \Theta_A[z_i] dy = \int \left| \frac{z}{|z|} - R_\theta a|z|^2 \right|^{-2n} (Q^{p-1} z_j z_i) \left( \frac{z - R_\theta a|z|^2}{|z|} - R_\theta a|z|^2 \right) dz$$
In analogous way, one has (6.5). We start from the first estimate. Let From Lemma 3.1, we see that

Moreover, To estimate \( g \) we have

In dimensions 5 and 6, one has (2.18), (2.19) gives that, for some constant \( C \),

The last equality follows from Lemma 3.4 in the Appendix. This concludes the proof of (6.4).

**Proof of (6.5).** We start from the first estimate. Let \( g_i = \Delta Z_i + pV^{p-1}Z_i \). By definition of \( Z_i \), we have \( g_i = p \left( V^{p-1} - Q_A^{p-1} \right) Z_i \). A close analysis of the functions \( z_j \) in (2.15), (2.16), (2.17), (2.18), (2.19) gives that, for some constant \( C \),

\[
|z_j(y)| \leq \frac{C}{1 + |y|^{n-2}}, \quad y \in \mathbb{R}^n.
\]

From Lemma 3.1 we see that

\[
V(y) = Q_A(y) - \gamma_n^{-1}Q(-R\theta a)H(\sqrt{\varepsilon y}, \xi)\varepsilon^{n/2} - F(\tau, a, \theta)\varepsilon^{n/2} + R(y)
\]

with

\[
|R(y)| \leq C\varepsilon^{n/2}\left( \frac{\varepsilon^{n/2}}{|y|^{n-2}} + \lambda^2 + \varepsilon^{n/2} \right).
\]

Thus we have the following estimate

\[
(6.8) \quad |[\Delta Z_i + pV^{p-1}Z_i]\phi| \leq C Q_A^{-2} \left( \varepsilon \frac{n-2}{2} + \frac{\varepsilon^{n/2}}{|y|^{n-2}} \right) |Z_i\phi|.
\]

To estimate \( \int_{D_{\varepsilon}} g_i \phi \), we estimate separately \( \int_{I_{\varepsilon}} g_i \phi \), and \( \int_{O_{\varepsilon}} g_i \phi \). Consider first the case \( p \geq 2 \). In dimensions 5 and 6, one has

\[
\left| \int_{I_{\varepsilon}} [\Delta Z_i + pV^{p-1}Z_i]\phi \right| \leq C\varepsilon^{n/2} \|\phi\|_{\rho} \int_{I_{\varepsilon}} \frac{1}{|y|^{4n+6+\rho}} dy
\]

\[
\leq C\varepsilon^{n/2} \varepsilon^{-\frac{n}{2} + 4} |\phi|_{\rho} \int_{\sqrt{\varepsilon} < |y| < 1} \frac{1}{|y|^{n-\sigma}} dy
\]

\[
\leq C\varepsilon^{1-\frac{n}{2} - \frac{\sigma}{2}} |\phi|_{\rho}.
\]

Moreover,

\[
\left| \int_{O_{\varepsilon}} [\Delta Z_i + pV^{p-1}Z_i]\phi \right| \leq C\varepsilon^{n/2} \|\phi\|_{\rho} \int_{O_{\varepsilon}} \frac{1}{1 + |y|^{6-\rho}} dy = \varepsilon^{2-\frac{n}{2}} |\phi|_{\rho}.
\]

In analogous way, one has

\[
\left| \int_{I_{\varepsilon}} [\Delta Z_i + pV^{p-1}Z_i]\phi \right| \leq C \begin{cases} 
\varepsilon^{2-\frac{n}{2} - \frac{\sigma}{2}} |\phi|_{\rho} & \text{if } n = 4; \\
\varepsilon^{\frac{1}{2}} |\phi|_{\rho} & \text{if } n = 3,
\end{cases}
\]

and

\[
\left| \int_{O_{\varepsilon}} [\Delta Z_i + pV^{p-1}Z_i]\phi \right| \leq C \begin{cases} 
\varepsilon |\phi|_{\rho} & \text{if } n = 4; \\
\varepsilon^{\frac{1}{2}} |\phi|_{\rho} & \text{if } n = 3,
\end{cases}
\]
as $\varepsilon \to 0$, in dimensions 4 and 3. Similar estimates hold also in dimensions 4 and 3. Thus the first estimate in (6.5) holds true in dimensions 3 to 6.

Let us consider now $n \geq 7$, that is $p < 2$. Define $R_{\varepsilon} = \{y \in D_{\varepsilon} : |Q_{\lambda}(y)| \leq \varepsilon\}$. We have
\[
|\Delta Z_i + pV^{p-1}Z_i|_{\phi_{\varepsilon}} \leq C\varepsilon^{p-1}Z_j|_{\phi_{\varepsilon}}, \quad \text{in } R_{\varepsilon}
\]
and
\[
|\Delta Z_i + pV^{p-1}Z_i|_{\phi_{\varepsilon}} \leq C|Q_{\lambda}|^{p-1}\varepsilon^{-1}\left(\varepsilon^{\frac{n-2}{2}} + \frac{\varepsilon^{\frac{n-2}{2}}}{|y|^{n-2}}\right)Z_j|_{\phi_{\varepsilon}}, \quad \text{in } D_{\varepsilon} \setminus R_{\varepsilon}
\]
Thus, we get
\[
\left|\int_{I_\varepsilon \cap R_{\varepsilon}} g_i \phi \right| \leq C\varepsilon^{p-1}\|\phi\|_\rho, \quad \left|\int_{O_\varepsilon \cap R_{\varepsilon}} g_i \phi \right| \leq C\varepsilon^{p-1}\|\phi\|_\rho,
\]
and
\[
\left|\int_{I_\varepsilon \cap R_{\varepsilon}} g_i \phi \right| \leq C\varepsilon^{1-\frac{2}{n}}a\|\phi\|_\rho, \quad \left|\int_{O_\varepsilon \cap R_{\varepsilon}} g_i \phi \right| \leq C\varepsilon^{\frac{n-4}{2}}\|\phi\|_\rho,
\]
for some $a > 0$ small. Thus we get the validity of the first estimate in (6.5). Let us discuss now the second estimate in (6.5). We write
\[
\int_{\partial D_{\varepsilon}} Z_i \frac{\partial \phi}{\partial \nu} = \int_{\partial D_{\varepsilon} \cap I_\varepsilon} Z_i \frac{\partial \phi}{\partial \nu} + \int_{\partial D_{\varepsilon} \cap O_\varepsilon} Z_i \frac{\partial \phi}{\partial \nu}.
\]
We observe that
\[
\left|\int_{\partial D_{\varepsilon} \cap I_\varepsilon} Z_i \frac{\partial \phi}{\partial \nu}\right| \leq C\|\phi\|_\rho \int_{\partial B(0,\varepsilon)} \frac{1}{|y|^{n-3+\rho}} \leq C\varepsilon^{1-\frac{2}{n}}\|\phi\|_\rho,
\]
and
\[
\left|\int_{\partial D_{\varepsilon} \cap O_\varepsilon} Z_i \frac{\partial \phi}{\partial \nu}\right| \leq C\|\phi\|_\rho \int_{\partial D_{\varepsilon} \cap O_\varepsilon} \varepsilon^{\frac{n-2}{2}} \varepsilon^{\frac{\rho}{2}} \leq C\varepsilon^{1-\frac{2}{n}}\|\phi\|_\rho.
\]
The second estimate in (6.5) is thus proven.

Proof of (6.6). We directly see that
\[
\left|\int_{D_{\varepsilon}} h_{\varepsilon} Z_i \right| \leq C \left(\int_{I_\varepsilon} \frac{dy}{|y|^{n-2}} + \int_{O_\varepsilon} \frac{1}{(1 + |y|^{n+2})} dy\right) \|h\|_{**} \leq C\|h\|_{**}.
\]
Let $G_{\varepsilon}$ denotes the Green’s function of $D_{\varepsilon}$. We have for $x \in D_{\varepsilon}$
\[
\begin{equation}
\phi_{\varepsilon}(x) = \frac{1}{\varepsilon} \int_{D_{\varepsilon}} G_{\varepsilon}(x,y)V^{p-1}\phi_{\varepsilon} dy - \int_{D_{\varepsilon}} G_{\varepsilon}(x,y)h_{\varepsilon} dy - \sum_{j} \int_{D_{\varepsilon}} V^{p-1}Z_j G_{\varepsilon}(x,y) dy \tag{6.9}
\end{equation}
\]
We claim that
\[
|\phi_{\varepsilon}(x)| \leq C \begin{cases} 
\frac{\|\phi_{\varepsilon}\|_{\rho} + \|h_{\varepsilon}\|_{**}}{|x|^{n-4}} \quad \text{if } n \geq 5; \\
\frac{\|\phi_{\varepsilon}\|_{\rho} + \|h_{\varepsilon}\|_{**}}{|x|^{n}} \quad \text{if } n = 4; \\
\|\phi_{\varepsilon}\|_{\rho} + \|h_{\varepsilon}\|_{**} \quad \text{if } n = 3,
\end{cases}
\]
and

\[
|\tilde{\phi}_\varepsilon(x)| \leq C \begin{cases} 
\frac{\|\phi_\varepsilon\|_{\rho} + \|h_\varepsilon\|_{**}}{1 + |x|^2} & \text{if } n \geq 5; \\
\frac{\|\phi_\varepsilon\|_{\rho} + \|h_\varepsilon\|_{**}}{1 + |x|^2} & \text{if } n = 4; \quad \text{for } |x| > 1. \\
\frac{\|\phi_\varepsilon\|_{\rho} + \|h_\varepsilon\|_{**}}{1 + |x|^2} & \text{if } n = 3,
\end{cases}
\]

(6.11)

\[
|\partial_x \tilde{\phi}_\varepsilon(x)| \leq C \begin{cases} 
\frac{\|\phi_\varepsilon\|_{\rho} + \|h_\varepsilon\|_{**}}{|x|^{n-3}} & \text{if } n \geq 5; \\
\frac{\|\phi_\varepsilon\|_{\rho} + \|h_\varepsilon\|_{**}}{|x|^{n-1}} & \text{if } n = 4; \quad \text{for } |x| < 1. \\
\|\phi_\varepsilon\|_{\rho} + \|h_\varepsilon\|_{**} & \text{if } n = 3,
\end{cases}
\]

(6.12)

\[
|\partial_x \phi_\varepsilon(x)| \leq C \begin{cases} 
\frac{\|\phi_\varepsilon\|_{\rho} + \|h_\varepsilon\|_{**}}{1 + |x|^2} & \text{if } n \geq 5; \\
\frac{\|\phi_\varepsilon\|_{\rho} + \|h_\varepsilon\|_{**}}{1 + |x|^2} & \text{if } n = 4; \quad \text{for } |x| > 1. \\
\|\phi_\varepsilon\|_{\rho} + \|h_\varepsilon\|_{**} & \text{if } n = 3,
\end{cases}
\]

(6.13)

Assume for the moment the validity of these estimates. Since \( \rho \) is arbitrarily small and \( \|\phi_\varepsilon\|_{\rho} = 1 \), the estimates above imply that that \( \|\tilde{\phi}_\varepsilon\|_{L^\infty(B(0,R_1) \setminus B(0,R_2))} > \gamma \) for certain \( R_1 > R_2 > 0 \) and \( \gamma > 0 \) independent of \( \varepsilon \). Then local elliptic estimates and the bounds above yield that, up to a subsequence, \( \tilde{\phi}_\varepsilon \) converges uniformly over compacts of \( \mathbb{R}^n \) to a nontrivial solution \( \tilde{\phi} \) of

\[
\Delta \tilde{\phi} + p|Q|^{p-1} \tilde{\phi} = 0,
\]

(6.14)

which besides satisfies

\[
|\tilde{\phi}(x)| \leq C|x|^{-\beta}, \quad \beta = 2, \quad \text{if } n \geq 5, \quad \beta = 2 - \sigma, \quad \text{if } n = 4
\]

(6.15)

In dimension \( n = 3 \) this means \( |\tilde{\phi}(x)| \leq C|x|^2 \). In higher dimension, a bootstrap argument of \( \tilde{\phi} \) solution of (6.14), using estimate (6.15), gives \( |\tilde{\phi}(x)| \leq C|x|^{2-n} \). Thanks to non degenerate result in [18], this implies that \( \tilde{\phi} \) is a linear combination of the functions \( z_j \), defined in (2.15), (2.16), (2.17), (2.18) and (2.19). On the other hand, dominated convergence Theorem gives that the orthogonality conditions \( \int_{D_\varepsilon} \phi_\varepsilon V^{p-1} z_j = 0 \) pass to the limit, thus getting

\[
\int_{\mathbb{R}^n} |Q|^{p-1} z_j \tilde{\phi} = 0 \quad \text{for all } j = 0, 1, \ldots, 3n - 1.
\]

Hence the only possibility is that \( \tilde{\phi} \equiv 0 \), which is a contradiction which yields the proof of \( \|\phi_\varepsilon\|_{\rho} \to 0 \). Moreover, we observe that

\[
\|\phi_\varepsilon\|_{**} \leq C(\|h_\varepsilon\|_{**} + \|\phi_\varepsilon\|_{\rho}),
\]

hence \( \|\phi_\varepsilon\|_{**} \to 0 \).

We shall show the validity of (6.10) and (6.11). To get (6.12) and (6.13), one proceeds in a similar way, using the fact the function \( \phi_\varepsilon \) is of class \( C^1 \) and

\[
\partial_{x_\xi} \phi_\varepsilon(x) = p \int_{D_\varepsilon} \partial_{x_\xi} G_\varepsilon(x,y) V^{p-1} \phi_\varepsilon dy - \int_{D_\varepsilon} \partial_{x_\xi} G_\varepsilon(x,y) h_\varepsilon dy
\]
Using the definitions of the norm in (5.11), we get that, for $|x| \leq 1$,

$$|g_2(x)| \leq C \|h_\varepsilon\|_{\ast \ast} \left( \int_{I_\varepsilon} \frac{1}{|x-y|^{n-2}} \frac{1}{|y|^{n-2}} dy + \int_{O_\varepsilon} \frac{1}{|x-y|^{n-2}} \frac{1}{1+|y|^4} dy \right)$$

$$\leq C \begin{cases} \frac{\|h_\varepsilon\|_{\ast \ast}}{|x|^{n-2}} & \text{if } n \geq 5; \\
\|h_\varepsilon\|_{\ast \ast} & \text{if } n = 4; \\
\|h_\varepsilon\|_{\ast \ast} & \text{if } n = 3,
\end{cases}$$

(6.16)

as consequence of (8.12), in Lemma 8.3. Consider now $|x| > 1$. In this region we have

$$|g_2(x)| \leq C \begin{cases} \frac{\|h_\varepsilon\|_{\ast \ast}}{1+|x|^2} & \text{if } n \geq 5; \\
\|h_\varepsilon\|_{\ast \ast} & \text{if } n = 4; \\
\|h_\varepsilon\|_{\ast \ast} & \text{if } n = 3,
\end{cases}$$

(6.17)

as consequence of (8.13), in Lemma 8.3. Arguing similarly, and using (6.7), we see that, for $|x| < 1$,

$$\left| \sum_j c_j \int_{D_\varepsilon} V^{n-1} Z_j \partial_x G_\varepsilon(x, y) dy \right| \leq C \begin{cases} \|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{\ast \ast} & \text{if } n \geq 4; \\
\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{\ast \ast} & \text{if } n = 3,
\end{cases}$$

(6.18)

and, for $|x| > 1$,

$$\left| \sum_j c_j \int_{D_\varepsilon} V^{n-1} Z_j \partial_x G_\varepsilon(x, y) dy \right| \leq C \begin{cases} \|\phi_\varepsilon\|_\rho + \frac{\|h_\varepsilon\|_{\ast \ast}}{1+|x|^2} & \text{if } n \geq 5; \\
\|\phi_\varepsilon\|_\rho + \frac{\|h_\varepsilon\|_{\ast \ast}}{1+|x|^2} & \text{if } n = 4; \\
\|\phi_\varepsilon\|_\rho + \frac{\|h_\varepsilon\|_{\ast \ast}}{1+|x|^2} & \text{if } n = 3,
\end{cases}$$

(6.19)

In order to estimate $g_1$, we consider first $n \geq 5$. For $|x| \leq 1$, we use (8.13) to get

$$|g_1(x)| \leq C \|\phi_\varepsilon\|_\rho \left( \int_{I_\varepsilon} \frac{1}{|x-y|^{n-2}} \frac{dy}{|y|^{n-4+\rho}} + \int_{O_\varepsilon} \frac{1}{|x-y|^{n-2}} \frac{dy}{1+|y|^{6+\rho}} \right) C \frac{\|\phi_\varepsilon\|_\rho}{|x|^{n-4}},$$

(6.20)

and (8.12) to get, for $|x| > 1$

$$|g_1(x)| \leq C \frac{\|\phi_\varepsilon\|_\rho}{1+|x|^2},$$

(6.21)

In a similar way, we have, for $|x| > 1$,

$$|g_1(x)| \leq C \frac{\|\phi_\varepsilon\|_\rho}{|x|^2} \text{ if } n = 4, \quad |g_1(x)| \leq C \|\phi_\varepsilon\|_\rho \text{ if } n = 3$$

(6.22)

and

$$|g_1(x)| \leq C \frac{\|\phi_\varepsilon\|_\rho}{1+|x|^{2-\sigma}} \text{ if } n = 4, \quad |g_1(x)| \leq C \frac{\|\phi_\varepsilon\|_\rho}{1+|x|} \text{ if } n = 3.$$

(6.23)
Collecting together estimates (6.16)–(6.23), we obtain the validity of (6.10) and (6.11).

**Step 2:** The existence of solution to (5.10). To do this, let us consider the space

\[ H = \left\{ \phi \in H^1_0(D_\varepsilon) \mid \int_{D_\varepsilon} V^{p-1}Z_j \phi = 0, \ \forall j = 0, 1, \ldots, 3n - 1 \right\} \]

endowed with the usual inner product \([\phi, \psi] = \int_{D_\varepsilon} \nabla \phi \nabla \psi\). Problem (5.10) expressed in weak form is equivalent to that of finding a \(\phi \in H\) such that

\[ [\phi, \psi] = \int_{D_\varepsilon} \left( pV^{p-1}\phi - h \right) \psi  \quad \forall \psi \in H. \]

With the aid of Riesz’s representation theorem, this equation gets rewritten in \(H\) in the operational form

\[ (6.24) \quad \phi = L_\varepsilon(\phi) + \tilde{h} \]

with certain \(\tilde{h} \in H\) which depends linearly in \(h\) and where \(L_\varepsilon\) is a compact operator in \(H\). Fredholm’s alternative guarantees unique solvability of this problem for any \(h\) provided that the homogeneous equation \(\phi = T_\varepsilon(\phi)\) has only the zero solution in \(H\). Assume it has a nontrivial solution \(\phi = \phi_\varepsilon\), which with no loss of generality may be taken so that \(\|\phi_\varepsilon\|_* = 1\). But for what we proved before, necessarily \(\|\phi_\varepsilon\|_* \to 0\). This is certainly a contradiction that proves that this equation only has the trivial solution in \(H\). We conclude then that for each \(h\), problem (5.10) admits a unique solution. Standard arguments give then the validity of (5.17).

We next analysis the dependence of the solution \(\phi\) to (5.10) on the parameters \(A' = (d, \tau, a, \theta)\). Let us define \(A' = (A_1, A_2, \ldots, A_{3n})\) the components of the vector \(A'\). Let us differential \(\phi\) with respect to \(A_s\), for some \(s = 1, \ldots, 3n\). We set formally \(Z = \frac{\partial}{\partial A_s} \phi\). Then \(Z\) satisfies the following equation

\[ \Delta Z + pV^{p-1}Z = -p\partial_{A_s}(V^{p-1})\phi + \sum_j e_j V^{p-1}Z_j + c_j \partial_{A_s}(V^{p-1}Z_j) \quad \text{in} \quad D_\varepsilon \]

Here \(e_j = \partial_{A_s}c_j\). Besides, from differentiating the orthogonality condition \(\int_{D_\varepsilon} V^{p-1}Z_j \phi dy = 0\), we get

\[ \int_{D_\varepsilon} \partial_{A_s}(V^{p-1}Z_j)\phi dy + \int_{D_\varepsilon} V^{p-1}Z_j Z dy = 0. \]

Let us consider constants \(b_i\) such that

\[ (6.25) \quad \int_{D_\varepsilon} V^{p-1}Z_j Z - \sum_i b_i \int_{D_\varepsilon} V^{p-1}Z_j Z_i = 0. \]

These relations amount to

\[ \sum_i b_i \int_{D_\varepsilon} V^{p-1}Z_j Z_i = \int_{D_\varepsilon} V^{p-1}Z_j Z. \]

Since this system is diagonal dominant with uniformly bounded coefficients, we use it is uniquely solvable and that

\[ b_i = O(\|\phi\|_*). \]

Let us set \(\eta = Z - \sum_i b_i Z_j\), thus \(\eta \in H^1_0(D_\varepsilon)\) and

\[ (6.26) \quad \int_{D_\varepsilon} V^{p-1}Z_j \eta = 0 \quad \text{for all} \ j. \]
On the other hand, we have that
\[
\Delta \eta + p V^{p-1} \eta = f + \sum_j e_j V^{p-1} Z_j \quad \text{in } D_\varepsilon,
\]
where \( e_j = \frac{\partial}{\partial A_s} c_j \) and
\[
f = \sum_j b_j (-\Delta + p V^{p-1}) Z_j + c_j \partial A_s (V^{p-1} Z_j) - p \partial A_s (V^{p-1} \phi),
\]
Thus we have that \( \eta = T_\varepsilon(f) \). Moreover, we easily see that
\[
\| \phi \partial A_s (V^{p-1}) \|_{*} \leq C \| \phi \|_n.
\]
On the other hand
\[
| \partial A_s (V^{p-1} Z_i (x)) | \leq C |x|^{-n-4},
\]
hence
\[
\| c_i \partial A_s V^{p-1} Z_i \|_{*} \leq C \| h \|_n
\]
since we have that \( c_i = O(\| h \|_n) \). We conclude that
\[
\| f \|_{*} \leq C \| h \|_{*}.
\]
Reciprocally, if we define
\[
Z = T_\varepsilon(f) + \sum_j e_j V^{p-1} Z_j,
\]
with \( b_j \) given by relations (6.25) and \( f \) by (6.28), we check that indeed \( Z = \partial A_s \phi \). In fact \( Z \) depends continuously on the parameters \( A' \) and \( h \) for the norm \( \| \|_n \) and \( \| Z \|_n \leq C \| h \|_n \) for parameters in the considered region.

In other words, we proved that \( (d, \tau, a, \theta) \mapsto T_\varepsilon \) is of class \( C^1 \) in \( L(L_{**}^{\infty}, L_\infty^{\infty}) \) and, for instance,
\[
(D A_s T_\varepsilon)(h) = T_\varepsilon(f) + \sum b_j Z_j,
\]
where \( f \) is given by (6.28) and \( b_j \) by (6.25). This concludes the proof. \( \square \)

7. The non-linear Problem: proof of Proposition 5.2

Proof of Proposition 5.2. We write the equation in (5.19) as
\[
\Delta \phi + p V^{p-1} \phi = -N(\phi) - E + \sum_j c_j V^{p-1} Z_j \quad \text{in } D_\varepsilon
\]
where \( N(\phi) \) and \( E \) are defined respectively by (5.6) and (5.7). We already showed in (5.13) that \( \| E \|_{*} \leq C \varepsilon^{\frac{n-2}{2}} \). To estimate \( N(\phi) \), it is convenient, and sufficient for our purposes, to assume \( \| \phi \|_n \leq 1 \). Note that, if \( n \leq 6 \), then \( p \geq 2 \) and we can estimate
\[
|N(\phi)| \leq C|V|^{p-2} |\phi|^2
\]
and hence
\[
|N(\phi)| (x) \leq C \begin{cases} \frac{\| \phi \|^2}{|x|^2} & \text{if } n \geq 4; \\ \| \phi \|^2 & \text{if } n = 3, \end{cases} \quad \text{for } |x| < 1,
\]
and

\[
|N(\phi)|(x) \leq C \begin{cases} 
\frac{\|\phi\|}{1+|x|^2} & \text{if } n \geq 5; \\
\varepsilon^{-\frac{n}{2}} \frac{\|\phi\|}{1+|x|^2} & \text{if } n = 4; \\
\|\phi\| & \text{if } n = 3,
\end{cases}
\]

Assume now that \( n > 6 \). If \( |\phi| \geq \frac{1}{2}V \) we have

\[
|N(\phi)| \leq C|\phi|^p,
\]

and thus

\[
|N(\phi)(x)| \leq C \begin{cases} 
\varepsilon^p \frac{\|\phi\|^*}{|y|^n} & \text{if } |x| < 1; \\
\varepsilon^p \frac{\|\phi\|^*}{1+|x|^2} & \text{if } |x| > 1.
\end{cases}
\]

Let us consider now the case \( |\phi| \leq \frac{1}{2}V \). In this case, we have that \( |N(\phi)| \leq C|V|^{p-1}|\phi| \), for some constant \( C \). Thus, for \( |x| < 1 \), we get

\[
N(\phi) \leq C \varepsilon^2 \frac{\|\phi\|^*}{|y|^n-4} \leq C \varepsilon \frac{\|\phi\|^*}{|y|^n-2}
\]

while for \( |x| > 1 \),

\[
|N(\phi)| \leq C \varepsilon^2 \frac{\|\phi\|^*}{1+|x|^2} \leq C \varepsilon \frac{\|\phi\|^*}{1+|x|^4}.
\]

Combining these relations we get

\[
\|N(\phi)\|^* \leq \begin{cases} 
C \|\phi\|^2 & \text{if } n \leq 6, n \neq 4 \\
C \varepsilon^2 \|\phi\|^2 & \text{if } n = 4 \\
C \varepsilon^{-\frac{n}{2}} \|\phi\|^2 & \text{if } n > 6.
\end{cases}
\]  

(7.1)

Now, we are in position to prove that problem (5.19) has a unique solution \( \phi = \tilde{\phi} + \tilde{\psi} \), with  

(7.2)

\[
\tilde{\psi} := -T_\varepsilon(E),
\]

with the required properties. Here \( T_\varepsilon \) denotes the linear operator defined by Proposition 5.1, namely \( \phi = T_\varepsilon(h) \) solves (5.10). We see that problem (5.19) is equivalent to solving a fixed point problem. Indeed \( \phi = \phi + \psi \) is a solution of (5.19) if and only if

\[
\tilde{\phi} = -T_\varepsilon(N(\tilde{\phi} + \tilde{\psi})) = A_\varepsilon(\tilde{\phi}).
\]

We proceed to prove that the operator \( A_\varepsilon \) defined above is a contraction inside a properly chosen region. Since \( \|E\|^* \leq C \varepsilon^{-\frac{n-2}{2}} \), the result of Proposition 5.2 gives that

\[
\|\tilde{\psi}\|^* \leq C \varepsilon^{-\frac{n-2}{2}}
\]

and

\[
\|N(\tilde{\psi} + \eta)\|^* \leq \begin{cases} 
C \|\eta\|^2 & \text{if } n \leq 6, n \neq 4 \\
C \varepsilon^2 \|\eta\|^2 & \text{if } n = 4 \\
C \varepsilon^{-\frac{n}{2}} \|\eta\|^2 & \text{if } n > 6.
\end{cases}
\]  

(7.3)

Call

\[
F = \{ \eta \in H^1_0 : \|\eta\|^* \leq R \varepsilon^{-\frac{n-2}{2}} \}.
\]

From Proposition 5.2 and (7.3) we conclude that, for \( \varepsilon \) sufficiently small and any \( \eta \in F \) we have

\[
\|A_\varepsilon(\eta)\|^* \leq C \varepsilon^{-\frac{n-2}{2}}.
\]
If we choose $R$ big enough in the definition of $F$, we get then that $A_\varepsilon$ maps $F$ in itself. Now we will show that the map $A_\varepsilon$ is a contraction, for any $\varepsilon$ small enough. That will imply that $A_\varepsilon$ has a unique fixed point in $F$ and hence problem (5.19) has a unique solution. For any $\eta_1, \eta_2$ in $F$ we have
\[
\|A_\varepsilon(\eta_1) - A_\varepsilon(\eta_2)\| \leq C\|N_\varepsilon(\tilde{\psi} + \eta_1) - N_\varepsilon(\tilde{\psi} + \eta_2)\|_{**},
\]
hence we just need to check that $N$ is a contraction in its corresponding norms. By definition of $N$
\[
D_\eta N_\varepsilon(\bar{\eta}) = p[(V + \bar{\eta})^{p-1} - V^{p-1}].
\]
Arguing as before, we get $c \in (0, 1)$ such that
\[
\|N(\tilde{\psi} + \eta_1) - N(\tilde{\psi} + \eta_2)\|_{**} \leq c\|\eta_1 - \eta_2\|_*.
\]
This concludes the proof of existence of $\phi$ solution to (5.19), and the first estimate in (5.20).

The regularity of the map $(d, \tau, a, \theta) \mapsto \phi$ can be proved by standard arguments involving the implicit function, and then we get the estimate (5.21), which can be seen in [18].

\[\square\]

8. Appendix

Let us recall that the existence of $Q$ was obtained in [8, 9], and also we will use some facts and computations in [18]. We have
\[
Q(y) = U(y) - \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y)
\]
with
\[
U(y) = \alpha_n \left( \frac{1}{1 + |y|^2} \right)^{\frac{n-2}{2}}, \quad U_j(y) = \mu_k \frac{n-2}{2} U(\mu_k^{-1}(y - \xi_j)),
\]
and the function $\tilde{\phi}$ is described in (2.7). Let us now define the following functions
\[
\pi_\alpha(y) = \frac{\partial}{\partial y_\alpha} \tilde{\phi}(y), \quad \text{for } \alpha = 1, 2, \cdots, n; \quad \pi_0(y) = \frac{n-2}{2} \tilde{\phi}(y) + \nabla \tilde{\phi}(y) \cdot y.
\]
Observe that the function $\pi_0$ is even in each of its variables, namely
\[
\pi_0(y_1, \cdots, y_j, \cdots, y_n) = \pi_0(y_1, \cdots, -y_j, \cdots, y_n), \quad \text{for all } j = 1, 2, \cdots, n,
\]
while $\pi_\alpha$, for $\alpha = 1, 2, \cdots, n$ is odd in the $y_\alpha$ variable, while it is even in all the other variables. Furthermore, all functions $\pi_\alpha$ are invariant under rotation of $\frac{2\pi}{k}$ in the first two coordinates, namely they satisfy (2.4).

Now let us define the functions
\[
Z_0(y) = \frac{n-2}{2} U(y) + \nabla U(y) \cdot y,
\]
and
\[
Z_\alpha(y) = \frac{\partial}{\partial y_\alpha} U(y), \quad \text{for } \alpha = 1, 2, \cdots, n.
\]
We note that, by symmetry and (8.5) in [18], we have
\[
\int_{\mathbb{R}^n} U(y)^{p-1} Z_0(y)^2 dy = \int_{\mathbb{R}^n} U(y)^{p-1} Z_\alpha(y)^2 dy = 2^{\frac{n-4}{2}} n(n-2)^2 \frac{\Gamma(\frac{n}{2})^2}{\Gamma(n+2)} := \tilde{c},
\]
for $\alpha = 1, 2, \cdots, n$.

We have the following results.
Lemma 8.1. Let the functions $z_i$ be defined in (2.15)-(2.19), and $\mu = \mu_k$ be defined in (1.3) and satisfies (1.9). It holds that,

$$
(8.3) \quad \int_{\mathbb{R}^n} |Q(y)|^{p-1} z_i(y)^2 \, dy = (k + 1) \tilde{c} + \begin{cases} O\left(k^{(1-\frac{n}{q})\max\{1, \frac{4}{n-1}\}}\right), & \text{if } n \geq 4, \\ O(|\log k|^{-1}), & \text{if } n = 3,
\end{cases}
$$

for $i = 0, 1, 2, \cdots, 3n - 1$, where $\frac{n}{q} < q < n$ and $\tilde{c}$ is as in (8.2). Moreover, there exists $C > 0$ such that

$$
(8.4) \quad |z_i(y)| \leq C \frac{1}{1 + |y|^{n-2}}, \quad \text{for } y \in \mathbb{R}^n.
$$

Proof. We will give the proof for the case $i = 0$ in (8.3), and the others can be obtained in the same way. Moreover, (8.4) follows directly from the definition of $z_i$ and the results of Proposition 2.1 in [13]. We have

$$
\begin{align*}
\int_{\mathbb{R}^n} |Q(y)|^{p-1} z_0(y)^2 \, dy &= \int_{\mathbb{R}^n} |Q(y)|^{p-1} \left[ \frac{n-2}{2} Q(y) + \nabla Q(y) \cdot y \right]^2 \, dy \\
&= \int_{\mathbb{R}^n} |Q(y)|^{p-1} \left[ \left( \frac{n-2}{2} U(y) + \nabla U(y) \cdot y \right) - \sum_{j=1}^{k} \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right) + \pi_0(y) \right]^2 \, dy \\
&= \int_{\mathbb{R}^n} |Q(y)|^{p-1} \left( \frac{n-2}{2} U(y) + \nabla U(y) \cdot y \right)^2 \, dy \\
&\quad + \sum_{j=1}^{k} \int_{\mathbb{R}^n} |Q(y)|^{p-1} \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right)^2 \, dy \\
&\quad + \int_{\mathbb{R}^n} |Q(y)|^{p-1} |\pi_0(y)|^2 \, dy \\
&\quad + 2 \int_{\mathbb{R}^n} |Q(y)|^{p-1} \left( \frac{n-2}{2} U(y) + \nabla U(y) \cdot y \right) \times \left[ - \sum_{j=1}^{k} \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right) + \pi_0(y) \right] \, dy \\
&\quad + 2 \sum_{i \neq j} \int_{\mathbb{R}^n} |Q(y)|^{p-1} \left( \frac{n-2}{2} U_i(y) + \nabla U_i(y) \cdot y \right) \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right) \, dy \\
&=: A_1 + A_2 + A_3 + A_4 + A_5.
\end{align*}
$$

Next we estimate each term as follows.

Estimate of $A_1$: We have

$$
A_1 = \int_{\mathbb{R}^n} \left| U(y) - \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y) \right|^{p-1} \left( \frac{n-2}{2} U(y) + \nabla U(y) \cdot y \right)^2 \, dy \\
= \int_{\mathbb{R}^n} \left[ |U(y)|^{p-1} + \sum_{j=1}^{k} |U_j(y)|^{p-1} + |\tilde{\phi}(y)|^{p-1} + |U(y)|^{q} \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y)|^{p-1-\gamma} \right] Z_0(y)^2 \, dy
$$
\[ = \int_{\mathbb{R}^n} |U(y)|^{p-1} Z_0(y)^2 dy + \sum_{j=1}^{k} \int_{\mathbb{R}^n} |U_j(y)|^{p-1} Z_0(y)^2 dy \]
\[ + \int_{\mathbb{R}^n} \left[ |\tilde{\phi}(y)|^{p-1} + |U(y)|^{\gamma} \left| \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y) \right|^{p-1-\gamma} \right] Z_0(y)^2 dy. \]

Since
\[ \sum_{j=1}^{k} \int_{\mathbb{R}^n} |U_j(y)|^{p-1} Z_0(y)^2 dy = (n - 2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \int_{\mathbb{R}^n} \frac{\mu_k^2}{(\mu_k^2 + |y - \xi_j|^2)^2} \left( 1 - \frac{|y|^2}{|\mu_k|^2} \right) dy \]
\[ = (n - 2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \mu_k^{n-2} \int_{\mathbb{R}^n} \frac{1}{(1 + |z|^2)^2} \left( 1 - \frac{|\mu_k z + \xi_j|^2}{|\mu_k|^2} \right) dz \]
\[ = (n - 2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \mu_k^{n-2} \int_{|z| \leq \frac{1}{4|\mu_k|}} \frac{1}{(1 + |z|^2)^2} \left( 1 - \frac{|\mu_k z + \xi_j|^2}{|\mu_k|^2} \right) dz \]

and by \((2.7)\) and \((2.12)\), we have
\[ |\tilde{\phi}(y)| \leq \frac{C}{(1 + |y|)^{n-2}} \begin{cases} \kappa^{1-\gamma}, & \text{if } n \geq 4, \\ |\log k|^{-1}, & \text{if } n = 3. \end{cases} \]

Then
\[ \left| \int_{\mathbb{R}^n} \left[ |\tilde{\phi}(y)|^{p-1} + |U(y)|^{\gamma} \left| \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y) \right|^{p-1-\gamma} \right] Z_0(y)^2 dy \right| \]
\[ \leq C \begin{cases} \kappa^{(1-\gamma) \frac{n}{n-2}}, & \text{if } n \geq 4, \\ |\log k|^{-4}, & \text{if } n = 3. \end{cases} \]

Thus
\[ A_1 = \tilde{c} + \begin{cases} O(\kappa^{(1-\gamma) \frac{n}{n-2}}), & \text{if } n \geq 4, \\ O(|\log k|^{-4}), & \text{if } n = 3. \end{cases} \]
Here \( \hat{c} \) is as in (8.2).

**Estimate of \( A_2 \):** We have

\[
A_2 = \int_{\mathbb{R}^n} \left| U(y) - \sum_{j=1}^{k} U_j(y) + \hat{\phi}(y) \right|^{p-1} \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right)^2 \, dy
\]

\[
= \int_{\mathbb{R}^n} \left[ |U(y)|^{p-1} + \sum_{j=1}^{k} |U_j(y)|^{p-1} + |\hat{\phi}(y)|^{p-1} + |U(y)|^\gamma |\sum_{j=1}^{k} U_j(y) + \hat{\phi}(y)|^{p-1-\gamma} \right]
\times \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right)^2 \, dy
\]

\[
= \int_{\mathbb{R}^n} |U(y)|^{p-1} \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right)^2 \, dy
\]

\[
+ \sum_{j=1}^{k} \int_{\mathbb{R}^n} |U_j(y)|^{p-1} \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right)^2 \, dy
\]

\[
+ \int_{\mathbb{R}^n} \left[ |\hat{\phi}(y)|^{p-1} + |U(y)|^\gamma \left| \sum_{j=1}^{k} U_j(y) + \hat{\phi}(y) \right|^{p-1-\gamma} \right] \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right)^2 \, dy.
\]

Since

\[
\int_{\mathbb{R}^n} |U(y)|^{p-1} \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right)^2 \, dy
\]

\[
= (n-2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \mu_j^{n-2} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^2} \frac{\mu_j^2 + |y - \xi_j|^2 - 2 \sum_{i=1}^{n} y_i(y - \xi_j)_i}{(\mu_j^2 + |y - \xi_j|^2)^n} \, dy
\]

\[
= (n-2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \int_{\{z \in \mathbb{R}^n \}} \frac{1}{(1 + |\mu_j z + \xi_j|^2)^2} \frac{1 + |z|^2 - 2 \sum_{i=1}^{n} z_i(z - \xi_j)_i}{(1 + |z|^2)^n} \, dy
\]

\[
= (n-2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \int_{\{z \leq \frac{1}{2\mu_j} \}} \frac{1}{(1 + |\mu_j z + \xi_j|^2)^2} \frac{1 + |z|^2 - 2 \sum_{i=1}^{n} z_i(z - \xi_j)_i}{(1 + |z|^2)^n} \, dy
\]

\[
+ (n-2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \int_{\{z \geq \frac{1}{2\mu_j} \}} \frac{1}{(1 + |\mu_j z + \xi_j|^2)^2} \frac{1 + |z|^2 - 2 \sum_{i=1}^{n} z_i(z - \xi_j)_i}{(1 + |z|^2)^n} \, dy
\]

\[
= O(\mu_j^{-2}) + O \left( \sum_{j=1}^{k} \mu_j^{-4} \int_{\{|z| \geq \frac{1}{2\mu_j} \}} \frac{1 - |z|^2}{|z|^4 (1 + |z|^2)^n} \, dy \right)
\]

\[
= O(\mu_j^{-2}).
\]

and

\[
\sum_{j=1}^{k} \int_{\mathbb{R}^n} |U_j(y)|^{p-1} \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right)^2 \, dy
\]

\[
= (n-2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \mu_j^2 \int_{\mathbb{R}^n} \frac{\mu_j^2 + |y - \xi_j|^2 - 2 \sum_{i=1}^{n} y_i(y - \xi_j)_i}{(\mu_j^2 + |y - \xi_j|^2)^{n+2}} \, dy
\]
\[(n - 2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \int_{\mathbb{R}^n} \frac{1 + |z|^2 - 2 \sum_{i=1}^{n} z_i(z - \xi_i \mu_k)_i}{(1 + |z|^2)^{n+2}} dy \]
\[= (n - 2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \int_{\{|z| \leq \frac{1}{2\mu_k}\}} \frac{1 + |z|^2 - 2 \sum_{i=1}^{n} z_i(z - \xi_i \mu_k)_i}{(1 + |z|^2)^{n+2}} dy \]
\[+ (n - 2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \int_{\{|z| \geq \frac{1}{2\mu_k}\}} \frac{1 + |z|^2 - 2 \sum_{i=1}^{n} z_i(z - \xi_i \mu_k)_i}{(1 + |z|^2)^{n+2}} dy \]
\[= (n - 2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \int_{\mathbb{R}^n} \frac{1 - |z|^2}{(1 + |z|^2)^{n+2}} dy + O(\mu_n^{p+2}) \]
\[= k\tilde{c} + O(\mu_k^{n+2}). \]

Moreover,
\[
\left| \int_{\mathbb{R}^n} \left[ |\tilde{\phi}(y)|^{p-1} + |U(y)|^\gamma \right] \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y)^{p-1-\gamma} \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right)^2 dy \right| \\
\leq C \begin{cases} 
  k^{(1-\frac{n}{q}) \frac{4}{n-2}}, & \text{if } n \geq 4, \\
  |\log k|^{-4}, & \text{if } n = 3.
\end{cases}
\]

Therefore, we obtain
\[
(8.7) \quad A_2 = k\tilde{c} + \begin{cases} 
  O(k^{(1-\frac{n}{q}) \frac{4}{n-2}}), & \text{if } n \geq 4, \\
  O(|\log k|^{-4}), & \text{if } n = 3.
\end{cases}
\]

**Estimate of A_3:** We have
\[
A_3 = \int_{\mathbb{R}^n} |Q(y)|^{p-1} \pi_0(y)^2 dy \\
\leq C \int_{\mathbb{R}^n} |Q(y)|^{p-1} \frac{1}{(1 + |y|)^{2(n-2)}} dy \left\{ O(k^{2(1-\frac{n}{q})}), \text{ if } n \geq 4, \right. \\
\left. O(|\log k|^2), \text{ if } n = 3, \right\} \\
\leq C \begin{cases} 
  k^{2(1-\frac{n}{q})}, & \text{if } n \geq 4, \\
  |\log k|^{-2}, & \text{if } n = 3.
\end{cases} \quad (8.8)
\]

**Estimate of A_4:** We have
\[
(8.9) \quad |A_4| \leq C \begin{cases} 
  k^{1-\frac{n}{q}}, & \text{if } n \geq 4, \\
  |\log k|^{-1}, & \text{if } n = 3.
\end{cases}
\]

**Estimate of A_5:** We have
\[
(8.10) \quad |A_5| \leq C k\mu_k^2.
\]

Thus (8.3) holds for \(i = 0\) follows from (8.5)-(8.10). \(\square\)
Lemma 8.2. Let the functions $z_j$ be defined in (2.15)-(2.19), and $\mu = \mu_k$ be defined in (1.3) and satisfies (1.4). It holds that, for $i \neq j$,

$$
\int_{\mathbb{R}^n} |Q(y)|^{p-1} z_i(y) z_j(y) \, dy = \begin{cases} 
(k+1)\hat{c} + \begin{cases} 
O(k^{1-\frac{q}{4}}), & \text{if } n \geq 4, \\
O(|\log k|^{-1}), & \text{if } n = 3,
\end{cases} & \text{for } i = 1, j = n + 2, \\
(k+1)\hat{c} + \begin{cases} 
O(k^{1-\frac{q}{4}}), & \text{if } n \geq 4, \\
O(|\log k|^{-1}), & \text{if } n = 3,
\end{cases} & \text{for } i = 2, j = n + 3, \\
O(\mu^{\frac{n-2}{2}}), & \text{otherwise},
\end{cases}
$$

(8.11)

where $q \in (\frac{2}{3}, n)$ and $\hat{c}$ is a positive constant, which is defined in (8.2).

Proof. We will only consider the case $i = 1, j = n + 2$, and the case $i = 2, j = n + 3$ can be proved in a similar way. Moreover, we omit the proof for the other cases, which can be obtained easily by using the definition of $z_i$ and the symmetry. We have

$$
\int_{\mathbb{R}^n} |Q(y)|^{p-1} z_1(y) z_{n+2}(y) \, dy 
= \int_{\mathbb{R}^n} |Q(y)|^{p-1} z_1(y) \left( -2y_1 z_0(y) + |y|^2 z_1(y) \right) \, dy 
= -2 \int_{\mathbb{R}^n} |Q(y)|^{p-1} y_1 \frac{\partial Q(y)}{\partial y_1} \left( \frac{n-2}{2} Q(y) + \nabla Q(y) \cdot y \right) \, dy + \int_{\mathbb{R}^n} |Q(y)|^{p-1} |y|^2 \left( \frac{\partial Q(y)}{\partial y_1} \right)^2 \, dy 
=: L_1 + L_2.
$$

Since

$$
L_1 = -2 \int_{\mathbb{R}^n} |Q(y)|^{p-1} y_1 \frac{\partial Q(y)}{\partial y_1} \left( \frac{n-2}{2} Q(y) + \nabla Q(y) \cdot y \right) \, dy 
= -2 \int_{\mathbb{R}^n} \left| U(y) - \sum_{j=1}^k U_j(y) + \tilde{\phi}(y) \right|^{p-1} \left( y_1 \partial_{y_1} U(y) - \sum_{j=1}^k y_1 \partial_{y_1} U_j(y) + y_1 \pi_1(y) \right) \, dy 
\times \left[ \frac{n-2}{2} U(y) + \nabla U(y) \cdot y - \sum_{j=1}^k \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right) + \pi_0(y) \right] \, dy 
= -2 \int_{\mathbb{R}^n} \left[ |U(y)|^{p-1} + \left| \sum_{j=1}^k |U_j(y)|^{p-1} + \left| \tilde{\phi}(y) \right|^{p-1} + |U(y)|^{\gamma} \left| \sum_{j=1}^k U_j(y) + \tilde{\phi}(y) \right|^{p-1-\gamma} \right] \, dy 
\times \left[ y_1 \partial_{y_1} U(y) - \sum_{j=1}^k y_1 \partial_{y_1} U_j(y) + y_1 \pi_1(y) \right] 
\times \left[ \frac{n-2}{2} U(y) + \nabla U(y) \cdot y - \sum_{j=1}^k \left( \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right) + \pi_0(y) \right] \, dy 
= -2 \int_{\mathbb{R}^n} |U(y)|^{p-1} y_1 \partial_{y_1} U(y) \left[ \frac{n-2}{2} U(y) + \nabla U(y) \cdot y \right] \, dy 
- 2 \sum_{j=1}^k \int_{\mathbb{R}^n} |U_j(y)|^{p-1} y_1 \partial_{y_1} U_j(y) \left[ \frac{n-2}{2} U_j(y) + \nabla U_j(y) \cdot y \right] \, dy
$$
Moreover, we find

\[
(n - 2)^2 \alpha_n^{p+1} \int_{\mathbb{R}^n} \frac{y_1^2(1 - |y|^2)}{(1 + |y|^2)^{n+2}} dy
\]

\[= (n - 2)^2 \alpha_n^{p+1} \sum_{j=1}^{k} \mu_k^2 \int_{\mathbb{R}^n} \frac{y_1(y_1 - \xi_{j,1})(\mu_k^2 + |y - \xi_j|^2 - 2 \sum_{i=1}^{n} y_i(y - \xi_j)_i)}{\mu_k^2 + |y - \xi_j|^2} dy
\]

\[+ O(k^{1-\frac{p}{2}}), \quad \text{if } n \geq 4,
\]

\[O(|\log k|^{-1}), \quad \text{if } n = 3.
\]

Thus we find

\[
\sum_{j=1}^{k} \mu_k \int_{\mathbb{R}^n} \frac{y_1(y_1 - \xi_{j,1})(\mu_k^2 + |y - \xi_j|^2 - 2 \sum_{i=1}^{n} y_i(y - \xi_j)_i)}{\mu_k^2 + |y - \xi_j|^2} dy
\]

\[= \sum_{j=1}^{k} \mu_k \int_{\mathbb{R}^n} \frac{(\mu_k z_1 + \xi_{j,1})z_1(1 + |z|^2 - 2 \sum_{i=1}^{n} (\mu_k z_i + \xi_{j,i})z_i)}{(1 + |z|^2)^{n+2}} dz
\]

\[= O\left( \sum_{j=1}^{k} \int_{\{|z| \leq \frac{1}{\mu_k}\}} \frac{z_1^2}{\mu_k^2} \left(1 + |z|^2\right)^{n+2} dz \right) + \sum_{j=1}^{k} \int_{\{|z| \geq \frac{1}{\mu_k}\}} \frac{z_1^2(1 - |z|^2)}{(1 + |z|^2)^{n+2}} dz
\]

\[= k \int_{\mathbb{R}^n} \frac{y_1^2(1 - |y|^2)}{(1 + |y|^2)^{n+2}} dy + O(k\mu_k^{n+2}).
\]

Thus we find

\[L_1 = (k + 1) \int_{\mathbb{R}^n} (n - 2)^2 \alpha_n^{p+1} \frac{y_1^2(1 - |y|^2)}{(1 + |y|^2)^{n+2}} dy + \left\{ \begin{array}{ll}
O(k^{1-\frac{p}{2}}), & \text{if } n \geq 4,
O(|\log k|^{-1}), & \text{if } n = 3.
\end{array} \right.
\]

Moreover

\[L_2 = \int_{\mathbb{R}^n} |Q(y)|^{p-1} |y|^2 \left( \frac{\partial Q(y)}{\partial y_1} \right)^2 dy
\]

\[= \int_{\mathbb{R}^n} \left| U(y) - \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y) \right|^{p-1} |y|^2 \left( \partial_{y_1} U(y) - \sum_{j=1}^{k} \partial_{y_1} U_j(y) + \pi_1(y) \right)^2 dy
\]

\[= \int_{\mathbb{R}^n} \left[ |U(y)|^{p-1} + \sum_{j=1}^{k} |U_j(y)|^{p-1} + |\tilde{\phi}(y)|^{p-1} + |U(y)|^{\gamma} \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y) \right]^{p-1} dy.
\]
Lemma 8.3. For any constant $a > 0$, there exists $C > 0$ such that

\begin{equation}
\int_{\mathbb{R}^n} \frac{1}{|y|^n} \frac{1}{(1 + |z|^2)^{2+a}} \, dz \leq \frac{C}{1 + |y|^a}.
\end{equation}

For any $0 < b < n$, there exists a constant $C > 0$ such that

\begin{equation}
\int_{B(0,1)} \frac{1}{|y - z|^n} \frac{1}{|z|^{n-2}} \, dz \leq \frac{C}{|y|^{n-2-b}}.
\end{equation}

**Proof.** Proof of (8.12). We just need to give the estimate for $|y| \geq 2$. Let $d = \frac{1}{2} |y|$. For $z \in B_d(0)$, we have $|z| \leq d$ and $|y - z| \geq |y| - |z| \geq d$, then

\[
\int_{B_d(0)} \frac{1}{|y - z|^n} \frac{1}{(1 + |z|)^{2+a}} \, dz \leq \frac{C}{d^n} \int_{B_d(0)} \frac{1}{(1 + |z|)^{2+a}} \, dz \\
\leq \frac{C}{d^n} \frac{d^{n-2}}{d^{2+a}} \leq \frac{C}{d^a}.
\]

For $z \in B_d(y)$, we have $|y - z| \leq d$ and $|z| = |y - (y - z)| \geq |y| - |z| \geq d$, we then have

\[
\int_{B_d(y)} \frac{1}{|y - z|^n} \frac{1}{(1 + |z|)^{2+a}} \, dz \leq \frac{C}{d^n} \int_{B_d(y)} \frac{1}{(1 + |z|)^{2+a}} \, dz \\
\leq \frac{C}{d^n} \frac{d^{n-2}}{d^{2+a}} \leq \frac{C}{d^a}.
\]

For $z \in \mathbb{R}^n \setminus (B_d(0) \cup B_d(y))$, we have $|z - y| \geq \frac{1}{2} |y|$ and $|z| \geq \frac{1}{2} |y|$. If $|z| \geq 2 |y|$, we have $|z - y| \geq |z| - |y| \geq \frac{1}{2} |z|$. Thus

\begin{equation}
\frac{1}{|y - z|^n} \frac{1}{(1 + |z|)^{2+a}} \leq \frac{C}{|z|^{n-2}(1 + |z|)^{2+a}}.
\end{equation}
If $|z| \leq 2|y|$, then
\begin{equation}
\frac{1}{|y-z|^{n-2}} \leq \frac{1}{C|y|^{n-2}(1+|z|)^{2+a}} \leq \frac{1}{C' |z|^{n-2}(1+|z|)^{2+a}}.
\end{equation}
From (8.14) and (8.15), we get that for $z \in \mathbb{R}^n \setminus (B_d(0) \cup B_d(y))$,
\begin{equation}
\frac{1}{|y-z|^{n-2}} \leq \frac{1}{C |z|^{n-2}(1+|z|)^{2+a}}.
\end{equation}
Then
\begin{equation}
\int_{\mathbb{R}^n \setminus (B_d(0) \cup B_d(y))} \frac{1}{|y-z|^{n-2}} \frac{1}{(1+|z|)^{2+a}} dz \leq \int_{\mathbb{R}^n \setminus (B_d(0) \cup B_d(y))} C \frac{1}{|z|^{n-2}(1+|z|)^{2+a}} dz \leq C \frac{d^n}{d^b}.
\end{equation}

**Proof of (8.13).** Assume $y = r \hat{y}$, with $r = |y|$. Direct computation gives
\begin{equation}
\int_{B(0,1)} \frac{1}{|y-z|^{n-2}} \frac{1}{|z|^{n-b}} dz = \frac{1}{r^{n-2} \int_{B(0,1)} \frac{1}{|y-r \hat{y}|^{n-2}} \frac{1}{|z|^{n-b}} dz}
\end{equation}
\begin{equation}
= \frac{1}{r^{n-2-b} \int_{B(0,1)} \frac{1}{|y-w|^{n-2}} \frac{1}{|w|^{n-b}} dz}
\end{equation}
\begin{equation}
\leq \frac{C}{|y|^{n-2-b}},
\end{equation}
where
\begin{equation}
C = \sup_{\epsilon \in S^n} \int_{\mathbb{R}^n} \frac{1}{|\epsilon - w|^{n-2}} \frac{1}{|w|^{n-b}} dz.
\end{equation}
\[\square\]

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