A MACKEY-FUNCTOR THEORETIC INTERPRETATION OF BISET FUNCTORS

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Abstract. In this article, we consider a formulation of biset functors using the 2-category of finite sets with variable finite group actions. We introduce a 2-category $\mathcal{S}$, on which a biset functor can be regarded as a special kind of Mackey functors. This gives an analog of Dress’ definition of a Mackey functor, in the context of biset functors.

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1. Introduction and Preliminaries

For a fixed finite group $G$ and a commutative coefficient ring $R$, an $R$-linear Mackey functor can be defined in three ways, which give essentially the same notion ([4]):

- Naive definition, which defines a Mackey functor $M$ as a family $M = (M(H), \text{ind}^G_H, \text{res}^G_H, c_g, H)$ of $R$-modules and homomorphisms.
- Bifunctorial definition by Dress, which defines $M$ as a pair of functors $M = (M^*, M_*)$ on the category $G\text{set}$ of finite $G$-sets to the category of $R$-modules $R\text{Mod}$, satisfying some compatibilities with respect to coproducts and fibered products in $G\text{set}$.
- The way shown by Lindner [6], which regards $M$ as a functor $T \rightarrow R\text{Mod}$ preserving finite products, from a category $T$ constructed from the span category of $G\text{set}$, to $R\text{Mod}$.

A Mackey functor is a useful tool to describe how an algebraic system associated to finite groups (such as Burnside rings or representation rings, or cohomology groups, etc.) behaves under the change of subgroups of a fixed group $G$. Moreover if one expects to be freed from the constraint of the container group $G$, we may consider all finite groups, and inclusions among them. This leads to the notion of a global Mackey functor.

Recently, Bouc [3] has defined the notion of a biset functor, which moreover enables us to deal with the behavior of algebraic systems named as above, with respect to all group homomorphisms between all finite groups. An $R$-linear biset functor $B$ is defined to be an $R$-linear functor $B: B_R \rightarrow R\text{Mod}$, from the biset category $B_R$ to $R\text{Mod}$. The biset category which we deal with in this article is the following one.

**Definition.** An $R$-linear category $B_R$ is defined as follows.

(1) An object in $B_R$ is a finite group.

(2) For objects $G, H$ in $B_R$, consider a set of the isomorphism classes of finite $H$-$G$-bisets. This forms a commutative monoid with addition $\sqcup$ and unit $\emptyset$, and thus we can take its additive completion $B(G, H)$. We define $B_R(G, H)$ by $B_R(G, H) = B(G, H) \otimes R$. This is the set of morphisms from $G$ to $H$ in $B_R$.

An $H$-$G$-biset $U$ is written as $HU_G$. The composition of two consecutive bisets $HU_G$ and $KV_H$ is given by

$$V \times_H U = (V \times U)/\sim,$$

where the equivalence relation is defined as

- $(v, u), (v', u') \in V \times U$ are equivalent if there exists $h \in H$ satisfying $v = v'h$ and $u' = hu$.

This defines the composition of morphisms in $B_R$, by linearity.

When $R = \mathbb{Z}$, we denote $B_{\mathbb{Z}}$ simply by $B$. This is a preadditive category.

We denote the category of $R$-linear biset functors by $\text{Biset}^{R}$. This is naturally equivalent to the category $\text{Add}(B, R\text{Mod})$ of additive functors from $B$ to $R\text{Mod}$.

Remark that an $H$-$G$-biset $U$ is identified with an $H \times G$-set, with the action

$$(h, g)u = hug^{-1} \quad (\forall(h, g) \in H \times G, \forall u \in U).$$
If $U$ is transitive as an $H \times G$-set, then it can be decomposed as follows [3]:

$$U \cong \text{Ind}_C^H \times \text{Inf}_{C/D}^G \times \text{Iso}(f) \times \text{Def}_B^G \times \text{Res}_{B/A}^G,$$

using a sequence of inclusions, quotients, and an isomorphism of groups

(1.1) \[ H \leftarrow C \rightarrow C/D \xrightarrow{f} B/A \leftarrow B \rightarrow G. \]

Thus a biset functor is regarded as a family of $R$-modules $\{B(G)\}_{G \in \text{Ob}(\text{Grp})}$ equipped with operations associated to elementary bisets ([3])

- $\text{Ind}_C^H = cG_H$, for a subgroup $H \leq G$,
- $\text{Res}_H^G = hG_G$, for a subgroup $H \leq G$,
- $\text{Inf}_N^G = c(G/N)(G/N)$, for a normal subgroup $N \triangleleft G$,
- $\text{Def}_N^G = (G/N)(G/N)_G$, for a normal subgroup $N \triangleleft G$,
- $\text{Iso}(f) = hH_G$, for a group isomorphism $f: G \xrightarrow{\sim} H$.

These operations satisfy some fundamental relations ([3]), together with an extra relation corresponding to

(1.2) \[ \text{Def}_N^G \times \text{Inf}_N^G \cong \text{Id} \]

for any normal subgroup $N \triangleleft G$.

Remark that the sequence (1.1) can be flipped up by taking fibered product to obtain a sequence

which can be regarded as a span of group homomorphisms

$$H \xleftarrow{\varphi} F \xrightarrow{\psi} G$$

up to some isomorphism. Since it is not always possible to ‘flip down’ conversely, spans of group homomorphisms are treating a bit wider class than bisets. In analogy with the ‘three definitions’ of Mackey functors, this observation gives us an impression that a biset functor is defined by some compound of ‘naive’ and ‘Lindner-type’ definitions. In this article, as a platform for further developments of biset functor theory, we introduce an analog of Dress’ definition for biset functors.

The central mechanism for the Dress’ definition of Mackey functors was that, for a fixed finite group $G$, a $G$-set can be regarded as a parallel array of subgroups of $G$ by taking stabilizers. A $G$-map then corresponds to a parallel array of inclusions of subgroups. In the case of biset functors, it will be natural to prepare a category which can encode all finite groups and all homomorphisms not only the inclusions of subgroups.

To realize this, we define a category $\mathcal{C}$ whose object is a pair $(G, X)$ of a finite group $G$ and a finite $G$-set $X$. By taking stabilizers, an object in $\mathcal{C}$ can be regarded as a parallel array of finite groups. Moreover, with an appropriate definition of morphisms, we can regard a morphism in $\mathcal{C}$ as a parallel array of group
homomorphisms between them, classified up to some conjugates. If one could show \( \mathcal{C} \) admits fibered products and coproducts, then it would be possible to find some analog of Dress’ definition.

However, it soon turns out that \( \mathcal{C} \) does not have strict fibered products, and that it is more natural to use a 2-categorical framework. Thus we introduce a 2-category \( \mathcal{S} \) whose classifying category is equal to \( \mathcal{C} \). We will show \( \mathcal{S} \) admits 2-coproducts and 2-fibered products, and thus we can define a ‘Mackey functor’ on \( \mathcal{S} \). Because of the ‘flipping-gap’ between spans and bisets, biset functors correspond to some special kind of Mackey functors on \( \mathcal{S} \), which we call deflative Mackey functors. A Mackey functor is called deflative if it satisfies a condition corresponding to (1.2). As a main theorem (Theorem 6.31), we establish an equivalence between the category of deflative Mackey functors on \( \mathcal{S} \) and the category of biset functors.

Throughout this article, any group \( G \) is assumed to be finite. The category of finite groups and homomorphisms is denoted by \( \text{Grp} \). The unit of a (finite) group will be denoted by \( e \). Abbreviately we denote the trivial group by \( e \), instead of \( \{e\} \).

For an element \( g \) in a group \( G \) and its subgroup \( H \leq G \), we denote the conjugation map by \( \sigma_g : H \to gHg^{-1} ; x \mapsto gxg^{-1} \). For a group \( G \), the symbol \( G\text{-set} \) denotes the category of finite \( G \)-sets and \( G \)-equivariant maps. A one-point set is denoted by \( 1 \). Abbreviately, any map from any set \( X \) to \( 1 \) is denoted by \( 1 : X \to 1 \).

In this article, a biset is always assumed to be finite. A monoid is always assumed to be unitary and commutative. Similarly a ring is assumed to be commutative, with an additive unit 0 and a multiplicative unit 1. We denote the category of monoids by \( \text{Mon} \), the category of rings by \( \text{Ring} \). A monoid homomorphism preserves units, and a ring homomorphism preserves 1. For any category \( \mathcal{K} \) and any pair of objects \( X \) and \( Y \) in \( \mathcal{K} \), the set of morphisms from \( X \to Y \) in \( \mathcal{K} \) is denoted by \( \mathcal{K}(X,Y) \).

Any 2-category is assumed to be strict ([1], [7]). For a 2-category \( \mathcal{C} \), the entity of 0-cells (respectively 1-cells, 2-cells) is denoted by \( \mathcal{C}_0 \) (resp. \( \mathcal{C}_1 \), \( \mathcal{C}_2 \)). For a pair of 0-cells \( X, Y \) in \( \mathcal{C} \), the set of 1-cells from \( X \) to \( Y \) in \( \mathcal{C} \) is denoted by \( \mathcal{C}(X,Y) \).

2. THE 2-CATEGORY OF FINITE SETS WITH GROUP ACTIONS

In this article, we work on (2-)categories whose objects are finite sets equipped with group actions. First, we introduce a naive one.

2.1. Category \( \text{GrSet} \).

**Definition 2.1.** The category \( \text{GrSet} \) is defined as follows.

1. An object in \( \text{GrSet} \) is a pair of a finite group \( G \) and a finite \( G \)-set \( X \). We denote this pair by \( X^G \).
2. If \( X^G \) and \( Y^H \) are two objects in \( \text{GrSet} \), then a morphism \( \theta : X^G \to Y^H \) is a pair of a map \( \theta : X \to Y \) and a map \( \theta : X \to \text{Map}(G,H) ; x \mapsto \theta_x \) (namely, \( \theta \) is a family of maps \( \{\theta_x\}_{x \in X} \) satisfying
   (i) \( \alpha(gx) = \theta_x(g)\alpha(x) \)
   (ii) \( \theta_x(gg') = \theta_{g'}(g)\theta_x(g') \)
   for any \( x \in X \) and any \( g, g' \in G \).

\footnote{This notation is thanks to Professor Serge Bouc.}
\( \theta \) is called the \textit{acting part} or the \textit{denominator} of \( \frac{\alpha}{\beta} \).

For any consecutive pair of morphisms
\[ X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\tau} K, \]
we define their composition \( (\beta \circ \tau) \circ (\alpha \circ \theta) = \beta \circ \alpha \circ \tau \circ \theta \) by
\[ \beta \circ (\alpha \circ \tau) \circ \theta : X \rightarrow Z \]
is the usual composition of maps of sets,
\( \tau \circ \theta \) is defined by
\[ (\tau \circ \theta)_{x}(g) = \tau_{\alpha(x)}(\theta_{x}(g)) \quad (\forall g \in G), \]
for any \( x \in X \) and any \( g \in G \).

**Remark 2.2.** In GrSet, the object \( \frac{\emptyset}{G} \) is terminal. Besides, for any finite group \( G \), the object \( \frac{\emptyset}{G} \) is initial in GrSet. In particular, there is a (unique) isomorphism
\[ \frac{\emptyset}{G} \cong \frac{\emptyset}{H} \]
for any pair of finite groups \( G \) and \( H \). We will often denote this isomorphic initial object simply by \( \frac{\emptyset}{\cdot} \in \text{Ob}(\text{GrSet}) \).

The definition of a morphism \( \frac{\alpha}{\beta} : \frac{X}{G} \rightarrow \frac{Y}{H} \) in GrSet can be also described in terms of a map \( G \times X \rightarrow H \times Y \) using only commutativity of diagrams, as follows: In the following remark (Remark 2.3), for each object \( \frac{X}{G} \),
- \( \mu : G \times G \rightarrow G \) denotes the multiplication of the group,
- \( \text{ac} : G \times X \rightarrow X \) denotes the group action,
- \( p_{2} : G \times X \rightarrow X \) denotes the projection onto the 2nd component,
- \( p_{23} : G \times G \times X \rightarrow G \times X \) denotes the projection onto the \((2,3)\)-components.

Also remark that
\[ G \times G \times X \xrightarrow{p_{23}} G \times X \]
(2.1)
gives a fibered product of sets.

**Remark 2.3.** To give a morphism \( \frac{X}{G} \rightarrow \frac{Y}{H} \) in GrSet is equivalent to give a map
\[ T : G \times X \rightarrow H \times Y \]
which satisfies the following conditions.

\[ \begin{array}{c}
\text{The author wishes to thank Professor Ergün Yalçın for making him aware of this.} \\
\end{array} \]
(0) \( T \) satisfies \( T(\{e\} \times X) \subseteq \{e\} \times Y \). Namely, there exists a (unique) map \( \alpha : X \to Y \) which makes the following diagram commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\downarrow & & \downarrow \\
\{e\} \times X & \check{\cap} & \{e\} \times Y \\
\downarrow & & \downarrow \\
G \times X & \xrightarrow{T} & H \times Y \\
\end{array}
\]

(1) \( T \) makes the following diagram commutative.

\[
\begin{array}{ccc}
X & \xleftarrow{ac} & G \times X \\
\downarrow \alpha & & \downarrow p_2 \\
Y & \xleftarrow{ac} & H \times Y \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \xleftarrow{ac} & G \times X \\
\downarrow \alpha & & \downarrow p_2 \\
Y & \xleftarrow{ac} & H \times Y \\
\end{array}
\]

From this, it immediately follows that the diagram

\[
\begin{array}{ccc}
G \times G \times X & \xrightarrow{T \circ p_2} & H \times Y \\
\downarrow T \circ (G \times ac) & & \downarrow ac \\
H \times Y & \xrightarrow{p_2} & Y \\
\end{array}
\]

becomes commutative. Hence, by the universality of the fibered product \([2.11]\) for \( \underline{Y} \), there exists a unique map \( \tilde{T} : G \times G \times X \to H \times H \times Y \) which makes the following diagram commutative.

\[
\begin{array}{ccc}
G \times G \times X & \xleftarrow{T \circ p_2} & G \times X \\
\downarrow T \circ (G \times ac) & & \downarrow ac \\
H \times Y & \xrightarrow{p_2} & Y \\
\end{array}
\]

(2) \( \tilde{T} \) makes the following diagram commutative.

\[
\begin{array}{ccc}
G \times G \times X & \xrightarrow{\mu \times X} & G \times X \\
\downarrow \tilde{T} & & \downarrow T \\
H \times H \times Y & \xrightarrow{\mu \times Y} & H \times Y \\
\end{array}
\]

Thus \( \alpha, T, \tilde{T} \) fit into the following commutative diagram.

\[
\begin{array}{ccc}
G \times G \times X & \xrightarrow{\mu \times X} & G \times X \\
\downarrow \tilde{T} & & \downarrow T \\
H \times H \times Y & \xrightarrow{\mu \times Y} & H \times Y \\
\end{array}
\]

\[
\begin{array}{ccc}
G \times X & \xrightarrow{ac} & X \\
\downarrow \alpha & & \downarrow p_2 \\
H \times Y & \xrightarrow{ac} & Y \\
\end{array}
\]

\[
\begin{array}{ccc}
G \times X & \xrightarrow{p_2} & G \times ac \\
\downarrow G \times ac & & \downarrow p_2 \\
H \times Y & \xrightarrow{p_2} & H \times ac \\
\end{array}
\]

\[
\begin{array}{ccc}
G \times X & \xrightarrow{ac} & X \\
\downarrow \alpha & & \downarrow p_2 \\
H \times Y & \xrightarrow{ac} & Y \\
\end{array}
\]
**Definition 2.4.** Let \( f : G \to H \) be a group homomorphism. A morphism \( \frac{\alpha}{f} : \frac{X}{G} \to \frac{Y}{H} \) is \( f\)-equivariant if it satisfies \( \theta_x = f \) for any \( x \in X \). (Remark that, condition (2) (ii) in Definition 2.1 is automatically satisfied.) In this case, we simply write the morphism as \( \frac{\alpha}{f} \).

When \( f = \text{id}_G \), we say the morphism is equivariant, or \( G\)-equivariant if we specify the group \( G \), and denote it by \( \frac{\alpha}{G} : \frac{X}{G} \to \frac{Y}{G} \). In this case, \( \alpha \) is nothing but a usual \( G \)-map \( \alpha : X \to Y \).

**Remark 2.5.**

1. The identity morphism for \( \frac{X}{Y} \) in GrSet is given by \( \text{id}_{\frac{X}{Y}} = \frac{\text{id}_X}{G} \).
2. We sometimes express a morphism \( \frac{\alpha}{Y} \) simply by \( \alpha : \frac{X}{Y} \to \frac{Y}{Y} \). This abbreviation does not mean that \( \theta \) is determined by \( \alpha \). For example, if \( X = Y = 1 \) and \( \alpha = 1 : 1 \to 1 \), then \( \frac{\alpha}{G} : \frac{1}{G} \to \frac{1}{Y} \) becomes a morphism in GrSet for any group homomorphism \( f : G \to H \). (See Proposition 2.8.)
3. If \( \frac{\alpha}{G} : \frac{X}{G} \to \frac{Y}{G} \) is a morphism, then for each \( G \)-orbit \( X_0 \subseteq X \), its image \( \alpha(X_0) \) is contained in some single \( H \)-orbit in \( Y \).
4. If \( \frac{\alpha}{G} : \frac{X}{G} \to \frac{Y}{G} \) is a morphism in GrSet, then for any \( x \in X \), the restriction of \( \theta_x : G \to H \) to the stabilizer \( G_x \) gives a group homomorphism

\[ \theta_x : G_x \to H_{\alpha(x)}. \]

In particular, we always have \( \theta_x(e) = e \) for any \( x \in X \).

5. If \( \frac{\alpha}{G} : \frac{X}{G} \to \frac{Y}{H} \xrightarrow{\tau} \frac{Z}{H} \) is a sequence of morphisms, then we have

\[
(\mu \circ \tau \circ \theta_x)(g) = (\mu \circ \tau)(\alpha(x))(\theta_x(g)) = \mu_{\beta(\alpha(x))}(\tau_{\alpha(x)}(\theta_x(g))) = \mu_{\beta(\alpha(x))}(\tau \circ \theta)_x(g)
\]

for any \( x \in X \) and \( g \in G \). This shows the associativity of the composition.

**Remark 2.6.** Let \( \frac{\alpha}{G} : \frac{X}{G} \to \frac{Y}{H} \) be a morphism in GrSet. Remark that for \( x \in X \), the \( G \)-orbit \( Gx \) is isomorphic to \( G/G_x \) as a \( G \)-set by

\[ Gx \xrightarrow{\cong} G/G_x ; gx \mapsto gG_x. \]

Similarly for \( \alpha(x) \in Y \), we have an isomorphism of \( H \)-sets

\[ H\alpha(x) \xrightarrow{\cong} H/H_{\alpha(x)} ; h\alpha(x) \mapsto hH_{\alpha(x)}. \]

Then \( \theta_x \) gives a map

\[ \Theta_{\alpha,x} : G/G_x \to H/H_{\alpha(x)} : gG_x \mapsto \theta_x(g)H_{\alpha(x)}, \]

which is compatible with \( \alpha \) and the above isomorphisms:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\text{U|} & & \text{U|} \\
Gx & \text{G/Gx} & \xrightarrow{\Theta_{\alpha,x}} H/H_{\alpha(x)} \\
\cong & & \cong \\
\text{H} & & \text{H}
\end{array}
\]

**Proposition 2.7.** Let \( G \) be any finite group. The following correspondence gives a faithful (but not full) functor

\[ \bullet : G\text{-set} \to \text{GrSet}. \]
To any \( X \in \text{Ob}(\mathcal{G} \text{-set}) \), we associate \( X \in \text{Ob}(\text{GrSet}) \).

To any \( \alpha \in \mathcal{G} \text{-set}(X,Y) \), we associate \( \alpha \in \text{GrSet}(\frac{X}{\mathcal{G}}, \frac{Y}{\mathcal{G}}) \).

Proof. This is straightforward. \( \square \)

**Proposition 2.8.** The following correspondence gives a fully faithful functor

\[
\frac{1}{\cdot} : \text{Grp} \rightarrow \text{GrSet}.
\]

- To any \( G \in \text{Ob}(\text{Grp}) \), we associate \( \frac{1}{G} \in \text{Ob}(\text{GrSet}) \).
- To any \( f \in \text{Grp}(G,H) \), we associate \( \frac{1}{f} \in \text{GrSet}(\frac{1}{G}, \frac{1}{H}) \).

Proof. This is straightforward. \( \square \)

What we really need is a category \( \mathcal{C} \) obtained by modifying \( \text{GrSet} \). Indeed, we will define \( \mathcal{C} \) to be a category satisfying \( \text{Ob}(\mathcal{C}) = \text{Ob}(\text{GrSet}) \), whose morphisms are equivalence classes of morphisms in \( \text{GrSet} \) with respect to an equivalence relation defined later (Definition 2.19).

In order to make this construction works well, we use a formalism of 2-categories.

### 2.2. 2-category \( \mathcal{S} \) and category \( \mathcal{C} \)

We add a class of 2-cells to \( \text{GrSet} \), so as to make it into a 2-category \( \mathcal{S} \). With this view, from now on we regard an object in \( \text{GrSet} \) as a 0-cell in \( \mathcal{S} \), and a morphism in \( \text{GrSet} \) as a 1-cell in \( \mathcal{S} \).

**Definition 2.9.** Let \( \frac{\alpha}{\theta} \frac{\alpha'}{\theta'} : \frac{X}{\mathcal{G}} \rightarrow \frac{Y}{\mathcal{H}} \) be any pair of 1-cells. A 2-cell \( \varepsilon : \frac{\alpha}{\theta} \Rightarrow \frac{\alpha'}{\theta'} \) is a map

\[
\varepsilon : X \rightarrow H ; \ x \mapsto \varepsilon_x
\]

satisfying

(i) \( \alpha'(x) = \varepsilon_x \alpha(x) \),

(ii) \( \varepsilon_{gx}\theta_x(g)\varepsilon_x^{-1} = \theta'_x(g) \)

for any \( x \in X \) and \( g \in G \).

If we are given a consecutive pair of 2-cells

\[
\frac{\alpha}{\theta} \frac{\alpha'}{\theta'} \circ \frac{\alpha''}{\theta''}
\]

then their vertical composition \( \varepsilon' \cdot \varepsilon : \frac{\alpha}{\theta} \Rightarrow \frac{\alpha''}{\theta''} \) is defined by

\( (\varepsilon' \cdot \varepsilon)_x = \varepsilon'_x \varepsilon_x \) \quad (\forall x \in X). \]

This becomes indeed a 2-cell, since we have

\[
\alpha''(x) = \varepsilon'_x \alpha(x) = \varepsilon'_x \varepsilon_x \alpha(x),
\]

\[
\varepsilon'_{gx}\varepsilon_x\theta_x(g)\varepsilon_x^{-1}\varepsilon_x^{-1} = \varepsilon'_{gx}\varepsilon'_x(g)\varepsilon_x^{-1} = \theta'_x(g)
\]

for any \( x \in X \) and \( g \in G \).

Associativity of this vertical composition is trivially satisfied. The identity 2-cell \( \text{id} : \frac{\text{id}}{\theta} \Rightarrow \frac{\text{id}}{\theta} \) is given by \( \text{id}_x = e \) \quad (\forall x \in X).
Remark 2.10. In the above definition, if $\frac{X}{G} = \emptyset$ and $\frac{\alpha}{\theta} = \frac{\alpha'}{\theta'}$ is the unique morphism $\emptyset \to \frac{Y}{H}$, then the 2-cell between them is also unique, which is regarded as the identity 2-cell.

Remark 2.11. For any 2-cell $\varepsilon: \frac{\alpha}{\theta} \Rightarrow \frac{\alpha'}{\theta'}$ (as in the notation in Definition 2.9), we have the following.

1. $\varepsilon$ is invertible with respect to the vertical composition. Indeed, its inverse $\varepsilon^{-1}: \frac{\alpha'}{\theta'} \Rightarrow \frac{\alpha}{\theta}$ is given by
   $$(\varepsilon^{-1})_x = \varepsilon^{-1}_x \in H \quad (\forall x \in X)$$
   where $\varepsilon^{-1}_x$ is the inverse element of $\varepsilon_x$ in $H$.

2. $\varepsilon$ preserves orbits. Namely, for any $G$-orbit $X_0 \subseteq X$, its images $\alpha(X_0)$ and $\alpha'(X_0)$ are contained in the same $H$-orbit in $Y$.

3. For any $x \in X$, the group homomorphisms
   $$\theta_x: G_x \to H_{\alpha(x)},$$
   $$\theta'_x: G_x \to H_{\alpha'(x)}$$

obtained in Remark 2.5 are related by the conjugation by $\varepsilon_x \in H$. In fact we have the following commutative diagram of group homomorphisms.

$$\begin{array}{ccc}
G_x & \xrightarrow{\theta_x} & H_{\alpha(x)} \\
\downarrow & & \downarrow \\
H_{\alpha'(x)} & \xleftarrow{\sigma_{\varepsilon_x}} & G_x
\end{array}$$

Definition 2.12. Let $\frac{X}{G} \xrightarrow{\beta} \frac{Y}{H} \xrightarrow{\delta} \frac{Z}{K}$ be a sequence of 1-cells.

1. For a 2-cell
   $$\varepsilon: \frac{\alpha}{\theta} \Rightarrow \frac{\alpha'}{\theta'}$$
   define $(\beta \circ \varepsilon): (\beta) \circ (\frac{\alpha}{\theta}) \Rightarrow (\beta \circ \frac{\alpha'}{\theta'})$ by
   $$(\beta \circ \varepsilon)_x = (\beta_x) \circ (\alpha_x^{-1} \varepsilon_x) \quad (\forall x \in X).$$

2. For a 2-cell
   $$\delta: \frac{\alpha}{\theta} \Rightarrow \frac{\alpha'}{\theta'}$$
   define $\delta \circ (\frac{\alpha}{\theta}) : (\delta) \circ (\frac{\alpha}{\theta}) \Rightarrow (\delta \circ \frac{\alpha'}{\theta'})$ by
   $$(\delta \circ (\frac{\alpha}{\theta}))_x = \delta_{\alpha(x)} \quad (\forall x \in X).$$
Remark 2.13. By the same abbreviation as in Remark 2.5, we abbreviate \((\frac{\delta}{\theta}) \circ \varepsilon\) and \(\delta \circ (\frac{\delta}{\theta})\) to \(\beta \circ \varepsilon\) and \(\delta \circ \alpha\). Thus equations (2.2), (2.3) are written as

\[
(\beta \circ \varepsilon)_x = \tau_{\alpha(x)}(\varepsilon_x), \quad (\delta \circ \alpha)_x = \delta_{\alpha(x)} \quad (\forall x \in X).
\]

Claim 2.14. In the notation in Definition 2.12, the following holds.

1. \(\beta \circ \varepsilon; \beta \circ \alpha \Rightarrow \beta \circ \alpha'\) is in fact a 2-cell.
2. \(\delta \circ \alpha; \beta \circ \alpha \Rightarrow \beta' \circ \alpha\) is in fact a 2-cell.

Proof. (1) For any \(x \in X\), we have

\[
(\beta \circ \alpha')(x) = \beta(\varepsilon_x \alpha(x)) = \tau_{\alpha(x)}(\varepsilon_x) \cdot (\beta \circ \alpha(x)) = (\beta \circ \varepsilon)_x \cdot (\beta \circ \alpha(x)).
\]

Remark that the equation

\[
e = \tau_{\alpha'(x)}(e) = \tau_{\alpha'(x)}(\varepsilon_x \varepsilon_x^{-1}) = \tau_{\alpha'(x)}(\varepsilon_x) \cdot \tau_{\alpha'(x)}(\varepsilon_x^{-1}) = \tau_{\alpha(x)}(\varepsilon_x) \cdot \tau_{\alpha'(x)}(\varepsilon_x^{-1})
\]

implies

\[
(2.4) \quad \tau_{\alpha(x)}(\varepsilon_x)^{-1} = \tau_{\alpha'(x)}(\varepsilon_x^{-1}).
\]

Thus we have

\[
(\beta \circ \varepsilon)_{g_x} \cdot (\tau \circ \theta)_x(g) \cdot (\beta \circ \varepsilon)^{-1} = \tau_{\alpha(g_x)}(\varepsilon_{g_x}) \cdot \tau_{\alpha(x)}(\theta_x(g)) \cdot \tau_{\alpha(x)}(\varepsilon_x)^{-1} = \tau_{\alpha(g_x)}(\varepsilon_{g_x}) \cdot \tau_{\alpha(x)}(\theta_x(g)) \cdot \tau_{\alpha(x)}(\varepsilon_x)^{-1} = \tau_{\alpha(x)}(\varepsilon_{g_x} \theta_x(g)) \cdot \tau_{\alpha'(x)}(\varepsilon_x^{-1}) = \tau_{\alpha'(x)}(\varepsilon_{g_x} \theta_x(g) \varepsilon_x^{-1}) = \tau_{\alpha'(x)}(\theta'_x(g)) = (\tau' \circ \theta')_x(g)
\]

for any \(x \in X\) and \(g \in G\).

(2) This follows from

\[
\beta' \circ \alpha(x) = \delta_{\alpha(x)} \cdot (\beta \circ \alpha(x)) \quad (\forall x \in X)
\]

and

\[
\delta_{\alpha(g_x)} \cdot (\tau \circ \theta)_x(g) \cdot \delta_{\alpha(x)}^{-1} = \delta_{\alpha(g_x)}(\theta_x(g)) \cdot \delta_{\alpha(x)}(\theta_x(g)) \cdot \delta_{\alpha(x)}^{-1} = \tau'_{\alpha(x)}(\theta_x(g)) = (\tau' \circ \theta)_x(g) \quad (\forall x \in X, \forall g \in G).
\]

To show that the category GrSet together with these 2-cells forms a 2-category \(\mathcal{S}\), it remains to show the following.

Proposition 2.15. For any diagram

\[
(2.5) \quad \begin{array}{c}
\alpha' \\
\downarrow \\
\beta \\
\downarrow \\
\gamma \\
\downarrow \\
\delta \\
\downarrow \\
\eta \\
\end{array} 
\]

Diagram (2.5)
where \( \varepsilon \) and \( \tau \) are 2-cells, we have
\[
(\delta \circ \alpha') \cdot (\beta \circ \varepsilon) = (\beta' \circ \varepsilon) \cdot (\delta \circ \alpha).
\]
Namely, the following diagram of 2-cells is commutative.
\[
\begin{array}{ccc}
\beta \circ \alpha & \xrightarrow{\beta \circ \varepsilon} & \beta \circ \alpha' \\
\delta \circ \alpha & \Downarrow & \delta \circ \alpha' \\
\beta' \circ \alpha & \xrightarrow{\beta' \circ \varepsilon} & \beta' \circ \alpha'
\end{array}
\]

Proof. Since \( \delta : \frac{\beta}{\tau} \Rightarrow \frac{\beta'}{\tau'} \) is a 2-cell, it satisfies
\[
\delta : h \tau y \Rightarrow \tau' y (h) = \tau' y (\delta y)
\]
for any \( y \in Y \) and \( h \in H \). Thus we obtain
\[
(\delta \circ \alpha') x \cdot (\beta \circ \varepsilon) x = \delta_{\alpha'(x)} \tau_{\alpha(x)} (\varepsilon_x) = \delta_{\varepsilon, \alpha(x)} \tau_{\alpha(x)} (\varepsilon_x) = \tau'_{\alpha(x)} (\varepsilon_x) \delta_{\alpha(x)} = (\beta' \circ \varepsilon) x \cdot (\delta \circ \alpha) x
\]
for any \( x \in X \). \( \square \)

By Proposition 2.15, we define horizontal composition \( \delta \circ \varepsilon \) of 2-cells \( \delta \) and \( \varepsilon \) (as in diagram (2.5)) by
\[
(\delta \circ \varepsilon)_x = \delta_{\alpha'(x)} \tau_{\alpha(x)} (\varepsilon_x) = \tau'_{\alpha(x)} (\varepsilon_x) \delta_{\alpha(x)} \quad (\forall x \in X).
\]

The arguments so far allow us the following definition.

**Definition 2.16.** 2-category \( S \) is defined as follows.

1. \( S^0 = \text{Ob}(\text{GrSet}) \).
2. For any 0-cells \( \frac{X}{G} \) and \( \frac{Y}{H} \),
   \[
   S^1 \left( \frac{X}{G}, \frac{Y}{H} \right) = \text{GrSet} \left( \frac{X}{G}, \frac{Y}{H} \right).
   \]
3. For any 1-cells \( \frac{\alpha}{G}, \frac{\alpha'}{G} : \frac{X}{G} \rightarrow \frac{Y}{H} \), 2-cells \( \varepsilon : \frac{\alpha}{G} \Rightarrow \frac{\alpha'}{G} \) are those defined in Definition 2.9. Thus any 2-cell in \( S \) is invertible with respect to the vertical composition.

**Definition 2.17.** Let \( \alpha : \frac{X}{G} \rightarrow \frac{Y}{H} \) be a 1-cell.

1. \( \alpha \) is an **equivalence** if there is a 1-cell \( \beta : \frac{Y}{H} \rightarrow \frac{X}{G} \) and 2-cells
   \[
   \rho : \beta \circ \alpha \Rightarrow \text{id}_{\frac{X}{G}}, \quad \lambda : \alpha \circ \beta \Rightarrow \text{id}_{\frac{Y}{H}}.
   \]
   \( \beta \) is called a **quasi-inverse** of \( \alpha \).
2. \( \alpha \) is an **adjoint equivalence** if there is a 1-cell \( \beta \) and 2-cells \( \rho, \lambda \) as above, which moreover satisfy
   \[
   \alpha \circ \rho = \lambda \circ \alpha, \quad \rho \circ \beta = \beta \circ \lambda
   \]
   in the diagram

\[
\begin{array}{ccc}
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H} \\
\downarrow{\rho} & \searrow{\beta} & \downarrow{\rho} \\
\frac{X}{G} & \xleftarrow{\beta} & \frac{Y}{H} \\
\downarrow{\rho} & \swarrow{\alpha} & \downarrow{\rho} \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H}
\end{array}
\]
(3) $\alpha$ is an isomorphism if there is a 1-cell $\beta: \frac{Y}{H} \to \frac{X}{G}$ which satisfies

$$\beta \circ \alpha = \text{id}_{\frac{X}{G}}, \quad \alpha \circ \beta = \text{id}_{\frac{Y}{H}}.$$  

This is equivalent to that $\alpha$ is an isomorphism in $\text{GrSet}$.

Remark that there are implications

(isomorphism) \Rightarrow (adjoint equivalence) \Rightarrow (equivalence).

Remark 2.18.

(1) A $G$-equivariant 1-cell $\frac{\alpha}{G}: \frac{X}{G} \to \frac{Y}{G}$ is an isomorphism if $\alpha: X \xrightarrow{\cong} Y$ is an isomorphism in $G\text{-set}$. (See also Corollary 4.8.)

(2) For a 1-cell $\frac{f}{1}: \frac{1}{G} \to \frac{1}{H}$, the following are equivalent.

(i) $f$ is an isomorphism of groups.

(ii) $\frac{f}{1}$ is an isomorphism in $S$.

(iii) $\frac{f}{1}$ is an adjoint equivalence in $S$.

(iv) $\frac{f}{1}$ is an equivalence in $S$.

Proof. (1) follows from Proposition 2.7.

(2) (i)$\Rightarrow$(ii) follows from Proposition 2.8. It remains to show (iv)$\Rightarrow$(i). Suppose $\frac{f}{1}: \frac{1}{G} \to \frac{1}{H}$ has a quasi-inverse $\frac{1}{q}: \frac{1}{H} \to \frac{1}{G}$. By the existence of 2-cells $\frac{\varepsilon}{1}: \frac{\alpha}{\theta} \Rightarrow \frac{\alpha'}{\theta'}$ and $\frac{1}{f} \circ \frac{1}{q} \Rightarrow \text{id}$, there are $g \in G$ and $h \in H$ which satisfy

$$\sigma_g \circ q \circ f = \text{id}_G, \quad \sigma_h \circ f \circ q = \text{id}_H.$$  

This imply

$$\sigma_g \circ q \circ \sigma_h^{-1} = (\sigma_g \circ q \circ \sigma_h^{-1} \circ (\sigma_h \circ f \circ q)) \circ q = q,$$

namely $\sigma_g \circ q = q \circ \sigma_h$. If we put $r = \sigma_g \circ q$, then

$$r \circ f = \sigma_g \circ q \circ f = \text{id}_G,$$

$$f \circ r = f \circ q \circ \sigma_h = \text{id}_H$$

holds, which means $f: G \to H$ is a group isomorphism.

\[\square\]

Category $\mathcal{C}$ is defined to be the classifying category of $S$, as follows.

**Definition 2.19.** Category $\mathcal{C}$ is defined as follows.

(i) $\text{Ob}(\mathcal{C}) = S^0 = \text{Ob}(\text{GrSet})$.

(ii) For any pair of objects $\frac{X}{G}, \frac{Y}{H} \in \text{Ob}(\mathcal{C})$, we define an equivalence relation on $S^1(\frac{X}{G}, \frac{Y}{H})$ as follows.

- 1-cells $\frac{\alpha}{\theta}, \frac{\alpha'}{\theta'} \in S^1(\frac{X}{G}, \frac{Y}{H})$ are equivalent if there exists some 2-cell $\varepsilon: \frac{\alpha}{\theta} \Rightarrow \frac{\alpha'}{\theta'}$.

The set of morphisms $\mathcal{C}(\frac{X}{G}, \frac{Y}{H})$ is defined to be the quotient of $S^1(\frac{X}{G}, \frac{Y}{H})$ by this equivalence:

$$\mathcal{C}(\frac{X}{G}, \frac{Y}{H}) = S^1(\frac{X}{G}, \frac{Y}{H}) / \text{2-cells}.$$

The equivalence class of $\frac{\alpha}{\theta}$ is denoted by $[\frac{\alpha}{\theta}]$, or simply by $\frac{\alpha}{\theta}$.
A general argument on 2-categories (with invertible 2-cells) shows that $\mathcal{C}$ becomes in fact a category. The composition of two consecutive morphisms

$$
\left(\frac{\alpha}{\theta}\right): \frac{X}{G} \to \frac{Y}{H} \quad \text{and} \quad \left(\frac{\beta}{\tau}\right): \frac{Y}{H} \to \frac{Z}{K}
$$

is given by

$$
\left(\frac{\beta}{\tau}\right) \circ \left(\frac{\alpha}{\theta}\right) = \left(\frac{\beta \circ \alpha}{\tau \circ \theta}\right).
$$

Identity morphism for $\frac{X}{G} \in \text{Ob}(\mathcal{C})$ is given by $\left(\frac{id_X}{id_G}\right) = \left(\frac{id_X}{id_G}\right)$.

**Remark 2.20.**

1. A 1-cell $\frac{\eta}{\vartheta}: \frac{X}{G} \to \frac{Y}{H}$ is an equivalence in $\mathbb{S}$ if and only if $\left(\frac{\eta}{\vartheta}\right)$ is an isomorphism in $\mathcal{C}$.
2. There is a natural functor $\text{GrSet} \to \mathcal{C}$ which sends $\frac{\alpha}{\vartheta}: \frac{X}{G} \to \frac{Y}{H}$ in $\text{GrSet}$ to $\left(\frac{\alpha}{\vartheta}\right): \frac{X}{G} \to \frac{Y}{H}$ in $\mathcal{C}$.

3. **First properties of $\mathbb{S}$ and $\mathcal{C}$**

In this section, we investigate first categorical properties satisfied by $\mathbb{S}$ and $\mathcal{C}$.

### 3.1. Ind-equivalence.

**Definition 3.1.** Let $\iota: H \hookrightarrow G$ be a monomorphism of groups. For any $X \in \text{Ob}(Hset)$, we define $\text{Ind}, X \in \text{Ob}(Gset)$ by

$$
\text{Ind}, X = (G \times X)/\sim,
$$

where the equivalence relation $\sim$ is defined by

- $(\xi, x)$ and $(\xi', x')$ in $G \times X$ are equivalent if there exists $h \in H$ satisfying 

$$
x' = hx, \quad \xi = \xi' \iota(h).
$$

We denote the equivalence class of $(\xi, x)$ by $[\xi, x] \in \text{Ind}, X$. The $G$-action on $\text{Ind}, X$ is defined by

$$
g[\xi, x] = [g\xi, x]
$$

for any $g \in G$ and $[\xi, x] \in \text{Ind}, X$.

**Proposition 3.2.** Let $\iota: H \hookrightarrow G$ be a monomorphism of groups. For any $X \in \text{Ob}(Hset)$, if we define a map $v: X \to \text{Ind}, X$ by

$$
v(x) = [e, x] \quad (\forall x \in X),
$$

then the 1-cell

$$
\frac{v}{\iota}: \frac{X}{H} \to \frac{\text{Ind}, X}{G}
$$

becomes an adjoint equivalence.
Claim 3.3.

\[ \xi, x \]

Claim 3.3. It can be easily checked that \( \xi \) is in fact a 1-cell. We construct a quasi-inverse of \( \upsilon \). Take a coset decomposition of \( G \) by \( \upsilon(H) \)

\[ G = g_1\upsilon(H) \Pi \cdots \Pi g_s\upsilon(H) \]

with \( g_1, \ldots, g_s \in G \), satisfying \( g_1 = e \). Then for any \( g \in G \), there uniquely exist \( 1 \leq i \leq s \) and \( h \in H \) satisfying \( g = g_i\upsilon(h) \). We denote these by

\[ a(g) = g_i, \quad b(g) = h \]

for each \( g \in G \). This gives maps \( a : G \rightarrow G \) and \( b : G \rightarrow H \), which satisfy

\[
\begin{align*}
  g &= a(g)\upsilon(b(g)), \quad b(g\upsilon(h)) = b(g)h, \\
  a(g\upsilon(h)) &= a(g), \quad b(a(g)) = e
\end{align*}
\]

for any \( g \in G \) and \( h \in H \).

We define \( \# : \frac{\text{Ind}_t X}{\upsilon} \rightarrow \frac{\text{Ind}_t X}{\upsilon} \) by

\[
\alpha([\xi, x]) = b(\xi)x \quad (\forall [\xi, x] \in \text{Ind}_t X), \\
\theta_{[\xi, x]}(g) = b(g\xi) \cdot b(\xi)^{-1} \quad (\forall [\xi, x] \in \text{Ind}_t X, \forall g \in G).
\]

It can be easily checked that \( \alpha([\xi, x]) \) and \( \theta_{[\xi, x]}(g) \) are well-defined, independently from the choice of a representative of \([\xi, x]\).

It suffices to show the following.

Claim 3.3.

1. \( \# : \frac{\text{Ind}_t X}{\upsilon} \rightarrow \frac{\text{Ind}_t X}{\upsilon} \) is a 1-cell.
2. \( \left( \frac{\#}{\upsilon} \right) \circ \left( \frac{\#}{\upsilon} \right) = \text{id}_{\frac{\text{Ind}_t X}{\upsilon}} \).
3. There exists a 2-cell \( \varepsilon : \left( \frac{\#}{\upsilon} \right) \circ \left( \frac{\#}{\upsilon} \right) \Rightarrow \text{id}_{\frac{\text{Ind}_t X}{\upsilon}} \).
4. \( \varepsilon \circ \upsilon = \text{id} \) and \( \alpha \circ \upsilon = \text{id} \) hold in the following diagram.

Proof. (1) For any \([\xi, x] \in \text{Ind}_t X\) and \( g, g' \in G \), we have

\[
\begin{align*}
  \alpha(g[\xi, x]) &= \alpha([g\xi, x]) = b(g\xi)x \\
  &= b(g\xi)b(\xi)^{-1}b(\xi)x = \theta_{[\xi, x]}(g) \cdot \alpha([\xi, x]), \\
  \theta_{[\xi, x]}(gg') &= b(gg'\xi)b(\xi)^{-1} \\
  &= b(gg'\xi)b(g'\xi)^{-1} \cdot b(g'\xi)b(\xi)^{-1} \\
  &= \theta_{[g'\xi, x]}(g) \cdot \theta_{[\xi, x]}(g').
\end{align*}
\]

(2) For any \( x \in X \) and \( h \in H \), we have

\[ \alpha \circ \upsilon(x) = \alpha([e, x]) = x, \]

\[ (\theta \circ \upsilon)(h) = \theta_{[e, x]}(\upsilon(h)) = b(\upsilon(h)) \cdot b(e)^{-1} = h. \]

(3) If we define \( \varepsilon \) by

\[ \varepsilon_{[\xi, x]} = a(\xi) \quad (\forall [\xi, x] \in \text{Ind}_t X), \]
then we have
\[
\epsilon_{[\xi,x]} \cdot (\nu \circ \alpha([\xi,x])) = a(\xi)[e,b(\xi)x] = [a(\xi)\iota(b(\xi)),x] = [\xi,x],
\]
\[
\epsilon_g[\xi,x] \cdot (\iota \circ \theta([\xi,x])(g) \cdot \epsilon_{[\xi,x]}^{-1}) = a(g\xi) \cdot \iota(b(g\xi)) \cdot \iota(b(\xi))^{-1} a(\xi)^{-1} \cdot (g\xi)\xi^{-1} = g
\]
for any \([\xi,x] \in \text{Ind}_X\) and \(g \in G\). Thus \(\epsilon\) gives a 2-cell \(\epsilon : (\nu \iota) \circ (\frac{\alpha}{\theta}) \Rightarrow \text{id}_{\text{Ind}_X}G\).

(4) For any \([\xi,x] \in \text{Ind}_X\), we have
\[
(\epsilon \circ \nu)_x = \epsilon_{[e,x]} = a(e) = e.
\]
For any \([\xi,x] \in \text{Ind}_X\), we have
\[
(\alpha \circ \epsilon)_{[\xi,x]} = \theta_{[e,b(\xi)x]}(\epsilon_{[\xi,x]}) = \theta_{[e,b(\xi)x]}(a(\xi)) = b(a(\xi)) = e.
\]
□

Remark 3.4. The adjoint equivalence in Proposition 3.2 can be thought of as “reduction of the fraction”: For any sequence of subgroups \(K \leq H \leq G\), we have adjoint equivalences
\[
\frac{(G/K)}{G} \simeq \frac{(H/K)}{H} \simeq \frac{(K/K)}{K} = 1_K.
\]

Corollary 3.5. Let \(G\) be a finite group and let \(X\) be a transitive finite \(G\)-set. Then there exists a finite group \(H\) and an adjoint equivalence
\[
1_H \simeq \frac{X}{G}.
\]
This \(H\) is unique up to group isomorphism.

Proof. For any \(x \in X\), existence of an adjoint equivalence \(\frac{1}{H} \simeq \frac{X}{G}\) follows from Remark 3.3. Uniqueness follows from Remark 2.18. □

Corollary 3.6. Under the same assumption as in Proposition 3.3
\[
(\nu) : \frac{X}{H} \to \text{Ind}_X G
\]
gives an isomorphism in \(\mathcal{C}\).

3.2. 2-coproducts and 2-fibered products in \(\mathcal{S}\). From now on, to avoid lack of Greek letters, we usually denote the acting part of 1-cell \(\alpha\) by \(\theta_\alpha\). Thus the abbreviated expression like “Let \(\alpha : \frac{X}{G} \to \frac{Y}{H}\) be a 1-cell” will mean that a family of maps \(\theta_\alpha = \{\theta_{\alpha,x} : G \to H\}_{x \in X}\) is implicitly given as a part of the defining datum for this 1-cell.

First we recall the definition of 2-coproducts, 2-products and 2-fibered products.

Definition 3.7. Let \(\mathcal{C}\) be a 2-category with invertible 2-cells. For any \(A_1\) and \(A_2\) in \(\mathcal{C}^0\), their \(2\)-\text{coproduct} \((A_1 \amalg A_2, \iota_1, \iota_2)\) is defined to be a triplet of \(A_1 \amalg A_2 \in \mathcal{C}^0\) and
\[
\iota_1 \in \mathcal{C}^1(A_1, A_1 \amalg A_2), \quad \iota_2 \in \mathcal{C}^1(A_2, A_1 \amalg A_2),
\]
satisfying the following conditions.
(i) For any \( X \in \mathbb{C}^0 \) and \( f_i \in \mathbb{C}^1(A_i, X) \) \((i = 1, 2)\), there exist \( f \in \mathbb{C}^1(A_1 \amalg A_2, X) \) and \( \xi_i \in \mathbb{C}^2(f \circ \iota_i, f_i) \) \((i = 1, 2)\) as in the following diagram.

![Diagram](image)

(ii) For any triplets \((f, \xi_1, \xi_2)\) and \((f', \xi'_1, \xi'_2)\) as in (i), there exists a unique 2-cell \( \eta \in \mathbb{C}^2(f, f') \) such that \( \xi'_i \cdot (\eta \circ \iota_i) = \xi_i \) \((i = 1, 2)\), namely, the following diagram of 2-cells is commutative.

![Diagram](image)

**Remark 3.8.**

(1) Since 2-cells are invertible, condition (ii) is only have to be checked for a fixed \((f, \xi_1, \xi_2)\). Namely, it is equivalent to the following.
- For some fixed triplet \((f, \xi_1, \xi_2)\), for any triplet \((f', \xi'_1, \xi'_2)\) there exists a unique 2-cell \( \eta \in \mathbb{C}^2(f, f') \) such that \( \xi'_i \cdot (\eta \circ \iota_i) = \xi_i \) \((i = 1, 2)\).

(2) By its universality, 2-coproduct is unique up to adjoint equivalences.

(3) If there are adjoint equivalences \( \eta_1 : A_1 \xrightarrow{\simeq} A'_1 \) and \( \eta_2 : A_2 \xrightarrow{\simeq} A'_2 \), then an adjoint equivalence \( A_1 \amalg A_2 \xrightarrow{\simeq} A'_1 \amalg A'_2 \) is obtained. In fact, if

\[
\begin{array}{c}
A'_1 \xrightarrow{\iota'_1} A'_1 \amalg A'_2 \xrightarrow{\iota'_2} A'_2
\end{array}
\]

is a 2-coproduct, then

\[
\begin{array}{c}
A_1 \xrightarrow{\iota_1 \circ \eta_1} A'_1 \amalg A'_2 \xrightarrow{\iota_2 \circ \eta_2} A_2
\end{array}
\]

gives a 2-coproduct.

**Remark 3.9.** 2-product is defined dually, by reversing the directions of 1-cells. Remark that the directions of 2-cells do not matter, since they are invertible.

**Definition 3.10.** Let \( \mathbb{C} \) be a 2-category with invertible 2-cells. For any \( A_1, A_2, B \in \mathbb{C}^0 \) and \( f_i \in \mathbb{C}^1(A_i, B) \) \((i = 1, 2)\), 2-fibered product of \( f_1 \) and \( f_2 \) is defined to be a quartet \((A_1 \times_B A_2, \pi_1, \pi_2, \kappa)\) as in the diagram

![Diagram](image)

which satisfies the following conditions.
(i) For any diagram in $C$

\[
\begin{array}{c}
X \xrightarrow{g_2} A_2 \\
g_1 \downarrow \Rightarrow \downarrow f_2, \\
A_1 \xrightarrow{f_1} B
\end{array}
\]

there exist $g, \xi_1, \xi_2$ as in the diagram

\[
\begin{array}{c}
X \xrightarrow{g} A_1 \times_B A_2 \xrightarrow{\pi_2} A_2 \\
g_1 \downarrow \Rightarrow \downarrow f_2, \\
\xi_1 \xrightarrow{\kappa} \xi_2 \\
A_1 \xrightarrow{f_1} B
\end{array}
\]

satisfying $\varepsilon \cdot (f_1 \circ \xi_1) = (f_2 \circ \xi_2) \cdot (\kappa \circ g)$, namely making the following diagram of 2-cells commutative.

\[
\begin{array}{c}
f_1 \circ \pi_1 \circ g \xrightarrow{\kappa \circ g} f_2 \circ \pi_2 \circ g \\
f_1 \circ \xi_1 \downarrow \Rightarrow \downarrow f_2 \circ \xi_2 \\
f_1 \circ g_1 \xrightarrow{\kappa \circ g_2}
\end{array}
\]

(ii) For any triplets $(g, \xi_1, \xi_2)$ and $(g', \xi_1', \xi_2')$ as in (i), there exists a unique 2-cell $\zeta \in C^2(g, g')$ which satisfies $\xi_i' \cdot (\pi_i \circ \zeta) = \xi_i (i = 1, 2)$.

**Remark 3.11.** Similar properties as in Remark 3.8 are also satisfied by 2-fibered products.

$S$ admits 2-coproducts, as follows.

**Proposition 3.12.** Let $G$ be any finite group. For any $X, Y \in \text{Ob}(G\text{-set})$, let $X \amalg Y \in \text{Ob}(G\text{-set})$ be the usual coproduct of $G$-sets. If we denote the inclusions by $\nu_X : X \hookrightarrow X \amalg Y$, $\nu_Y : Y \hookrightarrow X \amalg Y$,

then

\[
\begin{array}{c}
\xrightarrow{\nu_X} \xrightarrow{\nu_Y}
\end{array}
\]

gives a 2-coproduct of $\frac{X}{G}$ and $\frac{Y}{G}$ in $S$.

**Proof.** We confirm conditions (i) and (ii) in Definition 3.7.

(i) Suppose we are given 1-cells

\[
\alpha : \frac{X}{G} \to \frac{W}{L} \quad \text{and} \quad \beta : \frac{Y}{G} \to \frac{W}{L}
\]

to some 0-cell $\frac{W}{L}$. If we take the usual union of maps

$\alpha \cup \beta : X \amalg Y \to W$

and the disjoint union of families

$\theta_{\alpha \cup \beta} = \theta_\alpha \amalg \theta_\beta = \{\theta_{\alpha,x}\}_{x \in X} \amalg \{\theta_{\beta,y}\}_{y \in Y}$,
then it can be easily shown that \( \alpha \cup \beta : X \cup Y \to W_L \) becomes a 1-cell which makes the following diagram commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{v_X} & X \cup Y \\
\downarrow^{\alpha} & & \downarrow^{\alpha \cup \beta} \\
W & \xrightarrow{\cup} & Y \\
\end{array}
\]

(ii) Suppose there also exist a 1-cell \( \gamma : X \cup Y \to W_L \) and 2-cells \( \lambda : \gamma \circ v_X \Rightarrow \alpha \), \( \rho : \gamma \circ v_Y \Rightarrow \beta \) as in

\[
\begin{array}{ccc}
X & \xrightarrow{v_X} & X \cup Y \\
\downarrow^{\alpha} & & \downarrow^{\alpha \cup \beta} \\
W & \xrightarrow{\cup} & Y \\
\end{array}
\]

Then the family of maps

\[
\lambda \Pi \rho = \{ \lambda_x \}_{x \in X} \Pi \{ \rho_y \}_{y \in Y}
\]

gives a 2-cell \( \lambda \Pi \rho : \gamma \Rightarrow \alpha \cup \beta \), which makes the following diagrams of 2-cells commutative.

\[
\begin{array}{ccc}
\gamma \circ v_X & \xrightarrow{\lambda \Pi \rho \circ v_X} & (\alpha \cup \beta) \circ v_X \\
\downarrow^{\lambda} & & \downarrow^{\alpha} \\
W & \xrightarrow{\cup} & Y \\
\end{array}
\]

\[
\begin{array}{ccc}
\gamma \circ v_Y & \xrightarrow{\lambda \Pi \rho \circ v_Y} & (\alpha \cup \beta) \circ v_Y \\
\downarrow^{\rho} & & \downarrow^{\beta} \\
W & \xrightarrow{\cup} & Y \\
\end{array}
\]

Uniqueness of such a 2-cell can be checked immediately.

Corollary 3.13. Under the same assumption as in Proposition 3.12,

\[
\begin{array}{ccc}
X & \xrightarrow{v_X} & X \Pi Y \\
\downarrow^{\alpha} & & \downarrow^{\alpha \Pi \beta} \\
W & \xrightarrow{\Pi} & Y \\
\end{array}
\]

gives a coproduct of \( \frac{X}{G} \) and \( \frac{Y}{H} \) in \( \mathcal{C} \).

Proposition 3.14. Let \( \frac{X}{G} \) and \( \frac{Y}{H} \) be any pair of 0-cells in \( S \). Denote the monomorphisms

\[
\begin{array}{ccc}
G & \to & G \times H \\
\downarrow & & \downarrow \\
H & \to & G \times H
\end{array}
\]

given by \( \iota^{(G)} \) and \( \iota^{(H)} \) respectively, and denote the natural maps

\[
\begin{array}{ccc}
X & \to & \text{Ind}_i(G)X \Pi \text{Ind}_i(H)Y \\
x & \mapsto & [e,x] \in \text{Ind}_i(G)X \\
Y & \to & \text{Ind}_i(G)X \Pi \text{Ind}_i(H)Y \\
y & \mapsto & [e,y] \in \text{Ind}_i(H)Y
\end{array}
\]

by \( v_X \) and \( v_Y \). Then

\[
\begin{array}{ccc}
X & \xrightarrow{\iota^{(G)}} & \text{Ind}_i(G)X \Pi \text{Ind}_i(H)Y \\
\downarrow & & \downarrow \\
G \times H & \xrightarrow{\iota^{(H)}} & Y \\
\end{array}
\]

gives a 2-coproduct of \( \frac{X}{G} \) and \( \frac{Y}{H} \) in \( S \).

Proof. This immediately follows from Proposition 3.12 and 3.12.

□
Corollary 3.15. Under the same assumption as in Proposition 3.14,
\[
\begin{array}{c}
\xrightarrow{\nu_X} \xrightarrow{\nu_Y} \\
\xrightarrow{\text{Ind}_G} \xleftarrow{\text{Ind}_H} \\
\xrightarrow{\nu_{Y/H}} \xleftarrow{\nu_{X/G}}
\end{array}
\]
gives a coproduct of \( \frac{X}{G} \) and \( \frac{Y}{H} \) in \( \mathcal{C} \).

\( \mathcal{S} \) admits 2-fibered products, as follows.

Proposition 3.16. Let \( \alpha: \frac{X}{G} \rightarrow \frac{Z}{K} \) and \( \beta: \frac{Y}{H} \rightarrow \frac{Z}{K} \) be any pair of 1-cells in \( \mathcal{S} \).

Denote the natural projection homomorphisms by
\[
\text{pr}(G): G \times H \rightarrow G, \quad \text{pr}(H): G \times H \rightarrow H.
\]

If we
- put \( F = \{(x, y, k) \in X \times Y \times K \mid \beta(y) = k\alpha(x)\} \), and put
\[
\varphi_X: F \rightarrow X; (x, y, k) \mapsto x,
\]
\[
\varphi_Y: F \rightarrow Y; (x, y, k) \mapsto y,
\]
- equip \( F \) with a \( G \times H \)-action
\[
(g, h)(x, y, k) = (gx, hy, \theta_{\beta,y}(h)k\theta_{\alpha,x}(g)^{-1})
\]
\[(\forall (g, h) \in G \times H, \forall (x, y, k) \in F),
\]
- define a 2-cell \( \kappa: \alpha \circ \varphi_X \Rightarrow \beta \circ \varphi_Y \) by
\[
\kappa(x, y, k) = k,
\]
then the diagram
\[
\begin{array}{ccc}
F & \xrightarrow{\varphi_X} & X/G \\
\downarrow{\varphi_Y} & & \downarrow{\nu_X} \\
Y/H & \xrightarrow{\nu_{Y/H}} & Z/K
\end{array}
\]
gives a 2-fibered product in \( \mathcal{S} \).

Proof. For any \( f = (x, y, k) \in F \) and \( (g, h) \in G \times H \), we have
\[
\varphi_X((g, h)f) = \varphi_X(gx, hy, \theta_{\beta,y}(h)k\theta_{\alpha,x}(g)^{-1})
\]
\[= gx = g\varphi_X(f),
\]
\[
\varphi_Y((g, h)f) = hy = h\varphi_Y(f),
\]
\[
\beta \circ \varphi_Y(f) = \beta(y) = k\alpha(x) = \kappa_f \cdot (\alpha \circ \varphi_X(f)),
\]
\[
\kappa_{(g,h)f} \cdot (\theta_{\alpha} \circ \text{pr}(G)((g, h))) \cdot \kappa_f^{-1}
\]
\[= \theta_{\beta,y}(h)k\theta_{\alpha,x}(g)^{-1} \cdot \theta_{\alpha,x}(g) \cdot k^{-1}
\]
\[= \theta_{\beta,y}(h) = (\theta_{\beta} \circ \text{pr}(H))((g, h)),
\]
which mean that \( \varphi_X, \varphi_Y \) are 1-cells, and \( \kappa \) is a 2-cell.

We confirm conditions (i), (ii) in Definition 3.10.

(i) Suppose we are given a diagram
\[
\begin{array}{ccc}
W & \xrightarrow{\delta} & Y/H \\
\downarrow{\gamma} & & \downarrow{\beta} \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Z}{K}
\end{array}
\]
Moreover, we have
\[
\gamma(\ell w) = \theta_{\gamma, w}(\ell) \gamma(w) \quad \text{and} \quad \delta(\ell w) = \theta_{\delta, w}(\ell) \delta(w),
\]
for any \(w \in W\) and \(\ell, \ell' \in L\). If we define
\[
\gamma_* \delta : W \to F \quad \text{and} \quad \theta_{\gamma_* \delta, w} = \{\theta_{\gamma_* \delta, w} : L \to G \times H\}_{w \in W}
\]
by
\[
(\gamma_* \delta)(w) = (\gamma(w), \delta(w), \varepsilon_w),
\]
\[
\theta_{\gamma_* \delta, w} = (\theta_{\gamma, w}, \theta_{\delta, w}) : L \to G \times H
\]
for any \(w \in W\), then \(\theta_{\gamma_* \delta, w} : W \to \mathcal{H} \to \mathcal{F} \to L \to G \times H \) becomes a 1-cell. Indeed, for any \(w \in W\) and \(\ell, \ell' \in L\), we have
\[
(\gamma_* \delta)(\ell w) = (\gamma(\ell w), \delta(\ell w), \varepsilon_{\ell w})
\]
\[
= (\theta_{\gamma, w}(\ell) \gamma(w), \theta_{\delta, w}(\ell) \delta(w), \theta_{\gamma_* \delta, w}(\ell) \cdot \varepsilon_w \cdot \theta_{\alpha, \gamma(w)}(\theta_{\gamma, w}(\ell))^{-1})
\]
\[
= (\theta_{\gamma, w}(\ell), \theta_{\delta, w}(\ell)) \cdot (\gamma(w), \delta(w), \varepsilon_w)
\]
\[
\theta_{\gamma_* \delta, w}(\ell \ell') = (\theta_{\gamma_* \delta, w}(\ell), \theta_{\gamma_* \delta, w}(\ell'))
\]
\[
= (\theta_{\gamma, w}(\ell) \theta_{\gamma, w}(\ell'), \theta_{\delta, w}(\ell) \theta_{\delta, w}(\ell'))
\]
Moreover, we have
\[
(\frac{\varphi_X}{\varphi_Y} \circ \frac{\gamma_* \delta}{\gamma_* \delta}) = \frac{\gamma}{\delta},
\]
and the diagram
satisfies
\[
(\kappa \circ (\gamma_* \delta))_w = \kappa(\gamma(w), \delta(w), \varepsilon_w) = \varepsilon_w \quad (\forall w \in W),
\]
which means the commutativity of the following diagram.
\[
\begin{array}{c}
\alpha \circ \varphi_X \circ (\gamma_* \delta) \\
\kappa \circ (\gamma_* \delta) \\
\end{array}
\]
\[
\begin{array}{c}
\beta \circ \varphi_Y \circ (\gamma_* \delta) \\
\end{array}
\]
(ii) Suppose that the diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\phi} & \mathcal{R} \\
\downarrow{\delta} & & \downarrow{\beta} \\
\mathcal{A} & \xrightarrow{\alpha} & \mathcal{B}
\end{array}
\]

also makes the following diagram commutative.

\[
\begin{array}{ccc}
\alpha \circ \mathcal{X} \circ \phi & \xrightarrow{\kappa \circ \phi} & \beta \circ \mathcal{Y} \circ \phi \\
\alpha \circ \gamma & \xrightarrow{\epsilon} & \beta \circ \delta
\end{array}
\]

Express \( \phi \) with its components by

\[\phi(w) = (x_w, y_w, k_w) \in F.\]

By definition, these satisfy

\[\gamma(w) = \mu_w x_w, \quad \delta(w) = \nu_w y_w,\]
\[\varepsilon(w) \circ \theta_{\alpha, x_w} (\mu_w) = \theta_{\beta, y_w} (\nu_w) \cdot k_w,\]
\[\theta_{\gamma, w}(\ell) = \mu_{\ell w} \cdot \text{pr}(G)(\theta_{\phi, w}(\ell)) \cdot \mu_w^{-1},\]
\[\theta_{\delta, w}(\ell) = \nu_{\ell w} \cdot \text{pr}(H)(\theta_{\phi, w}(\ell)) \cdot \nu_w^{-1}\]

for any \( w \in W \) and \( \ell \in L \).

Remark that a 2-cell \( \zeta: \phi \Rightarrow \gamma \circ \delta \), if it exists, makes

\[
\begin{array}{ccc}
\mathcal{X} \circ \phi & \xrightarrow{\mu} & \mathcal{Y} \circ \phi \\
\mathcal{X} \circ (\gamma \circ \delta) & \xrightarrow{\gamma} & \mathcal{Y} \circ (\gamma \circ \delta) \\
\mathcal{X} \circ \phi & \xrightarrow{\nu} & \mathcal{Y} \circ \phi
\end{array}
\]

commutative if and only if

\[
\begin{array}{ccc}
L \xrightarrow{\mu_w} & \mathcal{A} & \xrightarrow{\zeta_w} \\
G \times H & \xrightarrow{\text{pr}(G)} & G
\end{array}
\]

and

\[
\begin{array}{ccc}
L \xrightarrow{\nu_w} & \mathcal{B} & \xrightarrow{\zeta_w} \\
G \times H & \xrightarrow{\text{pr}(H)} & H
\end{array}
\]

are commutative for each \( w \in W \). Thus there is no other choice than

\[\zeta_w = (\mu_w, \nu_w): L \to G \times H.\]
It remains to show that this \( \zeta = \{(\mu_w, \nu_w)\}_{w \in W} \) in fact forms a 2-cell \( \zeta : \phi \Rightarrow \gamma \). However, this follows from

\[
(\gamma \ast \delta)(w) = (\gamma(w), \delta(w), \varepsilon_w) = (\mu_w x_w, \nu_w y_w, \theta_{\beta, y_w}(\nu_w) \cdot k_w \cdot \theta_{\alpha, x_w}(\mu_w)^{-1}) = (\mu_w, \nu_w) \cdot (x_w, y_w, k_w) = \zeta_w \cdot \phi(w) \quad (\forall w \in W)
\]

and

\[
\zeta_{\ell w} \cdot \theta_{\phi, w}(\ell) \cdot \zeta_{w}^{-1} = (\mu_{\ell w}, \nu_{\ell w}) \cdot (\text{pr}(G)(\theta_{\phi, w}(\ell)), \text{pr}(H)(\theta_{\phi, w}(\ell))) \cdot (\mu_w, \nu_w)^{-1} = (\theta_{\gamma, w}(\ell), \theta_{\delta, w}(\ell)) = \theta_{\gamma \ast \delta, w}(\ell) \quad (\forall w \in W, \forall \ell \in L).
\]

\[\square\]

**Corollary 3.17.** A 2-product of 0-cells \( \overline{X}^G, \overline{Y}^H \) in \( \mathcal{S} \) is given by

\[
\begin{array}{ccc}
X & \xrightarrow{\phi_X} & X \times Y \\
G & \xrightarrow{\text{pr}(G)} & G \times H \\
\downarrow & \downarrow \circ \downarrow & \downarrow \beta \\
\overline{X}^G & \xrightarrow{\alpha} & \overline{Y}^H \\
\end{array}
\]

where \( \phi_X : X \times Y \rightarrow X \), \( \phi_Y : X \times Y \rightarrow Y \) are the projections.

**Proof.** If we take \( \overline{Z}^K = \frac{1}{\ell} \) in Proposition 3.16 then we obtain a 2-product of \( \overline{X}^G \) and \( \overline{Y}^H \). In this case, we have a natural identification of \( G \times H \)-sets \( F = X \times Y \). \[\square\]

**Corollary 3.18.** In the notation of Proposition 3.16, if the \( K \)-orbits generated by \( \alpha(X) \) and \( \beta(Y) \) in \( Z \) are disjoint, namely if

\[ K\alpha(X) \cap K\beta(Y) = \emptyset \]

holds as a subset of \( Z \), then the 2-fibered product is given by

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{\beta} & \overline{Y}^H \\
\downarrow & \downarrow \circ \downarrow & \downarrow \beta \\
\overline{X}^G & \xrightarrow{\alpha} & \overline{Y}^H \\
\end{array}
\]

**Proof.** This immediately follows from Proposition 3.16. \[\square\]

**Caution 3.19.** Proposition 3.16 does not mean

\[
\begin{array}{ccc}
F & \xrightarrow{\phi_Y} & Y \\
G \times H & \xrightarrow{\text{pr}(H)} & H \\
\downarrow & \downarrow \circ \downarrow & \downarrow \beta \\
\overline{X}^G & \xrightarrow{\alpha} & \overline{Y}^H \\
\end{array}
\]

is a fibered product in \( \mathcal{C} \). In fact, this is only a weak fibered product.

Nevertheless by Remark 3.11 these weak fibered products which come from 2-fibered products are closed under isomorphisms in \( \mathcal{C} \), and thus form a natural distinguished class in the whole weak fibered products.
Definition 3.20. A weak fibered product in \( \mathcal{C} \)

\[
\begin{array}{ccc}
W & \xrightarrow{\delta} & Y \\
\Downarrow^\gamma & \circ & \Downarrow^\beta \\
X \otimes L & \xrightarrow{\delta} & Z \otimes K
\end{array}
\]

is called a natural weak pullback (of \( \alpha \) and \( \beta \)) if it comes from some 2-fibered product in \( \mathcal{S} \). We write as

\[
\begin{array}{ccc}
W & \xrightarrow{\delta} & Y \\
\Downarrow^\gamma & \mbox{nwp} & \Downarrow^\beta \\
X \otimes L & \xrightarrow{\delta} & Z \otimes K
\end{array}
\]

to indicate it is a natural weak pullback.

Lemma 3.21. Let \( \alpha: \frac{X}{G} \to \frac{Y}{H} \) be an adjoint equivalence, and

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\Downarrow^\lambda & \Downarrow^\beta & \Downarrow^\lambda \\
X & \xrightarrow{\alpha} & Y
\end{array}
\]

be a diagram which satisfies

\[\alpha \circ \rho = \lambda \circ \alpha \quad \text{and} \quad \rho \circ \beta = \beta \circ \lambda.\]

For any \( h \in H \) and \( x \in X \), if we put

\[g = \theta_{\beta,\alpha}(x) \cdot \rho^{-1}_x,\]

then the following holds.

1. \( \theta_{\alpha,x}(g) = \lambda^{-1}_{\alpha}(x) \cdot h.\)
2. \( \theta_{\alpha,gx}(g^{-1}) = h^{-1} \lambda_{\alpha}(x).\)

Proof. (1) This follows from

\[
\begin{align*}
\theta_{\alpha,x}(g) &= \theta_{\alpha,x}(\theta_{\beta,\alpha}(x) \cdot \rho^{-1}_x) \\
&= \theta_{\alpha,\beta,\alpha}(x)(\theta_{\beta,\alpha}(x)(h)) \cdot \theta_{\alpha,x}(\rho^{-1}_x) \\
&= \theta_{\alpha,\beta,\alpha}(x)(h) \cdot (\alpha \circ \rho)^{-1}_x \\
&= \theta_{\alpha,\beta,\alpha}(x)(h) \cdot (\lambda \circ \alpha)^{-1}_x \\
&= \theta_{\alpha,\beta,\alpha}(x)(h) \cdot \lambda^{-1}_{\alpha}(x) \\
&= \lambda^{-1}_{\alpha}(x) \cdot h.
\end{align*}
\]

(2) This follows from (1) and

\[e = \theta_{\alpha,x}(g^{-1}) = \theta_{\alpha,gx}(g^{-1}) \cdot \theta_{\alpha,x}(g).\]
Proposition 3.22. If $\alpha: \frac{X}{G} \to \frac{Y}{H}$ is an adjoint equivalence, then

$$
\begin{array}{ccc}
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H} \\
\downarrow \text{id} & \circ & \downarrow \alpha \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H}
\end{array}
$$

is a 2-fibered product.

Proof. Since $\alpha$ is an adjoint equivalence, there is a diagram

$$
\begin{array}{ccc}
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H} \\
\downarrow \text{id} & \circ & \downarrow \alpha \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H}
\end{array}
$$

which satisfies

$\alpha \circ \rho = \lambda \circ \alpha$ and $\beta \circ \lambda = \rho \circ \beta$.

We confirm conditions (i), (ii) in Definition 3.10

(i) Suppose we are given a diagram

$$
\begin{array}{ccc}
\frac{W}{L} & \xrightarrow{\delta} & \frac{X}{G} \\
\downarrow \gamma & \circ & \downarrow \alpha \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H}
\end{array}
$$

in $S$. Then we see that the diagram

$$
\begin{array}{ccc}
\frac{W}{L} & \xrightarrow{\delta} & \frac{X}{G} \\
\downarrow \gamma & \circ & \downarrow \alpha \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H}
\end{array}
$$

with $\eta = (\rho \circ \delta) \cdot (\beta \circ \varepsilon) \cdot (\rho^{-1} \circ \gamma)$, satisfies

$$
\alpha \circ \eta = (\alpha \circ \rho \circ \delta) \cdot (\alpha \circ \beta \circ \varepsilon) \cdot (\alpha \circ \rho^{-1} \circ \gamma) = (\lambda \circ \alpha \circ \delta) \cdot (\alpha \circ \beta \circ \varepsilon) \cdot (\lambda^{-1} \circ \alpha \circ \gamma) = \varepsilon.
$$

(ii) Suppose that the diagram

$$
\begin{array}{ccc}
\frac{W}{L} & \xrightarrow{\pi} & \frac{X}{G} \\
\downarrow \gamma & \circ & \downarrow \alpha \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H}
\end{array}
$$
also satisfies
\[
\varepsilon \cdot (\alpha \circ \xi) = \alpha \circ \zeta.
\]
It suffices to show the existence and the uniqueness of \( \varpi : \pi \Rightarrow \gamma \) which satisfies
\[
\varpi = \xi \quad \text{and} \quad \eta \cdot \varpi = \zeta.
\]
Since such \( \varpi \) is trivially unique (\( = \xi \)), it remains to show that (3.1) implies \( \zeta = \eta \cdot \xi \).

By taking \( g \) to be \( \zeta_w \) and \( x \) to be \( \pi(w) \), we obtain
\[
\zeta_w = \rho_{\pi(w)} \cdot \theta_{\beta \circ \alpha, x}(g) \cdot \rho^{-1}_x \quad (\forall x \in X, \forall g \in G).
\]

Proposition 3.23. Let \( G \) be a finite group. If
\[
\begin{array}{ccc}
X \times Z & \xrightarrow{\gamma} & Y \\
\downarrow \alpha & & \downarrow \beta \\
X & \xrightarrow{\Delta} & Z
\end{array}
\]
is a fibered product in \( G \text{-set} \), then
\[
\begin{array}{ccc}
\frac{X \times Z}{G} & \xrightarrow{\Delta} & \frac{Y}{G} \\
\downarrow \Psi & & \downarrow \Phi \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Z}{G}
\end{array}
\]
is a 2-fibered product in \( G \text{-set} \). Thus the functor \( \bullet : G \text{-set} \to \text{GrSet} \) sends fibered products in \( G \text{-set} \) to 2-fibered products in \( \mathcal{S} \).

Proof. Let
\[
\begin{align*}
\text{pr}^{(1)} & : G \times G \to G, \\
\text{pr}^{(2)} & : G \times G \to G, \\
\Delta & : G \to G \times G,
\end{align*}
\]
be the projections onto 1st and 2nd components, and the diagonal homomorphism respectively.
By Proposition 3.16 we have a 2-fibered product of $\frac{\mathcal{F} \times \mathcal{G}}{\mathcal{G}}$ and $\frac{\mathcal{G}}{\mathcal{G}}$

\[
\begin{array}{c}
\frac{\mathcal{F} \times \mathcal{G}}{\mathcal{G}} \\
\xrightarrow{\rho} \\
\xrightarrow{\pi \circ \chi} \\
\xrightarrow{\lambda} \\
\frac{\mathcal{G}}{\mathcal{G}} \\
\xrightarrow{\iota} \\
\xrightarrow{\psi} \\
\frac{\mathcal{Y}}{\mathcal{G}}
\end{array}
\]

as in the notation of Proposition 3.16. Remark that $F$ is defined by

\[
F = \{ (x, y, g) \in X \times Y \times G \mid \beta(y) = g \alpha(x) \},
\]
on which $G \times G$ acts by

\[
(g_1, g_2) \cdot (x, y, g) = (g_1 x, g_2 y, g_2 g_1^{-1}) \quad (\forall (g_1, g_2) \in G \times G, \forall (x, y, g) \in F).
\]

If we define maps $\pi$ and $\chi$ by

\[
\pi : F \to X \times Y ; \quad (x, y, g) \mapsto (gx, y)
\]

\[
\chi : X \times Y \to F ; \quad (x, y) \mapsto (x, y, e),
\]

then $\frac{\pi \circ \chi}{\mathcal{G}}$ and $\frac{\chi}{\mathcal{G}}$ become 1-cells.

By Remark 3.11 it suffices to show that $\pi$ and $\chi$ give an adjoint equivalence $\frac{F}{\mathcal{G}} \simeq \frac{X \times Y}{\mathcal{G}}$. It can be easily checked that we have $\pi \circ \chi = \text{id}_{\frac{X \times Y}{\mathcal{G}}}$. If we define $\lambda : \chi \circ \pi \Rightarrow \text{id}_{\frac{X \times Y}{\mathcal{G}}}$ by

\[
\lambda_f = (g^{-1}, e) \quad (\forall f = (x, y, g) \in F),
\]

then $\lambda$ becomes in fact a 2-cell, which satisfies

\[
(\lambda \circ \chi)(x, y) = \lambda(x, y, e) = e \quad (\forall (x, y) \in X \times Y),
\]

\[
(\pi \circ \lambda)f = \text{pr}^{(2)}(\lambda_f) = e \quad (\forall f \in F)
\]
in the following diagram.

\[
\begin{array}{c}
\frac{X \times Y}{\mathcal{G}} \\
\xrightarrow{\chi} \\
\xrightarrow{\pi} \\
\xrightarrow{\lambda} \\
\frac{X \times Y}{\mathcal{G}}
\end{array}
\]

This means $\chi$ is an adjoint equivalence, with quasi-inverse $\pi$. \qed

**Corollary 3.24.** Let $\xrightarrow{\iota} \frac{X}{\mathcal{G}}$ be any 0-cell. Let $\iota_1 : X_1 \hookrightarrow X$ and $\iota_2 : X_2 \hookrightarrow X$ be inclusions of finite $G$-sets. If we denote the inclusions $X_1 \cap X_2 \hookrightarrow X_1$ and $X_1 \cap X_2 \hookrightarrow X_2$ by $\iota_1'$ and $\iota_2'$ respectively, then

\[
\begin{array}{c}
\frac{X_1 \cap X_2}{\mathcal{G}} \\
\xrightarrow{\iota_1'} \\
\xrightarrow{\iota_2'} \\
\xrightarrow{\iota_1} \\
\xrightarrow{\iota_2} \\
\frac{X_2}{\mathcal{G}}
\end{array}
\]
gives a 2-fibered product. Especially, remark that we have the following.

1. If $X_1 = X_2$, then $\iota_1'$ and $\iota_2'$ are identities.
(2) If \( X_1 \cap X_2 = \emptyset \), then \( \frac{X_1 \cap X_2}{G} = \emptyset \).

**Proof.** This immediately follows from Proposition 3.23 \( \square \)

### 4. Stabilizerwise image

As Remark 2.5 suggests, a 1-cell \( \alpha : X \to Y \) can be thought as a parallel array of group homomorphisms on stabilizers \( \theta_{\alpha,x} : G_x \to H_{\alpha(x)} \). With this view, we can consider analogs of images of group homomorphisms and factorizations through them, for 1-cells in \( \mathcal{S} \).

#### 4.1. Stab-surjective 1-cells.

**Definition 4.1.** A 1-cell \( \alpha : X \to Y \) is called *surjective on stabilizers* or shortly *stab-surjective*, if the following conditions are satisfied.

(i) For any \( y \in Y \), there exist \( x \in X \) and \( h \in H \) satisfying

\[
y = h\alpha(x).
\]

(ii) If \( x, x' \in X \) and \( h, h' \in H \) satisfy \( h\alpha(x) = h'\alpha(x') \), then there exists \( g \in G \) which satisfies

\[
x' = gx \quad \text{and} \quad h = h'\theta_{\alpha,x}(g).
\]

**Remark 4.2.** If \( \alpha : X \to Y \) is stab-surjective, then for any \( x \in X \), the restriction of \( \theta_{\alpha,x} \) onto \( G_x \) gives a surjective homomorphism

\[
\theta_{\alpha,x} : G_x \to H_{\alpha(x)}.
\]

**Proof.** This follows from condition (ii) in Definition 4.1 \( \square \)

**Example 4.3.** Let \( G \) be a finite group, and let \( N \trianglelefteq G \) be a normal subgroup. Let

\[
p : G \to G/N : g \mapsto \overline{g}
\]

denote the quotient homomorphism. Then for any \( Z \in \text{Ob}((G/N)\text{-set}) \), the 1-cell

\[
id_Z : \text{Inf}^G_Z \to \frac{Z}{(G/N)}
\]

is stab-surjective.

**Proof.** For any \( z \in Z \), we have \( e \cdot \text{id}_Z(z) = z \). Moreover if \( z_1, z_2 \in Z \) and \( \overline{g_1} \overline{g_2} \in G/N \) satisfy \( \overline{g_1}z_1 = \overline{g_2}z_2 \), then \( g_2^{-1}g_1 \in G \) satisfies

\[
(g_2^{-1}g_1)z_1 = \overline{g_2^{-1}g_1}z_1 = \overline{g_2^{-1}g_2}z_2 = z_2,
\]

\[
\overline{g_2} \cdot p(g_2^{-1}g_1) = \overline{g_1}.
\]

\( \square \)

**Proposition 4.4.** Let \( \alpha : X \to Y \) be a 1-cell in \( \mathcal{S} \).

(1) If there exists a 2-cell \( \delta : \alpha' \Rightarrow \alpha \) from a stab-surjective 1-cell \( \alpha' : X' \to Y' \), then so is \( \alpha \). Namely the stab-surjectivity does not depend on representatives of the equivalence class \( \alpha \). Thus we can speak of the stab-surjectivity of morphisms in \( \mathcal{C} \).

(2) If \( \alpha \) is an adjoint equivalence, then \( \alpha \) is stab-surjective.
Proof. (1) By definition, \( \alpha \) and \( \alpha' \) are related by
\[
\alpha(x) = \delta_x \alpha'(x) \quad (\forall x \in X),
\delta_x \cdot \theta_{\alpha',x}(g) \cdot \delta_x^{-1} = \theta_{\alpha,x}(g) \quad (\forall x \in X, \forall g \in G).
\]
We confirm conditions (i), (ii) in Definition 4.1.

(i) For any \( y \in Y \), there exist \( x \in X \) and \( h \in H \) satisfying \( y = h \alpha'(x) \). Thus we have
\[
y = (h \delta_x^{-1}) \alpha(x).
\]

(ii) Suppose \( x_1, x_2 \in X \) and \( h_1, h_2 \in H \) satisfy
\[
h_1 \alpha(x_1) = h_2 \alpha(x_2),
\]
i.e., \( h_1 \delta_x \alpha'(x_1) = h_2 \delta_x \alpha'(x_2) \), then by the stab-surjectivity of \( \alpha' \), there exists \( g \in G \) which satisfies
\[
x_2 = gx_1 \quad \text{and} \quad h_1 \delta_{x_1} = h_2 \delta_{x_2} \theta_{\alpha',x_1}(g).
\]
Since \( \delta_{x_2} \theta_{\alpha',x_1}(g) \delta_{x_1}^{-1} = \theta_{\alpha,x_1}(g) \), we obtain
\[
x_2 = gx_1 \quad \text{and} \quad h_1 = h_2 \theta_{\alpha,x_1}(g).
\]

(2) Take a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\downarrow{id} & & \downarrow{id} \\
G & \xrightarrow{\beta} & G
\end{array}
\]

in \( S \), satisfying \( \alpha \circ \rho = \lambda \circ \alpha \) and \( \rho \circ \beta = \beta \circ \lambda \). We confirm conditions (i) and (ii) in Definition 4.1.

(i) For any \( y \in Y \), we have
\[
y = \lambda_y \cdot \alpha(\beta(y)).
\]

(ii) Suppose \( x_1, x_2 \in X \) and \( h_1, h_2 \in H \) satisfy \( h_1 \alpha(x_1) = h_2 \alpha(x_2) \). If we put
\[
g_i = \theta_{\beta,\alpha(x_1)}(h_i) \cdot \rho_{x_i}^{-1} \quad (i = 1, 2),
\]
then we have
\[
g_1 x_1 = \theta_{\beta,\alpha(x_1)}(h_1) \cdot \rho_{x_1}^{-1} x_1 = \theta_{\beta,\alpha(x_1)}(h_1) \cdot (\beta \circ \alpha(x_1)) = \beta(h_1 \alpha(x_1)) = \beta(h_2 \alpha(x_2)) = \theta_{\beta,\alpha(x_2)}(h_2) \cdot (\beta \circ \alpha(x_2)) = \theta_{\beta,\alpha(x_2)}(h_2) \cdot \rho_{x_2}^{-1} x_2 = g_2 x_2
\]
and, by Lemma 3.21
\[
\theta_{\alpha,x_1}(g_2^{-1} g_1) = \theta_{\alpha,g_1 x_1}(g_2^{-1}) \cdot \theta_{\alpha,x_1}(g_1) = \theta_{\alpha,g_2 x_2}(g_2^{-1}) \cdot \theta_{\alpha,x_1}(g_1) = h_2^{-1} \cdot \lambda_{h_2 \alpha(x_2)} \cdot \lambda_{h_1 \alpha(x_1)}^{-1} \cdot h_1 = h_2^{-1} h_1.
\]
Thus \( g = g_2^{-1} g_1 \in G \) satisfies
\[
x_2 = gx_1 \quad \text{and} \quad h_1 = h_2 \cdot \theta_{\alpha,x_1}(g).\]
Proposition 4.5. Let \( \frac{X}{G} \xrightarrow{\alpha} \frac{Y}{H} \xrightarrow{\beta} \frac{Z}{K} \) be a sequence of 1-cells in \( S \). If \( \alpha \) and \( \beta \) are stab-surjective, then so is \( \beta \circ \alpha \).

Proof. We confirm conditions (i), (ii) in Definition 4.1.

(i) For any \( z \in Z \), there exist \( y \in Y \) and \( k \in K \) satisfying \( z = k\beta(y) \) by the stab-surjectivity of \( \beta \). Then by the stab-surjectivity of \( \alpha \), there exist \( x \in X \) and \( h \in H \) satisfying \( y = h\alpha(x) \). Thus we obtain

\[
z = k\beta(h\alpha(x)) = k\theta_{\beta,\alpha}(h) \cdot (\beta \circ \alpha(x)).
\]

(ii) Suppose \( x_1, x_2 \in X \) and \( k_1, k_2 \in K \) satisfy

\[
k_1\beta(\alpha(x_1)) = k_2\beta(\alpha(x_2)).
\]

By the stab-surjectivity of \( \beta \), there exists \( h \in H \) satisfying

\[
\alpha(x_2) = h\alpha(x_1) \quad \text{and} \quad k_1 = k_2 \cdot \theta_{\beta,\alpha}(x_1)(h).
\]

Then by the stab-surjectivity of \( \alpha \), there exists \( g \in G \) satisfying

\[
x_2 = gx_1 \quad \text{and} \quad h = \theta_{\alpha,x_1}(g).
\]

Thus we have

\[
k_1 = k_2 \cdot \theta_{\beta,\alpha}(x_1)(\theta_{\alpha,x_1}(g)) = k_2 \cdot \theta_{\beta\circ\alpha,x_1}(g).
\]

\qed

Stab-surjective 1-cells are stable under 2-pullbacks, as follows.

Proposition 4.6. Let

\[
\begin{array}{ccc}
W & \xrightarrow{\delta} & Y \\
\gamma & \downarrow & \downarrow \beta \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Z}{K}
\end{array}
\]

be a 2-fibered product in \( S \). If \( \beta \) is stab-surjective, then so is \( \gamma \).

Proof. We use the notation in Proposition 3.16. By Remark 3.11 Proposition 4.4 and 4.5 it suffices to confirm conditions (i), (ii) in Definition 4.1 for \( \varphi_X \) in the 2-fibered product

\[
\begin{array}{ccc}
\frac{F_{X \times H}}{ \varphi_Y \circ \varphi_X } & \xrightarrow{\delta} & \frac{Y}{H} \\
\uparrow \varphi_Y \circ \varphi_X & \downarrow \varphi_{\beta} & \uparrow \varphi_{\alpha} \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Z}{K}
\end{array}
\]

constructed in Proposition 3.16.

(i) For any \( x \in X \), since \( \beta \) is stab-surjective, there exist \( y \in Y \) and \( k \in K \) satisfying \( \alpha(x) = k\beta(y) \). Thus we obtain an element \( (x, y, k^{-1}) \in F \), which satisfies

\[
x = \varphi_X(x, y, k^{-1}).
\]

(ii) Suppose \( f_1 = (x_1, y_1, k_1), f_2 = (x_2, y_2, k_2) \in F \) and \( g_1, g_2 \in G \) satisfy

\[
g_1\varphi_X(f_1) = g_2\varphi_X(f_2).
\]
Then we have \( \theta_{\alpha,x_1}(g_1)\alpha(x_1) = \theta_{\alpha,x_2}(g_2)\alpha(x_2) \), namely
\[
\theta_{\alpha,x_1}(g_1)k_1^{-1}\beta(y_1) = \theta_{\alpha,x_2}(g_2)k_2^{-1}\beta(y_2).
\]
Since \( \beta \) is stab-surjective, there exists \( h \in H \) which satisfies
\[
y_2 = hy_1 \quad \text{and} \quad \theta_{\alpha,x_1}(g_1)k_1^{-1} = \theta_{\alpha,x_2}(g_2)k_2^{-1}\theta_{\alpha,y_1}(h).
\]
Then \( a = (g_2^{-1}g_1, h) \in G \times H \) satisfies
\[
a \cdot (x_1, y_1, k_1) = (g_2^{-1}g_1x_1, hy_1, \theta_{\beta,y_1}(h)k_1 \cdot \theta_{\alpha,x_1}(g_1^{-1}g_2^{-1}))
\]
\[
= (x_2, y_2, \theta_{\beta,y_1}(h)k_1 \theta_{\alpha,x_1}(g_1^{-1})\theta_{\alpha,x_2}(g_2))
\]
and
\[
g_1 = g_2 \cdot \text{pr}(G)(a).
\]

**Proposition 4.7.** Let \( \alpha: \frac{X}{G} \rightarrow \frac{Y}{H} \) be a stab-surjective 1-cell in \( \mathcal{S} \). Let \( X = X_1 \sqcup \cdots \sqcup X_s \) be the decomposition of \( X \) into \( G \)-orbits. If we put
\[
Y_i = H\alpha(X_i) = \{ h\alpha(x) \mid h \in H, x \in X_i \},
\]
then we have the following.

1. For \( i \neq j \), we have \( Y_i \cap Y_j = \emptyset \).
2. Each \( Y_i \) is \( H \)-transitive.
3. \( Y = Y_1 \sqcup \cdots \sqcup Y_s \) gives the decomposition of \( Y \) into \( H \)-orbits. In particular, \( X \) and \( Y \) have the same number of orbits.
4. For any \( 1 \leq i \leq s \), the restriction of \( \alpha \)
\[
\alpha_i = \alpha|_{X_i}: X_i \rightarrow Y_i
\]
is stab-surjective.
5. \( Y_i \) does not depend on the choice of representatives of \( \alpha \).

**Proof.** (1) If there is an element \( y \in Y_i \cap Y_j \) for \( i \neq j \), then there exist \( x \in X_i, x' \in X_j \) and \( h, h' \in H \) satisfying
\[
y = h\alpha(x) = h'\alpha(x').
\]
Then by the stab-surjectivity of \( \alpha \), there should be \( g \in G \) which satisfies \( gx = x' \), which contradicts to the fact that \( X_i \) and \( X_j \) are distinct \( G \)-orbits.

(2) For any \( y, y' \in Y_i \), there exist \( x, x' \in X_i \) and \( h, h' \in H \) satisfying
\[
y = h\alpha(x), \quad y' = h'\alpha(x')
\]
by definition of \( Y_i = H\alpha(X_i) \). Since \( X_i \) is \( G \)-transitive, there is \( g \in G \) satisfying \( x' = gx \). Thus we obtain
\[
y' = h'\alpha(gx) = h'\theta_{\alpha,x}(g)\alpha(x)
\]
\[
= (h'\theta_{\alpha,x}(g)h^{-1}) \cdot y.
\]
(3) By (1) and (2), it remains to show
\[
Y = Y_1 \sqcup \cdots \sqcup Y_s.
\]
However, this is obvious from the stab-surjectivity of \( \alpha \).

(4) This is trivial. (5) This follows from Remark 2.11. \( \square \)
Corollary 4.8. For a $G$-equivariant 1-cell $\xymatrix{X \ar[r]^\alpha & Y}$, the following are equivalent.

1. $\alpha$ is a $G$-equivariant isomorphism.
2. $\alpha$ is an adjoint equivalence.
3. $\alpha$ is stab-surjective.

These are also equivalent to the following.

4. $\alpha$ is an equivalence.
5. $\alpha$ is an isomorphism in $\mathcal{C}$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and (2) $\Rightarrow$ (4), and (4) $\Leftrightarrow$ (5) are already shown. It remains to show (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1).

(3) $\Rightarrow$ (1) Suppose $\xymatrix{X \ar[r]^\alpha & Y}$ is stab-surjective. By Proposition 4.7, we may assume $X$ and $Y$ are transitive. By Remark 4.2, $\alpha$ induces a surjection on stabilizers. This means $\alpha$ is isomorphic.

(4) $\Rightarrow$ (1) Here we involve a somewhat noncanonical argument using finiteness of $X$ and $Y$. Assume $\alpha$ is an equivalence, and let $\beta: \xymatrix{Y \ar[r] & X}$ be its quasi-inverse. By definition, there exist 2-cells $\lambda: \alpha \circ \beta \Rightarrow \text{id}$ and $\rho: \beta \circ \alpha \Rightarrow \text{id}$. By the symmetry of $\alpha$ and $\beta$, it suffices to show $\alpha$ is surjective. If we modify $\beta$ to obtain a map $\gamma: Y \to X; y \mapsto \lambda y \beta(y)$, then we have

$\alpha \circ \gamma(y) = \alpha(\lambda y \beta(y)) = \lambda y \alpha(\beta(y)) = y$

for any $y \in Y$, namely we obtain $\alpha \circ \gamma = \text{id}_Y$. Thus $\alpha$ is surjective. $\square$

Proposition 4.9. Let $\xymatrix{X \ar[r]^\alpha & Y}$ be a 1-cell, where $X$ is $G$-transitive. Then the following are equivalent.

1. $\alpha$ is stab-surjective.
2. There exist a section $N \triangleleft G_0 \leq G$ and $Z \in \text{Ob}((G_0/N)\text{-set})$ and a diagram

\[
\begin{array}{ccc}
X & \xymatrix{\ar[r]^-\alpha & } & Y \\
G & \xymatrix{\ar[r]^-\xi & } & G_0/N \\
\xymatrix{\ar[u]^-\xi & } & \xymatrix{\ar[u]^-\eta & } & \\
\xymartr{\text{Inf}_{G_0}Z} & \xymartr{\text{Inf}_{G_0}Z} & \\
G & \xymartr{\ar[r]^-p & } & G_0/N \\
\end{array}
\]

where

(i) $\xi$ and $\eta$ are adjoint equivalences.
(ii) $p: G_0 \to G_0/N$ is the quotient homomorphism.

Moreover, $Z$ in (2) can be taken as $Z = 1$.

Proof. (2) $\Rightarrow$ (1) follows from Example 4.3 and Proposition 4.7. It suffices to show (1) $\Rightarrow$ (2).

Suppose $\alpha$ is stab-surjective. Remark that $Y$ becomes transitive by Proposition 4.7. Take $x_0 \in X$, and put $G_0 = G_{x_0}, \ y_0 = \alpha(x_0), \ H_0 = H_{y_0}$.


Then by Remark 2.6 we have a commutative diagram

\[
\begin{array}{ccc}
X/G & \xrightarrow{\alpha} & Y/H \\
\cong & & \cong \\
G/G_0 & \circ & H/H_0 \\
\cong & & \cong \\
G_0 & H_0 \\
\end{array}
\]

where the vertical arrows are (equivariant) isomorphisms. Since we have \(G/G_0 = \text{Ind}^G_{G_0}(G_0/G_0)\) and \(H/H_0 = \text{Ind}^H_{H_0}(H_0/H_0)\), there are adjoint equivalences

\[
\frac{1}{G_0} \cong \frac{G/G_0}{G},
\]

\[
\frac{1}{H_0} \cong \frac{H/H_0}{H}
\]
as in Proposition 3.2. Moreover, since \(\alpha\) is stab-surjective, it induces a surjective group homomorphism

\[
\theta_{\alpha,x_0}|_{G_0}: G_0 \to H_0,
\]

which induces a group isomorphism \(\eta: G_0/Ker(\theta_{\alpha,x_0}|_{G_0}) \xrightarrow{\sim} H_0\). Thus if we put \(N = Ker(\theta_{\alpha,x_0]|_{G_0}) < G_0\), we obtain the following commutative diagram.

\[
\begin{array}{ccc}
X/G & \xrightarrow{\alpha} & Y/H \\
\cong & & \cong \\
G/G_0 & \circ & H/H_0 \\
\cong & & \cong \\
G_0 & \{(G_0/N) \}
\end{array}
\]

\[\square\]

4.2. Factorization through \(\text{SIm}\).

**Definition 4.10.** Let \(\alpha: \frac{X}{G} \to \frac{Y}{H}\) be any 1-cell in \(S\).

1. Define \(\text{SIm} \alpha = \text{SIm}(\frac{\alpha}{\theta_{\alpha}}) \in \text{Ob}(H\text{set})\) by

\[
\text{SIm} \alpha = (H \times X)/\sim,
\]

where the relation \(\sim\) is defined as follows.

- \((\eta, x), (\eta', x') \in H \times X\) are equivalent if there exists \(g \in G\) satisfying

\[
x' = gx \quad \text{and} \quad \eta = \eta' \theta_{\alpha,x}(g).
\]

We denote the equivalence class of \((\eta, x)\) by \([\eta, x]\). The \(H\)-action on \(\text{SIm} \alpha\) is given by

\[
h[\eta, x] = [h\eta, x].
\]

We call \(\text{SIm} \alpha\) the *stabilizerwise image* of \(\alpha = \frac{\alpha}{\theta_{\alpha}}\).

2. Define a map \(\nu_\alpha: X \to \text{SIm} \alpha\) by

\[
\nu_\alpha(x) = [e, x] \quad (\forall x \in X)
\]

and put \(\theta_{\nu_\alpha} = \theta_\alpha\). Then

\[
\nu_\alpha = \frac{\nu_\alpha}{\theta_\alpha}: \frac{X}{G} \to \frac{\text{SIm} \alpha}{H}
\]
becomes a 1-cell.

**Proposition 4.11.** For any 1-cell \( \alpha: \frac{X}{G} \to \frac{Y}{H} \), the induced 1-cell \( v_\alpha: X \to \text{Sim}_\theta \) is stab-surjective.

**Proof.** Conditions (i), (ii) in Definition 4.1 are confirmed as follows.

(i) For any \([\eta, x] \in \text{Sim}_\alpha\), we have
\[
[\eta, x] = \eta[e, x] = \eta v_\alpha(x).
\]
(ii) If \( x, x' \in X \) and \( h, h' \in H \) satisfy
\[
h v_\alpha(x) = h' v_\alpha(x'),
\]
i.e., \([h, x] = [h', x']\), then by definition of \( \text{Sim}_\alpha \), there exists \( g \in G \) which satisfies
\[
x' = gx \quad \text{and} \quad h = h' \theta_{\alpha, x}(g).
\]
\[\square\]

**Remark 4.12.** \( \text{Sim}(\frac{X}{G}) \) essentially depends only on the acting part \( \theta_{\alpha} \).

**Remark 4.13.** If \( \alpha \) is \( \iota \)-equivariant for some monomorphism \( \iota: G \to H \), then \( \text{Sim}_\alpha = \text{Sim}(\frac{X}{G}) \) is nothing but \( \text{Ind}_\alpha X \). In this case, \( v_\alpha: \frac{X}{G} \to \frac{\text{Ind}_\iota X}{H} \) is an adjoint equivalence, as shown in Proposition 3.2.

**Lemma 4.14.** Let \( \alpha: \frac{X}{G} \to \frac{Y}{H} \) and \( \beta: \frac{X}{G} \to \frac{Z}{H} \) be 1-cells satisfying \( \theta_{\alpha} = \theta_{\beta} = \theta \). If we define a map \( \overline{\beta}: \text{Sim}_\theta \to Z \) by
\[
\overline{\beta}([\eta, x]) = \eta \beta(x) \quad (\forall [\eta, x] \in \text{Sim}_\alpha),
\]
then we obtain a commutative diagram of 1-cells
\[
\begin{array}{ccc}
\frac{X}{G} & \xrightarrow{\overline{\beta}} & \frac{Z}{H} \\
\downarrow \alpha & & \downarrow \beta \\
\overline{\beta} & & \overline{\beta}_\alpha \\
\end{array}
\]

**Proof.** Well-definedness of \( \overline{\beta} \) follows from the equation
\[
\eta \beta(gx) = \eta \theta_{\alpha}(g) \beta(x) \quad (\forall [\eta, x] \in \text{Sim}_\alpha, \forall g \in G).
\]
Commutativity of the diagram can be checked immediately. \[\square\]

**Proposition 4.15.** For any 1-cell \( \alpha: \frac{X}{G} \to \frac{Y}{H} \), we have a commutative diagram of 1-cells
\[
\begin{array}{ccc}
\frac{X}{G} & \xrightarrow{v_\alpha} & \frac{Y}{H} \\
\downarrow \alpha & & \downarrow \overline{\beta} \\
\overline{\beta}_\alpha & & \overline{\beta}_\alpha \\
\end{array}
\]

We call this the \( \text{Sim} \)-factorization of \( \alpha \).

**Proof.** This immediately follows from Lemma 4.14. \[\square\]

**Proposition 4.16.** In \( \mathcal{S} \), let \( \mathcal{S} \) be the class of stab-surjective 1-cells, and let \( \mathcal{E} \) be the class of equivariant 1-cells. Then \( (\mathcal{S}, \mathcal{E}) \) forms a 2-factorization system, in the following sense (cf. [5]).

(0) Each of \( \mathcal{S} \) and \( \mathcal{E} \) is closed under compositions.
(1) Any 1-cell $\alpha: \frac{X}{G} \to \frac{Y}{H}$ can be written as a composition

\[
\begin{array}{c}
\frac{X}{G} \\
\downarrow s \\
\frac{Y}{H} \\
\end{array}
\begin{array}{c}
\alpha \\
\downarrow u \\
\frac{Y}{H} \\
\end{array}
\]

where $s$ is stab-surjective and $u$ is $H$-equivariant.

(2) If in the diagram

\[
\begin{array}{c}
\frac{X}{G} \\
\downarrow \alpha \\
\frac{Y}{H} \\
\downarrow \epsilon \\
\frac{Y'}{H} \\
\end{array}
\begin{array}{c}
\beta \\
\downarrow \delta \\
\frac{Y'}{H} \\
\end{array}
\]

$\alpha$ and $\beta$ are stab-surjective, then the following holds.

(i) There exists a triplet $(\omega, \mu, \nu)$

\[
\begin{array}{c}
\frac{X}{G} \\
\downarrow \alpha \\
\frac{Y}{H} \\
\downarrow \omega \\
\frac{Y'}{H} \\
\end{array}
\begin{array}{c}
\beta \\
\downarrow \nu \\
\frac{Y'}{H} \\
\end{array}
\]

satisfying $\delta \circ \mu = \epsilon \cdot (\nu \circ \alpha)$.

(ii) For any other triplet $(\omega', \mu', \nu')$ as in (i), there exists a unique 2-cell $\zeta: \omega \Rightarrow \omega'$ which satisfies

$\mu' \cdot (\zeta \circ \alpha) = \mu$ and $\nu' \cdot (\delta \circ \zeta) = \nu$.

(3) $\omega$ in (2) is an adjoint equivalence. More precisely, this $\omega$ can be taken as an $H$-isomorphism $\omega: \frac{Y}{\sim} \to \frac{Y'}{\sim}$.

Proof. (0) This follows from Proposition 4.5.

(1) This follows from Proposition 4.15.

(2) Suppose diagram (4.1) is given. We confirm conditions (i), (ii).

(i) For any $y \in Y$, take $x \in X$ and $\eta \in H$ satisfying $y = \eta \alpha(x)$. If we define $\omega(y)$ by

$$\omega(y) = \eta \varepsilon_{x}^{-1} \beta(x),$$

then this gives a well-defined $H$-equivariant map $\omega: \frac{Y}{\sim} \to \frac{Y'}{\sim}$. Indeed if $x_1, x_2 \in X$ and $\eta_1, \eta_2 \in H$ satisfy

$$y = \eta_1 \alpha(x_1) = \eta_2 \alpha(x_2),$$

then, since there is $g \in G$ satisfying

$$x_2 = gx_1 \quad \text{and} \quad \eta_1 = \eta_2 \theta_{\alpha, x_1}(g),$$

we obtain

$$\eta_2 \varepsilon_{x_2}^{-1} \beta(x_2) = \eta_2 \theta_{\alpha, x_1}(g) \varepsilon_{x_1}^{-1} \theta_{\beta, x_1}(g)^{-1} \beta(x_2) = \eta_2 \theta_{\alpha, x_1}(g) \varepsilon_{x_1}^{-1} \beta(x_1) = \eta_1 \varepsilon_{x_1}^{-1} \beta(x_1).$$

$H$-equivariance is obvious.
Moreover, for the 2-cell $\varepsilon : \omega \circ \alpha \Rightarrow \beta$, triplet $(\omega, \varepsilon, \text{id})$

\[
\begin{array}{c}
X \\
\alpha \downarrow \quad \beta \downarrow \\
Y \quad G \\
\varepsilon \quad \text{id} \\
Y' \\
\downarrow \quad \omega \circ \alpha \Rightarrow \beta \\
H \\
\downarrow \\
Z \\
\end{array}
\]

satisfies the desired property.

(ii) Suppose there is another triplet $(\omega', \mu', \nu')$. By assumption, we have

\[(4.2) \quad \mu'_x = \varepsilon_x \cdot \nu'_{\alpha(x)} \]

for any $x \in X$. It suffices to show the existence and the uniqueness of a 2-cell $\zeta : \omega \Rightarrow \omega'$ satisfying

\[
\nu' \circ (\delta \circ \zeta) = \text{id} \quad \text{and} \quad \mu' \circ (\zeta \circ \alpha) = \varepsilon.
\]

By (4.2), we can rephrase this condition as

\[
\nu' \circ (\delta \circ \zeta) = \text{id} \quad \text{and} \quad \mu' \circ (\zeta \circ \alpha) = \varepsilon
\]

\[
\iff \quad \nu'_y \cdot \zeta_y = \text{id} \quad \text{and} \quad \mu'_x \cdot \zeta_{\alpha(x)} = \varepsilon_x \quad (\forall x \in X, \forall y \in Y)
\]

\[
\iff \quad \zeta_y = \nu'^{-1}_y \quad \text{and} \quad \mu'_x \cdot \nu'^{-1}_{\alpha(x)} = \varepsilon_x \quad (\forall x \in X, \forall y \in Y)
\]

\[
\iff \quad \zeta_y = \nu'^{-1}_y \quad (\forall y \in Y).
\]

This last condition is satisfied only by $\zeta = \{\zeta_y = \nu'^{-1}_y\}_{y \in Y}$. This in fact becomes a 2-cell, since we have

\[
\nu'^{-1}_y \omega(y) = \nu'^{-1}_y \eta_x \beta(x) = \nu'^{-1}_y \eta_x \mu'_x \omega'_{\alpha(x)}
\]

\[
= \nu'^{-1}_y \eta \nu'^{-1}_{\alpha(x)} \omega'_{\alpha(x)} = \omega'(\eta x)
\]

for any $y = \eta \alpha(x) \in Y$.

(3) This is shown by a canonical argument, by applying (2) twice. A closer look at the construction of $\omega$ in the proof of (2) shows it can be taken as an $H$-isomorphism. (cf. Corollary 4.8.)

**Corollary 4.17.** For any $\alpha : X \overset{G}{\rightarrow} Y$, its stabilizerwise image $\text{Sim}(\alpha)$ is characterized up to the adjoint equivalence (more precisely, $H$-isomorphism), by the factorization property in Proposition 4.15.

**Proof.** This immediately follows from Proposition 4.16. □

**Corollary 4.18.** Let $\overrightarrow{\alpha} \rightarrow \overrightarrow{\beta} \rightarrow \overrightarrow{\alpha}$ be a sequence of 1-cells in $S$.

1. If $\alpha$ is stab-surjective, then we have an isomorphism of $K$-sets $\text{Sim}(\beta \circ \alpha) \cong \text{Sim}(\beta)$. In particular, we have $\text{Sim}(\beta \circ \alpha) \cong \text{Sim}(\beta)$ if $\alpha$ is an adjoint equivalence (for example, Ind-equivalence).

2. If $H = K$ and $\beta$ is $H$-equivariant, then we have an isomorphism of $H$-sets $\text{Sim}(\beta \circ \alpha) \cong \text{Sim}(\alpha)$. Thus in particular we have an isomorphism of finite $G$-sets $\text{Sim}(\alpha) \cong X$ for any $G$-equivariant 1-cell $\alpha : X \overset{G}{\rightarrow} Y$.

**Proof.** This immediately follows from Proposition 4.16. □
Proposition 4.19. Let $G, K$ be finite groups, and let $\alpha: \frac{X}{G} \to \frac{Z}{K}$ and $\beta: \frac{Y}{H} \to \frac{Z}{K}$ be any pair of 1-cells. Then for the union map $\alpha \cup \beta: \frac{X \amalg Y}{G} \to \frac{Z}{K}$ (Proposition 3.12), we have an isomorphism of $G$-sets

$$\text{SIm}(\alpha \cup \beta) \cong \text{SIm} \alpha \amalg \text{SIm} \beta.$$

Proof. This follows from the definition of $\text{SIm}$. \qed

Corollary 4.20. For any pair of 1-cells $\alpha: \frac{X}{G} \to \frac{Z}{K}$ and $\beta: \frac{Y}{H} \to \frac{Z}{K}$ in $S$, if we take the 1-cell

$$\alpha \cup \beta: \frac{X \amalg Y}{G} \to \frac{Z}{K}$$

obtained by the universality of the 2-coproduct, then we have an isomorphism of $K$-sets

$$\text{SIm}(\alpha \cup \beta) \cong \text{SIm} \alpha \amalg \text{SIm} \beta.$$

Proof. This follows from Proposition 3.14, Corollary 4.18 and Proposition 4.19. \qed

Proposition 4.21. Let

$$\begin{array}{ccc}
\frac{W}{L} & \xrightarrow{\delta} & \frac{Y}{H} \\
\gamma \downarrow & & \beta \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Z}{K}
\end{array}$$

be a 2-fibered product in $S$. If we factorize $\alpha$ and $\beta$ as

$$\begin{array}{ccc}
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Z}{K} \\
v_\alpha \downarrow & & \downarrow \\
\text{SIm} \alpha & \xrightarrow{\tilde{\alpha}} & \frac{\tilde{Z}}{K}
\end{array}, \quad \begin{array}{ccc}
\frac{Y}{H} & \xrightarrow{\beta} & \frac{Z}{K} \\
v_\beta \downarrow & & \downarrow \\
\text{SIm} \beta & \xrightarrow{\tilde{\beta}} & \frac{\tilde{Z}}{K}
\end{array},
$$

and if we take the fibered product of $\tilde{\alpha}$ and $\tilde{\beta}$

$$S = \text{SIm} \alpha \times_Z \text{SIm} \beta \xrightarrow{p_\beta} \text{SIm} \beta,$$

in $K$-set, then there is an isomorphism of $K$-sets

$$\phi: \text{SIm}(\alpha \circ \gamma) \xrightarrow{\sim} \text{SIm} \alpha \times_Z \text{SIm} \beta.$$

Proof. By Proposition 3.23.
becomes a 2-fibered product in $\mathcal{S}$. By taking 2-fibered products $F_1$ and $F_2$, we obtain the following diagram.

\[
\begin{array}{cccccccc}
F \downarrow \quad & F_1 \downarrow & F_2 \downarrow & p' \downarrow & Y \downarrow \quad & p \downarrow \quad & S \downarrow \quad & v' \downarrow \\
G \quad & G \quad & K \quad & \mathcal{S} \quad & \mathcal{S} \quad & \mathcal{S} \quad & \mathcal{S} \quad & \mathcal{S} \\
\beta \quad & \beta \quad & \beta \quad & \beta \quad & \beta \quad & \beta \quad & \beta \quad & \beta \\
\end{array}
\]

By the universality of the 2-fibered product, the 2-fibered product of $v'_\alpha$ and $v'_\beta$ should be adjoint equivalent to $W_L$. Thus we obtain a 2-fibered product

\[
\begin{array}{cccccccc}
W \downarrow \quad & v'' \downarrow & v' \downarrow & F_2 \downarrow \quad & F_1 \downarrow \quad & v'' \downarrow \quad & v' \downarrow \quad & v' \downarrow \\
L \quad & L \quad & L \quad & K \quad & K \quad & K \quad & K \quad & K \\
\end{array}
\]

together with 2-cells $p'_\alpha \circ v''_\beta \Rightarrow \gamma$ and $p'_\beta \circ v''_\alpha \Rightarrow \delta$.

Remark that $v'_\alpha$ and $v'_\beta$ are stab-surjective by Proposition 4.6 and thus so is $v'_\alpha \circ v''_\beta$ by Proposition 4.5. Thus we obtain an $(\mathcal{S}, \mathcal{E})$-factorization

\[
\begin{array}{cccccccc}
W \downarrow \quad & v'' \downarrow & v' \downarrow & F_2 \downarrow \quad & F_1 \downarrow \quad & v'' \downarrow \quad & v' \downarrow \quad & v' \downarrow \\
L \quad & L \quad & L \quad & K \quad & K \quad & K \quad & K \quad & K \\
\end{array}
\]

which implies that $\text{SIm}(\alpha \circ \gamma)$ and $S$ are $K$-isomorphic, by Corollary 4.17.

5. Mackey Functors on $\mathcal{S}$

5.1. Definition. We define the notions of a (semi-)Mackey functor on $\mathcal{S}$ and on $\mathcal{C}$, which turn out to be the same.

**Definition 5.1.** A semi-Mackey functor $M = (M^*, M_*)$ on $\mathcal{C}$ is a pair of a contravariant functor

\[ M^*: \mathcal{C} \to \text{Set} \]

and a covariant functor

\[ M_*: \mathcal{C} \to \text{Set} \]

which satisfies the following.

1. $M^*(\frac{X}{G}) = M_*(\frac{X}{G})$ for any object $\frac{X}{G} \in \text{Ob}(\mathcal{C})$. We denote this simply by $M(\frac{X}{G})$.

2. [Additivity] For any pair of objects $\frac{X}{G}$ and $\frac{Y}{H}$ in $\mathcal{C}$, if we take their coproduct

\[
\begin{array}{cccccccc}
\frac{X}{G} \downarrow \quad & \frac{X}{G} \downarrow & \frac{X}{G} \downarrow & \frac{Y}{H} \downarrow & \frac{Y}{H} \downarrow & \frac{Y}{H} \downarrow & \frac{Y}{H} \downarrow & \frac{Y}{H} \\
\end{array}
\]
in $\mathcal{C}$, then the natural map

$$(M^*(\nu_X), M^*(\nu_Y)) : M(\frac{X}{G} \coprod \frac{Y}{H}) \to M(\frac{X}{G}) \times M(\frac{Y}{H})$$

is bijective. Also, $M(\emptyset)$ is a singleton.

(2) [Mackey condition] For any natural weak pullback

$$\begin{array}{ccc}
W & \xrightarrow{\delta} & Y \\
\downarrow \gamma & & \downarrow \beta \\
X & \xrightarrow{\alpha} & Z
\end{array}$$

in $\mathcal{C}$, the following diagram in $\text{Set}$ becomes commutative.

$$\begin{array}{ccc}
M(W) & \xrightarrow{M^*(\delta)} & M(Y) \\
\downarrow M(\gamma) & & \downarrow M(\beta) \\
M(X) & \xrightarrow{M^*(\alpha)} & M(Z)
\end{array}$$

We can also formulate this by using $\mathcal{S}$. In the following definition, when we speak of a 2-functor from $\mathcal{S}$ to $\text{Set}$, we regard $\text{Set}$ as a 2-category equipped only with identity 2-cells. Thus a 2-functor $\mathcal{S} \to \text{Set}$ is nothing but a functor $\mathcal{S} \to \text{Set}$.

**Definition 5.2.** A semi-Mackey functor $M = (M^*, M_*)$ on $\mathcal{S}$ is a pair of a contravariant 2-functor

$$M^* : \mathcal{S} \to \text{Set}$$

and a covariant 2-functor

$$M_* : \mathcal{S} \to \text{Set}$$

which satisfies the following.

(0) $M^*(\xi) = M_*(\xi)$ for any 0-cell $\xi \in \mathcal{S}^0$. We denote this simply by $M(\xi)$.

(1) [Additivity] For any pair of 0-cells $\xi$ and $\eta$ in $\mathcal{S}$, if we take their 2-coproduct

$$\begin{array}{ccc}
\frac{X}{G} & \xrightarrow{\nu_X} & \frac{X}{G} \coprod \frac{Y}{H} \\
\downarrow \delta & & \downarrow \nu_Y \\
\frac{X}{G} & \xrightarrow{\mu} & \frac{Y}{H}
\end{array}$$

in $\mathcal{S}$, then the natural map

$$(M^*(\nu_X), M^*(\nu_Y)) : M(\frac{X}{G} \coprod \frac{Y}{H}) \to M(\frac{X}{G}) \times M(\frac{Y}{H})$$

is bijective. Also, $M(\emptyset)$ is a singleton.

(2) [Mackey condition] For any 2-fibered product

$$\begin{array}{ccc}
W & \xrightarrow{\delta} & Y \\
\downarrow \gamma & & \downarrow \beta \\
X & \xrightarrow{\alpha} & Z
\end{array}$$

in $\mathcal{C}$, then the natural map

$$(M^*(\nu_X), M^*(\nu_Y)) : M(\frac{X}{G} \coprod \frac{Y}{H}) \to M(\frac{X}{G}) \times M(\frac{Y}{H})$$

is bijective. Also, $M(\emptyset)$ is a singleton.
in $S$, the following diagram in $\text{Set}$ becomes commutative.

$$
\begin{array}{ccc}
M\left(\frac{W}{F}\right) & \xrightarrow{M^*(\delta)} & M\left(\frac{Y}{F}\right) \\
M_s(\gamma) & \circ & M_s(\beta) \\
M\left(\frac{X}{G}\right) & \xrightarrow{M^*(\alpha)} & M\left(\frac{Z}{K}\right)
\end{array}
$$

(5.3)

Definition 5.2 is just a rephrasement of Definition 5.1. We thus do not distinguish these two notions. With this view, for any morphism $\alpha$ in $C$, we write $M^*(\alpha)$ and $M_s(\alpha)$ simply as $M^*(\alpha)$ and $M_s(\alpha)$.

**Remark 5.3.** Let $M$ be a semi-Mackey functor on $S$ (= semi-Mackey functor on $C$). Let $\alpha: \frac{X}{G} \to \frac{Y}{H}$ be a 1-cell. Then the following holds.

1. If $\alpha$ is an equivalence, then both $M^*(\alpha)$ and $M_s(\alpha)$ are bijective.
2. Moreover if $\alpha$ is an adjoint equivalence, then $M^*(\alpha)$ and $M_s(\alpha)$ are mutually inverse to each other.

**Proof.** (1) For a quasi-inverse $\beta$ of $\alpha$, we have

$$
M^*(\beta) \circ M^*(\alpha) = M^*(\beta \circ \alpha) = M^*(\text{id}) = \text{id},
$$

$$
M^*(\alpha) \circ M^*(\beta) = M^*(\alpha \circ \beta) = M^*(\text{id}) = \text{id},
$$

and thus $M^*(\alpha)$ is a bijection. Similarly for $M_s(\alpha)$.

(2) By (1), both $M^*(\alpha)$ and $M_s(\alpha)$ are bijective. Moreover, since

$$
\begin{array}{ccc}
\frac{X}{G} & \xrightarrow{\text{id}} & \frac{X}{G} \\
\frac{X}{G} & \circ & \frac{Y}{H} \\
\frac{X}{G} & \xrightarrow{\alpha} & \frac{Y}{H}
\end{array}
$$

is a 2-fibered product by Proposition 3.22, we have

$$
M^*(\alpha) \circ M_s(\alpha) = M_s(\text{id}) \circ M^*(\text{id}) = \text{id}.
$$

This means $M_s(\alpha) = M^*(\alpha)^{-1}$. □

**Definition 5.4.** Let $M$ and $N$ be semi-Mackey functors on $S$. A morphism $\varphi: M \to N$ of semi-Mackey functors is a family of maps

$$
\varphi = \{ \varphi_X: M\left(\frac{X}{G}\right) \to N\left(\frac{X}{G}\right) \}_{X \in S^0}
$$

compatible with contravariant and covariant parts. Namely, it gives natural transformations

$$
\varphi: M^* \Rightarrow N^* \quad \text{and} \quad \varphi: M_s \Rightarrow N_s.
$$

With the usual composition of natural transformations, we obtain the category of semi-Mackey functors denoted by $\text{SMack}(S)$.

**Remark 5.5.**

1. Let $M$ be a semi-Mackey functor on $S$. Let $\frac{X}{G}$ be any 0-cell in $S$. If we denote the coproduct by

$$
\begin{array}{ccc}
\frac{X}{G} & \xrightarrow{v_1} & \frac{X}{G} \coprod \frac{X}{G} \\
\frac{X}{G} & \xrightarrow{v_2} & \frac{X}{G}
\end{array}
$$
and the folding map by
\[ \nabla: \frac{X \amalg X}{G} \to \frac{X}{G}, \]
then the composition of
\[ M\left(\frac{X}{G}\right) \times M\left(\frac{X}{G}\right) \xrightarrow{(M^*(\nu_1),M^*(\nu_2))^{-1}} M\left(\frac{X \amalg X}{G}\right) \xrightarrow{M_\nabla} M\left(\frac{X}{G}\right) \]
gives an addition on \( M\left(\frac{X}{G}\right) \). With this addition and the unit given by
\[ M\left(\frac{\emptyset}{G}\right) \xrightarrow{\iota_X} M\left(\frac{X}{G}\right) \]
where \( \iota_X: \emptyset \to \frac{X}{G} \) is the unique map, \( M\left(\frac{X}{G}\right) \) becomes a monoid.

(2) Let \( \varphi: \mathcal{M} \to \mathcal{N} \) be a morphism of semi-Mackey functors on \( \mathcal{S} \). For any 0-cell \( \frac{X}{G} \) in \( \mathcal{S} \),
\[ \varphi_{\frac{X}{G}}: M\left(\frac{X}{G}\right) \to N\left(\frac{X}{G}\right) \]
becomes a monoid homomorphism.

Thus \( M^* \) and \( M_\star \) can be regarded as functors to \( \text{Mon} \), and \( \varphi \) becomes a natural transformation between such functors.

**Definition 5.6.** A semi-Mackey functor \( \mathcal{M} \) on \( \mathcal{S} \) is a Mackey functor if the monoid \( M\left(\frac{X}{G}\right) \) is an additive group for any \( \frac{X}{G} \in \mathcal{S}^0 \). The full subcategory of Mackey functors in \( \mathcal{SMack}(\mathcal{S}) \) is denoted by \( \text{Mack}(\mathcal{S}) \).

**Remark 5.7.** \( M \in \text{Ob}(\mathcal{SMack}(\mathcal{S})) \) belongs to \( \text{Mack}(\mathcal{S}) \) if and only if both \( M^* \) and \( M_\star \) are functors to \( \text{Ab} \).

This allows us the following definition. Compare with Definition 5.2. In this definition, \( \text{RMod} \) denotes the category of \( \text{R} \)-modules. A 2-functor from \( \mathcal{S} \) to \( \text{RMod} \) is nothing but a functor from \( \mathcal{C} \) to \( \text{RMod} \).

**Definition 5.8.** Let \( \text{R} \) be a commutative ring. An \( \text{R} \)-linear Mackey functor \( \mathcal{M} = (M^*, M_\star) \) on \( \mathcal{S} \) is a pair of a contravariant 2-functor
\[ M^*: \mathcal{S} \to \text{RMod} \]
and a covariant 2-functor
\[ M_\star: \mathcal{S} \to \text{RMod}, \]
which satisfies the following.

1. \[ M^*(\frac{X}{G}) = M_\star(\frac{X}{G}) = M(\frac{X}{G}) \] for any 0-cell \( \frac{X}{G} \in \mathcal{S}^0 \).
2. [Additivity] For any pair of 0-cells \( \frac{X}{G} \) and \( \frac{Y}{G} \) in \( \mathcal{S} \), the natural map \( \text{b} \) is an isomorphism.
3. \[ M(\emptyset) = 0 \] is the zero module.
4. [Mackey condition] For any 2-fibered product \( \text{c} \) in \( \mathcal{S} \), the diagram \( \text{d} \) is a commutative diagram in \( \text{RMod} \).

A morphism \( \varphi: \mathcal{M} \to \mathcal{N} \) of \( \text{R} \)-linear Mackey functors is a family \( \varphi = \{ \varphi_{\frac{X}{G}} \}_{\frac{X}{G} \in \mathcal{S}^0} \) of \( \text{R} \)-homomorphisms compatible with contravariant and covariant parts. We denote the category of \( \text{R} \)-linear Mackey functors by \( \text{Mack}^\text{R}(\mathcal{S}) \), or by \( \text{Mack}^\text{R}(\mathcal{C}) \).

**Remark 5.9.** Remark that the additive completion of monoids gives a functor \( K_0: \text{Mon} \to \text{Ab} \). From any semi-Mackey functor \( \mathcal{M} = (M^*, M_\star) \), by composing \( K_0 \) we obtain a Mackey functor \( K_0\mathcal{M} = (K_0 \circ M^*, K_0 \circ M_\star) \) on \( \mathcal{S} \). This gives
a functor $K_0: SMack(\mathcal{S}) \to Mack(\mathcal{S})$, which is left adjoint to the inclusion functor $Mack(\mathcal{S}) \hookrightarrow SMack(\mathcal{S})$.

Furthermore, since tensoring $R$ gives an additive functor $- \otimes_R R: Ab \to RMod$. From any semi-Mackey functor $M = (M^*, M_*)$, by composing $- \otimes_R R$ and $K_0$, we obtain an $R$-linear Mackey functor $M^R = ((- \otimes_R R) \circ K_0 \circ M^*, (- \otimes_R R) \circ K_0 \circ M_*)$ on $\mathcal{S}$. This gives a functor $(-)^R: SMack(\mathcal{S}) \to Mack^R(\mathcal{S})$, which is left adjoint to the forgetful functor $Mack^R(\mathcal{S}) \to SMack(\mathcal{S})$.

**Remark 5.10.** For a fixed finite group $G$, the functor $\overset{\bullet}{-}: Gset \to GrSet$ in Proposition 2.7 induces a functor

$Mack(\mathcal{S}) \to Mack^G(\mathcal{S}) ; M = (M^*, M_*) \mapsto (M^*(\overset{\bullet}{G}), M_*(\overset{\bullet}{G}))$,

where $Mack^G(\mathcal{S})$ denotes the category of (ordinary) Mackey functors on $G$.

**Proof.** This follows from Proposition 3.12 and 3.24. □

**Remark 5.11.** Let $G$ be a fixed finite group. Mackey functors $M$ on $G$ obtained in Remark 5.10 form a special class in $Mack^G(\mathcal{S})$, since $M$ satisfies $M^*(\alpha) = M^*(\alpha'), \ M_*(\alpha) = M_*(\alpha')$ for any $\alpha, \alpha' \in Gset(X, Y)$ satisfying $\alpha = \alpha'$ in $\mathcal{C}$. This can be explained more precisely as follows.

1. Let $GrSet|_G$ denote the subcategory of $GrSet$, whose objects are $\overset{\bullet}{X}$ for some $X \in Ob(Gset)$, and morphisms are $G$-equivariant maps. Then obviously we have $GrSet|_G = Gset$.

2. Let $\mathcal{C}|_G$ denote the subcategory of $\mathcal{C}$ obtained as the quotient image of $GrSet|_G$ under the functor $GrSet \to \mathcal{C}$ in Remark 2.20. Then we have $\mathcal{C}|_G \simeq G-set$, where the right hand side denotes the category of finite fused $G$-sets (2). Thus the functor $\overset{\bullet}{-}: GrSet \to \mathcal{C}$ factors through $G-set$.

3. Since $\mathcal{C}|_G$ is closed under coproducts and natural weak pullbacks by Corollary 3.13 and Proposition 3.23, any Mackey functor $M$ on $\mathcal{C}$ can be restricted to give a Mackey functor $M$ on $G-set$, which is called fused Mackey functor on $G$ (2).

**Proof.** Since (1) is obvious and (3) follows from (2), we only show (2).

Let $\alpha, \alpha': \overset{\bullet}{X} \to \overset{\bullet}{Y}$ be $G$-equivariant 1-cells. Then a 2-cell $\varepsilon: \alpha \Rightarrow \alpha'$ is, by definition, a map $\varepsilon: X \to G$ satisfying

\begin{equation}
\varepsilon(x) = \varepsilon_x \alpha(x) \quad (\forall x \in X)
\end{equation}

and

\begin{equation}
\varepsilon_g g \varepsilon_x^{-1} = g \quad (\forall g \in G, \forall x \in X).
\end{equation}

\(^3\)This question is raised by Professor Fumihito Oda.
Remark that (5.5) is equivalent to that $\varepsilon$ is an element of $G\text{-set}(X, G^c)$, where $G^c$ is the set $G$ on which $G$ acts by conjugate. This condition does not depend on 1-cells $\alpha, \alpha'$. Also remark that the vertical composition of 2-cells gives a group structure on $G\text{-set}(X, G^c)$. Condition (5.5) means that this group $G\text{-set}(X, G^c)$ acts on the set of morphisms $\text{GrSet}|(\bar{X}, \bar{Y}) = G\text{-set}(X, Y)$.

Since $\mathcal{C}|(\bar{X}, \bar{Y})$ is the quotient of $\text{GrSet}|(\bar{X}, \bar{Y})$ by 2-cells, it agrees with the quotient of $G\text{-set}(X, Y)$ by this group action. Namely, we have

$$\mathcal{C}|(\bar{X}, \bar{Y}) \cong G\text{-set}(X, Y).$$

This gives an equivalence $\mathcal{C}|(\bar{X}, \bar{Y}) \simeq G\text{-set}$. □

5.2. Functors on span category.

**Definition 5.12.** Let $\bar{X}$ and $\bar{Y}$ be 0-cells in $\mathcal{S}$. A span $S$ to $\bar{X}$ from $\bar{Y}$ in $\mathcal{S}$ is a pair of 1-cells from some 0-cell $W$ in $\mathcal{S}$,

$$S = (\bar{X} \xrightarrow{\rho_S} W_S \xrightarrow{\beta_S} \bar{Y})$$

in $\mathcal{S}$. We sometimes simply write this as $\bar{X} \xrightarrow{\rho_S} \bar{Y}$. The span $(\bar{X} \xleftarrow{\rho'_S} \bar{Y} \xrightarrow{\beta'_S} \bar{X})$ is denoted by $\text{Id} = \bar{X} \xrightarrow{\rho'_S} \bar{Y}$, and called the identity span.

**Definition 5.13.** Let $\bar{X}$ and $\bar{Y}$ be any pair of 0-cells in $\mathcal{S}$. Then a 2-category $\text{Span}(\bar{X}, \bar{Y})$ is defined as follows.

1. A 0-cell in $\text{Span}(\bar{X}, \bar{Y})$ is a span $S$ to $\bar{X}$ from $\bar{Y}$.
2. A 1-cell in $\text{Span}(\bar{X}, \bar{Y})$ from $S = (\bar{X} \xrightarrow{\rho_S} W_S \xrightarrow{\beta_S} \bar{Y})$ to $T = (\bar{X} \xrightarrow{\rho_T} W_T \xrightarrow{\beta_T} \bar{Y})$ is a triplet $(\varphi, \mu_X, \mu_Y)$ of a 1-cell $\varphi$ and 2-cells $\mu_X, \mu_Y$ in $\mathcal{S}$ as in the following diagram.

(2) If $(\varphi, \mu_X, \mu_Y): S \to T$ and $(\varphi', \mu'_X, \mu'_Y): S \to T$ are 1-cells in $\text{Span}(\bar{X}, \bar{Y})$, then a 2-cell $\varepsilon: (\varphi, \mu_X, \mu_Y) \Rightarrow (\varphi', \mu'_X, \mu'_Y)$ in $\text{Span}(\bar{X}, \bar{Y})$ is a 2-cell $\varepsilon: \varphi \Rightarrow \varphi'$ in $\mathcal{S}$, which makes the following diagrams commutative.

$$\alpha_S \circ \varphi = \beta_S \circ \varphi', \quad \alpha_T \circ \varphi = \beta_T \circ \varphi'$$

$$\mu_X \circ \varphi = \mu'_X \circ \varphi', \quad \mu_Y \circ \varphi = \mu'_Y \circ \varphi'$$
Composition of 1-cells

$$(\varphi, \mu_X, \mu_Y): \left( X \overset{\alpha_S}{\underset{L_S}{\leftrightarrow}} W_S \overset{\beta_S}{\rightarrow} Y \right) \rightarrow \left( X \overset{\alpha_T}{\underset{L_T}{\leftrightarrow}} W_T \overset{\beta_T}{ightarrow} Y \right)$$

and

$$(\psi, \nu_X, \nu_Y): \left( X \overset{\alpha_T}{\underset{L_T}{\leftrightarrow}} W_T \overset{\beta_T}{ightarrow} Y \right) \rightarrow \left( X \overset{\alpha_P}{\underset{L_P}{\leftrightarrow}} W_P \overset{\beta_P}{ightarrow} Y \right)$$

is defined to be

$$(\psi \circ \varphi, \mu_X \cdot (\nu_X \circ \varphi), \mu_Y \cdot (\nu_Y \circ \varphi)).$$

Vertical composition of 2-cells

Vertical composition of 2-cells

is defined to be $\varepsilon' \cdot \varepsilon$, using the vertical composition in $S$.

Horizontal composition of 2-cells

Horizontal composition of 2-cells

is defined to be $\delta \circ \varepsilon$, using the horizontal composition in $S$.

Then $\text{Span} \frac{X}{Y}$ becomes in fact a 2-category.

**Definition 5.14.** Let $\frac{X}{Y}$ and $\frac{T}{Y}$ be 0-cells in $S$. Two spans

$$S = \left( X \overset{\alpha_S}{\underset{L_S}{\leftrightarrow}} W_S \overset{\beta_S}{\rightarrow} Y \right),$$

$$T = \left( X \overset{\alpha_T}{\underset{L_T}{\leftrightarrow}} W_T \overset{\beta_T}{\rightarrow} Y \right)$$

are adjoint equivalent if there exists an adjoint equivalence

in $\text{Span} \frac{X}{Y}$. Remark that this implies in particular $\varphi$ is an adjoint equivalence in $S$.

We denote the adjoint equivalence class of $S$ by $[S]$. 
Definition 5.15. For any 1-cell \( \alpha : \frac{X}{G} \to \frac{Y}{H} \) in \( S \), we define the adjoint equivalence classes \( R_\alpha \) and \( T_\alpha \) by
\[
R_\alpha = \left\{ \begin{array}{c}
\frac{X}{G} \xrightarrow{\alpha} \frac{Y}{H} \\
\frac{Y}{H} \xleftarrow{\alpha} \frac{X}{G}
\end{array} \right\} \quad \text{(in Span}_{\frac{X}{G}}{Y})
\]
\[
T_\alpha = \left\{ \begin{array}{c}
\frac{Y}{H} \xrightarrow{\alpha} \frac{X}{G} \\
\frac{X}{G} \xleftarrow{\alpha} \frac{Y}{H}
\end{array} \right\} \quad \text{(in Span}_{\frac{Y}{H}}{X}).
\]

Proposition 5.16. Span \( \frac{X}{G} \to \frac{Y}{H} \) admits 2-coproducts induced from those in \( S \).

Proof. For any pair of 0-cells \( S = (\frac{X}{G} \xleftarrow{\alpha_S} \frac{W_S}{L_S} \xrightarrow{\beta_S} \frac{Y}{H}) \) and \( T = (\frac{X}{G} \xleftarrow{\alpha_T} \frac{W_T}{L_T} \xrightarrow{\beta_T} \frac{Y}{H}) \) in Span \( \frac{X}{G} \to \frac{Y}{H} \), if we take the 2-coproduct of \( \frac{W_S}{L_S} \) and \( \frac{W_T}{L_T} \)
\[
\frac{W_S}{L_S} \xleftarrow{v_{W_S}} \frac{W_T}{L_T} \quad \frac{W_T}{L_T} \xrightarrow{v_{W_T}} \frac{W_S}{L_S}
\]
in \( S \), then by its universality, we obtain a diagram

This gives a 2-coproduct
\[
S \xleftarrow{(v_{W_S}, \alpha_S, \beta_S)} (\frac{X}{G} \xleftarrow{\alpha_S} \frac{W_S}{L_S} \xrightarrow{\beta_S} \frac{Y}{H}) = (\frac{W_T}{L_T} \xrightarrow{\beta_T} \frac{Y}{H}) \xleftarrow{v_{W_T}, \lambda_X, \lambda_Y} T
\]
in Span \( \frac{X}{G} \to \frac{Y}{H} \). \( \square \)

Definition 5.17. Let \( \frac{X}{G} \) and \( \frac{Y}{H} \) be 0-cells in \( S \). For spans in \( S \)
\[
S = (\frac{X}{G} \xleftarrow{\alpha_S} \frac{W_S}{L_S} \xrightarrow{\beta_S} \frac{Y}{H})
\]
and
\[
T = (\frac{X}{G} \xleftarrow{\alpha_T} \frac{W_T}{L_T} \xrightarrow{\beta_T} \frac{Y}{H}),
\]
their sum is defined to be the 2-coproduct
\[
S + T = (\frac{X}{G} \xleftarrow{\alpha_S, \beta_T} \frac{W_S}{L_S} \xrightarrow{\beta_S} \frac{Y}{H})
\]
\[
\text{Remark 5.18. Sum of the spans does not depend on the representatives of the adjoint equivalence classes in Span}_{\frac{X}{G}}{Y}. Thus } [S] + [T] = [S + T] \text{ is well-defined.}
Definition 5.19. Let
\[ S = \left( \begin{array}{ccc} Y & \xymatrix{ W_S & X \ar[r]^-{\beta} & G } \\ H & L_S \end{array} \right) \in (\text{Span}_{Y/H}^{G})^0 \]
\[ T = \left( \begin{array}{ccc} Z & \xymatrix{ W_T & Y \ar[r]^-{\beta} & H } \\ K & L_T \end{array} \right) \in (\text{Span}_{Z/K}^{Y/H})^0 \]
be two consecutive spans in S. We define their composition
\[ T \circ S = \left( \begin{array}{ccc} Z & \xymatrix{ W_T \circ S & X \ar[r]^-{\beta} & G } \\ K & L_T \end{array} \right) \]
as follows.

- Take a 2-fibered product

\[
\begin{array}{ccc}
F & \xymatrix{ \phi W_T & W_S } \\
L_T \times L_S & \xymatrix{ \circ S & Y } \\
\phi W_T & L_T & Y \end{array}
\]

and put
\[ L_{T \circ S} = L_T \times L_S , \quad W_{T \circ S} = F , \]
\[ \alpha_{T \circ S} = \alpha_T \circ \phi W_T , \quad \beta_{T \circ S} = \beta_S \circ \phi W_S , \]
as in the following diagram.

The adjoint equivalence class \([T \circ S]\) does not depend on representatives of adjoint equivalence classes of spans \([S]\) and \([T]\). Consequently, we obtain the following category.

Definition 5.20. The span category \(\text{Sp}\) of \(S\) is defined as follows.

1. \(\text{Ob}(\text{Sp}) = S^0 = \text{Ob}(\mathcal{E})\).
2. For any pair of objects \(X/G\) and \(Y/H\), a morphism from \(X/G\) to \(Y/H\) is a adjoint equivalence class \([S]\) of a span \(Y/S_X \xrightarrow{\phi} X/F \in (\text{Span}_{X/F}^{Y/H})^0\). When we want to emphasize it is a morphism in \(\text{Sp}\), we will denote it by 
\[ [S]: \xymatrix{ X \ar[r]^-{\phi} & Y } \]
The composition of morphisms is defined by the composition of spans, and the identity span gives the identity morphism.

Remark 5.21. For any pair of objects \(X/G\) and \(Y/H\) in \(\text{Sp}\), the set of morphisms \(\text{Sp}(X/G, Y/H)\) has a structure of monoid with the addition obtained in Definition 5.17. Unit for this addition is given by \(0 = [Y/H \xleftarrow{\emptyset} \emptyset \rightarrow X/G]\).

The following are shown in the same way as in \([6]\) and \([8]\).
Proposition 5.22. Let $\frac{X}{G}$, $\frac{Y}{H}$ be any pair of objects in $\text{Sp}$. If we take their 2-coproduct

$$\frac{X}{G} \rightharpoonup \frac{X}{G} \amalg \frac{Y}{H} \leftarrow \frac{Y}{H}.$$

in $\mathcal{S}$, then

$$\frac{X}{G} \overset{R_{\psi X}}{\rightharpoonup} \frac{X}{G} \amalg \frac{Y}{H} \overset{R_{\psi Y}}{\leftarrow} \frac{Y}{H}$$

is a product of $\frac{X}{G}$ and $\frac{Y}{H}$ in $\text{Sp}$.

Definition 5.23. Category $\mathcal{T}$ is defined as follows.

1. $\text{Ob}(\mathcal{T}) = \text{Ob}(\text{Sp})$.

2. For any objects $\frac{X}{G}, \frac{Y}{H}$ in $\mathcal{T}$,

$$\mathcal{T}(\frac{X}{G}, \frac{Y}{H}) = K_0(\text{Sp}(\frac{X}{G}, \frac{Y}{H})).$$

Thus a morphism $\frac{X}{G} \rightarrow \frac{Y}{H}$ in $\mathcal{T}$ is written as a difference

$$[S] - [T]: \frac{X}{G} \rightarrow \frac{Y}{H}$$

of $[S], [T] \in \text{Sp}(\frac{X}{G}, \frac{Y}{H})$. Composition of morphisms is defined by extending the composition in $\text{Sp}$ by linearity. Also in $\mathcal{T}$,

$$\frac{X}{G} \overset{R_{\psi X}}{\rightharpoonup} \frac{X}{G} \amalg \frac{Y}{H} \overset{R_{\psi Y}}{\leftarrow} \frac{Y}{H}$$

gives a product of $\frac{X}{G}$ and $\frac{Y}{H}$.

Since adjoint equivalences in $\mathcal{S}$ preserve the numbers of orbits by Proposition 4.7, it can be easily shown that the natural maps $\text{Sp}(\frac{X}{G}, \frac{Y}{H}) \rightarrow \mathcal{T}(\frac{X}{G}, \frac{Y}{H})$ is a monomorphism. These form a faithful functor $c: \text{Sp} \rightarrow \mathcal{T}$.

Definition 5.24.

1. Denote the category of functors $E: \text{Sp} \rightarrow \text{Set}$ preserving finite products by $\text{Add}(\text{Sp}, \text{Set})$. Morphisms are natural transformations.

2. Similarly, denote the category of functors $F: \mathcal{T} \rightarrow \text{Set}$ preserving finite products by $\text{Add}(\mathcal{T}, \text{Set})$. Morphisms are natural transformations.

Remark 5.25.

1. For any $E \in \text{Ob}(\text{Add}(\text{Sp}, \text{Set}))$ and for any $\frac{X}{G} \in \text{Ob}(\text{Sp})$, the set $E(\frac{X}{G})$ becomes a monoid with respect to the addition

$$E(\frac{X}{G}) \times E(\frac{X}{G}) \equiv E(\frac{X \amalg X}{G}) \overset{E(\nabla)}{\rightarrow} E(\frac{X}{G}),$$

where $\nabla: \frac{X \amalg X}{G} \rightarrow \frac{X}{G}$ is the folding map. Similarly, $F(\frac{X}{G})$ becomes an abelian group for any $F \in \text{Ob}(\text{Add}(\mathcal{T}, \text{Set}))$ and any $\frac{X}{G} \in \text{Ob}(\mathcal{T})$.

2. Composition of the natural functor $c: \text{Sp} \rightarrow \mathcal{T}$ yields a functor

$$\text{Add}(\mathcal{T}, \text{Set}) \rightarrow \text{Add}(\text{Sp}, \text{Set}); F \mapsto F \circ c.$$ 

This is a fully faithful functor, and $E \in \text{Ob}(\text{Add}(\text{Sp}, \text{Set}))$ comes from some $F \in \text{Ob}(\text{Add}(\mathcal{T}, \text{Set}))$ if and only if $E(\frac{X}{G})$ is an abelian group for any $\frac{X}{G} \in S^0$. 
Example 5.26. For any 1-cell $\frac{X}{G}$ in $\mathbb{S}$, the representable functor

$$T(\frac{X}{G}, -): T \to \text{Set}$$

preserves finite products, and thus becomes an object in $\text{Add}(T, \text{Set})$. Similarly for representable functors on $\text{Sp}$.

If $F$ is an object in $\text{Add}(T, \text{Set})$, it can be regarded as a functor to $\text{Ab}$. Similarly as in Definition 5.8, we can also define $R$-linear case as follows.

Definition 5.27. We denote the category of functors $F: T \to R\text{Mod}$ preserving finite products by $\text{Add}(T, R\text{Mod})$. Morphisms are natural transformations. Since $T$ is an additive category, this is nothing but the category of additive functors, in the usual sense.

Proposition 5.28. To give a (resp. semi-)Mackey functor $M$ on $\mathbb{S}$ is equivalent to give a functor $F: T \to \text{Set}$ (resp. $\text{Sp} \to \text{Set}$) preserving finite products. More precisely, there are equivalences of categories

$$\text{SMack}(\mathbb{S}) \xrightarrow{\cong} \text{Add}(\text{Sp}, \text{Set}),$$

$$\text{Mack}(\mathbb{S}) \xrightarrow{\cong} \text{Add}(T, \text{Set}),$$

which makes the following diagram commutative.

$$\text{Mack}(\mathbb{S}) \xrightarrow{\cong} \text{Add}(T, \text{Set})$$

$$\xrightarrow{\circ}$$

$$\text{SMack}(\mathbb{S}) \xrightarrow{\cong} \text{Add}(\text{Sp}, \text{Set})$$

Proof. This is shown in the same way as in [6] and [8]. The only different point is that we are using natural weak pullbacks instead of fibered products. We only state the correspondence of $M$ and $F$.

- For an object $\frac{X}{G}$,
  $$F(\frac{X}{G}) = M(\frac{X}{G}).$$

- For any 1-cell $\frac{\alpha}{Y} \to \frac{\beta}{X}$ in $\mathbb{S}$,
  $$M_*(\alpha) = F(T_\alpha), \quad M^*(\alpha) = F(R_\alpha).$$

- For any span $S = (\frac{Y}{T} \xleftarrow{\alpha_S} \frac{W_S}{L_S} \xrightarrow{\beta_S} \frac{X}{G}),$
  $$F([S]) = M_*(\alpha_S) \circ M^*(\beta_S).$$

This only depends on the adjoint equivalence class $[S]$, since for an adjoint equivalence of spans

$$\frac{Y}{T} \xleftarrow{\alpha_S} \frac{W_S}{L_S} \xrightarrow{\beta_S} \frac{X}{G},$$

$$\frac{\alpha_X}{\alpha_T} \xleftarrow{\nu_X} \frac{W_T}{L_T} \xrightarrow{\nu_Y} \frac{Y}{T},$$

$$\frac{\beta_X}{\beta_T}.$$
we have
\[ M_*(\alpha_S) \circ M^*(\beta_S) = M_*(\alpha_T) \circ M_*(\varphi) \circ M^*(\beta_T) = M_*(\alpha_T) \circ M^*(\beta_T) \]
by Remark 5.3.

The same correspondence gives the following equivalence.

**Proposition 5.29.** There is an equivalences of categories
\[ \text{Mack}^R(\mathcal{S}) \xrightarrow{\cong} \text{Add}(\mathcal{T}, \mathcal{R}\text{-Mod}). \]

5.3. **Deflative Mackey functors.** We define a special class of Mackey functors, which will be shown to correspond to biset functors later.

**Definition 5.30.** A semi-Mackey functor \( M \) on \( \mathcal{S} \) is called deflative if for any stab-surjective 1-cell \( \alpha: \frac{X}{G} \to \frac{Y}{H} \) in \( \mathcal{S} \), the equality
\[ M_*(\alpha) \circ M^*(\alpha) = \text{id}_{M(\frac{Y}{H})} \]
is satisfied. A (\( R \)-linear) Mackey functor is called deflative if it is deflative as a semi-Mackey functor.

The full subcategory of deflative semi-Mackey functors is denoted by \( \text{SMack}^d(\mathcal{S}) \subseteq \text{SMack}(\mathcal{S}) \). Similarly, the full subcategory of deflative Mackey functors is denoted by \( \text{Mack}^d(\mathcal{S}) \subseteq \text{Mack}(\mathcal{S}) \). In the \( R \)-linear case, similarly we denote as \( \text{Mack}^R^d(\mathcal{S}) \subseteq \text{Mack}^R(\mathcal{S}) \).

**Proposition 5.31.** For an \( R \)-linear (resp. semi-)Mackey functor \( M \) on \( \mathcal{S} \), the following are equivalent.

1. \( M \) is deflative.
2. For any finite group \( G \) and its normal subgroup \( N \triangleleft G \), if we denote the quotient homomorphism by \( p: G \to G/N \), then the equality
\[ M_*(\frac{1}{p}) \circ M^*(\frac{1}{p}) = \text{id}_{M(\frac{G}{N})} \]
is satisfied for the 1-cell \( \frac{1}{p}: \frac{1}{G} \to \frac{1}{(G/N)} \).

**Proof.** This follows from Proposition 4.7, 4.9 and Remark 5.3.

**Corollary 5.32.** Let \( M \) be an \( R \)-linear (resp. semi-)Mackey functor on \( \mathcal{S} \), and let \( F \) be a corresponding object in \( \text{Add}(\mathcal{T}, \mathcal{R}\text{-Mod}) \) (resp. \( \text{Add}(\mathcal{Sp}, \mathcal{Set}) \)). Then the following are equivalent.

1. \( M \) is deflative.
2. For any stab-surjective 1-cell \( \alpha: \frac{X}{G} \to \frac{Y}{H} \) in \( \mathcal{S} \), we have
\[ F([\frac{Y}{H} \xleftarrow{\alpha} \frac{X}{G} \xrightarrow{\alpha} \frac{Y}{H}]) = \text{id}_{F(\frac{Y}{H})}. \]
3. For any finite group \( G \) and its normal subgroup \( N \triangleleft G \),
\[ F([\frac{1}{G} \xleftarrow{p} \frac{1}{G} \xrightarrow{p} \frac{1}{(G/N)}]) = \text{id} \]
holds for the quotient homomorphism \( p: G \to G/N \).
Proof. This follows from the fact that for any 1-cell \( \alpha : \frac{X}{G} \rightarrow \frac{Y}{H} \), we have

\[
M_* (\alpha) \circ M^*(\alpha) = F(T_\alpha) \circ F(R_\alpha) = F(T_\alpha \circ R_\alpha)
\]

\[
= F([\frac{Y}{H} \xleftarrow{\alpha} \frac{X}{G} \xrightarrow{\alpha} \frac{Y}{H}]).
\]

By this corollary, we define as follows.

**Definition 5.33.** An object \( F \) in \( \text{Add}(\mathcal{T}, \text{RMod}) \) (resp. \( \text{Add}(\mathcal{T}, \text{Set}), \text{Add}(\text{Sp}, \text{Set}) \)) is called deflative if for any stab-surjective 1-cell \( \alpha : \frac{X}{G} \rightarrow \frac{Y}{H} \),

\[
F([\frac{Y}{H} \xleftarrow{\alpha} \frac{X}{G} \xrightarrow{\alpha} \frac{Y}{H}]) = \text{id}_{F([\frac{Y}{H}])}
\]

holds. We denote the full subcategory of deflative objects by \( \text{Add}_{dfl}(\mathcal{T}, \text{RMod}) \) (resp. \( \text{Add}_{dfl}(\mathcal{T}, \text{Set}), \text{Add}_{dfl}(\text{Sp}, \text{Set}) \)).

### 5.4. Bigger Burnside rings

We introduce an example of Mackey functor, which is not deflative.

**Example 5.34.** We have an object \( T(\frac{1}{e}, -) \) in \( \text{Add}(\mathcal{T}, \text{Set}) \). We call the corresponding Mackey functor the **bigger Burnside functor**, and denote it by \( \Omega_{\text{big}} \in \text{Ob}(\text{Mack}(\mathcal{S})) \).

**Remark 5.35.** By Proposition 5.28 and Yoneda’s lemma, there is a natural isomorphism of abelian groups

\[
\text{Mack}(\mathcal{S})(\Omega_{\text{big}}, M) \cong M(\frac{1}{e})
\]

for any \( M \in \text{Ob}(\text{Mack}(\mathcal{S})) \). When \( M = \Omega_{\text{big}} \), this gives an isomorphism for the endomorphism ring of \( \Omega_{\text{big}} \)

\[
\text{Mack}(\mathcal{S})(\Omega_{\text{big}}, \Omega_{\text{big}}) \cong \Omega_{\text{big}}(\frac{1}{e}).
\]

It can be easily shown that \( \Omega_{\text{big}} \) is not deflative. In fact, for any \( \frac{X}{G} \in \mathcal{S}^0 \) with \( X \neq \emptyset \), the representable functor \( \mathcal{T}(\frac{X}{G}, -) \in \text{Ob}(\text{Add}(\mathcal{T}, \text{Set})) \) becomes non-deflative. For simplicity, we only show in the following case.

**Claim 5.36.** For \( \frac{X}{G} \in \mathcal{S}^0 \), if \( X \) is \( G \)-transitive, then \( F = \mathcal{T}(\frac{X}{G}, -) \) is non-deflative.

**Proof.** By Corollary 5.32, replacing \( G \) if necessary, we may assume \( \frac{X}{G} \) is of the form \( \frac{1}{G} \) from the first. Take a finite group \( G' \) and a surjective group homomorphism \( p: G' \rightarrow G \) satisfying \( |G'| > |G| \), and let \( aS_p \) be the span \( S_p = (\frac{1}{G'} \xleftarrow{\frac{1}{G'}} \frac{1}{G} \xrightarrow{\frac{1}{G'}} \frac{1}{G'}) \). Since \( \frac{1}{G'} \) and \( \frac{1}{G} \) are never equivalent in \( \mathcal{S} \) by Remark 2.18, we have \( [S_p] \neq \text{id}_{\frac{1}{G'}} \).

This means that the endomorphism on \( F(\frac{1}{G'}) = \mathcal{T}(\frac{1}{G'}, \frac{1}{G'}) \)

\[
F([S_p]) = [S_p] \circ - : \mathcal{T}(\frac{1}{G'}, \frac{1}{G'}) \rightarrow \mathcal{T}(\frac{1}{G'}, \frac{1}{G'})
\]

satisfies

\[
F([S_p])(\text{id}_{\frac{1}{G'}}) = [S_p] \neq \text{id}_{\frac{1}{G'}} = \text{id}_{F(\frac{1}{G'})}(\text{id}_{\frac{1}{G'}}),
\]

and thus \( F([S_p]) \neq \text{id}_{F(\frac{1}{G'})} \). By Corollary 5.32, this means \( F \) is non-deflative. \( \square \)
By definition, for any 0-cell \( \frac{X}{G} \in S^0 \) we have

\[
\Omega_\text{big}(\frac{X}{G}) = T(\frac{1}{e}, \frac{X}{G}) = K_0(\text{Sp}(\frac{1}{e}, \frac{X}{G})).
\]

A closer look at this shows that \( \Omega_\text{big}(\frac{X}{G}) \) has a structure of a commutative ring related to the ordinary Burnside ring (Proposition \ref{proposition}). With this view, we call \( \Omega_\text{big}(\frac{X}{G}) \) the \textit{bigger Burnside ring} over \( \frac{X}{G} \).

By (5.6), we see that \( \Omega_\text{big}(\frac{X}{G}) \) arises from the following 2-category. (This is \( \text{Span}_{\frac{X}{G}} \), or in other words ‘the left half’ of the 2-category defined in Definition \ref{definition}.)

\textbf{Definition 5.37.} Let \( \frac{X}{G} \) be any 0-cell in \( S \). Then a 2-category \( S/\frac{X}{G} \) is defined as follows.

1. A 0-cell in \( S/\frac{X}{G} \) is a 1-cell \((\frac{A}{K} \xrightarrow{\alpha} \frac{X}{G}) \) in \( S \), from some \( \frac{A}{K} \in S^0 \).
2. A 1-cell in \( S/\frac{X}{G} \) from \((\frac{A}{K} \xrightarrow{\alpha} \frac{X}{G}) \) to \((\frac{B}{H} \xrightarrow{\beta} \frac{X}{G}) \) is a pair \((\varphi, \mu)\) of a 1-cell \( \varphi \) and a 2-cell \( \mu \) in \( S \) as in the following diagram.

\[
\begin{array}{ccc}
\frac{A}{K} & \xrightarrow{\varphi} & \frac{B}{H} \\
\alpha & \searrow & \downarrow \\
\multicolumn{2}{c}{\frac{X}{G}} & \\
\mu & \swarrow & \\
\beta & & \\
\end{array}
\]

Composition of 1-cells

\[
(\frac{A}{K} \xrightarrow{\alpha} \frac{X}{G} \xrightarrow{\varphi, \mu} \frac{B}{H} \xrightarrow{\beta} \frac{X}{G} \xrightarrow{\psi, \nu} \frac{C}{L} \xrightarrow{\gamma} \frac{X}{G})
\]

is defined to be

\[
(\psi \circ \varphi, \mu \cdot (\nu \circ \varphi)): (\frac{A}{K} \xrightarrow{\alpha} \frac{X}{G}) \xrightarrow{\psi \circ \varphi} \xrightarrow{\mu \cdot (\nu \circ \varphi)} (\frac{C}{L} \xrightarrow{\gamma} \frac{X}{G}).
\]

Vertical composition of 2-cells

\[
(\frac{A}{K} \xrightarrow{\alpha} \frac{X}{G} \xrightarrow{\varphi, \mu} \frac{B}{H} \xrightarrow{\beta} \frac{X}{G} \xrightarrow{\psi, \nu} \frac{C}{L} \xrightarrow{\gamma} \frac{X}{G})
\]

is defined to be \( \varepsilon' \cdot \varepsilon \), using the vertical composition in \( S \).
Horizontal composition of 2-cells

\[
\begin{array}{ccc}
(A \xrightarrow{\alpha} X) & \xrightarrow{(\xi, \nu)} & (B \xrightarrow{\beta} X) \\
\phantom{(A \xrightarrow{\alpha} X)} & \downarrow & \phantom{(B \xrightarrow{\beta} X)} \\
\phantom{(A \xrightarrow{\alpha} X)} & \xrightarrow{\phi, \mu} & \phantom{(B \xrightarrow{\beta} X)} \\
\end{array}
\]

is defined to be \(\delta \circ \varepsilon\), using the horizontal composition in \(S\).

Then \(S/\mathcal{H}\) becomes in fact a 2-category. Moreover, the following is also shown by a general argument on 2-categories.

**Proposition 5.38.**

1. \(S/\mathcal{H}\) admits 2-coproducts induced from those in \(S\).
2. \(S/\mathcal{H}\) admits 2-products induced from 2-fibered products over \(\mathcal{H}\) in \(S\).

**Proof.** (1) This is a special case of Proposition 5.16. For any pair of 0-cells \((A \xrightarrow{\alpha} X)\) and \((B \xrightarrow{\beta} X)\) in \(S/\mathcal{H}\), if we take their 2-coproduct

\[
\begin{array}{ccc}
A & \xrightarrow{\nu_A} & B \\
\xrightarrow{\alpha} & \Pi & \xleftarrow{\beta} \\
\xrightarrow{\lambda_A} & \xleftarrow{\lambda_B} & \\
\xrightarrow{\alpha \cup \beta} & \xleftarrow{\beta} & \\
\xrightarrow{\varepsilon} & \xleftarrow{\delta} & X \\
\end{array}
\]

in \(S\), then by its universality, we obtain a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\nu_A} & B \\
\xrightarrow{\alpha} & \Pi & \xleftarrow{\beta} \\
\xrightarrow{\varepsilon} & \xleftarrow{\delta} & X \\
\xrightarrow{\alpha \cup \beta} & \xleftarrow{\beta} & \\
\xrightarrow{\lambda_A} & \xleftarrow{\lambda_B} & \\
\xrightarrow{\lambda} & \xleftarrow{\lambda} & \\
\xrightarrow{\varepsilon} & \xleftarrow{\delta} & X \\
\end{array}
\]

This gives a 2-coproduct

\[
\begin{array}{ccc}
(A \xrightarrow{\alpha} X) & \xrightarrow{(\nu_A, \lambda_A)} & (A \xrightarrow{\alpha} X) \\
\xrightarrow{\phi, \mu} & \Pi & \xleftarrow{\beta} \\
\xrightarrow{\lambda_A} & \xleftarrow{\lambda_B} & \\
\xrightarrow{\alpha \cup \beta} & \xleftarrow{\beta} & \\
\xrightarrow{\varepsilon} & \xleftarrow{\delta} & X \\
\end{array}
\]

in \(S/\mathcal{H}\).

(2) For any pair of 0-cells \((A \xrightarrow{\alpha} X)\) and \((B \xrightarrow{\beta} X)\) in \(S/\mathcal{H}\), if we take their 2-fibered product

\[
\begin{array}{ccc}
F & \xrightarrow{\nu_B} & B \\
\xrightarrow{\beta} & \Pi & \xleftarrow{\beta} \\
\xrightarrow{\nu_A} & \xleftarrow{\nu_A} & \\
\xrightarrow{\alpha} & \xleftarrow{\alpha} & \\
\xrightarrow{\beta} & \xleftarrow{\beta} & X \\
\end{array}
\]

in \(S\), then

\[
\begin{array}{ccc}
F & \xrightarrow{\nu_B} & B \\
\xrightarrow{\beta} & \Pi & \xleftarrow{\beta} \\
\xrightarrow{\nu_A} & \xleftarrow{\nu_A} & \\
\xrightarrow{\alpha} & \xleftarrow{\alpha} & \\
\xrightarrow{\beta} & \xleftarrow{\beta} & X \\
\end{array}
\]

gives a 2-product of \((A \xrightarrow{\alpha} X)\) and \((B \xrightarrow{\beta} X)\) in \(S/\mathcal{H}\). Also remark that \((\text{id}, \chi): (\xrightarrow{\nu_B, \beta B, \beta B, \beta B}) \xrightarrow{\text{adj.}} (\xrightarrow{\nu_B, \beta B, \beta B})\) is an adjoint equivalence in \(S/\mathcal{H}\).  \(\square\)
Corollary 5.39. We say two 0-cells \((\alpha : A \rightarrow X, \beta : B \rightarrow X)\) in \(\mathbb{S}/X\) are adjoint equivalent if there exists an adjoint equivalence \((\varphi, \mu) : (\alpha : A \rightarrow X) \Rightarrow (\beta : B \rightarrow X)\). Then the set of adjoint equivalence classes 
\[(\mathbb{S}/X)_0/\text{adjoint equivalence} = \text{Sp}(\frac{1}{e}, X/G)\]
forms a semi-ring with the addition and the multiplication induced from 2-coproducts and 2-products. In the same notation as in Definition 5.14, we denote the adjoint equivalence class of \((\alpha : A \rightarrow X, \beta : B \rightarrow X)\) by \([\alpha : A \rightarrow X]\).

Proposition 5.40. Let \(G\) be any finite group, and let \(X\) be any finite \(G\)-set. To any 1-cell in \(\mathbb{S}/X\)
\[(\varphi, \mu) : (\alpha : A \rightarrow X, \beta : B \rightarrow X),\]
we associate a map \(s_{(\varphi, \mu)} = f : \text{SIm} \alpha \rightarrow \text{SIm} \beta\) defined by
\[f([\xi, a]) = [\xi \mu, \varphi(a)] \quad (\forall [\xi, a] \in \text{SIm} \alpha = (G \times A)/\sim ).\]
Then we have the following.
1. \(f\) is well-defined. Moreover, if there is a 2-cell \(\omega : (\varphi, \mu) \Rightarrow (\varphi', \mu')\), then the associated maps \(f = s_{(\varphi, \mu)}\) and \(f' = s_{(\varphi', \mu')}\) are equal.
2. \(f\) is a \(G\)-map, which makes the following diagram in \(G\)-set commutative, where \(\tilde{\alpha}\) and \(\tilde{\beta}\) are those obtained in Proposition 4.15.
\[
\begin{array}{ccc}
\text{SIm} \alpha & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\text{SIm} \beta & \xrightarrow{\beta} & \end{array}
\]
3. \(\mu : (\frac{\varphi}{\varphi'} \circ \frac{\mu}{\mu'}) \Rightarrow (\frac{\varphi}{\varphi'} \circ \frac{\mu}{\mu'})\) is a 2-cell in \(\mathbb{S}\). Thus we have the following diagram in \(\mathbb{S}\).
\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & X \\
\downarrow & & \downarrow \\
B & \xrightarrow{\beta} & \\
\end{array}
\]
(4) If \((\varphi, \mu)\) is an adjoint equivalence, then \(f\) is an isomorphism of \(G\)-sets.
Proof. (1) Remark that
- By definition, \(\omega : \varphi \Rightarrow \varphi'\) satisfies \(\mu' \cdot (\beta \circ \omega) = \mu\).
- $[\xi, a] = [\xi', a']$ as an element in $\text{SIm} \alpha$ if and only if there exists $k \in K$ satisfying $a' = ka$ and $\xi = \xi' \theta_{\alpha, a}(k)$.

Then from the equations

\[
\mu'_{\alpha}, \theta_{\beta, \varphi(a')}(\omega_{a'}) = \mu_{a'}, \\
\mu_{ka} \theta_{\beta, \varphi(a)}(\theta_{\varphi, a}(k)) \mu_a^{-1} = \theta_{\alpha, a}(k) \quad (\forall k \in K),
\]

we obtain

\[
f([\xi, a]) = [\xi \mu_a, \varphi(a)] = [\xi' \theta_{\alpha, a}(k) \mu_a, \varphi(a)] = [\xi' \mu_{ka}, \theta_{\varphi, a}(k) \varphi(a)] = [\xi' \mu'_{\alpha}, \omega_{a'} \varphi(a')]
\]

for any $[\xi, a] = [\xi', a'] \in \text{SIm} \alpha$. This shows the well-definedness of $f$, and the equation $f = f'$.

(2) $f$ is a $\text{Map}$-map, since we have

\[
f(g[\xi, a]) = f([g \xi, a]) = [g \xi \mu_a, \varphi(a)] = g[\xi \mu_a, \varphi(a)] = g \cdot (f([\xi, a]))
\]

for any $[\xi, a] \in \text{SIm} \alpha$ and $g \in G$. The commutativity follows from

\[
\bar{\beta} \circ f([\xi, a]) = \bar{\beta}([\xi \mu_a, \varphi(a)]) = \xi \mu_a \cdot (\beta \circ \varphi(a)) = \xi \alpha(a) = \bar{\alpha}([\xi, a]) \quad (\forall [\xi, a] \in \text{SIm} \alpha).
\]

(3) For any $a \in A$ and $k \in K$, we have

\[
\mu_a \cdot (\nu_{\beta} \circ \varphi)(a) = \mu_a \cdot [e, \varphi(a)] = [\mu_a, \varphi(a)] = f([e, a]) = f \circ \nu_a(a), \\
\mu_{ka} \cdot (\theta_{\beta} \circ \theta_{\varphi})(a)(k) \cdot \mu_a^{-1} = \theta_{\alpha, a}(k).
\]

(4) For the identity morphism

\[
(id, id) : \left[ A \xrightarrow{\alpha} \frac{X}{G} \right] \rightarrow \left[ A \xrightarrow{\alpha} \frac{X}{G} \right],
\]

we have

\[
s_{(id, id)}([\xi, a]) = [\xi, a]
\]

for any $[\xi, a] \in \text{SIm} \alpha$, which means $s_{(id, id)} = \text{id}_{\text{SIm} \alpha}$.

Suppose $(\varphi, \mu) : \left( \frac{A}{K} \xrightarrow{\alpha} \frac{X}{G} \right) \rightarrow \left( \frac{B}{F} \xrightarrow{\beta} \frac{X}{G} \right)$ is an adjoint equivalence with a quasi-inverse $(\psi, \nu) : \left( \frac{B}{F} \xrightarrow{\beta} \frac{X}{G} \right) \rightarrow \left( \frac{A}{K} \xrightarrow{\alpha} \frac{X}{G} \right)$. Recall that we have

\[
(\psi, \nu) \circ (\varphi, \mu) = (\psi \circ \varphi, \mu \circ \nu)(\varphi).
\]

Thus for any $[\xi, a] \in \text{SIm} \alpha$, we have

\[
s_{(\psi, \nu) \circ (\varphi, \mu)}([\xi, a]) = s_{(\psi, \nu) \circ (\varphi, \mu)}([\xi, a]) = s_{(\psi, \nu) \circ (\varphi, \mu)}([\xi, a]) = s_{(\psi, \nu) \circ (\varphi, \mu)}([\xi, a]).
\]

By (1), it follows $s_{(\psi, \nu) \circ (\varphi, \mu)} = s_{(id, id)} = \text{id}$. Similarly we have $s_{(\varphi, \mu)} \circ s_{(\psi, \nu)} = \text{id}$. □
Remark 5.41. The assumption for (4) in Proposition 5.40 can be a bit weakened. In fact, if $(\varphi, \mu)$ is an equivalence (not necessarily an adjoint equivalence), then $f$ becomes isomorphic.

Proposition 5.42. Let $X$ be any 0-cell in $S$.

1. The correspondence
\[ \Omega_G(X) \to \Omega_{\text{big}} \left( \frac{X}{G} \right) ; \ [A \xrightarrow{p} X] \mapsto [A \xrightarrow{\varphi} \frac{X}{G}] \]
preserves additions and multiplications, and thus induces a ring homomorphism. The left hand side is the ordinary Burnside ring over $X$, namely the Grothendieck ring of the category $G_{set}/X$.

2. The correspondence
\[ \Omega_{\text{big}} \left( \frac{X}{G} \right) \to \Omega_G(X) ; \ [A \xrightarrow{\alpha} \frac{X}{G}] \mapsto [\text{Sim} \xrightarrow{\tilde{\alpha}} X] \]
obtained in Proposition 5.40 preserves additions and multiplications, and thus induces a ring homomorphism.

3. The composition of homomorphisms in (1) and (2)
\[ \Omega_G(X) \to \Omega_{\text{big}} \left( \frac{X}{G} \right) \to \Omega_G(X) \]
is identity.

Proof. Remark that the ring structure on $\Omega_{\text{big}} \left( \frac{X}{G} \right)$ is given by 2-coproducts and 2-products obtained in Proposition 5.38.

1. This follows from Proposition 3.12 and 3.23.
2. This follows from Corollary 4.20 and Proposition 4.21.
3. This follows from Corollary 4.18 (2).

Remark 5.43. In Proposition 5.42, the homomorphism obtained in (2) is shown to be a morphism of Mackey functors, while the homomorphism in (1) is not.

6. Interpretation of biset functors

6.1. Range of a span.

Remark 6.1. Let $G, H$ be a finite group. A $G$-$H$-biset $U$ can be regarded as a finite $G \times H$-set with the action
\[ (g, h)u = guh^{-1} \quad (\forall u \in U, \forall (g, h) \in G \times H). \]

We can associate a span to any biset as follows.

Definition 6.2. Let $G, H$ be finite groups. For any $G$-$H$-biset $U$, we associate a span $S_U$ to $\frac{1}{G}$ from $\frac{1}{H}$ by
\[ S_U = \left( \frac{1}{G} \xrightarrow{1} \frac{1}{G \times H} \xrightarrow{U \times \text{pr}_H} \frac{1}{H} \right), \]
where $\text{pr}^{(G)} : G \times H \to G$ and $\text{pr}^{(H)} : G \times H \to H$ are the projections.

Remark 6.3. Let $U, U'$ be two $G$-$H$-bisets. Then, we have the following.
1. $[S_{U \cup U'}] = [S_U] + [S_{U'}].$
2. If there is an isomorphism of $G$-$H$-bisets $U \cong U'$, then $[S_U] = [S_{U'}].$
Proof. (1) This is straightforward. 

(2) Assume there is an isomorphism of \( G \)-\( H \)-bisets \( f : U \xrightarrow{\sim} U' \). This yields an equivariant isomorphism \( \frac{f}{G \times H} : \frac{U}{G \times H} \xrightarrow{\sim} \frac{U'}{G \times H} \), which makes the following diagram commutative.

\[
\begin{array}{ccc}
\frac{1}{G} & \xleftarrow{\mu_G(g)} & \frac{1}{G} \\
\downarrow & & \downarrow \\
\frac{U}{G \times H} & \xrightarrow{\mu_{G \times H}(g)} & \frac{U'}{G \times H} \\
\downarrow & & \downarrow \\
\frac{1}{H} & \xleftarrow{\mu_H(h)} & \frac{1}{H}
\end{array}
\]

Conversely, we associate a biset to any span, which we call the range of the span.

**Definition 6.4.** Let \( S = (X \xleftarrow{\alpha_S} W_S \xrightarrow{\beta_S} Y) \) be any span in \( S \). By Corollary \[3.17\] we obtain a 1-cell

\[
\gamma_S : \frac{W_S}{L_S} \to \frac{G \times Y}{G \times H}
\]

defined by

\[
\gamma_S(w) = (\alpha_S(w), \beta_S(w)) \quad (\forall w \in W_S), \\
\theta_{\gamma_S}(\ell) = (\theta_{\alpha_S}(\ell), \theta_{\beta_S}(\ell)) \quad (\forall w \in W_S, \forall \ell \in L_S).
\]

Then the range of \( S \) is defined to be

\[
\mathcal{R}(S) = \text{Shm}_{\gamma_S}(G \times H \times W_S)/\sim,
\]

regarded as a \( G \)-\( H \)-biset by

\[
g[\xi, \eta, w]h = (g, h^{-1})[\xi, \eta, w] = [g\xi, h^{-1}\eta, w] \quad (\forall g \in G, \forall h \in H, \forall [\xi, \eta, w] \in \mathcal{R}(S)).
\]

**Remark 6.5.** By definition, the range \( \mathcal{R}(S) \) is

\[
\mathcal{R}(S) = (G \times H \times W_S)/\sim,
\]

where \((\xi, \eta, w), (\xi', \eta', w') \in G \times H \times W_S\) are equivalent (i.e. \([\xi, \eta, w] = [\xi', \eta', w']\)) if and only if there exists \( \ell \in L_S \) satisfying

\[
w' = \ell w, \quad \xi = \xi'\theta_{\alpha_S}(\ell), \quad \eta = \eta'\theta_{\beta_S}(\ell).
\]

**Remark 6.6.** For any span \( S = (X \xleftarrow{\alpha_S} W_S \xrightarrow{\beta_S} Y) \), we have the following commutative diagram.

\[
\begin{array}{ccc}
W_S & \xrightarrow{v_S} & \mathcal{R}(S) \\
\downarrow & & \downarrow \\
X \times Y & \xrightarrow{\gamma_S} & Y
\end{array}
\]
Proposition 6.7. If two spans in $S$ to $X$ from $Y$ are adjoint equivalent, then their ranges $\mathcal{R}(S)$ and $\mathcal{R}(T)$ are isomorphic as $G$-$H$-bisets.

Proof. Since $S$ is adjoint equivalent to $T$, there exists a diagram

and a diagram

with

which satisfy

for any $w \in W_S$. Similar equations hold for $\rho_T$.

If we define a map $f_\lambda: \mathcal{R}(S) \rightarrow \mathcal{R}(T)$ by

for any $[\xi, \eta, w] \in \mathcal{R}(S)$, then the following holds.

Claim 6.8. $f_\lambda$ is a well-defined map of $G$-$H$-bisets.

Suppose Claim 6.8 is shown. By symmetry, we also have a $G$-$H$-map $f_\kappa: \mathcal{R}(T) \rightarrow \mathcal{R}(S)$ defined by

for any $[x, y, v] \in \mathcal{R}(T)$. 

for any \([x, y, v] \in \mathcal{R}(T)\). Then we obtain

\[
f_{\lambda} \circ f_{\kappa}([\xi, \eta, w]) = f_{\kappa}([\xi \nu_{X, w}, \eta \nu_{Y, w}, \lambda(w)])
\]

\[
= [\xi \nu_{X, w} \mu_{X, \lambda(w)}, \eta \nu_{Y, w} \mu_{Y, \lambda(w)}, \kappa(\lambda(w))]
\]

\[
= [\xi \theta_{\alpha S, w}(\rho S_w), \eta \theta_{\beta S, w}(\rho S_w), \rho_{S_w}^{-1} w]
\]

\[
= [\xi, \eta, w]
\]

for any \([\xi, \eta, w] \in \mathcal{R}(S)\), namely, \(f_{\kappa} \circ f_{\lambda} = \text{id}_{\mathcal{R}(S)}\). Similarly we have \(f_{\lambda} \circ f_{\kappa} = \text{id}_{\mathcal{R}(T)}\), and thus \(f_{\lambda}\) is an isomorphism of \(G\)-\(H\)-bisets.

Thus it remains to show Claim 6.3. To show the well-definedness of \(f_{\lambda}\), suppose \((\xi, \eta, w), (\xi', \eta', w') \in G \times H \times W_S\) satisfy \([\xi, \eta, w] = [\xi', \eta', w']\) in \(\mathcal{R}(S)\). By definition there exists \(\ell \in L_S\) which satisfies

\[w' = \ell w, \quad \xi = \xi' \theta_{\alpha S, w}(\ell), \quad \eta = \eta' \theta_{\beta S, w}(\ell).\]

Then we have

\[
\xi' \nu_{X, w'} = \xi \theta_{\alpha S, w}(\ell)^{-1} \nu_{X, \ell w}
\]

\[
= \xi \nu_{X, w} \theta_{\alpha T, \lambda(w)}(\ell)^{-1}
\]

\[
= \xi \nu_{X, w} \theta_{\alpha T, \lambda(w)}(\theta_{\lambda, w}(\ell))^{-1}
\]

and similarly

\[
\eta' \nu_{Y, w'} = \eta \nu_{Y, w} \theta_{\beta T, \lambda(w)}(\theta_{\lambda, w}(\ell))^{-1}
\]

Thus we obtain

\[
[\xi' \nu_{X, w'}, \eta' \nu_{Y, w'}, \lambda(w')]
\]

\[
= [\xi \nu_{X, w} \theta_{\alpha T, \lambda(w)}(\theta_{\lambda, w}(\ell))^{-1}, \eta \nu_{Y, w} \theta_{\beta T, \lambda(w)}(\theta_{\lambda, w}(\ell))^{-1}, \theta_{\lambda, w}(\ell)(\lambda(w))]
\]

in \(\mathcal{R}(T)\), which shows the well-definedness of \(f_{\lambda}\). The \(G\)-\(H\)-equivariance is obvious. 

\[
\square
\]

**Proposition 6.9.** For any object \(F\) in \(\text{Add}(T, \text{RMod})\), the following are equivalent.

1. \(F\) is deflative.
2. For any pair of finite groups \(G, H\) and any span \(S = (\mathcal{O} \xrightarrow{\alpha_s} W_S \xrightarrow{\beta_S} \mathcal{F})\) to \(\mathcal{O}\) from \(\mathcal{F}\), the equality \(F([S]) = F([S_{\mathcal{R}(S)}])\) holds.

**Proof.** Let \(S = (\mathcal{O} \xrightarrow{\alpha_s} W_S \xrightarrow{\beta_S} \mathcal{F})\) be any span to \(\mathcal{O}\) from \(\mathcal{F}\). If \(F\) is deflative, then since \(v_{\gamma S}\) is stab-surjective in diagram (6.1), we have

\[
F([S]) = F(T_{\alpha S}) \circ F(R_{\beta S})
\]

\[
= F(T_{\alpha X \circ \gamma S}) \circ F(T_{\nu_{\gamma S}}) \circ F(R_{\nu_{\gamma S}}) \circ F(R_{\nu_{\gamma S}})
\]

\[
= F(T_{\alpha X \circ \gamma S}) \circ F(R_{\nu_{\gamma S}}) = F([S_{\mathcal{R}(S)}]).
\]

Conversely, assume \(F\) satisfies (2). We confirm the condition (3) in Corollary 6.3. For any surjective group homomorphism \(p: G \rightarrow H\), take a span

\[
S_p = (\mathcal{O} \xrightarrow{\frac{1}{H}} \mathcal{F} \xrightarrow{\frac{1}{G}} \mathcal{E} \xrightarrow{\frac{1}{H}})
\]
and let $\gamma = (\frac{1}{g}, \frac{1}{p})$ be the 1-cell obtained by the universality of the 2-product. The following diagram is commutative.

It suffices to show $F([S_p]) = \text{id}$. Let $\Delta: H \to H \times H$ denote the diagonal morphism, and let 

$$\nu: 1 = H/H \to (H \times H)/\Delta(H)$$

be the map defined by $\nu(eH) = (e,e)\Delta(H)$. Then $\nu\Delta : 1 \to (H \times H)/\Delta(H)$ gives an adjoint equivalence (Ind-equivalence), and the composition of 1-cells

$$1 \xrightarrow{\nu} 1 \xrightarrow{\nu\Delta} (H \times H)/\Delta(H)$$

becomes equal to $\gamma$. Since $\nu\Delta \circ \frac{1}{\nu}$ is stab-surjective, this means $\text{SIm}\gamma \cong \frac{(H \times H)/\Delta(H)}{H \times H}$, and thus $R(S_p) \cong (H \times H)/\Delta(H)$ as $H$-$H$-bisets. By assumption we obtain

$$F([S_p]) = F([S\nu(S_p)]) = F([\frac{1}{H} \xrightarrow{\nu\Delta} (H \times H)/\Delta(H) \xrightarrow{\nu\Delta} 1/H]).$$

Moreover, since

$$\frac{1}{H} \xrightarrow{\nu\Delta} (H \times H)/\Delta(H) \xrightarrow{\nu\Delta} 1/H$$

is commutative, we have

$$F\left(\frac{1}{H} \xrightarrow{\nu\Delta} (H \times H)/\Delta(H) \xrightarrow{\nu\Delta} 1/H\right) = F(\frac{1}{H}) \circ F(\frac{1}{\nu\Delta}) = F(\frac{1}{H}) \circ F(\frac{1}{\nu\Delta}),$$

Proposition 6.10. For any consecutive spans

$$S = (Y \xrightarrow{\gamma} W \xrightarrow{\beta} X),$$

$$T = (Z \xrightarrow{\delta} W \xrightarrow{\gamma} X),$$

there exists a natural isomorphism of $K$-$G$-bisets

$$R(T) \times_H R(S) \cong R(T \circ S).$$
\textbf{Proof.} By definition, composition $T \circ S$ is defined by using a 2-fibered product as

\[
\begin{array}{ccc}
Z & \xrightarrow{F} & X \\
\downarrow{\phi_W} & & \downarrow{\phi_W} \\
L_T \times L_S & \xrightarrow{\gamma} & L_T \times L_S \\
\downarrow{\alpha_T} & & \downarrow{\alpha_S} \\
W_T & \xrightarrow{\beta_T} & W_S \\
\downarrow{\gamma_T} & & \downarrow{\gamma_S} \\
H & \xrightarrow{\gamma_T} & H \\
\end{array}
\]

Namely,

\[T \circ S = (Z, F, \gamma, \delta, X, G)\]

is given by

\[F = \{ (b, a, h) \in W_T \times W_S \times H \mid \beta_T(b) = h\alpha_S(a) \},\]
\[
\gamma: F \to Z; (b, a, h) \mapsto \alpha_T(b),
\]
\[
\delta: F \to X; (b, a, h) \mapsto \beta_S(a).
\]

$F$ is equipped with a $L_T \times L_S$-action

\[(t, s) \cdot (b, a, h) = (tb, sa, \theta_{\alpha_S, a}(s)h\theta_{\beta_T, b}(t)^{-1})\]
\[
(\forall (t, s) \in L_T \times L_S, \forall (b, a, h) \in F).
\]

For any $f = (b, a, h) \in F$, the acting part of $\gamma$ and $\delta$ satisfy

\[\theta_{\gamma, f}: L_T \times L_S \to K; (t, s) \mapsto \theta_{\alpha_T, a}(t),\]
\[\theta_{\delta, f}: L_T \times L_S \to G; (t, s) \mapsto \theta_{\beta_S, a}(s).\]

Thus its range $R(T \circ S)$ should be

\[R(T \circ S) = (K \times G \times F)/\sim,\]

where $(\kappa, \xi, (b, a, h))$ and $(\kappa', \xi', (b', a', h'))$ in $K \times G \times F$ are equivalent if and only if there exists $(t, s) \in L_T \times L_S$ satisfying the following condition.

\textbf{Condition 6.11.}

\[((b', a', h')) = (tb, sa, \theta_{\alpha_S, a}(s)h\theta_{\beta_T, b}(t)^{-1}),\]
\[
\kappa = \kappa' \cdot \theta_{\alpha_T, a}(t), \quad \xi = \xi' \cdot \theta_{\beta_S, a}(s).
\]

The $K$-$G$-biset structure on $R(T \circ S)$ is defined by

\[k[\kappa, \xi, f]g = [k\kappa, g^{-1}\xi, f];\]
\[(\forall k \in K, \forall g \in G, \forall [\kappa, \xi, f] \in R(T \circ S)).\]

On the other hand, $R(T) \times_H R(S)$ is given by

\[R(T) \times_H R(S) = \left\{ ([\kappa, \eta_1, b], [\eta_2, \xi, a]) \mid \begin{array}{c} [\kappa, \eta_1, b] \in R(T) \\
\eta_2, \xi, a \in R(S) \end{array} \right\} / \sim,
\]

where $([\kappa, \eta_1, b], [\eta_2, \xi, a]) \sim ([\kappa', \eta_1', b'], [\eta_2', \xi', a'])$ holds if and only if there exists $h \in H$ satisfying

\[|\kappa', \eta_1', b'| = [\kappa, \eta_1, b]h^{-1} = [\kappa, h\eta_1, b] \quad \text{in} \quad R(T),\]
\[|\eta_2', \xi', a'| = h|\eta_2, \xi, a| = [h\eta_2, \xi, a] \quad \text{in} \quad R(S).\]
By decoding the equivalence relations defining \( \mathcal{R}(S) \) and \( \mathcal{R}(T) \), the set (6.2) can be identified with
\[
(K \times H \times W_T \times H \times G \times W_S) / \sim,
\]
where \((\kappa, \eta_1, b, \eta_2, \xi, a)\) and \((\kappa', \eta_1', b', \eta_2', \xi', a')\) in \( K \times H \times W_T \times H \times G \times W_S \) are equivalent
\[
(\kappa, \eta_1, b, \eta_2, \xi, a) \sim (\kappa', \eta_1', b', \eta_2', \xi', a')
\]
if and only if there exist \( h \in H, s \in L_S, t \in H \) satisfying
\[
a' = sa, \quad b' = tb, \\
\xi = \xi' \theta_{S,a}(s), \quad h\eta_1 = \eta_1' \theta_{T,b}(t), \\
h\eta_2 = \eta_2' \theta_{T,a,s}(s), \quad \kappa = \kappa' \theta_{T,b}(t).
\]
Since \( h \) is uniquely determined by
\[
h = \eta_2' \theta_{T,a,s}(s) \eta_2^{-1} = \eta_1' \theta_{T,b}(t) \eta_1^{-1}
\]
in these relations, we may ignore it. Namely, \((\kappa, \eta_1, b, \eta_2, \xi, a) \sim (\kappa', \eta_1', b', \eta_2', \xi', a')\) holds if and only if there exists \((t, s) \in L_T \times L_S\) which satisfies the following condition.

**Condition 6.12.**
\[
a' = sa, \quad b' = tb, \\
\xi = \xi' \theta_{S,a}(s), \quad \kappa = \kappa' \theta_{T,b}(t), \\
\eta_2^{-1} \eta_1' = \theta_{S,a}(s) \eta_2^{-1} \eta_1' \theta_{T,b}(t)^{-1}.
\]
Thus we can identify an element in \( \mathcal{R}(T) \times_H \mathcal{R}(S) \) with \([\kappa, \eta_1, b, \eta_2, \xi, a]\), which denotes the equivalence class of \((\kappa, \eta_1, b, \eta_2, \xi, a)\) with respect to \( \sim \). By Condition 6.12 especially we have
\[
[\kappa, \eta_1, b, \eta_2, \xi, a] = [\kappa, \eta_2^{-1} \eta_1, b, e, \xi, a]
\]
for any \([\kappa, \eta_1, b, \eta_2, \xi, a] \in \mathcal{R}(T) \times_H \mathcal{R}(S)\). The \( K \)-\( G \)-action on \( \mathcal{R}(T) \times_H \mathcal{R}(S) \) is given by
\[
k[\kappa, \eta_1, b, \eta_2, \xi, a] g = [k\kappa, \eta_1, b, \eta_2, g^{-1} \xi, a] \\
(\forall k \in K, \forall g \in G, [\kappa, \eta_1, b, \eta_2, \xi, a] \in \mathcal{R}(T) \times_H \mathcal{R}(S)).
\]
Now we define the maps
\[
\varphi : \mathcal{R}(T \circ S) \to \mathcal{R}(T) \times_H \mathcal{R}(S), \\
\psi : \mathcal{R}(T) \times_H \mathcal{R}(S) \to \mathcal{R}(T \circ S)
\]
by
\[
\varphi([\kappa, \xi, (b, a, h)]) = [\kappa, h, b, e, \xi, a] \quad (\forall [\kappa, \xi, (b, a, h)] \in \mathcal{R}(T \circ S)), \\
\psi([\kappa, \eta_1, b, \eta_2, \xi, a]) = [\kappa, \xi, (b, a, \eta_2^{-1} \eta_1)] \quad (\forall [\kappa, \eta_1, b, \eta_2, \xi, a] \in \mathcal{R}(T) \times_H \mathcal{R}(S)).
\]
By comparing Condition 6.11 and Condition 6.12 we can easily confirm that \( \varphi \) and \( \psi \) are well-defined \( K \)-\( G \)-maps. It remains to show \( \varphi \circ \psi = \text{id} \) and \( \psi \circ \varphi = \text{id} \).

For any \([\kappa, \eta_1, b, \eta_2, \xi, a] \in \mathcal{R}(T) \times_H \mathcal{R}(S)\), we have
\[
\varphi \circ \psi([\kappa, \eta_1, b, \eta_2, \xi, a]) = [\kappa, \eta_2^{-1} \eta_1, b, e, \xi, a] = [\kappa, \eta_1, b, \eta_2, \xi, a].
\]
For any \([\kappa, \xi, (b, a, h)] \in \mathcal{R}(T \circ S)\), we have
\[
\psi \circ \varphi([\kappa, \xi, (b, a, h)]) = \psi([\kappa, h, b, e, \xi, a]) = [\kappa, \xi, (b, a, h)].
\]
\[\square\]

**Remark 6.13.** Let \(G, H\) be finite groups. For any \(H\)-\(G\)-biset \(U\), the range of the span
\[
S_U = \left( \frac{1}{H} \xrightarrow{\text{pr}(H)} \frac{1}{H \times G} \xrightarrow{\text{pr}(G)} \frac{1}{G} \right)
\]
is calculated by

\[
\begin{array}{c}
\xrightarrow{U} \\
\xrightarrow{\text{pr}(G)} \\
\xrightarrow{\text{pr}(G/N)}
\end{array}
\]

and thus we have \(\mathcal{R}(S_U) = U\) as an \(H\)-\(G\)-biset. Remark that thus we have
\[
S_U = \left( \frac{1}{H} \xrightarrow{\text{pr}(H)} \mathcal{R}(S_U) \xrightarrow{\text{pr}(G)} \frac{1}{G} \right).
\]

**Proposition 6.14.** Let \(G\) be a finite group, and let \(N \trianglelefteq G\) be its normal subgroup. Let \(p: G \to G/N\) denote the quotient homomorphism, and let

\[
\text{pr}^{(G)}: G \times (G/N) \to G,
\]

\[
\text{pr}^{(G/N)}: G \times (G/N) \to G/N
\]

be the projections. Then there is an adjoint equivalence of spans as follows, which implies \(\left[ R_p \right] = [S_{\text{Inf}}^G_N] \). (Here, \(\text{Inf}_N^G\) denotes the biset \(G(G/N)(G/N)\) as in the introduction.)

\[
\begin{array}{c}
\xrightarrow{id} \\
\xrightarrow{\text{pr}^{(G)}} \\
\xrightarrow{\text{pr}^{(G/N)}}
\end{array}
\]

**Proof.** Let \(\iota: G \mapsto G \times (G/N)\) be the monomorphism defined by
\[
\iota(g) = (g, \overline{g}) \quad (\forall g \in G),
\]
where \(\overline{g}\) denotes the residue class of \(g\) in \(G/N\). Then we have an isomorphism of \(G \times (G/N)\)-sets
\[
\text{Ind}_\iota(1) = \text{Ind}_\iota(G/G) \xrightarrow{\cong} G/N.
\]
Thus by Proposition 3.2 we have an adjoint equivalence
\[ \nu: \frac{1}{G} \xrightarrow{\sim} \frac{(G/N)}{G \times (G/N)} \]
defined by \( \nu(1) = e \). (Here, 1 denotes the unique point in \( 1 = G/G \).) This \( \nu \) makes diagram (6.3) commutative. \( \square \)

**Remark 6.15.** Under the same assumption as in Proposition 6.14, we also have an adjoint equivalence of spans as follows, which implies \( [T_{\frac{1}{G}}] = [S_{\text{Def}_{\text{G}}}] \).

\[ \begin{array}{ccc}
\frac{1}{G/N} & \xrightarrow{\beta} & \frac{G}{(G/N) \times U} \\
\downarrow & & \downarrow \\
\frac{1}{G} & \cong & \frac{1}{G}
\end{array} \]

**Corollary 6.16.** In particular for \( N = e \), we obtain \( [S_{\text{Id}_{\text{G}}}] = [\text{Id}] \).

**Proposition 6.17.** Let \( S \) and \( T \) be any pair of spans to \( Y \) from \( X \). Then there is an isomorphism
\[ \mathcal{R}(S + T) \cong \mathcal{R}(S) \amalg \mathcal{R}(T) \]
of \( H \)-\( G \)-bisets.

**Proof.** For \( S = (\hat{Y} \xleftarrow{\beta_S} W \xrightarrow{\alpha_S} \hat{X}) \) and \( T = (\hat{Y} \xleftarrow{\beta_T} W \xrightarrow{\alpha_T} \hat{X}) \), we have
\[ \mathcal{R}(S + T) = \frac{(H \times G \times (W_S \amalg W_T))/ \sim}{(H \times G \times W_S)/ \sim} \amalg \frac{(H \times G \times W_T)/ \sim}{(H \times G \times W_T)/ \sim}. \]

It is straightforward to check this is isomorphic to
\[ \mathcal{R}(S) \amalg \mathcal{R}(T) = (\frac{(H \times G \times W_S)/ \sim}{(H \times G \times W_S)/ \sim}) \amalg (\frac{(H \times G \times W_T)/ \sim}{(H \times G \times W_T)/ \sim}). \]
\( \square \)

### 6.2. From Mackey functors to biset functors.

**Proposition 6.18.** Let \( F \) be an object in \( \text{Add}_{\text{df}}(T, \text{RMod}) \). Then an object \( B_F \) in \( \text{Add}(\mathcal{B}, \text{RMod}) \) is associated to \( F \) as follows.

(i) For any finite group \( G \), put \( B_F(G) = F(\frac{1}{G}) \).

(ii) For any \( H \)-\( G \)-biset \( U \), \( B_F(U) = F([S_U]) \). By the linearity, this is extended to any morphism in \( \mathcal{B} \).

**Proof.** By Remark 6.3, (ii) is well-defined. For any finite group \( G \), we have
\[ B_F(\text{Id}_G) = F([S_{\text{Id}_G}]) = F([\text{Id}]) = \text{id}_{F(\frac{1}{G})} = \text{id}_{B_F(G)}. \]
by Corollary 6.16.

By Proposition 6.10 and Remark 6.13, for any consecutive pair of bisets \( HU_G \) and \( KV_H \) we have
\[ B_F(V) \circ B_F(U) = F([S_V]) \circ F([S_U]) = F([S_V \circ S_U]) \]
\[ = F([S_{\beta_{SV}}]) = F([S_{(S_{(SV)} \times_H \beta_{SV})}]) \]
\[ = F([S_{(SV \times_H U)}]) = B_F(V \times_H U). \]

By linearity, it follows that \( B_F \) becomes in fact an additive functor. \( \square \)
Proposition 6.19. Let $\varphi : E \to F$ be a morphism in $Add_{dfl}(\mathcal{T}, R\text{Mod})$. Then a morphism $B\varphi : B_E \to B_F$ is associated as follows.

- For any $G \in \text{Ob}(B)$,

$$
(B\varphi)_G : B_E(G) \to B_F(G)
$$

is defined to be $\varphi_1 : E(\frac{1}{G}) \to F(\frac{1}{G})$.

Proof. For any $H$-$G$-biset $U$, we have a commutative diagram

$$
\begin{array}{ccc}
E(\frac{1}{G}) & \xrightarrow{E([U])} & E(\frac{1}{H}) \\
\varphi \downarrow & \circ & \downarrow \varphi \\
F(\frac{1}{G}) & \xrightarrow{F([U])} & F(\frac{1}{H})
\end{array}
$$

Thus

$$
B_E(G) \xrightarrow{B_E(U)} B_E(H) \\
(B\varphi)_G \circ \uparrow \downarrow \circ \uparrow \downarrow
B_F(G) \xrightarrow{B_F(U)} B_F(H)
$$

is commutative for any $H$-$G$-biset $U$. By linearity, this implies $B\varphi : B_E \to B_F$ is in fact a natural transformation. \qed

Corollary 6.20. We obtain a functor

$$
\Phi : Add_{dfl}(\mathcal{T}, R\text{Mod}) \to Add(B, R\text{Mod}) ; F \mapsto B_F.
$$

Proof. This follows from Proposition 6.18 and 6.19. \qed

6.3. From biset functors to Mackey functors.

Definition 6.21. Let $\frac{X}{G}$ be a 0-cell, and let $x_1, \ldots, x_s \in X$ be a complete set of representatives of $G$-orbits of $X$. Since

$$
\text{Ind}_{G_{x_i}}^G (G_{x_i}/G_{x_i}) = G/G_{x_i} \cong Gx_i
$$

as $G$-sets, we have an adjoint equivalence

$$
\eta_i^{(x)} : \frac{1}{G_{x_i}} \xrightarrow{\sim} Gx_i \xrightarrow{G} \frac{X}{G}.
$$

By composing this with the inclusion $Gx_i \hookrightarrow X$, we define a 1-cell $\eta_i^{(x)} : \frac{1}{G_{x_i}} \to \frac{X}{G}$.

Remark 6.22. The union of $\eta_i^{(x)} (1 \leq i \leq s)$ gives an adjoint equivalence

$$
\eta = \bigcup_{1 \leq i \leq s} \eta_i^{(x)} : \prod_{1 \leq i \leq s} \left( \frac{1}{G_{x_i}} \right) \xrightarrow{\sim} \frac{X}{G}.
$$

Thus for any $F \in \text{Ob}(Add(\mathcal{T}, R\text{Mod}))$,

$$
\begin{pmatrix}
F(R_{\eta_1^{(x)}}) \\
\vdots \\
F(R_{\eta_{s}^{(x)}})
\end{pmatrix} : \frac{X}{G} \to \bigoplus_{1 \leq i \leq s} F(\frac{1}{G_{x_i}})
$$
becomes an isomorphism of $R$-modules, with the inverse
\[
(F(T_{n}(x)) \cdots F(T_{n}(x))): \bigoplus_{1 \leq i \leq s} F\left(\frac{1}{G_{x_{i}}}\right) \to F\left(\frac{X}{G}\right).
\]

**Definition 6.23.** Let $\frac{X}{G}$, $\frac{Y}{H}$ be 1-cells. Let $x_{1}, \ldots, x_{s} \in X$ and $y_{1}, \ldots, y_{t} \in Y$ be complete sets of representatives of orbits of $X$ and $Y$. For any span $S$ to $\frac{Y}{H}$ from $\frac{X}{G}$, for any $1 \leq i \leq s$ and $1 \leq j \leq t$, define an adjoint equivalence class of span $\Gamma_{ji}^{(S)}$ to $\frac{1}{G_{x_{i}}} = \frac{1}{H_{y_{j}}}$ from $\frac{X}{G}$ to $\frac{Y}{H}$ by
\[
\Gamma_{ji}^{(S)} = R_{n}(y) \circ [S] \circ T_{n}(x).
\]

**Remark 6.24.** If $X = 1$ and $Y = 1$, then $\Gamma_{11}^{(S)} = [S]$.

**Remark 6.25.** By the well-definedness of compositions of morphisms in $Sp$, we have
\[
[S] = [T] \implies \Gamma_{ji}^{(S)} = \Gamma_{ji}^{(T)} (\forall i, j)
\]
for any pair of spans $S$ and $T$ to $\frac{Y}{H}$ from $\frac{X}{G}$.

**Lemma 6.26.** Let $\frac{X}{G}$ be a 0-cell, and let $x_{1}, \ldots, x_{s} \in X$ be a complete set of representatives of orbits of $X$. For the identity span $\frac{X}{G} = \frac{1}{G}$, we have
\[
\Gamma_{ji}^{(Id)} = R_{n}(x) \circ [Id] \circ T_{n}(x) = \begin{cases} 0 & (i \neq j), \\ [Id] & (i = j) \end{cases}
\]
for any $1 \leq i, j \leq s$.

**Proof.** This follows from Proposition 3.22 and Corollary 3.24.

**Lemma 6.27.** Let $\frac{X}{G}$, $\frac{Y}{H}$, $\frac{Z}{K}$ be 0-cells, and let
\[
x_{1}, \ldots, x_{s} \in X, \ y_{1}, \ldots, y_{t} \in Y, \ z_{1}, \ldots, z_{u} \in Z
\]
be sets of representatives of orbits. For an consecutive pair of spans $\frac{X}{G} = \frac{Y}{H} = \frac{Z}{K}$, we have
\[
\sum_{j=1}^{t} \Gamma_{ji}^{(T)} \circ \Gamma_{ji}^{(S)} = \Gamma_{ji}^{(T \circ S)},
\]
for any $1 \leq i \leq s$ and $1 \leq j \leq u$.

**Proof.** This follows from
\[
\sum_{j=1}^{t} \Gamma_{ji}^{(T)} \circ \Gamma_{ji}^{(S)} = \sum_{j=1}^{t} \left( R_{n}(x) \circ [T] \circ T_{n}(y) \circ R_{n}(y) \circ [S] \circ T_{n}(x) \right)
\]
\[
= R_{n}(x) \circ [T] \circ \sum_{j=1}^{t} \left( T_{n}(y) \circ R_{n}(y) \circ [S] \circ T_{n}(x) \right)
\]
\[
= R_{n}(x) \circ [T] \circ [Id] \circ [S] \circ T_{n}(x)
\]
\[
= R_{n}(x) \circ [T \circ S] \circ T_{n}(x) = \Gamma_{ji}^{(T \circ S)}.
\]

**Proposition 6.28.** For any object $B$ in $Add(B, RMod)$, we can associate an object $F_{B}$ in $Add(T, RMod)$ as follows. Moreover, $F_{B}$ becomes deflative.
(i) For any $\frac{X}{G} \in \text{Ob}(\mathcal{T})$, take a set of representatives of $G$-orbits $x_1, \ldots, x_s$, and put

$$F_B(\frac{X}{G}) = B(G_{x_1}) \oplus \cdots \oplus B(G_{x_s}).$$

(ii) Let $\frac{X}{G}$ and $\frac{Y}{H}$ be objects in $\mathcal{T}$. Let $x_1, \ldots, x_s \in X$ and $y_1, \ldots, y_t \in Y$ be the set of representatives chosen in (i). For any $[S] \in \text{Sp}(\frac{X}{G}, \frac{Y}{H})$, define

$$F_B([S]) : F_B(\frac{X}{G}) \to F_B(\frac{Y}{H})$$

to be the matrix

$$M_B^{(S)} = (m_{ji}^{(S)})_{ji},$$

with $(j, i)$-component $m_{ji}^{(S)} : B(G_{x_i}) \to B(H_{y_j})$ given by

$$m_{ji}^{(S)} = B(\mathcal{R}(\Gamma_{ji}^{(S)})).$$

This extends to an morphism in $\mathcal{T}$ by linearity.

**Proof.** By Remark 6.25 matrix $M_B^{(S)}$ is well-defined. For any $\frac{X}{G} \in \text{Ob}(\mathcal{T})$, we have

$$m_{ji}^{(\text{Id})} = B(\mathcal{R}(\Gamma_{ji}^{(\text{Id})})) = \begin{cases} B(\mathcal{R}(0)) = 0 & (i \neq j), \\ B(\mathcal{R}(\text{Id})) = \text{id} & (i = j) \end{cases}$$

by Lemma 6.26 which means $F_B(\text{Id}) = \text{id}_{F_B(\mathcal{T})}$. For any consecutive spans $\frac{S}{\frac{T}{\mathcal{R}}} \frac{X}{\mathcal{R}}$ and $\frac{S}{\frac{T}{\mathcal{R}}} \frac{Y}{\mathcal{R}}$, we have

$$\sum_{j=1}^{t} m_{ij}^{(T)} \circ m_{ji}^{(S)} = \sum_{j=1}^{t} B(\mathcal{R}(\Gamma_{ij}^{(T)})) \circ B(\mathcal{R}(\Gamma_{ji}^{(S)}))$$

$$= B(\prod_{1 \leq j \leq t} (\mathcal{R}(\Gamma_{ij}^{(T)})) \times_H \mathcal{R}(\Gamma_{ji}^{(S)})))$$

$$= B(\prod_{1 \leq j \leq t} \mathcal{R}(\Gamma_{ij}^{(T)} \circ \Gamma_{ji}^{(S)}))$$

$$= B(\mathcal{R}(\sum_{j=1}^{t} \Gamma_{ij}^{(T)} \circ \Gamma_{ji}^{(S)}))$$

$$= B(\mathcal{R}(\Gamma_{T \circ S}^{(T \circ S)})) = m_{i}^{(T \circ S)}$$

for any $1 \leq i \leq s$ and $1 \leq \ell \leq u$ by Remark 6.23, Proposition 6.10, 6.17 and Lemma 6.27. This means $M_B^{(T \circ S)} = M_B^{(T)} \circ M_B^{(S)}$, and thus we obtain

$$F_B([T] \circ [S]) = F_B([T]) \circ F_B([S]).$$

By linearity, this implies $F_B$ preserves compositions for arbitrary morphisms in $\mathcal{T}$. Thus $F_B : \mathcal{T} \to R\text{Mod}$ is in fact a functor. By construction, $F_B$ preserves finite products.

To show $F_B$ is deflative, let $G$ be any finite group, and let $N \triangleleft G$ be any normal subgroup. Let $p : G \to G/N$ denote the quotient homomorphism. Then by
Proposition 6.14 and Remark 6.13, 6.15, 6.24, we have

\[ F_B\left(\begin{array}{c}
\frac{1}{G/N} \\
\frac{1}{G} \\
\frac{1}{G/N}
\end{array}\right) = F_B(T_p) \circ F_B(R_p) = F_B([S_{\text{Def}}]_N) \circ F_B([S_{\text{Inf}}]_N) = B(\text{def}(G/N)) \circ B(\text{inf}(G/N)) = B(\text{def}(G/N)) \circ B(\text{inf}(G/N)) = \text{id}. \]

□

Proposition 6.29. For any morphism \( \varphi: B \to B' \) in \( \text{Add}(B, R\text{Mod}) \), we can associate a morphism \( F \varphi: F_B \to F_{B'} \) in \( \text{Add}_d(T, R\text{Mod}) \) as follows.

- For any object \( \frac{X}{G} \in \text{Ob}(T) \) with a set of representatives of orbits \( x_1, \ldots, x_s \), we define \( (F \varphi)_{\frac{X}{G}}: F_B(\frac{X}{G}) \to F_{B'}(\frac{X}{G}) \) to be

\[ \varphi_{Gx_1} \oplus \cdots \oplus \varphi_{Gx_s}: B(G_{x_1}) \oplus \cdots \oplus B(G_{x_s}) \to B'(G_{x_1}) \oplus \cdots \oplus B'(G_{x_s}). \]

Proof. It suffices to show that \( F \varphi \) is in fact a natural transformation. By linearity, it is enough to show the commutativity of

\[ F_B(\frac{X}{G}) \xrightarrow{(F \varphi)_{\frac{X}{G}}} F_{B'}(\frac{X}{G}) \]

\[ F_B(\frac{Y}{H}) \xrightarrow{(F \varphi)_{\frac{Y}{H}}} F_{B'}(\frac{Y}{H}) \]

for any span \( \frac{Y}{H} \xrightarrow{\varphi} \frac{X}{G} \). However this follows from the commutativity of

\[ B(G_{x_1}) \oplus \cdots \oplus B(G_{x_s}) \xrightarrow{\varphi_{Gx_1} \oplus \cdots \oplus \varphi_{Gx_s}} B'(G_{x_1}) \oplus \cdots \oplus B'(G_{x_s}) \]

\[ B(H_{y_1}) \oplus \cdots \oplus B(H_{y_t}) \xrightarrow{\varphi_{H_{y_1}} \oplus \cdots \oplus \varphi_{H_{y_t}}} B'(H_{y_1}) \oplus \cdots \oplus B'(H_{y_t}) \]

□

Corollary 6.30. We obtain a functor

\[ \Psi: \text{Add}(B, R\text{Mod}) \to \text{Add}_d(T, R\text{Mod}); B \mapsto F_B. \]

Proof. This follows from Proposition 6.28 and 6.29.

□

Theorem 6.31. There is an equivalence of categories

\[ \text{Mack}_d^R(S) \simeq \text{BisetFtr}^R. \]

Proof. By Proposition 6.29 it suffices to show the equivalence \( \text{Add}_d(T, R\text{Mod}) \simeq \text{Add}(B, R\text{Mod}) \). So far we constructed functors

\[ \Phi: \text{Add}_d(T, R\text{Mod}) \to \text{Add}(B, R\text{Mod}); F \mapsto B_F, \]

and

\[ \Psi: \text{Add}(B, R\text{Mod}) \to \text{Add}_d(T, R\text{Mod}); B \mapsto F_B. \]

It suffices show \( \Phi \circ \Psi \simeq \text{id} \) and \( \Psi \circ \Phi \simeq \text{id} \).

□
(i) $\Phi \circ \Psi \cong \text{Id}$. Let $B$ be any object in $Add(B, R\text{Mod})$. For any finite group $G$, we have

$$B_{F_{\Phi}}(G) = F_B\left(\frac{1}{G}\right) = B(G).$$

For any $H$-$G$-biset $U$, we have

$$B_{F_{\Phi}}(U) = F_B(\mathcal{S}_U) = B(\mathcal{R}(\mathcal{S}_U)) = B(U).$$

Thus there is an isomorphism $B_{F_{\Phi}} \cong B$. This gives a natural isomorphism $\Phi \circ \Psi \cong \text{Id}$.

(ii) $\Psi \circ \Phi \cong \text{Id}$. Let $F$ be any object in $Add_{dfl}(T, R\text{Mod})$. For any 1-cell $\frac{X}{G} \in \text{Ob}(T)$ with a set of representatives $x_1, \ldots, x_s \in X$ of orbits, we have an adjoint equivalence

$$\eta^{(X)} = \bigcup_{1 \leq i \leq s} \eta^{(x_i)} : \bigoplus_{1 \leq i \leq s} \left(\frac{1}{G x_i}\right) \cong \frac{X}{G},$$

as in Definition 6.21, and thus an isomorphism

$$\tau^{X,G} = \begin{pmatrix}
F(\mathcal{R}(\eta^{(x_1)})) \\
\vdots \\
F(\mathcal{R}(\eta^{(x_s)}))
\end{pmatrix} : F\left(\frac{X}{G}\right) \cong \bigoplus_{1 \leq i \leq s} F\left(\frac{1}{G x_i}\right) = F_{B_{\Phi}}\left(\frac{X}{G}\right).$$

Let $S$ be any span to $\frac{Y}{H}$ from $\frac{X}{G}$, where $\frac{Y}{H}$ is equipped with a set of representatives of orbits $y_1, \ldots, y_t \in Y$. Remark that by definition of $\Gamma^{(S)}_{ji}$ we have a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{1 \leq i \leq s} F\left(\frac{1}{G x_i}\right) & \overset{\cong}{\longrightarrow} & F\left(\frac{X}{G}\right) \\
(F(\Gamma^{(S)}_{ji})) & \circlearrowleft & F(\mathcal{S}) \\
\bigoplus_{1 \leq j \leq t} F\left(\frac{1}{H y_j}\right) & \overset{\cong}{\longrightarrow} & F\left(\frac{Y}{H}\right)
\end{array}$$

As in Proposition 6.28, the map $F_{B_{\Phi}}(S)$ is defined by using

$$M^{(S)}_{B_{\Phi}} = (m^{(S)}_{ji}),$$

where $m^{(S)}_{ji} = B_{\Phi}(\mathcal{R}(\Gamma^{(S)}_{ji}))$. Since $F$ is deflative, we have

$$F(\Gamma^{(S)}_{ji}) = F(\mathcal{R}(\mathcal{S}(\Gamma^{(S)}_{ji}))) = B_{\Phi}(\mathcal{R}(\mathcal{S}(\Gamma^{(S)}_{ji}))) = m^{(S)}_{ji} \quad (\forall i, j),$$

and thus we obtain $F_{B_{\Phi}}(S) = (F(\Gamma^{(S)}_{ji}))_{ji}$. By

$$\left(F(\mathcal{T}_{\eta^{(x_i)}}) \cdots F(\mathcal{T}_{\eta^{(x_s)}})\right) = \left(\begin{pmatrix}
F(\mathcal{R}(\eta^{(x_1)})) \\
\vdots \\
F(\mathcal{R}(\eta^{(x_s)}))
\end{pmatrix}^{-1}
\right),$$

we have $F_{B_{\Phi}}(S) = (F(\Gamma^{(S)}_{ji}))_{ji}$. By

$$\left(F(\mathcal{T}_{\eta^{(x_i)}}) \cdots F(\mathcal{T}_{\eta^{(x_s)}})\right) = \left(\begin{pmatrix}
F(\mathcal{R}(\eta^{(x_1)})) \\
\vdots \\
F(\mathcal{R}(\eta^{(x_s)}))
\end{pmatrix}^{-1}
\right),$$

we have $F_{B_{\Phi}}(S) = (F(\Gamma^{(S)}_{ji}))_{ji}$. By
diagram (6.4) means the commutativity of

$$
\begin{array}{ccc}
F_B F(X) & \overset{\tau_X}{\sim} & F(X) \\
\downarrow F_B([S]) & & \downarrow F([S]) \\
F_B F(Y) & \overset{\tau_Y}{\sim} & F(Y)
\end{array}
$$

By the linearity, this shows that $\tau$ becomes an isomorphism $\tau: F \overset{\sim}{\rightarrow} F_B$ of objects in $Add_{df}(T, RMod)$. It can be easily checked this gives a natural isomorphism $\Psi \circ \Phi \cong \text{Id}$. \hfill \Box

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References

[1] Borceux, F.: Handbook of categorical algebra. 1. Basic category theory, Encyclopedia of Mathematics and its Applications, 50. Cambridge University Press, Cambridge, 1994. xvi+345 pp.
[2] Bouc, S.: Fused Mackey functors, Geometriae Dedicata, DOI:10-1007/s10711-014-9965-3, to appear.
[3] Bouc, S.: Biset functors for finite groups, Lecture Notes in Mathematics, 1990, Springer-Verlag, Berlin (2010).
[4] Bouc, S.: Green functors and $G$-sets, Lecture Notes in Mathematics, 1671, Springer-Verlag, Berlin (1997).
[5] Dupont, M.; Vitale, E.M.: Proper factorization systems in 2-categories, J. Pure Appl. Algebra 179 (2003) no. 1–2, 65–86.
[6] Lindner, H.: A remark on Mackey-functors, Manuscripta math. 18 (1976), 273-278.
[7] Mac Lane, S.: Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, (1998). xii+314 pp.
[8] Panchadcharam, E.; Street, R.: Mackey functors on compact closed categories. J. Homotopy Relat. Struct. 2 (2007) no. 2, 261–293.

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