Hyers-Ulam Stability for Linear Differences with Time Dependent and Periodic Coefficients: The Case When the Monodromy Matrix Has Simple Eigenvalues

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Abstract: Let \( q \geq 2 \) be a positive integer and let \((a_j), (b_j)\) and \((c_j)\) (with \( j \) nonnegative integer) be three given \( \mathbb{C}\)-valued and \( q\)-periodic sequences. Let \( A(q) := A_{q-1} \cdots A_0 \), where \( A_j \) is defined below. Assume that the eigenvalues \( x, y, z \) of the "monodromy matrix" \( A(q) \) verify the condition \( (x - y)(y - z)(z - x) \neq 0 \). We prove that the linear recurrence in \( \mathbb{C}\ x_{n+3} = a_nx_{n+2} + b_nx_{n+1} + c_nx_n, \quad n \in \mathbb{Z}_+ \) is Hyers–Ulam stable if and only if \( (|x| - 1)(|y| - 1)(|z| - 1) \neq 0 \), i.e., the spectrum of \( A(q) \) does not intersect the unit circle \( \Gamma := \{ w \in \mathbb{C} : |w| = 1 \} \).

Keywords: difference and differential equations; discrete dichotomy; Hyers–Ulam stability

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1. Introduction

Exponential dichotomy and its links with the unconditional stability of differential dynamics systems were first highlighted by O. Perron in 1930 [1]. The reader can find details on the subsequent evolution of this topic in Coppel’s monograph [2]. The history of the Ulam problem (concerning the stability of a functional equation) and of stability in the sense of Hyers–Ulam is well known. In particular, Hyers–Ulam stability for linear recurrences and for systems of linear recurrences is considered in [3–16], and the references therein.

The relationship between exponential stability and Hyers–Ulam stability has been studied in the articles [3,8,9,17,18], and this article continues these studies.

2. Notations and Definitions

By \( \mathbb{C} \), we denote the set complex numbers and \( \mathbb{Z}_+ \) is the set of all nonnegative integers. Now, \( \mathbb{C}^m \) (with \( m \) a given positive integer) is the set of all vectors \( v = (\xi_1, \cdots, \xi_m)^T \) with \( \xi_j \in \mathbb{C} \) for every integers \( 1 \leq j \leq m \); here and in as follows \( ^T \) denotes the transposition. The norm on \( \mathbb{C}^m \) is the well-known Euclidean norm defined by \( \|v\| := (|\xi_1|^2 + \cdots + |\xi_m|^2)^{1/2} \). In addition, \( \mathbb{C}^{m \times n} \) (with \( m \) and \( n \) given positive integers) denotes the set of all \( m \) by \( n \) matrices with complex entries. In particular, \( \mathbb{C}^{m \times m} \) becomes a Banach algebra when it is endowed with the (Euclidean) matrix norm defined by \( ||M|| := \sup_{\|v\| \leq 1} ||Mv||, \quad v \in \mathbb{C}^m, \quad M \in \mathbb{C}^{m \times m} \). As is usual, the rows and columns of a matrix
$M \in \mathbb{C}^{m \times n}$ are identified by vectors of the corresponding dimensions and in that case its norm is the vector norm. The entry $m_{ij}$ of a matrix $M$ (i.e., the entry in $M$ located at the intersection between the $i$th row and the $j$th column) is denoted by $|M|_{ij}$. As is usual, the uniform norm of a $\mathbb{C}^m$-valued and bounded sequence $g = (g_n)$ is defined and denoted by $\|g\|_\infty := \sup_{n \in \mathbb{Z}_+} \|g_n\|$.

Let $\varepsilon > 0$ be given. We recall (see also [8] for the two-dimensional case) that a scalar valued sequence $(y_j)$ is an $\varepsilon$-approximative solution of the linear recurrence

$$x_{n+3} = a_n x_{n+2} + b_n x_{n+1} + c_n x_n, \quad n \in \mathbb{Z}_+ \quad (1)$$

if

$$|y_{n+3} - a_n y_{n+2} - b_n y_{n+1} - c_n y_n| \leq \varepsilon, \quad \forall n \in \mathbb{Z}_+. \quad (2)$$

The recurrence in Equation (1) is Hyers–Ulam stable if there exists a positive constant $L$ such that for every $\varepsilon > 0$ and every $\varepsilon$-approximative solution $y = (y_j)$ of Equation (1) there exists an exact solution $\theta = (\theta_j)$ of Equation (1) such that $\|y - \theta\|_\infty \leq L \varepsilon$.

**Remark 1.** Since any $\varepsilon$-approximative solution of the recurrence in Equation (1) can be seen as a solution of the nonhomogeneous equation

$$x_{n+3} - a_n x_{n+2} - b_n x_{n+1} - c_n x_n = f_{n+1}, \quad n \in \mathbb{Z}_+. \quad (3)$$

for some scalar valued sequence $(f_n)$ with $f_0 = 0$ and $\|f(0, f_1)\|_\infty \leq \varepsilon$, one has that Equation (1) is Hyers–Ulam stable if and only if there exists a positive constant $L$ such that for every $\varepsilon > 0$, every sequence as above, and every initial condition $Y_0 = (z_0, v_0, w_0)^T \in \mathbb{C}^3$, there exists an initial condition $X_0 = (x_0, x_1, x_2)^T \in \mathbb{C}^3$ such that

$$|\phi(n, Y_0, (f_k)) - \phi(n, X_0, (0))| \leq L \varepsilon. \quad (4)$$

Here, and in what follows, $(\phi(n, Y_0, (f_k))$ denotes the solution of the nonhomogeneous linear recurrence in Equation (3) initiated from $Y_0$.

**Proof.** See the proof of Proposition 3.1 in [9]. □

3. Background, Previous Results and the Main Result

**Proposition 1.** ([19]) Let $A$ be a $3 \times 3$ matrix whose spectrum (i.e., the set of its eigenvalues $\sigma(A) := \{x, y, z\}$) satisfies the condition

$$(x - y)(x - z)(y - z) \neq 0. \quad (5)$$

Then, for every nonnegative integer $n$, one has

$$A^n = x^n B + y^n C + z^n D \quad (6)$$

where

$$B = \frac{(A - yI_3)(A - zI_3)}{(x - y)(x - z)}, \quad C = \frac{(A - xI_3)(A - zI_3)}{(y - x)(y - z)} \quad (7)$$

and

$$D = \frac{(A - xI_3)(A - yI_3)}{(z - x)(z - y)}. \quad (8)$$

**Remark 2.** (i) The matrices $B, C$ and $D$ in Equation (6) are orthogonal projections, that is

$$BC = BD = CD = 0_3; \text{ the null matrix of order three}, \quad (9)$$

and

$$B^2 = B, \quad C^2 = C, \text{ and } D^2 = D. \quad (10)$$
(ii) In addition, B, C, and D are nonzero matrices.

Proof. Under assumption in Equation (5), the characteristic polynomial $P_A$ and the minimal polynomial $m_A$ of $A$ coincide and $P_A(\lambda) = (\lambda - x)(\lambda - y)(\lambda - z)$. Thus, from the Hamilton–Cayley Theorem we have $P_A(A) = (A - xI_3)(A - yI_3)(A - zI_3) = 0_3$, and Equation (9) becomes clear.

To prove Equation (10), it is enough to see that

$$B^2 - B = \frac{(A - yI_3)(A - zI_3)(A - xI_3)(A - (y + z - x)I_3)}{(x - y)(x - z)^2};$$

the details are clear thus omitted. Then, we apply the Hamilton–Cayley theorem and obtain Equation (10).

Finally, assuming that $B = 0_3$, the polynomial

$$Q(\lambda) = \frac{(\lambda - y)(\lambda - z)}{(x - y)(x - z)}$$

is annulated by $A$ and its degree is equal 2 and is a contradiction with the minimality of the degree of $m_A$. □

Let $q, (a_j), (b_j), (c_j)$ be as above. Recall that

$$A(q) := A_{q-1} \cdots A_0, \text{ where } A_j := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_j & b_j & a_j \end{pmatrix}, \quad j \in \mathbb{Z}_+.$$ (11)

Our main result reads as follows.

**Theorem 1.** Assume that the eigenvalues $x, y, z$ (of $A(q)$) satisfy the condition in Equation (5). Then, the following two statements are equivalent:

1. The linear recurrence in $\mathbb{C}$

$$x_{n+3} = a_n x_{n+2} + b_n x_{n+1} + c_n x_n, \quad n \in \mathbb{Z}_+.$$ (12)

is Hyers–Ulam stable.

2. The eigenvalues of $A(q)$ verify the condition

$$|x| - 1)(|y| - 1)(|z| - 1) \neq 0.$$ (13)

The proof of the implication 2 $\Rightarrow$ 1 is covered (for the most part) in the existing literature. We present the ideas and complete the details. For unexplained terminology, we refer the reader to [8,9]. The following result is taken directly from the second section of [9].

Let $X$ be a complex, finite dimensional Banach space and let $B = \{B_n\}_{n \in \mathbb{Z}_+}$ and $P = \{P_n\}_{n \in \mathbb{Z}_+}$ be two families of linear operators acting on $X$. Assume that:

[A1] $B_n q = B_n$ and $P_{n+q} = P_n$, for all $n \in \mathbb{Z}_+$ and some positive integer $q$.

[A2] $P_n^2 = P_n$, for all $n \in \mathbb{Z}_+$, that is, $P$ is a family of projections.

[A3] $B_n P_n = P_{n+1} B_n$, for all $n \in \mathbb{Z}_+. \text{ In particular, this yields that } B_n x \in \ker(P_{n+1}) \text{ for each } x \in \ker(P_n).$

[A4] For each $n \in \mathbb{Z}_+$, the map

$$x \mapsto B_{[n]} x := B_n x : \ker(P_n) \rightarrow \ker(P_{n+1})$$

is invertible. Denote by $(B_{[n]})^{-1}$ its inverse.
We say that the family $B$ is $\mathcal{P}$-dichotomic if there exist four positive constants $N_1$, $N_2$, $\nu_1$ and $\nu_2$ such that

(i) $\|U_B(n, k)P_k\| \leq N_1 e^{-\nu_1(n-k)}$ for all $n \geq k \geq 0$.

(ii) $\|U_B(n, k)(I - P_k)\| \leq N_2 e^{\nu_2(n-k)}$ for all $0 \leq n < k$.

Here, $U_B(n, k) = B_{n-1} \cdots B_k$ when $n > k$, $U_B(k, k) = I$-the identity operator on $X$, and $U_B(n, k) := (B_{k})^{-1} \cdots (B_{n-1})^{-1}$ when $n < k$.

**Theorem 2.** ([9]) Assume that the families $B$ and $\mathcal{P}$ satisfy [A1]–[A4] above. The following four statements are equivalent:

1. The monodromy operator $B(q) := B_{q-1} \cdots B_0$ is hyperbolic (that is, the spectrum of $B(q)$ does not intersect the unit circle $\Gamma = \{ w \in \mathbb{C} : |w| = 1 \}$, or equivalently (with the terminology in [9]) it possesses a discrete dichotomy.

2. The family $B$ is $\mathcal{P}$-dichotomic.

3. For each bounded sequence $(G_n)_{n \in \mathbb{Z}}$, $G_0 = 0$ (of $X$-valued functions) there exists a unique bounded solution (starting from $\ker(P_0)$) of the difference equation.

$$x_{n+1} = B_n x_n + G_{n+1}, \quad n \in \mathbb{Z}.$$  

4. The family $B$ is Hyers–Ulam stable.

We mention that the equivalence between (2) and (3) still works when $X$ is an infinite dimensional Banach space (see [20]). We use Theorem 2 to prove $2 \Rightarrow 1$ in Theorem 1.

The main ingredient in the proof of the implication $1 \Rightarrow 2$ in Theorem 1 is the following Lemma. With $A$ we denote the set of all matrices $A_j$ (with $j \in \mathbb{Z}_+$), where $A_j$ is given in Equation (11) and the matrix $U_A(n, k)$ is defined above.

**Lemma 1.** If the spectrum of $A(q)$ intersects the unit circle then for each $\epsilon > 0$ there exists a $\mathbb{C}$-valued sequence $(f_j)_{j \in \mathbb{Z}}$, with $f_0 = 0$ and $\|((f_j))\|_{\infty} \leq \epsilon$ such that for every initial condition $Z_0 = (z_0, y_0, z_0)^T \in \mathbb{C}^3$, the $\mathbb{C}$-valued sequence

$$\left( \left\| U_A(n, 0)Z_0 + \sum_{k=1}^n U_A(n, k)F_k \right\|_{\infty} \right)_{n \in \mathbb{Z}_+}$$  

(with $F_k = (0, 0, f_k)^T$), is unbounded.

4. **Proofs**

**Proof of Lemma 1.** We first use Proposition 1 with $A(q)$ instead of $A$. Assume that the eigenvalue $x$ has modulus 1. Let $P_x$ be the Riesz projection associated to $A(q)$ and $x$; that is

$$P_x = \frac{1}{2\pi i} \int_{C(x, r)} (wI_3 - A(q))^{-1} \, dw,$$

where $C(x, r)$ is the circle centered at $x$ of radius $r$, and $r$ is small enough that $y$ and $z$ are located outside of the circle. Using Dunford calculus (see [21]), it is easy to see that $P_x A(q)^n = x^n B$, for each $n \in \mathbb{Z}_+$. Consider the matrix $B$ from Equation (8), of the form:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$  

The solution of the system
\[ X_{n+1} = A_n X_n + F_{n+1}, \quad n \in \mathbb{Z}_+, \quad (15) \]

initiated from \( Z_0 \), where \( X_n = (z_n \ v_n \ w_n)^T \in \mathbb{C}^3, F_n = \left( \begin{array}{ccc} 0 & 0 & f_n \end{array} \right)^T \) and \( A_n = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_n & b_n & a_n \end{array} \right) \)

is given by

\[ \Phi_n := \Phi(n, Z_0, (F_k)) = U_A(n, 0) Z_0 + \sum_{k=1}^{n} U_A(n, k) F_k. \quad (16) \]

Denote by \( (\varphi(n, Z_0, (f_k)) \) the solution of Equation (1). An obvious calculation yields

\[ \varphi_n := \varphi(n, Z_0, (f_k)) = \left[ U_A(n, 0) Z_0 + \sum_{k=1}^{n} U_A(n, k) F_k \right]_{11}. \quad (17) \]

In fact, one has \( \Phi_n = \left( \begin{array}{ccc} \varphi_n & \varphi_{n+1} & \varphi_{n+2} \end{array} \right)^T. \)

**Case 1.1.** Let \( b_{13} \neq 0 \). Set

\[ F_k = \left\{ \begin{array}{ll} x^{j/q} u_0, & \text{if } k = nq \\ 0, & \text{if } k \text{ is not a multiple of } q. \end{array} \right. \quad (18) \]

where \( u_0 = \left( \begin{array}{ccc} 0 & 0 & c_0 \end{array} \right)^T \) and \( c_0 \) is a randomly chosen nonzero complex scalar with \( |c_0| < \varepsilon \).

Successively, one has

\[ \Phi_{nj} = U_A(nq, 0) Z_0 + \sum_{k=1}^{nq} U_A(nq, k) F_k = U_A(nq, 0) Z_0 + U_A(nq, 0) F_0 + U_A(nq, q) F_q + \cdots + U_A(nq, nq) F_{nq} = U_A(nq, 0) Z_0 + \sum_{j=1}^{nq} U_A(nq, jq) F_{jq} = A(q)^{nq} Z_0 + \sum_{j=1}^{nq} x^{j/q} A(q)^{nq-j} u_0, \]

that yields

\[ P_x \left[ U_A(nq, 0) Z_0 + \sum_{k=1}^{nq} U_A(nq, k) F_k \right] = P_x A(q)^{nq} Z_0 + \sum_{j=1}^{nq} x^{j/q} P_x A(q)^{nq-j} u_0 = x^n B Z_0 + \sum_{j=1}^{nq} x^{j/q} B u_0 = x^n B Z_0 + n x^n B u_0. \]

Since the sequence \( (x^n B Z_0)_n \) is bounded, it is enough to prove that the sequence \( (n x^n B u_0)_{11} \) is unbounded, and note

\[ |(n x^n B u_0)_{11}| = n |b_{13} c_0| \to \infty \text{ as } n \to \infty. \]

**Case 1.2.** Let \( b_{23} \neq 0 \). Arguing as above we can show that \( (\varphi_{n+1}) \) is unbounded, that is that \( (\varphi_n) \) is unbounded as well.

**Case 1.3.** Analogously, we can treat the case \( b_{33} \neq 0 \).

**Case 1.** Let \( b_{13} = b_{23} = b_{33} = 0 \) and \( b_{12} \neq 0 \). Set

\[ F_k = \left\{ \begin{array}{ll} x^{j/q} A_{q-1} u_0, & \text{if } k = nq \\ 0, & \text{if } k \text{ is not a multiple of } q, \end{array} \right. \quad (19) \]

where \( u_0 \) and \( c_0 \) are taken as above. We obtain
\[ \Phi_{nq} = A(q)^n Z_0 + \sum_{j=1}^{n} x^j A(q)^{n-j} A_{q-1} u_0 \]

which leads to

\[ |q_{nq}| = \left| \sum_{j=1}^{n} x^j P_A(q)^{n-j} A_{q-1} u_0 \right|_{11} \]

\[ = \left| \sum_{j=1}^{n} x^j n^q A_{q-2} A_{q-1} u_0 \right|_{11} \]

\[ = n^q b_{12} c_0 \]

\[ \Rightarrow |n^q b_{12} c_0| = n |b_{12} c_0| \to \infty \text{ as } n \to \infty. \]

**Case 2.2.** Let \( b_{22} \neq 0 \). Similar to the previous case, we can show that \( (q_{n+1}) \) is unbounded, that is, \( (q_n) \) is unbounded as well.

**Case 3.** Let \( b_{12} = b_{13} = 0 \) and \( b_{11} \neq 0 \). Then, set

\[ F_k = \begin{cases} x^k A_{q-2} A_{q-1} u_0, & \text{if } k = nq \\ 0, & \text{if } k \text{ is not a multiple of } q, \end{cases} \] (20)

with \( u_0 \) and \( c_0 \) as above.

As in the previous cases, we obtain

\[ \left| \sum_{j=1}^{n} x^j P_A(q)^{n-j} A_{q-2} A_{q-1} u_0 \right|_{11} = |n^q b_{12} c_0| = n |b_{12} c_0| \to \infty \text{ as } n \to \infty, \]

therefore \( (q_n) \) is again unbounded.

Finally, we remark that the matrix \( B \) cannot be of the form

\[ \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}. \]

Indeed, if this is the case, all eigenvalues of \( B \) are equal to 0 and the Hamilton–Cayley Theorem yields \( B^3 = 0_3 \). Since \( B^2 = B \) we obtain \( B^2 = 0_3 \), that is, \( B = 0_3 \). This contradicts the statement in Remark 2, (ii).

**Proof of Theorem 1.** \( \Rightarrow \). We argue by contradiction. Suppose that \( \sigma(A(q)) \) intersects the unit circle. Without loss of generality, assume that \( x \) is an eigenvalue of \( A(q) \) and \( |x| = 1 \). Let \( Y_0 \) and \( X_0 \) be as in the Remark 1. From Lemma 1, it follows that the sequence in Equation (14) with \((Y_0 - X_0)\) instead of \(Z_0\) is unbounded and this contradicts Equation (4).

\( \Rightarrow \). From the assumption and Theorem 2, it follows that the system \( X_{n+1} = A_n X_n \) is Hyers–Ulam stable. Thus, for a certain positive constant \( L \), every \( \epsilon > 0 \), every sequence \((f_n)\), every \( Y_0 \) and some \( X_0 \) one has

\[ |\phi(n, Y_0, (f_k)) - \phi(n, X_0, (0))| \]

\[ \leq \left| \left[ \sum_{k=1}^{n} U_A(n, k) F_k \right] \right|_{11} \]

for all \( n \in \mathbb{Z}_+ \). Now, the assertion follows from Remark 1.

**5. An Example**

The following example illustrates our theoretical result.

**Example 1.** The linear recurrence of order three

\[ x_{n+3} = \sin \frac{2n\pi}{3} x_{n+2} + \cos \frac{2n\pi}{3} x_{n+1} + c_n x_n, \quad n \in \mathbb{Z} \] (21)
\[ c_n = \begin{cases} 1, & \text{if } n \text{ is a multiple of } 3 \\ 0, & \text{elsewhere} \end{cases} \]

is Hyers–Ulam stable. Indeed, with the above notation one has
\[
A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},
\]

and
\[
A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}.
\]

Now, the monodromy matrix associated to Equation (21) is
\[
A(3) = A_2 A_1 A_0 = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.
\]

The characteristic equation associated to \(A(3)\) is
\[
\lambda^3 - \left(1 + \frac{3\sqrt{3}}{4}\right) \lambda^2 + \frac{\sqrt{3}}{4} - \frac{1}{4} = 0
\]

and the absolute value of each of its solutions is different to 1.

**Remark 3.** Reading [22], we note that an interesting question is if the spectral condition 
\[ |x| - 1 \left( |y| - 1 \right) \left( |z| - 1 \right) \neq 0 \]

is equivalent to Hyers–Ulam stability of the recurrence in Equation (12) with \(\mathbb{Z}_+\) replaced by \(\mathbb{Z}\).

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