A proposed gravitodynamic theory

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Abstract

This paper proposes a gravitodynamic theory because there are similarities between gravitational theory and electrodynamics. Based on Einstein’s principle of equivalence, two coordinate conditions are proposed into the four-dimensional line element and transformations. As a consequence, the equation of motion for gravitational force or inertial force has a form similar to the equation of Lorentz force on a charge in electrodynamics. The inertial forces in a uniformly rotating system are calculated, which show that the Coriolis force is produced by a magnetic-type gravitational field. We have also calculated the Sagnac effect due to the rotation. These experimental facts strongly support our proposed coordinate conditions. In addition, the gravitodynamic field equations are briefly discussed. Since only four gravitational potentials (3+1 split) enter the metric tensor, the gravitodynamic field equations in “3+1 split” form would be analogous to Maxwell’s equations.

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1. Introduction

In a recent review paper, Norton\cite{1} has shown that the principle of general
covariance has been disputed as physically vacuous. The debate over this principle
persists today. In general relativity, space-time coordinates can be arbitrarily
chosen,\cite{2,3} and gravitational potentials are represented by the metric tensor, $g_{\mu\nu}$,
in the four-dimensional (4-D) line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$  \hspace{1cm} (1)

where $g_{\mu\nu}$ has ten independent components, and Greek letters $\mu$ and $\nu$ run from
0 to 3. In Eq.(1), $ds^2$ is an invariant under space-time coordinate transformation.
However, in order to get unique solutions of the Einstein field equation, certain
coordinate conditions must be imposed. Ref.[4] is a recent example that discusses
the choice of coordinate conditions. In this paper, we emphasize that constraints
on coordinate conditions are imposed by the principle of equivalence.

Einstein illustrated the principle of equivalence using the following simple ex-
ample. Let $\Sigma$ be an inertial reference system; let another system $K$ be uniformly
accelerated with respect to $\Sigma$. Then relative to $K$, all free bodies have equal and
parallel accelerations. They behave just as if a gravitational field were present and
$K$ were unaccelerated.\cite{2,3} In another words, inertial force is a type of experiment-
ally confirmed “gravitational force”.

In order to formulate the above simple equivalent gravitational field suggested
by Einstein, let us start with an inertial system $\Sigma$. Let $X^\mu = (cT,X,Y,Z) = (cT,$
$R)$ be the pseudo-Cartesian coordinate in the system $\Sigma$. Then the 4-D line element
takes the form

$$ds^2 = (cdT)^2 - dX^2 - dY^2 - dZ^2$$  \hspace{1cm} (2)

We now consider a system $K$ with coordinates $x^\mu = (ct,x,y,z) = (x^\rho, r)$, which
has a constant acceleration, $a$, with respect to $\Sigma$. Since the Lorentz transformations
are not valid between $\Sigma$ and $K$, the following quasi-Galilean transformations are
usually introduced:\cite{5}

$$\begin{align*}
R &= r + \frac{1}{2}at^2; \quad T = t
\end{align*}$$  \hspace{1cm} (3)

A simple calculation for Eqs.(2) and (3) gives

$$ds^2 = (1 - (at)^2/c^2)(dx^\rho)^2 - 2(at/c) \cdot d\rho dx^\rho - dr^2$$  \hspace{1cm} (4)

Comparing (4) with (1), we obtain $g_{\rho\rho} = (1 - a^2t^2/c^2)$; $g_{\rho i} = -at/c$; $g_{ij} = -\delta_{ij}$;
Latin letters $i$ and $j$ run from 1 to 3.

We substitute these components of $g_{\mu\nu}$ into the geodesic equation giving

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\lambda} \frac{dx^\rho}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$  \hspace{1cm} (5)
In the case of low velocity approximation, Eq. (5) reduces to \( \frac{d^2r}{dt^2} = -a \). This indicates that a particle with a rest mass \( m_0 \) in a uniformly accelerated system should experience a uniformly inertial force \( (-m_0a) \). Although this result is well known, the above calculation shows that certain constraints on space-time coordinates and transformations should be imposed by the principle of equivalence. As is well known, the inertial force is a type of measurable force. Just like other forces, inertial force is a vector, which can be reduced or balanced by other forces. However, if arbitrary curved coordinates are chosen, in general, the force components cannot be singled out from the expression of Eq. (5). Then the physical meaning would be lost in the mathematical maze. This is a major reason why the debate on the principle of general covariance still persists.\(^1\) Only when the equation of motion in a vector form is derived, curvilinear spatial coordinates can be introduced for different applications.

2. Two Proposed Coordinate Conditions

Based on the principle of equivalence, we propose two coordinate conditions into the 4-D line element and transformations in our formulation:

(i) In any reference system, the 4-D line element takes the standard form

\[
\mathit{ds}^2 = \Psi_o(dx^o)^2 - 2(\Psi \cdot dr)dx^o - dr^2
\]

Comparing Eq. (6) with \( \mathit{ds}^2 = g_{\mu\nu}dx^\mu dx^\nu \), we have \( g_{oo} = \Psi_o, g_{oi} = -\Psi_i, g_{ij} = -\delta_{ij} \).

(ii) Between two reference systems, \( K \) and \( K' \), the value of \( \mathit{ds}^2 \) is an invariant, but space-time coordinates take infinitesimal Galilean transformations

\[
\mathit{dr}' = \mathit{dr} - vdt; \quad t' = t
\]

where \( v = v(r,t) \) is the 3-D relative velocity between two systems \( K \) and \( K' \). As a special case, one reference system can be an inertial frame which has Minkowski metrics.

The physical implication of the above proposed conditions is described by Weinberg.\(^6\) He has emphasized that a local gravitational field has two kinds of sources. One is nearby mass; another is all the mass in the universe. It can be anticipated that the inertial systems are determined by the mean background gravitational field produced by all the matter of the universe. This background gravitational field (or simply called ether) provides a base for us to propose the above two specific coordinate conditions.

Landau et al\(^7\) and Møller\(^5\) have rewritten the 4-D line element, Eq. (1), in another form:

\[
\mathit{ds}^2 = (\gamma_0dx^0 - \gamma_i dx^i)^2 - d\sigma^2
\]
Comparing Eq. (8) with Eq. (6), \( \gamma_o = \sqrt{g_{oo}} = \sqrt{\Psi_o}; \gamma_i = -g_{oi}/\sqrt{g_{oo}} = \Psi_i/\sqrt{\Psi_o}; \) 
\( \gamma_{ik} = -g_{ik} + \gamma_l \gamma_k = \delta_{ik} + \Psi_l \Psi_k/\Psi_o. \) \( d\sigma^2 \) is the 3-D measured spatial interval in system \( K. \) The proper time is defined as \( d\tau = ds/c, \) which is an invariant and indicates the measured time interval in system \( K. \) For slow motion and weak field approximation, it is easily seen that the coordinate time interval \( dt \approx d\tau, \) and the coordinate spatial interval \( dr^2 \approx d\sigma^2. \)

Notice that \( \Psi_o \) and \( \Psi_i \) are components of metric tensor \( g_{\mu\nu} \) in Eq. (6). Although they have properties of gravitational potentials, they do not form a 4-D vector. From Eq. (6), it is easy to derive the transformations of these potentials

\[
\Psi'_o = \Psi_o - \frac{v^2}{c^2} - \Psi \cdot \frac{v}{c}; \quad \Psi' = \Psi + \frac{v}{c}
\] (9)

These transformations form a group with the parameters \( v. \)

3. Gravitational Equation of Motion

In this section, we will rewrite the general equation of motion, Eq. (5), into an explicit form in terms of the proposed two conditions. We consider the variational principle \[7\]
\[
\delta \int (-m_o c) ds = \delta \int L(x^i, u^i, t) dt
\] (10)
where \( L(x^i, u^i, t) \) is the Lagrangian. Using Eq. (6), we obtain

\[
L(x^i, u^i, t) = (-m_o c^2) \sqrt{\Psi_o - 2 \Psi \cdot u/c - u^2/c^2}
\] (11)
where \( u = dr/dt \) is the velocity of a particle in system \( K. \)

Let \( \Psi_o = (1 + 2\phi). \) Substituting the Lagrangian (11) into the Lagrange equation,\[7\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial u} \right) = \frac{\partial L}{\partial t}
\] (12)
we obtain the equation of motion:

\[
\frac{d(mu)}{dt} + c^2 \Psi dm/dt = m(g + u/c \times h)
\] (13)
where \( g \) and \( h \) are defined as

\[
g = c^2(-\nabla \phi - \frac{1}{c} \frac{\partial \Psi}{\partial t}); \quad h = c^2(\nabla \times \Psi)
\] (14)

Here, \( g \) is gravitational intensity and \( h \) is magnetic-type gravitational intensity. Both of them have the units of \( cm/sec^2. \) Therefore, the gravitational force is
analogous to the Lorentz force on a charged particle in an electromagnetic field. As is discussed in section 1, force is a vector, which can be reduced or balanced by other forces. We now have clearly singled the force term out in the right hand side of Eq. (13). However, there are two different terms between the Lorentz formula and Eq. (13): (i) a moving mass \( m = m_0 \Gamma \), where \( \Gamma = (\Psi_0 - 2\Psi \cdot u/c - u^2/c^2)^{-1/2} \), while the electric charge is a constant; (ii) a small additional term on the left hand side of Eq. (13), which is related to the changing rate of a moving mass. We stress that Eq. (13) is the explicit formulation of the gravitational equation of motion, which is also valid to describe the magnetic-type gravitational force as well as inertial force in non-inertial systems.

We now consider the total energy of a moving particle with a velocity \( u \) in a gravitational field. According to Lagrange’s formulation, the total energy of a particle is defined as \(^{[7]}\)

\[
E_t = \frac{\partial L}{\partial u} \cdot u - L
\]

Substituting the Lagrangian expression (11) into (15), one obtains

\[
E_t = m_0 \Gamma c^2 [(1 + 2\phi) - \Psi \cdot u/c]
\]

For a weak field, \( \phi, \Psi \ll 1 \), an approximation of Eq. (16) is

\[
E_t = \frac{m_0 c^2 (1 + \phi)}{\sqrt{1 - u^2/c^2}}
\]

Eq. (17) is essentially the same as the formula derived by Landau and Lifshitz. \(^{[7]}\)

4. Inertial Force in Accelerated Systems

In terms of the formulation described above, inertial force is a type of experimentally confirmed gravitational force. We now calculate two typical cases of inertial force.

(a). A Uniformly Linear Accelerated System

In the introduction of this paper, we discussed the inertial force in a uniformly linear accelerated system. From Eqs. (4) and (6), we have \( \phi = -a^2 t^2/2c^2, \Psi = at/c \). By using Eq. (14), we obtain

\[
g = -c \frac{\partial \Psi}{\partial t} = -a; \quad h = 0
\]

Notice that this equivalent gravitational intensity \( g \) in Eq. (18) is produced by the 3-D gravitational vector potential.

(b). A Uniformly Rotating System
Suppose a uniformly rotating system has a constant angular velocity \( \Omega = \Omega \mathbf{k} \) with respect to an inertial system \( \Sigma \). Let the infinitesimal Galilean transformation be
\[
\mathbf{dr} = d\mathbf{R} - (\Omega \times \mathbf{R}) \, dT;
\]
\( t = T \) \hspace{1cm} (19)
Substituting Eq.(19) into Eq.(2), we obtain the 4-D line element in the rotating system \( K \) as follows:
\[
ds^2 = \left(1 - \left(\frac{\Omega \times r}{c}\right)^2\right) dx^o dt^2 - 2 \frac{\Omega \times r}{c} \cdot drdx^o - dr^2 \hspace{1cm} (20)
\]
From Eq.(20), we have \( \phi = -\frac{(\Omega \times r)^2}{2c^2} \), \( \Psi = \frac{\Omega \times r}{c} \). Using Eq.(14), we obtain
\[
g = \frac{1}{2} \nabla \left(\Omega \times r\right)^2 = \Omega^2 r^\parallel; \quad h = \nabla \times \left(\Omega \times r\right) = 2c\Omega \hspace{1cm} (21)
\]
where \( r^\parallel = xi + yj \). Eq.(21) gives the centrifugal force, \( F_a = mg = m\Omega^2 r^\parallel \), and the Coriolis force, \( F_b = mu \times h/c = 2mu \times \Omega \) in the rotating system. Therefore, the Coriolis force is produced by a magnetic-type gravitational field. Møller\'s\(^{[5]} \) derived result is only approximate because he did not explicitly choose the coordinate conditions.

5. Sagnac Effect

In a sagnac interferometer, the light beam coming from a source is split into two sub-beams. One sub-beam circulates a loop in a clockwise direction, and another one circulates the same loop in a counter-clockwise direction. When the whole interferometer with light source and fringe detector is set in rotation with a rate of \( \Omega \) rad/sec, a fringe shift \( \Delta \) with respect to the fringe position for the stationary interferometer is observed, as first reported by Sagnac in 1913\(^{[8,9]} \). The formula is given as
\[
\Delta = 4\Omega Af/c^2 \hspace{1cm} (22)
\]
We now derive the above formula simply using the 4-D element in Eq. (20). As is well known, the propagation of light has zero value of the 4-D line element, that is \( ds = 0 \). If a light beam travels along a circle with radius \( R \) in a uniformly rotating system, Eq. (20) can be simplified as follows: \( d\mathbf{r} = Rd\theta \phi, \Omega \times \mathbf{r} = \Omega R \theta \phi \), then we have
\[
ds^2 = \left(c^2 - (\Omega R)^2\right) d\theta^2 - 2(\Omega R^2) d\theta dt - R^2 (d\theta)^2 = 0 \hspace{1cm} (23)
\]
Solving the quadratic equation (23) for \( Rd\theta/dt \), we obtain the speed of light in the frame of the rotating disk:
\[ c_\theta = Rd\theta/dt = c \mp \Omega R \]  

(24)

The sign of minus or plus in Eq.(24) depends on the direction of the light beam. When one beam circulates a loop in a clockwise direction, and another one circulates in the opposite direction, the time difference is

\[ \Delta t = \frac{2\pi R}{c - \Omega R} - \frac{2\pi R}{c + \Omega R} \simeq 4\pi R^2 \Omega / c^2 \]  

(25)

Since the shift of fringes equals \( \Delta t \) multiplying the frequency of the light, Eq.(25) leads to \( \Delta = 4\Omega A f / c^2 \), which is the same as Eq.(22).

As a matter of fact, the Sagnac effect has clearly shown that (i). one-way speed of light is direction-dependent in a rotating system, which is given in Eq.(24); (ii). the transformation of space and time coordinates between an inertial system and a rotating system obeys the infinitesimal Galilean transformations, instead of the Lorentz transformations.

6. Brief Discussion on the gravitodynamic Field Equations

In this paper, the term “gravitodynamics” is used as a replacement for the term “gravitation” since there are similarities between gravitational theory and electrodynamics. For instance, in the systems with weak gravitational fields and low mass velocities (\( u \ll c \)), Einstein’s field equations can be written in a form analogous to Maxwell’s equations:[10–14]

\[ \nabla \cdot g = -4\pi G \rho_m \]  

(26a)

\[ \nabla \times h = N (-\frac{4\pi G}{c^2} \rho_m u + \frac{1}{c} \partial g \partial t) \]  

(26b)

\[ \nabla \times g = -\frac{1}{c} \partial h \partial t \]  

(26c)

\[ \nabla \cdot h = 0 \]  

(26d)

In Eqs.(26), 4-D space-time is sliced into 3-D space plus 1-D time (3+1 split). Because of the different approach employed to approximate Einstein’s field equations, the deduced Maxwell-type field equations in Ref.[10-14] have small differences with Eqs.(26) in high order terms. For instance, the right hand side term in (26c) is a high order term, which is often set to zero. Regardless of these small differences, Thorne[10] pointed out that there are three major differences between Eqs.(26) and Maxwell’s equations: (i) minus signs in the source term of Eqs.(26a) and (26b) are due to the fact that the gravitational field is attractive rather than repulsive; (ii) a factor \( N=4 \) on the right hand side of Eq.(26b) presumably is due to the fact that
gravitational potential $g_{\mu\nu}$ has ten independent components; (iii) a charge density is replaced by mass density $\rho_m$ times Newton’s gravitational constant. Furthermore, Thorne emphasized the practical importance of such “3 + 1 split” for experimental physicists. As a matter of fact, preparations are already underway for experiments to search magnetic-type gravitational field.\textsuperscript{[10,11,15]} In principle, the properties of gravitational waves can be derived from the field equation (26). However, derivation of the velocity of gravitational waves should require using rigorous forms of the gravitodynamic field equations. As an example, it is easily seen that the value of $N$ in Eq.(26) would affect the velocity of gravitational waves.

Following the regular procedures to derive Einstein’s gravitational field equations, one could apply the variation principle:

$$
\delta (S_{mf} + S_f) = \delta \int (\mathcal{L}_{mf} + \mathcal{L}_f)dVdx^o = 0
$$

(27)

where $dVdx^o=dx dy dz dx^o$ is 4-D volume element. $\mathcal{L}_{mf}$ is the part of Lagrangian density that depends on the interaction between particles and gravitodynamic field, while $\mathcal{L}_f$ is the part that depends on the gravitodynamic field itself. Both of them could be derived from Eq.(6) and its metric tensor.

We emphasize that the metric tensor in our formulation has only four non-constant components, $\Psi_o$ and $\Psi$. Therefore the derived gravitodynamic field equations would have four component equations (3+1 split), instead of ten equations as in general relativity. For this reason, the gravitodynamic field equations for a weak field would reduce to a form analogous to Maxwell’s equations. Such equations are approximately shown in Eqs.(26) with a constant $N=1$ in our formulation.

Generally speaking, the rigorous “3 + 1 split” form of the gravitodynamic field equations should be a set of non-linear differential equations in terms of $\Psi_o$, $\Psi$, $g$ and $h$. This approach deserves further investigation. Furthermore, since the gravitational equation of motion and gravitodynamic field equations in our formulation can be written in 3-D vector forms, one still has freedom to choose 3-D curvilinear spatial coordinates in a given reference system. At this stage, the process of general covariance has its limited application.

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Appendix

A Derivation of the Gravitodynamical Field Equations

In the following, a possible form of the gravitodynamical field equations will be derived from Eq.(27). First, $L_{mf}$ can be obtained from Eq.(11).

\[ L_{mf} = (\rho_m c^2)\sqrt{\Psi_o - 2\Psi \cdot u/c - u^2/c^2} \quad (28) \]

For the weak field and low mass velocities, Eq.(28) can be approximately rewritten as

\[ L_{mf} = -\rho_m (-c^2 + u^2/2 - c^2 \phi + c(\Psi \cdot u)) \quad (29) \]

Eq.(29) is quite similar to the $L_{mf}(EM)$ in electrodynamics.

Secondly, regarding the function of $L_f$, we need to consider the differences between the “actual” gravitational field and the “equivalent” gravitational field produced by acceleration in a non-inertial system. Here we only intend to derive the field equations for the “actual” gravitational field produced by gravitational mass. For this purpose, we should study the transformations of field intensities here.

Start with Eq.(9), the transformations of potentials $\Psi_o$ and $\Psi$. It can be rewritten as

\[ \phi' = \phi - \frac{1}{2} \frac{v^2}{c^2} - \frac{v}{c} \cdot \Psi; \quad \Psi' = \Psi + \frac{v}{c} \quad (30) \]

where $v$ is the relative velocity of system $K'$ with respect to system $K$. Since we only consider the actual gravitational field in both systems, a constant $v$ is assumed here. Using the definition of $g$ and $h$ in Eq.(14), Eq.(30) leads to

\[ g' = g + \beta \times h; \quad h' = h \quad (31) \]

Therefore, the magnetic-type gravitational intensity is an invariant 3-D vector. From Eqs.(30) and (31), it is easy to prove another invariant 3-D vector $\tilde{g}$, which is defined as

\[ \tilde{g} = g - \Psi \times h = g' - \Psi' \times h' \quad (32) \]

In terms of the above invariant, various kinds of $L_f$ can be introduced. Comparing the function of $L_f(EM)$ in electrodynamics, we choose the following $L_f$ for discussion:

\[ L_f = -(8\pi G)^{-1}(\tilde{g} \cdot \tilde{g} - h \cdot h) \quad (33) \]
Substituting Eq.(32) into (33), and neglecting the high order terms, we obtain

$$\mathcal{L}_f = -(8\pi G)^{-1}(g \cdot g - 2g \cdot (\Psi \times h) - h \cdot h)$$

(34)

Substituting the expressions of $\mathcal{L}_{mf}$ in Eq.(29) and $\mathcal{L}_{mf}$ in Eq.(34) into the Eq.(27), and following the same procedures to derive the field equations in electrodynamics [7], we can obtain two gravitodynamic field equations:

$$\nabla \cdot (g - \Psi \times h) = -4\pi G\rho_m$$

(35a)

$$\nabla \times (h - \Psi \times g) = -\frac{4\pi G}{c} \rho_m u + \frac{1}{c} \frac{\partial}{\partial t}(g - \Psi \times h)$$

(35b)

When comparing Eq.(35a, b) with Maxwell’s equations, it is easy to see that there are additional terms, $\Psi \times h$ and $\Psi \times g$, in Eq.(35a, b).

Using the mathematical properties of the operator $\nabla$, other two gravitational field equations can be easily derived from Eq.(14):

$$\nabla \times g = -\frac{1}{c} \frac{\partial h}{\partial t}$$

(35c)

$$\nabla \cdot h = 0$$

(35d)

Furthermore, from Eq.(35a) and Eq.(35b), the equation of continuity can be obtained

$$\frac{1}{c} \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m u) = 0$$

(36)
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