A Universal upper bound on Graph Diameter
based on Laplacian Eigenvalues

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Abstract

We prove that the diameter of any unweighted connected graph \(G\) is \(O(k \log n / \lambda_k)\), for any \(k \geq 2\). Here, \(\lambda_k\) is the \(k\) smallest eigenvalue of the normalized laplacian of \(G\). This solves a problem posed by Gil Kalai.

1 Introduction

Let \(G = (V, E)\) be a connected, undirected and unweighted graph, and let \(d(v)\) be the degree of vertex \(v\) in \(G\). Let \(D\) be the diagonal matrix of vertex degrees and \(A\) be the adjacency matrix of \(G\). The normalized laplacian of \(G\) is the matrix \(L = I - D^{-1/2}AD^{-1/2}\), where \(I\) is the identity matrix. The matrix \(L\) is positive semi-definite. Let

\[
0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2
\]

be the eigenvalues of \(L\). For any pair of vertices \(u, v \in G\), we define their distance, \(\text{dist}(u, v)\), to be the length of the shortest path connecting \(u\) to \(v\). The diameter of the graph \(G\) is the maximum distance between all pairs of vertices, i.e.,

\[
\text{diam}(G) := \max_{u,v} \text{dist}(u, v).
\]

The following question is asked by Gil Kalai in a personal communication [Kal12]. Is it true that for any connected graph \(G\), and any \(k \geq 2\), \(\text{diam}(G) = O(k \log(n)/\lambda_k)\). We remark that for \(k = 2\), the conjecture is already known to hold, since the mixing time of the lazy random walk on \(G\) is \(O(\log n / \lambda_2)\). Therefore, this conjecture can be seen as a generalization of the \(k = 2\) case.

In this short note we answer his question affirmatively and we prove the following theorem

**Theorem 1.1.** For any unweighted, connected graph \(G\), and any \(k \geq 2\),

\[
\text{diam}(G) \leq \frac{48k \log n}{\lambda_k}.
\]

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Our proof uses the easy direction of the higher order cheeger inequalities (see e.g. [LOT12]). For a set $S \subseteq V$, let $E(S, \overline{S}) := \{\{u, v\} : |\{u, v\} \cap S| = 1\}$ be the set of edges with exactly one endpoint in $S$, and let $N(S)$ be the set of neighbors of the set $S$. Let $\text{vol}(S) := \sum_{v \in S} d(v)$ be the volume of the set $S$, and let

$$\phi(S) := \frac{|E(S, \overline{S})|}{\min\{\text{vol}(S), \text{vol}(\overline{S})\}}$$

be the conductance of $S$.

Let $\phi_k(G)$ be the worst conductance of any $k$ disjoint subsets of $V$, i.e.,

$$\phi_k(G) := \min_{\text{disjoint } S_1, S_2, \ldots, S_k} \max_{1 \leq i \leq k} \phi(S_i).$$

The following theorem is proved in [LOT12]; it shows that for any graph $G$, $\phi_k(G)$ is well characterized by the $\lambda_k$.

**Theorem 1.2** (Lee et al.[LOT12]). For any graph $G$, and any $k \geq 2$,

$$\frac{\lambda_k}{2} \leq \phi_k(G) \leq O(k^2) \sqrt{\lambda_k}.$$ 

We use the left side of the above inequality (a.k.a. easy direction of higher order cheeger inequality) to prove Theorem 1.1.

## 2 Proof

In this section we prove Theorem 1.1. We construct $k$ disjoint sets $S_1, \ldots, S_k$ such that for each $1 \leq i \leq k$, $\phi(S_i) \leq O(k \log n / \text{diam}(G))$, and then we use Theorem 1.2 to prove the theorem.

First, we find $k+1$ vertices $v_0, \ldots, v_k$ such that the distance between each pair of the vertices is at least $\text{diam}(G)/2k$. We can do that by taking the vertices $v_0$ and $v_k$ to be at distance $\text{diam}(G)/2k$.

Then, we consider a shortest path connecting $v_0$ to $v_k$ and take equally spaced vertices on that path.

For a set $S \subseteq V$, and radius $r \geq 0$ let

$$B(S, r) := \{v : \min_{u \in S} \text{dist}(v, u) \leq r\}$$

be the set of vertices at distance at most $r$ from the set $S$. If $S = \{v\}$ is a single vertex, we abuse notation and use $B(v, r)$ to denote the ball of radius $r$ around $v$. For each $i = 0, \ldots, k$, consider the ball of radius $\text{diam}(G)/6k$ centered at $v_i$, and note that all these balls are disjoint. Therefore, at most one of them can have a volume of at least $\text{vol}(V)/2$. Remove that ball from consideration, if present. So, maybe after renaming, we have $k$ vertices $v_1, \ldots, v_k$ such that the balls of radius $\text{diam}(G)/6k$ around them, $B(v_1, \text{diam}(G)/6k), \ldots, B(v_k, \text{diam}(G)/6k)$, are all disjoint and all contain at most a mass of $\text{vol}(V)/2$.

The next claim shows that for any vertex $v_i$ there exists a radius $r_i < \text{diam}(G)/6k$ such that $\phi(B(v_i, r_i)) \leq 24k \log n / \text{diam}(G)$.

**Claim 2.1.** For any vertex $v \in V$ and $r > 0$, if $\text{vol}(B(v, r)) \leq \text{vol}(V)/2$, then for some $0 \leq i < r$, $\phi(B(v, i)) = 4\log n/r$. 

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Proof. First observe that for any set $S \subseteq V$, with $\text{vol}(S) \leq \text{vol}(V)/2$, 
\[
\text{vol}(B(S, 1)) = \text{vol}(S) + \text{vol}(N(S)) \geq \text{vol}(S) + |E(S, \overline{S})| = \text{vol}(S)(1 + \phi(S))
\]
where the inequality follows from the fact that each edge $\{u, v\} \in E(S, \overline{S})$ has exactly one endpoint in $N(S)$, and the last equality follows from the fact that $\text{vol}(S) \leq \text{vol}(V)/2$. Now, since $B(v, r) \leq \text{vol}(V)/2$, by repeated application of (1) we get, 
\[
\text{vol}(B(v, r)) \geq \text{vol}(B(v, r - 1))(1 + \phi(B(v, r - 1))) \geq \ldots \geq \prod_{i=0}^{r-1} (1 + \phi(B(v, i))) \geq \exp \left( \frac{1}{2} \sum_{i=0}^{r-1} \phi(B(v, i)) \right).
\]
where the last inequality uses the fact that $\phi(S) \leq 1$ for any set $S \subseteq V$. Since $G$ is unweighted, $\text{vol}(B(v, r)) \leq \text{vol}(V) \leq n^2$. Therefore, by taking logarithm from both sides of the above inequality we get, 
\[
\sum_{i=0}^{r-1} \phi(B(v, i)) \leq 2 \log(\text{vol}(B(v, r))) \leq 4 \log n.
\]
Therefore, there exists $i < r$ such that $\phi(B(v, i)) \leq 4 \log n/r$. \hfill \qed

Now, for each $1 \leq i \leq k$, let $S_i := B(v_i, r_i)$. Since $r_i < \text{diam}(G)/6k$, $S_1, \ldots, S_k$ are disjoint. Furthermore, by the above claim $\phi(S_i) \leq 24k \log n/\text{diam}(G)$. Therefore, $\phi_k(G) \leq 24k \log n/\text{diam}(G)$. Finally, using Theorem 1.2, we get 
\[
\lambda_k \leq 2\phi_k(G) \leq \frac{48k \log n}{\text{diam}(G)}.
\]
This completes the proof of Theorem 1.1.

References

[Kal12] Gil Kalai. Personal Communication, 2012. 

[LOT12] James R. Lee, Shayan Oveis Gharan, and Luca Trevisan. Multi-way spectral partitioning and higher-order cheeger inequalities. In STOC, pages 1117–1130, 2012. 

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