EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPLICITY 2 BASED ON DOUBLE FOURIER–LEGENDRE SERIES SUMMARIZED BY PRINGSHEIM METHOD

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Abstract. The article is devoted to the expansion of iterated Stratonovich stochastic integrals of second multiplicity into the double series of products of standard Gaussian random variables. The proof of expansion is based on the application of double Fourier–Legendre series and double trigonometric Fourier series summarized by Pringsheim method. The results of the article are generalized to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t,T])$, where $\psi_1(\tau), \psi_2(\tau)$ are weight functions of the iterated Stratonovich stochastic integral of second multiplicity. The considered expansion can be applied to the numerical integration of Itô stochastic differential equations. Some recent results on the expansion of iterated Stratonovich stochastic integrals of multiplicities 3 to 6 are given.

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1. Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, let \(\{\mathcal{F}_t, t \in [0, T]\}\) be a nondecreasing right-continuous family of \(\sigma\)-algebras of \(\mathcal{F}\), and let \(\mathbf{f}_t\) be a standard \(m\)-dimensional Wiener stochastic process, which is \(\mathcal{F}_t\)-measurable for any \(t \in [0, T]\). We assume that the components \(\mathbf{f}_t^{(i)} (i = 1, \ldots, m)\) of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

\[
x_t = x_0 + \int_0^t a(x_\tau, \tau) d\tau + \int_0^t B(x_\tau, \tau) d\mathbf{f}_\tau, \quad x_0 = x(0, \omega).
\]

Here \(x_t\) is some \(n\)-dimensional stochastic process satisfying the equation \((1)\). The nonrandom functions \(a : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n, B : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times m}\) guarantee the existence and uniqueness up to stochastic equivalence of a solution of the Ito SDE \((1)\). The second integral on the right-hand side of \((1)\) is interpreted as an Ito stochastic integral. Let \(x_0\) be an \(n\)-dimensional random variable, which is \(\mathcal{F}_0\)-measurable and \(\mathbb{E}\{|x_0|^2\} < \infty\) (\(\mathbb{E}\) denotes a mathematical expectation). We assume that \(x_0\) and \(\mathbf{f}_t - \mathbf{f}_0\) are independent when \(t > 0\).

It is well known that one of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions \([2]-[5]\). Moreover, one of the most important features of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

\[
J[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \cdots \int_t^{t_1} \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} \cdots d\mathbf{w}^{(i_k)}_{t_k},
\]

\[
J^*[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \cdots \int_t^{t_1} \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} \cdots d\mathbf{w}^{(i_k)}_{t_k},
\]

where every \(\psi_l(\tau) (l = 1, \ldots, k)\) is a nonrandom function at the interval \([t, T]\), \(\mathbf{w}^{(i)}_{\tau} = \mathbf{f}_{\tau}^{(i)}\) for \(i = 1, \ldots, m\) and \(\mathbf{w}^{(0)}_{\tau} = \tau, i_1, \ldots, i_k = 0, 1, \ldots, m,\)

\[
\int_t^T \text{ and } \int_t^{t_1}
\]

denote Ito and Stratonovich stochastic integrals, respectively. In this paper we use the definition of the Stratonovich stochastic integral from \([3]\).

Note that usually in applications the functions \(\psi_l(\tau) \equiv 1 (l = 1, \ldots, k)\) are equal to 1 or have a binomial form. More precisely, \(\psi_l(\tau) \equiv 1 (l = 1, \ldots, k)\) and \(i_1, \ldots, i_k = 0, 1, \ldots, m\) in \([2]-[6]\). At the same time \(\psi_l(\tau) \equiv (\tau - \tau)^{q_l} (l = 1, \ldots, k, q_1, \ldots, q_k = 0, 1, 2, \ldots)\) and \(i_1, \ldots, i_k = 1, \ldots, m\) in \([7]-[12]\).

Effective solution of the problem of combined mean-square approximation for collections of iterated Stratonovich stochastic integrals \([3]\) of second multiplicity composes the subject of this article.

It is well known that the mean-square approximation of iterated Ito and Stratonovich stochastic integrals \([2], [3]\) using multiple integral sums requires significant computational costs \([11]\) since this approach implies the partitioning of the integration interval \([t, T]\) of the iterated stochastic integrals
(2), (3) into parts ($T - t$ is already a sufficiently small value since $T - t$ plays the role of an integration step in numerical methods for solving Ito SDEs).

More efficient approximation methods for the iterated stochastic integrals (2), (3) use Fourier series, and they do not require the interval $[t, T]$ to be subdivided into smaller parts. One such method was proposed in [2] and elaborated in [3, 4]. This method, which received widespread use, is based on the Karhunen–Loève expansion of the Brownian bridge process [2] in the eigenfunctions of its covariance, which form a complete orthonormal trigonometric basis in the space $L_2([t, T])$.

Note that in [11] (2006) the more general and effective method (the so-called method of generalized multiple Fourier series) for the mean-square approximation of iterated Ito stochastic integrals (2) was proposed. This method is based on the generalized multiple Fourier series that converge in the sense of norm in Hilbert space $L_2([t, T]^k)$, where $[t, T]^k$ is the hypercube $[t, T] \times \ldots \times [t, T]$ ($k$ times) and $k$ is the multiplicity of the iterated Ito stochastic integral. The method of generalized multiple Fourier series was developed in [12]-[41], [43]-[61].

An important feature of the method of generalized multiple Fourier series is that various complete orthonormal systems of functions in the space $L_2([t, T])$ can be used (the method proposed in [2] admits only the trigonometric system of functions). Hence, we can state the problem of comparing the efficiency of using different complete orthonormal systems of functions in the space $L_2([t, T])$ in the context of numerical solution of Ito SDEs. This problem has been solved in [37], [38] (also see [25]-[28]). In particular, in [25]-[28], [37], [38] it was shown that the optimal system of basis functions in the framework of numerical solution of Ito SDEs is the system of Legendre polynomials. This fact is true at least for high-order strong numerical methods with orders of convergence $1.5, 2.0, \ldots$. That is why the part of this article is devoted to the expansions of iterated Stratonovich stochastic integrals with multiplicity 2 based on multiple Fourier–Legendre series.

Usage of Fourier series with respect to the system of Legendre polynomials for approximation of iterated stochastic integrals took place for the first time in [7] (1997), [8] (1998), [9] (also see [10]-[41], [43]-[59]).

The results of [7] (also see [8]-[41], [43]-[59]) convincingly testify that there is a doubtless relation between multiplier factor $1/2$, which is typical for Stratonovich stochastic integral and included into the sum connecting Stratonovich and Ito stochastic integrals, and the fact that in the point of finite discontinuity of piecewise smooth function $f(x)$ its Fourier series converges to the value

$$\frac{f(x + 0) + f(x - 0)}{2}.$$

In addition, in [7], [8], [10]-[20], [23]-[29], [31], [33], [35], [39], [49]-[51], [54], [58] several theorems on expansion of iterated Stratonovich stochastic integrals were formulated and proved. As shown in these papers, the final formulas for expansions of iterated Stratonovich stochastic integrals are more compact than their analogues for iterated Ito stochastic integrals.

This paper continues the study of the relationships between generalized multiple Fourier series and iterated stochastic integrals. We use the double Fourier–Legendre series and double trigonometric Fourier series (summarized by Pringsheim method) for the proof of Theorem 5.3 [24] or Theorem 2.1 [25] (also see [26], [28]). As shown below the conditions of these theorems can be weakened.

Moreover, the mentioned theorems are generalized to the case of an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$.

2. Method of Generalized Multiple Fourier Series

Let us consider an approach to the expansion of iterated Ito stochastic integrals [11]-[41], [43]-[61] (the so-called method of generalized multiple Fourier series).
Suppose that \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \). Define the following function on the hypercube \([t, T]^k\)

\[
K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \cdots \psi_k(t_k) & \text{for } t_1 < \ldots < t_k \\
0 & \text{otherwise}
\end{cases}, \quad t_1, \ldots, t_k \in [t, T], \ k \geq 2,
\]

and \( K(t_1) \equiv \psi_1(t_1), \ t_1 \in [t, T] \).

Suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of functions in the space \( L_2([t, T]) \).

The function \( K(t_1, \ldots, t_k) \) belongs to the space \( L_2([t, T]^k) \). At this situation it is well-known that the generalized multiple Fourier series of \( K(t_1, \ldots, t_k) \in L_2([t, T]^k) \) is converging to \( K(t_1, \ldots, t_k) \) in the hypercube \([t, T]^k\) in the mean-square sense, i.e.,

\[
\lim_{p_1, \ldots, p_k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_j(t_l) dt_1 \cdots dt_k \right\|_{L_2([t, T]^k)} = 0,
\]

where

\[
C_{j_k \ldots j_1} = \int_{[t, T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_j(t_l) dt_1 \cdots dt_k,
\]

\[
\|f\|_{L_2([t, T]^k)} = \left( \int_{[t, T]^k} f^2(t_1, \ldots, t_k) dt_1 \cdots dt_k \right)^{1/2}.
\]

Consider the partition \( \{\tau_j\}_{j=0}^{N} \) of the interval \([t, T]\) such that

\[
t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta \tau_j \to 0 \text{ if } N \to \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\]

Theorem 1 [11] (2006) (also see [12]-[11], [33]-[61]). Suppose that every \( \psi_l(\tau) \ (l = 1, \ldots, k) \) is a continuous nonrandom function on \([t, T]\) and \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of continuous functions in \( L_2([t, T]) \). Then

\[
J[\psi^{(k)}]_{T,t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \phi_{j_l}(\tau_l) \Delta f_{l1}^{(i_1)} \cdots \phi_{j_k}(\tau_k) \Delta f_{lk}^{(i_k)} \right) - \lim_{N \to \infty} \sum_{(l_1, \ldots, l_k) \in G_k} \phi_{j_1}(\tau_{l_1}) \Delta w_{l_1}^{(i_1)} \cdots \phi_{j_k}(\tau_{l_k}) \Delta w_{l_k}^{(i_k)},
\]

where \( J[\psi^{(k)}]_{T,t} \) is defined by [2],

\[
G_k = H_k \setminus L_k, \quad H_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N-1, \},
\]

\[
L_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N-1; l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k, \}.
\]
l.i.m. is a limit in the mean-square sense, \(i_1, \ldots, i_k = 0, 1, \ldots, m\),

\[
\zeta_j^{(i)} = \int_t^T \phi_j(s) \, dw_s^{(i)}
\]

are independent standard Gaussian random variables for various \(i \neq j\) (if \(i \neq 0\), \(C_{j_i} \ldots j_k\) is the Fourier coefficient [43], \(\Delta w^{(i)}_{T, j} = w^{(i)}_{T, j+1} - w^{(i)}_{T, j}\) \((i = 0, 1, \ldots, m)\), \(\{\tau_j\}_{j=0}^N\) is a partition of the interval \([t, T]\), which satisfies the condition [43].

Note that the condition of continuity of the functions \(\phi_j(x)\) \((j = 0, 1, \ldots)\) can be weakened (see [11]-[20], [23]-[28]). Another versions and generalizations of Theorem 1 can be found in [12]-[41], [43]-[59].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for \(k = 1, \ldots, 5\) [11]-[11], [43]-[59].

\[
J[\psi^{(1)}]_{T, t} = \text{l.i.m.} \sum_{j_1 = 0}^{p_1} C_{j_1} \zeta_j^{(1)}
\]

\[
J[\psi^{(2)}]_{T, t} = \text{l.i.m.} \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_1 j_2} \left( \zeta_j^{(1)} \zeta_j^{(2)} - 1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} \right)
\]

\[
J[\psi^{(3)}]_{T, t} = \text{l.i.m.} \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} \sum_{j_3 = 0}^{p_3} C_{j_1 j_2 j_3} \left( \zeta_j^{(1)} \zeta_j^{(2)} \zeta_j^{(3)} - 1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} \zeta_j^{(1)} - 1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} \zeta_j^{(2)} - 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} \zeta_j^{(1)} \right)
\]

\[
J[\psi^{(4)}]_{T, t} = \text{l.i.m.} \sum_{j_1 = 0}^{p_1} \cdots \sum_{j_4 = 0}^{p_4} C_{j_1 \cdots j_4} \left( \prod_{t=1}^{4} \zeta_j^{(i)} - 1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} \zeta_j^{(i)} \zeta_j^{(i_2)} \zeta_j^{(i_4)} - 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} \zeta_j^{(i)} \zeta_j^{(i_2)} \zeta_j^{(i_4)} - 1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} \zeta_j^{(i)} \zeta_j^{(i_1)} \zeta_j^{(i_4)} - 1_{\{i_2 = i_4 \neq 0\}} 1_{\{j_2 = j_4\}} \zeta_j^{(i)} \zeta_j^{(i_1)} \zeta_j^{(i_3)} \zeta_j^{(i_1)} \zeta_j^{(i_4)} + 1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} 1_{\{i_3 \neq 0\}} 1_{\{j_3 = j_4\}} + 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} 1_{\{i_2 = i_4 \neq 0\}} 1_{\{j_2 = j_4\}} + 1_{\{i_1 = i_4 \neq 0\}} 1_{\{j_1 = j_4\}} 1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} \right)
\]

\[
J[\psi^{(5)}]_{T, t} = \text{l.i.m.} \sum_{j_1 = 0}^{p_1} \cdots \sum_{j_5 = 0}^{p_5} C_{j_1 \cdots j_5} \left( \prod_{t=1}^{5} \zeta_j^{(i)} - 1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} 1_{\{i_3 \neq 0\}} 1_{\{j_3 = j_4\}} + 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} 1_{\{i_2 = i_4 \neq 0\}} 1_{\{j_2 = j_4\}} + 1_{\{i_1 = i_4 \neq 0\}} 1_{\{j_1 = j_4\}} 1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} \right)
\]
In order to do this, let us introduce some notations. Consider the unordered set $\{i_1, \ldots, i_g\}$, separate it into two parts: the first part consists of unordered pairs (sequence order of these pairs does not matter) $\{i_1^1, i_2^1\}, \ldots, \{i_{r}^1, i_{r}^2\}$, the second part consists of unordered pairs $\{i_{r+1}^1, \ldots, i_{r+q}^1\}, \{i_{r+1}^2, \ldots, i_{r+q}^2\}$.

We will say that (14) is a partition and consider the sum with respect to all possible partitions.

\[
\sum_{\{(i_1, i_2), \ldots, (i_{r+q-1}, i_{r+q})\} \in \{1, \ldots, k\}^{r+q-1}} \cdot \sum_{\{(i_1^1, i_2^1), \ldots, (i_{r}^1, i_{r}^2)\} \in \{1, \ldots, k\}^{r}} \cdot \sum_{\{(i_{r+1}^1, \ldots, i_{r+q}^1), (i_{r+1}^2, \ldots, i_{r+q}^2)\} \in \{1, \ldots, k\}^{q-1}}
\]

where $a_{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}}$ is an arbitrary multiplicity $k (k \in \mathbb{N})$ of the iterated Ito stochastic integral $J_{\psi(k)}^{T,t}$ defined by (2).

In order to do this, let us introduce some notations. Consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of $r$ unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

\[
\sum_{\{(i_1, i_2), \ldots, (i_{r+q-1}, i_{r+q})\} \in \{1, \ldots, k\}^{r+q-1}} a_{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}}
\]

where

\[
\{(g_1, g_2), \ldots, (g_{2r-1}, g_{2r})\} \in \{1, \ldots, k\}^{2r-1}
\]

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (14) is a partition and consider the sum with respect to all possible partitions.

\[
\sum_{\{(i_1, i_2), \ldots, (i_{r+q-1}, i_{r+q})\} \in \{1, \ldots, k\}^{r+q-1}} a_{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}}
\]

where $a_{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}} \in \mathbb{R}$.

Below there are several examples of sums in the form (14)

\[
\sum_{\{(i_1, i_2)\} \in \{1, 2\}} a_{i_1, i_2}
\]
\[
\sum_{\{(s_1,s_2),(s_3,s_4)\}} a_{g_1 g_2 g_3 g_4} = a_{1234} + a_{1324} + a_{2314},
\]

\[
\sum_{\{(s_1,s_2),(s_1,s_2)\}} a_{g_1 g_2 q_1 q_2} =
\]

\[
= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},
\]

\[
\sum_{\{(s_1,s_2),(s_1,s_2),(s_3,s_4)\}} a_{g_1 g_2 q_1 q_2 q_3} =
\]

\[
= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +
\]

\[
+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},
\]

\[
\sum_{\{(s_1,s_2),(s_1,s_2),(s_3,s_4),(s_1)\}} a_{g_1 g_2 g_3 g_4,q_1} =
\]

\[
= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +
\]

\[
+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +
\]

\[
+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.
\]

Now we can write (7) as

\[
J[^{(k)}]\tau = \text{l.i.m.} \sum_{p_1,\ldots,p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \frac{c^{(i)}}{s_{j_l}} \right) + \sum_{r=1}^{[k/2]} (-1)^r \times
\]

\[
\prod_{s=1}^{r} 1_{(i_{g_{2s-1}} = i_{g_{2s}} \neq 0)} 1_{(j_{g_{2s-1}} = j_{g_{2s}})} \prod_{t=1}^{k-2r} \frac{c^{(i)}}{s_{j_t}}
\]

\[(16) \times \]

where \([x]\) is an integer part of a real number \(x\), \(\prod_{\emptyset} \equiv 1\), \(\sum_{\emptyset} \equiv 0\); another notations are the same as in Theorem 1.

In particular, from (16) for \(k = 5\) we obtain

\[
J[^{(5)}]\tau = \text{l.i.m.} \sum_{p_1,\ldots,p_5 \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_5=0}^{p_5} C_{j_5 \ldots j_1} \left( \prod_{l=1}^{5} \frac{c^{(i)}}{s_{j_l}} \right) -
\]
functions on system of Legendre polynomials or trigonometric functions in the space \( \psi \) the case of multiplicity 2 have been adapted for iterated Stratonovich stochastic integrals (3) of multiplicities 2 to 6 (also see multiple Fourier series (see [11], Chapter 5, Theorem 5.1, pp. 235-245 or [25], Chapter 1 for details).

\[ \phi(\text{idea [11] (2006) on the expansion of the kernel (4) or (61). The results of [62], as well as the results of [25] (Sect. 1.11), [46] (Sect. 15), [61] are based on our Note that we use another notations [25] (Sect. 1.11), [46] (Sect. 15), [61] in comparison with [62].

The last equality obviously agrees with [13].

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \).

**Theorem 2** [25] (Sect. 1.11), [46] (Sect. 15), [61]. Suppose that \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \) and \( \{\phi_j(x)\}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L_2([t, T]) \). Then the following expansion

\[
J[\psi(k)]_{T,t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1 = 0}^{p_1} \cdots \sum_{j_k = 0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times \right)
\[
\times \sum_{j_2, \ldots, j_k \in \mathbb{Z}} \prod_{s=1}^{r} 1_{\{i_{j_2, \ldots, j_k} = 0\}} 1_{\{\tau_{j_2, \ldots, j_k} = 0\}} \prod_{l=1}^{k-2r} \zeta_{j_l}^{(i_l)}
\]

converging in the mean-square sense is valid, where \( [x] \) is an integer part of a real number \( x \), \( \prod_{\emptyset} \) def 1,

\[ \sum_{\emptyset} \def 0; \text{another notations are the same as in Theorem 1}. \]

It should be noted that an analogue of Theorem 2 was considered in [62] using Hermite polynomials. Note that we use another notations [25] (Sect. 1.11), [46] (Sect. 15), [61] in comparison with [62]. Moreover, the proof from [62] is different from the proof given in [25] (Sect. 1.11), [46] (Sect. 15), [61]. The results of [62], as well as the results of [25] (Sect. 1.11), [46] (Sect. 15), [61] are based on our idea [11] (2006) on the expansion of the kernel [41] (or \( \Phi(t_1, \ldots, t_k) \in L_2([t, T)^k] \)) into a generalized multiple Fourier series (see [11], Chapter 5, Theorem 5.1, pp. 235-245 or [25], Chapter 1 for details).

3. **Theorem on Expansion of Iterated Stratonovich Stochastic Integrals of Second Multiplicity. Some Old Results**

In a number of works of the author [16]-[20], [23]-[29], [31]-[33], [35], [49], [51], [54], [58] Theorems 1, 2 have been adapted for iterated Stratonovich stochastic integrals [33] of multiplicities 2 to 6 (also see the case of multiplicity \( k \) (\( k \in \mathbb{N} \)) in [25] (Sect. 2.10), [29], [33], [47]). For example, we can formulate the following theorem for iterated Stratonovich stochastic integrals of second multiplicity.

**Theorem 3** [16]-[20], [23]-[29], [31], [33], [35], [49], [51], [54], [58]. Suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \). At the same time \( \psi_2(\tau) \) is a continuously differentiable function on \([t, T]\) and \( \psi_1(\tau) \) is twice continuously differentiable functions on \([t, T]\). Then
\begin{equation}
J^*[\psi^{(2)}]_{T,t} = \lim_{p_1,p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2,j_1} \phi_{j_1}^{(i_1)} \phi_{j_2}^{(i_2)} \quad (i_1,i_2 = 1, \ldots, m),
\end{equation}

where \( J^*[\psi^{(2)}]_{T,t} \) is defined by [3]; another notations are the same as in Theorem 1.

Note that the proof of Theorem 3 is based on the proof of the following equality

\begin{equation}
\frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 = \sum_{j_1=0}^{\infty} C_{j_1,j_1},
\end{equation}

where \( C_{j_1,j_1} \) is defined by (5) for \( k = 2 \) and \( j_1 = j_2; \) \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t,T]). \)

According to the standard relation between Ito and Stratonovich stochastic integrals, we can write w. p. 1 (with probability 1)

\begin{equation}
J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2} \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1,
\end{equation}

where we assume that the functions \( \psi_1(\tau), \psi_2(\tau) \) are continuous at the interval \([t,T]. \) This condition is related to the definition of the Stratonovich stochastic integral that we use [3] (also see Sect. 2.1.1 [25]).

From the other hand according to (10), we obtain

\begin{equation}
J[\psi^{(2)}]_{T,t} = \lim_{p_1,p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2,j_1} \phi_{j_1}^{(i_1)} \phi_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1 = i_2 \neq 0\}} \mathbf{1}_{\{j_1 = j_2\}} \sum_{j_1=0}^{\infty} C_{j_1,j_1}.
\end{equation}

From (18)–(20) we get (17). Note that the existence of the limit on the right-hand side of (18) will be proved below (see Lemma 2 and Theorem 7).

The proof of Theorem 3 [16]–[20], [23]–[25], [31], [33], [35], [49], [51], [54], [58] is based on double (iterated ) Fourier–Legendre series and analogous trigonometric Fourier series. This proof uses double integration by parts, which leads to the requirement of double continuous differentiability of the function \( \psi_1(\tau) \) at the interval \([t,T]. \)

In this article, we formulate and prove the analogue of Theorem 3 (Theorem 6, see below) but under the weakened conditions: the functions \( \psi_1(\tau), \psi_2(\tau) \) are assumed to be continuously differentiable only one time at the interval \([t,T]. \) At that we will use double Fourier–Legendre series and double trigonometric Fourier series summarized by Pringsheim method for the proof of Theorem 6 (see below).

In Sect. 5 (Theorem 7), we generalize the equality (18) to the case of an arbitrary complete orthonormal system of functions in the space \( L_2([t,T]). \) and \( \psi_1(\tau), \psi_2(\tau) \in L_2([t,T]). \)
4. Proof of the Equality (13). The Case of Legendre Polynomials and
Trigonometric Functions as well as Continuously Differentiable Functions
ψ1(τ), ψ2(τ)

Let \( P_j(x) \) \((j = 0, 1, 2, \ldots)\) be the Legendre polynomial. Consider the function \( f(x, y) \) defined for \((x, y) \in [-1, 1]^2\).

Consider the double Fourier–Legendre series summarized by Pringsheim method and corresponding to the function \( f(x, y) \)

\[
\lim_{n,m \to \infty} \sum_{j=0}^{n} \sum_{i=0}^{m} \frac{\sqrt{(2j+1)(2i+1)}}{2} C_{ij}^* P_i(x) P_j(y) \overset{\text{def}}{=} \sum_{i,j=0}^{\infty} \frac{\sqrt{(2j+1)(2i+1)}}{2} C_{ij}^* P_i(x) P_j(y),
\]

where

\[
C_{ij}^* = \frac{\sqrt{(2j+1)(2i+1)}}{2} \int_{[-1,1]^2} f(x, y) P_i(x) P_j(y) dx dy.
\]

Let us consider the generalization for the case of two variables (63) of the theorem on equiconvergence for the Fourier–Legendre series (64).

**Theorem 4** (63). Let \( f(x, y) \in L_2([-1, 1]^2) \) and the function

\[
f(x, y)(1 - x^2)^{-1/4}(1 - y^2)^{-1/4}
\]

is integrable on the square \([-1, 1]^2\). Moreover, let

\[
|f(x, y) - f(u, v)| \leq G(y)|x - u| + H(x)|y - v|,
\]

where \( G(y), H(x) \) are bounded functions on the square \([-1, 1]^2\). Then for all \((x, y) \in (-1, 1)^2\) the following equality is satisfied

\[
\lim_{n,m \to \infty} \left( \sum_{j=0}^{n} \sum_{i=0}^{m} \frac{\sqrt{(2j+1)(2i+1)}}{2} C_{ij}^* P_i(x) P_j(y) - \right.
\]

\[
-(1 - x^2)^{-1/4}(1 - y^2)^{-1/4} S_{nm}(\arccos x, \arccos y, F) \left. \right) = 0,
\]

where \( S_{nm}(\theta, \varphi, F) \) is a partial sum of the double trigonometric Fourier series of the auxiliary function

\[
F(\theta, \varphi) = \sqrt{|\sin \theta|} \sqrt{|\sin \varphi|} f(\cos \theta, \cos \varphi), \quad \theta, \varphi \in [0, \pi],
\]
the Fourier coefficient $C_{ij}^*$ has the form (22). At that, the convergence in (23) is uniform on the rectangle

$$[-1 + \varepsilon, 1 - \varepsilon] \times [-1 + \delta, 1 - \delta]$$

for any $\varepsilon, \delta > 0$.

From Theorem 4, for example, follows that for all $(x, y) \in (-1, 1)^2$ the following equality is fulfilled

$$\lim_{n,m \to \infty} \left( \sum_{j=0}^{n} \sum_{i=0}^{m} \frac{\sqrt{2j + 1}(2i + 1)}{2} C_{ij}^* P_i(x)P_j(y) - f(x, y) \right) = 0$$

and convergence in (24) is uniform on the rectangle

$$[-1 + \varepsilon, 1 - \varepsilon] \times [-1 + \delta, 1 - \delta]$$

for any $\varepsilon, \delta > 0$ if the corresponding conditions of convergence of the double trigonometric Fourier series of the auxiliary function

$$g(x, y) = f(x, y)(1 - x^2)^{1/4}(1 - y^2)^{1/4}$$

are satisfied.

Note that Theorem 4 does not imply any conclusions on the behavior of the double Fourier–Legendre series on the boundary of the square $[-1, 1]^2$.

For each $\delta > 0$ let us call the exact upper edge of the difference $|f(t') - f(t'')|$ in the set of all points $t'$, $t''$ (which belong to the domain $D$) as the module of continuity of the function $f(t)$ ($t = (t_1, \ldots, t_k)$) in the $k$-dimensional domain $D$ ($k \geq 1$) if the distance $\rho(t', t'')$ between $t$ and $t''$ satisfies the condition $\rho(t', t'') < \delta$.

We will say that the function of $k$ ($k \geq 1$) variables $f(t)$ ($t = (t_1, \ldots, t_k)$) belongs to the Hölder class with the parameter $\alpha \in (0, 1]$ ($f(t) \in C^\alpha(D)$) in the domain $D$ if the module of continuity of the function $f(t)$ ($t = (t_1, \ldots, t_k)$) in the domain $D$ has orders $O(\delta^\alpha)$ ($\alpha \in (0, 1)$) and $O(\delta)$ ($\alpha = 1$).

In 1967, Zhizhiashvili L.V. proved that the rectangular sums of multiple trigonometric Fourier series of the function of $k$ variables in the hypercube $[t, T]^k$ converge uniformly in the hypercube $[t, T]^k$ to this function if it belongs to the class $C^\alpha([t, T]^k)$, $\alpha > 0$ (definition of the Holder class with the parameter $\alpha > 0$ can be found in the well-known mathematical analysis tutorials; see, for example, [65]). More precisely, the following theorem is correct.

**Theorem 5 [65].** If the function $f(x_1, \ldots, x_n)$ is periodic with period $2\pi$ with respect to each variable and belongs in $\mathbb{R}^n$ to the Hölder class $C^\alpha$ for any $\alpha > 0$, then the rectangular partial sums of the multiple trigonometric Fourier series of the function $f(x_1, \ldots, x_n)$ converge to this function uniformly in $\mathbb{R}^n$.

**Lemma 1.** Let the function $f(x, y)$ satisfies to the following condition

$$|f(x, y) - f(x_1, y_1)| \leq C_1|x - x_1| + C_2|y - y_1|,$$

where $C_1, C_2 < \infty$ and $(x, y), (x_1, y_1) \in [-1, 1]^2$. Then the following inequality is fulfilled

$$|g(x, y) - g(x_1, y_1)| \leq K \rho^{1/4},$$
where \( g(x, y) \) has the form (25),

\[
\rho = \sqrt{(x - x_1)^2 + (y - y_1)^2},
\]

\((x, y) \) and \((x_1, y_1) \in [-1, 1]^2, K < \infty.\)

**Proof.** First, we assume that \( x \neq x_1, y \neq y_1. \) In this case we have

\[
|g(x, y) - g(x_1, y_1)| = |(1 - x^2)^{1/4}(1 - y^2)^{1/4}(f(x, y) - f(x_1, y_1)) + f(x_1, y_1)((1 - x^2)^{1/4}(1 - y^2)^{1/4} - (1 - x_1^2)^{1/4}(1 - y_1^2)^{1/4})| \leq C_1|x - x_1| + C_2|y - y_1| + C_3(1 - x^2)^{1/4}(1 - y^2)^{1/4} - (1 - x_1^2)^{1/4}(1 - y_1^2)^{1/4}|, \quad C_3 < \infty.
\]

Moreover,

\[
|(1 - x^2)^{1/4}(1 - y^2)^{1/4} - (1 - x_1^2)^{1/4}(1 - y_1^2)^{1/4}| = \\
= |(1 - x^2)^{1/4}((1 - y^2)^{1/4} - (1 - y_1^2)^{1/4}) + (1 - y_1^2)^{1/4}((1 - x^2)^{1/4} - (1 - x_1^2)^{1/4})| \leq \\
\leq |(1 - y^2)^{1/4} - (1 - y_1^2)^{1/4}| + |(1 - x^2)^{1/4} - (1 - x_1^2)^{1/4}|,
\]

\[
|(1 - x^2)^{1/4} - (1 - x_1^2)^{1/4}| = \\
= |((1 - x)^{1/4} - (1 - x_1)^{1/4})(1 + x)^{1/4} + (1 - x_1)^{1/4}((1 + x)^{1/4} - (1 + x_1)^{1/4})| \leq \\
\leq K_1(|(1 - x)^{1/4} - (1 - x_1)^{1/4}| + |(1 + x)^{1/4} - (1 + x_1)^{1/4}|), \quad K_1 < \infty.
\]

It is not difficult to see that

\[
|(1 \pm x)^{1/4} - (1 \pm x_1)^{1/4}| = \\
= \frac{|(1 \pm x) - (1 \pm x_1)|}{((1 \pm x)^{1/2} + (1 \pm x_1)^{1/2})((1 \pm x)^{1/4} + (1 \pm x_1)^{1/4})} = \\
= |x_1 - x|^{1/4} \frac{|x_1 - x|^{1/2}}{(1 \pm x)^{1/2} + (1 \pm x_1)^{1/2}} \cdot \frac{|x_1 - x|^{1/4}}{(1 \pm x)^{1/4} + (1 \pm x_1)^{1/4}} \leq |x_1 - x|^{1/4}.
\]

The last inequality follows from the obvious inequalities.
From (27)–(30) we obtain
\[
|g(x, y) - g(x_1, y_1)| \leq C_1|x - x_1| + C_2|y - y_1| + C_4(|x_1 - x|^{1/4} + |y_1 - y|^{1/4}) \leq C_5 \rho + C_6 \rho^{1/4} \leq K \rho^{1/4},
\]
where $C_5, C_6, K < \infty$.

The cases $x = x_1, y \neq y_1$ and $x \neq x_1, y = y_1$ can be considered analogously to the case $x \neq x_1, y \neq y_1$. At that, the consideration begins from the inequalities
\[
|g(x, y) - g(x_1, y_1)| \leq K_2(1 - y^2)^{1/4}f(x, y) - (1 - y_1^2)^{1/4}f(x_1, y_1)|
\]
for $(x = x_1, y \neq y_1)$ and
\[
|g(x, y) - g(x_1, y_1)| \leq K_2(1 - x^2)^{1/4}f(x, y) - (1 - x_1^2)^{1/4}f(x_1, y_1)|
\]
for $(x \neq x_1, y = y_1)$, where $K_2 < \infty$. Lemma 1 is proved.

Lemma 1 and Theorem 5 imply that rectangular partial sums of the double trigonometric Fourier series of the function $g(x, y)$ (in the case of periodic continuation of the function $g(x, y)$) converge uniformly in the square $[-1, 1]^2$ to the function $g(x, y)$. This means that the equality (24) holds.

**Theorem 6** [25–28, 39, 50]. Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Moreover, $\psi_1(\tau), \psi_2(\tau)$ are continuously differentiable functions on $[t, T]$. Then for the iterated Stratonovich stochastic integral
\[
J^s[\psi_2]_{T,t} = \int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) d\mathcal{F}^{(i_1)}_{t_2} d\mathcal{F}^{(i_2)}_{t_2} \quad (i_1, i_2 = 1, \ldots, m)
\]
the following expansion
\[
J^s[\psi_2]_{T,t} = \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 = \sum_{j_1 = 0}^{\infty} C_{j_1 j_1} \psi_1^{(j_1)} \psi_2^{(j_2)}
\]
that converges in the mean-square sense is valid, where the notations are the same as in Theorem 3.

**Proof.** Let us prove the equality
\[
\frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 = \sum_{j_1 = 0}^{\infty} C_{j_1 j_1},
\]
where $C_{j_1j_2}$ is defined by the formula (3) for $k = 2$ and $j_1 = j_2$. At that $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$.

Consider the auxiliary function

$$K'(t_1, t_2) = \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \geq t_2 \\ \psi_1(t_1)\psi_2(t_2), & t_1 \leq t_2 \end{cases}, \quad t_1, t_2 \in [t, T]$$

and prove that

$$|K'(t_1, t_2) - K'(t_1^*, t_2^*)| \leq L(|t_1 - t_1^*| + |t_2 - t_2^*|), \quad (32)$$

where $L < \infty$, and $(t_1, t_2), (t_1^*, t_2^*) \in [t, T]^2$.

By the Lagrange formula for the functions $\psi_1(t_1^*), \psi_2(t_2^*)$ at the interval $[\min\{t_1, t_1^*\}, \max\{t_1, t_1^*\}]$ and for the functions $\psi_1(t_2^*), \psi_2(t_2^*)$ at the interval $[\min\{t_2, t_2^*\}, \max\{t_2, t_2^*\}]$ we obtain

$$|K'(t_1, t_2) - K'(t_1^*, t_2^*)| \leq \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \geq t_2 \\ \psi_1(t_1)\psi_2(t_2), & t_1 \leq t_2 \end{cases} - \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1^* \geq t_2 \\ \psi_1(t_1)\psi_2(t_2), & t_1 \leq t_2 \end{cases} + \begin{cases} 0, & t_1 \geq t_2, t_1^* \geq t_2^* \quad \text{or} \quad t_1 \leq t_2, t_1^* \leq t_2^* \\ \psi_2(t_1)\psi_1(t_2) - \psi_1(t_1)\psi_2(t_2), & t_1 \geq t_2, t_1^* \leq t_2^* \\ \psi_1(t_1)\psi_2(t_2) - \psi_2(t_1)\psi_1(t_2), & t_1 \leq t_2, t_1^* \geq t_2^* \end{cases} \quad (33)$$

We have

$$\begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \geq t_2 \\ \psi_1(t_1)\psi_2(t_2), & t_1 \leq t_2 \end{cases} = \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1^* \geq t_2^* \\ \psi_1(t_1)\psi_2(t_2), & t_1^* \leq t_2^* \end{cases}$$

$$\begin{cases} \psi_2(t_1)\psi_1(t_2) - \psi_1(t_1)\psi_2(t_2), & t_1 \geq t_2, t_1^* \leq t_2^* \\ \psi_1(t_1)\psi_2(t_2) - \psi_2(t_1)\psi_1(t_2), & t_1 \leq t_2, t_1^* \geq t_2^* \end{cases} \quad (34)$$

By the Lagrange formula for the functions $\psi_1(t_2), \psi_2(t_2)$ at the interval $[\min\{t_1, t_2\}, \max\{t_1, t_2\}]$ we get the estimate

$$\begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \geq t_2 \\ \psi_1(t_1)\psi_2(t_2), & t_1 \leq t_2 \end{cases} - \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1^* \geq t_2^* \\ \psi_1(t_1)\psi_2(t_2), & t_1^* \leq t_2^* \end{cases} \leq \begin{cases} \psi_2(t_1)\psi_1(t_2), & t_1 \geq t_2, t_1^* \geq t_2^* \\ \psi_1(t_1)\psi_2(t_2), & t_1 \leq t_2, t_1^* \leq t_2^* \end{cases}.$$
Let us show that if \( t_1 \leq t_2, \ t_1^* \geq t_2^* \) or \( t_1 \geq t_2, \ t_1^* \leq t_2^* \), then the following inequality is satisfied

\[
|t_2 - t_1| \leq |t_1^* - t_1| + |t_2^* - t_2|.
\]

First, consider the case \( t_1 \geq t_2, \ t_1^* \leq t_2^* \). For this case

\[
t_2 + (t_1^* - t_2^*) \leq t_2 \leq t_1.
\]

Then

\[
(t_1^* - t_1) - (t_2^* - t_2) \leq t_2 - t_1 \leq 0
\]

and (35) is satisfied.

For the case \( t_1 \leq t_2, \ t_1^* \geq t_2^* \) we have

\[
t_1 + (t_2^* - t_1^*) \leq t_1 \leq t_2.
\]

Then

\[
(t_1 - t_1^*) - (t_2 - t_2^*) \leq t_1 - t_2 \leq 0
\]

and (35) is also satisfied.

From (35) and (36) we obtain

\[
\begin{align*}
&\{ \psi_2(t_1)\psi_1(t_2), \ t_1 \geq t_2 \} - \{ \psi_1(t_1)\psi_2(t_2), \ t_1 \leq t_2 \} \\
&\leq L_3(|t_1^* - t_1| + |t_2^* - t_2|) \\
&\leq \begin{cases} 
0, & t_1 \geq t_2, \ t_1^* \geq t_2^* \text{ or } t_1 \leq t_2, \ t_1^* \leq t_2^* \\
1, & t_1 \leq t_2, \ t_1^* \geq t_2^* \text{ or } t_1 \geq t_2, \ t_1^* \leq t_2^*
\end{cases}
\end{align*}
\]

(37) \[ = L_3(|t_1^* - t_1| + |t_2^* - t_2|). \]
From (33) and (37) we get (32). Let
\[ t_1 = \frac{T - t}{2} x + \frac{T + t}{2}, \quad t_2 = \frac{T - t}{2} y + \frac{T + t}{2}, \]
where \( x, y \in [-1, 1] \). Then
\[
K'(t_1, t_2) \equiv K^*(x, y) = \begin{cases} 
\psi_2(h(x)) \psi_1(h(y)), & x \geq y \\
\psi_1(h(x)) \psi_2(h(y)), & x \leq y 
\end{cases}
\]
where \( x, y \in [-1, 1] \) and
(38) \[
h(x) = \frac{T - t}{2} x + \frac{T + t}{2}.
\]
Inequality (32) can be written in the form
(39) \[
|K^*(x, y) - K^*(x^*, y^*)| \leq L^* (|x - x^*| + |y - y^*|),
\]
where \( L^* < \infty \) and \((x, y), (x^*, y^*) \in [-1, 1]^2\) .
Thus, the function \( K^*(x, y) \) satisfies the conditions of Lemma 1 and hence for the function
\[
K^*(x, y)(1 - x^2)^{1/4}(1 - y^2)^{1/4}
\]
the inequality (26) is fulfilled.
Due to the continuous differentiability of the functions \( \psi_1(h(x)) \) and \( \psi_2(h(x)) \) at the interval \([-1, 1]\) we have \( K^*(x, y) \in L_2([-1, 1]^2) \). In addition
\[
\int_{[-1, 1]^2} \frac{K^*(x, y) dxdy}{(1 - x^2)^{1/4}(1 - y^2)^{1/4}} \leq C \left( \int_{-1}^{1} \frac{1}{(1 - x^2)^{1/4}} \int_{-1}^{x} \frac{1}{(1 - y^2)^{1/4}} dy dx + \right. \\
+ \left. \int_{-1}^{1} \frac{1}{(1 - x^2)^{1/4}} \int_{x}^{1} \frac{1}{(1 - y^2)^{1/4}} dy dx \right) < \infty, \quad C < \infty.
\]
Thus, the conditions of Theorem 4 are fulfilled for the function \( K^*(x, y) \). Note that the mentioned properties of the function \( K^*(x, y), x, y \in [-1, 1] \) also correct for the function \( K'(t_1, t_2), t_1, t_2 \in [t, T] \).

**Remark 1.** On the basis of (32) it can be argued that the function \( K'(t_1, t_2) \) belongs to the Holder class with parameter 1 in \([t, T]^2\). Hence by Theorem 5 this function can be expanded into the uniformly convergent double trigonometric Fourier series in the square \([t, T]^2\), which summarized by Pringsheim method. However, the expansions of iterated stochastic integrals obtained by using the system of Legendre polynomials are essentially simpler than their analogues obtained by using the trigonometric system of functions (see Sect. 7).
Let us expand the function $K'(t_1, t_2)$ into a double Fourier–Legendre series or double trigonometric Fourier series in the square $[t, T]^2$. This series is summable by the method of rectangular sums (Pringsheim method), i.e.

$$K'(t_1, t_2) = \lim_{n_1, n_2 \to \infty} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \int_t^T \int_t^T K'(t_1, t_2) \phi_{j_1}(t_1) \phi_{j_2}(t_2) dt_1 dt_2 \cdot \phi_{j_1}(t_1) \phi_{j_2}(t_2) =$$

$$= \lim_{n_1, n_2 \to \infty} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \left( \int_t^T \psi_2(t_2) \phi_{j_2}(t_2) \int_t^T \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 + \int_t^T \psi_1(t_2) \phi_{j_2}(t_2) \int_t^T \psi_2(t_1) \phi_{j_1}(t_1) dt_1 \right) dt_2 \phi_{j_1}(t_1) \phi_{j_2}(t_2) =$$

$$= \lim_{n_1, n_2 \to \infty} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} (C_{j_2 j_1} + C_{j_1 j_2}) \phi_{j_1}(t_1) \phi_{j_2}(t_2), \quad (t_1, t_2) \in (t, T)^2.$$ (40)

Moreover, the convergence of the series (40) is uniform on the rectangle

$$[t + \varepsilon, T - \varepsilon] \times [t + \delta, T - \delta] \quad \text{for any} \quad \varepsilon, \delta > 0 \quad \text{(in particular, we can choose} \quad \varepsilon = \delta).$$

In addition, the series (40) converges to $K'(t_1, t_2)$ at any inner point of the square $[t, T]^2$. Note that Theorem 4 does not answer the question of convergence of the series (40) on a boundary of the square $[t, T]^2$. In obtaining (40) we replaced the order of integration in the second iterated integral.

Let us substitute $t_1 = t_2$ into (40). After that, let us rewrite the limit on the right-hand side of (40) as two limits. Let us replace $j_1$ with $j_2$, $j_2$ with $j_1$, $n_1$ with $n_2$, and $n_2$ with $n_1$ in the second limit. Thus, we get

$$\lim_{n_1, n_2 \to \infty} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_1) = \frac{1}{2} \psi_1(t_1) \psi_2(t_1), \quad t_1 \in (t, T).$$ (41)

According to the above reasoning, the equality (41) holds uniformly on the interval $[t + \varepsilon, T - \varepsilon]$ for any $\varepsilon > 0$. Additionally, (41) holds at each interior point of the interval $[t, T]$.

Let us fix $\varepsilon > 0$ and integrate the equality (41) at the interval $[t + \varepsilon, T - \varepsilon]$. Due to the uniform convergence of the series (41) we can swap the series and the integral

$$\lim_{n_1, n_2 \to \infty} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} C_{j_2 j_1} \int_{t+\varepsilon}^{T-\varepsilon} \phi_{j_1}(t_1) \phi_{j_2}(t_1) dt_1 = \frac{1}{2} \int_{t+\varepsilon}^{T-\varepsilon} \psi_1(t_1) \psi_2(t_1) dt_1.$$ (42)

Lemma 2. Under the conditions of Theorem 6 the following limit

$$\lim_{n \to \infty} \sum_{j_1=0}^{n} C_{j_1, j_1}$$
exists and is finite, where $C_{j_1j_1}$ is defined by (5) for $k = 2$ and $j_1 = j_2$, i.e.

$$C_{j_1j_1} = \int_t^{t_2} \psi_2(t_2)\phi_{j_1}(t_2) \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2.$$  

The proof of Lemma 2 will be given further in this section. Using the equality (42) for $n_1 = n_2 = n$ and Lemma 2, we get

$$\frac{1}{2} \int_{t+\varepsilon}^{T-\varepsilon} \psi_1(t_1)\psi_2(t_1)dt_1 = \lim_{n \to \infty} \sum_{j_1,j_2=0}^n C_{j_2j_1} \int_{t+\varepsilon}^{T-\varepsilon} \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 =$$

$$= \lim_{n \to \infty} \sum_{j_1,j_2=0}^n C_{j_2j_1} \left( \int_{t}^{T} \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 - \int_{t}^{t+\varepsilon} \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 - \int_{T-\varepsilon}^{T} \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 \right) =$$

$$= \lim_{n \to \infty} \sum_{j_1,j_2=0}^n C_{j_2j_1} \left( \sqrt{\varepsilon} \sum_{j_1,j_2=0}^n C_{j_2j_1} \left( \phi_{j_1}(\theta)\phi_{j_2}(\theta) + \phi_{j_1}(\lambda)\phi_{j_2}(\lambda) \right) \right) =$$

$$= \lim_{n \to \infty} \sum_{j_1=0}^n C_{j_1j_1} - \varepsilon \lim_{n \to \infty} \sum_{j_1,j_2=0}^n C_{j_2j_1} \left( \phi_{j_1}(\theta)\phi_{j_2}(\theta) + \phi_{j_1}(\lambda)\phi_{j_2}(\lambda) \right),$$  

where $\theta \in [t, t+\varepsilon]$, $\lambda \in [T-\varepsilon, T]$. In obtaining (43) we used the theorem on the mean value for the Riemann integral and orthonormality of the functions $\phi_j(x)$ for $j = 0, 1, 2\ldots$

Applying (43), we obtain

$$\varepsilon \lim_{n \to \infty} \sum_{j_1,j_2=0}^n C_{j_2j_1} \left( \phi_{j_1}(\theta)\phi_{j_2}(\theta) + \phi_{j_1}(\lambda)\phi_{j_2}(\lambda) \right) =$$

$$= \lim_{n \to \infty} \sum_{j_1=0}^n C_{j_1j_1} - \lim_{n \to \infty} \sum_{j_1,j_2=0}^n C_{j_2j_1} \int_{t+\varepsilon}^{T-\varepsilon} \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1,$$

where the limits

$$\lim_{n \to \infty} \sum_{j_1=0}^n C_{j_1j_1}, \quad \lim_{n \to \infty} \sum_{j_1,j_2=0}^n C_{j_2j_1} \int_{t+\varepsilon}^{T-\varepsilon} \phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1$$

exist and are finite (see Lemma 2 and the equality (42)). This means that the limit

$$\varepsilon \lim_{n \to \infty} \sum_{j_1,j_2=0}^n C_{j_2j_1} \left( \phi_{j_1}(\theta)\phi_{j_2}(\theta) + \phi_{j_1}(\lambda)\phi_{j_2}(\lambda) \right)$$

also exists and is finite.
Suppose that the following relations

\[(44) \quad \left| \sum_{j_1, j_2=0}^{n} C_{j_2 j_1} \phi_{j_2}(T) \phi_{j_1}(T) \right| \leq K < \infty, \quad \left| \sum_{j_1, j_2=0}^{n} C_{j_2 j_1} \phi_{j_2}(t) \phi_{j_1}(t) \right| \leq K < \infty\]

are satisfied (the relations (44) will be proved further in this section); constant \(K\) does not depend on \(n\).

Note that

\[\varepsilon \lim_{n \to \infty} \sum_{j_1, j_2=0}^{n} C_{j_2 j_1} \left( \phi_{j_1}(\theta) \phi_{j_2}(\theta) + \phi_{j_1}(\lambda) \phi_{j_2}(\lambda) \right) = \varepsilon \lim_{n \to \infty} \sum_{j_1, j_2=0}^{n} C_{j_2 j_1} \phi_{j_1}(\theta) \phi_{j_2}(\theta) = \varepsilon \lim_{n \to \infty} \sum_{j_1, j_2=0}^{n} C_{j_2 j_1} \phi_{j_1}(\lambda) \phi_{j_2}(\lambda) \]

Using (41) \((n_1 = n_2 = n)\) and (44), we obtain

\[\varepsilon \left| \sum_{j_1, j_2=0}^{n} C_{j_2 j_1} \phi_{j_1}(\theta) \phi_{j_2}(\theta) + \sum_{j_1, j_2=0}^{n} C_{j_2 j_1} \phi_{j_1}(\lambda) \phi_{j_2}(\lambda) \right| \leq \varepsilon \left| \sum_{j_1, j_2=0}^{n} C_{j_2 j_1} \phi_{j_1}(\theta) \phi_{j_2}(\theta) + \sum_{j_1, j_2=0}^{n} C_{j_2 j_1} \phi_{j_1}(\lambda) \phi_{j_2}(\lambda) \right| \leq 2\varepsilon K_1 \to 0\]

if \(\varepsilon \to +0\), where \(\theta \in [t, t + \varepsilon], \lambda \in [T - \varepsilon, T]\), constant \(K_1\) is independent on \(n\).

Performing the passage to the limit \(\varepsilon \to +0\) in the equality (43) and taking into account (45), (46), we get

\[(47) \quad \frac{1}{2} \int_{t}^{T} \psi_1(t_1) \psi_2(t_1) dt_1 = \sum_{j_1=0}^{\infty} C_{j_2 j_1}.\]

Thus, to complete the proof of Theorem 6, it is necessary to prove (44) and Lemma 2. To prove (44) and Lemma 2, as well as for further consideration, we need some well known properties of the Legendre polynomials (64), (67).

The complete orthonormal system of Legendre polynomials in the space \(L_2([t, T])\) looks as follows

\[(48) \quad \phi_j(x) = \sqrt{\frac{2j + 1}{T - t}} P_j \left( \left( x - \frac{T + t}{2} \right) \frac{2}{T - t} \right), \quad j = 0, 1, 2, \ldots,\]

where \(P_j(x)\) is the Legendre polynomial.

It is known that the Legendre polynomial \(P_j(x)\) is represented, for example, as

\[P_j(x) = \frac{1}{2^j j!} \frac{d^j}{dx^j} (x^2 - 1)^j.\]
At the boundary points of the interval of orthogonality the Legendre polynomials satisfy the following relations

\[ P_j(1) = 1, \quad P_j(-1) = (-1)^j, \]

\[ P_{j+1}(1) - P_{j-1}(1) = 0, \quad P_{j+1}(-1) - P_{j-1}(-1) = 0, \]

\[ P_{j+1}(1) - P_j(1) = 0, \quad P_{j+1}(-1) + P_j(-1) = 0, \]

where \( j = 0, 1, 2, \ldots \)

Relation of the Legendre polynomial \( P_j(x) \) with derivatives of the Legendre polynomials \( P_{j+1}(x) \) and \( P_{j-1}(x) \) is expressed by the following equality

\[ P_j(x) = \frac{1}{2j+1} \left( P'_{j+1}(x) - P'_{j-1}(x) \right), \quad j = 1, 2, \ldots \]  

(49)

The recurrent relation has the form

\[ xP_j(x) = \frac{(j+1)P_{j+1}(x) + jP_{j-1}(x)}{2j+1}, \quad j = 1, 2, \ldots \]

Orthogonality of Legendre polynomial \( P_j(x) \) to any polynomial \( Q_k(x) \) of lesser degree we write in the following form

\[ \int_{-1}^{1} Q_k(x)P_j(x)dx = 0, \quad k = 0, 1, 2, \ldots, j - 1. \]

From the property

\[ \int_{-1}^{1} P_k(x)P_j(x)dx = \begin{cases} 0 & \text{if } k \neq j \\ 2/(2j+1) & \text{if } k = j \end{cases} \]

it follows that the orthonormal on the interval \([-1, 1]\) Legendre polynomials determined by the relation

\[ P^*_j(x) = \sqrt{\frac{2j+1}{2}} P_j(x), \quad j = 0, 1, 2, \ldots \]

It is well known that there is an estimate

\[ |P_j(y)| < \frac{K}{\sqrt{j + 1(1 - y^2)^{1/4}}}, \quad y \in (-1, 1), \quad j = 1, 2, \ldots, \]

(50)

where constant \( K \) does not depends on \( y \) and \( j \).

Moreover,

\[ |P_j(x)| \leq 1, \quad x \in [-1, 1], \quad j = 0, 1, \ldots \]  

(51)
The Christoffel–Darboux formula has the form

\begin{equation}
\sum_{j=0}^{n}(2j+1)P_j(x)P_j(y) = (n+1)\frac{P_n(x)P_{n+1}(y) - P_{n+1}(x)P_n(y)}{y-x}.
\end{equation}

Let us prove (44). From (52) for \( x = \pm 1 \) we obtain

\begin{equation}
\sum_{j=0}^{n}(2j+1)P_j(y) = (n+1)\frac{P_{n+1}(y) - P_n(y)}{y-1},
\end{equation}

\begin{equation}
\sum_{j=0}^{n}(2j+1)(-1)^jP_j(y) = (n+1)(-1)^nP_{n+1}(y) + P_n(y)\frac{y+1}{y+1}.
\end{equation}

From the other hand (see (49))

\begin{equation}
\sum_{j=0}^{n}(2j+1)P_j(y) = 1 + \sum_{j=1}^{n}(2j+1)P_j(y) = 1 + \left(\sum_{j=1}^{n}(P_{j+1}(y) - P_{j-1}(y))\right)'
\end{equation}

\begin{equation}
= 1 + \left(\sum_{j=1}^{n}(P_{j+1}(y) - P_{j-1}(y))\right)'
\end{equation}

\begin{equation}
= 1 + \left((n+1)(P_{n+1}(x) - P_n(x)) - x - 1\right)' = (P_n(x) + P_{n+1}(x))'.
\end{equation}

and

\begin{equation}
\sum_{j=0}^{n}(2j+1)(-1)^jP_j(y) = 1 + \sum_{j=1}^{n}(-1)^j(2j+1)P_j(y) = 1 + \left(\sum_{j=1}^{n}(-1)^j(P_{j+1}(y) - P_{j-1}(y))\right)'
\end{equation}

\begin{equation}
= 1 + \left((n+1)(P_{n+1}(x) - P_n(x)) - x + 1\right)' = (P_{n+1}(x) - P_n(x))'.
\end{equation}

Applying (53)–(56), we get

\begin{equation}
(n+1)\frac{P_{n+1}(y) - P_n(y)}{y-1} = (P_n(x) + P_{n+1}(x))',
\end{equation}

\begin{equation}
(n+1)\frac{P_{n+1}(y) + P_n(y)}{y+1} = (P_{n+1}(x) - P_n(x))'.
\end{equation}

Let us prove the boundedness of the first sum in (44). We have

\[ \sum_{j_1,j_2=0}^{n} C_{j_1j_2} \phi_{j_2}(T)\phi_{j_1}(T) = \]
\[
\frac{1}{4} \sum_{j_2=0}^{n} \sum_{j_1=0}^{n} (2j_2 + 1)(2j_1 + 1) \int_{-1}^{1} \psi_2(h(y)) P_{j_2}(y) \int_{-1}^{y} \psi_1(h(y_1)) P_{j_1}(y_1) dy_1 dy = \\
= \frac{1}{4} \int_{-1}^{1} \psi_2(h(y)) \sum_{j_2=0}^{n} (2j_2 + 1) P_{j_2}(y) \int_{-1}^{y} \psi_1(h(y_1)) \sum_{j_1=0}^{n} (2j_1 + 1) P_{j_1}(y_1) dy_1 dy = \\
= \frac{1}{4} \int_{-1}^{1} \psi_2(h(y)) \left( \int_{-1}^{y} \psi_1(h(y_1)) d(P_{n+1}(y_1) + P_n(y_1)) \right) d(P_{n+1}(y) + P_n(y)) = \\
= \frac{1}{4} \int_{-1}^{1} \psi_1(h(y)) \left( \int_{-1}^{y} \psi_1(h(y_1)) d(P_{n+1}(y_1) + P_n(y_1)) \right) d(P_{n+1}(y) + P_n(y)) + \\
+ \frac{1}{4} \int_{-1}^{1} \Delta(h(y)) \left( \int_{-1}^{y} \psi_1(h(y_1)) d(P_{n+1}(y_1) + P_n(y_1)) \right) d(P_{n+1}(y) + P_n(y)) = \\
= \frac{1}{4} I_1 + \frac{1}{4} I_2,
\]

where

\[\Delta(h(y)) = \psi_2(h(y)) - \psi_1(h(y)), \quad h(y) = \frac{T - t}{2}y + \frac{T + t}{2}.\]

Further,

\[I_1 = \frac{1}{2} \left( \int_{-1}^{1} \psi_1(h(y)) d(P_{n+1}(y) + P_n(y)) \right)^2 = \]

\[= \frac{1}{2} \left( 2\psi_1(T) - \int_{-1}^{1} (P_{n+1}(y) + P_n(y)) \psi_1'(h(y)) \frac{T - t}{2} dy \right)^2 < C_1 < \infty,\]

where \(\psi_1'\) is a derivative of the function \(\psi_1\) with respect to the variable \(y\), constant \(C_1\) does not depend on \(n\).

By the Lagrange formula we obtain

\[\Delta(h(y)) = \psi_2 \left( \frac{1}{2} (T - t)(y - 1) + T \right) - \psi_1 \left( \frac{1}{2} (T - t)(y - 1) + T \right) = \]

\[= \psi_2(T) - \psi_1(T) + (y - 1) \left( \psi_2'(\xi_y) - \psi_1'(\theta_y) \right) \frac{1}{2} (T - t) = \]

\[= C_1 + \alpha_y(y - 1),\]

where \(|\alpha_y| < \infty\) and \(C_1 = \psi_2(T) - \psi_1(T)\).

Let us substitute (60) into the integral \(I_2\)
where
\[
I_3 = \int_{-1}^{1} \alpha_y(y-1) \left( \int_{-1}^{y} \psi_1(h(y_1))d(P_{n+1}(y_1) + P_n(y_1)) \right) d(P_{n+1}(y) + P_n(y)),
\]
\[
I_4 = C_1 \int_{-1}^{1} \left( \int_{-1}^{y} \psi_1(h(y_1))d(P_{n+1}(y_1) + P_n(y_1)) \right) d(P_{n+1}(y) + P_n(y)).
\]
Integrating by parts and using (50), we obtain
\[
I_3 = \int_{-1}^{1} \alpha_y(y-1)(n+1)(P_{n+1}(y) - P_n(y)) \frac{1}{y-1} \left( \psi_1(h(y))(P_{n+1}(y) + P_n(y)) - \int_{-1}^{y} (P_{n+1}(y_1) + P_n(y_1)) \psi_1'(h(y_1)) \frac{1}{2}(T-t)dy_1 \right) dy.
\]

Applying the estimate (50) and taking into account the boundedness of \(\alpha_y\) and \(\psi_1'(h(y_1))\), we have that \(|I_3| < \infty\).

Using the integration order replacement in \(I_4\), we get
\[
I_4 = C_1 \int_{-1}^{1} \psi_1(h(y_1)) \left( \int_{-1}^{y_1} d(P_{n+1}(y) + P_n(y)) \right) d(P_{n+1}(y_1) + P_n(y_1)) =
\]
\[
=C_1 \int_{-1}^{y_1} \psi_1(h(y_1))d(P_{n+1}(y_1) + P_n(y_1)) \int_{-1}^{1} d(P_{n+1}(y) + P_n(y)) -
\]
\[
-C_1 \int_{-1}^{y_1} \psi_1(h(y_1)) \left( \int_{-1}^{1} d(P_{n+1}(y) + P_n(y)) \right) d(P_{n+1}(y_1) + P_n(y_1)) =
\]
\[
= I_5 - I_6.
\]
Consider \(I_5\)
\[
I_5 = 2C_1 \int_{-1}^{1} \psi_1(h(y_1))d(P_{n+1}(y_1) + P_n(y_1)) =
\]
\[
= 2C_1 \left( 2\psi_1(T) - \int_{-1}^{1} (P_{n+1}(y_1) + P_n(y_1)) \psi_1'(h(y_1)) \frac{1}{2}(T-t)dy_1 \right).
\]

Applying the estimate (51) and using the boundedness of \(\psi_1'(h(y_1))\), we obtain that \(|I_5| < \infty\).
Thus, the boundedness of the first sum in (44) is proved.

where \( |\beta| < \infty \) and \( C_2 = \psi_1(T) \), then

\[
I_6 = C_3 \int_{-1}^{1} \left( \int_{-1}^{y_1} d(P_{n+1}(y) + P_n(y)) \right)^2 dy_1 =
\]

Using the estimate (50) and taking into account the boundedness of \( \beta_{y_1} \), we obtain that \( |I_6| < \infty \).

Let us prove the boundedness of the second sum in (44). We have

\[
\sum_{j_1,j_2=0}^{n} C_{j_2,j_1} \phi_{j_2}(t) \phi_{j_1}(t) =
\]

\[
= \frac{1}{4} \int_{-1}^{1} \psi_2(h(y)) \sum_{j_2=0}^{n} \sum_{j_1=0}^{n} (2j_2 + 1)(2j_1 + 1)(-1)^{j_1+j_2} \left( \int_{-1}^{y_1} d(P_{n+1}(y) - P_n(y)) \right) \psi_1(h(y_1)) P_{n+1}(y_1) dy_1 dy =
\]

\[
= \frac{(-1)^{2n}}{4} \int_{-1}^{1} \psi_2(h(y)) \left( \int_{-1}^{y_1} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) \right) d(P_{n+1}(y) - P_n(y)) =
\]

\[
= \frac{1}{4} \int_{-1}^{1} \psi_1(h(y)) \left( \int_{-1}^{y_1} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) \right) d(P_{n+1}(y) - P_n(y)) +
\]
\begin{align*}
+ \frac{1}{4} \int_{-1}^{1} \Delta(h(y)) \left( \int_{-1}^{y} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) \right) d(P_{n+1}(y) - P_n(y)) = \\
= \frac{1}{4} J_1 + \frac{1}{4} J_2,
\end{align*}

where $\Delta(h(y))$, $h(y)$ are defined by (59).

Further,

\begin{equation}
J_1 = \frac{1}{2} \left( \int_{-1}^{1} \psi_1(h(y)) d(P_{n+1}(y) - P_n(y)) \right)^2 = \frac{1}{2} \left( 2(-1)^n \psi_1(t) - \int_{-1}^{1} (P_{n+1}(y) - P_n(y)) \psi_1'(h(y)) \frac{T-t}{2} dy \right)^2 < K_1 < \infty,
\end{equation}

where $\psi_1'$ is a derivative of the function $\psi_1$ with respect to the variable $y$, constant $K_1$ is independent of $n$.

By the Lagrange formula we obtain

\begin{align*}
\Delta(h(y)) &= \psi_2 \left( \frac{1}{2} (T-t)(y+1) + t \right) - \psi_1 \left( \frac{1}{2} (T-t)(y+1) + t \right) = \\
&= \psi_2(t) - \psi_1(t) + (y+1) \left( \psi_2'(\mu_y) - \psi_1'(\rho_y) \right) \frac{1}{2} (T-t) = \\
&= K_2 + \gamma_y (y+1),
\end{align*}

where $|\gamma_y| < \infty$ and $K_2 = \psi_2(t) - \psi_1(t)$.

Consider $J_2$

\begin{align*}
J_2 &= \int_{-1}^{1} \Delta(h(y)) d(P_{n+1}(y) - P_n(y)) \int_{-1}^{1} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) - \\
&= \int_{-1}^{1} \Delta(h(y)) \left( \int_{-1}^{y} \psi_1(h(y_1)) d(P_{n+1}(y_1) - P_n(y_1)) \right) d(P_{n+1}(y) - P_n(y)) = \\
&= J_3 J_4 - J_5.
\end{align*}

The integral $J_4$ was considered earlier (see $J_1$ and (61)), i.e. it has already been shown that $|J_4| < \infty$. Analogously, we have that $|J_3| < \infty$.

Let us substitute (62) into the integral $J_5$

\begin{equation}
J_5 = J_6 + J_7,
\end{equation}
where
\[ J_6 = \int_{-1}^{1} \gamma_y(y + 1) \left( \int_{y}^{1} \psi_1(h(y_1))d(P_{n+1}(y_1) - P_n(y_1)) \right) d(P_{n+1}(y) - P_n(y)), \]
\[ J_7 = K_2 \int_{-1}^{1} \left( \int_{y}^{1} \psi_1(h(y_1))d(P_{n+1}(y_1) - P_n(y_1)) \right) d(P_{n+1}(y) - P_n(y)). \]

Integrating by parts and using (58), we get
\[ J_6 = \int_{-1}^{1} \frac{\gamma_y(y + 1)(n + 1)(P_{n+1}(y) + P_n(y))}{y + 1} \left( -\psi_1(h(y))(P_{n+1}(y) - P_n(y)) - \right. \]
\[ \left. - \int_{y}^{1} (P_{n+1}(y_1) - P_n(y_1))\psi_1'(h(y_1)) \frac{1}{2}(T-t)dy_1 \right) dy. \]

Applying the estimate (50) and taking into account the boundedness of \( \gamma_y \) and \( \psi_1'(h(y_1)) \), we have that \( |J_6| < \infty. \)

Using the integration order replacement in \( J_7 \), we obtain
\[ J_7 = K_2 \int_{-1}^{1} \psi_1(h(y_1)) \left( \int_{y_1}^{1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)) = \]
\[ = K_2 \int_{-1}^{1} \psi_1(h(y_1))d(P_{n+1}(y_1) - P_n(y_1)) \int_{y_1}^{1} d(P_{n+1}(y) - P_n(y)) - K_2J_8 = \]
\[ = K_2J_42(-1)^n - K_2J_8, \]
where
\[ J_8 = \int_{-1}^{1} \psi_1(h(y_1)) \left( \int_{y}^{1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)). \]

Since (see (62))
\[ \psi_1(h(y)) = \psi_1 \left( \frac{1}{2}(T-t)(y + 1) + t \right) = \]
\[ = \psi_1(t) + (y + 1)\psi_1'(\rho_y) \frac{1}{2}(T-t) = K_3 + \varepsilon_y(y + 1), \]
where \( |\varepsilon_y| < \infty \) and \( K_3 = \psi_1(t) \), then
\[ J_8 = K_3 \int_{-1}^{1} \left( \int_{y_1}^{1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)) + \]
\[
+ \int_{-1}^{1} \varepsilon(y + 1) \left( \int_{y_1}^{1} d(P_{n+1}(y) - P_n(y)) \right) d(P_{n+1}(y_1) - P_n(y_1)) =
\]
\[
= \frac{K_3}{2} \left( \int_{-1}^{1} d(P_{n+1}(y) - P_n(y)) \right)^2 +
\]
\[
+ \int_{-1}^{1} \varepsilon(y_1 + 1)(n + 1)(P_{n+1}(y_1) + P_n(y_1))(P_n(y_1) - P_{n+1}(y_1))dy =
\]
\[
= 2K_3 + \int_{-1}^{1} \varepsilon(y_1)(n + 1)(P_{n+1}(y_1) + P_n(y_1))(P_n(y_1) - P_{n+1}(y_1))dy.
\]

When obtaining the equality (64), we used (58). Applying the estimate (50) and taking into account the boundedness of \(\varepsilon y_1\), we obtain that \(|J_8| < \infty\). Thus, the boundedness of the second sum in (44) is proved. The relations (44) are proved.

Let us prove Lemma 2. We will prove that
\[
\sum_{j_1=0}^{n} C_{j_1j_1}
\]
is the Cauchy sequence for the cases of Legendre polynomials and trigonometric functions.

Consider the case of Legendre polynomials and fix \(n > m\) \((n, m \in \mathbb{N})\). We have
\[
\sum_{j_1=m+1}^{n} C_{j_1j_1} = \sum_{j_1=m+1}^{n} \int_{t}^{T} \psi_2(s)\phi_{j_1}(s) \int_{t}^{x} \psi_1(\tau)\phi_{j_1}(\tau)d\tau ds =
\]
\[
= \frac{T-t}{4} \sum_{j_1=m+1}^{n} (2j_1 + 1) \int_{-1}^{1} \psi_2(h(x))P_{j_1}(x) \int_{-1}^{x} \psi_1(h(y))P_{j_1}(y)dydx =
\]
\[
= \frac{T-t}{4} \sum_{j_1=m+1}^{n} \int_{-1}^{1} \psi_1(h(x))\psi_2(h(x)) (P_{j_1+1}(x)P_{j_1}(x) - P_{j_1}(x)P_{j_1-1}(x)) dx-
\]
\[
- \frac{(T-t)^2}{8} \sum_{j_1=m+1}^{n} \int_{-1}^{1} \psi_2(h(x))P_{j_1}(x) \int_{-1}^{x} (P_{j_1+1}(y) - P_{j_1-1}(y)) \psi_1'(h(y))dydx =
\]
\[
= \frac{T-t}{4} \int_{-1}^{1} \psi_1(h(x))\psi_2(h(x)) \sum_{j_1=m+1}^{n} (P_{j_1+1}(x)P_{j_1}(x) - P_{j_1}(x)P_{j_1-1}(x)) dx-
\]
\[-\frac{(T-t)^2}{8} \sum_{j_1=m+1}^{n-1} \int (P_{j_1+1}(y) - P_{j_1-1}(y)) \psi'_1(h(y)) \int \limits_{y} P_{j_1}(x)\psi_2(h(x)) dx dy =\]

\[= T - t \int \limits_{-1}^{1} \psi_1(h(x))\psi_2(h(x)) (P_{n+1}(x)P_n(x) - P_{m+1}(x)P_m(x)) dx +\]

\[+ \frac{(T-t)^2}{8} \sum_{j_1=m+1}^{n} \int \limits_{-1}^{1} (P_{j_1+1}(y) - P_{j_1-1}(y)) \psi'_1(h(y)) \times\]

\[\times (P_{j_1+1}(y) - P_{j_1-1}(y)) \psi_2(h(y)) +\]

\[
\frac{T - t}{2} \int \limits_{y} (P_{j_1+1}(x) - P_{j_1-1}(x)) \psi'_2(h(x)) dx \bigg) dy, (65)\]

where \(\psi'_1, \psi'_2\) are derivatives of the functions \(\psi_1(\tau), \psi_2(\tau)\) with respect to the variable \(h(y)\) (see (38)).

Applying the estimate (50) and taking into account the boundedness of the functions \(\psi_1(\tau), \psi_2(\tau)\) and their derivatives, we finally obtain

\[
\left| \sum_{j_1=m+1}^{n} C_{j_1,j_1} \right| \leq C_1 \left( \frac{1}{n} + \frac{1}{m} \right) \int \limits_{-1}^{1} \frac{dx}{(1-x^2)^{1/2}} +
\]

\[
+C_2 \sum_{j_1=m+1}^{n} \frac{1}{j_1^2} \left( \int \limits_{-1}^{1} \frac{dy}{(1-y^2)^{1/2}} + \int \limits_{-1}^{1} \frac{1}{(1-y^2)^{1/4}} \int \limits_{y}^{1} \frac{dx}{(1-x^2)^{1/4}} dy \right) \leq
\]

\[
\leq C_3 \left( \frac{1}{n} + \frac{1}{m} + \sum_{j_1=m+1}^{n} \frac{1}{j_1^2} \right) \rightarrow 0 \]

if \(n, m \rightarrow \infty\ (n > m)\), where constants \(C_1, C_2, C_3\) do not depend on \(n\) and \(m\).

Consider the trigonometric case. Below in this section we write \(\lim \limits_{n,m \rightarrow \infty}\) instead of \(\lim \limits_{n,m \rightarrow \infty} \left( n > m \right)\). Fix \(n > m\ (n, m \in \mathbb{N})\). Denote

\[S_{n,m} \overset{\text{def}}{=} \sum_{j_1=m+1}^{n} C_{j_1,j_1} = \sum_{j_1=m+1}^{n} \int \limits_{t}^{T} \psi_2(t_2) \phi_{j_1}(t_2) \int \limits_{t}^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2.\]

By analogy with (65) we obtain
\[ S_{2n,2m} = \sum_{j_1=2m+1}^{2n} \int_t^T \psi_2(t_2) \phi_{j_1}(t_2) \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 = \]

\[ = \frac{2}{T-t} \sum_{j_1=m+1}^{n} \left( \int_t^T \psi_2(t_2) \sin \frac{2\pi j_1(t_2-t)}{T-t} \int_t^{t_2} \psi_1(t_1) \sin \frac{2\pi j_1(t_1-t)}{T-t} dt_1 dt_2 + \right. \]

\[ + \left. \int_t^T \psi_2(t_2) \cos \frac{2\pi j_1(t_2-t)}{T-t} \int_t^{t_2} \psi_1(t_1) \cos \frac{2\pi j_1(t_1-t)}{T-t} dt_1 dt_2 \right) = \]

\[ = \frac{T-t}{2\pi^2} \sum_{j_1=m+1}^{n} \frac{1}{j_1^2} \left( \psi_1(t) \left( \psi_2(t) - \psi_2(T) + \int_t^T \psi_2(t_2) \cos \frac{2\pi j_1(t_2-t)}{T-t} dt_2 \right) - \right. \]

\[ \left. - \int_t^T \psi_1(t_1) \cos \frac{2\pi j_1(t_1-t)}{T-t} \left( \psi_2(t) - \psi_2(t_1) \cos \frac{2\pi j_1(t_1-t)}{T-t} \right) \right. \]

\[ \left. + \int_t^T \psi_1(t_1) \sin \frac{2\pi j_1(t_1-t)}{T-t} \left( \psi_2(t_1) \sin \frac{2\pi j_1(t_1-t)}{T-t} + \right. \right. \]

\[ \left. \left. + \int_t^T \psi_2(t_2) \sin \frac{2\pi j_1(t_2-t)}{T-t} dt_2 \right) dt_1 \right) \right), \]

where \( \psi_1^\prime(\tau), \psi_2^\prime(\tau) \) are derivatives of the functions \( \psi_1(\tau), \psi_2(\tau) \) with respect to the variable \( \tau \).

From (67) we get

\[ |S_{2n,2m}| \leq C \sum_{j_1=m+1}^{n} \frac{1}{j_1^2} \rightarrow 0 \]

if \( n, m \rightarrow \infty \) \((n > m)\), where constant \( C \) does not depend on \( n \) and \( m \).

Further,

\[ S_{2n-1,2m} = S_{2n,2m} - \]

\[ - \frac{2}{T-t} \int_t^T \psi_2(t_2) \cos \frac{2\pi n(t_2-t)}{T-t} \int_t^{t_2} \psi_1(t_1) \cos \frac{2\pi n(t_1-t)}{T-t} dt_1 dt_2, \]
S_{2n,2m} = S_{2n,2m} + 

\frac{2}{T-t} \int_t^T \psi_2(t_2) \cos \frac{2 \pi m (t_2 - t)}{T-t} \int_t^{t_2} \psi_1(t_1) \cos \frac{2 \pi m (t_1 - t)}{T-t} dt_1 dt_2,

S_{2n-1,2m} = S_{2n,2m-1} - 

- \frac{2}{T-t} \int_t^T \psi_2(t_2) \cos \frac{2 \pi n (t_2 - t)}{T-t} \int_t^{t_2} \psi_1(t_1) \cos \frac{2 \pi n (t_1 - t)}{T-t} dt_1 dt_2 =

= S_{2n,2m} + \frac{2}{T-t} \int_t^T \psi_2(t_2) \cos \frac{2 \pi m (t_2 - t)}{T-t} \int_t^{t_2} \psi_1(t_1) \cos \frac{2 \pi m (t_1 - t)}{T-t} dt_1 dt_2 -

(71)

\frac{2}{T-t} \int_t^T \psi_2(t_2) \cos \frac{2 \pi n (t_2 - t)}{T-t} \int_t^{t_2} \psi_1(t_1) \cos \frac{2 \pi n (t_1 - t)}{T-t} dt_1 dt_2.

Integrating by parts in (69)–(71), we obtain

(72) \quad |S_{2n-1,2m}| \leq |S_{2n,2m}| + \frac{C_1}{n},

(73) \quad |S_{2n,2m-1}| \leq |S_{2n,2m}| + \frac{C_1}{m},

(74) \quad |S_{2n,2m} - 1| \leq |S_{2n,2m}| + C_1 \left( \frac{1}{m} + \frac{1}{n} \right),

where constant C_1 does not depend on n and m.

The relations (68), (72)–(74) imply that

(75) \quad \lim_{n,m \to \infty} |S_{2n,2m}| = \lim_{n,m \to \infty} |S_{2n-1,2m}| = \lim_{n,m \to \infty} |S_{2n,2m-1}| = \lim_{n,m \to \infty} |S_{2n-1,2m-1}| = 0.

From (75) we get

(76) \quad \lim_{n,m \to \infty} |S_{n,m}| = 0.

Lemma 2 is proved. Theorem 6 is proved.
5. Proof of the Equality. The Case of an Arbitrary Complete Orthonormal System of Functions in the Space $L_2([t, T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$

**Theorem 7.** Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$ and $\psi_1(\tau), \psi_2(\tau) \in L_2([t, T])$. Then the following equality

\[
\sum_{j=0}^{\infty} \int_{t}^{T} \psi_2(t_2) \phi_j(t_2) \int_{t}^{t_2} \psi_1(t_1) \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int_{t}^{T} \psi_1(\tau) \psi_2(\tau) d\tau
\]

is fulfilled.

**Proof.** First consider the case $\psi_1(\tau) \equiv \psi_2(\tau)$ or

\[
\psi_1(\tau) = \psi_2(\tau) \int_{t}^{\tau} g(\theta) d\theta,
\]

where $\tau \in [t, T]$ and $\psi_1(\tau), \psi_2(\tau), g(\tau) \in L_2([t, T])$.

First suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$ such that $\phi_j(x)$ for $j < \infty$ is continuous at the interval $[t, T]$ except may be for the finite number of points of the finite discontinuity. Furthermore, let $\psi_1(\tau) \equiv \psi_2(\tau)$ or the equality (78) is satisfied. Here we suppose that $\psi_1(\tau), \psi_2(\tau), g(\tau)$ are continuous functions at the interval $[t, T]$.

Using the integration order replacement and the Parseval equality, we have (see (78))

\[
\sum_{j=0}^{\infty} \int_{t}^{T} \psi_2(t_2) \phi_j(t_2) \int_{t}^{t_2} \psi_1(t_1) \phi_j(t_1) dt_1 dt_2 =
\]

\[
= \sum_{j=0}^{\infty} \int_{t}^{T} \psi_2(t_2) \phi_j(t_2) \int_{t}^{t_2} \psi_2(t_1) \phi_j(t_1) \int_{t}^{t_1} g(\tau) d\tau dt_1 dt_2 =
\]

\[
= \sum_{j=0}^{\infty} \int_{t}^{T} g(\tau) \int_{t}^{T} \psi_2(t_1) \phi_j(t_1) \int_{t}^{t_1} \psi_2(t_2) \phi_j(t_2) dt_2 dt_1 d\tau =
\]

\[
\frac{1}{2} \sum_{j=0}^{\infty} \int_{t}^{T} g(\tau) \left( \int_{t}^{T} \psi_2(t_1) \phi_j(t_1) dt_1 \right)^2 d\tau =
\]

\[
\frac{1}{2} \int_{t}^{T} g(\tau) \sum_{j=0}^{\infty} \left( \int_{t}^{T} 1_{\{\tau < t_1\}} \psi_2(t_1) \phi_j(t_1) dt_1 \right)^2 d\tau =
\]

\[
= \frac{1}{2} \int_{t}^{T} g(\tau) \int_{t}^{T} 1_{\{\tau < t_1\}} \psi_2^2(t_1) dt_1 d\tau =
\]
where the transition from (79) to (80) is based on the Dini Theorem (using the continuity of the functions \(u_q(\tau)\) (see below), the nondecreasing property of the functional sequence

\[
u_q(\tau) = \sum_{j=0}^{q} \left( \int_{\tau}^{T} \psi_2(t_1) \phi_j(t_1) dt_1 \right)^2,
\]

and the continuity of the limit function

\[
u(\tau) = \int_{\tau}^{T} \psi_2^2(t_1) dt_1
\]

according to Dini’s Theorem, we have the uniform convergence \(u_q(\tau)\) to \(u(\tau)\) at the interval \([t, T]\).

From the other hand, using the integration order replacement and the generalized Parseval equality as well as (81), we get

\[
\sum_{j=0}^{\infty} \int_{t}^{T} \psi_1(t_2) \phi_j(t_2) \int_{t}^{t_2} \psi_2(t_1) \phi_j(t_1) dt_1 dt_2 =
\]

\[
= \sum_{j=0}^{\infty} \int_{t}^{T} \psi_2(t_2) \phi_j(t_2) \int_{t}^{t_2} g(\tau) d\tau \int_{t}^{t_2} \psi_2(t_1) \phi_j(t_1) dt_1 dt_2 =
\]

\[
= \sum_{j=0}^{\infty} \int_{t}^{T} \psi_2(t_1) \phi_j(t_1) \int_{t_1}^{T} \psi_2(t_2) \phi_j(t_2) \int_{t}^{t_2} g(\tau) d\tau dt_2 dt_1 =
\]

\[
= \sum_{j=0}^{\infty} \int_{t}^{T} \psi_2(t_1) \phi_j(t_1) dt_1 \int_{t}^{T} \psi_2(t_2) \phi_j(t_2) \int_{t}^{t_2} g(\tau) d\tau dt_2 dt_1 -
\]

\[- \sum_{j=0}^{\infty} \int_{t}^{T} \psi_2(t_1) \phi_j(t_1) \int_{t}^{T} \psi_2(t_2) \phi_j(t_2) \int_{t}^{t_2} g(\tau) d\tau dt_2 dt_1 =
\]
\[
\int_t^T \psi_2(t_1) \cdot \psi_2(t_1) \int_t^{t_1} g(\tau) d\tau dt_1 - \frac{1}{2} \int_t^T \psi_2^2(t_1) \int_t^{t_1} g(\tau) d\tau dt_1 = \\
= \frac{1}{2} \int_t^T \psi_2^2(t_1) \int_t^{t_1} g(\tau) d\tau dt_1 = \frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1.
\]

(83)

In addition, for the case \(\psi_1(\tau) \equiv \psi_2(\tau)\), using the Parseval equality, we obtain

\[
\sum_{j=0}^\infty \int_t^T \psi_1(t_2) \phi_j(t_2) \int_t^{t_1} \psi_1(t_1) \phi_j(t_1) dt_1 dt_2 = \\
= \frac{1}{2} \sum_{j=0}^\infty \left( \int_t^T \psi_1(t_1) \phi_j(t_1) dt_1 \right)^2 = \\
\frac{1}{2} \int_t^T \psi_1^2(t_1) dt_1.
\]

(84)

By interpreting the integrals in the above formulas as Lebesgue integrals, using Fubini’s theorem and Lebesgue’s Dominated Convergence Theorem in the above reasoning, we get the equality (77) for the case of an arbitrary complete orthonormal system of functions in the space \(L_2([t, T])\) and \(\psi_1(\tau), \psi_2(\tau), g(\tau) \in L_2([t, T])\).

Suppose that

\[
\psi_2(\tau) = (\tau - t)^l, \quad g(\tau) = k(\tau - t)^{k-1},
\]

where \(l = 0, 1, 2, \ldots, k = 1, 2, \ldots\)

From (83) we have

\[
\psi_1(\tau) = \psi_2(\tau) \int_t^\tau g(\theta) d\theta = k(\tau - t)^l \int_t^\tau (\theta - t)^{k-1} d\theta = (\tau - t)^{l+k}.
\]

Taking into account (82)–(84), we obtain

\[
\sum_{j=0}^\infty \int_t^T (t_2 - t)^l \phi_j(t_2) \int_t^{t_1} (t_1 - t)^{l+k} \phi_j(t_1) dt_1 dt_2 = \\
= \sum_{j=0}^\infty \int_t^T (t_2 - t)^{l+k} \phi_j(t_2) \int_t^{t_1} (t_1 - t)^l \phi_j(t_1) dt_1 dt_2 = \\
= \frac{1}{2} \int_t^T (\tau - t)^{2l+k} d\tau,
\]

(85)
where \( k, l = 0, 1, 2, \ldots \).

The equality similar to (85) was obtained in [68] using other arguments. In addition, the formula similar to (85) was used in [68] to generalize the equality (77) to the case of an arbitrary complete orthonormal system of functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \psi_2(\tau) \in L_2([t, T]) \). Consider this approach [68] in more detail.

Let us rewrite the equality (85) in the following form

\[
\sum_{j=0}^{\infty} \int_{t}^{T} \int_{t}^{T} (t_2 - t)^j \phi_j(t_2) \left( t_1 - t \right)^m \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int_{t}^{T} (\tau - t)^j (\tau - t)^m d\tau,
\]

where \( l, m = 0, 1, 2, \ldots \).

Since the equality (85) is valid for monomials with respect to \( \tau - t \ (\tau \in [t, T]) \), it will obviously also be valid for Legendre polynomials that form a complete orthonormal system of functions in the space \( L_2([t, T]) \) and finite linear combinations of Legendre polynomials.

Let \( \psi_1(\tau), \psi_2(\tau) \in L_2([t, T]) \) and \( \psi_1^{(p)}(\tau), \psi_2^{(q)}(\tau) \) be approximations of the functions \( \psi_1(\tau), \psi_2(\tau) \), respectively, which are partial sums of the corresponding Fourier–Legendre series. Then we have (see [68])

\[
\sum_{j=0}^{\infty} \int_{t}^{T} \int_{t}^{T} \psi_1^{(p)}(\tau) \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int_{t}^{T} \psi_1^{(p)}(\tau) \psi_1^{(q)}(\tau) d\tau,
\]

where \( p, q \in \mathbb{N} \), the series converges absolutely and its sum does not depend on a basis system \( \{\phi_j(x)\}_{j=0}^{\infty} \).

Let us fix \( q \) in (87). The right-hand side of (87) for a fixed \( q \) defines (as a scalar product in \( L_2([t, T]) \)) a linear bounded (and therefore continuous) functional in \( L_2([t, T]) \), which is given by the function \( \psi_2^{(q)}(\tau) \). The left-hand side of the equality (87) has the same properties. Let us implement the passage to the limit \( \lim_{p \to \infty} \) in (87)

\[
\sum_{j=0}^{\infty} \int_{t}^{T} \int_{t}^{T} \psi_1^{(p)}(\tau) \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int_{t}^{T} \psi_1(\tau) \psi_2^{(q)}(\tau) d\tau,
\]

where \( q \in \mathbb{N} \). The equality (88) defines a linear bounded functional in \( L_2([t, T]) \) given by the function \( \psi_1 \). Let us implement the passage to the limit \( \lim_{q \to \infty} \) in (88)

\[
\sum_{j=0}^{\infty} \int_{t}^{T} \int_{t}^{T} \psi_2(t_2) \phi_j(t_2) \int_{t}^{T} \psi_1(\tau) \phi_j(t_1) dt_1 dt_2 = \frac{1}{2} \int_{t}^{T} \psi_1(\tau) \psi_2(\tau) d\tau,
\]

where \( \{\phi_j(x)\}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \psi_2(\tau) \in L_2([t, T]) \).

Thus we have the following theorem.

**Theorem 8** [25]. Suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \psi_2(\tau) \in L_2([t, T]) \). Then for the iterated Stratonovich stochastic integral...
\[ J^*[\psi^{(2)}]_{T,t} = \int_t^T \psi_2(t) \int_t^{t_2} \psi_1(t_1) d\mathbf{f}^{(i_2)}_{t_1} d\mathbf{f}^{(i_2)}_{t_2} \ (i_1, i_2 = 1, \ldots, m) \]

the following expansion

\[
J^*[\psi^{(2)}]_{T,t} = \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} c^{(i_1)}_{j_1} c^{(i_2)}_{j_2}
\]

that converges in the mean-square sense is valid, where the notations are the same as in Theorem 3.

6. Some Recent Results on Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3 to 6

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [25] (Sect. 2.10–2.16), [29] (Sect. 13–19), [33] (Sect. 5–11), [47] (Sect. 7–13), [48]. Let us formulate four theorems that were obtained using this approach.

**Theorem 9** [25], [29], [33], [47]. Suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t,T]) \). Furthermore, let \( \psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \) are continuously differentiable nonrandom functions on \( [t,T] \). Then for the iterated Stratonovich stochastic integral of third multiplicity

\[
J^*[\psi^{(3)}]_{T,t} = \int_t^T \int_t^{t_3} \int_t^{t_2} \psi_3(t_3) \psi_2(t_2) \psi_1(t_1) d\mathbf{w}^{(i_3)}_{t_3} d\mathbf{w}^{(i_2)}_{t_2} d\mathbf{w}^{(i_1)}_{t_1} \ (i_1, i_2, i_3 = 0, 1, \ldots, m)
\]

the following relations

\[
J^*[\psi^{(3)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1, j_2, j_3=0}^{p} C_{j_3 j_2 j_1} c^{(i_1)}_{j_1} c^{(i_2)}_{j_2} c^{(i_3)}_{j_3}
\]

are fulfilled, where \( i_1, i_2, i_3 = 0, 1, \ldots, m \) in (90) and \( i_1, i_2, i_3 = 1, \ldots, m \) in (91), constant \( C \) is independent of \( p \),

\[
C_{j_3 j_2 j_1} = \int_t^T \int_t^{t_3} \int_t^{t_2} \int_t^{t_1} \psi_3(t_3) \phi_{j_3}(t_3) \psi_2(t_2) \phi_{j_2}(t_2) \phi_{j_1}(t_1) dt_3 dt_2 dt_1 dt_1
\]
and

$$\zeta_j^{(i)} = \int_0^T \phi_j(\tau) dF^{(i)}_\tau$$

are independent standard Gaussian random variables for various \(i\) or \(j\) (in the case when \(i \neq 0\)); another notations are the same as in Theorems 1, 2.

**Theorem 10** [25, 29, 33, 47]. Let \(\{\phi_j(x)\}_{j=0}^{\infty}\) be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([0, T])\). Furthermore, let \(\psi_1(\tau), \ldots, \psi_4(\tau)\) be continuously differentiable nonrandom functions on \([0, T]\). Then for the iterated Stratonovich stochastic integral of fourth multiplicity

$$J^{[\psi(4)]}_{T, t} = \int_t^T \psi_4(t_4) \int_t^{t_4} \psi_3(t_3) \int_t^{t_3} \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) dt_1 dt_2 dt_3 dt_4$$

the following relations

$$J^{[\psi(4)]}_{T, t} = \lim_{p \to \infty} \sum_{j_1, j_2, j_3, j_4=0}^p C_{j_1, j_2, j_3, j_4} \zeta_j^{(i_1)} \zeta_j^{(i_2)} \zeta_j^{(i_3)} \zeta_j^{(i_4)},$$

$$M \left( J^{[\psi(4)]}_{T, t} - \sum_{j_1, j_2, j_3, j_4=0}^p C_{j_1, j_2, j_3, j_4} \zeta_j^{(i_1)} \zeta_j^{(i_2)} \zeta_j^{(i_3)} \zeta_j^{(i_4)} \right)^2 \leq \frac{C}{p^{1+\varepsilon}}$$

are fulfilled, where \(i_1, \ldots, i_4 = 0, 1, \ldots, m\) in (92), (93) and \(i_1, \ldots, i_4 = 1, \ldots, m\) in (94), constant \(C\) does not depend on \(p, \varepsilon\) is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space \(L_2([0, T])\) and \(\varepsilon = 0\) for the case of complete orthonormal system of trigonometric functions in the space \(L_2([0, T])\),

$$C_{j_1, j_2, j_3, j_4} =$$

$$= \int_t^T \psi_4(t_4) \phi_{j_4}(t_4) \int_t^{t_4} \psi_3(t_3) \phi_{j_3}(t_3) \int_t^{t_3} \psi_2(t_2) \phi_{j_2}(t_2) \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4;$$

another notations are the same as in Theorem 9.

**Theorem 11** [25, 29, 33, 47]. Assume that \(\{\phi_j(x)\}_{j=0}^{\infty}\) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([0, T])\) and \(\psi_1(\tau), \ldots, \psi_5(\tau)\) are continuously differentiable nonrandom functions on \([0, T]\). Then for the iterated Stratonovich stochastic integral of fifth multiplicity

$$J^{[\psi(5)]}_{T, t} = \int_t^T \psi_5(t_5) \ldots \int_t^{t_5} \psi_1(t_1) dt_1 dt_2 dt_3 dt_4$$

the following relations

$$= \int_t^T \psi_5(t_5) \phi_{j_5}(t_5) \int_t^{t_5} \psi_4(t_4) \phi_{j_4}(t_4) \int_t^{t_4} \psi_3(t_3) \phi_{j_3}(t_3) \int_t^{t_3} \psi_2(t_2) \phi_{j_2}(t_2) \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 dt_5;$$

$$\zeta_j^{(i)} = \int_0^T \phi_j(\tau) dF^{(i)}_\tau$$

are independent standard Gaussian random variables for various \(i\) or \(j\) (in the case when \(i \neq 0\)); another notations are the same as in Theorems 1, 2.
\[ J^*[\psi^{(5)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1, \ldots, j_5 = 0}^{p} C_{j_5 \ldots j_1} \psi^{(i_1)}_{j_1} \cdots \psi^{(i_5)}_{j_5}, \]

\[ M \left\{ \left( J^*[\psi^{(5)}]_{T,t} - \sum_{j_1, \ldots, j_5 = 0}^{p} C_{j_5 \ldots j_1} \psi^{(i_1)}_{j_1} \cdots \psi^{(i_5)}_{j_5} \right)^2 \right\} \leq \frac{C}{p^{1-\varepsilon}} \]

are fulfilled, where \( i_1, \ldots, i_5 = 0, 1, \ldots, m \) in (95), (96) and \( i_1, \ldots, i_5 = 1, \ldots, m \) in (97), constant \( C \) is independent of \( p, \varepsilon \) is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space \( L_2([t,T]) \) and \( \varepsilon = 0 \) for the case of complete orthonormal system of trigonometric functions in the space \( L_2([t,T]) \),

\[ C_{j_5 \ldots j_1} = \int_{t}^{T} \psi_5(t_5) \phi_{j_5}(t_5) \cdots \int_{t}^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \cdots dt_5; \]

another notations are the same as in Theorems 9, 10.

**Theorem 12** [25], [29], [33], [47]. Suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t,T]) \). Then, for the iterated Stratonovich stochastic integral of sixth multiplicity

\[ J^*[(i_1 \cdots i_6)]_{T,t} = \int_{t}^{T} \ldots \int_{t}^{t_2} dW^{(i_1)}_{t_1} \ldots dW^{(i_6)}_{t_6} \]

the following expansion

\[ J^*[(i_1 \cdots i_6)]_{T,t} = \lim_{p \to \infty} \sum_{j_6, \ldots, j_1 = 0}^{p} C_{j_6 \ldots j_1} \phi^{(i_1)}_{j_1} \cdots \phi^{(i_6)}_{j_6} \]

that converges in the mean-square sense is valid, where \( i_1, \ldots, i_6 = 0, 1, \ldots, m \),

\[ C_{j_6 \ldots j_1} = \int_{t}^{T} \phi_{j_6}(t_6) \cdots \int_{t}^{t_2} \phi_{j_1}(t_1) dt_1 \cdots dt_6; \]

another notations are the same as in Theorems 9–11.

### 7. Expansions of Iterated Stratonovich Stochastic Integrals of First and Second Multiplicity Based on Multiple Fourier–Legendre Series

We will use the following notations for iterated Stratonovich stochastic integrals of first and second multiplicities
for example, in \[3\]).

High-order strong numerical methods for Ito SDEs (definition of a strong numerical method see, Stratonovich expansion \[42\] (also see \[25\]-\[26\]). This expansion can be used for construction of the second, the stochastic integrals (99) and (100) are included in the so-called unified Taylor–Stratonovich expansion (31) or (89).

\[ I^{(i_1)}_{(l_1)T,t} = \int_t^T (t - t_1)^{l_1} dI^{(i_1)}_{t_1}, \]

\[ I^{(i_1 i_2)}_{(l_1 l_2)T,t} = \int_t^T (t - t_2)^{l_2} \int_{t_2}^T (t - t_1)^{l_1} dI^{(i_1)}_{t_1} dI^{(i_2)}_{t_2}, \]

where \(l_1, l_2 = 0, 1, \ldots; i_1, \ldots, i_k = 1, \ldots, m\).

Note that together with the iterated Stratonovich stochastic integrals of higher multiplicities than the second, the stochastic integrals (99) and (100) are included in the so-called unified Taylor–Stratonovich expansion (31) or (89). This expansion can be used for construction of high-order strong numerical methods for Ito SDEs (definition of a strong numerical method see, for example, in \[3\]).

Consider the expansions of some stochastic integrals (99) and (100) obtained by using Theorems 6, 8 (see \[31\] or \[30\]).

\[ I^{(i_1)}_{(0)T,t} = \sqrt{T - t}c^{(i_1)}_0, \]

\[ I^{(i_1)}_{(1)T,t} = -\frac{(T - t)^{3/2}}{2} \left( c^{(i_1)}_0 + \frac{1}{\sqrt{3}} c^{(i_1)}_1 \right), \]

\[ I^{(i_1)}_{(2)T,t} = \frac{(T - t)^{5/2}}{3} \left( c^{(i_1)}_0 + \frac{\sqrt{3}}{2} c^{(i_1)}_1 + \frac{1}{2\sqrt{3}} c^{(i_1)}_2 \right), \]

\[ I^{(i_1 i_2)}_{(00)T,t} = \frac{T - t}{2} \left( c^{(i_1)}_0 s^{(i_2)}_0 + \sum_{i=1}^\infty \frac{1}{\sqrt{4i^2 - 1}} \left( r^{(i_1)}_{2i-1}s^{(i_2)}_{2i-1} - r^{(i_1)}_{2i}s^{(i_2)}_{2i} \right) \right), \]

\[ I^{(i_1 i_2)}_{(01)T,t} = -\frac{T - t}{2} I^{(i_1 i_2)}_{(00)T,t} - \frac{(T - t)^2}{4} \left( \frac{c^{(i_1)}_0 s^{(i_2)}_0}{\sqrt{3}} + \sum_{i=0}^\infty \left( \frac{(i + 2)c^{(i_1)}_{2i + 2}s^{(i_2)}_{2i + 2} - (i + 1)c^{(i_1)}_{2i}c^{(i_2)}_{2i}}{(2i + 1)(2i + 2)(2i + 3)} - \frac{c^{(i_1)}_{2i}c^{(i_2)}_{2i}}{(2i - 1)(2i + 3)} \right) \right), \]

\[ I^{(i_1 i_2)}_{(10)T,t} = -\frac{T - t}{2} I^{(i_1 i_2)}_{(00)T,t} - \frac{(T - t)^2}{4} \left( \frac{c^{(i_2)}_0 c^{(i_1)}_1}{\sqrt{3}} + \sum_{i=0}^\infty \left( \frac{(i + 1)c^{(i_2)}_{2i + 2}s^{(i_1)}_{2i + 2} - (i + 2)c^{(i_2)}_{2i}c^{(i_1)}_{2i}}{(2i + 1)(2i + 2)(2i + 3)} + \frac{c^{(i_1)}_{2i}c^{(i_2)}_{2i}}{(2i - 1)(2i + 3)} \right) \right), \]

\[ (104) \]
\[ I_{(02)}^{(11)T,t} = \frac{(T-t)^2}{4} I_{(00)}^{(11)T,t} - \frac{3}{10}(T-t) \left( I_{(01)}^{(11)T,t} + \frac{(T-t)^3}{8} \left( \frac{2c_{(1)} c_{(2)}}{3 \sqrt{5}} \right) \right) + \frac{1}{3} \zeta_{(2)} - \frac{1}{3} \zeta_{(1)} + \sum_{i=0}^{\infty} \frac{(i+2)(i+3)}{\sqrt{(2i+1)(2i+7)(2i+3)(2i+5)}} \left( i(i+2)(i+3) \zeta_{(2)} - (i+1)(i+2) \zeta_{(1)} \right) + \frac{1}{3} \zeta_{(1)} \zeta_{(1)} \right),

\[ I_{(10)}^{(11)T,t} = \frac{(T-t)^2}{4} I_{(00)}^{(11)T,t} - \frac{3}{10}(T-t) \left( I_{(01)}^{(11)T,t} + \frac{(T-t)^3}{8} \left( \frac{2c_{(1)} c_{(2)}}{3 \sqrt{5}} \right) \right) + \frac{1}{3} \zeta_{(2)} - \frac{1}{3} \zeta_{(1)} + \sum_{i=0}^{\infty} \frac{i(i+2)(i+3)}{\sqrt{(2i+1)(2i+7)(2i+3)(2i+5)}} \left( i(i+2)(i+3) \zeta_{(2)} - (i+1)(i+2) \zeta_{(1)} \right) + \frac{1}{3} \zeta_{(1)} \zeta_{(1)} \right),

\[ I_{(01)}^{(11)T,t} = \frac{(T-t)^2}{4} I_{(00)}^{(11)T,t} - \frac{3}{10}(T-t) \left( I_{(01)}^{(11)T,t} + \frac{(T-t)^3}{8} \left( \frac{2c_{(1)} c_{(2)}}{3 \sqrt{5}} \right) \right) + \frac{1}{3} \zeta_{(2)} - \frac{1}{3} \zeta_{(1)} + \sum_{i=0}^{\infty} \frac{1}{\sqrt{(2i+1)(2i+7)(2i+3)(2i+5)}} \left( i(i+2)(i+3) \zeta_{(2)} - (i+1)(i+2) \zeta_{(1)} \right) + \frac{1}{3} \zeta_{(1)} \zeta_{(1)} \right),

\[ I_{(00)}^{(11)T,t} = \frac{(T-t)^2}{4} I_{(00)}^{(11)T,t} - \frac{3}{10}(T-t) \left( I_{(01)}^{(11)T,t} + \frac{(T-t)^3}{8} \left( \frac{2c_{(1)} c_{(2)}}{3 \sqrt{5}} \right) \right) + \frac{1}{3} \zeta_{(2)} - \frac{1}{3} \zeta_{(1)} + \sum_{i=0}^{\infty} \frac{1}{\sqrt{(2i+1)(2i+7)(2i+3)(2i+5)}} \left( i(i+2)(i+3) \zeta_{(2)} - (i+1)(i+2) \zeta_{(1)} \right) + \frac{1}{3} \zeta_{(1)} \zeta_{(1)} \right),

where

\[ (105) \quad \zeta_{(i)} = \int_{t}^{T} \phi_j(\tau) \, df_{(i)} \quad (i = 1, \ldots, m) \]

are independent standard Gaussian random variables for various \( i \) or \( j \).

Note the simplicity of the formulas (101), (102). For comparison, we present analogs of the formulas (101), (102) obtained in [4] (also see [3]) using the method proposed in [2].
\[
I_{(1)T,t}^{(i_1)q} = -(T-t)^{3/2} \frac{3}{2} \left( \frac{r^{(i_1)}}{s_0} - \frac{\sqrt{2}}{\pi} \left( \sum_{r=1}^{q} \frac{1}{r} \varphi_{s_2r-1}^{(i_1)} + \sqrt{\alpha_q} \xi_q^{(i_1)} \right) \right),
\]

\[
I_{(2)T,t}^{(i_1)q} = (T-t)^{5/2} \left( \frac{1}{3} \right) \left( \sum_{r=1}^{q} \frac{1}{r^2} \varphi_{s_2r}^{(i_1)} + \sqrt{\beta_q} \mu_q^{(i_1)} \right) - \frac{1}{\sqrt{2\pi}} \left( \sum_{r=1}^{q} \frac{1}{r^2} \varphi_{s_2r-1}^{(i_1)} + \sqrt{\alpha_q} \xi_q^{(i_1)} \right),
\]

where \( \xi_q^{(i)} \) is defined by the formula \( (105) \), \( \phi_j(s) \) is a complete orthonormal system of trigonometric functions in the space \( L_2([t,T]) \), and \( s_0, \varphi_{s_2r-1}, \varphi_{s_2r}, \xi_q, \mu_q^{(i_1), q} (r = 1, \ldots, q, \ i = 1, \ldots, m) \) are independent standard Gaussian random variables, \( i_1 = 1, \ldots, m \).

Another example of obvious advantage of the Legendre polynomials over the trigonometric functions (in the framework of the considered problem) is the truncated expansion of the iterated Stratonovich stochastic integral \( I_{(10)T,t}^{(i_1)q} \) obtained by Theorems 6, 8 in which instead of the double Fourier–Legendre series is taken the double trigonometric Fourier series

\[
I_{(10)T,t}^{(i_1)q} = -(T-t)^2 \left( \frac{1}{6} \frac{r^{(i_1)}}{s_0} \right) - \frac{1}{2\sqrt{2\pi}} \sqrt{\alpha_q} \xi_q^{(i_1)} \varphi_{s_0}^{(i_1)} +
\]

\[
+ \frac{1}{2\sqrt{2\pi}} \sqrt{\beta_q} \left( \mu_q^{(i_1)} \xi_q^{(i_1)} - 2\mu_q^{(i_1)} \varphi_{s_0}^{(i_1)} \right) +
\]

\[
+ \frac{1}{2\sqrt{2\pi}} \sum_{r=1}^{q} \frac{1}{\pi r^2} \left( \varphi_{s_2r-1}^{(i_1)} \varphi_{s_2r}^{(i_1)} - 2\varphi_{s_2r}^{(i_1)} \varphi_{s_2r-1}^{(i_1)} \right) -
\]

\[
- \frac{1}{2\pi} \sum_{r=1}^{q} \frac{1}{\pi r^2} \left( \varphi_{s_2r}^{(i_1)} \varphi_{s_2r}^{(i_1)} \right) +
\]

\[
+ \frac{1}{4\pi} \left( \varphi_{s_2r-1}^{(i_1)} \varphi_{s_2r-1}^{(i_1)} - \varphi_{s_2r}^{(i_1)} \varphi_{s_2r}^{(i_1)} \right) +
\]

\[
+ \frac{1}{8\pi^2} \left( 3\varphi_{s_2r-1}^{(i_1)} \varphi_{s_2r-1}^{(i_1)} + \varphi_{s_2r}^{(i_1)} \varphi_{s_2r}^{(i_1)} \right),
\]
where the meaning of notations included in (106), (7) is saved.

An analogue of the formula (108) (for the case of Legendre polynomials) is (according to (103) and (104)) the following representation

\[
I^s(00)_{(10)T,t} = \frac{T - t}{2} \left( \frac{c^{(i_1)}_{\zeta_0} \cdot \zeta_0}{3} + \sum_{i=1}^{q} \frac{1}{\sqrt{4i^2 - 1}} \left( \frac{c^{(i_1)}_{\zeta_{i-1}} c^{(i_2)}_{\zeta_i} - c^{(i_1)}_{\zeta_{i-1}} c^{(i_2)}_{\zeta_i}}{2(2i - 1)(2i + 3)} \right) \right),
\]

(109)

which is obviously substantially simpler than (108).

Here it is necessary to pay a special attention on the fact that the representation (109) includes a single sum with the upper summation limit \(q\) while the representation (108) includes the double sum with the same summation limit. In numerical simulation, obviously, the formula (109) is more economical in terms of computational cost than its analogue (108).

There is another feature that should be noted in connection with the formula (108). This formula was first obtained in [4] by the method from [2]. As we noted in Sect. 1, the method [2] of approximation of iterated stochastic integrals is based on the trigonometric series expansion of the Brownian bridge process. So, this method leads to iterated application of the operation of limit transition (in contrast to Theorems 1, 2, 6, and 8–12 in which limit transition is performed only once). This means, generally speaking, that the mean-square convergence of \(I^s(00)_{(10)T,T}\) (see (108)) to \(I^s(00)_{(10)T,T}\) does not follow if \(q \to \infty\) for the method [2]. The same applies to some others approximations of iterated Stratonovich stochastic integrals obtained in [3] by the method [2] (see discussion in Sect. 8 for details).

The validity of the formula

\[
\lim_{q \to \infty} \mathbb{M} \left\{ \left( I^s(00)_{(10)T,t} - I^s(00)_{(10)T,t} \right)^2 \right\} = 0,
\]

where \(I^s(00)_{(10)T,T}\) is defined by (108), follows from Theorems 3, 6, and 8.

8. Theorems 1, 2, 6, 8–12 FROM POINT OF VIEW OF THE WONG–ZAKAI APPROXIMATION

The iterated Ito stochastic integrals and solutions of Itô SDEs are complex and important functionals from the independent components \(f_s^{(i)}, \ i = 1, \ldots, m\) of the multidimensional Wiener process \(f_s\), \(s \in [0, T]\). Let \(f_s^{(i)p}\), \(p \in \mathbb{N}\) be some approximation of \(f_s^{(i)}, \ i = 1, \ldots, m\). Suppose that \(f_s^{(i)p}\) converges to \(f_s^{(i)}\), \(i = 1, \ldots, m\) if \(p \to \infty\) in some sense and has differentiable sample trajectories.

A natural question arises: if we replace \(f_s^{(i)}\) by \(f_s^{(i)p}\), \(i = 1, \ldots, m\) in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components \(f_s^{(i)}, \ i = 1, \ldots, m\) of the multidimensional Wiener process \(f_s\)? The answer to this question is negative in the
general case. However, in the pioneering works of Wong E. and Zakai M. [69], [70], it was shown that under the special conditions and for some types of approximations of the Wiener process the answer is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [69]-[71] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $f_s, s \in [0, T]$ be an $m$-dimensional standard Wiener process with independent components $f_s^{(i)}$, $i = 1, \ldots, m$. It is well known that the following representation takes place [72], [73]

$$f_s^{(i)} - f_t^{(i)} = \sum_{j=0}^{\infty} \int_t^t \phi_j(s) \, ds \, \zeta_j^{(i)}, \quad \zeta_j^{(i)} = \int_t^T \phi_j(s) \, df_s^{(i)},$$  

where $\tau \in [t, T]$, $t \geq 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various $i$ or $j$. Moreover, the series (110) converges for any $\tau \in [t, T]$ in the mean-square sense.

Let $f_s^{(i)p} - f_t^{(i)p}$ be the mean-square approximation of the process $f_s^{(i)} - f_t^{(i)}$, which has the following form

$$f_s^{(i)p} - f_t^{(i)p} = \sum_{j=0}^{p} \int_t^t \phi_j(s) \, ds \, \zeta_j^{(i)}.$$  

From (111) we obtain

$$df_s^{(i)p} = \sum_{j=0}^{p} \phi_j(\tau) \zeta_j^{(i)} \, d\tau.$$  

Consider the following iterated Riemann–Stieltjes integral

$$\int_t^T \psi_k(t_k) \cdots \int_t^{t_1} \psi_1(t_1) \, dw_{t_1}^{(i_1)p_1} \cdots dw_{t_k}^{(i_k)p_k},$$  

where $p_1, \ldots, p_k \in \mathbb{N}$, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$dw_{\tau}^{(i)p} = \begin{cases} df_{\tau}^{(i)p} & \text{for } i = 1, \ldots, m, \\ d\tau & \text{for } i = 0 \end{cases},$$  

and $df_{\tau}^{(i)p}$ in defined by the relation (112).

Let us substitute (112) into (113)

$$\int_t^T \psi_k(t_k) \cdots \int_t^{t_1} \psi_1(t_1) \, dw_{t_1}^{(i_1)p_1} \cdots dw_{t_k}^{(i_k)p_k} = \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)}.$$  

where
\[
\zeta_j^{(i)} = \int_t^T \phi_j(s) d\mathbf{w}_s^{(i)}
\]
are independent standard Gaussian random variables for various \(i\) or \(j\) (in the case when \(i \neq 0\)), \(\mathbf{w}_s^{(i)} = \mathbf{f}_s^{(i)}\) for \(i = 1, \ldots, m\) and \(\mathbf{w}_s^{(0)} = s\),

\[
C_{j_k \ldots j_1} = \int_t^T \psi_k(t_k) \phi_{j_k}(t_k) \ldots \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_k
\]
is the Fourier coefficient.

To best of our knowledge [69]-[71] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [71] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (111) were not considered in [69], [70] (also see [71], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [71] for approximations of the Wiener process based on its series expansion (110) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (115) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers [69], [70] (also see [71], Theorems 7.1, 7.2).

From the other hand, Theorems 1, 2, 6, 8–12 can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1–5 and \(k (k \in \mathbb{N})\) based on the approximation (111) of the Wiener process. At that, the Riemann–Stieltjes integrals (113) converge (according to Theorems 1, 2, 6, 8–12) to the appropriate Stratonovich stochastic integrals (3). Recall that \(\{\phi_j(x)\}_{j=0}^{\infty}\) (see (110), (111), and Theorems 6, 9–12) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([t,T])\).

To illustrate the above reasoning, consider two examples for the case \(k = 2, \psi_1(s), \psi_2(s) \equiv 1; i_1, i_2 = 1, \ldots, m\).

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [69]-[71]).

Let \(b_\Delta^{(i)}(t), t \in [0, T]\) be the piecewise linear approximation of the \(i\)th component \(f_t^{(i)}\) of the multidimensional standard Wiener process \(f_t, t \in [0, T]\) with independent components \(f_t^{(i)}, i = 1, \ldots, m\), i.e.

\[
b_\Delta^{(i)}(t) = f_\Delta^{(i)} + \frac{t - k\Delta}{\Delta} \Delta f_\Delta^{(i)},
\]

where

\[
\Delta f_\Delta^{(i)} = f_{(k+1)\Delta}^{(i)} - f_{k\Delta}^{(i)}, \quad t \in [k\Delta, (k+1)\Delta), \quad k = 0, 1, \ldots, N - 1.
\]

Note that w. p. 1

\[
\frac{db_\Delta^{(i)}}{dt}(t) = \frac{\Delta f_\Delta^{(i)}}{\Delta}, \quad t \in [k\Delta, (k+1)\Delta), \quad k = 0, 1, \ldots, N - 1.
\]

Consider the following iterated Riemann–Stieltjes integral.
Using (116) and additive property of the Riemann–Stieltjes integral, we can write w. p. 1

\[ T \int_0^s \int_0^b (i_1) db_{\Delta}^{(i_2)}(s), \quad i_1, i_2 = 1, \ldots, m. \]

Using (117) and additive property of the Riemann–Stieltjes integral, we can write

\[
\begin{align*}
&\int_0^T \int_0^s db_{\Delta}^{(i_1)}(\tau)db_{\Delta}^{(i_2)}(s) = \int_0^T \int_0^s \frac{db_{\Delta}^{(i_1)}}{d\tau}(\tau)d\tau \frac{db_{\Delta}^{(i_2)}}{ds}(s)ds = \\
&= \sum_{l=0}^{N-1} \frac{(l+1)\Delta}{\Delta} \int_0^{s} \Delta f_{\Delta}^{(i_1)} \Delta t_{\Delta}^{(i_2)} + \frac{1}{\Delta^2} \sum_{l=0}^{N-1} \Delta f_{\Delta}^{(i_1)} \Delta f_{\Delta}^{(i_2)} \int_0^{l\Delta} \int_0^{s} \Delta \tau ds = \\
&= \sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta f_{\Delta}^{(i_1)} \Delta f_{\Delta}^{(i_2)} + \frac{1}{2} \sum_{l=0}^{N-1} \Delta f_{\Delta}^{(i_1)} \Delta f_{\Delta}^{(i_2)}.
\end{align*}
\]

(117)

where \( \Delta \to 0 \) if \( N \to \infty \) \((N\Delta = T)\).

Obviously, (118) agrees with Theorem 7.1 (see [71], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (110) for \( t = 0 \), where \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space L_2([0, T]).

Consider the following iterated Riemann–Stieltjes integral

\[
\int_0^T \int_0^s d\zeta_{\Delta}^{(i_1)} d\zeta_{\Delta}^{(i_2)} \int_0^s \frac{d\zeta_{\Delta}^{(i_1)}}{d\tau}(\tau) d\tau \frac{d\zeta_{\Delta}^{(i_2)}}{d\tau}(\tau) ds =
\]

(119)

where \( d\zeta_{\Delta}^{(i_2)} \) is defined by the relation (112).

Let us substitute (112) into (119).
\[(120)\]
\[
\int_0^T \int_0^s df^{(i_1)}_s df^{(i_2)}_s = \sum_{j_1, j_2=0}^P C_{j_2 j_1} \zeta_j^{(i_1)} \zeta_j^{(i_2)},
\]

where
\[
C_{j_2 j_1} = \int_0^T \phi_{j_2}(s) \int_0^s \phi_{j_1}(\tau) d\tau ds
\]
is the Fourier coefficient; another notations are the same as in (115).

As we noted above, approximations of the Wiener process that are similar to (111) were not considered in [69], [70] (also see Theorems 7.1, 7.2 in [71]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [71] to the case under consideration is not obvious.

However, the authors of the works [3] (Sect. 5.8, pp. 202-204), [4] (pp. 438-439), [74] (pp. 82-84), [75] (pp. 263-264) use the Wong–Zakai approximation [69]-[71] (without rigorous proof) within the frames of the approach [2] based on the series expansion of the Brownian bridge process.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [25]-[26]. More precisely, using Theorem 6 from this paper we obtain from (120) the desired result
\[(121)\]
\[
\lim_{p \to \infty} \int_0^T \int_0^s df^{(i_1)}_s df^{(i_2)}_s = \lim_{p \to \infty} \sum_{j_1, j_2=0}^P C_{j_2 j_1} \zeta_j^{(i_1)} \zeta_j^{(i_2)} =
\]
\[
\int_0^s \int_0^T df^{(i_1)}_t df^{(i_2)}_s.
\]

From the other hand, by Theorems 1, 2 (see (10)) for the case \(k = 2\) we obtain from (120) the following relation
\[(122)\]
\[
\lim_{p \to \infty} \int_0^T \int_0^s df^{(i_1)}_s df^{(i_2)}_s = \lim_{p \to \infty} \sum_{j_1, j_2=0}^P C_{j_2 j_1} \zeta_j^{(i_1)} \zeta_j^{(i_2)} =
\]
\[
\lim_{p \to \infty} \sum_{j_1, j_2=0}^P C_{j_2 j_1} \left( \zeta_j^{(i_1)} \zeta_j^{(i_2)} - 1_{\{i_1 = i_2\}} 1_{\{j_1 = j_2\}} \right) + 1_{\{i_1 = i_2\}} \sum_{j_1=0}^\infty C_{j_1 j_1} =
\]
\[
\int_0^s \int_0^T df^{(i_1)}_t df^{(i_2)}_s + 1_{\{i_1 = i_2\}} \sum_{j_1=0}^\infty C_{j_1 j_1}.
\]

Since
\[
\sum_{j_1=0}^\infty C_{j_1 j_1} = \frac{1}{2} \sum_{j_1=0}^\infty \left( \int_0^T \phi_j(\tau) d\tau \right)^2 =
\]
\[
\frac{1}{2} \left( \int_0^T \phi_0(\tau) d\tau \right)^2 = \frac{1}{2} \int_0^T ds,
\]
then from (19) and (122) we obtain (121).

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