QCD sum rules with nonlocal condensates and the spacelike pion form factor

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Abstract

We present a detailed investigation of the spacelike pion’s electromagnetic form factor using three-point QCD sum rules that exclusively involve nonlocal condensates. Our main methodological tools are a spectral density which includes $O(\alpha_s)$ corrections, a suitably improved Gaussian ansatz to model the distribution of the average momentum of quarks in the QCD vacuum, and a perturbative scheme that avoids Landau singularities. Using this framework, we obtain predictions for the pion form factor together with error estimates originating from intrinsic theoretical uncertainties owing to the perturbative expansion and the nonperturbative method applied. We also discuss our results in comparison with other calculations, in particular, with those following from the AdS/QCD correspondence. We find good agreement of our predictions with measurements in the range of momenta covered by the existing experimental data between $1 – 10 \text{ GeV}^2$.

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I. INTRODUCTION

The description of hadron form factors within QCD represents a major challenge because in such exclusive processes intact hadrons appear in the initial and final states, bearing the quark-gluon binding effects controlled by nonperturbative dynamics. Essential to implementing perturbation theory is the factorization assumption of a short-distance part from a large-distance remainder, the latter being mainly nonperturbative. It was proven long ago [1, 2, 3, 4] that the pion’s electromagnetic form factor can be factorized at large $Q^2$ in the form of a convolution of a hard-scattering amplitude $T_H$ and two distribution amplitudes (DA) $\varphi_\pi$ [5] to describe the initial and final pion states.

Following this type of approach, it has been shown by many authors (see, e.g., [6, 7, 8, 9, 10] and references cited therein) that the factorizable (hard) part of the electromagnetic pion form factor is too small to afford for a good agreement with the available experimental data [11, 12]. Admittedly, the quality of the existing data at higher $Q^2$ values is too poor to draw any definitive conclusions, though—at least the tendency—is clear: if at all, the perturbative regime seems to be far outside the currently probed momentum scales. For that reason, it seems reasonable to pursue alternative methods to compute the pion’s electromagnetic form factor which do not rely on perturbation theory.

Our present work is in the context of a three-point AAV (A for axial, V for vector current) correlator, having recourse to the operator product expansion (OPE) and a dispersion relation. Our techniques are closely related to the QCD sum-rule approach of [13, 14], but with the important difference that we use exclusively nonlocal condensates. In particular, a quark-gluon-antiquark nonlocal condensate is employed in which all three inter-parton separations are nonlocal.

Moreover, we use a spectral density which includes terms of $O(\alpha_s)$. The influence of this next-to-leading-order (NLO) contribution to the pion form factor turns out to be quite important, reaching the level of 20%. To describe the momentum distribution of vacuum quarks, we go beyond the minimal Gaussian model, used before [15, 16, 17], and consider also a model, obtained more recently by two of us [18], that has the following methodological advantages: (a) it satisfies the QCD equations of motion and (b) it minimizes the non-transversality of the VV correlator.

The principal results of this paper are the following:

(i) We give a general formalism for the calculation of the pion’s electromagnetic form factor that contains several methodological improvements relative to all previous approaches employing QCD sum rules with vacuum (local and nonlocal) condensates.

(ii) We suggest a way how to determine the threshold in the local-duality approach for intermediate values of $Q^2$ where measurements have already been carried out or are planned.

(iii) We present predictions for the pion form factor, including also inherent theoretical uncertainties, that are in good agreement with new lattice results and real experimental data from the Cornell and JLab Collaborations.

The plan of this paper is as follows. In the next section we recall the standard QCD description of the pion’s electromagnetic form factor according to the factorization theorem. A brief report on the current status of the obtained predictions is given, avoiding technicalities. In Sec. III, we discuss the method of QCD sum rules with local and nonlocal condensates in the calculation of the pion form factor. The analysis of the present work is detailed in Sec. IV. We present our computations and the obtained results and compare them with other theoretical predictions, recent lattice simulations, and experimental data. This section contains also estimates of the inherent theoretical uncertainties of the applied method. In Sec. V we state our conclusions and summarize our main results.
II. CONVOLUTION SCHEME FOR THE PION FORM FACTOR IN QCD

Applying the factorization theorem [2, 4], the pion’s electromagnetic form factor can be written in the form (for reviews see, for instance, [19, 20, 21])

\[
F_\pi(Q^2) = \int_0^1 \int_0^1 \varphi_\pi(x, \mu^2) T_H(x, y; Q^2, \mu^2) \varphi_\pi(y, \mu^2) \, dx \, dy .
\] (2.1)

The hard amplitude is the sum of all Feynman diagrams in which the struck quark is connected to the spectator via highly off-shell gluon propagators, meaning that the transverse interquark distance is rather small, i.e., of the order of the inverse large momentum transfer \(Q\), and that both partons share comparable fractions of longitudinal momentum \(x_i = p^+_i / P^+\), where the parton four-momentum in light-cone coordinates is \(p_i = (p^+_i, p^-_i, k^\perp_i)\) and \(P\) is the initial four-momentum of the pion. Momentum conservation under the assumption of exact \(u-d\) symmetry implies that \(\sum_{i=1}^2 x_i = 1\) and \(\sum_{i=1}^2 k^\perp_i = 0\). In our case, \(x_1 = x\) and \(x_2 = 1 - x \equiv \bar{x}\).

The unknown binding effects of the pion state have been absorbed into the pion DA which at the leading-twist level two is represented by the valence-state wave function on the light cone averaged over transverse momenta up to the factorization scale \(\mu\) of the process. It is defined by the following universal operator matrix element (see, e.g., [19] for a review)

\[
\langle 0 | \bar{d}(z) \gamma^\mu \gamma_5 C(z, 0) u(0) | \pi(P) \rangle \bigg|_{z^2 = 0} = i f_\pi P^\mu \int_0^1 dx e^{ix(zP)} \varphi_\pi(x, \mu_0^2) \] (2.2)

with the normalization

\[
\int_0^1 \varphi_\pi(x, \mu_0^2) \, dx = 1 ,
\] (2.3)

where \(f_\pi = 130.7 \pm 0.4\) MeV [22] is the pion decay constant defined by

\[
\langle 0 | \bar{d}(0) \gamma^\mu \gamma_5 u(0) | \pi^+(P) \rangle = i P_\mu f_\pi .
\] (2.4)

Here

\[
C(z, 0) = \mathcal{P} \exp \left[ -ig_s \int_0^z t^a A_\mu^a(y) dy^\mu \right]
\] (2.5)

is the Fock–Schwinger phase factor (termed in [23] the color “connector”), path-ordered along the straight line joining the points 0 and \(z\), to preserve gauge invariance. Note that the scale \(\mu_0^2\), called the normalization scale of the pion DA, is related to the ultraviolet (UV) regularization of the quark-field operators on the light cone whose product becomes singular for \(z^2 = 0\). The derivation of hadron distribution amplitudes at finite momentum transfer is outside perturbative QCD; it requires nonperturbative approaches, like QCD sum rules with vacuum condensates [19], or lattice calculations. The latter method has recently reached a high level of precision in calculating the first and second moment of the pion DA [24, 25, 26], though the calculation of the fourth moment is still pending, while recently a prediction for its value was worked out [27] by combining the CLEO data [28] and the lattice calculation for the second moment. Here, an alternative method proposed by Braun and Müller [29] may be proven useful. On the other hand, QCD sum rules incorporate information about the non-trivial structure of the QCD vacuum and, therefore, constitute a useful analytic tool for determining hadron distribution amplitudes using local constraints.
on their moments (inverse-scattering method). Chernyak and Zhitnitsky [19] extracted the lowest few moments \( \langle \xi^N \rangle \equiv \int_0^1 \varphi(x)(2x-1)^N \, dx \) of \( \varphi \) using correlators of two axial currents with local operators containing \( N \) covariant derivatives [1, 5]. However, it was pointed out later [15, 30, 31] that QCD sum rules with local condensates strongly emphasize the endpoint region, because the condensate terms are strongly peaked just at these endpoints \( x = 0, 1 \), where one of the quarks has a vanishing virtuality. Therefore, the authors of Refs. [15, 30, 31] suggested to introduce nonlocal condensates that can account for the possibility that vacuum quarks may have a finite virtuality (a nonzero average transverse momentum). As a result, Feynman-type configurations, in which one of the quark carries almost the entire available momentum while the spectators are “wee” (with almost vanishing virtualities), are separated out from the pion DA and treated separately in a non-factorizing (soft) contribution to the form factor (for a discussion of the Feynman mechanism in this context, see [32]).

More recently, a QCD sum-rule analysis was carried out [17] which employs nonlocal condensates parameterized for simplicity in terms of a single mass-scale parameter \( \lambda_2^2 = \langle k^2 \rangle \) with values in the range [33, 34, 35, 36]

\[
\lambda_2^2 = 0.35 - 0.55 \text{ GeV}^2.
\]

This way, the first ten moments of \( \varphi \) at the initial scale \( \mu_0^2 = 1.35 \text{ GeV}^2 \) were determined with the following values (theoretical errors in parentheses) [17]

\[
\langle \xi^2 \rangle = 0.265(20), \quad \langle \xi^4 \rangle = 0.115(12), \quad \langle \xi^N \rangle \approx 0, (N = 6, 8, 10).
\]

Recasting the pion DA in terms of the Gegenbauer polynomials \( C_n^{3/2}(2x-1) \), which are the one-loop eigenfunctions of the ERBL kernel [2, 4], one finds

\[
\varphi_\pi(x, \mu^2) = \varphi_{\text{as}}(x) \left[ 1 + \sum_{n \geq 1} a_{2n}(\mu^2) C_n^{3/2}(2x-1) \right], \quad I_{\pi-1}^\pi(\mu^2) = 3 \left[ 1 + \sum_{n \geq 1} a_{2n}(\mu^2) \right],
\]

where the asymptotic pion DA has the form

\[
\varphi_{\text{as}}(x) = 6x(1-x).
\]

The values of the Gegenbauer coefficients, encoding the nonperturbative dynamics, of some characteristic pion DAs are given in Table I, while the corresponding profiles are displayed in Fig. 1.

It is worth noting that the rate of convergence \( \varphi_\pi(x, \mu^2) \to \varphi_{\text{as}}(x) \), or \( a_{2n}(Q^2) \to 0 \), at \( Q^2 \to \infty \) is not fast and is determined in the one-loop approximation by the logarithmic law

\[
a_{2n}(\mu^2) = a_{2n}(\mu_0^2) \left[ \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right]^{\gamma_{2n}^{(0)}}.
\]

| DA  | \( a_2(\text{1 GeV}^2) \) | \( a_4(\text{1 GeV}^2) \) | \( I_{\pi-1}^\pi(\text{1 GeV}^2) \) | Shape                   |
|-----|----------------------|----------------------|----------------------|---------------------|
| Asy | 0.00                 | 0.00                 | 3.00                 | convex              |
| CZ  | 0.56                 | 0.00                 | 4.68                 | double-humped, end-point enhanced |
| BMS | 0.20                 | -0.14                | 3.18                 | double-humped, end-point suppressed |

TABLE I: Main characteristics of three different DAs, abbreviated by acronyms: CZ stands for Chernyak–Zhitnitsky [19] and BMS for Bakulev–Mikhailov–Stefanis [17]. Here \( a_2 \) and \( a_4 \) are the second and fourth Gegenbauer coefficients, respectively, whereas \( I_{\pi-1} \) denotes the inverse moment of \( \varphi_\pi \).
FIG. 1: Comparison of selected pion DAs labeled by obvious acronyms: $\varphi_{\text{as}}$ (dotted line), $\varphi_{\text{CZ}}$ (dashed line) [19], and $\varphi_{\text{BMS}}$ (solid line) [17], defined by the coefficients $a_2$ and $a_4$ in Table I. All DAs are normalized at the same scale $\mu_0^2 \approx 1 \text{ GeV}^2$.

Here $\gamma_2^{(0)} = 0.62$, $\gamma_4^{(0)} = 0.90$, whereas all other anomalous dimensions $\gamma_{2n}^{(0)} \geq 1$ for $n \geq 3$. Numerical estimates show that if one has at some typical initial scale $\mu^2 \approx 1 \text{ GeV}^2$ a coefficient $a_2(1 \text{ GeV}^2) = 0.25$, as indicated by the CLEO data [28] and recent lattice simulations [37, 38], its value would become 3 times smaller only at the tremendous scale of $\mu^2 \sim 75400 \text{ GeV}^2$, which is certainly outside the reach of any experimental measurement. The situation at intermediate momentum transfers $20 \text{ GeV}^2 \geq Q^2 \geq 4 \text{ GeV}^2$ is more delicate. In this region, factorization partially fails owing to a quite sizable contribution of the soft part which is non-factorizable. Hence, one has to calculate this part of the pion form factor using either phenomenological models [7, 8, 39, 40], or by employing some nonperturbative concepts, like the method of QCD sum rules (SR) [13, 41, 42]. Still another option is to apply the local quark-hadron duality approach [14, 43, 44]. On the other hand, at the one-loop level and at asymptotically large $Q^2$, the pion form factor turns out to be [45, 46]

$$F_{\pi}^\text{pert}(Q^2) = \frac{8\pi\alpha_s(Q^2)f_\pi^2}{9Q^2} \left| I_{-1}(Q^2) \right|^2 \quad \text{with} \quad I_{-1}(Q^2) = \int_0^1 \frac{\varphi_\pi(x, Q^2)}{x} dx.$$  \hspace{1cm} (2.11)

The precise value of $Q^2$ at which this perturbative expression should start to prevail cannot be predicted (determined) accurately. The estimates for the cross-over momentum scale range from $100 \text{ GeV}^2$ [7, 8, 47] down to values around $20 \text{ GeV}^2$ [9, 10, 32]. But even this latter relatively small momentum is hopelessly far away from the capabilities of any operating or planned accelerator facility.

III. QCD VACUUM AND THE PION FORM FACTOR

In view of these facts, one may look for alternative methods to calculate the pion form factor in the experimentally accessible region of momentum transfer. Even after the commissioned 12 GeV upgrade for Continuous Electron Beam Accelerator Facility (CEBAF), the expected high-precision experimental data will not be sufficient to enter the perturbative regime. Hence, instead of predicated to the elusive hope for still higher energies in the future, we discuss an alternative approach, based on three-point QCD SR to the pion form factor [41, 42]. One of the advantages of this technique is that the shape of the pion DA is irrelevant, reducing this way the theoretical uncertainty.
The construction of the Borel SR for the pion form factor from the three-point AAV correlator
\[
\int \int d^4x \, d^4y \, e^{i(x \cdot p_2 - y \cdot p_2)} \langle 0 \mid T[J_5^\alpha(y) J^\mu(x) J_{5\alpha}(0)] \rangle \langle 0 \rangle ,
\]
where \( q \) corresponds to the photon momentum \( (q^2 = -Q^2) \) and \( p_2 \) is the outgoing pion momentum, was described in detail in [43, 48] for the case of local condensates and in [13] for the non-local condensate (NLC) case. In this correlator, \( J^\mu(x) = e_u \, \pi(x) \gamma^\mu u(x) + e_d \, \bar{d}(x) \gamma^\mu d(x) \) is the electromagnetic current, while \( J_{5\alpha}(x) = \bar{d}(x) \gamma_5 \gamma_\alpha u(x) \) and \( J_{5\alpha}^+(x) = \bar{\pi}(x) \gamma_5 \gamma_\alpha \partial(x) \) are axial-vector currents, where \( e_u = 2/3 \) and \( e_d = -1/3 \) are the electric charges of the \( u \) and \( d \) quarks, respectively. Herewith one obtains the following SR:
\[
f_\pi^2 F_\pi(Q^2) = \int_0^{s_0} ds_1 \, ds_2 \, \rho_3(s_1, s_2, Q^2) \, e^{-(s_1 + s_2)/M^2} \, + \Phi_G(Q^2, M^2) + \Phi_{\bar{q}q}(Q^2, M^2),
\]
where the quark condensate contribution
\[
\Phi_{\bar{q}q}(Q^2, M^2) = \Phi_{4Q}(Q^2, M^2) + \Phi_{2V}(Q^2, M^2) + \Phi_{\bar{q}Aq}(Q^2, M^2)
\]
contains the four-quark condensate (4Q), the bilocal vector-quark condensate (2V), and the antiquark-gluon-quark condensate term \( \langle \bar{q}Aq \rangle \). The gluon-condensate contribution to SR (3.2) is presented by the \( \Phi_G(Q^2, M^2) \) term. The graphical illustration of the corresponding diagrams is shown in Fig. 2. The perturbative three-point spectral density entering the SR above reads
\[
\rho_3^{(1)}(s_1, s_2, Q^2) = \left[ \rho_3^{(0)}(s_1, s_2, Q^2) + \frac{\alpha_s(Q^2)}{4\pi} \Delta \rho_3^{(1)}(s_1, s_2, Q^2) \right].
\]
Recall that the leading-order spectral density
\[
\rho_3^{(0)}(s_1, s_2, t) = \frac{3}{4\pi^2} \left[ \frac{t^2}{dt^2} + \frac{t^3}{3 \, dt^3} \right] \frac{1}{\sqrt{(s_1 + s_2 + t) - 4 \, s_1 s_2}}
\]
has been calculated in the early eighties [43, 48], whereas the explicit (but too complicated to show it here) expression of the analogous next-to-leading order (NLO) version \( \Delta \rho_3^{(1)}(s_1, s_2, Q^2) \) has been obtained quite recently in [49]. Note that the contribution from higher-resonances (HR), \( F_{\text{HR}} \), is usually modeled with the help of the same spectral density
\[
\rho_{\text{HR}}(s_1, s_2) = [1 - \theta(s_1 < s_0) \theta(s_2 < s_0)] \, \rho_3(s_1, s_2, Q^2)
\]
and using the continuum threshold parameter \( s_0 \).

A novelty of the present investigation is the use of a running coupling that shows by construction analytic behavior in \( Q^2 \), i.e., has no Landau singularities. This is done by adopting the arguments and techniques used in our previous works in [10, 44]. To be specific, we will use the one-loop analytic Shirkov–Solovtsov coupling [50]
\[
\alpha_s(Q^2) = \frac{4\pi}{b_0} \left( \frac{1}{\ln(Q^2/\Lambda_{\text{QCD}}^2)} - \frac{\Lambda_{\text{QCD}}^2}{Q^2 - \Lambda_{\text{QCD}}^2} \right),
\]
with \( b_0 = 9 \) and \( \Lambda_{\text{QCD}} = 300 \) MeV. The interested reader can find more about this subject in the recent reviews [51, 52, 53]. The nonperturbative terms \( \Phi_G \) and \( \Phi_{\bar{q}q} \) in the local-
condensate case are well-known [43, 48]:

$$
\Phi_{\text{loc}}^G(M^2) = \frac{(\alpha_s GG)}{12 \pi M^2}, \quad \Phi_{\text{loc}}^{\bar{q}q}(Q^2, M^2) = \frac{104 A_0}{M^4} \left( 1 + \frac{2 Q^2}{13 M^2} \right).
$$

(3.7)

The standard QCD SRs [43, 48] for the pion form factor are based on these local expressions which show a wrong scale behavior at large $Q^2$. This means that the quark contribution contains both a linearly growing term as well as a constant one, while the gluon contribution is just a constant (see Table II). At the same time, the first, i.e., the perturbative, term on the right-hand side of Eq. (3.2) behaves at large $Q^2$ as $s_0/Q^4$ or $M^2/Q^4$. For this reason, the SR becomes unstable for $Q^2 > 3 \text{ GeV}^2$. Hence, in order to be able to extract predictions from a SR of this sort in the $Q^2$ region between $3 - 10 \text{ GeV}^2$, we have first to improve the quality of the SR.

| Approach                          | Accuracy | Condensates                      | $Q^2$-behavior of $\Phi_{\text{OPE}}$                     |
|-----------------------------------|----------|----------------------------------|----------------------------------------------------------|
| Standard QCD SR [43, 48]          | LO       | local                            | $c_1 + Q^2/M^2$                                           |
| QCD SR with NLCs [13]             | LO       | local + nonlocal                 | $(c_2 + Q^2/M^2) \left( e^{-c_3 Q^2 \lambda_q^2/M^4} + c_4 \right)$ |
| LD QCD SR [41, 47, 49]            | NLO      | no                               | 0                                                        |
| Here                              | NLO      | nonlocal                         | $(c_1 + Q^2/M^2) e^{-c_3 Q^2 \lambda_q^2/M^4}$            |

The fact that the condensate contributions in the SR for the pion form factor are constant, or even growing with $Q^2$, is somewhat surprising because exactly the corresponding diagrams should actually generate decreasing contributions as $Q^2$ increases. But recall that the diagrams generating the condensate contributions differ from the ordinary Feynman diagrams of QCD perturbation theory. They result from the replacement of some of the propagators by constant factors which represent condensates. For example, the quark propagator $\langle T(q(z)\bar{q}(0)) \rangle$ is substituted by the quark condensate $\langle \bar{q}(0)q(0) \rangle$. As a result, instead
of obtaining a $Q^2$-dependent contribution, one gets a constant one. The dependence on $Q^2$ appears when one calculates the contributions of higher-dimension operators of the type $\langle \bar{q}(0) D^2 q(0) \rangle$, $\langle \bar{q}(0) (D^2)^2 q(0) \rangle$ etc., entailed by the Taylor expansion of the original nonlocal condensate (NLC), where $\langle \bar{q}(0) q(z) \rangle$ is the nonperturbative part of the quark propagator. The resulting total condensate contribution decreases for large $Q^2$. However, each term of the standard OPE has the structure $(Q^2/M^2)^n$, and one should, therefore, resum them to get a meaningful result. Our strategy is to avoid the original Taylor expansion and deal instead directly with the NLC. This leads to a modified diagram technique involving new lines and vertices that correspond to the NLC. Then, the simplest contribution to $\Phi_{\langle \bar{q}q \rangle}$ is due to the vector condensate $M_\mu$ (cf. (A.2)); viz.,

$$\Delta \Phi_V(M^2, Q^2) = \frac{8 A_0}{M^4} \left( 2 + \frac{Q^2}{M^2 - \lambda_q^2} \right) \exp \left[ \frac{-Q^2 \lambda_q^2}{2 M^2 (M^2 - \lambda_q^2)} \right].$$  

(3.8)

Obviously, this term indeed vanishes for large $Q^2$, as expected. Moreover, the larger the nonlocality parameter $\lambda_q^2$, the faster this contribution decreases with $Q^2$. The value $Q^2_*$ at which this decrease starts, strongly depends on the value of $M^2$. Adopting for the nonlocality $\lambda_q^2 = 0.4$ GeV$^2$, one finds for $M^2 = 1, 1.5, 2$ GeV$^2$, $Q^2_* = 2, 6, 13$ GeV$^2$, respectively.

![FIG. 3: Illustration of the distance dependence of the three-point condensate in terms of the mutual inter-parton separations.](image)

A first attempt [13] to generalize the QCD SR to the NLC case employed the specific model (3.9) for the three-point quark-gluon-quark NLC (with the NLC $M_i(x^2, y^2, (x-y)^2)$ given by Eq. (A.2)) and assuming a nonlocality only with respect to two inter-parton separations, say, $x^2$ and $(x-y)^2$, out of the three possible separations $x^2$, $y^2$, and $(x-y)^2$ (see Fig. 3). Hence, this approach is only partially nonlocal. The following parametric functions $(\Lambda = \lambda_q^2/2)$ were used:

$$f_i^{BR}(\alpha, \beta, \gamma) = \delta (\alpha - x_{i1} \Lambda) \delta (\beta - x_{i2} \Lambda) \delta (\gamma - x_{i3} \Lambda),$$  

(3.9)

$$x_{ij} = \begin{pmatrix} 0.4 & 0 & 0.4 \\ 0 & 1 & 0.4 \\ 0 & 0.4 & 0.4 \end{pmatrix}.$$  

The zero elements in the matrix $x_{ij}$ indicate the absence of nonlocality effects either for the quark-antiquark separation $y^2$ ($i = 1, j = 2$) or for the antiquark-gluon separation $x^2$ ($i = 2, 3$ and $j = 1$)—see Fig. 3. Therefore, also this approach partially suffers from the same shortcomings as the standard QCD SR.

Fortunately, the NLC contributions to the pion form factor taken into account in the SR of [13], have a model-independent form, allowing us to use them in connection with more
improved versions of the quark-gluon NLC. To be more specific, we apply here QCD SR employing the minimal (A.3) [15, 16, 17] and the improved (A.4) [18] Gaussian models of NLC. The corresponding parameters of the NLC are listed in Appendix A. An advantage of employing NLCs is that their use considerably enlarges the region of applicability of the QCD SR to momenta as high as 10 GeV$^2$. In addition, we extend the accuracy of the perturbative spectral density to the NLO level by including contributions of $O(\alpha_s)$ [49]—in contrast to previous works [13, 43, 48] in which only a LO perturbative spectral density was taken into account. It turns out that the NLO contribution influences the prediction for the pion form factor, calculated with the described method, reaching the level of 20%.

IV. PION FORM FACTOR ANALYSIS

A. Calculation and predictions

The strategy to further process the obtained SR is standard: At each fixed value of $Q^2$, SR (3.2) gives us the pion form factor $F_\pi(Q^2, M^2, s_0)$ as a function of two additional parameters, notably, the auxiliary Borel parameter $M^2$ and the continuum threshold $s_0$. The parameter $s_0$ can be interpreted as the boundary between the pion and the higher resonances ($A_1$, $\pi'$, etc.). We assume that $s_0$ should not be lower than the middle point, 0.6 GeV$^2$, of the interval between $m^2_\pi = 0$ and $m^2_{A_1} \approx 1.6$ GeV$^2$.

The specific value of $s_0(Q^2)$ at a given value of $Q^2$ is determined by the condition implied by the minimal sensitivity of the function $F_\pi(M^2, s_0)$ on the auxiliary parameter $M^2$ in the fiducial interval of the SR. We derive these intervals and the values of the pion decay constant $f_\pi$ from the corresponding two-point NLC QCD SR, employing both the minimal and the improved model for the vacuum nonlocality—see Table III for details. Note here that the value of the Borel parameter $M^2$ in the three-point SR roughly corresponds to the Borel parameter in the two-point SR, having, however, twice its magnitude: $M^2_{\text{three-point}} = 2M^2_{\text{two-point}}$, as given in Table III. In the left panel of Fig. 4, we show how the scaled pion form factor $Q^2 F_\pi(Q^2, M^2, s_0)$ depends on $M^2$ for three different values of the threshold parameter: $s_0 = 0.65, 0.75, \text{and} 0.85$ GeV$^2$ at $Q^2 = 5$ GeV$^2$. As a rule, the higher the value of $s_0$, the larger the form factor because the perturbative input increases.

Using the root-mean-square deviation $\chi^2(Q^2, s_0)$, given by Eq. (B.1), we determine that continuum threshold $s^\text{SR}_0(Q^2)$ which minimizes the dependence of the right-hand side of (3.2) on the Borel parameter $M^2 \in [M^2_-, M^2_+]$ at each value of $Q^2$. For an illustration, we refer to the right panel of Fig. 4.

As one sees from the right panel of Fig. 4 the long-dashed line—which corresponds to the minimal NLC model at $Q^2 = 5$ GeV$^2$—has no minimum in the relevant $s_0$ interval. Therefore, we cannot extract a reliable continuum-threshold value $s^\text{SR}_0(Q^2 \geq 4$ GeV$^2)$ for this model using the root-mean-square deviation criterion.

| Model         | $f_\pi$         | $M^2_-$ | $M^2_+$ |
|---------------|-----------------|---------|---------|
| Minimal [17]  | 0.137 GeV$^2$   | 1 GeV$^2$ | 1.7 GeV$^2$ |
| Improved [18] | 0.142 GeV$^2$   | 1 GeV$^2$ | 1.9 GeV$^2$ |
FIG. 4: Left panel: Dependence of the pion form factor $Q^2 F_{\pi}(Q^2)$ at $Q^2 = 5$ GeV$^2$ on the auxiliary Borel parameter $M^2$ in the improved NLC model. The solid blue line corresponds to $s_0 = 0.75$ GeV$^2$, whereas the dashed lines refer to $s_0 = 0.65$ GeV$^2$ (upper curve) and $s_0 = 0.85$ GeV$^2$ (lower curve). Right panel: Root-mean-square deviation $\chi^2(Q^2, s_0)$ for the minimal NLC model at $Q^2 = 3$ GeV$^2$ (long-dashed blue line) and at $Q^2 = 5$ GeV$^2$ (short-dashed red line). The improved model of NLC is shown as a solid blue line at $Q^2 = 5$ GeV$^2$.

Note, however, that the values $\min_s \chi^2(Q^2, s)$ and $\chi^2(Q^2, s_0) = s_0^{LD(1)}(Q^2) \simeq 0.63$ GeV$^2$ are very close to each other, the relative difference for $Q^2 = 4 - 10$ GeV$^2$ being of the order of $10 - 15\%$. For this reason, we will use in the minimal model $s_0^{SR}(Q^2) = s_0^{LD(1)}(Q^2)$ as the continuum threshold. In contrast, in the improved case (the solid line in the right panel of Fig. 4), the root-mean-square deviation has a minimum approximately at the same point $s_0^{SR}(Q^2) \approx 0.75$ GeV$^2$ for any $Q^2$ value within the considered interval. The continuum thresholds $s_0^{SR}(Q^2)$ for the minimal (dashed line) and the improved (solid line) NLC model, obtained this way, are shown in Fig. 5.

On that basis, we can define the SR result for the pion form factor as the average value

FIG. 5: Continuum threshold $s_0(Q^2)$ [GeV$^2$] for the minimal (dashed line) and for the improved (solid line) NLC model.

Here $s_0^{LD(1)}(Q^2)$ is the standard Local-Duality prescription for the continuum threshold, see the discussion after Eq. (4.5).
of the right-hand side of (3.2) with respect to the Borel parameter $M^2 \in [M_2^-, M_2^+]$:

$$F_{\pi}^{SR}(Q^2) = \frac{1}{M_2^+ - M_2^-} \int_{M_2^-}^{M_2^+} F(Q^2, M^2, s_0^{SR}(Q^2)) dM^2. \quad (4.1)$$

Our main predictions for the minimal and the improved NLC model, using in both cases $\lambda_q^2 = 0.4$ GeV$^2$, are shown in Fig. 6 in the form of two bands, each one corresponding to the particular NLC model used, with the width of each band denoting the inherent theoretical uncertainties of the underlying QCD SR method. The band within the dashed lines contains the predictions for the minimal model. Its counterpart for the improved model is limited by the solid lines. For the central curves (dashed—minimal model; solid—improved model) in Fig. 6, as well as in both panels of Fig. 7, we used the following interpolation formulas:

$$F_{\pi;\text{min}}^{SR}(Q^2 = x \, \text{GeV}^2) = e^{1.402x^{0.525}} \left(1 + 0.182x + \frac{0.0219x^3}{1 + x}\right), \quad (4.2a)$$

$$F_{\pi;\text{imp}}^{SR}(Q^2 = x \, \text{GeV}^2) = e^{1.171x^{0.536}} \left(1 + 0.0306x + \frac{0.0194x^3}{1 + x}\right), \quad (4.2b)$$

valid for $Q^2 \in [1, 10]$ GeV$^2$, i.e., for $x \in [1, 10]$. The two broken vertical lines in Fig. 6 denote the strict fidelity window of the NLC QCD sum rules. We have limited this window from above by the requirement that the predictions obtained with both NLC models have an overlap. As one sees, the central curve for the minimal model starts departing from the band of the improved model just around 7 GeV$^2$. But the predictions remain useful (though less accurate) even at higher $Q^2$ values close to 10 GeV$^2$. Our results compare favorably with the lattice calculation of [56], shown as a monopole fit with associated error bars between the two thick lines at lower $Q^2$. We observe a similarly good agreement with the existing

![Image of Fig. 6](image-url)
experimental data of the Cornell [11, 54, 55] (triangles) and the JLab Collaboration [12] (diamonds).

Figure 7 shows our results in comparison with the predictions obtained with various theoretical models (left panel), whereas the right panel compares our results with predictions from AdS/QCD models. Specifically, in the left panel of Fig. 7 we display as a short-dashed green line the prediction from [57], whereas the long-dashed green line is from [58]. The short heavy solid red line at low $Q^2$ represents the standard QCD SR result with local condensates [43, 48], while the dash-dotted red line gives the more recent estimate of the Local Duality QCD SR of [47]. Note here that the results of [10], which are not shown in this figure, are approximately 20% higher than those represented by the dash-dotted red curve. This is due to (i) the inclusion of the $O(\alpha_s^2)$ correction (10%) and (ii) the uncertainty of the matching procedure, used in [10], (10%) (see the discussion in Sect. IV B and the graphics in Fig. 9). Finally, the dotted line below 4 GeV$^2$ displays the result extracted from the model of [59] which uses the Bethe–Salpeter equation.

In the right panel of Fig. 7 we collect recent estimates from the following AdS/QCD models: The heavy long-dashed line displays the result obtained with an AdS/QCD pion DA [60]. The dash-dot-dotted green line represents a Hirn–Sanz-type holographic model, discussed in [61]. The dotted green line gives the prediction of the AdS/QCD soft model [62]. Finally, the two top broken red lines show the results obtained from the improved soft-wall (heavy dash-dotted line) and the hard-wall (fine dash-dotted line) AdS/QCD background approach [63], respectively. One sees that the dotted and the dash-dot-dotted green

![Graph](image)

**FIG. 7:** Comparison of theoretical predictions for the scaled pion form factor $Q^2 F_\pi(Q^2)$ obtained with a variety of theoretical methods and models. Our estimates are shown in both panels as shaded bands corresponding to the minimal NLC model (dashed blue lines) and the improved NLC model (solid blue lines). In both cases $\lambda_0 = 0.4$ GeV$^2$ has been used and the associated error range is indicated by the width of the shaded bands. Left panel: The first short-dashed green line above to the dash-dotted red one is from [57] and is based on a large-$N_c$ Regge model. The long-dashed green line is from [58], obtained by including radiative and higher-twist effects within the framework of resummed pQCD. The short solid red line at low $Q^2$ shows the result from the standard QCD SR with local condensates [43, 48]. The dash-dotted red line denotes the estimate from [47] which was derived from Local Duality QCD SR. The dotted line represents the model of [59], based on the Bethe–Salpeter equation. Right panel: The heavy long-dashed line shows the result obtained with an AdS/QCD pion DA in [60]. The dash-dot-dotted green line shows a Hirn–Sanz-type holographic model result, discussed in [61]. The dotted green line gives the prediction of the AdS/QCD soft-wall model [62]. The heavy and the fine dash-dotted red lines show, respectively, the predictions derived from the improved soft and hard wall AdS/QCD background approach [63].
FIG. 8: Contributions to the pion form factor from the $O(1)$ and $O(\alpha_s)$ spectral-density terms (LO and NLO). The specific contributions are denoted by labels referring to their particular origin: 4-quarks (scalar) condensate—($4Q$), bilocal vector quark condensate—($2V$), tri-local quark-gluon-quark condensate—($\bar{q}Aq$), and gluon condensate—($G$) in the minimal (left panel) and the improved (right panel) model.

lines are in compliance with our predictions, favoring the results of [61], as these leave some space to add radiative corrections.

Figure 8 serves to illustrate the origin of the differences between the minimal (dashed line—left panel) and the improved (solid line—right panel) Gaussian NLC model with respect to their corresponding predictions for the pion form factor. The difference between them can be traced back to the quark-gluon-quark contribution $\Phi_{\bar{q}Aq}(Q^2, M^2)$ which is different for each model (top lines in both panels below the final results). This, in turn, has influence on the continuum threshold $s_0(Q^2)$ entailing changes in the corresponding LO and NLO terms, albeit mild ones. This figure exhibits in detail how the various contributions to the pion form factor are accumulated. Each curve from bottom to top is the sum of all previous terms, while the heavy lines on the top denote the full result for each model.

Note that the calculations were made for a nonlocality parameter $\lambda_q^2 = 0.4$ GeV$^2$, a value receiving support from a recent comprehensive analysis [64, 65] of the CLEO data on the pion-photon transition. Using higher values of this parameter, would entail a decrease of the pion form factor owing to a stronger influence of the nonlocality effects. Evidently, using smaller values of $\lambda_q^2$ would cause an increase of the pion form factor. The predictions shown in our figures, discussed above, provide further support for a value of $\lambda_q^2$ in the range compatible with the result extracted from the CLEO-data analysis.

### B. Local duality approach

Sum rules based on local duality have no condensate contributions due to the $M^2 \to \infty$ limit. On the other hand, the determination of the threshold $s_0$ is not possible using this method, in contrast to QCD SR. One could, nevertheless, try to extract it from (local duality) sum rules for the pion-decay constant $f_\pi$, but this would correspond to a value of the pion form factor at a small $Q^2$ value because of the Ward identity [47].

As originally argued in Refs. [7, 41, 43, 48, 66], the dominant contribution to the pion form factor at low up to moderate values of the momentum transfer $Q^2 \leq 10$ GeV$^2$ originates mainly from the soft part that involves no hard-gluon exchanges but may be attributed to the Feynman mechanism. Using the Local Duality (LD) approach to calculate the soft contribution, it is assumed that the pion form factor is dual to the free quark spectral
density [14, 43]. Then, staying within the \((l+1)\)-loop order, one has

\begin{align}
F_{\pi}^{\text{LD};(l)}(Q^2) &= F_{\pi}^{\text{LD};(l)}(Q^2, s_0^{\text{LD};(l)}(Q^2)), \\
F_{\pi}^{\text{LD};(l)}(Q^2, S) &= \frac{1}{f_\pi^2} \int_0^S \int_0^S \rho_3^{(l)}(s_1, s_2, Q^2) \, ds_1 \, ds_2,
\end{align}

where \(s_0^{\text{LD};(l)}(Q^2)\) is the LD effective threshold parameter for the higher states in the axial channel and the three-point \((l+1)\)-loop spectral density is \(\rho_3^{(l)}(s_1, s_2, Q^2)\). In leading order, we know \(\rho_3^{(0)}(s_1, s_2, Q^2)\) from Eq. (3.4), so that

\[ F_{\pi}^{\text{LD};(0)}(Q^2, S) = \frac{S}{4\pi^2 f_\pi^2} \left( 1 - \frac{Q^2 + 6S}{Q^2 + 4S} \sqrt{\frac{Q^2}{Q^2 + 4S}} \right). \tag{4.4} \]

The LD prescription for the corresponding correlator [14, 67] implies the relations

\[ s_0^{\text{LD};(0)}(0) = 4\pi^2 f_\pi^2 \quad \text{and} \quad s_0^{\text{LD};(1)}(0) = \frac{4\pi^2 f_\pi^2}{1 + \alpha_s(Q_0^2)/\pi}, \tag{4.5} \]

where \(Q_0^2\) is of the order of \(s_0^{\text{LD};(0)}(0)\). This prescription is a strict consequence of the Ward identity for the AAV correlator due to the vector-current conservation. In principle, the \(Q^2\) dependence of the LD parameter \(s_0^{\text{LD}}(Q^2)\) (4.3) should be determined from the QCD SR at \(Q^2 \gtrsim 1\ \text{GeV}^2\). As we have already explained in Sec. III, the standard QCD SR becomes unstable at \(Q^2 > 3\ \text{GeV}^2\) because of the appearance of terms in the condensate contributions linearly growing with \(Q^2\) [41, 42]. For this reason, this dependence was known only for \(Q^2 \leq 3\ \text{GeV}^2\) and, therefore, most authors usually used the constant approximation \(s_0^{\text{LD};(0)}(Q^2) \approx s_0^{\text{LD};(0)}(0)\), like in [10, 43, 44, 49], or a slightly \(Q^2\)-dependent approximation \(s_0^{\text{LD};(1)}(Q^2) \approx 4\pi^2 f_\pi^2/(1 + \alpha_s(Q^2)/\pi)\), like in [47].

Lacking profound knowledge about the exact structure of the NLO spectral density \(\rho_3^{(1)}(s_1, s_2, Q^2)\), two of us (A.B. and N.G.S.) with collaborators have suggested in [10] to use for the full pion form factor the information about the factorizable part, \(F_{\pi}^{\text{pQCD};(2)}(Q^2)\), which is computable within perturbative QCD (pQCD) (here evaluated at the two-loop order). But it should be noted that the pQCD term has the wrong limit at \(Q^2 = 0\), calling for a correction of this behavior in order to maintain the Ward identity (WI) \(F_{\pi}(0) = 1\). To achieve this goal, we apply a matching procedure, introduced in [10], namely,

\[ F_{\pi}^{\text{WI};(2)}(Q^2) = F_{\pi}^{\text{LD};(0)}(Q^2) + \left( \frac{Q^2}{2s_0^{(2)} + Q^2} \right)^2 F_{\pi}^{\text{pQCD};(2)}(Q^2), \tag{4.6} \]

with \(s_0^{(2)} \approx 0.6\ \text{GeV}^2\). This approximation was used to ‘glue’ together the LD model for the soft part, \(F_{\pi}^{\text{LD};(0)}(Q^2)\) (which is dominant at small \(Q^2 \leq 1\ \text{GeV}^2\)), with the perturbative hard-rescattering part, \(F_{\pi}^{\text{pQCD};(2)}(Q^2)\) (which provides the leading perturbative \(O(\alpha_s) + O(\alpha_s^2)\) corrections and is dominant at large \(Q^2 \gg 1\ \text{GeV}^2\)), in such a way as to ensure the validity of the Ward identity \(F_{\pi}^{\text{WI};(2)}(0) = 1\). In order to test the quality of the matching prescription

\[ \footnotetext{1}{The triangle diagram at the one-loop level does not include any radiative corrections from the very beginning. For this reason, the \(O(\alpha_s^4)\)-correction for this diagram appears first at the \((l+1)\)-loop order.} \]
given by Eq. (4.6), we propose to compare it with the LD model (4.3) evaluated at the one-loop order (i.e., in the $O(\alpha_s)$-approximation [47, 49]). To this end, we construct the analogous $O(\alpha_s)$-model

$$F_{\pi;WI}^{(1)}(Q^2) = F_{\pi;LD}^{(0)}(Q^2) + \frac{\alpha_s(Q^2)}{\pi} \frac{2Q^2 s_0^{LD;(0)}(0)}{(2s_0^{LD;(1)}(Q^2) + Q^2)^2}, \quad (4.7a)$$

where we made use of the asymptotic form factor [45, 46]

$$F_{\pi;QCD}^{(1)}(Q^2) = \frac{8\pi f_{\pi}^2 \alpha_s(Q^2)}{Q^2} = \frac{\alpha_s(Q^2)}{\pi} \frac{2s_0^{LD;(0)}(0)}{Q^2}, \quad (4.7b)$$

implying the same prescription for the effective LD threshold as in [47]:

$$s_0^{LD;(1)}(Q^2) = \frac{4\pi^2 f_{\pi}^2}{1 + \alpha_s(Q^2)/\pi}. \quad (4.7c)$$

It is worth noting that the model $F_{\pi;WI}^{(1)}(Q^2)$, following from the matching procedure (4.7a) suggested in [10], works quite well, albeit it was proposed without the knowledge of the exact two-loop spectral density, which became available only somewhat later [49]. Recall that the key feature of the matching prescription is that it uses information on $F_{\pi}(Q^2)$ in two asymptotic regions:

1. $Q^2 \to 0$, where the Ward identity dictates $F_{\pi}(0) = 1$ and hence $F_{\pi}(Q^2) \simeq F_{\pi;LD}^{(0)}(Q^2)$,

2. $Q^2 \to \infty$, where $F_{\pi}(Q^2) \simeq F_{\pi;QCD}^{(1)}(Q^2)$

in order to join properly the hard tail of the pion form factor with its soft part. Numerical analysis of (4.7a) shows that the applied prescription yields a pretty accurate result, with a relative error varying in the range 5% at $Q^2 = 1$ GeV$^2$ to 9% at $Q^2 = 3 - 30$ GeV$^2$. The graphical comparison of $F_{\pi;WI}^{(1)}(Q^2)$ (solid blue line) with the LD result (4.3) (black dots), employing the two-loop spectral density [49], is displayed in the left panel of Fig. 9. This figure also shows the purely perturbative part $Q^2 F_{\pi;QCD}^{(1)}(Q^2)$ (dashed red line)—not corrected by the factor $(Q^2/(2s_0 + Q^2))^2$ to the right low-$Q^2$ behavior.\[ In similar context a matching procedure, as that described above, was also used in a related work [58] on the electromagnetic pion and kaon form factor including radiative and higher-twist effects.\]

Once the spectral density $\rho_{3}^{(1)}(s_1, s_2, Q^2)$ was calculated [49], it made it possible to improve the representation of the LD part in (4.7a) by taking into account the leading $O(\alpha_s)$ correction in the electromagnetic vertex. On that basis, we suggest the following improved WI model for $F_{\pi;imp}^{(1)}(Q^2, s_0^{LD;(1)}(Q^2))$:

$$F_{\pi;imp}^{(1)}(Q^2, S) = F_{\pi;LD}^{(0)}(Q^2, S) + \frac{S}{4\pi^2 f_{\pi}^2} \left\{ \frac{\alpha_s(Q^2)}{\pi} \left( \frac{2S}{2S + Q^2} \right)^2 + F_{\pi;QCD}^{(1)}(Q^2) \left( \frac{Q^2}{2S + Q^2} \right)^2 \right\}. \quad (4.8)$$

We explicitly display the dependence on the threshold $S$ in Eq. (4.8)—the aim being to apply it later on with $S = s_0^{LD-eff}(Q^2)$, the latter value being extracted by comparing the

\[ For this reason, this expression tends to the finite value 0.21 GeV$^2$ at $Q^2 \to 0$ and does not vanish.\]
NLC QCD SR results with the LD approximation. Note that this model has the right limit at \( Q^2 \to 0 \), provided one sets \( S = s_0^{\text{LD};(1)}(Q^2) = 4\pi^2 f_\pi^2/(1 + \alpha_s(Q^2)/\pi) \). Indeed, one has \( F_{\pi;\text{LD};(0)}(0, S) = S/(4\pi^2 f_\pi^2) \) by virtue of the fact that the term \( F_{\pi;\text{LD};(1)}(Q^2) \) is canceled by the factor \([Q^2/(2S + Q^2)]^2 \) in this limit, so that the net result is\(^5\)

\[
F_{\pi;\text{imp};(0)}(0, s_0^{\text{LD};(1)}(0)) = \frac{s_0^{\text{LD};(1)}(0)}{4\pi^2 f_\pi^2} \left[ 1 + \frac{\alpha_s(s_0)}{\pi} \right] = 1.
\]

The graphical evaluation of this new WI model in comparison with the exact LD result in the one-loop approximation is displayed in the right panel of Fig. 9. We can see from this graphics that the quality of the matching condition, given by (4.7a), is improved. Indeed, the relative error is reduced, varying between 4% at \( Q^2 = 1 - 10 \) GeV\(^2\) and 3% at \( Q^2 = 30 \) GeV\(^2\).

Proceeding along similar lines of reasoning, we construct the two-loop WI model \( F_{\pi;\text{LD};(2)}(Q^2, s_0^{\text{LD};(2)}(Q^2)) \) for the pion form factor to obtain

\[
F_{\pi;\text{LD};(2)}(Q^2, S) = F_{\pi;\text{LD};(0)}(Q^2, S) + \frac{S}{4\pi^2 f_\pi^2} \left\{ \frac{\alpha_s(Q^2)}{\pi} \left( \frac{2S + Q^2}{2S + Q^2} \right)^2 + F_{\pi;\text{FA};(2)}(Q^2) \left( \frac{Q^2}{2S + Q^2} \right)^2 \right\}, \tag{4.9}
\]

where \( F_{\pi;\text{FA};(2)}(Q^2) \) is the analyticized expression generated from \( F_{\pi;\text{LD};(2)}(Q^2) \) using Fractional Analytic Perturbation Theory (FAPT) (see Refs. [68, 69, 70, 71, 72] and, in particular, [52, 73]) to get a result which appears to be very close to the outcome of the default scale setting \( \mu_\text{R}^2 = \mu_\text{S}^2 = Q^2 \), investigated in detail in [10]. The explicit expression for \( F_{\pi;\text{FA};(2)}(Q^2) \) is given in Appendix C. This model provides the possibility to implement the perturbative QCD \( O(\alpha_s^2) \)-results for the pion form factor without performing an explicit three-loop calculation of the three-point spectral density, a welcome advantage, as this calculation is very

\(^5\) One usually applies some “freezing” assumption \( s_0^{\text{LD};(1)}(Q^2) = 4\pi^2 f_\pi^2/(1 + \alpha_s(s_0)/\pi) \) for \( Q^2 \leq s_0 \approx 0.6 \) GeV\(^2\). In this case, the same “freezing” should be used in Eq. (4.8) for the argument of \( \alpha_s \) as well: \( \alpha_s(Q^2) \to \alpha_s(s_0) \) for \( Q^2 \leq s_0 \). Note that in order to treat the electromagnetic radius of the pion (i.e., the derivative of the pion form factor in the low \( Q^2 \) domain) correctly, one has to apply a different form of the OPE and the corresponding SR—see for details in [41].
tedious. Moreover, the case of the one-loop approximation with the WI model (4.8) indicates that the relative error of this procedure is of the order of 10%. This level of accuracy allows us to estimate the weight of the \(O(\alpha_s^2)\)-correction. Hence, the relative error of our estimate is of the order of 1%—provided we take into account the \(O(\alpha_s)\)-correction exactly via the specific choice of \(s_0(Q^2)\), as done in (4.10). In order to use this formula, we only need to improve our knowledge about the effective LD thresholds \(s_{0,\text{LD-eff}}^{\text{LD};(1)}(Q^2)\).

As we also stated at the beginning of this section, the problem of adjusting the continuum threshold parameter \(s_0^{\text{LD}}(Q^2)\) is of high importance for the LD approach. From our point of view, it should be determined by comparing the LD results with those derived via the Borel QCD SRs. In the previous section, we processed our NLC QCD SR and obtained the interpolation expressions (4.2a), (4.2b) which are valid for \(Q^2 \in [1,10] \text{ GeV}^2\). Now we can determine the corresponding effective thresholds \(s_{0,\text{LD-eff}}^{\text{LD-eff}}(Q^2)\) and \(s_{0,\text{imp}}^{\text{LD-eff}}(Q^2)\) in the minimal and the improved NLC model, respectively, to find

\[
F_{\pi;\text{imp}}^{\text{WI};(1)}(Q^2, s_{0,\text{LD-eff}}^{\text{LD-eff}}(Q^2)) = F_{\pi;\text{min}}^{\text{SR}}(Q^2),
F_{\pi;\text{imp}}^{\text{WI};(1)}(Q^2, s_{0,\text{imp}}^{\text{LD-eff}}(Q^2)) = F_{\pi;\text{imp}}^{\text{SR}}(Q^2). \tag{4.10}
\]

The solutions of these equations, namely, \(s_{0,\text{LD-eff}}^{\text{LD-eff}}(Q^2)\) and \(s_{0,\text{imp}}^{\text{LD-eff}}(Q^2)\), are shown in Fig. 10; they can be represented in this range of \(Q^2\) by the following interpolation formulas:

\[
s_{0,\text{min}}^{\text{LD-eff}}(Q^2 = x \text{ GeV}^2) = 0.57 + 0.307 \tanh(0.165 x) - 0.0323 \tanh(775 x), \tag{4.11a}
\]

\[
s_{0,\text{imp}}^{\text{LD-eff}}(Q^2 = x \text{ GeV}^2) = 0.57 + 0.461 \tanh(0.0954 x). \tag{4.11b}
\]

We see that both thresholds turn out to increase only moderately with \(Q^2\).

The results obtained for the pion form factor with our two-loop model, i.e., Eq. (4.9), and using the effective LD thresholds \(s_{0,\text{LD-eff}}^{\text{LD-eff}}(Q^2)\) and \(s_{0,\text{imp}}^{\text{LD-eff}}(Q^2)\), are displayed in Fig. 11. We see from this figure that the main effect of the NNLO correction peaks at \(Q^2 \gtrsim 4 \text{ GeV}^2\), reaching the level of \(3 - 10\%\), and can be estimated by taking recourse to the results of the FAPT analysis in [10, 52].

C. Estimation of uncertainties

Even with our methodologically improved approach, there are still intrinsic uncertainties that influence our results. Therefore we consider in this subsection two potential sources

![FIG. 10: Effective continuum thresholds \(s_{0,\text{imp}}^{\text{LD-eff}}(Q^2)\) (solid blue line) and \(s_{0,\text{min}}^{\text{LD-eff}}(Q^2)\) (dashed blue line) that approximate the NLC QCD SR results using the LD \(O(\alpha_s(Q^2))\)-formulas.](attachment:fig10.png)
FIG. 11: We show as a narrow dash-dotted red strip the predictions for the pion form factor, obtained in the two-loop WI model, Eq. (4.9), using the minimal (left panel) and the improved (right panel) Gaussian model. The width of the strip is due to the variation of the Gegenbauer coefficients $a_2$ and $a_4$ (needed to calculate the collinear part $F_2^{QCD(2)}(Q^2)$) in the corresponding shaded bands for the pion DA in both models (indicated by the central blue lines). The normalization of the strip is also affected by the effective continuum thresholds $s_{0,\text{min}}^{LD-eff}(Q^2)$ (left panel) and $s_{0,\text{imp}}^{LD-eff}(Q^2)$ (right panel).

of uncertainties: (i) the choice of $\Lambda_{QCD}$ and (ii) those uncertainties originating from our nonperturbative assumptions.

• We performed our numerical estimates using a value of $\Lambda_{QCD} = 300$ MeV for 3 flavors of active quarks. Let us here discuss how our results change as we vary the value of this parameter. First, we note that the relative contribution of the NLO correction to the value of the pion form factor is of the order of 20%, as we already mentioned at the end of Section III. Next, the relative difference between $\alpha_s(Q^2, \Lambda_{QCD} = 300$ MeV) and $\alpha_s(Q^2, \Lambda_{QCD} = 200$ MeV) is of the order of 15% for $Q^2 = 1 - 10$ GeV$^2$. Hence, varying the QCD scale parameter $\Lambda_{QCD}$ in the range 200–300 MeV induces uncertainties of our predictions of the order of about 3%.

• As it was shown in [18], the minimal Gaussian model for the nonlocal quark condensate does not satisfy the QCD equations of motion and the transversality condition for the two-point correlator of the vector currents. For this reason, using the minimal model can induce some artificial term in the OPE part of the QCD SR. Bearing this in mind, the improved model was constructed in such a way as to minimize this unphysical contribution for the first five lowest-order conformal moments of the pion DA. Speaking in terms of a Taylor expansion of the nonlocal condensates, this improvement pertains to the correction of the low-dimensional local condensates—which represent coefficients in this expansion. The dark side of this improvement is that it could render the definition of higher-dimensional local condensates ill-defined relative to the minimal model. This is the reason why we refrained from applying the improved model to the extraction of the higher conformal moments of the pion DA and the high-momentum behavior of the pion form factor. As we can see from Fig. 8, in the case of the improved model the quark-gluon term $(\bar{q}Aq)$ grows with $Q^2$ in the considered momentum region—in contrast to the minimal model. We conclude from this that the $\bar{q}Aq$-term in the improved model is not well-defined in the high $Q^2$ region. One can realize from Fig. 7 that the difference between the minimal and the improved model tends to become larger relative to the accuracy of the SR at $Q^2 \approx 7$ GeV$^2$. This means that the validity of the nonperturbative model used is lost at this point. Therefore, we should anticipate a
reduced accuracy of the obtained results on the pion form factor in the high-momentum regime. To overcome this problem, one should eventually set stricter constraints on the model using experimental data—a promising option after the planned upgrade of the CEBAF accelerator. Still another possibility comes from theory, notably, current conservation and the equations of motion. Such attempts are currently under scrutiny.

V. DISCUSSION AND CONCLUSIONS

In this work we have calculated the pion’s electromagnetic form factor using a three-point QCD sum rule with nonlocal condensates. The main scope of our investigation was on the methodological side, though the obtained predictions compare well with both the existing experimental data and lattice simulations. The applied method offers the following key advantages:

(a) The profile of the pion distribution amplitude is irrelevant. This removes a serious source of theoretical uncertainty entailed at the normalization point of the order of 1 GeV and simplifies the computation because the technically demanding task of the NLO evolution of the pion DA to higher $Q^2$ values to compare with data is absent.

(b) Because this type of approach tends to become unstable above $Q^2 > 3$ GeV$^2$, we had to improve the quality of the sum rules considerably. This was achieved by dealing not with a Taylor expansion but directly with the nonlocal condensates, avoiding any local contribution. As a result, we were able to enlarge the region of applicability of the QCD SR towards momenta as high as 10 GeV$^2$.

(c) In order to minimize the transversality violations of the two-point correlator of vector currents, we considered in our analysis not only the minimal Gaussian model for the distribution of the quark average momentum in the vacuum, but also an improved version of it introduced earlier in [18].

(d) A spectral density was used in the dispersion integral of the sum rules which includes terms of $O(\alpha_s)$. The influence of this next-to-leading-order (NLO) contribution to the pion form factor turns out to be quite important.

(e) An analytic running coupling was used in the calculations pertaining to perturbation theory that helps surmount the problem of a Landau singularity at $Q^2 = \Lambda_{QCD}^2$.

The phenomenological findings of our analysis and their implications are the following.

(i) Our main predictions for $F_\pi(Q^2)$ were shown in Fig. 6 in comparison with the existing experimental data from the early Cornell [11, 54, 55] and the more recent JLab Collaboration [12]. We found that the $O(\alpha_s)$-contribution to the spectral density influences the pion form factor at the level of 20%. This estimate contrasts with the result obtained in [47, 49], which was found to be somewhat larger.

(ii) Inspection of Fig. 6 reveals that the central-line curve of the improved model lies within the error range of the minimal model up to values $Q^2 \approx 7$ GeV$^2$, indicating a comparable quality of both models in this momentum region. We pointed out that the main differences between these two models for the vacuum nonlocality can be attributed to the different contributions related to the quark-gluon-quark contribution $\Phi_{qAq}(Q^2, M^2)$ (see Fig. 8).
In Fig. 7 we compared our predictions with a variety of theoretical models for the electromagnetic pion form factor—including also results derived from holographic models based on the AdS/QCD correspondence. We found that the models of [61] and [62] are within the error bands of the minimal and the improved model, respectively.

Applying a matching method of the factorized pion form factor, calculable within perturbative QCD, to its value at $Q^2 = 0$, subject to the Ward identity, we were able to model the pion form factor at the two-loop level without having recourse to the spectral density—appealing only to the local duality concept. This method was previously developed and applied successfully in [10] and was used quite recently with an appropriate modification in [58] in the calculation of the pion and the kaon form factors at the twist-three level. The key element in our approach is the extraction of the effective continuum thresholds $s_0^{LD}(Q^2)$ from the results of the NLC QCD SR.

The results obtained with the local duality model, just described, were found to be inside the error band of the QCD sum rules with the nonlocal condensates (Fig. 11) providing support for the consistency of both approaches. Both methods yield predictions comparing well with the experimental data.

Comparison of our predictions with those found with the LD approach [47] reveals that they are systematically higher than these. The reason for this difference is that the effective LD threshold $s_0^{LD}(Q^2)$ has a well-defined value only in the small-$Q^2$ region. For higher $Q^2$ values, it is not firmly fixed. The authors of [47] proposed therefore a logarithmically increasing threshold

$$s_0^{LD}(Q^2) = \frac{4\pi^2 f_\pi^2}{1 + \alpha_s(Q^2)/\pi},$$

which is about 0.67 GeV$^2$ at $Q^2 \approx 10$ GeV$^2$. In order to imitate the NLC QCD SR results within the LD approach, one needs to use $s_0^{LD}(Q^2 = 10$ GeV$^2) = 0.87$ GeV$^2$. This means that the $s_0^{LD}$ uncertainty in the region of $Q^2 = 10$ GeV$^2$ is of the order of 20%.

Bottom line: The use of three-point QCD sum rules with nonlocal condensates provides a reliable alternative to calculate the spacelike pion’s electromagnetic form factor in that momentum region which is accessible to current and planned experiments.

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APPENDIX A: PARAMETRIZATION OF THE NONLOCAL CONDENSADES

We use, as usual in the QCD sum-rules approach, the fixed-point (Fock–Schwinger) gauge $x^\nu A_\nu(x) = 0$. For this reason, all connectors $\mathcal{C}(x,0) \equiv \mathcal{P} \exp \left[ -ig_d \int_0^x t^\alpha A_\alpha^a(y)dy^\alpha \right] = 1$,
assuming for the integration contour a straight line going from 0 to \( x \). For the scalar and the vector condensates, we apply the same minimal model, as used in \([16, 17]\):

\[
\langle \bar{q}(0)q(z) \rangle = \langle \bar{q}q \rangle e^{-|z|^2/\lambda^2_q/8}; \quad \langle \bar{q}(0)\gamma_\mu q(z) \rangle = \frac{i z_\mu z^2}{4} A_0 e^{-|z|^2/\lambda^2_q/8},
\]

(A.1)

where \( A_0 = 2\alpha_s \pi \langle \bar{q}q \rangle^2/81 \). The nonlocality parameter \( \lambda^2_q = \langle k^2 \rangle \) characterizes the average momentum of quarks in the QCD vacuum and has been estimated in QCD SR \([34, 74]\) and on the lattice \([35, 36]\) to have a value around \( \lambda^2_q = 0.45 \pm 0.1 \text{ GeV}^2 \). For the vector and the axial-vector quark-gluon-antiquark condensate we use a parametrization suggested in \([15, 31]\):

\[
\langle \bar{q}(0)\gamma_\mu (-g \hat{A}_\nu(y))q(x) \rangle = (y_\mu x_\nu - g_{\mu\nu} (y \cdot x)) \overline{M}_1(x^2, y^2, (y - x)^2)
\]

\[
+ (y_\mu y_\nu - g_{\mu\nu} y^2) \overline{M}_2(x^2, y^2, (y - x)^2),
\]

\[
\langle \bar{q}(0)\gamma_5 \gamma_\mu (-g \hat{A}_\nu(y))q(x) \rangle = i \varepsilon_{\mu\nu\lambda} \overline{M}_3(x^2, y^2, (y - x)^2)
\]

with

\[
\overline{M}_i(x^2, y^2, z^2) = A_i \int_0^\infty d\alpha d\beta d\gamma \; f_i(\alpha, \beta, \gamma) e^{(\alpha x^2 + \beta y^2 + \gamma z^2)/\Lambda}.
\]

(A.2)

Note that here the following abbreviation \( A_{1,2,3} = A_0 \times (-\frac{3}{2}, 2, \frac{3}{2}) \) has been used. The minimal model of the nonlocal QCD vacuum relies upon the following ansatz:

\[
f_i(\alpha, \beta, \gamma) = \delta (\alpha - \Lambda) \; \delta (\beta - \Lambda) \; \delta (\gamma - \Lambda)
\]

(A.3)

with \( \Lambda = \lambda^2_q/2 \). By construction, this type of model faces problems with the QCD equations of motion and the deficiency of the two-point correlator of the vector currents to satisfy the transversality condition. In order to fulfill the QCD equations of motion and at the same time to minimize the non-transversality of the \( VV \) correlator, an improved model of the QCD vacuum was suggested in \([18]\) that uses a modified Gaussian ansatz, viz.,

\[
f_i^{\text{imp}}(\alpha, \beta, \gamma) = (1 + X_i \partial_x + Y_i \partial_y + Z_i \partial_z) \delta (\alpha - x \Lambda) \; \delta (\beta - y \Lambda) \; \delta (\gamma - z \Lambda),
\]

(A.4a)

where \( z = y, \Lambda = \frac{1}{2} \lambda^2_q \) and

\[
X_1 = +0.082; \quad X_2 = -1.298; \quad X_3 = +1.775; \quad x = 0.788,
\]

\[
Y_1 = -2.243; \quad Y_2 = -0.239; \quad Y_3 = -3.166; \quad y = 0.212.
\]

(A.4b)

By virtue of the QCD equations of motion, these parameters satisfy the supplementary conditions:

\[
12 (X_2 + Y_2) - 9 (X_1 + Y_1) = 1, \quad x + y = 1.
\]

(A.5)

On the other hand, the vacuum condensates of the four-quark operators are reduced to a product of two scalar quark condensates (A.1) by virtue of the vacuum-dominance hypothesis \([67]\).
APPENDIX B: QCD SUM-RULE PARAMETERS

The parameters to characterize the nonlocal condensates are $\Lambda = \lambda_q^2/2 = 0.2$ GeV$^2$, $\langle \alpha_s G G \rangle/\pi = 0.012$ GeV$^4$ and $\alpha_s \langle \bar{q}q \rangle^2 = 1.83 \cdot 10^{-4}$ GeV$^6$. The nonlocal gluon-condensate contribution $\Phi_G(M^2)$ produces a very complicated expression. In analogy to the quark case, we model it by an exponential factor [13, 75]: $\Phi_G(M^2) = \Phi_{\text{loc}}(M^2) e^{-\lambda_g^2 Q^2/M_4^2}$ with $\lambda_g^2 = 0.4$ GeV$^2$.

In order to determine the best value of the threshold $s_0$, we define a $\chi^2$ function for each value of $Q^2$ and $s_0$:

$$
\chi^2(Q^2, s_0) = \frac{\varepsilon^{-2}}{N_M} \left[ \sum_{i=0}^{N_M} Q^4 F(Q^2, M_i^2, s_0) - \left( \frac{\sum_{i=0}^{N_M} Q^2 F(Q^2, M_i^2, s_0)}{N_M + 1} \right)^2 \right],
$$

where we used $M_i^2 = M^2 + i \Delta_M$, $\Delta_M = (M_+^2 - M_-^2)/N_M$, $N_M = 20$, and with $\varepsilon$ denoting the desired accuracy for $\chi^2 \simeq 1$ (the actual value in the computation is $\varepsilon = 0.07$ GeV$^2$.)

APPENDIX C: FACTORIZABLE PART OF THE PION FORM FACTOR IN FRACTIONAL ANALYTIC PERTURBATION THEORY

Here, we provide some technical details about the FAPT analytization procedure in the computation of the factorizable part of the pion form factor using perturbation theory. In perturbative QCD at the NLO level and with the scale setting $\mu_R^2 = Q^2$ and $\mu_F^2 = \text{const}$ (where the subscripts F and R denote, respectively, the factorization and the renormalization scale), one obtains

$$
F_{\pi}^{\alpha_s(2)}(Q^2; \mu_F^2) = \alpha_s^{(2)}(Q^2) F_{\pi}^{\text{LO}}(Q^2; \mu_F^2) + \left[ \frac{\alpha_s^{(2)}(Q^2)}{\pi} \right]^2 F_{\pi}^{\text{NLO}}(Q^2; \mu_F^2),
$$

where the superscript $(2)$ in $\alpha_s^{(2)}(Q^2)$ denotes the two-loop order of the coupling, with

$$
F_{\pi}^{\text{LO}}(Q^2; \mu_F^2) \equiv \frac{8 \pi f_{\pi}^2}{Q^2} [1 + a_2^{\text{LO}}(\mu_F^2) + a_4^{\text{LO}}(\mu_F^2)]^2
$$

and

$$
F_{\pi}^{\text{NLO}}(Q^2; \mu_F^2) \equiv b_0 F_{\pi}^{(1,\beta)}(Q^2; \mu_F^2) + F_{\pi}^{(1,\text{FG})}(Q^2; \mu_F^2) + C_F F_{\pi}^{(1,\text{F})}(Q^2; \mu_F^2) \ln \left[ \frac{Q^2}{\mu_F^2} \right].
$$

In Eq. (C.1), and below, $a_2^{\text{LO}}$ and $a_4^{\text{LO}}$ are the LO Gegenbauer coefficients of the pion DA, whereas the individual contributions in Eq. (C.3a) read (with further details and explanations
improved result recently [52], this procedure was further improved by setting

This type of approach was analyzed in [71] and it was shown that the analytization procedure

Here

The analyticized NNLO contribution (which includes radiative corrections of $O(\alpha_s^2)$) is then computed to be

\[
F_{\pi}^{\text{FAPT},(2)}(Q^2) = A_1^{(2)}(Q^2) F_{\pi}^{\text{LO}}(Q^2, \mu_F^2) + \frac{1}{\pi} L_{2,1}^{(2)}(Q^2) F_{\pi}^{(1,F)}(Q^2; \mu_F^2)
+ \frac{1}{\pi} A_2^{(2)}(Q^2) \left[ F_{\pi}^{\text{NLO}}(Q^2, \mu_F^2) - F_{\pi}^{(1,F)}(Q^2; \mu_F^2) \ln \frac{Q^2}{\Lambda_{\text{QCD}}^2} \right].
\]  

Here $A_1(Q^2)$, $A_2(Q^2)$, and $L_{2,1}(Q^2)$ are the analytic images of $\alpha_s(Q^2)$, $\alpha_s^2(Q^2)$, and $\alpha_s^2(Q^2) \ln(Q^2/\Lambda_{\text{QCD}}^2)$ [52, 70, 71]:

\[
A_1(Q^2) \equiv A_E \left[ \alpha_s(Q^2) \right],
\]  

\[
A_2(Q^2) \equiv A_E \left[ \alpha_s^2(Q^2) \right],
\]  

\[
L_{2,1}(Q^2) \equiv A_E \left[ \alpha_s^2(Q^2) \ln \left( \frac{Q^2}{\Lambda_{\text{QCD}}^2} \right) \right].
\]  

with

\[
A_E \left[ f(Q^2) \right] \equiv \int_0^\infty \frac{\text{Im} f(-\sigma)}{\sigma + Q^2} \, d\sigma.
\]  

This type of approach was analyzed in [71] and it was shown that the analytization procedure significantly reduces the dependence of the results on the factorization-scale setting. More recently [52], this procedure was further improved by setting $\mu_F^2 = Q^2$. This renders the logarithmic term to cancel out, but swamps the appearance of complicated expressions like $A_E \left[ \alpha_s^p(Q^2) a_{2n}^{\text{LO}}(Q^2) a_{2m}^{\text{LO}}(Q^2) \right]$ with $p = 1, 2$ and $n, m = 0, 1, 2$. Nevertheless, the obtained results are very close to those previously found in [71] and in [10] with $\mu_R^2 = \mu_F^2 = Q^2$. The improved result $F_{\pi}^{\text{FAPT},(2)}(Q^2)$ was recently obtained in [52].

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