Analytical and numerical studies of resonant wave run-up on a plane structure

G Andadari* and I Magdalena

Department of Mathematics, Faculty of Mathematics and Natural Science, Institut Teknologi Bandung, Indonesia

*Corresponding authors: gita.andadari@gmail.com,

Abstract. Wave run-up is the vertical extent of wave up rushed on a structure. To describe wave run-up characteristic, we tend to relate to its maximum height. It was a common belief that the leading wave will usually reach the maximum run-up. However, it turns out that this is not always the case. Resonance is a phenomenon when the incident wave shares the same frequency as the natural frequency. In this article, the natural frequency of a semi-enclosed basin on a plane structure is derived using the variable separation technique. Next, the staggered conservative scheme is used to test these natural frequencies through simulations of wave run-up on a plane structure. The result shows that when the incoming wave frequency is close to the natural frequency, the largest run-up height is not the leading wave, but the second, third, or fourth waves. This indicates the occurrence of resonance phenomena. Sensitivity analysis was applied to show the dependence of the maximum run-up height to the structure's slope, as well as the incident wave frequency. Further, a physical simulation was conducted to examine whether the resonance phenomenon appears in the actual events.

1. Introduction
Researchers have done studies about wave run-up for over 50 years with a constant improvement made throughout the process. One of the challenges in the study is coping with moving shoreline. In 1958, an important contribution came from Carrier and Greenspan [1] which used the hodograph transformation to find analytical solutions of the shallow water equations (SWE). Since then, many researchers used the Carrier and Greenspan transformation to study run-up characteristics of various waves. In 2007, Antuono [2,3] used this transformation to solve the boundary value problem of SWE in order to find the run-up height formula. However, there are not many studies that discuss the resonance phenomena even though this may arise in certain situations. For example, when the incoming wave frequency is close to the structural natural frequency, the resonance phenomenon may appear. Stefanakis [4] found that resonance may enhance the run-up of the nonleading wave. This causes the run-up height of the following wave higher than the leading wave. Resonance of long waves on the composite structure was also encountered.
by Ezersky in [5]. In this research, we study the resonance run-up phenomena using Non-linear Shallow Water Model. We derive the analytical solution for the Linear Shallow Water Equations for single slope structure to obtain the natural wave frequency. A staggered finite volume method is implemented to yield a computational check. There are seven sections in this paper. In Section 2, a natural frequency for the semi-enclosed structure is derived analytically. The staggered conservative scheme is reviewed in section 3. In section 4, the numerical simulation that has been done to support this paper is explained. The occurrence of resonance phenomena is investigated in section 5. Conclusions are given in section 6 and the references in the last section.

2. Analytical solution for natural frequency of long waves over a sloping structure

In this section we will derive the natural frequency for waves on a sloping structure using the variable separation technique. Consider the following linear shallow water equations

$$\eta_t + (hu)_x = 0$$  \hspace{1cm} (1)
$$u_t + g\eta_x = 0$$  \hspace{1cm} (2)

where $\eta$ is surface elevation, $u$ horizontal velocity, $h$ undisturbed water depth, and $g$ gravitational acceleration. Eliminating $u$ from (1) and (2) will give us

$$\eta_{tx} - (gh(x)\eta_x)_x = 0.$$  \hspace{1cm} (3)

which is the homogeneous linear wave equation. If we have a structure such as $d(x) = -\alpha x$ where $\alpha$ denotes the slope of the structure and $h(x) = -d(x)$, the equation (3) becomes

$$\eta_{tx} + \alpha g\eta_x + \alpha g\eta_{xx} = 0.$$  \hspace{1cm} (4)

Assume that the incoming wave is a monochromatic wave, so we can write

$$\eta(x, t) = X(x)e^{-i\omega t}.$$  \hspace{1cm} (5)

where $\omega$ is the wave frequency. Consider a sloping structure with semi-enclosed boundary condition on a domain $-x_1 \leq x \leq 0$. Using (5), equation (4) reduces to an equation of $X(x)$ reads

$$g\alpha(xX_{xx} + X_x) - \omega^2 X = 0, 0 < x < x_1.$$  \hspace{1cm} (6)

Solutions of (6) are

$$X(x) = C_1 J_0 \left( \frac{-4\omega^2 x}{g\alpha} \right) + C_2 Y_0 \left( \frac{-4\omega^2 x}{g\alpha} \right),$$

where $J_0$ and $Y_0$ are the zero-th order of the first and second kind of Bessel functions, respectively. Moreover, under the assumption that wave amplitude is bounded, we should have $\lim_{x \to 0} \eta(x, t)$ is finite. That means the constant $C_2$ should be zero. Next, the other boundary condition is $\eta(-x_1, t)$ finite. That means the constant $C_1$ should be zero. Let $2z_k$ be the k-root of $J_0(x)$ on the interval $-x_1 \leq x \leq 0$, to be explicit they read

$$z_1 = 1.2025, z_2 = 2.760, z_3 = 4.3270, ...$$  \hspace{1cm} (7)

Hence, the natural frequency for waves on a plane beach is

$$\omega_k = z_k \sqrt{\frac{g\alpha}{x_1}}, k = 1,2,3,...$$  \hspace{1cm} (8)
Later, we use this natural frequency formula to investigate resonance phenomena that may occur in the case of wave run-up on a sloping structure. Note that at the shore, we have \( \eta(0, t) = C_1 e^{j\omega t} \) which has maximum value \( C_1 \). For the rest of the paper, we called \( C_1 \) as maximum run-up height and denote it by \( R \).

3. The staggered conservative scheme of Non Linear SWE

In this section, we describe a numerical scheme that have been explained by Stelling and Duinmeijer [6], Magdalena et.al [7], Pudjaprasetya and Magdalena [9], Stelling and Zijlema [8] to simulate wave run-up phenomena using numerical finite volume method on a staggered grid. Consider shallow water equations (1) and (2), which hold for relatively long waves in a shallow region. To cover breaking waves, we propose to use a nonlinear model. Let \( \eta(x, t) \) be the surface elevation and \( u(x, t) \) be the depth average horizontal velocity. The nonlinear SWE read as:

\[
\eta_t + (hu)_x = 0 \tag{9}
\]
\[
uu + uu_x + g\eta_x = 0 \tag{10}
\]
with \( g \) is gravitational constant. Notation \( h(x, t) = \eta(x, t) + d(x, t) \) denotes water thickness, where \( d \) is bottom topography.

To solve equation (9) and (10) numerically, we will use Leapfrog numeric scheme with staggered conservative method. Consider the computational domain \([-L, L]\) which is divided into \( N_x \) cells with equal \( \Delta x \) length, a staggered partition points are

\[
x_{1/2} = -L, \quad x_{3/2} = -L + \Delta x, \ldots, \quad x_{(j+1)/2} = -L + j\Delta x, \ldots, \quad x_{N_x+1/2} = -L + N_x\Delta x = L
\]

with \( j = 1, 2, \ldots, N_x \). On this staggered partitions, \( u \) are computed on staggered grid whereas \( h \) (which also means \( \eta \)) are computed on full grids, as illustrated in Figure 1.

**Figure 1.** Staggered grid partition of a computational domain \([-x_j, x_{j+1/2}]\)

Assume \( q = uh \) denotes the flux with length of the space partition \( \Delta x \) and length of the time partition \( \Delta t \). With FTCS, consider the discrete form of the mass conservation equation below

\[
\frac{\eta_j^{n+1} - \eta_j^n}{\Delta t} + \frac{q_{j+1/2}^n - q_{j-1/2}^n}{\Delta x} = 0 \tag{11}
\]

\[
\eta_j^{n+1} = \eta_j^n + \frac{\Delta t}{\Delta x} \left( q_{j+1/2}^n - q_{j-1/2}^n \right), \quad j = 1, 2, \ldots, N_x \tag{12}
\]

with

\[
q_{j+1/2}^n = h_{j+1/2} u_{j+1/2}^n \tag{13}
\]
where (*) denotes the variable with unknown value. Thus, we will perform an upwind approximation which considers the flow direction of the water. Therefore, we have:

\[
h(x) = \begin{cases} 
    h_j, & u_{i+1/2} \geq 0 \\
    h_{j+1}, & u_{i+1/2} \leq 0,
\end{cases}
\]  

(14)

Consider the momentum equation (10). With BTCS, this equation is discretized as follows:

\[
\frac{u^{n+1}_{i+1/2} - u^n_{i-1/2}}{\Delta t} + g \frac{h_{i+1}^{n+1} - h_{i}^{n+1}}{\Delta x} + \text{adv}|_{j+1/2} = 0
\]

(15)

\[
u^{n+1}_{i+1/2} = u^n_{i+1/2} + \frac{g\Delta t}{\Delta x} \left( \frac{h_{i+1}^{n+1} - h_{i}^{n+1}}{\Delta x} \right) - \Delta t \frac{\text{adv}|_{j+1/2}}{\Delta x} = 1,2,\ldots,N_x
\]

(16)

with \(\text{adv}\) is the advection variable \(uu_x\) which is

\[
uu_x = \frac{1}{h} \left( \frac{\partial(uu)}{\partial x} - u \frac{\partial q}{\partial x} \right)
\]

(17)

Next, we discretize equation (17) into

\[
(uu_x)_{i+1/2} = \frac{1}{\tilde{h}_{i+1/2}} \left( \bar{q}_{i+1} u_{i+1} - \bar{q}_i * u_i - u_{i+1/2} \frac{\bar{q}_{i+1} - \bar{q}_i}{\Delta x} \right)
\]

(18)

with

\[
\tilde{h}_{i+1/2} = \frac{1}{2} (h_i + h_{i+1})
\]

(19)

\[
\bar{q}_i = \frac{1}{2} \left( q_{i+1/2} + q_{i-1/2} \right)
\]

(20)

\[
* u_i = \begin{cases} 
    u_{i+1/2}, & \bar{q}_i \geq 0 \\
    u_{i+1/2}, & \bar{q}_i < 0
\end{cases}
\]

(21)

For simulation of wave over a sloping beach, the numerical scheme should be able to accommodate wet-dry region. Here the wet-dry procedure is simply, computing the discrete formula (16) only if the water depth at the momentum cell \([x_{i-1/2}, x_{i+1/2}]\) is greater than a minimum threshold depth \(h_{\text{min}}\). Ideally, this threshold depth \(h_{\text{min}}\) is zero, but to avoid computation difficulties of division using small number, we adopt \(h_{\text{min}} \approx 10^{-2}\), or other small number depending on the particular problem we are dealing with. By adopting this simple requirement, the staggered scheme (12,16) can handle simulations that involve dry areas.

4. Numerical Simulation

In this section, the staggered scheme (12,16) is implemented to simulate waves run-up over a sloping sea wall structure that also has been produced by Kanoglu and Synolakis in [10]. The single slope used is described in figure 2. The vertical extend of shoreline during run-up process is denoted as \(R\), whereas \(R_{\text{max}}\) denotes its maximum run-up. Now we are going to run numerical simulation with slope \(\alpha = 1/4\).
initial water level $h_0 = 0.53\text{m}$ in the domain $[-x_1, x_2]$, where $x_2 = 1.59\text{m}$. Further, we employ the incident wave from the left by taking

$$\eta\left(-\frac{x_1}{2}, t\right) = A \sin \omega t$$

where $A$ and $\omega$ denote the wave amplitude and frequency, respectively.

The simulation conducted using $\Delta x=0.1\text{ m}$, and $\Delta t = \frac{\Delta x}{2\sqrt{gh}} = 0.021\text{ s}$, to maintain stability. This first simulation uses $A = 0.0287\text{ m}$ and $\omega = 6.28\text{ s}^{-1}$, snapshots of surface wave at subsequent times are given in Figure 3.

5. **Resonance on a plane structure**

Under the same set up as the previous, $x_1 = 1.59\text{m}, \alpha = \frac{1}{4}$, and $h_0 = 0.53\text{m}$ here we conduct a simulation of wave run-up using different frequency $\omega = 1.493\text{ s}^{-1}$. We record the shoreline position as a function of time and show it in Figure 4 (right) for the resonance phenomenon and Figure 4 (left) for the non-resonance phenomenon. On the left, it can be seen that the first wave yielded the highest run-up at $R_{\text{max}} = 0.2689$. On the other hand, on the Figure 4 (right), the first wave is not the one that yielded the highest run-up anymore. Run up height of wave $\omega = 1.493\text{s}^{-1}$ exhibits resonance behavior: its value is increases from the first wave, the second and so on. During the 20s time interval, the run-up value continues to increase until the fifth wave that reaches the maximum of $\frac{R_{\text{max}}}{h_0} = 0.7736$. 

![Figure 2. Snapshots of numerical simulation on a sloping sea wall](image)

![Figure 3. Snapshots of numerical simulation on a sloping sea wall](image)
Figure 4. Plot wave run-up $R/h_0$ as a function of time for sloping structure with slope $\alpha = \frac{1}{4}$ and wave frequency (left) $\omega = 6.28 \text{s}^{-1}$, (right) $\omega = 1.493 \text{s}^{-1}$.

From this simulation, it can be seen that resonance phenomenon does amplify the maximum run-up. Therefore, if we are building a plane structure or artificial beach, we should avoid building them with a specification that can support the resonance phenomenon. Observe the usual incoming wave and make sure that it doesn’t have the same wave frequency as the structure’s natural frequency.

5.1 Maximum Run-Up

In this section, we study the dependence of maximum run—up with respect to wave frequency. For this purpose, we use three beach slopes, i.e. $\alpha = 0.13, 0.26, 0.3$ with the same structure length. For each slope, we conduct simulations using various wave frequencies. For every beach slope, run-up height $R$ of each wave frequency $\omega$ is measured, and the result is plotted in Figure 5. It is shown that for all three slopes, the first peak of the maximum run-up graph is achieved at 1.2, followed by the second and third peak that achieved at 2.8 and 4.4, respectively. This proved that our numerical results confirm the analytical formula of the normalized frequency as derived before, which is half of the zeroes of the Bessel function $J_0$ as provided in (7).

Figure 5. Maximum run-up versus normalized frequency
6. Conclusion
The run-up event of wave over a sloping beach has been investigated within one dimensional framework. Natural frequency of the semi-enclosed structure has been derived analytically. Implementation of the staggered conservative method to the shallow water equations has provide us a robust code appropriate for simulating wave oscillation over a sloping structure. For breaking wave, the analytical solution is too rigorous. Therefore, we extend our computational model in order to cope with breaking wave case. Our simulations could predict the amplification of maximum run-up height due to resonance.

References
[1] Carrier G F and Greenspan H P 1958 *J Fluid Mech* **4**(1) 97
[2] Antuono M and Brochini M 2007 *Stud Appl Math* **119** 7393
[3] Antuono M and Brochini M 2008 *Coastal Engineering* **55** 73240
[4] Stefanakis T S, Dias F, and Dutykh D 2011 *Phys Rev Lett* **107**(12) 1245021
[5] Ezersky A, Abcha N and Pelinovsky E 2013 *Nonlin Processes Geophys* **20** 3540
[6] Stelling G S and Duinmeijer S P 2003 *Int J Numer Methods Fluids* **43** 132954
[7] Magdalena I, Erwina N and Pudjaprasetya S R 2015 *J Sci Comput* **65**(3) 867874
[8] Stelling, Guus and Zijlema, Marcel 2003 *Int J Numer Methods Fluids* **43** 1
[9] Pudjaprasetya S R and Magdalena I 2014 *East Asian J Applied Math* **4**(2) 152
[10] Kanoglu U and Synolakis C 2006 *Physical Review Letters* **97**(14) 148501