WHEN POLARIZATIONS GENERATE

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Abstract. Let $G$ be a reductive complex algebraic group and $V$ a finite-dimensional $G$-module. From elements of the invariant algebra $\mathbb{C}[V]^G$ we obtain by polarization elements of $\mathbb{C}[kV]^G$, where $k \geq 1$ and $kV$ denotes the direct sum of $k$ copies of $V$. For $G$ simple our main result is the classification of the $G$-modules $V$ and integers $k \geq 2$ such that polarizations generate $\mathbb{C}[kV]^G$.

1. Introduction

Our base field is $\mathbb{C}$, the field of complex numbers. Throughout this paper, $G$ will denote a reductive algebraic group. All our $G$-modules are assumed to be finite-dimensional and rational. Let $V$ be a $G$-module and let $f \in \mathbb{C}[V]^G$ be homogeneous of degree $d$. For $v_1, v_2, \ldots, v_k \in V$, consider the function $f(\sum_i s_i v_i)$ where the $s_i$ are indeterminates. Then

$$f(\sum_i s_i v_i) = \bigoplus_{\alpha \in (\mathbb{Z}^+)^k, |\alpha| = d} s^\alpha f_\alpha(v_1, \ldots, v_k)$$

where the $f_\alpha \in \mathbb{C}[kV]^G$ are multihomogeneous of the indicated degrees $\alpha$. Here for $\alpha = (a_1, \ldots, a_k) \in (\mathbb{Z}^+)^k$ we have $s^\alpha = s_1^{a_1} \cdots s_k^{a_k}$ and $|\alpha| = a_1 + \cdots + a_k$.

We call the $f_\alpha$ the polarizations of $f$. Let $\text{pol}_k(V)^G$ denote the subalgebra of $\mathbb{C}[kV]^G$ generated by polarizations. We say that $V$ has the $k$-polarization property if $\text{pol}_k(V)^G = \mathbb{C}[kV]^G$.

For $G = O_n$ and $V$ the standard action on $\mathbb{C}^n$ one has the $k$-polarization property for all $k$. We have the same result when $G = S_n$ and $V$ is the standard action on $\mathbb{C}^n$ by Weyl [Weyl46] and also for the standard actions of the Weyl groups of type $B$ and $C$ by Hunziker [Hum97]. From Wallach [Wal93] we learn that the 2-polarization property is false for the Weyl group of type $D_4$.

One can also ask if $\mathbb{C}[kV]^G$ is finite over $\text{pol}_k(V)^G$. In case $V$ is a module for SL$_2$ which does not contain a copy of the irreducible two-dimensional module, then $\mathbb{C}[kV]^G$ is finite over $\text{pol}_k(V)^G$ for all $k$. See Kraft-Wallach [KrWa04] (V irreducible) and Losik-Michor-Popov [LMP06] for the general case. The same references prove the $k$-polarization property for all $k$ when $G$ is a torus. However, for the adjoint representation of a simple group of rank at least 2, finiteness fails for $k \geq 2$ [LMP06]. See Draisma-Kemper-Wehlau [DKW06] for questions about polarization and separation of orbits.

Example 1.1. Let $V := \mathbb{C}^2$ be the two-dimensional irreducible $G := \text{SL}_2$-module. Then $\mathbb{C}[V \oplus V]^G$ is generated by the determinant function. The polarizations of this generator give rise to three of the six determinant generators of $\mathbb{C}[4V]^G$. Hence
V ⊕ V does not have the 2-polarization property. Of course, neither does V, since 
\( \mathbb{C}[V]^G = \mathbb{C} \) while \( \mathbb{C}[V ⊕ V]^G ≠ \mathbb{C} \).

Our main aim is to classify the G-modules which have the k-polarization property, \( k ≥ 2 \), when G is a simple linear algebraic group. Along the way we establish general criteria for a representation to have the k-polarization property. Note the trivial fact that the \((k + 1)\)-polarization property fails if the k-polarization property fails. The crucial case to consider is usually that of \( k = 2 \).

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2. Slices

We establish some tools for obtaining our classification. Let V be a G-module and let \( f ∈ \mathbb{C}[V]^G \) be homogeneous of degree d. If \( k = 2 \) or 3, then we will denote the polarizations of \( f \) as \( \{ f_{i,j} \}_{i+j=d} \) and \( \{ f_{i,j,k} \}_{i+j+k=d} \), respectively. We leave the proof of the following to the reader.

**Lemma 2.1.** Let V be a G-module with the k-polarization property. Then any G-submodule of V has the k-polarization property.

Let V be a G-module. Then \( \mathbb{C}[V]^G \) is finitely generated, and we denote by \( \pi: V → V/G \) the morphism of affine varieties dual to the inclusion \( \mathbb{C}[V]^G ⊂ \mathbb{C}[V] \).

Let V be a G-module and \( v ∈ V \) such that the orbit \( G · v \) is closed. Then the isotropy group \( G_v \) is reductive and there is a splitting \( V = S ⊕ T_v(G · v) \) of \( G_v \)-modules. The representation \( G_v → GL(S) \) is called the slice representation at \( v \). We can arrange that \( v ∈ S \). We have a canonical map \( ϕ: G * G_v → S ) → G · S \) which is equivariant. Here \( G * G_v \) is the quotient of \( G × S \) by the \( G_v \)-action sending \( (g, s) → (gh^{-1}, h · s) \) for \( h ∈ G_v \), \( g ∈ G \) and \( s ∈ S \). The \( G_v \)-orbit of \( (g, s) \) is denoted \( [g, s] \). Then \( ϕ: G * G_v → V \) sends \( [g, s] \) to \( g(v + s) \). Replacing \( S \) by an appropriate \( G_v \)-stable neighborhood of \( 0 ∈ S \) one has Luna’s slice theorem [La73]. But here we only need one consequence of this theorem. Namely, that the induced mapping \( ϕ//G: (G * G_v )/G ≃ S//G_v → V//G \) induces an isomorphism of the Zariski cotangent spaces at the points \( 0 \) and \( G · v \) in the quotients.

**Lemma 2.2.** Let \( v, G_v \), etc. be as above. Suppose that V has the k-polarization property. Then so does the \( G_v \)-module S.

**Proof.** We treat the case \( k = 2 \) and leave the general case to the reader. A \( G_v \)-stable complement to \( T_{(v,0)}(G · (v,0)) \) in \( V ⊕ V \) is \( S ⊕ V \). Let \( I := \{ f ∈ \mathbb{C}[V ⊕ V]^G : f(v, 0) = 0 \} \), let \( J := \{ f ∈ \mathbb{C}[S ⊕ V]^G : f(0, 0) = 0 \} \) and let K := \( \{ f ∈ \mathbb{C}[S ⊕ S]^G_v : f(0, 0) = 0 \} \). Let \( ψ: G * G_v (S ⊕ V) → (V ⊕ V) \) be the canonical map. Then, as indicated above, Luna’s slice theorem implies that \( ψ^* \) induces an isomorphism of \( I/I^2 \) with \( J/J^2 \), and clearly \( J/J^2 → K/K^2 \) is surjective. Thus \( I → K/K^2 \) is surjective.

Let \( f ∈ \mathbb{C}[V]^G \) be homogeneous of degree \( d > 0 \). Then \( f − f(v) ∈ I ∩ \mathbb{C}[V]^G \) and its image in \( K ∩ \mathbb{C}[S]^G_v \) is the mapping sending \( s ∈ S \) to \( f(v + s) − f(v) \). We have \( f(v + s) − f(v) = \sum_{i+j=d, j>0} f_{i,j}(v, s) \). Thus each homogeneous component \( f_{i,j}(v, s), j > 0 \), lies in \( K ∩ \mathbb{C}[S]^G_v \). If we polarize in the second argument we obtain the collection of functions \( \{ f_{i,j,k}(v, s, s') : j + k > 0 \} \subset K \) where \( s, s' ∈ S \). On the other hand, if we take the polarizations \( f_{i,j,k} \) of \( f \) where \( k > 0 \), then they are in \( I \) and their images in \( K \) are sums of the elements of \( \{ f_{i,j,k}(v, s, s') : k > 0 \} \). Thus the images of the \( f_{i,j} − δ_{0j} f(v) \) in \( K \) are polarizations of elements of \( K ∩ \mathbb{C}[S]^G_v \). Since V has the 2-polarization property, the \( f_{i,j} − δ_{0j} f(v) \) generate I (as one varies
f), hence $K/K^2$ is generated by the polarizations of elements of $K \cap \mathbb{C}[S]^G$. But functions in $K$ which span $K/K^2$ generate $\mathbb{C}[S \oplus S]^G$, so $S$ has the 2-polarization property.

Remarks 2.3.  
(1) The $G_v$-fixed part of $S$ plays no role. If $S'$ is the sum of the nontrivial isotypic components of $S$, then the interesting fact is that $\mathbb{C}[S' \oplus S]^G$ is generated by polarizations.

(2) Suppose that $\dim \mathbb{C}[V]^G = 1$ and the representation is stable (i.e., there is a non-empty open set of closed orbits). Then for any closed non-zero orbit $G \cdot v$, the slice representation is trivial, so that the Lemma is of no help.

3. Representations without the polarization property

One can say that “most” representations do not have the 2-polarization property. This is born out by the following sequence of lemmas.

Lemma 3.1. Suppose that $V = V_1 \oplus V_2$ where the $V_i$ are $G^0$-stable and the elements of $G$ preserve the $V_i$ or interchange them. For example, the $V_i$ could be the isotypic components corresponding to nontrivial irreducible $G^0$-modules. Suppose further that $\mathbb{C}[V_1 \oplus V_2]^G$ has a minimal bihomogeneous generator $f$ of degree $(a, b)$ where $ab \geq 2$. Then $V$ does not have the 2-polarization property.

Proof. Let $v_i, v'_i$ denote elements in $V_i$, $i = 1, 2$. Set $d = a + b \geq 3$. Consider the polarization $f_{d-2,2}$ of $f$. We can write $f_{d-2,2}(v_1, v_2, v'_1, v'_2)$ as a sum of terms $f^{2,0} + f^{1,1} + f^{0,2}$ where $f^{i,j}(v_1, v_2, v'_1, v'_2)$ has homogeneity $(i, j)$ in $v'_1$ and $v'_2$. The $f^{i,j}$ are $G'$-invariant, where $G'$ is the subgroup of $G$ preserving the $V_i$. Clearly, up to a scalar, $G$ leaves $f^{2,0} + f^{0,2}$ and $f^{1,1}$ invariant. Hence the functions are $G$-invariant (since their sum is $G$-invariant) and nonzero (since $d \geq 3$). Let $I$ denote the ideal of elements of $\mathbb{C}[V]^G$ vanishing at 0. Suppose that $\alpha f^{1,1} + \beta(f^{2,0} + f^{0,2}) \in I^2$ for some $\alpha$ and $\beta$. We may assume that $f^{2,0} \neq 0$, i.e., that $a \geq 2$. Then evaluating at points $(v_1, v_2, v_1, 0)$ we see that $\beta f \in I^2$, hence $\beta = 0$. Now one evaluates at points $(v_1, v_2, v_1, v_2)$ to see that $\alpha = 0$. We have shown that $f^{1,1}$ and $f^{2,0} + f^{0,2}$ are linearly independent modulo $I^2$.

If $f^{1,1}$ is in the subalgebra generated by polarizations of elements of $\mathbb{C}[V]^G$, then $f^{1,1}$ is a sum of terms $q r_{-2,2}$ and $s t_{-1,1} u_{-1,1}$ for appropriate homogeneous $q, \ldots, u \in \mathbb{C}[V]^G$. Since $f^{1,1}$ is a minimal generator, our sum has to contain terms of the form $r_{d-2,2}$. Thus we may assume that $f^{1,1} \in r_{d-2,2} + I^2$ for some $r$. Restituting we see that $(ab)f \in (\binom{d}{2})r + I^2 \cap \mathbb{C}[V]^G$. Hence we have that $r + I^2 = cf + I^2$ for some $c \neq 0$. It follows that $f^{1,1} \in c(f^{1,1} + f^{2,0} + f^{0,2}) + I^2$ which implies that $f^{1,1}$ and $f^{2,0} + f^{0,2}$ are linearly dependent modulo $I^2$. This is a contradiction, hence $V$ does not have the 2-polarization property.

Lemma 3.2. Suppose that $G$ acts on $G^0$ by inner automorphisms and that $V$ is a $G$-module which contains an irreducible symplectic $G^0$-submodule $U$. Further suppose that $\mathbb{C}[U]^G$ has generators of even degree. Then $V$ does not have the 2-polarization property.

Proof. We may suppose that, as $G^0$-module, $V$ is the isotypic component of type $U$. A central torus of $G^0$ must act trivially on $U$, so we can reduce to the case that $G^0$ is semisimple. Set $H := Z_G(G^0)$. Then $H$ is finite and $G = HG^0$ where $H \cap G^0 = Z(G^0)$. Now $W := \text{Hom}(U, V)^{G^0}$ is an $H$-module (via the action of $H$ on
Proof. For $G$-polarization, act as an appropriate basis of $V$ such that $G$-spaces, counting multiplicity.

Corollary 3.3. Suppose that $G^0 = SL_2$ and that $V$ is a $G$-module which contains a $G^0$-submodule $R_j$ where $j$ is odd. Then $V$ does not have the 2-polarization property.

Proof. For $j$ odd, $R_j$ is a symplectic representation of $SL_2$. Moreover, $\pm I \in SL_2$ act as $\pm 1$ on $R_j$, so that all elements of $C[R_j]^{SL_2}$ have even degree. Finally, all automorphisms of $SL_2$ are inner. Thus we can apply Lemma 3.2.

Now assume that $G^0 = C^*$. Let $\nu_j$ denote the irreducible $C^*$-module with weight $j$. We denote by $m\nu_j$ the direct sum of $m$ copies of $\nu_j$. Assume that $V$ is a $G^0$-module such that the multiplicity of each $\nu_j$ is the same as that of $-\nu_j$ for all $j$. We say that $V$ is balanced and we let $q(V)$ denote half the number of nonzero weight spaces, counting multiplicity.

Proposition 3.4. Suppose that $G^0 = C^*$. Let $V$ be a balanced $G$-module with $q(V) \geq 2$. Then $V$ does not have the 2-polarization property.

Proof. First assume that there is a nonzero weight $j$ of multiplicity $m \geq 2$. Then the $C^*$-submodule $m(\nu_j \oplus -\nu_j)$ is $G$-invariant, so that we may assume that it is all of $V$. We may then also assume that $j = 1$. Set $V_1 := m\nu_1$ and $V_2 := m\nu_{-1}$. If $f$ is a minimal homogeneous generator of degree at least 3, then we are done by Lemma 3.1. Thus we may suppose that all the minimal homogeneous generators of $C[V_1 \oplus V_2]^{G^0}$ have degree 2. Let $G' := Z_G(G^0)$. Then $V_1$ and $V_2$ are $G'$-modules.

Write $V_1 = \oplus W_i$ where the $W_i$ are irreducible $G'$-submodules, and similarly write $V_2 = \oplus U_j$. Then the quadratic $G'$-invariants correspond to pairs $W_i$ and $U_j$ such that $U_j \simeq W_i^*$. But $C[V]^{G^0}$ has to be finite over $C[V]^{G'}$ and this forces that $V_1 \simeq mW$ and $V_2 \simeq mW^*$ for some irreducible one-dimensional representation $W$ of $G'$. It follows that the image of $G'$ in $GL(V)$ is that of $G^0 = C^*$, so we may assume that $G' = G^0$. If $G = G^0$, then one can easily see that there are more quadratic generators in $C[2V]^G$ than those coming from polarizations. If $G \neq G^0$, then $G$ is generated by $G^0$ and an element $\alpha$ such that $\alpha t = t^{-1}$ for $t \in G^0$ and $\alpha^2 \in G^0$. Now $\alpha^2$ is fixed under conjugation by $\alpha$ so that $\alpha^2 = \pm 1$. For an appropriate basis $v_1, \ldots, v_m$ of $V_1$ and $w_1, \ldots, w_m$ of $V_2$ we have that $\alpha(v_i) = w_i$ and $\alpha(w_i) = \pm v_i$, $i = 1, \ldots, m$. If $\alpha^2 = -1$, then one can see that quadratic invariants do not generate $C[V]^{G'}$. If $\alpha^2 = 1$, then we just have $m$-copies of the standard representation of $O_2$. Since $m \geq 2$, one easily sees that there are more generators of $C[V \oplus V]^{G}$ than polarizations.
Now suppose that $V$ contains two different pairs of weights. Then we can assume that $V = m_1(v_p \oplus v_{-p}) \oplus m_2(v_q \oplus v_{-q})$ as $\mathbb{C}^*$-module where $p$ and $q$ are relatively prime and $m_1, m_2 \geq 1$. There is then clearly a bihomogeneous minimal $G$-invariant of degree $(a, b)$ where $ab \geq 2$, so that we can again apply Lemma 3.1.

If $V$ is a $G$-module where $G^0$ is simple of rank 1 then we define $q(V)$ as before, relative to the action of a maximal torus.

**Corollary 3.5.** Suppose that $G^0$ is simple of rank 1 and that $V$ is a $G$-module with $q(V) \geq 3$. Then $V$ does not have the 2-polarization property.

**Proof.** By Corollary 3.3 we may assume that, as $G^0$-module, $V$ is the direct sum of copies of $R_j$, $j$ even. Let $v \in V$ be a nonzero zero weight vector. Then the $G$-orbit though $w$ is closed with isotropy group a finite extension of $\mathbb{C}^*$ and slice representation $V'$ where $q(V') \geq 2$. By Proposition 3.4, $(V', G_v)$ does not have the 2-polarization property, hence neither does $V$.

4. **The Main Theorem**

Recall that a $G$-module is called coregular if $\mathbb{C}[V]^G$ is a regular $\mathbb{C}$-algebra.

**Proposition 4.1.** Let $V$ be an irreducible representation of the simple algebraic group $G$. If $V$ is not coregular, then $V$ does not have the 2-polarization property.

**Proof.** The representations $R_j$ of SL$_2$ which are not coregular have $q(R_j) \geq 3$, hence they do not have the 2-polarization property by Corollary 3.3. By [Sch78, Remark 5.2] we know that if $V$ is not coregular and the rank of $G$ is at least 2, then one of the following occurs

1. There is a closed orbit $G \cdot v$ such that $G_v$ has rank 1. Let $G_v \to$ GL$(S)$ be the slice representation. Then $S$ is balanced. If $G^0_v$ is simple, then $q(S) \geq 4$ and if $G^0_v \cong \mathbb{C}^*$, then $q(S) \geq 2$.
2. $V = S^3(\mathbb{C}^*)$ and $G =$ SL$_4$.

By Proposition 3.4 and Corollary 3.5 any representation in (1) above does not have the 2-polarization property. Thus the only remaining case is (2). Here the laziest thing to do is to use the program LiE [vL94] to compute some low degree invariants of one or two copies of $V$. The first generator of $S^*(V^*)^G$ occurs in $S^8(V^*)$ and the dimension of the fixed space is 1. In $S^2(V^*) \otimes S^6(V^*)$ there is a two-dimensional space of invariants, so that $V$ does not have the 2-polarization property.

We would not have had to use LiE if the following could be established.

**Conjecture 4.2.** Let $H \subset$ GL$(V)$ where $H$ is finite and not generated by reflections. Then $V$ does not have the 2-polarization property.

In the following we use the notation of [Sch78] for the simple groups and their representations. We list such representations as pairs $(V, G)$.

**Theorem 4.3.** Let $V$ be an irreducible nontrivial representation of the simple algebraic group $G$. If $V$ has the 2-polarization property, then, up to (possibly outer) isomorphism, the pair $(V, G)$ is on the following list.

1. $(\varphi_1, A_n)$, $n \geq 2$.
2. $(\varphi_2, A_n)$, $n \geq 1$.
3. $(\varphi_3, A_n)$, $n \geq 4$. 

(4) \((\varphi_1, B_n), n \geq 2\) and \((\varphi_1, D_n), n \geq 3\).
(5) \((\varphi_1, G_2)\).
(6) \((\varphi_3, B_3)\).
(7) \((\varphi_1, E_6)\).

**Proof.** One can verify from [Sch78] that all the listed representations have the 2-polarization property. We must rule out all other cases. The list of coregular representations is due to KAC-POPOV-VINBERG [KPV76], see also [Sch78].

There are several easy ways to see that an irreducible coregular representation fails to have the 2-polarization property. One of the following can occur:

(i) \(V\) is a symplectic representation of \(G\).
(ii) \(\mathbb{C}[V]^G\) is minimally generated by homogeneous elements of degrees \(m_1, \ldots, m_k\) where \(\sum_i (m_i + 1) < \dim \mathbb{C}[2V]^G\) or \(\sum_i (m_i + 1) = \dim \mathbb{C}[2V]^G\) and \((2V,G)\) is not coregular.
(iii) \((V,G)\) is the adjoint representation of \(G\) where \(G\) has rank at least two.

Here there is a slice representation whose effective part is the adjoint representation of \(A_2, B_2\) or \(G_2\). Then we can apply (ii).

Examples of (i) are the representations \((\varphi_3, A_5)\), \((\varphi_3, C_3)\) and \((\varphi_5, B_5)\). The representations \((\varphi_3, A_4), n = 6, 7\) are examples of (ii) as is \((\varphi_1, F_4)\). Now we mention the remaining irreducible coregular representations that can’t be decided by the criteria above.

(a) \((\varphi_8, D_8)\). Here \(\mathbb{C}[V]^G\) has generators in degree 2, 8, \ldots while \(\mathbb{C}[2V]^G\) has three bihomogeneous invariants of degree \((2,2)\).
(b) \((\varphi_3, A_5)\). Here \(\mathbb{C}[V]^G\) has generators in degrees 12, 18, \ldots while \(\mathbb{C}[2V]^G\) has a bihomogeneous generator in degree \((3,3)\).
(c) \((\varphi_1^2, B_n), n \geq 2\) or \((\varphi_1^2, D_n), n \geq 3\). Up to a trivial factor, these are just the representations of the groups \(\text{SO}_n\) on \(S^2(\mathbb{C}^n)\), \(n \geq 5\). Let \(e_1, \ldots, e_n\) be the standard basis of \(\mathbb{C}^n\). The slice representation at the point \(e_1^4 + \cdots + e_{n-3}^2\) is, up to trivial factors, the sum \(S^2(\mathbb{C}^{n-3}) \oplus S^2(\mathbb{C}^{3})\), \(\text{SO}_3\) and the latter representation does not have the 2-polarization property by (ii).
(d) \((\varphi_2, C_n), n \geq 3\). In case that \(n = 3\), the representation fails to have the 2-polarization property by (ii). For \(n \geq 4\), the representations have a slice representation which contains a factor \((\varphi_2, C_3)\), similarly to case (c).

The remaining representations \((\varphi_4, A_7)\) and \((\varphi_4, C_4)\) are handled as in (a) and (b).

One can now use the tables of [Sch78] to see which irreducible representations have the \(k\)-polarization property for \(k \geq 3\).

**Corollary 4.4.** Let \(G\) be simple and \(V\) irreducible with the \(k\)-polarization property, \(k \geq 3\). Then \((V,G)\) and \(k\) are on the following list.

(1) \((\varphi_1, A_n), n \geq 2\) and \(3 \leq k < n + 1\).
(2) \((\varphi_1, B_n), n \geq 2\) and \(3 \leq k < 2n + 1\).
(3) \((\varphi_1, D_n), n \geq 3\) and \(3 \leq k < 2n\).
(4) \((\varphi_3, B_3)\) and \(k = 3\).

It is also easy to determine the reducible representations with the \(k\)-polarization property for \(k \geq 2\). One uses Lemma 3.1 the criterion (ii) above and the tables of [Sch78].
Corollary 4.5. Let $V$ be the direct sum of at least two irreducible nontrivial $G$-modules, where $G$ is simple. If $V$ has the $k$-polarization property, $k \geq 2$, then, up to isomorphism, $G = \text{SL}_n$, $n \geq 5$ and $V = jC^n$ where $2 \leq j < n/k$.

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