HOMOTOPY GROUPS OF SPHERES AND DIMENSION QUOTIENTS

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To the memory of John R. Stallings, 1935–2008

Abstract. We construct for every prime \( p \) a finitely presented group \( G \) in which the dimension quotient \( (G \cap (1 + \omega^n))/\gamma_n(G) \) has \( p \)-torsion for some \( n \). The construction uses Serre’s element of order \( p \) in the homotopy group \( \pi_{2p}(S^2) \), and derives a result analogous to the main claim in the context of Lie algebras.

1. Introduction

This paper solves the classical dimension subgroup problem with the help of homotopy groups of spheres: a deep topological phenomenon is translated to algebraic form and used in pure group theory.

Consider a group \( G \) naturally embedded in its integral group ring \( \mathbb{Z}G \). For the natural filtrations \((\gamma_n(G))\) of \( G \) by its lower central series and \((\omega^n)\) of \( \mathbb{Z}G \) by powers of its augmentation ideal, we have an induced map \( G/\gamma_n(G) \to \mathbb{Z}G/\omega^n \), whose injectivity is known as the dimension problem for \( G \).

Set \( \delta_n(G) := G \cap (1 + \omega^n) = \ker(G \to \mathbb{Z}G/\omega^n) \), the \( n \)th dimension subgroup; then \( \gamma_n(G) \leq \delta_n(G) \), and the dimension problem asks to understand the dimension quotient \( \delta_n(G)/\gamma_n(G) \). If it is trivial for all \( n \), one says that \( G \) has the dimension property.

The quotients \( \delta_n(G)/\gamma_n(G) \) are abelian, and Jan Sjogren proved in [31] that they have finite exponent, bounded by a function of \( n \) only: there exists an explicit \( s(n) \in \mathbb{N} \) (roughly \((n!)^n\)) with \( \delta_n(G) s(n) \subseteq \gamma_n(G) \) for all groups \( G \). Inder Bir Passi [24] gave \( s(4) = 2 \).

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This can be improved in the case of metabelian groups: then Narain Gupta proved in [11] that \( s(n) \) may be chosen to be a power of 2, so \( \gamma_n = \delta_n \) if \( G \) is a metabelian \( p \)-group.

He claimed that the same bound holds for all groups (a proof is published in [13]) and even that it may be improved to \( s(n) = 2 \), see [12]. However his arguments contain many unintelligible parts, and we shall see that his claims may not hold.

Our main result is the construction, for every prime \( p \), of a group \( G \) and an index \( n \) such that \( \delta_n(G)/\gamma_n(G) \) contains \( p \)-torsion; see Theorem 1.1 below. Thus Sjogren’s function cannot be bounded, or even constrained to a finite collection of primes.

An analogous question may be asked for Lie rings; namely, Lie algebras over \( \mathbb{Z} \). Every Lie ring \( A \) embeds in its universal enveloping algebra \( U(A) \), which also admits an augmentation ideal. The dimension subrings are defined analogously by \( \delta_n(A) = A \cap \mathbb{Z}^n \), see [1]. Again \( \delta_n(A) = \gamma_n(A) \) when \( n \leq 3 \), and there is a Lie ring \( A \) with \( \delta_n(A)/\gamma_n(A) = \mathbb{Z}/2 \). Sjogren’s bound also holds for Lie rings [30], and many details are simpler in the category of Lie rings.

Even though we are not aware of any direct construction of a group from a Lie ring or vice versa that preserves dimension quotients, it often happens that a presentation involving only powers and commutators, which may therefore be interpreted either as group or Lie algebra presentation, yields isomorphic dimension quotients.

To give a quick taste of dimension quotients in Lie rings, we reproduce first an example due to Pierre Cartier of a Lie algebra over a commutative ring \( k \) not embedding in its universal envelope [4]: consider \( k = \mathbb{F}_2[x_0, x_1, x_2]/(x_0^2, x_1^2, x_2^2) \), and

\[
A = \langle e_0, e_1, e_2 \mid x_0e_0 + x_1e_1 + x_2e_2 = 0 \rangle \text{ qua } k \text{-Lie algebra.}
\]

Then \( \alpha := x_0x_1[e_0, e_1] + x_0x_2[e_0, e_2] + x_1x_2[e_1, e_2] \) is non-trivial in \( A \), but in any associative algebra it maps to \( (x_0e_0 + x_1e_1 + x_2e_2)^2 = 0 \).

Rips’ example, or rather its Lie algebra variant [1] Theorem 4.7, is of a similar spirit. In \( k = \mathbb{Z} \) one can of course not choose \( x_i \) nilpotent; but one may choose \( x_i \) a large power of 2 and impose relations that guarantee that elements of large 2-valuation are mapped far in the lower central series: set \( x_i = 2^{2i+1} \) and consider

\[
A = \langle e_0, e_1, e_2, \cdots \mid 2^{2i+2}e_i \in \gamma_2 \text{ for all } i \in \{0, 1, 2\}, \]
\[
x_1x_k e_i + x_1x_{k'} e_j \in 2^{2k+2}\gamma_2 + \gamma_3 \text{ for all } \{i, j, k\} = \{0, 1, 2\} \rangle
\]

with the element \( \alpha = \sum_{0 \leq i < j \leq 2} x_i x_j [e_i, e_j] \). Then the relations imply \( \alpha \in A \cap (\gamma_2(A) \cdot \gamma_2(A) + A \cdot \gamma_3(A)) \subseteq \delta_4(A) \), while it is easy to make choices of elements in \( \gamma_2 \) and \( 2^{2+2k}\gamma_2 + \gamma_3 \) that yield, by direct computation, that \( \alpha \) has a non-trivial image in the quotient \( A/\gamma_4(A) \).

It is even possible to write a 3-related Lie algebra, based on [21] Example 2.3, that satisfies (1) and \( \alpha \in \delta_4(A) \setminus \gamma_4(A) \): \( A = \langle e_0, e_1, e_2, z \mid 2^4e_0 = [z, e_1 + 2e_2], 2^4e_1 = [z, -e_0 + 4e_2], 2^6e_2 = [z, -2e_0 - 4e_1] \rangle \) with as before \( \alpha = 2^3[e_0, e_1] + 2^6[e_0, e_2] + 2^7[e_1, e_2] \),

This Lie algebra presentation may also be interpreted as a group presentation,

\[
G = \langle e_0, e_1, e_2, z \mid e_0^4 = [z, e_1][z, e_2]^2, e_1^6 = [z, e_0]^{-1}[z, e_2]^4, e_2^6 = [z, e_0]^{-2}[z, e_1]^{-4}, \]

in which the element \( \alpha = [e_0, e_1][e_0, e_2][e_1, e_2]^{128} \) belongs to \( \delta_4(G) \setminus \gamma_4(G) \).
1.1. Main statement and sketch of proof. We write iterated commutators as left-normed: \([x_1, x_2, \ldots, x_d] = \cdots [x_1, x_2] \cdots [x_1, x_2, \ldots, x_d].\) In a group or Lie algebra presentation, we introduce the following notation: for \(d \in \mathbb{N},\) when we write a generator \(x_i^{(d)}\) of degree \(d\) we mean a shorthand for the iterated commutator \(x_i := [x_i, \ldots, x_i, d]\) of usual generators \(x_{i,1}, \ldots, x_{i,d}\).

**Theorem 1.1.** Consider a prime \(p,\) set \(\ell = (2p)(2p-1)/2,\) and define the following group \(G:\)
\[
\langle x_0^{(\ell)}, \ldots, x_{2p-1}, y_0^{(\ell+1)}, \ldots, y_{2p-1}^{(\ell+2p)} \mid x_0 \cdots x_{2p-1} = 1, x_i^{p^i} = y_i \text{ for } i = 0, \ldots, 2p-1 \rangle.
\]
Then there is an element of order \(p\) in \(\delta_{2p(\ell+1)+\ell}(G)/\gamma_{2p(\ell+1)+\ell}(G).\)

**Theorem 1.2.** Consider a prime \(p,\) and define the following Lie algebra \(A\) over \(\mathbb{Z}:\)
\[
\langle x_0, \ldots, x_{2p-1}, y_0^{(2)}, \ldots, y_{2p-1}^{(2p+1)} \mid x_0 + \cdots + x_{2p-1} = 0, p^i x_i = y_i \text{ for } i = 0, \ldots, 2p-1 \rangle.
\]
Set \(\ell = (2p)(2p-1)/2;\) then there is an element of order \(p\) in \(\delta_{3p+\ell}(A)/\gamma_{3p+\ell}(A).\)

The ingredients of the proof are as follows. The homotopy groups of spheres \(\pi_{2p}(S^2)\) contain an element of \(p\)-torsion due to Jean-Pierre Serre. On the other hand, \(\pi_{2p}(S^2)\) may be expressed as a quotient of normal subgroups in a free group \(F = \langle x_0, \ldots, x_{2p-1} \mid x_0 \cdots x_{2p-1} = 1 \rangle,\) following Wu. We write the \(p\)-torsion element as a free group element, and derive some of its properties. In particular, it is a product \(\alpha_p\) of commutators of weight \(2p,\) it does not belong to the symmetric commutator of the normal subgroups \(\langle x_0 \rangle^F, \ldots, \langle x_{2p-1} \rangle^F,\) but \(\alpha_p - 1\) belongs to the symmetric product of the ideals \((x_0 - 1)^F, \ldots, (x_{2p-1} - 1)^F\). The relations \(x_i^{p^i} = y_i\) allow the \(p^i\)th power of \(\alpha_p\) to be “pushed down” the dimension and lower central series; but only symmetric products/commutators (namely, those involving each \(\langle x_i \rangle^F\) a single time) may be pushed down maximally. Thus the maximal depth achievable by \(\alpha_p\) in the lower central series is strictly less that the depth achievable by \(\alpha_p - 1\) in the group ring.

It is natural to ask whether we need an explicit form of the Serre element written in terms of a free group or a free Lie ring via Wu’s. For the proof of our main results we do not need an explicit form, but we do need to infer numerous properties, such as the structure of commutators that appear in the expression, its length, number of repetitions of generators, etc. We make use of an explicit form for \(p = 2\) and \(p = 3\) to obtain smaller examples, in particular for \(p = 2\) we obtain straightforward constructions, for arbitrary \(n \geq 4,\) of Lie algebras in which \(\delta_n/\gamma_n\) contains 2-torsion, and for \(p = 3\) we obtain a Lie algebra and a group in which \(\delta_7/\gamma_7\) contains 3-torsion. These examples have been checked using computer algebra programs.

Stallings already recognized the value of homological arguments (in particular the Curtis spectral sequence) towards studying dimension quotients, in a programme carried out by Sjogren. To the extend of our knowledge, however, Theorems 1.1 and 1.2 are the first instances of the solution of an open problem in abstract algebra by using classical homotopy theory, to wit the homotopy groups of spheres.

2. Homotopy groups of spheres

We give in this section an explicit generator of the \(p\)-torsion in \(\pi_{2p}(S^2)\) discovered by Serre, using the group-theoretic formulation of this homotopy group. It will be used for the construction of the required element in Theorems 1.2 and 1.1.
2.1. Groups. Fix an integer \( n \geq 1 \) and let \( F = \langle x_0, \ldots, x_n \mid x_0 \cdots x_n \rangle \) be a free group of rank \( n \). Consider its normal subgroups
\[
R_i := \langle x_i \rangle^F \text{ for } i = 0, \ldots, n.
\]
Note that \( F \) is the fundamental group of a 2-sphere with \( n + 1 \) punctures, and \( R_i \) contains the conjugacy class of a loop around the \( i \)th puncture; the operation of filling-in the \( i \)th puncture induces the map \( F \to F/R_i \) on fundamental groups.

Denote by \( \Sigma_{n+1} \) the symmetric group on \( \{0, \ldots, n\} \), and define the symmetric commutator product of the above subgroups by
\[
[R_0, \ldots, R_n]_{\Sigma} := \prod_{\rho \in \Sigma_{n+1}} [R_{\rho(0)}, \ldots, R_{\rho(n)}].
\]
Here and below the iterated commutators are assumed to be left-normalized, namely
\[
[R_0, R_1, R_2] = [[R_0, R_1], R_2] \text{ etc.}
\]
We view the circle \( S^1 \) as a simplicial set. Milnor’s \( F \) construction produces a group complex, having in degree \( n \) a free group on the degree-\( n \) objects of \( S^1 \) subject to a single relation \( (s_0^n(+) = 1) \) and the same boundaries and degeneracies as \( S^1 \).

According to a formula due to Jie Wu [7, 35], considered in the standard basis of Milnor’s \( F[S^1] \)-construction, homotopy groups of the sphere \( S^2 \) can be presented in the following way:
\[
\pi_{n+1}(S^2) \simeq R_0 \cap \cdots \cap R_n \frac{[R_0, \ldots, R_n]}{[R_0, \ldots, R_n]_{\Sigma}}.
\]

Consider now for \( i = 0, \ldots, n \) the ideals \( r_i := (R_i - 1)\mathbb{Z}[F] \) in the free group ring \( \mathbb{Z}[F] \), and their symmetric product
\[
(r_0, \ldots, r_n)_{\Sigma} := \sum_{\rho \in \Sigma_{n+1}} r_{\rho(0)} \cdots r_{\rho(n)}
\]
which is also an ideal in \( \mathbb{Z}[F] \).

**Proposition 2.1.** For \( n \geq 3 \) we have \( R_0 \cap \cdots \cap R_n = F \cap (1 + (r_0, \ldots, r_n)_{\Sigma}) \) when considered in \( \mathbb{Z}[F] \).

**Proof.** It is shown in [21] that the quotient \( \frac{R_0 \cap \cdots \cap R_n}{[R_0, \ldots, R_n]_{\Sigma}} \) can be viewed as the \( n \)th homotopy group of the simplicial abelian group \( \mathbb{Z}[F[S^1]] \), and the map \( F \to \mathbb{Z}[F] \) given by \( f \mapsto f - 1 \) induces the following commutative diagram
\[
\begin{array}{ccc}
R_0 \cap \cdots \cap R_n & \longrightarrow & R_0 \cap \cdots \cap R_n \\
[R_0, \ldots, R_n]_{\Sigma} & \longrightarrow & (r_0, \ldots, r_n)_{\Sigma} \\
\pi_{n+1}(S^2) & \longrightarrow & H_n(\Omega S^2).
\end{array}
\]
The lower map is the \( n \)th Hurewicz homomorphism for the loop space \( \Omega S^2 \). Since all homotopy groups \( \pi_n(\Omega S^2) \) are finite for \( n \geq 3 \), but all homology groups \( H_n(\Omega S^2) \) are infinite cyclic (\( H_1(\Omega S^2) \) is the tensor algebra generated by the homology of \( S^2 \) in dimension one), we conclude that, for \( n \geq 3 \), the map in the above diagram is zero. \( \square \)
2.2. Lie algebras. One obtains an analogous picture in the case of Lie algebras over \( \mathbb{Z} \). The homotopy groups of the simplicial Lie algebra

\[
L[S^1] := \bigoplus_i \gamma_i(F[S^1])/\gamma_{i+1}(F[S^1])
\]

are equal to the direct sum of terms in rows of the \( E^1 \)-term of the Curtis spectral sequence

\[
E^1_{i,j} := \pi_j(\gamma_i(F[S^1])/\gamma_{i+1}(F[S^1])) \Rightarrow \pi_{j+1}(S^2).
\]

The mod-\( p \)-lower central series spectral sequence is well-studied, see for example, the foundational paper [3]. The integral case which we consider here has similar properties, see [2, 14]. Here we will only need elementary properties of this spectral sequence and will consider essentially the (pre)image of the Serre elements.

Observe that the \( E^1 \)-page of the above spectral sequence consists of derived functors \( \mathbb{L}_j \) in the sense of Dold-Puppe, applied to Lie functors: if \( \mathcal{L}^i \) denotes the \( i \)th Lie functor in the category of abelian groups, then

\[
\pi_j(\gamma_i(F[S^1])/\gamma_{i+1}(F[S^1])) = \mathbb{L}_j \mathcal{L}^i(\mathbb{Z}, 1).
\]

Proposition 2.2. For \( n \geq 3 \) we have \( I_0 \cap \cdots \cap I_n = L \cap (I_0, \ldots, I_n)_\Sigma \) when considered in the universal enveloping algebra.
Proof. Similarly to the group case, the natural map \( L \to U(L) \) induces
\[
\begin{array}{ccc}
I_0 \cap \cdots \cap I_n & \longrightarrow & x_0 U(L) \cap \cdots \cap x_n U(L) \\
[\{I_0, \ldots, I_n\}] & \longrightarrow & (x_0 U(L), \ldots, x_n U(L)) \Sigma \\
\bigoplus_{i \geq 1} E_{i,n}^1 & \longrightarrow & H_n(U(L[S^1])).
\end{array}
\]
By [27] the \( E_{i,j}^1 \)-terms of the lower central series spectral sequence for \( S^2 \) are finite for all \( j \geq 3 \), while the universal enveloping simplicial algebra \( U(L[S^1]) \) has infinite cyclic homology groups in all dimensions. It follows that the map is 0. \( \square \)

2.3. Homotopy groups of \( S^2 \). Let \( p \) be a prime. In this subsection we describe explicitly a copy of \( \mathbb{Z}/p \) in \( \pi_{2p}(S^2) \) due to Serre [29], by computing its (pre)image \( \alpha_p \) in the \( E^1 \)-term of the lower central spectral sequence associated to \( F[S^1] \). There is a single \( (\mathbb{Z}/p) \)-term in dimension \( 2p-1 \) of the spectral sequence
\[
p\text{-torsion } \left( \frac{I_0 \cap \cdots \cap I_{2p-1}}{[I_0, \ldots, I_{2p-1}]} \right) = \mathbb{L}_{2p-1}^{2p}(\mathbb{Z}, 1) = \mathbb{Z}/p,
\]
and \( \alpha_p \) will be a generator of this subgroup.

**Theorem 2.3.** Let \( x_i \) for \( i = 0, \ldots, 2p-2 \) be free generators of a free Lie algebra, and consider the following element
\[
\alpha_p = \sum_{\rho \in \Sigma_{2p-2} \text{ a } 2^{p-1}\text{-shuffle}} (-1)^p [x_{\rho(0)}, x_{2p-2}], [x_{\rho(1)}, x_{2p-2}], [x_{\rho(2)}, x_{\rho(3)}], \ldots, [x_{\rho(2p-4)}, x_{\rho(2p-3)}];
\]
the sum is taken over all permutations \( (\rho(0), \ldots, \rho(2p-3)) \in \Sigma_{2p-2} \) satisfying \( \rho(0) < \rho(1), \ldots, \rho(2p-4) < \rho(2p-3) \) as well as \( \rho(1) < \rho(3) < \cdots < \rho(2p-5) \). Then \( \alpha_p \) represents a generator of the \( p \)-torsion in \( \mathbb{L}_{2p-1}^{2p}(\mathbb{Z}, 1) \).

**Proof.** Consider the free abelian simplicial group \( K(\mathbb{Z}, 2) \): it has a single generator \( \sigma \) in degree 2, and its other generators may be chosen to be all iterated degeneracies of \( \sigma \). We will use the dual notation for generators: for \( k > 2 \) the free abelian group \( K(\mathbb{Z}, 2)_k \) is generated by ordered sequences of two elements
\[
(i_1, i_2) := s_{k-1} \cdots s_{i_2} \cdots s_{i_1} \cdots s_0(\sigma)
\]
with \( 0 \leq i_1 < i_2 < k \). For example, \( K(\mathbb{Z}, 2)_5 \) has generators
\[
\begin{align*}
(01) & := s_4 s_3 s_2(\sigma), (02) := s_4 s_3 s_1(\sigma), (03) := s_4 s_2 s_1(\sigma), (04) := s_3 s_2 s_1(\sigma), \\
(12) & := s_4 s_3 s_0(\sigma), (13) := s_4 s_2 s_0(\sigma), (14) := s_3 s_2 s_0(\sigma), (23) := s_4 s_1 s_0(\sigma), \\
(24) & := s_3 s_1 s_0(\sigma), (34) := s_2 s_1 s_0(\sigma).
\end{align*}
\]
For \( n \geq 1 \), define the functor \( J^n \) as the metabelianization of the \( n \)th Lie functor \( \mathbb{L}^n \). For a group \( A \), there is a natural epimorphism
\[
\mathbb{L}^p(A) \to J^p(A)
\]
with kernel generated by Lie brackets of the form \([*, *], [*, *])\). The elements of \( J^p \) can also be written as linear combinations of Lie brackets, namely as elements of
the Lie functor $\mathcal{L}^p$, but there is additional rule which holds in $J^p$ but not hold in $\mathcal{L}^p$ in general:

\[ [a_1, a_2, \ldots, a_p] = [a_1, a_2, a_{\rho(3)}, \ldots, a_{\rho(p)}] \]

for arbitrary $a_i$ and permutation $(\rho(3), \ldots, \rho(p))$ of $\{3, \ldots, p\}$. For $p = 3$, the functors $\mathcal{L}^3$ and $J^3$ are equal.

For $n \geq 1$, denote by $S^n$ the $n$th symmetric power functor

\[ S^n : \text{Abelian groups} \to \text{Abelian groups}. \]

For a free abelian group $A$, there is a natural short exact sequence \[27\] Proposition 3.2)

\[ 0 \to J^n(A) \to S^{n-1}(A) \otimes A \to S^n(A) \to 0, \]

where the left-hand map is given by

\[ [b_1, \ldots, b_n] \mapsto b_2 b_3 \ldots b_n \otimes b_1 - b_1 b_3 \ldots b_n \otimes b_2 \text{ for } b_i \in A. \]

Applying the functors $J^p \hookrightarrow S^{p-1} \otimes id \to S^p$ to the simplicial abelian group $K(\mathbb{Z}, 2n)$, and taking the homotopy groups, we get the long exact sequence

\[ \pi_{2pn}(S^{p-1}K(\mathbb{Z}, 2n) \otimes K(\mathbb{Z}, 2n)) \to \mathbb{L}_{2pn}S^p(\mathbb{Z}, 2n) \to \mathbb{L}_{2pn-1}J^p(\mathbb{Z}, 2n) \to \pi_{2pn-1}(S^{p-1}K(\mathbb{Z}, 2n) \otimes K(\mathbb{Z}, 2n)). \]

It follows from \[6\] p. 307] that the above sequence has the following form:

\[ \pi_{2pn}(S^{p-1}K(\mathbb{Z}, 2n) \otimes K(\mathbb{Z}, 2n)) \quad \mathbb{Z} \quad \mathbb{L}_{2pn}S^p(\mathbb{Z}, 2n) \quad \mathbb{L}_{2pn-1}J^p(\mathbb{Z}, 2n) \quad \mathbb{Z}/p. \]

By \[27\] Proposition 4.7, the natural epimorphism $\mathcal{L}^p \to J^p$ gives a natural isomorphism of derived functors

\[ \mathbb{L}_{2pn-1}\mathcal{L}^p(\mathbb{Z}, 2n) \xrightarrow{\sim} \mathbb{L}_{2pn-1}J^p(\mathbb{Z}, 2n) \xrightarrow{\sim} \mathbb{Z}/p. \]

Let us first find a simplicial generator of $\mathbb{L}_{2p}S^p(\mathbb{Z}, 2)$. For this, we observe that the inclusion of the symmetric power into the tensor power $S^p \hookrightarrow \otimes^p$ induces an isomorphism of derived functors

\[ \mathbb{L}_{2p}S^p(\mathbb{Z}, 2) \to \mathbb{L}_{2p}\otimes^p(\mathbb{Z}, 2). \]

A simplicial generator of $\mathbb{L}_{2p}\otimes^p(\mathbb{Z}, 2)$ can be given by the Eilenberg-Zilber shuffle-product theorem. Using interchangeably the notation $\rho(i)$ and $\rho_i$, this is the element

\[ \sum_{\rho \in \Sigma_{2p} \text{ a } 2^p\text{-shuffle}} (-1)^\rho(\rho_0 \rho_1) \otimes (\rho_2 \rho_3) \otimes \cdots \otimes (\rho_{2p-2} \rho_{2p-1}). \]

It follows immediately from the definition of $2^p$-shuffles that the symmetric group $\Sigma_p$, acting by permutation on blocks $\{2i, 2i + 1\}$ of size 2, acts on $2^p$-shuffles. A generator of $\mathbb{L}_{2p}S^p(\mathbb{Z}, 2)$ can be chosen by keeping only a single element per $\Sigma_p$-orbit, and replacing tensor products by symmetric products:

\[ \beta := \sum_{\rho \in \Sigma_{2p} \text{ a } 2^p\text{-shuffle}} (-1)^\rho(\rho_0 \rho_1) \cdot (\rho_2 \rho_3) \cdots (\rho_{2p-2} \rho_{2p-1}). \]
The conditions imply $\rho(2p-1)=2p-1$. For example, for $p=3$ we get the element
\[(0 1)(2 3)(4 5) - (0 1)(2 4)(3 5) + (0 1)(3 4)(2 5) - (0 2)(3 4)(1 5) - (0 2)(1 3)(4 5)
+ (0 3)(2 4)(1 5) - (0 3)(1 4)(2 5) + (1 2)(0 3)(4 5) - (2 3)(0 4)(1 5) - (1 2)(0 4)(3 5)
+ (1 2)(3 4)(0 5) - (1 3)(2 4)(0 5) + (2 3)(1 4)(0 5) + (0 2)(1 4)(3 5) + (1 3)(0 4)(2 5).
\]
Now we lift the element from $S^pK(Z,2)_{2p}$ to $(S^{p-1}K(Z,2) \otimes K(Z,2))_{2p}$ in a standard way:
\[
\tilde{\beta} := \sum_{\rho \in \Sigma_{2p} \text{ a } 2^p\text{-shuffle}} (-1)^\rho (\rho_0 \rho_1 \cdots (\rho_{2p-4} \rho_{2p-3}) \otimes (\rho_{2p-2} \rho_{2p-1}).
\]
Observe that we have
\[
d_j(i_1 i_2) = \begin{cases} (i_1 \ i_2) & \text{if } i_2 < j, \\ (i_1 \ i_2 - 1) & \text{if } i_1 < j \leq i_2, \\ (i_1 - 1 \ i_2 - 1) & \text{if } j \leq i_1 \end{cases}
\]
with the understanding that $(i \ i) = 0$, that we use the same notation $(i_1 i_2)$ for elements of varying degree, and that $d_0(0 i_2) = 0$ and $d_1(i_1 i_2) = 0$ if $\deg(i_1 i_2) = j = i_2 - 1$. Thus e.g. $d_0(0 4) = d_2(0 4) = 0$ and $d_1(0 4) = d_2(0 4) = d_4(0 4) = (0 3)$ while $d_0(2 3) = d_1(2 3) = d_2(2 3) = (1 2)$ and $d_3(2 3) = 0$ and $d_4(2 3) = d_5(2 3) = (2 3)$.

Clearly $d_0(\tilde{\beta}) = d_{2p-1}(\tilde{\beta}) = 0$. If $j < 2p-2$, then we express $\tilde{\beta}$ as a sum over all possible values of $r := \rho(2p-2)$ (remembering $\rho(2p-1) = 2p-1$) and obtain
\[
d_j(\tilde{\beta}) = \sum_{r=0}^{2p-2} (-1)^r (d_j(\cdots) \otimes (r 2p-1) + (\cdots) \otimes d_j(r 2p-1)).
\]
Now the sum in $(\cdots)$ is a symmetric product similar to $\beta$, but with $p-1$ instead of $p$ factors, so $(\cdots)$ is exact. The second terms telescope, so we get $d_j(\tilde{\beta}) = 0$ when $j < 2p-2$. However, $\tilde{\beta}$ is not a cycle in $S^{p-1}(Z,2) \otimes K(Z,2)$, because $d_{2p-2}(\tilde{\beta})$ is not zero: we compute
\[
d_{2p-2}(\tilde{\beta}) = \sum_{\rho \in \Sigma_{2p} \text{ a } 2^p\text{-shuffle}} (-1)^\rho (\rho_0 \rho_1 \cdots (\rho_{2p-4} \rho_{2p-3}) \otimes (\rho_{2p-2} 2p-2).
\]

We use the long exact sequence associated with (2) to obtain a cycle in $J^p(Z,2)_{2p-1}$. The ascending $2^p$-shuffles $(\rho(0), \ldots, \rho(2p-1))$ appearing in the sum can in fact be viewed as $2^{p-1}$-shuffles $(\rho(0), \rho(1), \ldots, \rho(2p-5), \rho(2p-4, \rho(2p-2))$ or $(\rho(0), \rho(1), \ldots, \rho(2p-6), \rho(2p-5), \rho(2p-2), \rho(2p-4))$, depending on whether $\rho(2p-2) < \rho(2p-4)$ or not, and in all cases completed by the values $(2p-2, 2p-1)$. Furthermore, these two shuffles come with opposite signs, and can be combined, via (4), into
\[
\sum_{\rho \in \Sigma_{2p-2} \text{ a } 2^{p-1}\text{-shuffle}} (-1)^\rho [(\rho_{2p-3} 2p-2), (\rho_{2p-4} 2p-2), (\rho_0 \rho_1), \ldots, (\rho_{2p-6} \rho_{2p-5})].
\]
We now consider the simplicial map $K(Z,2) \to L^2 K(Z,1)$, given by $\sigma \mapsto [s_0(\sigma'), s_1(\sigma')]$, where $\sigma'$ is the generator of $K(Z,1)_1$; it is a homotopy equivalence of complexes.
The abelian group $K(\mathbb{Z}, 1)_k$ is $k$-dimensional, with generators
\[
x_i := s_k \cdots \hat{s}_i \cdots s_0(\sigma')
\]
for all $0 \leq i < k$, and we have $(i_1 i_2) \mapsto [x_{i_1}, x_{i_2}]$ under this homotopy equivalence.
Thus $\mathcal{L}^k(\mathbb{Z}, 2)_{2p-1}$ is a free Lie algebra on $2p - 1$ generators. There is an induced map
\[
\mathcal{L}^p K(\mathbb{Z}, 2) \to \mathcal{L}^p \circ \mathcal{L}^2 K(\mathbb{Z}, 1) \to \mathcal{L}^{2p} K(\mathbb{Z}, 1)
\]
which also is a homotopy equivalence of complexes. The image of the element (4) is
\[
\sum_{\rho \in \Sigma_{2p-2} \text{ a } 2^{p-1}\text{-shuffle}} \rho(1) \rho(2) \cdots \rho(2p-3), x_{2p-2}, x_{2p-4}, x_{2p-2}, [x_p(0), x_p(1)], \ldots, [x_p(2p-6), x_p(2p-5)].
\]
Up to sign and renumbering, this is exactly our element $\alpha_p$. \qed

Note that we considered, in the beginning of this section, a free Lie algebra of rank $2p - 1$ with $2p$ generators $x_0, \ldots, x_{2p-1}$ subject to the relation $x_0 = 0$. Any choice of $2p - 1$ out of these $2p$ generators yields a free Lie algebra on $2p - 1$ generators, and an expression $\alpha_p$. The point being made is that every such expression involves one of the generators (here $x_{2p-2}$) twice, and omits another (here $x_{2p-1}$).

We summarize as follows the properties of the element $\alpha_p$ that will be useful to us:

**Proposition 2.4.** For every prime $p$ there is an element $\tilde{\alpha}_p$ in the free group $\langle x_0, \ldots, x_{2p-1} | x_0 \cdots x_{2p-1} = 1 \rangle$ with the properties:

- $\tilde{\alpha}_p - 1 \in (t_0, \ldots, t_{2p-1})_\Sigma$;
- $\tilde{\alpha}_p \notin [R_0, \ldots, R_{2p-1}]_\Sigma$;
- $\tilde{\alpha}_p \in [R_0, \ldots, R_{2p-1}]_\Sigma$.

Furthermore, $\tilde{\alpha}_p - 1 \in ([t_0, t_1], \ldots, [t_{2p-2}, t_{2p-1}])_\Sigma$, namely in the sum of all $p$-fold associative products of brackets of $t_i$ in any of the $(2p)!$ orderings.

**Proof.** The first claim follows from Proposition 2.1 since $\alpha_p$ represents an element of $\pi_{2p}(S^2)$. The second claim holds because this element is non-trivial in $\pi_{2p}(S^2)$. The third claim holds because it has order $p$ in $\pi_{2p}(S^2)$. The last claim follows from general facts: $\mathbb{L}_n \mathcal{L}^n(\mathbb{Z}, 1) = 0$ for odd $n$, and $\mathbb{L}_n \mathcal{L}^{2n}(\mathbb{Z}, 1) = \mathbb{L}_n \mathcal{L}^n(\mathbb{Z}, 2)$. \qed

The same statement holds for Lie algebras; we omit the proof.

**Proposition 2.5.** For every prime $p$ there is an element $\alpha_p$ in the free Lie algebra $\langle x_0, \ldots, x_{2p-1} | x_0 + \cdots + x_{2p-1} = 0 \rangle$ with the properties:

- $\alpha_p \in (I_0, \ldots, I_{2p-1})_\Sigma$;
- $\alpha_p \notin [I_0, \ldots, I_{2p-1}]_\Sigma$;
- $p \alpha_p \in [I_0, \ldots, I_{2p-1}]_\Sigma$.

Furthermore, $\alpha_p \in ([R_0, R_1], \ldots, [R_{2p-2}, R_{2p-1}])_\Sigma$. \qed

**Example 2.6.** Here is an explicit generator of $\pi_4(S^2) = \mathbb{Z}/2$. If we consider $p = 2$ in Theorem 2.3 we have only one $[2]$-shuffle and the element $\alpha_2$ is $[[x_0, x_2], [x_1, x_2]]$. Reintroducing $x_3 = -x_0 - x_1 - x_2$, we can easily check that $\alpha_2 \in (I_0, I_1, I_2, I_3)_\Sigma$:

\[
\alpha_2 := [[x_0, x_2], [x_1, x_2]] = [x_2, x_3] \cdot [x_0, x_1] + [x_1, x_2] \cdot [x_0, x_3] - [x_0, x_2] \cdot [x_1, x_3] .
\]
Applying to it the Dynkin idempotent \( w \cdot v \mapsto \frac{1}{2}[w', v] \) gives then \( 2\alpha_2 \in [I_0, I_1, I_2, I_3]\).

It is only slightly harder to write a generator of \( \pi_4(S^2) \) in the language of groups. We may lift \( \alpha_2 \) to \( \tilde{\alpha}_2 \in F \), the free group \( \langle x_0, x_1, x_2, x_3 \mid x_0 \cdots x_3 \rangle \), as

\[
\tilde{\alpha}_2 = [[x_0, x_2], [x_0 x_1, x_2]],
\]

since then the Hall-Witt identities give \( \tilde{\alpha}_2 = [[x_0, x_2], [x_3^{-1}, x_2]^{-1}] = [[x_0, x_2], [x_0, x_2]^{x_1} [x_1, x_2]] = [[x_0, x_2], [x_1, x_2]] \cdot [[x_0, x_2], [x_0, x_2]^{x_2}] \) so \( \tilde{\alpha}_2 \in R_0 \cap \cdots \cap R_3 \). We have thus produced a non-trivial cycle \( \tilde{\alpha}_2 \in (R_0 \cap R_1 \cap R_2 \cap R_3)/(R_0, R_1, R_2, R_3)^\Sigma \).

**Example 2.7.** Here is a generator of the 3-torsion in \( \pi_6(S^2) \). For \( p = 3 \), we have six \([2, 2]\)-shuffles in Theorem 2.3

\[
(0, 1, 2, 3) \text{ with sign } = 1, \quad (0, 2, 1, 3) \text{ with sign } = -1, \\
(0, 3, 1, 2) \text{ with sign } = 1, \quad (2, 3, 0, 1) \text{ with sign } = 1, \\
(1, 3, 0, 2) \text{ with sign } = -1, \quad (1, 2, 0, 3) \text{ with sign } = 1.
\]

The element \( \alpha_3 \) representing 3-torsion in \( \pi_6(S^2) \) is

\[
\alpha_3 := [[x_0, x_4], [x_1, x_4], [x_2, x_3]] - [[x_0, x_4], [x_2, x_4], [x_1, x_3]] \\
+ [[x_0, x_4], [x_3, x_4], [x_1, x_2]] + [[x_1, x_4], [x_2, x_4], [x_0, x_3]] \\
- [[x_1, x_4], [x_3, x_4], [x_0, x_2]] + [[x_2, x_4], [x_3, x_4], [x_0, x_1]].
\]

It may be expressed as a sum of 30 associative products of the form \( \pm [x_a, x_b] \cdot [x_c, x_d] \cdot [x_e, x_f] \) with \( \{a, b, c, d, e, f\} = \{0, 1, 2, 3, 4, 5\} \).

Again it is possible (but now with considerably more effort) to lift \( \alpha_3 \) to a generator of \( \pi_6(S^2) \) in terms of free groups. Here is a lift of \( \alpha_3 \) to the free group \( \langle x_0, \ldots, x_5 \mid x_0 \cdots x_5 \rangle \) which defines a simplicial cycle, i.e. which lies in the intersection \( R_0 \cap \cdots \cap R_5 \): it is the product of the following fourteen elements

\[
\tilde{\alpha}_3 = [[x_0, x_4], [x_2, x_4], [x_1, x_3]^{[x_0, x_4]}]\cdot [[x_1, x_4], [x_2, x_4], [x_0, x_3]^{[x_1, x_4]}] \\
\cdot [[x_1, x_4], [x_2, x_3], [x_0, x_4]^{[x_1, x_4]}]^{-1} \cdot [[x_0, x_4], [x_2, x_3], [x_1, x_4]^{[x_0, x_4]}] \\
\cdot [[x_2, x_4], [x_0, x_4], [x_1, x_3]^{[x_2, x_4]}] \cdot [[x_2, x_4], [x_1, x_4], [x_0, x_3]^{[x_2, x_4]}]^{-1} \\
\cdot [[x_2, x_3], [x_1, x_4], [x_0, x_4]^{[x_2, x_3]}] \cdot [[x_2, x_3], [x_0, x_4], [x_1, x_4]^{[x_2, x_3]}]^{-1} \\
\cdot [[x_3, x_4], [x_1, x_4], [x_0, x_2]^{[x_3, x_4]}] \cdot [[x_3, x_4], [x_0, x_4], [x_1, x_2]^{[x_3, x_4]}]^{-1} \\
\cdot [[x_3, x_4], [x_2, x_4], [x_0, x_1]^{[x_3, x_4]}] \cdot [[x_3, x_4], [x_1, x_4], [x_0, x_3]^{[x_1, x_4]}]^{-1} \\
\cdot [[x_0, x_4], [x_3, x_4], [x_1, x_2]^{[x_0, x_4]}] \cdot [[x_2, x_4], [x_3, x_4], [x_0, x_1]^{[x_2, x_4]}].
\]

One can directly check that \( \tilde{\alpha}_3 \) defines a simplicial cycle and that modulo the seventh term of the lower central series it represents exactly the element \( \alpha_3 \).
Remark 2.8. We have $\pi_6(S^2) = \mathbb{Z}/3 \times \mathbb{Z}/4$, and it is also possible to give an explicit generator of the 4-torsion. It is

$$\tilde{\alpha}_4 = \left[\left[[x_3, x_1], [x_3, x_2]\right], \left[[x_4, x_0], [x_4, x_2]\right]\right]$$

\[\cdot \left[\left[[x_4, x_1], [x_4, x_2]\right], \left[[x_3, x_0], [x_3, x_2]\right]\right] \cdot \left[\left[[x_4, x_1], [x_4, x_2]\right], \left[[x_4, x_0], [x_4, x_2]\right]\right] \cdot \left[\left[[x_4, x_1], [x_4, x_2]\right], \left[[x_4, x_0], [x_4, x_1]\right]\right]$$

This element does not have the form “one letter repeats, all the others appear once”.

Note that, contrary to prime torsion, this element will not (at least, easily) lead to 4-torsion in a dimension quotient. Indeed $\tilde{\alpha}_4^2$ is, up to the symmetric commutator $[R_0, \ldots, R_5]$, equal to

\[(6) \quad \left[[[x_0, x_1], [x_0, x_2]], \left[[x_0, x_1], [x_0, x_3]\right]\right], \left[[[x_0, x_1], [x_0, x_2]], \left[[x_0, x_1], [x_0, x_4]\right]\right]\]

and this element does not have the form “one letter repeats, all the others appear once”.

Here is a brief explanation of the origin of $\tilde{\alpha}_4$. The elements of the $E^1$-page of the spectral sequence can be coded by generators of lambda-algebra. Serre elements, which we study, correspond to the elements $\lambda_1$. The element $\tilde{\alpha}_4$ corresponds to $\lambda_2 \lambda_1$ of the lambda-algebra. $E_{5,5}^\infty$-terms $S^2$ have the following terms: $E_{5,5}^\infty = \mathbb{Z}/2$ (generator $\lambda_2 \lambda_1$), $E_{5,5}^\infty = \mathbb{Z}/3$ (generator $\lambda_1$ for $p = 3$), $E_{5,5}^\infty = \mathbb{Z}/2$ (generator $\lambda_1^2$). The 4-torsion in $\pi_6(S^2)$ is glued from two terms in $E^\infty$: $\lambda_1^2$ and $\lambda_2 \lambda_1$. A representative of $\lambda_1^2$ is the bracket (6), see for example [7]. To show that $\tilde{\alpha}_4$ represents the 4-torsion, we observe first that it is a cycle, namely that it lies in $R_0 \cap \cdots \cap R_5$, and secondly we show that, modulo $\gamma_9$, it represents the element $\lambda_2 \lambda_1$ of the simplicial
Lie algebra, given as a sum
\[ [x_3, x_1, [x_3, x_2]], [x_4, x_0, [x_4, x_2]] \]
\[ + [x_4, x_1, [x_4, x_2]], [x_3, x_0, [x_3, x_2]] \]
\[ + [[x_3, x_2], [x_3, x_1]], [x_4, x_2, [x_4, x_0]] \]
\[ + [[x_4, x_2], [x_4, x_1]], [x_3, x_2, [x_3, x_0]] \]
\[ + {[x_4, x_2], [x_3, x_0]}, [x_4, x_2, [x_3, x_1]] \]
\[ + {[x_4, x_1], [x_3, x_2]}, [x_4, x_2, [x_3, x_0]] \]
\[ + [[x_4, x_2], [x_3, x_1]], [x_4, x_0, [x_3, x_2]] \]
\[ + [[x_4, x_3], [x_2, x_1]], [x_4, x_2, [x_3, x_0]] \]
\[ + [[x_4, x_3], [x_2, x_0]], [x_4, x_1, [x_3, x_2]] \]
\[ + {[x_4, x_3], [x_2, x_0]}, [x_4, x_1, [x_3, x_2]] \]
\[ + {[x_4, x_0], [x_3, x_2]}, [x_4, x_3, [x_2, x_1]] \]
\[ + {[x_4, x_0], [x_3, x_2]}, [x_4, x_3, [x_2, x_1]] \].

3. Proof of Theorems 1.2 and 1.1

We will prove our main theorems by using the Lie algebra element \( \alpha_p \) constructed in Theorem 2.3 respectively a lift \( \tilde{\alpha}_p \) to a free group. Recall that \( \alpha_p \) belongs to \( I_0 \cap \cdots \cap I_{2p-1} \setminus [I_0, \ldots, I_{2p-1}] \Sigma \), and \( \tilde{\alpha}_p \) belongs to \( R_0 \cap \cdots \cap R_{2p-1} \setminus [R_0, \ldots, R_{2p-1}] \Sigma \).

In fact, we shall use the fact that \( \alpha_p \) is a linear combination of Lie brackets of the form \([I_0, I_j, I_2, I_{2p-1}, R_i] \Sigma \) with \( i \neq j \), namely that one index repeats twice and another doesn’t appear at all; and similarly for \( \tilde{\alpha}_p \).

We further constrain the Lie brackets that may appear in it as follows:

**Proposition 3.1.** Consider as above a free group \( F = \langle x_0, \ldots, x_n \mid x_0 \cdots x_n \rangle \), let \( R_i \) be the normal subgroup generated by \( x_i \), and consider an element \( \beta \in R_0 \cap \cdots \cap R_n \) satisfying \( \beta \notin [R_0, \ldots, R_n] \Sigma + \gamma_{n+2}(F) \). Then we also have

\[
\beta \notin [R_0, \ldots, R_n] \Sigma \cdot \prod_{\{i_0, \ldots, i_n\} \in \{0, \ldots, n\}^{n+1} \# \{i_0, \ldots, i_n\} \leq n-1} [R_{i_0}, \ldots, R_{i_n}] \Sigma \cdot \gamma_{n+2}(F).
\]

Similarly, consider a free Lie algebra \( L = \langle x_0, \ldots, x_n \mid x_0 + \cdots + x_n \rangle \), let \( I_i \) be the ideal generated by \( x_i \), and consider an element \( \beta \in I_0 \cap \cdots \cap I_n \) satisfying \( \beta \notin [I_0, \ldots, I_n] \Sigma + \gamma_{n+2}(L) \). Then we also have

\[
\beta \notin [I_0, \ldots, I_n] \Sigma + \sum_{\{i_0, \ldots, i_n\} \in \{0, \ldots, n\}^{n+1} \# \{i_0, \ldots, i_n\} \leq n-1} [I_{i_0}, \ldots, I_{i_n}] \Sigma + \gamma_{n+2}(L).
\]

In other words, \( \beta \) is non-trivial modulo all symmetrized brackets involving the same index at least three times, or involving two indices at least twice.

**Proof.** We only prove the Lie algebra case; the group case proceeds entirely analogously. Assume that we have

\[
\beta \in [I_0, \ldots, I_n] \Sigma + \sum_{\{i_0, \ldots, i_n\} \in \{0, \ldots, n\}^{n+1} \# \{i_0, \ldots, i_n\} \leq n-1} [I_{i_0}, \ldots, I_{i_n}] \Sigma + \gamma_{n+2}(L).
\]

Choose two indices \( i, j \in \{0, \ldots, n\} \), and note that \( \langle x_0, \ldots, x_j, \ldots, x_n \rangle \) is free on its generators.
By our assumption, we may write $\beta = \beta_1 \beta_2$, with $\beta_1$ consisting of symmetrized commutators in $\{I_0, \ldots, I_n\} \setminus \{I_i, I_j\}$ and $\beta_2 \in I_i$. We also have $\beta \in I_i$ by assumption; and $\beta_1 \in N := \gamma_{n+1}(L) \cap \langle x_0, \ldots, \hat{x}_i, \hat{x}_j, \ldots, x_n \rangle$. Now $N \cap I_i = [N, I_i]$ because these are two ideals of $L$ generated by disjoint subsets of its basis $\{x_0, \ldots, \hat{x}_j, \ldots, \hat{x}_n\}$. Therefore we have $\beta_1 \in \gamma_{n+2}$. We continue with $\beta_2$ and all other choices of indices $i, j$ to arrive at $\beta \in [I_0, \ldots, I_n]_{\Sigma} + \gamma_{n+2}(L)$. 

\[ \text{Remark 3.2.} \] Obviously, $R_0 \cap \cdots \cap R_n \subseteq \gamma_n(F)$, respectively $I_0 \cap \cdots \cap I_n \subseteq \gamma_n(L)$ in the Lie algebra case. A simple analysis shows in fact that, for $n > 1$, we have $R_0 \cap \cdots \cap R_n \subseteq \gamma_{n+1}(F)$, respectively $I_0 \cap \cdots \cap I_n \subseteq \gamma_{n+1}(L)$ in the Lie algebra case (the terms $E_{n,n}$ are zero for $n > 2$). This means that the hypothesis of Proposition 3.1 is satisfied only in the case of elements of $\pi_{n+1}(S^2)$ which come from $E_{n+1,n}^1$, respectively elements from $E_{n+1,n}^1$. All terms of the $E^1$-page of the lower central sequence are described in terms of the $E$-algebra [1, 1, 14]. The terms $E_{n+1,n}^1$ are nonzero only when $n + 1 = 2p^k$ for some prime $p$ and $k \geq 0$. For $k > 1$, the $E$-algebra generator of the $E_{n+1,n}^1$ term is $\lambda_2 \lambda_4 \cdots \lambda_{2^{k-1}} \lambda_{2^{k-1}}$ in the case $p = 2$, and $\mu_1 \mu_2 \cdots \mu_{p^k-1} \lambda_{p^k}$ in the case of odd $p$. However, for $k > 1$, these terms vanish on the $E^2$-page of the spectral sequence. This follows from the fact that $d^1$ of these terms is nonzero. The Serre element, which we study here, and is in the language of $E$-algebras, is lifted to the homotopy class $\pi_{2p}(S^2)$. This follows from the form of the spectral sequence: in the line $E^1_{*,2p-1}$, we have only one torsion term, which we describe in terms of free Lie algebras, in lower dimensions $E^1_{*,<2p-1}$ there are no torsion elements at all. In the next line $E^1_{*,2p}$ we don’t have torsion elements at all for odd prime $p$. For $p = 2$, there is one element in $E^1_{*,4}$, but it lives in higher degree, so it cannot kill the element which we study. In other words, the homotopy element is actually given by an $E^1$-term, in the case of Serre elements. Therefore, Serre’s $p$-torsion element comes not only from Wu’s formula, but also from its linearization, namely

$$\mathbb{Z}/p \subseteq \frac{R_0 \cap \cdots \cap R_{2p-1}}{[R_0, \ldots, R_{2p-1}]_{\Sigma} \gamma_{2p+1}(G) \cap R_0 \cap \cdots \cap R_{2p-1}}.$$
free Lie ring $L$ generated by $S := \{x_0, \ldots, x_{2p-1}, y_0, \ldots, y_{2p-1}\}$. It is a graded ring, in which we give weight 1 to the each generator $x_i$ and weight $i+2$ to generator $y_i$.

Suppose that after applying the relations $p^i x_i = y_i$, the element $\omega$ can be made to lie in $\gamma_{4p+\ell}(A)$. Then, $\omega$ may be presented as a linear combination of brackets of length 2 of the form $[z_0, \ldots, z_{2p-1}]$ with each $z_i \in S$. Since $L$ is free, each such bracket $[z_0, \ldots, z_{2p-1}]$ must have weight at least $4p+\ell$. By Proposition 3.1, we may also assume that, writing each $z_i \in \{x_{\sigma(i)}, y_{\sigma(i)}\}$, we have that $\sigma$ is either a rearrangement of $(0, \ldots, 2p-1)$ or of $(0, \ldots, i, \ldots, j, j, \ldots, 2p-1)$ for some indices $i \neq j$.

In order to make $[z_0, \ldots, z_{2p-1}]$ of degree at least $4p+\ell$, some of the original $x_{\sigma(k)}$ letters in the summand of $\omega$ bracket must have been substituted into $y_{\sigma(k)}$. Each such substitution “costs” a factor of $p^\sigma(k)$ and brings the bracket down $\sigma(k)+1$ steps along the central series. We shall see that the bracket can be brought to degree $4p+\ell$ only in the first case, when $\sigma$ is a permutation of $(0, \ldots, 2p-1)$.

Let us then consider the second case. If $j > i$, then $\sigma(0)+\cdots+\sigma(2p-1) > \ell$, so not all relations $p^\sigma(k)x_{\sigma(k)} = y_{\sigma(k)}$ could be applied, and therefore at least one of the $z_k$ is $x_{\sigma(k)}$. Let $J \subseteq \{0, \ldots, 2p-1\}$ be the set of indices $k$ on which the substitution $p^\sigma(k) x_{\sigma(k)} \rightarrow y_{\sigma(k)}$ was applied. Then on the one hand we have $\sum_{i \in J} \sigma(i) \leq \ell$, and on the other hand the total degree of $[z_0, \ldots, z_{2p-1}]$ is $\sum_{i \in J} (\sigma(i)+2) + \sum_{i \notin J} (1) \leq 4p+\ell - \#(\{0, \ldots, 2p-1\} \setminus J) < 4p+\ell$. Consider next the case $i > j$; then the total degree of $[z_0, \ldots, z_{2p-1}]$ is $4p+\ell + j - i < 4p+\ell$.

It follows that all the brackets appearing in an expression of $\omega$, and therefore of $\alpha_p$, are of the form $[z_0, \ldots, z_{2p-1}]$ with $\sigma$ a permutation of $(0, \ldots, 2p-1)$; so $\alpha_p \in [I_0, \ldots, I_{2p-1}]\omega$, a contradiction.

3.2. The group case. The difference with the Lie algebra case is that we give weight $\ell$ to each generator $x_i$ and weight $\ell+1+i$ to each generator $y_i$. Let us write $m := 2p(\ell+1)+\ell$, and note that $m = (\ell+1) + \cdots + (\ell+2p)$ is the total weight of the $y_i$ generators. We construct the element $\omega$ as above as $\tilde{\alpha}_p^\ell$, with $\tilde{\alpha}_p$ given by Proposition 2.3. To show that we have $\omega \in \delta_m(G)$, we note the identities

$$x_i^p - 1 = (1+x_i-1)^p - 1 = p^i(x_i-1) + \binom{p^i}{2} (x_i-1)^2 + \cdots + p^i(x_i-1) \quad (\text{mod } \omega^{2\ell});$$

and furthermore $x_i - 1 \in \omega^{\ell}$. It follows that we have

$$[x_{\sigma(0)}, \ldots, x_{\sigma(2p-1)}]^{p^\ell} - 1 \equiv p^i[x_{\sigma(0)} - 1, \ldots, (x_{\sigma(2p-1)} - 1)]$$

$$\equiv [p^\sigma(0)(x_{\sigma(0)} - 1), \ldots, p^\sigma(2p-1)(x_{\sigma(2p-1)} - 1)]$$

$$\equiv [y_{\sigma(0)} - 1, \ldots, y_{\sigma(2p-1)}]$$

$$\in \omega^{\ell+1+\sigma(0)+\cdots+\ell+1+\sigma(2p-1)} = \omega^m.$$  

The other claims — that $\omega^p \in \gamma_m(G)$ and $\omega \notin \gamma_m(G)$ — are exactly the same as in the Lie algebra case and need not be repeated.

4. Examples

The examples presented above yielded with relatively little computational effort Lie algebras and groups with $p$-torsion in their dimension quotients. Using more
computational resources, we were able to find \( p \)-torsion in lower degree for \( p = 2 \) and \( p = 3 \).

A general simplification (see Propositions 2.4 and 2.5) is that we can start by an element \( \alpha_p \) of degree \( p \) an not \( 2p \), by writing generators \( x_{ij} \) in place of \( [x_i, x_j] \). Indeed all the computations that express \( \alpha_p \) as a symmetrized associative product actually take place in \( \mathcal{L}_p \mathcal{L}_2(\mathbb{Z}^{2p}) \subset \mathcal{L}_p(\mathbb{Z}^{2p}) \). In fact, this amounts to working in Milnor’s simplicial construction \( F[S^2] \), whose geometric realization is \( \Omega S^3 \), and in its Lie analog \( L[S^2] \). Observe that, for higher spheres \( S^n \) with \( n > 3 \), as well as of Moore spaces, there is a description of homotopy groups as centers of explicitly defined finitely generated groups [22]. However, these groups are not as easily defined as in the case of \( S^2 \), when we quotient by the symmetric commutator. An application of homotopy groups of higher spheres in group-theoretical questions such as the problems considered here is obviously possible, however, it will involve more complicated constructions.

4.1. \( p = 2 \). The construction given in the proof of Theorem 1.2 has generators \( x_0, x_1, x_2 \) and \( x_3 := -x_0 - x_1 - x_2 \). The element \( \omega \) belongs to \( \delta_1(A) \setminus \gamma_1(A) \). It is possible to be a little bit more economical, by keeping the nilpotency degrees of the \( y_i \) more under control: consider

\[
A = \langle x_0, x_1, x_2, x_3, y_0^{(1)}, y_1^{(2)}, y_2^{(3)}, y_3^{(4)} \mid x_0 + x_1 + x_2 + x_3 = 0, x_0 = 2^7 y_0, 2^1 x_1 = y_1, 2^2 x_2 = y_2, 2^4 x_3 = y_3 \rangle
\]

and the element \( \omega = [x_0, x_2], [x_1, x_2] \). In that Lie algebra, we have \( \omega \in \delta_10(A) \setminus \gamma_10(A) \) and \( 2\omega \in \gamma_10(A) \). This can be checked by computer using the program \texttt{lienq} by Csaba Schneider [15, 28].

Rewriting \([x_i, x_j]\) as \( x_{ij} \) and simplifying somewhat, we obtain

\[
A = \langle x_{01}, x_{02}, x_{03}, x_{12}, x_{13}, x_{23}, y_0^{(2)}, y_0^{(3)}, y_0^{(4)}, y_1^{(5)}, y_1^{(6)} \mid x_{01} + x_{02} + x_{03} = -x_{01} + x_{12} + x_{13} = -x_{02} - x_{12} + x_{23} = 0, \\
2^0 x_{01} = y_{01}, 2^1 x_{02} = y_{02}, 2^2 x_{12} = y_{12}, \\
2^3 x_{03} = y_{03}, 2^4 x_{13} = y_{13}, 2^5 x_{23} = y_{23} \rangle
\]

with \( 2^5[x_{02}, x_{12}] \in \delta_8(A) \setminus \gamma_8(A) \).

There is in fact substantial flexibility in this example: suppose that the element \( \omega \) is \( 2^d[x_{02}x_{12}] \) and we want to show that it belongs to \( \delta \ell(A) \setminus \gamma \ell(A) \). Using the associative rewriting \([x_{02}, x_{12}] = x_{23}x_{01} + x_{12}x_{03} - x_{02}x_{13} \) from [5], we will have \( \omega \in \delta \ell(A) \) as soon as \( A \) has relations of the form \( 2^d x_{23} = 2^2 y_{23} \) and \( 2^{23} x_{01} = y_{01} \) with \( \deg(y_{23}) + \deg(y_{01}) = \ell \), and similarly for the other generators. The condition \( \omega \notin \gamma \ell(A) \) can be checked by a direct calculation, e.g. using \texttt{lienq}. For instance, we have (eliminating the \( x_{13} \))

\[
A = \langle x_{01}, x_{02}, x_{12}, y_0^{(2)}, y_0^{(3)}, y_0^{(4)}, y_1^{(5)}, y_1^{(6)}, y_1^{(7)} \mid 2^2 x_{01} = y_{01}, 2^3 x_{02} = y_{02}, 2^4 x_{12} = y_{12}, \\
2^4(-x_{01} - x_{02}) = 2^2 y_{03}, 2^4(x_{01} - x_{12}) = 2^2 y_{13}, 2^4(x_{02} + x_{12}) = 2^1 y_{23} \rangle
\]

with \( 2^4[x_{02}, x_{12}] \in \delta_9(A) \setminus \gamma_9(A) \).

Another modification of the examples often leads to lower values of \( \ell \): we may replace the variables \( x_{ij} \) by \( 2^{b_{ij}} x_{ij} \) for well-chosen \( b_{ij} \). This amounts, essentially,
to letting the $x_{ij}$ have distinct, negative degrees. For instance, replacing $x_{ij}$ by $2^{i+j}x_{ij}$ in the previous example, we get

$$A = \langle x_{01}, x_{02}, x_{13}, y_{01}, y_{02}, y_{03}, y_{12}, y_{13}, y_{23} |$$

$$2^2x_{01} = y_{01}, \quad 2^4x_{02} = y_{02}, \quad 2^6x_{12} = y_{12},$$

$$2^6(-x_{01} - 2x_{02}) = 2^6y_{03}, \quad 2^5(x_{01} - 4x_{12}) = 2^4y_{13}, \quad 2^5(x_{02} + 2x_{12}) = 2^2y_{23} \rangle$$

with $\omega := 2^5[x_{02}, x_{12}] \in \delta_4(A) \setminus \gamma_4(A)$. We have $\omega = 2^5[x_{01}, x_{02}] + 2^6[x_{01}, x_{12}] + 2^7[x_{02}, x_{12}]$ modulo $\gamma_4(A)$; note the similarity with Rips's original example ([1]). We may also choose $\ell \geq 2$, let $y_{0i}$ have degree $\ell$ for all $i$ and in this manner obtain examples with 2-torsion in $\delta_{\ell+2}(A)/\gamma_{\ell+2}(A)$.

It is straightforward to convert the example above into a group: it will be

$$G = \langle x_{01}, x_{02}, x_{13}, y_{01}, y_{02}, y_{03}, y_{12}, y_{13}, y_{23} |$$

$$x_{01}^4 = y_{01}, \quad x_{02}^{16} = y_{02}, \quad x_{12}^{12} = y_{12},$$

$$x_{01}^{64}x_{02}^{128} = y_{03}, \quad x_{01}^{32}x_{12} = y_{13}, \quad x_{02}^{64}x_{12} = y_{23} \rangle$$

and the element $\omega = [e_0, e_1]^{32}[e_0, e_2]^{64}[e_1, e_2]^{128}$ belongs to $\delta_4(G) \setminus \gamma_4(G)$. Increasing the degree of the $x_{ij}$ and $y_{ij}$ leads, for every $\ell \geq 4$, to a group $G$ with 2-torsion in $\delta_\ell(G)/\gamma_\ell(G)$.

4.2. $p = 3$. As in the $p = 2$ example, we may construct a Lie algebra with generators $x_{ij}$ as follows:

$$A = \langle x_{ij}, y_{ij}^{(i+j+1)} | 0 \leq i < j \leq 5,$$

$$x_{01} + x_{02} + x_{03} + x_{04} + x_{05} = 0,$$

$$-x_{01} + x_{12} + x_{13} + x_{14} + x_{15} = 0,$$

$$-x_{02} - x_{12} + x_{23} + x_{24} + x_{25} = 0,$$

$$-x_{03} - x_{13} - x_{23} + x_{34} + x_{35} = 0,$$

$$-x_{04} - x_{14} - x_{24} - x_{34} + x_{45} = 0,$$

$$3^{i+j}x_{ij} = y_{ij} \text{ for } 0 \leq i < j \leq 5 \rangle$$

and the element $\omega = 3^{15}(x_{04}, x_{14}, x_{23})-[x_{04}, x_{24}, x_{13}]+[x_{04}, x_{34}, x_{12}]+[x_{14}, x_{24}, x_{03}]-[x_{14}, x_{34}, x_{02}]+[x_{24}, x_{34}, x_{01}])$ which belongs to $\delta_{18}(A) \setminus \gamma_{18}(A)$.

Again there is substantial flexibility in this example: the degrees of the $y_{ij}$ may be adjusted, and the last relations may be changed to $3^{a_{ij}}x_{ij} = 3^{c_{ij}}y_{ij}$ for well-chosen $a_{ij}, c_{ij}$. The variables $x_{ij}$ themselves may be replaced by $3^{b_{ij}}x_{ij}$ for well-chosen $b_{ij}$. Finally, some extra linear conditions may be imposed on the variables, such as $x_{02} = x_{13} = x_{15} = x_{24} = x_{34} = 0$. After some experimentation, we arrived at the following reasonably small example:

$$(7)$$

$$A = \langle e_0, e_1, e_2, e_3, y_i^{(2)} | i \in \{0, \ldots, 3\}, y_i^{(3)} \text{ for } 0 \leq i < j \leq 3 |$$

$$3^{2i}e_i = y_i, \quad 3^{12-i}e_i + 3^{12-2i}e_i = 3^{12-2i}y_{ij} \text{ for } (i, j) \in \{(0, 1), (0, 2), (1, 3), (2, 3)\}$$

with $\omega = 3^9[e_2, e_1, e_0]$.

**Proposition 4.1.** For the Lie ring $A$ defined in (7) we have $\omega \in \delta_T(A) \setminus \gamma_T(A)$ and $3\omega \in \gamma_T(A)$. 
Proof. Expanding \( \omega \) associatively, we get
\[
\omega = 3^9(e_0e_1e_2 - e_0e_2e_1 - e_1e_2e_0 + e_2e_1e_0).
\]
We may rewrite it as
\[
\omega = -e_0(3^9e_2 + 3^{10}e_3)e_1 - e_1(3^9e_2 + 3^{10}e_3)e_0
+ e_0(3^9e_1 + 3^{11}e_3)e_2 + e_2(3^9e_1 + 3^{11}e_3)e_0
+ (3^{10}e_0 + 3^{12}e_2)e_3e_1 + e_1e_3(3^{10}e_0 + 3^{12}e_2)
- (3^{11}e_0 + 3^{12}e_1)e_3e_2 - e_2e_3(3^{11}e_0 + 3^{12}e_1).
\]
Each of the summands belongs to \( \omega^7(A) \): they are all products of \( e_k, e_\ell \) and \( 3^{12-i}e_j + 3^{12-j}e_i \) for some \( \{i, j, k, \ell\} = \{0, 1, 2, 3\} \). The binomial term equals
\[
3^{12-2i-2j}y_{ij} = 3^{2k}e_k + 3^{2\ell}e_\ell \text{ and } y_{ij},
\]
namely the product of \( y_k, y_i, y_{ij} \), of respective degrees 2, 2, 3.

To check that \( \omega \) does not belong to \( \gamma_7(A) \) but that \( 3\omega \) does, we compute nilpotent quotients of \( A \). We did the calculation using two different programs: \texttt{lienq} by Csaba Schneider and \texttt{LieRing} [10] for \texttt{GAP} [8] by Willem de Graaf and Serena Cicalò.

In the next subsection, we give a direct proof that the associated group has 3-torsion in \( \delta_7/\gamma_7 \).

4.3. A small, finite 3-group \( G \) with \( \delta_7(G) \neq \gamma_7(G) \). We consider the group \( G \) given by the presentation (7), namely
\[
(8)
G = \langle e_0, e_1, e_2, e_3, y_i^{(2)} \rangle \text{ for } i \in \{0, \ldots, 3\}, y_i^{(3)} \text{ for } 0 \leq i < j \leq 3 \mid
\]
\[
eq e_i^{2^i} = y_i, e_i^{3^{12-i}} = y_i^{3^{12-2i}} \text{ for } (i, j) \in \{(0, 1), (0, 2), (1, 3), (2, 3)\}\]
with \( \omega = [e_2, e_1, e_0]^{3^9} \).

**Proposition 4.2.** In the group defined by (8) we have \( \omega \in \delta_7(G) \).

**Proof.** We will use the following well-known identity, which holds for any element \( x \in G \) and \( d \geq 2 \):
\[
(9)
x^d - 1 = \sum_{k=1}^{d} \binom{d}{k} (x - 1)^k.
\]
We compute modulo \( \omega^7(G) \), and from now on write \( \equiv \) to mean equivalence modulo \( \omega^7(G) \). We get
\[
\omega - 1 \equiv 3^9([e_2, e_1, e_0] - 1) \text{ since } [e_2, e_1, e_0] \in \gamma_2(G)
= 3^9[e_1, e_2]e_0^{-1}([(e_2, e_1) - 1)(e_0 - 1) - (e_0 - 1)([e_2, e_1] - 1)]
= 3^9[e_1, e_2]e_0^{-1}(e_0^{-1}e_1^{-1}((e_2 - 1)(e_1 - 1) - (e_1 - 1)(e_2 - 1))(e_0 - 1)
- (e_0 - 1)e_2^{-1}e_1^{-1}((e_2 - 1)(e_1 - 1) - (e_1 - 1)(e_2 - 1)));
\]
and since \( 3^9 \) is divisible by the product of exponent of \( e_0, e_1, e_2 \) modulo \( \gamma_2(G) \),
\[
\equiv 3^9((e_0 - 1)(e_1 - 1)(e_2 - 1) - (e_0 - 1)(e_2 - 1)(e_1 - 1)
- (e_1 - 1)(e_2 - 1)(e_0 - 1) + (e_2 - 1)(e_1 - 1)(e_0 - 1)).
\]

\( \square \)
Let us write \( f_i := e_i - 1 \) for \( i \in \{0, 1, 2, 3\} \). Then, as in the proof of Proposition 4.1, we have
\[
\omega - 1 = 3^0(f_0 f_1 f_2 - f_0 f_2 f_1 - f_1 f_2 f_0 + f_2 f_1 f_0)
\]
\[
= -f_0(3^9 f_2 + 3^{10} f_3) f_1 - f_1(3^9 f_2 + 3^{10} f_3) f_0
\]
\[
+ f_0(3^9 f_1 + 3^{11} f_2) f_2 + f_2(3^9 f_1 + 3^{11} f_3) f_0
\]
\[
+ (3^{10} f_0 + 3^{12} f_2) f_3 f_1 + f_1 f_3(3^{10} f_0 + 3^{12} f_2)
\]
\[
- (3^{11} f_0 + 3^{12} f_1) f_3 f_2 - f_2 f_3(3^{11} f_0 + 3^{12} f_1).
\]

Next, using (9) we have
\[
e_i^{312-j} e_j^{312-i} - 1 = (e_i^{312-j} - 1) + (e_j^{312-i} - 1)(e_i^{312-j} - 1)
\]
\[
= 3^{12-j} f_i + \left(\frac{3^{12-j}}{2}\right) f_i^2 + \left(\frac{3^{12-j}}{3}\right) f_i^3 + \cdots
\]
\[
+ 3^{12-i} f_j + \left(\frac{3^{12-i}}{2}\right) f_j^2 + \left(\frac{3^{12-i}}{3}\right) f_j^3 + \cdots
\]
\[
+ 3^{24-i-j} f_i f_j + \cdots
\]
\[
= y_j^{312-2i-2j} - 1 = 3^{12-2i-2j}(y_{ij} - 1) + \left(\frac{3^{12-2i-2j}}{2}\right)(y_{ij} - 1)^2 + \cdots
\]

Again using (9), the relations \( e_i^{3^2} = y_i \) imply \( 3^2 f_i \in \varpi^2 \). Now \( 3^{12-2i-2j} \) divides \( \left(\frac{3^{12-i}}{2}\right) f_i^2 \) in \( 12-2i-2j \varpi^3 \). Similarly, \( 3^{12-2i-2j} \) divides \( \left(\frac{3^{12-j}}{3}\right) f_j^3 \). The same holds for all terms in the last two rows. We therefore have
\[
3^{12-j} f_i + 3^{12-i} f_j \in 3^{12-2i-2j} \varpi^3 + \varpi^5.
\]

We note \( 3^{12-2i-2j} = 3^{2k+2\ell} \), for \( k, \ell \in \{0, 1, 2, 3\} \). Returning to computations modulo \( \varpi^7 \), we consider a typical summand \( f_k(3^{12-j} f_i + 3^{12-i} f_j) f_\ell \) in our decomposition of \( \omega - 1 \). We write \( 3^{12-j} f_i + 3^{12-i} f_j = 3^{12-2i-2j} u + v \) with \( u \in \varpi^3, v \in \varpi^5 \) to get
\[
f_k(3^{12-j} f_i + 3^{12-i} f_j) f_\ell = f_k(3^{2k+2\ell} u + v) f_\ell = (3^{2k} f_k) u(3^{2\ell} f_\ell) + f_k v f_\ell
\]
where each summand belongs to \( \varpi^7 \). \( \square \)

**Proposition 4.3.** In the group defined by (8), the element \( \omega \) defined above does not belong to \( \gamma_7(G) \), but its cube does.

**Proof.** The proof is computer-assisted. It suffices to exhibit a quotient \( \overline{G} \) of \( G \) in which the image of \( \omega \) does not belong to \( \gamma_7(\overline{G}) \) but its cube does, and we shall exhibit a finite 3-group as quotient.

To make the computations more manageable, we replace the generators \( y_i \) and \( y_{ij} \) by generators \( z_0, \ldots, z_3 \), and impose the choices
\[
y_0 = [z_0, z_1], \quad y_1 = [z_0, z_2], \quad y_2 = [z_0, z_3], \quad y_3 = [z_1, z_2],
\]
\[
y_{01} = 1, \quad y_{02} = [z_1, z_3, z_2], \quad y_{12} = [z_1, z_3, z_1], \quad y_{23} = [z_1, z_3, z_0].
\]

In this manner, we obtain an 8-generated group \( (e_0, \ldots, e_3, z_0, \ldots, z_3) \). We next impose extra commutation relations: \( [e_2, z_2], [e_3, z_3], [e_1, z_1], [e_2, z_3], [e_3, z_1] \).
We compute a basis of left-normed commutators of length at most 6 in that group; notice that $\omega$ may be expressed as $[z_3, z_2, z_3, z_1, z_1, e_3]^3$, and impose extra relations making $\gamma_6$ cyclic and central.

The resulting finite presentation may be fed to the program $pq$ by Eamonn O’Brien [23], to compute the maximal quotient of 3-class 17. This is a group of order $3^{3996}$, and can (barely) be loaded in the computer algebra system GAP [8] so as to check (for safety) that the relations of $G$ hold, and that the element $\omega$ has a non-trivial image in it.

Finally, the order of the group may be reduced by iteratively quotienting by maximal subgroups of the centre that do not contain $\omega$.

The resulting group, which is the minimal-order 3-group with non-trivial dimension quotient that we could obtain, has order $3^{494}$.

It may be loaded in any GAP distribution by downloading the ancillary file 3group.gap to the current directory and running `Read("3group.gap");` in a GAP session.

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