Good wild harmonic bundles and good filtered Higgs bundles

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Abstract

We prove the Kobayashi-Hitchin correspondence between good wild harmonic bundles and polystable good filtered Higgs bundles satisfying a vanishing condition. We also study the correspondence for good wild harmonic bundles with the homogeneity with respect to a group action, which is expected to provide another way to construct Frobenius manifolds.

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1 Introduction

Let $X$ be a smooth projective variety with a simple normal crossing hypersurface $H$. Let $L$ be an ample line bundle on $X$. We shall prove the following theorem, that is the Kobayashi-Hitchin correspondence for good wild harmonic bundles and good filtered Higgs bundles.

Theorem 1.1 (Corollary 2.17) The following objects are equivalent.

• Good wild harmonic bundles on $(X, H)$.

• $\mu_L$-polystable filtered Higgs bundles $(\mathcal{P}_\ast\mathcal{V}, \theta)$ on $(X, H)$ satisfying $\int \text{par}\cdot c_1(\mathcal{P}_\ast\mathcal{V}) c_1(L)^{\dim X-1} = 0$ and $\int \text{par}\cdot \text{ch}_2(\mathcal{P}_\ast\mathcal{V}) c_1(L)^{\dim X-2} = 0$.

We shall recall the precise definitions of the objects in §2.

In [47], we have already proved that good wild harmonic bundles on $(X, H)$ induce $\mu_L$-polystable good filtered Higgs bundles satisfying the vanishing condition. Indeed, more generally, for any complex number $\lambda$, good wild harmonic bundles induce $\mu_L$-polystable good filtered $\lambda$-flat bundles satisfying a similar vanishing condition. Note that $0$-flat bundles are equivalent to Higgs bundles, and $1$-flat bundles are equivalent to flat bundles in the ordinary sense. Moreover, we studied an analogue of Theorem 1.1 in the case $\lambda = 1$, i.e., the correspondence between good wild harmonic bundles and $\mu_L$-polystable good filtered flat bundles satisfying a similar vanishing condition [47, Theorem 16.1]. It was applied to the study of the correspondence between semisimple algebraic holonomic $D$-modules and pure twistor $D$-modules.

There is no new essential difficulty to prove Theorem 1.1 after our studies [43, 44, 45, 47] on the basis of [57, 58]. Moreover, in some parts of the proof, the arguments can be simplified in the Higgs case. However, because the Higgs case is also particularly important, it would be useful to explain a rather detailed proof of the correspondence.

1.1 Kobayashi-Hitchin correspondences

1.1.1 Kobayashi-Hitchin correspondence for vector bundles

We briefly recall a part of the history of this type of correspondences. (See also [24, 32, 38].) For a holomorphic vector bundle $E$ on a compact Riemann surface $C$, we set $\mu(E) := \text{deg}(E)/\text{rank}(E)$, which is called the slope of $E$. A holomorphic bundle $E$ is called stable (resp. semistable) if $\mu(E') < \mu(E)$ (resp. $\mu(E') \leq \mu(E)$) holds for any holomorphic subbundle $E' \subset E$ such that $0 < \text{rank}(E') < \text{rank}(E)$. It is called polystable if it is a direct sum of stable subbundles with the same slope. This stability, semistability and polystability conditions were introduced by Mumford [52] for the construction of the moduli spaces of vector bundles with
reasonable properties. Narasimhan and Seshadri \cite{narasimhan_seshadri} established the equivalence between unitary flat bundles and polystable bundles of degree 0 on compact Riemann surfaces.

Let \((X, \omega)\) be a compact connected Kähler manifold. For any torsion-free \(\mathcal{O}_X\)-module \(\mathcal{F}\), the slope of \(\mathcal{F}\) with respect to \(\omega\) is defined as

\[
\mu_\omega(\mathcal{F}) := \frac{\int_X c_1(\mathcal{F})\omega^{\dim X-1}}{\text{rank}\mathcal{F}}.
\]

If the cohomology class of \(\omega\) is the first Chern class of an ample line bundle \(L\), then \(\mu_\omega(\mathcal{F})\) is also denoted by \(\mu_L(\mathcal{F})\). Then, a torsion-free \(\mathcal{O}_X\)-module \(\mathcal{F}\) is called \(\mu\)-stable if \(\mu_\omega(\mathcal{F}) < \mu_\omega(\mathcal{F}')\) holds for any saturated subsheaf \(\mathcal{F}' \subset \mathcal{F}\) such that \(0 < \text{rank}(\mathcal{F}') < \text{rank}(\mathcal{F})\). This condition was first studied by Takegami \cite{takegami} \cite{takegami2}. It is also called \(\mu\)-stability, or slope stability. Slope semistability and slope polystability are naturally defined.

Bogomolov \cite{bogomolov} introduced a different stability condition for torsion-free sheaves on connected projective surfaces, and he proved the inequality of the Chern classes \(c_2(E) - (r - 1)c_1^2/2r < 0\) for any unstable bundle \(E\) of rank \(r\) in his sense. Gieseker \cite{gieseker} proved the inequality for slope semistable bundles. The inequality is called Bogomolov-Gieseker inequality.

Inspired by these works, Kobayashi \cite{kobayashi} introduced the concept of Hermitian-Einstein condition for metrics of holomorphic vector bundles. Let \((E, \overline{\partial}_E)\) be a holomorphic vector bundle on a Kähler manifold \((X, \omega)\). Let \(h\) be a Hermitian metric of \(E\). Let \(R(h)\) denote the curvature of the Chern connection \(\nabla_h = \overline{\partial}_E + \partial_E h\), associated to \(h\) and \(\overline{\partial}_E\). Then, \(h\) is called Hermitian-Einstein if \(\Lambda R(h)^\perp = 0\), where \(R(h)^\perp\) denote the trace-free part of \(R(h)\).

In \cite{kobayashi}, he particularly studied the case where the tangent bundle of a compact Kähler manifold has a Hermitian-Einstein metric, and he proved that such bundles are not unstable in the sense of Bogomolov. Kobayashi \cite{kobayashi1} \cite{kobayashi2} and Lübke \cite{lbuecke} \cite{lbuecke2} proved that a holomorphic vector bundle on a compact connected Kähler manifold satisfies the slope polystability condition if it has a Hermitian-Einstein metric. Moreover, Lübke \cite{lbuecke2} established the so called Kobayashi-Lübke inequality for the first and the second Chern forms associated to Hermitian-Einstein metrics, which is reduced to the inequality \(\text{Tr}((R(h)^\perp)^2)\omega^{\dim X - 2} \geq 0\) in the form level. It particularly implies the Bogomolov-Gieseker inequality for holomorphic vector bundles \((E, \overline{\partial}_E)\) with a Hermitian-Einstein metric \(h\) on compact Kähler manifolds \((X, \omega)\). Moreover, if \(c_1(E) = 0\) and \(\int_X c_2(E)\omega^{\dim X - 2} = 0\) are satisfied for such \((E, \overline{\partial}_E, h)\), and if we impose \(\det(h)\) is flat, then the Kobayashi-Lübke inequality implies that \(R(h) = 0\), i.e., \(\nabla_h\) is flat.

Independently, in \cite{hitchin}, Hitchin proposed a problem to ask an equivalence of the stability condition and the existence of a metric \(h\) such that \(\Lambda R(h) = 0\), under the vanishing of the first Chern class of the bundle. (See \cite{bogomolov} for more precise.) It clearly contains the most important essence. He also suggested possible applications on the vanishings. His problem stimulated Donaldson whose work on this topic brought several breakthroughs to whole geometry.

In \cite{donaldson}, Donaldson introduced the method of global analysis to reprove the theorem of Narasimhan-Seshadri. In \cite{donaldson2}, by using the method of the heat flow associated to the Hermitian-Einstein condition, he established the equivalence of the slope polystability condition and the existence of a Hermitian-Einstein metric for holomorphic vector bundles on any complex projective surface. The important concept of Donaldson functional was also introduced in \cite{donaldson2}.

Eventually, Donaldson \cite{donaldson3} and Uhlenbeck-Yau \cite{uhlenbeck_yau} established the equivalence on any dimensional complex projective manifolds. Note that Uhlenbeck-Yau proved it for any compact Kähler manifolds, more generally. The correspondence is called with various names: Kobayashi-Hitchin correspondence, Hitchin-Kobayashi correspondence, Donaldson-Hitchin-Uhlenbeck-Yau correspondence, etc. In this paper, we adopt Kobayashi-Hitchin correspondence.

As a consequence of the Kobayashi-Hitchin correspondence and the Kobayashi-Lübke inequality, we also obtain an equivalence between unitary flat bundles and slope polystable holomorphic vector bundles \(E\) satisfying \(\mu_\omega(E) = 0\) and \(\int_X c_2(E)\omega^{\dim X - 2} = 0\). Note that Mehta and Ramanathan \cite{mehta_ramanathan} \cite{mehta_ramanathan2} deduced the equivalence on complex projective manifolds directly from the equivalence in the surface case due to Donaldson \cite{donaldson2}.

1.1.2 Higgs bundles

Such correspondences have been also studied for vector bundles equipped with something additional, which are also called Kobayashi-Hitchin correspondences in this paper. One of the most rich and influential is the case of Higgs bundles, pioneered by Hitchin and Simpson.
Let \((E, \mathcal{J}_E)\) be a holomorphic vector bundle on a compact Riemann surface \(C\). A Higgs field of \((E, \mathcal{J}_E)\) is a holomorphic section \(\theta\) of \(\text{End}(E) \otimes \Omega^1_{\mathcal{J}_E}\). Let \(h\) be a Hermitian metric of \(E\). We obtain the Chern connection \(\nabla + \partial_{E,h}\) and its curvature \(R(h)\). Let \(\theta^1_h\) denote the adjoint of \(\theta\). In \([23]\), Hitchin introduced the following equation, called the Hitchin equation:

\[
R(h) + [\theta, \theta^1_h] = 0. 
\]

(1)

Such \((E, \mathcal{J}_E, \theta, h)\) is called a harmonic bundle. He particularly studied the case \(E = 2\). Among many deep results in \([23]\), he proved that a Higgs bundle \((E, \mathcal{J}_E, \theta)\) has a Hermitian metric \(h\) satisfying \((1)\) if and only if it is polystable of degree 0. Here, a Higgs bundle \((E, \mathcal{J}_E, \theta)\) is called stable (resp. semistable) if \(\mu(E') < \mu(E)\) (resp. \(\mu(E') \leq \mu(E)\)) holds for any holomorphic subbundle \(E' \subset E\) such that \(\theta(E') \subset E' \otimes \Omega^1_{\mathcal{J}_E}\), and that \(0 < \text{rank}(E') < \text{rank}(E)\), and a Higgs bundle is called polystable if it is a direct sum of stable Higgs subbundles with the same slope. By this equivalence and another equivalence due to Donaldson \([10]\) between irreducible flat bundles and twisted harmonic maps, Hitchin obtained that the moduli space of polystable Higgs bundles of degree 0 and the moduli space of semistable flat bundle are isomorphic. Together with another equivalence due to Donaldson \([10]\) between irreducible flat bundles and twisted harmonic maps, Hitchin’s work showed that the moduli spaces of Higgs bundles and flat bundles have extremely rich structures.

The higher dimensional case was studied by Simpson \([57]\). Note that Simpson started his study independently motivated by a new way to construct variations of Hodge structure, which we shall mention later in \([12]\). For a holomorphic vector bundle \((E, \mathcal{J}_E)\) on a complex manifold \(X\) with arbitrary dimension, a Higgs field \(\theta\) is defined to be a holomorphic section of \(\text{End}(E) \otimes \Omega^1_X\) satisfying the additional condition \(\theta \wedge \theta = 0\). Suppose that \(X\) has a Kähler form. Let \(h\) be a Hermitian metric of \(E\). Let \(F(h)\) denote the curvature of the connection \(\nabla_h + \theta + \theta^1_h\).

A Hermitian metric \(h\) of a Higgs bundle \((E, \mathcal{J}_E, \theta)\) is called Hermitian-Einstein if \(\Lambda F(h)^{-1} = 0\). When \(X\) is compact, the slope stability, semistability and polystability conditions for Higgs bundles are naturally defined in terms of the slopes of Higgs subsheaves. Simpson established that a Higgs bundle \((E, \mathcal{J}_E, \theta)\) on a compact Kähler manifold \((X, \omega)\) has a Hermitian-Einstein metric if and only if it is slope polystable. Moreover, he generalized the Kobayashi-Lübke inequality for the Chern forms to the context of Higgs bundles, which is reduced to the inequality \(\text{Tr}((F(h)^{-1})^p)\omega^{\text{dim}X - 2} \geq 0\) in the form level for any Hermitian-Einstein metric \(h\) of \((E, \mathcal{J}_E, \theta)\).

This correspondence is not only really interesting, but also a starting point of the further investigations. Simpson pursued the comparison of flat bundles and Higgs bundles in deeper levels \([59]\), and developed the non-abelian Hodge theory \([61]\).

1.1.3 Filtered case

It is interesting to generalize such correspondences for objects on complex quasi-projective manifolds. We need to impose a kind of boundary condition, that is parabolic structure.

Mehta and Seshadri \([42]\) introduced the concept of parabolic structure of vector bundles on compact Riemann surfaces. Let \(C\) be a compact Riemann surface with a finite subset \(D \subset C\). Let \(E\) be a holomorphic vector bundle on \(C\). A parabolic structure of \(E\) at \(D\) is a tuple of filtrations \(F_* (E|_P)\) \((P \in D)\) indexed by \([-1, 0]\) satisfying \(F_a(E|_P) = \bigcap_{b > a} F_b(E|_P)\). Set \(\text{Gr}^a(E|_P) := F_a(E)/F_{<a}(E)\), and

\[
\text{deg}(E, F) := \text{deg}(E) - \sum_{P \in D} \sum_{-1 < a < 0} a \dim \text{Gr}^a(E|_P).
\]

We set \(\mu(E, F) := \text{deg}(E, F)/\text{rank}(E)\). For any subbundle \(E' \subset E\), filtrations \(F(E'|_P)\) on \(E'|_P\) are induced as \(F_a(E'|_P) := F_a(E|_P) \cap E'|_P\). Then, \((E, F)\) is called stable if \(\mu(E', F) < \mu(E, F)\) for any subbundle \(E' \subset E\) with
variety of is proper over $C$ and Seshadri proved an equivalence of unitary flat bundles on $C \setminus D$ and parabolic vector bundles $(E, F)$ with $\mu(E, F) = 0$ on $(C, D)$.

For some purposes, it is more convenient to replace parabolic bundles with filtered bundles introduced by Simpson [57, 58]. Let $V$ be a locally free $O_C$-module. A filtered bundle $P, \mathcal{V}$ over $V$ is a tuple of lattices $P_a \mathcal{V}$ such that (i) $P_a \mathcal{V} = \mathcal{V}$, (ii) the restriction of $P_a \mathcal{V}$ to a neighbourhood of $P \in D$ depends only on $a$, (iii) $P_{a+n} \mathcal{V} = P_a \mathcal{V} \sum n_P F_{P}$ for any $a \in R^D$ and $n \in Z^D$, (iv) for any $a \in R^D$, there exists $\epsilon \in R^D$ such that $P_\alpha \mathcal{V} = P_{\alpha+\epsilon} \mathcal{V}$. Let $\mathcal{V}$ denote $(0, \ldots, 0) \in R^D$. Then, $P_0 \mathcal{V}$ is equipped with the parabolic structure $F$ induced by the images of $P_\alpha \mathcal{V}_D \rightarrow P_\alpha \mathcal{V}_D$. It is easy to observe that filtered bundles are equivalent to parabolic bundles. We set $\mu(P, \mathcal{V}) := \mu(P_0 \mathcal{V}, F)$ for filtered bundles $P, \mathcal{V}$.

Simpson [57, 58] generalized the theorem of Mehta-Seshadri to the correspondences of tame harmonic bundles, regular filtered Higgs bundles and regular filtered Higgs bundles on compact Riemann surfaces. A harmonic bundle was not expected in those days.

As mentioned, in [47], the author studied the wild harmonic bundles on any dimensional varieties. We obtained that good wild harmonic bundles induce $\mu_L$-polystable good filtered Higgs bundles $\mu_L$-polystable good filtered flat bundles satisfying the vanishing conditions. Moreover, we proved that the construction induces an equivalence of good harmonic bundles and slope polystable good filtered flat bundles satisfying the vanishing condition. Such an equivalence for meromorphic flat bundles is particularly interesting because we
may apply it to prove a conjecture of Kashiwara \[28\] on semisimple algebraic holonomic \(\mathcal{D}\)-modules. See \[49\] for more details on this application.

In \[47\], we did not give a proof of the equivalence for wild harmonic bundles on the Higgs side because it is rather obvious that a similar argument can work even in the Higgs case after \[43\ 44\ 45\ 47\] on the basis of \[57\ 58\]. But, because the Higgs case is also important, it would be better to have a reference in which a rather detailed proof is explained. It is one reason why the author writes this manuscript. As another reason, in the next subsection, we shall explain an application to the correspondence for good wild harmonic bundles with homogeneity, which is expected to be useful in the generalized Hodge theory.

### 1.2 Homogeneity with respect to group actions

#### 1.2.1 Variation of Hodge structure

As mentioned, Simpson \[57\] was motivated by the construction of polarized variation of Hodge structure. Let us recall the definition of polarized complex variation of Hodge structure given in \[57\], instead of the original definition of polarized variation of Hodge structure due to Griffiths. A complex variation of Hodge structure of weight \(w\) is a graded \(C^\infty\)-vector bundle \(V = \bigoplus_{p+q=w} V^{p,q}\) equipped with a flat connection \(\nabla\) satisfying the Griffiths transversality condition, i.e., \(\nabla^{0,1}(V^{p,q}) \subset \Omega^{0,1} \otimes (V^{p+1,q-1} \otimes V^{p,q})\) and \(\nabla^{1,0}(V^{p,q}) \subset \Omega^{1,0} \otimes (V^{p-1,q+1} \otimes V^{p,q})\), where \(\nabla^{p,q}\) denote the \((p,q)\)-part of \(\nabla\). A polarization of a complex variation of Hodge structure is a flat Hermitian pairing \(\langle \cdot, \cdot \rangle\) satisfying the following conditions: (i) the decomposition \(V = \bigoplus V^{p,q}\) is orthogonal with respect to \(\langle \cdot, \cdot \rangle\), (ii) \(\langle \sqrt{-1} \rangle^{p-q} \langle \cdot, \cdot \rangle\) is positive definite on \(V^{p,q}\).

A polarization of pure Hodge structure typically appears when we consider the Gauss-Manin connection associated to a smooth projective morphism \(f: X \to Y\). Namely, the family of vector spaces \(H^w(f^{-1}(y))\) \((y \in Y)\) naturally induces a flat bundle on \(Y\). With the Hodge decomposition, it is a variation of Hodge structure of weight \(w\). A relatively ample line bundle induces a polarization on the variation of Hodge structure.

Simpson discovered a completely different way to construct polarized variation of Hodge structure. Let \((V = \bigoplus V^{p,q}, \nabla)\) be a complex variation of Hodge structure. Note that \(\nabla^{0,1}\) induces holomorphic structures \(\partial_{V^{p,q}}: V^{p,q} \to V^{p,q} \otimes \Omega^{0,1}\) of \(V^{p,q}\). We set \(\partial_V := \bigoplus \partial_{V^{p,q}}\). Then, \((V = \bigoplus V^{p,q}, \partial_V)\) is a graded holomorphic vector bundle. We also note that \(\nabla^{1,0}\) induces linear maps \(V^{p,q} \to V^{p-1,q+1} \otimes \Omega^{1,0}\), and hence \(\theta: V \to V \otimes \Omega^{1,0}\). It is easy to check that \(\theta\) is a Higgs field of \((V, \partial_V)\). Such a graded holomorphic bundle \(V = \bigoplus_{p+q=w} V^{p,q}\) with a Higgs field \(\theta\) such that \(\theta \cdot V^{p,q} \subset V^{p-1,q+1} \otimes \Omega^{1,0}\) is called a Hodge bundle of weight \(w\). In general, we cannot construct a complex variation of Hodge structure from a Hodge bundle. However, Simpson discovered that if a Hodge bundle \((V = \bigoplus V^{p,q}, \theta)\) on a compact Kähler manifold satisfies the stability condition and the vanishing condition, then there exists a flat connection \(\nabla\) and a flat Hermitian pairing \(\langle \cdot, \cdot \rangle\) such that (i) \((V = \bigoplus V^{p,q}, \nabla)\) is a complex variation of Hodge structure which induces the Hodge bundle, (ii) \(\langle \cdot, \cdot \rangle\) is a polarization of \((V = \bigoplus V^{p,q}, \nabla)\). Indeed, according to the equivalence of Simpson between Higgs bundles and harmonic bundles, there exists a pluri-harmonic metric \(h\) of \((V, \theta)\). It turns out that the flat connection \(\nabla_h + \theta + \theta^*\) satisfies the Griffiths transversality. Moreover, the decomposition \(V = \bigoplus V^{p,q}\) is orthogonal with respect to \(h\), and flat Hermitian pairing \(\langle \cdot, \cdot \rangle\) is constructed by the relation \(\langle \sqrt{-1} \rangle^{p-q} \langle \cdot, \cdot \rangle_{V^{p,q}} = h_{V^{p,q}}\).

Note that a Hodge bundle is regarded as a Higgs bundle \((V, \partial_V, \theta)\) with an \(S^1\)-homogeneity, i.e., \((V, \partial_V)\) is equipped with an \(S^1\)-action such that \(t \cdot \theta \circ t^{-1} = t \cdot \theta\) for any \(t \in S^1\). It roughly means that Hodge bundles correspond to the fixed points in the moduli space of Higgs bundles with respect to the natural \(S^1\)-action induced by \(t(E, \partial_E, \theta) = (E, \partial_E, t\theta)\).

By the deformation \((E, \partial_E, \alpha\theta)\) \((\alpha \in \mathbb{C}^*)\), any Higgs bundles is deformed to an \(S^1\)-fixed point in the moduli space, i.e., a Hodge bundle as \(\alpha \to 0\). Note that the Higgs field of the limit is not necessarily 0. Hence, by the equivalence between Higgs bundles and flat bundles, it turns out that any flat bundle is deformed to flat bundle underlying a polarized variation of Hodge structure.

Simpson \[57\] particularly applied these ideas to construct uniformizations of some types of projective manifolds. He also applied it to prove that some type of discrete groups cannot be the fundamental group of any projective manifolds in \[59\].
1.2.2 TE-structure

We recall that a complex variation of Hodge structure on $X$ induces a TE-structure in the sense of Hertling [20], i.e., a holomorphic vector bundle $V$ on $\mathcal{X} := \mathbb{C}^*_\lambda \times X$ with a meromorphic flat connection

$$\nabla : V \rightarrow V \otimes \mathcal{O}_X(\mathcal{X}^0) \otimes \Omega^1_X(\log \mathcal{X}^0),$$

where $\mathcal{X}^0 := \{0\} \times X$. Indeed, for a complex variation of Hodge structure $(V = \bigoplus_{p,q \geq 0} V^{p,q}, \nabla)$, $F^p(V) := \bigoplus_{p \leq q} V^{p,q}$ are holomorphic subbundles with respect to $\nabla^{0,1}$. Thus, we obtain a decreasing filtration of holomorphic subbundles $F^p(V)$ ($p \in \mathbb{Z}$) satisfying the Griffiths transversality $\nabla^{1,0} F^p(V) \subset F^{p-1}(V) \otimes \Omega^{1,0}$. Let $p : \mathbb{C}^*_\lambda \times X \rightarrow X$ denote the projection. We obtain the induced flat bundle $(p^* V, p^* \nabla)$. By the Rees construction, $p^* V$ is extended to a locally free $\mathcal{O}_X$-module $\nabla$, on which $\nabla := p^* \nabla$ is a meromorphic flat connection satisfying the condition $\nabla \nabla \subset V \otimes \mathcal{O}_X(\mathcal{X}^0) \otimes \Omega^1_X(\log \mathcal{X}^0)$.

It is recognized that a TE-structure appears as a fundamental piece of interesting structures in various fields of mathematics. For instance, TE-structure is an ingredient of Frobenius manifold, which is important in the theory of primitive forms due to K. Saito [50], the topological field theory of Dubrovin [17], the tt*-geometry of Cecotti-Vafa [7, 8], the Gromov-Witten theory, the theory of Landau-Ginzburg models, etc. For the construction of Frobenius manifolds, it is an important step to obtain TE-structures. Abstractly, TE-structure is also an important ingredient of semi-infinite variation of Hodge structure [1, 9, 25], TERP structure [20, 21, 22], integrable variation of twistor structure [55], etc. (See also [46, 48].)

1.2.3 Homogeneous harmonic bundles

As Simpson applied his Kobayashi-Hitchin correspondence to construct complex variation of Hodge structure, we may apply Theorem 1.1 to construct TE-structure with something additional. It is done through harmonic bundles with homogeneity as in the Hodge case.

Let $X$ be a complex manifold equipped with an $S^1$-action. Let $(E, \overline{\nabla}_E)$ be an $S^1$-equivariant holomorphic vector bundle. Let $\theta$ be a Higgs field of $(E, \overline{\nabla}_E)$, which is homogeneous with respect to the $S^1$-action, i.e., $t^* \theta = t^m \theta$ for some $m \neq 0$. Let $h$ be an $S^1$-invariant pluri-harmonic metric of $(E, \overline{\nabla}_E, \theta)$. Then, as studied in [48, §3], we naturally obtain a TE-structure. More strongly, it is equipped with a grading in the sense of [9, 25], and it also underlies a polarized integrable variation of pure twistor structure of weight 0 [55]. Moreover, if there exists an $S^1$-equivariant isomorphism between $(E, \overline{\nabla}_E, \theta, h)$ and its dual, the TE-structure is enhanced to a semi-infinite variation of Hodge structure with a grading [1, 9, 25]. If the $S^1$-action on $X$ is trivial, this is the same as the construction of a variation of Hodge structure from a Hodge bundle with a pluri-harmonic metric for which the Hodge decomposition is orthogonal.

Let $H$ be a simple normal crossing hypersurface of $X$. If we are given an $S^1$-homogeneous good wild harmonic bundle $(E, \overline{\nabla}_E, \theta, h)$ on $(X, H)$, as mentioned above, we obtain a TE-structure with a grading on $X \setminus H$. Moreover, it is extended to a meromorphic TE-structure on $(X, H)$ as studied in [48, §3]. We obtain the mixed Hodge structure as the limit objects at the boundary, which is useful for the study of more detailed property of the TE-structure.

1.2.4 An equivalence

Let $X$ be a complex projective manifold with a simple normal crossing hypersurface $H$ and an ample line bundle $L$, equipped with a $\mathbb{C}^*$-action. We may define a good filtered Higgs bundle $(\mathcal{P}_* \mathcal{V}, \theta)$ is called $\mathbb{C}^*$-homogeneous if $\mathcal{P}_* \mathcal{V}$ is $\mathbb{C}^*$-equivariant and $t^* \theta = t^m \cdot \theta$ for some $m \neq 0$. Then, we obtain the following theorem by using Theorem 1.1 (See [8.1.2] for the precise definition of the stability condition in this context.)

**Theorem 1.2 (Corollary 8.10)** There exists an equivalence between the following objects.

- $\mu_L$-polystable $\mathbb{C}^*$-homogeneous good filtered Higgs bundles $(\mathcal{P}_* \mathcal{V}, \theta)$ on $(X, H)$ satisfying
  $$\int_X \text{par-ch}_1(\mathcal{P}_* \mathcal{V})c_1(L)^{\text{dim } X - 1} = \int_X \text{par-ch}_2(\mathcal{P}_* \mathcal{V})c_1(L)^{\text{dim } X - 2} = 0.$$

- $S^1$-homogeneous good wild harmonic bundles on $(X, H)$.
As mentioned in §1.2.3, Theorem 1.2 allows us to obtain a meromorphic TE-structure on \((X, H)\) with a grading from a \(\mu_L\)-polystable \(C^\ast\)-equivariant good filtered Higgs bundle satisfying the vanishing condition. We already applied it to a classification of solutions of the Toda equations on \(C^\ast\) [51]. It seems natural to expect that this construction would be another way to obtain Frobenius manifolds.

Although we explained the homogeneity with respect to an \(S^1\)-action, Theorem 1.2 is generalized for \(G\)-homogeneous good wild harmonic bundles as explained in [58], where \(G\) is any compact Lie group.

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2 Good filtered Higgs bundles and wild harmonic bundles

2.1 Filtered sheaves and filtered Higgs sheaves

2.1.1 Filtered sheaves

Let \(X\) denote a complex manifold with a simple normal crossing hypersurface \(H\). Let \(H = \bigcup_{i \in \Lambda} H_i\) denote the irreducible decomposition. For any \(P \in H\), a holomorphic coordinate neighbourhood \((X_P, z_1, \ldots, z_n)\) around \(P\) is called admissible if \(H_P := H \cap X_P = \bigcup_{i=1}^{\ell(P)} \{z_i = 0\}\). For such an admissible coordinate neighbourhood, there exists the map \(\rho_P : \{1, \ldots, \ell(P)\} \to \Lambda\) determined by \(H_{\rho_P(i)} \cap X_P = \{z_i = 0\}\). We obtain the map \(\kappa_P : \mathbb{R}^\Lambda \to \mathbb{R}^{\ell(P)}\) by \(\kappa_P(a) = (a_{\rho(1)}, \ldots, a_{\rho(\ell(P))})\).

Let \(\mathcal{E}\) be any coherent torsion free \(\mathcal{O}_X(*H)\)-module. A filtered sheaf over \(\mathcal{E}\) is defined to be a tuple of coherent \(\mathcal{O}_X\)-submodules \(\mathcal{P}_a \mathcal{E} \subseteq \mathcal{E}(a \in \mathbb{R}^\Lambda)\) satisfying the following conditions.

- \(\mathcal{P}_a \mathcal{E} \subseteq \mathcal{P}_b \mathcal{E}\) if \(a \leq b\), i.e., \(a_i \leq b_i\) for any \(i \in \Lambda\).
- \(\mathcal{P}_a \mathcal{E}(\ast H) = \mathcal{E}\) for any \(a \in \mathbb{R}^\Lambda\).
- \(\mathcal{P}_{a+n} \mathcal{E} = \mathcal{P}_a \mathcal{E} \left(\sum_{i \in \Lambda} n_i H_i\right)\) for any \(a \in \mathbb{R}^\Lambda\) and \(n \in \mathbb{Z}^\Lambda\).
- For any \(a \in \mathbb{R}^\Lambda\) there exists \(\epsilon \in \mathbb{R}^\Lambda_{>0}\) such that \(\mathcal{P}_{a+\epsilon} \mathcal{E} = \mathcal{P}_a \mathcal{E}\).
- For any \(P \in H\), we take an admissible coordinate neighbourhood \((X_P, z_1, \ldots, z_n)\) around \(P\). Then, for any \(a \in \mathbb{R}^\Lambda\), \(\mathcal{P}_a \mathcal{E}|_{X_P}\) depends only on \(\kappa_P(a)\).

For any coherent \(\mathcal{O}_X(*H)\)-submodule \(\mathcal{E}' \subset \mathcal{E}\), we obtain a filtered sheaf \(\mathcal{P}_a \mathcal{E}'\) over \(\mathcal{E}'\) by \(\mathcal{P}_a \mathcal{E}' := \mathcal{P}_a \mathcal{E} \cap \mathcal{E}'\). If \(\mathcal{E}'\) is saturated, i.e., \(\mathcal{E}'' := \mathcal{E}/\mathcal{E}'\) is torsion-free, we obtain a filtered sheaf \(\mathcal{P}_a \mathcal{E}''\) over \(\mathcal{E}''\) by \(\mathcal{P}_a \mathcal{E}'' := \text{Im}(\mathcal{P}_a \mathcal{E} \to \mathcal{E}'')\).

A morphism of filtered sheaves \(f : \mathcal{P}_a \mathcal{E}_1 \to \mathcal{P}_a \mathcal{E}_2\) is defined to be a morphism \(f : \mathcal{E}_1 \to \mathcal{E}_2\) of \(\mathcal{O}_X(*H)\)-modules such that \(f(\mathcal{P}_a \mathcal{E}_1) \subset \mathcal{P}_a \mathcal{E}_2\) for any \(a \in \mathbb{R}^\Lambda\).

**Remark 2.1** The concept of filtered bundles on curves was introduced by Mehta and Seshadri [12] and Simpson [57, 58]. A higher dimensional version was first studied by Maruyama and Yokogawa [39] for the purpose of the construction of the moduli spaces.
2.1.2 Reflexive filtered sheaves

A filtered sheaf $\mathcal{P},\mathcal{E}$ on $(X, H)$ is called reflexive if each $\mathcal{P}_a\mathcal{E}$ is a reflexive $\mathcal{O}_X$-module. Note that it is equivalent to the “reflexive and saturated” condition in [43, Definition 3.17] by the following lemma.

Lemma 2.2 Suppose that $\mathcal{P}_*\mathcal{E}$ is reflexive. Let $a \in \mathbb{R}^\Lambda$. We take $a_i-1 < b \leq a_i$, and let $a' \in \mathbb{R}^\Lambda$ be determined by $a'_j = a_j$ ($j \neq i$) and $a'_i = b$. Then, $\mathcal{P}_a\mathcal{E}/\mathcal{P}_{a'}\mathcal{E}$ is a torsion-free $\mathcal{O}_{H_i}$-module.

Proof Let $s$ be a section of $\mathcal{P}_a\mathcal{E}/\mathcal{P}_{a'}\mathcal{E}$ on an open set $U \subset D_i$. There exists an open subset $\tilde{U} \subset X$ and a section $\tilde{s}$ of $\mathcal{P}_a\mathcal{E}$ on $\tilde{U}$ such that $\tilde{U} \cap D_i = U$ and that $\tilde{s}$ induces $s$. Note that there exists $Z \subset \tilde{U}$ of codimension 2 such that $\tilde{s}|_{\tilde{U} \setminus Z}$ is a section of $\mathcal{P}_{a'}\mathcal{E}|_{\tilde{U} \setminus Z}$. Because $\mathcal{P}_{a'}\mathcal{E}$ is reflexive, there exists a section $\tilde{s}'$ of $\mathcal{P}_{a'}\mathcal{E}$ on $\tilde{U}$ such that $\tilde{s}'|_{\tilde{U} \setminus Z} = \tilde{s}|_{\tilde{U} \setminus Z}$. Hence, we obtain that $\tilde{s}$ is a section of $\mathcal{P}_a\mathcal{E}$, i.e., $s = 0$.

The following lemma is clear.

Lemma 2.3 Let $\mathcal{P}_*\mathcal{E}$ be a reflexive filtered sheaf on $(X, H)$. Then a coherent $\mathcal{O}_X(*H)$-submodule $\mathcal{E}' \subset \mathcal{E}$ is saturated if and only if the induced filtered sheaf $\mathcal{P}_*\mathcal{E}'$ is reflexive.

2.1.3 Filtered Higgs sheaves

Let $\mathcal{E}$ be a coherent torsion-free $\mathcal{O}_X(*H)$-module. Let $\theta: \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ be an $\mathcal{O}_X$-linear morphism. Note $\theta \wedge \theta: \mathcal{E} \to \mathcal{E} \otimes \Omega^2_X$ is induced by the composition of morphisms and the wedge product. If $\theta \wedge \theta = 0$ is satisfied, $\theta$ is called a Higgs field of $\mathcal{E}$. When a Higgs field $\theta$ is given, a Higgs subsheaf of $\mathcal{E}$ means a coherent $\mathcal{O}_X$-submodule $\mathcal{E}' \subset \mathcal{E}$ such that $\theta(\mathcal{E}') \subset \mathcal{E}' \otimes \Omega^1_X$. A pair of a filtered sheaf $\mathcal{P}_*\mathcal{E}$ over $\mathcal{E}$ and a Higgs field $\theta$ of $\mathcal{E}$ is called a filtered Higgs sheaf. It is called reflexive if $\mathcal{P}_*\mathcal{E}$ is reflexive.

2.2 $\mu_L$-Stability condition for filtered Higgs sheaves

Let $X$ be a connected projective manifold with a simple normal crossing hypersurface $H = \bigcup_{i \in \Lambda} H_i$. Let $L$ be an ample line bundle.

2.2.1 Slope of filtered sheaves

Let $\mathcal{P}_*\mathcal{E}$ be a filtered sheaf on $(X, H)$ which is not necessarily a filtered bundle. Recall that par-$c_1(\mathcal{P}_*\mathcal{E})$ is defined as follows. Let $\eta_i$ be the generic point of $H_i$. Note that $\mathcal{O}_{X,\eta_i}$-modules $(\mathcal{P}_a\mathcal{E})_{\eta_i}$ depends only on $a_i$, which is denoted by $\mathcal{P}_a(\mathcal{E}_{\eta_i})$. We obtain $\mathcal{O}_{H_i,\eta_i}$-modules $\text{Gr}_0(\mathcal{P}_a(\mathcal{E}_{\eta_i})) := \mathcal{P}_a(\mathcal{E}_{\eta_i})/\mathcal{P}_{<a}(\mathcal{E}_{\eta_i})$. Then, we have

$$\text{par-c}_1(\mathcal{P}_*\mathcal{E}) = c(\mathcal{P}_a\mathcal{E}) - \sum_{i \in \Lambda} \sum_{a_i-1 < a \leq a_i} \text{rank} \text{Gr}_0(\mathcal{P}_a(\mathcal{E}_{\eta_i}))[H_i].$$

We set

$$\mu_L(\mathcal{P}_*\mathcal{E}) := \frac{1}{\text{rank} \mathcal{E}} \int_X \text{par-c}_1(\mathcal{P}_*\mathcal{E}) \cdot c_1(L)^{n-1}.$$

It is called the slope of $\mathcal{P}_*\mathcal{E}$ with respect to $L$. The following is proved in [43, Lemma 3.7].

Lemma 2.4 Let $f: \mathcal{P}_*\mathcal{E}^{(1)} \to \mathcal{P}_*\mathcal{E}^{(2)}$ be a morphism of filtered sheaves which is generically an isomorphism, i.e., the induced morphism $\mathcal{E}^{(1)}_{\eta(X)} \to \mathcal{E}^{(2)}_{\eta(X)}$ at the generic point of $X$ is an isomorphism. Then, $\mu_L(\mathcal{P}_*\mathcal{E}^{(1)}) \leq \mu_L(\mathcal{P}_*\mathcal{E}^{(2)})$ holds. If the equality holds, $f$ is an isomorphism in codimension one, i.e., there exists an algebraic subset $Z \subset X$ such that (i) the codimension of $Z$ is larger than 2, (ii) $f^{|X\setminus Z}: \mathcal{P}_*|_{X\setminus Z} \to \mathcal{P}_*|_{X\setminus Z}$ is an isomorphism.
2.2.2 $\mu_L$-Stability condition

A filtered Higgs sheaf $(P, E, \theta)$ on $(X, H)$ is called $\mu_L$-stable (resp. $\mu_L$-semistable) if the following holds.

- Let $E' \subset E$ be any Higgs $O_X(*H)$-submodule such that $E' \neq 0$ and $E' \neq E$. Then, $\mu_L(P, E') < \mu_L(P, E)$ (resp. $\mu_L(P, E') \leq \mu_L(P, E)$) holds.

A filtered Higgs sheaf $(P, E, \theta)$ is called $\mu_L$-polystable if the following holds.

- $(P, E, \theta)$ is $\mu_L$-semistable.
- $(P, E, \theta) = \bigoplus (P, E_i, \theta_i)$, where each $(P, E_i, \theta_i)$ is $\mu_L$-stable.

The following is standard. (See [43, §3.1.3] and [45, §2.1.4].)

Lemma 2.5 Suppose that $(P, E, \theta)$ is a $\mu_L$-polystable reflexive filtered Higgs sheaf. Then, there exists a decomposition $(P, E, \theta) = \bigoplus_{i=1}^N (P, E_i, \theta_i) \otimes C^n(i)$ such that (i) $(P, E_i, \theta_i)$ are $\mu_L$-stable, (ii) $\mu_L(P, E_i) = \mu_L(P, E)$, (iii) $(P, E_i, \theta_i) \neq (P, E_j, \theta_j)$ (i $\neq$ j).

Remark 2.6 In [43, §3.1.3], “the inequality $\text{par-deg}_L(E') < \text{par-deg}_L(E_*)$” should be corrected to “the inequality $\mu_L(E'_*) < \mu_L(E_*)$”.

2.3 Filtered bundles

2.3.1 Filtered bundles in the local case

We explain the notion of filtered bundle in the local case. We shall explain it in the global case in 2.3.3. Let $U$ be a neighbourhood of $(0, \ldots, 0)$ in $C^n$. We set $H_U, i := U \cap \{z_i = 0\}$, and $H_U := \bigcup_{i=1}^r H_U, i$ for some $0 \leq \ell \leq n$. Let $V$ be a locally free $O_U(*H_U)$-module. A filtered bundle $P, V$ over $V$ is a tuple of locally free $O_U$-submodules $P_a V$ ($a \in \mathbb{R}^\ell$) such that the following holds.

- $P_a V \subset P_b V$ if $a \leq b$, i.e., $a_i \leq b_i$ for any $i = 1, \ldots, \ell$.
- There exist a frame $v = (v_1, \ldots, v_r)$ of $V$ and tuples $a(v_j) \in \mathbb{R}^\ell$ ($j = 1, \ldots, r$) such that

$$P_b V = \bigoplus_{j=1}^r O_U \left( \sum_{i} [b_i - a_i(v_j)] H_{U, i} \right) \cdot v_j,$$

where we set $[c] := \max\{p \in \mathbb{Z} | p \leq c\}$ for any $c \in \mathbb{R}$.

2.3.2 Pull back, push-forward and descent with respect to ramified coverings in the local case

Let $\varphi : C^n \rightarrow C^n$ be given by $\varphi(\zeta_1, \ldots, \zeta_n) = (\zeta_1^{m_1}, \ldots, \zeta_{\ell+1}^{m_{\ell+1}}, \ldots, \zeta_n)$. We set $U' := \varphi^{-1}(U)$, $H_{U', i} := \varphi^{-1}(H_{U, i})$ and $H_{U'} := \varphi^{-1}(H_U)$. The induced ramified covering $U' \rightarrow U$ is also denoted by $\varphi$.

For any $b \in \mathbb{R}^\ell$, we set $\varphi^*(b) = (m_i b_i) \in \mathbb{R}^\ell$. For any filtered bundle $P_a V_1$ on $(U, H_U)$, we define a filtered bundle $P_a V_1'$ on $(U', H_{U'})$ as follows:

$$P_a V_1' = \sum_{\varphi^*(b) + n \leq a} \varphi^*(P_b V_1) \left( \sum n_i H_{U', i} \right).$$

We set $\varphi^*(P_a V_1) := P_a V_1'$. Thus, we obtain the pull back functor $\varphi^*$ from the category of filtered bundles on $(U, H_U)$ to the category of filtered bundles on $(U', H_{U'})$.

For any $b \in \mathbb{R}^\ell$, we set $\varphi_*(b) = (m_i^{-1} b_i)$. For any filtered bundle $P_a V_2$ on $(U', H_{U'})$, we obtain the following filtered bundle

$$P_b \varphi_*(V_2) := \varphi_*(P_b V_2).$$

In this way, we obtain a functor $\varphi_*$ from the category of filtered bundles on $(U', H_{U'})$ to the category of filtered bundles on $(U, H_U)$. 

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We set $G := \prod_{i=1}^{\ell} \{ \mu_i \in \mathbb{C}^* \ | \mu_i^{m_i} = 1 \}$. We define the action of $G$ on $U'$ by $(\mu_1, \ldots, \mu_\ell)(\zeta_1, \ldots, \zeta_\ell) = (\mu_1 \zeta_1, \ldots, \mu_\ell \zeta_\ell, \zeta_{\ell+1}, \ldots, \zeta_n)$. We identify $G$ as the Galois group of the ramified covering $U' \to U$. Let $\mathcal{P}_H\mathcal{V}_3$ be a $G$-equivariant filtered bundles on $(U', H_{U'})$. Then, $\mathcal{P}_H\mathcal{V}_3$ is equipped with $G$-action. We obtain a filtered bundle $(\mathcal{P}_H\mathcal{V}_3)^G$ on $(U, H_U)$ as the $G$-invariant part of $\mathcal{P}_H\mathcal{V}_3$, which is called the descent of $\mathcal{P}_H\mathcal{V}_3$ with respect to the $G$-action. In this way, we obtain a functor from the category of $G$-equivariant filtered bundles on $(U', H_{U'})$ to the category of filtered bundles on $(U, H_U)$.

### 2.3.3 Filtered bundles in the global case

We use the notation in [2.1.1]. Let $\mathcal{V}$ be a locally free $\mathcal{O}_X(\ast H)$-module. A filtered bundle $\mathcal{P}_H\mathcal{V} = (\mathcal{P}_a \mathcal{V} \ | \ a \in \mathbb{R}^\Lambda)$ be a sequence of locally free $\mathcal{O}_X$-submodules $\mathcal{P}_a \mathcal{V}$ of $\mathcal{V}$ such that the following holds.

- For any $P \in H$, we take an admissible coordinate neighbourhood $(X_P, z_1, \ldots, z_n)$ around $P$. Then, for any $a \in \mathbb{R}^\Lambda$, $\mathcal{P}_a \mathcal{V}|_{X_P}$ depends only on $\kappa_P(a)$. We denote $\mathcal{P}_{\kappa_P(a)}(\mathcal{V}|_{X_P}) := \mathcal{P}_a(\mathcal{V})|_{X_P}$.

- The sequence $(\mathcal{P}_b(\mathcal{V}|_{X_P}) \ | \ b \in \mathbb{R}^{\ell(P)})$ is a filtered bundle over $\mathcal{V}|_{X_P}$ in the sense of [2.3.1].

Clearly, a filtered bundle is a special type of filtered sheaf in [2.1.1].

### Remark 2.7

The higher dimensional version of filtered bundles was introduced in [44] with a different formulation. See also [5, 6]. In this paper, we follow Iyer and Simpson [26].

### 2.3.4 The induced bundles and filtrations

For any $I \subset \Lambda$, let $\delta_I \in \mathbb{R}^\Lambda$ be the element whose $j$-th component is $0$ ($j \notin I$) or $1$ ($j \in I$). We also set $H_I := \cap_{i \in I} H_i$.

Let $\mathcal{P}_H\mathcal{V}$ be a filtered bundle on $(X, H)$. Take $i \in \Lambda$. Let $a \in \mathbb{R}^\Lambda$. For any $a_i - 1 \leq b \leq a_i$, we set $a(b) := a + (b - a_i)\delta_i$. We set

$$\mathcal{F}_b(\mathcal{P}_a(\mathcal{V}|_{H_I})) := \mathcal{P}_a(\mathcal{V})/\mathcal{P}_{a(a_i - 1)}(\mathcal{V}).$$

It is naturally regarded as a locally free $\mathcal{O}_{H_I}$-module. Moreover, it is a subbundle of $\mathcal{P}_a(\mathcal{V}|_{H_I})$. In this way, we obtain a filtration $\mathcal{F}$ of $\mathcal{P}_a(\mathcal{V}|_{H_I})$ indexed by $|a_i - 1, a_i|$. We obtain the induced filtrations $\mathcal{F}$ of $\mathcal{P}_a(\mathcal{V}|_{H_I})$ if $i \in I$. Let $a_I \in \mathbb{R}^I$ denote the image of $a$ by the projection $\mathbb{R}^\Lambda \to \mathbb{R}^I$. Set $|a_I - \delta_I, a_I| := \prod_{i \in I}|a_i - 1, a_i|$. For any $b \in |a_I - \delta_I, a_I|$, we set

$$\mathcal{F}_b(\mathcal{P}_a(\mathcal{V}|_{H_I})) := \bigcap_{i \in I} \mathcal{F}_b(\mathcal{P}_a(\mathcal{V}|_{H_i})).$$

By the condition of filtered bundles, the following compatibility condition holds.

- Let $P$ be any point of $H_I$. There exists a neighbourhood $X_P$ of $P$ in $X$ and a decomposition

$$\mathcal{P}_a\mathcal{V}|_{X_P \cap H_I} = \bigoplus_{b \in |a_I - \delta_I, a_I|} \mathcal{G}_P(b)$$

such that the following holds for any $c \in |a_I - \delta_I, a_I|$:

$$\mathcal{F}_c(\mathcal{P}_a(\mathcal{V}|_{H_I \cap X_P})) = \bigoplus_{b \leq c} \mathcal{G}_P(b).$$

For any $c \in |a_I - \delta_I, a_I|$, we obtain the following locally free $\mathcal{O}_{H_I}$-modules:

$$\text{Gr}^F_c(\mathcal{P}_a\mathcal{V}) := \frac{\mathcal{F}_c(\mathcal{P}_a\mathcal{V}|_{H_I})}{\sum_{b \leq c} \mathcal{F}_b(\mathcal{P}_a\mathcal{V}|_{H_I})}.$$

Here, $b \leq c$ means “$b \leq c$ and $b \neq c$”.

We introduce some notation. We set $\mathcal{P}ar(\mathcal{P}_a\mathcal{V}, I) := \{ b \in \mathbb{R}^I \ | \text{Gr}^F_b(\mathcal{P}_a\mathcal{V}) \neq 0, \exists a \in \mathbb{R}^\Lambda \}$. Let $\text{Irr}(H_I)$ be the irreducible decomposition of $H_I$. For $C \in \text{Irr}(H_I)$, let $\text{Gr}^F_c(\mathcal{P}_a\mathcal{V})$ denote the restriction of $\text{Gr}^F_c(\mathcal{P}_a\mathcal{V})$ to $C$.  

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2.3.5 First and second Chern characters for filtered bundles

Let \( P, V \) be a filtered bundle over \((X, H)\). Take any \( a \in \mathbb{R}^A \). We set

\[
\text{par-ch}_2(P, V) := ch_2(P_a V) - \sum_{i \in A} a_i \sum_{1 < b \leq a_i} \text{rank}^i \text{Gr}^i_F(\mathcal{E}|_{H_i}) \cdot [H_i] \in \mathbb{H}^2(X, \mathbb{R}).
\]

Here, \([H_i]\) denote the cohomology class induced by \( H_i \). It is easy to see that \( \text{par-ch}_2(P, V) \) is independent of a choice of \( a \in \mathbb{R}^A \). We also obtain the following element in \( \mathbb{H}^4(X, \mathbb{R}) \):

\[
\text{par-ch}_1(P, V) := ch_1(P_a V) - \sum_{i \in A} a_i \sum_{1 < b \leq a_i} \text{rank}^i \text{Gr}^i_F(\mathcal{E}|_{H_i}) \cdot [H_i] \]

\[
+ \frac{1}{2} \sum_{i \in A} a_i \sum_{1 < b \leq a_i} b^2 \text{rank}^i \text{Gr}^i_F(\mathcal{E}) \cdot [H_i]^2
\]

\[
+ \frac{1}{2} \sum_{(i,j) \in A^2 C \in \text{Ir}(H_i \cap H_j)} \sum_{a_i-1 < c_i \leq a_i} \sum_{a_j-1 < c_j \leq a_j} c_i \cdot c_j \text{rank}^i \text{Gr}^i_F(\mathcal{E}) \cdot [C].
\]

Here, \( i : H^2(H_i, \mathbb{R}) \rightarrow H^4(X, \mathbb{R}) \) denote the Gysin map induced by \( i : H_i \rightarrow X \), and \([C]\) denote the cohomology class induced by \( C \).

**Remark 2.8** The higher Chern character for filtered sheaves was defined by Iyer and Simpson [20] in a systematic way. In this paper, we adopt the definition of \( \text{par-ch}_1(P, V) \) in [26].

2.4 Good filtered Higgs bundles

Let \( X \) be a complex manifold with a simple normal crossing hypersurface \( H = \bigcup_{i \in A} H_i \).

2.4.1 Good set of irregular values at \( P \)

Let \( P \) be any point of \( H \). We take an admissible holomorphic coordinate neighbourhood \((X_P, z_1, \ldots, z_n)\) around \( P \). Let \( f \in O_X(*H)_{P} \). If \( f \in O_{X,P} \), we set \( \text{ord}(f) := (0, \ldots, 0) \in \mathbb{R}^{|P|} \). If there exists \( n \in \mathbb{Z}_{\leq 0}^{(P)} \setminus \{0, \ldots, 0\} \) such that (i) \( g := f \prod_i z_i^{n_i} \in O_{X,P} \), (ii) \( g(P) \neq 0 \), then we set \( \text{ord}(f) := n \). Otherwise, \( \text{ord}(f) \) is not defined.

For any \( a \in O_X(*H)_{P}/O_{X,P} \), we take a lift \( \tilde{a} \in O_X(*H)_{P} \). If \( \text{ord}(\tilde{a}) \) is defined, we set \( \text{ord}(a) := \text{ord}(\tilde{a}) \). Otherwise, \( \text{ord}(a) \) is not defined. Note that it is independent of the choice of a lift \( \tilde{a} \).

Let \( I_P \subset O_X(*H)_{P}/O_{X,P} \) be a finite subset. We say that \( I_P \) is a good set of irregular values if the following holds.

- \( \text{ord}(a) \) is defined for any \( a \in I_P \).
- \( \text{ord}(a - b) \) is defined for any \( a, b \in I_P \).
- \( \{\text{ord}(a - b) | a, b \in I\} \) is totally ordered with respect to the order \( \leq_{\mathbb{Z}^{P}} \). Here, we define \( n \leq n' \) if \( n_i \leq n'_i \) for any \( i \).

2.4.2 Good filtered Higgs bundles

Let \( V \) be a locally free \( O_X(*H) \)-module with a Higgs field \( \theta \). Let \( P, V \) be a filtered bundle over \( V \). We say that \((P_a V, \theta)\) is unramifiedly good at \( P \) if the following holds.

- There exist a good set of irregular values \( I_P \subset O_X(*H)_{P}/O_{X,P} \), an admissible holomorphic coordinate neighbourhood \((X_P, z_1, \ldots, z_n)\) around \( P \), and a decomposition

\[
(P_a V, \theta)_{|X_P} = \bigoplus_{a \in I_P} (P_a V_a, \theta_a)
\]
such that $\theta_a - d\tilde{a} \text{id}_{V_a}$ are logarithmic with respect to the lattice $\mathcal{P}_a V_a$ for any $a \in \mathbb{R}^{\ell(P)}$ and $a \in \mathcal{I}_P$, i.e.,

$$(\theta_a - d\tilde{a} \text{id}_{V_a})\mathcal{P}_a V_a \subset \mathcal{P}_a V_a \otimes \Omega^1_X \log(H).$$

Here $\tilde{a}$ denote lifts of $a$ to $O_X(\ast H)_P$.

We say that $(\mathcal{P}_*, \mathcal{V}, \theta)$ is good at $P$ if the following holds.

- There exist a neighbourhood $X_P$ of $P$ in $X$ and a covering map $\varphi_P : X'_P \to X_P$ ramified over $H_P = H \cap X_P$ such that $\varphi_P^*(\mathcal{P}_*, \mathcal{V}, \theta)$ is unramifiedly good at any point of $\varphi_P^{-1}(H_P)$.

We say that $(\mathcal{P}_*, \mathcal{V}, \theta)$ is good (resp. unramifiedly good) if it is good (resp. unramifiedly good) at any point of $H$.

### 2.5 Prolongation of holomorphic vector bundles with a Hermitian metric

Let $X$ be any complex manifold with a simple normal crossing hypersurface $H = \bigcup_{i \in \Lambda} H_i$. Let $(E, \mathcal{O}_E)$ be a holomorphic vector bundle on $X \setminus H$ with a Hermitian metric $h$. Let us recall the construction of $O_X(\ast H)$-module $\mathcal{P}^h E$ and $O_X$-modules $\mathcal{P}^h a E$ ($a \in \mathbb{R}^\Lambda$).

Let $a \in \mathbb{R}^\Lambda$. For any open subset $U \subset X$, let $\mathcal{P}_a^h E(U)$ be the space of holomorphic sections of $E_{|U \setminus H}$ satisfying the following condition.

- For any point $P$ of $U \cap H$, take an open admissible holomorphic coordinate neighbourhood $(X_P, z_1, \ldots, z_n)$ around $P$ such that $X_P$ is relatively compact in $U$. Set $c = \kappa_P(a)$. Then,

$$|s|_h = O \left( \prod_{i=1}^{\ell(P)} |z_i|^{c_i - \varepsilon} \right)$$

holds on $X_P \setminus H$ for any $\varepsilon > 0$.

Thus, we obtain an $O_X$-module $\mathcal{P}_a^h E$. We set $\mathcal{P}^h E := \bigcup_{a \in \mathbb{R}^\Lambda} \mathcal{P}^h a E$ which is an $O_X(\ast H)$-module.

**Remark 2.9** In general, $\mathcal{P}_a^h E$ are not necessarily coherent $O_X$-modules.

**Definition 2.10** Let $\mathcal{P}_*, \mathcal{V}$ be a filtered bundle over $(X, H)$. Let $(E, \mathcal{O}_E)$ be the holomorphic vector bundle obtained as the restriction of $\mathcal{V}$ to $X \setminus H$. A Hermitian metric $h$ is called adapted to $\mathcal{P}_*, \mathcal{V}$ if $\mathcal{P}^h a E = \mathcal{P}_a E$ in $\iota_*(E) = \iota_*(\mathcal{V}|_{X \setminus H})$, where $\iota : X \setminus H \to X$ denotes the inclusion.

**2.5.1 A sufficient condition to be filtered bundles**

We mention a useful sufficient condition for $\mathcal{P}_a^h E$ to be a filtered bundle, although we do not use it in this paper. Let $g_{X \setminus H}$ be a Kähler metric satisfying the following condition $[\mathbf{I}]$:

- For any $P \in H$, we take an admissible holomorphic coordinate neighbourhood $(X_P, z_1, \ldots, z_n)$ around $P$ such that $X_P$ is isomorphic to $\prod_{i=1}^n \{ |z_i| < 1 \}$ by the coordinate system. Set $X'_P := \prod_{i=1}^n \{ |z_i| < 1/2 \}$. Then, $g_{X'_P \setminus H}$ is mutually bounded with the restriction of the Poincaré metric

$$\sum_{i=1}^{\ell(P)} \frac{d z_i \bar{d} z_i}{|z_i|^2 (\log |z_i|^2)^2} + \sum_{i=\ell(P)+1}^n d z_i \bar{d} z_i.$$ 

A Hermitian metric $h$ of $(E, \mathcal{O}_E)$ is called acceptable if the curvature of the Chern connection is bounded with respect to $h$ and $g_{X \setminus H}$. The following theorem is proved in [17, Theorem 21.3.1].

**Theorem 2.11** If $h$ is acceptable, then $\mathcal{P}_a^h E$ is a filtered bundle, and $\mathcal{P}^h E$ is a locally free $O_X(\ast H)$-module.
2.6 Wild harmonic bundles

2.6.1 Harmonic bundles

Let $Y$ be any complex manifold. Let $(E, \overline{\nabla}_E, \theta)$ be a Higgs bundle on $Y$. Let $h$ be a Hermitian metric of $E$. We obtain the Chern connection $\nabla_h = \overline{\nabla}_E + \partial_{E,h}$. Let $R(h)$ denote the curvature of $\nabla_h$. We also obtain the adjoint $\theta_h^\dagger$ of $\theta$ with respect to $h$. The metric $h$ is called pluri-harmonic if

$$((\overline{\nabla}_E + \theta_h^\dagger)^2 = 0, \quad R(h) + [\theta, \theta_h^\dagger] = 0, \quad (\partial_{E,h} + \theta)^2 = 0.$$ 

A Higgs bundle $(E, \overline{\nabla}_E, \theta)$ with a pluri-harmonic metric $h$ is called a harmonic bundle.

2.6.2 Wild harmonic bundles

Let $X$ be a complex manifold with a simple normal crossing hypersurface $H = \bigcup_{i \in \Lambda} H_i$. Let $(E, \overline{\nabla}_E, \theta, h)$ be a harmonic bundle on $X \setminus H$. It is called wild on $(X, H)$ if the following holds.

- Let $\Sigma_\theta \subset T^* (X \setminus H)$ denote the spectral cover of $\theta$, i.e., $\Sigma_\theta$ denotes the support of the coherent $\mathcal{O}_{T^* (X \setminus H)}$-module induced by $(E, \overline{\nabla}_E, \theta)$. Then, the closure of $\Sigma_\theta$ in the projective completion of $T^* X$ is complex analytic.

A wild harmonic bundle $(E, \overline{\nabla}_E, \theta, h)$ is called unramifiedly good at $P \in H$ if the following holds.

- There exists a good set of irregular values $\mathcal{I}_P \subset \mathcal{O}_X(^* H)/\mathcal{O}_X, P$, a neighbourhood $X_P$, and a decomposition

$$(E, \overline{\nabla}_E, \theta)_{|X_P \setminus H} = \bigoplus_{a \in \mathcal{I}_P} (E_a, \overline{\nabla}_{E_a}, \theta_a)$$

such that the closure of the spectral cover $\Sigma_a$ of $\theta_a - d a \text{id}_{E_a}$ in $T^* X_P (\log (X_P \cap H))$ is proper over $X_P$, where $\tilde{a}$ denote lifts of $a$ to $\mathcal{O}_X(^* H)$.

A wild harmonic bundle $(E, \overline{\nabla}_E, \theta, h)$ is called good at $P \in H$ if the following holds.

- There exist a neighbourhood $X_P$ and a covering $\varphi_P : X'_P \longrightarrow X_P$ ramified along $H'_P$ such that the pull back $\varphi_P^{-1} (E, \overline{\nabla}_E, \theta, h)_{|X_P}$ is unramifiedly good wild at any point of $\varphi_P^{-1} (H)$.

We say that $(E, \overline{\nabla}_E, \theta, h)$ is good wild (resp. unramifiedly good wild) on $(X, H)$ if it is good wild (resp. unramifiedly good wild) at any point of $H$.

Note that not every wild harmonic bundle on $(X, H)$ is good on $(X, H)$. But, the following is known [50, Corollary 15.2.8].

**Theorem 2.12** Let $(E, \overline{\nabla}_E, \theta, h)$ be a wild harmonic bundle on $(X, H)$. Then, there exists a proper birational morphism $\varphi : X' \longrightarrow X$ of complex manifolds such that (i) $H' := \varphi^{-1} (H)$ is simple normal crossing, (ii) $X' \setminus H' \simeq X \setminus H$, (iii) $\varphi^{-1} (E, \overline{\nabla}_E, \theta, h)$ is good wild on $(X', H')$.

The following is one of the fundamental theorem in the study of wild harmonic bundles [47, Theorem 7.4.3].

**Theorem 2.13** If $(E, \overline{\nabla}_E, \theta, h)$ is a good wild harmonic bundle on $(X, H)$, then $(\mathcal{P}_h^k E, \theta)$ is a good filtered Higgs bundle on $(X, H)$.

The following is a consequence of the norm estimate for good wild harmonic bundles [47, Theorem 11.7.2].

**Theorem 2.14** If $h_1$ is another pluri-harmonic metric of $(E, \overline{\nabla}_E, \theta)$ such that $\mathcal{P}_h^{k_1} E = \mathcal{P}_h^{k_1} E$. Then, $h$ and $h_1$ are mutually bounded.
2.6.3 Prolongation of good wild harmonic bundles in the projective case

Suppose that $X$ is projective and connected. Let $L$ be any ample line bundle on $X$. The following is proved in [47 Proposition 13.6.1, Proposition 13.6.4].

**Proposition 2.15** Let $(E, \overline{\Omega}_E, \theta, h)$ be a good wild harmonic bundle on $(X, H)$.

- $(\mathcal{P}_E, h)\text{-polystable}$ with $\mu_L(\mathcal{P}_E) = 0$.

- Let $h'$ be another pluri-harmonic metric of $(E, \overline{\Omega}_E, \theta, h)$ such that $\mathcal{P}_E = \mathcal{P}_E^h$. Then, there exists a decomposition of the Higgs bundle $(E, \overline{\Omega}_E, \theta) = \bigoplus (E_j, \overline{\Omega}_{E_j}, \theta_j)$ such that (i) the decomposition is orthogonal with respect to both $h$ and $h'$, (ii) $h_{|E_j} = a_i \cdot h_{|E}$, for some $a_i > 0$.

- Let $(\mathcal{P}_\mathcal{V}, \mathcal{V}_1, \theta_1)$ be any direct summand of $(\mathcal{P}_E, \mathcal{V}, \theta)$. Let $(E_1, \overline{\Omega}_{E_1}, \theta_1)$ be the Higgs bundle on $X \setminus H$ obtained as the restriction of $(\mathcal{V}_1, \theta_1)$, and let $h_1$ be the metric of $E_1$ induced by $h$. Then, $(E_1, \overline{\Omega}_{E_1}, \theta_1, h_1)$ is a harmonic bundle. In particular, we have $\par-c_1(\mathcal{P}_\mathcal{V}, \mathcal{V}_1) = 0$ and $\int_X \par-c_2(\mathcal{P}_\mathcal{V}, \mathcal{V}_1)c_1(L)\dim X - 2 = 0$. 

2.7 Main existence theorem in this paper

Let $X$ be a smooth connected projective complex manifold with a simple normal crossing hypersurface $H$. Let $L$ be any ample line bundle on $X$. Let $(\mathcal{P}_\mathcal{V}, \theta)$ be a good filtered Higgs bundle on $(X, H)$. Let $(E, \overline{\Omega}_E, \theta)$ be the Higgs bundle obtained as the restriction of $(\mathcal{P}_\mathcal{V}, \theta)$ to $X \setminus H$.

**Theorem 2.16** Suppose that $(\mathcal{P}_\mathcal{V}, \theta)$ is $\mu_L$-polystable, and the following vanishing:

$$\mu_L(\mathcal{P}_\mathcal{V}) = 0, \quad \int_X \par-c_2(\mathcal{P}_\mathcal{V})c_1(L)\dim X - 2 = 0. \tag{4}$$

Then, there exists a pluri-harmonic metric $h$ of $(E, \overline{\Omega}_E, \theta)$ such that the isomorphism $(\mathcal{V}, \theta)|_{X \setminus H} \simeq (E, \theta)$ is extended to $(\mathcal{P}_\mathcal{V}, \theta) \simeq (\mathcal{P}_E, \mathcal{V}, \theta)$.

We proved a similar theorem for good filtered flat bundles in [47 Theorem 16.1.1]. Theorem 2.16 can be proved similarly and more easily on the basis of the fundamental theorem of Simpson [57] after [43, 45]. We shall explain a proof in [47]. Note that the one dimensional case is due to Biquard-Boalch [3].

**Corollary 2.17** We have the equivalence of the following objects.

- Good wild harmonic bundles on $(X, H)$.

- $\mu_L$-polystable good filtered Higgs bundles $(\mathcal{P}_\mathcal{V}, \theta)$ satisfying the vanishing condition (4).

3 Preliminaries

3.1 Hermitian-Einstein metrics of Higgs bundles

Let $Y$ be a Kähler manifold with a Kähler form $\omega$. Let $(E, \overline{\Omega}_E, \theta)$ be a Higgs bundle on $Y$ with a Hermitian metric. We set $D^1 := \nabla h + \theta + \theta_h^\dagger$. Let $F(h)$ denote the curvature of $D^1$, i.e.,

$$F(h) = (\overline{\Omega}_E + \theta_h^\dagger + \overline{\partial}_E h + \theta)^2 = \overline{\Omega}_E \theta_h^\dagger + \partial_E h \theta + R(h) + [\theta, \theta_h^\dagger].$$

Note that $(E, \overline{\Omega}_E, \theta, h)$ is a harmonic bundle if and only if $F(h) = 0$.

Recall that $h$ is a Hermitian-Einstein metric of the Higgs bundle if $\Lambda_\omega F(h)^\perp = 0$, where $F(h)^\perp$ denote the trace-free part of $F(h)$, and $\Lambda_\omega$ denote the adjoint of the multiplication of $\omega$ (see [32 §3.2]). The following is a generalization of Kobayashi-Lübke inequality to the context of Higgs bundles due to Simpson [57, Proposition 3.4].
Proposition 3.1 (Simpson) If \( h \) is a Hermitian-Einstein metric, there exists \( C(n) > 0 \) depending only on \( n = \dim Y \) such that the following holds:

\[
\text{Tr} \left( (F(h)\overline{F})^2 \right) \omega^{n-2} = C(n)|F(h)\overline{F}|^2 \omega^n.
\]

As a result, if \( \text{Tr} \left( (F(h)\overline{F})^2 \right) \omega^{n-2} = 0 \), then we obtain \( F(h)\overline{F} = 0 \).

Corollary 3.2 (Simpson) If \( Y \) is compact, and if a Higgs bundle \((E, J_E, \theta)\) on \( Y \) has a Hermitian-Einstein metric \( h \), then the Bogomolov-Gieseker type inequality holds:

\[
\int_Y c_2(E)\omega^{n-2} - \frac{\int_Y c_1^2(E)\omega^{n-2}}{2 \text{rank } E} \leq 0.
\]

3.2 Rank one case

Let \( X \) be an \( n \)-dimensional smooth connected projective variety with a simple normal crossing hypersurface \( H \). Let \( \omega \) be a Kähler form. Let \( \Lambda_\omega \) denote the adjoint of the multiplication of \( \omega \). We have the irreducible decomposition \( H = \bigcup_{i \in \Lambda} H_i \). We take a \( C^\infty \)-Hermitian metric \( g_i \) of the line bundle \( \mathcal{O}(H_i) \). Let \( \sigma_i \) denote the section of \( \mathcal{O}_X(H_i) \) induced by the inclusion \( \mathcal{O}_X \rightarrow \mathcal{O}_X(H_i) \).

Let \((\mathcal{P}, \mathcal{V}, \theta)\) be a good filtered Higgs bundle on \((X, H)\) of rank one. For each \( i \in \Lambda \), there exists unique \( a_i \in [-1, 0] \cap \text{Par}(\mathcal{P}, \mathcal{V}, i) \). Let \( A \) be the constant determined by \( A \int_X \omega^n = 2\pi n \int_X c_1(\mathcal{P}, \mathcal{V})\omega^{n-1} \). The following proposition is standard.

Proposition 3.3 There exists a Hermitian metric \( h \) of the line bundle \( E := \mathcal{V}|_{X \setminus D} \) such that (i) \( \sqrt{-1} \Lambda_\omega F(h) = A \), (ii) \( h \prod_{i \in \Lambda} |\sigma_i|^{-2a_i} \) is a Hermitian metric of \( \mathcal{P}_a(\mathcal{V}) \) of \( C^\infty \)-class. Such a metric is unique up to the multiplication of positive constants. Moreover, if \( c_1(\mathcal{P}, E) = 0 \), then \( R(h) = 0 \) holds, and hence \( h \) is a pluri-harmonic metric of \((E, \theta)\).

Proof Note that \( F(h) = R(h) \) holds in the rank one case. Let \( h_0 \) be a \( C^\infty \)-metric of \( \mathcal{P}_aE \). We obtain the metric \( h_0 := h_0' \cdot \prod_{i \in \Lambda} |\sigma_i|^{-2a_i} \) of \( E|_{X \setminus H} \). It is well known that \( \frac{\sqrt{-1}}{2\pi} R(h_0) \) is naturally extended to a closed \((1, 1)\)-form on \( X \) of \( C^\infty \)-class which represents \( c_1(\mathcal{P}, E) \). By the condition of \( A \), we have \( \int_X (\sqrt{-1} \Lambda R(h_0)) \omega^n = 0 \).

Note that \( \sqrt{-1} \Lambda R(h_0 e^\varphi) = \sqrt{-1} \Lambda R(h_0) + \sqrt{-1} \partial R \varphi \). Hence, there exists an \( \mathbb{R} \)-valued \( C^\infty \)-function \( \varphi_0 \) such that \( \sqrt{-1} \Lambda R(h_0 e^{\varphi_0}) - A \equiv 0 \). The metric \( h = h_0 e^{\varphi_0} \) has the desired property. The uniqueness is clear.

Suppose that \( c_1(\mathcal{P}, E) = 0 \). In the rank one case, a Hermitian metric of \( E \) is a pluri-harmonic metric of \((E, J_E, \theta)\), if and only if \( R(h) = 0 \). Because the cohomology class of \( R(h_0) \) is 0, there exists an \( \mathbb{R} \)-valued \( C^\infty \)-function \( \varphi_0 \) such that \( R(h_0 e^{\varphi_0}) = 0 \) by the standard \( \partial \bar{\partial} \)-lemma.

Note that \( \frac{\sqrt{-1}}{2\pi} R(h) \) induces a closed \((1, 1)\)-form on \( X \) of \( C^\infty \)-class which represents \( c_1(\mathcal{P}, E) \).

3.3 \( \beta \)-subobject and socle for reflexive filtered Higgs sheaves

Let \( X \) be a complex projective connected manifold with a simple normal crossing hypersurface \( H = \bigcup_{i \in \Lambda} H_i \) and an ample line bundle \( L \). For any coherent \( \mathcal{O}_X \)-module \( \mathcal{M} \), we set \( \deg_{\mathcal{L}}(\mathcal{M}) := \int_X c_1(\mathcal{M}) c_1(L)^{\dim X - 1} \).

3.3.1 \( \beta \)-subobjects

Let \((\mathcal{P}, \mathcal{V}, \theta)\) be a reflexive filtered Higgs sheaf on \((X, H)\). For any \( A \in \mathbb{R} \), let \( \mathcal{S}(\mathcal{P}_0\mathcal{V}, A) \) denote the family of saturated subsheaves \( \mathcal{F} \) of \( \mathcal{P}_0\mathcal{V} \) such that \( \deg_{\mathcal{L}}(\mathcal{F}) \geq -A \) and that \( \mathcal{F}^{*(H)} \) is a Higgs subsheaf of \( \mathcal{V} \). Any \( \mathcal{F} \in \mathcal{S}(\mathcal{P}_0\mathcal{V}, A) \) induces a reflexive filtered Higgs sheaf \( \mathcal{P}_*(\mathcal{F}^{*(H)}) \) by \( \mathcal{P}_*(\mathcal{F}^{*(H)}) := \mathcal{P}_0\mathcal{V} \cap \mathcal{F}^{*(H)} \) for any \( c \in \mathbb{R}^\Lambda \). We set \( f_A(\mathcal{F}) := \mu_L(\mathcal{P}_*\mathcal{F}^{*(H)}) \). Thus, we obtain a function \( f_A \) on \( \mathcal{S}(\mathcal{P}_0\mathcal{V}, A) \).

Lemma 3.4 The image \( f_A(S(\mathcal{P}_0\mathcal{V}, A)) \) is a finite subset of \( \mathbb{R} \). In particular, \( f_A \) has the maximum.
Proof According to [19] Lemma 2.5, \( S(\mathcal{P}_0 V, A) \) is bounded. Hence, it is easy to see that there exists a finite decomposition \( S(\mathcal{P}_0 V, A) = \prod_{i=1}^{N} S(\mathcal{P}_0 V, A) \) such that \( f_A \) is constant on each \( S(\mathcal{P}_0 V, A) \).

It is standard that any reflexive filtered Higgs sheaf has a \( \beta \)-subobject, i.e., the following holds.

**Proposition 3.5** For any reflexive filtered Higgs sheaf \((\mathcal{P}, V, \theta)\), there uniquely exists a non-zero Higgs subsheaf \( V_0 \subset V \) such that the following holds for any non-zero reflexive Higgs subsheaf \( V' \subset V \).

- \( \mu_L(\mathcal{P} \cdot V') \leq \mu_L(\mathcal{P} \cdot V_0) \) holds.
- If \( \mu_L(\mathcal{P} \cdot V') = \mu_L(\mathcal{P} \cdot V_0) \) holds, then \( \text{rank } V' \leq \text{rank } V_0 \) holds.
- If \( \mu_L(\mathcal{P} \cdot V') = \mu_L(\mathcal{P} \cdot V_0) \) and \( \text{rank } V' = \text{rank } V_0 \) hold, then \( V' = V_0 \) holds.

**Proof** There exists \( N > 0 \) such that for any saturated subsheaf \( F \subset \mathcal{P}_0 V \):

\[
|\deg_L(F) - \text{rank}(F)\mu_L(\mathcal{P}, F(\ast H))| < N.
\]

We set \( A_0 := |\deg_L(\mathcal{P}_0 V)| + 10N \). Let \( B_0 \) denote the maximum of \( f_{A_0} \). Then, it is easy to see that \( \mu_L(\mathcal{P} \cdot V') \leq B_0 \) for any saturated Higgs subsheaf \( V' \subset V \). Moreover, if \( \mu_L(\mathcal{P} \cdot V') = B_0 \), then \( (\mathcal{P} \cdot V', \theta') \) is \( \mu_L \)-semistable, where \( \theta' \) denote the Higgs field induced by \( \theta \).

Suppose that the Higgs subsheaves \( V_i \subset V \) \((i = 1, 2)\) satisfy \( \mu_L(\mathcal{P} \cdot V_i) = B_0 \). We obtain the subsheaf \( V_1 + V_2 \subset V \). Because \( V_1 + V_2 \) is a quotient of \( V_1 \oplus V_2 \), we obtain a filtered sheaf \( \mathcal{P} \cdot (V_1 + V_2) \) over \( V_1 + V_2 \), induced by \( \mathcal{P} \cdot V_1 \oplus \mathcal{P} \cdot V_2 \). Then, by the \( \mu_L \)-semistability of \( (\mathcal{P} \cdot V_i, \theta_i) \), we obtain that \( B_0 = \mu_L(\mathcal{P} \cdot V_1 \oplus \mathcal{P} \cdot V_2) \leq \mu_L(\mathcal{P} \cdot (V_1 + V_2)) \). Let \( V_0 \) denote the saturated subsheaf of \( V \) generated by \( V_1 + V_2 \). We obtain a filtered sheaf \( \mathcal{P} \cdot V_0 \) by \( \mathcal{P} \cdot V_3 = \mathcal{P} \cdot V_0 \cap V_3 \). Because the natural morphism \( \mathcal{P} \cdot (V_1 + V_2) \to \mathcal{P} \cdot V_0 \) is generically an isomorphism, we obtain \( \mu_L(\mathcal{P} \cdot (V_1 + V_2)) \leq \mu_L(\mathcal{P} \cdot V_0) \leq B_0 \) by Lemma 2.4. Hence, we obtain \( \mu_L(\mathcal{P} \cdot V_0) = B_0 \). Then, the claim of the lemma is clear.

### 3.3.2 Socle

Let \((\mathcal{P}, V, \theta)\) be a \( \mu_L \)-semistable reflexive filtered Higgs sheaf on \((X, H)\). Let \( \mathcal{T} \) denote the family of saturated Higgs subsheaves \( V' \subset V \) such that the induced filtered Higgs sheaf \((\mathcal{P} \cdot V', \theta') \) is \( \mu_L \)-stable. Let \( V_1 \) be the saturated \( \mathcal{O}_X(\ast H) \)-submodule of \( V \) generated by \( \sum_{V' \in \mathcal{T}} V' \). It is a Higgs subsheaf of \( V \).

**Proposition 3.6** \((\mathcal{P} \cdot V_1, \theta_1)\) is equal to the direct sum \( \bigoplus_{k=1}^\ell (\mathcal{P} \cdot V^{(k)} \cdot \theta^{(k)}) \) for a tuple of \( \mu_L \)-stable filtered Higgs subsheaves of \((\mathcal{P} \cdot V, \theta)\). In particular, \((\mathcal{P} \cdot V_1, \theta_1)\) is \( \mu_L \)-polystable, and \( V_1 = \sum_{V' \in \mathcal{T}} V' \) holds. The filtered Higgs subsheaf \((V_1, \theta_1)\) is called the socle of \((\mathcal{P}, V, \theta)\).

**Proof** Let \( V^{(i)} \) \((i = 1, 2)\) be saturated Higgs subsheaves of \( V \) such that (i) \( \mu_L(\mathcal{P} \cdot V^{(i)}) = \mu_L(\mathcal{P} \cdot V) \), (ii) \((\mathcal{P} \cdot V^{(1)}, \theta^{(1)})\) is \( \mu_L \)-semistable, (iii) \((\mathcal{P} \cdot V^{(2)}, \theta^{(2)})\) is \( \mu_L \)-stable.

**Lemma 3.7** Either \( V^{(2)} \subset V^{(1)} \) or \( V^{(1)} \cap V^{(2)} = 0 \) holds.

**Proof** Let us consider the morphism \( \iota_1 - \iota_2 : V^{(1)} \oplus V^{(2)} \to V \), where \( \iota_i : V^{(i)} \to V \) denote the inclusions. Let \( K \) denote the kernel. We obtain a filtered sheaf \( \mathcal{P} \cdot K \) over \( K \) by \( \mathcal{P} \cdot K := K \cap \mathcal{P} \cdot (V_1 \oplus V_2) \). The projection \( V^{(1)} \oplus V^{(2)} \to V^{(2)} \) induces \( K \simeq V^{(1)} \cap V^{(2)} =: I \). It induces a morphism of filtered Higgs sheaves \( g : (\mathcal{P}, K, \theta_K) \to (\mathcal{P} \cdot V^{(2)}, \theta^{(2)}) \). We set \( \mu_0 := \mu_L(\mathcal{P} \cdot K) \). Because \( \bigoplus_{i=1,2} (\mathcal{P} \cdot V^{(i)}, \theta^{(i)}) \) and \((\mathcal{P}, V, \theta)\) are \( \mu_L \)-semistable with the same slope \( \mu_0 \), we obtain that \((\mathcal{P}, K, \theta)\) is also \( \mu_L \)-semistable with \( \mu_L(\mathcal{P} \cdot K) = \mu_0 \).

Suppose that \( K \neq 0 \), i.e., \( I \neq 0 \). Because \( I \) is a subsheaf of \( V^{(2)} \), we also obtain a filtered sheaf \( \mathcal{P} \cdot I \) induced by \( \mathcal{P} \cdot V^{(2)} \). Because \( I \simeq K \), we obtain a filtered sheaf \( \mathcal{P} \cdot I \) over \( I \) induced by \( \mathcal{P} \cdot K \). Then, we obtain

\[
\mu_0 = \mu_L(\mathcal{P} \cdot K) = \mu_L(\mathcal{P} \cdot I) \leq \mu_L(\mathcal{P} \cdot V^{(2)}) \leq \mu_L(\mathcal{P} \cdot V^{(2)}) = \mu_0.
\]

Because \((\mathcal{P} \cdot V^{(2)}, \theta)\) is \( \mu_L \)-stable and because \( I \neq 0 \), we obtain that \( \text{rank}(I) = \text{rank}(V^{(2)}) \), i.e., \( I \) and \( V^{(2)} \) are generically isomorphic. Because \( \mu_L(\mathcal{P} \cdot I) = \mu_L(\mathcal{P} \cdot V^{(2)}) \), Lemma 2.4 implies that \( \mathcal{P} \cdot I \to \mathcal{P} \cdot V^{(2)} \) is an isomorphism in codimension 1. Hence, there exists a closed algebraic subset \( Z \subset X \) such that (i) the codimension of \( Z \) is larger than 2, (ii) \( V^{(2)} \mid_{X \setminus Z} \subset V^{(1)} \mid_{X \setminus Z} \). Because \( V^{(1)} \) is reflexive we obtain that \( V^{(2)} \subset V^{(1)} \). 

Let us study the case where $\mathcal{V}^{(1)} \cap \mathcal{V}^{(2)} = 0$. Let $\mathcal{V}^{(3)}$ denote the saturated Higgs subsheaf of $\mathcal{V}$ generated by $\mathcal{V}^{(1)} + \mathcal{V}^{(2)}$. Let $\mathcal{P}_*\mathcal{V}^{(3)}$ denote the filtered sheaf over $\mathcal{V}^{(3)}$ induced by $\mathcal{P}_*\mathcal{V}$.

**Lemma 3.8** $(\mathcal{P}_*\mathcal{V}^{(3)}, \theta^{(3)})$ is $\mu_L$-semistable, and the induced morphism $g : \mathcal{P}_*\mathcal{V}^{(1)} \oplus \mathcal{P}_*\mathcal{V}^{(2)} \rightarrow \mathcal{P}_*\mathcal{V}^{(3)}$ is an isomorphism in codimension one.

**Proof** We obtain $\mu_0 = \mu_L(\mathcal{P}_*(\mathcal{V}^{(1)} \oplus \mathcal{V}^{(2)})) \leq \mu_L(\mathcal{P}_*\mathcal{V}^{(3)}) \leq \mu_L(\mathcal{P}_*\mathcal{V}) = \mu_0$. Hence, we obtain that $\mu_L(\mathcal{P}_*\mathcal{V}) = \mu_0$ and that $(\mathcal{P}_*\mathcal{V}^{(3)}, \theta^{(3)})$ is $\mu_L$-semistable. Because $g : \mathcal{P}_*\mathcal{V}^{(1)} \oplus \mathcal{P}_*\mathcal{V}^{(2)} \rightarrow \mathcal{P}_*\mathcal{V}^{(3)}$ is generically an isomorphism, and because they have the same slope, $g$ is an isomorphism in codimension one by Lemma 2.4.

By Lemma 3.8 it is easy to observe that there exists a finite sequence of reflexive Higgs subsheaves $\mathcal{V}_j'$ $(j = 1, \ldots, m)$ such that (i) the induced filtered Higgs sheaves $(\mathcal{P}_*\mathcal{V}_j', \theta_j')$ are $\mu_L$-stable, (ii) the image of the induced morphism $g : \tilde{\mathcal{V}} := \bigoplus \mathcal{V}_j' \rightarrow \mathcal{V}_1$ is generically an isomorphism. Because $\mu_0 = \mu_L(\mathcal{P}_*\tilde{\mathcal{V}}) \leq \mu_L(\mathcal{P}_*\mathcal{V}_1) \leq \mu_L(\mathcal{P}_*\mathcal{V}) = \mu_0$, we obtain that $\mu_L(\mathcal{P}_*\mathcal{V}_1) = \mu_L(\mathcal{P}_*\mathcal{V})$. Hence, $g$ is an isomorphism in codimension one by Lemma 2.4. Because both $\mathcal{P}_*\tilde{\mathcal{V}}$ and $\mathcal{P}_*\mathcal{V}_1$ are reflexive, we obtain that $\mathcal{P}_*\tilde{\mathcal{V}} \simeq \mathcal{P}_*\mathcal{V}_1$. Thus, we obtain Proposition 3.9.

### 3.4 Mehta-Ramanathan type theorem

Let $X$ be a smooth connected projective variety with a simple normal crossing hypersurface $H$. Let $L$ be an ample line bundle on $X$.

**Proposition 3.9** Let $\mathcal{P}_*\mathcal{V}$ be a filtered sheaf on $(X, H)$ with a meromorphic Higgs field $\theta$. Suppose that $(\mathcal{P}_*\mathcal{V}, \theta)$ is $\mu_L$-stable. Then, it is $\mu_L$-stable (resp. $\mu_L$-semistable) if and only if the following holds.

- For any $m_1 > 0$, there exists $m > m_1$ such that $(\mathcal{P}_*\mathcal{V}, \theta)_{|Y}$ is $\mu_L$-stable (resp. $\mu_L$-semistable) where $Y$ denotes the 1-dimensional complete intersection of generic hypersurfaces of $L^\otimes m$.

**Proof** We can prove this proposition by the argument in [43 §3.4], which closely follows the arguments of Mehta-Ramanathan [40 41] and Simpson [59]. We use $W = \Omega^1(ND)$ for a large $N$ instead of $\Omega^1(\log D)$ in [43 §3.4]. (See also [47 §13.2].)

### 3.5 Residues of good filtered Higgs bundles

#### 3.5.1 Residues in the local and unramified case

Let $U$ be a neighbourhood of $(0, \ldots, 0)$ in $\mathbb{C}^n$. Set $H_{U, j} := U \cap \{z_i = 0\}$ and $H_U := \bigcup_{i=1}^j H_{U, i}$. Let $(\mathcal{P}_*\mathcal{V}, \theta)$ be a good filtered Higgs bundle on $(X, H)$. Suppose that there exists a decomposition

$$(\mathcal{P}_*\mathcal{V}, \theta) = \bigoplus_{a \in \mathcal{I}} (\mathcal{P}_a\mathcal{V}_a, \theta_a)$$

(5)

such that $\theta_a - d\mathbf{a} \text{id}_{\mathcal{V}_a}$ are logarithmic with respect to the lattices $\mathcal{P}_a\mathcal{V}_a$. We obtain the endomorphism

$$\text{Res}(\theta_a - d\mathbf{a} \text{id}_{\mathcal{V}_a})$$

of $^i\text{Gr}_b^F(\mathcal{P}_a\mathcal{V}_a)$. By taking the direct sum, we obtain the following endomorphism of $^i\text{Gr}_b^F(\mathcal{P}_a\mathcal{V})$:

$$\text{Res}_i(\theta) := \bigoplus_a \text{Res}(\theta_a - d\mathbf{a} \text{id}_{\mathcal{V}_a}).$$

Note that $\text{Res}_i(\theta)_{|H_I}$ preserves the induced filtrations $^iF \ (j \in I \setminus \{i\})$ of $^i\text{Gr}_b^F(\mathcal{P}_a\mathcal{V})_{|H_I}$.

We set $\partial H_I := \bigcup_{j \notin I} (H_j \cap H_I)$. Let $\pi_I : \mathbb{R}^I \rightarrow \mathbb{R}^I$ be the projection. We obtain the following filtered bundle on $(H_I, \partial H_I)$:

$$^i\text{Gr}_b^F(\mathcal{P}_a\mathcal{V}) := \left( ^i\text{Gr}_b^F(\mathcal{P}_a\mathcal{V}) \mid \mathbf{a} \in \pi_{i-1}^{-1}(b) \right).$$
Define by the projection $I$. Let $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be given by $\varphi(\xi_1, \ldots, \xi_n) = (\xi_1^{m_1}, \ldots, \xi_{\ell+1}^{m_{\ell}}, \ldots, \xi_n)$. Let $U' := \varphi^{-1}(U)$. The induced map $U' \rightarrow U$ is also denoted by $\varphi$. Set $H_{U'} := \varphi^{-1}(H_U)$. We obtain the good filtered Higgs bundle $(\mathcal{P}_U, \mathcal{V}_U, \theta_U) := \varphi^*(\mathcal{P}_V, \mathcal{V}, \theta)$ on $(U', H_{U'})$ obtained as the pull back. We obtain the endomorphisms $\text{Res}_i(\theta_1)(i \in I)$ of the filtered bundles $I\text{Gr}^F_b(\mathcal{P}_U, \mathcal{V}_U)$.

Note that $I\text{Gr}^F_b(\mathcal{P}_U, \mathcal{V})$ is the descent of $I\text{Gr}^F_{\varphi^*}(b)(\mathcal{P}_V, \mathcal{V})$. We obtain endomorphisms $\text{Res}_i(\theta_1)'(i \in I)$ of $I\text{Gr}^F_{\varphi^*}(b)(\mathcal{P}_V, \mathcal{V})$ obtained as the descent of $\text{Res}_i(\theta_1)$. By the relation $d\xi_i/\xi_i = m_i \varphi^*(dz_i/z_i)$, we obtain the following relation:

$$\text{Res}_i(\theta) = \frac{1}{m_i} \text{Res}_i(\theta_1)'(i \in I).$$

### 3.5.2 Residue in the local and ramified case

Let $(\mathcal{P}, \mathcal{V}, \theta)$ be a good filtered Higgs bundle on $(U, H_U)$. There exists a ramified covering $\varphi : (U', H_{U'}) \rightarrow (U, H_U)$ such that $(\mathcal{P}_V, \mathcal{V}_V, \theta) := \varphi^*(\mathcal{P}, \mathcal{V}, \theta)$ has a decomposition as in $\square$. For any $I \subset \{1, \ldots, \ell\}$, we obtain the endomorphisms $\text{Res}_i(\theta_1)(i \in I)$ of the filtered bundles $I\text{Gr}^F_b(\mathcal{P}_V, \mathcal{V}_V)$ on $(H', \partial H')$. We obtain the endomorphism $\text{Res}_i(\theta_1)'(i \in I)$ of $I\text{Gr}^F_b(\mathcal{P}_V, \mathcal{V}_V)$ as the descent of $\text{Res}_i(\theta_1)$. We set

$$\text{Res}_i(\theta) := \frac{1}{m_i} \text{Res}_i(\theta_1)'(i \in I).$$

It is easy to check that $\text{Res}_i(\theta)$ are independent of the choice of a ramified covering $U' \rightarrow U$. In particular, we obtain endomorphisms $\text{Res}_i(\theta)$ of $I\text{Gr}^F_b(\mathcal{P}_\mathcal{V}_\mathcal{A})$ for any $\mathcal{A} \in \pi_{\mathcal{A}}^{-1}(b)$.

The above construction is independent of a holomorphic coordinate system.

### 3.5.3 Global case

Let $(\mathcal{P}, \mathcal{V}, \theta)$ be a good filtered Higgs bundle on $(X, H)$. Then, by gluing the residues locally obtained in $\square$ for any $I \subset \Lambda$, we obtain the endomorphisms $\text{Res}_i(\theta)(i \in I)$ of $I\text{Gr}^F_b(\mathcal{P}_\mathcal{V}_\mathcal{A})$ for any $\mathcal{A} \in \pi_{\mathcal{A}}^{-1}(b)$.

### 3.6 Perturbation of good filtered Higgs bundles

#### 3.6.1 Gap of filtered bundles

Let $X$ be a complex manifold with a simple normal crossing hypersurface $H = \bigcup_{i \in \Lambda} H_i$. For simplicity, we assume that $\Lambda$ is finite. Let $(\mathcal{P}, \mathcal{V}, \theta)$ be a good filtered Higgs bundle on $(X, H)$.

We take $\mathcal{A} \in \mathbb{R}^\Lambda$ such that $a_i \notin \text{Par}(\mathcal{P}, \mathcal{V}, i)$. We set $\text{Par}(\mathcal{P}, \mathcal{V}, \mathcal{A}, i) := \text{Par}(\mathcal{P}, \mathcal{V}, i)\cap|a_i - 1, a_i|$. We also set

$$\text{gap}(\mathcal{P}, \mathcal{V}, \mathcal{A}, i) := \min\left\{\left|b_1 - b_2\right| \bigg| b_1, b_2 \in \text{Par}(\mathcal{P}, \mathcal{V}, \mathcal{A}, i), b_1 \neq b_2\right\} \bigcup \left\{\left|b - a_i\right| \bigg| b \in \text{Par}(\mathcal{P}, \mathcal{V}, i)\right\}.$$ We set $\text{gap}(\mathcal{P}, \mathcal{V}, \mathcal{A}) := \min_{i \in \Lambda} \text{gap}(\mathcal{P}, \mathcal{V}, \mathcal{A}, i)$. Recall that $\Lambda$ is assumed to be finite.

#### 3.6.2 Curve case

Let $C$ be a complex curve with a finite subset $D \subset C$. Let $(\mathcal{P}, \mathcal{V}, \theta)$ be a good filtered Higgs bundle on $(C, D)$. We take $\mathcal{A} \in \mathbb{R}^D$ such that $a(P) \notin \text{Par}(\mathcal{P}, \mathcal{V}, i)$ for each $P \in D$. We choose $\eta > 0$ such that $10\text{rank}(\mathcal{V})(\eta < \text{gap}(\mathcal{P}, \mathcal{V}, \mathcal{A})$. For any $0 < \epsilon < \eta$, let $\psi, \epsilon_P$ be a map $\text{Par}(\mathcal{P}, \mathcal{V}, \mathcal{A}, P) \rightarrow \mathbb{R}$ such that $|\psi_P(b) - b| < 2\epsilon$. We define $\varphi_P : \mathbb{Z} \times \text{Par}(\mathcal{P}, \mathcal{V}, \mathcal{A}, P) \rightarrow \mathbb{R}$ by

$$\varphi_P(k, b) := \psi_P(b) + \epsilon k.$$

For any $(k, b) \in \mathbb{Z} \times \text{Par}(\mathcal{P}, \mathcal{V}, \mathcal{A}, P)$, we obtain the subspace $W_k F(b)(\mathcal{P}_\mathcal{V}_P)$ as the pull back of $W_k \text{Gr}^F_{\varphi^*}(\mathcal{P}_\mathcal{V}_P)$ by the projection $F(b)(\mathcal{P}_\mathcal{V}_P) \rightarrow \text{Gr}^F_{\varphi^*}(\mathcal{P}_\mathcal{V}_P)$). We define the filtration $\bar{F}^c$ on $\mathcal{P}_\mathcal{V}_P$ indexed by $|a(P) - 1, a(P)|$ as follows:

$$\bar{F}^c \mathcal{P}_\mathcal{V}_P := \sum_{(k, b) \in \mathbb{Z} \times \text{Par}(\mathcal{P}, \mathcal{V}, \mathcal{A}, P)} W_k F(b)(\mathcal{P}_\mathcal{V}_P).$$
We have the corresponding good filtered Higgs bundle \((P_\epsilon^x)\mathcal{V}, \theta)\).

We clearly have \(\lim_{\epsilon \to 0} \text{par-c}_1(P_\epsilon^x)\mathcal{V}) = \text{par-c}_1(P_\epsilon)\mathcal{V})\). The following is standard.

**Lemma 3.10** Suppose that \(C\) is compact and that \((P_\epsilon)\mathcal{V}, \theta)\) is stable (resp. polystable). Then, if \(\epsilon\) is sufficiently small, \((P_\epsilon^x)\mathcal{V}, \theta)\) is also stable (resp. polystable).

**Proof** See [43, Proposition 3.28] for the stability. If \((P_\epsilon)\mathcal{V}, \theta) = \bigoplus P_\epsilon(V_i, \theta_i)\), then we have \((P_\epsilon^x)\mathcal{V}, \theta) = \bigoplus P_\epsilon^x(V_i, \theta_i)\). Moreover, \(\deg(P_\epsilon V_i) = \deg(P_\epsilon^x V_i)\) holds. Hence, we obtain the claim for the polystability.

### 3.6.3 Surface case

Let \(X\) be a complex projective surface with a simple normal crossing hypersurface \(H = \bigcup_{i \in A} H_i\). Let \((P_\epsilon)\mathcal{V}, \theta)\) be a good filtered Higgs bundle on \((X, H)\). We shall explain a similar perturbation of good filtered Higgs bundles. We take \(a \in \mathbb{R}^A\) such that \(a_i \notin \text{Par}(P_\epsilon V, i)\) for any \(i \in A\). We choose \(\eta > 0\) such that \(0 < 10 \text{rank}(V) \eta < \text{gap}(P_\epsilon V, a)\).

For any \(0 < \epsilon < \eta\), let \(\psi_{x,i}\) be a map \(\text{Par}(P_\epsilon V, a, i) \rightarrow \mathbb{R}\) such that \(|\psi_{x,i}(b) - b| < 2\epsilon\). We define \(\varphi_{x,i}(k, b) := \psi_{x,i}(b) + c k\).

Note that the eigenvalues of the endomorphism \(\text{Res}_i(\theta)\) on \(\text{Gr}^b_i(P_\epsilon V|_{H_i})\) are constant on \(H_i\) because \(H_i\) are compact. Hence, we have the well defined nilpotent part \(N_{i,b}\) of \(\text{Res}_i(\theta)\). Note that there exists a finite subset \(Z_i \subset H_i\) such that the conjugacy classes of the nilpotent part of \(N_{i,b,Q}\) \((Q \in H_i \setminus Z_i)\) are constant. We obtain the filtration \(W\) of \(\text{Gr}^b_i(P_\epsilon V|_{H_i \setminus Z_i})\) by algebraic vector subbundles whose restriction to \(Q \in H_i \setminus Z_i\) are the weight filtration of \(N_{i,b,Q}\). By the valuative criterion, it is uniquely extended to a filtration of \(\text{Gr}^b_i(P_\epsilon V|_{H_i})\) by holomorphic subbundles, which is also denoted by \(W\).

For any \((k, b) \in \mathbb{Z} \times \text{Par}(P_\epsilon V, a, i)\), let \(W_k F_b(P_\epsilon V|_{H_i})\) denote the subbundle of \(P_\epsilon V|_{H_i}\), obtained as the pull back of \(W_k \text{Gr}^b_i(P_\epsilon V|_{H_i})\) by the projection \(F_b(P_\epsilon V|_{H_i}) \rightarrow \text{Gr}^b_i(P_\epsilon V|_{H_i}))\). We define the filtration \(\tilde{F}^x(\epsilon)\) on \(\text{Gr}^b_i(P_\epsilon V|_{H_i})\) as follows:

\[\tilde{F}^x(\epsilon) P_\epsilon V|_{H_i} := \sum_{(k, b) \in \mathbb{Z} \times \text{Par}(P_\epsilon V, a, i)} W_k F_b(P_\epsilon V|_{H_i}).\]

We have the corresponding good filtered Higgs bundle \((P_\epsilon^x)\mathcal{V}, \theta)\).

We clearly have \(\lim_{\epsilon \to 0} \text{par-c}_1(P_\epsilon^x)\mathcal{V}) = \text{par-c}_1(P_\epsilon)\mathcal{V})\) and \(\lim_{\epsilon \to 0} \text{par-ch}_2(P_\epsilon^x)\mathcal{V}) = \text{par-ch}_2(P_\epsilon)\mathcal{V})\). The following is standard, and similar to Lemma 3.10 (See also [43, Proposition 3.28].)

**Lemma 3.11** Suppose that \((P_\epsilon)\mathcal{V}, \theta)\) is stable (resp. polystable). Then, if \(\epsilon\) is sufficiently small, \((P_\epsilon^x)\mathcal{V}, \theta)\) is also stable (polystable).

### 3.7 Some families of auxiliary metrics on a punctured disc

#### 3.7.1 Perturbation of filtrations

Let \(X = \Delta = \{z \in \mathbb{C} \mid |z| < 1\}\) and \(D = \{0\}\). Let \((P_\epsilon)\mathcal{V}, \theta)\) be a good filtered Higgs bundle on \((X, D)\). Let \((E, \overline{\partial} E, \theta)\) be the Higgs bundle obtained as the restriction of \((P_\epsilon)\mathcal{V}, \theta)\) to \(X \setminus D\).

We take \(a \in \mathbb{R}\) such that \(a \notin \text{Par}(P_\epsilon V)\). We take \(\eta_1, \eta_2 > 0\) such that \(10 \text{rank}(V) \eta_1 < \text{gap}(P_\epsilon V, a)\) and \(10 \text{rank}(V) \eta_2 < \eta_1\). We construct a family of filtered bundles \((P_\epsilon^x)\mathcal{V}, \theta)\) as in [3.6.2] by taking a function \(\psi_\epsilon\). Let \(g_\epsilon := (\eta_1^2 |z|^{2\eta_2} + \epsilon^2 |z|^{2\epsilon-2}) \, dz \wedge \overline{dz}\) be the Kähler metric on \(X \setminus D\). We shall prove the following proposition in [3.7.2, 3.7.3].

**Proposition 3.12** There exists a family of Hermitian metrics \(h_\epsilon (0 \leq \epsilon < \eta_2)\) of \((E, \overline{\partial} E, \theta)\) such that (i) \(P_\epsilon^x E = P_\epsilon^x \mathcal{V}\), (ii) there exists \(C_i > 1 (i = 0, 1, 2)\) which are independent of \(\epsilon\), such that

\[|F(h_\epsilon)|_{g_\epsilon, h_\epsilon} \leq C_0, \quad C_1^{-1} |z|^C r^{h_\epsilon} \leq h_\epsilon \leq C_1 |z|^{-C r^{h_0}}.\]
3.7.2 Example of a family of harmonic bundles of rank 2

Let \( \mathcal{V} = \bigoplus_{i=1,2} \mathcal{O}_X(\star D)v_i \). Let \( \theta \) be the Higgs field given by \( \theta(v_1) = v_2 \, dz/z \) and \( \theta(v_2) = 0 \). Let \((E, \mathcal{F}_E, \theta)\) be the Higgs bundle on \( X \setminus D \) obtained as the restriction of \((\mathcal{V}, \theta)\) to \( X \setminus D \). For any \( \epsilon > 0 \), we set \( L_\epsilon(z) := \epsilon^{-1}(|z|^{-\epsilon} - |z|^\epsilon) \). We also set \( L_0 := -\log|z|^2 \).

Lemma 3.13 \( L_0(z) \leq L_\epsilon(z) \leq |z|^{-\epsilon} L_0(z) \).

Proof As proved in [15 §4.2], \( L_0(z) \leq L_\epsilon(z) \) holds. We set \( g(\epsilon) := -\log|z|^2 - (1 - |z|^{2\epsilon}) \) for any \( z \in \Delta^* \) and for \( \epsilon > 0 \). It is easy to check that \( \partial_\epsilon g(\epsilon) \geq 0 \) and \( \lim_{\epsilon \to 0} g(\epsilon) = 0 \). Hence, we obtain \( L_\epsilon(z) \leq |z|^{-\epsilon} L_0(z) \).

Let \( h_\epsilon \) be the \( C^\infty \)-metric of \( E \) given by

\[
h_\epsilon(v_1, v_1) = L_\epsilon, \quad h_\epsilon(v_2, v_2) = L_\epsilon^{-1}, \quad h_\epsilon(v_1, v_2) = 0.
\]

Lemma 3.14 \((E, \mathcal{F}_E, \theta, h_\epsilon)\) are harmonic bundles.

Proof Let \( H_\epsilon \) be the matrix valued function on \( X \setminus D \) determined by \((H_\epsilon)_{i,j} := h_\epsilon(v_i, v_j)\). Then, the following holds:

\[
\overline{\partial}(H_\epsilon^{-1} \partial H_\epsilon) = \begin{pmatrix} 0 & -\partial_\epsilon \log L_\epsilon \\ -\partial_\epsilon \log L_\epsilon & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \epsilon^2 |z|^{-2} \frac{dz \, dz}{(|z|^{-\epsilon} - |z|^{\epsilon})^2} = -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} L_\epsilon^{-2} \frac{dz \, dz}{|z|^2}.
\]

Let \( \Theta \) be the matrix valued function representing \( \theta \) with respect to the frame \((v_1, v_2)\), i.e., \( \theta(v_1, v_2) = (v_1, v_2) \Theta \). Let \( \theta_\epsilon^\dagger \) denote the adjoint of \( \theta \) with respect to \( h_\epsilon \). Let \( \Theta_\epsilon^\dagger \) denote the matrix valued function representing \( \theta_\epsilon^\dagger \).

The following holds:

\[
\Theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{dz}{z}, \quad \Theta_\epsilon^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} L_\epsilon^{-2} \frac{dz \, dz}{|z|^2}.
\]

Hence, we obtain

\[
[\Theta, \Theta_\epsilon^\dagger] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot L_\epsilon^{-2} \cdot \frac{dz \, dz}{|z|^2}.
\]

It implies that

\[
\overline{\partial}(H_\epsilon^{-1} \partial H_\epsilon) + [\Theta, \Theta_\epsilon^\dagger] = 0.
\]

It is exactly the Hitchin equation for \((E, \mathcal{F}_E, \theta, h_\epsilon)\).

Let \((\mathcal{P}_\epsilon^{(c)} E, \theta)\) denote the associated filtered Higgs bundle. We have

\[
\mathcal{P}_\epsilon^{(c)} E = \mathcal{O}_Xv_1 \oplus \mathcal{O}_X v_2, \quad \mathcal{P}_\epsilon^{(c)} E = \mathcal{O}_X(-\{0\}) v_1 \oplus \mathcal{O}_X v_2 \quad \mathcal{P}_\epsilon^{(c)} E = \mathcal{O}_X(-\{0\}) v_1 \oplus \mathcal{O}_X(-\{0\}) v_2.
\]

Let \( s_\epsilon \) be determined by \( h_\epsilon = h_0 s_\epsilon \).

Lemma 3.15 There exists \( C > 1 \) such that \( |s_\epsilon|_{h_0} \leq C|z|^{-\epsilon} \) and \( |s_\epsilon^{-1}|_{h_0} \leq C|z|^{-\epsilon} \) for any \( 0 \leq \epsilon < 1/10 \).

3.7.3 Families of equivariant harmonic bundles with nilpotent Higgs fields

Let \( G := \{ \mu \in \mathbb{C}^* \mid \mu^\ell = 1 \} \) for some \( \ell \). Let \( V \) be a finite dimensional \( \mathbb{C} \)-vector space equipped with a \( G \)-action and a \( G \)-invariant nilpotent endomorphism \( N \). We set \( \mathcal{V} = V \otimes \mathcal{O}_X(\star D) \) with the Higgs field \( \theta = N \frac{dz}{z} \). Let \( W \) denote the weight filtration of \( N \) on \( V \). We fix \( 0 < \eta \) such that \( 10 \operatorname{rank}(\mathcal{V}) \eta < 1 \). Take \( c \in \mathbb{R} \) and \( c(\epsilon) \in \mathbb{R} \) \((0 \leq \epsilon \leq \eta)\) such that \( |c(\epsilon) - c| \leq 2\epsilon \). We set

\[
\mathcal{P}_{a(\epsilon, c(\epsilon))} \mathcal{V} := \sum_j W_j \otimes \mathcal{O}_X([a - c(\epsilon) - j\epsilon] D) \subseteq \mathcal{V}.
\]

We consider the \( G \)-action on \( X \) by the multiplication on the coordinate. Then, \((\mathcal{P}_{a(\epsilon,c(\epsilon))} \mathcal{V}, \theta)\) is naturally \( G \)-equivariant.
Lemma 3.16 There exists a family of $G$-invariant harmonic metrics $h_{\epsilon,c(\epsilon)}(0 \leq \epsilon \leq \eta)$ of $(\mathcal{V}, \theta)|_{X \setminus D}$ such that the following holds.

- $h_{\epsilon,c(\epsilon)}$ is adapted to $\mathcal{P}_{*}^{{c(\epsilon)}}(\mathcal{V})$.
- $\lim_{\epsilon \to 0} h_{\epsilon,c(\epsilon)} = h_{0,c}$ in the $C^\infty$-sense locally on $X \setminus D$. Moreover, there exists $C > 1$ such that
  
  $$C^{-1}|z|^{2\epsilon \text{rank} V} h_{0,c} < h_{\epsilon,c(\epsilon)} < C|z|^{-2\epsilon \text{rank} V} h_{0,c}$$

for any $0 \leq \epsilon < \eta$.

Proof We set $G' := \text{Hom}(G, \mathbb{C}^*)$. For each $\chi \in G'$, let $\mathbb{C}_\chi$ denote the irreducible $G$-representation corresponding to $\chi$. There exists the canonical decomposition $(V, N) = \bigoplus (V^\chi, N^\chi) \otimes \mathbb{C}_\chi$, where $(V^\chi, N^\chi)$ denote finite dimensional $G$-vector spaces with a nilpotent endomorphism.

We set $V_0 := \mathbb{C} v_1 \oplus \mathbb{C} v_2$. We have $N_0 \in \text{End}(V_0)$ given by $N_0(v_1) = v_2$ and $N_0(v_2) = 0$. We set $V_0 := V_0 \otimes \mathcal{O}_X(*)D$ and $\theta_0 := N_0 dz/z$. Note that $(V_0, \theta_0)$ is naturally $G$-equivariant by $\mu^*(f(z)v) := f(\mu z)v$.

For any finite dimensional vector space $U$, let $\text{Sym}^l(U)$ denote the $l$-th symmetric tensor product of $U$. For any nilpotent endomorphism $N_U$ on $U$, let $\text{Sym}^l(N_U)$ denote the endomorphism of $\text{Sym}^l(U)$ induced by the Leibniz rule.

There exist $\ell_{\chi,i} \in \mathbb{Z}_{>0}$ $(i = 1, \ldots, m(\chi))$ and an isomorphism $(V^\chi, N^\chi) \simeq \bigoplus_{i=1}^{m(\chi)} \text{Sym}^\ell_{\chi,i}(V_0, \theta_0)$. We have the induced $G$-equivariant isomorphism:

$$(\mathcal{V}, \theta) \simeq \bigoplus_{\chi \in G'} \bigoplus_{i=1}^{m(\chi)} \text{Sym}^\ell_{\chi,i}(V_0, \theta_0) \otimes \mathbb{C}_\chi.$$  

For each $0 \leq \epsilon < \eta$, the harmonic metric $h_\epsilon$ of $(V_0, \theta)$ induce a $G$-invariant harmonic metric $h_{\epsilon,0}$ of $(\mathcal{V}, \theta)$. We set $h_{\epsilon,c(\epsilon)} := |z|^{-2\epsilon(\epsilon)}h_{\epsilon,0}$. Then, the family has the desired property.

3.7.4 Example of family of equivariant unramifiedly good wild harmonic bundles

Let $X$, $D$ and $G$ be as in 3.7.3. Note that $G$ acts on $z^{-1}\mathbb{C}[z^{-1}]$ by the pull back. Let $a \in z^{-1}\mathbb{C}[z^{-1}]$. We set $G \cdot a := \{\mu^* a | \mu \in G\}$. Let $V$ be a finite dimensional $G$-vector space equipped with a nilpotent endomorphism $N$, a grading

$$(V, N) = \bigoplus_{b \in G \cdot a} (V_b, N_b),$$

and a $G$-action such that $\mu \circ N = N \circ \mu$ for any $\mu \in G$, and $\mu V_b = V_{\mu^* b}$ for any $\mu \in G$ and $b \in G \cdot a$.

We set $\mathcal{V} := V \otimes \mathcal{O}_X(*)D$ and $V_b := V_b \otimes \mathcal{O}_X(*)D$. We have the decomposition $\mathcal{V} = \bigoplus_{b \in G \cdot a} \mathcal{V}_b$. Let $a \in \mathbb{C}$. Let $\theta$ be the Higgs field of $\mathcal{V}$ given by

$$\theta = \left( \bigoplus_{b \in G \cdot a} db \cdot \text{id}_{V_b} \right) + (\alpha \text{id}_V + N) dz/z = \bigoplus_{b \in G \cdot a} \left( (db + \alpha dz/z) \cdot \text{id}_{V_b} + N_b dz/z \right).$$

We have the decomposition $(\mathcal{V}, \theta) = \bigoplus_{b \in G \cdot a} (\mathcal{V}_b, \theta_b)$. Let $W$ denote the weight filtration on $V$ with respect to $N$. Take $\eta > 0$ such that $10 \text{rank}(V)\eta < 1$. Take $c \in \mathbb{R}$ and $c(\epsilon) \in \mathbb{R}$ $(0 \leq \epsilon \leq \eta)$ such that $|c(\epsilon) - c| \leq 2\epsilon$. For $a \in \mathbb{R}$, we set

$$\mathcal{P}_{a}^{c(\epsilon)}(\mathcal{V}) := \bigoplus_{j} W_j \otimes \mathcal{O}_X \left( [a - c(\epsilon) - j\epsilon] D \right) \subset \mathcal{V}.$$  

Lemma 3.17 There exists a family of harmonic metrics $h_{\epsilon,c(\epsilon)}(0 \leq \epsilon \leq \eta)$ of $(\mathcal{V}, \theta)|_{X \setminus D}$ such that the following holds

- $h_{\epsilon,c(\epsilon)}$ is adapted to $\mathcal{P}_{*}^{c(\epsilon)}(\mathcal{V})$.  

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• \( \lim_{\epsilon \to 0} h_{\epsilon, c(\epsilon)} = h_{0, c} \) in the \( C^\infty \)-sense locally on \( X \setminus D \). Moreover, there exists \( C > 1 \) such that

\[
C^{-1} |z|^{2\epsilon \text{rank}_V} h_{0,c} < h_{\epsilon, c(\epsilon)} < C |z|^{-2\epsilon \text{rank}_V} h_{0,c}
\]

for any \( 0 < \epsilon < \eta \).

**Proof** Set \( G_a := \{ \mu \in G | \mu^*a = a \} \). Then, \( (V_a, N_a) \) is naturally \( G_a \)-equivariant. Let \( h_{a, c(\epsilon)} \) be a family of \( G_a \)-invariant harmonic metrics of \( (V_a, \theta_a) \) as in Lemma 3.16. By the isomorphisms \( \mu^*V_a \simeq V_{\mu^*a} \), we obtain harmonic metrics \( h_{b, c(\epsilon)} \) for \( (V_b, \theta_b) \). We set \( h_{\epsilon, c(\epsilon)} := \bigoplus_b h_{b, c(\epsilon)} \). Then, the family of the harmonic metrics has the desired property.

### 3.7.5 Proof of Proposition 3.12

Let \( \varphi : \mathbb{C} \to \mathbb{C} \) be the map determined by \( \varphi(\zeta) = \zeta^\ell \) for some \( \ell \). We set \( X' := \varphi^{-1}(X) \). We may assume to have \( I \subset \zeta^{-1}\mathbb{C} \) and a decomposition

\[
\varphi^*(P_a V, \theta) = \bigoplus_{a \in I} \bigoplus_{\alpha \in \mathbb{C}} (P_a V_{a, \alpha}, \theta_{a, \alpha}),
\]

where \( \theta_{a, \alpha} - (da + adz/z) \text{id} \) are logarithmic, and the eigenvalues of the residues are 0. There exists the natural action of \( G := \{ \mu \in \mathbb{C}^* | \mu^\ell = 1 \} \) on \( X' \) given by the multiplication on the coordinate. Because \( \varphi^*(P_a V, \theta) \) is naturally \( G \)-equivariant, there exists the natural \( G \)-action on \( I \). We obtain the orbit decomposition \( I = \bigsqcup_{i=1}^m G \cdot a_i \). We set

\[
(P_i V, \theta_i) := \bigoplus_{a \in G \cdot a_i \alpha \in \mathbb{C}} (P_a V_{a, \alpha}, \theta_{a, \alpha}).
\]

It is naturally \( G \)-equivariant.

For \( (b, \alpha) \in \mathbb{C}a - 1, a \times \mathbb{C} \), we obtain the \( G \cdot a_i \)-graded vector space

\[
\bigoplus_{a \in G \cdot a_i} \text{Gr}_b^F (P_a V_{a, \alpha})
\]

equipped with the induced \( G \)-action such that \( \mu^* \text{Gr}_b^F (P_a V_{a, \alpha}) = \text{Gr}_b^F (P_{\mu^*a} V_{\mu^*a, \alpha}) \), and the \( G \)-invariant nilpotent endomorphism \( N \) which is compatible with the grading. We set \( b(\epsilon) := \psi(\epsilon) \). By applying Lemma 3.17, we obtain a family of \( G \)-equivariant unramifiedly good filtered Higgs fields

\[
(P^{(\epsilon, b(\epsilon))}_c V', \theta') = \bigoplus_{i=1}^m \bigoplus_{a \in G \cdot a_i \alpha \in \mathbb{C}} (P^{(\epsilon, b(\epsilon))}_{a, \alpha} V'_{a, b, \alpha}, \theta'_{a, b, \alpha}),
\]

equipped with a family of harmonic metrics \( h^{(\epsilon, b(\epsilon))}_{i, b, \alpha} \) satisfying the conditions in Lemma 3.17. We set

\[
(P^{(c)} V', \theta') = \bigoplus_{i=1}^m \bigoplus_{a \in G \cdot a_i \alpha \in \mathbb{C}} (P^{(c)}_{a, \alpha} V'_{a, b, \alpha}, \theta'_{a, b, \alpha}).
\]

**Lemma 3.18** There exists a \( G \)-equivariant isomorphism of filtered bundles \( \tilde{f} : P^{(0)} V' \simeq P_* V \) such that

\[
(f \circ \theta' - \theta \circ \tilde{f}) P^{(0)} V' \subset P_{<c} V \otimes \Omega^1(\log D)
\]

for any \( c \in \mathbb{R} \).

**Proof** Let \( F^{(0)} \) denote the filtration on \( P^{(0)} V'_{a, b, \alpha} \) induced by the filtered bundle \( P^{(0)} V' \). By the construction, we have the isomorphisms \( f_{a, b, \alpha} : \text{Gr}_b^F (P^{(0)} V'_{a, b, \alpha}) \simeq \text{Gr}_b^F (P_a V_{a, \alpha}) \), which is \( G_a \)-equivariant. There exists a \( G_a \)-equivariant isomorphism

\[
\tilde{f}_{a, \alpha} : \bigoplus_{a-1 < b \leq a} P^{(0, b)} V'_{a, b, \alpha} \simeq P_* V_{a, \alpha}
\]
which induces $f_{a, b, c}$. It induces a $G$-equivariant isomorphism

$$\tilde{f}_{i, a} : \bigoplus_{b \in G_a} \bigoplus_{a - 1 < b \leq a} P^{(0, b)}_i \simeq \bigoplus_{b \in G_a} P_i \mathcal{V}_{b, i, a}.$$ 

We set $\tilde{f} := \bigoplus_{i, a} \tilde{f}_{i, a}$. Then, $\tilde{f}$ has the desired property by the construction.

We identify the filtered bundles $P_\ast \mathcal{V}$ and $P^{(0)}_\ast \mathcal{V}^\prime$ by the isomorphism $\tilde{f}$. We obtain a family of Hermitian metrics $h_\varepsilon := \bigoplus h^{(\varepsilon)}_{a, b, c}$ of $E$. Set $\Phi := \theta' - \theta$. Let $\Phi_{h_\varepsilon, g_0}$ be the norm of $\Phi$ with respect to $h_\varepsilon$ and $g_0 = dz d\bar{z}$. We set $\theta_0' := \theta' - \Theta(\Theta + \theta d\bar{z}/z) d\bar{z}$. The following is clear by the construction.

**Lemma 3.19** There exists $C > 0$ such that $|\Phi|_{h_\varepsilon, g_0} \leq C|z|^{5q_{1,2} + 2 \text{rank}(\mathcal{V}) - 1}$ and $|\theta_0'|_{h_\varepsilon, g_0} \leq C|z|^{-1}$ for any $0 \leq \varepsilon \leq \eta_2$.

We also remark that $[\theta', h^\dagger_{h_\varepsilon}] = [\theta_0', h^\dagger_{h_\varepsilon}]$ and $[(\theta'_{h_\varepsilon})^\dagger, \Phi] = [(\theta_0')^\dagger_{h_\varepsilon}, \Phi]$. Because $R(h) + [\theta', (\theta')^\dagger_{h_\varepsilon}] = 0$, we obtain

$$R(h) + [\theta, (\theta')^\dagger_{h_\varepsilon}] = [\theta_0', h^\dagger_{h_\varepsilon}] + [\Phi, (\theta_0')^\dagger_{h_\varepsilon}] + [\Phi, h^\dagger_{h_\varepsilon}].$$

Hence, we obtain the claim of the proposition.

## 4 Existence and continuity of harmonic metrics in the curve case

### 4.1 Some statements

#### 4.1.1 Existence of pluri-harmonic metric

Let $C$ be a compact Riemann surface. Let $D \subset C$ be a finite subset. Let $(P_\ast \mathcal{V}, \theta)$ be a stable good filtered Higgs bundle on $(C, D)$ with $\text{deg}(P_\ast \mathcal{V}) = 0$. Let $(E, \mathcal{V}, \theta)$ be the Higgs bundle on $C \setminus D$ obtained as the restriction of $(\mathcal{V}, \theta)$. Let $h_{\det(\mathcal{V})}$ be a Hermitian metric of $\det(E)$ such that (i) $R(h_{\det(\mathcal{V})}) = 0$, (ii) $h_{\det(\mathcal{V})}$ is adapted to $P$, i.e., $P^\perp_{\det(\mathcal{V})} \det(E) \simeq P \det(\mathcal{V})$.

**Theorem 4.1 (Biquard-Boalch)** There exists a Hermitian-Einstein metric $h$ of $(E, \theta)$ adapted to $P_\ast \mathcal{V}$ such that $\det(h) = h_{\det(\mathcal{V})}$.

We give an outline of the proof in [4.2] based on the fundamental theorem of Simpson [57, Theorem 1] because we obtain a consequence on the Donaldson functional from the proof, which will be useful in the proof of Proposition 4.2.

#### 4.1.2 Continuity of harmonic metrics with respect to the perturbation

We take $a = (a P \mid P \in D)$ such that $a P \notin \text{Par}(P_\ast \mathcal{V}, P)$. We take $0 < \eta_i (i = 1, 2)$ such that $10 \text{rank}(\mathcal{V}) \eta_i < \text{gap}(P_\ast \mathcal{V}, a)$ and $10 \text{rank}(\mathcal{V}) \eta_2 < \eta_1$. For any $0 < \varepsilon < \eta_2$, we obtain a family of good filtered Higgs bundles $(P^{(\varepsilon)}_\ast \mathcal{V}, \theta)$ by applying the construction in [4.6.2]. We assume the following for each $P \in D$:

$$\sum_{c(a P) - 0 < c \leq a(P)} \psi_{P, c}(c) \text{rank} \mathcal{G}^F_{\varepsilon} (P_\ast \mathcal{V}_P) = \sum_{c(a P) - 0 < c \leq a(P)} c \text{rank} \mathcal{G}^F_{\varepsilon} (P_\ast \mathcal{V}_P).$$

In particular, $\text{deg}(P_\ast \mathcal{V}) = \text{deg}(P^{(\varepsilon)}_\ast \mathcal{V})$ holds. By making $\eta_2$ smaller, we may assume that $(P^{(\varepsilon)}_\ast \mathcal{V}, \theta)$ is stable for any $0 \leq \varepsilon \leq \eta_2$. According to Theorem 4.1 there exists a harmonic metric $h^{(\varepsilon)}$ of $(E, \mathcal{V}, \theta)$ such that $P^{h^{(\varepsilon)}} E = P^{(\varepsilon)} \mathcal{V}$ and $\det(h^{(\varepsilon)}) = \det(h^{(0)})$. The following proposition is a variant of [15, Proposition 4.1], for which we shall explain the outline of the proof in [4.3].

**Proposition 4.2** For any sequence $\varepsilon_i \to 0$, the sequence $h^{(\varepsilon_i)}$ is convergent to $h^{(0)}$ locally on $C \setminus D$ in the $C^\infty$-sense.
4.1.3 Convergence of some families of Hermitian metrics

For each \( P \in D \), we take a holomorphic coordinate neighbourhood \((C_P, z_P)\) around \( P \) such that \( z_P(P) = 0 \). Set \( C_P := C_P \setminus \{ P \} \). Fix \( N > 10 \). Let \( g_i \) be a sequence of \( C^\infty \)-metrics of \( C \setminus D \), such that

\[
g_i|C_P = (\epsilon^{N+2}|z_P|^{2\epsilon} + |z_P|^2) \frac{dz_P d\bar{z}_P}{|z_P|^2}.
\]

The following proposition is a variant of [45, Proposition 5.1].

**Proposition 4.3** Let \( h_1^{(i)} (i = 1, 2, \ldots) \) be a sequence of Hermitian metrics of \( E \) satisfying the following conditions.

- \( \det h_1^{(i)} = h_{\det(E)} \).
- \( |F(h_1^{(i)})|_{L^2_2} \to 0 \) as \( i \to \infty \).
- \( |b^{(i)}(h_1^{(i)})|_{L^2} \) is self-adjoint with respect to \( h^{(i)} \) and determined by \( h_1^{(i)} = h^{(i)} b^{(i)} \).

Then, \( b^{(i)}(h_1^{(i)}) \) and \( (b^{(i)})^{-1} \) are bounded with respect to \( h^{(i)} \) on \( C \setminus D \). We do not assume the uniform estimate.

Then, the sequence \( \{ h_i \} \) is convergent to \( \text{id}_E \) in \( L_1^2 \) locally on \( C \setminus D \). Moreover, there exists \( A > 0 \) such that \( |b^{(i)}(h_i^{(i)})| < A \) and \( |(b^{(i)})^{-1}|_{h_i^{(i)}} < A \) for any \( i \).

**Proof** We have only to apply the argument in the proof of [45, Proposition 5.1] by replacing \( G(h) \) and \( D^\lambda \) with \( F(h) \) and \( \theta + \theta \), respectively.

4.2 Proof of Theorem 4.1 and a consequence on the Donaldson functional

4.2.1 Proof of Theorem 4.1

Take \( a \in \mathbb{R}^D \) such that \( a_P \notin \text{Par}(P, \mathcal{V}, P) \) for any \( P \in D \). Let \((C_P, z_P)\) be a holomorphic coordinate neighbourhood around \( P \) such that \( z_P(P) = 0 \). Set \( C_P := C_P \setminus \{ P \} \). Take \( \eta > 0 \) such that \( 10 \text{rank}(\mathcal{V})\eta < \text{gap}(P, a) \). We take a Kähler metric \( g_{C \setminus D, \eta} \) of \( C \setminus D \) satisfying the following condition.

- \( g_{C \setminus D, \eta} \) is mutually bounded with \( |z_P|^{-\eta} d_2 d\bar{z}_2 \) on \( C_P \) for each \( P \in D \).

Recall that the Kähler manifold \((C \setminus D, g_{C \setminus D, \eta})\) satisfies the assumptions given in [57, §2], according to [57, Proposition 2.4].

**Lemma 4.4** There exists a Hermitian metric \( h_0 \) of \( E \) such that the following holds.

(a) \((E, \mathcal{O}_E, h_0)\) is acceptable, and \( \mathcal{P}^0_{\lambda} E = \mathcal{P} \).

(b) \( F(h_0) \) is bounded with respect to \( h_0 \) and \( g_{C \setminus D, \eta} \).

(c) \( \det(h_0) = h_{\det(E)} \)

**Proof** By applying Proposition 3.12 only in the case \( \epsilon = 0 \), we obtain a Hermitian metric \( h_0 \) of \( E \) satisfying (a) and (b). We define the function \( \varphi : C \setminus D \to \mathbb{R} \) by \( h_{\det(E)} = \det(h_0^\lambda) e^{\varphi} \). Then, \( \varphi \) induces a \( C^\infty \)-function on \( C \). We set \( h_0 := h_0^\lambda e^{\varphi/\text{rank}(E)} \). Then, the metric \( h_0 \) has the desired property.

For any holomorphic Higgs subbundle \( E' \subset E \), let \( h_0' \) denote the Hermitian metric of \( E' \) induced by \( h_0 \). Let \( \theta' \) denote the Higgs field of \( E' \) obtained as the restriction of \( \theta \). We have the Chern connection \( \nabla_{h_0'} \) and the adjoint \( \theta_{h_0'}^\dagger \) of \( \theta' \) with respect to \( h_0' \). Let \( R(h_0') \) denote the curvature of \( \nabla_{h_0'} \). Let \( F(E', \theta', h_0') \) denote the curvature of \( \nabla_{h_0'} + \theta' + \theta_{h_0'}^\dagger \). We set

\[
\deg(E', h_0) := \frac{-1}{2\pi} \int_{C \setminus D} \text{Tr} F(E', \theta', h_0') = \frac{-1}{2\pi} \int_{C \setminus D} \text{Tr} R(h_0').
\]
Let $\Lambda_{g_{C \setminus D, \eta}}$ denote the adjoint of the multiplication of the Kähler form associated to $g_{C \setminus D, \eta}$. Because $F(h_0)$ is bounded with respect to $h_0$ and $g_{C \setminus D, \eta}$, $\deg(E', h_0)$ is well defined in $\mathbb{R} \cup \{-\infty\}$ by the Chern-Weil formula [57, Lemma 3.2]:

$$\deg(E', h_0) = \frac{\sqrt{-1}}{2\pi} \int \text{Tr}(\Lambda_{g_{C \setminus D, \eta}} F(h_0) \pi_{E'}) - \frac{1}{2\pi} \int |\partial \Sigma_{E'}|^2 - \frac{1}{2\pi} \int |[\theta, \pi_{E'}]|^2.$$  \hfill (6)

Here, $\pi_{E'}$ denotes the orthogonal projection $E \to E'$ with respect to $h_0$.

**Lemma 4.5** $\deg(E', h_0)/\text{rank}(E') < \deg(E, h_0)/\text{rank}(E)$ holds. Namely, $(E, \overline{\partial}_E, \theta, h_0)$ is analytically stable in the sense of [58, §6].

**Proof** By [58, Lemma 6.1], we have $\deg(E, h_0) = \deg(P_{\theta, h_0} E) = 0$. Let $0 \neq E' \subseteq E$ be a Higgs subbundle on $C \setminus D$. By [58, Lemma 6.2], if $\deg(E', h_0) \neq -\infty$, $E'$ is extended to a filtered subbundle $P_{\theta, h_0} E' \subset P_{\theta, h_0} E$, and we have $\deg(E', h_0) = \deg(P_{\theta, h_0} E')$. Because $(P_{\theta, h_0} E, \theta)$ is assumed to be stable, we have $\deg(E', h_0)/\text{rank} E' < \deg(P_{\theta, h_0} E')/\text{rank} E = 0$. Hence, we obtain that $(E, \overline{\partial}_E, \theta, h_0)$ is analytically stable.

According to the existence theorem of Simpson [57, Theorem 1], there exists a harmonic metric $h$ of $(E, \overline{\partial}_E, \theta)$ such that $\det(h) = \det(h_0)$ and that $h$ and $h_0$ are mutually bounded. Thus, we obtain Theorem 4.1.

### 4.2.2 Complement on the Donaldson functional

Let $\mathcal{P}(h_0)$ be the space of $C^\infty$-Hermitian metrics $h_1$ of $E$ satisfying the following condition.

- Let $u_1$ be the endomorphism of $E$ such that (i) $h_1 = h_0 e^{u_1}$, (ii) $u_1$ is self-adjoint with respect to both $h_0$ and $h_1$. Then, $\sup_{Q \in C \setminus D} |u_1|_{h_0} (Q) + \| (\overline{\partial} + \theta) u_1 \|_{L^2} + \| (\overline{\partial} + \theta)(\overline{\partial} + \theta^\dagger) u_1 \|_{L^1} < \infty$. Here, we consider the $L^p$-norms induced by $h_0$ and $g_{C \setminus D, \eta}$.

The Donaldson functional $M(h_0, \bullet): \mathcal{P}(h_0) \to \mathbb{R}$ is defined as in [57, §5].

**Proposition 4.6** Let $h$ be the harmonic metric in Theorem 4.1. Then, $h$ is contained in $\mathcal{P}(h_0)$, and $M(h_0, h) \leq 0$ holds.

**Proof** Let $b$ be the automorphism of $E$ which is self-adjoint with respect to both $h$ and $h_0$, and determined by $h = h_0 \cdot b$. The theorem of Simpson [57, Theorem 1] implies that $b$ and $b^{-1}$ are bounded, and that $(\overline{\partial} + \theta) b$ is $L^2$ with respect to $h_0$ and $g_{C \setminus D, \eta}$. By [57, Lemma 3.1], we also obtain $(\overline{\partial} + \theta)(\overline{\partial} + \theta^\dagger) b$ is $L^1$. Hence, $h$ is contained in $\mathcal{P}(h_0)$. In the proof of [57, Theorem 1], the metric $h$ is constructed as the limit of a subsequence of the heat flow $h_t$ ($t \geq 0$) for which $\partial_t M(h_0, h_t) \leq 0$ holds. Because $M(h_0, h_0) = 0$ by the construction, we obtain $M(h_0, h_t) \leq 0$, and hence $M(h_0, h) \leq 0$.

### 4.3 Proof of Proposition 4.2

For $0 \leq \epsilon \leq \eta_2$, let $g_{C \setminus D, \epsilon, \eta_1}$ be the Kähler metric on $C \setminus D$ such that the following holds on $C^\epsilon_P$ for any $P \in D$:

$$g_{C \setminus D, \epsilon, \eta_1}(C_{P}) = (\epsilon^2 |z_P|^2 + \eta_1^2 |z_P|^{2 \eta_1}) |z_P|^{-2} d\bar{z}_P d\bar{z}_P.$$  

Let $\Lambda_{\omega, \epsilon}$ denote the adjoint of the multiplication of the Kähler form $\omega_{C \setminus D, \epsilon, \eta_1}$, associated to $g_{C \setminus D, \epsilon, \eta_1}$.

By using families of Hermitian metrics as in Proposition 3.12, we construct a family of metrics $h_{\epsilon}^{(\epsilon)}$ ($0 \leq \epsilon \leq \eta_2$) of $E$ such that the following holds:

- $h_{\epsilon}^{(\epsilon)}$ is adapted to $P_{\epsilon}^{(\epsilon)} \mathcal{V}$.
- $\text{det} h_{\epsilon}^{(\epsilon)} = \text{det} E$.
- $h_{\epsilon}^{(\epsilon)} \to h_{\epsilon}^{(0)}$ locally on $C \setminus D$ in the $C^\infty$-sense as $\epsilon \to 0$.  

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Lemma 4.7  Let \(w^{(\epsilon_i)}(\epsilon_i \to 0)\) be automorphisms of \(E\) which are self-adjoint with respect to \(h_{\infty}^{(\epsilon_i)}\) such that the following holds:

- \(\text{Tr}(w^{(\epsilon_i)}) = 0\).
- \(h_{\infty}^{(\epsilon_i)} e^{w^{(\epsilon_i)}} \in \mathcal{P}(\mathcal{H}(\epsilon_i))\), i.e., \(\sup \|u^{(\epsilon_i)}\|_{h_{\infty}^{(\epsilon_i)}} + \|\overline{(\partial + \theta)}u^{(\epsilon_i)}\|_{L^2} + \|\overline{(\partial + \theta)}(\partial + \theta)^{1/2}u^{(\epsilon_i)}\|_{L^1} < \infty\), where the \(L^p\)-norms are taken with respect to \(h_{\infty}^{(\epsilon_i)}\) and \(g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i}\). We do not assume that the estimate is uniform in \(i\).
- \(\|\Lambda_{\epsilon_i, \eta_i} F(h_{\infty}^{(\epsilon_i)} e^{w^{(\epsilon_i)}})\|_{h_{\infty}^{(\epsilon_i)}} < C_1\) for any \(i\).

Then, there exists \(C_3, C_4 > 0\) such that the following holds for any \(\epsilon_i\)

\[
\sup_{Q \in \mathcal{C} \setminus D} \|b_1^{(\epsilon_i)}\|_{h_{\infty}^{(\epsilon_i)}} < C_3 + C_4 M(h_{\infty}^{(\epsilon_i)}, h_{\infty}^{(\epsilon_i)} e^{w^{(\epsilon_i)}}).
\]

Proof  We have only to apply the argument in the proof of [15] Lemma 2.45] by replacing \(G(h)\) with \(F(h)\).

Let \(b_1^{(\epsilon)}\) be the automorphism of \(E\) which is self-adjoint with respect to \(h_{\infty}^{(\epsilon)}\), and determined by \(h_{\infty}^{(\epsilon)} = h_{\infty}^{(\epsilon)} h_{\infty}^{(\epsilon)}\). Note that \(\det(b_1^{(\epsilon)}) = 1\). Take any sequence \(\epsilon_i \to 0\). By Proposition [12] and Lemma [4.7] there exists a constant \(C_{10} > 0\) such that the following holds for any \(i\): \[
\sup_{Q \in \mathcal{C} \setminus D} \|b_{1/Q}^{(\epsilon_i)}\|_{h_{\infty}^{(\epsilon_i)}} < C_{10}.
\]

Lemma 4.8  \[
\int \Lambda_{\epsilon_i, \eta_i} (\overline{\partial \partial} \text{Tr}(b_1^{(\epsilon_i)})) \omega_{\mathcal{C} \setminus D, \epsilon_i, \eta_i} = 0
\]

Proof  Because \(\overline{\partial \partial} (\partial + \theta)^{1/2}b_1^{(\epsilon_i)}\) is \(L^1\) with respect to \(h_{\infty}^{(\epsilon_i)}\) and \(g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i}\), we obtain that \(\Lambda_{\epsilon_i, \eta_i} \overline{\partial \partial} \text{Tr}(b_1^{(\epsilon_i)}))\) is \(L^1\) with respect to \(g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i}\). Because \(\overline{\partial \partial} (\partial + \theta)^{1/2}b_1^{(\epsilon_i)}\) is \(L^2\) with respect to \(g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i}\) and \(h_{\infty}^{(\epsilon_i)}\), \(\partial \text{Tr}(b_1^{(\epsilon_i)}))\) is \(L^2\) with respect to \(g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i}\). Therefore, we obtain the claim of the lemma by using [57] Lemma 5.2.

By [57] Lemma 3.1, the following holds:

\[
\sqrt{-1} \Lambda_{\epsilon_i, \eta_i} \overline{\partial \partial} \text{Tr}(b_1^{(\epsilon_i)}) = - \text{Tr}(b_1^{(\epsilon_i)} \Lambda_{\epsilon_i, \eta_i} F(h_{\infty}^{(\epsilon_i)})) - \|\overline{\partial \partial} + \theta\|((\partial + \theta)^{1/2}b_1^{(\epsilon_i)})^{1/2} h_{\infty}^{(\epsilon_i)} g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i}.
\]

Therefore, there exists \(C_{12} > 0\) such that the following holds for any \(i\):

\[
\int |\overline{\partial \partial} + \theta|^{1/2} h_{\infty}^{(\epsilon_i)} g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i} < C_{12}.
\]

Take \(Q \in \mathcal{C} \setminus D\). Let \((C_Q, z_Q)\) be a holomorphic coordinate neighbourhood around \(Q\) which is relatively compact in \(\mathcal{C} \setminus D\). Because \(b_1^{(\epsilon_i)}\) are self-adjoint with respect to \(h_{\infty}^{(\epsilon_i)}\), we have \(\overline{\partial \partial} b_1^{(\epsilon_i)} h_{\infty}^{(\epsilon_i)} g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i} = |\partial E, h_{\infty}^{(\epsilon_i)} b_1^{(\epsilon_i)}| h_{\infty}^{(\epsilon_i)} g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i}|.\) Hence, by (1), there exists \(C_{13}(Q) > 0\) such that the following holds for any \(i\):

\[
\int_{C_Q} |\partial E, h_{\infty}^{(\epsilon_i)} b_1^{(\epsilon_i)}| h_{\infty}^{(\epsilon_i)} g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i} < C_{13}(Q).
\]

According to a variant of Simpson’s main estimate (for example, see [13] Proposition 2.10]), there exists \(C_{14}(Q) > 0\) such that the following holds on \(C_Q\) for any \(i\):

\[
|\partial^2 h_{\infty}^{(\epsilon_i)} g_{\mathcal{C} \setminus D, \epsilon_i, \eta_i} < C_{14}(Q).
\]
Because $R(h^{(c_i)}) + [\theta, \theta h^{(c_i)}] = 0$, we obtain the following estimate on $C_Q$ for any $i$:

$$|R(h^{(c_i)})|_{h^{(c_i)}} g_{C \setminus D, \epsilon_i, \eta} < 10C_{14}(Q).$$

Because $R(h^{(c_i)}) - R(h^{(c_i)}) = \overline{\partial}(b_1^{(c_i)})^{-1} \partial_{E, h^{(c_i)}} b_1^{(c_i)}$, there exists $C_{15}(Q) > 0$ such that

$$\|\partial_{E, h^{(c_i)}} b_1^{(c_i)}\|_{L_2^1} < C_{15}(Q)$$

for any $i$. By a bootstrapping argument, for any $p \geq 2$, there exists $C_{16}(Q, p) > 0$ such that $\|\partial_{E, h^{(c_i)}} b_1^{(c_i)}\|_{L_1^p} < C_{16}(Q, p)$ for any $i$. There exists a subsequence $c'_j$ such that the sequence $b_1^{(c'_j)}$ is weakly convergent locally on $C \setminus D$ in $L_2^p$ for any $p$. Let $b_1^{(c)}$ be the weak limit. Then, $h^{(0)} := h^{(0)} b_1^{(c)}$ is a harmonic metric of $(E, \overline{\partial}_E, \theta)$ such that (i) $h^{(0)}$ and $h^{(0)}$ are mutually bounded on $C \setminus D$, (ii) $\det(h^{(0)}) = h_{\det E}$. Then, by the uniqueness, we obtain that $b_1^{(c)} = \text{id}_E$. Namely, $h^{(c'_j)}$ is weakly convergent to $h^{(0)}$ locally in $L_2^p$ for any $p$. By a bootstrapping argument, we obtain that $h^{(c'_j)}$ is convergent to $h^{(0)}$ locally in the $C^\infty$-sense. Then, the claim of the proposition follows.

4.4 Continuity of family of harmonic metrics

Let $\pi : C \to \Delta$ be a smooth projective family of complex curves. Let $D \subset C$ be a smooth hypersurface such that the induced map $D \to \Delta$ is proper and locally bi-holomorphic. For each $t \in \Delta$, we set $C_t := \pi^{-1}(t)$ and $D_t := C_t \cap D$.

Let $(P, V, \theta)$ be a good filtered Higgs bundle on $(C, D)$. The induced good filtered Higgs bundles $(P_t, V_t, \theta_t)$ are denoted by $(P, V, \theta)_t$.

Let $(E, \overline{\partial}_E, \theta)$ be the Higgs bundle on $C \setminus D$ obtained as the restriction of $(P, V, \theta)$ to $C \setminus D$. Let $(E_t, \overline{\partial}_E, \theta_t)$ be the Higgs bundle on $C_t \setminus D_t$ obtained as the restriction of $(E, \overline{\partial}_E, \theta)$. Suppose the following:

- There exists a Hermitian metric $h_{\det E}$ of $E$ such that (i) $R(h_{\det E}) = 0$, (ii) $h_{\det E}$ is adapted to $P_\pi(\det V)$.
- Each $(P_t, V_t, \theta_t)$ is stable of degree 0.

According to Theorem 4.11, there exists harmonic metrics $h_t$ of $(E_t, \overline{\partial}_E, \theta_t)$ adapted to $P_t V_t$ such that $\det(h_t) = h_{\det(E_t)}|_{C_t \setminus D_t}$. We obtain the Hermitian metric $h$ of $E$ determined by $h_t|_{C_t \setminus D_t} = h_t$. We obtain the following proposition by using Proposition 4.6 and an argument similar to the proof of Proposition 4.2 (See also 4.4 Proposition 4.2.)

**Proposition 4.9** $h$ is continuous. Moreover, any derivatives of $h$ in the fiber direction are continuous.

5 Preliminary existence theorem for Hermitian-Einstein metrics

5.1 Statements

5.1.1 Kähler metrics

Let $X$ be a smooth projective surface with a simply normal crossing hypersurface $H = \bigcup_{i \in A} H_i$. Let $L$ be an ample line bundle on $X$. Let $g_X$ be the Kähler metric of $X$ such that the associated Kähler form $\omega_X$ represents $c_1(L)$.

We take Hermitian metrics $g_i$ of $O(H_i)$. Let $\sigma_i : O_X \to O_X(H_i)$ denote the canonical section. Take $N > 10$. There exists $C > 0$ such that the following form defines a Kähler form on $X \setminus H$ for any $0 \leq \epsilon < 1/10$:

$$\omega_\epsilon := \omega_X + \sum_{i \in A} C \cdot \epsilon^{N+2} \cdot \sqrt{-1} \partial \overline{\partial} |\sigma_i|^2 g_i.$$  

It is easy to observe that $\int_X \omega_\epsilon^2 = \int_X \omega_X^2$ and that $\int_X \omega_\epsilon \tau = \int_X \omega_X \tau$ for any closed $C^\infty$-(1,1)-form $\tau$ on $X$.  

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5.1.2 Condition for good filtered Higgs bundles and initial metrics

Let \((\mathcal{P}, \mathcal{V}, \theta)\) be a good filtered Higgs bundle on \((X, H)\) satisfying the following condition.

Condition 5.1

- There exists \(c \in \mathbb{R}^\Lambda\) and \(m \in \mathbb{Z}_{>0}\) such that \(\text{Par}(\mathcal{P}, \mathcal{V}, i) = \{c_i + n/m \mid n \in \mathbb{Z}\}\) for each \(i \in \Lambda\).
- The nilpotent part of \(\text{Res}_i(\theta)\) on \(i \text{Gr}_1^\varphi(\mathcal{P}_a \mathcal{V})\) are 0 for any \(i \in \Lambda\), \(a \in \mathbb{R}^\Lambda\) and \(b \in |a_i - 1, a_i|\).

Let \((E, \mathcal{D}_E, \theta)\) denote the Higgs bundle on \(X \setminus H\) obtained as the restriction of \((\mathcal{P}, \mathcal{V}, \theta)\).

Let \(P\) be any point of \(H \setminus \bigcup_{j \neq i} H_j\). Let \((X_P, z_1, z_2)\) be a holomorphic coordinate neighbourhood around \(P\) such that \(H \cap X_P = \{z_1 = 0\}\). There exists an open subset \(X'_P \subset \mathbb{C}^2\) \(\{((\zeta_1, \zeta_2))\}\) such that the map \(\varphi_P : X'_P \longrightarrow X_P\) given by \(\varphi_P(\zeta_1, \zeta_2) = (\zeta_1^n, \zeta_2)\) is a ramified covering. We set \(H'_P := \{\zeta_1 = 0\} \cap X'_P\). We obtain the induced good filtered Higgs bundle \((\mathcal{P}_P, \varphi_P^* \mathcal{V}, \varphi^* \theta)\) on \((X'_P, H'_P)\) such that \(\text{Par}(\mathcal{P}_P, \varphi_P^* \mathcal{V}) = \{m \cdot c_i\} + \mathbb{Z}\).

Definition 5.2 A Hermitian metric \(h_P\) of \(E|_{X_P \setminus H}\) is called strongly adapted to \(\mathcal{P}_* \mathcal{V}|_{X_P}\) if there exists a \(C^\infty\) Hermitian metric \(h'_P\) of \(\mathcal{P}_{mc}(\varphi_P^* \mathcal{V})\) on \(X'_P\) such that \(\varphi^{-1}(h_P) = |\zeta_1|^{-2mc_i} h'_P\).

Let \(P\) be any point of \(H_i \setminus H_j \ (i \neq j)\). Let \((X_P, z_1, z_2)\) be a holomorphic coordinate neighbourhood around \(P\) such that \(X_P \cap H_i = \{z_1 = 0\}\) and \(X_P \cap H_j = \{z_2 = 0\}\). There exists an open subset \(X'_{P}\) in \(\mathbb{C}^2\) \(\{((\zeta_1, \zeta_2))\}\) such that the map \(\varphi_P : X'_{P} \longrightarrow X_P\) given by \(\varphi_P(\zeta_1, \zeta_2) = (\zeta_1^n, \zeta_2^n)\) is a ramified covering. We set \(H'_P := \{\zeta_1 \zeta_2 = 0\} \cap X'_P\). We obtain the induced good filtered Higgs bundle \((\mathcal{P}_P, \varphi_P^* \mathcal{V}, \varphi^* \theta)\) on \((X'_P, H'_P)\) such that \(\text{Par}(\mathcal{P}_P, \varphi_P^* \mathcal{V}, 1) = \{m \cdot c_i\} + \mathbb{Z}\) and \(\text{Par}(\mathcal{P}_P, \varphi_P^* \mathcal{V}, 2) = \{m \cdot c_j\} + \mathbb{Z}\).

Definition 5.3 A Hermitian metric \(h_P\) of \(E|_{X_P \setminus H}\) is called strongly adapted to \(\mathcal{P}_* \mathcal{V}|_{X_P}\) if there exists a \(C^\infty\) Hermitian metric \(h'_P\) of \(\mathcal{P}_{(mc, cmc)}(\varphi_P^* \mathcal{V})\) such that \(\varphi^*(h_P) = |\zeta_1|^{-mc_i} |\zeta_2|^{-cm_j} h'_P\).

Definition 5.4 A Hermitian metric \(h\) of \(E\) is called strongly adapted to \(\mathcal{P}_* \mathcal{V}\) if the following holds.

- For any \(P \in H\), there exists a small neighbourhood \(X_P\) of \(P\) such that \(h|_{X_P \setminus H}\) is strongly adapted to \(\mathcal{P}_* \mathcal{V}|_{X_P}\) in the sense of Definition 5.2 and Definition 5.3.

Lemma 5.5 Let \(h\) be a Hermitian metric of \(E\) strongly adapted to \(\mathcal{P}_* \mathcal{V}\). Then, the following holds:

\[
\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X \setminus H} \text{Tr}(R(h)^2) = 2 \int_X \text{par-ch}_2(\mathcal{P}_* \mathcal{V}).
\]

Proof It is the equality (36) in the proof of [43] Proposition 4.18.

For each \(i \in \Lambda\), we choose \(b_i \in \text{Par}(\mathcal{P}_i \text{det} \mathcal{V}, i)\). Set \(b = (b_i) \in \mathbb{R}^\Lambda\). We take a Hermitian metric \(h_{\text{det}(E)}\) of \(\text{det}(E)\) such that \(h_{\text{det}(E)} \prod_{i \in \Lambda} \varphi_i^{\text{det}(b_i)}\) induces a Hermitian metric of \(\mathcal{P}_b \text{det} \mathcal{V}\) of \(C^\infty\)-class. We shall prove the following proposition in [5.4] after preliminaries in [5.2, 5.3].

Proposition 5.6 There exists a Hermitian metric \(h_{in}\) of \(E\) such that the following holds.

- \(h_{in}\) is strongly adapted to \(\mathcal{P}_* \mathcal{V}\).
- \(F(h_{in})\) is bounded with respect to \(h_{in}\) and \(\omega_{\epsilon}\), where \(\epsilon := m^{-1}\).
- The following holds
  \[
  \int_{X \setminus H} \text{Tr}(R(h_{in})^2) = \int_{X \setminus H} \text{Tr}(F(h_{in})^2).
  \]
- \(\text{det}(h_{in}) = h_{\text{det}(E)}\).

Such a Hermitian metric \(h_{in}\) is called an initial metric of \((\mathcal{P}, \mathcal{V}, \theta)\).
5.1.3 Preliminary existence theorem for Hermitian-Einstein metrics

Let \((\mathcal{P}, \mathcal{V}, \theta)\) be a good filtered Higgs bundle satisfying Condition 5.1. Let \(h_{in}\) be an initial metric for \((\mathcal{P}, \mathcal{V}, \theta)\) as in Proposition 5.6.

**Theorem 5.7** Suppose that \((\mathcal{P}, \mathcal{V}, \theta)\) is \(\mu_L\)-stable. Then, there exists a Hermitian-Einstein metric \(h_{HE}\) of \((E, \overline{\mathcal{V}}_E, \theta)\) with respect to the Kähler form \(\omega_\varepsilon (\varepsilon := m^{-1})\) satisfying the following conditions.

(i) \(h_{HE}\) and \(h_{in}\) are mutually bounded.

(ii) \((\overline{D} + \theta)(h_{HE} \cdot h_{in}^{-1})\) is \(L^2\) with respect to \(h_{in}\) and \(\omega_\varepsilon\).

(iii) \(\det(h_{HE}) = \det(h_{in})\) holds. In particular, \(\text{Tr}(F(h_{HE})) = \text{Tr}(F(h_{in})) = \text{Tr}(R(h_{in}))\) holds.

(iv) The following equalities hold:

\[
\left(\frac{-1}{2\pi}\right)^2 \int_{X \setminus H} \text{Tr}(F(h_{HE})^2) = 2 \int_X \text{par-ch}_2(\mathcal{P}, E). \tag{9}
\]

5.2 Around cross points

Let \(X_0 := \{(z_1, z_2) \in \mathbb{C}^2 | |z_1| < 1\}\). We set \(H_i := X_0 \cap \{z_i = 0\}\) and \(H := H_1 \cup H_2\). Let \((\mathcal{P}, \mathcal{V}, \theta)\) be a good filtered Higgs bundle on \((X_0, H)\). We choose \(b_i \in \text{Par}(\mathcal{P}, V, i)\) \((i = 1, 2)\), and set \(b = (b_1, b_2)\). We also choose any Hermitian metric \(h_{det(E)}\) of \(det(E)\) such that \(h_{det(E)}|z_1|^{2b_1}|z_2|^{2b_2}\) is a Hermitian metric of \(\mathcal{P}_b(det V)\) of \(C^\infty\)-class.

5.2.1 Unramified case

Suppose that \((\mathcal{P}, \mathcal{V}, \theta)\) satisfies the following condition.

**Condition 5.8**

- There exists \(c = (c_1, c_2) \in \mathbb{R}^2\) such that \((i) -1 < c_i \leq 0\), (ii) \(\text{Par}(\mathcal{P}, V, i) = \{c_i + n | n \in \mathbb{Z}\}\).

- There exists a decomposition of good filtered Higgs bundles

\[
(\mathcal{P}_* \mathcal{V}, \theta) = \bigoplus_{a \in I} \bigoplus_{\alpha \in \mathbb{C}^2} (\mathcal{P}_* \mathcal{V}_a, \mathcal{V}_a, \theta_{a, \alpha})
\]

such that \(\theta_{a, \alpha} = (da + \sum \alpha_i dz_i/z_i)\) induce holomorphic Higgs fields of \(\mathcal{P}_c \mathcal{V}_a, \alpha\).

We take any holomorphic frame \(v = (v_j)\) of \(\mathcal{P}_c \mathcal{V}\) compatible with the decomposition. For \(j = 1, \ldots, r\), we have \((a, \alpha_i)\) determined by \(v_i \in \mathcal{P}_c \mathcal{V}_a, \alpha_i\). Let \(h_0\) be the metric of \(\mathcal{V}|_{X_0 \setminus H}\) determined by \(h_0(v_i, v_j) = |z_1|^{-2c_1}|z_2|^{-2c_2}\) and \(h_0(v_i, v_j) = 0 (i \neq j)\). We have \(\partial v = v(- \sum k=1,2 \alpha_k dz_k/z_k)I\), where \(I\) denotes the identity matrix. We have the description \(\theta v = v(\Lambda_0 + \Lambda_1)\) such that the following holds.

- \((\Lambda_0)_{ij} = (da_i + \sum k=1,2 \alpha_{ik} dz_k/z_k)\) and \((\Lambda_0)_{ij} = 0 (i \neq j)\).

- \((\Lambda_1)_{ij}\) are holomorphic 1-forms for any \(i\) and \(j\). Moreover, \((\Lambda_1)_{ij} = 0\) holds unless \((a_i, \alpha_i) = (a_j, \alpha_j)\).

We have \(\theta^b_{h_0} v = v(\overline{\Lambda}_0 + \overline{\Lambda}_1)\). We have \(\partial \theta_{h_0} v = 0\). We have \(\theta^b_{h_0} v = v[\Lambda_1, \overline{\Lambda}_1]\), where the entries of \([\Lambda_1, \overline{\Lambda}_1]\) are \(C^\infty\) on \(X_0\). We have \((\partial \theta_{h_0} v = v(\partial \Lambda_1),\) where any entries of \(\partial \Lambda_1\) are holomorphic 2-forms, and \((\partial \Lambda_1)_{ij} = 0\) unless \((a_i, \alpha_i) = (a_j, \alpha_j)\).

Note that there exists a \(C^\infty\)-function \(u\) on \(X_0\) such that \(\det(h_0) = e^u h_{det(E)}\). We set \(h_{in} := h_0 e^{-u/\text{rank} E}\).

**Lemma 5.9** \([\theta, \theta^b_{h_{in}}], \partial h_{in} \theta\) and \(\partial \theta^b_{h_{in}}\) are bounded with respect to \(h_{in}\) and \(\sum k=1,2 dz_k d\overline{z}_k\).
5.2.2 Ramified case
Let \( \varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) be given by \( \varphi(\zeta_1, \zeta_2) = (\zeta_1^{m_1}, \zeta_2^{m_2}) \). We set \( X'_0 := \varphi^{-1}(X_0), \) \( H'_0 := X'_0 \cap \varphi^{-1}(H_1) \) and \( H' := H'_1 \cup H'_2. \) We set \( \text{Gal}(\varphi) := \{(\kappa_1, \kappa_2) \in \mathbb{C}^2 | \kappa_1^{m_1} = 1 \} \), which acts on \( X'_0 \) by \( (\kappa_1, \kappa_2)(\zeta_1, \zeta_2) = (\kappa_1 \zeta_1, \kappa_2 \zeta_2) \).

Suppose that \( \varphi^*(P, \mathcal{V}, \vartheta) \) satisfies Condition 5.8 on \( (X', H') \). We construct a \( C^\infty \)-metric \( h'_0 \) of \( \varphi^*(E)|_{X'_0 \setminus H'_0} \) as in the previous subsection. We may assume that \( h'_0 \) is Gal(\( \varphi \))-invariant. Note that there exists a Gal(\( \varphi \))-invariant \( C^\infty \)-function \( u \) on \( X'_0 \) such that \( \det(h'_0) = e^u \varphi^{-1}(\det(E)|_{X'_0 \setminus H'_0}). \) We set \( h''_0 := h'_0 e^{-u/\text{rank}(E)}. \) Because it is Gal(\( \varphi \))-invariant, we obtain the induced metric \( h''_0 \) of \( E. \)

Let \( gX'_0 \) denote the Kähler metric \( \sum_{k=1,2}^k d\zeta_k \overline{d\zeta_k} \) on \( X'_0 \). Because \( \varphi : X'_0 \setminus H' \rightarrow X_0 \setminus H \) is a covering map, it induces a Kähler metric \( \varphi^*(gX'_0) \) of \( X_0 \setminus H. \)

Lemma 5.10 \( [\theta, \theta^\dagger_{h''_0}], \partial_{h''_0} \theta \) and \( \overline{\partial}_h^\dagger \) are bounded with respect to \( (h''_0, \varphi^*(gX'_0)). \)

5.2.3 An estimate
We set \( Y(\epsilon) := \{(z_1, z_2) \in X_0 \mid \min(|z_i|) = \epsilon \}. \)

Lemma 5.11 We have \( \lim_{\epsilon \to 0} \int_{Y(\epsilon)} \text{Tr}(\theta \overline{\partial}_h^\dagger) = 0 \) and \( \lim_{\epsilon \to 0} \int_{Y(\epsilon)} \text{Tr}(\theta^\dagger \partial_h \theta) = 0. \)

Proof It is enough to consider the case where Condition 5.8 is satisfied for \( (P, \mathcal{V}, \vartheta) \). Let \( f \) be any anti-holomorphic function on \( X_0 \). Let us consider \( \int_{Y(\epsilon)} df d\overline{z}_1 dz_2. \) We set \( Y_1(\epsilon) := \{|z_1| = \epsilon, |z_2| \geq \epsilon \} \) and \( Y_2(\epsilon) := \{|z_2| = \epsilon, |z_1| \geq \epsilon \}. \) We have

\[
\int_{Y_1(\epsilon)} df d\overline{z}_1 dz_2 = \int_{Y_2(\epsilon)} dz_2 d\overline{z}_1 f dz_1 d\overline{z}_2.
\]

It is of the form

\[
\int_{Y_1(\epsilon)} \frac{b(z_1, z_2)}{z_1 \overline{z}_2} f(\overline{z}_1, z_2) dz_1 d\overline{z}_2.
\]

Here, \( b \) is a holomorphic function. We consider the Taylor expansion of \( b \) and \( f. \) Then, the contributions of the terms

\[
\frac{z_1^{k_1} \overline{z}_1^{m_1}}{z_1^{\ell_1}} dz_1 \overline{z}_1 \overline{z}_2^{k_2} \overline{z}_2^{m_2} dz_2
\]

(11) to \( \int_{Y_1(\epsilon)} \) is 0 unless \( k_1 - \ell_1 - m_1 = 1 \) and \( k_2 - \ell_2 - m_2 = 0. \) If the equalities hold, we have \( k_1 - \ell_1 + m_1 = 2m_1 + 1 \geq 1 \) and \( k_2 - \ell_2 + m_2 = 2m_2 \geq 0. \) Hence, we obtain \( \lim_{\epsilon \to 0} \int_{Y_1(\epsilon)} df d\overline{z}_1 dz_2 = 0. \) Similarly, we obtain \( \lim_{\epsilon \to 0} \int_{Y_2(\epsilon)} df d\overline{z}_1 dz_2 = 0. \) Similarly and more easily, we obtain \( \lim_{\epsilon \to 0} \int_{Y(\epsilon)} (\alpha \overline{d} z_1 /\overline{z}_1) f dz_1 d\overline{z}_2 = 0. \) Then, the claim of the lemma follows.

5.3 Around smooth metrics
We set \( X_0 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_i| < 1 \} \) and \( H := \{z_1 = 0 \}. \) Let \( \nu : X_0 \setminus H \rightarrow \mathbb{R}_{>0} \) be a \( C^\infty \)-function such that \( \nu|_{z_1 = 1}^{-1} \) induces a nowhere vanishing function on \( X_0 \) of \( C^\infty \)-class. Let \( (P, \mathcal{V}, \vartheta) \) be a good filtered Higgs bundle on \( (X_0, H). \) Let \( (E, \varphi_E, \vartheta) \) be the Higgs bundle obtained as the restriction of \( (P, \mathcal{V}, \vartheta) \) to \( X_0 \setminus H. \) We choose \( b \in \text{Par}(P_\epsilon) \) and a Hermitian metric \( h_{\det(E)} \) of \( \det(E) \) such that \( h_{\det(E)} \nu^{2c} \) induces a \( C^\infty \) metric of \( P_\epsilon(\det \mathcal{V}). \)

5.3.1 Unramified case
Suppose that \( (P, \mathcal{V}, \vartheta) \) satisfies Condition 5.12

Condition 5.12

- There exists \( -1 < c \leq 0 \) such that \( \text{Par}(P, \mathcal{V}) = \{c + n \mid n \in \mathbb{Z}\}. \)
There exists a decomposition of good filtered Higgs bundles
\[(P, \mathcal{V}, \theta) = \bigoplus_{a \in \mathcal{I}} \bigoplus_{\alpha \in \mathbb{C}} (P_a \mathcal{V}_a, \theta_{a, \alpha}).\]

\[\theta_{a, \alpha} - (da + adz_1/z_1)\text{ are holomorphic Higgs fields of } P_a \mathcal{V}_a.\]

We take \(C^\infty\)-metrics \(h_{a, \alpha}\) of \(P_a \mathcal{V}_a\), and we set \(h_0 := \bigoplus \nu^{-2} h_{a, \alpha}\). We may assume that \(\det(h_0) = h_{\det(c)}\).

Let \(v = (v_1, \ldots, v_r)\) be any holomorphic frame of \(P_a \mathcal{V}\) compatible with the decomposition. For each \(i, a\) and \(\alpha\), \(v_i\) is determined by the condition that \(v_i\) is a section of \(P_a \mathcal{V}_{a, \alpha}\). There exist matrix valued \(C^\infty-(1, 0)\)-forms \(A_{a, \alpha}\) such that

\[\partial_{h_0} v = v \left((-c \cdot \partial \log \nu^2) I + \bigoplus A_{a, \alpha}\right),\]

where \(I\) denotes the identity matrix, and \((A_{a, \alpha})_{i,j} = 0\) unless \((a_i, \alpha_i) = (a_j, \alpha_j)\). Let \(\Lambda\) denote the matrix valued holomorphic 1-form determined by \(\theta v = \nu \Lambda\). There exists the decomposition \(\Lambda = \Lambda_0 + \Lambda_1\) such that the following holds.

- \((\Lambda_0)_{i,j} = (d a_i + \alpha_i dz_1/z_1)\) if \(i = j\), and \((\Lambda_0)_{i,j} = 0\) if \(i \neq j\).
- \((\Lambda_1)_{i,j}\) are holomorphic 1-forms, and \((\Lambda_1)_{i,j} = 0\) unless \((a_i, \alpha_i) = (a_j, \alpha_j)\).

There exists a matrix valued \(C^\infty\) \((0, 1)\)-form \(\Lambda_2\) such that \(\partial_{h_0} v = v \left(\Lambda_0 + \Lambda_2\right)\). Moreover, \((\Lambda_2)_{i,j} = 0\) holds unless \((a_i, \alpha_i) = (a_j, \alpha_j)\).

We have \(R(h_0) = (-c \partial \log \sigma^2) I + \bigoplus R(h_{a, \alpha})\), where \(R(h_{a, \alpha})\) are \(C^\infty\). Note that \(d \Lambda_0 = 0\) and \([\Lambda_0, \Lambda_i] = [\Lambda_0, \Lambda_i] = 0\). Hence, \([\theta, \Lambda_0], \partial_{h_0} \theta\) and \(\partial_{h_0} \theta\) are \(C^\infty\). We also have

\[\left(\partial_{h_0} \theta\right) v = v \left(\partial \Lambda_1 + \left(\bigoplus A_{a, \alpha}, \Lambda_1\right)\right), \quad \left(\partial_{h_0} \theta\right) v = v \left(\Lambda_2\right).\]

We set \(w_1 = z_1/z_1^{-1} \nu\) and \(w_2 = z_2\). Then, it is easy to check that \((w_1, w_2)\) is a \(C^\infty\) complex coordinate system. Clearly, \(dz_1 = dw_1 + w_1 \kappa_1\). We set \(Y(\epsilon) = \{\nu = \epsilon\} = \{|w_1| = \epsilon\}\).

Lemma 5.13 \(\lim_{\epsilon \to 0} \int_{Y(\epsilon)} \text{Tr}(\Lambda_0 \overline{\Lambda}_2) = 0\) and \(\lim_{\epsilon \to 0} \int_{Y(\epsilon)} \text{Tr}(\Lambda_1 \overline{\Lambda}_2) = 0\) hold.

Proof It is enough to prove \(\lim_{\epsilon \to 0} \int_{Y(\epsilon)} \text{Tr}(\Lambda_1 \overline{\Lambda}_2) = 0\). It is easy to see that \(\lim_{\epsilon \to 0} \int_{Y(\epsilon)} \text{Tr}(\Lambda_0 \overline{\Lambda}_2) = 0\). Let us study \(\int_{Y(\epsilon)} \text{Tr}(\Lambda_0 \overline{\Lambda}_2)\). For any \(C^\infty\)-function \(g\), we consider the following integral:

\[\int_{Y(\epsilon)} g(da \cdot d\overline{z}_1, d\overline{z}_2) = \int_{Y(\epsilon)} (g(\gamma) \cdot da) \cdot d\overline{w}_1 d\overline{w}_2 + \int_{Y(\epsilon)} g(\overline{w}_1) \cdot da \kappa d\overline{w}_2.\]

(12)

We can rewrite the first term in the right hand side of (12) as follows, for some non-negative integer \(\ell\) and for a \(C^\infty\)-function \(b\):

\[\int_{Y(\epsilon)} (g(\gamma) \cdot da) d\overline{w}_1 d\overline{w}_2 = \int_{Y(\epsilon)} (g(\gamma b) w_1^{-1} d\overline{w}_1 d\overline{w}_2.\]

Take \(N > \ell\). We consider the expansion

\[g(\gamma b) = \sum_{k, m \geq 0, k + m \leq N} (g(\gamma b))_{k,m}(w_2) w_1^k w_2^m + O(|w_1|^N).\]

Here, \((g(\gamma b))_{k,m}(w_2)\) are \(C^\infty\)-functions of \(w_2\). The contributions

\[\int_{Y(\epsilon)} (g(\gamma b))_{k,m}(w_2) w_1^k w_2^m d\overline{w}_1 d\overline{w}_2\]
are 0 unless \( k - \ell - m = 1 \). If \( k - \ell - m = 1 \), then \( k - \ell + m = 2m + 1 \geq 1 \) holds. Hence, we obtain
\[
\lim_{\epsilon \to 0} \int_{Y(e)} (g^2) \, d\bar{w}_1 \, dw_2 = 0.
\]
We rewrite the second term in the right hand side of (12) as follows, for some \( C^\infty \)-functions \( f_i \) (i = 1, 2) and a non-negative integer \( \ell \):
\[
\int_{Y(e)} g^2 \, d\bar{w}_1 \, dw_2 = \int_{Y(e)} f_1 w_1^{\ell-1} w_1 \, dw_1 \, dw_2 + \int_{Y(e)} f_2 w_1^{\ell} w_1 \, dw_1 \, dw_2. \tag{13}
\]
Take \( N > \ell + 1 \). Consider the expansions \( f_i = \sum (k, m, w_2) w_1^k w_1^m + O(|w_1|^N) \). The contributions
\[
\int_{Y(e)} (f_1)_{k, m, w_2} w_1^k w_1^m \, dw_1 \, dw_2
\]
are 0 unless \( k - (\ell + 1) - (m + 1) = -1 \). If \( k - (\ell + 1) - (m + 1) = -1 \) holds, then we have \( k - (\ell + 1) + (m + 1) = 2m + 1 \geq 1 \). The contributions
\[
\int_{Y(e)} (f_2)_{k, m, w_2} w_1^k w_1^m \, dw_1 \, dw_2
\]
are 0 unless \( k - \ell - (m + 1) = 1 \). If \( k - \ell - (m + 1) = 1 \) holds, then we have \( k - \ell + (m + 1) = 2(m + 1) + 1 \geq 3 \). Hence, we obtain
\[
\lim_{\epsilon \to 0} \int_{Y(e)} g^2 \, d\bar{w}_1 \, dw_2 = 0.
\]
Similarly and more easily, we obtain \( \lim_{\epsilon \to 0} \int_{Y(e)} g(adz_1/z_1) \, d\bar{w}_1 \bar{w}_2 = 0 \) for any \( \alpha \in \mathbb{C} \) and for any \( C^\infty \)-function \( g \). Thus, we obtain the claim of the lemma.

\[5.3.2 \text{ Ramified case}
\]
Let \( \varphi : \mathbb{C}^2 \to \mathbb{C}^2 \) be given by \( \varphi(\zeta_1, \zeta_2) = (\zeta_1^m, \zeta_2) \). We set \( X_0' := \varphi^{-1}(X_0) \) and \( H' := \varphi^{-1}(H) \). Let \( \text{Gal}(\varphi) := \{ \mu \in \mathbb{C} | \mu^m = 1 \} \), which acts on \( X_0' \) by \( \mu \cdot (\zeta_1, \zeta_2) = (\mu \zeta_1, \zeta_2) \).

Suppose that \( \varphi(\mathcal{P}, V, \theta) \) satisfies Condition \(5.12\). We construct a Hermitian metric \( h_0' \) for \( \varphi(\mathcal{P}, V, \theta) \) as in the previous subsection. We may assume that \( h_0' \) is \( \text{Gal}(\varphi) \)-invariant. There exists a \( C^\infty \)-function \( f \) on \( X_0' \) determined by \( \text{det}(h_0') = e^f \cdot \varphi^{-1}(h_{\text{det}(E)}) \). We set \( h_{in}' := h_0' e^{-f \cdot \text{rank}(E)} \). Because \( h_{in}' \) is \( \text{Gal}(\varphi) \)-invariant, we obtain a Hermitian metric \( h_{in} \) of \( E \) induced by \( h_{in}' \). Let \( \varphi_*(gX_0') \) denote the Kähler metric of \( X_0 \setminus H_0 \) induced by \( \sum_{k=1}^m d\bar{w}_k d\bar{w}_k \).

\[5.14 \text{ Lemma} \]
\( R(h_{in}), [\theta, \theta]_{h_{in}}, \partial_{h_{in}} \theta \) and \( \overline{\partial} h_{in} \theta \) are bounded with respect to \( \varphi_*(gX_0') \) and \( h_0 \). We also have
\(
\lim_{\epsilon \to 0} \int_{Y(e)} \text{Tr}(\partial h_{in} \theta) = 0, \quad \lim_{\epsilon \to 0} \int_{Y(e)} \text{Tr}(\partial h_{in} \theta) = 0.
\]

\[5.4 \text{ Proof of Proposition} \ 5.6
\]
Let \( X, H \) and \( L \) be as in \(5.1.1\). Let \( (\mathcal{P}, V, \theta) \) be a good filtered Higgs bundle on \( (X, H) \) satisfying Condition \(5.1\). Note that \( (\mathcal{P}, V, \theta) \) is as in \(5.2.2\) around any cross point of \( H \), and \( (\mathcal{P}, V, \theta) \) is as in \(5.3.2\) around any smooth points of \( H \). There exists a Hermitian metric \( h_{in} \) of \( E \) such that (i) \( \text{det}(h_{in}) = h_{\text{det}(E)} \), (ii) the restriction of \( h_{in} \) around any points of \( H \) are as in \(5.2.2\) or \(5.3.2\) and (iii) the restriction of \( h_{in} \) around any points of \( H \) are as in \(5.2.2\) or \(5.3.2\). By the construction, \( h_{in} \) is strongly adapted to \( \mathcal{P}, V \). By Lemma \(5.10\) and Lemma \(5.14\), we obtain that \( R(h_{in}), [\theta, \theta]_{h_{in}}, \partial_{h_{in}} \theta \) and \( \overline{\partial} h_{in} \theta \) are bounded with respect to \( h_{in} \) and \( \omega_\epsilon \). As in the proof of \[4.3 \text{ Proposition} \ 4.18\], we have
\[
\text{Tr}(F(h_{in})^2) = \text{Tr}(R(h_{in})^2) + d(\text{Tr}(\partial h_{in} \theta) + \text{Tr}(\partial h_{in} \theta)).
\]
Then, we obtain \[5.3 \text{ from Lemma} \ 5.11 \text{ and Lemma} \ 5.14\]. Thus, we obtain Proposition \(5.6\).
5.5 Proof of Theorem 5.7

Let $E' \subset E$ be any coherent Higgs $O_{X \setminus H}$-subsheaf. We assume that $E'$ is saturated, i.e., $E/E'$ is torsion-free. Let $(E', \overline{\partial} E', \theta')$ be the induced Higgs sheaf on $X \setminus H$. There exists a discrete subset $Z \subset X \setminus H$ such that $E|_{X \setminus (H \cup Z)}^{'\prime}$ is a subbundle of $E|_{X \setminus (H \cup Z)}$. Let $h'$ denote the metric of $E|_{X \setminus (H \cup Z)}^{'\prime}$ induced by $h_i$. We obtain the Chern connection $\nabla_{h'}$ of $(E', \overline{\partial} E', h')$ and the adjoint of the Higgs field $\theta'|^{\dagger}_{h'}$. Let $R(E', h')$ denote the curvature of $\nabla_{h'}$, and let $F(E', \theta', h')$ denote the curvature of the connection $\nabla_{h'} + \theta' + \theta'|^{\dagger}_{h'}$. Following \cite{57}, we define

$$\deg_{\omega_i}(E', h_i) := \frac{\sqrt{-1}}{2\pi} \int_{X \setminus H} \text{Tr}(\Lambda_{\omega_i} F(E', \theta', h')) \, \text{dvol}_{\omega_i}.$$ 

It is well defined in $\mathbb{R} \cup \{-\infty\}$ by the Chern-Weil formula \cite{57} Lemma 3.2:

$$\deg_{\omega_i}(E', h_i) = \frac{\sqrt{-1}}{2\pi} \int_{X \setminus H} \text{Tr}(\pi_{E'} \Lambda_{\omega_i} F(h_i)) - \frac{1}{2\pi} \int_{X \setminus H} |\overline{\partial} \pi_{E'}|^2_{h_i, \omega_i} - \frac{1}{2\pi} \int_{X \setminus H} |[\theta, \pi_{E'}]|^2_{h_i, \omega_i}.$$ 

Here, $\pi_{E'}$ denotes the orthogonal projection of $E|_{X \setminus (H \cup Z)}$ onto $E|_{X \setminus (H \cup Z)}^{'\prime}$.

Lemma 5.15 If $\deg_{\omega_i}(E', \theta) \neq -\infty$, then $E'$ is extended to a filtered subsheaf $\mathcal{P}^{h'} E'$ of $\mathcal{P}_* \mathcal{V}$ and

$$\deg_{\omega_i}(E', h_i) = \int_X \text{par-c}_1(\mathcal{P}^{h'} E') \omega_X$$

holds. As a result, $(E, \overline{\partial} E, \theta, h_i)$ is analytically stable in the sense of \cite{57}.

Proof If $\deg_{\omega_i}(E', h_i) \neq -\infty$, we obtain $\int |\overline{\partial} \pi_{E'}|^2 < \infty$. As studied in \cite{33, 35} on the basis of \cite{32}, we obtain a coherent $O_X(H)$-submodule $\mathcal{P}^{h'}(E') \subset \mathcal{V}$ as an extension of $E'$. Moreover, as proved in \cite{43} Lemma 4.20, we obtain the equality $\deg_{\omega_i}(E', h_i) = \int_X \text{par-c}_1(\mathcal{P}^{h'} E') \omega_X$.

According to the fundamental theorem of Simpson \cite{57} Theorem 1], there exists a Hermitian-Einstein metric $h_{HE}$ of $(E, \overline{\partial} E, \theta)$ satisfying the conditions (i), (ii) and (iii). By \cite{57} Proposition 3.5 and \cite{57} Lemma 7.4], we obtain

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X \setminus H} \text{Tr}(F(h_{HE})^2) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_{X \setminus H} \text{Tr}(F(h_i)^2).$$

It is equal to $2 \int_X \text{par-ch}_2(\mathcal{P}_* \mathcal{V})$ by Lemma 3.3 and Proposition 5.6. Thus, Theorem 5.7 is proved.

6 Bogomolov-Gieseker inequality

Let $X$ be any dimensional smooth connected projective variety with a simple normal crossing hypersurface $H = \bigcup_{i \in \Lambda} H_i$. Let $L$ be any ample line bundle on $X$.

Theorem 6.1 Let $(\mathcal{P}_* \mathcal{V}, \theta)$ be a $\mu_L$-polystable good filtered Higgs bundle on $(X, H)$. Then, the Bogomolov-Gieseker inequality holds:

$$\int_X \text{par-ch}_2(\mathcal{P}_* \mathcal{V}) c_1(L)^{\dim X - 2} \leq \frac{\int_X \text{par-c}_1(\mathcal{P}_* \mathcal{V})^2 c_1(L)^{\dim X - 2}}{2 \text{rank} \mathcal{V}}.$$ 

Proof By the Mehta-Ramanathan type theorem (Proposition 4.9), it is enough to study the case $\dim X = 2$, which we shall assume in the rest of the proof. We use the notation in \cite{36.3} We take $a \in \mathbb{R}^3$ such that $a_i \not\in \text{Par}(\mathcal{P}_* \mathcal{V}, i)$ for any $i \in \Lambda$. We choose $\eta > 0$ such that $0 < 10 \text{rank}(\mathcal{V}) \eta < \text{gap}(\mathcal{P}_* \mathcal{V}, a)$.

Let $m \in \mathbb{Z}_{>0}$, and set $\epsilon := m^{-1} < \eta$. For $b \in \text{Par}(\mathcal{P}_* \mathcal{V}, a, i)$, we set $b(\epsilon) := \max\{d \in \epsilon \mathbb{Z} \mid d < b\}$. We set

$$c_i := \frac{1}{\text{rank} E} \sum_{b \in \text{Par}(\mathcal{P}_* \mathcal{V}, a, i)} (b - b(\epsilon)) \text{rank}^i \text{Gr}_b^E(\mathcal{P}_a \mathcal{V}).$$

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We have $0 \leq c_i \leq \epsilon$. We set $\psi_{c_i}(b) := b(\epsilon) + c_i$. Then, we have $|\psi_{c_i}(b) - b| < 2\epsilon$, and the following holds:

$$\sum_{b \in \mathcal{P}(\mathcal{P}_r, \mathcal{V}, a, i)} \psi_{c_i}(b) \rank \Gr^f_0 (\mathcal{P}_a \mathcal{V}) = \sum_{b \in \mathcal{P}(\mathcal{P}_r, \mathcal{V}, a, i)} \rank \Gr^f_0 (\mathcal{P}_a \mathcal{V}).$$

Moreover, we have $\psi_{c_i}(b) - c_i \in \epsilon \mathbb{Z}$.

Applying the construction in 3.6.3, we obtain a good filtered Higgs bundle $(\mathcal{P}_r, \mathcal{V}, \theta)$ on $(X, H)$. By the construction, it satisfies Condition 5.1. There exists $m_0$ such that $(\mathcal{P}_r, \mathcal{V}, \theta)$ is $\mu_L$-stable if $m \geq m_0$. Let $(E, \mathcal{U}_E, \theta)$ be the Higgs bundle obtained as the restriction of $(\mathcal{P}_r, \mathcal{V}, \theta)$ to $X \setminus H$. We use the Kähler metric $g_e$ of $X \setminus H$ as in 5.1.1. There exists a Hermitian-Einstein metric $h_{HE}$ of the Higgs bundle $(E, \mathcal{U}_E, \theta)$ as in Theorem 5.7 for the good filtered Higgs bundle $(\mathcal{P}_r, \mathcal{V}, \theta)$. By Proposition 3.1, the equality (3), and the equality \( \sqrt{\frac{-1}{2\pi}} \text{Tr} F(h_{HE}) = \sqrt{\frac{-1}{2\pi}} R(h_{det E}) \), we obtain

$$\int_X \par-ch_2(\mathcal{P}_r, \mathcal{V}) \leq \frac{\int_X \par-c_1(\mathcal{P}_r, \mathcal{V})^2}{2 \rank \mathcal{V}}.$$

By taking the limit as $m \to \infty$, i.e., $\epsilon \to 0$, we obtain the desired inequality.

\[\text{Corollary 6.2}\]

Let $(\mathcal{P}_r, \mathcal{V}, \theta)$ be a $\mu_L$-polystable good filtered Higgs bundle on $(X, H)$. Suppose that

$$\int_X \par-c_1(\mathcal{P}_r, \mathcal{V}) c_1(L)^{\dim X - 1} = 0, \quad \int_X \par-ch_2(\mathcal{P}_r, \mathcal{V}) c_1(L)^{\dim X - 2} = 0.
$$

Then, $\par-c_1(\mathcal{P}_r, \mathcal{V}) = 0$ holds.

Moreover, for any decomposition $(\mathcal{P}_r, \mathcal{V}, \theta) = \bigoplus (\mathcal{P}_r, \mathcal{V}_i, \theta_i)$ into $\mu_L$-stable good filtered Higgs bundles, we have $\par-c_1(\mathcal{P}_r, \mathcal{V}_i) = 0$ and $\int_X \par-ch_2(\mathcal{P}_r, \mathcal{V}_i) c_1(L)^{\dim X - 2} = 0$.

\[\text{Proof}\] On one hand, because of the Hodge index theorem and $\int_X \par-c_1(\mathcal{P}_r, \mathcal{V}) c_1(L)^{\dim X - 1} = 0$, we obtain

$$\int_X \par-c_1(\mathcal{P}_r, \mathcal{V})^2 c_1(L)^{\dim X - 2} \leq 0,$$

and the equality holds if and only if $\par-c_1(\mathcal{P}_r, \mathcal{V}) = 0$. On the other hand, by the Bogomolov-Gieseker inequality and $\int_X \par-ch_2(\mathcal{P}_r, \mathcal{V}) c_1(L)^{\dim X - 2} = 0$, we obtain

$$\int_X \par-c_1(\mathcal{P}_r, \mathcal{V})^2 c_1(L)^{\dim X - 2} \geq 0.$$

Hence, we obtain $\par-c_1(\mathcal{P}_r, \mathcal{V}) = 0$.

Let $(\mathcal{P}_r, \mathcal{V}, \theta) = \bigoplus (\mathcal{P}_r, \mathcal{V}_i, \theta_i)$ be a decomposition into $\mu_L$-stable good filtered Higgs bundles. We have $\int_X \par-c_1(\mathcal{P}_r, \mathcal{V}_i) c_1(L)^{\dim X - 1} = 0$. Hence, by the Hodge index theorem, we obtain

$$\int_X \par-c_1(\mathcal{P}_r, \mathcal{V}_i)^2 c_1(L)^{\dim X - 2} \leq 0.$$

By the Bogomolov-Gieseker type inequality, we obtain

$$\int_X \par-ch_2(\mathcal{P}_r, \mathcal{V}_i) c_1(L)^{\dim X - 2} \leq 0.$$

Because $\sum \int_X \par-ch_2(\mathcal{P}_r, \mathcal{V}_i) c_1(L)^{\dim X - 2} = \int_X \par-ch_2(\mathcal{P}_r, \mathcal{V}) = 0$, we obtain $\int_X \par-ch_2(\mathcal{P}_r, \mathcal{V}_i) c_1(L)^{\dim X - 2} = 0$. Thus, we obtain the claim of the corollary.
7 Existence theorem of pluri-harmonic metrics

7.1 Statement

Let us prove Theorem 2.16. According to Corollary 6.3, it is enough to study the case where \((P, V, \theta)\) is a \(\mu_L\)-stable good filtered Higgs bundle on \((X, H)\) such that

\[
\text{par-c}_1(P, V) = 0, \quad \int_X \text{par-ch}_2(P, V)c_1(L)^{\dim X - 2} = 0.
\]

Let \((E, \overline{\theta}_E, \theta)\) be the Higgs bundle obtained as the restriction \((P, V, \theta)|_{X \setminus H}\). Let \(h_{\det(E)}\) denote the pluri-harmonic metric of \((\det(E), \overline{\theta}_{\det(E)}, \text{Tr} \theta)\) strongly adapted to \(P_s(\det(E))\). For the proof of Theorem 2.16, it is enough to prove the following theorem.

**Theorem 7.1** There exists a unique pluri-harmonic metric \(h\) of the Higgs bundle \((E, \overline{\theta}_E, \theta)\) such that \(P^h E = P_s V\) and \(\det(h) = h_{\det(E)}\).

The proof is given in the rest of this section.

7.2 Surface case

Let us study the case \(\dim X = 2\). The following argument is essentially the same as the proof of [45, Theorem 5.5]. Let \((P, V, \theta)\) be as in 7.1. We use the notation in the proof of Theorem 6.1.

For large \(m \in \mathbb{Z}_{>0}\), we set \(\epsilon := m^{-1}\). We have the perturbations \((P_s^c V, \theta)\). We use the Kähler metrics \(g_s\) of \(X \setminus H\) as in 5.5. Let \(\tilde{E}\) denote the Hermitian-Einstein metrics \(h_{\text{HE}}\) of \((E, \overline{\theta}_E, \theta)\) adapted to \((P_s^c V, \theta)\) such that \(\det(h_{\text{HE}}) = h_{\det(E)}\).

**Proposition 7.2** For any sequence \(m_i \to \infty\), we set \(\epsilon_i := m_i^{-1}\). Then, the sequence \(h_{\text{HE}}^{(\epsilon_i)}\) is convergent almost everywhere on \(X \setminus H\), and the limit \(h\) is a pluri-harmonic metric of the Higgs bundle \((E, \overline{\theta}_E, \theta)\) such that \(P^h E = P_s V\) and \(\det(h) = h_{\det E}\).

7.2.1 Family of ample hypersurfaces

Take a sufficiently large integer \(M\). We set \(M := H^0(X, L^\otimes M) \setminus \{0\}\). It is equipped with a natural \(C^*\)-action. Let \(P_i\) denote the projection of \(X \times 3_M\) onto the \(i\)-th component. There exists the universal section \(s\) of \(P_1^1(L^\otimes M)\).

Let \(\mathcal{X}_M\) denote the scheme obtained as \(s^{-1}(0)\). Let \(P_1 : \mathcal{X}_M \to X\) and \(P_2 : \mathcal{X}_M \to 3_M\) denote the morphism induced by \(P_i\). For each \(s \in 3_M\), let \(X_s\) denote the fiber product of \(P_2\) and the inclusion \(\{s\} \to 3_M\).

There exists the \(C^*\)-invariant maximal Zariski open subset \(3_M^o \subseteq 3_M\) such that (i) the induced morphism \(P_2^o : \mathcal{X}_M^o \times 3_M^o \to 3_M^o\) is smooth, (ii) \(X_s \cup H\) is normal crossing for any \(s \in 3_M^o\). Let \(P_1^o\) denote the restriction of \(P_1\) to \(\mathcal{X}_M^o\). For any \(Q \in \mathcal{X}_M^o\), we obtain the subspace \(T_{P_1^o(Q)}X_{P_1^o(Q)} \subseteq T_{P_1^o(Q)}X\) of codimension 1. It determines a point in \(\mathbb{P}(T_{P_1^o(Q)}X)\). Hence, we obtain the natural morphism \(\mathcal{F}_1^o : \mathcal{X}_M^o \to \mathbb{P}(T^*X)\). If \(M\) is sufficiently large, \(\mathcal{F}_1^o\) and \(\mathcal{F}_1^o\) are surjective.

By the Mehta-Ramanathan type theorem, there exists a \(C^*\)-invariant Zariski open subset \(3_M^\Delta\) of \(3_M^o\) such that the following holds.

- For each \(s \in 3_M^\Delta\), \(P_s V, \theta)\) is stable.

We set \(\mathcal{X}_M^\Delta := \mathcal{X}_M^o \times 3_M^\Delta\). Note that \(W_M := X \setminus P_1^o(\mathcal{X}_M^\Delta)\) is a finite set. For each \(P \in X \setminus (H \cup W_M)\), the intersection \(P_1^o(\mathcal{X}_M^\Delta) \cap \mathbb{P}(T^*X)\) in \(\mathbb{P}(T^*X)\) is Zariski dense in \(\mathbb{P}(T^*X)\).

We set \(H_s := X_s \cap H\). Let \((E_s, \overline{\theta}_E, \theta_s)\) denote the Higgs bundle on \(X_s \setminus H_s\). For each \(s \in 3_M^\Delta\), there exists a pluri-harmonic metric \(h_s\) of \((E_s, \overline{\theta}_E, \theta_s)\) such that (i) \(h_s\) is adapted to \(P_s V|_{X_s}\), (ii) \(\det(h_s) = h_{\det(E)|X_s \setminus H_s}\).

Let \(P_1^\Delta : \mathcal{X}_M^\Delta \to X\) be the induced map. Let \(3_M^\Delta := (P_1^\Delta)^{-1}(H)\). We set \((E^\Delta, \overline{\theta}_E^\Delta, \theta^\Delta) := (P_1^\Delta)^{-1}(E, \overline{\theta}_E, \theta)\) on \(\mathcal{X}_M^\Delta \setminus 3_M^\Delta\). We also obtain Hermitian metrics \(h^{\Delta(\epsilon_i)} := (P_1^\Delta)^{-1}(h_{\text{HE}}^{(\epsilon_i)})\). By Proposition 4.9, the family of pluri harmonic metrics \(h_s\) \((s \in 3_M^\Delta)\) induces a continuous Hermitian metric of \(E^\Delta\).
7.2.2 Local holomorphic coordinate systems

Let $P \in X \setminus W_M$. We take $s_\infty \in \mathcal{Z}_M^\Delta$ such that $P \not\in X_{s_\infty}$. The following is clear because $\mathcal{P}_1(X^\Delta_M) \cap \mathbb{P}(T_pX)$ is dense in $\mathbb{P}(T_pX)$.

**Lemma 7.3** There exist $s_i \in \mathcal{Z}_M^\Delta$ (i = 1, 2) and $\epsilon > 0$ such that the following holds.

- $P \in X_{s_i}$ (i = 1, 2).
- $X_{s_1}$ and $X_{s_2}$ are transversal at $P$.
- $\{s_1 + as_\infty | |a| < \epsilon\}$, $\{s_2 + as_\infty | |a| < \epsilon\}$, $\{s_1 + s_2 + as_\infty | |a| < \epsilon\}$ and $\{s_1 + \sqrt{1}s_2 + as_\infty | |a| < \epsilon\}$ are contained in $\mathcal{Z}_M^\Delta$.

We set $x_i := s_i/s_\infty$ (i = 1, 2). There exists a neighbourhood $U_P$ of $P$ in $X \setminus H$ such that $(x_1, x_2)$ is a holomorphic coordinate system on $U_P$. Note that $\{\sum b_i x_i = c\} \cap U_P$ is equal to $U_P \cap X_{b_1s_1 + b_2s_2 - cs_\infty}$.

7.2.3 Proof of Proposition 7.2

Take a sequence $m_i \to \infty$ in $\mathbb{Z}$. We set $\epsilon_i := m_i^{-1}$. By Proposition 4.1, we obtain the following convergence:

$$\lim_{i \to \infty} \int_{X \setminus H} |F(h_{\epsilon_i}^{(1)})|_{h_{\epsilon_i}^{(1)}}^2 \omega_{m_i} = 0.$$ 

Let $\omega_{m_i,s}$ denote the Kähler form of $X_s \setminus H_s$ induced by $\omega_{m_i}$. Let $h_{\epsilon_i}^{(1)}$ denote the restriction of $h_{\epsilon_i}$ to $X_s \setminus H_s$. By Fubini’s theorem, there exists a $\mathbb{C}^*$-invariant subset $\mathcal{Z}_M^\Delta \subset \mathcal{Z}_M^\Delta$ with the following property.

- $\lim_{i \to \infty} \int_{X \setminus H} |F(h_{\epsilon_i}^{(1)})|_{h_{\epsilon_i}^{(1)}}^2 \omega_{m_i,s} = 0$ holds for any $s \in \mathcal{Z}_M^\Delta$.

- The Lebesgue measure of $\mathcal{Z}_M^\Delta \setminus \mathcal{Z}_M^\Delta$ is 0.

By Proposition 4.3, for any $s \in \mathcal{Z}_M^\Delta$, the sequence $h_{\epsilon_i}^{(1)}$ is weakly convergent to $h_s$ in $L^2_1$ locally on $X_s \setminus H_s$. In particular, $h_{\epsilon_i}^{(1)}$ is convergent to $h_s$ almost everywhere on $X_s \setminus H_s$.

**Lemma 7.4** There exists a $\mathbb{C}^*$-invariant subset $\mathcal{X}_M^\Delta \subset \mathcal{X}_M^\Delta \times \mathcal{Z}_M^\Delta$ such that the following holds:

- The measure of $\mathcal{X}_M^\Delta \setminus \mathcal{X}_M^\Delta$ is 0.
- For each $Q \in \mathcal{X}_M^\Delta$, the sequence $h_{\epsilon_i}^{(1)}(Q)$ is convergent to $h_{\epsilon_i}^{\Delta}(Q)$.

**Proof** We define $\mathcal{X}_M^\Delta$ as the set of points $Q \in \mathcal{X}_M^\Delta$ such that the sequence $h_{\epsilon_i}^{\Delta}(Q)$ is convergent to $h_{\epsilon_i}^{\Delta}(Q)$. It is a measurable set. Then, we obtain the claim of the lemma by Fubini’s theorem.

We set $\mathcal{X}_i := \rho_{\epsilon_i}^{\Delta}(\mathcal{X}_M^\Delta)$. Then, we obtain that the sequence $h_{\epsilon_i}^{(1)}(Q_{\epsilon_i})$ is convergent to a Hermitian metric $h_\infty$ of $E|_{X_1}$.  

**Lemma 7.5** $h_\infty$ induces a Hermitian metric of $E|_{X \setminus (H_s \cup W_M)}$ of $C^1$-class. The $C^1$-Hermitian metric is also denoted by $h_\infty$.

**Proof** Let $P$ be any point of $X \setminus W_M$. Let $(U_P, x_1, x_2)$ be a holomorphic coordinate neighbourhood as in (7.2.2). By using Proposition 4.3, we define the continuous Hermitian metric $h_{P}^{(1)}$ of $E|_{U_P}$ by the condition that $h_{P}^{(1)}(x_i = a)$ is equal to the restriction of $h_{s_1+a}$ of $X_{s_1+a}$. By the construction of $h_\infty$, we have $h_\infty|_{U_P \cap X_1} = h_{P}^{(1)}|_{U_P \cap X_1}$. Hence, we obtain that $h_\infty|_{U_P \cap X_1}$ induces a continuous Hermitian metric $h_{P,\infty}$ of $E|_{U_P}$, $h_{P}^{(1)} = h_{P,\infty} = h_{P}^{(2)}$ hold. Moreover, by Proposition 4.3, any derivative of $h_{P}^{(1)}$ with respect to $\partial_{x_i}$ and $\partial_{y_j}$ (i ≠ j) are continuous. Hence, we obtain that $h_{P,\infty}$ is $C^1$. Thus, we obtain the claim of the lemma.

The curvature $R(h_\infty)$ of the Chern connection is defined as a current. We also obtain the adjoint of Higgs field $\nabla{h_\infty}$ as a $C^1$-section of $\text{End}(E) \otimes \Omega^{0,1}$. We obtain $F(h_\infty)^{(1,1)} := R(h_\infty) + [\theta, \nabla{h_\infty}]$ as a current.
Lemma 7.6 \( F(h_\infty)^{(1,1)} = 0 \) on \( X \setminus (H \cup W_M) \).

Proof Let \( P \in X \setminus (H \cup W_M) \). Let \( (U_P, x_1, x_2) \) be a holomorphic coordinate neighbourhood as in Lemma 7.2. We have the expression
\[
F(h_\infty)^{(1,1)} = F(h_\infty)_{11} dx_1 d\overline{x}_1 + F(h_\infty)_{12} dx_1 d\overline{x}_2 + F(h_\infty)_{21} dx_2 d\overline{x}_1 + F(h_\infty)_{22} dx_2 d\overline{x}_2.
\]
Because \( h_\infty|_{x_i=a} \) is equal to \( h_{s_i+\alpha x_\infty} \), we obtain \( F(h_\infty)_{ii} = 0 \) for \( i = 1, 2 \).

By considering the holomorphic coordinate system \((w_1, w_2) = (x_1 + x_2, x_1 - x_2)\) and the coefficient of \( dw_1 d\overline{w}_1 \) in \( F(h_\infty)^{(1,1)} \), we obtain \( F(h_\infty)_{12} + F(h_\infty)_{21} = 0 \). By considering the holomorphic coordinate system \((z_1, z_2) = (x_1 + \sqrt{-1}x_2, x_1 - \sqrt{-1}x_2)\) and the coefficient of \( dz_1 d\overline{z}_1 \) in \( F(h_\infty)^{(1,1)} \), we obtain \( F(h_\infty)_{12} - F(h_\infty)_{21} = 0 \). Therefore, we obtain that \( F(h_\infty)_{ij} = 0 \).

Lemma 7.7 We have \( \Lambda F(h_\infty) = 0 \) on \( X \setminus (H \cup W_M) \). As a result, \( h_\infty \) is \( C^\infty \) on \( X \setminus (H \cup W_M) \).

Proof The first claim immediately follows from Lemma 7.6. We obtain the second claim by the elliptic regularity and a standard bootstrapping argument.

Lemma 7.8 \( \partial_{h_\infty} \theta \) is convergent to \( \partial h_\infty \theta \) everywhere.

Proof Let \( b^*(\epsilon_i) \) be the automorphism of \( E|_{X\setminus W_M} \) which is self-adjoint with respect to \( h_\infty \) and \( h_{HE}^{(\epsilon_i)} \) and determined by \( h_{HE}^{(\epsilon_i)} = h_\infty b^*(\epsilon_i) \). Because the sequence \( h_{HE}^{(\epsilon_i)} \) is weakly convergent locally on \( X_\epsilon \setminus H \) in \( L^2 \) for \( s \in X_M \), we obtain that \( \partial_{h_{HE}^{(\epsilon_i)}} \theta \) is convergent to 0 almost everywhere. Hence, \( \partial_{h_\infty} b^*(\epsilon_i) \) is convergent to 0 almost everywhere. Because \( \partial_{h_{HE}^{(\epsilon_i)}} \theta = \partial_{h_\infty} + (b^*(\epsilon_i))^{-1} \partial_{h_\infty} b^*(\epsilon_i) \), we obtain the claim of the lemma.

Lemma 7.9 \( \partial h_\infty \theta = 0 \).

Proof Note that
\[
0 \leq \int_{X \setminus H} |\partial_{h_{HE}^{(\epsilon_i)}} \theta|_{h_{HE}^{(\epsilon_i)}}^2 \omega_{m_i} \leq \int_{X \setminus H} |F(h_{HE}^{(\epsilon_i)})|_{h_{HE}^{(\epsilon_i)}}^2 \omega_{m_i} = -8\pi^2 \int_X \text{par-ch}_2(\mathcal{P}_s^{(\epsilon_i)}) \mathcal{V}.
\]
We also have \( \lim_{i \to \infty} \int_X \text{par-ch}_2(\mathcal{P}_s^{(\epsilon_i)}) \mathcal{V} = 0 \). By Lemma 7.8 we have the following convergence almost everywhere on \( X \setminus H \):
\[
\lim_{i \to \infty} |\partial_{h_{HE}^{(\epsilon_i)}} \theta|_{h_{HE}^{(\epsilon_i)}}^2 \omega_{m_i} = |\partial h_\infty \theta|_{h_\infty \omega_X}^2.
\]
Therefore, we obtain \( \int |\partial h_\infty \theta|^2_{h_\infty \omega_X} = 0 \) by Fatou’s lemma.

Thus, we obtain that \( h_\infty \) is a pluri-harmonic metric of \((E, \overline{\partial}_E, \theta)|_{X \setminus (H \cup W_M)}\).

Lemma 7.10 \( h_\infty \) induces a \( C^\infty \)-metric of \( E \) on \( X \setminus H \), and hence it is a pluri-harmonic metric of \((E, \overline{\partial}_E, \theta)\).

Proof Take \( P \in W_M \setminus H \). We take a holomorphic coordinate neighbourhood \((X_P, z_1, z_2)\) around \( P \) in \( X \setminus H \). We may assume that \( X_P \) is bi-holomorphic with \( \{(z_1, z_2) ||z_i| < 2\} \) by the coordinate system, and that \( P \) corresponds to \((0,0)\). Let \( C_{i,a} = \{z_i = a\} \cap X_P \). Let \( g_P \) denote the metric \( \sum d z_i dz_i \). We have the expression \( \theta = f_1 dz_1 + f_2 dz_2 \). According to a variant of Simpson’s main (for example, see Proposition 2.10), there exist \( \epsilon > 0 \) such that \( |f_1|_{C_{i,a}} < A \) and \( |f_2|_{C_{i,a}} < A \) for any \( a \in \mathbb{C} \{0 < |a| < 1\} \). Hence, we obtain that \( |f_1| < A \) and \( |f_2| < A \) on \( X \setminus \{P\} \). We obtain that \( |R(h)|_{X_P \setminus \{P\}}|_{g_P, h} < BA^2 \), for a constant \( B > 0 \) depending only on \( \text{rank}(E) \).

We take a \( C^\infty \)-metric \( h_{0,P} \) of \( E|_{X_P} \). Let \( b_P \) be the automorphism of \( E|_{X_P} \) which is self-adjoint with respect to both \( h_{\infty P} \) and \( h_{0,P} \) and determined by \( h_{\infty P} = h_{0,P} b_P \). By the norm estimate for tame harmonic bundles \( [44] \), we obtain that \( b_P \) and \( b_P^{-1} \) are bounded with respect to \( h_{0,P} \).
By [57, Lemma 3.1], we have the following equality on \( C_{2,a} \) (\( a \neq 0 \)):

\[
-2 \partial z_i \partial z_j \text{Tr}(b_p|_{C_{2,i}}) = \text{Tr}(-b_p F(h_0|_{C_{2,a}})) - \frac{1}{2} (\partial E_{i,j} b_p + [f_1, b_p]) \cdot b_p^{1/2} |_{h_0|_{C_{2,a}}}.
\]

Hence, there exists a positive constant \( A_1 \) such that the following holds for any \( 0 < |a| < 1 \):

\[
\int_{C_{2,a}} |\partial E_{i,j} b_p|_{h_0,p}^2 |dz_1 dz_2| < A_1.
\]

Therefore, we obtain that \( |\partial E_{i,j} b_p|_{h_0,p} \) is \( L^2 \) on \( X_P \). Similarly, we obtain that \( |\partial E_{i,j} b_p|_{h_0,p} \) is \( L^2 \) on \( X_P \). Hence, we obtain that \( b_p^{-1} \partial h_{0,p} b_p \) is \( L^2 \). Because \( \partial (b_p^{-1} \partial h_{0,p} b_p) = R(h_\infty|X_P) - R(h_0|X_P) \), we obtain that \( \partial (b_p^{-1} \partial h_{0,p} b_p) \) is bounded on \( X_P \setminus \{P\} \). It is easy to observe that \( \partial (b_p^{-1} \partial h_{0,p} b_p) = R(h_\infty|X_P) - R(h_0|X_P) \) holds \( X_P \) as distributions. By the elliptic regularity, we obtain that \( b_p^{-1} \partial h_{0,p} b_p \) is \( L^2 \) for any \( p > 1 \). By using

\[
\overline{\partial h_{0,p} b_p - \partial (b_p^{-1} \partial h_{0,p} b_p)} = b_p (R(h_\infty|X_P) - R(h_0|X_P))
\]

and the elliptic regularity, we obtain that \( b_p \) is \( L^2 \) for any \( p > 1 \). Then, by using [15] and the standard bootstrapping argument, we obtain that \( b_p \) is \( C^\infty \) on \( X_P \).

Because \( (E, \mathcal{T}_E, \theta, h_{\infty}) \) is a good wild harmonic bundle on \((X, H)\), we obtain that a good filtered Higgs bundle \((\mathcal{P}^h\mathcal{E}, \mathcal{E}, \theta, h_{\infty})\) on \((X, H)\). We put \( H^{[2]} = \bigcup_{i \neq j} (H_i \cap H_j) \). For any \( P \in H \setminus (W_M \cup H^{[2]}) \), there exists \( s \in \mathcal{T}_M \) such that \( P \in X_s \). By the construction, \( h_\infty|X_s \setminus H_s = h_s \). Hence, we obtain \( \mathcal{P}^h\mathcal{E}|X_s = \mathcal{P}_s \mathcal{E}|X_s \).

Let \( Y := (H \cap W_M) \setminus H^{[2]} \), which is a finite subset of \( H \). We obtain that \( \mathcal{P}^h\mathcal{E}|Y = \mathcal{P}_s \mathcal{E}|Y \). By Hartogs theorem, we obtain that \( \mathcal{P}^h\mathcal{E}|Y \simeq \mathcal{P}_s \mathcal{E}|Y \). Thus, the proof of Proposition 7.2 is completed.

### 7.3 Higher dimensional case

Let us prove Theorem 7.1 in the case \( \dim X \geq 3 \) by an induction on \( \dim X \). Take a sufficiently large integer \( M \). We set \( \mathcal{X}_M := H^0(X, L_M^a) \setminus \{0\} \), and \( \mathcal{X}_M \subset X \times \mathcal{X}_M \) be defined as \( s^{-1}(0) \) as in [7, 27, 28]. For any \( s \in \mathcal{X}_M \), set \( X_s := s^{-1}(0) \). Let \( \mathbb{P}(T^* X) \) denote the projectivization of the cotangent bundle of \( X \). If \( M \) is sufficiently large, there exists a Zariski open subset \( \mathcal{X}'_M \subset \mathcal{X}_M \) such that the following holds.

- \( \mathcal{P}^2 : \mathcal{X}'_M := \mathcal{X}_M \times \mathcal{X}_M \to \mathcal{X}'_M \) is smooth.
- \( \mathcal{X}'_M \cup (H \times \mathcal{X}_M) \) is simply normal crossing. Moreover, the intersections of any tuple of irreducible components are smooth over \( \mathcal{X}'_M \).
- The induced map \( p_1 : \mathcal{X}'_M \to X \) is surjective. Moreover, the induced morphism \( \overline{p}_1 : \mathcal{X}'_M \to \mathbb{P}(T^* X) \) is surjective.

Let \( p_{i,j} \) denote the projection of \( X \times \mathcal{X}_M^2 \) onto the product of the \( i \)-th component and the \( j \)-th component. For \( j = 2,3 \), let \( (\mathcal{X}'_M^{(j)}) \) denote the pull back of and \( \mathcal{X}'_M^{(j)} \) by \( p_{1,j} \). There exists a Zariski open subset \( \mathcal{U}_M \subset S^2_M \times S^3_M \) such that the following holds.

- Let \( (\mathcal{X}'_M^{(j)}) \cup (H \times \mathcal{U}_M) \) denote the fiber product of \( (\mathcal{X}'_M^{(j)}) \) and \( \mathcal{U}_M \) over \( S^2_M \times S^3_M \). Then, \( (\mathcal{X}'_M^{(j)}) \cup (H \times \mathcal{U}_M) \) is simply normal crossing. Moreover, the intersection of any tuple of irreducible components are smooth over \( \mathcal{U}_M \).

By the Mehta-Ramanathan type theorem (Proposition 6.9), there exists a Zariski open subset \( \mathcal{U}_M^\triangle \subset \mathcal{U}_M \) such that the following holds.

- For \( s = (s_1, s_2) \in \mathcal{U}_M^\triangle \), we set \( X_s := X_{s_1} \cap X_{s_2} \). Then, the restriction \( \mathcal{P}_s \mathcal{E} \times H_{X_s} \) is a \( \mu_L \)-stable good filtered Higgs bundle on \((X_s, H \cap X_s)\).

Hence, Zariski open subset \( S^\triangle_M \subset S^3_M \) such that the following holds.

- For any \( s \in S^\triangle_M \), \( \mathcal{P}_s \mathcal{E} \times H_{X_s} \) is a \( \mu_L \)-stable good filtered Higgs bundle on \((X_s, H \cap X_s)\).
For any \( s_1, s_2 \in \mathfrak{g}^\Delta_M \), there exists a Zariski open subset \( \mathfrak{g}(s_1, s_2) \subset \mathfrak{g}^\Delta_M \) such that \((P_sV, \theta)_{|X_{s_1} \cap X_{s_2}} (i = 1, 2)\) are \( \mu_L \)-stable for any \( s_3 \in \mathfrak{g}(s_1, s_2) \).

We set \( \mathfrak{g}^\Delta_M := \mathfrak{g}_M \times \mathfrak{g}_M^\Delta \). Let \( P_2^\Delta : \mathfrak{g}^\Delta_M \to X \) denote the naturally induced morphism. Then, \( W_M := X \setminus P_2^\Delta(\mathfrak{g}^\Delta_M) \) is a finite subset.

For any \( P \in X \setminus (H \cup W_M) \), there exists \( s \in \mathfrak{g}^\Delta_M \) such that \( P \subset X_s \). Then, \((P_sV, \theta_s) := (P_sV, \theta)_{|X_s} \) is \( \mu_L \)-stable, and the following holds:

\[
\int_{X_s} \text{par-ch}_1(P_sV_s)c_1(L_{|X_s}) \dim X_s - 1 = 0, \quad \int_{X_s} \text{par-ch}_2(P_sV_s)c_1(L_{|X_s}) \dim X_s - 2 = 0.
\]

There exists a pluri-harmonic metric \( h_s \) of \((E_s, \overline{\partial}_{E_s}, \theta_s) := (E, \overline{\partial}_{E}, \theta)_{|X_s \setminus H} \) adapted to \( P_sV_s \) such that \( \det(h_s) = h_{\det(E_{|X_s \setminus H})} \). Take another \( s' \in \mathfrak{g}^\Delta_M \) such that \( P \subset X_{s'} \). There exists a pluri-harmonic metric \( h_{s'} \) of \((E_{s'}, \overline{\partial}_{E_{s'}}, \theta_{s'}) \) adapted to \( P_sV_{s'} \) such that \( \det(h_{s'}) = h_{\det(E_{|X_{s'} \setminus H})} \).

**Lemma 7.11** \( h_{s|P} = h_{s'|P} \).

**Proof** Suppose that \( X_s \cup X_{s'} \cup H \) is simply normal crossing. We set \( X_{s,s'} := X_s \cap X_{s'} \). It is smooth and connected. We obtain a good filtered Higgs bundle \((P_sV, \theta)_{|X_{s,s'}}, \text{ and } h_{s|X_{s,s'}}, \theta_{X_{s,s'}} \) are adapted to \( P_sV_{|X_{s,s'}} \). Let \( b_{s,s'} \) be the automorphism of \( E_{X_{s,s'}} \), which is self-adjoint with respect to both \( h_{s|X_{s,s'}} \) and \( h_{s'|X_{s,s'}} \), and determined by \( h_{s'|X_{s,s'}} = h_{s|X_{s,s'}} \cdot b_{s,s'} \). There exists a decomposition

\[
(P_sV, \theta)_{|X_{s,s'}} = \bigoplus (P_sV_i, \theta_i),
\]

which is orthogonal with respect to both \( h_{s|X_{s,s'}} \) and \( h_{s'|X_{s,s'}} \), and \( b_{s,s'} = \sum a_i \text{id}_{V_i} \) for some positive constants \( a_i \).

There exists \( s_1 \in \mathfrak{g}(s,s') \). Then, \((P_sV, \theta)_{|X_{s_1}}, \text{ and } (P_sV, \theta)_{|X_{s_1'}} \) are \( \mu_L \)-stable. Therefore, we have

\[
h_{s|X_{s,s'}} = h_{s_1|X_{s_1,s}}, \quad \text{and} \quad h_{s'|X_{s,s'}} = h_{s_1'|X_{s_1,s'}}.
\]

We obtain that \( h_{s|X_{s_1} \cap X_{s_1'}} = h_{s'|X_{s_1} \cap X_{s_1'}} \). It implies that \( a_i = 1 \), and hence \( h_{s|P} = h_{s|P} \).

In general, there exists \( s_2 \in \mathfrak{g}^\Delta_M \) such that (i) \( P \in X_{s_2} \), (ii) \( X_s \cup X_{s_2} \cup H \) and \( X_{s'} \cup X_{s_2} \cup H \) are simply normal crossing. By the above consideration, we obtain \( h_{s|P} = h_{s_2|P} = h_{s'|P} \).

Therefore, we obtain Hermitian metrics \( h_P \) of \( E_P \) \((P \in X \setminus (H \cup W_M)) \). By using the argument in Lemma 7.11, we can prove that they induce a Hermitian metric \( h \) of \( E_{|X \setminus (H \cup W_M)} \). We obtain \( F(h) \) as a current. Because \( h_{|X_s} \) \((s \in \mathfrak{g}^\Delta_M) \) are pluri-harmonic metrics of \((E, \overline{\partial}_{E}, \theta)_{|X_s \setminus H} \), we obtain that \( F(h) = 0 \). It also implies that \( h \) is a \( C^\infty \) on \( X \setminus (H \cup W_M) \). By using the argument in the proof of Lemma 7.10 we obtain that \( h \) induces a pluri-harmonic metric of \((E, \overline{\partial}_{E}, \theta) \) on \( X \setminus H \). Then, as in the proof of Proposition 7.4 we can conclude that \( P^\Delta_0(E) = P_sV \). Thus, we obtain Theorem 7.1.

### 8 Homogeneity with respect to group actions

#### 8.1 Preliminary

**8.1.1 Homogeneous harmonic bundles**

Let \( Y \) be a complex manifold. Let \( K \) be a compact Lie group. Let \( \rho : K \times Y \to Y \) be a \( K \)-action on \( Y \) such that \( \rho_k : Y \to Y \) is holomorphic for any \( k \in K \). Let \( \kappa : K \to S^1 \) be a homomorphism of Lie groups.

Let \((E, \overline{\partial}_E, \theta, h) \) be a harmonic bundle on \( Y \). It is called \((K, \rho, \kappa)\)-homogeneous if \((E, \overline{\partial}_E, \theta) \) is \( K \)-equivariant and \( k^* \theta = \kappa(k) \theta \).

**Remark 8.1** According to [60], harmonic bundles are equivalent to polarized variation of pure twistor structure of weight \( w \), for any given integer \( w \). As studied in [38] [3], by choosing a vector \( v \) in the Lie algebra of \( K \) such that \( \text{de}(\psi) \neq 0 \), we obtain the integrability of the variation of pure twistor structure from the homogeneity of harmonic bundles.
8.1.2 Homogeneous filtered Higgs sheaves and the stability condition with respect to the action

Let $X$ be a connected complex projective manifold with a simple normal crossing hypersurface $H$. Let $G$ be a complex reductive algebraic group. Let $\rho : G \times Y \rightarrow Y$ be an algebraic $G$-action on $Y$ which preserves $H$. Let $\kappa : G \rightarrow \mathbb{C}^*$ be a homomorphism of complex algebraic groups.

Let $(P, V, \theta)$ be a filtered Higgs sheaf on $(Y, H)$. It is called $(G, \rho, \kappa)$-homogeneous if $P, V$ is $G$-equivariant and $g' \theta = \kappa(g) \theta$ for any $g \in G$.

Let $L$ be a $G$-equivariant ample line bundle on $X$. A $(G, \rho, \kappa)$-homogeneous filtered Higgs sheaf $(P, V, \theta)$ on $(X, H)$ is called $\mu_L$-semistable (resp. $\mu_L$-semistable) with respect to the $G$-action if the following holds.

- Let $V'$ be a $G$-invariant saturated Higgs subsheaf of $V$ such that $0 < \operatorname{rank} V' < \operatorname{rank} V$. Then, $\mu_L(P, V') < \mu_L(P, V)$ (resp. $\mu_L(P, V') \leq \mu_L(P, V)$) holds.

A $(G, \rho, \kappa)$-homogeneous filtered Higgs sheaf $(P, V, \theta)$ on $(X, H)$ is called $\mu_L$-polystable with respect to the $G$-action if it is $\mu_L$-semistable with respect to the $G$-action and isomorphic to a direct sum of $(G, \rho, \kappa)$-homogeneous filtered sheaves $\bigoplus (P, V, \theta_i)$, where each $(P, V, \theta_i)$ is $\mu_L$-stable with respect to the $G$-action.

**Lemma 8.2** $(P, V, \theta)$ is $\mu_L$-semistable if and only if $(P, V, \theta)$ is $\mu_L$-semistable with respect to the $G$-action.

**Proof** The “only if” part is clear. Let us prove that the “if” part. Let $V_0 \subset V$ be the subsheaf as in Proposition 3.3. Because $g^*V_0$ also has the same property in Proposition 3.3, we obtain that $V_0$ is $G$-invariant. Then, the claim of the proposition is clear.

The following lemma is clear.

**Lemma 8.3** If $(P, V, \theta)$ is $\mu_L$-stable, then $(P, V, \theta)$ is $\mu_L$-stable with respect to the $G$-action.

**Lemma 8.4** If $(P, V, \theta)$ is $\mu_L$-stable with respect to the $G$-action, then $(P, V, \theta)$ is $\mu_L$-polystable. Moreover, there exists a $\mu_L$-stable good filtered Higgs bundle $(P, V_0, \theta_0)$ and a finite dimensional vector space $U$ with an isomorphism $(P, V, \theta) \simeq (P, V_0, \theta_0) \otimes U$.

**Proof** According to Lemma 8.2 $(P, V, \theta)$ is $\mu_L$-semistable. Let $V_1$ be the socle of $(P, V, \theta)$ as in Proposition 3.6. Because $g^*V_1$ also has the same property, $V_1$ is $G$-invariant. Moreover, $\mu_L(P, V_1) = \mu_L(P, V)$ holds. Hence, we obtain $V_1 = V$. According to Proposition 3.6 $(P, V, \theta)$ is $\mu_L$-polystable. Moreover, the canonical decomposition $(P, V, \theta)$ in Lemma 2.5 is preserved by the $G$-action. Hence, we obtain the claim of the lemma.

**Remark 8.5** In general, even if $(P, V, \theta)$ is $\mu_L$-stable with respect to the $G$-action, $(P, V, \theta)$ is not necessarily $\mu_L$-stable.

8.1.3 Actions of a complex reductive group and its compact real form

Let $X$ be a complex projective manifold equipped with an algebraic action of a complex reductive group $G$. Let $L$ be a $G$-equivariant ample line bundle on $X$. Let $K$ be a compact real form of $G$.

Let $(E, \mathcal{F}_E)$ be a $G$-equivariant holomorphic vector bundle on $X$. Then, as the restriction, we may naturally regard $(E, \mathcal{F}_E)$ as a $K$-equivariant holomorphic vector bundle on $X$.

**Lemma 8.6** The above procedure induces an equivalence between $G$-equivariant holomorphic vector bundles and $K$-equivariant holomorphic vector bundles on $X$.

**Proof** Let $(E, \mathcal{F}_E)$ be a $K$-equivariant holomorphic vector bundle on $X$. There exists $m_0 > 0$ such that $E \otimes L^\otimes m_0$ is globally generating. We set $G_0 := H^0(X, E \otimes L^\otimes m_0) \otimes (L^\otimes m_0)^{-1}$. There exists a naturally induced epimorphism of $\mathcal{O}_X$-modules $G_0 \rightarrow E$. Let $\mathcal{K}$ denote the kernel. There exists $m_1 > 0$ such that $\mathcal{K} \otimes L^\otimes m_1$ is globally generating. We set $G_1 := H^0(X, \mathcal{K} \otimes L^\otimes m_1) \otimes (L^\otimes m_1)^{-1}$. There exists a naturally induced epimorphism $G_1 \rightarrow \mathcal{K}$. Thus, we obtain a resolution $G_1 \rightarrow G_0$ of $E$. Because $E$ is $K$-equivariant, $H^0(X, E \otimes L^\otimes m_0)$ is naturally a $K$-representation, $G_0$ is a $K$-equivariant holomorphic vector bundle on $X$, and
\[ G_0 \rightarrow E \] is \( K \)-equivariant. Hence, \( K \) is a \( K \)-equivariant holomorphic vector bundle. Similarly \( H^0(X, K \otimes L^m) \) is a \( K \)-representation, and \( G_1 \) is \( K \)-equivariant holomorphic vector bundle, and \( G_1 \rightarrow K_2 \) is \( K \)-equivariant.

The \( K \)-representations on \( H^0(X, E \otimes L^m) \) and \( H^0(X, K \otimes L^m) \) naturally induce \( G \)-representations on \( H^0(X, E \otimes L^m) \) and \( H^0(X, K \otimes L^m) \). Hence, \( G_i \) are naturally algebraic \( G \)-equivariant vector bundles on \( X \). Moreover, the morphism \( G_1 \rightarrow G_0 \) is \( G \)-equivariant and algebraic. Hence, \( E \) is a \( G \)-equivariant algebraic vector bundle on \( X \).

### 8.2 An equivalence

#### 8.2.1 Good filtered Higgs bundles associated to homogeneous good wild Higgs bundles

Let \( X \) be a connected complex projective manifold with a simple normal crossing hypersurface \( H \). Let \( G \) be a complex reductive group acting on \( (X, H) \). Let \( K \) be a compact real form of \( G \). The actions of \( G \) and \( K \) on \( X \) are denoted by \( \rho \). Let \( \kappa : G \rightarrow \mathbb{C}^* \) be a character. The induced homomorphism \( K \rightarrow S^1 \) is also denoted by \( \kappa \).

Let \( (E, \bar{\nabla}_E, \theta, h) \) be a \((K, \rho, \kappa)\)-homogeneous harmonic bundle on \( X \setminus H \) which is good wild on \( (X, H) \). We obtain a good filtered Higgs bundle \( (P^h_E, \theta) \) on \( (X, H) \). Because each \( P^h_E \) is naturally a \( K \)-equivariant holomorphic vector bundle on \( X \), \( P^h_E \) is naturally \( G \)-equivariant by Lemma 8.3. Because \( k^* \theta = \kappa(g) \theta \) for any \( k \in K \), we obtain \( g^* \theta = \kappa(g) \theta \) for any \( g \in G \). Therefore, \( (P^h_E, \theta) \) is a \((G, \rho, \kappa)\)-homogeneous good filtered Higgs bundle on \( (X, H) \).

Let \( L \) be a \( G \)-equivariant ample line bundle on \( X \).

**Proposition 8.7** \((P^h_E, \theta)\) is \( \mu_L \)-polystable with respect to the \( G \)-action, i.e., there exists a decomposition \( (E, \bar{\nabla}_E, \theta, h) = \bigoplus (E_i, \bar{\nabla}_{E_i}, \theta_i, h_i) \) of \((G, \rho, \kappa)\)-homogeneous harmonic bundles such that each \((P^h_i, \theta_i)\) is \( \mu_L \)-stable with respect to the \( G \)-action.

**Proof** Because \((P^h_E, \theta)\) is \( \mu_L \)-polystable, we obtain that \((P^h_i, \theta_i)\) is \( \mu_L \)-semistable with respect to the \( G \)-action. Let \( V_1 \subset P^h_E \) be a saturated Higgs \( \mathcal{O}_X \times H \)-submodule such that \( \mu_L(P^h_i, V_1) = \mu_L(P^h_E) = 0 \). Let \( E_1 \) be the Higgs subsheaf of \( E \) obtained by \( V_1 \) to \( X \setminus H \). Then, by the argument in the proof of [8.3](#), Proposition 13.6.1, we obtain that \( E_1 \) is a subbundle, and the orthogonal complement \( E_2 := E_1^\perp \) is also a holomorphic subbundle. Moreover, \( \theta(E_2) \subset E_2 \otimes \Omega^1_{X \setminus H} \) and \( E_2 \) is \( K \)-equivariant. Hence, we obtain a decomposition \( (E, \bar{\nabla}_E, \theta, h) = (E_1, \bar{\nabla}_{E_1}, \theta_1, h_1) \oplus (E_2, \bar{\nabla}_{E_2}, \theta_2, h_2) \) of \((K, \rho, \kappa)\)-homogeneous harmonic bundles. Then, the claim of the proposition is clear.

#### 8.2.2 Uniqueness

Let \((E, \bar{\nabla}_E, \theta, h)\) be a \((K, \rho, \kappa)\)-homogeneous harmonic bundle on \( X \setminus H \) which is good wild on \((X, H)\). Let \( h' \) be another pluri-harmonic metric of \((E, \bar{\nabla}_E, \theta)\) such that (i) \( h' \) is \( K \)-invariant, (ii) \( P^{h'}_E = P^h_E \). The following is clear from Proposition 2.11.

**Proposition 8.8** There exists a decomposition \((E, \bar{\nabla}_E, \theta) = \bigoplus_{i=1}^n (E_i, \bar{\nabla}_{E_i}, \theta_i)\) such that (i) the decomposition is orthogonal with respect to both \( h \) and \( h' \), (ii) there exist \( a_i > 0 \) \((i = 1, \ldots, m)\) such that \( h'|_{E_i} = a_i h_{E_i} \), (iii) the decomposition \( E = \bigoplus E_i \) is preserved by the \( K \)-action.

#### 8.2.3 Existence theorem

Let \((P, \mathcal{V}, \theta)\) be a \((G, \rho, \kappa)\)-homogeneous good filtered Higgs bundle on \((X, H)\) such that

\[
\int_X \text{par-c}_1(P, \mathcal{V}) c_1(L)^{\text{dim} X - 1} = 0, \quad \int_X \text{par-ch}_2(P, \mathcal{V}) c_1(L)^{\text{dim} X - 2} = 0.
\]

Let \((E, \bar{\nabla}_E, \theta)\) be the Higgs bundle on \( X \setminus H \) obtained as the restriction of \((P, \mathcal{V}, \theta)\).

**Theorem 8.9** Suppose that \((P, \mathcal{V}, \theta)\) is \( \mu_L \)-stable with respect to the \( G \)-action. Then, there exists a \( K \)-invariant pluri-harmonic metric \( h \) of \((E, \bar{\nabla}_E, \theta)\) such that \( P^h_E = P^h \). If \( h' \) is another \( K \)-invariant pluri-harmonic metric of \((E, \bar{\nabla}_E, \theta)\), there exists a positive constant \( a \) such that \( h' = ah \).
Proof By Lemma 8.4 there exists a \( \mu_L \)-stable good filtered Higgs bundle \((P_*, \mathcal{V}_0, \theta_0)\) and a finite dimensional vector space \(U\) with an isomorphism \((P_*, \mathcal{V}_0, \theta_0) \otimes U \simeq (P_*, \mathcal{V}, \theta)\). Let \((E_0, \mathcal{J}_{E_0}, \theta_0)\) be the Higgs bundle on \(X \setminus H\) obtained as the restriction of \((P_*, \mathcal{V}_0, \theta_0)\). There exists a pluri-harmonic metric \(h_{E_0}\) of \((E_0, \mathcal{J}_{E_0}, \theta_0)\) such that \(P^*_h E_0 = P_* \mathcal{V}_0\). Let \(Herm(U)\) denote the space of Hermitian metrics of \(U\).

Let \(h_1\) be any pluri-harmonic metric of \((E, \mathcal{J}_E, \theta)\) such that \(P^*_h E = P_* \mathcal{V}\). By Proposition 2.15 there uniquely exists \(h_0(h_1) \in Herm(U)\) such that \(h_1 = h_{E_0} \otimes h_0(h_1)\). Let \(K \subset K\). Note that \(k^*(h_1)\) is a pluri-harmonic metric of \((E, \mathcal{J}_E, \kappa(k) \theta)\). Because \(|\kappa(k)| = 1\), \(k^*(h_1)\) is a pluri-harmonic metric of \((E, \mathcal{J}_E, \theta)\). Moreover, \(P^{k^* h_1} E = P_* \mathcal{V}\) holds. Hence, there uniquely exists \(h_U(k^* h_1) \in Herm(U)\) such that \(k^*(h_1) = h_{E_0} \otimes h_U(k^* h_1)\). Because \(k^*(h_1)\) is continuous with respect to \(K\), we obtain a continuous map \(K \rightarrow Herm(U)\). By using the convexity of \(Herm(U)\), we obtain \(\int_K \Psi dk \in Herm(U)\), where \(dk\) is the Haar measure of \(K\) such that \(\int_K dk = 1\).

We obtain a Hermitian metric \(h := \int k^* h_1 dk\) of \(E\). Then, \(h\) is \(K\)-invariant. Moreover, \(h = h_{E_0} \otimes \int_K \Psi dk\) holds. Hence, \(h\) is a pluri-harmonic metric of \((E, \mathcal{J}_E, \theta)\) such that \(P^*_h E = P_* \mathcal{V}\). Hence, we obtain the claim of the theorem. The uniqueness is clear.

Corollary 8.10 We obtain the equivalence between the isomorphism classes of the following objects.

- \((K, \rho, \kappa)\)-homogeneous good wild harmonic bundles on \((X, H)\).
- \((G, \rho, \kappa)\)-homogeneous good filtered Higgs bundles \((P_*, \mathcal{V}, \theta)\) such that \(i)\) it is \(\mu_L\)-polystable with respect to the \(G\)-action, \(ii)\) \(\mu_L(P_*, \mathcal{V}) = 0\), \(\int_X \text{par-ch}_2(P_*, \mathcal{V}) c_1(L)^{\text{dim}X-2} = 0\).

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