Abstract

Abstract mathematical theory of an observer is elaborated upon the basis of A.Poincare’s ideas on the nature of geometry and the role of observer’s perceptive space. The said theory is generalizing reference frames theory in GR. Physical structure (\(P\)-structure) and corresponding physical geometry (\(P\)-geometry) notions, representing properties invariance of some physical objects and their relations, are introduced. \(P\)-structure of classical physical time and its corresponding chronogeometry is considered as an example. Some quantitative characteristics of observer’s visual space geometry are experimentally determined. The affine model of visual geometry is offered to interpret experimentally sampled data. The connection of the obtained results with some problems of theoretical physics is being discussed.

1 Introduction

The creation of special relativity (SR) and general relativity (GR) has been a powerful incentive for applying geometrical ideas to the basement of modern theoretical physics [7, 4, 2, 1, 8]. No physical theory pretending at present to describe nature’s fundamental ground principles is free to certain extent from geometrical ingredients. Though as our knowledge about physical world structure is deepening aided by accelerators and mighty telescopes, detail description is getting more and more intricate and refined, geometrical ideas and methods, taking shape of more modern and abstract forms, remain basic working language of theoretical and mathematical physics.

At the beginning of the last century A.Poincare, possessing ingenious geometrical intuition, was one of the first to question the nature of geometry’s axioms [17]. Analyzing geometries known at that time, their origin and correlation with experience, he has come to conclusion that geometry was an artifact of physical objects properties. He wrote in particular:
"Where do primary geometry principles originate from? Are they prescribed by logic? Lobachevsky, having created non-Euclidean geometries, proved it wrong. Don't we discover space with our perception? Again, not, since the space of our perception can teach us is totally different from that of geometrist's. Does geometry stem from experience at all? Profound research shows us that it's not so. We can conclude from here that these principles are no more than conditional . . . and if we were transported into some other world (I call it non-Euclidean and am trying to depict it), then we would draw our attention to other foundations" [17, p.10].

Thus, according to Poincare's thought, different geometries are different languages expressing some invariant physical sense which, as a matter of fact, is an object of physicists' and mathematicians' researches.

As Poincare's profound analysis has shown, observer's visual space, being a subspace of broader perceptive space, plays exceptional role in geometry axioms formation and is by its substantial properties different from geometrical space.

"The space of imagination is just an image of geometrical space — an image changed by some sort of prospect; we do not imagine therefore external bodies in geometrical space, but we are speculating about these bodies as if they were placed in geometrical space" [17, p.10].

Developing this Poincare's thought, it can be assumed that there exists some observer's inner geometry which turns out to be imperceptibly built into our observations, laws, equations, and physical principles. In fact, this is not, if understood in broader sense, a new thought and was already developed in sound perception laws discovered XIX century. So, famous Weber-Fechner law, stating that perceived (seeming) sound signal force is proportional to logarithm of its true force, actually asserts non-Euclidean nature of hearing subspace of an observer's perceptive space. Since an observer receives more than 80% of information through eyesight, analogous researches of an observer's visual space are of interest. Information on quantitative characteristics of visual (or more general — perceptive) space can be used in a wide class of problems: from computer display engineering and technologies, architecture and design to alternative interpretations of some observational experiments in cosmology and unsolved problem of quantum theory [19, 16] (see also Conclusion).

In the first half of the present article we are suggesting concrete mathematical realization of Poincare's ideas. Different physical geometries are being concluded from the sole geometry of perceptive manifold and fundamental properties of physical objects — particles, bodies, fields. Besides quoted above Poincare's ideas on geometry's conditional role and its dependence upon physical context, we are using a number of other ideas and observations, part of which were expressed by some authors earlier, disregarding of our consideration.

1. **Observer's role in physical theory.** This question has drawn attention mostly because of the problems connected to the search for quantum mechanics' due interpretation and observables' problem in GR [5, 14]. From the very beginning we are introducing a formalized notion of an observer
— his perceptive space and Newton’s mapping roughly modelling observation process together with objects of observation: hyperbodies and their images in perceptive space — ordinary bodies’ histories. It should be emphasized that observer’s presence and his properties play a role already at classical physics construction in our approach since hyperbodies world’s laws can have a different character than their images in perceptive space.

2. **An essence of a physics law.** To reveal it, we are formulating $\mathcal{P}$-structure (physical structure) notion. It embraces quite general understanding of a physical law: some property observed in perceptive space is invariant to selection of some observers’ subset. It intersects as analogy in a sense of terminology and ideology but doesn’t coincide with Yu.I. Kulakov’s physical structures theory [11, 13].

3. **Physical geometry’s secondary meaning.** We are inferring $\mathcal{P}$—geometry notion from $\mathcal{P}$—structure under certain assumptions. This $\mathcal{P}$—geometry is realized on observers’ set transformation group space [9].

4. **Normal divisor role.** Here is an observation we make: there is a normal divisor of Euclidean metric isometry group $R^3 \rtimes O(3)$ — translations’ subgroup $R^3$, which can be identified as physical space itself — Euclidean space $E^3$ — by fixing one origin point. Analogously, there is normal divisor — translations’ subgroup $R^4$ in isometry group $R^4 \rtimes O(1, 3)$ of Minkowski metric. This subgroup is isomorphic to physical space of special relativity — Minkowski space $M_{1,3}$. We are generalizing these facts and building $\mathcal{P}$—geometry, which has been drawn on in previous clause, as normal divisor’s inner geometry of the corresponding group if such divisor exists. However, we have not assumed divisor’s abelian property as against given examples (see also [18]).

We are demonstrating the approach developed here when inferring chronogeometry.

In the second part of the article we are presenting statistically processed data, obtained from the experiment that’s been carried out to determine some quantitative characteristics of observer’s visual space geometry. We suggest visual manifold’s affine model that satisfactorily approximates experimental data within studied distances range. Achieved results are subjected to short general discussion in Conclusion.

Most of the definitions and symbols used in the first part of the article are standard. Particularly, we denote:

- $\text{Dom}(f)$ — domain of some mapping $f$;
- $\text{Im}(f)$ — range of images of some mapping $f$;
- $A \leq B$ — $A$ is subgroup of group $B$;

As new signs and definitions appear, their meaning is explained.
2 \( \mathcal{P} \)-structure and \( \mathcal{P} \)-geometry definitions

2.1 Properties algebra over arbitrary set

In this section we are developing some initial foundations for abstract properties theory. This theory being of interest in itself will be used further on in defining \( \mathcal{P} \)-structure notion.

Let’s assume \( \mathcal{A} \) and \( \Omega \) — are some element sets. We will say that \( \Omega \) constructs properties system over \( \mathcal{A} \) set and write it down like \( \Omega = \mathcal{P}(\mathcal{A}) \), if mapping \( \chi : \mathcal{A} \times \Omega \rightarrow \mathbb{Z}_2 = \{0, 1\} \) is given. Element \( a \in \mathcal{A} \) will be called possessing the property \( \omega \in \Omega \), if \( \chi(a, \omega) = 1 \) and non-possessing such property \( \omega \), if \( \chi(a, \omega) = 0 \).

Let \( \omega \) be some fixed element of the set \( \Omega \). Designate \( A_\omega \) and \( \overline{A}_\omega \) subsets of \( \mathcal{A} \), satisfying the following conditions:

\[
a \in A_\omega \iff \chi(a, \omega) = 1; \quad b \in \overline{A}_\omega \iff \chi(b, \omega) = 0.
\]

Apparently \( A_\omega \cup \overline{A}_\omega = \mathcal{A} \) and \( A_\omega \cap \overline{A}_\omega = \emptyset \). Analogously, let \( a \) be some fixed element of the set \( \mathcal{A} \). Designate \( \Omega_a \) and \( \overline{\Omega}_a \) subsets of \( \Omega \), satisfying the conditions:

\[
\omega \in \Omega_a \iff \chi(a, \omega) = 1; \quad \delta \in \overline{\Omega}_a \iff \chi(a, \delta) = 0.
\]

Thus, we can say that \( A_\omega \) — a set of all \( \mathcal{A} \) elements possessing property \( \omega \), \( \overline{A}_\omega \) — a set of all \( \mathcal{A} \) elements, not possessing property \( \omega \), \( \Omega_a \) — a set of all properties possessed by an element \( a \) and \( \overline{\Omega}_a \) — a set of all properties not possessed by an element \( a \).

Our preceding and following description of the sets of elements and properties over them reveals the following property of duality: same mapping \( \chi \) determines a set \( \Omega \) as set of properties over \( \mathcal{A} \), but the same mapping determines \( \mathcal{A} \) as a set of properties on its own properties \( \Omega \). Therefore we can define a set \( \mathcal{A} \) as \( \tilde{\mathcal{P}}(\Omega) \) — a set of properties over \( \Omega \), where \( \tilde{\chi}(\omega, a) = \chi^T(\omega, a) \equiv \chi(a, \omega) \). Here duality relations are valid:

\[
\tilde{\chi}(\varphi(\mathcal{A})) = \mathcal{A}; \quad \varphi(\tilde{\chi}(\Omega)) = \Omega.
\]

At first, in light of the above relations, identification of the sets \( \mathcal{A} \) and \( \Omega \) as elements of sets and properties on them becomes matter of agreement, i.e. relative and conditional and we will stick to the once chosen manner. Secondly, all definitions and relations, pertaining to one of the sets (of elements or properties), not connected with introducing additional structures on these sets, allow dual wording on complementary set. We will mention it in plain form only if necessary.

Let’s call property \( \omega_0 \in \Omega \) trivial on \( \mathcal{A} \), if \( A_{\omega_0} = \mathcal{A} \), and property \( \omega_\emptyset \) empty on \( \mathcal{A} \), if \( A_{\omega_\emptyset} = \emptyset \). Dual analogies for trivial and empty properties are universal element \( a_0 \) and transcendent element \( a_\emptyset \), which are identified by the relations \( \Omega_{a_0} = \Omega \), and \( \Omega_{a_\emptyset} = \emptyset \). Note, that the existence of trivial and empty properties and their dual analogs is not obligatory.
Since function $\chi$ puts into correspondence each element $\omega \in \Omega$ subsets $A_\omega$, on which boolean operations are naturally determined, we can introduce mappings on $\Omega$, induced by boolean algebra on $A$. Thus, let $T_A$, $\bot_A$, $q_A$ — correspondingly, binary operation, unary operation, and binary relation on $\mathcal{B}(A)$, where $\mathcal{B}(A)$ — set of all subsets of $A$, called boolean. Then it is possible to determine the corresponding one- and two-component functions and binary relation: $T_\Omega : \Omega \times \Omega \to \Omega''$, $\bot_\Omega : \Omega \to \Omega'$, $q_\Omega$ according to following rules:

\[
\begin{align*}
\omega_1 T_\Omega \omega_2 = \omega_3 & \iff A_{\omega_3} = A_{\omega_1} T_A A_{\omega_2}, \text{ for all } \omega_1, \omega_2 \in \Omega; \\
\bot_\Omega \omega_1 = \omega_2 & \iff A_{\omega_2} = \bot_A A_{\omega_1}, \text{ for all } \omega_1 \in \Omega; \\
\omega_1 q_\Omega \omega_2 & \iff A_{\omega_1} q_A A_{\omega_2} \text{ for all } \omega_1, \omega_2 \in \Omega.
\end{align*}
\]

Here $\Omega'' \supseteq \Omega$, $\Omega' \supseteq \Omega$ — some new sets of properties, which can be produced from the initial set $\Omega$ by completion over corresponding one or two component mappings.

Let’s designate $\Omega_{\infty}$, completion of $\Omega$ by whole boolean algebra, generated by elements $\{A_\omega\}_{\omega \in \Omega}$ and let’s call it complete properties system over $A$, induced by $\Omega$. Initial set $\Omega$ will be called generating for $\Omega_{\infty}$. Note, that set $\mathcal{A}$ becomes topological space with topology $\{A_\omega\}_{\omega \in \Omega_{\infty}}$ with prebase $\{A_\omega\}_{\omega \in \Omega}$.

We are going to introduce the following correspondence rules between boolean operations designations in $\mathcal{A}$ and the corresponding operations and relations in $\Omega_{\infty}$:

\[
\mathcal{A} \to 1; \quad \emptyset_A \to 0; \quad \cup \to +; \quad \cap \to ; \quad \subseteq \to \geq; \quad \neg \to 1; \quad \setminus \to =; \quad =\to = . \quad (1)
\]

The properties, connected by a relation "=" will be called equivalent on $A$, and if $\omega_1 \geq \omega_2$, we’ll say, that $\omega_1$ is not weaker than $\omega_2$ on $A$. It is easy to see that the introduced earlier set $\mathcal{A}_\omega = A_{\omega_1\omega}$. Obviously, the properties similar to those of boolean operations and relations in $\mathcal{A}$ are satisfied by all introduced operations and ratios between elements of $\Omega_{\infty}$ (commutativity, associativity, distributivity, etc). For instance, it easy to check the following identities of the obtained properties algebra:

\[
(\omega_1 \pm \omega_2) \cdot \omega_3 = \omega_1 \cdot \omega_3 \pm \omega_2 \cdot \omega_3; \quad \omega^2 = \omega; \quad (1 - \omega) \cdot \omega = 0.
\]

Note, that in $\Omega_{\infty}$, $\omega_0 = 1$, $\omega_\emptyset = 0$.

Concluding this paragraph, let’s introduce a notion of determinable elements set $D \subset \mathcal{A}$, whose characteristic is that there exists (perhaps not the unique!) family of properties $\{\omega_i\}_{i=1,\ldots,N} \subset \Omega$, $N < \infty$, that $D = A_{f(\omega_1,\ldots,\omega_N)}$, where $f(\omega_1,\ldots,\omega_N)$ — finite superposition of $[\Pi]$ with $\{\omega_i\}_{i=1,\ldots,N}$. In other words, determinable elements set $D$ can be determined (described) using finite number of properties from $\Omega$. By properties set definition, some sets from $\mathcal{A}$ are always determinable, therefore any $\mathcal{A}$ is partially determinable. Accordingly, let’s call $\mathcal{A}$ quite determinable, if a system of determinable sets $\{A_\alpha\}$ coincides with boolean $\mathcal{B}(A)$. 

5
2.2 Newton’s mapping and hyperclasses.

The next point of our $\Psi$-structure defining will be two sets: world set $\mathcal{M}$ and perceptive set of events $\mathcal{N}$. The former is going to contain elements of "true" physical world, the latter — elements of its perception by some observer.

Since space and time perceptions exist to a significant extent independently, we assume that perceptive set of events $\mathcal{N}$ is direct product $\mathcal{T} \times \mathcal{V}$, where $\mathcal{T}$ — time perceptive set, $\mathcal{V}$ — visual perceptive set, whose connection with physical time and space is to be defined later on. Set $\mathcal{V}$ is in bijective correspondence with simultaneous event space $\mathcal{V}_T$, that we define as section $\{T\} \times \mathcal{V}$ of set $\mathcal{N}$ by some element of time set $T \in \mathcal{T}$. The sets $\mathcal{T}$ and $\mathcal{V}$ will be considered to be metrized with metrics $\tau: \mathcal{T} \times \mathcal{T} \to R$ and $\eta: \mathcal{V} \times \mathcal{V} \to R$.

Assume further that $\mathcal{N}$ is an image of some surjective mapping $f: \mathcal{M} \to \mathcal{N}$, which, in physical language, describes a process of observation of some subset of world set made by some observer. We are going to call this mapping Newton’s one.

Let’s define hyperbody $\mathcal{B}$ as some subset of a set $\text{Dom}(f) \subseteq \mathcal{M}$. Its image $f(\mathcal{B}) \subseteq \mathcal{T}_B \times \mathcal{B}$, where $\mathcal{T}_B = (\pi_1 \circ f)(\mathcal{B}) \equiv f_T(\mathcal{B})$, $B = (\pi_2 \circ f)(\mathcal{B}) \equiv f_V(\mathcal{B}) \equiv \cup_{T \in \mathcal{T}} \mathcal{B}_T$, $\pi_1, \pi_2$ — projections $\mathcal{N} \to \mathcal{T}$, $\mathcal{N} \to \mathcal{V}$ correspondingly and a designation $\mathcal{B}_T$ for instant body at a time point $T$: $\mathcal{B}_T \equiv \mathcal{V}_T \cap f(\mathcal{B})$ is introduced. This image can be understood as some subset of direct product $\mathcal{T} \times \mathcal{B}(\mathcal{V})$, which, in turn, defines mapping $\mathcal{T} \ni T \mapsto \mathcal{B}_T \in \mathcal{B}(\mathcal{V})$. A graph of this mapping will be referred to as $f$-history of body $B$ induced by a hyperbody $\mathcal{B}$.

Using mapping $f$ on the set of all hyperbodies in $\mathcal{M}$, we can introduce canonical equivalency: two hyperbodies $\mathcal{B}_a$ and $\mathcal{B}_\beta$ are $f$-equivalent, if $f(\mathcal{B}_a) = f(\mathcal{B}_\beta)$. Equivalency class $[\mathcal{B}]_f$ of some hyperbody $\mathcal{B}$ will be called its $f$-hyperclass. Particularly, elementary event $p \in \mathcal{N}$ preimage is some hyperbody $\mathcal{M}_p$, thus $\mathcal{M}' = \text{Dom}(f)/\mathcal{M}_p$ is canonically isomorphic $\mathcal{N}$. It is also possible to introduce the sets $\mathcal{M}_T \equiv f^{-1}(\mathcal{T}_V)$ and $\mathcal{M}_V \equiv f^{-1}(\mathcal{T}_V)$, where $\mathcal{T}_V \equiv \mathcal{T} \times \{V\}$ — the history of a point $V \in \mathcal{V}$. If $P = (T, V) \in \mathcal{N}$, than it’s obvious that $\mathcal{M}_P = \mathcal{M}_T \cap \mathcal{M}_V$.

Thus, any member of $f$-hyperclass gives identical $f$-history in $\mathcal{N}$, that, in turn, can be understood as “movement graphics” of some body $B$ in $\mathcal{V}$, perceived by an observer. Alongside with hyperbodies and hyperclasses, we can consider as well their unions $\cup\alpha \mathcal{B}_\alpha$ and their corresponding $f$-hyperclasses, which will generally speaking be transformed by Newton’s mapping into compound history of a bodies system $B_\alpha$: $\mathcal{f}(\cup\alpha \mathcal{B}_\alpha) \subseteq \cup\alpha (\mathcal{T}_B_\alpha \times \mathcal{B}_\alpha)$. It is natural to call these hyperbodies compound. Later on, if it’s not specially stipulated, it will be dealt only with hyperclasses, omitting square brackets, where it is appropriate and will not lead to a mess.

Note, that using $\tau$ and $\eta$ and mapping $f$, it’s possible to introduce functions $\tau^*$ and $\eta^*$: $\text{Dom}(f) \times \text{Dom}(f) \to R$,

$$
\tau^*(p, q) \equiv \tau(f_T(p), f_T(q)); \quad \eta^*(p, q) \equiv \eta(f_V(p), f_V(q))
$$

for all $p, q \in \text{Dom}(f)$, which can be considered as metrics on factor-set $\mathcal{M}'$. 
2.3 Newton’s mappings transformation group

Up to now we have dealt with only one observer, to be more exact — with his Newton’s mapping. The existence of observers’ set as well as a possibility of one observer’s state change (for example his motion) will be highlighted in our construction by means of some subgroup \( G \) of general group of automorphisms \( \text{Aut}(\mathcal{N}) \). We are not specifying this subgroup at this stage but major role in applications will be played by finite-dimensional Lie groups acting on \( \mathcal{N} \).

In fact, the very group character of mappings from \( G \) enables to make a number of important conclusions regarding behaviour of the objects introduced in previous paragraph with \( G \) acting.

The Proposition 2.1 Automorphisms \( g \in G \) induce transformations of Newton’s mapping:

\[
f \rightarrow f_g = g \circ f.
\]

We are postulating that any automorphism \( g \) from \( G \), acting on some Newton’s mapping, results again in Newton’s mapping. Thus, we can say that observers family is a homogeneous space with respect to a group \( G \) action. Let’s designate this family \( \mathcal{O}_f \equiv \{ f_g | g \in G \} \). Any two families \( \mathcal{O}_f \) and \( \mathcal{O}_{f'} \) either coinside or not intersect. More exactly, relation of \( \mathcal{O}_f \) and \( \mathcal{O}_{f'} \) is ruled by

The Proposition 2.2 Any \( \mathcal{O}_f \) and \( \mathcal{O}_{f'} \) coinside only if \( \text{zer}(f) = \text{zer}(f') \), where \( \text{zer} \) — fibers decomposition of domain of some mapping.

Proof. Let \( \mathcal{O}_{f'} = \mathcal{O}_f \), then \( f' = g \circ f \) for some \( g \in G \). Going to its factors, we have:

\[
\text{fact } f' = \text{fact}(g \circ f) = g \circ \text{fact } f, \tag{2}
\]

since \( g \) — automorphism. Here \( \text{fact } f: \mathcal{M}/\text{zer}(f) = \mathcal{M}' \rightarrow \mathcal{N} \). It implies \( \text{zer}(f) = \text{zer}(f') \), since \( \text{zer} = \text{Dom} (\text{fact}) \). Inversely, if \( \text{zer}(f) = \text{zer}(f') \), then there exists \( g \in \text{Aut}(\mathcal{N}) \), such that (2) takes place. However, \( g \) can be lying in \( \text{Aut}(\mathcal{N}) \setminus G \), so condition of the statement is only necessary, but not sufficient. □

As a consequence we have

The Proposition 2.3 The set \( \mathcal{M}' \) is invariant under \( G \) action.

Proof. It follows from the isomorphism \( \mathcal{M}' \sim \mathcal{N} \) that \( g \in G \) induces automorphism \( g_* : \mathcal{M}' \rightarrow \mathcal{M}' \), where \( g_* = (\text{fact } f)^{-1} \circ g \circ \text{fact } f \). □

Let \( g(\mathcal{N}) = \mathcal{N}' = \mathcal{T}' \times \mathcal{V}' \). We will be saying that \( g \) is automorphism of \( \mathcal{T} \)-type \( (g \in G_T) \), if for any \( T \in \mathcal{T} \) there exists the only \( T' \in \mathcal{T}' \), such as \( g(T, \pi_2 V_T) = (T', \pi_2 V_{T'}) \), i.e. \( g \) defines bijection between the sets of simultaneous events in \( \mathcal{N} \) and \( \mathcal{N}' \). Similarly, we will call \( g \) automorphism of \( \mathcal{V} \)-type \( (g \in G_V) \), if for any \( V \in \mathcal{V} \) there can be found the only \( V' \in \mathcal{V}' \), such as \( g(\pi_1 T_V, V) = (\pi_1 T'_{V'}, V') \), i.e. \( g \) defines bijection between histories of point-events in \( \mathcal{N} \) and \( \mathcal{N}' \). And finally, we’ll call \( g \) automorphism of \( \mathcal{T} \mathcal{V} \)-type \( (g \in G_{\mathcal{T} \mathcal{V}}) \) if at the same time \( g \in G_T \) and \( g \in G_V \). It is easy to show that the above described bijection can be plainly written down as follows:
\[ g_T = \begin{cases} T' = T'(T); \\ V' = V'(T, V), \end{cases} \]
\[ g_V = \begin{cases} T' = T'(T, V); \\ V' = V'(V), \end{cases} \]
\[ g_{TV} = \begin{cases} T' = T'(T); \\ V' = V'(V), \end{cases} \]

This representation obviously shows that each type of automorphisms form subgroup of \( \mathcal{G} \) (see also [3]).

### 2.4 Sets \( \text{Cont}(\mathcal{N}) \), \( M_D \), \( \mathfrak{P} \)-structures and \( \mathfrak{P} \)-geometries.

Let us introduce a set \( \text{Cont}(\mathcal{N}) \) in the category \( \text{SETS} \) as a maximal subset of morphisms epicones beginnings\(^1\) \( \text{EpiC}^*(\mathcal{N}) \subset \text{Mor}(\mathfrak{A}, \mathcal{N}) \), where \( \mathfrak{A} \in \text{Ob SETS} \), each element \( \mathfrak{A} \) of which:

a) is connected with \( \mathcal{N} \) by (perhaps nonunique) epimorphism \( \sigma \in \text{EpiC}^*(\mathcal{N}) \);

b) is \( (T, V, R) \)-autonomous, i.e. can be defined by some universal symbolic formula of the following kind:

\[ \mathfrak{A} = \Lambda(T, V, R), \tag{3} \]

where \( \Lambda = \alpha_1 \circ \cdots \circ \alpha_N \) \( (N < \infty) \) and every morphism \( \alpha_i \) is either universal, or \( R \)-morphism of the category \( \text{SETS} \). Morphism \( \alpha \) of \( \text{SETS} \) we’ll call universal, if its definition don’t refer to any additional structures on the sets of the class \( \text{Ob SETS} \). Morphism \( \alpha \) we’ll call \( R \)-morphism, if \( \text{Im} \alpha \subseteq \mathbb{R}^n \) or \( \text{Dom} \alpha \subseteq \mathbb{R}^n \) under some \( n < \infty \). Futhermore for the brevity sake we’ll write (3) in the compact form \( \mathfrak{A} = \Lambda(\mathcal{N}) \).

Let us make some comments to the above given definitions. Laws of nature, which we open analyzing our experiments and working out theories, may be rather complicated to be formulated in terms of primary observers perceptions, i.e. in terms of \( \mathcal{N}, T \) and \( V \). For example, if we temporarily assume \( T = E^1, V = E^3 \), then Lagrange mechanics of \( n \) matter points is formulated in terms of tangent bundle \( \nabla(E^3)^\times n \), while Hamiltonian mechanics — in terms of cotangent bundle \( \nabla^*(E^3)^\times n \) (Lagrange and Hamilton functions are defined on this manifolds). Solid dynamics in classical mechanics can be described by means of Lagrange function, defined on manifold \( \nabla E_3 \times \nabla \text{SO(3)} \), where \( \text{SO(3)} \) — proper orthogonal group of \( E_3 \), isomorphic to a space of solids angle positions. Classical electrodynamics demands consideration of vector and tensor bundles over Minkowski space \( M_{1,3} \). Abstract set \( \text{Cont}(\mathcal{N}) \), which has been defined above, is wide (in some sense maximal) arena for formulation of physical laws with any

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\(^1\) Lets remind that for any category \( \mathfrak{R} \) morphisms cocone \( C^*(A) \) with vertex \( A \in \text{Ob} \mathfrak{R} \) is a system of morphisms \( \alpha_i : A_i \to A, i \in I \). Cocone is called dense or epicone, if from the \( \varphi \circ \alpha_i = \psi \circ \alpha_i \) for all \( i \in I \) it follows \( \varphi = \psi \). Here “\( \circ \)” means standard (from right to left) composition of morphisms. In other words, epicone \( \text{EpiC}^*(A) \) consist of all epimorphisms \( \text{Epi} \mathfrak{R} \) of the category \( \mathfrak{R} \) with end \( A \). Within the sets category \( \text{SETS} \) epimorphisms coincide with surjections. The notions of cocone and epicone are dual to cone and monocone ones within the given category \( \mathfrak{R} \) [20].
degree of complexity. In our examples Cartesian product "×", is example of universal morphism $A \to A \times B$, while tangentialization $\top$ and cotangentialization $\top^*$ operations are important particular cases of nonuniversal $R$-morphisms of the category SETS, in fact, acting in category DIFF of smooth manifolds, which is subcategory of SETS. By means of coordinate homeomorphisms the mapping $\top$ and $\top^*$ always can be realized as family of embeddings $R^n \to R^{2n}$ for some $n$.

Our construction, of course, is based on some particular role of the set of real number $R$ within class of objects of the category SETS. Cause of this particularity is clarified by the following definition. For any $\Lambda$ from (3) and every natural $D$ lets define the set:

$$M_D(\Lambda) \equiv \text{Maps}(\text{Im}(\Lambda), R^D) \equiv \{f : \mathfrak{A} \to R^D \mid \mathfrak{A} = \Lambda(N)\}$$ (4)

of all mappings of the sets $\Lambda(N)$ in $R^D$ with fixed $\Lambda$ and $D$. Let designate

$$\mathfrak{Z}_N \equiv \bigcup_{D=0}^{\infty} \bigcup_{\Lambda} M_D(\Lambda).$$

The set $\mathfrak{Z}_N$ is, in physical context, the collections of various "arithmetizations" and "metrics" that enable us to fix measuring devices data and measurements’ results in the form of number tables, functions, graphs, correlations and to derive quantitative (value) results from the abstract theories, formulated in terms of some sets from $\text{Cont}(N)$.

Let’s define homomorphism

$$\Delta_{(\Lambda,D)} : \mathcal{G} \to \text{Aut}(M_D(\Lambda)),$$

designed as follows. Let $\mathfrak{A} = \Lambda(N)$, where $\Lambda$ — function of type (3), defining $\mathfrak{A}$ by means of some finite set of suitable morphisms. If under $g \in \mathcal{G}$, $N \to N' = g(N)$, then homomorphism $\chi_g : \mathfrak{A} \to \mathfrak{A}' = \Lambda(N')$, is defined, such as for diagram:

$$\mathfrak{A} \xrightarrow{x_g} \mathfrak{A}'$$

$$\Lambda \uparrow \Lambda$$

$$N \xrightarrow{g} N'$$

commutativity condition is met: $\Lambda \circ g = x_g \circ \Lambda$. We define then $\Delta_{(\Lambda,D)}(g) \varphi \equiv \varphi \circ x_g$ for any $\varphi \in M_D(\Lambda)$.

Let’s designate the set $\mathfrak{A}_B \equiv (\Lambda \circ f)(B) \equiv \Lambda_B(N)$, consisting of those elements of $\mathfrak{A}$, whose preimage under mapping $\Lambda$ were lying in $f(B) \subseteq N'$, where $B$ is some hyperbody from $\mathcal{M}$, and $f$ — Newton’s mapping of some observer. Consider as well the set of properties $\mathcal{P}(M_D(\Lambda))$ over $M_D(\Lambda)$. Now we can formulate the notion of $\mathcal{P}$-structure.

Lets call the collection $(\{B^\omega\}, f, \mathcal{G}^\omega, \Lambda, M_D(\Lambda_\omega), \omega)$, $\mathcal{P}$-structure induced by a property $\omega$ or, in short, $\mathcal{P}^\omega$-structure, if

$$\text{id}_{M_D(\Lambda)} \neq \Delta_{(\Lambda,D)}(\mathcal{G}^\omega) \leq \text{Aut}(M_D(\Lambda)).$$ (5)
In the above mentioned collection:
\( \{ B^\omega \} \) — collection of some hyperbodies from \( \mathcal{M} \);
\( f \) — some Newton’s mapping;
\( G^\omega \) — some subgroup of the group \( G \in \text{Aut}(\mathcal{N}) \);
\( \Lambda \) — some function from \( \mathcal{N} \) in \( \text{Cont}(\mathcal{N}) \);
\( M_D(\Lambda_\omega) \) — subset of mappings from \( M_D(\Lambda) \), acting from \( \mathcal{A}_\mathcal{B}^\omega \) into \( R^D \) and having a property \( \omega \);
\( \omega \) — an element from \( \wp(M_D(\Lambda)) \).

In other words, \( \wp \)-structure is formed every time when for any member of the family of hyperbodies \( \{ B^\omega \} \) the property \( \omega \) over \( M_D(\Lambda) \) remains invariant with respect to some nontrivial subgroup of automorphisms \( \Delta(\Lambda,D)(G^\omega) \leq \text{Aut}(M_D(\Lambda)) \).

We are going to say that \( \wp^\omega \)-structure induces \( \wp^\omega \)-geometry, if \( G^\omega \):
1) has a structure of semidirect product: \( G^\omega = N^\omega \rtimes S^\omega \), where \( N^\omega \leq G^\omega \) — normal divisor, \( S^\omega \leq G^\omega \).
2) is an isometry of some metric \( \rho^\omega : N^\omega \times N^\omega \to R \), defined on \( N^\omega \), i.e.
\[
\rho^\omega(gg_1, gg_2) = \rho^\omega(g_1, g_2),
\] (6) for any \( g_1, g_2 \in N^\omega, g \in G^\omega \).

Let us clarify the implication of the introduced notions and definitions by means of the following chain scheme, connecting objects of the outer physical world on the left side and our geometrical notions about them:

- hyperclasses \( [\mathcal{B}] \) \( \xrightarrow{f} \) the set of observed histories \( \wp^\omega \)
- \( \wp^\omega \)-structure \( \xrightarrow{G^\omega} \)
- \( \wp^\omega \)-geometry

First arrow plainly introduces an observer into the scope of physical consideration, or to be more exact, his apparatus of surrounding world perception, mathematically formalized via Newton’s mapping. It can turn out that the answer to the question about specific type of this mapping doesn’t lie entirely within the frames of a certain physical theory or physical experiment but should be based as well upon the data of other branches of science: neurobiology, perception psychophysiology, and, perhaps, psychology (see, for example [10, 16]).

Second arrow shows that the most general (abstract) physical structures, which contain, among other things, the category called laws of physics, are formed under two necessary conditions: 1) hyperclasses are not arbitrary, but structured in a certain way in \( \mathcal{M} \); 2) this structuredness possesses a certain (and non-trivial) steadiness with respect to the change of one observer with the other. First condition can in plain language be attributed to the properties of objective reality of the outer world, the second — to the identity of perception apparatus of different observers and to the notion of invariance, which, in one way or the other, accompany any physical law. The third arrow is closely connected to Poincare’s ideas that have been touched upon in the Introduction: geometry, within which we are constructing our models of physical reality, has two foundations in itself: outer — physical properties of the objects we are dealing with in theory and practice, and inner — non-empty intersection of the groups of automorphisms of perceptive manifold and symmetry groups of the corresponding \( \wp \)-structure.
The appropriate $\Psi$-metric, i.e. physical geometry, appears as a notion derived from the observed properties of some hyperclasses that are at first formulated on the set $\text{Cont}(\mathcal{N})$ and the properties (topological, metrical, etc.) of perceptive space $\mathcal{N}$.

3 Group space metrization

We remind that a non-negative function $\rho : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$, meeting the following characteristics:

1. $\rho(a, a) = 0$, for all $a \in \mathcal{A}$;
2. $\rho(a, b) = \rho(b, a)$, for all $a, b \in \mathcal{A}$;
3. $\rho(a, b) \leq \rho(a, c) + \rho(c, b)$, for all $a, b, c \in \mathcal{A}$

is called metric on an arbitrary set $\mathcal{A}$. In our case, the role of a set $\mathcal{A}$ is played by group space of normal divisor $N^\omega$ of group $G^\omega$.

At the first stage of considering various $\Psi$-structures, there appears the group $G^\omega$, which is a solution to property invariance equations (5). Further, in case this group is split into normal divisor $N^\omega$ and the group of its external automorphisms $S^\omega$, metric is found from the condition of isometry (6), which supplements conditions 1, 2, 3, defining metric. To solve the appropriate functional equations using analysis method, it is necessary to demand enough smoothness for function $\rho$. Everywhere below we will assume $\rho \in C^k \times C^k$, $k \geq 2$.

Let us formulate a number of propositions enabling to construct the families of metrics given that some initial function with special properties is known. Namely, assume that function $\rho_0$, meeting isometry condition (6), is found.

**The Proposition 3.1** In case function $\rho_0$ meets metricity condition:

$$|\rho_0(c, c)| \leq |\rho_0(a, c)| + |\rho_0(c, b)| - |\rho_0(a, b)|,$$

(7)

for all $a, b, c \in \mathcal{A}$, then function

$$\rho(a, b) \equiv C[|\rho_0(a, b)| + |\rho_0(b, a)| - |\rho_0(a, a)| - |\rho_0(b, b)|],$$

(8)

($C \in R_+$) meets conditions 1, 2, 3 and, therefore, is a metric, satisfying isometry condition (6).

Function $\rho_0$ will further be referred to as generating, and generating function, satisfying theorem’s condition metric.

**The Proposition 3.2** If smooth function $\varphi : R_+ \to R_+$ of class $C^k$, $(k \geq 2)$, satisfying the following requirements:

1. $\varphi(0) = 0$;
2. $\varphi$ - isotonic, i.e. $\varphi(x) \leq \varphi(y)$, always, when $x \leq y$, for all $x, y \in R_+$;
3. \( \varphi - \text{lowering} \) i.e. \( \varphi(x + y) \leq \varphi(x) + \varphi(y) \), for all \( x, y \in R_+ \)
is given, then, if \( \rho \) is metric on \( A \), so \( \rho_\varphi \equiv \varphi \circ \rho \) - metric on \( A \) as well.

The Proposition 3.3 Let smooth function \( \Phi : R_n^+ \to R_+ \) of class \( C^k, k \geq 2 \) satisfies the following requirements:

1. \( \Phi(0, \ldots, 0) = 0; \)
2. \( \Phi - \text{isotonic}, \) i.e.
   \[ \Phi(x_1, \ldots, x_n) \leq \Phi(y_1, \ldots, y_n), \]
   always, when \( x_i \leq y_i, \) for all \( i = 1, \ldots, n; \)
3. \( \Phi - \text{lowering}, \) i.e.
   \[ \Phi(x_1 + y_1, \ldots, x_n + y_n) \leq \Phi(x_1, \ldots, x_n) + \Phi(y_1, \ldots, y_n), \]
   for all \( \{x_i\}, \{y_i\} \in R_n^+. \)

Further, let \( \{\rho_i\}_{i=1, \ldots, n} \) - a set of metrics on \( A \), meeting isometry condition (5). Then
\( \Phi(\rho_1, \ldots, \rho_n) \) - metric on \( A \), satisfying isometry condition.

Propositions 3.1-3.3 are proved by straight check-up on satisfiability of conditions of metric 1,2,3.

The Proposition 3.4 Assume that function \( \Phi_0 : R^n \to R \) meets the conditions:

1. \( \Phi_0 \in C^k(R^n_+), \) \( k \geq 2; \)
2. \( \frac{\partial \Phi_0}{\partial x_i} > 0, \) for all \( \{x_i\} \in R^n_+; \)
3. Quadratic form \( \partial^2 \Phi_0 \) is negatively defined on \( TR^n_+ \times TR^n_+ \).

Then function \( \Phi \equiv (\Phi_0 - \Phi_0(0)) \bigg|_{R^n_+} \) satisfies the conditions of the theorem 5.3.

The idea of proving holds that with the conditions of the theorem satisfied, function \( \Phi \) graph about any point \( P \) appears like a piece of “convex up” surface, for which it is straitly checked if the condition of proposition 3.3 holds true locally. Global validity of the theorem follows from transitive property of the relation "\( \leq \)."
4 Example: Physical Time of the classical mechanics

4.1 $\mathfrak{P}$-structure of physical time in classical mechanics

$\mathfrak{P}$-structure of classical time is based upon three fundamental facts, which are directly detected by an observer $f$ in $\mathcal{N}$: the existence of elementary bodies, the existence of physical patterns of time and time arrow. Let us introduce the appropriate mathematical wording.

1) A set $E \subseteq \mathfrak{B}(\mathcal{M})$, any element $B_1$ of which satisfies the following: $f(B_1) = \gamma$, where $\gamma$ — continious curve: $\mathbb{R} \rightarrow \mathcal{N}$, such as $\pi_1 \circ \gamma = \mathcal{T}$ and $\pi_1|\gamma$ — homeomorphism:

$$
\pi_1|\gamma \in \text{Hom}(\gamma, \mathcal{T}).
$$

will be referred to as a family of elementary hyperbodies. In other words, elementary hyperbodies are inverse images of ordinary mass points’ world lines.

The Proposition 4.1 Elementary hyperbodies define $\mathfrak{P}$-structure in regard to the group of transformations $\mathcal{G}_T$.

Proof. Let $\mathcal{G}^c$ — group of invariance of elementarity properties. Let $\gamma$ — observable image of some elementary hyperbody in $\mathcal{N}$ and $\gamma_g \equiv g(\gamma)$ — its image under action of $g \in \mathcal{G}^c$. By definition of $\mathcal{G}^c$ and elementarity property we have the following chain of equalities:

$$
\mathcal{T} = \pi_1 \circ \gamma_g = \pi_1 \circ g \circ \pi_1|^{-1} \mathcal{T},
$$

that gives $\pi_1 \circ g \circ \pi_1|^{-1} \in \text{Aut}(\mathcal{T})$. It means, that $g$ transform fibers of simultaneous events to itself, i.e. $\mathcal{G}^c \subseteq \mathcal{G}_T$. Inverse inclusion follows from (10) together with continuity of projections and Lie group action.

In our case (9) — is a determining property of elementary hyperbodies and $\mathfrak{A} = \mathcal{T}, D = 1$. Elementary hyperbodies by definition allow us to consider physical hyperbodies $\{\cup_{\alpha} B_{\alpha}\}$.

2) The possibility of determining physical time is connected with the possibility of measuring it using special bodies with hyperbodies, whose observable evolution we call periodic. Let $\text{Per}(\mathcal{T})$ — a family of continuous periodic functions, determined on $\mathcal{T}$. Let us define a family of ”standard” hyperbodies $\{\mathcal{B}^{\text{per}}\}$, such as for any $\mathcal{B}^{\text{per}}$ and any $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}^{\text{per}}$, it occurs:

$$
\eta(b^T_1, b^T_2) \subseteq \text{Per}(\mathcal{T}),
$$

where $b^T_{1,2} = \pi_2(\mathcal{V}_T \cap f(B_{1,2}^T))$ for all $T \in \mathcal{T}$.

The Proposition 4.2 The family of standard hyperbodies $\{\mathcal{B}^{\text{per}}\}$ defines $\mathfrak{P}$-structure with regard to the group $A(1, R) \times \mathcal{G}_n \subseteq \mathcal{G}_T$, where $A(1, R)$

---

2Hereafter we assume that $\mathcal{N}$ has a structure of 4-dimensional differentiable manifold. It means, in particular, that we can use, if necessary, local real coordinates. Also, we assume, if not otherwise specified, that $\mathcal{G}$ — Lie group.
- nonhomogeneous affine group acting on coordinate space \( \tau : \mathcal{T} \to R \), and \( \mathcal{G}_\eta \leq \mathcal{G} \) - isometry group of metric \( \eta \).

**Proof.** Let for some physical body (11) takes place. It means, that left-hand side of (11) can be represented as the following Fourier row:

\[
\eta(b'_1, b'_2) = \sum_{n=0}^{\infty} a_n e^{in\omega \tau(T)},
\]

where \( \tau(T) \) — perceptive time coordinate, \( \omega \) — some main frequency. We can rewrite this relation in terms of transformed by \( g \) values: \( T' = g_0(T), b'^T = g_0(b^T) \):

\[
\eta(g^{-1}_0(b'_1)g^{-1}_0(T'), g^{-1}_0(b'_2)g^{-1}_0(T')) = \sum_{n=0}^{\infty} a_n e^{in\omega \tau(g^{-1}_0(T'))}.
\]

By definition of \( g \), together with rule \( \tau \circ g_0(\bullet) = \alpha \tau(\bullet) + \beta, (\alpha, \beta \) — affine group parameters) we obtain:

\[
\eta(b'_1, b'_2) = \sum_{n=0}^{\infty} a_n' e^{in\omega' \tau(T')}
\]

— relation, similar to (11). Here \( b'_1 = \pi_2(\mathcal{V}_{g^{-1}_0(T')} \cap f_0(\mathcal{B}^e_1)), a_n' = a_n e^{-in\omega \beta / \alpha}, \omega' = \omega / \alpha. \Box

3) We are fixing natural partial order "\( \prec \)" on manifold \( \mathcal{N} \) in the following way:

\[
P_1 \prec P_2 \iff \tau(P_1) \prec \tau(P_2) \quad \text{for all} \quad P_1, P_2 \in \mathcal{N},
\]

where \( \tau(P_1), \tau(P_2) \) — absolute (perceptive) time coordinates of events \( P_1, P_2 \), ordered as elements of \( R \).

**The proposition 4.3** Partial order "\( \prec \)" defines \( \mathfrak{P} \)-structure with regard to subgroup \( \mathcal{G}_T^+ \subset \mathcal{G}_T \), for which \( d\tau_{\mathcal{G}} / dr > 0 \).

**Proof.** Let \( \tau_g = \tau \circ \pi_1 \circ g \) — result of action of order group symmetry, representing in local coordinates. Using identity:

\[
f(x) - f(y) = \langle \overline{\partial f}, (x - y) \rangle,
\]

where \( x, y \) — points of \( R^n \), \( f \) — smooth function \( R^n \to R \), \( \overline{\partial f} \) — overaged over segment \( xy \) covector \( \partial f / \partial \xi_i, \langle, \rangle \) — standard pairing of forms and vectors, we get:

\[
\tau_g(P_1) - \tau_g(P_2) = \alpha_0(\tau_1 - \tau_2) + \alpha_i(X^i_1 - X^i_2) > 0,
\]

\((i = 1, 2, 3)\). Here \( \{X^i_{1,2}\} \) — local coordinates of \( P_1 \) and \( P_2 \) on \( \mathcal{V}, \alpha_\mu = \partial \tau_g / \partial X^\mu \), \( (\mu = 0, 1, 2, 3) \), \( X^0 = \tau \). If \( \alpha_i \neq 0 \), then by suitable choice of space perceptive coordinates \( \{X^i_{1,2}\} \) can be always violated. So, requirement of ordering conservation leads to the condition \( \alpha_i = 0 \). By continuity of partial derivatives it gives
\[ \frac{\partial g}{\partial X^i} = 0. \] It means, that we are restricted by \( \mathcal{G}_T \). For conservation of it is also necessary, that for all \( P_1 \) and \( P_2 \) the inequality \( \alpha_0 > 0 \) be satisfied. By continuity of partial derivatives it gives: \( \frac{d\tau g}{d\tau} > 0. \square \)

Note, that as against the previous \( \mathcal{P} \)-substructures, we are introducing time arrow here irrelatively of properties of any "asymmetrical in time" physical bodies and therefore time arrow is an inner property of an observer. This is mainly connected with well known fact that basic laws of classical mechanics are invariant with regard to time inversion.

Thus, complete \( \mathcal{P} \)-chron-structure of time is given by family of elementary hyperbodies and their unions with properties of continuity, periodicity, and time order. Appropriate common subgroup \( \mathcal{G} \leq \mathcal{G} \) is an intersection of all subgroups, defined by \( \mathcal{P} \)-substructures it is made up of and has a form of direct product \( A^+ (1, R) \times \mathcal{G}_\eta \).

### 4.2 \( \mathcal{P} \)-geometry of physical time

Direct product, obtained in previous paragraph, belongs to \( \mathcal{T} \)-class of diffeomorphisms and each multiplier is in itself a normal divisor of direct product. \( \mathcal{P} \)-chron-geometry, corresponding to the \( \mathcal{G}_\eta \), relates to 3-dimensional space of classical physics and turns out to be "very indistinct" at this stage. Experimental approach to a visual 3-geometry investigation we’ll consider in the second part of the paper. For the moment being we are going to concentrate on analyzing geometry of physical time, contained in affine group \( A^+ (1, R) \). Its group space is an open two dimensional manifold, homeomorphic \( R^+ \setminus \{0\} \times R \), with cut edge. For any two elements \( a, b \in A^+ (1, R) \) group composition law is:

\[ a \circ b = (a_1, a_2) \circ (b_1, b_2) = (a_1 b_1, a_1 b_2 + a_2). \]

Let us determine generating function \( \rho_0 \) for group \( A^+ (1, R) \). Isometry condition (6) becomes:

\[ \rho_0 (\xi x_1, \xi x_2 + \eta, \xi y_1, \xi y_2 + \eta) = \rho_0 (x_1, x_2, y_1, y_2), \] (14)

where \((\xi, \eta), (x_1, x_2), (y_1, y_2) \in A^+ (1, R)\). Differentiating both parts of (14) by \( \xi \) and \( \eta \), we are getting defining system of differential equations of the first order:

\[
\begin{align*}
&x_1 \frac{\partial \rho_0}{\partial x_1} + x_2 \frac{\partial \rho_0}{\partial x_2} + y_1 \frac{\partial \rho_0}{\partial y_1} + y_2 \frac{\partial \rho_0}{\partial y_2} = 0; \quad \frac{\partial \rho_0}{\partial x_2} + \frac{\partial \rho_0}{\partial y_2} = 0.
\end{align*}
\]

Their common solutions respectively are:

\[ \rho_0 = f \left( \frac{x_2}{x_1}, \frac{y_1}{x_1}, \frac{y_2}{x_1} \right); \quad \tilde{\rho}_0 = h(x_1, y_1, x_2 - y_2), \]

where \( f \) and \( h \) — arbitrary differentiated functions. Uniting them, we finally get:

\[ \rho_0 = \rho_0 \left( \frac{y_1}{x_1}, \frac{x_2 - y_2}{x_1} \right). \] (15)

\[ ^3 \] Of course, we can say that the observers themselves are a special family of "bodies", each member of which is capable of inducing time arrow (see also \[13\]).
Consider two independent generating functions:

\[ \rho_0^1 = \frac{y_1}{x_1}; \quad \rho_0^2 = \frac{x_2 - y_2}{x_1} \]

and check them for metricity.

**The Proposition 4.4** Generating function \( \rho_0^1 \) satisfies metricity condition (7) only on sections \( \text{const} \times R \) of group space \( A^+(1,R) \), where \( \rho_0^1 \equiv 1 \).

**The Proposition 4.5** Generating function \( \rho_0^2 \) satisfies metricity condition (7) in two and only two cases:

1. on sections \( \text{const} \times R \), with \( \rho_0^2 = \text{const} \cdot (y_2 - x_2) \).
2. on sections \( R^+ \times \text{const} \), with \( \rho_0^2 \equiv 0 \).

Excluding trivial cases \( \rho = \text{const} \), we are getting, basing on propositions of section 3 to the following general expression of physical \( \Psi_{\text{chron}} \)-metrics:

\[ \rho(t_1, t_2) = \varphi(|t_2 - t_1|), \quad (16) \]

where \( \varphi \) — differentiable function, satisfying condition of proposition 3.4. Most important in practical sense and having technically the simplest construction, metrics class from (16) is given by:

\[ \rho(t_2, t_1) = C|t_2 - t_1|^\alpha, \quad 0 < \alpha \leq 1. \quad (17) \]

From the physical viewpoint constant \( C \) defines physical units of time. Linear Euclidean case, commonly used in classical mechanics is obtained from (17) under \( \alpha = 1 \).

## 5 Experimental research of visual manifold geometry

### 5.1 Experiment description

The idea of determining quantitative characteristics of visual manifold geometry\(^4\) is simple: it is necessary to find correlation between geometrical characteristics’ estimates by statistically considerable number of observers for given objects and true characteristics of these objects, determined by measuring instruments. Despite its simplicity, practical realization of this idea is quite a labor consuming problem. It is worth noting here just a few general problems arising on our way along with some significant details of the carried out experiment.

\(^4\)Instead of symbolic term ”space” we are going to use more exact term ”manifold” in the second part.
1. In visual manifold geometry, there is a natural length scale $\ell$ — observer’s own size. Because of this, whole visual manifold is naturally divided into charts $\{V^\text{near}_i\}$, $\{V^\text{mid}_j\}$, $\{V^\text{far}_k\}$, covering near ($d \ll \ell$) middle ($d \gtrsim \ell$) and far ($d \gg \ell$) zones correspondingly, where $d$ — characteristic distance from observers’ eyes to some internal point within a map. In our experiment we, due to technical reasons, have limited ourselves to middle zone: $d \sim 1.5\text{m}–15\text{m}$.

2. Complete research of visual manifold’s geometrical properties comprises of studying its at least three relatively independent aspects: topology, connection and metrics. In the present work we have only studied metric properties, since they do not require sophisticated equipment for their measurement and are more common.

3. Since visual manifold is a part of observer’s united perceptive manifold, it is natural to expect that geometrical characteristics of an observed object will be influenced by the properties of those observed objects that pertain to other subspaces of perceptive space (for example, object’s color which is object’s non-geometrical characteristic). To exclude those "undesirable interactions" of different perceptive subspaces from our work, we have prepared the artificial objects of unified format — 8 blue rectangles on white background differing in sizes and proportions. They have been shown to the probationers — "observers" — under relatively the same conditions (university’s auditorium) at various distances. The exact dimensions of the rectangles are given in Table 1 of Appendix. The observers have been proposed to estimate the samples dimensions and the distance to them. After that, they have been asked to enter their estimates into special card.

4. To eliminate subjectivity at estimating objects’ linear sizes, it’s been necessary to question quite a number of the "observers" to smoothen individual perception peculiarities after averaging data and, on the contrary, to make common characteristics of visual manifold geometry appear. Due to technical reasons, only 80 observers have been involved in our experiment. As our study has shown, this value is, in fact, a lower limit at which general laws of perception start to show. When processing the results, we have ignored the data, obtained from the same individual, but severely fluctuated around object’s true dimensions and distances to them. We have interpreted the situations like this as influence by purely psychological factor — such "observers" have not been very responsible to our experiment.

5. It is well known to neurophysiologists that different areas of brain crust are responsible for perception of vertical and horizontal dimensions. This

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\textsuperscript{5}We are assuming here that characteristic map size is much less than a distance between a map and observers’ eyes.

\textsuperscript{6}The majority of “observers” have been graduate students of YSPU who get a little tired at the end of the day. Fortunately, we have faced the situation described above only a few times.
tips to the idea that metric properties of visual manifold can discover anisotropy. To be able to study it (anisotropy), we have distinguished (for ourselves) between objects’ orientation and their vertical and horizontal dimensions.

In the course of experiment, the students have been divided into groups each consisting of 16 people. Each group’s members have occupied one row consisting of 8 tables each. The tables have been at fixed distances from the showed rectangles. These distances can be found in Table 2 of the Appendix. All students have been given the cards. The cards had objects’ numbers on them and opposite each object’s number there have been three blank spaces designated by letters V, H, and D standing for ”estimated vertical dimension”, ”estimated horizontal dimension”, ”estimated distance to the object” correspondingly. Having been shown an object and having fixed its’ dimensions in the card, the students circularly changed their seats according to the following scheme: 1 → 2 → 3 → 4 → 5 → 6 → 7 → 8 → 1. Thus, each group has finally given two estimates on each object’s characteristic and at each distance to the object. We have overall tested about five such groups so on each object’s characteristic and at each distance to the object we got sampling consisting of 10 estimates we could later do averaging on. In the Appendix the Table 3 of the averaged experimental data is presented.

Basing on experimental data summarized in the table, we have drawn the dependencies \( \vec{X}(\vec{x}) \), where \( \vec{X} = \{ H, V, R \} \) and \( \vec{x} = \{ h, v, r \} \) — vectors of estimated and true characteristics — heights, widths, and distances to the objects correspondingly. The interpretation of the obtained dependencies has been done on the basis of affine model of visual manifold geometry. It is assumed within the frames of this model that the averaged data, obtained from the students, are linear functions on objects’ true characteristics:

\[
X_i = k_i r + X_{i0}, \quad i = 1, 2, 3,
\]

where \( k_i \) — linear radial overstating coefficient of width \((i = 1)\), height \((i = 2)\) and of a distance to the object \((i = 3)\). Parameter \( X_{i0} \) — ideal parameter, having a sense of correspondent dimension estimate ”at zero distance”. In some sense it is more convenient to analyze a behavior of differences \( \Delta_i = X_{i0} - x_i \) and dimensionless ratios \( \varepsilon_i = \Delta_i / X_{i0} \). For the most dependencies it is possible to introduce one important characteristic \( r^* \) — ideal distance, defined from the equation:

\[
\vec{X}(r^*) = \vec{X}_{true},
\]

i.e. such a distance at which objects’ estimated characteristics coincide with the true ones. Finally, let us introduce one more notion — radial anisotropy coefficient: \( \sigma = 1 - k_2 / k_1 \), where \( k_1 \) — radial overstatement coefficient for objects’ horizontal characteristics, and \( k_2 \) — for the vertical ones.

All of the obtained characteristics for linear model are shown in overall Table 4 in the Appendix. Let us analyze the acquired results setting up the following rather natural assumptions from our daily experience as trial hypothesis:
1. For objects having sides proportion near to 1 and large area, their dimensions are overstated, and the distances to them are understated;

2. The dimensions of little objects at long distances are understated, and the distances from them — are overstated;

3. Perception of one side for substantially anisotropic objects influences the perception of the other;

4. It is possible that there exist some stable proportions, perception of which is sufficiently different from that of the other proportions.

Basing on table 4 data, it can be checked whether the above statements are right. First two lines of the table clearly show that despite hypothesis 1 and 2 radial sizes overestimation coefficient is positive for all objects except the object 6. This coefficient is particularly great for objects 3, 4, 7. The difference between vertical and horizontal overestimation coefficients for anisotropic objects 3, 6 and 7 verifies hypothesis 3 and corresponds to its particular case when greater side is being more times overestimated. For the symmetric object 4 overestimation coefficients practically coincide (it supports hypothesis 1 and 3).

Practically exact coincidence of overestimation coefficients for symmetrical (or nearly symmetrical) objects 4 and 8 is natural. Objects 1 and 2 also reflect aforementioned variant of hypothesis 3 but to weaker degree. Objects 5 and 6 are exclusions. First of them has a ratio between overestimation coefficients which corresponds to inverse variant of hypothesis 3: greater side is being overestimated by fewer times. The second one has been underestimated in both vertical and horizontal sizes, and, what is interesting, this is according to direct variant of hypothesis 3. It’s worth noting that both objects have largest areas (approximately 103 cm$^2$ and 140 cm$^2$ correspondingly) and proportions of the first are close to 3:2 (exactly 1.57), and the second’s — to golden section $(3 - \sqrt{5})/2 \approx 0.381$ (exactly 0.369). We are going to make sure further that objects 5 and 6 possess a number of other ”strange” properties, apparently according to hypothesis 1 and 4.

The table’s bottom line reflects the above mentioned as well. Besides, in this line there have been unexpectedly strong anisotropy for perception of vertical object 7 having sides proportions close to 1:2.

As against radial overestimation coefficients of objects’ sizes, pertaining to their picture plane, radial overestimation coefficients of distances $k_3 - 1$ are all negative but the smallest object 8. In full accordance with hypothesis 1, the objects 3, 5, 6 with largest areas, have maximum underestimation coefficients. Objects 1, 7, 8 with smallest areas, have minimum underestimation coefficients, where object 8 verifies hypothesis 2.

Estimates at zero distance $\Delta_i$ all turn out to be either negative or null. In conjunction with the first two lines, this implies that nearly all objects have positive ”ideal distance”. Various dimensionless estimates $\varepsilon_i$, correspond to anisotropic objects, and isotropic objects are being corresponded by ”close” ones. Objects 5 and 6 are again the exclusions. The first one has $\varepsilon_1 > 0, \varepsilon_2 < 0,$
and the second — $\varepsilon_1 = 0, \varepsilon_2 = 0.12$. This anomaly, along with the first two, leads to anomalously big negative "ideal distances" for object 5 and to anomalously big positive ones for object 6.

Parameter $R_0$ fluctuates around zero and is insignificant for all objects.

Symmetrical (4,8) or almost symmetrical (1,7) objects obviously have an "ideal distance" (4-5 m), at which the perception of their vertical and horizontal dimensions approximately corresponds to their real values. It is worth mentioning that inaccuracy in radial perception remains. In accordance with hypothesis[3] the most significant divergence in ideal distances takes place for anisotropic objects: 2 and 3 and anomalous object 6. Anomalous object 5 "has no" horizontal ideal distance at all. As to ideal distance to estimate the distances themselves, it does exist (i.e. notable exceeds zero) only for the smallest object 8 and equals, surprisingly, to 28m.

Thus, we can conclude for all radial dependencies with the following remarks which substantially specify and complete our initial a priori hypotheses:

1. Estimates for distances to the objects mostly satisfy hypothesis[1] and[2].

2. Estimates for horizontal and vertical dimensions do not generally satisfy hypothesis[1] and[2] in a sense that almost all radial overestimation coefficients exceed zero and satisfy hypothesis in other sense: big dimensions are overestimated to a bigger extend and the smaller ones — to a smaller, according as well to general hypothesis[3].

3. Objects 5 and 6, having proportions 3:2 and "golden section" (and probably 7 with proportion 1:2), and largest areas, fall out of general law by a number of factors and evidently play a special role in geometry of perception in accordance with hypothesis[3].

4. There is an ideal distance (4-6m) for symmetrical or almost symmetrical objects. At this distance, their perception adequately reflects their real dimensions.

5. The perception of extended objects is distorted at any distances.

Let us briefly describe the characteristics of visual manifold basing on other relations obtained from Table 4. The relations between perceived proportions of sides and their true proportions are for all distances well approximated by linear function with a coefficient being a little less or equal to 1. Since, proportion is a ratio of horizontal dimension to vertical one, the result we have obtained points to the fact that in general, vertical and horizontal perception "distortions" are a little bit different. Horizontal dimensions are on the average overestimated a little less than the vertical.

3D graphs of perceptive width and height dependencies on their true values are presented in Figure 1 (see Appendix). Comparison of this two graphs support conclusion about anisotropy of visual manifold. In first, distortion of vertical sizes perception are expressed more strongly, than distortion of horizontal sizes perception. This is illustrated by horizontal sections of these surfaces.
in Figure 2. In second, shape of distortions of vertical and horizontal perceptive geometries from Euclidean geometry are different. We illustrate it in Figure 3. It present two pairs of 2-dimensional dependencies: experimental (bright) taken from Figure 1, and Euclidean (dark) of type \( H = h \quad V = v \). Boundaries of bright and dark areas — "ideal" curves, i.e. going through a set of parameters, for which true and perceptive objects’ characteristics coincide. We see, that that while horizontal characteristics are described by a weakly up-convex "perception surface", vertical characteristics are described by apparently down-convex "perception surface". Thus, we conclude, that vertical estimates have no "ideal" range for perception (height) (are always overestimated), while horizontal ones this "ideal" range is about 3-5m and horizontal sizes about 5-15cm.

We would like to stress that all our conclusions are approximate and require further correction (for example, by increasing testers’ number).

6 Conclusion

We have made an attempt basing on Poincare’s ideas to find a tie connecting perceptive space geometry, physical objects, and physical geometry which is used when formulating and analyzing the laws of physics. Basic ideas of this approach have been demonstrated using physical chronogeometry as an example. Experimental research of visual perceptive space geometry’s peculiarities has shown its’ non trivial character even within the limited area (middle zone) that attracted our attention. Obviously, our affine model should be considered as linear approximation of more general, nonlinear model which is necessary for obtaining complete geometrical picture of the visual perceptive space.

Let us make some general remarks in conclusion.

1. As it can be seen from the basic points of the approach being set forth, observer’s conception is necessary for building physical picture of the world even at the level of classical physics. Without an observer — his Newton’s mapping, world and perceptive manifolds remain absolutely detached and isolated. In other words, an observer is built in the surrounding world in such a way that he is not only (and not so much) a passive spectator but an active participant in forming and uncovering laws of physics.

2. Despite this circumstance, and in spite of the thing that the geometry of perceptive space should apparently play its role at any stage of our physical reasoning, the laws of the nature can be formulated in such a manner as if there was no ubiquitous geometry-mediator at all. For instance, when deducing chronogeometry metrics \( [17] \), at the interim stage of the argumentation we only used a fact of existence of some perceptive geometry, but its specific properties turned out to be unsubstantial for deducing formula \( [17] \). Evidently, this situation is typical for all of the laws of classical dependencies, represented in Figure 3 are nonlinear, so in what follows we discuss some general features of nonlinear perceptive space model.
physics of XIX century, in which the role of the observer is disguised by the notions of absolute space, time and a number of others. Some sort of the physical laws' dependence on an observer starts to reveal itself in SR and GR. The role of the observer in these theories is played by the notion of the reference frame\textsuperscript{[14]}\textsuperscript{8}. Likewise, within Maxwell’s electrodynamics in Minkowski space-time, angular velocity of the reference frame’s rotation can imitate magnetic charges’ density; in Friedmann-Robertson-Walker’s cosmology, the observed volume of the universe can be both finite and infinite depending on reference frame even within the frames of the same cosmological model. Though, covariant approach, reflecting, apart from invariance idea, some general scientific-philosophical aims of the modern scientific thinking, again takes observer’s perceptive space out of context when formulating the laws of relativity physics. The further development of our approach into the field of quantum phenomena will probably enable to attribute some part of "strangenesses" and oddities of microworld to the observer’s perception geometry (see, for example, \textsuperscript{[12]}). The detailed evolution and development of our approach into relativity physics area could as well serve as a foundation for the new interpretation of some observation cosmology facts.

3. In the present article we have just touched upon space-time aspect of perceptive space. The said space, taking into account the full range of sensations and perceptions, is much broader and includes auditory, haptic, motor, gustatory, olfactory, thermal and a number of the other subspaces which, in addition, are connected with each other by the complex, as a matter of fact, non-functional dependencies. We are convinced that the combined research of the perception space by the methods of mathematics, physics and traditional sciences about a human being could be a durable basis for building up a uniform language for describing different phenomena of the outer (surrounding) world and a man — as its in many respects unique representative.

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A  Figures

Figure 1. 2-dimensional dependencies of perceptive parameters (width and height) on correspondent true parameters and distant to the objects.

Figure 2. Horizontal sections, corresponding to dependencies in Figure 1. They shows, that vertical perceptive geometry differ from Euclidean geometry in more extent, than horizontal one.

Figure 3. Sections of experimental dependencies (bright surfaces) by planes $V = v$ and $H = h$ respectively (dark surfaces), describing Euclidean perceptive geometry.

B  Data tables

Table 1. Rectangular’s sizes.
| Number | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|--------|----|----|----|----|----|----|----|----|
| Sizes, cm | 5,4×2,2 | 0,9×10,2 | 23,2×2,6 | 6,5×6,1 | 12,7×8,1 | 7,2×19,5 | 4,5×9,4 | 1,7×1,7 |

**Table 2.** Set of distances in a middle zone.

| Number | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|--------|----|----|----|----|----|----|----|----|
| Distances, m | 0.8 | 2.2 | 3.6 | 4.95 | 6.25 | 7.5 | 8.75 | 9.9 |

**Table 3.** Averaged experimental data.

| Number | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|--------|----|----|----|----|----|----|----|----|
| w-1    | 5.2 | 5.2 | 4.6 | 5.4 | 5.4 | 5.7 | 5.6 | 5.9 |
| v-1    | 2.0 | 2.2 | 1.8 | 2.3 | 2.4 | 2.3 | 2.5 | 2.9 |
| d-1    | 0.7 | 1.8 | 3.0 | 4.0 | 5.4 | 7.1 | 8.8 | 9.2 |
| w-2    | 1.0 | 0.9 | 1.2 | 1.2 | 1.2 | 1.5 | 1.4 | 1.4 |
| v-2    | 10.1 | 8.9 | 9.1 | 9.6 | 10.5 | 10.0 | 10.1 | 9.9 |
| d-2    | 0.8 | 1.6 | 3.1 | 4.3 | 5.9 | 7.2 | 6.9 | 9.5 |
| w-3    | 20.9 | 20.4 | 21.5 | 24.7 | 22.6 | 24.2 | 22.1 | 22.2 |
| v-3    | 2.6 | 2.7 | 2.5 | 2.6 | 2.7 | 2.7 | 3.4 | 2.4 |
| d-3    | 0.8 | 1.6 | 3.0 | 5.0 | 6.0 | 6.2 | 8.3 | 7.8 |
| w-4    | 5.6 | 6.3 | 6.7 | 6.2 | 5.7 | 6.8 | 7.5 | 7.7 |
| v-4    | 5.6 | 6.3 | 6.7 | 6.2 | 5.7 | 6.8 | 7.5 | 7.7 |
| d-4    | 0.8 | 1.6 | 3.0 | 5.0 | 6.0 | 6.2 | 8.3 | 7.8 |
| w-5    | 13.3 | 13.8 | 14.4 | 13.4 | 12.8 | 16.0 | 12.8 | 14.2 |
| v-5    | 7.7 | 8.7 | 8.9 | 7.9 | 7.7 | 8.9 | 8.1 | 9.6 |
| d-5    | 0.8 | 2.0 | 3.3 | 4.0 | 6.0 | 6.6 | 8.2 | 8.2 |
| w-6    | 7.7 | 7.9 | 5.7 | 6.8 | 7.0 | 6.1 | 7.1 | 7.5 |
| v-6    | 22.6 | 23.5 | 19.5 | 20.3 | 21.0 | 19.1 | 21.0 | 23.2 |
| d-6    | 0.8 | 2.0 | 3.0 | 4.6 | 5.0 | 7.1 | 7.0 | 9.5 |
| w-7    | 3.7 | 4.1 | 4.0 | 3.9 | 3.8 | 4.3 | 4.3 | 5.7 |
| v-7    | 8.5 | 8.9 | 9.7 | 10.5 | 8.8 | 10.8 | 10.8 | 13.0 |
| d-7    | 0.8 | 1.9 | 3.2 | 4.2 | 5.7 | 5.8 | 8.1 | 10.3 |
| w-8    | 1.4 | 1.6 | 1.4 | 1.6 | 1.8 | 1.8 | 2.2 | 2.0 |
| v-8    | 1.4 | 1.6 | 1.4 | 1.5 | 1.8 | 1.7 | 2.2 | 2.0 |
| d-8    | 0.7 | 2.0 | 3.2 | 4.5 | 4.6 | 7.0 | 9.0 | 10.1 |

**Table 4.** Main number values of an affine model of perceptive geometry.

| Number | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|--------|----|----|----|----|----|----|----|----|
| k₁₁, 10⁻² | 9  | 5.6 | 22 | 18.8 | 4.8 | -4.1 | 13.8 | 8.3 |
| k₂₂, 10⁻² | 7.8 | 7.7 | 3.1 | 18.7 | 9.1 | -9.4 | 38.6 | 8.0 |
| k₃₁ - 1 | -0.022 | -0.084 | -0.146 | -0.083 | -0.137 | -0.1 | -0.027 | 0.021 |
| H₀ - h, cm | -0.5 | 0.0 | -2.1 | -1.0 | 0.9 | 0.0 | -1.0 | -0.5 |
| v₀ - v, cm | -0.3 | -0.9 | -0.1 | -0.6 | -0.2 | 2.3 | -1.4 | -0.5 |
| ε₁, 10⁻² | -9 | 0 | -9 | -15 | 7 | 0 | -22 | -29 |
| ε₂, 10⁻² | -14 | -9 | -4 | -10 | -2 | 12 | -15 | -29 |
| R₀, m | -0.48 | -0.25 | 0.05 | -0.11 | 0.03 | -0.18 | -0.48 | -0.60 |
| r₁ *, m | 5.8 | -0.2 | 9.8 | 5.3 | -18 | -0.7 | 7.4 | 5.5 |
| r₂ *, m | 4.2 | 10.5 | 3.0 | 3.7 | 1.9 | 24 | 3.8 | 5.7 |
| r₃ *, m | -22 | -3 | 0.3 | -1.35 | 0.24 | -1.8 | -18 | 28 |
| σ | 0.13 | -0.38 | 0.86 | 0.005 | -0.9 | -1.3 | -1.8 | 0.04 |