THE PICARD GROUP OF SIMPLY CONNECTED REGULAR VARIETIES AND STRATIFIED LINE BUNDLES

LARS KINDLER

Abstract. We prove that the Picard group of a regular simply connected variety over an algebraically closed field of arbitrary characteristic is finitely generated. The main difficulty to overcome is the unavailability of resolution of singularities. From this we deduce that in positive characteristic there exist no nontrivial stratified line bundles on such a variety, and we present a complex analog.

1. Introduction

Let $k$ be an algebraically closed field of characteristic $p \geq 0$, and $U$ a regular connected $k$-scheme of finite type. Assume that $U$ is simply connected, i.e. $\pi_1^\text{ét}(U, \bar{u}) = 1$ for some geometric point $\bar{u}$ of $U$. If there is a dominant open immersion of $k$-schemes $\iota: U \rightarrow X$, for some regular proper $k$-scheme $X$, then $\text{Pic} U$ is finitely generated. In fact, by the regularity assumptions we have surjections $\text{Pic} X \rightarrow \text{Pic} U$ and $\pi_1(U, \bar{u}) \rightarrow \pi_1(X, \iota \bar{u})$, so $X$ is also simply connected and it suffices to show that $\text{Pic} X$ is finitely generated. By the properness of $X$ the relative Picard functor for $X/k$ is representable by a $k$-group scheme $\text{Pic}_{X/k}$ locally of finite type, and the connected component of the origin with its reduced structure $\text{Pic}_{0,X/k}^{\text{red}}$ is an abelian variety. The Kummer sequence in étale cohomology then shows that $\text{Pic}_{0,X/k}^{\text{red}}[\ell^n] = \text{hom}_{\text{cont}}(\pi_1(X, \iota \bar{u}), \mathbb{Z}/\ell^n \mathbb{Z}) = 1$ for any prime $\ell \neq p$. This implies that $\text{Pic}_{0,X/k}^{\text{red}}(k)$ is trivial, as for an abelian variety $A$ of dimension $g$ over $k$, we know that $A[\ell^n] \cong (\mathbb{Z}/\ell^n \mathbb{Z})^{2g}$. Hence the Picard group $\text{Pic}(X) = \text{Pic}_{X/k}^{\text{red}}(k)$ is equal to the Néron-Severi group $\text{NS}(X) := \text{Pic}_{X/k}(k)/\text{Pic}_{0,X/k}^{\text{red}}(k)$, which is finitely generated.

The first main result of this note is a generalization of the above fact, proven in Section 3.

Theorem 1.1. Let $k$ be an algebraically closed field of characteristic $p \geq 0$. If $U$ is a connected, regular, separated $k$-scheme of finite type, and if the maximal abelian pro-$\ell$ quotient $\pi_1(U)^{\text{ab},(\ell)}$ is trivial for some $\ell \neq p$, then $\text{Pic} U$ is finitely generated.

Note that it is not known in general, whether $U$ as in the theorem can be embedded into a regular proper $k$-scheme, because resolution of singularities is not known to hold. To circumvent this, we use de Jong’s theory of alterations and simplicial techniques. More precisely, in Section 2 we study the simplicial Picard group and the simplicial Néron-Severi group of a simplicial scheme $X_\bullet$, and prove finiteness statements similar to the nonsimplicial case. This relies on results of [Ram01] and [BVS01]. We apply these statements in Section 3 to prove the theorem.

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As an application, we study in Section 4 stratified line bundles on $U$, i.e. line bundles coming with a $\mathcal{D}_{U/k}$-action, where $\mathcal{D}_{U/k}$ is the sheaf of differential operators of $X$. Extending an argument from [EM10], we prove the following statement.

**Theorem 1.2.** Let $k$ be an algebraically closed field of characteristic $p > 0$. If $U$ is a regular, connected $k$-scheme of finite type such that the maximal abelian pro-prime-to-$p$ quotient $\pi_1(U)^{ab,(p')}$ of $\pi_1(U, \bar{u})$ is trivial, then every stratified line bundle on $U$ is trivial.

In [EM10], this theorem is proven for stratified bundles of arbitrary rank, under the additional assumptions that $U$ is projective and $\pi_1(U) = 1$.

Proving this theorem was the original motivation to study Theorem 1.1, and it originated out of work for my thesis, in which possible extensions of the results of [Em10] to non-projective varieties are studied.

Finally, in Section 5 we reproduce an argument of Hélène Esnault, showing that a complex analog of Theorem 1.2 is “not quite” correct. More precisely, we show that on a simply connected, regular, complex variety $X$, the additional assumptions that $U$ is a proper, simplicial $X$-scheme, and $X$ is of the form $(\mathcal{O}_U, d + \omega)$, with $\omega$ a closed 1-form. The reason for this discrepancy is that in positive characteristic, simply connectedness is a “stronger” condition than in characteristic 0, see Remark 5.5.

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## 2. Simplicial Picard Groups

For background on the simplicial techniques used in this section, we refer to [De74].

In [BVS01, 4.1] and [Ram01, 3.1], the simplicial Picard group and the simplicial Picard functor are defined as follows:

**Definition 2.1.** Let $S$ be a scheme. If $\delta_i^k : X_i \to X_{i-1}$ denote the face maps of a simplicial $S$-scheme $X_\bullet$, then $\text{Pic}(X_\bullet)$ is defined to be the group of isomorphism classes of pairs $(L, \alpha)$ consisting of a line bundle $L$ on $X_0$ and an isomorphism $\alpha : (\delta_i^k)^*L \to (\delta_i^k)^*L$ on $X_1$, satisfying cocycle condition $(\delta_0^2)^*(\alpha) - (\delta_1^2)^*(\alpha) = (\delta_1^3)^*(\alpha)$ on $X_2$. The simplicial Picard functor $\text{Pic}_{X_\bullet/S}$ is obtained by fpqc-sheafifying the functor $T \mapsto \text{Pic}(X_\bullet \times_S T)$.

It turns out that $\text{Pic}(X_\bullet)$ is canonically isomorphic to $\mathbb{H}^1(X_\bullet, \mathcal{O}_{X_\bullet}^{\times})$ and to the group of isomorphism classes of invertible $\mathcal{O}_{X_\bullet}$-modules, see [BVS01, A.3]. We will use the following representability and finiteness statements.

**Theorem 2.2.** Let $k$ be an algebraically closed field, $X$ a proper $k$-scheme of finite type, and $X_\bullet$ a proper, simplicial $k$-scheme of finite type (which means that all the $X_n$ are proper, and of finite type over $k$).

1. The relative Picard functor associated to $X \to \text{Spec} k$ is representable by a separated commutative group scheme $\text{Pic}_{X/k}$, locally of finite type over $k$, which is the disjoint union of open, quasi-projective subschemes, see [SGA6, Cor. XII.1.2].

2. The Néron-Severi group $\text{NS}(X) = \text{Pic}_{X/k}(k)/\text{Pic}_{X/k}^0(k)$ is a finitely generated abelian group, see [SGA6, Thm. XIII.5.1].

3. If $X$ also normal, then the connected component $\text{Pic}_{X/k}^0$ of the origin is projective, see [Kle05, Thm. 5.4, Rem. 5.6], so passing to the reduced structure $\text{Pic}_{X/k}^{0,\text{red}}$ gives an abelian variety.

4. The simplicial Picard functor is representable by a group scheme $\text{Pic}_{X_\bullet/k}$, locally of finite type over $k$, see [Ram01, Thm. 3.2].
Proposition 2.3. Let \( k \) be an algebraically closed field. For a proper reduced simplicial \( k \)-scheme \( X_\bullet \), with \( X_n \) of finite type for all \( n \) and \( X_0 \) normal, the simplicial Néron-Severi group \( \text{NS}(X_\bullet) := \text{Pic}(X_\bullet/k)/\text{Pic}^0_{X_\bullet/k}(k) \) is finitely generated.

Proof. Let \( \tau : X_\bullet \to \text{Spec } k \) denote the structure morphism. The spectral sequence (see e.g. [Del74, (5.2.3.2)])

\[
E_1^{pq} = H^q(X_p, \mathcal{O}_{X_p}) \implies H^{p+q}(X_\bullet, \mathcal{O}_{X_\bullet})
\]
gives rise to the exact sequence

\[
0 \to E_2^{1,0} \to \mathbb{H}^1(X_\bullet, \mathcal{O}_{X_\bullet}) \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0}
\]
and thus, as in [Ram01] p. 284, after sheafifying we get an exact sequence of fpqc-sheaves (in fact group schemes by the representability of \( \text{Pic}^0_{X_\bullet/k} \))

\[
0 \to T \to \text{Pic}^0_{X_\bullet/k} \to K \xrightarrow{d_2} W,
\]
where \( K := \ker(\delta_0^* - \delta_1^*: \text{Pic}^0_{X_0/k} \to \text{Pic}^0_{X_1/k}) \).

\[
T := \ker((\tau_1)_* \mathbb{G}_{m,X_1} \xrightarrow{\delta_0^* - \delta_1^* + \delta_2^*} (\tau_2)_* \mathbb{G}_{m,X_2}) \xrightarrow{\operatorname{im}((\tau_0)_* \mathbb{G}_{m,X_0} \xrightarrow{\delta_0^* - \delta_1^*} (\tau_1)_* \mathbb{G}_{m,X_1})}
\]
and \( W \) is affine. The scheme \( T \) is an affine \( k \)-scheme with finitely many connected components, and a \( k \)-torus as neutral component (compare [Ram01] top of p. 284). This follows from the fact that \( \tau_n \) is proper and \( X_n \) reduced, since this implies that for every \( k \)-scheme \( S \) we have \( \tau_n_* \mathbb{G}_{m,X_n}(S) = \mathcal{O}_{X_n \times_k S}(X_n \times_k S) = \mathbb{G}_{m,k}(S)^{\tau_n}(X_n) \), see [EGAI3 Prop. 7.8.6].

As \( X_0 \) is normal, \( \text{Pic}^0_{X_0/k} \) is an abelian variety, and hence so is \( K^0 \). Since \( W \) is affine, any homomorphism \( K^0 \to W \) is trivial, so \( K^0 \subset \ker(d_2^0) \). But \( \ker(d_2^0) \subset K^0 \), so we have equality. Moreover, \( \text{Pic}^0_{X_\bullet/k} \) maps surjectively to \( K^0 \). This shows that the kernel of the map

\[
\text{Pic}^0_{X_\bullet/k}(k)/\text{Pic}^0_{X_\bullet/k}(k) = \text{NS}(X_\bullet) \to K(k)/K^0(k)
\]
is

\[
\frac{T(k)\text{Pic}^0_{X_\bullet/k}(k)}{\text{Pic}^0_{X_\bullet/k}(k)}
\]
because, if \( L \) maps to \( M \in K^0(k) \), then there is some \( L^0 \in \text{Pic}^0_{X_\bullet/k}(k) \) also mapping to \( M \), and the difference is in \( T(k) \).

As \( T \) has only finitely many connected components, this kernel is finite.

The group \( K(k)/K^0(k) \) maps to the group of connected components \( \text{NS}(X_0) \) of \( \text{Pic}^0_{X_0/k} \). The kernel of this map is \( (\text{Pic}^0_{X_\bullet/k}(k) \cap K(k))/K^0(k) \), which is finite, as \( \text{Pic}^0_{X_\bullet/k}(k) \cap K \) has only finitely many connected components, and the subgroup \( K^0 \subset \text{Pic}^0_{X_\bullet/k} \) is the neutral component of \( \text{Pic}^0_{X_\bullet/k} \cap K \). Hence \( K(k)/K^0(k) \) is finitely generated, because \( \text{NS}(X_0) \) is finitely generated by Theorem [2.2] This shows that \( \text{NS}(X_\bullet) \) is finitely generated.

\[\square\]
Recall that to an \( X \)-scheme \( X_0 \) one can associate an \( X \)-augmented simplicial scheme \( \cosk_0(X_0)_\bullet \), the \( 0 \)-coskeleton, defined by taking \( \cosk_0(X_0)_n \) to be the \( n \)-fold fiber product of \( X_0 \) over \( X \), with the necessary maps given by the various projections (resp. diagonals) to (resp. from) \( \cosk_0(X_0)_{n-1} \). The \( 0 \)-coskeleton has the following universal property: If \( Y_\bullet \) is a simplicial scheme with augmentation to \( X \), then there is a bifunctorial bijection \( \hom_X(Y_0, X_0) \cong \hom_X(Y_\bullet, \cosk_0(X_0)_\bullet) \).

In particular, if \( X_\bullet \) is a simplicial scheme, then there is a unique simplicial \( X \)-morphism \( \gamma : X_\bullet \to \cosk_0(X_0)_\bullet \), with \( \gamma_0 = \text{id}_{X_0} \). For the general construction see, e.g., [Del74, 5.1.1].

**Lemma 2.4.** If \( \tau : X_\bullet \to X \) is an augmented simplicial \( k \)-scheme such that \( X_n \) is separated and of finite type over \( k \) for all \( n \), and \( \gamma : X_\bullet \to \cosk_0(X_0)_\bullet \) the morphism of augmented simplicial schemes such that \( \tau_0 = \text{id}_{X_0} \), then the kernel of the induced morphism \( \Pic(\cosk_0(X_0)_\bullet) \to \Pic X_\bullet \) is finitely generated.

**Proof.** We have the following situation:

\[
\begin{array}{c}
g_0 : X_\bullet \longrightarrow \cosk_0(X)_\bullet \\
\end{array}
\]

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\gamma_1} & X'_{0} \\
\downarrow \delta_i & & \downarrow p_i \\
X_0 & \xrightarrow{\gamma_0 = \text{id}} & X_0 \\
\downarrow & & \downarrow \\
X & \xrightarrow{} & X
\end{array}
\]

where \( X'_0 := X_0 \times_X X_0 \) and \( p_i \) the projection to the \( i \)-th factor.

If \( (L, \alpha : p_1^*L \cong p_2^*L) \in \Pic(\cosk_0(X_0)_\bullet) \) pulls back to the trivial element in \( \Pic(X_\bullet) \), then there is some isomorphism \( \beta : L \to \mathcal{O}_{X_0} \), such that the diagram

\[
\begin{array}{ccc}
\delta_1^*L & \xrightarrow{\gamma_1 \alpha} & \delta_1^*L \\
\downarrow \delta_i^* \beta & & \downarrow \delta_i^* \beta \\
\mathcal{O}_{X_1} & \xrightarrow{\text{id}} & \mathcal{O}_{X_1}
\end{array}
\]

commutes, and \((p_2^* \beta) \alpha (p_1^* \beta)^{-1} \) is an automorphism of \( \mathcal{O}_{X_0} \), pulling back the identity on \( X_1 \). Hence \((p_2^* \beta) \alpha (p_1^* \beta)^{-1} \) is an element of

\[
\ker(\gamma_1^* : \Gamma(X'_0, \mathcal{O}^{\times}_{X'_0}) \longrightarrow \Gamma(X_1, \mathcal{O}^{\times}_{X_1})).
\]

Note that \((p_2^* \beta) \alpha (p_1^* \beta)^{-1} = 1 \) if, and only if, \((L, \alpha) \) is already trivial in \( \Pic(\cosk_0(X_0)_\bullet) \).

Replacing \( \beta \) by \( \beta \lambda \) for some \( \lambda \in \ker(\delta_i^* - \delta_i^* : \Gamma(X_0, \mathcal{O}^{\times}_{X_0}) \to \Gamma(X_1, \mathcal{O}^{\times}_{X_1})) \) gives a new trivialization \( \gamma^*(L, \alpha) \cong (\mathcal{O}_{X_0}, \text{id}) \), and any trivialization can be reached like this (trivializations of the line bundle \( L \) are a \( \mathbb{G}_m \)-torsor, and to get a trivialization of the pair \( \gamma^*(L, \alpha) = (L, \gamma_1^\alpha) \), the condition that \( 1 = (\delta_i^* \lambda)(\delta_i^* \lambda)^{-1} \) is necessary and sufficient). Next, observe that \( p_1^* - p_2^* \) (or rather \( p_1^*/p_2^* \)) induces a map

\[
\ker(\Gamma(X_0, \mathcal{O}^{\times}_{X_0}) \xrightarrow{\delta_i^* - \delta_i^*} \Gamma(X_1, \mathcal{O}^{\times}_{X_1})) \to \ker(\Gamma(X'_0, \mathcal{O}^{\times}_{X'_0}) \xrightarrow{\gamma_1^*} \Gamma(X_1, \mathcal{O}^{\times}_{X_1})).
\]
Putting all of this together, we see that we obtain an injective map
\[
\ker(\text{Pic}(\cosk_0 X_0^\bullet) \to \text{Pic}(X_\bullet)) \to \ker(\Gamma(X'_0, \mathcal{O}_{X'_0}^\times) \xrightarrow{\gamma_1} \Gamma(X_1, \mathcal{O}_{X_1}^\times)) \to (p_1 - p_2)^*\ker(\Gamma(X_0, \mathcal{O}_{X_0}^\times) \xrightarrow{\delta} \Gamma(X_1, \mathcal{O}_{X_1}^\times)).
\]
This implies that \(\ker(\text{Pic}(\cosk_0(X_0^\bullet) \to \text{Pic}(X_\bullet))\) is finitely generated. In fact, pulling back units from \(k^\times\) by \(\gamma_1\) is injective, as \(k \to \Gamma(X_1, \mathcal{O}_{X_1})\) is injective. Thus \(\ker(\Gamma(X'_0, \mathcal{O}_{X'_0}^\times) \xrightarrow{\gamma_1} \Gamma(X_1, \mathcal{O}_{X_1}^\times)) \subseteq \Gamma(X'_0, \mathcal{O}_{X'_0}^\times)/k^\times\), which is a finitely generated abelian group. To see this we use the separatedness of \(X_0\) to ensure the existence of a Nagata compactification of \(X'_0\), so that we can apply, e.g., [Kah06, Lemme 1].

**Proposition 2.5.** Let \(U\) be a regular, connected \(k\)-scheme of finite type, and \(\tau : U^\bullet \to U\) a smooth, proper hypercovering such that \(\tau_0 : U_0 \to U\) is an alteration (i.e., proper, surjective and generically finite) and \(U_0\) is connected. Then the kernel of \(\tau^* : \text{Pic}\ U \to \text{Pic}\ U^\bullet\) is finitely generated. In particular, \(\text{Pic}\ U^\bullet\) is finitely generated, then so is \(\text{Pic}\ U\).

**Proof.** If \(V \subseteq U\) is the biggest open subset of \(U\) such that \(\tau_0\) restricted to \(V_0 := \tau_0^{-1}(V)\) is flat, then \(V \neq \emptyset\), and the complement \(U \setminus V\) has codimension \(\geq 2\). In fact, as \(U_0 \to U\) is surjective, for any \(\eta\) mapping to a codimension 1 point \(\xi \in U\), the morphism \(\mathcal{O}_{U,\xi} \to \mathcal{O}_{U_0,\eta}\) is injective, so \(\mathcal{O}_{U_0,\eta}\) is a torsion free \(\mathcal{O}_{U,\xi}\)-module. But as \(U\) is regular, \(\mathcal{O}_{U,\xi}\) is a discrete valuation ring, so \(\tau_0\) is flat at \(\eta\), and \(\xi \in V\). Thus \(\text{Pic}(U) = \text{Pic}(V)\), and \(\tau_0|_{V_0}\) is faithfully flat.

Giving an element of \(\text{Pic}(\cosk_0(U_0^\bullet))\) is the same thing as giving an (isomorphism class of) a pair \((L, \alpha)\) with \(L\) a line bundle on \(U_0\) and \(\alpha\) a descent datum of \(L\) relative to \(U\).

Finally, we see that if a line bundle \(L\) on \(U\) pulls back to the trivial descent datum, then restricting it to \(V_0\) and using faithful flatness shows that \(L|_V\) is trivial, so \(L\) is trivial, as \(U \setminus V\) has codimension \(\geq 2\). Hence \(\text{Pic}\ U \to \text{Pic}(\cosk_0(U_0^\bullet))\) is injective, and by Lemma 2.4 this implies that the kernel of \(\tau^* : \text{Pic}\ U \to \text{Pic}\ U^\bullet\) is finitely generated. \(\square\)

**Proposition 2.6.** Let \(j : U^\bullet \to X^\bullet\) be a morphism of \(k\)-simplicial schemes, such that
\begin{enumerate}
\item \(X_p\) is regular and proper over \(k\) for every \(p\),
\item \(j_p : U_p \to X_p\) is an open immersion with dense image,
\item the face maps \(X_{i+1} \to X_i\) map \(X_{i+1} \setminus U_{i+1}\) to \(X_i \setminus U_i\).
\end{enumerate}
Then the cokernel of the induced map \(j^* : \text{Pic}\ X^\bullet \to \text{Pic}\ U^\bullet\) is finitely generated.

**Proof.** Let \(K_X^i := H^0(X_i, \mathcal{O}_{X_i}^\times)\) and make it into a complex of abelian groups \((K_X^i, \delta_i)\) via \(\delta_i := \sum_{t=0}^{i+1} (-1)^t \delta_{i,t} : K_X^{i+1} \to K_X^i\). Define \(K_U\) in the analogous fashion (where, to simplify notation, we write \(\delta_i\) for the faces of \(X^\bullet\) and for the faces of \(U^\bullet\)). Note that the complexes \(K_X^i\) and \(K_U\) have finitely generated cohomology groups: For even \(i > 0\) we have \(k^\times \subseteq \ker(\delta_{i-1})\), so \(H^i(K_X) = \ker(\delta_i)/\ker(\delta_{i-1})\) is a subquotient of \(\Gamma(X_i, \mathcal{O}_{X_i}^\times)/k^\times\), which is finitely generated, see e.g. [Kah06, Lemme 1]. The same argument holds for \(K_U\). For odd \(i\), we have \(k^\times \cap \ker(\delta_i) = 1\), so \(\ker(\delta_i) \subseteq \Gamma(X_i, \mathcal{O}_{X_i}^\times)/k^\times\), and hence \(H^i(K_X)\) and \(H^i(K_U)\) are finitely generated as well.
The morphism $j$ induces a morphism of spectral sequences
\[
E^{p,q}_{1,X} = H^q(X_p, O_{X_p}^\times) \longrightarrow H^{p+q}(X, O_X^\times)
\]
\[
E^{p,q}_{1,U} = H^q(U_p, O_{U_p}^\times) \longrightarrow H^{p+q}(U, O_U^\times),
\]
(note that $K_X = E^{1,0}_{1,X}$, and similarly for $K_U$) from which we obtain the morphism of short exact sequences
\[
0 \longrightarrow E^{1,0}_{2,X} = H^1(K_X) \longrightarrow \text{Pic}(X) \longrightarrow \ker(d_{2,X}) \longrightarrow 0
\]
\[
0 \longrightarrow E^{1,0}_{2,U} = H^1(K_U) \longrightarrow \text{Pic}(U) \longrightarrow \ker(d_{2,U}) \longrightarrow 0,
\]
where $d_{2,X}$ is the differential
\[
E^{0,1}_{2,X} = \ker(\text{Pic}(X) \xrightarrow{\delta_1 \delta_2} \text{Pic}(X_1)) \longrightarrow H^2(K_X) = E^{0,2}_{2,X},
\]
and similarly for $d_{2,U}$. Since $H^1(K_U)$ is finitely generated, $\coker(H^1(K_X) \rightarrow H^1(K_U))$ is also finitely generated, so to finish the proof of the proposition, it remains to show that $\coker(\ker(d_{2,X}) \rightarrow \ker(d_{2,U}))$ is finitely generated.

Consider the diagram
\[
0 \longrightarrow \ker(d_{2,X}) \longrightarrow \ker(\text{Pic}(X) \xrightarrow{\delta_1 \delta_2} \text{Pic}(X_1)) \xrightarrow{d_{2,X}} \text{im}(d_{2,X}) \longrightarrow 0
\]
\[
0 \longrightarrow \ker(d_{2,U}) \longrightarrow \ker(\text{Pic}(U) \xrightarrow{\delta_1 \delta_2} \text{Pic}(U_1)) \xrightarrow{d_{2,U}} \text{im}(d_{2,U}) \longrightarrow 0.
\]
As $\text{im}(d_{2,X}) \subset H^2(K_X)$, we know that $\ker(\text{im}(d_{2,X}) \rightarrow \text{im}(d_{2,U}))$ is finitely generated, so by the Snake Lemma, to finish the proof it suffices to show that the middle vertical map $\phi_0 : E^{0,1}_{2,X} \rightarrow E^{0,1}_{2,U}$, $\phi_0(L) = L|_{U_0}$ from above has a finitely generated cokernel.

By our regularity assumptions, we have for each $i$ an exact sequence
\[
0 \longrightarrow \mathcal{Y}_i \longrightarrow \text{Pic}(X_i) \longrightarrow \text{Pic}(U_i) \longrightarrow 0,
\]
where $\mathcal{Y}_i$ is the subgroup of $\text{Pic}(X_i)$ generated by the classes of the (finite number of) codimension 1 points of $X_i \setminus U_i$. In particular, this induces a map
\[
\ker(\delta_1^* : \text{Pic}(U_i) \rightarrow \text{Pic}(U_{i+1})) \longrightarrow \mathcal{Y}_{i+1}/\delta_1^*\mathcal{Y}_i,
\]
where $\delta_1^* = \sum_{i=0}^{r+1}(-1)^i\delta_i^*$. Indeed, we may extend $L \in \ker(\delta_1^* : \text{Pic}(U_i) \rightarrow \text{Pic}(U_{i+1})$ to some $\hat{L} \in \text{Pic}(X)$, and map it to $\text{Pic}(X_{i+1})$, where it has support contained in $X_{i+1} \setminus U_{i+1}$, i.e. it is mapped to $\mathcal{Y}_{i+1}$. To get a well-defined map on $\text{Pic}(U_i)$, we have to account for the choice of the extension of $L$ to $X_i$, that is we have to divide out by the image of $\mathcal{Y}_i$ under $\delta_1^*$ which is contained in $\mathcal{Y}_{i+1}$ by assumption.

Next, we show that the kernel of this map is precisely the image of the restriction
\[
\phi_i : \ker(\text{Pic}(X_i) \rightarrow \text{Pic}(X_{i+1})) \rightarrow \ker(\text{Pic}(U_i) \rightarrow \text{Pic}(U_{i+1})).
\]
If $L \in \ker(\text{Pic}(U_i) \rightarrow \text{Pic}(U_{i+1})$ maps to $\delta_1^* M$, for some $M \in \mathcal{Y}_i$, then there is some extension $\hat{L}$ of $L$ to $X_i$ such that $\delta_1^*(\hat{L} \otimes M^{-1}) \cong O_{X_{i+1}}$, so $\hat{L} \otimes M^{-1} \in \ker(\text{Pic}(X_i) \rightarrow \text{Pic}(X_{i+1})$. This shows $\hat{L} \cong \phi_i(L \otimes M^{-1})$, as $M$ is supported on $X_i \setminus U_i$. Conversely, if some $L \in \ker(\text{Pic}(U_i) \rightarrow \text{Pic}(U_{i+1})$ can be extended to $\hat{L} \in \ker(\text{Pic}(X_i) \rightarrow \text{Pic}(X_{i+1})$, then by definition $\hat{L}$ maps to 0 in $\mathcal{Y}_{i+1}/\delta_1^*\mathcal{Y}_i$. 


This finishes the proof: Specializing the last calculation to \( i = 1 \), we see that 
\( \ker(\phi_0) \) can be embedded into the finitely generated group \( \mathbb{Y}_1/(\delta_0^1 - \delta_1^1)\mathbb{Y}_0 \). \( \square \)

3. The Picard Group of Simply Connected Varieties

By a simply connected scheme we mean an irreducible scheme \( X \) such that 
\( \pi_1^n(X, x) = 1 \) for some (or any) geometric point \( x \) of \( X \). Often we will suppress notation of base points and write \( \pi_1 \) for \( \pi_1^n \). If \( X \) is a \( k \)-scheme, for some field \( k \), then \( k \) is necessarily algebraically closed. We will mostly be interested in the case \( \text{char}(k) = p > 0 \).

**Proposition 3.1.** If \( X \) is a normal, proper, connected \( k \)-scheme of finite type, such that \( \pi_1^n(X)^{ab,(\ell)} = 1 \) for some \( \ell \neq p \), and \( X_\bullet \to X \) a proper hypercovering with \( X_0 \) normal, and \( X_n \) reduced for all \( n \), then \( \text{NS}(X_\bullet) = \text{Pic}(X_\bullet) \). In particular, \( \text{Pic}(X_\bullet) \) is finitely generated.

**Proof.** This is a consequence of cohomological descent: There is an isomorphism 
\[ 0 = H^1_k(X, \mu_n) \cong H^1(X, \mu_n, X_\bullet) \] (see e.g. [BO78] Lemma 5.1.3), so \( \text{Pic}_{X_\bullet/k}^\text{red}(k) \) has no \( \ell \)-torsion, and thus is trivial, as \( \text{Pic}_{X_\bullet/k}^\text{red} \) is semi-abelian by Theorem 2.2. In fact, if a semi-abelian variety has no \( \ell \)-torsion, then it is an abelian variety, as a nontrivial subtorus would have nontrivial \( \ell \)-torsion. But an abelian variety with trivial \( \ell \)-torsion is trivial. Hence \( \text{NS}(X_\bullet) = \text{Pic}(X_\bullet) = \text{Pic}_{X_\bullet/k}(k) \). This group is finitely generated by Proposition 2.3. \( \square \)

We are ready to prove the first main theorem.

**Proof of Theorem [2.7]** By Nagata’s theorem there exists a proper variety \( X \) admitting \( U \) as a dense open subscheme, and since \( U \) is normal we may assume \( X \) to be normal. By [16] there exists an augmented proper hypercovering \( X_\bullet \to X \) with \( X_n \) regular and proper over \( k \), such that the part \( Z_n \) of \( X_n \) lying over \( X \setminus U \) is a strict normal crossing divisor. Write \( U_n := X_n \setminus Z_n \). As \( U \) is connected we can pick \( X_\bullet \) such that \( X_0 \) and \( U_0 \) are connected. Also note that \( \pi_1(U)^{ab,(\ell)} \) surjects onto \( \pi_1(X)^{ab,(\ell)} \) (see, e.g., [SGA1] Prop. V.6.9), so \( \pi_1(X)^{ab,(\ell)} = 1 \). We have shown that \( \text{Pic}(X_\bullet) \) is finitely generated (Proposition 3.1), that \( \text{Pic}(X_\bullet) \) maps to \( \text{Pic}(U_\bullet) \) with finitely generated cokernel (Proposition 2.6) and that this implies that \( \text{Pic}(U) \) is finitely generated (Proposition 2.3). \( \square \)

4. Stratified Line Bundles on Regular Simply Connected Varieties

We continue to denote by \( k \) an algebraically closed field of characteristic \( p \geq 0 \).

Let \( U \) be a \( k \)-scheme. Grothendieck defined in [EGA4 \( \S \)16] the sheaf \( \mathcal{D}_{U/k} \) of differential operators of \( U \) over \( k \). In characteristic 0, \( \mathcal{D}_{U/k} \) is (locally) the enveloping algebra of \( \mathcal{O}_U \) and the sheaf of derivations \( \text{Der}_{U/k}(\mathcal{O}_U) \). This is false in positive characteristic. For details, see e.g. [BO78, Ch. 2].

**Definition 4.1.** A \( \mathcal{O}_U \)-coherent module \( E \) is called stratified bundle, if \( E \) has a \( \mathcal{D}_U \)-action, compatible with its \( \mathcal{O}_U \)-structure. A horizontal morphism of stratified bundles \( E \to E' \) is a morphism of \( \mathcal{O}_U \)-modules, which is also a morphism of \( \mathcal{D}_U \)-modules. A stratified bundle is trivial, if it is isomorphic to \( \mathcal{O}_U^\oplus n \) for some \( n \), and if the \( \mathcal{D}_U \)-action is the sum of the canonical actions on \( \mathcal{O}_U \).

Note that if \( U \) is regular, then a stratified bundle is automatically locally free, [BO78 2.17], which justifies the name.

In positive characteristic, Katz gives a nice description of stratified bundles.
Theorem 4.2 (Katz, [Gie75, Thm. 1.3]). Let $k$ be an algebraically closed field of positive characteristic $p$. If $U$ is a regular, finite type $k$-scheme, then the category of stratified bundles on $U$ is equivalent to the category of sequences of pairs $(E_n, \sigma_n)_{n \in \mathbb{N}}$, where $E_n$ is a locally free sheaf of finite rank on $U$, and $\sigma_n$ an isomorphism $F^*E_{n+1} \to E_n$, with $F : X \to X$ the absolute Frobenius.

A morphism $(E_n, \sigma_n) \to (E_n', \sigma'_n)$ in the latter category is given by a sequence morphisms $\phi_n : E_n \to E'_n$ compatible with the $\sigma_n, \sigma'_n$.

A trivial stratified bundle is corresponds to a sequence of pairs $(\mathcal{O}_U, \text{id}_{\mathcal{O}_U})_n$.

We will use this characterization to prove Theorem 1.2 but first we need a statement about global functions on simply connected schemes.

Proposition 4.3. If $U$ is a connected normal $k$-scheme of finite type, such that the maximal abelian pro-$\ell$-quotient $\pi_1(U)^{\text{ab},(\ell)}$ is trivial for some $\ell \neq p$, then $H^0(U, \mathcal{O}_U^*) = k^*$. If $k$ has positive characteristic $p$, and $\pi_1(U)^{(p)} = 1$, then $H^0(U, \mathcal{O}_U) = k$.

Proof. The argument for the first assertion is due to Hélène Esnault. Asssume $f \in H^0(U, \mathcal{O}_U^*) \setminus \{0\}$. Then $f$ induces a dominant morphism $f' : U \to G_{m,k} \cong k^1 \setminus \{0\}$, as $f'$ is given by the map $k[x^{\pm 1}] \to H^0(U, \mathcal{O}_U)$, $x \mapsto f$, which is injective if, and only if, $f$ is transcendental over $k$. Thus $f'$ induces an open morphism $\pi_1(U) \to \pi_1(G_{m,k})$, see e.g. [St72, Lemma 4.2.10]. But under our assumption, the maximal abelian pro-$\ell$-quotient of the image of this morphism is trivial, so the image of $\pi_1(U)$ cannot have finite index in the group $\pi_1(G_{m,k})$, as in fact $\pi_1(G_{m,k})^{(\ell)} \cong \hat{\mathbb{Z}}^{(\ell)} = \hat{\mathbb{Z}}_\ell$.

For the second assertion, if $f \in H^0(U, \mathcal{O}_U) \setminus \{0\}$, then by the same arguments as above, $f$ induces a dominant morphism $U \to A^1_k$, and hence an open morphism $\pi_1(U) \to \pi_1(A^1_k)$. For $k$ of positive characteristic it is known that $\pi_1(A^1_k)$ has an infinite maximal pro-$p$-quotient (in fact it is a free pro-$p$-group of infinite rank: by [Kat80, 1.4.3, 1.4.4] we have $H^2(\pi_1(U), \mathcal{F}_p) = 0$, so $\pi_1(U)^{(p)}$ is free pro-$p$ of rank $\dim_{\mathbb{F}_p} H^1(\hat{\mathbb{A}}^1_k, \mathcal{F}_p) = \# k$). Thus the image of $\pi_1(U)$ in this group can only have finite index, if $\pi_1(U)^{(p)} \neq 1$. \hfill $\square$

Remark 4.4. Proposition 4.3 gives a proof of the well known fact that over a field $k$ of positive characteristic, unlike in characteristic 0, no affine $k$-scheme of positive dimension is simply connected.

Lemma 4.5. Let $U$ be a connected normal $k$-scheme of finite type. If $\pi_1(U)^{\text{ab},(\ell)} = 1$ for some prime $\ell \neq p$, then the isomorphism class of a stratified line bundle $L = (L_n, \sigma_n)_n$ is uniquely determined by the isomorphism classes of the $L_n$.

Proof. This follows from Proposition 4.3 and the argument is essentially contained in the proof of [Gie75, Prop. 1.7]: Let $M := (M_n, \tau_n)$ be a second stratified line bundle on $U$ and $u_n : L_n \to M_n$ isomorphisms of $\mathcal{O}_X$-modules. We will construct an isomorphism of stratified line bundles $L \to M$. Consider the following diagram:

$$
\begin{array}{ccc}
L_0 & \xrightarrow{\sigma_0} & F^*L_1 \\
\downarrow{u_0} & \cong & \downarrow{\tau_0} \\
M_0 & \xrightarrow{\tau_0} & F^*M_1
\end{array}
$$

The automorphism $\lambda := \tau_0 u_0 \sigma_0^{-1} F^*(u_1^{-1})$ of $F^*M_1$ corresponds to a global unit $\lambda \in \Gamma(U, \mathcal{O}_U^*)$. By Proposition 4.3 $U$ has only constant global units, so there is a $p$-th root $\lambda^{1/p}$ of $\lambda$, which defines an automorphism of $M_1$ such that $F^*\lambda^{1/p} = \lambda$. Defining $f_0 := u_0$ and $f_1 := \lambda^{1/p} u_1$ gives the first two steps of defining an isomorphism of stratified bundles $f : L \to M$. We can continue this process. \hfill $\square$

The following fact from group theory is elementary.
Lemma 4.6. Let $G$ be a finitely generated abelian group and $p$ a prime number. A nontrivial element $L \in G$ is infinitely $p$-divisible if and only if $L$ has finite order prime to $p$.

We can now easily prove the main result of this section.

Proof of Theorem 5.2. This is an adaptation of an argument from the introduction of [EM10]. By Lemma 4.8 we only need to show that the classes of $L_n$ in $\text{Pic} U$ are all trivial. Note that $L_n$ is infinitely $p$-divisible in $\text{Pic} U$ for all $n$. For regular $U$ as in the assertion, we know from Theorem 1.1 that $\text{Pic} U$ is finitely generated, and hence by Lemma 4.6 it follows that $L_n$ is torsion of order prime to $p$ in $\text{Pic} U$. But Kummer theory shows that $\text{Pic} U$ does not have nontrivial prime-to-$p$ torsion.

5. A Complex Analog

In this section, we reproduce a complex analog of Theorem 1.2 due to Hélène Esnault.

Definition 5.1. Let $\text{Pic}^\nabla(U)$ denote the group of isomorphism classes of pairs $(L, \nabla)$, where $L$ is an invertible $\mathcal{O}_U$-module and $\nabla$ an integrable connection on $L$. The group structure is given by tensor product connections.

Remark 5.2. For a regular scheme over a field of characteristic 0, the notion of stratified bundles is equivalent to the notion of vector bundles with integrable connection.

Let $\Omega^\nabla_{U/C}$ denote the complex of abelian sheaves in the Zariski topology
$$\mathcal{O}_X^\times \to\Omega^1_{U/C} \to \Omega^2_{U/C} \to \cdots,$$
and note that we obtain a short exact sequence of complexes
$$0 \to \Omega^2_{U/k} \to \Omega^\nabla_{U/k} \to \mathcal{O}_U^\times \to 0.$$ From this we obtain a homomorphism $\tilde{\epsilon}_1 : \text{Pic} U \to \mathbb{H}^2(U, \Omega^2_{U/k})$, which can be described explicitly as follows: Let $L \in \text{Pic} U$ be a line bundle. As $\Omega^2_{U/C}$ is a complex of coherent sheaves, we can compute its hypercohomology directly via Čech theory. Let $U = \{U_i\}$ be an open affine covering of $U$ trivializing $L$, such that $L|_{U_i} = e_i \mathcal{O}_{U_i}$, and $e_{ij} \xi_{ij} = e_j$ on $U_i \cap U_j =: U_{ij}$, with $\xi_{ij} \in \mathcal{O}_{U_{ij}}^\times(U_{ij})$. If $C^p(U, \Omega^q_{U/C}) = \prod_{i_1, \ldots, i_p} \Omega^q_{U/C}(U_{i_1 \ldots i_p})$ denotes the associated Čech double complex, then $\tilde{\epsilon}_1$ is induced by
$$L \mapsto ((\text{dlog} \xi_{ij}, 0) \in C^1(U, \Omega^1_{U/C}) \otimes C^0(U, \Omega^2_{U/C}) = \text{Tot}^2(C^\bullet(U, \Omega^2_{U/C})).$$ where Tot denotes the total complex of a double complex. From this description it follows that $\tilde{\epsilon}_1$, composed with the natural map $\mathbb{H}^2(U, \Omega^2_{U/k}) \to \mathbb{H}^2(U, \Omega^\nabla_{U/k}) = H^2(U, \mathcal{C})$, is the usual complex first Chern class.

Proposition 5.3. If $U$ is a regular, finite type $\mathbb{C}$-scheme, then the natural morphism $\text{Pic}^\nabla(U) \to \text{Pic}(U)$, $(L, \nabla) \mapsto L$, fits in a short exact sequence
$$0 \to H^0(U, \Omega^1_{U/C, \text{cls}})/\text{dlog} H^0(U, \mathcal{O}_U^\times) \to \text{Pic}^\nabla(U) \to \ker(\tilde{\epsilon}_1) \to 0,$$
where $\Omega^1_{U/C, \text{cls}}$ denotes the closed 1-forms.

Proof. Consider the map $\varphi : H^0(U, \Omega^1_{U/C, \text{cls}}) \to \text{Pic}^\nabla(U)$ given by $\omega \mapsto (\mathcal{O}_U, d + \omega)$. Every integrable connection on $\mathcal{O}_U$ is given by $d + \omega$ for a closed 1-form $\omega$, so $H^0(U, \Omega^1_{U/C, \text{cls}})$ maps onto the kernel of $\text{Pic}^\nabla(U) \to \text{Pic} U$. The kernel of $\varphi$ consists
precisely of those 1-forms $\omega$ such that there is a $\lambda \in \mathcal{O}_U^\times$ with $d(1) = 0 = d\lambda + \lambda \omega$, i.e. $\ker(\varphi) = \text{dlog} \mathcal{O}_U^\times$. We have proven that the sequence

$$0 \to H^0(U, \Omega^1_{U/\mathcal{C},\text{ets}})/\text{dlog} H^0(U, \mathcal{O}_U^\times) \to \text{Pic}^\nabla(U) \to \text{Pic}(U)$$

is exact.

It remains to determine the image of $\text{Pic}^\nabla(U) \to \text{Pic}(U)$. Let $(L, \nabla)$ be a line bundle with integrable connection. Let $\mathcal{U} = \{U_i\}$ be an open affine covering of $U$ trivializing $L$, such that $L|_{U_i} = e_i$, and $e_i \xi_{ij} = e_j$ on $U_i \cap U_j =: U_{ij}$, with $\xi_{ij} \in \mathcal{O}_{U_{ij}}^\times$. We have seen that $\tilde{c}_1$ is given by

$$L \mapsto ((\text{dlog} \xi_{ij})_{ij}, 0) \in \mathcal{C}^1(\mathcal{U}, \Omega^1_{U/\mathcal{C}}) \oplus \mathcal{C}^0(\mathcal{U}, \Omega^2_{U/\mathcal{C}}) = \text{Tot}^2(\mathcal{C}^*(\mathcal{U}, \Omega^2_{U/\mathcal{C}})).$$

Now if we write $\omega_i := \nabla(e_i) \in \Omega^1_{U/\mathcal{C},\text{ets}}(U_i)$, then we get on $U_{ij}$

$$\omega_{ij} \xi_{ij} e_j = \omega_i e_i = \nabla(e_i) = \nabla(\xi_{ij} e_j) = (\xi_{ij} w_j + di \xi_{ij}) e_j$$

and thus $\omega_i - \omega_j|_{U_{ij}} = \text{dlog}(\xi_{ij})$, which means that $(L, \nabla)$ maps to $\ker(\tilde{c}_1)$.

Conversely, if $L$ is a line bundle with $\tilde{c}_1(L) = 0$, then there is an open affine covering $\mathcal{U} = \{U_i\}$ trivializing $L$ with $L|_{U_i} = e_i \mathcal{O}_{U_i}$, and $e_i \xi_{ij} = e_j$ on $U_{ij}$, such that $\tilde{c}_1(L) = ((\text{dlog} \xi_{ij})_{ij}, 0)$ can be represented as $((\omega_i|_{U_{ij}} - \omega_j|_{U_{ij}})_{ij}, 0) \in \mathcal{C}^1(\mathcal{U}_{ij}^\times) \oplus \mathcal{C}^0(\mathcal{U}_{ij}^\times, \mathcal{O}_{U_{ij}})$ with $\omega_i \in \Omega^1_{U/\mathcal{C}}(U_i)$. Defining $\nabla(e_i) = \omega_i$, then defines an integrable connection on $L$, so the image of $\text{Pic}^\nabla(U) \to \text{Pic}(U)$ is precisely $\ker(\tilde{c}_1)$. □

**Theorem 5.4.** If $U$ is a regular, finite type $\mathcal{C}$-scheme, such that $\pi_1(U)_{\text{et,ab}} = 0$, then $H^0(U, \Omega^1_{U/\mathcal{C},\text{ets}}) \cong \text{Pic}^\nabla(U)$, via the map $\omega \mapsto (\mathcal{O}_U, d + \omega)$.

**Proof.** By Proposition 5.3 we have $H^0(U, \mathcal{O}_U^\times) = \mathbb{C}^\times$, so Proposition 5.3 shows that it suffices to prove that $\ker(\tilde{c}_1) = 0$, and for this it is enough to prove that $c_1 : \text{Pic}(U) \to H^2(U, \mathcal{C})$ is injective.

Let $L$ be a line bundle with $c_1(L) = 0$. Let $X$ be a regular compactification of $U$, such that $Y := X \setminus U$ is a normal crossings divisor, and choose an extension $\mathcal{T} \in \text{Pic}X$ of $L$. Note that $\pi_1(X)_{\text{ab}} = 1$, and that this implies that $c_1 : \text{Pic}(X) \hookrightarrow H^2(X, \mathcal{C})$ is injective, as $\text{Pic}(X) = \text{NS}(X)$ is a free abelian group of finite rank. Thus $c_1(\mathcal{T}) \in \ker(H^2(X, \mathcal{C}) \to H^2(U, \mathcal{C}))$. Since $\pi_1(U)_{\text{ab}}$ is trivial, we have $\ker(H^2(X, \mathcal{C}) \to H^2(U, \mathcal{C})) = H^2_X(U, \mathcal{C})$, and by purity $H^2_X(U, \mathcal{C}) \cong \bigoplus_i c_1(Y_i) \mathcal{C}$ (see [SGA4, 2], Cycle, 2.2.6, 2.2.8, 2.1.4), if $Y_1, \ldots, Y_n$ are the regular components of $Y$. This implies that there is some $M \in \text{Pic}(X)$, such that $M = \sum_i a_i Y_i$, with $c_1(\mathcal{T} \otimes M) = 0$, so $M = \mathcal{T}^{-1}$. But $M|_U = \mathcal{O}_U$, so $L = \mathcal{O}_U$.

**Remark 5.5.** The difference between Theorem 5.2 and Theorem 5.4 is related to the failure of the second assertion of Proposition 5.3 in characteristic 0: There are simply connected complex varieties $U$ with nonconstant global functions, and on such $U$ the conclusion of Theorem 5.2 fails, $H^0(U, \Omega^1_{U/\mathcal{C},\text{ets}}) \neq 0$.

On the other hand, from Deligne’s Riemann-Hilbert correspondence, we know that there are no nontrivial regular singular stratified bundles on $U$.

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Universität Duisburg-Essen, Mathematik, 45117 Essen, Germany
E-mail address: lars.kindler@uni-due.de