A JOINT CENTRAL LIMIT THEOREM FOR THE SUM-OF-DIGITS FUNCTION, AND ASYMPTOTIC DIVISIBILITY OF CATALAN-LIKE SEQUENCES

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Abstract. We prove a central limit theorem for the joint distribution of
\( s_q(A_jn) \),  
1 ≤ j ≤ d, where \( s_q \) denotes the sum-of-digits function in base \( q \) and the \( A_j \)'s are positive integers relatively prime to \( q \). We do this in fact within the framework of quasi-additive functions. As application, we show that most elements of “Catalan-like” sequences — by which we mean integer sequences defined by products/quotients of factorials — are divisible by any given positive integer.

1. Introduction

In [5], Burns shows that most of the ubiquitous Catalan numbers \( C_n := \frac{1}{n+1} \binom{2n}{n} \) (cf. [17, Ex. 6.19]) are divisible by \( p \), where \( p \) is some given prime number. Let \( v_p(N) \) denote the \( p \)-adic valuation of the integer \( N \), which by definition is the maximal exponent \( \alpha \) such that \( p^\alpha \) divides \( N \). In view of Legendre’s formula [10, p. 10] for the \( p \)-adic valuation of factorials,

\[
v_p(n!) = \frac{1}{p-1} \left(n - s_p(n)\right),
\]

where \( s_p(N) \) denotes the \( p \)-ary sum-of-digits function

\[
s_p(N) = \sum_{j \geq 0} \varepsilon_j(N),
\]

with \( \varepsilon_j(N) \) denoting the \( j \)-th digit in the \( p \)-adic representation of \( N \), we have

\[
v_p \left( \frac{1}{n+1} \binom{2n}{n} \right) = \frac{1}{p-1} \left(2s_p(n) - s_p(2n)\right) - v_p(n+1).
\]

Thus, one sees that the above and many more asymptotic divisibility results — such as the divisibility of most of the Catalan numbers, or even of most of the Fuß–Catalan numbers — follow via a central limit theorem. As application, we show that most elements of “Catalan-like” sequences — by which we mean integer sequences defined by products/quotients of factorials — are divisible by any given positive integer.

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numbers (cf. [1, pp. 59–60]) by any given prime power — can be proved if one has sufficiently precise results on the distribution of the vector

\[(s_p(A_1 n), s_p(A_2 n), \ldots, s_p(A_d n)), \quad n < N.\]  

Indeed, for \(p = 2\), Schmidt [14] and Schmid [13] showed that for pairwise different positive odd integers \(A_1, A_2, \ldots, A_d\) the vector (1.2) satisfies a \(d\)-dimensional central limit theorem with asymptotic mean vector \((1/2, \ldots, 1/2) \cdot \log_2 N\) and asymptotic covariance matrix \(\Sigma \cdot \log_2 N\) with

\[\Sigma = \left( \frac{\gcd(A_i, A_j)^2}{4A_i A_j} \right)_{1 \leq i, j \leq d}.
\]

We want to mention that the distribution of the vector (1.2) in residue classes was intensively studied in [6]. Furthermore we also emphasize that many results on divisibility properties of non-central binomial coefficients are available, too, see for example [16].

The first goal of the present paper is to generalize the central limit theorem by Schmidt [14] and Schmid [13] to arbitrary primes \(p\), and even to arbitrary bases \(q\). We do this in Theorem 1 in Section 2, by using an even more general concept, namely the concept of \(q\)-quasi-additive functions.

We then apply this result in Section 3 (see Theorem 4 and Corollary 5) to prove the somewhat non-intuitive fact that most elements of any sequence \((S(n))_{n \geq 0}\) of integers given by a (non-trivial) formula

\[
P(n) \prod_{i=1}^r (C_i n)! \quad \frac{Q(n) \prod_{i=1}^s (D_i n)!}{P(n)}
\]

are divisible by any given prime power, and thus by any given positive integer. Here, \(P(n)\) and \(Q(n)\) are polynomials in \(n\) over the integers, where \(Q(n)\) is a product of linear factors, and the \(C_i\)'s and \(D_i\)'s are positive integers with \(\sum_{i=1}^r C_i = \sum_{i=1}^s D_i\). The attribute “non-trivial” means that the set of \(C_i\)'s is different from the set of \(D_i\)'s. As is pointed out in more detail in Section 3, numerous (mainly combinatorial) sequences that appear in the literature in various contexts are of this form.

2. A CENTRAL LIMIT THEOREM

Let \(q \geq 2\) be a given integer. It is well known that the sum-of-digits function \(s_q(n)\) satisfies a central limit theorem of the form

\[
\frac{1}{N} \# \left\{ n < N : s_q(n) \leq \mu_q \log_q N + t \sqrt{\sigma_q^2 \log_q N} \right\} = \Phi(t) + o(1),
\]

uniformly in \(t\), where \(\mu_q = (q-1)/2\), \(\sigma_q^2 = (q^2-1)/12\), and \(\Phi(t)\) denotes the distribution function of the standard Gaussian distribution. This result is easy to prove since the digits \(\varepsilon_j(n), 0 \leq j < \log_q(N)\), behave almost as i.i.d. random variables if \(n\) varies between 0 and \(N - 1\). Actually much more is known (see for example [2]). Suppose that \(P(x)\) is a polynomial of degree \(D \geq 1\) with non-negative integer coefficients. Then we also have

\[
\frac{1}{N} \# \left\{ n < N : s_q(P(n)) \leq \mu_q \log_q P(N) + t \sqrt{\sigma_q^2 \log_q P(N)} \right\} = \Phi(t) + o(1).
\]
Note that the value $P(N)$ can be replaced by $N^D$ without changing the validity of the statement.

This result applies in particular to linear polynomials $P_j(n) = A_j n$ (with integers $A_j \geq 1$). In what follows, we will consider linear combinations of the form
\[
f(n) = c_1 s_q(A_1 n) + c_2 s_q(A_2 n) + \cdots + c_d s_q(A_d n), \quad n < N, \tag{2.2}
\]
with real numbers $c_j$ and integers $A_j \geq 1$, $1 \leq j \leq d$. Clearly, the central limit result of Schmidt [14] and Schmid [13] mentioned in the introduction is equivalent to the fact that $f(n)$ as in (2.2) with $q = 2$, $n < N$, satisfies a one-dimensional central limit theorem with asymptotic mean $\frac{1}{2}(c_1 + c_2 + \cdots + c_d) \cdot \log_2 N$ and asymptotic covariance $\mathbf{c} \Sigma \mathbf{c}^t \cdot \log_2 N$, where $\mathbf{c} = (c_1, c_2, \ldots, c_d)$.

It is also clear that the results of [13, 14] should directly transfer to a general basis $q \geq 2$ so that we can cover general $f(n)$. We will establish this generalization, however, with a completely different (and in fact more modern) proof.

**Theorem 1.** Let $q \geq 2$ be an integer, and let $A_1, A_2, \ldots, A_d$ be positive integers. Then the vector
\[
(s_q(A_1 n), s_q(A_2 n), \ldots, s_q(A_d n)), \quad 0 \leq n < N, \tag{2.3}
\]
satisfies a $d$-dimensional central limit theorem with asymptotic mean vector $((q - 1)/2, \ldots, (q - 1)/2) \cdot \log_2 N$ and asymptotic covariance matrix $\Sigma \cdot \log_2 N$, where $\Sigma$ is positive semi-definite.

If we further assume that $q$ is prime and that the integers $A_1, A_2, \ldots, A_d$ are not divisible by $q$, then $\Sigma$ is explicitly given by
\[
\Sigma = \left( \frac{(q^2 - 1) \gcd(A_i, A_j)^2}{12 \ A_i A_j} \right)_{1 \leq i,j \leq d}. \tag{2.4}
\]

For the proof we make use of the (recent) concept of quasi-additivity which is thoroughly discussed in [8]. There, a function $f$ defined on the non-negative integers is called $q$-quasi-additive, if there exists $r \geq 0$ such that
\[
f(q^{k+r} a + b) = f(a) + f(b) \quad \text{for all } b < q^k. \tag{2.5}
\]

We note that if (2.5) holds for some $r \geq 0$, then it holds as well for every larger $r$. This also shows that linear combinations of $q$-quasi-additive functions are $q$-quasi-additive, too. We further note that $s_q(n)$ is $q$-quasi-additive with parameter $r = 0$.

One of the main results of the paper [8] is that any $q$-quasi-additive function $f(n)$ of at most logarithmic growth satisfies a central limit theorem of the form
\[
\frac{1}{N} \# \left\{ n < N : f(n) \leq \mu \log_q N + t \sqrt{\sigma^2 \log_q N} \right\} = \Phi(t) + o(1),
\]
for appropriate constants $\mu$ and $\sigma^2$.

Our first observation is that $f(n)$ given in (2.2) is $q$-quasi-additive. The logarithmic growth property is trivially satisfied since $s_q(n) \leq (q - 1) \log_q n$.

**Lemma 2.** Let $A$ and $r$ be positive integers with $q^r \geq A$. Then $g(n) = s_q(A n)$ is $q$-quasi-additive (with parameter $r$).

**Proof.** Suppose that $b < q^k$. Then $Ab < q^{k+r}$, and consequently
\[
g(q^{k+r} a + b) = s_q(q^{k+r} A a + Ab) = s_q(Aa) + s_q(Ab) = g(a) + g(b). \quad \square
\]
Since linear combinations of \(q\)-quasi-additive functions are \(q\)-quasi-additive, it directly follows that \(f(n)\), as given by (2.2), satisfies a central limit theorem of the prescribed form. This also implies that the vector (2.3) satisfies a \(d\)-dimensional central limit theorem with asymptotic mean \(((q-1)/2, \ldots, (q-1)/2)\)-log \(N\). (This follows from the fact that a random vector \(X = (X_1, \ldots, X_d)\) is Gaussian with mean vector \((\mu_1, \ldots, \mu_d)\) and covariance matrix \(\Sigma\) if and only if every projection \(c_1X_1 + \cdots + c_dX_d\) with real \(c_1, \ldots, c_d\) is univariate Gaussian with mean \(c_1\mu_1 + \cdots + c_d\mu_d\) and variance \((c_1, \ldots, c_d)\Sigma(c_1, \ldots, c_d)^t\).)

It remains to compute the covariance matrix in the case, where \(q\) is a prime number.

**Lemma 3.** Let \(q \geq 2\) be a prime number, let \(A_1, A_2\) be positive integers that are not divisible by \(q\), and set \(D = \text{gcd}(A_1, A_2)\). Then, uniformly for \((\log N)^{1/3} \leq i, j \leq (\log N)^{1/3}\) and \(a, b \in \{0, 1, \ldots, q - 1\}\), we have

\[
\frac{1}{N} \# \{n < N : \varepsilon_i(A_1n) = a, \varepsilon_j(A_2n) = b\} = \begin{cases} \frac{1}{q^2} + O\left((\log N)^{-C}\right), & \text{if } i \neq j, \\ \frac{1}{q^2} + \frac{D^2}{A_1A_2} \sum_{\ell \neq 0} \frac{1}{4\pi^2\ell^2} & e\left(R\left(\frac{\ell A_1 a}{qD}\right) - e\left(-\frac{\ell A_2(a+1)}{qD}\right)\right) e\left(\frac{\ell A_1 b}{qD}\right) - e\left(-\frac{\ell A_2(b+1)}{qD}\right) \\ + O\left((\log N)^{-C}\right), & \text{if } i = j, \end{cases}
\]

for any given \(C > 0\). Here, \(e(x) = e^{2\pi ix}\).

**Proof.** We adapt the method of [2] to the present situation. However, in order to make the presentation more transparent, we first present a slightly simplified approach. First we note that \(\varepsilon_j(n) = a\) if and only if \(\{nq^{-j-1}\} \in [a/q, (a+1)/q]\), where \(\{x\} = x - \lfloor x\rfloor\) denotes the fractional part of \(x\). We also note that the Fourier series of the characteristic function \(1_{[a/q, (a+1)/q]}(x)\) is given by

\[
1_{[\frac{a}{q}, \frac{a+1}{q}]}(x) = \sum_m d_m(a)e(mx) \quad \text{with} \quad d_m(a) = \begin{cases} \frac{1}{q^2} e\left(-\frac{ma}{q}\right) - e\left(-\frac{m(a+1)}{q}\right) & \text{if } m = 0, \\ \frac{1}{2\pi im} & \text{if } m \neq 0. \end{cases}
\]

This Fourier series is not absolutely convergent. This is the major reason that we have to be more precise in a second round. Observe that \(d_m(a) = 0\) if \(m \neq 0\) and \(m \equiv 0 \mod q\).

We have

\[
\# \{n < N : \varepsilon_i(A_1n) = a, \varepsilon_j(A_2n) = b\} = \sum_{n < N} 1_{[\frac{a}{q}, \frac{a+1}{q}]}(\frac{A_1n}{q^{j+1}}) 1_{[\frac{b}{q}, \frac{b+1}{q}]}(\frac{A_2n}{q^{j+1}}) = \sum_{m_1, m_2} d_{m_1}(a) d_{m_2}(b) \sum_{n < N} e\left(\frac{A_1m_1}{q^{j+1}} + \frac{A_2m_2}{q^{j+1}}\right) n.
\]

Since

\[
\left| \sum_{n < N} e(\alpha n) \right| \leq \frac{2}{|1 - e(\alpha)|},
\]

we may neglect all exponential sums where \(\alpha = \frac{A_1m_1}{q^{j+1}} + \frac{A_2m_2}{q^{j+1}}\) is not an integer. At this moment, this step is not rigorous since the Fourier series is not absolutely convergent.
Next suppose that \( \alpha \) is an integer. If \( i \neq j \), the number \( \frac{A_{1m_1}}{q_1^{m_1}} + \frac{A_{2m_2}}{q_2^{m_2}} \) can be an integer only if \( m_1 = m_2 = 0 \) since we also assume that \( A_1 \) and \( A_2 \) are not divisible by \( q \). Thus we should get

\[
\# \{ n < N : \varepsilon_1(A_1n) = a, \varepsilon_2(A_2n) = b \} = d_0(a)d_0(b)N + o(N) = \frac{N}{q^2} + o(N).
\]

If \( i = j \), then the assumption \( \frac{A_{1m_1}}{q_1^{m_1}} + \frac{A_{2m_2}}{q_2^{m_2}} = k \) for an integer \( k \) leads to \( |m_1| \geq \frac{1}{2A_1}|k|q^{i+1} \geq \frac{1}{2A_1}|k|q^{(\log N)^{1/3}} \) or \( |m_2| \geq \frac{1}{2A_2}|k|q^{j+1} \geq \frac{1}{2A_2}|k|q^{(\log N)^{1/3}} \) so that the corresponding terms are negligible (if the Fourier series would be absolutely convergent).

Thus we should get (again by observing that all summations for which \( \alpha \) is an integer can be put into an error term)

\[
\# \{ n < N : \varepsilon_j(A_1n) = a, \varepsilon_j(A_2n) = b \} = \sum \ell d_{\ell A_2/D}(a)d_{-\ell A_1/D}(b) N + o(N)
\]

\[
= \frac{N}{q^2} + \sum_{\ell \neq 0} \frac{1}{4\pi^2\ell^2} \left( e \left( -\frac{\ell A_2 a}{qD} \right) - e \left( -\frac{\ell A_2(a+1)}{qD} \right) \right) \left( e \left( \frac{\ell A_1 b}{qD} \right) - e \left( -\frac{\ell A_1(b+1)}{qD} \right) \right) N + o(N).
\]

In order to make the above heuristics rigorous, we proceed as in [2]. We replace the characteristic function \( 1_{[a/q,(a+1)/q]}(x) \) by a smoothed version. Let \( \chi_{a, \Delta}(x) \) be defined by

\[
\chi_{a, \Delta}(x) := \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} 1_{[a/q, (a+1)/q]}(\{x + z\}) dz,
\]

The Fourier coefficients of the Fourier series \( \chi_{a, \Delta}(x) = \sum_{m \in \mathbb{Z}} d_{m, \Delta}(a)e(mx) \) are given by

\[
d_{0, \Delta}(a) = \frac{1}{q},
\]

and for \( m \neq 0 \) by

\[
d_{m, \Delta}(a) = \frac{e \left( -\frac{ma}{q} \right) - e \left( -\frac{m(a+1)}{q} \right)}{2\pi im} \cdot \frac{e \left( \frac{m\Delta}{2} \right) - e \left( -\frac{m\Delta}{2} \right)}{2\pi im\Delta}.
\]

Note that \( d_{m, \Delta}(a) = 0 \) if \( m \neq 0 \) and \( m \equiv 0 \mod q \), and that

\[
|d_{m, \Delta}(a)| \leq \min \left( \frac{1}{\pi|m|}, \frac{1}{\Delta \pi m^2} \right).
\]

By definition, we have \( 0 \leq \chi_{a, \Delta}(x) \leq 1 \) and

\[
\chi_{a, \Delta}(x) = \begin{cases} 
1, & \text{if } x \in \left[ \frac{a}{q} + \Delta, \frac{a+1}{q} - \Delta \right], \\
0, & \text{if } x \in [0, 1] \setminus \left[ \frac{a}{q} - \Delta, \frac{a+1}{q} + \Delta \right].
\end{cases}
\]

In particular, we set \( \Delta = (\log N)^{-C} \) for some (sufficiently large) constant \( C \). Of course we have to take into account all error terms. The smoothing error can be handled with
the help of a discrepancy estimate (see [2]). Putting the resulting estimates together — we leave the details to the reader —, one obtains

\[
\# \{ n < N : \varepsilon_i(A_1n) = a, \varepsilon_j(A_2n) = b \} = d_{0,\Delta}(a) d_{0,\Delta}(b) N + O \left( N (\log N)^{-C} \right) \]

\[
= \frac{N}{q^2} + O \left( N (\log N)^{-C} \right)
\]

for \( i \neq j \), and

\[
\# \{ n < N : \varepsilon_j(A_1n) = a, \varepsilon_j(A_2n) = b \} = \sum_\ell d_{\ell A_2/D,\Delta}(a) d_{-\ell A_1/D,\Delta}(b) N
\]

\[
+ O \left( N (\log N)^{-C} \right)
\]

for \( i = j \), where all estimates are uniform for \( (\log N)^{1/3} \leq i, j \leq \log_q N - (\log N)^{1/3} \).

Since

\[
d_{m,\Delta}(a) = d_m(a) \frac{\sin(\pi m \Delta)}{\pi m \Delta} = d_m(a) \left( 1 + O \left( \frac{1}{m \Delta} \right) \right)
\]

for \( 1 \leq |m| \leq 1/\Delta \), we obtain (with \( A = \max\{A_1, A_2\} \))

\[
\sum_{1 \leq |\ell| \leq 1/(A \Delta)} d_{\ell A_2/D,\Delta}(a) d_{-\ell A_1/D,\Delta}(b) = \sum_{1 \leq |\ell| \leq 1/(A \Delta)} d_{\ell A_2/D}(a) d_{-\ell A_1/D}(b)
\]

\[
+ O \left( \Delta \log(1/\Delta) \right),
\]

and

\[
\sum_{|\ell| > 1/(A \Delta)} d_{\ell A_2/D,\Delta}(a) d_{-\ell A_1/D,\Delta}(b) = O \left( \Delta \right),
\]

\[
\sum_{|\ell| > 1/(A \Delta)} d_{\ell A_2/D}(a) d_{-\ell A_1/D}(b) = O \left( \Delta \right).
\]

Thus,

\[
\sum_{\ell} d_{\ell A_2/D,\Delta}(a) d_{-\ell A_1/D,\Delta}(b) = \sum_{\ell} d_{\ell A_2/D}(a) d_{-\ell A_1/D}(b) + O \left( \frac{\log \log N}{(\log N)^C} \right)
\]

This completes the proof of the lemma.

It is now not difficult to complete the computation of the covariance matrix (which also completes the Proof of Theorem II). By definition, the covariance of \( s_q(A_1n) \) and \( s_q(A_2n) \) is given by

\[
\text{Cov} = \frac{1}{N} \sum_{n < N} s_q(A_1n) s_q(A_2n) - \frac{1}{N} \sum_{n < N} s_q(A_1n) \cdot \frac{1}{N} \sum_{n < N} s_q(A_2n).
\]

In order to apply Lemma 3, we neglect the digits \( \varepsilon_j \) with \( j \leq (\log N)^{1/3} \) or \( j \geq \log_q N - (\log N)^{1/3} \) and denote by \( \overline{s}_q \) the sum of digits of the remaining digits \( \varepsilon_j \) with \( (\log N)^{1/3} < j < \log_q N - (\log N)^{1/3} \). Then the corresponding approximate covariance \( \overline{\text{Cov}} \) satisfies

\[
\overline{\text{Cov}} - \text{Cov} = O \left( (\log N)^{5/6} \right),
\]
which can be shown with the help of the Cauchy–Schwarz inequality. Hence, by re-writing \( \text{Cov} \) with the help of the numbers \( \frac{1}{N} \# \{ n < N : \varepsilon_i(A_1n) = a, \varepsilon_j(A_2n) = b \} \) (from Lemma 3) and the numbers

\[
\frac{1}{N} \# \{ n < N : \varepsilon_j(A_i n) = a \} = \frac{1}{q} + O \left( (\log N)^{-C} \right)
\]

(note that the fact that this asymptotic property holds uniformly for \( (\log N)^{1/3} \leq j \leq \log_q N - (\log N)^{1/3} \), \( a \in \{0, \ldots, q-1\} \), and \( i = 1, 2 \) follows from Lemma 3), we immediately get

\[
\text{Cov} = L \frac{D^2}{A_1 A_2} \sum_{a,b=0}^{q-1} \sum_{\ell \not\equiv 0 \bmod q} \frac{1}{4\pi^2 \ell^2} \cdot \left( e \left( -\frac{\ell A_1 a}{q} \right) - e \left( -\frac{\ell A_2 (a+1)}{q} \right) \right) \left( e \left( \frac{\ell A_1 b}{q} \right) - e \left( -\frac{\ell A_1 (b+1)}{q} \right) \right) + O \left( (\log N)^2^{-C} \right)
\]

\[
= L \frac{D^2}{A_1 A_2} \frac{q^2 - 1}{12} + O \left( (\log N)^2^{-C} \right),
\]

where \( L = [\log_q N - 2(\log N)^{1/3}] \), and where we have used the identity

\[
\sum_{a=0}^{q-1} a e(ak/q) = \frac{q}{e(k/q) - 1},
\]

which is valid for integers \( k \) that are not divisible by \( q \). We can choose \( C \) appropriately — for example \( C = 2 \) — and finally obtain

\[
\text{Cov} = \frac{q^2 - 1 \gcd(A_i, A_j)^2}{12 A_i A_j} \log_q N + O \left( (\log N)^{5/6} \right),
\]

which completes the proof of Theorem 1.

3. Asymptotic divisibility of Catalan-like integer sequences

The main result in this section concerns divisibility of “Catalan-like” sequences by prime powers.

**Theorem 4.** Let \( p \) be a given prime number, \( \alpha \) a positive integer, \( P(n) \) a polynomial in \( n \) with integer coefficients, and \( (C_i)_{1 \leq i \leq r}, (D_i)_{1 \leq i \leq s}, (E_i)_{1 \leq i \leq t}, (F_i)_{1 \leq i \leq t} \) given integer sequences with \( C_i, D_i > 0 \) and \( p \not\mid \gcd(E_i, F_i) \) for all \( i, \sum_{i=1}^r C_i = \sum_{i=1}^s D_i \), and \( \{C_i : 1 \leq i \leq r\} \neq \{D_i : 1 \leq i \leq s\} \). If all elements of the sequence \( (S(n))_{n \geq 0} \), defined by

\[
S(n) := \frac{P(n)}{\prod_{i=1}^r (E_i n + F_i) \prod_{i=1}^s (D_i n)!},
\]

are integers, then

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ n < N : S(n) \equiv 0 \pmod{p^\alpha} \} = 1.
\]
We note that (3.2) remains true if $\alpha$ increases slowly with $N$. In particular, we can choose $\alpha = \lfloor \eta \log N \rfloor$ for an appropriate $\eta > 0$. We will actually show in the proof of Theorem 4 that
\[ \# \{ n < N : v_p(S(n)) < \eta \log N \} = O \left( \frac{N}{\sqrt{\log N}} \right) \]
if $\eta > 0$ is sufficiently small.

Furthermore we note that the assumption $p \nmid \gcd(E_i, F_i)$ is not really necessary since we can always reduce the problem to this case by separating the factors $p^\nu_p(\gcd(E_i, F_i))$. Thus, we immediately obtain the following corollary.

**Corollary 5.** Let $S(n)$ be given as in Theorem 4 (without assuming the condition $p \nmid \gcd(E_i, F_i)$). Then, for all positive integers $m$, we have
\[ \lim_{N \to \infty} \frac{1}{N} \# \{ n < N : S(n) \equiv 0 \pmod{m} \} = 1. \quad (3.3) \]

We call integer sequences of the form as in (3.1) — that is, integer sequences given by a product/quotient of factorials multiplied by a rational function — “Catalan-like” since the Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$ represent obviously such a sequence, but as well many other sequences that one finds in the literature (and in the On-Line Encyclopedia of Integer Sequences [15]).

**Examples.** All of the following sequences are “Catalan-like” in the sense of (3.1).

1. **Binomial coefficients** such as the central binomial coefficients $\binom{2n}{n}$, or more generally $\binom{a+b}{an}$ for positive integers $a$ and $b$, including variations such as $\binom{2n}{n-1}$, etc.
2. **Multinomial coefficients** such as $\frac{((a_1+a_2+\ldots+a_k)n)!}{(a_1)! (a_2)! \cdots (a_k)!}$, etc.
3. **Fuß-Catalan numbers.** These are defined by (cf. [11, pp. 59–60]) $\frac{1}{n} \binom{(m+1)n}{n-1}$, where $m$ is a given positive integer.
4. **Gessel’s [17] super ballot numbers** (often also called super-Catalan numbers) $\frac{(2n)! (2m)!}{n! m! (m+n)!}$ for non-negative integers $m$, or for $m = an$ with $a$ a positive integer.
5. **Many counting sequences in tree and map enumeration** (cf. [12] for a survey) such as $\frac{m+1}{n(m-1)n+2} \binom{mn}{n-1}$ (m-ary blossom trees with $n$ white nodes; cf. [11, Sec. 3]), $\frac{2}{m+2}(2n)(2n+1)$ (number of rooted planar maps with $n$ edges; cf. [20]), $\frac{2}{(3n-1)(3n-2)} \binom{3n-1}{n}$ (number of rooted non-separable planar maps with $n$ edges; cf. [4]), $\frac{1}{2(n+1)(n+1)} \binom{4n+1}{n}$ (number of rooted planar triangulations with $n+3$ vertices; cf. [18]), $\frac{1}{2(n+2)(n+1)} \binom{2n}{n}$ (number of rooted Hamiltonian maps with $2n$ vertices; cf. [19]), to mention just a few.

What Theorem 4 says is that, for any of these sequences, most elements (in the sense that the proportion of those in the set of all elements tends to 1) are divisible by $p^\alpha$, for a given prime number $p$ and given positive integer $\alpha$.

We should at this point remind the reader of Landau’s criterion [9], which says that
\[ U(n) := \frac{\prod_{i=1}^r (C_i n)!}{\prod_{i=1}^r (D_i n)!} \]
is an integer for all \(n\) if and only if
\[
\sum_{i=1}^{r} \lfloor C_i x \rfloor - \sum_{i=1}^{s} \lfloor D_i x \rfloor \geq 0 \tag{3.4}
\]
for all real numbers \(x\). (Here, we still assume that \(\sum_{i=1}^{r} C_i = \sum_{i=1}^{s} D_i\).) In view of [3, Lemma 3.3], which says that if \(U(n)\) is non-integral for some \(n\) then, for almost all primes \(p\), there exists an \(n\) such that \(v_p(U(n)) < 0\), this means (more or less; we do not believe that the polynomial \(P(n)\) can “correct” non-integrality of \(U(n)\) for all \(n\)) that (3.4) is an implicit assumption in Theorem 4.

For the proof of Theorem 4, we consider the integer interval \([0, N-1]\) as a probability space, with each integer equally likely, precisely as in Section 2. For notational convenience, the corresponding probability function will be denoted by \(P_N\). Functions on the non-negative integers are then interpreted as random variables \(X\) on this space by restricting them to \([0, N-1]\). The expectation of \(X\) on the space, that is, \(1/N \sum_{i=0}^{N-1} X(i)\), will be denoted by \(E_N(X)\), the variance will be denoted by \(\text{Var}_N(X)\), and the covariance of two functions by \(\text{Cov}_N(X, Y)\).

We need two auxiliary lemmas. The first concerns asymptotic mean and variance for the \(p\)-adic valuation of a linear function.

**Lemma 6.** Let \(E\) and \(F\) be integers, \(E > 0\), not both divisible by the prime \(p\). If \(v_p(En + F)\) is considered as a random variable for \(n\) in the integer interval \([0, N-1]\), then
\[
E_N(v_p(En + F)) = \begin{cases} 0, & \text{if } p \mid E, \\ \frac{1}{p-1} + o(1), & \text{if } p \nmid E, \end{cases} \quad \text{as } N \to \infty, \tag{3.5}
\]
and
\[
\text{Var}_N(v_p(En + F)) = \begin{cases} 0, & \text{if } p \mid E, \\ \frac{1}{(p-1)^2} + o(1), & \text{if } p \nmid E, \end{cases} \quad \text{as } N \to \infty. \tag{3.6}
\]

**Proof.** The first assertion in the case distinction in (3.5) is obvious since our assumptions imply that \(En + F \not\equiv 0 \pmod{p}\) if \(p \mid E\). If \(p \nmid E\), then the congruence \(En + F \equiv 0 \pmod{p^\alpha}\) has a unique solution for \(n\) modulo \(p^\alpha\) for any given positive integer \(\alpha\). Thus, we have
\[
E_N(v_p(an + b)) = \frac{1}{N} \sum_{\ell=1}^{\lfloor \log_p N \rfloor} \left( \frac{N}{p^\ell} + O(1) \right) = \frac{1}{p-1} + o(1), \quad \text{as } N \to \infty.
\]
Similarly, still assuming \(p \nmid E\), for the variance we have
\[
\text{Var}_N(v_p(an + b)) = \frac{1}{N} \sum_{\ell=1}^{\lfloor \log_p N \rfloor} (2\ell - 1) \left( \frac{N}{p^\ell} + O(1) \right) - \left( E_N(v_p(an + b)) \right)^2
\]
\[= \frac{p+1}{(p-1)^2} - \frac{1}{(p-1)^2} + o(1), \quad \text{as } N \to \infty,
\]
establishing also (3.6). \(\square\)

The second auxiliary lemma provides an asymptotic upper bound on the covariance of a linear function and the sum-of-digits function of a linear function.
The variance of $s_p(Cn)$ and $v_p(En + F)$ are considered as random variables for $n$ in the integer interval $[0, N - 1]$, then

$$\text{Cov}_N(s_p(Cn), v_p(En + F)) = O\left(\log_p^{1/2}(N)\right), \quad \text{as } N \to \infty. \quad (3.7)$$

**Proof.** By the Cauchy–Schwarz inequality in probabilistic setting, we have

$$\text{Cov}_N\left(s_p(Cn), v_p(En + F)\right) \leq \text{Var}_N\left(s_p(Cn)\right)^{1/2} \text{Var}_N\left(v_p(En + F)\right)^{1/2}.$$ 

The variance of $s_p(Cn) = s_p(Cp^{-v_p(C)n})$ has been (implicitly) given in (2.1) (see the line below that equation; see also (2.2) with $q = p$ and $A_i = A_j = Cp^{-v_p(C)}$) and turned out to be of the order $\log_p(N)$, while the variance of $v_p(En + F)$ has been given in (3.6) and turned out to be bounded. The assertion of the lemma is hence obvious. \qed

**Proof of Theorem 4.** With $S(n)$ given in (3.1), we have

$$v_p(S(n)) = v_p(P(n)) - \sum_{i=1}^t v_p(E_i n + F_i) + \sum_{i=1}^r v_p((C_i n)!) - \sum_{i=1}^s v_p((D_i n)!)$$

$$\leq -\sum_{i=1}^t v_p(E_i n + F_i) - \frac{1}{p - 1} \sum_{i=1}^r s_p(C_i n) + \frac{1}{p - 1} \sum_{i=1}^s s_p(D_i n). \quad (3.8)$$

Here, we used Legendre’s formula (1.1) and the assumption that $\sum_{i=1}^r C_i = \sum_{i=1}^s D_i$.

Now, it follows from [3, Lemma 3.5 and its proof], that under the integrality and non-triviality assumption for $S(n)$ of the theorem, we have $r < s$.

The expression on the right-hand side of (3.8) is a linear combination of the functions $v_p(E_i n + F_i)$, $s_p(C_i n)$, and $s_p(D_i n)$, which we view again as random variables on $[0, N - 1]$. For convenience, let us denote the function on the right-hand side of (3.8) by $T(n)$. By Theorem 1, (3.6), and (3.7), we have

$$\text{E}_N\left(T(n)\right) = \Omega\left(\log_p(N)\right), \quad \text{as } N \to \infty.$$ 

The reader should observe that the inequality $r < s$ is used here crucially. On the other hand, the variance of $T(n)$ is bounded above by the sum of the pairwise covariances of the involved random variables (functions). By Theorem 1, (3.6), and (3.7), we see that

$$\text{Var}_N\left(T(n)\right) = O\left(\log_p(N)\right), \quad \text{as } N \to \infty.$$ 

Given a random variable $X$, Chebyshev’s inequality reads

$$\text{P}\left(|X - \text{E}(X)| < \varepsilon\right) > 1 - \frac{1}{\varepsilon^2} \text{V}(X). \quad (3.9)$$

Choosing $\varepsilon = \left(\log_p(n)\right)^{3/4}$ and $X = T(n)$, we get

$$\text{P}_N\left(|T(n) - \text{E}_N\left(T(n)\right)| < \log_p^{3/4}(N)\right) = 1 + O\left(\log_p^{-1/2}(N)\right), \quad \text{as } N \to \infty.$$ 

Thus, for $N$ large enough, we have

$$T(n) > \text{E}_N\left(T(n)\right) - \log_p^{3/4}(N) = \Omega\left(\log_p(N)\right) > \alpha,$$

with probability $1 + O\left(\log_p^{-1/2}(N)\right)$. If we use this information in (3.8), then the assertion of the theorem follows immediately. \qed
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