ON THE $\sigma$-PAIR CORRELATION DENSITY OF QUADRATIC SEQUENCES MODULO ONE

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ABSTRACT. In this note we study the $\sigma$-pair correlation density

\[ R_2^\sigma([a,b],\{\theta_n\}_n,N) = \frac{1}{N^{2-\sigma}} \# \{ 1 \leq j \neq k \leq N \mid \theta_j - \theta_k \in \left[ \frac{a}{N^\sigma}, \frac{b}{N^\sigma} \right] + \mathbb{Z} \} \]

of a sequence $\{\theta_n\}_n$ that is equidistributed modulo one for $0 \leq \sigma < 2$.

The case $\sigma = 1$ is commonly referred to as the pair correlation density and the sequence $\{n^2\alpha\}_n$ has been of special interest due to its connection to a conjecture of Berry and Tabor on the energy levels of generic completely integrable systems.

We prove that if $\alpha$ is Diophantine of type $3 - \epsilon$ for every $\epsilon > 0$, then for any $0 \leq \sigma < 1$

\[ R_2^\sigma([a,b],\{an^2\}_n,N) \to b - a, \quad \text{as} \quad N \to \infty. \]

In this case, we say that the sequence exhibits $\sigma$-pair correlation.

In addition to this, we show that for any $0 \leq \sigma < \frac{1}{4}(9 - \sqrt{17}) = 1.21922...$ there is a set of full Lebesgue measure such that the sequence $\{\alpha n^2\}_n$ exhibits $\sigma$-pair correlation.

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1. INTRODUCTION

1.1. The pair correlation density for a sequence of $N$ numbers $\theta_1, \ldots, \theta_N$ which are equidistributed modulo one as $N$ tends to infinity, measures the distribution of spacings between the elements $\theta_n$ of the sequence at distances of order of the mean spacing $N^{-1}$. More precisely, the pair correlation function is defined as follows

\[ R_2([a,b],\{\theta_n\}_n,N) = \frac{1}{N} \# \{ 1 \leq j \neq k \leq N \mid \theta_j - \theta_k \in \left[ \frac{a}{N}, \frac{b}{N} \right] + \mathbb{Z} \}, \]
for any interval \([a, b]\).

The pair correlation function \(R_2(\cdot, \{\theta_n\}_n, N)\) is not a probability measure in general. However, if the sequence \(\theta_n\) arises from independent and uniformly distributed random variables on \([0, 1]\), then it is well-known that

\[
R_2([a, b], \{\theta_n\}_n, N) \to b - a, \quad \text{as } N \to \infty.
\]

(1.1)

Thus, we will say that a deterministic sequence \(\{\theta_n\}_n\) exhibits Poissonian pair correlation if (1.1) holds for all intervals \([a, b]\).

At this point let us also remark the connection between a sequence exhibiting Poissonian pair correlation and equidistribution modulo one: the former condition implies the latter (see [GL17; ALP18]) but not vice-versa.

1.2. In 1916, Weyl [Wey16] proved that the sequence of fractional parts \(\{\alpha n^d\}_n\) is equidistributed in \([0, 1]\). A finer question in the context of the pseudo-randomness of such a sequence is whether the corresponding spacings behave like those of independent and uniformly distributed random variables on \([0, 1]\).

For \(d = 1\) the answer is no and in fact the consecutive spacings have at most three values; this result is commonly referred to as the Steinhaus conjecture or the three distance theorem (see [Sós57; Swi59; MS17]).

For \(d \geq 2\), this is believed to be true for the pair correlation statistics depending only on the Diophantine properties of \(\alpha\): We say that a real number \(\alpha\) is Diophantine of type \(\kappa\) if there is a constant \(c > 0\) such that

\[
\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^\kappa}
\]

for all \(p \in \mathbb{Z}\) and \(q \in \mathbb{N}_{>0}\). We say that \(\alpha\) is Diophantine if it is Diophantine of type \(2 + \epsilon\) for any \(\epsilon > 0\). For reference, a rational number \(\alpha\) is of type \(\kappa = 1\); an irrational number \(\alpha\) is of type \(\kappa \geq 2\); an algebraic irrational number \(\alpha\), due to Roth’s theorem, is of type \(\kappa = 2 + \epsilon\) for all \(\epsilon > 0\).

Rudnick and Sarnak observed (see [RS98] Remark 1.2 or [Hea10]) that if \(\alpha\) is not Diophantine of type \(d + 1\), then \(\{\alpha n^d\}_n\) cannot exhibit Poissonian pair correlation (1.1). In the special case \(d = 2\), it seems likely to be true that \(\{\alpha n^2\}_n\) exhibits Poissonian pair correlation if \(\alpha\) is Diophantine of type \(3 - \epsilon\) for every \(\epsilon > 0\).

However, one can find a set \(P \subset \mathbb{R}\) of full Lebesgue measure such that for any \(\alpha \in P\), the sequence \(\{\alpha n^d\}_n\) exhibits Poissonian pair correlation (1.1). Results of this nature are commonly referred to as metric Poisson pair correlation. This was first proved by Rudnick and Sarnak [RS98], but let us also mention proofs using different methods in the special case \(d = 2\) by Marklof and Strömbergsson [MS03] as well as Heath-Brown [Hea10].

Yet, as of this moment there is no specific \(\alpha\) for which the Poissonian pair correlation property (1.1) has been verified. The most compelling result so far for an explicit \(\alpha\) was proved by Heath-Brown [Hea10], who showed that
for a number $\alpha$ of type $\kappa = \frac{9}{4}$ the sequence $\{\alpha n^2\}_n$ satisfies

$$R_2(L[-1,1]\{\alpha n^2\}_n, N) = 2L + O(L^{\frac{7}{2}}) + O((\log N)^{-1}),$$

(1.2)

for $1 \leq L \leq \log N$ as $N$ and $L$ tend to infinity.

In light of our result below, it seems plausible to expect $\{\alpha n^2\}_n$ to exhibit Poissonian pair correlation if $\alpha$ is Diophantine of type $3 - \epsilon$ for any $\epsilon > 0$. In fact, this would follow from a conjecture due to Truelsen [Tru10] on the average value of $\tau_{M,N}(n) = \#\{(a,b) \in \mathbb{N}^2 | a \leq M, b \leq N, ab = n\}$ in short arithmetic progressions.

1.3. Heath-Brown’s result [1.2] motivates the following notion, already introduced by Nair and Pollicott [NP07]. Namely, for $0 \leq \sigma < 2$ let us set

$$R_2^\sigma([a,b],\{\theta_n\}_n, N) = \frac{1}{N^{\sigma - 2}} \#\{1 \leq j \neq k \leq N \bigg| \theta_j - \theta_k \in \left[ \frac{a}{N^\sigma} - \frac{b}{N^\sigma} \right] + \mathbb{Z} \}.$$

We say that a deterministic sequence $\{\theta_n\}_n$ has $\sigma$-pair correlation if

$$R_2^\sigma([a,b],\{\theta_n\}_n, N) \to b - a, \text{ as } N \to \infty,$$  

(1.3)

holds for all intervals $[a,b]$. The case $\sigma = 0$ corresponds vaguely speaking to that of equidistribution on $[0,1]$ and the case $\sigma = 1$ to that of Poissonian pair correlation. Moreover, note that for any two $0 \leq \sigma_1, \sigma_2 < 2$

$$R_2^\sigma([a,b],\{\theta_n\}_n, N) = \frac{1}{N^{\sigma_1 - \sigma_2}} R_2^{\sigma_1}(N^{\sigma_1 - \sigma_2}[a,b],\{\theta_n\}_n, N)$$

and thus if a sequence $\{\theta_n\}_n$ exhibits $\sigma$-pair correlation for some $\sigma$, then

$$R_2(N^{1-\sigma}[a,b],\{\theta_n\}_n, N) \sim N^{1-\sigma}(b-a), \text{ as } N \to \infty.$$  

Analogously to the case $\sigma = 1$, it is known that a sequence exhibiting $\sigma$-pair correlation for some $0 < \sigma < 1$ is equidistributed in $[0,1]$ (see [Ste20]).

In this note we obtain for the sequence $\theta_n = \alpha n^2$ the following result

**Theorem 1.1.** Let $0 \leq \sigma < 1$ and $2 \leq \kappa < 1 + \frac{2}{7}$. Suppose $\alpha$ is Diophantine of type $\kappa$, then $\{\alpha n^2\}_n$ exhibits $\sigma$-pair correlation (1.3), that is

$$R_2^\sigma([a,b],\{\alpha n^2\}_n, N) \to b - a, \text{ as } N \to \infty,$$  

(1.4)

for all intervals $[a,b]$.

Let us note that the same method can be applied to other sequences of interest like the fractional parts of $\{\alpha n^2\}_{1 \leq n \leq N}$ (see [Mar10]).

In this context it is natural to understand the set of $0 \leq \sigma < 2$ for which $\{\theta_n\}_n$ exhibits $\sigma$-pair correlation. One can ask how large $\sigma$ can be taken as to still expect the sequence $\{\theta_n\}_n$ to exhibit $\sigma$-pair correlation. Thus, let us introduce the following quantity

$$\sigma(\{\theta_n\}_n) := \sup \left\{ 0 \leq \sigma < 2 \bigg| \{\theta_n\}_n \text{ exhibits } \sigma\text{-pair correlation} \right\}.$$

With this notation, Theorem 1.1 clearly implies
Corollary 1.2. Let $\alpha \in \mathbb{R}$ be Diophantine of type $3 - \epsilon$ for any $\epsilon > 0$, then $\sigma(\{\alpha n^2\}) \geq 1$.

Moreover, for almost every $\alpha$ we obtain using the approach of Rudnick and Sarnak [RS98].

Theorem 1.3. For any $0 \leq \sigma < \frac{1}{4}(9 - \sqrt{17}) = 1.21922...$ there is a set $P \subset \mathbb{R}$ of full Lebesgue measure such that $\{\alpha n^2\}_n$ exhibits $\sigma$-pair correlation for all $\alpha \in P$.

Similarly as above, this implies

Corollary 1.4. For almost all $\alpha \in \mathbb{R}$ we have $\sigma(\{\alpha n^2\}_n) \geq \frac{1}{4}(9 - \sqrt{17})$.

More generally, for the sequence $\{\alpha n^d\}_n$ with $d \geq 3$, we obtain in analogy to Theorem 1.3.

Theorem 1.5. For any $0 \leq \sigma < 2 + 2^{-d} - \sqrt{1 + 4^{-d}}$ there is a set $P \subset \mathbb{R}$ of full Lebesgue measure such that $\{\alpha n^d\}_n$ exhibits $\sigma$-pair correlation for all $\alpha \in P$. Note that $2 + 2^{-d} - \sqrt{1 + 4^{-d}} > 1$ for all $d \in \mathbb{N}$ and this expression converges to 1 as $d$ tends to infinity.

The proof of Theorem 1.5 goes along the exact same lines as that of Theorem 1.3 with some minor modifications. A question that arises out of this study – albeit beyond the scope of this note – is whether there are numbers $\alpha$ for which the sequence $\{\alpha n^d\}_n$ exhibits $\sigma$-pair correlation for some $\sigma \geq 2 + 2^{-d} - \sqrt{1 + 4^{-d}}$.

1.4. The strategy of the proof of Theorem 1.4 is classical and relies on studying the frequency side of the counting problem

$$\frac{1}{N^2-\sigma} \# \left\{ 1 \leq j \neq k \leq N \mid \alpha(j^2 - k^2) \in \left[ \frac{a}{N^\sigma}, \frac{b}{N^\sigma} \right] + \mathbb{Z} \right\}.$$ 

As this counting problem lies on the torus, it translates on the frequency side to sums of theta sums. For context, the classical problem of counting values of quadratic forms at integral points, which lies on the real line, translates on the frequency side to integrals of theta sums (see [BD58; Mar03; Göt04; But+22]).

We can view this problem as a counting problem regarding the distribution of values of the quadratic form $\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$ modulo one at integral points away from the isotropic rational subspaces.

From this point of view, two difficulties arise: On the one hand, the stabilizer of this form is a torus and thus we cannot exploit some standard techniques from homogeneous dynamics. On the other hand, the main difficulty in our case is the avoidance of isotropic rational subspaces. More precisely, the counting problem above leads us to study sums of the form

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{(x,y) \in \mathbb{Z}^2} e^{i\Omega_n(x,y)},$$

where $\Omega_n(x,y)$ is given by

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{(x,y) \in \mathbb{Z}^2} e^{i\Omega_n(x,y)}.$$
where $\Omega_n$ is a quadratic form in the Siegel upper half-space of degree two such that the rational isotropic subspaces of the real part are precisely \{(x, y) \in \mathbb{Z}^2 \mid x = \pm y\}. In Lemma 3.3 we show that such an expression can essentially be rewritten as the difference of two theta series. However, we are unable to exploit this difference, which is the likely culprit of the deficiency in Theorem 1.1. Nevertheless, building upon a method going back to the seminal work of Götze [Göt04], the bounds we obtain for both of these theta series are related to the number of Diophantine approximations $\alpha$ up to a certain height. Vaguely speaking, the proof of Theorem 1.1 reduces to the following counting problem

$$\frac{1}{N} \# \left\{ v \in \left( N^{2\frac{\sigma}{T}} \cap \mathbb{Z}^2 \right) \left| \| v \| \leq N^{\frac{\sigma}{T}} \right. \right\},$$

which can be seen to be of order $O(N^{\sigma-1})$ as long as $\alpha$ is Diophantine of type specified in Theorem 1.1.

1.5. Let us mention a few related results and further extensions regarding the pair correlation problem.

Zelditch [Zel98b; Zel98a] studied the pair correlation for sequences of the form $\alpha N \phi \left( \frac{n}{N} \right) + \beta n$, where $\phi$ is a fixed polynomial satisfying $\phi'' \neq 0$ on $[-1, 1]$ and obtained metric results, similar to that of Sarnak and Rudnick [RS98], with the caveat of dealing with the averaged pair correlation function. Marklof and Yesha [MY18] were able to prove and expand on Zelditch’s averaged pair correlation for the special case $(n - \alpha)^2$ for explicit $\alpha$ satisfying a Diophantine condition.

Boca and Zaharescu [BZ00] studied the pair correlation of sequences of the form \{f(n) mod p\}_{1 \leq n \leq N}, where $f$ is a rational function with integer coefficients.

Rudnick and Zaharescu [RZ99] obtained metric results for sequences of the form \{a(n)\}, where $a(n)$ is a lacunary sequence. This motivated the study of the pair correlation problem from the point of view of the so-called additive energy (for a set $A$ of real numbers, the additive energy $E(A)$ is defined to be $\#\{(a, b, c, d) \in A^4 \mid a + b = c + d\}$), which has proven to yield fruitful results. Let us mention in this regard results of Aistleitner, Larcher and Lewko [ALL17], Bloom and Walker [BW20], Aistleitner, El-Baz and Munsch [AEM21].

The pair correlation of the sequence $\{\alpha n^\theta\}$ has attracted special attention. For $\theta = \frac{1}{2}$, Elkies and McMullen [EM04] showed, using techniques from homogeneous dynamics, that the gap distribution of $\{\sqrt{n}\}$ is not Poissonian. Surprisingly, after removing the perfect squares out of this sequence, El-Baz, Marklof and Vinogradov [EMV15] showed that the resulting sequence does exhibit Poissonian pair correlation. For $\theta \in (0, \frac{14}{17})$ and $\alpha > 0$, Lutsko, Sourmelidis and Technau [LST21] show that the sequence $\{\alpha n^\theta\}$ exhibits Poissonian pair correlation without any further restrictions on $\alpha$. 
In particular their result gives an example of a sequence that exhibits Poissonian pair correlation, but not triple pair correlation.

1.6. Finally let us elaborate more on the underlying motivation to this problem. The special case \( d = 2 \) is of particular interest due to its connection between number theory and theoretical physics. Namely, the distribution of spacings of the sequence \( \{\alpha n^2\} \) modulo one are related to the spacings between the energy levels of the boxed oscillator [BT77], a particle in a two-dimensional potential well with hard walls in one direction and harmonic binding in the other.

For a sequence of numbers \( \theta_1, \ldots, \theta_N \) modulo one, denote the corresponding order statistics on \([0, 1]\) by \( \theta(1) \leq \cdots \leq \theta(N) \). The corresponding spacing measure is given by

\[
\mu_2(\{\theta_n\}, N) := \frac{1}{N} \sum_{n=1}^{N} \delta_{N(\theta_{(n+1)} - \theta_{(n)})},
\]

where we set \( \theta_{(N+j)} = \theta_{(j)} \) and \( \delta_x \) denotes the Dirac measure at \( x \). If the sequence \( \theta_n \) arises from a sequence of independent and uniformly distributed random variables on \([0, 1]\), then it is well-known that

\[
\mu_2(\{\theta_n\}, N) \to e^{-x} \, dx, \quad \text{as} \quad N \to \infty.
\] (1.5)

We say that a deterministic sequence \( \{\theta_n\} \) is Poissonian if (1.5) holds.

The standard approach to the analysis of spacings is through higher correlation functions defined analogously as before by

\[
R_k(\prod_{i=1}^{k-1} [a_i, b_i], \{\theta_n\}, N) = \frac{1}{N} \# \left\{ \mathbf{x} \in [N]^k \mid \theta_{x_i} - \theta_{x_{i+1}} \in [a_i, b_i] \text{ for all } i \right\},
\]

where \([N]^k\) denotes the set of all \( k \)-tuples \( \mathbf{x} = (x_1, \ldots, x_k) \) of distinct integers in \( \{1, \ldots, N\} \). One can show that if all correlation functions are Poissonian, in the sense that

\[
R_k(\prod_{i=1}^{k-1} [a_i, b_i], \{\theta_n\}, N) \to \prod_{i=1}^{k-1} (b_i - a_i), \quad \text{as} \quad N \to \infty,
\]

then \( \{\theta_n\} \) is Poissonian.

In this generality, Rudnick, Sarnak and Zaharescu [RSZ01] proved that if \( \alpha \) is not Diophantine of type 3 and the denominators of the corresponding rational approximations (for which the Diophantine property fails) are essentially square free, then \( \{\alpha n^2\} \) is Poissonian along a subsequence \( N_j \). Of course, as mentioned above, such a sequence cannot possibly be Poissonian along the entire sequence \( N \in \mathbb{N} \); In fact, if \( \alpha \) is not Diophantine of type 3, then there is a subsequence \( N_l \) for which \( \mu_2(\{\alpha n^2\}, N_l) \) converges to a measure supported on \( \mathbb{N}_0 \). Nevertheless, their analysis leads them to conjecture that \( \{\alpha n^2\} \) is Poissonian if \( \alpha \) is Diophantine of type \( 2 + \epsilon \) for every \( \epsilon > 0 \) and the denominators of the convergents to \( \alpha \) are essentially square free (see [RSZ01] for details).
Acknowledgments. I thank I. Khayutin for very helpful discussions and G. Margulis for introducing me to many of the ideas contained in this note.

2. The Pair Correlation Functional and Test Functions

2.1. Let \( \{\theta_n\}_n \) be a sequence and \( 0 \leq \sigma < 2 \). For sufficiently fast decaying functions \( f : \mathbb{R} \to \mathbb{R} \) and \( \psi : \mathbb{R}^2 \to \mathbb{R} \) we define the \( \sigma \)-pair correlation functional to be

\[
R_2^\sigma(f, \psi, \{\theta_n\}_n, N) := \frac{1}{N^{2-\sigma}} \sum_{j, k, \in \mathbb{Z}, |j| \neq |k|} \psi\left(\frac{1}{N} \left(\frac{j}{k}\right)\right) \sum_{n \in \mathbb{Z}} f(N^{\sigma}(\theta_j - \theta_k + n)).
\]

Note that with this definition we have

\[
R_2^\sigma([a, b], \{\theta_n\}_n, N) = 4R_2^\sigma(1_{[a, b]}, 1_{[-1, 1]^2}, \{\theta_n\}_n, N),
\]

for any interval \([a, b]\).

2.2. We shall approximate the indicator function corresponding to the interval \([a, b]\) and the square \([-1, 1]^2\) from above and from below by a class of functions that are well-suited to this problem. More precisely, we will call a pair of functions \( f : \mathbb{R} \to \mathbb{R}, \psi : \mathbb{R}^2 \to \mathbb{R} \) test functions of class \( \mathcal{P} \) if

\[
\hat{f} \in C_c(\mathbb{R}) \quad \text{or} \quad \hat{f}(t) = e^{-2\pi|t|} \quad \text{and} \quad \hat{\psi} \in C_c(\mathbb{R}^2).
\]

The following approximation lemma by functions whose Fourier transform is compactly supported is well-known and modified slightly from [Mar03] (see §8.6.2-8.6.4). We provide a proof for completeness.

Lemma 2.1. Let \( I \subset \mathbb{R} \) be a closed (finite) interval, \( g : \mathbb{R} \to \mathbb{R}_{\geq 0} \) a non-negative and continuous function and \( \epsilon > 0 \). There exist functions \( h_- \) and \( h_+ \) such that

\[
\hat{h}_+ \in C_c(\mathbb{R}), \quad \frac{\epsilon}{\pi(1 + s^2)} \leq h_-(s) \leq g(s) 1_I(s) \leq h_+(s), \quad \text{for all } s \in \mathbb{R}, \quad (2.2)
\]

\[
\int_I g(s) \, ds \leq \hat{h}_+(0) \leq \int_I g(s) \, ds + \epsilon,
\]

\[
\int_I g(s) \, ds - \epsilon \leq \hat{h}_-(0) \leq \int_I g(s) \, ds.
\]

Proof. Let \( \chi_\pm \in C_c^\infty(\mathbb{R}) \) be compactly supported and continuous functions such that

\[
0 \leq \chi_- \leq g \cdot 1_I \leq \chi_+,
\]

\[
\int_{\mathbb{R}}(\chi_+ - \chi_-)ds < \epsilon.
\]
Let us set
\[ h_{\pm,\epsilon}(s) = \chi_{\pm}(s) \pm \frac{\epsilon}{\pi(1 + s^2)}. \]

Then,
\[ 0 \leq h_{-\epsilon}(s) + \frac{\epsilon}{\pi(1 + s^2)} \leq g(s) \leq h_{+\epsilon}(s) - \frac{\epsilon}{\pi(1 + s^2)}, \]
\[ \hat{h}_{\pm,\epsilon}(u) = \chi_{\pm}(u) \pm \epsilon e^{-2\pi|u|}. \]

Next, fix a continuous and compactly supported function \( \chi \in C_c(\mathbb{R}) \) such that
\[ \text{supp}(\chi) \subseteq [-2, 2], \]
\[ 0 \leq \chi \leq 1, \]
\[ \chi|_{[-1,1]} \equiv 1. \]

For \( P \geq 1 \), to be determined below, set
\[ \hat{h}_{\pm,\epsilon,P}(u) = \hat{h}_{\pm,\epsilon}(u)\chi\left(\frac{u}{P}\right) \]

and observe that \( \hat{h}_{\pm,\epsilon,P} \in C_c(\mathbb{R}) \) is continuous and compactly supported with \( \text{supp}(\hat{h}_{\pm,\epsilon,P}) \subseteq [-2P, 2P] \). We claim that we can choose \( P \geq 1 \), depending on \( \epsilon \) and \( \chi_{\pm} \) only, such that
\[ |h_{\pm,\epsilon,P}(s) - h_{\pm,\epsilon}(s)| \leq \frac{\epsilon}{\pi(1 + s^2)}, \quad \text{for any } s \in \mathbb{R}. \]

Indeed, for some \( C \) depending on \( \epsilon \) and \( \chi_{\pm} \) we have on the one hand
\[ |h_{\pm,\epsilon,P}(s) - h_{\pm,\epsilon}(s)| \leq \int_{\mathbb{R}} \left| \hat{h}_{\pm,\epsilon}(u)\right| \left(1 - \chi\left(\frac{u}{P}\right)\right) du \]
\[ \leq \int_{|u| \geq P} \left| \hat{h}_{\pm,\epsilon}(u)\right| du \leq \frac{C}{P}, \]
on the other hand, after integrating by parts twice,
\[ |h_{\pm,\epsilon,P}(s) - h_{\pm,\epsilon}(s)| \]
\[ \leq \frac{1}{4\pi^2 s^4} \left( \int_{|u| \geq P} \left| \hat{h}_{\pm,\epsilon}''(u)\right| du \right) \]
\[ + \frac{2}{P} \int_{\mathbb{R}} \left| \hat{h}_{\pm,\epsilon}'(u)\chi\left(\frac{u}{P}\right)\right| du \]
\[ + \frac{1}{P^2} \int_{\mathbb{R}} \left| \hat{h}_{\pm,\epsilon}(u)\chi''\left(\frac{u}{P}\right)\right| du \]
\[ \leq \frac{C}{Ps^2} \]

and thus
\[ |h_{\pm,\epsilon,P}(s) - h_{\pm,\epsilon}(s)| \leq \frac{C}{P} \min \left\{ 1, \frac{1}{s^2} \right\} \leq \frac{\epsilon}{\pi(1 + s^2)}, \]
whenever \( P \geq 2C\pi\epsilon^{-1} \). Hence, we conclude that
\[
g(s) \mathbb{1}_I(s) \geq h_{-\epsilon} P(s) \geq g(s) \mathbb{1}_I(s) - (\chi_+(s) - \chi_-(s)) - 2\frac{\epsilon}{\pi(1 + s^2)}
\]
\[
g(s) \mathbb{1}_I(s) \leq h_{+\epsilon} P(s) \leq g(s) \mathbb{1}_I(s) + (\chi_+(s) - \chi_-(s)) + 2\frac{\epsilon}{\pi(1 + s^2)},
\]
for all \( s \in \mathbb{R} \), which implies
\[
\frac{g \cdot \mathbb{1}_I(0)}{\mathbb{1}_I(0)} \leq \int_{\mathbb{R}} h_{+\epsilon} P(s) \, ds \leq \frac{g \cdot \mathbb{1}_I(0)}{\mathbb{1}_I(0)} + 3\epsilon,
\]
\[
\frac{g \cdot \mathbb{1}_I(0)}{\mathbb{1}_I(0)} - 3\epsilon \leq \int_{\mathbb{R}} h_{-\epsilon} P(s) \, ds \leq \frac{g \cdot \mathbb{1}_I(0)}{\mathbb{1}_I(0)}.
\]

Finally, also note by construction that
\[
\frac{h_{-\epsilon}}{\mathbb{1}_I(0)} P \geq -\frac{\epsilon}{\pi(1 + s^2)} + h_{-\epsilon} = \chi_+ - \frac{2\epsilon}{\pi(1 + s^2)} \geq -\frac{2\epsilon}{\pi(1 + s^2)},
\]
where we use that \( \chi_- \) is non-negative.

### 2.3.

The sufficiency of the class of test functions introduced above is established by the following

**Lemma 2.2.** Suppose that for any two test functions \( f : \mathbb{R} \to \mathbb{R} \) and \( \psi : \mathbb{R}^2 \to \mathbb{R} \) of class \( \mathcal{P} \) (see (2.1)) the following identity holds
\[
\lim_{N \to \infty} R^2_N(f, \psi \cdot e^{-\|\cdot\|_2^2}, \{\theta_j\}_j, N) = \widehat{f}(0)(\mathcal{F}(\psi \cdot e^{-\|\cdot\|_2^2}))(0),
\]
where \( \cdot \) denotes pointwise multiplication of two functions. Then, for any \( a < b \)
\[
\lim_{N \to \infty} R^2_N([a, b], \{\theta_j\}_j, N) = b - a.
\]

**Proof.** Let \( \epsilon > 0 \). According to Lemma 2.1 we can find \( f_\pm : \mathbb{R} \to \mathbb{R} \) such that
\[
\begin{align*}
\hat{f}_\pm \in C_c(\mathbb{R}), \\
-\frac{\epsilon}{\pi(1 + s^2)} \leq f_-(s) \leq 1_{[a, b]}(s) \leq f_+(s), \quad &\text{for all } s \in \mathbb{R}, \\
b - a \leq \hat{f}_+(0) \leq b - a + \epsilon, \\
b - a - \epsilon \leq \hat{f}_-(0) \leq b - a.
\end{align*}
\]

as well as \( \psi_\pm : \mathbb{R}^2 \to \mathbb{R} \) satisfying
\[
\begin{align*}
\psi_- \in C_c(\mathbb{R}^2), \\
\psi_-(v) \leq 1_{[-1, 1]^2}(v) e^{\|v\|^2} \leq \psi_+(v), \quad &\text{for all } v \in \mathbb{R}^2 \\
\int_{[-1, 1]^2} e^{\|v\|^2} \, dv \leq \psi_+(0) \leq \int_{[-1, 1]^2} e^{\|v\|^2} \, dv + \epsilon, \\
\int_{[-1, 1]^2} e^{\|v\|^2} \, dv - \epsilon \leq \psi_-(0) \leq \int_{[-1, 1]^2} e^{\|v\|^2} \, dv.
\end{align*}
\]
By assumption, there is \( N_0 \in \mathbb{N} \) such that for any \( N \geq N_0 \)
\[
|R_2^\sigma(f_\pm, \psi_\pm \cdot e^{-\|\cdot\|^2}, \{\theta_j\}_j, N) - \tilde{f}_\pm(0)(\mathcal{F}(\psi_\pm \cdot e^{-\|\cdot\|^2}))(0)| < \epsilon, \tag{2.5}
\]
\[
|R_2^\sigma\left(\frac{2\epsilon}{\pi(1 + (\cdot)^2)}, \psi_\pm \cdot e^{-\|\cdot\|^2}, \{\theta_j\}_j, N\right) - 2\epsilon \mathcal{F}(\psi_\pm \cdot e^{-\|\cdot\|^2})(0)| < \epsilon, \tag{2.6}
\]
where we use that
\[
\int_{\mathbb{R}} e^{-2\pi i u s} ds = e^{-2\pi |u|}.
\]
Let \( N \geq N_0 \). Using (2.3), (2.4), (2.5) and (2.6), we obtain the upper bound
\[
4R_2^\sigma([a, b], \{\theta_j\}_j, N) - 4(b - a)
\]
\[
\leq R_2^\sigma(f_+, \psi_+ \cdot e^{-\|\cdot\|^2}, \{\theta_j\}_j, N) - \tilde{f}_-(-\epsilon)(\mathcal{F}(\psi_+ \cdot e^{-\|\cdot\|^2}))(0)
\]
\[
\leq \epsilon + \tilde{f}_+(0)(\mathcal{F}(\psi_+ \cdot e^{-\|\cdot\|^2}))(0) - \tilde{f}_-(0)(\mathcal{F}(\psi_- \cdot e^{-\|\cdot\|^2}))(0)
\]
\[
= \epsilon + (\tilde{f}_+(0) - \tilde{f}_-(0))(\mathcal{F}(\psi_+ \cdot e^{-\|\cdot\|^2}))(0)
\]
\[
+ \tilde{f}_-(0)\int_{\mathbb{R}^2} (\psi_+(v) - \psi_-(v)) e^{-\|v\|^2} dv
\]
\[
\leq \epsilon + 2\epsilon(12 + \epsilon) + 2\epsilon(b - a).
\]
The corresponding lower bound is more delicate as \( f_- \) is not everywhere positive. However, due to Lemma 2.1, \( f_- + \frac{2\epsilon}{\pi(1 + (\cdot)^2)} \geq 0 \) as well as (2.3), (2.4), (2.5) and (2.6), we obtain
\[
4R_2^\sigma([a, b], \{\theta_j\}_j, N) - 4(b - a)
\]
\[
- R_2^\sigma\left(\frac{2\epsilon}{\pi(1 + (\cdot)^2)}, 1_{[-1,1]^2}, \{\theta_j\}_j, N\right)
\]
\[
\geq R_2^\sigma(f_, \psi_- \cdot e^{-\|\cdot\|^2}, \{\theta_j\}_j, N) - \tilde{f}_+(0)(\mathcal{F}(\psi_+ \cdot e^{-\|\cdot\|^2}))(0)
\]
\[
- R_2^\sigma\left(\frac{2\epsilon}{\pi(1 + (\cdot)^2)}, (\psi_+ - \psi_-) \cdot e^{-\|\cdot\|^2}, \{\theta_j\}_j, N\right)
\]
\[
\geq -\epsilon + \tilde{f}_-(0)(\mathcal{F}(\psi_- \cdot e^{-\|\cdot\|^2}))(0) - \tilde{f}_+(0)(\mathcal{F}(\psi_+ \cdot e^{-\|\cdot\|^2}))(0)
\]
\[
- R_2^\sigma\left(\frac{2\epsilon}{\pi(1 + (\cdot)^2)}, (\psi_+ - \psi_-) \cdot e^{-\|\cdot\|^2}, \{\theta_j\}_j, N\right)
\]
\[
- \tilde{f}_-(0)\int_{\mathbb{R}^2} (\psi_+(v) - \psi_-(v)) e^{-\|v\|^2} dv
\]
\[
- R_2^\sigma\left(\frac{2\epsilon}{\pi(1 + (\cdot)^2)}, (\psi_+ - \psi_-) \cdot e^{-\|\cdot\|^2}, \{\theta_j\}_j, N\right)
\]
\[
\geq -2\epsilon - (\tilde{f}_+(0) - \tilde{f}_-(0))(\mathcal{F}(\psi_+ \cdot e^{-\|\cdot\|^2}))(0)
\]
\[
- \tilde{f}_-(0)\int_{\mathbb{R}^2} (\psi_+(v) - \psi_-(v)) e^{-\|v\|^2} dv
\]
\[
\geq -\epsilon - 2\epsilon(12 + \epsilon) - 2\epsilon(b - a) - 4\epsilon^2.
\]
2.4. The following lemma allows us to establish the connection to the Geometry of Numbers in the next section. In fact, it shows that the main error term is roughly speaking an incomplete theta sum.

**Lemma 2.3.** Let $f : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ be test functions of class $\mathcal{P}$ (see (2.1)). Then,

$$R_2^\sigma(f, \psi \cdot e^{-\| \cdot \|^2}, \{\theta_j\}_j, N)$$

$$= \hat{f}(0) \mathcal{F}(\psi \cdot e^{-\| \cdot \|^2})(0)$$

$$+ \hat{f}(0) \left( \frac{1}{N^2} \sum_{j, k \in \mathbb{Z}} (\psi \cdot e^{-\| \cdot \|^2}) \left( \frac{1}{N} \left( \begin{array}{c} j \\ k \end{array} \right) \right) - \hat{f}(0) \mathcal{F}(\psi \cdot e^{-\| \cdot \|^2})(0) \right)$$

$$+ \frac{1}{N^2} \int_{\mathbb{R}^2} \hat{\psi}(\xi) \sum_{n \neq 0} \hat{f} \left( \frac{n}{N^\sigma} \right) S(\{\theta_j\}_j, N, n, \xi) d\xi,$$

where $S(\{\theta_j\}_j, N, n, \xi)$ is given by

$$\sum_{j, k \in \mathbb{Z}} \exp \left\{ - \frac{1}{N^2} (j^2 + k^2) + 2\pi i (\theta_j - \theta_k) + 2\pi i \langle \xi, \frac{1}{N} \left( \begin{array}{c} j \\ k \end{array} \right) \rangle \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Let us remark that the second sum is $o(1)$ as $N \to \infty$.

**Proof.** Let $f : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ be test functions of class $\mathcal{P}$. The Poisson summation formula implies that $R_2^\sigma(f, \psi \cdot e^{-\| \cdot \|^2}, \{\theta_j\}_j, N)$ can be rewritten as

$$\hat{f}(0) \frac{1}{N^2} \sum_{j, k \in \mathbb{Z}} (\psi \cdot e^{-\| \cdot \|^2}) \left( \frac{1}{N} \left( \begin{array}{c} j \\ k \end{array} \right) \right)$$

$$+ \frac{1}{N^2} \sum_{n \neq 0} \hat{f} \left( \frac{n}{N^\sigma} \right) \sum_{j, k \in \mathbb{Z}} \psi \left( \frac{1}{N} \left( \begin{array}{c} j \\ k \end{array} \right) \right) e^{-\pi^2 (j^2 + k^2)}.$$

Finally, it suffices to note that

$$\psi \left( \frac{1}{N} \left( \begin{array}{c} j \\ k \end{array} \right) \right) = \int_{\mathbb{R}^2} \hat{\psi}(\xi) e^{\left( \langle \xi, \frac{1}{N} \left( \begin{array}{c} j \\ k \end{array} \right) \rangle \right)} d\xi$$

and switch the order of integration and summation to obtain the claim. \qed

Let us record the following observation: Suppose that any two test functions $f : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{P}$ (see (2.1)) satisfy

$$\lim_{N \to \infty} \int_{\mathbb{R}^2} \hat{\psi}(\xi) \sum_{n \neq 0} \hat{f} \left( \frac{n}{N} \right) S(\{\theta_j\}_j, N, n, \xi) d\xi = 0,$$
Lemma 3.1. Let \( \chi \) be a bijection and with this notation note that \( \{ (x, y) \in \mathbb{Z}^2 \mid |x| \neq |y| \} \) can be partitioned as follows

\[
(Z_0 \times Z_1) \cup (Z_1 \times Z_0) \cup \left\{ (x, y) \in \mathbb{Z}_0^2 \cup \mathbb{Z}_1^2 \mid \chi_1^+(x, y)\chi_2^+(x, y) \neq 0 \right\}.
\]
For \((x, y) \in (\mathbb{Z}_0 \times \mathbb{Z}_1) \cup (\mathbb{Z}_1 \times \mathbb{Z}_0)\) we have
\[
\Omega_{\delta, \xi}(x, y) = \Omega_{\delta, \xi}(\chi_1^-(x, y) + \chi_2^-(x, y), \chi_1^-(x, y) - \chi_2^-(x, y) - 1),
\]
and for \((x, y) \in \mathbb{Z}_0^2 \cup \mathbb{Z}_1^2\) we have
\[
\Omega_{\delta, \xi}(x, y) = \Omega_{\delta, \xi}(\chi_1^+(x, y) + \chi_2^+(x, y), \chi_1^+(x, y) - \chi_2^+(x, y)).
\]

□

Lemma 3.2. Let \(\delta \in \mathbb{H}\) and \(\xi \in \mathbb{R}^2\). Then,
\[
\sum_{(x, y) \in \mathbb{Z}^2 \atop |x| \neq |y|} e^{\Omega_{\delta, \xi}(x, y)} = \Psi_{\delta, \xi}(0) + \sum_{x, y \in \mathbb{Z}^2} \Psi_{\delta, \xi}(x, y) - \sum_{y \in \mathbb{Z}} \Psi_{\delta, \xi}(0, y) - \sum_{x \in \mathbb{Z}} \Psi_{\delta, \xi}(x, 0)
\]
\[
+ \sum_{x, y \in \mathbb{Z}^2} \left( \Psi_{\delta, \xi}\left(\frac{x}{2}, \frac{y}{2}\right) - \Psi_{\delta, \xi}(x, \frac{y}{2}) - \Psi_{\delta, \xi}(\frac{x}{2}, y) \right).
\]

Note here, that \(\Psi_{\delta, \xi}(0) = 1\).

Proof. For simplicity let us write \(\Omega_{\delta, \xi} = \Omega\) and \(\Psi_{\delta, \xi} = \Psi\). According to Lemma 3.1 we have
\[
\sum_{(x, y) \in \mathbb{Z}^2 \atop |x| \neq |y|} \exp \left\{ \Omega(x, y) \right\} = \sum_{(x, y) \in \mathbb{Z}^2 \atop xy \neq 0} \Psi(x, y) + \sum_{(x, y) \in \mathbb{Z}^2} \Psi(x - \frac{1}{2}, y + \frac{1}{2}).
\]

Let us first prove the following claim
\[
\sum_{(x, y) \in \mathbb{Z}^2} \Psi\left(x - \frac{1}{2}, y + \frac{1}{2}\right) = \sum_{(x, y) \in \mathbb{Z}^2 \atop xy \neq 0} \left( \Psi(x, y) - \Psi\left(\frac{x}{2}, y\right) - \Psi(x, \frac{y}{2}) + \Psi\left(\frac{x}{2}, \frac{y}{2}\right) \right).
\]

Fix \(x \in \mathbb{Z}\), then
\[
\sum_{y \in \mathbb{Z}} \Psi\left(x - \frac{1}{2}, y + \frac{1}{2}\right) = \sum_{y \in \mathbb{Z} \setminus \{0\}} \left( \Psi(x - \frac{1}{2}, \frac{y}{2}) - \Psi\left(x - \frac{1}{2}, y\right) \right)
\]
and summing this expression over \(x \in \mathbb{Z}\) yields
\[
\sum_{(x, y) \in \mathbb{Z}^2} \Psi\left(x - \frac{1}{2}, y + \frac{1}{2}\right) = \sum_{(x, y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})} \left( \Psi(x - \frac{1}{2}, \frac{y}{2}) - \Psi\left(x - \frac{1}{2}, y\right) \right).
\]

Finally, if we fix \(y \in \mathbb{Z} \setminus \{0\}\), then
\[
\sum_{x \in \mathbb{Z}} \Psi\left(x - \frac{1}{2}, y\right) = \sum_{x \in \mathbb{Z} \setminus \{0\}} \Psi\left(\frac{x}{2}, y\right) - \sum_{x \in \mathbb{Z} \setminus \{0\}} \Psi(x, y),
\]
and
\[
\sum_{x \in \mathbb{Z}} \Psi\left(x - \frac{1}{2}, y\right) = \sum_{x \in \mathbb{Z} \setminus \{0\}} \Psi\left(\frac{x}{2}, y\right) - \sum_{x \in \mathbb{Z} \setminus \{0\}} \Psi(x, y),
\]
which proves the claim after summing these last two expressions over \( y \in \mathbb{Z} \setminus \{0\} \). Hence,

\[
\sum_{(x,y) \in \mathbb{Z}^2 \atop x \neq y} e^{\Omega(x,y)} = \sum_{(x,y) \in \mathbb{Z}^2 \atop x \neq y} \left( 2\Psi(x,y) - \Psi\left(\frac{x}{2}, y\right) - \Psi\left(x, \frac{y}{2}\right) + \Psi\left(\frac{x}{2}, \frac{y}{2}\right) \right).
\]

Fix \( x \in \mathbb{Z} \setminus \{0\} \), then

\[
\sum_{y \in \mathbb{Z} \setminus \{0\}} \left( 2\Psi(x,y) - \Psi\left(\frac{x}{2}, y\right) - \Psi\left(x, \frac{y}{2}\right) + \Psi\left(\frac{x}{2}, \frac{y}{2}\right) \right) = \sum_{y \in \mathbb{Z}} \left( 2\Psi(x,y) + \Psi\left(\frac{x}{2}, y\right) - \Psi\left(x, \frac{y}{2}\right) + \Psi\left(\frac{x}{2}, \frac{y}{2}\right) \right) - \Psi(x, 0)
\]

and summing this expression over \( x \in \mathbb{Z} \setminus \{0\} \) yields

\[
\sum_{(x,y) \in \mathbb{Z}^2 \atop x \neq y} \left( 2\Psi(x,y) - \Psi\left(\frac{x}{2}, y\right) - \Psi\left(x, \frac{y}{2}\right) + \Psi\left(\frac{x}{2}, \frac{y}{2}\right) \right) = \sum_{(x,y) \in \mathbb{Z}^2} \left( 2\Psi(x,y) - \Psi\left(\frac{x}{2}, y\right) - \Psi\left(x, \frac{y}{2}\right) + \Psi\left(\frac{x}{2}, \frac{y}{2}\right) \right)
\]

\[
- \sum_{y \in \mathbb{Z}} \Psi(0,y) - \sum_{x \in \mathbb{Z}} \Psi(x, 0) + \Psi(0).
\]

\[\square\]

**3.2.** For any \( z = a + ib \in \mathbb{H} \) let us define for \( b \in \{0,1\} \)

\[
g_{z,b} := \left( \frac{1}{\sqrt{2b}} \begin{pmatrix} 1 & 2a \\ \sqrt{2b} & 1 \end{pmatrix} \right), \quad (3.1)
\]

\[
\Lambda_{z,b} := g_{z,b} \mathbb{Z}^2, \quad (3.2)
\]

\[
\Lambda_{z,b}^* := \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \Lambda_{z} \mid v_2 \neq 0 \right\}, \quad (3.3)
\]

and in addition to this set

\[
\mathcal{C}_b(\zeta, \xi, \eta) := \sqrt{\frac{2}{b}} \sum_{v \in \Lambda_{z,b}^*} e^{-\pi \|v + \frac{i}{\sqrt{2b}} \xi\|^2 + 2\pi i \frac{1}{\sqrt{2b}} \langle v, \eta \rangle} \quad (3.4)
\]

for any two \( \xi, \eta \in \mathbb{R}^2 \). Note that \( \sqrt{b} \mathcal{C}_b(\zeta, \xi, \eta) \) is majorized (up to a constant) from above by the Siegel transform of \( e^{-\pi \|\cdot\|^2} \) over the affine lattice \( \Lambda_{z,b} + \frac{i}{\sqrt{2b}} \xi \).
Lemma 3.3. Let $\zeta = a + ib \in \mathbb{H}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ an integrable function which is even in the second variable, that is, $F(\xi_1, -\xi_2) = F(\xi_1, \xi_2)$. Then,

$$
\int_{\mathbb{R}^2} F(\xi) \left( \sum_{(x,y) \in \mathbb{Z}^2} e^{\Omega_{\xi}(x,y)} \right) \, d\xi
= \int_{\mathbb{R}^2} F(\xi) \left( \psi_{\frac{1}{2} , \xi}(0) - \sqrt{2} \sum_{x \in \mathbb{Z}} e^{-\frac{i}{\sqrt{2}(x+1)} (\xi, 1)^2} \right) \, d\xi
+ \int_{\mathbb{R}^2} F(\xi) \left( C_0(\beta, \langle \xi, 1 \rangle) e_1, \langle \xi, 1 \rangle e_2 - C_1(\beta, \langle \xi, 1 \rangle) e_1, \langle \xi, 1 \rangle e_2 \right) \, d\xi,
$$

where $\bar{1} = \left( \frac{1}{1} \right)$, $\bar{\mathbf{1}} = \left( \frac{1}{-1} \right)$ and $C_0(\beta, \xi, \eta)$ is defined as in (3.4).

Proof. Let $\zeta = a + ib \in \mathbb{H}$ and $\xi \in \mathbb{R}$. Recall from Lemma 3.2 that

$$
\sum_{(x,y) \in \mathbb{Z}^2} e^{\Omega_{\xi}(x,y)}
= \psi(0) + \left( 2 \sum_{x,y \in \mathbb{Z}^2} \psi_{\frac{1}{2}, \xi}(x,y) - \sum_{y \in \mathbb{Z}} \psi_{\frac{1}{2}, \xi}(0,y) - \sum_{x \in \mathbb{Z}} \psi_{\frac{1}{2}, \xi}(x,0) \right)
+ \sum_{x,y \in \mathbb{Z}^2} \left( \psi_{\frac{1}{2}, \xi}(\frac{x}{2}, \frac{y}{2}) - \psi_{\frac{1}{2}, \xi}(\frac{x}{2}) - \psi_{\frac{1}{2}, \xi}(\frac{y}{2}) \right).
$$

A straightforward calculation shows that

$$
\Omega_{\xi}(x+y, x-y) = \pi i \left( a 4xy + i2b(x^2 + y^2) \right) + 2\pi i \langle \xi, \bar{1} \rangle + 2\pi iy \langle \xi, 1 \rangle,
$$

where $\bar{1} = \left( \frac{1}{1} \right)$ and $\bar{\mathbf{1}} = \left( \frac{1}{-1} \right)$. For any $r, s > 0$ let us state the following identities

$$
\sum_{x,y \in \mathbb{Z}} \psi_{\frac{1}{2}, \xi}(rx, sy) = \frac{1}{r\sqrt{2b}} \sum_{x,y \in \mathbb{Z}} e^{-\pi r^2 s^2 y^2 / (r^2 2b) (x+2rsy+r\langle \xi, 1 \rangle)^2 + 2\pi i y (s \xi, \bar{1})},
$$

$$
\sum_{x,y \in \mathbb{Z}} \psi_{\frac{1}{2}, \xi}(rx, sy) = \frac{1}{s \sqrt{2b}} \sum_{x,y \in \mathbb{Z}} e^{-\pi r^2 s^2 b x^2 / (s^2 2b) (y+2rsx+s\langle \xi, 1 \rangle)^2 + 2\pi iz (r \xi, \bar{1})},
$$

$$
\sum_{x \in \mathbb{Z}} \psi_{\frac{1}{2}, \xi}(rx, 0) = \frac{1}{r \sqrt{2b}} \sum_{x \in \mathbb{Z}} e^{-\pi r^2 s^2 x^2 / (r^2 2b) (x+r\langle \xi, 1 \rangle)^2},
$$

$$
\sum_{s \in \mathbb{Z}} \psi_{\frac{1}{2}, \xi}(0, sy) = \frac{1}{s \sqrt{2b}} \sum_{y \in \mathbb{Z}} e^{-\pi r^2 s^2 b y^2 / (s^2 2b) (y+s\langle \xi, 1 \rangle)^2}.
$$

Let us only prove the first identity as the second one is analogous to the first one and the latter two can be inferred from the former two. Indeed, note
that

\[ \sum_{x,y \in \mathbb{Z}} \Psi_{\delta,\xi}(rx, sy) = \sum_{y \in \mathbb{Z}} e^{-2\pi s^2y^2 + 2\pi iy(x\xi, 1)} \left( \sum_{x \in \mathbb{Z}} e^{\pi i(2x^2b^2x^2 + 2\pi ix(2rsy + (r\xi, 1)))} \right) \]

\[ = \frac{1}{r\sqrt{2b}} \sum_{x,y \in \mathbb{Z}} e^{-2\pi x^2s^2 y^2 + 2\pi iy(x\xi, 1)} - \frac{x}{2b^2} (x + 2rsy + (r\xi, 1))^2, \]

where we applied Poisson’s summation formula to the inner sum over \( x \).

We first deduce, using the above identities, that

\[ 2 \sum_{x,y \in \mathbb{Z}} \Psi_{\delta,\xi}(x, y) = \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} \Psi_{\delta,\xi}(x, 0) - \sum_{y \in \mathbb{Z}} \Psi_{\delta,\xi}(0, y) \]

\[ = \frac{1}{\sqrt{2b}} \sum_{(x,y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})} e^{-\pi 2b y^2 - \frac{\pi}{2b} (x + 2ay + (\xi, 1))^2} + 2\pi iy(x\xi, 1) \]

\[ + \frac{1}{\sqrt{2b}} \sum_{(x,y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}} e^{-\pi 2bx^2 - \frac{\pi}{2b} (y + 2ax + (\xi, 1))^2} + 2\pi iz(x\xi, 1). \]

Hence, if \( F \) is an even function in the second variable, it is plain to see that

\[ \int_{\mathbb{R}^2} F(\xi) \left( 2 \sum_{x,y \in \mathbb{Z}} \Psi_{\delta,\xi}(x, y) \right) \left( \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} \Psi_{\delta,\xi}(x, 0) - \sum_{y \in \mathbb{Z}} \Psi_{\delta,\xi}(0, y) \right) \, d\xi \]

\[ = \frac{\sqrt{\frac{2}{b}}}{\sqrt{2\pi}} \int_{\mathbb{R}^2} \hat{\psi}(\xi) \sum_{(x,y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})} e^{-\pi 2b y^2 - \frac{\pi}{2b} (x + 2ay + (\xi, 1))^2} + 2\pi iy(x\xi, 1) \, d\xi. \]

On the other hand, we have

\[ \frac{1}{2} \sum_{x,y \in \mathbb{Z}} \Psi_{\delta,\xi}(x, y) \]

\[ = \frac{1}{\sqrt{2b}} \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi 2b (\frac{x^2}{2})^2 - \frac{\pi}{2b} (2x + 2a\frac{x}{2} + (\xi, 1))^2} + 2\pi 1\frac{x}{2}(x\xi, 1) \]

and

\[ \sum_{x,y \in \mathbb{Z}} \Psi_{\delta,\xi}(x, y) \]

\[ = \frac{1}{\sqrt{2b}} \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi 2b (\frac{y^2}{2})^2 - \frac{\pi}{2b} (x + 2a\frac{y}{2} + (\xi, 1))^2} + 2\pi 1\frac{y}{2}(x\xi, 1) \]

\[ = \frac{1}{2} \sum_{x,y \in \mathbb{Z}} \Psi_{\delta,\xi}(x, y) + \frac{1}{\sqrt{2b}} \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi 2b (\frac{x^2}{2})^2 - \frac{\pi}{2b} (2x + 1 + 2a\frac{x}{2} + (\xi, 1))^2} + 2\pi 1\frac{x}{2}(x\xi, 1). \]
Similarly, by symmetry from the previous case, we deduce
\[
\frac{1}{2} \sum_{x,y \in \mathbb{Z}} \Psi_{b,\xi}(x/2, y/2) = \frac{1}{\sqrt{2b}} \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi 2b\left(\frac{x}{2}\right)^2 - \frac{\pi}{20} (2y + 2a + 2\sqrt{x})^2 + 2\pi i \frac{b}{\sqrt{2}}(\xi, 1)}
\]
and
\[
\sum_{x,y \in \mathbb{Z}} \Psi_{b,\xi}(x/2, y) = \frac{1}{2} \sum_{x,y \in \mathbb{Z}} \Psi_{b,\xi}(x/2, y/2) + \frac{1}{\sqrt{2b}} \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi 2b\left(\frac{x}{2}\right)^2 - \frac{\pi}{20} (2y + 1 + 2a + 2\sqrt{x})^2 + 2\pi i \frac{b}{\sqrt{2}}(\xi, 1)}.
\]
Thus, we conclude that
\[
\sum_{x,y \in \mathbb{Z}} \left( \Psi_{b,\xi}(x/2, y/2) - \Psi_{b,\xi}(x/2, y) - \Psi_{b,\xi}(x/2, y) \right)
= -\frac{1}{\sqrt{2b}} \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi 2b\left(\frac{x}{2}\right)^2 - \frac{\pi}{20} (2y + 1 + 2a + 2\sqrt{x})^2 + 2\pi i \frac{b}{\sqrt{2}}(\xi, 1)}
- \frac{1}{\sqrt{2b}} \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi 2b\left(\frac{x}{2}\right)^2 - \frac{\pi}{20} (2y + 1 + 2a + 2\sqrt{x})^2 + 2\pi i \frac{b}{\sqrt{2}}(\xi, 1)}.
\]
and after integrating this expression against $F$, we obtain
\[
\int_{\mathbb{R}^2} F(\xi) \sum_{x,y \in \mathbb{Z}} \left( \Psi_{b,\xi}(x/2, y/2) - \Psi_{b,\xi}(x/2, y) - \Psi_{b,\xi}(x/2, y) \right) d\xi
= -\int_{\mathbb{R}^2} F(\xi) \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi 2b\left(\frac{x}{2}\right)^2 - \frac{\pi}{20} (2y + 1 + 2a + 2\sqrt{x})^2 + 2\pi i \frac{b}{\sqrt{2}}(\xi, 1)} d\xi.
\]
It is then easy to see that
\[
\int_{\mathbb{R}^2} F(\xi) \sum_{(x,y) \in \mathbb{Z}^2, |x| \neq |y|} e^{\Omega_{b,\xi}(x,y)} d\xi
= \int_{\mathbb{R}^2} F(\xi) \left( \Psi_{b,\xi}(0) + \sqrt{\frac{2}{b}} \sum_{(x,y) \in \mathbb{Z} \setminus \{0\}} e^{-\pi 2b y^2 - \frac{\pi}{20} (x + 2ay + (\xi, 1))^2 + 2\pi iy(\xi, 1)} \right)
- \sqrt{\frac{2}{b}} \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}} e^{-\pi 2b\left(\frac{x}{2}\right)^2 - \frac{\pi}{20} (2y + 1 + 2a + 2\sqrt{x})^2 + 2\pi i \frac{b}{\sqrt{2}}(\xi, 1)} d\xi.
\]
\[\Box\]

**Lemma 3.4.** Let $\theta_j = \alpha j^2$ and let $f : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ be test functions of class $\mathcal{P}$. Then, for any $\epsilon > 0$ there is $N_0 = N_0(\epsilon)$ such that for
any \( N \geq N_0 \)

\[ E_\sigma^f(f, \psi, \{\theta_j\}_j, N) \ll \frac{1}{N} + E_\sigma^\psi(\psi, \{\theta_j\}_j, N), \]

where \( E_\sigma^\psi(\psi, \{\theta_j\}_j, N) \) is defined as

\[ \frac{1}{N^2} \int_{\mathbb{R}^2} |\hat{\psi}(\xi)| \sum_{1 \leq |n| \leq N^{\sigma+\epsilon}} \sum_{b \in \{0,1\}} C_b \left( \delta_n^\alpha, \sqrt{\frac{\pi}{2}} N \langle \xi, \bar{1} \rangle e_1, 0 \right) d\xi, \]

and

\[ \delta_n^\alpha = 2n\alpha + i \frac{1}{\pi N^2} \in \mathbb{H}. \]

Proof. Recall that \( E_\sigma^f(f, \psi, \{\theta_j\}_j, N) \) is given by

\[ \left| \frac{1}{N^2} \int_{\mathbb{R}^2} \hat{\psi}(\xi) \sum_{n \neq 0} \hat{f} \left( \frac{n}{N^\sigma} \right) S(\{\theta_j\}_j, N, n, \xi) d\xi \right|, \]

where \( S(\{\theta_j\}_j, N, n, \xi) \) is defined as in Lemma 2.3. It is easy to see that

\[ S(\{\theta_j\}_j, N, n, \xi) = \sum_{(x,y) \in \mathbb{Z}^2} \exp \left\{ \Omega_{\delta_n^\alpha, \xi, \eta}(x, y) \right\}, \]

where

\[ \delta_n^\alpha = 2n\alpha + i \frac{1}{\pi N^2} \in \mathbb{H}. \]

Since, \( \hat{\psi} \) has compact support, we can choose \( N_0 \) such that for all \( N \geq N_0, \)

\[ \| \xi \|^2 - \frac{1}{2} \]

for all \( \xi \in \text{supp}(\hat{\psi}) \) and in addition

\[ 2\pi N \sum_{x \in \mathbb{Z}} e^{-\pi^2 N(2x^2 + (\xi, \bar{1})^2) \leq 1.} \]

Thus, according to Lemma 3.3, \( E_\sigma^f(f, \psi, \{\theta_j\}_j, N) \) is bounded from above, up to a constant depending on \( \psi, \)

\[ \frac{1}{N} + \frac{1}{N^2} \int_{\mathbb{R}^2} |\hat{\psi}(\xi)| \sum_{n \in \mathbb{Z}\setminus\{0\}} \left| \hat{f} \left( \frac{n}{N^\sigma} \right) \right| \sum_{b \in \{0,1\}} C_b \left( \delta_n^\alpha, \sqrt{\frac{\pi}{2}} N \langle \xi, \bar{1} \rangle e_1, 0 \right) d\xi, \]

for all \( N \geq N_0. \) Let us now eliminate the tail of the sum over \( n \) for the right-hand side expression in the former equation. Since \( \hat{f} \) is compactly supported or equal to \( e^{-2\pi |\cdot|^{-\epsilon}}, \) for any \( \epsilon > 0, \) we can assume that

\[ \left| \hat{f} \left( \frac{n}{N^\sigma} \right) \right| \leq e^{-N^{\sigma - \frac{\epsilon}{2}}}, \]

for all \( |n| \geq N^{\sigma+\epsilon}. \)

Thus, a geometric series argument together with the trivial upper bound

\[ |C(\delta_n^\alpha, \xi, \eta)| \ll N^2, \]

implies that

\[ \frac{1}{N^2} \int_{\mathbb{R}^2} |\hat{\psi}(\xi)| \sum_{|n| \geq N^{\sigma+\epsilon}} \left| \hat{f} \left( \frac{n}{N^\sigma} \right) \right| \sum_{u \in U} C(\delta_n^\alpha, u) \ll \psi N^\sigma e^{-N^\epsilon}. \]
Thus, we can enlarge $N_0$ (depending on $\epsilon$) once again, to require
\[
\frac{1}{N^2} \int_{\mathbb{R}^2} |\hat{\psi}(\xi)| \sum_{|n| \geq N^{\sigma+\epsilon}} \left| \sum_{b \in \{0,1\}} C_b \left( \frac{a_n}{\sqrt{2}}, \sqrt{\frac{\pi}{2}} \frac{1}{a_n} \frac{\hat{f}(n)}{\sqrt{a_n}} \right) \right| \, d\xi \leq \frac{1}{N},
\]
for all $N \geq N_0$. \qed

4. An Argument from the Geometry of Numbers

4.1. For a two-dimensional unimodular lattice $\Delta \subset \mathbb{R}^2$ denote by $a(\Delta)$ the first successive minimum (its reciprocal is commonly referred to as the height) of $\Delta$ (i.e. the length of the shortest non-zero vector in $\Delta$); It is well-known that $a(\Delta) \leq \frac{2}{\sqrt{3}}$. For any lattice $\Delta \subset \mathbb{R}^2$ and any $\mu > 0$ let us denote by
\[
N_\Delta(\mu) := \left\{ v \in \Delta \mid \|v\| \leq \mu \right\},
\]
the number of lattice points contained in the disk of radius $\mu$. The following three Lemmas are basic results in the Geometry of Numbers. They can all be generalized to higher dimensions, but we restrict it here to the case of two dimensional unimodular lattices.

4.2. The Lipschitz principle in the Geometry of Numbers \cite{Dav51} is a well-known counting method for the number of lattice points in a bounded region in terms of the successive minima:

**Lemma 4.1.** Let $\Delta$ be a two-dimensional unimodular lattice and $\mu > 0$. Then,
\[
N_\Delta(\mu) \ll \begin{cases} 
1 + \mu a(\Delta)^{-1}, & \text{if } a(\Delta) \leq \mu < a(\Delta)^{-1} \\
1 + \mu^2, & \text{if } \mu \geq a(\Delta)^{-1}.
\end{cases}
\]
The implicit constant is independent of $\Delta$.

**Proof.** Let us write $\Delta = g \mathbb{Z}^2$ with $g \in SL_2(\mathbb{R})$. It is well-known that there are $\gamma \in SL_2(\mathbb{Z})$, $k \in SO(2)$ and $s \in [-\frac{1}{2}, \frac{1}{2}]$ such that
\[
g \gamma = k \begin{pmatrix} a(\Delta) & \ast \\
\ast & a(\Delta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\
1 & 1 \end{pmatrix}.
\]
Thus, if
\[
v = k \begin{pmatrix} a(\Delta) & \ast \\
\ast & a(\Delta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\
1 & 1 \end{pmatrix} \begin{pmatrix} x \\
y \end{pmatrix} \in \Delta, \ x, y \in \mathbb{Z}
\]
satisfies $\|v\| \leq \mu$, then
\[
a(\Delta) |x + sy| \leq \mu, \\
a(\Delta)^{-1} |y| \leq \mu.
\]
If $a(\Delta) \leq \mu < a(\Delta)^{-1}$, then $y = 0$ and thus $|x| \leq a(\Delta)^{-1} \mu$, which proves the first assertion. If $\mu \geq a(\Delta)^{-1}$, then there are at most $1 + 2a(\Delta)\mu$ possibilities
for \( y \); And for each such \( y \) any \( x \in \mathbb{Z} \) satisfying \( a(\Delta)|x + sy| \leq \mu \) lies in an interval of length \( 2a(\Delta)^{-1}\mu \), from which we deduce that there are at most \( 1 + 2a(\Delta)^{-1}\mu \) such integers. Thus, we conclude that there are at most \((1 + 2a(\Delta))\mu + 6\mu^2\) possible lattice points \( v \in \Delta \) satisfying \( \|v\| \leq \mu \) in this case. \( \square \)

4.3. In the following it will be convenient to introduce the following notation: For any \( P, Q > 0 \) and \( \alpha \in \mathbb{R} \) let us introduce the lattice

\[
\Delta_{P,Q}^\alpha := \left( P \begin{array}{c} 1 \\ \alpha \end{array} \right) \mathbb{Z}^2.
\]

Moreover, we shall simply write \( \Delta_{P}^\alpha \) to denote \( \Delta_{P,P-1}^\alpha \), i.e.

\[
\Delta_{P}^\alpha := \left( P \begin{array}{c} 1 \\ \alpha \end{array} \right) \mathbb{Z}^2.
\]

The height of such lattices is known to exhibit a better upper bound in terms of \( P \) than the trivial one when \( \alpha \) is Diophantine.

**Lemma 4.2.** Let \( \alpha \in \mathbb{R} \) be Diophantine of type \( \kappa \). Then, for any \( P > \frac{2}{\sqrt{3}} \), we have

\[
a(\Delta_{P}^\alpha)^{-1} \leq c^{-\frac{1}{\kappa}} P^{1-\frac{2}{\kappa}}.
\]

**Proof.** Let \( (p, q) = 1 \) be a coprime integer pair such that

\[
a(\Delta_{P}^\alpha)^2 = P^2(p + \alpha q)^2 + P^{-2}q^2,
\]

and note that \( q \neq 0 \) as long as \( P > \frac{2}{\sqrt{3}} \). Thus,

\[
a(\Delta_{P}^\alpha) \geq P|p + \alpha q| \geq \frac{cP}{q^{\frac{1}{\kappa}-1}} \geq \frac{ca(\Delta_{P}^\alpha)^{-\kappa+1}}{P^{\kappa-2}},
\]

which proves the assertion. \( \square \)

**Lemma 4.3.** Let \( P, Q, \alpha \in \mathbb{R}, C > 0 \) and \( u \in \mathbb{R}^2 \). Then, for any \( Z, \zeta > 0 \)

\[
\sum_{v \in \Delta_{P,Q}^\alpha} e^{-C\|v+u\|^2} \leq \left( 1 + N_{\Delta_{(Z\zeta^{-1})^{1/2}}^\alpha}(\sqrt{2}(Z\zeta)^{\frac{1}{2}}) \right) \sum_{v \in \Delta_{P,Q}^\alpha} e^{-C\|v\|^2}.
\]

In particular, if \( P\zeta \gg 1 \) and \( QZ \gg 1 \), then

\[
\sum_{v \in \Delta_{P,Q}^\alpha} e^{-C\|v+u\|^2} \ll 1 + N_{\Delta_{(Z\zeta^{-1})^{1/2}}^\alpha}(\sqrt{2}(Z\zeta)^{\frac{1}{2}}).
\]

**Proof.** For simplicity we prove this for \( C = 1 \). Any element \( v \in \Delta_{P,Q}^\alpha \) is of the form

\[
v = \left( P(x + \alpha z) \begin{array}{c} 1 \\ Qz \end{array} \right), \ x, z \in \mathbb{Z}
\]
and so
\[ \|v + u\|^2 = P^2\left(x + \alpha z + \frac{u_1}{P}\right)^2 + Q^2\left(z + \frac{u_2}{Q}\right)^2 \]
\[ = (P\zeta)^2 \frac{1}{\zeta^2}\left(x + \alpha z + \frac{u_1}{P}\right)^2 + (QZ)^2 \frac{1}{Z^2}\left(z + \frac{u_2}{Q}\right)^2. \]

For any \(x, z \in \mathbb{Z}\) (and consequently \(v\)) there is a unique \(m = (m_1, m_2) \in \mathbb{Z}^2\) such that
\[ \frac{1}{\zeta}\left(x + \alpha z + \frac{u_1}{P}\right) \in \left[m_1 - \frac{1}{2}, m_1 + \frac{1}{2}\right), \]
\[ \frac{1}{Z}\left(z + \frac{u_2}{Q}\right) \in \left[m_2 - \frac{1}{2}, m_2 + \frac{1}{2}\right). \]

Observe, in this case, that
\[ \left|\frac{1}{\zeta}\left(x + \alpha z + \frac{u_1}{P}\right)\right| \geq \frac{|m_1|}{2}, \]
\[ \left|\frac{1}{Z}\left(z + \frac{u_2}{Q}\right)\right| \geq \frac{|m_2|}{2}. \]

Let us denote by \(\mathcal{B}_m(\zeta, Z)\) the collection of \(v \in \Delta\) satisfying (4.2), then
\[ \sum_{v \in \Delta} e^{-\|v + u\|^2} \leq \sum_{m \in \mathbb{Z}^2} \sum_{v \in \mathcal{B}_m(\zeta, Z)} e^{-\|v + u\|^2} \leq \sum_{m \in \mathbb{Z}^2} \#\mathcal{B}_m(\zeta, Z) e^{-\frac{1}{4}\left((P\zeta)^2m_1^2 + (QZ)^2m_2^2\right)}.

For each \(m \in \mathbb{Z}^2\), either \(\#\mathcal{B}_m(\zeta, Z) \leq 1\) or \(\mathcal{B}_m(\zeta, Z)\) contains at least two elements. In the latter case, fix an element of the form
\[ w_0 = \left(\frac{1}{\zeta}\left(x_0 + \alpha z_0 + \frac{u_1}{P}\right), \frac{1}{Z}\left(z_0 + \frac{u_2}{Q}\right)\right), \quad x_0, z_0 \in \mathbb{Z} \]
satisfying (4.2), then for any other
\[ w = \left(\frac{1}{\zeta}\left(x + \alpha z + \frac{u_1}{P}\right), \frac{1}{Z}\left(z + \frac{u_2}{Q}\right)\right), \quad x, z \in \mathbb{Z}, \]
satisfying (4.2) note that
\[ w - w_0 = \left(\frac{1}{\zeta}\left((x - x_0) + \alpha(z - z_0)\right)\right) \in [-1, 1]^2. \]

Thus, if \(\mathcal{B}_m(\zeta, Z)\) contains at least two elements, then
\[ \#\mathcal{B}_m(\zeta, Z) \leq \#\left\{\left(\frac{x}{z}\right) \in \mathbb{Z}^2 \mid |x + \alpha z| \leq \zeta, |z| \leq Z\right\}. \]
Finally observe that the inequalities
\[ |x + \alpha z| \leq \zeta, \quad |z| \leq Z \]
can be rewritten as
\[ \left( \frac{Z}{\zeta} \right)^{\frac{1}{2}} |x + \alpha z| \leq (Z \zeta)^{\frac{1}{2}}, \quad \left( \frac{\zeta}{Z} \right)^{\frac{1}{2}} |z| \leq (Z \zeta)^{\frac{1}{2}} \]
and hence
\[ \# \left\{ \left( \frac{x}{z} \right) \in \mathbb{Z}^2 \left| |x + \alpha z| \leq \zeta, \quad |z| \leq Z \right. \right\} = \# \left\{ v \in \Delta_{\zeta, (Z-1)/2}^\alpha \left| v \in \left[ -(Z \zeta)^{\frac{1}{2}}, (Z \zeta)^{\frac{1}{2}} \right] \right. \right\} \]
\[ \leq N \Delta_{\zeta, (Z-1)/2}^\alpha (\sqrt{2}(Z \zeta)^{\frac{1}{2}}). \]

4.4. The following lemma is the main step towards Theorem 1.1.

Lemma 4.4. Let \( \sigma > 0, b \in \{0,1\} \) and let \( \alpha \in \mathbb{R} \) be Diophantine of type \( \kappa \). Then, for any \( \epsilon > 0 \) and any \( u \in \mathbb{R} \)
\[ \frac{1}{N} \sum_{1 \leq n \leq N^{\sigma+\epsilon}} \sum_{v \in \Lambda_{\eta_{n,N},b}^\alpha} e^{-\pi \|v + ue_1\|^2} \ll \epsilon \ N^{\sigma-1+\epsilon} + N^{\sigma - \frac{2+\epsilon}{\kappa} + \epsilon(1 - \frac{1}{\kappa})}, \]
where \( \eta_{n,N}^\alpha \) is defined in Lemma 3.4 and \( \Lambda_{\eta_{n,N},b}^\alpha \subset \Lambda_{\eta_{n,N},b}^\kappa \) is defined as in (3.2) and (3.3). Note that the right-hand side decays to zero as \( N \to \infty \) as long as \( \sigma < 1 \) and \( \kappa < 1 + \frac{2}{\sigma} \).

Proof. Let us prove this for \( b = 0 \) only, as the case \( b = 1 \) is exactly the same. Let \( \epsilon > 0 \) to be determined below. Denote by \( \tau(\cdot) \) the divisor function and observe that
\[ \frac{1}{N} \sum_{1 \leq n \leq N^{\sigma+\epsilon}} \sum_{v \in \Lambda_{\eta_{n,N}}^\alpha} e^{-\pi \|v + u e_1\|^2} \]
\[ = \frac{1}{N} \sum_{1 \leq n \leq N^{\sigma+\epsilon}} \sum_{(x,y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})} e^{-2\pi^2 N^2 (x + 4\alpha y + \frac{u}{\sqrt{2\pi N}})^2 - 2 \frac{u^2}{N^2}} \]
\[ \leq \frac{1}{N} \sum_{1 \leq n \leq N^{\sigma+\epsilon}} \sum_{(x,y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})} e^{-2\pi^2 N^2 (x + 4\alpha y + \frac{u}{\sqrt{2\pi N}})^2 - \frac{1}{8} \frac{(4\alpha y)^2}{N^{2(1+\sigma+\epsilon)}}} \]
\[ \leq \frac{2}{N} \sum_{z=1}^{\infty} \tau(z) \sum_{x \in \mathbb{Z}} e^{-2\pi^2 N^2 (x + \alpha z + \frac{u}{\sqrt{2\pi N}})^2 - \frac{1}{8} \frac{(4\alpha z)^2}{N^{2(1+\sigma+\epsilon)}}}. \]
Using the fact that $\tau(z) \ll e^{\frac{1}{2}z^{2(1+\sigma+2\epsilon)}}$, we first deduce
\[
\sum_{1 \leq z \leq N^{2(1+\sigma+2\epsilon)}} \tau(z) \sum_{x \in \mathbb{Z}} e^{-2\pi^2 N^2 (x+az+\frac{u}{\sqrt{2\pi N}})^2 - \frac{1}{8} N^{2(1+\sigma+\epsilon)}} \ll \epsilon, \sigma \sum_{1 \leq z \leq N^{2(1+\sigma+2\epsilon)}} \sum_{x \in \mathbb{Z}} e^{-2\pi^2 N^2 (x+az+\frac{u}{\sqrt{2\pi N}})^2 - \frac{1}{8} N^{2(1+\sigma+\epsilon)}}
\]
but we also have
\[
\ll \epsilon, \sigma \sum_{z \geq N^{2(1+\sigma+2\epsilon)}} \sum_{x \in \mathbb{Z}} e^{-2\pi^2 N^2 (x+az+\frac{u}{\sqrt{2\pi N}})^2 - \frac{1}{8} N^{2(1+\sigma+\epsilon)}}
\]
where we use that for $z \geq N^{2(1+\sigma+2\epsilon)}$
\[
z^{2(1+\sigma+2\epsilon)} e^{-\frac{1}{16} N^{2(1+\sigma+\epsilon)}} \leq 1,
\]
for all $N$ sufficiently large. Thus, we find that
\[
\frac{1}{N} \sum_{1 \leq n \leq N^{\sigma+\epsilon}} \sum_{v \in \Lambda^*_{\gamma_0,n}} e^{-\pi \|v+ue_1\|^2} \ll \epsilon, \sigma \frac{1}{N^{1-\epsilon}} \sum_{v \in \Delta^*_{\gamma_0,1/N^{1+\sigma+\epsilon}-1}} e^{-\frac{1}{16} \|v+ue_1\|^2}. \tag{4.3}
\]
Let us apply Lemma 4.3 with $P = N, Q = (N^{1+\sigma+\epsilon})^{-1}, \zeta = N^{-1}, \Lambda = N^{1+\sigma+\epsilon}$ to obtain the following estimate
\[
\sum_{v \in \Delta^*_{\gamma_0,1/N^{1+\sigma+\epsilon}-1}} e^{-\frac{1}{16} \|v+ue_1\|^2} \ll \left(1 + N_{\Delta^0_{\gamma_0,1/N^{1+\sigma+\epsilon}}/2} \left(\sqrt{2} N \frac{2+\sigma+\epsilon}{2}\right)\right). \tag{4.4}
\]
Moreover, according to Lemma 4.2, since $\alpha$ is Diophantine of type $\kappa$, we have
\[
a(\Delta^0_{\gamma_0,1/N^{1+\sigma+\epsilon}}/2)^{-1} \ll N^{2+\frac{\sigma+\epsilon}{2}} \left(1 - \frac{2}{\kappa}\right).
\]
Hence, Lemma 4.1 implies
\[
N_{\Delta^0_{\gamma_0,1/N^{1+\sigma+\epsilon}}/2} \left(\sqrt{2} N \frac{2+\sigma+\epsilon}{2}\right) \ll N^{\sigma+\epsilon} + N^{2+\frac{\sigma+\epsilon}{2}} \left(1 - \frac{2}{\kappa}\right) + \frac{\sigma+\epsilon}{2}, \tag{4.5}
\]
The assertion follows now from (4.3), (4.4) and (4.5). \qed

**Proof of Theorem 7.1.** According to Lemma 2.2 it suffices to show
\[
\lim_{N \to \infty} R^2(f, \psi \cdot e^{-\|\cdot\|^2}, \{\theta_j, N\}, N) = \hat{f}(0) \left(\mathcal{F}(\psi \cdot e^{-\|\cdot\|^2})\right)(0),
\]
for any two test functions \(f : \mathbb{R} \to \mathbb{R}\) and \(\psi : \mathbb{R}^2 \to \mathbb{R}\) of class \(\mathcal{P}\). According to Lemma 3.4, this last identity follows if 
\[
\lim_{N \to \infty} E_{\sigma}^\epsilon(f, \psi, \{\alpha_j^2\}, N) = 0,
\]
for some fixed \(\epsilon > 0\). It is plain to see that 
\[
E_{\sigma}^\epsilon(f, \psi, \{\alpha_j^2\}, N) \ll_{\psi, f} \sup_{u \in \mathbb{R}} \left| \sum_{b \in \{0, 1\}} \frac{1}{N} \sum_{1 \leq n \leq N^{\sigma+\epsilon}} \sum_{v \in \Lambda_{n, N, b}} e^{-\pi \|v+ue_1\|^2} \right|.
\]
and the right-hand side is bounded above by 
\[
N^{\sigma-1+\epsilon} + N^{\sigma-\frac{2\kappa \sigma}{\kappa}} + (1-\frac{1}{\kappa}) \] 
by Lemma 4.4. We clearly require \(\sigma < 1\) and \(\kappa < 1 + \frac{2}{\sigma}\). \(\square\)

5. Upper Bound for \(\sigma(\{n^2 \alpha\}_n)\) for Almost Every \(\alpha\)

5.1. This section contains the proof of Theorem 1.3. It follows essentially the same strategy as that of Rudnick and Sarnak [RS98] up to some small modifications.

5.2. The first step consists in showing that the variance of the \(\sigma\)-pair correlation is small. Let \(f : \mathbb{R} \to \mathbb{R}\) be a test function such that \(\hat{f} \in C_c(\mathbb{R})\) has compact support and set 
\[
R_{\sigma}^\epsilon(f, \{ j^2 \alpha \}, N) := R_{\sigma}^\epsilon(f, 1_{[1, N]^2}, \{ j^2 \alpha \}, N),
\]
as in Section 2.1. Following Rudnick and Sarnak [RS98] we define 
\[
X_N(\alpha) := R_{\sigma}^\epsilon(f, \{ j^2 \alpha \}, N) - \hat{f}(0).
\]

Lemma 5.1. For any \(\epsilon > 0\), we have 
\[
\|X_N\|_{L^2([0,1])}^2 \ll_{\psi, f} \frac{1}{N^{2-\sigma}}
\]
where \(\| \cdot \|_{L^2([0,1])}\) denotes the standard norm on the \(L^2\)-space on \([0,1]\) with respect to the Lebesgue measure.

Proof. Note that \(X_N\) is a periodic function in the variable \(\alpha\). Thus, we can express it as a Fourier series 
\[
X_N(\alpha) = \sum_{l \in \mathbb{Z}} c_l(N)e(l\alpha),
\]
where 
\[
c_l = \frac{1}{N^2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{1 \leq j \neq k \leq N} \frac{\hat{f} \left( \frac{n}{N^{\sigma}} \right)}{(j^2-k^2)n=l},
\]
and 
\[
c_0 = O\left(\frac{1}{N}\right), \text{ as } N \to \infty.
\]
Note that $c_l(N) \ll \frac{\tau(|l|)^2}{N^2}$ for any $l \neq 0$, where $\tau$ denotes the divisor function, and hence for any fixed $\epsilon > 0$

$$c_l(N) \ll \epsilon \frac{|l|}{N^2} \text{ for any } l \neq 0.$$  

Moreover, for any fixed $\delta > 0$ and all sufficiently large $N$

$$c_l(N) = 0 \text{ for all } l \geq N^{2+\sigma+\delta}.$$  

It follows from these two observations together with Parseval’s identity that

$$\|X_N\|_{L^2([0,1])}^2 \ll f, \epsilon \frac{1}{N^{2-\sigma-\epsilon}}. \quad (5.1)$$

\[\square\]

### 5.3.

The second step consists in noticing that small variance leads to almost everywhere convergence of the $\sigma$-pair correlation along a sparse subsequence of Planck constants. Fix $0 < \delta < 1$ such that

$$(2 - \sigma)(1 + \delta) > 1 \quad (5.2)$$

and choose $\epsilon > 0$ small enough such that $(2 - \sigma - \epsilon)(1 + \delta) > 1$. Let $\{N_m\}_m$ be a sequence of integers with

$$N_m \sim m^{1+\delta}.$$  

Then, according to Lemma 5.1, we find

$$\lim_{m \to \infty} X_{N_m}(\alpha) = 0 \text{ for almost every } \alpha \in [0,1]. \quad (5.3)$$

We can now choose a set $P(f)$ of full measure in $[0,1]$ such that for any $\alpha \in P(f)$, $\alpha$ is Diophantine and satisfies (5.3).

### 5.4.

The final step consists in proving that for any $\alpha \in P(f)$, due to the Diophantine nature of $\alpha$, the oscillations $X_n(\alpha) - X_{N_m}(\alpha)$ along the sparse subsequence $\{N_m\}_m$ are small for all $N_m \leq n < N_{m+1}$. As $n - N_m \leq N_{m+1} - N_m \ll N_m^{\delta}$ it suffices to prove the following

**Lemma 5.2.** Let $0 \leq \sigma < 2$ and $\delta > 0$ such that

$$-2 + \sigma + 2\delta < 0, \text{ and}$$

$$-2 + \frac{1}{2} + \sigma + \delta < 0, \quad (5.4)$$

and let $\alpha$ be Diophantine. Then,

$$\sup_{0 \leq l \leq N^\delta} |X_{N+l}(\alpha) - X_N(\alpha)| \to 0, \text{ as } N \to \infty.$$
In view of (5.3) and Lemma 5.2, it follows that
\[
\lim_{N \to \infty} X_N(\alpha) = 0,
\]
for all \(\alpha \in P(f)\), as long as \(\sigma\) and \(\delta\) satisfy both (5.2) and (5.4). It is plain to see that these two requirements are satisfied if
\[
0 \leq \sigma \leq 1 \quad \text{and} \quad 0 < \delta < 1 - \frac{\sigma}{2}, \quad \text{or} \quad 1 < \sigma < \frac{1}{4}(9 - \sqrt{17}) \quad \text{and} \quad \frac{\sigma - 1}{2 - \sigma} < \delta < \frac{3}{2} - \sigma.
\]
Theorem 1.3 follows then easily from this observation. The rest of this section will be devoted towards the proof of Lemma 5.2.

**Proof of Lemma 5.2** The number \(\epsilon\) will denote a small positive quantity, whose value will be adapted as needed during the argument. Let us set \(M = N^{\sigma + \epsilon}\). Then, for all sufficiently large \(N\) we have
\[
X_N(\alpha) = \frac{1}{N^2} \sum_{0 < |n| \leq M} \hat{f}\left(\frac{n}{N^\sigma}\right) \sum_{1 \leq j \neq k \leq N} e(n\alpha(j^2 - k^2)), \quad \text{as well as}
\]
\[
X_{N+l}(\alpha) = \frac{1}{(N+l)^2} \sum_{0 < |n| \leq M} \hat{f}\left(\frac{n}{(N+l)^\sigma}\right) \sum_{1 \leq j \neq k \leq N} e(n\alpha(j^2 - k^2)).
\]
for all \(0 \leq l \leq N^\delta\). Moreover, for any \(0 \leq l \leq N^\delta\), we have
\[
\frac{1}{(N+l)^2} = \frac{1}{N^2} + O(N^{-3+\delta}),
\]
\[
\hat{f}\left(\frac{n}{(N+l)^\sigma}\right) = \hat{f}\left(\frac{n}{N^\sigma}\right) + O(N^{-1+\delta+\epsilon}),
\]
where the last identity follows from \(\frac{n}{(N+l)^\sigma} = \frac{n}{N^\sigma} + O\left(\frac{M^\delta}{N}\right)\). Thus, it is easy to see that for \(0 \leq l \leq N^\delta\) and \(N\) sufficiently large,
\[
\left|X_{N+l}(\alpha) - \frac{1}{N^2} \sum_{0 < |n| \leq M} \hat{f}\left(\frac{n}{N^\sigma}\right) \sum_{1 \leq j \neq k \leq N+l} e(n\alpha(j^2 - k^2))\right| 
\leq N^{-2+\sigma+\delta+2\epsilon} \frac{1}{N^{3-\delta-\epsilon}} \sum_{0 < n \leq M} |S_\alpha(n, N+l)|^2,
\]
where
\[
S_\alpha(n, N) := \sum_{1 \leq j \leq N} e(n\alpha j^2).
\]
At this point let us note that we require
\[
-2 + \sigma + \delta < 0.
\]
5.4.1. Let us provide a short proof of the fact that

$$\sum_{n=1}^{M} |S_\alpha(n, N)|^2 \ll \epsilon (NM)^{1+\epsilon}, \text{ for any } \epsilon > 0,$$

(5.7)

for a Diophantine number \(\alpha\). Indeed, it is easy to see that (e.g. Lemma 3.1 in [Dav05])

$$|S_\alpha(n, N)|^2 \ll N + \sum_{v=1}^{N} \min \left\{ N, \frac{1}{\|2\alpha n v\|_Z} \right\},$$

where \(\| \cdot \|_Z\) denotes the distance to the nearest integer. Moreover,

$$\sum_{n=1}^{M} \sum_{v=1}^{N} \min \left\{ N, \frac{1}{\|2\alpha n v\|_Z} \right\} \ll \epsilon N^{\epsilon/2} \sum_{z=1}^{2MN} \min \left\{ N, \frac{1}{\|\alpha z\|_Z} \right\} \ll N^{1+\epsilon/2} \left( \sum_{1 \leq z \leq 2MN \|\alpha z\|_Z \leq N^{-1}} 1 \right) + N^{\epsilon/2} \sum_{1 \leq z \leq 2NM \|\alpha z\|_Z > N^{-1}} \frac{1}{\|\alpha z\|_Z}.$$

First observe that

$$\sum_{1 \leq z \leq 2MN \|\alpha z\|_Z \leq N^{-1}} 1 \leq N_{\Delta \sqrt{2MN}}(2\sqrt{M}) \ll (MN)^{\epsilon/2},$$

where we use the notation introduced in Section 4.1 and Section 4.3, as well as Lemma 4.2. Similarly, notice that

$$\sum_{1 \leq z \leq 2NM \|\alpha z\|_Z > N^{-1}} \frac{1}{\|\alpha z\|_Z} \ll \sum_{r=0}^{\log_2(N)} 2^r \sum_{1 \leq z \leq 2NM \|\alpha z\|_Z \leq 2^{-r}} 1 \leq \sum_{r=0}^{\log_2(N)} 2^r N_{\Delta \sqrt{2^{r+1}MN}}(\sqrt{2^{-r+2}MN}) \ll (MN)^{1+\epsilon/2}.$$

5.4.2. Let us return to the proof Lemma 5.2. In view of (5.5) and (5.7) we find that

$$\left| X_{N+l}(\alpha) - \frac{1}{N^2} \sum_{0 < |n| \leq M} \hat{f}(\frac{n}{N^\sigma}) \sum_{1 \leq j \neq k \leq N+l} e(n\alpha(j^2 - k^2)) \right| \ll \epsilon N^{-2+\sigma+\delta+\epsilon}$$

(5.8)
for any $\epsilon > 0$. In order to relate this last estimate to the oscillation $|X_{N+l}(\alpha) - X_N(\alpha)|$, notice that

$$\left| \sum_{1 \leq j \neq k \leq N+l} e(n\alpha(j^2 - k^2)) - \sum_{1 \leq j \neq k \leq N} e(n\alpha(j^2 - k^2)) \right| \ll \sum_{k=N+1}^{N+l} e(-n\alpha k^2) \left| \sum_{j=1}^N e(-n\alpha j^2) \right| + \left| \sum_{N+1 \leq j \neq k \leq N+l} e(n\alpha(j^2 - k^2)) \right| \ll l|S_\alpha(n,N)| + l^2.$$  

Thus,

$$|X_{N+l}(\alpha) - X_N(\alpha)| \ll \epsilon, f N^{-2+\sigma+\delta+\epsilon} + \frac{M l^2}{N^2} + \frac{l}{N^2} \sum_{n=1}^M |S_\alpha(n,N)| \ll N^{-2+\sigma+2\delta+\epsilon} + N^{-2+\frac{1}{2}+\sigma+\delta+\epsilon},$$

where we use that $0 \leq l \leq N^{-\delta}$ and apply the Cauchy-Schwarz inequality to the sum in order to utilize the estimate (5.7). Clearly, at this point we require $-2+\sigma+2\delta < 0$ as well as $-2+\frac{1}{2}+\sigma+\delta < 0$, where $0 \leq \sigma < 2$ and $\delta > 0$. Finally, observe that the requirement (5.4) implies (5.6).

□

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