Thermodynamics of toroidal black holes

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The thermodynamical properties of toroidal black holes in the grand canonical ensemble are investigated using York’s formalism. The black hole is enclosed in a cavity with finite radius where the temperature and electrostatic potential are fixed. The boundary conditions allow one to compute the relevant thermodynamical quantities, e.g. thermal energy, entropy and specific heat. This black hole is thermodynamically stable and dominates the grand partition function. This means that there is no phase transition, as the one encountered for spherical black holes.

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I. INTRODUCTION

In recent years there has been an increasing interest in the study of black holes with non-trivial topologies. Black holes whose event horizon have toroidal topology have been found, see [1,2]. It has been shown that they can be formed from gravitational collapse [3]. There is also a generalization of these black holes to other topologies [4,5]. These black holes are solutions of the Einstein equations with negative cosmological constant. It is the presence of a negative cosmological constant that allows the violation of the theorems of general relativity forbidding non-spherical black hole topologies.

It is interesting to study the thermodynamical properties of these black holes and compare them to their spherically symmetric counterparts. In this paper we consider a static and charged black hole with toroidal event horizon found in [2] and analyze its thermodynamical behaviour, see also [3,6]. We use York’s formalism to study its thermodynamics in the grand canonical following the same procedure as in [7] for its spherical counterpart, the Reissner-Nordström-anti-de Sitter black hole.

In section II we compute the reduced action of the toroidal black hole in York’s formalism. Using this action we evaluate the main thermodynamical quantities: energy, mean value of the charge and entropy of the toroidal black hole. In section III we evaluate the black hole solutions, i.e. the event horizon radius and charge of the black hole formed for the temperature and the electrostatic potential fixed by the boundary conditions. We then study the local and global stability of the black hole solutions in section IV. Finally, in section V we consider the limit where the boundary is taken to infinite. The results obtained are compared to the ones found in [7] for the Reissner-Nordström-anti-de Sitter black hole. Conclusions are drawn in section VI.

II. THERMODYNAMICS IN THE GRAND CANONICAL ENSEMBLE

In this section we will compute the reduced action of the black hole. We will follow [8].
We consider a general static metric with toroidal symmetry of the form

$$ds^2 = b^2 d\tau^2 + a^2 dy^2 + r^2 (d\theta^2 + d\varphi^2) ,$$

(1)

where $a$, $b$, and $r$ are only function of the radial coordinate $y$. The Euclidean time $\tau$ and the angular coordinates $\theta$ and $\varphi$ have period $2\pi$. For convenience we choose $y \in [0, 1]$ so that the event horizon is given by $y = 0$ and has radius $r_+ = r(0)$ and area $A_+ = 4\pi^2 r_+^2$. The boundary is given by $y = 1$ and at this boundary the thermodynamical variables defining the ensemble are fixed. The boundary is a 2-torus with area $A_B = 4\pi^2 r_B^2$, where $r_B = r(1)$.

In order to obtain the reduced action from (1) we use the usual regularity conditions and we impose the proper constraints \[8,9\], i.e. Hamiltonian constraint

$$G^\tau_\tau + \Lambda g^\tau_\tau = 8\pi T^\tau_\tau ,$$

(2)

which corresponds to the first of Einstein equations, and the Gaussian constraint

$$F^{\mu\nu};_\nu = 0 ,$$

(3)

which corresponds to the Maxwell equations.

The reduced action is given by

$$I^* = -\beta r_B \sqrt{\pi^2 \alpha^2 r_B^2 - \pi^2 \alpha^2 r_+^2 - \frac{e^2}{r_+ r_B} + \frac{e^2}{r_B^3} - \pi^2 r_+^2 - e \beta \phi - I_{\text{subtr}} .}$$

(4)

Where $\beta$ is the inverse temperature at the boundary, $\phi$ is the difference in electrostatic potential between the boundary and the horizon, $e$ the charge of the black hole, $\alpha^2 = -\frac{\Lambda}{3}$, $\Lambda$ is the cosmological constant and $I_{\text{subtr}}$ is an arbitrary term that can be used to define the zero of the energy.

Using the same procedure as in \[8,9\], we can compute $I_{\text{subtr}}$. We choose for convenience the thermal energy of anti-de Sitter spacetime $E_{\text{ADS}} = E(r_+ = 0, e = 0) = 0$ to define the zero of the energy. Therefore obtaining

$$I_{\text{subtr}} = \beta r_B \sqrt{\pi^2 \alpha^2 r_B^2} .$$

(5)
Now substituting (3) in (4) we obtain the reduced action in the form
\[ I^* = \beta r_B \left( \sqrt{\pi^2 \alpha^2 r_B^2} - \pi^2 \alpha^2 r_B^2 - \pi^2 \alpha^2 \frac{r_+^3}{r_B^3} - \frac{e^2}{r_+ r_B} + \frac{e^2}{r_B^3} - \pi^2 r_+^2 - e \beta \phi \right). \] (6)

We can use the reduced action given in (3) to compute all the thermodynamical quantities of interest (like the energy, the entropy and the mean value of charge of the black hole). This is done using the relation between the reduced action and the grand canonical potential of thermodynamics \[ I = \beta F. \] (7)

From the grand canonical potential \( F \), we can compute the thermodynamical quantities using the common laws of thermodynamics, see for example [10].

The thermal energy is given by
\[ E = F + \beta \left( \frac{\partial F}{\partial \beta} \right)_{\phi, r_B} - \left( \frac{\partial F}{\partial \phi} \right)_{\beta, r_B} - \phi \left( \frac{\partial I}{\partial \beta} \right)_{\phi, r_B} - \phi \beta \left( \frac{\partial I}{\partial \phi} \right)_{\beta, r_B} = r_B \left( \sqrt{\pi^2 \alpha^2 r_B^2} - \pi^2 \alpha^2 r_B^2 - \pi^2 \alpha^2 \frac{r_+^3}{r_B^3} - \frac{e^2}{r_+ r_B} + \frac{e^2}{r_B^3} \right). \] (8)

The mean value of the charge is given by
\[ Q = -\left( \frac{\partial F}{\partial \phi} \right)_{\beta, r_B} = -\frac{1}{\beta} \left( \frac{\partial I}{\partial \phi} \right)_{\beta, r_B} = e. \] (9)

The entropy is
\[ S = \beta^2 \left( \frac{\partial F}{\partial \phi} \right)_{\phi, r_B} = \beta \left( \frac{\partial I}{\partial \beta} \right)_{\phi, r_B} - I = \pi^2 r_+^2. \] (10)

Since \( \pi^2 r_+^2 = A_+/4 \), where \( A_+ \) is the area of the event horizon, we have \( S = \frac{A_+}{4} \). This is the usual Hawking-Bekenstein entropy [11], which means this law is still valid for black holes with toroidal symmetry.

**III. THE BLACK HOLE SOLUTIONS**

The black hole solutions are determined by computing the extrema of the reduced action. As the variables \( \beta, \phi, r_B \) and \( \alpha \) are fixed by the boundary conditions, the reduced action (3)
is a function of only two parameters: \( r_+ \) the event horizon radius and \( e \) the electric charge. Inverting the equation \( \nabla I^*(r_+, e) = 0 \), we obtain the black hole solutions as function of the boundary conditions, i.e. \( r_+ = r_+ (\beta, \phi, r_B, \alpha) \) and \( e = e (\beta, \phi, r_B, \alpha) \). Equation \( \nabla I^* = 0 \) yields

\[
\frac{\partial I^*}{\partial r_+} = -\frac{1}{2} \beta \left( -3 \pi^2 \alpha^2 r_+^2 + \frac{e^2}{r_+^2} \right) \left( \pi^2 \alpha^2 r_B^2 - \pi^2 \alpha^2 \frac{r_+^3}{r_B} - \frac{e^2}{r_+ r_B} + \frac{e^2}{r_B^2} \right)^{-\frac{1}{2}} - 2 \pi^2 r_+ = 0 ,
\]

and

\[
\frac{\partial I^*}{\partial r_+} = -\beta \left( -\frac{e}{r_+} + \frac{e}{r_B} \right) \left( \pi^2 \alpha^2 r_B^2 - \pi^2 \alpha^2 \frac{r_+^3}{r_B} - \frac{e^2}{r_+ r_B} + \frac{e^2}{r_B^2} \right)^{-\frac{1}{2}} - \beta \phi = 0 .
\]

We can invert equation (11) to obtain the inverse temperature of the black hole

\[
\beta = \frac{4 \pi^2 r_+^3}{3 \pi^2 \alpha^2 r_B^4 - e^2} \sqrt{\pi^2 \alpha^2 r_B^2 - \pi^2 \alpha^2 \frac{r_+^3}{r_B} - \frac{e^2}{r_+ r_B} + \frac{e^2}{r_B^2}} .
\]

This is the Hawking temperature times the redshift factor due to the Tolman effect [12].

Inverting equation (12), one obtains the electrostatic potential as

\[
\phi = \left( \frac{e}{r_+} - \frac{e}{r_B} \right) \left( \pi^2 \alpha^2 r_B^2 - \pi^2 \alpha^2 \frac{r_+^3}{r_B} - \frac{e^2}{r_+ r_B} + \frac{e^2}{r_B^2} \right)^{-\frac{1}{2}} .
\]

This is the difference in electrostatic potential between the horizon and the boundary “redshifted” to the boundary.

In order to invert equations (13) and (14), we are going to define the new variables

\[
\alpha = \pi \alpha r_B , \quad x = \frac{r_+}{r_B} , \quad q = \frac{e}{r_B} , \quad \beta = \frac{\beta}{4 \pi^2 r_B} .
\]

Using these new variables and inverting equation (14) we obtain

\[
q^2 = \frac{\alpha^2 \phi^2 \left( 1 + x + x^2 \right) x^2}{1 - x + \phi^2 x} .
\]

Inverting equation (13) and using equation (16), we obtain the equation

\[
\sigma^2 \phi^4 + 4 \sigma^2 \phi^4 x + (10 \sigma^2 \phi^4 - 6 \sigma^2 \phi^2 - 1) x^2 + \phi^2 (12 \sigma^2 \phi^2 - 12 \sigma^2 - 1) x^3
\]

\[
+ (9 \sigma^2 \phi^4 - 18 \sigma^2 \phi^2 - 9 \sigma^2 - \phi^2) x^4 + (1 - \phi^2) x^5 = 0 .
\]
Where we have used a new variable

\[ \sigma = \alpha \beta = \frac{\alpha \beta}{4\pi}. \]  

(18)

Notice that from equation (17) the event horizon radius does not depend on \( \beta \) and \( \alpha \), but on their product \( \sigma \). Something that does not happen for the Reissner-Nordström-anti-de Sitter black hole [7], the spherical counterpart of this black hole.

Solving equation (17), we obtain \( r_+ \) as a function of the boundary conditions and replacing this solution into equation (16), we obtain the respective charge \( e \). However not every solution of equation (17) is a physical solution corresponding to a black hole. Effectively, the black hole is inside the cavity so \( r_+ < r_B \), therefore the solutions must obey \( x < 1 \). Moreover the charged black hole has two horizons but we are only interested in the event horizon, which verifies condition

\[ 3 \pi^2 \alpha^2 r_+^4 - e^2 > 0. \]  

(19)

This condition implies the inverse temperature (13) is real and positive and that the electrostatic potential (14) is also real and positive and furthermore verifies

\[ \phi^2 < \frac{3 x^2}{1 + 2 x + 3 x^2}. \]  

(20)

Therefore only the solutions of equation (17) that obey condition (20) are physical solutions. In figures 1 and 2 the solutions of (17) that verify this condition are presented.

FIG. 1. Solutions of equation (17) which obey condition (20), for fixed values of \( \sigma = 0.1, 0.5, 1, 5, 10 \).
In figure 1, the curves have fixed values of the variable $\sigma$ and the values of $x$ are presented as function of the electrostatic potential. Notice that, due to condition (20), the electrostatic potential is always $\phi < \sqrt{0.5} \simeq 0.7$.

In order to present in graphics all possible values of $\sigma$, we define the new variable

$$s = \frac{2}{\pi} \arctan \sigma .$$  \hspace{1cm} (21)

It is this new variable that is used in figure 2, where the black hole solutions are again presented as functions of $\sigma$ and $\phi$.

![Diagram](image)

FIG. 2. Solutions of equation (17) which obey condition (20). The variable $s$ is defined in (21).

**IV. STABILITY**

To study the stability of the solutions found in the previous section we compute the local minima of the reduced action (3). We will follow the same procedure as [8]. The conditions of local stability are, see [8],

$$\beta \left( \frac{\partial \phi}{\partial e} \right)_{S,r_B} \geq 0 ,$$  \hspace{1cm} (22)

$$C_{\phi,r_B} = \beta \left( \frac{\partial S}{\partial \beta} \right)_{\phi,r_B} \geq 0 ,$$  \hspace{1cm} (23)
where \( C_{\phi,r_B} \) is the heat capacity at constant \( \phi \) and \( r_B \). Computing these functions we obtain

\[
\beta \left( \frac{\partial \phi}{\partial e} \right)_{s,r_B} = \frac{4 \pi^4 \alpha^2 (r_B^3 - r_+^3) r_+^3}{(3 \pi^2 \alpha^2 r_+^4 - e^2)(\pi^2 \alpha^2 r_+ r_B (r_B^2 + r_+ r_B + r_+^2) - e^2)} \geq 0 \quad (24)
\]

and

\[
C_{\phi,r_B} = \frac{4 \pi^4 \alpha^2 r_+^3 (r_B^3 - r_+^3) (3 \pi^2 \alpha^2 r_+^4 - e^2)}{e^4 + 2 \pi^2 \alpha^2 e^2 r_B (r_B^2 - 2 r_+ r_B - 2 r_+^2) + 3 \pi^4 \alpha^4 r_+^5 (2 r_B^3 + r_+^3)} \geq 0 \quad (25)
\]

These conditions are satisfied for every value of \( r_+ \) and \( e \) that verify conditions \( r_+ < r_B \) and (19). Therefore all physical black hole solutions are locally stable. This means that the toroidal black holes are more stable than the spherical ones, since the Reissner-Nordström-anti-de Sitter black hole is unstable for a wide regions of values of \( \beta, \phi \) and \( \alpha \). However these solutions are not necessarily global minima of the reduced action. In this case they do not dominate the grand partition function and the zero-loop approximation being used here does not hold [13].

The reduced action given in (6) goes to infinity in the non-compact directions where \( r_+ \) or \( e \) go to infinity. Therefore the global minimum of the reduced action is either at the local minimum or at \( r_+ = e = 0 \). At this latter point the reduced action is null. Therefore the condition for global stability of the solutions computed in the previous section is that the classical action, i.e. the reduced action evaluated at the local minimum, is negative. This is indeed the case for all physical solutions of equation (17). We conclude that the solutions presented in figures 1 and 2 are globally stable and dominate the grand partition function. Again, we can say that the toroidal black hole is more stable than its spherical counterpart, the Reissner-Nordström-anti-de Sitter black hole, which is not dominant in a certain region of values of \( \beta, \phi \) and \( \alpha \). [7]

V. TAKING THE BOUNDARY TO INFINITY

As for the spherical counterpart of this black hole [7], there are two ways of taking the limit \( r_B \to \infty \), (i) fixing the black hole solutions, i.e. fixing the values of \( r_+ \) and \( e \) and (ii) fixing the boundary conditions, i.e. fixing the values of \( \beta \) and \( \phi \).
Fixing the black hole solutions and taking the limit \( r_B \to \infty \), the temperature \( T = \beta^{-1} \) and the electrostatic potential go to zero as \( r_B^{-1} \), see equations (13) and (14). In this case the classical action, is given by, see equation (3),

\[
I = \frac{\pi^2 r_+^2 (e^2 + \pi^2 \alpha^2 r_+^4)}{e^2 - 3 \pi^2 \alpha^2 r_+^4}.
\]

(26)

This is always negative as long as the \( r_+ \) obeys the necessary condition to represent the event horizon, i.e. condition (19). Therefore the locally stable solutions are also globally stable and dominate the grand partition function. This means that for this black hole there is no phase transition as the one found for spherical black holes [14,7]. The thermal energy goes to zero as \( m / \pi \alpha r_B \), where \( m \) is the mass of the black hole given by

\[
m = \frac{\pi}{2} \left( \frac{e^2}{\pi^2 r_+} + \alpha^2 r_+^3 \right).
\]

(27)

The heat capacity is given by, see equation (25),

\[
C_\phi = 2 \pi^2 r_+^2 \left( 1 - \frac{2 e^2}{e^2 + 3 \pi^2 \alpha^2 r_+^4} \right).
\]

(28)

The heat capacity is positive as long as \( r_+ \) obeys condition (19), which means these solutions are all stable.

Fixing the boundary conditions and taking the limit \( r_B \to \infty \), we obtain solutions that diverge. This can be seen using equation (17) and taking this limit, the event horizon radius \( r_+ \) goes to infinity as \( x r_B \). All other thermodynamical quantities: mean value of the charge, entropy, action, energy and heat capacity (see equations (10), (10), (8), (8) and (25), respectively), diverge as \( r_B^2 \). Therefore this way of taking the limit seems to be of less physical interest than the previous.

VI. CONCLUSIONS

We have studied the thermodynamics of the charged, static and toroidal black hole (studied in [2]) closed in a box with finite radius. We conclude that the Hawking-Bekenstein law for entropy is still valid for black holes with this symmetry. Furthermore we find that
in the grand canonical ensemble, with temperature and electrostatic potential fixed at the boundary, there is a black hole solution that is globally stable, which means it dominates the grand partition function. These results are generally different from the results obtained for the spherical counterpart of this black hole, the Reissner-Nordström-anti-de Sitter black hole, for which there were found one or two solutions, that can be stable or unstable, and do not necessarily dominate the grand partition function \[7,15\]. This means that, contrary to the Reissner-Nordström-anti de Sitter black hole, for the toroidal black hole no phase transition was found.

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[1] J. P. S. Lemos, Class. Quantum Grav. 12, 1081 (1995); J. P. S. Lemos, Phys. Lett. B 353, 46 (1995); C. G. Huang and C. B. Liang, Phys. Lett. A 201, 27 (1995).

[2] J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D 54, 3840 (1996).

[3] J. P. S. Lemos, Phys. Rev. D 57, 4600 (1998).

[4] S. Åminneborg, I. Bengtsson, S. Holst, and P. Peldán, Class. Quantum Grav. 13, 2707 (1996); W. L. Smith and R. B. Mann, Phys. Rev. D 56, 4942 (1997).

[5] D. R. Brill, J. Louko, and P. Peldán, Phys. Rev. D 56, 3600 (1997).

[6] L. Vanzo, Phys. Rev. D 56, 6475 (1997); A. DeBenedictis, gr-qc/9808023 (1998).

[7] C. S. Peça and J. P. S. Lemos, gr-qc/9805004 (1998).
[8] H. W. Braden, J. D. Brown, B. F. Whiting, and J. W. York, Phys. Rev. D 42, 3376 (1990).

[9] J. W. York, Physica A 158, 425 (1989).

[10] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics* (John Wiley & Sons, New York, 1985).

[11] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975); J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).

[12] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Dover, New York, 1987), p. 313.

[13] B. F. Whiting and J. W. York, Phys. Rev. Lett. 61, 1336 (1988).

[14] S. W. Hawking and D. N. Page, Commun. Math. Phys. 87, 577 (1983).

[15] J. Louko and S. N. Winters-Hilt, Phys. Rev. D 54, 2647 (1996).