Competition between paramagnetism and diamagnetism in charged Fermi gases

Xiaoling Jian, Jihong Qin, and Qiang Gu

Department of Physics, University of Science and Technology Beijing, Beijing 100083, China
(Dated: June 29, 2010)

The charged Fermi gas with a small Lande-factor $g$ is expected to be diamagnetic, while that with a larger $g$ could be paramagnetic. We calculate the critical value of the $g$-factor which separates the dia- and para-magnetic regions. In the weak-field limit, $g_c$ has the same value both at high and low temperatures, $g_c = 1/\sqrt{12}$. Nevertheless, $g_c$ increases with the temperature reducing in finite magnetic fields. We also compare the $g_c$ value of Fermi gases with those of Boltzmann and Bose gases, supposing the particle has three Zeeman levels $\sigma = \pm 1, 0$, and find that $g_c$ of Bose and Fermi gases is larger and smaller than that of Boltzmann gases, respectively.

PACS numbers: 05.30.Fk, 51.60.+a, 75.10.Lp, 75.20.-g

I. INTRODUCTION

Magnetism of electron gases has been considerably studied in condensed matter physics. In magnetic field, the magnetization of a free electron gas consists of two independent parts. The spin magnetic moment of electrons results in the paramagnetic part (the Pauli paramagnetism), while the orbital motion due to charge degree of freedom in magnetic field induces the diamagnetic part (the Landau diamagnetism)\cite{1}. The Pauli paramagnetism and the Landau diamagnetism compete with each other. For electrons whose Lande-factor $g = 2$, the zero-field paramagnetic susceptibility is two times stronger than the diamagnetic susceptibility, so the free electron gas exhibits paramagnetism altogether.

Magnetic properties of relativistic Fermi gases have also been under extensive investigation. Daicic et al. developed statistical mechanics for the magnetized pair-fermion gases and found that the intrinsic spin causes important effects upon the relativistic para- and diamagnetism\cite{2}.

The study of ultracold atoms has stimulated renewed research interest in the magnetism of quantum gases\cite{3– 4}. When atomic gases are confined in optical traps\cite{2,3,4}, their spin degree of freedom becomes active, leading to the manifestation of magnetism. Theoretically, the paramagnetism\cite{5} and ferromagnetism\cite{6,7} in a neutral spin-1 Bose gas have ever been studied. In experiments, magnetic domains have been directly observed in $^{87}$Rb condensate, a typical ferromagnetic spinor condensate\cite{5}. Very recently, the exploration of magnetism in quantum gases has been extended to the Fermi gas. It is already observed that an ultracold two-component Fermi gas of neutral $^6$Li atoms exhibits ferromagnetism caused by repulsive interactions between atoms\cite{6}.

Furthermore, it is possible to realize charged quantum gases from neutral ultracold atoms. So far, cold plasma has been created by photoionization of cold atoms\cite{11}. The temperatures of electrons and ions are as low as 100 mK and 10 µK, respectively. The ions can be regarded as charged Bose or Fermi gases. Once the quantum gas has both the spin and charge degrees of freedom, there arises the competition between paramagnetism and diamagnetism, as in electrons. Different from electrons, the $g$-factor for different magnetic ions is diverse, ranging from 0 to 2.

The $g$-factor is a characteristic parameter to measure the intensity of paramagnetic effect. It is expected that the quantum gas displays diamagnetism in small $g$ region, but paramagnetism in large $g$ region. The main purpose of this paper is to calculate the critical value of $g$. We also present a comparison with results of charged spin-1 Bose gases which have been obtained previously\cite{12}.

II. THE MODEL

We consider a charged Fermi gas of $N$ particles, with the effective Hamiltonian written as

$$\hat{H} = \mu N = D_L \sum_{j,k_z,\sigma} (\epsilon_{j,k_z} + \epsilon_\sigma - \mu) n_{j,k_z,\sigma},$$

(1)

where $\mu$ is the chemical potential and $\epsilon_{j,k_z}$ is the quantized Landau energy in the magnetic field $B$,

$$\epsilon_{j,k_z} = \left(\frac{1}{2} + j\right)\hbar \omega + \frac{\hbar^2 k_z^2}{2m^*}$$

(2)

with $j = 0, 1, 2, \ldots$ labeling different Landau levels and $\omega = qB/(m^*c)$ being the gyromagnetic frequency of a fermion with charge $q$ and mass $m^*$. $D_L = qBS_{xy}/(2\pi\hbar c)$ marks the degeneracy of each Landau level with $S_{xy}$ the total section area in $x, y$ directions of the system. $\epsilon_\sigma$ denotes the Zeeman energy,

$$\epsilon_\sigma = -g\frac{\hbar q}{m^*c}B = -g\sigma\hbar \omega,$$

(3)

where $g$ is the Lande-factor and $\sigma$ refers to the spin-$z$ index of Zeeman state $|F, \sigma\rangle$.

The grand thermodynamic potential of the Fermi gas is expressed as

$$\Omega_{T \neq 0} = -\frac{1}{\beta} D_L \sum_{j,k_z,\sigma} \ln[1 + e^{-\beta(\epsilon_{j,k_z} + \epsilon_\sigma - \mu)}],$$

(4)
where $\beta = (k_B T)^{-1}$. Performing Taylor expansions and then integrating out $k_z$, Eq. (1) becomes

$$
\Omega_{T \neq 0} = -\frac{\omega V}{\hbar^2} \left(\frac{m^*}{2\pi \beta}\right)^{3/2} \times \sum_{l=1}^{\infty} \sum_{\sigma} \frac{(-1)^l + 1}{1 - e^{-l\beta \hbar \omega}}.
$$

where $V$ is the volume of the system. For simplicity, the following notation is introduced,

$$
F^\sigma_\tau[\alpha, \delta] = \sum_{l=1}^{\infty} \frac{(-1)^l + 1}{1 - e^{-l\tau}} F^\sigma_\tau[-3, 0],
$$

where $x = \beta \hbar \omega$ and $\eta_\sigma = ((\hbar \omega/2 - \mu + \epsilon_\sigma)/(\hbar \omega))$. Then Eq. (6) is rewritten as

$$
\Omega_{T \neq 0} = -\frac{\omega V}{\hbar^2} \left(\frac{m^*}{2\pi \beta}\right)^{3/2} \sum_{\sigma} F^\sigma_0[-3, 0].
$$

For a system with the given particle density $n = N/V$, the chemical potential is obtained according to the following equation

$$
n = -\frac{1}{V} \frac{\partial \Omega}{\partial \mu} \bigg|_{T,V} = x \left(\frac{m^*}{2\pi \beta \hbar^2}\right)^{3/2} \sum_{\sigma} F^\sigma_0[1, 0].
$$

A similar treatment has been employed to study diamagnetism of the charged spinless Bose gas\[13\] and extended to the study of competition of diamagnetism and paramagnetism in charged spin-1 Bose gases\[12\]. This method is more applicable at relatively high temperatures.

To determine $g_c$, we need calculate the magnetization density $M$ as a function of the magnetic field $B$ and temperature $T$. The system is paramagnetic when $M > 0$ while diamagnetic when $M < 0$. $M$ can be derived from the thermodynamic potential by the standard procedure

$$
M_{T \neq 0} = -\frac{1}{V} \frac{\partial \Omega}{\partial \mu} \bigg|_{T,V},
$$

which yields

$$
M_{T \neq 0} = \frac{\hbar q}{m c} \left(\frac{m^*}{2\pi \beta \hbar^2}\right)^{3/2} \sum_{\sigma} \left\{ F^\sigma_0[-3, 0] + x(\sigma - \frac{1}{2}) F^\sigma_0[-1, 0] - x F^\sigma_0[-1, 1] \right\}.
$$

For carrying out numerical calculations, it is useful to make the parameters dimensionless, such as $\overline{M} = m^* c M/(\hbar q)$, $\overline{\omega} = \hbar \omega/(k_B T^*)$, and $\overline{t} = T/T^*$. Here $T^*$ is the characteristic temperature of the system, which is given by $k_B T^* = 2\pi \hbar^2 n^*/m^*$. Then we have $x = \overline{\omega}/\overline{t}$, $\eta_\sigma = 1/2 - \mu/\overline{\omega} - g_\sigma$ and $\mu' = \mu/(k_B T^*)$ in the notation $F^\sigma_\tau[\alpha, \delta]$. Equations (8) and (9) can be re-expressed as

$$
1 = \overline{\omega}^{1/2} \sum_{\sigma} F^\sigma_0[-1, 0]
$$

FIG. 1: (a) The total magnetization density ($\overline{M}$), (b) the paramagnetization density ($\overline{M}_p$), and (c) the diamagnetization density ($\overline{M}_d$) as a function of $g$ at $t = 0.2$. The dotted and dashed lines correspond to $\overline{\omega} = 5$ and 10, respectively.

and

$$
\overline{M}_{T \neq 0} = t^{3/2} \sum_{\alpha} \left\{ F^\sigma_0[-3, 0] + x(\sigma - \frac{1}{2}) F^\sigma_0[-1, 0] - x F^\sigma_0[-1, 1] \right\}.
$$

In following discussions, $\overline{\omega}$ is used to indicate the magnitude of magnetic field since it is proportional to $B$.

III. RESULTS AND DISCUSSIONS

First, we look at a spin-$\frac{1}{2}$ Fermi gas, setting $\sigma = \pm 1$ to present the two Zeeman levels. The dimensionless magnetization density $\overline{M}$ as a function of $g$ is shown in Fig. (1a). As expected, $\overline{M}$ is negative in the small $g$ region, which means that the diamagnetism dominates. For each given value of $\overline{\omega}$, $\overline{M}$ grows with $g$ and changes its sign from negative to positive in the larger $g$ region, indicating that the paramagnetic effect is enhanced due to increase of $g$. This phenomenon is also observed in the Bose system\[12\]. $g_c$ is just the value of $g$ where $\overline{M} = 0$. 
The total magnetization $\overline{M}$ shown in Fig. 1(a) consists of both the paramagnetic and diamagnetic contributions. Figure 1(b) depicts the pure paramagnetic contribution to $\overline{M}$, which is calculated by $M_p = gm$ where $m = n_1 - n_{-1}$. Figure 1(c) plots the diamagnetic contribution to $\overline{M}$, $M_d = \overline{M} - M_p$. For each fixed value of $\overline{\sigma}$, the diamagnetization $M_d$ is slightly weakened with increasing $g$. Interestingly, for a charged spin-1 Bose gas in a constant magnetic field, the diamagnetism is not suppressed but enhanced as $g$ becomes larger [12]. Comparing Figs 1(a), 1(b) and 1(c), it can be seen clearly that the increase of $\overline{M}$ with $g$ is mainly owing to the paramagnetization $M_p$.

$g_c$ is an important parameter to describe the competition between the diamagnetism and paramagnetism. Figure 2 plots $g_c$ for the charged spin-$\frac{1}{2}$ Fermi gas. The temperature is described by $1/t$ and the magnetic fields are chosen to be relatively larger, since the method adopted here is more applicable at higher temperatures or in stronger fields. In the high temperature limit, $g_c$ tends to a universal value, $g_c|_{t \to \infty} = 0.28868$, regardless of the strength of the magnetic field. In a fixed magnetic field, $g_c$ increases monotonically as the temperature falls down. But the trend of increasing is slowed down if the magnetic field is weakened.

Present method can not produce valid results in the low temperature region if a small magnetic field is chosen. In this case that $g\overline{\sigma} \ll t \ll \mu'$, the grand thermodynamic potential in Eq. (1) can be calculated using the Euler-Maclaurin formula

\[
\sum_{j=0}^{\infty} \psi(j + \frac{1}{2}) \approx \int_{0}^{\infty} \psi(x)dx + \frac{1}{24} \psi'(0),
\]

and then Eq. (4) is transformed into

\[
\Omega_{T \neq 0} = -\frac{V}{\beta \lambda^3} \sum_{\sigma} f_{\frac{1}{2}}(z) + \frac{V\beta}{24\lambda^3}(h\omega)^2 \sum_{\sigma} f_{\frac{1}{2}}(z),
\]

where $\lambda = h\beta^2/(2\pi m^*)^{\frac{3}{2}}$, $z = e^{\beta(\mu-t)}$ and the Fermi-Dirac integral $f_{\frac{1}{2}}(z)$ is normally defined as

\[
f_{\frac{1}{2}}(z) = \frac{1}{\Gamma(n)} \int_{0}^{\infty} \frac{x^{n-1}}{z^{-1}e^x + 1}dx,
\]

where $\Gamma(n)$ is a usual gamma function and $x = \beta \epsilon$. Then the magnetization density, $M$, is obtained from grand thermodynamic potential in Eq. (13),

\[
\overline{M}_{T \neq 0} = g t^{3/2} \sum_{\sigma} \sigma f_{\frac{1}{2}}(z') - \frac{g t^{1/2}}{12} \sum_{\sigma} f_{\frac{1}{2}}(z') - g t^{-1/2} \sum_{\sigma} \sigma f_{\frac{1}{2}}(z'),
\]

where $z' = e^{(\mu + \sigma \overline{\sigma})/t}$. After some algebra, we get from Eq. (15) that $g_c|_{t \to 0} = 1/\sqrt{12} \approx 0.28868$. So in the weak field limit, $g_c$ has the same value both at the high and low temperature limit. This implies that the $g_c - 1/t$ curve is likely to flatten out in the weak-field limit.

To proceed, we make a comparison between Fermi and Bose gases. Considering that $g_c$ for charged spin-1 Bose gas has already been studied [12], we discuss a Fermi gas with three sublevels, $\sigma = \pm 1, 0$ (a pseudo-spin-1 Fermi gas) to ensure the comparability. Results for the spin-1 Boltzmann gas are also obtained. Figure 3 shows the $g_c - 1/t$ curves for the three kinds of charged spin-1 gases. In a given value of $\overline{\sigma}$, $g_c$ of all the three gases displays similar temperature-dependence: $g_c$ increases monotonously as $t$ reduces. In the high temperature limit, $g_c$ goes to the same value in all magnetic fields, $g_c|_{t \to \infty} = 1/\sqrt{8}$, reflecting that the Bose-Einstein (BE) and Fermi-Dirac (FD) statistics coincide with the Maxwell-Boltzmann (MB) statistics in this case.

Figure 3 also demonstrates the difference among the three kinds of statistics. For each given fixed value of $\overline{\sigma}$, the --

![Diagram](https://via.placeholder.com/150)

**FIG. 2:** Plots of the critical value of $g$-factor as a function of $1/t$. The magnetic field is chosen as $\overline{\sigma} = 20$ (solid line), 10 (dashed line) and 5 (dotted line), respectively.

**FIG. 3:** The $g_c - 1/t$ curves for charged spin-1 gases obeying the Bose-Einstein (BE, dotted line), Maxwell-Boltzmann (MB, solid line) and Fermi-Dirac (FD, dashed line) statistics, respectively. The magnetic field is chosen as $\overline{\sigma} = 10$ and 5.
the \( g_c - 1/t \) curves of Bose and Fermi gases always locate at the two sides of that of the Boltzmann gas. Given the same temperature and magnetic field, \( g_c \) of Fermi gas is the smallest. According to our previous research\[12\], \( g_c|_{t \to 0} = 1/2 \) for the Boltzmann gas regardless of the magnetic field. This means that \( g_c \) of the Fermi gas does never exceed 1/2 in the low temperature, no matter how strong the field is.

IV. SUMMARY

In summary, we study the interplay between paramagnetism and diamagnetism of charged Fermi gases supposing that the Lande-factor \( g \) is a variable. The gas undergoes a shift from diamagnetism to paramagnetism at the critical value of \( g \) and \( g_c \) increases monotonically as the temperature \( t \) decreases in a fixed magnetic field \( \varpi \), and the rise in \( g_c \) is lowered as \( \varpi \) is reduced. We conjecture that \( g_c \) holds a constant at all temperatures in the weak field limit. For a spin-1/2 Fermi gas, \( g_c|_{\varpi \to 0} = 1/\sqrt{12} \). We also briefly compare \( g_c \) of charged spin-1 gases obeying the Fermi-Dirac, Bose-Einstein and Maxwell-Boltzmann statistics. The \( g_c - 1/t \) curves of Boltzmann gases are always between those of Bose and Fermi gases in the same magnetic field. In the high temperature limit, \( g_c \) of all the three gases tends to the same value.

This work was supported by the Fok Ying Tung Education Foundation of China (No. 101008), the Key Project of the Chinese Ministry of Education (No. 109011), and the Fundamental Research Funds for the Central Universities of China.

[1] L.D. Landau, E.M. Lifshitz, Statistical Physics. Part 1, Butterworth-Heinemann, Oxford, 1980.
[2] J. Daicic, N.E. Frankel, R.M. Gailis, V. Kowalenko, Phys. Rep. 63 (1994) 237 and references therein.
[3] D.M. Stamper-Kurn, M.R. Andrews, A.P. Chikkatur, S. Inouye, H.-J. Miesner, J. Stenger, W. Ketterle, Phys. Rev. Lett. 80 (1998) 2027.
[4] J. Stenger, S. Inouye, D.M. Stamper-Kurn, H.-J. Miesner, A.P. Chikkatur, W. Ketterle, Nature 396 (1998) 345.
[5] L.E. Sadler, J.M. Higbie, S.R. Leslie, M. Vengalattore, D.M. Stamper-Kurn, Nature 436 (2005) 312.
[6] G.B. Jo, Y.R. Lee, J.H. Choi, C.A. Christensen, T.H. Kim, J.H. Thywissen, D.E. Pritchard, W. Ketterle, Science 325 (2009) 1521.
[7] K. Yamada, Prog. Theor. Phys. 76 (1982) 443; M.V. Simkin, E.G.D. Cohen, Phys. Rev. A 59 (1999) 1528.
[8] T.-L. Ho, Phys. Rev. Lett. 81 (1998) 742; T. Ohmi, K. Machida, J. Phys. Soc. Jpn. 67 (1998) 1822.
[9] Q. Gu, R.A. Klemm, Phys. Rev. A 68 (2003) 031604(R); C. Tao, P. Wang, J. Qin, Q. Gu, Phys. Rev. B 78 (2008) 134403.
[10] K. Kis-Szabo, P. Szepfalusy, G. Szirmai, Phys. Rev. A 72 (2005) 023617; S. Ashhab, J. Low Temp. Phys. 140 (2005) 51.
[11] T.C. Killian, S. Kulin, S.D. Bergeson, L.A. Orozco, C. Orzel, S.L. Rolston, Phys. Rev. Lett. 83 (1999) 4776.
[12] X. Jian, J. Qin, Q. Gu, unpublished.
[13] G.B. Standen, D.J. Toms, Phys. Rev. E 60 (1999) 5275.