q-Bernoulli Numbers and Polynomials
Associated with Gaussian Binomial Coefficient

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Abstract. The first purpose of this paper is to present a systemic study of some families of multiple \( q \)-Bernoulli numbers and polynomials by using multivariate \( q \)-Volkenborn integral (= \( p \)-adic \( q \)-integral) on \( \mathbb{Z}_p \). From the studies of these \( q \)-Bernoulli numbers and polynomials of higher order we derive some interesting \( q \)-analogs of Stirling number identities.

\section{Introduction}

Let \( q \) be regarded as either a complex number \( q \in \mathbb{C} \) or a \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \), then we always assume \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume \( |1 - q|_p < p^{-\frac{1}{p-1}} \), which implies that \( q^x = \exp(x \log_q) \) for \( |x|_p \leq 1 \). Here, \( | \cdot |_p \) is the \( p \)-adic absolute value in \( \mathbb{C}_p \) with \( |p|_p = \frac{1}{p} \). The \( q \)-basic natural number are defined by

\[
\left[ n \right]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}, \quad (n \in \mathbb{N}),
\]

and \( q \)-factorial are also defined as

\[
[n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q.
\]

In this paper we use the notation of Gaussian binomial coefficient as follows:

\[
\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q!}.
\]

Note that \( \lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{n!} \). The Gaussian coefficient satisfies the following recursion formula:

\[
\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q, \quad \text{cf.} \ [1-23].
\]

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From this recursion formula we derive

\[
\binom{n}{k}_q = \sum_{d_0 + \cdots + d_k = n-k, d_i \in \mathbb{N}} q^{d_1 + 2d_2 + \cdots + kd_k}, \text{ see [15, 20, 21].}
\]

Let \( p \) be a fixed prime. Throughout this paper \( \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}, \) and \( \mathbb{C}_p \) will, respectively, denote the ring of \( p \)-adic rational integers, the field of \( p \)-adic rational numbers, the complex number field, and the completion of algebraic closure of \( \mathbb{Q}_p \). For \( d \) a fixed positive integer \((p, d) = 1\), let

\[
X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N \mathbb{Z}, \text{ and } X_1 = \mathbb{Z}_p,
\]

\[
X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} (a + dp \mathbb{Z}_p),
\]

\[
a + dp^N \mathbb{Z}_p = \{x \in X | x \equiv a \pmod{dp^N}\},
\]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \), cf. [8-18]. For \( x \in \mathbb{C}_p \), we use the notation \([x]_q = \frac{1}{1-q^x}\), cf. [1-6].

We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \) and denote this property by \( f \in UD(\mathbb{Z}_p) \), if the difference quotients \( F_f(x, y) = \frac{f(x) - f(y)}{x - y} \) have a limit \( l = f'(a) \) as \( (x, y) \to (a, a) \). For \( f \in UD(\mathbb{Z}_p) \), let us start with the expression

\[
\frac{1}{[pN]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p),
\]

representing a \( q \)-analogue of Riemann sums for \( f \), cf. [8, 21-23]. The integral of \( f \) on \( \mathbb{Z}_p \) will be defined as limit \((n \to \infty)\) of those sums, when it exists. The \( q \)-Volkenborn integral (= \( p \)-adic \( q \)-integral) of the function \( f \in UD(\mathbb{Z}_p) \) is defined by

\[
I_q(f) = \int_X f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[pN]_q} \sum_{0 \leq x < dp^N} f(x) q^x,
\]

see [8].

The Carlitz’s \( q \)-Bernoulli numbers \( \beta_{k, q} \) can be determined inductively by

\[
\beta_{0, q} = 1, \quad q(q^k + 1)^k - \beta_{k, q} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases}
\]

with the usual convention of replacing \( \beta^i \) by \( \beta_{i, q} \)(see [2, 3, 24, 25]).

In [8], it was shown that the Carlitz’s \( q \)-Bernoulli numbers can be represented by \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) as follows:

\[
\int_{\mathbb{Z}_p} [x]_q^m d\mu_q(x) = \int_X [x]_q^m d\mu_q(x) = \beta_{m, q}, \quad m \in \mathbb{Z}_+. \]
The $k$-th order factorial of the $q$-number $[x]_q$, which is defined by

$$[x]_{k,q} = [x]_q \cdot [x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)^k},$$

is called $q$-factorial of $x$ of order $k$, cf.[15, 19, 20]. From this we note that $\binom{x}{k}_q = \frac{[x]_{k,q}}{[k]_q!}$, cf.[4-16]. The theory of $q$-number and the factorial of $q$-number are applicable in the many areas related to mathematics, mathematical physics and probability. For example, we consider a sequence of Bernoulli trials and assume that the conditionally probability of success at the $n$-th trial, given that $k$ successes occur before that trial varies geometrically with $n$ and $k$. Specifically, suppose that the probability of success at the $n+1$-th trial, given that $k$ success occur up the $n$-th trial, is given by

$$\lambda_{n,k} = q^{an+bk+c}(\in \mathbb{R}), \ k = 0, 1, 2, \cdots, n, \ n = 0, 1, 2, \cdots,$$

with $a$, $b$ and $c$ such that $0 \leq \lambda_{n,k} \leq 1$. The particular case $b = 0$ corresponds to the assumption that the probability of success at any trial depends only on the number of previous trials, while the other particular case $a = 0$ corresponds to the assumption that the probability of success at only trial depends only on the number of previous success. The purpose of this paper is to present a systemic study of some families of multiple $q$-Bernoulli numbers and polynomials by using multivariate $q$-Volkenborn integral (= $p$-adic $q$-integral) on $\mathbb{Z}_p$. From the studies of these $q$-Bernoulli numbers and polynomials we derive some interesting $q$-anals of Stirling number identities. That is, the $q$-anals of many classical Stirling number identities are formulated and their interesting features are revealed in this paper.

§2. $q$-Bernoulli numbers associated with $q$-Stirling number identities

In this section we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. From the definition of $[x]_q$ we can easily derive the following equation.

$$(6) \quad q^n[x-n]_q = \frac{q^n - 1 + 1 - q^x}{1-q} = [x]_q - [n]_q, \text{ and } [-x]_q = \frac{1}{q^x} \frac{q^x - 1}{1-q} = -\frac{1}{q^x} [x]_q.$$  

Let $(Eh)(x) = h(x+1)$ be the shift operator. Then we consider the $q$-difference operator as follows:

$$(7) \quad \Delta_q^n = \prod_{i=1}^{n} (E - q^{i-1}I), \text{ where } (Ih)(x) = h(x).$$

From (6) and (7), we note that

$$(8) \quad f(x) = \sum_{n \geq 0} \binom{x}{n}_q \Delta_q^n f(0),$$
where

\[(9) \quad \Delta_q^n f(0) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} f(n - k).\]

The \(q\)-Stirling number of the second kind is defined by Carlitz as follows:

\[(10) \quad s_2(n, k, q) = \frac{q - (\frac{k}{2})}{[k]_q!} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [k - j]_q^n, \text{ see } [3].\]

By (9) and (10) we easily see that

\[(11) \quad s_2(n, k, q) = \frac{q - (\frac{k}{2})}{[k]_q!} \Delta_q^n 0^n.\]

From (11) we can also derive the following equation.

\[(12) \quad [x]^n = \sum_{k=0}^{n} \binom{x}{k}_q [k]_q! s_2(k, n - k, q) q^{\binom{k}{2}} = \sum_{k=0}^{n} [x]_{k,q} \frac{q^{\binom{k}{2}} q^{-(\frac{n-k}{2})}}{[n-k]_q!} \Delta_q^{n-k} 0^k.\]

By (2), we easily see that

\[(13) \quad \int_{\mathbb{Z}_p} \binom{x}{n}_q d\mu_q(x) = \frac{(-1)^n}{[n+1]_q} q^{(n+1)-(\frac{n+1}{2})}.\]

From (5), (12) and (13) we can derive the following theorem.

**Theorem 1.** For \(m \in \mathbb{Z}_+\), we have

\[(14) \quad \beta_{m,q} = q \sum_{k=0}^{m} \frac{[k]_q!}{[k+1]_q} (-1)^k s_2(k, n - k, q),\]

where \(\beta_{m,q}\) are \(m\)-th Carlitz \(q\)-Bernoulli numbers.

The \(q\)-Stirling numbers of the first kind is defined as

\[(15) \quad (1 - q)^n [x]_{n,q} = \prod_{i=1}^{n} (1 - q^{x-n+1} q^{i-1}) = \sum_{l=0}^{n} \binom{n}{l}_q q^{\binom{l}{2}} (-1)^l q^{l(x-n+1)}.\]

It is easy to see that

\[(16) \quad q^{lx} = ([x]_q (q - 1) + 1)^l = \sum_{m=0}^{l} \binom{l}{m} (q - 1)^m [x]_q^m.\]
From (16) we note that

\[
\frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{(l)} (-1)^{l} q^{l(x-n+1)}
\]

(17) \\
\[
= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{(l)} (-1)^{l-n} \sum_{m=0}^{l} \binom{l}{m} (q-1)^m [x]_q^m
\]

\[
= \frac{1}{(1-q)^n} \sum_{m=0}^{n} (q-1)^m \left( \sum_{l=m}^{n} \binom{n}{l} q^{(l)} (-1)^{l-n} \binom{l}{m} (q-1)^m \right) [x]_q^m.
\]

By (12), (15) and (17) we obtain the following theorem.

**Theorem 2.** For \( n \in \mathbb{Z}_+ \) we have

\[
\beta_{n,q} = \sum_{l=0}^{n} s_2(l, n-l, q) q^{(l)} \sum_{m=0}^{l} \frac{1}{(1-q)^{l-m}} \left( \sum_{i=m}^{l} \binom{l}{i} q^{(i)} (-1)^{i-1} \right) \beta_{m,q}.
\]

In [3] Carlitz has given the following relation.

(18) \\
\[
s_2(n, k, q) = (q-1)^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k+n}{j} \binom{j+n}{j}_q,
\]

and

\[
\binom{n}{k}_q = \sum_{j=0}^{n} \binom{n}{j}_q (q-1)^{j-k} s_2(k, j-k, q).
\]

By simple calculation we easily see that

(19) \\
\[
q^{nt} = \sum_{k=0}^{n} (q-1)^k q^{(k)} \binom{n}{k}_q [t]_{k,q} = \sum_{m=0}^{n} \left( \sum_{k=m}^{n} (q-1)^k \binom{n}{k}_q s_1(k, m, q) \right) [t]_q^m.
\]

By using \( p\)-adic \( q\)-integral on \( \mathbb{Z}_p \) we have

(20) \\
\[
\int_{\mathbb{Z}_p} q^{nt} d\mu_q(t) = \sum_{m=0}^{n} \binom{n}{m}_q (q-1)^m \beta_{m,q}.
\]

By using (19) and (20) we derive

(21) \\
\[
\binom{n}{m}_q = \sum_{k=m}^{n} (q-1)^{-m+k} \binom{n}{k}_q s_1(k, m, q).
\]
From the definition of the first kind Stirling number we note that

\begin{equation}
q^{(n)}_{\frac{1}{2}} \binom{x}{n} q^n = [n]_{n,q} q^{(n)}_{\frac{1}{2}} = \sum_{k=0}^{n} s_1(n, k, q)[x]^k,
\end{equation}

By (13) and (22) we have

\begin{equation}
\frac{1}{[n+1]} = \frac{q^{-1}}{[n]_{n,q} q^n} \sum_{k=0}^{n} (-1)^{n-k} s_1(n, k, q) \beta_{k,q}.
\end{equation}

By (14), (15), (17) and (23) we obtain the following theorem.

**Theorem 3.** For \( n, j \in \mathbb{Z}_+ \) we have

\begin{equation}
s_1(n, j, q) = \frac{q^{(n)}}{(q-1)^{n-j}} \sum_{k=j}^{n} (-1)^{n-k} q^{(k+1)}_{\frac{1}{2}} - nk \binom{n}{k} q^{(j)}_{\frac{1}{2}}.
\end{equation}

Moreover,

\begin{equation}
\frac{1}{[n+1]} = \frac{q^{-1}}{[n]_{n,q} q^n} \sum_{k=0}^{n} (-1)^{n-k} s_1(n, k, q) \beta_{k,q}.
\end{equation}

§3. Multivariate \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) associated with \( q \)-Stirling numbers

In this section we also assume that \( q \in \mathbb{C}_p \) with \( |1-q|_p < p^{-\frac{1}{v-1}} \). For any positive integers \( k, m \), we consider the following multivariate \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) related to \( q \)-Bernoulli polynomials of higher order as follows:

\begin{equation}
\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^{n}} \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^{k}(k-l+i)x_l} d\mu_q(x_1) \cdots d\mu_q(x_k).
\end{equation}

In the special case \( x = 0 \), \( \beta_{n,q}^{(k)}(0) = \beta_{n,q}^{(k)} \) will be called the \( q \)-Bernoulli numbers of order \( k \). From (25) we note that

\begin{equation}
\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^{n}} \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \frac{(i+k)!}{[i+k]_q} \cdots \frac{(i+1)q^x}{[i+1]_q}.
\end{equation}

Thus, we obtain the following theorem.

**Theorem 4.** For \( m, k \in \mathbb{Z}_+ \), we have

\begin{equation}
\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^{n}} \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \frac{(i+k)!}{[i+k]_q} \frac{k!}{[k]_q q^i}.
\end{equation}
Now we also define $\beta_{n,q}^{(-k)}(x)$ as follows:

$$
\beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{k}(k-l+i)x_i} d\mu_q(x_1) \cdots d\mu_q(x_k),
$$

where $n, k$ are positive integers. From (27) we note that

$$
\beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{i+k} \frac{[k]!}{k!} q^i x^i.
$$

It is easy to see that

$$
\binom{k}{j} \binom{2j+n}{n} n! = \binom{k+n}{-j} = \frac{(k+n)\cdots(k+1)k\cdots(k-j+1)}{(j+n)!(k+n)\cdots(k+1)} = \frac{(k+n)}{(k+1)}.
$$

By (27), (28) and (29) we obtain the following theorem.

**Theorem 5.** For $n, k \in \mathbb{Z}_+$, we have

$$
\beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{i+k} \frac{[k]!}{k!} q^i x^i.
$$

From (18) and (30) we derive

$$
s_2(n, k, q) = \binom{k+n}{n} \frac{n!}{[n]_q!} \beta_{k,q}^{(-k)}(0).
$$

That is,

$$
\frac{1}{(1-q)^k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} q^{\sum_{i=1}^{k}(n+i)x_i} d\mu_q(x_1) \cdots d\mu_q(x_n) = \frac{[n]_q!}{(k+n)_n!} s_2(n, k, q).
$$

Thus, we note that

$$
\beta_{0,q}^{(-k)}(0) = \left( \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{k}(k-i)x_i} d\mu_q(x_1) \cdots d\mu_q(x_k) \right)^{-1} = \frac{[k]!}{k!}.
$$

By the same method we see that $\beta_{1,q}^{(2)}(0) = \frac{a+2}{2a+3}_q, \beta_{0,q}^{(-k)}(0) = \frac{[k]!}{k!}$. Thus, we have

$$
s_2(k, 0, q) = k! \beta_{0,q}^{(-k)}(0) = \frac{k! [k]!}{[k]_q!} = 1.
$$
From the definition of $\beta_{m,q}^{(k)}$ we can also derive the following equality.

$$\sum_{i=0}^{m} \binom{m}{i} (q-1)^i \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^i q^{\sum_{l=1}^{k} (k-l)x_l} d\mu_q(x_1) \cdots d\mu_q(x_k)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{m+k-1}x_1 + \cdots + (m+1)x_{k-1} + mx_k d\mu_q(x_1) \cdots d\mu_q(x_k) = \frac{(m+k)}{(m+k)_q} k! \cdot [k]_q!.$$

Therefore we obtain the following:

$$\sum_{i=0}^{m} \binom{m}{i} (q-1)^i \beta_{i,q}^{(k)} = \frac{(m+k)}{(m+k)_q} k! \cdot [k]_q!.$$

Finally, we observe that

$$q^{(n)}[x]_{n,q} = [x]_q [x-1]_q \cdots q^{n-1}[x-n+1]_q = [x]_q \cdot ([x]_q - 1) \cdots ([x]_q - [n-1]_q).$$

Thus, we have

$$q^{(n)}(x)_q = \frac{1}{[n]_q!} \prod_{k=0}^{n} ([x]_q - [k]_q) = \frac{1}{[n]_q!} \sum_{k=0}^{n} s_1(n, k, q)[x]_q^k.$$

§4. Further Remarks and Observations

In this section, let $p$ be a fixed odd prime number. For $n \in \mathbb{N}, k \in \mathbb{Z}_+$, we consider the following $q$-Euler numbers of higher order.

$$E_k^{(n)}(x, q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \sum_{i=1}^{n} x_i + x \right] q^{\sum_{j=1}^{n} x_j(n-j)} d\mu_q(x_1) \cdots d\mu_q(x_n),$$

where $\mu_q(x + p^n\mathbb{Z}_p) = \frac{1+q^n}{1+q^n}(-q)^x = \frac{(q^n)^x}{[p^n]_{-q}}$, see [10]. From (32) we note that

$$E_k^{(n)}(x, q) = \frac{[2]_q^n}{(1-q)^k} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^l q^lx_l}{(1+q^{n+l}) \cdots (1+q^{l+1})}.$$

The $q$-binomial formulae are known as

$$\prod_{i=1}^{n} (a + bq^{-i-1}) = \sum_{k=0}^{n} \binom{n}{k}_q q^{n-k} y^k,$$

and

$$\prod_{i=1}^{n} (a - bq^{-i-1}) = \sum_{k=0}^{n} \binom{n+k-1}{k}_q b^k.$$

By (32) and (33) we obtain the following:
Proposition 6. For \( n \in \mathbb{N}, k \in \mathbb{Z}_+ \), we have
\[
E_k^{(n)}(x, q) = \frac{[2q]^n}{(1-q)^k} \sum_{l=0}^{k} \binom{k}{l} (-1)^l q^{lx} \sum_{i=0}^{\infty} \binom{n+i-1}{i} q^{(l+1)i}.
\]

Seeking to define a suitable polynomial analogue for negative value of \( n \), we give the definition as follows:
\[
E_k^{(-n)}(x, q) = \frac{1}{(1-q)^k} \frac{1}{[2q]^n} \sum_{l=0}^{k} \binom{k}{l} (-1)^l q^{lx} \left( \prod_{i=1}^{n} (1 + q^{l+i}) \right).
\]

From (33) and (34) we can also derive the following Eq.(35).
\[
E_k^{(-n)}(x, q) = \frac{1}{(1-q)^k} \frac{1}{[2q]^n} \sum_{l=0}^{k} \binom{k}{l} (-1)^l q^{lx} \sum_{i=0}^{n} \binom{n}{i} q^{(l+1)i}.
\]

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