Delayed Dynamics with Transient Oscillations

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December 13, 2022

Abstract
Recently, we have studied a delay differential equation which has a coefficient that is a linear function of time. The equation has shown the oscillatory transient dynamics appear and disappear as the delay is increased between zero to asymptotically large delay. We here propose and study another equation that shows similar transient oscillations. It has an extra exponential gaussian factor on the delayed feedback term. It is shown that this equation is analytically tractable with the use of Lambert W function. This equation is also studied numerically to confirm some of the properties inferred from the analytical solution. We also have found that the amplitude of transient oscillation changes and goes through a maximum as we increase the value of the delay. In this sense, the proposed equation is one of the simplest dynamical equations that brings out a resonant behavior without any external oscillating inputs.

1 Introduction

Delays exist in many control and mutually interacting systems and have been investigated in various fields including mathematics, biology, physics, engineering, and economics. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Typically, delays cause instability of stable fixed points leading to oscillatory and more complex dynamics. A representative example is the Mackey–Glass equation [8], which shows the sequence of the monotonic convergence, transient oscillations, persistent oscillations, and chaotic dynamics with increasing feedback delay. The path to the complex behaviors of many systems with delays, including this model, is a difficult subject, and understandings have been gradually gained (e.g. [15]). There are, however, more to be explored, particularly concerning the nature of time-dependent dynamical trajectories.

“Delay Differential Equations (DDE)” are the main mathematical approaches and modeling tools for such systems. Typically, DDEs with constant coefficients have been investigated. Recently, we have studied a DDE which has a coefficient that is a linear function of time [16]. Even though this is formally a small
change in the equation, it shows the oscillatory transient dynamics appear and
disappear as the delay is increased between zero to asymptotically large delay.
Also, a new type of resonating behavior has been observed contrasting to the
constant-coefficient case.

Here, we propose and study another equation that shows similar transient
oscillations in this paper. It has an extra exponential gaussian factor on the
delayed feedback term. Though it makes this equation more complex, we will
show that it is analytically more tractable with the use of the Lambert W
function. This equation is also studied numerically to confirm some of the
properties inferred from the analytical solution. Also, we have found that the
amplitude of transient oscillation changes and goes through a maximum as we
increase the value of the delay. In this sense, the proposed equation is one of
the simplest dynamical equations that brings out a resonant behavior without
external oscillating inputs.

It should be noted that we are not investigating stability switching pheno-
mena (e.g.\cite{17}) with the delay as the bifurcation parameter. Indeed, in our
analysis of the proposed model in this and previous work\cite{16}, the asymptotic
stability of the fixed point never changes with increasing delay. Instead, the
shapes of dynamical trajectories approaching the stable fixed point change with
the above-mentioned resonant phenomena.

We close the paper with a brief discussion of transient oscillations from delay
differential equations of similar types.

2 Main equation and its properties

The general form of the equation we are interested in is given by the following.
\[
\frac{dX(t)}{dt} + atX(t) = f(t, X(t), X(t-\tau))
\] (1)
where \( a \geq 0, \tau \geq 0 \) are real parameters, and \( \tau \) is interpreted as a delay. This
is a slight extension of the constant-coefficient linear delay differential equation
describing the dynamics of the variable \( X(t) \). The notable difference is that we
have \( at \) instead of \( a \) in the second term of the equation. Though this appears to
be a small change, the behavior of \( X(t) \) becomes quite different. In the special
case the function \( f \) is identically zero, this can be viewed as the equation for the
ground state of the quantum simple harmonic oscillator with the interpretation
of \( t \) as a position rather than time (e.g.\cite{18}).

For example, we have proposed and studied the following simple special case,
\[
\frac{dX(t)}{dt} + atX(t) = bX(t-\tau)
\] (2)
so that the right-hand side is a simple linear function of \( X(t-\tau) \) with a
constant real parameter coefficient \( b \). This is a simple modification of much-
studied Hayes’s equation, first-order delay differential equation with constant
coefficients\cite{4}. We have shown, however, that its behavior is quite different. It
gives rise to the behavior where oscillatory transient dynamics appear and disappear as the value of delay increases. The resonant phenomena with respect to the delay have also been observed.

In this paper, we extend the above equation (2) as
\[
\frac{dX(t)}{dt} + atX(t) = be^{-a\tau t}X(t - \tau).
\]
With the exponent factor inserted in the right-hand side of the equation, it appears more complex to analyze. We, nevertheless, investigate this equation and show that previous analytical knowledge using the Lambert W-functions can be employed.

2.1 Analysis

We start with the case that \( b = 0 \). With the initial condition \( X(t = 0) = X_0 \), the solution to the equation is given as
\[
X(t) = X_0 e^{-\frac{1}{2}at^2}
\]
Thus, its dynamics have a trajectory with a gaussian shape. Also, if we consider \( t \) as a position rather than time, this is the ground state of the quantum simple harmonic oscillator.

On the other hand, the case that \( a = 0 \) becomes
\[
\frac{dX(t)}{dt} = bX(t - \tau).
\]
This is the simplest first-order delay differential equation with constant coefficients and is a special case of the Hayes equation [4].

This equation has been much studied and we know the following.

- It is known that the dynamics of \( X(t) \) monotonically approach to the asymptotically stable origin \( X = 0 \) in the range of
\[
0 > b > -1/e\tau.
\]
With \( b < -1/e\tau \), the oscillatory dynamics begin to appear.
- Including the above, in the range of
\[
0 > b > -\pi/2\tau,
\]
\( X = 0 \) is asymptotically stable.
- Hence, for \( b < 0 \), the critical delay \( \tau_c \) for the loss of the stability of the origin \( X = 0 \) is
\[
\tau_c = -\pi/2b.
\]
At this point, \( X(t) \) has a stationary sinusoidal solution with constant amplitude with the angular frequency \( \omega_c = |b| \). Equivalently, the critical period of the oscillation is
\[
T_c = 2\pi/\omega_c = 4\tau_c.
\]
Further, the general solution of (5) can be expressed by using the Lambert W function [19, 20], which is defined as a multivalued complex function $W : C \rightarrow C$ satisfying

$$W(z)e^{W(z)} = z$$  \hspace{1cm} (10)

The branches of the W function are expressed as $W_k, k = 0, \pm 1, \pm 2, \ldots, \pm \infty$. Using this function, the general solutions can be written as the following.

$$X(t) = \sum_{k=-\infty}^{\infty} C_k e^{\lambda_k t}, \quad \lambda_k = \frac{1}{\tau} W_k(b \tau).$$  \hspace{1cm} (11)

We note that $\lambda_k$ are the roots of the transcendental characteristic equation of (5),

$$\lambda = be^{-\tau \lambda},$$  \hspace{1cm} (12)

and that $C_k$ are the constant coefficients determined by the initial interval condition $X(t) = \phi(t), [-\tau, 0]$ of (5).

The general solution of (3) can now be obtained by combining the above two cases. Namely, we set

$$X(t) = e^{-\frac{1}{2} at^2} \tilde{X}(t),$$  \hspace{1cm} (13)

then, it is easy to show that $\tilde{X}(t)$ satisfies the case of $a = 0$ given above by (9). This leads to the general solution of (3) using (11) and (13) as

$$X(t) = e^{-\frac{1}{2} at^2} \sum_{k=-\infty}^{\infty} C_k e^{\frac{1}{\tau} W_k(b \tau) t}.$$  \hspace{1cm} (14)

We can infer qualitatively some properties of this solution.

- The first gaussian factor dominates as $t \to \infty$. Thus, for $a > 0$, the asymptotic stability of the origin is kept regardless of the value of the delay.
- The oscillatory behavior of the solution arises due to the second factor. Thus, as we have mentioned, the dynamics of $X(t)$ monotonically approach to the asymptotically stable origin $X = 0$ in the range of

$$0 > b > -1/e\tau.$$  \hspace{1cm} (15)

With $b < -1/e\tau$, the oscillatory dynamics begin to appear.

In the next section, we investigate equation (3) numerically to confirm these characteristics. Also, we observe that oscillatory behaviors appear and disappear as we increase the value of the delay $\tau$, which can be considered as a resonating phenomenon.
2.2 Numerical simulations

We studied equation (3) numerically. The typical dynamics for the case of $a > 0, b < 0$ are shown in Figure 1. As we noted in the previous section, asymptotic stability of the origin is kept even for large delays. Also, we note that the amplitude of the oscillation changes and goes through the maximum as we increase the value of the delay (Figure 2). In this sense, we have resonant phenomena with the delay as a tuning parameter. Thus, equation (3) is one of the simplest dynamical equations showing a resonance without any external oscillatory inputs. These properties are in contrast to the case of $a = 0$, where the stability of the origin is lost beyond the critical delay, and the amplitude of the oscillation is a monotonic increasing function of the delay.

On the other hand, some of the properties are shared with the case of $a = 0$. They are shown in Figure 3 and Figure 4. In the former, the beginning of the non-monotonic approach to the stable origin appears at $\tau_o \approx -1/eb$. In the latter, the period of oscillation near the critical delay, $\tau_c = -\pi/2b$ is approximately $4\tau_c$ consistent with (9).

![Figure 1: Representative dynamics of the main equation (3) with different values of the delays, $\tau$. The parameters are set at $a = 0.01, b = -2.0$ with the initial interval condition as $X(t) = 0.1(-\tau \leq t \leq 0)$. The values of the delays $\tau$ are (A)0.2, (B)0.5, (C)0.8, (D)1.0, (E)1.5, (F)2.0, (G)5, (H)30.](image-url)
Figure 2: The resonant curve showing the maximum oscillatory amplitude, \( \max[X] \), with values of the delays \( \tau \) as the tuning parameter. The parameters are set as the same as Fig.1; \( a = 0.01, b = -2.0 \) with the initial interval condition as \( X(t) = 0.1(-\tau \leq t \leq 0) \).

Figure 3: Representative examples of change from monotonic to non-monotonic approach to the stable origin. The delay for this onset of the oscillation is at \( \tau_o \approx -1/eb = 0.368 \) with \( a = 0.01, b = -1.0 \). The values of the delays \( \tau \) are (A)0.34, (B)0.35, (C)0.36, (D)0.37, (E)0.38, (F)0.39.
Figure 4: The ratio of the period $T^*$ of dominant oscillation mode to the delay $\tau$. The parameters are set as the same as Fig.1; $a = 0.01, b = -2.0$ with the initial interval condition as $X(t) = 0.1(-\tau \leq t \leq 0)$. The values of the critical delay $\tau_c = -\pi/2b = 0.785$ is indicated by the arrow at which $T^* \approx 4.0\tau_c$.

3 Discussion

In this paper, we have investigated the delay differential equation (3) both analytically and numerically. Its solution is analytically tractable in the sense it can be expressed in the form of equation (13) using the Lambert $W$ function. Numerically, we confirmed that the asymptotic stability of the origin is kept even with a larger delay due to the quadratic exponential factor.

At the same time, transient oscillations are observed. Notably, the amplitude changes with changing delay showing resonant behavior with respect to delay. The resonant behaviors have been observed with the simpler equation (2) without an exponential factor. There, however, it is the peak of the power spectrum that went through the maximum, showing “regular” oscillations. In this sense, even though equations (2) and (3) both show transient oscillation behavior, the resonant phenomena are different in nature. It is also different from the delay induced transient oscillation (DITO)[21, 22]. The DITO is a solution of a coupled set of delay differential equations. With DITO, the transient oscillations do not get suppressed as in our model, but oscillatory behaviors keep a prolonged duration of oscillation with increasing delay.

Similar transient oscillations are observed with some other equations in the form of equation (1). Our preliminary numerical investigation using a monotone decreasing (“negative feedback”) and Mackey–Glass (“mixed feedback”) functions[8, 23, 24] show transient oscillations. The detailed nature of these equations and associated behaviors are left for future studies.
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