Comment on
“Fourier transform of hydrogen-type atomic orbitals”,
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Podolsky and Pauling (Phys. Rev. 34, 109 - 116 (1929)) were the first ones to derive an explicit expression for the Fourier transform of a bound-state hydrogen eigenfunction. Yükcü and Yükcü, who were apparently unaware of the work of Podolsky and Pauling or of the numerous other earlier references on this Fourier transform, proceeded differently. They expressed a generalized Laguerre polynomial as a finite sum of powers, or equivalently, they expressed a bound-state hydrogen eigenfunction as a finite sum of Slater-type functions. This approach looks very simple, but it leads to comparatively complicated expressions that cannot match the simplicity of the classic result obtained by Podolsky and Pauling. It is, however, possible to reproduce not only the Podolsky and Pauling formula for the bound-state hydrogen eigenfunction, but to obtain results of similar quality also for the Fourier transforms of other, closely related functions such as Sturmians, Lambda functions or Guseinov’s functions by expanding generalized Laguerre polynomials in terms of so-called reduced Bessel functions.

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I. INTRODUCTION

Yükcü and Yükcü [2] derived explicit expressions for the Fourier transform of a bound-state hydrogen eigenfunction. Their article creates the impression that their results [2, Eqs. (15) and (16)] are new. This is wrong. Moreover, their explicit expressions are less compact and therefore also less useful than those already described in the literature.

In 1929, Podolsky and Pauling [1, Eq. (28)] were the first ones to derive an explicit expression via a direct Fourier transformation of a generating function of the generalized Laguerre polynomials. In 1932, Hylleras [3, Eqs. (11c) and (12)] derived this Fourier transformation algebraically by solving a differential equation for the momentum space eigenfunction. In 1935, Fock [4] re-formulated the momentum space Schrödinger equation for the hydrogen atom as a 4-dimensional integral equation, whose solutions – the 4-dimensional hyperspherical harmonics – are nothing but the Fourier transforms of bound-state hydrogen eigenfunctions in disguise (see for example [5, Section VI] or the books by Avery [6, 7], Avery and Avery [8], and Avery, Rettrup, and Avery [9] and references therein).

The Fourier transform of a bound state hydrogen eigenfunction has been treated in numerous books and articles. Examples are the books by Bethe and Salpeter [10, Eq. (8.8)], Englefield [11, Eqs. (5.5) and (5.6)], or Biedenharn and Louck [12, Eq. (7.4.69)] or the relatively recent review by Hill [13, Eq. (9.55)]. This Fourier transform was even discussed in a Wikipedia article [14], which cites the book by Brandsen and Joachain [15, Eq. (A5.34)] as its source. There are also articles by Klein [16] and by Hey [17, 18], which discuss properties of the momentum space hydrogen functions. In [5, Section IV], I presented a different and remarkably simple derivation of the Fourier transform of a bound state hydrogen eigenfunction and of related functions which will play a major role in Section VI.

In Section VI, basic properties of the generalized Laguerre polynomials and of bound state hydrogen eigenfunctions are reviewed. As discussed in Section III, bound state hydrogen eigenfunctions are in contrast to several other similar function sets not complete in the Hilbert space $L^2(\mathbb{R}^3)$ of square integrable functions. This makes bound state hydrogen eigenfunctions useless in expansions. Apparently, Yükcü and Yükcü are unaware of this well known and very consequential fact.

The explicit expressions derived by Yükcü and Yükcü [2, Eqs. (15) and (16)] are less useful than those mentioned above (compare the discussion in Section V). This is a direct consequence of their derivation: Yükcü and Yükcü expressed generalized Laguerre polynomials in terms of better suited alternative function sets, the so-called reduced Bessel functions.

II. GENERALIZED LAGUERRE POLYNOMIALS AND BOUND-STATE HYDROGEN EIGENFUNCTIONS

The generalized Laguerre polynomials $L_n^{(\alpha)}(z)$ with $\Re(\alpha) > -1$ and $n \in \mathbb{N}_0$ are the classical orthogonal polynomials associated with the integration interval $[0, \infty)$ and the weight function $w(z) = z^\alpha \exp(-z)$. They are of considerable importance in mathematics and also in theoretical...
physics. There is a detailed literature which is far too extensive to be cited here. Those interested in the historical development with a special emphasis on quantum physics should consult an article by Mawhin and Ronveaux [19]. Generalized Laguerre polynomials also played a major role in my own research [5, 20–24].

It is recommendable to use the modern mathematical definition of the generalized Laguerre polynomials \( L_n^{(\alpha)}(z) \) with \( n \in \mathbb{N}_0 \) and \( \alpha, z \in \mathbb{C} \), which are defined either via their Rodrigues’ relationship [25, Eq. (18.5.5) and Table 18.5.1] or as a terminating confluent hypergeometric series \( F_1 \) [25, Eq. (18.5.12)]:

\[
L_n^{(\alpha)}(z) = \frac{z^{-\alpha} e^z}{n!} \frac{d^n}{dz^n} \left[ e^{-z} z^{n+\alpha} \right] = \frac{(\alpha + 1)_n}{n!} F_1(-n; \alpha + 1; z). \tag{2}
\]

Further details can be found in books on special functions.

Dating back from the early days of quantum mechanics, an antiquated notation is still frequently used mainly in atomic theory. For example, Bethe and Salpeter [10, Eq. (3.5)] introduced so-called associated Laguerre functions \( L_n^m(z) \) with \( m, n \in \mathbb{N}_0 \) via the Rodrigues-type relationships

\[
\begin{align*}
[L_n^m(z)]_{\text{BS}} &= \frac{d^m}{dz^m} \left[ L_n(z) \right]_{\text{BS}}, \tag{3a} \\
[L_n(z)]_{\text{BS}} &= e^z \frac{d^n}{dz^n} \left[ e^{-z} z^n \right]. \tag{3b}
\end{align*}
\]

This convention is also used in the books by Condon and Shortley [26, Eqs. (6) and (9) on p. 115] and Condon and Odabasi [27, Eq. (2) on p. 189].

Generalized Laguerre polynomials with integral superscript \( \alpha = m \in \mathbb{N}_0 \) and the associated Laguerre functions \(3\) are connected via

\[
L_n^{(m)}(z) = \frac{(-1)^m}{(n+m)!} [L_{n+m}^m(z)]_{\text{BS}}, \quad m, n \in \mathbb{N}_0. \tag{4}
\]

The notation for associated Laguerre functions is less intuitive than the notation for the generalized Laguerre polynomials, whose subscript \( n \) corresponds to the polynomial degree and whose superscript \( \alpha \) characterizes the weight function \( w(z) = z^{\alpha} \exp(-z) \). The worst drawback of the functions \(3\) is that they cannot express generalized Laguerre polynomials \( L_n^{(\alpha)} \) with non-integral superscripts \( \alpha \) which also occur in quantum physics. The eigenfunctions \( \Omega_{n,\ell}^{(\beta)}(r) \) of the Hamiltonian \( \beta^{-2} \nabla^2 - \beta^2 r^2 \) of the three-dimensional isotropic harmonic oscillator contain generalized Laguerre polynomials in \( r^2 \) with half-integral superscripts (see for example \(3\), Eq. (5.4)) and references therein). Similarly, the eigenfunctions of the Dirac equation for the hydrogen atom contain generalized Laguerre polynomials with in general non-integral superscripts [13, Eqs. (9.84) and (9.85)].

If the modern mathematical notation is used, the bound-state eigenfunctions of a hydrogenlike ion with nuclear charge \( Z \) in spherical polar coordinates is essentially the product of an exponential and a generalized Laguerre polynomial, both depending on \( r \), and a regular solid harmonic \( Y_{\ell m}^p(\theta, \phi) = r^\ell Y_{\ell m}^p(\theta, \phi) \) (see for example [12, Eqs. (7.4.41) - (7.4.43)] or [13, Eqs. (9.2) and (9.10)):

\[
W_{n,\ell}^m(Z, r) = (\frac{2Z}{n})^{3/2} \left[ (n - \ell - 1)! \right]^{1/2} \\
\times e^{-Zr/n} L_{n-\ell}^{2(\ell+1)}(2Zr/n)^m (2Zr/n), \quad n \in \mathbb{N}, \ell \in \mathbb{N}_0 \leq n - 1, -\ell \leq m \leq \ell. \tag{5}
\]

Yükçü and Yükçü [2] define the radial part of the bound-state eigenfunctions \(5\) via their Eq. (3), which is incompatible with their definition of the generalized Laguerre polynomials via their Eq. (11). It can be shown that their Eq. (11) is equivalent to Eq. (4) which implies that Yükçü and Yükçü also use the modern mathematical notation. In addition, their Ref. [26] for their Eq. (11) is incorrect. The so-called Bateman Manuscript Project [28–32] was named to honor Harry Bateman who had died in 1946, i.e., long before these books had been completed. Thus, the correct reference for Eq. (11) of Yükçü and Yükçü [2] would be [29, Eq. (7) on p. 188].

### III. INCOMPLETENESS OF THE BOUND-STATE HYDROGEN EIGENFUNCTIONS

Expansions of a given function in terms of suitable function sets are among the most useful techniques of mathematical physics. This approach requires that the function set being used is complete and preferably also orthogonal in the corresponding Hilbert space. As for example discussed in [23] or in [33], non-orthogonal expansions can easily have pathological properties.

Bound-state hydrogenic eigenfunctions \(5\) are orthonormal with respect to an integration over the whole \( \mathbb{R}^3 \),

\[
\int [W_{n,\ell}^m(Z, r)]^* W_{n',\ell'}^{m'}(Z, r) \, d^3r = \delta_{nn'} \delta_{\ell\ell'} \delta_{mm'}, \tag{6}
\]

but they are not complete in the Hilbert space

\[
L^2(\mathbb{R}^3) = \left\{ f : \mathbb{R}^3 \to \mathbb{C} \mid \int |f(r)|^2 \, d^3r < \infty \right\} \tag{7}
\]

of square integrable functions without the inclusion of the technically very difficult continuum eigenfunctions, described for instance in [10, pp. 21 - 25], in [25, Chapter 33 Coulomb Functions] or in the recent article [34]. Yükçü and Yükçü are apparently not aware of this incompleteness.

In the literature, this incompleteness, which was first described in [28] by Hylleraas [35, p. 469], is sometimes overlooked — often with catastrophic consequences. For example, Yükçü and Yükçü cited as their Ref. [4] an article by Yamaguchi [36] in order to demonstrate the usefulness of bound-state hydrogenic eigenfunctions in expansions. However, Yamaguchi’s article had been severely criticized in [21] for simply neglecting the troublesome continuum eigenfunctions. Already in 1955, Shull and Löwdin [37] had emphasized the importance of the continuum eigenfunctions and tried to estimate the magnitude of the error due to their omission.
At first sight, this incompleteness may seem surprising since the completeness of the generalized Laguerre polynomials \( L_{n}^{(\alpha)}(z) \) in the weighted Hilbert space

\[
L_{n}^{2}(z, \alpha, \infty) = \left\{ f : \mathbb{C} \rightarrow \mathbb{C} \mid \int_{0}^{\infty} e^{-z} z^{\alpha} |f(z)|^2 \, dz < \infty \right\}
\]

is a classic result of mathematical analysis (see for example the books by Higgins [38, p. 33], Sansone [39, pp. 349 - 351], Szegő [40, pp. 108 - 110], or Tricomi [41, pp. 235 - 238]). Thus, every function \( f \in L_{n}^{2}(z, \alpha, \infty) \) can be expressed by a Laguerre series

\[
f(z) = \sum_{n=0}^{\infty} \lambda_{n}^{(\alpha)} L_{n}^{(\alpha)}(z), \tag{9a}
\]

\[
\lambda_{n}^{(\alpha)} = \frac{n!}{\Gamma(\alpha + n + 1)} \int_{0}^{\infty} z^{\alpha} e^{-z} L_{n}^{(\alpha)}(z) f(z) \, dz, \tag{9b}
\]

which converges in the mean with respect to the norm of the Hilbert space \( L_{n}^{2}(z, \alpha, \infty) \). For a condensed discussion of Laguerre expansions, see [22, Section 2].

How can the incompleteness of the bound-state hydrogen eigenfunctions [5] be explained? The culprit is their \( n \)-dependent scaling parameter \( 2Z/n \). Fock [42, Eq. (6.17) on p. 200] showed that the confluent hypergeometric function

\[
1F_1(-n + \ell + 1; 2\ell + 2; 2Zr/n) = \sum_{\nu=0}^{n-\ell-1} \frac{(-n + \ell + 1)\nu}{(2\ell + 2)\nu} \frac{(2Zr/n)^\nu}{\nu!}
\]

occurring in Eq. (5) can in the limit \( n \rightarrow \infty \) be represented by a Bessel function \( J_{2\ell+1}(\sqrt{8Zr}) \) of the first kind, which is an oscillatory function that decays too slowly to be square integrable (compare also [23, Eq. (18.11.6)]). In the limit \( n \rightarrow \infty \), the exponential \( \exp(-Zr/n) \) in Eq. (5) loses its exponential decay as \( r \rightarrow \infty \). Consequently, the bound state hydrogen eigenfunctions [5] become oscillatory as \( n \rightarrow \infty \), which means that they are no longer square integrable. Instead, they belong to the continuous spectrum. Thus, the so-called bound-state eigenfunctions are no longer bound-state functions if the principal quantum number \( n \) becomes very large. This implies that the bound-state eigenfunctions cannot form a basis for the Hilbert space \( L^2(\mathbb{R}^3) \) of square integrable functions (compare [42, text following Eq. (6.19)] on p. 201).

Because of the incompleteness of the bound-state hydrogen eigenfunction, it is now common to use in expansions alternative function sets also based on the generalized Laguerre polynomials that possess more complete completeness properties. Closely related to the bound-state hydrogeneigenfunctions are the so-called Coulomb Sturmians or Sturmians which were already used in [1928] by Hylleraas [35, Eq. (25)] on p. 478:

\[
\Psi_{n,\ell}^{m}(\beta, r) = (2\beta)^{3/2} \left[ \frac{(n - \ell - 1)!}{2n(n + \ell)!} \right]^{1/2} e^{-\beta r} L_{n-\ell-1}^{(2\ell+1)}(2\beta r) Y_{\ell}^{m}(2\beta r), \tag{11}
\]

Here, the notation of [5, Eq. (4.6)] is used. We obtain bound-state hydrogen eigenfunctions [5] with a correct normalization factor if we make in Eq. (11) the substitution \( \beta \rightarrow Z/n \) (compare the discussion following [5, Eq. (4.12)]):

\[
\Psi_{n,\ell}^{m}(Z/n, r) = W_{n,\ell}^{m}(Z, r). \tag{12}
\]

This is a non-trivial result. Sturmians are complete and orthonormal in the in the Sobolev space \( W_{2}^{(1)}(\mathbb{R}^3) \) (for the definition of Sobolev spaces plus further references, see [5, Section III]), whereas bound state hydrogen functions are orthonormal but incomplete in the Hilbert space \( L^2(\mathbb{R}^3) \).

Sturmians occur in the context of Fock’s treatment of the hydrogen atom [4], albeit in a somewhat disguised form (compare [5, Section VII]). There is a classic review by Rotenberg [43]. A fairly detailed discussion of their properties was given by Novosadov [44]. Sturmians also play a major role in books by Aver [6, 7], Avery and Avery [8], and Avery, Rettrup, and Avery [9]. We used Sturmians for the construction for an addition theorem of the Yukawa potential [45] with the help of weakly convergent orthogonal and biorthogonal expansions for the plane wave introduced in [5, Section III].

Lambda functions were introduced already in [1929] by Hyllemaas [46, Footnote * on p. 349], and later by Shull and Löwdin [37] and by Löwdin and Shull [47, Eq. (46)]:

\[
\Lambda_{n,\ell}^{m}(\beta, r) = (2\beta)^{3/2} \left[ \frac{(n - \ell - 1)!}{(n + \ell + 1)!} \right]^{1/2} \times e^{-\beta r} L_{n-\ell-1}^{(2\ell+2)}(2\beta r) Y_{\ell}^{m}(2\beta r). \tag{13}
\]

Here, the notation of [5, Eq. (4.4)] is used. The use of Lambda functions in electronic structure theory was suggested by Kutzelnigg [48] and Smeyers [49] in 1963 and 1964 respectively. Filter and Steinborn [50] used them for the derivation of one-range addition theorems of exponentially decaying functions, and I used both Sturmians and Lambda functions for the construction of weakly convergent expansions of a plane wave [5].

Both Sturmians and Lambda functions defined by Eqs. (11) and (13) have a fixed scaling parameter \( \beta > 0 \) that does not depend on the principal quantum number \( n \). Consequently, these functions are orthogonal and complete in suitable Hilbert and Sobolev spaces. A detailed discussion of the mathematical properties of the functions \( \Psi_{n,\ell}^{m}(\beta, r) \) and \( \Lambda_{n,\ell}^{m}(\beta, r) \) was given in [5, Section IV] or in [23, Section 2].

IV. THE WORK OF PODOLSKY AND PAULING

The Fourier transform of an irreducible spherical tensor of integral rank yields a Hankel-type radial integral multiplied by a spherical harmonic if the so-called Rayleigh expansion of a plane wave (compare for instance [12, p. 442]) is used:

\[
e^{\pm i\mathbf{r} \cdot \mathbf{y}} = 4\pi \sum_{\ell=0}^{\infty} (\pm i)^{\ell} j_{\ell}(x y)
\]

\[
\times \sum_{m=-\ell}^{\ell} \left[ Y_{\ell}^{m}(x/|y|) \right]^{*} Y_{\ell}^{m}(y/|y|), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3. \tag{14}
\]
With the help of the orthonormality of the spherical harmonics and the definition of the spherical Bessel functions $j_n(xy)$ (see for example [25, Eq. (10.47.3)]), we obtain the following expression for the Fourier transformation of a bound-state hydrogen eigenfunction in the usual books on special function theory. Even today, I could not find the required expression for the Fourier transformation of a Sturmian function. Inserting this generating function of the generalized Laguerre polynomials into the radial integral in Eq. (15), we need an explicit expression for the integral

$$I_n^{(\alpha, \mu, \nu)}(a, b) = \int_0^\infty y^\mu e^{-\alpha y} J_\nu(b y) L_n^{(\alpha)}(2 a y) \, dy.$$  

In [1929], when Podolsky and Pauling [1] tried to derive an expression for the Fourier transform of a bound-state hydrogen eigenfunction, no explicit expression for this integral was known. Even today, I could not find the required expression in the usual books on special function theory.

Podolsky and Pauling [1, Eq. (6)] found a very elegant solution to this problem. Their starting point was the generating function [51, p. 242]

$$\exp\left(\frac{x t}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n, \quad |t| < 1.$$  

Inserting this generating function of the generalized Laguerre polynomials into the radial integral in Eq. (15) yields:

$$\int_0^\infty \int_0^\infty r^{\ell + 3/2} e^{-\beta r} J_{\ell + 1/2}(pr) L_n^{(2\ell + 1)}(2\beta r) \, dr = (1-t)^{-2\ell - 2} \times \int_0^\infty e^{-\beta r \frac{1+t}{2}} r^{\ell + 3/2} J_{\ell + 1/2}(pr) \, dr.$$  

The radial integral on the right-hand side can be expressed in closed form. We use [52, Eq. (2) on p. 385]

$$\int_0^\infty e^{-\alpha y} J_\nu(b y) y^{\mu-1} \, dy = \frac{(b^2/2)^\nu \Gamma(\mu + \nu)}{\alpha^{\nu+\nu} \Gamma(\nu + 1)} \times 2F_1\left(\frac{\mu + \nu - 1}{2}; \nu + 1; -\frac{b^2}{\alpha^2}\right), \quad \Re(\alpha + ib) > 0,$$  

to obtain

$$\int_0^\infty e^{-\beta r \frac{1+t}{2}} r^{\ell + 3/2} J_{\ell + 1/2}(pr) \, dr = \frac{(2\ell + 2)!}{\Gamma(\ell + 3/2)} \frac{(p/2)^{\ell + 1/2}}{[\beta(1+t)/(1-t)]^{2\ell + 3}} \times 2F_1\left(\ell + 3/2, \ell + 2; \ell + 3/2; -\frac{p^2}{\beta^2} (1-t)^2\right).$$

This Gaussian hypergeometric series $2F_1$ is actually a binomial series

$$F_0(\ell + 2; z) = \sum_{m=0}^{\infty} \frac{(\ell + 2)_m z^m}{m!} = (1-z)^{-\ell - 2}$$

with $z = -p^2 (1-t)^2 / [\beta^2 (1+t)^2]$ [25, Eq. (15.4.6)]. Thus, we obtain for the right-hand side of Eq. (18):

$$\int_0^\infty e^{-\beta r \frac{1+t}{2}} r^{\ell + 3/2} J_{\ell + 1/2}(pr) \, dr = \frac{(2\ell + 2)!}{\Gamma(\ell + 3/2)} \frac{(p/2)^{\ell + 1/2} \beta (1-t)^2}{[\beta^2 (1+t)^2 + p^2 (1-t)^2]^{\ell + 2}}.$$  

The denominator can be simplified further, using $\beta^2 (1+t)^2 + p^2 (1-t)^2 = (\beta^2 + p^2) \{1 + [2(\beta^2 - p^2)]/\beta^2 + p^2\} t + t^2$, yielding

$$\int_0^\infty e^{-\beta r \frac{1+t}{2}} r^{\ell + 3/2} J_{\ell + 1/2}(pr) \, dr = \frac{(2\ell + 2)!}{\Gamma(\ell + 3/2)} \frac{(p/2)^{\ell + 1/2} \beta (1-t)^2}{[\beta^2 + p^2] (1+t)^2 t + t^2}.$$  

The rational function on the right-hand side closely resembles the generating function [51, p. 222]

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) t^n, \quad |t| < 1,$$  

of the Gegenbauer polynomials. Podolsky and Pauling only had apply the differential operator $t^{1-\lambda} \partial / \partial t^\lambda$ to Eq. (23). This yields the following modified generating function of the Gegenbauer polynomials (compare [1, Eq. (25)])

$$\frac{1 - t^2}{(1 - 2xt + t^2)^{\lambda + 1}} = \sum_{n=0}^{\infty} \frac{\lambda + n}{\lambda} C_n^\lambda(x) t^n,$$  

which I could not find in the usual books on special function theory. The rational function on the right-hand side of Eq. (22) is of the same type as the left-hand side of this modified generating function. If we make in Eq. (24) the substitutions $x \rightarrow (p^2 - \beta^2)/(p^2 + \beta^2)$ and $\lambda \rightarrow \ell + 1$, we obtain the following expansion in terms of Gegenbauer polynomials:

$$\frac{1 - t^2}{t^{\ell + 1}} \left[1 - \frac{2p^2 - \beta^2}{p^2 + \beta^2} t + t^2\right]^{\ell + 2} = \sum_{n=0}^{\infty} \frac{n + \ell + 1}{\ell + 1} C_n^{\ell + 1}\left(\frac{p^2 - \beta^2}{p^2 + \beta^2}\right) t^n.$$  

Inserting this into Eq. (22) yields:

$$\int_0^\infty e^{-\beta r \frac{1+t}{2}} r^{\ell + 3/2} J_{\ell + 1/2}(pr) \, dr = \frac{(p/2)^{\ell + 1/2}(2\ell + 2)!}{(\ell + 1)\Gamma(\ell + 3/2) \beta} \times \sum_{n=0}^{\infty} \frac{n + \ell + 1}{\ell + 1} C_n^{\ell + 1}\left(\frac{p^2 - \beta^2}{p^2 + \beta^2}\right) t^n.$$
Thus, we finally obtain the following explicit expression for the Fourier transform of an unnormalized Sturmian:

\[
(2\pi)^{-3/2} \int e^{i\mathbf{p} \cdot \mathbf{r}} e^{-\beta r} L_{n-\ell-1}^{2(\ell+1)}(2\beta r) Y_{\ell m}(2\beta r) \, d^3 r
\]

\[
= \frac{(2/\pi)^{1/2} 2^{2\ell+1} \ell! \beta^{\ell+1} n}{[p^2 + \beta^2]^{\ell+2}} \times C^{\ell+1}_{n-\ell-1} \left( \frac{p^2 - \beta^2}{p^2 + \beta^2} \right) Y_{\ell m}(-i\mathbf{p}).
\]

(27)

To obtain the Fourier transform of a normalized Sturmian defined by Eq. (11), we multiply Eq. (27) by the normalization factor \((2\beta)^{3/2} \,(n-\ell-1)!/[2n(n+\ell)!]^{1/2}\), yielding [5, Eq. (4.24)):

\[
\Psi_{n,\ell}(\beta, \mathbf{r}) = (2\pi)^{-3/2} \int e^{i\mathbf{p} \cdot \mathbf{r}} \Psi_{n,\ell}(\beta, \mathbf{r}) \, d^3 r
\]

\[
= 2^\ell \ell! \left[ \frac{2\beta n(n-\ell-1)!}{\pi(n+\ell)!} \right]^{1/2} \left[ \frac{2\beta}{p^2 + \beta^2} \right]^{\ell+2} \times C^{\ell+1}_{n-\ell-1} \left( \frac{p^2 - \beta^2}{p^2 + \beta^2} \right) Y_{\ell m}(-i\mathbf{p}).
\]

(28)

To obtain the Fourier transform of a bound-state hydrogen eigenfunction, we only have to use Eq. (12) and make the substitution \(\beta \to Z/n\). Thus, we obtain [1, Eq. (28)):

\[
\Psi_{n,\ell}(Z, \mathbf{p}) = (2\pi)^{-3/2} \int e^{i\mathbf{p} \cdot \mathbf{r}} \Psi_{n,\ell}(Z, \mathbf{r}) \, d^3 r
\]

\[
= 2^\ell \ell! \left[ \frac{2Z(n-\ell-1)!}{\pi(n+\ell)!} \right]^{1/2} \left[ \frac{2Zn}{n^2p^2 + Z^2} \right]^{\ell+2} \times C^{\ell+1}_{n-\ell-1} \left( \frac{n^2p^2 - Z^2}{n^2p^2 + Z^2} \right) Y_{\ell m}(-i\mathbf{p}).
\]

(29)

This Fourier transformation was in principle also derived by Rotenberg [43, Eq. (26) on p. 241] in disguised form. However, Rotenberg’s results are misleading because of an unfortunate definition of the Sturmians (compare [5, p. 283]).

If we compare Eq. (29) with formulas published by other authors, we find some discrepancies. In the formula given by Podolsky and Pauling [1, Eq. (28)], a phase factor \((-i)^{\ell}\) is missing. The same error was reproduced by Bethe and Salpeter [19, Eq. (8.8)]. The formula given by Englefield [11, Eqs. (5.5) and (5.6)] differs from Eq. (28) by a phase factor \((-1)^{\mu}\). Finally, in the expression given by Biedenharn and Louck [12, Eq. (7.4.6)], a factor \(\pi^{-1/2}\) is missing.

Kaisjer and Smith [53, pp. 50 - 52] showed that the generating function approach of Podolsky and Pauling can be extended to the Fourier transform of a Lambda function defined by Eq. (13). However, the approach of Podolsky and Pauling [1, Eq. (28)] and Kaisjer and Smith [53] requires considerable manipulative skills. In Section VII I will show how the Fourier transforms of bound-state hydrogen eigenfunctions, Sturmians, and Lambda functions and of other Laguerre-type functions can be constructed in an almost trivially simple way by expanding generalized Laguerre polynomials in terms of so-called reduced Bessel functions (compare [5, Section IV]).

V. THE WORK OF YÜKÜ AND YÜKÜ

Podolsky and Pauling [1] faced the problem that no simple closed form expression for the Hankel-type integral in Eq. (15) was known. They solved this problem by computing instead the Fourier transform of the generating function [17], which leads to the comparatively simple and explicitly known Hankel-type integral in Eq. (18). In this way, Podolsky and Pauling only had to perform a series expansion of the radial integral in Eq. (18) to derive the explicit expression (29) for the Fourier transform of a bound state hydrogen eigenfunction.

Yükü and Yükü [2, 3], who were apparently unaware of the work by Podolsky and Pauling [1] or of the whole extensive literature on this topic, proceeded differently. They utilized the fact that a generalized Laguerre polynomial \(L_n^{(\alpha)}(z)\) is according to Eq. (2) a polynomial of degree \(n\) in \(z\). Thus, the generalized Laguerre polynomial \(L_{n-\ell-1}^{(2\ell+1)}(2\beta r)\) occurring in Eq. (11) can be expressed as a sum of powers:

\[
L_{n-\ell-1}^{(2\ell+1)}(2\beta r) = \sum_{\nu=0}^{n-\ell-1} \frac{(-n+\ell+1)_\nu (2\beta r)^\nu}{(2\ell+\nu+1)! \nu!}.
\]

(30)

To achieve what they believe to be a further simplification, Yükü and Yükü [2, Eqs. (10) - (12)] combined Eq. (30) with the Laguerre multiplication theorem [51, p. 249]

\[
L_n^{(\alpha)}(z) = \sum_{m=0}^n \binom{n+\alpha}{m} \binom{1-z}{m-n} L_m^{(\alpha)}(x).
\]

(31)

However, the combination of Eqs. (30) and (31) leads to the same Hankel-type integrals as the direct use of Eq. (30). Thus, this combination accomplishes nothing and only introduces a completely useless additional inner sum. Therefore, I will only consider the direct use of Eq. (30).

In 1930, Slater [54] introduced the so-called Slater-type functions, which had an enormous impact on atomic electronic structure theory and which in normalized form are expressed as follows:

\[
\chi_{n,\ell}^{\text{Sl}}(a, r) = (ar)^{-n-\ell} e^{-ar} Y_{\ell m}(\theta, \phi) = (ar)^{-n-\ell} e^{-ar} Y_{\ell m}(ar), \quad a > 0.
\]

(32)

I always tacitly assume that the principal quantum number \(n\) is a positive integer \(n \in \mathbb{N}\) satisfying \(n-\ell \geq 1\).

With the help of Slater-type functions, an unnormalized Sturmian function [11] with fixed \(\beta > 0\) can be expressed as follows:

\[
e^{-\beta r} L_{n-\ell-1}^{(2\ell+1)}(2\beta r) Y_{\ell m}(2\beta r) = \frac{(n+\ell)!}{(n-\ell-1)!} \times \sum_{\nu=0}^{n-\ell-1} \frac{(-n+\ell+1)_\nu 2^\nu}{(2\ell+\nu+1)! \nu!} \chi_{n,\ell}^{\text{Sl}}(2\beta r).
\]

(33)

The idea of expressing functions based on the generalized Laguerre polynomial by finite sums of Slater-type functions
is not new. To the best of my knowledge, it was introduced by Smeyers [49] in 1966, who expressed Lambda functions defined by Eq. (13) as linear combinations of Slater-type functions. Smeyers constructed in this way one-range addition theorems of Slater-type functions, which were expansion in terms of Lambda functions. Thus, their expansion coefficients are overlap integrals [49, Section 3]. In 1978, Guseinov [55, Eqs. (6) - (8)] adopted Smeyers’ approach and consistently used it in his countless later publications, without ever giving credit to Smeyers [49].

Smeyers’ approach is undoubtedly very simple. Nevertheless, it is not good. In Eq. (30) there are strictly alternating signs. Therefore, in sums of the type of Eq. (33), which inherit the alternating signs from Eq. (30), numerical instabilities are to be expected in the case of larger summation indices. This had already been emphasized in 1982 by Trivedi and Steinborn [56, pp. 116 - 117]. For a more detailed discussion plus additional references, see [23, pp. 32 - 34].

Fourier transformation is a linear operation. Consequently, Eq. (33) implies that the Fourier transformation of a Sturmian – or of any of the various other functions based on generalized Laguerre polynomials – can be expressed as a finite linear combination of Fourier transforms of Slater-type functions with integral principal quantum numbers (compare [2, Eq. (12)]).

There is an extensive literature on Fourier transforms of Slater-type functions. I am aware of articles by Geller [57, 58], Silverstone [59, 60], Edwards, Gottlieb, and Doddrell [61], Henneker and Cade [62], Kajser and Smith [53], Weniger and Steinborn [20], Niukkanen [63], Belkić and Taylor [64], and by Akdemir [65]. In addition, there is a Wikipedia article [66], whose principal reference is the article by Belkić and Taylor [64]. Yükcü and Yükcü [2] only mentioned Niukkanen [63] as their Ref. [8].

The Rayleigh expansion [24] leads to an expression of the Fourier transform of a Slater-type function as a Hankel-type radial integral:

\[
\frac{\chi_{n,\ell}(\alpha, p)}{\lambda_{n,\ell}(\alpha)} = \frac{2}{\lambda_{n,\ell}(\alpha)} = \int e^{-ipr} \chi_{n,\ell}(\alpha, r) \, dr = \int e^{-ipr} \chi_{n,\ell}(\alpha, r) \, dr = \lambda_{n,\ell}(\alpha)
\]

\[
\int_0^{\infty} e^{-at} J_{\ell}(bt) t^{\nu-1} \, dt = \frac{\Gamma(\mu+\nu)}{(a \pm ib)^{\nu+1}} \Gamma(\nu+1)
\]

\[
\times F_1\left(\nu + \frac{1}{2}, \mu + \nu; 2\nu + 1; \; \pm 2ib \; \pm 2ib \right).
\]

This yields the following Fourier transformation:

\[
\frac{\chi_{n,\ell}(\alpha, p)}{\lambda_{n,\ell}(\alpha)} = \frac{(n + \ell + 1)!}{(2\pi)^{1/2}(1/2)_{\ell+1}} \frac{\alpha^{\nu-1}}{(\alpha \pm ip)^{\nu+1}} \frac{\nu^{\ell}\chi_{\ell}(\alpha)}{\lambda_{n,\ell}(\alpha)}
\]

\[
\times F_1\left(\nu + \frac{1}{2}, \mu + \nu; 2\nu + 1; \; \pm 2ib \; \pm 2ib \right).
\]

A slightly less general result had been obtained by Belkić and Taylor [64, Eq. (15)]. They used only the upper signs in the expression involving the \( J_1 \) given above. In this way, Belkić and Taylor [64, Eq. (15)] obtained only the upper signs on the right-hand side of Eq. (37).

The Hankel-type integral in Eq. (36) is real if \( \mu, \nu, p \in \mathbb{R} \). Thus, the right-hand side of Eq. (36) also has to be real, or equivalently, it has to be equal to its complex conjugate. This implies:

\[
F_1\left(\nu + \frac{1}{2}, \mu + \nu; 2\nu + 1; \; \pm 2ib \; \pm 2ib \right) = \frac{\alpha \pm ib}{\alpha \pm ib}
\]

\[
\times F_1\left(\nu + \frac{1}{2}, \mu + \nu; 2\nu + 1; \; \pm 2ib \; \pm 2ib \right).
\]

Analogous symmetries also exist in Eq. (37) and in all other expressions of that kind derived later.

The hypergeometric series \( F_1 \) in Eqs. (35) and (37) do not converge for all \( p \geq 0 \). We either have \( \pm p^2 \alpha^2 \to \infty \) as \( p \to \infty \), or \( \pm 2ib/|\alpha \pm ip| \to 2 \) as \( p \to \infty \). Thus, Eqs. (35) and (37) are not sufficient for computational purposes. They are, however, convenient starting points for the construction of alternative expressions with better numerical properties. In the case of the \( F_1 \) in Eq. (35), this had already been emphasized in the first edition of Watson’s classic book [52, Eq. (3.0) on p. 385] which appeared in 1922.

For the construction of analytic continuation formulas, it makes sense to use the highly developed transformation theory of the Gaussian hypergeometric function \( F_1 \) as an ordering principle (see for example [51, pp. 47 - 51] or [23, §15.8 Transformations of Variable]). This leads to a vast number of alternative expressions (far too many to be presented here). Therefore, I will only concentrate on illustrative examples.

The first author – N. Yükcü – should be aware of the relevance of analytic continuation formulas of hypergeometric functions because of his recent article Hypergeometric Functions in Mathematics and Theoretical Physics [67].

The simplest transformations of \( F_1 \) are the Euler and
Pfaff transformations (see for example [51, p. 47]):

\[ \begin{align*}
2F_1(a, b; c; z) & = (1 - z)^{c-a-b}2F_1(c - a, c - b; c; z) \\
& = (1 - z)^{-a}2F_1(a, c - b; c; z/(z - 1)) \\
& = (1 - z)^{-b}2F_1(c - a, b; c; z/(z - 1)).
\end{align*} \tag{39, 40, 41} \]

The application of the Euler transformation (39) to the \(2F_1\) in Eqs. (35) and (37) yields [20, Eq. (3.11)]

\[ \begin{align*}
\chi_{n,ℓ}^m(a, p) &= \frac{(n + ℓ + 1)!}{(2π)^{1/2} (1/2)_{ℓ+1}} \frac{α^{n−1}}{n^2 + p^2} \times \frac{(α ± ip)ℓ+1 (α ± ip)n+1}{2F_1 \left( \ell - n, ℓ + 1; 2ℓ + 2; \frac{±2ip}{α ± ip} \right)}.
\end{align*} \tag{42, 43} \]

Since we assume \( n, ℓ \in \mathbb{N}, n - ℓ - 1 \geq 0, n - ℓ \) and either \((ℓ - n)/2 \) or \((ℓ + n)/2 \) are positive integers. Accordingly, the \(2F_1\)s in Eqs. (42) and (43) terminate, which represents a substantial improvement compared to the non-terminating \(2F_1\)s in Eqs. (35) and (37). Both Eqs. (42) and (43) allow a convenient evaluation of \(\chi_{n,ℓ}^m(a, p)\) for all \( p \in \mathbb{R}^3 \). For recurrence formulas of the Gaussian hypergeometric function \(2F_1(a, b; c; z)\), where two or three of the parameters \(a, b, \) and \(c\) change simultaneously, see [68, Appendix C].

We can also employ the Pfaff transformations (40) and (41). In the case of Eq. (35), this yields hypergeometric series with argument \(p^2/(α^2 + p^2)\) that either terminate or converge for all \( p \geq 0 \) [20, Eqs. (3.16) and (3.17)]. In the case of Eq. (43), we only obtain complex conjugates of known radial parts.

But this is not yet the end of the story. By systematically exploiting the known transformation properties of the Gaussian hypergeometric function \(2F_1\), many other terminating or non-terminating expressions can be derived. For example, we could also use one of the linear transformations that accomplish the variable transformations \( z \mapsto 1 - z, z \mapsto 1/z, \) respectively, by expressing a given \(2F_1\) in terms of two other \(2F_1\)s (see for example [51, pp. 47 - 49]). Normally, these transformations lead to comparatively complicate expressions which can safely be ignored.

An exception is the following expression obtained by a transformation \( z \mapsto 1/(1 - z) \) [20, Eqs. (3.19) and (3.20)]:

\[ \begin{align*}
\chi_{n,ℓ}^m(a, p) &= \frac{(π/2)^{1/2} (n + ℓ + 1)!}{α^{n−1} (α^2 + p^2)^{(n+ℓ+2)/2}} \times \frac{Γ[(n + ℓ + 3)/2]Γ[(ℓ - n + 1)/2]}{2F_1 \left( \left( \frac{n + ℓ + 3}{2} \right)^2 - \frac{α^2}{2^2 + p^2} \right)}
\end{align*} \tag{44} \]

This expression is simpler than it looks. If \( n - ℓ \) is even, the second part of the right-hand side vanishes because of the gamma function \( Γ((ℓ - n)/2)\), and if \( n - ℓ \) is odd, the first part vanishes because of the gamma function \( Γ((ℓ - n + 1)/2)\).

In addition to linear transformations, a Gaussian hypergeometric function \(2F_1\) may also satisfy so-called quadratic transformations (see for example [51, pp. 49 - 51] or [25, §15.8(iii) Quadratic Transformations]). Unlike the linear transformations considered so far, quadratic transformations do not exist for a \(2F_1(a, b; c; z)\) with completely arbitrary parameters \(a, b,\) and \(c\). They only exist for special values of the parameters \(a, b,\) and \(c\) [25, Table 15.8.1].

The hypergeometric series in Eq. (35) is of the general type \(2F_1(a, a + 1/2; c; z)\). This suggests the application of the following quadratic transformations [51, p. 50] to this \(2F_1\):

\[ \begin{align*}
2F_1 \left( a, a + 1/2; c; z \right) &= (1 - z)^{-a}2F_1 \left( 2a, 2c - 2a - 1; c; \frac{\sqrt{1 - z} - 1}{2\sqrt{1 - z}} \right) \tag{45} \\
&= (1 ± \sqrt{z})^{-2a}2F_1 \left( 2a, 2c - 1/2; 2c - 1; ± \frac{2\sqrt{z}}{1 + \sqrt{2}} \right) \tag{46} \\
&= \frac{1 + \sqrt{1 - z}}{2}^{-2a} \times 2F_1 \left( 2a, 2c - 1/2; c; \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right) \tag{47}
\end{align*} \]

Application of Eqs. (45 - 47) to the \(2F_1\) in Eq. (35) yields the following alternative expressions:

\[ \begin{align*}
\chi_{n,ℓ}^m(a, p) &= \frac{(n + ℓ + 1)!}{(2π)^{1/2} (1/2)_{ℓ+1}} \frac{α^{n−1}}{α^2 + p^2} \times \frac{Γ[(n + ℓ + 3)/2]Γ[(ℓ - n + 1)/2]}{2F_1 \left( \left( \frac{n + ℓ + 3}{2} \right)^2 - \frac{α^2}{2^2 + p^2} \right)}
\end{align*} \tag{48} \]

\[ \begin{align*}
&= \frac{(n + ℓ + 1)!}{(2π)^{1/2} (1/2)_{ℓ+1}} \frac{α^{n−1}}{α ± ip} \times \frac{Γ[(n + ℓ + 3)/2]Γ[(ℓ - n + 1)/2]}{2F_1 \left( \left( \frac{n + ℓ + 3}{2} \right)^2 - \frac{α^2}{2^2 + p^2} \right)}
\end{align*} \tag{49} \]
\[ \times _2 F_1 \left( n + \ell + 2, n + \frac{3}{2}; \ell + \frac{3}{2}; \frac{\alpha - \sqrt{\alpha^2 + p^2}}{\alpha + \sqrt{\alpha^2 + p^2}} \right). \] (50)

Equations (37) and (49) are identical. The derivation of \( F_2 \) in Eq. (48) terminates. As a remedy, we can apply the Euler transformation (39) to the non-terminating \( F_1 \) in Eqs. (49) and (50), yielding

\[ \frac{\chi_{n,\ell}^\lambda(\alpha, p)}{(2\pi)^{1/2}(1/2)_{\ell+1}} \frac{\alpha^{\ell+1}}{(\alpha + i p)\ell + 1} \left( \frac{\alpha^2 + p^2}{\alpha^2 + p^2} \right)^{n+1/2} \] (51)

The terminating \( F_1 \) in Eq. (48) can be expressed as a Gegenbauer polynomial via [51, p. 220]

\[ C_n(x) = \frac{(2\lambda)_n}{n!} F_1 \left( -n, n + 2\lambda; \lambda + \frac{1}{2}; \frac{1 - x}{2} \right), \] (53)

yielding

\[ \frac{\chi_{n,\ell}^\lambda(\alpha, p)}{(2\pi)^{1/2}(1/2)_{\ell+1}} \frac{\alpha^{\ell+1}}{(\alpha + i p)\ell + 1} \left( \frac{\alpha^2 + p^2}{\alpha^2 + p^2} \right)^{n+1/2} \times \frac{\alpha^{\ell+1}}{(\alpha^2 + p^2)^{\ell+1}} C_n^{\ell+1} \left( \frac{\alpha}{\sqrt{\alpha^2 + p^2}} \right), \] (54)

The Gegenbauer polynomial representation (54) corresponds to the second representation given by Yüksel and Yüksel [2] in their Eqs. (13) and (15). As their source, Yüksel and Yüksel [2] give the book by Gradsteyn and Rhyzhik [69] as their Ref. [25], without specifying a page or equation number. Unfortunately, I was not able to find the corresponding expression in the book by Gradsteyn and Rhyzhik [69].

In earlier articles by Yavuz, Yüksel, Öztek, Yilmaz, and Dündür [70, Eq. (12)] and Yüksel [71, Eq. (32)], the Gegenbauer polynomial representation (54) had been attributed to Guseinov [72]. Google Scholar gave me the title of Guseinov’s article, but I was not able to obtain a copy. Personal contacts to the Wuhan Institute of Physics and Mathematics of the Chinese Academy of Sciences could not help, either.

But Guseinov was not the first one to derive the Gegenbauer polynomial representation (54). To the best of my knowledge, this had been achieved by Niukkanen [63] in 1984 who introduced a fairly large class of exponentially decaying functions [63, Eqs. (2) and (3)], which contain all function sets considered in this article as special cases. The radial part of the Fourier transform of Niukkanen’s function can be expressed in terms of an Appell function \( F_2 \) [63, Eq. (21)], which is an hypergeometric function in two variables [25, Eq. (16.13.2)]. By means of a reduction formula in combination with a suitable quadratic transformation of a \( F_2 \), Niukkanen [63, Eq. (55)] obtained the Gegenbauer polynomial representation (54). This Gegenbauer representation had also been derived by Belkic and Taylor [64, Eq. (21)] in 1989 in connection with their restricted version of Eq. (55) [64, Eq. (15)].

Yüksel and Yüksel [2] used either a representation given by their Eqs. (14) and (16) involving a non-terminating \( F_1 \), which correspond to Eq. (55), or alternatively a Gegenbauer polynomial representation given by their Eqs. (13) and (15), which correspond to Eq. (54). The non-terminating \( F_1 \) in Eq. (55) converges only for \( |p^2/\alpha^2| < 1 \), whereas the Gegenbauer polynomial in Eq. (54) is meaningful for all \( |p| \in \mathbb{R}^3 \).

Thus, Yüksel and Yüksel had to prove that their Gegenbauer polynomial representation provides an analytic continuation of their representation involving a non-terminating \( F_1 \) with a finite radius of convergence to all \( |p| \in \mathbb{R}^3 \). They did this by showing in [2, Table 1] that the radial parts of these representation give for a variety of quantum numbers \( n \) and \( \ell \) and for certain values of \( p \) identical numerical results. This highly pedestrian approach is no substitute for a rigorous mathematical proof.

So far, I only showed that representations involving a \( F_1 \) with real argument can be obtained from representations involving a \( F_1 \) with complex argument (compare Eqs. (37) and (43)). However, the inverse operations are also possible. For example, the application of the quadratic transformation [51, p. 51]

\[ F_1 (a, b; 2b; z) = (1 - z^2)^{-a} \]

\[ \times F_1 \left( \frac{a}{2} - 1, b + \frac{1}{2}; \frac{z^2}{2|z|^2} \right) \] (55)

to the \( F_1 \) in Eqs. (37) and (44) yields Eqs. (55) and (42).

By suitably combining linear and quadratic transformations, many explicit expressions for the Fourier transform of a Slater-type function can be derived. However, this is not yet the
end of the story. Those Gaussian hypergeometric functions \(_{2}F_{1}\), for which a quadratic transformation exists, can also be expressed in terms of Legendre functions [51, pp. 51 - 54]. Since, however, Legendre functions can be viewed to be nothing but special hypergeometric series \(_{2}F_{1}\) [25, §14.3 Definitions and Hypergeometric Representations], I will refrain from considering Legendre function representations explicitly. This would only lead to a repetition of known hypergeometric expressions in disguise. Let me just mention that Gradshteyn and Rhyzhik [68, Eq. (6.621.1)] expressed the Hankel-type integral in Eq. (19) also in terms of Legendre functions.

My incomplete list of representations of the Fourier transform of a Slater-type function should suffice to convince even a skeptical reader that the highly developed transformation theory of the Gaussian hypergeometric function \(_{2}F_{1}\) is extremely useful in this context. It allows the derivation of a large variety of different representations, which are all analytic continuations of the basic expressions (35) and (37).

The derivation and classification of the various expressions for the Fourier transforms of Slater-type functions is certainly an achievement in its own right. Nevertheless, one should not forget that in the context of the Fourier transform of a bound-state hydrogen eigenfunction or of other functions based on the generalized Laguerre polynomials, these Slater results are essentially irrelevant. The formulas presented in this Section confirm once more what I had already emphasized in [23, p. 29]: although extremely simple in the coordinate representation, Slater-type functions are comparatively complicated objects in momentum space. Their Fourier transforms have the same level of complexity as the Fourier transforms of bound state hydrogen eigenfunctions (see [5, Section IV]).

Therefore, it cannot be a good idea to express the Fourier transform of a bound-state hydrogen eigenfunction as a linear combination of Fourier transforms of Slater-type functions. Because of strictly alternating sings, Eq. (33) as well as all formulas derived from it become numerically unstable for large quantum numbers \(n\). In addition, these linear combinations of the Fourier transforms of Slater-type functions are for large \(n\) hopelessly inefficient compared to the classic result (29) derived by Podolsky and Pauling [1, Eq. (28)]. To the best of my knowledge, nobody has ever been able to construct Eq. (29) from a linear combination of Fourier transforms of Slater-type functions.

If we evaluate the Fourier transform of a bound-state hydrogen eigenfunction or of related functions via linear combinations of the Fourier transforms of Slater-type functions, we have to deal with extensive intrinsic cancellations. I learned the hard way from my work on convergence acceleration and the summation of divergent series (see for example [73, 74] or [25, §3.9(v) Levin’s and Weniger’s Transformations]) that expansions, which are plagued by substantial intrinsic cancellations, can easily become numerically problematic. It is always desirable to use only those expressions for computational purposes, whose cancellations had been done analytically.

VI. EXPANSION IN TERMS OF REDUCED BESSEL FUNCTIONS

A single power \(z^n\) is obviously simpler than a generalized Laguerre polynomial \(L^{(α)}_n(z)\). Therefore, it is tempting to believe that powers produce simpler Hankel-type integrals than corresponding generalized Laguerre polynomials. However, simplicity is a very elusive concept, and the results in Section VI show that this seemingly obvious assumption is not true.

If we want to evaluate the Fourier transforms of bound-state hydrogen eigenfunctions or of related functions by expanding the generalized Laguerre polynomials, we must find alternative expansion functions that have more convenient properties than powers. The so-called reduced Bessel functions and their an-isotropic generalization, the so-called \(B\) functions produce the desired expansions. Based on previous work by Shavitt [75, Eq. (55) on p. 15], \(B\) functions were defined in [1978] by Filter and Steinborn [76, Eq. (2.14)] as follows:

\[
B_{n,ℓ}^m(\beta, r) = \frac{k_{n-1/2}^m(\beta r)}{2^{n+ℓ}(n + ℓ)!} Y_{n}^m(βr).
\]

Here, \(\beta > 0, n ∈ \mathbb{Z}\), and \(k_{n-1/2}\) is a reduced Bessel function. If \(K_\nu(z)\) is a modified Bessel function of the second kind [25, Eq. (10.27.4)], the reduced Bessel function is defined as follows [77, Eqs. (3.1) and (3.2)]:

\[
\hat{k}_\nu(z) = (2/π)^{1/2} z^\nu K_\nu(z), \quad \nu, z ∈ \mathbb{C}.
\]

If the order \(\nu\) is half-integral, \(\nu = n + 1/2\) with \(n ∈ \mathbb{N}_0\), the reduced Bessel function can be expressed as an exponential multiplied by a terminating confluent hypergeometric series \(_1F_1\) (see for example [78, Eq. (3.7)]):

\[
\hat{k}_{n+1/2}(z) = 2^n (1/2)_n e^{-z} \Gamma(n+1/2, 2z) + 1/2F_1 (-n; -2n; 2z).
\]

A condensed review of the history of \(B\) functions including numerous references can be found in [79]. Reduced Bessel and \(B\) functions had been the topic of my Diploma [80] and my PhD thesis [81].

Equations (56) – (58) indicate that \(B\) functions are fairly complicated mathematical objects. Therefore, it is not at all obvious why \(B\) functions should offer any advantages. However, the Hankel-type integral [52, Eq. (2) on p. 410])

\[
\int_0^\infty K_\mu(αt) J_\nu(βt) t^{μ+ν+1} dt = \frac{\Gamma(μ + ν + 1)}{2^{μ+ν} ℓ^2} \frac{α^μ β^ν}{(α^2 + β^2)^{μ+ν+1}},
\]

\[
|\Re(μ + ν)| > |\Re(μ)|, \quad |\Re(α)| > |\Re(β)|,
\]

implies that a \(B\) function possesses a Fourier transform of exceptional simplicity:

\[
\overline{B_{n,ℓ}^m(β, p)} = (2\pi)^{-3/2} ∫ e^{-iρ r} B_{n,ℓ}^m(β, r) d^3r = (2/π)^{1/2} \frac{β^{2n+ℓ+1}}{[β^2 + p^2]^{n+ℓ+1}} Y_{n}^m(-iρ).
\]
This is the most consequential and also the most often cited result of my PhD thesis [81, Eq. (7.1-6) on p. 160]. Later, the Fourier transform (60) was published in [20, Eq. (3.7)]. Independently and almost simultaneously, Eq. (60) was also derived by Niukkanen [63, Eqs. (57)-(58)].

It follows from Eq. (58) that a $B$ function can be expressed as a finite sum of Slater-type functions, or equivalently, that the Fourier transform (60) of a $B$ function can be expressed as a linear combination of the Fourier transforms of Slater-type functions, just as Yürükçü and Yürükçü [2] had done it in the case of bound-state hydrogen eigenfunctions (compare Section V).

Yürükçü [71] used this seemingly simple approach of expressing a $B$ function as a linear combination of Slater-type functions (71) Eq. (21)]. For the Fourier transform of a Slater-type function – his Eqs. (32), (39), and (40) – he used the same expressions as the ones used by Yürükçü and Yürükçü [2, Eqs. (13)- (16)]. This leads to explicit expressions [71, Eqs. (41) and (42)] that are, however, much more complicated and therefore much less useful than the remarkably compact Fourier transform (60).

We do not know for sure whether Yürükçü and Yürükçü [2] were aware of the Fourier transform (29) derived by Podolsky and Pauling [1, Eq. (28)] or of the other earlier references mentioned in Section I. Maybe, Yürükçü and Yürükçü genuinely believed that their results for the Fourier transform of a bound-state hydrogen eigenfunctions are actually the best possible. However, Yürükçü [71] did not only present his fairly complicated Eqs. (41) and (42) for the Fourier transform of a $B$ function, but as his Eq. (28) also the very compact expression (60).

It is hard to imagine that anyone would want to use Yürükçü’s complicated Eqs. (41) and (42) instead of the much simpler Eq. (60). Not all expressions, which are mathematically correct, are useful and deserve to be published.

The exceptionally simple Fourier transform (60) gives $B$ functions a special position among exponentially decaying functions. It explains why other exponentially decaying functions as for example Slater-type functions with integral principal quantum numbers, bound state hydrogen eigenfunctions, and other functions based on generalized Laguerre polynomials can be expressed in terms of finite linear combinations of $B$ functions (for details, see [5, Section IV] or [62, Section 4]).

The Fourier transform (60) was extensively used by Safouhi and co-workers for the evaluation of molecular multicenter integrals with the help of numerical quadrature combined with extrapolation techniques. Many references of the Safouhi group can be found in the PhD thesis of Slevinsky [83].

Apart from the Fourier transform (60), the most important expression of this Section is the expansion of a generalized Laguerre polynomial in terms of reduced Bessel functions with half-integral indices [81, Eq. (3.3-35) on p. 45]:

$$e^{-z} L_{\nu}^{(\alpha)}(2z) = (2n + 1) \times \sum_{\nu=0}^{n} \frac{(-2)^{\nu} \Gamma(n + \alpha + \nu + 1)}{\nu!(n - \nu)!(\alpha + 2\nu + 2)} \hat{k}_{\nu+1/2}(z).$$

This relationship was used by Filter and Steinborn [50, Eq. (3.17)] for the construction of addition theorems and other expansions in terms of Lambda functions.

With the help of Eq. (61), it is trivially simple to express Sturmian and Lambda functions as finite linear combinations of $B$ functions [8, Eqs. (4.19) and (4.20)]:

$$\Psi_{n,\ell}^{m}(\beta, r) = \frac{(2\beta)^{\ell/2} 2^{\ell}}{(2\ell + 1)!!} \left[ \frac{2n(n + \ell)!}{(n - \ell - 1)!} \right]^{1/2} \sum_{\nu=0}^{n-\ell-1} \frac{(-n + \ell + 1)_{\nu} (n + \ell + 1)_{\nu}}{\nu!(\ell + 3/2)_{\nu}} B_{\nu+1,\ell}(\beta, r),$$

$$\Lambda_{n,\ell}^{m}(\beta, r) = \frac{(2\beta)^{\ell/2} 2^{\ell}}{(2\ell + 3)!!} \left[ \frac{2n(n + \ell + 1)!}{(n - \ell - 1)!} \right]^{1/2} \sum_{\nu=0}^{n-\ell-1} \frac{(-n + \ell + 1)_{\nu} (n + \ell + 2)_{\nu}}{\nu!(\ell + 5/2)_{\nu}} B_{\nu+1,\ell}(\beta, r).$$

Now, we only need the Fourier transform (60) of a $B$ function to obtain explicit expressions for the Fourier transforms of a Sturmian or of a Lambda function. By combining Eqs. (60), (62) and (63), we obtain the following hypergeometric representations:

$$\Psi_{n,\ell}^{m}(\beta, p) = (2\pi)^{-3/2} \int e^{-ipr} \Psi_{n,\ell}^{m}(\beta, r) d^3r$$

$$= \frac{1}{(2\ell + 1)!!} \left[ \frac{2\beta}{\pi} \frac{2n(n + \ell)!}{(n - \ell - 1)!} \right]^{1/2} \left[ \frac{2\beta}{\beta^2 + p^2} \right]^{\ell/2} J_{n+\ell}^{(-\beta)} 2F_1 \left( -n + \ell + 1, n + \ell + 1; \ell + \frac{3}{2}; \frac{\beta^2}{\beta^2 + p^2} \right),$$

$$\Lambda_{n,\ell}^{m}(\beta, p) = (2\pi)^{-3/2} \int e^{-ipr} \Lambda_{n,\ell}^{m}(\beta, r) d^3r$$

$$= \frac{(2n + 1)}{(2\ell + 3)!!} \left[ \frac{\beta (n + \ell + 1)!}{\pi (n - \ell - 1)!} \right]^{1/2} \left[ \frac{2\beta}{\beta^2 + p^2} \right]^{\ell/2} J_{n+\ell}^{(-\beta)} 2F_1 \left( -n + \ell + 1, n + \ell + 2; \ell + \frac{5}{2}; \frac{\beta^2}{\beta^2 + p^2} \right).$$

The terminating $2F_1$ in Eq. (64) can according to Eq. (53) be replaced as a Gegenbauer polynomial, yielding Eq. (28) [5, Eq. (4.24)], and the terminating $2F_1$ in Eq. (65) can be expressed as a Jacobi polynomial [51, p. 212] via

$$P_{n}^{(\alpha, \beta)}(x) = \left( \frac{n + \alpha}{n} \right) 2F_1 \left( -n, \alpha + \beta + n + 1; \alpha + 1; \frac{1-x}{2} \right),$$

for $x 

\begin{align*}
\Psi_{n,\ell}^{m}(\beta, r) & = \frac{(2\beta)^{\ell/2} 2^{\ell}}{(2\ell + 1)!!} \left[ \frac{2n(n + \ell)!}{(n - \ell - 1)!} \right]^{1/2} \sum_{\nu=0}^{n-\ell-1} \frac{(-n + \ell + 1)_{\nu} (n + \ell + 1)_{\nu}}{\nu!(\ell + 3/2)_{\nu}} B_{\nu+1,\ell}(\beta, r), \\
\Lambda_{n,\ell}^{m}(\beta, r) & = \frac{(2\beta)^{\ell/2} 2^{\ell}}{(2\ell + 3)!!} \left[ \frac{2n(n + \ell + 1)!}{(n - \ell - 1)!} \right]^{1/2} \sum_{\nu=0}^{n-\ell-1} \frac{(-n + \ell + 1)_{\nu} (n + \ell + 2)_{\nu}}{\nu!(\ell + 5/2)_{\nu}} B_{\nu+1,\ell}(\beta, r).
\end{align*}
yielding the following explicit expressions for the Fourier transforms of a Lambda function [5, Eq. (4.25)]:

$$\Lambda_{n,\ell}^m(\beta, p) = \frac{2}{(2\pi)^n} \left\{ \frac{\beta (n + \ell + 1)!}{(n - \ell - 1)!} \right\}^{1/2} \frac{e^{-\beta p^2}}{\beta^{\ell+2}} \int_{-\infty}^{\infty} \frac{\gamma_{\ell+1/2, \ell+3/2, p^2}}{\pi} J_{n-\ell-1}^{\ell+1/2}(p^2) \right. \right.$$  \[67\]

The orthogonality relationships satisfied by the Fourier transforms of Sturmians and Lambda functions with respect to an integration over the whole three-dimensional momentum space can be deduced directly from the known orthogonality properties of the Gegenbauer and Jacobi polynomials [3 Eqs. (4.31) - (4.37)].

My approach, which is based on the Eqs. (60) and (61), can also be employed in the case of other, more complicated exponentially decaying functions. In [84, Abstract or Eqs. (1) and (2)], Guseinov introduced a large class of complete and orthonormal functions. In terms of the polynomials $[L_q^p(x)]_{BS}$ defined in Eq. (5), Guseinov’s functions can be expressed as follows:

$$\Psi_{n\ell m}(\zeta, r) = (-1)^\alpha \left[ \frac{(2\zeta)^{n-\ell-1}}{(2n\alpha)^{n-\ell-1}} \right]^{1/2} \times \left( 2\zeta r \right)^\alpha e^{-\zeta r} \left[ L_{n\ell+1-\alpha}^{2\ell+2-2\alpha}(2\zeta r) \right] S_{m+1}(\theta, \varphi). \quad (68)$$

Here, $\zeta > 0$ is a scaling parameter, and $S_{m+1}(\theta, \varphi)$ is either a real or a complex spherical harmonic (Guseinov did not provide an exact definition of $S_{m+1}(\theta, \varphi)$).

The additional parameter $\alpha$, which Guseinov calls fractional or self-frictional quantum number, was originally chosen to be an integer satisfying $\alpha = 1, 0, -1, -2, \ldots$ [84 Abstract]. In the text following [84, Eq. (3)], Guseinov remarked that for fixed $\alpha = 1, 0, -1, -2, \ldots$ the functions (68) form a complete orthonormal set.

This statement is meaningless. Completeness is not a generally valid property of a given function set. It only guarantees that functions belonging to a suitable Hilbert space, which has to be specified, can be expanded by this function set, and that the resulting expansions converge with respect to the norm of this Hilbert space (for further details, I recommend a book by Higgins [38] or a review by Klahn [33]).

Guseinov’s original definition (68) implies that his functions are according to [84, Eq. (4)] orthogonal with respect to the weight function $w(r) = [n'/(\zeta r)^\alpha$ [84, Eq. (4)]:

$$\int \left| \Psi_{n'\ell' m'}(\zeta, r) \right|^\alpha \left( \frac{n'!}{\zeta r} \right)^\alpha \Psi_{n\ell m}(\zeta, r) \; d^3r = \delta_{n'n'} \delta_{\ell\ell'} \delta_{m'm'} \quad \text{(69)}.$$  

In the theory of classical orthogonal polynomials, which is intimately linked to Hilbert space theory, it is common practice to introduce on the basis of their orthogonality relationships suitable inner products $\langle f | g \rangle_w = \int A \; w(x) f(x)^* g(x)dx$ with a positive weight function $w: [a, b] \rightarrow \mathbb{R}^+$. These weighted inner products then lead to the corresponding weighted Hilbert spaces $\mathcal{H}_w$, in which the orthogonal polynomials under consideration are complete and orthogonal.

In the case of Guseinov’s orthogonality relationship (69), this approach does not work. The weight function $w(r) = [n'/(\zeta r)^\alpha$ cannot be used to define a Hilbert space because both $\zeta$ and $n'$ are in general undefined. Thus, instead of incorporating $\zeta$ and $n'$ into the weight function, they should be incorporated in the normalization factor.

A further disadvantage of Guseinov’s original definition (68) is its use of the polynomials $[L_q^p(x)]_{BS}$ defined by Eq. (3), which can only have integral superscripts. As an alternative, I suggested the following definition, which uses the modern mathematical notation for the generalized Laguerre polynomials (see for example [85, Eq. (4.16)]) or [23, Eq. (2.13)]:

$$k \Psi_{n\ell m}^m(\beta, r) = \left[ \frac{(2\beta)^{n-\ell-1}}{(2\pi)^{n-\ell-1}} \right]^{1/2} \times e^{-\beta r} L_{n-\ell-1}^{2\ell+3k+2}(2\beta r) Y_{m}^m(\beta, r). \quad (70)$$

The indices satisfy $n \in \mathbb{N}, \ell \in \mathbb{N}_0, n - \ell - m \leq \ell$, and the scaling parameter satisfies $\beta > 0$.

In my original definition in [85, Eq. (4.16)] or [23, Eq. (2.13)], I had assumed that $k$ is a positive or negative integer satisfying $k = -1, 0, 1, 2, \ldots$, which corresponds to the straightforward translation $-\alpha \mapsto k$ of Guseinov’s original condition $\alpha = 1, 0, -1, -2, \ldots$ [84, Eq. (4)]. Therefore, my original definition in [85, Eq. (4.16)] or [23, Eq. (2.13)] assumed $k$ being integral and contained $(n + \ell + k + 1)!$ instead of $\Gamma(n + \ell + k + 2)$.

However, in the text following [85, Eq. (4.16)] on p. 11 or in the text following [23, Eq. (2.13) on p. 27], I had emphasized that the condition $k = -1, 0, 1, 2, \ldots$ is unnecessarily restrictive and that it can be generalized to $k \in [-1, \infty)$. My criticism of Guseinov’s original definition (68) was explicitly confirmed by Guseinov himself. In his later articles [86, 87], Guseinov generalized his so-called frictional quantum number from originally $\alpha = 1, 0, -1, -2, \ldots$ to $\alpha \in (-\infty, 3]$, which corresponds to $k \in (-3, \infty)$ in my notation. This change could not be done with Guseinov’s original definition (68).

Therefore, Guseinov finally had to use the modern mathematical notation for his functions (compare [86, Eqs. (1) - (5)] or [67, Abstract]). To disguise the obvious, Guseinov used in these formulas instead of a generalized Laguerre polynomial a terminating confluent hypergeometric series $\mathbf{F}_1$. Because of Eq. (7), Guseinov’s formulas are equivalent to my definition (70). Characteristically, Guseinov did not acknowledge my contributions [85, Eq. (4.16)] or [23, Eq. (2.13)] to his functions. Because of [85], Guseinov cannot claim to be unaware of [85].

For fixed $k \in (-3, \infty)$, Guseinov’s functions defined by
Eq. (70) satisfy the orthonormality relationship
\[
\int \left[ k \Psi_{n,\ell}^m(\beta, r) \right]^* r^k k \Psi_{n',\ell'}^m(\beta, r) \, d^3 r = \delta_{nn'} \delta_{\ell \ell'} \delta_{mm'},
\]
which implies that they are complete and orthonormal in the weighted Hilbert space
\[
L^2_{r_k}(\mathbb{R}^3) = \left\{ f: \mathbb{R}^3 \to \mathbb{C} \mid \int r^k |f(r)|^2 \, d^3 r < \infty \right\}.
\]
For \( k = 0 \), the functions \( k \Psi_{n,\ell}^m(\beta, r) \) are identical to the Lambda functions defined by Eq. (13). Thus, they are complete and orthonormal in the Hilbert space \( L^2(\mathbb{R}^3) \) of square integrable functions defined by Eq. (7).

With the help of Eq. (61), Guseinov’s functions can be expressed as a finite sum of \( B \) functions [23, Eq. (2.22)]:
\[
k \Psi_{n,\ell}^m(\beta, r) = \left\{ \frac{\beta^{k+3} (n + \ell + k + 1)!}{2^{k+1} (n - \ell - 1)!} \right\}^{1/2} \frac{(2n + k + 1) \Gamma(1/2) (\ell + 1)!}{\Gamma(\ell + 2 + k/2) \Gamma(\ell + [k + 5]/2)} \times \sum_{\nu=0}^{n-\ell-1} \frac{(-n + \ell + 1)_\nu (n + \ell + k + 2)_\nu (\ell + 2)_\nu}{\nu! (\ell + 2 + k/2)_\nu (\ell + [k + 5]/2)_\nu} B_{\nu+1,\ell}^m(\beta, r).
\]
Now, we only need the Fourier transform (60) to obtain an explicit expression for the Fourier transform of Guseinov’s function:
\[
\overline{k \Psi_{n,\ell}^m(\beta, p)} = (2\pi)^{-3/2} \int e^{-ip \cdot r} k \Psi_{n,\ell}^m(\beta, r) \, d^3 r
= \left\{ \frac{\beta^{k+1} (n + \ell + k + 1)!}{\pi 2^k (n - \ell - 1)!} \right\}^{1/2} \frac{(2n + k + 1) \Gamma(1/2) (\ell + 1)!}{2^{\ell+2} \Gamma(\ell + 2 + k/2) \Gamma(\ell + [k + 5]/2)} \frac{1}{\beta^2 + p^2} \sum \Gamma_m^{n+\nu} (-ip) \cdot \frac{(2\beta)}{\beta^2 + p^2} \cdot 3F_2 \left( \begin{array}{c} n + \ell + 1, n + \ell + 2, \ell + 2; \ell + 2 + \frac{k}{2}, \ell + \frac{k + 5}{2}; \frac{\beta^2}{\beta^2 + p^2} \end{array} \right).
\]
In this Fourier transform, the radial part is essentially a terminating generalized hypergeometric series \( 3F_2 \), which simplifies for either \( k = -1 \) or \( k = 0 \) to yield the terminating Gaussian hypergeometric series \( 2F_1 \) in the hypergeometric representations (64) or (65) for the Fourier transforms of Sturmians and Lambda functions, respectively. Thus, the Fourier transform of Guseinov’s function \( k \Psi_{n,\ell}^m(\beta, r) \) with \( k \neq -1, 0 \) is more complicated than the Fourier transforms of either Sturmian or Lambda functions.

VII. SUMMARY AND CONCLUSIONS

The Fourier transform [29] for a bound-state hydrogen eigenfunction [3] is a classic result of quantum physics already derived in 1929 by Podolsky and Pauling [11, Eq. (28)] with the help of the generating function (17). I am not aware of a more compact and more useful expression for this Fourier transform.

Yükçü and Yükçü [2], who were apparently unaware not only of Podolsky and Pauling [3] but also of the other references listed in Section I, proceeded differently. As discussed in Section VI, Yükçü and Yükçü [2] expressed a generalized Laguerre polynomial as a finite sum of powers according to Eq. (30), or equivalently, they expressed a bound-state hydrogen eigenfunction as a finite sum of Slater-type functions. Since Fourier transformation is a linear operation, this leads to an expression of the Fourier transformation of a bound-state hydrogen eigenfunction as a finite sum of Fourier transforms of Slater-type functions, for which many explicit expressions are known in the literature (compare Section VI).

At first sight, this approach, which requires no mathematical skills, looks like a good idea. Unfortunately, the simplicity of Slater-type functions in the coordinate representation is deceptive. As already emphasized in [23, p. 29], the Fourier transforms of bound-state hydrogen eigenfunctions and Slater-type function have the same level of complexity. Consequently, it cannot be a good idea to express the Fourier transform of a bound-state hydrogen eigenfunction as a lin-
ear combination of Fourier transforms of Slater-type functions. Moreover, in the case of large principal quantum numbers \( n \), these finite sums tend to become numerically unstable. This is a direct consequence of the alternating signs in Eq. (30).

In principle, it should be possible to derive the Podolsky and Pauling formula (29) from the comparatively complicated linear combinations presented by Yüçü and Yüçü. However, the Fourier transforms of Slater-type functions discussed in Section [V] are all fairly complicated objects. Therefore, it is very difficult or even practically impossible to obtain the remarkably compact Podolsky and Pauling formula (29) in this way. I am not aware of anybody who achieved this.

It is nevertheless possible to derive the Podolsky and Pauling formula (29) by expanding generalized Laguerre polynomials, albeit in terms of some other, less well-known polynomials. This was shown in Section [VI]. The key relationships in Section [VI] are the exceptionally simple Fourier transform (60) of a \( B \) function and the expansion (61) of a generalized Laguerre polynomial in terms of reduced Bessel functions (58) with half-integral indices. With the help of Eqs. (60) and (61), it is trivial to derive the Podolsky and Pauling formula (29). This derivation is much simpler than the original derivation by Podolsky and Pauling (1), which is discussed in Section [IV] and which required the skillful use of the generating function (17) of the generalized Laguerre polynomials.

Hylleraas [35] observed already in 1928 that bound-state hydrogen eigenfunctions without the inclusion of the mathematically very difficult continuum eigenfunctions are incomplete in the Hilbert space of square integrable functions (compare Section [III]). This is a highly consequential fact, which Yüçü and Yüçü were apparently not aware of. In combination with the difficult nature of the continuum eigenfunctions, this incompleteness greatly limits the practical usefulness of bound-state eigenfunctions as mathematical tools. It is certainly not a good idea to do expansions in terms of an incomplete function set.

Therefore, attention has shifted away from hydrogen eigenfunctions to other, related function sets also based on the generalized Laguerre polynomials, which, however, have more convenient completeness properties. The best known examples are the so-called Sturmians (11), which had been introduced by Hylleraas [35] already in 1928 and which can be obtained from the bound-state hydrogen eigenfunctions by the substitution \( Z/n \mapsto \beta \) according to Eq. (12), and the so-called Lambda functions (13), which were also introduced by Hylleraas [46] in 1929.

With the help of the expansion (61), it is a trivial matter to express both Sturmians and Lambda functions as linear combinations of \( B \) functions, yielding Eqs. (62) and (63). Then, one only needs the Fourier transform (60) to convert these linear combinations to compact explicit expressions for the Fourier transforms of Sturmians and Lambda functions, respectively.

In 2002, Guseinov [84] introduced a large class of complete and orthonormal functions defined by Eq. (65), which used an antiquated notation for the Laguerre polynomials. Guseinov’s functions contain an additional parameter \( \alpha \) called frictional or self-frictional quantum number, which was originally assumed to be integral. Depending on this \( \alpha \), Guseinov’s functions contain Sturmians and Lambda functions as special cases.

Guseinov’s original notation (65) does not allow non-integral values of \( \alpha \). In order to rectify this obvious deficiency, I introduced in [23, 85] the alternative definition (70) which uses the modern mathematical notation for the generalized Laguerre polynomials. Later, Guseinov [86, 87] was forced to change to my notation because he wanted to consider non-integral self-frictional quantum numbers \( \alpha \).

With the help of Eqs. (60) and (61), it is again a trivial matter to construct the Fourier transform (74) of a Guseinov function in the notation of Eq. (71). Equation (74) contains the Fourier transforms of Sturmians and Lambda functions as special cases, but is more complicated.

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