Spatial behaviour in dynamical thermoelasticity backward in time for porous media *

Gerardo Iovane, Francesca Passarella †

Department of Information Engineering and Applied Mathematics (DIIMA),
University of Salerno, 84084 Fisciano (Sa) Italy

Abstract – The aim of this paper is to study the spatial behaviour of the solutions to the boundary–final value problems associated with the linear theory of elastic materials with voids. More precisely the present study is devoted to porous materials with a memory effect for the intrinsic equilibrated body forces. An appropriate time–weighted volume measure is associated with the backward in time thermoelastic processes.

Then, a first–order partial differential inequality in terms of such measure is established and further is shown how it implies the spatial exponential decay of the thermoelastic process in question.

1 Introduction

The boundary–final value problems associated with the linear thermoelasticity have been studied by Ames and Payne [1] in connection with the continuous dependence of the thermoelastic processes backward in time with respect to the final data. It is well known that this is an improperly posed problem. A further study on this subject was recently developed by Ciarletta [2].

More recently, Chirita and Ciarletta have developed an exhaustive description for the spatial behaviour of solutions in linear thermoelasticity forward in time [3, 4, 5]. In this connection some appropriate time–weighted surface power functions are introduced. On this basis some spatial decay estimates of Saint–Venant type are established for bounded bodies; while for unbounded bodies some alternatives of Phragmén–Lindelöf type are obtained. In the present paper, we aim to extend the above context to porous materials for backward in time processes. We shall refer to the well-known theory by Nunziato and Cowin, in which

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†E-mail address: iovane@diima.unisa.it, passarella@diima.unisa.it
the presence of small pores (or voids) in the conventional continuum model is introduced by assigning an additional degree of freedom, namely, the fraction of elementary volume that is possibly found void of matter [6, 7, 8]. Following a previous paper by Goodman and Cowin on granular (flowing) materials, the bulk mass density of the material is represented as a product of two fields, the void volume fraction and the mass density of the matrix material [6]. Starting from the fundamental work of Iesan in [9], we consider the porous materials with the memory effects for the intrinsic equilibrated body forces and we associate with such a model the boundary-final value problem as in [8] and [10]. Then, we study the spatial behaviour of the thermoelastic processes backward in time by using an appropriate time-weighted measure.

In Section 2, we present the basic equations of the linear dynamic theory of porous body developed in [8] and [10]. Some constitutive assumptions and other useful results are also presented. In Section 3 a time–weighted volume measure is defined and a first–order partial differential inequality is established in terms of such measure. On this basis a spatial decay estimate of Saint–Venant type is proved; our decay results look quite similar to those established by Lin and Payne for the backward heat equation defined on a semi–infinite cylinder [11].

2 Preliminaries

We shall denote by \( B \) the smooth domain of the physical space (\( \equiv \mathbb{R}^3 \)) occupied by an anisotropic, homogeneous and porous body in a fixed, natural reference configuration. Identified \( \mathbb{R}^3 \) with the associated vector space, an orthonormal frame of reference is introduced. The vectors and tensors will have components denoted by Latin subscripts (ranging over \( \{1,2,3\} \), unless otherwise specified). Summation over repeated subscripts and other typical conventions for differential operations are implied, such as a superposed dot or a comma followed by a subscript to denote partial derivative with respect to time or the corresponding coordinate. Occasionally, we shall use bold–face character and typical notations for vectors and operations upon them.

We study the boundary–final value problems associated with the linear thermoelasticity. Following [8, 10], the local balance equations become

\[
\begin{align*}
S_{ji,j} + \rho f_i &= \rho \ddot{u}_i, & \text{balance of momentum}, \\
h_{i,i} + g + \rho \ell &= \rho \chi \ddot{\phi}, & \text{balance of equilibrated stress}, \\
\rho \theta \delta \eta &= q_{i,i} + \rho r, & \text{in } B \times (-\infty, 0) & \text{energy equation.}
\end{align*}
\]

In these equations, \( \mathbf{u} \) is the displacement vector fields; \( \phi \) is the change in volume fraction starting from the reference configuration; \( \theta \) is the temperature variation from the uniform reference temperature \( \theta_0 (>0) \). Moreover, \( \mathbf{S} \) and \( \mathbf{f} \) are the stress tensor and body force, respectively; \( \mathbf{h}, g \) and \( \ell \) are the equilibrated stress vector, intrinsic and extrinsic equilibrated...
body force, respectively; \( \eta, q \) and \( r \) are the specific entropy, the heat flux vector and the extrinsic heat supply, respectively. Finally, \( \rho \) and \( \chi \) are the bulk mass density and equilibrated inertia in the reference state, respectively. We assume that \( f, \ell \) and \( r \) are the continuous functions on \( \bar{B} \times (-\infty, 0] \), with \( \bar{B} \) closure of \( B \).

By denoting \( U \equiv \{ u, \varphi, \theta \} \), it follows the strain fields are
\[
e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \gamma_i = \varphi, i, \quad \kappa_i = \theta, i, \quad \text{in } \bar{B} \times (-\infty, 0]. \tag{2}
\]
The surface tractions \( s \), the surface equilibrated stress \( h \) and the boundary heat flux \( q \) are
\[
s_i = S_{ji}n_j, \quad h = h_jn_j, \quad q = q_jn_j, \tag{3}
\]
where \( n \) is the outward unit normal vector to the boundary surface.

We consider the porous bodies with the memory effect for the intrinsic equilibrated body forces. Our attention is on the materials, which are initially free from stress and the intrinsic equilibrated body force, the entropy and the heat flux rate are equal zero. The constitutive equations are given by
\[
S_{ij} = C_{ijrs}e_{rs} + D_{ij}\gamma_s + B_{ij}\varphi - M_{ij}\theta, \\
h_i = D_{rsi}e_{rs} + A_{ij}\gamma_j + b_i\varphi - a_i\theta, \\
g = -\tau\dot{\varphi} + G, \\
G = -B_{ij}e_{ij} - b_i\gamma_i - \xi\varphi + m\theta, \\
\rho\eta = M_{kl}e_{kl} + m\varphi + a_i\gamma_i + a\theta, \\
q_i = K_{ij}\kappa_j, \quad \text{in } \bar{B} \times (-\infty, 0]. \tag{4}
\]
The material coefficients satisfy the relations
\[
C_{ijrs} = C_{rsij} = C_{jirs}, \quad D_{ijr} = D_{jir}, \quad A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \quad M_{ij} = M_{ji}, \tag{5}
\]
and
\[
\tau \geq 0. \tag{6}
\]
Throughout this article, we assume the bulk mass density \( \rho \), the equilibrated inertia \( \chi \) and the constant heat \( a \) are strictly positive.

The constant conductivity tensor \( K \) is a symmetric positive definite tensor; thus, there exist the positive constants \( k_m \) and \( k_M \) such that
\[
k_m\kappa_i\kappa_i \leq K_{ij}\kappa_i\kappa_j \leq k_M\kappa_i\kappa_i \quad \forall \kappa. \tag{7}
\]
The constants \( k_m \) and \( k_M \) are the minimum and maximum conductivity moduli for \( K \).

It follows from Schwarz’s inequality and the last inequality that
\[
q_iq_i = K_{ij}\kappa_jq_i \leq (K_{ij}\kappa_i\kappa_j)^{1/2}(K_{rs}q_rq_s)^{1/2} \leq (K_{ij}\kappa_i\kappa_j)^{1/2}(k_Mq_rq_s)^{1/2}, \tag{8}
\]
so that
\[ q_i q_i \leq k M K_{ik} \kappa_i \kappa_j. \] (9)

For what follows, it is useful to introduce the vector space \( \mathcal{E} \) of all vector fields of the form
\[ \mathbf{E} \equiv \{ E_{ij}, \chi_1 \pi_i, \psi \}, \quad \text{with} \quad E_{ij} = E_{ji}, \quad \text{and} \quad \chi_1 = \sqrt{\chi}. \] (10)

Moreover, for each \( \mathbf{E} \in \mathcal{E} \) we define the vector field \( \hat{\mathbf{S}}(\mathbf{E}) \) as
\[ \hat{\mathbf{S}}(\mathbf{E}) \equiv \left\{ \hat{S}_{ji}, \chi_1 \left( \frac{1}{\chi} \hat{h}_i \right), -\hat{G} \right\}, \]
where
\[ \hat{S}_{ij} = C_{ijrs} E_{rs} + D_{ij} \pi_s + B_{ij} \psi; \]
\[ \hat{h}_i = D_{rsi} E_{rs} + A_{ij} \pi_j + b_i \psi; \]
\[ \hat{G} = -B_{ij} E_{ij} - b_i \pi_i - \xi \psi, \] (11)
and the coefficients obey the symmetry relations (5). The vector field \( \hat{\mathbf{S}}(\mathbf{E}) \) belongs to \( \mathcal{E} \), too.

Now, for any \( \mathbf{E}, \bar{\mathbf{E}} \in \mathcal{E} \), we consider the following bilinear form
\[ 2 \mathcal{F}(\mathbf{E}, \bar{\mathbf{E}}) = C_{ijrs} E_{ij} \bar{E}_{rs} + \xi \psi \bar{\psi} + A_{ij} \pi_i \pi_j + B_{ij} (E_{ij} \bar{\psi} + \bar{E}_{ij} \psi) + 
+ D_{ij} (E_{ij} \bar{\pi}_s + \bar{E}_{ij} \pi_s) + b_i (\psi \bar{\pi}_i + \bar{\psi} \pi_i), \] (12)
where \( \bar{\mathbf{E}} \equiv \{ E_{ij}, \chi_1 \bar{\pi}_i, \bar{\psi} \} \). From (11) we have
\[ 2 \mathcal{F}(\mathbf{E}, \bar{\mathbf{E}}) = [\hat{S}_{ji} \bar{E}_{ij} + \hat{h}_i \bar{\pi}_i - \hat{G} \bar{\psi}] \quad \text{and} \quad 2 \mathcal{F}(\mathbf{E}, \mathbf{E}) = [\hat{S}_{ji} E_{ij} + \hat{h}_i \pi_i - \hat{G} \psi]. \] (13)

By the constitutive equations, we prove
\[ \mathcal{F}(\mathbf{E}, \bar{\mathbf{E}}) = \mathcal{F}(\bar{\mathbf{E}}, \mathbf{E}), \quad \forall \mathbf{E}, \bar{\mathbf{E}} \in \mathcal{E}. \] (14)

The Cauchy-Schwarz's inequality implies
\[ \mathcal{F}(\mathbf{E}, \bar{\mathbf{E}}) \leq [\hat{\mathcal{W}}(\mathbf{E})]^{1/2} [\hat{\mathcal{W}}(\bar{\mathbf{E}})]^{1/2}; \quad \forall \mathbf{E}, \bar{\mathbf{E}} \in \mathcal{E}, \] (15)
where \( \hat{\mathcal{W}} \) is the quadratic form associated to \( \mathcal{F} \)
\[ 2 \hat{\mathcal{W}}(\mathbf{E}) = 2 \mathcal{F}(\mathbf{E}, \mathbf{E}) = C_{ijrs} E_{ij} E_{rs} + \xi \psi^2 + A_{ij} \pi_i \pi_j + 2B_{ij} \psi E_{ij} + 
+ 2D_{ij} \pi_s E_{ij} + 2b_i \psi \pi_i. \] (16)

We assume that \( \hat{\mathcal{W}}(\mathbf{E}) \) is a positive definite quadratic form and, consequently, we have
\[ \mu_m \left( E_{ij} E_{ij} + \chi \pi_i \pi_i + \psi^2 \right) \leq 2 \hat{\mathcal{W}}(\mathbf{E}) \leq \mu_M \left( E_{ij} E_{ij} + \chi \pi_i \pi_i + \psi^2 \right). \] (17)
By setting $E = \hat{S}(E)$ in (17) and (13), we prove

$$2\hat{W}(\hat{S}(E)) \leq \mu_M \left( \hat{S}_{ji} \hat{S}_{ji} + \frac{1}{\chi} \hat{h}_i \hat{h}_i + \hat{G}^2 \right),$$

(18)

and

$$\hat{S}_{ji} \hat{S}_{ji} + \frac{1}{\chi} \hat{h}_i \hat{h}_i + \hat{G}^2 = 2\mathcal{F}(E, \hat{S}(E)) \leq 2[\hat{W}(E)]^{1/2}[\hat{W}(\hat{S}(E))]^{1/2} \leq$$

$$\leq 2[\hat{W}(E)]^{1/2} \mu_M^{1/2} \left( \hat{S}_{ji} \hat{S}_{ji} + \frac{1}{\chi} \hat{h}_i \hat{h}_i + \hat{G}^2 \right)^{1/2};$$

(19)

so that this inequality implies

$$\hat{S}_{ji} \hat{S}_{ji} + \frac{1}{\chi} \hat{h}_i \hat{h}_i + \hat{G}^2 \leq 2\mu_M \hat{W}(E).$$

(20)

If $E = \{e_{ij}, \chi_1 \gamma_i, \varphi\}$, then we introduce the following notations

$$2W^* = 2\hat{W}(E) = C_{ijrs} e_{ij} e_{rs} + \xi \varphi^2 + A_{ij} \gamma_i \gamma_j + 2B_{ij} \varphi e_{ij} + 2D_{ijrs} \gamma_i \gamma_s + 2b_{ij} \varphi \gamma_i,$$

$$\tilde{S}_{ij} = C_{ijrs} e_{rs} + D_{ijrs} \gamma_s + B_{ij} \varphi, \quad \tilde{h}_i = D_{rsi} e_{rs} + A_{ij} \gamma_j + b_i \varphi,$$

$$\tilde{G} = -B_{ij} e_{ij} - b_i \gamma_i - \xi \varphi.$$ 

(21)

By eqs. (11), (13) and (21), we can see that

$$2W^* = [\tilde{S}_{ji} e_{ij} + \tilde{h}_i \gamma_i - \tilde{G} \varphi], \quad \hat{W}^* = [\hat{S}_{ji} e_{ij} + \hat{h}_i \gamma_i - \hat{G} \varphi].$$

(22)

For any positive number $\epsilon$ and each second-order tensors, $\mathbf{L}$ and $\mathbf{F}$, we have the inequality

$$(L_{ij} + F_{ij})(L_{ij} + F_{ij}) \leq (1 + \epsilon)L_{ij}L_{ij} + (1 + \frac{1}{\epsilon})F_{ij}F_{ij}.$$ 

(23)

By aid of eqs. (4), (21) and (23), we obtain

$$S_{ij} = \tilde{S}_{ij} - M_{ij} \theta, \quad h_i = \tilde{h}_i - a_i \theta, \quad G = \tilde{G} + m \theta,$$

(24)

and

$$S_{ij}S_{ij} + \frac{1}{\chi} h_i h_i = (\tilde{S}_{ij} - M_{ij} \theta)(\tilde{S}_{ij} - M_{ij} \theta) + \frac{1}{\chi} (\tilde{h}_i - a_i \theta)(\tilde{h}_i - a_i \theta) \leq$$

$$(1 + \epsilon)\tilde{S}_{ij} \tilde{S}_{ij} + (1 + \frac{1}{\epsilon})M_{ij}M_{ij} \theta^2 + \frac{1}{\chi} [(1 + \epsilon)\tilde{h}_i \tilde{h}_i + (1 + \frac{1}{\epsilon})a_i a_i \theta^2] \leq$$

$$\leq (1 + \epsilon)2\mu_M W^* + (1 + \frac{1}{\epsilon})M^2 \theta^2,$$

(25)

in which

$$M^2 = \max_B (M_{ij} M_{ij} + \frac{1}{\chi} a_i a_i).$$

(26)
We consider the boundary-final value problem $\mathcal{P}$ defined by the equations of motion (1), the geometrical equations (2) and the constitutive equations (4) and the following final-boundary conditions

\begin{align}
&u_i(x,0) = u_0^i(x), \quad \dot{u}_i(x,0) = \dot{u}_0^i(x), \quad \varphi(x,0) = \varphi^0(x), \\
&\dot{\varphi}(x,0) = \dot{\varphi}^0(x), \quad \theta(x,0) = \theta^0(x), \quad x \in B,
\end{align}

(27)

and

\begin{align}
&u_i = u_i^* \quad \text{on } \Sigma_1 \times (-\infty, 0], \quad s_i = s_i^* \quad \text{on } \Sigma_2 \times (-\infty, 0], \\
&\varphi = \varphi^* \quad \text{on } \Sigma_3 \times (-\infty, 0], \quad h = h^* \quad \text{on } \Sigma_4 \times (-\infty, 0], \\
&\theta = \theta^* \quad \text{on } \Sigma_5 \times (-\infty, 0], \quad q = q^* \quad \text{on } \Sigma_6 \times (-\infty, 0],
\end{align}

(28)

where $\Sigma_i \ (i = 1, \ldots, 6)$ are the subsets of $\partial B$ such that

$$\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \Sigma_5 \cup \Sigma_6 = \partial B, \quad \Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Sigma_5 \cap \Sigma_6 = \emptyset.$$  

The terms on the right-hand of eqs. (27) and (28) are prescribed continuous functions.

For further convenience we use an appropriate change of time variable and notations to transform the boundary-final value problem $\mathcal{P}$ in the boundary-initial value problem $\mathcal{P}^*$ defined by the following equations

\begin{align}
&S_{ji,j} + \rho f_i = \rho \ddot{u}_i, \\
&h_{i,i} + g + \rho \ell = \rho \chi \ddot{\varphi}, \quad -\rho \theta \dot{\eta} = q_{i,i} + \rho r, \quad \text{in } B \times (0, +\infty), \\
&e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \gamma_i = \varphi_{,i}, \quad \kappa_i = \theta_{,i}, \quad \text{in } \bar{B} \times [0, +\infty),
\end{align}

(30)

and

\begin{align}
&S_{ij} = C_{ijrs} e_{rs} + D_{ij} \gamma_s + B_{ij} \varphi - M_{ij} \theta, \\
&h_i = D_{rs} e_{rs} + A_{ij} \gamma_j + b_i \varphi - a_i \theta, \\
&g = \tau \dot{\varphi} + G, \\
&G = -B_{ij} e_{ij} - b_i \gamma_i - \xi \varphi + m \theta, \\
&\rho \eta = M_{kl} e_{kl} + a_i \gamma_i + m \varphi + a \theta, \\
&q_i = K_{ij} \kappa_j, \quad \text{in } \bar{B} \times [0, +\infty),
\end{align}

(31)

with the initial conditions

\begin{align}
&u_i(x,0) = u_0^i(x), \quad \dot{u}_i(x,0) = \dot{u}_0^i(x), \quad \varphi(x,0) = \varphi^0(x), \\
&\dot{\varphi}(x,0) = \dot{\varphi}^0(x), \quad \theta(x,0) = \theta^0(x), \quad x \in B,
\end{align}

(32)
and the boundary conditions
\[
\begin{align*}
    u_i &= u_i^* \text{ on } \Sigma_1 \times [0, +\infty), \quad s_i = s_i^* \text{ on } \Sigma_2 \times [0, +\infty), \\
    \varphi &= \varphi^* \text{ on } \Sigma_3 \times [0, +\infty), \quad h = h^* \text{ on } \Sigma_4 \times [0, +\infty), \\
    \theta &= \theta^* \text{ on } \Sigma_5 \times [0, +\infty), \quad q = q^* \text{ on } \Sigma_6 \times [0, +\infty).
\end{align*}
\] (33)

We define as solution of the boundary-initial value problem \( \mathcal{P}^* \) a process \( \pi = \{ u, e, S, \varphi, \gamma, h, g, \theta, \kappa, q, \eta \} \) that satisfies eqs. (29)–(33) and
\[
\begin{align*}
    &\text{i. } u_i, \varphi \in C^{2,2}(\overline{B} \times [0, +\infty)), \quad \theta \in C^{1,1}(\overline{B} \times [0, +\infty)); \\
    &\text{ii. } e_{ij} = e_{ji}, \quad \gamma_i, \kappa_i \in C^{1,1}(\overline{B} \times [0, +\infty)); \\
    &\text{iii. } S_{ij} = S_{ji}, \quad h_i, q_i, g \in C^{1,0}(\overline{B} \times [0, +\infty)); \quad \eta \in C^{0,1}(\overline{B} \times [0, +\infty)).
\end{align*}
\]

### 3 Saint-Venant’s Principle

Let us consider a given time \( T \in (0, +\infty) \) and a given (external) data \( \mathcal{D} = \{ f, \ell, r; u_i^0, \dot{u}_i^0, \varphi^0, \dot{\varphi}^0, \theta^0; u_i^*, s_i^*, \varphi^*, h^*, \theta^*, q^* \} \) in the problem \( \mathcal{P}^* \). We denote by \( \overline{D}_T \) the support of the initial and boundary data, the body force, the extrinsic equilibrated body force and the heat supply on the time interval \([0, T]\), i.e. the set of all \( x \in \overline{B} \) such that:

\[
\begin{align*}
    &\text{i. if } x \in B, \text{ then } \\
    &\quad \begin{cases}
        u_i^0(x) \neq 0 \text{ or } \dot{u}_i^0(x) \neq 0, \quad \text{or } \varphi^0(x) \neq 0, \quad \text{or } \dot{\varphi}^0(x) \neq 0, \quad \text{or } \theta^0(x) \neq 0, \\
        \text{or } f_i(x, s) \neq 0, \quad \text{or } \ell(x, s) \neq 0, \quad \text{or } r(x, s) \neq 0 \text{ for some } s \in [0, T];
    \end{cases}
\end{align*}
\] (34)

\[
\begin{align*}
    &\text{ii. if } x \in \partial B, \text{ then } \\
    &\quad \begin{cases}
        u_i^*(x, s) \neq 0 \text{ for some } (x, s) \in \Sigma_1 \times [0, T], \\
        s_i^*(x, s) \neq 0 \text{ for some } (x, s) \in \Sigma_2 \times [0, T], \\
        \varphi^*(x, s) \neq 0 \text{ for some } (x, s) \in \Sigma_3 \times [0, T], \\
        h^*(x, s) \neq 0 \text{ for some } (x, s) \in \Sigma_4 \times [0, T], \\
        \theta^*(x, s) \neq 0 \text{ for some } (x, s) \in \Sigma_5 \times [0, T], \\
        q^*(x, s) \neq 0 \text{ for some } (x, s) \in \Sigma_6 \times [0, T].
    \end{cases}
\end{align*}
\] (35)

We assume that \( \overline{D}_T \) is a bounded set. We consider a non-empty bounded regular region \( \overline{D}_T^* \) such that \( \overline{D}_T \subset \overline{D}_T^* \subset B \). We note that

\[
\begin{align*}
    &\text{i. if } \emptyset \neq \overline{D}_T, \text{ then we choose } \overline{D}_T^* \text{ to be the smallest bounded regular region in } \overline{B} \text{ that includes } \overline{D}_T; \text{ in particular, we set } \overline{D}_T^* = \overline{D}_T \text{ if } \overline{D}_T \text{ is also a regular region;} \\
    &\text{ii. if } \overline{D}_T = \emptyset, \text{ then } \overline{D}_T^* \text{ may be chosen in an arbitrary way.}
\end{align*}
\]

Now, we mean the set \( D_r \), by
\[
D_r = \{ x \in \overline{B} : \overline{D}_T^* \cap \Sigma(x, r) \neq \emptyset \}, \quad r \geq 0,
\] (36)
where \( \Sigma(x, r) \) is the closed ball with radius \( r \) and center at \( x \). Clearly, \( \bar{D}_T \subseteq \bar{D}_r^* = D_0 \subset D_r \) (\( r > 0 \)). Further, we set \( B_r = B \setminus D_r \); we have \( B_{r_2} \supset B_{r_1} \), and \( B(r_1, r_2) = B_{r_2} \setminus B_{r_1} \) for \( r_1 > r_2 \). Let \( L \) be the diameter of \( B_0 \). The surface \( S_r \) is the subsurface of \( \partial B_r \) contained inside \( B \) and whose outward unit normal vector is oriented to the exterior of \( D_r \).

In what follows we need the below lemma

**Lemma 1.** Let \( \pi = \{ u, e, S, \varphi, \gamma, h, g, \theta, \kappa, q, \eta \} \) be a the solution of \( P^* \). Then, for every regular region \( P \subseteq B \) with regular boundary \( \partial P \) and for each \( t \in [0, T] \), we obtain

\[
\int_0^t \int_P e^{\lambda s} \left[ \frac{1}{2} [\rho \dot{u}_i(s) \dot{u}_i(s) + \rho \chi \dot{\varphi}^2(s) + a \theta^2(s) + 2W^*(s)] + \tau \dot{\varphi}^2(s) + \right.
\]
\[
+ \frac{1}{\theta_0} K_{ij} \kappa_i(s) \kappa_j(s) \] \( ds \) \( dv \) \[ + 2W^*(t) \] \( dv \) \[ - \int_0^t \int_P e^{\lambda s} \left[ s_i(s) \dot{u}_i(s) + h(s) \dot{\varphi}(s) - \frac{q_i}{\theta_0} \theta(s) \right] ds \] \( dv \) \[ - \frac{1}{2} \left[ \rho \dot{u}_i^0 \dot{u}_i^0 + \rho \chi \dot{\varphi}^0 \dot{\varphi}^0 + a \theta^0 \theta^0 + 2W^*(0) \right] dv, \]

where \( \lambda \) is a prescribed positive parameter.

**Proof.** By eqs. (29)–(31) we deduce that

\[
\frac{\partial}{\partial s} \left\{ \frac{1}{2} [\rho \dot{u}_i(s) \dot{u}_i(s) + \rho \chi \dot{\varphi}^2(s) + a \theta^2(s) + 2W^*(s)] \right\} = \tau \dot{\varphi}^2(s) + \\
+ \frac{1}{\theta_0} K_{ij} \kappa_i(s) \kappa_j(s) + \rho f_i(s) \dot{u}_i(s) + \rho \ell \dot{\varphi}(s) - \frac{\rho}{\theta_0} \theta(s) + \\
+ \left[ S_{ji}(s) \dot{u}_i(s) + h_j(s) \dot{\varphi}(s) - \frac{q_j}{\theta_0} \theta(s) \right]_j. \]

Thus, we can see that

\[
\frac{\partial}{\partial s} \left\{ e^{\lambda s} \frac{1}{2} [\rho \dot{u}_i(s) \dot{u}_i(s) + \rho \chi \dot{\varphi}^2(s) + a \theta^2(s) + 2W^*(s)] \right\} = \\
e^{\lambda s} \frac{\lambda}{2} [\rho \dot{u}_i(s) \dot{u}_i(s) + \rho \chi \dot{\varphi}^2(s) + a \theta^2(s) + 2W^*(s)] + e^{\lambda s} \tau \dot{\varphi}^2(s) + \\
+ e^{\lambda s} \frac{1}{\theta_0} K_{ij} \kappa_i(s) \kappa_j(s) + e^{\lambda s} \left[ \rho f_i(s) \dot{u}_i(s) + \rho \ell \dot{\varphi}(s) - \frac{\rho}{\theta_0} \theta(s) \right] + \\
+ e^{\lambda s} \left[ S_{ji}(s) \dot{u}_i(s) + h_j(s) \dot{\varphi}(s) - \frac{q_j}{\theta_0} \theta(s) \right]_j. \]

If we integrate this relation over \( P \times [0, T] \), then we obtain the desired result with the help of the divergence theorem and eqs. (3). \( \bullet \)

For a prescribed strictly positive parameter \( \lambda \) and for any \( r \in [0, L], t \in [0, T] \), we associate with the solution \( \pi \) the following time–weighted volume measure \( E(r, t) \) (\( > 0 \))
where and

The parameter \( \lambda \), in the above function, is characteristic for the considered measure.

Taking into account that, for \( r_1 \geq r_2 \)

\[
E(r_1, t) - E(r_2, t) = -\int_0^t \int_{B(r_1, r_2)} e^{\lambda s} \left\{ \frac{\lambda}{2} [\rho \hat{u}_i(s) \hat{u}_i(s) + \rho \chi \hat{\varphi}^2(s) + a \theta^2(s) + 2W^*(s)] + \tau \hat{\varphi}^2(s) + \frac{1}{\theta_0} K_{ij} \kappa_i(s) \kappa_j(s) \right\} dv ds.
\]

(41)

it is a simple matter to prove the following lemma.

**Lemma 2.** Let \( \pi \) be a solution of initial-boundary-value problem \( \mathcal{P}^* \) and \( \widehat{D}_T \) be the bounded support of the external data \( D \) on the time interval \([0, T]\). Then, the corresponding time–weighted volume measure satisfies the following properties

(i) \( E(r, t) \) is a non–increasing function with respect to \( r \), i.e.

\[
E(r_1, t) \leq E(r_2, t), \quad \text{with } r_1 \geq r_2, t \in [0, T].
\]

(42)

(ii) \( E(r, t) \) is a continuous differentiable function on \( r \in [0, L] \), \( t \in [0, T] \) and

\[
\frac{\partial}{\partial r} E(r, t) = -\int_0^t \int_{S_r} e^{\lambda s} \left\{ \frac{\lambda}{2} [\rho \hat{u}_i(s) \hat{u}_i(s) + \rho \chi \hat{\varphi}^2(s) + a \theta^2(s) + 2W^*(s)] + \tau \hat{\varphi}^2(s) + \frac{1}{\theta_0} K_{ij} \kappa_i(s) \kappa_j(s) \right\} dv ds,
\]

(43)

\[
\frac{\partial}{\partial t} E(r, t) = \int_{B_r} e^{\lambda t} \left\{ \frac{\lambda}{2} [\rho \hat{u}_i(t) \hat{u}_i(t) + \rho \chi \hat{\varphi}^2(t) + a \theta^2(t) + 2W^*(t)] + \tau \hat{\varphi}^2(t) + \frac{1}{\theta_0} K_{ij} \kappa_i(t) \kappa_j(t) \right\} dv.
\]

(44)

**Lemma 3.** Let \( \pi \) be a solution of the initial-boundary-value problem \( \mathcal{P}^* \) and \( \widehat{D}_T \) be the bounded support of the external data \( D \) on the time interval \([0, T]\). Then, \( E(r, t) \) satisfies the following first–order differential inequality

\[
E(r, t) \leq -\frac{\zeta}{\lambda} \frac{\partial}{\partial r} E(r, t) + \frac{1}{\lambda} \frac{\partial}{\partial t} E(r, t), \quad \forall r \in [0, L], t \in [0, T],
\]

(45)

where

\[
\zeta(\lambda) = \sqrt{\frac{\mu M}{\rho}(1 + \varepsilon)},
\]

(46)

and

\[
1 + \varepsilon = \frac{1}{2} + \frac{M^2}{2a_\rho \mu M} + \frac{\lambda k_M}{4\theta_0 a_\mu M} + \sqrt{\left( \frac{1}{2} - \frac{M^2}{2a_\rho \mu M} - \frac{\lambda k_M}{4a_\theta \mu M} \right)^2 + \frac{M^2}{a_\rho \mu M}}.
\]

(47)
Proof. Taking $P = B_r$ into Lemma 1, we have

$$
\mathcal{E}(r, t) = \int_{B_r} e^{\lambda t/2} [\rho \dot{u}(t) \dot{s}(t) + \rho \chi \dot{\varphi}^2(t) + a \theta^2(t) + 2W^*(t)] dv - \int_0^t \int_{S_r} e^{\lambda s} \left[ s(t) \dot{u}(t) + h(t) \dot{\varphi}(t) - \frac{q}{\theta_0} \dot{\theta}(t) \right] dadt.
$$

(48)

It follows from (44) and (48)

$$
\int_{B_r} e^{\lambda t/2} [\rho \dot{u}(t) \dot{s}(t) + \rho \chi \dot{\varphi}^2(t) + a \theta^2(t) + 2W^*(t)] dv \leq \frac{1}{\lambda} \frac{\partial \mathcal{E}}{\partial t}(r, t).
$$

(49)

If we use Schwarz’s inequality and the arithmetic–geometric mean inequality, we obtain

$$
|S_{ji}(s) n_j \dot{u}_i(s) + h_j(s) n_j \dot{\varphi}(s) - \frac{1}{\theta_0} \theta(s) q_j(s) n_j| \leq
$$

$$
\leq \frac{1}{\lambda \epsilon_1} \left[ \frac{\lambda}{2} \rho \dot{u}_i(s) \dot{u}_i(s) \right] + \frac{\epsilon_1}{\lambda \rho} \left[ \frac{\lambda}{2} S_{ji}(s) S_{ji}(s) \right] +
$$

$$
+ \frac{1}{\lambda \epsilon_1} \left[ \frac{\lambda}{2} (\rho \dot{\varphi} + \frac{2\tau}{\lambda}) \dot{\varphi}^2(s) \right] + \frac{\epsilon_1}{\lambda (\rho + \frac{2\tau}{\lambda} \chi)} \left[ \frac{\lambda h_j(s) h_j(s)}{2\chi} \right] +
$$

$$
+ \frac{1}{\theta_0 \lambda \epsilon_2} \left[ \frac{\lambda}{2} a \theta^2(s) \right] + \frac{\epsilon_2}{2a \theta_0} \left[ 1 \right] q_j(s) q_j(s), \quad \forall \epsilon_1, \epsilon_2 > 0.
$$

(50)

By eqs. (9), (25) and (50), we deduce that

$$
|S_{ji}(s) n_j \dot{u}_i(s) + h_j(s) n_j \dot{\varphi}(s) - \frac{1}{\theta_0} \theta(s) q_j(s) n_j| \leq
$$

$$
\leq \frac{1}{\lambda \epsilon_1} \left[ \frac{\lambda}{2} (\rho \dot{u}_i(s) \dot{u}_i(s) + \rho \chi \dot{\varphi}^2(s)) + \tau \dot{\varphi}^2(s) \right] + \frac{\epsilon_1 (1 + \epsilon) \mu_M}{\lambda \rho} \left[ \lambda W^*(s) \right] +
$$

$$
+ \frac{\epsilon_1 M^2}{\lambda \rho a} \left[ 1 + \frac{1}{\epsilon} \right] + \frac{1}{\lambda \theta_0 \epsilon_2} \left[ \frac{\lambda}{2} a \theta^2(s) \right] + \frac{\epsilon_2 k_M}{2a \theta_0} \left[ \frac{1}{\theta_0} K_{ij} \kappa_i(s) \kappa_j(s) \right], \quad \forall \epsilon, \epsilon_1, \epsilon_2 > 0.
$$

(51)

If we consider $\epsilon, \epsilon_1, \epsilon_2$ such that

$$
\frac{1}{\lambda \epsilon_1} = \frac{\epsilon_1 (1 + \epsilon) \mu_M}{\lambda \rho} = \frac{\epsilon_1 M^2}{\lambda \rho a} \left( 1 + \frac{1}{\epsilon} \right) + \frac{1}{\lambda \theta_0 \epsilon_2} = \frac{\epsilon_2 k_M}{2a \theta_0},
$$

(52)

then

$$
\epsilon_1 = \frac{1}{\zeta(\lambda)} \quad \text{and} \quad \epsilon_2 = \frac{2a \zeta(\lambda)}{\lambda k_M},
$$

(53)

and $\epsilon$ satisfies to (47), i.e. $\epsilon$ is the positive root of the algebraic equation

$$
\epsilon^2 + 2\epsilon \left[ 1 - \frac{M^2}{2a \rho \mu_M} - \frac{\lambda k_M}{4 \theta_0 a \mu_M} \right] = \frac{M^2}{a \rho \mu_M} = 0.
$$

(54)
By eqs. (51), (53) we get for any \( r \in [0, L] \) and \( t \in [0, T] \)
\[
- \int_0^t \int_{S_r} e^{\lambda s}[s_i(s) \dot{u}_i(s) + h_i(s)n_j\dot{\varphi}(s) - \frac{1}{\theta_0}(s)q_j(s)n_j]dads \\
\leq |\int_0^t \int_{S_r} e^{\lambda s}[s_i(s) \dot{u}_i(s) + h_i(s)n_j\dot{\varphi}(s) - \frac{1}{\theta_0}(s)q_j(s)n_j]dads| \leq -\frac{\zeta \partial \mathcal{E}}{\lambda \partial r}(r, t).
\]
(55)

Therefore, the relations (48), (49), (55) lead to (45). Thus, the proof is complete. •

**Theorem 1.** Let \( \pi \) be a solution of initial-boundary-value problem \( P^* \) and \( \hat{D}_T \) be the bounded support of the external data \( D \) on the time interval \([0, T]\). For \( \lambda \) (sufficiently large), \( t_0 \in [0, T], r_0 \in [0, L] \) such that
\[
L \leq \zeta(\lambda)t_0 + r_0 \leq \zeta(\lambda)T,
\]
(56)
we have
\[
\mathcal{E}(r, t_0 + \frac{r_0 - r}{\zeta(\lambda)}) \leq \mathcal{E}(0, t_0 + \frac{r_0}{\zeta(\lambda)}) \exp\left(-\frac{\lambda}{\zeta(\lambda)}r\right), \quad \text{for } r \in [0, L].
\]
(57)

**Proof.** We put
\[
\mathcal{I}(r, t) = \exp\left(\frac{\lambda}{\zeta(\lambda)}r\right)\mathcal{E}(r, t), \quad \forall r \in [0, L], \ t \in [0, T];
\]
(58)
the inequality (45) takes the following form
\[
\frac{\partial \mathcal{I}}{\partial r}(r, t) - \frac{1}{\zeta(\lambda)} \frac{\partial \mathcal{I}}{\partial t}(r, t) \leq 0, \quad \forall r \in [0, L], \ t \in [0, T].
\]
(59)

From eqs. (46) and (47) it is trivial to see \( \zeta(\lambda) \sim \lambda^{1/2} \) for \( \lambda \to \infty \) and so \( \zeta(\lambda) \) is an increasing function for sufficiently large values of \( \lambda \). Thus, we can choose \( t_0 \) and \( r_0 \) satisfying (56) and hence
\[
0 \leq t_0 + \frac{r_0 - r}{\zeta(\lambda)} \leq T \quad \forall r \in [0, L].
\]
(60)

By setting \( t = t_0 + \frac{r_0 - r}{\zeta(\lambda)} \) in (59), it follows
\[
\frac{d}{dr}[\mathcal{I}(r, t_0 + \frac{r_0 - r}{\zeta(\lambda)})] \leq 0, \quad \forall r \in [0, L],
\]
(61)
and therefore,
\[
0 \leq \mathcal{I}(r, t_0 + \frac{r_0 - r}{\zeta(\lambda)}) \leq \mathcal{I}(0, t_0 + \frac{r_0}{\zeta(\lambda)}), \quad \forall r \in [0, L].
\]
(62)
The relations (58) and (62) lead to (57) and the proof is complete. •
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