Resonance regimes of scattering by small bodies with impedance boundary conditions

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Received 26 June 2010, in final form 25 August 2010
Published 27 September 2010
Online at stacks.iop.org/JPhysA/43/415205

Abstract

The paper concerns scattering of plane waves by a bounded obstacle with complex-valued impedance boundary conditions. We study the spectrum of the Neumann-to-Dirichlet operator for small wave numbers and long wave asymptotic behavior of the solutions of the scattering problem. The study includes the case when $k=0$ is an eigenvalue or a resonance. The transformation from the impedance to the Dirichlet boundary condition as impedance grows is described. A relation between poles and zeroes of the scattering matrix in the non-self-adjoint case is established. The results are applied to a problem of scattering by an obstacle with a springy coating. The paper describes the dependence of the impedance on the properties of the material, that is on forces due to the deviation of the boundary of the obstacle from the equilibrium position.

PACS numbers: 43.20.Fn, 43.25.Jh, 02.30.Tb
Mathematics Subject Classification: 35P25, 35Qxx, 78A45

1. Introduction

We consider the scattering of plane waves by a bounded obstacle $\mathcal{O} \in \mathbb{R}^3$ with smooth boundary $\partial \mathcal{O} \in C^2$ and impedance boundary conditions. The scattered field $u = u(r)$, $r = (x,y,z)$ satisfies the Helmholtz equation in $\Omega = \mathbb{R}^3 \setminus \mathcal{O}$ and radiation conditions

\begin{equation}
\begin{cases}
\Delta u(r) + k^2 u(r) = 0, & r \in \Omega, \quad k > 0, \\
\int_{|r|=R} |\frac{\partial u(r)}{\partial |r|} - iku(r)| dS = o(1), & R \to \infty.
\end{cases}
\end{equation}

The Robin boundary condition holds at the boundary

\begin{equation}
\frac{\partial u}{\partial n} - \gamma(k) u = - \left( \frac{\partial e^{ik\cdot\alpha}}{\partial n} - \gamma(k)e^{ik\cdot\alpha} \right), \quad r \in \partial \Omega,
\end{equation}

where $\alpha$ is the unit outward normal to $\partial \Omega$. The complex-valued function $\gamma(k)$ is the impedance of the boundary, and $\alpha$ is the angle between the incident field $e^{ik\cdot\alpha}$ and the normal $n$.

The study includes the case when $k=0$ is an eigenvalue or a resonance. The transformation from the impedance to the Dirichlet boundary condition as impedance grows is described. A relation between poles and zeroes of the scattering matrix in the non-self-adjoint case is established. The results are applied to a problem of scattering by an obstacle with a springy coating. The paper describes the dependence of the impedance on the properties of the material, that is on forces due to the deviation of the boundary of the obstacle from the equilibrium position.
where $\alpha \in S^2$ is the direction of the incident plane wave, and $n$ is the exterior normal for $\Omega$ (directed into $\Omega_1$). When a solution of problem (1), (2) is considered for complex $k$, it is understood as an analytic in $k$ continuation of solutions for $k > 0$ (see [18, 19]).

In mathematical literature, the impedance is usually assumed to be purely imaginary with negative imaginary part (see, e.g., [5, 8, 16]). In [12, 13] one can find the theorem on the solution to problem (1), (2) with $\Im \gamma \leq 0$ and the long wave asymptotics away from resonance regimes. We are not imposing any restrictions on the real or imaginary parts of $\gamma$. Recall that $\gamma$ is determined by the forces at the boundary of the obstacle. Existence of an elastic component of the force (proportional to the displacement) defines the real part of $\gamma$. Friction (the force proportional to the velocity of the displacement) defines the imaginary part of $\gamma$. A more detailed analysis of the dependence of $\gamma$ on the forces is given in the appendix. Thus, all the values of the impedance are of interest, including those which correspond to artificial forces such as ‘negative friction’ or negative compressibility coefficient.

For the sake of transparency, we assume that the impedance does not depend on the point of the boundary. We also assume that $\gamma = \gamma(k)$ is analytic in $k$ in a neighborhood of the point $k = 0$. The latter preserves the analyticity of the scattering matrix and will allow us to discuss poles. Using perturbation arguments one can easily extend the main results below to the case of sufficiently smooth $\gamma(k)$.

Let us recall that any solution of problem (1) has the following asymptotic behavior at infinity:

$$u(r) = u_{\infty}(\theta) \frac{\mathrm{e}^{ik|r|}}{|r|} + o\left(\frac{1}{|r|}\right), \quad \theta = \frac{r}{|r|}, \quad |r| \to \infty. \quad (3)$$

The function $u_{\infty} \in L^2(S^2)$ is called the scattering amplitude (it depends also on $k$ and $\alpha$), and the square of its norm

$$\sigma_k(u) = \int_{S^2} |u_{\infty}(\theta)|^2 \, dS$$

is called the total cross-section.

Problem (1), (2) can be easily reformulated in terms of the Neumann-to-Dirichlet operator $D = D(k)$ which maps the normal derivative $\frac{\partial u}{\partial n}|_{\partial \Omega}$ of the Neumann problem for equations (1) into the value $u|_{\partial \Omega}$ of the solution at the boundary. Let $u(r) \in H^{1/2}_{\text{loc}}(\Omega)$ satisfy (1). Then $\frac{\partial u}{\partial n}|_{\partial \Omega} \in H^{1/2}(\partial \Omega)$, $u|_{\partial \Omega} \in H^{3/2}(\partial \Omega)$ and $D(k)$ is a bounded operator from $H^{1/2}(\partial \Omega)$ into $H^{3/2}(\partial \Omega)$. In fact, it is a pseudo-differential operator of order $-1$ (see, e.g., [17]) and will be considered as a compact operator in $L^2(\partial \Omega)$:

$$D(k) : L^2(\partial \Omega) \to L^2(\partial \Omega), \quad k \geq 0. \quad (5)$$

If the operator $D(k)$ is considered for complex $k \in \mathbb{C}$ it is understood as an analytic continuation of (5).

From the Green’s formula it follows that

$$u(r) = -\int_{\partial \Omega} G_k(r, s) \frac{\partial u(s)}{\partial n} \, dS(s), \quad r, s \in \partial \Omega, \quad (6)$$

where $G_k(r, s)$ is the Green’s function of the Neumann problem (see [15] for the existence of $G_k$). Thus, problem (1), (2) for $u \in H^1_{\text{loc}}(\Omega)$ is equivalent to the equation

$$(I - \gamma(k)D) v = -\left(\frac{\partial u}{\partial n} - \gamma(k) \mathrm{e}^{ik(\alpha \cdot r)}\right)|_{\partial \Omega}, \quad v \in H^{1/2}(\partial \Omega), \quad (7)$$

and the corresponding field is given by (6) with $\frac{\partial u}{\partial n} = v$. 

2
Let \( \{\gamma_j(k), j = 0, 1, 2, \ldots\} \) be the set of eigenvalues of the operator \( D(k) \), and let \( \{\psi_j(k)\} \) be the corresponding eigenfunctions. Usually, we do not mark the dependence of these functions \( \psi_j(k) = \psi_j(k, r) \) on \( r \in \partial/\Omega_1 \).

We study the spectrum of the operator \( D(k) \), and the solvability and properties of the solution of the scattering problem (1), (2) and of the total cross-section \( \sigma_k(u) \). In particular, it is shown that the eigenvalues \( \{\gamma_j(k)\} \) of the operator \( D(k) \) are analytic in a neighborhood of \( k = 0 \), and the inverse values \( \{\gamma_j^{-1}(k)\} \), \( k \geq 0 \), belong to a half-strip in the upper complex half-plane \( \{z : 0 < \Re z < c(k), \Im z < d(k)\} \). Long wave asymptotic of the solution of (1), (2) is obtained and the location of resonances is described. We show convergence of the impedance problem to the Dirichlet problem when the impedance grows along any ray through the origin different from \((0, -\infty)\). These general results are applied to a scattering problem for an obstacle coating by a springy layer. Some other results related to the topics discussed below can be found in [3, 10].

The paper is organized as follows. The main results are stated in the next section. The proofs are given in section 3 followed by an appendix containing additional discussions concerning springy covers.

2. Main results

The following theorem is rather simple, but very essential. A proof will be given in the next section.

**Theorem 1.**

1. The condition \( \frac{1}{\gamma(k)} \not\in \text{Sp}(D(k)) \) implies the existence and uniqueness of the solution of problem (1), (2).

The operator \( D(k) \) is meromorphic in \( k \in \mathbb{C} \) with all the poles located in the lower half-plane. Let \( \gamma(k) \) be analytic in some connected domain \( U \subset \mathbb{C} \), and \( \frac{1}{\gamma(k)} \not\in \text{Sp}(D(k')) \) for some value of \( k = k' \in U \). Then the solution of problem (1), (2) exists and is meromorphic in \( k \in U \) as an element of \( H_{loc}^2(\Omega) \).

2. The eigenvalues \( \{\gamma_j(k)\} \) and the eigenfunctions \( \{\psi_j(k), j = 0, 1, 2, \ldots\} \) of \( D(k) \) are analytic in some neighborhoods of the origin \( k = 0 \). To be more exact, for any \( \varepsilon > 0 \) there exists a neighborhood \( U_{\varepsilon} \) of the origin in the complex \( k \)-plane such that \( L^2(\partial/\Omega_1) \) can be represented as a sum

\[
L^2(\partial/\Omega_1) = L_{1,\varepsilon}(k) + L_{2,\varepsilon}(k), k \in U_{\varepsilon},
\]

where the projection operators \( P_i(k) : L^2(\partial/\Omega_1) \to L^2_{i,\varepsilon}(k) \) are analytic in \( k \in U_{\varepsilon} \) and orthogonal when \( k = 0 \), and the first space is finite dimensional and has the form

\[
L_{1,\varepsilon}(k) = \text{span}\{\psi_j(k), 0 \leq j \leq N_{\varepsilon}\}.
\]

Here, the eigenfunctions \( \psi_j(k) \) are analytic in \( k \in U_{\varepsilon} \) (together with their eigenvalues \( \gamma_j(k) \)), linearly independent for each \( k \) and orthogonal when \( k = 0 \). The second space is also invariant for \( D(k) \), and the norm of the restriction of \( D(k) \) on \( L^2_{2,\varepsilon}(k) \) does not exceed \( \varepsilon : ||D(k)||_{L^2_{2,\varepsilon}(k)}| < \varepsilon, k \in U_{\varepsilon} \).

3. For each \( k > 0 \), the set of inverse values \( \{\gamma_j^{-1}(k)\} \) belongs to a half-strip in the upper complex half-plane \( \{z : 0 < \Re z < c(k), \Im z < d(k)\} \), with the only limiting point at \( -\infty \).

The last statement can be found in [1], 4.2. We will show that the last statement of the theorem leads to the convergence of the solutions of the impedance problem (1), (2) to the
solution of the corresponding Dirichlet problem if the impedance grows along any ray through the origin different from \((0,-\infty)\). Namely, the following theorem holds.

**Theorem 2.** Let \(u_t\) be the solution of (1), (2) with \(\gamma = te^{\delta t}\), where \(-\pi < \delta \leq \pi\), and let \(w\) be the scattered field in the case of the Dirichlet boundary condition, i.e. \(w\) satisfies (1) and \(w = -e^{ikr - \sigma_0}\) on \(\partial \Omega\). Then

\[
\lim_{t \to \infty} \|u_t - w\|_{L^2(\partial \Omega)} = 0, \quad \lim_{t \to \infty} \|u'_t - w'_{\infty}\|_{L^2(\partial \Omega)} = 0.
\] (9)

An analog of theorem 2 with \(k = 0\) was proved recently in [2].

Now we are going to discuss the long wave asymptotics \((k \to 0)\) of the solution of problem (1), (2). Since the problem is equivalent to (7), it is obvious that the result depends on the asymptotics of functions \(1 - \gamma(k)\zeta_j(k)\) as \(k \to 0\), where \(\zeta = \zeta_j(k), j = 0, 1, 2, \ldots\), are the eigenvalues of operator (5). The impedance \(\gamma(k)\) can be an arbitrary function, and therefore first we study the asymptotics of the functions \(\zeta = \zeta_j(k)\). Then we impose certain conditions on the difference \(1 - \gamma(k)\zeta_j(k)\) and obtain some results for the solution of (1), (2).

Note that the operator \(D(0)\) is symmetric, and therefore numbers \(\zeta = \zeta_j(0)\) are real (which is not necessarily true for \(\zeta_j(k), k > 0\)). Moreover, \(\zeta_j(0) < 0\) since the operator \(D(0)\) is negative, see (26). We can always enumerate the eigenvalues in such a way that the sequence \(\{\zeta_j(0)\}\) is not decreasing. Recall that the eigenfunctions \(\varphi_j(0)\) of the operator \(D(0)\) form an orthonormal basis in \(L^2(\partial \Omega)\). They also can be chosen to be real valued. The eigenfunctions \(\varphi_j(k)\) for complex \(k, |k| < 1\), are analytic in \(k\) due to theorem 1. Let \(u_0\) be the unit function on \(\partial \Omega\) and let \(c_j\) be the scalar projection of \(u_0\) on \(\varphi_j\), i.e.

\[
c_j := \int_{\partial \Omega} \varphi_j(0) dS, \quad j = 0, 1, 2, \ldots
\] (10)

Then

\[
\sum_{j=0}^{\infty} c_j^2 = |\partial \Omega|.
\] (11)

Note that the eigenfunctions \(\varphi_j(0)\) which correspond to a multiple eigenvalue of the operator \(D(0)\) cannot be chosen absolutely arbitrary, since the existence of analytic in \(k\) continuation leads to some restriction in their choice.

**Theorem 3.**

1. The smallest eigenvalue \(\zeta_0(0)\) is simple and \(c_0 \neq 0\). The following estimate holds:

\[
|\zeta_0(0)| \leq \frac{S}{4\pi C},
\] (12)

where \(S\) is the area of \(\partial \Omega\) and \(C\) is its electrostatic capacity.

2. The following relation holds:

\[
\zeta_j'(0) = \frac{-i}{4\pi} (c_j)^2, \quad j = 0, 1, 2, \ldots
\]

3. Let the impedance \(\gamma(k)\) be analytic in a neighborhood of \(k = 0\) and \(\frac{1}{\gamma(j0)} \notin \{\zeta_j(0), j = 0, 1, \ldots\}\). Then the solution of (1), (2) exists in some neighborhood of \(k = 0\) and is analytic in \(k\) as an element of \(H^1_{loc}(\Omega)\).

Let \(\gamma(k)\) be analytic in a neighborhood of \(k = 0\) and \(\frac{1}{\gamma(0)} = \zeta_p(0)\), where \(\zeta_p(0) = \zeta_{p+1}(0) = \ldots = \zeta_{p+m-1}(0)\) is an eigenvalue of \(D(0)\) of multiplicity \(m \geq 1\), and \(\varphi_j, p \leq j \leq p + m - 1\), are the corresponding eigenfunctions. Let

\[
\gamma'(0) \neq \frac{i(c_j)^2}{4\pi (\zeta_j(0))^2}, \quad p \leq j \leq p + m - 1.
\] (13)
Then the solution of (1), (2) exists and depends meromorphically (as an element of $H^2_{loc}(\Omega)$) on $k$ in some neighborhood of $k = 0$. It has a pole of the first order at $k = 0$, and the value of $u$ at the boundary has the form

$$u(r) = \frac{1}{k} \sum_{j=p}^{p+m-1} b_j \varphi_j(0, r) + O(1), \quad r \in \partial \Omega, \quad k \to 0,$$

(14)

where

$$b_j = \frac{-c_j}{(\gamma \zeta_j'(0))} = \frac{c_j \zeta_j(0)}{\frac{1}{4\pi}(c_j)^2 - \gamma'(0)(\zeta_j(0))^2}.$$

(15)

The scattering amplitude has the form

$$u_\infty(\theta) = \frac{\gamma(0)}{4\pi k} \sum_{j=p}^{p+m-1} c_j b_j + O(1), \quad k \to 0.$$

(16)

The total cross-section has the form

$$\sigma_k = \sigma^0 + O\left(\frac{1}{k}\right), \quad k \to 0; \quad \sigma^0 = \frac{1}{4\pi} \left| \sum_{j=p}^{p+m-1} \gamma(0)c_j b_j \right|^2.$$

(17)

4. For any $\varepsilon > 0$ and $U_\varepsilon, N_\varepsilon$ defined in theorem 1, there exists a $\gamma$-independent (if $\gamma$ is analytic in some fixed neighborhood of $k = 0$) domain $V_\varepsilon \subset U_\varepsilon$ such that the solution $u$ of the scattering problem (1), (2) and the total cross-section $\sigma_k(u)$ have poles at all points $k \in V_\varepsilon$, where $\gamma(k) \zeta_j(k) = 1$ for some $j \leq N_\varepsilon$ with $c_j \neq 0$.

Remark. 1. When $k = 0$ is a pole, the main part of the scattering amplitude does not depend on the angle of scattering $\theta \in S^2$ (see (16)), so the scattering is isotropic.

2. In the case of a sphere, the growth of the total cross-section in resonance regimes is well studied in acoustics (see, e.g., [11]). It was noted that the scattering of acoustic waves by small air bubbles in the water has total cross-section which is more than 500 times larger than geometrical cross-section of the bubble.

Order of the poles in (14) and (17) at $k = 0$ was restricted by assumption (13). In general, these poles may have any order which is defined by the order with which functions $1 - \gamma(k) \zeta_j(k)$ vanish at $k = 0$. There is another condition which restricts the order of these poles.

Recall that the scattering matrix is the operator

$$S = I + \frac{ik}{2\pi} F : L_2(S^2) \to L_2(S^2),$$

(18)

where $S^2$ is the unit sphere in $R^3$, and $F$ is the operator with the integral kernel $u_\infty = u_\infty(\theta, \alpha)$:

$$(F h)(\theta) = \int_{S^2} u_\infty(\theta, \alpha) h(\alpha) dS.$$

The operator $S$ is unitary when $\gamma$ is real, $k > 0$ (see, e.g., [6]). Obviously, the latter is equivalent to the relation

$$SF^* = \left( I + \frac{ik}{2\pi} F \right) F^* = F, \quad 3\gamma = 0, \quad k > 0.$$

(19)

The unitarity of $S$ and (18), (4) immediately imply the following statement.
**Theorem 4.** Let \( \gamma(k) \) be analytic in a neighborhood of \( k = 0 \) and \( \Im \gamma(k) = 0 \), \( k \geq 0 \). Then the scattering amplitude has a pole at \( k = 0 \) of at most the first order and the total cross-section has a pole at \( k = 0 \) of at most the second order.

**Example.** Consider the case when \( \mathcal{O} = B \) is the ball of radius \( R = 1 \) centered at the origin. Recall that functions 

\[
\psi_{n,m}(kr) = h_n(k|r|)Y_{n,m}(\theta),
\]

satisfy (1), and their restrictions on the sphere \( \partial B \) are eigenfunctions of \( \mathcal{D}(k) \). Here \( Y_{n,m}, m \leq n \), are the spherical functions (eigenfunctions of the Laplace–Beltrami operator with the eigenvalues \( \mu_n = -n(n+1) \)) and \( H_{n+1/2}^{(1)} \) are the Hankel functions. The solution of problem (1), (2) can be obtained by the Fourier method:

\[
u = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{f_{n,m}(k)}{kh'_n(k) - \gamma(k)h_n(k)} u_{n,m}(kr),
\]

where \( f_{n,m} \) are the coefficients of the expansion of the right-hand side \( f \) in equation (7) in the basis \( \{Y_{n,m}\} \). The same result will be obtained if problem (1), (2) is solved by means of (7), (6).

It is convenient to enumerate the eigenfunctions \( \varphi_{n,m} \) of \( \mathcal{D}(k) \) by two indexes: \( \varphi_{n,m} = Y_{n,m}/||Y_{n,m}|| \), and introduce \( c_{n,m} \) instead of \( c_p \) (see (12)). Then \( c_0 = c_{0,0} = 4\pi \) and \( c_{n,m} = 0 \) for \( (n, m) \neq (0, 0) \). Since \( f = 1 \) when \( k = 0 \), we have \( f_{0,0} = 1, f_{n,m} = 0, (n, m) \neq (0, 0) \).

The operator \( \mathcal{D}(k) \) has eigenvalues

\[
\varsigma_n(k) = h_n(k)/kh'_n(k)
\]

of multiplicity 2\( n + 1 \). In particular, \( \varsigma_0(k) = 1/(-1 + ik) \) and

\[
\varsigma_0(0) = -\frac{1}{n + 1}.
\]

The field \( u \) may have a pole of any order at \( k = 0 \) if \( \gamma(k) \) is not real. For example, a pole of a high order can be obtained if \( \gamma(k) = -1 + ik + O(k^N) \). If \( \gamma(k) \equiv \varsigma_0(0) \) then the order of the pole is restricted by theorem 4 and the result depends on whether \( n = 0 \) or \( n > 0 \). If \( \gamma(k) \equiv -1 \), (i.e. \( n = 0 \)), then the field and the scattering amplitude have poles of the first order, and therefore the total cross-section \( \sigma_k(u) \) has a pole of the second order. If \( \gamma(k) \equiv -(n+1), n > 0 \), the scattering amplitude and the total cross-section do not have poles at \( k = 0 \). The latter is a consequence of the fact that \( c_{n,m} = 0, (n, m) \neq (0, 0) \).

### 2.1. Scattering matrix

It is well known that in the self-adjoint case (in particular, for impedance boundary conditions with a real-valued impedance) the scattering matrix \( \mathcal{S} \), \( k > 0 \), is unitary, and the relation \( \mathcal{S}(k)\mathcal{S}^*(k) = I, k > 0 \), implies that the set of poles and the set of zeroes of \( \mathcal{S}, k \in \mathbb{C} \), are complex conjugate. The next statement generalizes this fact.

We use the subindex \( \gamma \) to indicate the dependence of the scattering matrix \( \mathcal{S} \) and the operator \( \mathcal{F} \) on the impedance.

**Theorem 5.** 1. Let the impedance \( \gamma(k) \) be an entire function (not necessarily real valued on \( \mathbb{R} \)).

Then \( \gamma(k) = \overline{\gamma(k)} \) is an entire function, and
(1) the following relation replaces (19):
\[ S^*_\gamma (F_{\gamma}) = F_{\gamma}, \quad k > 0. \]

(2) The following two statements are equivalent:
- \( S_{\gamma}(k) \) has a pole at \( k = k_0 \in \mathbb{C} \).
- \( S_{\gamma,1}(k) \) has a non-trivial kernel at \( k = k_0 \).

(3) \( S_{\gamma}(k) \) may have a kernel at real \( k = k_0 > 0 \) only in the case of absorbing impedance, \( \Im \gamma < 0 \).

Remark. The existence of a non-trivial kernel of \( S_{\gamma}(k) \) at a point \( k = k_0, 0 < k_0 \ll 1 \), allows one to concentrate energy at a boundary of a small obstacle using an incident wave for which the scattered wave has zero amplitude, see [7] for a practical implementation of this effect. In fact, let \( v \) be the solution of (1), (2) with \(-ik\) in radiation condition replaced by \( ik \).

Let \( v_\infty = v_\infty(\theta, \alpha) \) be the scattering amplitude of this solution. It is defined in (3) with \( v \) instead of \( u \) and \(-ik\) instead of \( k \). Then, for each \( \alpha \),
\[ S_{\gamma}(\gamma) v_\infty(-\theta, \alpha) = u_\infty(\theta, \alpha). \]

This relation is valid for real \( \gamma \), and therefore it is valid for complex \( \gamma \) due to the analyticity of both sides in \( \gamma \). (To show (20) for real \( \gamma \) we note that (19) and the reciprocity identity for \( F \) lead to \( S_{\gamma}(\gamma) v_\infty(-\theta, -\alpha) = u_\infty(\theta, \alpha), k > 0 \), which justifies (20) for \( k > 0 \), and therefore for all \( k \in \mathbb{C} \).)

The following is an equivalent form of (20). Let \( u \) be a solution of the problem
\[
\begin{align*}
\Delta u(r) + k^2 u(r) &= 0, \quad r \in \Omega, k > 0, \\
\frac{\partial u}{\partial n} - \gamma(k) u &= 0, \quad r \in \partial \Omega, \\
u(r) &= u_{in}(\theta) \frac{e^{-ikr}}{|r|} - u_{out}(\theta) \frac{e^{ikr}}{|r|} + o(1/|r|), \quad \theta = \frac{r}{|r|}, \quad |r| \to \infty.
\end{align*}
\]
(21)

Then
\[ S_{\gamma}(\gamma) u_{in}(-\theta) = u_{out}(\theta), \quad k > 0. \]

(22)

Let now \( \Omega \) be the exterior of the ball of radius \( r_0 \ll 1 \) centered at the origin, and let \( \gamma = -ik_0 \). Then \( u = \frac{e^{-ikr}}{|r|} \) satisfies (21), \( u_{in} \equiv 1, u_{out} \equiv 0 \) and the field \( u \) and the density of the energy in a small neighborhood of the boundary are much bigger than in any other point if \( r_0 \) is small enough.

2.2. An obstacle with a springy coating

The last statement of this section concerns one applied problem: acoustic scattering by an obstacle coated by a springy material. It is modeled (see the appendix) by problem (1), (2) with the impedance
\[ \gamma(k) = -Zk^2, \quad Z \gg 1. \]

The value of \( Z \) depends on the relative characteristics of the cover layer and the exterior medium. For example, \( Z \gg 1 \) if there is a gas layer around the obstacle with an elastic exterior membrane (for example, rubber) and radial walls in the layer attached to both the obstacle and the membrane, and the whole construction is submerged into a liquid. The walls are needed in order to achieve a springy character of the layer (to localize the output of a point exterior pressure). The attachment contains calculation of \( Z \) for this particular construction.
The last statement of theorem 3 implies:

**Theorem 6.** For each $i_0 < \infty$ there exists $Z_0$ such that the total cross-section of problem (1), (2) with impedance $\gamma(k) = -Zk^2$, $Z \geq Z_0$, has poles at points

$$k^+_i = \pm \frac{1}{\sqrt{Z|\zeta_i(0)|}} + \frac{\zeta_i'(0)}{2Z|\zeta_i(0)|^2} + O \left( \frac{1}{Z^{3/2}} \right), \quad Z \to \infty,$$

for all $i \leq i_0$ with $c_i \neq 0$.

Note that the value of $\Im \zeta_i'(0) < 0$ in the above formula is given in item 2 of theorem 3. Hence, generally (when the number of non-zero coefficients $c_i$ is infinite) the number of poles in any neighborhood of $k = 0$ tends to infinity when $Z \to \infty$.

Theorem 6 shows a similarity between standard Helmholtz resonators and the construction discussed above which we will call ‘a gas layer in a liquid’. The main feature of Helmholtz resonators is the presence of poles of the scattering cross-section as close to the point $k = 0$ as we please. The same is true for $k^+_i$ in the case of a gas layer in a liquid. Moreover, one can deform the Helmholtz resonator into a springy covered obstacle by changing simultaneously the shape of the resonator and the impedance in such a way that the scattering cross-sections of all the intermediate problems have poles at the same distance from the origin.

### 3. Proofs

We start with a couple of general formulas needed below. First, let us recall the well-known formulas for the scattering amplitude and total cross-section:

$$u_\infty(\theta) = \frac{1}{4\pi} \int_{|r|=R} \left( ik \left( \theta \cdot \frac{r}{R} \right) u + u_r \right) e^{-ik(\theta \cdot r)} dS(r), \quad \sigma_k(u) = \frac{1}{k} \Im(u_n, u), \quad (24)$$

where $k > 0$ and $R$ is large enough, so that the ball $|r| < R$ contains the obstacle.

The Green’s formula for the solutions of (1) implies

$$\int_{\Omega} [-|\nabla u|^2 + k^2 |u|^2] dx = \int_{\partial \Omega} u_n \overline{u} dS - ik\sigma_k(u), \quad k > 0. \quad (25)$$

Note that the second formula in (24) is an obvious consequence of (25). Formula (25) remains valid if $k = 0$ and the second condition in (1) is replaced by the decay of $u$ at infinity. In this case,

$$- \int_{\Omega} |\nabla u|^2 dx = \int_{\partial \Omega} u_n \overline{u} dS, \quad k = 0. \quad (26)$$

Thus the operator $D(0)$ is negative.

**Proof of theorem 1.** The operator $D$ is compact in both spaces $L_2(\partial \Omega)$ and $H^{1/2}(\partial \Omega)$, and the condition $\frac{1}{\zeta(0)} \notin \text{Sp}(D(k))$ implies the solvability of (7), and, therefore, the existence of the solution of problem (1), (2).

Let $u$ be a solution of the non-homogeneous problem (1) with the right-hand side of the equation in the space $L_{2\text{com}}(\Omega)$ (functions from $L_2$ with compact supports) and with zero Neumann boundary condition. The function $u$, being considered as an element of the Sobolev space $H^{1/2}_{\text{loc}}(\Omega)$, admits a meromorphic continuation to the whole complex plane $k \in \mathbb{C}$ with all the poles located in the lower half-plane, see [18, 19]. Indeed, a pole at a point $k = k_0$, $\Im k_0 > 0$, would lead to a complex eigenvalue of the Neumann Laplacian, a real pole $k = k_0 \neq 0$ would lead to a non-uniqueness of the solution of problem (1) with the Neumann boundary condition and a pole at the origin would lead to the existence of a decaying at infinity harmonic function.
in Ω with zero Neumann boundary condition, see [19]. These properties of u immediately imply that the operator
\[ D(k) : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega) \]  
(27) is meromorphic in the whole \( k \)-plane with all the poles in the lower half-plane. (Recall that the operator \( D(k) \) for complex \( k \) is understood as an analytic continuation from the operator on semiaxis \( k > 0 \).) To obtain the same analytic properties of the operator \( D(k) \) in \( L_2(\partial\Omega) \), one needs to study the kernel \( G_k \) of the operator (see (6)). A standard approach to the construction of the Green’s function (reduction to an integral equation on the boundary) allows one to show that \( G_k = G_0 + (G_k - G_0) \) where \( G_0 \) is the Green’s function of the Neumann problem for the Laplacian, the function \( G_k(x, s) - G_0(x, s) \) has a weak singularity at \( x = s \) and the operator in \( L_2(\partial\Omega) \) with the kernel \( G_k(x, s) - G_0(x, s) \) is meromorphic in \( k \in \mathbb{C} \). Poles in the region \( \Im k \geq 0 \) do not exist by the same reason as for operator (27).

The analytic Fredholm theorem implies that \( (I - \gamma(k)D(k))^{-1} \) is meromorphic in \( k \in U \) if the latter operator is bounded for some value of \( k \in U \). In fact, one needs to refer to a Fredholm theorem for a meromorphic family of operators, see [4].

Let us prove the second statement of the theorem. The kernel \( G_k \) is real and symmetric when \( k = i\rho, \rho \geq 0 \). Thus the operator \( D(i\rho) \) is symmetric. If a family of compact operators (which is \( D(k) \) in our case) is symmetric on a ray and analytic in its neighborhood, then the eigenfunctions and eigenvalues of this family are analytic on the ray, see [9]. In particular, the eigenfunctions and eigenvalues of \( D(k) \) are analytic at \( k = 0 \). Let us give a little more detail.

Formula (26) implies that \( \zeta_j(0) < 0 \). Besides, \( \zeta_j(0) \to 0 \) as \( j \to \infty \) since the operator \( D(0) \) is compact. Let us fix \( \alpha \in (\varepsilon/2, \varepsilon) \) which is not an eigenvalue of \( D(0) \). Let \( \Gamma_1 \) be a bounded contour in the half-plane \( \Re k < -\alpha \) which encloses all the eigenvalues \( \zeta_j(0) < -\alpha \) and let \( \Gamma_2 \) be a circle of radius \( \alpha \) centered at \( k = 0 \) (which encloses the remaining eigenvalues). Obviously, operators
\[ P_i(k) = \frac{1}{4\pi i} \int_{\Gamma_i} (z - D(k))^{-1} dz, \quad i = 1, 2, \]
are analytic in \( k \) in a small neighborhood \( U_\varepsilon \) of the origin, commute with \( D(k) \) and \( P_1(k) + P_2(k) = I \) is the identity operator. We define \( L_{1,\varepsilon}(k) \) as the range of the operator \( P_1(k) \). Spaces \( L_{1,\varepsilon}(i\rho), \rho \geq 0 \), are spanned by the corresponding sets of eigenfunctions of the symmetric operator \( D(i\rho) \), and \( P_1(i\rho), \rho \geq 0 \), are the orthogonal projections, and therefore they are self-adjoint. Thus eigenfunctions of \( P_1(i\rho) \) (which are also eigenfunctions of \( D(i\rho) \)) admit an analytic continuation (see [9]). We refer to the same source [9] but in a simpler situation of a finite-dimensional operator (one also could reduce the statement above to the statement for a matrix which is symmetric on a segment and analytic in a neighborhood of the segment). The norm of \( D(k)P_1(k) \) does not exceed \( \alpha < \varepsilon \) when \( k = 0 \), and therefore it does not exceed \( \varepsilon \) when \( k \in U_\varepsilon \) if \( U_\varepsilon \) is small enough. The second statement of the theorem is proved.

Let us prove the last statement. It is well known that the operator \( \Sigma : u|_{\partial\Omega} \to u_{\infty} \) is bounded (and compact) in \( L_2, i.e.
\[ \sigma_k(u) \leq C(k)||u||^2_{L_2(\partial\Omega)}, \quad k > 0. \]
(28) Indeed, consider two operators which map Dirichlet data on \( \partial\Omega \) into the solution \( u \) of the Dirichlet problem and its derivative \( u_r \) on the sphere \( r = R, R \gg 1 \). These operators are given by formulas similar to (6). The kernels of these operators are smooth when \( s \in \partial\Omega, |r| = R, \) and therefore the operators are bounded (and compact). Thus an application of the first formula in (24) implies (28). On the other hand, from the second formula in (24) it follows that
\[ \Im \zeta_j^{-1}(k)||u_j||^2 = k\sigma_k(u_j), \]
where \( u_j \) is the solution of problem (1) whose Neumann data are an
eigenfunction of $D(k)$ with the eigenvalue $\varsigma_j(k)$. This and (28) imply that $0 < \Im \varsigma_j^{-1} < c(k)$ for some $c(k) < \infty$ and all $j$. The estimate $\Re \varsigma_j^{-1} < d(k)$ follows from the representation

$$D^{-1}(k) = D^{-1}(0) + \{D^{-1}(k) - D^{-1}(0)\},$$

where the first operator on the right-hand side is negative (see (26)) and the second one is bounded (and compact). The compactness of the second operator can be easily derived from the fact that the kernel of the operator $D^{-1}(k)$ is the normal derivative of the Green’s function of the Dirichlet problem. The theorem is proved.

\[\square\]

**Proof of theorem 2.** Condition (2) implies

$$\langle N(k) - te^{i\delta} \rangle u' = f_1 + te^{i\delta} f, \quad f = e^{i(k \alpha)}, \quad f_1 = -\frac{\partial e^{i(k \alpha)}}{\partial n},$$

(29)

where $N(k) = D^{-1}(k)$ is the unbounded operator in $L^2(\partial/\Omega_1)$ which corresponds to the Dirichlet-to-Neumann map. Let us write it in the form $N = A + iB$, where the operators $A$ and $B$ are self-adjoint. Hence, we can rewrite (24) as $(Bu, u) = \frac{1}{2}\sigma(u)$, and (28) implies that the operator $B$ is bounded.

We rewrite (29) in the form

$$u' = (N(k) - te^{i\delta})^{-1}(f_1 + te^{i\delta} f) = (I + i(A - te^{i\delta})^{-1}B)^{-1}(A - te^{i\delta})^{-1}(f_1 + te^{i\delta} f).$$

(30)

Consider first the case of $\sin \delta \neq 0$. Since the operator $A$ is self-adjoint, we have

$$\|A^{-1}\|_{L^2(\partial/\Omega_1)} \leq \frac{1}{\|\sin(\delta)\|}, \quad \sin(\delta) \neq 0.$$  

(31)

Thus

$$(I + i(A - te^{i\delta})^{-1}B)^{-1} \rightarrow I, \quad t \rightarrow \infty, \quad \sin(\delta) \neq 0.$$  

(32)

From (31) and the identity

$$te^{i\delta}(A - te^{i\delta})^{-1}h = (A + te^{i\delta})^{-1}Ah - h$$

which is valid for any $h$ in the domain of $A$, it also follows that

$$s\lim_{t \rightarrow \infty} te^{i\delta}(A - te^{i\delta})^{-1} = I, \quad \sin(\delta) \neq 0.$$  

(33)

Since functions $f, f_1$ are smooth and belong to the domain of $A$, relations (30), (31) and (33) imply the first of relations (9). The second relation follows from the first one and (28).

The case of $\delta = 0$ is treated similarly. One needs only to note that the operator $A$ is bounded from above ([1], 4.2) and, therefore, estimate (31) holds with the right-hand side replaced by $1/(t - d), t > d$, where $d$ is a constant.

The proof is complete. \[\square\]

The following lemma will be needed for the proof of theorem 3. Recall that the operator $D(k)$ is analytic, i.e.

$$D = \sum_{m=0}^{\infty} D_m k^m.$$  

Obviously, the kernel of the operator $D_0$ is the Green’s function of the Neumann problem for the Laplacian in the exterior of $\Omega$. Let us show that $D_1$ is the one-dimensional operator of the projection on the constant $u_0$.

**Lemma 7.**

$$D_1 v = -\frac{i}{4\pi} (v, u_0) u_0, \quad v \in L^2(\partial\Omega, d\sigma).$$
Proof. It is enough to prove the statement above for smooth $v$ or $v \in H^{1/2}(\partial \Omega)$. Let $u$ be a solution of problem (1) with the Neumann data at the boundary equal to $v$, i.e.

$$(\Delta + k^2)u = 0, \quad x \in \Omega; \quad \frac{\partial u}{\partial n} \bigg|_{\partial \Omega} = v, \quad (34)$$

We also assume that $u$ satisfies the radiation condition (1). Green’s formula implies that

$$u(r_0) = \int_{\partial \Omega} \left[ \frac{\partial}{\partial n} \left( \frac{\omega^{|r-r_0|}}{4\pi |r-r_0|} \right) u - \frac{\omega^{|r-r_0|}}{4\pi |r-r_0|} v \right] dS, \quad r_0 \in \Omega.$$ 

If $v \in H^{1/2}(\partial \Omega)$, then the solution $u = \sum_{k=1}^{\infty} k^j u_j$ is analytic in $k$ as an element of $H^2(\Omega)$, see [18]. We expand the left- and right-hand sides in the above formula in the Taylor series and equate the linear terms. Then we arrive at

$$u_1(r_0) = \int_{\partial \Omega} \left[ \frac{\partial}{\partial n} \left( \frac{1}{4\pi |r-r_0|} \right) u_0 - \frac{1}{4\pi} v \right] dS, \quad r_0 \in \Omega. \quad (35)$$

From this equation it follows that $|u_1| < C < \infty$ as $r_0 \to \infty$. On the other hand, (34) implies that $\Delta u_1 = 0$ in $\Omega$, $(u_1)_n = 0$ on $\partial \Omega$. Thus, $u_1$ is constant in $\Omega$. The integral of the first term on the right-hand side of (35) decays at infinity, and the integral of the second term is a constant. Hence the first integral is zero, and

$$u_1 = -\frac{1}{4\pi} \int_{\partial \Omega} v dS, \quad r_0 \in \Omega.$$ 

It remains to note that $D_1 v = u_1|_{\partial \Omega}$. The proof is complete. \qed

**Proof of theorem 3.** The integral kernel of the operator $-D(0)$ is positive (see, e.g., ([14], example 3, XIII.12)). Thus, from the Perron–Frobenius theorem it follows that $\zeta_0$ is simple and the sign of the corresponding eigenfunction is not changing. The latter implies that $c_0 \neq 0$.

Let us prove (12). Let $u$ be a harmonic function in $\Omega$ equal to $u_0 \equiv 1$ on $\partial \Omega$. Then

$$C = -\frac{1}{4\pi} \int_{\partial \Omega} \frac{\partial u}{\partial n} dS = \frac{1}{4\pi} (-N^* u_0, u_0), \quad N^* = D^{-1}(0).$$

Thus,

$$-\frac{1}{\zeta_0(0)} = \min_{\varphi \neq 0} \left( \frac{-N^* \varphi, \varphi}{||\varphi||^2} \right) \leq \left( \frac{-N^* u_0, u_0}{||u_0||^2} \right) = \frac{4\pi C}{S}.$$ 

The first statement is proved. Let us prove the second statement.

Since the operator $D(k)$ is analytic and $D(0)$ is symmetric, the standard perturbation theory implies that

$$\zeta'_0(0) = (D'(0)\varphi_0(0), \varphi_0(0)) = (D_1 \varphi_0(0), \varphi_0(0)),$$

and lemma 7 justifies statement 2.

Let us prove the last two statements. Since operators (27) are bounded and analytic in $k$, operators

$$D(k) : H^{1/2}(\partial \Omega) \to H^{3/2}(\partial \Omega)$$

are compact and analytic in $k$. Thus, if $\frac{1}{\varpi(0)} \notin Sp(D(0))$, then the analytic Fredholm theorem implies that the solution $v \in H^{1/2}(\partial \Omega)$ of equation (7) exists and is analytic in $k$ in some neighborhood of $k = 0$, and therefore (see [18]) the solution $u \in H^2_{loc}(\Omega)$ of (1), (2) exists in some neighborhood of $k = 0$ and is analytic in $k$.

Let now $\frac{1}{\varpi(0)} = \zeta'_0(0)$ and (13) hold. We fix $\varepsilon = -\gamma(0)/2$, split $L_2(\partial \Omega)$ according to (8) and represent the right-hand side $f$ in equation (7) in the form

$$f = f_1 + f_2, \quad f_1 = P_1(k)f \in L_{1,\varepsilon}, \quad f_2 = P_2(k)f \in L_{2,\varepsilon}.$$
Then the solution \( v \in H^{1/2}(\partial \Omega) \) has the form

\[
v = v_1 + v_2, \quad v_i = (I - \gamma(k) D(k))^{-1} P_i(k) f,
\]

where the subindex \( i \) in the first operator on the right indicates the restriction of the operator on the space \( L_{\varepsilon,i}(k) \):

\[
(I - \gamma(k) D(k))^{-1} : L_{\varepsilon,i} \to L_{\varepsilon,i}.
\]

From the choice of \( \varepsilon \) it follows that the operator

\[
(I - \gamma(k) D(k))^{-1} = \sum_{j=0}^{\infty} (\gamma(k) D(k))^j
\]

is analytic in a neighborhood of \( k = 0 \), and therefore the same is true for \( v_2 \in H^{1/2}(\partial \Omega) \). In order to find \( v_1 \), we write

\[
P_1(k) f = \sum_{j=0}^{N} a_j(k) \varphi_j(k), \quad a_j(0) = \gamma(0)c_j.
\]

Then, as \( k \to 0 \), we have

\[
v_1(r) = \sum_{j=0}^{N} \frac{1}{1 - \gamma(k) D(k)} a_j(k) \varphi_j(k, r)
\]

\[
= \sum_{j=0}^{N} \frac{1}{1 - \gamma(k) D(k)} (\gamma(0) c_j \varphi_j(0, r) + O(k)).
\]

Hence,

\[
v = \frac{1}{k} \sum_{p} c_j \gamma(0) \varphi_j(0, r) + O(1), \quad k \to 0.
\]

Since \( u = Dv, \quad r \in \partial \Omega \), (37) immediately implies (14).

In order to obtain (17), we note that Green’s formula allows us to rewrite the first equality in (24) in the form

\[
u_\infty(\theta) = \frac{1}{4\pi} \int_{\partial \Omega} \left( u \frac{\partial}{\partial n} e^{-ik(\theta \cdot r)} - \psi e^{-ik(\theta \cdot r)} \right) dS.
\]

This, (37) and (14) imply that, as \( k \to 0 \),

\[
u_\infty(\theta) = \frac{1}{4\pi} \int_{\partial \Omega} u dS + O(1) = -\frac{1}{4\pi} \sum_{p} \frac{-c_j \gamma(0)}{\gamma(0)} \varphi_j(0) + O(1),
\]

where the coefficients \( b_j \) are defined in (15). This proves (17). The last statement of the theorem follows from analyticity of \( v_2 \), (36) and linear independence of functions \( \varphi_j \).

The proof is complete.

**Proof of theorem 5.** The first statement is an analytic in \( \gamma \) extension of (19) (\( \gamma_1 \) appears in that statement because the integral kernel of the operator \( F^* \) contains complex conjugation). Let us prove the second statement. From (24) it follows that the integral kernel \( u_\infty \) of the operator \( F \) is smooth with respect to \( \theta, \alpha \) and meromorphic in \( k \), i.e. (18) is a meromorphic family of Fredholm operators. Consider \( S^{-1}_\gamma(k) \). From the relation \( S^{-1}_\gamma(k) S^{-1}_\gamma(k) = I \) it follows that \( S^{-1}_\gamma(k) \) has a pole at \( k = k_0 \) if and only if \( S^{-1}_\gamma(k) \) has a non-trivial kernel at this point.

12
Consider now the operator $\hat{S}_\gamma(k)$ which is defined as follows. If $S_{\gamma}^{-1}(k)f(\theta) = g(\theta)$, then $\hat{S}_\gamma(k)f(-\theta) = g(-\theta)$. Obviously, the operators $S_{\gamma}^{-1}(k)$ and $\hat{S}_\gamma(k)$ have non-trivial kernels at the same points $k \in \mathbb{C}$.

Let us construct the operator $\hat{S}_\gamma(k)$. First we assume that $k > 0$. From (22) it follows that $S_{\gamma}^{-1}(k)u_{\text{out}}(\theta) = u_{\text{in}}(-\theta), k > 0$.

Thus $\hat{S}_\gamma(k)u_{\text{out}}(-\theta) = u_{\text{in}}(\theta), k > 0$.

On the other hand, after complex conjugation in (21) and application of (22), we get $S_{\gamma_1}(k)u_{\text{out}}(-\theta) = \bar{u}_{\text{in}}(\theta), \gamma_1 = \bar{\gamma}, k > 0$.

Hence, $\hat{S}_\gamma(k)f(\theta) = \overline{S_{\gamma_1}(k)f(\theta)}, \gamma_1 = \bar{\gamma}, k > 0$.

We extend the last relation analytically in the complex plane $k$. If $\gamma_1 = \bar{\gamma}(k), k \in \mathbb{C}$, we obtain that $\hat{S}_\gamma(k)f(\theta) = \overline{S_{\gamma_1}(k)f(\theta)}, k \in \mathbb{C}$.

This proves the second statement of the theorem. The last statement follows immediately from theorem 1. The proof is complete.

Acknowledgments

The authors are very grateful to N Grinberg, A Kirsch and M A Mironov for numerous and useful discussions. In particular, the appendix could not have been written without input from M A Mironov. The work of the first author was supported in part by Centre for Research on Optimization and Control (CEOC) from the ‘Fundação para a Ciência e a Tecnologia’ (FCT), cofinanced by the European Community Fund FEDER/POCTI, and by the FCT research project PTDC/MAT/113470/2009. The work of the second author was supported in part by the NSF grant DMS-0706928.

Appendix

A.1. Reduction of acoustic equations to an impedance problem

The general acoustic equations in $\Omega = R^3 \setminus O$ have the form

$$\Delta p(r) + k^2 p(r) = 0, -i\omega \rho_l(\nabla u) = -\nabla p, \quad (A.1)$$

where $p$ satisfies the radiation conditions at infinity. Here $Re(p(r)e^{-ist})$ is the pressure, $u$ is the velocity potential ($v = \nabla u$), and $\rho_l > 0$ is the density of the media. The second equation above follows from the Newton law: density of the media times acceleration equals the negative of the gradient of the pressure, where the minus sign is needed since the direction of the force corresponds to the decay of the pressure.

Now we derive boundary conditions for equations (A.1) assuming that the obstacle is covered by a springy layer with mutually independent ‘springs’, and the thickness $h \sim 0$ of the layer is negligible. Since the normal velocity of the media at the boundary coincides with the normal velocity of the cover of the obstacle (there are no voids between media and the cover), Hooke’s law implies

$$\frac{1}{-i\omega} \frac{\partial u}{\partial n}(r) = -\beta p(r), \quad r \in \partial \Omega. \quad (A.2)$$
Here \( n \) is the interior unit normal vector to \( \partial \Omega \), the left-hand side is the radial displacement (the integral with respect to time of the normal velocity), \( \beta > 0 \) is the compressibility coefficient (for 'springs') and the minus sign on the right indicates that the layer shrinks when \( \beta > 0 \) increases. Note that \( \beta \) could depend on the point \( r \in \partial \Omega \).

The second equation in (A.1) implies that

\[
\frac{\partial u}{\partial n} = \frac{1}{\omega \rho \omega} \frac{\partial p}{\partial n} \tag{A.3}
\]

From here and Hooke's law it follows that

\[
\frac{\partial p}{\partial n}(r) + \beta \rho \omega^2 p(r) = 0, \quad r \in \partial \Omega. \tag{A.4}
\]

We obtained the impedance boundary condition for \( p \), where impedance has the form (23) with \( Z = \beta \rho \).

**Appendix A.2. Evaluation of the compressibility coefficient**

Consider a rigid obstacle \( O \) covered by an elastic membrane with a gas (for example, air) layer between them, and numerous rigid walls fixed to the obstacle and the membrane, which partition the gas layer into small chambers. We assume that the sizes of chambers are small enough so that the impedance can be considered as local. The whole construction is submerged into a liquid. We show that the compressibility coefficient \( \beta \) is given by the following expression:

\[
\beta = \frac{\gamma g \rho g c_g^2}{\rho \mu}, \tag{A.5}
\]

where \( \rho_g \) is the density of the gas in the chambers, \( c_g \) is the speed of the sound propagation there, \( h \) is the distance between the obstacle and the membrane and \( \gamma_g = C_p/C_v \) is the ratio of heat capacities of the gas with constant pressure and volume respectively.

In fact, Kliperon's law implies that

\[
P V = \frac{m}{\mu} R T, \tag{A.6}
\]

where \( P, V \) are the values of unperturbed pressure and volume in the chambers, \( \mu \) is molecular weight of the gas and \( R \) is the universal gas constant. Let \( \Delta P, \Delta V \) be the change in \( P \) and \( V \) respectively. Assuming that the temperature is constant, we get

\[
\frac{\Delta V}{\Delta P} = \frac{V}{P}. \quad \text{Let } V = Sh, \quad \Delta V = S \Delta h, \quad \text{where } S \text{ is the area of the chamber base. Then}
\]

\[
\beta = - \frac{\Delta h}{\Delta P} = \frac{h}{P}. \tag{A.7}
\]

From (A.6) it follows that

\[
P = \frac{m}{V} \frac{RT}{\mu} = \frac{\rho_g RT}{\mu},
\]

and since

\[
c_g = \sqrt{\frac{\gamma_g RT}{\mu}},
\]

we obtain

\[
P = \frac{\rho_g (c_g)^2}{\gamma_g}. \tag{A.8}
\]
We substitute the latter expression into (A.7) and arrive at (A.5).

From (A.5) and (A.4), taking into account that \( \omega = k c_1 \), we obtain the following final expression for the impedance in the problem under consideration:

\[
\gamma = -\frac{\rho_l}{\rho_g} \left( \frac{c_g}{c_l} \right)^2 h k^2.
\] (A.9)

Let us find this value assuming that the air is used as a gas and water with zero temperature is outside of the construction. Then

\[
\frac{\gamma_g \rho_g c_g \rho_l c_l}{1.383} = \frac{1000}{331.3} \frac{1390}{1000} = 4.383 \text{ m}^{-1}.
\] (A.10)

and therefore

\[
\gamma \sim -25 800 h k^2.
\] (A.10)

Note that both \( k \) and \( \gamma \) are measured in units of \((\text{length})^{-1}\).

Appendix A.3. Impedance when friction is present

Assume that oscillations of a springy cover are accompanied by friction. Usually, a force due to friction is proportional to the velocity: \( F_{fr} = -\varepsilon \frac{du}{dn} \). In natural circumstances, \( \varepsilon \geq 0 \), but one can also create artificially a situation when \( \varepsilon < 0 \) (negative friction). Let us write an analog of (A.2).

By equating all the forces, we obtain

\[
-\varepsilon \frac{du}{dn} + \frac{1}{\beta} \left( \frac{du}{dn} \right)(r) = p(r), \quad r \in \partial \Omega.
\] (A.11)

Using (A.3), we obtain

\[
\frac{dp}{dn}(r) + (\beta \rho_l) \omega^2 \frac{1 + i \varepsilon \omega \beta}{1 + (\varepsilon \omega \beta)^2} p(r) = 0, \quad r \in \partial \Omega.
\] (A.12)

One can see that the presence of friction creates an imaginary part of \( \gamma \), and natural friction corresponds to \( \Im \gamma < 0 \), and ‘negative friction’ corresponds to \( \Im \gamma > 0 \).

References

[1] Agranovich M S 2002 Spectral problems for second-order strongly elliptic systems in domains with smooth and nonsmooth boundaries Russ. Math. Surv. 57 847–920
[2] Alber H-D and Ramm A G 2009 Asymptotics of the solution to Robin problem J. Math. Anal. Appl. 349 156–64
[3] Aleksenko A, Cruz P and Lakshtanov E 2008 High-frequency limit of the transport cross section in scattering by an obstacle with impedance boundary conditions J. Phys. A: Math. Theor. 41 255203
[4] Blekher P 1972 Operators that depend meromorphically on a parameter Mosc. Univ. Math. Bull. 24 21–6
[5] Colton D and Kress R 1998 Inverse Acoustic and Electromagnetic Scattering Theory (Applied Mathematical Sciences vol 93) (Berlin: Springer)
[6] Grinberg N and Kirsch A 2004 The factorization method for obstacles with a-priori separated sound-soft and sound-hard parts Math. Comput. Simul. 66 267–79
[7] Gureev M and Mironov M 2007 Superfocusing of a spherical waves: theory and experiment Acoust. Phys. 53 774–8
[8] Isakov V 2006 Inverse problems for Partial Differential Equations (Applied Mathematical Sciences vol 127)
[9] Kato T 1966 Perturbation Theory for Linear Operators (Berlin: Springer)
[10] Lakshtanov E 2010 Spectral properties of the Dirichlet-to-Neumann operator for exterior Helmholtz problem and its applications to scattering theory (submitted)
[11] Leighton T G 1994 The Acoustic Bubble (New York: Academic)
[12] Ramm A G 1986 Scattering by Obstacles (Dordrecht: Reidel)
[13] Ramm A G 2005 Scattering by Small Bodies of Arbitrary Shapes (Singapore: World Scientific)
[14] Reed M and Simon B 1978 Methods of Modern Mathematical Physics, IV (New York: Academic)
[15] Tychonov A N and Samarsky A A 1963 *Equations of Mathematical Physics* (Oxford: Pergamon)

[16] Uhlmann G 1992 Inverse boundary value problems and applications *Astérisque* 207 153–211

[17] Vainberg B R and Grushin V 1967 Uniformly nonelliptic problems *Math. USSR-Sb.* 2 111–33

[18] Vainberg B 1973 On exterior elliptic problems polynomially depending on a spectral parameter, and the asymptotic behaviour for large time of solutions of nonstationary problems *Math. USSR Sb.* 21 221–39

[19] Vainberg B 1975 On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t \to \infty$ of solutions of non-stationary problems *Russ. Math. Surv.* 30 1–58