A simple approach to the wave uniqueness problem

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Abstract

We propose a new approach for proving uniqueness of semi-wavefronts in generally non-monotone monostable reaction-diffusion equations with distributed delay. This allows to solve an open problem concerning the uniqueness of non-monotone (hence, slowly oscillating) semi-wavefronts to the KPP-Fisher equation with delay. Similarly, a broad family of the Mackey-Glass type diffusive equations is shown to possess a unique (up to translation) semi-wavefront for each admissible speed.

Keywords: monostable equation, non-monotone reaction, uniqueness, KPP-Fisher delayed equation, Mackey-Glass type diffusive equation

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1. Introduction and main results

The uniqueness of traveling waves for the monostable delayed or non-local reaction-diffusion equations is an important and ‘largely open’ question of the theory of partial functional differential equations. In consequence, different strategies have been elaborated so far to tackle the uniqueness problem, e.g. see [1, 2, 4, 5, 6, 7, 8, 9, 11, 12, 14, 18, 21, 24, 26, 27, 28, 31, 33]. Broadly speaking, the cited works show that the wave uniqueness can be established when either the evolution equation or the waves are monotone, or when the Lipschitz constant of nonlinear reaction term is dominated by its derivative at the unstable equilibrium (the Diekmann-Kaper condition). On the other hand, recent studies [8, 17, 25] reveal that the uniqueness property can fail to hold even for monotone waves of some non-local monostable equations. To get a clearer picture of the situation, consider the following KPP-Fisher delayed equation (see [2, 10, 14, 15, 17, 18, 19, 20, 32] for more detail and references concerning this model):

\[
\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)(1-u(t-h,x)).
\]  

This partial functional differential equation does not meet quasi-monotonicity assumptions neither in the sense of Wu-Zou [32] nor in the sense of Martin-Smith [23]. Moreover, its reaction term does not satisfy the Diekmann-Kaper dominance condition at 0. In addition, if \( h \geq 0.57 \) then all non-constant positive wave solutions to (1) are slowly oscillating in the space variable, see [10, 18]. In this situation, neither the comparison techniques nor
Identifying each constant function equation with delay or for the May diffusive baleen whale model, allows to provide a complete solution to the uniqueness problem for the KPP-Fisher which we believe can also be useful for other diffusive systems. In particular, our method only two zeros on \( f \) and suppose that \( f \) is differentiable at 0. Moreover, for some positive \( c_1 \) and delays \( h \leq 0.57 \), this equation has monotone wavefronts \( u(t, x) = \phi(x + ct) \), their uniqueness (up to translation) was proved in \([14, 18]\). Noteworthily, in the recent e-print \([2]\), this result was complemented and the uniqueness of all fast waves, \( c \geq 2\sqrt{2} \), for \([11]\) (including non-monotone waves) was deduced from their global stability on semi-infinite intervals.

A similar situation is also observed for another popular delayed model, the Mackey-Glass type diffusive equation \([11, 14, 21, 24, 26, 28, 30, 31, 33, 34]\)

\[
\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} - u(t, x) + g(u(t - h, x)), \ u \geq 0.
\] (2)

Here \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) is unimodal (i.e. hump-shaped) bounded \( C^1 \)-smooth function possessing exactly two non-negative fixed points \( u_1 \equiv 0 < u_2 \equiv \kappa \). For \([2]\), the uniqueness question is completely (i.e. for all \( h \geq 0 \) and for all admissible speeds) answered only if either \( g \) is monotone on \([0, \kappa] \), or \( |g'(s)| \leq g(0), \ s \in \mathbb{R} \) (this amounts to the Diekmann-Kaper condition at the equilibrium 0 for \([2]\)). This is for instance the case of the Nicholson’s diffusive equation \((g(u) = pue^{-u})\); however, other population models (like R. May’s sei whale model \([3]\), where \( g(u) = \max(\{pu(1 - u^2)/k^2, 0\}, \text{for some } z > 1) \) do not fit into the frameworks of the above mentioned theories.

In the present work, we propose a novel approach for proving uniqueness of semi-wavefronts in a general non-monotone monostable reaction-diffusion equations with distributed delay of the following form

\[
\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + f(u(t, x)), \ u \geq 0,
\] (3)

where \( f : C[-h, 0] \to \mathbb{R} \) is a continuous functional, \( C := C[-h, 0] \) is the Banach space of all continuous scalar functions defined on \([-h, 0] \) and \( u(t, x) = u(t + s, x), \ s \in [-h, 0] \), belongs to \( C \) for every fixed \( x \in \mathbb{R} \). This approach is based on a relatively simple idea which we believe can also be useful for other diffusive systems. In particular, our method allows to provide a complete solution to the uniqueness problem for the KPP-Fisher equation with delay or for the May diffusive baleen whale model.

In the sequel, we always assume the following natural and easily verifiable conditions:

\((M)\) Identifying each constant function \( x \in C \) with a real number \( x \in \mathbb{R} \), set \( f^*(x) = f(x) \) and suppose that \( f^*(x) \) satisfies the standard monostability requirements (i.e. \( f^* \) has only two zeros on \( \mathbb{R}_+ \), \( x_1 = 0 \) and \( x_2 = \kappa \); moreover, \( f^*(x) > 0 \) on \((0, \kappa)) \).

\((S)\) Functional \( f : C \to \mathbb{R} \) is continuous, transforms bounded sets into bounded sets, and it is differentiable at 0. Moreover, for some positive \( \alpha, \delta, K \), and the max-norm \( |\phi|_C \),

\[
|f(\psi) - f(\phi) - f'(0)(\psi - \phi)| \leq K|\psi - \phi|_C(|\phi|_C^2 + |\psi|_C^2), \ \text{for all } |\phi|_C < \delta, |\psi|_C < \delta. \quad (4)
\]

Using the Jordan decomposition theorem, we can write \( f'(0)\phi \) as

\[
f'(0)\phi = \int_{-h}^{0} \phi(s) d\mu_+(s) - \int_{-h}^{0} \phi(s) d\mu_-(s),
\]
where \( \mu_{\pm} \) are non-decreasing functions on \([-h, 0]\).

**Corollary 2.** We will assume that \( \int_{-h}^{0} \phi(s) d\mu_{-}(s) = q\phi(0) \) for some \( q \geq 0 \).

By (J), we obtain that
\[
f'(0)\phi = -q\phi(0) + \int_{-h}^{0} \phi(s) d\mu_{+}(s).
\]

(ND) Set \( p := \int_{-h}^{0} d\mu_{+}(s) \), it follows from (M) that \( p \geq q \). In addition, we will assume the following non-degeneracy condition: \( p > q \).

(UB) For each \( \phi, \psi \in C \) satisfying \( 0 < \phi(s) \leq \psi(s) \), \( s \in [-h, 0] \), it holds that \( f(\psi) - f(\phi) \leq f'(0)(\psi - \phi) \).

(LB) Moreover, for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( \phi \in C \) satisfying \( 0 < \phi(s) \leq \delta, s \in [-h, 0] \), it holds that
\[
q\phi(0) + f(\phi) \geq (1 - \epsilon) \int_{-h}^{0} \phi(s) d\mu_{+}(s).
\]

Our main result is the following theorem.

**Theorem 1.** Let \( u_{1}(x, t) = \phi(x+ct), u_{2}(x, t) = \psi(x+ct) \) be two positive semi-wavefronts (i.e. \( \phi(-\infty) = \psi(-\infty) = 0 \) and \( \phi(t), \psi(t) \) are positive and bounded on \( \mathbb{R} \)) of equation \( \mathcal{J} \).

If assumptions (M), (S), (J), (ND), (LB) and (UB) are satisfied then \( \phi(t+t') \equiv \psi(t), t \in \mathbb{R} \), for some finite \( t' \in \mathbb{R} \).

**Corollary 2.** For each \( c \geq 2 \) and for each \( h \geq 0 \), the KPP-Fisher delayed equation \( \mathcal{J} \) has a unique (up to translation) semi-wavefront.

**Proof.** By \( \mathbb{R} \), for each \( c \geq 2 \) and \( h \geq 0 \), equation \( \mathcal{J} \) has at least one semi-wavefront. For \( \phi \in C \), set \( f(\phi) = \phi(0)(1 - \phi(-h)) \), then \( f'(0)\phi = \phi(0), p = 1 > q = 0 \), and
\[
f(\psi) - f(\phi) = \psi(0) - \phi(0) + \phi(0)\phi(-h) - \psi(0)\psi(-h) = f'(0)(\psi - \phi) + \phi(0)(\phi(-h) - \psi(-h)) + \psi(-h)(\phi(0) - \psi(0)) \leq f'(0)(\psi - \phi) \text{ for } 0 < \phi(s) \leq \psi(s), s \in [-h, 0];
\]
\[
|f(\psi) - f(\phi) - f'(0)(\psi - \phi)| \leq |\psi - \phi|_{C}(|\phi|_{C} + |\psi|_{C}) \text{ for all } \phi, \psi \in C.
\]
Thus assumptions (M), (S), (J), (UB) and (LB), (ND) are clearly satisfied. Therefore the statement of the corollary follows from Theorem \( \mathbb{R} \) \( \Box \)

To present a similar result for the Mackey-Glass type diffusive equation \( \mathcal{J} \), we need the following auxiliary assertion.

**Lemma 3.** Assume (J), (ND) and set \( \chi(z, c) = z^{2} - cz + f'(0)e^{cz} \). Then there exists \( c_{*} > 0 \) such that \( \chi(z, c) \) has exactly two positive zeros (counting multiplicity) \( \lambda_{1}(c) \leq \lambda_{2}(c) \) if and only if \( c \geq c_{*} \). These zeros are simple if \( c > c_{*} \), while \( \lambda_{1}(c_{*}) = \lambda_{2}(c_{*}) \) is a double zero. Next, for \( c \geq c_{*} \) every different zero \( \lambda_{j}(c) \) of \( \chi(z, c) \) satisfies \( R\lambda_{j}(c) < \lambda_{1}(c) \).

\( ^{1} \) It is instructive to note that, by its essence, ‘quasi-monotonicity’ condition (UB) is completely different from the Wu-Zou quasi-monotonicity condition (A2) introduced in \( \mathbb{R} \).
The proof of Lemma 5 uses standard arguments of the complex analysis, for the convenience of the reader it is given in the appendix.

Hence, we have the following existence and uniqueness result for equation (2):

**Corollary 4.** Suppose that the real Lipschitz continuous function \(-x + g(x)\) satisfies the monostability and smoothness conditions of (M), (S) where the space \(C\) is replaced with \(R\). If, furthermore, \(g(0) > 1\) and \(g(x_2) - g(x_1) \leq g(0)(x_2 - x_1)\) for all \(x_1 < x_2\), then for each \(c \geq c_*\) and for each \(h \geq 0\), the diffusive delayed equation (2) has a unique (up to translation) semi-wavefront.

**Proof.** For (2), the semi-wavefront existence (for all \(c \geq c_*\) and for all \(h \geq 0\)) was proved in [13, Theorem 18]. With \(f(\phi) = -\phi(0) + g(\phi(-h))\), \(f'(0)\phi = -\phi(0) + g'(0)\phi(-h)\), and for all \(x \in \mathbb{R}\), verification of other assumptions of Theorem 1 is an easy task. □

Corollary 4 does apply to the above mentioned May diffusive baleen whale model.

2. Proof of Theorem 1

The proof of Theorem 1 is divided into the following four parts.

2.1. **Proof of the exponential decay of wave profiles at \(-\infty\).**

By the definition of a semi-wavefront \(u(t, x) = \phi(x + ct)\), it holds that \(\phi(-\infty) = 0\). To prove the uniqueness of \(\phi\), we need to derive more detailed information concerning asymptotic behavior of \(\phi\) at \(-\infty\). In this subsection, under assumptions imposed in the introduction, we establish that \(\phi(t)\) decays exponentially at \(-\infty\). This property is well known from [16] in the case when \(f\) is \(C^1\)-smooth and bounded, together with its Fréchet derivative, in some vicinity of 0 \(\in C\) and when, in addition, \(\chi(z, c)\) does not have zeros on the imaginary axis (clearly, this is true for the KPP-Fisher delayed equation). However, for some admissible pairs \((c, f'(0))\), function \(\chi(z, c)\) can have purely imaginary zeros, and therefore we should prove exponential decay of \(\phi\) at \(-\infty\) even if 0 is non-hyperbolic equilibrium of the profile equation

\[
\phi''(t) - c\phi'(t) + f'(\phi_t) = 0, \quad \phi(-\infty) = 0, \quad \phi(t) > 0, \quad \sup_{t \in \mathbb{R}} |\phi(t)| < \infty. \tag{5}
\]

Here \(\phi_t \in C\) is defined by \(\phi_t(s) = \phi(t + cs), \quad s \in [-h, 0]\). Observe that the analysis of the rate of decay of wave profiles at \(-\infty\) is an important part of proofs of almost all wave uniqueness theorems (e.g. cf. [1, 2, 3, 21, 31, 53]).

**Lemma 5.** Assume (J), (ND) and (LB), (UB). Then for each semi-wavefront profile \(\phi\) there exists \(\gamma > 0\) such that \(\phi(t) + |\phi'(t)| = O(e^{\gamma t})\) as \(t \to -\infty\).

**Proof.** Since the wave profile \(\phi\) is a bounded function, it satisfies the integral equation

\[
\phi(t) = \int_{-\infty}^{+\infty} K(t - s)(1 + q)\phi(s) + f'(\phi_s)ds, \quad t \in \mathbb{R}, \tag{6}
\]

\[\text{In is easy to show that in this model each semi-wavefront } \phi \text{ satisfies the inequality } \phi(t) < k, \quad t \in \mathbb{R}.\]
where $K$ is the positive Green function (the fundamental solution, cf. [29]) of the equation 
\[ y'(t) - cy'(t) - (1 + q)y(t) = 0. \]
Take $\epsilon > 0$ in (LB) so small that $(1 - \epsilon)p > q$ and let $s'$ be such that $\phi(s) < \delta = \delta(\epsilon)$ for all $s \leq s'$. Consider
\[ G(s) := (1 - \epsilon) \int_{-h}^{0} K(s + cs) \, ds + K(s), \quad \int_{-\infty}^{+\infty} G(s) \, ds = \frac{1 + (1 - \epsilon)p}{1 + q} > 1, \]
and take $N > ch$ large enough to satisfy $\int_{-N}^{N} G(s) \, ds > 1$. In view of (LB), for each $t < s' - N - ch$, it holds that
\[ \phi(t) \geq \int_{t - N}^{t + N + c} K(t - s) [\phi(s) + (1 - \epsilon) \int_{-h}^{0} \phi(s + cs) \, ds] \, ds \geq \int_{t - N}^{t + N} G(t - s) \phi(s) \, ds. \]
Thus for $t' < t < s' - 2N$, we obtain that
\[ \int_{t'}^{t} \phi(v) \, dv \geq \int_{-N}^{N} G(s) \int_{t'}^{t} \phi(v - s) \, dv \, ds, \quad \text{where} \quad \int_{-N}^{N} G(s) \, ds > 1. \quad (7) \]
As it was shown in [1, Theorem 1], inequality (7) implies that $\int_{-\infty}^{0} e^{-\gamma s} \phi(s) \, ds$ converges for some positive $\gamma$.

Next, due to (UB), we obtain from (5) that
\[ \phi(t) \leq \int_{-\infty}^{+\infty} K(t - s) [(1 + q)\phi(s) + f'(0) \phi_s] \, ds = \int_{-\infty}^{+\infty} G_1(t - s) \phi(s) \, ds, \quad t \in \mathbb{R}, \quad (8) \]
where
\[ G_1(s) := \int_{-h}^{0} K(s + cs) \, ds + K(s), \quad s \in \mathbb{R}. \]
Since the bounded function $K$ satisfies $K(s) = O(e^{cs}), \; t \to -\infty$, we have that $G_1(s) \leq A e^{cs}, \; s \in \mathbb{R}$, for some positive $A$. In consequence, for $\gamma \in (0, c)$ as above, we obtain that $G_1(s) \leq B e^{cs}, \; s \in \mathbb{R}$, for some $B > 0$. Thus $\phi(t) = O(e^{ct})$ as $t \to -\infty$ because of
\[ \phi(t) e^{-\gamma t} \leq \int_{-\infty}^{+\infty} G_1(t - s) e^{-\gamma (t-s)} \phi(s) e^{-\gamma s} \, ds \leq B \int_{-\infty}^{+\infty} e^{-\gamma s} \phi(s) \, ds =: D < \infty, \quad t \in \mathbb{R}. \]
Similarly, solving (5) with respect to $\phi'(t)$, we find that
\[ \phi'(t) = \int_{t}^{+\infty} e^{ct-s} f(\tilde{\phi}_s) \, ds, \quad t \in \mathbb{R}. \quad (9) \]
Next, (LB), (UB) and the exponential estimate for $\phi(t)$ implies that, for some $T_1 \in \mathbb{R}$, $D_1 > 0$,
\[ |f(\tilde{\phi}_s)| \leq q \phi(s) + \int_{-h}^{0} \phi(s + cs) \, ds \leq D_1 e^{cs}, \quad s \leq T_1. \]
Since $|f(\tilde{\phi}_s)|$ is a bounded function on $\mathbb{R}$ (cf. (S)), for some $D_2 > 0$, we conclude that $|f(\tilde{\phi}_s)| \leq D_2 e^{cs}, \; s \in \mathbb{R}$. Then (11) implies the following:
\[ |\phi'(t)| e^{-\gamma t} \leq \int_{t}^{+\infty} e^{ct-s} D_2 e^{-\gamma (t-s)} \, ds = \frac{D_2}{e^{\gamma}}, \quad t \in \mathbb{R}. \]
This completes the proof of Lemma 5. \hfill \Box
2.2. Non-existence of super-exponentially decaying solutions at $-\infty$.

We will also need the following nonlinear version of Lemma 3.6 in \cite{30}. It excludes the existence of small solutions to asymptotically autonomous delayed differential equations at $-\infty$:

**Lemma 6.** Suppose that $L : C([-h, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is continuous linear operator and $M : (-\infty, 0] \times C([-h, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous function such that $|M(t, \phi)| \leq \mu(t)|\phi|_C$ for some non-negative $\mu(t) \to 0$ as $t \to -\infty$. Then the system

$$
x'(t) = Lx_t + M(t, x_t), \quad x_t(t) := x(t+s), \quad s \in [-h, 0],
$$

(10)
do not have nontrivial exponentially small solutions at $-\infty$ (i.e. non-zero solutions $x : \mathbb{R} \to \mathbb{R}^n$ such that for each $\gamma \in \mathbb{R}$ it holds that $x(t)e^{\gamma t} \to 0$, $t \to -\infty$).

**Proof.** The proof of Lemma \ref{lem:nonexistence} is a slight modification of the proof of Lemma 3.6 in e-print \cite{30}; for the reader’s convenience, it is included in this paper. So, on the contrary, suppose that there exists a small solution $x(t)$ of (10) at $-\infty$. Take some $b > h$. It is straightforward to see that the property $|x(t)e^{\gamma t} \to 0$, $t \to -\infty$, is equivalent to $|x_t|e^{\gamma t} \to 0$, $t \to -\infty$, where $|x_t| := \max_{s \in [-h, 0]} |x(t+s)|$. We claim that smallness of $x(t)$ implies that $\inf_{t \leq 0} |x_t|/|x_t| = 0$. Indeed, otherwise there is $K > 0$ such that $|x_t|/|x_t| \geq K$, $t \leq 0$, and therefore, setting $\nu := b^{-1} \ln K$, we obtain the following contradiction:

$$
0 < |x_t|e^{\nu t} \leq |x_t-b|e^{\nu (t-b)} \leq |x_t-2b|e^{\nu (t-2b)} \leq \cdots \leq |x_t-mb|e^{\nu (t-mb)} \to 0, \quad m \to +\infty.
$$

Hence, for $b = 3h$ there is a sequence $t_j \to -\infty$ such that $|x_{t_j-3h}|/|x_{t_j}| \to 0$ as $j \to \infty$. Clearly, $|x_{t_j}|/|x_{t_j-s_j}|$ for some $s_j \in [t_j - 3h, t_j]$ and, for all large $j$, it holds $|x(s)| \geq |x(s)|$, $s \in [t_j - 6h, t_j]$. Since $0 \leq t_j - s_j \leq 3h$, without loss of generality we can assume that $\theta_j := t_j - 3h$.

Now, for sufficiently large $j$, consider the sequence of functions

$$
y_j(t) = \frac{x(t + t_j)}{|x(s_j)|}, \quad t \in [-6h, 0], \quad |y_j(-\theta_j)| = 1, \quad |y_j(t)| \leq 1, \quad t \in [-6h, 0].
$$

For each $j$, $y_j(t)$ satisfies the equations

$$
y_j'(t) = Ly_t + \frac{M(t + t_j, x_{t+j})}{|x(s_j)|}, \quad y_j(t) = y_j(-\theta_j) + \int_{-\theta_j}^t \left( Ly_u + \frac{M(u + t_j, x_{u+t_j})}{|x(s_j)|} \right) du,
$$

and therefore $|y_j(t)| \leq 1$, $|y_j'(t)| \leq \|L\| + \sup_{s \leq t_j} \mu(s) \leq \|L\| + \sup_{s \leq 0} \mu(s)$, $t \in [-5h, 0]$, $j \in \mathbb{N}$ (here $\|\cdot\|$ denotes the operator norm). Thus, due to the Arzelà-Ascoli theorem, there exists a subsequence $y_{j_k}(t)$ converging, uniformly on $[-5h, 0]$, to some continuous function $y_*(t)$ such that $|y_*(-\theta_0)| = 1$,

$$
y_*(t) = y_*(-\theta_0) + \int_{-\theta_0}^t L(y_*) du, \quad t \in [-4h, 0].
$$

In particular, $y'_{*}(t) = Ly_t$, $t \in [-4h, 0]$. Since $y_*(t) = 0$ for all $t \in [-5h, -3h]$, the existence and uniqueness theorem applied to the initial value problem $y'(t) = Ly_t$, $t \in [-3h, 0]$, $y_{-3h} = 0$, implies that also $y_*(t) = 0$ for all $t \in [-3h, 0]$. However, this contradicts that $|y_*(-\theta_0)| = 1$. The proof of Lemma \ref{lem:nonexistence} is completed. \qed
2.3. Asymptotic representations of semi-wavefronts at $-\infty$.

The estimate obtained in Lemma 5 can be considerably improved:

**Lemma 7.** Assume (J), (ND), (S) and (LB), (UB). Then there exists some $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ such that

(i) if $c > c_*$ then $\phi(t + t_0), \phi'(t + t_0)) = (1, \lambda_1(c)e^{\lambda_1(c)t} + O(e^{(\lambda_1(c)+\varepsilon)t}), \ t \to -\infty$;

(ii) if $c = c_* \ then \ \phi(t + t_0), \phi'(t + t_0)) = (1, \lambda_1(c)e^{\lambda_1(c)t}(t + O(1)), \ t \to -\infty$.

**Proof.** Clearly, $\phi(t)$ satisfies the linear inhomogeneous equation

$$\phi''(t) - c\phi'(t) - q\phi(t) + \int_{-h}^{t_0} \phi(t + cs)d\mu_+(s) = Q(t),$$

where, due to assumptions (S) and (UB), for some $T_2 \in \mathbb{R}$ and $C > 0$,

$$Q(t) := f'(0)\hat{\phi}_t - f(\hat{\phi}_t) \geq 0, \ t \in \mathbb{R}; \ |Q(t)| \leq C|\phi|^{1+\varepsilon}, \ t \leq T_2.$$  

Then, in view of the positivity of $\phi(t)$, Lemma 5 together with Lemma 28 in [14, Lemma 28] assure that

$$\phi(t) = -\text{Re}_{z=\lambda_1(c)} \left[ \frac{e^{zt}}{\chi(z,c)} \int_{-\infty}^{+\infty} e^{-z+s} Q(s)ds \right] + O(e^{(\lambda_1(c)+\varepsilon)t}), \ t \to -\infty.$$  

A straightforward calculation of the above residue (cf. [1, 14]) implies the asymptotic formulas for $\phi(t)$ in both cases, $c = c_*$ and $c > c_*$, whenever

$$\int_{-\infty}^{+\infty} e^{-\lambda_1(c)s} Q(s)ds > 0.$$  

(12)

Now, suppose that (12) does not hold. Then $Q(t) \equiv 0$ on $\mathbb{R}$ and, consequently, $\phi(t)$ solves the homogeneous equation

$$\phi''(t) - c\phi'(t) - q\phi(t) + \int_{-h}^{t_0} \phi(t + cs)d\mu_+(s) = 0, \ t \in \mathbb{R}.$$  

By Lemma 5 this equation does not have nontrivial small solutions at $-\infty$. But then Lemma 28 and Lemma 5 assure that $\phi(t)$ is a linear combination of the eigenfunctions $e^{\lambda_1(c)t}, e^{\lambda_2(c)t}$. This means that $\phi(t)$ is unbounded on $\mathbb{R}$, a contradiction proving inequality (12).

Finally, assuming $c > c_*$ and integrating (11) over $(-\infty, t)$, we find that

$$\phi'(t) = c\phi(t) + \int_{-h}^{+\infty} (q\phi(s) - \int_{-h}^{0} \phi(s + cs)d\mu_+(s) + Q(s))ds = \lambda_1(c)e^{\lambda_1(c)t} + V(t),$$  

where $V(t) = O(e^{(\lambda_1(c)+\varepsilon)t}), \ t \to -\infty$. A similar computation in the case $c = c_*$ ends the proof of Lemma 7. \hfill $\Box$

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3Lemma 28 in [14] was proved for the case of a single discrete delay, however its proof is valid without significant modifications for the case of delays distributed on a fixed finite interval.
2.4. Proof of the semi-wavefront uniqueness.

Suppose that \( \phi, \psi \) satisfy (13) and for some \( \lambda > 0 \), set \( y(t) = (\psi(t) - \phi(t))e^{-\lambda t} \). Then
\[
y''(t) - (c - 2\lambda)y'(t) + (\lambda^2 - c\lambda - q)y(t) + \int_{-h}^{0} y(t + cs)e^{\lambda cs}d\mu_+(s) + R(t, \bar{y}_t) = 0, \tag{13}
\]
where
\[
R(t, \bar{y}_t) = e^{-\lambda t} \left[ f(\bar{y}_t) - f(\bar{\phi}_t) - f'(0)(\bar{\psi}_t - \bar{\phi}_t) \right].
\]
Then condition (S) implies that, for some \( T \) and \( K > 0 \),
\[
|R(t, \bar{y}_t)| \leq K|y(t + c\cdot)|_{C([\bar{\phi}_t], |\bar{\psi}_t|_0^c)}, \quad t \leq T.
\]
First, we consider the non-critical case when \( c > c_* \) and \( \lambda_1(c) < \lambda_2(c) \). Choose some \( \lambda \in (\lambda_1(c), \lambda_2(c)) \). By Lemma 7, without loss of generality, we can assume that
\[
(\psi(t), \psi'(t)) \text{ and also } (\phi(t), \phi'(t)) = (1, \lambda_1(c))e^{\lambda_1(c)t} + O(e^{(\lambda_1(c) + t)t}), \quad t \to -\infty.
\]
Clearly, this implies that \( y(t) = O(e^{(\lambda_1(c) - \lambda_1(c))t}) \) and \( R(t, \bar{y}_t) = O(e^{\lambda_1(c)\alpha t} |\bar{y}_t|_{\alpha}) \) as \( t \to -\infty \). Then \( y(-\infty) = 0 \) and arguing as in \([22, \text{Proposition 6.1}]\), we conclude that
\[
y(t) = Ae^{(\lambda_2(c) - \lambda_1(c)t} + O(e^{(\lambda_2(c) - \lambda_1(c))t}), \quad t \to -\infty.
\]
By interchanging, if necessary, the roles of \( \phi \) and \( \psi \), we may assume that \( A \geq 0 \).

Suppose first that \( A = 0 \), then \( y(t) = O(e^{(\lambda_2(c) - \lambda_1(c)\alpha t}) \), \( t \to -\infty \). Note that the eigenvalues of the homogeneous part of equation (13) coincide with the zeros of \( \chi(z, c) \) shifted by \( -\lambda \). Therefore, since \( \chi(z, c) \) does not have eigenvalues with \( \Re \lambda_j > \lambda_2(c) \), by \([22, \text{Proposition 6.1}]\), we conclude that \( y(t) \) is a small solution of the asymptotically autonomous (at \( -\infty \)) linear equation (13). Invoking now Lemma 6, we conclude that \( y(t) \equiv 0 \). This means that \( \phi(t) \equiv \psi(t) \) in the case when \( A = 0 \).

Next, suppose that \( A > 0 \). Then \( y(t) > 0 \) on some maximal interval \( \mathcal{J} = (-\infty, \theta), \theta \in \mathbb{R} \cup \{+\infty\} \). Since \( y(-\infty) = y(\theta) = 0 \), we conclude that \( y(t), \ t \in \mathcal{J} \), reaches its positive absolute maximum at some point \( \zeta \in \mathcal{J} \), where \( y(\zeta) > 0 \) and
\[
y''(\zeta) \leq 0, \quad y'(\zeta) = 0, \quad \int_{-h}^{0} y(\zeta + cs)e^{\lambda cs}d\mu_+(s) \leq \int_{-h}^{0}e^{\lambda cs}d\mu_+(s)y(\zeta), \quad R(\zeta, \bar{y}_\zeta) \leq 0.
\]
However, then (13) yields the following contradiction:
\[
0 = y''(\zeta) - (c - 2\lambda)y'(\zeta) + (\lambda^2 - c\lambda - q)y(\zeta) + \int_{-h}^{0} y(\zeta + cs)e^{\lambda cs}d\mu_+(s) + R(\zeta, \bar{y}_\zeta) \leq
\]
\[
(\lambda^2 - c\lambda - q + \int_{-h}^{0} e^{\lambda cs}d\mu_+(s))y(\zeta) = \chi(\lambda, c)y(\zeta) < 0.
\]
This proves the uniqueness of every non-critical semi-wavefront.

Finally, we consider critical case, \( c = c_* \) (usually more difficult, cf. \([1, \text{Lemma 3}]\)). Then \( \lambda_1(c) = \lambda_2(c) \) and we take \( \lambda = \lambda_1(c) \). We will need the following equivalent form of the relations \( \chi(\lambda, c) = \chi'(\lambda, c) = 0 \):
\[
\lambda^2 - c\lambda - q + \int_{-h}^{0} e^{\lambda cs}d\mu_+(s) = 0, \quad 2\lambda - c + \int_{-h}^{0} cse^{\lambda cs}d\mu_+(s) = 0. \tag{14}
\]
Again invoking Lemma 7, without loss of generality we can assume that 
\[(\psi(t), \psi'(t)) \text{ and also } (\phi(t), \phi'(t)) = -(1, \lambda_1(c))e^{\lambda_1(c)t}(t + O(1)), \ t \to -\infty.\]

Clearly, this implies that \((y(t), y'(t)) = O(1)\) while \(R(t, \tilde{y}_t) = O(\|\tilde{y}_t\|e^{\lambda_1(c)\alpha t})\) as \(t \to -\infty.\)

By Proposition 6.1, we conclude that, for some small \(\epsilon > 0,\)
\[y(t) = B + O(e^{\epsilon t}), \ y'(t) = O(e^{\epsilon t}), \ t \to -\infty, \ y(+\infty) = 0.\]

By interchanging, if necessary, the roles of \(\phi\) and \(\psi,\) we may assume that \(B \geq 0.\) If \(B = 0,\) then the same argument as in the non-critical case with \(A = 0\) shows that \(y(t) \equiv 0\) proving the uniqueness of the semi-wavefront profile. If \(B > 0,\) then \(y(t) > 0\) on some maximal interval \(J = (-\infty, \theta), \ \theta \in \mathbb{R} \cup \{+\infty\}.\) Using (14), we can rewrite equation (13) as follows:

\[0 = y''(t) - (c - 2\lambda)y'(t) + (\lambda^2 - c\lambda - q)y(t) + \int_{-h}^{0} y(t + cs)e^{\lambda cs}d\mu_+(s) + R(t, \tilde{y}_t) =\]

\[y''(t) - (c - 2\lambda)y'(t) + \int_{-h}^{0} [y(t + cs) - y(t)]e^{\lambda cs}d\mu_+(s) + R(t, \tilde{y}_t) =\]

\[y''(t) - \int_{-h}^{0} cs\lambda e^{\lambda cs}d\mu_+(s)y'(t) + \int_{-h}^{0} \int_{0}^{1} csy'(t + cs\sigma)d\sigma e^{\lambda cs}d\mu_+(s) + R(t, \tilde{y}_t) =\]

\[y''(t) + \int_{-h}^{0} \int_{0}^{1} cs[y'(t + cs\sigma) - y'(t)]d\sigma e^{\lambda cs}d\mu_+(s) + R(t, \tilde{y}_t) = 0. \quad (15)\]

Thus
\[y''(t) + \int_{-h}^{0} (cs)^2e^{\lambda cs}d\mu_+(s) \int_{0}^{1} \sigma d\sigma \int_{0}^{1} y''(t + cs\sigma)d\tau + R(t, \tilde{y}_t) = 0. \quad (16)\]

Because of (15), similarly to \(y'(t)\) and \(R(t, \tilde{y}_t),\) the second derivative \(y''(t)\) has exponential decay at \(-\infty\) and \(+\infty.\) Therefore, integrating (16) with respect to \(t\) on \((-\infty, \theta]\) and using Fubini’s theorem, we find that

\[0 = y'(\theta) + \int_{-h}^{0} (cs)^2e^{\lambda cs}d\mu_+(s) \int_{0}^{1} \sigma d\sigma \int_{0}^{1} y'(t + cs\sigma)d\tau + \int_{-\infty}^{\theta} R(t, \tilde{y}_t)dt =\]

\[y'(\theta) + \int_{-h}^{0} cs\lambda e^{\lambda cs}d\mu_+(s) \int_{0}^{1} [y(t + cs\sigma) - y(t)]d\sigma + \int_{-\infty}^{\theta} R(t, \tilde{y}_t)dt =\]

\[y'(\theta) + \int_{-h}^{0} cs\lambda e^{\lambda cs}d\mu_+(s) \int_{0}^{1} y(t + cs\sigma)d\sigma + \int_{-\infty}^{\theta} R(t, \tilde{y}_t)dt.\]

Since all three terms in the latter line are non-negative, we conclude that
\[y'(\theta) = \int_{-h}^{0} cs\lambda e^{\lambda cs}d\mu_+(s) \int_{0}^{1} y(t + cs\sigma)d\sigma = \int_{-\infty}^{\theta} R(t, \tilde{y}_t)dt = 0.\]
In addition, we know that continuous function $R$ satisfies $R(t, \tilde{y}t) \leq 0$ on $(-\infty, \theta]$. This means that $R(t, \tilde{y}t) = 0$ for all $t \in (-\infty, \theta]$ so that $y(t)$ is a bounded solution of the linear equation
\[
y''(t) - (c - 2\lambda)y'(t) + (\lambda^2 - c\lambda - q)y(t) + \int_{-h}^{0} y(t + cs)e^{\lambda cs} d\mu_+(s) = 0, \quad t \leq \theta.
\]
Thus $y(t) \equiv B$ on $(-\infty, \theta]$ and, in particular, $y(\theta) = B > 0$. The obtained contradiction shows that actually $B = 0$ that completes the proof of Theorem 1. □

Appendix

This section contains the proof of Lemma 1.

By (ND) and (14), equation $\chi(z, c) = 0$ does not have real roots when $c = 0$. If $c > 0$, then it can be rewritten in the following equivalent form
\[
z + q - \epsilon z^2 = \int_{-h}^{0} e^{zs} d\mu_+(s), \quad z := \lambda c, \quad \epsilon = c^{-2}. \tag{17}
\]

On the left [respectively, right] side of (17) we have a strictly convex upward [respectively, downward] function, so that equation (17) can have at most two real roots counting multiplicity. In fact, since $p > q$ and positive function $\int_{-h}^{0} e^{zs} d\mu_+(s)$ is non-increasing in $z$, we deduce the existence of $\epsilon_* > 0$ such that (17) has exactly two simple real roots if and only if $\epsilon \in (0, \epsilon_*)$ and has a double real root if and only if $\epsilon = \epsilon_*$. The above analysis implies all conclusions of Lemma 1 except for the last (and key) assertion concerning the dominance of the real zeros $\lambda_1(c) \leq \lambda_2(c)$. Actually, the vertical strip $\lambda_1(c) \leq \Re z \leq \lambda_2(c)$ does not contain any complex zero $w = a + ib, \ b \neq 0$ of $\chi(z, c)$ for otherwise, with $z_1 < 0 \leq z_2$ denoting the roots of $z^2 - cz - q = 0$, we get the following contradiction: $|w^2 - cw - q| = |w - z_1||w - z_2| > |a - z_1||a - z_2| = ca + q - a^2 \geq \int_{-h}^{0} e^{acs} d\mu_+(s) \geq \int_{-h}^{0} e^{wcs} d\mu_+(s)$.

Note that $z_1 < \lambda_1(c) \leq \lambda_1(c) < z_2$.

Now, consider $c > c_*$ and choose $\nu$ such that $\lambda_1(c) < \nu < \lambda_2(c)$. The above argument shows that $|w^2 - cw - q| > \int_{-h}^{0} e^{wcs} d\mu_+(s)$ for all $w \in \Gamma$, where $\Gamma$ is the boundary of the rectangle $[\nu, \zeta] \times [-k, k] \in \mathbb{R}^2 \simeq \mathbb{C}$ with $\zeta$ and $k$ sufficiently large. By Rouché’s theorem this implies that $\chi(z, c)$ and $z^2 - cz - q$ have the same number of zeros with $\Re z > \nu$, i.e. exactly one zero. Finally, if equation $\chi(z, c_*) = 0$ has at least one root $z_0$ with $\Re z_0 > \lambda_1(c_*)$, then by Hurwitz’s theorem from the complex analysis, we conclude that $\chi(z, c)$ also has at least one root $\tilde{z}_0$ with $\Re \tilde{z}_0 > \lambda_1(c)$ for all $c > c_*$ close to $c_*$, a contradiction. This completes the proof of Lemma 1. □

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