The infinity Laplace equation

\[ \langle \nabla u, \nabla |\nabla u|^2 \rangle = 0, \]

was introduced by Aronsson in the 1960’s. The solutions are called infinity harmonic functions. Geometrically, \( u \) is infinity harmonic if and only if the integral curves of its gradient field are parameterized with constant speed.

The equation was largely ignored until the 1990’s when Crandall, Ishii, and Lions’ theory of viscosity solutions for fully nonlinear problems [CIL] gave impetus to a revival that lead to many important works [ACJ], [BB], [Ba], [BLW1], [BLW2], [BEJ], [Bl], [CE], [CEG], [CIL], [CY], [Ex], [EG], [EY], [J], [JK], [JLM1], [JLM2], [LM1], [LM2], [Ob], and [Yu]. In the meantime, some very interesting applications of the infinity Laplace equation have been found in areas such as image processing [CMS], [Sa], mass transfer problems [EG], and shape metamorphism [CEPB].

Here we initiate the study of infinity harmonicity on Riemannian manifolds. Although there are many interesting examples of infinity harmonic functions on Riemannian manifolds, we will work in the broader context of infinity harmonic maps.

**Definition.** A map \( \psi : (M, g) \to (N, h) \) is said to be infinity harmonic if and only if

\[ \Delta_\infty (\psi) \equiv \frac{1}{2} d\psi (\nabla |d\psi|^2) = 0, \]

where

\[ |d\psi|^2 \equiv \sum_{i=1}^{n} h(d\psi(e_i), d\psi(e_i)) \]

is called the energy density of \( \psi \), and \( \{e_i\} \) is an orthonormal basis for \( T_x M \).

This generalizes the concept of infinity harmonic functions on Euclidean space. The definition can also be viewed as the limiting case of the well-known notion of \( p \)-harmonic map [BG] as \( p \to \infty \). (See Proposition 1.2 for details.)

In Section 1, we give some examples of infinity harmonic maps which include some very important and well-known classes such as metric projection onto an orbit of an isometric group action, projections of multiply warped products, totally geodesic maps, isometric immersions, Riemannian submersions, and eigenmaps between spheres.
Section 2 begins with some examples that show that infinity harmonicity is not preserved under composition. Motivated by this and the theory of $p$-harmonicity, we introduce a subclass of infinity harmonic maps called infinity harmonic morphisms, which preserve solutions to the $\infty$-Laplace equation in the following sense.

**Definition.** A map between Riemannian manifolds is said to be an infinity harmonic morphism if and only if it pulls back locally defined infinity harmonic functions to infinity harmonic functions.

This is motivated by the categorically analogous definition of $p$-harmonic morphism ([Fu], [Is], [Lo], and [BL]), and is therefore very appealing. On the other hand, it is a difficult condition to verify. Fortunately, we will provide an alternative characterization of infinity harmonic morphisms that is easier to check. To this end we recall [BE], [BW]

**Definition.** A map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is horizontally weakly conformal with dilation $\lambda : M \to [0, \infty)$ if apart from the points where $d\varphi = 0$, $d\varphi_x$ is onto and

$$h(d\varphi_x(X), d\varphi_x(Y)) = \lambda^2(x)g_x(X, Y)$$

for all horizontal vector fields on $M$.

A horizontally weakly conformal map with dilation $\lambda$ having vertical gradient is a horizontally homothetic map. A horizontally weakly conformal map without a critical point is called a horizontally conformal submersion and a horizontally homothetic map without a critical point is called a horizontally homothetic submersion.

**Theorem 0.2.** A map between Riemannian manifolds is an infinity harmonic morphism if and only if it is a horizontally weakly conformal, infinity harmonic map, which is precisely a horizontally homothetic map.

In Section 3, we give several methods to construct infinity harmonic maps into Euclidean spaces, characterize those immersions which are infinity harmonic maps, and show that isometrically immersing the target manifold of a map into another manifold does not change the infinity harmonicity of the map. Section 4 is devoted to constructions of infinity harmonic maps into spheres. We use ideas similar to those of Smith’s in finding harmonic maps into spheres to find infinity harmonic maps into spheres by reduction of partial differential equations into ordinary differential equations. Finally, in Section 5 we examine the effect of a conformal change on the infinity Laplacian to obtain formulas for the infinity Laplace equation on spheres and on hyperbolic spaces in terms of the infinity Laplacian on Euclidean space.

1. **Examples of Infinity Harmonic Maps**

For $p > 1$, a $p$-harmonic map is a map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds such that $\varphi|\Omega$ is a critical point of the $p$-energy

$$E_p(\varphi, \Omega) = \frac{1}{p} \int_{\Omega} |d\varphi|^p \, dx$$
for every compact subset \( \Omega \) of \( M \). Locally, \( p \)-harmonic maps are solutions of the following systems of PDEs:

\[
\Delta_p(\varphi) = |d\varphi|^{p-2} \Delta_2(\varphi) + (p-2)|d\varphi|^{p-4} d\varphi(\text{grad}|d\varphi|) = 0,
\]

where \( \Delta_2(\varphi) = \text{Trace}_g \nabla d\varphi \) denotes the tension field of \( \varphi \). When \( p = 2 \), we get the familiar notion of harmonic maps which include geodesics, harmonic functions, minimal isometric immersions, and Riemannian submersions with minimal fibers as special cases (See [EL1], [EL2], [EL3], and [SY]).

The definition of infinity harmonic map can be viewed as the limiting case of the notion of \( p \)-harmonic map as \( p \to \infty \) in the following sense.

**Proposition 1.2.** For any \( C > 0 \),

\[
\lim_{p \to \infty} \sup_{\varphi \in H^p_C} \Delta_\infty(\varphi) = 0
\]

where \( H^p_C \) is the class of all \( p \)-harmonic maps \( \varphi \) with \( |d\varphi|^2 |\Delta_2(\varphi)| \leq C \).

**Proof.** Dividing the \( p \)-harmonic equation by \( (p-2)|d\varphi|^{p-4} \) gives

\[
\frac{|d\varphi|^2 \Delta_2(\varphi)}{(p-2)} + \frac{1}{2} d\varphi(\text{grad}|d\varphi|^2) = 0.
\]

So within the class of \( p \)-harmonic maps \( \varphi \) with \( |d\varphi|^2 |\Delta_2(\varphi)| \leq C \), we can make the \( \left| \frac{1}{2} d\varphi(\text{grad}|d\varphi|^2) \right| \) as small as we please by letting \( p \to \infty \). \( \square \)

Another relationship between \( p \)-harmonic and infinity harmonic maps is the following.

**Proposition 1.3.** If a map is \( p \)-harmonic for two different \( p \) values, then it is \( \infty \)-harmonic; An infinity harmonic map is also a harmonic map if and only if it is a \( p \)-harmonic map for any \( p \neq 2 \).

Some very important and familiar classes of maps are infinity harmonic.

**Example 1.4. [Infinity harmonic functions]** A real-valued function

\[
u : (M, g) \rightarrow \mathbb{R}
\]

on a Riemannian manifold is infinity harmonic if and only if \( u \) is a solution of infinity Laplace equation:

\[
\Delta_\infty u = \frac{1}{2} d\nu(\text{grad}|\text{grad} u|^2)
\]

\[= \frac{1}{2} g(\text{grad} u, \text{grad}|\text{grad} u|^2) = 0.
\]

For \( u : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R} \), this becomes Aronsson’s infinity Laplace equation.

**Proposition 1.6.** Let \( G \) act isometrically on \( M \). Then metric projection onto an orbit of \( G \) is an infinity harmonic map, wherever it is defined and smooth.
Proof. Let $O$ be an orbit of $G$, and let metric projection, $\pi : N \rightarrow O$ be defined and smooth on the subset $N$ of $M$. Since isometries take geodesic segments to geodesic segments, $G$ acts by symmetries of $\pi$. In particular, the vertical and horizontal spaces of $\pi$ are preserved by $G$.

If $\gamma$ is a horizontal curve for $\pi$, then

$$g_t \gamma (0) = \gamma (t),$$

for some one parameter family of isometries $g_t \in G$. Since multiplication by $g_t$ preserves the horizontal and vertical distributions of $\pi$, it follows that

$$|d\pi|^2 (x) = |d\pi|^2 (g_t x)$$

for all $t$. In particular, grad$|d\pi|^2$ is vertical. □

The maps that come from this proposition are typically not infinity harmonic morphisms. (Cf Example 1.10.)

Example 1.7. [Maps with constant energy density] Any map with constant energy density,

$$|d\phi|^2 = \text{constant},$$

is infinity harmonic. This class includes

- any affine map between Euclidean spaces $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\phi (X) = AX + b$, where $A = (a_{ij})$ is an $n \times m$ matrix and $b \in \mathbb{R}^n$ is fixed. In this case, the energy density is $|d\phi|^2 = \sum_{i=1, j=1}^{n, m} a_{ij}^2$;

- any totally geodesic map between Riemannian manifolds. Recall that a map $\phi : (M^n, g) \rightarrow (N^n, h)$ is totally geodesic if its second fundamental form vanishes identically, i.e., $\nabla d\phi = 0$. It is not difficult to see that $\phi$ is totally geodesic if and only if it carries geodesics to geodesics. It is well known [ER] that a totally geodesic map has constant rank and constant energy density;

- any eigenmap between spheres $\phi : S^m \rightarrow S^n$. Recall that an eigenmap is a harmonic map between spheres with constant energy density, which can be characterized as the restriction to $S^m$ of a map $F : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{n+1}$ whose components are harmonic homogeneous polynomial of a common degree [ER];

- any isometric immersion $\phi : (M^n, g) \rightarrow (N^n, h)$. In this case, the energy density $|d\phi|^2 = m$. Recall that an isometric immersion is a harmonic map if and only if the immersion is minimal;

- any Riemannian submersion $\phi : (M^n, g) \rightarrow (N^n, h)$. In this case, the energy density $|d\phi|^2 = n$. Note that a Riemannian submersion is a harmonic map if and only if it has minimal fibers;

- the globally defined nonlinear complex-valued functions $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^2 \equiv \mathbb{C}$ with $\phi (x_1, \ldots, x_m) = \lambda_1 e^{ix_1} + \ldots + \lambda_m e^{ix_m}$, where $\lambda_k (k = 1, \ldots, m)$ are constant real numbers. One can easily check that this map has constant energy density $|d\phi|^2 = \sum_{k=1}^{m} |\lambda_k|^2$. Note that a map of this class does not belong to any of the above classes, for instance, $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by $\phi (x, y, z) = (\cos x + \cos y + \cos z, \sin x + \sin y + \sin z)$ is a globally defined smooth nonlinear infinity harmonic map which is neither an isometric immersion nor a Riemannian submersion;
Let \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m \) be defined by
\[
\varphi(x_1, \ldots, x_m) = (\cos x_1 + \sin x_2, \cos x_2 + \sin x_3, \ldots, \cos x_{m-1} + \sin x_m, \cos x_m + \sin x_1).
\]
A straightforward computation gives the energy density \( |d\varphi|^2 = m \).

**Example 1.9.** [Infinity harmonic curves] Any regular curve \( \gamma : (a, b) \rightarrow (M^m, g) \) is an infinity harmonic map provided it is parametrized by arc length.

The following example provides a large class of infinity harmonic maps with nonconstant energy density.

**Example 1.10.** [Projection of multiply warped products] Recall that a multiply warped product of Riemannian manifolds \((B, g_B)\) and \((F_1, h_1), \ldots, (F_k, h_k)\) is the smooth manifold \( M = B \times F_1 \times \ldots \times F_k \) with the metric
\[
g_B + \lambda_1^2 h_1 + \ldots + \lambda_k^2 h_k,
\]
where \( \lambda_1, \ldots, \lambda_k : B \rightarrow (0, \infty) \) are called warping functions, and we have written \( h_i \) for the pull back of \( h_i \) under \( \pi_B : B \times F_1 \times \ldots \times F_k \rightarrow B \). We denote the resulting Riemannian manifolds by \( B \times \lambda_1^2 F_1 \times \ldots \times \lambda_k^2 F_k \).

Let
\[
\pi : B \times \lambda_1^2 F_1 \times \ldots \times \lambda_k^2 F_k \rightarrow (F_1 \times \ldots \times F_k, h_1 + \ldots + h_k)
\]
be projection. A simple computation gives that the energy density of \( \pi \) is
\[
|d\pi|^2 = \lambda_1^{-2} + \ldots + \lambda_k^{-2}.
\]
Since the gradients of all of the \( \lambda_i \)'s are tangent to the “B-factors” they are all vertical for \( \pi \), and \( \pi \) is infinity harmonic. In particular we have

- the projection \( \pi : (\mathbb{R}^3, g_{Sol}) \rightarrow (\mathbb{R}^2, dx^2 + dy^2) \) with \( \pi(x, y, z) = (x, y) \) is an infinity harmonic harmonic map, where \((\mathbb{R}^3, g_{Sol})\) denotes the Sol space, one of Thurston’s eight 3-dimensional geometries, which can be viewed as
\[
(\mathbb{R}^3, g_{Sol}) = (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, e^{2z}dx^2 + e^{-2z}dy^2 + dz^2).
\]
- the projection from 3-sphere onto the Clifford torus
\[
\varphi : S^3 \setminus \{\Gamma_1, \Gamma_2\} \equiv ((0, \frac{\pi}{2}) \times S^1 \times S^1, dt^2 + \sin^2 t \, d\theta_1^2 + \cos^2 t \, d\theta_2^2) \rightarrow S^1 \times S^1
\]
with \( \varphi(t, \theta_1, \theta_2) = (\theta_1, \theta_2) \) is an infinity harmonic submersion with nonconstant energy density \( |d\varphi|^2 = \frac{1}{\sin^2 \tau} + \frac{1}{\cos^2 \tau} \).

**Proposition 1.11.** A submersion \( \pi : (M^m, g) \rightarrow (N^n, h) \) is infinity harmonic if and only if the gradient of the energy density of \( \pi \) is vertical. In particular, a horizontally conformal submersion is infinity harmonic if and only if it is a horizontally homothetic submersion.

**Proof.** Suppose that \( \pi \) is a submersion. Since \( \pi \) is infinity harmonic if and only if
\[
d\pi(\nabla |d\pi|^2) = 0,
\]
it follows that \( \pi \) is infinity harmonic if and only if the gradient of its energy density is vertical.

If \( \pi \) is a horizontally conformal submersion with dilation function \( \lambda \), then
\[
|d\pi|^2 = \lambda^2 \dim (N).
\]
So \( \pi \) is infinity harmonic if and only if \( \nabla \lambda \) is vertical, i.e. \( \pi \) is a horizontally homothetic submersion. \( \square \)
2. Infinity Harmonic Morphisms

As the following example shows, infinity harmonicity is not preserved under composition of infinity harmonic maps.

Example 2.1. Let \((M, g)\) and \((N, h)\) be Riemannian manifolds, \(I\) an interval with the metric \(dt\), and \(\alpha : (M, g) \to I\) and \(\beta : (N, h) \to I\) warping functions, giving us a doubly warped product metric \((I \times_\alpha M \times_\beta N, dt + \alpha^2 g + \beta^2 h)\).

Let 
\[
\psi : (I \times_\alpha M \times_\beta N, dt + \alpha^2 g + \beta^2 h) \to (M \times N, g + h)
\]
be the projection map on \(M \times N\). We saw in Example 1.10 that \(\psi\) is infinity harmonic.

Now let \(\text{dist}_{(m,n)} : M \times N \to \mathbb{R}\) be a distance function. Let \((1, m', n)\) be a perturbation in the \(M\)-direction of \((1, m, n)\) and \((1, m, n')\) a perturbation in the \(N\)-direction of \((1, m, n)\). The differential, \(d(\text{dist}_{(m,n)} \circ \psi)\) sends vectors in \(T_{(1,m',n)}(1) \times M \times \{n\}\) of length \(|\alpha|\) to vectors of length 1 and vectors in \(T_{(1,m,n')}(1) \times \{m\} \times N\) of length \(|\beta|\) to vectors of length 1. So \(|\nabla(\text{dist} \circ \psi)|_{(1, m', n)} = \frac{1}{|\alpha|}\) and \(|\nabla(\text{dist} \circ \psi)|_{(1, m, n')} = \frac{1}{|\beta|}\). If we move along a horizontal curve from \((1, m', n)\) to \((1, m, n')\), \(|\nabla(\text{dist} \circ \psi)|\) changes so \(|\nabla(\text{dist} \circ \psi)|\) must have a point where its not vertical. So \(\text{dist}_{(m,n)} \circ \psi\) is not infinity harmonic.

Knowing that infinity harmonicity is not preserved under composition of infinity harmonic maps and recalling that a \(p\)-harmonic morphism is defined to be a map between Riemannian manifolds that pulls back (locally defined) \(p\)-harmonic functions to \(p\)-harmonic functions, we recall from our introduction

Definition 2.2. A map \(\varphi : (M, g) \to (N, h)\) is said to be an infinity harmonic morphism if it pulls back locally defined infinity harmonic functions to infinity harmonic functions.

In this section we prove the characterization theorem of infinity harmonic morphisms stated in the introduction.

Theorem 2.3. A map between Riemannian manifolds is an infinity harmonic morphism if and only if it is a horizontally weakly conformal, infinity harmonic map, which is precisely a horizontally homothetic map.

Using this characterization, we see that the following are examples of infinity harmonic morphisms.

Example 2.4. Riemannian Submersions are infinity harmonic morphisms.

Example 2.5. The projection of a warped product onto the base is a Riemannian submersion. The projection onto the fiber is a horizontally homothetic submersion and hence an infinity harmonic morphism. The next three examples are important special cases of this idea.

Example 2.6. Radial projection of \(\mathbb{R}^{n+1} \setminus \{0\}\) onto \(S^n\) is an infinity harmonic morphism. In this example, \(\lambda(x) = \frac{1}{|x|}\).

Example 2.7. \(S^2 \setminus \{\text{Poles}\}\) onto a circle with \(\lambda(\phi) = \frac{1}{\sin \phi}\), where \(x = (r, \theta, \phi)\) are the spherical coordinates of \(x\).
Example 2.8. Projection of a cone without an apex onto a circle with \( \lambda(r) = \frac{1}{r} \) where \( x = (r, \theta) \) are the polar coordinates of \( x \).

Example 2.9. Infinity harmonic functions on Riemannian manifolds can be viewed as horizontally weakly conformal, infinity harmonic maps, and hence are infinity harmonic morphisms.

We refer the reader to [OW] for other examples of horizontally homothetic submersions, where it is shown

Theorem 2.10. Any nonconstant horizontally homothetic submersion from a compact nonnegatively curved manifold is a rescaling of a Riemannian submersion.

We prove Theorem 2.3 with three lemmas. The first of which is as follows.

Lemma 2.11. If \( \pi : E \rightarrow B \) is an infinity harmonic morphism, then \( \pi \) is a horizontally weakly conformal map.

Before proving this we study the linear case.

Proposition 2.12. A linear map \( \varphi : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n \) is an infinity harmonic morphism if and only if it is a horizontally conformal submersion. In other words, it can be written as the composition of a homothety, an isometry, and an orthogonal projection.

In fact, if \( \varphi \) is onto but not horizontally weakly conformal, then \( \varphi^* (\text{dist} (0, \cdot)) \) is not infinity harmonic and

\[
\limsup_{p \to 0} |\Delta_\infty [\varphi^* (\text{dist} (0, \cdot))]|_p = \infty.
\]

Proof. First we consider the case when \( \varphi : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n \) is onto.

\[
\langle \nabla (f \circ \varphi), w \rangle = d(f \circ \varphi)(w) = df[d\varphi(w)] = \langle \nabla f, d\varphi(w) \rangle
\]

So

\[
\nabla (f \circ \varphi) = \sum_i \langle \nabla f, d\varphi(e_i) \rangle e_i
\]

and

\[
|\nabla (f \circ \varphi)|^2 = \sum_i (\nabla (f \circ \varphi), e_i)^2 = \sum_i (\nabla f, d\varphi(e_i))^2.
\]

Thus

\[
\left\langle \nabla |\nabla (f \circ \varphi)|^2, e_\alpha \right\rangle = d |\nabla (f \circ \varphi)|^2 [e_\alpha]
\]

\[
= \sum_i d \left( (\nabla f, d\varphi(e_i))^2 \right) [e_\alpha]
\]

\[
= 2 \sum_i \langle \nabla f, d\varphi(e_i) \rangle [d (\nabla f, d\varphi(e_i))] [e_\alpha]
\]

\[
= 2 \sum_i \langle \nabla f, d\varphi(e_i) \rangle [(\nabla_{e_\alpha} (\nabla f \circ \varphi), d\varphi(e_i)) + \langle \nabla f, \nabla_{e_\alpha} d\varphi(e_i) \rangle],
\]
where \( \langle \nabla f \circ \phi, \nabla_{e\alpha} \nabla f, d\phi, (e_i) \rangle \) and \( \nabla_{e\alpha} d\phi (e_i) \) are being viewed as vector fields along \( \phi \).

Since \( \phi \) is linear, \( d\phi (e_i) \) is a constant vector field, and the second term vanishes. So

\[
\langle \nabla |\nabla (f \circ \phi)|^2, e\alpha \rangle = 2 \sum_i \langle \nabla f, d\phi (e_i) \rangle \left( \langle \nabla_{e\alpha} (\nabla f \circ \phi), d\phi (e_i) \rangle \right)
\]

Combining this with

\[
\nabla |\nabla (f \circ \phi)|^2 = 2 \sum_{\alpha,i} \langle \nabla f, d\phi (e_i) \rangle \left( \langle \nabla_{e\alpha} (\nabla f \circ \phi), d\phi (e_i) \rangle \right) e_\alpha
\]

we get

\[
\Delta_\infty [\phi^* (f)] = \frac{1}{2} \left( \langle \nabla |\nabla (f \circ \phi)|^2, \nabla (f \circ \phi) \rangle \right)
\]

(2.12)  

Now set

\[
f \left( \overline{x} \right) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}
\]

Then

\[
\nabla f = \frac{1}{f} (x_1, x_2, \ldots, x_n)
\]

Since \( f \) is a distance function we have

\[
\nabla \nabla f = \nabla \frac{\nabla f}{f}
\]

for all \( z \perp \nabla f \).

To evaluate \( \Delta_\infty (f \circ \phi) \) at \( \overline{\phi} \in \mathbb{R}^{n+k} \) using (2.12) choose an orthonormal basis \( \{v_0, v_1, v_2, \ldots, v_{n+k-1}\} \) for \( T_{\overline{\phi}} \mathbb{R}^{n+k} \) so that

\[
\langle d\phi (v_i), \nabla f \rangle |_{\phi(\overline{\phi})} = 0, \text{ for } i = 1, 2, 3, \ldots, n + k - 1
\]

Then

\[
\Delta_\infty (f \circ \phi)_{\overline{\phi}} = \frac{1}{2} \left( \langle \nabla |\nabla (f \circ \phi)|^2, \nabla (f \circ \phi) \rangle \right)_{\overline{\phi}}
\]

\[
= \sum_{\alpha,i} \langle \nabla f, d\phi (v_i) \rangle \left( \langle \nabla_{e\alpha} (\nabla f \circ \phi), d\phi (v_i) \rangle \right) \langle \nabla f, d\phi (v_\alpha) \rangle
\]

Since span\{\(v_1, \ldots, v_{n+k-1}\}\} contains the vertical space for \( \phi \), \( v_0 \) is horizontal for \( \phi \). It follows that

\[
\Delta_\infty (f \circ \phi)_{\overline{\phi}} = \langle \nabla f, d\phi (v_0) \rangle^2 \left( \langle \nabla d\phi (v_0), \nabla f \rangle, d\phi (v_0) \rangle \right).
\]

Since \( \nabla_{d\phi (v_0)} \nabla f \) is proportional to the component, \( d\phi (v_0) \perp \), of \( d\phi (v_0) \) that is perpendicular to \( \nabla f \), it follows that \( \Delta_\infty (f \circ \phi)_{\overline{\phi}} = 0 \) if and only if \( d\phi (v_0) \) is proportional to \( \nabla f |_{\overline{\phi}} \). This is equivalent to saying that \( d\phi \) maps the orthogonal spaces \( \text{span} \{v_0\} \) and \( \text{span} \{v_1, v_2, \ldots, v_{n+k-1}\} \) to the orthogonal spaces \( \text{span} \{\nabla f |_{\overline{\phi}}\} \) and
span $\{(\nabla f)_p\}$. By varying $p$ we can make $\nabla f|_p$ point in any direction, and it follows that $d\varphi = \varphi$ preserves all angles in its horizontal space. So it is a weakly conformal submersion as desired.

If $\varphi$ is onto but not horizontally weakly conformal, then as we have seen $\Delta_{\infty}(f \circ \varphi)_{\tilde{p}} \neq 0$ for some $\tilde{p}$, and

$$\Delta_{\infty}(f \circ \varphi)_{\tilde{p}} = \langle \nabla f, d\varphi(v_0) \rangle^2 \left[ \langle \nabla d\varphi(v_0) \nabla f, d\varphi(v_0) \rangle \right]$$

$$= \langle \nabla f, d\varphi(v_0) \rangle^2 \left[ \langle \nabla d\varphi(v_0) \nabla f, d\varphi(v_0) \rangle \right]$$

where we have again used the fact that $\nabla d\varphi(v_0)\nabla f$ is proportional to $d\varphi(v_0)\perp$. Therefore

$$\Delta_{\infty}(f \circ \varphi)_{\tilde{p}} = \frac{1}{\theta} \langle \nabla f, d\varphi(v_0) \rangle^2 |d\varphi(v_0)|^2 \sin^2 \langle (d\varphi(v_0), \nabla f) \rangle$$

Letting $\tilde{p}$ approach the origin along a radial line, all quantities on the right hand side stay fixed, except, $\frac{1}{\theta}$ which goes to $\infty$. It follows that

$$\limsup_{\tilde{p} \to 0} \left| \Delta_{\infty}(f \circ \varphi)_{\tilde{p}} \right| = \infty.$$ 

Now suppose $\varphi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ is a linear infinity harmonic morphism, that is not onto. We may post compose with an orthogonal transformation $\mathbb{R}^n \to \mathbb{R}^n$ to obtain a linear infinity harmonic morphism whose image is contained in a coordinate subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$. Applying the result just proven to $\varphi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ we see that $\varphi$ is a horizontally conformal linear submersion.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the distance function from $(0, \ldots, 0, \varepsilon) \notin \text{Im}(\varphi)$. Then the curve

$$c : t \mapsto (t, 0, \ldots, 0)$$

has the same image as an integral curve, $\gamma$, of $\nabla f|_{\mathbb{R}^k \times \{0\}}$, only the velocity field of $\gamma$ at $c(t)$ is

$$\left( \frac{t}{\sqrt{t^2 + \varepsilon}}, 0, 0, \ldots, 0 \right)$$

Notice in particular that this integral curve of $\nabla f|_{\mathbb{R}^k \times \{0\}}$ is not parameterized by arc length, and that $\nabla (\varphi^*(f))$ is a horizontal lift of $\nabla f|_{\mathbb{R}^k \times \{0\}}$. Since $\varphi : \mathbb{R}^{n+k} \to \mathbb{R}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ is a horizontally conformal linear submersion it follows that $\nabla (\varphi^*(f))$ will also have an integral curve that is not parameterized by arc length. So $\varphi^*(f)$ is not infinity harmonic and $\varphi$ is not an infinity harmonic morphism.

To estimate the infinity Laplacian of $\varphi^*(f)$ in this event, we compute

$$|\nabla f|_{\mathbb{R}^k \times \{0\}}|^2 = \frac{t^2}{t^2 + \varepsilon}$$

$$\nabla |\nabla f|_{\mathbb{R}^k \times \{0\}}|^2 = \left( \frac{2t (t^2 + \varepsilon) - 2t^2}{(t^2 + \varepsilon)^2}, 0, \ldots, 0 \right)$$

$$= \left( \frac{2t \varepsilon}{(t^2 + \varepsilon)^2}, 0, \ldots, 0 \right)$$

$$\langle \nabla |\nabla f|_{\mathbb{R}^k \times \{0\}}|^2, \nabla f|_{\mathbb{R}^k \times \{0\}} \rangle = \frac{2t \varepsilon}{(t^2 + \varepsilon)^{3/2}}$$
So if \( t^2 = \varepsilon \) we have
\[
\left\langle \nabla \left| \nabla f|_{\mathbb{R}^k \times \{0\}} \right|^2, \nabla f|_{\mathbb{R}^k \times \{0\}} \right\rangle = O\left( \frac{\varepsilon^2}{\varepsilon^{5/2}} \right) = O\left( \frac{1}{\varepsilon^{1/2}} \right)
\]

So if \( \varphi \) is linear, horizontally weakly conformal, and not onto, then we can find infinity harmonic functions on the target that pull back to functions with arbitrarily large infinity laplacians.

Conversely, if \( \varphi : \mathbb{R}^{n+k} \to \mathbb{R}^n \) is an linear horizontally conformal submersion and \( f : \mathbb{R}^n \to \mathbb{R} \) is infinity harmonic, then \( \nabla \left( \varphi^* (f) \right) \) is obtained as a horizontal lift of \( \nabla f \). Since the integral curves of \( \nabla f \) are parameterized by arc length and \( \varphi : \mathbb{R}^{n+k} \to \mathbb{R}^n \) is a linear horizontally conformal submersion, it follows that the integral curves of \( \nabla \left( \varphi^* (f) \right) \) are also parameterized by arc length and hence that \( \varphi^* (f) \) is infinity harmonic and that \( \varphi \) is an infinity harmonic morphism. \( \square \)

To prove Lemma 2.11 we combine our characterization of linear infinity harmonic morphisms with the principle that the set of infinity harmonic functions is closed in the \( C^1 \)-topology.

This principle is embodied in the next two propositions.

**Proposition 2.13.** The set of infinity harmonic functions is closed in the \( C^1 \)-topology. I.e. if \( f : U \to \mathbb{R} \) is any locally defined real valued \( C^2 \)-function on \( M \) that is not infinity harmonic, then there is an \( \varepsilon > 0 \) so that if \( h \) is any \( C^2 \)-function with
\[
|f(x) - h(x)| < \varepsilon, \quad \text{and} \quad |df(v) - dh(v)| < \varepsilon
\]
for all \( x \in U \) and all unit vectors \( v \in TM \), then \( h \) is not infinity harmonic.

**Proof.** Since \( f \) is not infinity harmonic, there is an integral curve \( \gamma : [a, b] \to U \) of \( \nabla f \) and a constant \( M > 0 \) so that
\[
\left| \nabla f_{\gamma(t)} \right| - \left| \nabla f_{\gamma(t)} \right| > M (b - a).
\]  

By continuity, there is a neighborhood \( V_a \) of \( \gamma(a) \) so that any integral curve \( c \) of \( \nabla f \) passing through \( V_a \) can be parameterized on \( [a, b] \) and satisfies
\[
\left| \nabla f_{c(t)} \right| - \left| \nabla f_{c(t)} \right| > \frac{M}{2} (b - a)
\]
\[
\left| \nabla f_{c(t)} \right| - \left| \nabla f_{c(t)} \right| \leq \frac{M}{100} (b - a).
\]

Let \( \Phi \) be the flow of \( \nabla f \) and set
\[
V = \cup_{t \in [a, b]} \Phi_t(V_a).
\]

Set \( \varepsilon < \frac{M}{100} (b - a) \), and choose \( h \) so that
\[
|f(x) - h(x)| < \varepsilon, \quad \text{and} \quad |df(v) - dh(v)| < \varepsilon
\]
for all \( x \in U \) and all unit vectors \( v \in TM \). Require further that \( h \) is close enough to \( f \) in the \( C^1 \)-topology so that the integral curve \( \beta \) of \( \nabla h \) that starts at \( \gamma(a) \) is
parameterized on \([a, b]\) and stays in \(V\). Then
\[
||\nabla h_{\beta(b)}| - |\nabla h_{\beta(a)}|| \geq ||\nabla f_{\beta(b)}| - |\nabla f_{\beta(a)}|| - 2\epsilon \\
\geq ||\nabla f_{\gamma(b)}| - |\nabla f_{\gamma(a)}|| - 4\epsilon \\
\geq \frac{M}{4} (b - a).
\]

So \(h\) is not infinity harmonic. \(\square\)

**Definition 2.14.** We call a map \(\Psi : M \rightarrow N\) and \(\epsilon\)-isometry provided,
\[
||d\Psi(v)| - 1| < \epsilon
\]
for all unit vectors \(v\).

An argument similar to the previous proposition gives us

**Proposition 2.15.** Let \(f\) be a locally defined real valued \(C^2\)-function on \(M\) that is not infinity harmonic. There is an \(\epsilon > 0\) so that if \(\Psi : E \rightarrow B\) is an \(\epsilon\)-isometry, then \(\Psi^* (f)\) is not infinity harmonic.

More generally, if \(\pi : E \rightarrow B\) is a \(C^2\)-map and \(f : B \rightarrow \mathbb{R}\) is such that \(\pi^* (f)\) is not infinity harmonic, then there is an \(\epsilon > 0\) so that if \(\Phi : E \rightarrow E\) and \(\Psi : B \rightarrow B\) are \(\epsilon\)-isometries, then \((\Psi \circ \pi \circ \Phi)^* (f)\) is not infinity harmonic.

We can now offer the proof of Lemma 2.11.

**Proof.** Suppose \(\pi : E \rightarrow B\) is a map that is not horizontal weakly conformal. Let \(\pi (\tilde{p}) = p\) and suppose that \(d\pi_{\tilde{p}}\) is onto, but not horizontal weakly conformal. Let
\[
f : T_{\tilde{p}}E \rightarrow \mathbb{R}
\]
be
\[
f = \text{dist} (0, \cdot) \circ d\pi_{\tilde{p}}
\]
and
\[
h : T_{\tilde{p}}E \rightarrow \mathbb{R}
\]
be
\[
h = \text{dist}_p \circ \pi \circ \exp_{\tilde{p}}
\]
From the proof of Proposition 2.12 \(f\) is not infinity harmonic. The \(C^1\)-distance between \(f|_{B(0, r)}\) and \(h|_{B(0, r)}\) goes to 0 like \(O (r^2)\) as \(r \rightarrow 0\); so \(h\) is not infinity harmonic.

Combining the facts that \(h\) is not infinity harmonic, \(\exp_{\tilde{p}}^{-1}\) is an \(O (r^2)\)-isometry on \(B (0, r)\), and
\[
\left(\exp_{\tilde{p}}^{-1}\right)^* (h) = \pi^* (\text{dist} (p, \cdot))
\]
we see that \(\pi^* (\text{dist}_p)\) is not infinity harmonic as desired.

The reader may be concerned that we “run out of room” for this argument, since we have to take \(r\) very small to make it work. This is not a concern, since \(\Delta_{\infty} (f)\) becomes arbitrarily large (in places) near the origin.

Now suppose that \(d\pi_{\tilde{p}}\) is nonzero and not onto. The above argument shows that it is horizontally weakly conformal. As in the proof of Proposition 2.12 we take \(v \in T_{\tilde{p}}B\), to be perpendicular to \(\text{Im} [d\pi_{\tilde{p}}]\) and very small. We saw that the infinity laplacians of \(\text{dist} (v, \cdot)\) and
\[
f : T_{\tilde{p}}E \rightarrow \mathbb{R}
\]
be
\[
f = \text{dist}_v \circ d\pi_{\tilde{p}}
\]
both can be made arbitrarily large by choosing the norm of $v$ to be small enough. Now let
\[
h : T_{p}E \to \mathbb{R} \text{ be } h = \text{dist}_v \circ \exp_p^{-1} \circ \pi \circ \exp_p.\]
The $C^1$-distance between $f|_{B(0,r)}$ and $h|_{B(0,r)}$ is $O(r^2)$ as $r \to 0$; so $h$ is not infinity harmonic. Since
\[
(\exp_p^{-1})^* (h) = \pi^* (\text{dist}_v \circ \exp_p^{-1}),
\]
and $\exp_p^{-1}$ is an $O(r^2)$-isometry on $B(0,r)$ we see that $\pi^* (\text{dist}_v \circ \exp_p^{-1})$ is not infinity harmonic, and in fact has infinity laplacians that are arbitrarily large if the norm of $v$ is small enough.

Finally notice that the $C^1$-distance between $\pi^* (\text{dist}_v \circ \exp_p^{-1})$ and $\pi^* \left( (\text{dist}_{\exp_p(v)}) \right)$ converges to 0 as the norm of $v$ goes to zero. So $\pi^* \left( (\text{dist}_{\exp_p(v)}) \right)$ can not be infinity harmonic, even though $\text{dist}_{\exp_p(v)}$ is infinity harmonic. So infinity harmonic morphisms are horizontally weakly conformal maps. \hfill \Box

**Lemma 2.16.** If $\pi : E \to B$ is an infinity harmonic morphism, then $\pi$ is an infinity harmonic map.

**Proof.** By Lemma 2.11, $\pi$ is a horizontally weakly conformal map. Let $\lambda$ be the dilation of $\pi$. Then, for any function $f$ locally defined on $B$, we have
\[
|\nabla (f \circ \pi)|^2 = g^{ij} (f \circ \pi)_i (f \circ \pi)_j = g^{ij} f_\alpha \pi^\alpha \pi^\beta g_{\alpha\beta}
\]
\[
= \lambda^2 (h^{\alpha\beta} \circ \pi) f_\alpha f_\beta = \lambda^2 (|\nabla f|^2 \circ \pi),
\]
where the third equality was obtained by using the horizontal weak conformality equation $g^{ij} \pi_i^{\alpha} \pi_j^{\beta} = \lambda^2 (h^{\alpha\beta} \circ \pi)$. It follows that
\[
\nabla (|\nabla (f \circ \pi)|^2) = (\nabla \lambda^2) (|\nabla f|^2 \circ \pi) + \lambda^2 \nabla (|\nabla f|^2 \circ \pi),
\]
and hence
\[
\Delta^M_\infty (f \circ \pi) = \frac{1}{2} \langle \nabla (f \circ \pi), \nabla |\nabla (f \circ \pi)|^2 \rangle
\]
(2.16)
\[
= \frac{1}{2} \langle \nabla (f \circ \pi), (\nabla \lambda^2)(|\nabla f|^2 \circ \pi) \rangle
\]
\[
+ \frac{1}{2} \langle \nabla (f \circ \pi), \lambda^2 \nabla (|\nabla f|^2 \circ \pi) \rangle.
\]
\[
= \frac{1}{2} (\langle |\nabla f|^2 \circ \pi \rangle df (d\pi (\nabla \lambda^2))
\]
\[
+ \frac{1}{2} \lambda^4 (\nabla f, \nabla |\nabla f|^2) h \circ \pi
\]
for any function (locally) defined on $B$.

Now choose $f$ to be a (locally defined) distance function. The second term vanishes by the infinity harmonicity of $f$. Since $f \circ \pi$ is infinity harmonic,
\[
0 = \Delta^M_\infty (f \circ \pi)
\]
\[
= \frac{1}{2} df (d\pi (\nabla \lambda^2)),
\]
for any locally defined distance function $f$ on $B$. Therefore
\[ d\pi(\nabla \lambda^2) = 0, \]
and $\pi$ is a horizontally homothetic map. Applying Proposition 1.11 we obtain the lemma.

\begin{lemma}
A horizontally weakly conformal, infinity harmonic map is an infinity harmonic morphism.
\end{lemma}

\begin{proof}
Suppose $\pi : E \to B$ is a horizontally weakly conformal infinity harmonic map, and $f : B \to \mathbb{R}$ is any (locally defined) function on $B$. It follows from Proposition 1.11 that $\pi$ is horizontally weakly conformal with dilation $\lambda$ having vertical gradient. At points where $\pi$ is submersive, we have as in the proof of Lemma 2.16
\[ |d(f \circ \pi)|^2 = \lambda^2(|df|^2 \circ \pi). \]
At points where $\pi$ is critical, $d\pi$ is 0 so $|d(f \circ \pi)|^2 = 0$. Since $\lambda$ is also zero at these points we have
\[ |d(f \circ \pi)|^2 = \lambda^2(|df|^2 \circ \pi). \]
in all cases.

Using Equation (2.16) and the fact that $\lambda$ has vertical gradient we have
\[ \Delta^M_\infty(f \circ \pi) = \frac{1}{2} \lambda^4 \langle \nabla f, \nabla |\nabla f|^2 \rangle_h \circ \pi, \]
for any $f$ (locally) defined on $B$. This implies that $\pi$ pulls back infinity harmonic functions to infinity harmonic functions and hence, by definition, $\pi$ is an infinity harmonic morphism.
\end{proof}

\begin{proof} (of Theorem 2.3): Together Lemmas 2.11, 2.16, 2.17, and Proposition 1.11 give Theorem 2.3 \end{proof}

We conclude this section by pointing out that a proof similar to that of Lemmas 2.16 and 2.17 gives the following

\begin{proposition}
A map between Riemannian manifolds is an infinity harmonic morphism if and only if it pulls back infinity harmonic maps to infinity harmonic maps.
\end{proposition}

\section{Constructions of Infinity Harmonic Maps}

In this section we give several methods to construct infinity harmonic maps into Euclidean space. We characterize those immersions which are infinity harmonic maps. We also show that isometrically immersing the target manifold of a map into another manifold does not change the infinity harmonicity of the map. Coupled with Nash’s embedding theorem this suggests that it is particularly interesting to study the infinity harmonic maps into a Euclidean space.

\begin{lemma}
[Infinity harmonic maps into a Euclidean space]
A map $\varphi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi(x_1, \ldots, x_m) = (\varphi^1(x), \ldots, \varphi^n(x))$ is an infinity
harmonic map if and only if it is a solution of the following system of PDEs:
\[
\begin{align*}
\langle \nabla \varphi^1, \nabla |\nabla \varphi^1|^2 \rangle + \langle \nabla \varphi^1, \nabla |\nabla \varphi^2|^2 \rangle + \cdots + \langle \nabla \varphi^1, \nabla |\nabla \varphi^n|^2 \rangle &= 0 \\
\langle \nabla \varphi^2, \nabla |\nabla \varphi^1|^2 \rangle + \langle \nabla \varphi^2, \nabla |\nabla \varphi^2|^2 \rangle + \cdots + \langle \nabla \varphi^2, \nabla |\nabla \varphi^n|^2 \rangle &= 0 \\
\vdots & \\
\langle \nabla \varphi^n, \nabla |\nabla \varphi^1|^2 \rangle + \langle \nabla \varphi^n, \nabla |\nabla \varphi^2|^2 \rangle + \cdots + \langle \nabla \varphi^n, \nabla |\nabla \varphi^n|^2 \rangle &= 0.
\end{align*}
\]

Proof. Since
\[
\begin{align*}
|d\varphi|^2 &= \sum_i |\nabla \varphi^i|^2, \\
\nabla |d\varphi|^2 &= \sum_i \nabla |\nabla \varphi^i|^2,
\end{align*}
\]
Since the horizontal space of \(\varphi\) is spanned by \(\{\nabla \varphi^1, \nabla \varphi^2, \ldots, \nabla \varphi^n\}\), the infinity harmonic condition \(d\varphi \left(\nabla |d\varphi|^2\right) = 0\) becomes
\[
\begin{align*}
\langle \sum_i \nabla |\nabla \varphi^i|^2, \nabla \varphi^1 \rangle &= 0, \\
\langle \sum_i \nabla |\nabla \varphi^i|^2, \nabla \varphi^2 \rangle &= 0, \\
&\vdots \\
\langle \sum_i \nabla |\nabla \varphi^i|^2, \nabla \varphi^n \rangle &= 0
\end{align*}
\]
as desired. \(\square\)

Proposition 3.2. Let \(\varphi : (M, g) \to (N, h)\) be a map and \(i : (N, h) \to (Q, k)\) an isometric immersion. Then, the composition \(i \circ \varphi : (M, g) \to (Q, k)\) is infinity harmonic if and only if \(\varphi\) is infinity harmonic.

Proof. Since \(i\) is an isometric immersion,
\[
|d(i \circ \varphi)|^2 = |d\varphi|^2.
\]
So \(i \circ \varphi\) is infinity harmonic if and only if \(\varphi\) is infinity harmonic. \(\square\)

The previous argument also works for homothetic immersions. An example of this is the following.

Proposition 3.4. [Infinity harmonic lines in \(\mathbb{R}^n\)] Let \(u : (M, g) \to R\) be an infinity harmonic function on a Riemannian manifold. Consider the map \(\varphi : (M, g) \to R^n\) defined by \(\varphi(x) = u(x)(a_1, \ldots, a_n)\), where \((a_1, \ldots, a_n)\) is a fixed point in \(R^n\). Then, \(\varphi\) is an infinity harmonic map.

Proposition 3.5. For any infinity harmonic functions \(\varphi^1, \ldots, \varphi^n : (M, g) \to R\) with constant energy density, the map \(\varphi : (M, g) \to R^n\) defined by \(\varphi(x) = (\varphi^1(x), \ldots, \varphi^n(x))\) is a map of constant energy density and hence an infinity harmonic map into Euclidean space.

Remark 3.6. Functions with constant energy density are often called Eikonal functions.

Proof. The proposition follows from
\[
|d\varphi|^2 = \sum_{k=1}^n |\nabla \varphi^k|^2 = \text{constant.}
\]
\(\square\)
Example 3.7. Let $\varphi : R^2 \setminus \{0\} \rightarrow R^2$ be given by $\varphi(x, y) = (ax + by + c, \sqrt{x^2 + y^2})$, where $a, b, c$ are constant. Then $\varphi$ is an infinity harmonic map with constant energy density. Note that $\varphi$ is not an affine map, neither is it an isometric immersion nor a Riemannian submersion.

It is well known that a map into Euclidean space is harmonic if and only if it is a Riemannian submersion. It is easily checked that this is not true for infinity harmonic maps in general. Nevertheless, we have the following method to construct infinity harmonic maps into Euclidean space using infinity harmonic functions.

Proposition 3.8. Let $u : (M, g) \rightarrow R$ and $v : (N, h) \rightarrow R$ be two infinity harmonic functions. Then, $\varphi : (M \times N, g + h) \rightarrow R^2$ with $\varphi(x, y) = (u(x), v(y))$ for any $(x, y) \in M \times N$ is an infinity harmonic map.

Proof. Since the domain manifold is provided with the product metric $G = g + h$, we can easily check that

$$|d\varphi|^2 = g^{ij}\varphi_i^\alpha \varphi_j^\beta \delta_{\alpha\beta} + h^{ab}\varphi_a^\alpha \varphi_b^\beta \delta_{\alpha\beta} = \sum_{\alpha=1}^{2} (g^{ij}\varphi_i^\alpha \varphi_j^\alpha + h^{ab}\varphi_a^\alpha \varphi_b^\alpha).$$

Noting that $\varphi^1 = u(x)$ depends only on $x \in M$ and $\varphi^2 = v(y)$ depends only on $y \in N$ we have

$$|d\varphi|^2 = |\text{grad } u|^2_g + |\text{grad } v|^2_h.$$ 

Therefore,

$$G(\text{grad } \varphi^1, \text{grad } |d\varphi|^2) = G(\text{grad } \varphi^1, \text{grad } |\text{grad } u|^2_g + |\text{grad } v|^2_h) = g(\text{grad } u, \text{grad } |\text{grad } u|^2_g) = 0,$$

since $u$ is an infinity harmonic function. Similarly, we have

$$G(\text{grad } \varphi^2, \text{grad } |d\varphi|^2) = G(\text{grad } \varphi^2, \text{grad } |\text{grad } v|^2_h) = h(\text{grad } u, \text{grad } |\text{grad } v|^2_h) = 0,$$

since $v$ is an infinity harmonic function.

Since $\{\text{grad } \varphi^1, \text{grad } \varphi^2\}$ span the orthogonal complement of the kernel of $d\varphi$, $\varphi$ is infinity harmonic as claimed. \qed

Example 3.11. It is well known \cite{At10} that $u(x_1, x_2) = x_1^{4/3} - x_2^{4/3}$ is an infinity harmonic function on $R^2$. By Theorem 3.8, we have a globally defined infinity harmonic map $\varphi : R^4 \rightarrow R^2$ given by $\varphi(x_1, x_2, x_3, x_4) = (x_1^{4/3} - x_2^{4/3}, x_3^{4/3} - x_4^{4/3})$ which has nonconstant energy density $|d\varphi|^2 = \frac{16}{3}(x_1^{2/3} + x_2^{2/3} + x_3^{2/3} + x_4^{2/3})$.

Proposition 3.12. [Direct sum construction] Let $\varphi : (M, g) \rightarrow R^n$ and $\psi : (N, h) \rightarrow R^n$ be two infinity harmonic maps into Euclidean space. Then, their direct sum $\varphi \oplus \psi : (M \times N, g + h) \rightarrow R^n$ defined by $(\varphi \oplus \psi)(p, q) = \varphi(p) + \psi(q)$ is an infinity harmonic map.
Proof. Use local coordinates \( \{x^i, \frac{\partial}{\partial x^i}\} \) and \( \{y^a, \frac{\partial}{\partial y^a}\} \) around points \( p \in M \) and \( q \in N \) respectively. Then, \( \{(x^i, y^a), (\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a})\} \) can be used as local coordinates around \((p, q)\in M \times N\). Let \( G = g + h \) and \( \Lambda = \varphi + \psi \).

Then, the component function \( \Lambda^\alpha(x, y) = \varphi^\alpha(x) + \psi^\alpha(y) \), and

\[
\text{grad} \Lambda^\alpha = g^{ij} \frac{\partial \Lambda^\alpha}{\partial x^i} \frac{\partial}{\partial x^j} + h^{ab} \frac{\partial \Lambda^\alpha}{\partial y^a} \frac{\partial}{\partial y^b} = \text{grad} \varphi^\alpha + \text{grad} \psi^\alpha.
\]

It follows that

\[
|d \Lambda|^2 = \sum_{\alpha=1}^n G(\text{grad} \Lambda^\alpha, \text{grad} \Lambda^\alpha) = \sum_{\alpha=1}^n G(\text{grad} \varphi^\alpha + \text{grad} \psi^\alpha, \text{grad} \varphi^\alpha + \text{grad} \psi^\alpha) = \sum_{\alpha=1}^n g(\text{grad} \varphi^\alpha, \text{grad} \varphi^\alpha) + \sum_{\alpha=1}^n h(\text{grad} \psi^\alpha, \text{grad} \psi^\alpha) = |d \varphi|^2 + |d \psi|^2.
\]

Combining the previous two equations we have

\[
G(\text{grad} \Lambda^\alpha, \text{grad} |d \Lambda|^2) = G(\text{grad} \varphi^\alpha + \text{grad} \psi^\alpha, \text{grad} |d \varphi|^2 + \text{grad} |d \psi|^2) = g(\text{grad} \varphi^\alpha, \text{grad} |d \varphi|^2) + h(\text{grad} \psi^\alpha, \text{grad} |d \psi|^2) = 0.
\]

Since \( \{\text{grad} \Lambda^\alpha\}_\alpha \) spans the orthogonal complement of the kernel of \( d\Lambda \), \( \Lambda \) is infinity harmonic as claimed.

\(\square\)

Proposition 3.15. An immersion \( \iota : (M^m, g) \rightarrow (N^n, h) \) is infinity harmonic if and only if the energy density of \( \iota \) is constant. In particular, a conformal immersion is infinity harmonic if and only if it is a homothetic immersion.

Proof. By definition, \( \iota \) is infinity harmonic if and only if \( d(\text{grad} |d\iota|^2) = 0 \). If \( \iota \) is an immersion then \( d\iota \) is injective and it follows that an immersion \( \iota \) is infinity harmonic if and only if \( \text{grad} |d\iota|^2 = 0 \), which is equivalent to saying that the energy density \( |d\iota|^2 \) is constant. Now if \( \iota \) is a conformal immersion with \( \iota^*h = \lambda^2 g \), then it is easy to check that

\[
|d\iota|^2 = g^{ij} \iota_i^\alpha \iota_j^\beta h_{\alpha\beta} = g^{ij} (\lambda^2 g_{ij}) = m \lambda^2.
\]

It follows that \( \iota \) is infinity harmonic if and only if \( \lambda \) is constant, in other words, \( \iota \) is a homothetic immersion.

\(\square\)

Corollary 3.17. Let \( g \) and \( h \) be two Riemannian metrics on a manifold \( M \). Then, the identity map \( 1 : (M, g) \rightarrow (M, h) \) is infinity harmonic if and only if \( \text{Trace}_g h = \text{constant} \). In particular, \( 1 : (M, g) \rightarrow (M, \lambda^2 g) \) is infinity harmonic if and only if \( \lambda \) is constant.
Proof. By (1) of Theorem 3.15, \(1\) is infinity harmonic if and only if it has constant energy density

\[
|d1|^2 = g^{ij}1^\alpha_j \partial_\alpha 1^\beta_i = g^{ij}1^\alpha_j \partial_\alpha 1^\beta_i = g^{\alpha\beta}h_{\alpha\beta} = \text{Trace}_gh.
\]

\[\square\]

Example 3.19. Let \(B\) denote the unit ball centered at \((2, \ldots, 2) \in \mathbb{R}^m\). Then the identity map

\[
(3.20) \quad 1 : (B, \sum_{i=1}^m dx_i^2) \rightarrow (B, \sum_{i=1}^m \frac{x_i^2}{|x|^2} dx_i^2)
\]
is an infinity harmonic map by Corollary 3.17.

4. Infinity Harmonic Maps Into Spheres

It is well known that in the presence of sufficient symmetry the harmonic map equation can be reduced to an ordinary differential equation. In this section we use ideas similar to those of Schoen and Yau’s and Smith’s about harmonic maps to find infinity harmonic maps into spheres.

Let \(i : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}\) be the injection \(i(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0)\), and let \((r, \theta)\) denote the polar coordinates on the unit ball \(B^n\) and \((\rho, \phi)\) the geodesic coordinates on the unit sphere \(S^n\), where \(\rho\) is the distance from the north pole of \(S^n\) and \(\phi \in S^{n-1}\). Following Schoen and Yau’s idea ([SY]) we try to solve the Dirichlet Problem for rotationally symmetric infinity harmonic maps.

Theorem 4.1. A rotationally symmetric map \(\varphi : B^n \rightarrow S^n \subset \mathbb{R}^{n+1}\) of the form

\[
\varphi : B^n \rightarrow S^n,
\]

\[\varphi(r, \theta) = (\rho(r), \theta) \quad \text{with} \quad \rho(1) = \pi/2.
\]
is an infinity harmonic map if and only if either \(\rho = \pi/2\) and \(\varphi\) is the equator map

\[
\varphi : B^n \setminus \{0\} \rightarrow S^{n-1} \subset S^n,
\]

\[\varphi(x) = x/|x|,
\]
or \(\rho\) satisfies the ordinary differential equation

\[
(4.2) \quad \rho^2 + \frac{n-1}{r^2} \sin^2 \rho = \text{constant},
\]

and \(\varphi\) has constant energy density.

Proof. Using polar coordinates \((r, \theta)\) on \(B^n\) and geodesic coordinates \((\rho, \phi)\) on \(S^n\) we can write the metrics as

\[
g_{B^n} = dr^2 + r^2 d\theta^2, \quad \text{and} \quad g_{S^n} = d\rho^2 + \sin^2 \rho d\phi^2.
\]

Let \(k_{ab}\) and \(k^{ab}\) be the covariant and contravariant components of the standard metric on \(S^{n-1}\). Then,

\[
(g_{B^n})^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r}k^{ab} \end{pmatrix}, \quad (g_{S^n})_{\alpha\beta} \circ \varphi = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \rho k_{\alpha\beta} \end{pmatrix},
\]

Since

\[
d\varphi = (\varphi_i^1) = \begin{pmatrix} \rho' & 0 \\ 0 & \text{id} \end{pmatrix},
\]

\[
g^{ij}1^\alpha_j \partial_\alpha 1^\beta_i = g^{\alpha\beta}h_{\alpha\beta} = \text{Trace}_gh.
\]
we have
\[ |d\varphi|^2 = (g_{B^n})^{ij}_{\alpha\beta} \varphi_i^\alpha \varphi_j^\beta (g_{S^n})_{\alpha\beta} \circ \varphi = \rho^2 + \frac{n-1}{\rho^2} \sin^2 \rho. \]
and
\[ \nabla |d\varphi|^2 = \left( \rho^2 + \frac{n-1}{\rho^2} \sin^2 \rho \right)' \partial_r. \]
So \( \varphi \) is infinity harmonic if and only if
\[ 0 = d\varphi \left( \nabla |d\varphi|^2 \right) = \rho' \left( \rho^2 + \frac{n-1}{\rho^2} \sin^2 \rho \right)' . \]
It follows that either \( \rho = \text{constant} \) and hence \( \rho = \pi/2 \) by boundary condition, or \( \rho \) is a solution of the ODE
\[ \rho^2 + \frac{n-1}{\rho^2} \sin^2 \rho = \text{constant}. \]
The first case corresponds to the map \( \varphi(r, \theta) = (\pi/2, \theta) \) which, in Cartesian coordinates, can be expressed as \( \varphi : B^n \setminus \{0\} \to S^{n-1} \subset S^n, \varphi(x) = x/|x|, \) the equator map. In the second case we have
\[ |d\varphi|^2 = \rho^2 + \frac{n-1}{\rho^2} \sin^2 \rho = \text{constant} \]
as desired. \( \Box \)

Let \( S^2 \) be the unit sphere in \( R^3 \) parametrized by spherical polar coordinates:
\[(\alpha, \beta) \mapsto (\cos \alpha, \sin \alpha e^{i\beta}) \in \mathbb{R} \oplus \mathbb{C}, \ (\alpha, \beta) \in \mathbb{R} \times \mathbb{R} \]
Parametrized the cylinder as \( R \times S^1 = \{(s, e^{it}) : (s, t) \in \mathbb{R} \times \mathbb{R}\}, \) and consider the rotationally symmetric map \( \varphi : R \times S^1 \to S^2 \) of the form
\[ \varphi(s, t) = (\cos \alpha(s), \sin \alpha(s)e^{ikt}), \]
where \( \alpha : R \to R \) is a smooth function and \( k \) a non-zero integer. Smith \( \text{(Sm)} \) proved that \( \varphi \) is harmonic if and only if \( \alpha \) is a solution of the ordinary differential equation
\[ \alpha'' = k^2 \sin \alpha \cos \alpha, \]
and by solving this equation of pendulum with constant gravity and no damping he was able to find some interesting harmonic maps from torus into a sphere (see also \( \text{BW} \) for a detailed explanation).

Our next theorem shows that Smith’s method can also be used to find infinity harmonic maps into a sphere.

**Theorem 4.11.** The rotationally symmetric map \( \varphi : R \times S^1 \to S^2, \)
\[ \varphi(s, t) = (\cos \alpha(s), \sin \alpha(s)e^{ikt}) \]
is an infinity harmonic map if and only if
1. \( \alpha = \text{constant} \) and \( \varphi \) is the projection onto the second factor followed by a homothetic immersion, or
2. \( \alpha(s) = 2 \arctan(e^{k(s+A)}) - \pi/2, \) where \( A \) is any constant, or
(3) $2\alpha$ is a solution of the pendulum equation (Equation 28.74 in [TP])

\[ \frac{d^2 \theta}{dt^2} + k^2 \sin \theta = 0. \]

In this case, the map $\varphi$ factors to an infinity harmonic map from the torus $R/(T) \times S^1$ to $S^2$, where $T$ is the period of $\alpha$.

Note that the ODE (4.13) differs from the harmonic map equation (4.10) by a negative sign.

**Proof.** The infinity harmonicity of $\varphi : R \times S^1 \rightarrow S^2$ is the same as the infinity harmonicity of $\varphi : R \times S^1 \rightarrow S^2 \hookrightarrow R^3$. Changing from complex to real notation we have

\[ \varphi(s, t) = (\cos \alpha(s), \sin \alpha(s) e^{ikt}) = (\cos \alpha(s), \sin \alpha(s) \cos kt, \sin \alpha(s) \sin kt), \]

so

\[ \begin{align*}
\nabla \varphi^1 &= (-\alpha'(s) \sin \alpha(s), 0), \\
\nabla \varphi^2 &= (\alpha'(s) \cos \alpha(s) \cos kt, -k \sin \alpha(s) \sin kt), \\
\nabla \varphi^3 &= (\alpha'(s) \cos \alpha(s) \sin kt, k \sin \alpha(s) \cos kt), \text{ and} \\
|d\varphi|^2 &= \sum_{i=1}^3 |\nabla \varphi^i|^2 = \alpha'(s)^2 + k^2 \sin^2 \alpha(s).
\end{align*} \]

By Lemma 3.1, $\varphi$ is infinity harmonic if and only if

\[ \begin{align*}
-\alpha'(s) \sin \alpha(s)(\alpha'^2 + k^2 \sin^2 \alpha)' &= 0 \\
\alpha'(s) \cos \alpha(s) \cos kt(\alpha'^2 + k^2 \sin^2 \alpha)' &= 0 \\
\alpha'(s) \cos \alpha(s) \sin kt(\alpha'^2 + k^2 \sin^2 \alpha)' &= 0.
\end{align*} \]

It follows that $\varphi$ is infinity harmonic if and only if either $\alpha' = 0$ and hence $\alpha$ is constant and $\varphi$ is the projection onto the second factor followed by a homothetic immersion, or,

\[ (\alpha'^2 + k^2 \sin^2 \alpha)' = 0, \]

which is equivalent to

\[ \alpha'^2 + k^2 \sin^2 \alpha = C. \]

When $C = k^2$, we can solve the previous equation and get

$\alpha = 2 \arctan(e^{k \pi}) - \pi/2$. When $C > k^2$,

\[ \alpha'^2 + k^2 \sin^2 \alpha = C. \]

is equivalent to

\[ (2\alpha)'' + k^2 \sin(2\alpha) = 0, \]

which means that $2\alpha$ is a solution of the pendulum equation (4.13). By the theory of the solutions of the pendulum equation (see, e.g., [TP]) we obtain the statement (3). $\square$
5. The Effect of a Conformal Change on the Infinity Laplacian

In this section we study the effect of a conformal change on the Infinity Laplacian to derive formulas for the infinity Laplacian of spheres and hyperbolic spaces in terms of the infinity Laplacian on Euclidean space.

Given Riemannian metrics $g$ and $\tilde{g}$ on a smooth manifold $M$, we let $\nabla$, $|\cdot |$, and $\Delta_\infty$ denote the gradient, the norm, and the infinity Laplacian with respect to $g$ and we let $\tilde{\nabla}$, $|\cdot |_{\tilde{g}}$, and $\tilde{\Delta}_\infty$ denote the gradient, the norm, and the infinity Laplacian with respect to $\tilde{g}$.

**Theorem 5.1.** Let $\tilde{g} = F^{-2}g$ be a metric conformal to $g$ on $M$. Then

$$\tilde{\Delta}_\infty u = F^4 \Delta_\infty u + F^3 |\nabla u|^2 g(\nabla u, \nabla F).$$

**Proof.** A direct computation using $\tilde{g} = F^{-2}g$ gives

$$\tilde{\nabla} |\nabla u|_{\tilde{g}}^2 = \nabla |F^2 \nabla u|^2_{\tilde{g}} = \nabla \left( F^4 F^{-2} |\nabla u|^2_g \right) = 2 F^3 |\nabla u|^2_g \nabla F + F^4 \nabla |\nabla u|^2_g.$$

It follows that

$$\tilde{\Delta}_\infty u = \frac{1}{2} \tilde{g}(\nabla u, \nabla |\nabla u|^2_{\tilde{g}})$$

$$= \frac{1}{2} F^{-2} g(F^2 \nabla u, 2 F^3 |\nabla u|^2_g \nabla F + F^4 \nabla |\nabla u|^2_g)$$

$$= F^4 \Delta_\infty u + F^3 |\nabla u|^2 g(\nabla u, \nabla F).$$

As an application of Theorem 5.1 we can study infinity Laplace equation in spheres or hyperbolic spaces by studying conformal infinity Laplace equations in Euclidean spaces.

**Corollary 5.3.** [Infinity Laplacian on spheres] Let $(S^m, g_{can})$ be the $m$-dimensional sphere with the standard metric. Identify $(S^m \setminus \{N\}, g_{can})$ with $(R^m, F^{-2} \delta_{ij})$, where $F = \frac{1}{2}(1 + |x|^2)$. With respect to these coordinates, the infinity Laplace equation in the sphere $S^m$ is

$$\Delta_{\tilde{g}}^m u + \frac{2 |\nabla u|^2}{1 + |x|^2} \langle x, \nabla u \rangle = 0, \quad x \in R^m,$$

where $|\cdot |$ and $\nabla$ denote the norm and the gradient defined by the standard metric $\langle \cdot, \cdot \rangle$ on Euclidean space $R^m$.

**Proof.** Starting with

$$F^4 \Delta_\infty u + F^3 |\nabla u|^2 g(\nabla u, \nabla F) = 0$$

we divide by $F^4$ to get

$$\Delta_\infty u + \frac{|\nabla u|^2}{F} \langle \nabla u, \nabla F \rangle = 0$$

Substituting $\frac{1}{2}(1 + |x|^2)$ for $F$ and $x$ for $\nabla F$ we get

$$\Delta_\infty u + \frac{|\nabla u|^2 \langle \nabla u, x \rangle}{\frac{1}{2}(1 + |x|^2)} = 0$$
or
\[ \Delta_{\infty} u + \frac{2|\nabla u|^2}{(1 + |x|^2)} \langle \nabla u, x \rangle = 0. \]

A similar argument gives us

**Corollary 5.5.** [Infinity Laplacian on hyperbolic space \(B^m\)] Let \((B^m, g^H)\) be the \(m\)-dimensional hyperbolic space with open-ball model, where \(B^m = \{ x \in R^m : |x| < 1 \}\) and \(g^H = F^{-2}\delta_{ij}\) with \(F = \frac{1}{2}(1 - |x|^2)\). Then, the infinity Laplace equation in the hyperbolic space \((B^m, g^H)\) is the conformal infinity Laplace equation in the Euclidean space \((R^m, \delta_{ij})\), which can be written as

\[ \Delta_{\infty}^{R^m} u - \frac{2|\nabla u|^2}{1 - |x|^2} \langle x, \nabla u \rangle = 0, \quad x \in \mathbb{R}^m, \]

where \(|.|\) and \(\nabla\) denote the norm and the gradient defined by the Euclidean metric \(\langle ., . \rangle\) on \(B^m \subset R^m\).

**Example 5.7.** Let \(u : \Omega \subset (S^2 \setminus \{N\}, g_{can}) \cong (R^2, F^{-2}\delta_{ij}) \rightarrow R\) be given by \(u(x_1, x_2) = \arctan \frac{x_1}{x_2}\). Then, we know (see [Ar1]) that \(u\) is an infinity harmonic function on \(\Omega \subset R^2\), so \(\Delta_{\infty}^{R^2} u = 0\). On the other hand, we can easily check that \(\langle x, \nabla u \rangle = 0\). Therefore, \(u\) satisfies Equation (5.4) and hence it is an infinity harmonic function on sphere \(S^2\). A more geometric way to see this is via the isometric \(\mathbb{R}\)-action that rotates the \(2\)-sphere and Proposition 1.6.

Note that the function \(u(x_1, x_2) = \arctan \frac{x_1}{x_2}\) is also an infinity harmonic function on hyperbolic space \((B^2, g^H)\) wherever it is defined.

The following example give families of infinity harmonic functions on hyperbolic space.

**Example 5.8.** Let \((B^m, g^H)\) be the \(m\)-dimensional hyperbolic space with open-ball model as in Corollary 5.5. Then, for constants \(a_1, \ldots, a_{m-1}\), the function \(u : (B^m, g^H) \rightarrow R\) given by \(u(x_1, \ldots, x_m) = \arctan \frac{a_1 x_1 + \ldots + a_{m-1} x_{m-1}}{1 + |x|^2 - 2x_m}\) is an infinity harmonic function. This follows from Theorem 2.9 in [Ou2].

Let \(i : S^m \rightarrow R^{m+1}\) be the standard inclusion and \(u : R^{m+1} \rightarrow R\) be a function. Set \(\tilde{u} = u \circ i = u|_{S^m}\). Then, it is well known (see e.g., [ER]) that we have the following relationship between the spherical and Euclidean Laplace operators.

\[ \Delta_{S^m} \tilde{u} = (\Delta_{R^{m+1}} u - \frac{\partial^2 u}{\partial r^2} - m \frac{\partial u}{\partial r}) \circ i. \]

For infinity Laplacian, we have

**Proposition 5.9.**

\[ \Delta_{\infty}^{S^m} \tilde{u} = \left( \Delta_{\infty}^{R^{m+1}} u - \frac{1}{2} |\nabla u| \left( \frac{\partial u}{\partial r} \right)^2 - \frac{1}{2} \frac{\partial u}{\partial r} \frac{\partial}{\partial r} \left( |\nabla u|^2 - \left( \frac{\partial u}{\partial r} \right)^2 \right) \right) \circ i, \]

where \(\frac{\partial}{\partial r}\) denotes radial partial differentiation.
Proof. At any point \( x \in S^m \) choose an orthonormal basis \( e_1, \ldots, e_m \) for \( T_x S^m \) and let \( e_{m+1} = x \) so that \( e_1, \ldots, e_m \) and \( e_{m+1} = x \) form an orthonormal basis for \( T_x R^{m+1} \). Then, \( du(x) = \langle x, \nabla u \rangle = x(u) = \frac{\partial u}{\partial r} \). It follows that

\[
(5.11) \quad |d\tilde{u}|^2 = |d(u \circ i)|^2 = \sum_{i=1}^{m} |du(e_i)|^2
\]

Therefore,

\[
\Delta^m_{\infty} \tilde{u} = \frac{1}{2} g(\text{grad } \tilde{u}, \text{grad } |d\tilde{u}|^2)
\]

\[
(5.12) \quad = \frac{1}{2} \sum_{k=1}^{m} (D_{e_k} \tilde{u})(D_{e_k} |d\tilde{u}|^2)
\]

\[
= \frac{1}{2} \sum_{k=1}^{m} (D_{e_k} \tilde{u})(D_{e_k} |du|^2) - \frac{1}{2} \sum_{k=1}^{m} (D_{e_k} \tilde{u})D_{e_k}(\frac{\partial u}{\partial r})^2
\]

\[
= (\Delta_{\infty}^{R^{m+1}} u) \circ i - \frac{1}{2} D_x(u)D_x(|du|^2) - \frac{1}{2} \sum_{k=1}^{m} (D_{e_k} \tilde{u})D_{e_k}(\frac{\partial u}{\partial r})^2
\]

Since

\[
\nabla(\frac{\partial u}{\partial r})^2 = \sum_{k=1}^{m} D_{e_k} (\frac{\partial u}{\partial r})^2 e_k + \frac{\partial}{\partial r} (\frac{\partial u}{\partial r})^2 e_{m+1}
\]

\[
\nabla u = \sum_{k=1}^{m} (D_{e_k} \tilde{u}) e_k + \frac{\partial u}{\partial r} e_{m+1}
\]

we get

\[
\Delta^m_{\infty} \tilde{u} = (\Delta_{\infty}^{R^{m+1}} u) \circ i - \frac{1}{2} (\nabla u, \nabla(\frac{\partial u}{\partial r})^2) + \frac{1}{2} \frac{\partial u}{\partial r} \left[ \frac{\partial}{\partial r} (\frac{\partial u}{\partial r})^2 \right] - \frac{1}{2} \frac{\partial u}{\partial r} (|du|^2)
\]

\[
= (\Delta_{\infty}^{R^{m+1}} u) \circ i - \frac{1}{2} (\nabla u, \nabla(\frac{\partial u}{\partial r})^2) - \frac{1}{2} \frac{\partial u}{\partial r} (|du|^2 - (\frac{\partial u}{\partial r})^2) \circ i.
\]

\[\square\]

References

[Ar1] G. Aronsson, Extension of functions satisfying Lipschitz conditions, Ark. Mat. 6 (1967) 551–561.

[Ar2] G. Aronsson, On the partial differential equation \( u^2_x u_{xx} + 2 u_x u_y u_{xy} + u^2_y u_{yy} = 0 \). Ark. Mat. 7 (1968) 395–425.

[Ar3] G. Aronsson, On certain singular solutions of the partial differential equation \( u^2_x u_{xx} + 2 u_x u_y u_{xy} + u^2_y u_{yy} = 0 \), Manuscripta Math. 47 (1984), 133–151.

[ACJ] G. Aronsson, M. Crandall, and P. Juutinen, A tour of the theory of absolutely minimizing functions Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 4, 439–505.

[BE] P. Baird and J. Eells, A conservation law for harmonic maps, E. Looijenga, D. Siersma, and F. Takens, editors, Geometry Symposium. Utrecht 1980, volume 894 of Lecture Notes in Math., Springer, Berlin, Heidelberg, New York, 1981, pp. 1–25.
[BG] P. Baird and S. Gudmundsson, \( p \)-Harmonic maps and minimal submanifolds, Math. Ann., 294 (1992), 611-624.

[BW] P. Baird and J. C. Wood, Harmonic morphisms between Riemannian manifolds, London Math. Soc. Monogr. (N.S.) No. 29, Oxford Univ. Press (2003).

[BB] G. Barles and J. Busca, Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term, Comm. Partial Differential Equations 26 (2001), no. 11-12, 2323–2337.

[Ba] E. N. Barron, Viscosity solutions and analysis in \( L^\infty \), in Nonlinear Analysis, Differential Equations and Control (ed. by Clarke and Stern), Kluwer Academic Publishers, 1999, 1-60.

[BLW1] E. N. Barron, R. Jensen, and C. Y. Wang, Lower Semicontinuity of \( L^\infty \) functionals, Ann. Inst. H. Poincaré Anal. Non Linéaire, 18 (2001), no. 4, 495–517.

[BLW2] E. N. Barron, R. Jensen, and C. Y. Wang, Euler equations and absolute minimizers of \( L^\infty \) functionals, Arch. Ration. Mech. Anal. 157 (2001), no. 4, 255-283.

[BEJ] E. N. Barron, L. C. Evans, and R. Jensen, The infinity Laplacian, Aronsson’s equation and their generalizations, Trans. Amer. Math. Soc., 360 (2008), no. 1, 77–101 (electronic).

[Bh] T. Bhattacharya, A note on non-negative singular infinity-harmonic functions in he half-space. Rev. Mat. Complut. 18 (2005), no. 2, 377–385.

[BL] J. M. Burel and E. Loubeau, \( p \)-harmonic morphisms: the \( 1 < p < 2 \) case and a non-trivial example, Contemp. Math. 308(2002), 21-37.

[CMS] V. Caselles, J.-M. Morel, and C. Sbert, An axiomatic approach to image interpolation, IEEE Trans. Image Process. 7 (1998), no. 3, 376–386.

[CEPB] G. Cong, M. Esser, B. Parvin, and G. Bebis, Shape metamorphosis using \( p \)-Laplacian equation, Proceedings of the 17th International Conference on Pattern Recognition, (2004), Vol. 4, 15-18.

[CE] M. G. Crandall and L. C. Evans, A remark on infinity harmonic functions. Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viña del Mar-Valparaiso, 2000), 123–129.

[CEG] M. G. Crandall, L. C. Evans, and R. F. Gariepy, Optimal Lipschitz extensions and the infinity Laplacian. Calc. Var. Partial Differential Equations 13 (2001), no. 2, 123–139.

[CIL] M. G. Crandall, H. Ishii, and P.-L. Lions, User’s guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1–67.

[CY] M. G. Crandall and J. Zhang, Another way to say harmonic, Trans. Amer. Math. Soc. 355 (2003), 241-263.

[DU] F. Dobarro, and B. Ünal, Curvature of multiply warped products, J. Geom. Phys. 55 (2005), no. 1, 75–106.

[EL1] J. Eells and L. Lemaire, A report on harmonic maps, Bull. London Math. Soc., 10:1–68, 1978.

[EL2] J. Eells and L. Lemaire, Selected topics in harmonic maps, volume 50 of CBMS Regional Conf. Ser. in Math., Amer. Math. Soc., Providence, R.I., 1983.

[EL3] J. Eells and L. Lemaire, report on harmonic maps, Bull. London Math. Soc., 20:385–524, 1988.

[ER] J. Eells and A. Ratto Andrea Harmonic maps and minimal immersions with symmetries. Methods of ordinary differential equations applied to elliptic variational problems. Annals of Mathematics Studies, 130. Princeton University Press, Princeton, NJ, 1993.

[Ev] L. C. Evans, Estimates for smooth absolutely minimizing Lipschitz extensions, Electron. J. Differential Equations, no. 3, 1993.

[EG] L. C. Evans and W. Gangbo, Differential equations methods for the Monge-Kantorovich mass transfer problem, Mem. Amer. Math. Soc. 137 (1999), no. 653.

[EY] L. C. Evans and Y. Yu, Various properties of solutions of the infinity-Laplacian equation, Comm. Partial Differential Equations, 30 (2005), no. 7-9, 1401–1428.

[Fu] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Ann. Inst. Fourier (Grenoble), vol 28 (1978), 107-144.

[Fu2] B. Fuglede, A criterion of non-vanishing differential of a smooth map, Bull. London Math. Soc. 14 (1982), 98–102.

[Gu] S. Gudmundsson, Harmonic morphisms between spaces of constant curvature, Proc. Edinburgh Math. Soc. (2) 36 (1993), no. 1, 133–143.
[Is] T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, J. Math. Kyoto Univ., 19(2) (1979), 215-229.

[J] R. Jensen, Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient, Arch. Rational Mech. Anal. 123 (1993), no. 1, 51–74.

[JK] P. Juutinen and B. Kawohl, On the evolution governed by the infinity Laplacian, Math. Ann., 335 (2006), no. 4, 819–851.

[JLM1] P. Juutinen, P. Lindqvist, and J. Manfredi, The infinity Laplacian: examples and observations. Papers on analysis, 207–217, Rep. Univ. Jyväskylä Dep. Math. Stat., 83, Univ. Jyväskylä, Jyväskylä, 2001.

[JLM2] P. Juutinen, P. Lindqvist, and J. Manfredi, The $\infty$-eigenvalue problem, Arch. Ration. Mech. Anal. 148 (1999), no. 2, 89–105.

[LM1] P. Lindqvist and J. Manfredi, The Harnack inequality $\infty$-harmonic functions, Electron. J. Differential Equations (1995), No. 04, approx. 5 pp.

[LM2] P. Lindqvist and J. Manfredi, Note on $\infty$-superharmonic functions, Rev. Mat. Univ. Complut. Madrid 10 (1997), no. 2, 471–480.

[Lo] E. Loubeau, On $p$-harmonic morphisms, Diff. Geom. Appl., 12(2000), 219-229.

[Ob] A. M. Oberman, A convergent difference scheme for the infinity Laplacian: construction of absolutely minimizing Lipschitz extensions, Math. Comp. 74 (2005), no. 251, 1217–1230 (electronic).

[Ou1] Y. -L. Ou, $p$-Harmonic morphisms, minimal foliations, and rigidity of metrics, J. Geom. Phys. 52 (2004), no. 4, 365–381.

[Ou2] Y. -L. Ou, $p$-Harmonic functions and the minimal graph equation in a Riemannian manifold, Illinois J. Math., 49 (2005), no. 3, 911–927 (electronic).

[OW] Y. -L. Ou and F. Wilhelm, Horizontally homothetic submersions and nonnegative curvature, Indiana Univ. Math. J., 56 (2007), no. 1, 243–261.

[Sa] G. Sapiro, Geometric partial differential equations and image analysis, Cambridge University Press, Cambridge, 2001.

[SY] R. Schoen and S. T. Yau, Lectures on harmonic maps. Conference Proceedings and Lecture Notes in Geometry and Topology, II, International Press, Cambridge, MA, 1997.

[Sm] R. T. Smith, Harmonic mappings of spheres, Thesis, Warwick University, 1972.

[Ta] H. Takeuchi, Some conformal properties of $p$-harmonic maps and regularity for sphere-valued $p$-harmonic maps, J. Math. Soc. Japan, 46(1994), 217-234.

[TP] M. Tenenbaum and H. Pollard, Ordinary differential equations, Harper and Row, Publishers, New York, 1963.

[Tr] T. Troutman, Critical points of infinity-harmonic functions on Riemannian manifolds, in preparation.

[Yu] Y. Yu, A remark on $C^2$-solutions of the infinity Laplace equation, Electronic Journal of Differential Equations, no. 122 (2006), 1–4.

Department of Mathematics, Texas A&M University-Commerce, Commerce, TX. 75429-3011, Yelin_Ou@tamuc-commerce.edu

Department of Mathematics, Bradley University, Peoria, IL 61625, Ttroutman@bradley.edu

Department of Mathematics, University of California, Riverside, Riverside, CA 92521, fred@math.ucr.edu