Soliton–antisoliton pair production in particle collisions

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We propose general semiclassical method for computing the probability of soliton–antisoliton pair production in particle collisions. The method is illustrated by explicit numerical calculations in \((1+1)\)-dimensional scalar field model. We find that the probability of the process is suppressed by an exponentially small factor which is almost constant at high energies.

Ever since the discovery of topological solitons, a question remains \cite{1,2} of whether soliton–antisoliton (SA) pair can be produced at sizable probability in collision of two quantum particles. This process, which involves a transition from perturbative two–particle state to a non–perturbative state containing SA pair, eludes treatment by any of the standard methods. A general expectation \cite{1,3} is that the probability of the process is exponentially suppressed in weak coupling regime,

\[ \mathcal{P}(E) \sim e^{-F(E)/g^2}, \tag{1} \]

where \(E\) is the total energy, \(g \ll 1\) is the coupling constant. Indeed, crudely speaking, one can think of solitons as bound states of \(N_S \sim 1/g^2\) particles \cite{1}. Then the suppression \cite{1} is due to multiparticle production \cite{3}.

In this Letter we propose general semiclassical method for computing the leading suppression exponent \(F(E)\) of the inclusive process “two particles \(\rightarrow\) SA pair + particles.” As a by–product, we calculate the exponent \(F_N(E)\) of the same process with \(N\) initial particles. In our method the problem is deformed by introducing a small parameter \(\delta\rho\) which turns the process of SA pair production into a well–known tunneling process. To the best of our knowledge, no method of this kind has ever been proposed before.

For definiteness we consider \((1+1)\)-dimensional scalar field theory with action \cite{4}

\[ S[\phi] = \frac{1}{g^2} \int d^2x \left[ -\frac{1}{2} \phi \Box \phi - V(\phi) \right]. \tag{2} \]

This model possesses topological solitons if the scalar potential \(V(\phi)\) has a pair of degenerate minima \(\phi_-\) and \(\phi_+\), see the inset in Fig. 1 solid line. Soliton and antisoliton solutions interpolate between the minima; their profiles are shown in Fig. 2.

An obstacle to the semiclassical description of SA pair production is related to the fact that soliton and antisoliton attract each other and annihilate classically into \(N_{SA} \sim 1/g^2\) particles. Thus, there is no potential barrier separating SA pair from the particle sector and the process under study cannot be treated as potential tunneling.

We get around this obstacle by introducing the potential barrier between SA pair and perturbative states.

Namely, we add negative energy density \((-\delta\rho)\) to the vacuum \(\phi_+\), see dashed line in the inset in Fig. 1. This turns \(\phi_-\) and \(\phi_+\) into false and true vacua, respectively; the process of SA pair production is now interpreted as false vacuum decay \cite{2} induced by particle collisions. The latter is a well–studied tunneling process \cite{6,7}. In the end of calculation we will take the limit \(\delta\rho \to 0\).

The height of the potential barrier between the false and true vacua is given by the energy \(E_{cb}\) of the critical bubble \cite{5} — unstable static solution “sitting” on top of the barrier. The pressure \(\delta\rho\) inside this bubble is balanced by the soliton–antisoliton attraction. Let us estimate the critical bubble size \(d_{cb}\) at small \(\delta\rho\). The attractive force \(F_{att}\) between the soliton and antisoliton is proportional to the Yukawa exponent \(\exp(-m_+d_{cb})\), where \(m^2_+ = V''(\phi_+)\). Setting \(F_{att} = \delta\rho\), one finds \(d_{cb} \sim -\log(\text{const} \cdot \delta\rho)/m_+\). We see that at \(\delta\rho \to 0\) the critical bubble turns into a widely separated SA pair and \(E_{cb} \to 2M_S\), where \(M_S\) is the soliton mass.

Another difficulty is met in the case of \(N = 2\) particles in the initial state, since states with few quanta cannot be described semiclassically. We solve this problem by Rubakov–Son–Tinyakov (RST) method \cite{8,9}.

![FIG. 1. Solutions in \((E, N)\) plane at \(\delta\rho = 0.4\). Numbers near the lines with empty and filled points are the values of \(T\) and \(\theta\), respectively. Inset: Potential density \(V(\phi)\).](image-url)
Semiclassical method for the calculation of the multiparticle probability at \( N \gg 1 \) and \( g \ll 1 \) has been developed in Ref. [8]. This method is based on the saddle-point evaluation of the path integral for the probability. Below we list the boundary conditions for the complex saddle-point configuration \( \phi_s(t, x) \in \mathbb{C} \) and give expression for \( F_N(E) \); see Refs. [3] for derivation.

Configuration \( \phi_s(t, x) \) satisfies classical field equation \( \delta S/\delta \phi = 0 \) along the contour in complex time shown in Fig. 2b, where the Euclidean part corresponds to tunneling. In the asymptotic past \( \phi_s(t, x) \) is a collection of linear waves above the false vacuum \( \phi_- \):

\[
\phi_s \to \phi_- + \int \frac{dk}{\sqrt{2\omega_k}} \left[ a_k e^{-i\omega_k t + ikx} + b_k^* e^{i\omega_k t - ikx} \right] \quad \text{as } t \to iT \to -\infty ,
\]

where \( \omega_k = k^2 + m_\omega^2 \), \( m_\omega = V''(\phi_-) \). The first boundary condition is relation between the amplitudes [3],

\[
a_k = e^{-2\omega_k T - \theta} b_k ,
\]

where \( T \) and \( \theta \) are real parameters. In the asymptotic future \( \phi_s(t, x) \) contains a bubble of true vacuum. The second boundary condition is asymptotic real–valuedness:

\[
\text{Im } \phi_s, \text{ Im } \partial_t \phi_s \to 0 \quad \text{as } t \to +\infty .
\]

Equations (6) are sufficient to specify complex solution \( \phi_s(t, x) \) for given values of \( T \) and \( \theta \) in Eq. (6a).

Parameters \( T \) and \( \theta \) are related to \( (E, N) \) by the saddle–point conditions [8, 7]

\[
g^2 E = 2\pi \int dk \omega_k a_k b_k^* , \quad g^2 N = 2\pi \int dk a_k^* b_k^* .
\]

Given the saddle–point configuration \( \phi_s(t, x) \), one evaluates the suppression exponent,

\[
F_N(E) = g^2 \left( 2\text{Im } S[\phi_s] - 2ET - N\theta \right) ,
\]

where the last two terms are due to non-trivial initial state, see Ref. [7]. Note that \( \phi_s(t, x) \) and \( F_N(E) \) depend on \( g, E, N \) only via combinations \( g^2 N, g^2 E \), see Eqs. [7].

To summarize, the recipe for calculating the suppression exponent of SA pair production in two–particle collisions is as follows. One starts at \( \delta \rho > 0 \) by finding two–parametric family of saddle–point configurations \( \phi_s(t, x) \) which satisfy the classical field equation and boundary conditions [8]. Then one computes the values of \( E, N \) and \( F_N(E) \), Eqs. (7), (8), for each of these configurations. The result for the suppression exponent \( F(E) \) in Eq. (11) is recovered in the limits \( \delta \rho \to 0 \) and \( g^2 N \to 0 \).

We support the method by performing explicit calculations in the model [23] with potential shown in the inset in Fig. 1.

\[
V(\phi) = \frac{1}{2}(\phi + 1)^2 \left[ 1 - v W \left( \frac{\phi - 1}{a} \right) \right] ,
\]
where \( W(x) = e^{-x^2} (x + x^3 + x^5) \), \( a = 0.4 \) and \( v \) is tuned to provide the required value of \( \delta \rho \). We do not use the standard \( \phi^4 \) potential because of the chaotic properties of \( \phi^4 \) kink–antikink dynamics which lead to difficulties in the semiclassical analysis.

We solve the semiclassical boundary value problem numerically using methods of Refs. [7, 15]. Our starting point is bounce Euclidean solution describing spontaneous decay of false vacuum at \( E = N = 0 \). By using the classical field equation we continue the bounce to the Minkowski parts of the contour in Fig. 2b.

Then, changing \( (T, \theta) \) (and thus \( (E, N) \)) in small steps and solving numerically the boundary value problem, we construct the continuous family of saddle–point configurations \( \phi_s(t, x) \) at \( E < E_{cb} \). Each configuration is represented by a point in the left part of Fig. 1. The points form the lines \( \theta = \text{const} \) (filled points) and \( T = \text{const} \) (empty points).

Solutions at \( E < E_{cb} \), \( E_{cb} \approx 2M_S \) have the form of distorted bounces, see Fig. 2b. Wave packets in the left part of the figure represent particles moving in the initial state; after collision, the particles back–react on the Euclidean part of solution. Using the semiclassical solutions, we calculate the multiparticle exponent \( F_N(E) \) at \( E < E_{cb} \).

We evaluate the two–particle exponent \( F(E) \) by extrapolating \( F_N(E) \) to \( g^2N = 0 \) with quadratic polynomials in \( g^2N \), cf. Eq. (4). The accuracy of extrapolation is 5%.

The probability of SA pair production must vanish below the kinematic threshold \( E = 2M_S \). Let us show that, indeed, \( F(E) \to +\infty \) as \( \delta \rho \to 0 \) in the region \( E < E_{cb} \to 2M_S \). We recall that at small \( \delta \rho \) the thin–wall approximation is applicable and the two–particle exponent \( F(E) \) can be evaluated analytically modulo \( O(\delta \rho^0) \) corrections [6].

\[
F = \frac{g^4M_S^2}{\delta \rho} \left( 2 \arccos \frac{E}{2M_S} - \frac{E}{M_S} \sqrt{1 - \frac{E^2}{4M_S^2}} \right). \tag{10}
\]

We see that \( F(E) \propto 1/\delta \rho \) at \( E < 2M_S \); this property disappears for \( E \geq 2M_S \) where the thin–wall approximation breaks down.

Our numerical results for \( \delta \rho \cdot F(E) \) at \( g^2N = 0 \) (Fig. 3, dashed lines) approach Eq. (10) (Fig. 3, solid line) as \( \delta \rho \) decreases and coincide with it after extrapolation to \( \delta \rho = 0 \) (points). This gives support to our method.

The fact that \( \delta \rho \cdot F(E) \to 0 \) as \( E \to 2M_S \), \( \delta \rho \to 0 \) hints that the properties of semiclassical solutions become qualitatively different at \( E > E_{cb} \). This is indeed the case; in fact, our numerical procedure does not produce solutions at \( E > E_{cb} \) at all. By inspecting \( \phi_s(t, x) \) with \( E \approx E_{cb} \) Fig. 2d, we see the reason for that: this solution has long, almost static part where it is close to the critical bubble. The instability of the latter makes the numerical techniques inefficient.
and different values of $\delta \rho$ (dashed lines). The graphs are almost indistinguishable; thus, the limit $\delta \rho \to 0$ exists. Extrapolating $F_N(E)$ to $\delta \rho = 0$, we obtain the final result for the suppression exponent of SA pair production in N–particle collisions (solid line in Fig.3). The two–particle exponent $F(E)$ is recovered by extrapolating $F_N(E)$ to $g^2 N = 0$ (upper graph in Fig.3).

At high energies our suppression exponents are almost constant which is a typical behavior for collision–induced suppressions [24, 18, 23]. We expect that this feature holds in other models.

We end up by arguing that the method proposed in this Letter is applicable in multidimensional ($D > 2$) and gauge theories. The problem with soliton–antisoliton attraction can be solved in the general case by introducing small constant force $F$, an analog of $\delta \rho$, which drags soliton and antisoliton apart. At $|F| \neq 0$ SA pair is separated from the perturbative sector by a potential barrier; tunneling through this barrier is a Schwinger process of spontaneous soliton–antisoliton pair creation in the external field [27]. For example, in the case of t’Hooft–Polyakov monopoles the external force is introduced by adding uniform magnetic field. Other steps of the method — RST conjecture [11] and semiclassical equations (8), (9), (10) — are explicitly general, cf. Ref. [8]. In particular, suppression exponent of Schwinger process is proportional to $1/|F|$ [27]; this enhancement vanishes [28] at $E = 2 M_S$. The final result for the semiclassical exponent of collision–induced SA pair production should be obtained by taking the limit $|F| \to 0$ at $E > 2 M_S$.

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