Estimate of the Fundamental Solution for Parabolic Operators with Discontinuous Coefficients

Michele Di Cristo∗ Kyoungsun Kim† Gen Nakamura‡

Abstract. We will show that the same type of estimates known for the fundamental solutions for scalar parabolic equations with smooth enough coefficients hold for the first order derivatives of fundamental solution with respect to space variables of scalar parabolic equations of divergence form with discontinuous coefficients. The estimate is very important for many applications. For example, it is important for the inverse problem identifying inclusions inside a heat conductive medium from boundary measurements.

Mathematics Subject Classification(2000): 35R30.

1 Introduction

Let $\mathcal{L}$ be a parabolic operator of the form

$$\mathcal{L} = \partial_t - \nabla \cdot A \nabla$$ (1.1)

with an $n \times n$ matrix $A = (a_{ij}) \in L^\infty(D)$ and a bounded domain $D \subset \mathbb{R}^n$ with boundary $\partial D$ of Lipschitz class. $D$ has compactly embedded subdomain $D_m (m = 1, 2, \ldots, L)$ with boundaries

∗Department of Mathematics, Polytechnic University of Milan, Milan 20133, Italy. This work was supported by JSPS Postdoctoral Fellowship for Foreign Researchers (PE 08039)

†Department of Mathematics, Ehwa Womans University, Seoul 120-750, Korea. This work was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD)(KRF-2006-214-C00007).

‡Department of Mathematics, Hokkaido University, Sapporo 080-061, Japan. This work was partially supported by Grant-in-Aid for Scientific Research (B)(No. 19340028) of Japan Society for Promotion of Science.
\[ \partial D_m (m = 1, 2, \cdots, L) \text{ of } C^{1, \alpha} \text{ class such that } \overline{D}_\ell \cap \overline{D}_m = \emptyset (\ell \neq m), \overline{D}_m \subset D \text{ (1 \leq m \leq L)}, \] where \( 0 < \alpha < 1 \). Moreover, there exists a constant \( \delta > 0 \) such that
\[ \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \delta \sum_{i=1}^n \xi_i^2 \quad (\xi \in \mathbb{R}^n, \text{ a.e. } x \in D). \tag{1.2} \]
Also, we assume that \( A \in C^\mu(\overline{D}_m) \) (1 \leq m \leq L) with 0 < \( \mu < 1 \).

We want to show the following theorem which is quite important in inverse problems for heat equations with discontinuous coefficients. (See the remark after the following theorem.)

**Theorem 1.1.** There exists a fundamental solution \( \Gamma(x, t; y, s) \) of \( \mathcal{L} \) with the estimates
\[ 0 < \Gamma(x, t; y, s) \leq \frac{C}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{C(t-s)}}, \tag{1.3} \]
\[ |\nabla_x \Gamma(x, t; y, s)| \leq \frac{C}{(t-s)^{(n+1)/2}} e^{-\frac{|x-y|^2}{C(t-s)}} \tag{1.4} \]
for any \( t, s \in \mathbb{R}, t > s \) and almost every \( x, y \in D \), where \( C > 0 \) is a constant depending only on \( A \) and \( n \).

**Remark 1.2.**

(i) For the simplicity of notations, we confined our argument to scalar parabolic operators of divergence form without zeroth order term. However, our argument can be generalized not only to more general scalar parabolic operators, but also to parabolic systems.

(ii) The estimate (1.3) is the well known estimate of the fundamental solution (2). We will call the estimate (1.4) gradient estimate. This gradient estimate is quite crucial for the dynamical probe method (3) and stability estimate for the inverse boundary value problem (3). Here, the dynamical probe method is a reconstruction scheme for the inverse boundary value problem identifying an unknown discontinuities of a medium inside a known heat conductor from our measurements on the boundary which are so called Dirichlet to Neumann map or the Neumann to Dirichlet map. The graphs of these maps are nothing but the set of infinitely many Cauchy data of the solutions to the forward problem of this inverse problem. Also, the stability estimate is the estimate of continuity of the map which maps the set of Cauchy data to the unknown this continuity of media.

We will show later that the gradient estimate of the fundamental solution follows from the following interior estimate following the argument given in (3).
Theorem 1.3 (Main Theorem). Let $0 < r < T$, $\overline{r\Omega} \subset D$ and $u \in W(r\Omega \times (-r^2, r^2))$; $u \in L^2((-r^2, r^2); H^1_0(r\Omega))$; $\partial_t u \in L^2((-r^2, r^2); H^{-1}(r\Omega))$ be a solution of $(\partial_t - \nabla \cdot A\nabla)u = 0$ in $r\Omega \times (-r^2, r^2)$. Then, there exists a constant $c > 0$ such that for any $0 < \rho < r/2$ and $(x, t) \in (r - 2\rho)\Omega \times (-r^2 + 4\rho^2, r^2)$, we have

$$\|\nabla_x u\|_{L^\infty(\rho\Omega(x) \times (-\rho^2 + t, t))} \leq c \frac{\|u\|_{L^2(\rho\Omega(x) \times (-4\rho^2 + t, t))}}{\rho^{n/2 + 2}},$$

where $\Omega(x) := \{y = (y_1, \cdots, y_n) \in \mathbb{R}^n; |x_i - y_i| < 1 (1 \leq i \leq n)\}$ with $x = (x_1, \cdots, x_n)$ and $\Omega := \Omega(0)$.

Remark 1.4. In [3] there is also a statement of the corresponding main theorem. Here we will give a proof of the main theorem based on the argument of [7] for second order elliptic systems of divergence forms which could provide a proof of the main theorem even in the case the inclusions touch.

By translation, rotation and scaling, it is enough to prove the following.

Proposition 1.5. Let $A$ as in Theorem 1.3. Then, for any solution $u \in W(\Omega \times (-1, 1))$ of $(\partial_t - \nabla \cdot A\nabla)u = 0$ in $\Omega \times (-1, 1)$, there exists a constant $c > 0$ depending only on the bound of $A$ and $n$ such that

$$|\nabla_x u(0, \tau)| \leq c \|u\|_{L^2(\frac{3}{4}\Omega \times (\frac{1}{3} - \frac{1}{4}, \tau))} \quad (\tau \in (\frac{1}{3}, 1))$$

whenever $\nabla_x u(0, \tau)$ exists.

The rest of this paper is organized as follows. In the next section we will give the proof of Proposition 1.5 by assuming the existence of a Green function in a two layered cube with Dirichlet boundary condition and its estimates. The proof of the gradient estimate of a fundamental solution is given in section 3. Finally, in Appendix, we give a construction and estimates of the Green function in the two layered cube with Dirichlet boundary condition.

2 Proof of Proposition 1.5

We will adapt the proof of Li-Nireberg’s paper [7]. To begin with, we note that Lemma 4.3 in [7] also holds if $\|\cdot\|_{Y^{s+\varepsilon, 2}}$ is replaced by $\|\cdot\|_{Y^{s+\varepsilon, p}}$ for any $p > 1$. That is
Lemma 2.1 (Lemma 4.3'). For $0 < \alpha' \leq \min\{\mu, \frac{\alpha}{p(\alpha+1)}\}$, there exists a constant $E > 0$ depending only on $\|A\|_{C^{\alpha'}(D_m)}$ for these $D_m$’s which intersect with $\Omega$ such that

$$\|A - \overline{A}\|_{Y^{1+\alpha',p}} \leq E,$$

where $\overline{A}$ is defined as in [7] for the very special case i.e. the two layered cube $\Omega$.

Proof. This can be easily proved by checking the proof in [7]. \qed

Remark 2.2. Hereafter, names of theorems in parentheses correspond to those of the theorem in [7]. For example, Lemma 2.1 (Lemma 4.3’) correspond to Lemma 4.3 in [7].

Next we generalize Lemma 3.1 in [7] in a special way to the parabolic operator (1.1). That is we have the following:

Lemma 2.3 (Lemma 3.1’). For $0 < \varepsilon < 1$, let the previous $A$ satisfy

$$\|A - \overline{A}\|_{Y^{1+\alpha',p}} \leq \varepsilon,$$

where $p$ will be specified in the proof. Here, we note that we have properly scaled $A$ so that by applying Lemma 2.1 (Lemma 4.3’), (2.2) is satisfied. Then, for any $f \in L^2((-1, \tau); L^2(\Omega))$ and solution $u \in W(\Omega \times (-1, \tau))$ of

$$(\partial_t - \nabla \cdot A \nabla)u = \nabla \cdot f \quad \text{in} \quad \Omega \times (-1, \tau),$$

there exists a solution $v \in W(\frac{3}{4} \Omega \times (\frac{7}{16} \tau - \frac{9}{16}, \tau))$ of

$$(\partial_t - \nabla \cdot \overline{A} \nabla)v = 0 \quad \text{in} \quad \frac{3}{4} \Omega \times (\frac{7}{16} \tau - \frac{9}{16}, \tau)$$

and $\frac{3}{4} < \sigma_0 < 1$ such that for any $0 < \sigma \leq \sigma_0$

$$\|u - v\|_{L^2(\sigma \Omega \times (\frac{7}{16} \tau - \frac{9}{16}, \tau))} \leq C\left(\|f\|_{L^2(\Omega \times (-1, \tau))} + e^{1/2}||u||_{L^2(\Omega \times (-1, \tau))}\right),$$

where $C > 0$ depends only on $n$ and $A$.

Proof. It is enough to prove the estimate for the case that $\Omega$ is divided into two parts by the boundary $\partial D_m$ of one of $D_m$ (1 ≤ $m$ ≤ $L$) such that the center $0 \in \partial D_m$ of $\Omega$ and $\Omega$ does not contain any portion of $\partial D_\ell$ ($\ell \neq m$, 1 ≤ $\ell$ ≤ $m$).
Take a cutoff function \( \zeta \in C^0_0(\Omega \times (-1, 2\tau)) \). By the definition of weak solution, we have
\[
\int_{-1}^{\tau} \{ (\partial_t u, \varphi) + (A \nabla u, \nabla \varphi) \} dt = - \int_{-1}^{\tau} (f, \nabla \varphi) dt
\]
for any \( \varphi \in H^1_0(\Omega) \).

If we take \( \varphi = \zeta^2 u \), then we have
\[
\int_{-1}^{\tau} \{ (\partial_t u, \zeta^2 u) + (A \nabla u, \nabla (\zeta^2 u)) \} dt = - \int_{-1}^{\tau} (f, \nabla (\zeta^2 u)) dt.
\]
Here, we have
\[
\text{LHS} \geq \frac{1}{2} \int_{-1}^{\tau} (\zeta^2 u^2)(\cdot, \tau) dx + \int_{-1}^{\tau} \int_{\Omega} \zeta \xi u^2 dx dt
\]
\[
+ \delta \int_{-1}^{\tau} \int_{\Omega} \zeta^2 |\nabla u|^2 dx dt - C \int_{-1}^{\tau} \int_{\Omega} (\zeta |\nabla u|)(u|\nabla \zeta|) dx dt,
\]
where \( C > 0 \) denotes any general constant. We further have
\[
\text{LHS} \geq \frac{\delta}{2} \int_{-1}^{\tau} \int_{\Omega} \zeta^2 |\nabla u|^2 dx dt - C(\delta) \int_{-1}^{\tau} \int_{\Omega} u^2 dx dt,
\]
where \( C(\delta) > 0 \) denotes any general constant depending on \( \delta \).

On the other hand, we have
\[
\text{RHS} = - \int_{-1}^{\tau} \int_{\Omega} (\zeta f \cdot (\zeta \nabla u) + uf \cdot \nabla (\zeta^2)) dx dt
\]
\[
\leq \frac{\delta}{4} \int_{-1}^{\tau} \int_{\Omega} \zeta^2 |\nabla u|^2 dx dt + C(\delta) \int_{-1}^{\tau} \int_{\Omega} (u^2 + |f|^2) dx dt,
\]
Hence,
\[
\frac{\delta}{4} \int_{-1}^{\tau} \int_{\Omega} \zeta^2 |\nabla u|^2 dx dt \leq C(\delta) \int_{-1}^{\tau} \int_{\Omega} (u^2 + |f|^2) dx dt.
\]
By \( |\nabla (\zeta u)|^2 \leq 2\zeta^2 |\nabla u|^2 + 2u^2 |\nabla \zeta|^2 \), we have
\[
\frac{\delta}{8} \int_{-1}^{\tau} \int_{\Omega} |\nabla (\zeta u)|^2 dx dt \leq C(\delta) \int_{-1}^{\tau} \int_{\Omega} (u^2 + |f|^2) dx dt.
\]
Further, let $\zeta$ to satisfy $\zeta = 1$ in a neighborhood of $\frac{2}{3}\Omega \times (\frac{9}{25} \tau - \frac{16}{25}, \tau)$, we have

$$
\|u\|_{L^2((1-\sigma_0^2)\tau - \sigma_0^2, \tau); H^1(\partial(\sigma_0 \Omega))} \leq C(\delta)(\|u\|_{L^2(\Omega \times (-1, \tau))} + \|f\|_{L^2(\Omega \times (-1, \tau))}).
$$

By (2.4) and the Fubini theorem, there exists $\frac{3}{4} \leq \sigma_0 < 1$ such that

$$
\|u\|_{L^2((1-\sigma_0^2)\tau - \sigma_0^2, \tau); H^1(\partial(\sigma_0 \Omega))} \leq C(\delta)(\|u\|_{L^2(\Omega \times (-1, \tau))} + \|f\|_{L^2(\Omega \times (-1, \tau))}).
$$

Let $v \in W(\sigma \Omega \times (-1, \tau))$ be the solution to

$$
\begin{cases}
(\partial_t - \nabla \cdot \overline{A} \nabla)v = 0 & \text{in } \sigma_0 \Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau), \\
v = u & \text{on } \partial(\sigma_0 \Omega) \times ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau), \\
v = u & \text{on } \sigma_0 \Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau).
\end{cases}
$$

Then, $w := u - v$ satisfies

$$
\begin{cases}
(\partial_t - \nabla \cdot \overline{A} \nabla)w = \nabla \cdot ((A - \overline{A}) \nabla u) + \nabla \cdot f & \text{in } \sigma_0 \Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau), \\
w = 0 & \text{on } \partial(\sigma_0 \Omega) \times ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau), \\
w = 0 & \text{on } \sigma_0 \Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau).
\end{cases}
$$

Further, let $G^*(x, t; y, s)$ be the Green function such that for $y \in \sigma \Omega$ and $s \in \mathbb{R}$. That is, $G^*(x, t; y, s)$ is a distribution which satisfies

$$
\begin{cases}
(\partial_t + \nabla \cdot \overline{A} \nabla)G^*(x, t; y, s) = 0 & \text{in } \sigma_0 \Omega \times (-\infty, s), \\
G^*(x, t; y, s) = 0 & \text{on } \partial(\sigma_0 \Omega) \times (-\infty, s), \\
\lim_{t \uparrow s} \int_{\sigma_0 \Omega} G^*(x, t; y, s) \varphi(x) dx = \varphi(y) \quad (\varphi \in C_0^\infty(\sigma_0 \Omega)).
\end{cases}
$$

Later in the appendix, we will provide the construction of $G^*(x, t; y, s)$ and proof of its estimate:

$$
|\partial_\alpha^\gamma G^*(x, t; y, s)| \leq c_\alpha (s - t)^{-\frac{n+|\alpha|}{2}} e^{-\frac{|x-y|^2}{2(t-s)}} \quad (t, s \in \mathbb{R}, \ t < s, \ a.e. \ x, y \in \sigma_0 \Omega).
$$

for any $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq 1$ with some constant $c_\alpha > 0$. 

6
By the Green formula, we have

$$w(x, t) = -\int_{(1-\sigma_0^2)\tau-\sigma_0^2}^\tau \int_{\sigma_0\Omega} \left\{(A - \overline{A})\nabla u(y, s) + f(y, s)\right\} \cdot \nabla_y G^*(y, s; x, t) dy ds.$$  \hspace{1cm} (2.6)

for \(x \in \sigma_0\Omega, t \in ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau).

To proceed further, we need the following Lemma 2.4 which is well-known in Fourier analysis.

**Lemma 2.4.** Let \((X_i, M_i, m_i) (i = 1, 2)\) be \(\sigma\) finite complete measure spaces and \((X_1 \times X_2, M_1 \otimes M_2, m_1 \times m_2)\) be the product space with complete measure \(m_1 \times m_2\). Also, let \(p_1, p_2\) and \(q \in [1, \infty)\) satisfy \(\frac{1}{p_1} + \frac{1}{q} = \frac{1}{p_1} + 1\) and a measurable function \(K(x_1, x_2)\) on \(X_1 \times X_2\) satisfy

$$\int_{X_1} |K(x_1, x_2)|^q m_1(dx_1) \leq L_1 \quad (a.e. \: x_2 \in X_2),$$

$$\int_{X_2} |K(x_1, x_2)|^q m_2(dx_2) \leq L_2 \quad (a.e. \: x_1 \in X_1).$$

Then, for any \(f(x_2) \in L^{p_2}(X_2)\), we have

$$\|Kf\|_{L^{p_1}(X_1)} \leq L_1^{1/p_1} L_2^{1-1/p_2}\|f\|_{L^{p_2}(X_2)},$$

where

$$(Kf)(x_1) = \int_{X_2} K(x_1, x_2) f(x_2) \, m_2(dx_2).$$

We apply this Lemma 2.4 to the kernel \((A - \overline{A})(y, s)\nabla_y G^*(y, s; x, t) ((x, t), (y, s) \in \sigma_0\Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau), s < t)\) by taking \(q = 1, p_1 = p_2 = 2\) and \(X_1 = X_2 = \sigma_0\Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau)\).

Let

$$I_1(y, s) = \int_{(1-\sigma_0^2)\tau-\sigma_0^2}^\tau \int_{\sigma_0\Omega} \left|(A - \overline{A})(y, s)\nabla_y G^*(y, s; x, t)\right| dx dt,$$

and

$$I_2(x, t) = \int_{(1-\sigma_0^2)\tau-\sigma_0^2}^\tau \int_{\sigma_0\Omega} \left|(A - \overline{A})(y, s)\nabla_y G^*(y, s; x, t)\right| dy ds.$$

Since \(A, \overline{A}\) are bounded and \(\int_{\mathbb{R}_+} e^{-\frac{|x|^2}{\alpha \omega}} dx = O((t - s)^{n/2}) \: (s < t),\)

$$I_1(y, s) \leq C \int_s^\tau (t - s)^{-1/2} \, dt = 2C(\tau - s)^{1/2} \leq C.$$
Similarly, by taking \( p > n + 2 \),
\[
I_2(x, t) \leq \left( \int_0^T \int_{(1-\sigma_0^2)\tau - \sigma_0^2 \partial^2_{\sigma_0} \Omega} \left| (A - \overline{A})(y, s) \right|^p dy ds \right)^{1/p} \left( \int_0^T \int_{(1-\sigma_0^2)\tau - \sigma_0^2 \partial^2_{\sigma_0} \Omega} \left| \nabla G'(y, s; x, t) \right|^{p'} dy ds \right)^{1/p'}
\]
\[
\leq C(\tau - (1 - \sigma_0^2 )\tau + \sigma_0^2 )^{1/p} \| A - \overline{A} \|_{L^p(\Omega)} \left( \int_0^T (t - s)^{(n/2 - (n + 1)p^*)/2} ds \right)^{1/p}.
\]
with \( 1/p + 1/p^* = 1 \). Here, by \( p > n + 2, n/2 - (n + 1)p^*/2 > -1 \). Hence, \( I_2(x, t) \leq C \| A - \overline{A} \|_{L^p(\Omega)} \).

Therefore, by Lemma (2.4) and (2.4),
\[
\| w \|_{L^2((\sigma_0 \Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2 \partial^2_{\sigma_0} \tau), \tau))} \leq C\left( \| A - \overline{A} \|_{L^p(\Omega)}^{1/2} \| \nabla u \|_{L^2((\sigma_0 \Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2 \partial^2_{\sigma_0} \tau)), \tau))} + \| f \|_{L^2((\sigma_0 \Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2 \partial^2_{\sigma_0} \tau), \tau))} \right).
\]

Since \( \| A - \overline{A} \|_{L^{1,p},p} < \varepsilon \) implies \( \| A - \overline{A} \|_{L^p(\Omega)} < C\varepsilon \), we have
\[
\| w \|_{L^2((\sigma_0 \Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2 \partial^2_{\sigma_0} \tau), \tau))} \leq C\left( \varepsilon^{1/2} \| u \|_{L^2(\Omega \times (-1,1), \tau))} + \| f \|_{L^2(\Omega \times (-1,1), \tau))} \right).
\]
Further, by applying (2.4) to \( w \), we have for a smaller \( \sigma_0 \)
\[
\| w \|_{L^2(((1 - \sigma^2)\tau - \sigma^2 \partial^2_{\sigma} \tau), H^1(\sigma_0 \Omega))} \leq C\left( \varepsilon^{1/2} \| u \|_{L^2(\Omega \times (-1,1), \tau))} + \| f \|_{L^2(\Omega \times (-1,1), \tau))} \right) \quad (0 < \sigma \leq \sigma_0).
\]

This ends the proof. □

Proof of (1.6). We adapt the proof of Proposition 4.1 in [7] to our case.

Let \( \sigma_k = \frac{3}{4^k + 1} \), \( \overline{\sigma}_k = \frac{2}{4^k} \), \( \overline{\sigma}_k = \frac{1}{4^k} \) (\( k = 0, 1, 2, \cdots \)) and \( M = \| u \|_{L^2(\frac{1}{4^k} \Omega \times (\frac{3}{4^k}, \frac{7}{4^k}))} \). We will prove by induction that there exist \( w_k \in W(\sigma_k \Omega \times ((1 - \sigma_k^2)\tau - \sigma_k^2 \partial^2_{\sigma_k} \tau)) \) (\( k = 0, 1, 2, \cdots \)) which satisfy
\[
(\partial_t - \nabla \cdot \overline{A} \nabla) w_k = 0 \quad \text{in} \quad \sigma_k \Omega \times ((1 - \sigma_k^2)\tau - \sigma_k^2 \partial^2_{\sigma_k} \tau), \quad (4.3)'
\]
\[
\| w_k \|_{L^2((\overline{\sigma}_k \Omega \times ((1 - \overline{\sigma}_k^2)\tau - \overline{\sigma}_k^2 \partial^2_{\overline{\sigma}_k} \tau), \tau))} \leq CM^4 \frac{(k + 4 + 2n')}{2}, \quad (4.4)_{1,k}'
\]
\[
\| \nabla w_k \|_{L^\infty((\overline{\sigma}_k \Omega \times ((1 - \overline{\sigma}_k^2)\tau - \overline{\sigma}_k^2 \partial^2_{\overline{\sigma}_k} \tau), \tau))} \leq CM^4 \frac{\kappa'^2}{\alpha'}, \quad (4.4)_{2,k}'
\]
and
\[
\| u - \sum_{j=0}^k w_j \|_{L^\infty((\overline{\sigma}_k \Omega \times ((1 - \overline{\sigma}_k^2)\tau - \overline{\sigma}_k^2 \partial^2_{\overline{\sigma}_k} \tau), \tau))} \leq M^4 \frac{(k + 1)k + 2n'}{2}. \quad (4.5)'
\]

Before starting the induction argument, we note that for any \( \varepsilon_0 > 0 \), we have \( \| A - \overline{A} \|_{L^{1,p},p} \leq \varepsilon_0 \) by considering a dilated \( \Omega \) instead of \( \Omega \), and \( \| A - \overline{A} \|_{L^p(\Omega)} \leq \| A - \overline{A} \|_{L^{1,p},p} \).
Since \( u \in W(\mathcal{L} \times (\frac{1}{4} \tau - \frac{1}{4}, \tau)) \) solves \((\partial_t - \nabla \cdot A \nabla)u = 0\) in \(\mathcal{L} \times (\frac{1}{4} \tau - \frac{1}{4}, \tau)\), we have from Lemma 2.3 that there exists a solution \( w_0 \in W(\sigma_0 \mathcal{L} \times ((1 - \sigma_0^2) \tau - \sigma_0^2 \tau, \tau)) \) of \((\partial_t - \nabla \cdot A \nabla) w_k = 0\) in \(\sigma_0 \mathcal{L} \times ((1 - \sigma_0^2) \tau - \sigma_0^2 \tau, \tau)\) with the estimates:

\[
\|u - w_0\|_{L^2(\mathcal{L} \times ((1 - \sigma_0^2) \tau - \sigma_0^2 \tau, \tau))} \leq C_0 \varepsilon^{1/2} M \leq M 4^{-\alpha + 2 \alpha'} (4.5)'_0
\]

by taking \( \varepsilon_0 > 0 \) small enough to satisfy \( C_0 \varepsilon^{1/4} \leq 1, \varepsilon_0^{1/4} \leq 4^{-\alpha + 2 \alpha'} \) and hence

\[
\|w_0\|_{L^2(\sigma_0 \mathcal{L} \times ((1 - \sigma_0^2) \tau - \sigma_0^2 \tau, \tau))} \leq 2M, (4.4)'_1,0
\]

and from the interior estimate for \( \partial_t - \nabla \cdot A \nabla \), we have

\[
\|\nabla_x w_0\|_{L^2(\sigma_0 \mathcal{L} \times ((1 - \sigma_0^2) \tau - \sigma_0^2 \tau, \tau))} \leq C_0 M.
\]

Hereafter, \( C_0 > 0 \) is a general constant for the estimate of solutions for our parabolic operators which is independent of the general constant \( C \) in the estimates (4.4)'-(4.5)'.

Suppose (4.3)'-(4.5)' hold up to \( k \geq 0 \). Then, we will prove them for \( k + 1 \). Let

\[
W(x, t) = \left( u - \sum_{j=0}^k w_j \right)(\sigma_k x, \sigma_k^2 t + (1 - \sigma_k^2) \tau) \quad ((x, t) \in \mathcal{L} \times (-1, \tau)),
\]

\[
A_{k+1}(x) = A(\sigma_k x), \quad \overline{A}_{k+1}(x) = \overline{A}(\sigma_k x)
\]

and

\[
f_{k+1}(x, t) = \overline{\sigma}_0(A_{k+1} - \overline{A}_{k+1})(x) \sum_{j=0}^k \nabla w_j(\sigma_k x, \sigma_k^2 t + (1 - \sigma_k^2) \tau).
\]

Then, it is not hard to see that \( (\partial_t - \nabla \cdot A_{k+1} \nabla) W = \nabla \cdot f_{k+1} \) in \(\mathcal{L} \times (-1, \tau)\). Further, we have

\[
\sum_{j=0}^k |(\nabla_x w_j)(\overline{\sigma}_k x, \overline{\sigma}_k^2 t + (1 - \overline{\sigma}_k^2) \tau)| \leq CM \sum_{j=0}^k 4^{-j \alpha'} \leq \frac{CM}{1 - 4^{-\alpha'}} \quad ((x, t) \in \mathcal{L} \times (-1, \tau))
\]

by (4.4)'_2, and

\[
\|W\|_{L^2(\mathcal{L} \times (-1, \tau))} \leq M 4^{-(k+1)(1 + \alpha')}
\]

by (4.5)'_k. Observe that

\[
\|A_{k+1} - \overline{A}_{k+1}\|_{L^2(\mathcal{L})} \leq 4^{-(k+1)\alpha'} \|A - \overline{A}\|_{Y_{1+\alpha', 2}} \leq 4^{-(k+1)\alpha'} \|A - \overline{A}\|_{Y_{1+\alpha', \varphi}} \leq 4^{-(k+1)\alpha'} \varepsilon_0.
\]

Together with this and (4.4)'_2, we have

\[
\|f_{k+1}\|_{L^2(\mathcal{L} \times (-1, \tau))} \leq CM 4^{-(k+1)(1 + \alpha')} \varepsilon_0 \sum_{j=0}^k 4^{-j \alpha'} \leq \frac{CM}{1 - 4^{-\alpha'}} \varepsilon_0.
\]
By Lemma 2.3 there exists a solution $v_{k+1} \in W(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))$ of $(\partial_t - \nabla \cdot \mathbf{A}_k \nabla) v_{k+1} = 0$ in $\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau)$ with the estimate

$$
\|W - v_{k+1}\|_{L^2(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))} \leq C_0(\|f_{k+1}\|_{L^2(\Omega \times (-1,\tau))} + 4^{-k+1} \varepsilon_0 \|W\|_{L^2(\Omega \times (-1,\tau))})
$$

$$
\leq C_0 M 4^{-k+1}(1+a') \left( \frac{C}{1 - 4^{-\alpha'} \varepsilon_0 + \varepsilon_0^{1/2}} \right).
$$

Let $w_{k+1}(x, t) = v_{k+1}(\sigma_k^{-1} x, \sigma_k^{-2} t + (1 - \sigma_k^{-2} \tau)) ((x, t) \in \sigma_{k+1} \Omega \times ((1 - \sigma_{k+1}^2)\tau - \sigma_{k+1}^2, \tau))$. Then, it is easy to see that $(\partial_t - \nabla \cdot \mathbf{A}_k \nabla) w_{k+1} = 0$ in $\sigma_{k+1} \Omega \times ((1 - \sigma_{k+1}^2)\tau - \sigma_{k+1}^2, \tau))$. Further, by

$$
(W - v_{k+1})(x, t) = (u - \sum_{j=0}^{k+1} w_j)(\sigma_k x, \sigma_k^{-2} t + (1 - \sigma_k^{-2})\tau),
$$

i.e.

$$
(u - \sum_{j=0}^{k+1} w_j)(x, t) = (W - v_{k+1})(\sigma_k^{-1} x, \sigma_k^{-2} t + (1 - \sigma_k^{-2})\tau).
$$

Since $\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau) \subset \sigma_{k+1} \Omega \times ((1 - \sigma_{k+1}^2)\tau - \sigma_{k+1}^2, \tau)$ and $(x, t) \in \sigma_{k+1} \Omega \times ((1 - \sigma_{k+1}^2)\tau - \sigma_{k+1}^2, \tau)$ which equivalents to $(\sigma_k^{-1} x, \sigma_k^{-2} t + (1 - \sigma_k^{-2})\tau) \in \sigma_0 \Omega \times ((1 - \sigma_0^2)\tau - \sigma_0^2, \tau)$, we have

$$
\left\| u - \sum_{j=0}^{k+1} w_j \right\|_{L^2(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))} = \sigma_k^{-2r/2} \| W - v_{k+1} \|_{L^2(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))}
$$

and hence by the estimate of $\| W - v_{k+1} \|_{L^2(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))}$, we have

$$
\left\| u - \sum_{j=0}^{k+1} w_j \right\|_{L^2(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))} \leq C_0 M 4^{-(k+1)(1+a')}(1+a') \sigma_k^{-(r+2)/2} \left( \frac{C}{1 - 4^{-\alpha'} \varepsilon_0 + \varepsilon_0^{1/2}} \right)
$$

$$
\leq 2C_0 M \varepsilon_0^{1/4} \max \left( \frac{C}{1 - 4^{-\alpha'}}, 1 \right)^{4^{-(k+1)(1+a')} - 1}.
$$

Therefore, by taking $\varepsilon_0 > 0$ small enough to satisfy $2C_0 \varepsilon_0^{1/4} \max \left( \frac{C}{1 - 4^{-\alpha'}}, 1 \right) \leq 1$, we have (4.5)$_{k+1}$.

By the estimate of $\| W - v_{k+1} \|_{L^2(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))}$ and the interior estimate for $\partial_t - \nabla \cdot \mathbf{A}_k \nabla$,

$$
\| \nabla v_{k+1} \|_{L^\infty(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))} \leq C_0 \| v_{k+1} \|_{L^2(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))}
$$

$$
\leq C_0 \| W \|_{L^2(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))} + \| W - v_{k+1} \|_{L^2(\sigma_0 \Omega \times ((1-\sigma_0^2)\tau - \sigma_0^2, \tau))}
$$

$$
\leq C_0 M \left[ 4^{-(k+1)(1+a')} + C_0 4^{-(k+1)(1+a')} \left( \frac{C}{1 - 4^{-\alpha'} \varepsilon_0 + \varepsilon_0^{1/2}} \right) \right]
$$

$$
\leq C_0 M 4^{-(k+1)(1+a')} \left[ 1 + 2C_0 \varepsilon_0^{1/2} \max \left( \frac{C}{1 - 4^{-\alpha'}}, 1 \right) \right].
$$
Hence, if we adjust C and \(\varepsilon_0\) to satisfy \(C \geq C_0[1 + 2C_0\varepsilon_0^{1/2}\max(\frac{C}{1-4^{-\nu}}, 1)]\), then we have (4.4)′\(_{2,k+1}\). Finally, to see (4.4)′\(_{1,k+1}\), we have from the above estimate of \(\|v_k\|_{L^2(\overline{\sigma}_0\Omega \times ((1-\overline{\sigma}_0^2)\tau_0, \tau])}\) and \((x, t) \in \overline{\sigma}_{k+1}\Omega \times ((1-\overline{\sigma}_{k+1}^2)\tau_0, \tau)\) which is equivalent to \((\overline{\sigma}_k^2 t, 1 + (1 - \overline{\sigma}_k^2)\tau_0 \tau) \in \overline{\sigma}_0\Omega \times ((1 - \overline{\sigma}_0^2)\tau - \overline{\sigma}_0^2, \tau).\) we have

\[
\|w_k\|_{L^2(\overline{\sigma}_{k+1}\Omega \times ((1-\overline{\sigma}_{k+1}^2)\tau_0, \tau))} \leq C_0M4^{-(k+1)(1+\alpha')/2} \left[1 + 2C_0\varepsilon_0^{1/2}\max\left(\frac{C}{1-4^{-\nu}}, 1\right)\right].
\]

Therefore, if we further adjust C and \(\varepsilon_0\) to satisfy \(C \geq C_0[1 + 2C_0\varepsilon_0^{1/2}\max(\frac{C}{1-4^{-\nu}}, 1)]\), then we have (4.4)′\(_{1,k+1}\). Thus, we have proven (4.3)′ – (4.5)′.

Let C be a general constant which is different from the general constant C in (4.4)′ – (4.5)′. As an easy consequence of (4.4)′\(_{2,k}\), we have

\[
\|w_k\|_{L^\infty(\overline{\sigma}_2\Omega \times ((1-\overline{\sigma}_2^2)\tau_0, \tau))} \leq CM4^{-k(1+\alpha')}.
\]

(4.6)′

Together with this and (4.4)′\(_2\),

\[
\left| \sum_{j=0}^k w_j(x, t) - \sum_{j=0}^\infty w_j(0, \tau) \right| \leq CM \sum_{j=0}^k 4^{-j\alpha'} |(x, t - \tau)| + CM \sum_{j=k+1}^\infty 4^{-j(1+\alpha')}
\]

\[
\leq CM |(x, t - \tau)| + CM4^{-k(1+\alpha')}.
\]

Hence, we have

\[
\left\| u - \sum_{j=0}^\infty w_j(0, \tau) \right\|_{L^2(\overline{\sigma}_1\Omega \times ((1-\overline{\sigma}_1^2)\tau_0, \tau))} \leq \left\| u - \sum_{j=0}^k w_j \right\|_{L^2(\overline{\sigma}_1\Omega \times ((1-\overline{\sigma}_1^2)\tau_0, \tau))} + \left\| \sum_{j=0}^\infty w_j - \sum_{j=0}^k w_j(0, \tau) \right\|_{L^2(\overline{\sigma}_1\Omega \times ((1-\overline{\sigma}_1^2)\tau_0, \tau))}
\]

\[
\leq M4^{-\frac{(k+1)(n+2+2\alpha')}{2}} + CM \left[ \int \int_{(1-\overline{\sigma}_1^2)\tau_0, \overline{\sigma}_1\Omega} \left( |x|^2 + (t-\tau)^2 + 4^{-2k(1+\alpha')} \right) dx \right]^{1/2}.
\]

Hence, by \((t - \tau)^2 \leq \overline{\sigma}_k^4 \tau \leq 4^{-4(k+1)} \tau \), we can absorb \((t - \tau)^2\) into \(4^{-2k(1+\alpha')}\). Since \(\int_{\overline{\sigma}_1\Omega} |x|^2 dx \leq C4^{-(k+1)(n+2)}\), \(\int_{\overline{\sigma}_1\Omega} 4^{-2k(1+\alpha')} dx \leq C4^{-2k(1+\alpha')-(k+1)n}\) and \(4^{-2k(1+\alpha')-(k+1)n} \leq 4^{-(k+1)(n+2)}\) for large k, we have

\[
\left\| u - \sum_{j=0}^\infty w_j(0, \tau) \right\|_{L^2(\overline{\sigma}_1\Omega \times ((1-\overline{\sigma}_1^2)\tau_0, \tau))} \leq CM4^{-\frac{(k+1)(n+2)}{2}} \overline{\sigma}_k.
\]

Therefore, \(u(0, \tau) = \sum_{j=0}^\infty w_j(0, \tau)\) and \(\nabla_x u(0, \tau) \leq CM\) if \(\nabla_x u(0, \tau)\) exists.
3 Gradient Estimate of Fundamental Solution

In this section, as we already mentioned in the introduction, we will give an estimate of $\nabla_x \Gamma(x, t; y, s)$ for a fundamental solution $\Gamma(x, t; y, s)$ of the operator $L$ as an application of our main theorem (Theorem 1.3) by following the argument given in [3]. For the readers’ convenience, we repeat the argument.

It is well known that there exists a fundamental solution $\Gamma(x, t; y, s)$ with the estimate

$$\Gamma(x, t; y, s) \leq C \left[\frac{\pi}{4(t-s)}\right]^{n/2} e^{-\frac{|x-y|^2}{4(t-s)}} \chi_{[s, \infty)} (t, s \in \mathbb{R}, t > s, \text{a.e. } x, y \in D),$$

(3.1)

which is positive for $t > s$, where $C > 0$ is a constant which depends only on $A, n$ and $\chi_{[s, \infty)}$ is the characteristic function of $[s, \infty)$. (See [2].)

Now we state the estimate of $\nabla_x \Gamma(x, t; y, s)$.

**Proposition 3.1.** Let $\Gamma(x, t; y, s)$ be the previous fundamental solution of the operator $\partial_t - \nabla \cdot A \nabla$. There exists a constant $C > 0$ depending only on $A$ and $n$ such that

$$|\nabla_x \Gamma(x, t; y, s)| \leq \frac{C}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}},$$

(3.2)

for any $t, s \in \mathbb{R}, t > s$ and almost every $x, y \in D$.

**Remark 3.2.** We recall that a fundamental solution $G^*(x, t; y, s)$ of the operator $\partial_t + \nabla \cdot A \nabla$ can be given by

$$G^*(x, t; y, s) = \Gamma(y, s; x, t) \quad ((x, t), (y, s) \in Q := D \times \mathbb{R}, (x, t) \neq (y, s)).$$

(3.3)

Hence, estimates similar to (3.1) and (3.2) hold for $G^*(x, t; y, s)$.

Before proving Proposition 3.1, we give the following estimate which is necessary for the proof.

**Proposition 3.3.** Let $Q_\rho(x_0, t_0) = B_\rho(x_0) \times (t_0 - \rho^2, t_0)$, $B_\rho(x_0) := \{x \in \mathbb{R}^n; |x - x_0| < \rho\}$. There exists a constant $C > 0$ depending only on $A$ and $n$ such that the following inequality holds.

$$\int_{Q_\rho(x_0, t_0)} |\Gamma(x, t; \xi, \tau)|^2 dx dt \leq C \frac{\rho^n}{(t_0 - \tau)^{n-1}} e^{-\frac{|x_0 - \xi|^2}{4(t_0 - \tau)}} \chi_{[0, \infty)} (\tau < t_0, \text{a.e. } \xi \in D),$$

(3.4)

where $\rho = \frac{1}{4}[|x_0 - \xi|^2 + t_0 - \tau]^{1/2}$. 

12
Proof. From the inequality (3.1) we have
\[
\int_{Q_{\rho}(x_0, t_0)} |\Gamma(x, t; \xi, \tau)|^2 dxdt \leq C_1 \int_{Q_{\rho}(x_0, t_0)} \frac{1}{(t-\tau)^n} e^{-\frac{|x-x'|^2}{2(t-\tau)}} \chi_{[\tau, +\infty)} dxdt,
\]
(3.5)
where \(C_1 > 0\) is a constant depending only on \(A\) and \(n\). In what follows we denote by \(I\) the integral at the right-hand side of (3.5). We distinguish two cases
i) \(t_0 - \rho^2 \leq \tau < t_0\),
ii) \(\tau < t_0 - \rho^2\).

Let us consider case i). It is easy to see that there exists a constant \(C > 0\) such that
\[
C^{-1} \rho \leq |x - \xi| \leq C \rho \quad (x \in B_{\rho}(x_0)).
\]
(3.6)
By (3.6) we have
\[
I \leq c_n \rho^n \int_0^{t_0-\tau} s^{-n} e^{\frac{\rho^2}{2s}} ds,
\]
(3.7)
where \(c_n > 0\) is a constant depending only on \(n\) and \(C_2 > 0\) is a constant depending only on \(A\) and \(n\). Now we assume \(0 < t_0 - \tau < \frac{\rho^2}{nC_2}\). Since \(s^{-n} e^{\frac{\rho^2}{2s}}\) is an increasing function in \((0, \frac{\rho^2}{nC_2})\), we have by (3.7)
\[
I \leq \frac{c_n \rho^n}{(t_0 - \tau)^{n-1}} e^{-\frac{\rho^2}{2(t_0 - \tau)}}.
\]
(3.8)
Further, if we assume \(\frac{\rho^2}{nC_2} \leq t_0 - \tau \leq \rho^2\), then
\[
\rho^{-n} I \leq C_1 \int_0^{\rho^2} s^{-n} e^{-\frac{\rho^2}{2s}} ds \leq \frac{C}{(t_0 - \tau)^{n-1}} e^{-\frac{\rho^2}{2(t_0 - \tau)}}
\]
due to the equivalence of \(t - \tau\) and \(\rho^2\), where \(c_n\) and \(C_2\) are the same kind of constants as before. Hence, by the last inequality and (3.8), we have the Proposition in case i).

Let us consider case ii). It is easy to see that
\[
6\rho^2 \leq |x - \xi|^2 + t - \tau \leq 60\rho^2,
\]
(3.9)
for every \((x, t) \in Q_{\rho}(x_0, t_0)\). Moreover, denoting
\[
M_{\rho} = \max \left\{ e^{-\frac{|x-x'|^2}{2(t-\tau)}}; (x, t) \in Q_{\rho}(x_0, t_0) \right\}
\]
where \(A > 0\) is a constant depending only on \(A\) and \(n\), and 0 < \(\rho < \min\{2\rho_0, 1\}\) for some \(\rho_0 > 0\). Thus, by (3.7) we have
\[
I \leq c_n \rho^n \int_0^{t_0-\tau} s^{-n} e^{\frac{\rho^2}{2s}} ds \leq C_1 \int_{Q_{\rho}(x_0, t_0)} \frac{1}{(t-\tau)^n} e^{-\frac{|x-x'|^2}{2(t-\tau)}} \chi_{[\tau, +\infty)} dxdt,
\]
(3.5)
and taking into account (3.9), we have

\[ M_\rho \leq C \left( \frac{C_1}{\rho^2} \right)^n, \tag{3.10} \]

where \( C > 0 \) is a constant depending only on \( n \). Now, since \( \tau < t_0 - \rho^2 \), we have

\[ \frac{|x_0 - \xi|^2}{t_0 - \tau} \leq 16. \tag{3.11} \]

Therefore, by (3.10) and (3.11), we have the Proposition in case ii) as well. \( \Box \)

**Proof of Proposition 3.7** By applying our main theorem to the function \( \Gamma(\cdot ; \xi, \tau) \), we have

\[ \| \nabla \Gamma(\cdot ; \xi, \tau) \|_{L^\infty(Q_{t_0}(x_0,t_0))} \leq C \rho^n \left[ \int_{Q_{t_0}(x_0,t_0)} |\Gamma(x,t;\xi,\tau)|^2 dxdt \right]^{1/2}. \tag{3.12} \]

Further, applying Proposition 3.3 to the right-hand side of (3.12) we have

\[ \| \nabla \Gamma(\cdot ; \xi, \tau) \|_{L^\infty(Q_{t_0}(x_0,t_0))} \leq C \rho^n \left( \frac{\rho^n}{(t_0 - \tau)^{n-1}} e^{\frac{|x_0 - \xi|^2}{2(t_0 - \tau)}} \right)^{1/2}. \]

Then, we immediately have (3.2), because

\[ \frac{1}{\rho^2} \leq \frac{C}{t_0 - \tau}. \]

\( \Box \)

### 4 Appendix A: Construction of Green Function in two Layered Cube

In this section we will construct the Green function \( G^*(x,t; y, s) \) of our operator \( \partial_t + \nabla \cdot A \nabla \) in \( \sigma_0 \Omega \times \mathbb{R} \) with Dirichlet boundary condition on \( \partial(\sigma_0 \Omega) \times \mathbb{R} \). If \( G(x,t; y, s) \) is the Green function of the operator \( L = \partial_t - \nabla \cdot A \nabla \) in \( \sigma_0 \Omega \times \mathbb{R} \) with Dirichlet boundary condition on \( \partial(\sigma_0 \Omega) \times \mathbb{R} \), we have

\[ G^*(x,t; y, s) = G(y, s; x, t). \tag{4.1} \]

Hence, it is enough to construct the Green function \( G(x,t; y, s) \).
First we construct a fundamental solution $\Gamma(x, t; y, s)$ of $\mathcal{L}$. We divide the construction into two cases. They are $y_n > 0$ and $y_n < 0$. We first consider the case $y_n > 0$. Let $A, B$ be positive definite symmetric constant matrices. $A = (a_{ij})_{1 \leq i, j \leq n}, B = (b_{ij})_{1 \leq i, j \leq n}$. Define $\overline{A} = A + (B - A)\chi_-(\xi)$, where

$$\chi_-(\xi) = \begin{cases} 0, & \xi_n > 0, \\ 1, & \xi_n < 0. \end{cases}$$

Let $\Gamma(x, t; y, s)$ be the fundamental solution for $\partial_t - \nabla \cdot (\overline{A} \nabla x)$, that is,

$$\partial_t \Gamma(x, t; y, s) - \nabla \cdot (\overline{A} \nabla \Gamma(x, t; y, s)) = \delta(x - y)\delta(t - s).$$

(4.2)

Note that $\Gamma(x, t; y, s)$ is also the fundamental of the Cauchy problem at $t = s$ for the operator $\partial_t - \nabla \cdot \overline{A} \nabla$. Let $\hat{\Gamma}$ be the Laplace transform of $\Gamma$ with respect to $t$, that is,

$$\hat{\Gamma}(x, \tau; y, s) = \int_0^\infty e^{-\tau t} \Gamma(x, t; y, s) dt.$$  (4.3)

Then $\hat{\Gamma}$ satisfies

$$\tau \hat{\Gamma}(x, \tau; y, s) - \nabla \cdot (\overline{A} \nabla \hat{\Gamma}(x, \tau; y, s)) = \delta(x - y)e^{-\tau s}.$$  (4.4)

Now, we denote $\Gamma$ for different regions as follows:

$$\Gamma = \begin{cases} \Gamma^{11} & \text{for } x_n > y_n, \\ \Gamma^{12} & \text{for } y_n > x_n > 0, \\ \Gamma^2 & \text{for } 0 > x_n. \end{cases}$$  (4.5)

For $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$0 = \int_{\mathbb{R}^n} [\tau \hat{\Gamma} \varphi + \overline{A} \nabla \hat{\Gamma} \cdot \nabla \varphi - e^{\tau s}\delta(x - y)\varphi]dx$$

$$= \int_{\mathbb{R}^n} [\tau \hat{\Gamma} \varphi - e^{\tau s}\delta(x - y)\varphi]dx$$

$$+ \int_{x_n > y_n} A \nabla \hat{\Gamma}^{11} \cdot \nabla \varphi + \int_{y_n > x_n > 0} A \nabla \hat{\Gamma}^{12} \cdot \nabla \varphi + \int_{0 > x_n} B \nabla \hat{\Gamma}^2 \cdot \nabla \varphi$$

$$= \int_{\mathbb{R}^n} \tau \hat{\Gamma} \varphi dx - \int_{x_n = y_n} e^{\tau s}\delta(x' - y')\varphi dx' - \int_{x_n = y_n} A \nabla \hat{\Gamma}^{11} \cdot e_n \varphi dx' - \int_{x_n > y_n} \nabla \cdot (A \nabla \hat{\Gamma}^{11}) \varphi dx$$
+ \int_{x_n = y_n} A \nabla \hat{\Gamma}^{12} \cdot e_n \varphi dx' - \int_{x_n = 0} A \nabla \hat{\Gamma}^{12} \cdot e_n \varphi dx' - \int_{y_n > x_n > 0} \nabla \cdot (A \nabla \hat{\Gamma}^{12}) \varphi \\
+ \int_{x_n = 0} B \nabla \hat{\Gamma}^{2} \cdot e_n \varphi dx' - \int_{0 > x_n} \nabla \cdot (B \nabla \hat{\Gamma}^{2}) \varphi \\
= \int_{\mathbb{R}^n} \tau \hat{\Gamma} \varphi dx - \int_{x_n > y_n} \nabla (A \cdot \nabla \hat{\Gamma}^{11}) \varphi - \int_{y_n > x_n > 0} \nabla \cdot (A \nabla \hat{\Gamma}^{12}) \varphi - \int_{0 > x_n} \nabla \cdot (B \nabla \hat{\Gamma}^{2}) \varphi \\
+ \int_{x_n = y_n} [-e^{-\tau s} \delta(x' - y') - A \nabla \hat{\Gamma}^{11} \cdot e_n + A \nabla \hat{\Gamma}^{12} \cdot e_n] \varphi dx' + \int_{x_n = 0} [-A \nabla \hat{\Gamma}^{12} \cdot e_n + B \nabla \hat{\Gamma}^{2} \cdot e_n] \varphi dx'.

Therefore, we have the following transmission problem

\begin{align*}
\nabla \cdot (A \nabla \hat{\Gamma}^{11}) - \tau \hat{\Gamma}^{11} &= 0 \quad \text{in } x_n > y_n \\
\nabla \cdot (A \nabla \hat{\Gamma}^{12}) - \tau \hat{\Gamma}^{12} &= 0 \quad \text{in } y_n > x_n > 0 \\
\nabla \cdot (B \nabla \hat{\Gamma}^{2}) - \tau \hat{\Gamma}^{2} &= 0 \quad \text{in } 0 > x_n \\
\hat{\Gamma}^{11} - \hat{\Gamma}^{12} &= 0 \quad \text{on } x_n = y_n \\
A \nabla (\hat{\Gamma}^{11} - \hat{\Gamma}^{12}) \cdot e_n &= -e^{-\tau s} \delta(x' - y') \quad \text{on } x_n = y_n \\
\hat{\Gamma}^{12} - \hat{\Gamma}^{2} &= 0 \quad \text{on } x_n = 0 \\
A \nabla \hat{\Gamma}^{12} \cdot e_n - B \nabla \hat{\Gamma}^{2} \cdot e_n &= 0 \quad \text{on } x_n = 0,
\end{align*}

where $e_n = (0, \cdots, 0, 1)$. Let $\phi^{11,12,2}$ be the Fourier transforms of $\hat{\Gamma}^{11,12,2}$ for $x' = (x_1, \cdots, x_{n-1})$. From now on, we use $\xi' = (\xi_1, \cdots, \xi_{n-1})$ to denote the Fourier variable associated with $x'$. Then,
we have

\[
\begin{aligned}
& a_{nn} \frac{\partial^2 \phi^{11}}{\partial x_n^2} + 2i \left( \sum_{j=1}^{n-1} a_{jn} \xi_j \right) \frac{\partial \phi^{11}}{\partial x_n} - (\tilde{A} \xi' \cdot \xi' + \tau) \phi^{11} = 0 \quad \text{in } \{x_n > y_n\}, \\
& a_{nn} \frac{\partial^2 \phi^{12}}{\partial x_n^2} + 2i \left( \sum_{j=1}^{n-1} a_{jn} \xi_j \right) \frac{\partial \phi^{12}}{\partial x_n} - (\tilde{A} \xi' \cdot \xi' + \tau) \phi^{12} = 0 \quad \text{in } \{y_n > x_n > 0\}, \\
& b_{nn} \frac{\partial^2 \phi^2}{\partial x_n^2} + 2i \left( \sum_{j=1}^{n-1} b_{jn} \xi_j \right) \frac{\partial \phi^2}{\partial x_n} - (\tilde{B} \xi' \cdot \xi' + \tau) \phi^2 = 0 \quad \text{in } \{x_n < 0\},
\end{aligned}
\]

(4.6)

where \(\tilde{A} = (a_{ij})_{1 \leq i,j \leq n-1}\), \(\tilde{B} = (b_{ij})_{1 \leq i,j \leq n-1}\). In addition, we put another conditions

\[
\lim_{x_n \to +\infty} \phi^{11} = 0, \quad \lim_{x_n \to +\infty} \phi^2 = 0.
\]

(4.7)

For simplicity of notations, let us put

\[
\begin{aligned}
& a = \sum_{j=1}^{n-1} a_{jn} \xi_j, \quad b = \sum_{j=1}^{n-1} b_{jn} \xi_j, \\
& \Theta_A = \left[ a_{nn}(\tilde{A} \xi' \cdot \xi' + \tau) - a^2 \right]^{1/2}, \quad \Theta_B = \left[ b_{nn}(\tilde{B} \xi' \cdot \xi' + \tau) - b^2 \right]^{1/2},
\end{aligned}
\]

where the real parts of \(\Theta_A\) and \(\Theta_B\) are positive. From the first three differential equations in (4.6), we have

\[
\begin{aligned}
& \phi^{11} = C_1 \exp \left[ \frac{-ia - \Theta_A}{a_{nn}} x_n \right], \\
& \phi^{12} = C_2 \exp \left[ \frac{-ia - \Theta_A}{a_{nn}} x_n \right] + C_3 \exp \left[ \frac{-ia + \Theta_A}{a_{nn}} x_n \right], \\
& \phi^2 = C_4 \exp \left[ \frac{-ib + \Theta_B}{b_{nn}} x_n \right].
\end{aligned}
\]

Conditions on \(x_n = y_n\) and \(x_n = 0\) imply that

\[
\begin{aligned}
& C_1 - C_2 - C_3 \exp \left[ \frac{2\Theta_A}{a_{nn}} y_n \right] = 0, \\
& (C_1 - C_2)(-ia - \Theta_A) - C_3(-ia + \Theta_A) \exp \left[ \frac{2\Theta_A}{a_{nn}} y_n \right] = -e^{-\tau s - iy' \cdot \xi'} \exp \left[ \frac{ia + \Theta_A}{a_{nn}} y_n \right], \\
& C_2 + C_3 - C_4 = 0, \\
& C_2 \Theta_A - C_3 \Theta_A + C_4 \Theta_B = 0.
\end{aligned}
\]
Therefore, we have the following forms for $\sigma > 0$

\[ C_1 = \frac{1}{2\Theta_A} e^{-rs - iy' \xi'} \exp\left[ ia + \Theta_A y_n \right] + \frac{\Theta_A - \Theta_B}{2\Theta_A(\Theta_A + \Theta_B)} e^{-rs - iy' \xi'} \exp\left[ ia - \Theta_A y_n \right], \]

\[ C_2 = \frac{\Theta_A - \Theta_B}{2\Theta_A(\Theta_A + \Theta_B)} e^{-rs - iy' \xi'} \exp\left[ ia - \Theta_A y_n \right], \]

\[ C_3 = \frac{1}{2\Theta_A} e^{-rs - iy' \xi'} \exp\left[ ia - \Theta_A y_n \right], \]

\[ C_4 = \frac{1}{\Theta_A + \Theta_B} e^{-rs - iy' \xi'} \exp\left[ ia - \Theta_A y_n \right]. \]

Hence, we have

\[ \phi^{11} = \frac{1}{2\Theta_A} e^{-rs - iy' \xi'} \exp\left[ -ia - \Theta_A x_n + ia + \Theta_A y_n \right] \]

\[ + \frac{\Theta_A - \Theta_B}{2\Theta_A(\Theta_A + \Theta_B)} e^{-rs - iy' \xi'} \exp\left[ -ia - \Theta_A x_n + ia - \Theta_A y_n \right], \]

\[ \phi^{12} = \frac{\Theta_A - \Theta_B}{2\Theta_A(\Theta_A + \Theta_B)} e^{-rs - iy' \xi'} \exp\left[ -ia + \Theta_A x_n + ia - \Theta_A y_n \right] \]

\[ + \frac{1}{2\Theta_A} e^{-rs - iy' \xi'} \exp\left[ -ia + \Theta_A x_n + ia - \Theta_A y_n \right], \]

\[ \phi^2 = \frac{1}{\Theta_A + \Theta_B} e^{-rs - iy' \xi'} \exp\left[ -ib + \Theta_B x_n + ia - \Theta_A y_n \right]. \]

Therefore, we have the following forms for $\Gamma$

\[ \Gamma^{11}(x, t; y, s) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} e^{i(x' - y') \xi'} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\tau(t-s)} V^{11}(x_n, y_n, \xi', \tau) d\tau d\xi' \]

\[ \Gamma^{12}(x, t; y, s) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} e^{i(x' - y') \xi'} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\tau(t-s)} V^{12}(x_n, y_n, \xi', \tau) d\tau d\xi' \]

\[ \Gamma^{2}(x, t; y, s) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} e^{i(x' - y') \xi'} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\tau(t-s)} V^{2}(x_n, y_n, \xi', \tau) d\tau d\xi', \]

where $\sigma > 0$ and

\[ V^{11}(x_n, y_n, \xi', \tau) = \frac{1}{2\Theta_A} \exp\left[ -ia - \Theta_A x_n + ia + \Theta_A y_n \right] \]

\[ + \frac{\Theta_A - \Theta_B}{2\Theta_A(\Theta_A + \Theta_B)} \exp\left[ -ia - \Theta_A x_n + ia - \Theta_A y_n \right], \]

\[ V^{12}(x_n, y_n, \xi', \tau) = \frac{\Theta_A - \Theta_B}{2\Theta_A(\Theta_A + \Theta_B)} \exp\left[ -ia - \Theta_A x_n + ia - \Theta_A y_n \right] \]

\[ + \frac{1}{2\Theta_A} \exp\left[ -ia + \Theta_A x_n + ia - \Theta_A y_n \right], \]
Let \( \hat{\Gamma} \) represent one of the functions \( \sigma > 0 \) and \( \sigma > 1 \), respectively. Therefore, we have the following forms for \( \Gamma \) with respect to \( n \) and \( \phi \):

\[
\Gamma_{1} = \begin{cases} 
\Gamma_{1}^1 & \text{for } x_n > 0, \\
\Gamma_{21} & \text{for } 0 > x_n > y_n, \\
\Gamma_{22} & \text{for } y_n > x_n.
\end{cases}
\]

(4.8)

Let \( \hat{\Phi}_{1,21,22} \) be the Laplace transform of \( \Gamma_{1,21,22} \) with respect to \( t \) and \( \phi_{1,21,22} \) be the Fourier transforms of \( \hat{\Phi}_{1,21,22} \) with respect to \( x' = (x_1, \cdots, x_{n-1}) \). Here we used the notation \( \Gamma_{1,21,22} \) for example to represent one of \( \Gamma_{1}, \Gamma_{21}, \Gamma_{22} \).

Then, by a similar argument as we did for the case \( y_n > 0 \), we have

\[
\phi_{1} = \frac{1}{\Theta_A + \Theta_B} e^{-\tau_s - i\phi'} \exp \left[ -ib + \frac{\Theta_B}{b_{nn}} \right] x_n + \frac{ib + \Theta_B}{b_{nn}} y_n, \\
\phi_{21} = \frac{1}{2(\Theta_B - \Theta_A)} e^{-\tau_s - i\phi'} \exp \left[ -ib - \frac{\Theta_B}{b_{nn}} \right] x_n + \frac{ib - \Theta_B}{b_{nn}} y_n, \\
\phi_{22} = \frac{1}{2(\Theta_B - \Theta_A)} e^{-\tau_s - i\phi'} \exp \left[ -ib + \frac{\Theta_B}{b_{nn}} \right] x_n + \frac{ib + \Theta_B}{b_{nn}} y_n.
\]

Therefore, we have the following forms for \( \Gamma \):

\[
\Gamma_{1}(x, t; y, s) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^2} e^{i(s'-y')\xi'} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\tau(t-s)} V_{1}(x_n, y_n, \xi', \tau) d\tau d\xi'
\]

\[
\Gamma_{21}(x, t; y, s) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^2} e^{i(s'-y')\xi'} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\tau(t-s)} V_{21}(x_n, y_n, \xi', \tau) d\tau d\xi'
\]

\[
\Gamma_{22}(x, t; y, s) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^2} e^{i(s'-y')\xi'} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\tau(t-s)} V_{22}(x_n, y_n, \xi', \tau) d\tau d\xi',
\]

where \( \sigma > 0 \) and

\[
V_{1}(x_n, y_n, \xi', \tau) = \frac{1}{\Theta_A + \Theta_B} \exp \left[ -ia - \frac{\Theta_B}{a_{nn}} x_n + \frac{ib + \Theta_B}{b_{nn}} y_n \right],
\]

\[
V_{21}(x_n, y_n, \xi', \tau) = \frac{1}{2\Theta_B} \exp \left[ -ib - \frac{\Theta_B}{b_{nn}} x_n + \frac{ib + \Theta_B}{b_{nn}} y_n \right].
\]
Let us distinguish the argument given in [8]. For example, consider a face \( x_1 = -\sigma_0 \) of \( \sigma_0 \Omega \). For the simplicity of notations, we introduce \( \tilde{\Gamma}(x_1, x''; t; y, s) = \Gamma(x_1 - \sigma_0, x''; t; y_1 - \sigma_0, y'', s) \) with \( x'' = (x_2, \cdots, x_n) \). Then, \( \tilde{\Gamma} \) solves
\[
\begin{align*}
\left\{ \begin{array}{l}
(\partial_t - \nabla \cdot A \nabla) \tilde{\Gamma} = 0 & \text{in } \mathbb{R}^n \times (s, \infty) \\
\lim_{t \uparrow \infty} \int_{\mathbb{R}^n} \tilde{\Gamma}(x, t; y, s) \phi(y) dy = \phi(x_1 - \sigma_0, x'') & (\phi \in C^\infty_0(\mathbb{R}^n))
\end{array} \right.
\end{align*}
\]
(4.9)

Let us distinguish \( A \) here by denoting it by \( \tilde{A} \). Now, we extend \( \tilde{A} = (a_{ij}^{\tilde{A}}) \) in \( x_1 > 0 \) denoted by \( \tilde{A}_+ \) to \( x_1 < 0 \) as follows. That is we define \( \tilde{A}_- = (a_{ij}^{\tilde{A}}) \) by \( a_{11}^{\tilde{A}} = a_{11}^{\tilde{A}_+}, a_{ij}^{\tilde{A}} = a_{ij}^{\tilde{A}_+} (2 \leq i, j \leq n), a_{1j}^{\tilde{A}} = -a_{1j}^{\tilde{A}_+} (2 \leq j \leq n) \). Then, if we define \( \tilde{\Gamma}'(x, t; y, s) \) by \( \tilde{\Gamma}'(x_1, x''; t; y, s) = \tilde{\Gamma}(\pm x_1, x'', t; y, s) (\pm x_1 > 0) \), then \( \tilde{\Gamma}' \) satisfies
\[
\begin{align*}
\left\{ \begin{array}{l}
(\partial_t - \nabla \cdot \tilde{A} \nabla) \tilde{\Gamma}' = 0 & \text{in } \mathbb{R}^n \times (s, \infty) \\
\lim_{t \uparrow \infty} \int_{\mathbb{R}^n} \tilde{\Gamma}'(x, t; y, s) \phi(y) dy = \phi(x_1 - \sigma_0, x'') & (\phi \in C^\infty_0(\{x_1 > 0\} \times \mathbb{R}^{n-1}))
\end{array} \right.
\end{align*}
\]
(4.10)

and \( \tilde{\Gamma}'(x, t; y, s) = \tilde{\Gamma}'(-x_1, x'', t; y, s) (t > s, \text{a.e. } x, y \in \mathbb{R}^n) \). Hence, \( \tilde{\Gamma}'(x, t; y, s) - \tilde{\Gamma}'(x, t; -y_1, y'', s) \) for \( x_1, y_1 > 0 \) is the Green function in the domain \( \{x_1 > 0\} \) satisfying the Dirichlet boundary condition on \( x_1 = 0 \). Repeating this argument for other faces of \( \sigma_0 \Omega \), we can construct the Green function \( G(x, t; y, s) \) for \( x, y \in \sigma_0 \Omega \). It is clear from its construction that \( G(x, t; y, s) \) satisfies the estimate (2.5) and by (4.1), \( G'(x, t; y, s) \) also satisfies the same estimate.

5 Appendix B: Estimate of the Green Function

In order to give its meaning to the fundamental solution \( \Gamma(x, t; y, s) \) constructed in the previous section and estimate it, we need the following theorem.

Theorem 5.1 (Lemmas 2 and 3 in [11]). For each \( \rho \geq 0 \), let \( g(\xi', \eta; \rho) \) be holomorphic function of \( (\xi', \eta) \) in \( L^{n-1}_\mu \subset \mathbb{C}^{n-1} \times \mathbb{C} \) for some \( \mu > 0 \) where
\[
L^{n-1}_\mu = \{ (\xi', \eta) \in \mathbb{C}^{n-1} \times \mathbb{C}; \text{Re} \eta < \mu(\text{Re} \eta) + |\text{Re} \xi'|^2 - \mu^{-1} |\text{Im} \xi'|^2 \}.
\]
We assume the following estimate for \( g(\xi', \eta; \rho) \). That is, there exist some constants \( C > 0 \) and \( c > 0 \) such that

\[
|g(\xi', \eta; \rho)| \leq C(|\xi'| + |\eta|^{1/2})^l \exp(-c\rho(|\xi'| + |\eta|^{1/2})) \exp(C|\text{Im}\xi'|) \quad (5.1)
\]

for \((\xi', \eta) \in L^{n-1}_\mu, l < 0, \) and \( \rho \geq 0. \) Then

\[
|G(x', t; \rho)| \leq C\tau^{-\frac{n-1}{2}-1} \exp\left[-c\frac{|x'|^2 + \rho^2}{t}\right]. \quad (5.2)
\]

where we set

\[
G(x', t; \rho) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \int_{-\infty-iq}^{\infty-iq} e^{im\rho g(\xi', \eta; \rho)} d\eta d\xi'
\]

with an arbitrarily fixed positive number \( q \) for \( \rho > 0 \) and

\[
G(x', t; 0) \equiv \lim_{\rho \downarrow 0} G(x', t; \rho).
\]

**Remark 5.2.**

(i) The theorem still holds even in the case the amplitude \( g \) depends on \((x_n, y_n)\).

(ii) In [1] the factor \( \exp(C|\text{Im}\xi'|) \) does not exist in the estimate \( (5.1) \). However, a slight modification of the proof given in [1] can include the case when the factor exists in the estimate. This was pointed out by Dr. S. Nagayasu.

We will apply Theorem 5.1 to estimate \( \Gamma^{11}, \Gamma^{12}, \Gamma^2 \) in Appendix A. For this, we have to show that the assumptions of Theorem 5.1 are satisfied in our case. To begin with we show that \( V^{11}(x_n, y_n, \xi', \eta), V^{12}(x_n, y_n, \xi', \eta), V^2(x_n, y_n, \xi', \eta) \) are holomorphic in \((\xi', \eta) \in L^{n-1}_\mu\) uniformly for \((x_n, y_n)\) with those \( x_n, y_n \) satisfying the conditions attached to the definitions of \( V^{11}, V^{12}, V^2 \). For simplicity we refer this property as uniform analyticity in \( L^{n-1}_\mu \) of \( V \). Let \( \gamma = (\gamma_{ij}) \) be either \( A \) or \( B \). Then, the characteristic equation of each equation of \((4.6)\) with respect to \( \lambda \) has the form in terms of \( \xi_n = i\lambda \)

\[
p_0\xi_n^2 + p_1(\xi')\xi_n + (p_2(\xi') - \tau) = 0,
\]

where \( p_0 = -\gamma_{nn}, p_1(\xi') = -2\sum_{j=1}^{n-1} \gamma_{nj}\xi_j, p_2(\xi') = -\sum_{j=1}^{n-1} \gamma_{ij}\xi_i\xi_j \) and \( \tau = i\eta \). Then, the roots are given as \( \xi_n = \frac{-p_1 \pm z^\pm}{2p_0} \), where \( z^\pm = \sqrt{p_1^2 - 4p_0(p_2 - \tau)} \). By the ellipticity, for some constant \( c' > 0 \), we have

\[
p_1^2 - 4p_0p_2 < -c'(|\xi'|^2) \quad (\xi' \in \mathbb{R}^2 \setminus \{0\}, \; \xi' \in U),
\]
where \( U \) is an bounded open set in which \( p_0, p_1, p_2 \) are smooth.

From the construction of \( V^{11}, V^{12}, V^2 \), the uniform analyticity of \( V \) in \( L^{n-1}_\mu \) easily follows from the following lemma.

**Lemma 5.3.** There exists \( \mu > 0 \) such that \( p_1^2 - 4p_0p_2 + 4p_0i\eta \notin [0, \infty) \) for \( (\xi', \eta) \in L^{n-1}_\mu \) and \((x_n, y_n)\) with those \( x_n, y_n \) satisfying the condition attached to the definition of \( V^l \).

We will prove Lemma 5.3 by a contradiction argument. We first note that

\[
L^{n-1}_\mu \ni (\xi', \eta) \iff \text{Im} \eta < \mu (|\text{Re} \eta| + |\text{Re} \xi'|^2) - \mu^{-1} |\text{Im} \xi'|^2.
\] (5.3)

Suppose that for \((\xi', \eta) \in L^{n-1}_\mu\), there is \( m \geq 0 \) such that \( p_1^2 - 4p_0p_2 + 4p_0i\eta = m \). Put \( \alpha = -\gamma_m \), \( \beta = (\beta_1, \ldots, \beta_{n-1}) := -2(\gamma_{n1}, \ldots, \gamma_{nm-1}) \) and \( \bar{\gamma} = (\bar{\gamma}_{ij})_{1 \leq i, j \leq n-1} := (-\gamma_{ij})_{1 \leq i, j \leq n-1} \). Then, we have

\[
p_0 = \alpha < 0, \quad p_1(\xi') = \beta \cdot \xi' \]
\[
p_2(\xi') = (\bar{\gamma}\xi') \cdot \xi' < 0 \quad \text{for} \quad \xi' \in \mathbb{R}^{n-1} \setminus \{0\}
\]
\[
p_1^2 = \sum_{j,k=1}^{2} \beta_\beta j \xi_j \xi_k = (\beta \otimes \beta) : (\xi' \otimes \xi')
\]
\[
m = (\beta \otimes \beta) : (\xi \otimes \xi') - 4\alpha(\bar{\gamma}\xi') \cdot \xi' + 4i\eta.
\]

For simplicity, we denote \( \xi_R' = \text{Re} \xi' \) and \( \eta_R = \text{Re} \eta \) etc. Then, we have

\[
m = (\beta \otimes \beta) : (\xi_R' \otimes \xi'_R) - (\beta \otimes \beta) : (\xi_I' \otimes \xi'_I) + 2i(\beta \otimes \beta) : (\xi_R' \otimes \xi'_I)
\]
\[
-4\alpha((\bar{\gamma}\xi'_R) \cdot \xi'_R - (\bar{\gamma}\xi'_I) \cdot \xi'_I) - 8i\alpha(\bar{\gamma}\xi'_R) \cdot \xi'_I + 4i\eta.
\]

That is, we have

\[
(\beta \otimes \beta) : (\xi_R' \otimes \xi'_R) - (\beta \otimes \beta) : (\xi_I' \otimes \xi'_I) - 4\alpha((\bar{\gamma}\xi'_R) \cdot \xi'_R - (\bar{\gamma}\xi'_I) \cdot \xi'_I) - 4\alpha \eta_I = m
\]
\[
(\beta \otimes \beta) : (\xi_R' \otimes \xi'_I) - 4\alpha(\bar{\gamma}\xi'_R) \cdot \xi'_I + 2\alpha \eta_R = 0.
\] (5.4)

From the first equation, we have

\[
-(\beta \otimes \beta) : (\xi'_R \otimes \xi'_R) + 4\alpha(\bar{\gamma}\xi'_R) \cdot \xi'_R + [(\beta \otimes \beta) : (\xi'_I \otimes \xi'_I) - 4\alpha(\bar{\gamma}\xi'_I) \cdot \xi'_I] = -m - 4\alpha \eta_I.
\] (5.5)

The left hand side (LHS) of (5.5) has the estimate \( \text{LHS} > c'|\xi_R'|^2 - c''|\xi'_I|^2 \) for some positive constants \( c' \) and \( c'' \). For the right hand side (RHS) of (5.5), by the definition of \( L^{n-1}_\mu \), we have from the second
uniformly for \((x, t, y, s)\) by \(\partial_{x_j} (1 \leq j \leq n - 1)\), then the integrand is multiplied by \(i\xi_j\). Then, multiply what we got by \(x_k - y_k (1 \leq k \leq n - 1)\) and then integrate by parts with respect to \(\xi_k\). By these procedures, we end up with an integrand which satisfies the same type of estimate as (5.6). Also, if we multiply by \(x_n - y_n\) instead of multiplying by \(x_j - y_j (1 \leq j \leq n - 1)\), we also have the same type of estimate as (5.6), because \((x_n - y_n)(|\xi'| + |\eta|^{1/2}) \exp \left( -c|x_n - y_n||\xi'| + |\eta|^{1/2} \right)\) is bounded. Hence, we have

\[
|\nabla_{x\nu} \Gamma(x, t; y, s)| \leq C(t - s)^{-\frac{n+2}{2}} \exp \left( -c' \frac{|x - y|^2}{t - s} \right) \quad (t > s).
\]

We can handle the derivative \(\partial_{x_n} \Gamma(x, t; y, s)\) in a similar way. Therefore, we have

\[
|\nabla_{y\nu} \Gamma(x, t; y, s)| \leq C(t - s)^{-\frac{n+1}{2}} \exp \left( -c' \frac{|x - y|^2}{t - s} \right) \quad (t > s).
\]

**Acknowledgement** The authors thank Dr. Sei Nagayasu for the useful discussions.
References

[1] R. Arima, On general boundary value problem for parabolic equations, J. Math. Kyoto Univ. 4-1 (1964) 207-243.

[2] D.G. Aronson, Non-negative solutions of linear parabolic equations, Annal. Scuola Norm. Sup. Pisa, C1 Sci. 22, 607-694, 1968.

[3] M. Di Cristo and S. Vessella, Stable determination of the discontinuous conductivity coefficient of a parabolic equation, 2009, preprint.

[4] V. Isakov, K. Kim and G. Nakamura, Reconstruction of an unknown inclusion by thermography, 2009, preprint.

[5] O. A. Ladyženskaja, V. J. Rivkind and N. N. Uralceva, The classical solvability result for diffraction problems, Proc. Steklov Inst. Math., 98, 132–166, 1966.

[6] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Uralceva, Linear and Quasi-linear Equations of Parabolic Type, American Mathematical Society, 23 (1968).

[7] Y. Y. Li and L. Nirenberg, Estimates for elliptic systems from composite material, Comm. Pure and Appl. Math., 56 (7), 892–925, 2003.

[8] L. Riahi, Green function bound and parabolic potentials on a half-space, Potential Analysis, 15, 133–150, 2001.