Entanglement and coherence in quantum state merging

A. Streltsov,1,2,∗ E. Chitambar,3 S. Rana,1 M. N. Bera,1 A. Winter,4,5 and M. Lewenstein1,5

1ICFO – Institut de Ciencies Fotòniques, The Barcelona Institute of Science and Technology, ES-08860 Castelldefels, Spain
2Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, D-14195 Berlin, Germany
3Department of Physics and Astronomy, Southern Illinois University, Carbondale, Illinois 62901, USA
4Física Teòrica: Informació i Fenòmens Quàntics, Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain
5ICREA – Institució Catalana de Recerca i Estudis Avançats, Pg. Lluis Companys 23, ES-08010 Barcelona, Spain

Understanding the resource consumption in distributed scenarios is one of the main goals of quantum information theory. A prominent example for such a scenario is the task of quantum state merging where two parties aim to merge their parts of a tripartite quantum state. In standard quantum state merging, entanglement is considered as an expensive resource, while local quantum operations can be performed at no additional cost. However, recent developments show that some local operations could be more expensive than others: it is reasonable to distinguish between local incoherent operations and local operations which can create coherence. This idea leads us to the task of incoherent quantum state merging, where one of the parties has free access to local incoherent operations only. In this case the resources of the process are quantified by pairs of entanglement and coherence. Here, we develop tools for studying this process, and apply them to several relevant scenarios. While quantum state merging can lead to a gain of entanglement, our results imply that no merging procedure can gain entanglement and coherence at the same time. We also provide a general lower bound on the entanglement-coherence sum, and show that the bound is tight for all pure states. Our results also lead to an incoherent version of Schumacher compression: in this case the compression rate is equal to the von Neumann entropy of the diagonal elements of the corresponding quantum state.

Introduction. While coherence has long been known in classical physics as a fundamental waves property [1], in quantum mechanics coherent superposition is elevated to a universal principle governing all processes. Indeed, the fact that all matter exhibits wave behavior was first understood by de Broglie [2], which became the basis of the now standard formulation of quantum mechanics in Schrödinger’s wave equation [3]. The universality of the superposition principle, i.e. the tenet that any two valid states of a system can be superposed to form a new valid state, marks a radical departure from classical physics. It is at the heart of the many counterintuitive features of quantum theory, perhaps most famously in Schrödinger’s Gedankenexperiment of the cat [4]. Quantum entanglement can be considered as a particular manifestation of coherence, and both of these nonclassical phenomena have led to extensive debates in the early days of quantum mechanics [5, 6].

While the study of the resource theory of entanglement has a long tradition [7, 8], the resource theory of quantum coherence has been formulated only recently [9, 10], although other attempts in this direction have been also presented earlier [11–16]. The basis of any resource theory are free states, these are states which can be created at no cost. In entanglement theory, these are all separable states. In coherence theory these are incoherent states [9], i.e., states which are diagonal in a fixed basis |i⟩. The second important ingredient of any resource theory are free operations, i.e., operations which can be performed at no additional cost. In entanglement theory this is usually the set of local operations and classical communication, although other more general sets such as separable operations [17, 18] and asymptotically nonentangling operations [19, 20] have also been considered. In coherence theory, free operations are called incoherent operations. These are precisely the quantum operations which have incoherent Kraus operators, i.e., K|M⟩∝|n⟩, where |M⟩ and |n⟩ are elements of the incoherent basis [9].

Triggered by these recent developments, much effort is put into understanding the role of coherence as a resource in quantum theory [21–38]. Several new quantifiers of coherence have been proposed [39–52], and the dynamics of some of these quantities under noisy evolution has been investigated [53–57]. Several works also study maximally coherent states [58, 59], the role of coherence in spin models [60, 61], cohering power of quantum channels [62–64], and relations between coherence and other measures of quantumness [65–71]. Coherence also plays an important role in quantum thermodynamics [72–82], and its investigation in biological systems is an important step towards finding quantum phenomena in living objects [83–86]. Additionally, a distinction between “speakable” and “unspeakable” coherence has also been introduced recently [87]. Here we are describing coherence in a speakable sense whereas unspeakable coherence is the resource captured in resource theories of asymmetry [15].

Contrary to entanglement, which inherently implies a scenario of at least two separated parties, the resource theory of coherence has been initially introduced for one party only. Very recently, there were several approaches to extend the notion of coherence to more than one party [53, 65, 68, 70, 88–93]. Here, we build on the methods presented in [89–91], aiming to study the interplay between entanglement and coherence in the task known as quantum state merging [94, 95].

In standard quantum state merging, two parties – their names are traditionally Alice and Bob – share a mixed quantum state. Alice aims to send her part of the state to Bob via an additional quantum channel. The difficulty of the task arises from an extra requirement: the process has to be performed in such a way that the overall purification of the state remains

∗ streltsov.physics@gmail.com
intact. As was shown in [94, 95], the singlet rate required for this process is equal to the conditional entropy \( S(\rho^{AB}) - S(\rho^B) \), where \( S(\rho) = -\text{Tr}[\rho \log_2 \rho] \) is the von Neumann entropy. To be precise, if the conditional entropy is positive, then merging is possible with singlets at rate \( S(\rho^{AB}) - S(\rho^B) \), and merging is not possible if less singlets are available. Moreover, if the conditional entropy is negative, the process is possible without any entanglement. Apart from merging the state for free, Alice and Bob can additionally gain singlets at rate \( S(\rho^B) - S(\rho^{AB}) \).

Here, we consider the task of incoherent quantum state merging. This task is very similar to standard quantum state merging, up to the fact that Bob has free access to incoherent operations only, i.e., he has to pay for operations which are not incoherent. There are at least two motivations for this: On the one hand, we would like to understand better the local quantum(!) operations that Alice and in particular Bob have to perform in merging. On the other hand, coherence seems to be the resource of choice to consider here, as entanglement and coherence are both resources of superposition, one in correlation, the other locally. Thus, while the cost of standard quantum state merging is quantified by the required entanglement rate \( E \), the cost of incoherent quantum state merging will be quantified by a pair of entanglement and coherence rate \((E, C)\). Solving the problem of incoherent quantum state merging requires the characterization of all optimal pairs \((E, C)\). These are pairs of entanglement and coherence for which merging is possible, but neither entanglement nor coherence of the pair can be reduced.

In this paper we define the task of incoherent quantum state merging and develop methods to study it. For arbitrary mixed states we provide a powerful lower bound on the entanglement-coherence sum \( E + C \). For pure states we show that this bound is tight by explicitly evaluating the minimal singlet rate needed for merging in the absence of local coherence. For a family of fully separable mixed states we solve the question of incoherent quantum state merging completely by presenting all optimal pairs of entanglement and coherence. Finally, we provide a discussion on the interplay between coherence and entanglement, also presenting evidence that a large amount of local coherence might be saved by using little extra entanglement in the merging procedure.

At this point we note that the term "coherence" used in this and other recent papers is, of course, also used in atomic and molecular physics, where "coherences" denote off-diagonal elements of the density matrix, typically in the basis of energy eigenstates. Note, however, that in quantum optics the term "coherence" is also used in the context on classical and quantum electrodynamics, where it describes the factorization property of certain correlation functions, ultimately related to the prominent Glauber-Sudarshan "coherent states" [96, 97]. Off-diagonal elements of the density matrix in this latter sense, are related rather to "non-classicality" of states of photos, phonons, bosons etc. (cf. [98–100] and references therein).

**Incoherent quantum state merging.** We consider the scenario where three parties, Alice, Bob, and a referee, share a joint quantum state \( \rho = \rho^{RAB} \). In the task of incoherent quantum state merging, Alice and Bob aim to merge their parts of the total state on Bob’s side by using local quantum-incoherent operations and classical communication (LQICC) [89]. Additionally, Alice and Bob have access to singlets at rate \( E \) and maximally coherent states at rate \( C \) on Bob’s side.

In the following, we are interested in achievable pairs \((E, C)\), these are pairs of coherence and entanglement for which the aforementioned task can be performed in the asymptotic scenario. Similar to standard quantum state merging [94, 95] we consider the most general situation, where Alice and Bob can make catalytic use of entanglement and coherence [101]. We call \( E_i \) the entanglement rate which is initially shared by Alice and Bob, and \( E_f \) will be the final amount of entanglement between them. Similarly, \( C_i \) and \( C_f \) will be the initial and the final amount of Bob’s local coherence.

An entanglement-coherence pair \((E, C)\) is achievable if there exist entanglements \( E_i \), \( E_f \), and \( C_i \), \( C_f \), with \( E = E_i - E_f \) and \( C = C_f - C_i \) such that for any \( \varepsilon > 0 \) and any \( \delta > 0 \) for all sufficiently large integers \( n \geq n_0 \) there exists an LQICC protocol \( \Lambda \) between Alice and Bob such that [102]

\[
\| |\rho_i^{\otimes n} \otimes \Phi_2^\delta(\varepsilon n) \otimes \Psi_2^\delta(C, \delta n) \rangle \rangle - \rho_i^{\otimes n} \otimes \Phi_2^\delta(\varepsilon n) \otimes \Psi_2^\delta(C, \delta n) \rangle \rangle \| 
\leq \varepsilon.
\]

(1)

Here, \( \rho_i = \rho^{RAB} \otimes |0\rangle\langle 0| \) is the total initial state, where \( B \) is an additional particle in Bob’s hands with dimension \( d_B = \delta d_A \).

\( |\Phi_2\rangle = \sqrt{\frac{1}{2}}(|00\rangle + |11\rangle) \) is a maximally entangled two-qubit state shared by Alice and Bob, and \( |\Psi_1\rangle = \sqrt{\frac{1}{2}}(|0\rangle + |1\rangle) \) is maximally coherent single-qubit state on Bob’s side. The target state \( \rho_i = \rho^{RAB} \otimes |0\rangle\langle 0| \) is the same as \( \rho_i \) up to relabeling the parties \( A \) and \( B \), and \( ||M||_1 = \text{Tr} \sqrt{M^\dagger M} \) is the trace norm.

The achievable region is a closed and convex set, due to the timesharing principle [103, 104]. Namely, on block length \( n \), and for \( 0 < p < 1 \), we can break the \( n \) systems into two blocks of \( k = [pn] \) and \( \ell = [(1 - p)n] \), and run a first protocol with asymptotic rate \( (E_i, C_i) \) on the \( k \)-block, and a second protocol with asymptotic rate \( (E_2, C_2) \) on the \( \ell \)-block. The tensor product of these protocols is evidently an asymptotically error-free merging protocol, and achieves the rate pair \( (E, C) = (pE_i + (1 - p)E_2, pC_i + (1 - p)C_2) \).

As in standard quantum state merging, the quantities \( E \) and \( C \) can be positive or negative. If \( E \) (\( C \)) is positive, it means that the merging procedure consumes entanglement (coherence) at rate \( E \) (\( C \)). If the corresponding quantity is negative, the process can be performed without the corresponding resource, and additionally singlets (maximally coherent states) are gained. Crucially, as we will see below in this paper, the latter gain is not possible for both entanglement and coherence at the same time: if entanglement is gained in the process, coherence has to be consumed, and vice versa.

Clearly, if a pair \((E, C)\) is achievable, then any other pair \((E', C')\) is also achievable for \( E' \geq E \) and \( C' \geq C \). A pair \((E, C)\) will be called optimal if it is achievable and if the pairs \((E', C')\) and \((E', C')\) are not achievable for any \( C' < C \) and \( E' < E \). Since via LQICC operations a singlet can be converted into a maximally coherent state on Bob’s side [89], with every achievable pair \((E, C)\), also \( (E + t, C - t) \) is achievable
for $t > 0$. Thus, it is always possible to perform incoherent merging with $C = 0$, and the corresponding optimal pair will be denoted $(E_0, 0)$. Another important pair is the one with the minimal amount of entanglement $E_{\text{min}}$ among all protocols. We denote it $(E_{\text{min}}, C_{\text{max}})$, since it also has the maximal amount of coherence among all optimal pairs [105].

A full solution of incoherent quantum state merging implies determining all optimal pairs for a given tripartite state. The following proposition provides a bound on the entanglement-coherence sum $E + C$.

**Proposition 1.** Given a tripartite quantum state $\rho = \rho^{RAB}$, any achievable pair $(E, C)$ fulfills the following inequality:

$$E + C \geq S\left(\text{id}^B \otimes \Delta^A [\rho]\right) - S\left(\text{id}^{RA} \otimes \Delta^B [\rho]\right),$$

(2)

where $\Delta^X [\rho]$ denotes full decoherence of the state $\rho$ in the incoherent basis of a (possibly multipartite) subsystem $X$:

$$\Delta^X [\rho] = \sum_i |i^X\rangle\langle i^X|.$$  

(3)

We refer the reader to Appendix A for the proof, which is based on monotonicity of QI relative entropy under LQICC operations [89].

It is instructive to compare these results to standard quantum state merging as presented in [94, 95]. In standard quantum state merging, the entanglement rate required for merging a pure state $|\psi\rangle^{RAB}$ is given by the conditional entropy of the reduced state $\rho^{AB}$, which can be either positive or negative. In the negative case, quantum state merging is possible without entanglement and additional singlets are produced. Since the right-hand side of Eq. (2) cannot be negative, it follows that the sum $E + C$ is also nonnegative. While each of the quantities $E$ or $C$ can still be negative individually, they cannot be both negative at the same time. Thus, there is no merging procedure where entanglement and coherence are gained simultaneously. This statement is true for all mixed states $\rho^{RAB}$.

Having presented the general framework, we will now focus on the situation where the total state is pure. Note that understanding of the pure-state scenario also gives insights for general mixed states. In particular, if a pair $(E, C)$ is achievable for a pure state $|\psi\rangle^{RAB}$, the same pair is also achievable for any state $\rho^{RAB}$ with the same reduction such that $\rho^{AB} = \text{Tr}_R[\rho^{RAB}]$.

**Incoherent merging of pure states.** We will now consider incoherent quantum state merging for general pure states. By state merging [95, 106] we have

$$E \geq E_{\text{min}} = S(\rho^{AB}) - S(\rho^B).$$

(4)

Moreover, for pure states Proposition 1 reduces to

$$E + C \geq S(\rho^{AB}) - S(\rho^B),$$

(5)

where we introduced the notation $\rho^X = \Delta^X [\rho^X]$ for the dephased state. As we will see in the following theorem, this bound is saturated.

**Theorem 2.** Any pure state $|\psi\rangle^{RAB}$ can be merged with the pair $(E_0, C = 0)$, where

$$E_0 = S(\rho^{AB}) - S(\rho^B).$$

(6)

Moreover, the pair $(E_0, C = 0)$ is optimal.

We refer to Appendix B for the proof, which is based on an adaptation of the Slepian-Wolf distributed compression of the decohered - classical! - source. Note that $\rho^{AB}$ is a classical state, and its conditional entropy, according to the Slepian-Wolf theorem [107], is precisely the amount of classical communication required to inform Bob about Alice’s register. In fact, the proof of this theorem in Appendix B uses the Slepian-Wolf protocol as a building block.

The above theorem implies that for pure states $|\psi^{RAB}\rangle$ the minimal entanglement-coherence sum $E + C$ required for merging is equal to the conditional entropy of the decohered state $\rho^{AB}$. We also mention that for pure states of the form $|\psi^{RAB}\rangle \otimes |0\rangle^B$, the procedure described here can be seen as the incoherent version of Schumacher compression [108]. In particular, Theorem 2 proves that any state $\rho$ can be faithfully compressed at rate $S(\Delta[p])$, under the assumption that the de-compression is performed with incoherent operations only.

A final comment is in order concerning the applicability of Proposition 1 and Theorem 2 to different operational classes. Beyond the incoherent operations considered in this letter, one can consider the more general class of “maximal” incoherent operations (MIO), which consists of all non-coherence-generating maps [10, 32]. As we discuss in Appendix A, the lower bound of 1 holds as well for MIO. On the achievability end, the rate of Theorem 2 is still achievable when Bob is limited to so-called strictly incoherent operations (SIO) [10, 32], and even if he is further restricted to the class of physical incoherent operations (PIO) [49]. Also, Alice’s measurement in Theorem 2 can always be made incoherent since the protocol is one-way with her final state being incoherent. Thus our result also applies to the scenario of bipartite local incoherent operations and classical communications (LICC) [90, 91].

**Coherence-entanglement tradeoff.** The development so far revealed some facts about the landscape of the achievable pairs $(E, C)$ for incoherent merging of a state $\rho^{RAB}$. Most importantly, there are two inaccessible regions given by the inequalities $E + C \geq S(\Delta^R [\rho]) - S(\Delta^B [\rho])$ and $E \geq E_{\text{min}}$. For a pure state, these simplify to $E + C \geq S(A|B)_{\rho}$ and $E \geq S(A|B)_{\rho}$, and the lower bound is tight as $(E = E_0 = S(A|B)|_{\rho}, C = 0)$ is achievable. Furthermore, since with every achievable pair $(E, C)$, also $(E + t, C - t)$ is achievable for $t > 0$, we find a boundary of the achievable region in the line of slope $-1$ from $(E_0, 0)$ to the right, see Fig. 1. We do not know at this point whether this boundary line continues with slope $-1$ also to the left of that point. The biggest open question is the characterization of $C_{\text{max}}$, which is the coherence rate required for the minimum possible entanglement rate $E_{\text{min}}$. Naturally, if we could show that $(E = E_{\text{min}}, C = E_0 - E_{\text{min}})$ is achievable, we would have characterized the entire achievable region, showing that it is delimited by the two above mentioned linear inequalities. On the other hand, it is quite conceivable that in general $C_{\text{max}} \gg E_0 - E_{\text{min}}$. 
where for definiteness a rate of 1 indeed, the previous procedure of Bob requires asymptotically applies computational basis and communicating any entanglement, which consists of Alice measuring in the \( E_1 \) (attained by simply teleporting Alice’s qubit) and \( \sum_{i<j} \langle \psi_{ij}, \psi_{ij} \rangle \).

We are now going to present an example indicative of the second option inspired by the “flower states” \[110\]:

\[
|\psi\rangle_{RAB} = \frac{1}{\sqrt{2d}} \sum_{j=0}^{d-1} (U_0^i |j\rangle + \sum_{j=0}^{d-1} (U_0^i |j\rangle |y\rangle^A |j\rangle^B
\]

where for definiteness \( U_0 = 1, U_1 = \text{QFT} \) is the quantum Fourier transform, and \( |\Phi_j\rangle = \sum_i |ii\rangle / \sqrt{d} \) is the maximally entangled state. One checks that for this family of states, \( E_0 = 1 \) (attained by simply teleporting Alice’s qubit) and \( E_{\text{min}} = 0 \). Indeed, there is a simple exact merging protocol not using any entanglement, which consists of Alice measuring in the computational basis and communicating \( i \) to Bob. Bob in turn applies \( U_1^i \) after which he is left with the maximally entangled state \( |\Phi_j\rangle^{BB} \) with the reference; now he creates the state \( +\rangle^B = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \) and recovers the state \( |\psi\rangle^{BB} \) by the controlled unitary \( |0\rangle \otimes U_0 + |1\rangle \otimes U_1 \). Note that while \( U_0 \) is trivial, \( U_1 \) requires a large amount of coherence to be implemented, indeed, the previous procedure of Bob requires asymptotically a rate of \( 1 + \frac{1}{2} \log d \) of coherence. Conversely, we have the following lower bound:

**Theorem 3.** Merging the state in Eq. (7) via one-way LQICC without any initial entanglement, i.e. not only \( E_i = 0 \) but also \( \delta = 0 \) in Eq. (1), requires a rate of coherence at least \( C \geq 1 + \frac{1}{2} \log d \gg 1 \).

We refer to Appendix C for the proof. While we proved the theorem for the case where classical communication only goes in one direction, it is reasonable to believe that this result can be extended to arbitrary LQICC protocols. We also note another limitation of the result: Our proof covers only the case that entanglement is exactly zero initially. It is not clear if this result also applies when considering more general merging procedure where entanglement vanishes only in the asymptotic limit. Nevertheless, this result provides strong evidence that in the task of quantum state merging it is possible to save a large amount of local coherence by using little extra entanglement.

**Application: A family of separable states.** We will now apply the results presented so far to the following family of states:

\[
\rho = \sum_{i,j} p_{ij} |i\rangle \langle j|^{RR} \otimes |\psi_{ij}angle^A \otimes |j\rangle^B,
\]

where the states \( |\psi_{ij}\rangle \) are mutually orthogonal for different \( j \), i.e., \( \langle \psi_{ij}|\psi_{jk}\rangle = \delta_{jk} \). As is shown in Appendix D, for this type of states all optimal pairs are given by

\[
(E, C) = (aC_{\max}, [1 - a]C_{\max})
\]

with \( a \geq 0 \) and \( C_{\max} = \sum_{i,j} p_{ij} S(\Delta(\psi_{ij})) \).

**Conclusions.** In the present paper we introduced and studied the task of incoherent quantum state merging. This task is the same as standard quantum state merging, up to the fact that one of the parties has free access to local incoherent operations only, and has to consume a coherent resource for more general operations. The amount of resources needed for merging is quantified by an entanglement-coherence pair \( (E, C) \). In general, we showed that the entanglement-coherence sum \( E + C \) is nonnegative, which means that no merging procedure can gain entanglement and coherence at the same time. For pure states we gave a protocol of incoherent quantum state merging by finding the minimal entanglement-coherence sum \( E + C \), which turns out to be the conditional entropy of the decohered state \( \rho^{AB} \).

Our results include an incoherent version of Schumacher compression. In particular, if we require that the decompression is performed via incoherent operations only, then the optimal compression rate is given by \( S(\Delta(\rho)) \). This rate is in general larger than the standard compression rate \( S(\rho) \), which comes from the fact that coherence is required for the decompression in the standard case.

We have also made first steps towards understanding of the precise tradeoff between entanglement and coherence for the task of LQICC merging. While this remains a major open problem in general, we have given strong indications that in certain situations the equivalent of one ebit can be an arbitrary amount of coherence, which we could prove in a setting of one-way LQICC and a situation where we want to reduce the
available entanglement exactly (and not only asymptotically) to zero.

Another open question is the relation of LQICC merging to the results presented in [111]. In particular, the authors of [111] study the work cost for erasing a system $A$ which is (quantum) correlated with another observer $B$ in an environment at temperature $T$. As was shown in [111], this work cost is bounded above by $S(A|B)kT \ln(2)$, where $k$ is the Boltzmann constant. At this point it is natural to ask if our results can be applied to understand the role of coherence in the era-

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Appendix A: Proof of Proposition 1

Here we will prove that any achievable pair \((E, C)\) fulfills the following inequality:

\[
E + C \geq S \left( \Delta^{AB} [\rho] \right) - S \left( \Delta^{B} [\rho] \right). \tag{A1}
\]

For proving this statement, we will use the QI relative entropy which can be written as \([89]\):

\[
C_{r}^{XY} (\rho^{XY}) = S \left( \Delta^{Y} [\rho^{XY}] \right) - S (\rho^{XY}) . \tag{A2}
\]

The definition of an achievable pair in Eq. (1) of the main text together with the continuity of the QI relative entropy \([89]\) implies that there exist nonnegative numbers \(E_{i}, E_{t}, C_{i}, C_{t}\) with \(E = E_{t} - E_{t}\) and \(C = C_{t} - C_{t}\) such that for any \(0 < \epsilon \leq 1/2\) and any \(\delta > 0\) there is an integer \(n \geq 1\) and an LQICC protocol \(\Lambda\) such that

\[
C_{r}^{RABB} \left( \Lambda \left[ \rho_{i}^{\otimes n} \otimes \Phi_{2}^{\oplus (E_{t} + \delta)n} \otimes \Psi_{2}^{\oplus (C_{t} + \delta)n} \right] \right) \geq \epsilon \tag{A3}
\]

\[
C_{r}^{RABB} \left( \rho_{i}^{\otimes n} \otimes \Phi_{2}^{\oplus (E_{t}n)} \otimes \Psi_{2}^{\oplus (C_{t}n)} \right) - 2n \log_{2} d_{tot} - 2h(\epsilon). \tag{A4}
\]

In the next step we will introduce the number \(d'\) as follows:

\[
d' = d_{RABB} \times 4^{E_{t} + E_{t} + \delta + 1} \times 2^{C_{t} + C_{t} + \delta + 1}, \tag{A5}
\]

and it can be verified by inspection that \((d')^{n} \geq d_{tot}\). Together with Eq. (A3) this leads us to the following inequality:

\[
C_{r}^{RABB} \left( \Lambda \left[ \rho_{i}^{\otimes n} \otimes \Phi_{2}^{\oplus (E_{t} + \delta)n} \otimes \Psi_{2}^{\oplus (C_{t} + \delta)n} \right] \right) \geq \epsilon \tag{A6}
\]

Since the QI relative entropy is additive and does not increase under LQICC operations \([89]\), it follows that

\[
C_{r}^{RABB} (\rho_{i}) + \left[ \frac{[E_{t} + \delta] n}{n} \right] + \left[ \frac{[C_{t} + \delta] n}{n} \right] \geq C_{r}^{RABB} (\rho_{i}) + \frac{[E_{t} n] + [C_{t} n]}{n} - 2n \log_{2} d' - \frac{2}{n} h(\epsilon). \tag{A7}
\]

The desired statement follows by using the relations:

\[
C_{r}^{RABB} (\rho_{i}) = S (\Delta^{B} (\rho)) - S (\rho), \tag{A8}
\]

\[
C_{r}^{RABB} (\rho_{i}) = S (\Delta^{AB} (\rho)) - S (\rho) \tag{A9}
\]

together with the facts that \([x] \leq x\) and \([x] \geq x\).

Note that in this proof, the restriction to LQICC operations is needed only to ensure monotonicity of the QI relative entropy. This monotonicity follows from the fact that LQICC operations preserve the set of so-called quantum-incoherent QI states: i.e. states of the form \(\rho^{XY} = \sum_{x} p_{x} |y\rangle \langle y| \otimes |y\rangle \langle y|\), where the \(p_{x}\) are arbitrary and \(|y\rangle\) is the incoherent basis for system \(Y\). The QI relative entropy is therefore a monotone for any other class of operations that also preserve the QI set of states.

The most general class of QI-preserving operations are formed by so-called “maximal” incoherent operations (MIO) on Bob’s side and arbitrary operations on Alice’s. Recall that a completely positive trace-preserving (CPTP) map \(\mathcal{E}\) belongs to the class MIO if \(\mathcal{E}(\rho) \in I\) for any \(\rho \in I\), where \(I\) denotes the set of incoherent states. A MIO measurement that produces classical outcomes \(i\) can be represented by the CPTP map \(\rho^{B} \mapsto \sum_{i} E_{i}(\rho^{B}) \otimes |i\rangle \langle i|\) with each \(E_{i}\) being an incoherent CP map. Hence the incoherent operations \(\{K_{i}\}\), studied in this paper are special MIO maps of the form \(\rho \mapsto \sum_{i} K_{i} \rho K_{i}^\dagger \otimes |i\rangle \langle i|\). It is easy to see that MIO acts invariantly on the set of QI states. Thus, the QI relative entropy is monotonic under MIO, and we see that Proposition 1 also holds for MIO performed on Bob’s side.

Appendix B: Proof of Theorem 2

Here we will prove that any pure state \(|\psi\rangle^{RAB}\) can be merged with the pair \((E_{0}, C = 0)\), where

\[
E_{0} = S (A |B \rangle_{\mathcal{F}}) = S (\rho^{AB}) - S (\rho^{B}), \tag{B1}
\]

and \(\rho^{X} = \Delta^{X} [\rho^{X}]\) denotes the dephased state.

For proving this, note that any pure state \(|\psi\rangle^{RAB}\) can be written in the following form:

\[
|\psi\rangle^{RAB} = \sum_{xy} a_{xy} |\mu_{xy}^{R} \otimes |x\rangle \otimes |y\rangle^{B}, \tag{B2}
\]

with complex coefficients \(a_{xy}\) and arbitrary referee’s states \(|\mu_{xy}^{R}\rangle^{R}\). We will now show that the state merging transformation \(|\psi\rangle^{RAB} \rightarrow |\psi\rangle^{RAB}\) can performed asymptotically by LQICC at an entanglement consumption rate of \(S (A |B \rangle_{\mathcal{F}}) = S (\rho^{AB}) - S (\rho^{B})\).

By the structure of the state \(|\psi\rangle^{RAB}\), we see that \(S (A |B \rangle_{\mathcal{F}}) = H (X |Y)\), where \(X\) and \(Y\) are random variables joint distributed according to \(p_{X,Y} = |\mu_{xy}^{R}\rangle \langle \mu_{xy}^{R}|\). The state merging protocol is essentially Slepian-Wolf data compression \([107]\) of the source \(X\) with side information \(Y\) at the decoder, run in coherent superposition. The resulting protocol will turn out to be fully incoherent, for both Alice and Bob.

To be precise, fix a code for block length \(n\), consisting of compression and decompression functions

\[
f : [X^{n}] \rightarrow [N], \tag{B3}
\]

\[
g : [N] \times [Y^{n}] \rightarrow [X^{n}], \tag{B4}
\]

such that \(\log N = n(H (X |Y) + \delta)\) and

\[
Pr \{X^{n} \neq g(f(X^{n}), Y^{n})\} \leq \epsilon. \tag{B5}
\]

By the Slepian-Wolf theorem \([103, 107]\), for every \(\epsilon, \delta > 0\), such a code exists for all sufficiently large \(n\). For the purposes of the quantum protocol, define \(G(v, y^{n}) := (g(v, y^{n}), y^{n})\), such that

\[
Pr \{X^{n} \neq G(f(X^{n}), Y^{n})\} \leq \epsilon. \tag{B6}
\]
Because of this, there exists a subset $S \subset \mathcal{X}^n \times \mathcal{Y}^n$ such that 
\[ \Pr \{ (X^n, Y^n) \in S \} \geq 1 - \epsilon \] and $(x^n, y^n) = G(f(x^n), y^n)$ for all $(x^n, y^n) \in S$. We can therefore introduce the one-to-one function $\tilde{G} : [N] \times \mathcal{Y}^n \rightarrow \mathcal{X}^n \times \mathcal{Y}^n \cup \mathcal{R}$ (with $\mathcal{R} = [N] \times \mathcal{Y}^n$) by
\[ \tilde{G}(x^n, y^n) = \begin{cases} G(v, y^n) & \text{if } G(v, y^n) \in S, \\ (v, y^n) & \text{otherwise.} \end{cases} \] (B7)

Note that by construction
\[ \Pr \{ (X^n, Y^n) \neq \tilde{G}(f(X^n), Y^n) \} \leq \epsilon. \]

Now we can describe the quantum protocol: Define the incoherent(!) inommences
\[ U : |x^n\rangle^{A^n} \rightarrow |x^n\rangle^{A^n} |f(x^n)\rangle^{A_0}, \] (B8)
and
\[ V : |y^n\rangle^{B_0} \rightarrow |\tilde{G}(x^n, y^n)\rangle^{(B_0 + A^n)^B}. \] (B9)

The first three steps of the protocol are easy: Alice applies $U$ to her register $A^n$, then sends the register $A_0$ to Bob by teleportation [114] using $\log N = n(E + \delta)$ qubits, which receives it in his register $B_0$ and applies $V$ to $B_0B^n$. The resulting state is
\[ |\varphi\rangle^{R^n A^n(B_0 + A^n)^B} = (|R^n \otimes V|A_0 = B_0U) |\psi\rangle^{\otimes n} \]
\[ = \sum_{x^n,y^n} a_{x^n,y^n} |\mu_{x^n,y^n}\rangle^{R^n} \otimes |\phi\rangle^{\otimes n} \otimes |\tilde{G}(x^n, y^n)\rangle^{(B_0 + A^n)^B} \]
\[ = \sum_{(x^n,y^n) \in S} a_{x^n,y^n} |\mu_{x^n,y^n}\rangle^{R^n} \otimes |\phi\rangle^{\otimes n} \otimes |\tilde{G}(x^n, y^n)\rangle^{(B_0 + A^n)^B} \]
\[ + \sum_{(x^n,y^n) \notin S} a_{x^n,y^n} |\mu_{x^n,y^n}\rangle^{R^n} \otimes |\phi\rangle^{\otimes n} \otimes |\tilde{G}(x^n, y^n)\rangle^{(B_0 + A^n)^B}. \] (B10)

Note that the overall amplitude of the second summation is
\[ < \epsilon \] since $\Pr(S) \geq 1 - \epsilon$. Furthermore, the $|y^n\rangle$ are orthogonal to the $|\phi\rangle^n$. Thus by defining
\[ |\tilde{\psi}\rangle^{RA^n B^n} := \sum_{x,y} a_{x,y} |\mu_{x,y}\rangle^{R^n} \otimes |\phi\rangle^{\otimes n} \otimes |\tilde{G}(x^n, y^n)\rangle^{(B_0 + A^n)^B}, \] (B11)
we have
\[ \text{Tr} |\psi\rangle^{\otimes n} |\Phi\rangle \geq 1 - \epsilon, \] (B12)
by the Slepian-Wolf property of the maps $f$ and $\tilde{G}$. In other words,
\[ |\varphi\rangle = \sqrt{1 - \epsilon} |\tilde{\psi}\rangle^{\otimes n} + \sqrt{\epsilon} |\theta\rangle, \] (B13)
with a (sub-)normalized vector $|\theta\rangle$. It remains to decouple the register $A^n$, which however is easily done due to the structure of $|\theta\rangle$ as a generalized GHZ-state: Indeed, for $d = |X| = |A|$, consider the conjugate basis
\[ |\alpha\rangle = \frac{1}{\sqrt{d}} \sum_{x=0}^{d-1} e^{i 2\pi ax/d} |x\rangle, \]
then one can confirm by direct calculation
\[ \langle \alpha | \tilde{\psi} \rangle = \frac{1}{\sqrt{d}} \sum_{x,y} a_{x,y} e^{-i 2\pi ax/d} |x\rangle^{\otimes n} \otimes |\phi\rangle^{\otimes n} \]
\[ = \frac{1}{\sqrt{d}} (I_R \otimes Z^{-\alpha} \otimes I_B) |\psi\rangle^{RA^n B^n}, \] (B14)
and thus
\[ \langle \alpha | \tilde{\psi} \rangle^{\otimes n} = \frac{1}{d^{n/2}} \prod_{i=1}^{n} (I_R \otimes Z^{-\alpha} \otimes I_B) |\psi\rangle^{RA^n B^n}. \] (B15)

I.e., to transform $(\tilde{\psi}^{RA^n B^n})^{\otimes n}$ to $(|\psi\rangle^{RA^n B^n})^{\otimes n}$, Alice will destructively measure each of the $n$ $A$-systems in the conjugate basis $|\alpha\rangle$ — which is an incoherent operation — then communicates the outcomes $\alpha^n = \alpha_1 \ldots \alpha_n$ to Bob who applies the diagonal unitaries $Z^{\alpha_i}$ (to the $i$-th $A'$-system). To be precise, the incoherent operation that Alice performs is given by Kraus operators $K_{\alpha} = |0\rangle\langle\alpha|$, which map her system to the incoherent state $|0\rangle$ for every outcome $\alpha$. By applying this procedure to $\varphi$ instead, they obtain a final state $|\psi\rangle^{\otimes n}$ with
\[ \| |\psi\rangle^{\otimes n} - (|\psi\rangle^{RA^n B^n})^{\otimes n} \| \leq \| |\psi\rangle^{\otimes n} - |\tilde{\psi}\rangle^{\otimes n} \| \leq 2 \sqrt{\epsilon}. \] (B16)

As $\epsilon$ can be made arbitrarily small for increasing $n$, this concludes the proof.

Note that in this protocol Bob simply performs permutations and diagonal unitaries. Since these operations belong to the classical physical incoherent operations (PIO) [49] and the more general class of strictly incoherent operations (SIO) [32], we see that Theorem 2 also holds when Bob is restricted to PIO/SIO.

### Appendix C: Proof of Theorem 3

We now show that for states of the form
\[ |\psi\rangle^{AB} = \frac{1}{\sqrt{2d}} \sum_{i=0}^{d-1} |U_i \rangle \langle j| |i\rangle^A |j\rangle^B \]
\[ = \frac{1}{\sqrt{2}} \left( (|0\rangle^A \otimes (I \otimes U_0) \Phi_d) + |1\rangle^A \otimes (I \otimes U_1) \Phi_d \right), \] (C1)
any one-way LQICC protocol that does not use any entanglement, requires a rate of coherence of at least $C' \geq 1 + \frac{1}{2} \log d \gg 1$. Namely, to succeed in merging of $\psi^{\otimes n}$, Alice’s measurement needs to leave Bob approximately with a maximal entangled state $R$, a state $|\phi\rangle^{AB}$ within trace distance $\epsilon$ from $|I \otimes V \rangle |\Phi_d\rangle^{\otimes n}$, with a unitary $U$ on $B'$ (at least for most outcomes). Bob’s local operation $T$ then must take $\phi$ to within $\epsilon$ of $|\psi\rangle^{AB}$, which means that $T$ in a certain precise sense has to approximate the action of $U^{\otimes n} V^\dagger$, with the isometry $U = \sqrt{1/\epsilon} (|0\rangle^B \otimes U_0 + |1\rangle^B \otimes U_1)$:
\[ (id \otimes T) \phi \approx |\psi\rangle^{\otimes n} \]
\[ = (I \otimes U_0^{\otimes n}) (I \otimes V) \Phi_d^{\otimes n} (I \otimes V) \]
\[ \approx (I \otimes V) \Phi_d^{\otimes n} (I \otimes V)^\dagger, \] (C2)
where the \( \approx \) sign means that the respective states are at trace distance \( \leq \epsilon \). Hence we get

\[
\| (\text{id} \otimes T) (\mathbb{I} \otimes V) \Phi_d^{\otimes n} (\mathbb{I} \otimes V)^\dagger - \psi^{\otimes n} \|_1 \leq 2\epsilon,
\]

which implies

\[
\frac{1}{d^n} \sum_{j^{n}=j_{1}...j_{n}} \| T(j^{n}\otimes j^{n}) - U^{\otimes n} V^{\dagger} j^{n}\otimes j^{n} | V(U^{\otimes n}) \|_1 \leq 2\epsilon,
\]

and applying \( \Delta \) to each of the \( n \) BB systems, we get

\[
\frac{1}{d^n} \sum_{j^{n}=j_{1}...j_{n}} \| \Delta^{\otimes n} (T(j^{n}\otimes j^{n})) - \Delta^{\otimes n} (U^{\otimes n} V^{\dagger} j^{n}\otimes j^{n} | V(U^{\otimes n})^{\dagger}) \|_1 \leq 2\epsilon.
\]

Now, we claim that for every state \( \sigma \) on \( B^n \),

\[
S (\Delta^{\otimes n} (U^{\otimes n} \sigma (U^{\otimes n})^{\dagger})) \geq n \left( 1 + \frac{1}{2} \log d \right).
\]

This is easy to see for \( n = 1 \), since

\[
M(\sigma) := \Delta(U\sigma U^{\dagger}) = \frac{1}{2} |0\rangle \langle 0| \otimes \Delta(U_0\sigma U_0^{\dagger}) + \frac{1}{2} |1\rangle \langle 1| \otimes \Delta(U_1\sigma U_1^{\dagger}),
\]

whose entropy is lower bounded for every state by \( 1 + \frac{1}{2} \log d \), according to the Maassen-Uffink entropic uncertainty relation [115]; namely, note that \( \Delta(U_i\sigma U_i^{\dagger}) \) correspond to measuring \( \sigma \) in one of two mutually unbiased bases. To obtain eq. (C6), we observe that the state of which we need the entropy is \( \sigma_{\text{max}}^{\otimes n} \), and by the additivity of the minimum output entropy of entanglement-breaking channels [116, 117], this is at least \( n \) times the single-copy bound. Thus, by eq. (C4) and the asymptotic continuity of \( C_r \), the relative entropy of coherence [10], we get

\[
\frac{1}{d^n} \sum_{j^{n}=j_{1}...j_{n}} C_r (T(j^{n}\otimes j^{n})) \geq n \left( 1 + \frac{1}{2} \log d \right) - 2n\epsilon - 2.
\]

What this says is that \( T \) is capable of generating a large amount of coherence, namely at least \( n \left( 1 + \frac{1}{2} \log d \right) - O(n\epsilon) \). Clearly, this implies that to implement \( T \), Bob must consume at least that amount of pure coherence, so that in the limit of \( n \to \infty \) and \( \epsilon \to 0 \), we obtain \( C' \geq 1 + \frac{1}{2} \log d \) for the coherence rate required.

We note that the proof also works for the case where general MIO operations are performed on Bob’s side. Thus, the statement of the theorem also holds in this scenario.

**Appendix D: Proof of Eq. (9)**

Here we will consider the following family of states:

\[
\rho = \sum_{i,j} p_{ij} |i\rangle \langle i| \otimes |\psi_{ij}\rangle \langle \psi_{ij}| \otimes |\bar{i}\rangle \langle \bar{i}|,
\]

where the states \( |\psi_{ij}\rangle \) are mutually orthogonal for different \( j \), i.e., \( \langle \psi_{ij}| \psi_{\bar{j}k} \rangle = \delta_{jk} \). We will now show that for these states all optimal pairs are given by

\[
(E, C) = (a C_{\text{max}}, [1 - a] C_{\text{max}})
\]

with \( a \geq 0 \) and \( C_{\text{max}} = \sum_{i,j} p_{ij} S(\Delta(\psi_{ij})) \).

For proving this we first invoke Proposition 1 in the main text, which implies that any achievable pair is bounded below by

\[
E + C \geq \sum_{i,j} p_{ij} S(\Delta(\psi_{ij})).
\]

In the next step note that for this family of states merging is achievable without entanglement, and thus \( E_{\text{min}} = 0 \). From Eq. (D3) it follows that

\[
C_{\text{max}} \geq \sum_{i,j} p_{ij} S(\Delta(\psi_{ij})).
\]

Now note that \((0, \sum_{i,j} p_{ij} S(\Delta(\psi_{ij}))\) is an achievable pair, which can be achieved if Bob performs a von Neumann measurement in the basis \( \{|\bar{i}\rangle\} \) and communicates the result of the outcome to Alice. Depending on the outcome \( i \) of Bob’s measurement, Alice performs a von Neumann measurement in the basis \( \{|\psi_{ij}\rangle\} \), and communicates her outcome to Bob. Depending on the outcomes \( i \) and \( j \), Bob prepares his additional system \( B \) in the state \( |\psi_{ij}\rangle \otimes \bar{B} \), and the merging procedure is complete. Since the coherence cost of preparing the state \( |\psi_{ij}\rangle \) is \( S(\Delta(\psi_{ij})) \) [10], this reasoning proves that

\[
C_{\text{max}} = \sum_{i,j} p_{ij} S(\Delta(\psi_{ij})).
\]

Due to the facts that \((0, C_{\text{max}})\) is an achievable pair and that via LQICC operations Alice and Bob can convert a singlet into a maximally coherent single-qubit state on Bob’s side [89], it follows that \((a C_{\text{max}}, [1 - a] C_{\text{max}})\) is also achievable for all \( a \geq 0 \). Moreover, all these pairs must be optimal due to Eq. (D3). It remains to show that all optimal pairs have this form. For this, note that any optimal pair \((E, C)\) must have coherence \( C \leq C_{\text{max}} \), and thus we can always write \( C = [1 - a] C_{\text{max}} \). Then, in order for the pair to be optimal, its entanglement must be \( E = a C_{\text{max}} \). In particular, the pair is not achievable if entanglement is below \( a C_{\text{max}} \), and the pair is not optimal if entanglement is above this value.