PARABOLIC JOHN-NIRENBERG SPACES

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ABSTRACT. We introduce a parabolic version of John-Nirenberg space with exponent $p$ and show that it is contained in local weak-$L^p$ spaces.

1. Introduction

In the classical paper of F. John and L. Nirenberg [10], where functions of bounded mean oscillation (BMO) were introduced, they also studied a class satisfying a weaker BMO type condition

$$K_f^p := \sup_{\{Q_j\}_j} \sum_j |Q_j| \left( \int_{Q_j} |f - f_{Q_j}| \, dx \right)^p < \infty,$$

where the supremum is taken over all partitions $\{Q_j\}_j$ of a given cube $Q_0$ into pairwise non-overlapping subcubes. The functional $f \mapsto K_f$ defines a seminorm and the class of functions satisfying $K_f < \infty$, which we denote by $JN_p(Q_0)$ for John-Nirenberg space with exponent $p$, can be seen as a generalization of BMO. Indeed, BMO is obtained as the limit case of $JN_p$ in the sense that

$$\lim_{p \to \infty} K_f = \sup_{Q \subseteq Q_0} \frac{1}{Q} \int_{Q} |f - f_Q| \, dx = ||f||_{\text{BMO}(Q_0)}.$$

In contrast to the exponential integrability of BMO functions, $K_f < \infty$ implies that $f$ belongs to the space weak-$L^p(Q_0)$. This was already observed by John and Nirenberg. Precisely, they showed that for $\lambda > 0$, we have

$$|\{x \in Q_0 : |f(x) - f_{Q_0}| > \lambda\}| \leq C \left( \frac{K_f}{\lambda} \right)^p,$$

where the constant $C$ depends on $n$ and $p$. Simpler proofs and generalizations have appeared in [11, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14]. In this
note we show that an analogous result holds in the context of parabolic BMO spaces.

2. PARABOLIC JOHN-NIRENBERG SPACE

We shall introduce some notation and terminology. Given an Euclidean cube $Q = \prod_{i=1}^{n} [a_i, a_i + h]$, we define the forward in time translation

$$Q^+ := \prod_{i=1}^{n-1} [a_i, a_i + h] \times [a_n + h, a_n + 2h].$$

Moreover, we use the notation $Q^{+,2} := (Q^+)^+$. We write $f \in BMO^+(\mathbb{R}^n)$, if we have

$$\|f\|_{BMO^+(\mathbb{R}^n)} := \sup_Q \int_Q (f - f_{Q^+})^+ \, dx < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. It should be observed that despite the notation, the quantity defined by (2.1) is not actually a norm.

The one-dimensional $BMO^+(\mathbb{R})$ class was first introduced by F. J. Martín-Reyes and A. de la Torre [13], who showed that this class possesses many properties similar to the standard BMO space. Even though steps towards a multidimensional theory has been taken (see [4]), a satisfactory theory has only been developed in dimension one. In the classical elliptic setting, one of the cornerstones of theory of BMO functions is the celebrated John-Nirenberg inequality, which shows that logarithmic growth is the maximum possible for a BMO function. A corresponding result holds for the class $BMO^+(\mathbb{R})$, and a slightly weaker version of this result for $BMO^+(\mathbb{R}^n)$ was obtained in [4].

In this setting we define John-Nirenberg spaces as follows. We write $f \in JN^+_p(\mathbb{R}^n)$ if

$$\left(K^+_f\right)^p := \sup_{\{Q_j\}} \left( \sum_j |Q_j| \left( \int_{Q_j \cup Q^+_j} (f - f_{Q^+_j})^+ \, dx \right)^p \right) < \infty,$$

where the supremum is taken over countable families $\{Q_j\}$ of pairwise non-overlapping cubes satisfying $\sum_j |Q_j| < \infty$. The definition is reasonable in the sense that the $BMO^+(\mathbb{R}^n)$ condition may be seen as the limit case of (2.2) as $p \to \infty$. Precisely,

$$\lim_{p \to \infty} K^+_f = \sup_Q \int_{Q \cup Q^+_2} (f - f_{Q^+})^+ \, dx,$$
where the quantity on the right-hand side is equivalent (up to a multiplication by a universal constant) to the BMO norm of $f$, defined by (2.1).

The following theorem is a parabolic version of the weak distribution inequality of John and Nirenberg.

**Theorem.** Assume $f \in JN_p^+(\mathbb{R}^n)$. Then, for every cube $Q_0$ and $\lambda > 0$, we have

$$(2.3) \quad |\{x \in Q_0 : (f(x) - f_{Q_0^+})^+ > \lambda\}| \leq C \left( \frac{K_j^+}{\lambda} \right)^p,$$

where $C$ only depends on $n$ and $p$.

### 3. Proof of the theorem

We follow the argument used in [1]. Given a non-negative $f$ and a cube $Q_0$, denote by $\Delta = \Delta(Q_0)$ the family of all dyadic subcubes obtained from $Q_0$ by repeatedly bisecting the sides into two parts of equal length. We shall make use of the “forward in time dyadic maximal function” defined by

$$M_{Q_0^+}^{+,d}f(x) := \sup_{Q \in \Delta} \int_{Q^+} f \, dx.$$

A standard stopping-time argument shows that we have

$$\{x \in Q_0 : M_{Q_0^+}^{+,d}f(x) > \lambda\} = \bigcup_j Q_j,$$

where $Q_j$’s are the maximal dyadic subcubes of $Q_0$ satisfying

$$(3.1) \quad \int_{Q_j^+} f \, dx > \lambda.$$

Maximality implies that the cubes $Q_j$ are pairwise non-overlapping. Moreover, if $\lambda \geq f_{Q_0^+}$, then $Q_0$ doesn’t satisfy (3.1). Consequently, in this case every $Q_j$ is contained in a larger dyadic subcube $Q_{j-}$ of $Q_0$ which does not satisfy (3.1). Since $Q_j^{+,2} \subset Q_{j-}^+$, we conclude

$$(3.2) \quad \int_{Q_j^{+,2}} f \, dx \leq 2^n \lambda.$$
provided $\lambda \geq f_{Q_0^+}$. Standard arguments imply a weak type estimate for $M_{Q_0^+}$. Indeed, we have

$$|\{x \in Q_0 : M_{Q_0^+} f(x) > \lambda\}| = \sum_j |Q_j|.$$  

While the cubes $Q_j$ are non-overlapping, the cubes $Q_j^+$ may not be. Let us replace $\{Q_j^+\}_j$ by the maximal non-overlapping subfamily $\{\tilde{Q}_j^+\}_j$ which we form by collecting those $Q_j^+$ which are not properly contained in any other $Q_j^+$. Maximality of $\{\tilde{Q}_j^+\}_j$ enables us to partition the family $\{Q_j\}_j$ as follows. Given $\tilde{Q}_j^+$, we define $I_j := \{i : Q_i^+ \subseteq \tilde{Q}_j^+\}$, and we may write $\{Q_j\}_j = \bigcup_j \{Q_i : i \in I_j\}$. Now, whenever $i \in I_j$, we have $Q_i \subseteq \tilde{Q}_j \cup \tilde{Q}_j^+$ and we get the estimate

$$\sum_j |Q_j| = \sum_j \sum_{i \in I_j} |Q_i|$$

$$\leq 2 \sum_j |\tilde{Q}_j^+|$$

$$\leq 2 \frac{\lambda}{\lambda} \int_{Q_0 \cup Q_0^+} f \, dx.$$  

Combining the previous estimates, we arrive at

$$(3.3) \quad |\{x \in Q_0 : M_{Q_0^+} f(x) > \lambda\}| \leq \frac{2}{\lambda} \int_{Q_0 \cup Q_0^+} f \, dx.$$  

We begin by proving the following good $\lambda$ inequality for the forward in time dyadic maximal operator.

**Lemma.** Assume $f \in JN_p^+(\mathbb{R}^n)$ and take $0 < b < 2^{-n}$. Then, given a cube $Q_0$, we have

$$|\{x \in Q_0 : M_{Q_0}^{+,d} f(x) - f_{Q_0^+}^+ > \lambda\}|$$

$$\leq \frac{aK^+}{\lambda} |\{x \in Q_0 : M_{Q_0}^{+,d} f(x) - f_{Q_0^+}^+ > b\lambda\}|^{1/q},$$

whenever

$$b\lambda \geq \int_{Q_0^+} (f - f_{Q_0^+}^+) \, dx.$$  

Here $a = 4(1 - 2^n b)^{-1}$ and $q$ is the conjugate exponent of $p$. 
Proof. Setting
\[ E_Q(\lambda) := \{ x \in Q : M_Q^{+,d}(f - f_{Q_0^+}^+)(x) > \lambda \}, \]
we may write the statement as
\[ (3.4) \quad |E_Q(\lambda)| \leq \frac{4K_f^+}{(1 - 2^n b)\lambda} |E_Q(b\lambda)|^{1/q}. \]
Consider the function \((f - f_{Q_0^+}^+)^+\) and form the decomposition as above at level \(b\lambda\) to obtain a family of pairwise non-overlapping dyadic sub-cubes with
\[ E_Q(b\lambda) = \bigcup_j Q_j. \]
Since \(b\lambda < \lambda\), we have \(E_Q(\lambda) \subset E_Q(b\lambda)\). It now follows that
\[ (3.5) \quad E_Q(\lambda) = \bigcup_j E_{Q_j}(\lambda). \]
We claim that for every \(j\),
\[ (3.6) \quad |E_{Q_j}(\lambda)| \leq \frac{2}{(1 - 2^n b)\lambda} \int_{Q_j \cup Q_j^+} (f - f_{Q_j^+}^+)^+ \, dx. \]
Consider the functions \(g_j := (f - f_{Q_j^+}^+)^+\). To prove (3.6) it suffices to show that
\[ (3.7) \quad E_{Q_j}(\lambda) \subset \{ x \in Q_j : M_{Q_j}^{+,d}g_j(x) > (1 - 2^n b)\lambda \}. \]
Indeed, (3.6) then follows at once from the weak type estimate (3.3) applied to the functions \(g_j\) with \(\lambda\) replaced by \((1 - 2^n b)\lambda\). Let \(x \in E_{Q_j}(\lambda)\) for some \(j\). Then there exists a dyadic subcube \(Q\) of \(Q_j\) containing \(x\) and satisfying
\[ \int_{Q^+} (f - f_{Q_0^+}^+)^+ > \lambda \]
From (3.2) we have
\[ \int_{Q_j^+} (f - f_{Q_0^+}^+)^+ \leq 2^n b\lambda. \]
Combining these, we obtain

\[ (1 - 2^n b) \lambda < \int_{Q_0^+} (f - f_{Q_0^+})^+ \, dx - \int_{Q_j^+} (f - f_{Q_j^+})^+ \, dx \]

\[ \leq \int_{Q_0^+} (f - f_{Q_0^+})^+ \, dx - \left( \int_{Q_j^+} f - f_{Q_0^+} \, dx \right)^+ \]

\[ = \int_{Q_0^+} (f - f_{Q_0^+})^+ - (f_{Q_j^+} - f_{Q_0^+})^+ \, dx \]

\[ \leq \int_{Q_0^+} (f - f_{Q_0^+})^+ \, dx \]

\[ \leq M_{Q_0^+} g_j(x). \]

Having now seen that (3.6) holds, we use (3.5) and sum over all \( j \) to obtain

\[ |E_{Q_0}(\lambda)| = \sum_j |E_{Q_j}(\lambda)| \]

\[ \leq \frac{2}{(1 - 2^n b) \lambda} \sum_j \int_{Q_j \cup Q_j^+} (f - f_{Q_j^+})^+ \, dx \]

\[ = \frac{2}{(1 - 2^n b) \lambda} \sum_j |Q_j|^{1/q} |Q_j|^{-1/q} \int_{Q_j \cup Q_j^+} (f - f_{Q_j^+})^+ \, dx \]

\[ \leq \frac{4K_j^+}{(1 - 2^n b) \lambda} \left( \sum_j |Q_j| \right)^{1/q}, \]

where the last inequality follows from the Hölder inequality and the definition of \( K_j^+ \). Remembering also that \( E_{Q}(b \lambda) = \bigcup_j Q_j \), we obtain the desired estimate.

We now complete the proof of the theorem by iterating the previous lemma. Except for a few details, this is just a repetition of the argument used in [1].

**Proof of the Theorem.** Using the same notation as in the proof of the lemma, we shall show

\[ |E_{Q_0}(\lambda)| \leq C \left( \frac{K_j^+}{\lambda} \right)^p. \]
Let us choose 
\[ \lambda_0 := \frac{2K_f^+}{b|Q_0|^{1/p}} \]
and assume \( \lambda > \lambda_0 \). Then take \( N \in \mathbb{Z}_+ \) such that
\[ b^{-N}\lambda_0 \leq \lambda < b^{-(N+1)}\lambda_0 = \frac{2b^{-(N+2)}K_f^+}{|Q_0|^{1/p}}. \]

By the definition of \( K_f^+ \), we have
\[ \frac{1}{|Q_0|} \int_{Q_0 \cup Q_0^+} (f - f_{Q_0^+})^+ \, dx \leq \frac{2K_f^+}{|Q_0|^{1/p}} = b\lambda_0. \]

In particular, this implies
\[ \frac{1}{b} \int_{Q_0^+} (f - f_{Q_0^+})^+ \, dx \leq \lambda_0 \leq b^{-1}\lambda_0 \leq \ldots \leq b^{-N}\lambda_0, \]
allowing us to apply the previous lemma successively \( N \) times to estimate the left-hand side of (3.8) as follows:

\[ |E_{Q_0}(\lambda)| \leq |E_{Q_0}(b^{-N}\lambda_0)| \]
\[ \leq aK_f^+ \cdot \left( \frac{aK_f^+}{b^{-N+1}\lambda_0} \right)^{1/q} \cdot \ldots \cdot \left( \frac{aK_f^+}{b^{-1}\lambda_0} \right)^{1/q^{N-1}} |E_{Q_0}(\lambda_0)|^{1/q^N} \]
\[ \leq aK_f^+ \cdot \left( \frac{aK_f^+}{b^{2}\lambda} \right)^{1/q} \cdot \ldots \cdot \left( \frac{aK_f^+}{b^{N}\lambda} \right)^{1/q^{N-1}} \cdot \left( \frac{2}{\lambda_0} \int_{Q_0 \cup Q_0^+} (f - f_{Q_0^+})^+ \, dx \right)^{1/q^N}, \]

where the last inequality follows from the weak type estimate (3.3) and the first inequality in (3.9). By the choice of \( \lambda_0 \) and (3.10) we further estimate
\[ |E_{Q_0}(\lambda)| \leq \left( \frac{aK_f^+}{\lambda} \right)^{1+q^{-1}+\ldots+q^{-(N-1)}} \cdot b^{-(1+2q^{-1}+\ldots+Nq^{-(N-1)})+q^{-N}} \cdot (2b|Q_0|)^{1/q^N} \]
\[ = \left( \frac{aK_f^+}{\lambda} \right)^{p-p/q^N} \cdot b^{-(1+2q^{-1}+\ldots+Nq^{-(N-1)})+q^{-N}} \cdot 2^{1/q^N} \cdot |Q_0|^{1/q^N}. \]

Since both \( 1+2q^{-1}+\ldots+Nq^{-(N-1)} \) and \( p-p/q^N \) remain bounded as \( N \to \infty \), we have
\[ |E_{Q_0}(\lambda)| \leq C|Q_0|^{1/q^N} \left( \frac{1}{\lambda} \right)^{p-p/q^N}. \]
Finally, we notice that from the second inequality in (3.9) we get

$$|Q_0|^{1/q^N} \left( \frac{1}{\lambda} \right)^{-p/q^N} = \lambda^{p/q^N} |Q_0|^{1/q^N} \leq 2^{p/q^N} b^{-(N+2)p/q^N} \leq C$$

with $C$ independent of $N$. Thus we have arrived at the desired estimate.

For $0 < \lambda \leq \lambda_0$ we use the trivial estimate

$$|\{ x \in Q_0 : (f(x) - f_{Q_0^{1/2}})^+ > \lambda \}| \leq |Q_0| = \frac{2^p(K_0^+)^p}{b^p \lambda_0^p} \leq C \left( \frac{K_0^+}{\lambda} \right)^p.$$

**Acknowledgement.** The author was supported by the Finnish Cultural Foundation. The author wishes to thank J. Kinnunen for proposing the problem.

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