There are no stable points for continuum-wise expansive homeomorphisms

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Abstract

We obtain some results about continuum-expansive expansive homeomorphisms, such as non-existence of stable points and presence of non-trivial connected components within the local stable and unstable sets. These facts have been of importance in theorems of classification of expansive homeomorphisms (see below). They are achieved now, however, by new and self contained techniques that hold on more general metric spaces.

1 Introduction

A homeomorphism $f$ on a compact metric space $(X, d)$ is said to be expansive if there exists a positive constant $\alpha > 0$, called an expansivity constant for $f$, such that $\sup \{d(f^n(x), f^n(y)) : n \in \mathbb{Z} \} > \alpha$ for all $x \neq y$ in $X$. This class of homeomorphisms includes, among other examples, subshifts of finite type, pseudo-Anosov homeomorphisms on surfaces and Anosov diffeomorphisms. Expansiveness is invariant under topological conjugation.

One may easily construct trivial examples of expansive dynamics, such as the restriction of the North Pole-South Pole diffeomorphism to the closure of a single (non trivial) orbit. However, the presence of some topological properties on the space $X$ imposes severe restrictions to the behavior of an expansive homeomorphism $f$, and viceversa. For instance, if $X = M^n$ is an $n$-dimensional orientable compact closed manifold, then the fact that $f$ is expansive on $M$ implies:

1. $n \neq 1$.
2. If $n = 2$ then either $M = \mathbb{T}^2$, in which case $f$ is conjugated to an Anosov diffeomorphism, or else $g(M) > 1$ and $f$ is then conjugated to a pseudo-Anosov diffeomorphism $\mathbb{H}, \mathbb{L}_2$.
3. If $n = 3$ and $f \in C^{1+\varepsilon}$, then $\Omega(f) = M$ implies $f$ is conjugated to an Anosov diffeomorphism and $M = \mathbb{T}^3$ $\mathbb{V}$.
4. $M$ cannot be simply connected $\mathbb{F}$.

Results in $\mathbb{H}, \mathbb{L}_2, \mathbb{V}$ make strong use of the fact that such homeomorphisms admit no stable points and that, furthermore, there are non-trivial, uniformly sized, connected pieces within both the local stable and unstable sets of each point (definitions are in next paragraph).

On $\mathbb{K}$, a weaker notion of expansiveness is introduced, which is the following: a homeomorphism $f : X \to X$ is said to be continuum-wise expansive (cw-expansive) if there exists a constant $\alpha > 0$ such that each non-trivial continuum $C$ satisfies $\sup_{n \in \mathbb{Z}} \text{diam } f^n(C) > \alpha$. 

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This class contains the class of expansive homeomorphisms, but it is strictly larger, as may be seen in the following example:

Considering $\mathbb{S}^2$ as the quotient space of the projection map $\pi : \mathbb{T}^2 \to \mathbb{S}^2$ defined for each $x \in \mathbb{T}^2$ by $\pi^{-1}(\pi(x)) = \{x, -x\}$, and providing $\mathbb{S}^2$ with the metric $d_{\mathbb{S}^2}(\xi, \eta) = \min\{d_{\mathbb{T}^2}(x, y) : \pi(x) = \xi, \pi(y) = \eta\}$; it is not difficult to show that $\text{diam}_{\mathbb{S}^2} \pi(C) \geq \frac{1}{2} \text{diam}_{\mathbb{T}^2} C$ for any continuum $C \subset \mathbb{T}^2$ such that $\text{diam}_{\mathbb{T}^2} C < \frac{1}{2}$. Now, if $f$ is a hyperbolic automorphism defined on $\mathbb{T}^2$, then $f$ is cw-expansive. The previous comment implies that $F = \pi \circ f \circ \pi^{-1}$ is a cw-expansive homeomorphism on $\mathbb{S}^2$ which cannot be expansive due to [H, L2].

The purpose of this paper is to provide self contained proofs of the presence of non trivial connected pieces within the local stable and unstable sets of each point for cw-expansive homeomorphisms acting on locally connected compact metric spaces. The same result was obtained in [H, L2] for expansive homeomorphisms, though we use less sophisticated machinery. As a particular case, it follows that there are no stable points for this kind of dynamics, a property which is also known in the literature as sensitive dependence on the initial conditions. The incidence of different forms of connectedness in these results is studied, obtaining counterexamples in some cases.

These new tools leave the possibility of improving classification results for this class of homeomorphisms.

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1.1 Main results

If $f$ is a homeomorphism on a compact metric space, the $\varepsilon$-local stable set of a point $x$ in $X$ is defined as the set

$$W^s_\varepsilon(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \varepsilon \quad \forall n \geq 0\}$$

We shall say that $x \in X$ is a stable point (in the future) if $\{W^s_\varepsilon(x)\}_{\varepsilon>0}$ is a neighborhood basis for $x$, that is, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup_{n \to \infty} d(f^n(x), f^n(y)) \leq \varepsilon$ if $d(x, y) \leq \delta$. The notions of $\varepsilon$-local unstable set and stable point in the past are defined similarly. Denoting by $X'$ the set of accumulation points of $X$, we show

**Theorem 1.1** If $f$ is a cw-expansive homeomorphism on a locally connected compact metric space $X$, then there are no stable points for $f$ in $X'$.

Previous theorem is used to show a much stronger fact, namely, that each point in $X'$ belongs to a non-trivial connected component of its own stable and unstable set. This is an essential step in [H, L2, V].

In fact, denoting by $CW^\sigma_\varepsilon(x)$ the connected component of $x$ in the set $W^\sigma_\varepsilon(x)$ with $\sigma = s, u$, we show:
Theorem 1.2 If $f$ is a cw-expansive homeomorphism on a locally connected compact metric space $X$, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that
\[
\inf_{x \in X'} \text{diam } CW^s_\varepsilon(x) \geq \delta \quad \text{and} \quad \inf_{x \in X'} \text{diam } CW^u_\varepsilon(x) \geq \delta
\]

Let us remark that in [RR] there is an example showing that Theorems 1.1 and 1.2 do not hold on general continua. Moreover, Theorem 1.2 does not hold on general continua even under the assumption of non existence of stable points. An example showing this is built in §3.

2 Connectedness and stability

2.1 Expansive homeomorphisms

In this paragraph we shall prove that there are no stable accumulation points for an expansive homeomorphism acting on a locally connected metric space $X$ (Theorem 2.7). Many of the steps involved, however, are valid under much weaker hypotheses on $X$. Namely, we shall see

1. Any stable recurrent point is periodic.
2. The set of stable points is open.
3. Any stable point having a point of local connectedness in its $\alpha$-limit point is recurrent, and hence periodic.

These statements are valid on any compact metric space. The three of them together, in a locally connected setting, imply non existence of stable accumulation points. The following property is basic in the whole proof.

Lemma 2.1 $f$ is uniformly expansive, i.e. for each $\varepsilon > 0$ there exists a positive integer $N_\varepsilon$ such that for either $x, y$ in $X$
\[
d(x, y) \geq \varepsilon \quad \Rightarrow \quad \sup_{|n| \leq N_\varepsilon} d(f^n(x), f^n(y)) > \alpha
\]

Proof. Otherwise, one could choose $\varepsilon > 0$ and pairs of points $x_j, y_j \in X$ with $d(x_j, y_j) \geq \varepsilon$, so that $d(f^n(x_j), f^n(y_j)) \leq \alpha$ for all $|n| \leq j$, and $j \in \mathbb{Z}^+$. Assuming that $x_j \to x_*$ and $y_j \to y_*$, we would obtain that $\varepsilon \leq \sup_{n \in \mathbb{Z}} d(f^n(x_*), f^n(y_*)) \leq \alpha$, contradicting expansiveness.

An immediate, though interesting, corollary of this is that $\alpha$-local stable sets are uniformly shrunk under the action of $f$. Indeed, if $y \in W^s_\alpha(x)$ then, for each $n \geq N_\varepsilon$ and $|j| \leq N_\varepsilon$ we have $d(f^{n+j}(x), f^{n+j}(y)) \leq \alpha$. Lemma 2.1 implies $d(f^n(x), f^n(y)) < \varepsilon$. This proves:

Corollary 2.2 For all $\varepsilon > 0$ and $x \in X$, $f^n(W^s_\alpha(x)) \subset B_\varepsilon(f^n(x))$ if $n \geq N_\varepsilon$.

In particular, $\omega(y) = \omega(x)$ for all $y \in W^s_\alpha(x)$, where $\omega(x)$ denotes, as usual, the set of $\omega$-limit points of the orbit of $x$. We may obtain from this fact the following relationship between stability and recurrence. Recall that $x$ is said to be a recurrent point if $x \in \omega(x)$.

Proposition 2.3 A stable recurrent point is always periodic.
Proof. It suffices to consider $\delta > 0$ so that $B_\delta(x) \subset W_\alpha(x)$, and $m \geq N_{\delta/4}$ so that $f^m(x) \in B_{\delta/4}(x)$. Then $f^m(B_\delta(x)) \subset f^m(W_\alpha(x)) \subset B_{\delta/4}(f(x)) \subset B_{\delta/2}(x)$. This implies the existence of a periodic point $z = \bigcap_{k \geq 0} f^{km}(B_\delta(x))$ in $B_{\delta/2}(x)$, which means $x \in \omega(x) = \omega(z) = \alpha(z)$ is actually a periodic point.

For the sake of simplicity, we shall denote

$$W_{\varepsilon,N}(x) = \bigcap_{n=0}^{N} \{ y \in X : d(f^n(x), f^n(y)) \leq \varepsilon \} \quad (2.1)$$

We have the following characterizations of stable points.

**Lemma 2.4** Each condition below is equivalent to $x$ being a stable point.

1. $W_{\alpha}(x)$ is a neighborhood of $x$

2. There exists $0 < \delta_0 < \alpha$ such that for each $0 < \delta \leq \delta_0$ we have $W_{\delta,N_k}(x) = W_{\delta}(x)$.

**Proof.** It follows by taking $\delta_0 > 0$ so that $B_{\delta_0}(x) \subset W_{\alpha}(x)$ and applying Corollary 2.2.\]

As a result, if $x$ is a stable point and $0 < \delta < \min\{\delta_0, \alpha/2\}$, then all points $y$ in $\text{int}(W_{\delta}(x))$ satisfy $W_{\alpha}(y) \supset \text{int}(W_{\delta}(x))$. Item [3] in proposition above implies $\text{int}(W_{\delta}(x))$ consists of stable points. Hence,

**Corollary 2.5** The set of stable points is open.

We shall impose now more restrictions on the set $X$. The following property is key in showing Theorem 2.7. We shall denote by $CB_x(x)$ the component of $x$ in the ball $B_x(x)$

**Proposition 2.6** Let $x$ be a stable point. If $\alpha(x)$ contains a point of local connectedness of $X$, then $x \in \omega(x)$, and hence it is periodic.

**Proof.** The proof is reduced to proving there exists $\rho > 0$ such that:

$$B_\rho(z) \subset W_{\alpha}(f^{-nk}(x)) \quad \forall k \geq 0$$

where $z = \lim_{k \to \infty} f^{-nk}(x)$ is a point of local connectedness of $X$. Indeed, this would give us for each $\varepsilon > 0$, a pair $n_k > m_k \geq N_k$ such that $f^{-mk}(x) \in B_\rho(z) \subset W_{\alpha}(f^{-nk}(x))$. Using Corollary 2.2 we would immediately have that $f^{-mk}(x) \in B_\varepsilon(x)$ whence $x \in \omega(x)$.

If a $\rho > 0$ as described above did not exist, we would find a subsequence $f^{-nk}(x) \in CB_{1/k}(z)$ such that $CB_{1/k}(z) \not\subset W_{\alpha}(f^{-nk}(x))$. Choosing $0 < \delta < \alpha$ as in Lemma 2.4, so that $W_{\delta}(x) = W_{\delta,N_k}(x)$, we would find, via connectedness, a point $f^{-nk}(y_k)$ in $CB_{1/k}(z) \cap \partial W_{\delta}(f^{-nk}(x))$, that would satisfy $\sup_{n \geq n_k} d(f^n(x), f^n(y_k)) = d(f^{-mk+n_k}(x), f^{-mk+n_k}(y_k)) = \delta$ for some $m_k \geq 0$ verifying $m_k \rightarrow \infty$, due to continuity of $f$.

We may assume that $f^{-mk+n_k}(x), f^{-mk+n_k}(y_k)$ converge, respectively, to $x^*, y^*$, whence it would follow that $d(x^*, y^*) = \delta$. On the other hand, for each $l \in \mathbb{Z}$, we would have

$$d(f(lx^*), f(l^*y^*)) = \lim_{k \to \infty} d(f^{l+mk-n_k}(x), f^{l+mk-n_k}(y_k)) \leq \sup_{n \geq n_k} \{ d(f^n(x), f^n(y_k)) \} = \delta$$

contradicting the hypothesis of expansiveness.\]

As an immediate corollary we obtain:
Theorem 2.7

If $f$ is an expansive homeomorphism over a locally connected compact metric space $X$, then there are no stable points for $f$ in $X'$.

We observe that, though it is beyond the scope of this paper, Proposition 2.6 could also be useful in identifying “basins of attraction” in general continua. For instance, in such a basin, all but possibly one point shall have $\alpha$-limit sets consisting of non locally connected points.

2.2 Continuum-wise expansive homeomorphisms

If $f : X \to X$ is an $\alpha$ cw-expansive homeomorphism on a compact metric space $X$, then a little make-up allows us to extend Theorems 2.7 to the cw-expansive case. Essentially the same steps will be followed, though special care is required at some steps.

Lemma 2.8

$f$ is uniformly cw-expansive, that is, for each $\varepsilon > 0$ there exists a positive integer $N_\varepsilon$ such that $\max_{1 \leq n \leq N_\varepsilon} \text{diam } f^n(C) > \alpha$ for any continuum $C$ satisfying $\text{diam } C \geq \varepsilon$.

The proof follows substantially as in Lemma 2.1, as well as the following corollary:

Corollary 2.9

For all $\varepsilon > 0$ and $x \in X$, $f^n(CW_{\alpha/2}(x)) \subset B_\varepsilon(f^n(x))$ if $n \geq N_\varepsilon$. In particular, $\omega(y) = \omega(x)$ for all $y \in CW_{\alpha/2}(x)$.

As in §2, we shall prove items 1., 2. and 3. stated in page 3, however, we must take into account that, unlike in the expansive case, locally connectedness is required. Observe that if $X$ is locally connected at $x$, then $x$ being a stable point is equivalent to the fact that $\{CW_{\varepsilon}(x)\}_{\varepsilon > 0}$ is a neighborhood basis of $x$. Having this in mind, a procedure very similar to that in proof of Proposition 2.3 yields

Proposition 2.10

Any stable recurrent point of local connectedness of $X$ is periodic.

Observe that if $X$ is locally connected at $x$, then all $y \in \text{int } CW_{\alpha/2}(x)$ are stable. Indeed, for each $\varepsilon > 0$ there exists $\rho > 0$ satisfying $B_\rho(y) \subset \text{int } CW_{\alpha/2}(x) \cap W^s_{\varepsilon,N_\varepsilon/2}(y)$. Corollary 2.9 implies $\sup_{n \geq 0} d(f^n(y), f^n(z)) \leq \varepsilon$. This proves:

Proposition 2.11

If $X$ is locally connected at a stable point $x$, then there exists a neighborhood of $x$ consisting of stable points. In particular, the set of stable points is open if $X$ is locally connected.

The following characterization of stable points shall also be used

Lemma 2.12

If $X$ is locally connected at $x$, each condition below is equivalent to the stability of the point $x$.

1. $CW_{\alpha/2}(x)$ is a neighborhood of $x$

2. There exists $0 < \delta_0 < \alpha$ such that for all $0 < \delta \leq \delta_0$, $W^s_{\delta,N_\delta}(x) = W^s_{\delta}(x)$

The following proposition directly implies Theorem 1.1.

Proposition 2.13

If $X$ is locally connected, then any stable point $x$ is recurrent and hence, periodic.
Proof. Following the spirit of the proof of Proposition 2.6, it will suffice to see that there is some \( \rho > 0 \) satisfying \( B_\rho(f^{-n}(x)) \subset CW\alpha/2(f^{-n}(x)) \) for all \( n \geq 0 \).

If it were not the case, we would find a subsequence \( n_k \to \infty \) such that \( CB_{1/k}(f^{-n_k}(x)) \setminus CW\alpha/2(f^{-n_k}(x)) \) is not empty. Choosing \( \delta > 0 \) as in Lemma 2.12 so that \( W_\delta(x) \subset CW\alpha/2(x) \) we would find, due to connectedness of \( CB_{1/k}(f^{-n_k}) \), a point \( f^{-n_k}(y_k) \in CB_{1/k}(f^{-n_k}(x)) \) belonging to \( \partial CW_\delta(f^{-n_k}(x)) \). Local connectedness of \( X \) would imply that \( f^{-n_k}(y_k) \) actually belongs to \( \partial W_\delta(f^{-n_k}(x)) \), and hence \( d(f^{m_k-n_k}(x), f^{m_k-n_k}(y_k)) = \delta \) for some \( m_k \geq 0 \), which, because of continuity, would satisfy \( m_k \to \infty \). This would mean that \( C_k = f^{m_k}(CW_\delta(f^{-n_k}(x))) \) is a continuum satisfying

\[
\delta \leq \sup_{n \geq m_k} \text{diam} f^n(C_k) \leq 2\delta \leq \alpha
\]

Assuming that \( C_k \) converges in the Hausdorff topology to a continuum \( C_* \), we would obtain that \( \delta \leq \sup_{n \in \mathbb{N}} \text{diam} f^n(C_*) \leq \alpha \), what would mean that \( f \) is not \( c.w. \)-expansive. □

Now we turn our attention to proving Theorem 1.2. The following Lemma is straightforward:

**Lemma 2.14** If \( X \) is a locally connected compact metric space, then for all \( 0 < \varepsilon < \alpha \) and \( \delta > 0 \) there exists \( K = K_{\varepsilon, \delta} > 0 \) such that all \( x \in X' \) satisfy

\[
CB_\delta(x) \not\subset CW^s_{\varepsilon, K}(x) \quad \text{and} \quad CB_\delta(x) \not\subset CW^u_{\varepsilon, K}(x)
\]

We are now in conditions to prove

**Theorem 2.15** If \( X \) is a locally connected compact metric space, then for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \inf_{x \in X'} \text{diam} CW^\sigma_\varepsilon(x) \geq \delta \) for \( \sigma = s, u \).

**Proof.** Let \( 0 < \varepsilon < \alpha \), and choose \( 0 < \delta < \varepsilon \) so that, for each \( x \in X \)

\[
d(x, y) \leq \delta \quad \Rightarrow \quad d(f^n(x)f^n(y)) < \varepsilon \quad \text{for all} \quad |n| \leq N_\varepsilon \quad (2.2)
\]

where \( N_\varepsilon \) is as in Lemma 2.8. Let us denote \( CW^\sigma_{\varepsilon, n}(x) = cc(W^\sigma_{\varepsilon, n}(x), x) \), for each \( n \geq 0 \). We shall see that \( CW^s_{\varepsilon, n}(x) \cap \partial B_\delta(x) \) is not empty if \( n \geq \max\{N_\varepsilon, K_{\varepsilon, \delta}\} \). From this it follows that \( CW^s_\varepsilon(x) = \lim_{n \to \infty} CW^s_{\varepsilon, n}(x) \) satisfies \( \text{diam}(CW^s_\varepsilon(x)) \geq \delta \).

From Lemma 2.14 we have that \( f^{-n}(CB_\delta(f^n(x))) \) is not included in \( CW^s_{\varepsilon, n}(x) \) if \( n \geq K_{\varepsilon, \delta} \). Due to connectedness, there is a point \( y_0 \) in \( f^{-n}(CB_\delta(f^n(x))) \) belonging to \( \partial CW^s_{\varepsilon, n}(x) \) which is included in \( \partial W^s_{\varepsilon, n}(x) \) because of local connectedness of \( X \). This implies

\[
d(f^k(x), f^k(y_0)) = \varepsilon \quad \text{for some} \quad k = 0, \ldots, n - 1
\]

Now, if \( n \geq N_\varepsilon \), then

- \( k \) cannot be in \( [n - N_\varepsilon, n - 1] \) because of (2.2).
- \( k \) cannot be in \( [N_\varepsilon, n - N_\varepsilon] \), for otherwise there would exist \( 0 \leq j \leq n \) such that \( \text{diam}(CW^s(x)) > \alpha > 2\varepsilon \).

Therefore \( k \) is mandatorily in \( [0, N_\varepsilon] \), resulting from (2.2) that \( d(x, y_0) > \delta \). □
3 Example

Here we show that $X$ being a continuum is not a sufficient condition for an expansive $f$ to have local stable or unstable sets of non trivial diameter for all points in $X$.

Take a linear Anosov automorphism $f$ on $T^2$, and choose a fixed point $p \in T^2$. Let us also choose a stable and an unstable separatrix of $p$, say $W_s^+(p)$ and $W_u^+(p)$. Lift copies $\tilde{p}$, $\tilde{W}_s^+(p)$, and $\tilde{W}_u^+(p)$ respectively, in $\mathbb{R}^3$, so that $\tilde{p} \in \tilde{W}_s^+(p) \cap \tilde{W}_u^+(p)$ be asymptotic to $W_\sigma^+(p)$, $\sigma = s, u$ (see Figure 1.a). We can introduce an expansive dynamic on this set extending the original hyperbolic diffeomorphism. In this way we obtain an expansive homeomorphism with stable points (in the future and the past), none of which are fixed or periodic. This is essentially the example of [RR].

If $G$ is any small local perturbation of $F$ at $\tilde{p}$, so that there are no fixed points in $L = W_s^+(p) \cup W_u^+(p)$, and $\omega(x) \cup \alpha(x) \subset T^2$ for each point in $L$, then $G$ is expansive, and has no stable points, while $W_\varepsilon^s(x) \cup W_\varepsilon^u(x) = \{x\}$ for all $x$ in $L$ (see Figure 1.b).

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