CUBIC RINGS AND THEIR IDEALS

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Abstract. We give an explicit description of cubic rings over a
discrete valuation ring, as well as a description of all ideals of such
rings.

INTRODUCTION

Ideals of local rings have been studied by a lot of authors from quite
different viewpoints. One of the questions that arise with this respect
is on the number of parameters \( \text{par}(C) \) defining the ideals of such a
ring \( C \) up to isomorphism, especially when it is reduced and of Krull
dimension 1. Certainly, it makes sense if the residue field \( k \) is infinite.
In [8] it was shown that \( \text{par}(C) = 0 \), i.e. \( C \) has a finite number
of ideals (up to isomorphism), if and only if \( C \) is Cohen–Macaulay finite,
i.e. has a finite number of indecomposable non-isomorphic Cohen–
Macaulay modules (in the 1-dimensional reduced case they coincide
with torsion free modules). Then Schappert [12] proved that a plane
curve singularity has at most 1-parameter families of ideals if and only
if it dominates one of the strictly unimodal plane curve singularities
in the sense of [14], or, the same, unimodal and bimodal plane curve
singularities in the sense of [1]. In [7] this result was generalized to all
curve singularities. Note that this time \( \text{par}(C) = 1 \) does not imply that
\( C \) is Cohen–Macaulay tame, i.e. has at most 1-dimensional families of
indecomposable Cohen–Macaulay modules. Tameness means that \( C \)
dominates a singularity of type \( T_{pq} \) [5]. The case \( \text{par}(C) > 1 \) had
not been studied before the second author described the one branch
singularities of type \( W \) such that \( \text{par}(C) \leq 2 \) [13].

In this paper we study the cubic rings. We describe all such rings,
their ideals and, in particular, establish the value \( \text{par}(C) \) for any cubic
ring \( C \). As a consequence, we show that a cubic ring is Gorenstein if
and only if it is a plane curve singularity (i.e. its embedding dimension
equals 2).

1. Generalities

We denote by \( D \) a discrete valuation ring with the ring of fractions \( K \),
the maximal ideal \( m = tD \) and the residue field \( k = D/tD \). A cubic
ring over \( D \) is, by definition, a \( D \)-subalgebra \( C \) in a 3-dimensional

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semisimple $K$-algebra $L$, which is a free $D$-module of rank 3. We also denote $A$ the integral closure of $D$ in $L$ and always suppose that $A$ is finitely generated as $C$-module. Equivalent condition (see, for instance, [3]): the $m$-adic completion $\hat{C}$ of the ring $C$ has no nilpotent elements. It is always the case if the algebra $L$ is separable, for instance, if $\text{char } K = 0$. We also set $A_m = t^mA + D$ and $J_m = tA_{m-1} = \text{rad } A_m$ ($m > 0$).

In what follows, an ideal means a fractional $C$-ideal in $K$, i.e. a finitely generated $C$-submodule $M \subseteq K$ such that $KM = K$. Then $M$ is a free $D$-module of rank 3. We are going to describe all ideals of cubic rings up to isomorphism. It is known (see, for instance, [9]) that there is a one-to-one correspondence between $C$-ideals and $\hat{C}$-ideals, mapping $M$ to its $m$-adic completion. This correspondence reflects isomorphisms, i.e. maps non-isomorphic ideals to non-isomorphic. So, in what follows we may (and will) suppose that $D$ is complete with respect to the $m$-adic topology.

Recall also that the embedding dimension $\text{edim } C$ of a local noetherian ring $C$ with the maximal ideal $J$ and the residue field $k$ is defined as $\text{dim}_k J/J^2$. If $C$ is of Krull dimension 1 and $\text{edim } C = 2$, $C$ is called a plane curve singularity. In the geometric case, when $C$ contains a subfield of representatives of $k$, it actually means that there is a plane curve $C$ such that $C$ is the completion of the local ring of a singular point $x \in C$.

From the general theory of ramification in finite extensions we see that the following cases can happen:

**One branch, ramified case**: $L$ is a field, the maximal ideal of $A$ equals $\tau A$, $A/\tau A \simeq k$ and $tA = \tau^3 A$.

**One branch, non-ramified case**: $L$ is a field, the maximal ideal of $A$ equals $tA$ and $A/tA = k[\bar{\theta}]$ is a cubic extension of the field $k$, where $\bar{\theta}$ is a root of an irreducible cubic polynomial $f(x) \in k[x]$.

**Two branches, ramified case**: $L = K_1 \times K$, where $K_1$ is a quadratic extension of $K$, $A = D_1 \times D_1$, the maximal ideal of $D_1$ is $\tau D_1$, $D_1/\tau D_1 \simeq k$ and $tD_1 = \tau^2 D_1$.

**Two branches, non-ramified case**: $L = K_1 \times K$, where $K_1$ is a quadratic extension of $K$, $A = D_1 \times D$, the maximal ideal of $D_1$ is $tD_1$ and $D_1/\tau D_1 = k[\bar{\theta}]$ is a quadratic extension of the field $k$, where $\bar{\theta}$ is a root of an irreducible quadratic polynomial $f(x) \in k[x]$.

**Three branches case**: $L = K^3$, $A = D^3$.

We recall [10, 2] that, for any cubic ring $C$, every ideal of $C$ is isomorphic either to an over-ring of $C$, i.e. a cubic ring $B$ such that $C \subseteq B \subseteq L$, or to the dual ideal $B^* = \text{Hom}_D(B,D)$ of such an over-ring. Hence, to describe all ideals of $C$, we only need to describe
2.1. One branch, ramified case.

Therefore, to describe all cubic rings (so their ideals as well), we have to describe the over-rings of $A_m$. If $B$ is an over-ring of $C$, they also say that $B$ dominates $C$.

Since the unique (up to isomorphism) $A$-ideal is $A$ itself, we proceed by induction: supposing that all over-rings of $A_m$ are known, we find all over-rings of $A_{m+1}$. If $C$ is an over-ring of $A_{m+1}$, then $B = CA_m$ is an over-ring of $A_m$, $tB \subset C$ and $C/tB$ is a $k$-subalgebra in $B/tB$. If $B \supset A_{m-1}$, then $tB \supset J_m$, hence, $C \supset J_m + D = A_m$. Therefore, the following procedure gives all over-rings of $A_{m+1}$ which are not over-rings of $A_m$:

Procedure.

- For every over-ring $B$ of $A_m$, which is not an over-ring of $A_{m-1}$, calculate $B = B/tB$. Set $A = (A_m + tB)/tB \subseteq B$.
- Find all proper subalgebras $S \subset B$ such that $\bar{AS} = B$. Equivalently, the natural map $S \rightarrow B/BJ_m$ must be surjective.
- For each such $S$ take its preimage in $B$.

2. Calculations

2.1. One branch, ramified case. We set

$$C_{2r}(\alpha) = D + t^r \alpha D + t^{2r} A, \quad \text{where } v(\alpha) = 1,$$

$$C_{2r+1}(\alpha) = D + t^r \alpha D + t^{2r+1} A, \quad \text{where } v(\alpha) = 2,$$

where $v$ is the discrete valuation related to the ring $A$, i.e. $v(\alpha) = k$ means that $\alpha \in \tau^k D \setminus \tau^{k+1} D$. Note that $C_0(\alpha) = A$. Obviously, $\alpha$ can be uniquely chosen as $\tau + a \tau^2$ for $C_{2r}$, and as $\tau^2 + a \tau$ for $C_{2r+1}$, where $a \in D$ is defined modulo $t^r$.

**Theorem 2.1.** Every over-ring of $A_m$ coincides with $t^k C_r(\alpha) + D$ for some $k, r$ such that $r + k \leq m$ and some $\alpha$. The rings $C_r(\alpha)$ are just all plane curve singularities in this case.

**Proof.** For $m = 1$ it is easy and known [8, 11]. So, we use the Procedure for $m > 1$, setting $B = t^k C_r(\alpha) + D$, where $k + r = m$. Then the basis of $B$ consists of the classes of the elements $\{1, t^h \alpha, t^m \tau^s\}$, where $h = k + [r/2]$, $s \in \{1, 2\}$ and $s \equiv r \pmod{2}$. Since $t^h \alpha \notin J_m$, the subalgebra $S$ necessarily contains the class of $t^h \alpha + c t^m \tau^s$ for some $c \in D$. If $k = 0$, then $m = r$ and $v(t^m \tau^s) = 2v(t^h \alpha)$. Therefore, $B$ has no proper subalgebra containing the class of $t^h \alpha + c t^m \tau^s$. If $k > 0$, the preimage of $S$ is $D + (t^h \alpha + c t^m \tau^s) D + t^{m+1} A$. It coincides with $t^{k-1} C_{r+2}(\alpha') + D$ where $\alpha' = \alpha + c t^{m-h} \tau^s$.

Now one easily checks that $\edim C_r(\alpha) = 2$, while $\edim C = 3$ for all other rings. \qed
2.2. One branch, non-ramified case. We set $C_r(\alpha) = D + t^r \alpha D + t^{2r} A_0$, where $\alpha \in A^* \setminus D$. Again $C_0(\alpha) = A_0$. Note that $\alpha$ can be uniquely chosen as $\theta + a\theta^2$, where $\theta$ is a fixed preimage of $\overline{\theta}$ in $D_1$ and $a \in D$ is defined modulo $t^r$.

**Theorem 2.2.** Every over-ring of $A_m$ coincides with $t^k C_r(\alpha) + D$ for some $k, r$ and $\alpha$ with $2r + k \leq m$. The rings $C_r(\alpha)$ are just all plane curve singularities in this case.

**Proof.** For $m = 1$ it is obvious. So, using the Procedure for $m > 1$, we set $B = t^k C_r(\alpha) + D$ with $2r + k = m$. Then a basis of $B$ consists of the classes of elements $\{1, t^{r+k} \alpha, t^{m} \alpha^2\}$ for some $\alpha^2 \in A^* \setminus (D + \alpha D)$. Since $t^{r+k} \alpha \notin J_m$, $S$ must contain the class of an element $t^{r+k} \alpha' = t^{r+k} \alpha + ct^m \alpha^2$ for some $c \in D$. As above, it is impossible if $k = 0$. If $k > 0$, then the preimage of $S$ is $D + t^{r+k} \alpha' + t^{m+1} A = t^{k-1} C_{r+1}(\alpha') + D$.

Now one easily checks that $\text{edim} C_r(\alpha) = 2$, while $\text{edim} C = 3$ for all other rings. \hfill \Box

2.3. Two branches, ramified case. We denote by $v$ the valuation defined by the ring $D_1$, by $e$ the idempotent in $A$ such that $e A = D_1$ and set

$$C_{l,q}(\alpha) = D + t^l(e + t^q \alpha)D + t^r A, \quad \text{where } r = 2l + q,$$

$$C_r(\alpha) = D + t^r \alpha D + t^{2r+1} A.$$

In both cases $\alpha \in D_1$ and $v(\alpha) = 1$, where $v$ is the valuation defined by the ring $D_1$. Obviously, $\alpha$ can be uniquely chosen as $a\tau$, where $a \in D$ is defined modulo $r$. Note that $C_{0,q}(\alpha) = D + e_1 D + t^q A$ are just all decomposable rings in this case and $C_{0,0}(\alpha) = A$.

**Theorem 2.3.** Every over-ring of $A_m$ coincides with either $t^k C_{l,q}(\alpha) + D$ or $t^k C_r(\alpha) + D$, where $k + r \leq m$. The rings $C_{l,q}(\alpha)$ and $C_r(\alpha)$ are just all plane curve singularities in this case.

**Proof.** The case $m = 1$ is obvious. So, using the Procedure, we suppose that $m > 1$ and $k + r = m$. If $B = t^k C_{l,q}(\alpha) + D$, a basis of $B$ consists of the classes of $\{1, t^{k+l}(e + t^q \alpha), t^m \tau\}$. Since $t^{k+l}(e + t^q \alpha) \notin J_m$, the subalgebra $S$ must contain the class of $t^{k+l}(e + t^q \alpha')$ for some $\alpha' \in D_1$ with $v(\alpha') = 1$. Again the case $k = 0$ is impossible. If $k > 0$, the preimage of $S$ coincides with $t^{k-1} C_{l+1,q} + D$. If $B = t^k C_r(\alpha) + D$, the calculations are quite similar.

Now one easily checks that $\text{edim} C_{l,q}(\alpha) = \text{edim} C_r(\alpha) = 2$, while $\text{edim} C = 3$ for all other rings. \hfill \Box

2.4. Two branches, non-ramified case. We set

$$C_{l,q}(\alpha) = D + t^l(e_1 + t^q \alpha)D + t^r A, \quad \text{where } r = 2l + q$$

and $\alpha \in D_1 \setminus (e_1 D + tD)$.  

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Then $\alpha$ can be chosen as $a\theta$, where $\theta$ is a fixed preimage of $\bar{\theta}$ in $D_1$ and $a \in D$ is uniquely defined modulo $t^l$. Again $C_{0,q}(\alpha) = D + e_1D + t^qA$ are just all decomposable rings in this case. Especially, $C_{0,0}(\alpha) = A$.

**Theorem 2.4.** Every over-ring of $A_m$ coincides with one of the rings $t^kC_{l,q}(\alpha) + D$, where $k + r \leq m$. The rings $C_{l,q}(\alpha)$ are just all plane curve singularities in this case.

We omit the proof in this case, since it practically repeats the calculations in the other cases.

2.5. **Three branches case.** We set

$$C_{l,q}(\alpha) = D + t^l\alpha D + t^rA,$$

where $\alpha = e + t^qae'$, $e \neq e'$ are primitive idempotent in $A$, $r = 2l + q$, $a \in D^\times$ and $a \not\equiv 1 \pmod{t}$ if $q = 0$. Obviously, $a$ is unique modulo $t^l$. Again $C_{0,q}(\alpha) = D + eD + t^qA$ are just all decomposable rings in this case and $C_{0,0} = A$. Note also that if $C = D + t^l\alpha D + t^rA$, where $\alpha = e + ae'$ as above with $a \equiv 1 \pmod{t^l}$, then, for $a \equiv 1 \pmod{t}$, $C = t^lC_{0,q}(1 - e - e') + D$, and for $a \equiv 1 \pmod{t^q}$ with $0 < q < l$, $C = C_{l,q}(\alpha')$ for some $\alpha'$.

**Theorem 2.5.** Every over-ring of $A_m$ coincides with $t^kC_{l,q}(\alpha) + D$ for some $\alpha$ and some $l, q$ with $k + r \leq m$. The rings $C_{l,q}(\alpha)$ are just all plane curve singularities in this case.

We also omit the proof in this case, since it practically repeats the calculations in the other cases.

2.6. **Table of plane curve cubic singularities.** We present in Table 1 below all plane curve cubic singularities. In this table $s$ is the number of branches, $\ast$ marks the unramified cases (related to the residue field extensions, hence impossible if $k$ is algebraically closed); $x, y$ are generators of the maximal ideal, $v(a)$ denotes the multivaluation of an element $a \in A$, i.e. the vector of valuations of its components with respect to the decomposition of $A$ into the product of discrete valuation rings. The column “type” shows the correspondence with the Arnold’s classification [1, §15]. If char $k = 0$ and $A$ is ramified, it actually shows the place of the rings in this classification. If char $k = 0$ and $C$ is non-ramified, it shows the place of the ring in this classification after the natural extension of the field $k$. The validation of this column is given in [7, Section 2.3]. Note that, following [7], we denote by $E_{l,q}$ the singularities $J_{l,q}$ in the sense of [1]. Such notations seem more uniform. Note also that the singularities of types $E_1$ and $E_2$ are actually not cubic, but quadratic, and coincide with those of types $A_1$ and $A_2$ of [1]. Finally, the last column, “par” shows the number of parameters $p$ from the residue field $k$ which define a unique ring of this type. We will consider this value in the last section. It does not coincide with the modality in the sense of [1]; the latter equals $p - 1$. 

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Table 1.

| s | name                  | v(x) | v(y)     | type     | par |
|---|-----------------------|------|----------|----------|-----|
| 1 | $C_{2r}(\alpha)$      | (3)  | (3r + 1) | $E_{6r}$ | r   |
|   | $C_{2r+1}(\alpha)$    | (3)  | (3r + 2) | $E_{6r+2}$| r   |
| 1*| $C_r(\alpha)$         | (1)  | (r)      | $E_{r,0}^*$| r   |
| 2 | $C_r(\alpha)$         | (2,1)| (2r + 1, $\infty$) | $E_{6r+1}$| r   |
|   | $C_{l,q}(\alpha)$     | (2,1)| (2l, $\infty$)| $E_{l,2q+1}$| l   |
| 2*| $C_{l,q}(\alpha)$     | (1,1)| (l, $\infty$) | $E_{l,2q}^*$| l   |
| 3 | $C_{l,q}(\alpha)$     | (1,1,1)| (l, l + q, $\infty$) | $E_{l,2q}$| l   |

Remark. The tame cubic plane curve singularities $T_{3,q}$ $(q \geq 6)$ [4, 5] coincide with those of types $E_{2,q-6}$.

3. Ideals

As we have mentioned above, every ideal of a cubic ring $C$ is isomorphic either to an over-ring $B \supseteq C$ or to its dual $B^* = \text{Hom}_D(B, D)$. If $C$ is Gorenstein (for instance, if it is a plane cubic singularity), then $C^* \simeq C$, thus $B^* \simeq \text{Hom}_C(B, C)$. Therefore, to calculate $B^*$, one has to choose a Gorenstein subring $C \subseteq B$ and to calculate

$$
\text{Hom}_C(B, C) \simeq \{ \lambda \in L \mid \lambda B \subseteq C \} = \{ \lambda \in C \mid \lambda B \subseteq C \}
$$

(the latter equality holds since $1 \in B$). This remark easily leads to the following result.

Theorem 3.1. The duals to the cubic rings are as follows:

One branch ramified case: If $B = D + t^kC_r(\alpha)$, then $B^* \simeq D + t^{(r/2)}\alpha D + t^{k+r}A$.

One branch non-ramified case: If $B = D + t^kC_r(\alpha)$, then $B^* \simeq D + t^r\alpha D + t^{k+2r}A$.

Two branches ramified case: (1) If $B = D + t^kC_{l,q}(\alpha)$, then $B^* \simeq D + t^{l(e + t^q\alpha)}D + t^{k+2l+q}A$.

(2) If $B = D + t^kC_r(\alpha)$, then $B^* \simeq D + t^r\alpha D + t^{k+2r+1}A$. 
Two branches non-ramified case: If \( B = D + t^k C_{l,q}(\alpha) \), then 
\( B^* \simeq D + t^l(e + t^q\alpha)D + t^{k+2l+q}A \).

Three branches case: If \( B = D + t^k C_{l,q}(\alpha) \), then 
\( B^* \simeq D + t^l\alpha D + t^{k+2l+q}A \).

Proof. The proof is immediate if we choose for a Gorenstein subring 
\( C \subseteq B \) the plane curve singularity 
\( C = C_{k+r}(\alpha) \) or \( C_{k+l,q}(\alpha) \) depending 
on the shape of \( B \). For instance, in two branches ramified case, if 
\( B = D + t^k C_{l,q}(\alpha) \) and \( C = D + C_{l+k,q}(\alpha) \), then 
\( B^* \simeq \{ \lambda \in C \mid \lambda B \subseteq C \} = t^k D + t^{k+l}(e + t^q\alpha)D + t^{2k+2l+q}A \)
\simeq D + t^l(e + t^q\alpha)D + t^{k+2l+q}A. \)

Corollary 3.2. If a cubic ring is Gorenstein, it is a plane curve singularity.

Note that it is no more the case for the extensions of bigger degrees. For instance, the rings \( P_{pq} \) from [5], which are quartic, are Gorenstein (they are complete intersections) but of embedding dimension 3.

4. Geometric case. Number of parameters

In this section we suppose that our rings are of geometric nature, 
i.e. \( D = k[[t]] \), where \( k \) is algebraically closed. Then one can consider 
the number of parameters \( \text{par}(C) \) defining \( C \)-ideals (see [4, Section 2.2] or [6, Section 3], where it is denoted by \( \text{par}(1; C, A) \)). Actually, it coincides with the minimal possible number \( p \) for which there is a finite set of families of ideals \( I_k \) \( (1 \leq k \leq m) \) of dimensions at most \( p \) such 
that every \( C \)-ideal is isomorphic to one belonging to some family \( I_k \). 
Equivalently, it is the maximal possible \( p \) such that is a \( p \)-dimensional 
family of ideals \( I \) where every isomorphism class of ideals only occurs 
finitely many times. In [7] a criterion was established in order that 
\( \text{par}(C) \leq 1 \). For cubic rings it means that \( C \) dominates a singularity of type \( E_m \) \( (18 \leq m \leq 20) \) or \( E_{3,i} \). The following results give the exact 
value of \( \text{par}(C) \) for all cubic rings of geometric nature. (Note that no unramified case can occur for such rings.)

Theorem 4.1. If \( C \) is a cubic ring of geometric nature, \( \text{par}(C) \leq n \) 
if and only if \( C \) dominates one of the singularities of type \( E_{12n+i} \) \( (6 \leq i \leq 8) \) or \( E_{2n+1,q} \) \( (q \geq 0) \).

Proof. Certainly, we have to prove that

(1) every ring of one of the listed types have at most \( n \)-parameter families of ideals;

(2) if \( C \) dominates no ring of the listed types, it has \( (n+1) \)-
parameter families of ideals.

Since the calculations in all cases are similar, we only consider the one branch ramified case. Note first that the rings \( C_{2r}(\alpha) \) as well as
\( C_{2r+1}(\alpha) \) form a \( r \)-parametric family. Indeed, we can choose in the first case \( \alpha = \tau + a\tau^2 \), and in the second one \( \alpha = \tau^2 + a\tau^4 \), where \( a \in D \) is defined modulo \( t^r \), and such a presentation is unique. The same is true also for \( t^k C_{2r}(\alpha) + D \) and \( t^k C_{2r+1}(\alpha) + D \) for any \( k \). Since \( C_{2r}(\alpha) \supseteq A_{2r} \) for all \( \alpha \), we get \( \text{par}(A_{2r}) \geq r \).

Let \( C \) dominate neither a ring of type \( E_{12n+6} \), i.e. \( C_{4n+2}(\alpha) \), nor a ring of type \( E_{12n+8} \), i.e. \( C_{4n+3}(\alpha) \). Then it contains no element of valuation smaller than \( 6n + 6 \), so \( C \subseteq A_{2n+2} \). Hence, \( \text{par}(C) \geq n + 1 \).

On the other hand, consider the ring \( C_{2r+q}(\alpha) \), where \( q \in \{0, 1\} \). Its over-rings are of the kind \( D + t^k C_{2m+q}(\beta) \), where \( k + m \leq r \) and \( k + 2m \leq 2r \). Moreover, let \( \alpha = \tau^q + a\tau^{2q+2} \) and \( \beta = \tau^{q+1} + b\tau^{2q+2} \). Then \( b \) is defined modulo \( t^m \) and \( b \equiv a \pmod{t^m-k} \). Therefore, the over-rings with the fixed \( m, k \) form a \( p \)-parameter family, where \( p = \min(m, r-m-k) \). Hence, \( 2p \leq r \) and \( p \leq [r/2] \). If we set \( r = 2n+1 \), we get that \( \text{par}(C_{4n+2}(\alpha)) \leq n \) and \( \text{par}(C_{4n+3}(\alpha)) \leq n \) for all possible \( \alpha \). It accomplishes the proof. \( \square \)

Obvious considerations give the number of parameters for special rings.

**Corollary 4.2.**

\[
\begin{align*}
\text{par}(C_r(\alpha)) & = \left[ \frac{r}{2} \right], \\
\text{par}(C_{l,q}(\alpha)) & = \left[ \frac{l}{2} \right], \\
\text{par}(A_m) & = \left[ \frac{m}{2} \right].
\end{align*}
\]

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