Dynamical analysis of Brans-Dicke Universe with inverse power-law effective potential

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We study Brans-Dicke cosmology with an inverse power-law effective potential. By using dynamical analyses, we search for fixed points corresponding to the radiation-like matter and dark energy-dominated era of our Universe, and the stability of fixed points is also investigated. We find phase space trajectories which are attracted to the stable point of the dark energy-dominated era from unstable fixed points like matter-dominated era of the Universe. The dark energy comes from effective potentials of the Brans-Dicke field, whose variation (related to the time-variation of the gravitational coupling constant) is shown to be in good agreement with observational data.

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1. Introduction

The recent acceleration of our Universe is thought to be caused by the mysterious dark energy, which composes about 68%\textsuperscript{1,4}. Roughly 27% of the Universe consists of dark matter\textsuperscript{5,6} and the remainder ordinary matter. One of the simplest candidate for dark energy is the well-known cosmological constant. The so-called \textit{ΛCDM} model is consistent with the current observational data.\textsuperscript{7} Nonetheless, there still remain fine tuning problems\textsuperscript{8} like the cosmological constant\textsuperscript{9} and the (anthropic) cosmic coincidence problem\textsuperscript{10} to be understood. To suppress these problems, researchers have studied alternative models such as quintessence\textsuperscript{11} k-essence\textsuperscript{12} tachyon\textsuperscript{13} scalar-tensor theories including Brans-Dicke gravity\textsuperscript{14} and other theories. (See Ref.[15] and Ref.[16] for reviews of these models.)
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In the standard model of particle physics Higgs-like fields have been studied to explain the primordial inflation and the late-time acceleration of the Universe. Extended Higgs models containing the Einstein tensor coupled, kinetic energy term have been examined. Scalar-tensor theories have been also studied to explain the late-time acceleration of the Universe. Specifically, the recent acceleration of the Universe could be explicated by the scalar field responsible for the early inflation, which is a quintessence having an exponential potential or an inverse power-law potentials. They might be most viable candidates to alleviate the coincidence problem. However, such potentials are not computed from a fundamental principle but are given by hand. The dynamical analysis is an useful method to treat autonomous system, while comparing with the observational data about the dark energy and so on. The method has been applied to scalar-tensor theories like Brans-Dicke gravity in Refs. [33 - 42] to describe the early or the late-time Universe. (See Refs.[43 - 68] for other applications.)

In this paper, as in Ref. [69] we consider Brans-Dicke gravity with mutual interactions of the Brans-Dicke field and a heavy field. In Sect. 2, we derive a low-energy effective potential of the Brans-Dicke field, when the temperature of our Universe is much lower than the heavy field mass. In Sect. 3, we set up our model to analyze the Brans-Dicke Universe as a dynamical system and find fixed points with various cosmological parameters. In Sect. 4, with the inverse power-law effective potential we analyze the dynamical system for cases of some \( \omega \)-values and investigate the stability around the fixed points. Also, with invariant submanifolds we reanalyze the dynamical system and investigate the stability around the fixed points. In Sect. 5, we study the de Sitter case of a specific fixed point to describe the late-time Universe. In Sect. 6, we summarize our results.

2 Effective potential

In this section, we briefly review the derivation of an effective potential from a high-energy theory by means of the low-energy effective theory formalism. We consider the action for a high-energy theory

\[
S(\phi, h) = \int d^4x \sqrt{-g} [\phi^2 R - \omega g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi)] + S_m, \tag{1}
\]

\[
S_m(\phi, h) = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha h \partial_\beta h - V(h) + u \phi^k h^l \right] \tag{2}
\]

Here \( \omega \) is related to the original Brans-Dicke coupling constant \( \omega_{bd} \) as \( \omega = 4\omega_{bd} \). \( h \) is a heavy field, and \( \phi \) is the Brans-Dicke field playing the role of a light field in the low-energy effective field theory. \( L_{om} \) is the lagrangian for the other matter. We consider the potential for a (Higgs-like) heavy field, \( V(h) = \frac{m_h^2 h^2}{2} + \frac{\lambda h^4}{4} \), and the
second last term in Eq. (2) is an interaction between the heavy field and the light Brans-Dicke field.

When the freedoms associated with a heavy field are concealed from direct observation at a late-time of the Universe of temperature lower than the heavy field mass, within the tree-level approximation we have the following equation by applying the low-energy effective theory formalism\cite{69,71} to Eqs. (1) and (2).

\[\frac{1}{\sqrt{-g}} \frac{\delta S(\phi, h)}{\delta h} = g^{\alpha \beta} \nabla_\alpha \partial_\beta h - \nabla' (h) + u \phi^k h^{l-1} = 0. \tag{3}\]

In the low-energy limit \( \partial_\beta h \ll m_h^2 h \), we can obtain \( h(\phi) \) from Eq. (3) and an effective potential \( V_{\text{eff}}(\phi) = m_h^2 h^2(\phi) \) dependent on the Brans-Dicke field only (when \( \lambda = 0 \)). In the case of the renormalizable interaction term\cite{69} with \( k = 1 \) and \( l = 3 \), \( h(\phi) \) and \( V_{\text{eff}}(\phi) \) can be written as

\[h(\phi) \simeq \frac{m_h^2}{3u\phi}, \tag{4}\]

\[V_{\text{eff}}(\phi) \simeq \frac{m_h^6}{54u^2\phi^2}. \tag{5}\]

Consequently, from Eqs. (1)-(5) we obtain the low-energy effective theory action dependent on the Brans-Dicke field \( \phi \) only and the other matter.

\[S(\phi, h(\phi)) = \int d^4x \sqrt{-g}[\phi^2 R - \omega g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi)] + \int d^4x \sqrt{-g}[-\frac{1}{2} g^{\alpha \beta} \partial_\alpha h(\phi) \partial_\beta h(\phi) - \nabla(\phi) + u \phi h^3(\phi)] + \int d^4x \sqrt{-g}L_{\text{om}}. \tag{6}\]

3. Set up autonomous system

In the flat Friedmann-Robertson-Walker (FRW) metric, \( g_{\mu \nu} = \text{Diag.}(-1, a^2(\tau), a^2(\tau), a^2(\tau)) \) with a scale factor \( a(\tau) \), the equations derived from Eq. (6) are given by

\[3H^2 = \frac{\rho}{2\dot{\phi}^2}, \tag{7}\]

\[-(2\dot{H} + 3H^2) = \frac{p}{2\dot{\phi}^2}, \tag{8}\]

\[2\phi R + 2\omega(-\ddot{\phi} - 3H\dot{\phi}) - V(\phi),_\phi = 0, \tag{9}\]

where \( H = \frac{\dot{a}(\tau)}{a(\tau)} \), the dot, \( \dot{\phantom{0}} \), denotes a derivative with respect to the cosmic time \( \tau \), and \( V(\phi),_\phi \equiv \frac{dV(\phi)}{d\phi} \). The total energy density and pressure can be written as\cite{69}

\[\rho = \rho_{\text{bd}} + \rho_{\text{eff}} + \rho_{\text{om}}, \quad p = p_{\text{bd}} + p_{\text{eff}} + p_{\text{om}}. \tag{10}\]
where
\[ \rho_{bd} = \omega \dot{\phi}^2 - 12H \phi \dot{\phi} + V(\phi), \]
\[ p_{bd} = \omega \dot{\phi}^2 + 4(\dot{\phi}^2 + \phi \ddot{\phi} + 2H \phi \dot{\phi}) - V(\phi), \]
\[ \rho_{\text{eff}} = \frac{1}{2} \dot{h}(\phi)^2 + V_{\text{eff}}(\phi), \]
\[ p_{\text{eff}} = \frac{1}{2} \dot{h}(\phi)^2 - V_{\text{eff}}(\phi), \]
(11)
\[ \rho_{om} \] is the energy density for the other matter, and \( p_{om} \) is the pressure. Eq. (7) can be rewritten as
\[ 1 = \frac{1}{6H^2\phi^2} [\rho_{om} + \omega \dot{\phi}^2 - 12H \phi \dot{\phi} + V + \frac{1}{2} \dot{h}(\phi)^2 + V_{\text{eff}}(\phi)]. \]
(12)
With dimensionless variables
\[ x^2 \equiv \frac{\dot{\phi}^2}{6H^2\phi^2}, \quad y^2 \equiv \frac{\ddot{\phi}^2}{H^2\phi^3}, \quad z^2 \equiv \frac{V(\phi)}{6H^2\phi^2}, \quad t^2 \equiv \frac{V_{\text{eff}}(\phi)}{6H^2\phi^2}, \]
(13)
where the constant \( c \equiv \frac{m_s^4}{108u^2} \), Eq. (12) becomes
\[ 1 = \Omega_{om} + \omega x^2 - 2\sqrt{6}x + z^2 + y^2 + t^2. \]
(14)
From Eqs. (8) and (9) we define other dimensionless variables \( A \equiv \frac{\ddot{\phi}}{H\dot{\phi}} \) and \( B \equiv -\frac{\dddot{\phi}}{H^2} \), which are dependent on each other as
\[ \frac{2}{3}B - w_{om}\Omega_{om} - (\omega + 4)x^2 - \frac{2}{3}\sqrt{6}Ax \]
\[ - \frac{4\sqrt{6}}{3}x + z^2 - y^2 + t^2 - 1 = 0, \]
(15)
\[ 24x - 12xB - 2\sqrt{6}\omega Ax^2 - 6\sqrt{6}\omega x^2 - 12Dz^2x \]
\[ - 2\sqrt{6}Ay^2 + 24y^2x - 6\sqrt{6}y^2 - 12Et^2x = 0, \]
(16)
where \( D = \frac{V'(\phi)}{2V(\phi)} \), \( E = \frac{V''(\phi)}{2V_{\text{eff}}(\phi)} \).

The ratio of the energy density of the other matter relative to \( 6H^2\phi^2 \) and that of the Brans-Dicke field can be expressed as
\[ \Omega_{om} \equiv \frac{\rho_{om}}{6H^2\phi^2} = 1 - \omega x^2 + 2\sqrt{6}x - z^2 - y^2 - t^2, \]
(17)
\[ \Omega_{\phi} = \omega x^2 - 2\sqrt{6}x + z^2 + y^2 + t^2. \]
(18)
Eqs. (17) and (18) give us constraints, \( 0 \leq \Omega_{\phi} \leq 1 \) and \( 0 \leq \Omega_{om} \leq 1 \). The equation of state for the total energy and pressure and the equation of state regarding to the
then we have to analyze 6D autonomous system because \( V \) derived by the low-energy effective theory formalism. In this case, \( D \) is Brans-Dicke field are given by

\[
w_m = \frac{p}{\rho} = \frac{p_{\text{om}} + p_{\text{bd}} + p_{\text{eff}}}{\rho_{\text{om}} + p_{\text{bd}} + \rho_{\text{eff}}}
\]

\[
w_{\phi} = \frac{p_{\phi}}{\rho_{\phi}} = \frac{p_{\text{bd}} + \rho_{\text{eff}}}{\rho_{\text{bd}} + \rho_{\text{eff}}}
\]

\[
= \frac{\omega(\omega + 4)x^2 + 2\sqrt{6}\, xA + 4\sqrt{6}/3\ x - z^2 + y^2 - t^2}{\omega x^2 - 2\sqrt{6}x + z^2 + y^2 + t^2}
\]

with \( w_{\text{om}} = \frac{p_{\text{om}}}{\rho_{\text{om}}} \). Note that our Universe is accelerating if the equation of state for the total energy and pressure \( w_m < -\frac{1}{3} \). If specially \( w_m = -1 \), then the Universe must be accelerating because of the influence of the cosmological constant. On the other hand, if \( -1 < w_m < -\frac{1}{3} \), then we have an accelerating Universe due to the presence of dark energy like quintessence. (With the equation of state for the total energy and pressure \( w_m = p/\rho \), \( \ddot{a}/a (= H + H^2 = -(\rho + 3p)/(12\phi^2)) = -(1 + 3w_m)p/(12\phi^2) \) and \( \dot{H} = -(1 + w_m)p/(4\phi^2) \) from Eqs. (7) and (8).)

Using Eqs. (12)-(16), we can rewrite our autonomous system in Eqs. (7)-(9) as

\[
x' = x[A + B - \sqrt{6}x],
\]

\[
y' = y[A + B - 3\sqrt{6}x],
\]

\[
z' = z[\sqrt{6}Dx + B - \sqrt{6}x],
\]

\[
t' = t[\sqrt{6}Ex + B - \sqrt{6}x],
\]

\[
D' = 2\sqrt{6}D^2x[\frac{1}{2D} + \Gamma - 1],
\]

\[
E' = 2\sqrt{6}E^2x[\frac{1}{2E} + \Theta - 1],
\]

where \( \tau \) denotes the derivative with respect to \( N = \ln a(\tau) \). We further define dimensionless variables as \( \Gamma = \frac{V(\phi)\dot{V}(\phi)}{V(\phi)^2} \), and \( \Theta = \frac{V_{\text{eff}}(\phi)\dot{V}_{\text{eff}}(\phi)}{V_{\text{eff}}(\phi)^2} \).

In this paper, we take \( V(\phi) \propto \phi^n \) which is a power-law potential regarding to Brans-Dicke field and \( V_{\text{eff}}(\phi) = \frac{m^2_{\text{pl}}\phi}{\sqrt{8\pi G}} \) which is an inverse power-law potential derived by the low-energy effective theory formalism. In this case, \( D = 0 \) and \( E' = 0 \) since \( D = \frac{2}{3} \) and \( E = \frac{2}{3} \) where \( n \) and \( j \) are constant.\(^a\)

\(^a\)Note that if the scalar field with (inverse) power-law potentials is not the Brans-Dicke field, then we have to analyze 6D autonomous system because \( V'_{\text{eff}}(\phi)/V(\phi) \) with \( \kappa = \sqrt{8\pi G} \) (and \( V_{\text{eff}}(\phi)/\sqrt{V(\phi)} \)) that should be studied is dependent on \( \phi \) as in Ref. [28]. However in Brans-Dicke gravity the Brans-Dicke field is related with the gravitational constant \( G \propto \phi^{-1/2} \)\(^b\) and thus \( D = \frac{\dot{V}'}{2\dot{V}} \phi \) and \( E = \frac{\dot{V}'}{2\dot{V}} (\phi) \phi \) are constants in cases of (inverse) power-law potentials.\(^c\)\(^d\)\(^e\)\(^f\)\(^g\)\(^h\)
3.1. Stability analysis of fixed points

In this subsection, we determine the linear stability of a fixed point \((x = x_0, y = y_0, z = z_0, t = t_0)\) with a perturbation \(x = x_0 + \delta x, y = y_0 + \delta y, z = z_0 + \delta z, t = t_0 + \delta t\) as

\[
\begin{pmatrix}
\delta x' \\
\delta y' \\
\delta z' \\
\delta t'
\end{pmatrix} = M \begin{pmatrix}
\delta x \\
\delta y \\
\delta z \\
\delta t
\end{pmatrix},
\]

where \(M\) is given by

\[
M = \begin{pmatrix}
\frac{\partial x'}{\partial x'} & \frac{\partial x'}{\partial y'} & \frac{\partial x'}{\partial z'} & \frac{\partial x'}{\partial t'} \\
\frac{\partial y'}{\partial x'} & \frac{\partial y'}{\partial y'} & \frac{\partial y'}{\partial z'} & \frac{\partial y'}{\partial t'} \\
\frac{\partial z'}{\partial x'} & \frac{\partial z'}{\partial y'} & \frac{\partial z'}{\partial z'} & \frac{\partial z'}{\partial t'} \\
\frac{\partial t'}{\partial x'} & \frac{\partial t'}{\partial y'} & \frac{\partial t'}{\partial z'} & \frac{\partial t'}{\partial t'}
\end{pmatrix}_{(x=x_0,y=y_0,z=z_0,t=t_0)}
\]

to be calculated from Eqs. (21)-(24). The above has four eigenvalues. When all eigenvalues are negative, the fixed point is stable. When all eigenvalues are positive, the fixed point is unstable. On the other hand, if some of eigenvalues are negative and the others positive, then the fixed point is saddle. If the determinant of the matrix \(M\) is negative and real parts of the eigenvalue are negative, then the fixed point is a stable spiral.

In sections 3 and 4, we analyze the 4-dimensional dynamical system by investigating the fixed points and their stability and show some physically meaningful trajectories around fixed points. However, we find an useful mathematical method by which a dynamical system can be described more appropriately in low-dimensional phase spaces: Invariant submanifolds are parts of entire phase space, which evolve to themselves under the dynamics, and each of them is not connected to any other areas.

We can find invariant submanifolds by looking at the structure of our dynamical system of Eqs. (21) - (24) so that \(x = 0\) (without the kinetic term), \(y = 0\) (without the effective kinetic term), \(z = 0\) (without the potential term), or \(t = 0\) (without the effective potential term), also the vacuum case \(\Omega_{om} = 0\) is an invariant submanifold, respectively. This implies that a global attractor exists.
when $x = y = z = t = 0$, but we cannot determine whether our dynamical system has a global attractor since a divergent singularity appears in Eqs. (27) and (28) (when $x = 0$ and $y = 0$). Also, in section 5, assuming that $B = 0$ and $A \neq 0$ which satisfy directly Eqs. (15) and (16) without using Eqs. (27) and (28), we investigate the stability of the fixed point, $(x_0 = 0, y_0 = 0)$, corresponding to the de Sitter Universe. In subsections 3.2-3.5, we summarize various cosmological parameters of each fixed point obtained from Eqs. (21)-(24), the equation of state regarding to the Brans-Dicke field $w_\phi$, the total equation of state $w_m$, the density ratio of the Brans-Dicke field $\Omega_\phi$, and eigenvalues $\lambda^i$ with $i = 1, 2, 3, 4$.

### 3.2. Fixed points of $(x = 0, y \neq 0, z = 0, t = 0)$ type

Among fixed points of Eqs. (21)-(24), we have this type (a).

#### 3.2.1. $(x_a, y_a, z_a, t_a) = (0, \pm 1, 0, 0)$

This type of fixed points with $\Omega_\phi = 1, w_m = 1$, and $w_\phi = 1$ describes the stiff matter-dominated era of our Universe. Eigenvalues are given by

$$
\lambda^1_a = 0, \lambda^2_a = 3 - 3w_{\text{om}}, \lambda^3_a = 3, \lambda^4_a = 3.
$$

It is a normally unstable point (Non-Hyperbolic).\(^b\)

### 3.3. Fixed points of $(x \neq 0, y = 0, z = 0, t = 0)$ type

Among fixed points of Eqs. (21)-(24), we obtain types (b).

#### 3.3.1. $(x_{b1\pm}, y_{b1}, z_{b1}, t_{b1}) = \left(\frac{\sqrt{6} \pm \sqrt{6 + 2\omega}}{\omega}, 0, 0, 0\right)$

With $\Omega_{\phi b1} = 1, w_{m b1} = \frac{24 + 3\omega \pm 6\sqrt{6 + 2\omega}}{3\omega}, w_{\phi b1} = \frac{24 + 3\omega \pm 4\sqrt{6 + 2\omega}}{3\omega}$, eigenvalues are given by

$$
\lambda^1_{b1\pm} = \frac{-2(12\sqrt{6} \pm \sqrt{6 + 2\omega})}{\omega(\sqrt{6} \pm \sqrt{6 + 2\omega})}, \lambda^2_{b1\pm} = \frac{6 + D + 3\omega \pm \sqrt{6 + 2\omega} \pm \sqrt{6} \sqrt{6 + 2\omega}}{\omega}, \lambda^3_{b1\pm} = \frac{5 + 6E + 3\omega \pm \sqrt{6 + 2\omega} \pm \sqrt{6} \sqrt{6 + 2\omega}}{\omega}, \lambda^4_{b1\pm} = \frac{12 + 3\omega \pm 2\sqrt{6} \sqrt{6 + 2\omega} - 3\omega w_{\text{om}}}{\omega}.
$$

For the '+' case of fixed points 3.3.1, stability conditions that all eigenvalues are negative are $E < -1, w_{\text{om}} > \frac{4 + \omega}{3} \left(\frac{36 + 6\omega}{\omega} - D\right)$ and $D < 2 - 1 \sqrt{\frac{36\omega^2 + 6\omega^3}{\omega^2}}, 0 < \omega < \frac{1}{18}(-30 - 24E + 6E^2) + \frac{1}{18}\sqrt{900 + 1440E + 216E^2 - 288E^3 + 36E^4}$, and the '-' case of the fixed points 3.3.1 is unstable (saddle).

#### 3.3.2. $(x_{b2}, y_{b2}, z_{b2}, t_{b2}) = \left(\frac{\sqrt{6} - 3\sqrt{6} w_{\text{om}}}{12 + 3\omega - 3\omega w_{\text{om}}}, 0, 0, 0\right)$

With $\Omega_{\phi b2} = -\frac{2(-1 + 3w_{\text{om}})(-24 + \omega(-5 + 3w_{\text{om}}))}{3(1 - 4\omega(-1 + w_{\text{om}}))}, w_{m b2} = \frac{\sqrt{6} - 3\sqrt{6} w_{\text{om}}}{12 + 3\omega - 3\omega w_{\text{om}}}, w_{\phi b2} = \frac{-8 + \omega(-2 + w_{\text{om}})(-1 + 3w_{\text{om}})}{(-24 + \omega(-5 + 3w_{\text{om}}))}$, eigenvalues are given by

\(^b\)To complete analysis of stability for the fixed point where one of its eigenvalues is 0, we should consider a center manifold analysis. However, we don't analyze such a deeper dynamical analysis in present paper, which is denoted by Non-Hyperbolic.
With \( \Omega \lambda x \) among fixed points of Eqs. (21)-(24), we have a type (c).

All the fixed points of 3.3.2 are unstable (saddle) with a constraint 0 \( \leq \Omega_{\phi,c} \leq 1. \)

3.4. Fixed points of \( (x \neq 0, y = 0, z \neq 0, t = 0) \) type

Among fixed points of Eqs. (21)-(24), we have a type (c).

3.4.1. \((x,c_1, y,c_1, z,c_1, t,c_1) = (-\sqrt[3]{(1+w_{om})}, 0, 2\sqrt[3]{(1+w_{om})}, 0) \)

With \( \Omega_\phi = 1, \ w_m = 2-18D+4D^2-3\omega, \ w_\phi = 2-18D+4D^2-3\omega, \) eigenvalues are given by

\[
\lambda_1^c = -3 + \frac{2(-2+D)(1+D)}{2+2D+\omega}, \lambda_2^c = -3 + \frac{4(2+D)D}{2+2D+\omega} - 3w_{om}, \]

\[
\lambda_3^c = \frac{2(-2+D)(-D-3)}{2+2D+\omega}, \lambda_4^c = \frac{4(2+D)D}{2+2D+\omega}.
\]

For fixed points of 3.4.1, stability conditions that all eigenvalues are negative with \((-10-8D+2D^2-3\omega)(6+\omega) < 0 \) are 0 < \( D < 2, \ w_{om} < -1, -2-2 \omega < \omega < -6+14D+4D^2-6w_{om}-6w_{om} \leq -1, \omega > -2-2D, E < D \) or \( D > 2, \ w_{om} < -1, -6+14D+4D^2-6w_{om}-6w_{om} < \omega < -2-2D, E < D \) or \( D > 2, \ w_{om} < -1, -6+14D+4D^2-6w_{om}-6w_{om} < \omega < -2-2D, E < D \).

3.4.2. \((x,c_2, y,c_2, z,c_2, t,c_2) = (-\sqrt[3]{(1+w_{om})}, 0, 2\sqrt[3]{(1+w_{om})}, 0) \)

Here \( F_2 = D^2(8-24w_{om})-3(2+\omega)(4+\omega)(-1+\omega)(1-2D)(16+\omega)(-5+6w_{om}+3w_{om}^2) \).

With \( w_{om} = -1+(1+D)w_{om}, \ \Omega_{\phi,c_2} = 3(2+\omega)(1+w_{om}+2D(7+3w_{om}))/4D^2, \)

\[
w_{\phi,c_2} = \frac{3(2+\omega)w_{om}+2D(1+w_{om})+3w_{om}^2}{4D(1+w_{om})+2D(1+3w_{om})}, \text{ eigenvalues are given by}
\]

\[
\lambda_1^c = \frac{3D^2(6+\omega)(2+2D+\omega)(1+D(-1+D)+w_{om})}{4D^2(6+\omega)(2+2D+\omega)} - 3\sqrt[3]{F_2},
\lambda_2^c = \frac{3D^2(6+\omega)(2+2D+\omega)(1+D(-1+D)+w_{om})}{4D^2(6+\omega)(2+2D+\omega)} + 3\sqrt[3]{F_2},
\lambda_3^c = \frac{3D(-E)(1+w_{om})}{4D^2}, \lambda_4^c = \frac{3(2+\omega)(1+w_{om})}{4D^2},
\]

where \( F = D(6+\omega)(2+2D+\omega)(32D^3(-1+3w_{om})+D^2(34-42w_{om}(10+3w_{om})+3w_{om}(7+9w_{om})-3(1+w_{om})(7+9w_{om})(-54+\omega(-37-6w_{om}+6w_{om}+\omega(-17+w_{om}(19+6w_{om})))) \)).

For fixed point 3.4.2, conditions \( F_2 > 0 \) are \( -2-2D > 0 \) are required.

3.5. Fixed points of \( (x \neq 0, y = 0, z = 0, t \neq 0) \) type

We obtain fixed points of types (d) from Eqs. (21)-(24).

3.5.1. \((x,d_1, y,d_1, z,d_1, t,d_1) = (-\sqrt[3]{(1+w_{om})}, 0, 2\sqrt[3]{(1+w_{om})}, 0) \)

With \( \Omega_{\phi,d_1} = 1, \ w_{om} = 2-18E+4E^2-3\omega, \ w_{\phi,d_1} = 2-18E+4E^2-3\omega, \) eigenvalues are given with the substitution \( D \rightarrow E \) for the fixed points in the subsection 3.4.1.
3.5.2. \((x_{1d}, y_{1d}, z_{1d}, t_{1d}) = (-\sqrt{\frac{3}{2} \frac{1+w_{om}}{2E}}, 0, 0, \pm \sqrt{\frac{3}{2} E \sqrt{\frac{3}{2} E + \omega}})\)

Here \(F_{l2} = E^2 (8-24 w_{om}) - 3(2+\omega)(-4+\omega(-1+w_{om}))(1+w_{om}) - 2E(-16+\omega(-5+6w_{om} + 3w_{om}^2))\).

With \(w_{m_{12}} = \frac{-24 E (1+w_{om})}{-12 E - 6 E \omega + 12 E \omega^2 + 6 E \omega^3 + 6 E \omega^4 + 3 E \omega^5 + 3 E \omega^6 + \omega^7}\),
\(w_{p_{12}} = \frac{3(2+\omega)(1+w_{om})+2E(2+5w_{om}+3w_{om}^2)}{3(2+\omega)(1+w_{om})+2E(1+3w_{om})}\),
\(w_{\phi_{12}} = \frac{E(1+w_{om})+2E(-2+5w_{om}+3w_{om}^2)}{3(2+\omega)(1+w_{om})+2E(1+3w_{om})}\),

eigenvalues are given with the substitution \(E \rightarrow E - \omega\) for the fixed points in the subsection 3.4.2.

In the subsections 3.2-3.5, we have found fixed points with general values \(E, D, w_{om}\), and \(\omega\). Among them, for example, the \(\omega = -4\) case corresponds to an effective theory of string theory and the fixed point with \(w_{om} = 0, E = -1, D = 2, \Omega = 1, w_m = -1, w_\phi = -1\), and eigenvalues \([0, 0, -3, -3]\) in the subsection 3.4.1 are normally stable (Non-Hyperbolic). The \(\omega = -6\) case corresponds to conformally invariant models and the fixed point with \(w_{om} = 0, E = -1, D = 1, \Omega = 1, w_m = -1, w_\phi = -1\), and eigenvalues \([2, 2, -1, -1]\) in the subsection 3.4.1 is saddle. (The \(\omega > 160000\) case satisfies cosmological constraints and solar-system test and the fixed point with \(w_{om} = 0, E = -1, D = 1, \Omega = 1, w_m = -1, w_\phi = -1\), and all eigenvalues are negative in the subsection 3.4.1 and then this fixed point is stable.)

In the next section we analyze physically meaningful, specified more fixed points.

4. Dynamical analysis

We consider cases with such special values as \(E = -1, D = 1\) (which give us \(V_{eff}(\phi) \propto \phi^{-2}, V(\phi) \propto \phi^2\), and \(w_{om} = 0\) for other non-relativistic matter, i.e. ordinary matter is dust \(p_{om} = 0\)). This \(j = 0\) case corresponds to that given in Eq. (5). We regard both cases with the Brans-Dicke coupling constant \(\omega < 0\) and \(\omega > 0\). Specific examples with \(\omega = -3\) and \(\omega = 5\) only are studied for convenience, and possible trajectories from the fixed point corresponding to the (effectively) radiation-like matter dominated era to the (effectively) dark energy-dominated era of our Universe are to be found.

4.1. \(\omega = -3\) case

When \(\omega = -3\), by using the results in the sections 3.2-3.5 we obtain explicit properties of fixed points like cosmological parameters relevant to them.

Requiring the constraint, \(0 \leq \Omega \leq 1\), we have written down (selected) realistic fixed points \(P_{A1}, P_{A2,3}, P_{A5,6}\) in Table 1. We represent possible paths:

1. \(P_{A1} \rightarrow P_{A5,6}\)
2. \(P_{A2,3} \rightarrow P_{A5,6}\)

It is shown that the paths in a phase space pass well from the radiation-like matter era to dark energy-dominated era of the Universe, as one can see in Figs. 1 and 2.
Fig. 1. The figure exhibits the phase space trajectories on the $xz$-plane for the case where $j = 0$, $\omega = -3$, $E = -1$, and $D = 1$, among fixed points given in the subsection 4.1. The paths go from $P_{A2}$ to $P_{A5,6}$. The stable (attractor) points $P_{A5}$ and $P_{A6}$ are related to the late-time accelerating Universe, and $P_{A2}$ is an unstable point corresponding to the radiation-like matter dominated era.

Fig. 2. The figure exhibits a part of phase space trajectories on the $xyz$-space for the case where $j = 0$, $\omega = -3$, $E = -1$, and $D = 1$. It shows the trajectory starting from the fixed point $P_{A1}$ is attracted toward the stable fixed point $P_{A5}$. 

Authors’ Names
Table 1. Fixed points (for $\omega = -3$, $E = -1$, $D = 1$, $w_{\text{om}} = 0$, and $0 \leq \Omega_\phi \leq 1$), their eigenvalues, and stability.

| Point  | $x$ | $y$ | $z$ | $t$ | Eigenvalues | Stability   |
|--------|-----|-----|-----|-----|-------------|-------------|
| $P_{A1}$ | 0   | $\pm 1$ | 0   | 0   | (3, 3, 3, 0) | unstable    |
| $P_{A2}$ | $\frac{1}{3(\sqrt{3} - \sqrt{6})}$ | 0   | 0   | 0   | (3, 1.8, 1.8, 1.17) | unstable |
| $P_{A3}$ | $\frac{1}{3(\sqrt{3} + \sqrt{6})}$ | 0   | 0   | 0   | (6.8, -3.8, -3.8, 3) | saddle     |
| $P_{A5,6}$ | $\frac{1}{\sqrt{2}}$ | 0 | $\pm \sqrt{7}$ | 0 | $(-7, -7, 4, -4)$ | stable     |

4.2. $\omega = 5$ case

When $\omega = 5$, we also obtain explicit properties of fixed points like cosmological parameters related to them, by using the results in the sections 3.2-3.5.

Requiring the constraint, $0 \leq \Omega_\phi \leq 1$ again, we have realistic fixed points $P_{B1}$, $P_{B2,3}$, $P_{B5,6}$ and $P_{B9,10}$ in Table 2. We represent possible paths:

3. $P_{B1} \rightarrow P_{B9,10} \rightarrow P_{B5,6}$
4. $P_{B2} \rightarrow P_{B9,10} \rightarrow P_{B5,6}$

We show also that these trajectories pass well from the radiation-like matter era to the dark energy-dominated era (as $\Lambda CDM$ cosmological model), as one can see in Figs. 3-5.

![Fig. 3.](image)

Fig. 3. The figure exhibits the phase space trajectories on the $xt$-space for the case where $j = 0$, $\omega = 5$, $E = -1$, and $D = 1$, among fixed points given in the subsection 4.2. The paths go from $P_{B2}$ to $P_{B9,10}$. The unstable point $P_{B2}$ corresponds to the radiation-like matter dominated era.

We briefly investigate the obsevational constraints in Brans-Dicke cosmology at the stable fixed points given in Tables 1 and 2. (By considering Ref. [77], we assume $\phi(\tau) = \phi_0(\tau/\tau_0)^{\sigma}$ and $a(\tau) = a_0(\tau/\tau_0)^{\beta}$ where $\tau_0$ is current cosmic time, $\sigma$ and $\beta$ are constant, and we have obtained $\tau_0 = \sqrt{\beta(\omega + 6)/(3\Omega_\text{m})} \ H_0^{-1}$ with relation to $\sigma = (2 - 2\beta)/3$, which is dependent on the Brans-Dicke parameter.)
The variability of the gravitational constant in this Brans-Dicke theory is given by $|\dot{\mathcal{G}}_0| \approx 2.9 \times 10^{-10}/\text{yr}$ for $P_{A5,6}$ and $|\dot{\mathcal{G}}_0| \approx 3.2 \times 10^{-11}/\text{yr}$ for $P_{B5,6}$ in late-time Universe, respectively. These are consistent with the observational results.

Fig. 4. The figure exhibits a part of phase space trajectories on the $xyt$-space for the case where $j = 0$, $\omega = 5$, $E = -1$, and $D = 1$. The figure shows the trajectory starting from the fixed point $P_{B1}$ is attracted toward the saddle fixed point $P_{B9}$.

Fig. 5. The figure exhibits a part of phase space trajectories on the $xzt$-space for the case where $j = 0$, $\omega = 5$, $E = -1$, and $D = 1$. The figure shows the trajectory starting from the fixed point $P_{B9}$ is attracted toward the stable fixed point $P_{B5}$.

$$c_x \equiv \frac{\dot{\phi}}{\sqrt{6H\phi}} = -\frac{\dot{\mathcal{G}}}{2\sqrt{6HG}}$$ because $\frac{\dot{\mathcal{G}}}{\mathcal{G}} = \frac{-2\dot{\phi}/\phi^3}{1/\phi} = -\frac{2\dot{\phi}}{\phi}$. 

4.3. Dynamical system with invariant submanifolds

In this section, we thus investigate some invariant submanifolds to study dark energy, which make us to exhibit more naturally the physically meaningful attractors and trajectories in 2-dimensional phase space as in Figs. 6-9.

4.3.1. Vacuum case $\Omega_{\text{om}} = 0$

With $\Omega_{\text{om}} = 0$, Eq. (14) becomes $1 = \omega x^2 - 2\sqrt{6}x + z^2 + y^2 + t^2$. Therefore, Eqs. (21) – (24) are reduced to equations of a 3 dimensional dynamical system. For $E = -1, D = 1, w_{\text{om}} = 0$, and $\omega = 6$, fixed points with various cosmological parameters are given below.

| Point  | $x$  | $y$  | $z$  | $t$  | Eigenvalues                  | Stability |
|--------|------|------|------|------|------------------------------|-----------|
| $P_{C1}$ | 0    | ±1   | 0    | 0    | (3, 3, 0)                    | unstable  |
| $P_{C2}$ | $\frac{1}{5(\sqrt{6} - \sqrt{11})}$ | 0    | 0    | 0    | (3, 2.15, 2.15, 0.85)        | unstable  |
| $P_{C3}$ | $\frac{1}{5(\sqrt{6} + \sqrt{11})}$ | 0    | 0    | 0    | (8.6, 8.6, -5.6, 3)          | saddle    |
| $P_{C5,6}$ | $\frac{\sqrt{2}}{2}$ | 0    | ±$\sqrt{\frac{13}{9}}$ | 0    | (-3.4, -3.4, -0.4, -0.4)    | stable    |
| $P_{C9,10}$ | $\sqrt{3}$ | 0    | 0    | ±$\sqrt{\frac{13}{9}}$ | (-3, -2.4, 2.4, -0.6) | saddle    |

Fig. 6. The figure exhibits the phase space trajectory on the $xz$-plane for the case where $w_{\text{om}} = 0, \omega = 6, E = -1$, and $D = 1$. The figure shows the fixed point $P_{C2,3}$ is stable, corresponding to the cosmological-constant dominated era.

$P_{C1} = (0, 0, 0) : y = 1, w_{\text{m}} = w_{\phi} = 1$. This seems to describe a stiff-matter dominated era of Universe. The eigenvalues are (3, 3, 0), which mean it is normally unstable (Non-Hyperbolic).
4.3.2. Invariant submanifold \( y = 0 \) case

When \( y = 0 \), we summarize various cosmological parameters of each fixed point by using the results in the sections 3.2–3.5. We take \( E = -1 \), \( D = 1 \), \( w_{\text{om}} = 0 \) for such a reason below section. For the case \( \omega = 2 \) fixed points with various cosmological parameters are given below.

\[
P_{D1,2} = (-1, \pm \sqrt{3}, 0) : \Omega_\phi = 1, \quad w_m = -1, \quad w_\phi = -1, \quad \text{eigenvalues} \quad (-\frac{2}{3}, -\frac{4}{3}, -\frac{5}{3}), \quad \text{which are stable}.
\]

\[
P_{D3,4} = (1, \pm \sqrt{3}, 0) : \Omega_\phi = 1, \quad w_m = 3, \quad w_\phi = 3, \quad \text{eigenvalues} \quad (6, -3, 3), \quad \text{which are saddle}.
\]

\[
P_{D5,6} = (-\frac{2}{3}, \pm \sqrt{3}, 0) : \Omega_\phi = 1, \quad w_m = \frac{1}{2} (30 \pm 16 \sqrt{3}), \quad w_\phi = \frac{1}{2} (30 \pm 16 \sqrt{3}), \quad \text{eigenvalues} \quad (\frac{1}{2} (18 \pm 8 \sqrt{3}), 3, \frac{1}{2} (18 \pm 8 \sqrt{3})), \quad \text{which are unstable}.
\]

\[
P_{D7,8} = (-\frac{2}{3}, \pm \sqrt{3}, 0) : \Omega_\phi = 1, \quad w_m = -1, \quad w_\phi = -\frac{2}{3}, \quad \text{eigenvalues} \quad (3, -\frac{11}{3}, \pm \frac{11}{3}), \quad \text{which are saddle}.
\]

\[
P_{D9,10} = (\sqrt{3}, 0, \pm \sqrt{3}) : \Omega_\phi = -\frac{1}{2}, \quad w_m = 1, \quad w_\phi = -2, \quad \text{eigenvalues} \quad (3, \frac{1}{2} (-6 + 3 \sqrt{3}), \frac{1}{2} (-6 - 3 \sqrt{3})), \quad \text{which are saddle}.
\]

\[
P_{D11} = (\frac{1}{5\sqrt{6}}, 0, 0) : \Omega_\phi = -\frac{17}{2}, \quad w_m = \frac{2}{3}, \quad w_\phi = -\frac{6}{17}, \quad \text{eigenvalues} \quad (\frac{11}{6}, \frac{7}{6}, -\frac{11}{6}), \quad \text{which are saddle}.
\]

Requiring the constraint, \( 0 \leq \Omega_\phi \leq 1 \), we write down (selected) realistic fixed points \( P_{D1,2}, P_{D3,4}, P_{D5,6} \) and represent possible paths.

5. \( P_{D3,4,5,6} \rightarrow P_{D1,2} \)

It is shown that the paths in the phase space pass well from the radiation-like matter dominated era to dark energy dominated era of the Universe, as one can see in Fig. 7.

For the case \( \omega = -7 \) fixed points with various cosmological parameters are given below.

\[
P_{E1,2} = (-\sqrt{3}, \pm \frac{2}{3}, 0) : \Omega_\phi = 1, \quad w_m = -1, \quad w_\phi = -1, \quad \text{eigenvalues} \quad (\frac{4}{3}, -\frac{5}{3}, -\frac{5}{3}), \quad \text{which are saddle}.
\]

\[
P_{E3,4} = (-\sqrt{3}, 0, \pm \frac{1}{\sqrt{3}}) : \Omega_\phi = 1, \quad w_m = -\frac{15}{7}, \quad w_\phi = -\frac{15}{7}, \quad \text{eigenvalues} \quad (-\frac{12}{7}, -3, -\frac{33}{7}), \quad \text{which are stable}.
\]
Instructions for Typing Manuscripts (Paper’s Title)

\[ P_{E5} = \left(-\frac{\sqrt{2}}{3}, 0, 0\right) : \Omega_\phi = \frac{22}{27}, w_m = -\frac{4}{9}, w_\phi = -\frac{6}{11}, \] eigenvalues \((\frac{5}{6}, \frac{13}{6}, -\frac{5}{6})\), which are saddle.

Requiring the constraint, \(0 \leq \Omega_\phi \leq 1\), we write down (selected) realistic fixed points \(P_{E1,2}, P_{E3,4}, P_{E5}\) and represent possible paths.

6. \(P_{E1,2,5} \rightarrow P_{E3,4}\)
It is shown that the paths in the phase space pass well from the radiation-like matter era to dark energy dominated era of the Universe, as one can see in Fig. 8.

Fig. 7. The figure exhibits the phase space trajectory on the \(xz\)-plane for the case where \(w_m = 0, \omega = 2, E = -1, \) and \(D = 1\). The figure shows the fixed point \(P_{D1,2}\) is stable, corresponding to the cosmological-constant dominated era.

Fig. 8. The figure exhibits the phase space trajectory on the \(xt\)-plane for the case where \(w_m = 0, \omega = -7, E = -1, \) and \(D = 1\). The figure shows the fixed point \(P_{E3,4}\) is stable, corresponding to a phantom dark energy.
4.3.3. Invariant submanifold \( x = 0 \) and \( z = 0 \) case

When \( x = 0 \) and \( z = 0 \), we summarize various cosmological parameters of each fixed point by using the results in the sections 3.2 - 3.5 which are reduced to 2 dimensional system. We summarize various cosmological parameters of each fixed point below

\[ P_{F1} = (0,0) : \Omega_\phi = 0. \] This is the ordinary matter dominated case with \( w_m = w_{om} \), \( w_\phi = \text{indeterminate} \). Eigenvalues are \( (\frac{3(w_{om}-1)}{2},\frac{3(w_{om}+1)}{2}) \) and the condition for stability is \( w_{om} < -1 \).

\[ P_{F2,3} = (\pm 1,0) : \Omega_\phi = 1, \ w_m = 1, \ w_\phi = 1, \text{ eigenvalues } (3,3-3w_{om}), \text{ which are unstable(saddle)}. \]

\[ P_{F4,5} = (0,\pm 1) : \Omega_\phi = 1, \ w_m = -1, \ w_\phi = -1, \text{ eigenvalues } (-3,-3(w_{om}+1)). \] The condition for stability that all eigenvalues are negative are \( w_{om} > -1 \).

Requiring the constraint, \( 0 \leq \Omega_\phi \leq 1, \) we write (selected) realistic fixed points \( P_{F1}, P_{F2,3}, P_{F4,5} \) and represent possible paths.

7. \( P_{F1,2,3} \rightarrow P_{F4,5} \)

It is shown that the paths in the phase space pass well from the radiation-like matter era to dark energy dominated era of the Universe, as one can see in Fig. 9.

![Figure 9](image)

Fig. 9. The figure exhibits the phase space trajectory on the \( yt \)-plane for the case where \( \omega \) is arbitrary, \( w_{om} = 0, \ E = -1, \) and \( D = 1 \). The figure shows the the fixed point \( P_{F4,5} \) is stable, mimicking the cosmological constant.

5. de Sitter Universe

In this section, we study the stability of de Sitter Universe, in which the cosmological scale factor has an exponential form. When \( A(\neq 0) \) is a constant and \( B = 0 \), the fixed points with \( x_0 = 0 \) and \( y_0 = 0 \) satisfy Eqs. (21)-(26), which are now reduced
to

\[ x' = Ax - \sqrt{6}x^2, \]
\[ y' = Ay - 3\sqrt{6}xy, \]
\[ z' = \sqrt{6}Dxz - \sqrt{6}xz, \]
\[ t' = \sqrt{6}Ext - \sqrt{6}xt, \]
\[ D' = 2\sqrt{6}D^2x[\frac{1}{2D} + \Gamma - 1], \]
\[ E' = 2\sqrt{6}E^2x[\frac{1}{2E} + \Theta - 1]. \]

The equation of state regarding to the Brans-Dicke field \( w_\phi = -1 \), the equation of state for the total energy density and pressure \( w_m = -z^2-t^2 \) with no other matter \( w_{om} = 0 \), and the density ratio regarding to the Brans-Dicke field \( \Omega_\phi = z^2 + t^2 \).

Note that \( B \) can be written as

\[ B \equiv -\frac{\dot{H}}{H} = \frac{3}{2}(1 + w_m) \]

from Eqs. (7) and (8). The fact that \( B = 0 \) in de Sitter spacetime is consistent with \( w_m = -1 \) and \( z^2 + t^2 = 1 \). From Eqs. (29)-(32) we can obtain eigenvalues of this fixed point, \( (A, A, 0, 0) \). When we assume that (with \( B = 0 \), \( x = 0 \), and \( y = 0 \))

\[ a(\tau) = a_0 e^{H_0\tau}, \phi(\tau) = \phi_0 f^\tau \]

where \( H_0 \) and \( f \) are constants, \( f = 0 \) because \( x \equiv \dot{x}/\sqrt{6}\dot{x} (\propto f/\tau) = 0 \) and \( y \equiv \sqrt{6}\dot{\phi}/\dot{\phi}^3 (\propto f^{f-2f-1}) = 0 \). Since \( A \equiv \dot{x}/\dot{\phi} = (f - 1)/H\tau \), \( A \) becomes \(-1 \) in the late-time when \( H_0 \simeq \tau_0^{-1} \) is used with the age of the Universe \( \tau_0 \). Thus the eigenvalues are normally stable (Non-Hyperbolic). Therefore, de Sitter Universe has solutions of the late-time attractor for the dark energy, which is dependent on the potentials.

6. Conclusions

We have studied cosmology in a Brans-Dicke gravity theory with the inverse power-law potential derived from the low-energy effective theory formalism\(^{[29,41]}\)\(^{[29,41]}\) by using the dynamical system method. Analyzing the evolution of our Universe as a dynamical system, we have got fixed points with various values of the cosmological parameters, \( E, D, \omega, \) and \( w_{om} \), in the sections 3 and 4. We have investigated the stability around the fixed points when \( \omega > 0 \) and \( \omega < 0 \) also. In the special \( \omega = -3 \) and \( \omega = 5 \) cases, we have described in phase spaces the evolution of the whole Universe from (unstable fixed point) the radiation-like matter to the (stable) dark energy dominated era. In addition, we have shown in Footnote c of the section 4 that a theoretical constraint for the variability \( x \) of the gravitational coupling constant in our Brans-Dicke theory is in good agreement with the experimental results\(^{[29]}\).

In the section 4.3, we have studied dynamical system with an invariant submanifold such as vacuum case \( \Omega_{om} = 0 \). Moreover, we have analyzed specific dynamical
systems such as $x = 0$, $y = 0$, and $z = 0$ case. We have got stable fixed points $P_{C2,3}, P_{D1,2}$ composed of only the kinetic and potential terms for Brans-Dicke field. The stable point $P_{E3,4}$ seems to correspond to a phantom dark energy composed of the kinetic term for the Brans-Dicke field and the effective potential for the scalar field. The $x = 0$ and $z = 0$ case with stable point $P_{F4,5}$ can be thought as quintessence-like model, composed of effective kinetic and potential terms for the scalar field.

In the section 5, for the specific, cosmic solution (with an arbitrary $\omega$-value) which corresponds to de Sitter Universe we have demonstrated that it is the stable fixed point corresponding to the late-time Universe.

In summary, we have shown that our cosmological model in a Brans-Dicke theory with inverse power-law potentials derived from the low-energy effective theory formalism can describe well the late-time Universe dominated by dark energy as a stable fixed point, which is evolved from the radiation-like matter dominated era (unstable fixed point). It would be interesting to perform sophisticated analyses with more general cases including $j \neq 0$ inverse power-law and exponential effective potentials as well as a more detailed comparison to recent cosmological observations.

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References

1. A. G. Riess et al., Astron. J. 116, 1009 (1998)
2. S. Perlmutter et al., Astrophys. J. 517, 565 (1999)
3. G. F. Smoot et al., Astrophys. J. 396, L1 (1992)
4. E. Komatsu et al., Astrophys. J. Suppl. Ser 192, 18 (2011)
5. P. D. Mannheim, Astrophys. J. 479, 659 (1997)
6. T. H. Lee, B. J. Lee, Phys. Rev. D 69, 127502 (2004)
7. P. A. R. Ade et al., Astron. Astrophys. 594, A13 (2016)
8. J. Lee, J. M. Overduin, T. H. Lee, P. Oh, Phys. Rev. D 90, 123003 (2014)
9. V. Sahni, Class. Quantum Gravity 19, 3435 (2002)
10. P. J. E. Peebles, B. Ratra, Rev. Mod. Phys. 75, 559 (2003)
11. R. R. Caldwell, R. Dave, P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998)
12. C. Armendariz-Picon, V. Mukhanov, P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2000)
13. A. Sen, JHEP 04, 048 (2002)
14. C. Brans, R. H. Dicke, Phys. Rev 124, 925 (1961)
15. E. J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006); arXiv:hep-th/0603057
16. S. Bahamonde, C. G. Böhmer, S. Carloni, E. J. Copeland, W. Fang, N. Tamanini, arXiv:1712.03107
17. F. Bezrukov, M. Shaposhnikov, Phys. Lett. B 659, 703 (2008)
18. J. Sim, T. H. Lee, J. Korean Phys. Soc. 68, 725 (2016)
19. J. Matsumoto, S. V. Sushkov, JCAP 11, 047 (2015)
20. C. Germani, A. Kehagias, Phys. Rev. Lett. 105, 011302 (2010)
21. S. Lee, J. Sim, T. H. Lee, J. Korean Phys. Soc. 64, 611 (2014)
22. M. A. Skugoreva, S. V. Sushkov, A. V. Toporensky, Phys. Rev. D 88, 083539 (2013)
23. Y. Huang, Q. Gao, Y. Gong, Eur. Phys. J. C 75, 143 (2015)
24. L. Amendola., Phys. Rev. D 62, 043511 (2000)
25. A. A. Sen, S. Sethi, Phys. Lett. B 532, 159 (2002)
26. Z. K. Guo, Y. S. Piao, Y. Z. Zhang, Phys. Lett. B 568, 1 (2003)
27. A. de la Macorra, G. Piccinelli, Phys. Rev. D 61, 123503 (2000)
28. S. C. C. Ng, N. J. Nunes, F. Rosati, Phys. Rev. D 64, 083510 (2001)
29. W. Zimdahl, D. Pavón, L. P. Chimento, Phys. Lett. B 521, 133 (2001)
30. E. J. Copeland, A. R. Liddle, D. Wands, Phys. Rev. D 57, 4686 (1998)
31. Y. Gong, Phys. Lett. B 731, 342 (2014)
32. T. Barreiro, E. J. Copeland, N. J. Nunes, Phys. Rev. D 61, 127301 (2000)
33. D. J. Holden, D. Wands, Phys. Rev. D 61, 043506 (2000)
34. O. Hrycyna, M. Szydowski, M. Kamionka, Phys. Rev. D 90, 124040 (2014)
35. O. Hrycyna, M. Szydowski, Phys. Rev. D 88, 064018 (2013)
36. G. Papagiannopoulos, J. D. Barrow, S. Basilakos, A. Giacomini, A. Paliathanasis, Phys. Rev. D 95, 024021 (2017)
37. X. M. Liu, Z. X. Zhai, K. Xiao, W. B. Liu, Eur. Phys. J. C 72, 2057 (2012)
38. A. Cid, G. Leon, Y. Levyva, JCAP 02, 027 (2016)
39. N. Roy, N. Banerjee, Phys. Rev. D 95, 064048 (2017)
40. L. M. Reyes, S. E. P. Bergliaffa, Eur. Phys. J. C 78, 17 (2018)
41. O. Hrycyna, M. Szydowski, JCAP 12, 016 (2013)
42. A. Paliathanasis, M. Tsamparlis, S. Basilakos, J. D. Barrow, Phys. Rev. D 91, 123535 (2015)
43. O. Hrycyna, M. Szydowski, JCAP 11, 013 (2015)
44. L. N. Granda, D. F. Jimenez, Eur. Phys. J. C 77, 679 (2017)
45. F. F. Bernardi, Ricardo C. G. Landim, Eur. Phys. J. C 77, 290 (2017)
46. M. Shahalam, S. D. Pathak, S. Li, R. Myrzakulov, A. Wang, Eur. Phys. J. C 77, 686 (2017)
47. V. Gupta, R. Kabir, A. Mukherjee, D. Lohiya, Int. J. Mod. Phys. D 24, 1550068 (2015); arXiv:1501.01604
48. N. Frusciante, M. Raveri, A. Silvestri, JCAP 02, 026 (2014)
49. G. Leon, J. Saavedra, E. N. Saridakis, Class. Quantum Gravity 30, 135001 (2013)
50. D. S. Odintsov, K. V. Oikonomou, Phys. Rev. D 98, 024013 (2018)
51. X. Chen, E. N. Saridakis, G. Leon, JCAP 07, 005 (2012)
52. D. S. Odintsov, K. V. Oikonomou, Phys. Rev. D 97, 124042 (2018)
53. Ricardo C. G. Landim, Int. J. Mod. Phys. D 24,1550085 (2015)
54. J. Dutta, W. Khylelep, E. N. Saridakis, N. Tamanini, S. Vagnozzi, JCAP 02, 041 (2018)
55. H. Zomumawia, W. Khylelep, N. Roy, J. Dutta, N. Tamanini, Phys. Rev. D 96, 083527 (2017)
56. G. Kofinas, G. Leon, E. N. Saridakis, Class. Quantum Gravity 31, 175011 (2014)
57. M. Skugoreva, E. N. Saridakis, A. Toporensky, Phys. Rev. D 91, 044023 (2015)
58. Carlos R. Fadragas, G. Leon, E. N. Saridakis, Class. Quantum Gravity 31, 075018 (2014)
59. V. K. Oikonomou, Int. J. Mod. Phys. D. 27, 1850059 (2018)
60. S. D. Odintsov, V. K. Oikonomou, Phys. Rev. D 96, 104049 (2017)
61. Ricardo C. G. Landim, Eur. Phys. J. C 76, 31 (2016)
62. J. Dutta, W. Khylelep, N. Tamanini, JCAP 01, 038 (2018)
63. G. Leon, E. N. Saridakis, JCAP 03, 025 (2013)
64. G. Leon, E. N. Saridakis, JCAP 11, 006 (2009)
65. G. Leon, E. N. Saridakis, Class. Quantum Gravity 28, 065008 (2011)
66. Ricardo C. G. Landim, Eur. Phys. J. C, 76, 480 (2016)
67. G. Leon, E. N. Saridakis, JCAP 04, 031 (2015)
68. X. Chen, Y. Gong, E. N. Saridakis, JCAP 04, 001 (2009)
69. B. Damdinsuren, J. Sim, T. H. Lee, Class. Quantum Gravity 34, 175012 (2017)
70. C. Lee, T. H. Lee, H. Min, Phys. Rev. D 39, 1681 (1989)
71. C. Lee, T. H. Lee, H. Min, Phys. Rev. D 39, 1701 (1989)
72. A. Avilez, C. Skordis, Phys. Rev. Lett 113, 011101 (2014)
73. V. Faraoni, *Cosmology in Scalar-Tensor Gravity*, Fundamental Theories of Physics (Kluwer Academic Publishers, Dordrecht, Boston, 2004), Vol. 139
74. S. Deser, Ann. Phys. NY 59, 248 (1970)
75. J. P. Uzan, Living Rev. Rel 14, 2 (2011)
76. Bertotti, B., Iess, L., Tortora, P., Nature 425, 374 (2003)
77. O. Bertolami, P.J. Martins, Phys. Rev. D 61, 064007 (2000)