EXTENSION OF LIPSCHITZ MAPS
DEFINABLE IN HENSEL MINIMAL STRUCTURES

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Abstract. In this paper, we establish a theorem on extension of Lipschitz maps definable in Hensel minimal, non-trivially valued fields $K$ of equicharacteristic zero. It may be regarded as a definable, non-Archimedean, non-locally compact version of Kirszbraun's theorem.

1. Introduction

We are concerned with a 1-h-minimal, non-trivially valued field $K$ of equicharacteristic zero. This means that $K$ is a model of a 1-h-minimal theory $T$ in an expansion $\mathcal{L}$ of the pure valued field language $(K, 0, 1, +, -, \cdot, \mathcal{O}_K)$. For the axiomatic concept of Hensel minimal theories, recently introduced by Cluckers–Halupczok–Rideau, we refer the reader to their papers [3, 4].

We establish a theorem on extension of Lipschitz maps definable in Hensel minimal, non-trivially valued fields $K$ of equicharacteristic zero. It may be regarded as a definable, non-Archimedean, non-locally compact version of Kirszbraun’s theorem [10].

To our best knowledge, the only definable, non-archimedean version of Kirszbraun’s theorem was achieved by Cluckers–Martin [6] in the $p$-adic, thus locally compact case; and more precisely, for Lipschitz extension of functions which are semi-algebraic, subanalytic or definable in an analytic structure on a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers (cf. [7, 8, 9]). Note that the easier case of Lipschitz extension of definable $p$-adic functions on the affine line $\mathbb{Q}_p$ was treated in [11].

Cluckers–Martin prove the extension theorem along with the existence of a definable Lipschitz retraction (with Lipschitz constant 1)
for any closed definable subset $A$ of $\mathbb{Q}_p^n$, proceeding with simultaneous
induction on the dimension $n$ of the ambient space. Their construction
of definable retractions makes use of some definable Skolem functions.

We should also mention that local Lipschitz continuity does not im-
ply piecewise Lipschitz continuity in the absence of the condition that
algebraic closure and definable closure coincide (cf. [1, 2, 3]). This
makes the problem of definable Lipschitz extension subtler yet.

Denote by
\[ \mathcal{O}_K, \quad \mathcal{M}_K, \quad K^v = \overline{K} := \mathcal{O}_K/\mathcal{M}_K, \]
the valuation ring, its maximal ideal and residue field, respectively.
Let $K^\times : = K \setminus \{0\}$ and $\mathcal{O}_K^\times$ be the set of units in the ring $\mathcal{O}_K$. Use
multiplicative notation for the value group and valuation map:
\[ vK = \Gamma^\times_K := K^\times/\mathcal{O}_K^\times, \quad | \cdot | : K^\times \to vK, \]
and put $\Gamma_K := \Gamma^\times_K \cup \{0\}$. Finally, define the ultra-
metric norm on the affine space $K^n$ as the maximum norm
\[ |(x_1, \ldots, x_n)| := \max \{|x_1|, \ldots, |x_n|\}. \]

The auxiliary sort:
\[ RV(K) := G(K) \cup \{0\}, \quad G(K) := K^\times/(1 + \mathcal{M}_K), \]
plays an essential role in geometry and model theory of Henselian fields.
In particular, it provides parameters for definable families of sets and
maps. This is of great importance as, for instance, cell decomposi-
tion requires generally the concept of a reparametrized cell, which is a
definable family of ordinary cells parametrized by $RV$-parameters.

We adopt the following convention: the words 0-definable and $X$-
definable shall mean $L$-definable and $L_X$-definable; "definable" will
refer to definable in $L$ with arbitrary parameters. In our geometric
approach, most essential is which (not how) sets are definable.

A Lipschitz continuous map $f : A \to K^m$, $A \subset K^n$, with Lipschitz
constant $\epsilon \in |K|$, $\epsilon \neq 0$, will be called an $\epsilon$-Lipschitz map.

Actually, the following theorem on extension of definable 1-Lipschitz
maps, being the main aim of this paper, embraces two cases according
as the subset of elements $>1$ in the value group $vK$ has the minimal
element $\epsilon$ (multiplicative notation) or otherwise. They provide distinct
estimates of the Lipschitz constant.

**Theorem 1.1.** Let $f : A \to K^m$ be a 0-definable 1-Lipschitz map on a
(possibly non-closed) subset $A \subset K^n$ of dimension $k$. 
I. Suppose the value group $vK$ has no minimal element among the elements $> 1$. Then, for any $\epsilon \in |K|$, $\epsilon > 1$, $f$ extends to a $0$-definable $\epsilon$-Lipschitz map $F : K^n \to K^m$.

II. Suppose the value group $vK$ has the minimal element $\epsilon$ among the elements $> 1$. Then $f$ extends to a $0$-definable $\epsilon$-Lipschitz map $F : K^n \to K^m$.

Remark 1.2. Since the non-Archimedean affine spaces $K^m$ are with the usual maximum norm, it suffices to prove the above theorem for the case $m = 1$.

We increase the Lipschitz constant in order to replace $f$ by a function $g$ which satisfies the following condition

\[(1.1) \quad rv(g(x_1, \ldots, x_{i-1}, x_i + y, x_{i+1}, \ldots, x_n) - g(x)) = rv(y)\]

for some variables $x_i, i \in I \subset \{1, \ldots, n\}$. Hence $g$ with respect to those variables separately is a risometry onto its image (for the concept of a risometry, we refer the reader to \[9\]). To this end, take an element $\varepsilon \in K$ such that $|\varepsilon| = \epsilon$ and put

$$g : A \to K, \quad g(x) := \sum_{i \in I} x_i + 1/\varepsilon \cdot f(x).$$

Clearly, once we find a $0$-definable $1$-Lipschitz extension $G : K^n \to K$ of $g$, the function

$$F(x) := \varepsilon \left( G(x) - \sum_{i \in I} x_i \right)$$

is an extension we are looking for. So from now on it will be assumed that a given function $f(x)$ satisfies a condition of type \[11\].

For the sake of the proof, we need the following stronger version, which is uniform with respect to parameters from the sort $RV$. We state it only for the case I.

**Theorem 1.3.** Consider a $0$-definable family

$$f_\lambda : A_\lambda \to K, \quad \lambda \in \Lambda \subset (RV)^*,$$

of $1$-Lipschitz functions on subsets of $K^n$. Then, for any $\epsilon \in |K|$, $\epsilon > 1$, there exists a $0$-definable family

$$F_\lambda : K^n \to K, \quad \lambda \in \Lambda$$

of $\epsilon$-Lipschitz functions such that $f_\lambda$ is the restriction of $F_\lambda$ to $A_\lambda$ for every $\lambda \in \Lambda$. 


We shall prove the above extension theorem by double induction on the dimension $n$ of the ambient space $K^n$ and on the dimension $k$ of the subset $A \subset K^n$. To be more precise, each induction step of type $(n, k)$, with $n > 1$ and $1 < k \leq n$, requires the induction hypothesis of type $(k-1, k-1)$ (uniform version) and type $(n, k-2)$; and each induction step of type $(n, 1)$ requires the induction hypothesis of type $(1, 1)$ (uniform version) and type $(n, 0)$.

Further observe that while the uniform version of the induction hypothesis is applied to globally extend a 0-definable family of 1-Lipschitz centers to the appropriate affine spaces, the usual one is applied to do the restriction of a given 1-Lipschitz function $f$ from a 0-definable subset of codimension 2 in the affine space under study.

The following two remarks refer to the increase of the Lipschitz constant while extending definable Lipschitz maps. They should be taken into account throughout the proof of Theorem 1.3, and actually, along with the above description of the induction procedure, prove the estimates of the Lipschitz constant in its conclusions.

**Remark 1.4.** In each induction step of the first kind, we use $\epsilon$-Lipschitz extensions of centers, and we can return to 1-Lipschitz centers after substitution of the dilatation $x/\epsilon$ for $x$. The function $f(x/\epsilon)$, obtained after this substitution, should be replaced by

$$f_1(x) := \epsilon f(x/\epsilon)$$

which is a risometry with respect to some variables separately, as the initial function $f(x)$ was. If $F_1(x)$ is a global 1-Lipschitz extension of $f_1(x)$, then

$$F(x) := 1/\epsilon \cdot F_1(\epsilon x)$$

is a global 1-Lipschitz extension of $f(x)$. Therefore induction steps of the first kind do not affect the Lipschitz constant in the proof of the extension theorem.

**Remark 1.5.** Each induction step of the second kind increases the Lipschitz constant by any factor $\epsilon > 1$ if the value group $vK$ has no minimal element among the elements $> 1$, and by factor $\epsilon^2$ if $\epsilon$ is the minimal element from among the elements $> 1$.

Finally, let us mention that geometry of Hensel minimal structures, with an additional condition imposed on the auxiliary sort $RV$, is studied in our recent paper [17]. Among the main results achieved there are the theorem on existence of the limit, curve selection, the closedness
theorem, several non-Archimedean versions of the Lojasiewicz inequalities, the theorems on extending continuous definable functions and on existence of definable retraction as well as an embedding theorem for regular definable spaces and the definable ultranormality of definable Hausdorff LC-spaces. Some of those results for Henselian fields, without and with an analytic structure, were provided in our previous papers \[13, 15, 16, 14\]. Let us also mention that our investigation into non-Archimedean geometry was inspired by our joint paper \[12\] devoted to some problems of real and \(p\)-adic algebraic geometry.

2. Auxiliary results

We begin by introducing some necessary terminology. For \(m \leq n\), denote by \(\pi_{\leq m}\) or \(\pi_{<m+1}\) the projection \(K^n \to K^m\) onto the first \(m\) coordinates; put \(x_{\leq m} = \pi_{\leq m}(x)\). Let \(C \subset K^n\) be a non-empty 0-definable set, \(j_i \in \{0, 1\}\) and \(c_i : \pi_{<i}(C) \to K\) be 0-definable functions \(i = 1, \ldots, n\). Then \(C\) is called a 0-definable cell with center tuple \(c = (c_i)_{i=1}^n\) and of cell-type \(j = (j_i)_{i=1}^n\) if it is of the form:

\[
C = \{x \in K^n : (rv(x_i - c_i(x_{<i})))_{i=1}^n \in R\},
\]

for a (necessarily 0-definable) set

\[
R \subset \prod_{i=1}^n j_i \cdot G(K),
\]

where \(0 \cdot G(K) = 0 \subset RV(K)\) and \(1 \cdot G(K) = G(K) \subset RV(K)\). One can similarly define \(A\)-definable cells.

In the absence of the condition that algebraic closure and definable closure coincide in \(T = Th(K)\), i.e. the algebraic closure \(acl(A)\) equals the definable closure \(dcl(A)\) for any Henselian field \(K' \equiv K\) and every \(A \subset K'\), the following concept of reparameterized cells must come into play.

Consider a 0-definable function \(\sigma : C \to RV(K)^s\). Then \((C, \sigma)\) is called a 0-definable reparameterized (by \(\sigma\)) cell if each set \(\sigma^{-1}(\xi), \xi \in \sigma(C),\) is a \(\xi\)-definable cell with some center tuple \(c_\xi\) depending definably on \(\xi\) and of cell-type independent of \(\xi\). We have the following theorem on Lipschitz cell decomposition compatible with \(RV\)-parameters (cf. \[3, Theorem 5.7.3\]).
Theorem 2.1. For every 0-definable sets
\[ X \subset K^n \quad \text{and} \quad P \subset X \times RV(K)^t, \]
there exists a finite decomposition of \( X \) into 0-definable reparametrized cells \((C_k, \sigma_k)\) such that the fibers of \( P \) over each twisted box of each \( C_k \) are constant or, equivalently, the fiber of \( P \) over each \( \xi \in RV(K)^t \) is a union of some twisted boxes from the cells \( C_k \).

Furthermore, one can require that each \( C_k \) is, after some coordinate permutation, a reparametrized cell of type \((1, \ldots, 1, 0, \ldots, 0)\) with 1-Lipschitz centers
\[ c_\xi = (c_{\xi,1}, \ldots, c_{\xi,n}), \quad \xi \in \sigma(C). \]
Such cells \( C_k \) shall be called 0-definable reparametrized Lipschitz cells.

Under the assumptions of the above theorem, we may regard \( P \) as a 0-definable family of subsets of \( X \) parametrized by \( RV(K)^t \), and say that the cell decomposition into cells \( C_k \) is compatible with that family. This also ensures the existence of cell decompositions compatible with imaginary parameter from the auxiliary sort \( RV \). We shall use it in Section 4 and 5 for construction of a skeleton of a 0-definable Lipschitz cell.

Now we give some further results which are indispensable in our proof of the extension theorem.

Proposition 2.2. Let \( A \) and \( B \) be arbitrary 0-definable subsets of \( K^n \) and \( f : A \cup B \to K \) a 0-definable 1-Lipschitz function which vanishes on \( B \). If the restriction \( f|A \) of \( f \) to \( A \) extends to a 0-definable 1-Lipschitz function \( F : K^n \to K \), so does \( f \) to a 0-definable 1-Lipschitz function \( G : K^n \to K \).

Proof. We begin by formulating two symmetric conditions for the points of \( x \in K^n \):
\begin{align*}
(1) & \quad \forall b \in B \ \exists a \in A \quad |x - a| < |x - b|; \\
(2) & \quad \forall a \in A \ \exists b \in B \quad |x - b| < |x - a|. 
\end{align*}

Note that the condition
\[ (3) := \neg[(1) \lor (2)] \]
is equivalent to
\[ \exists b_0 \in B, a_0 \in A \quad \forall a \in A, b \in B \quad |x - a| \geq |x - b_0| \land |x - b| \geq |x - a_0|, \]
and thus condition (3) means that
\[ \exists b_0 \in B, a_0 \in A \quad \forall a \in A, b \in B \quad |x - a|, |x - b| \geq |x - a_0| = |x - b_0|. \]
It is easy to check that condition (1) holds on $A \setminus B$, condition (2) on $B \setminus A$, and condition (3) on $A \cap B$. Hence the function

$$G(x) = \begin{cases} F(x) & \text{if } x \text{ satisfies } [(1) \land \neg(2)], \\ 0 & \text{if } x \text{ satisfies } [(2) \lor (3)]. \end{cases}$$

is an extension of $f$.

We shall show that $G$ is a 1-Lipschitz function. It suffices to analyze the value $|G(x) - G(y)|$ in the only two non-trivial cases where:

I. $x$ satisfies $[(1) \land \neg(2)]$ and $y$ satisfies (2),

or

II. $x$ satisfies $[(1) \land \neg(2)]$ and $y$ satisfies (3).

**Case I.** We first show that

$$\exists b \in B \mid y - b \leq |x - y|.$$ 

Otherwise

$$\forall b \in B \mid y - b > |x - y|.$$ 

By condition (2), for any $a \in A$ there is an element $b \in B$ such that $|y - b| < |y - a|$. Then

$$|y - b| = |x - b|, \quad |y - a| = |x - a| \quad \text{and} \quad |x - b| < |x - a|.$$ 

Therefore $x$ satisfies condition (2), which is a contradiction.

By condition (1), we get $|x - a| < |x - b|$ for some $a \in A$. Then

$$|F(x) - f(a)| \leq |x - a| < |x - b|,$$

$$|f(a)| = |f(a) - f(b)| \leq |a - b| = |x - b| \quad \text{and} \quad |F(x)| \leq |x - b|.$$ 

Hence

$$|G(x) - G(y)| = |F(x) - 0| = |F(x)| \leq |x - b| \leq |x - y|,$$

as desired.

**Case II.** Since $y$ satisfies condition (3), it follows that

$$\exists b_0 \in B, a_0 \in A \quad \forall a \in A, b \in B \quad |y - a|, |y - b| \geq |y - a_0| = |y - b_0|.$$ 

Were

$$|x - y| < |y - a_0| = |y - b_0|,$$

we would get for every $a \in A$ the inequalities

$$|x - y| < |y - a_0| \leq |y - a| \quad \text{and} \quad |x - y| < |y - b_0| = |x - b_0|.$$ 

Hence

$$|x - a| = |y - a| \geq |y - b_0| = |x - b_0|,$$

and thus $x$ does not satisfy condition (1), which is a contradiction.
Therefore
\[ |x - y| \geq |y - a_0| = |y - b_0|, \]
and thus
\[ |x - a_0| \leq |x - y|, \quad |x - b_0| \leq |x - y| \quad \text{and} \quad |a_0 - b_0| \leq |x - y|. \]

Hence
\[ |F(x) - f(a_0)| \leq |x - a_0|, \quad |f(a_0)| = |f(a_0) - f(b_0)| \leq |a_0 - b_0|, \]
and thus
\[ |F(x)| \leq \max \{|x - a_0|, |a_0 - b_0|\} \leq |x - y|. \]

Therefore
\[ |G(x) - G(y)| = |F(x)| \leq |x - y|, \]
as desired. This finishes the proof. \(\square\)

Observe more precisely that the points \(x\) from
\[ \overline{A} \setminus B, \quad B \setminus \overline{A}, \quad (\overline{A} \setminus A) \cap (B \setminus B), \quad (\overline{A} \setminus A) \cap B, \quad A \cap (B \setminus B), \quad A \cap B, \]
satisfy respectively the following conditions
\[ (1) \land \lnot (2), \quad (2) \land \lnot (1), \quad (1) \land (2), \quad (2) \land \lnot (1), \quad (1) \land \lnot (2), \quad (3); \]
here \(\overline{A}\) denotes the topological closure of a set \(A\). Therefore the value \(G(x)\) at the points \(x\) from the above subsets of \(K^n\) is respectively equal to
\[ F(x), \quad 0, \quad 0, \quad F(x), \quad 0. \]

Hence
\[ G|(A \setminus B) = F|(A \setminus B), \quad G|(B \setminus \overline{A}) = 0, \quad G|((\overline{A} \setminus A) \cap B) = 0, \quad G|(A \cap (B \setminus B)) = F|(A \cap (B \setminus B)), \quad G|(A \cap B) = 0. \]

We say that a 0-definable subset \(A\) of \(K^n\) has the Lipschitz extension property, or the LE-property for short, if every 0-definable 1-Lipschitz function \(f : A \to K\) extends to a 0-definable 1-Lipschitz function \(F : K^n \to K\).

**Corollary 2.3.** (Partition Lemma) Consider a finite number of 0-definable subsets \(A_1, \ldots, A_s\) of \(K^n\). If each of them has the LE-property, so does their union \(A_1 \cup \cdots \cup A_s\).

**Proof.** Induction with respect to the number \(s\) reduces the proof to the case \(s = 2\). Then the conclusion follows directly from Proposition 2.2. Indeed, let \(f : A_1 \cup A_2 \to K\) be a 0-definable 1-Lipschitz function, \(F_2\) be a 0-definable 1-Lipschitz extension of the restriction \(f|A_2\). Then the function
\[ g := f - F_2|(A_1 \cup A_2) \]
vanishes on $A_2$, and thus, by Proposition 2.2, $g$ extends to a 0-definable 1-Lipschitz function $F_1 : K^n \to K$. Then the function $F := F_1 + F_2$ is an extension of the initial function $f$, concluding the proof. □

We still need the following

**Proposition 2.4.** Every 0-definable subset $A \subset K^n$ is a finite disjoint union of 0-definable sets $E$ that are, after some permutation of the variables, disjoint unions of the form

$$E = \bigcup_{\xi \in \sigma(D)} \bigcup_{j=1}^{d} \text{graph}\phi_{\xi,j} \subset K^n,$$

where

$$\phi_{\xi,j} : D_\xi \to K_{x_{>k}}, \quad \xi \in \Xi \subset RV(K)^t, \quad j = 1, \ldots, d,$$

are $\xi$-definable 1-Lipschitz functions and

$$D = \bigcup_{\xi \in \sigma(D)} D_\xi \subset K_{x_{\leq k}}^k$$

is a 0-definable reparametrized Lipschitz open cell in $K_{x_{\leq k}}^k$.

**Proof.** By cell decomposition, we may assume that $A$ is, after some coordinate permutation, a 0-definable reparametrized Lipschitz cell $C$ of dimension $k$ of type $(1, \ldots, 1, 0, \ldots, 0)$. Put

$$D := \pi_{\leq k}(C) = \bigcup_\xi D_\xi \quad \text{with} \quad D_\xi := \pi_{\leq k}(C_\xi) \subset K_{x_{\leq k}}.$$

The restriction

$$p : \bigcup_\xi \text{graph}(c_{\xi,k+1}, \ldots, c_{\xi,n}) \to D$$

of the projection $\pi_{\leq k}$ has of course finite, thus uniformly bounded fibres; say, of maximum cardinality $l$. Since $D$ is the union of the interiors $\text{int}(D_d)$ of the sets

$$D_d := \{x_1 \in D : \# p^{-1}(x_1) = d\}, \quad d \leq l,$$

and of a set of dimension $< k$, we can assume, via a routine induction argument, that $D = \text{int}(D_d)$, and thus that the constant cardinality of the fibers of $p$ is $d$. Therefore the 0-definable family

$$D_\xi := \left\{x_1 \in \bigcap_{j=1}^{d} D_{\xi_j} : \# \{c_{\xi_{1,>k}}(x_1), \ldots, c_{\xi_{d,>k}}(x_1)\} = d \right\},$$
where $\bar{\xi} = (\xi_1, \ldots, \xi_d) \in \sigma(C)^d$, covers $D$, and even is its partition. Indeed, the first assertion follows from that we have

$$D_\lambda = \bigcup_{\lambda \in \xi} D_{\xi},$$

for $\lambda \in \sigma(C)$, and the second from that a reparametrized cell is a disjoint union of cells.

Now it follows from Theorem 2.1 that $D$ is a finite disjoint union of 0-definable reparametrized Lipschitz cells $D_t$ compatible with the second family. Obviously, we can assume that

$$D = \bigcup \{D_\eta : \eta \in \tau(D)\},$$

is one of those cells. Since each cell $D_\eta$ is contained in exactly one of the sets $D_{\xi}$, the conclusion of the proposition follows immediately. \hfill \Box

3. Lipschitz extension on the affine line

Consider a 0-definable 1-Lipschitz function $f : A \to K$ with $A \subset K$. If $A$ is of dimension zero, then $A$ is finite and it is easy to check that the function

$$F : K \ni x \mapsto \sum_{x} f(A(x)) \frac{\# A}{A(x)} \in K,$$

where

$$A(x) := \{a \in A : |x - a| = \min \{|x - s| : s \in A\}\},$$

is a 0-definable 1-Lipschitz extension of $f$ we are looking for.

Hence and by Corollary 2.3, we can assume that $A = C = \bigcup_{c} C_{c}$ is an 0-definable reparametrized open cell; and further, as explained before, that $f : C \to K$ satisfies condition 1.1, which means that $f$ is a risometry onto its image.

The concept of a 0-definable reparametrized open cell is especially simply on the affine line (and only for dimension 1). Indeed, the set of centers $c_\xi \in K$ of $C$ is of course finite, and consists of some $d$ distinct points $\{c_1, \ldots, c_d\} \subset K$. Put

$$C_i := \bigcup \{C_{c_\xi} : c_\xi = c_i\} \subset K$$

and

$$R_i := \bigcup \{R_{c_\xi} : c_\xi = c_i\} \subset G(K) \subset RV(K);$$

clearly each set $C_i$ and $R_i$ is $c_i$-definable for $i = 1, \ldots, d$. But the auxiliary sort $RV$ is stably embedded (cf. 3. Proposition 2.6.12) and even more, as indicated below.
Remark 3.1. For any $a \in K$, every $\{a\}$-definable set $X \subset RV(K)^n$ is $\chi(a)$-definable for a 0-definable map 
$$\chi : K \to RV(K)^t$$
with some $t \in \mathbb{N}$. This follows directly from the proof of loc.cit.

Hence each $C_i$ is a $\xi_i$-definable open cell for some $\xi_i \in RV(K)^t$, and thus we have proved the following

Proposition 3.2. Every 0-definable reparametrized open cell $C$ in $K$ is actually the disjoint union of a finite 0-definable family of open cells $C_i$, $i = 1, \ldots, d$.

We are now going to introduce the concept of a skeleton of $C$. Though this can be done in arbitrary Hensel minimal structures, with no additional conditions, we first suppose for simplicity that the infima 

(3.1) $$\rho(c_i) := \inf \{|x - c_i| : x \in C_i\} \in |K|$$

are well defined for $i = 1, \ldots, d$. Write the 0-definable set of the values $\rho(c_i)$ in the ascending order:

$$r_0 := 0 < r_1 < r_2 < \cdots < r_s;$$

take $r_0 = 0$ if 0 occurs among the $\rho(c_i)$. For $i = 0, \ldots, s$, $j = 1, \ldots, s_i$, let $c_{i,j}$ be the centers with $\rho(c_{i,j}) = r_i$; obviously, $s \leq d$ and

$$s_0 + s_1 + \cdots + s_s = d.$$

Remark 3.3. For each $i = 0, \ldots, s$, the finite set $\{c_{i,1}, \ldots, c_{i,s_i}\}$ is 0-definable, because so is each singleton $r_i$.

We can canonically define, by modifying the initial centers, a skeleton $S(C)$ of $C$, which is a 0-definable set of centers satisfying the following metric condition:

(3.2) $$\rho(c_{i,j}) = r_i \quad \text{and} \quad |c_{i,j} - c_{p,k}| > r_p \quad \text{for} \quad i \leq p.$$

- At the level 0, all the centers $c_{0,j}$ will be added to the skeleton without modification.
- At the level 1, first remove from the centers $c_{1,j}$, for construction of skeleton only, all those for which

$$|c_{1,j} - c_{0,k}| \leq r_1$$

for some $k$. Note that, for any such removed center $c_{1,j}$, the cell $C_{1,j}$ is also a cell with center $c_{0,k}$ (with the same set $R_{1,j}$).

Next notice that the equivalence relation

$$a \sim b \iff |a - b| \leq r_1$$
partitions the set of the remaining centers from among \(c_{1,j}\) into equivalence classes. Any equivalence class is a maximal subset of that set, say \(c_{1,j_1}, \ldots, c_{1,j_l}\), such that

\[
|c_{1,j_i} - c_{1,k}| \leq r_1 \quad \text{for all } 1 \leq j, k \leq l.
\]

It is easy to check that either the arithmetic average

\[
c := \frac{c_{1,j_1} + \ldots + c_{1,j_l}}{l}
\]

lies in no cell \(C_{1,j_1}, \ldots, C_{1,j_l}\), or it lies in a ball of radius \(r_1\) from exactly one of those cells, say \(C_{1,j_k}\). We can thus replace each center \(c_{1,j_p}\) from such a maximal set with \(c\) or \(c_{1,j_k}\) (without change of the set \(R_{1,j_p}\)), according as the former or latter condition holds, and add such a new center to the skeleton. In this manner, any such equivalence class determines one point to be added to the skeleton. By Remark 3.3, the set of centers from the skeleton, obtained at this level 1, form a 0-definable set satisfying condition 3.2 for \(i \leq p \leq 1\).

- At the level 2, first remove from the centers \(c_{2,j}\), for construction of skeleton only, all those for which

\[
|c_{2,j} - c_{0,k}| \leq r_2 \quad \text{or} \quad |c_{2,j} - c_{1,k}| \leq r_2
\]

for some \(k\). Next reason as before.

- Further, repeat the above construction at the level 3 for the centers \(c_{3,j}\), and so on. This canonical process leads eventually to a unique 0-definable set of centers \(S(C)\) satisfying condition 3.2. Then we shall also say that the reparametrized cell \(C\) is with a skeleton \(S(C)\).

**Remark 3.4.** In the construction of skeleton, the range of parameters \(\xi\) for a given 0-definable reparametrized open cell \(C\) does not change while modifying some centers in the construction of skeleton. Note also that \(C\) is a finite, but not necessarily disjoint, union of cells centered at the points of the skeleton \(S(C)\). Lack of disjointness is caused, to some extent, by the absence of definable choice in non-Archimedean geometry of Hensel minimal structures.

It is easy to see that

\[
S(C) \cap C = \emptyset.
\]

It remains to explain the case where the infima \(\xi\) do not exist. Then the above reasoning can be repeated verbatim provided that \(\rho(c_i), i = 1, \ldots, d,\) are replaced by the Dedekind cuts determined by the
closed downward subsets of all elements of $|K|$ which are smaller than the sets
\[ \{ |x - c_i| : x \in \bigcup \{ \sigma^{-1}(\xi) : c_\xi = c_i \} \} \]
and smaller than $\rho(c_i)$ whenever this infimum exists.

**Remark 3.5.** For any Dedekind cut $\rho$ on the value group $|K|$ and an $r \in |K|$, the strong inequalities $r < \rho$ and $\rho < r$ have a clear meaning. The equality $\rho = r$ means that the cut $\rho$ represents the value $r$.

**Remark 3.6.** Observe that a skeleton $S(C)$ of a reparametrized cell $C$ depends on the initial set of centers of $C$. Even more, there are finitely many configurations, say
\[ C_1, \ldots, C_N, \quad N = N(d), \]
of the initial $d$-tuples $(c_1, \ldots, c_d)$ of centers $c_i$ or, more precisely, of $d$-tuples of pairs $(c_i, \rho(c_i))_{i=1}^d$, for each of which the skeleton is given by a unique formula (built from some arithmetic averages after a suitable choice, as described above) depending only on a given configuration. Our preference to configurations with respect to $d$-tuples $(c_1, \ldots, c_d)$ rather than their sets $\{c_1, \ldots, c_d\}$ is for the sake of application later on in the affine spaces of higher dimensions.

In the sequel, we shall still need the following lemma whose proof uses the concept of skeleton.

**Proposition 3.7.** Let $\mathcal{B}$ be a 0-definable family of disjoint open balls in $K$ which satisfies the following condition: for all balls $B_1, B_2 \in \mathcal{B}$, $B_1 \neq B_2$, with radii $r_1, r_2 \in |K| \setminus \{0\}$, respectively, we have
\[ |b_1 - b_2| = \max\{r_1, r_2\} \quad \text{for every} \quad b_1 \in B_1, \ b_2 \in B_2. \]
Then $\mathcal{B}$ is a 0-definable cell in $K$. Since Hensel minimality admits adding constants, the conclusion holds for a $\xi$-definable family as well.

**Proof.** The union of $\mathcal{B}$ is a finite union of 0-definable reparametrized open cells $C_i$. It follows from the assumed condition that the skeleton of each of those reparametrized cells consist of one point, and that all those points must coincide. Hence the conclusion follows. \qed

We now apply the above results to risometries. Since 0-h-minimality implies spherical completeness (cf. [3, Lemma 2.1.7]), the image $f(B)$ of an open ball $B$ under a definable risometry $f : B \to K$ is an open ball of the same radius (cf.[3, Lemma 2.33]). Hence and by Proposition 3.7, we get
Corollary 3.8. Let \( f : C \to K \) be a 0-definable risometry of a 0-definable open cell \( C \subset K \), with a center \( c \), onto its image. Then \( f(C) \) is a 0-definable cell with a 0-definable center \( d \), and the function
\[
\tilde{f}(x) = \begin{cases} 
  f(x) & \text{if } x \in C, \\
  d & \text{if } x = c.
\end{cases}
\]
is a risometry onto its image too. The same conclusion holds for a \( \xi \)-definable risometry as well.

We also obtain the following

Corollary 3.9. Let \( f : C \to K \) be a 0-definable risometry of a 0-definable reparametrized open cell \( C \), with a skeleton \( S(C) \), onto its image. By Proposition 3.2, \( C \) is actually a finite disjoint union of open cells \( C_i, i = 1, \ldots, d \). Then the image \( D := f(C) \) is a 0-definable reparametrized open cell which is the finite disjoint union of open cells
\[
D_i := f(C_i), \quad i = 1, \ldots, d.
\]
Further \( f \) induces a unique 0-definable function
\[
f_s : S(C) \to S(D)
\]
between the skeletons of \( C \) and \( D \), respectively, and the function
\[
\tilde{f}(x) = \begin{cases} 
  f(x) & \text{if } x \in C, \\
  f_s(x) & \text{if } x \in S(C).
\end{cases}
\]
is a risometry of \( C \cup S(C) \) onto \( D \cup S(D) \) too.

Proof. While the first conclusion immediately follows directly from Corollary 3.8, the second one is a straightforward consequence of the very construction of skeleton.

Now we turn to the extension problem. The function \( f : C \to K \) under study is a risometry onto its image. Consider its extension
\[
\tilde{f} : C \cup S(C) \to D \cup S(D)
\]
from Corollary 3.9. Then the function
\[
F(x) = \begin{cases} 
  \tilde{f}(x) & \text{if } x \in C \cup S(C), \\
  \sum \tilde{f}(S(x)) & \text{otherwise},
\end{cases}
\]
where
\[
S(x) := \{ a \in S(C) : |x - a| = \min \{|x - s| : s \in S(C)\}\}
\]
is a 0-definable 1-Lipschitz extension of \( f \) we are looking for.
The above extension procedure can be split into the following two steps, which will be especially convenient for higher dimensional ambient spaces. First observe that the function

\[ g : K \ni x \mapsto \sum \tilde{f}(S(x)) \in K \]

is 1-Lipschitz continuous, and next that so is the extension of \( \tilde{f} - g|_{(C \cup S(C))} \) by zero through the complement \( K \setminus (C \cup S(C)) \).

Since the foregoing construction is canonical, the uniform version of the extension theorem (Theorem [3.3]) follows directly via a routine compactness argument.

4. Lipschitz extension on the affine plane

Consider a 0-definable 1-Lipschitz function \( f : A \to K \) with \( A \subset K^2 \). By Corollary [2.3], we can assume that \( A = C \) is a 0-definable reparametrized Lipschitz cell of dimension \( k = 0, k = 1 \) or \( k = 2 \).

Case I. Suppose \( k = \dim C = 0 \), and then the set \( A \) is finite, say

\[ A = \{ (u_i, v_{i,j}) \in K^2 : i = 1, \ldots, s, j = 1, \ldots, d_i \} \]

denote by

\[ A_{u_i} := \{ v_{i,1}, \ldots, v_{i,d_i} \} \]

the fiber of \( A \) over \( u_i \). Put

\[ \delta_1 := \min\{|u_i - u_j| : i \neq j\}, \quad \delta_2 := \min\{|u_i - u_j| : |u_i - u_j| > \delta_1\}, \]

and so on. In this fashion, we get a finite increasing sequence

\[ \delta_1 < \delta_2 < \ldots < \delta_t < \delta_{t+1} = \infty, \quad t \leq s. \]

We shall canonically extend the function \( f \) to the successive \( \delta_k \)-neighbourhoods \( A^\delta \) of \( A \), where

\[ A^\delta := \bigcup_{x \in A} B(x, \delta), \quad B(x, \delta) := \{ y \in K^2 : |y - x| < \delta \}. \]

Observe that if \( \Phi : E \to K \) is a 1-Lipschitz function on a finite subset \( E \) of \( K \), then so is its extension

\[ \omega(\Phi) : K \ni v \mapsto \sum f(E(v)) \in K, \quad (4.1) \]

where

\[ E(v) := \{ w \in E : |v - w| = \min \{|v - s| : s \in E\} \}, \]
is a 1-Lipschitz extension of $\phi$.

For 1-Lipschitz functions $\phi_i : E_i \rightarrow K$ on finite sets $E_i \subset K$, $i = 1, 2, \ldots, p$ and $\delta \in |K|$, define the function

$$\Phi = \Phi(\phi_1, \delta, \phi_2, \ldots, \phi_p) : \bigcup_{i=1}^{p} E_i \rightarrow K$$

by putting

$$\Phi(v) = \phi_1(v) \text{ if } v \in E_1,$$

and

$$\Phi(v) = \frac{\sum_{i=2}^{p} \sum \phi_i(B \cap E_i)}{\sum_{i=2}^{p} \#(B \cap E_i)} \text{ if } v \in B \cap \bigcup_{i=2}^{p} E_i,$$

for any open ball

$$B \subset \bigcup_{i=2}^{p} E_i^\delta \setminus E_1^\delta$$

of radius $\delta$.

We begin by extending the 0-definable 1-Lipschitz function $f : A \rightarrow K$ on the finite set $A$ to the $\delta_1$-neighbourhood of $A$. Consider any largest subset $B_1$ of $\pi_{<2}(A) = \{u_1, \ldots, u_s\}$ that satisfies the following condition

$$(\Lambda_1) \ |u - u'| = \delta_1 \text{ for every } u, u' \in B, \ u \neq u';$$

say, $B = \{u_1, \ldots, u_p\}$. Consider the functions

$$\phi_i : A_{u_i} \rightarrow K, \ \phi_i(v_{i,j}) := f(u_i, v_{i,j}), \ i = 1, \ldots, p, \ j = 1, \ldots, d_i$$

and

$$\Phi(v_{i,j}) = \Phi(\phi_1, \delta_1, \phi_2, \ldots, \phi_p)(v_{i,j}), \ i = 1, \ldots, p, \ j = 1, \ldots, d_i.$$ 

Put

$$\tilde{f}_{B_1,1}(u, v) = \omega(\Phi)(v), \ (u, v) \in \bigcup_{k=1}^{p} \bigcup_{j=1}^{d_k} B(u_1, \delta_1) \times B(v_{k,j}, \delta_1);$$

note that the function $\tilde{f}(u, v)$ is in fact constant with respect to the variable $u \in B(u_1, \delta_1)$.

Similarly, one can canonically define the functions

$$\tilde{f}_{B_1,i}(u, v) = \omega(\Phi)(v), \ (u, v) \in \bigcup_{k=1}^{p} \bigcup_{j=1}^{d_i} B(u_i, \delta_1) \times B(v_{k,j}, \delta_1),$$
the function \( \tilde{f}_{B_1} \) by gluing the functions \( \tilde{f}_{B_{1,1}}, \ldots, \tilde{f}_{B_{1,p}} \), and then a unique function \( \tilde{f}_1 \) by gluing the functions \( \tilde{f}_{B_1} \) where \( B_1 \) runs over all largest sets that satisfy condition \( \Lambda_1 \).

It is not difficult to check that all the functions \( \tilde{f}_{B_1} \) are 1-Lipschitz on the \( \lambda_1 \)-neighbourhood of the finite set 
\[
A(B_1) := B_1 \times \{v_{i,j} : i = 1, \ldots, p, j = 1, \ldots, d_i\},
\]
and \( \tilde{f}_1 \) is a 0-definable 1-Lipschitz extension of \( f \) to the \( \delta_1 \)-neighbourhood of a finite subset of \( K^2 \) containing \( A \).

Next consider any largest subset \( B_2 \) of \( \pi_{<2}(A) = \{u_1, \ldots, u_s\} \) that satisfies the following condition
\[
(\Lambda_2) \quad |u - u'| \leq \delta_2 \quad \text{for every } u, u' \in B, \ u \neq u'.
\]
Every such subset \( B_2 \) is a union of subsets \( B_1 \) considered above. Since the sets \( A(B_1) \) have constant fibres over the points of \( B_1 \), we can now handle subsets \( B_1 \) of the sets \( B_2 \) likewise we did points of the sets \( B_1 \) before. In this fashion, again we can canonically achieve 1-Lipschitz extensions \( \tilde{f}_{B_2} \) of \( \tilde{f}_1 \) on the \( \lambda_2 \)-neighbourhood of a finite set \( A(B_2) \) with constant fibres over the points of \( B_2 \), and then a unique 0-definable 1-Lipschitz extension \( \tilde{f}_2 \) of \( f \) to the \( \delta_1 \)-neighbourhood of a finite subset of \( K^2 \) containing \( A \). Repeating the process, we eventually achieve a unique 0-definable 1-Lipschitz extension \( \tilde{f}_{t+1} \) of \( f \) to the affine space \( K^2 \), which is the desired result.

Since the foregoing construction is canonical, the uniform version of the extension theorem (Theorem 1.3) follows again via a routine compactness argument. Actually, a parametrized family of finite sets is uniformly bounded, and thus we encounter only a finite number of configurations of points and formulae which determine a unique family of canonical extensions.

**Case II.** Suppose \( k = \dim C = 1 \). Making use of Corollary 2.3 and Proposition 2.4, we can assume that \( A = E \) is a disjoint union of the form
\[
C = \bigcup_{\xi \in \Xi} \bigcup_{j=1}^d \text{graph } \phi_{\xi,j} \subset K^2,
\]
where
\[
D = \bigcup_{\xi \in \Xi} D_\xi \subset K^1_{x_1}
\]
is a 0-definable reparametrized open cell in \( K_{x_1} \) with a skeleton
\[
S(D) = \{S_1, \ldots, S_s\} \subset K_{x_1},
\]
and
\[ \phi_{\xi,j} : D_\xi \to K_{x_2}, \; \xi \in \Xi, \; j = 1, \ldots, d, \]
are \( \xi \)-definable 1-Lipschitz functions. (By Proposition 3.2, \( D \) is actually a finite disjoint union of definable open cells.)

In view of Remark 1.4 and by the uniform extension theorem (Theorem 1.3) on the affine line, we can assume that the above 0-definable family of 1-Lipschitz functions is a 0-definable family of 1-Lipschitz functions on the affine line \( K^1_{x_1} \). Consider the 0-definable finite set \( O(C) \) of origins of \( C \):

\[ \{ O_{\xi,j} := (S_i, \phi_{\xi,j}(S_i)) : \xi \in \Xi, \; c_\xi = S_i, \; i = 1, \ldots, s, \; j = 1, \ldots, d \} = \{ O^1, \ldots, O^q \} =: O(C) \subset K^2. \]

As explained in Section 1 after Theorem 1.1, we can require that, after a suitable modification, \( f \) satisfy condition 1.1 with respect to the variable \( x_1 \). Then the function \( f(x_1, \phi_{\xi,j}(c_{x_1}(x_1))) \) is a risometry onto its image for each \( \xi \in \Xi, \; j = 1, \ldots, d \). Since the image of any open ball is a ball of the same radius (cf. Section 3), the images \( E^i := f(C^i), \; i = 1, \ldots, q \), of the sets \( C^i := \bigcup \{ \text{graph} \phi_{\xi,j} : \xi \in \Xi, \; (S_i, \phi_{\xi,j}(S_i)) = O^i, \; j = 1, \ldots, d \} \), satisfy the condition of Proposition 3.7. Therefore the definable sets \( E^i \) are open cells with some centers \( e_i \in K, \; i = 1, \ldots, q \).

Further, it is not difficult to verify that the function
\[ \tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in C, \\ e_i & \text{if } x = O^i, \; (i = 1, \ldots, q). \end{cases} \]
is a 0-definable 1-Lipschitz extension of \( f \) satisfying condition 1.1. Here \( E := \bigcup_{i=1}^q E^i \).

But we have already considered the case where \( A \) is a subset of the affine plane of dimension 0. Therefore there exists a 0-definable 1-Lipschitz function \( g : K^2 \to K \) that agrees with the function \( f \) at the origins \( O(C) \). This reduces the problem to the case where \( f \) is a 1-Lipschitz function on \( C \cup O(C) \) which vanishes on \( O(C) \). Now let
\[ f_{x_1} : C_{x_1} \ni x_2 \mapsto f(x) \in K, \; x_1 \in D, \]
be the restriction of \( f \) to the fiber \( C_{x_1} \) of \( C \) over \( x_1 \). Then the function \( F : K^2 \to K \) given by the formula
\[ F(x) = \begin{cases} \omega(f_{x_1})(x_2) & \text{if } x_1 \in D, \\ 0 & \text{if } x_1 \notin D, \end{cases} \]
is a 1-Lipschitz extension of \( f \) we are looking for; here the function \( \omega(\Phi) \) is given by formula 4.1. For verification apply the following two estimates. If \( \pi_2(x) \in D_\xi \) and \( \pi_2(y) \in D_\eta \) with \( \xi \neq \eta \), then
\[
|f(y) - f(x)| \leq \max \{|f(y)|, |f(x)|\} \leq \max \{|y_1 - c_\eta, 1|, |x_1 - c_\xi, 1|\} \leq |y_1 - x_1| \leq |y - x|.
\]
And similarly, if \( \pi_2(x) \in D_\xi \) and \( \pi_2(y) \in D \), then
\[
|f(x) - 0| = |f(x)| \leq |x_1 - c_\xi, 1| \leq |x_1 - y_1| \leq |x - y|.
\]
We thus established the case where \( A = C \) is a cell of dimension 2.

**Case III.** Finally, suppose \( k = \dim C = 2 \). Then \( A = C \) is a 0-definable reparametrized open cell in \( K^2 \):
\[
C = \bigcup_{\xi} C_\xi \subset K^2, \quad C_\xi = \sigma^{-1}(\xi), \quad \xi \in \sigma(D),
\]
where \( \sigma : C \to (K^2)^s \) is a 0-definable function. Each set \( C_\xi := \sigma^{-1}(\xi), \quad \xi \in \sigma(C), \) is a \( \xi \)-definable Lipschitz open cell with some center tuple \( c_\xi \) of the form
\[
(4.2) \quad C_\xi = \{ x \in K^2 : (rv(x_1 - c_\xi, 1), rv(x_2 - c_\xi, 2(x_1))) \in R_\xi \}
\]
with some 0-definable families of center tuples \( c_\xi = (c_\xi, 1, c_\xi, 2) \) and of sets \( R_\xi \subset G(K)^2 \).

Now we are going to define a **skeleton** of the reparametrized Lipschitz open cell \( C \). It will be done via a suitable finite 0-definable partitioning of the set
\[
D := \pi_1(C) = \bigcup_{\xi} D_\xi, \quad \text{where} \quad D_\xi := \pi_1(C_\xi) \subset K_{x_1},
\]
up to a subset of lower dimension, which will not affect our solution to the extension problem in view of Corollary 2.3 and the induction hypothesis. The restriction
\[
p : \bigcup_{\xi} \text{graph}(c_\xi, 2) \to D
\]
of the projection \( \pi_1 \) has of course finite, thus uniformly bounded fibres; say, of maximum cardinality \( l \). Since \( D \) is the union of the interiors \( \text{int}(D_d) \) of the sets
\[
D_d := \{ x_1 \in D : \# p^{-1}(x_1) = d \}, \quad d \leq l,
\]
and of a (finite) set of dimension $< 1$, we can assume that $D = \text{int}(D_d)$, and thus that the constant cardinality of the fibers of $p$ is $d$. Therefore the 0-definable family

$$D_\xi := \left\{ x_1 \in \bigcap_{j=1}^d D_{\xi_j} : \# \{ c_{\xi_1,2}(x_1), \ldots, c_{\xi_d,2}(x_1) \} = d \right\},$$

where $\xi = (\xi_1, \ldots, \xi_d) \in \sigma(C)^d$, covers $D$. Indeed, we have

$$D_\lambda = \bigcup_{\lambda \in \xi} D_\xi$$

for $\lambda \in \sigma(C)$. At the cost of a further finite 0-definable partitioning of $D$, we can also require that all (ordered) fibers

$$(c_{\xi_1,2}(x_1), \ldots, c_{\xi_d,2}(x_1))$$

of $p$, regarded as $d$-tuples, have over $D_\xi$ the same configuration with respect to construction of skeleton.

We are now going to find a 0-definable cell decomposition of $D$ which is finer than the covering $\{ D_\xi : \xi \in \sigma(C)^d \}$. To this end consider a 0-definable family

$$\Lambda_a := \{ \xi \in \sigma(C)^d : a \in D_\xi \}, \quad a \in D,$$

of subsets of $\sigma(C)^d$. What will be used here is the fact that the auxiliary sort $RV$ is stably embedded (cf. [3, Proposition 2.6.12]). By Remark 3.1 and model theoretical compactness, there exists a 0-definable map

$$\chi : K \to RV(K)^\sigma$$

such that the family

$$\Lambda_{\chi(a)} := \Lambda_a, \quad \chi(a) \in \chi(D),$$

is 0-definable too. Therefore the 0-definable equivalence relation

$$R(\chi(a), \chi(b)) \iff \Lambda_{\chi(a)} = \Lambda_{\chi(b)}, \quad a, b \in D,$$

allows us to code the same fibres from among $\Lambda_{\chi(a)}$ with $\chi(a) \in \chi(D)$, by the imaginary elements (equivalence classes) induced by $R$:

$$[\Lambda_{\chi(a)}]_R = [\chi(a)]_R = [\chi(a)].$$

In other words, we can parametrize the same fibres from among $\Lambda_{\chi(a)}$ by the imaginary elements $[\chi(a)]$ with $\chi(a) \in \chi(D)$. Consider the two 0-definable families

$$S_a = S_{\chi(a)} = S_{[\chi(a)]} := \bigcap_{\lambda \in \xi} D_\lambda = \bigcap_{\xi \in \Lambda_a} D_\xi = \bigcap_{\xi \in [\Lambda_{\chi(a)}]} D_\xi$$
and
\[ T_a = T_{\chi(a)} = T_{[\chi(a)]]} := \{ b \in S_{[\chi(a)]]} : \Lambda_{\chi(b)} = \Lambda_{\chi(a)} \} \]

parametrized by \( \chi(a) \in \chi(D) \), and also by the imaginary elements \([\chi(a)]] \in \chi(D)/R \). The second family is of course a partition of \( D \).

It follows from Theorem 2.1 that \( D \) is a finite disjoint union of 0-definable reparametrized Lipschitz cells \( D_t \) compatible with the second family. By Corollary 2.3 and the induction hypothesis on the dimension \( k \) of the subset \( A \), we can assume that
\[ D = \bigcup \{ D_\eta : \eta \in \tau(D) \}. \]
is one of those cells. Then each set \( T_a = T_{[\chi(a)]]} \) is a union of some twisted boxes of \( D \).

Now we can replace the initial 0-definable reparametrized Lipschitz open cell \( C \) by the 0-definable reparametrized Lipschitz open cell
\[ \bigcup \{ C_\eta : \eta \in \tau(D) \} \text{ with } C_\eta := C \cap \pi_1^{-1}(D_\eta). \]

Concluding, the (ordered) centers
\[ (c_{\xi_1,2}(a), \ldots, c_{\xi_d,2}(a)), \quad \xi = (\xi_1, \ldots, \xi_d) \in [\chi(a)], \]
of \( C \), regarded as \( d \)-tuples, have the same configuration over \( T_{[\chi(a)]]} \) and, a fortiori, over every \( D_\eta \) contained in \( T_{[\chi(a)]]} \). Therefore the punctual construction of skeleton from Section 3, performed here over parameters \( x_1 \in D \), leads to new \( e \) 1-Lipschitz centers with a number \( e \leq d \) independent of \( x_1 \), which are produced from the initial \( d \) centers \( c_{\xi_1}, \ldots, c_{\xi_d} \). Thus the graphs of those new centers
\[ (x_1, \tilde{c}_{\xi_1,2}(x_1)), \ldots, (x_1, \tilde{c}_{\xi_d,2}(x_1)), \]
form a skeleton of \( p^{-1}(x_1) \) over every \( D_\eta \) contained in \( T_{[\chi(a)]]} \). We shall then say that \( C \) has a skeleton and, more precisely, that the union of the graphs of all the new centers is a skeleton of \( C \) at the level 2 (i.e. with respect to the variable \( x_2 \)).

Finally, we can construct a skeleton of the cell \( D \) as well. We shall then say that \( C \) has a total skeleton.

Remark 4.1. The above construction of a 0-definable partition into Lipschitz open cells (after a suitable finite 0-definable partitioning up to a subset of lower dimension), which is parametrized by the auxiliary sort and compatible with a given 0-definable covering by open cells, can be repeated almost verbatim in the affine spaces \( K^k \). Therefore, omitting the details, it will be applied again in Section 5 for defining the concept of skeleton in the affine spaces \( K^k \).
At this stage, we are able to assume that $f : C \to K$ is a 0-definable 1-Lipschitz function on a reparametrized Lipschitz open cell $C$ with a total skeleton $(c_\xi)_\xi$. In view of Remark 1.4 and by the uniform extension theorem (Theorem 1.3) on the affine line, we can assume that the above 0-definable family of 1-Lipschitz centers is a 0-definable family of 1-Lipschitz functions on the affine line $K^1_{x_1}$.

As before, we can require that, after a suitable modification, $f$ satisfy condition 1.1 with respect to the variables $x_1, x_2$. Suppose first that the center-tuple $c_\xi$ of an open cell $C_\xi$ vanishes, partition the affine space $K^2$ into the following 2 subsets:

$$\Delta_1 := \{x \in K^2 : |x_1| < |x_2|\}, \quad \Delta_2 := \{x \in K^2 : |x_2| \leq |x_1|\},$$

and put

$$C^1_\xi := C_\xi \cap \Delta_1, \quad C^2_\xi := C_\xi \cap \Delta_2.$$  

By Corollary 3.9, condition 1.1 implies that the images

$$f(C^1_\xi \cap \{(x_2) \times K\}) \text{ and } f(C^2_\xi \cap (K \times \{x_1\}))$$

are cells definable over $\{\xi, x_2\}$ and $\{\xi, x_1\}$, respectively, unless they are empty sets. By model theoretical compactness, the images

$$(4.3) \quad f(C^1_\xi) = \bigcup_{x_2 \in K} f(C^1_\xi \cap \{(x_2) \times K\}) \text{ and } f(C^2_\xi) = \bigcup_{x_1 \in K} f(C^2_\xi \cap (K \times \{x_1\}))$$

are two families of $\xi$-definable cells with some $\xi$-definable centers $d^1_\xi(x_2)$ and $d^2_\xi(x_1)$, respectively. Clearly, the domains of those centers are the following sets

$$\text{dom}(d^1_\xi) = \{x_2 \in K : \exists \lambda_1 \in G(K), |\lambda_1| < |x_2|, (\lambda_1, rv(x_2)) \in R_\xi\}$$

and

$$\text{dom}(d^2_\xi) = \{x_1 \in K : \exists \lambda_2 \in G(K), |\lambda_2| \leq |x_1|, (rv(x_1), \lambda_2) \in R_\xi\};$$

observe that they are some unions of open balls.

For $\lambda_1, \lambda_2 \in G(K)$, the infima

$$\rho_{\xi,1}(\lambda_2) := \min \{|\lambda_1| : \lambda_1 \in G(K), |\lambda_1| < |\lambda_2|, (\lambda_1, \lambda_2) \in R_\xi\}$$

and

$$\rho_{\xi,2}(\lambda_1) := \min \{|\lambda_2| : \lambda_2 \in G(K), |\lambda_2| \leq |\lambda_1|, (\lambda_1, \lambda_2) \in R_\xi\}$$

may generally not exist. Nevertheless, in the reasonings here and also in Section 5, we can then argue replacing those infima by the Dedekind
cuts determined by the closed downwards subsets of all elements of $|K|$ which are smaller than the sets
\[ \{ |\lambda_1| : \lambda_1 \in G(K), |\lambda_1| < |\lambda_2|, (\lambda_1, \lambda_2) \in R_\xi \} \]
and
\[ \{ |\lambda_2| : \lambda_2 \in G(K), |\lambda_2| \leq |\lambda_1|, (\lambda_1, \lambda_2) \in R_\xi \}, \]
respectively. Note that we have already used Dedekind cuts in the construction of a skeleton in Section 3 (see also Remark 3.3).

Clearly, if $\rho_{\xi,1}(\lambda_2) = 0$, then the set
\[ \{ 0 \} \times \{ x_2 \in K : rv(x_2) = \lambda_2 \} \]
is contained in the closure of $C_1^\xi$; similarly, if $\rho_{\xi,2}(\lambda_1) = 0$, then the set
\[ \{ 0 \} \times \{ x_1 \in K : rv(x_1) = \lambda_1 \} \]
is contained in the closure of $C_2^\xi$.

By the valuative Jacobian property (cf. [3, Lemma 2.8.5]), there exists a finite $\xi$-definable set $Z_\xi \subset K$ such that the quotients
\[
\frac{|d_1^\xi(y) - d_1^\xi(z)|}{|y - z|} \quad \text{and} \quad \frac{|d_2^\xi(y) - d_2^\xi(z)|}{|y - z|}
\]
are constant for all points $y, z \in B$, $y \neq z$, where $B$ is any ball next to $Z_\xi$ such that
\[ B \subset \text{dom } (d_1^\xi) \quad \text{or} \quad B \subset \text{dom } (d_2^\xi), \]
respectively.

Now consider any balls
\[ B_1 = \{ x_2 : rv(x_2) = \lambda_2 \} \subset \text{dom } (d_1^\xi), \quad \lambda_2 \in \pi_2(R_\xi), \]
and
\[ B_2 = \{ x_1 : rv(x_1) = \lambda_1 \} \subset \text{dom } (d_2^\xi), \quad \lambda_1 \in \pi_1(R_\xi). \]
Since $f$ satisfies condition \[\square\] we get
\[
|d_1^\xi(y) - d_1^\xi(z)| \leq |y - z| \quad \text{and} \quad |d_2^\xi(y) - d_2^\xi(z)| \leq |y - z|
\]
for all $y, z$ in $B_1$ with $|y - z| \geq \rho_{\xi,1}(\lambda_2)$ or in $B_2$ with $|y - z| \geq \rho_{\xi,2}(\lambda_1)$, respectively.

In the case where the value group $vK$ has no minimal element among the elements $> 1$, it follows directly from estimates \[\square\] that the two functions $d_1^\xi$ and $d_2^\xi$ are 1-Lipschitz on every ball $B$ next to $Z_\xi$, $B \subset B_1$ or $B \subset B_2$, and of radius $\geq \rho_{\xi,1}(\lambda_2)$ or $\geq \rho_{\xi,2}(\lambda_1)$, respectively.

In the other case, however, those functions are only $\epsilon$-Lipschitz on every such ball, which gives rise to increasing the Lipschitz constant in the extension theorem yet once more. This is taken into account in
the statement of the extension theorem, and we shall further consider only the first case, as explained in the Introduction (cf. Remarks \[1.4\] and \[1.5\]).

Hence and by estimate \[4.5\], those functions are 1-Lipschitz on the sets

\[ B_1 \setminus \bigcup_{a \in Z} \{ x_2 \in K : |x_2 - a| < \rho_{\xi,1}(\lambda_2) \} \]

and

\[ B_2 \setminus \bigcup_{a \in Z} \{ x_1 \in K : |x_1 - a| < \rho_{\xi,2}(\lambda_1) \} , \]

respectively.

Now modify the functions \(d_1^\xi\) and \(d_2^\xi\) on each ball \(\{ x_2 \in K : |x_2 - a| < \rho_{\xi,1}(\lambda_2) \}\) or \(\{ x_1 \in K : |x_1 - a| < \rho_{\xi,2}(\lambda_1) \}\), \(a \in Z\), by putting the constant value \(d_1^\xi(a)\) or \(d_2^\xi(a)\), \(a \in Z\). Clearly, after such modification, the functions \(d_1^\xi\) and \(d_2^\xi\) remain \(\xi\)-definable 1-Lipschitz centers of two families of cells \[4.3\]. It is easy to verify that, after gluing such modifications for all balls \(B_1\) and \(B_2\) as above, we achieve new \(\xi\)-definable 1-Lipschitz centers

\( \tilde{d}_1^\xi : \text{dom}(d_1^\xi) \to K \) and \( \tilde{d}_2^\xi : \text{dom}(d_2^\xi) \to K \)

of the two families of cells. For simplicity we drop tilde over \(d_1^\xi\) and \(d_2^\xi\).

Now we give two general observations used here and in the sequel.

**Remark 4.2.** In the general situation, we shall partition the affine space \(K^n\) along the coordinate hyperplanes by putting

\[ \Delta_1 := \{ x \in K^k : |x_1| < |x_2|, \ldots, |x_n| \} , \]

\[ \Delta_2 := \{ x \in K^k : |x_2| \leq |x_1|; |x_2| < |x_3|, |x_4|, \ldots, |x_n| \} , \]

\[ \Delta_3 := \{ x \in K^k : |x_3| \leq |x_1|, |x_2|; |x_3| < |x_4|, |x_5|, \ldots, |x_n| \} , \]

\[ \ldots \ldots \]

\[ \Delta_n := \{ x \in K^k : |x_n| \leq |x_1|, |x_2|, \ldots, |x_{n-1}| \} . \]

**Remark 4.3.** Consider a \(\xi\)-definable open cell \(C_\xi \subset K^n\) with a center tuple \(c = (c_i)_{i=1}^n\), and suppose that those centers extend to global \(\xi\)-definable 1-Lipschitz functions on the ambient affine spaces. Then the map

\[ (4.6) \quad \Phi_\xi : K^n \to K^n, \]

\[ x \mapsto (x_1 - c_{\xi,1}, x_2 - c_{\xi,2}(x_1), x_3 - c_{\xi,3}(x_3), \ldots, x_n - c_{\xi,n}(x_{<n})), \]
is a $\xi$-definable bi-Lipschitz homeomorphism of $K^n$, which maps the graph
$$\text{graph } (c_i(x_{\neq i})) \subset K^n$$
of each center $c_i(x_{<i}) = c_i(x_{\neq i})$, $i = 1, \ldots, n$, regarded here as a function of all variables $x = (x_1, \ldots, x_n)$ but the $i$-th, onto the coordinate hyperplanes
$$H_i := \{x \in K^n : x_i = 0\}.$$
Note also that the canonical projections
$$\text{graph } (c_i(x_{\neq i})) \to H_i, \ i = 1, \ldots, n,$$
are bi-Lipschitz homeomorphisms.

We state yet another key observation with a straightforward proof. It will be used both here and in the next section while defining the functions $d^1_\xi$ for arbitrary 1-Lipschitz centers. The domain of each function $d^1_\xi$ is namely the union of the graph of the restrictions of $c_{\xi,i}$ to some twisted boxes considered in the following

**Lemma 4.4.** Consider a twisted box
$$X := \{(rv(x_i - c_i(x_{\xi,<i})))_{i=1}^n = \lambda\} \subset K^n$$
for some $\lambda = (\lambda_1, \ldots, \lambda_n) \in G(K)^n$, and the projection $\pi_i : K^n \to K$ onto the $x_i$-axis. If $|\lambda_i| \leq |\lambda_j|$ for all $j = 1, \ldots, n$, then the restriction
$$\pi_i|X : X \to K_{x_i}$$
has a common fiber which is the twisted box with the center tuple
$$c_{\xi,1}, c_{\xi,2}(x_1), \ldots, c_{\xi,i-1}(x_{<i-1}), c_{\xi,i+1}(c_{\xi,<i+1}), \ldots, c_{\xi,n}(c_{\xi,<n})$$
determined by
$$(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n) \in G(K)^{n-1}.$$

Applying Lemma 4.4 with $n = 2$ and the map $\Phi_\xi$ from Remark 1.3, we can carry over the above reasonings for cells in $K^2$ with vanishing centers to those with arbitrary 1-Lipschitz centers. Hence we achieve $\xi$-definable 1-Lipschitz functions $d^1_\xi$ and $d^2_\xi$ whose domains are subsets of the following two sets (of dimension 1):
$$\text{graph } c_{\xi,2} \subset K^2 \text{ and } \{x \in K^2 : x_1 = c_{\xi,1}\} \subset K^2,$$
respectively.

Since $C$ is a reparametrized Lipschitz open cell with total skeleton, it is not difficult to verify that
$$\text{dom } (d^1_\xi) \cap C = \emptyset \text{ and } \text{dom } (d^2_\xi) \cap C = \emptyset,$$
and that the function
\[ \tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in C, \\ d_1^\xi(x) & \text{if } x \in \text{dom}(d_1^\xi), \\ d_2^\xi(x) & \text{if } x \in \text{dom}(d_2^\xi), \end{cases} \]
is a 0-definable 1-Lipschitz extension of \( f \) satisfying condition [1.1].

Since we have already considered the case where the set \( A \) is of dimension \(< 2\), there exists a 0-definable \( \epsilon \)-Lipschitz function
\[ g : K^2 \to K \]
that agrees with the function \( \tilde{f} \) on the set
\[ d(C) := \bigcup_\xi (\text{dom}(d_1^\xi) \cup \text{dom}(d_2^\xi)). \]

In this manner, the problem has been reduced to the case where \( f \) is an \( \epsilon \)-Lipschitz function which vanishes on set [17]. Again, since the reparametrized open cell \( C \) is with total skeleton, the function
\[ F : K^n \to K \]
given by the formula
\[ F(x) = \begin{cases} f(x) & \text{if } x \in C \cup d(C), \\ 0 & \text{otherwise}, \end{cases} \]
is a 0-definable \( \epsilon \)-Lipschitz extension of the function \( f \) we are looking for.

The uniform version of the extension theorem follows as before via a routine compactness argument, because the foregoing construction is canonical. This completes the proof of Theorem [13] for the affine plane.

5. Lipschitz extension for higher dimensions

To establish the extension theorem for higher dimension, we shall proceed with double induction on the dimension \( n \) of the ambient space and the dimension \( k \) of the subset \( A \). Now we very briefly outline our strategy, which is here more complicated than for the affine plane. The canonical construction of an extension in the case \( k = 0 \) can be established similarly as in the case of the affine plane.

For \( k > 0 \), however, it is easier to explain the extension process for \( k = n \) than for \( 0 < k < n \). First, we consider a total skeleton of a 0-definable reparametrized open cell in \( K^k \) under consideration. Further, we extend a given 1-Lipschitz function \( f : A \to K \) to some 0-definable
subset of $K^n$ of dimension $k-1$ with finite projection onto $K^{k-1}$, which is related to the total skeleton. This is also connected with the use of Proposition 2.4 when $k < n$. Thus the proof of the case $k = n$ is easier to present, although the applied ideas and techniques are similar in both the cases. In this fashion, a final extension to the affine space $K^n$ is eventually provided.

Therefore we shall first give the proof in the case $k = n$, where the problem is reduced to extending 0-definable 1-Lipschitz functions from a reparametrized open cell in $K^n$.

We begin with the concept of a skeleton of a 0-definable reparametrized open cell $C$. This concept on the affine line was introduced in Section 3 by means of a canonical construction. We remind the reader that the distance between any two centers $c_\xi$ and $c_\lambda$ from the skeleton was greater than the minimum of the radii of the balls from the cells $C_\xi$ and $C_\lambda$.

The canonical construction of a total skeleton of a 0-definable reparametrized Lipschitz open cell $C$ on the affine plane was given in Section 4. First, after a suitable refining of $\pi_{<2}(C)$ up to a lower dimensional subset, the construction for the variable $x_2$ has been performed punctually with respect to the parameters $x_1$ (level 2). Next, a skeleton for the variable $x_1$ has been defined (level 1).

In the similar fashion (cf. Remark 4.1), we can construct a total skeleton in the case of a 0-definable reparametrized Lipschitz open cell $C$ in the affine space $K^k$. First, after a suitable refined decomposition of $\pi_{<k}(C)$ into reparametrized open Lipschitz cells in $K^{k-1}_{x<k}$ up to a lower dimensional subset, the construction for the variable $x_k$ is performed punctually, regarding the variables $x_1, \ldots, x_{k-1}$ as parameters (level $k$). Next we go downstairs, successively repeating the construction of a skeleton at the level $k-1$, $k-2$ and eventually 1. In this manner, a total skeleton of $C$ is achieved.

Again, likewise in Section 4, we shall make use of Remarks 4.3, 4.2 and Lemma 1.4. We shall follow similar lines of reasonings, but the proof now will be technically more complicated in comparison with the plane case from Section 4.

Case I. For pedagogical reasons mentioned above, we first prove the case $k = n$, assuming that the extension theorem holds both in the affine spaces of dimensions $< n$, and for subsets $A \subset K^n$ of dimension $k < n$. So consider a 0-definable 1-Lipschitz function $f : A \to K$ on a set $A \subset K^n$ of dimension $n$. By Corollary 2.3, we can thus assume
that \( A = C \subset K^n \) is a 0-definable reparametrized open cell:

\[
C = \bigcup_{\xi} C_\xi \subset K^n, \quad C_\xi = \sigma^{-1}(\xi), \quad \xi \in \sigma(D),
\]

where \( \sigma : C \to (Kv)^s \) is a 0-definable function. Each set

\[
C_\xi := \sigma^{-1}(\xi), \quad \xi \in \sigma(C),
\]

is thus a \( \xi \)-definable open Lipschitz cell with some center tuple \( c_\xi \) of the form

\[
(5.1) \quad C_\xi = \{ x \in K^n : (rv(x_i - c_\xi,i(x_{<i})))_{i=1}^n \in \mathcal{R}_\xi \}
\]

with \( \mathcal{R}_\xi \subset G(K)^n \).

As before, we can require that, after a suitable modification, \( f \) satisfy condition 1.1 with respect to the variables \( x \).

In view of Remark 1.4 and by the uniform extension theorem (Theorem 1.3) for the affine spaces of dimensions \(< n \), we can assume that the 0-definable family of 1-Lipschitz center tuples \( c_\xi \) is a 0-definable family of global 1-Lipschitz functions on the appropriate affine spaces.

As before, we can require that, after a suitable modification, \( f \) satisfy condition 1.1 with respect to the variables \( x \).

Suppose first that the center-tuple \( c_\xi \) of an open cell \( C_\xi \) vanishes, partition the affine space \( K^n \) into the \( n \) subsets \( \Delta_i \) along the coordinate hyperplanes \( H_i \), as in Remark 4.2, and put

\[
C^i_\xi := C_\xi \cap \Delta_i, \quad i = 1, \ldots, n.
\]

Similarly as images 4.3, the images

\[
(5.2) \quad f(C^i_\xi) = \bigcup_{x_i \in K^n} f(C^i_\xi \cap \pi^{-1}_i(x_{\neq i})), \quad i = 1, \ldots, n,
\]

are \( n \) families of \( \xi \)-definable cells with some \( \xi \)-definable centers \( d^i_\xi(x_{\neq i}) \).

It is clear that the domains of those centers are the following sets

\[
\text{dom} (d^i_\xi) = \{ x_{\neq i} \in K^{n-1} : \exists \lambda_i \in G(K) : (rv(x_1), \ldots, rv(x_{i-1}), \lambda_i, rv(x_{i+1}), \ldots, rv(x_n)) \in \mathcal{R}_\xi, \]

\[
|\lambda_i| \leq |x_1|, \ldots, |x_{i-1}|, |\lambda_i| < |x_{i+1}|, \ldots, |x_n| \};
\]

observe that they are some unions of boxes.

As in Section 4 (see also Remark 4.3), for any

\[
\lambda_{\neq i} \in G(K)^{n-1}, \quad i = 1, \ldots, n,
\]
let \( \rho_{\xi,i}(\lambda_{\neq i}) \) denote the Dedekind cut determined by the closed downwards subset of all elements of \(|K|\) which are smaller than the set
\[
\{|\lambda_i| \in |K|: \lambda_i \in G(K), \lambda = (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_n) \in R_{\xi}, |\lambda_i| \leq |\lambda_1|, \ldots, |\lambda_{i-1}|, |\lambda_i| < |\lambda_{i+1}|, \ldots, |\lambda_n| \}.
\]

Now the valuative Jacobian property (cf. [7, Lemma 2.8.5]) will be applied, likewise for estimates [4.4] in Section 4. Here, however, we will use its parameter version, which can be obtained from the original one via a routine model theoretical compactness argument. In this manner, there exist \( \xi \)-definable subsets
\[
Z_{\xi,i,j} \subset K_{\neq x_i}^{n-1} \quad \text{for} \quad i, j \in \{1, \ldots, n\}, i \neq j,
\]
such that the canonical projections
\[
\pi_{\neq i,j}: Z_{\xi,i,j} \to K_{\neq x_i,x_j}^{n-2}
\]
have finite fibers, and which satisfies the following condition:

Consider any set \( B \subset \text{dom} (d_{\xi}^i) \subset K_{\neq x_i}^{n-1} \) which is a ”ball” with respect to the variable \( x_j \), i.e. \( \pi_j(B) \) is a ball and \( \pi_{\neq i,j}(B) = \{b\} \) is a singleton. Suppose that \( B \) is next to the finite fiber
\[
Z_{\xi,i,j}(b) := Z_{\xi,i,j} \cap \pi_{\neq i,j}^{-1}(b)
\]
of \( Z_{\xi,i,j} \) over \( b \); we shall then say that the \( x_j \)-ball \( B \) is next to \( Z_{\xi,i,j} \). Then the quotients
\[
(5.3) \quad \frac{|d_{\xi}^i(y) - d_{\xi}^i(z)|}{|y - z|}
\]
are constant for all points \( y, z \in B, y \neq z \). Obviously, we have
\[
y_{\neq i,j} = z_{\neq i,j} = b \quad \text{and} \quad |y - z| = |y_j - z_j|.
\]

Now fix an \( i = 1, \ldots, n \) and consider any box
\[
B_i = \{x_{\neq i} : rv(x_{\neq i}) = \lambda_{\neq i}\} \subset \text{dom} (d_{\xi}^i), \lambda_{\neq i} \in \pi_{\neq i}(R_{\xi}).
\]
Since \( f \) satisfies condition \( \Box \), we get
\[
(5.4) \quad |d_{\xi}^i(y) - d_{\xi}^i(z)| \leq |y - z| \quad \text{for all} \quad y, z \in B_i, |y - z| \geq \rho_{\xi,i}(\lambda_{\neq i}).
\]

Put
\[
\tilde{Z}_{\xi,i,j} := \bigcup_{a \in Z_{\xi,i,j}} \{x : |x_j - a_j| < \rho_{\xi,i}(\lambda_{\neq i})\};
\]
for each \( j \neq i \) the set \( \tilde{Z}_{\xi,i,j} \) is a neighbourhood of \( Z_{\xi,i,j} \) of radius \( \rho_{\xi,i}(\lambda_{\neq i}) \) in direction of the \( x_j \)-axis. Now we reason likewise in Section 4 after estimates \( \Box \).
In the case where the value group $vK$ has no minimal element among the elements $> 1$, it follows directly from estimates $5.3$ that, for each $j \neq i$, the function $d^i_\xi$ is $1$-Lipschitz on every $x_j$-ball $B \subset B_i$ of radius $\geq \rho_{\xi,i}(\lambda \neq i)$, which is next to $Z_{\xi,i,j}$.

In the other case, however, the function $d^i_\xi$ is only $\epsilon$-Lipschitz on every such ball, which gives rise to increasing the Lipschitz constant in the extension theorem yet once more. This is taken into account in the statement of the extension theorem, and we shall further consider only the first case, as explained in the Introduction (cf. Remarks $1.4$ and $1.5$).

Therefore the function $d^i_\xi$ is $1$-Lipschitz on the set

$$B_i \setminus \bigcup_{j \neq i} \tilde{Z}_{\xi,i,j}.$$ 

But the extension theorem is assumed to hold for all dimensions $k < n$. We can thus extend the restriction of $f$ to the set

$$C^i_\xi \cap \pi^{-1}_{\neq i} \left( \bigcup_{j \neq i} Z_{\xi,i,j} \right)$$

to a global $\xi$-definable $\epsilon$-Lipschitz function $g_{\xi,i} : K^n \to K$. Observe that this can be done uniformly with respect to the parameter $\xi$ by model theoretical compactness.

Now modify the function $d_{\xi,i}|B_i$ by putting

$$d^i_{\xi}|B_i(x) = \begin{cases} 
  d^i_\xi(x) & \text{if } x \in B_i \setminus \bigcup_{j \neq i} \tilde{Z}_{\xi,i,j}, \\
  g_{\xi,i}(x) & \text{if } x \in \bigcup_{j \neq i} \tilde{Z}_{\xi,i,j}.
\end{cases}$$

After such modification, the function $\tilde{d}^i_\xi|B_i$ remains a $\xi$-definable $\epsilon$-Lipschitz center of the $i$-th family of cells from among the families $5.2$. It is easy to verify that, after gluing such modifications for all boxes $B_i$ as above, we achieve new $\xi$-definable $\epsilon$-Lipschitz centers

$$\tilde{d}^i_\xi : \text{dom } (d^i_\xi) \to K$$

doing that family of cells. For simplicity we drop tilde over $d^i_\xi$. The cost of increasing the Lipschitz constant at this stage of the proof was taken into account in the statement of the extension theorem, as explained in the Introduction. We can thus assume that the above new centers are $1$-Lipschitz.

At this stage, we shall proceed likewise in the proof of the extension theorem for the affine plane in Section 4. Applying Lemma $4.4$ and
the map $\Phi_\xi$ from Remark 4.3, we can as before define $\xi$-definable 1-Lipschitz functions $d^i_\xi$ whose domains are subsets of the following $n$ sets (of dimension $n - 1$):

$$\text{graph} \left( c_i(x_{\neq i}) \right) \subset K^n, \quad i = 1, \ldots, n.$$ 

Since $C$ is a reparametrized Lipschitz open cell with total skeleton, we see as before that

$$\text{dom} \left( d^i_\xi \right) \cap C = \emptyset, \quad i = 1, \ldots, n,$$

and that the function

$$\tilde{f}(x) = \begin{cases} 
    f(x) & \text{if } x \in C, \\
    d^i_\xi(x) & \text{if } x \in \text{dom} \left( d^i_\xi \right), \quad i = 1, \ldots, n
\end{cases}$$

is a 0-definable 1-Lipschitz extension of $f$ satisfying condition 1.1.

The cost of increasing the Lipschitz constant at this stage of the proof was taken into account in the statement of the extension theorem, as explained in the Introduction. So we shall consider only the case where there is no minimal element among the elements $> 1$. Since we assume that the extension theorem holds in the case where the set $A$ is of dimension $< n$, there exists a 0-definable $\epsilon$-Lipschitz function $g : K^n \to K$ that agrees with the function $\tilde{f}$ on the set

$$(5.5) \quad d(C) := \bigcup_{\xi} \bigcup_{i=1}^n \text{dom} \left( d^i_\xi \right).$$

This reduces the problem to the case where $f$ is a $\epsilon$-Lipschitz function which vanishes on set 5.4. Again, since the reparametrized open cell $C$ is with total skeleton, the function $F : K^n \to K$ given by the formula

$$F(x) = \begin{cases} 
    f(x) & \text{if } x \in C \cup d(C), \\
    0 & \text{otherwise},
\end{cases}$$

is a 0-definable $\epsilon$-Lipschitz extension of the function $f$ we are looking for.

The uniform version of the extension theorem follows as before via a routine compactness argument, because the foregoing construction is canonical, concluding the case $k = n$.

**Case II.** Fix $k \in \{1, \ldots, n - 1\}$ and assume that the extension theorem holds both in the affine spaces of dimensions $< n$, and for subsets $A \subset K^n$ of dimension $< k$.

So consider a 0-definable 1-Lipschitz function $f : A \to K$ on a set $A \subset K^n$ of dimension $k$. By Corollary 2.3 and Proposition 2.4, we can
assume that $A$ is a disjoint union of the form

$$E = \bigcup_{\xi \in \Xi} \bigcup_{j=1}^{d} \text{graph } \phi_{\xi,j} \subset K^n,$$

where

$$\phi_{\xi,j} : D_\xi \to K_{x \leq k}^{n-k}, \quad \xi \in \Xi \subset RV(K)^t, \quad j = 1, \ldots, d,$$

are $\xi$-definable 1-Lipschitz functions and

$$D = \bigcup_{\xi \in \Xi} D_\xi \subset K_{x \leq k}^k$$

is a 0-definable reparametrized open cell in $K_{x \leq k}^k$. It will be convenient to denote the coordinates in $K_{x \leq k}^k$ by $u = (u_1, \ldots, u_k)$.

Again, we can require that, after a suitable modification, $f$ satisfy condition (1.1) with respect to the variables $u$. Now we consider open cells $D_\xi$ in $K^k$ with center-tuples $c_\xi$. When a center tuple $C_\xi$ vanishes, we partition, as before, the affine space $K^k$ into the $k$ subsets $\Delta_i$ along the coordinate hyperplanes $H_i$, as in Remark 4.2, and put

$$D^i_\xi := D_\xi \cap \Delta_i, \quad i = 1, \ldots, k.$$

In Case I, we analyzed $n$ families of $\xi$-definable cells with some $\xi$-definable centers $d^i_\xi(x_{\neq i})$ (cf. 5.2 ff.). But here we shall analyze $k \cdot d$ images

$$(f \circ \tilde{\phi}_{\xi,j})(C^i_\xi) = \bigcup_{u_i \in K} (f \circ \tilde{\phi}_{\xi,j})(C^i_\xi \cap p_i^{-1}(u_{\neq i})), \quad i = 1, \ldots, k, \quad j = 1, \ldots, d,$$

where $\tilde{\phi}_{\xi,j}(u) := (u, \phi_{\xi,j}(u))$ and $p_i : K^k_u \to K$ is the canonical projection onto the $u_i$-axis. Again, by Lemma 3.7 and model theoretical compactness, we get $k \cdot d$ families of $\xi$-definable cells with some $\xi$-definable centers $d^i_{\xi,j}(u_{\neq i})$. Their domains are the following sets

$$\text{dom } (d^i_{\xi,j}) = \{u_{\neq i} \in K^{k-1} : \exists \lambda_i \in G(K) :$$

$$rv(u_1), \ldots, rv(u_{i-1}), \lambda_i, rv(u_{i+1}), \ldots, rv(u_k) \in R_\xi,$$

$$|\lambda_i| \leq |u_1|, \ldots, |u_{i-1}|, \quad |\lambda_i| < |u_{i+1}|, \ldots, |u_k|\},$$

which depend only on $\xi$ and $i$, and are some unions of boxes.

However, we encounter here a subtle point. For some fixed indices $\xi$ and $i = 1, \ldots, k$, and for some points $u_{\neq i}$, the cells

$$(f \circ \tilde{\phi}_{\xi,j})(C^i_\xi \cap p_i^{-1}(u_{\neq i})), \quad j = 1, \ldots, d,$$
may possess a common center; say for some \( j \) and \( l \). We should require that
\[
d_{\xi,j}(u_{\neq i}) = d_{\xi,l}(u_{\neq i})
\]
whenever this happens. This definable condition will not affect the line of reasonings from Case I, and will ensure the control over the metric behaviour of those functions.

As in Case I, those centers can be corrected, taking into account Remark 1.4, to new 1-Lipschitz ones by means of the parametric version of the valuative Jacobian property. Further, we can mutatis mutandis carry over the reasoning from Case I to achieve a 0-definable 1-Lipschitz extension
\[
f(x) = \begin{cases} f(x) & \text{if } x \in C, \\ d_{\xi,j}(x) & \text{if } x \in \text{dom}(d_{\xi,i}), \ i = 1, \ldots, n, \ j = 1, \ldots, d, \end{cases}
\]
of \( f \) satisfying condition 1.1.

Hence and by the induction assumption on the dimension \( k \), the problem is again reduced to the case where \( f \) vanishes on the set
\[
d(C) := \bigcup_{\xi} \bigcup_{i=1}^{k} \bigcup_{j=1}^{d} \text{dom}(d_{\xi,i}).
\]

Then we can extend \( f \) by zero through the set
\[
(K^k \setminus D) \times K^{n-k}
\]
and fiberwise over the open cell \( D \) in \( K^k \). The latter is obtained by the canonical construction of an extension from a subset of \( K^{n-k} \) of cardinality \( d \) applied to the set
\[
\bigcup_{\xi \in \Xi} \bigcup_{j=1}^{d} \text{graph} \phi_{\xi,j} \subset K^n,
\]
regarded here as a family of finite subsets of \( K^{n-k} \) with respect to the parameters \( u = x_{\leq k} \in D \). We leave the details to the reader.

Again, the uniform version of the extension theorem follows via a routine compactness argument, because the foregoing construction is canonical. This completes the proof of Theorem 1.3.
References

[1] R. Cluckers, G. Comte, F. Loeser, Non-Archimedean Yomdin–Gromov parametrizations and points of bounded height, Forum Math. Pi 3 (2015), e5.

[2] R. Cluckers, A. Forey, F. Loeser, Uniform Yomdin–Gromov parametrizations and points of bounded height in valued fields, Algebra Number Theory 14 (2020), 1423–1456.

[3] R. Cluckers, I. Halupczok, S. Rideau, Hensel minimality I, arXiv:1909.13792 [math.LO] (2019).

[4] R. Cluckers, I. Halupczok, S. Rideau, Hensel minimality II: mixed characteristic and a Diophantine application, arXiv:2104.09475 [math.LO] (2021).

[5] R. Cluckers, L. Lipshitz, Fields with analytic structure, J. Eur. Math. Soc. 13 (2011), 1147–1223.

[6] R. Cluckers, F. Martin, A definable p-adic analogue of Kirszbraun’s theorem on extension of Lipschitz maps, J. Inst. Math. Jussieu 17 (2018), 39–57.

[7] J. Denef, L. van den Dries, p-adic and real subanalytic sets, Ann. Math. 128 (1988), 79–138.

[8] L. van den Dries, A. Macintyre, D. Marker, The elementary theory of restricted analytic fields with exponentiation, Ann. Math. 140 (1994), 183–205.

[9] I. Halupczok, Non-Archimedean Whitney stratifications, Proc. London Math. Soc. (3) 109 (2014), 1304–1362.

[10] M.D. Kirszbraun, Über die zusammenziehende und Lipschitzsche Transformationen, Fund. Math. 22 (1934), 77–108.

[11] T. Kuijpers, Lipschitz extension of definable p-adic functions Math. Log. Q. 61 (2015), 151–158.

[12] J. Kollár, K. Nowak, Continuous rational functions on real and p-adic varieties, Math. Zeit. 279 (2015), 85–97.

[13] K.J. Nowak, Some results of algebraic geometry over Henselian rank one valued fields, Sel. Math. New Ser. 23 (2017), 455–495.

[14] K.J. Nowak, Definable transformation to normal crossings over Henselian fields with separated analytic structure, Symmetry 11 (7) (2019), 934.

[15] K.J. Nowak, A closedness theorem and applications in geometry of rational points over Henselian valued fields, J. Singul. 21 (2020), 212–233.

[16] K.J. Nowak, A closedness theorem over Henselian fields with analytic structure and its applications. In: Algebra, Logic and Number Theory, Banach Center Publ. 121, Polish Acad. Sci. (2020), 141–149.

[17] K.J. Nowak, Tame topology and desingularization in Hensel minimal structures, arXiv:2103.01836 [math.AG] (2021).

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