BEYOND THE ERDŐS MATCHING CONJECTURE

PETER FRANKL AND ANDREY KUPAVSKII

Abstract. A family $\mathcal{F} \subset \binom{[n]}{k}$ is $U(s,q)$ of for any $F_1, \ldots, F_s \in \mathcal{F}$ we have $|F_1 \cup \ldots \cup F_s| \leq q$. This notion generalizes the property of a family to be $t$-intersecting and to have matching number smaller than $s$.

In this paper, we find the maximum $|\mathcal{F}|$ for $\mathcal{F}$ that are $U(s,q)$, provided $n > C(s,q)k$ with moderate $C(s,q)$. In particular, we generalize the result of the first author on the Erdős Matching Conjecture and prove a generalization of the Erdős–Ko–Rado theorem, which states that for $n > s^2k$ the largest family $\mathcal{F} \subset \binom{[n]}{k}$ with property $U(s,s(k-1)+1)$ is the star and is in particular intersecting. (Conversely, it is easy to see that any intersecting family in $\binom{[n]}{k}$ is $U(s,s(k-1)+1)$.)

We investigate the case $k = 3$ more thoroughly, showing that, unlike in the case of the Erdős Matching Conjecture, in general there may be 3 extremal families.

1. Introduction

Let $[n] = \{1, \ldots, n\}$ be the standard $n$-element set and let $\binom{[n]}{k}$ denote the collection of all its $k$-subsets, $1 \leq k < n$. A $k$-graph (or a $k$-uniform family) $\mathcal{F}$ is simply a subset of $\binom{[n]}{k}$. Let us recall two fundamental results from extremal set theory.

**Theorem 1** (Erdős–Ko–Rado Theorem [6]). Let $t$ be a positive integer, $t \leq k$ and suppose that $|F \cap F'| \geq t$ for all pairs of edges of the $k$-graph $\mathcal{F}$. Then

$$|\mathcal{F}| \leq \binom{n-t}{k-t}$$

holds for all $n \geq n_0(k,t)$.

The family $\{F \in \binom{[n]}{k} : [t] \subset F\}$ shows that (1) is the best possible.

Let $p, r$ be non-negative integers with $p \geq r$. Define

$$\mathcal{A}_{p,r} := \mathcal{A}(p,r,n,k) := \{A \in \binom{[n]}{k} : |A \cap [p]| \geq r\}.$$  

(We omit $n, k$ when they are clear from the context.) The Erdős–Ko–Rado family mentioned above is simply $\mathcal{A}_{r,r}$. These families arise in several important results and open questions in extremal set theory.

The following extension of the Erdős–Ko–Rado theorem was conjectured by Frankl [8], proved in many cases by Frankl and Füredi [13], and nearly 20 years later proved in full generality by Ahlswede and Khachatrian [1].

**Theorem 2** (Complete Intersection Theorem [1]). Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is $t$-intersecting, $n \geq 2k-t$. Then

$$|\mathcal{F}| \leq \max_{0 \leq i \leq k-t} |\mathcal{A}_{2i+t,i+t}|.$$  

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Moreover, unless \( n = 2k, t = 1 \) or \( F \) is isomorphic to \( A_{2i+t,i+t} \), the inequality is strict.

For a \( k \)-graph \( F \) let \( \nu(F) \) denote its matching number, that is, the maximum number of pairwise disjoint edges in \( F \). Obviously, for every positive integer \( s \), \( \nu\left(\binom{[s+1]k}{k-1}\right) = s \).

The Erdős Matching Conjecture (the EMC for short) is one of the central open problems in extremal set theory.

**Conjecture 1** (EMC\([4]\)). Let \( n \geq (s + 1)k \) and \( F \subset \binom{[n]}{k} \). If \( \nu(F) \leq s \) then

\[
|F| \leq \max\{ |A_{s,1}|, |A_{(s+1)k-1,k}| \}.
\]

We note that both families appearing on the right hand side have matching number \( s \). It is known to be true for \( k \leq 3 \) (cf. \([5], [12]\)).

We should mention that Erdős proved \((1)\) for \( n \geq n_0(k, s) \). For such values of \( n \) the bound is

\[
|F| \leq \binom{n}{k} - \binom{n-s}{k}.
\]

Improving earlier bounds \([3], [16], [15]\) was proved by the first author in \([11]\) for \( n \geq (2s+1)k \). For \( s \geq s_0 \) this was further improved by the present authors to \( n \geq \frac{1}{7}sk \) (cf. \([15]\)).

Both the above results forbid certain intersection patterns (two sets intersecting in less than \( t \) elements or \( s+1 \) sets having pairwise empty intersection). In the present paper, we study restrictions on the maximum size of the union, rather than intersections, of \( s+1 \) edges of \( F \). This setting permits to unify the above two results and to formulate a natural new problem.

**Definition 3.** Let \( k, s \geq 2 \) and \( k \leq q < sk \) be integers. A \( k \)-graph \( F \subset \binom{[n]}{k} \) is said to have property \( U(s, q) \) if

\[
|F_1 \cup \ldots \cup F_s| \leq q
\]

for all choices of \( F_1, \ldots, F_s \in F \). For shorthand, we will also say \( F \) is \( U(s, q) \) to refer to this property.

With this definition, \( F \) being \( U(2, 2k-t) \) is equivalent to \( F \) being \( t \)-intersecting and, similarly, \( F \) being \( U(s + 1, (s + 1)k - 1) \) is equivalent to \( \nu(F) \leq s \).

**Definition 4.** Let \( n, k, s, q \) be integers, \( n > q \geq k, sk > q, s \geq 2 \). Define \( m(n, k, s, q) \) as the maximum of \(|F|\) over all \( F \subset \binom{[n]}{k} \), where \( F \) has property \( U(s, q) \).

With this terminology, Theorem \([2]\) states that

\[
m(n, k, 2, 2k-t) = \max_{0 \leq i \leq k-t} |A_{2i+t,i+t}|,
\]

and the EMC can be formulated as

\[
m(n, k, s+1, (s+1)k - 1) = \max\{ |A_{s,1}|, |A_{(s+1)k-1,k}| \} = \max\left\{ \binom{n}{k} - \binom{n-s}{k}, \binom{k(s+1)-1}{k} \right\}.
\]

For all choices of \( A_1, \ldots, A_s \in A_{p,r} \), we have

\[
|A_1 \cup \ldots \cup A_s| \leq p + s(k-r).
\]

Thus, the EMC and the Complete Intersection Theorem may be seen as particular cases of the following general conjecture.

**Conjecture 2.** For all choices \( n, k, s, q \) one has \( m(n, k, s, q) = |A_{p,r}| \) for an appropriate choice of \( p, r > 0 \) with \( A_{p,r} \) having property \( U(s, q) \). More precisely, if \( q = (k-r)s+p \) with \( r \leq p \leq s+r-2 \), then \( m(n, k, s, q) = \max_{0 \leq i \leq k-r} |A_{p+i,p+i}| \).
In particular, Theorem\(^2\) is the case \(p = r = t, s = 2\) of the conjecture.

Let us remark that the non-uniform version of Conjecture\(^2\) goes back to the PhD dissertation of the first author (cf. also \([7]\) and \([9]\) where it is proved in a certain range). Let us also mention that the \(s = 2\) case of the non-uniform case is a classical result of Katona \([17]\). Let us give some additional motivation for the question. Recall the following definition.

**Definition 1.** Consider two sets \(F_i = (a^i_1, \ldots, a^i_k)\) with \(a^i_1 < a^i_2 < \ldots < a^i_k\) for \(i = 1, 2\). Then \(F_1 <_s F_2\) iff \(a^1_j \leq a^2_j\) for every \(j \in [k]\). We say that a family \(\mathcal{F} \subseteq \binom{[n]}{k}\) is shifted if \(F \in \mathcal{F}\) and \(G <_s F\) implies \(G \in \mathcal{F}\).

For many extremal problems, including the ones mentioned above, it is sufficient to prove the statements for shifted families. Let \(\mathcal{F} \subseteq \binom{[n]}{k}\) be a shifted family satisfying \(\nu(\mathcal{F}) \leq s\). In dealing with such families one often considers subfamilies of the form \(\mathcal{F}([\bar{r}]) := \{F \in \mathcal{F} : F \cap [r] = \emptyset\}\). As we noted above, \(\nu(\mathcal{F}) < s\) is equivalent to \(\mathcal{F}\) being \(U(s, sk - 1)\). Due to shiftedness, \(\mathcal{F}([\bar{r}])\) has property \(U(s, ks - 1 - r)\).

There is a certain hierarchy for properties \(U(s, ks - r)\) in the range \(1 \leq r < s\). To explain it, recall that \(A_{p,1}\) has property \(U(s, (k - 1)s + p)\) for \(1 \leq p < s\).

**Proposition 5.** If \(m(n, k, s, (k - 1)s + p)) = \binom{n}{k} - \binom{n-p}{k}\) then
\[
m(n + 1, k, s, (k - 1)s + p + 1) = \binom{n + 1}{k} - \binom{n - p}{k}.
\]

**Proof.** Let \(\mathcal{F} \subseteq \binom{[n+1]}{k}\) be a shifted family satisfying property \(U(s, (k - 1)s + p + 1)\). Then \(\mathcal{F}(1)\) is \(U(s, (k - 1)s + p)\) by shiftedness, and so \(|\mathcal{F}(1)| \leq \binom{n}{k} - \binom{n-p}{k}\) follows. Since \(|\mathcal{F}(1)| \leq \binom{n}{k-1}\), we get \(|\mathcal{F}| = |\mathcal{F}(1)| + |\mathcal{F}(1)| \leq \binom{n+1}{k} - \binom{n-p}{k}\). \(\square\)

In the following theorem, we determine \(m(n, k, s + 1, q)\) for all \(q\) and \(n > C(s)k\). Naturally, we are only interested in the values \(q > k\).

**Theorem 6.** Fix some integers \(n, k, s, p, r,\) such that \(1 \leq r \leq k\) and \(r \leq p \leq s + r - 1\). Suppose that \(\mathcal{F} \subseteq \binom{[n]}{k}\) has property \(U(s + 1, q)\) for \(q = (k - r)(s + 1) + p\). If \(n \geq p + 1 + (s + f(s, p, r))(k - r)\), where
\[
f(s, p, r) := \frac{s(s+1)}{\max\{1, r-1\}} \cdot \sum_{j=0}^{r-1} \frac{s^{r-1-j} \binom{r}{j}}{\binom{p}{j}},
\]
then
\[|\mathcal{F}| \leq |A_{p,r}|.\]

In particular, for \(r = 1\) and \(p = s\) we retrieve the bound on the size of the family as in the Erdős Matching Conjecture, while for \(r = t\) and \(p = 0\) we get the bound from the Erdős–Ko–Rado theorem. Note that \(|A_{p,r}| = \sum_{i=r}^{k} \binom{p}{i} \binom{n-p}{k-i} \sim \binom{p}{r} \binom{n-p}{k-r}\).

**Remarks.** The function \(f(s, p, r)\) looks complicated, which is partially due to the generality of the statement. Let us illustrate it on a few examples. First, if one substitutes \(r = 1\) and \(p = s\), then \(f(s, p, r) = s(s+1)\frac{1}{s} = s + 1\), and we get the bound \(n \geq s + 1 + (2s + 1)(k - 1)\), exactly the restriction in \([11]\) Theorem 1.1], which guarantees that the Erdős Matching Conjecture holds in this range. For \(r = 1\) and \(1 \leq p \leq s\), we get that \(f(s, p, r) = s(s+1)/p\), and, roughly speaking, Theorem \([4]\) holds for \(n \geq 2s^2k/p\).

\(^1\)One may simply say \(n > C(s)k\) instead, if the exact form of the dependence is not important.
For $r = p = 1$, the theorem implies that for $n \geq s(s+1)k$ any family satisfying the $UP(s + 1, (s+1)(k-1) + 1)$ condition is at most the full star (which is the largest intersecting family). Thus, Theorem 6 can be seen as a sharpening of the Erdős–Ko–Rado theorem. (In any choice of $k$-element sets $E_1, \ldots, E_{s+1}$ satisfying $|E_1 \cup \ldots \cup E_{s+1}| > (k-1)(s+1) + 1$ there must be two that are disjoint, but not vice versa.)

**Sharpness.** The aforementioned rough bound $n \geq 2s^2k/p$ is sharp up to a small constant factor: it is not difficult to see that $\mathcal{A}_{s+p,2}$ is bigger than $\mathcal{A}_{p,1}$ for $n < cs^2k/p$ with, say, $c = 1/4$. We will have more precise results for $k = 3$ in Theorem 19.

The proof of Theorem 6 is given in Section 3. In the next section, we resolve the problem for some small values of $k, s, q$. Most of the results are devoted to the case $k = 3$.

## 2. Results for Small $k, s, q$

### 2.1. The complete solution for $k = 2$

Throughout this section $\mathcal{F} \subset \binom{[n]}{2}$ is a 2-graph having property $U(s, s+r)$ with $1 \leq r < s, n > s+r$. Recall the definition (2). Then $\mathcal{A}(r, 1, n, 2)$ and $\mathcal{A}(s+r, 2, n, 2)$ are $U(s, s+r)$.

**Theorem 7.** For all values of $n, s, r, n > s+r, s > r \geq 1$

$$|\mathcal{F}| \leq \max \{|\mathcal{A}(r, 1, n, 2)|, |\mathcal{A}(s+r, 2, n, 2)|\}. \quad (9)$$

The case $r = s - 1$ of the theorem is a classical result of Erdős and Gallai, determining the maximum number of edges in a graph without $s$ pairwise disjoint edges.

**Proof.** Let $\cup \mathcal{F} := \cup_{F \in \mathcal{F}} F$. In the case $|\cup \mathcal{F}| \leq s+r$, we have $|\mathcal{F}| \leq \binom{s+r}{2}$. From now on we suppose that $|\cup \mathcal{F}| > s+r$.

**Claim 8.** $\nu(\mathcal{F}) \leq r$.

**Proof.** Indeed, if we can find $F_1, \ldots, F_{r+1}$ with $|F_1 \cup \ldots \cup F_{r+1}| = 2(r+1)$, then $|\cup \mathcal{F}| \geq s+r+1$ enables us to find $s-r-1$ edges $F_{r+2}, \ldots, F_s$ such that $|F_1 \cup \ldots \cup F_s| \geq |F_1 \cup \ldots \cup F_{r+1}| + s + r - 1 = s + r + 1$. This contradicts $U(s, s+r)$.

Applying the Erdős–Gallai theorem 5 (for a short proof, see 2 or the next section) to $\mathcal{F}$ yields $$|\mathcal{F}| \leq \max \left\{ \binom{2r+1}{2}, \binom{n}{2} - \binom{n-r}{2} \right\}$$

and proves (9).

### 2.2. The case $q \leq 2s + k - 3$

Let us first state a very simple result.

**Claim 9.** $m(n, k, s, q) = \binom{n}{k}$ for $k \leq q \leq k + s - 2$.

**Proof.** Let $\mathcal{F} \subset \binom{[n]}{k}$ have property $U(s, q)$. Consider $X := \cup_{F \in \mathcal{F}} F$. If $|X| > q$ holds then we can easily find $F_1, \ldots, F_s$ satisfying $|F_1 \cup \ldots \cup F_s| \geq k+s-1 > q$, a contradiction. Thus $|\mathcal{F}| \leq \binom{|X|}{k} \leq \binom{n}{k}$,

**Theorem 10.** Let $\mathcal{F} \subset \binom{[n]}{k}$ be $U(s, s+t+k-3)$ for $t \in [2, s]$. Then

$$|\mathcal{F}| \leq \max \{ |\mathcal{A}(s+t+k-3, k, n, k)|, |\mathcal{A}(t+k-3, k-1, n, k)| \}.$$ 

Let us note that this theorem for $k = 2$ includes the aforementioned Erdős–Gallai result, as well as the result from the previous subsection. The proof is a generalization of the proof in 2.
Proof. W.l.o.g. assume that \( \mathcal{F} \) is shifted. Consider the collection of sets \( \{(1, \ldots, k-2, i+k-2, 2t+1-i+k-2) : i \in \{t\}\} \cup \{(1, \ldots, k-1, j) : j \in \{2t+k-1, s+t+k-2\}\} \). These are \( t+(s-t)=s \) sets in total. Moreover, their union is \([s+t+k-2]\), and, therefore, one of these sets is not in \( \mathcal{F} \).

- Since \( \mathcal{F} \not\subset \mathcal{A}(s+t+k-3,k,n,3) \) we get \((1, \ldots, k-1, s+t+k-2) \in \mathcal{F} \), which implies that all sets of the form \((1, \ldots, k-1, t')\), \(k \leq t' \leq s+t+k-2\) are in \( \mathcal{F} \);
- since \( \mathcal{F} \not\subset \mathcal{A}(t+k-3,k-1,n,k) \) we get \((1, \ldots, k-2, t+k-2, t+k-1) \in \mathcal{F} \).

Therefore, one of the sets \((1, \ldots, k-2, \ell+k-2, 2t+1-\ell+k-2) \) for some \( \ell \in [2, t-1] \) is missing. This implies that \( \mathcal{F} \subset \binom{[2t-\ell+k-2]}{k} \cup \binom{[\ell+k-3]}{k-1} \times \binom{[2t-\ell+k-1,n]}{k} \). Thus
\[
|\mathcal{F}| \leq \binom{2t-\ell+k-2}{k} + \binom{\ell+k-3}{k-1} (n-2t+\ell-k+2) =: f(\ell).
\]

We note that \( f(t) = |\mathcal{A}(t+k-3,k-1,n,k)| \), while \( f(1) = \binom{2t+k-3}{k} \leq |\mathcal{A}(s+t+k-3,k,n,3)| \).

Therefore, to conclude the proof of the theorem, it suffices to show that \( f(\ell) \) is an integer convex function for \( \ell \in [t] \). (Recall that \( g(x) \) is an integer convex function on a certain interval if \( 2g(x) \leq g(x-1) + g(x+1) \) on that interval.) Note that if function on an interval is convex then it is integer convex on the same interval.

Let us note that \( g'(x) = -g'(x) \), yielding \( g''(x) = g''(x) \). The function \( \binom{x}{k} \) is convex for \( x \geq k \), and thus the first term \( \binom{2t-\ell+k-2}{k} \) is convex for \( \ell \in [t] \). The second term \( f_1(\ell) \) in the definition of \( f(\ell) \) is integer convex. Indeed, for any \( \ell \in [t] \), we have \( f_1(\ell-1) + f_1(\ell+1) \geq \left( \binom{\ell-1+k-3}{k-1} + \binom{\ell+1+k-3}{k-1} \right) (n-2t+\ell-k+2) \), which is bigger than \( 2\binom{\ell+k-3}{k-1} (n-2t+\ell-k+2) \) due to integer convexity of \( \binom{x}{k-1} \) for \( x \in \mathbb{Z} \). We conclude that \( f(\ell) \) is integer convex for \( \ell \in [t] \).

2.3. The case \( k = 3 \), \( q = 2s+1 \). For \( k = 2 \), (each of) the theorems from the previous two sections resolve the problem completely. For \( k = 3 \), Theorem 10 covers the cases with \( q \leq 2s \). Thus, the case mentioned in the title of this section is the “first” case, not covered by Theorem 10. As we would see, the situation changes quite significantly: instead of having two potential extremal families, we shall have three.

Consider a family \( \mathcal{F} \subset \binom{[n]}{3} \) that is \( U(s, 2s+1) \). There are three natural candidates for extremal families here (cf. \( \mathcal{F} \)):
\[
\mathcal{F}_1 := \mathcal{A}(1,1,n,3), \quad \mathcal{F}_2 := \mathcal{A}(s+1,2,n,3), \quad \mathcal{F}_3 := \mathcal{A}(2s+1,3,n,3).
\]

In particular, \( \mathcal{F}_1 \) contains all sets that contain 1 and \( \mathcal{F}_3 := \binom{[2s+1]}{3} \). It is easy to see that all three families are \( U(s, 2s+1) \). Let us make the following conjecture.

Conjecture 3. If \( \mathcal{F} \subset \binom{[n]}{3} \) is \( U(s, 2s+1) \) then \( |\mathcal{F}| \leq \max_{i \in [3]} |\mathcal{F}_i| \).

Unlike in the case of the Erdős Matching Conjecture, each of the three families is the largest for each \( s \geq 3 \) in a certain interval depending on \( n \). Let us show that. We have
\[
|\mathcal{F}_1 \setminus \mathcal{F}_2| = \binom{n-s-1}{2}, \quad |\mathcal{F}_2 \setminus \mathcal{F}_1| = \binom{s}{2} \binom{n-s-1}{1} + \binom{s}{3},
\]
therefore, \( |\mathcal{F}_1| \geq |\mathcal{F}_2| \) for roughly \( n \geq s^2 \), and smaller otherwise. Next, we have
\[
|\mathcal{F}_2 \setminus \mathcal{F}_3| = \binom{s+1}{2} \binom{n-2s-1}{1}, \quad |\mathcal{F}_3 \setminus \mathcal{F}_2| = \binom{s}{3} + \binom{s}{2} \binom{s+1}{1},
\]
and \( |\mathcal{F}_3| \geq |\mathcal{F}_2| \) iff \( 3(s+1)(n-3s) \leq (s-1)(s-2) \). For large \( s \), this happens roughly for \( n \leq \frac{10}{3}s \).
**Theorem 11.** Assume that $F \subset \binom{n}{3}$ is $U(s, 2s + 1)$ and, moreover, $n \leq 3s$ and $s \geq 10$. Then

$$|F| \leq |F_3|.$$ 

Let us also note that Theorem 6 gives $|F| \leq |F_1|$ for $n \geq 2s^2 + 4s + 2$. In the Section 2.5 we will show (in a more general setting) that in a certain range $|F| \leq |F_2|$ for any $F$ that is $U(s, 2s + 1)$.

**Proof.** Let $F \subset \binom{n}{3}$ be shifted and $U(s, 2s + 1)$, $|F| > \max |F_i|$, $i = 1, 2, 3$. In particular, $(1, 2, 2s + 2) \in F$, $(2, 3, 4) \in F$.

Consider $G_2 := F_2 \setminus F_3$, $G_1 := F_1 \setminus (F_2 \cup F_3)$. Then

$$G_2 = \{(a, b, c) : b \leq s + 1, 2s + 1 < c \leq n\},$$
$$G_1 = \{(1, b, c) : b \geq s + 2, c \geq 2s + 2\}.$$ 

**Claim 12.** $(1, s + 3, 2s + 2) \notin F$. Consequently, $|G_1 \cap F| \leq n - 2s - 1$.

**Proof.** Otherwise, $(1, s + 3 - \ell, 2s + 2 - \ell) \in F$ for $\ell = 0, 1, \ldots, s - 2$. Together with $(2, 3, 4)$ these are $s$ sets with union $[2s + 2]$. As for the second part, only sets from $G_1$ with $b = s + 2$ may be included in $F$. 

**Remark.** Should $(2, 3, s + 3) \in F$, $G_1 \cap F = \emptyset$ would follow in a similar way.

**Claim 13.** We have $\nu(F) \leq \frac{s + 1}{2}$.

**Proof.** Indeed, otherwise there are $\lceil s/2 \rceil + 1$ sets in $F$, whose union is $\lceil 3s/2 \rceil + 3$. Together with $\lceil s/2 \rceil - 1$ sets $(1, 2, 3s/2 + 4), \ldots, (1, 2, 2s + 2)$, we get $s$ sets whose union is $[2s + 2]$. 

Since the Erdős Matching Conjecture holds for $k = 3$ (cf. [12]), we have $|F_3 \cap F| \leq \left(\frac{2s + 1}{3}\right) - \left(\frac{2s + 1 - \frac{s + 1}{2}}{3}\right)$.

To prove $|F| \leq |F_3|$ we need to show

$$\left(\frac{2s + 1}{3} - \frac{s + 1}{2}\right) > |G_2| + |G_1 \cap F|. \tag{10}$$

We have $|G_2| \leq \left(\frac{s + 1}{2}\right)(n - 2s - 1)$. Using Claim 12 we get that the right hand side of (10) is at most

$$\left(\left(\frac{s + 1}{2}\right) + 1\right)(n - 2s - 1).$$

On the other hand, the left hand side of (10) is

$$\left(\frac{9}{16}s^2 - \frac{1}{16}\right)(s - 1).$$

Given that $n \leq 3s$, the left hand side of (10) is bigger than the right hand side if

$$\left(\frac{s + 1}{2}\right) + 1 \leq \frac{9}{16}s^2 - \frac{1}{16},$$

which holds for any $s \geq 10$. 

□
2.4. The complete solution for $k = s = 3$, $q = 7$. In this section, we prove a stronger form of Conjecture for these parameters.

**Theorem 14.** Let $\mathcal{F} \subset \binom{[n]}{3}$ be shifted and satisfy $U(3, 7)$. Then

$$\mathcal{F} < \max_{i \in [3]} |\mathcal{F}_i|,$$

(11)

unless $\mathcal{F} \in \{\mathcal{F}_i : i \in [3]\}$.

Simple computation shows that $\max_{i \in [3]} |\mathcal{F}_i|$ is given by $|\mathcal{F}_3|$ for $n \leq 9$, $|\mathcal{F}_2|$ for $10 \leq n \leq 11$, and $|\mathcal{F}_1|$ for $n \geq 12$.

**Proof of Theorem 14.** The statement is obvious for $n \leq 7$ since $\mathcal{F}_3 = \binom{[7]}{3}$ is $U(3, 7)$. In what follows, we assume that $n \geq 8$. Arguing indirectly, assume that $\mathcal{F}$ is not contained in any of $\mathcal{F}_i$.

Then, by shiftedness, this implies that

(i) $(1, 2, 8) \in \mathcal{F}$;
(ii) $(1, 5, 6) \in \mathcal{F}$;
(iii) $(2, 3, 4) \in \mathcal{F}$.

**Claim 15.** $(1, 6, 8) \notin \mathcal{F}$ and $(2, 4, 8) \notin \mathcal{F}$.

**Proof.** Using shiftedness, if $(1, 6, 8) \in \mathcal{F}$ then $(1, 5, 7) \in \mathcal{F}$. Together with (iii) this contradicts $U(3, 7)$.

Similarly, if $(2, 4, 8) \in \mathcal{F}$ then $(1, 3, 7) \in \mathcal{F}$ and (ii) gives the contradiction with $U(3, 7)$. □

**Claim 16.** $|\mathcal{F} \cap \binom{[n]}{3}| \leq \binom{7}{3} - 10 = 25$.

**Proof.** There are 10 pairs $F, F' \in \binom{[7]}{3}$ with $F \cup F' = [2, 7]$, and both sets in each pair cannot appear together in $\mathcal{F}$ due to (i). □

**Claim 17.** $(1, 4, 7) \in \mathcal{F}$.

**Proof.** The contrary would imply $(a, b) \in \binom{[9]}{2}$ for $(a, b, c) \in \mathcal{F}$ with $c \geq 7$. Consequently, $|\mathcal{F}| \leq \binom{6}{3} + 3(n - 6)$.

However, the right hand side of (11) is at least $|\mathcal{F}_2| = 4 + 6(n - 4) = 16 + 6(n - 6) > \binom{6}{3} + 3(n - 6)$ for $n \geq 8$.

Combined with (ii), we get the following corollary.

**Corollary 18.** $(2, 3, 8) \notin \mathcal{F}$.

Now $(1, 6, 8) \notin \mathcal{F}$ implies

$$|\mathcal{F}(n)| \leq 4 \quad \text{for all } n \geq 8.$$

Using Claim 16 we infer that

$$|\mathcal{F}| \leq 25 + 4(n - 7).$$

The right hand side is strictly less than $|\mathcal{F}_3| = 35$ for $n = 8, 9$ and strictly less than $|\mathcal{F}_2| = 22 + 4(n - 7)$ for $n \geq 10$. This concludes the proof. □
2.5. The case $k = 3$, $q = 2s + t$ for small $t$. In this section, we study families $\mathcal{F} \subset \binom{[n]}{3}$ that are $U(s, 2s + t)$ for $t \leq \epsilon s$ with some small $\epsilon > 0$. We show in particular that $\mathcal{F}_2$ from Section 2.3 is the largest in a certain range. Reusing the notation from Section 2.3, let us define the following families:

$$\mathcal{F}_1 := \mathcal{A}(t, 1, n, 3), \quad \mathcal{F}_2 := \mathcal{A}(s + t, 2, n, 3), \quad \mathcal{F}_3 := \mathcal{A}(2s + t, 3, n, 3).$$

**Theorem 19.** Let $\mathcal{F} \subset \binom{[n]}{3}$ be $U(s, 2s + t)$ with $t \geq 1$. Assume that $5(s + t) \leq n \leq \frac{(s+t)^2}{3t}$. Then $|\mathcal{F}| \leq |\mathcal{F}_2|$.

As we have seen in Theorems 6 and 11, both the upper and the lower bounds on $n$ are tight up to constants.

**Proof.** Take $\mathcal{F}$ as in the theorem and w.l.o.g. assume that $\mathcal{F}$ is shifted and $\mathcal{F} \not\subset \mathcal{F}_2$. This implies $(1, s + t + 1, s + t + 2) \in \mathcal{F}$.

Let us define the following $t + 1$ sets $B_1, \ldots, B_{t+1}$:

$$B_i := (i, 2t + 3 - i, 2s + t + 2 - i), \quad i = 1, \ldots, t + 1.$$

Clearly,

$$B_1 \cup \ldots \cup B_{t+1} = [2t + 2] \cup [2s + 1, 2s + t + 1].$$

Next, define $A_1, \ldots, A_{s-t-1}$ by

$$A_i := (1, 2t + 2 + i, 2s + 1 - i), \quad i = 1, \ldots, s - t - 1.$$

Then $A_{s-t-1} = (1, s + t + 1, s + t + 2) \in \mathcal{F}$ and

$$A_1 \cup \ldots \cup A_{s-t-1} = \{1\} \cup [2t + 3, 2s].$$

In particular, the union of the $s$ sets $B_1, \ldots, B_{t+1}$ and $A_1, \ldots, A_{s-t-1}$ is $[2s + t + 1]$. Consequently, at least one of them is missing from $\mathcal{F}$.

We prove the theorem separately according to whether $B_i \not\in \mathcal{F}$ for some $i$ or $A_j \not\in \mathcal{F}$ for some $j$.

**Proposition 20.** If not all $B_i$, $1 \leq i \leq t + 1$, are in $\mathcal{F}$ then $\nu(\mathcal{F} \setminus \mathcal{F}_3) \leq t$.

**Proof.** Note first that $F \in \mathcal{F} \setminus \mathcal{F}_3$ if and only if $F = (a, b, c)$ with $c \geq 2s + t + 1$. It is easy to see that $\nu(\mathcal{F} \setminus \mathcal{F}_3) \geq t + 1$ implies that there are sets $F_1, \ldots, F_{t+1} \in \mathcal{F} \setminus \mathcal{F}_3$ such that $F_1 \cup \ldots \cup F_{t+1} = [2t + 2] \cup \{2s + t + 1\}$.

It is not difficult to see that, for any $B_i$, $i \in [t + 1]$, there is $F_j$ such that $B_i \prec s F_j$, and thus $B_i \in \mathcal{F}$, a contradiction. \qed

Since $n > 6t$, we have $|\mathcal{F} \setminus \mathcal{F}_3| \leq \binom{n}{3} - \binom{n-t}{3}$ and $|\mathcal{F}| \leq \binom{n}{3} - \binom{n-t}{3} + \binom{2s+t}{3}$. On the other hand, we have $|\mathcal{F}_2| = \binom{s+t}{3} + \binom{s+t}{2} (n - s - t)$. An easy, but tedious, calculation shows that the first inequality below holds.

$$|\mathcal{F}_2| - |\mathcal{F}| \geq \frac{(s + t)^3}{6} + \frac{(s + t)^2}{2} (n - s - t) - t \binom{n}{2} - \frac{(2s + t)^3}{6} \geq \frac{n}{2} ((s + t)^2 - tn) - \frac{5}{3} (s + t)^3.$$

Since $5(s + t) \leq n \leq \frac{(s+t)^2}{3t}$, the right hand side is at least $\frac{5}{2} (s + t) \frac{2(s+t)^2}{3} - \frac{5}{3} (s + t)^3 \geq 0$.

Thus, we may assume that $A_1 = 1(2t + 2 + \ell, 2s + 1 - \ell)$, $\ell \in [s - t - 2]$, is not in $\mathcal{F}$. We want to conclude the proof by showing that $|\mathcal{F}_2 \setminus \mathcal{F}| \geq |\mathcal{F} \setminus \mathcal{F}_2|$.

For any $(a, b, c) \in \mathcal{F} \setminus \mathcal{F}_2$, we have $s + t < b < c \leq 2s - \ell$. Thus, we can bound

$$|\mathcal{F} \setminus \mathcal{F}_2| \leq \binom{s - t - \ell}{3} + (s + t) \binom{s - t - \ell}{2}.$$
On the other hand, \((a, b, c) \in \mathcal{F}_2 \setminus \mathcal{F}\) if \(2t + 2 + \ell \leq b \leq s + t, 2s + 1 - \ell \leq c \leq n\). The number of choices of \(c\) is \(n - 2s\), the number of choices of \((a, b)\) is \(\binom{s + t}{2} - \binom{2t + 1 + \ell}{2}\). Thus, to show that \(|\mathcal{F} \setminus \mathcal{F}_2| \leq |\mathcal{F}_2 \setminus \mathcal{F}|\), we need to show
\[
\binom{s - t - \ell}{3} + (s + t)\binom{s - t - \ell}{2} \leq (n - 2s)\left(\binom{s + t}{2} - \binom{2t + 1 + \ell}{2}\right).
\]
Let us show this for some range of \(\ell\) by (reverse) induction on \(\ell\). For \(\ell = s - t - 2\), the left hand side is \(s + t\), and the right hand side is \((n - 2s)(s + t - 1)\), and thus the inequality is satisfied. When passing from \(\ell\) to \(\ell - 1\), the LHS increases by \((s - t - \ell) + (s + t)(s - t - \ell)\leq (s - t - \ell)\frac{3s + t + \ell}{2}\), and the RHS increases by \((n - 2s)(2t + 1 + \ell)\). Since \(n \geq 5(s + t)\), we are good as long as \(2t + 1 + \ell \geq (s - t)\). If this inequality does not hold, then the RHS of the displayed inequality is at least \((n - 2s)\frac{3}{4}\binom{s + t}{2}\), which is bigger than the LHS as long as \(\frac{3}{4}(n - 2s) \geq \frac{s}{2} + s = \frac{4}{3}s\). This holds for \(n \geq 5(s + t) \geq 5s\).

\[\square\]

3. Proof of Theorem 6

Since the \(U\)-property is preserved by shifting, we may w.l.o.g. assume that \(\mathcal{F}\) is shifted.

Denote \(\mathcal{G}(B) := \{G \setminus [p + 1] : G \in \mathcal{G}, G \cap [p + 1] = B\}\). Then
\[|\mathcal{F}(B)| \leq |\mathcal{A}_{p,r}(B)| \text{ holds for all } B \subset [p + 1], |B| \geq r + 1. \tag{13}\]

**Claim 21.** For any \(B \subset [p + 1]\), \(|B| \leq r - 1\), we have \(\nu(\mathcal{F}(B)) \leq s\).

**Proof.** Indeed, the opposite gives \(s + 1\) members of \(\mathcal{F}\) whose union has size \(|B| + (k - |B|)(s + 1) \geq r - 1 + (k - r + 1)(s + 1) = s + r + (k - r)(s + 1) > p + (k - r)(s + 1)\), a contradiction with the \(U(s, q)\) property.

Similarly, we can obtain the following claim.

**Claim 22.** For fixed \(B \in \binom{[p + 1]}{r - 1}\), any \(s + 1\) families \(\mathcal{F}\{i_1\} \cup B, \ldots, \mathcal{F}\{i_s\} \cup B\), where \(i_1, \ldots, i_s \in [p + 1] \setminus B\) and \(\bigcup_{j \in [s]} \{i_j\} = [p + 1] \setminus B\) (note that some of the \(i_j\) will coincide for \(p < s + r - 1\)), are cross-dependent.

**Proof.** Indeed, the opposite gives \(s\) members of \(\mathcal{F}\) whose union has size \(p + 1 + (k - r)(s + 1)\), a contradiction.

Let us recall that the following theorem was proved in [11].

**Theorem 23 ([11]).** If \(\mathcal{G} \subset \binom{[m]}{r - 1}\) satisfies \(\nu(\mathcal{G}) \leq s\) then \(s|\partial \mathcal{G}| \geq |\mathcal{G}|\).

Note that, for \(G\) and \(G'\) of the same size and such that \(G \prec_s G'\) (cf. Definition 1), we have \(\mathcal{F}(G) \supset \mathcal{F}(G')\) due to the fact that \(\mathcal{F}\) is shifted. Similarly,
\[\partial \mathcal{F}(B) \subset \mathcal{F}(B \cup \{i\}) \text{ for any } B \subset [p + 1] \text{ and } i \in [p + 1] \setminus B,\]
and, combined with Theorem 23, we get that
\[|\mathcal{F}(B)| \leq s|\mathcal{F}(B \cup \{i\})| \text{ for any } B \subset [p + 1] \text{ and } i \in [p + 1] \setminus B. \tag{14}\]
Iteratively applying (14), we get that, for any \(i \geq 0,\)
\[
\sum_{B \subset [p], |B| \leq i} |\mathcal{F}(B)| \leq \sum_{j=0}^{i} \frac{s^{i-j} \binom{p}{j}}{\binom{p}{i}} \sum_{B \in \binom{[p]}{i}} |\mathcal{F}(B)| \leq s \sum_{j=0}^{i} \frac{s^{i-j} \binom{p}{j}}{\binom{p}{i}} \sum_{B \in \binom{[p]}{i}} |\mathcal{F}(B \cup \{p + 1\})|.\]
Similarly,
\[
\sum_{B \subseteq [p], |B| \leq r} |\mathcal{F}(B \cup \{p + 1\})| \leq \sum_{j=0}^{i} \frac{s^{i-j}(r)}{(r-1)!} \sum_{B \in \binom{[p]}{i}} |\mathcal{F}(B \cup \{p + 1\})|.
\]
Summing these two expressions for \(i = r - 1\), we get that
\[
\sum_{B \subseteq [p], |B| \leq r-1} |\mathcal{F}(B)| + |\mathcal{F}(B \cup \{p + 1\})| \leq M \sum_{B' \in \binom{[p+1]}{r}} |\mathcal{F}(B')|,
\]
where
\[
M := (s+1) \sum_{j=0}^{r-1} \frac{s^{r-1-j}(r)}{(r-1)!}.
\]
The following lemma (in the particular case \(u = s + 1\)) was proved in \[11\] (see \[11\] Theorem 3.1) and also \[14\] Lemma 5).

**Lemma 24.** Let \(N \geq (u + s)l\) for some \(u \in \mathbb{Z}, u \geq s + 1\), and suppose that \(G_1, \ldots, G_{s+1} \subseteq \binom{[N]}{l}\) are cross-dependent and nested. Then
\[
|G_1| + |G_2| + \ldots + |G_s| + u|G_{s+1}| \leq s\binom{N}{l}.
\]

Fix \(B \in \binom{[p]}{r-1}\) and assume that \([p] \setminus B = \{j_1, \ldots, j_{p-r+1}\}\). For each \(z \in [p-r+1]\), the \((s+1)\)-tuple
\[
\mathcal{T}(B, z) := (\mathcal{F}(\{j_1\} \cup B), \ldots, \mathcal{F}(\{j_{p-r+1}\} \cup B), \mathcal{F}(\{p+1\} \cup B), \mathcal{F}(\{j_2\} \cup B), \ldots, \mathcal{F}(\{j_z\} \cup B)),
\]
(note that \(\mathcal{F}(\{j_z\} \cup B)\) occurs \(s - p + r\) times in this tuple in total) is cross-dependent by Claim 22. Moreover, after reordering, it is nested with the smallest family being \(\mathcal{F}(\{p+1\} \cup B)\). Apply Lemma 24 to \(\mathcal{T}(B, z)\) for each \(z \in [p-r+1]\) and sum up the corresponding inequalities. Note that \(\mathcal{F}(\{j_z\} \cup B)\) gets coefficient \(s\) in this summation \((s - p + r\) from \(\mathcal{T}(B, z)\) and \(1\) from each of the \(\mathcal{T}(B, z')\), \(z' \in [p-r+1] \setminus \{z\}\)). Thus, for fixed \(B \in \binom{[p]}{r-1}\) and provided
\[
n \geq p + 1 + (s + u)(k - r),
\]
we get
\[
s \sum_{i \in [p] \setminus B} |\mathcal{F}(\{i\} \cup B)| + (p - r + 1)u|\mathcal{F}(\{p+1\} \cup B)| \leq s(p - r + 1)\binom{n - p - 1}{k - r}.
\]
Next, we sum this inequality over all \(B \in \binom{[p]}{r-1}\). Note that, for each \(B' \subseteq \binom{[p]}{r}\), it will appear in \(r\) summands as above, and, for each \(B' \in \binom{[p+1]}{r \backslash [p]}\), it will appear in \(\max\{1, r - 1\}\) summands. That is, we get
\[
sr \sum_{B' \in \binom{[p]}{r}} |\mathcal{F}(B')| + (p - r + 1) \cdot \max\{1, r - 1\} \cdot u \sum_{B' \in \binom{[p+1]}{r \backslash [p]}} |\mathcal{F}(B')| \leq
\]
\[
s(p - r + 1)\binom{p}{r-1} \binom{n - p - 1}{k - r} = sr \binom{p}{r} \binom{n - p - 1}{k - r}.
\]
Put \(\mathcal{A}_{r,p}' := \{A \in \mathcal{A}_{r,p} : |A \cap [p+1]| \leq r\}\). Note that \(|\mathcal{A}_{r,p}' \cap \{F : |F \cap [p+1]| = r\}| = \binom{p}{r} \binom{n-p-1}{k-r}\), that is, up to a multiplicative factor \(sr\), the right hand side of the last formula above. Therefore,
we divide both parts by \( sr \) and rewrite this formula as follows.

\[
\sum_{B' \in \binom{[p]}{r}} |\mathcal{F}(B')| + u' \sum_{B' \in \binom{[p+1]}{r} \setminus \binom{[p]}{r}} |\mathcal{F}(B')| \leq |A'_{r,p}|,
\]

(18)

where

\[ u' = \frac{(p-r+1) \cdot \max\{1, r-1\} \cdot u}{sr}. \]

(19)

Therefore, if \( u' \geq M \), then, using (15), the inequality (18) implies that

\[
\sum_{B' \subseteq [p+1], |B'| \leq r} |\mathcal{F}(B')| \leq |A'_{r,p}|,
\]

(20)

which, together with (13), completes the proof of the theorem. Finally, the inequality \( u' \geq M \) is equivalent to

\[
u \geq \frac{sr}{(p-r+1) \cdot \max\{1, r-1\}} \cdot (s+1) \sum_{j=0}^{r-1} \frac{s^{r-1-j}(p)}{(p-1)^j} = \frac{s(s+1)}{\max\{1, r-1\}} \sum_{j=0}^{r-1} \frac{s^{r-1-j}(p)}{(p-1)^j} = f(s, p, r).
\]

Since \( n \geq p+1+(s+f(s, p, r))(k-r) \), we may take \( u = f(s, p, r) \). Then the inequality above, as well as the inequality (17) is satisfied. This completes the proof.

4. Final remarks

The question of determining \( m(n, k, s, q) \) in general seems to be very hard since it includes some difficult questions, notably the Erdős Matching Conjecture, as a subcase. However, such a generalization of the problem might be easier to deal with by means of induction.

We believe that the first natural case to settle completely is the \( k = 3 \) case. The first author proved the Erdős Matching Conjecture for \( k = 3 \) and any \( n \) in [12]. We have obtained some partial results in Theorems [10] [11] and [19] which notably show that each of the 3 potential extremal families suggested by Conjecture [2] are extremal in some ranges. However, a full resolution requires much more work. It seems that the case of large \( s \) may be simpler, in particular because some tools are available for large \( s \) (cf. [15]). Concluding, we suggest the following particular case of Conjecture [2].

**Problem.** Determine \( m(n, 3, s, q) \) for all \( s \geq s_0 \) and all meaningful \( n, q \).

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Rényi Institute, Budapest, Hungary; Email: peter.frankl@gmail.com

University of Oxford and Moscow Institute of Physics and Technology; Email: kupavskii@ya.ru.