Recursive construction of a Nash equilibrium in a two-player nonzero-sum stopping game with asymmetric information

Roiy Jacobovic

Abstract

We study a discrete-time finite-horizon two-players nonzero-sum stopping game where the filtration of Player 1 is richer than the filtration of Player 2. A major difficulty which is caused by the information asymmetry is that Player 2 may not know whether Player 1 has already stopped the game or not. Furthermore, the classical backward-induction approach is not applicable in the current setup. This is because when the informed player decides not to stop, he reveals information to the uninformed player and hence the decision of the uninformed player at time \( t \) may not be determined by the play after time \( t \), but also on the play before time \( t \). In the current work we initially show that the expected utility of Player 2 will remain the same even if he knows whether Player 1 has already stopped. Then, this result is applied in order to prove that, under appropriate conditions, a recursive construction in the style of Hamadène and Zhang (2010) converges to a pure-strategy Nash equilibrium.

1 Introduction

1.1 Stopping games

Dynkin [11] considered a game with two players who observe a stochastic process \( (R_t, X_t)_{t=0}^\infty \). Player 1 (resp. 2) picks a stopping time \( \tau \) (resp. \( \nu \)).
such that $P(X_\tau \leq 0, X_\nu > 0) = 1$. The purpose of Player 1 (resp. 2) is to maximize (resp. minimize) the expected value of $R_{\min}\{\tau, \nu\}$. Dynkin proved that once $\sup_t R_t$ is integrable, the game has a value. Kifer [27] and Neveu [37] considered a similar game in which two players observe a three-dimensional stochastic process $(X_t, Y_t, Z_t)_{t=0}^\infty$. Each player chooses a stopping time and the purpose of Player 1 (resp. 2) is to maximize (resp. minimize) the expected value of

$$X_\tau 1_{\{\tau < \nu\}} + Y_\nu 1_{\{\nu > \tau\}} + Z_\tau 1_{\{\nu = \tau\}}.$$  

(1)

In particular, it is possible that the players choose stopping times $\tau$ and $\nu$ such that $P(\tau = \nu) > 0$. In two-player nonzero-sum stopping games, the processes $(X_t)$, $(Y_t)$ and $(Z_t)$ are two dimensional; the first (resp. second) coordinate is interpreted as the payoff of Player 1 (resp. Player 2). In general, the existence of an equilibrium point in these games is ensured when the joint distribution of the payoff processes satisfies some conditions. For example, see Morimoto [36], Mamer [35], Ohtsubo [39] and Hamadène et al [25]. Also, some relevant literature reviews are given by Sakaguchi [47], Novak et al. [38] and Jaśkiewicz et al. [26].

Another strand of literature studies stopping games with random stopping times. See, e.g., Yasuda [50], Rosenberg et al. [45], Shmaya et al. [49] for the discrete-time case and Laraki et al. [31] for the continuous time case. Notably, there is an extensive literature about continuous time versions of stopping games. For example, consider the classical works of Bismut [5] and Lepeltier et al. [34]. More recent works in this direction are those of: Attard [2, 3], De Angelis et al. [9, 10], Hamadène et al. [25]. In addition, a continuous (resp. discrete) version of stopping games with more than two players was discussed by Hamadène et al. [24] (resp. Hamadène et al. [23]).

Stopping games are motivated by various applications. Traditionally, a task that has stimulated mathematicians to analyse models with unique implications on optimal-stopping theory is the celebrated secretary’s problem (for surveys, see e.g., Ferguson [15] and Freeman [16]). Some versions of this problem with more than one decision maker were analyzed by Bruss et al. [6], Chen et al. [7], Enns et al. [12], Fushimi [17], and Sakaguchi [46]. Some other applications of stopping games are: Shrinking markets (see, e.g., Ghemawat et al. [20]), duels (see, e.g., Lang [29]) and financial markets with a recallable option (see, e.g., Hamadène et al. [22, 25] and Kifer [28]).
1.2 Asymmetric information

Recently, there is a growing attention towards stopping games with asymmetric information. Lempa et al. [33] considered a stopping game with a finite-horizon which is an exponentially distributed random variable, where only one player is exposed to the value of this random variable. Esmaeeli et al. [13] and Grün [21] presented two models in which the asymmetry of information is modelled as in the classical work of Aumann and Maschler [4]. Namely, there is a finite set of states of nature in which the game can take place. The true state is a random variable whose distribution is a common knowledge. The asymmetry in information is due to the assumption that the true state is exposed only to Player 1. An extension of this framework was discussed by Renault [44]. De Angelis et al. [8] considered a game with payoffs which are diffusion processes, where only one player knows the exact value of the drift and the diffusion coefficient. Mamer [35] and Ohtsubo [40] considered a class of games with a monotone payoff structure where players have information flows that are modelled as two possibly different filtrations. Gapaev et al. [18] considered a game with asymmetric information in the context of a financial market. In particular Gapaev et al. [18] refer to a solution concept which is called Stackelberg equilibrium (for an analysis of this solution concept in other stopping games see, e.g., the works of Ektröm et al. [14] and Peshkir [41]). Gensbittel [19] considered a two-player zero-sum stopping game with payoff processes which are determined by a Markov chain. The evolution of this chain is observed only by one of the Players. Ashkenazi-Golan et al. [1] analyzed a two-player two-state Markov game such that only one of the players knows when the state changes.

One major challenge in the analysis of stopping games with asymmetric information is due to the fact that they cannot be analysed by the classical backward-induction approach. To see why, notice that when an informed player decides not to stop, he reveals information to the uninformed players. Indeed, he reveals that his payoff by stopping in early stages is not too high. In particular, the decision of the uninformed player at time \( t \) may not be determined by the play after time \( t \), but also on the play before time \( t \) (see also Skarupski et al. [48]). Evidently, as it was shown by Laraki [30], a zero-sum stopping games with asymmetric information may not have a value even under the standard assumptions. However, there are also other cases in which an equilibrium result may be derived. For example, Laraki et al. (see Section 2.8.4.3 of [32]) mentioned that for finite-horizon zero-sum stopping games with incomplete information on one side, the value exists in discrete-time using Sion’s minmax theorem.
1.3 The current work

We study finite-horizon two-player nonzero-sum stopping games in discrete-time, where \( Z \equiv X \): If the players stop simultaneously, the payoff is the same as if Player 1 stops alone. Similar assumptions appear in the works of Hamadène et al. [25] and Ramsey [43]. Formally, consider an integer \( 0 < T < \infty \) with some two discrete-time processes \((X^1_t, Y^1_t)_{t=0}^T\) and \((X^2_t, Y^2_t)_{t=0}^T\). We assume that Player 1 observes all four processes while Player 2 observes only \((X^2_t, Y^2_t)_{t=0}^T\). In addition, Player 1 (resp. 2) picks a stopping time \( \tau \) (resp. \( \nu \)) with respect to his information flow and his goal is to maximize the expectation of

\[
X^1_\tau 1_{\{\tau \leq \nu\}} + Y^1_\nu 1_{\{\tau > \nu\}} \quad (\text{resp. } X^2_\nu 1_{\{\nu < \tau\}} + Y^2_\tau 1_{\{\tau \leq \nu\}}).
\]  

One main difficulty in the analysis of this game is the fact that Player 1 may pick a stopping time \( \tau \) which is not a stopping time with respect to the information flow of Player 2. Consequently, it is possible that Player 2 should decide whether to stop or keep playing without knowing whether Player 1 has already stopped. It is reasonable to expect that if Player 2 knew that Player 1 has not stopped yet, he could learn something new which might help him to gain greater utility. With some extent of surprise, we will show that this intuition is wrong. Namely, Player 2 may always assume that Player 1 has not stopped yet and by doing so he will get the maximal expected utility that could be obtained under the assumption that he is informed about the stopping epoch of Player 1. Then, this result is applied in order to prove that once \( X^1_t \geq Y^1_t \) for every \( 0 \leq t \leq T \), a recursive construction in the style of Hamadène et al. [25] yields a pure-strategy Nash equilibrium. To see a crucial difference between the current work and Hamadène et al. [25], recall that Hamadène et al. [25] studied stopping games with symmetric information. Thus, their construction yields stopping times which are measurable with respect to (w.r.t.) the information of Player 1 but not with respect to the information of Player 2. Hence, a proper modification of the construction is needed.

The rest of this paper is organized as follows: Section 2 contains a detailed presentation of the model assumptions with the precise statements of the main results. In addition, this section contains a sketch of the proof of the equilibrium result. This sketch is based on some benchmarks which are given with some intuitive explanations. For clarity, the technical proofs are supplied in Section 3.
2 Game description and the main results

Let $T \in (0, \infty)$ be a positive integer and consider a probability space $(\Omega, \mathcal{F}, P)$ with two filtrations $\mathbb{F} \equiv (\mathcal{F}_t)_{t=0}^T$ and $\mathbb{G} \equiv (\mathcal{G}_t)_{t=0}^T$ such that $\mathcal{G}_t \subseteq \mathcal{F}_t$ for every $0 \leq t \leq T$. Denote the set of all $\mathbb{F}$ (resp. $\mathbb{G}$)-stopping times by $\mathcal{T}(\mathbb{F})$ (resp. $\mathcal{T}(\mathbb{G})$). Let $\mathcal{D}$ be the space of all $\mathbb{F}$-adapted $\mathbb{R}$-valued processes $(S_t)_{t=0}^T$ such that

$$E \left( \max_{0 \leq t \leq T} |S_t| \right) < \infty. \quad (3)$$

Let $X^1, X^2, Y^1, Y^2 \in \mathcal{D}$ such that $X^2$ and $Y^2$ are adapted to $\mathbb{G}$. Consider a two-player stopping game $\Gamma$ in which:

- Player 1 picks $\tau \in \mathcal{T}(\mathbb{F})$ and Player 2 picks $\nu \in \mathcal{T}(\mathbb{G})$.
- For every $\tau \in \mathcal{T}(\mathbb{F})$ and $\nu \in \mathcal{T}(\mathbb{G})$ the expected utility of Player 1 is given by
  $$J_1(\tau, \nu) \equiv E \left[ X^1_{\tau} 1_{\{\tau \leq \nu\}} + Y^1_{\tau} 1_{\{\tau > \nu\}} \right], \quad (4)$$

and the expected utility of Player 2 is given by
  $$J_2(\tau, \nu) \equiv E \left[ X^2_{\nu} 1_{\{\nu < \tau\}} + Y^2_{\tau} 1_{\{\tau \leq \nu\}} \right]. \quad (5)$$

**Remark 1** Note that the assumption that $X^2$ and $Y^2$ are adapted to $\mathbb{G}$ practically means that Player 2 observes the evolution of his payoff processes over time.

**Remark 2** By the definitions of $J_1$ and $J_2$, Player 1 has a priority in stopping: The payoff if the two players stop simultaneously is the same as Player 1 stops alone.

**Definition 1** A pair $(\tau^*, \nu^*) \in \mathcal{T}(\mathbb{F}) \times \mathcal{T}(\mathbb{G})$ constitutes a pure-strategy Nash equilibrium in $\Gamma$ if and only if (iff)

$$J_1(\tau, \nu^*) \leq J_1(\tau^*, \nu^*), \quad \forall \tau \in \mathcal{T}(\mathbb{F}), \quad (6)$$

and

$$J_2(\tau^*, \nu) \leq J_2(\tau^*, \nu^*), \quad \forall \nu \in \mathcal{T}(\mathbb{G}). \quad (7)$$

In the current work, we are going to prove that if

$$X^1_t \geq Y^1_t, \quad \forall 0 \leq t \leq T, \ P\text{-a.s.,} \quad (8)$$

5
then \( \Gamma \) admits a pure-strategy Nash equilibrium which follows from a construction in the style of Hamadéne et al. [25].

In the rest of this section, there is an elaboration which is necessary in order to make this statement more precise. In addition, we will review some important benchmarks like the analysis of the best response of Player 2 and the recursive construction which yields the equilibrium. Especially, the results which appear in the following Subsection 2.1 and Subsection 2.2 do not require the condition which appears in (8).

2.1 The best response of Player 2

For every \( 0 \leq t \leq T \), let \( T_t(F) \) (resp. \( T_t(G) \)) be the set of all \( F \) (resp. \( G \))-stopping times \( \tau \) such that \( P(t \leq \tau \leq T) = 1 \). Furthermore, for every \( \tau \in F \) define the filtration \( G^\tau \equiv (G^\tau_t)_{t=0}^T \) as follows:

\[
G_t^\tau \equiv G_t \vee \sigma \{1_{\{\tau \leq s\}}; 0 \leq s \leq t\}, \quad \forall 0 \leq t \leq T.
\]

(9)

Note that once Player 1 picks \( \tau \in T_t(F) \), the only difference between Player 2 and an observer with a filtration \( G^\tau \) is the knowledge of whether Player 1 has already stopped. Thus, the following theorem states that even if Player 2 knows whether Player 1 has already stopped, his expected utility will remain the same. This is because at any moment \( t \) such that \( P(\tau > t) > 0 \), Player 2 (whose information flow is described by the filtration \( G \)) may conduct his decision-making under the assumption that Player 1 has not stopped yet.

Theorem 1 Let \( \tau \in T_t(F) \) and \( t \in [0, T - 1] \). Then, there exists \( \hat{\nu}_t \in T_t(G) \) such that the next equality holds P-a.s.

\[
E \left[ X^2_\nu \mathbf{1}_{\{\nu < \tau\}} + Y^2_\nu \mathbf{1}_{\{\tau \leq \nu\}|G_t^\tau} \right] = \text{ess sup} \nu \in T_t(G) \left[ X^2_\nu \mathbf{1}_{\{\nu < \tau\}} + Y^2_\nu \mathbf{1}_{\{\tau \leq \nu\}|G_t^\tau} \right] = Y^2_\tau \mathbf{1}_{\{\tau \leq \nu\}} + \mathbf{1}_{\{\tau > \nu\}} \text{ess sup} \nu \in T_t(G) \left[ X^2_\nu \mathbf{1}_{\{\nu < \tau\}} + Y^2_\nu \mathbf{1}_{\{\tau \leq \nu\}|G_t} \right].
\]

(10)

The proof of Theorem 1 is supplied in Subsection 3.1.
2.2 Recursive construction

The recursive construction is as follows: Let $\tau_1 = \nu_1 = T$ and for every $n \geq 1$ define:

\[
W_{t}^{1,n} \equiv \text{ess sup}_{\tau \in \mathcal{T}(\mathbb{F})} E \left[ X_{\tau}^{1} 1_{\{\tau < \nu_n\}} + (X_{\tau}^{1} 1_{\{\nu_n = T\}} + Y_{\nu_n}^{1} 1_{\{\nu_n < T\}}) 1_{\{\tau \geq \nu_n\}} | \mathcal{F}_{t} \right], \quad \forall t \in [0, T],
\]

\[
\tilde{\tau}_{n+1} \equiv \inf \left\{ t \geq 0; W_{t}^{1,n} = X_{t}^{1} \right\} \wedge \nu_n,
\]

\[
\tau_{n+1} \equiv \begin{cases} \tilde{\tau}_{n+1}, & \tilde{\tau}_{n+1} < \nu_n, \\ \tau_n, & \text{o.w.} \end{cases}
\]

\[
W_{t}^{2,n} \equiv \text{ess sup}_{\nu \in \mathcal{T}(\mathbb{G})} E \left[ X_{\nu}^{2} 1_{\{\nu < \tau_n\}} + Y_{\tau_n}^{2} 1_{\{\tau_n \leq \nu\}} | \mathcal{G}_{t}^{\tau_n} \right], \quad \forall t \in [0, T],
\]

\[
\tilde{\nu}_{n+1} \equiv \inf \left\{ t \geq 0; W_{t}^{2,n} = X_{t}^{2} \right\} \wedge \tau_n,
\]

\[
\nu_{n+1} \equiv \begin{cases} \tilde{\nu}_{n+1}, & \tilde{\nu}_{n+1} < \tau_n, \\ \nu_n, & \text{o.w.} \end{cases}
\]

By construction, both of the sequences $(\tau_n)_{n=1}^{\infty}$ and $(\nu_n)_{n=1}^{\infty}$ are nonincreasing and hence they have limits $\tau^*$ and $\nu^*$ as $n \to \infty$. When $\mathbb{F}$ and $\mathbb{G}$ are the same, Hamadène et al. [25] showed (in an analogue continuous-time model) that $(\tau^*, \nu^*)$ is a pure-strategy Nash equilibrium. In our model, the following Proposition 1 is an analogue statement. Its proof is provided in Subsection 3.2.

**Proposition 1** There exist limits $\tau^* \equiv \lim_{n \to \infty} \tau_n$ and $\nu^* \equiv \lim_{n \to \infty} \nu_n$ such that:

1. $\tau^* \in \mathcal{T}(\mathbb{F})$ and $\nu^* \in \mathcal{T}(\mathbb{F})$.

2. $J_1(\tau, \nu^*) \leq J_1(\tau^*, \nu^*), \quad \forall \tau \in \mathcal{T}(\mathbb{F}).$ (17)

3. $J_2(\tau^*, \nu) \leq J_2(\tau^*, \nu^*), \quad \forall \nu \in \mathcal{T}(\mathbb{G}).$ (18)

Importantly, $\nu^*$ is not necessarily a $\mathbb{G}$-stopping time, and hence an equilibrium result will not follow from the above-mentioned construction unless
some modifications are applied. Specifically, for each \( n \geq 1 \) and \( 0 \leq t \leq T \) define

\[
V^n_t \equiv \begin{cases} 
\text{ess sup}_{\theta \in \Theta_t(G)} E_P(\{\tau_n > t\}) \left[ X^2_{\theta} 1_{\{\tau_n < \theta\}} + Y^2_{\tau_n} 1_{\{\tau_n \leq \theta\}} | G_t \right], & P(\tau_n > t) > 0, \\
\infty, & \text{o.w.,}
\end{cases}
\]

(19)

\[
\theta_{n+1} \equiv \inf \{ t \geq 0; V^n_t = X^2_t \} \wedge T.
\]

(20)

In particular, \( \theta^* \equiv \inf_{n \geq 2} \theta_n \) is a \( G \)-stopping time as an infimum of a sequence of \( G \)-stopping times. Moreover, for each \( n \geq 1 \), Theorem 1 implies that

\[
\tilde{\nu}_{n+1} = \inf \{ t \geq 0; W^{2,n}_t = X^2_t 1_{\{t \leq \tau_n\}} + Y^2_{\tau_n} 1_{\{\tau_n \leq t\}} \}
\]

\[
= \inf \{ t \geq 0; Y^2_{\tau_n} 1_{\{\tau_n \leq t\}} + V^n_t 1_{\{t \leq \tau_n\}} = X^2_t 1_{\{t \leq \tau_n\}} + Y^2_{\tau_n} 1_{\{\tau_n \leq t\}} \}
\]

\[
= \inf \{ t \geq 0; V^n_t = X^2_t \} \wedge \tau_n
\]

\[
= \theta_{n+1} \wedge \tau_n, \quad P\text{-a.s.}
\]

(21)

and hence we shall deduce that

\[
\nu^* = \inf_{n \geq 1} \{ \theta_{n+1}; \theta_{n+1} < \tau_n \} \wedge T, \quad P\text{-a.s.}
\]

(22)

This identity yields the next Corollary 1 which is going to be useful for the proof of the equilibrium result in the next subsection.

**Corollary 1**

\[
\nu^* < \tau^* \iff \theta^* < \tau^*, \quad P\text{-a.s.}
\]

(23)

The proof for this corollary is given in Subsection 3.3.

### 2.3 Equilibrium result

**Theorem 2** if the condition (8) is satisfied, then \((\tau^*, \theta^*)\) is a pure-strategy Nash equilibrium.

The following Lemma 1 states that it is enough to prove Theorem 2 under the assumption that \( X^1_t > Y^1_t \), P-a.s. for every \( 0 \leq t \leq T \). The proof for this Lemma appears in Subsection 3.3.

**Lemma 1** Assume that the condition (8) is satisfied and let \( \epsilon > 0 \). In addition, define a two-player stopping game \( \Gamma_{\epsilon} \) which is identical to \( \Gamma \) except that the expected utility of Player 1 is given by
\[ J_{1,\epsilon}(\tau, \nu) \equiv E \left[ (X^1_\tau + \epsilon) I_{\{\tau \leq \nu\}} + Y^1_\nu I_{\{\tau > \nu\}} \right], \quad \forall \tau \in \mathcal{T}(\mathbb{F}), \nu \in \mathcal{T}(\mathbb{G}). \tag{24} \]

If a pair \((\tau_0, \nu_0) \in \mathcal{T}(\mathbb{F}) \times \mathcal{T}(\mathbb{G})\) is a pure-strategy Nash equilibrium in \(\Gamma_{\epsilon}\), then it is also a pure-strategy Nash equilibrium in \(\Gamma\).

**Proof of Theorem 2**

Assume by contradiction that \(P(\tau^* > \nu^*) > 0\). Since \(X^1_t > Y^1_t\) for every \(t \in [0, T]\), then \(J_1(\tau^*, \nu^*) < J_1(\nu^* \land \tau^*, \nu^*)\) and hence the second part of Proposition 1 implies that \(\nu^* \land \tau^* \notin \mathcal{T}(\mathbb{F})\). This is a contradiction with the first part of Proposition 1 and hence \(\tau^* \leq \nu^*, P\text{-a.s.}\). As a result, Corollary 1 implies that \(\tau^* \leq \theta^* \leq \nu^*, P\text{-a.s.}\) and hence \(J_2(\tau^*, \nu^*) = J_2(\tau^*, \theta^*)\).

Thus, the third part of Proposition 1 yields that \(\theta^*\) is a best response of Player 2 to \(\tau^*\).

Assume by contradiction that there exists \(\tau_0 \in \mathcal{T}(\mathbb{F})\) such that \(J_1(\tau_0, \theta^*) > J_1(\tau_0, \nu^*)\). Denote \(\tau' \equiv \tau_0 \land \theta^*, \) and note that \(\tau' \in \mathcal{T}(\mathbb{F})\). Since \(X^1_t > Y^1_t\) for every \(0 \leq t \leq T\), deduce that

\[ J_1(\tau', \theta^*) \geq J_1(\tau_0, \theta^*) > J_1(\tau_0, \nu^*). \tag{25} \]

Now, \(\tau' \leq \theta^* \leq \nu^*, P\text{-a.s.}\), and hence it does not matter whether Player 2 picks either \(\theta^*\) or \(\nu^*\), when picking \(\tau'\), Player 1 is the one who is responsible for stopping, i.e., \(J_1(\tau', \theta^*) = J_1(\tau', \nu^*)\). On the other hand, we have already proved that \(\tau^* \leq \theta^* \leq \nu^*, P\text{-a.s.}\) and hence \(J_1(\tau^*, \theta^*) = J_1(\tau^*, \nu^*)\). This leads to a contradiction because \(\tau^*\) is the best response of Player 1 to \(\nu^*\). ■

**3 Proofs of Theorem 1 and some auxiliary results**

**3.1 Proof of Theorem 1**

Let \(\tau \in \mathcal{T}(\mathbb{F})\) and \(0 \leq t \leq T\). The following lemma includes a characterization of the structure of a general event in \(\mathcal{G}_t^\tau\). Intuitively speaking, it states that an event belongs to \(\mathcal{G}_t^\tau\) if and only if conditioning on the available information about \(\tau\) yields an event which belongs to \(\mathcal{G}_t\).

**Lemma 2** Let \(\tau \in \mathcal{T}(\mathbb{F})\) and \(0 \leq t \leq T\). \(A \in \mathcal{G}_t^\tau\) iff there exist \(A^0 \in \mathcal{G}_t\) and \(\{A_s; 0 \leq s \leq t\} \subseteq \mathcal{G}_t\) for which

\[ A = \left( \{\tau > t\} \cap A^0 \right) \cup \left[ \bigcup_{s=0}^t \{\{\tau = s\} \cap A_s\} \right]. \tag{26} \]
Proof: If \( A \) has the structure of (26), then immediately deduce that \( A \in \mathcal{G}_t^\tau \) because \( \{ \tau > t \} \in \mathcal{G}_t^\tau \) and \( \{ \tau = s \} \in \mathcal{G}_t^\tau \) for every \( s \in [0, t] \). In order to prove the other direction, observe that every

\[
A \in \mathcal{P} \equiv \{ B \cap \{ \tau \leq s \}; B \in \mathcal{G}_t, 0 \leq s \leq t \} \cup \mathcal{G}_t
\]  

(27)

has the structure of (26). Specifically, if \( A = B \cap \{ \tau \leq s \} \) for some \( B \in \mathcal{G}_t \) and \( s \in [0, t] \), then \( A_u = B \) for every \( u \in [0, s] \) and \( A^0 = A_{s+1} = \ldots = A_t = \emptyset \). Similarly, if \( A \in \mathcal{G}_t \), then it is possible to set \( A^0 = A_1 = \ldots = A_t = A \). In addition, it may be verified that \( \mathcal{P} \) is a \( \pi \)-system. Now, let \( D \) be the collection of all \( A \in \mathcal{F} \) which have the structure of (26) and observe that:

1. An insertion of \( A^0 = A_0 = A_1 = \ldots = A_t = \Omega \) implies that \( \Omega \in D \).
2. Assume that \( A_1, A_2 \in D \) are such that

\[
A_i = (\{ \tau > t \} \cap A^0_i) \cup \left[ \bigcup_{s=0}^{t} (\{ \tau = s \} \cap A_{s,i}) \right], \quad i = 1, 2
\]  

(28)

and \( A_1 \subseteq A_2 \). As a result, deduce that

\[
A_2 \setminus A_1 = \left( \{ \tau > t \} \cap \tilde{A}^0 \right) \cup \left[ \bigcup_{s=0}^{t} \left( \{ \tau = s \} \cap \tilde{A}_s \right) \right]
\]  

(29)

where \( \tilde{A}^0 = A_2^0 \setminus A_1^0 \in \mathcal{G}_t \) and \( \tilde{A}_s = A_{s,2} \setminus A_{s,1} \in \mathcal{G}_t \) for every \( s \in [0, t] \). Therefore, deduce that \( A_2 \setminus A_1 \in D \).
3. Assume that \( A_1 \subseteq A_2 \subseteq \ldots \) is a sequence of events which belong to \( D \) such that

\[
A_i = (\{ \tau > t \} \cap A^0_i) \cup \left[ \bigcup_{s=0}^{t} (\{ \tau = s \} \cap A_{s,i}) \right], \quad i \geq 1.
\]  

(30)

Then,

\[
\bigcup_{i \geq 1} A_i = \left( \{ \tau > t \} \cap \tilde{A}^0 \right) \cup \left[ \bigcup_{s=0}^{t} \left( \{ \tau = s \} \cap \tilde{A}_s \right) \right]
\]  

(31)

such that \( \tilde{A}^0 = \bigcup_{i \geq 1} A^0_i \in \mathcal{G}_t \) and \( \tilde{A}_s = \bigcup_{i \geq 1} A_{s,i} \in \mathcal{G}_t \) for every \( s \in [0, t] \). Therefore, deduce that \( \bigcup_{i \geq 1} A_i \in D \).

This means that \( D \) is a Dynkin system such that \( \mathcal{P} \subseteq D \) and hence Dynkin’s \( \pi - \lambda \) theorem yields that \( \mathcal{G}_t^\tau = \sigma \{ \mathcal{P} \} \subseteq D \) from which the result follows.

Now we are ready to provide the proof of Theorem
Proof of Theorem 1

For every \( t \leq u \leq T - 1 \), let \( \nu^*_u \) be the minimal-optimal stopping time which solves

\[
\text{ess sup}_{\nu \in \mathcal{T}_u(\mathcal{G}^* \cap G^r)} E \left[ X^2 \mathbf{1}_{\{\nu < \tau\}} + Y^2 \mathbf{1}_{\{\tau \leq \nu\}} \middle| \mathcal{G}^*_u \right].
\]  

(32)

Thus, Lemma 2 implies that for every \( t \leq u \leq T - 1 \), there are \( A^0_u \in \mathcal{G}_u \) and \((A_{s,u})_{s=0}^u \subseteq \mathcal{G}_u\) such that

\[
\{\nu^*_u = u\} = \left( A^0_u \cap \{\tau > u\} \right) \cup \left( \bigcup_{s=0}^u \left( \{\tau = s\} \cap A_{s,u} \right) \right).
\]

(33)

As a result, deduce that

\[
\{\tau > t\} \cap \{\nu^*_t = t\} = \{\tau > t\} \cap A^0_t.
\]

(34)

Now, consider the case when \( u > t \) and observe that for every \( 0 \leq l \leq s \) such that \( t \leq s \leq u - 1 \):

\[
\{\tau > u\} \subseteq \{\tau \neq l\},
\]

(35)

\[
\{\tau > u\} \cap \{\tau \leq s\} = \emptyset.
\]

(36)

Therefore, if we use the notation \( \overline{A} \equiv \Omega \setminus A \) for every \( A \subseteq \Omega \), then an insertion of (33) and applying standard algebra of sets imply that:

\[
\{\tau > u\} \cap \{\nu^*_s = u\} = \{\tau > u\} \cap \{\nu^*_s = u\} \cap \left( \bigcap_{s=t}^{u-1} \{\nu^*_s > s\} \right)
\]

(37)

\[
= \{\tau > u\} \cap \{\nu^*_s = u\} \cap \left( \bigcap_{s=t}^{u-1} \{\nu^*_s = s\} \right)
\]

\[
= \{\tau > u\} \cap A^0_u \cap \left[ \bigcap_{s=t}^{u-1} \left( \bigcup_{s=t}^{u-1} \{\tau > s\} \right) \cup \left( \bigcup_{s=0}^{u-1} \left( \{\tau = s\} \cap A_{s,u} \right) \right) \right]
\]

\[
= \{\tau > u\} \cap A^0_u \cap \left[ \bigcap_{s=t}^{u-1} \left( \bigcup_{s=t}^{u-1} \{\tau > s\} \right) \cup \left( \bigcup_{s=0}^{u-1} \left( \{\tau = s\} \cap A_{s,u} \right) \right) \right]
\]

\[
= \{\tau > u\} \cap A^0_u \cap \left[ \bigcap_{s=t}^{u-1} \left( \bigcup_{s=t}^{u-1} \{\tau > s\} \right) \cup \left( \bigcup_{s=0}^{u-1} \left( \{\tau = s\} \cap A_{s,u} \right) \right) \right]
\]

(38)

In particular, (36) implies that for every \( t \leq s \leq u - 1 \),

\[
\{\tau > u\} \cap \left( \{\tau \leq s\} \cap \left( \bigcap_{s=0}^{u-1} \left( \{\tau \neq l\} \cup \overline{A}_{l,s} \right) \right) \right) \subseteq \{\tau > u\} \cap \{\tau \leq s\} = \emptyset.
\]

(39)

In addition, (35) implies that for every \( 0 \leq l \leq s \) such that \( t \leq s \leq u - 1 \),

\[
\{\tau > u\} \cap \left( \{\tau \neq l\} \cup \overline{A}_{l,s} \right) = \{\tau > u\},
\]

(40)
Combining (37), (38) and (40) all together yields that
\[ \{\tau > u\} \cap \{v_t^* = u\} = \{\tau > u\} \cap A_0^u \cap \left( \cap_{s=t}^{u-1} A_s^u \right). \] (41)

Therefore, (34) and (41) imply that for every \( t \) and \( \nu \) holds for every \( \tau \)
\[ \{\tau > u\} \cap \{v_t^* = u\} = \{\tau > u\} \cap K_u \] (42)

where
\[ K_u \equiv A_u^0 \setminus \cup_{s=t}^{u-1} A_s^0. \] (43)

In particular, observe that
\[ K_u \in \mathcal{G}_u, \ \forall t \leq u \leq T - 1 \] (44)

and
\[ K_u_1 \cap K_u_2 = \emptyset, \ \forall t \leq u_1 < u_2 \leq T - 1. \] (45)

Therefore, it is possible to define a random variable \( \hat{\nu}_t : \Omega \rightarrow [t, T - 1] \) such that
\[ \{\hat{\nu}_t = u\} = K_u, \ \forall t \leq u \leq T - 1 \] (46)

and \( \{\hat{\nu}_t = T\} = \{t \leq \hat{\nu}_t \leq T - 1\} \). As a result, by construction we get that \( \hat{\nu}_t \in \mathcal{T}_t(G) \) and
\[ \{\tau > u\} \cap \{\nu_t^* = u\} = \{\tau > u\} \cap \{\hat{\nu}_t = u\}, \ \forall t \leq u \leq T - 1. \] (47)

In addition, the identity
\[ X_\nu^2 1_{\{\nu < \tau\}} + Y_\tau^2 1_{\{\tau \leq \nu\}} = \sum_{u=t}^{T-1} X_u^2 1_{\{\tau > u\}} + Y_\tau^2 \left( 1 - \sum_{u=t}^{T-1} 1_{\{\tau > u\}} \right) \] (48)

holds for every \( \nu \in \mathcal{T}_t(G^\tau) \) (especially, it holds for both \( \nu_t^* \) and \( \hat{\nu}_t \)) and hence (47) implies the first part of (10).

For the proof of the second part of (10), consider some arbitrary \( \nu \in \mathcal{T}_t(G^\tau) \) and assume that \( P(\tau > t) > 0 \). Let \( A \in \mathcal{G}_t \) and observe that:
\[ E \{1_A 1_{\{\tau > t\}} | \mathcal{G}_t\} \left[ X_\nu^2 1_{\{\nu < \tau\}} + Y_\tau^2 1_{\{\tau \leq \nu\}} | \mathcal{G}_t\right] \] (49)

\[ = E \{1_A 1_{\{\tau > t\}} | \mathcal{G}_t\} \left[ \left( X_\nu^2 1_{\{\nu < \tau\}} + Y_\tau^2 1_{\{\tau \leq \nu\}} \right) 1_A | \mathcal{G}_t\right] \]
\[ = P(\tau > t) E_{P(\tau > t)} \left[ E_{P(\tau > t)} \left[ \left( X_\nu^2 1_{\{\nu < \tau\}} + Y_\tau^2 1_{\{\tau \leq \nu\}} \right) 1_A | \mathcal{G}_t\right] \right] \]
\[ = P(\tau > t) E_{P(\tau > t)} \left[ \left[ X_\nu^2 1_{\{\nu < \tau\}} + Y_\tau^2 1_{\{\tau \leq \nu\}} \right] 1_A \right] \]
\[ = E \left[ \left[ X_\nu^2 1_{\{\nu < \tau\}} + Y_\tau^2 1_{\{\tau \leq \nu\}} \right] 1_A 1_{\{\tau > t\}} \right], \ \text{P-a.s.} \]
In addition, \( \nu \geq t \), \( P \)-a.s. and hence \( P \)-a.s., \( \tau \leq t \) implies that \( \tau \leq \nu \). This and (50) yield that

\[
E \left\{ \mathbf{1}_A \left[ Y_\nu^2 \mathbf{1}_{\{\nu < t\}} + \mathbf{1}_{\{\tau > t\}} E_P(\{\tau > t\}) \left( X_\nu^2 \mathbf{1}_{\{\nu < \tau\}} + Y_\tau^2 \mathbf{1}_{\{\tau \leq \nu\}} | \mathcal{G}_t \right) \right]\right\}
\]

(50)

\[
= E \left\{ \mathbf{1}_A \left[ Y_\tau^2 \mathbf{1}_{\{\tau \leq t\}} \mathbf{1}_{\{\tau \leq \nu\}} + \mathbf{1}_{\{\tau > t\}} E_P(\{\tau > t\}) \left( X_\nu^2 \mathbf{1}_{\{\nu < \tau\}} + Y_\tau^2 \mathbf{1}_{\{\tau \leq \nu\}} | \mathcal{G}_t \right) \right]\right\}
\]

\[
= E \left\{ \left[ X_\nu^2 \mathbf{1}_{\{\nu < \tau\}} + Y_\tau^2 \mathbf{1}_{\{\tau \leq \nu\}} \right] \mathbf{1}_A \right\}.
\]

Note that in the last equality is valid because of the fact that \( \nu \geq t \), \( P \)-a.s. and hence \( \mathbf{1}_{\{\nu < \tau\}} \mathbf{1}_{\{\tau > t\}} = \mathbf{1}_{\{\nu < \tau\}} \), \( P \)-a.s. Furthermore, notice that (50) holds for every

\[
A \in \mathcal{P} \equiv \{ B \cap \{ \tau \leq s \}; B \in \mathcal{G}_t, 0 \leq s \leq t \} \cup \mathcal{G}_t.
\]

(51)

In addition, let \( \mathcal{D} \) be the set of all \( A \in \mathcal{F} \) for which (50) holds. Then, observe that:

1. \( \Omega \in \mathcal{G}_t \) and hence (50) implies that \( \Omega \in \mathcal{D} \).
2. Assume that \( A \in \mathcal{D} \). Since \( \mathbf{1}_{\Omega \setminus A} = \mathbf{1}_\Omega - \mathbf{1}_A \) and \( \Omega, A \in \mathcal{D} \), then the linearity of conditional expectation yields that \( \Omega \setminus A \in \mathcal{D} \).
3. Assume that \( (A_n)_{n=1}^\infty \subset \mathcal{D} \) such that \( A_i \cap A_j = \emptyset \) for every \( i \neq j \). Then, \( \mathbf{1}_\bigcup_{n=1}^\infty A_n = \sum_{n=1}^\infty \mathbf{1}_{A_n} \) and hence the linearity of conditional expectation yields that \( \bigcup_{n=1}^\infty A_n \in \mathcal{D} \).

Thus, \( \mathcal{D} \) is a Dynkin system. Since \( \mathcal{P} \) is a \( \pi \)-system which is a subset of \( \mathcal{D} \), Dynkin’s \( \pi - \lambda \) theorem implies that \( \mathcal{G}^*_t = \sigma(\mathcal{P}) \subseteq \mathcal{D} \). As a result, the definition of conditional expectation implies that

\[
E \left[ X_\nu^2 \mathbf{1}_{\{\nu < \tau\}} + Y_\tau^2 \mathbf{1}_{\{\tau \leq \nu\}} \bigg| \mathcal{G}_t \right] = Y_\tau^2 \mathbf{1}_{\{\tau \leq t\}} + \mathbf{1}_{\{\tau > t\}} E_P(\{\tau > t\}) \left[ X_\nu^2 \mathbf{1}_{\{\nu < \tau\}} + Y_\tau^2 \mathbf{1}_{\{\tau \leq \nu\}} \bigg| \mathcal{G}_t \right], \quad P\text{-a.s.}
\]

(52)

Now, recall that it has already been proved that \( \hat{\nu}_t \) solves the essential supremum problem of the LHS of (52) over the domain \( \mathcal{T}_t(\mathcal{G}^*_t) \). Thus, \( P \)-a.s. \( \hat{\nu}_t \) solves the essential supremum problem of the RHS of (52) on the same domain, i.e.,

\[
E_P(\{\tau > t\}) \left[ X_\nu^2 \mathbf{1}_{\{\nu < \tau\}} + Y_\tau^2 \mathbf{1}_{\{\tau \leq \nu\}} \bigg| \mathcal{G}_t \right] = \operatorname{ess} \sup_{\nu \in \mathcal{T}_t(\mathcal{G}^*_t)} E_P(\{\tau > t\}) \left[ X_\nu^2 \mathbf{1}_{\{\nu < \tau\}} + Y_\tau^2 \mathbf{1}_{\{\tau \leq \nu\}} \bigg| \mathcal{G}_t \right], \quad P\text{-a.s.}
\]

(53)
Consequently, since \( \hat{\nu}_t \in T_t(G) \subseteq T_t(G^\tau) \), deduce that
\[
\text{ess sup}_{\nu \in T_t(G)} E_{P | \{ \tau > t \}} \left[ X_{\hat{\nu}_t}^2 1_{\{ \nu < \tau \}} + Y_{\hat{\nu}_t}^2 1_{\{ \tau \leq \nu \}} | G_t \right] = \text{es} \sup_{\nu \in T_t(G)} E_{P | \{ \tau > t \}} \left[ X_{\nu}^2 1_{\{ \nu < \tau \}} + Y_{\nu}^2 1_{\{ \tau \leq \nu \}} | G_t \right], \quad P\text{-a.s.} \tag{54}
\]

This, with the first equality in (10), implies that the following equation holds \( P\text{-a.s.} \)
\[
\text{ess sup}_{\nu \in T_t(G^\tau)} E_{P | \{ \tau > t \}} \left[ X_{\nu}^2 1_{\{ \nu < \tau \}} + Y_{\nu}^2 1_{\{ \tau \leq \nu \}} | G_t \right] = Y_{\nu}^2 1_{\{ \tau \leq \nu \}} \tag{55}
\]
and the result follows. \( \blacksquare \)

3.2 Proof of Proposition 1

Proposition 1 follows from the same arguments which were made by Hamadéne et al. [25] with certain modifications. Now, we elaborate on these modifications via several lemmata.

Lemma 3 For each \( n \geq 1 \), \( \tau_n \in T(F) \) and \( \nu_n \in T(G^\tau) \). In addition, both sequences \( (\tau_n)_{n \geq 1} \) and \( (\nu_n)_{n \geq 1} \) are \( P\text{-a.s.} \) nonincreasing.

Proof: For \( n = 1 \), the claim is immediate. Assume that it holds for some \( n \geq 1 \) and notice that this assumption yields that \( \nu_n \in T(F) \). Now, define
\[
U^{1,n}_t \equiv X^{1} t_{\{ t \leq \nu_n \}} + \left( X^{1}_{\nu_n = T} + Y^{1}_{\nu_n} 1_{\{ \nu_n < T \}} \right) 1_{\{ t \geq \nu_n \}}, \quad \forall t \in [0, T] \tag{56}
\]
and \( (U^{1,n}_t)_{t=0}^{T} \) satisfies the pre-conditions of Theorem 1.2 in [42] w.r.t. the filtration \( F \). Thus, deduce that \( \tilde{\tau}_{n+1} \) is the minimal-optimal stopping time of the problem
\[
\text{ess sup}_{\tau \in T(F)} E \left[ (U^{1,n}_\tau | F_0) \right], \tag{57}
\]
and hence \( \tilde{\tau}_{n+1} \in T(F) \). Then, showing that \( \tau_{n+1} \in T(F) \) follows by the same arguments which appear in the proof of Lemma 3.1 in [25]. Similarly, define
\[
U^{2,n}_t \equiv X^{2} t_{\{ t < \tau_n \}} + Y^{2}_{t} 1_{\{ t \leq \tau_n \}}, \quad \forall t \in [0, T], \tag{58}
\]
and notice that the current model assumptions yield that \( (U^{2,n}_t)_{t=0}^{T} \) satisfies the pre-conditions of Theorem 1.2 in [42] w.r.t. \( G^{\tau_n} \). Then, showing that
\( \nu_{n+1} \in \mathcal{T}(\mathcal{G}^{\tau_n}) \) and hence \( \nu_{n+1} \in \mathcal{T}(\mathcal{G}^{\tau_n}) \) follows by the same arguments which appear in the proof of Lemma 3.1 in Hamadène et al. [25]. Finally, the monotonicity of the sequences \( (\tau_n)_{n \geq 1} \) and \( (\nu_n)_{n \geq 1} \) is justified by construction and it is explained in the proof of Lemma 3.1 in Hamadène et al. [25].

**Lemma 4** For every \( n \geq 2 \), one has:

1. \[ J_1(\tau, \nu_n) \leq J_1(\tau_n, \nu_n), \quad \forall \tau \in \mathcal{T}(\mathcal{F}). \] (59)

2. \[ J_2(\tau_n, \nu) \leq J_2(\tau_n, \nu_n), \quad \forall \nu \in \mathcal{T}(\mathcal{G}). \] (60)

**Proof:** The proof is the same as the proof of Lemma 3.3 in [25]. The only thing that should be noticed (for the proof of the second part) is that \( \nu \in \mathcal{T}(\mathcal{G}) \) implies that \( \nu \land \tau_n \in \mathcal{T}(\mathcal{G}^{\tau_n}) \).

Observe that the discrete-time modelling with the condition (43) in the definition of \( \mathcal{D} \) implies that a standard convergence theorem argument might be used in order to justify the following convergences:

1. \( J_1(\tau_n, \nu) \to J_1(\tau^*, \nu) \) as \( n \to \infty \) for every \( \nu \in \mathcal{T}(\mathcal{G}) \).

2. \( J_2(\tau_n, \nu) \to J_2(\tau_n, \nu^*) \) as \( n \to \infty \) for every \( \tau \in \mathcal{T}(\mathcal{F}) \).

3. For each \( i = 1, 2 \), \( J_i(\tau_n, \nu_n) \to J_i(\tau^*, \nu^*) \) as \( n \to \infty \).

Finally, combining these results with the previous lemmata in this subsection yields the proof of Proposition [1].

### 3.3 Proof of Corollary [1]

Observe that the first direction is an immediate consequence of (22). In order to prove the other direction, assume that \( \theta^* < \tau^* \). Thus, there exists \( n' \geq 1 \) (which is a random variable) such that \( \theta_{n'+1} < \tau_n \) for every \( n \geq 1 \). In addition, according to (21), \( \nu_{n'+1} = \theta_{n'+1}, \quad P\text{-a.s.} \). Recall that \( (\tau_n)_{n \geq 1} \) and \( (\nu_n)_{n \geq 1} \) are nonincreasing sequences. As a result, by construction, \( \tau_{n'} = \tau_{n'+1} = \ldots \) and hence

\[ \tau^* = \lim_{n \to \infty} \tau_n > \theta_{n'+1} \geq \nu^*, \quad P\text{-a.s.} \] (61)

which makes (23) follows.
3.4 Proof of Lemma [1]

The best response of Player 2 is the same in both games and hence it is sufficient to show that $\tau_0$ is a best response of Player 1 to $\nu_0$ in $\Gamma$. To this end, observe that (8) yields that $X_1^t + \epsilon > Y_1^t$ for every $0 \leq t \leq T$. Therefore, since Player 1 has a priority in stopping and $\tau_0$ is a best response of Player 1 to $\nu_0$ in $\Gamma$, deduce that $\tau_0 \leq \nu_0$, $P$-a.s. (otherwise $\tau_0 \wedge \nu_0$ is a better response). Now, assume by contradiction that there exists $\tau' \in T(F)$ such that $J_1(\tau_0, \nu_0) < J_1(\tau', \nu_0)$. Then, (8) with the priority in stopping of Player 1 implies that $\tau' \wedge \nu_0$ is also a better response of Player 1, i.e.,

$$J_{1,\epsilon}(\tau' \wedge \nu_0, \nu_0) - \epsilon = J_1(\tau' \wedge \nu_0, \nu_0) \geq J_1(\tau', \nu_0) > J_1(\tau_0, \nu_0) = J_{1,\epsilon}(\tau_0, \nu_0) - \epsilon$$

from which a contradiction follows.

Acknowledgement: The author would like to express his deep gratitude to Yan Dolinsky and Eilon Solan for insightful discussions about the problem as well as for their valuable comments on early versions of the current work. In addition, special thanks are devoted to Reviewer 1 for detecting a mathematical gap in the original manuscript and to Reviewer 2 whose suggestions lead to some major changes in the structure.

References

[1] Ashkenazi-Golan, G., Rainer, C., & Solan, E. (2020). Solving two-state Markov games with incomplete information on one side. Games and Economic Behavior, 122, 83-104.

[2] Attard, N. (2017). Nash equilibrium in nonzero-sum games of optimal stopping for Brownian motion. Advances in Applied Probability, 49, 430-445.

[3] Attard, N. (2018). Nonzero-sum games of optimal stopping for Markov processes. Applied Mathematics AND Optimization, 77, 567-597.

[4] Aumann, R. J., Maschler, M., & Stearns, R. E. (1995). Repeated games with incomplete information. MIT press.

[5] Bismut, J. M. (1977). Sur un probleme de Dynkin. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 39, 31-53.
[6] Bruss, F. T., Drmota, M., & Louchard, G. (1998). The complete solution of the competitive rank selection problem. *Algorithmica*, **22**, 413-447.

[7] Chen, R. W., Rosenberg, B., & Shepp, L. A. (1997). A secretary problem with two decision makers. *Journal of Applied Probability*, **34**, 1068-1074.

[8] De Angelis, T., Ekström, E., & Glover, K. (2021). Dynkin games with incomplete and asymmetric information. *Mathematics of Operations Research*.

[9] De Angelis, T., & Ferrari, G. (2018). Stochastic nonzero-sum games: a new connection between singular control and optimal stopping. *Advances in Applied Probability*, **50**, 347-372.

[10] De Angelis, T., Ferrari, G., & Moriarty, J. (2018). Nash equilibria of threshold type for two-player nonzero-sum games of stopping. *The Annals of Applied Probability*, **28**, 112-147.

[11] Dynkin, E. (1967). Game variant of a problem on optimal stopping. *Soviet Math. Dokl*. **10**, 270-274.

[12] Enns, E. G., & Ferenstein, E. (1985). Horse game. *Journal of the Operations Research Society of Japan*, **28**, 51-62.

[13] Esmaeeli, N., Imkeller, P., & Nzengang, V. (2019). Dynkin game with asymmetric information. *Bulletin of the Iranian Mathematical Society*, **45**, 283-301.

[14] Ekström, E., & Peskir, G. (2008). Optimal stopping games for Markov processes. *SIAM Journal on Control and Optimization*, **47**, 684-702.

[15] Ferguson, T. S. (1989). Who solved the secretary problem? *Statistical science*, **4**, 282-289.

[16] Freeman, P. R. (1983). The secretary problem and its extensions: A review. *International Statistical Review/Revue Internationale de Statistique*, 189-206.

[17] Fushimi, M. (1981). The secretary problem in a competitive situation. *Journal of the Operations Research Society of Japan*, **24**, 350-359.
[18] Gapeev, P. V., & Rodosthenous, N. (2021). Optimal stopping games in models with various information flows. *Stochastic Analysis and Applications, 39*, 1050-1094.

[19] Gensbittel, F., & Grün, C. (2019). Zero-sum stopping games with asymmetric information. *Mathematics of Operations Research, 44*, 277-302.

[20] Ghemawat, P., & Nalebuff, B. (1985). Exit. *The RAND Journal of Economics, 16*, 184-194.

[21] Grün, C. (2013). On Dynkin games with incomplete information. *SIAM Journal on Control and Optimization, 51*, 4039-4065.

[22] Hamadène, S. (2006). Mixed zero-sum stochastic differential game and American game options. *SIAM Journal on Control and Optimization, 45*, 496-518.

[23] Hamadène, S., & Hassani, M. (2014). The multi-player nonzero-sum Dynkin game in discrete-time. *Mathematical Methods of Operations Research, 79*, 179-194.

[24] Hamadène, S., & Mohammed, H. (2014). The multiplayer nonzero-sum Dynkin game in continuous time. *SIAM Journal on Control and Optimization, 52*, 821-835.

[25] Hamadène, S., & Zhang, J. (2010). The continuous time nonzero-sum Dynkin game problem and application in game options. *SIAM Journal on Control and Optimization, 48*(5), 3659-3669.

[26] Jaśkiewicz, A., & Nowak, A. S. (2018). Non-zero-sum stochastic games. *Handbook of dynamic game theory*, 1-64.

[27] Kifer, Y. I. (1971). Optimal stopped games. *Theory of Probability and Its Applications, 16*, 185-189.

[28] Kifer, Y. (2000). Game options. *Finance and Stochastics, 4*, 443-463.

[29] Lang, J. P., & Kimeldorf, G. (1975). Duels with continuous firing. *Management Science, 22*, 470-476.

[30] R. Laraki. (2000). *Repeated games with incomplete information: a variational approach*. UPMC, Paris, France.
[31] Laraki, R., & Solan, E. (2013). Equilibrium in two-player non-zero-sum Dynkin games in continuous time. *Stochastics An International Journal of Probability and Stochastic Processes, 85*, 997-1014.

[32] Laraki, R., & Sorin, S. (2015). Advances in zero-sum dynamic games. *Handbook of game theory with economic applications 4* pp. 27-93. Elsevier.

[33] Lempa, J., & Matomäki, P. (2013). A Dynkin game with asymmetric information. *Stochastics An International Journal of Probability and Stochastic Processes, 85*, 763-788.

[34] Lepeltier, J. P., & Maingueneau, E. M. (1984). Le jeu de Dynkin en théorie générale sans l’hypothèse de Mokobodski. *Stochastics: An International Journal of Probability and Stochastic Processes, 13*, 25-44.

[35] Mamer, J. W. (1987). Monotone stopping games. *Journal of Applied Probability, 24*, 386-401.

[36] Morimoto, H. (1986). Non-zero-sum discrete parameter stochastic games with stopping times. *Probability theory and related fields, 72*, 155-160.

[37] Neveu, J. (1975). *Discrete-parameter martingales*. North-Holland, Amsterdam.

[38] Nowak, A. S., & Szajowski, K. (1999). Nonzero-sum stochastic games. In *Stochastic and differential games* (pp. 297-342). Birkhäuser, Boston, MA.

[39] Ohtsubo, Y. (1987). A nonzero-sum extension of Dynkin’s stopping problem. *Mathematics of Operations Research, 12*, 277-296.

[40] Ohtsubo, Y. (1991). On a discrete-time non-zero-sum Dynkin problem with monotonicity. *Journal of applied probability, 28*, 466-472.

[41] Peskir, G. (2009). Optimal stopping games and Nash equilibrium. *Theory of Probability and Its Applications, 53*, 558-571.

[42] Peskir, G., & Shiryaev, A. (2006). *Optimal stopping and free-boundary problems*. Birkhäuser Basel.

[43] Ramsey, D. M. (2007). A model of a 2-player stopping game with priority and asynchronous observation. *Mathematical Methods of Operations Research, 66*, 149-164.
[44] Renault, J. (2006). The value of Markov chain games with lack of information on one side. *Mathematics of Operations Research, 31*, 490-512.

[45] Rosenberg, D., Solan, E., & Vieille, N. (2001). Stopping games with randomized strategies. *Probability theory and related fields, 119*, 433-451.

[46] Sakaguchi, M. (1980). Non-zero-sum games related to the secretary problem. *Journal of the Operations Research Society of Japan, 23*, 287-293.

[47] Sakaguchi, M. (1995). Optimal stopping games: A review. *Mathematica japonicae, 42*, 343-351.

[48] Skarupski, M., & Szajowski, K. (2018). Full vs. no information best choice game with finite horizon. *arXiv preprint arXiv:1809.02955*.

[49] Shmaya, E., & Solan, E. (2004). Two-player nonZero-sum stopping games in discrete-time. *The annals of probability, 32*, 2733-2764.

[50] Yasuda, M. (1985). On a randomized strategy in Neveu’s stopping problem. *Stochastic Processes and their Applications, 21*, 159-166.