ON POLYGONAL MEASURES WITH VANISHING HARMONIC MOMENTS

DMITRII V. PASECHNIK AND BORIS SHAPIRO

To the memory of Andrei Zelevinsky

Abstract. A polygonal measure is the sum of finitely many real constant density measures supported on triangles in \( \mathbb{C} \). Given a finite set \( S \subset \mathbb{C} \), we study the existence of polygonal measures spanned by triangles with vertices in \( S \), which have all harmonic moments vanishing. For \( S \) generic, we show that the dimension of the linear space of such measures is \( \left( |S| - 3 \right) / 2 \).

We also investigate the situation where the resulting density attains only values 0 or \( \pm 1 \), which corresponds to pairs of polygons of unit density having the same logarithmic potential at \( \infty \). We show that such a signed measure does not exist if \( |S| \leq 5 \), but for each \( n \geq 6 \) there exists an \( S \), with \( |S| = n \), giving rise to such a signed measure.

1. Introduction and main results

Inverse problems in logarithmic potential theory have attracted substantial attention since the publication of the fundamental paper [15], where P.S. Novikov, in particular, proved that two convex (or, more generally, star-shaped) domains in \( \mathbb{C} \) with unit density cannot have the same logarithmic potential near \( \infty \). Notice that the knowledge of the germ of a logarithmic potential of a finite compactly supported Borel measure \( \mu \) at \( \infty \) is equivalent to the knowledge of the sequence of its harmonic moments \( m_j(\mu), \ j = 0, 1, \ldots, \) where the \( j \)-th harmonic moment of \( \mu \) is defined by:

\[
m_j(\mu) = \int_{\mathbb{C}} z^j d\mu(z).
\]

More precisely, if

\[
u_{\mu}(z) := \int_{\mathbb{C}} \ln |z - \xi| d\mu(\xi)
\]

is the logarithmic potential of \( \mu \) and

\[
\mathcal{C}_\mu(z) := \int_{\mathbb{C}} \frac{d\mu(\xi)}{z - \xi} - \frac{\partial \nu_{\mu}(z)}{\partial z}
\]

is its Cauchy transform then the Taylor expansion of \( \mathcal{C}_\mu(z) \) at \( \infty \) has the form:

\[
\mathcal{C}_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \ldots.
\]

Thus Novikov’s result can be reformulated as the statement that two convex domains in \( \mathbb{C} \) with unit density cannot have coinciding sequences of harmonic moments. It is well-known that already for non-convex domains with unit density the uniqueness in this problem no longer holds. For instance, examples of pairs of non-convex polygons with the same logarithmic potential near \( \infty \) can be found on [3, p. 333], see Fig. 1 below. The class of general polygons as well as domains...
bounded by lemniscates has attracted a substantial attention in this area. Several authors have also considered the class of polynomial densities instead of the unit density.

By a **convex polygon** we mean the convex hull of finite many points in the plane, at least 3 of which are non-collinear. A general **polygon** is the set-theoretic union of finitely many convex polygons. By a **vertex** of a polygon we mean a point of its boundary such that its sufficiently small $\epsilon$-neighborhood in the polygon is different from a half-disk of radius $\epsilon$.

Given an open set $D \subset \mathbb{C}$, define its **standard measure**

$$\mu_D = \chi_D \, dx \, dy,$$

where $\chi_D$ is the characteristic function of $D$. The same measure is associated with the closure of $D$. We say that two polygons in $\mathbb{C}$ are **equipotential** if their standard measures create coinciding logarithmic potential outside their union. Below we present one of the simplest examples of pairs of equipotential polygons given in [6, Example 1].

**Example 1.** Consider the 6-tuples $T = \{ \pm \sqrt{3} \pm I, \pm 2I \}$ and $T' = \{ \pm \frac{1 \pm \sqrt{3}I}{2}, \pm 1 \}$.

Let $F \subset \mathbb{C}$ be the difference of the convex hull of $T$ and the union of the set of 6 triangles obtained as the orbit of the triangle with nodes $(\sqrt{3} + I, \sqrt{3} - I, 1)$ under the rotation by $\frac{\pi}{3}$, see Fig. 1. Let $F' \subset \mathbb{C}$ be the difference of the convex hulls of $T$ and of $T'$. Then $F$ and $F'$ have the same logarithmic potential.

![Figure 1. Two equipotential polygons: $F$ on the left, $F'$ on the right.](image)

Notice that if different polygons with constant (but not necessarily unit) density have the same logarithmic potential near $\infty$ then they must have the same set of vertices, see [6, Corollary 2 and Lemma 2]. (The coincidence of the logarithmic potential near $\infty$ implies even more restrictions on the polygons than just the coincidence of their set of vertices, cf. [6].)

Taking this fact into account we pose the following **classical inverse logarithmic potential problem for polygons in $\mathbb{C}$**.

**Problem 1.** Given a finite set $S \subset \mathbb{C}$, determine whether there exist two equipotential polygons whose sets of vertices coincide with $S$.

One can show that for generic $S$ no pairs of equipotential polygons exist.

**Definition 1.** A complex (respectively, real) **polygonal measure** $\mu := \mu(D)$ is the sum

$$\mu := \sum_{\Delta \in D} c_{\Delta} \mu_{\Delta}, \quad c_{\Delta} \in \mathbb{C} \text{ (respectively, } c_{\Delta} \in \mathbb{R} \text{)},$$

where $D$ is a finite set of closed triangles in the plane. The set of vertices of the triangles $\Delta \in D$ with $c_{\Delta} \neq 0$ in (1.1) is called the set of nodes of this decomposition.
Notice that the decomposition (1.1) of a given \( \mu \) need not be unique, and different choices of \( D \) can lead to different sets of nodes.

Besides the nodes of decompositions (1.1) of \( \mu \) it is natural to talk about the \textit{vertices} of \( \mu \). They are \( v \in \mathbb{C} \) such that for any sufficiently small \( \epsilon > 0 \) the restriction of the density of \( \mu \) to the \( \epsilon \)-disk centered at \( v \) is neither constant, nor there exists a line through \( v \) dividing the disk into two halves with different constant densities.

Obviously, the set of vertices of \( \mu \) is a subset of the set of intersections of sides of the triangles in \( D \). There exists a finite collection \( \mathcal{D} \) of triangles with pairwise empty intersections of interiors, such that \( \mu = \mu(\mathcal{D}) \), and nodes and vertices of \( \mu \) coincide. However, such a representation of \( \mu \) need not be the most economic one, cf. e.g. Example 1.

Namely, in notation of Example 1 consider \( \tilde{\mu} := \mu_F - \mu_V \). Observe that \( \tilde{\mu} \) can be represented using only 6 nodes, although the polygons themselves have 12 vertices! This also illustrates the non-uniqueness of representation of \( \tilde{\mu} \) in the form (1.1). Indeed,

\[
2\tilde{\mu} = \sum_{0 \leq j \leq 5} \mu_{\exp(\frac{\pi j}{3})}(\sqrt{3}+1, \sqrt{3}-1, -2I) - \sum_{0 \leq j \leq 1} \mu_{3+((-1)^j I, \sqrt{3}-(-1)^j I, -\sqrt{3}+(-1)^j I)}.
\]

Let an \( S \) admit a pair of equipotential polygons. Taking the difference of their standard measures, one obtains a polygonal measure supported on the convex hull \( \text{conv}(S) \) of \( S \) with density attaining only values 0, \( \pm1 \) and with all harmonic moments vanishing. Conversely, if one can find a polygonal measure with all vanishing harmonic moments, and such that its density attains only values 0, \( \pm1 \), then one obtains a pair of equipotential polygons by taking the differences of \( \text{conv}(S) \) with the sets where the density attains value 1, respectively \( -1 \).

If we weaken the condition that the density of a polygonal measure attains only values 0, \( \pm1 \) then we arrive at the setup of the present paper. Given a \textit{spanning} set \( S \) (i.e. \( S \) contains at least 3 non-collinear points), we introduce the linear spaces \( \mathcal{M}_{\mathbb{R}}(S) \subset \mathcal{M}_{\mathbb{C}}(S) \) of real-valued, respectively, complex-valued polygonal measures obtained as real, respectively, complex linear spans of the standard measures of all triangles with vertices in \( S \). Obviously, \( \mathcal{M}_{\mathbb{C}}(S) = \mathbb{C} \otimes \mathcal{M}_{\mathbb{R}}(S) \).

We make a further step in the study of (non-)uniqueness in logarithmic potential theory by considering the following question.

\textbf{Problem 2.} Given a finite set \( S \subset \mathbb{C} \), determine the linear subspace \( \mathcal{M}_{\mathbb{R}}(S) \subset \mathcal{M}_{\mathbb{R}}(S) \) of real-valued polygonal measures (resp. of complex-valued polygonal measures \( \mathcal{M}_{\mathbb{C}}(S) \subset \mathcal{M}_{\mathbb{C}}(S) \)) with all harmonic moments vanishing.

The main technical tool we use is the \textit{normalized generating function} \( \Psi_\mu(u) \) for harmonic moments of a measure \( \mu \), defined by

\[
\Psi_\mu(u) = \sum_{j=0}^{\infty} \binom{j+2}{2} m_j(\mu) u^j. \tag{1.2}
\]

Notice that \( \Psi_\mu(u) \) is closely related to the Cauchy transform \( \mathcal{C}_\mu(z) \) at \( \infty \). Namely,

\[
\Psi_\mu(u) = \frac{1}{2} \frac{d^2}{du^2} \left( \sum_{j=0}^{\infty} m_j(\mu) u^{j+2} \right).
\]

At the same time for a compactly supported measure \( \mu \) and sufficiently large \( |z| \), \( z \mathcal{C}_\mu(z) = \sum_{j=0}^{\infty} m_j(\mu) |z|^j \). Thus for \( |u| \) sufficiently small,

\[
\Psi_\mu(u) = \frac{1}{2} \frac{d^2}{du^2} \left( u \mathcal{C}_\mu \left( \frac{1}{u} \right) \right).
\]
Similar multivariate generating functions were recently considered in [13]. Important in our consideration are the following observations.

**Proposition 1.** For measures $\mu$ with compact support,

$$
\Psi_\mu(u) = \sum_{j=0}^{\infty} \left( \frac{j+2}{2} \right) m_j(\mu) u^j = \int \frac{d\mu(z)}{(1-uz)^2}.
$$

(1.3)

The normalized generating function $\Psi_\Delta(u)$ of (the standard measure of) the triangle $\Delta \subset \mathbb{C}$ whose vertices are located at $a, b, c$ is given by

$$
\Psi_\Delta(u) = \frac{\text{Area } \Delta}{(1-au)(1-bu)(1-cu)}.
$$

Note that the integral transform in (1.3) appears to be a variant of Fantappiè transformation, cf. [4].

**Definition 2.** We say that a finite set $S = \{z_0, z_1, \ldots, z_n\}$ of points in $\mathbb{C}$ is **non-degenerate** if no three of its points are collinear.

**Proposition 2.** For any non-degenerate set $S = \{z_0, z_1, \ldots, z_n\}$, $n \geq 2$ of points in $\mathbb{C}$ and any fixed non-negative integer $j \leq n$, the set of (standard measures of) all triangles with a node at $z_j$ is a basis of the spaces $\mathcal{M}^R(S)$ and $\mathcal{M}^C(S)$. In particular,

$$
\dim_{\mathbb{R}} \mathcal{M}^R(S) = \dim_{\mathbb{C}} \mathcal{M}^C(S) = \left( \begin{array}{c} n+1 \\ \frac{n}{2} \end{array} \right).
$$

We are interested in linear subspaces $\mathcal{M}^R_{null}(S) \subset \mathcal{M}^R(S)$ (resp. $\mathcal{M}^C_{null}(S) \subset \mathcal{M}^C(S)$) of real-valued (resp. complex-valued) measures having all vanishing harmonic moments.

The main results of this paper are as follows.

**Proposition 3.** For any non-degenerate set $S = \{z_0, z_1, \ldots, z_n\}$, $n \geq 2$ of points in $\mathbb{C}$,

$$
\dim_{\mathbb{C}} \mathcal{M}^C_{null} = \left( \begin{array}{c} n-1 \\ 2 \end{array} \right).
$$

**Example 2.** For $n = 3$ the space $\mathcal{M}^C_{null}(S)$ is spanned by the complex-valued measure $\tilde{\mu}$ whose densities with respect to the basis of triangles $\Delta_{012}, \Delta_{013}, \Delta_{023}$ are given by:

$$
\begin{cases}
    d_{012} = (z_1 - z_2)/[012] \\
    d_{013} = (z_3 - z_1)/[013] \\
    d_{023} = (z_2 - z_3)/[023]
\end{cases},
$$

where $[i,j,k] = \det \begin{pmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{pmatrix}$ stands for twice the signed area of the triangle with nodes $z_i, z_j, z_k$ and $z_j = x_j + y_j I$, $I$ being the imaginary unit.

**Remark 1.** For $S$ non-degenerate, the space $\mathcal{M}^C_{null}(S)$ projects isomorphically on the linear subspace of $\mathcal{M}^C(S)$ spanned by all triangles $\Delta_{0,1,j}$ where $2 \leq i < j \leq n$. In other words, assigning arbitrarily complex-valued densities $d_{0,1,j}$, $2 \leq i < j \leq n$ we can uniquely determine the densities $d_{0,1,j}$, $j = 2, \ldots, n$ to get a measure belonging to $\mathcal{M}^C_{null}(S)$.

**Theorem 1.** For any non-degenerate set $S = \{z_0, z_1, \ldots, z_n\}$, $n \geq 2$ of points in $\mathbb{C}$,

$$
\dim_{\mathbb{R}} \mathcal{M}^R_{null}(S) = \left( \begin{array}{c} n-2 \\ 2 \end{array} \right).
$$
Remark 2. For $S$ non-degenerate, the space $\mathcal{M}^R_{null}(S)$ projects isomorphically on the linear subspace of $\mathcal{M}^R(S)$ spanned by all triangles $\Delta_{0,i,j}$ where $3 \leq i < j \leq n$. In other words, arbitrarily real-valued densities $d_{0,i,j}$, $3 \leq i < j \leq n$, uniquely determine the densities $d_{0,1,j}$, $j = 2, \ldots, n$ and $d_{0,2,j}$, $j = 3, \ldots, n$ of a measure belonging to $\mathcal{M}^R_{null}(S)$.

Theorem 2. For any non-degenerate 5-tuple $S = \{z_0, z_1, z_2, z_3, z_4\}$, the space $\mathcal{M}^R_{null}(S)$ is spanned by the real measure $\tilde{\mu}$ with densities with respect to the basis of triangles $\Delta_{012}, \Delta_{013}, \Delta_{014}, \Delta_{023}, \Delta_{024}, \Delta_{034}$ given by:

$$
\begin{align*}
\ell_{012} &= ||z_1 - z_2||^2[134][234]/[012] \\
\ell_{013} &= ||z_1 - z_3||^2[124][34]/[013] \\
\ell_{014} &= ||z_1 - z_4||^2[123][43]/[014] \\
\ell_{023} &= -||z_2 - z_3||^2[124][134]/[023] \\
\ell_{024} &= -||z_2 - z_4||^2[134][124]/[024] \\
\ell_{034} &= -||z_3 - z_4||^2[123][134]/[034] \\
\end{align*}
$$

Example 3. For the 5-tuple $\{0, 2, 3 + I, 1 + 3I, 2I\}$ the measure $3\tilde{\mu}$ is shown in Fig. 2 below. (In this case $3\tilde{\mu}$ has integer densities which are easier to show TexXnically.)

![Figure 2: Measure $3\tilde{\mu}$ spanning $\mathcal{M}^R_{null}(0, 2, 3 + I, 1 + 3I, 2I)$.](image)

Remark 3. Suppose that the densities of a polygonal measure $\mu \in \mathcal{M}^R_{null}(S)$ with respect to the basic triangles containing a fixed node (say $z_0$) are known. It is still desirable to find the densities in all its chambers, for instance in view of the classical Problem 1. Here by a chamber we mean a connected component of $conv(S) \backslash Arr(S)$, $Arr(S)$ being the union of all lines connecting pairs of points in $S$. (Integers in Fig. 2 show the densities in the chambers they are placed in.) Each chamber is contained in a number of basic triangles and the density of a given chamber equals the sum of the densities of all basic triangles containing it. Containment of chambers in triangles (and more generally in simplices in $\mathbb{R}^d$) can be coded by an appropriate incidence matrix whose rows correspond to simplices and columns correspond to chambers. If a simplex contains a chamber then the corresponding entry of the incidence matrix equals 1, otherwise the entry equals 0. Examples of incidence matrices are given in the proof of Theorem 3 below.
This incidence matrix of chambers and simplices in $\mathbb{R}^d$ was for the first time studied in [3] and later in [1, 2]. It has rather delicate properties and already the number of chambers is a complicated function of the initial non-degenerate set $S$. In particular, this number can change if we deform $S$ within the class of non-degenerate sets. This observation partially explains why results of the present paper do not automatically solve Problem 1.

Remark 4. Notice that if $S = \{z_0, \ldots, z_n\}$ consists of complex numbers having only rational real and imaginary parts then one can choose a basis of $\mathfrak{M}_{\text{nat}}(S)$ consisting of polygonal measures with integer densities.

Using Example 1 together with Theorem 2 we can prove the following result related to the classical Problem 1.

**Theorem 3.** For each $n \geq 6$ there exists $S$, with $|S| = n$, admitting a pair of equipotential polygons. No such $S$ exists if $|S| \leq 5$. The essential part of the proof of Theorem 3 is to deal with the case $|S| = 5$.

Our final result concerns a natural cone spanned by the standard measures of triangles with nodes in $S$. Namely, for an arbitrary non-degenerate set $S = \{z_0, z_1, \ldots, z_n\}$ denote by $\mathcal{R}(S) \subset \mathfrak{M}(S)$ the $\binom{n}{2}$-dimensional cone obtained by taking non-negative linear combinations of the standard measures of all triangles with nodes in $S$. (Recall that $\mathfrak{M}(S)$ is the linear span of these measures.)

**Theorem 4.** Extreme rays of $\mathcal{R}(S)$ are spanned by (the standard measures) of triangles which do not contain any point of $S$ different from its own nodes. In particular, if $S$ is a convex configuration, (i.e. each $z_j$ belongs to the convex hull of $S$) then every triangle with nodes in $S$ spans an extreme ray of $\mathcal{R}(S)$. We finish the introduction with a conjectural description of all faces of $\mathcal{R}(S)$. We say that a pair of triangles with vertices in $S$ forms a flip if they have a common side and their convex hull is a 4-gon. With any pair of triangles forming a flip we associate their flipped pair obtained by removing the opposite diagonal from their convex hull, see Case a) Fig. 3 below. (On this figure the pairs of triangles $(\Delta_{013}, \Delta_{123})$ and $(\Delta_{012}, \Delta_{023})$ form a flip and each pair is the flipped one to the other pair.)

**Conjecture 1.** A collection $Col$ of triangles having no internal vertices spans a face of $\mathcal{R}(S)$ if and only if for each pair of triangles from $Col$ forming a flip its flipped pair of triangles is also contained in $Col$. The necessity of the stated condition is quite obvious and its sufficiency might follow from the results of [3].

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2. Proofs

**Proof of Proposition 4.** First, we prove (1.3). Indeed,

$$\int \frac{d\mu(z)}{(1-u z)^k} = \sum_{k \geq 0} u^k \int (\frac{k+2}{2}) \frac{1}{z} d\mu(z) = \sum_{k \geq 0} u^k (\frac{k+2}{2}) m_k(\mu) = \Psi_u(n),$$

where $m_k(\mu) = \int z^k d\mu(z)$ for $k \geq 0$.
as required. By \([7, (1)]\), for any \(f(z)\) analytic in the closure of \(\Delta\), we have

\[
\frac{1}{2 \text{Area} \Delta} \int_{\Delta} f''(z) \, dx \, dy = \sum_{k=1, j \neq i \in \{1, 2, 3\} \setminus \{k\}} f(z_k) \frac{1}{(z_k - z_i)(z_k - z_j)}.
\]

Applying the latter identity and \([1.3]\) to \(f(z) = \frac{1}{2} u_2 u_1 (1 - uz)\), we get the claimed formula.

\[\Box\]

To prove Proposition 2 we need to recall some basic notions. First we present a description of all linear dependences among the standard measures of all triangles with vertices in a non-degenerate set \(S\). Namely, any 4-tuple of points (say, \(\{z_0, z_1, z_2, z_3\}\)) in \(S\) has 4 triangles with vertices at these points. To study linear dependences between these 4 triangles, one has to distinguish between two cases.

Consider the convex hull of \(\{z_0, z_1, z_2, z_3\}\), which is either a quadrangle or a triangle, see Fig. 3. Obviously, in Case a) we have (up to permutation of the vertices)

\[\mu_{\Delta_013} + \mu_{\Delta_123} = \mu_{\Delta_023} + \mu_{\Delta_012} .\]

Analogously, in Case b) we have (up to permutation of the vertices) the relation \(\mu_{\Delta_012} = \mu_{\Delta_013} + \mu_{\Delta_123} + \mu_{\Delta_023}\).

To complete the proof of Proposition 2 we need to show that if \(S\) is non-degenerate then the set of (the standard measures of) all triangles containing a given vertex \(z_j \in S\) spans \(\mathfrak{M}^3(S)\) and that this set is linearly independent. The former immediately follows from the discussion preceding Fig. 3. It remains to show the latter. We need more notions.

**Definition 3.** By a 2-chain \(C^{(2)}\) we mean a formal linear combination

\[
C^{(2)} = \alpha_1 \Delta_1 + \alpha_2 \Delta_2 + \ldots + \alpha_s \Delta_s
\]

of triangles \(\Delta_1, \ldots, \Delta_s\) in \(\mathbb{C}\) with real or complex coefficients where each triangle is equipped with the standard orientation induced from \(C\).

By using the standard pairing

\[
(f \, dx \, dy, C^{(2)}) = \int_{C^{(2)}} f \, dx \, dy = \sum_{j=1}^s \alpha_j \int_{\Delta_j} f \, dx \, dy
\]

one sees that a 2-chain \((2.1)\) defines a linear functional on the space \(\Omega^{(2)}\) of smooth 2-forms on \(\mathbb{C}\).

**Definition 4.** Analogously, by a 1-chain \(C^{(1)}\) we mean a formal linear combination

\[
C^{(1)} = \beta_1 I_1 + \beta_2 I_2 + \ldots + \beta_t I_t
\]

of oriented finite intervals \(I_1, \ldots, I_s\) in \(\mathbb{C}\) with real or complex coefficients.
Again, by using the standard pairing
\[
\langle w, C(1) \rangle = \int_{C(1)} w = \sum_{j=1}^{t} \beta_j \int_{I_j} w,
\]
where \( w \) is an arbitrary smooth 1-form, one sees that a 1-chain \( C(1) \) defines a linear functional on the space \( \Omega(1) \) of smooth 1-forms on \( C \).

**Definition 5.** For a given triangle \( \Delta \) with vertices \( a, b, c \) where triple \( (a, b, c) \) is counterclockwise oriented we define its boundary \( \partial \Delta \) as the sum of three oriented intervals \([ab] + [bc] + [ca] \). As usual, we extend the boundary operator \( \partial \) by linearity to the linear space of all 2-chains.

**Definition 6.** A 2-chain (resp. a 1-chain) is called vanishing if it defines the zero linear functional on \( \Omega(2) \) (resp. \( \Omega(1) \)).

**Lemma 1.** A 2-chain \( C(2) \) is vanishing if and only if its boundary \( \partial C(2) \) is a vanishing 1-chain.

**Proof.** Stokes theorem says that \( \int_{\partial \Delta} w = \int_{\Delta} dw \), where \( w \in \Omega(1) \), \( \Delta \) is an arbitrary triangle, \( \partial \Delta \) is its boundary and \( dw \) is the differential of \( w \). (Recall that if \( w = F(x, y)dx + G(x, y)dy \) then \( dw = (G'_x − F'_y)dx dy \).) Observe that any 2-form \( f(x, y)dx dy \) can be represented as \( dw \), where \( w_x = F(x, y)dx \) and \( F(x, y) \) is the primitive function of \(-f(x, y)\) along vertical lines. Analogously, \( f(x, y)dx dy \) equals \( dw_y \) where \( w_y = G(x, y)dy \) and \( G(x, y) \) is the primitive function of \( f(x, y) \) along horizontal lines. Thus
\[
\int_{C(2)} f dx dy = \int_{\partial C(2)} w_x = \int_{\partial C(2)} w_y.
\]
If the l.h.s. vanishes for all \( f dx dy \) then \( \partial C(2) \) should vanish and vice versa. \( \square \)

**Proof of Proposition 2** We need to show that for any non-degenerate \( S \) the standard measures of all triangles containing \( z_0 \) are linearly independent. Indeed, by Lemma 1 a 2-chain \( C(2) \) of triangles vanishes if and only its boundary \( \partial C(2) \) vanishes. But if \( S \) is non-degenerate then each triangle \( \Delta_{0, i, j} \) has its unique edge \((z_i, z_j)\) in the boundary and no chain of the form \( \beta_{i,j}(z_i, z_j) \) with non-trivial \( \beta_{i,j} \) can be vanishing. Therefore the standard measures of triangles \( \Delta_{0, i, j} \) form a basis in \( \mathcal{M}(S) \) and \( \mathcal{M}^h(S) \).

**Remark 5.** Proposition 2 has an interesting and immediate corollary, that the linear dependencies among the standard measures of all triangles with vertices in \( S \) are generated by the linear dependencies shown on Fig. 3 which come from all possible 4-tuples of vertices in \( S \).

It is a special case of [3] Theorem 1]. (Unfortunately, it seems that a proof of this important statement is missing in the available literature.) J.A. De Loera informed us that it can be derived from results in [8] (e.g. in the plane case one can use Lawson Theorem), or [9].

**Proof of Proposition 4** The case \( n = 2 \) is trivial, so we assume \( n \geq 3 \). Given a non-degenerate \( S = \{z_0, z_1, \ldots, z_n\} \), consider the complex-valued measure \( \mu \) obtained by assigning (complex) densities \( a_{0ij} \), \( 1 \leq i < j \leq n \) to triangles \( \Delta_{0ij} \). Set \( m_{ij} = a_{0ij} \text{Area } \Delta_{0ij} \). Then the normalized generating function \( \Psi_{\mu}(u) \) for harmonic moments of \( \mu \) is given by
\[
\Psi_{\mu}(u) = \sum_{1 \leq i < j \leq n} a_{0ij} \Psi_{\Delta_{0ij}}(u) = \sum_{1 \leq i < j \leq n} \frac{m_{ij}}{(1 - z_0u)(1 - z_iu)(1 - z_ju)} = \frac{1}{1 - z_0u} \prod_{i=1}^{n} (1 - z_iu)^{-1}, \tag{2.3}
\]
where \( P(u) \) is a polynomial of degree at most \( n - 2 \). Its coefficients at 1, \( u, u^2, \ldots, u^{n-2} \) are the consecutive entries of the vector \( \mathcal{M}_n^C \cdot \mathbf{m}_n \), where

\[
\mathbf{m}_n = (m_{12}, m_{13}, \ldots, m_{n-1, n})^T
\]

with \( m_{ij} \)'s ordered lexicographically, and \( \mathcal{M}_n^C \) is the \((n - 1) \times \binom{n}{2}\)-matrix with columns corresponding to \( m_{ij} \). Such a column contains consecutive elementary symmetric functions of the \((n - 2)\)-tuple \((-\bar{z}_1, -\bar{z}_2, \ldots, -\bar{z}_i, \ldots, -\bar{z}_j, \ldots, -\bar{z}_n)\), where \( \bar{z}_i \) and \( \bar{z}_j \) stands for the omission of these points.

**Example 4.** For \( n = 4 \) the coefficients at \((1, u, u^2)\) of the numerator \( P(u) \) of (2.3) are the consecutive entries of the vector \( \mathcal{M}_4^C \cdot \mathbf{m}_4 \) where

\[
\mathbf{m}_4 = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})^T
\]

and

\[
\mathcal{M}_4^C = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
-z_2 - z_4 & -z_2 - z_4 & -z_2 - z_4 & -z_2 - z_4 & -z_2 - z_4 & -z_2 - z_4 \\
- \bar{z}_2 \bar{z}_4 & - \bar{z}_2 \bar{z}_4 & - \bar{z}_2 \bar{z}_4 & - \bar{z}_2 \bar{z}_4 & - \bar{z}_2 \bar{z}_4 & - \bar{z}_2 \bar{z}_4 \\
\end{pmatrix}.
\]

In other words,

\[
P(u) = \sum_{1 \leq i < j \leq 4, k < t, (ij) \cap (kt) = \emptyset} (m_{ij} - (z_i + z_j) m_{kt} u + z_i z_j m_{kt} u^2).
\]

Consider the maximal minor \( \text{Min}_n^C \) of \( \mathcal{M}_n^C \) formed by the columns corresponding to \( m_{12}, \ldots, m_{1n} \), i.e. the first \( n - 1 \) columns of \( \mathcal{M}_n^C \).

**Lemma 2.** \( \det_n = (-1)^{n-1} \det(\text{Min}_n^C) = (-1)^{n-1} \prod_{2 \leq i < j \leq n} (z_i - z_j) \).

**Proof.** Indeed, the degree of \( \det(\text{Min}_n^C) \) as a polynomial in \( z_2, \ldots, z_n \) equals \( (n-1) \). We need to show that it vanishes if and only if \( z_i = z_j \). The 'if' part is obvious since the column corresponding to \( m_{1i} \) will coincide with the column corresponding to \( m_{1j} \). To see the remaining part, argue by contradiction and assume that \((\alpha_{12}, \ldots, \alpha_{nn})\) is a nontrivial linear dependence among the columns of \( \text{Min}_n^C \). The 1\(k\)-th column consists of the coefficients of the polynomial \( g_{1k}(u) = \frac{1 - z_1 u}{u - z_k} \), and our linear dependence is a linear dependence among such polynomials. Evaluate these at \( \frac{1}{z_k} \) and note that \( g_{1k}(\frac{1}{z_k}) \) vanish whenever \( k \neq j \). Thus \( \alpha_{1j} = 0 \), a contradiction. Thus \( \det(\text{Min}_n^C) \) is divisible by \( \prod_{2 \leq i < j \leq n} (z_i - z_j) \). Substituting \( z_2 = 0, z_3 = 1, \ldots, z_n = n - 2 \) we can check that the normalizing factor equals \((-1)^{n-1}\). \( \square \)

**Remark 6.** By using Cramer’s rule, it is not difficult to give an explicit formula for the inverse \((\text{Min}_n^C)^{-1}\).

From Lemma 2 we know that for any, not necessarily non-degenerate, \( S = \{ z_0, z_1, \ldots, z_n \} \) with pairwise distinct points the rank of \( \mathcal{M}_n^C \) equals \( n - 1 \). Thus the kernel of \( \mathcal{M}_n^C \), which by definition coincides with \( \mathfrak{M}_n \), has dimension \( \binom{n}{2} - (n - 1) = \binom{n-2}{2} \). \( \square \)

**Proof of Theorem 4.** The space \( \mathfrak{M}_n \subset \mathfrak{M}_n(S) \) is the maximal by inclusion real subspace of the complex kernel. In other words, it can be interpreted as the real kernel of the real matrix \( \mathcal{M}_n^R \) obtained by taking the real and imaginary parts of all rows of \( \mathcal{M}_n^C \).

The case \( n = 2 \) is trivial. The case \( n = 3 \) can be dealt with by explicitly computing the kernel of \( \mathcal{M}_3^C \) and seeing that it does not contain real vectors if \( S \) is non-degenerate. Thus we assume \( n \geq 4 \). Since the first row of \( \mathcal{M}_n^C \) equals \((1, 1, \ldots, 1)\) the matrix \( \mathcal{M}_n^R \) has size \((2n - 3) \binom{n}{2} \), see (2.3). Ordering \( m_{ij} \)'s lexicographically, consider the maximal minor \( \text{Min}_n^R \) of \( \mathcal{M}_n^R \) formed by the columns
corresponding to \((2n-3)\) variables \(m_{12}, m_{13}, \ldots, m_{1n}, m_{23}, m_{24}, \ldots, m_{2n}\), \(\text{i.e.}\) the first \((2n-3)\) columns of \(\mathcal{M}_n^R\).

**Lemma 3.** \(\det{\operatorname{Min}_n^R} = C[123][124] \cdots [12n] \prod_{3 \leq i < j \leq n} |z_i - z_j|^2,\) \(0 \neq C \in \mathbb{R}\).

**Proof.** We begin by showing that \(\Theta := \det{\operatorname{Min}_n^R}\) is divisible by \([12k]\) for any \(3 \leq k \leq n\). As \([12k]\) is an irreducible quadratic polynomial in \(x_1, x_2, x_k\) and \(y_1, y_2, y_k\), it suffices to show that vanishing of \([12k]\) implies vanishing of \(\Theta\). Vanishing of \([12k]\) is equivalent to existence of \(a \in \mathbb{R}\) satisfying \(z_k = a z_1 + (1-a) z_2\). The latter implies that \(\operatorname{Min}_n^R\) has linearly dependent columns 12, 1k, and 2k. Indeed, they consist, respectively, of the coefficients of

\[
\begin{align*}
ge_{12}(u) &= (1 - a z_1 u - (1-a) z_2 u) x (1 - z_3 u) \cdots (1 - z_{k-1} u) (1 - z_{k+1} u) \cdots (1 - z_n u) \\
ge_{1k}(u) &= (1 - z_2 u) x (1 - z_3 u) \cdots (1 - z_{k-1} u) (1 - z_{k+1} u) \cdots (1 - z_n u) \\
ge_{2k}(u) &= (1 - z_1 u) x (1 - z_3 u) \cdots (1 - z_{k-1} u) (1 - z_{k+1} u) \cdots (1 - z_n u)
\end{align*}
\]

which are linearly dependent: \(g_{12}(u) = (a-1)g_{1k}(u) - ag_{2k}(u)\).

To show that \(\Theta\) is divisible by \(|z_i - z_j|^2 = (z_i - z_j)(z_i - z_j)\) for any \(3 \leq i < j \leq n\), observe that \(z_i = z_j\) implies \(g_{ij}(u) = g_{ij}(u)\) for \(k = 1, 2\).

It remains to see that \(\Theta\) is not identically 0. Arguing by contradiction, let \((\alpha_{12}, \alpha_{13}, \ldots, \alpha_{1n}, \alpha_{23}, \ldots, \alpha_{2n})\) be the coefficients of a nontrivial real linear dependence among the columns of \(\operatorname{Min}_n^R\). The latter columns correspond to the coefficients of \(g_{ij}(u)\). Evaluating these at \(u = \frac{1}{z_k}\), for \(3 \leq k \leq n\), makes all of them but \(g_{1k}\) and \(g_{2k}\) vanish. Thus

\[
\alpha_{1k} g_{1k}(z_k^{-1}) + \alpha_{2k} g_{2k}(z_k^{-1}) = 0,\]

implying \(\alpha_{1k} = -\alpha_{2k} z_k - z_2\) and \(z_k - z_2 \in \mathbb{R}\).

A direct computation shows that the rightmost relation is equivalent to \([12k] = 0\), a contradiction. \(\square\)

Lemma 3 implies that for any non-degenerate \(S\) the matrix \(\mathcal{M}_n^R\) has rank equal to \(2n-3\). Therefore, \(\dim{\mathfrak{M}_n^{\text{null}}(S)} = \binom{n}{2} - (2n-3) = \binom{n-2}{2}.\) \(\square\)

**Proof of Theorem 2** For \(S = \{0, z_1, z_2, z_3, z_4\}\) the space \(\mathfrak{M}_n^{\text{null}}(S)\) is given by the system

\[
\mathcal{M}_4^R \cdot m_4 = 0, \quad \text{where } m_4 = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})^T \quad \text{and}
\]

\[
\mathcal{M}_4^R = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
-x_3 x_4 & -x_2 x_4 & -x_2 x_3 & -x_1 x_2 & -x_1 x_3 & -x_1 x_4 \\
-x_3 y_4 & -y_2 y_4 & -y_2 y_3 & -y_1 y_2 & -y_1 y_3 & -y_1 y_4 \\
x_3 x_4 y_3 y_4 & x_2 x_4 y_2 y_4 & x_2 x_3 y_2 y_3 & x_1 x_4 y_1 y_4 & x_1 x_3 y_1 y_3 & x_1 x_2 y_1 y_2 \\
x_3 x_4 y_3 y_4 & x_2 x_4 y_2 y_4 & x_2 x_3 y_2 y_3 & x_1 x_4 y_1 y_4 & x_1 x_3 y_1 y_3 & x_1 x_2 y_1 y_2
\end{pmatrix}
\]

Recall that a \(k \times (k+1)\)-matrix \(T\) of rank \(k\) has right kernel spanned by the vector \((T^{(1)} \cdots T^{(k+1)})^T\), where \(T^{(j)}\) is the minor of \(T\) with \(j\)th column removed multiplied by \((-1)^j\). Thus \((2,4)\) has a unique (up to a scaling) solution of the form:

\[
\begin{align*}
m_{12} &= |z_1 - z_2|^2 \binom{[134][234]}{1} \\
m_{13} &= |z_1 - z_3|^2 \binom{[124][234]}{1} \\
m_{14} &= |z_1 - z_4|^2 \binom{[123][234]}{1} \\
m_{23} &= |z_2 - z_3|^2 \binom{[124][134]}{1} \\
m_{24} &= |z_2 - z_4|^2 \binom{[124][134]}{1} \\
m_{34} &= |z_3 - z_4|^2 \binom{[123][124]}{1}
\end{align*}
\]

It is easy to prove this. We give a sketch here for \(m_{12}\). Note that \(m_{12}\) equals to the determinant of the matrix \(A^{(12)}\) obtained from \(\mathcal{M}_4^R\) by removing the 1st
column. Then, \( \det A^{(12)} \) is divisible by \( ||z_1 - z_2||^2 \), as the rank of \( A^{(12)} \) drops when \( z_1 = z_2 \), and as \( ||z_1 - z_2||^2 = (z_1 - z_2)(\overline{z_1} - \overline{z_2}) \) is the product of two irreducible polynomials with complex coefficients.

Similarly, \( \det A^{(12)} \) is divisible by \( [234] \) (and a very similar argument applies to \([134]\)). To see this, note that, as \([234]\) is irreducible, it suffices to show that its vanishing implies vanishing of \( \det A^{(12)} \). To this end, assume that \( z_4 = a z_2 + (1 - a) z_3 \), with \( a \in \mathbb{R} \), and make this substitution into \( A^{(12)} \). The last 3 columns of \( A^{(12)} \) become

\[
\begin{pmatrix}
1 & -a x_2 + a x_3 - x_1 - x_3 & -x_1 - x_3 - x_1 - x_2 \\
1 & -a y_2 + a y_3 - y_1 - y_3 & -y_1 - y_3 - y_1 - y_2 \\
1 & a x_1 x_2 - a x_1 x_3 - a y_1 y_2 + a y_1 y_3 + x_1 x_3 - y_1 y_3 & x_1 x_3 - y_1 y_3 - x_1 x_2 - y_1 y_2 \\
1 & a x_1 y_2 - a x_1 y_3 + a x_2 y_1 - a x_3 y_1 + x_1 y_3 + x_3 y_1 & x_1 y_3 + x_3 y_1 - x_1 y_2 + x_2 y_1
\end{pmatrix}
\]

They are linearly dependent with coefficients \((1, a - 1, -a)\).

\[ \square \]

**Proof of Theorem 3.** To prove the first part, we recall that Example 1 settles the case \(|S| = 2\). To settle the case \(|S| = 6 + q\), we modify the latter Example. Add \(q\) points \(P_1, \ldots, P_q\) outside \(\text{conv}(T)\), so that so that \(P_1, \ldots, P_q\) and \(\sqrt{3} \pm I\) are in the convex position, and \(Q\) is the resulting convex \(q + 2\). Then \(F \cup Q\) and \(F' \cup Q\) are equipotential \((6 + q)\)-gons, by additivity of the measure.

To prove the second part, we have consider the cases \(|S| = 3, 4, 5\), one by one. Cases \(|S| = 3, 4\) follow from Theorem 1.

It remains to deal with the only non-trivial case \(|S| = 5\). We have to consider the incidence matrices between the chambers and the basic simplices for all possible non-degenerate 5-tuples of points \(S\). One can easily see that for non-degenerate 5-tuples there are (up to permutation of the vertices) only 3 different cases to consider depending on the shape of \(\text{conv}(S)\) which can be a 5-gon, a 4-gon, or a triangle. The corresponding incidence matrices \(\text{Inc}_1, \text{Inc}_2, \text{Inc}_3\) are given below using the labeling presented in Fig. 4 and 5 for these cases. (Greek letters in Fig. 4 denote the vertices of the inner 5-gon. They will be needed below.) We show that in none of these case one can find a pair of equipotential polygons.

![Figure 4. Chambers and their labeling for conv(S) a 5-gon.](image-url)
Lemma 4. We need an elementary fact: the area of triangle $\triangle I012$ is equal to the area of triangle $\triangle I024$. Indeed, they are obtained from removing heights.

For brevity, we introduce notation $\frac{1}{2}||K||$ for the area of a polygon $K$. First, we need an elementary

Lemma 4. For an arbitrary triangle $\triangle ABC$ and arbitrary secants $\alpha\delta\gamma$, see Fig. 6, the area of triangle $\triangle ABC$ is bigger than that of $\triangle BCD$:

$$||\triangle ABC|| > ||\triangle BCD||.$$

Proof. Indeed, draw the line $\alpha\kappa$ parallel to $\beta\gamma$ and extend $\beta\delta$ till it hits $\alpha\kappa$. (The intersection point of the latter lines is denoted by $\eta$.) Triangles $\triangle ABC$ and $\triangle BCD$ have equal area. Indeed, they are obtained from $\triangle ABC$ and $\triangle BCD$, respectively, by removing $\triangle ABC$. Notice that $\triangle ABC$ and $\triangle BCD$ have the same base $\alpha\eta$ and equal heights.

Case 2. Using labeling on the left part of Fig. 6 and 7, we conclude that densities $d_{012}, d_{014}, d_{023}, d_{034}$ are positive while $d_{013}, d_{024}$ are negative. From chambers $E$ and $C$ we conclude $d_{023} = d_{012} = 1$. Then from chamber $D$ we have that either $d_{024} \geq -1$ or $-2$. The second case leads to $d_{013} = 0$, contradiction. Thus $d_{024} \leq -1$ which from chamber $I$ gives $d_{013} = -1$. Finally, $d_{034} = d_{014} = 1$. Thus chambers $A, C, E, G$ have density 1, chamber $I$ has density $-1$ and the remaining chambers have vanishing density. We need to show that $||I|| < ||A|| + ||C|| + ||E|| + ||G||$. We will show that actually $||I|| < ||C|| + ||G||$. Cut $I$ into two triangles by drawing its diagonal connecting $z_4$ with non-neighboring vertex $p$ of $I$ (lying strictly above $z_4$ in the left part of Fig. 6). Extending $z_3p$ and $z_0z_4$ we get a triangle containing $G$ and the left half of $I$ and we can apply Lemma 3. Analogously, extending $z_2p$
Lemma 4. Thus the required measure does not exist.

**Case 3.** Using labeling on the right part of Fig. 6 and formulas (1.4) we again conclude that densities \( d_{012}, d_{014}, d_{024} \) are negative, while \( d_{013}, d_{023}, d_{034} \) are positive. Similar considerations as above give \( d_{012} = d_{014} = d_{024} = -1 \) and \( d_{013} = d_{023} = d_{034} = 1 \). Thus, the densities of \( A, C, E \) are \(-1\) and the density of \( G \) is \( 1 \). In fact, \( |G| < |A| \) already. Indeed, extending the interval \( z_0z_4 \) and \( z_1z_3 \) till they intersect at a point, say \( p \) we get the triangle \( z_0z_1p \) to which we apply Lemma 4 Thus the required measure does not exist.

**Case 1.** Using labeling on Fig. 6 and (1.4) we see that densities \( d_{012}, d_{014}, d_{023}, d_{034} \) are positive while \( d_{013}, d_{024} \) are negative. Assuming that the densities of all chambers attain only values \( 0, \pm 1 \) and looking at chambers \( C, E, G \) we get that \( d_{023} = d_{012} = d_{034} = 1 \). Looking at chamber \( D \) we conclude that \( d_{024} = -1 \). (It might be equal \(-2\) as well but then looking at chamber \( K \) we have to conclude that \( d_{013} = 0 \) which is impossible.) From chamber \( K \) we get \( d_{013} = -1 \) and from chamber \( I \) we get \( d_{014} = 1 \). Thus the density in chambers \( A, C, E, G, I \) equals \( 1 \), in chamber \( K \) it equals \(-1\) and it vanishes in the remaining chambers. Notice that the total mass of the measure should vanish. To see that this cannot happen, we show that \( |K| < |A| + |C| + |E| + |G| + |I| \). Using Lemma 4 we conclude that \( |A| > |\Delta_{\alpha\beta\gamma}|, |C| > |\Delta_{\alpha\beta\gamma}|, |E| > |\Delta_{\beta\gamma\delta}|, |G| > |\Delta_{\delta\gamma\eta}| \), and \( |I| > |\Delta_{\alpha\delta}| \), see Fig. 4 Triangles \( \Delta_{\alpha\beta\gamma}, \Delta_{\alpha\delta\gamma}, \Delta_{\beta\gamma\delta}, \Delta_{\delta\gamma\eta}, \Delta_{\alpha\delta} \) pairwise overlap. These overlapping consists of 5 smaller triangles inside \( K \). The complement in \( K \) to the union of triangles \( \Delta_{\alpha\beta\gamma}, \Delta_{\alpha\delta\gamma}, \Delta_{\beta\gamma\delta}, \Delta_{\delta\gamma\eta}, \Delta_{\alpha\delta} \) is a small 5-gon inside \( K \). Now we can use these 5 small triangles to cover the small 5-gon inside \( K \). We get exactly the same situation as the original one and we can apply the same argument as we did and cover a substantial part of the small 5-gon etc. Continuing this process we will in infinitely many steps exhaust the original 5-gon \( K \). Thus the required measure does not exist. □

To prove Theorem 4 we need the following observation.

**Lemma 5.** The convex hull of the standard measures of 4 triangles as in Case a) Fig. 6 i.e. two pairs forming a flip is a plane quadrangle. The convex hull of the standard measures of 4 triangles as in Case b) Fig. 5 is a plane triangle.

*Proof.* Obvious from the relations given above Fig. 5 □
Proof of Theorem 4. Indeed if a triangle $\Delta$ contains an interior point other than its vertices than $\mu_\Delta$ is the sum of three triangles in which it is subdivided by an inner vertex, see Lemma 5. (Recall that $S$ is non-degenerate by assumption.) Thus $\mu_\Delta$ is not an extreme ray. On the other hand, assume that no point in $S$ other than its vertices is contained in $\Delta$ and $\mu_\Delta$ is a linear combination of the standard measures of some other triangles with vertices in $S$ with positive coefficients. Since no such triangle can be contained strictly inside $\Delta$ by assumption and all coefficients are positive we get that any such linear combination necessarily has positive density somewhere outside $\Delta$, contradiction.

3. Open problems

1. Theorem 1 gives the dimension of $M_{\text{null}}^R(S)$ for non-degenerate $S$. Its dimension for arbitrary $S$ is unclear. On one hand, if $S$ is degenerate then $\dim M^R(S)$ decreases. On the other hand, the number of equations imposed on the densities might also decrease. It seems highly plausible that $\dim M_{\text{null}}^R(S)$ for an arbitrary $S$ depends only on non-oriented matroid associated to this set, see e.g. [10]. An algorithm calculating this dimension is given in [2].

2. Besides the cone $\mathcal{R}(S) \subset M^R(S)$ one can introduce a more important, bigger, cone $\mathcal{R}_{\text{pos}}(S) \subset M^R(S)$ where $\mathcal{R}_{\text{pos}}(S) \supset \mathcal{R}(S)$ consists of all non-negative measures from $M^R(S)$.

Conjecture 2. The combinatorial structure of $\mathcal{R}_{\text{pos}}(S)$ depends only on the oriented matroid associated to $S$.

Already for generic configurations $S$ with 6 points the combinatorial structure of $\mathcal{R}_{\text{pos}}(S)$ and, in particular, the set of its extreme rays seems to be quite complicated. We plan to study this fascinating subject in the future.

3. Notice that we have a natural linear map $\Psi_\mu : M^R(S) \to \text{Rat}_n$ obtained by associating to each measure $\mu \in M^R(S)$ its normalized generating function (1.2). Here $\text{Rat}_n$ is the linear space of rational functions of the form $R(u) = \sum_{\deg P(u) \leq n-2} P(u) \prod_{j=0}^{n-2} (1-z_j u)$ having real constant term. Obviously, $\dim \text{Rat}_n = 2n-3$ and using Theorem 1 we obtain that $M^R(S)$ is mapped onto $\text{Rat}_n$. The following question is very natural in connection with the inverse problem for the class of non-negative measures.

Problem 3. Describe the extreme rays/faces of the image cones $\Psi_\mu(\mathcal{R}(S))$ and $\Psi_\mu(\mathcal{R}_{\text{pos}}(S))$ in $\text{Rat}_n$.

4. We have an example of a pair of equipotential polygons with $|S| = 6$, see Fig. 1.

Problem 4. Describe all 6-tuples $S$ admitting a pair of equipotential polygons.

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School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, 637371 Singapore
E-mail address: dima@ntu.edu.sg

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
E-mail address: shapiro@math.su.se