Monogamy and polygamy for multi-qubit entanglement using Rényi entropy

Jeong San Kim and Barry C. Sanders
Institute for Quantum Information Science, University of Calgary, Alberta T2N 1N4, Canada
E-mail: jekim@ucalgary.ca, sandersb@ucalgary.ca

Abstract. Using Rényi-$\alpha$ entropy to quantify bipartite entanglement, we prove monogamy of entanglement in multi-qubit systems for $\alpha \geq 2$. We also conjecture a polygamy inequality of multi-qubit entanglement with strong numerical evidence for $0.83 - \epsilon \leq \alpha \leq 1.43 + \epsilon$ with $0 < \epsilon < 0.01$.

PACS numbers: 03.67.-a, 03.65.Ud
1. Introduction

One distinct property of quantum entanglement from other classical correlations is its restricted sharability. For instance, if a pair of parties in a multipartite quantum system share maximal entanglement, then they can share neither entanglement [1, 2] nor classical correlations [3] with the rest. This is known as the Monogamy of Entanglement (MoE) [4], and it has been shown that this restricted sharability of quantum entanglement can be used as a resource to distribute a secret key which is secure against unauthorized parties [5, 6].

Whereas MoE is a restricted property of entanglement in multipartite quantum systems, the sharability itself is about the bipartite entanglements among the parties in multipartite systems. In other words, it is inevitable to have a proper way of quantifying bipartite entanglement for a good description of the monogamy nature in multipartite quantum systems. For this reason, certain criteria of bipartite entanglement measure were recently proposed for a good description of the monogamy nature of entanglement in multipartite quantum systems [7]; that is,

(i) **Monotonicity**: the property that ensures entanglement cannot be increased under local operations and classical communications.

(ii) **Separability**: capability of distinguishing entanglement from separability.

(iii) **Monogamy**: upper bound on a sum of bipartite entanglement measures thereby showing that bipartite sharing of entanglement is bounded.

The first mathematical characterization of MoE was shown in three-qubit systems [1] using **concurrence** [8] as the bipartite entanglement measure. It is known as **CKW-inequality** named after its establishers, Coffman, Kundu and Wootters, and this CKW-type inequality was also shown for arbitrary multi-qubit systems later [2]. In other words, concurrence is a good entanglement measure for multi-qubit-systems that satisfies the criteria proposed in [7].

However, monogamy inequality using concurrence is known to fail in its generalization for higher-dimensional quantum systems [9, 7]. Furthermore, although MoE in multi-qubit systems is mathematically well-characterized in terms of concurrence, it is not generally true for other entanglement measures such as **Entanglement of Formation** (EoF) [10]. In other words, MoE does not have CKW-type characterization in terms of EoF, and this exposes the importance of the choice of a bipartite entanglement measure to characterize MoE even in multi-qubit systems. Moreover, for possible generalization of monogamy inequality into higher-dimensional quantum systems, it is undoubtedly one of the most important and necessary tasks to have a proper way of quantifying bipartite entanglement.

Rényi-\(\alpha\) entropy [11] is a generalization of Shannon entropy [12] in terms of the non-negative real parameter \(\alpha\), and it has been widely used in the study of quantum information theory such as quantum entanglement and correlations [13, 14, 15]. As a way to quantify the uncertainty of probability distribution, Rényi-\(\alpha\) entropy shows...
its selective characters with respect to $\alpha$: The higher values of $\alpha$, Rényi-\(\alpha\) entropy is increasingly determined by consideration of only the highest probability events, whereas it increasingly weights all possible events more equally, regardless of their probabilities for lower values of $\alpha$. For the case when $\alpha$ tends to 1, Rényi-\(\alpha\) entropy converges to Shannon entropy.

Here, we show that the distinct character of Rényi-\(\alpha\) entropy with respect to the parameter $\alpha$ also appears in establishing monogamy inequalities of entanglement in multipartite quantum systems. Although EoF that is based on Shannon entropy is known to fail for usual CKW-type characterization of MoE, Rényi-\(\alpha\) entropy can still be shown to have CKW-type monogamy inequality for all case of $\alpha$ if it exceeds a certain threshold. We first provide an analytic formula for the bipartite entanglement measure based on Rényi-\(\alpha\) entropy namely Rényi-\(\alpha\) entanglement in two-qubit systems for $\alpha \geq 1$, and we show that multi-qubit entanglement shows the usual CKW-type monogamy inequalities in terms of Rényi-\(\alpha\) entanglement for $\alpha \geq 2$. For $\alpha < 2$, we conjecture with strong numerical evidence that Rényi-\(\alpha\) entropy can provide a possible dual monogamy or polygamy inequality of multi-qubit entanglement for $0.83 - \epsilon \leq \alpha \leq 1.43 + \epsilon$ with $0 < \epsilon < 0.01$.

This paper is organized as follows. In Section 2.1, we recall the definition of Rényi-\(\alpha\) entropy, and Rényi-\(\alpha\) entanglement for bipartite quantum states. In Section 2.2, we provide an analytic formula of Rényi-\(\alpha\) entanglement for arbitrary two-qubit states. In Section 3.1, we derive a monogamy inequality of multi-qubit entanglement in terms of Rényi-\(\alpha\) entanglement for $\alpha \geq 2$, and we conjecture a polygamy inequality of multi-qubit entanglement using Rényi-\(\alpha\) entropy with strong numerical evidences in Section 3.2. Finally, we summarize our results in Section 4.

2. Rényi-\(\alpha\) Entropy and Entanglement Measures

2.1. Definition

For a probability distribution $P = \{p_i\}$ where $0 \leq p_i \leq 1$ for all $i$ and $\sum_i p_i = 1$, its classical Rényi-\(\alpha\) entropy is defined as

$$H_\alpha(P) = \frac{1}{1-\alpha} \log \sum_i p_i^\alpha,$$

for any positive $\alpha$ such that $\alpha \neq 1$. Throughout this paper, the logarithmic function is assumed to have base two otherwise specified. In the limiting case where $\alpha$ tends to 1, $H_\alpha(P)$ converges to Shannon entropy, that is,

$$\lim_{\alpha \to 1} H_\alpha(P) = - \sum_i p_i \log p_i = H(P).$$

For any quantum state $\rho$, its quantum Rényi-\(\alpha\) entropy is defined as

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \text{tr} \rho^\alpha,$$
for any $\alpha > 0$ and $\alpha \neq 1$ [16]. For the quantum state $\rho$ with its spectral decomposition $\rho = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i|$, we have

$$S_\alpha(\rho) = H_\alpha(X),$$

where $X = \{\lambda_i\}$ is the spectrum of $\rho$, and thus, $S_\alpha(\rho)$ converges to the von Neumann entropy of $\rho$ for the case when $\alpha \to 1$. In other words, Shannon entropy and von Neumann entropy are the singular points of classical and quantum Rényi entropies respectively, and those singularities are removable. For this reason, we will just consider $H_1(P) = H(P)$ and $S_1(\rho) = S(\rho)$ for any probability distribution $P$ and quantum state $\rho$.

For a bipartite pure state $|\psi\rangle_{AB}$, the von Neumann entropy of the reduced density matrix $\rho_A = \text{tr}_B |\psi\rangle_{AB} \langle \psi|$ is known to be a good bipartite entanglement measure

$$E(|\psi\rangle_{AB}) = S(\rho_A) = S(\rho_B).$$

With noticing that von Neumann entropy quantifies the uncertainty of the quantum state, this way of quantifying bipartite entanglement is based on the uncertainty of subsystem: More uncertainty on subsystems implies stronger quantum correlation between subsystems.

A well-known way to generalize this concept of entanglement measure into mixed states is taking the minimum (or infimum) of the average entanglements

$$E_t(\rho_{AB}) = \min \sum_i p_i E(|\psi_i\rangle_{AB})$$

over all possible pure state decompositions of the mixed state $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$. This generalization is known as convex-roof extension, and $E_t(\rho_{AB})$ is called the entanglement of formation of $\rho_{AB}$.

As a generalization of EoF into the full spectrum of Rényi-\(\alpha\) entropy [17], Rényi-\(\alpha\) entanglement of a bipartite pure state $|\psi\rangle_{AB}$ is defined as

$$E_\alpha(|\psi\rangle_{AB}) = S_\alpha(\rho_A),$$

where $\rho_A = \text{tr}_B |\psi\rangle_{AB} \langle \psi|$, and for a mixed state $\rho_{AB}$, its Rényi-\(\alpha\) entanglement is defined as,

$$E_\alpha(\rho_{AB}) = \min \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}),$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$. Similar to EoF for bipartite quantum states, Rényi-\(\alpha\) entanglement is defined based on the uncertainty of subsystems, which has EoF as a special case when $\alpha \to 1$.

It is direct to check that $E_\alpha(\rho_{AB}) = 0$ if and only if $\rho_{AB}$ is a separable state, and furthermore, Rényi-\(\alpha\) entanglement is also known as entanglement monotone: it is not increased under local operations and classical communications.
2.2. Analytical formula for two-qubit systems

Let us recall the definition of concurrence. For any bipartite pure state \(|\phi\rangle_{AB}\), its concurrence, \(C(|\phi\rangle_{AB})\) is defined as

\[
C(|\phi\rangle_{AB}) = \sqrt{2(1 - \text{tr} \rho^2_A)},
\]

where \(\rho_A = \text{tr}_B(|\phi\rangle_{AB}\langle\phi|)\), and for any mixed state \(\rho_{AB}\), its concurrence is defined as

\[
C(\rho_{AB}) = \min \sum_k p_k C(|\phi_k\rangle_{AB}),
\]

where the minimum is taken over all possible pure state decompositions, \(\rho_{AB} = \sum_k p_k|\phi_k\rangle_{AB}\langle\phi_k|\).

For any two-qubit mixed state \(\rho_{AB}\) in \(B(\mathbb{C}^2 \otimes \mathbb{C}^2)\), its concurrence is known to have an analytic formula \[8\], that is,

\[
C(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},
\]

where \(\lambda_i\)'s are the eigenvalues, in decreasing order, of \(\sqrt{\rho_{AB}\bar{\rho}_{AB}\sqrt{\rho_{AB}}}\) and \(\bar{\rho}_{AB} = \sigma_y \otimes \sigma_y \rho_{AB}^* \sigma_y \otimes \sigma_y\) with the Pauli operator \(\sigma_y\). Furthermore, the relation between concurrence and EoF of a two-qubit mixed state \(\rho_{AB}\) (or a pure state \(|\psi\rangle_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^d\)), can be given as a monotone increasing, convex function \(E\) \[8\], such that

\[
E_I(\rho_{AB}) = E(C_{AB}),
\]

where

\[
E(x) = H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - x^2}\right), \quad \text{for } 0 \leq x \leq 1,
\]

with the binary entropy function \(H(t) = -t \log t - (1 - t) \log(1 - t)\). In other words, the analytic formula of concurrence as well as its functional relation with EoF lead us to an analytic formula of EoF for two-qubit states.

For any two-qubit pure state (or any pure state with Schmidt-rank less than or equal to two) with its Schmidt decomposition \(|\psi\rangle_{AB} = \sqrt{\lambda_0}|00\rangle_{AB} + \sqrt{\lambda_1}|11\rangle_{AB}\), its Rényi-\(\alpha\) entanglement is

\[
E_{\alpha}(|\psi\rangle_{AB}) = S_{\alpha}(\rho_A) = \frac{1}{1 - \alpha} \log (\lambda_0^\alpha + \lambda_1^\alpha).
\]

Now, for each \(\alpha > 0\), by defining an analytic function

\[
f_\alpha(x) := \frac{1}{1 - \alpha} \log \left[\left(\frac{1 - \sqrt{1 - x^2}}{2}\right)^\alpha + \left(\frac{1 + \sqrt{1 - x^2}}{2}\right)^\alpha\right]
\]

on \(0 \leq x \leq 1\), it can be directly checked that

\[
E_{\alpha}(|\psi\rangle_{AB}) = f_\alpha(C(|\psi\rangle_{AB})),
\]

where \(C(|\psi\rangle_{AB})\) is the concurrence of \(|\psi\rangle_{AB}\). Thus, for each \(\alpha > 0\), we have a functional relation between the concurrence and Rényi-\(\alpha\) entanglement for two-qubit pure state.
Here, we note that for the case when $\alpha$ tends to 1, $f_\alpha(x)$ converges to the function $E(x)$ in Eq. (13); that is,

$$\lim_{\alpha \to 1} f_\alpha(x) = E(x).$$  \hspace{1cm} (17)

For a mixed state $\rho_{AB}$, we have the following theorem.

**Theorem 1.** For each $\alpha > 0$, if

$$f_\alpha(x) = \frac{1}{1-\alpha} \log \left[ \left( \frac{1-\sqrt{1-x^2}}{2} \right)^{\alpha} + \left( \frac{1+\sqrt{1-x^2}}{2} \right)^{\alpha} \right]$$  \hspace{1cm} (18)

is a monotonically increasing and convex function, then

$$E_\alpha (\rho_{AB}) = f_\alpha (C(\rho_{AB})), \hspace{1cm} (19)$$

for any two-qubit state $\rho_{AB}$ where $C(\rho_{AB})$ is the concurrence and $E_\alpha (\rho_{AB})$ is the Rényi-$\alpha$ entanglement of $\rho_{AB}$.

**Proof.** Suppose $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i| \rangle$ is an optimal decomposition for $E_\alpha (\rho_{AB})$ such that $E_\alpha (\rho_{AB}) = \sum_i p_i E_\alpha (|\psi_i\rangle_{AB})$, then

$$E_\alpha (\rho_{AB}) = \sum_i p_i E_\alpha (|\psi_i\rangle_{AB})$$

$$= \sum_i p_i f_\alpha (C(|\psi_i\rangle_{AB}))$$

$$\geq f_\alpha \left( \sum_i p_i C(|\psi_i\rangle_{AB}) \right)$$

$$\geq f_\alpha (C(\rho_{AB})) \hspace{1cm} (20)$$

where the second equality is by the relation between concurrence and Rényi-$\alpha$ entanglement for pure states in Eq. (16), the first inequality is by the convexity of $f_\alpha$, and the last inequality is by the monotonicity of $f_\alpha$ and the definition of $C(\rho_{AB})$.

Due to the analytic formula of concurrence for two-qubit states [8], we can always assume the existence of an optimal decomposition $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i| \rangle$ such that

$$C(\rho_{AB}) = \sum_i p_i C(|\psi_i\rangle_{AB}) \hspace{1cm} (21)$$

and

$$C(|\psi_i\rangle_{AB}) = C(\rho_{AB}) \hspace{1cm} (22)$$

for all $i$. Now, we have

$$f_\alpha (C(\rho_{AB})) = f_\alpha \left( \sum_i p_i C(|\psi_i\rangle_{AB}) \right)$$

$$= \sum_i p_i f_\alpha (C(|\psi_i\rangle_{AB}))$$

$$= \sum_i p_i E_\alpha (|\psi_i\rangle_{AB})$$

$$\geq E_\alpha (\rho_{AB}) \hspace{1cm} (23)$$
where the second equality is by Eq. (22) and the inequality is by the definition of $E_\alpha (\rho_{AB})$.

Thus, by Eqs. (20) and (23), we have
\[ E_\alpha (\rho_{AB}) = f_\alpha (C(\rho_{AB})), \tag{24} \]
which completes the proof.

Theorem 1 implies that for any positive $\alpha$ such that $f_\alpha (x)$ in Eq. (15) is monotonically increasing and convex, the analytic formula of concurrence for a two-qubit state $\rho_{AB}$ in Eq. (11) together with the functional relation between $C(\rho_{AB})$ and $E_\alpha (\rho_{AB})$ in Eq. (19) provide us an analytic formula for Rényi-$\alpha$ entanglement of $\rho_{AB}$.

The rest of this section is mainly contributed to the analytic proof for the range of $\alpha$ where $f_\alpha (x)$ is monotonically increasing and convex.

Theorem 2. For any real $\alpha \geq 1$,
\[ f_\alpha (x) = \frac{1}{1 - \alpha} \log \left( \frac{\left( 1 - \sqrt{1 - x^2} \right)^\alpha}{2} + \frac{\left( 1 + \sqrt{1 - x^2} \right)^\alpha}{2} \right) \tag{25} \]
is a monotonically increasing and convex function for $0 \leq x \leq 1$. Furthermore, the monotonicity and convexity of $f_\alpha (x)$ are strict in the sense that $f_\alpha (x) < f_\alpha (x')$ and $f_\alpha (\lambda x + (1 - \lambda)y) < \lambda f_\alpha (x) + (1 - \lambda)f_\alpha (y)$ for any $0 < \lambda < 1$ and $0 \leq x, x', y \leq 1$ such that $x < x'$.

Proof. For the case when $\alpha$ tends to 1, $f_\alpha (x)$ converges to $E(x)$ in Eq. (13) which is already known to be monotonically increasing and convex \cite{8}. Here, we consider the case where $\alpha > 1$.

Let us define a function
\[ g_\alpha (x) := - \log \left( \left( 1 - \sqrt{1 - x^2} \right)^\alpha + \left( 1 + \sqrt{1 - x^2} \right)^\alpha \right), \tag{26} \]
then we have
\[ f_\alpha (x) = \frac{1}{\alpha - 1} g_\alpha (x) + \frac{\alpha}{\alpha - 1}, \tag{27} \]
which implies that the convexity and the monotonicity of $f_\alpha (x)$ follow from those of $g_\alpha (x)$ for $\alpha > 1$. Furthermore, $g_\alpha (x)$ is an analytic function on $0 \leq x \leq 1$, thus the convexity and monotonicity of $g_\alpha (x)$ can be assured by showing its first and second derivatives are nonnegativity.

By taking the first derivative of $g_\alpha (x)$, we have
\[ \frac{dg_\alpha (x)}{dx} = \alpha x \left[ \frac{(1 + \sqrt{1 - x^2})^{\alpha - 1} - (1 - \sqrt{1 - x^2})^{\alpha - 1}}{(1 - \sqrt{1 - x^2})^\alpha + (1 + \sqrt{1 - x^2})^\alpha} \right] \geq 0 \tag{28} \]
for all $\alpha > 1$ and the equality holds only at the boundary, that is $x = 0$ or $x = 1$. In other words, $g_\alpha (x)$ is a strictly monotone-increasing function for $0 \leq x \leq 1$. 


For the second derivative of $g_{\alpha}(x)$, we have
\[
\frac{d^2 g_{\alpha}(x)}{dx^2} = B \left( \frac{A_1^{\alpha-1} - A_2^{\alpha-1}}{\sqrt{1-x^2}} (A_1^\alpha + A_2^\alpha) \right. \\
+ B \left[ x^2 (A_1^{\alpha-1} - A_2^{\alpha-1})^2 - 4(\alpha-1)x^{2\alpha-2} \right]
\]
where $A_1 = 1 + \sqrt{1-x^2}$, $A_2 = 1 - \sqrt{1-x^2}$ and $B = \alpha/ \left[ (1-x^2) (A_1^\alpha + A_2^\alpha)^2 \right]$. The binomial series for $A_1^{\alpha-1}$ and $A_2^{\alpha-1}$ lead us to
\[
A_1^{\alpha-1} = \left( 1 + \sqrt{1-x^2} \right)^{\alpha-1} \\
= 1 + (\alpha-1)\sqrt{1-x^2} + \frac{(\alpha-1)(\alpha-2)}{2!} (\sqrt{1-x^2})^2 + \cdots
\]
and
\[
A_2^{\alpha-1} = \left( 1 - \sqrt{1-x^2} \right)^{\alpha-1} \\
= 1 - (\alpha-1)\sqrt{1-x^2} + \frac{(\alpha-1)(\alpha-2)}{2!} (\sqrt{1-x^2})^2 - \cdots,
\]
and thus
\[
A_1^{\alpha-1} - A_2^{\alpha-1} = 2(\alpha-1)\sqrt{1-x^2} + C_1 \geq 2(\alpha-1)\sqrt{1-x^2},
\]
\[
A_1^{\alpha-1} + A_2^{\alpha-1} = 2 + C_2 \geq 2,
\]
for some non-negative $C_1$ and $C_2$. Now, let
\[
\Gamma_1 = \frac{(A_1^{\alpha-1} - A_2^{\alpha-1}) (A_1^\alpha + A_2^\alpha)}{\sqrt{1-x^2}}, \quad \Gamma_2 = x^2 (A_1^{\alpha-1} - A_2^{\alpha-1})^2,
\]
then by Eq. (32), we have
\[
\Gamma_1 \geq 4(\alpha-1), \quad \Gamma_2 \geq 4(\alpha-1)^2x^2(1-x^2),
\]
and
\[
\frac{d^2 g_{\alpha}(x)}{dx^2} = B \left[ \Gamma_1 + \Gamma_2 - 4(\alpha-1)x^{2\alpha-2} \right] \\
\geq B \left[ 4(\alpha-1) - 4(\alpha-1)x^{2\alpha-2} + 4(\alpha-1)^2x^2(1-x^2) \right] \\
\geq 0,
\]
for any $\alpha > 1$ where the equality holds if and only if $x = 1$. Thus, the first and second derivatives of $g_{\alpha}(x)$ are nonnegative for $0 \leq x \leq 1$ and strictly positive for $0 < x < 1$, which implies the strict monotonicity and convexity of $f_{\alpha}(x)$ on $0 \leq x \leq 1$ for $\alpha > 1$. □

Now, we have the following corollary, which is the first main result of this paper.

**Corollary 1.** For any $\alpha \geq 1$, a two-qubit state $\rho_{AB}$ has an analytic formula for its Rényi-$\alpha$ entanglement such that $E_{\alpha}(\rho_{AB}) = f_{\alpha}(C(\rho_{AB}))$ where $C(\rho_{AB})$ is the concurrence of $\rho_{AB}$, and
\[
f_{\alpha}(x) = \frac{1}{1-\alpha} \log \left[ \left( \frac{1-\sqrt{1-x^2}}{2} \right)^{\alpha} + \left( \frac{1+\sqrt{1-x^2}}{2} \right)^{\alpha} \right].
\]
In fact, \( \frac{ds_{\alpha}(x)}{dx} \) can be easily checked to be nonpositive for \( 0 < \alpha < 1 \), and together with Eq. (27), we have \( \frac{df_{\alpha}(x)}{dx} = \frac{1}{\alpha-1} \frac{ds_{\alpha}(x)}{dx} \geq 0 \) for \( 0 < \alpha < 1 \). In other words, the function \( f_{\alpha}(x) \) in Eq. (15) is a monotone-increasing function on the domain of \( 0 \leq x \leq 1 \) for any positive \( \alpha \).

For the convexity of \( f_{\alpha}(x) \) with \( 0 < \alpha < 1 \), we first note that the continuity of \( \frac{d^2s_{\alpha}(x)}{dx^2} \) in Eq. (29) with respect to \( \alpha \) assures the positivity of \( \frac{d^2s_{\alpha}(x)}{dx^2} \) for \( \alpha \) slightly less than 1. However, for general \( \alpha \) between 0 and 1, we tried a numerical way of calculations for the second derivative of \( f_{\alpha}(x) \) which is illustrated in Figure 1.

We have tested \( \frac{d^2s_{\alpha}(x)}{dx^2} \) for various values of \( \alpha \) between 0 and 1 and observed that \( \frac{d^2s_{\alpha}(x)}{dx^2} \) is nonnegative for \( \alpha \geq 0.83 \) (Figure 1 (a) and (b)), whereas its positivity is violated for \( \alpha = 0.82 \) with \( x \) around 1 (Figure 1 (c)). In other words, \( f_{\alpha}(x) \) is numerically observed to be still convex for \( 0.83 - \epsilon < \alpha < 1 \) with some positive \( \epsilon \) such that \( 0 \leq \epsilon < 0.01 \). Thus, together with Theorem 2, we have the following conjecture.

**Conjecture 1.** For any positive \( \alpha > 0.83 - \epsilon \) with some positive \( \epsilon \) such that \( 0 \leq \epsilon < 0.01 \),

\[
f_{\alpha}(x) = \frac{1}{1-\alpha} \log \left( \frac{1{}^{-\sqrt{1-x^2}}}{2} \right)^{\alpha} + \left( \frac{1{}^{+\sqrt{1-x^2}}}{2} \right)^{\alpha}
\]

(37)

is a convex function for \( 0 \leq x \leq 1 \). Furthermore, for this range of \( \alpha \), any two-qubit state \( \rho_{AB} \) has an analytic formula for its Rényi-\( \alpha \) entanglement such that \( E_{\alpha}(\rho_{AB}) = f_{\alpha}(C(\rho_{AB})) \) where \( C(\rho_{AB}) \) is the concurrence of \( \rho_{AB} \).

### 3. Entanglement Constraint in Multi-party Quantum Systems

In this section, we establish a mathematical formulation for the monogamous and polygamous properties of multi-qubit entanglement in terms of Rényi-\( \alpha \) entanglement.
First, we show an important property of the function $f_\alpha(x)$ in Eq. (15). By using the property of $f_\alpha(x)$ as well as the functional relation between Rényi-$\alpha$ entanglement and concurrence for two-qubit states in previous section, we derive a monogamy inequality of multi-qubit entanglement in terms of Rényi-$\alpha$ entanglement for $\alpha \geq 2$. We also conjecture a polygamy inequality of multi-qubit entanglement in terms of Rényi-$\alpha$ entanglement for $\alpha$ around 1.

3.1. Monogamy of multi-qubit entanglement

For a three-qubit pure state $|\psi\rangle_{ABC}$ CKW-inequality was shown as [1],

$$C^2_{A(BC)} \geq C^2_{AB} + C^2_{AC},$$

(38)

where $C_{A(BC)} = C(|\psi\rangle_{A(BC)})$ is the concurrence of a 3-qubit state $|\psi\rangle_{A(BC)}$ with respect to the bipartite cut between $A$ and $BC$, and $C_{AB}$ and $C_{AC}$ are the concurrences of the reduced density matrices onto subsystems $AB$ and $AC$ respectively. Later, Eq. (38) was generalized into arbitrary multi-qubit systems as

$$C^2_{A_1(A_2\cdots A_n)} \geq C^2_{A_1A_2} + \cdots + C^2_{A_1A_n},$$

(39)

for an $n$-qubit state $\rho_{A_1\cdots A_n}$ [2].

In this section, we provide a class of monogamy inequalities in multi-qubit systems in terms of Rényi-$\alpha$ entanglement. Before we prove this, we first show a property of the function $f_\alpha(x)$ that plays a crucial role in the proof of Rényi-$\alpha$ entanglement monogamy.

**Theorem 3.** For any real $\alpha \geq 2$ and the function,

$$f_\alpha(x) = \frac{1}{1-\alpha} \log \left[ \left( \frac{1 - \sqrt{1-x^2}}{2} \right)^\alpha + \left( \frac{1 + \sqrt{1-x^2}}{2} \right)^\alpha \right],$$

(40)

we have

$$f_\alpha\left(\sqrt{x^2+y^2}\right) \geq f_\alpha(x) + f_\alpha(y)$$

(41)

for $0 \leq x, y \leq 1$ such that $0 \leq x^2 + y^2 \leq 1$.

**Proof.** Let $D = \{(x,y) | 0 \leq x, y, x^2 + y^2 \leq 1\}$ and $h_\alpha(x,y)$ be a function defined on the domain $D$ such that

$$h_\alpha(x,y) := f_\alpha\left(\sqrt{x^2+y^2}\right) - f_\alpha(x) - f_\alpha(y).$$

(42)

Then, it is enough to show that $h_\alpha(x,y)$ is a non-negative function on $D$.

Because $h_\alpha(x,y)$ is continuous on the domain $D$ and analytic in the interior of $D$, its maximum or minimum arises at the critical points or boundary of $D$. Here, we will show that $h_\alpha(x,y)$ has neither vanishing gradient in the interior of $D$ nor negative function values on the boundary of $D$, which implies that $h_\alpha(x,y)$ is a non-negative function on the domain $D$.

First, let us consider the gradient of $h_\alpha(x,y)$,

$$\nabla h_\alpha(x,y) = \left( \frac{\partial h_\alpha(x,y)}{\partial x}, \frac{\partial h_\alpha(x,y)}{\partial y} \right).$$

(43)
where the first-order partial derivatives of \( h_\alpha(x, y) \) are

\[
\frac{\partial h_\alpha(x, y)}{\partial x} = \frac{C x \left[ (1 + \sqrt{1 - x^2 - y^2})^{\alpha-1} - (1 - \sqrt{1 - x^2 - y^2})^{\alpha-1} \right]}{\sqrt{1 - x^2 - y^2} \left[ (1 + \sqrt{1 - x^2 - y^2})^{\alpha} + (1 + \sqrt{1 - x^2 - y^2})^{\alpha} \right]}
\]

\[
- \frac{C x \left[ (1 + \sqrt{1 - x^2})^{\alpha-1} - (1 - \sqrt{1 - x^2})^{\alpha-1} \right]}{\sqrt{1 - x^2} \left[ (1 - \sqrt{1 - x^2})^{\alpha} + (1 + \sqrt{1 - x^2})^{\alpha} \right]},
\]

(44)

and

\[
\frac{\partial h_\alpha(x, y)}{\partial y} = \frac{C y \left[ (1 + \sqrt{1 - x^2 - y^2})^{\alpha-1} - (1 - \sqrt{1 - x^2 - y^2})^{\alpha-1} \right]}{\sqrt{1 - x^2 - y^2} \left[ (1 + \sqrt{1 - x^2 - y^2})^{\alpha} + (1 + \sqrt{1 - x^2 - y^2})^{\alpha} \right]}
\]

\[
- \frac{C y \left[ (1 + \sqrt{1 - y^2})^{\alpha-1} - (1 - \sqrt{1 - y^2})^{\alpha-1} \right]}{\sqrt{1 - y^2} \left[ (1 - \sqrt{1 - y^2})^{\alpha} + (1 + \sqrt{1 - y^2})^{\alpha} \right]},
\]

(45)

with \( C = \frac{\alpha}{\alpha-1} \). Now, suppose \( \nabla h_\alpha(x_1, y_1) = (0, 0) \) for some \((x_1, y_1)\) in the interior of \( \mathcal{D} \), that is, \( 0 < x_1, y_1 < 1 \) and \( 0 < x_1^2 + y_1^2 < 1 \). Because both \( x_1 \) and \( y_1 \) are nonzero, from Eqs. (44) and (45), we have

\[
\frac{\left[ (1 + \sqrt{1 - x_1^2})^{\alpha-1} - (1 - \sqrt{1 - x_1^2})^{\alpha-1} \right]}{\sqrt{1 - x_1^2} \left[ (1 - \sqrt{1 - x_1^2})^{\alpha} + (1 + \sqrt{1 - x_1^2})^{\alpha} \right]} = \frac{\left[ (1 + \sqrt{1 - y_1^2})^{\alpha-1} - (1 - \sqrt{1 - y_1^2})^{\alpha-1} \right]}{\sqrt{1 - y_1^2} \left[ (1 - \sqrt{1 - y_1^2})^{\alpha} + (1 + \sqrt{1 - y_1^2})^{\alpha} \right]}.
\]

(46)

Furthermore, by defining a function \( l_\alpha(x) \) on \( 0 < x \leq 1 \) such that

\[
l_\alpha(x) := \frac{\left[ (1 + \sqrt{1 - x^2})^{\alpha-1} - (1 - \sqrt{1 - x^2})^{\alpha-1} \right]}{\sqrt{1 - x^2} \left[ (1 - \sqrt{1 - x^2})^{\alpha} + (1 + \sqrt{1 - x^2})^{\alpha} \right]},
\]

(47)

Eq. (46) can be rewritten as

\[
l_\alpha(x_1) = l_\alpha(y_1).
\]

(48)

Here, we note that the first derivative of \( l_\alpha(x) \) is

\[
\frac{dl_\alpha(x)}{dx} = B \left[ \frac{(A_1^\alpha + A_2^\alpha) (A_1^{\alpha-1} - A_2^{\alpha-1})}{\sqrt{1 - x^2}} + (A_1^{\alpha-1} - A_2^{\alpha-1})^2 \right]
\]

\[
- 4B (\alpha - 1) x^{2\alpha-4}
\]

(49)
with $A_1 = 1 + \sqrt{1 - x^2}$, $A_2 = 1 - \sqrt{1 - x^2}$ and $B = \frac{x}{(1-x^2)(A_1^\alpha + A_2^\alpha)^2}$, and from the inequalities in Eq. (52), we have

$$\frac{(A_1^\alpha + A_2^\alpha)(A_1^{\alpha - 1} - A_2^{\alpha - 1})}{\sqrt{1 - x^2}} \geq 4(\alpha - 1),$$

$$\frac{(A_1^{\alpha - 1} - A_2^{\alpha - 1})}{x} \geq 4(\alpha - 1)^2(1 - x^2),$$

which implies

$$\frac{dl_\alpha(x)}{dx} \geq 4B \left[ (\alpha - 1) + (\alpha - 1)^2(1 - x^2) - (\alpha - 1)x^{2\alpha - 4} \right].$$

For the region of $\alpha \geq 2$ and $0 < x < 1$, $\alpha - 1$ in the second term of the right-hand side of Eq. (51) is always larger than or equal to 1, whereas $x^{2\alpha - 4}$ in the last term is always less than or equal to 1, thus

$$\frac{dl_\alpha(x)}{dx} \geq 4B \left[ (\alpha - 1) + (\alpha - 1)^2(1 - x^2) - (\alpha - 1)x^{2\alpha - 4} \right] \geq 4B \left[ (\alpha - 1) + (\alpha - 1)(1 - x^2) - (\alpha - 1) \right] = 4B(\alpha - 1)(1 - x^2) \geq 0,$$ (52)

and the last inequality is strict for $0 < x < 1$. In other words, $l_\alpha(x)$ is a strictly monotone-increasing function on $0 < x < 1$, and therefore Eq. (18) implies $x_1 = y_1$. (Also note that the possible range of $x_1$ now becomes $0 \leq x_1 \leq \frac{1}{\sqrt{2}}$.) However, from Eqs. (14) and (15), $\nabla h_\alpha(x_1, y_1) = (0, 0)$ together with $x_1 = y_1$ implies that

$$l_\alpha(\sqrt{2}x_1) = l_\alpha(x_1),$$

which contradicts to the strict monotonicity of $l_\alpha(x)$ for $0 < x < 1$. Thus, $\nabla h_\alpha(x_1, y_1) \neq (0, 0)$ for any $(x_1, y_1)$ in the interior of $\mathcal{D}$.

Now, let us consider the function value of $h_\alpha(x, y)$ on the boundary of $\mathcal{D}$, that is $x = 0$ or $y = 0$, or $x^2 + y^2 = 1$. If one of $x$ or $y$ is 0, then it is clear to check that $h_\alpha(x, y) = 0$. For the case when $x^2 + y^2 = 1$, we have

$$h_\alpha(x, y) = f_\alpha(\sqrt{x^2 + y^2}) - f_\alpha(x) - f_\alpha(y) = f_\alpha(1) - f_\alpha(x) - f_\alpha(\sqrt{1-x^2}) = 1 - \frac{2\alpha}{\alpha - 1} + \frac{1}{\alpha - 1}\log \left[ (1 + x^2)^\alpha + (1 - x^2)^\alpha \right] + \frac{1}{\alpha - 1}\log \left[ (1 - \sqrt{1 - x^2})^\alpha + (1 + \sqrt{1 - x^2})^\alpha \right] =: m_\alpha(x)$$

where $m_\alpha(x)$ is a one-parameter real-valued analytic function defined on $0 \leq x \leq 1$. For $\alpha \geq 2$, it is also straightforward to check that the function $m_\alpha(x)$ has only one critical point at $x = \frac{1}{\sqrt{2}}$ through the domain $0 \leq x \leq 1$. Furthermore, all the function values of $m_\alpha(x)$ at the critical point and the boundary are nonnegative, that is, we have

$$h_\alpha(x, y) = m_\alpha(x) \geq 0$$ (55)
Proof. From Eq. (39), we have
\[
E \rho = \text{bipartite cut between} \ \text{of the reduced density matrix} \ \rho.
\]

Now, by using Theorem 3 together with Corollary 1 in previous section, we have the following theorem, which is the secondary result of this paper.

**Theorem 4.** For \( \alpha \geq 2 \) and any multi-qubit state \( \rho_{A_1A_2...A_n} \) in \( \mathcal{B}(C^{2^{\otimes n}}) \), we have
\[
E_\alpha (\rho_{A_1(A_2...A_n)}) \geq E_\alpha (\rho_{A_1A_2}) + \cdots + E_\alpha (\rho_{A_1A_n}),
\]
where \( E_\alpha (\rho_{A_1(A_2...A_n)}) \) is the Rényi-\( \alpha \) entanglement of \( \rho_{A_1A_2...A_n} \) with respect to the bipartite cut between \( A_1 \) and the others, and \( E_\alpha (\rho_{A_1A_i}) \) is the Rényi-\( \alpha \) entanglement of the reduced density matrix \( \rho_{A_1A_i} \) on two-qubit subsystem \( A_1A_i \) for \( i = 2, \ldots, n \).

**Proof.** From Eq. (39), we have
\[
C_{A_1(A_2...A_n)} \geq \sqrt{C_{A_1A_2}^2 + \cdots + C_{A_1A_n}^2},
\]
where \( C_{A_1(A_2...A_n)} \) and \( C_{A_1A_i} \) are the concurrences of \( \rho_{A_1(A_2...A_n)} \) and \( \rho_{A_1A_i} \) respectively.

Now, let \( \rho_{A_1(A_2...A_n)} = \sum_i p_i |\psi_i\rangle_{A_1(A_2...A_n)} \langle \psi_i| \) be an optimal decomposition for \( E_\alpha (\rho_{A_1(A_2...A_n)}) \) such that \( E_\alpha (\rho_{A_1(A_2...A_n)}) = \sum_i p_i E_\alpha (|\psi_i\rangle_{A_1(A_2...A_n)}) \). Because each \( |\psi_i\rangle_{A_1(A_2...A_n)} \) in the decomposition has Schmidt-rank less than or equal to two (they are \( 2 \otimes d \) pure states for \( d = 2^{\otimes n-1} \)), its concurrence and Rényi-\( \alpha \) entanglement are related by the function \( f_\alpha(x) \) in Eq. (19), that is,
\[
E_\alpha (|\psi_i\rangle_{A_1(A_2...A_n)}) = f_\alpha (C(|\psi_i\rangle_{A_1(A_2...A_n)}))
\]
for each \( i \). Thus, we have
\[
E_\alpha (\rho_{A_1(A_2...A_n)}) = \sum_i p_i E_\alpha (|\psi_i\rangle_{A_1(A_2...A_n)})
\]
\[
= \sum_i p_i f_\alpha (C(|\psi_i\rangle_{A_1(A_2...A_n)}))
\]
\[
\geq f_\alpha (\sum_i p_i C(|\psi_i\rangle_{A_1(A_2...A_n)}))
\]
\[
\geq f_\alpha (C_{A_1(A_2...A_n)}),
\]
where the first inequality is by the convexity of \( f_\alpha(x) \), and the second inequality is by the definition of concurrence and the monotonicity of \( f_\alpha(x) \). Furthermore, by Eq. (57) together with Theorem 3, we have
\[
f_\alpha (C_{A_1(A_2...A_n)}) \geq f_\alpha (\sqrt{C_{A_1A_2}^2 + \cdots + C_{A_1A_n}^2})
\]
\[
\geq f_\alpha (C_{A_1A_2}) + f_\alpha (\sqrt{C_{A_1A_3}^2 + \cdots + C_{A_1A_n}^2})
\]
\[
\vdots
\]
\[
\geq f_\alpha (C_{A_1A_2}) + \cdots + f_\alpha (C_{A_1A_n})
\]
\[
= E_\alpha (\rho_{A_1A_2}) + \cdots + E_\alpha (\rho_{A_1A_n}),
\]
where the first inequality is by the monotonicity of $f_{\alpha}(x)$, the other inequalities are by iterative use of Theorem \ref{thm:monogamy_1} and the last equality is by the functional relation of Rényi-$\alpha$ entanglement and concurrence for two-qubit states. Thus, Eqs. (59) together with (60) complete the proof.

In fact, it was recently shown that Rényi-$\alpha$ entanglement for $\alpha = 2$ can be used to establish a monogamy inequality for multi-qubit systems by straightforward calculation \cite{18}. However, Theorem \ref{thm:monogamy_2} says that the monogamous property of entanglement in multi-qubit systems can be mathematically characterized in terms of Rényi-$\alpha$ entanglement for all positive real number $\alpha$ larger than or equal to 2.

### 3.2. Polygamy of multi-qubit entanglement

In previous section, we have established the monogamy inequalities of multi-qubit entanglement in terms of Rényi-$\alpha$ entanglement for all positive real number $\alpha \geq 2$. Here, we consider the case when $0 < \alpha < 2$, and claim another kind of entanglement constraint in multi-qubit systems using Rényi-$\alpha$ entropy.

Let us first recall the definition of *Entanglement of Assistance* (EoA) \cite{19} for a bipartite state $\rho_{AB}$, that is,

$$
E^\alpha(\rho_{AB}) = \max_k \sum p_k E(|\psi_k\rangle_{AB}),
$$

(61)

where the maximum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_k p_k |\psi_k\rangle_{AB}\langle \psi_k|$. Here, we note that EoA in Eq. (61) is clearly a mathematical dual to EoF in Eq. (6) because one of them is the maximum average entanglement over all possible pure state decompositions whereas the other is the minimum. Moreover, by introducing a third party $C$ that has the purification of $\rho_{AB}$, $E^\alpha(\rho_{AB})$ can also be considered as the maximum achievable entanglement between $A$ and $B$ assisted by $C$. (This is the reason why it is called the assistance.) In other words, $E^\alpha(\rho_{AB})$ is the maximal entanglement that can be *distributed* between $A$ and $B$ assisted by the environment; therefore, EoA is also physically dual to the concept of *formation*.

Similar to the duality between EoF and EoA, we can also have a dual concept to concurrence: *Concurrence of Assistance* (CoA) \cite{20} for a bipartite state $\rho_{AB}$ is defined as

$$
C^\alpha(\rho_{AB}) = \max_k \sum p_k C(|\psi_k\rangle_{AB}),
$$

(62)

where the maximum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_k p_k |\psi_k\rangle_{AB}\langle \psi_k|$. Furthermore, it was shown that there exists a different kind of entanglement constraint in multi-qubit systems in terms of CoA \cite{21}. More precisely, for any pure state $|\psi\rangle_{A_1 A_2 \ldots A_n}$ in an $n$-qubit system, we have

$$
C^2_{A_1 A_2 \ldots A_n} \leq (C^a_{A_1 A_2})^2 + \cdots + (C^a_{A_1 A_n})^2,
$$

(63)
where $C_{A_1(A_2\cdots A_n)}$ is the concurrence of $|\psi\rangle_{A_1\cdots A_n}$ with respect to the bipartite cut between $A_1$ and $A_2\cdots A_n$, and $C_{A_1A_i}^a$ is the CoA of the reduced density matrix $\rho_{A_1A_i}$ for $i = 2, \ldots, n$.

Eq. (63) is known as the dual monogamy or polygamy inequality of entanglement in multi-qubit systems: Whereas concurrence can be used to characterize the monogamy of multipartite entanglement as in Eq. (39), its dual quantity, CoA can also be used for the dual monogamy of multipartite entanglement. Later, it was shown that polygamy of multi-qubit entanglement can also be characterized in terms of EoA, that is, for any multi-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$,

$$E\left(|\psi\rangle_{A_1(A_2\cdots A_n)}\right) \leq E^a(\rho_{A_1A_2}) + \cdots + E^a(\rho_{A_1A_n}),$$

(64)

where $E\left(|\psi\rangle_{A_1(A_2\cdots A_n)}\right) = S(\rho_A)$ is the entanglement of $|\psi\rangle_{A_1\cdots A_n}$ with respect to the bipartite cut $A_1$ and $A_2\cdots A_n$, and $E^a(\rho_{A_1A_i})$ is the EoA of the reduced density matrix $\rho_{A_1A_i}$ for $i = 2, \ldots, n$ [22]. More recently, a general polygamy inequality of multipartite entanglement in arbitrary dimensional quantum systems was proposed by using an analytical upper bound of CoA [23].

Here, we claim that the polygamous property of multi-qubit entanglement can also be shown by using Rényi-$\alpha$ entropy for $\alpha$ around 1. In fact, we can intuitively expect this: Due to the continuity of Rényi-$\alpha$ entropy with respect to $\alpha$, we can easily expect that the inequality in Eq. (64) would also be true by using Rényi-$\alpha$ entropy, instead of von Neumann entropy, where $\alpha$ is in a region of small perturbation from 1. However, we conjecture, with strong numerical evidences, a specific region of $\alpha$ where the polygamy inequality can be obtained using Rényi-$\alpha$ entropy.

For any bipartite state $\rho_{AB}$, let us first define a dual quantity to Rényi-$\alpha$ entanglement as

$$E^a_{\alpha}(\rho_{AB}) := \max_k \sum p_k E^\alpha\left(|\psi_k\rangle_{AB}\right),$$

(65)

where the maximum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_k p_k |\psi_k\rangle_{AB}\langle\psi_k|$, and call it Rényi-$\alpha$ Entanglement of Assistance (REoA).

Similar to the functional relation between concurrence and Rényi-$\alpha$ entanglement for two-qubit states in Eq. (19), the same function $f_\alpha(x)$ can also relate REoA of a two-qubit state with its CoA. For a two-qubit state $\rho_{AB}$, let $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$ be an optimal decomposition for its CoA, that is,

$$C^a(\rho_{AB}) = \sum_i p_i C\left(|\psi_i\rangle_{AB}\right),$$

(66)

where $C^a(\rho_{AB})$ is the CoA of $\rho_{AB}$ defined in Eq. (62). By Theorem 2, $f_\alpha(x)$ is convex for the range of $\alpha \geq 1$, thus, for this range of $\alpha$, we have

$$f_\alpha\left(C^a(\rho_{AB})\right) = f_\alpha\left(\sum_i p_i C\left(|\psi_i\rangle_{AB}\right)\right) \leq \sum_i p_i f_\alpha(C\left(|\psi_i\rangle_{AB}\right))$$
Monogamy and polygamy for multi-qubit entanglement using Rényi entropy

1.5
1.0
1.0
0.0
0.5
0.5
0.0
10
10
−5
0
5
0
2
10
−12
0.5
1.0
C
0.0
C
0.5
0.75
1.0
0.0
10
−3
2.5
0.5
0.0
10
−3
2.5
0.5
0.0
0.75
1.0
0.0

(a) Graph of $h_{1.9}(x, y)$ for $x^2 + y^2 \leq 0.008$.

(b) Graph of $h_{1.44}(x, y)$.

(c) Graph of $h_{1.43}(x, y)$.

Figure 2. (a), (b) and (c) illustrate the graphs of $h_\alpha(x, y)$ for $\alpha = 1.9$, 1.44 and 1.43 respectively. In each picture, the variables $x$ and $y$ are reparameterized as $x = C \cos t$ and $y = C \sin t$ for $0 \leq C \leq 1$ and $0 \leq t \leq \pi/2$. The vertical axes in (a), (b) and (c) are scaled by $10^{-12}$, $10^{-3}$ and $10^{-3}$ respectively.

\[ = \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}) \]
\[ \leq E_\alpha^a(\rho_{AB}), \]  \hspace{1cm} (67)

where the first inequality is by the convexity of $f_\alpha(x)$ and the last inequality is by the definition of REoA. Furthermore, based on Conjecture 1 we also claim that Eq. (67) is true for $\alpha > 0.83 - \varepsilon$ with small $\varepsilon$.

Now, let us consider a property of $h_\alpha(x, y)$ in Eq. (42). From Theorem 3 we first note that the continuity of $h_\alpha(x, y)$ in Eq. (42) with respect to $\alpha$ assures its positivity for $\alpha$ slightly less than 2. In other words, the monogamy inequality in Eq. (56) of Theorem 4 that is analytically proven for $\alpha \geq 2$ is also true for this region of $\alpha$. However, $h_\alpha(x, y)$ has negative function values for most case of $0 < \alpha < 2$, which is illustrated in Figure 2 (a). In Figure 2 (a), negative function values of $h_\alpha(x, y)$ are observed near $(x, y) = (0, 0)$ when $\alpha = 1.9$. Furthermore, the region in the domain of $h_\alpha(x, y)$ with negative function values is getting larger as $\alpha$ decreases. (Figure 2 (b))

In fact, for the limiting case of $\alpha \to 1$, $h_\alpha(x, y)$ was analytically shown to be a non-positive function [22], that is,

\[ \mathcal{E}(\sqrt{x^2 + y^2}) \leq \mathcal{E}(x) + \mathcal{E}(y), \]  \hspace{1cm} (68)

for non-negative $x$ and $y$ such that $0 \leq x^2 + y^2 \leq 1$ where $\mathcal{E}(x) = \lim_{\alpha \to 1} f_\alpha(x)$ is defined in Eq. (13). Again, due to the continuity of $h_\alpha(x, y)$ with respect to $\alpha$, we can assure that $h_\alpha(x, y)$ is nonpositive everywhere when $\alpha$ is around 1. For a specific region of $\alpha$, we have numerically tested for various values of $\alpha$, and $h_\alpha(x, y)$ is observed to have non-positive function values for $0 \leq \alpha \leq 1.43 + \varepsilon$ with $0 < \varepsilon < 0.01$, which is illustrated in Figure 2 (c).
**Conjecture 2.** For any positive $0 \leq \alpha \leq 1.43 + \epsilon$ with some positive $\epsilon$ such that $0 \leq \epsilon < 0.01$ and the function

$$f_\alpha(x) = \frac{1}{1 - \alpha} \log \left[ \left( \frac{1 - \sqrt{1 - x^2}}{2} \right)^\alpha + \left( \frac{1 + \sqrt{1 - x^2}}{2} \right)^\alpha \right],$$

(69)
defined on the domain $D = \{(x, y)|0 \leq x, y, x^2 + y^2 \leq 1\}$, we have

$$f_\alpha(x^2 + y^2) \leq f_\alpha(x) + f_\alpha(y).$$

(70)

Now, Conjectures 1 and 2 lead us to the following theorem, which is the last main result of this paper.

**Theorem 5.** For a multi-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$ and any real $\alpha$ such that $0.83 - \epsilon \leq \alpha \leq 1.43 + \epsilon$ with $0 < \epsilon < 0.01$, we have

$$E_\alpha \left(|\psi\rangle_{A_1(A_2\cdots A_n)}\right) \leq E_\alpha^a(\rho_{A_1 A_2}) + \cdots + E_\alpha^a(\rho_{A_1 A_n}),$$

(71)

where $E_\alpha \left(|\psi\rangle_{A_1(A_2\cdots A_n)}\right) = S_{\alpha}(\rho_A)$ is the Rényi-$\alpha$ entanglement of $|\psi\rangle_{A_1(A_2\cdots A_n)}$ with respect to the bipartite cut $A_1$ and $A_2 \cdots A_n$, and $E_\alpha^a(\rho_{A_i A_i})$ is the REoA of the reduced density matrix $\rho_{A_i A_i}$ for $i = 2, \cdots, n$.

**Proof.** The proof method follows the construction used in [22]. From the polygamy inequality of entanglement in multi-qubit systems in Eq. (63) gives us an inequality

$$C_{A_1(A_2\cdots A_n)} \leq \sqrt{(C_{A_1 A_2})^2 + \cdots + (C_{A_1 A_n})^2}.$$ (72)

First, let us assume that $(C_{A_1 A_2})^2 + \cdots + (C_{A_1 A_n})^2 \leq 1$, then we have

$$E_\alpha \left(|\psi\rangle_{A_1(A_2\cdots A_n)}\right) = f_\alpha(C_{A_1(A_2\cdots A_n)})$$

$$\leq f_\alpha \left(\sqrt{(C_{A_1 A_2})^2 + \cdots + (C_{A_1 A_n})^2}\right)$$

$$\leq f_\alpha \left(C_{A_1 A_2}^a\right) + f_\alpha \left(\sqrt{(C_{A_1 A_3})^2 + \cdots + (C_{A_1 A_n})^2}\right)$$

$$\leq f_\alpha \left(C_{A_1 A_2}^a\right) + f_\alpha \left(C_{A_1 A_3}^a\right) + \cdots + f_\alpha \left(C_{A_1 A_n}^a\right)$$

$$\leq E_\alpha^a(\rho_{A_1 A_2}) + \cdots + E_\alpha^a(\rho_{A_1 A_n}),$$

(73)

where the first equality is by the functional relation between the concurrence and the Rényi-$\alpha$ entanglement for $2 \otimes d$ pure states, the first inequality is due to the monotonicity of the function $f_\alpha(x)$, the second, third and forth inequalities are obtained by iterative use of Eq. (70) in Conjecture 2 and the last inequality is by Eq. (67).

Now, assume that $(C_{A_1 A_2}^a)^2 + \cdots + (C_{A_1 A_n}^a)^2 > 1$. Since $E_\alpha \left(|\psi\rangle_{A_1(A_2\cdots A_n)}\right) = S_{\alpha}(\rho_A) \leq 1$ for any multi-qubit pure state $|\psi\rangle_{A_1(A_2\cdots A_n)}$, it is enough to show that $E_\alpha^a(\rho_{A_1 A_2}) + \cdots + E_\alpha^a(\rho_{A_1 A_n}) \geq 1$. Here, we note that there exist $k \in \{2, \ldots, n-1\}$ that satisfies

$$(C_{A_1 A_2}^a)^2 + \cdots + (C_{A_1 A_k}^a)^2 \leq 1, \ (C_{A_1 A_2}^a)^2 + \cdots + (C_{A_1 A_{k+1}}^a)^2 > 1.$$ (74)
Monogamy and polygamy for multi-qubit entanglement using Rényi entropy

By letting
\[ T := (C_{A_1A_2}^a)^2 + \cdots + (C_{A_1A_{k+1}}^a)^2 - 1, \] (75)
we have
\[ 1 = f_\alpha \left( \sqrt{(C_{A_1A_2}^a)^2 + \cdots + (C_{A_1A_{k+1}}^a)^2 - T} \right) \]
\[ \leq f_\alpha \left( \sqrt{(C_{A_1A_2}^a)^2 + \cdots + (C_{A_1A_k}^a)^2} \right) + f_\alpha \left( \sqrt{(C_{A_1A_{k+1}}^a)^2 - T} \right) \]
\[ \leq f_\alpha (C_{A_1A_2}^a) + \cdots + f_\alpha (C_{A_1A_k}^a) + f_\alpha (C_{A_1A_{k+1}}^a) \]
\[ \leq E_\alpha^a(\rho_{A_1A_2}) + \cdots + E_\alpha^a(\rho_{A_1A_n}), \] (76)
where the first inequality is by using Eq. (70) with respect to \((C_{A_1A_2}^a)^2 + \cdots + (C_{A_1A_k}^a)^2\) and \((C_{A_1A_{k+1}}^a)^2 - T\), the second inequality is by iterative use of Eq. (70) on \((C_{A_1A_2}^a)^2 + \cdots + (C_{A_1A_k}^a)^2\), and the last inequality is by Eq. (67).

4. Conclusion

By using Rényi-\(\alpha\) entropy, we have established a class of monogamy and polygamy inequalities of multi-qubit entanglement. We have shown that monogamy of multi-qubit entanglement can have CKW-type characterization in terms of Rényi-\(\alpha\) entanglement for \(\alpha \geq 2\), and conjectured the possible region of \(\alpha\) for polygamy inequality in terms of REoA. Although the specific region of \(\alpha\) for polygamy inequality is supported by numerical evidences, the existence of such region is based on the continuity of Rényi-\(\alpha\) entropy, which is analytically provable. Thus, the conjecture we claimed here is highly reasonable because it has both numerical and analytical reasons.

Multipartite entanglement is known to have many inequivalent classes, which are not convertible to each other under Stochastic Local operations and classical communications (SLOCC) [24]. Furthermore, the number of inequivalent classes increases dramatically as the number of parties increase [25]. Not like bipartite entanglement, the existence of inequivalent classes of multipartite entanglement implies that the states from different classes are hardly comparable to each other in such a way of comparing a single parameter that quantifies their entanglement. This is one of the main difficulties in the study of multipartite entanglement.

Whereas the interconvertibility of quantum states under SLOCC gives us an operational way to classify multipartite entanglement, entangled states from different classes can also reveal different characters with respect to their monogamy and polygamy properties. For example, three-qubit systems are known to have two inequivalent classes of genuine three-qubit entanglement, the Greenberger-Horne-Zeilinger (GHZ) class [26] and the W-class [24]. In terms of monogamy and polygamy relations, CKW and its dual inequalities are saturated by W-class states, while the differences between terms in the inequalities can assume their largest values for GHZ-class states. In other words, monogamy and polygamy of multipartite entanglement can also be used for an analytical characterization of entanglement in multipartite quantum systems.
The class of monogamy and polygamy inequalities of multi-qubit entanglement we provided here consists of infinitely many inequalities parameterized by $\alpha$, and each inequality is based on the distinct character of Rényi-$\alpha$ entropy with respect to $\alpha$. We believe that this selective choice of our monogamy and polygamy inequalities will leads us to an efficient way of analytic classification of multi-qubit entanglement. Moreover, our result will also provide useful tools and strong candidates for general monogamy and polygamy relations of entanglement in multipartite higher-dimensional quantum systems, which is one of the most important and necessary topics in the study of multipartite quantum entanglement.

Acknowledgments

This work was supported by iCORE, MITACS, and USARO. BSC is a CIFAR Associate.

References

[1] Coffman V, Kundu J and Wootters W K 2000 Phys. Rev. A 61 052306
[2] Osborne V and Verstraete F 2006 Phys. Rev. Lett. 96 220503
[3] Koashi M and Winter A 2004 Phys. Rev. A 69 022309
[4] Terhal B M 2004 IBM J. Research and Development 48 71
[5] Renes J M and Grassl M 2006 Phys. Rev. A 74 022317
[6] Masanes L 2009 Phys. Rev. Lett. 102 140501
[7] Kim J S, Das A and Sanders B C 2009 Phys. Rev. A 79 012329
[8] Wootters W K 1998 Phys. Rev. Lett. 80 2245
[9] Ou Y C 2007 Phys. Rev. A 75 034305
[10] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Phys. Rev. A 54 3824
[11] Rényi A 1960 Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability (Berkeley University Press, Berkeley, CA) 547
[12] Shannon C E 1951 The Bell System Technical Journal 30 50–64
[13] Bovino F A, Castagnoli G, Ekert A, Horodecki P, Alves C M and Sergienko A V 2005 Phys. Rev. Lett. 95 240407
[14] Terhal B M 2002 J. Theor. Comp. Sci. 287 313
[15] Lévay P, Nagy S and Pipek J 2005 Phys. Rev. A 72 022302
[16] Horodecki R, Horodecki P and Horodecki M 1996 Phys. Lett. A 210 377
[17] Vidal G 2000 J. Mod. Opt. 47 355
[18] Cornelio M F and de Oliveira M C 2009 arXiv:0906.0332
[19] Cohen O 1998 Phys. Rev. Lett. 80 2493
[20] Laustsen T, Verstraete F and van Enk S J 2003 Quantum Inf. Comput. 3 64
[21] Gour G, Bandyopadhay S and Sanders B C 2007 J. Math. Phys. 48 012108
[22] Buscemi F, Gour G and Kim J S 2009 Phys. Rev. A 80 012324
[23] Kim J S 2009 Phys. Rev. A 80 022302
[24] Dür W, Vidal G and Cirac J I 2000 Phys. Rev. A 62 062314
[25] Osterloh A and Siewert J 2005 Phys. Rev. A 72 012337
[26] Greenberger D M, Horne M A and Zeilinger A 1989 Bell’s Theorem, Quantum Theory, and Conceptions of the Universe edited by M. Kafatos (Kluwer, Dordrecht) 69