ON THE GEOMETRY OF $S_2$

GILBERTO BINI AND CLAUDIO FONTANARI

ABSTRACT. We investigate topological properties of the moduli space of spin structures over genus two curves. In particular, we provide a combinatorial description of this space and give a presentation of the (rational) cohomology ring via generators and relations.

The authors warn the reader that an erratum to the present paper has been kindly posted on arXiv by Sebastian Krug.

1. Introduction

The moduli space $S_g$ of spin curves of genus $g$ has been introduced in [13] in order to compactify the moduli space of pairs

(smooth genus $g$ complex curve $C$, theta-characteristic on $C$).

The interest in this space has been increasing in the last few years due to a wide spectrum of applications, which range from Mathematical Physics to Arithmetic Geometry passing through Algebraic Geometry and Combinatorics (see, for instance, [1], [7], [9], [10], [11], [12], [20], [23]).

Nonetheless, very little is known about the topological properties of $S_g$. The first results in this direction can be found in [17] and [18]. In these pioneering articles, the stability of the homology groups and some low degree cohomology groups of $S_g$ are investigated.

In the present paper, we compute the virtual cohomological dimension of $S_g$ (see Lemma 2.2). Next, we focus on the compactification $\overline{S}_g$, especially on the first relevant case $g = 2$. In particular, we determine the Hodge diamond of $\overline{S}_2$ (see Proposition 3.5) and describe the cohomology ring of $\overline{S}_2$ via generators and relations (see Theorem 3.12 and Theorem 3.13).

At first sight, it might seem that the topology of the moduli space of spin curves is not really different from that of the moduli space of stable curves. In fact, the constructions of these two spaces do not differ in a substantial way. On the other hand, $\overline{S}_g$ turns out to have a richer geometric structure, which requires a more careful analysis. Indeed, $\overline{S}_g$ is the union of two connected components, $\overline{S}_g^+$ and $\overline{S}_g^-$, which correspond to even and odd spin curves, respectively. In the case $g = 2$, we separately investigate the cohomology of each component and discover two non-isomorphic ring structures.

In order to accomplish this task, we combine an inductive approach inspired by [2] with direct calculations in the style of [22]. This leads us to introduce suitable moduli spaces of pointed spin curves and to address several related problems. In particular, we obtain an explicit affine stratification of $\overline{S}_2$ (see Appendix A).

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An erratum to the present paper has been kindly posted on arXiv by Sebastian Krug (see [25]).

Throughout we work over the field \( \mathbb{C} \) of complex numbers.

2. THE MODULI SPACE OF CURVES WITH SPIN STRUCTURES

In this section, we recall some basic definitions about spin structures. Here we follow closely [13], with which we refer for more details. For a more general approach see, for instance [1].

Let \( X \) be a Deligne-Mumford semistable curve and let \( E \) be a complete, irreducible subcurve of \( X \). The curve \( E \) is said to be exceptional when it is smooth, rational, and intersects the other components in exactly two points. Moreover, \( X \) is said to be quasi-stable when any two distinct exceptional components of \( C \) are disjoint. In the sequel, \( \tilde{X} \) will denote the subcurve \( X \setminus \cup E_i \) obtained from \( X \) by removing all the exceptional components.

A spin curve of genus \( g \) (see [13], § 2) is the datum of a quasi-stable genus \( g \) curve \( X \) with an invertible sheaf \( \zeta_X \) of degree \( g - 1 \) on \( X \) and a homomorphism of invertible sheaves \( \alpha_X : \zeta_X^{\otimes 2} \to \omega_X \) such that i) \( \zeta_X \) has degree 1 on every exceptional component of \( X \), and ii) \( \alpha_X \) is not zero at a general point of every non-exceptional component of \( X \). Therefore, \( \alpha_X \) vanishes identically on all exceptional components of \( X \) and induces an isomorphism \( \tilde{\alpha}_X : \zeta_X^{\otimes 2} |_{\tilde{X}} \to \omega_{\tilde{X}} \). In particular, when \( X \) is smooth, \( \zeta_X \) is just a theta-characteristic on \( X \). Two spin curves \( (X, \zeta_X, \alpha_X) \) and \( (X', \zeta_{X'}, \alpha_{X'}) \) are isomorphic if there are isomorphisms \( \sigma : X \to X' \) and \( \tau : \sigma^*(\zeta_{X'}) \to \zeta_X \) such that \( \tau \) is compatible with the natural isomorphism between \( \sigma^*(\omega_{X'}) \) and \( \omega_X \).

A family of spin curves is a flat family of quasi-stable curves \( f : \mathcal{X} \to S \) with an invertible sheaf \( \zeta_f \) on \( \mathcal{X} \) and a homomorphism \( \alpha_f : \zeta_f^{\otimes 2} \to \omega_f \) such that the restriction of these data to any fiber of \( f \) gives rise to a spin curve. Two families of spin curves \( f : \mathcal{X} \to S \) and \( f' : \mathcal{X}' \to S \) are isomorphic if there are isomorphisms \( \sigma : \mathcal{X} \to \mathcal{X}' \) and \( \tau : \sigma^*(\zeta_{f'}) \to \zeta_f \) such that \( f = f' \circ \sigma \) and \( \tau \) is compatible with the natural isomorphism between \( \sigma^*(\omega_{f'}) \) and \( \omega_f \).

Let \( \mathcal{S}_g \) be the moduli space of isomorphism classes of spin curves of genus \( g \). Denote by \( S_0 \) the open subset consisting of classes of smooth curves. As shown in [13], § 5, \( \mathcal{S}_g \) has a natural structure of analytic orbifold given as follows. For any spin curve \( X \), there is a neighbourhood \( U \) of \( [X] \) such that \( U \cong B_X / \text{Aut}(X) \), where \( B_X \) is a 3g - 3-dimensional polydisk and \( \text{Aut}(X) \) is the automorphism group of the spin curve \( X \). Alternatively, \( \mathcal{S}_g \) may be viewed as a projective normal variety with finite quotient singularities.

The moduli space of spin curves can be slightly generalized as follows. For all integers \( g, n, m_1, . . . , m_n \), such that \( 2g - 2 + n > 0 \), \( 0 \leq m_i \leq 1 \) for every \( i \), and \( \sum_{i=1}^{n} m_i \) is even, we define
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$$S_{g,n}^{(m_1,\ldots,m_n)} := \left\{ [ (C, p_1, \ldots, p_n; \zeta; \alpha) ] : (C, p_1, \ldots, p_n) \text{ is a genus } g \right.$$ quasi-stable projective curve with $n$ marked points; 
$$\zeta$$ is a line bundle of degree $g - 1 + \frac{1}{2} \sum_{i=1}^{n} m_i$ on $C$ having degree 1 on every exceptional component of $C$, and 
$$\alpha : \zeta \otimes \omega_{C}^2 \to \omega_C \left( \sum_{i=1}^{n} m_i p_i \right)$$ is a homomorphism which is not zero at a general point of every non-exceptional component of $C$. 

In order to put an analytic structure on $S_{g,n}^{(m_1,\ldots,m_n)}$, we notice that Cornalba’s construction in [13] can be easily adapted to $S_{g,n}^{(m_1,\ldots,m_n)}$. Indeed, from the universal deformation of the stable model of $(C, p_1, \ldots, p_n)$ we obtain exactly as in [13], § 4, a universal deformation $U_X \to B_X$ of $X = (C, p_1, \ldots, p_n; \zeta; \alpha)$. Next, we put on $S_{g,n}^{(m_1,\ldots,m_n)}$ the structure of the quotient analytic space $B_X / \text{Aut}(X)$ following [13], § 5. Alternatively, we can regard $S_{g,n}^{(m_1,\ldots,m_n)}$ as the coarse moduli space associated to the stack of $r$-spin curves (in the easiest case $r = 2$), which has been constructed by Jarvis in [19] and revisited by Abramovich and Jarvis in [1].

Analogously to $S_g$ (see [13], Proposition 5.2), the spaces $S_{g,n}^{(m_1,\ldots,m_n)}$ are normal projective varieties of complex dimension $3g - 3 + n$. If $m_1 = \ldots = m_n = 0$, then $S_{g,n} := S_{g,n}^{(0,\ldots,0)}$ splits into two disjoint irreducible components $S_{g,n}^+ \text{ and } S_{g,n}^-$ that consist of the even and the odd spin curves, respectively (see [13], Lemma 6.3). Similarly to Lemma 1 in [7], we point out the following

**Proposition 2.1.** Let $\text{Pic}(S_{g,n}^{(m_1,\ldots,m_n)}) := H^1(S_{g,n}^{(m_1,\ldots,m_n)}, \mathcal{O}^*)$. There is a natural isomorphism

$$\text{Pic}(S_{g,n}^{(m_1,\ldots,m_n)}) \otimes \mathbb{Q} \cong A_{3g-4+n}(S_{g,n}^{(m_1,\ldots,m_n)}) \otimes \mathbb{Q}.$$ 

**Proof.** Since $S_{g,n}^{(m_1,\ldots,m_n)}$ is normal, there is an injection:

$$\text{Pic}(S_{g,n}^{(m_1,\ldots,m_n)}) \otimes \mathbb{Q} \hookrightarrow A_{3g-4+n}(S_{g,n}^{(m_1,\ldots,m_n)}) \otimes \mathbb{Q}.$$ 

Moreover, from the construction of $S_{g,n}^{(m_1,\ldots,m_n)}$ it follows that the singularities of $S_{g,n}^{(m_1,\ldots,m_n)}$ are of finite quotient type, so every Weil divisor is $\mathbb{Q}$-Cartier and there is a surjective morphism:

$$\text{Pic}(S_{g,n}^{(m_1,\ldots,m_n)}) \otimes \mathbb{Q} \twoheadrightarrow A_{3g-4+n}(S_{g,n}^{(m_1,\ldots,m_n)}) \otimes \mathbb{Q}.$$ 

Hence the claim follows.

We end this section with some results on the topology of $S_g$, which will be applied in the next Section when $g = 2$. In what follows, all homology and cohomology groups are intended to have rational coefficients.
Let us recall some general notions from [16], § 4. A group $\Gamma$ is virtually torsion-free if it has a subgroup $G$ of finite index which is torsion free. If $\Gamma$ is virtually torsion free, the virtual cohomological dimension of $\Gamma$ is the cohomological dimension of a torsion free subgroup $G$ of finite index in $\Gamma$. A theorem of Serre states that this number is independent of the choice of $G$ (see [16], p. 173).

It is well-known that the mapping class group $\Gamma_g$ is virtually torsion-free (see for instance [16], p. 172). Moreover, Harer proved that the virtual cohomological dimension of $\Gamma_g$ is $4g - 5$ (see [16], Theorem 4.1). Since the Teichmüller space $T_g$ is contractible, the rational homology of $M_g$ and $\Gamma_g$ is the same, so the group $H_k(M_g, \mathbb{Q})$ is zero for $k > 4g - 5$ (see [16], Corollary 4.3).

Let now $S$ be a smooth orientable surface of genus $g$ and let $Q$ be a $\mathbb{Z}/2\mathbb{Z}$-quadratic form on $H_1(S, \mathbb{Z}/2\mathbb{Z})$. The isomorphism class of $Q$ is determined by its Arf invariant $\varepsilon = 0$ or 1 (see [17], p. 324). Call $Q_0$ (resp. $Q_1$) the even (resp. the odd) quadratic form and let $G_g(Q_i)$ be the subgroup of the mapping class group $\Gamma_g$ which preserves $Q_i$. Then the following holds:

**Lemma 2.2.** For $i = 0, 1$, $G_g(Q_i)$ is virtually torsion-free and its virtual cohomological dimension is $4g - 5$.

**Proof.** Let $\chi : \Gamma_g \to \text{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ be the following homomorphism. The image under $\chi$ of an isotopy class $[\gamma]$ of orientation-preserving diffeomorphism of $S$ is the automorphism induced by $\gamma$ on $H_1(S, \mathbb{Z}/2\mathbb{Z})$ (see [3], Chapter 14, (2.1)). Since the target group is finite, the kernel $K$ of $\chi$ has finite index in $\Gamma_g$. It is a straightforward consequence of the definitions that $K \subseteq G_g(Q_i)$, hence we may deduce that $G_g(Q_i)$ has finite index in $\Gamma_g$. Now, pick a torsion-free subgroup $H$ of $\Gamma_g$ of finite index and consider $G_i := H \cap G_g(Q_i)$. Since both $H$ and $G_g(Q_i)$ have finite index in $\Gamma_g$, it follows that $G_i$ has finite index in $\Gamma_g$. Since $H$ is torsion-free, we have that $G_i$ is torsion-free. As mentioned before, all torsion-free subgroups of finite index in $\Gamma_g$ have the same cohomological dimension. Therefore, from the previous two facts and Harer’s theorem we may deduce that the cohomological dimension of $G_i$ is $4g - 5$. Since $G_i$ has finite index in $\Gamma_g$, it has finite index in $G_g(Q_i)$ too. Thus, it computes the virtual cohomological dimension of $G_g(Q_i)$.

Recall that $S_g$ is the disjoint union of $S^+_g$, which contains curves with even theta-characteristics, and $S^-_g$, which contains those with odd theta-characteristics. Further, we have $S^+_g = T_g/G_g(Q_0)$ and $S^-_g = T_g/G_g(Q_1)$ (see [17], p. 324). Analogously to $M_g$, the following holds.

**Theorem 2.3.** We have $H_k(S_g) = 0$ for $k > 4g - 5$.

3. The Rational Cohomology of $\overline{S}_2$

In this section, we specialize to the case $g = 2$ and we investigate the additive and multiplicative structure of the cohomology algebra of $\overline{S}_2$. Let $\nu : \overline{S}_2 \to \overline{M}_2$ be the morphism which forgets the spin structure and passes to the stable model. It is a $16 : 1$ ramified covering of normal threefolds. Denote by $\nu^+ : \overline{S}^+_2 \to \overline{M}_2$ (resp. $\nu^- : \overline{S}^-_2 \to \overline{M}_2$) the restriction of $\nu$ to
the moduli space of curves with even spin structures $S_2^+$ (resp. odd spin structures $S_2^-$).

**Remark 3.1.** The stratification of $S_2$ by topological type is affine. Indeed, as shown in [22], the stratification of $\mathcal{M}_2$ by graph type is affine and since $\nu$ is finite and preserving topological type, $\overline{S}_2$ turns out to be stratified as the union of affine subvarieties.

For an explicit description of the stratification in Remark 3.1, see Appendix A.

**Theorem 3.2.** We have $H^1(S_2) = H^3(S_2) = 0$.

**Proof.** Consider the following morphisms:

$$f^+: \mathcal{M}_{0,6} \to S_2^+,$$

$$f^-: \mathcal{M}_{0,6} \to S_2^-.$$

In order to define $f^+$ and $f^-$, let $(C; p_1, \ldots, p_6)$ be a 6-pointed, stable, genus zero curve. The morphism $f^+$ (respectively $f^-$) associates to $(C; p_1, \ldots, p_6)$ the admissible covering $Y$ of $C$ which is branched at the $p_i$’s. Generically, it comes equipped with the line bundle $O_Y(q_1 + q_2 - q_3)$ (respectively $O_Y(q_1)$), where $q_i$ denotes the point of $Y$ lying above $p_i$, and the morphism extends to the boundary since $\mathcal{M}_{0,6}$ is normal and $\overline{S}_2$ is finite over $\mathcal{M}_2$.

By (1), we have $H^k(S_2) \cong H^k(\mathcal{M}_{0,6})$.

The claim follows from [21] since $H^k(\mathcal{M}_{0,n}) = 0$ for every $n$ and every odd $k$. $\square$

In order to determine $H^2(S_2)$, we first look at algebraic cycles of codimension one in $S_2$. We recall that the boundary $\partial S_g = S_g \setminus S_g$ is the union of the irreducible components $A_i^+, B_i^- \in S_g^+$ (contained in $S_g^+$) and $A_i^-, B_i^+ \in S_g^-$ (contained in $S_g^-$), which are completely described in [13], § 7, by their general members:

- $A_i^+$ (resp. $B_i^+$), $i > 0$: two smooth components $C_1$ and $C_2$ of genera $i$ and $g - i$, joined at points $p \in C_1$ and $q \in C_2$ by a $\mathbb{P}^1$, with even (resp. odd) theta-characteristics on $C_1$ and $C_2$
- $A_i^-$ (resp. $B_i^-$), $i > 0$: as above, but with an even (resp. odd) theta-characteristic on $C_1$ and an odd (resp. even) one on $C_2$
- $A_0^+$ (resp. $A_0^-)$: an irreducible curve of genus $g$ with only one node, with an even (resp. odd) spin structure
- $B_0^+$ (resp. $B_0^-$): an irreducible curve of genus $g$ with only one node, blown up at the node (so to add an exceptional component $E \cong \mathbb{P}^1$), with an even (resp. odd) spin structure glued to $O_E(1)$ on $E$.

Let $\alpha_i^+, \beta_i^+, \alpha_i^-, \beta_i^-$ denote the corresponding classes in $A_{3g-4}(S_g)$ for $i = 0, 1$. In [13], p. 585, the class $\beta_i^-$ is defined to be zero, hence we always
omit it. We point out that the arguments provided in [13] in order to prove Proposition 7.2 imply that $\alpha^+_i$, $\beta^+_i$, $\alpha^-_i$, $\beta^-_i$ are independent for $g = 2$, too.

**Proposition 3.3.** The Chow group $A_2(S_2)$ is generated by boundary classes.

**Proof.** By restricting $f^+$ (resp. $f^-$) over $S^+_2$ (resp. $S^-_2$), we obtain finite surjective morphisms from $\mathcal{M}_{0,6}$ to $S_2$. By [21], $A_2(\mathcal{M}_{0,6})$ is generated by boundary classes. Thus, the exact sequence

$$A_2(\mathcal{M}_{0,6} \setminus \mathcal{M}_{0,6}) \to A_2(\mathcal{M}_{0,6}) \to A_2(M_{0,6})$$

yields $A_2(M_{0,6}) = 0$. By [13], Lemma A, we conclude that $A_2(S_2) = 0$, so the map $A_2(S_2 \setminus S_2) \to A_2(S_2)$ is onto.

Next, we are going to prove that the whole cohomology of $S_2$ is indeed algebraic. In order to do so, we need the following auxiliary result.

**Lemma 3.4.** $\overline{S}_1,_{n(m_1,\ldots,m_n)}$ is a unirational variety for $n \leq 2$.

**Proof.** Since the structure sheaf is the unique odd theta-characteristic on an elliptic curve, we have $S_{1,n}^{0,\ldots,0,-} \cong \mathcal{M}_{1,n}$ for every $n$, and $\mathcal{M}_{1,n}$ is rational for $n \leq 10$, as shown in [6]. Next, we are going to define three surjective morphisms by defining them generically as in the proof of Theorem 3.2. The first one is

$$h_4 : \mathcal{M}_{0,4} \longrightarrow S_{1,1}^{(0),+}.$$  

Let $(C; p_1, p_2, p_3, p_4)$ be a 4-pointed stable genus zero curve. If $E$ is the admissible covering of $C$ that is branched at the $p_i$’s and $q_1$ denotes the preimage of $p_1$ under such covering, then the morphism $h$ maps the pointed curve $(C; p_1, p_2, p_3, p_4)$ to $(E; q_1)$ with the even theta-characteristic $O_E(q_1 - q_2)$.

In a similar fashion, we define a second morphism as follows:

$$h_5 : \mathcal{M}_{0,5} \longrightarrow S_{1,2}^{(0,0),+}.$$  

Let $(C; p_1, p_2, p_3, p_4, p)$ be a 5-pointed stable genus zero curve. The image under $h_5$ is the pointed curve $(F; r_1, r)$ with the even theta-characteristic $O_F(r_1 - r_2)$. The curve $F$ is the admissible covering of $C$ that is branched at the points $p_i$’s. The point $r$ is the preimage of $p_1$, while the point $r$ is one of the two points in the preimage of $p$. Notice that a different choice will yield the same point in $S_{1,2}^{(0,0),+}$. Finally, we have a third surjective morphism

$$t_5 : \mathcal{M}_{0,5} \longrightarrow S_{1,2}^{(1,1),+},$$

which is defined as follows. With the same notation used for $h_5$, the image under $t_5$ of $(C; p_1, p_2, p_3, p_4, p)$ is given by $(E; r_2, r_3)$ with the odd theta-characteristic $O_E(r_1)$. The points $r_2$ and $r_3$ lie in the preimage of $p$ under $t_5$. Note that the pointed curve $(E; r_3, r_2)$ and $O_E(r_1)$ will yield the same element in $S_{1,2}^{(1,1),+}$.

Since the moduli spaces $\mathcal{M}_{0,n}$ are rational, the claim is completely proved.
Proposition 3.5. The Hodge diamond of $\overline{S}_2$ is given by

![Hodge Diamond](image)

Figure 3: The Hodge Diamond of $\overline{S}_2$

Proof. By Theorem 3.2 and Proposition 3.3, all we need to show is that $h^{2,0}(\overline{S}_2) = 0$. As recalled in Remark 3.1, each connected component of $S_2$ is an affine variety. Hence, $H^k(S_2) = 0$ for $k \geq \dim(S_2) + 1 = 4$. By Poincaré duality, we have $H^k_c(S_2) = 0$ for $0 \leq k \leq 2$. By the exact sequence

$$
\ldots \rightarrow H^k_c(S_2) \rightarrow H^k(\overline{S}_2) \rightarrow H^k(\partial \overline{S}_2) \rightarrow \ldots
$$

we get an injective morphism $H^k(S_2) \rightarrow H^k(\overline{S}_2)$, which is compatible with the Hodge structures (exactly as in [2], p. 102); so there is an injection $H^{p,0}(\overline{S}_2) \hookrightarrow H^{p,0}(\partial \overline{S}_2)$ for $p \leq 2$. Notice that each irreducible component of $\partial \overline{S}_2$ is the image of a map from either $\overline{S}_{1,1} \times \overline{S}_{1,1}$ or $\overline{S}_{1,2}^{(0,0)}$ or $\overline{S}_{1,2}^{(1,1)}$. Analogously to Lemma 2.6 in [2], it suffices to check that $h^{p,0}(\overline{S}_{1,n}^{(m_1,\ldots,m_n)}) = 0$ for $n \leq 2$. By Lemma 3.4, we get a dominant rational map $\mathbb{P}^n \dashrightarrow \overline{S}_{1,n}^{(m_1,\ldots,m_n)}$. Thus, as in [15], p. 494, we have an injective morphism $H^{p,0}(\overline{S}_{1,n}^{(m_1,\ldots,m_n)}) \hookrightarrow H^{p,0}(\mathbb{P}^n)$. In particular, $h^{2,0}(\overline{S}_2) = 0$, and the claim follows. □

Remark 3.6. As pointed out by the referee, the vanishing of $h^{2,0}(\overline{S}_2)$ is also a direct consequence of (1) and the rationality of $\overline{M}_{0,6}$.

Theorem 3.7. The rational cohomology of $\overline{S}_2$ is algebraic. In particular, the following hold:

i) a basis for the cohomology group $H^2(\overline{S}_2)$ is given by the boundary classes $\alpha_i^+, \alpha_i^-, \beta_i^+, \beta_i^-$;

ii) a basis for the cohomology group $H^4(\overline{S}_2)$ is given by the products $\Delta \alpha_i^+, \Delta \alpha_i^-, \Delta \beta_i^+, \Delta \beta_i^-$, where $\Delta$ is the sum of all boundary classes.
Proof. By the standard exponential sequence, we have a long exact sequence
\[ \cdots \to H^1(\mathcal{S}_2, \mathcal{O}) \otimes \mathbb{Q} \to H^1(\mathcal{S}_2, \mathcal{O}^*) \otimes \mathbb{Q} \to H^2(\mathcal{S}_2, \mathbb{Q}) \to H^2(\mathcal{S}_2, \mathcal{O}) \otimes \mathbb{Q} \to \cdots \]

By Proposition 3.3, we have \( H^1(\mathcal{S}_2, \mathcal{O}) \otimes \mathbb{Q} = (0) = H^2(\mathcal{S}_2, \mathcal{O}) \otimes \mathbb{Q} \). Hence the (rational) cohomology group \( H^2(\mathcal{S}_2) \) is isomorphic to \( \text{Pic}(\mathcal{S}_2) \otimes \mathbb{Q} \). Thus, i) follows from Proposition 2.1 and Proposition 3.3. As for ii), note that \( \Delta \) is ample because \( \mathcal{S}_2 \) is affine (cfr. Remark 3.1). Accordingly, multiplication by \( \Delta \) induces an isomorphism between \( H^2(\mathcal{S}_2) \) and \( H^2(\mathcal{S}_2) \) by the Hard Lefschetz Theorem, which holds in the category of projective orbifolds - see [24], Theorem 1.13. By Theorem 3.7, the claim is completely proved. \[ \square \]

Corollary 3.8. \( A^*(\mathcal{S}_2^+) \cong A^*(\mathcal{S}_2^-) \cong \mathbb{Q} \).

Remark 3.9. Theorem 3.4 gives generators for the rational cohomology of \( \mathcal{S}_2^+ \) (the classes with superscript +) and \( \mathcal{S}_2^- \) (the classes with superscript –).

Corollary 3.10. The Euler characteristic of the normal variety \( \mathcal{S}_2 \) is 18. In particular, \( e(\mathcal{S}_2^+) = 10 \) and \( e(\mathcal{S}_2^-) = 8 \).

For the ring structure of \( H^*(\mathcal{S}_2) \), we need some results on the cohomology of genus one spin curves. Denote by \( \alpha_0^+ \), \( \alpha_1^+ \) (resp. \( \alpha_0^- \), \( \alpha_1^- \)) the (Poincaré) dual cohomology class of the loci \( A_0^+ \), \( A_1^+ \) (resp., \( A_0^- \), \( A_1^- \)) in \( \mathcal{S}_{1,n} \) described above in the case \( n = 0 \) (the only difference here is that the rational component of the general member carries \( n \) marked points). Next, define by \( \lambda_1^+ \) (resp. \( \lambda_1^- \)) the pull-back on \( \mathcal{S}_{1,n}^+ \) (resp. \( \mathcal{S}_{1,n}^- \)) of the cohomology class \( \lambda \) on \( \overline{\mathcal{M}}_{1,n} \). Clearly, these classes can be expressed in terms of boundary divisors. In fact, we have \( \lambda = \frac{1}{12} \delta_{\text{irr}} \) on \( \overline{\mathcal{M}}_{1,n} \). Moreover, the space \( \mathcal{S}_{1,1}^+ \) is isomorphic to \( \mathbb{P}^1 \) and if \( \gamma_{1,2}^+ : \mathcal{S}_{1,2}^+ \to \overline{\mathcal{M}}_{1,2} \) is the \( 3 : 1 \) covering of \( \overline{\mathcal{M}}_{1,2} \), then we have \( (\gamma_{1,2}^+)^*(\delta_{\text{irr}}) = 3\alpha_0^+ \). Hence,

\[ \lambda_1^+ = \frac{1}{4}\alpha_0^+, \quad \lambda_1^- = \frac{1}{12}\alpha_0^- \quad \text{(2)} \]

Remark 3.11. We have \( (\lambda_1^+)^2 = (\lambda_1^-)^2 = 0 \). Indeed, as pointed out in \[ \text{[9]}, \text{Lemma 3.2.6}, \delta_{\text{irr}}^2 = 0 \text{ on } \overline{\mathcal{M}}_{1,n} \text{ since the boundary divisor } \Delta_{\text{irr}} \text{ is the pull-back of a point in } \overline{\mathcal{M}}_{1,1} \cong \mathbb{P}^1 \text{ via the natural forgetful morphism.} \]

Theorem 3.12. The cohomology ring \( H^*(\mathcal{S}_2^+) \) is isomorphic to the quotient ring \( \mathbb{Q}[\alpha_0^+, \alpha_1^+, \beta_0^+, \beta_1^+]/J^+ \), where \( J^+ \) is the ideal generated by the following elements:

\[ \alpha_1^+ \beta_1^+; \quad \beta_0^+ \beta_1^+; \quad \alpha_0^+ \alpha_1^+ - \beta_0^+ \alpha_1^+; \]

\[ \alpha_0^+ \alpha_1^+ + 8 (\alpha_1^+)^2; \quad \alpha_0^+ \beta_1^+ + 24 (\beta_1^+)^2; \quad 4(\beta_0^+)^2 + 8\alpha_1^+ \beta_0^+ - 3\alpha_0^+ \beta_0^+; \]

\[ (\alpha_0^+)^2 \alpha_1^+; \quad (\alpha_0^+)^2 \beta_0^+; \quad 3(\alpha_0^+)^3 + 22(\alpha_0^+)^2 \beta_1^+. \]

Proof. The relations \( \alpha_1^+ \beta_1^+, \beta_0^+ \beta_1^+ \) follow immediately from the definition of \( \beta_1^+ \). In fact, the intersection of the corresponding cycles are empty by parity reasons (for instance, if \( B_0^+ \cap B_1^+ \) were non-empty, then it would be one-dimensional and its general element should carry an even theta characteristic.
restricting to an even one on the genus 0 component and to an odd one to the genus 1 component, which is clearly impossible).

Let \( \theta \) be the boundary map

\[
\theta : \mathbb{S}^+_{1,1} \times \mathbb{S}^+_{1,1} \to \mathbb{S}^+_{2}
\]
defined as follows. A point in the domain is the datum of two 1-pointed genus one curves with even spin structures. The image point is obtained by glueing the two curves along the two marked points and then blowing-up the node. The exceptional component \( C \) is given a spin structure via \( \mathcal{O}_C(1) \).

We have

\[
(\alpha_0^+ - \beta_0^+) \alpha_1^+ = \frac{1}{2} \theta_* (\theta^*(\alpha_0^+ - \beta_0^+)) = 0
\]
since \( \theta^*(\alpha_0^+ - \beta_0^+) = 0 \). Indeed, the classes \( \alpha_0^+ \) and \( \beta_0^+ \) coincide on \( \mathbb{S}^+_{1,1} \cong \mathbb{P}^1 \), hence \( 1 \otimes \alpha_0^+ + \alpha_0^+ \otimes 1 - 1 \otimes \beta_0^+ - \beta_0^+ \otimes 1 = 0 \).

We recall from [22], p. 321, that 10 \( \lambda \) is the pull-back of \( \delta_{\text{irr}} \) on \( \mathcal{M}_2 \). We denote by \( \lambda_2^+ \) the pull-back of \( \lambda \) on \( \mathbb{S}^+_{2} \). By Proposition (7.2), i) and ii), we have \( (\nu^+)^*(\delta_{\text{irr}}) = \alpha_0^+ + 2\beta_0^+ \) and \( (\nu^+)^*(\delta_1) = 2\alpha_1^+ + 2\beta_1^+ \), hence we obtain

\[
\lambda_2^+ = \frac{1}{10} (\alpha_0^+ + 2\beta_0^+ + 12\beta_1^+ + 4\alpha_1^+).
\]

On the one hand, we get \( \lambda_2^+ \alpha_1^+ = \frac{1}{10} \alpha_0^+ \alpha_1^+ + \frac{2}{5} (\alpha_1^+)^2 \) by applying the previous relations. On the other hand, if we recall [2] and apply the push-pull formula we see that

\[
\lambda_2^+ \alpha_1^+ = \frac{1}{2} \theta_* (\theta^*(\lambda_2)) = \frac{1}{2} \theta_* (\lambda_1^+ \otimes 1 + 1 \otimes \lambda_1^+)
\]

\[
= \frac{1}{8 \alpha_0^+ \alpha_1^+}.
\]

Thus, the codimension two relation \( \alpha_0^+ \alpha_1^+ + 8 (\alpha_1^+)^2 = 0 \) holds.

As shown in [22], p. 321, the following relation holds in \( H^1(\mathcal{M}_2) \):

\[
\lambda \delta_1 = \frac{1}{12} \delta_{\text{irr}} \delta_1.
\]

By subtracting (6), the pull-back of \( (\nu^+)^* \) yields \( \lambda_2^+ \beta_1^+ = \frac{1}{12} \alpha_0^+ \beta_1^+ \). By (4), the class \( \lambda_2^+ \beta_1^+ \) is also equal to \( 1/10 (\alpha_0^+ \beta_1^+ + 4\alpha_1^+ \beta_1^+ + 4(\beta_1^+)^2) \); so the codimension two relation

\[
\alpha_0^+ \beta_1^+ + 24(\beta_1^+)^2 = 0
\]

follows. Notice that

\[
3 (\alpha_0^+ \alpha_1^+ + 8 (\alpha_1^+)^2) + \alpha_0^+ \beta_1^+ + 24(\beta_1^+)^2 = 0
\]
is the pull-back of \( \delta_{\text{irr}} \delta_1 + 12 \delta_1^2 \) on \( \mathcal{M}_2 \), which is zero - see [22], p. 321.

In order to compute the last relation, let \( \eta \) be the boundary map

\[
\eta : \mathbb{S}^+_{1,2} \to \mathbb{S}^+_{2},
\]

which is defined as follows. A point in the domain is the datum of a two pointed genus one curve with an even spin structure. The corresponding image point is obtained by glueing the two marked points and blowing-up the node. The resulting quasi-stable curve is given a spin structure
via the spin structure of the domain curve plus \( \mathcal{O}_F(1) \) on the exceptional component \( F \). By definition we have \( \beta_0^+ = \frac{1}{4} \eta_*(1) \). Notice that the order of the automorphism group of a generic curve in the image of \( \eta \) is four. By (2), we have

\[
\lambda_2^+ \beta_0^+ = \frac{1}{4} \eta_* (\lambda_1^+) = \frac{1}{4} \eta_* \left( \frac{1}{4} \alpha_0^+ \right) = \frac{1}{4} \alpha_0^+ \beta_0^+.
\]

By (4), the class \( \lambda_2^+ \beta_0^+ \) is also equal to \( 1/10 (\alpha_0^+ \beta_0^+ + 2(\beta_0^+)^2 + 4\alpha_1^+ \beta_0^+) \). Hence, the last codimension two relation follows.

Let us now determine the codimension three relations. By (8), we get

\[
(\lambda_2^+)^2 \beta_0^+ = \frac{1}{4} \lambda_2^+ \beta_0^+ \alpha_0^+ = \frac{1}{16} (\alpha_0^+)^2 (\beta_0^+).
\]

We remark that \( (\lambda_2^+)^2 \alpha_0^+ = (\lambda_2^+)^2 \beta_0^+ = 0 \). Indeed, restricting \( (\lambda_2^+)^2 \) to the divisors corresponding to \( \alpha_0^+ \) and \( \beta_0^+ \) is equivalent to computing \( (\lambda_1^+)^2 \) on \( \overline{S}_{1,2} \), which is zero by Remark 3.11 This yields \( (\alpha_0^+)^2 (\beta_0^+) = 0 \). By plugging (4) in this last identity, we get \( (\alpha_0^+)^2 \alpha_1^+ = 0 \). This, together with the codimension two relations above, yields that all degree three monomials in \( H^6(\overline{S}_{2}) \) vanish except for \( (\alpha_0^+)^3 \), \( (\alpha_0^+)^2 \beta_1^+ \), \( \alpha_0^+ (\beta_1^+)^2 \). By (4), we have \( (\beta_1^+)^2 \alpha_0^+ = -\frac{1}{22} \alpha_0^+ \beta_1^+ \). Finally, by the relation \( (\lambda_2^+)^2 \alpha_0^+ = 0 \), we get \( 3(\alpha_0^+)^3 + 22(\alpha_0^+)^2 \beta_1^+ = 0 \).

A calculation with computer program Macaulay [5] yields that the ring \( \mathbb{Q}[\alpha_0^+, \alpha_1^+, \beta_0^+, \beta_1^+]/J^+ \) is finite dimensional and has Betti numbers 1, 4, 4, 1. Since this ring surjects onto \( H^* (\overline{S}_{2}) \) and the Betti numbers coincide, they are isomorphic.

Notice in particular that \( (\alpha_0^+)^2 \beta_1^+ = \frac{2}{3} [p] \), where \([p] \) is the class of a point on \( \overline{S}_{2} \). This follows from the intersection number \( \delta_3^3 = \frac{1}{576} \) on \( \overline{M}_{2} \).

**Theorem 3.13.** The cohomology ring \( H^* (\overline{S}_{2}) \) is isomorphic to the quotient ring \( \mathbb{Q}[\alpha_0^-, \alpha_1^-, \beta_0^-]/J^- \), where \( J^- \) is the ideal generated by the following polynomials:

\[
3(\beta_0^-)^2 + 6\alpha_1^- \beta_0^- - \alpha_0^- \beta_0^- , \quad 2\alpha_1^- \beta_0^- - \alpha_1^- \alpha_0^- , \quad 12(\alpha_1^-)^2 + \alpha_1^- \alpha_0^- , \quad (\alpha_0^-)^2 \beta_0^- , \quad 3(\alpha_0^-)^3 + 32\alpha_1^- (\alpha_0^-)^2.
\]

**Proof.** Analogously to (5), we can prove that

\[
\lambda_2^{-} \beta_0^- = \frac{1}{6} \alpha_0^- \beta_0^- , \quad \lambda_1^- \beta_0^- = \frac{1}{10} (\alpha_0^- + 2\beta_0^- + 4\alpha_1^-) .
\]

If we multiply by \( \beta_0^- \) and compare with (5), we get the relation \( 3(\beta_0^-)^2 + 6\alpha_1^- \beta_0^- - \alpha_0^- \beta_0^- = 0 \) holds.
Similarly to (3), define the boundary map \( \iota : S_{1,1}^+ \times S_{1,1}^- \rightarrow S_2^- \). Since \( \alpha_0^+ = \beta_0^- \) on \( S_{1,1}^+ \approx \mathbb{P}^1 \), we have:

\[
\beta_0^- \alpha_1^- = \iota_* \tau^* (\beta_0^-) = \iota_* (\beta_0^+ \otimes 1) = \iota_* (\alpha_0^+ \otimes 1) = \iota_* (\iota^* (\alpha_0^-) - 1 \otimes \alpha_0^-) = \frac{1}{2} \alpha_0^+ \alpha_1^-.
\]

Note that \( \iota_* (1 \otimes \alpha_0^-) = \frac{1}{2} \alpha_0^+ \alpha_1^- \). In fact, the automorphism group of the generic curve in the intersection of the divisors corresponding to \( \alpha_0^- \) and \( \alpha_1^- \) has order two. Hence, the codimension two relation \( 2 \alpha_1^- \beta_0^- - \alpha_0^- \alpha_1^- = 0 \) follows. Finally, if we pull-back \( \delta_{irr} \delta_1 + 12 \delta_1^2 \) on \( S_2^- \), we get \( 48(\alpha_1^-)^2 + 2 \alpha_1^- \alpha_0^- + 4 \alpha_1^- \beta_0^- \), which yields the relation \( 12(\alpha_1^-)^2 + \alpha_1^- \alpha_0^- = 0 \).

If we use the codimension two relations, we obtain the following relations in codimension three:

\[
144(\alpha_1^-)^3 - \alpha_1^- (\alpha_0^-)^2 = 0, \quad 54(\beta_0^-)^3 - 6(\alpha_0^-)^2 \beta_0^- + 45 \alpha_1^- (\alpha_0^-)^2 = 0,
\]

\[
12(\alpha_1^-)^2 \alpha_0^- + \alpha_1^- (\alpha_0^-)^2 = 0, \quad 24(\alpha_1^-)^2 \beta_0^- + \alpha_1^- (\alpha_0^-)^2 = 0,
\]

\[
\alpha_0^- (\beta_0^-)^2 + \alpha_1^- (\alpha_0^-)^2 = 0, \quad 4(\beta_0^-)^2 \alpha_1^- - (\alpha_0^-)^2 \alpha_1^- = 0,
\]

\[
2 \alpha_1^- \alpha_0^- \beta_0^- - \alpha_1^- (\alpha_0^-)^2 = 0.
\]

By Remark 3.11, we have \( (\lambda_2^-)^2 \beta_0^- = (\lambda_2^-)^2 \alpha_0^- = 0 \). By (9), we get \( (\alpha_0^-)^2 \beta_0^- = 0 \). If we plug in \( \lambda_2^- = \frac{1}{10} (\alpha_0^- + 2 \beta_0^- + 4 \alpha_1^-) \) in \( (\lambda_2^-)^2 \alpha_0^- = 0 \) and use the relations above, we also have \( 3(\alpha_0^-)^3 + 32 \alpha_1^- (\alpha_0^-)^2 = 0 \).

A calculation with computer program Macaulay [3] yields that the quotient ring \( \mathbb{Q}[\alpha_0^-, \alpha_1^-, \beta_0^-]/J^- \) is finite dimensional and has Betti numbers \( 1,3,3,1 \). Since this ring surjects onto \( H^*(S_2^-) \) and the Betti numbers coincide, they are isomorphic.

Finally, notice that \( \alpha_1^- (\alpha_0^-)^2 = \frac{1}{10} [q] \), where \( [q] \) is the class of a point on \( S_2^- \). This follows easily from the intersection number \( \delta_{irr} \delta_1^2 = -\frac{1}{10} \) on \( \mathcal{M}_2 \).

\[\square\]

APPENDIX A

We recall that the moduli space of stable curves has a natural stratification by topological type. An explicit description in genus 2 can be found in [22]: there are seven strata \( \Delta_{G_i} \), which correspond to the graphs \( G_i \) in Figure 1.

We shall list the components of \( \nu^{-1}(\Delta_{G_i}) \) by describing their general member.

The preimage of \( \Delta_{G_1} \) yields two strata, namely \( S_g^+ \) and \( S_g^- \).

The eight strata of the preimages \( \nu^{-1}(\Delta_{G_2}) \) and \( \nu^{-1}(\Delta_{G_3}) \) are described in [13]. Their closures are denoted by \( A_g^+, A_g^-, B_g^+, B_g^- \).

The preimage \( \nu^{-1}(\Delta_{G_4}) \) is the union of the following strata:

- \( C^- \): an irreducible curve with two nodes and an even spin structure;
- \( C^+ \): an irreducible curve with two nodes and an odd spin structure;
Figure 1. Graph-types.

Figure 2. Curves in $C^+$ and $C^-$.  

- $D^+$: a rational irreducible curve with two nodes - blown-up at one of them - with an even spin structure;  
- $D^-$: as above, but with an odd spin structure;  

Figure 3. Curves in $D^+$ and $D^-$.  

- $E$: a rational irreducible curve with two nodes - blown-up at both nodes - with an even spin structure.  

The preimage $\nu^{-1}(\Delta_{G_5})$ is the disjoint union of the following strata:  
- $X^+$: a smooth elliptic curve $F_1$ and an irreducible nodal curve $F_2$ - joined via a smooth rational curve at two points $y_1$ and $y_2$ - with even theta-characteristics on $F_1$ and $F_2$;  
- $X^-$: as above, but with an even theta-characteristic on $F_1$ and an odd theta-characteristic on $F_2$;
• \( Y^+ \): as above, but with odd theta-characteristics on \( F_1 \) and \( F_2 \);
• \( Y^- \): as above, but with an odd theta-characteristic on \( F_1 \) and an even theta-characteristic on \( F_2 \);

\[ \begin{array}{c}
\text{Figure 4. Curves in } E.
\end{array} \]

• \( Z^+ \): an irreducible nodal curve \( T_2 \) (blown-up at the node) and a smooth elliptic curve \( T_1 \) - joined via a smooth rational curve - with an even theta-characteristic on \( T_1 \) and an even spin structure on \( T_2 \);
• \( Z^- \): as above, but with an odd theta-characteristic on \( T_1 \) and an even spin structure on \( T_2 \).

\[ \begin{array}{c}
\text{Figure 5. Curves in } X^+, X^-, Y^+ \text{ and } Y^-.
\end{array} \]

The preimage \( \nu^{-1}(\Delta_{G_6}) \) has seven strata, i.e.,
• \( L^+ \): two smooth rational curves that intersect in three points (blown-up at one of them) with an even spin structure\(^1\);
• \( L^- \): as above, but with an odd spin structure;
• \( M \): two smooth rational curves \( K_1 \) and \( K_2 \) that intersect in three points \( p_1, p_2, p_3 \) (blown-up at the intersection points) with an even spin structure.

\(^1\)Notice that blowing-up at a different point yields an isomorphic spin curve.
The preimage $\nu^{-1}(\Delta_{G_7})$ has seven strata (points), i.e.,

- $P^+$: two irreducible nodal curves $C_1$ and $C_2$ of (arithmetic) genus 1 - joined at points $p$ and $q$ by a smooth rational curve - with an even theta-characteristics on $C_1$ and $C_2$;
- $Q^+$: as above, but with odd theta-characteristics on $C_1$ and $C_2$;
- $P^- = Q^-$: as above, but with an odd theta-characteristic on $C_1$ and an even theta-characteristic on $C_2$;
- $R$: two irreducible curves with one node (each of them blown-up at the node) that are joined by a smooth rational curve at two distinct points, with an even spin structure;
- $U^+$: an irreducible curve $V_2$ with one node (blown-up at the node) and an irreducible nodal curve $V_1$ of arithmetic genus one - joined via a smooth rational curve at two distinct points - with an even theta-characteristic on $V_1$ and an even spin structure on $V_2$;
- $U^-$: as above, but with an odd theta-characteristic on $V_1$ and an even spin one on $V_2$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7}
\caption{Curves in $L^+$ and $L^-$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8}
\caption{Curves in $M$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9}
\caption{Curves in $P^+, Q^+$ and $P^- = Q^-$.}
\end{figure}
Figure 10. Curves in $R$.

Figure 11. Curves in $U^+$ and $U^-$.

References

[1] D. Abramovich and T. J. Jarvis: Moduli of twisted spin curves. Proc. Amer. Math. 131 (2003), 685–699.
[2] E. Arbarello, M. Cornalba: Calculating cohomology groups of moduli spaces of curves via algebraic geometry. Inst. Hautes Études Sci. Publ. Math. 88 (1998), 97–127.
[3] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris: Geometry of Algebraic Curves. Vol. I. Grund. der Math. Wiss., 267, Springer-Verlag, New York, 1985.
[4] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris: Geometry of Algebraic Curves. Vol. II. In preparation.
[5] D. Bayer, M. Stillman: Macaulay - a software system for algebraic geometry and commutative algebra. Available at [http://www.math.columbia.edu/~bayer/Macaulay](http://www.math.columbia.edu/~bayer/Macaulay).
[6] P. Belorousski: Chow rings of moduli spaces of pointed elliptic curves. Ph.D Dissertation, Chicago, 1998.
[7] G. Bini, C. Fontanari: Moduli of curves and spin structures via algebraic geometry. Trans. Amer. Math. Soc. 358 (2006), 3207–3217.
[8] M. Boggi, M. Pikaart: Galois covers on moduli of curves. Comp. Math. 120 (2000), 171–191.
[9] C. Casagrande, L. Caporaso: Combinatorial properties of stable spin curves. Comm. in Algebra: Special issue in Honor of Professor Steven Kleiman. 31 (2003), 3653–3672.
[10] C. Casagrande, L. Caporaso, M. Cornalba: Moduli of roots of line bundles on curves. Preprint [math.AG/0404078](http://arxiv.org/abs/math.AG/0404078). To appear in Trans. Amer. Math. Soc.
[11] A. Chiodo: Roots of line bundles on curves and Néron models. Preprint [math.AG/0603689](http://arxiv.org/abs/math.AG/0603689).
[12] A. Chiodo: Stable twisted curves and their $r$-spin structures. Preprint [math.AG/0603687](http://arxiv.org/abs/math.AG/0603687).
[13] M. Cornalba: Moduli of curves and theta-characteristics. In: Lectures on Riemann surfaces (Trieste, 1987), 560–589, Teaneck, NJ: World Sci. Publishing, 1989.
[14] C. Faber: Chow rings of moduli spaces of curves. I. The Chow ring of $\overline{M}_3$. Ann. of Math. 132 (1990), 331–419.
[15] P. Griffiths and J. Harris: Principles of Algebraic Geometry. New York: John Wiley and Sons, Inc., 1978.
[16] J. Harer: The virtual cohomological dimension of the mapping class group of an orientable surface. Inv. Math. 84 (1986), 157–176.
[17] J. Harer: Stability of the homology of the moduli spaces of Riemann surfaces with spin structure. Math. Ann. 287 (1990), 323–334.
[18] J. Harer: The rational Picard group of the moduli space of Riemann surfaces with spin structure. Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math. 150 (1993), 107–136.
[19] T. J. Jarvis: Geometry of the moduli of higher spin curves. Internat. J. Math. 11 (2000), 637–663.
[20] T. J. Jarvis, T. Kimura, A. Vaintrob: Moduli spaces of higher spin curves and integrable hierarchies. Comp. Math. 126 (2001), 157–212.
[21] S. Keel: Intersection theory of moduli space of stable $n$ pointed curves of genus zero. Trans. Amer. Math. Soc. 330 (1992), 545–574.
[22] D. Mumford: Towards an enumerative geometry of the moduli space of curves, Progress in Math. (1983), 271–328.
[23] M. Pacini, Spin structures over non stable curves. Preprint [math.AG/0602420]
[24] J. H. M. Steenbrink: Mixed Hodge structure on the vanishing cohomology. Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
[25] S. Krug: Rational cohomology of $\mathbb{P}^2$ (and $\mathbb{S}^2$). arXiv:1012.5191 (2010).

E-mail address: gilberto.bini@mat.unimi.it
Current address: DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA C. SALDINI 50, 20133 MILANO, ITALY.
E-mail address: claudio.fontanari@polito.it
Current address: DIPARTIMENTO DI MATEMATICA, POLITECNICO DI TORINO, CORSO DUCA DEGLI ABRUZZI 24, 10129 TORINO, ITALY.