THE INTERNAL STRUCTURAL ADJUSTMENT DUE TO TIDAL HEATING OF SHORT-PERIOD INFLATED GIANT PLANETS

PIN-GAO GU
Institute of Astronomy and Astrophysics, Academia Sinica, P.O. Box 23141, Taipei 106, Taiwan

AND

PETER H. BODENHEIMER AND DOUGLAS N. C. LIN
University of California Observatories/Lick Observatory, University of California, 373 Interdisciplinary Sciences, Santa Cruz, CA 95064

Received 2003 December 5; accepted 2004 March 5

ABSTRACT

Several short-period Jupiter-mass planets have been discovered around nearby solar-type stars. During the circularization of their orbits, the dissipation of tidal disturbance by their host stars heats the interior and inflates the sizes of these planets. Based on a series of internal structure calculations for giant planets, we examine the physical processes that determine their luminosity-radius relation. In the gaseous envelope of these planets, efficient convection enforces a nearly adiabatic stratification. During their gravitational contraction, the planets’ radii are determined, through the condition of a quasi-hydrostatic equilibrium, by their central pressure. In interiors of mature, compact, distant planets, such as Jupiter, degeneracy pressure and the nonideal equation of state determine their structure. However, in order for young or intensely heated gas giant planets to attain quasi-hydrostatic equilibria, with sizes comparable to or larger than $2R_J$, their interiors must have sufficiently high temperature and low density such that degeneracy effects are relatively weak. Consequently, the effective polytropic index monotonically increases, whereas the central temperature increases and then decreases with the planets’ size. These effects, along with a temperature-sensitive opacity for the radiative surface layers of giant planets, cause the power index of the luminosity’s dependence on radius to decrease with increasing radius. For planets larger than twice Jupiter’s radius, this index is sufficiently small that they become unstable to tidal inflation. We make comparisons between cases of uniform heating and cases in which the heating is concentrated in various locations within the giant planet. Based on these results, we suggest that accurate measurement of the sizes of close-in young Jupiters can be used to probe their internal structure under the influence of tidal heating.

Subject headings: planetary systems — planets and satellites: general

On-line material: color figure

1. INTRODUCTION

One of the surprising findings in the search for planetary systems around other stars is the discovery of extrasolar planets with periods down to 3 days (Mayor & Queloz 1995). Nearly all planets with periods less than 7 days have nearly circular orbits. In contrast, known extrasolar planets with periods longer than 2–3 weeks have nearly a uniform eccentricity distribution. The shortest period planets and their host stars induce tidal perturbations on each other. When these disturbances are dissipated, angular momentum is exchanged between the planets and their host stars, leading toward both a spin synchronization and orbital circularization (Rasio et al. 1996).

Bodenheimer et al. (2001, hereafter Paper I) considered the effect of tidal dissipation of energy during the synchronization of these planets’ spin and the circularization of their orbits. In that analysis, they compute a series of numerical models for the interior structures of weakly eccentric Jovian planets at constant orbital distances under the influence of interior tidal heating and stellar irradiation. In these previous calculations, the interior heating rate per unit mass was imposed to be constant in time and uniformly distributed within the planet. Under these assumptions, they showed that Jovian planets can be inflated to equilibrium sizes considerably larger than those deduced for gravitationally contracting and externally heated planets. For the transiting planet around HD 209458, they suggested that, provided that its dimensionless dissipation $Q$-value is comparable to that inferred for Jupiter (Yoder & Peale 1981), a small eccentricity ($e \approx 0.03$) would provide adequate tidal heating to inflate it to its observed size (Brown et al. 2001). Since the orbital circulation timescale is expected to be shorter than the life span of the planet, they also suggested that this eccentricity may be excited by another planet with a longer orbital period. The prediction of a small eccentricity and the existence of another planet are consistent with existing data (Bodenheimer et al. 2003).

There are at least two other scenarios for the unexpectedly large size of HD 209458b. Heating by stellar irradiation reduces the temperature gradient and the radiative flux in the outer layers of short-period planets. This process could significantly slow down the Kelvin-Helmholtz contraction of the planet and explain the large size (Burrows et al. 2000). However, even though the stellar flux onto the planet’s surface is 5 orders of magnitude larger than that released by the gravitational contraction and cooling of its envelope, this heating effect alone increases the radius of the planet by about 10%, not by 40% as observed (Guillot & Showman 2002). An alternative source is the kinetic heating induced by the dissipation of the gas flow in the atmosphere, which occurs because of the pressure gradient between the day and night sides (Guillot & Showman 2002). In order to account for the observed size of the planet, conversion of only 1% of the incident radiative flux may be needed, provided that the
dissipation of induced kinetic energy into heat occurs at sufficiently deep layers (tens to 100 bars). Showman & Guillot (2002) suggest that the Coriolis force associated with a synchronously spinning planet may induce the circulation to penetrate that far into the planet’s interior and that dissipation could occur through, for example, Kelvin-Helmholtz instability. A follow-up analysis suggests that this effect may be limited (Burkert et al. 2003; C. Jones & D. N. C. Lin 2004, in preparation).

In order to distinguish between these three scenarios, the effect of tidal heating for planets with modest to large eccentricities was considered. In a follow-up paper (Gu et al. 2003, hereafter Paper II) we showed that the size \( R_p \) of compact Jupiter-mass planets slowly increases with the tidal dissipation rate. For computational simplicity, we adopted a conventional equilibrium tidal model that describes the planets’ continuous structural adjustment in order for them to maintain a state of quasi-hydrostatic equilibrium in the varying gravitational potential of their orbital companion. In this prescription, a phase lag into the response is introduced to represent the lag being proportional to the tidal forcing frequency and attributable to the viscosity of the body. The phase lag gives rise to a net tidal torque and dissipation of energy and the efficiency of the tidal dissipation can be parameterized, whatever its origin, by a specific dissipation function or \( Q \)-value (quality factor; Goldreich & Soter 1966). External perturbation can also induce dynamical tidal responses through the excitation of \( g \)-modes (Ioannou & Lindzen 1993a, 1993b) or inertial waves (Ogilvie & Lin 2004), which can be damped by viscous dissipation in the interior (Goldreich & Nicholson 1977) or radiative and nonlinear dissipation in the atmosphere of the planet (Lubow et al. 1997).

In the prescription for both equilibrium and dynamical responses, the tidal dissipation rate is a rapidly increasing function of the planets’ radius. However, their surface luminosity increases even faster with \( R_p \) such that planets with relatively small eccentricities and modest to long periods attain a state of thermal equilibrium in which the radiative loss on their surface is balanced by the tidal dissipation in their interior. For planets with short periods and modest to large eccentricities, the rate of interior heating is sufficiently large that their \( R_p \) may inflate to more than \( 2R_J \). In this limit, the surface luminosity of the planet becomes a less sensitive function of its \( R_p \) and the eccentricity damping rate is smaller than the expansion rate of the planet so that the increases in their surface cooling rate cannot keep pace with the enhanced dissipation rate due to their inflated sizes. These planets are expected to undergo runaway inflation and mass loss. We suggested that the absence of ultra–short-period Jupiter-mass planets with \( P < 3 \) days, which corresponds to an orbital semimajor axis \( a \) of 0.04 AU, may be due to mass loss through Roche lobe overflow resulting from such a tidal inflation instability. Other scenarios have been proposed to explain the lack of Jupiter-mass planets with \( P < 3 \) days: truncation of the inner part of a disk (Lin et al. 1996; Kuchner & Lecar 2002), orbital migration due to the spin-orbit tidal interaction between the close-in planet and the parent star (Rasio et al. 1996; Witte & Savonije 2002; Paetold & Rauer 2002; Jiang et al. 2003), and Roche-overflowing planets with the help of disk–planet interaction but without tidal inflation (Trilling et al. 1998).

In this contribution, we continue our investigation on the internal structure of tidally heated short-period planets. The main issues to be examined here are as follows: (1) How does tidal energy dissipation actually lead to the expansion of the envelope? (2) How does the internal structure of the planet depend on the distribution of their internal tidal dissipation rate? (3) What are the important physical effects that determine the tidal inflation stability of the planets? These issues are important in determining the mass-radius relation of short-period planets, which is directly observable.

Structural adjustments may also modify the efficiency of dissipation and the planets’ \( Q \)-value for both equilibrium and dynamical tides. In the extended convective envelopes of gaseous giant planets and low-mass stars, turbulence can lead to dissipation of the motion that results from the continual adjustment of the equilibrium tide. However, the turbulent viscosity estimated from the mixing-length theory ought to be reduced by a frequency-dependent factor as a result of the fact that the convective turnover timescale is usually much longer than the period of the tidal forcing. Based on the present-day structure of Jupiter, Goldreich & Nicholson (1977) estimated \( Q \approx 5 \times 10^{13} \). However, within the intensely heated (by tidal dissipation) short-period extrasolar planets, convection is expected to be more rigorous with higher frequencies, whereas their tidal forcing frequencies are smaller than that of Jupiter. The \( Q \)-value for the equilibrium tide within extrasolar planets is likely to be much smaller than that within Jupiter.

For the dynamical response of short-period extrasolar planets, the forcing frequencies are typically comparable to their spin frequencies but are small compared to their dynamical frequencies. Convective regions of the planets support inertial waves, which possess a dense or continuous frequency spectrum in the absence of viscosity, while any radiative regions support generalized Hough waves (Ioannou & Lindzen 1993a, 1993b). Inertial waves provide a natural avenue for efficient tidal dissipation in most cases of interest. The resulting value of \( Q \) depends, in principle, in a highly erratic way on the forcing frequency. Since the planets’ spin frequency adjusts with their \( R_p \), which is a time-delayed function of the tidal dissipation rate within them, the efficiency of tidal dissipation may fluctuate while the overall evolution is determined by a frequency-averaged \( Q \)-value (Ogilvie & Lin 2004).

In § 2 we briefly recapitulate the basic equations that determine the quasi-static evolution of the planets’ structure. We show the simulation results for inflated giant planets in the case of constant internal heating per unit mass in § 3 and analyze the results, in the Appendix, in terms of polytropic models that allow us to conclude that the onset of the tidal runaway inflation instability is regulated by a transition in the equation of state for the interior gas from a partially degenerate/nonideal state toward a more ideal gas state. We examine the dependence of planetary adjustment on different locations of the energy dissipation in § 4. Finally, we summarize the results and discuss their observational implications in § 5.

2. THE PLANETARY STRUCTURE EQUATIONS AND NUMERICAL METHODS

The internal structure of an inflated Jupiter is constructed in this paper with the same numerical scheme as that used in Paper I. The code employs the tabulated equation of state and adiabatic gradient \( \nabla_{\text{ad}} \) described by Saumon et al. (1995), and the values of opacities are derived from those provided by Alexander & Ferguson (1994). With this scheme, we compute distributions of thermodynamical parameters as a function of radius for a spherically symmetric planet (i.e., one-dimensional calculation). The surface temperature of the planet is assumed to be maintained by the irradiation from a solar-type star.
located 0.04 AU away. The structure of the planet is assumed to be hydrostatic and not to be affected by rotation since the nearly synchronized short-period planets spin at least several times slower than Jupiter and their rotational energy is quite small compared to their gravitational energy (Paper I).

The models include a tidal heating function, whose physical basis is not well understood. Thus, in § 3 we consider simulations in which the planet has a constant internal heating that is uniform in mass. In § 4 we consider a set of models in which the heating is nonuniform in mass. As discussed by Ogilvie & Lin (2004), different mechanisms may operate in radiative or convective layers. One should, in principle, allow for three avenues of tidal dissipation: viscous dissipation of the equilibrium tide, viscous dissipation of inertial waves, and emission of Hough waves in the radiative zone. Rotational effects can also affect the behavior of tidal dissipation. In view of the uncertainties, we do not model specific mechanisms but simply parameterize the heating rate.

In our calculations, we assume that convection is so efficient that the temperature gradient behaves adiabatically within a convective zone (Guillot et al. 2003). In other words, we solve the following equations for the radius $r$, the density $\rho$, the pressure $P$, the temperature $T$, and the intrinsic luminosity $L$ (see, e.g., Kippenhahn & Weigert 1990):

\[ \frac{\partial r}{\partial m} = \frac{1}{4\pi r^3 \rho}, \]  
\[ \frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}, \]  
\[ \frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4} \nabla, \]  
\[ \nabla = \left\{ \begin{array}{ll} \nabla_{\text{rad}} & \text{if radiative}, \\ \nabla_{\text{ad}} & \text{if convective}, \end{array} \right. \]  
\[ \frac{\partial L}{\partial m} = \epsilon - c_P \frac{\partial T}{\partial t} + \frac{\delta}{\rho} \frac{\partial P}{\partial t}, \]  
where $m$ is the Lagrangian mass coordinate, $\delta = \frac{\partial \ln \rho}{\partial \ln T}$, $c_P$ is the specific heat at constant pressure, $\epsilon$ is the heating rate per unit mass, and $\nabla$ is defined as $d\ln T/d\ln P$.

The above structure equations are solved simultaneously with a one-dimensional implicit Lagrangian scheme that uses the mass coordinate $m$ as an independent variable. In the case in which the heating rate is uniformly distributed with mass (i.e., $\epsilon = $ const), the results of our numerical calculations usually show that the planet interior is largely convective. Consequently, without the aid of the energy equation (5), equations (1)–(4) imply that the adiabatic assumption gives rise to a unique radial stratification for a given planet’s size and mass, regardless of how strong the internal heating rate is or whether the planet is in thermal equilibrium. Equation (5) then indicates how fast the planet expands or contracts as a result of thermal imbalance. These adjustments proceed through a series of quasi-hydrostatic and quasi-thermal equilibria.

The initial condition is obtained from calculations for the formation phases of planets of the appropriate masses, as described in Bodenheimer et al. (2000). Just after accretion ends, those calculations show that the radius is about $2R_p$, which is the initial condition used here. This value is somewhat uncertain, and it changes quite rapidly during the earliest part of the planet’s cooling phase, but the exact value makes little difference here because most of the models reach a thermal equilibrium that is practically independent of the initial state.

For the inner boundary, we consider models with or without cores. For the models with cores, we assume that they have a constant density $\rho_{\text{core}} = 5$ g cm$^{-3}$. Their temperature is also assumed to be the same as that of the envelope immediately outside the core. The luminosity generated in the cores is assumed to be negligible. In most of our models, the central temperature is around $\sim 3 \times 10^4$ K, and the heavy elements in the cores are likely to be soluble (Paper I). Since we have already shown that coreless models lead to more inflated planetary structure, the planetary radii determined for the core models represent a lower limit. There are some uncertainties in the equation of state at the temperatures and pressures of the central regions, where the hydrogen-helium gas is partially degenerate and nonideal. We show below that the degree of degeneracy and the nonideal effects are important in determining the stability against the tidal runaway instability.

Near the planet’s surface, the gas becomes radiative when $\nabla_r < \nabla_{\text{ad}}$, where $\nabla_r$ is computed using equation (4) and $\nabla_{\text{ad}}$ is obtained from a tabulated equation of state. The depth of the radiative zone is sensitive to the opacity. The temperature of the surface layer is sufficiently low for grains to condense. We adopt the standard opacity table (Alexander & Ferguson 1994), which is computed for the interstellar gas. However, in Jupiter’s atmosphere the sedimentation of grains leads to a much reduced opacity that modifies the structure of the radiative zone. For short-period planets, however, the side facing the host star is heated to $\sim 1500$ K, which is near or above the sublimation temperature of most grain species. In addition, a large-scale circulation flow may also modify the composition and the heat transport process near the surface. It remains to be determined whether grain sedimentation occurs (Burkert et al. 2003). In order to take into account this possibility, we consider some models under the assumption of extreme grain depletion, i.e., without any grain opacity, so that the effective opacity $\kappa_{\text{ad}}$ is reduced from the normal opacity $\kappa_0$ (see Table 1).

At the surface of the planet, we include the irradiation from the star in the boundary condition for the temperature $T_{\text{surf}}$. We neglect the difference between the day and night sides of the planet. For most of the models, we set the irradiation temperature $T_0 = 1500$ K, which corresponds to a planet with a 3 day period around a solar-type star. For two models, we also considered somewhat lower temperatures at the photosphere, the normal boundary conditions for pressure $P = 2G M_p / 3 r^2$ and for the intrinsic luminosity $L = 4\pi r^2 (T_{\text{surf}} - T_0^4)$ are adopted.

### 3. CONSTANT INTERNAL HEATING PER UNIT MASS

In this section we show the numerical results for an internal heating rate that is constant in time and uniform in mass; that is, $\epsilon$ in equation (5) takes a constant value. The calculations are carried out for three different masses: $0.63 M_J$, $1 M_J$, and $3 M_J$.

The time span for each calculation covers a few times $10^6$ yr. At the end of each computation, the planets have attained a thermal equilibrium. In this state, the integrated energy term $\epsilon$ in equation (5) balances the intrinsic radiated luminosity. Normally when a planet contracts and cools, the
time-dependent terms in equation (5) dominate, and a thermal balance is never reached. The same is true even if the planet is strongly irradiated by the star. However, the present results show that an internal tidal energy dissipation rate on the order of $10^{27}-10^{30}$ ergs s$^{-1}$, depending on the mass, is sufficient to stop the contraction and cooling at some radius in the range of a few $R_J$.

In Figure 1 we show that there exists a radius-luminosity relation for a Jovian planet with a given mass $M_p$. The planet’s intrinsic luminosity (i.e., excluding the luminosity due to stellar irradiation) emitted from the photosphere is denoted by $L$. Discrete points shown in the figure are our computational results, which are fitted by three quadratic curves associated with three different masses. For a 0.63 $M_J$ coreless planet, the solid curve in the left panel can be approximated by

$$\log \left( \frac{L}{10^{27} \text{ ergs s}^{-1}} \right) = -11.7039 \left[ \log \left( \frac{R_p}{R_J} \right) \right]^2 + 12.8994 \log \left( \frac{R_p}{R_J} \right) - 2.12471. \quad (6)$$

For the same mass planet with a core (solid curve, right panel), we can approximate

$$\log \left( \frac{L}{10^{27} \text{ ergs s}^{-1}} \right) = -8.78501 \left[ \log \left( \frac{R_p}{R_J} \right) \right]^2 + 10.004 \log \left( \frac{R_p}{R_J} \right) - 1.05358. \quad (7)$$

**TABLE 1**

| Model | $\epsilon$ | $\epsilon_0/\epsilon$ | $\beta$ | $m_0/M_p$ | $\Delta m/M_p$ | $\nu_0/\nu_0$ | $T_e/T_0$ | Figure |
|-------|------------|---------------------|---------|------------|----------------|--------------|-----------|--------|
| 1     | $\epsilon_1$ | 1          | 0       | ...        | ...            | 1            | 1         | 5, 8, 9 |
| 2     | $\epsilon_2$ | 1          | 0       | 0.9999     | 0.00015        | 1            | 1         | 1       |
| 3     | $\epsilon_3$ | 1          | 0.95    | 0.05       | 0.05           | 1            | 1         | 5       |
| 4     | $\epsilon_4$ | 1          | 0.90    | 0.05       | 0.05           | 1            | 1         | 8       |
| 5     | $\epsilon_5$ | 1          | 0.70    | 0.05       | 0.05           | 1            | 1         | 5       |
| 6     | $\epsilon_6$ | 1          | 0.05    | 0.05       | 0.05           | 1            | 1         | 5       |
| 7     | $\epsilon_7$ | 1          | 2       | ...        | ...            | 1            | ...       | ...     |
| 8     | $\epsilon_8$ | 1          | 2       | ...        | ...            | 1            | ...       | ...     |
| 9     | $\epsilon_9$ | 1          | 0       | ...        | ...            | ...          | ...       | ...     |
| 10    | $\epsilon_{10}$ | 1          | 0       | 0.9999     | 0.05           | 1            | 1         | 6       |
| 11    | $\epsilon_{11}$ | 1          | 0       | 0.90       | 0.05           | 1            | 1         | 6       |
| 12    | $\epsilon_{12}$ | 1          | 0       | 0.70       | 0.05           | 1            | 1         | 6       |
| 13    | $\epsilon_{13}$ | 1          | 0       | 0.90       | 0.05           | 1            | 1         | 8       |
| 14    | $\epsilon_{14}$ | 1          | 0       | 0.05       | 0.05           | 1            | 1         | 8       |
| 15    | $\epsilon_{15}$ | 1          | 0       | 0.90       | 0.05           | 1            | 1         | 6       |
| 16    | $\epsilon_{16}$ | 1          | 0       | 0.70       | 0.05           | 1            | 1         | 9       |
| 17    | $\epsilon_{17}$ | 0          | 0       | ...        | ...            | ...          | ...       | ...     |
| 18    | $\epsilon_{18}$ | $\propto R_p^{9/2}$ | 0     | ...        | ...            | ...          | ...       | ...     |
| 19    | $\epsilon_{19}$ | $\propto R_p^{9/2}$ | 0     | ...        | ...            | ...          | ...       | ...     |
| 20    | $\epsilon_{20}$ | $\propto R_p^{9/2}$ | 0     | ...        | ...            | ...          | ...       | ...     |

**Notes.**—Here $\epsilon_0 = 5 \times 10^{-5}$, $L_\infty = 1 M_\odot$, and $T_0 = 1500$ K. The last column shows the figures, in terms of their labels, in which the model is plotted. For instance, model 1 appears in Figs. 5, 8, and 9.
Fig. 2.—Temperature profiles against the pressure inside a planet of $1M_J$ with (right) and without (left) a core. Different curves are plotted for different $R_p$: the interior structure evolves from the lowest curve to the uppermost one as the planet expands.

Similarly, the model without a core (dashed curve, left panel) and with a core (dashed curve, right panel) for a $1M_J$ planet can be approximated by

$$\log \left( \frac{L}{10^{27} \text{ ergs s}^{-1}} \right) = -7.7909 \left[ \log \left( \frac{R_p}{R_f} \right) \right]^2 + 10.3338 \log \left( \frac{R_p}{R_f} \right) - 1.20809 \quad (8)$$

and

$$\log \left( \frac{L}{10^{27} \text{ ergs s}^{-1}} \right) = -6.74289 \left[ \log \left( \frac{R_p}{R_f} \right) \right]^2 + 8.92021 \log \left( \frac{R_p}{R_f} \right) - 0.573055, \quad (9)$$

respectively. Finally, $3M_J$ models without a core (dotted curve, left panel) and with a core (dotted curve, right panel) can be approximated by

$$\log \left( \frac{L}{10^{27} \text{ ergs s}^{-1}} \right) = -8.10772 \left[ \log \left( \frac{R_p}{R_f} \right) \right]^2 + 10.6893 \log \left( \frac{R_p}{R_f} \right) - 0.29812, \quad (10)$$

and

$$\log \left( \frac{L}{10^{27} \text{ ergs s}^{-1}} \right) = -10.525 \left[ \log \left( \frac{R_p}{R_f} \right) \right]^2 + 12.3851 \log \left( \frac{R_p}{R_f} \right) - 0.508571, \quad (11)$$

respectively.

For large values of $R_p$, these three curves all become flattened as the planet is inflated; i.e., as the planet expands, the slope ($\gamma \equiv d \log L / d \log R_p$) decreases. In the coreless cases, $\gamma = 5$ at $R_p = 2.175R_f$ for $M_p = 0.63M_J$, $R_p = 2.199R_f$ for $M_p = 1M_J$, and $R_p = 2.243R_f$ for $M_p = 3M_J$. In the core cases, $\gamma = 5$ at $R_p = 1.927R_f$ for $M_p = 0.63M_J$, $R_p = 1.953R_f$ for $M_p = 1M_J$, and $R_p = 2.243R_f$ for $M_p = 3M_J$. In addition, the luminosity for a given $R_p$ is an increasing function of $M_p$.

The results in Figure 1 do not correspond to the exact thermal equilibrium solutions. Although the planets quickly establish a hydrostatic equilibrium, $L$ does not necessarily equal the internal heating rate. In these calculations, $L$ is the instantaneous intrinsic luminosity associated with the corresponding $R_p$. We use various initial conditions for $R_p$ and $\epsilon$ for our computation, but the $L$-versus-$R_p$ curves remain nearly identical to those shown in Figure 1. This invariance is due to the adiabatic structure as we have stated in the previous section. The number marked on each data point is the degree of electron degeneracy at the center of the planet in the coreless cases or at the surface of the core in the core cases. The degree of electron degeneracy is expressed by the ratio of the Fermi energy to the thermal energy

$$D \equiv \frac{E_{\text{Fermi}}}{kT} \approx \frac{\rho}{6 \times 10^{-9} T^{3/2}}. \quad (12)$$

As shown in the figure, the degeneracy is lifted as the planet expands.

In Figure 2 we plot the temperature against the pressure inside a planet with $M_p = 1M_J$. Each curve corresponds to a different value of $R_p$. While the upper right parts of the curves, where $T$ and $P$ are higher, represent the conditions for planet interiors, the lower left portions of the curves show the $T$ and $P$ distribution near the planet’s surface. The thick line across the upper parts of the curves marks the “plasma phase transition (PPT)” line that segregates hydrogen molecules and metallic hydrogen atoms. This line is approximately drawn based on an extrapolation of a hydrogen phase diagram (Saumon et al. 1995). As indicated by the slope of the curves, $\nabla_{\text{ad}}$ has a value $\approx \frac{1}{2}$ in the interiors where hydrogen atoms are in metallic form and then drops down to around 0.2 as we
follow the curves to the outer region of the planet where the pressure decreases below 10 bars. This transition occurs near the base of the radiative envelope. The flattening of $\nabla$ in the radiative envelope results from the external heating due to the stellar irradiation. The overall behavior of the curves for core and coreless cases is very similar. In the coreless case, a decrease in the central pressure $P_c$ with the planet’s size can be fitted with the following power law:

$$P_c \propto R_p^{-2.24}. \quad (13)$$

The above relation is less steep than what is expected from a purely polytropic structure in which $P_c \propto R_p^{-1}$. In addition, the central temperature $T_c$ increases with $R_p$ when $R_p \lesssim 2.3R_J$ but decreases with $R_p$ when $R_p \gtrsim 2.3R_J$. This variation of temperature is nearly independent of the strength of internal heating rates; that is, the variation of temperature profiles is almost the same for a given planet’s size no matter whether the planet is expanding, contracting, or even in a thermal equilibrium. This tendency arises because, as stated in the previous section, adiabatic profiles are imposed in the convection zone, which amounts to 95% of the total radius. The change of internal temperature in this manner is not driven by a thermal imbalance, but it is caused by an intrinsic change in the physical properties of the fluid in the convection zone as a result of the reduction of electron degeneracy (see § A2).

Figure 3 displays the mass density profiles as a function of the temperature for various planet sizes. The theoretical PPT line is also marked in the plot. Resembling the flattened $P-T$ curves in Figure 2, the steepening of the $T$-$\rho$ curves for $\rho$ less than about $10^{-4}$ g cm$^{-3}$, which roughly corresponds to the region of the radiative envelope, is caused by the stellar irradiation. We estimate the degree of degeneracy at the center of the planet with no core and at the surface of the core for the planet with a core. As the planet with no core (with a core) expands from 1.777$R_J$ to 2.826$R_J$ (from 1.6$R_J$ to 3.9$R_J$), the value $D$ declines from 21 to 9 (from 21 to 5.3) primarily as a result of the central density decrease. We fit the relation between $\rho_c$ and $R_p$ for the planet with no core by using the following power law:

$$\rho_c \propto R_p^{-1.6}. \quad (14)$$

As is the case for the $R_p$-$P_n$ relation, which is described by equation (13), the power 1.6 in equation (14) is smaller than what is expected from a purely polytropic structure, in which the power is 3.

The interior model of Jupiter is usually approximated by an $n = 1$ polytrope, which results from the combination of Coulomb effects and electron degeneracy in the metallic hydrogen plasma (Hubbard 1984; Burrows et al. 2001). This simplification is confirmed by the simulation as shown in Figure 4, which plots $P/\rho^2$ (in cgs units) as a function of the comoving mass coordinate $m$ for various planet sizes. The hypothetical plasma phase transition line is denoted by PPT and is marked by a thick line. The almost constant value of $P/\rho^2$ throughout the region of metallic hydrogen indicates an $n \approx 1$ polytropic structure. As we follow these curves to the outer regions of the planet where the molecular hydrogen dominates, the slopes of these curves increase from zero, meaning that the polytropic index is increased. The fractional region of the $n = 1$ polytropic structure shrinks as the planet’s size becomes larger. This tendency suggests that the interior structure at various evolutionary stages of an inflated planet does not proceed in a self-similar manner but behaves differentially as a consequence of the evolution of the equation of state. This motivates us to employ the simple polytrope approach to sketch the physical properties of an inflated planet associated with different values of the polytropic index $n$. By fitting the curves in the region of convection zones in Figure 4 for the coreless cases with a polytropic index, we show in Table 2 that the polytropic index increases with $R_p$. As demonstrated by the polytropic approach compared to the simulations in the Appendix, the increase in $n$, and therefore the decrease in the adiabatic exponent $\Gamma$ (see eq. [A19] and the right panel of Fig. 10 in the Appendix), with $R_p$ for the interior structure arises from a migration of an equation of state from a more degenerate and nonideal phase to a state with less degenerate and more ideal properties. Consequently, for a given planet mass, the rate of $R_p$ change in response to the change of total entropy becomes more sensitive as $n$ increases (see eq. [A21] for the planet compressibility $C_M$ and the description in the Appendix); namely, a larger planet is more elastic with respect to the change of entropy and pressure. This explains the simulation results that the increase in the intrinsic luminosity.
is reduced as the planet expands as shown in Figure 1 and equation (A17), making a larger planet vulnerable to the tidal inflation instability if the dimensionless parameter for the tidal dissipation $Q$ remains unchanged during the inflation.

Although the above analysis is based on the simulation for uniform heating rate per unit mass, the concept of planet elasticity depending on the planet’s size can be extended to the off-center heating cases. This is discussed in the next section.

4. DEPENDENCE ON THE LOCATION OF THE TIDAL DISSIPATION

4.1. Prescription for Tidal Dissipation Rate

In general, the distribution of tidal energy dissipation is a function of the radius, i.e., $\epsilon = \epsilon(r)$. As we indicated above, the actual functional form $\epsilon(r)$ depends not only on the location of wave excitation and propagation but also on the nature of dissipation. In stars and gaseous planets, the tidal perturbation of a companion not only distorts their equilibrium shape but also excites gravity and inertial waves (Zahn 1977, 1989; Ioannou & Lindzen 1993a, 1993b, 1994; Ogilvie & Lin 2004). The dissipation of the waves, which occurs throughout the planets’ convective envelope, also leads to the deposition of angular momentum, which in general leads to differential rotation (Goldreich & Nicholson 1977, 1989). It is not clear how the interior of the gaseous planets and stars may readjust after the angular momentum deposition may have introduced a differential rotation in their interior and how the energy stored in the shear may be dissipated (Korycansky et al. 1991).

In this paper we are focused on the response of the planets’ envelope to the release of energy associated with the tidal dissipation. To determine the actual dissipation distribution is beyond the scope of the present work. In light of these uncertainties, we adopt three ad hoc prescriptions for the positional dependence in $\epsilon(r)$ such that

$$
c_1(m) = \frac{(1 + \beta) \epsilon_0}{M_p} \left(\frac{m}{M_p}\right)^{\beta}, \tag{15}
$$

$$
c_2(m) = \frac{(1 + \beta) \epsilon_0}{M_p} \left(\frac{M_p - m}{M_p}\right)^{\beta}, \tag{16}
$$

$$
c_3(m) = \frac{2 \epsilon_0}{\sqrt{\pi} M_p} \exp \left[-\frac{(m - m_0)^2}{\Delta m^2}\right], \tag{17}
$$

### Table 2

| $R_p/R_J$ | $n$ | $\left(k_0\right)^n$ |
|----------|-----|----------------------|
| 1.777    | 2.058 | 0.098 |
| 1.973    | 2.141 | 0.102 |
| 2.314    | 2.222 | 0.107 |
| 2.711    | 2.358 | 0.118 |
| 2.826    | 2.392 | 0.122 |

Fig. 4. $P/r^2$ (in cgs units) as a function of the comoving radial coordinate $m/M_p$ for various planets’ sizes in the cases of a planet with a core (right) and without a core (left). Boundaries $m/M_p = 0$ and 1 represent the location of the planet’s center and the bottom of its photosphere, respectively. The interior structure evolves from the lowest curve to the uppermost curve as the planet expands to different sizes indicated by $R_p/R_J$ shown at the upper left corner of each panel.
where $M_p$ is the planets’ total mass, $\beta$ is a power index, and $m_0$ and $\Delta m$ determine the Gaussian profile. For $\beta > 0$, $\epsilon_1$ corresponds to a concentrated dissipation near the planet’s surface such as that due to the atmospheric nonlinear dissipation or radiative damping. $\epsilon_2$ corresponds to intense dissipation within the planets’ envelope such as that expected from turbulent dissipation and nonlinear shock of resonant inertial waves, and $\epsilon_3$ is designed to highlight the dissipation near some transition zone, such as the convective radiative transition front, or near the corotation radius. In the notion of equilibrium tides with constant lag angle, the total tidal energy dissipation rate within the planet in its rest frame (Eggleton et al. 1998; Mardling & Lin 2002; Papers I and II) is

$$
\epsilon_0 = \dot{E}_\text{tide} = - \left( \frac{9\mu a^2}{2Q_p} \right) \left( \frac{M_p}{M_p^*} \right) \left( \frac{R_p}{a} \right)^5 \times \left[ \Omega_p^2 h_3(e) - 2n\Omega_p h_4(e) + n^2 h_5(e) \right], \quad (18)
$$

where $M_p$ and $M_p^*$ are the mass of the stars and their planets, $\mu = M_\ast M_p/(M_\ast + M_p)$ is the reduced mass, $\Omega_p$ and $R_p$ are the planets’ spin frequency and size, and $n$, $a$, and $e$ are the mean motion, semimajor axis, and eccentricity of the planets’ orbit, $h_3(e) = (1 + 3e^2 + 3e^4/8)(1 - e^2)^{-9/2}$, $h_4(e) = (1 + 15e^2/2 + 45e^4/8 + 5e^6/16)(1 - e^2)^{-6}$, and $h_5(e) = (1 + 31e^2/2 + 255e^4/8 + 185e^6/16 + 25e^8/64)(1 - e^2)^{-15/2}$. In the limit of low eccentricities, equation (18) can be approximated as follows:

$$
\epsilon_0 \approx \frac{e^2 GM_p M_p^*}{a \tau_d} \approx 6.1 \times 10^{29} e^2 \left( \frac{M_p}{M_\odot} \right) \left( \frac{M_p^*}{M_\odot} \right) \left( \frac{R_p}{a} \right) \left( \frac{20 \text{ Myr}}{\tau_d} \right) \text{ergs s}^{-1}, \quad (19)
$$

where

$$
\tau_d \approx 20 \left( \frac{Q_p}{10^9} \right) \left( \frac{M_p}{M_\odot} \right) \left( \frac{M_p^*}{M_\odot} \right)^{3/2} \left( \frac{a}{10 R_\odot} \right)^{13/2} \left( \frac{2R_p}{R_p} \right)^5 \text{Myr}, \quad (20)
$$

and $Q_p$ is the dimensionless parameter that quantifies tidal deformation and dissipation of a planet (Paper II).

An equilibrium tidal model with a constant lag angle for all components of the tide may not necessarily be the most reliable model. All the uncertainties associated with the physical processes are contained in the $Q_p$ values, and we parameterize our results in terms of it. For a fiducial value, we note that the $Q_p$ value inferred for Jupiter from Io’s orbital evolution is $5 \times 10^4 < Q_p < 2 \times 10^6$ (Yoder & Peale 1981). With this $Q_p$ value, orbits of planets with $M_p$ and $R_p$ comparable to those of Jupiter and with a period less than a week are circularized within the main-sequence life span of solar-type stars as observed.

For identical forcing to spin frequency ratios $f_{\text{tide}}$, the magnitude of $\epsilon_0$ for dynamical tides has the same power-law dependence on $R_p$ as that for equilibrium tide in equation (18). However, the dissipation rate and the $Q$-value vary sensitively with $f_{\text{tide}}$, which is modulated by the changes in $R_p$ (Ogilvie & Lin 2004). Numerical calculation and analytic approximation show that the relevant frequency-averaged $Q$-value is comparable to that inferred for Jupiter and may be asymptotically independent of the viscosity in the limit of small viscosity or equivalently Ekman number.

In the present context we assume that $\epsilon_0$ is constant in time in most of our models. However, for models 18–20, we consider the possibility that the damping timescale for the eccentricity ($\tau_e$) is longer than the thermal expansion timescale ($\tau_R$) for the planets to inflate. The expression for $\tau_R$ is given by the equation (Paper II)

$$
\tau_R = \frac{\epsilon_0}{e} \tau_d, \quad e_R = \left( \frac{q_p M_p a}{\beta M_p R_p} \right)^{1/2} \approx 0.18 \left( \frac{q_p}{0.75 \beta} \right) \left( \frac{M_p}{M_\odot} \right) \left( \frac{R_p}{R_\odot} \right) \left( \frac{a}{0.04 \text{ AU}} \right)^{1/2}. \quad (21)
$$

Hence, in these models the heating rate $\epsilon_0$ increases as $R_p^5$ according to equation (18) with a constant eccentricity.

### 4.2. Numerical Models

In order to explore the dominant effect of tidal inflation, we consider several models (see Table 1) for a $1M_\odot$ planet being inflated from the same initial size of $1.9R_\odot$. The parameters for these models are chosen to represent a wide range of possibilities.

In the first and second series of models, mild (strong) heating is deposited in different locations in models 1–8 (9–12). With models 13–15 in the third series, we consider the possibility of smaller opacities, due to grains’ sedimentation, in the radiative envelope. With a lower $T_e$, we consider the possibility of less intense stellar irradiation in models 16 and 17 in the fourth series. Simple forms of time-varying tidal dissipation are also considered in the fifth series of models 18–20. We consider that the damping timescale of eccentricity $\tau_e$ is longer than the expansion timescale, so the heating rate in models 18–20 takes the tidal form $\epsilon \propto R_p^5$. We only deal with the coreless cases for all of these models shown in Table 1 except for model 19, in which the simulations for both core and coreless models are carried out.

#### 4.2.1. Dependencies on the Heating Location

In this series of models, we consider the effect of nonuniform heating on the planets’ radius-luminosity relation. The relation that stays approximately invariant for different locations of heat deposit is the radius-adiabat relation (Stahler 1988). In Figure 5 we plot the density profiles of a $1M_\odot$ inflated giant planet in a thermal equilibrium. Models 1 and 6 are represented by the dotted curve because their interior structure is almost the same for these two models as the heating is concentrated in the inner regions of the planet. Models 2 and 5 are represented by solid and dashed curves, respectively.

These models indicate that as the location of maximal heating moves from the planet’s center toward the photosphere ($m_0/M_\odot = 0.05 \rightarrow 0.7 \rightarrow 0.9999$), the final equilibrium size of the planet decreases. This tendency indicates that less $PdV$ work on expansion is achieved when the heating shell is closer to the photosphere because the heating is also more efficiently lost near the planet’s surface. Equivalently, less specific entropy content is retained by the convective region of the planet. Given a mass, a size, and the same heating rate, a planet with the heating shell closer to its photosphere has larger intrinsic luminosity $L$ as a result of the difficulty in transporting entropy.
into the planet’s interior via both radiative and convective transport. Our numerical results indicate that the region beneath the main heating zone continues to adjust slowly without gaining any entropy. Even without a substantial increase in entropy, we show in the next paragraph for models 11 and 12 that the inner convective region can swell in order to adjust to a new hydrostatic equilibrium. This adjustment is due to a drop in the boundary pressure in the limit that the spatial extent of the heating shell spreads significantly. Although this result seems to be in qualitative agreement with the radius-adiabat relation, it is still unclear that the relation should precisely hold for a giant planet heated in different locations because different equations of state and therefore different $PdV$ work are involved for a given entropy input to different locations.

Figure 6 displays the evolution of expansion for model 11 (top panels) and model 12 (bottom panels). The three solid curves from right to left in each of the temperature and density diagrams are the profiles corresponding to the phases marked by the data points 1, 6, 11 in the $R_p$-versus-$t$ diagram for model 11 and the phases labeled by the data points 1, 4, 10 in the $R_p$-versus-$t$ diagram for model 12. The vertical dashed lines in the temperature and density diagrams denote the location of the center of the Gaussian heating zones: $m_0/M_p = 0.9$ for model 11 and $m_0/M_p = 0.7$ for model 12. In model 11, a heating front is generated at $m_0/M_p = 0.9$, as is illustrated in the temperature diagram (top middle panel), and then propagates inward. The front corresponds to the region with the temperature inversion. It also acts as a rarefaction structure, as
is shown in the density diagram. For comparison, we also draw the temperature and density profiles for model 9 (dot-dashed curves) and for model 10 (dotted curves). In model 12, a heating front is generated at \( m/M_p = 0.7 \). As it travels inward (bottom middle panel), the front decreases the density analogous to a rarefaction wave (bottom right panel).

It is interesting to point out that except for the data points from 1 to 4 in model 12, all of the other data points in the \( R_p \)-versus-\( t \) diagrams correspond to a state of quasi-thermal equilibrium; i.e., the intrinsic luminosity \( L \) is equal to the dissipation rate. However, the planet is not in a global thermal equilibrium because of the presence of a temperature inversion in the region just below the shell. This heat front very gradually propagates inward and is associated with a gradual overall expansion. After quasi-thermal equilibrium is reached, the rate of expansion is very slow compared to its rate before this time (compare the data before and after point 4 in the bottom left panel of Fig. 6). Clearly, an analogous expansion phase does not occur in the uniformly heated case (model 9).

The underlying physical cause for expansion of \( R_p \) without any gain of internal entropy (or very little gain of entropy as a result of the tail of a Gaussian profile) can be described as follows. Prior to reaching a quasi-thermal equilibrium, a giant planet experiences a fast inflation due to a high dissipation rate focused in a spherical envelope (such as the stage before the data point 4 for model 12 in Fig. 6). In models 11 and 12, about 10%-30% of planet gas by mass lies above the heating zone. This gas is inflated significantly as a result of a strong underlying shell-heating source as prescribed by the model. The huge inflation on the top of the heating shell reduces the pressure and therefore causes the whole region beneath the heating zone to expand and to adjust to a new hydrostatic equilibrium. The expansion induced by a shell-heating source is less efficient than that induced by a uniform heating throughout the envelope since more energy is radiated away for a given \( R_p \), as we have already stated in the previous paragraph. The surface radiation becomes more intensified as the heating zone approaches to the photosphere.

After the entire shell-heated planet has reached its quasi-thermal equilibrium, the region around the heating front is not yet in the thermal equilibrium locally. This arises because the region just below the heating front (i.e., at the bottom of the temperature inversion region) gains entropy via convection from the bottom and through radiation from its top. However, the region just above the heating front (i.e., at the top of the temperature inversion region) loses entropy as a result of radiation from its bottom and convection from its top. On the average, the net increase in entropy vanishes. Thus, the rarefaction heating wave is driven by a local entropy transport that averages to a zero net flux. Since the heating front is a result of the temperature inversion that transports entropy inward by radiative diffusion and causes the entropy disturbance, the timescale for the heating front to cross the planet’s interior is roughly equal to

\[
\frac{R_p}{\nu_{\text{raref}}/\Delta r} \approx 40 \left( \frac{R_p}{3R_J} \right) \left( \frac{7.5 \times 10^4 \text{ K}}{T} \right)^3 \times \left( \frac{\kappa}{10^3 \text{ cm}^2 \text{ g}^{-1}} \right) \left( \frac{\rho}{0.3 \text{ g cm}^{-3}} \right)^2 \left( \frac{c_p}{K} \right) \text{ Gyr},
\]

where the radiative diffusivity \( \nu_{\text{raref}} = 4\alpha c T^3/3\kappa\rho c_p \) and \( \Delta r \) is the width of the heating front. The reference values for \( \Delta r, T, \kappa, \) and \( \rho \) shown in the above estimate are taken from the simulation for model 12. The large value of \( \nu_{\text{raref}} \) is consistent with our numerical results that the shell-heating planet expands very slowly after it has reached the quasi-thermal equilibrium (see the top left panel and the stage after the data point 4 in the bottom left panel in Fig. 6). In reality the slow expansion of the planet due to the inward propagation of the rarefaction wave is unimportant because the damping time for tidal dissipation is much shorter than \( \nu_{\text{raref}} \), and also because \( \nu_{\text{raref}} \) is much longer than the lifetime of the star.

The \( L-R_p \) relation for model 12 before the planet reaches a quasi-thermal equilibrium can be fitted with a quadratic approximation:

\[
\log \left( \frac{L}{10^{27} \text{ ergs s}^{-1}} \right) = -9.36888 \left[ \log \left( \frac{R_p}{R_J} \right) \right]^2 + 11.9403 \log \left( \frac{R_p}{R_J} \right) - 1.33544
\]

which indicates that \( \gamma \) decreases with \( R_p \) and \( \gamma = 5 \) when \( R_p = 2.346R_J \). In contrast to the uniform heating cases, in which the planet’s interior gains entropy and expands, the degree of degeneracy at the planet’s center, in model 12, remains essentially constant \( (D \approx 17) \) during the evolution (see Fig. 6). This tendency arises largely because the planet interior expands without gaining much entropy. This result implies that the flattening of \( L \) with \( R_p \) in the shell-heating cases is not due to the reduction of electron degeneracy of the metallic interior but is primarily caused by the gradual phase transformation from nonideal properties to the ideal state within and above the shell-heating zone. This interpretation is well illustrated in Figure 7 in terms of \( J\phi (=1/\Gamma) \); see eq. [A19] and the description in the Appendix for data point 1 (solid curve) and data point 4 (dotted curve): while in the interior \( J\phi \) hardly changes with \( R_p \), \( J\phi \) indeed increases with \( R_p \) around and above the shell-heating zone.
4.2.2. Variations in Other Model Parameters

In addition to the cases in which most of the dissipation is concentrated in a narrow spherical shell, we also considered the cases with relatively flat heating profiles for models 7 and 8, we choose the power index to be $\beta = 2$. We found that the interior structure is almost the same for models 7, 8, and 1. This similarity arises because the heating per unit volume is concentrated in the high-density inner region of the planet in all these models, including model 7 with $\beta = 2$. For much larger values of $\beta$, the internal structure of models 7 and 8 should resemble that of models 2 and 6, respectively.

Figure 8 depicts the density profiles of the planet in a thermal equilibrium with the regular and the reduced opacities for grains. The solid and dashed curves represent models 1 and 4, respectively. In both models, $\kappa_d = \kappa_0$. The corresponding models 13 and 14 for $\kappa_d = 10^{-3} \kappa_0$ are marked by solid and dashed curves, respectively. This reduction of opacity by a factor of $10^3$ is implemented in the grain region where $T < 2100$ K. This region roughly corresponds to the radiative envelope of the planet. Since the radiative envelope is a bottleneck for the outward energy transport, the reduced opacity in the radiative envelope allows photons to escape from the planet much more easily. Consequently, the planet attains a much smaller size when a thermal equilibrium is established. The dashed curves have slightly smaller $R_p$ than the solid curves as a result of more radiative loss for the shell-heating cases as reasoned in the previous paragraph. Since the location of the heating shell is close to the planet’s center, the simulation for model 15 is almost the same as that for model 13.

We also computed the thermal equilibrium solution for models with different irradiation temperatures $T_0$. In Figure 9 we plot the temperature and opacity distributions for models 1, 16, and 17 by solid, dashed, and dotted curves, respectively. Since the interior structures for these three cases are almost the same, we only show the temperature and opacity profiles in the radiative envelopes, which are affected by the stellar irradiation. The planets with less irradiation (models 16 and 17) have slightly larger equilibrium sizes than the one with more irradiation (model 1). A direct comparison between models 16 and 17.
and 17 indicates that higher surface temperatures result in not only a slight increase in the density scale height of the planet’s atmosphere but also an enhanced opacity. Both effects cause model 16 to attain a larger $R_p$ than model 17.

However, $R_p$ for model 1 is smaller than that for model 16 even though the photospheric temperature for the former is higher than that of the latter. This difference arises because the temperature of the envelope in model 1 is sufficiently high for most of the silicate grains to sublime. Consequently, opacity in the radiative envelope in model 1 is well below that in model 16. The correlation between $R_p$ and $\kappa$ is already established in the previous discussion on models 13–15. However, the difference in the size between models 1 and 16 is only about 1%, in accord with the usual notion that the size of an optically thick object should not be strongly altered by external irradiation.

4.2.3. Self-Consistent Tidal Heating Models and Inflation Instability

We also consider, in model 18, a self-consistent calculation in which the energy dissipation is proportional to $R_p^5$ in accordance with equation (18). In this model we assume that the energy dissipation is uniformly distributed in mass. The dissipation is normalized to $\epsilon_0 = 5 \times 10^{-6} L_\odot$ at $2R_p$ (this rate corresponds to $a = 10 R_\odot$, and $\epsilon \approx 0.18$ in eq. [19]; note that $\tau_R \sim \tau_d$ in this case according to eq. [21]). The final equilibrium size $R_p = 1.857R_\odot$ is slightly smaller than that ($R_p = 1.977R_\odot$) for model 1. The planet would be slightly bigger ($R_p = 1.983R_\odot$) than that for model 1 if the normalization constant were increased by 5%. Since $\gamma < 5$ as $R_p \gtrsim 2R_\odot$, the planet is expected to be vulnerable to the tidal inflation instability once its size expands beyond $2R_p$. Our numerical results show that a further increase in the normalization constant by another 5% (therefore $e \sim 0.2$ and $\tau_R \sim 0.9\tau_d$ when $R_p = 2R_\odot$ and $a = 10 R_\odot$) by eq. [21]) causes $R_p$ to increase to at least $4R_\odot$.

In comparison to the uniform heating in model 18, we consider in models 19 and 20 the possibility of nonuniform heating with $m = 0.9$ and 0.9999, respectively. Similar to model 18, the total heating rate is proportional to $R_p^5$. In model 19 with the heating rate normalized to 2.1 times larger than $5 \times 10^{-6} L_\odot$ at $R_p = 2R_\odot$ (this corresponds to $e \approx 0.26$ at $a = 10 R_\odot$ according to eq. [19]), the planet is thermally unstable and swells from $2R_p$ to $3R_p$ in about 2 Myr (shorter than the eccentricity damping timescale $\tau_d \approx 2.7$ Myr according to eq. [20]) evaluated at $R_p = 3R_\odot$). When the heating rate is normalized to 4 times larger than $5 \times 10^{-6} L_\odot$ at $R_p = 2R_\odot$ (this corresponds to $e \approx 0.07$ at $a = 0.03$ AU according to eq. [19]) in model 19, the planet is thermally unstable and expands from $2R_p$ to $3R_p$ in about 0.27 Myr (the eccentricity damping timescale $\tau_d \approx 0.15$ and 1.17 Myr evaluated at $R_p = 3R_\odot$ and $R_p = 2R_\odot$, respectively).

When the normalized heating is set to $10^{-3} L_\odot$ for $a = 0.03$ AU and $R_p = 2R_\odot$, equation (19) is no longer a fair approximation and equation (27) in Paper II gives $e \approx 0.294$ for this heating rate. In this case the planet with or without a core expands from $2R_p$ to $3.6R_p$ in less than 40,000 yr, which is shorter than the eccentricity damping time $\tau_d \approx 47,800$ yr for $R_p = 3.6R_\odot$ according to equation (12) in Paper II. Since the planet’s size $3.6R_\odot$ is actually beyond the Roche radius $R_p \approx 3R_\odot$ for $a = 0.03$ AU and $e = 0.294$, the inflated planet would overflow the inner Lagrangian point in this case.

The expansion rate is drastically reduced for model 20 in which the dissipation is largely deposited at $m = 0.9999$, roughly the location of the radiation-convection boundary. A $1M_J$ planet with the normalized heating $5 \times 10^{-6} L_\odot$ at $R_p = 2R_\odot$ can only reach the final size of $\approx 1R_\odot$. A normalized heating rate that is twice as big as $5 \times 10^{-6} L_\odot$ results in a final size $\approx 1.9R_\odot$. The planet with the normalized heating 4 times as large as $5 \times 10^{-6} L_\odot$ in model 20 expands very slowly to $2R_\odot$ from $1.9R_\odot$ over several tens of million years, which is comparable with the eccentricity damping timescale $20$ Myr at $a = 10 R_\odot$ and is much longer than the eccentricity damping timescale ($\approx 1.17$ Myr) at $a = 0.03$ AU. The planet in model 20 obviously requires a larger normalized heating rate than model 19 to reach the critical size beyond which the planet is thermally unstable in response to the $R_p^5$ heating rate.

5. SUMMARY AND DISCUSSION

In this paper we continue our investigation on the adjustment of a planetary interior as a consequence of intense tidal heating. As a giant planet’s interior is heated and inflated, we showed in Paper II that its interior remains mostly convective. Efficient energy transport leads to an adiabatic stratification. With a constant heating rate per unit mass, we deduced a unique luminosity-radius relation regardless of how intense the heating rate is. The planet’s luminosity increases with its radius. However, the growth rate of $L$ is a decreasing function of $R_p$. At the same time, the tidal dissipation heats the interior of the planet at a rate that increases rapidly with $R_p$. At around $2R_p$, $L$ can no longer sustain sufficient growth to maintain a thermal equilibrium with the tidal dissipation rate. Thereafter, the planet’s inflation becomes unstable and it overflows its Roche radius and becomes tidally disrupted.

Here we show that the change of luminosity during the planet’s expansion is directly linked to the evolution of its interior, in particular, the equation of state. We employ the polytrope approach to investigate the interior structure of an inflated giant planet. According to the simulation, interior profiles deviate away from $P \propto \rho^n$ as the planet expands. The central temperature $T_c$ increases and then decreases with the size of the planet. Furthermore, $P_c$ and $\rho_c$ are less steep functions of $R_p$ than the polytrope theory with a constant polytropic index $n$ indicates. All of these effects suggest that the planet interior does not evolve in a self-similar manner, but $n$ gets larger as $R_p$ increases. In conjunction with the numerical results that the degeneracy $D$ decreases and that $T_c$ rises and then drops during the course of inflation, the result of a positive value of $dn/dR_p$ can be interpreted as a manifestation of a reduction in degeneracy during the expansion.

We reason that the coefficient of thermal expansion $\delta$ increases in response to a decrease in degeneracy and nonideal effects, leading to an increase in $n$ through equation (A19). Consequently, the planet compressibility at constant mass $C_M$ increases with $R_p$ (see eq. [A21]). This pattern can be translated into the phenomenon of a decrease in luminosity growth during the inflation, as a consequence of a one-to-one relation between $K$ and $L$ in the case of uniform heating in mass under the conditions of the polytropic interior and hydrostatic equilibrium. We also compare the results between a planet with a core and without a core. To be inflated to the same size, a planet with a core, therefore possessing a larger gravitational binding energy, needs a larger intrinsic luminosity $L$ than a planet with no core and the same mass. We also show that the opacity in the radiative envelope has a drastic effect on the final equilibrium size of an inflated planet: the size would be much smaller if grains are depleted in the radiative envelope.
We also consider the possibility of localized tidal dissipation. Such a process may occur in differentially rotating planets or near the interface between the convective and radiative zones where the wavelength associated with dynamical tidal response is comparable to the density scale height. Localized dissipation may also occur through the dissipation of resonant inertial waves or radiative damping in the atmosphere. In the strong shell-heating models, the one-to-one relation between $K$ and $\dot{\mathcal{L}}$ disappears because of the existence of a radiative region caused by temperature inversion beneath the shell-heating zone. The unheated planet’s interior in such cases might still be inflated as a result of a significant expansion of the gas above the heating zone, although the overall expansion rate is less efficient than that in the uniform heating cases as a result of a greater amount of radiative loss from the planet’s photosphere. Without gaining entropy, the expanding interior cannot lift its degeneracy and therefore cannot increase its elasticity. However, the gas above the shell-heating zone can lift its nonideal properties and hence enhance its elasticity, leading to a decrease in $\Gamma$ in that region and thereby diminishing the growth rate of $\mathcal{L}$ as the planet expands.

Finally, we consider the self-consistent response, taking into account the modification of heating rate due to a planet’s expansion. In this paper we adopt a constant-$Q$ prescription for equilibrium tides in which the tidal dissipation rate is assumed to be proportional to $R_p^3$. The results for the uniform heating model suggest that a young gaseous planet of $1M_J$ without a solid core can be thermally unstable and inflated from $2R_J$ to a size beyond $4R_J$ if $e \sim 0.2$ at $a = 10 \, R_J$. If the dissipation rate is proportional to $R_p^3$ and if most of the tidal perturbation is deposited at $m/M_p = 0.9$, a coreless young planet of $1M_J$ would be thermally unstable and inflated from $2R_J$ to a size beyond $3R_J$ for $e > 0.07$ at $a = 0.03 \, AU$ or $e > 0.26$ at $a = 10 \, R_J$.

With the same heating concentration $m/M_p = 0.9$ and $R_p$ dependence in the dissipation rate, a young planet with a core at $a = 0.03 \, AU$ with an initial eccentricity $e > 0.294$ can be inflated from $2R_J$ to a size beyond its Roche radius. We have assumed that the convective flow still behaves adiabatically even though the heating shell causes a narrow radiative zone. However, the condition away from adiabaticity implies that the internal heat is not transported away as efficiently as in the case of adiabatic convection, leading to a more severe reduction in nonideal properties of the gas and therefore an even faster decrease of $\gamma$ as the planet expands.

Note that the Eddington approximation for the surface boundary condition is used in these models rather than more detailed frequency-dependent model atmospheres. This approximation is not necessarily valid for the strongly irradiated atmospheres studied here (Guillot & Showman 2002). However, it is unlikely to make much difference for the main results discussed here, namely, the behavior of the planet’s radius as a function of tidal dissipation energy. It could, however, lead to errors in other kinds of predictions, such as the radius as a function of opacity.

The equilibrium tidal dissipation formula is based on an ad hoc assumption of a constant lag angle. In reality, the dynamical tidal response of a planet through both gravity and inertial waves near the planet’s surface and convective envelope may be much more intense, especially through global normal modes. Their dissipation may provide the dominant angular momentum transfer mechanism for the orbital evolution and heating sources for the internal structure of close-in extrasolar planets. In the limit of small viscosity, the intensity of tidal dissipation is highly frequency dependent (Ogilvie & Lin 2004). When the forcing and response frequencies are in resonance, the energy dissipation rate is intense, whereas between the resonances it is negligible. As the planets undergo structure adjustments, their spin frequency, Brunt–Väisälä frequency distribution, adiabatic index, and equation of state also evolve. Since all of these physical effects contribute to the planets’ dynamical response to the tidal perturbation from their host stars, their response and resonant frequencies are continually modified. The results in this paper indicate that the structure of the planet adjusts on a radiation transfer timescale that generally differs from the timescale for a planet to evolve through the nonresonant region. In addition, the tidal forcing frequency also changes as the planets evolve toward a state of synchronous spins and circular orbits. Therefore, it is more appropriate to consider a frequency-averaged tidal dissipation rate. In the limit of small viscosity, the frequency-averaged dissipation rate converges (Ogilvie & Lin 2004) such that the equilibrium tidal dissipation formula may be a reasonable approximation. Nevertheless, we cannot yet rule out the possibility that some close-in planets may attain some nonresonant configuration and stall their orbital evolution. Therefore, accurate measurement of the sizes of close-in young Jupiters via planet transit surveys can be used to constrain the theories of tidal dissipation and hence internal structure for these objects.

We wish to thank G. Laughlin, N. Murray, G. Ogilvie, and E. Vishniac for useful conversation. We also thank the referee T. Guillot for invaluable comments. Part of this work was completed when one of us (P. G.) was a visitor at the UCO/Lick Observatory, and he is grateful to K.-Y. Lo for the support of this project. This work is supported by NSF and NASA through grants AST 99-87417 and NCC2-5418.

**APPENDIX**

### A POLYTROPE MODEL

In order to identify the dominant physical effects that determine the internal structure of planets, we consider an inflated giant planet that consists of a polytropic interior and a thin envelope. With this model we can construct analytic solutions. For this analysis, we assume that the polytropic interior comprises almost all of the mass and radius and that the thin envelope overlying the interior is radiative and is composed of a nondegenerate ideal gas. Therefore, the polytropic equation $P = K \rho^{1+1/n}$, together with the condition for hydrostatic equilibrium, specifies the value of $K$ for a given planet’s mass $M_p$ and a given planet’s size $R_p$ (Cox & Giuli 1968):

$$K = k_b M_p^{1-1/n} R_p^{-1+3/n},$$

(A1)
where \( k_n \) is a function of \( n \). All of the thermodynamic quantities must be continuous across the boundary between the convective interior and the radiative envelope, such that

\[
P_b = \frac{\mathcal{R} \rho_b T_b}{\mu}, \quad \text{(A2)}
\]

\[
P_b = K \rho_b^{1+1/n}, \quad \text{(A3)}
\]

\[
\nabla_{\text{polytrope}} = \nabla_{\text{rad}} \approx \frac{3}{16\pi c G} \frac{P_b \kappa_b \mathcal{L}}{T_b^4 M_p}, \quad \text{(A4)}
\]

where the subscript \( b \) denotes the values evaluated at the boundary. The magnitude of \( \nabla_{\text{polytrope}} \) is determined by the equation of state of the gas in the interior such that it is a function of \( n \) rather than directly dependent on the magnitude of \( R_p \) and \( M_p \). Note that in equation (A4) we have assumed that \( \mathcal{L} \) does not vary greatly across the radiative envelope after it emerges from the polytropic interior. This assumption should be a reasonable approximation for the thin radiative envelope so long as there is no localized source of intense heating there. The above four equations give rise to the relation

\[
\mathcal{L} \propto M_p T_b^3 \propto \frac{k_n^2 M_p^n (T_b R_p)^{3-n}}{\rho_b \kappa_b} \left( \frac{\mu}{\mathcal{R}} \right)^n, \quad \text{(A5)}
\]

where \( \mathcal{R} \) is the gas constant and \( \mu \) is the mean molecular weight. From the opacity table, we find that the opacity at the boundary may be roughly approximated by a power law

\[
\kappa_b \propto T_b^a, \quad \text{(A6)}
\]

with \( a = 4.08 \).

### A1. Completely Degenerate Interior

The equation of state for a completely degenerate gas in the nonrelativistic regime is given by \( P \propto \rho^{5/3} \), which corresponds to the \( n = 3/2 \) polytrope with a constant \( K \). Therefore, equation (A1) leads to the well-known mass-radius relation for low-mass white dwarfs:

\[
R_p \propto M_p^{-1/3}, \quad \text{(A7)}
\]

which does not vary with \( \mathcal{L} \). This independence is equivalent to the expression

\[
\gamma = \frac{d \ln \mathcal{L}}{d \ln R_p} \rightarrow \infty. \quad \text{(A8)}
\]

### A2. Partially Degenerate Interior with \( n = 1 \) Polytrope

We now consider a model in which the planet has a sufficiently large \( R_p \) such that its interior is partially degenerate with an \( n = 1 \) equation of state (\( P \propto \rho^2 \)). Partial degeneracy occurs when \( D \approx 1 \). Therefore, by setting \( D = 1 \) in equation (12), the partially degenerate interior can be described by

\[
\rho = c T^{3/2}, \quad \text{(A9)}
\]

where \( c \) is a constant. Under these conditions, \( \nabla_{\text{ad}} = \frac{1}{b} \), which is consistent with the numerical solution for the region dominated by the pressure-ionized hydrogen atoms as shown in Figure 2.

At the boundary, equations (A1), (A2), (A3), and (A9) uniquely determine the value of \( T_b \) for a given planet’s size \( R_p \) without having to consider the photosphere:

\[
T_b = \left( \frac{\mathcal{R}}{\mu} \right)^2 \frac{1}{K^2 c^2} = \left( \frac{\mathcal{R}}{\mu \kappa_1} \right)^2 \frac{1}{R_p^2 c^2} \approx 2300 \text{ K} \left( \frac{2 R_j}{R_p} \right)^4 \left( \frac{2 \times 10^{-7} \text{ g cm}^{-3} \text{ K}^{-3/2}}{c} \right)^2 \left( \frac{1}{\mu} \right)^2. \quad \text{(A10)}
\]

With equations (A5) and (A6), it follows that

\[
\mathcal{L} \propto M_p R_p^{10.32} c^{4.16}. \quad \text{(A11)}
\]

Figure 3 shows that the degree of degeneracy at the center of the planet decreases from 21 to 9 as its radius expands from 1.777\( R_j \) to 2.826\( R_j \). The value \( c \) decreases as the degree of partial degeneracy is reduced (\( c \approx 6 \times 10^{-9} \text{ g cm}^{-3} \text{ K}^{-3/2} \) when Fermi energy equals thermal energy). Therefore, in accordance with equation (A9), the temperature of the planet can increase with, even though \( \rho \) decreases with, \( R_p \).\(^1\) Temperature indeed increases until \( R_p \) is increased up to \( \approx 2.3 R_p \). The temperature then decreases with \( R_p \) after that, as seen in our numerical results.

\(^1\) Extrapolating from eq. (A9) to the case of high degree of partial degeneracy, we find that, as energy is added to the system, \( \rho \) in eq. (A9) remains essentially unchanged while \( T \) increases as \( c \) decreases. The input energy is mostly converted to an increase in temperature instead of doing the \( P dV \) work in a degenerate state.
Figure 10 depicts the coefficient of thermal expansion at constant pressure \(C_{14}\) \(\frac{\partial \ln P}{\partial \ln T}\), \(\frac{\partial \ln P}{\partial \ln T}\), and \(J\) related to the adiabatic index; see eq. (A18) for a planet of 0.63\(M_J\) as a function of the comoving mass coordinate \(m/M_p\). The boundaries \(m/M_p = 0\) and 1 represent the location of the planet’s center and its photosphere, respectively. The curves in Figure 10 are not smooth because the calculation involves several numerical derivatives. The magnitude of these thermal expansion coefficients decreases as the gas changes its phase from the ideal to the nonideal regime and approaches zero as a fluid increases its degeneracy, meaning that \(PdV\) work is of less importance in a more degenerate gas as indicated in equation (5). Figure 10 shows that \(\delta\) and \(\chi_T\) monotonically increase with \(m/M_p\), resulting from a transition from partial degeneracy in the inner region featuring the pressure-ionized hydrogen gas, to the nonideal phase of dense molecular hydrogen in the middle range of \(m/M_p\), and then to the ideal gas regime in the very outer region (not able to be shown because of the small scale) where \(\delta = \chi_T = 1\). The magnitudes of \(\delta\) and \(\chi_T\) are larger in the case of a relatively large size planet (with \(R_p = 3.22R_J\)) because the equation of state for its interior is less degenerate and more ideal than that for a smaller planet (with \(R_p = 1.66R_J\)). This correlation is consistent with the consequence that the temperature variation results from the shift of degeneracy. Since \(c\) drops as \(R_p\) increases, the term \(c^{4.16}\) in equation (A11) indicates that the exponent of \(R_p\) should become less than 10.32 as the degree of partial degeneracy is reduced.

A3. ADIABATIC INTERIOR COMPOSED OF AN IDEAL GAS

For planets with extremely large \(R_p (\gg 2R_J)\), the density near the center of the planet becomes sufficiently low that degeneracy is lifted and the equation of state is better approximated by that of an ideal gas for an \(n = 1.5\) polytrope. In this limit, the radiative envelope is also relatively extensive. Integrating the radiative diffusion equation over the radiative envelope, we find the ratio of the temperature at the boundary \(T_b\) to the temperature at the photosphere \(T_{ph}\) (Cox & Giuli 1968) to be

\[
\frac{T_b}{T_{ph}} = \left[\frac{(1 + n_{eff})\nabla_{ph} - 1}{(1 + n_{eff})\nabla_{ad} - 1}\right]^{\frac{1}{(m + s + 2)}},
\]

where \(\kappa \propto \rho^{m+1}\), \(n_{eff} = (s + 3)/(m + 1)\), and the value of \(\nabla\) at the photosphere is

\[
\nabla_{ph} = \frac{3}{16\pi acG} \frac{P_{ph} c_{ph} \mathcal{L}}{T_{ph}^4 M_p} = \frac{1}{8} \frac{\mathcal{L}}{4\pi R_p^2 \sigma T_{ph}^4},
\]

where \(\mathcal{L}\) is the luminosity of the planet.
In deriving the above equation, we have used \( P_{\text{ph}} = \frac{2GM_p}{3R_p^2} \). Since \( \mathcal{L} \) is usually much smaller than the stellar irradiation \( 4\pi R_p^2 \sigma T_{\text{ph}}^4 \) and therefore \( \nabla_{\text{ph}} \ll 1 \), the ratio \( T_b/T_{\text{ph}} \) is not a sensitive function of \( \mathcal{L} \) and \( M_p \). Since \( T_{\text{ph}} \) is determined by the stellar irradiation, neither it nor \( T_b \) varies significantly with \( \mathcal{L} \) and \( M_p \).

There are some uncertainties for the values of \( m \) and \( s \). If we evaluate \( \kappa \) for the radiative/convective interface, we find from equation (A6) that \( m = 0 \) and \( s = -4.08 \). The temperature near the surface layer is 2000–3000 K such that the diatomic molecular hydrogen attains \( n = 5/2 \) and \( \nabla_{\text{ad}} = \frac{2}{n} \). With these parameters, we find from equation (A12) \( T_b \approx 1.326T_{\text{ph}} \approx 2150 \) K when \( T_{\text{ph}} = 1620 \) K. Near the photosphere, the grains may also provide the dominant opacity source, in which case \( s = -1 \) but other parameters have the same values. In this case, we find from equation (A12) that \( T_b \approx 1.917T_{\text{ph}} \approx 3090 \) K, which is too high for grain opacity to be relevant. The actual values of \( s \) and \( T_b \) are probably between these extreme cases. Finally, with equations (A5) and (A6) we find that for \( n = 5/2 \) an ideal gas equation of state leads to

\[
\mathcal{L} \propto M_p^{5/2} R_p^{3/2}.
\]

(A14)

A4. POLYTROPIC EVOLUTION AND PLANET COMPRESSIBILITY

As we can see clearly from equations (A8), (A11), and (A14), the values of \( \gamma \) decrease from \( \infty \) to 10.32 to 0.5 as the degeneracy is lifted for increasing values of \( R_p \). The physical properties responsible for this change can be attributed to the “compressibility” of the planet. It is well known that the compressibility of a gas is linked to the effective exponent \( \Gamma_{\text{eff}} \equiv d \ln P / d \ln \rho \), which varies with different thermal conditions in a situation of thermal equilibrium. However, in the case of inflated giant planets, \( \Gamma_{\text{eff}} \) is always equal to the adiabatic value \( \Gamma_{\text{ad}} = (d \ln P / d \ln \rho)_{\text{ad}} \) as a result of an efficient energy transport by convection. Consequently, \( \Gamma_{\text{eff}} \) varies with different equations of state that are characterized by the index of polytrope \( n \) in our case. We elucidate this point in this section.

In general, equations (A5) and (A6) can be approximated with a simple relation

\[
\mathcal{L} \propto \frac{M_p}{\rho_b T_b}.
\]

(A15)

The factor \( M_p \) comes into the above expression because the larger \( M_p \) is, the larger gravity is in the thin radiative envelope, and therefore the stronger \( \mathcal{L} \) is as a result of the steeper temperature gradient in the radiative envelope. Unlike the relation with \( M_p \), \( \mathcal{L} \) decreases as \( \rho_b \) increases simply because the radiative energy flux decreases as the density and therefore the optical depth increase.

The numerical solutions for the constant \( m \) models show that \( T_b \) increases with \( R_p \) when the degree of partial degeneracy is relatively high. However, \( T_b \) decreases with \( R_p \) when the degree of partial degeneracy is modest \( (R_p \) is larger than \( 2.3R_J) \). This tendency suggests that \( T_b \), a thermal quantity introduced by the equation of state in the framework of a polytropic analysis, primarily varies with \( c \) as shown in equation (A9).

Figure 11 illustrates the mass density at the interface between the radiative envelope and the convective interior, \( \log \rho_b \) as a function of the planet size \( \log R_p \). As the planet expands, \( \log \rho_b \) decreases but its slope gets flattened. The similarity between Figures 1 and 11 suggests that the flattening of \( \log \rho_b \) is the major cause of the flattening of \( \log \mathcal{L} \) as the planet is inflated. Equation (A5) implies that

\[
\rho_b \propto T_b^m M_p^{1-n} \left( \frac{R_p^{n-3}}{R_J^n} \right) \left( \frac{R_p}{R_J} \right)^n.
\]

(A16)

As demonstrated in Figure 4, the simulation shows that the fraction of the volume of the planet that deviates from the \( n = 1 \) polytrope increases with \( R_p \). This dependence is roughly equivalent to the polytropic index being an increasing function of \( R_p \). We

![Fig. 11.—Density at the interface between the radiative envelope and the convective interior \( \rho_b \) as a function of the planet’s radius \( R_p \). While discrete points are the simulation data, the fitting curves are illustrated by a solid curve (0.63 \( M_J \)), a dashed curve (1 \( M_J \)), and a dotted curve (3 \( M_J \)). [See the electronic edition of the Journal for a color version of this figure.]


fit the curves in the region of convection zones in Figure 4 for the coreless cases with a polytropic index and show the results in Table 2. The terms \( (R_p/\mu)^2 \) and \( k_{2D}^0 \) in the above equation increase with \( n \), but their influence is much weaker than the terms \( R_p^{-3} \) and \( M_p^{-1} \) because \( R_p \) and \( M_p \) are much larger. As a result, the simple polytropic approach roughly suggests that \( L \propto R_p^{3-n}M_p^n \), and hence the value of \( \gamma \) (as defined by eq. [A8]) decreases with \( R_p \) as \( 3-n \) decreases with \( R_p \). Equation (A5) gives a more precise representation of \( \gamma \) as a function of \( R_p \):

\[
\frac{d\gamma}{d\log R_p} = 2 \frac{dn}{d\log R_p} \left( \frac{d\log k_n}{d\log R_p} - \frac{d\log T_b}{d\log R_p} - 1 \right) + \frac{n d^2\log k_n}{d(\log R_p)^2} - \frac{(a-3+n) d^2\log T_b}{d(\log R_p)^2} + \frac{d^2n}{d(\log R_p)^2} \log \left( \frac{M_p k_n \mu}{R_p R_T b} \right)
\]

(A17)

The approximations made for the last expression in the above derivation are that we ignore the terms associated with the second derivative with respect to \( \log R_p \) and that the terms \( d \log k_n / d \log R_p \) and \( d \log T_b / d \log R_p \) are less than unity.

The magnitude of \( L \) increases with \( M_p \) for a given \( R_p \) as suggested by the factor \( M_p^n \), which is consistent with the results shown in Figure 1. An increase of \( n \) with \( R_p \) in the polytropic analysis can be confirmed by the numerical results, which show that the central density \( \rho_c \) and the central pressure \( P_c \) do not scale as fast as \( R_p^{-3} \) and \( R_p^{-4} \), respectively. However, \( \rho_c \propto R_p^{-1.6} \) and \( P_c \propto R_p^{-2.24} \), i.e., the ratio of the central density to the average density \( \rho_c / (\rho) \) increases with \( R_p \), meaning that \( n \) also increases with \( R_p \). Hence, the fact that an inflating planet loses its partial degeneracy can be interpreted as an increase of \( n \) from 1 in the polytropic analysis. The overall effect on the intrinsic luminosity \( L \) for its dependence on \( R_p \) is that \( \gamma \) (\( \equiv d \ln L / d \ln R_p \)) decreases with \( R_p \) as a result of the flattening of the value of \( \rho_c R_T b \) as the planet inflates and loses its degeneracy.

Another piece of information that suggests a correlation between electron degeneracy and \( \gamma \) is the comparison of the evolution of \( \gamma \) between the core and coreless cases. As shown in \( \S \) 3, \( \gamma \) passes 5 at a smaller \( R_p \) for a less massive planet \( (M_p = 0.63, 1M_f) \) with a core than for that without a core. In the case of a less massive planet, a planet with a core requires more internal heating (see Fig. 1) than that with no core to be inflated to the same size, resulting in a lower degeneracy of the planet interior and therefore a faster decrease in \( \gamma \) even though the core does not expand at all and the planet with a core is more gravitationally bound. On the other hand, \( \gamma \) decreases below 5 when \( R_p \) expands beyond 2.243\( R_f \) for a massive planet \( (M_p = 3M_f) \) in both core and coreless cases. The reason for this independence of the core structure is because a 3\( M_f \) planet without a core has a central density comparable to the 5 g cm\(^{-3} \) that we impose for the density of the core in our numerical prescription; i.e., the interior structure of a massive coreless planet is comparable to that of the planet with a core, resulting in the similar \( L \) and \( D \) required for the core and coreless cases to be inflated to the same size (see Fig. 1). In summary, given a size of an inflated less massive giant planet and assuming that the core is not heated and does not radiate, the planet with a core is less degenerate than the one without a core, and hence the planet with a core has a smaller \( \gamma \) and expands more for a given amount of entropy input (see eq. [A21] and detailed explanations below).

If we trace back through the above derivation, we note that the relation \( L \propto R_p^{3-n}M_p^n \) originates from equation (A1). The parameter \( K \) might be related to the total entropy content of a polytropic object. The energy equation can be expressed as follows (Kippenhahn & Weigert 1990):

\[
ds = c_p J \left( d \ln P - \frac{1}{T_b} d \ln \rho \right)
\]

(A18)

where \( J \equiv -\nabla_{ad} + 1/\chi_T \) and \( \chi_T \equiv (\partial \ln P / \partial \ln T)_\rho \). Therefore, the adiabatic exponent \( \Gamma \) and the polytropic index might be written in terms of \( \delta \) and \( \chi_T \):

\[
1 + \frac{1}{n} = \Gamma = \frac{1}{T_b}
\]

(A19)

For a completely degenerate gas, \( \chi_T \rightarrow 0 \), \( \delta \rightarrow 0 \), and \( 1/\delta \) can be written as the expression \( d \ln P / d \ln \rho \), which equals 5/3 in the nonrelativistic case. As the degree of degeneracy is reduced, both \( \delta \) and \( \chi_T \) increase away from zero, and \( J \), whose evolution is dominated by \( 1/\chi_T \), decreases from infinity. In the region of the middle and larger values of \( m/M_p \) where molecular hydrogen is abundant, both \( \delta \) and \( \chi_T \) increase with \( R_p \) as the nonideal effect is lifted (Saumon et al. 1995). This qualitative evolution of \( \delta \) and \( \chi_T \) is in agreement with the plots obtained for a simulation with an inflated planet of 0.63\( M_f \), (Fig. 10, left and middle panels). The right panel of Figure 10 shows that the evolution of the product \( Jb \) is dominated by \( \delta \) and therefore increases as the planet expands, leading to an increase in the polytropic index \( n \).

Motivated by the entropy equation (A18), which indicates that the specific entropy \( s \propto \ln (P/\rho^3) = \ln K \), we associate the issue of entropy to the polytropic approach with a constant \( n \) by rewriting equation (A1) as

\[
d \ln R_p = C_M d \ln K + C_K d \ln M_p,
\]

(A20)

\(^2\) The effect that \( \delta \) increases as the molecular gas reduces its nonideal properties can be seen from Fig. 8 in Saumon et al. (1995) by noting that \( \delta = \chi_T/\chi_s \), where \( \chi_s \equiv (\partial \ln P/\partial \ln \rho)_T \). The rise of \( \chi_T \) above 1 during the inflation of a young giant planet is caused by the effects of molecular dissociation (see Fig. 7 in Saumon et al. 1995 for more details).
where we have defined the compressibilities $C_M$ at constant mass and $C_K$ at constant entropy as follows:

\[ C_M \equiv \left( \frac{d \ln R_p}{d \ln K} \right)_{M_p} = \frac{n}{3-n}, \tag{A21} \]

\[ C_K \equiv \left( \frac{d \ln R_p}{d \ln M_p} \right)_K = \frac{1-n}{3-n}. \tag{A22} \]

For a completely degenerate gas, $K = \text{const}$ [hence the result $n/(3-n)$ shown in eq. (A21) breaks down], $C_K = -\frac{1}{4}$ (since $n = 1.5$), and the term associated with $C_M$ vanishes in the nonrelativistic case. In the cases of terrestrial planets and asteroids where the atomic/molecular interaction dominates and therefore the density is on the similar order regardless of mass, volume, and entropy, the interior structure can be roughly described by the $n = 0$ polytrope. When $n = 0$, $C_K = \frac{1}{2}$, $C_M = 0$, and $\Gamma = \infty$. These relations reflect a nearly constant density ($C_K = \frac{1}{2}$) for the planetary interior, as well as the difficulty of producing any nonnegligible density gradient with an entropy ($C_M = 0$) and gravity ($\Gamma = \infty$) distribution. In the special case of partial degeneracy, $C_K \approx 0$ and $C_M$ increases as $n$ rises from 1.\(^3\) The zero value of $C_K$ indicates a maximum size of the planet at $n = 1$ solely in response to the mass change (Hubbard 1984). Phenomenologically this result arises from a transition from degeneracy ($M_p \propto R^{-3}$) to the nonideal equation of state ($M_p \propto R^3$) due to atom-atom/molecule-molecule repulsion (Burrows et al. 2001; Shu 1992). In terms of the polytropic index, the state of partial degeneracy with $n \approx 1$ is just a transition from an $n = 3/2$ degenerate state to an $n = 0$ constant density state. The rise of $C_M$ with $n$ means that, for a given mass, the rate of $R_p$ change in response to the change of total entropy gets more sensitive as $n$ increases, leading to the radius-adiabat relation and therefore the radius-luminosity relation for constant heating per unit mass. It is physically straightforward to see why $C_M$ increases with $n$ (and therefore decreases with $\Gamma$). It is because the adiabatic exponent $\Gamma = \Gamma_{\text{ad}}$ is just the bulk modulus for adiabatic expansion/compression. Equation (A21) can be rearranged to have the following form:

\[ \frac{1}{C_M} = 3(\Gamma - \Gamma_{M_p}), \tag{A23} \]

where we have defined the bulk modulus at constant planet’s mass $\Gamma_{M_p} \equiv (\partial \ln P_c/\partial \ln \rho_c)_{M_p}$, which equals the constant 4/3. The adiabatic bulk modulus $\Gamma_{\text{ad}}$ should decrease as the planet’s elasticity $C_M$ increases as a result of the reduction of electron degeneracy and nonideal effects. Figure 12 illustrates the isothermal curves for $\Gamma_{\text{ad}}$ as a function of $\log \rho$ in the case of hydrogen (left panel) and helium (right panel). The number marked on each curve denotes the logarithmic value of temperature $\log T(K)$. The plots are drawn based on the tabulated equations of state (Saumon et al. 1995). The peaks of $\Gamma_{\text{ad}}$ ($\approx 2$) around $\rho \simeq 1$ g cm\(^{-3}\) at the temperature $T \sim 10^4 – 10^5$ K roughly correspond to the pressure-ionized regime of a Jupiter interior. This effect may be compared to the sudden rise of $\Gamma_{\text{ad}}$ for $\log T(K) = 3.22$ at high densities due to nonideal effects in the dense molecular (for hydrogen) or atomic (for helium) regime (see also Figs. 8 and 15 in Saumon et al. 1995). The variations of $\chi_p \equiv \partial \log P/\partial \log \rho_c$ and of $\Gamma_{\text{ad}}$ are quite similar; the rise of $\Gamma_{\text{ad}}$ in the pressure-ionized phase in the case of higher temperatures should also result from the nonideal effects due to the interactions between densely packed hydrogen and helium atoms, increasing the rigidity of the fluid. After all, equation (A20) describes the simple fact that the size of a planet is in general determined by its gravity ($M_p$), its entropy content ($K$), and the elastic properties of the planet responding to gravity (described by $C_K$) and to the entropy content (quantified by $C_M$).

\(^3\) Eq. (A21) gives the relation $dC_M/d \log R_p = C_M^2(3-n^2) d \log R_p/d \log R_p > 0$. A more precise expression derived from eq. (A1) is given by the equation $dC_M/d \log R_p = C_M^2(6/n^2) d \log R_p/d \log R_p - (2/n^2)(d \log R_p/d \log R_p) \Gamma_3 \log (M_p/R_p^3) + (1/n^2)(d \log R_p/d \log R_p) \Gamma_3 \log (M_p/R_p^3) + d^2 \log k_e/d \log (R_p)^2$, which approximately equals $C_M^2(6/n^2) d \log R_p$ when the other terms associated with $d^2/d \log (R_p)^2$ and $1/(\log R_p)^2$ are neglected.
REFERENCES

Alexander, D. R., & Ferguson, J. W. 1994, ApJ, 437, 879
Bodenheimer, P. H., Hubickyj, O., & Lissauer, J. J. 2000, Icarus, 143, 2
Bodenheimer, P. H., Laughlin, G., & Lin, D. N. C. 2003, ApJ, 592, 555
Bodenheimer, P. H., Lin, D. N. C., & Mardling, R. A. 2001, ApJ, 548, 466
(Paper I)
Brown, T. M., Charbonneau, D., Gilliland, R. L., Noyes, R. W., & Burrows, A. 2001, ApJ, 552, 699
Burkert, A., Lin, D. N. C., Bodenheimer, P., Jones, C. A., & Yorke, H. 2003, ApJ, submitted
Burrows, A., Guillot, T., Hubbard, W. B., Marley, M., Saumon, D., Lunine, J. I., & Sudarsky, D. 2000, ApJ, 534, L97
Burrows, A., Hubbard, W. B., Lunine, J. I., & Liebert, J. 2001, Rev. Mod. Phys., 73, 719
Cox, J. P., & Giuli, R. T. 1968, Principles of Stellar Structure (New York: Gordon & Breach)
Eggleton, P. P., Kiseleva, L. G., & Hut, P. 1998, ApJ, 499, 853
Goldreich, P., & Nicholson, P. D. 1977, Icarus, 30, 301
———. 1989, ApJ, 342, 1079
Goldreich, P., & Soter, S. 1966, Icarus, 5, 375
Goodman, J., & Oh, S. P. 1997, ApJ, 486, 403
Gu, P.-G., Lin, D. N. C., & Bodenheimer, P. H. 2003, ApJ, 588, 509 (Paper II)
Guillot, T., Hubbard, W. B., Stevenson, D. J., & Saumon, D. 2003, in Jupiter: The Planet, Satellites, and Magnetosphere, ed. F. Bagenal, W. McKinnon, & T. Dowling (Tucson: Univ. Arizona Press), in press
Guillot, T., & Showman, A. P. 2002, A&A, 385, 156
Hubbard, W. B. 1984, Planetary Interiors (New York: van Nostrand Reinhold)
Ioannou, P. J., & Lindzen, R. S. 1994a, ApJ, 406, 252
———. 1993b, ApJ, 406, 266

Gu, Bodenheimer, & Lin