Contribution of evanescent waves to the effective medium of disordered waveguides

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Abstract — We consider a wave propagating through a thin disordered slab inside a wire or waveguide of finite width. In the dense weak-scattering limit, the statistics for the complex reflection and transmission coefficients (the coherent field) is found to depend dramatically on the contribution of evanescent modes or closed channels, leading to an effective refractive index whose real part is quite sensitive to the closed-channels inclusion. In contrast, evanescent modes play no role in the statistical average of transmittances and reflectances. The theoretical predictions, based on the perturbative Born series expansion, are in excellent agreement with numerical simulations in disordered wires.

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Introduction. — The coherent transport of electromagnetic, electronic and acoustic waves through random media has long been a central issue in physics [1]. In recent years, there has been a revival of interest in the impact of inhomogeneities and defects in the transmission properties in quasi-one-dimensional (Q1D) systems [2] and is the subject of intense research including the electronic conductance of nanowires, nanotubes and nanoribbons [3–5], the propagation of slow light through disordered photonic-crystal waveguides [6–9] or acoustic propagation [10].

A correct description of wave transport through inhomogeneous media involves not only “propagating” traveling modes (“open” channels) but it is also very sensitive to near-field “evanescent” modes (“closed” channels) [11,12]. The contribution of evanescent modes has been known for a long time in the analysis of inhomogeneous waveguides and periodic structures like multilayered frequency selective surfaces [13], being particularly important near the thresholds of new propagating modes where the coupling between open and closed channels leads to geometric resonance effects [14–16]. However, the statistical properties of transport through disordered Q1D systems, obtained from numerical calculations, do not depend on the evanescent modes [17,18] and they are in good agreement with theoretical scaling approaches [2,17,19–23], including the celebrated Dorokhov-Mello-Pereyra-Kumar (DMPK) [20] and nonlinear sigma-model [21] approaches, which do not take into account explicitly evanescent modes. Why are evanescent modes critical for a given configuration while it seems that they play no role after statistical averaging over different configurations? One of our main goals here is to provide an answer to this open question.

In this letter, we use numerical simulations and a perturbative analytical approach to analyze the influence of evanescent waves on the statistical properties of the transport coefficients of a thin disordered slab inside a wire or waveguide. The influence of the evanescent modes is analyzed in the asymptotic region for the far field and no near to the physical interface of the disordered region, where it is well known that the evanescent modes affect strongly the statistics of the near field [11]. The wave number \(k\) is considered halfway between the threshold of the last propagating mode and the first evanescent mode; this avoids the strong influence of the first evanescent mode on the statistical scattering properties when \(k\) is near to the threshold of a new propagating mode [14].

Assuming a weak-scattering, non-dissipative, random medium with white-noise statistics, we find that the
statistical averages of (power) transmittances and reflectances are solely determined by the mean free paths (MFPs) (in agreement with the scaling theory). As we will show, in the so-called dense weak-scattering limit (DWSL) [17,23], the MFPs themselves do not depend on the evanescent modes. In striking contrast, the statistical average of the scattered field (coherent field) present a completely different behavior (which was not captured by previous scaling approaches). The propagation of the coherent wave field [24] is characterized by an effective wave number, whose real part determines the speed of propagation, while its imaginary part represents the losses due to scattering (often known as waveguide extrinsic losses). As the wave propagates, the amplitude of the coherent part decays exponentially, whose rate decrease is the scattering MFP. We will see that the scattering MFP is insensitive to evanescent modes, while changes in the phase of the coherent field (i.e. in the real part of the effective wave number) are solely related to evanescent modes.

Model system and numerical results. – Consider a wave propagating along the x-direction in the two-dimensional (2D) waveguide sketched in fig. 1. In this work we restrict ourselves to the problem of scalar waves following the equation \( \nabla^2 \Psi + k^2 \Psi = U \Psi \). For electron transport, \( \Psi \) and \( U \) represent the wave function and the random potential (for electromagnetic waves, this would correspond to random inclusions with permittivity slightly larger and lower than the background permittivity).

specified by its potential

\[
U_r(\rho) = u_r(y) \Theta \left( \frac{\delta}{2} - |x - x_r| \right),
\]

where the integer number \( b \) labels the wave modes, also referred to as scattering channels. Modes with \( b \leq N (\leq kW/\pi) \), are propagating modes with longitudinal wave number \( k_b = \sqrt{k^2 - (b\pi/W)^2} \) real; \( N \) denotes the number of traveling modes supported by the waveguide. Those modes with \( b > N \) have an imaginary wave number and represent evanescent modes.

The disordered slab is constructed as a sequence of \( n \) statistically independent and identically distributed thin scattering units of thickness \( \delta \), with \( k_b \ll 1 \). The scattering units are separated one from the other by a fixed distance \( d \gg \delta \) in the wave propagation direction \( x \), with \( L = nd + \delta \), as is sketched in fig. 1. The \( r \)-th scattering unit \( (r = 0, 1, 2, \cdots, n) \), centered at \( x_r = rd \), is

![Fig. 1: (Color online) (a) Sketch of the disordered slab in a waveguide. The fluctuating potential is considered as a sequence of thin random slices (b) with zero mean scattering potential (for electromagnetic waves, this correspond to random inclusions with permittivity slightly larger and lower than the background permittivity).](image-url)
the numerical results). Independently of $N'$ the GSM approach guarantees flux conservation ($T_{a_0} + R_{a_0} = 1$) for each realization.

**Evanescent modes and statistical averages.** In order to illustrate the effect of the evanescent modes on the statistical averages of different quantities, we consider a waveguide that supports $N = 2$ propagating modes, with $kW/\pi = 2.5$ far away from a new propagating mode; in this case, the scattering matrix of the disordered system, $S = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix}$, is a $4 \times 4$ matrix, while its transmission $t$ and reflection $r$ blocks are $2 \times 2$ matrices themselves. For this system, we perform four numerical simulations, each one considering a different number of evanescent modes $N' = 0, 1, 2, 3$. The results for $a_0 = a = 2$ are summarized in fig. 2 (a complete summary for all the other coefficients can be found in the supplemental information given in [26]), where the statistical averages are plotted as a function of the slab thickness $L$ in units of the mean free path $\ell$. The mean free path $\ell$ is numerically obtained ($k\ell \approx 100$) from the slope of the conductance curve at $L = 0$ ($\langle g \rangle_L \approx N(1 - L/\ell + \cdots)$) [5,26]. All the expectation values involve an ensemble average of $10^6$ different realizations of the microscopic random potential with $k d = 0.1$, $k \delta = 0.001$ and $u_0^2 = k\delta/(4\sqrt{3})$. Our results show that the ensemble averages of both transmission, $\langle T_{a_0 a_0} \rangle$ (fig. 2(a)) and reflectance, $\langle R_{a_0 a_0} \rangle$ (fig. 2(b)) are not sensitive to the number of closed channels that are taken into account in the calculations, a remarkable unexplained result already pointed out in previous numerical simulations for those quantities [17,18]. In contrast (fig. 2(c)–(f)), the complex transmission and reflection coefficients are strongly dependent on the evanescent modes field but in a peculiar way: while the real (imaginary) part of the transmission (reflection) coefficients are rather insensitive to the number of evanescent modes (at least for small thicknesses), both $\text{Im} \langle t_{22} \rangle$ and $\text{Re} \langle t_{22} \rangle$ show a clear dependence with the number of evanescent modes. Before discussing the origin of these, apparently, puzzling results, it is interesting to notice that the intensity of the averaged transmitted and reflected fields (the so-called “coherent” intensity), defined as $|\langle t_{a_0 a_0} \rangle|^2$ and $|\langle r_{a_0 a_0} \rangle|^2$, is also independent of the evanescent modes as shown in fig. 2(a), (b) (the difference $\langle T_{a_0 a_0} \rangle - |\langle t_{a_0 a_0} \rangle|^2 = \langle |\Delta T_{a_0 a_0} |^2 \rangle$ corresponds to the diffuse field intensity). This suggests that evanescent modes only affect the phase of the averaged (coherent) fields [26].

In order to understand the numerical findings, let us first consider the transmission, $t_r$, and reflection, $r_r$, matrices for a single slice centered at $x = x_r$. Expanding the wave function in transverse eigenfunctions and assuming that the wave function inside the slice is constant along $x$, it is easy to obtain

$$t_r = I - \frac{i}{2} \varphi_+^r T_r \varphi_+^r, \quad r_r = -\frac{i}{2} \varphi_+^r T_r \varphi_+^r, \quad \text{(6)}$$

![Fig. 2: (Color online) Numerical results. (a), (b): transmission ($T_{22}$), reflectance ($R_{22}$) and their coherent intensities $\langle t_{22} \rangle^2$, $\langle r_{22} \rangle^2$. (c)–(f) Complex coefficients ($t_{22}$) and ($r_{22}$). Each simulation considers $N = 2$ propagating modes and different evanescent modes ($N' = 0, 1, 2, 3$).]
\( k_{\text{eff}} \) inside a thin slab by comparison between the average transmitted coherent field, \( \langle \Psi \rangle \), to the result of an homogeneous uniform media \([24,28]\),

\[
\langle t_{a \rightarrow a} \rangle \approx 1 - i \frac{1}{d} \left[ \frac{\langle (T)_{a \rightarrow a} \rangle}{2k_{ao}} \right] L = 1 + i k_{\text{eff}}^2 - k^2 \frac{2}{L} \tag{11}
\]

\[
k_{\text{eff}}^2 = k^2 - \frac{1}{d} \left[ \frac{\langle (T)_{a \rightarrow a} \rangle}{k_{ao}} \right] = k^2 n_{ao}^2 \tag{12}
\]

where \( n_{ao}^2 = 1 - \langle (T)_{a \rightarrow a} \rangle / k^2 d \), defines the —mode-dependent— refractive index of the traveling mode \( a_0 \).

For weak-scattering units, eq. (12) leads to an effective longitudinal wave number inside the thin slab given by \((k_{\text{eff}})_{ao} \approx k_{ao} + \Delta k_{ao} \), being

\[
\Delta k_{ao} = \frac{\langle (T)_{a \rightarrow a} \rangle}{2k_{ao} d} \approx \frac{1}{\ell_{ao}} + i \frac{1}{\ell_{ao}} \tag{13}
\]

In eq. (13), we have identified \( \ell_{ao}^{-1} = \text{Im}(k_{\text{eff}})_{ao} \) as the inverse of the scattering MFP for the incoming open channel \( a_0 \), \( \ell_{ao} \) can be related to the channel-channel mean free paths \( \ell_{aa} \), which are associated with the incoherent sum of channels from channel \( a_0 \) to channel \( a \),

\[
\frac{1}{\ell_{aa}} = \frac{1}{d} \left[ \langle (T)_{a \rightarrow a} \rangle \right]^2 = \frac{1}{d} \left[ \langle (T)_{a \rightarrow a} \rangle \right]^2. \tag{14}
\]

\[
\frac{1}{\ell_{ao}} = \sum_{a=1}^{N} \frac{1}{\ell_{aa}} = \frac{1}{d} \frac{\text{Im} \langle (T)_{a \rightarrow a} \rangle}{2k_{ao}}. \tag{15}
\]

The last identity, in agreement with eq. (13), is a direct consequence of the OT for a single slice, see eq. (10). Notice that our definition of the scattering MFP, \( \ell_{ao} \), commonly used in the scaling theory of transport \([2]\), differs by a factor of 2 from that of the MFP \( \ell_{ao}^* \) of kinetic theory, i.e. \( \ell_{ao} = 2\ell_{ao}^* \).

**Weak-scattering limit.** In the weak-scattering limit, \( T \rightarrow \delta u_{cb} - \delta u_{a\cdot G_{cb}} + \cdots \) (the two first terms corresponding to the second-order Born approximation \([16]\)), from eq. (13), we easily obtain

\[
\frac{1}{\ell_{aa}} \approx \frac{1}{d} \sum_{b=1}^{N} \frac{\langle (T)_{b \rightarrow a} \rangle_{ba}}{2k_{ao}} \overline{G}_{bb} \tag{16}
\]

\[
\frac{1}{\ell_{ao}} = \sum_{a=1}^{N} \frac{1}{\ell_{aa}} \approx \frac{1}{d} \sum_{a=1}^{N} \frac{\langle (T)_{a \rightarrow a} \rangle_{a}}{4k_{ao}} \tag{17}
\]

Near the onset of a new propagating channel, \( \overline{G}_{bb} \) diverges and the weak-scattering approximation is no longer valid. However, as we have mentioned before, the wave number has been considered halfway for the threshold from the last open channel and the first closed channel, i.e., \( kW/\pi = N+1/2 \). In this regime, while the scattering MFP \( \ell_{ao}^{-1} = \text{Im}(k_{\text{eff}})_{ao} \) is independent of the evanescent modes, changes in \( 1/\ell_{ao} \) come solely from the evanescent modes. This result, valid for arbitrary transversal correlations inside each slab, is a key important outcome of the present work. If we restrict ourselves to short-range correlations (see eqs. (3) and (4)), \( k_{\text{eff}} \) can be rewritten as

\[
k_{\text{eff}}^2 = k^2 + \int_0^W dy \chi^2_{ao}(y) \int G(\rho, \rho') \Gamma(\rho, \rho') d^2 \rho' \tag{18}
\]

with \( \Gamma(\rho, \rho') \equiv u^2 \Theta(\delta/2 - |x - x'|) \Theta(\delta/2 - |y - y'|) \), which, in the limit \( W \rightarrow \infty \), is equivalent to that obtained for a homogeneous infinite medium, with sub-wavelength short-range correlations \([29]\).

In order to obtain a physical description of the macroscopic statistics beyond the \( L \rightarrow 0 \) limit, we make use of the Born series in the DWSL. Since the Born approach generates a double series expansion in powers of \( L/\ell \) and in inverse powers of \( k\ell \) (here \( k \) denotes symbolically any possible wave number \( k_0 \), while \( \ell \) does represent any physical parameter \( \ell_{aa} \), \( \ell_{ao} \) or \( \ell_{ao}^* \)), we work in the short-wavelength approximation (SWLA) \([17]\), which fixes the relation between the characteristic lengths: \( k\delta \ll 1 \ll L \ll k\ell \). After a lengthy calculation, we find

\[
\langle T_{aa} \rangle = \left[ \frac{L}{\ell_{aa}} + \sum_{b=1}^{N} \frac{L^2}{\ell_{ab}/\ell_{ba}} \right] + \left( \frac{1}{\ell_{aa}} + \frac{1}{\ell_{ao}} \right) + \ldots
\]

\[
\langle R_{aa} \rangle = \left[ \frac{L}{\ell_{aa}} - \left( \frac{1}{\ell_{aa}} + \frac{1}{\ell_{ao}} + \frac{1}{\ell_{ao}^*} \right) L^2 \right] + \ldots
\]

which shows that the transmittances and reflectances only depend on the MFPs and, as a consequence, they are independent of the number of evanescent modes. As is shown in fig. 3(a), in the ballistic regime \( L < \ell \) (with \( 1/\ell = N^{-1} \sum_{a=1}^{N} 1/\ell_{ao} \), there is an excellent agreement between the theoretical predictions and the numerical
results of fig. 2(a), (b). The complex coefficients, given by
\[
\langle t_{aa} \rangle = \delta_{aa} \left[ 1 + i \Delta k_a L + (i \Delta k_a)^2 \frac{L^2}{2!} + \cdots \right] + O \left( \frac{1}{k L} \right),
\]
\[
\langle r_{aa} \rangle = \delta_{aa} \frac{\Delta k_a}{2k_a} \left( e^{2ik_a L} - 1 \right)
- \frac{1}{2}\Delta k_a + \frac{1}{2} \left[ \frac{1}{2i} L e^{2ik_a L} + \cdots \right] + O \left( \frac{1}{k L} \right)^2,
\]
are also in excellent agreement with the numerical results, as can be seen in fig. 3(b)–(d). The results for \( \langle t_{aa} \rangle \) suggest an exponential behavior \( \sim \delta_{aa} e^{i\Delta k_a L} \) which differs from the simple exponential decay predicted by the scaling theory given in ref. [23]. In contrast, \( \langle r_{aa} \rangle \) oscillates \( \propto e^{2ik_a L} \) around a constant background \( \sim -\delta_{aa} \Delta k_a/2k_a \) which explains the peculiar behavior of the reflection coefficients in fig. 2(e), (f) [26].

**Conclusions.** – The numerical and theoretical results of the present work demonstrate that even in the asymptotic region and far away from the threshold of the first evanescent mode, the evanescent modes could affect the statistical scattering properties of disordered waveguides. Equations (19), (20) together with (13), (16) and (17) summarize the main theoretical results of this letter. They show that in the dense weak-scattering limit, the mean free paths \( \ell_{aa} \) do not depend on the evanescent modes which provide a simple explanation of the numerical results and the success of the scaling approach to wave transport. In contrast, we have shown that the so-called coherent field and its effective wave number depend both on propagating and evanescent modes which could be specially relevant to understand the influence of disorder in the propagation of slow waves in a waveguide or when dealing with a device where interferences between different wires or waveguide arms are relevant. Our predictions could be tested, for example, in microwave waveguides where the analysis of transport coefficients and intensities is currently performed to obtain accurate values of the permittivity of complex solid and liquid dielectrics [30] or by direct measurement of both the real and imaginary parts of the electric fields [31]. Equivalent measurements in the visible or infrared would require waveguide heterodyne methods [32].

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