GLOBAL WELL-POSEDNESS OF A 3D STOKES-MAGNETO EQUATIONS WITH FRACTIONAL MAGNETIC DIFFUSION

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Abstract. This paper is devoted to the global well-posedness of a three-dimensional Stokes-Magneto equations with fractional magnetic diffusion. It is proved that the equations admit a unique global-in-time strong solution for arbitrary initial data when the fractional index \( \alpha \geq \frac{3}{2} \). This result might have a potential application in the theory of magnetic relaxation.

1. Introduction and the main result. In this paper, we consider the following equations

\[
-\nu \Delta u + \nabla p_* = b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b + \eta (-\Delta)^\alpha b = b \cdot \nabla u, \\
\nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\
b|_{t=0} = b_0,
\]

in the whole space \( \mathbb{R}^3 \). Here \( u \) is the velocity field, \( b \) is the magnetic field, \( p_* = p + \frac{1}{2} |b|^2 \) is the total pressure, \( p \) is the pressure, \( \nu > 0 \) is the viscosity coefficient and \( \eta > 0 \) is the magnetic resistivity coefficient. The fractional operator \( (-\Delta)^\alpha \) with \( \alpha > 0 \) is defined by the Fourier transform, namely, \( (-\Delta)^\alpha = \mathcal{F}^{-1} |\xi|^{2\alpha} \mathcal{F} \).

Equation (1)-(3) is obtained by removing the adventive terms \( \partial_t u + u \cdot \nabla u \) from the \( u \) equation of the magnetohydrodynamics (MHD) equations and then replacing \( -\Delta \) by \( (-\Delta)^\alpha \) in \( b \) equation. It is well-known that MHD equations govern the motion of the electrically conducting fluids arising from plasmas, liquid metals, and electrolytes, etc (see [6]). Since the first derivation by Alfvén, MHD equations have become one of the most important equations in the study of many phenomena in geophysics, astrophysics, cosmology and engineering (see, e.g., [1,3]).

When \( \eta = 0 \), Moffatt [9] made a method of magnetic relaxation to argue that, if the topology of the magnetic field is non-trivial, (1)-(3) on a smooth bounded domain \( \Omega \) should produce a magnetostatic equilibrium \( b^E(x) \) that satisfies

\[
j^E \times b^E = \nabla p^E, \quad j^E = \nabla \times b^E, \quad \nabla \cdot b^E = 0 \quad \text{in} \; \Omega, \quad b^E \cdot n = 0 \quad \text{on} \; \partial \Omega. \quad (5)
\]
This equation almost shares the same form with the steady Euler equation:
\[ u^E \times u^E = \nabla h^E, \quad \omega^E = \nabla \times u^E, \quad \nabla \cdot u^E = 0 \quad \text{in} \quad \Omega, \quad u^E, n = 0 \quad \text{on} \partial \Omega, \quad (6) \]
if one “identifies” \( b^E \) with velocity field \( u^E \). It indicates that the study of (1)-(3) might be helpful to understand the unstable Euler flows. Note that, in 1980s, he studied in [8] that the steady state of some non-resistive MHD equations should also obey (5). However, there is no rigorous proof that the magnetic relaxation will yield a steady Euler flow. One of the reasons is that the global well-posedness of MHD equations in three dimensions remains open (see [7] and references therein).

For a limiting state point of view, the dynamical model used to obtain the above steady state is not particularly important (see [4,7,9]). In fact, the advantage of (1)-(3) can be illustrated by this sentence coming from [7] (or [4]): “...Moffatt [9] argued that dropping the acceleration terms from the equation and working with a “Stokes” model might prove more mathematically amenable.” In recent years, the well-posedness of (1)-(4) and related models have attracted great attention. When \( \alpha = 1 \), the existence of weak solutions was obtained by McCormick et al. in [7], where the uniqueness of weak solutions for the two-dimensional case was also proved (see [2,4] for the local-in-time existence of regular solutions of 3D non-resistive MHD equations in Besov spaces). Furthermore, they showed that weak solutions of the 2D equations become regular if \( b_0 \) is smooth (see [7]). Recently, we [11] established a optimal regularity criterion for (1)-(4) with \( \alpha = 1 \), and we also studied the global-in-time existence of regular solutions when the initial data satisfy some small conditions in critical Sobolev spaces, as well as in critical Besov spaces.

This paper is devoted to the global well-posedness of (1)-(4) with arbitrary initial data. Before the statement of the main result, let us introduce some notations. \( c \) represents a generic positive constant whose value may change at each occurrence. \( A \lesssim B \) denotes the inequality \( A \leq cB \). We consider function spaces on \( \mathbb{R}^3 \), for instance, \( C_c^\infty := C_c^\infty (\mathbb{R}^3) \), \( L^p := L^p (\mathbb{R}^3) \), \( H^s := H^s (\mathbb{R}^3) \). \( L^{p, \infty} \) denotes the weak \( L^p \) space. We will use \( f := \int_{\mathbb{R}^3} ||f||_{L^p} \) and \( ||f||_2 := ||f||_2 \) for convenience. We define \( \mathcal{D}_\sigma = \{ f \in C_c^\infty : \nabla \cdot f = 0 \} \). Let \( L^2_\sigma \) and \( H^1_\sigma \) be the closure of \( \mathcal{D}_\sigma \) in the \( L^2 \) and \( H^1 \) norm, respectively.

**Definition 1.1.** Let \( T > 0 \) and let \( b_0 \in L^2_\sigma \). A function \((u, b)\) is called a weak solution of the equation (1)-(4) on \((0, T)\), if
\begin{itemize}
  \item[(i)] \( u \in L^\infty (0, T; L^2; \infty) \cap L^2 (0, T; H^1) \) and \( b \in L^\infty (0, T; L^2) \cap L^2 (0, T; H^\alpha) \),
  \item[(ii)] \((u, b)\) verifies:
  \[ \int_0^T \nu \nabla u : \nabla \phi_1 + (b \cdot \nabla) \phi_1 \cdot b dx = 0, \]
  \[ \int_0^T \int_\Omega b_0 \cdot \phi_2 (0) dx - \int_0^T \int_\Omega b \cdot \partial_t \phi_2 - \eta \Lambda^\alpha b 
  \Lambda^\alpha \phi_2 + (u \cdot \nabla) \phi_2 \cdot b - (b \cdot \nabla) \phi_2 \cdot u dx dt = 0, \]
end{itemize}
for all test functions \( \phi_1, \phi_2 \in C_c^\infty ([0, T]; \mathcal{D}_\sigma) \).

**Definition 1.2.** Let \( T > 0 \) and let \( b_0 \in H^1_\sigma \). A function \((u, b)\) is called a strong solution of the equation (1)-(4) on \([0, T]\) if
\begin{itemize}
  \item[(i)] \((u, b)\) is a weak solution of (1)-(4) on \((0, T)\),
  \item[(ii)] \((u, b)\) satisfies \( u \in C([0, T]; H^2) \cap L^2 (0, T; H^2) \) and \( b \in C([0, T]; H^1) \cap L^2 (0, T; H^{1+\alpha}) \).
end{itemize}

The main result of this paper is stated as follows:
Theorem 1.3. Let \( \alpha \geq \frac{3}{2} \) and let \( b_0 \in H^1(\mathbb{R}^3)^3 \) with \( \nabla \cdot b_0 = 0 \). Then the initial value problem (1)-(4) admits a unique strong solution on \([0, \infty)\).

Remark 1. If \((u, b)\) is a strong solution of (1)-(4), then \((u, b)\) is a classical solution. That is, for any \( s \geq 2 \) and any \( m \in \mathbb{Z}^+ \), it is easy to show that \( \partial_t^m u \in C([0, \infty); H^{s+1}) \) and \( \partial_t^m b \in C([0, \infty); H^s) \cap L^2(0, \infty; H^{s+\alpha}) \) by the use of the \( L^2(0, T; H^2) \) estimates of \( u \) and \( b \).

Remark 2. It has been proved that weak solutions of the 3D generalized viscous Navier-Stokes equations

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nu(-\Delta)^{\alpha_1} u + \nabla p &= 0, \\
\nabla \cdot u &= 0,
\end{align*}
\]

remain smooth for all time when \( \alpha_1 \geq \frac{3}{4}, \) see e.g., [10]. This result has been extended to \( d \)-dimensional generalized MHD equations

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nu(-\Delta)^{\alpha_2} u + \nabla p &= b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b + \eta(-\Delta)^{\alpha_3} b &= b \cdot \nabla u, \\
\nabla \cdot u &= 0, \quad \nabla \cdot b = 0,
\end{align*}
\]

when \( \alpha_2 \geq 1 + \frac{d}{4} \) and \( \alpha_3 \geq 1 + \frac{d}{4}, \) see e.g., [12]. Thus it is natural to ask whether the global-in-time existence of strong solution of (1)-(4) is still valid for \( \alpha \geq \frac{3}{4}. \) However, the effort towards to establish this result seems failed because in the situation \( \alpha \in (\frac{3}{4}, \frac{3}{2}) \), the dissipative term \( -\nu \Delta u \) together with the resistive term \( \eta(-\Delta)^{\alpha} b \) of (1)-(2) are not strong enough to bound the nonlinearities of these equations. In fact, compared with the global well-posedness of (8) in [12], the main result (Theorem 1.3) seems to be optimal in the sense that the summation of the fractional indices of the operators \( -\nu \Delta u \) and \( \eta(-\Delta)^{\alpha} b \) equals to that of the indices of \( \nu(-\Delta)^{\alpha_2} u \) and \( \eta(-\Delta)^{\alpha_3} b \) in (8) when \( \alpha = \frac{3}{2}, \alpha_2 = \alpha_3 = \frac{5}{4}. \)

2. Proof of Theorem 1.3. The proof of the existence of weak solutions \((u, b)\) almost parallels to the manipulation of [7] and thus omitted here. It is sufficient to prove a priori estimate of \((u, b)\) in \( L^\infty(0, \infty; H^{\frac{d}{2}}) \cap L^2(0, \infty; H^2) \times L^\infty(0, \infty; H^1) \cap L^2(0, \infty; H^{1+\alpha}). \) Let \( T > 0 \) be arbitrary.

Firstly, we show that

\[
u \in L^\infty(0, T; L^2(0, T; H^1)) \cap L^2(0, T; H^1), \quad b \in L^\infty(0, T; L^2) \cap L^2(0, T; H^\alpha).
\]

Taking the inner product of (1) and (2) with \( u \) and \( b \) respectively, then integrating in \( \mathbb{R}^3 \) and summing the resultant equations, we use integration by parts and \( \nabla \cdot b = 0 \) to obtain that

\[
\frac{1}{2} \frac{d}{dt} \| b \|^2 + \nu \| \nabla u \|^2 + \eta \| \Lambda^\alpha b \|^2 = \int (b \cdot \nabla b) \cdot u \, dx + \int (b \cdot \nabla u) \cdot b \, dx \\
= - \int (b \cdot \nabla u) \cdot b \, dx + \int (b \cdot \nabla u) \cdot b \, dx = 0.
\]

Integrating (10) with respect to \( t \), we deduce that for any \( t \in [0, T], \)

\[
\| b(t) \|^2 + 2 \int_0^t (\nu \| \nabla u(\tau) \|^2 + \eta \| \Lambda^\alpha b(\tau) \|^2) \, d\tau \leq \| b_0 \|^2.
\]
To prove $L^{\frac{3}{2},\infty}$ estimate of $u$, we consider the following nonhomogeneous Stokes equation:

\[
\begin{cases}
-\nu \Delta u + \nabla p = b \cdot \nabla b, \\
\nabla \cdot u = 0.
\end{cases}
\]  

(12)

By the theory of [5], the solution of (12) is solved by

\[
u(t,x) = \int U(x-y) \cdot (b \cdot \nabla b)(t,y)dy
\]

and

\[
p^*(t,x) = \int q(x-y) \cdot (b \cdot \nabla b)(t,y)dy,
\]

where $(U(\cdot), q(\cdot))$ is the fundamental solution of Stokes equations. Note that $\nabla U \in L^{\frac{3}{2},\infty}$ (see [5, 7, 11]). Thus, by $\nabla \cdot b = 0$ and Young inequality in weak $L^p$ spaces, we deduce that for any $t \in [0, T]$,

\[
\|u(t)\|_{L^{\frac{3}{2},\infty}} = \left\| \int_{\mathbb{R}^3} \nabla U(x-y)(b \otimes b)(t,y)dy \right\|_{L^{\frac{3}{2},\infty}} \\
\lesssim \|\nabla U\|_{L^{\frac{3}{2},\infty}} \|b \otimes b(t)\|_1 \\
\lesssim \|b_0\|^2.
\]  

(13)

Furthermore, by interpolation inequality, Sobolev embedding, (11) and (13), we get

\[
\int_0^T \|u(\tau)\|^2 d\tau \lesssim \int_0^T \|u(\tau)\|^\frac{3}{2}_{L^{\frac{3}{2},\infty}} \|u(\tau)\|^\frac{3}{2}_6 d\tau \\
\lesssim T^{\frac{2}{3}} \|b_0\|^2 \left( \int_0^T \|\nabla u(\tau)\|^2 d\tau \right)^{\frac{1}{3}} < \infty.
\]

This equation, together with (11), implies that $u \in L^2(0, T; H^1)$. Secondly, we prove that $u \in L^2(0, T; H^2)$ and $b \in L^\infty(0, T; H^1) \cap L^2(0, T; H^{1+\alpha})$. In view of (9), it suffices to show that

\[
u \in L^2(0, T; H^2) \text{ and } b \in L^\infty(0, T; H^1) \cap L^2(0, T; H^{1+\alpha}).
\]

(14)

Taking the inner product of (1) and (2) with $-\Delta u$ and $-\Delta b$ respectively, integrating in $\mathbb{R}^3$ and then summing the resultant equations, we use $\nabla \cdot u = 0$ to obtain that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla b\|^2 + \nu \|\Delta u\|^2 + \eta \|\Lambda^{1+\alpha}\|b\|^2 \\
= \int \nabla (b \cdot \nabla b) : \nabla u dx - \int \nabla (u \cdot \nabla b) : \nabla b dx + \int \nabla (b \cdot \nabla u) : \nabla b dx \\
\equiv I_1 + I_2 + I_3.
\]  

(15)

By $\nabla \cdot u = 0$, we deduce that

\[
I_2 = \sum_{j,k,l=1}^3 \int \partial_i u_j \partial_j b_k \partial_l b_l dx \\
= \sum_{j,k,l=1}^3 \int \left( \partial_i u_j \partial_j b_k \partial_l b_l + u_j \partial_i b_k \partial_l b_l \right) dx \\
= \sum_{j,k,l=1}^3 \int \partial_i u_j \partial_j b_k \partial_l b_l dx.
\]
On the other hand, applying integration by parts and $\nabla \cdot b = 0$, we find that

$$I_1 + I_3$$

$$= \sum_{j,k,l=1}^{3} \int \partial_t (b_j \partial_j b_k) \partial_t u_k dx + \sum_{j,k,l=1}^{3} \int \partial_t (b_l \partial_l u_k) \partial_j b_k dx$$

$$= \sum_{j,k,l=1}^{3} \int \left( \partial_t b_j \partial_j b_k \partial_t u_k + b_j \partial_j b_k \partial_t u_k \right) dx + \int \left( \partial_t b_l \partial_l u_k \partial_j b_k + b_l \partial_j u_k \partial_j b_k \right) dx$$

$$= \sum_{j,k,l=1}^{3} \int \left( \partial_t b_j \partial_j b_k \partial_t u_k - b_j \partial_j b_k \partial_t u_k \right) dx + \int \left( \partial_t b_l \partial_l u_k \partial_j b_k + b_l \partial_j u_k \partial_j b_k \right) dx$$

$$= \sum_{j,k,l=1}^{3} \int \left( \partial_t b_j \partial_j b_k \partial_t u_k + \partial_t b_j \partial_j b_k \partial_t u_k \right) dx.$$

Hence we deduce that

$$|I_1 + I_2 + I_3| \leq \|\nabla u\|_6 \|\nabla b\|_\frac{10}{3},$$

$$\leq \frac{\nu}{2} \|\Delta u\|^2 + c \|\nabla b\|_\frac{4}{3}.$$  

Since there hold the interpolation inequalities

$$\|\nabla b\|_\frac{4}{3} \lesssim \|\nabla b\|^{1-\frac{5}{2\alpha}} \|\Lambda^{1+\alpha} b\|^\frac{5}{2\alpha},$$

$$\|\nabla b\|_\frac{4}{3} \lesssim \|b\|^{1-\frac{5}{2\alpha}} \|\Lambda^{\alpha} b\|^\frac{5}{2\alpha}.$$

Hence we have

$$|I_1 + I_2 + I_3| \leq \frac{\nu}{2} \|\Delta u\|^2 + c \|\nabla b\|^{2-\frac{5}{2\alpha}} \|\Lambda^{1+\alpha} b\|^{\frac{5}{2\alpha}} \|b\|^2 \frac{5}{2-\frac{5}{2\alpha}} \|\Lambda^{\alpha} b\|^{\frac{5}{2\alpha}}$$

$$\leq \frac{\nu}{2} \|\Delta u\|^2 + \frac{\eta}{2} \|\Lambda^{1+\alpha} b\|^2 + c \|b\|^{\frac{10(4\alpha-5)}{4\alpha+5}} \|\Lambda^{\alpha} b\|^{\frac{10}{4\alpha+5}} \|\nabla b\|^2.$$  

(16)

Substituting (16) into (15) and subtracting the resultant equation of $\frac{\nu}{2} \|\Delta u\|^2 + \frac{\eta}{2} \|\Lambda^{1+\alpha} b\|^2$, we deduce that

$$\frac{d}{dt} \|\nabla b\|^2 + \nu \|\Delta u\|^2 + \eta \|\Lambda^{1+\alpha} b\|^2 \lesssim \|b\|^{\frac{10(4\alpha-5)}{4\alpha+5}} \|\Lambda^{\alpha} b\|^{\frac{10}{4\alpha+5}} \|\nabla b\|^2.$$

Integrating the previous equation with respect to $t$ and using (11), we deduce that for any $t \in [0, T]$

$$\|\nabla b(t)\|^2 + \int_0^t (\nu \|\Delta u(\tau)\|^2 + \eta \|\Lambda^{1+\alpha} b(\tau)\|^2) d\tau$$

$$\leq \|\nabla b_0\|^2 + c \int_0^t \|b(\tau)\|^{\frac{10(4\alpha-5)}{4\alpha+5}} \|\Lambda^{\alpha} b(\tau)\|^{\frac{10}{4\alpha+5}} \|\nabla b(\tau)\|^2 d\tau$$

$$\leq \|\nabla b_0\|^2 + c \|b_0\|^{\frac{10(4\alpha-5)}{4\alpha+5}} \int_0^t \|\Lambda^{\alpha} b(\tau)\|^{\frac{10}{4\alpha+5}} \|\nabla b(\tau)\|^2 d\tau.$$  

(17)
Using the Gronwall’s inequality, \( \alpha \geq \frac{3}{2} \) and (11), we get

\[
\sup_{t \in [0,T]} \| \nabla b(t) \| \leq \| \nabla b_0 \| \exp \left( c \| b_0 \| \frac{2(4\alpha-3)}{4\alpha-1} \int_0^T \| \Lambda^\alpha b(\tau) \| \frac{10}{4\alpha-1} d\tau \right)
\]

\[
\leq \| \nabla b_0 \| \exp \left( c \| b_0 \| \frac{2(4\alpha-3)}{4\alpha-1} T \frac{10}{4\alpha-1} \right) \int_0^T \| \Lambda^\alpha b(\tau) \|^2 d\tau
\]

\[
\leq \| \nabla b_0 \| \exp \left( c \| b_0 \| \frac{2(4\alpha-3)}{4\alpha-1} T \frac{10}{4\alpha-1} \right) < \infty,
\]

(18)

here \( T \frac{10}{4\alpha-1} = 1 \) if \( \alpha = \frac{3}{2} \). This proves \( b \in L^\infty(0,T; H^1) \). Setting \( t = T \) in (17) and using (18) again, we find that \( u \in L^2(0,T; H^2) \) and \( b \in L^2(0,T; H^{1+\alpha}) \). This proves (14).

Thirdly, we prove that \( u \in L^\infty(0,T; H^{\frac{3}{2}}) \).

Applying \( \Lambda^{-\frac{3}{2}} \) to (1) and taking the inner product of the resultant equation with \( \Lambda^{\frac{3}{2}} u \), then integrating in \( \mathbb{R}^3 \), we deduce that

\[
\nu \| \Lambda^{\frac{3}{2}} u(t) \|^2 \leq \| \Lambda^{\frac{3}{2}} u(t) \| \| \Lambda^{-\frac{3}{2}} (b \cdot \nabla b)(t) \| \text{ for any } t \in [0,T].
\]

(19)

Dividing (19) by the factor \( \| \Lambda^{\frac{3}{2}} u(t) \| \) and using Hardy-Littlewood-Sobolev inequality, we obtain that

\[
\nu \| \Lambda^{\frac{3}{2}} u(t) \| \leq \| b \cdot \nabla b \|^{\frac{3}{2}}
\]

\[
\leq \| b(t) \| \| \nabla b(t) \|
\]

\[
\leq \| \nabla b(t) \|^2 < \infty.
\]

(20)

Moreover, applying the interpolation inequality (see Lemma 2.2 in [7]), (13) and the embedding \( \dot{H}^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow BMO(\mathbb{R}^3) \), it follows that for any \( t \in [0,T] \),

\[
\| u(t) \| \leq \| u(t) \|_{L^{\frac{3}{2},\infty}} \| u(t) \|_{BMO}^{\frac{3}{4}}
\]

\[
\leq \| b_0 \|^{\frac{3}{4}} \| \Lambda^{\frac{3}{2}} u(t) \|^{\frac{1}{4}} < \infty.
\]

(21)

Thus (19)-(21) implies that \( u \in L^\infty(0,T; H^{\frac{3}{2}}) \).

Uniqueness. Let \((u, b)\) and \((\tilde{u}, \tilde{b})\) be two solutions of (1)-(4) with the same initial data, that is, \( b(0) = \tilde{b}(0) = b_0 \). Then \((u - \tilde{u}, b - \tilde{b})\) satisfies the following equations:

\[
-\nu \Delta (u - \tilde{u}) = P(b \cdot \nabla b - \tilde{b} \cdot \nabla \tilde{b}),
\]

(22)

\[
\partial_t (b - \tilde{b}) + \eta (-\Delta)^\alpha (b - \tilde{b}) = -(u \cdot \nabla b - \tilde{u} \cdot \nabla \tilde{b}) + (b \cdot \nabla u - \tilde{b} \cdot \nabla \tilde{u}),
\]

(23)

\[
\nabla \cdot u = \nabla \cdot \tilde{u} = 0, \quad \nabla \cdot b = \nabla \cdot \tilde{b} = 0,
\]

(24)

\[
(b - \tilde{b})|_{t=0} = 0.
\]

(25)

Here \( P = I - \nabla \Delta^{-1} \nabla \cdot \) is the Leray projection. Taking the inner product of (22) (resp. (23)) with \( u - \tilde{u} \) (resp. \( b - \tilde{b} \)) and integrating the resultant equations in \( \mathbb{R}^3 \), we use integrations by parts to obtain that

\[
\nu \| \nabla (u - \tilde{u}) \|^2 = \int P(b \cdot \nabla b - \tilde{b} \cdot \nabla \tilde{b}) \cdot (u - \tilde{u}) dx \equiv I_1,
\]

(26)
and
\[ \frac{1}{2} \frac{d}{dt} \| b - \tilde{b} \|^2 + \eta \| \Lambda^\alpha (b - \tilde{b}) \|^2 \]
\[ = \int - (u \cdot \nabla b - \tilde{u} \cdot \nabla \tilde{b}) \cdot (b - \tilde{b}) \, dx + \int (b \cdot \nabla u - \tilde{b} \cdot \nabla \tilde{u}) \cdot (b - \tilde{b}) \, dx \]
\[ = I_2 + I_3. \] (27)

By the $L^2$ boundedness of $P$, $H^s \hookrightarrow L^\infty$ for $s > \frac{3}{2}$ and $c$-Young inequality, we use integration by parts and $\nabla \cdot b = \nabla \cdot \tilde{b} = 0$ to deduce that
\[ |I_1| = \int P[(b - \tilde{b}) \otimes b + \tilde{b} \otimes (b - \tilde{b})] : \nabla (u - \tilde{u}) \, dx \]
\[ \leq \| \nabla (u - \tilde{u}) \| \| P[(b - \tilde{b}) \otimes b + \tilde{b} \otimes (b - \tilde{b})] \| \]
\[ \leq \| \nabla (u - \tilde{u}) \| \| b - \tilde{b} \| (\| b \|_\infty + \| \tilde{b} \|_\infty) \]
\[ \leq \frac{\nu}{4} \| \nabla (u - \tilde{u}) \|^2 + c (\| b \|_{H^{1+\alpha}}^2 + \| b \|_{H^{1+\alpha}}^2) \| b - \tilde{b} \|^2. \] (28)

Substituting (28) into (26), we obtain that
\[ \| \nabla (u - \tilde{u}) \| \lesssim (\| b \|_{H^{1+\alpha}} + \| \tilde{b} \|_{H^{1+\alpha}}) \| b - \tilde{b} \|. \] (29)

Similarly, the bounds of $I_2$, $I_3$ are obtained as follows:
\[ |I_2| \leq \| b - \tilde{b} \| \| u \cdot \nabla b - \tilde{u} \cdot \nabla \tilde{b} \|
\leq \| b - \tilde{b} \| (\| u - \tilde{u} \|_a \| \nabla b \|_3 + \| \tilde{u} \|_{3^{-1}} \| \nabla (b - \tilde{b}) \|_{a^{-1}})
\leq \frac{\nu}{4} \| \nabla (u - \tilde{u}) \|^2 + \frac{\eta}{4} \| \Lambda^\alpha (b - \tilde{b}) \|^2 + c (\| u \|_{H^1}^2 + \| b \|_{H^{1+\alpha}}^2) \| b - \tilde{b} \|^2, \] (30)

and
\[ |I_3| \leq \| \nabla (b - \tilde{b}) \|_{a^{-1}} \| b \otimes u - \tilde{b} \otimes \tilde{u} \|_{a^2}
\leq \| \Lambda^\alpha (b - \tilde{b}) \| (\| b - \tilde{b} \|_a \| u \|_{\frac{3}{2}} + \| \tilde{b} \|_a \| u - \tilde{u} \|_{\frac{3}{2}})
\leq \frac{\eta}{4} \| \Lambda^\alpha (b - \tilde{b}) \|^2 + c (\| u \|_{H^1}^2 + \| b \|_{H^{1+\alpha}}^2 + \| b_0 \|_a^2) \| u - \tilde{u} \| \| L^{\frac{2}{3}} \| \| \nabla (u - \tilde{u}) \|_{L^{\frac{2}{3}}} \|_{\frac{21}{3} \alpha \cdot \alpha}, \] (31)

after the use of $H^1 \hookrightarrow L^{\frac{2}{3} \alpha \cdot \alpha}$ for $\alpha \geq \frac{3}{2}$ and the interpolation inequality
\[ \| u - \tilde{u} \|_{L^{\frac{2}{3} \alpha \cdot \alpha}} \lesssim \| u - \tilde{u} \|_{L^{\frac{2}{3} \alpha \cdot \alpha}} \| \nabla (u - \tilde{u}) \|_{L^{\frac{21}{3} \alpha \cdot \alpha}}. \]

Similar as the derivation of (13), we have
\[ \| u - \tilde{u} \|_{L^{\frac{2}{3} \alpha \cdot \alpha}} \lesssim \| b_0 \| \| b - \tilde{b} \|. \] (32)

Substituting (32) into (31) then using (29), we get
\[ |I_3| \leq \frac{\eta}{4} \| \Lambda^\alpha (b - \tilde{b}) \|^2 + c \left( \| u \|_{H^1}^2 + \| b_0 \|_{\frac{4}{3} \alpha} \| b \|_{H^{1+\alpha}}^4 + \| \tilde{b} \|_{H^{1+\alpha}}^4 \| \frac{21}{3} \alpha \cdot \alpha \right) \| b - \tilde{b} \|^2. \] (33)

Thus substituting (30) and (33) into (27) and then summing the resultant equation with (29), we obtain that
\[ \frac{d}{dt} \| b - \tilde{b} \|^2 + \nu \| \nabla (u_1 - u_2) \|^2 + \eta \| \nabla (b_1 - b_2) \|^2 \]
\[ \lesssim (\| u, \tilde{u} \|_{H^1}^2 + \| b \|_{H^\alpha}^2 + \| b, \tilde{b} \|_{H^{1+\alpha}}^2 + \| b_0 \|_{\frac{4}{3} \alpha} \| (b, \tilde{b}) \|_{H^{1+\alpha}}^4 \| \frac{21}{3} \alpha \cdot \alpha \| b - \tilde{b} \|^2. \] (34)
Here $\|(u, \bar{u})\|_{H^1} := \sqrt{\|u\|_{H^1}^2 + \|\bar{u}\|_{H^1}^2}$, $\|(b, \bar{b})\|_{H^{1+\alpha}} := \sqrt{\|b\|_{H^{1+\alpha}}^2 + \|\bar{b}\|_{H^{1+\alpha}}^2}$. Since $\frac{4(3-\alpha)}{3} \leq 2$ when $\alpha \geq \frac{3}{2}$ and $u, \bar{u} \in L^2(0, T; H^1)$, $b, \bar{b} \in L^2(0, T; H^{1+\alpha})$, integrating (34) with respect to $t$ yields for any $t \in [0, T]$

$$\|\bar{b}(t) - \bar{b}(t)\|^2 \lesssim \|b_0 - \bar{b}_0\|^2 \exp \int_0^T (\|(u, \bar{u})\|_{H^1} + \|b\|_{H^{1+\alpha}}^2 + \|(b, \bar{b})\|_{H^{1+\alpha}}^2)$$

$$+ \|b_0\|^{\frac{4}{3}} \|(b, \bar{b})\|_{H^{1+\alpha}}^{\frac{4(3-\alpha)}{3}} d\tau = 0. \quad (35)$$

This implies that $b = \bar{b}$ almost everywhere on $[0, T] \times \mathbb{R}^3$.

Finally, let us show $u = \bar{u}$. By the interpolation inequality applied to derive (21), (32) and (35), we deduce that for any $t \in [0, T]$,

$$\|u(t) - \bar{u}(t)\| \lesssim \|u(t) - \bar{u}(t)\|_\infty^{\frac{3}{4}} \|u(t) - \bar{u}(t)\|_{BMO}^{\frac{1}{4}}$$

$$\lesssim \|b_0\|^{\frac{3}{4}} \|b(t) - \bar{b}(t)\|^{\frac{3}{4}} \|(u, \bar{u})(t)\|_{H^\alpha}^{\frac{1}{4}} = 0. \quad (36)$$

Thus we have $u = \bar{u}$ almost everywhere on $[0, T] \times \mathbb{R}^3$. This completes the proof of Theorem 1.3.

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