Compton scattering by pion: there is no room for the off–shell effects.

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Abstract

We show that the off–shell contributions in Compton scattering by pion may exist only in the single exchange diagrams. These contributions are canceled completely in the total gauge–invariant amplitude as it confirmed by one–loop calculations. It explains, in particular, some results of the chiral pertubation theory in the order $p^4$ but this cancelation has no relation to chiral symmetry.

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1 Introduction

The problem of incorporating of the internal structure of particles in electromagnetic (electroweak) processes has different aspects and a long history. Here we would like to discuss one particular question from this area concerned the off-shell effects in Compton scattering by a pion. First of all to attach the exact meaning to above phrase let’s look at the Born diagrams of Fig. 1.

![Figure 1: Born contributions for Compton scattering](image)

The internal structure of pions and strong interactions can significantly modify the amplitude, nevertheless the threshold theorem of Low [1] guarantees that the Born contribution dominates near the threshold. Corrections to the Born terms (polarizabilities and the higher structure constants in the threshold expansion) are defined by different physical effects. The so called off-shell effects arise because the exchange pion in diagrams of Fig. 1 is virtual. It leads to appearance in vertices of two form-factors which depend on the virtuality and can, generally speaking, modify the amplitude.

The subject under discussion is directly related with two-photon experiments on $e^+e^-$ colliders studying $\gamma\gamma \rightarrow \pi^+\pi^-$ reaction near threshold. In the framework of S-matrix description for this process (see, for instance, [3, 4]) the off-shell contributions usually are not included (or one should believe that they are properly absorbed by existing arbitrary parameters). But there exist some attempts (e.g. [5]) of simple accounting (parametrization) of these effects.

From more theoretical point of view, the contributions under discussion were considered [6] in the framework of chiral perturbation theory. It was found that there exists few effective lagrangians (physically equivalent, the authors call them representations) which generate the same Compton amplitudes at different off-shell formfactors in the electromagnetic vertex. The authors came to conclusion that the off-shell effects are not only model dependent but also representation dependent.

\footnote{Some details and history of the low-energy theorem including the structure corrections may be found in [2].}
However the results of [6] (recall that they are dealing with the momentum expansion up to the order $p^4$) lead to thought about total disappearance of the off-shell contributions by some reasons. It turns out that this is truth and the reasons are rather simple.

Our main point may be easily clarified without any formulae. The off-shell effects in a vertex and propagator appear after loop summation in field theory, in other words it means the substitution of full vertices and propagators instead of bare ones as it shown in Fig. 2a,b. To keep the gauge invariance, one should change the contact term too (Fig.2c) but one needs the concrete model for it. The Ward–Takahashi identity leads to a partial (but not total) cancellation of the loop effects (see Fig. 3 for illustration and details below). Now it’s instructive to look into the full vertex using some field theory.

The one-loop contributions in the simplest model (interaction of pions with scalar $\sigma$-meson) may be seen in Fig. 4. Note that Fig. 3 is equivalent to contributions of Fig. 4a–c. Cutting the $\pi\sigma$ loop in diagram 4a, we shall have $\pi\sigma$ real intermediate state with spin $J = 0$ since it is connected with pion line. But everyone knows that the partial wave decomposition of Compton amplitude $\gamma\pi \rightarrow \gamma\pi$ starts from $J = 1$. The obtained contradiction has simple solution: contributions from intermediate state $J = 0$ exist only in single (gauge non-invariant) diagrams. In the total amplitude, the S-wave
contributions must cancel each other, so the off–shell effects should disappear from gauge–invariant amplitude. We will demonstrate below by direct calculation that the S–wave discontinuity on \( s \) really cancels in sum of diagrams 4a–d.

\[
\begin{align*}
\Gamma_{R}^{\mu} = (p_f + p_i)^{\mu} \cdot F(q^2, p_f^2, p_i^2) + (p_f - p_i)^{\mu} \cdot G(q^2, p_f^2, p_i^2), \quad q = p_f - p_i 
\end{align*}
\]
From T–invariance:

$$F(q^2, p_f^2, p_i^2) = F(q^2, p_i^2, p_f^2), \quad G(q^2, p_f^2, p_i^2) = -G(q^2, p_i^2, p_f^2).$$  \hfill (2)$$

The Ward—Takahashi identity \cite{7, 8} relates a vertex and propagator:

$$q_\mu \Gamma_R^\mu = \Delta^{-1}(p_f^2) - \Delta^{-1}(p_i^2),$$

or

$$(p_f^2 - p_i^2) \cdot F(q^2, p_f^2, p_i^2) + q^2 \cdot G(q^2, p_f^2, p_i^2) = \Delta^{-1}(p_f^2) - \Delta^{-1}(p_i^2).$$  \hfill (4)$$

Here $\Delta(p^2)$ is the renormalized irreducible propagator of $\pi$– meson and $\mu = m_\pi$.

$$\Delta^{-1}(p^2) = p^2 - \mu^2 - J(p^2) = p^2 - \mu^2 - (p^2 - \mu^2)^2 \cdot \tilde{J}(p^2),$$

$$J(\mu^2) = J'(\mu^2) = 0$$  \hfill (5)$$

If the photon and one of pions are on the mass–shell, the Ward—Takahashi identity reduces to the simple relation.

$$\Delta(p_f^2) \cdot F(0, p_f^2, \mu^2) = \frac{1}{p_f^2 - \mu^2}$$  \hfill (6)$$

Note that an amplitude we are concern $\gamma\pi^+ \rightarrow \gamma\pi^+$ for real photons will not contain the formfactor $G$, which is accompanied by the momentum $q^\mu$ and will disappear after multiplying by polarization vector.

### 3 Amplitude with full vertices and propagators

We consider the Compton scattering by pion:

$$\gamma(\mu, q_1) + \pi^+(p_1) \rightarrow \gamma(\nu, q_2) + \pi^+(p_2),$$

$$s = (p_1 + q_1)^2, \quad t = (q_1 - q_2)^2, \quad u = (q_1 - p_2)^2, \quad q_1^2 = q_2^2 = 0$$

The Compton amplitude for spinless particle has a standard decomposition which includes two invariant functions, free of kinematical singularities and zeros.

$$T^{\mu\nu} = \sum_{i=1}^{2} L_i^{\mu\nu} \cdot B_i(t, s, u)$$  \hfill (7)$$
It is convenient to use the following independent momenta: \( q_1, q_2, \) and \( P = (p_1 + p_2)/2 \). The gauge–invariant tensor structures are of the form:

\[
L_1^{\mu\nu} = q_2^\mu q_1^\nu - (q_1q_2) g^{\mu\nu}
\]

\[
L_2^{\mu\nu} = -P^\mu P^\nu(q_1q_2) + qP^\mu q_2^\nu(q_2P) + q_2^\mu P^\nu(q_1P) - g^{\mu\nu}(q_1P)(q_2P).
\]  

(8)

Let us start from the standard Born amplitude for point–like pions.

\[
T^{\mu\nu} = e^2 \left\{ - (2p_1 + q_1)^\mu \frac{1}{s - \mu^2} (2p_2 + q_2)^\nu - (2p_2 - q_1)^\mu \frac{1}{u - \mu^2} (2p_1 - q_2)^\nu + 2g^{\mu\nu} \right\}
\]  

(9)

One can easily verify the gauge–invariance of this expression.

\[
q_1^{\mu}T^{\mu\nu} = q_2^{\nu}T^{\mu\nu} = 0
\]  

(10)

To account the off–shell effects, we have modify both vertices and propagators, since the same physical effects generate the full vertices and propagators.

Vertices (see (1)):

\[
(2p_1 + q_1)^\mu \rightarrow (2p_1 + q_1)^\mu F(0, s, \mu^2) - q_1^\mu G(0, s, \mu^2) \quad \text{and so on.}
\]  

(11)

Propagator:

\[
\frac{1}{s - \mu^2} \rightarrow \Delta(s)
\]  

(12)

One can see that the contact term should be changed too to keep gauge invariance. Introducing some unknown generalized contact term \( T_P^{\mu\nu} \) we have the modified Born contribution ( here the symmetry properties (2) were taken into account )

\[
\frac{1}{e^2} T^{\mu\nu} = - \left[ (2P + q_2)^\mu F(s) + q_1^\mu q_2^\nu G(s) \right] \Delta(s) \left[ (2P + q_1)^\nu F(s) + q_2^\nu G(s) \right]
\]

\[
- \left[ (2P - q_2)^\mu F(u) - q_1^\mu G(u) \right] \Delta(u) \left[ (2P - q_1)^\nu F(u) - q_2^\nu G(u) \right]
\]

\[+ T_P^{\mu\nu}
\]

\[
F(s) \equiv F(0, s, \mu^2), \quad G(s) \equiv G(0, s, \mu^2).
\]  

(13)

Having in mind the Ward—Takahashi identity (6) we can rewrite the amplitude:

\[
\frac{1}{e^2} T^{\mu\nu} = - (2P + q_2)^\mu \frac{F(s)}{s - \mu^2} (2P + q_1)^\nu - (2P - q_2)^\mu \frac{F(u)}{u - \mu^2} (2P - q_1)^\nu
\]

\[+ T_P^{\mu\nu} + \text{(terms} \sim q_1^\mu \text{or} q_2^\nu)\]

(15)
So we can see that the Ward—Takahashi identity leads to partial cancellation of the off–shell effects (see Fig. 3). As for the generalized contact term $T_{\mu\nu} P$, its form can not be restored unambiguously from the gauge invariance requirement because we always can add a gauge–invariant expression to any obtained answer.

4 One–loop calculations

Dressing of π–meson (i.e. turning of bare propagator into full) proceeds due to the three–pion intermediate state. It is generally believed that the three–pion state is saturated by the quasi–two–particle πρ and πσ ones. To demonstrate the off–shell effects cancellation we will use the simplest model: interaction of π–mesons with a scalar σ–meson (nevertheless, it’s not the σ–model).

$$L_{int} = g \sigma(x) \vec{\pi}(x) \vec{\pi}(x)$$ (16)

All the one–loop contributions are depicted in Fig. 4. The diagrams 4e and 4f have no relation with the off–shell corrections and are the gauge–invariant together (it’s seen due to presence of the pole $1/(t - m^2)$), so we can forget about them. We shall consider the discontinuity of amplitude by s, originated from the πσ intermediate state, only diagrams 4a–4d have such discontinuities. These contributions are of the form:

$$\Delta_s T_{\mu\nu} = i f \int d^4 l \delta(l^2 - \mu^2) \left\{ \frac{(2p_1 + q_1)^\mu(2p_2 + q_2)^\nu}{(Q^2 - \mu^2)^2} - \frac{(2l + q_1)^\mu(2p_2 + q_2)^\nu}{(Q^2 - \mu^2)(l + q_1)^2 - \mu^2} \right\} \delta((l + Q)^2 - m^2).$$

Here $f = e^2 g^2/(2\pi)^2$, $\mu = m_\pi$, $m = m_\sigma$, $Q = p_1 + q_1$, $Q^2 = s$.

Passing to the set $q_1, q_2, P$, let us rewrite the discontinuity in another form.

$$\Delta_s T_{\mu\nu} = i f \int d^4 l \delta(l^2 - \mu^2) \delta((l + Q)^2 - m^2) \cdot \frac{1}{4(lq_1)(lq_2)} \left\{ \frac{4(2p_1 + q_2)^\mu(2p_1 + q_1)^\nu}{(s - \mu^2)^2} - \frac{2(lq_2)(2l + q_1)^\mu(2p_1 + q_1)^\nu}{(s - \mu^2)} \right\}. $$

3By the way, a cancellation of the one–loop off–shell corrections in Compton amplitude was noted a long time ago in the chiral model. Note that in chiral models there are definite relations between parameters in contrast to our calculations.
\[-2(lq_1)(2P + q_2)^\mu(2l + q_2)^\nu + (2l + q_1)^\mu(2l + q_2)^\nu\] \hspace{1cm} (18)

One can verify easily that this expression is gauge–invariant. The subsequent actions consist in projection of (18) onto the tensor structure of interest and calculation of arising integrals. All this is rather standard so we shall omit some details. The answer will be expressed in term of the following scalar integrals.

\[
I_0 = \int d^4 l \delta(l^2 - \mu^2) \delta((l + Q)^2 - m^2) \frac{1}{(lq_1)}
\]

\[
J_1 = \int d^4 l \delta(l^2 - \mu^2) \delta((l + Q)^2 - m^2) \frac{1}{(lq_2)}
\]

\[
J_2 = \int d^4 l \delta(l^2 - \mu^2) \delta((l + Q)^2 - m^2) \frac{1}{(lq_1)(lq_2)}
\]

\[
V = \int d^4 l \delta(l^2 - \mu^2) \delta((l + Q)^2 - m^2) \frac{1}{(lq_1)(lq_2)}
\]

\hspace{1cm} (19)

The \(q_2^\mu q_1^\nu\) term in (18).

The off–shell corrections of diagrams 4a–4c are very simple and we will write the answer for them, not cancelling different terms.

\[
\Delta_s T_{abc} = if \{ \frac{I_0}{(s - \mu^2)^2} - \frac{I_0}{(s - \mu^2)^2} - \frac{I_0}{(s - \mu^2)^2} \}
\]

\hspace{1cm} (20)

Here three terms in brackets correspond to diagrams 4a—4c. Calculating the integral, we have

\[
I_0 = \pi K \theta(s - (m + \mu)^2), \quad \text{where} \quad K^2 = \frac{[s - (m + \mu)^2][s - (m - \mu)^2]}{4s}.
\]

\hspace{1cm} (21)

K is a momentum of \(\pi \sigma\) state in the center mass system. Since every term in (20) is proportional to CM momentum they are the S–wave contributions. We see that first term cancels with second or third, which corresponds to Ward–Takahashi identity (6).

Completely the coefficient at \(q_2^\mu q_1^\nu\) (with accounting that \(J_2 = J_1\)) is:

\[
T_1 = -\frac{if}{4}(K_I \cdot I_0 + K_J \cdot J_1 + K_V \cdot V).
\]

\hspace{1cm} (22)

The coefficients, calculated with help of REDUCE:

\[
K_I = \frac{ts}{4(s - \mu^2)^2 \Delta^2} \left[t^3 s + t^2 (\mu^2 - 6\mu^2 s + 5s^2) + 4t(-\mu^6 + 4\mu^4 s - 5\mu^2 s^2 + 2s^3) + 4(s - \mu^2)^4\right]
\]
\[ K_J = \frac{t^2(t - 4\mu^2)}{8\Delta^2}(s - \mu^2)(s + \mu^2 - m^2) \]
\[ K_V = \frac{t}{32\Delta^2}[t^3(-\mu^4 + 2\mu^2m^2 - m^4 + 2m^2s - s^2) + 2t^2\mu^2(3\mu^4 - 4\mu^2m^2 - 2\mu^2s + 2m^4 - 4m^2s - s^2) + 8t\mu^2(-\mu^6 + 3\mu^2s^2 - 3s^3) - 8\mu^2(s - \mu^2)^4]. \]  

(23)

Here \( \Delta = \frac{t(su - \mu^4)}{4} \).

It is convenient to make the further calculations in the center mass system \( \vec{Q} = 0 \), using \( s \) and scattering angle \( c = \cos \theta \) as the variables. The remaining scalar integrals are:

\[ J_1 = \frac{\pi}{(s - \mu^2)} \ln \left( \frac{1 + \tau}{1 - \tau} \right) = \frac{\pi}{(s - \mu^2)} \left[ 2\tau + \frac{2}{3}\tau^3 + \ldots \right] \]
\[ V = \frac{4\pi s}{(s - \mu^2)(s + \mu^2 - m^2)} \left[ 2\tau + \frac{2}{3}\tau^3(c + 2) + \ldots \right], \]

where

\[ \tau = -\frac{[(s - (m + \mu)^2)(s - (m - \mu)^2)]^{1/2}}{s + \mu^2 - m^2}. \]

We didn’t show here rather complicated expression for \( V \), restricting ourselves by the threshold expansion of corresponding logarithm.

After substitution into (22), we can convince ourselves that the S–wave contributions, proportional to \( K \), really cancel and decomposition starts from D–wave,

\[ \frac{1}{i\hbar}T_1 = \delta^3 \cdot \frac{\pi \sqrt{m\mu}}{3} \cdot \left[ \frac{8\mu^2 + 14m\mu + 7m^2 + c(m^2 + 2m\mu)}{m^2\mu(m + \mu)^4(m + 2\mu)^3} \right] + O(\delta^5), \]  

(24)

where \( \delta = \sqrt{s - (m + \mu)^2} \).

The \( P^\mu P^\nu \) term in (18).

After similar calculations we found again that the S–wave contributions cancel in the sum of diagrams 4a–4c . The discontinuity in this structure is of the form

\[ \frac{1}{i\hbar}T_2 = -\delta^3 \cdot \frac{4\pi \sqrt{m\mu}}{3} \cdot \left( \frac{1 - c}{m\mu(m + \mu)^2(m + 2\mu)^4} \right) + O(\delta^5). \]  

(25)

An additional control of calculations consists in construction of the center of mass helicity amplitudes from (24), (25) with accounting of (8):

\[ T_{0+,0-} = -\left[ B_1 \cdot \frac{s}{2} - B_2 \cdot \frac{s(s - 4\mu^2)}{16} \right], \]
\[ T_{0+,0+} = -B_2 \cdot \frac{su - \mu^4}{4}. \]  

(26)
Partial decomposition of helicity amplitudes is well known:

\[
T_{0+,0\pm} = \sum_{J=1}^{\infty} (2J + 1) T^{J}_{\pm}(s) d^{J}_{1,\pm}(\cos \theta) = \frac{3}{2} T^{1}_{\pm}(s) (1 \pm \cos \theta) + \ldots
\]  

(27)

One can verify that angular dependence of the helicity amplitudes built from (24), (25) indeed correspond to the lower partial waves in (27).

5 Conclusion

We checked by direct calculation that the off-shell effects which appear in the exchange diagrams cancel in the total amplitude. It confirms the general arguments indicated in Introduction about disappearance of the off-shell contributions in the total gauge–invariant expression for Compton amplitude. Note that we did not use the chiral symmetry; the reason for the disappearance is rather kinematical and model–independent.

Finally, let us note that the corrections to QED contribution may be investigated in the cross–channel \( \gamma \gamma \rightarrow \pi^+ \pi^- \) near the threshold. Most of experiments performed up to now studied the invariant mass region \( M_{\pi\pi} \geq 1 \) GeV. As for the near–threshold region \( M_{\pi\pi} < 0.5 \) GeV, where the Born term dominates, only a few points of MARK II \([10]\) there exist. Experiments at DAΦNE will allow to investigate this region more accurately (see, e.g.,\([11]\)), so as a result of our consideration we have a good news for DAΦNE: there are no off–shell effects in the Compton amplitude.

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