CAN WE IDENTIFY LENSED GAMMA RAY BURSTS?

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ABSTRACT

A gravitationally lensed gamma-ray burst (GRB) would appear as multiple bursts with identical light curves, separated in time and differing only by the scaling of their amplitudes. In reality, the detected bursts will be noisy, and therefore they may be difficult to identify as lensed images. Furthermore, faint, intrinsically similar, yet distinct light curves may be falsely identified as lensing events. In this paper we develop two statistical tests to distinguish noisy burst light curves. We use Fourier analysis techniques to analyze the signals for both intrinsic variability and variability due to noise. We are able to determine the noise level, and we compare the bursts only at frequency channels that are signal dominated. Utilizing these methods, we are able to make quantitative statements about whether two bursts are distinct. We apply these statistics to scaled versions of two subbursts of GRB 910503—subbursts previously investigated by Wambsganss (1993) using a different statistical test. We find that our methods are able to distinguish these bursts at slightly smaller amplitudes than those at which Wambsganss’s method succeeds. We then apply our techniques to “candidate” lensing events taken from the BATSE catalogue, and we find that nearly all of them, except for the very shortest ones (durations \( \lesssim 0.3 \) s), are distinguishable. We therefore expect that a majority of bursts will be distinguishable from one another.

1. Introduction

At the Second Huntsville Gamma Ray Burst Workshop, the question of whether gamma ray bursts (GRBs) are galactic or cosmological in origin was put to a vote, and the attendants were split 60%-40% in favor of the cosmological scenario (cf. Wang 1994). More than 20 years after their discovery, and after more than a hundred proposed models (cf. Nemiroff 1994a), not even the location of the bursts are agreed upon! There are strong arguments to support both views (cf. Blaes 1994 and Hartmann 1994 for reviews of the observations and theories). However, simple standard-candle cosmological scenarios, wherein GRBs have a constant burst rate per comoving volume per comoving time, appear to fit the data quite well (e.g. Fenimore et al. 1993). These models are attractive because they naturally explain the observed isotropy and flux distribution of the bursts. These models, however, only show consistency with the data, and do not provide proof of the cosmological scenario.

As first suggested by Paczyński (1986), perhaps the only unambiguous verification of a cosmological origin for GRBs would be the detection of a gravitationally lensed burst. The multiple images of a lensed GRB would have identical light curves (to within the noise and an overall magnification) that arrive from the same position on the sky; however, the
individual bursts would be separated in time. A number of authors have investigated the possibility of GRB lensing. Mao (1992) considered lensing by galaxies and calculated the expected time delays between lensed bursts. He concluded that the Burst and Transient Source Experiment (BATSE) aboard the Compton Gamma Ray Observatory (CGRO) has a significant chance of observing a lensing event during its lifetime. More recently, Grossman & Nowak (1994) have refined the calculation of Mao (1992) and they find a substantially smaller probability that BATSE will observe GRBs lensed by galaxies (although a suitably designed instrument likely could detect a lensed burst). Blaes & Webster (1992) considered lensing by point masses, Narayan & Wallington (1992) investigated the possibility of determining lens parameters from an observed event, and Nemiroff et al. (1993a,b) have used the absence of any observed lensing events to put rather loose constraints upon the distances to bursts and upon the density of $10^6 M_\odot - 10^8 M_\odot$ objects in the universe.

[Other authors—namely Paczyński (1987), Gould (1992), and Mao (1993)—have considered micro- and “femto-” lensing. These events may be manifest within a single light curve in some cases. In this work, however, we only consider the possibility of identifying lensing events in distinct GRB light curves.]

The discussion above presumes that we will actually recognize a lensing event when we see it. As first pointed out by Wambsganss (1993), such recognition may be very difficult. The BATSE light curves are comprised of an intrinsic source signal, a background, and noise. Only the source component will be identical (to within an overall scale factor) among the lensed images. It is possible that the background and noise may overwhelm our ability to identify a lensing event. Another potential problem is that two distinct but intrinsically similar light curves, if faint enough, may be falsely identified as a lensing event. To address these issues, Wambsganss (1993) developed a test to determine whether or not two gamma ray bursts with similar time profiles could be statistically distinguished from each other. His method essentially compared the variance of the difference of the two light curves (where one light curve is scaled and time-shifted compared to the other) with the variance due to Poisson noise. If the variance was consistent with that due to Poisson noise only, the bursts were said to be statistically identical. Wambsganss then applied his test to two intrinsically similar but distinct BATSE light curves and found that as the amplitudes of the bursts were reduced (keeping the backgrounds constant, and adding an appropriate amount of counting noise), the light curves became indistinguishable. Furthermore, the light curves became indistinguishable at a level well above the BATSE detection threshold. Wambsganss (1993) thus concluded that it will be very difficult in practice to identify lensed GRBs.

Wambsganss’s method, however, used every time bin of the light curves, whether the bins were noise dominated or not. No attempts were made to minimize the noise. Furthermore, Wambsganss’s criteria for burst distinction were more subjective than quantitative. (No significance values for the comparisons are presented in Wambsganss [1993].) In this
paper, we use Fourier analysis techniques to analyze bursts for both intrinsic variability and variability due to noise. We are able to determine the noise level in the Fourier domain, and compare burst light curves only at frequency channels that are signal dominated. We thus do not dilute the signal with excessively noisy data. We quantify the confidence with which we distinguish two light curves. In §2 we use the properties of Fourier transforms to define the two statistics that we use to compare light curves. In §3 we apply our statistics to the same scaled light curves that Wambsganss (1993) considered. Utilizing our methods, we can distinguish these two bursts at slightly smaller amplitudes than those at which Wambsganss’s method succeeded. We also discuss how the scaled amplitudes compare to the BATSE detection threshold. In §4 we apply our techniques to “candidate” lensing events taken from the BATSE burst catalogue. We find that nearly all of these bursts, except for the very shortest ones (durations $\lesssim 0.3$ s), are easily distinguishable. In §5 we summarize our results and discuss their implications for lensing calculations.

2. Comparison of Noisy Light Curves

Assume that we have two light curves, $s_g(t)$ and $s_h(t)$, comprised respectively of intrinsic signals, $g(t)$ and $h(t)$, and noise, $n_g(t)$ and $n_h(t)$. We assume that the noise is counting noise, and therefore it is uncorrelated among time bins. We have

$$s_g(t) = g(t) + n_g(t) ,$$
$$s_h(t) = h(t) + n_h(t) .$$

We shall assume that the above signals have the (assumed known) background subtracted. If the intrinsic signals are scaled copies of each other such that $g(t) = A h(t + \tau_d)$, where $A$ and $\tau_d$ are constants, then the normalized correlation function, defined by

$$c(\tau) \equiv \frac{\int_{-\infty}^{\infty} g(t)h(t - \tau) \, dt}{\sqrt{\int_{-\infty}^{\infty} g(t)g(t) \, dt} \sqrt{\int_{-\infty}^{\infty} h(t)h(t) \, dt}} ,$$

is equal to unity if $\tau = -\tau_d$. The measured signals $s_g(t)$ and $s_h(t)$ include uncorrelated noise, and the resulting correlation will be less than unity. The question we need to address is: how much of a deviation from unity is significant, indicating that $g(t)$ and $h(t)$ are not copies of each other?

Analysis of the correlation function is simplest in the Fourier frequency domain. Using the Correlation Theorem (cf. Davenport & Root 1978), we define the cross power spectral density (CPD),

$$C(f) \equiv \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} g(t)h(t - \tau) \, dt \right) e^{i2\pi ft} \, d\tau ,$$

where $G(f)$ and $H(f)$ are the complex Fourier transforms of $g(t)$ and $h(t)$, respectively. (Here and throughout we shall use capital letters to denote functions in the frequency
domain, and lower case letters to denote functions in the time domain. We will suppress the functional dependences on $f$ and $t$ whenever there is little chance of confusion.) If $g(t) = A h(t + \tau_d)$, then
\[
C(f) = A|H(f)|^2 e^{-i2\pi f \tau_d}.
\] (2.4)

Note that the CPD time delay, defined as the complex phase of the CPD divided by $2\pi f$, is constant. In addition, the power spectral densities (PSDs, i.e. squared Fourier amplitudes) of the individual signals are proportional to each other. In analogy to the normalized correlation function, we find it convenient to use a Fourier transform normalization such that the amplitude of the transform is independent of the amplitude of the signal in the time domain (cf. Appendix A). Therefore, in the absence of noise, lensed bursts will have identical PSDs.

In reality, the measured signals will be noisy, and therefore the PSDs will differ from one another and the CPD time delay will not be constant. The advantage of working in the frequency domain is that we can estimate the level of the noise (which we do in Appendix A), and determine which frequencies are above this noise level. The criteria for lensed signals, that is consistent Fourier amplitudes and coherent phases, must hold for the signal dominated frequencies. Below we derive error statistics that measure the statistical significance of deviations from perfect correlation.

In the following sections we shall construct “optimized” Fourier transforms from the measured transforms, and then use these optimized transforms in our statistical tests. From our knowledge of the statistical properties of the noise (cf. Appendix A) and our measurements of the signal, we can construct “optimal filters” that when multiplied by our measured transforms yield estimates of the “true” transforms. These optimal filters statistically minimize the difference between the true and estimated transforms. The literature and history of optimal filtering is vast (cf. Whalen 1971); however, nothing presented in this work goes beyond the scope of the discussion found in Press et al. (1993). Below we consider separately applications of optimal filtering to the PSDs and CPDs of a pair of light curves.

\textit{a) Comparison via Power Spectral Densities}

The Fourier transforms of equation (2.1) are given by
\[
S_g(f) = G(f) + N_g(f) \\
S_h(f) = H(f) + N_h(f).
\] (2.5)

In the following analysis, we adopt a Fourier transform normalization in a light curve, based upon the integrated counts, such that PSD amplitudes are independent of the intensity of the burst. Thus, for lensed bursts we expect $G^2 - H^2 = 0$. We measure only $S_g^2$ and $S_h^2$, however, and obtain only statistical estimates of $N_g^2$ and $N_h^2$ (cf. Appendix A). From our
measurements we wish to construct “best estimates” of the intrinsic PSDs, which we call \( \tilde{G}^2 \) and \( \tilde{H}^2 \). These estimates are related to the measured signals according to

\[
\tilde{G}^2 \equiv S_g \, \Phi_g \\
\tilde{H}^2 \equiv S_h^2 \, \Phi_h ,
\]

where the optimal filters, \( \Phi_g \) and \( \Phi_h \), are functions that minimize \( \langle (\tilde{G}^2 - G^2)^2 \rangle \) and \( \langle (\tilde{H}^2 - H^2)^2 \rangle \). The brackets denote an ensemble average. Expanding these functions we find

\[
\langle (\tilde{G}^2 - G^2)^2 \rangle = \left[ (G^2 + \langle N_g^2 \rangle)^2 + 2G^2 \langle N_g^2 \rangle \right] \, \Phi_g^2 - 2G^2(G^2 + \langle N_g^2 \rangle) \, \Phi_g + G^4 ,
\]

where we have used the fact that \( \langle N_g \rangle = 0 \). Equation (2.7) is minimized at each frequency if

\[
\Phi_g = \frac{G^2(G^2 + \langle N_g^2 \rangle)}{(G^2 + \langle N_g^2 \rangle)^2 + 2G^2 \langle N_g^2 \rangle} \approx \frac{G^2 S_g^2}{S_g^2 + 2G^2 \langle N_g^2 \rangle} . \tag{2.8}
\]

The approximation above uses \( S_g^2 \approx G^2 + \langle N_g^2 \rangle \). We obtain a similar expression for \( \Phi_h \). Note that \( \Phi_g \sim 1 \) in the signal dominated regime, where \( S_g \approx G \gg N_g \), and \( \Phi_g \sim G^2/\langle N_g^2 \rangle \) in the noise dominated regime, where \( S_g \approx N_g \gg G \). In practice, we calculate the filters by taking \( G^2 \approx S_g - \langle N_g^2 \rangle \) and \( H^2 \approx S_h - \langle N_h^2 \rangle \). Substituting the optimal filter (2.8) into equation (2.7), we obtain

\[
\langle (\tilde{G}^2 - G^2)^2 \rangle = \frac{2 \, G^6 \langle N_g^2 \rangle}{(G^2 + \langle N_g^2 \rangle)^2 + 2G^2 \langle N_g^2 \rangle} \approx \frac{2 \, G^6 \langle N_g^2 \rangle}{S_g^4 + 2G^2 \langle N_g^2 \rangle} . \tag{2.9}
\]

This defines the residual uncertainty in the PSD due to noise which we use to test the significance of deviations between two estimated PSDs.

If two bursts are lensed copies, we expect \( \tilde{G}^2 - \tilde{H}^2 \approx 0 \). We measure the significance of deviations from zero by defining the \( \chi^2 \) statistic:

\[
\chi^2 = \frac{1}{N_f - 2} \sum_f \frac{(\tilde{G}^2 - \tilde{H}^2)^2}{\langle (\tilde{G}^2 - G^2 + H^2 - \tilde{H}^2)^2 \rangle} , \tag{2.10}
\]

where the sum is over the \( N_f \) frequency bins not dominated by the noise (here defined as \( \tilde{G}^2/\langle N_g^2 \rangle \geq 3 \) and \( \tilde{H}^2/\langle N_h^2 \rangle \geq 3 \)). We also define the degrees of freedom to be \( \nu \equiv N_f - 2 \), since we subtract two degrees of freedom due to the two constraints provided by the filters, \( \Phi_g \) and \( \Phi_h \). We require at least three frequency channels that meet our signal dominated criterion, and this places a limit to how faint a burst to which we can apply this statistic.

b) Comparison via Cross Power Spectral Densities

Just as we defined an optimal filter for the PSDs, we now define an optimal filter, \( \Phi_{gh} \), for the CPD, given by

\[
\tilde{C} = \tilde{G} \tilde{H}^* \equiv S_g S_h^* \, \Phi_{gh} , \tag{2.11}
\]
such that \( \langle (G\mathcal{H}^\ast - GH^\ast)^2 \rangle \) is minimized. Following the same procedures as above, we can show that
\[
\Phi_{gh} = \frac{G^2H^2}{G^2H^2 + G^2\langle N^2_h \rangle + H^2\langle N^2_g \rangle + \langle N^2_g \rangle \langle N^2_h \rangle} \approx \frac{G^2H^2}{S^2_g S^2_h} .
\tag{2.12}
\]

Similar to the above, we take
\[
G^2H^2 \approx S^2_g S^2_h - \tilde{G}^2\langle N^2_h \rangle - \tilde{H}^2\langle N^2_g \rangle - \langle N^2_g \rangle \langle N^2_h \rangle.
\]

The CPD is a complex quantity, having both an amplitude and a (typically) non-zero phase. We write the phase of the intrinsic signal as \( 2\pi \tau_0(f) \), and the phase of the optimized signal as \( 2\pi \tau_m(f) \). We write the optimized and intrinsic CPDs as
\[
\tilde{C} = \tilde{G}\mathcal{H}^\ast = C_m e^{i 2\pi \tau_m(f) f} \\
C = GH^\ast = C_0 e^{i 2\pi \tau_0 f} .
\tag{2.13}
\]

If the intrinsic signals are scaled, time-shifted versions of each other, then \( \tau_0(f) \) is constant; however, \( \tau_m(f) \) will vary due to noise. We wish to determine how large a variation is expected in the presence of noise under the assumption that \( \tau_0 \) is indeed a constant.

If the light curves are lensed copies of each other, we expect that \( \tilde{C} - C \approx 0. \) In the limit of small noise, we can also take \( C_m \sim C_0 \), and therefore the squared difference between the measured and true transforms can be written as
\[
\frac{(\tilde{C} - C)^2}{C^2} \approx 4 \sin^2 \pi [\tau_m(f) - \tau_0] f \approx 4\pi^2 [\tau_m(f) - \tau_0]^2 f^2 ,
\tag{2.14}
\]
which measures the phase difference between the optimized and true CPDs. Unfortunately, we do not know the true time delay, \( \tau_0 \), so instead we calculate the “true” phase by taking \( \tau_0 \) to be the mean of the measured delays, \( i.e. \tau_0 \approx \langle \tau_m(f) \rangle \).

We can quantify the dispersion of the measured CPD phase due to noise from our knowledge of the statistical properties of our optimized transforms. Specifically, using equation (2.12) we can show
\[
\frac{\langle (\tilde{C} - C)^2 \rangle}{C^2} = (1 - \Phi_{gh}) .
\tag{2.15}
\]

With this as the measure of the uncertainty of the phase resulting from noise, we define the \( \chi^2 \) statistic:
\[
\chi^2_{\nu}/\nu \equiv \frac{1}{N_f - 2} \sum_f \frac{(\tilde{C} - C)^2}{\langle (\tilde{C} - C)^2 \rangle} = \frac{4\pi^2}{N_f - 2} \sum_f \frac{[\tau_m(f) - \langle \tau_m \rangle]^2 f^2}{(1 - \Phi_{gh})} ,
\tag{2.16}
\]
where as before we sum over the signal dominated frequency channels, and \( \nu = N_f - 2 \) is the degrees of freedom (we subtract two degrees of freedom for the constraints provided
by \( \Phi_{gh} \) and \( \langle \tau_m \rangle \). Equations (2.10) and (2.16) together are the statistical tests that we shall apply to real burst light curves.

3. Application to Simulated Burst Data

a) Application to Scaled Bursts

We apply the statistics developed above to the problem of distinguishing two similar but distinct burst profiles. Following Wambsganss (1993), we choose burst GRB 910503 for our test. This burst consists of two subbursts: a bright, spikey burst (which we call subburst A) and a weaker, smoother burst (which we call subburst B). Each subburst has a duration of roughly 10 s. The background subtracted burst light curves are shown in Figure 1. We have chosen this burst for the same reasons as Wambsganss; namely, this burst is one of the brightest BATSE bursts and therefore should be easily distinguishable. Most bursts, however, are much fainter; therefore, one should check the possibility that subbursts A and B could have been falsely identified as a lensing event if they were fainter. In addition, by choosing this event we can make direct comparisons between our method and that of Wambsganss.

We follow the same methods that Wambsganss (1993) used to rescale the burst light curves to lower amplitudes. We assume that the unscaled, background subtracted burst profiles are the “true” \( i.e., \) noiseless) burst profiles. First, these profiles are shifted by a random fraction of a data bin (each bin is 64 ms, and the photons are assumed to be evenly distributed within each bin). The burst is then rescaled to a fainter level and the background is readded. Gaussian noise is then added to the profiles, simulating the noisy signal of a fainter burst, and the background is resubtracted.

We compare the bursts using information provided by their PSDs and CPDs. In Figure 2, we plot the PSDs (based upon a 256 point FFT, spanning a duration of 16.4 s) of subbursts A and B for various rescalings. As a consequence of our PSD normalization, the amplitude of the PSD in the signal dominated frequencies is independent of scaling \( cf. \) Appendix A). The amplitude of the noise dominated channels, however, increases by roughly a factor of 10 for each factor of 3 reduction of the signal amplitude in the time domain. There are two other things to notice here. First, the PSD’s of the two subbursts are intrinsically very similar, approximately following a \( 1/f^2 \) power law (the form for a fast rise and exponential decay [FRED]). The similarities are a consequence of the two bursts having comparable rise and decay times. Second, even for rescalings to much fainter levels, the low frequency power is well defined and roughly unchanged. This gives hope that the method will be useful even for faint bursts. In Figure 3 we plot the time delay \( i.e., \tau_m(f) \) calculated from the CPD of subburst A and B. We immediately notice that this time delay is inconsistent with a constant, even at the lowest frequencies. As seen in Figure 3, the fourth lowest frequency bin, corresponding to \( \sim 0.24 \) Hz, has a time delay that is much larger and of the opposite sign from the three lowest frequency bins.
Figure 1: Background subtracted time profiles for subbursts A (top) and B (bottom) of GRB 910503. The solid line corresponds to photon count per 64 ms time bin, and the dashed line to the burst signal to noise ratio for 2.048 s time bins.
**Figure 2:** Power spectral densities for the 256 point FFTs of subbursts A (left) and B (right), for several rescalings of the burst amplitude. The PSD normalization is as defined in Appendix A. Dotted lines are the expected amplitudes for the noise PSD when the subbursts are scaled to 3%, 10%, 30%, and 100%.

**Figure 3:** Time delay (defined as the phase of the cross power spectral density divided by Fourier frequency) between subbursts A and B. The solid line is the computed value, and dashed lines are the 2 $\sigma$ noise limits.
We compare subbursts A and B scaled respectively to levels (32%, 32%), (8%, 16%), (4%, 8%), and (2%, 8%). We also compare A to itself using the rescalings 32%, 8%, 4%, 2%, and B to itself using the rescalings 32%, 16%, 8%. For each comparison we generate 1000 noisy light curve pairs, and we calculate for each pair the degrees of freedom (i.e., number of frequency channels above noise minus 2) and the corresponding reduced $\chi^2$ for the PSD and CPD statistics. Histograms of the results are presented in Figures 4 and 5.

In Figure 4 for the PSD statistic, the top row shows the reduced $\chi^2$ histograms for the A vs. B comparisons, the middle row shows the histograms for the A vs. A comparisons, and the bottom row shows the histograms for the B vs. B comparisons. From left to right, the bursts are rescaled to fainter levels. Each panel gives the number of statistically well-defined simulations (i.e., those with 1 or more degrees of freedom), the mean degrees of freedom, $\bar{\nu}$, and the mean reduced $\chi^2$, $\chi^2/\bar{\nu}$. The means are calculated only from the well-defined simulations. Plotted on top of the histograms are true reduced $\chi^2$ probability distributions (cf. Abramowitz and Stegun 1972), with the degrees of freedom taken to be $\bar{\nu}$ for that particular histogram. These distributions are normalized such that their integrated areas are identical to those of the histograms.

It is obvious from visual inspection alone that when subbursts A and B are not reduced below $\sim 10\%$ of their original flux, their $\chi^2$’s are well above that expected from noise alone and they are easily distinguishable by the PSD statistic. In particular, most of the area in the A vs. B histograms at (32%, 32%) and (8%, 16%) fall far outside the true reduced $\chi^2$ distributions. For the comparison (4%, 8%), which Wambsgansss (1993) considered too similar to be unambiguously distinguished, the overlap between the reduced $\chi^2$ distribution and the calculated histogram is indeed substantial. Nevertheless, a fair fraction of the histogram lies outside the distribution. For the case (2%, 8%), which Wambsgansss termed “impossible”, it is indeed very difficult to distinguish the reduced $\chi^2$ distribution from the PSD statistic histogram.

The comparisons of A with itself and B with itself should show agreement to within the noise. These comparisons are useful in determining to what extent the PSD statistic follows a true reduced $\chi^2$ distribution. From visual inspection, it appears that measured distributions with more degrees of freedom have histograms that are closer to a true re-

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**Figure 4 (following page):** Reduced $\chi^2$ histograms for the PSD statistic (equation [2.9]) based upon 1000 Monte Carlo simulations for each rescaling. **Top row:** From left to right, A vs. B at (32%, 32%), (8%, 16%), (4%, 8%), (2%, 8%). **Middle row:** From left to right, A vs. itself at 32%, 8%, 4%, 2%. **Bottom row:** B vs. itself at 32%, 16%, 8%. Each panel lists the number of simulations with well defined statistics (i.e. three or more frequency bins above the noise, cf. §2), the mean degrees of freedom ($\bar{\nu}$, the average number of frequency bins above noise minus two), and the mean reduced $\chi^2$ ($\chi^2/\bar{\nu}$). Solid lines in each panel correspond a true reduced $\chi^2$ distribution with $\bar{\nu}$ degrees of freedom (cf. Abramowitz and Stegun 1972), normalized to have the same area as the histogram.
duced \( \chi^2 \) distribution. As the degrees of freedom decrease (with the downscaling of the subbursts), the histograms tend to be skewed to values greater than a true reduced \( \chi^2 \) distribution. In fact, the histograms for A vs. A at 2% and B vs. B at 8% are quite similar to the A vs. B at (2%, 8%) histogram. This suggests that the latter comparison is mostly consistent with identical light curves differing only by the noise. For distributions with few degrees of freedom, we are hesitant to assign a firm "significance value" to our measured \( \chi^2 \) since the probability distribution does not follow a true \( \chi^2 \) distribution. Any such numbers should be indicative of a trend, but their significances are not directly quantifiable. For example, a 5\( \sigma \) \( \chi^2 \) might be in reality 4 – 6\( \sigma \), but large \( \chi^2 \)'s are definitely significant.

Table 1 quantifies the above discussion. In it we present the percentage of (statistically well-defined) simulations that are statistically different at \( \geq 1 \sigma \), \( \geq 2 \sigma \), and \( \geq 3 \sigma \) confidence levels, as calculated from a true reduced \( \chi^2 \) distribution. These confidence levels correspond to the most extreme 32%, 4.5%, or 0.3% of the distribution, respectively. For A vs. B at (32%, 32%), 100% of the simulations are different with 3 \( \sigma \) confidence. For A vs. B at (8%, 16%), 81% are different with 3 \( \sigma \) confidence. The percentage drops markedly for A vs. B at (4%, 8%); however, 26% of the defined simulations (19% of all the simulations) are different with 3 \( \sigma \) confidence. This is greater, by a factor of 2.5, than the most skewed A vs. A or B vs. B comparisons, indicating greater differences than can be understood from noise alone. This suggests that the statistic, though becoming difficult to calibrate, is still useful at such a low rescaling. The A vs. B at (2%, 8%) simulations are defined only 19% of the time, and only 8% of these simulations are greater than 3 \( \sigma \) significant. This percentage is comparable to the A vs. A at 2% and B vs. B at 8% comparisons, and therefore most of the greater than 3 \( \sigma \) results cannot be considered significant.

With an average of 16.0 and 7.9 degrees of freedom respectively, the A vs. A at 32% and A vs. A at 8% simulations are the closest to a true reduced \( \chi^2 \) distribution. All the lower amplitude simulations show pronounced excesses of large deviations, with 15-35 times too many 3 \( \sigma \) events. In light of this, it would be prudent to regard a 3 \( \sigma \) deviation (as measured by a true \( \chi^2 \) distribution) in the PSD statistic as the lower bound for a "significant deviation" for distinguishing two bursts.

In Figure 5 we present histograms for the CPD statistic. As before, the comparisons between A and B show that they are fairly distinct when the subbursts are brighter than \( \sim 10\% \) their original flux. At lower levels the comparisons become more consistent with deviations due to noise only. Unlike the PSD statistic, however, the self comparison (A vs. A and B vs. B) histograms are significantly narrower than a true reduced \( \chi^2 \) distribution. As with the previous statistic, fewer degrees of freedom broadens the histogram, so that in this case the CPD histogram becomes more similar to a true reduced \( \chi^2 \) distribution. These graphical results are quantified in Table 2, which is the analogue of Table 1 for the CPD statistic.
As before, we define the $1\sigma$, $2\sigma$, and $3\sigma$ confidence levels as the most extreme 38%, 4.5%, and 0.3% points of the true reduced $\chi^2$ distribution. All the A vs. B comparisons show more $1\sigma$, $2\sigma$, and $3\sigma$ deviations than either the self comparisons or the true reduced $\chi^2$ distributions. This is true of even the “impossible” A vs. B at (2%, 8%) (for which 5.5% of the total simulations were distinguishable at $\geq 2\sigma$, and 2% of the total simulations distinguishable at $\geq 3\sigma$—making it merely exceedingly difficult to distinguish). If we adopt the criteria of requiring a $\geq 2\sigma$ significance in the CPD statistic, then nearly 20% of the A vs. B at (4%, 8%) simulations are “distinguishable”. This is comparable to what was found for the PSD statistic with a $3\sigma$ criterion.

At this point we should note that the PSD and CPD statistics are independent, since noise in the Fourier phase is uncorrelated with noise in the Fourier amplitude. This is demonstrated in Figure 6 where we plot the two statistics against each other for A vs. A at 4% and B vs. B at 8%. The absence of correlation justifies using the PSD and CPD statistics jointly as even more powerful discriminants for distinguishing bursts. If we adopt the criterion that a burst is distinguished if either the PSD statistic is $\geq 3\sigma$ significant or the CPD statistic is $\geq 2\sigma$ significant, then for the A vs. B at (4%, 8%), 47% of the well-defined simulations are distinguished (34% of all the simulations). This is to be compared to 11% of the well-defined A vs. A at 4% and 12% of the B vs. B at 8% simulations being falsely distinguished.

It is not surprising that our method is slightly more successful at distinguishing bursts than Wambsgans’s method. In his method, all time bins in a light curve are considered, whether they are dominated by noise or signal. As the bursts are scaled downward, more and more time bins become noise dominated until the statistic itself becomes noise dominated. As Wambsgans noted in his paper, if he rebinned the data by combining 4 time bins at a time, the sensitivity of his statistic improved. This is not surprising as the signal to noise ratio increases in each new bin, and in some ways this is analogous to using the lower frequency bins in the Fourier domain. As an extreme case, each burst can be rebinned into a single time bin, and then a best fit scale factor and time delay can be computed. Each burst then can be rebinned into two time bins, and the constancy— with respect to the noise levels— of the scale factor and time delay can be checked. The procedure can be repeated for four, eight, sixteen, etc., time bins, until the bins become noise dominated. However, Wambsgans only used the finest resolution time bins and thus diluted his error measure with more noise than necessary. Our procedure is essentially equivalent to a rebinning that uses the finest time bins allowed by our signal to noise criteria, hence enhancing the discriminatory power of our statistics.

Figure 5 (following page): Same as Figure 4, except for the CPD statistic.
Figure 6: The reduced $\chi^2$ for the CPD statistic $[(\chi^2_\nu/\nu)_{CPD}]$ vs. the reduced $\chi^2$ for the PSD statistic $[(\chi^2_\nu/\nu)_{PSD}]$. Top: Subburst A vs. itself at 4%. Bottom: Subburst B vs. itself at 8%. Note the lack of correlation.
b) Interpretation in Terms of Signal to Noise Ratios

To understand what the above results mean for real burst data, it is necessary to discuss their interpretation in terms of the BATSE signal to noise ratio (cf. Fishman et al. 1992). BATSE consists of eight detectors, essentially forming the eight faces of an octahedron, hence a burst will illuminate two to four detectors. Each detector is capable of observing a burst in a variety of spectral ranges; however, the coarsest spectral resolution (which is used for triggering the detectors) consists of only four energy channels: 20-50 keV, 50-100 keV, 100-300 keV, and 300+ keV (we will refer to these as channels 1-4, respectively). BATSE triggers when the signal in the second most illuminated detector exceeds the background noise level by a factor of 5.5. The signal and noise are measured from the sum of channels 2 and 3 only. BATSE can trigger on timescales of 64 ms, 256 ms, or 1024 ms, the latter being the most sensitive for sufficiently long and smooth bursts. Below, we shall consider only the 1024 ms trigger threshold, and any burst whose peak flux (in 1024 ms, between 50-300 keV, in one detector) is at the detection threshold will be said to have $C_{\text{max}}/C_{\text{min}} = 1$. $C_{\text{max}}$ and $C_{\text{min}}$ are, respectively, the maximum detected count rate and the minimum detectable count rate. Any burst with an intrinsic flux such that $C_{\text{max}}/C_{\text{min}} = 1$ has a $\sim 50\%$ probability of triggering BATSE. (Noise fluctuations will reduce the observed flux below the threshold roughly half the time.) In order for the burst to trigger BATSE 100% of the time, the intrinsic flux must correspond to $C_{\text{max}}/C_{\text{min}} \approx 1.5$ (cf. BATSE trigger tables, Fishman et al. 1992).

The particular light curve used by us and Wambsganss (1993) is the summed 20-50 keV (channel 1) data of four BATSE detectors, making direct comparison between the burst rescalings used above and the true BATSE detection threshold difficult. In order to make these comparisons, we shall pretend that the single energy channel light curve that we used actually corresponds to a hypothetical complete light curve pieced together from all four available energy channels and two to four on-burst detectors. The signal to noise ratio of the light curve in the energy channels that trigger BATSE are typically less than the signal to noise ratio in the full light curve (channels 1-4 summed). We therefore must determine what fraction of our channel 1 light curves would correspond to the BATSE trigger if they were in reality complete light curves (2-4 detectors, channels 1-4, summed). To do this, we use the full light curve (i.e., summed over all energy channels) of GRB 910503 as an example of a “typical” gamma ray burst. We use this burst to determine the relationship between the signal to noise ratio in the trigger channels 2-3, and the (larger) signal to noise ratio in the full burst. We assume that our channel 1 light curves would show the same relationship if they were in reality complete light curves. We use these estimates as a means of assigning to the rescalings we performed above a $C_{\text{max}}/C_{\text{min}}$, which here shall always refer to the peak count rate (over 1024 ms) divided by the threshold in the 50-300 keV band of the second most illuminated detector.
Assuming 144 photons per 64 ms as the “canonical” background rate in the 50-300 keV energy band, the minimum detectable $5.5\sigma$ signal will be 264 photons per 1024 ms in one detector ($264 = 5.5\sqrt{1024\text{ms}/64\text{ms}} \times 144$). Realistically, we will have at least one more detector, and an additional 40% photon flux due to the 20-50 keV and 300+ keV band passes. (The 40% increase is based upon the spectrum of GRB 910503.) The minimum summed signal is therefore 740 photons per 1024 ms. As our background, we take a typical value to be 300 photons per 64 ms per detector (again, based upon the spectrum of the background of GRB 910503), which yields a Poisson noise level of 98 photons per 1024 ms (2 detectors, channels 1-4). The minimum signal to noise ratio in the summed burst is therefore $740/98 = 7.6$. For the channel 1 light curves that we used, the peak signal to noise ratio on the 1024 ms time scale is approximately $32300X_A/84 \approx 385 X_A$ for subburst A and $13300X_B/84 \approx 158 X_B$ for subburst B, where $X_A$ and $X_B$ are the factors by which the subbursts are scaled (84 photons per 1024 ms is the noise level in channel 1, and 32300 photons per 1024 ms and 13300 photons per 1024 ms are the peak count rates in this channel for subbursts A and B respectively). In terms of $C_{\text{max}}/C_{\text{min}}$, subbursts A and B would have $C_{\text{max}}/C_{\text{min}} \sim 50.7 X_A$ and $C_{\text{max}}/C_{\text{min}} \sim 20.8 X_B$, respectively, if these channel 1 light curves were in reality complete light curves (2 detectors, channels 1-4).

Comparing these results to our rescalings, we see that $C_{\text{max}}/C_{\text{min}} = 1$ (50% probability of detection) corresponds to A and B at (2%, 4.8%), and $C_{\text{max}}/C_{\text{min}} = 1.5$ ($\sim$100% probability of detection) corresponds to A and B at (3%, 7.2%). Note that this is for only two detectors. If the burst is seen in three detectors with roughly equal flux, then the signal to noise ratio in the summed burst light curve increases by a factor of $\sqrt{3}/2$. For this case, the 100% probability of detection ($C_{\text{max}}/C_{\text{min}} \approx 1.5$) corresponds to A and B at (3.6%, 8.8%). For four equally illuminated detectors, the $\sim$100% probability of detection corresponds to A and B at (6%, 9.6%). Wambsganss’s interpretation of A vs. B scaled at (2%, 8%) being at the BATSE detection threshold is something of a worst case scenario. A liklely lower limit to bursts that would trigger BATSE corresponds to A vs. B at (4%, 8%), which as we demonstrated before yields 34% of the comparisons being distinguishable by at least one of our statistical tests. Depending upon the energy spectra of individual burst light curves and the background, it may be possible to choose more optimal combinations of detectors and energy channels so as to achieve even greater signal to noise ratios.

We saw above that for A vs. B at (8%, 16%), each individual statistic was capable of distinguishing $\sim 80\%$ of the pairs. Therefore, $\sim 96\%$ of the pairs are distinguishable by at least one of the tests. This is the level of rescaling at which we have strong confidence in our tests. In terms of $C_{\text{max}}/C_{\text{min}}$, this rescaling corresponds to: $C_{\text{max}}/C_{\text{min}} \approx 4.1$ for A, $C_{\text{max}}/C_{\text{min}} \approx 3.3$ for B (two detectors); $C_{\text{max}}/C_{\text{min}} \approx 3.3$ for A, $C_{\text{max}}/C_{\text{min}} \approx 2.7$ for B (three detectors); $C_{\text{max}}/C_{\text{min}} \approx 2.0$ for A, $C_{\text{max}}/C_{\text{min}} \approx 1.7$ for B (four detectors). We therefore expect that when $C_{\text{max}}/C_{\text{min}} \lesssim 3$ for two intrinsically similar bursts, they will not be consistently distinguishable. For the 211 bursts from the burst catalogue
(Fishman et al. 1992) that have a known $C_{\text{max}}/C_{\text{min}}$ on the 1024 ms timescale, 59% have $C_{\text{max}}/C_{\text{min}} < 3$ and 45% have $C_{\text{max}}/C_{\text{min}} < 2$. Thus, roughly half of all bursts will not be consistently distinguishable, within the noise, from another similar, but different, light curve.

We should note that these limits apply to intrinsically similar bursts. The light curves of subbursts A and B are very similar on the coarsest time scales. Returning to Figures 2 and 3, we see that for the three of the four lowest frequencies (0.06 Hz, 0.12 Hz, 0.24 Hz) the PSD amplitudes for the two subbursts are nearly identical. For the three lowest frequencies, the CPD time delay is small and near zero. This is just a statement that the coarsest features (i.e., rise and decay times, broad spikes, etc.) are very similar from subburst A to B. These subbursts will appear to be statistically identical for burst rescalings wherein only the few lowest frequency bins are above the noise. The only way to determine how common such similar but distinct pairs are would be to compare pairs of bursts whose angular separations are much larger than the positional errors. If indistinguishable pairs are uncommon, then we would have greater confidence that an indistinguishable pair within the same error box was due to lensing.

Our statistics also require at least three frequency bins above the noise. The shortest Fourier transform that yields three or more distinct frequency channels is the 8 point FFT (we only consider transforms whose length are a power of 2, cf. Press et al. 1992). This FFT yields a PSD at the frequencies: $(1/4) f_{\text{ny}}, (1/2) f_{\text{ny}}, (3/4) f_{\text{ny}}, f_{\text{ny}}$; where $f_{\text{ny}}$ is the Nyquist frequency. The Nyquist frequency is the the highest frequency calculable by an FFT, and for an 8 point FFT it is given (in Hz) by $f_{\text{ny}} = 4/T$, where $T$ is the duration of the data. We expect that in order to have a well-defined FFT, half of our time bins must be above the noise. In Figure 1, we show the signal to noise ratio for subbursts A and B rebinned into 8 time bins. At 100% scaling, the bursts are well above the noise. However, when A is scaled to 4%, three time bins are clearly below the noise, two are marginal at 4-6 times the noise, and only three are well above the noise. When A is scaled to 2%, four time bins have signal to noise less than three (our usual criteria for a detectable signal), one is marginal at 3 times the noise, and only three are well above the noise. This reduction of signal to noise is reflected in the fact that only 73% of the A vs. B at (4%, 8%) comparisons have well-defined statistics, and only 19% of the A vs. B at (2%, 8%) have well-defined statistics. Again, the only way to know how often the statistic will be defined in practice is to compare a large number of real burst pairs. In the next section, we do this for a small number of lensing event candidates.

4. Application to Real Burst Pairs

As discussed above, in order to fully calibrate our statistic, or any other statistic that attempts to distinguish burst profiles, one should compare all possible burst pairs and see how many of these pairs are indistinguishable. Pairs that are well outside of each others
positional error boxes cannot be lensing events. Therefore the fraction of such events that are indistinguishable would give a good indication of the number of “false positives” among lensing candidates. With 260 bursts in the BATSE catalogue, and hence 33670 possible pairs, this is a daunting task. One could restrict the comparison to bursts of similar duration, such as only using pairs of bursts whose durations are within a factor of two of each other. A convenient measure of a burst’s duration is given by its $t_{50}$ and $t_{90}$ times, which are defined as the time intervals during which the burst’s integrated photon counts go from 25% to 75% and 5% to 95%, respectively, of the total photon counts (Fishman et al. 1992). With this restriction, which we adopt below, there are 1648 possible pairs, which is still a rather large number of comparisons.

Instead, we apply our statistic to the subset of the 1648 pairs where the positional error boxes of the pair overlap. Each burst’s positional error is taken to be $4^\circ$ (the BATSE systematic error) added in quadrature with its positional error as listed in the BATSE catalog (Fishman et al., 1992). The positional errors for the two bursts are then added in quadrature, giving us a total angular error between the two. Only 25 pairs of bursts with similar durations are separated by an angle less than this error. Table 3 lists these 25 pairs (45 separate bursts). They are potential lensing candidates. In Table 3, bursts are identified by their BATSE trigger number (Fishman et al. 1992). In addition we include two pairs—bursts (235, 1447) and (493, 1430)—that did not show up in our search of the BATSE catalogs. Burst 1430 is not listed in the duration tables, and the bursts (235, 1447) are farther apart than their positional errors. Both pairs, however, are listed in Nemiroff et al. (1993a) as potential lensing candidates. We include these pairs in order to compare our statistics to those of Nemiroff et al..

We have applied both of our statistics to each of the 27 pairs listed in Table 3. All the burst light curves are constructed from the sum of all energy channels and all on-burst detectors. We chose a length for our Fourier transforms such that the number of 64 ms time bins was the smallest power of 2 (cf. Press et al. 1993) such that the duration of the transform was longer than the $t_{90}$ time of the longest burst. The background for each burst was assumed to be a constant and was calculated by averaging 50 points beyond the end of each transform. This background was subtracted before performing the Fourier transforms. Several of the light curves had levels at the beginning of the data set that were inconsistent with the background at the end of the data set. This could be due to a time-varying background, or the beginning of the burst may be absent from the data set. (BATSE stores only 2.048 s before each trigger, so bursts that have durations longer than this before triggering may have portions of the light curve missing.) These bursts are noted with a superscript $b$ in Table 3.

Our statistics are applicable to 19 out of the 27 pairs, and all of these pairs are distinguishable. The most similar pair are bursts (235, 1447), which, although indistinguishable by the CPD statistic, have a formal probability of only $10^{-3}$ of being identical.
to within noise by the PSD statistic. This pair was indistinguishable by Nemiroff et al. (1993a), but was not seriously considered to be a lensing event due to the large angular separation between the two and the fact that one of the bursts (235) was found to have a position-consistent precursor in other BATSE data. Pair (493, 1430), which Nemiroff et al. (1993a) calculated as having only a probability of \(10^{-2}\) of being statistically identical, is distinguished with even greater confidence by our statistics. Although this pair is indistinguishable by the CPD statistic, the PSD statistic gives a probability of only \(6 \times 10^{-6}\) of the pair being statistically identical. All of the remaining 17 pairs to which we were able to apply our statistics are distinguishable with greater than 3 \(\sigma\) confidence.

We were not able to apply our statistics to 8 pairs. One of these pairs, bursts (942, 1298), was too noisy to apply our statistics, having only one frequency channel (yielding -1 degrees of freedom) above the noise. The remaining 7 pairs were too short to compare. At least one burst within each of these pairs had three or fewer time bins more than 3 \(\sigma\) above the noise. As discussed above, we require a light curve with at least 4 time bins clearly above the noise, to which we can apply an 8 (or more) point FFT. It is not likely that any statistic based upon the temporal profile alone will be able to adequately distinguish bursts with only 3 time bins above the noise. Fortunately, 90\% of the bursts have \(t_{90}\) times greater than 0.256 ms (i.e. four 64 ms time bins).

As an example of the difficulty of distinguishing short bursts, we show pair (138, 444), which is completely consistent with lensing. Not only are the light curves nearly identical, the bursts are also achromatic, as is shown in Figures 7a,b. In Figure 7a we plot the two bursts, summed over all detectors and energy channels, on top of one another. Burst 444 is scaled so that its peak is identical to the peak of burst 138. In Figure 7b we show the color-color diagrams for each time bin that is at least 2 \(\sigma\) above the noise. (We loosen our signal to noise criteria somewhat so as to have more time bins to compare.) The colors are defined as the ratios of energy (channel 2)/(channel 1) and (channel 3)/(channel 2). The overlapping time bins have colors that are consistent within the noise. The brighter of the two bursts has a third time bin with well defined colors, but its counterpart does not. Being that the brighter burst arrives after the fainter burst, this pair cannot be produced by a circular lens (cf. Narayan & Wallington 1992), but could be consistent with an elliptical lens. We emphasize, however, that this pair is only consistent with lensing because of a lack of information, not because of abundant similarities. Thus, we do not consider it a lensing candidate. This example points out the difficulties of being certain that there are no lensing events in the burst catalogue, except for the subset with durations \(\gtrsim 0.3\) s.

5. Summary

If lensed gamma ray burst light curves are to be used to demonstrate the cosmological origin of GRBs, we must be able to identify such events. As Wambsganss (1993) has pointed out, such events may be difficult to distinguish from two noisy light curves that
Figure 7a,b: a) Top: Background subtracted light curves for BATSE triggers 138 and 444. Burst 444 is scaled to have the same peak amplitude as burst 138. Numbered points correspond to points in the color-color diagram to the right. b) Bottom: Color-color diagram for bursts 138 and 444. The vertical axis color corresponds to BATSE energy (channel 1)/(channel 2) \[ (20-50 \text{ keV})/(50-100 \text{ keV}) \]. The horizontal axis color corresponds to BATSE energy (channel 2)/(channel 3) \[ (50-100 \text{ keV})/(100-300 \text{ keV}) \]. Error bars represent 1 \( \sigma \) uncertainties.
are similar but distinct. In this work, we developed two statistical tests that are capable of distinguishing noisy light curves. These tests used Fourier transforms to compare bursts. Comparisons were made at frequency channels that are not noise dominated. If the light curves had, to within the noise, comparable Fourier amplitudes and phases consistent with a constant time delay, they were said to be statistically identical.

We then applied these statistics to two subbursts of GRB 910503, the burst that Wambsganss (1993) considered. We showed that as long as the subbursts were not scaled to \( \lesssim 10\% \) of their original amplitude, they were distinguishable. Just as Wambsganss (1993) noted, these bursts became harder to distinguish as they were further reduced. However, our results were not as pessimistic, since we found that bursts that were only 50\% above the BATSE detection threshold were still distinguishable one third of the time. In addition to our tests being slightly more successful than Wambsganss (1993), they are more quantitative in that they give a significance value of the burst differences and an indication of the information content of the comparison (through the degrees of freedom, the number of frequency channels above noise minus 2). We also interpreted our results in terms of the BATSE detection thresholds. These intrinsically similar subbursts were nearly 100\% distinguishable for \( C_{\text{max}}/C_{\text{min}} \approx 3 \) (i.e., three times the detection threshold), 34\% distinguishable for \( C_{\text{max}}/C_{\text{min}} \approx 1.5 \), and 5\% distinguishable for \( C_{\text{max}}/C_{\text{min}} \approx 1 \).

We then applied our statistics to 27 lensing candidate burst pairs, 19 of which were found to be easily distinguishable. Of the remaining 8 pairs, one was too noisy to apply our statistics to, and the others were too short, having fewer than four time bins above the noise. By this duration criterion, our statistic should be applicable to more than 90\% of all bursts. These calculations could be improved in several ways. In addition to the 64 ms time binning, BATSE “time tags” photon arrival times with a precision of 2 ms (Fishman et al. 1992). Thus, the brighter short duration bursts may be able to be examined at a higher time resolution, allowing more detailed comparisons. Our statistics being inapplicable to 10\% of all bursts therefore represents a worst case. We have also taken the simplest approximation to the burst background, that is a constant. More detailed modelling may allow us to distinguish the bursts labelled \( b \) in Table 3 with even greater confidence. Finally, we have not checked [with the exception of bursts (138, 444)] that the burst pairs are achromatic. This undoubtedly will only enhance our ability to distinguish light curves (perhaps even dramatically, according to some preliminary tests [Nemiroff 1994b]).

We are less pessimistic than Wambsganss (1993), and believe that a substantial fraction of the observed bursts should be distinguishable from one another. The best test of the sensitivity of our statistics (or any statistic that attempts to distinguish GRB light curves) would be to apply it to all burst pairs that do not have overlapping error boxes. Due to the enormity of this task (over one thousand pairs), we have not attempted such a calibration. The question then remains: how many unidentified lensed pairs should exist in the BATSE data? Recent results of Grossman & Nowak (1994) indicate that if the only lenses are galaxies, then it is very unlikely that there are or will be any events observed by BATSE. However, other possibilities, such as lensing by massive black holes or compact dark matter, may produce observable events (Mao 1992, Blaes & Webster 1992). Even if a lensing event is unlikely, the implications of one’s discovery would be profound. The search is therefore worthwhile, and the techniques described above can aid in this search.
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Appendix A. Fourier Analysis in the Presence of Noise

The light curves that BATSE measures will be comprised of an intrinsic signal (which we assume to be scaled by lensing), and background, with noise superimposed on top of these components. In this Appendix, we compute the statistical properties of noise in the Fourier domain, and we define a normalization for our Fourier transforms such that the Fourier amplitudes are independent, to within the noise, of any uniform scaling of the intrinsic signal. Detailed derivations of the probability distribution of power spectral densities (PSDs, i.e., squared Fourier amplitudes) in the presence of Poisson noise can be found in Leahy et al. (1983) and Groth (1975). These references use slightly different PSD normalizations (wherein the level of the noise is fixed at a constant value). Here we derive the statistics for our choice of normalization.

The Noise PSD

We have an intrinsic signal which we call \( x^I(t) \) (for our purposes, photon counts per second). Assuming that we evenly sample this signal at \( N \) time bins, \( t_k \), of duration \( \Delta t \), we obtain a series of \( N \) measured signals \( m_k \) (photon counts). The measured counts will be different from the intrinsic counts, \( n_k \approx x^I(t_k)\Delta t \), due to counting noise. Data obtained from the BATSE experiment is comprised of the sum of 1–4 energy channels and 2–4 “on burst” detectors; therefore, by the central limit theorem, the measured signal is gaussian distributed about \( n_k \) with variance \( n_k \), according to

\[
P_k(m_k) = \frac{1}{\sqrt{2\pi n_k}} e^{-\frac{(m_k-n_k)^2}{2n_k}} .
\]  

(A.1)

To study the data in the Fourier domain, we take the discrete transform of the measured data.

\[
X^m(f_j) = \sum_{k=0}^{N-1} m_k e^{2\pi if_j t_k} \Delta t ,
\]  

(A.2)
where \( f_j = j/(N\Delta t) \), and \( j = -N/2, -N/2 + 1, \ldots, N/2 \). The transform of the intrinsic signal is

\[
X^I(f_j) = \sum_{k=0}^{N-1} n_k e^{2\pi i f_j t_k} \Delta t .
\]  

(A.3)

The PSD of the noise is simply the difference between the true and measured PSD, \( X^m(f_j)^2 - X^I(f_j)^2 \). Expanding the PSDs, we find for the measured signal

\[
X^m(f_j)^2 = \sum_{k=0}^{N-1} m_k^2 \Delta t^2 + \sum_{k,l=0, k \neq l}^{N-1} m_k m_l e^{2\pi i f_j (t_k - t_l)} \Delta t^2 ,
\]  

(A.4)

and likewise for the intrinsic signal. Averaging the measured signal over the probability distribution, keeping in mind that from time bin to time bin the noise is uncorrelated, we find

\[
\langle X^m(f_j)^2 \rangle = \left[ \prod_k P_k(m_k) \right] X^m(f_j)^2
\]

\[
= \sum_{k=0}^{N-1} \left[n_k^2 + n_k\right] \Delta t^2 + \sum_{k,l=0, k \neq l}^{N-1} n_k n_l e^{2\pi i f_j (t_k - t_l)} \Delta t^2 .
\]  

(A.5)

The first term of the first summation and the second summation together are \( X^I(f_j)^2 \), so that

\[
\langle X^m(f_j)^2 \rangle - X^I(f_j)^2 = \sum_{k=0}^{N-1} n_k \Delta t^2 = \left\langle \sum_{k=0}^{N-1} m_k \Delta t^2 \right\rangle .
\]  

(A.6)

This difference is the mean power spectral density due to noise, and is simply \( N_\gamma \Delta t^2 \) where

\[
N_\gamma = \sum_{k=0}^{N-1} n_k = \left\langle \sum_{k=0}^{N-1} m_k \right\rangle ,
\]  

(A.7)

is the total photon count. Note that the mean PSD due to noise is constant, independent of frequency.

**PSD Normalization**

We choose a one-sided PSD normalization (cf. Press et al. 1992) such that the intrinsic power spectral density is given by

\[
X^{II}(f_j) \equiv 2 \frac{N X^I(f_j)^2}{N_\gamma^2 \Delta t} ,
\]  

(A.8)

where we recall that \( N \) is the number of time or frequency bins. With this normalization, summing the PSD over frequency yields the mean square signal divided by the square of
the mean signal, $\langle n_k^2 \rangle / \langle n_k \rangle^2$. This PSD is therefore independent of any uniform scaling of the total signal, source plus background.

In the above, $n_k$ is the intrinsic photon count, which consists of both a signal from a source, $n_k^s$, and the signal from the background, $b_k$. We wish to modify our normalization so that the PSD is independent of any uniform scaling of the source signal alone. This is done simply by replacing $N_\gamma$ in (A.8) with

$$N_s = \sum_{k=0}^{N-1} n_k - b_k = \left\langle \sum_{k=0}^{N-1} m_k - b_k \right\rangle , \quad (A.9)$$

where $N_s$ is the source photon count. (We subtract the background from the signal before performing the transform.)

The noise level in our PSD will still be based upon the sum of the source and the background. Combining (A.7) with our normalization gives the noise PSD as

$$2\Delta t \frac{\langle n_k \rangle}{\langle n_k - b_k \rangle^2} . \quad (A.10)$$

This is the expression that we use to determine our noise level when calculating our optimal filters (§2) and for determining when the signal to noise level is $\geq 3$. When the background count rate, $b_k$, dominates the total count rate, $n_k = n_k^s + b_k$, then a factor of ten decrease in the source count rate leads to a factor of one hundred increase in the level of the noise PSD. This behavior is seen in Figure 2.
REFERENCES

Abramowitz, M., & Stegun, I. A. 1972, Handbook of Mathematical Functions, (New York:Dover)
Blaes, O. M. 1994, ApJS, in press
Blaes, O. M., & Webster, R. L. 1992, ApJ, 391, L63
Davenport, W. B., Jr., & Root, W. L. 1958, An Introduction to the Theory of Random Signals and Noise, (New York:McGraw-Hill)
Fenimore, E. E., et al. 1993, Nature, 366, 40
Fishman, G. J., et al. 1992, BATSE Burst Catalog, GROSSC
Groth, E. J. 1975, ApJS, 29, 285
Leahy, D. A., et al. 1983, ApJ, 266, 160
Gould, A. 1992, ApJ, 386, L5
Grossman, S. A., & Nowak, M. A. 1994, ApJ, submitted
Hartmann, D. H. 1994, to be published in “High Energy Astrophysics”, ed. J. Matthews (World Scientific)
Mao, S. 1992, ApJ, 389, L41
Mao, S. 1993, ApJ, 402, 382
Narayan, R., & Wallington, S. 1992, ApJ, 399, 368
Nemiroff, R. J., et al. 1993a, Proceedings of the Compton Gamma-Ray Observatory Conference, (New York:American Institute of Physics), p. 974
Nemiroff, R. J., et al. 1993b, ApJ, 414, 36
Nemiroff, R. J. 1994a, Comm. on Astr., to be published
Nemiroff, R. J. 1994b, private communication
Paczyński, B. 1986, ApJ, 308, L43
Paczyński, B. 1987, ApJ, 317, L51
Press, W., et al. 1992, Numerical Recipes (Cambridge: Cambridge University Press)
Wambsganss, J. 1993, ApJ, 406, 29
Wang, J. L. 1994, Comm. on Astr., 17, in press
Whalen, A. D. 1971, Detection of Signals in Noise, (New York:Academic Press)
### Table 1: Results of Power Spectra Simulations

| Burst a | Burst b | $\bar{\nu}$ | % > $1\sigma$ | % > $2\sigma$ | % > $3\sigma$ |
|---------|---------|-------------|--------------|--------------|--------------|
| A at 32% | B at 32% | 4.5         | 100.0        | 100.0        | 100.0        |
| A at 8%  | B at 16% | 2.2         | 99.6         | 96.8         | 81.2         |
| A at 4%  | B at 8%  | 1.2         | 94.4 (69.2)  | 61.7 (45.2)  | 25.6 (18.8)  |
| A at 2%  | B at 8%  | 1.1         | 82.9 (16.0)  | 37.3 (7.2)   | 7.8 (1.5)    |

| Burst a | Burst b | $\bar{\nu}$ | % > $1\sigma$ | % > $2\sigma$ | % > $3\sigma$ |
|---------|---------|-------------|--------------|--------------|--------------|
| A at 32% | A at 32% | 16.0        | 20.8         | 2.7          | 0.1          |
| A at 8%  | A at 8%  | 7.9         | 53.2         | 12.8         | 1.3          |
| A at 4%  | A at 4%  | 2.5         | 65.5 (64.4)  | 23.4 (23.0)  | 7.2 (7.1)    |
| A at 2%  | A at 2%  | 1.1         | 75.2 (8.8)   | 28.2 (3.3)   | 9.4 (1.1)    |

| Burst a | Burst b | $\bar{\nu}$ | % > $1\sigma$ | % > $2\sigma$ | % > $3\sigma$ |
|---------|---------|-------------|--------------|--------------|--------------|
| B at 32% | B at 32% | 3.4         | 62.8         | 19.3         | 5.1          |
| B at 16% | B at 16% | 1.7         | 78.8         | 32.0         | 10.4         |
| B at 8%  | B at 8%  | 1.0         | 83.9 (65.6)  | 34.5 (27.0)  | 7.4 (5.8)    |

Table 1: Percentage of simulations above a given confidence limit, based on 1000 simulations of the PSD statistic. The $1\sigma$, $2\sigma$, and $3\sigma$ thresholds correspond to the 68%, 95.5%, and 99.7% levels for a reduced $\chi^2$ distribution with $\bar{\nu}$ degrees of freedom. Percentages are based upon those simulations with well defined statistics ($\leq 1000$). Numbers in parentheses are based on all 1000 simulations.

### Table 2: Results of Cross Power Spectra Simulations

| Burst a | Burst b | $\bar{\nu}$ | % > $1\sigma$ | % > $2\sigma$ | % > $3\sigma$ |
|---------|---------|-------------|--------------|--------------|--------------|
| A at 32% | B at 32% | 4.5         | 100.0        | 100.0        | 100.0        |
| A at 8%  | B at 16% | 2.2         | 90.0         | 84.4         | 78.8         |
| A at 4%  | B at 8%  | 1.2         | 57.3 (42.0)  | 27.0 (19.8)  | 11.6 (8.5)   |
| A at 2%  | B at 8%  | 1.1         | 63.2 (12.2)  | 28.5 (5.5)   | 9.8 (1.9)    |

| Burst a | Burst b | $\bar{\nu}$ | % > $1\sigma$ | % > $2\sigma$ | % > $3\sigma$ |
|---------|---------|-------------|--------------|--------------|--------------|
| A at 32% | A at 32% | 16.0        | 0.1          | 0.0          | 0.0          |
| A at 8%  | A at 8%  | 7.9         | 4.0          | 0.1          | 0.0          |
| A at 4%  | A at 4%  | 2.5         | 28.5 (28.0)  | 3.9 (3.8)    | 0.4          |
| A at 2%  | A at 2%  | 1.1         | 51.3 (6.0)   | 12.0 (1.4)   | 2.6 (0.3)    |

| Burst a | Burst b | $\bar{\nu}$ | % > $1\sigma$ | % > $2\sigma$ | % > $3\sigma$ |
|---------|---------|-------------|--------------|--------------|--------------|
| B at 32% | B at 32% | 3.4         | 16.7         | 0.6          | 0.0          |
| B at 16% | B at 16% | 1.7         | 28.4         | 2.2          | 0.2          |
| B at 8%  | B at 8%  | 1.0         | 46.0 (36.0)  | 3.7 (2.9)    | 0.0          |

Table 2: Percentage of simulations above a given confidence limit, based on 1000 simulations of the PSD statistic. The $1\sigma$, $2\sigma$, and $3\sigma$ thresholds correspond to the 68%, 95.5%, and 99.7% levels for a reduced $\chi^2$ distribution with $\bar{\nu}$ degrees of freedom. Percentages are based upon those simulations with well defined statistics ($\leq 1000$). Numbers in parentheses are based on all 1000 simulations.
Table 3: Statistical Comparison of Burst Light Curves

| A  | B   | Δt (days) | N_f  | ν    | $\langle \chi^2/\nu \rangle_{PSD}$ | $Q_{PSD}$ | $\langle \chi^2/\nu \rangle_{CPD}$ | $Q_{CPD}$ |
|----|-----|-----------|------|------|-------------------------------|----------|-------------------------------|----------|
| 111 | 114 | 0.5       | 2048 | 10   | 10.3                          | $1 \times 10^{-17}$ | 37.3                          | 0.00     |
| 133 | 1152| 221.      | 4096 | 10   | 60.5                          | 0.00      | 82.9                          | 0.00     |
| 138 | 444 | 54.8      | –    | –    | –                             | –         | –                             | –        |
|     | 575 | 83.6      | –    | –    | –                             | –         | –                             | –        |
| 179 | 555 | 69.4      | 256  | 14   | 5.8                           | $2 \times 10^{-11}$ | 4.1                           | $4 \times 10^{-7}$ |
| 223 | 678 | 82.8      | 1024 | 4    | 4.3                           | $2 \times 10^{-3}$ | 5.3                           | $3 \times 10^{-4}$ |
|     | 824 | 126.      | 1024 | 1    | 18.6                          | $2 \times 10^{-5}$ | 16.9                          | $4 \times 10^{-5}$ |
|     | 1456| 283.      | 1024 | 3    | 4.6                           | $3 \times 10^{-3}$ | 22.3                          | $2 \times 10^{-14}$ |
| 235 | 1447| 280.      | 512  | 2    | 6.9                           | $1 \times 10^{-3}$ | 0.9                           | 0.43     |
| 249 | 469 | 28.5      | 1024 | 21   | 137                           | 0.00      | 190.                          | 0.00     |
| 404 | 1318| 221.      | 4096 | 9    | 50.0                          | 0.00      | 108.                          | 0.00     |
| 444 | 568 | 26.2      | -    | -    | -                             | -         | -                             | -        |
| 480 | 1308| 202.      | -    | -    | -                             | -         | -                             | -        |
| 493 | 1430| 231.      | 128  | 2    | 11.9                          | $6 \times 10^{-6}$ | 0.1                           | 0.89     |
| 503 | 676 | 36.0      | 2048 | 5    | 786                           | 0.00      | 36.7                          | $1 \times 10^{-37}$ |
| 547 | 1128| 135.      | -    | -    | -                             | -         | -                             | -        |
| 549 | 398 | 28.2      | 512  | 5    | 7.6                           | $4 \times 10^{-7}$ | 13.2                          | $8 \times 10^{-13}$ |
|     | 741 | 42.5      | 512  | 3    | 8.9                           | $7 \times 10^{-6}$ | 6.4                           | $3 \times 10^{-4}$ |
| 559 | 1384| 204.      | 1024 | 3    | 9.5                           | $3 \times 10^{-6}$ | 18.3                          | $7 \times 10^{-12}$ |
| 606 | 1167| 133.      | 512  | 5    | 6.3                           | $7 \times 10^{-6}$ | 6.2                           | $9 \times 10^{-6}$ |
| 809 | 1461| 165.      | -    | -    | -                             | -         | -                             | -        |
| 906 | 1128| 44.2      | -    | -    | -                             | -         | -                             | -        |
| 942 | 1298| 79.7      | 128  | -1   | -                             | -         | -                             | -        |
| 1086 | 1087 | 0.2     | 512  | 3    | 15.4                          | $5 \times 10^{-10}$ | 54.4                          | $4 \times 10^{-35}$ |
| 1125 | 1303 | 49.8   | 512  | 4    | 26.6                          | $3 \times 10^{-17}$ | 16.2                          | $2 \times 10^{-10}$ |
| 1126 | 1452 | 93.3   | 512  | 4    | 25.7                          | $2 \times 10^{-21}$ | 6.2                           | $5 \times 10^{-5}$ |
| 1167 | 1200 | 8.1    | 512  | 1    | 21.6                          | $3 \times 10^{-6}$ | 1.4                           | 0.24     |

*a* Burst has less than 4 time bins 3 σ above noise.

*b* Beginning of burst may be missing from data set.

Table 3: Summary of statistical comparison between burst light curves. Bursts were chosen to lie within each others positional error boxes, as well as have similar durations (i.e. the $t_{50}$ and $t_{90}$ times [cf. Fishman et al. 1992] of one burst were within a factor of 2 of those for the other). Columns A and B identify bursts by their BATSE trigger number. A long dash indicates an entry identical to the preceding one. $\Delta t$ is the time delay (in days) between bursts. $N_f$ is the number of Fourier frequency channels used in the comparison. $\nu$ is the degrees of freedom (defined as the number of frequency channels above noise minus 2). $\langle \chi^2/\nu \rangle_{PSD}$ and $\langle \chi^2/\nu \rangle_{CPD}$ are the reduced $\chi^2$’s derived from the power spectral density and cross power spectral density, respectively. $Q_{PSD}$ and $Q_{CPD}$ are the associated probabilities that two statistically identical signals would show reduced $\chi^2$ as large or larger than those derived (assuming they obey a reduced $\chi^2$ distribution).
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