Deviation inequalities for centered additive functionals of recurrent Harris processes having general state space

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Abstract

Let $X$ be a Harris recurrent strong Markov process in continuous time with general Polish state space $E$, having invariant measure $\mu$. In this paper we use the regeneration method to derive non asymptotic deviation bounds for

$$P_x \left( \left| \int_0^t f(X_s) ds \right| \geq t^{\frac{1}{2} + \eta \varepsilon} \right)$$

in the positive recurrent case, for nice functions $f$ with $\mu(f) = 0$ ($f$ must be a charge). We generalize these bounds to the fully null-recurrent case in the moderate deviations regime. We obtain a Gaussian concentration bound for all functions $f$ which are a charge. The rate of convergence is expressed in terms of the deterministic equivalent of the process. The main ingredient of the proof is Nummelin splitting in continuous time which allows to introduce regeneration times for the process on an enlarged state space.

Key words: Harris recurrence, Nummelin splitting, continuous time Markov processes, special functions, additive functionals, deviation inequalities, deterministic equivalent for additive functionals.

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1 Introduction

Consider a Harris recurrent strong Markov process $X = (X_t)_{t \geq 0}$ in continuous time with invariant measure $\mu$, taking values in a Polish space $E$.

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If the total mass of $\mu$ is finite, $X$ is called positive recurrent, null-recurrent otherwise. In the case of positive recurrence it is well known that for certain functions $f$ such that $\mu(f) = 0$ we have a central limit theorem for
\[
\left( \frac{1}{\sqrt{n}} \int_0^{nt} f(X_s)ds \right)_{t \geq 0}^{n \rightarrow \infty}
\]
as $n$ goes to $\infty$, see for instance H"opfner and L"ocherbach [14] and Touati [28].

In the null-recurrent case, the re-normalization $\sqrt{n}$ of (1.1) has to be changed. For that sake we have to consider what is called deterministic equivalent of additive functionals. The deterministic equivalent has been introduced for Markov chains by Chen [5] and then been generalized to the context of continuous time diffusion models by Loukianova and Loukianov [20] and to any continuous time recurrent Markov process by L"ocherbach and Loukianova in [16]. It is a deterministic function $t \mapsto v(t)$ such that $v(t) \rightarrow \infty$ as $t \rightarrow \infty$ and such that for any integrable additive functional $A_t$,
\[
\lim_{M \rightarrow \infty} \liminf_{t \rightarrow \infty} P_\pi \left( \frac{1}{M} \leq A_t/v(t) \leq M \right) = 1
\]
for any initial measure $\pi$. $v(t)$ can be defined as follows. Take any fixed positive special function $g$ of the process having $\mu(g) > 0$ (see definition [24] below for the exact definition of special functions, for strong Feller processes, any bounded function having compact support is special) and define
\[
v(t) := E_\eta \left( \int_0^t g(X_s)ds \right),
\]
where $\eta$ is an arbitrary initial measure. Then the strong Chacon-Ornstein theorem implies that for any other special function $g'$ and any other initial measure $\eta'$,
\[
\lim_{t \rightarrow \infty} \frac{E_\eta \left( \int_0^t g(X_s)ds \right)}{E_{\eta'} \left( \int_0^t g'(X_s)ds \right)} = \frac{\mu(g)}{\mu(g')}
\]
Hence the deterministic equivalent is unique up to a constant in the sense that for two choices of the deterministic equivalent, $v$ and $v'$, we have that $\lim_{t \rightarrow \infty} v(t)/v'(t) = c$, where $c$ is a positive constant. In regular models, $v(t) \sim t^{\alpha} \cdot l(t)$, where $l$ is a function that varies slowly at infinity. For example, for Brownian motion in dimension one, we have $\alpha = 1/2$. The explosion rate $v_t$ is in general slower than the ergodic (positive recurrent) rate $t$.

In the null-recurrent regular case, we have convergence in law of
\[
\left( \frac{1}{\sqrt{v(n)}} \int_0^{nt} f(X_s)ds \right)_{t \geq 0}^{n \rightarrow \infty}
\]
to $B \circ W^\alpha$, where $B$ is a one-dimensional Brownian motion and $W^\alpha$ the Mittag-Leffler process of index $\alpha$, $0 < \alpha \leq 1$, for certain functions $f$ having $\mu(f) = 0$, see Touati [28] and H"opfner and L"ocherbach [14] for similar results for martingale additive functionals. In the fully null-recurrent case (i.e. null-recurrent, but not regular) no convergence in law holds true, which is a consequence of the famous theorem of Darling-Kac.

Continuing in the spirit of the limit theorems of (1.1) and (1.2), the aim of this paper is to study deviation inequalities for additive functionals $\int_0^{nt} f(X_s)ds$ for a certain class of centered functions.
$f$ (the class of bounded special functions that will be introduced below). More precisely we study the deviation bounds for

$$P_x \left( | \int_0^t f(X_s) ds | \geq t^{\frac{1}{2} + \eta} x \right)$$

for $\eta > 0$ in the positive recurrent case, and in the general null-recurrent case, including the non-regular case, of

$$P_x \left( | \int_0^t f(X_s) ds | \geq v(t)^{\frac{1}{2} + \eta} x \right).$$

Thus we are interested in deviation inequalities in the moderate deviations regime. We will work in the most general situation, our results hold for any recurrent Markov process taking values in any Polish state space, without restrictions on the quality of recurrence. As a counterpart of this general approach, we have to restrict attention to the class of test functions which are the bounded centered specials functions. In the case of strong Feller processes, that means that we consider centered functions being of compact support.

Our approach is based on the regeneration method. Regeneration times allow to split the trajectory of the process into i.i.d. excursions. In the one-dimensional case, such regeneration times are usually introduced as successive visits to recurrent points. Our aim is, however, to work in the frame of a general state space. In this general frame, recurrent points usually do not exist, and we use Nummelin splitting in continuous time, as developed in L"ocherbach and Loukianova [16], in order to overcome this difficulty. This technique allows to introduce a recurrent atom for the process on an enlarged state space and thus to mimic the idea of recurrent points in higher dimensions. Even if the technical details concerning Nummelin splitting are somehow cumbersome, the evident advantage of this method is that it is conceptually very easy (regeneration means that we are able to work with i.i.d. variables) and that it works in any dimension and for any state space.

For positive recurrent processes $X$, we obtain the following deviation upper bound. For any bounded special function $f$ with $\mu(f) = 0$, we have

$$P_\pi \left( | \int_0^t f(X_s) ds | \geq t^{\frac{1}{2} + \eta} x \right) \leq C_1 \exp \left( -C_2 t^{2\eta} (x^2 \wedge x) \right) + R_t(x), \quad (1.3)$$

where $\pi$ is any initial measure, for any $0 < \eta \leq \frac{1}{2}$. Here, $C_1$ and $C_2$ are explicit positive constants. $R_t(x)$ is a remainder term which is explicitly known and which is of order $O(\exp \left( -\sqrt{t}(x \vee 1) \right))$. In the general null-recurrent case, including the non-regular case, we have

$$P_\pi \left( | \int_0^t f(X_s) ds | \geq v(t)^{\frac{1}{2} + \eta} x \right) \leq C_1 \exp \left( -C_2 v(t)^{\eta} (x^2 \wedge x) \right) + R_t(x), \quad (1.4)$$

$R_t$ being of order $O(\exp \left( -\sqrt{v(t)}(x \vee 1) \right))$. Note that (1.3) and (1.4) yield Gaussian concentration in $x$. This is due to the fact that we consider functions $f$ which are bounded and special. For this class of functions, the contribution of the additive functional up to the first regeneration time can be controlled uniformly in the the starting point of the process, and it is this uniform dependence that is the basis of the Gaussian behaviour. In particular, for uniformly contracting Markov processes, Gaussian concentration is obtained for any bounded function $f$ that need not be of compact support.
and (1.4) represent a first and important step towards the study of moderate deviations in the fully general case of any recurrent process. The problem of obtaining such non-asymptotic bounds is of major importance for many applications. Let us cite just some of them: model selection or other non asymptotic problems for statistics of Harris processes, particle approximations of Gibbs measures, .... Our work is strongly motivated by applications to statistics (see for example [19]), in particular model selection, and it is important for such applications to obtain bounds that are valid for fixed \( t \) such that the constants involved are as explicit as possible. This is what we try to achieve in this paper. Note also that our results hold in any dimension and under any starting measure \( P_\pi \).

Our deviation inequalities are stated in the moderated deviations regime. Note that the moderate deviations for additive functionals have been widely studied firstly in the case of discrete time, i.e. for Markov chains. We refer the reader to the work of Guillin ([12]), Djellout and Guillin ([8]), Chen and Guillin ([6]) for a survey on the subject. In the case of time continuous observations, less results are known. In the ergodic situation, Guillin and Liptser ([13]) have studied the moderate deviations of \( \int_0^t f(X_s) ds \) where \( X \) is a multidimensional ergodic diffusion and \( f \) any centered function belonging to \( L^1(\mu) \). They use techniques of the stochastic calculus which are well-adapted to the particular diffusion case. Douc, Fort and Guillin ([9]) seem to be the first who deal with the general case of bounded additive functionals in the continuous time situation, but under assumptions implying sub-exponential ergodicity of the process (and in particular, positive recurrence) and assuming irreducibility of some skeleton chain (i.e. assuming minorization of \( P_m \) for some \( m \) on a petite set). All these results have been achieved in the positive recurrent case, specifying the quality of ergodicity – whereas in the present paper we work in the fully general recurrent but not necessarily positive recurrent case.

Finally, let us cite a recent work by Loukianova et al. [18] dealing with the same type of questions in the framework of positive recurrent diffusions in dimension one. In this case regeneration times are hitting times, and a precise control of these hitting times is provided in [18]. These kinds of results are not available within the general frame of the present article.

The paper is organized as follows. Since we are using heavily the method of Nummelin splitting and the concept of deterministic equivalent, section 2 gives a review of all known results concerning this technique that will be needed in the sequel. Section 3 gives the main results, for both positive and null-recurrent cases (Theorem 3.1). In the case of null-recurrent but regular models, i.e. in the case when \( v_t \sim t^\alpha \) for some \( 0 < \alpha < 1 \), some finer control of the Laplace transform of the length of the regeneration time yields better results than in the general null-recurrent case (see Theorem 3.3). Sections 4 and 5 are devoted to the proofs. Here, we also state a technical proposition which is a sort of generalization of Kac’s formula, but more cumbersome in the context of Nummelin splitting (Proposition 4.1). Finally, section 6 gives an idea of possible applications of our results in the framework of some interacting particle systems.

## 2 Notation

Consider a probability space \((\Omega, \mathcal{A}, (P_x)_x)\). Let \( X = (X_t)_{t \geq 0} \) be a process defined on \((\Omega, \mathcal{A}, (P_x)_x)\) which is strong Markov, taking values in a locally compact Polish space \((E, \mathcal{E})\), with càdàg
paths. $X_0 = x P_x$—almost surely. We write $L$ for the generator and $(P_t)_t$ for the semi group of $X$ and we suppose that $X$ is recurrent in the sense of Harris, with invariant measure $\mu$, unique up to multiplication with a constant. This means that for any set $A \in \mathcal{E}$ such that $\mu(A) > 0$, $\limsup_{t \to \infty} 1_A(X_t) = 1$ almost surely. Moreover, we shall write $(\mathcal{F}_t)_t$ for the filtration generated by the process.

We impose the following condition on the transition semi-group $(P_t)_t$ of $X$:

**Assumption 2.1** There exists a sigma-finite positive measure $\Lambda$ on $(\mathcal{E}, \mathcal{E})$ such that for every $t > 0$, $P_t(x, dy) = p_t(x, y)\Lambda(dy)$, where $(t, x, y) \mapsto p_t(x, y)$ is jointly measurable.

**Example 2.2**

1. In general, it is difficult to check whether a given Markov process is recurrent or not, and the most used criterion for recurrence is the existence of a so-called Lyapunov-function for the generator of the process, see for example Meyn and Tweedie ([21], [22], [23]).

   We say that $V \in \text{dom}(L)$ is a Lyapunov-function, if $V \geq 1$ and if there exists a constant $a > 0$, a constant $b$ and a closed petite set $C$ such that for all $x$,

   $$LV(x) \leq -aV(x) + b1_C(x).$$

   The existence of Lyapunov-functions implies exponential ergodicity. This concept can be extended to obtain slower rates of convergence, still in the positive recurrent case, see Douc, Fort and Guillin [9].

2. In the context of interacting particle systems, (see Galves et al. ([11] and section 6 below), recurrence can be shown via arguments using the dual process of the system.

### 2.1 On the deterministic equivalent of additive functionals

In the general recurrent not necessarily positive recurrent case, rates of convergence of additive functionals are given by what is called *deterministic equivalent* of additive functionals. This object has been introduced for Markov chains by Chen [5] and then been generalized to the context of continuous time diffusion models by Loukianova and Loukianov [20] and to any continuous time recurrent Markov process by Löcherbach and Loukianova in [16]. In the context of one dimensional diffusion models, similar ideas to the notion of *deterministic equivalent* have also been developed in Delattre et al. ([7]). We start by resuming the most relevant results of [16] on the deterministic equivalent that will be needed in the sequel. We first recall the notion of additive functionals.

**Definition 2.3** An additive functional of the process $X$ is a $\mathbb{R}_+ -$valued, adapted process $A = (A_t)_{t \geq 0}$ such that

1. Almost surely, the process is non-decreasing, right-continuous, having $A_0 = 0$.

2. For any $s, t \geq 0$, $A_{s+t} = A_t + A_s \circ \theta_t$ almost surely. Here, $\theta$ denotes the shift operator.
Examples for additive functionals are $A_t = \int_0^t f(X_s)ds$ where $f$ is a positive measurable function. Such an additive functional is said to be integrable, if $\mu (f) < \infty$. The deterministic equivalent of any integrable additive functional is a deterministic function $v \mapsto v(t)$ such that $v(0) = 0, v(.)$ is non-decreasing and $v(t) \to \infty$ as $t \to \infty$. It satisfies that for any integrable additive functional $A_t$, $A_t/v(t)$ is bounded and bounded away from zero in probability. In order to define the deterministic equivalent, we have to recall the notion of a special function (see also [26], [4]):

**Definition 2.4** A measurable function $f : E \to \mathbb{R}_+$ is called special if for all bounded and positive measurable functions $h$ such that $\mu (h) > 0$, the function

$$x \mapsto E_x \int_0^\infty \exp \left[ - \int_0^t h(X_s)ds \right] f(X_t)dt$$

is bounded.

Note that in the case of strong Feller processes having locally compact Polish state space, any bounded function having compact support is special.

By [16], any special function $g$ of $X$ with $\mu (g) > 0$ defines a version of the deterministic equivalent via

$$v(t) = E_\pi \int_0^t g(X_s)ds,$$

(2.5)

for any arbitrary initial measure $\pi$. $v(t)$ is called deterministic equivalent due to the following result (corollary 2.19 of [16]).

**Theorem 2.5** For any additive functional $A$ of the process having $\mu (A_1) \in ]0, \infty[, for any initial measure $\pi$, we have

$$\lim_{M \to \infty} \liminf_{t \to \infty} P_\pi \left( \frac{1}{M} \leq \frac{1}{v(t)} A_t \leq M \right) = 1.$$ 

**Remark 2.6** The deterministic equivalent is unique up to a constant : for two choices of the deterministic equivalent, $v$ and $v'$, we have that $\lim_{t \to \infty} v(t)/v'(t) = c$, where $c$ is a positive constant.

In the sequel, depending on the situation, we shall fix a suitable choice of $v(t)$. In the positive recurrent case, evidently $v(t) = t$, up to multiplication with a constant. In order to avoid too cumbersome notation, we sometimes also write $v_t = v(t)$.

### 2.2 On Nummelin splitting in continuous time

The proof of the deviation inequality is based on a very simple idea : the use of regeneration times that allow to divide the trajectory of the process into (almost) i.i.d. excursions. In the one-dimensional case, regeneration times are introduced as successive visits to recurrent points. For
Harris recurrent Markov processes with general state space, points are in general not recurrent. That is why we have to use the Nummelin splitting in continuous time, as developed in [16], in order to introduce a recurrent atom for the process. An atom is a set that, roughly speaking, behaves as a point for the process. Once a recurrent atom exists, we can introduce regeneration times that split the trajectory of the process into (almost) i.i.d. excursions.

We recall briefly the construction of Nummelin splitting in continuous time.

Introduce a sequence \((\sigma_n)_{n \geq 1}\) of i.i.d. \(\exp(1)\)-waiting times, independent of the process \(X\) itself. Let \(T_0 := 0, T_n := \sigma_1 + \ldots + \sigma_n\) and \(\bar{X}_n := X_{T_n}\). Then the chain \(\bar{X} = (\bar{X}_n)_{n}\) is recurrent in the sense of Harris and its one-step transition kernel \(U^1(x, dy) := \int_0^\infty e^{-t}P_t(x, dy)dt\) satisfies the

\[
U^1(x, dy) \geq \alpha 1_C(x)\nu(dy),
\]

(2.6)

where \(0 < \alpha < 1, \mu(C) > 0\) and \(\nu\) a probability measure equivalent to \(\mu(\cdot \cap C)\) (cf [26], [14], proposition 6.7). The set \(C\) can be chosen to be compact. Note that the above minorization holds always under the only condition of Harris recurrence. We do not impose any further condition, neither irreducibility of some skeleton chain nor aperiodicity of the process.

Remark 2.7 In some cases, the measure of assumption 2.1 \(\Lambda\) satisfies \(\Lambda \sim \mu\), and the densities \(p_t(x, y)\) are explicitly known, for example in the case of \(k\)-dimensional diffusions, under suitable regularity assumptions. If one can specify some set \(C\) and some time interval \([s, t]\) such that

\[
\inf_{(x, y) \in C \times \bar{C}, u \in [s, t]} p_u(x, y) > 0, \quad \Lambda(C) > 0,
\]

(w.l.o.g. also \(\Lambda(C) \leq 1\)) then (2.6) holds true with

\[
\alpha = \left[e^{-s} - e^{-t}\right] \Lambda(C) \left(\inf_{(x, y) \in C \times \bar{C}, u \in [s, t]} p_u(x, y) \wedge 1\right) \quad \text{and} \quad \nu = \Lambda(\cdot \cap C)/\Lambda(C).
\]

In particular, for multi-dimensional diffusions satisfying Hörmander’s condition on some set \(\Gamma\), the classical results of Kusuoka and Stroock [13] allow us to conclude that any choice of a compact set \(C \subset \Gamma\) will be possible.

Then it is possible to define on an extension of the original space \((\Omega, A, (P_n))\) a Markov process \(Z = (Z_t)_{t \geq 0}\), taking values in \(E \times [0, 1] \times E\) such that the \(T_n\) are jump times of the process and such that under \(P_x, ((Z^1_t)_t, (T^1_n)_n)\) has the same distribution as \(((X_t)_t, (T_n)_n)\). Here are the details of this construction.

First of all, define the following transition kernel \(Q((x, u), dy)\) from \(E \times [0, 1]\) to \(E\) :

\[
Q((x, u), dy) = \begin{cases} 
\nu(dy) & \text{if } (x, u) \in C \times [0, \alpha] \\
\frac{1}{1 - \alpha} (U^1(x, dy) - \alpha \nu(dy)) & \text{if } (x, u) \in C \times [\alpha, 1] \\
U^1(x, dy) & \text{if } x \notin C
\end{cases}
\]

(2.7)

We now recall the construction of \(Z_t = (Z_t^1, Z_t^2, Z_t^3)\) taking values in \(E \times [0, 1] \times E\) as given in [16]. Write \(u^1(x, x') := \int_0^\infty e^{-t}p_t(x, x')dt\). Let \(Z_0^1 = X_0 = x\). Choose \(Z_0^3\) according to the uniform distribution \(U\) on \([0, 1]\). On \(Z_0^3 = u\), choose \(Z_0^3 \sim Q((x, u), dx')\). Then inductively in \(n \geq 0\), on \(T_n = (x, u, x')\) :
1. Choose a new jump time $\sigma_{n+1}$ according to
\[
e^{-t} \frac{p_t(x, x')}{u^1(x, x')} dt \text{ on } \mathbb{R}_+,
\]
where we define $0/0 := a/\infty := 1$, for any $a \geq 0$, and put $T_{n+1} := T_n + \sigma_{n+1}$.

2. On $\{\sigma_{n+1} = t\}$, put $Z^2_{T_{n+1}} := u$, $Z^3_{T_{n+1}} := x'$ for all $0 \leq s < t$.

3. For every $s < t$, choose
\[
Z^1_{T_{n+1}} := p_s(x, y)p_{t-s}(y, x') \Lambda(dy).
\]
Choose $Z^1_{T_{n+1}} := x_0$ for some fixed point $x_0 \in E$ on $\{p_t(x, x') = 0\}$. Moreover, given $Z^1_{T_{n+1}} = y$, on $s + u < t$, choose
\[
Z^1_{T_{n+1} + s} := p_{u}(y, y')p_{t-s-u}(y', x') \Lambda(dy').
\]
Again, on $\{p_{t-s}(y, x') = 0\}$, choose $Z^1_{T_{n+1} + s} = x_0$.

4. At the jump time $T_{n+1}$, choose $Z^1_{T_{n+1}} := Z^2_{T_{n+1}} = x'$. Choose $Z^2_{T_{n+1}}$ independently of $Z_{s} s < T_{n+1}$, according to the uniform law $U$. Finally, on $\{Z^2_{T_{n+1}} = u'\},$ choose $Z^3_{T_{n+1}} \sim Q((x', u'), dx'')$.

Note that by construction, given the initial value of $Z$ at time $T_n$, the evolution of the process $Z^1$ during $[T_n, T_{n+1}]$ does not depend on the chosen value of $Z^2_{T_n}$.

We will write $P_{\pi}$ for the measure related to $X$, under which $X$ starts from the initial measure $\pi(dx)$, and $\mathbb{P}_\pi$ for the measure related to $Z$, under which $Z$ starts from the initial measure $\pi(dx) \otimes U(du) \otimes Q((x, u), dy)$. In the same spirit we denote $E_\pi$ the expectation with respect to $P_{\pi}$ and $E_\pi$ the expectation with respect to $\mathbb{P}_\pi$. Moreover, we shall write $\mathcal{F}$ for the filtration generated by $Z$, $\mathcal{G}$ for the filtration generated by the first two coordinates $Z^1$ and $Z^2$ of the process, and $\mathcal{F}^X$ for the sub-filtration generated by $X$ interpreted as first coordinate of $Z$.

Write
\[
A := C \times [0, \alpha] \times E.
\]
$A$ is the recurrent atom of the process. Now we put
\[
S_0 := 0, R_0 := 0, S_{n+1} := \inf\{T_m > R_n : Z_{T_m} \in A\}, R_{n+1} := \inf\{T_m : T_m > S_{n+1}\}.
\]
Then the sequence of $\mathcal{F}$–stopping times $R_n$ generalizes the notion of life-cycle decomposition in the following sense.

**Proposition 2.8** [Proposition 2.13 of [16]]

a) $Z_{R_{n+}}$ is independent of $\mathcal{F}_{S_n}$ for all $n \geq 1$.

b) $Z_{R_n} \sim \nu(dx)U(du)Q((x, u), dx')$ for all $n \geq 1$.

c) The sequence of $(Z_{R_n})_{n \geq 1}$ is i.i.d.
Proposition 2.9 [Proposition 2.20 of [10]]
Let $A_t$ be any integrable additive functional of $X$. Then, up to multiplication by a constant, for any initial measure $\pi$ and any $n \geq 1$,
$$E_\pi(A_{R_{n+1}} - A_{R_n}) = E_\nu(A_{R_1}) = E_\mu(A_1).$$

From now on, we shall fix a version $\mu$ of the invariant measure such that always
$$\mu(f) = \mathbb{E}_\pi \int_{R_1}^{R_2} f(X_s) ds. \quad (2.8)$$

Moreover, we have the following:

Proposition 2.10 [Proposition 4.4 of [17]]
Let $f$ be a measurable $\mu$–integrable function. Put
$$\xi_n := \int_{R_{n-1}}^{R_n} f(X_s) ds, n \geq 1.$$ 

Then the sequence $(\xi_n)_n$ is a stationary ergodic sequence under $\mathbb{P}_\nu$. Moreover, for $n \geq 2$, $\xi_n$ is independent of $\mathcal{F}_{R_{n-2}}$.

Remark 2.11 In the usual one-dimensional case, regeneration times $R_n$ allow to split in such a way that $\xi_n$ is independent of $\xi_{n-1}$. Our situation, however, is more complicated, since by construction, at a regeneration time, $Z_{R_n}$ depends on the state of the process one jump time before, i.e. on $Z_{S_n}$. This is due to the structure of continuous time and due to the use of Markov bridges (step 3. of the construction).

Finally, let us recall the following useful result.

Proposition 2.12 [Proposition 2.16 of [10]]
Let $f$ be a special function of the process $X$. Then
$$C(f) := \sup_{x \in E} \mathbb{E}_x \int_0^{R_1} |f|(X_s) ds < +\infty.$$ 

3 The deviation inequality

In this section, we state our main result which is a deviation inequality for $\int_0^t f(X_s) ds$, where $f$ is a special function of the process having $\mu(f) = 0$. Write
$$N_t := \sup\{n : R_n \leq t\}.$$ 

In the following, we shall also use the following version of the deterministic equivalent.
$$v_t^* = \mathbb{E}_\nu(N_t) + 1, \quad (3.9)$$ 

where $\nu = L(Z_{R_n})$ is given in (2.6). The following is our main result.
Theorem 3.1 Let $f$ be a bounded special function such that $\mu(f) = 0$. Recall that $C(f) = \sup_x \mathbb{E}_x \int_0^{R_1} |f(X_s)|ds$ and put

$$K(f) := \|f\|_\infty + C(f), \quad B(f) := \max(K^2(f), K(f)).$$

1. Suppose that $X$ is positive recurrent. Let

$$\hat{F}(\lambda) = \mathbb{E}(e^{-\lambda(R_2-R_1)})$$

be the Laplace transform of the length of a life cycle and put $m = \mathbb{E}(R_2 - R_1)$. Then for any $u < m$,

$$\Lambda^*(u) = \sup_{\lambda > 0} \left[-\lambda u - \log \hat{F}(\lambda)\right] > 0,$$  \hfill (3.10)

and we have the following results.

(i) For any $0 < \eta \leq \frac{1}{2}$, for all $x$, for any initial measure $\pi$ and for all $t > 4m$,

$$P_\pi \left( |\int_0^t f(X_s)ds| \geq t^{\frac{1}{2}+\eta} \left( \frac{2}{m} \right)^{\frac{1}{2}+\eta} x \right)$$

$$\leq 4 \exp \left( -t^{2n} \frac{1}{42mB(f)} (x^2 \land x) \right) + 4e \exp \left( -\frac{2^{\frac{1}{2}+\eta}}{6K(f)m^{\frac{1}{2}+\eta}} t^{\frac{1}{2}+\eta} x \right)$$

$$+ 8 \exp \left( -t(x \lor 1) \frac{3}{4m} \Lambda^*(\frac{2}{3}m) \right).$$  \hfill (3.11)

(ii) For all $x$ and for any initial measure $\pi$, for all $t > 4m$,

$$P_\pi \left( |\int_0^t f(X_s)ds| \geq \sqrt{t} x \frac{\sqrt{2}}{\sqrt{m}} \right) \leq 4 \exp \left( -\frac{1}{42 B(f)} (x^2 \land x) \right)$$

$$+ 4e \exp \left( -\frac{\sqrt{2}}{6K(f)\sqrt{m}} \sqrt{t} x \right) + 8 \exp \left( -t(x \lor 1) \frac{3}{4m} \Lambda^*(\frac{2}{3}m) \right).$$  \hfill (3.12)

2. Suppose that $X$ is null-recurrent. Then there exists $t_0$, such that for all $t \geq t_0$, for all $x$, for any $0 < \eta \leq \frac{1}{2}$, for any initial measure $\pi$ and for $v^*_t$ as in (3.9),

$$P_\pi \left( |\int_0^t f(X_s)ds| \geq (v^*_t)^{\frac{1}{2}+\eta} x \right) \leq 4 \exp \left( -\frac{1}{42 B(f)} (v^*_t)^{\eta} (x^2 \land x) \right)$$

$$+ 4e \exp \left( -\frac{1}{6K(f)} (v^*_t)^{\frac{1}{2}+\eta} x \right) + 8 \exp \left( -\frac{1}{2}(v^*_t)^{\eta} (x \lor 1) \right).$$  \hfill (3.13)

$t_0$ is defined through $v^*_{t_0} = 1$.

The bounds obtained in the above theorem are in general not optimal. But they are valid in a non-asymptotic framework. The quantities which are involved are almost all known or known up to some constants. Note also that as a function of the deviation level $x$, the above bounds $\exp(-Ct^{2n}(x^2 \land x))$ and $\exp(-C(v^*_t)^{\eta}(x^2 \land x))$ are of Gaussian type. This is due to the fact that we consider special functions allowing for exponential moments even in cases where the underlying process is not exponentially ergodic, see also Proposition 4.1 below.
Remark 3.2 In (3.9) we do not use the usual form of the deterministic equivalent as chosen in (2.5). But we have the following comparison result:

1. For the choice of \( v_t = E_\nu \int_0^t g(X_s) ds \) as in (2.5), we have
   \[
   v_t \leq C(g) + \mu(g) \cdot v^*_t.
   \] (3.14)

This can be seen as follows. Since \( g > 0 \), we have clearly that
   \[
   v_t \leq \sum_{n \geq 0} E_\nu \left( 1_{\{ R_n \leq t \}} \int_{R_n}^{R_{n+1}} g(X_s) ds \right)
   \leq C(g) + \sum_{n \geq 1} E_\nu \left( 1_{\{ R_{n-1} \leq t \}} \int_{R_n}^{R_{n+1}} g(X_s) ds \right).
   \]

Using Markov’s property with respect to \( F_{R_{n-1}} \) in the last expression and (2.8), we obtain that
\[
 v_t \leq C(g) + \mu(g) \cdot \left[ E_\nu(N_t) + 1 \right],
\]
and thus,
\[
 v_t \leq C(g) + \mu(g) \cdot v^*_t.
\]

2. Moreover, we have the following lower bound.
   \[
   v_t \geq \mu(g) \cdot v^*_t - 2C(g). \] (3.15)

Indeed,
\[
 v_t \geq \left( \sum_{n \geq 0} E_\nu \left( 1_{\{ R_n \leq t \}} \int_{R_n}^{R_{n+2}} g(X_s) ds \right) \right) - E_\nu \left( \int_{R_{N_1}}^{R_{N_2}+2} g(X_s) ds \right)
   \geq [E_\nu(N_t) + 1] \mu(g) - 2C(g) = v^*_t \cdot \mu(g) - 2C(g),
\]

since \( E_\nu \left( \int_{R_{N_1}}^{R_{N_2}+2} g(X_s) ds \right) \leq 2C(g) \).

The above Theorem 3.1 can be improved in the null-recurrent but regular case. Here, in accordance with Theorem 3.15 of Höpfner and Löcherbach, [14], we call a process regular, if for \( 0 < \alpha < 1 \) and a function \( l \) varying slowly at \( \infty \), the following is true. For any measurable and positive function \( g \) with \( 0 < \mu(g) < \infty \), we have regular variation of resolvants
\[
 R_{1/t} g(x) = E_x \left( \int_0^\infty e^{-\frac{s}{l(t)}} g(X_s) ds \right) \sim t^{\alpha} \frac{1}{l(t)} \mu(g) \text{ as } t \to \infty,
\] (3.16)

for \( \mu \)-almost all \( x \). Here we do not consider the case \( \alpha = 1 \). We will show in proposition 5.3 below that (3.16) is equivalent to the following:
\[
 \mathbb{P}(R_2 - R_1 > x) \sim \frac{x^{-\alpha} l(x)}{\Gamma(1 - \alpha)} \text{ as } x \to \infty.
\] (3.17)
Thus we are in the situation where $R_2 - R_1$ belongs to the domain of attraction of a stable law. In this case a finer control of $N_t$ is possible and it is well-known that

$$ v(t) \sim t^{\alpha} \frac{1}{l(t)} \text{ as } t \to \infty, $$

see for instance Höpfner and Löcherbach [14], theorem 5.6.A. Moreover,

$$ 1 - \hat{F}(\lambda) \sim \lambda^{\alpha} l(1/\lambda) \text{ as } \lambda \to 0, $$

(3.18)

see for instance Bingham et al. [2], corollary 8.1.7).

Then we get the following version of the deviation theorem.

**Theorem 3.3** Suppose that (3.17) holds. Let $f$ be a bounded special function such that $\mu(f) = 0$. Then there exists $t_0 \geq 0$ which is given explicitly in (5.45) below, a function $L$ varying slowly at infinity, such that for all $x$ and for any initial measure $\pi$, for all $t \geq t_0$, for all $0 < \eta \leq \frac{\alpha}{2}$,

$$ P_{\pi} \left( | \int_0^t f(X_s)ds | \geq t_0^{\frac{\alpha}{2} + \eta} x \right) \leq 4 \exp \left( -\frac{1}{42} B(f) t^{2n/(2-\alpha)} (x^2 \wedge x) L(t) \right) 
+ 4 \epsilon \exp \left( -\frac{1}{6} K(f) t_0^{\frac{\alpha}{2} + \eta} x \right) + 8 \exp \left( -\frac{1}{2} t^{2n/(2-\alpha)} (x \vee 1) \Lambda^* t \right), $$

where

$$ \Lambda^*_t = \sup_{\lambda > 0} \left[ -\log \hat{F}(\lambda) t^{\alpha-\alpha} \frac{1}{2} \frac{\alpha}{2} \frac{1}{L(t)} - \lambda t^{1-n} \right] > 0 $$

is positive and does not depend on $t$ asymptotically:

$$ \lim_{t \to \infty} \inf \Lambda^*_t \geq (1 - \alpha) \frac{\alpha}{(1 - \alpha)} > 0. $$

**Remark 3.4** Note that in the above theorem, for $\eta = \alpha/2$, we obtain a rate of decay of the order of $\exp(-t^{\alpha/(2-\alpha)})$. For $\alpha = 2$, this gives the rate of convergence $\exp(-t^{1/3})$ which is better than the rate $\exp(-t^{1/4})$ obtained in Theorem 2.1 item 2. (for $\eta = \frac{1}{2}$).

The above Theorems 3.1 and 3.3 do not use the Legendre transform of $\xi = \int_{R_n} f(X_s)ds$ since in general the law of the abstract regeneration times $R_n$ is not explicitly known. That is why the above deviation inequalities involve the constant $K(f)$. In the following, we show how to compare $K(f)$ and $\mu(f)$.

**Remark 3.5**

1. By proposition 2.16 of [10] we know the following. Recall the definition of the set $C$ of (2.7), see also remark 2.7. Let $S$ be the first jump time of a Poisson process having rate $1_C(X_s)$, and let $K$ be a constant such that for all $x$,

$$ E_x \int_0^S |f|(X_s)ds \leq K, \quad |f(x)| \leq K. $$

Then we have that

$$ C(f) \leq K + \frac{3K}{\alpha} \text{ and hence } K(f) \leq 2K + \frac{3K}{\alpha}. $$
2. We are now going to explore the relationship between $C(f)$ and the invariant measure $\mu(|f|)$ in some special cases. Suppose that the process $X$ is strong Feller and positive recurrent. Let $F$ be a compact set such that the support of $f$ is contained in $F$. Let

$$T_C := \inf \{ t \geq 0 : X_t \in C \}$$

be the entrance time in the set $C$. Recall that the measure $\nu$ is concentrated on $C$. Then

$$C(f) = \sup_{x \in F} \mathbb{E}_x \int_0^{R_1} |f|(X_s)ds \leq \|f\|_\infty \sup_{x \in F} E_x T_C + \sup_{x \in F \cap C} \mathbb{E}_x \int_0^{R_1} |f|(X_s)ds. \quad (3.19)$$

By continuity of the map $x \mapsto \mathbb{E}_x \int_0^{R_1} |f|(X_s)ds$, there exists $x_0 \in C \cap F$ such that

$$\mathbb{E}_x \int_0^{R_1} |f|(X_s)ds = \mathbb{E}_{x_0} \int_0^{R_1} |f|(X_s)ds.$$

Moreover, there exists $\varepsilon > 0$, such that for all $x \in B_\varepsilon(x_0)$,

$$\mathbb{E}_x \int_0^{R_1} |f|(X_s)ds \geq \frac{1}{2} \mathbb{E}_{x_0} \int_0^{R_1} |f|(X_s)ds.$$

Then we have that

$$\mu(|f|) = \mathbb{E}_\nu \int_0^{R_1} |f|(X_s)ds \geq \int_{B_\varepsilon(x_0)} \nu(dx) \mathbb{E}_x \int_0^{R_1} |f|(X_s)ds$$

$$\geq \frac{1}{2} \cdot \nu(B_\varepsilon(x_0)) \cdot \mathbb{E}_{x_0} \int_0^{R_1} |f|(X_s)ds. \quad (3.20)$$

Putting together (3.19) and (3.20), we conclude that

$$C(f) \leq \|f\|_\infty \sup_{x \in F} E_x T_C + \frac{2}{\nu(B_\varepsilon(x_0))} \mu(|f|). \quad (3.21)$$

3. Suppose that $X$ is a positive recurrent one-dimensional diffusion process

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

In the case of dimension one, we can avoid Nummelin splitting since successive visits of recurrent points allow to split the trajectory into i.i.d. excursions. More precisely, let $a < b$ be two recurrent points of $X$ and define a sequence of stopping times $(S_n)_n, (R_n)_n$ as follows. $S_0 = R_0 = 0,$

$$S_1 = \inf \{ t \geq 0 : X_t = b \}, \quad R_1 = \inf \{ t \geq S_1 : X_t = a \},$$

and for any $n \geq 1,$ $S_{n+1} = R_n + S_1 \circ \theta_{R_n},$ $R_{n+1} = R_n + R_1 \circ \theta_{R_n}$. Using this sequence $(R_n)_n$, under additional regularity assumptions (see [19] for the details), we have that

$$C(f) \leq \kappa \mu(|f|),$$

for any function $f$ having compact support. Here, the constant $\kappa$ is explicitly known.
4 Proof of Theorem 3.1

The following proposition will be crucial in the sequel. It states that additive functionals built from bounded special functions admit a certain number of exponential moments.

**Proposition 4.1** Let $f$ be a bounded special function. Recall that $K(f) = ||f||_{\infty} + C(f)$. Then we have for any initial measure $\pi$ and any $n \geq 1$, that

$$E_\pi(||\xi_n||^p) \leq p!K(f)^p. \tag{4.23}$$

In particular, we obtain for any $0 < \lambda < K(f)^{-1}$,

$$E_\pi(e^{\lambda \xi_n}) \leq 1 + \sum_{p \geq 1} \lambda^p K(f)^p = 1 - \frac{1}{\lambda K(f)},$$

and if $\mu(f) = 0$,

$$E_\pi(e^{\lambda \xi_n}) \leq 1 + \sum_{p \geq 2} \lambda^p K(f)^p = 1 + \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)}.$$

**Proof** Evidently, we have that

$$\xi_n^p = \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} f(X_{t_1}) \cdots f(X_{t_p}) dt_1 \cdots dt_p \tag{4.24}$$

$$\leq p! \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \cdots \leq t_p\}} |f(X_{t_1})| \cdots |f(X_{t_p})| dt_1 \cdots dt_p.$$

Taking expectation and conditional expectation with respect to $\mathcal{F}_{t_{p-1}}$, we get

$$E_\pi \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \cdots \leq t_p-1\}} |f(X_{t_1})| \cdots |f(X_{t_p-1})| dt_1 \cdots dt_{p-1} = \mathbb{E}_{Z_{t_{p-1}}} \left[ \int_0^{R_1} |f(X_s)| ds \right] dt_1 \cdots dt_{p-1}. \tag{4.25}$$

But note that for any fixed $z$, for $M = ||f||_{\infty}$,

$$E_z \left[ \int_0^{R_1} |f|(X_s) ds \right] \leq ME_z(T_1) + E_z \int_{T_1}^{R_1} |f|(X_s) ds$$

$$\leq ME_z(T_1) + E_z(E_{Z_{T_1}} \int_0^{R_1} |f|(X_s) ds)$$

$$\leq ME_z(T_1) + C(f).$$

(Compare to proposition 2.16 of [10].) But by construction, for $z = (x, u, x')$,

$$E_z(T_1) = \int_0^{\infty} t e^{-t} \frac{p_1(x, x')}{u^1(x, x')} dt,$$
and this expression does only depend on $x$ and $x'$. According to proposition 4.1 of [17], we have that

$$\mathcal{L}(Z_t^1 | Z_t^1 = x)(dx') = u^1(x, x')\Lambda(dx').$$

Taking now conditional expectation in (4.25) with respect to $\mathcal{F}_{tp-2}$ and putting all these results together, we obtain

$$\mathbb{E}_x \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \cdots \leq t_p\}} |f|(X_{t_1}) \cdots |f|(X_{t_p}) dt_1 \cdots dt_p$$

$$= \mathbb{E}_x \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \cdots \leq t_{p-1}\}} |f|(X_{t_1}) \cdots |f|(X_{t_{p-1}}) \mathbb{E}_{Z_{t_{p-1}}} \left[ \int_0^{R_1} |f|(X_s) ds \right] dt_1 \cdots dt_{p-1}$$

$$\leq \mathbb{E}_x \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \cdots \leq t_{p-1}\}} |f|(X_{t_1}) \cdots |f|(X_{t_{p-1}}) \left[ ME_{Z_{t_{p-1}}}(T_1) + C(f) \right] dt_1 \cdots dt_{p-1}$$

$$= \mathbb{E}_x \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \cdots \leq t_{p-2}\}} |f|(X_{t_1}) \cdots |f|(X_{t_{p-2}})$$

$$\mathbb{E}_{Z_{t_{p-2}}} \left[ \int_0^{R_1} |f|(X_s) (M + C(f)) ds \right] dt_1 \cdots dt_{p-2},$$

where the last equality follows from

$$\mathbb{E}_x \int_0^{R_1} |f|(X_s) \mathbb{E}_{Z_s}(T_1) ds = \mathbb{E}_x \int_0^{\infty} 1_{\{s < R_1\}} |f|(X_s) \left( \int u^1(X_s, x') \Lambda(dx') \mathbb{E}(x, x')(T_1) \right) ds$$

$$= \mathbb{E}_x \int_0^{\infty} 1_{\{s < R_1\}} |f|(X_s) ds,$$

since

$$\int u^1(x, x') \Lambda(dx') \int_0^{\infty} te^{-t} \frac{p_t(x, x')}{u^1(x, x')} dt = 1.$$

Taking successively conditional expectations with respect to $\mathcal{F}_{tp-3}, \ldots, \mathcal{F}_{t_1}$ yields the result. 

A first step in order to prove Theorem 3.1 is the following proposition. Let

$$v_t = \begin{cases} \frac{2}{m} t & \text{if } X \text{ is positive recurrent} \\ v_t^\prime & \text{if } X \text{ is null-recurrent} \end{cases}.$$ 

**Proposition 4.2** Let $f$ be a bounded special function such that $\mu(f) = 0$. Then there exists $t_0$, such that for all $x > 0$, for any initial measure $\pi$, for any $0 \leq \eta \leq \frac{1}{2}$, $0 \leq \delta \leq 2\eta$, for any fixed choice of a deterministic equivalent $v_t$ and for all $t \geq t_0$,

$$P_\pi \left( \left| \int_0^t f(X_s) ds \right| \geq v_t^{\frac{1}{2} + \delta} x \right) \leq 4 \exp \left( -\frac{1}{42} \frac{x^2 \wedge x}{\max(K^2(f), K(f))} v_t^{2\eta - \delta} \right)$$

$$+ 4 \exp \left( -\frac{1}{6K(f)} v_t^{\frac{1}{2} + \eta} x \right) + 4p_\pi(N_t \geq v_t^{1+\delta}(x \vee 1)).$$

Here, $t_0$ is given by the equation $v_{t_0} = 1.$

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Proof Write $\xi_n := \int_{R_{n-1}}^{R_n} f(X_s)ds$, then the $\xi_n, n \geq 2$, are identically distributed random variables having mean zero, such that $\xi_n$ and $\xi_{n+2}$ are independent. It can be proven that for any $p \geq 1$, $\mathbb{E}(|\xi_n|^p) < \infty$, by the properties of a special function. More precisely, we have that for any $n \geq 1$,

$$\mathbb{E}(|\xi_n|^p) \leq p!K(f)^p, \tag{4.26}$$

which will be shown in proposition 4.1 below. Thus for $\lambda > 0$, sufficiently small ($\lambda < K(f)^{-1}$ suffices),

$$Z(\lambda) := \mathbb{E}_\pi(e^{\lambda \xi_n}), \ n \geq 2, \tag{4.27}$$

exists and is finite (and does not depend on $n$).

Since the $\xi_n$ are not independent, but only $2-$independent, we have to proceed in the following way. Firstly, define a sequence $\xi_n^{(1)}$ by

$$\xi_n^{(1)} = \begin{cases} \xi_n & \text{if } n \text{ odd} \\ 0 & \text{elseif} \end{cases}. \tag{4.28}$$

Then define a second sequence $\xi_n^{(2)}$ by

$$\xi_n^{(2)} = \begin{cases} \xi_n & \text{if } n \text{ even} \\ 0 & \text{elseif} \end{cases}. \tag{4.28}$$

Note that $\xi_n$ is not independent of $\mathcal{F}_{R_{n-1}}$, but it is independent of $\mathcal{F}_{R_{n-2}}$. That is why we introduce the following two sub-filtrations, associated to the sum of odd and the sum of even terms. Let

$$\mathcal{G}_n^{(1)} := \sigma\{R_k, \xi_k^{(1)} : k \leq n, k \text{ odd}\},$$

and

$$\mathcal{G}_n^{(2)} := \sigma\{R_k, \xi_k^{(2)} : k \leq n, k \text{ even}\}.$$  

Moreover, let

$$N^{(1)}_t := \sup\{n : n \text{ odd}, R_n \leq t\}, \ N^{(2)}_t := \sup\{n : n \text{ even}, R_n \leq t\}.$$  

Then it is immediate that $N^{(1)}_t + 2$ is a $(\mathcal{G}^{(1)}_n)_n-$ stopping time and $N^{(2)}_t + 2$ a $(\mathcal{G}^{(2)}_n)_n-$ stopping time. Moreover, we evidently have that $N^{(1)}_t \leq N_t$, $N^{(2)}_t \leq N_t$.

Now, for $\lambda$ sufficiently small, let

$$M_n^1 := \exp(\lambda \sum_{k=2}^{n} \xi_k^{(1)}) \cdot Z(\lambda)^{-[(n-1)/2]}, \ M_n^2 := \exp(\lambda \sum_{k=2}^{n} \xi_k^{(2)}) \cdot Z(\lambda)^{-[n/2]},$$

where $[\cdot]$ denotes the integer part of a real number. Then $(M_n^1)_n$ and $(M_n^2)_n$ are discrete $\mathcal{G}_n^{(1)}-$martingales $(\mathcal{G}_n^{(2)}-$martingales, respectively). Hence using Doob’s stopping rule for positive super-martingales we get

$$\mathbb{E}_\pi\left[\exp(\lambda \sum_{k=2}^{N^{(1)}_t + 2} \xi_k^{(1)}) \exp(-[(N^{(1)}_t + 1)/2] \log Z(\lambda))\right] \leq 1 \tag{4.29}$$

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and also
\[
\mathbb{E}_\pi \left[ \exp(\lambda \sum_{k=2}^{N_t^{(2)}+2} \xi_k^{(2)}) \exp\left(-[(N_t^{(2)} + 2)/2] \log Z(\lambda)\right) \right] \leq 1. \quad (4.30)
\]

Now, we proceed as follows. Evidently,
\[
\mathbb{P}_\pi(\int_{0}^{R_{N_t^{(2)}+2}} f(X_s) ds \geq v_t^{\frac{1}{2} + \eta} x) \leq I_0 + I_1 + I_2,
\]

where \( I_0 = \mathbb{P}_\pi(\xi_1 \geq v_t^{\frac{1}{2} + \eta} x/3) \),
\[
I_1 = \mathbb{P}_\pi(\sum_{k=2}^{N_t^{(1)}+2} \xi_k^{(1)} \geq v_t^{\frac{1}{2} + \eta} x/3), \quad I_2 = \mathbb{P}_\pi(\sum_{k=2}^{N_t^{(2)}+2} \xi_k^{(2)} \geq v_t^{\frac{1}{2} + \eta} x/3).
\]

We start with a study of the first term \( I_0 \). Let
\[
\tilde{Z}(\lambda) = \mathbb{E}_\pi e^{\lambda \xi_1}.
\]

Then, for \( 0 < \lambda < K(f)^{-1} \),
\[
\mathbb{P}_\pi[\xi_1 \geq v_t^{\frac{1}{2} + \eta} x/3] \leq \exp \left(-v_t^{\frac{1}{2} + \eta} \lambda x/3 - \log \tilde{Z}(\lambda) \right).
\]

But due to (4.26), we have that \( \log \tilde{Z}(\lambda) \leq \frac{\lambda K(f)}{1 - \lambda C(f)} \), and thus, taking \( \lambda = \frac{1}{2} K(f)^{-1} \),
\[
I_0 = \mathbb{P}_\pi[\xi_1 \geq v_t^{\frac{1}{2} + \eta} xK(f)/3] \leq e^{\exp \left(-\frac{1}{6K(f)} v_t^{\frac{1}{2} + \eta} x \right)}. \quad (4.31)
\]

Let us now turn to the study of \( I_1 \) and \( I_2 \).

Note that for any \( k > 0 \),
\[
I_1 \leq \mathbb{P}_\pi(\sum_{k=2}^{N_t^{(1)}+2} \xi_k^{(1)} \geq v_t^{\frac{1}{2} + \eta} x/3; N_t \leq k) + \mathbb{P}_\pi(N_t > k).
\]

By (4.26), we have that for any \( 0 < \lambda < K(f)^{-1} \),
\[
Z(\lambda) \leq 1 + \sum_{n \geq 2} [\lambda K(f)]^n.
\]

Hence
\[
Z(\lambda) \leq 1 + \frac{\lambda^2 K(f)^2}{1 - \lambda C(f)}, \quad \text{thus } \log Z(\lambda) \leq \frac{\lambda^2 K(f)^2}{1 - \lambda C(f)}.
\]
Hence for any \(0 < \lambda < K(f)^{-1}\), using (4.29) and recalling that \(N_l^{(1)} \leq N_l\),

\[
\mathbb{P}_\pi \left[ \sum_{k=2}^{N_l^{(1)}+2} \xi^{(1)}_k \geq v_t^{1+\eta} x/3 ; N_l \leq k \right] \leq \mathbb{P}_\pi \left[ \sum_{k=2}^{N_l^{(1)}+2} \xi^{(1)}_k \geq v_t^{1+\eta} x/3 ; N_l^{(1)} \leq k \right]
\]

\[
\leq \mathbb{P}_\pi \left[ M_{N_l^{(1)}+2}^1 \geq \exp(\lambda v_t^{1+\eta} x/3 - \log Z(\lambda)[(N_l^{(1)} + 1)/2]) ; N_l^{(1)} \leq k \right]
\]

\[
\leq \mathbb{P}_\pi \left[ M_{N_l^{(1)}+2}^1 \geq \exp(\lambda v_t^{1+\eta} x/3 - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)}[(N_l^{(1)} + 1)/2]) ; N_l^{(1)} \leq k \right]
\]

\[
\leq \mathbb{P}_\pi \left[ M_{N_l^{(1)}+2}^1 \geq \exp(\lambda v_t^{1+\eta} x/3 - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)}[(k + 1)/2]) \right]
\]

\leq \exp \left( -(v_t^{1+\eta} \lambda x \frac{1}{3} - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} k) \right).
\]

In the same way we get that

\[
\mathbb{P}_\pi \left[ \sum_{k=2}^{N_l^{(2)}+2} \xi^{(2)}_k \geq v_t^{1+\eta} x/3 ; N_l \leq k \right] \leq \exp \left( -(v_t^{1+\eta} \lambda x \frac{1}{3} - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} k) \right).
\]

Now, take

\[
k = v_t^{1+\delta} (x \lor 1).
\]

(4.32)

Then we have that

\[
\mathbb{P}_\pi \left[ \sum_{k=2}^{N_l^{(1)}+2} \xi^{(1)}_k \geq v_t^{1+\eta} x/3 ; N_l \leq v_t^{1+\delta} \right] \leq \exp \left( -(v_t^{1+\delta} h(t, x)) \right),
\]

(4.33)

\[
\mathbb{P}_\pi \left[ \sum_{k=2}^{N_l^{(2)}+2} \xi^{(2)}_k \geq v_t^{1+\eta} x/3 ; N_l \leq v_t^{1+\delta} \right] \leq \exp \left( -(v_t^{1+\delta} h(t, x)) \right),
\]

where

\[
h(t, x) := (x \lor 1) \sup_{0 < \lambda < K(f)^{-1}} \left( \frac{\lambda (x \land 1)}{3v_t^{1+\delta-\eta}} - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} \right).
\]

(4.34)

It can be shown, see for example Birgé and Massart (3), lemma 8, pages 366 and 367, that

\[
\sup_{0 < \lambda < 1/v} \left( \lambda y - \frac{\lambda^2 v^2}{1 - \lambda v} \right) \geq \frac{y^2}{2vy + 4v^2}.
\]

(4.35)

This is seen as follows. A simple calculus shows that

\[
\sup_{0 < \lambda < 1/v} \left( \lambda y - \frac{\lambda^2 v^2}{1 - \lambda v} \right) = \lambda^* y - \frac{(\lambda^*)^2 v^2}{1 - \lambda^* v},
\]

where

\[
\lambda^* = \frac{1}{v} \left( 1 - \sqrt{\frac{v}{y + v}} \right) < \frac{1}{v}.
\]

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It follows that
\[ \lambda^* y - \frac{(\lambda^*)^2 v^2}{1 - \lambda^* v} = \left( \sqrt{\frac{1}{y + v}} - 1 \right)^2 = \frac{y^2}{yv + 2v^2 + 2v(1 + \frac{y}{2})^{1/2}}, \]
and using \((1 + y)^{1/2} \leq 1 + y/2\), one gets the desired inequality.

Using \((4.34)\) and \((4.35)\), we get for any \(t \geq t_0\) such that \((v_t)^{\frac{1}{2} + \delta - \eta} \geq 1\),
\[ h(t, x) \geq \frac{1}{3} (x \lor 1) \frac{(x^2 \lor 1)v_t^{-1+2\eta-2\delta}}{2K(f)v_t^{-\frac{1}{2}+\delta-\eta} + 12K^2(f)} \geq \frac{1}{42} \frac{(x^2 \lor x)v_t^{-1+2\eta-2\delta}}{\max(K^2(f), K(f))}. \tag{4.36} \]

Note that the right hand side of \((4.36)\) tends to zero at speed \(v_t^{-1+2\eta-2\delta}\). However, this right hand side has to be multiplied with \(v_t^{1+\delta}\), compare to \((4.33)\), which yields the term \(v_t^{2\eta-\delta}\) which does not tend to zero since by assumption, \(\delta \leq 2\eta\). Thus, together with \((4.31)\),
\[ \mathbb{P}_\pi \left[ \int_0^{R_{N_t+2}} f(X_s)ds \geq v_t x \right] \leq 2 \exp \left( -\frac{1}{42} \frac{x^2 \lor x}{\max(K^2(f), K(f))} v_t^{2\eta-\delta} \right) \]
\[ + e \exp \left( -\frac{1}{6K(f)} v_t^{1+\eta} x \right) + 2\mathbb{P}_\pi(N_t > v_t^{1+\delta}(x \lor 1)). \]

Applying the same argument to \(-f\) instead of \(f\) yields
\[ \mathbb{P}_\pi \left( \left| \int_0^{R_{N_t+2}} f(X_s)ds \right| \geq v_t x \right) \leq 4 \exp \left( -\frac{1}{42} \frac{x^2 \lor x}{\max(K^2(f), K(f))} v_t^{2\eta-\delta} \right) \]
\[ + 2e \exp \left( -\frac{1}{6K(f)} v_t^{1+\eta} x \right) + 4\mathbb{P}_\pi(N_t > v_t^{1+\delta}(x \lor 1)). \tag{4.37} \]

Moreover, note that
\[ \mathbb{P}_\pi \left( \left| \int_t^{R_{N_t+2}} f(X_s)ds \right| \geq v_t^{\frac{1}{2}+\eta} x \right) \leq \mathbb{P}_\pi \left( \left| \int_t^{R_{N_t+1}} f(X_s)ds \right| \geq \frac{1}{2} v_t^{\frac{1}{2}+\eta} x \right) \]
\[ + \mathbb{P}_\pi \left( \left| \int_{R_{N_t+1}}^{R_{N_t+2}} f(X_s)ds \right| \geq \frac{1}{2} v_t^{\frac{1}{2}+\eta} x \right) \]
and, using Markov’s property with respect to \(\mathcal{F}_t\) and to \(\mathcal{F}_{R_{N_t+1}}\),
\[ \mathbb{P}_\pi \left( \left| \int_t^{R_{N_t+1}} f(X_s)ds \right| \geq \frac{1}{2} v_t^{\frac{1}{2}+\eta} x \right) \leq \mathbb{E}_\pi \mathbb{E}_{X_t} e^{\lambda \int_0^{R_t} |f(X_s)ds} e^{-v_t^{\frac{1}{2}+\eta} \lambda x/2}. \tag{4.38} \]

Write
\[ \tilde{Z}(\lambda) = \mathbb{E}_\pi \mathbb{E}_{X_1} (\exp \lambda \int_0^{R_t} |f(X_s)ds). \]
As before we have that
\[ \log \tilde{Z}(\lambda) \leq \frac{\lambda K(f)}{1 - \lambda K(f)}, \]

and thus, taking $\lambda = \frac{1}{2}K(f)^{-1}$,
\[
\mathbb{P}_\pi \left( \left| \int_t^{R_{N_t+1}} f(X_s) ds \right| \geq \frac{1}{2} \nu_t^{\frac{1}{2}+\eta} x \right) \leq \exp \left( -\frac{1}{4K(f)} \nu_t^{\frac{1}{2}+\eta} x \right).
\]
In the same way we get that
\[
\mathbb{P}_\pi \left( \left| \int_t^{R_{N_t+2}} f(X_s) ds \right| \geq \frac{1}{2} \nu_t^{\frac{1}{2}+\eta} x \right) \leq \exp \left( -\frac{1}{4K(f)} \nu_t^{\frac{1}{2}+\eta} x \right).
\]

**Remark 4.3** The most delicate point in the above proof is (4.36) which gives a lower bound converging to zero. It is important to be able control the speed of convergence of this expression to zero. If we decided to work with the Legendre transform, then the above proof remains true by replacing (4.34) by
\[
h(t, x) = (x \lor 1) \sup_{\lambda > 0} \left( \lambda \frac{x \land 1}{\nu_t^{\frac{1}{2}+\delta-\eta}} - \log Z(\lambda) \right).
\]
In this form it is evident that $h(t, x) \to 0$ as $t \to \infty$, and developing $Z(\lambda) = 1 + \frac{\sigma^2}{2} \lambda^2 + o(\lambda^2)$, where
\[
\sigma^2 := \mathbb{E} \left( \int_{R_1}^{R_2} f(X_s) ds \right)^2,
\]
would yield the same speed of convergence. However, using this approach we would not be able to put hands on the exact form of the constants appearing in the remainder term $o(\lambda^2)$, and that is why we decided to use the upper bound of $\log Z(\lambda)$ as proposed in the proof.

In order to make use of Proposition 4.2 we have to control the number of life cycles before time $t$, $N_t$. We first study the positive recurrent case. Recall that $\nu_t = \frac{2}{m} t$ in this case.

**Proposition 4.4** Suppose that $X$ is positive recurrent. Then
\[
\mathbb{P}_\pi (N_t > \nu_t (x \lor 1)) \leq 2 \exp \left( -\frac{1}{2} \nu_t (x \lor 1) \sup_{\lambda} h_t(\lambda, x) \right),
\]
where
\[
h_t(\lambda, x) := - \left( 1 - \frac{m}{t(x \lor 1)} \right) \log \hat{F}(\lambda) - \lambda \frac{m}{2(x \lor 1)}
\]
and where
\[
\hat{F}(\lambda) = \mathbb{E}(e^{-\lambda(R_2-R_1)})
\]
is the Laplace transform of the length of a life cycle. Moreover we have for any $u < m$ that
\[
\Lambda^*(u) = \sup_{\lambda > 0} \left[ -\lambda u - \log \hat{F}(\lambda) \right] > 0. \tag{4.39}
\]
Proof of Proposition 4.4 Writing \( k_t := [v_t(x \lor 1)] \), for any \( \lambda > 0 \),

\[
\mathbb{P}_\pi(N_t > v_t(x \lor 1)) = \mathbb{P}_\pi(R_{k_t} \leq t) \leq \mathbb{P}_\pi(R_{k_t} - R_1 \leq t) \leq \mathbb{P}_\pi(e^{-\lambda(R_{k_t} - R_1)} \geq e^{-\lambda t}).
\]

But, using the definition of \( \xi_n^{(1)} \) and \( \xi_n^{(2)} \) as in (4.23), with \( f = 1 \), and the same technique as above,

\[
\mathbb{P}_\pi(e^{-\lambda(R_{k_t} - R_1)} \geq e^{-\lambda t}) \leq \mathbb{P}_\pi(e^{-\lambda \sum_{n=2}^{k_t} \xi_n^{(1)}} \geq e^{-\lambda t/2} + \mathbb{P}_\pi(e^{-\lambda \sum_{n=2}^{k_t} \xi_n^{(2)}} \geq e^{-\lambda t/2}) \leq 2 \hat{F}(\lambda)^{(k_t - 1)/2} e^{\lambda t/2} \leq 2 \hat{F}(\lambda) e^{(x \lor 1)t/2} e^{\lambda t/2}, \tag{4.40}
\]

where

\[
\hat{F}(\lambda) = \mathbb{E}(e^{-\lambda(R_2 - R_1)})
\]

is the Laplace transform of the length of a life cycle. So write

\[
h_t(\lambda, x) := -\left(1 - \frac{2}{v_t(x \lor 1)}\right) \log \hat{F}(\lambda) - \frac{\lambda t}{v_t(x \lor 1)}.
\]

Then

\[
\mathbb{P}_\pi(N_t > v_t(x \lor 1)) \leq 2e^{-\frac{3}{4}v_t(x \lor 1) \sup \lambda h_t(\lambda, x)}.
\]

Let us finally show that

\[
\Lambda^*(u) = \sup_{\lambda > 0} \left[-\lambda u - \log \hat{F}(\lambda)\right] > 0
\]

for any \( u < m \). This is evident using the fact that for \( \lambda \to 0 \), \( -\log \hat{F}(\lambda) \sim 1 - \hat{F}(\lambda) \sim m\lambda \).

This concludes the proof.

Proof of Theorem 3.1, item 1. The proof follows from Proposition 4.2 taking \( \delta = 0 \) and from Proposition 4.4 in the following way: A straightforward calculus shows that, for all \( t \geq 4m \), since \( x \lor 1 \geq 1 \),

\[
\sup \lambda h_t(\lambda, x) \geq \frac{3}{4} \sup \lambda \left(-\frac{2}{3} m \lambda - \log \hat{F}(\lambda)\right) = \frac{3}{4} \Lambda^*(\frac{2}{3} m).
\]

This concludes the proof of item 1. of Theorem 3.1.

In the null-recurrent case, recall the choice of a deterministic equivalent \( v_t^* \) of (3.9). Then the following proposition allows us to control the deviations for \( N_t \) in the null-recurrent case.

Proposition 4.5

\[
\mathbb{E}_\pi(N_t^p) \leq p!(v_t^*)^p.
\]

Proof

\[
N_t = \sum_{n \geq 1} 1_{\{R_n \leq t\}},
\]

and thus

\[
N_t^p = p! \sum_{1 \leq n_1 \leq n_2 \leq \cdots \leq n_p} 1_{\{R_{n_1} \leq t\}} \cdot \cdots \cdot 1_{\{R_{n_p} \leq t\}} = p! \sum_{1 \leq n_1 \leq n_2 \leq \cdots \leq n_p} 1_{\{R_n \leq t\}}.
\]
Hence, using successively Markov’s property with respect to $\mathcal{F}_{R_{np-1}}$ etc, we get
\[
E_\pi (N_t^p) \leq p! \sum_{1 \leq n_1 \leq n_2 \leq \ldots \leq n_{p-1}} E_\pi (1_{\left\{ R_{np-1} \leq t \right\}} E_Z_{R_{np-1}} (\sum_{n_p \geq 1} 1_{\left\{ R_p \leq t \right\}})).
\]
But note that for any $n$,
\[
E_Z_{R_n} (\sum_{k \geq 1} 1_{\left\{ R_k \leq t \right\}}) \leq 1 + E_Z_{R_n} (E_Z_{R_1} \sum_{k \geq 1} 1_{\left\{ R_k \leq t \right\}}) = 1 + E_\nu \sum_{k \geq 1} 1_{\left\{ R_k \leq t \right\}} = \nu^*(t),
\]
since $\mathcal{L}(Z_{R_1}) = \nu$. Induction on $n$ then yields the assertion.

We get the following corollary.

**Corollary 4.6** We have that
\[
\mathbb{P}_\pi (N_t > (\nu^*)^{1+\delta} (x \lor 1)) \leq 2 \exp \left( -\frac{1}{2} (\nu^*)^\delta (x \lor 1) \right).
\]

**Proof** By proposition 4.5,
\[
E_\pi (\exp(\lambda N_t)) \leq \frac{1}{1 - \lambda \nu_t^*},
\]
for $\lambda$ sufficiently small. Choosing $\lambda = (\nu^*)^{-1}/2$, we get that
\[
\mathbb{P}_\pi (N_t > (\nu_t^*)^{1+\delta} (x \lor 1)) \leq e^{-\lambda (\nu_t^*)^{1+\delta} (x \lor 1)} \mathbb{E}_\pi (\exp(\lambda N_t)) \leq 2 e^{-\frac{1}{2} (\nu_t^*)^\delta (x \lor 1)}.
\]

**Proof of Theorem 3.1** item 2. Choosing $\delta = \eta$, item 2. follows immediately from Proposition 4.2 and from (4.41) above.

**5 Proof of Theorem 3.3**

**Proof of Theorem 3.3** The proof is a slight modification of the proof of proposition 4.2. We go back to the proof of proposition 4.2. We fix the following choice of $k$ (compare to (4.32)):
\[
k = k_t = 2 + (x \lor 1) t \gamma \frac{1}{l(1/\lambda_t)}, \text{ where } \gamma = \alpha + 2 \eta \frac{1 - \alpha}{2 - \alpha}.
\]
Here, $\lambda_t \to 0$ will be defined in (5.44) below and $l$ is the function of (3.18).
As in (4.40), we have for any $\lambda > 0$,
\[
\mathbb{P}_\pi (N_t > k) \leq 2 e^{-\frac{1}{2} \left[ -\log F(\lambda (k-2) - \lambda) \right]},
\]
and therefore, due to the choice (5.42), we have to find
\[
\sup_{\lambda} \left[ -\log \hat{F}(\lambda)(x \lor 1) t^\alpha \frac{1}{l(1/\lambda t)} - \lambda t \right].
\]
Writing \( L(t) = l(1/\lambda t) \), this last expression can be rewritten as
\[
t^{\frac{1}{1 - \alpha}} (x \lor 1) \sup_{\lambda} \left[ -\log \hat{F}(\lambda) t^{\alpha \frac{1}{1 - \alpha}} \frac{1}{L(t)} - \lambda t^{\frac{1}{1 - \alpha}} \right] \geq t^{\frac{2\alpha}{1 - \alpha}} (x \lor 1) \Lambda^*_t,
\]
where
\[
\Lambda^*_t = \sup_{\lambda} \left[ -\log \hat{F}(\lambda) t^{\alpha \frac{1}{1 - \alpha}} \frac{1}{L(t)} - \lambda t^{\frac{1}{1 - \alpha}} \right].
\]
(Note that \( \frac{1}{1 - \alpha} = 1 - \frac{2\alpha}{1 - \alpha} \) ) Write for simplicity
\[
s_t := t^{\frac{1}{1 - \alpha}} = t^{1 - \frac{2\alpha}{1 - \alpha}}.
\]
Then,
\[
\Lambda^*_t = \sup_{\lambda > 0} \left[ -\log \hat{F}(\lambda) \frac{1}{L(t)} s_t^\alpha - \lambda s_t \right] = \sup_{\lambda > 0} \left[ -\log \hat{F}(\lambda) \frac{1}{l(1/\lambda t)} s_t^\alpha - \lambda s_t \right],
\]
and we have to show that \( \Lambda^*_t \) is positive. Note that since \( R_2 - R_1 \) does not possess any moments, we are not able to develop \( \hat{F}(\lambda) \) near 0 in the usual way. But we can use (3.18). That’s why we take \( \lambda \) of the form \( \lambda_t \to 0 \) at a speed that will be precised in (5.44) below. Note that
\[
\log \hat{F}(\lambda_t) = \log \left[ 1 - (1 - \hat{F}(\lambda_t)) \right].
\]
Due to (3.18) we have that
\[
-\log \hat{F}(\lambda_t) \frac{1}{l(1/\lambda_t)} s_t^\alpha - \lambda_t s_t \sim \lambda_t^\alpha s_t^\alpha - \lambda_t s_t,
\]
and hence
\[
\Lambda^*_t \geq -\log \hat{F}(\lambda_t) \frac{1}{l(1/\lambda_t)} s_t^\alpha - \lambda_t s_t \sim \lambda_t^\alpha s_t^\alpha - \lambda_t s_t \text{ as } t \to \infty. \quad (5.43)
\]
Maximizing \( \lambda^\alpha s_t^\alpha - \lambda_t s_t \) with respect to \( \lambda \) suggests the choice
\[
\lambda_t = \alpha^{1/(1 - \alpha)} s_t^{-1}. \quad (5.44)
\]
For this choice we get that
\[
\lambda_t^\alpha s_t^\alpha - \lambda_t s_t = (1 - \alpha) \alpha^{\alpha/(1 - \alpha)} > 0.
\]
This implies that \( \Lambda^*_t \) is strictly positive eventually and that
\[
\liminf_{t \to \infty} \Lambda^*_t \geq (1 - \alpha) \alpha^{\alpha/(1 - \alpha)} > 0.
\]
We continue the proof following the lines of the proof of proposition 4.2 (4.33) is true with \( v_t^{\frac{1}{4} + \eta} \) replaced by \( t^{\frac{1}{2} + \eta} \) and \( v_t^{1+\delta} \) by \( k_t \). Here,
\[
h(t, x) = \sup_{\lambda > 0} \left( \lambda \frac{x}{3k_t t^{-\frac{\eta}{2}}} - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} \right) = \sup_{\lambda > 0} \left( \lambda \frac{x \land 1}{3k_t t^{-\frac{\eta}{2}}} - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} \right),
\]
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where $\tilde{k}_t = \frac{2}{x_1} + t^\gamma / L(t)$. Let
\[ t_0 \text{ such that for all } t \geq t_0, \frac{t^\gamma}{L(t)} t^{-2-\eta} \geq 1. \] (5.45)

Then for $t \geq t_0$, with $B(f) = \max(K^2(f), K(f))$,
\[ h(t, x) \geq \frac{1}{3} (x^2 \wedge 1) L(t)^2 \left[ \frac{1}{2(x \wedge 1) K(f) + 12 K^2(f)} \right] t^{-2\gamma + \alpha + 2\eta} \geq \frac{x^2 \wedge 1}{42B(f)} L(t)^2 t^{-2\gamma + \alpha + 2\eta}. \]

**Remark 5.1** Note that in the regular case, we even have convergence in law of
\( (\frac{1}{\sqrt{v}} \int_0^t f(X_s) ds)_t \) to $\sigma B \circ W^\alpha$, where $B$ is a one-dimensional Brownian motion and $W^\alpha$ the Mittag-Leffler process of index $\alpha$, i.e. the process inverse of the stable subordinator of index $\alpha$, see for instance Touati [28].

There are various examples where the exact form of the Laplace transform is known. Brownian motion in dimension one is the most famous example.

**Example 5.2** Let $X$ be the one-dimensional standard Brownian motion. In this case, we can define the regeneration times without Nummelin splitting in the following very simple way. Let
\[ R_n = \inf\{t > S_n : X_t = 0\}, \quad S_n = \inf\{t > R_{n-1} : X_t = 1\}, \quad R_0 = 0. \]

Then
\[ \tilde{F}(\lambda) = e^{-2\sqrt{\lambda}}, \]
see Revuz-Yor ([27]). In this case it is possible to take $v_t = \sqrt{t}$ (see for example Höpfner and Löcherbach ([14]), theorem 5.6.A.).

We conclude this section with the following proposition showing that the notion of regularity (3.17) is intrinsic of the process and does not depend on the concrete splitting we are using.

**Proposition 5.3** Suppose that for $0 < \alpha < 1$ and a function $l$ varying slowly at $\infty$, the following is true. For any measurable and positive function $g$ with $0 < \mu(g) < \infty$, we have regular variation of resolvants
\[ R_1 / \mu g(x) = E_x \left( \int_0^\infty e^{-t s} g(X_s) ds \right) \sim t^\alpha \frac{1}{l(t)} \mu(g) \text{ as } t \to \infty, \] (5.46)
for $\mu$–almost all $x$. Then we have
\[ \mathbb{P}(R_2 - R_1 > x) \sim \frac{x^{-\alpha} l(x)}{\Gamma(1 - \alpha)} \text{ as } x \to \infty. \] (5.47)
Proof By theorem 3.15 of [14], we know that (5.46) implies weak convergence
\[
\frac{(A_{n_\alpha})_{\alpha \geq 0}}{n^\alpha/l(n)} \to E_\mu(A_1)W^\alpha,
\]
for any additive functional \( A_t \) of the process having \( 0 < E_\mu(A_1) < \infty \). This convergence holds true in \( D(\mathbb{R}_+, \mathbb{R}) \), under \( P_\pi \) for any initial measure \( \pi \). \( W^\alpha \) is the Mittag-Leffler process of index \( \alpha \).

From now on we fix \( \nu \) as initial measure. Interpreting \( X \) as first coordinate of \( Z \) and using Chacon-Ornstein’s theorem, we have in particular weak convergence of \( (N_n)/(n^\alpha/l(n)) \), under \( \mathbb{P}_\nu \). Now, write \( v(t) = t^\alpha/l(t) \) and let \( a(n) \) be its asymptotic inverse, i.e. \( a(n) \sim n^{1/\alpha}l(n) \). Note that \( a(n)/n \to \infty \) since \( \alpha < 1 \). \( N_t \) is the inverse process of the sequence of regeneration times \( R_n \), which implies that
\[
\frac{R_n}{a(n)} \text{ converges weakly as } n \to \infty, \text{ under } \mathbb{P}_\nu, \tag{5.48}
\]
where the limit is non-degenerated. But \( R_n = R_1 + (R_2 - R_1) + \ldots + (R_n - R_{n-1}) \). We would like to deduce from this that necessarily (5.47) is true. Unfortunately, due to the complex definition of the Nummelin splitting, the \( (R_k - R_{k-1}) \) are not independent, but they have all the same law. Independence holds true only after “nearly” exponential times, the reason for this being the choice of the new jump times in step 1. of the splitting algorithm.

That is why we proceed as follows. Define for any \( k \geq 0 \), \( \tilde{T}_k = \inf\{T_n : T_n > R_k\} \) the first jump of the process after the \( k \)-th regeneration time. Then it is straightforward to show that under \( \mathbb{P}_\nu \), \( \tilde{T}_k - R_k \) is exponentially distributed with parameter 1, and that the random times \( \tilde{T}_k - R_k, k \geq 0 \), are independent. Hence,
\[
\frac{\sum_{k=0}^{n-1}(\tilde{T}_k - R_k)}{n} \to 1, \text{ which implies that } \frac{\sum_{k=0}^{n-1}(\tilde{T}_k - R_k)}{a(n)} \to 0
\]
amost surely. This together with (5.48) yields weak convergence of
\[
\frac{\sum_{k=0}^{n-1}(R_{k+1} - \tilde{T}_k)}{a(n)}. \tag{5.49}
\]
But \( R_{k+1} - \tilde{T}_k, k \geq 0 \), is an i.i.d. sequence of random variables under \( \mathbb{P}_\nu \), thus (5.49) implies that necessarily the law of \( R_{k+1} - \tilde{T}_k \) belongs to the domain of attraction of a stable law, see for instance Feller, [10], XIII.6. This implies that
\[
\mathbb{P}_\nu(R_2 - \tilde{T}_1 > x) \sim x^{-\alpha}L(x)
\]
for some function \( L \) varying slowly at infinity. Since
\[
\mathbb{P}_\nu(R_2 - \tilde{T}_1 > x) \leq \mathbb{P}_\nu(R_2 - R_1 > x) \leq \mathbb{P}_\nu(R_2 - \tilde{T}_1 > x/2) + \mathbb{P}_\nu(\tilde{T}_1 - R_1 > x/2) = \mathbb{P}_\nu(R_2 - \tilde{T}_1 > x/2) + e^{-x/2},
\]
we deduce that
\[
x \mapsto \mathbb{P}_\nu(R_2 - R_1 > x) \text{ varies regularly at infinity with index } \alpha.
\]
This implies (5.47).
6 Application to some interacting particle systems

In some applications it is interesting to know at which speed empirical means\[ \frac{1}{t} \int_0^t f(X_s)ds \]
converge to the – in general unknown – invariant measure \( \mu(f) \). This is most often the case in statistical applications.

Consider for instance the following interacting particle system which has been studied in Galves et al. [11]. Particles are on positions (sites) of a finite subset \( V \subset \mathbb{Z}^d \), and any particle has either spin \(+1\) or \(-1\). So let \( A = \{-1,+1\} \) and let \( E = A^V \). Any element of \( E \) shall be called configuration of the system. Configurations will be denoted by letters \( \eta, \xi, \zeta \).

Any element \( i \) ∈ \( V \) is called a site. For any site \( i \) let \( \eta_i \) be the modified configuration \( \eta_i(i) = -\eta(i), \eta_i(j) = \eta(j) \) for all \( j \neq i \). We associate to any site \( i \) and any configuration \( \eta \) a spin-flip-rate \( c_i(\eta) \geq 0 \). Roughly speaking, site \( i \) will change its spin at rate \( c_i(\eta) \) whenever the overall configuration of particles is \( \eta \). We suppose that
\[
\sup_{\eta} c_i(\eta) \leq M_i
\]
for some constant \( M_i \). Then the associated interacting particle system is a Markov process \( X \) on \( E \) having generator
\[
L f(\eta) = \sum_{i \in V} c_i(\eta)[f(\eta_i) - f(\eta)].
\]

Let \( V_i(k) = \{j \in \mathbb{Z}^d; 0 \leq \|j - i\| \leq k\} \), where \( \|j\| = \sum_{u=1}^d |j_u| \) is the usual \( L_1 \)-norm of \( \mathbb{Z}^d \). In Galves et al. [11] the following criterion for recurrence of the system has been given:

**Proposition 6.1 (Theorem 3 of Galves et al.)** There exists a sequence \( \lambda_i(k), k \geq -1 \), associated to the spin-flip rates \( c_i \) such that: If
\[
\sup_{i \in V} \sum_{k \geq 0} |V_i(k)| \lambda_i(k) \leq 1,
\]
then the process is recurrent. If the above sum is strictly less than 1, then the process is uniformly exponentially ergodic. Here, \( |V_i(k)| \) is the number of sites belonging to \( V_i(k) \).

**Remark 6.2** The above theorem relies on the construction of a backward dual process \( C^{(i)}_s \), \( s \geq 0 \), such that for any site \( i \) ∈ \( V \), \( C^{(i)}_s \) denotes the set of sites at time \(-s\) that have to be known in order to determine the spin of site \( i \) at time 0. The cardinal of \( C^{(i)}_s \) can be compared to a classical branching process in continuous time, having reproduction mean \( \sum_{k \geq 0} |V_i(k)| \lambda_i(k) < 1 \), and thus being subcritical. We refer the reader to Galves et al. [11] for the details.

The sequence \( \lambda_i(k) \) can be constructed explicitly, we refer the reader to Galves et al. (2008) for the details. In [11], we were mainly concerned with the issue of perfect simulation of the invariant measure \( \mu \) of the process, under condition (6.50), based on the precise knowledge of the spin-flip rates \( c_i \). Suppose now, that we are in the converse situation, observing the process over some time interval \([0,t]\), without knowledge of the spin-flip rates, and that we want to deduce information about the associated invariant measure (and for example on the associated spin-flip rates, a posteriori). This is the classical situation of statistical inference.
Note that assumption 2.1 is satisfied and that the process is strongly Feller.

We apply Theorem 4.2 with \( \eta = \frac{1}{2} \). Since the process is uniformly exponentially ergodic, any constant function is special. In particular, we have for any bounded function \( f \), since \( f - \mu(f) \) is special, that

\[
P\left( \frac{1}{t} \int_0^t f(X_s)ds - \mu(f) > x \right) \leq ce^{-Ct(x^2 \wedge x)} + R_t(x),
\]

for some constants \( c, C \). This is a first step on the way towards estimating spin-flip rates and associated interaction schemes for interacting particles.

7 Final remarks

Our results obtained in the positive recurrent case (Theorem 3.1, item 1.) have to be compared to the results obtained by Loukianova et al. (18). The results stated in the present paper are different from the situation studied in Loukianova et al. (2009) since we consider centered functions that are special, i.e. satisfying \( \sup_x \mathbb{E}_x \int_0^R |f(X_s)ds < \infty \). Additive functionals built of special functions admit exponential moments independently of the degree of recurrence of the underlying process, and this is a very special feature about special functions that does in general not hold for other functions.

Namely, if \( f \) is any positive bounded function having compact support, then \( f \) is certainly special, but \( \bar{f} := f - \mu(f) \) in general won’t (constants are special only if the underlying process is uniformly ergodic).

Note however that our method works for any function \( f \) such that exponential moments (4.27) are finite. Thus if, for \( \lambda \) sufficiently small,

\[
\mathbb{E}_\pi(e^{\lambda(R_2-R_1)}) + \mathbb{E}_\pi(e^{\lambda R_1}) < \infty,
\]

then our result holds also for any function \( \bar{f} = f - \mu(f) \), where \( f \) is positive, bounded and of compact support. It will be the subject of some future work to give necessary conditions for (7.52).

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