The Landau-Pekar equations: Adiabatic theorem and accuracy

Nikolai Leopold, Simone Rademacher, Benjamin Schlein and Robert Seiringer

October 8, 2021

Abstract

We prove an adiabatic theorem for the Landau-Pekar equations. This allows us to derive new results on the accuracy of their use as effective equations for the time evolution generated by the Fröhlich Hamiltonian with large coupling constant $\alpha$. In particular, we show that the time evolution of Pekar product states with coherent phonon field and the electron being trapped by the phonons is well approximated by the Landau-Pekar equations until times short compared to $\alpha^2$.

I Introduction

We are interested in the dynamics of an electron in a ionic crystal. For situations in which the extension of the electron is much larger than the lattice spacing, Fröhlich [9] derived a model which treats the crystal as a continuous medium and describes the polarization of the lattice as the excitations of a quantum field, called phonons. If the coupling between the electron and the phonons is large it is expected that the dynamics of the system can be approximated by the Landau-Pekar equations, a set of nonlinear differential equations which model the phonons by means of a classical field. The coupling parameter of the Fröhlich model enters into the Landau-Pekar equations leading to a separation of time scales of the electron and the phonon field. This phenomenon, often referred to as adiabatic decoupling [23], is believed to be responsible for the classical behavior of the radiation field. The physical picture one has in mind is that the electron is trapped in a cloud of slower phonons which increase the effective mass of the electron [18].

The goal of this paper is to compare the time evolution generated by the Fröhlich Hamiltonian with the Landau-Pekar equations and to give a quantitative justification of the applied approximation. In particular, we will consider the evolution of factorized initial data, with a coherent phonon field and an electron trapped by the phonons and minimizing the corresponding energy. For such initial data, we show that the Landau-Pekar equations provide a good approximation of the dynamics, up to times short compared to $\alpha^2$, with $\alpha$ denoting the coupling between the electron and phonons (space and time units are chosen so that the electron is initially localized in a volume of order one and has speed of order one). This result improves previous bounds in [6, 7], which only hold up to times of order $\alpha$ (but for more general initial data). Also, it extends the findings of [11], which show a result similar to ours but only for initial data minimizing the Pekar energy functional (in this case, the solution of the Landau-Pekar equations remains constant). To prove our bound, we establish an adiabatic theorem for the solution of the Landau-Pekar equations. The idea of considering states with the electron trapped by the phonon field and showing an adiabatic theorem was
first proposed in [5, 8], where an adiabatic theorem is proved for a one-dimensional version of the Landau-Pekar equations. Apart from the restriction to the one-dimensional setting, the adiabatic theorem in [5, 8] differs from ours, because it compares the full solution of the Landau-Pekar equations with the solution of a limiting system of equations, independent of $\alpha$ (in Theorem II.1 below, on the other hand, we only compare the electron wave function with the ground state of the Schrödinger operator associated with the phonon field determined by the Landau-Pekar equations; this is sufficient for our purposes).

II Model and Results

We consider the Fröhlich model which describes the interaction between an electron and a quantized phonon field. The state of the phonon field is represented by an element of the bosonic Fock space $\mathcal{F} := \bigoplus_{n \geq 0} L^2(\mathbb{R}^3)^{\otimes n}_s$, where the subscript $s$ indicates symmetry under the interchange of variables. The system is described by elements $\Psi_t \in \mathcal{H}$ of the Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathcal{F}.$$  

(1)

Its time evolution is governed by the Schrödinger equation

$$i\partial_t \Psi_t = H_\alpha \Psi_t$$  

(2)

with the Fröhlich Hamiltonian

$$H_\alpha := -\Delta + \int d^3k \ |k|^{-1} \left( e^{i k \cdot x} a_k + e^{-i k \cdot x} a_k^* \right) + \int d^3k a_k^* a_k.$$  

(3)

Here, $a_k^*$ and $a_k$ are the creation and annihilation operators in the Fock space $\mathcal{F}$, satisfying the commutation relations

$$[a_k, a_{k'}^*] = \alpha^{-2} \delta(k - k'), \quad [a_k, a_{k'}] = [a_k^*, a_{k'}^*] = 0 \quad \text{for all } k, k' \in \mathbb{R}^3,$$

(4)

for a coupling constant $\alpha > 0$. One should note that the Hamiltonian is written in the strong coupling units, which gives rise to the $\alpha$ dependence in the commutation relations. These units are related to the usual ones by rescaling all lengths by $\alpha$ and time by $\alpha^{-2}$, see [6, Appendix A]. We will be interested in the limit $\alpha \to \infty$. Motivated by Pekar’s Ansatz, we consider the evolution of initial states of product form

$$\psi_0 \otimes W(\alpha^2 \varphi_0)\Omega.$$  

(5)

Here $\Omega$ denotes the vacuum of the Fock space $\mathcal{F}$ and $W(f)$ for $f \in L^2(\mathbb{R}^3)$ denotes the Weyl operator given by

$$W(f) = \exp \left[ \int d^3k \left( f(k) a_k^* - \overline{f(k)} a_k \right) \right].$$  

(6)

Note that the Weyl operator is unitary and satisfies the shifting property with respect to the creation and annihilation operator, i.e.

$$W^*(f)a_k W(f) = a_k + \alpha^{-2} f(k), \quad W^*(f)a_k^* W(f) = a_k^* + \alpha^{-2} \overline{f(k)}$$  

(7)

for all $f \in L^2(\mathbb{R}^3)$. Due to the interaction the system will develop correlations between the electron and the radiation field and the solution of (2) will no longer be of product form. However, for an appropriate class of initial states we will show that it can be approximated
up to times short compared to $\alpha^2$ (in the limit of large $\alpha$) by a product state $\psi_t \otimes W(\alpha^2 \varphi_t) \Omega$, with $(\psi_t, \varphi_t)$ being a solution of the Landau-Pekar equations

\[
\begin{align*}
    i\partial_t \psi_t &= -\Delta + \int d^3k \, |k|^{-1} \left( e^{ik \cdot x} \varphi_t(k) + e^{-ik \cdot x} \overline{\varphi_t(k)} \right) \psi_t(x), \\
    i\alpha^2 \partial_t \varphi_t(k) &= \varphi_t(k) + |k|^{-1} \int d^3x \, e^{-ik \cdot x} |\psi_t(x)|^2
\end{align*}
\]  

with initial data $(\psi_0, \varphi_0)$. We define for $\varphi \in L^2(\mathbb{R}^3)$ the potential

\[
V_{\varphi}(x) = \int d^3k \, |k|^{-1} \left( \varphi(k) e^{ik \cdot x} + \overline{\varphi(k)} e^{-ik \cdot x} \right).
\]  

We are interested, in particular, in initial data of the form (8) where the phonon field $\varphi$ is such that the Schrödinger operator

\[
h_{\varphi} := -\Delta + V_{\varphi}
\]  

has a non-degenerate eigenvalue at the bottom of its spectrum separated from the rest of the spectrum by a gap, and the electron wave function $\psi$ is a ground state vector of (10).

**Assumption II.1.** Let $\varphi_0 \in L^2(\mathbb{R}^3)$ such that

\[
e(\varphi_0) := \inf \{ \langle \psi, h_{\varphi_0} \psi \rangle : \psi \in H^1(\mathbb{R}^3), \|\psi\|_2 = 1 \} < 0.
\]  

This assumption ensures the existence of a unique positive ground state vector $\psi_{\varphi_0}$ for $h_{\varphi_0}$ with corresponding eigenvalue separated from the rest of the spectrum by a gap of size $\Lambda(0) > 0$. If we then consider solutions of (8) with initial data $(\psi_{\varphi_0}, \varphi_0)$ the spectral gap can be shown to persist at least for times of order $\alpha^2$.

**Lemma II.1.** Let $\varphi_0$ satisfy Assumption II.1 and let $(\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ denote the solution of the Landau-Pekar equations with initial value $(\psi_{\varphi_0}, \varphi_0)$. Moreover, let

\[
\Lambda(t) := \inf_{\lambda \in \text{spec}(h_{\varphi_t})} \{ e(\varphi_t) - \lambda \}.
\]  

Then, for all $\Lambda$ with $0 < \Lambda < \Lambda(0)$ there is a constant $C_{\Lambda} > 0$ such that, for all $|t| \leq C_{\Lambda} \alpha^2$, the Hamiltonian $h_{\varphi_t}$ has a unique positive and normalized ground state $\psi_{\varphi_t}$ with eigenvalue $e(\varphi_t) < 0$, which is separated from the rest of the spectrum by a gap of size $\Lambda(t) \geq \Lambda$.

The Lemma is proven in Subsection II.A. Using the persistence of the spectral gap, we can prove the following adiabatic theorem for the solution of the Landau-Pekar equations (8). As mentioned in the introduction, the idea of such result is based on [3, 8, 9], where an adiabatic theorem for the Landau-Pekar equations in one dimension is proved.

**Theorem II.1.** Let $T > 0$, $\Lambda > 0$ and $(\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ denote the solution of the Landau-Pekar equations with initial value $(\psi_{\varphi_0}, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Assume that the Hamiltonian $h_{\varphi_0}$ has a unique positive and normalized ground state $\psi_{\varphi_0}$ and a spectral gap of size $\Lambda(t) > \Lambda$ for all $|t| \leq T$. Then

\[
\|\psi_t - e^{-i \int_0^t du \, e(\varphi_u) \psi_{\varphi_u}}\|_2^2 \leq C \Lambda^{-4} \alpha^{-4} \left( 1 + (1 + \Lambda^{-1})^2 \alpha^{-4} |t|^2 \right), \quad \forall |t| \leq T.
\]  

**Remark II.1.** One also has $\|\psi_t - e^{-i \int_0^t du \, e(\varphi_u) \psi_{\varphi_t}}\|_2^2 \leq C \alpha^{-2} \Lambda^{-1} |t|$ for all $|t| \leq T$. 

3
Remark II.2. Note that the proof of the theorem only requires the existence of the spectral gap $\Lambda > 0$. Assuming $\Lambda$ to be of order one for times of order $\alpha^2$, the theorem shows that $\psi_t$ is well approximated by the ground state $\psi_{\varphi_0}$ for any $|t| \ll \alpha^4$.

Remark II.3. Lemma [11] shows that the existence of the ground state and the spectral gap for all times $|t| \leq C_{\alpha} \alpha^2$ can be inferred from Assumption [11]. In this case, [13] is valid for all $|t| \leq C_{\alpha} \alpha^2$ without any assumptions on $h_{\varphi_t}$ and $\Lambda(t)$ at times $t > 0$. Theorem II.1 implies that

\[
\|\psi_t - e^{-i \int_0^t dt \mathcal{E}(\varphi_u) \psi_{\varphi_t}}\|_2^2 \leq C_{\alpha} \alpha^{-4}
\]

for all $|t| \leq C_{\alpha} \alpha^2$.

Using Theorem II.1 we can show that the Landau-Pekar equations (3) provide a good approximation to the solution of the Schrödinger equation (2), for initial data of the form (5), with $\varphi_0$ satisfying Assumption II.1 and with $\psi_0 = \psi_{\varphi_0}$ being the ground state of the operator $h_{\varphi_0}$ defined as in (10).

Theorem II.2. Let $\varphi_0$ satisfy Assumption [11] and $\alpha_0 > 0$. Let $(\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ denote the solution of the Landau-Pekar equations with initial data $(\psi_{\varphi_0}, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and

\[
\omega(t) := \alpha^2 \text{Im} \langle \varphi_t, \partial_t \varphi_t \rangle + \|\varphi_t\|_2^2.
\]

Then, there exists a constant $C > 0$ such that

\[
\left\| e^{-i H_{\alpha t}} \psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) \Omega - e^{-i \int_0^t du \omega(u) \psi_t \otimes W(\alpha^2 \varphi_t) \Omega} \right\| \leq C \alpha^{-1} |t|^{1/2}
\]

for all $|t| \leq \alpha_0$. The constant $C > 0$ depends only on $\alpha_0 > 0$ and the initial condition, i.e. $(\psi_{\varphi_0}, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and the spectral gap $\Lambda(0)$ of $h_{\varphi_0}$.

Remark II.4. Theorem II.2 shows that the Pekar ansatz is a good approximation for times small compared to $\alpha^2$. Note that even though $(\psi_t, \varphi_t)$ stay close to their initial values for these times (as shown in Theorem II.1), it is essential to use the time-evolved version in (16). This is due to the large factor $\alpha^2$ in the Weyl operator $W(\alpha^2 \varphi_t)$, which leads to a very sensitive behavior of the state on $\varphi_t$.

During the publication process of this article it has been shown [13] that the Landau–Pekar equations approximately describe the Fröhlich evolution of the electron–and one-phonon reduced density matrices up to times of order $\alpha^2$. In order to approximate the many-body state in norm at times of order $\alpha^2$ it is, however, necessary to take quantum fluctuations into account; see [13] Theorem I.3 and Remark I.7] and [20] for initial data minimizing the Pekar energy functional. In this context we would also like to mention new results about the persistence of the spectral gap for the Landau–Pekar equations [4].

A first rigorous result concerning the effective evolution of the Fröhlich polaron was obtained in [9], where the product $\psi_t \otimes W(\alpha^2 \varphi_0) \Omega$, with $\psi_0$ solving the linear equation

\[
i \partial_t \psi_t = -\Delta \psi_t + \int dk \, |k|^{-1} \left( e^{ik \cdot x} \varphi_0(k) + e^{-ik \cdot x} \overline{\varphi_0(k)} \right) \psi_t(x)
\]

was proven to give a good approximation for the solution $e^{-i H_{\alpha t}} \psi_0 \otimes W(\alpha^2 \varphi_0) \Omega$ of the Schrödinger equation (2), up to times of order $\alpha^2$. This result was improved in [7], where

\[\text{In fact, a simple modification of the Gronwall argument in [5] leads to convergence for times } |t| \ll \alpha.\]
convergence towards \( \psi \) was established for all times \( |t| \ll \alpha \) (the analysis in that work also
gives more detailed information on the solution of Schrödinger equation \( 2 \), in particular, it
implies convergence of reduced density matrices). Note that the results of \( 6, 7 \) hold for gen-
eral initial data of the form \( 5 \), with no assumption on the relation between the initial electron
wave function \( \psi_0 \) and the initial phonon field \( \varphi_0 \). Theorem \( 11.2 \) shows therefore that, under the
additional assumption that \( \psi_0 = \psi \varphi_0 \) the Landau-Pekar equations \( 8 \) provide a good approx-
imation to \( 2 \) for longer times (times short compared to \( \alpha^2 \)). In this sense, Theorem \( 11.2 \)
also extends the result of \( 11 \), where the validity of the Landau-Pekar equations was established
for times short compared to \( \alpha^2 \), but only for initial data \( (\psi_0, \varphi_0) \) minimizing the Pekar energy
functional (in this case, the solution of \( 8 \) is stationary, i.e. \( (\psi_t, \varphi_t) = (e^{-itV(\varphi_0)}\psi_0, \varphi_0) \) for
all \( t \)). In fact, similarly to the analysis in \( 11 \), we use the observation that the spectral gap
above the ground state energy of \( h_{\varphi_0} \) allows us to obtain bounds that are valid on longer time
scales (it allows us to integrate by parts, after \( 60 \) and after \( 97 \); this step is crucial to save a factor of \( t \). The classical behavior of a quantum field does not only appear in the strong
coupling limit of the Fröhlich polaron but has also been studied in other situations. In \( 10 \)
it was shown in case of the Nelson model that a quantum scalar field behaves classically in a certain limit
where the number of field bosons becomes infinite while the coupling constant tends to zero. The emergence of classical radiation was also proven for the Nelson model with
ultraviolet cutoff \( 3, 14 \), the renormalized Nelson model \( 1 \) and the Pauli-Fierz Hamiltonian
\( 15 \) in situations in which a large number of particles weakly couple to the radiation field.
The articles \( 2, 12, 22 \) revealed in addition that quantum fields can sometimes be replaced
by two-particle interactions if the particles are much slower than the bosons of the quantum
field.

### III Preliminaries

In this section, we collect properties of the Landau-Pekar equations that are used in the proofs
of Theorem \( 11.1 \) and Theorem \( 11.2 \). For \( \psi \in L^2(\mathbb{R}^3) \) we define the function

\[
\sigma_\psi(k) = |k|^{-1} \int d^3x \, e^{-ik \cdot x} |\psi(x)|^2.
\]

The first and second derivative of the potential \( V_{\varphi_0} \) are given by \( 2 \)

\[
\partial_t V_{\varphi_0}(x) = V_{i\varphi_0}(x) = -\alpha^{-2} \int d^3k \, |k|^{-1} \left( e^{ik \cdot x \varphi_0} + e^{-ik \cdot x \varphi_0} \right)
- i\alpha^{-2} \int d^3k \, |k|^{-1} \left( e^{ik \cdot x \sigma_{\varphi_0}(k)} - e^{-ik \cdot x \sigma_{\varphi_0}(-k)} \right) = -\alpha^{-2} V_{\varphi_0}(x) \tag{19}
\]

and

\[
\partial_t V_{i\varphi_0}(x) = V_{i\varphi_0}(x) = \alpha^{-2} V_{\varphi_0}(x) + \alpha^{-2} V_{\sigma_{\varphi_0}}(x) \tag{20}
\]

We define the energy functional \( \mathcal{E} : H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to \mathbb{R} \)

\[
\mathcal{E}(\psi, \varphi) = \langle \psi, h_{\varphi} \psi \rangle + \|\varphi\|_2^2. \tag{21}
\]

Using standard methods one can show that the Landau-Pekar equations are well posed and
that the energy \( \mathcal{E}(\psi_t, \varphi_t) \) is conserved if \( (\psi_t, \varphi_t) \) is a solution of \( 8 \). For a proof of the following
Lemma see \( 11 \) Appendix C.\(^2\)

\(^2\)We use the notation \( f' \) to denote the derivative of a function \( f \) with respect to time.
Lemma III.1 ([7], Lemma 2.1). For any \((\psi_0, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\), there is a unique global solution \((\psi_t, \varphi_t)\) of the Landau-Pekar equations ([8]). The following conservation laws hold true
\[
\|\psi_t\|_2 = \|\psi_0\|_2 \quad \text{and} \quad \mathcal{E}(\psi_t, \varphi_t) = \mathcal{E}(\psi_0, \varphi_0) \quad \forall t \in \mathbb{R}^3.
\] (22)
Moreover, there exists a constant \(C\) such that
\[
\|\psi_t\|_{H^1(\mathbb{R}^3)} \leq C, \quad \|\varphi_t\|_2 \leq C
\] (23)
for all \(\alpha > 0\) and all \(t \in \mathbb{R}\).

The next Lemma (also proven in [7, Appendices B and C]) collects some properties of quantities occurring in the Landau-Pekar equations.

Lemma III.2. For \(V_\varphi\) being defined as in (9) there exists a constant \(C > 0\) such that for every \(\psi \in H^1(\mathbb{R}^3)\) and \(\varphi \in L^2(\mathbb{R}^3)\)
\[
\|V_\varphi\|_6 \leq C\|\varphi\|_2 \quad \text{and} \quad \|V_\varphi \psi\|_2 \leq C\|\varphi\|_2 \|\psi\|_{H^1(\mathbb{R}^3)}.
\] (24)
Furthermore, for every \(\delta > 0\) there exists \(C_\delta > 0\) such that
\[
\pm V_\varphi \leq -\delta \Delta + C_\delta,
\] (25)
thus there exists \(C > 0\) such that
\[
-\frac{1}{2}\Delta - C \leq h_\varphi \leq -2\Delta + C.
\] (26)

Let \(\sigma_\psi\) be defined as in (18). Then, there exists \(C > 0\) such that
\[
\|\sigma_\psi\|_2 \leq C\|\psi\|_{H^1(\mathbb{R}^3)}^2.
\] (27)

Remark III.1. Let \(T > 0\), \(\Lambda > 0\) and \((\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) denote the solution of the Landau-Pekar equations with initial value \((\psi_{\varphi_0}, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\). Assume that the Hamiltonian \(h_\varphi\) has a unique positive and normalized ground state \(\psi_\varphi\) and a spectral gap of size \(\Lambda(t) > \Lambda\) for all \(t \leq T\). Lemma III.2 then implies the existence of constant such that
\[
\|\psi_{\varphi_t}\|_{H^1(\mathbb{R}^3)} \leq C \quad \forall |t| \leq T.
\] (28)

Proof of Lemma III.2. Recall the definition (9) of the potential \(V_\varphi\). We write
\[
V_\varphi(x) = 2^{3/2} \pi^{-1/2} \text{Re} \int \frac{d^3y}{|x-y|^2} \hat{\varphi}(y),
\] (29)
where \(\hat{\varphi}\) denotes the inverse Fourier transform defined for \(\varphi \in L^1(\mathbb{R}^3)\) through
\[
\hat{\varphi}(x) = (2\pi)^{-3/2} \int d^3k e^{ik \cdot x} \varphi(k).
\] (30)
The first inequality follows directly from the Hardy-Littlewood-Sobolev inequality
\[
\|V_\varphi\|_6 \leq C\|\varphi\|_2.
\] (31)
In order to prove the second inequality we use the first one and the Hölder inequality
\[
\|V_\varphi \psi\|_2 \leq \|V_\varphi\|_6 \|\psi\|_3 \leq C\|\varphi\|_2 \|\psi\|_3.
\] (32)

\[6\]
IV. The ground state

IV.1 The ground state

Proof of the adiabatic theorem

and the Sobolev inequality imply (27). As before, the interpolation inequality together with the Hardy-Littlewood-Sobolev inequality implies (26) follows.

Before proving the adiabatic theorem, we show that the spectral gap of $H$ is positive and (26) follows.

The second operator inequality follows again from the Sobolev inequality. For this let $\psi \in H^1(\mathbb{R}^3)$, then for $\varepsilon > 0$

$$\langle \psi, V_\varphi \psi \rangle \leq C \| V_\varphi \|_6 \| \psi \|_{12/5}^2 \leq C \| V_\varphi \|_6 (\varepsilon \| \nabla \psi \|_2^2 + \varepsilon^{-1} \| \psi \|_2^2),$$

where we used the interpolation inequality implies the first inequality of the Lemma implies

$$\pm \langle \psi, V_\varphi \psi \rangle \leq \langle \psi, (-\varepsilon \Delta + C_\varepsilon) \psi \rangle,$$

and (26) follows.

The last inequality of the Lemma follows from the observation that

$$\| \sigma_\psi \|_2^2 = \int \frac{dk}{|k|^2} \int dxdy \left| \psi(x) \right|^2 \left| \psi(y) \right|^2 e^{ik(x-y)}$$

$$= 2\pi^2 \int dxdy \left| \psi(x) \right|^2 \left| \psi(y) \right|^2 \frac{1}{|x-y|} \leq C \| \psi \|_{6/5}^2,$$

where we used again the Hardy-Littlewood-Sobolev inequality. As before, the interpolation and the Sobolev inequality imply (27).

IV. Proof of the adiabatic theorem

IV.1 The ground state $\psi_{\varphi_1}$

Before proving the adiabatic theorem, we show that the spectral gap of $h_{\varphi_1}$ does not close for times of order $\alpha^2$. In particular, we show the existence of the ground state $\psi_{\varphi_1}$.

**Lemma IV.1.** Let $\varphi_0$ satisfy Assumption [17]. Then, there exists a unique positive and normalized ground state $\psi_{\varphi_1}$ of $h_{\varphi_1} = -\Delta + V_{\varphi_1}$. Let $(\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ denote the solution of the Landau-Pekar equations ([8]) with initial value $(\psi_{\varphi_0}, \varphi_0)$. There exists $C > 0$ such that for all $|t| \leq C\alpha^2$, there exists a unique, positive and normalized ground state $\psi_{\varphi_1}$ of $h_{\varphi_1} = -\Delta + V_{\varphi_1}$ with corresponding eigenvalue $e(\varphi_1) < 0$. It satisfies

$$\partial_t \psi_{\varphi_1} = \alpha^{-2} R_t V_{\varphi_1} \psi_{\varphi_1} \text{ with } R_t = q_t(h_{\varphi_1} - e(\varphi_1))^{-1} q_t,$$

where $q_t = 1 - |\psi_{\varphi_1}\rangle \langle \psi_{\varphi_1}|$ denotes the projection onto the subspace of $L^2(\mathbb{R}^3)$ orthogonal to the span of $\psi_{\varphi_1}$.

**Proof.** Lemma [III.1] and Lemma [III.2] imply that $V_{\varphi_1} \in L^6(\mathbb{R}^3)$ for all $t \in \mathbb{R}$. The existence of the ground state $\psi_{\varphi_1}$ at time $t = 0$ then follows from the negativity of the infimum of the spectrum; see [17] Theorem 11.5. In order to prove the existence of the ground state $\psi_{\varphi_1}$ at later times, it suffices to show that $e(\varphi_1)$ is negative. For this we pick the ground state $\psi_{\varphi_0}$ at time $t = 0$ and estimate

$$e(\varphi_1) \leq \langle \psi_{\varphi_0}, (-\Delta + V_{\varphi_1}) \psi_{\varphi_0} \rangle = e(\varphi_0) - \alpha^{-2} \int_0^t ds \langle \psi_{\varphi_0}, V_{\varphi_1} \psi_{\varphi_0} \rangle$$

$$\leq e(\varphi_0) + C \int_0^t ds \alpha^{-2} \| \varphi_s \|_{L^2(\mathbb{R}^3)}^2 \leq e(\varphi_0) + C|t|\alpha^{-2},$$

(39)
by means of \((\ref{eq:8})\), Lemma \((\ref{lem:lemma3.1})\) and Lemma \((\ref{lem:lemma3.2})\). Thus if we restrict our consideration to times \(|t| < C^{-1}|e(\varphi_0)|\alpha^2\), we conclude that \(e(\varphi_t) < 0\). The ground state \(\psi_{\varphi_t}\) satisfies

\[
0 = (h_{\varphi_t} - e(\varphi_t)) \psi_{\varphi_t}.
\]

(40)

Differentiating both sides of the equality with respect to the time variable leads to

\[
0 = \left( \hat{h}_{\varphi_t} - \dot{e}(\varphi_t) \right) \psi_{\varphi_t} + (h_{\varphi_t} - e(\varphi_t)) \dot{\psi}_{\varphi_t}.
\]

(41)

On the one hand \(\hat{h}_{\varphi_t} = V_{\varphi_t} = -\alpha^{-2} V_{\varphi_t}\) by means of \((\ref{eq:8})\) and on the other hand, the Hellmann-Feynman theorem implies

\[
\dot{e}(\varphi_t) = \langle \psi_{\varphi_t}, \dot{h}_{\varphi_t} \psi_{\varphi_t} \rangle = -\alpha^{-2} \langle \psi_{\varphi_t}, V_{\varphi_t} \psi_{\varphi_t} \rangle
\]

so that \((\ref{eq:8})\) becomes

\[
0 = -\alpha^{-2} q_t V_{\varphi_t} \psi_{\varphi_t} + (h_{\varphi_t} - e(\varphi_t)) \dot{\psi}_{\varphi_t}.
\]

(43)

Since \(\psi_{\varphi_t}\) is chosen to be real and normalized for all \(t \in \mathbb{R}\), it follows that \(\langle \psi_{\varphi_t}, \dot{\psi}_{\varphi_t} \rangle = 0\) for all \(t \in \mathbb{R}\). Hence,

\[
\dot{\psi}_{\varphi_t} = \alpha^{-2} q_t (h_{\varphi_t} - e(\varphi_t))^{-1} q_t V_{\varphi_t} \psi_{\varphi_t} = \alpha^{-2} R_t V_{\varphi_t} \psi_{\varphi_t},
\]

(44)

completing the proof. \(\blacksquare\)

Using the Lemma above, we prove Lemma \((\ref{lem:lemma3.1})\).

Proof of Lemma \((\ref{lem:lemma3.1})\). By the min-max principle \((\ref{thm:min-max-principle})\) Theorem 12.1, the first excited eigenvalue of \(h_{\varphi_t}\) (or the bottom of the essential spectrum) is given by

\[
e_1(t) = \inf_{A \in L^2(\mathbb{R}^3)} \sup_{\text{dim}A = 2} \left\{ \psi, h_{\varphi_t} \psi \right\}.
\]

(45)

For any \(\psi \in L^2(\mathbb{R}^3)\) with \(\|\psi\|_2 = 1\) we have by Lemma \((\ref{lem:lemma3.2})\)

\[
\langle \psi, h_{\varphi_t} \psi \rangle = \langle \psi, h_{\varphi_0} \psi \rangle - \alpha^{-2} \int_0^t ds \langle \psi, V_{\varphi_s} \psi \rangle
\]

\[
\geq \langle \psi, h_{\varphi_0} \psi \rangle - C|t| \alpha^{-2} \sup_{|s| \leq |t|} \|\nabla \psi\|_2 \|\psi\|_2 - C|t| \alpha^{-2} \sup_{|s| \leq |t|} \|\varphi_s\|_2 \|\psi\|_2
\]

\[
\geq (1 - 2C|t| \alpha^{-2}) \langle \psi, h_{\varphi_0} \psi \rangle - C|t| \alpha^{-2}
\]

(46)

Inserting in \((\ref{eq:45})\), we conclude that

\[
e_1(t) \geq (1 - 2C|t| \alpha^{-2}) e_1(0) - C|t| \alpha^{-2}.
\]

(47)

With \(e(\varphi_t) \leq e(\varphi_0) + C|t| \alpha^{-2}\) (see \((\ref{eq:39})\)), we obtain

\[
\Lambda(t) \geq \Lambda(0) - C|t| \alpha^{-2},
\]

(48)

completing the proof. \(\blacksquare\)

Using the persistence of the spectral gap, the resolvent \(R_t = q_t (h_{\varphi_t} - e(\varphi_t))^{-1} q_t\) can be estimated as follows.
Lemma IV.2. Let \( T > 0 \), \( \Lambda > 0 \) and \((\psi_t, \phi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) denote the solution of the Landau-Pekar equations with initial value \((\psi_{\varphi_0}, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\). Assume that the Hamiltonian \( h_{\varphi_0} \) has a unique positive and normalized ground state \( \psi_{\varphi_0} \) with \( e(\varphi_0) < 0 \) and a spectral gap of size \( \Lambda(t) > \Lambda \) for all \( t \leq T \). Then, for all \(|t| \leq T\)

\[
\|R_t\| \leq \Lambda^{-1}, \quad \|(-\Delta + 1)^{1/2} R_t^{1/2}\| \leq C(1 + \Lambda^{-1})^{1/2},
\]

and

\[
\|\dot{R}_t\| \leq C\Lambda^{-3/2}\alpha^{-2}(1 + \Lambda^{-1})^{1/2}
\]

with \( C > 0 \) depending only on \( \varphi_0 \).

Proof. Since the spectral gap is at least of size \( \Lambda > 0 \) for times \(|t| \leq T\), it follows that

\[
\|R_t\| \leq \Lambda^{-1}.
\]

To prove the second inequality, we estimate for arbitrary \( \psi \in L^2(\mathbb{R}^3) \)

\[
\|(-\Delta + 1)^{1/2} R_t^{1/2}\psi\|_2^2 = \langle \psi, R_t^{1/2}(-\Delta + 1)R_t^{1/2}\psi \rangle \leq C\langle \psi, R_t^{1/2}(h_{\varphi_t} + C)R_t^{1/2}\psi \rangle,
\]

where the last inequality follows from Lemma III.2. Thus,

\[
\|(-\Delta + 1)^{1/2} R_t^{1/2}\psi\|_2^2 \leq C\langle \psi, R_t^{1/2}(h_{\varphi_t} - e(\varphi_t) + e(\varphi_t) + C)R_t^{1/2}\psi \rangle
\]

\[
= C\langle \psi, (q_t + (e(\varphi_t) + C)R_t)\psi \rangle
\]

\[
\leq C\langle \psi, (1 + R_t)\psi \rangle
\]

since \( e(\varphi_t) < 0 \) for all \(|t| \leq T\) by assumption. The gap condition then implies

\[
\|(-\Delta + 1)^{1/2} R_t^{1/2}\psi\|_2^2 \leq C(1 + \Lambda^{-1}).
\]

In order to prove the third bound of the Lemma we calculate (with \( p_t = 1 - q_t \))

\[
\dot{R}_t = -\alpha^{-2}p_tV_{\varphi_t}R_t^2 - \alpha^{-2}R_t^2V_{\varphi_t}p_t + q_t(\partial_t(h_{\varphi_t} - e(\varphi_t))^{-1})q_t
\]

by means of the Leibniz rule and Lemma IV.1 With the resolvent identities, \[9\] and \[12\] this becomes

\[
\dot{R}_t = -\alpha^{-2}p_tV_{\varphi_t}R_t^2 - \alpha^{-2}R_t^2V_{\varphi_t}p_t - q_t(h_{\varphi_t} - e(\varphi_t))^{-1}\left(h_{\varphi_t} - \dot{e}(\varphi_t)\right)(h_{\varphi_t} - e(\varphi_t))^{-1}q_t
\]

\[
= -\alpha^{-2}p_tV_{\varphi_t}R_t^2 - \alpha^{-2}R_t^2V_{\varphi_t}p_t + \alpha^{-2}R_t(V_{\varphi_t} - \langle \psi_{\varphi_t}, V_{\varphi_t}\psi_{\varphi_t} \rangle) R_t.
\]

Hence, Lemma III.2 leads to

\[
\|\dot{R}_t\| \leq C\alpha^{-2}\|V_{\varphi_t}R_t\| \|R_t\| + C\alpha^{-2}\|R_t\|^2 \|\psi_{\varphi_t}\|_{H^1(\mathbb{R}^3)}^2
\]

\[
\leq C\alpha^{-2}\|(-\Delta + 1)^{1/2} R_t\| \|R_t\| + \alpha^{-2}\|R_t\|^2 \|\psi_{\varphi_t}\|_{H^1(\mathbb{R}^3)}^2
\]

\[
\leq C\Lambda^{-3/2}\alpha^{-2}(1 + \Lambda^{-1})^{1/2} + C\alpha^{-2}\Lambda^{-2},
\]

for all \(|t| \leq T\), where we used \[25\] and the second bound of the Lemma.
IV.2 Proof of Theorem II.1

In the following we denote \( \tilde{\psi}_{\phi_t} = e^{-i \int_0^t du \phi(u)} \psi_{\phi_t} \). The fundamental theorem of calculus implies that

\[
\| \psi_t - \tilde{\psi}_{\phi_t} \|_2^2 = - \int_0^t ds \frac{d}{ds} 2 \text{Re} \langle \psi_s, \tilde{\psi}_{\phi_s} \rangle \\
= - 2 \int_0^t ds \text{Re} (-ih_{\phi_s} \psi_s, \tilde{\psi}_{\phi_s}) \\
- 2 \int_0^t ds \text{Re} \langle \psi_s, (-ie(\phi_s) + \alpha^{-2} R_s V_i \phi_s) \tilde{\psi}_{\phi_s} \rangle \\
= - 2\alpha^{-2} \int_0^t ds \text{Re} \langle \psi_s, R_s V_i \phi_s \tilde{\psi}_{\phi_s} \rangle,
\]

where we used that \( h_{\phi_s} \tilde{\psi}_{\phi_s} = e(\phi_s) \tilde{\psi}_{\phi_s} \) and Lemma IV.1 to compute the derivative of the ground state \( \psi_{\phi_s} \). Using the Cauchy-Schwarz inequality, Lemma II.1 and Lemma III.2 together with Lemma IV.2 and (28), we obtain the inequality from Remark II.1, i.e. a bound of order \( \alpha^{-2} |t| \). In the following we shall improve this bound. We define \( \tilde{\psi}_s := e^{i \int_0^t d\tau e(\phi_{\tau})} \tilde{\psi}_{\phi_s} \) satisfying

\[
i \partial_s \tilde{\psi}_s = (h_{\phi_s} - e(\phi_s)) \tilde{\psi}_s
\]

and write (58) as

\[
\| \psi_t - \tilde{\psi}_{\phi_t} \|_2^2 = - 2\alpha^{-2} \int_0^t ds \text{Re} \langle q_s \tilde{\psi}_s, R_s V_i \phi_s \psi_{\phi_s} \rangle.
\]

Then, we exploit that the time derivative of \( \tilde{\psi}_s \) is of order one while the time derivatives of \( R_s, V_i \phi_s \) and \( \psi_{\phi_s} \) are of order \( \alpha^{-2} \); compare also with [23, p.9]. We observe that

\[
i \partial_s \left( \tilde{\psi}_s - \psi_{\phi_s} \right) = (h_{\phi_s} - e(\phi_s)) \tilde{\psi}_s - i\alpha^{-2} R_s V_i \psi_{\phi_s}
\]

Hence,

\[
q_s \tilde{\psi}_s = R_s i \partial_s \left( \tilde{\psi}_s - \psi_{\phi_s} \right) + i\alpha^{-2} R_s^2 V_i \psi_{\phi_s}.
\]

Plugging this identity into (60) and using that \( \text{Im} \langle \phi_s, V_i \psi_{\phi_s}, R_s^2 V_i \psi_{\phi_s} \rangle = 0 \), we obtain

\[
\| \psi_t - \tilde{\psi}_{\phi_t} \|_2^2 = - 2\alpha^{-2} \int_0^t ds \text{Im} \langle \partial_s (\tilde{\psi}_s - \psi_{\phi_s}), R_s^2 V_i \psi_{\phi_s} \rangle.
\]

Integrating by parts and using the initial condition \( \psi_0 = \psi_{\phi_0} \) lead to

\[
\| \psi_t - \tilde{\psi}_{\phi_t} \|_2^2 = 2\alpha^{-2} \int_0^t ds \text{Im} \left( \left( \tilde{\psi}_s - \psi_{\phi_s} \right), \partial_s (R_s^2 V_i \psi_{\phi_s}) \right) \\
- 2\alpha^{-2} \text{Im} \left( \left( \tilde{\psi}_t - \psi_{\phi_t} \right), R_t^2 V_i \psi_{\phi_t} \right).
\]
The Leibniz rule with (20) and Lemma [V.1] lead to
\[ \| \psi_t - \tilde{\psi}_\varphi_t \|_2^2 = -2\alpha^{-2} \text{Im}(R_t \left( \psi_t - \tilde{\psi}_\varphi \right), R_t V_{\varphi,\psi} \tilde{\psi}_\varphi) \] (65a)
\[ + 2\alpha^{-2} \int_0^t ds \text{Im}\left( \left( \psi_s - \tilde{\psi}_\varphi \right), \left( \partial_s R_s^2 \right) V_{\varphi,\psi} \tilde{\psi}_\varphi \right) \] (65b)
\[ + 2\alpha^{-4} \int_0^t ds \text{Im}(R_s \left( \psi_s - \tilde{\psi}_\varphi \right), R_s V_{\varphi,\psi} \tilde{\psi}_\varphi) \] (65c)
\[ + 2\alpha^{-4} \int_0^t ds \text{Im}(R_s \left( \psi_s - \tilde{\psi}_\varphi \right), (R_s V_{\varphi,\psi})^2 \tilde{\psi}_\varphi). \] (65d)

Using Lemma [III.2] the first term can be estimated by
\[ |(65a)| \leq C\alpha^{-2} \| R_t \|_2^2 \| V_{\varphi,\psi} \tilde{\psi}_\varphi \|_2 \| \psi_t - \tilde{\psi}_\varphi \|_2 \]
\[ \leq C\alpha^{-2} \| R_t \|_2^2 \| \tilde{\psi}_\varphi \|_{H^1(\mathbb{R}^3)} \| \varphi_t \|_2 \| \psi_t - \tilde{\psi}_\varphi \|_2. \] (66)

On the one hand Lemma [III.1] and (28) show that \( \| \varphi_t \|_2 \) and \( \| \tilde{\psi}_\varphi \|_{H^1(\mathbb{R}^3)} \) are uniformly bounded in time. On the other hand Lemma [IV.2] implies that the resolvent \( R_t \) is bounded for all times \( |t| \leq T \), so that we obtain
\[ |(65a)| \leq C\alpha^{-2} \| \psi_t - \tilde{\psi}_\varphi \|_2 \leq \frac{1}{2} \| \psi_t - \tilde{\psi}_\varphi \|_2^2 + C\alpha^{-4}, \quad \forall |t| \leq T. \] (67)

Similarly, we bound the second term by
\[ |(65b)| \leq C\alpha^{-2} \int_0^t ds \| R_s \| \| V_{\varphi,\psi} \tilde{\psi}_\varphi \|_2 \| \varphi_t \| \| \psi_s - \tilde{\psi}_\varphi \|_2 \]
\[ \leq C\alpha^{-4} \Lambda^{-5/2} (1 + \Lambda^{-1})^{1/2} \int_0^t ds \| \psi_s - \tilde{\psi}_\varphi \|_2 \] (68)
for all \( |t| \leq T \), using Lemma [III.1] Lemma [III.2] and Lemma [IV.2]. The third term \( (65c) \) can be bounded using \( \| \sigma \psi_t \|_2 \leq C\| \psi_t \|_{H^1(\mathbb{R}^3)}^2 \leq C \) by Lemma [III.1]. We find
\[ |(65c)| \leq C\alpha^{-2} \| \sigma \psi_t \| \int_0^t ds \| \psi_s - \tilde{\psi}_\varphi \|_2, \quad \forall |t| \leq T. \] (69)

Using the same ideas we estimate the last term by
\[ |(65d)| \leq C\alpha^{-4} \int_0^t ds \| R_s \| \| V_{\varphi,\psi} R_s \|_2 \| \psi_s - \tilde{\psi}_\varphi \|_2 \]
\[ \leq C\alpha^{-1} \alpha^{-4} \int_0^t ds \| V_{\varphi,\psi} R_s \|_2 \| \psi_s - \tilde{\psi}_\varphi \|_2. \] (70)

Lemma [III.2] implies that for all \( \psi \in H^1(\mathbb{R}^3) \)
\[ \| V_{\varphi,\psi} \psi \|_2 \leq C \| \varphi_t \|_2 \| \psi \|_{H^1(\mathbb{R}^3)} \] (71)
and therefore that \( \| V_{\varphi,\psi} (-\Delta + 1)^{-1/2} \| \leq C \) for all \( |t| \leq T \) by Lemma [III.2]. Hence
\[ \| V_{\varphi,\psi} R_s \| \leq C \| (-\Delta + 1)^{1/2} R_s \| \leq C\alpha^{-1/2} (1 + \Lambda^{-1})^{1/2}, \] (72)
for all $|t| \leq T$ where we used Lemma [IV.2] for the last inequality. Thus,

$$\|\psi_t - \tilde{\psi}_{\varphi_t}\|_2 \leq C \alpha^{-4} \Lambda^{-2} (1 + \Lambda^{-1}) \int_0^T ds \|\psi_s - \tilde{\psi}_{\varphi_s}\|_2, \quad \forall |t| \leq T,$$

(73)

In total, this leads to the estimate

$$\|\psi_t - \tilde{\psi}_{\varphi_t}\|_2^2 \leq C \Lambda^{-4} \alpha^{-4} + C \Lambda^{-2} (1 + \Lambda^{-1}) \alpha^{-4} \int_0^T ds \|\psi_s - \tilde{\psi}_{\varphi_s}\|_2.$$

(74)

A Gronwall type argument then leads to

$$\|\psi_t - \tilde{\psi}_{\varphi_t}\|_2^2 \leq C \Lambda^{-4} \alpha^{-4} + C \Lambda^{-4} (1 + \Lambda^{-1})^2 \alpha^{-8} |t|^2,$$

(75)

completing the proof. \qed

V Accuracy of the Landau-Pekar equations

V.1 Preliminaries

For notational convenience we define

$$\Phi_x = \int d^3 k |k|^{-1} \left( e^{ik \cdot x} a_k + e^{-ik \cdot x} a_k^\ast \right) = \Phi_x^+ + \Phi_x^-$$

(76)

with

$$\Phi_x^+ = \int d^3 k |k|^{-1} e^{ik \cdot x} a_k, \quad \text{and} \quad \Phi_x^- = \int d^3 k |k|^{-1} e^{-ik \cdot x} a_k^\ast.$$  

(77)

In addition we introduce for $f \in L^2(\mathbb{R}^3)$ the creation operator $a^\ast (f)$ and the annihilation operator $a(f)$ which are given by

$$a^\ast (f) = \int d^3 k f(k) a_k^\ast, \quad a(f) = \int d^3 k \overline{f(k)} a_k$$

(78)

and bounded with respect to the number of particles operator $\mathcal{N} = \int d^3 k a_k^\ast a_k$, i.e.

$$\|a(f)\xi\| \leq \|f\|_2 \|\mathcal{N}^{1/2} \xi\|, \quad \|a^\ast (f)\xi\| \leq \|f\|_2 \|\mathcal{N} + \alpha^{-2}\|^{1/2} \xi\|$$

(79)

for all $\xi \in \mathcal{F}$. Moreover, recall the definition (6) of the Weyl operator $W(f) = e^{a^\ast (f) - a(f)}$. For a time dependent function $f_t \in L^2(\mathbb{R}^3)$ the time derivative of the Weyl operator is given by

$$\partial_t W(f_t) = \frac{\alpha^{-2}}{2} (\langle f_t, \partial_t f_t \rangle - \langle \partial_t f_t, f_t \rangle) W(f_t) + (a^\ast \partial_t f_t) W(f_t).$$

(80)

The proof of this formula can be found in [7, Lemma A.3].
V.2 Proof of Theorem II.2

It should be noted that (15) is valid for all times which are at least of order $\alpha^2$ because both states in the inequality have norm one. To show its validity for shorter times we split the norm difference into two parts by the triangle inequality and use Remark II.1 to estimate

$$\left\| e^{-iH_{\alpha}t} \psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) - e^{-i \int_0^t du \omega(u)} \tilde{\psi}_{\varphi_t} \otimes W(\alpha^2 \varphi_t) \right\|^2$$

$$\leq 2 \left\| e^{-iH_{\alpha}t} \psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) - e^{-i \int_0^t du \omega(u)} \tilde{\psi}_{\varphi_t} \otimes W(\alpha^2 \varphi_t) \right\|^2$$

$$+ 2 \left\| \tilde{\psi}_{\varphi_t} \otimes W(\alpha^2 \varphi_t) - \psi_t \otimes W(\alpha^2 \varphi_t) \right\|^2$$

$$\leq C \alpha^{-2} |t| + 2 \left\| e^{-iH_{\alpha}t} \psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) - e^{-i \int_0^t du \omega(u)} \tilde{\psi}_{\varphi_t} \otimes W(\alpha^2 \varphi_t) \right\|^2$$

(81)

for all $|t| \leq C \Lambda \alpha^2$ where we used the notation $\tilde{\psi}_{\varphi_t} = e^{-i \int_0^t du \omega(u) \varphi_t} \psi_{\varphi_t}$. Therefore it remains to estimate the second term. For this, we introduce

$$\xi_s = e^{t \int_0^s du \omega(u)} W^*(\alpha^2 \varphi_s) e^{-iH_{\alpha}s} \psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) \Omega$$

(82)

to shorten the notation and compute

$$W^*(\alpha^2 \varphi_t) H_{\alpha} W(\alpha^2 \varphi_t) = h_{\varphi_t} + \| \varphi_t \|^2_2 + \mathcal{N} + \Phi_x + a(\varphi_t) + a^*(\varphi_t).$$

Using (7), this leads to

$$i \partial_s \xi_s = (i \partial_s W^*(\alpha^2 \varphi_s)) W(\alpha^2 \varphi_s) \xi_s + \left( W^*(\alpha^2 \varphi_s) H_{\alpha} W(\alpha^2 \varphi_s) - \omega(s) \right) \xi_s$$

$$= \left( h_{\varphi_s} + \Phi_x - a(\varphi_s) - a^*(\varphi_s) + \mathcal{N} \right) \xi_s.$$

(83)

Note that $\xi_0 = \psi_{\varphi_0} \otimes \Omega$. We apply Duhamel’s formula and use the unitarity of the Weyl operator $W(\alpha^2 \varphi_t)$ to compute

$$\left\| e^{-iH_{\alpha}t} \psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) - e^{-i \int_0^t du \omega(u) \tilde{\psi}_{\varphi_t} \otimes W(\alpha^2 \varphi_t) \right\|^2$$

$$= \left\| \xi_t - \tilde{\psi}_{\varphi_t} \otimes \Omega \right\|^2 = \int_0^t ds \partial_s \left\| \xi_s - \tilde{\psi}_{\varphi_s} \otimes \Omega \right\|^2 = -2 \text{Re} \int_0^t ds \partial_s \langle \xi_s, \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle.$$  

(84)

Recall that $\Phi_x = \int d^3k |k|^{-1} (e^{ik \cdot x} a_k + e^{-ik \cdot x} a_k^*)$. We obtain

$$\left\| e^{-iH_{\alpha}t} \psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) - e^{-i \int_0^t du \omega(u) \tilde{\psi}_{\varphi_t} \otimes W(\alpha^2 \varphi_t) \right\|^2$$

$$= 2 \text{Im} \int_0^t ds \left( \langle i \partial_s \xi_s, \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle - \langle \xi_s, i \partial_s \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle \right)$$

$$= 2 \text{Im} \int_0^t ds \langle \xi_s, \left( h_{\varphi_s} - e(\varphi_s) + \Phi_x - a(\varphi_s) - a^*(\varphi_s) + \mathcal{N} \right) \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle$$

$$- 2 \alpha^{-2} \text{Re} \int_0^t ds \langle \xi_s, R_s V_{\varphi_s} \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle$$

$$= 2 \text{Im} \int_0^t ds \langle \xi_s, (\Phi_x - a^*(\varphi_s)) \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle - 2 \alpha^{-2} \text{Re} \int_0^t ds \langle \xi_s, R_s V_{\varphi_s} \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle.$$  

(85)
Here we used the definition $R_s = q_s(h_{\varphi} - e(\varphi_s))^{-1}q_s$ and Lemma [IV.1]. Thus if we insert the identity $1 = p_s + q_s$ and note that $q_s a^*(\sigma_{\psi_s}) \tilde{\psi}_{\varphi_s} \otimes \Omega = 0$ and $q_s \tilde{\psi}_{\varphi_s} = 0$, we get
\[
\| e^{-iH_{\alpha}t} \tilde{\psi}_{\varphi_s} \otimes W(\alpha^2 \varphi_0)\Omega - e^{-i\int_0^t du \omega(u)} \tilde{\psi}_{\varphi_s} \otimes W(\alpha^2 \varphi_t)\Omega \|^2 = -2\alpha^{-2} \Re \int_0^t ds \langle \xi_s, R_s V_{i\varphi_s} \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle \\
+ 2\Im \int_0^t ds \langle \xi_s, p_s (\Phi_x - a^*(\sigma_{\psi_s})) \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle \\
+ 2\Im \int_0^t ds \langle \xi_s - \tilde{\psi}_{\varphi_s} \otimes \Omega, q_s \Phi_x \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle.
\]
(86a)
(86b)
(86c)

We observe that the first term (86a) is already of the right order, namely $\alpha^{-2} t$. To be more precise,
\[
\| e^{-iH_{\alpha}t} \tilde{\psi}_{\varphi_s} \otimes \Omega - e^{-i\int_0^t du \omega(u)} \tilde{\psi}_{\varphi_s} \otimes \Omega \|^2 \\
\leq 2\alpha^{-2} \int_0^t ds \| \xi_s \|_2 \| V_{i\varphi_s} \tilde{\psi}_{\varphi_s} \|_2 \\
\leq 2\alpha^{-2} \int_0^t ds \| \varphi_s \|_2 \| \tilde{\psi}_{\varphi_s} \|_{H^1(\mathbb{R}^3)} \\
\leq C\alpha^{-2} |t|
\]
(87)

for all $|t| \leq C\alpha^2$ where we used Lemma [IV.2], Lemma [IV.2], Lemma [III.1] and (28). We estimate the remaining two terms (86b) and (86c) separately.

**The term (86b)**

We have
\[
\| e^{-iH_{\alpha}t} \tilde{\psi}_{\varphi_s} \otimes \Omega - e^{-i\int_0^t du \omega(u)} \tilde{\psi}_{\varphi_s} \otimes \Omega \|^2 \\
\leq 2\int_0^t ds \langle \xi_s, \int d^3k a_k^* |k|^{-1} \left( \langle \tilde{\psi}_{\varphi_s}, e^{-ik} \tilde{\psi}_{\varphi_s} \rangle - \langle \psi_s, e^{-ik} \tilde{\psi}_s \rangle \right) \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle \\
\leq 2\int_0^t ds \left\| \int d^3k a_k^* |k|^{-1} \left( \langle \tilde{\psi}_{\varphi_s}, e^{-ik} \tilde{\psi}_{\varphi_s} \rangle - \langle \psi_s, e^{-ik} \tilde{\psi}_s \rangle \right) \tilde{\psi}_{\varphi_s} \otimes \Omega \right\|_2 \\
= 2\int_0^t ds \left\| \int d^3k a_k^* |k|^{-1} \left( \langle \tilde{\psi}_{\varphi_s}, e^{-ik} \tilde{\psi}_{\varphi_s} - \psi_s \rangle + \langle \tilde{\psi}_{\varphi_s} - \psi_s, e^{-ik} \psi_s \rangle \right) \tilde{\psi}_{\varphi_s} \otimes \Omega \right\|.
\]
(88)

Since $\| a^*(f) \psi \otimes \Omega \| = \alpha^{-1} \|f\|_2 \|\psi\|$ for all $f \in L^2(\mathbb{R}^3)$, we find
\[
\| e^{-iH_{\alpha}t} \tilde{\psi}_{\varphi_s} \otimes \Omega - e^{-i\int_0^t du \omega(u)} \tilde{\psi}_{\varphi_s} \otimes \Omega \|^2 \\
\leq C\alpha^{-1} \int_0^t ds \left[ \int d^3k \|k\|^{-2} \left( \| \tilde{\psi}_{\varphi_s}, e^{-ik} (\tilde{\psi}_{\varphi_s} - \psi_s) \|^2 + \| (\tilde{\psi}_{\varphi_s} - \psi_s), e^{-ik} \psi_s \|^2 \right) \right]^{1/2}.
\]
(89)

With the help of $|\cdot|^{-2}(x) = \pi^{-1} |x|^{-1}$ and the inequalities of Hardy-Littlewood-Sobolev and Hölder we obtain
\[
\int d^3k \|k\|^{-2} \langle \tilde{\psi}_{\varphi_s}, e^{-ik} (\tilde{\psi}_{\varphi_s} - \psi_s) \rangle^2 \\
= C \int d^3x \int d^3y \ |x - y|^{-1} (\tilde{\psi}_{\varphi_s} - \psi_s)(x) \tilde{\psi}_{\varphi_s}(y)(\tilde{\psi}_{\varphi_s} - \psi_s)(y) \\
\leq C \| \tilde{\psi}_{\varphi_s} - \psi_s \|_{L_0^6/5} \| \tilde{\psi}_{\varphi_s} \|_{L_2^5} \| \tilde{\psi}_{\varphi_s} \|_{L_3^2} \\
\leq C \| \tilde{\psi}_{\varphi_s} - \psi_s \|_{L_2^5} \| \tilde{\psi}_{\varphi_s} \|_{H^1(\mathbb{R}^3)} \leq C \| \tilde{\psi}_{\varphi_s} - \psi_s \|_{L_2^5}
\]
(90)
for all $|t| \leq C\alpha^2$ by (28). Similarly,

$$\int d^3k \ |k|^{-2} \left| \left\langle \tilde{\psi}_\varphi - \psi_s, e^{-ik \cdot \varphi_s} \right\rangle \right|^2 \leq C \left\| \psi_s \right\|_{L^2(\mathbb{R}^3)}^2 \left\| \psi_s - \tilde{\psi}_\varphi \right\|_2^2 \leq C \left\| \psi_s - \tilde{\psi}_\varphi \right\|_2^2$$

(91)

by Lemma III.1. Hence,

$$\left| (86b) \right| \leq C \alpha^{-1} \int_0^t ds \left\| \psi_s - \tilde{\psi}_\varphi \right\|_2$$

for all $|t| \leq C\alpha^2$. Applying Theorem II.1 leads to

$$\left| (86b) \right| \leq C\alpha^{-2} |t|$$

for all $|t| \leq C\alpha^2$. Applying Theorem II.1 leads to

(93)

The term (86c)

In order to continue we note that [21, Theorem X.71], whose assumptions can easily shown to be satisfied by Lemma III.2 guarantees the existence of a two parameter group $U_h(s; \tau)$ on $L^2(\mathbb{R}^3)$ such that

$$\frac{d}{ds} U_h(s; \tau) \psi = -i h_{\varphi_s} U_h(s; \tau) \psi, \quad U_h(\tau; \tau) \psi = \psi$$

for all $\psi \in H^1(\mathbb{R}^3)$. (94)

Moreover, we define

$$\tilde{U}_h(s; \tau) = e^{i \int_0^s du e(\varphi_u)} U_h(s; \tau).$$

(95)

We then have for all $s \in \mathbb{R}$

$$\frac{d}{ds} \tilde{U}_h^s(s; \tau) = \tilde{U}_h^s(s; \tau)i(h_{\varphi_s} - e(\varphi_u))$$

(96)

and

$$\tilde{U}_h^s(s; \tau) q_0 f_s = -i \frac{d}{ds} \left[ \tilde{U}_h^s(s; \tau) R_s f_s \right] + i \tilde{U}_h^s(s; \tau) \tilde{R}_s f_s + i \tilde{U}_h^s(s; \tau) R_s \partial_s f_s,$$

(97)

for $f_s \in L^2(\mathbb{R}^3)$. This allows us to express

$$\left| (86c) \right| = 2 \text{Im} \int_0^t ds \left\langle \xi_s - \tilde{\psi}_\varphi, \Omega, q_0 \Phi_x^{-} \tilde{\psi}_\varphi \otimes \Omega \right\rangle$$

$$= 2 \text{Im} \int_0^t ds \left\langle U_h^s(s; 0) \left( \xi_s - \tilde{\psi}_\varphi \otimes \Omega \right), \tilde{U}_h^s(s; 0) q_0 \Phi_x^{-} \tilde{\psi}_\varphi \otimes \Omega \right\rangle$$

(98)

by three integrals which contain a derivative with respect to the time variable. Note that we absorbed the phase factor of $\tilde{\psi}_\varphi$ in the dynamics $\tilde{U}_h^s(s; 0)$. Thus,

$$\left| (86c) \right| \leq 2 \left| \int_0^t ds \left\langle U_h^s(s; 0) \left( \xi_s - \tilde{\psi}_\varphi \otimes \Omega \right), \frac{d}{ds} \left[ U_h^s(s; 0) R_s \Phi_x^{-} \tilde{\psi}_\varphi \otimes \Omega \right] \right\rangle \right|$$

$$+ 2 \left| \int_0^t ds \left\langle U_h^s(s; 0) \left( \xi_s - \tilde{\psi}_\varphi \otimes \Omega \right), \tilde{U}_h^s(s; 0) \tilde{R}_s \Phi_x^{-} \tilde{\psi}_\varphi \otimes \Omega \right\rangle \right|$$

$$+ 2 \left| \int_0^t ds \left\langle U_h^s(s; 0) \left( \xi_s - \tilde{\psi}_\varphi \otimes \Omega \right), \tilde{U}_h^s(s; 0) R_s \Phi_x^{-} \partial_s \tilde{\psi}_\varphi \otimes \Omega \right\rangle \right|.$$

(99)
In the first term we integrate by parts and we use ξ₀ = ȧψ₀ ⊗ Ω. We find

\[ | [86c] | \leq 2 \left| \left< U_h^s(t; 0) (ξ₀ - ȧψ₀, Ω), U_h^s(t; 0) R_Φ ȧψ₀ ⊗ Ω \right> \right| \]
\[ + 2 \int_0^t ds \left\| \frac{d}{ds} \left[ U_h^s(s; 0) (ξ_s - ȧψ_s, Ω) \right], U_h^s(s; 0) R_Φ ȧψ_s ⊗ Ω \right\| \]
\[ + 2 \int_0^t ds \left\| \left< ξ_s - ȧψ_s, R_Φ ȧψ_s, Ω \right> \right\| \right| \]
\[ + 2 \int_0^t ds \left\| \left< ξ_s, R_Φ ȧψ_s, Ω \right> \right\| \]  \hspace{1em} (100) \]

In order to compute the time derivative occurring in the second summand, we use (83) and the notation

\[ δH_s = Φ_x - a(σ_{ψ_s}) - a^*(σ_{ψ_s}) + N \]

and get

\[ \frac{d}{ds} \left[ U_h^s(s; 0) (ξ_s - ȧψ_s, Ω) \right] = -i U_h^s(s; 0) δH_s ξ_s - α^{-2} U_h^s(s; 0) R_Φ ȧψ_s ⊗ Ω \]

as well as

\[ U_h^s(t; 0) (ξ_t - ȧψ_t, Ω) = -i \int_0^t ds U_h^s(s; 0) δH_s ξ_s - α^{-2} \int_0^t ds U_h^s(s; 0) R_Φ ȧψ_s, Ω. \]  \hspace{1em} (102) \]

Applying (102) to the first term of the r.h.s. of (100) and (102) to the second, we obtain

\[ \left| [86c] \right| \leq 2 \left| \int_0^t ds \left< δH_s ξ_s, R_Φ ȧψ_s, Ω \right> \right| \]
\[ + 2 \left| \int_0^t ds \left< δH_s ξ_s, U_h^s(t; s) R_Φ ȧψ_t, Ω \right> \right| \]
\[ + 2α^{-2} \left| \int_0^t ds \left< R_Φ ȧψ_s, Ω, R_Φ ȧψ_s, Ω \right> \right| \]
\[ + 2α^{-2} \left| \int_0^t ds \left< R_Φ ȧψ_s, Ω, U_h^s(t; s) R_Φ ȧψ_t, Ω \right> \right| \]
\[ + 2 \left| \int_0^t ds \left< \left< ξ_s - ȧψ_s, Ω \right>, R_Φ ȧψ_s, Ω \right> \right| \]
\[ + 2 \left| \int_0^t ds \left< ξ_s, R_Φ ȧψ_s, Ω \right> \right| \]  \hspace{1em} (104a-f) \]

The term (104a): According to the definition of δH_s, we decompose (104a) as

\[ (104a) \leq 2 \left| \int_0^t ds \left< ξ_s, N R_Φ ȧψ_s, Ω \right> \right| \]
\[ + 2 \left| \int_0^t ds \left< ξ_s, (σ(ψ_s) + a^*(σ_{ψ_s})) R_Φ ȧψ_s, Ω \right> \right| \]
\[ + 2 \left| \int_0^t ds \left< ξ_s, σ R_Φ ȧψ_s, Ω \right> \right| \]  \hspace{1em} (105a-c) \]
We notice that $[\mathcal{N}, R_s] = 0$ and that $\mathcal{N}\Psi = \alpha^{-2}\Psi$ if $\Psi \in \mathcal{H}$ is a one-phonon state and write the first line as

\begin{align}
\tag{105a}
\langle \xi_s, R_s \Phi_x \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle &= 2\alpha^{-2} \int_0^t ds \left| \langle \xi_s, R_s \Phi_x \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle \right| \\
&= 2\alpha^{-2} \int_0^t ds \left| \langle \xi_s, R_s \int d^3k \ |k|^{-1} e^{-ikx a^*_k \tilde{\psi}_{\varphi_s} \otimes \Omega} \rangle \right| .
\end{align}

By means of Lemma IV.2 and Lemma A.1 this becomes

\begin{align}
\tag{106a}
(105a) &\leq C\alpha^{-3} \int_0^t ds \left| \langle R_s(-\Delta + 1)^{1/2} \rangle \right| \| \psi_{\varphi_s} \|_2 \leq C\alpha^{-3} |t|
\end{align}

for all $|t| \leq C\lambda \alpha^2$. In a similar way, we calculate $[a(\sigma_{\psi_s}), a^*_k] = \alpha^{-2} \sigma_{\psi_s}(k)$ for all $k \in \mathbb{R}^3$ and estimate

\begin{align}
\tag{105b}
\langle \xi_s, (a(\sigma_{\psi_s}) + a^*(\sigma_{\psi_s})) \rangle &\leq 2 \int_0^t ds \left| \langle \xi_s, R_s \int d^3k \ |k|^{-1} e^{-ikx a^*_k \tilde{\psi}_{\varphi_s} \otimes \Omega} \rangle \right| \\
&+ 2\alpha^{-2} \int_0^t ds \left| \langle \xi_s, R_s \int d^3k \ |k|^{-1} e^{-ikx \sigma_{\psi_s}(k) \tilde{\psi}_{\varphi_s} \otimes \Omega} \rangle \right| .
\end{align}

Applying Lemma A.1, Lemma III.2 and Lemma IV.2 to the first line and using the same arguments as in Lemma A.1 for the second line this becomes

\begin{align}
\tag{105b}
(105b) &\leq C\alpha^{-2} \int_0^t ds \left| \langle \xi_s, R_s(-\Delta + 1)^{1/2} \rangle \right| \| \sigma_{\psi_s} \|_2 \| \psi_{\varphi_s} \|_2 \\
&\leq C\alpha^{-2} |t| \quad \text{for all } |t| \leq C\lambda \alpha^2.
\end{align}

Since $\Phi_x = \Phi_x^+ + \Phi_x^-$ we have

\begin{align}
\tag{105c}
&\langle \xi_s, R_s \Phi_x^+ \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle \\
&\leq 2 \int_0^t ds \left| \langle \xi_s, R_s \Phi_x^+ \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle \right| \\
&+ 2 \int_0^t ds \left| \langle \Phi_x^+ \xi_s, R_s \Phi_x^+ \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle \right| .
\end{align}

Making use of Lemma A.2 the first line can be estimated by

\begin{align}
\tag{110a}
(110a) &\leq C \int_0^t ds \left| \langle -\Delta + 1 \rangle^{1/2} R_s \Phi_x^+ \tilde{\psi}_{\varphi_s} \otimes \Omega \right| \\
&= C \int_0^t ds \left| \langle -\Delta + 1 \rangle^{1/2} R_s \mathcal{N}^{1/2} \Phi_x^+ \tilde{\psi}_{\varphi_s} \otimes \Omega \right| .
\end{align}

Since $\| (-\Delta + 1)^{1/2} R_s (-\Delta + 1)^{1/2} \| \leq C$ for all $|t| \leq C\lambda \alpha^2$ by Lemma IV.2 and $\mathcal{N}^{1/2}\Psi = \alpha^{-1}\Psi$ if $\Psi \in \mathcal{H}$ is a one-phonon state, we find

\begin{align}
\tag{110a}
&\leq C\alpha^{-1} \int_0^t ds \left| \langle -\Delta + 1 \rangle^{-1/2} \Phi_x^+ \tilde{\psi}_{\varphi_s} \otimes \Omega \right| .
\end{align}
for all $|t| \leq C_A \alpha^2$. With Lemma A.3, we arrive at

$$
\text{(110a)} \leq C \alpha^{-2} \int_0^t ds \| \psi_{\varphi, s} \|_2 \leq C \alpha^{-2} |t| \tag{113}
$$

for all $|t| \leq C_A \alpha^2$. In similar fashion we use Lemma A.2, Lemma A.1 and $N R_s \Phi_x \tilde{\psi}_{\varphi, s} \otimes \Omega = \alpha^{-2} R_s \Phi_x \tilde{\psi}_{\varphi, s} \otimes \Omega$ to estimate

$$
\text{(110b)} = 2 \int_0^t ds \left| \left\langle \Phi_x^+ \xi_s, R_s \Phi_x \tilde{\psi}_{\varphi, s} \otimes \Omega \right\rangle \right| \\
= 2 \int_0^t ds \left| \left\langle (\mathcal{N} + \alpha^{-2})^{-1/2} \Phi_x^+ \xi_s, (\mathcal{N} + \alpha^{-2})^{1/2} R_s \Phi_x \tilde{\psi}_{\varphi, s} \otimes \Omega \right\rangle \right| \\
\leq 2 \int_0^t ds \left\| (\mathcal{N} + \alpha^{-2})^{-1/2} \Phi_x^+ \xi_s \right\| \left\| (\mathcal{N} + \alpha^{-2})^{1/2} R_s \Phi_x \tilde{\psi}_{\varphi, s} \otimes \Omega \right\| \\
\leq C \alpha^{-1} \int_0^t ds \left\| (\Delta + 1)^{1/2} \xi_s \right\| \left\| R_s \Phi_x \tilde{\psi}_{\varphi, s} \otimes \Omega \right\| \\
\leq C \alpha^{-1} \int_0^t ds \left\| (\Delta + 1)^{1/2} \xi_s \right\| \left\| R_s (\Delta + 1)^{1/2} \right\| \left\| (\Delta + 1)^{-1/2} \Phi_x^+ \tilde{\psi}_{\varphi, s} \otimes \Omega \right\| \\
\leq C \alpha^{-2} \int_0^t ds \left\| (\Delta + 1)^{1/2} \xi_s \right\| \left\| \psi_{\varphi, s} \right\|_2 \\
= C \alpha^{-2} \int_0^t ds \left\| (\Delta + 1)^{1/2} e^{-iH_0 s} \tilde{\psi}_{\varphi, s} \otimes W(\alpha^2 \varphi_0) \Omega \right\| \tag{114}
$$

for all $|t| < C_A \alpha^2$. Thus, if we now use $-\Delta + 1 \leq C(H_0 + C)$ (see Lemma A.3) this becomes using the properties (7) of the Weyl operators

$$
\text{(110c)} \leq C \alpha^{-2} |t| \quad \text{and hence (104a)} \leq C \alpha^{-2} |t| \quad \text{for all } |t| < C_A \alpha^2. 
$$

**The term (104b):** For the next estimate, we recall the notation (101) to write (104b) as

$$
\text{(104b)} \leq 2 \int_0^t ds \left| \left\langle \xi_s, NU^*_h(t; s) R_t \Phi_x \tilde{\psi}_{\varphi, t} \otimes \Omega \right\rangle \right| \\
+ 2 \int_0^t ds \left| \left\langle \xi_s, (a(\sigma_{\psi, s}) + a^*(\sigma_{\psi, s})) U^*_h(t; s) R_t \Phi_x \tilde{\psi}_{\varphi, t} \otimes \Omega \right\rangle \right| \\
+ 2 \int_0^t ds \left| \left\langle \xi_s, \Phi_x U_h^*(t; s) R_t \Phi_x \tilde{\psi}_{\varphi, t} \otimes \Omega \right\rangle \right| \tag{116}
$$

Using $[\mathcal{N}, U^*_h(t; s)] = [a(\sigma_{\psi, s}), U^*_h(t; s)] = [a^*(\sigma_{\psi, s}), U^*_h(t; s)] = 0$ allows us to estimate the first two lines in exactly the same way as (105a) and (105b) and leaves us with

$$
\text{(104b)} \leq C \alpha^{-2} |t| + 2 \int_0^t ds \left| \left\langle \xi_s, \Phi_x U_h^*(t; s) R_t \Phi_x \tilde{\psi}_{\varphi, t} \otimes \Omega \right\rangle \right| \tag{117}
$$
for all $|t| < C_\Lambda \alpha^2$. The difficulty of this term is the fact that the operators $\Phi_x$ and $U_h^*(t; s)$ do not commute. Nevertheless, we can use $\Phi_x = \Phi_x^+ + \Phi_x^-$ to get

$$\frac{104b}{118a} \leq C \alpha^{-2} |t| + 2 \int_0^t ds \left| \langle \Phi_x^+ c, U_h^*(t; s) R_t \Phi_x^- \tilde{\psi}_{\varphi_t} \otimes \Omega \rangle \right|$$

$$+ 2 \int_0^t ds \left| \langle \xi, U_h^*(t; s) R_t \Phi_x^- \tilde{\psi}_{\varphi_t} \otimes \Omega \rangle \right|. \quad (118b)$$

Using the same estimates as in (113) and (115) we bound the first integral by

$$2 \int_0^t ds \left| \langle \Phi_x^+ c, U_h^*(t; s) R_t \Phi_x^- \tilde{\psi}_{\varphi_t} \otimes \Omega \rangle \right|$$

$$= 2 \int_0^t ds \left| \langle (N + \alpha^{-2})^{-1/2} \Phi_x^+ c, U_h^*(t; s) (N + \alpha^{-2})^{1/2} R_t \Phi_x^- \tilde{\psi}_{\varphi_t} \otimes \Omega \rangle \right|$$

$$\leq 2 \int_0^t ds \left| \langle (N + \alpha^{-2})^{-1/2} \Phi_x^+ c \rangle \right| \left| \langle (N + \alpha^{-2})^{1/2} R_t \Phi_x^- \tilde{\psi}_{\varphi_t} \otimes \Omega \rangle \right|$$

$$\leq C \alpha^{-2} |t| \quad \text{for all } |t| \leq C_\Lambda \alpha^2. \quad (119)$$

For the second term Lemma 1.2 and $U_h^*(t; s) = U_h(s; t)$ imply

$$\frac{118b}{118b} \leq C \int_0^t ds \left| \langle (\Delta + 1)^{1/2} \Phi_x^- \tilde{\psi}_{\varphi_t} \otimes \Omega \rangle \right|. \quad (120)$$

It follows from Lemma 1.2 and (19) that for $\xi \in L^2(\mathbb{R}^3) \otimes \mathcal{F}$

$$\langle \xi, U_h^*(s; \tau)(\Delta + 1) U_h(s; \tau) \xi \rangle \leq C \langle \xi, U_h^*(s; \tau)(\Delta + 1) U_h(s; \tau) \xi \rangle$$

$$= C \langle \xi, (\Delta + 1) \xi \rangle - \alpha^{-2} \int_{\tau}^t d\tau' \langle \xi, U_h^*(\tau'; \tau) \tilde{V}_{\varphi_{\tau'}} U_h(\tau'; \tau) \xi \rangle$$

$$\leq C \langle \xi, (\Delta + 1) \xi \rangle + \alpha^{-2} \int_{\tau}^t d\tau' \langle \xi, U_h^*(\tau'; \tau)(\Delta + 1) U_h(\tau'; \tau) \xi \rangle. \quad (121)$$

The Gronwall inequality yields

$$\| (\Delta + 1)^{1/2} U_h(s; \tau) \xi \| \leq C e^{\alpha^{-2} |s - \tau|} (\Delta + 1)^{1/2} \| \xi \| \leq C (\Delta + 1)^{1/2} \| \xi \| \quad (122)$$

for all $|s - \tau| \leq C_\Lambda \alpha^2$. Thus

$$\frac{118b}{118b} \leq C \int_0^t ds \left| (\Delta + 1)^{1/2} R_t \Phi_x^- \tilde{\psi}_{\varphi_t} \otimes \Omega \right| \leq C \alpha^{-2} |t| \quad (123)$$

for all $|t| \leq C_\Lambda \alpha^{-2}$, where we concluded by Lemma 1.1 and Lemma IV.2 as for the term \(110a\).

**The terms \(104c\) and \(104d\):** With the help of Lemma 1.2, Lemma 1.1, Lemma IV.2 and (38) one obtains

$$\frac{104c}{104c} = 2 \alpha^{-2} \int_0^t ds \left| \langle R_s \tilde{\psi}_{\varphi_{\tau}}, \tilde{\psi}_{\varphi_{\tau}} \otimes \Omega, R_s \int d^3k |k|^{-1} e^{-ikx} a_k^\star \tilde{\psi}_{\varphi_{\tau}} \otimes \Omega \rangle \right|$$

$$\leq 2 \alpha^{-2} \int_0^t ds \| R_s \| \| \psi_{\varphi_{\tau}} \| \| H^2(\mathbb{R}^3) \| \| R_s (\Delta + 1)^{1/2} \| \| \alpha^{-3} |t| \right|$$

$$\leq C \alpha^{-3} |t| \quad (124)$$

and \(104d\) $\leq C \alpha^{-3} |t|$ for all $|t| < C_\Lambda \alpha^2$. \hfill 19
The term (104e): Applying Lemma A.1 once more we estimate

\[ \begin{align*}
(104e) &= 2 \int_0^t ds \langle \xi_s, \tilde{\psi}_{\varphi_s} \otimes \Omega \rangle, \tilde{R}_s \int d^3k \ |k|^{-1} e^{-ik \cdot x} a^*_k \tilde{\psi}_{\varphi_s} \otimes \Omega \\
&\leq 4 \int_0^t ds \left\| \tilde{R}_s(-\Delta + 1)^{1/2} \right\| \left\| (-\Delta + 1)^{-1/2} \int d^3k \ |k|^{-1} e^{-ik \cdot x} a^*_k \tilde{\psi}_{\varphi_s} \otimes \Omega \right\|. 
\end{align*} \]

From (28), (56) and Lemma IV.2 we get

\[ (104e) \leq C \alpha^{-3} |t| \quad \text{for all } |t| < C_\Lambda \alpha^2. \]  

The term (104f): With the help of Lemma IV.1 and Lemma III.2 we get

\[ \begin{align*}
(104f) &= 2 \alpha^{-2} \int_0^t ds \langle \xi_s, R_s \Phi_\tilde{\varphi} R_s V_\varphi \varphi_s \otimes \Omega \rangle \\
&\leq 2 \alpha^{-2} \int_0^t ds \left\| R_s(-\Delta + 1)^{1/2} \right\| \left\| (-\Delta + 1)^{-1/2} \Phi_\tilde{\varphi} R_s V_\varphi \varphi_s \otimes \Omega \right\| \\
&\leq C \alpha^{-3} \int_0^t ds \left\| R_s(-\Delta + 1)^{1/2} \right\| \left\| R_s V_\varphi \varphi_s \right\|_2 \leq C \alpha^{-3} |t|. 
\end{align*} \]

Here we used again Lemma A.1 and Lemma IV.2. In total, we obtain

\[ |S_{6c}| \leq C \alpha^{-2} |t| \quad \text{for all } |t| < C_\Lambda \alpha^2. \]  

Summing up, we have shown that

\[ \left\| e^{-iH_{\alpha} t \psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) \Omega} - e^{-i \int_0^t ds a(a) \tilde{\psi}_{\varphi_s} \otimes W(\alpha^2 \varphi_0) \Omega} \right\|^2 \leq C \alpha^{-2} |t|, \]

for all \( |t| \leq C_\Lambda \alpha^2 \).

A Auxiliary estimates

Lemma A.1. There exists a constant \( C > 0 \) such that for all \( u \in L^2(\mathbb{R}^3) \) and \( f \in L^2(\mathbb{R}^3) \)

\[ \left\| (-\Delta + 1)^{-1/2} \int d^3k \ |k|^{-1} e^{-ik \cdot x} a^*_k u \otimes \Omega \right\| \leq C \alpha^{-1} \|u\|_2, \]

(130)

\[ \left\| (-\Delta + 1)^{-1/2} \int d^3k \ |k|^{-1} e^{-ik \cdot x} a^*_k f a_k^* u \otimes \Omega \right\| \leq C \alpha^{-2} \|u\|_2 \|f\|_2. \]

(131)

Proof. The commutation relations imply

\[ \left\| (-\Delta + 1)^{-1/2} \int d^3k \ |k|^{-1} e^{-ik \cdot x} a^*_k u \otimes \Omega \right\|^2 \]

\[ = \int d^3k \int d^3k' \ |k|^{-1} |k'|^{-1} \left\langle e^{-ik \cdot x} a^*_k u \otimes \Omega, (-\Delta + 1)^{-1} e^{-ik' \cdot x} a^*_k u \otimes \Omega \right\rangle \]

\[ \leq \alpha^{-2} \int d^3k \ |k|^{-2} \left\langle e^{-ik \cdot x} u, (-\Delta + 1)^{-1} e^{-ik \cdot x} u \right\rangle \]

\[ = \alpha^{-2} \int d^3k \ |k|^{-2} \langle u, ((-i \nabla - k)^2 + 1)^{-1} u \rangle \]

\[ = \alpha^{-2} \int d^3p \ |\hat{u}(p)|^2 \int d^3k \ \frac{1}{((p - k)^2 + 1)|k|^2}. \]

(132)
Since $| \cdot |^{-2}$ and $(| \cdot |^2 + 1)^{-1}$ are radial symmetric and decreasing functions we have
\[
\sup_{p \in \mathbb{R}^3} \int d^3k \frac{1}{((p - k)^2 + 1)|k|^2} = \int d^3k \frac{1}{(k^2 + 1)|k|^2} < \infty
\]
by the rearrangement inequality. Hence,
\[
\|(-\Delta + 1)^{-1/2} \int d^3k |k|^{-1} e^{-ik \cdot x} a_k^* u \otimes \Omega\|^2 \leq C\alpha^{-2} \int d^3p |\tilde{u}(p)|^2 = C\alpha^{-2}\|u\|_2.
\]
\[
\|(-\Delta + 1)^{-1/2} \int d^3k |k|^{-1} e^{-ik \cdot x} u \otimes \Omega\|^2 \leq C\alpha^{-2} \int d^3p |\tilde{u}(p)|^2 = C\alpha^{-2}\|u\|_2.
\]

The second bound of the Lemma follows from the first one and the bounds (79) for the creation and annihilation operators.

Lemma A.2 (Lemma 4, Lemma 10 in [6]). Let $\Phi^+ = \int d^3k \ |k|^{-1} e^{ik \cdot x} a_k^*$ and $\mathcal{N} = \int d^3k a_k^* a_k$.

Then
\[
\|\Phi^+ \Psi\| \leq C \|(-\Delta + 1)^{1/2} \mathcal{N}^{1/2} \Psi\| \quad \text{and} \quad \|\mathcal{N} + \alpha^{-2})^{-1/2} \Phi_x^+ \Psi\| \leq C \|(-\Delta + 1)^{1/2} \Psi\|.
\]

Proof. We split the operator $\Phi^+_x = \Phi^+_x \geq + \Phi^+_x <$, where
\[
\Phi^+_x \geq = \int_{|k|>\kappa} \frac{d^3k}{|k|} e^{ik \cdot x} a_k, \quad \Phi^+_x < = \int_{|k|<\kappa} \frac{d^3k}{|k|} e^{ik \cdot x} a_k
\]
for a constant $\kappa > 0$ of order one. Then, we deduce from [6] Lemma 10] that
\[
\|\Phi^+_x \geq \Psi\| \leq C \|(-\Delta + 1)^{1/2} \mathcal{N}^{1/2} \Psi\| \quad \text{and} \quad \|\mathcal{N} + \alpha^{-2})^{-1/2} \Phi_x^+_\geq \Psi\| \leq C \|(-\Delta + 1)^{1/2} \Psi\|.
\]
\[
\|\Phi^+_x < \Psi\| \leq C \|\mathcal{N}^{1/2} \Psi\| \quad \text{and} \quad \|\mathcal{N} + \alpha^{-2})^{-1/2} \Phi_x^+ < \Psi\| \leq C \|\Psi\|.
\]
for a constant $C > 0$ depending only on $\kappa$.

Lemma A.3 ([6], p.7). Let $\alpha > 0$. Let $H_\alpha$ denote the Fröhlich Hamiltonian defined in [6] and $\varepsilon > 0$. There exists a constant $C_\varepsilon$ (depending on $\alpha_0$), such that
\[
(1 - \varepsilon)(-\Delta + \mathcal{N}) - C_\varepsilon \leq H_\alpha \leq (1 + \varepsilon)(-\Delta + \mathcal{N}) + C_\varepsilon.
\]

for all $\alpha \geq \alpha_0$.

The proof is given in [6] (see Lemma 7 and p.7) and relies on arguments of Lieb and Yamazaki [19]; see [16] p.12 for a concise explanation.

Acknowledgments

N.L. and R.S. gratefully acknowledge financial support by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 694227). B.S. acknowledges support from the Swiss National Science Foundation (grant 200020_172623) and from the NCCR SwissMAP. N.L. would like to thank Andreas Deuchert and David Mitrouskas for interesting discussions. B.S. and R.S. would like to thank Rupert Frank for stimulating discussions about the time-evolution of a polaron.
References

[1] Z. Ammari and M. Falconi, *Bohr’s correspondence principle for the renormalized Nelson model*. SIAM J. Math. Anal. 49 (6), 5031–5095 (2017).

[2] E. B. Davies, *Particle-boson interactions and the weak coupling limit*. J. Math. Phys. 20 (3), 345–351 (1979).

[3] M. Falconi, *Classical limit of the Nelson model with cutoff*. J. Math. Phys. 54 (1), 012303 (2013).

[4] D. Feliciangeli, S. Rademacher and R. Seiringer, *Persistence of the spectral gap for the Landau–Pekar equations*. Lett. Math. Phys. 111 (19), 19 pp. (2021).

[5] R. L. Frank, *A non-linear adiabatic theorem for the Landau-Pekar equations*. Oberwolfach Reports, DOI: 10.4171, OWR/2017/27 (2017).

[6] R. L. Frank and B. Schlein, *Dynamics of a strongly coupled polaron*. Lett. Math. Phys. 104 (8), 911–929 (2014).

[7] R. L. Frank and Z. Gang, *Derivation of an effective evolution equation for a strongly coupled polaron*. Anal. PDE 10 (2), 379–422 (2017).

[8] R. L. Frank, Z. Gang, *A non-linear adiabatic theorem for the one-dimensional Landau-Pekar equations*. J. Funct. Anal. 279, no. 7, 108631, 43 pp. (2020).

[9] H. Fröhlich, *Theory of electrical breakdown in ionic crystals*. Proc. R. Soc. Lond. A 160 (901), 230–241 (1937).

[10] J. Ginibre, F. Nironi, and G. Velo, *Partially classical limit of the Nelson model*. Ann. H. Poincaré 7 (1), 21–43 (2006).

[11] M. Griesemer, *On the dynamics of polarons in the strong-coupling limit*. Rev. Math. Phys. 29 (10), 1750030 (2017).

[12] F. Hiroshima, *Weak coupling limit with a removal of an ultraviolet cutoff for a Hamiltonian of particles interacting with a massive scalar field*. Infin. Dimens. Anal. Qu. 1 (3), 407–423 (1998).

[13] N. Leopold, D. Mitrouskas, S. Rademacher, B. Schlein and R. Seiringer, *Landau–Pekar equations and quantum fluctuations for the dynamics of a strongly coupled polaron*. Preprint, arXiv:2005.02098 Pure Appl. Anal (in press).

[14] N. Leopold and S. Petrat, *Mean-field Dynamics for the Nelson Model with Fermions*. Ann. H. Poincaré 20(10), 3471–3508 (2019).

[15] N. Leopold and P. Pickl, *Derivation of the Maxwell-Schrödinger equations from the Pauli-Fierz Hamiltonian*. SIAM J. Math. Anal. 52(5), 4900–4936 (2020).

[16] E.H. Lieb and L.E. Thomas, *Exact ground state energy of the strong-coupling polaron*. Comm. Math. Phys. 183 (3), 511–519 (1997), Erratum: ibid. 188, no.2, 499 – 500 (1997).

[17] E.H. Lieb and M. Loss, *Analysis, second edition*, American Mathematical Society, 2001.

[18] E.H. Lieb and R. Seiringer, *Divergence of the effective mass of a polaron in the strong coupling limit*. J. Stat. Phys. 180, no.1-6, 23-33 (2020).
[19] E.H. Lieb and K. Yamazaki, *Ground-state energy and effective mass of the polaron*. Phys. Rev. 111, 728 – 733 (1958).

[20] D. Mitrouskas, *A note on the Fröhlich dynamics in the strong coupling limit*. Lett. Math. Phys. 111 (45), 24 pp. (2021).

[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II, Fourier Analysis, Self-adjointness*, Academic Press, 1975.

[22] S. Teufel, *Effective N-body dynamics for the massless Nelson model and adiabatic decoupling without spectral gap*. Ann. H. Poincaré 3 (5), 939–965 (2002).

[23] S. Teufel, *Adiabatic perturbation theory in quantum dynamics*, Lecture Notes in Mathematics 1821, Springer-Verlag, 2003.

(Nikolai Leopold) Institute of Science and Technology Austria (IST Austria)  
Am Campus 1, 3400 Klosterneuburg, Austria  
current address: University of Basel, Department of Mathematics and Computer Science  
Spiegelgasse 1, 4051 Basel, Switzerland  
E-mail address: nikolai.leopold@unibas.ch

(Simone Rademacher) Institute of Mathematics, University of Zurich  
Winterthurerstrasse 190, 8057 Zurich, Switzerland  
current address: Institute of Science and Technology Austria (IST Austria)  
Am Campus 1, 3400 Klosterneuburg, Austria  
E-mail address: simone.rademacher@ist.ac.at

(Benjamin Schlein) Institute of Mathematics, University of Zurich  
Winterthurerstrasse 190, 8057 Zurich, Switzerland  
E-mail address: benjamin.schlein@math.uzh.ch

(Robert Seiringer) Institute of Science and Technology Austria (IST Austria)  
Am Campus 1, 3400 Klosterneuburg, Austria  
E-mail address: robert.seiringer@ist.ac.at