ADAPTIVE ESTIMATION FOR SMALL DIFFUSION PROCESSES BASED ON SAMPLED DATA

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Abstract. We consider parametric estimation for multi-dimensional diffusion processes with a small dispersion parameter ε from discrete observations. For parametric estimation of diffusion processes, the main targets are the drift parameter α and the diffusion parameter β. In this paper, we propose two types of adaptive estimators for (α, β) and show their asymptotic properties under ε → 0, n → ∞ and the balance condition that (εn^p)^{-1} = O(1) for some ρ ≥ 1/2. In simulation studies, we examine and compare asymptotic behaviors of the two kinds of adaptive estimators. Moreover, we treat the SIR model which describes a simple epidemic spread for a biological application.

1. Introduction

We consider a d-dimensional small diffusion process satisfying the following stochastic differential equation (SDE):

\[
\begin{align*}
\frac{dX_t}{dt} &= b(X_t, \alpha)dt + \varepsilon \sigma(X_t, \beta)dW_t, \quad t \in [0, T], \\
X_0 &= x_0,
\end{align*}
\]

(1)

where \( W_t \) is the r-dimensional standard Wiener process, \( \alpha \in \Theta_\alpha \subset \mathbb{R}^p, \beta \in \Theta_\beta \subset \mathbb{R}^q, \theta = (\alpha, \beta), \Theta := \Theta_\alpha \times \Theta_\beta \) being compact and convex parameter space, \( b: \mathbb{R}^d \times \Theta_\alpha \to \mathbb{R}^d \) and \( \sigma: \mathbb{R}^d \times \Theta_\beta \to \mathbb{R}^d \otimes \mathbb{R}^r \) are known except for the parameter \( \theta \), and the initial value \( x_0 \in \mathbb{R} \) and the small coefficient \( \varepsilon > 0 \) are known. We assume the true parameter \( \theta_0 = (\alpha_0, \beta_0) \) belongs to \( \text{Int}(\Theta) \), and the data are discrete observations \( (X_{t_k}^n)_{k=0,\ldots,n} \), where \( t_k^n = kh_n \) and \( h_n = T/n \).

A family of small diffusion processes defined by (1) is an important class and called dynamical systems with small perturbations, see Azencott [1], Freidlin and Wentzell [2] and Yoshida [25]. For applications of small diffusion processes to mathematical finance and mathematical biology, see Yoshida [24], Uchida and Yoshida [22], Guy et al [6, 7] and references therein.

Asymptotic theory of parametric inference for small diffusion processes has been well developed. For continuous-time observations, see Kutoyants [11, 12] and Yoshida [25, 27]. As for discrete observations, Genon-catalot [3] studied minimum contrast estimation for the drift parameter and proved that this estimator has asymptotic efficiency under the assumption \( \varepsilon \sqrt{n} = O(1) \). Laredo [13] investigated the asymptotically efficient estimator by using interpolated process under the assumption \( (\varepsilon n^2)^{-1} \to 0 \). Sørensen and Uchida [18] studied the joint estimation for both drift and diffusion parameters based on minimum contrast estimators. They proved that the estimator for drift parameter is asymptotically efficient and the estimator for diffusion parameter is asymptotically normal under \( (\varepsilon n^4)^{-1} = O(1) \). Uchida [20] investigated the asymptotically efficient estimator for drift parameter by using the approximate martingale estimating function under \( (\varepsilon n^l)^{-1} \to 0 \), where \( l \) is a positive integer. For the asymptotically efficient estimator of drift parameter based on an approximate martingale estimating function of a one-dimensional small diffusion process under \( \varepsilon \to 0 \) and \( n \to \infty \), see Uchida [21]. Gloter and Sørensen [5] generalized the results in Sørensen and Uchida [18] and Uchida [20]. They proposed the minimum contrast estimators for both drift and diffusion parameters whose asymptotic covariance matrix equals to that of the estimators in Sørensen and Uchida [18] under \( (\varepsilon n^p)^{-1} = O(1) \), where \( \rho > 0 \).

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The adaptive inference for diffusion processes has been studied by many researchers. Since the adaptive method can divide the inference for \((\alpha, \beta)\) into that for the drift parameter \(\alpha\) and that for diffusion parameter \(\beta\), we expect that the adaptive parametric inference for diffusion processes is dealt with more accurately and quickly from the viewpoint of numerical analysis. For adaptive parametric estimation for ergodic diffusion processes, many researchers studied and obtained the asymptotic results, see Prakasa Rao [16] [17], Yoshida [26], Kessler [10], and Uchida and Yoshida [23]. Nakakita and Uchida [14] investigated the adaptive test for noisy ergodic diffusion processes. They derived the asymptotic null distribution of the adaptive test statistics based on the local means and consistency of the tests under alternatives. Nomura and Uchida [15] and Kaino and Uchida [9] proposed the adaptive Bayes type estimators and the hybrid estimators for both drift and diffusion parameters in small diffusion processes. They proved that the asymptotic efficiency for the estimator of the drift parameter and the asymptotic normality for the estimator of diffusion parameter whose asymptotic variance equals to that of the estimator in Sørensen and Uchida [18]. Moreover their estimators have convergence of moments under the assumption \((\varepsilon \sqrt{n})^{-1} = O(1)\).

In this paper, we utilize the results of Gloter and Sørensen [5] and propose the adaptive estimators for both drift and diffusion parameters in small diffusion processes under the assumption \((\varepsilon n^\rho)^{-1} = O(1)\), where \(\rho \geq 1/2\). In small diffusions, the convergence rates of estimators for the drift parameter and the diffusion parameter are \(\varepsilon^{-1}\) and \(\sqrt{n}\), respectively. Since it holds that \(\varepsilon^{-1}/\sqrt{n} \rightarrow \infty\) under this assumption with \(\rho > 1/2\), we first estimate drift parameter \(\alpha\) and next estimate diffusion parameter \(\beta\) in our adaptive estimation methods. The main result of this paper is that our proposed adaptive estimators have asymptotic efficiency for \(\alpha\) and asymptotic normality for \(\beta\) whose asymptotic covariance matrix is the same as that of the estimators proposed in Sørensen and Uchida [18] under the milder assumption than \((\varepsilon \sqrt{n})^{-1} = O(1)\). We also give the estimator for drift parameter \(\alpha\) which can be estimated independently from diffusion parameter \(\beta\) and show that the estimator has asymptotic normality under the assumption \((\varepsilon n^\rho)^{-1} = O(1)\).

The paper is organized as follows. In Section 2, notation and assumptions are introduced. The infinitesimal generator of the small diffusion process and its approximation used in constructing the contrast functions for adaptive estimators are defined. In Section 3, we propose two kinds of adaptive estimators and state the asymptotic properties of the proposed estimators. Note that the drift parameter can be estimated independent of the diffusion parameter. In Section 4, we give some examples and simulation results of the asymptotic performance for two types of adaptive estimators for multi-dimensional small diffusion processes. In model 1, we compare the adaptive estimators with the joint estimator proposed in Gloter and Sørensen [5]. In model 2, we treat the numerical simulation of the SIR model. In model 3, the difference of the asymptotic performances between the two adaptive estimators is examined. Section 5 is devoted to the proofs of the results presented in Section 4.

### 2. Notation and Assumptions

In this paper, we set \(\partial_{\alpha_i} := \partial/\partial\alpha_i\), \(\partial_{\beta_i} := \partial/\partial\beta_i\), \(\partial_{\alpha} := (\partial_{\alpha_1}, \ldots, \partial_{\alpha_p})^T\), \(\partial_{\beta} := (\partial_{\beta_1}, \ldots, \partial_{\beta_q})^T\), \(\partial_\alpha := \partial_{\alpha} \partial_{\alpha}^T\), \(\partial_\beta := \partial_{\beta} \partial_{\beta}^T\), \(\partial_{\alpha \beta} := \partial_{\alpha} \partial_{\beta}^T\), where \(\top\) is the transpose of a matrix. The symbols \(\overset{P}{\rightarrow}\) and \(\overset{d}{\rightarrow}\) indicate convergence in probability and convergence in distribution, respectively.

Let \((X^0_t)\) be the solution of the following ordinary differential equation (ODE) which is the case of \(\varepsilon = 0\) and \(\alpha = \alpha_0\) in the SDE (1):

\[
\begin{align*}
\frac{dX^0_t}{dt} &= b(X^0_t, \alpha_0) dt, \quad t \in [0, T], \\
X^0_0 &= x_0,
\end{align*}
\]

and \(I(\theta_0)\) be the \((p + q) \times (p + q)\)-matrix defined as

\[
I(\theta_0) := \begin{pmatrix}
I_b^{ij}(\theta_0) & 0 \\
0 & I_b^{ij}(\beta_0)
\end{pmatrix}_{1 \leq i, j \leq p, q},
\]
where
\[ I^{i,j}_b(\theta_0) = \int_0^T (\partial_{\alpha_i} b(X_s^0, \alpha_0)) (\partial_{\alpha_j} b(X_s^0, \alpha_0)) ds, \]
\[ I^{i,j}_\sigma(\beta_0) = \frac{1}{2T} \int_0^T \text{tr} \left[ (\partial_{\beta_i} [\sigma \sigma^T]) (\partial_{\beta_j} [\sigma \sigma^T]) (X_s^0, \beta_0) \right] ds. \]

Moreover, we define \( p \times p \)-matrices \( J_b(\alpha_0) = \left( J^{i,j}_b(\alpha_0) \right)_{1 \leq i, j \leq p} \) and \( K_b(\theta_0) = \left( K^{i,j}_b(\theta_0) \right)_{1 \leq i, j \leq p} \) as
\[ J^{i,j}_b(\alpha_0) = \int_0^T (\partial_{\alpha_i} b(X_s^0, \alpha_0)) (\partial_{\alpha_j} b(X_s^0, \alpha_0)) ds, \]
\[ K^{i,j}_b(\theta_0) = \int_0^T (\partial_{\alpha_i} b(X_s^0, \alpha_0)) (\partial_{\alpha_j} b(X_s^0, \alpha_0)) ds. \]

We make the following assumptions.

\[ \text{[A1]} \] For all \( \varepsilon > 0 \), SDE (1) with the true value of the parameter has a unique strong solution on some probability space \((\Omega, \mathcal{F}, P)\), and ODE [2] has a unique solution.

\[ \text{[A2]} \] \( b \in C^\infty(\mathbb{R}^d \times \Theta_\alpha), \sigma \in C(\mathbb{R}^d \times \Theta_\beta) \), and there exists an open convex subset \( \mathcal{U} \subset \mathbb{R}^d \) such that \( X_t^0 \in \mathcal{U} \) for all \( t \in [0, 1] \), and \( \sigma \in C^\infty(\mathcal{U} \times \Theta_\beta) \). Moreover \( [\sigma \sigma^T](x, \beta) \) is invertible on \( \mathcal{U} \times \Theta_\beta \).

\[ \text{[A3]} \] \( b(X_t^0, \alpha) = b(X_t^0, \alpha_0) \) for all \( t \in [0, 1] \) \( \Rightarrow \alpha = \alpha_0 \), \( \sigma(X_t^0, \beta) = \sigma(X_t^0, \beta_0) \) for all \( t \in [0, 1] \) \( \Rightarrow \beta = \beta_0 \).

\[ \text{[A4]} \] (i) \( I_b(\theta_0) \) and \( J_b(\alpha_0) \) are non-singular,

(ii) \( I_\sigma(\theta_0) \) is non-singular.

\[ \text{[B]} \] \( \varepsilon = \varepsilon_n \to 0 \) as \( n \to \infty \) and there exists \( \rho \geq \frac{1}{2} \) such that \( \lim_{n \to \infty} (\varepsilon n^\rho)^{-1} < +\infty \).

**Remark 1**

(i) Assumption [A2] is derived from a localization argument and hence [A2] is the mild condition. The strict assumption of [A2] and the relationship between the two assumptions are introduced in Section 5.

(ii) For [A4], if we assume a condition for \( \sigma \), then we can clarify the relationship between the regularity for \( I_b(\theta_0) \) and \( J_b(\alpha_0) \). Under [A2'] in Section 5, in particular, it holds true that if one is regular, then the other is also regular.

For [B], we set the approximation degree \( v \) as the integer such that \( v = \lfloor \rho + \frac{1}{2} \rfloor \), where \( \lfloor y \rfloor := \min \{ z \in \mathbb{Z} : y \leq z \} \). Let us define the operators \( \mathcal{L}_{\alpha_0}^v \) and \( \mathcal{L}_{\alpha}^v \). We denote by \( \mathcal{L}_{\alpha_0}^v \) the infinitesimal generator of the diffusion process \( X \): for any smooth function \( f \),
\[ \mathcal{L}_{\alpha_0}^v(f)(x) := \sum_{i=1}^d b^i(x, \alpha_0) \frac{\partial}{\partial x_i} f(x) + \frac{\varepsilon}{2} \sum_{i,j=1}^d [\sigma \sigma^T]_{i,j}(x, \beta_0) \frac{\partial^2}{\partial x_i \partial x_j} f(x), \]
and define the simple approximation of the generator as
\[ \mathcal{L}_{\alpha}^0(f)(x) := \sum_{i=1}^d b^i(x, \alpha) \frac{\partial}{\partial x_i} f(x). \]

Using the operator \( \mathcal{L}_{\alpha}^0 \), we set that for \( 2 \leq l \leq v \),
\[ P_{1,k}(\alpha) := X_{t_k^0} - X_{t_{k-1}^0} - h_k b(X_{t_{k-1}^0}, \alpha), \]
\[ P_{l,k}(\alpha) := P_{1,k}(\alpha) - Q_{l,k}(\alpha), \]
\[ Q_{l,k}(\alpha) := \sum_{j=1}^{l-1} \frac{h_j^{j+1}}{(j+1)!} (\mathcal{L}_{\alpha}^0)^j b(X_{t_{k-1}^0}, \alpha). \]
In particular,\[ P_{2,k}(\alpha) = X_{t_k^n} - X_{t_{k-1}^n} - h_n b(X_{t_{k-1}^n}, \alpha) - \frac{h_n^2}{2} \sum_{i=1}^d b'(X_{t_{k-1}^n}, \alpha) \frac{\partial}{\partial x_i} b(X_{t_{k-1}^n}, \alpha), \]
\[ Q_{2,k}(\alpha) = \frac{h_n^2}{2} \sum_{i=1}^d b'(X_{t_{k-1}^n}, \alpha) \frac{\partial}{\partial x_i} b(X_{t_{k-1}^n}, \alpha). \]

3. Adaptive estimation

In this section, we propose the following two types of adaptive estimators.

3.1. Type I estimator. First, we introduce the adaptive estimators which can be divided the parameter optimization for \( \theta = (\alpha, \beta) \) into the optimization of \( \alpha \) and that of \( \beta \).

Step 1. Let \[ U_{\varepsilon,n,v}^{(1)}(\alpha) := \varepsilon^{-2} h_n^{-1} \sum_{k=1}^n P_{v,k}(\alpha)^\top P_{v,k}(\alpha), \]
and \( \tilde{\alpha}_{\varepsilon,n}^{(1)} \) is defined as

\[ \tilde{\alpha}_{\varepsilon,n}^{(1)} := \arg \min_{\alpha \in \Theta} U_{\varepsilon,n,v}^{(1)}(\alpha). \]

Step 2. Set

\[ U_{\varepsilon,n,v}^{(2)}(\beta | \tilde{\alpha}) := \sum_{k=1}^n \left\{ \log \det (\sigma \sigma^\top)(X_{t_{k-1}^n}, \beta) + \varepsilon^{-2} h_n^{-1} P_{v,k}(\tilde{\alpha})^\top \sigma \sigma^\top^{-1}(X_{t_{k-1}^n}, \beta) P_{v,k}(\tilde{\alpha}) \right\}, \]
and \( \tilde{\beta}_{\varepsilon,n}^{(1)} \) is defined as

\[ \tilde{\beta}_{\varepsilon,n}^{(1)} := \arg \min_{\beta \in \Theta_{\beta}} U_{\varepsilon,n,v}^{(2)}(\beta | \tilde{\alpha}_{\varepsilon,n}^{(1)}). \]

Step 3. Let

\[ U_{\varepsilon,n,v}^{(3)}(\alpha | \tilde{\beta}) := \varepsilon^{-2} h_n^{-1} \sum_{k=1}^n P_{v,k}(\alpha)^\top \sigma \sigma^\top^{-1}(X_{t_{k-1}^n}, \tilde{\beta}) P_{v,k}(\alpha), \]
and \( \tilde{\alpha}_{\varepsilon,n}^{(2)} \) is defined as

\[ \tilde{\alpha}_{\varepsilon,n}^{(2)} := \arg \min_{\alpha \in \Theta} U_{\varepsilon,n,v}^{(3)}(\alpha | \tilde{\beta}_{\varepsilon,n}^{(1)}). \]

In Step 1, we have the following asymptotic properties.

**Lemma 1** Assume [A1]-[A3], [A4]-[i] and [B]. Then it holds that

\[ \varepsilon^{-(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0)} = O_P(1). \]

In particular, we have the convergence

\[ \varepsilon^{-1}(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0) \xrightarrow{d} N_p(0, \sigma \sigma^\top \times \kappa_0(\theta_0)^{-1} \kappa_0(\alpha_0)^{-1}) \]
as \( \varepsilon \to 0 \) and \( n \to \infty \).

**Remark 2** In order to prove the asymptotic normality of the estimator in Theorem 1 below, we need to show that \( \varepsilon^{-1}(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0) = O_P(1) \). However, using the estimator \( \tilde{\alpha}_{\varepsilon,n}^{(1)} \) and the results of Lemma 1 we can estimate the drift parameter \( \alpha \) independent of the diffusion parameter \( \beta \) and ensure that this estimator has asymptotic normality. Note that this estimator is not asymptotic efficient in general.

The main result for Type I method is as follows.
Theorem 1  Assume [A1]-[A4] and [B]. Then it follows that
\[ \hat{\theta}_{\varepsilon,n} \xrightarrow{P} \theta_0. \]
Moreover, we have the convergence
\[ (\varepsilon^{-1}(\hat{\alpha}_{\varepsilon,n} - \alpha_0))^{(1)} \] 
\[ \sqrt{n}(\hat{\beta}_{\varepsilon,n} - \beta_0) \xrightarrow{d} N_{p+q}(0, I(\theta_0)^{-1}) \]
as \( \varepsilon \to 0 \) and \( n \to \infty \).

3.2. Type II estimator. In the Type II method, we divide the Step 1 of Type I method into many steps. By this split, we expect that the estimators in Type II method are computed more quickly than those in the Type I method from the viewpoint of computation time.
Step 1. Set
\[ V_{\varepsilon,n,v}^{(1)}(\alpha) := \varepsilon^{-2}h_n^{-1} \sum_{k=1}^{n} P_{1,k}(\alpha)^\top P_{1,k}(\alpha), \]
and \( \hat{\alpha}_{\varepsilon,n}^{(1)} \) is defined as
\[ \hat{\alpha}_{\varepsilon,n}^{(1)} := \arg\min_{\alpha \in \Theta_n} V_{\varepsilon,n,v}^{(1)}(\alpha). \]

Step \( l = 2 \) to \( v \). Let
\[ V_{\varepsilon,n,v}^{(l)}(\alpha|\bar{\alpha}) := \varepsilon^{-2}h_n^{-1} \sum_{k=1}^{n} (P_{1,k}(\alpha) - Q_{l,k}(\bar{\alpha}))^\top (P_{1,k}(\alpha) - Q_{l,k}(\bar{\alpha})), \]
and \( \hat{\alpha}_{\varepsilon,n}^{(l)} \) is defined as
\[ \hat{\alpha}_{\varepsilon,n}^{(l)} := \arg\min_{\alpha \in \Theta_n} V_{\varepsilon,n,v}^{(l)}(\alpha|\hat{\alpha}_{\varepsilon,n}^{(l-1)}). \]

Step \( v + 1 \). Set
\[ V_{\varepsilon,n,v}^{(v+1)}(\beta|\bar{\alpha}) := \sum_{k=1}^{n} \left\{ \log \det[\sigma \sigma^\top](X_{t_{k-1}}, \beta) + \varepsilon^{-2}h_n^{-1}P_{v,k}(\bar{\alpha})^\top \sigma \sigma^\top^{-1}(X_{t_{k-1}}, \beta)P_{v,k}(\bar{\alpha}) \right\}, \]
and \( \hat{\beta}_{\varepsilon,n} := \hat{\beta}_{\varepsilon,n}^{(1)} \) is defined as
\[ \hat{\beta}_{\varepsilon,n} := \arg\min_{\beta \in \Theta_n} V_{\varepsilon,n,v}^{(v+1)}(\beta|\hat{\alpha}_{\varepsilon,n}^{(v)}). \]

Step \( v + 2 \). Let
\[ V_{\varepsilon,n,v}^{(v+2)}(\alpha|\bar{\alpha}, \bar{\beta}) := \varepsilon^{-2}h_n^{-1} \sum_{k=1}^{n} (P_{1,k}(\alpha) - Q_{v,k}(\bar{\alpha}))^\top \sigma \sigma^\top^{-1}(X_{t_{k-1}}, \bar{\beta})(P_{1,k}(\alpha) - Q_{v,k}(\bar{\alpha})), \]
and \( \hat{\alpha}_{\varepsilon,n} := \hat{\alpha}_{\varepsilon,n}^{(v+1)} \) is defined as
\[ \hat{\alpha}_{\varepsilon,n} := \arg\min_{\alpha \in \Theta_n} V_{\varepsilon,n,v}^{(v+2)}(\alpha|\hat{\alpha}_{\varepsilon,n}^{(v)}, \hat{\beta}_{\varepsilon,n}). \]

In Step \( v \), we have the following asymptotic properties.

Lemma 2  Assume [A1]-[A3], [A4]-[i] and [B]. Then it holds that
\[ \varepsilon^{-1}(\hat{\alpha}_{\varepsilon,n}^{(v)} - \alpha_0) = O_p(1). \]
In particular, we have the convergence
\[ \varepsilon^{-1}(\hat{\alpha}_{\varepsilon,n}^{(v)} - \alpha_0) \xrightarrow{d} N_p(0, J_\beta(\alpha_0)^{-1}K_\beta(\alpha_0)J_\beta(\alpha_0)^{-1}) \]
as \( \varepsilon \to 0 \) and \( n \to \infty \).

The main result for the Type II method is as follows.
**Theorem 2** Assume [A1]-[A4] and [B]. Then it follows that
\[ \hat{\theta}_{\varepsilon,n} \overset{p}{\to} \theta_0. \]
Moreover, we have the convergence
\[ \left( \varepsilon^{-1}(\hat{\alpha}_{\varepsilon,n} - \alpha_0) \right) \overset{d}{\to} N_p(0, I(\theta_0)^{-1}) \]
as \( \varepsilon \to 0 \) and \( n \to \infty \).

**Remark 3** The Type I and Type II estimators work well for the case of estimating both the drift parameter \( \alpha \) and the diffusion parameter \( \beta \). Moreover, we can also apply our adaptive methods in the following cases and the adaptive estimators have asymptotic efficiency for the drift parameter.

(i) Case of \( \sigma \sigma^T = I_d \) (see, for example, Model 3 in Section 4.3 below): We estimate only the drift parameter \( \alpha \).

Type I: Calculate only the estimator in Step 1.
Type II: Calculate the estimators in Step 1 to \( v \).

(ii) Case of \( \alpha = \beta \) and \( \rho > 1/2 \) (see, for example, Model 2 in Section 4.2 below): Since \( \varepsilon^{-1}/\sqrt{n} \to \infty \), we estimate the drift parameter \( \alpha \).

Type I: Calculate the estimator in Step 1. Next, run Step 3 with \( \tilde{\beta}_{\varepsilon,n} = \hat{\alpha}_{\varepsilon,n}^{(1)} \).
Type II: Calculate the estimators in Step 1 to \( v \). Next, run Step \( v + 2 \) with \( \tilde{\beta}_{\varepsilon,n} = \hat{\alpha}_{\varepsilon,n}^{(v)} \).

4. **Examples and simulations**

4.1. **Model 1 (Case of estimating both drift and diffusion parameters).**

First, we examine the asymptotic performance of Theorems 1 and 2. Consider the following two-dimensional model:
\[
\begin{align*}
\frac{dX_t}{X_0} &= \begin{pmatrix} -\alpha_1 X_{t,1} + 2 \cos(1 + \alpha_2 X_{t,2}) \\ 2 \sin(1 + \alpha_3 X_{t,1}) - \alpha_4 X_{t,2} \end{pmatrix} dt + \varepsilon \begin{pmatrix} \beta_1 (1 + X_{t,1}^2)^{-1} & 0.1 \\ 0.1 & \beta_2 (1 + X_{t,2}^2)^{-1} \end{pmatrix} dW_t, & t \in [0,1] \\
X_0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{align*}
\]
where \( \theta = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) \) are unknown parameters. The true parameter values are \( \theta_0 = (3, 6, 5, 4, 1, 0.5) \), and the parameter space is assumed to be \( \Theta = [0.01, 50]^6 \). We estimate these parameters by the joint estimation method in Gloter and Sorensen [5], the Type I method, and the Type II method. We choose the initial parameters \( \theta = \theta_0 \) or \( \theta = (6, 4, 6, 8, 2, 2) \) and treat the case of \( (\varepsilon, n) = (0.05, 100), (0.01, 100), (0.01, 1000) \). For the balance condition, we set \( \rho = 1 \), that is, the approximation degree \( v = 2 \). In the simulation, \texttt{optim()} \( \) is used with the "L-BFGS-B" method in R Language, and 10000 independent sample paths are estimated.

Tables 1 and 2 show the simulation results of parameter estimation with the two choice of the initial parameter values \( \theta_{\text{init}} = \theta_0 = (3, 6, 5, 4, 1, 0.5) \) or \( \theta_{\text{init}} = (6, 4, 6, 8, 2, 2) \). In Table 1, we see that all of the estimation methods have good performances and there is no notable difference among the three types of methods. In Table 2, however, the joint estimation method has considerable biases while the Type I and Type II methods still have good performances. This is because the joint estimation method calculates a six-dimensional optimization while Type I and Type II method calculates at most four-dimensional optimization. Therefore, the adaptive estimation methods are useful for decreasing the dimensions of the parameter optimization.
Table 1. Mean (S.D.) of the simulated values with the true initial parameters $\theta_{\text{init}} = \theta_0$.

| $\varepsilon$ | $n$ | Method | $\alpha_1 (3)$ | $\alpha_2 (6)$ | $\alpha_3 (5)$ | $\alpha_4 (4)$ | $\beta_1 (1)$ | $\beta_2 (0.5)$ |
|---------------|-----|--------|----------------|----------------|----------------|----------------|---------------|---------------|
| 0.05          | 100 | Joint  | 2.9999         | 6.0009         | 4.9996         | 4.0018         | 0.9643        | 0.4905        |
|               |     |        | (0.0839)       | (0.0801)       | (0.0341)       | (0.0671)       | (0.0670)      | (0.0343)      |
|               |     | Type I | 2.9999         | 6.0009         | 4.9996         | 4.0018         | 0.9700        | 0.4925        |
|               |     |        | (0.0839)       | (0.0801)       | (0.0341)       | (0.0671)       | (0.0673)      | (0.0343)      |
|               |     | Type II| 2.9998         | 6.0006         | 4.9997         | 4.0019         | 0.9698        | 0.4926        |
|               |     |        | (0.0842)       | (0.0808)       | (0.0342)       | (0.0672)       | (0.0673)      | (0.0343)      |
| 0.01          | 100 | Joint  | 2.9998         | 6.0006         | 5.0000         | 4.0002         | 0.9661        | 0.4904        |
|               |     |        | (0.0168)       | (0.0157)       | (0.0068)       | (0.0133)       | (0.0664)      | (0.0345)      |
|               |     | Type I | 2.9998         | 6.0006         | 5.0000         | 4.0002         | 0.9719        | 0.4924        |
|               |     |        | (0.0168)       | (0.0157)       | (0.0068)       | (0.0133)       | (0.0667)      | (0.0345)      |
|               |     | Type II| 2.9987         | 6.0004         | 5.0000         | 4.0002         | 0.9716        | 0.4924        |
|               |     |        | (0.0168)       | (0.0157)       | (0.0068)       | (0.0133)       | (0.0667)      | (0.0345)      |
| 0.01          | 1000| Joint | 3.0001         | 6.0002         | 5.0001         | 4.0000         | 0.9665        | 0.4988        |
|               |     |        | (0.0167)       | (0.0155)       | (0.0068)       | (0.0133)       | (0.0213)      | (0.0310)      |
|               |     | Type I | 3.0001         | 6.0002         | 5.0001         | 4.0000         | 0.9971        | 0.4991        |
|               |     |        | (0.0167)       | (0.0155)       | (0.0068)       | (0.0133)       | (0.0213)      | (0.0310)      |
|               |     | Type II| 3.0001         | 6.0002         | 5.0001         | 4.0000         | 0.9971        | 0.4991        |
|               |     |        | (0.0167)       | (0.0155)       | (0.0068)       | (0.0133)       | (0.0213)      | (0.0310)      |

Table 2. Mean (S.D.) of the simulated values with the initial parameters $\theta_{\text{init}} = (6, 4, 6, 8, 2, 2)$.

| $\varepsilon$ | $n$ | Method | $\alpha_1 (3)$ | $\alpha_2 (6)$ | $\alpha_3 (5)$ | $\alpha_4 (4)$ | $\beta_1 (1)$ | $\beta_2 (0.5)$ |
|---------------|-----|--------|----------------|----------------|----------------|----------------|---------------|---------------|
| 0.05          | 100 | Joint  | 2.7221         | 40.914         | 17.772         | 3.7052         | 4.5697        | 3.2030        |
|               |     |        | (0.1964)       | (13.420)       | (9.6504)       | (0.7528)       | (1.1162)      | (1.3278)      |
|               |     | Type I | 2.9999         | 6.0009         | 5.0000         | 4.0018         | 0.9700        | 0.4925        |
|               |     |        | (0.0839)       | (0.0801)       | (0.0341)       | (0.0671)       | (0.0675)      | (0.0343)      |
|               |     | Type II| 2.9977         | 6.1745         | 4.9995         | 4.0014         | 0.9929        | 0.4928        |
|               |     |        | (0.0902)       | (2.1176)       | (0.0343)       | (0.0677)       | (0.2911)      | (0.0344)      |
| 0.01          | 100 | Joint  | 2.6508         | 46.418         | 42.110         | 4.3134         | 22.673        | 17.687        |
|               |     |        | (0.0919)       | (1.9025)       | (2.6332)       | (0.4090)       | (0.5207)      | (0.3332)      |
|               |     | Type I | 2.9998         | 6.0006         | 5.0000         | 4.0002         | 0.9719        | 0.4925        |
|               |     |        | (0.0168)       | (0.0157)       | (0.0068)       | (0.0133)       | (0.0668)      | (0.0347)      |
|               |     | Type II| 2.9997         | 6.0004         | 5.0000         | 4.0002         | 0.9717        | 0.4925        |
|               |     |        | (0.0168)       | (0.0158)       | (0.0068)       | (0.0134)       | (0.0667)      | (0.0346)      |
| 0.01          | 1000| Joint | 2.5863         | 30.540         | 4.9727         | 3.9766         | 6.8759        | 0.5218        |
|               |     |        | (0.0839)       | (1.5777)       | (0.0340)       | (0.0488)       | (0.7533)      | (0.0422)      |
|               |     | Type I | 3.0001         | 6.0002         | 5.0001         | 4.0000         | 0.9971        | 0.4991        |
|               |     |        | (0.0167)       | (0.0155)       | (0.0068)       | (0.0133)       | (0.0214)      | (0.0112)      |
|               |     | Type II| 3.0001         | 6.0002         | 5.0001         | 4.0000         | 0.9971        | 0.4991        |
|               |     |        | (0.0167)       | (0.0155)       | (0.0068)       | (0.0133)       | (0.0215)      | (0.0109)      |
Figure 1. Histogram (left), empirical distribution (middle), and Q-Q plot (right) for estimating $\alpha_1$ (top), $\alpha_2$ (middle), and $\alpha_3$ (bottom). (Type II, true initial parameter, $\varepsilon = 0.01$, $n = 1000$)

Figure 2. Histogram (left), empirical distribution (middle), and Q-Q plot (right) for estimating $\alpha_4$ (top), $\beta_1$ (middle), and $\beta_2$ (bottom). (Type II, true initial parameter, $\varepsilon = 0.01$, $n = 1000$)
4.2. Model 2 (Case of having the same parameter).

Second, we consider the case that the model has the same parameter in the drift and diffusion coefficients, that is, the case of $\alpha = \beta$ in the SDE (1). For example, we introduce the following SIR model with the small diffusion coefficient proposed in Guy et al [6, 7]: let $X_t = (S_t, I_t)$, $t \in [0, T]$, and

$$
\begin{align*}
    dS_t &= -\beta S_t I_t \ dt + \varepsilon \sqrt{\beta S_t I_t} \ dW_{t,1}, \\
    dI_t &= (\beta S_t I_t - \gamma I_t) \ dt + \varepsilon \left( -\sqrt{\beta S_t I_t} \ dW_{t,1} + \sqrt{\gamma I_t} \ dW_{t,2} \right),
\end{align*}
$$

where both of the drift and diffusion coefficients have $\theta = (\beta, \gamma)$ as unknown parameters and $(s_0, i_0) \in (0, 1)^2$ is the fixed value. The SIR model describes the simple epidemic spread with the three mutually exclusive health states: Susceptible-Infectious-Removed from the infectious chain. The parameter $\beta$ implies the transmission rate and the parameter $\gamma$ implies the recovery rate. Moreover, the basic reproduction number $R_0 := \beta/\gamma$ implies the average number of secondary cases generated by one infectious. Therefore, when the value $R_0$ is greater than one, then the epidemic spreads and vice versa. We treat the case of $\theta_0 = (1.2, 1.0)$ and $\theta_0 = (1.0, 0.9)$. The first case corresponds to the case of $R_0 > 1$, and the second case corresponds to $R_0 < 1$.

In this simulation, we set $\varepsilon = 10^{-4}$ ($= 1/\sqrt{N}$, $N = 10^8$: population number), $(s_0, i_0) = (0.99999, 0.00001)$ and the parameter space $\Theta = [0.01, 100]^2$. In order to treat the 10 days, monthly and yearly data, we set $(n, T, h_n) = (10, 1, 1/10)$, $(30, 1, 1/30)$, $(360, 12, 1/30)$ and determine the balance coefficient $\rho = 4$. In this model, the Type I and Type II methods described in Remark 3 are used for estimation. It is remarked that this model does not have to consider the initial parameter problem. This is because the model has only two parameters, and the simulation does not fail the parameters optimization. Therefore, we set that the initial parameter is the true value. The simulation is repeated 10000 times for each settings.

Tables 3 and 4 show the simulation results with the two types of true parameters settings. In both tables, the sample means are close to the true value and the sample standard deviations are also close to the theoretical standard deviations. Overall, the simulations for both Type I and Type II methods have good behavior.

**Table 3.** Mean and standard deviation (S.D.) of the estimators in the case of $\theta_0 = (1.2, 1.0)$.

| $n$ | $T$ | Method | $\beta$ | S.D. | Theoretical S.D. | $\gamma$ | S.D. | Theoretical S.D. |
|-----|-----|--------|--------|------|-----------------|--------|------|-----------------|
| 10  | 1   | Type I | 1.199507 | 0.035437 | 0.034884 | 0.999758 | 0.032008 | 0.031845 |
|     |     | Type II| 1.199570 | 0.035440 | 0.034884 | 0.999802 | 0.032010 | 0.031845 |
| 30  | 1   | Type I | 1.199555 | 0.033724 | 0.033545 | 1.000083 | 0.030760 | 0.030622 |
|     |     | Type II| 1.199578 | 0.033726 | 0.033545 | 1.000099 | 0.030762 | 0.030622 |
| 360 | 12  | Type I | 1.199709 | 0.004906 | 0.004915 | 1.000173 | 0.004553 | 0.004486 |
|     |     | Type II| 1.199712 | 0.004906 | 0.004915 | 1.000176 | 0.004554 | 0.004486 |
Table 4. Mean and standard deviation (S.D.) of the estimators in the case of $\theta_0 = (0.9, 1.0)$.

| $n$ | $T$ | Method | Mean   | S.D.   | Theoretical S.D. | Mean | S.D. | Theoretical S.D. |
|-----|-----|--------|--------|--------|------------------|------|------|------------------|
| 10  | 1   | Type I | 0.899630 | 0.032820 | 0.032337 | 0.999791 | 0.034273 | 0.034086 |
|     |     | Type II| 0.899663 | 0.032819 | 0.032337 | 0.999833 | 0.034279 | 0.034086 |
| 30  | 1   | Type I | 0.899652 | 0.031437 | 0.031253 | 1.000083 | 0.033082 | 0.032944 |
|     |     | Type II| 0.899659 | 0.031440 | 0.031253 | 1.000092 | 0.033078 | 0.032944 |
| 360 | 12  | Type I | 0.899130 | 0.011284 | 0.011358 | 1.000537 | 0.012076 | 0.011972 |
|     |     | Type II| 0.899144 | 0.011286 | 0.011358 | 1.000555 | 0.012076 | 0.011972 |

4.3. Model 3 (Case of estimating only drift parameter).

Third, we consider the case of estimating only the drift parameter. In particular, we treat the case of the identical diffusion coefficient:

$$\begin{align*}
\frac{dX_t}{dt} & = \begin{pmatrix}
1 - \alpha_1 X_{t,1} - 5 \sin(\alpha_2 X_{t,2}^2) \\
2 - \alpha_3 X_{t,2} - 5 \sin(\alpha_4 X_{t,3}^2) \\
3 - \alpha_5 X_{t,3} - 5 \sin(\alpha_6 X_{t,1}^2)
\end{pmatrix}
\frac{dt}{\varepsilon} dW_t, \quad t \in [0, 1], \quad X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\end{align*}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ are unknown parameters. Assume that the parameter space is $\Theta_\alpha = [0.01, 30]^6$ and the true parameter values are $\alpha_0 = (3, 7, 2, 8, 1, 6)$. We treat the case of $(\varepsilon, n) = (0.01, 100), (0.001, 100), (0.001, 1000)$ and choose $v = 3$. We estimate the parameters with the Type I method and Type II method. Moreover, we estimate the initial parameters with the following the Uniform + optim() method:

Step 1. Generate 20000 uniform random numbers $\alpha_{0,m}$ ($m = 1, \ldots, 20000$) on $[0.01, 30]^6$.

Step 2. Compute

$$\hat{\alpha}_{m}^{(1)} = \arg \min_{\alpha} U_{\varepsilon,n,v}^{(1)}(\alpha), \quad (\text{case of Type I}),$$

$$\hat{\alpha}_{m}^{(1)} = \arg \min_{\alpha} V_{\varepsilon,n,v}^{(1)}(\alpha), \quad (\text{case of Type II}),$$

by means of optim() in the R language, where the uniform random number $\alpha_{0,m}$ is used as the initial value for optimization.

Step 3. Define the initial estimator $\hat{\alpha}_{\text{init}}^{(1)}$ or $\hat{\alpha}_{\text{init}}^{(1)}$ as

$$\hat{\alpha}_{\text{init}}^{(1)} = \arg \min_{\alpha} \left\{ U_{\varepsilon,n,v}^{(1)}(\hat{\alpha}_{1}), U_{\varepsilon,n,v}^{(1)}(\hat{\alpha}_{2}), \ldots, U_{\varepsilon,n,v}^{(1)}(\hat{\alpha}_{20000}) \right\}, \quad (\text{case of Type I}),$$

$$\hat{\alpha}_{\text{init}}^{(1)} = \arg \min_{\alpha} \left\{ V_{\varepsilon,n,v}^{(1)}(\hat{\alpha}_{1}), V_{\varepsilon,n,v}^{(1)}(\hat{\alpha}_{2}), \ldots, V_{\varepsilon,n,v}^{(1)}(\hat{\alpha}_{20000}) \right\}, \quad (\text{case of Type II}).$$

The simulations are repeated 1000 times for each estimation method.

Table 5 shows the sample means, standard deviations of the simulated estimator values and the computation times of estimation for one sample path. Regarding accuracy of the estimation, we see that both methods performed well for each setting. From the viewpoint of the computation time, however, the estimators in the Type II method are computed more quickly than those of the Type I method. This is because that the contrast function of the Type II method does not optimize the higher order term but put the estimated value $\bar{\alpha}$ into $Q_{t,k}(\bar{\alpha})$. As the result, we recommend using the Type II method.
Table 5. Mean (S.D.) of the simulated values and the computation times.

| $\varepsilon$ | $n$   | Method | $\alpha_1(3)$ | $\alpha_2(7)$ | $\alpha_3(2)$ | $\alpha_4(8)$ | $\alpha_5(1)$ | $\alpha_6(6)$ | Time (m) |
|----------------|-------|--------|---------------|---------------|---------------|---------------|---------------|---------------|-----------|
| 0.01           | 100   | Type I | 3.0032        | 7.0127        | 1.9983        | 7.9964        | 1.0049        | 5.9979       | 58        |
|                |       |        | (0.1036)      | (0.3040)      | (0.0127)      | (0.0047)      | (0.0227)      | (0.0131)      |           |
|                |       | Type II| 2.9891        | 7.0030        | 1.9987        | 7.9958        | 0.9898        | 6.0032       | 11        |
|                |       |        | (0.0570)      | (0.0127)      | (0.0124)      | (0.0068)      | (0.0088)      | (0.0090)      |           |
| 0.001          | 100   | Type I | 2.9997        | 6.9990        | 1.9986        | 7.9958        | 1.0083        | 6.0003       | 53        |
|                |       |        | (0.0079)      | (0.0028)      | (0.0045)      | (0.0052)      | (0.0240)      | (0.0121)      |           |
|                |       | Type II| 2.9977        | 7.0002        | 1.9987        | 7.9947        | 0.9994        | 6.0022       | 9         |
|                |       |        | (0.0051)      | (0.0024)      | (0.0050)      | (0.0081)      | (0.0010)      | (0.0014)      |           |
| 0.001          | 1000  | Type I | 2.99994       | 7.00001       | 1.99993       | 7.99997       | 0.99999       | 6.00005      | 204       |
|                |       |        | (0.00182)     | (0.00034)     | (0.00108)     | (0.00018)     | (0.00082)     | (0.00072)     |           |
|                |       | Type II| 2.99994       | 7.00001       | 1.99993       | 7.99997       | 0.99999       | 6.00005      | 21        |
|                |       |        | (0.00182)     | (0.00034)     | (0.00108)     | (0.00018)     | (0.00082)     | (0.00072)     |           |

5. Proofs

In this section, we treat the case of $T = 1$ without loss of generality. Let the $\sigma$-field $G^n_k := \sigma(X^n_k : s \leq t^n_k)$, and for any vector $u$ (matrix $A$), $u^i(A^{i,j})$ denotes the $i$-th element ($(i,j)$-th element) of the vector (matrix). For any positive sequence $u_n$, $R : R \times R^d \rightarrow R$ denotes a function with a constant $C > 0$ such that for all $x \in R^d$, $|R(u_n, x)| \leq u_n C (1 + |x|)^C$, and $R_d$ denotes the $d$-dimensional vector whose element satisfies the definition of the function $R$. In order to prove the theorems and the lemmas proposed in Section 2, we introduce the following restrictive condition of $[A2]$:

$[A2']$ For all $(x, \beta) \in R^d \times \Theta_{\beta}$, the matrix $[\sigma \sigma^T](x, \beta)$ is positive definite. Moreover, the functions $\sigma$, $[\sigma \sigma^T]^{-1}$ (respectively $b$) are bounded and smooth with bounded derivatives of any order on $R^d \times \Theta_{\beta}$ (respectively $R^d \times \Theta_{\alpha}$).

The following proposition enables us to prove the theorems and the lemmas under $[A1]$, $[A2']$, $[A3]$, $[A4]$ and $[B]$.

**Proposition 1** To prove that the conclusions of Theorems 1-2 and Lemmas 1-2 hold under $[A1]$-$[A4]$ and $[B]$, it is enough to prove that they hold under $[A1]$, $[A2']$, $[A3]$, $[A4]$ and $[B]$.

**Proof.** This result is obtained in an analogous manner to the proof of Proposition 1 in Gloter and Sørensen [5]. We omit the detailed proof. □

**Proof of Lemma 1** First, let us show $\hat{\alpha}^{(1)}_{\varepsilon,n} \overset{P}{\rightarrow} \alpha_0$. One deduces that

$$
U^{(1)}_{\varepsilon,n,v}(\alpha) = \varepsilon^{-2} n \sum_{k=1}^{n} (P_{v,k}(\alpha) - P_{v,k}(\alpha_0) + P_{v,k}(\alpha_0)) \top (P_{v,k}(\alpha) - P_{v,k}(\alpha_0) + P_{v,k}(\alpha_0))
$$

$$
= \varepsilon^{-2} n \sum_{k=1}^{n} (P_{v,k}(\alpha) - P_{v,k}(\alpha_0)) \top (P_{v,k}(\alpha) - P_{v,k}(\alpha_0))
$$

$$
+ 2 \varepsilon^{-2} n \sum_{k=1}^{n} (P_{v,k}(\alpha) - P_{v,k}(\alpha_0)) \top P_{v,k}(\alpha)
$$

$$
+ \varepsilon^{-2} n \sum_{k=1}^{n} P_{v,k}(\alpha_0) \top P_{v,k}(\alpha_0).
$$
Hence, it follows from [A2] and Lemma 4 in Gloter and Sørensen [5] that

\[
\epsilon^2 \left( U_{\epsilon,n,v}^{(1)}(\alpha) - U_{\epsilon,n,v}^{(1)}(\alpha_0) \right) = n \sum_{k=1}^{n} (P_{v,k}(\alpha) - P_{v,k}(\alpha_0))^{\top} (P_{v,k}(\alpha) - P_{v,k}(\alpha_0)) \\
+ 2n \sum_{k=1}^{n} (P_{v,k}(\alpha) - P_{v,k}(\alpha_0))^{\top} P_{v,k}(\alpha_0)
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} \left( b(X_{t_{k-1}}^{\epsilon}, \alpha_0) - b(X_{t_{k-1}}^{\epsilon}, \alpha) \right)^{\top} \left( b(X_{t_{k-1}}^{\epsilon}, \alpha_0) - b(X_{t_{k-1}}^{\epsilon}, \alpha) \right)
\]

\[
+ 2 \sum_{k=1}^{n} \left( b(X_{t_{k-1}}^{\epsilon}, \alpha_0) - b(X_{t_{k-1}}^{\epsilon}, \alpha) + R_d(n^{-1}, X_{t_{k-1}}^{\epsilon}) \right)^{\top} P_{v,k}(\alpha_0)
\]

\[
+ \frac{1}{n} \sum_{k=1}^{n} R(n^{-1}, X_{t_{k-1}}^{\epsilon}) \xrightarrow{P} U_1(\alpha, \alpha_0) \text{ uniformly in } \alpha,
\]

where

\[
U_1(\alpha, \alpha_0) := \int_{0}^{1} \left( b(X_{s}^{0}, \alpha_0) - b(X_{s}^{0}, \alpha) \right)^{\top} \left( b(X_{s}^{0}, \alpha_0) - b(X_{s}^{0}, \alpha) \right) ds.
\]

(6)

Let \( \omega \in \Omega \) be fixed. It follows from the compactness of \( \Theta_\alpha \) that for any sequence \( (\epsilon_m, n_m) \), there exists subsequence \( (\epsilon_m', n_m') \) such that

\[
\hat{\alpha}_{\epsilon_m, n_m}^{(1)}(\omega) \to \alpha_\infty \in \Theta_\alpha \quad (\epsilon_m' \to 0, n_m' \to \infty).
\]

(7)

From the continuity of \( U_1 \) and the definition of \( \hat{\alpha}_{\epsilon, n}^{(1)} \), one deduces that

\[
0 \geq \epsilon^2 \left( U_{\epsilon,n,v}^{(1)}(\hat{\alpha}_{\epsilon, n}^{(1)}(\omega)) - U_{\epsilon,n,v}^{(1)}(\alpha_0) \right)(\omega) \to U_1(\alpha_\infty, \alpha_0) \geq 0.
\]

Hence, we have \( \alpha_\infty = \alpha_0 \) from the identifiability condition [A3], and (7) means that \( \hat{\alpha}_{\epsilon, n}^{(1)} \xrightarrow{P} \alpha_0 \).

Second, let us prove \( \epsilon^{-1}(\hat{\alpha}_{\epsilon, n}^{(1)} - \alpha_0) = O_P(1) \). It follows from Taylor’s theorem that

\[
-\epsilon \partial_{\alpha} U_{\epsilon,n,v}^{(1)}(\alpha_0) = \left( \epsilon^2 \int_{0}^{1} \partial^2_{\alpha} U_{\epsilon,n,v}^{(1)}(\alpha_0 + u(\hat{\alpha}_{\epsilon, n}^{(1)} - \alpha_0)) du \right) \epsilon^{-1}(\hat{\alpha}_{\epsilon, n}^{(1)} - \alpha_0).
\]

(8)

For \( 1 \leq l \leq p \) and \( 1 \leq l_1, l_2 \leq p \), we deduce from [A2] and Lemma 4 in Gloter and Sørensen [5] that

\[
-\epsilon \partial_{\alpha} U_{\epsilon,n,v}^{(1)}(\alpha_0) = 2\epsilon^{-1} \sum_{k=1}^{d} \sum_{i=1}^{d} \left( \partial_{\alpha_i} b^i(X_{t_{k-1}}^{\epsilon}, \alpha_0) + R(n^{-1}, X_{t_{k-1}}^{\epsilon}) \right) P_{v,k}(\alpha_0) = O_P(1),
\]

(9)
and

\[ \varepsilon^2 \partial^2_{\alpha_1 \alpha_2} U_{\varepsilon,n}^{(1)}(\alpha) = -2 \sum_{k=1}^{n} \sum_{i=1}^{d} \left( \partial^2_{\alpha_1 \alpha_2} b'(X_{t_k}^0, \alpha) - R(n^{-1}, X_{t_k}^0) \right) \left( P^i_{\varepsilon,k}(\alpha) - P^i_{\varepsilon,k}(\alpha_0) \right) \]

\[ + 2n^{-1} \sum_{k=1}^{n} \sum_{i=1}^{d} \left( \partial^2_{\alpha_1 \alpha_2} b'(X_{t_k}^0, \alpha) - R(n^{-1}, X_{t_k}^0) \right) \left( \partial^2_{\alpha_1 \alpha_2} b'(X_{t_k}^0, \alpha) - R(n^{-1}, X_{t_k}^0) \right) \]

\[ = \frac{2}{n} \sum_{k=1}^{n} \sum_{i=1}^{d} \partial^2_{\alpha_1 \alpha_2} b'(X_{t_k}^0, \alpha) \left( b'(X_{t_k}^0, \alpha) - b'(X_{t_k}^0, \alpha_0) \right) \]

\[ + \frac{2}{n} \sum_{k=1}^{n} \sum_{i=1}^{d} \partial^2_{\alpha_1 \alpha_2} b'(X_{t_k}^0, \alpha) \partial^2_{\alpha_1 \alpha_2} b'(X_{t_k}^0, \alpha) + \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{d} R(n^{-1}, X_{t_k}^0) \]

\[ J^0_{\varepsilon} \rightarrow 2B_{1}^{1}J_{2}\alpha_0, \alpha_0 \] uniformly in \( \alpha \),

where

\[ B_{1}^{1}J_{2}\alpha_0, \alpha_0 := \int_{0}^{1} \left( \partial^2_{\alpha_1 \alpha_2} b(X_{s}^0, \alpha) \right)^T \left( b(X_{s}^0, \alpha) - b(X_{s}^0, \alpha_0) \right) ds + J^0_{\varepsilon}J_{2}\alpha_0, \alpha_0. \]

By noting that for all \( \lambda \in \mathbb{R}^d \setminus \{0\} \), it follows from \([A4]\) that

\[ \eta := 2\lambda^T B_1(\alpha_0, \alpha_0) \lambda = 2\lambda^T J_0^{1,2}(\alpha_0) \lambda > 0, \]

and one deduces that

\[ 1 = P \left( 2\lambda^T B_1(\alpha_0, \alpha_0) \lambda > \frac{\eta}{2} \right) \]

\[ \leq P \left( \lambda^T \left[ \int_{0}^{1} \left\{ B_1(\alpha_0, \alpha_0) - B_1(\alpha_0 + u(\tilde{\alpha}_{n}^{(1)} - \alpha_0), \alpha_0) \right\} du \right] \lambda > \frac{\eta}{12} \right) \]

\[ + P \left( \lambda^T \left[ \int_{0}^{1} \left\{ 2B_1(\alpha_0 + u(\tilde{\alpha}_{n}^{(1)} - \alpha_0), \alpha_0) - \varepsilon^2 \partial^2_{\alpha_0} U^{(1)}_{\varepsilon,n,v}(\alpha_0 + u(\tilde{\alpha}_{n}^{(1)} - \alpha_0)) \right\} du \right] \lambda > \frac{\eta}{6} \right) \]

\[ + P \left( \varepsilon^2 \int_{0}^{1} \partial^2_{\alpha_0} U^{(1)}_{\varepsilon,n,v}(\alpha_0 + u(\tilde{\alpha}_{n}^{(1)} - \alpha_0)) du \right) \lambda > \frac{\eta}{6} \right). \]

For a sequence \( \{r_n\}_{n \in \mathbb{N}} \) such that \( r_n \rightarrow 0 \ (n \rightarrow \infty) \), we define a set \( N_{n,\alpha} \) and an event \( A_{n,\alpha} \) as

\[ N_{n,\alpha} := \{ \alpha \in \Theta_{\alpha} \mid |\alpha - \alpha_0| \leq r_n \}, \quad A_{n,\alpha} := \{ \tilde{\alpha}_{n}^{(1)} \in N_{n,\alpha} \}. \]

We then obtain from \( \tilde{\alpha}_{n}^{(1)} \overset{P}{\rightarrow} \alpha_0 \) that \( P(A_{n,\alpha}) \rightarrow 1 \ (\varepsilon \rightarrow 0, n \rightarrow \infty) \). Therefore, for the right hand side of \([12]\), it follows from uniformly continuous of \( B_1 \) that

\[ P \left( \lambda^T \left[ \int_{0}^{1} \left\{ B_1(\alpha_0, \alpha_0) - B_1(\alpha_0 + u(\tilde{\alpha}_{n}^{(1)} - \alpha_0), \alpha_0) \right\} du \right] \lambda > \frac{\eta}{12} \right) \]

\[ \leq P \left( \sup_{\alpha \in N_{n,\alpha}} |B_1(\alpha_0, \alpha_0) - B_1(\alpha_0, \alpha_0)| > \frac{\eta}{12|\lambda|^2} \cap A_{n,\alpha} \right) + P(A_{n,\alpha}^{c}) \]

\[ \leq P \left( \sup_{\alpha \in N_{n,\alpha}} |B_1(\alpha_0, \alpha_0) - B_1(\alpha_0, \alpha_0)| > \frac{\eta}{12|\lambda|^2} \right) + P(A_{n,\alpha}^{c}) \]

\[ \rightarrow 0 \ (\varepsilon \rightarrow 0, n \rightarrow \infty), \]
and for (13), we deduce from the uniformly convergence (10) that

\[
P\left(\lambda^T \left[ \int_0^1 \left\{ 2B_1(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0), \alpha_0) - \varepsilon^2 \partial^2 u_{\varepsilon,n,v}^{(1)}(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0)) \right\} du \right] \lambda > \frac{\eta}{6} \right) \\
\leq P \left( \sup_{\alpha \in \Theta_\alpha} \left| 2B_1(\alpha, \alpha_0) - \varepsilon^2 \partial^2 U_{\varepsilon,n,v}^{(1)}(\alpha) \right| > \frac{\eta}{6} |\lambda|^2 \right) \\
\rightarrow 0 \quad (\varepsilon \to 0, n \to \infty).
\]

Consequently, we obtain

\[
P\left(\lambda^T \left( \varepsilon^2 \int_0^1 \partial^2 U_{\varepsilon,n,v}^{(1)}(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0)) du \right) \lambda > \frac{\eta}{6} \right) \rightarrow 1 \quad (\varepsilon \to 0, n \to \infty),
\]

and hence, in the equation (15), the results (9) and (15) lead to \( \varepsilon^{-1}(\tilde{\alpha}_{\varepsilon}^{(1)} - \alpha_0) = O_P(1) \).

Third, we prove the asymptotic normality of \( \tilde{\alpha}_{\varepsilon,n}^{(1)} \). In an analogous manner to Sorensen and Uchida [18], it is sufficient to show the following two properties:

\[
\sup_{u \in [0,1]} \left| \varepsilon^2 \partial^2 U_{\varepsilon,n,v}^{(1)}(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0)) - 2J_6(\alpha_0) \right| \overset{P}{\to} 0,
\]

\[
-\varepsilon \partial_{\alpha,U_{\varepsilon,n,v}^{(1)}}(\alpha_0) \overset{d}{\to} N(0, 4K_1(\theta_0)).
\]

For (16), it follows from (10), the consistency of \( \tilde{\alpha}_{\varepsilon,n}^{(1)} \) and [A2*] that for all \( \delta > 0 \),

\[
P\left( \sup_{u \in [0,1]} \left| \varepsilon^2 \partial^2 U_{\varepsilon,n,v}^{(1)}(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0)) - 2J_6(\alpha_0) \right| > \delta \right) \\
\leq P \left( \sup_{u \in [0,1]} \left| \varepsilon^2 \partial^2 U_{\varepsilon,n,v}^{(1)}(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0)) - 2B_1(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0), \alpha_0) \right| > \frac{\delta}{3} \right) \\
+ P \left( \sup_{u \in [0,1]} \left| 2B_1(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n}^{(1)} - \alpha_0), \alpha_0) - 2B_1(\alpha_0, \alpha_0) \right| > \frac{\delta}{3} \right) \\
+ P \left( \sup_{u \in [0,1]} \left| 2B_1(\alpha, \alpha_0) - 2B_1(\alpha_0, \alpha_0) \right| > \frac{\delta}{3} \right) \\
+ P \left( \sup_{\alpha \in \Theta_\alpha} \left| J_6(\alpha) \right| > \frac{\delta}{3} \right) \\
\rightarrow 0 \quad (\varepsilon \to 0, n \to \infty).
\]

Regarding (17), we have for \( 1 \leq l \leq p \),

\[
-\varepsilon \partial_{\alpha_1} U_{\varepsilon,n,v}^{(1)}(\alpha_0) = \sum_{k=1}^n c_{k,1}(\alpha_0) + \sum_{k=1}^n R_d(\varepsilon^{-1} n^{-1}, X_{t_{k-1}}^{n})^\top P_{v,k}(\alpha_0),
\]

where

\[
c_{k,1}(\alpha_0) := 2\varepsilon^{-1} \left( \partial_{\alpha_1} b(X_{t_{k-1}}^{n}, \alpha_0) \right)^\top P_{v,k}(\alpha_0).
\]
It follows from Lemma 1 in Gloter and Sørensen [5] that
\[
E \left[ R \left( \epsilon^{-1} n^{-1}, X_{l_{n-1}}^{v} \right)^{\top} P_{v,k}(\alpha_0) \right| G_{k-1}^{n} \right] = E \left[ \sum_{i=1}^{d} R \left( \epsilon^{-1} n^{-1}, X_{l_{n-1}}^{v} \right) P_{v,i}(\alpha_0) \left| G_{k-1}^{n} \right] \right]
\]
\[
\leq \sum_{i=1}^{d} E \left[ |P_{v,i}(\alpha_0)| |G_{k-1}^{n}| R \left( \epsilon^{-1} n^{-1}, X_{l_{n-1}}^{v} \right) \right]
\]
\[
\leq \sum_{i=1}^{d} E \left[ |P_{v,i}(\alpha_0)|^{2} |G_{k-1}^{n}| \right]^{\frac{1}{2}} R \left( \epsilon^{-1} n^{-1}, X_{l_{n-1}}^{v} \right)
\]
\[
\leq R \left( n^{-\frac{3}{2}}, X_{l_{n-1}}^{v} \right),
\]
and hence the second term of the right hand side in (18) converges to 0 in probability as \( \epsilon \to 0 \) and \( n \to \infty \).

From Theorems 3.2 and 3.4 in Hall and Heyde [8], it is sufficient to show the following convergences: for \( 1 \leq l_{1}, l_{2} \leq p \),
\[
\sum_{k=1}^{n} E \left[ c_{l_{1},1}(\alpha_0) |G_{k-1}^{n} \right] \xrightarrow{P} 0
\]
\[
\sum_{k=1}^{n} E \left[ c_{l_{1},1}(\alpha_0) c_{l_{2},1}(\alpha_0) |G_{k-1}^{n} \right] \xrightarrow{P} 4K_{l_{1},l_{2}}(\theta_0)
\]
\[
\sum_{k=1}^{n} E \left[ c_{l_{1},1}(\alpha_0) |G_{k-1}^{n} \right] E \left[ c_{l_{2},1}(\alpha_0) |G_{k-1}^{n} \right] \xrightarrow{P} 0
\]
\[
\sum_{k=1}^{n} E \left[ c_{l_{1},1}(\alpha_0) \right] ^{4} |G_{k-1}^{n} \right] \xrightarrow{P} 0.
\]

The above convergences are obtained by Lemma 1 and 4 in Gloter and Sørensen [5]. We omit the detailed proof.

**Proof of Theorem 1** 1st step. We prove the consistency of \( \hat{\beta}_{\epsilon,n} \). By the definition of the contrast function \( U_{\epsilon,n,v}(\beta|\tilde{a}) \), we have
\[
\frac{1}{n} \left( U_{\epsilon,n,v}(\beta|\tilde{a}_{(1),n}^{(1)}) - U_{\epsilon,n,v}(\beta|\tilde{a}_{(1),n}^{(2)}) \right)
\]
\[
= \frac{1}{n} \sum_{k=1}^{n} \left\{ \log det \left( [\sigma \sigma^{\top}] [X_{l_{n-1}}^{v}, \beta] [\sigma \sigma^{\top}]^{-1} [X_{l_{n-1}}^{v}, \beta] \right) \right\}
\]
\[
+ \epsilon^{-2} \sum_{k=1}^{n} P_{v,k}(\alpha_0) \left( [\sigma \sigma^{\top}]^{-1} [X_{l_{n-1}}^{v}, \beta] - [\sigma \sigma^{\top}]^{-1} [X_{l_{n-1}}^{v}, \beta] \right) P_{v,k}(\alpha_0)
\]
\[
+ 2\epsilon^{-2-n} \sum_{k=1}^{n} \left( nP_{v,k}(\tilde{a}_{(1),n}^{(1)}) - nP_{v,k}(\alpha_0) \right) \left( [\sigma \sigma^{\top}]^{-1} [X_{l_{n-1}}^{v}, \beta] - [\sigma \sigma^{\top}]^{-1} [X_{l_{n-1}}^{v}, \beta] \right) P_{v,k}(\alpha_0)
\]
\[
+ (\epsilon n)^{-1} \sum_{k=1}^{n} \left( nP_{v,k}(\tilde{a}_{(1),n}^{(1)}) - nP_{v,k}(\alpha_0) \right) \left( [\sigma \sigma^{\top}]^{-1} [X_{l_{n-1}}^{v}, \beta] - [\sigma \sigma^{\top}]^{-1} [X_{l_{n-1}}^{v}, \beta] \right)
\]
\[
\left( nP_{v,k}(\tilde{a}_{(1),n}^{(1)}) - nP_{v,k}(\alpha_0) \right)
\]

It follows from Lemmas 4 and 5 in Gloter and Sørensen [5] that the sum of (20) and (21) converges to \( U_{2}(\beta, \beta_0) \) in probability uniformly with respect to \( \beta \), where
\[
U_{2}(\beta, \beta_0) = \int_{0}^{1} \log det \left( [\sigma \sigma^{\top}] [X_{0}^{v}, \beta] [\sigma \sigma^{\top}]^{-1} [X_{0}^{v}, \beta_0] \right) ds + \int_{0}^{1} tr \left( [\sigma \sigma^{\top}]^{-1} [X_{0}^{v}, \beta] [\sigma \sigma^{\top}] [X_{0}^{v}, \beta_0] \right) ds - d.
\]

Noting that the condition [A2'] leads to the Lipschitz continuity of the functions \( nP_{v,k} \) and \( [\sigma \sigma^{\top}]^{-1} \), we deduce from Lemma 1-(5) in Gloter and Sørensen [5] and Lemma 1 in this paper that (22) and (23) converge.
to 0 in probability uniformly with respect to $\beta$. Therefore, one deduces that

$$\frac{1}{n} \left( U^{(2)}_{\varepsilon,n,v}(\beta|\hat{\alpha}^{(1)}_{\varepsilon,n}) - U^{(2)}_{\varepsilon,n,v}(\beta_0|\hat{\alpha}^{(1)}_{\varepsilon,n}) \right) \xrightarrow{P} U_2(\beta, \beta_0) \text{ uniformly in } \beta. \quad (25)$$

Let $\omega \in \Omega$ be fixed, then it follows from the compactness of $\Theta$ and the consistency of $\hat{\alpha}^{(1)}_{\varepsilon,n}$ that for any sequence $(\varepsilon_m, n_m)$, there exists subsequence $(\varepsilon'_m, n'_m)$ such that

$$\left( \hat{\alpha}^{(1)}_{\varepsilon_m,n'_m}(\omega), \hat{\beta}_{\varepsilon_m,n'_m}(\omega) \right) \to (\alpha_0, \beta_0) \in \Theta \quad (\varepsilon'_m \to 0, n'_m \to \infty). \quad (26)$$

From (25), the continuity of $U_2$ and the definition of $\tilde{\beta}_{\varepsilon,n}$, we obtain that

$$0 \geq \frac{1}{n} \left( U^{(2)}_{\varepsilon,n,v}(\tilde{\beta}_{\varepsilon,n}(\omega)|\hat{\alpha}^{(1)}_{\varepsilon_m,n'_m}(\omega)) - U^{(2)}_{\varepsilon,n,v}(\beta_0|\hat{\alpha}^{(1)}_{\varepsilon_m,n'_m}(\omega)) \right) \to U_2(\beta_0, \beta_0) \geq 0.$$

By [A3] and the proof of Lemma 17 in Genon-Catalot and Jacod [4], we have $\beta_\infty = \beta_0$, and (26) means that $\tilde{\beta}_{\varepsilon,n} \xrightarrow{P} \beta_0$.

2nd step. Next we show the consistency of $\hat{\alpha}_{\varepsilon,n}$. By the definition of the contrast function $U^{(3)}_{\varepsilon,n,v}(\alpha|\tilde{\beta})$, we have

$$\varepsilon^2 \left( U^{(3)}_{\varepsilon,n,v}(\alpha|\tilde{\beta}_{\varepsilon,n}) - U^{(3)}_{\varepsilon,n,v}(\alpha_0|\tilde{\beta}_{\varepsilon,n}) \right)$$

$$= 2n \sum_{k=1}^{n} \left( P_{v,k}(\alpha) - P_{v,k}(\alpha_0) \right)^{\top} [\sigma^T]^{-1}(X_{t_{k-1}^n, \tilde{\beta}_{\varepsilon,n}}) P_{v,k}(\alpha_0)$$

$$+ n \sum_{k=1}^{n} \left( P_{v,k}(\alpha) - P_{v,k}(\alpha_0) \right)^{\top} [\sigma^T]^{-1}(X_{t_{k-1}^n, \beta_0}) (P_{v,k}(\alpha) - P_{v,k}(\alpha_0))$$

$$+ n \sum_{k=1}^{n} \left( P_{v,k}(\alpha) - P_{v,k}(\alpha_0) \right)^{\top} \left( [\sigma^T]^{-1}(X_{t_{k-1}^n, \tilde{\beta}_{\varepsilon,n}}) - [\sigma^T]^{-1}(X_{t_{k-1}^n, \beta_0}) \right) (P_{v,k}(\alpha) - P_{v,k}(\alpha_0))$$

$$= 2n \sum_{k=1}^{n} \left( b(X_{t_{k-1}^n, 0} - b(X_{t_{k-1}^n, \alpha}) + R(n^{-1}, X_{t_{k-1}^n}^0) \right)^{\top} [\sigma^T]^{-1}(X_{t_{k-1}^n, \tilde{\beta}_{\varepsilon,n}}) P_{v,k}(\alpha_0)$$

$$+ \frac{1}{n} \sum_{k=1}^{n} \left( b(X_{t_{k-1}^n, \alpha}) - b(X_{t_{k-1}^n, 0}) \right)^{\top} [\sigma^T]^{-1}(X_{t_{k-1}^n, \beta_0}) (b(X_{t_{k-1}^n, \alpha}) - b(X_{t_{k-1}^n, 0}))$$

$$+ \frac{1}{n} \sum_{k=1}^{n} \left( b(X_{t_{k-1}^n, \alpha}) - b(X_{t_{k-1}^n, 0}) \right)^{\top} \left( [\sigma^T]^{-1}(X_{t_{k-1}^n, \tilde{\beta}_{\varepsilon,n}}) - [\sigma^T]^{-1}(X_{t_{k-1}^n, \beta_0}) \right) (b(X_{t_{k-1}^n, \alpha}) - b(X_{t_{k-1}^n, 0}))$$

$$+ \frac{1}{n} \sum_{k=1}^{n} R(n^{-1}, X_{t_{k-1}^n}). \quad (27)$$

It follows from Lemma 4-(2) in Gloter and Sørensen [5] for (27), and the consistency of $\tilde{\beta}_{\varepsilon,n}$ for (29) that the two terms converge to 0 in probability uniformly with respect to $\alpha$. Moreover, from Lemma 4-(1) in Gloter and Sørensen [5], (28) converges to $U_3(\alpha, \theta_0)$ in probability uniformly with respect to $\alpha$, where

$$U_3(\alpha, \theta_0) = \int_0^1 \left( b(X_s^0, \alpha) - b(X_s^0, 0) \right)^{\top} [\sigma^T]^{-1}(X_s^0, \beta_0) \left( b(X_s^0, \alpha) - b(X_s^0, 0) \right) ds. \quad (30)$$

Therefore, we have

$$\varepsilon^2 \left( U^{(3)}_{\varepsilon,n,v}(\alpha|\tilde{\beta}_{\varepsilon,n}) - U^{(3)}_{\varepsilon,n,v}(\alpha_0|\tilde{\beta}_{\varepsilon,n}) \right) \xrightarrow{P} U_3(\alpha, \theta_0) \text{ uniformly in } \alpha. \quad (31)$$

In an analogous manner to the proof of the consistency for $\hat{\alpha}^{(1)}_{\varepsilon,n}$ and $\tilde{\beta}_{\varepsilon,n}$, we have $\hat{\alpha}_{\varepsilon,n} \xrightarrow{P} \alpha_0$. 

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**3rd step.** We prove the asymptotic normality for $\tilde{\theta}_{\varepsilon,n}$. From Taylor’s theorem, we have the following expansions:

$$-\varepsilon \partial_\alpha U^{(3)}_{\varepsilon,n,v}(\alpha_0|\tilde{\beta}_{\varepsilon,n}) = \left( \varepsilon^2 \int_0^1 \partial_\alpha^2 U^{(3)}_{\varepsilon,n,v}(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n} - \alpha_0)|\tilde{\beta}_{\varepsilon,n})du \right) \varepsilon^{-1}(\tilde{\alpha}_{\varepsilon,n} - \alpha_0),$$

$$- \frac{1}{\sqrt{n}} \partial_\beta U^{(2)}_{\varepsilon,n,v}(\beta_0|\alpha_0) = \left( \frac{1}{n} \int_0^1 \partial_\beta^2 U^{(2)}_{\varepsilon,n,v}(\beta_0 + u(\tilde{\beta}_{\varepsilon,n} - \beta_0)|\tilde{\alpha}_{\varepsilon,n})du \right) \sqrt{n}(\tilde{\beta}_{\varepsilon,n} - \beta_0),$$

$$\varepsilon \partial_\alpha U^{(3)}_{\varepsilon,n,v}(\alpha_0|\tilde{\beta}_{\varepsilon,n}) - \varepsilon \partial_\alpha U^{(3)}_{\varepsilon,n,v}(\alpha_0|\beta_0) = \left( \frac{\varepsilon}{\sqrt{n}} \int_0^1 \partial_{\alpha\beta} U^{(3)}_{\varepsilon,n,v}(\alpha_0|\beta_0 + u(\tilde{\beta}_{\varepsilon,n} - \beta_0))du \right) \sqrt{n}(\tilde{\beta}_{\varepsilon,n} - \beta_0).$$

Using these expressions, we calculate that

$$\Gamma^{1}_{\varepsilon,n} = C^{1}_{\varepsilon,n}\Lambda^{1}_{\varepsilon,n},$$

where

$$\Gamma^{1}_{\varepsilon,n} := \left( -\varepsilon \partial_\alpha U^{(3)}_{\varepsilon,n,v}(\alpha_0|\beta_0) \right), \quad \Lambda^{1}_{\varepsilon,n} := \left( \varepsilon^{-1}(\tilde{\alpha}_{\varepsilon,n} - \alpha_0) \right),$$

$$C^{1}_{\varepsilon,n} := \frac{\varepsilon^2}{\sqrt{n}} \int_0^1 \partial_{\alpha\beta} U^{(3)}_{\varepsilon,n,v}(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n} - \alpha_0)|\tilde{\beta}_{\varepsilon,n})du \frac{1}{\sqrt{n}} \int_0^1 \partial_{\alpha\beta} U^{(2)}_{\varepsilon,n,v}(\beta_0 + u(\tilde{\beta}_{\varepsilon,n} - \beta_0)|\tilde{\alpha}_{\varepsilon,n})du.$$

In an analogous manner to the proof of Theorem 1 in Sørensen and Uchida [18], it is sufficient to show the following convergences:

$$\sup_{u \in [0,1]} \left| \frac{\varepsilon^2}{\sqrt{n}} \partial_{\alpha\beta} U^{(3)}_{\varepsilon,n,v}(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n} - \alpha_0)|\tilde{\beta}_{\varepsilon,n}) - 2I_0(\theta_0) \right| \xrightarrow{P} 0, \quad (32)$$

$$\sup_{u \in [0,1]} \left| \frac{1}{n} \partial_{\beta} U^{(2)}_{\varepsilon,n,v}(\beta_0 + u(\tilde{\beta}_{\varepsilon,n} - \beta_0)|\tilde{\alpha}_{\varepsilon,n}) - 2I_0(\theta_0) \right| \xrightarrow{P} 0, \quad (33)$$

$$\sup_{u \in [0,1]} \left| \frac{\varepsilon}{\sqrt{n}} \partial_{\alpha\beta} U^{(3)}_{\varepsilon,n,v}(\alpha_0 + u(\tilde{\alpha}_{\varepsilon,n} - \alpha_0)|\tilde{\beta}_{\varepsilon,n}) - \sqrt{n}(\tilde{\beta}_{\varepsilon,n} - \beta_0) \right| \xrightarrow{P} 0, \quad (34)$$

$$\Gamma^{1}_{\varepsilon,n} \xrightarrow{d} N_{p+q}(0, 4I(\theta_0)). \quad (35)$$

**Proof of (32).** By a simple calculation, it holds that for $1 \leq l_1, l_2 \leq p$,

$$\varepsilon^2 \partial_{\alpha l_1 l_2} U^{(3)}_{\varepsilon,n,v}(\alpha|\tilde{\beta}_{\varepsilon,n}) = 2n \sum_{k=1}^{n} \partial_{\alpha l_1 l_2} P_{v,k}(\alpha)[\tilde{\sigma}^T]^{-1}(X_{\tilde{t}_{\varepsilon,n}, k}) P_{v,k}(\alpha_0)$$

$$+ 2n \sum_{k=1}^{n} \partial_{\alpha l_1 l_2} P_{v,k}(\alpha)[\tilde{\sigma}^T]^{-1}(X_{\tilde{t}_{\varepsilon,n}, k}) (P_{v,k}(\alpha) - P_{v,k}(\alpha_0))$$

$$+ 2n \sum_{k=1}^{n} \partial_{\alpha l_1 l_2} P_{v,k}(\alpha) \left[ (\tilde{\sigma}^T)^{-1}(X_{\tilde{t}_{\varepsilon,n}, k}, \tilde{\beta}_{\varepsilon,n}) - [\tilde{\sigma}^T]^{-1}(X_{\tilde{t}_{\varepsilon,n}, k}, \beta_0) \right] (P_{v,k}(\alpha) - P_{v,k}(\alpha_0))$$

$$+ 2n \sum_{k=1}^{n} \partial_{\alpha l_1} P_{v,k}(\alpha)[\tilde{\sigma}^T]^{-1}(X_{\tilde{t}_{\varepsilon,n}, k}, \beta_0) \partial_{\alpha l_2} P_{v,k}(\alpha)$$

$$+ 2n \sum_{k=1}^{n} \partial_{\alpha l_1} P_{v,k}(\alpha) \left[ (\tilde{\sigma}^T)^{-1}(X_{\tilde{t}_{\varepsilon,n}, k}, \tilde{\beta}_{\varepsilon,n}) - [\tilde{\sigma}^T]^{-1}(X_{\tilde{t}_{\varepsilon,n}, k}, \beta_0) \right] \partial_{\alpha l_2} P_{v,k}(\alpha).$$

Note that $\partial_{\alpha l_1} P_{v,k}(\alpha) = -n^{-1} \partial_{\alpha l_1} b(X_{\tilde{t}_{\varepsilon,n}, k}, \alpha) + R_d(n^{-2}, X_{\tilde{t}_{\varepsilon,n}, k}) = R_d(n^{-1}, X_{\tilde{t}_{\varepsilon,n}, k})$ and $P_{v,k}(\alpha) - P_{v,k}(\alpha_0) = n^{-1} (b(X_{\tilde{t}_{\varepsilon,n}, k}, \alpha_0) - b(X_{\tilde{t}_{\varepsilon,n}, k}, \alpha)) + R_d(n^{-2}, X_{\tilde{t}_{\varepsilon,n}, k}) = R_d(n^{-1}, X_{\tilde{t}_{\varepsilon,n}, k})$, it follows from Lemma 4-(1) in
Gloter and Sørensen [5] that the sum of (37) and (39) converges to $2B_2^{(1,2)}(\alpha, \theta_0)$ in probability uniformly with respect to $\alpha$, where

$$B_2^{(1,2)}(\alpha, \theta_0) := \int_0^1 \left( \partial_{\alpha \alpha_1}^2 b(X_s^0, \alpha) \right)^T \left[ \sigma \sigma^T \right]^{-1}(X_s^0, \beta_0) \left( b(X_s^0, \alpha) - b(X_s^0, \alpha_0) \right) ds$$

$$+ \int_0^1 \left( \partial_{\alpha_1} b(X_s^0, \alpha) \right)^T \left[ \sigma \sigma^T \right]^{-1}(X_s^0, \beta_0) \left( \partial_{\alpha_2} b(X_s^0, \alpha) \right) ds.$$

On the other hand, by the consistency of $\hat{\beta}_{\epsilon,m}$ and Lemma 4 in Gloter and Sørensen [5], the remains terms (36), (38) and (40) converge to 0 in probability. Hence, we have

$$\sup_{\alpha \in \Theta_n} \left| \varepsilon^2 \partial_{\alpha \alpha_1}^2 U^{(3)}_{\epsilon,m,v}(\alpha) - 2B_2^{(1,2)}(\alpha, \theta_0) \right| \xrightarrow{p} 0. \quad (41)$$

By noting that $B_2(\alpha_0, \theta_0) = I_0(\theta_0)$, it follows from the consistency of $\hat{\alpha}_{\epsilon,n}$ and the uniform continuity of $B_2$ that for all $\delta > 0$,

$$P \left( \sup_{u \in [0,1]} \left| \varepsilon^2 \partial_{\alpha \alpha_1}^2 U^{(3)}_{\epsilon,m,v}(\alpha_0 + u(\hat{\alpha}_{\epsilon,n} - \alpha_0)) - 2I_0(\theta_0) \right| > \delta \right)$$

$$\leq P \left( \sup_{\alpha \in \Theta_n} \left| \varepsilon^2 \partial_{\alpha \alpha_1}^2 U^{(3)}_{\epsilon,m,v}(\alpha) - 2B_2(\alpha, \theta_0) \right| > \frac{\delta}{2} \right)$$

$$+ P \left( \sup_{u \in [0,1]} \left| 2B_2(\alpha_0 + u(\hat{\alpha}_{\epsilon,n} - \alpha_0), \theta_0) - 2I(\theta_0) \right| > \frac{\delta}{2} \right) \to 0 \quad (\varepsilon \to 0, n \to \infty).$$

This implies (32).

Proof of (32). We deduce that for $1 \leq m_1, m_2 \leq q$,

$$\frac{1}{n} \partial_{\beta_{m_1 m_2}}^2 U^{(2)}_{\epsilon,m,v}(\beta|\hat{\alpha}_{\epsilon,n})$$

$$= \frac{1}{n} \sum_{k=1}^n \partial_{\beta_{m_1 m_2}} \log \det \left[ \sigma \sigma^T \right](X_{t_{k-1}}^\epsilon, \beta)$$

$$+ \varepsilon^{-2} \sum_{k=1}^n P_{v,k}(\alpha_0)^T \left( \partial_{\beta_{m_1 m_2}}^2 \left[ \sigma \sigma^T \right]^{-1}(X_{t_{k-1}}^\epsilon, \beta) \right) P_{v,k}(\alpha_0) \quad (43)$$

$$+ 2\varepsilon^{-2} \sum_{k=1}^n \left( P_{v,k}(\hat{\alpha}_{\epsilon,n}) - P_{v,k}(\alpha_0) \right)^T \left( \partial_{\beta_{m_1 m_2}}^2 \left[ \sigma \sigma^T \right]^{-1}(X_{t_{k-1}}^\epsilon, \beta) \right) P_{v,k}(\alpha_0) \quad (44)$$

$$+ 2\varepsilon^{-2} \sum_{k=1}^n \left( P_{v,k}(\hat{\alpha}_{\epsilon,n}) - P_{v,k}(\alpha_0) \right)^T \left( \partial_{\beta_{m_1 m_2}}^2 \left[ \sigma \sigma^T \right]^{-1}(X_{t_{k-1}}^\epsilon, \beta) \right) \left( P_{v,k}(\hat{\alpha}_{\epsilon,n}) - P_{v,k}(\alpha_0) \right). \quad (45)$$

It follows from Lemma 4 and 5 in Gloter and Sørensen [5] that the sum of (42) and (43) converges to $B_3^{(1,2)}(\beta, \beta_0)$ in probability uniformly with respect to $\beta$, where

$$B_3^{m_1 m_2}(\beta, \beta_0) := - \int_0^1 \text{tr} \left[ \left( \left[ \sigma \sigma^T \right]^{-1} \left( \partial_{\beta_{m_1}} [\sigma \sigma^T] \right) \left[ \sigma \sigma^T \right]^{-1} \left( \partial_{\beta_{m_2}} [\sigma \sigma^T] \right) \right) (X_s^0, \beta) \right] ds$$

$$+ \int_0^1 \text{tr} \left[ \left( \left[ \sigma \sigma^T \right]^{-1} \left( \partial_{\beta_{m_2}}^2 [\sigma \sigma^T] \right) \right) (X_s^0, \beta) \right] ds$$

$$+ 2 \int_0^1 \text{tr} \left[ \left( \left[ \sigma \sigma^T \right]^{-1} \left( \partial_{\beta_{m_1}} [\sigma \sigma^T] \right) \left[ \sigma \sigma^T \right]^{-1} \left( \partial_{\beta_{m_2}} [\sigma \sigma^T] \right) \right) [\sigma \sigma^T]^{-1} (X_s^0, \beta) [\sigma \sigma^T] (X_s^0, \beta_0) \right] ds$$

$$- \int_0^1 \text{tr} \left[ \left( \left[ \sigma \sigma^T \right]^{-1} \left( \partial_{\beta_{m_1 m_2}}^2 [\sigma \sigma^T] \right) \right) [\sigma \sigma^T]^{-1} (X_s^0, \beta) [\sigma \sigma^T] (X_s^0, \beta_0) \right] ds.$$
We can show from Lemma 1 in this paper and Lemma 1 in Gloter and Sørensen [5] that (44) and (45) converge to 0 in probability. Therefore, it holds that

\[
\sup_{\beta \in \Theta_0} \left| \frac{1}{n} \partial_{\beta m}^2 U^{(2)}_{\varepsilon, n, v}(\beta | \hat{\alpha}_{e, n}^{(1)}) - B_3^{m_0} \right| \xrightarrow{P} 0.
\]

(46)

Noting that \( B_3(\beta_0, \beta_0) = 2I_m(\beta_0) \), in an analogous manner to the proof of (32), we deduce that for all \( \delta > 0 \),

\[
P \left( \sup_{u \in [0,1]} \left| \frac{1}{n} \partial_{\beta}^2 U^{(2)}_{\varepsilon, n, v}(\beta_0 + u(\hat{\beta}_{e, n} - \beta_0)| \hat{\alpha}_{e, n}^{(1)}) - 2I_m(\beta_0) \right| > \delta \right) \rightarrow 0 \quad (\varepsilon \rightarrow 0, n \rightarrow \infty).
\]

This implies (33).

Proof of (34). From Lemma 4-(2) in Gloter and Sørensen [5], it holds that for \( 1 \leq l \leq p \) and \( 1 \leq m \leq q \),

\[
\frac{\varepsilon}{\sqrt{n}} \partial_{\alpha_l \beta_m} U^{(3)}_{\varepsilon, n, v}(\alpha_0 | \beta_0) = \frac{\varepsilon}{\sqrt{n}} \sum_{k=1}^{n} \partial_{\alpha_l} P_{v, k}(\alpha_0) \partial_{\beta_m} [\sigma \sigma^T]^{-1}(X_{t_{k-1}^\varepsilon}, \beta_0) P_{v, k}(\alpha_0)
\]

\[
= \frac{2}{\sqrt{n}} \varepsilon \sum_{k=1}^{n} R_d(1, X_{t_{k-1}^\varepsilon})^\top P_{v, k}(\alpha_0)
\]

\[
\xrightarrow{P} 0 \quad \text{uniformly in } \beta.
\]

In particular, we have

\[
\sup_{u \in [0,1]} \left| \frac{\varepsilon}{\sqrt{n}} \partial_{\alpha_l \beta_m} U^{(3)}_{\varepsilon, n, v}(\alpha_0 | \beta_0 + u(\hat{\beta}_{e, n} - \beta_0)) \right| \xrightarrow{P} 0.
\]

Proof of (35). Using Lemma 4-(2) in Gloter and Sørensen [5], it holds that for \( 1 \leq l \leq p \),

\[
-\varepsilon \partial_{\alpha_l} U^{(3)}_{\varepsilon, n, v}(\alpha_0 | \beta_0) = -\frac{\varepsilon}{\sqrt{n}} \sum_{k=1}^{n} \partial_{\alpha_l} P_{v, k}(\alpha_0) [\sigma \sigma^T]^{-1}(X_{t_{k-1}^\varepsilon}, \beta_0) P_{v, k}(\alpha_0)
\]

\[
= \sum_{k=1}^{n} \xi_{l, k}(\theta_0) + o_P(1),
\]

(47)

where

\[
\xi_{l, k}(\theta_0) := 2\varepsilon^{-1} \partial_{\alpha_l} b(X_{t_{k-1}^\varepsilon}, \alpha_0)^\top [\sigma \sigma^T](X_{t_{k-1}^\varepsilon}, \beta_0) P_{v, k}(\alpha_0).
\]

(48)

On the other hand, it follows from Taylor’s theorem that

\[
\frac{-1}{\sqrt{n}} \partial_{\beta} U^{(2)}_{\varepsilon, n, v}(\beta_0 | \hat{\alpha}_{e, n}^{(1)}) = \frac{-1}{\sqrt{n}} \partial_{\beta} U^{(2)}_{\varepsilon, n, v}(\beta_0 | \alpha_0)
\]

\[
- \left( \frac{\varepsilon}{\sqrt{n}} \int_0^1 \partial_{\alpha \beta} U^{(2)}_{\varepsilon, n, v}(\beta_0 | \alpha_0) + u(\hat{\alpha}_{e, n}^{(1)} - \alpha_0) du \right) \varepsilon^{-1}(\hat{\alpha}_{e, n}^{(1)} - \alpha_0).
\]

In particular, it holds from Lemma 1 in this paper and Lemma 4-(2) in Gloter and Sørensen [5] that

\[
\frac{\varepsilon}{\sqrt{n}} \partial_{\alpha_l \beta_m} U^{(2)}_{\varepsilon, n, v}(\beta_0 | \alpha) = 2\varepsilon^{-1} \frac{\varepsilon}{\sqrt{n}} \sum_{k=1}^{n} \partial_{\alpha_l} P_{v, k}(\alpha) \partial_{\beta_m} [\sigma \sigma^T]^{-1}(X_{t_{k-1}^\varepsilon}, \beta_0) P_{v, k}(\alpha_0)
\]

\[
+ 2\varepsilon^{-1} \frac{\varepsilon}{\sqrt{n}} \sum_{k=1}^{n} \partial_{\alpha_l} P_{v, k}(\alpha) \partial_{\beta_m} [\sigma \sigma^T]^{-1}(X_{t_{k-1}^\varepsilon}, \beta_0) (P_{v, k}(\alpha) - P_{v, k}(\alpha_0))
\]

\[
\xrightarrow{P} 0 \quad \text{uniformly in } \alpha,
\]

and we have

\[
\sup_{u \in [0,1]} \left| \frac{\varepsilon}{\sqrt{n}} \partial_{\alpha_l \beta_m} U^{(2)}_{\varepsilon, n, v}(\alpha_0 + u(\hat{\alpha}_{e, n}^{(1)} - \alpha_0)) \right| \xrightarrow{P} 0.
\]
Therefore, by Lemma \[1\] we deduce that for \(1 \leq m \leq q\),
\[
\frac{1}{\sqrt{n}} \partial_{\beta_m} U^{(2)}_{\varepsilon,n,v} (\beta_0 \mid \alpha_0) = -\frac{1}{\sqrt{n}} \partial_{\beta_m} U^{(2)}_{\varepsilon,n,v} (\beta_0 \mid \alpha_0) + o_P(1)
\]
\[= \sum_{k=1}^{n} (\eta_{k,1}^{m}(\beta_0) + \eta_{k,2}^{m}(\theta_0)) + o_P(1), \tag{49}\]
where
\[
\eta_{k,1}^{m}(\beta_0) := -n^{-\frac{1}{2}} \text{tr} \left( [\sigma \sigma^T]^{-1} \partial_{\beta_m} [\sigma \sigma^T] \right) (X_{l_{k-1}}^{(2)}, \beta_0), \tag{50}\]
\[
\eta_{k,2}^{m}(\theta_0) := \varepsilon^{-2} n^{\frac{1}{2}} P_{\varepsilon,n,v}(\alpha_0) \left( [\sigma \sigma^T]^{-1} (\partial_{\beta_m} [\sigma \sigma^T]) [\sigma \sigma^T]^{-1} \right) (X_{l_{k-1}}^{(2)}, \beta_0). \tag{51}\]

In order to show \((35)\), by Theorems 3.2 and 3.4 in Hall and Heyde \[8\], it is sufficient to show the following convergences: for \(1 \leq l_1, l_2 \leq p\) and \(1 \leq m_1, m_2 \leq q\),
\[
\sum_{k=1}^{n} E \left[ \xi_{k,1}^{l_1}(\theta_0) | G_{k-1}^{n} \right] \xrightarrow{P} 0,
\]
\[
\sum_{k=1}^{n} E \left[ \eta_{k,1}^{m_1}(\beta_0) + \eta_{k,2}^{m_1}(\theta_0) | G_{k-1}^{n} \right] \xrightarrow{P} 0,
\]
\[
\sum_{k=1}^{n} E \left[ \xi_{k,1}^{l_1}(\theta_0) \xi_{k,1}^{l_2}(\theta_0) | G_{k-1}^{n} \right] - \sum_{k=1}^{n} E \left[ \xi_{k,1}^{l_1}(\theta_0) | G_{k-1}^{n} \right] E \left[ \xi_{k,1}^{l_2}(\theta_0) | G_{k-1}^{n} \right] \xrightarrow{P} 4 I_{l_1,l_2}^{1,1}(\theta_0),
\]
\[
\sum_{k=1}^{n} E \left[ \eta_{k,1}^{m_1}(\beta_0) + \eta_{k,2}^{m_1}(\theta_0)(\eta_{k,1}^{m_2}(\beta_0) + \eta_{k,2}^{m_2}(\theta_0)) | G_{k-1}^{n} \right] - \sum_{k=1}^{n} E \left[ \eta_{k,1}^{m_1}(\beta_0) + \eta_{k,2}^{m_1}(\theta_0) | G_{k-1}^{n} \right] E \left[ \eta_{k,1}^{m_2}(\beta_0) + \eta_{k,2}^{m_2}(\theta_0) | G_{k-1}^{n} \right] \xrightarrow{P} 4 I_{m_1,m_2}^{1,1}(\beta_0),
\]
\[
\sum_{k=1}^{n} E \left[ \xi_{k,1}^{l_1}(\theta_0) (\eta_{k,1}^{m_1}(\beta_0) + \eta_{k,2}^{m_1}(\theta_0)) | G_{k-1}^{n} \right] - \sum_{k=1}^{n} E \left[ \xi_{k,1}^{l_1}(\theta_0) | G_{k-1}^{n} \right] E \left[ \eta_{k,1}^{m_1}(\beta_0) + \eta_{k,2}^{m_1}(\theta_0) | G_{k-1}^{n} \right] \xrightarrow{P} 0,
\]
\[
\sum_{k=1}^{n} E \left[ \xi_{k,1}^{l_1}(\theta_0) | G_{k-1}^{n} \right] E \left[ \eta_{k,1}^{m_1}(\beta_0) + \eta_{k,2}^{m_1}(\theta_0) | G_{k-1}^{n} \right] \xrightarrow{P} 0,
\]
\[
\sum_{k=1}^{n} E \left[ (\eta_{k,1}^{m_1}(\beta_0) + \eta_{k,2}^{m_1}(\theta_0))^4 | G_{k-1}^{n} \right] \xrightarrow{P} 0.
\]

The above properties are obtained by Lemmas 1 and 4 in Gloter and Sørensen \[5\]. We omit the detailed proof. Therefore, we have \(A_{k,n} \xrightarrow{d} N_4(0, I(\theta_0)^{-1})\), which completes the proof. \(\square\)

**Proof of Lemma \[2\]** First, we prove that \(\delta_{\epsilon,n}^{(l)} \xrightarrow{P} \alpha_0\) for \(l = 1, \ldots, v\). When \(l = 1\), a simple computation shows that
\[
\varepsilon^2 \left( V_{\varepsilon,n,v}^{(1)}(\alpha) - V_{\varepsilon,n,v}^{(1)}(\alpha_0) \right) = \frac{1}{n} \sum_{k=1}^{n} (b(X_{l_{k-1}}^{(2)}, \alpha_0) - b(X_{l_{k-1}}^{(2)}, \alpha))^T b(X_{l_{k-1}}^{(2)}, \alpha_0) - b(X_{l_{k-1}}^{(2)}, \alpha))
\]
\[+ 2 \sum_{k=1}^{n} b(X_{l_{k-1}}^{(2)}, \alpha_0) - b(X_{l_{k-1}}^{(2)}, \alpha)^T P_{l,k}(\alpha_0). \]

By Lemma 2 in Sørensen and Uchida \[18\], the first term of the right hand side converges in probability to \(U_1(\alpha, \alpha_0)\) defined by \(46\) and the second term converges to 0 in probability. Moreover, these convergences
hold uniformly in \( \alpha \), and we have
\[
\varepsilon^2 \left( V^{(1)}_{\varepsilon,n,v}(\alpha) - V^{(1)}_{\varepsilon,n,v}(\alpha_0) \right) \overset{P}{\rightarrow} U_1(\alpha, \alpha_0) \quad \text{uniformly in } \alpha.
\] (52)

Therefore, in the same manner as the proof of the consistency of \( \hat{\alpha}^{(1)}_{\varepsilon,n} \) in Lemma 1, we can show \( \hat{\alpha}^{(1)}_{\varepsilon,n} \overset{P}{\rightarrow} \alpha_0 \).

When \( l \geq 2 \), assume that \( \hat{\alpha}^{(l-1)}_{\varepsilon,n} \) is consistent. Noting that \( Q_{l,k}(\alpha) = R_d(n^{-2}, X_{l_k-1}) \), we calculate that
\[
\varepsilon^2 \left( V^{(l)}_{\varepsilon,n,v}(\alpha|\hat{\alpha}^{(l-1)}_{\varepsilon,n}) - V^{(l)}_{\varepsilon,n,v}(\alpha_0|\hat{\alpha}^{(l-1)}_{\varepsilon,n}) \right) = n \sum_{k=1}^{n} \left( P_{1,k}(\alpha) - P_{1,k}(\alpha_0) \right)^\top \left( P_{1,k}(\alpha) - P_{1,k}(\alpha_0) \right)
+ 2n \sum_{k=1}^{n} \left( P_{1,k}(\alpha) - P_{1,k}(\alpha_0) \right)^\top \left( P_{1,k}(\alpha) - Q_{l,k}(\hat{\alpha}^{(l-1)}_{\varepsilon,n}) \right)
= \frac{1}{n} \sum_{k=1}^{n} \left( b(X_{l_k-1}, \alpha_0) - b(X_{l_k-1}, \alpha) \right)^\top \left( b(X_{l_k-1}, \alpha_0) - b(X_{l_k-1}, \alpha) \right)
+ 2 \sum_{k=1}^{n} \left( b(X_{l_k-1}, \alpha_0) - b(X_{l_k-1}, \alpha) \right)^\top \left( P_{1,k}(\alpha) - \hat{\alpha}^{(l)}_{\varepsilon,n} \right)
+ \frac{1}{n} \sum_{k=1}^{n} R(n^{-1}, X_{l_k-1}).
\]

Similarly, we have
\[
\varepsilon^2 \left( V^{(l)}_{\varepsilon,n,v}(\alpha|\hat{\alpha}^{(l-1)}_{\varepsilon,n}) - V^{(l)}_{\varepsilon,n,v}(\alpha_0|\hat{\alpha}^{(l-1)}_{\varepsilon,n}) \right) \overset{P}{\rightarrow} U_1(\alpha, \alpha_0) \quad \text{uniformly in } \alpha.
\] (53)

Hence, it follows from [A3] and the consistency of \( \hat{\alpha}^{(l-1)}_{\varepsilon,n} \) that \( \hat{\alpha}^{(l)}_{\varepsilon,n} \overset{P}{\rightarrow} \alpha_0 \).

Second, let us show \( \varepsilon^{-1} (\hat{\alpha}^{(1)}_{\varepsilon,n} - \alpha_0) \overset{P}{\rightarrow} 0 \). In the case of \( v = 1 \), this statement is the same as that of Lemma 1. Therefore, we assume \( v \geq 2 \). It follows from Taylor’s theorem that
\[
-\varepsilon^{2-\frac{1}{v}} \partial_{\alpha} V^{(1)}_{\varepsilon,n,v}(\alpha_0) = \left( \varepsilon^{2} \int_{0}^{1} \partial_{\alpha}^2 V^{(1)}_{\varepsilon,n,v}(\alpha_0 + \varepsilon \hat{\alpha}^{(1)}_{\varepsilon,n} - \alpha_0) d\varepsilon \right) \varepsilon^{-\frac{1}{v}} (\hat{\alpha}^{(1)}_{\varepsilon,n} - \alpha_0).
\]

In order to prove \( \varepsilon^{-\frac{1}{v}} (\hat{\alpha}^{(1)}_{\varepsilon,n} - \alpha_0) \overset{P}{\rightarrow} 0 \), by [A4]-i), it is sufficient to show the following properties:
\[
-\varepsilon^{2-\frac{1}{v}} \partial_{\alpha} V^{(1)}_{\varepsilon,n,v}(\alpha_0) \overset{P}{\rightarrow} 0,
\sup_{u \in [0,1]} \left| \varepsilon^{2} \partial_{\alpha}^2 V^{(1)}_{\varepsilon,n,v}(\alpha_0 + \varepsilon \hat{\alpha}^{(1)}_{\varepsilon,n} - \alpha_0) - 2J_b(\alpha_0) \right| \overset{P}{\rightarrow} 0.
\] (54) (55)

Proof of (54). By the definition of the contrast function \( V^{(1)}_{\varepsilon,n,v}(\alpha) \), we have
\[
-\varepsilon^{2-\frac{1}{v}} \partial_{\alpha} V^{(1)}_{\varepsilon,n,v}(\alpha_0) = \sum_{k=1}^{n} \psi_{k}^{(1)}(\alpha_0),
\]
where
\[
\psi_{k}^{(1)}(\alpha_0) = 2\varepsilon^{-\frac{1}{v}} \partial_\alpha b(X_{l_k-1}, \alpha_0)^\top P_{1,k}(\alpha_0).
\]
From Lemma 9 in Genon-Catalot and Jacod [4], it is sufficient to show the following two convergences.
\[
\sum_{k=1}^{n} \mathbb{E} \left[ \psi_{k}^{(1)}(\alpha_0) | G^n_{k-1} \right] \overset{P}{\rightarrow} 0,
\sum_{k=1}^{n} \mathbb{E} \left[ (\psi_{k}^{(1)}(\alpha_0))^2 | G^n_{k-1} \right] \overset{P}{\rightarrow} 0.
\] (56) (57)
By using Lemmas 1 and 4 in Gloter and Sørensen [5], the two terms on the left hand side in (59) and (57) are evaluated from above by \( O_P(\varepsilon^{2-\frac{1}{2}}) \) and \( O_P(\varepsilon^{2(1-\frac{1}{2})}) \), respectively. Therefore, it follows from \( v \geq 2 \) that the two properties hold.

Proof of (55). By Lemma 2 in Sørensen and Uchida [13], one deduces that

\[
\varepsilon^2 \partial^2_{\alpha} V^{(1)}_{\varepsilon,n,v}(\alpha) = -\frac{2}{n} \sum_{k=1}^{n} \left( \partial^2_{\alpha} b(X_{i_{k-1}}^n, \alpha) \right)^{\top} \left( b(X_{i_{k-1}}^n, \alpha) - b(X_{i_{k-1}}^n, \alpha) \right) + \frac{2}{n} \sum_{k=1}^{n} \left( \partial_{\alpha} b(X_{i_{k-1}}^n, \alpha) \right)^{\top} \left( \partial_{\alpha} b(X_{i_{k-1}}^n, \alpha) \right) - 2 \sum_{k=1}^{n} \left( \partial^2_{\alpha} b(X_{i_{k-1}}^n, \alpha) \right)^{\top} P_{1,k}(\alpha) \]

\[
P \rightarrow 2B_1(\alpha, \alpha_0) \quad \text{uniformly in } \alpha.
\]

By noting that \( B_1(\alpha_0, \alpha_0) = J_6(\alpha_0) \), it follows from the consistency of \( \hat{\alpha}^{(1)}_{\varepsilon,n} \) and the uniform continuity of \( B_1 \) that for all \( \delta > 0 \),

\[
P \left( \sup_{u \in [0,1]} \left| \varepsilon^2 \partial^2_{\alpha} V^{(1)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}^{(1)}_{\varepsilon,n} - \alpha_0)) - 2J_6(\alpha_0) \right| > \delta \right) \leq P \left( \sup_{\alpha \in \Theta_n} \left| \varepsilon^2 \partial^2_{\alpha} V^{(1)}_{\varepsilon,n,v}(\alpha) - 2B_1(\alpha, \alpha_0) \right| > \frac{\delta}{2} \right) + P \left( \sup_{u \in [0,1]} \left| 2B_1(\alpha_0 + u(\hat{\alpha}^{(1)}_{\varepsilon,n} - \alpha_0), \alpha_0) - 2J_6(\alpha_0) \right| > \frac{\delta}{2} \right) \rightarrow 0 \quad (\varepsilon \rightarrow 0, n \rightarrow \infty).
\]

This implies (55).

Third, we assume \( \varepsilon^{-\frac{l-1}{2}}(\hat{\alpha}^{(l-1)}_{\varepsilon,n} - \alpha_0) \rightarrow 0 \) and prove \( \varepsilon^{-\frac{l}{2}}(\hat{\alpha}^{(l)}_{\varepsilon,n} - \alpha_0) \rightarrow 0 \) for \( 2 \leq l \leq v - 1 \). From Taylor’s theorem, we deduce that

\[
-\varepsilon^{-\frac{l}{2}} \partial_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0)(\hat{\alpha}^{(l-1)}_{\varepsilon,n}) = \left( \varepsilon^{2} \int_{0}^{1} \partial^2_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}^{(l)}_{\varepsilon,n} - \alpha_0)) \hat{\alpha}^{(l-1)}_{\varepsilon,n} du \right) \varepsilon^{-\frac{l}{2}}(\hat{\alpha}^{(l)}_{\varepsilon,n} - \alpha_0),
\]

\[
\varepsilon^{-\frac{l}{2}} \partial_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0)(\hat{\alpha}^{(l-1)}_{\varepsilon,n}) = \varepsilon^{-\frac{l}{2}} \partial_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0)(\alpha_0) + \left( \varepsilon^{-\frac{l}{2}} \int_{0}^{1} \partial^2_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}^{(l-1)}_{\varepsilon,n} - \alpha_0)) du \right) \varepsilon^{-\frac{l}{2}}(\hat{\alpha}^{(l)}_{\varepsilon,n} - \alpha_0).
\]

Hence, one deduces that

\[
-\varepsilon^{-\frac{l}{2}} \partial_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0)(\alpha_0) - \left( \varepsilon^{-\frac{l}{2}} \int_{0}^{1} \partial^2_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}^{(l-1)}_{\varepsilon,n} - \alpha_0)) du \right) \varepsilon^{-\frac{l}{2}}(\hat{\alpha}^{(l)}_{\varepsilon,n} - \alpha_0) = \left( \varepsilon^{2} \int_{0}^{1} \partial^2_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}^{(l)}_{\varepsilon,n} - \alpha_0)) du \right) \varepsilon^{-\frac{l}{2}}(\hat{\alpha}^{(l)}_{\varepsilon,n} - \alpha_0).
\]  

(58)

In an analogous manner to the proof of \( \varepsilon^{-\frac{l}{2}}(\hat{\alpha}^{(l)}_{\varepsilon,n} - \alpha_0) \rightarrow 0 \), it is sufficient to show the following properties:

\[
-\varepsilon^{-\frac{l}{2}} \partial_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0)(\alpha_0) \rightarrow 0,
\]

\[
\sup_{u \in [0,1]} \left| \varepsilon^2 \partial^2_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}^{(l)}_{\varepsilon,n} - \alpha_0)) \hat{\alpha}^{(l-1)}_{\varepsilon,n} - 2J_6(\alpha_0) \right| \rightarrow 0,
\]

\[
\sup_{u \in [0,1]} \left| \varepsilon^{-\frac{l}{2}} \partial^2_{\alpha} V^{(l)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}^{(l-1)}_{\varepsilon,n} - \alpha_0)) \right| \rightarrow 0.
\]  

(59) (60) (61)
Proof of (59). By the definition of the contrast function $V_{\varepsilon,n,v}^{(l)}(\alpha|\hat{\alpha})$, we have

$$-\varepsilon^{2-\frac{l}{2}} \partial_{\alpha} V_{\varepsilon,n,v}^{(l)}(\alpha|\alpha_0) = \sum_{k=1}^{n} \psi_k^{(l)}(\alpha_0) + \varepsilon^{2-\frac{l}{2}} \frac{1}{n} \sum_{k=1}^{n} R_d(n^{-1}, X_{t_{k-1}^n}),$$  \hspace{1cm} (62)

where

$$\psi_k^{(l)}(\alpha_0) = 2\varepsilon^{-\frac{l}{2}} \partial_{\alpha} b(X_{t_{k-1}^n}, \alpha_0) \top P_1(\alpha_0).$$

Since the second term of the right hand side of (62) converges to 0 in probability, it is sufficient to show the following two properties:

1. $$\sum_{k=1}^{n} \mathbb{E} \left[ \psi_k^{(l)}(\alpha_0) | G_{k-1}^{n} \right] \xrightarrow{P} 0,$$  \hspace{1cm} (63)
2. $$\sum_{k=1}^{n} \mathbb{E} \left[ \psi_k^{(l)}(\alpha_0)^2 | G_{k-1}^{n} \right] \xrightarrow{P} 0.$$  \hspace{1cm} (64)

In an analogous manner to the case of $\psi_k^{(1)}$, the two terms on the left hand side in (63) and (64) are evaluated from above by $O_P(\varepsilon^{2-\frac{l}{2}})$ and $O_P(\varepsilon^{2(1-\frac{l}{2})})$, respectively. Therefore, it follows from $l < v$ that the two properties hold.

Proof of (60). By Lemma 2 in Sorensen and Uchida [18], one deduces that

$$\varepsilon^2 \partial_{\alpha}^2 V_{\varepsilon,n,v}^{(l)}(\alpha|\hat{\alpha}_{\varepsilon,n}^{(l-1)}) = -\frac{2}{n} \sum_{k=1}^{n} \left( \partial_{\alpha}^2 b(X_{t_{k-1}^n}, \alpha) \right) \top \left( b(X_{t_{k-1}^n}, \alpha_0) - b(X_{t_{k-1}^n}, \alpha) \right)$$

$$+ \frac{2}{n} \sum_{k=1}^{n} \left( \partial_{\alpha} b(X_{t_{k-1}^n}, \alpha) \right) \top \left( \partial_{\alpha} b(X_{t_{k-1}^n}, \alpha) \right)$$

$$- \frac{2}{n} \sum_{k=1}^{n} \left( \partial_{\alpha}^2 b(X_{t_{k-1}^n}, \alpha) \right) \top P_4(\alpha_0)$$

$$- \frac{2}{n} \sum_{k=1}^{n} \left( \partial_{\alpha} b(X_{t_{k-1}^n}, \alpha) \right) \top Q_{l,k}(\hat{\alpha}_{\varepsilon,n}^{(l-1)}) \xrightarrow{P} 2 B_1(\alpha, \alpha_0) \text{ uniformly in } \alpha.$$

Hence, it follows from the consistency of $\hat{\alpha}_{\varepsilon,n}^{(l)}$ and the uniform continuity of $B_1$ that for all $\delta > 0$,

$$P \left( \sup_{u \in [0,1]} \left| \varepsilon^2 \partial_{\alpha}^2 V_{\varepsilon,n,v}^{(l)}(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n}^{(l)} - \alpha_0)|\hat{\alpha}_{\varepsilon,n}^{(l-1)}) - 2 J_b(\alpha_0) \right| > \delta \right)$$

$$\leq P \left( \sup_{\alpha \in \Theta_n} \left| \varepsilon^2 \partial_{\alpha}^2 V_{\varepsilon,n,v}^{(l)}(\alpha|\hat{\alpha}_{\varepsilon,n}^{(l-1)}) - 2 B_1(\alpha, \alpha_0) \right| > \frac{\delta}{2} \right)$$

$$+ P \left( \sup_{u \in [0,1]} \left| 2 B_1(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n}^{(l)} - \alpha_0), \alpha_0) - 2 J_b(\alpha_0) \right| > \frac{\delta}{2} \right)$$

$$\to 0 \text{ (} \varepsilon \to 0, n \to \infty \text{)}.$$

This implies (60).
Lemma 4-(2) in Gloter and Sørensen [5] that

\[ \varepsilon^{2-\frac{1}{2}} \partial_{\alpha_1,\alpha_2}^2 V^{(l)}_{\varepsilon,n,v}(\alpha|\bar{\alpha}) = 2 \varepsilon^{-\frac{1}{2}} \sum_{k=1}^{n} \partial_{\alpha_1} b(X_{\varepsilon,n}^\top, \alpha)^\top \partial_{\alpha_2} Q_{l,k}(\bar{\alpha}) \]

\[ = 2n^{-\left(1-\frac{1}{p}\right)} \cdot (\varepsilon n^p)^{-\frac{1}{n}} \sum_{k=1}^{n} R(1, X_{\varepsilon,n}^\top) \]

\[ \xrightarrow{P} 0 \quad \text{uniformly in } (\alpha, \bar{\alpha}). \]

Therefore, we have (61).

Next, we prove \( \varepsilon^{-1}(\hat{\alpha}_{\varepsilon,n} - \alpha_0) = O_P(1) \). In the same manner as the derivation of (58), it holds from Taylor’s theorem that

\[ -\varepsilon \partial_v V^{(v)}_{\varepsilon,n,v}(\alpha_0|\alpha_0) - \left( \varepsilon^{2-\frac{1}{2}} \int_0^1 \partial_{\alpha}^2 V^{(v)}_{\varepsilon,n,v}(\alpha_0|\alpha_0 + u(\hat{\alpha}_{\varepsilon,n} - \alpha_0)) du \right) \varepsilon^{-\frac{1}{2}} (\hat{\alpha}_{\varepsilon,n} - \alpha_0) \]

\[ = \left( \varepsilon^2 \int_0^1 \partial_{\alpha}^2 V^{(v)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n} - \alpha_0)|\hat{\alpha}_{\varepsilon,n} - \alpha_0)) du \right) \varepsilon^{-1}(\hat{\alpha}_{\varepsilon,n} - \alpha_0). \]

In order to prove \( \varepsilon^{-1}(\hat{\alpha}_{\varepsilon,n} - \alpha_0) = O_P(1) \), it is sufficient to show the following properties:

\[ -\varepsilon \partial_v V^{(v)}_{\varepsilon,n,v}(\alpha_0|\alpha_0) = O_P(1), \]

\[ \sup_{u \in [0,1]} \left| \varepsilon^{2-\frac{1}{2}} \partial_{\alpha}^2 V^{(v)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n} - \alpha_0)|\hat{\alpha}_{\varepsilon,n} - \alpha_0)) - 2J_b(\alpha_0) \right| \xrightarrow{P} 0, \]

\[ \sup_{u \in [0,1]} \left| \varepsilon^{2-\frac{1}{2}} \partial_{\alpha}^2 V^{(v)}_{\varepsilon,n,v}(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n} - \alpha_0)) \right| \xrightarrow{P} 0. \]

In analogous manners to the proofs of (60) and (61), we can show (66) and (67). For (65), it follows from Lemma 4-(2) in Gloter and Sørensen [5] that

\[ -\varepsilon \partial_v V^{(v)}_{\varepsilon,n,v}(\alpha_0|\alpha_0) = 2 \varepsilon^{-1} \sum_{k=1}^{n} \partial_v b(X_{\varepsilon,n}^\top, \alpha)^\top \left( P_{l,k}(\alpha_0) - Q_{v,k}(\alpha_0) \right) \]

\[ = 2 \varepsilon^{-1} \sum_{k=1}^{n} \partial_v b(X_{\varepsilon,n}^\top, \alpha)^\top P_{v,k}(\alpha_0) \]

\[ = O_P(1). \]

Finally, we show the asymptotic normality of \( \hat{\alpha}_{\varepsilon,n} \) in the same manner as the proof of Lemma 1. Noting that

\[ -\varepsilon \partial_1 V^{(v)}_{\varepsilon,n,v}(\alpha_0|\alpha_0) = \sum_{k=1}^{n} \alpha_{k,1}(\alpha_0) \]

for \( 1 \leq l \leq p \), we can show \( -\varepsilon \partial_1 V^{(v)}_{\varepsilon,n,v}(\alpha_0|\alpha_0) \xrightarrow{P} \mathcal{N}_p(0, 4J_b(\theta_0)). \) From (66) and this convergence, we complete the proof. \( \square \)

**Proof of Theorem 2** We prove this theorem in the same order as the proof of Theorem 1.

**1st step.** Let us show \( \hat{\beta}_{\varepsilon,n} \xrightarrow{P} \beta_0 \). Noting that \( V^{(v+1)}_{\varepsilon,n,v}(\beta|\tilde{\alpha}) = U^{(2)}_{\varepsilon,n,v}(\beta|\tilde{\alpha}) \), we can utilize the proof of Theorem 1. In particular, by replacing \( U^{(2)}_{\varepsilon,n,v} \) and \( (\hat{\alpha}_{\varepsilon,n}, \tilde{\beta}_{\varepsilon,n}) \) with \( V^{(v+1)}_{\varepsilon,n,v} \) and \( (\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n}) \) in the 1st step of the proof of Theorem 1 we obtain the result.
2nd step. We prove \( \hat{\alpha}_{\varepsilon,n} \xrightarrow{P} \alpha_0 \). It follows from Lemma 4 in Gloter and Sørensen [18] and the consistency of \( \hat{\beta}_{\varepsilon,n} \) that
\[
\varepsilon^2 \left( V_{\varepsilon,n,v}^{(v+2)}(\alpha_0(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n}) - V_{\varepsilon,n,v}^{(v+2)}(\alpha_0(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n})) \right)
= 2 \sum_{k=1}^{n} \begin{pmatrix} b(X_{t_{k-1}}^\varepsilon, \alpha_0) - b(X_{t_{k-1}}^\varepsilon, \alpha) \end{pmatrix}^T \begin{pmatrix} \sigma \sigma^T \end{pmatrix}^{-1} (X_{t_{k-1}}^\varepsilon, \hat{\beta}_{\varepsilon,n}) P_{k,1}(\alpha_0)
- 2 \sum_{k=1}^{n} \begin{pmatrix} b(X_{t_{k-1}}^\varepsilon, \alpha) - b(X_{t_{k-1}}^\varepsilon, \alpha_0) \end{pmatrix}^T \begin{pmatrix} \sigma \sigma^T \end{pmatrix}^{-1} (X_{t_{k-1}}^\varepsilon, \hat{\beta}_{\varepsilon,n}) Q_{v,1}(\alpha_0)
+ \frac{1}{n} \sum_{k=1}^{n} \left( b(X_{t_{k-1}}^\varepsilon, \alpha) - b(X_{t_{k-1}}^\varepsilon, \alpha_0) \right) \begin{pmatrix} \sigma \sigma^T \end{pmatrix}^{-1} (X_{t_{k-1}}^\varepsilon, \hat{\beta}_{\varepsilon,n}) \left( b(X_{t_{k-1}}^\varepsilon, \alpha) - b(X_{t_{k-1}}^\varepsilon, \alpha_0) \right)
+ \frac{1}{n} \sum_{k=1}^{n} \left( b(X_{t_{k-1}}^\varepsilon, \alpha) - b(X_{t_{k-1}}^\varepsilon, \alpha_0) \right) \begin{pmatrix} \sigma \sigma^T \end{pmatrix}^{-1} (X_{t_{k-1}}^\varepsilon, \hat{\beta}_{\varepsilon,n}) \left( b(X_{t_{k-1}}^\varepsilon, \alpha) - b(X_{t_{k-1}}^\varepsilon, \alpha_0) \right)
\]
\( \xrightarrow{P} U_3(\alpha, \theta_0) \) uniformly in \( \alpha \).

Therefore, in an analogous manner to the proof of the consistency for \( \hat{\alpha}_{\varepsilon,n} \), we have \( \hat{\alpha}_{\varepsilon,n} \xrightarrow{P} \alpha_0 \).

3rd step. We prove the asymptotic normality for \( \hat{\beta}_{\varepsilon,n} \). Using Taylor’s theorem, we have the following expansions:
\[
- \varepsilon \partial_\alpha V_{\varepsilon,n,v}^{(v+2)}(\alpha_0(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n}) = \left( \varepsilon^2 \int_0^1 \partial_\alpha^2 V_{\varepsilon,n,v}^{(v+2)}(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n} - \alpha_0))(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n})du \right) \varepsilon^{-1}(\hat{\alpha}_{\varepsilon,n} - \alpha_0),
\]
\[
- \frac{1}{\sqrt{n}} \partial_\beta V_{\varepsilon,n,v}^{(v+1)}(\hat{\beta}_{\varepsilon,n} = \left( \frac{1}{n} \int_0^1 \partial_\beta^2 V_{\varepsilon,n,v}^{(v+1)}(\beta_0 + u(\hat{\beta}_{\varepsilon,n} - \beta_0))(\hat{\alpha}_{\varepsilon,n})du \right) \sqrt{n}(\hat{\beta}_{\varepsilon,n} - \beta_0),
\]
\[
\varepsilon \partial_\alpha V_{\varepsilon,n,v}^{(v+2)}(\alpha_0(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n}) = \left( \frac{\varepsilon}{\sqrt{n}} \int_0^1 \partial_{\alpha \beta}^2 V_{\varepsilon,n,v}^{(v+2)}(\alpha_0(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n} + u(\hat{\beta}_{\varepsilon,n} - \beta_0))du \right) \sqrt{n}(\hat{\beta}_{\varepsilon,n} - \beta_0).
\]

Using these expressions, we calculate that
\[
\Gamma_{\varepsilon,n}^2 = C_{\varepsilon,n}^2 \Lambda_{\varepsilon,n}^2,
\]
where
\[
\Gamma_{\varepsilon,n}^2 := \begin{pmatrix} -\varepsilon \partial_\alpha V_{\varepsilon,n,v}^{(v+2)}(\alpha_0(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n}) & \frac{1}{\sqrt{n}} \partial_\beta V_{\varepsilon,n,v}^{(v+1)}(\hat{\beta}_{\varepsilon,n} & \frac{\varepsilon}{\sqrt{n}} \partial_{\alpha \beta}^2 V_{\varepsilon,n,v}^{(v+2)}(\alpha_0(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n} + u(\hat{\beta}_{\varepsilon,n} - \beta_0))du \end{pmatrix},
\]
\[
\Lambda_{\varepsilon,n} := \begin{pmatrix} \varepsilon^{-1}(\hat{\alpha}_{\varepsilon,n} - \alpha_0) \end{pmatrix},
\]
\[
C_{\varepsilon,n} := \begin{pmatrix} \varepsilon^2 \int_0^1 \partial_\alpha^2 V_{\varepsilon,n,v}^{(v+2)}(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n} - \alpha_0))(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n})du \end{pmatrix} \begin{pmatrix} \frac{1}{n} \int_0^1 \partial_\beta^2 V_{\varepsilon,n,v}^{(v+1)}(\beta_0 + u(\hat{\beta}_{\varepsilon,n} - \beta_0))(\hat{\alpha}_{\varepsilon,n})du \end{pmatrix}.
\]

In an analogous manner to the proof of Theorem 1 in Sørensen and Uchida [18], it is sufficient to show the following convergences:
\[
\sup_{u \in [0,1]} \left| \varepsilon^2 \partial_\alpha^2 V_{\varepsilon,n,v}^{(v+2)}(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n} - \alpha_0))(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n}) - 2I_6(\theta_0) \right| \xrightarrow{P} 0,
\]
\[
\sup_{u \in [0,1]} \left| \frac{1}{n} \partial_\beta^2 V_{\varepsilon,n,v}^{(v+1)}(\beta_0 + u(\hat{\beta}_{\varepsilon,n} - \beta_0))(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n}) - 2I_4(\theta_0) \right| \xrightarrow{P} 0,
\]
\[
\sup_{u \in [0,1]} \left| \frac{\varepsilon}{\sqrt{n}} \partial_{\alpha \beta}^2 V_{\varepsilon,n,v}^{(v+2)}(\alpha_0(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n} + u(\hat{\beta}_{\varepsilon,n} - \beta_0))du \right| \xrightarrow{P} 0,
\]
\[
- \varepsilon \partial_\alpha V_{\varepsilon,n,v}^{(v+2)}(\alpha_0(\hat{\alpha}_{\varepsilon,n}, \hat{\beta}_{\varepsilon,n}) \xrightarrow{d} N_p(0, 4I_6(\theta_0)),
\]
\[
- \frac{1}{\sqrt{n}} \partial_\beta V_{\varepsilon,n,v}^{(v+1)}(\beta_0(\hat{\alpha}_{\varepsilon,n}) \xrightarrow{d} N_q(0, 4I_4(\theta_0)).
\]
By the definitions of $U_{n,v}^{(2)}$ and $V_{n,v}^{(v+1)}$, we have already shown (69) and (72) in the proof of Theorem 1.

Proof of (68). It follow from Lemma 2 in Sørensen and Uchida [18], Lemma 4 in Gloter and Sørensen [5], and the consistency of $\hat{\beta}_{\varepsilon,n}$ that

$$\varepsilon^2 \partial_{\alpha}^2 V_{\varepsilon,n,v}^{(v+2)}(\alpha|\hat{\alpha}_{\varepsilon,n}^{(v+1)}, \hat{\beta}_{\varepsilon,n}^{(v+1)})$$

$$= -2 \sum_{k=1}^{n} \left( \partial_{\alpha}^2 b(X_{t_{k-1}^n}, \alpha) \right) \top \left[ \sigma \sigma^\top \right]^{-1}(X_{t_{k-1}^n}, \hat{\beta}_{\varepsilon,n}) P_{1,k}(\alpha_0)$$

$$+ 2 \sum_{k=1}^{n} \left( \partial_{\alpha}^2 b(X_{t_{k-1}^n}, \alpha) \right) \top \left[ \sigma \sigma^\top \right]^{-1}(X_{t_{k-1}^n}, \hat{\beta}_{\varepsilon,n}) Q_{v,k}(\hat{\alpha}_{\varepsilon,n}^{(v)})$$

$$+ 2 \sum_{k=1}^{n} \left( \partial_{\alpha}^2 b(X_{t_{k-1}^n}, \alpha) \right) \top \left[ \sigma \sigma^\top \right]^{-1}(X_{t_{k-1}^n}, \beta_0) \left( b(X_{t_{k-1}^n}, \alpha) - b(X_{t_{k-1}^n}, \alpha_0) \right)$$

$$+ 2 \sum_{k=1}^{n} \left( \partial_{\alpha}^2 b(X_{t_{k-1}^n}, \alpha) \right) \top \left[ \sigma \sigma^\top \right]^{-1}(X_{t_{k-1}^n}, \beta_0) \left( \partial_{\alpha} b(X_{t_{k-1}^n}, \alpha) \right)$$

$$+ 2 \sum_{k=1}^{n} \left( \partial_{\alpha}^2 b(X_{t_{k-1}^n}, \alpha) \right) \top \left[ \sigma \sigma^\top \right]^{-1}(X_{t_{k-1}^n}, \beta_0) \left( \partial_{\alpha} b(X_{t_{k-1}^n}, \alpha) \right)$$

$$\overset{P}{\to} 2B_2(\alpha, \theta_0) \quad \text{uniformly in } \alpha.$$ 

In an analogous manner to the proof of (32), it follows from the consistency for $\hat{\alpha}_{\varepsilon,n}$ and the uniform continuity of $B_2$ that for all $\delta > 0$,

$$P \left( \sup_{\alpha \in [0,1]} \left| \varepsilon^2 \partial_{\alpha}^2 V_{\varepsilon,n,v}^{(v+2)}(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n} - \alpha_0)|\hat{\alpha}_{\varepsilon,n}^{(v)}, \hat{\beta}_{\varepsilon,n}^{(v)}) - 2I_0(\theta_0) \right| > \delta \right)$$

$$\leq P \left( \sup_{\alpha \in [0,1]} \left| \varepsilon^2 \partial_{\alpha}^2 V_{\varepsilon,n,v}^{(v+2)}(\alpha|\hat{\alpha}_{\varepsilon,n}^{(v)}, \hat{\beta}_{\varepsilon,n}^{(v)}) - 2B_2(\alpha, \theta_0) \right| > \frac{\delta}{2} \right)$$

$$+ P \left( \sup_{\alpha \in [0,1]} \left| 2B_2(\alpha_0 + u(\hat{\alpha}_{\varepsilon,n} - \alpha_0), \theta_0) - 2I_0(\theta_0) \right| > \frac{\delta}{2} \right)$$

$$\to 0 \quad (\varepsilon \to 0, n \to \infty).$$

This implies (68).

Proof of (70). By using the Lipschitz continuity of $n^2 Q_{v,k}(\alpha)$, Lemma 2 in this paper and Lemma 4-(2) in Gloter and Sørensen [5], it holds that

$$\frac{\varepsilon}{\sqrt{n}} \partial_{\alpha, \beta} V_{\varepsilon,n,v}^{(v+2)}(\alpha_0|\hat{\alpha}_{\varepsilon,n}^{(v)}, \beta) = -2(\varepsilon \sqrt{n})^{-1} \sum_{k=1}^{n} \left( \partial_{\alpha} b(X_{t_{k-1}^n}, \alpha) \right) \top \left( \partial_{\beta} [\sigma \sigma^\top]^{-1}(X_{t_{k-1}^n}, \beta) \right) P_{v,k}(\alpha_0)$$

$$+ 2(\varepsilon \sqrt{n})^{-1} \sum_{k=1}^{n} \left( \partial_{\alpha} b(X_{t_{k-1}^n}, \alpha) \right) \top \left( \partial_{\beta} [\sigma \sigma^\top]^{-1}(X_{t_{k-1}^n}, \beta) \right) \left( Q_{v,k}(\hat{\alpha}_{\varepsilon,n}^{(v)}) - Q_{v,k}(\alpha_0) \right)$$

$$\overset{P}{\to} 0 \quad \text{uniformly in } \beta.$$ 

Hence, we have (70).
Proof of (71). For 1 ≤ l ≤ p, it follows from the Lipschitz continuity of n²Q_{v,k}(\alpha), Lemma 2 in Gloter and Sørensen [5] that

\[ -\varepsilon \partial_{\alpha_l} V_{v,n,v}^{(v+2)}(\alpha_0|\hat{\alpha}_{v,n}^{(v)}, \beta_0) = 2\varepsilon^{-1} \sum_{k=1}^{n} \left( \partial_{\alpha_l} b(X_{t_k}^{n,v}, \alpha_0) \right)^\top [\sigma\sigma^\top]^{-1}(X_{t_k}^{n,v}, \beta_0)P_{v,k}(\alpha_0) \]

\[ - 2\varepsilon^{-1} \sum_{k=1}^{n} \left( \partial_{\alpha_l} b(X_{t_k}^{n,v}, \alpha_0) \right)^\top [\sigma\sigma^\top]^{-1}(X_{t_k}^{n,v}, \beta_0) \left( Q_{v,k}(\hat{\alpha}_{v,n}^{(v)}) - Q_{v,k}(\alpha_0) \right) \]

\[ = \sum_{k=1}^{n} \xi_{k,l}(\theta_0) + o_P(1). \]

Since the main term of \(-\varepsilon \partial_{\alpha_l} V_{v,n,v}^{(v+2)}(\alpha_0|\hat{\alpha}_{v,n}^{(v)}, \beta_0)\) is the same as that of \(-\varepsilon \partial_{\alpha_l} U_{v,n,v}^{(3)}(\alpha_0|\beta_0)\), we utilize the results in the proof of Theorem 1 and complete the proof. □

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