Monte Carlo studies of the spontaneous rotational symmetry breaking in dimensionally reduced super Yang-Mills models

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1 Introduction

2 Monte Carlo Simulations
   - Phase Quenched Model
   - Complex Action Problem
   - The Factorization Method
   - Simulations

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1 Introduction

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IIB Matrix Model: Overview

- A non-perturbative definition of string theory in the large $N$ limit
- A theory with only one scale, possibility to dynamically choose a unique vacuum
- Dynamical emergence of space–time and matter content
- Dynamical compactification of extra dimensions
- Tackle cosmological questions, like expansion of $3 + 1$ dimensional space–time, resolution of cosmic singularity
The IKKT or IIB Matrix Model

\[ Z = \int dA \, d\Psi \, e^{iS} \]

\[ S = -\frac{1}{4g^2} \text{tr} ([A_\mu, A_\nu][A^\mu, A^\nu]) - \frac{1}{2g^2} \text{tr} (\Psi_\alpha (C\Gamma^\mu)_{\alpha\beta} [A_\mu, \Psi_\beta]) \]

- \[ = S_B \]
- \[ = S_F \]

\( A_\mu (\mu = 0, \ldots, 9), \)
\( \Psi_\alpha (\alpha = 1, \ldots, 16) \) (10D Majorana-Weyl spinor),
\( (A_\mu)_{ij}, (\Psi_\alpha)_{ij}, i,j = 1, \ldots, N \) hermitian matrices.

- manifest SO(9,1) symmetry and SU(N) gauge invariance
- \( \mathcal{N} = 2 \) Supersymmetry
**Relation to String Theory**

- Matrix regularization of IIB string action in the large $N$ limit:
  
  $$S_{\text{Schild}} = - \int d^2 \sigma \sqrt{g} \left( \frac{1}{4} \{X_\mu(\sigma), X_\nu(\sigma)\}^2 + \frac{1}{2} \Psi(\sigma) C \Gamma^\mu \{X_\mu(\sigma), \Psi(\sigma)\} \right)$$

  $$X_\mu(\sigma) \rightarrow (A_\mu)_{ij} \quad \Psi_\alpha(\sigma) \rightarrow (\Psi_\alpha)_{ij}$$

  $$\{\cdot, \cdot\} \rightarrow -i \mathbb{L} [\cdot, \cdot] \quad \int d^2 \sigma \sqrt{g} \rightarrow \text{tr}$$

- non-commutative world sheet
- block structure in matrices $\rightarrow$ second quantized string theory
- reproduce interaction between D-branes at one loop level
- loop equation for Wilson loops $\rightarrow$ light cone IIB string field theory:
  $$w(C) = \text{tr} P \exp \left[ i \int_C k^\mu A_\mu \right] \rightarrow \Psi [k(\cdot)]$$

[Fukuma, Kawai, Kitazawa, Tsuchiya (’97)]
$N = 2$ Supersymmetry

\[
\begin{align*}
\delta^{(1)} A_\mu &= i\epsilon_1 C \Gamma_\mu \Psi \\
\delta^{(1)} \Psi &= \frac{i}{2} \Gamma^{\mu\nu} [A_\mu, A_\nu] \epsilon_1 \\
\delta^{(2)} A_\mu &= 0 \\
\delta^{(2)} \Psi &= \epsilon_2 1
\end{align*}
\]

and bosonic symmetry

\[
\begin{align*}
\delta^{(1)} A_\mu &= c_\mu 1 \\
\delta^{(1)} \Psi &= 0
\end{align*}
\]

Generators: $Q^{(1)}, Q^{(2)}, P_\mu$ resp.

\[
\tilde{Q}^{(1)} = Q^{(1)} + Q^{(2)}, \quad \tilde{Q}^{(1)} = i(Q^{(1)} - Q^{(2)})
\]

\[
[\epsilon_1 C \tilde{Q}^{(i)}, \epsilon_2 C \tilde{Q}^{(j)}] = -2\delta^{ij} \epsilon_1 C \Gamma^\mu \epsilon_2 P_\mu
\]

Identify as $D = 10, N = 2$ SUSY and $P_\mu$ as translations $\Rightarrow$ eigenvalues of $A_\mu, D = 10$ space-time coordinates [Aoki et al hep-th/9802985]
non-commutative space–time [Iso,Kawai 99, Ambjørn,KNA,Bietenholz,Hotta,Nishimura et al 00]
possibility of dynamical compactification of extra dimensions
possibility of built-in mechanism that generates $(3 + 1)$-dim space-time
Emergence of \((3 + 1)\) dimensional spacetime

- Simulations of Lorentzian model: no sign problem, introduce large scale
cutoffs in \(\text{tr}(A_0^2)\) and \(\text{tr}(A_i^2)\) \([\text{Kim},\text{Nishimura},\text{Tsuchiya 1108.1540}]\)
  - dynamical time from \(A^0\) (SUSY crucial)
  - expanding 3+1 universe after a critical time
- classical, expanding solutions at late times \([\text{Kim},\text{Nishimura},\text{Tsuchiya 1110.4803,1208.0711}]\)
- local field theory as fluctuations around classical solns representing
  commutative space-time \([1208.4910]\)
- constructively realize chiral fermions at finite-\(N\) by imposing conditions on
  extra dims \([1305.5547]\)
**Euclidean Model**

\[ A_0 \rightarrow iA_{10} \quad \Gamma^0 \rightarrow -i\Gamma^{10} \]

- **SO(10) rotational symmetry**
- **Finite: quantum effects, despite flat directions**

[Krauth, Nicolai, Staudacher 98, Austing, Wheater 01]

**Gaussian Expansion Method Calculations** [Nishimura, Sugino 02, Kawai et.al. 03,06]

show that:

- \( d = 3 \) configurations have lowest free energy
- extent of the shrunken dimensions \( r \) is independent of \( d \)
- the extent of the large dimensions \( R \) depends on \( d \) so that the 10 dimensional volume is a finite constant and independent of \( d \):
  \[ R^d r^{10-d} = l^{10} \]
- the ratio \( R/r \) remains finite in the large \( N \) limit

**Dynamical compactification by SSB of SO(10) \rightarrow SO(3)**
6 dimensional Euclidean model

- Need a simpler model to study the above results using Monte Carlo simulations
- $D = 4$ studied before, no SSB of SO(4) \cite{AmbjornK,NA,Be,Hot,Nishimura et al 00}
- $D = 6$ the simplest model with SSB of SO(6):

$$Z = \int dA \, d\psi \, d\bar{\psi} \, e^{-S_b - S_f}$$

$$S_b = -\frac{1}{4g^2} \text{tr}[A_\mu, A_\nu]^2 \quad S_f = -\frac{1}{2g^2} \text{tr} \left( \bar{\psi}_\alpha (\Gamma_\mu)_{\alpha \beta} [A_\mu, \psi_\beta] \right)$$

- $A_\mu$ are $N \times N$, hermitian, traceless, vectors w.r.t. SO(6)
- $\psi_\alpha, \bar{\psi}_\alpha$ are $N \times N$, grassmannian entries, Weyl spinors w.r.t. SO(6)

Similar to $D = 10$:
SO(6) rotational symmetry, $N = 2$ SUSY, SU($N$) symmetry
Dynamical Compactification

Order Parameter

$$T_{\mu \nu} = \frac{1}{N} \text{tr} (A_\mu A_\nu)$$

- Eigenvalues of $T_{\mu \nu}$: $\lambda_n, n = 1, \ldots, 6$

  $$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{10}$$

- Extended $d$–dimensions if e.g. in the large $N$ limit

  $$\langle \lambda_1 \rangle = \ldots = \langle \lambda_d \rangle \equiv R^2$$

  Shrunk $(6 - d)$–dimensions if e.g.

  $$\langle \lambda_{d+1} \rangle = \ldots = \langle \lambda_{10} \rangle \equiv r^2$$

- SSB of SO(6) invariance

  $$\text{SO}(6) \rightarrow \text{SO}(d)$$
Gaussian Expansion Method (GEM) - Improved Mean Field Approximation

- a systematic expansion method to study non-perturbative effects
- introduce Gaussian $S_0[M_\mu, A_{\alpha\beta}]$ where $M_\mu, A_{\alpha\beta}$ parameters

$$S = (S + S_0) - S_0 = (S_b + S_f + S_0) - S_0$$

$$S_0[M_\mu, A_{\alpha\beta}] = M_\mu \text{tr}(A^2_\mu) + A_{\alpha\beta} \text{tr}(\bar{\psi}_\alpha \psi_\beta)$$

- expand $\tilde{S} = S_0 + \epsilon S_b + \sqrt{\epsilon} S_f$ w.r.t $\epsilon$
- replace $M \rightarrow (1 - \epsilon)M$, $A \rightarrow (1 - \epsilon)A$
- reorganize series, truncate, set $\epsilon = 1$
- look for “plateaux” in parameter space $(M_\mu, A_{\alpha\beta})$, in practice by solving

$$\frac{\partial F}{\partial M_\mu} = 0 \quad \frac{\partial F}{A_{\alpha\beta}} = 0$$
GEM results

- Parameter space is very large: simplify by considering SO($d$) invariant ansätze, $2 \leq d \leq 5$

\[
\langle \lambda_1 \rangle_{SO(d)} = \ldots = \langle \lambda_d \rangle_{SO(d)} = (R_d)^2
\]

- Compute free energy and observables at solutions in the large $N$ limit
- Compare free energy of ansätze, minimum free energy for the $d = 3$ ansatz, i.e. conclude SO(6) $\rightarrow$ SO(3)
- The extent of the shrunken dimensions $r$ (“compactification scale”) is independent of $d$
- The extent of the large dimensions $R$ depends on $d$ so that the 6 dimensional volume is a finite constant and independent of $d$:

\[
R^d r^{6-d} = \ell^6 \quad r^2 \approx 0.223 \quad \ell^2 \approx 0.627
\]

- The ratio $R_d/r$ is finite

(in units where $g\sqrt{N} = 1$)
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Integrate out fermions first:

\[ Z = \int dA d\bar{\psi} d\psi e^{-S_b - S_f} = \int dA e^{-S_b} Z_f[A] \]

\[ Z_f[A] = \int d\bar{\psi} d\psi e^{-S_f} = \det \mathcal{M} \]

Monte Carlo simulations hard due to the strong complex action problem

\[ \det \mathcal{M} = |\det \mathcal{M}| e^{i\Gamma} \quad \text{is Complex} \]
The Algorithm

Phase Quenched Model: ignore the phase $e^{i\Gamma}$

$$Z_0 = \int dA \, e^{-S_0} \quad S_0 = S_b - \log |\det \mathcal{M}|$$

Simulate using Rational Hybrid Monte Carlo: use rational approximation

$$x^{-1/2} \simeq a_0 + \sum_{k=1}^{Q} \frac{a_k}{x + b_k}$$

- increased accuracy and range of $x$ requires higher $Q$
- coefficients $a_k$ and $b_k$ computed using Remez algorithm

[E. Remez 34, Clark and Kennedy 05 github.com/mikeaclark/AlgRemez]
The Algorithm

Define \( D = \mathcal{M}^\dagger \mathcal{M} \Rightarrow \det D^{1/2} = |\det \mathcal{M}| \), then we can approximate

\[
\det D^{1/2} \approx \int dF dF^* e^{-S_{PF}[F,F^*,A]} (F_\alpha)_{ij} \text{ pseudofermions}
\]

where

\[
S_{PF}[F,F^*,A] = \text{tr} \left\{ a_0 F^\dagger F + \sum_{k=1}^{Q} a_k F^\dagger (D + b_k)^{-1} F \right\}
\]

- spectrum of \( D \) determines \( Q \)
- rescale \( A, F \) to adjust spectrum to desired range
The Algorithm

\[ H = \frac{1}{2} \text{tr}\Pi^2 + \text{tr}\tilde{\Pi}^\dagger\tilde{\Pi} + S_{\text{eff}}[F, F^*, A] \]

where

\[ S_{\text{eff}}[F, F^*, A] = S_0[A] + S_{PF}[F, F^*, A] \]

- \( \Pi_{ij}^\mu = (\Pi^*)_{ji}^\mu, \tilde{\Pi}_{ij}^\alpha \) canonical momenta of \((A_{\mu})_{ij}, (F_{\alpha})_{ij}\)
- \( \int d\tilde{\Pi}d\tilde{\Pi}^*d\Pi dFdF^*dA e^{-H} = \int dFdF^*dA e^{-S_{\text{eff}}} \)
- \( \tau \)-evolution according to eom preserve \( H \):

\[
\begin{align*}
\frac{dA_{\mu}}{d\tau} &= \frac{\partial H}{\partial \Pi_{\mu}} = \Pi_{\mu}^*, \\
\frac{d\tilde{\Pi}_{\alpha}}{d\tau} &= \frac{\partial H}{\partial \tilde{\Pi}_{\beta}} = \tilde{\Pi}_{\beta}^*, \\
\frac{d\Pi_{\mu}}{d\tau} &= -\frac{\partial H}{\partial A_{\mu}} = -\frac{\partial S_{\text{eff}}}{\partial A_{\mu}}, \\
\frac{dF_{\beta}}{d\tau} &= -\frac{\partial H}{\partial F_{\beta}} = -\frac{\partial S_{\text{eff}}}{\partial F_{\beta}},
\end{align*}
\]
The Algorithm

- Discretize eom: $\tau_f = N_\tau \Delta \tau$
- Discretization errors: $\Delta H \sim O(\Delta \tau^2)$. To maintain detailed balance condition use a Metropolis accept/reject decision. Acceptance rate depends on $\Delta H$, tune parameters in order to maximize acc. ratio and minimize autocorrelation times.
- Main part of computational effort: terms $(\mathcal{D} + b_k)^{-1} F$. Replace by solutions $\chi$ of $(\mathcal{D} + b_k) \chi_k = F$
- Use conjugate gradient method for the smallest of $b_k$’s. ($O(N^3)$ ops if cleverly done)
- Use multimass Krylov solvers for other $b_k$ ($O(Q)$ gain).
- Conjugate gradient method needs $O(N^2)$ iterations to converge. (instead of $O(1)$ in typical LQCD)
In the large–N limit \( \langle \lambda_1 \rangle_0 = \ldots = \langle \lambda_6 \rangle_0 = \ell^2 \approx 0.627 \) consistent with GEM result

No SO(6) SSB \( \Rightarrow \) phase fluctuations are important in inducing SSB as expected
Complex Action Problem

\[ Z = \int dA \ e^{-S_0} \ e^{i\Gamma} \quad Z_f[A] = |\det M| \ e^{i\Gamma} \]

- no ordinary Monte Carlo importance sampling possible: not a positive definite probability measure
- A serious and important "technical" problem
  - Lattice QCD at high \( T/\text{finite} \mu \) [1302.3028]
  - Lattice QCD with \( \theta \)-vacua [0803.1593]
  - Real time QFT [hep-lat/0609058]
  - Electron structure calculation [PRL 71(93)1148, J.Chem.Phys 102,4495+109,6219]
  - Repulsive Hubbard model [PRB 41(90) 9301]
  - Nuclear shell model [Phys.Repts. 278(97)1]
  - Polymer theory [Phys.Repts. 336(00)167]
Possible approach: use the phase quenched model $Z_0 = \int dA e^{-S_0}$:

$$\langle \lambda_n \rangle = \frac{\langle \lambda_n e^{z\Gamma} \rangle_0}{\langle e^{z\Gamma} \rangle_0}$$

- $\langle e^{z\Gamma} \rangle_0$ decreases as $e^{-N^2\Delta f} \sim Z/Z_0$, $\Delta f > 0$.
- Need $O(e^{cN^2})$ statistics for given accuracy goal.
- Overlap problem: distribution of sampled configs in $Z_0$ has exponentially small overlap with $Z$.

Dominant configurations determined by competition of entropy, action and phase fluctuations.
Factorization Method \[\text{[KNA, Nishimura 01]}\]

\[\tilde{\lambda}_n = \frac{\lambda_n}{\langle \lambda_n \rangle_0}\]

- \(\langle \tilde{\lambda}_n \rangle_0 \equiv 1\), deviation from 1 is the effect of the phase
- Consider the distribution functions

\[\rho(x_1, \ldots, x_6) = \left\langle \prod_{k=1}^{6} \delta(x_k - \tilde{\lambda}_k) \right\rangle \]

\[\rho^{(0)}(x_1, \ldots, x_6) = \left\langle \prod_{k=1}^{6} \delta(x_k - \tilde{\lambda}_k) \right\rangle_0\]

- Consider the ensemble

\[Z_{x_1, \ldots, x_6} = \int dA \ e^{-S_0[A]} \ \prod_{k=1}^{6} \delta(x_k - \tilde{\lambda}_k)\]

then

\[\rho(x_1, \ldots, x_6) = \frac{1}{C} \rho^{(0)}(x_1, \ldots, x_6) \ w(x_1, \ldots, x_6)\]

where

\[w(x_1, \ldots, x_6) = \left\langle e^{i\Gamma} \right\rangle_{x_1, \ldots, x_6}\]

\[C = \langle e^{i\Gamma} \rangle_0\] not needed in the calculation.
\[ \langle \tilde{\lambda}_n \rangle = \int \prod_{k=1}^{6} dx_k \, x_n \rho(x_1, \ldots, x_6) \]

- In the large-\( N \) limit, dominating configs determined by minimum of the “free energy”:

\[ \mathcal{F}(x_1, \ldots, x_6) = - \frac{1}{N^2} \log \rho(x_1, \ldots, x_6) \]

\[ = - \frac{1}{N^2} \log \rho^{(0)}(x_1, \ldots, x_6) - \frac{1}{N^2} \log w(x_1, \ldots, x_6) + \frac{1}{N^2} \log C \]

- The minimum is determined by solutions of

\[ \frac{1}{N^2} \frac{\partial}{\partial x_n} \log \rho^{(0)}(x_1, \ldots, x_6) = - \frac{\partial}{\partial x_n} \frac{1}{N^2} \log w(x_1, \ldots, x_6) \quad \text{for} \quad n = 1, \ldots, 6 \]
Factorization Method

\[
\frac{1}{N^2} \frac{\partial}{\partial x_n} \log \rho^{(0)}(x_1, \ldots, x_6) = - \frac{\partial}{\partial x_n} \frac{1}{N^2} \log w(x_1, \ldots, x_6) \quad \text{for} \quad n = 1, \ldots, 6
\]

- each function has a well defined large-\(N\) limit
- dominating solution can be used as an estimator of \(\langle \tilde{\lambda}_n \rangle\)
- no need to know \(\rho(x_1, \ldots, x_6)\) everywhere to compute \(\langle \tilde{\lambda}_n \rangle\)
- RHS has complex action problem but scales fast with increasing \(N\) \(\Rightarrow\) extrapolation to larger \(N\)
- errors do not propagate exponentially with \(N\) as with a naive large \(N\) extrapolation
Factorization Method

- key in using the method: find the right observables to constrain
- determine the ones that are strongly correlated with the phase expectation values of all others computed at the saddle point solution: no sign problem! [KNA, Azuma, Nishimura 1009.4504, 1108.1534]

- $d$-dimensional configs:
  
  $d = 6 \Rightarrow \det \mathcal{M} \in \mathbb{C}$, $d = 5 \Rightarrow \det \mathcal{M} \in \mathbb{R}$, ($\mathbb{R}^+ \text{ dominates at large } N$)
  
  $d = 4, 3 \Rightarrow \det \mathcal{M} \in \mathbb{R}^+$, $d \leq 2 \Rightarrow \det \mathcal{M} = 0$

- phase is stationary w.r.t. perturbations around $d < 6$ configs
  [Nishimura, Vernizzi 00]

- strong evidence that $\lambda_1, \ldots, \lambda_6$ found to be the only ones strongly correlated with the phase: our choice for studying their distribution functions [1009.4504]

Strong complex phase fluctuations play central role in the SSB mechanism

[Nishimura, Vernizzi 00, KNA, Nishimura 01]
Simplifications

- hard to solve the saddle point equations in full 6D parameter space
- we study SO($d$) symmetric vacua $2 \leq d \leq 5$, compare to GEM
  \[ x_1 = \ldots = x_d > 1 > x_{d+1} = \ldots = x_6 \]
- we find that large evs, when sufficiently large, decorrelate from the phase
  \[ \Rightarrow \text{omit large evs from } \rho(x_1, \ldots, x_6) \]
- we find that small evs to acquire the same value in the large-$N$ limit
  \[ \Rightarrow \text{omit smallest evs from } \rho(x_1, \ldots, x_6) \]

Therefore, in order to study the SO($d$) vacuum, consider only $\rho(x_{d+1})$
Observables

We take $n = d + 1$ for the $\text{SO}(d)$ vacuum

- Define $w_n(x) = \langle e^{i\Gamma} \rangle_{n,x}$ w.r.t $Z_{n,x} = \int dA \, e^{-S_0[A]} \delta(x - \tilde{\lambda}_n)$
- Define $\rho_n^{(0)}(x) = \langle \delta(x - \tilde{\lambda}_n) \rangle_0$
- Let $\bar{x}_n$ be the solution to the saddle point equation

\[
\frac{1}{N^2} f_n^{(0)}(x) \equiv \frac{1}{N^2} \frac{d}{dx} \log \rho_n^{(0)}(x) = -\frac{d}{dx} \frac{1}{N^2} \log w_n(x)
\]

in the $x < 1$ region. Then we define the estimator

\[
\langle \tilde{\lambda}_n \rangle_{\text{SO}(d)} = \bar{x}_n, \quad n = d + 1
\]
Given $\bar{x}_n$ we also use the estimators

- $\langle \tilde{\lambda}_k \rangle_{SO(d)} = \langle \tilde{\lambda}_k \rangle_{n, \bar{x}_n}$
- Compute free energy

$$F_{SO(d)} = \int_{\bar{x}_n}^{1} \frac{1}{N^2} f^{(0)}(x) dx - \frac{1}{N^2} \log w_n(\bar{x}_n), \text{ where } n = d + 1$$

By computing $F_{SO(d)}$ for different $d$ we can in principle determine the true vacuum.
Simulations

We simulate the system

\[ Z_{n,V} = \int dA \ e^{-S_0[A] - V(\lambda_n[A])}, \quad V(z) = \frac{1}{2} \gamma (z - \xi)^2 \]

- \( \gamma \) large enough \( e^{-V} \rightarrow \delta(x - \tilde{\lambda}_n) \)
- in practice, we make sure that results are independent of \( \gamma \)
- study the distribution function

\[ \rho_{n,V}(x) = \left\langle \delta(x - \tilde{\lambda}_n) \right\rangle_{n,V} \propto \rho_{n}^{(0)}(x) \exp \{-V(x(\lambda_n)_0)\} \]
Simulations

- position of the peak of $\rho_{n,V}(x)$ solution of

$$0 = \frac{d}{dx} \log \rho_{n,V}(x) = f^{(0)}_{n}(x) - \langle \lambda_{n} \rangle_{0} V'(x \langle \lambda_{n} \rangle_{0})$$

- we take the peak sharp and use

$$x_{p} = \langle \tilde{\lambda}_{n} \rangle_{n,V}$$

- we define the estimators

$$w_{n}(x_{p}) = \langle \cos \Gamma \rangle_{n,V},$$

$$f^{(0)}_{n}(x_{p}) = \langle \lambda_{n} \rangle_{0} V'(\langle \lambda_{n} \rangle_{n,V}) = \gamma \langle \lambda_{n} \rangle_{0} (\langle \lambda_{n} \rangle_{n,V} - \xi).$$

- $\gamma$ too small, distribution of $\tilde{\lambda}_{n}$ wide, large error in $\langle \tilde{\lambda}_{n} \rangle_{n,V}$
- $\gamma$ too large, small error in $\langle \tilde{\lambda}_{n} \rangle_{n,V}$ propagates by factor of $\gamma$ to $f^{(0)}_{n}(x_{p})$

$$\langle \tilde{\lambda}_{n} \rangle_{n,V} - \xi \sim 1/\gamma$$
Simulations

- It is possible to compute $f_n^{(0)}(x)$, $w_n(x)$ for $x$ suppressed by many orders of magnitude in $Z_0$
- $w_n(x)$ hard due to the complex action problem, but

$$\Phi_n(x) = \lim_{N \to \infty} \frac{1}{N^2} \log w_n(x)$$

scales for small enough $N$
- $f_n^{(0)}(x)$, $w_n(x)$ computed by interpolation or fits. Fitting functions determined by simple scaling arguments for small $x$
- We find that the function $f_n^{(0)}(x)$ scales as $\frac{1}{N}f_n^{(0)}(x)$ for $x \gtrsim 0.4$, but as $\frac{1}{N^2}f_n^{(0)}(x)$ for smaller $x$. Need to subtract the $O(1/N)$ finite size effects in the calculations.
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Compute the solution to \( \frac{1}{N^2} f_n^{(0)}(x) = -\Phi'(x) \) (after subtracting finite size effects): Compare to the GEM result \( r^2/\ell^2 \approx 0.223/0.627 = 0.355 \)

\[
\langle \tilde{\lambda}_3 \rangle_{\text{SO}(2)} = \bar{x}_3 = 0.31(1) \quad \langle \tilde{\lambda}_4 \rangle_{\text{SO}(3)} = \bar{x}_4 = 0.35(1)
\]
Compute the solution to $\frac{1}{N^2} f_n^{(0)}(x) = -\Phi'(x)$ (after subtracting finite size effects): Compare to the GEM result $r^2/\ell^2 \approx 0.223/0.627 = 0.355$

$\langle \tilde{\lambda}_5 \rangle_{SO(4)} = \bar{x}_5 = 0.34(2)$  
$\langle \tilde{\lambda}_6 \rangle_{SO(5)} = \bar{x}_6 = 0.36(3)$
\[ \langle \lambda_k \rangle_{SO(d)},\ k \neq n = d + 1,\ is\ estimated\ from\ \langle \lambda_k \rangle_{x_p} = \langle \lambda_k \rangle_{n,V} \]

In order to minimize the finite size effects, we compute

\[ L_n^2(x) = \left( \prod_{k=1}^{6} \langle \lambda_k \rangle_{n,x} \right)^{\frac{1}{6}} \]

and find that \( L_n^2(x) \approx \ell^2 \approx 0.627 \) for \( 0.5 < x < 1 \).
Hard!

After subtracting finite size effects, we fit \( \frac{1}{N^2} f_n^{(0)}(x) = p_n e^{-q_n x} \).

Attempt e.g. to substitute in \( \mathcal{F}_{SO(d)} = \int_{\bar{x}_n}^{1} \frac{1}{N^2} f_n^{(0)}(x) dx - \frac{1}{N^2} \log w_n(\bar{x}_n) \) for \( \bar{x}_n \approx 0.355 \). Still working!! TBA...
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Conclusions

- Simulation from first principles 6D version of IIB matrix model
- Complex action problem very strong, use factorization method successfully
- Computed numerically the maxima of $\lambda_n$ distributions and estimated $\langle \lambda_n \rangle$ for SO($d$) vacua
- Large-$N$ and small-$x$ scaling properties of distribution functions play important role in the calculation
- Short distance, non-perturbative, dynamics of eigenvalues of matrices $A$ play crucial role in determining $r$
- Results are consistent with GEM prediction $R^d r^{6-d} = \ell^6$, $r^2 \approx 0.223$, $\ell^2 \approx 0.627$
- Consistent with the GEM scenario of dynamical compactification with SSB of SO(6)$\rightarrow$SO(3)
- Consistent with (euclidean) spacetime having volume independent of $d$ and $R/r$ finite
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