Abstract. A tensor description of perturbative Einsteinian gravity about an arbitrary background spacetime is developed. By analogy with the covariant laws of electromagnetism in spacetime, gravito-electromagnetic potentials and fields are defined to emulate electromagnetic gauge transformations under substitutions belonging to the gauge symmetry group of perturbative gravitation. These definitions have the advantage that on a flat background, with the aid of a covariantly constant timelike vector field, a subset of the linearised gravitational field equations can be written in a form that is fully analogous to Maxwell’s equations (without awkward factors of 4 and extraneous tensor fields). It is shown how the remaining equations in the perturbed gravitational system restrict the time dependence of solutions to these equations and thereby prohibit the existence of propagating vector fields. The induced gravito-electromagnetic Lorentz force on a test particle is evaluated in terms of these fields together with the torque on a small gyroscope. It is concluded that the analogy of perturbative gravity to Maxwell’s description of electromagnetism can be valuable for (quasi-)stationary gravitational phenomena but that the analogy has its limitations.

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Einstein's theory of gravitation remains a pinnacle in the evolution of theoretical physics. It offers an overarching description of phenomena ranging from the familiar behaviour of Newtonian gravitation to exotic astrophysical events at the extremes of space and time. Although its modern formulation is in terms of tensor fields on a manifold with a spacetime structure, its physical interpretation often benefits from a choice of observer and an appropriate reference frame. One of the traditional methods for extracting information from Einstein’s gravitational field equations is to exploit the properties of observers in some fiducial background spacetime in which gravitational physics is either absent or familiar. This approach has led to various approximations in such backgrounds. Further reduction is afforded by a “3+1” decomposition in which spacetime tensors are expressed in terms of a field of frames adapted to some local foliation of spacetime by spacelike hypersurfaces. More generally frames are afforded by timelike vector fields, the integral curves of which model ideal observers. In such a manner it becomes possible to contemplate different limits in which matter moves slowly or the gravitational field is weak relative to such observers. It is known that Newtonian gravitation follows from such a limit. Within the framework of weak gravity non-Newtonian gravitational effects may arise and a number of experiments have been devised in order to detect such phenomena as the “Lense-Thirring effect” due to “frame-dragging” produced by the earth’s rotation. The nature of this effect may be detectable by a small orbiting gyroscope and is analogous to that produced by the torque on a small magnetic dipole in the presence of the magnetic field of a fixed magnetic dipole. Indeed the component of weak gravity (additional to the dominant Newtonian gravitational field) responsible for this effect is now referred to as the gravito-magnetic field. The sensitivity of recently developed rotation sensors may also be increased to detect post-Newtonian effects in the future.

Several authors have noticed that a subset of the Einstein equations when perturbed about flat spacetime can be written in a form that looks remarkably similar to Maxwell’s equations with the Newtonian gravitational field corresponding to the gravito-electric field and mass-currents playing the role of electric currents. Since the laws of electromagnetism are well studied and understood this analogy has proved quite fruitful in the context particularly in astrophysical applications. Extended “astrophysical jet-structures” are now thought to have their origin in gravito-electromagnetic forces. In the details of astrophysical lensing have been explored in terms of parameters in the NUT metric.

It is also amusing to recall that one of the first theories of post-Newtonian gravitation was formulated by Heaviside in direct analogy with the then recently formulated theory of electromagnetism by Maxwell. In the language of the Poincaré isometry group it predicted that gravitation like electromagnetism was mediated by an independent vector field rather than a second degree tensor field associated with the metric of spacetime. This difference of course must imply that any analogy between weak gravity and electromagnetism is incomplete and most derivations of the gravito-electromagnetic field equations take care to point this out. However in our view the caveats are themselves often incomplete and a close examination of various derivations of the gravito-electromagnetic equations display significant differences in detail. The difficulty in making objective comparisons often arises due to the implicit use of a particular coordinate system (usually adapted to a flat spacetime background) or a partial gauge fixing. Indeed the question of the gauge transformations induced on the gravito-electromagnetic fields from the underlying gauge covariance of the perturbative Einstein equations is usually dealt with rather cursorily. This leads one to contemplate the most useful way to define the gravito-electromagnetic fields in terms of the perturbed components of the spacetime metric. Different choices are often responsible for the location of odd factors of 4 that permeate the gravito-electromagnetic equations compared with the Maxwell equations. Such choices also have implications for the form of the induced gravito-electromagnetic Lorentz force (and torque) in terms of the gravito-electromagnetic fields that enter into the equation for the motion of a massive point (spinning) particle. In gravito-electromagnetic gauge transformations are discussed from a perspective different from the one presented in this paper. Here such transformations are explicitly related to the gauge symmetry of perturbative gravitation and the definitions of gravito-electromagnetic fields in turn induce the notions of gravito-magnetic and gravito-electric coupling strengths.

In this article a tensor description of perturbative Einsteinian gravity about an arbitrary background spacetime is first constructed. By analogy with the covariant laws of electromagnetism in spacetime gravito-electromagnetic potentials and fields are then defined to emulate electromagnetic gauge transformations under
substitutions belonging to the gauge symmetry group of perturbative gravitation. These definitions have the advantage that on a flat background, with the aid of a covariantly constant timelike vector field, a subset of the linearised gravitational field equations can be written in a form that is fully analogous to Maxwell’s equations (without awkward factors of 4 and extraneous tensor fields. It is shown how the remaining equations in the perturbed gravitational system restrict the time dependence of solutions to these equations and thereby prohibit the existence of propagating vector fields. The induced gravito-electromagnetic Lorentz force on a test particle is evaluated by geodesic perturbation in terms of these fields together with the torque on a small gyroscope. It is concluded that the analogy of perturbative gravity to Maxwell’s description of electromagnetism can be valuable for (quasi-) stationary gravitational phenomena but that the analogy has its limitations. It has been argued that such limitations are absent in the approach to gravito-electromagnetic based on properties of the conformal tensor in a spacetime determined by Einstein’s equations. Although this reformulation makes no reference to perturbative methods the analogy with the structure of Maxwell’s equations is less direct. A tensorial description of this formulation is given in Appendix D. Throughout this article the language of tensor fields as multi-linear maps on vector and co-vector fields is adopted. Co-vector fields are manipulated using the exterior calculus of differential forms and Hodge maps. Manifolds are assumed smooth and tensor fields sufficiently differentiable as required. Notations based on the tools used are summarised in Appendix A and some technical computational details are relegated to Appendices B and C.

In order to facilitate comparisons with other authors certain field redefinitions are discussed in section 7 together with the changes induced by them in the gravito-electromagnetic field equations. These alternatives are discussed in the concluding section where the salient features of this paper are summarised.

2. The Maxwell Equations

Maxwell’s equations for the electromagnetic field can be written concisely (in a general spacetime with metric tensor \( g \)) in terms of the Faraday 2-form \( F \) and current 1-form \( J \) on spacetime:

\[
dF = 0, \tag{2.1a}
\]

\[
\delta F = J. \tag{2.1b}
\]

Equation (2.1a) implies that in a regular domain

\[
F = dA \tag{2.2}
\]

for some 1-form potential \( A \), and \( F \) clearly remains invariant under electromagnetic gauge transformations of \( A \) which take the form

\[
A \mapsto A + d\lambda, \tag{2.3}
\]

where \( \lambda \) is an arbitrary smooth function on spacetime. Equation (2.1b) implies that the current is conserved

\[
\delta J = 0. \tag{2.4}
\]

In terms of \( A \), (2.1a) may be written

\[
\Delta A + d\delta A = -J, \tag{2.5}
\]

where \( \Delta = -(\delta d + d\delta) \). In the Lorenz gauge defined by the condition

\[
\delta A = 0, \tag{2.6}
\]

this reduces to the Helmholtz wave equation

\[
\Delta A = -J. \tag{2.7}
\]

The equation of motion for a (spinless) test particle with mass \( m \) and charge \( q \) moving along a curve \( C(\tau) \), parameterized by proper-time \( \tau \), with tangent vector \( C'(\tau) \), in an arbitrary gravitational field and electromagnetic field \( F \) is

\[
m\nabla_{C'}C' + q (i_{C'}F)^{\sharp} = 0, \tag{2.8}
\]

where \( \nabla \) is the Levi-Civita connection associated with \( g \). The Lorentz force \( F_L \) is defined by

\[
F_L = -q (i_{C'}F)^{\sharp}. \tag{2.9}
\]
One can decompose $F$ with respect to some unit-normalized timelike vector field $V$ as

$$F = V^\flat \wedge e + \#b,$$

(2.10)

where $i_V e = 0$ and $i_V \#b = 0$. Similarly one may decompose $\mathcal{J}$ as

$$\mathcal{J} = \rho V^\flat + j$$

(2.11)

where $i_V j = 0$. Ideal observers may be associated with the integral curves of $V$.

The Maxwell field equations (2.1a) and (2.1b) can then be written in (3+1)-form in terms of $e$, $b$ and $V$ as

\begin{align*}
d_V \#b + \Omega V \wedge e &= 0, \\
D_V e + \mathcal{L}_V \#b &= 0, \\
d_V \#e - \Omega V \wedge b &= \rho \#1, \\
D_V b - \mathcal{L}_V \#e &= \#j,
\end{align*}

(2.12)

where the notation is given in the Appendix A. Such equations exhibit possible “pseudo-sources” measured by non-inertial observers associated with non-parallel vector fields $V$.

When $V$ is a parallel vector field (e.g., associated with an observer defining an inertial frame in flat spacetime) these reduce to the familiar Maxwell equations relating electromagnetic fields to their sources written in terms of differential forms. The metric tensor permits one to define the vector fields: $E = e^\flat$, $B = b^\flat$, $J = j^\flat$, and (for $V$ parallel) the Maxwell equations take their more familiar form

\begin{align*}
\text{div } B &= 0, \\
\text{curl } E + \frac{\partial B}{\partial t} &= 0, \\
\text{div } E &= \rho, \\
\text{curl } B - \frac{\partial E}{\partial t} &= J.
\end{align*}

(2.13)

Similarly decomposing the 4-velocity $C'$ with respect to some (unit timelike parallel) vector field $V$ as

$$C' = \frac{1}{\sqrt{1 - g(v, v)}} (V + v),$$

(2.14)

and assuming that the magnitude of $v$ is small ($v^2 = g(v, v) \ll 1$),

$$C' = V + v + O(v^2),$$

(2.15)

the equation of motion for a charged non-relativistic particle becomes

$$\frac{dv}{dt} = \frac{q}{m} (E + v \times B).$$

(2.16)

3. Perturbative Gravitation

The analogy between gravitation and electromagnetism to be discussed follows from a perturbation of the gravitational field about some “background” spacetime geometry. We explore the constraints on this geometry in order to execute this analogy to its fullest extent.

\footnote{Such a vector satisfies $g(V, V) = -1$ with the choice of signature for $g$ chosen throughout this article.}
3.1. The Perturbed Einstein Equations.

A perturbative approach to Einstein’s theory of gravitation can be based on a formal linearization of spacetime geometry about that determined by some solution of Einstein’s equations for the spacetime metric tensor. A generic perturbation will be identified with a class of linearizations that belong to the tangent space to the space of Einstein solutions. Following [24] it is convenient to introduce a 5-dimensional manifold \( M \) that can be foliated by hypersurfaces belonging to a one parameter (\( \epsilon \)) family of spacetimes. The geometry of each leaf \( M_\epsilon \) of the foliation is determined by some second degree symmetric tensor field \( g(\epsilon) \) on \( M \) where \( \epsilon \in [-1,1] \) that restricts to a Lorentzian metric tensor field on each leaf. Furthermore it is asserted that all points of one leaf can be connected to all points on a neighbouring leaf by a one parameter diffeomorphism \( \varphi_\epsilon : M \rightarrow M. \) Thus a tensor field \( T(0) \) on \( M_0 \) can be related to a tensor field \( T(\epsilon) \) on \( M_\epsilon \) according to

\[
\dot{\varphi}_\epsilon T(\epsilon) = T(0) + \epsilon \dot{T}_V + O(\epsilon^2)
\]

where, for some nowhere vanishing vector field \( V \) that is nowhere tangent to the spacetime leaves in \( M, \)

\[
\varphi_\epsilon = \exp(\epsilon V)
\]

induces \( \dot{\varphi}_\epsilon \) on tensors and

\[
\dot{T}_V = \mathcal{L}_V T(\epsilon)|_{\epsilon=0}.
\]

In a local chart with coordinates \( \{x^\mu, \epsilon\} \) adapted to the foliation one may write:

\[
V = \frac{\partial}{\partial \epsilon} + \epsilon(x) \frac{\partial}{\partial x^\mu}.
\]

The tensor \( \dot{T}_V \) is said to be a linearization of \( T(\epsilon) \) about \( T(0) \) with respect to a choice of the vector field \( V. \) Since the leaves \( M_\epsilon \) are diffeomorphic it is natural to identify points on distinct leaves that lie on the same integral curve of \( V. \) Different choices of \( V \) correspond to different identifications. If \( \dot{T}_{V_1} \) and \( \dot{T}_{V_2} \) are distinct linearizations then by construction their difference is generated by a vector field \( X \) on \( M_0: \)

\[
\dot{T}_{V_1} \mapsto \dot{T}_{V_2} = \dot{T}_{V_1} + \mathcal{L}_X T(0).
\]

This may be called a gauge transformation of \( \dot{T}_{V_1} \) induced by \( X. \) If \( \mathcal{L}_X T(0) = 0 \) for all \( X \) then \( T(\epsilon) \) is gauge invariant. In general there is no preferred positive (or negative) definite metric on \( M_0 \) that enables one to assign a natural norm to quantities that are not gauge invariant in this sense.

Linearized gravity proceeds by writing the covariant physical metric tensor \( g(\epsilon) \) field so that:

\[
\dot{\varphi}_\epsilon g(\epsilon) = g(0) + \epsilon \dot{g}_V + O(\epsilon^2)
\]

By common abuse of notation this is simply written:

\[
\dot{g} = g + h
\]

where \( h \) is of order \( \epsilon \) and higher order terms in \( \epsilon \) are subsequently neglected. This abbreviated notation hides the choice of \( V \) in the definition of \( h \) and an alternative choice arises from the gauge transformation

\[
h \mapsto h + \epsilon \mathcal{L}_X g
\]

for any vector field \( X \) on the spacetime with background metric \( g. \) Similarly we introduce the abbreviated notation \( \dot{T} = \dot{\varphi}_- T(\epsilon), \ T = T(0), \ \dot{T} = \epsilon \dot{T}_V, \) for some tensor \( T \) (thus \( \dot{T} \) is of order \( \epsilon \)).

The contravariant physical metric tensor field \( \dot{G} \) can be similarly written in terms of the induced contravariant background metric tensor \( G \) and a contravariant perturbation tensor \( H \)

\[
\dot{G} = G + H
\]

In a smooth local basis of vector fields on spacetime \( \{X_a\}, \) with dual cobasis \( \{e^b\} \) such that \( e^b(X_a) = \delta^b_a \)

for \( a = 0, 1, 2, 3, \) the induced perturbation tensor \( H \) can be written as

\[
H = -h_{ab}X^a \otimes X^b
\]

with \( g_{ac}g^{cb} = \delta^b_a \) (where \( g_{ac} \) are the components of \( G \) in the above cobasis) and \( X^a = g^{ab}X_b. \) The metric-dual of vector and 1-form fields, and the associated raising and lowering of indices are defined with respect to the background metric tensor. Likewise any operations that depend on the metric tensor (e.g. the Hodge map) will be defined with respect to the background metric unless indicated otherwise.
To derive the perturbed Einstein tensor in terms of $h$ (in some gauge) write the Levi-Civita connection $\hat{\nabla}$ with respect to the physical metric $\hat{g}$ in terms of the Levi-Civita connection $\nabla$ with respect to $g$ so that for any vector field $X$ on spacetime:

$$\hat{\nabla}X = \nabla X + \gamma X,$$

(3.11)

Since $\hat{\nabla}$ and $\nabla$ are torsion free

$$\gamma X Y = \gamma Y X,$$

(3.12)

for arbitrary vector fields $X$ and $Y$. It follows that

$$\gamma X f = 0,$$

(3.13)

for any function $f$ on spacetime. It is convenient to define a tensor $\gamma$ by

$$\gamma (X,Y,\alpha) = \alpha (\gamma X Y),$$

(3.14)

for any 1-form $\alpha$, then

$$\gamma X Y = \gamma (X,Y,-)$$

and

$$\gamma X \alpha = -\gamma (X,-,\alpha).$$

(3.15)

(3.16)

Since $\hat{\nabla}$ and $\nabla$ are compatible with respect to $\hat{g}$ and $g$ respectively $\gamma$ can be written in terms of $h$ as

$$\gamma (X,Y,\alpha) = \frac{1}{2} \{ (\nabla_X h)(Y,\alpha^2) + (\nabla_Y h)(X,\alpha^2) - (\nabla_{\alpha^2} h)(X,Y) \},$$

(3.17)

for arbitrary $X$, $Y$ and $\alpha$.

The curvature operators $\hat{R}$ and $R$ for the connections $\hat{\nabla}$ and $\nabla$ respectively are defined in the usual manner as

$$\hat{R}_{X Y} = [\hat{\nabla}_X, \hat{\nabla}_Y] - \hat{\nabla}_{[X,Y]},$$

$$R_{X Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

(3.18a)

(3.18b)

Thus

$$\hat{R}_{X Y} = R_{X Y} + \hat{\gamma}_{X Y},$$

(3.19)

and it follows that the perturbed curvature operator $\hat{\gamma}$ is given by

$$\hat{\gamma}_{X Y} = \nabla_X \gamma_Y - \nabla_Y \gamma_X + \gamma_X \nabla_Y - \gamma_Y \nabla_X - \gamma_{[X,Y]}.$$

(3.20)

When acting on any vector field $Z$ this simplifies to

$$\hat{\gamma}_{X Y} Z = (\nabla_X \gamma)(Y,Z,-) - (\nabla_Y \gamma)(X,Z,-),$$

(3.21)

and the perturbed curvature tensor $\hat{R}$ is defined by

$$\hat{R}(X,Y,Z,\alpha) = \alpha (\hat{\gamma}_{X Y} Z) = (\nabla_X \gamma)(Y,Z,\alpha) - (\nabla_Y \gamma)(X,Z,\alpha).$$

(3.22)

The perturbed Ricci tensor $\hat{\text{Ric}}$ follows by contraction

$$\hat{\text{Ric}}(X,Y) = \hat{R}(X_a,X,Y,e^a).$$

(3.23)

It is useful to introduce the trace-reverse map $\mu$ on covariant degree 2 tensors so that

$$\mu(T) = T - \frac{1}{2} \text{Tr}(T) g$$

(3.24)

where $\text{Tr}(T) = T(X_a,X^a)$ in any basis. The trace-reverse of $h$ is denoted by $\psi$:

$$\psi = \mu(h).$$

(3.25)

After some calculation $\hat{\text{Ric}}$ can be written in terms of $\psi$ and the Laplacian operator $\text{Lap} = \nabla\nabla$ as

$$\hat{\text{Ric}} = -\frac{1}{2} \mu(\text{Lap} \psi) + \text{Sym} \nabla\nabla \psi - C_\psi + S_\psi,$$

(3.26)
where
\[ C_\psi(X, Y) = R(\psi^a, X, Y, \psi^a), \]  
(3.27a)
\[ S_\psi(X, Y) = \frac{1}{2}(\text{Ric}(\psi^a_X, Y) + \text{Ric}(\psi^a_Y, X)) \]  
(3.27b)
and the convenient notation
\[ \psi_X = \psi(X, -), \]  
(3.28a)
\[ \psi^a = \psi^a_X = \psi(X^a, -) \]  
(3.28b)
is used.

Note that \(C_\psi\) and \(S_\psi\) have the same trace, namely
\[ \text{Tr}(C_\psi) = \text{Tr}(S_\psi) = \text{Ric}(\psi^a_X, X^a). \]  
(3.29)
The perturbed curvature scalar \(\dot{R}\) follows as
\[ \dot{R} = \text{Tr}(\text{Ric} - S_\psi + \frac{1}{2}\text{R}\psi), \]  
(3.30)
and with \(\text{Ein} = \text{Ric} - \frac{1}{2}\bar{g}\dot{R} = \text{Ein} + \text{Ein}\) the perturbed Einstein tensor \(\text{Ein}\) is
\[ \text{Ein} = \text{Ric} - \frac{1}{2}\text{R}h - \frac{1}{2}\text{R}\bar{g}, \]  
(3.31)
or
\[ \text{Ein} = -\frac{1}{2}\text{Lap}\psi + \mu(\text{Sym} \nabla \nabla \cdot \psi) - \mu(C_\psi) + S_\psi - \frac{1}{2}\text{R}\psi \]  
(3.32)
in terms of \(\psi\).

One may now express the perturbed Einstein equation in terms of \(\psi\) by writing the physical stress energy-momentum tensor \(\hat{T}\) as
\[ \hat{T} = \mathcal{T} + \hat{\mathcal{T}}, \]  
(3.33)
where the background stress energy-momentum tensor \(\mathcal{T}\) acts as a source for the background metric via the background Einstein equation
\[ \text{Ein} = \kappa \mathcal{T}, \]  
(3.34)
and \(\kappa = 8\pi G\) in units where \(c = 1\). The perturbed Einstein equation
\[ \text{Ein} = \kappa \hat{T}, \]  
(3.35)
becomes
\[ -\frac{1}{2}\text{Lap}\psi + \mu(\text{Sym} \nabla \nabla \cdot \psi) - \mu(C_\psi) + S_\psi - \frac{1}{2}\text{R}\psi = \kappa \hat{T}. \]  
(3.36)
In the next section this equation is simplified by exploiting a gauge symmetry of the Einstein equations.

### 3.2. Gauge Transformations and the Transverse Gauge Condition.

A gauge transform of \(h\) has been defined as a substitution of the form
\[ h \mapsto h + \mathcal{L}_V g, \]  
(3.37)
where \(\mathcal{L}_V\) is the Lie derivative with respect to some vector field \(V\) that maintains \(h\) perturbative with respect to \(g\). This substitution is used to determine the induced gauge transformation of tensors defined in terms of \(h\) (keeping the background geometry fixed). Thus the induced gauge transformation of \(\gamma_X\) follows as
\[ \gamma_X \mapsto \gamma_X + \mathcal{D}_X(V), \]  
(3.38)
where, for any vector field \(X\) the tensor derivation \(\mathcal{D}_X(V)\) is defined by
\[ \mathcal{D}_X(V) = [\mathcal{L}_V, \nabla_X] - \nabla_{\mathcal{L}_V X}. \]  
(3.39)
The operator \(\mathcal{D}_X(V)\) provides a useful tool when performing calculations involving both Lie derivatives and covariant derivatives (3.39).
It follows that the perturbed curvature operator transforms as
\[ \dot{R}_{XY} \mapsto \dot{R}_{XY} + \mathcal{R}_{XY}(V), \]
where the operator \( \mathcal{R}_{XY}(V) \) is defined by
\[ \mathcal{R}_{XY}(V) = [\mathcal{L}_V, R_{XY}] - R_{L_X Y} - R_{X L_Y}. \] (3.41)
A contraction shows that the perturbed curvature tensor transforms as
\[ \dot{R} \mapsto \dot{R} + \mathcal{L}_V R, \] (3.42)
from which one deduces
\[ \dot{\text{Ric}} \mapsto \dot{\text{Ric}} + \mathcal{L}_V \text{Ric} \] (3.43)
and
\[ \dot{\mathcal{R}} \mapsto \dot{\mathcal{R}} + \mathcal{L}_V \mathcal{R}. \] (3.44)
It follows that the perturbed Einstein tensor exhibits the induced gauge transformation
\[ \dot{\text{Ein}} \mapsto \dot{\text{Ein}} + \mathcal{L}_V \text{Ein}. \] (3.45)
This behaviour of \( \dot{\text{Ein}} \) dictates the behaviour of any (phenomenological) stress energy-momentum tensor under an induced gauge transformation in order to maintain the gauge covariance of (3.33).

It can be similarly shown that if \( \psi \mapsto \bar{\psi} = \psi + \mu(\mathcal{L}_V g) \) the divergence of \( \psi \) transforms as
\[ \nabla \cdot \psi \mapsto \nabla \cdot \bar{\psi} = \nabla \cdot \psi + (\mathcal{L}_V V)^\flat + \text{Ric}(V, -). \] (3.46)
Thus, if for some \( \psi \), one chooses \( V \) to satisfy the differential equation
\[ (\mathcal{L}_V V)^\flat + \text{Ric}(V, -) = -\nabla \cdot \psi, \] (3.47)
then this imposes the gauge condition
\[ \nabla \cdot \bar{\psi} = 0, \] (3.48)
referred to as the \textit{transverse gauge}. Further gauge transformations, generated by additional vector fields \( W \) satisfying the linear differential equation
\[ (\mathcal{L}_W W)^\flat + \text{Ric}(W, -) = 0, \] (3.49)
maintain \( \bar{\psi} \) divergenceless.

In the transverse gauge the perturbed Einstein equation immediately simplifies to
\[ -\frac{1}{2} \mathcal{L}_V \psi - \mu(\mathcal{C}_\psi) + \mathcal{S}_\psi - \frac{1}{2} \mathcal{R}_\psi = \kappa \mathcal{T}. \] (3.50)
It should be noted that this equation holds in any (background) metric that satisfies the (background) Einstein equation (3.34).

In the case where the background metric is Ricci-flat (or has a cosmological constant), the perturbed Einstein equation simplifies to
\[ -\frac{1}{2} \mathcal{L}_V \psi - \mathcal{C}_\psi = \kappa \mathcal{T}. \] (3.51)
which may be compared with the component form [17] and the abstract index form [25].

If the background metric is flat this simplifies further to
\[ \mathcal{L}_V \psi = -2\kappa \mathcal{T}. \] (3.52)
This equation is analogous to the Helmholtz equation for the electromagnetic potential 1-form in the electromagnetic Lorenz gauge and leads to the prediction of propagating gravitational perturbations.
4. Gravito-Electromagnetism

The perturbed Einstein equation about a flat background spacetime metric has a form that is similar to the Helmholtz equation for the electromagnetic potential. However the perturbed gravitational potential $\psi$ is a symmetric tensor field $\mathbf{(2,0)}$ tensor field, whereas the electromagnetic potential $A$ is a type $\mathbf{(1,0)}$ tensor field (a 1-form). The description of Maxwell’s equations in terms of electric and magnetic fields suggests that $\psi$ be decomposed relative to some fiducial vector field and split into a frame dependent 1-form (analogous to the 1-form potential in electromagnetism) plus an extra non-Maxwellian tensor field. We motivate this definition by inducing a gauge transformation described above on this 1-form and comparing the result with the structure of the electromagnetic gauge transformation (Section 4.2) of the 1-form potential. This will also show that restrictions must be placed on the fiducial vector field if a close analogy with electromagnetism is to hold.

4.1. Gravito-Electromagnetic Gauge Transformations.

The trace-reversed perturbation $\psi$ can be written, relative to some unit-normalized (to zero order in $\epsilon$) timelike vector field $\xi$, $(g(\xi,\xi) = 1 + O(\epsilon))$ and general background metric tensor $g$ as

$$\psi = \phi \xi g - \psi \otimes \xi^0 - \xi^a \otimes \psi_{\xi a} - \Sigma_{\xi}.$$  \hfill (4.1)

The tensor $\psi$ has been split into a 1-form part

$$\psi_{\xi} = \psi(\xi, -),$$  \hfill (4.2a)

with $\xi$ component

$$\phi \xi = \psi_{\xi}(\xi) = \psi(\xi, \xi),$$  \hfill (4.2b)

and a spacelike tensor part $\Sigma_{\xi}$ satisfying

$$\Sigma_{\xi}(\xi, -) = 0.$$  \hfill (4.2c)

Since

$$\psi \mapsto \psi + \mu (\mathcal{L}_V g)$$

under a gauge transformation $h \mapsto h + \mathcal{L}_V g$ generated by $V$, a contraction with $\xi$ gives

$$\psi_{\xi} \mapsto \psi_{\xi} + d(g(\xi, V)) + i_V d\xi^a - \nabla V \xi^a + (\mathcal{L}_V \xi)^a.$$  \hfill (4.4)

With a special choice for $V$ of the form

$$V = - \lambda \xi,$$  \hfill (4.5)

where $\lambda$ is any smooth function of order $\epsilon$ on spacetime, the induced transformation on the 1-form $\psi_{\xi}$ becomes

$$\psi_{\xi} \mapsto \psi_{\xi} + d\lambda + \lambda (\Theta_{\xi} \xi^a - A_{\xi a}),$$  \hfill (4.6)

where $\Theta_{\xi}$ is the expansion of $\xi$ and $A_{\xi a}$ is its acceleration (Appendix A). If $\lambda(\Theta_{\xi} \xi^a - A_{\xi a})$ is of higher order in $\epsilon$ than $\lambda$ the last term can be neglected and the gauge transformation of $\psi_{\xi}$ simplifies to

$$\psi_{\xi} \mapsto \psi_{\xi} + d\lambda.$$  \hfill (4.7)

This suggests that $\psi_{\xi}$ be interpreted as the analogue of the 1-form potential $A$ in electromagnetism and henceforth $\psi_{\xi}$ will be referred to as the gravito-electromagnetic 1-form potential.

Similarly, projecting (4.3) with $\Pi_{\xi}$ (Appendix A) one finds

$$\Sigma_{\xi} \mapsto \Sigma_{\xi} + \Pi_{\xi} \mathcal{L}_V g + 2(\nabla V + g(\xi, \nabla V))g_{\xi}.$$  \hfill (4.8)

Hence with $V$ as in (4.5), $\Sigma_{\xi}$ transforms as

$$\Sigma_{\xi} \mapsto \Sigma_{\xi} + 2\lambda (\sigma_{\xi} - \frac{2}{3} \Theta_{\xi} g_{\xi}),$$  \hfill (4.9)

where $\sigma_{\xi}$ is the shear of $\xi$ (Appendix A). If $\lambda(\sigma_{\xi} - \frac{2}{3} \Theta_{\xi} g_{\xi})$ is of higher order in $\epsilon$ than $\Sigma_{\xi}$ the last term is negligible and $\Sigma_{\xi}$ is gauge invariant under such transformations.

The class of gauge transformations generated by $- \lambda \xi$ such that $\psi_{\xi}$ transforms as in (4.7) with $\Sigma_{\xi}$ invariant will be said to contain gravito-electromagnetic gauge transformations. Clearly if $\xi$ is parallel (i.e. $\nabla \xi = 0$), or parallel to zero order in $\epsilon$ (since $\lambda$ is of order $\epsilon$), such transformations are guaranteed to be gravito-electromagnetic gauge transformations. For a given $h$ one may therefore define a class of vector fields $\{\xi\}$, members
of which are equivalent if they can be used to generate gravito-electromagnetic gauge transformations. Such fields will be said to define gravito-electromagnetic frames of reference. Not all background spacetimes will permit the existence of such vector fields (e.g. the black-hole Schwarzschild spacetime), however they are guaranteed to exist in Minkowski spacetime (and interestingly also in the Einstein static universe). In the next section these vector fields will be used to define a class of gauge equivalent gravito-electromagnetic fields.

4.2. The Gravito-Electromagnetic Field Equations.

On a flat background the linearized Einstein equation can be written as

$$\text{Lap} \psi = -2\kappa \mathcal{T}, \quad (4.10)$$

where the transverse gauge condition

$$\nabla \cdot \psi = 0, \quad (4.11)$$

has been imposed. Contracting (4.10) with a unit-normalized timelike vector field $\xi$ that can be used to define gravito-electromagnetic gauge transformations gives

$$\Delta \psi_\xi = 2\kappa J_\xi, \quad (4.12)$$

where the mass-current $J_\xi$ is defined by

$$J_\xi = -\dot{T}(\xi, -). \quad (4.13)$$

Contracting (4.12) on $\xi$ gives

$$\Delta \phi_\xi = -2\kappa \rho_\xi, \quad (4.14)$$

where the mass-density $\rho_\xi$ is defined by

$$\rho_\xi = -J_\xi(\xi) = \dot{T}(\xi, \xi). \quad (4.15)$$

Similarly acting on (4.10) with the projection operator $\Pi_\xi$ and using (4.1) gives

$$\Delta \phi_\xi g_\xi - \text{Lap} \Sigma_\xi = -2\kappa \mathcal{T}_\xi. \quad (4.16)$$

With the aid of (4.14) this becomes

$$\text{Lap} \Sigma_\xi = 2\kappa(\mathcal{T}_\xi - \rho_\xi g_\xi). \quad (4.17)$$

where $\mathcal{T}_\xi$ is the spacelike (with respect to $\xi$), part of the stress energy-momentum tensor

$$\mathcal{T}_\xi = \Pi_\xi \mathcal{T}. \quad (4.18)$$

The gravito-electromagnetic analogue of the Faraday 2-form is now defined as

$$F_\xi = d\psi_\xi, \quad (4.19)$$

hence

$$dF_\xi = 0. \quad (4.20)$$

The tensor $F_\xi$ is invariant under transformations in the class of gravito-electromagnetic gauge transformation defined above. The gravito-electric field $\mathcal{E}_\xi$, and the gravito-magnetic field $\mathcal{B}_\xi$ follow from a (3+1)-split (with respect to $\xi$):

$$F_\xi = \xi^i \wedge \mathcal{E}_\xi + \# \mathcal{B}_\xi. \quad (4.21)$$

With $\psi_\xi$ written in terms of its timelike and spacelike parts:

$$\psi_\xi = -\phi_\xi \xi^i + \Psi_\xi, \quad (4.22)$$

where $i_\xi \Psi_\xi = 0$, one can write the gravito-electric field as

$$\mathcal{E}_\xi = d_\xi \phi_\xi - \mathcal{L}_\xi \Psi_\xi, \quad (4.23a)$$

and the gravito-magnetic field as

$$\mathcal{B}_\xi = \# d_\xi \Psi_\xi \quad (4.23b)$$
in terms of the gravito-electromagnetic potential. The mass-current can be similarly split into its timelike and spacelike parts with respect to $\xi$ as
\[ J_\xi = \rho_\xi \xi^b + J_\xi, \] (4.24)
where $i_\xi J_\xi = 0$.

The transverse gauge condition in terms of $\xi$, $\psi_\xi$, $\phi_\xi$ and $\Sigma_\xi$ is:
\[ \nabla \cdot \psi = \delta \psi_\xi \xi^b + d\phi_\xi - \nabla_\xi \psi_\xi - \nabla \Sigma_\xi - \nabla_\xi \psi_\xi - \Theta_\xi \psi_\xi = 0. \] (4.25)

For a gravito-electromagnetic frame $\xi$ ($\nabla \xi = 0$ to order $\epsilon$) this reduces to
\[ \nabla \cdot \psi = \delta \psi_\xi \xi^b + d\phi_\xi - \nabla_\xi \psi_\xi - \nabla \Sigma_\xi = 0. \] (4.26)

Using (4.22), (A.19) and (A.30) it follows that $d\phi_\xi - \nabla_\xi \psi_\xi = d\xi \phi_\xi - L_\xi \Psi_\xi = E_\xi$, hence
\[ \nabla \cdot \psi = (\delta \psi_\xi \xi^b) + (E_\xi - \nabla \Sigma_\xi) = 0. \] (4.27)

The two bracketed terms are orthogonal so the transverse gauge condition is equivalent to the two equations
\[ \delta \psi_\xi = 0, \] (4.28a)
\[ E_\xi = \nabla \cdot \Sigma_\xi. \] (4.28b)

The first condition is analogous to the Lorenz gauge in electromagnetism, while the second condition has no electromagnetic analogue. The consequences of this second condition are explored below.

The equation (4.28a) implies (Appendix A) that (4.12) takes the Maxwell-like covariant form
\[ \delta F_\xi = -2\kappa J_\xi. \] (4.29)

The perturbative part of the stress energy-momentum tensor may be expressed in terms of the mass-current using (1.13), (1.15) and (1.18) as
\[ \tilde{T} = \rho_\xi \xi^b \otimes \xi^b - J_\xi \otimes \xi^b - \xi^b \otimes J_\xi + \tilde{T}_\xi, \] (4.30)
and since the background source is assumed zero ($\tilde{T} = 0$) the divergence condition becomes
\[ \nabla \cdot \tilde{T} = 0 \] (4.31)
hence
\[ (\delta J_\xi \xi^b) - (\nabla_\xi J_\xi - \nabla \tilde{T}_\xi) = 0. \] (4.32)

Since the two bracketed terms are orthogonal one has:
\[ \delta J_\xi = 0, \] (4.33a)
\[ \nabla_\xi J_\xi = \nabla \tilde{T}_\xi. \] (4.33b)

with the first condition expressing the conservation of mass-current in the background geometry.

The field equation (4.29) and the closure of $F_\xi$ (1.20) can now be written in terms of these decompositions:
\[ d_\xi \# B_\xi = 0, \] (4.34a)
\[ d_\xi E_\xi + L_\xi \# B_\xi = 0, \] (4.34b)
\[ d_\xi \# E_\xi = -2\kappa \rho_\xi \# 1, \] (4.34c)
\[ d_\xi B_\xi - L_\xi \# E_\xi = -2\kappa \# J_\xi. \] (4.34d)

Although the equations (1.34) involving the gravitational field have the mathematical structure of Maxwell’s equations for electromagnetism the two theories are not isomorphic since the gravitational fields must additionally satisfy (4.28b) in the transverse gauge. Not all solutions of Maxwell’s equations translate to perturbative gravitational fields that are compatible with this condition for a given $\Sigma_\xi$. There does exist
however a class of compatible solutions. Such solutions \( \psi \) are characterized by the existence of a gauge in which

\[
\nabla \cdot \psi = 0, \tag{4.35a}
\]
\[
\Pi_\xi \psi = 0, \tag{4.35b}
\]
in some gravito-electromagnetic frame \( \xi \) (Section 4.1). In this case such solutions will be said to belong to the gravito-electromagnetic limit. It follows from equation (4.1) that

\[
\Sigma_\xi = \phi_\xi g_\xi. \tag{4.36}
\]

Such a \( \xi \) will not be unique. If \( \zeta = \xi + v \), such that \( g(\zeta, \zeta) = -1 \) and \( v^2 = g(v, v) \ll 1 \), then \( \Pi_\xi \psi \) is of order \( \nu \epsilon \). Hence if \( \nu \lesssim \epsilon \), \( \zeta \) also defines the gravito-electromagnetic limit. However the fields \( (\psi_\zeta, \Sigma_\zeta, E_\zeta, B_\zeta) \) defined with respect to \( \zeta \) are identical (within this approximation) to the corresponding fields defined with respect to \( \xi \) (Appendix C).

The condition (4.35b) and the field equation (4.10) require

\[
T_\xi = 0. \tag{4.37}
\]

Equation (4.35b) implies \( \nabla \cdot \Sigma_\xi = d_\xi \phi_\xi \), hence in this limit condition (4.28b) becomes

\[
E_\xi = d_\xi \phi_\xi, \tag{4.38}
\]

or with (4.23a)

\[
L_\xi \Psi_\xi = 0, \tag{4.39}
\]

and using (4.28a) it follows that

\[
\xi^2 \phi_\xi = 0. \tag{4.40}
\]

Thus \( \Psi_\xi \) is time independent and it follows from (4.23b) that the gravito-magnetic field is also time independent, \( L_\xi B_\xi = 0 \). In this limit the field equations reduce to

\[
d_\xi # B_\xi = 0, \tag{4.41a}
\]
\[
d_\xi E_\xi = 0, \tag{4.41b}
\]
\[
d_\xi # E_\xi = -2\kappa \rho_\xi # 1, \tag{4.41c}
\]
\[
d_\xi B_\xi - L_\xi # E_\xi = -2\kappa # J_\xi, \tag{4.41d}
\]

along with the conditions (4.28a), (4.28b) which can be written as

\[
d_\xi # \Psi_\xi - \xi \phi_\xi = 0, \tag{4.42a}
\]
\[
L_\xi \Psi_\xi = 0. \tag{4.42b}
\]

respectively. Similarly it follows from (4.33a), (4.33b) and (4.37) that

\[
d_\xi # J_\xi + \xi \rho_\xi = 0, \tag{4.43a}
\]
\[
L_\xi J_\xi = 0. \tag{4.43b}
\]

With \( \xi = \frac{\partial}{\partial t} \) and the vector fields \( E_\xi = E^\xi_\xi, B_\xi = B^\xi_\xi, J_\xi = J^\xi_\xi \), the field equations (4.41) can be written in more familiar notation as

\[
\text{div } B_\xi = 0, \tag{4.44a}
\]
\[
\text{curl } E_\xi = 0, \tag{4.44b}
\]
\[
\text{div } E_\xi = -2\kappa \rho_\xi, \tag{4.44c}
\]
\[
\text{curl } B_\xi - \frac{\partial E_\xi}{\partial t} = -2\kappa J_\xi. \tag{4.44d}
\]
Similarly, with $A_\xi = \Psi \sharp \xi$, the gauge conditions (4.42a), (4.42b) become

$$\text{div} A_\xi - \frac{\partial \phi_\xi}{\partial t} = 0,$$

(4.45a)

and the conservation equations (4.43a), (4.43b) become

$$\text{div} J_\xi + \frac{\partial \rho_\xi}{\partial t} = 0,$$

(4.46a)

Thus stationary electric and magnetic field configurations and their sources have direct analogues in the theory of perturbative gravitation.

The Newtonian limit is defined as the gravito-electromagnetic limit supplemented by the condition

$$\Psi \xi = 0.$$  

(4.47)

From (4.12) this implies

$$J_\xi = 0.$$  

(4.48)

The Newtonian potential $\Phi_\xi$ is identified as

$$\Phi_\xi = -\frac{1}{4} \phi_\xi.$$  

(4.49)

From (4.42a) it must be time-independent

$$\xi \Phi_\xi = 0,$$

(4.50)

and equation (4.14) becomes

$$\Delta \Phi_\xi = 4\pi G \rho_\xi,$$

(4.51)

which is just the field equation for Newtonian gravitation.

In the following section the above framework is illustrated by calculating the gravito-electromagnetic fields arising as perturbations on the asymptotic gravitational field of a rotating source.

4.3. The gravito-electromagnetic Field of a Rotating Source.

The metric tensor at large distances from a compact rotating body, with Newtonian gravitational mass $M$ and angular momentum $J$, may be approximated (for $r \gg 2GM$ in units where $c = 1$) by

$$\hat{g} = -\left(1 - \frac{2GM}{r}\right) dt \otimes dt + \left(1 + \frac{2GM}{r}\right) (dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2 \theta d\phi \otimes d\phi)$$

$$+ \frac{2GJ}{r^2} \sin \theta (r \sin \theta d\phi \otimes dt + r \sin \theta dt \otimes d\phi).$$

(4.52)

In the cobasis $\{e^0 = dt, e^1 = dr, e^2 = r d\theta, e^3 = r \sin \theta d\phi\}$ the metric tensor can be rewritten as

$$\hat{g} = -\left(1 - \frac{R_S}{r}\right) e^0 \otimes e^0 + \left(1 + \frac{R_S}{r}\right) (e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3)$$

$$+ \frac{R_SR_K}{r^2} \sin \theta (e^0 \otimes e^3 + e^3 \otimes e^0),$$

(4.53)

in terms of the two length-scales $R_S = 2GM$ and $R_K = \frac{J}{M}$.

Defining the dimensionless coordinates $R$ and $T$ by

$$r = \frac{R_S}{\epsilon} R,$$

(4.54a)

$$t = \frac{R_S}{\epsilon} T,$$

(4.54b)
and the dimensionless cobasis $e^a$ by
\[
e^a = \frac{R_S}{\epsilon} e^a,
\]
the dimensionless physical metric tensor $\hat{g}$ defined by
\[
\hat{g} = \left( \frac{R_S}{\epsilon} \right)^2 \hat{g},
\]
and the perturbation $h$ about $g$ is then
\[
h = \frac{\epsilon}{R} \left( \epsilon^0 \otimes \epsilon^0 + \epsilon^1 \otimes \epsilon^1 + \epsilon^2 \otimes \epsilon^2 + \epsilon^3 \otimes \epsilon^3 \right).
\]
and the dimensionless metric-dual map associated with $g$ and $\Lambda = \frac{\Lambda}{\partial Z}$, the gravito-electric and -magnetic vector fields defined by $E_{X_0} = \epsilon^2$ and $B_{X_0} = \Sigma$, become
\[
E_{X_0} = -2 \frac{\epsilon}{R^2} e^1,
\]
and (with $\#1 = e^1 \wedge e^2 \wedge e^3$) the dimensionless gravito-magnetic field is
\[
B_{X_0} = \frac{\epsilon}{R^3} \Lambda \sin \theta e^1.
\]

In terms of the dimensionless gravito-Faraday 2-form $F_{X_0}$, defined by
\[
F_{X_0} = \frac{\epsilon}{R^2} \Lambda \sin \theta e^1.
\]
and taking the trace-reverse yields
\[
\psi = 2 \frac{\epsilon}{R^2} \Lambda \sin \theta e^1.
\]
In terms of the dimensionless Minkowski metric tensor
\[
g = -\epsilon^0 \otimes \epsilon^0 + \epsilon^1 \otimes \epsilon^1 + \epsilon^2 \otimes \epsilon^2 + \epsilon^3 \otimes \epsilon^3
\]
the perturbation $h$ about $g$ is then
\[
h = \epsilon \left( \epsilon^0 \otimes \epsilon^0 + \epsilon^1 \otimes \epsilon^1 + \epsilon^2 \otimes \epsilon^2 + \epsilon^3 \otimes \epsilon^3 \right) + \frac{\epsilon}{R^2} \Lambda \sin \theta \epsilon^3.
\]
In terms of the dimensionless gravito-Faraday 2-form $F_{X_0}$, defined by
\[
F_{X_0} = \frac{\epsilon}{R^2} \Lambda \sin \theta e^1.
\]
and the gravito-electric and -magnetic vector fields defined by $E_{X_0} = \epsilon^2$ and $B_{X_0} = \Sigma$, become
\[
E_{X_0} = -2 \frac{\epsilon}{R^2} e^1,
\]
and (with $\#1 = e^1 \wedge e^2 \wedge e^3$) the dimensionless gravito-magnetic field is
\[
B_{X_0} = \frac{\epsilon}{R^3} \Lambda \sin \theta e^1.
\]
is expected to follow from the weak field limit of equations presented in [1]. To facilitate a comparison with treatments that ignore Mathisson-Papapetrou type coupling to spacetime curvature the discussion is restricted to the separate pre-geodesic motion of a spinless test particle and the gyroscopic precession that follows from a parallel spin vector along such a motion.

5. Motion of a Test Particle in a Weak Gravitational Field

The history of an electrically neutral, spinless test particle in spacetime is modeled by a future-pointing timelike curve \( C(\tau) \) that satisfies

\[
\nabla_{C'}C' - \frac{1}{2} \frac{C'(\dot{g}(C',C'))}{\dot{g}(C',C')}C' = 0,
\]

for some general parameter \( \tau \). To compare the following gravito-electromagnetic equations with analogous equations from electromagnetism in Minkowski spacetime it is most convenient to parameterize the curve so that

\[
g(C',C') = -1.
\]

Using (3.7) and (3.11), to first order in \( \epsilon \) equation (5.1) is

\[
\nabla_{C'}C' + \gamma(C',C',-) - \frac{1}{2} C'(h(C',C'))C' = 0.
\]

Rewriting \( \gamma(C',C',-) \) in terms of \( F_{C'} \) and \( \Sigma_{C'} \) (see Appendix [2], (5.3)) becomes

\[
\nabla_{C'}C' + (i_{C'} F_{C'} + \frac{1}{2} d_{C'} \text{Tr}(\Sigma_{C'}))^{\sharp} = 0
\]

and, for a test particle with mass \( m \), the perturbed gravitational force \( F_G \), is then

\[
F_G = -m(i_{C'} F_{C'} + \frac{1}{2} d_{C'} \text{Tr}(\Sigma_{C'}))^{\sharp}.
\]

The first term on the right is analogous to the Lorentz force in electromagnetism, however the second term is non-Maxwellian. Since \( F_G \) depends only on \( F_{C'} \) and \( \Sigma_{C'} \) it is invariant under gravito-electromagnetic gauge transformations (Section 4.1).

To examine the above equations in the gravito-electromagnetic limit assume that some vector \( \partial_t \) defines a gravito-electromagnetic frame, and reparameterize \( C(\tau) \) in terms of \( t \), so that \( C(t) = C(\tau) \). Writing \( C'(t) \) as

\[
C' = \partial_t + v,
\]

where \( g(v, \partial_t) = 0 \), it then follows that

\[
g(C',C') = -(1 - v^2),
\]

where \( v^2 = g(v,v) \).

Writing \( C' = \frac{1}{\sqrt{1 - v^2}} C' \), and introducing the assumption that the speed \( v \) of the particle is non-relativistic, so that \( v \ll 1 \) (hence terms smaller than \( ev \) will be neglected), (5.4) can be written in terms of \( C' \) as

\[
\nabla_{C'}C' + v \partial_t v C' + (i_{C'} F_{C'} + \frac{1}{2} d_{C'} \text{Tr}(\Sigma_{C'}))^{\sharp} = 0.
\]

Expressing \( F_{C'} \) and \( \Sigma_{C'} \) in terms of \( \psi_{\partial_t} \) (see Appendix [2] with \( \zeta = C' \) and \( \xi = \partial_t \)) so that

\[
F_{C'} = F_{\partial_t} + d_{\partial_t} (\psi_{\partial_t}(v)) \wedge dt,
\]

\[
\Sigma_{C'} = (\phi_{\partial_t} + 2 \psi_{\partial_t}(v)) g_{\partial_t} - \psi_{\partial_t} \otimes v^b + v^b \otimes \psi_{\partial_t},
\]

it follows that

\[
i_{C'} F_{C'} = -E_{\partial_t} - \#(v^b \wedge B_{\partial_t}) - d_{\partial_t}(\psi_{\partial_t}(v)) + E_{\partial_t}(v) dt,
\]

\[
d_{C'} \text{Tr}(\Sigma_{C'}) = 3 E_{\partial_t} + 4 d_{\partial_t} (\psi_{\partial_t}(v)) + 3 \partial_t \phi_{\partial_t} v - 3 E_{\partial_t}(v) dt,
\]

where (4.38) has been used.
Using (5.6) the spatial part of the equation of motion (5.8) becomes
\[
\frac{dv}{dt} = \frac{1}{4} E^{\|}_\partial + \#(v^\vee \wedge B) \|^2 - \frac{3}{4} \partial_t \phi \partial_t v,
\] (5.11)
and if \( \phi \partial_t \) is time-independent this reduces to
\[
\frac{dv}{dt} = \frac{1}{4} E^{\|}_\partial + \#(v^\vee \wedge B) \|^2.
\] (5.12)
or in vector notation
\[
\frac{dv}{dt} = \frac{1}{4} E_{\partial} + v \times B_{\partial},
\] (5.13)
where \( E_{\partial} = E^{\|}_\partial \) and \( B_{\partial} = B^\|_{\partial} \).
In terms of these fields the perturbed gravitational force \( F_G \) is then
\[
F_G = m \left( \frac{1}{4} E_{\partial} + v \times B_{\partial} \right).
\] (5.14)
This is similar in structure to the Lorentz force law of electromagnetism, except for the factor of \( \frac{1}{4} \) multiplying the gravito-electric term (and the fact that the \( E_{\partial} \) and \( B_{\partial} \) fields couple universally to inertial mass). Note that rescaling \( E_{\partial} \) to remove the factor \( \frac{1}{4} \) would introduce a factor into the gravito-Maxwell equations above.
If one continues to the Newtonian limit then \( B_{\partial} = 0 \) and, using (4.38), (4.49) to express \( E_{\partial} \) in terms of the Newtonian potential \( \Phi_{\partial} \) (which will automatically be time-independent by the gauge condition \( \nabla \cdot \psi = 0 \)), (5.14) reduces to
\[
F_G = -m \text{grad} \Phi_{\partial},
\] (5.15)
which is just Newton’s force of gravity in terms of a gravitational potential \( \Phi_{\partial} \) satisfying (4.51).

6. PRECESSION OF A SMALL GYROSCOPE IN A WEAK GRAVITATIONAL FIELD

One may model the relativistic spin of a freely falling gyroscope by a unit spacelike vector field \( S \) along a future-pointing timelike curve \( C(\tau) \) that satisfies (5.1), such that \( S \) solves
\[
\hat{\nabla}_{C'} S = 0,
\] (6.1)
with
\[
\dot{g}(S, S) = 1,
\] (6.2a)
\[
\dot{g}(S, C') = 0,
\] (6.2b)
in terms of the physical metric tensor. A perturbative analysis can be given in terms of the vector \( s \) defined along \( C(\tau) \) by
\[
s = (1 + \frac{1}{2} h(S, S)) S - h(S, C') C',
\] (6.3)
so that (to first order in \( \epsilon \))
\[
g(s, s) = 1,
\] (6.4a)
\[
g(s, C') = 0,
\] (6.4b)
in terms of the background metric \( g \).
Equation (6.3) can be inverted and to first order in \( \epsilon \)
\[
S = (1 - \frac{1}{2} h(s, s)) s + h(s, C') C'.
\] (6.5)
Using the perturbed connection, (6.1) can be written in terms of \( s \) to first order in \( \epsilon \) as
\[
\nabla_{C'} s - \frac{1}{2} C' (h(s, s)) s + C'(h(s, C')) C' + \gamma(C', s, -) = 0.
\] (6.6)
Rewriting \( \gamma(C', s, -) \) in terms of \( F_{C'} \) and \( \Sigma_{C'} \) (see Appendix [3]), this becomes
\[
\nabla^F_{C'} s + \frac{1}{2} (\Pi_{C'} i_s F_{C'} - \Pi_s \nabla_{C'} (\Sigma_{C'} (s, -))) s = 0,
\] (6.7)
where for any vector field $X$, the Fermi-Walker connection $\nabla^F$ is defined on $C$ by
\[
\nabla^F_{C'} X = \nabla_{C'} X + g(C',X)A_{C'} - g(A_{C'},X)C',
\]
with $A_{C'} = \nabla_{C'} C'$. From (6.4) the effective gravitational torque $T_G$ on the gyroscope is
\[
T_G = -\frac{1}{2}(\Pi_{C'ves} + \Pi_5 \nabla C' (\Sigma C'(s,-)))^t.
\]

The vector $s$ is said to be non-rotating along the path $C$ when $T_G = 0$. As with the point particle, if this curve is parameterized in terms of $t$ so that $C'(t)$ is given by (5.6) (and dropping terms smaller than $\epsilon v$), (6.7) becomes
\[
\nabla^F_{C'} s + \frac{1}{2}(\Pi_{C'ves} + \Pi_5 \nabla C' (\Sigma C'(s,-)))^t = 0.
\]
The vector $s$ can be written in terms of its spacelike component $\sigma$ as
\[
s = g(\sigma,v) \partial_t + \sigma,
\]
which follows from (6.4a). Imposing the gravito-electromagnetic limit in the $\partial_t$ frame, and writing $F_{C'}$ and $\Sigma C'$ in terms of $E_{\partial_t}, B_{\partial_t}$ and $\phi_{\partial_t}$ (see Appendix C), the spacelike component of (6.10) then becomes
\[
\frac{d\sigma}{dt} = \frac{1}{2} \left( \sigma \times B_{\partial_t} + (\sigma \cdot v) E_{\partial_t} - \frac{1}{2} (\sigma \cdot E_{\partial_t}) v + (C' \phi_{\partial_t}) \right).
\]

Since the length $\sqrt{\sigma \cdot \sigma}$ of the vector $\sigma$ is dependent on $t$ (to the appropriate order), introduce the constant length gyroscopic spin vector $S$ by
\[
S = (1 - \frac{1}{2} \phi_{\partial_t}) \sigma - \frac{1}{2} (v \cdot \sigma) v.
\]
In terms of $S$, (6.12) can be written
\[
\frac{dS}{dt} = \frac{1}{2} S \times \left( B_{\partial_t} - \frac{3}{4} v \times E_{\partial_t} \right)
\]
which may be compared with the similar equation found in (11).

To this approximation the precession rate $\frac{1}{2} (\frac{3}{4} v \times E_{\partial_t} - B_{\partial_t})$ of $S$ is independent of the gyroscopic spin and could in principle be used to detect $B_{\partial_t}$.

7. Field Redefinitions

In order to facilitate a comparison with alternative formulations of weak gravity using the gravito-electromagnetic analogy it is worthwhile to effect certain field redefinitions. These are motivated by looking at the analogues of the transformations (3.37) with (4.5).

Let $h_\xi = h(\xi,-)$ and $\xi$ be a gravito-electromagnetic frame, then under a gravito-electromagnetic gauge transformation $h_\xi$ transforms as
\[
h_\xi \mapsto h_\xi + d\lambda - (\xi \lambda) \xi^\flat,
\]
which is not quite of the same form as (2.3). However, defining
\[
\Phi_h = -\frac{1}{2} h(\xi,\xi),
\]
and
\[
A_h = \Pi_\xi h_\xi^\flat,
\]
it follows, writing $\xi = \partial_t$, that
\[
\Phi_h \mapsto \Phi_h - \partial_t \lambda,
\]
\[
A_h \mapsto A_h + \text{grad} \lambda
\]
which is indeed analogous to the (3+1)-decomposed form of an electromagnetic gauge transformation.

Similarly $\psi_\xi$ transforms as in (4.7) and defining
\[
\Phi_\psi = -\phi_\xi = -\psi(\xi,\xi),
\]
the (3+1)-decomposed form of the gauge transformation (7.4) becomes

\[ \Phi_\psi \mapsto \Phi_\psi - \partial_t \lambda, \]
\[ A_\psi \mapsto A_\psi + \text{grad} \lambda. \] (7.5a, 7.5b)

Since (7.3), (7.5) are analogous to electromagnetic gauge transformations this suggests that a Maxwell type system of field equations may be constructed using \( \Phi_h \) and \( A_h \) instead of \( \Phi_\psi \) and \( A_\psi \).

From (4.1) and the above definitions of \( \Phi_h \), \( \Phi_\psi \), \( A_h \) and \( A_\psi \), it follows that

\[ \Phi_\psi = \Phi_h - \frac{1}{4} \text{Tr}(\Sigma), \]
\[ A_\psi = A_h, \] (7.6a, 7.6b)

where \( \Sigma = \Sigma_\xi \).

The Lorenz gauge condition in (4.28a) written in terms of \( \Phi_\psi \) and \( A_\psi \) becomes

\[ \text{div} A_\psi + \partial_t \Phi_\psi = 0, \] (7.7a)
which has the same form as in electromagnetism for all \( \Sigma \). In terms of \( \Phi_h \) and \( A_h \) this becomes

\[ \text{div} A_h + \partial_t \Phi_h = \frac{1}{4} \text{Tr}(\Sigma), \] (7.7b)

which is unlike the electromagnetic Lorenz gauge condition when the right-hand side is non-zero.

The gravito-electric and -magnetic fields \( E_\psi = E_\xi \) and \( B_\psi = B_\xi \) discussed earlier are related to \( \Phi_\psi \) and \( A_\psi \) by

\[ E_\psi = -\text{grad} \Phi_\psi - \partial_t A_\psi, \] (7.8a)
\[ B_\psi = \text{curl} A_\psi. \] (7.8b)

Alternative gravito-electromagnetic fields \( E_h \) and \( B_h \) can be defined in terms of \( \Phi_h \) and \( A_h \) as

\[ E_h = -\text{grad} \Phi_h - \partial_t A_h, \] (7.9a)
\[ B_h = \text{curl} A_h. \] (7.9b)

These are related to \( E_\psi \) and \( B_\psi \) by

\[ E_\psi = E_h + \frac{1}{4} \text{grad} \text{Tr}(\Sigma), \] (7.10a)
\[ B_\psi = B_h. \] (7.10b)

In terms of \( E_\psi \) and \( \Sigma \) the second gauge condition (4.28b) can be written as

\[ E_\psi = (\nabla \cdot \Sigma)^2, \] (7.11a)
or using \( E_h \) and \( \Sigma \) as

\[ E_h = (\nabla \cdot \Sigma)^2 - \frac{1}{4} \text{grad} \text{Tr}(\Sigma). \] (7.11b)

In the \( E_\psi \) and \( B_\psi \) notation , with \( \kappa = 8\pi G \), \( \rho = \rho_\xi \) and \( J = J_\xi \) the field equations (4.34) become

\[ \text{div} B_\psi = 0, \] (7.12a)
\[ \text{curl} E_\psi + \partial_t B_\psi = 0, \] (7.12b)
\[ \text{div} E_\psi = -16\pi G \rho, \] (7.12c)
\[ \text{curl} B_\psi - \partial_t E_\psi = -16\pi G J \] (7.12d)

for all \( \Sigma \). As stressed above these equations together with (4.17), (7.7a), (7.11a) summarise linearised Einsteinian gravitation.

Alternatively using (7.10a) and (7.10b) the above take the form

\[ \text{div} B_h = 0, \] (7.13a)
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\begin{align}
\text{curl} \mathbf{E}_h + \partial_t \mathbf{B}_h &= 0, \quad (7.13b) \\
\text{div} \mathbf{E}_h &= -16\pi G\rho - \frac{1}{4} \text{div grad Tr}(\Sigma), \quad (7.13c) \\
\text{curl} \mathbf{B}_h - \partial_t \mathbf{E}_h &= -16\pi G J + \frac{1}{4} \partial_t \text{grad Tr}(\Sigma) \quad (7.13d)
\end{align}

in terms of \( \mathbf{E}_h \) and \( \mathbf{B}_h \) and \( \Sigma \). In the gravito-electromagnetic limit, \( \text{Tr}(\Sigma) = -12\Phi_h \) so \( \Phi_\psi = 4\Phi_h \) and \( \mathbf{E}_\psi = 4\mathbf{E}_h \). Then the Lorenz gauge condition \( (7.7b) \) becomes

\begin{align}
\text{div} \mathbf{A}_h + 4\partial_t \Phi_h &= 0, \quad (7.14)
\end{align}

and (since \( \partial_t \mathbf{B}_h = 0 \) in the gravito-electromagnetic limit) the field equations \( (7.13) \) simplify to

\begin{align}
\text{div} \mathbf{B}_h &= 0, \quad (7.15a) \\
\text{curl} \mathbf{E}_h &= 0, \quad (7.15b) \\
\text{div} \mathbf{E}_h &= -4\pi G\rho, \quad (7.15c) \\
\text{curl} \mathbf{B}_h - 4\partial_t \mathbf{E}_h &= -16\pi G J \quad (7.15d)
\end{align}

which may be compared with \( (4.44) \) and the formulations given in \cite{23}, \cite{11}.

8. Discussion

In this article the analogy between Maxwell’s equations for the electromagnetic field, \( (2.1a), (2.1b) \) and the Einstein equations for weak gravitational fields in the transverse gauge, \( (4.20), (4.29) \) has been made in terms of tensor fields. While the former are valid in an arbitrary Lorentzian spacetime the latter have been developed in terms of perturbations about a flat spacetime background. A comparison has been made between the general equation describing the motion of electrically charged point particles \( (2.8) \) and the motion of massive point particles in a weak gravitational field \( (5.4) \). Equations have also been developed \( (6.14) \) describing the motion of a freely falling small gyroscope in terms of gravito-electromagnetic fields.

In general it is asserted that any analogy between electromagnetism and weak gravity is closest in a restricted class of reference frames related by suitable non-relativistic transformations and for stationary physical field configurations in such frames. In addition to gravito-electromagnetic fields the general equations of weak gravity involve a second degree symmetric tensor field \( \Sigma_{\xi} \) which has no electromagnetic analogue. The gravito-electromagnetic fields defined in \( (4.2) \) are coupled to \( \Sigma_{\xi} \) via \( (4.28b) \) and this field produces non-Maxwellian terms in the weak gravitational force and torque equations, \( (5.5) \) and \( (5.5) \) respectively.

In electromagnetism the Maxwell fields \( A, F \) and \( J \) are defined independent of any frame of reference. The latter is only required to define electric and magnetic fields and their sources in terms of electric charge and current density \( (2.10), (2.11) \). In gravito-electromagnetism the analogous fields \( \psi_\xi, \mathcal{F}_\xi \) and \( \mathcal{J}_\xi \) are manifestly frame-dependent.

The definition of these fields has been motivated by their behaviour under a class of gauge transformations belonging to the gauge symmetry of the weak Einstein equations. Unlike electromagnetism, \( \mathcal{F}_\xi \) is not gauge invariant under these transformations in general. However a subset of these transformations does exist for which \( \mathcal{F}_\xi \) remains invariant. These have been called gravito-electromagnetic gauge transformations by analogy with the gauge symmetry of Maxwell’s equations. Unlike electromagnetic interactions with electrically charged particles, weak gravitational interactions are not mediated by complex representations of these symmetries. In terms of the gravito-electromagnetic fields a subset of the linearised Einstein equations take a remarkable form that is isomorphic to Maxwell’s electromagnetic field equations. No limits or further approximations are required to establish this correspondence. The explicit appearance of the \( \Sigma_{\xi} \) tensor is confined to the remaining equations in the linearised system. This is a primary distinction of the approach adopted here compared with previous derivations of the gravito-electromagnetic field equations.

To exploit this reformulation and link physical field configurations with solutions to Maxwell’s equations further conditions must be imposed on the linearised system.

Conditions have been found that enable a useful analogy between weak gravitation and electromagnetism to be established. In the gravito-electromagnetic limit \( \Sigma_{\xi} \) depends only on the a gravito-electromagnetic
potential (4.36). Consequently, such weak gravitational fields can be described in terms of $\psi$ (or alternatively $E_\xi$ and $B_\xi$).

However as stressed above, in order to obtain Maxwell-like equations the transverse gauge condition (4.11) is imposed on $\psi$. Condition (4.27) induces an equivalent condition (4.28a) on $\psi_\xi$. (By contrast the electromagnetic gauge condition (2.6) is one of many that may be imposed on $A_\xi$.) The condition on $\psi$ also imposes the restriction (4.28b) which in the gravito-electromagnetic limit implies that $\psi_\xi$ has a restricted time dependence. Physical gravito-electromagnetism consequently shares more in common with electromagnetostatics than electromagnetism. Although both the Maxwell equations (2.12) and the equivalent gravito-electromagnetic equations (4.34) hold in any inertial frame (in Minkowski spacetime), in order to remain within the gravito-electromagnetic limit only a class of gravito-electromagnetic frames of reference is permitted (Appendix C).

By perturbing the equation of a physical timelike geodesic the relativistic equation of motion for a massive point particle in the weak gravitational field can be cast into a form containing a gravito-electromagnetic Lorentz force (5.5) and an additional non-Maxwellian term. It is worth pointing out that in the context of the gravito-electromagnetic fields defined in this paper the derivation of this equation of motion does not rely on the speed of the particle and the gravito-electromagnetic Lorentz-like force takes its natural form. In the gravito-electromagnetic and non-relativistic limit the particle acceleration is then determined by a non-relativistic Lorentz-like force (5.14) containing an additional factor of $\frac{1}{4}$ multiplying the gravito-electric field.

When working in the gravito-electromagnetic limit the field redefinitions presented in section 7 permit a comparison with the work of Thorne in [23] with the notation $\Phi_h = \Phi$, $A_h = \gamma$, $E_h = g$, and $B_h = H$, and with a current of the form $J = \rho v$. The work of [11], [16], [4], [24] and others may be related to that of [23] either by trivial field redefinitions or changes in metric signature. The analogy between electromagnetism and weak gravity developed by Wald [25] is similar to the one presented here. However he does not discuss how the gravito-electromagnetic 1-form-potential behaves under gauge transformations nor how the restricted time dependence arises from the transverse gauge condition.

This article offers an alternative description of weak field gravitation in the language of gravito-electromagnetic fields. We feel that it clarifies a number of issues concerning various other analogies between the equations of post-Newtonian gravity according to Einstein and Maxwell’s description of electromagnetism. In the absence of a gravito-electromagnetic limit formulations based on $\Phi_h$ and $A_h$ give rise to gauge conditions (7.7b), (7.11b) and field equations (7.13c), (7.13d) thereby exposing couplings between the gravito-electromagnetic potentials and fields and the tensor field $\Sigma_{\xi}$. In the approach adopted here such couplings are relegated to the gauge condition (7.11a). Many analogies coalesce in the gravito-electromagnetic limit modulo a re-shuffling of numerical factors that cannot be scaled away entirely. Although the mathematical analogy between weak gravity and the full system of Maxwells equations for electromagnetism in terms of covariant tensor fields on flat spacetime can be made close, the existence of gravitational gauge conditions limits the physical analogy to stationary phenomena. Despite this limitation the interpretation of weak gravity in terms of gravito-electromagnetic fields offers a fertile avenue of exploration for phenomena associated with the detection of the stationary gravito-magnetic field. The methods presented here are also applicable in principle to certain non-flat backgrounds and to weak field descriptions of non-Einsteinian gravitation (in which other geometrical fields may compete with metric-induced gravity) at the post-Newtonian level. These issues will be discussed elsewhere.

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2Fields derived from a potential $\phi_\xi$ that vary linearly with $t$ are deemed unphysical here.


Appendix A. Definitions and Notation

Throughout this article the geometry of spacetime is described in terms of a smooth Lorentzian metric tensor field and its associated Levi-Civita connection. The connection 1-forms $\omega^a_b$ associated with any such connection $\nabla$ are defined with respect to any local basis of vector fields $\{X_a\}$ where $a = 0, 1, 2, 3$ by

$$\nabla_{X_a} X_b = \omega^a_b(X_a) X_c.$$  \hfill (A.1)

If $T$ is a smooth tensor field on spacetime then in terms of this connection the covariant differential $\nabla$ of $T$ is defined by

$$(\nabla T)(X,-,...,-) = (\nabla_X T),$$  \hfill (A.2)

and if the first argument of $T$ is contravariant, the divergence of $T$ is

$$\nabla \cdot T = (\nabla_{X_a} T)(X_a,-,...,-),$$  \hfill (A.3)

where indices are raised with the components of the metric tensor in the usual way. Similarly with $\{e^a\}$ dual to $\{X_a\}$ (i.e. $e^a(X_b) = \delta^a_b$)

$$\nabla \cdot S = (\nabla_{X_a} S)(e^a,-,...,-),$$  \hfill (A.4)

for all tensors $S$ whose first argument is covariant. For any degree 2 covariant tensor $T$ the tensor $\text{Sym} T$ is defined by $\text{Sym} T(X,Y) = \frac{1}{2}(T(X,Y) + T(Y,X))$.

In terms of the exterior derivative $d$ on smooth differential $p$-forms on spacetime, the covariant exterior derivative is defined on mixed basis indexed $p$-forms by

$$dS^{a...b}_{c...d} = dS^{a...b}_{c...d} + \omega^a_{s} \wedge S^{a...b}_{s...c...d} + ... + \omega^b_{s} \wedge S^{a...b}_{a...s...c...d} - \omega^s_{c} \wedge S^{a...b}_{s...c...d} - ... - \omega^s_{d} \wedge S^{a...b}_{a...s...c...d}.$$  \hfill (A.5)

The curvature operator $\mathbf{R}$ for the connection $\nabla$ is

$$\mathbf{R}_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$  \hfill (A.6)

for all vector fields $X, Y$, and the (Riemann) curvature tensor $R$ defined by

$$R(X,Y,Z,\alpha) = \alpha(\mathbf{R}_{X,Y} Z).$$  \hfill (A.7)

Then in dual local bases $\{X_a\}$ and $\{e^a\}$ the curvature 2-forms $R^{a}_{b}$ are defined by

$$R = 2R^{a}_{b} \otimes e^b \otimes X_a.$$  \hfill (A.8)

Successive contractions (where $i_X$ is the interior operator with respect to the vector $X$) give the Ricci 1-forms

$$P_b = i_{X_a} R^{a}_{b},$$  \hfill (A.9)

and the curvature scalar $\mathcal{R}$

$$\mathcal{R} = i_{X_a} P_a.$$  \hfill (A.10)

With $g$ a metric tensor on vectors and $G$ the induced metric tensor on 1-forms, the metric-dual of a vector $X$ is given by $X^\flat = g(X,-)$ and that of and a 1-form $\alpha$ by $\alpha^\flat = G(\alpha,-)$.

If $U$ is a unit-normalized vector field ($g(U,U) = \lambda$ where $\lambda = \pm 1$), then the projection operator on contravariant tensor fields is defined as

$$\Pi_U = 1 - \lambda U \otimes U^\flat.$$  \hfill (A.11a)

By abuse of notation the same symbol is used to denote the projector on covariant tensors

$$\Pi_U = 1 - \lambda U^\flat \otimes U,$$  \hfill (A.11b)

and differential $p$-forms

$$\Pi_U = 1 - \lambda U^\flat \wedge i_U,$$  \hfill (A.11c)

since the domain should be clear from the context. The map $\Pi_U$ is a tensor homomorphism

$$\Pi_U(\alpha \otimes \ldots \otimes \beta) = \Pi_U \alpha \otimes \ldots \otimes \Pi_U \beta,$$  \hfill (A.11d)

for all $\alpha, \beta, ...$ and has the following properties:

$$\Pi_U \Pi_U = \Pi_U,$$  \hfill (A.12a)
\[ \Pi_U i_U = i_U, \quad \Pi_U V^b = 0. \]  

The projection tensor \( \Pi_V \) is used to define the \textit{spatial metric} on spacetime associated with the timelike vector field \( V \):

\[ g_V = \Pi_V g. \]  

One can write \( \nabla V^b \) in terms of \( g_V \) and \( \Pi_V \) as follows:

\[ \nabla V^b = \sigma_V + \Omega_V + \frac{1}{3} \Theta_V g_V - V^b \otimes A_V. \]  

where

\[ \sigma_V = \text{Sym} \Pi_V \nabla V^b - \frac{1}{3} \Theta_V g_V, \]

is the \textit{shear} of \( V \),

\[ \Theta_V = \nabla \cdot V, \]

is the \textit{expansion} of \( V \),

\[ \Omega_V = \Pi_V dV^b, \]

is the \textit{vorticity} of \( V \) and

\[ A_V = \nabla V \]

is the \textit{acceleration} of \( V \).

A vector field \( \xi \) is said to be \textit{parallel} (with respect to \( \nabla \)) if

\[ \nabla \xi = 0, \]  

in which case \( \nabla \xi^b = 0 \), so \( \xi \) has vanishing shear, vorticity, expansion and acceleration. Furthermore

\[ \text{Sym} \nabla \xi^b = 0, \]

or equivalently

\[ L_\xi g = 0 \]

in terms of the Lie derivative. Thus, if \( \xi \) is parallel then it is also a Killing vector and

\[ \nabla \xi = L_\xi. \]

For any tensor field \( T \) taking at least one covariant argument let \( T_\xi \) be defined by contraction such that

\[ T_\xi(-,\ldots,-) = T(-,\ldots,-,\xi,-\ldots,-). \]

If \( \xi \) is parallel it follows that

\[ (\nabla_X T_\xi)(-,\ldots,-) = (\nabla_X T)(-,\ldots,-,\xi,-\ldots,-) \]

for all vector fields \( X \). When acting on \( p \)-forms this is simply the rule:

\[ i_\xi \nabla_X = \nabla_X i_\xi. \]

The metric tensor \( g \) gives rise to a canonical \textit{volume} 4-form \( *1 \) on spacetime and an associated Hodge map \( * \) on \( p \)-forms. In terms of a local \( g \)-orthonormal local basis of 1-forms \( \{e^a\} \) one may write \( *1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \).

A volume 3-form \( #1 \) associated with the unit timelike vector field \( V \) is given in terms of \( *1 \) by

\[ *1 = V^b \wedge #1 \]

with \( i_V #1 = 0 \). Hence

\[ #1 = - * V^b \]

and \( #1 \) induces a spatial Hodge map, \( # \), on the image of forms under \( \Pi_V \). With these operations defined one can decompose the Hodge map of any spacetime form and its exterior derivative into spatial forms. Thus the Hodge dual of \textit{any} \( p \)-form \( \omega \) on spacetime may always be written

\[ *\omega = \left\{ #i_V \omega \right\} + V^b \wedge \left\{ #(\Pi_V \omega) \right\} \]

(A.25)
where $\alpha^p = (-1)^p \alpha$ for any $p$-form $\alpha$ and each of the terms in brackets is annihilated by $V$ (i.e. $i_V \{ \} = 0$). A projected Lie derivative with respect to $V$ is defined as

$$\mathcal{L}_V = \Pi_V \mathcal{L}_V \Pi_V$$  \hfill (A.26)

and for any $p$-form $\omega$ one can write

$$\mathcal{L}_V \omega = \left\{ \mathcal{L}_V \omega - \mathcal{A}_V^p \wedge i_V \omega \right\} - V^p \wedge \left\{ \mathcal{L}_V i_V \omega \right\}$$  \hfill (A.27)

where each of the terms in brackets is annihilated by $V$.

In terms of the projected exterior derivative

$$d_V = \Pi_V d \Pi_V,$$  \hfill (A.28)

and with $D_V$ defined by

$$D_V = d_V + \mathcal{A}_V^p \wedge$$  \hfill (A.29)

one may write

$$d\omega = \left\{ d_V \omega - \Omega_V \wedge i_V \omega \right\} - V^p \wedge \left\{ (\mathcal{L}_V \omega - D_V i_V \omega) \right\}$$  \hfill (A.30)

where each of the terms in brackets is annihilated by $V$. These formulae permit a local “3+1” decomposition of exterior differential equations with respect to the general observer field $V$ in a spacetime with an arbitrary Lorentzian metric and permit one to identify spatial fields parametrised by a local time associated with $V$.

With the Faraday $2$-form $F$ decomposed as

$$F = V^p \wedge e + \# b$$  \hfill (A.31)

where $i_V e = 0$, and $i_V b = 0$, it follows that $i_V F = -e$ and $\Pi_V F = \# b$. Similarly using (A.25)

$$*F = V^p \wedge b - \# e,$$  \hfill (A.32)

so $i_V * F = -b$ and $\Pi_V F = \# e$. Writing

$$\mathcal{J} = \rho V^p + j$$  \hfill (A.33)

where $i_V j = 0$, and using (A.27) it follows that

$$*\mathcal{J} = -\rho \# 1 - V^p \wedge \# j.$$  \hfill (A.34)

The co-derivative $\delta$ is defined on spacetime $p$-forms in terms of the Hodge map $*$ and the exterior derivative $d$ by

$$\delta = *^{-1} d * \eta$$  \hfill (A.35)

where $\eta \omega = \omega^\eta$ for any $p$-form $\omega$. The field equations (2.1a) and (2.1b) can be written as

$$dF = 0,$$  \hfill (A.36a)

$$d * F = * \mathcal{J}.$$  \hfill (A.36b)

Using (A.30) the Maxwell equations given in (2.12) follow immediately.
With the definitions:

\[ \text{div } \mathbf{X} = \left( \#d_{\mathbf{V}} \# \mathbf{X}^\flat \right)^\sharp, \]  
(A.37a)

\[ \text{curl } \mathbf{X} = \left( \#d_{\mathbf{V}} \mathbf{X}^\flat \right)^\sharp, \]  
(A.37b)

\[ \text{grad } \Phi = (d_{\mathbf{V}} \Phi)^\sharp, \]  
(A.37c)

\[ \mathbf{X} \times \mathbf{Y} = \left( \mathbf{X}^\flat \wedge \mathbf{Y}^\flat \right)^\sharp, \]  
(A.37d)

\[ \mathbf{X} \cdot \mathbf{Y} = g_{\mathbf{V}}(\mathbf{X}, \mathbf{Y}) = \#(\mathbf{X}^\flat \wedge \# \mathbf{Y}^\flat) \]  
(A.37e)

where \( \mathbf{X} \) and \( \mathbf{Y} \) are spacelike (with respect to \( \mathbf{V} \)) vectors, and \( \Phi \) is a 0-form on spacetime, exterior equations can be transcribed to Euclidean vector notation.

The Laplacian operator \( \text{Lap} \) associated with \( g \) and \( \nabla \) is defined by

\[ \text{Lap} = \nabla \cdot \nabla, \]  
(A.38)

and the associated Laplace-Beltrami operator \( \Delta \) on \( p \)-forms is

\[ \Delta = - (\delta d + d \delta). \]  
(A.39)

It can be shown that, when acting on any \( p \)-form \( \alpha \), \( \text{Lap} \) is related to the Laplace-Beltrami operator \( \Delta \) by

\[ \Delta \alpha = \text{Lap} \alpha + e_a^i \wedge i_{X^a} R_{X^b} X^i \alpha. \]  
(A.40)

Thus if \( \beta \) is a 1-form

\[ \Delta \beta = \text{Lap} \beta - \text{Ric}(\beta^i, -) \]  
(A.41)

and for any 0-form \( f \)

\[ \Delta f = \text{Lap} f. \]  
(A.42)

In a spacetime with a flat metric

\[ \Delta = \text{Lap} \]  
(A.43)

for all \( p \)-forms.
Appendix B. The Perturbed Connection and Gravitoelectromagnetic Fields

With the $(3,0)$ tensor field $\gamma^\flat$ defined by
\[
\gamma^\flat(X,Y,Z) = \gamma(X,Y,Z),
\] (B.1)
using (3.17) it follows that
\[
\gamma^\flat(\xi,-,-) = \frac{1}{2} \nabla_\xi h + \text{Alt}(\nabla h)(-,-,-),
\] (B.2)
where $\xi$ defines a gravito-electromagnetic frame (unit timelike and parallel, to zero order in $\epsilon$), and for any type $(2,0)$ tensor $T$, $\text{Alt} T$ is defined by $\text{Alt} T(X,Y) = \frac{1}{2} (T(X,Y) - T(Y,X))$.

To relate expression (B.2) to gravito-electromagnetic fields one may proceed as follows. By trace-reversing $\psi$ as given in (4.1), $h$ can be written as
\[
h = -\psi_\xi \otimes \xi^\flat - \xi^\flat \otimes \psi_\xi - \Sigma_\xi + \frac{1}{2} \text{Tr}(\Sigma_\xi)g,
\] (B.3)
and acting with $\nabla_\xi$ gives
\[
\nabla_\xi h = -\nabla_\xi \psi_\xi \otimes \xi^\flat - \xi^\flat \otimes \nabla_\xi \psi_\xi - \nabla_\xi \Sigma_\xi + \frac{1}{2} \xi \text{Tr}(\Sigma_\xi)g.
\] (B.4)
Since $\nabla_\xi$ is first order in $\epsilon$ it follows that
\[
(\nabla h)(-,-,\xi,-) = \nabla(h(\xi,-,-)),
\] (B.5)
and contracting (B.3) with $\xi$ gives
\[
h(\xi,-) = \psi_\xi - (\phi_\xi - \frac{1}{2} \text{Tr}(\Sigma_\xi))\xi^\flat.
\] (B.6)
Hence
\[
(\nabla h)(-,-,\xi,-) = \nabla \psi_\xi - d(\phi_\xi - \frac{1}{2} \text{Tr}(\Sigma_\xi)) \otimes \xi^\flat.
\] (B.7)
Antisymmetrizing this yields
\[
\text{Alt}(\nabla h)(-,-,\xi,-) = F_\xi + \xi^\flat \wedge d(\phi_\xi - \frac{1}{2} \text{Tr}(\Sigma_\xi)),
\] (B.8)
where $F_\xi = d\psi_\xi$.

Equation (B.2) can now be written as
\[
\gamma^\flat(\xi,-,-) = - (\nabla_\xi \psi_\xi \otimes \xi^\flat + \xi^\flat \otimes \nabla_\xi \psi_\xi + \nabla_\xi \Sigma_\xi)
+ \frac{1}{2} \xi \text{Tr}(\Sigma_\xi)g + F_\xi + \xi^\flat \wedge d(\phi_\xi - \frac{1}{2} \text{Tr}(\Sigma_\xi)),
\] (B.9)
and contracting on $\xi$ gives
\[
\gamma^\flat(\xi,\xi,-) = -\xi(\phi_\xi - \frac{1}{2} \text{Tr}(\Sigma_\xi))\xi^\flat + i_\xi F_\xi + \frac{1}{4} d_\xi \text{Tr}(\Sigma_\xi).
\] (B.10)

If $X$ is orthogonal to $\xi$, $g(\xi,X) = 0$, and $\nabla X$ is first order or higher in $\epsilon$, then $\gamma^\flat(\xi,X,-)$ can be rewritten as
\[
\gamma^\flat(\xi,X,-) = - \xi(\psi(X))\xi^\flat + \xi(\frac{1}{2} \text{Tr}(\Sigma_\xi) + \Sigma_\xi(X,X))X^\flat
+ i_X(i_\xi F_\xi + \frac{1}{4} d_\xi \text{Tr}(\Sigma_\xi)) \xi + \frac{1}{2} [\xi, i_X F_\xi] + [\xi, i_X \nabla_\xi \Sigma_\xi(X,-)].
\] (B.11)

Let $\xi$ be a vector field such that $\nabla_\xi = 0$ and $g(\xi,\xi) = -1$ (to at least first order in $\epsilon$). Using (B.11) and adding hats to quantities defined with respect to the physical metric $\hat{g}$ it follows that
\[
\hat{\nabla}_\xi = \frac{\xi}{\hat{\gamma}(\xi,-,-)},
\] (B.12)
or with $X^\flat = \hat{G}(X,-)$, that:
\[
\hat{\nabla}_\xi \xi^\flat = \gamma^\flat(\xi,-,-),
\] (B.13)
where $\gamma^\flat(X,Y,Z) = \gamma(X,Y,Z^\flat)$. Antisymmetrizing and using (B.9) yields
\[
d_\xi \xi^\flat = F_\xi + \xi^\flat \wedge d(\phi_\xi - \frac{1}{2} \text{Tr}(\Sigma_\xi)).
\] (B.14)
The left hand side can be contracted on $\xi$ and rewritten as
\[
i_\xi d_\xi \xi^\flat = \hat{A}_\xi \xi^\flat - \frac{1}{2} d(h(\xi,\xi)),
\] (B.15)
where \( \hat{\mathbf{A}}_\xi = \hat{\nabla}_\xi \xi \). Contracting (B.6) with \( \xi \) and using (B.14) it follows that
\[
\hat{\mathbf{A}}_\xi \mathbf{=} -e_\xi + \frac{1}{2} d_\xi \text{Tr} (\Sigma_\xi)) + \frac{1}{2} \xi \text{Tr} (\Sigma_\xi) \xi^b.
\] (B.16)

Defining
\[
\hat{\Omega}_\xi = \hat{\Pi}_\xi d_\xi^b,
\] (B.17)

and projecting (B.14) gives
\[
\hat{\Omega}_\xi = \# B_\xi.
\] (B.18)

In the gravito-electromagnetic limit and with \( \xi \phi \xi = 0 \) (B.16) simplifies to
\[
\hat{\mathbf{A}}_\xi = -\frac{1}{4} e_\xi.
\] (B.19)

Thus the gravito-electromagnetic fields can be interpreted in terms of the vorticity and acceleration of the vector field \( \xi \) with respect to \( \hat{g} \).
Let $\zeta$ and $\xi$ define two gravito-electromagnetic frames (as defined in section 4.1) such that
\[ \zeta = \xi + v, \] (C.1)
where $g(\xi, v) = 0$ and $v^2 = g(v, v) \ll 1$.

Since $\psi$ is independent of any frame of reference it may be expanded as
\[ \psi = \phi_\xi g - \psi_\xi \otimes \zeta^b - \zeta^b \otimes \psi_\xi - \Sigma_\xi, \] (C.2a)
in terms of $\zeta$ or as
\[ \psi = \phi_\xi g - \psi_\xi \otimes \zeta^b - \zeta^b \otimes \psi_\xi - \Sigma_\xi \] (C.2b)
in terms of $\xi$. Equating these two expressions yields
\[ \psi_\xi = \psi_\xi - \psi_\xi (v) \zeta^b \]
\[ + \left\{ \phi_\xi v^b - \Sigma_\xi (v, -) \right\}, \] (C.3a)
\[ \Sigma_\xi = \Sigma_\xi + 2 \psi_\xi (v) g_\xi - \psi_\xi \otimes v^b - v^b \otimes \psi_\xi \]
\[ - \left\{ \phi_\xi v^b - \Sigma_\xi (v, -) \right\} \otimes \zeta^b - \zeta^b \otimes \left\{ \phi_\xi v^b - \Sigma_\xi (v, -) \right\}. \] (C.3b)

Thus if $v$ is of order $\epsilon$ the fields $\psi_\xi$ and $\Sigma_\xi$ are invariant under a change of gravito-electromagnetic frame to first order in $\epsilon$.

If the gravito-electromagnetic limit is satisfied in the $\xi$ frame, the terms in braces vanish and the above simplify to
\[ \psi_\xi = \psi_\xi - \psi_\xi (v) \zeta^b, \] (C.4a)
\[ \Sigma_\xi = \left\{ \phi_\xi + 2 \psi_\xi (v) \right\} g_\xi - \psi_\xi \otimes v^b - v^b \otimes \psi_\xi. \] (C.4b)

If $v$ is of order $\epsilon$ then the gravito-electromagnetic limit is preserved and the above fields remain invariant.
Appendix D. An Alternative Analogy Between Einstein’s Equations and Electromagnetism Based on Properties of the Conformal Tensor

As mentioned in the introduction there exist alternative analogies between Einstein’s theory of gravitation and Maxwell’s electromagnetic equations that do not necessarily require a perturbative approach. One such analogy is summarised here in terms of the conformal tensor on spacetime since it highlights the differences accorded to gravito-electromagnetism by different authors [15], [7], [3], [8], [12].

The fourth order (Weyl) conformal tensor may be written as

$$C_{ab} = -2C_{ab} \otimes (e^a \wedge e^b)$$  \hspace{1cm} (D.1)

where (in any local cobasis \(\{e^a\}\)), the conformal 2-forms \(C_{ab}\) are defined in terms of the curvature 2-forms \(R_{ab}\), the Ricci 1-forms \(P_a\) and the curvature scalar \(R\):

$$C_{ab} = R_{ab} - \frac{1}{2}(P_a \wedge e_b - P_b \wedge e_a) + \frac{1}{6}R e_a \wedge e_b.$$  \hspace{1cm} (D.2)

The covariant exterior derivative of the conformal 2-forms is

$$DC_{ab} = -\frac{1}{2}(Y_a \wedge e_b - Y_b \wedge e_a),$$  \hspace{1cm} (D.3)

where the 2-forms \(Y_a\) are defined by

$$Y_a = D(P_a - \frac{1}{6}dR \wedge e_a).$$  \hspace{1cm} (D.4)

If the geometry of spacetime is determined by Einstein’s equations then these forms may be related to the stress energy-momentum tensor \(\mathcal{T}\) by introducing the 3-forms \(\tau_a\) such that

$$\mathcal{T} = (\ast^{-1}\tau_a) \otimes e^a.$$  \hspace{1cm} (D.5)

Since the connection is torsion-free:

$$Y_a = D(P_a - \frac{1}{6}dR \wedge e_a).$$  \hspace{1cm} (D.6)

It follows immediately from Einstein’s equations written in terms of the Ricci forms \(P_a\) that

$$Y_a = \frac{\kappa}{2}D(\ast^{-1}\tau_a - \frac{1}{3}(i_X \ast^{-1}\tau_c) e_a),$$  \hspace{1cm} (D.7)

where \(\kappa = 8\pi G\).

In terms of the covariant Lie derivative \(L_X\):

$$L_X = Di_X + i_X D,$$  \hspace{1cm} (D.8)

one may show that

$$L_X \ast C_{ab} = \frac{1}{2}Y_b$$  \hspace{1cm} (D.9)

and

$$L_X \ast Y_a = 0.$$  \hspace{1cm} (D.10)

These relations may be compared with the equations for the Faraday 2-form \(F\):

$$\delta F = \mathcal{J},$$  \hspace{1cm} (D.11)

from which there follows the conservation of electric current,

$$\delta \mathcal{J} = 0.$$  \hspace{1cm} (D.12)

In any local cobase \(F\) can be written in terms of its components as

$$F = F_{ab} e^a \wedge e^b,$$  \hspace{1cm} (D.13)

(compare with (D.1)) and \(\mathcal{J}\) as

$$\mathcal{J} = \mathcal{J}_a e^a.$$  \hspace{1cm} (D.14)
In terms of the vector-valued 0-forms $F_{ab}$ and $J_a$ (D.11) and (D.12) can be rewritten as
\begin{align}
L_X F_{ab} &= -\frac{1}{2} J_b, \quad (D.15) \\
L_X J_a &= 0 \quad (D.16)
\end{align}
which are analogous to (D.9) and (D.10).

The use of local coframes is not mandatory to see this correspondence. A purely tensorial formulation follows by writing
\begin{equation}
\nabla \cdot C = -Y, \quad (D.17)
\end{equation}
where
\begin{equation}
Y = e^a \otimes Y_a. \quad (D.18)
\end{equation}
This is the analogue of the Maxwell equation
\begin{equation}
\nabla \cdot F = -\frac{1}{2} J. \quad (D.19)
\end{equation}
Similarly one may write
\begin{equation}
\nabla \cdot Y = 0, \quad (D.20)
\end{equation}
to compare it with the conservation of electric current written in the form
\begin{equation}
\nabla \cdot J = 0. \quad (D.21)
\end{equation}

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