Exact Conformal Scalar Field Cosmologies*

João P. Abreu†
Paulo Crawford‡
and
José P. Mimoso§

Departamento de Física, Universidade de Lisboa
Campo Grande, Ed. C1, piso 4, 1700 Lisboa
Portugal

Abstract

New exact solutions of Einstein’s gravity coupled to a self-interacting conformal scalar field are derived in this work. Our approach extends a solution-generating technique originally introduced by Bekenstein for massless conformal scalar fields. Solutions are obtained for a Friedmann-Robertson-Walker geometry both for the cases of zero and non-zero curvatures, and a variety of interesting features are found. It is shown that one class of solutions tends asymptotically to a power-law inflationary behaviour $S(t) \sim t^p$ with $p > 1$, while another class exhibits a late time approach to the $S(t) \sim t$ behaviour of the coasting models. Bouncing models which avoid an initial singularity are also obtained. A general discussion of the asymptotic behaviour and of the possibility of occurrence of inflation is provided.

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†E-MAIL: fjpabreu@scosysv.cc.fc.ul.pt
‡E-MAIL: crawford@risc1.cc.fc.ul.pt
§E-MAIL: fmimoso@risc1.cc.fc.ul.pt
1 Introduction

The action describing the interaction between the gravitational field and a scalar field is generally constructed using the minimal coupling principle. Only such terms are included that give the curved generalization of the flat Minkowski spacetime form of the equation of motion for the scalar field, i.e., of the Klein-Gordon equation. Terms that contain the spacetime curvature are usually disregarded.

Recently, a non-minimal coupling between the spacetime curvature and a scalar field has attracted a great deal of attention as a possible improvement of the minimal coupling case \([1–10]\). In principle, the consideration of such an interaction allows one to take more properly into account the influence that, in the very early Universe, the extremely high value of the curvature potentially had in the dynamical behaviour of this coupled system. This point of view seems reinforced when the low energy limit of unified theories (e.g., Superstrings, Supergravity, Kaluza-Klein theories, . . . ) is considered. Upon compactification to four spacetime dimensions, the effective action obtained in those theories often exhibits the aforementioned feature \([11]\).

If one requires that such a non-minimal coupling should: (i) involve only the scalar field and not its derivatives, and (ii) be characterised by a dimensionless coupling parameter; one is led uniquely to the following action \([1,2,12]\)

\[
S[g^{ab}, \phi] = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{16\pi} R - \frac{1}{2} \xi \phi^2 R - \frac{1}{2} \partial_a \phi \partial^a \phi - V(\phi) \right],
\]

where \(g\) is the determinant of the metric \(g_{ab}\), \(M_{Pl} = G_N^{-1/2}\) is the Planck mass, \(R\) is the Ricci scalar curvature, \(\phi\) is a real scalar field singlet, \(V(\phi)\) is a general (effective) potential expressing the self-interaction of \(\phi\) and \(\xi\) is a dimensionless parameter that characterises the strength of this finite coupling between the scalar field and the curvature scalar. Note that the \(\frac{1}{2} \xi \phi^2 R\) term in the action (1) is the only conceivable local scalar coupling satisfying the requirements (i), (ii) stated above \([12]\). (Our sign conventions are that \(g_{ab}\) has signature \((-,+,+,+\)) and the Riemann and Ricci tensors are defined as \(R^a_{bcd} = \partial_c \Gamma^a_{bd} - \ldots\), \(R_{ab} = R^c_{acb}\). We use
units such that $\hbar = c = 1$, and usually we shall set $8\pi/M_{Pl}^2 = 8\pi G_N = 1$ for the sake of convenience.)

For $\xi = 0$ we recover from eq.(1) the usual action for a minimally coupled self-interacting scalar field. On the other hand, for $\xi = \frac{1}{6}$ we obtain the conformal coupling case, and if $V(\phi)$ does not contain dimensional scales (i.e., the potential is zero or a quartic power of $\phi$), it can be shown that the Klein-Gordon equation is invariant under a conformal rescaling of the metric and the field itself, i.e., is conformally invariant [13].

From expression (1), we find that its possible to define an effective Newton gravitational constant $G_N^{eff}$, which is given by the inverse of the coefficient of the curvature scalar, depending on the scalar field as

$$G_N^{eff} = \frac{1}{M_{Pl}^2 - 8\pi\phi^2}.$$  \hspace{1cm} (2)

Thus, if the scalar field is assumed homogeneous, the essential role of the non-minimal coupling is to induce a time variation of the Newton gravitational constant.

The theory based on (1) has been studied by a number of authors in several different contexts. Hosotani [1], has discussed how the back-reaction of gravity affects the stability of a scalar field, thus clarifying the conditions for this theory to have an absolutely stable ground state in which $\phi$ is constant and the spacetime background is maximally symmetric (in connection with this issue see also Refs. [2] and [3]). Dolgov and Ford [4], in turn, have investigated the possibility of a damping of the cosmological constant, in an attempt to solve the cosmological constant problem. In Ref. [5] the canonical quantization procedure is applied to (1) and a homogeneous and isotropic minisuperspace model is discussed. Finally, the prospects and consequences of inflation within this theory have been considered in a variety of works [6–8]. However, there aren’t many exact solutions in this context [9,10,14].

Here, we aim at addressing this question in the framework of the conformal coupling case. We obtain new exact solutions of the coupled Einstein–self-interacting conformal scalar field equations in the particular case of a spatially
homogeneous field propagating both in flat and curved Friedmann-Robertson-Walker (FRW) spacetime backgrounds. This is of considerable interest to assess the cosmological implications of this theory and is done by extending a solution-generating technique originally introduced by Bekenstein for massless conformal scalar fields, which is based on the use of a conformal transformation [14]. This device, discussed at length in Refs. [8,14,15], enables one to obtain a representation of the theory in which the original coupled system is described by Einstein’s gravity plus a redefined self-interacting scalar field, which is now a minimally coupled one. This new frame, sometimes called the Einstein frame, is particularly suitable to the search of new exact solutions (e.g., following the strategy recently adopted by Ellis and Madsen [16]). Given a solution in this new frame, the extended solution-generating technique allows one to get a solution in the physical frame.

This paper is organized as follows. In the next Section, we derive and review the general field equations for the non-minimal coupling model characterised by the action (1). We also present here the equations that govern the evolution of a FRW spacetime in the presence of a homogeneous self-interacting conformal scalar field. In Section 3, we present the extended solution-generating technique, reviewing in some detail the conformal transformation upon which it is build and emphasizing its usefulness regarding the derivation of exact conformal scalar field cosmologies. In Section 4, we apply the extended solution-generating technique to a number of classes of solutions in the Einstein frame that were recently obtained by Ellis and Madsen [16]. The classes of solutions obtained in the physical frame exhibit a diverse range of properties. For instance, one class of solutions tends asymptotically to a power-law inflationary behaviour, while another class exhibits a late time approach to the $S(t) \sim t$ behaviour of the coasting models. Bouncing models which avoid an initial singularity are also presented. Finally, we conclude with a Section devoted to an overall summary and discussion of the results.

2 The Field Equations
We begin this section by deriving and reviewing the general field equations for the non-minimal coupling model characterised by the action (1).

Variation of the action (1) with respect to the gravitational degrees of freedom yields Einstein’s equations

\[ R_{ab} - \frac{1}{2}g_{ab}R = T_{ab}, \tag{3} \]

where \( R_{ab} \) and \( T_{ab} \) are the usual Ricci and energy-momentum tensors. The later has the form

\[ T_{ab} = (1 - \xi \phi^2)^{-1}[(1 - 2\xi)\partial_a \phi \partial_b \phi + (2\xi - \frac{1}{2})g_{ab}\partial_c \phi \partial^c \phi - 2\xi \phi \nabla_a(\partial_b \phi) - g_{ab}V(\phi) + 2\xi g_{ab}\phi \Box \phi], \tag{4} \]

with \( \nabla_a \) standing for the covariant spacetime derivative associated with the metric \( g_{ab} \) and \( \Box = \nabla_a \nabla^a \) is the corresponding curved spacetime D’Alembertian. Variation with respect to the scalar field results in the Klein-Gordon equation

\[ \Box \phi - \xi \phi R - V'(\phi) = 0, \tag{5} \]

where \( V'(\phi) = dV(\phi)/d\phi \).

Next, we specialize the field equations for the particular case of a FRW spacetime background filled with a homogeneous self-interacting conformal scalar field.

The line element for a FRW spacetime can be written in standard normalized comoving coordinates \((t,r,\theta,\varphi)\) as

\[ ds^2 = -dt^2 + S^2(t) \left( \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right), \tag{6} \]

where \( S(t) \) represents the cosmic scale factor and \( k \) is the curvature index (\( k = 0, \pm 1 \) corresponds to the flat, closed and open FRW model, respectively). The independent components of Einstein’s equations (3)–(4) for this metric can be represented by the Friedmann and Raychaudhuri equations. In the conformal coupling case they become

\[ 3H^2 + 3K = (1 - \zeta^2 \phi^2)^{-1} \left[ 3\zeta^2 \dot{\phi}^2 + V(\phi) + \phi \ddot{\phi} \right], \tag{7} \]

\[ 3\dot{H} + 3H^2 = (1 - \zeta^2 \phi^2)^{-1} \left[ V(\phi) - 3\zeta^2 (\dot{\phi}^2 - \phi \dot{\phi} H - \phi \ddot{\phi}) \right], \tag{8} \]
where, in accordance with the FRW symmetry, we have assumed that $\phi = \phi(t)$. Here $K$ is the purely spatial part of the scalar curvature $K = k/S^2$, $\zeta = (\frac{1}{6})^{1/2}$ and, as usual, $H = \dot{S}/S$ denotes the Hubble parameter and the overdot means derivative with respect to the synchronous cosmic time, t. On the other hand, using eqs.(7) and (8), the Klein-Gordon equation (5) becomes

$$\ddot{\phi} + 3H\dot{\phi} + \zeta^2\phi(1 - \zeta^2\phi^2)^{-1}\left[4V(\phi) + 3\phi\dot{\phi}H + \phi\ddot{\phi}\right] + V'(\phi) = 0.$$  \hspace{1cm} (9)

It is worthwhile to point here that, for the case of a homogeneous scalar field propagating in a spatially homogeneous and isotropic spacetime background, the energy-momentum tensor (4) can be rewritten in the form of a perfect fluid for any value of $\xi$ [2,7]. The energy density and pressure of the equivalent perfect fluid are, in the conformal coupling case, given by

$$\rho_\phi = (1 - \zeta^2\phi^2)^{-1}[3\zeta^2\dot{\phi}^2 + V(\phi) + \phi\dot{\phi}H],$$  \hspace{1cm} (10)

and

$$p_\phi = (1 - \zeta^2\phi^2)^{-1}[\zeta^2(\dot{\phi}^2 - 2\phi\dddot{\phi} - 4\phi\dot{\phi}H) - V(\phi)],$$  \hspace{1cm} (11)

respectively. It is evident that neither the energy density nor the pressure are sign-definite. As a consequence, $\frac{1}{2}(\rho_\phi + 3p_\phi)$ has indeterminate sign, so that the strong energy condition [17]

$$(T_{ab} - \frac{1}{2}Tg_{ab})U^aU^b = \frac{1}{2}(\rho_\phi + 3p_\phi) \geq 0,$$  \hspace{1cm} (12)

where $U^\alpha = \partial^\alpha \phi/\sqrt{-\partial_\mu \phi \partial^\mu \phi}$ denotes the 4-velocity vector of the equivalent perfect fluid, may or may not be valid. In the later case, the Hawking-Penrose theorem on the existence of singularities [18] is not applicable and singularity-free solutions to equations (7)–(9) are then possible. In Section 4 a number of singularity-free solutions of those equations will be presented.

We will find it convenient to define a new scalar field variable $\psi$ such that $\psi = \zeta \phi$. Using this variable eqs.(7)–(9) become respectively

$$3H^2 + 3K = (1 - \psi^2)^{-1}[3\dot{\psi}^2 + V(\psi) + \zeta^{-2}\psi\dot{\psi}H],$$  \hspace{1cm} (13)

$$3\dot{H} + 3H^2 = (1 - \psi^2)^{-1}[V(\psi) - 3(\dot{\psi}^2 - \psi\dot{\psi}H - \psi\dddot{\psi})],$$  \hspace{1cm} (14)
\[
\ddot{\psi} + 3H \dot{\psi} + \psi(1 - \psi^2)^{-1}[4\zeta^2 V(\psi) + 3\psi \dot{\psi} H + \psi \ddot{\psi}] + \zeta^2 V'(\psi) = 0, \tag{15}
\]
where now \(V'(\psi) = dV(\psi)/d\psi\).

3 The Extended Solution-Generating Technique

As we remarked in the Introduction, the solution-generating technique employed in this work is based on the use of a conformal transformation of the spacetime metric, which reduces the theory under consideration to one containing canonical Einstein’s gravity plus a canonical scalar field. In what follows we present this solution-generating technique, reviewing in some detail the conformal transformation upon which it is build.

Let us perform the following Weyl conformal transformation

\[
\tilde{g}_{ab} = \Omega^2 g_{ab}, \quad \tilde{g}^{ab} = \Omega^{-2} g^{ab}, \quad \tilde{g} = \Omega^8 g, \tag{16}
\]
where

\[
\Omega^2 = |1 - \xi \phi^2|. \tag{17}
\]

Since the transformed scalar curvature is

\[
\tilde{R} = \Omega^{-2}[R - 6\Omega^{-1}\square \Omega], \tag{18}
\]

it is then straightforward to show that action (1) can be written in the Einstein frame as

\[
S[\tilde{g}^{ab}, \phi] = \int d^4 x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2} - \frac{1}{2} F^2(\phi) \partial_a \phi \partial^a \phi - \tilde{V}(\phi) \right], \tag{19}
\]
where

\[
F^2(\phi) = \frac{1 - \xi (1 - 6\xi) \phi^2}{(1 - \xi \phi^2)^2}, \tag{20}
\]
and

\[
\tilde{V}(\phi) = \frac{V(\phi)}{(1 - \xi \phi^2)^2}. \tag{21}
\]

In order to put the kinetic term of the scalar field in the canonical form, we define a new field \(\tilde{\phi}\) as

\[
\tilde{\phi} = \int d\phi F(\phi). \tag{22}
\]
The theory can thus be written in terms of these new variables as a minimally coupled theory with the action

\[ S[\tilde{g}^{ab}, \tilde{\phi}] = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2} - \frac{1}{2} \partial_a \tilde{\phi} \partial^a \tilde{\phi} - \tilde{V}(\tilde{\phi}) \right]. \]  

(23)

Taking \( \xi = \frac{1}{6} \) in eq.(22) and solving for \( \phi \) yields

\[ \phi(\tilde{\phi}) = \zeta^{-1} \tanh(\zeta \tilde{\phi}), \]  

(24)

and upon substitution of this result in eq.(17) one obtains

\[ \Omega^{-1}(\tilde{t}) = \cosh(\zeta \tilde{\phi}(\tilde{t})), \]  

(25)

and eq.(21) can be rewritten as

\[ V(\phi) = (1 - \zeta^2 \phi^2)^2 \tilde{V} \left[ \frac{\zeta^{-1}}{2} \ln \left( \frac{1 + \zeta \phi}{1 - \zeta \phi} \right) \right]. \]  

(26)

From the point of view of the search of exact solutions to the field equations (3)–(5) in the conformal coupling case, this changing frame process can be interpreted in the following way: if the set \((\tilde{g}^{ab}, \tilde{\phi}, \tilde{V}(\tilde{\phi}))\) forms a solution to the field equations in the Einstein frame, that is, a solution to eqs.(3)–(5) but with \( \xi = 0 \), then the set \((g^{ab}, \phi, V(\phi))\), where \( g^{ab}, \phi \) and \( V(\phi) \) are given, respectively, by eqs.(16), (24) and (26), with \( \Omega \) given by eq.(25), constitutes a solution to the same field equations but in the physical frame. The massless case of this solution-generating technique was given by Bekenstein [14].

Note that this solution-generating technique is valid for any metric \( g^{ab} \). In the particular case that concerns us here, \( g^{ab} \) is a FRW metric, whose line element was given in eq.(6). The line element of the conformal spacetime is, according to eq.(16), given by

\[ ds^2 = \Omega^2 d\tilde{s}^2. \]  

(27)

This can be brought into the same form as eq.(6) by the following change of variables

\[ d\tilde{t} = \Omega dt, \]  

(28)

\[ \tilde{S} = \Omega S. \]  

(29)
Then, we have in the Einstein frame
\[ ds^2 = -d\tilde{t}^2 + \tilde{S}^2(\tilde{t}) \left( \frac{dv^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right) . \] (30)
i.e., the metric in the Einstein frame $\tilde{g}_{ab}$ is also of the FRW type, but with a redefined scale factor.

Before proceeding further, it will be useful to pause in order to introduce some observational quantities of interest. The cosmological density parameter $\Omega_d$ (we adopt this notation slightly at variance with the standard usage to avoid any confusion with the conformal factor $\Omega$ ) is defined by
\[ \Omega_d = \frac{\rho_\phi}{3H^2}, \] (31)
and when combined with the Friedmann equation (7), can be rewritten as
\[ \Omega_d = 1 + \frac{k}{S^2}. \] (32)
The deceleration parameter $q$ is defined by
\[ q = -\frac{\ddot{S}}{H^2 S}. \] (33)
Following [19], we take the sign of the deceleration parameter as indicating whether a given cosmological model inflates or not. The positive sign corresponds to “standard” decelerating cosmological solutions whereas the negative sign indicates inflation. Although there are several definitions of what is actually meant by inflation, this is the most commonly used in the study of exact scalar field cosmologies.

Defining a “mixed” cosmic scale factor by $S(\tilde{t}) = S(t(\tilde{t}))$ we have
\[ \dot{S} = \Omega S' \] (34)
\[ \ddot{S} = \Omega (\Omega' S' + \Omega S''), \] (35)
where the prime denotes differentiation with respect to the cosmic time $\tilde{t}$ of the Einstein frame. A very convenient form for $\Omega_d$ and $q$ can then be obtained inserting these results into eqs.(32) and (33)
\[ \Omega_d = 1 + \frac{k}{\Omega^2 S^2}, \] (36)
8
\[ q = \frac{\Omega' S}{\Omega S'} - \frac{SS''}{S''}. \] (37)

For future reference, we end this section giving the equations of the extended solution-generating technique for a spatially homogeneous and isotropic metric (eqs. (24)–(26), (28)–(29)), rewritten in terms of the \( \psi \) scalar field variable

\[ S(\tilde{t}) = \cosh(\tilde{\psi}(\tilde{t})) \tilde{S}(\tilde{t}), \] (38)

\[ \frac{d\tilde{t}(\tilde{t})}{dt} = \cosh(\tilde{\psi}(\tilde{t})), \] (39)

\[ \psi(\tilde{t}) = \tanh(\tilde{\psi}(\tilde{t})), \] (40)

\[ V(\psi) = (1 - \psi^2)^2 \tilde{V} \left[ \frac{\zeta^{-1}}{2} \ln \left( \frac{1 + \psi}{1 - \psi} \right) \right]. \] (41)

This equations can be looked upon as a map between the space of Einstein frame solutions and the space of physical frame solutions.

\section{The Solutions}

We now apply the solution-generating technique developed in the preceding section to a number of interesting situations. We consider, in the Einstein frame, the solutions which were recently obtained by Ellis and Madsen [16]. Here only three of the five exact solutions found by Ellis and Madsen will be mapped to the physical frame.

The titles adopted in the following subsections refer to the behaviour of the solution in the Einstein frame.

\subsection{Power-Law Expansion}

First, let us consider the solution in the Einstein frame characterised by a power-law behaviour

\[ \tilde{S}(\tilde{t}) = A \tilde{t}^n, \] (42)

\[ \tilde{\psi}(\tilde{t}) = \ln \tilde{\psi}, \] (43)
\[ \tilde{V}(\tilde{\psi}) = V_0 \exp \left( -\frac{2}{p} \tilde{\psi} \right), \quad (44) \]

where \( A, n \) are strictly positive constants with \( n \neq 1 \), \( p(n) = (n/3)^{1/2} \), \( V_0(n) = n(3n-1) \), which is only possible for \( k = 0 \). This class of solutions is inflationary for \( n > 1 \), because if the power-law expansion continues for long enough, it will solve all the well known problems of the standard hot big-bang model (horizon, flatness, spectrum of primordial density perturbations, . . .) [20]. However, it behaves like a “standard” decelerating big-bang when \( 0 < n < 1 \).

Mapping the class of solutions (42)–(44) in the Einstein frame into the physical frame with the help of eqs.(38)–(41), we obtain the following parametric class of solutions in terms of the “unphysical” time variable \( \tilde{t} \)

\[ S(\tilde{t}) = \frac{A}{2}(\tilde{t}^{n+p} + \tilde{t}^{n-p}), \quad (45) \]

\[ t(\tilde{t}) = \begin{cases} \frac{1}{2(1+p)}\tilde{t}^{1+p} + \frac{1}{2(1-p)}\tilde{t}^{1-p} & \text{if } p \neq 1, \\ \frac{1}{4}\tilde{t}^2 + \ln \tilde{t} & \text{if } p = 1 \end{cases}, \quad (46) \]

\[ \psi(\tilde{t}) = \frac{\tilde{t}^{2p} - 1}{\tilde{t}^{2p} + 1}, \quad (47) \]

\[ V(\psi) = V_0(1 - \psi)^\alpha (1 + \psi)^\beta, \quad (48) \]

where \( \alpha(p) = 2 + 1/p \) and \( \beta(p) = 2 - 1/p \). This class of solutions is only consistent with a flat FRW geometry. As we shall find in what follows, although it depends on the two parameters \( A \) and \( n \), its behaviour is essentially determined by the \( n \) parameter.

The cosmological density parameter for this class of solutions is \( \Omega_d = 1 \), while the deceleration parameter is given in terms of the “unphysical” time variable \( \tilde{t} \) by

\[ q(\tilde{t}) = \frac{q_1\tilde{t}^{4p} + q_2\tilde{t}^{2p} + q_3}{(n + p)^2\tilde{t}^{4p} - q_2\tilde{t}^{2p} + (n - p)^2}, \quad (49) \]

where

\[ q_1(n) = (n + p)(1 - n), \quad (50) \]

\[ q_2(n) = 2(p^2 - n^2), \quad (51) \]

\[ q_3(n) = (n - p)(1 - n). \quad (52) \]
The behaviour of the class of solutions (45)–(48) is illustrated in Figure 1 where $S(t)$ and $V(\psi)$ are plotted for a representative set of values of the $A,n$ parameters.

Two points immediately stand out from eqs.(45) and (46). First, from eq.(46) we can see that for $0 < n < 3$ we have

$$0 < \tilde{t} < +\infty \Rightarrow 0 < t < +\infty,$$

while for $n \geq 3$ we have

$$0 < \tilde{t} < +\infty \Rightarrow -\infty < t < +\infty.$$

Secondly, from eq.(45) we can see that all the solutions of this class corresponding to the range $\frac{1}{3} \leq n < 3$ are big-bang solutions, that is, start expanding from a Friedmann singularity ($S = 0$), while those corresponding to the ranges $0 < n < \frac{1}{3}$ and $n > 3$ are singularity-free. Indeed, for all solutions with $\frac{1}{3} \leq n < 3$ the edge $S = 0$ is always reached in a finite proper time into the past ($S \to 0$ as $t \to 0$) (see Fig. 1(c)). On the other hand, there is no Friedmann singularity for $0 < n < \frac{1}{3}$ because these solutions always "bounce" at a finite proper time into the past given by (see Fig. 1(a))

$$t_{bounce} = \frac{1}{2(1 + p)} \left( \frac{p - n}{n + p} \right)^{\frac{1 + p}{2p}} + \frac{1}{2(1 - p)} \left( \frac{p - n}{n + p} \right)^{\frac{1 + p}{2p}},$$

neither for $n > 3$ because, in this case, an infinite proper time into the past is required to reach the edge $S = 0$ (see Fig. 1(e)).

The asymptotic behaviour of the class of solutions (45)–(48) as $\tilde{t} \to 0$ is given by

$$S(t) \sim \begin{cases} t^m & \text{if } 0 < n < 3 \\ (-t)^m & \text{if } n > 3 \end{cases},$$

$$\psi(t) \sim \begin{cases} -1 + \psi_0 t^s & \text{if } 0 < n < 3 \\ -1 + \psi_0 (-t)^s & \text{if } n > 3 \end{cases},$$

$$V(\psi) \sim (1 + \psi)^{\beta},$$
where

\[ m(n) = \frac{n - p}{1 - p}, \quad s(n) = \frac{2p}{1 - p}, \] (59)

and

\[ \psi_0 = (2|1 - p|)^s, \] (60)

the asymptotic behaviour of the \( n = 3 \) solution results from the above equations in the limit \( n \to 3 \). Hence, we see that for an initial inflationary phase to occur we must have \( m > 1 \iff 0 < n < \frac{1}{3} \) or \( n > 1 \). More precisely, the solutions corresponding to the range \( 1 < n < 3 \) exhibit an initial power-law inflationary phase of expansion \( (\dot{S} > 0, \ddot{S} > 0 \text{ and } \dot{H} < 0) \) which, in the limit \( n \to 3 \), yield to an initial phase of standard De Sitter exponential inflation; those corresponding to the range \( n > 3 \) display an initial phase of pole inflation (superinflationary expansion, \( \dot{S}, \ddot{S}, \dot{H} \) all positive); finally, those corresponding to the range \( 0 < n < \frac{1}{3} \) exhibit an initial phase of contraction characterised by \( \dot{S} < 0, \ddot{S} > 0 \) and \( \dot{H} > 0 \), which leads to a bounce. In this situation the criterion based on \( q \) to define inflation, does not hold, since \( \dot{S} < 0 \).

It is worth noticing that while in the minimal coupling case superinflation only occurs for the \( k = +1 \) models [21], in the conformal coupling framework that can occur for other values of the curvature index, namely for \( k = 0 \).

The late time behaviour \((\tilde{t} \to +\infty)\) of the class of solutions (45)–(48) is given by

\[ S(t) \sim t^l, \] (61)
\[ \psi(t) \sim 1 - \psi_\infty t^{-r}, \] (62)
\[ V(\psi) \sim (1 - \psi)^\alpha, \] (63)

where

\[ l(n) = \frac{n + p}{1 + p}, \quad r(n) = \frac{2p}{1 + p}, \] (64)

and

\[ \psi_\infty = (2(1 + p))^{-r}, \] (65)

so, we see that for a late time power-law inflationary phase to occur we must have \( l > 1 \iff n > 1 \), the limit \( n \to +\infty \) corresponding to a late time standard De
Sitter exponential inflationary phase. While in the minimal coupling case power-law inflation is normally driven by exponential potentials $V(\psi) \propto \exp(-\lambda \psi)$, with $\lambda$ constant $> 0$, we find from eqs.(61),(63) that in the conformal coupling case power-law inflation is driven by a polynomial potential.

All this conclusions on the asymptotic behaviour of the class of solutions (45)–(48) are corroborated by a direct analysis of the asymptotic behaviour of the deceleration parameter $q$ (cf. eq.(49)). Indeed, as $\tilde{t} \to 0$ asymptotes to

$$q(t) \sim \frac{q_3}{(n-p)^2} = \frac{1-n}{n-p},$$

yielding $q(t) < 0$ only when $0 < n < \frac{1}{3}$ or $n > 1$, while its late time behaviour ($\tilde{t} \to +\infty$) is given by

$$q(t) \sim \frac{q_1}{(n+p)^2} = \frac{1-n}{n+p},$$

yielding $q(t) < 0$ only when $n > 1$. Note that in these asymptotic inflationary behaviours the slow-rolling approximation (see Ref.[23]) is not valid.

An exceptional case occurs when $n = \frac{1}{3}$: on the one hand, the potential is flat $V(\psi) = 0$ and on the other, we can write the corresponding solution explicitly in terms of the physical cosmic time $t$

$$\frac{S(t)}{A} = -\frac{1}{2} + \sqrt{\frac{4t+3}{12}},$$

$$\psi(t) = 1 - 2\sqrt{\frac{6}{16t+6}}.$$  (68)

As $t \to +\infty$ the cosmic scale factor asymptotes to

$$S(t) \sim t^{1/2}.\quad (70)$$

Thus, from a volumetric point of view, a flat FRW geometry filled with a massless and non-interacting homogeneous conformally coupled scalar field exhibits as late time behaviour that of the standard flat FRW radiation-dominated solution. This should be compared with the analogous situation in the Einstein frame, where a homogeneous free scalar field propagating in a flat FRW background behaves as

$$S(t) \sim t^{1/3}.\quad (71)$$
To summarize all these results on the global and asymptotic behaviour of the class of solutions (45)–(48) we have collected them together in Table 1.

4.2 Linear Expansion

Next, consider the solution in the Einstein frame characterised by a linear expansion behaviour

\[ \tilde{S}(\tilde{t}) = A\tilde{t}, \quad (72) \]
\[ \tilde{\psi}(\tilde{t}) = \ln \tilde{t}^p, \quad (73) \]
\[ \tilde{V}(\tilde{\psi}) = V_0 \exp \left( -\frac{2}{p} \tilde{\psi} \right), \quad (74) \]

where \( A \) is a strictly positive constant,

\[ p(k, A) = \left( \frac{A^2 + k}{3A^2} \right)^{1/2}, \quad (75) \]

and \( V_0(p) = \zeta^{-2p^2} \), which is always possible for \( k = 0, +1 \) and can be made possible for \( k = -1 \) when subjected to the constraint \( A > 1 \). This class of coasting solutions may be viewed as a modern version of the Milne solution (which is characterised by \( p = \rho = 0 \) and \( k = -1 \) [17]), where the scalar field matter source allows it to be non-empty and with a generalization that any value of the curvature index \( k \) is allowed [16].

Mapping the class of solutions (72)–(74) in the Einstein frame into the physical frame with the help of eqs.(38)–(41), we obtain the following parametric class of solutions in terms of the “unphysical” time variable \( \tilde{t} \)

\[
S(\tilde{t}) = \frac{A}{2}(\tilde{t}^{1+p} + \tilde{t}^{1-p}), \quad (76)
\]
\[
t(\tilde{t}) = \begin{cases} 
\frac{1}{2(1+p)}\tilde{t}^{1+p} + \frac{1}{2(1-p)}\tilde{t}^{1-p} & \text{if } p \neq 1 \\
\frac{1}{4}\tilde{t}^2 + \frac{1}{2}\ln \tilde{t} & \text{if } p = 1
\end{cases}, \quad (77)
\]
\[ \psi(\tilde{t}) = \frac{\tilde{t}^{2p} - 1}{\tilde{t}^{2p} + 1}, \quad (78) \]
\[ V(\psi) = V_0(1 - \psi)^\alpha(1 + \psi)^\beta, \quad (79) \]
where $\alpha(p) = 2 + 1/p$ and $\beta(p) = 2 - 1/p$. This class of solutions also is consistent with all the FRW geometries, the $A$-amplitude being in the $k = -1$ case subjected to the same constraint as in the Einstein frame.

The cosmological density parameter for this class of solutions is given in terms of the “unphysical” time variable $\tilde{t}$ by

$$\Omega_d(\tilde{t}) = \frac{\omega_1 \tilde{t}^{4p} + \omega_2 \tilde{t}^{2p} + \omega_3}{(\omega_1 - k)\tilde{t}^{4p} + (\omega_2 - 2k)\tilde{t}^{2p} + (\omega_3 - k)},$$

where

$$\omega_1(k, A) = A^2(1 + p)^2 + k,$$
$$\omega_2(k, A) = 2A^2(1 - p^2) + 2k,$$
$$\omega_3(k, A) = A^2(1 - p)^2 + k;$$

and the deceleration parameter is given by

$$q(\tilde{t}) = -\frac{4A^2 p^2 \tilde{t}^{2p}}{(\omega_1 - k)\tilde{t}^{4p} + (\omega_2 - 2k)\tilde{t}^{2p} + (\omega_3 - k)}. $$

The behaviour of the cosmic density parameter of the class of solutions (76)–(79), eq.(80), is illustrated in Figure 2 for closed and open FRW geometries. For each of these cases a representative set of values of the $p$ parameter is plotted.

Two points immediately stand out from eqs.(76) and (77). First, from eq.(77) we can see that for $0 < p < 1 \iff A^2 > k/2$ (this inequality is trivially satisfied by all the flat and open solutions of the class, being also satisfied by the closed solutions with $A^2 > \frac{1}{2}$) we have

$$0 < \tilde{t} < +\infty \implies 0 < t < +\infty$$

while for $p \geq 1 \iff A^2 \leq k/2$ (this inequality is only satisfied by the closed solutions of the class with $A^2 \leq \frac{1}{2}$) we have

$$0 < \tilde{t} < +\infty \implies -\infty < t < +\infty.$$
because these solutions always “bounce” at a finite proper time into the past (see Fig. 3(a)), given by

\[ t_{\text{bounce}} = \frac{1}{2(1 + p)} \left( p - 1 \right) \left( p + 1 \right)^{1 + p} \left( 1 - \frac{1}{p} \right)^{1 - p}, \tag{87} \]

neither for \( p = 1 \) because, in this case, the corresponding solution starts expanding from a non-zero value of the cosmic scale factor \( S_\infty \) (see Fig. 3(b)).

The asymptotic behaviour of the class of solutions (76)-(79) as \( \tilde{t} \to 0 \) is given by

\[
S(t) \sim \begin{cases} 
  t & \text{if } p < 1 \\
  S_\infty & \text{if } p = 1 \ , \\
  -t & \text{if } p > 1
\end{cases} \tag{88}
\]

\[
\psi(t) \sim \begin{cases} 
  -1 + \psi_0 t^s & \text{if } p < 1 \\
  -1 + \exp(4t) & \text{if } p = 1 \ , \\
  -1 + \psi_0 (-t)^s & \text{if } p > 1
\end{cases} \tag{89}
\]

\[
V(\psi) \sim (1 + \psi)^\beta; \tag{90}
\]

where

\[
s(p) = \frac{2p}{1 - p}, \tag{91}
\]

\[
\psi_0 = (2\left|1 - p\right|)^s; \tag{92}
\]

while the late time behaviour (\( \tilde{t} \to +\infty \)) is given by

\[
S(t) \sim t, \tag{93}
\]

\[
\psi(t) \sim 1 - \psi_\infty t^{-r}, \tag{94}
\]

\[
V(\psi) \sim (1 - \psi)^\alpha; \tag{95}
\]

where

\[
r(p) = \frac{2p}{1 - p}, \tag{96}
\]

\[
\psi_\infty = (2(1 + p))^{-r}. \tag{97}
\]

Hence, we see that the class of solutions (76)-(79) does not exhibit an asymptotic inflationary phase neither as \( \tilde{t} \to 0 \) nor as \( \tilde{t} \to +\infty \) approaching instead, in both
cases and for general $p$, the $S(t) \sim t$ behaviour of the coasting solutions. This conclusion is corroborated by a direct analysis of the asymptotic behaviour of the deceleration parameter since $q \to 0$ both as $\tilde{t} \to 0$ and as $\tilde{t} \to +\infty$ (cf. eq.(84)).

Although the class of solutions (76)-(79) is not asymptotically inflationary according to the negative deceleration parameter criterion, it has been argued that the coasting solutions ought to be considered inflationary [21]. In fact, if the linear expansion continues for long enough it will solve all the well known kinematical problems of the standard hot big-bang model. Furthermore, as shown in Ref. [22], these models allow the generation and evolution of density perturbations that may be responsible for the large scale structure that we observe in the Universe. However, unlike what is usually expected from inflationary models the present models can lead to $\Omega_d \neq 1$ today [21]. This interesting feature of the coasting models, which may provide a solution to the cosmic dark matter problem [23], can be illustrated by the asymptotic behaviour of the cosmological density parameter corresponding to the class of solutions (76)-(79), since as $\tilde{t} \to 0$ we have (see Fig. 4)

$$\Omega_d(t) \sim \frac{\omega_3}{\omega_3 - k} = 1 + \frac{k}{A^2(1 - p)^2},$$

(98)

and as $\tilde{t} \to +\infty$ we have

$$\Omega_d(t) \sim \frac{\omega_1}{\omega_1 - k} = 1 + \frac{k}{A^2(1 + p)^2}.$$ 

(99)

4.3 De Sitter Expansion From a Singularity

Finally, let us consider the solution in the Einstein frame characterised by a “sinh” expansion from a Friedmann singularity ($\bar{S} = 0$)

$$\bar{S}(\tilde{t}) = A \sinh(\omega \tilde{t}),$$

(100)

$$\bar{\psi}(\tilde{t}) = \ln \left( \tanh^p \left( \frac{\omega \tilde{t}}{2} \right) \right),$$

(101)

$$\bar{V}(\bar{\psi}) = V_0 \left( \frac{1}{2} + p^2 \sinh^2 \left( \frac{1}{p} \bar{\psi} \right) \right),$$

(102)
where \(A, \omega\) are strictly positive constants,

\[
p(\omega, A) = \left(\frac{1}{3} + \frac{k}{3A^2\omega^2}\right)^{1/2},
\]
and \(V_0 = \zeta^{-2}\omega^2\), which is always possible for \(k = 0, +1\) and can be made possible for \(k = -1\) when subjected to the constraint \(A\omega > 1\). For large \(\tilde{t}\), this class of solutions approaches a standard De Sitter inflationary phase, the cosmic scale factor growing exponentially with time in this limit.

Mapping the class of solutions (100)–(102) in the Einstein frame into the physical frame with the help of eqs.(38)–(41), we obtain the following parametric class of solutions in terms of the “unphysical” time variable \(\tilde{t}\)

\[
S(\tilde{t}) = \frac{A}{2} \sinh(\omega \tilde{t}) \left(\tanh^p\left(\frac{\omega \tilde{t}}{2}\right) + \coth^p\left(\frac{\omega \tilde{t}}{2}\right)\right),
\]

\[
\frac{dt(\tilde{t})}{d\tilde{t}} = \frac{1}{2} \left(\tanh^p\left(\frac{\omega \tilde{t}}{2}\right) + \coth^p\left(\frac{\omega \tilde{t}}{2}\right)\right),
\]

\[
\psi(\tilde{t}) = \frac{\tanh^{2p}\left(\frac{\omega \tilde{t}}{2}\right) - 1}{\tanh^{2p}\left(\frac{\omega \tilde{t}}{2}\right) + 1},
\]

\[
V(\psi) = V_1(1 - \psi^2)^2 + V_2 \left((1 + \psi)\alpha(1 - \psi)^\beta + (1 + \psi)^\beta(1 - \psi)^\alpha\right),
\]

where

\[
\alpha(p) = 2 + \frac{1}{p} , \quad \beta(p) = 2 - \frac{1}{p},
\]

and

\[
V_1(\omega, A) = 2\omega^2 - \frac{1}{A^2} , \quad V_2(\omega, A) = \frac{1}{2} \left(\omega^2 + \frac{1}{A^2}\right).
\]

Here \(p\) is restricted to be a positive integer and \(A, \omega\) are fixed in such a way as to satisfy the following constraint equation

\[
A^2\omega^2 = \frac{1}{3p^2 - 1}.
\]

Although the starting class of solutions in the Einstein frame eqs.(100)–(102) is consistent with all the FRW geometries, this class is only consistent with a closed FRW geometry. This reduction in the number of admissible FRW geometric backgrounds, is a consequence of the integration of eq.(105) only being
possible when $p$ is a positive integer, which happens only when $k = +1$. However, the integrals of integer powers of the “tanh” and “coth” functions are given inductively [24] so, we can not write the solution of eq.(105) in closed form for general $p$. To accomplish this we have to restrict ourselves to a particular value of $p$.

In what follows we will focus, for simplicity, on the $p = 1$ solution. Taking $p = 1$ in eqs.(104)–(107), we get

\begin{align}
S(\tilde{t}) &= A \cosh(\omega \tilde{t}), \\
t(\tilde{t}) &= \frac{1}{\omega} \ln(\sinh(\omega \tilde{t})), \\
\psi^{-1}(\tilde{t}) &= -\cosh(\omega \tilde{t}), \\
V(\psi) &= V_0 (1 - \psi^4),
\end{align}

where $A^2 \omega^2 = \frac{1}{2}$ and $V_0 = 3/2A^2$.

Now, it’s interesting to note that eq.(112) can be solved for $t$, which enables one to eliminate the $\tilde{t}$ parameter between eqs.(111)–(113) and to write the solution explicitly in terms of the physical cosmic time $t$. When this is done, we obtain

\begin{align}
S(t) &= A(1 + \exp(2\omega t))^{1/2}, \\
\psi(t) &= -(1 + \exp(2\omega t))^{-1/2},
\end{align}

with $V(\psi)$ given by eq.(114).

The cosmological density parameter for this solution is given in terms of the physical cosmic time $t$ by

\begin{equation}
\Omega_d(t) = 1 + 2(\exp(-2\omega t) + \exp(-4\omega t)),
\end{equation}

and the deceleration parameter is given by

\begin{equation}
q(t) = -\left(\frac{1}{\omega^2} + \exp(-2\omega t)\right).
\end{equation}

For very large $t$ ($t \to +\infty$), the cosmic scale factor asymptotes to

\begin{equation}
S(t) \sim \exp(\omega t),
\end{equation}
while for $t \to -\infty$ we have

$$S(t) \sim A.$$

(120)

This is a non-singular inflationary solution that approaches asymptotically a standard De Sitter exponential inflationary phase. Such late time inflationary phase results here, as in the minimal coupling case, from the presence of the constant term $V_0$ in eq. (114), which is just an effective cosmological constant. On the other hand, the cosmological density parameter makes a smooth transition from infinity ($\Omega_d \to +\infty$ as $t \to -\infty$) to unity ($\Omega_d \to 1$ as $t \to +\infty$), so the solution is asymptotically flat.

It is worth noticing that the slow-rolling condition (see Ref. [23]) although asymptotically satisfied in the Einstein frame, fails to be true in the physical frame. The ratio $\Delta = |\dot{\psi}/\psi|/H$ is equal to unity instead of $\Delta \ll 1$. So, to a slow-rolling inflationary phase in the Einstein frame do not always corresponds a slow-rolling inflationary phase in the original frame. In fact, the condition for this to happen requires $\psi^2 \ll 1$ as shown in Ref. [8], which is violated in our case ($\psi^2 < 1$).

5 Summary and Conclusions

In this work we have found a number of new, exact classes of conformal scalar field cosmologies. The solution-generating technique used here extends the technique originally introduced by Bekenstein for massless conformal scalar fields [14], and is based on the conformal equivalence between the non-minimal coupling model characterised by the action (1) and canonical Einstein’s gravity plus a canonical scalar field. This technique has been used in the literature to map a number of familiar results from the Einstein frame, such as the reheating temperature and density spectrum, into the physical frame [25]. It has also been used to derive new exact cosmological solutions to some scalar-tensor theories [26]. Nevertheless, to our knowledge, the conformal transformation technique has never been used to derive self-interacting conformal scalar field cosmologies.
We have applied the extended solution-generating technique to the classes of solutions in the Einstein frame that were recently obtained by Ellis and Madsen [16]. For the classes of solutions obtained in the physical frame we have discussed the existence of singularities, the asymptotic behaviour and the possibility of occurrence of inflationary phases. As a result, we have found that one class of solutions (eqs.(45)–(48)) tends asymptotically to a power-law inflationary behaviour $S(t) \sim t^p$ with $p > 1$ yielding, in the limit $p \to +\infty$, to a late time phase of standard De Sitter exponential inflation; and that another class (eqs.(76)–(79)) exhibits a late time approach to the $S(t) \sim t$ behaviour of the coasting models, leading to any asymptotic value for the cosmological density parameter, in particular to $\Omega_d \neq 1$, depending on the choice of the initial conditions. Both of these classes comprise bouncing solutions which avoid an initial singularity. It was also found that, a flat FRW geometry filled with a massless and non-interacting homogeneous conformally coupled scalar field approaches asymptotically the typical $S(t) \sim t^{1/2}$ behaviour of the standard flat FRW radiation-dominated solution. Our results thus reveal that to simple behaviours of the cosmic scale factor in the Einstein frame corresponds a wealth of behaviours in the physical frame, namely, we find several different phases, with in some cases, a smooth transition between ordinary and inflationary expansion. Note that to a slow-rolling inflationary phase in the Einstein frame do not always corresponds a slow-rolling inflationary phase in the physical frame. This indicates that some caution is required in transposing to the physical frame results which are derived in the Einstein frame, in particular when regarding inflation.

This work also serves the purpose of establishing the classes of potentials in the physical frame which drive the various behaviours of the cosmic scale factor considered in the Einstein frame. This allows a comparison to be made with the corresponding potentials in the Einstein frame. For instance, while in the minimal coupling case a power-law inflationary phase is normally driven by exponential potentials $V(\psi) \propto \exp(-\lambda\psi)$, with $\lambda$ constant $> 0$, we have found that in the conformal coupling case it is driven by a polynomial potential(cf. eq.(63)).

Although we have focused on some classes of solutions in the Einstein frame,
the technique used here can also be utilised to generate other conformal scalar field cosmologies by starting from other minimal scalar field cosmologies. Furthermore, the extended solution-generating technique can also be applied to other geometric backgrounds. In a forthcoming paper we will apply it to a number of anisotropic scalar field cosmologies.

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Table Captions

Table 1: Summary of the global and asymptotic volumetric behaviour of the solutions of the power-law expansion class (eqs.(45)–(48)).

Figure Captions

Figure 1: The figure shows the scale factor plotted against the time $t$ and the potential plotted against $\psi$ for the power-law expansion class (eqs.(45)–(48)): (a)-(b) $n = \frac{1}{4}$ and $A = 2$ ($\Rightarrow p = \sqrt{\frac{1}{12}}$), (c)-(d) $n = \frac{1}{2}$ and $A = 2$ ($\Rightarrow p = \sqrt{\frac{1}{6}}$), (e)-(f) $n = 12$ and $A = 2$ ($\Rightarrow p = 2$). The graphs (a),(c) and (e) correspond to the scale factor. The graphs (b),(d) and (f) correspond to the potential.

Figure 2: The figure shows the time evolution of the cosmological density parameter $\Omega_d$ for the linear expansion class (eqs.(76)–(79)): (a) $k = +1$ and $p = \frac{2}{3}$ ($\Rightarrow A = \sqrt{3}$), (b) $k = +1$ and $p = 1$ ($\Rightarrow A = \frac{1}{\sqrt{2}}$), (c) $k = +1$ and $p = 2$ ($\Rightarrow A = \frac{1}{\sqrt{11}}$), (d) $k = -1$ and $p = \sqrt{\frac{3}{2}}$ ($\Rightarrow A = \frac{3}{2}$).

Figure 3: The figure shows the scale factor plotted against the time $t$ for the solutions of the linear expansion class (eqs.(76)–(79)) corresponding to the parameter values (a) $p = 2$ ($\Rightarrow A = \sqrt{\frac{1}{11}}$) and (b) $p = 1$ ($\Rightarrow A = \sqrt{\frac{1}{6}}$).

Figure 4: The figure shows the asymptotic values taken by the cosmological density parameter $\Omega_d$ as $\tilde{t} \to 0$ ($\Omega_0$) and as $\tilde{t} \to +\infty$ ($\Omega_\infty$) plotted against the $A$ parameter, for the solutions of the linear expansion class (eqs.(76)–(79)). The graphs (a) and (b) correspond to the closed ($k = +1$) model. In (a) the graph diverges to infinity as $A \to \sqrt{\frac{1}{2}}$, asymptotes to 4 as $A \to 0$ and to 1 as $A \to +\infty$. The graphs (c) and (d) correspond to the open ($k = -1$) model.
### Table 1

| Global Behaviour | Asymptotic Behaviour \( t \to 0 \) | Asymptotic Behaviour \( t \to +\infty \) |
|------------------|--------------------------------------|------------------------------------------|
| \( 0 < n < \frac{1}{3} \) | Bouncing Models | Contraction | Ordinary Expansion |
| \( \frac{1}{3} \leq n < 1 \) | “Standard” Big-Bang Models | Ordinary Expansion | Ordinary Expansion |
| \( n = 1 \) | “Standard” Big-Bang Model | Linear Expansion | Linear Expansion |
| \( 1 < n < 3 \) | Inflationary Big-Bang Models | Power-Law Inflation | Power-Law Inflation |
| \( n = 3 \) | Non-Singular Inflationary Model | De Sitter Expansion | Power-Law Inflation |
| \( n > 3 \) | Non-Singular Inflationary Models | Pole Inflation | Power-Law Inflation |
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