SU(N) Irreducible Schwinger Bosons

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We construct SU(N) irreducible Schwinger bosons satisfying certain U(N-1) constraints which implement the symmetries of SU(N) Young tableaux. As a result all SU(N) irreducible representations are simple monomials of $(N - 1)$ types of SU(N) irreducible Schwinger bosons. Further, we show that these representations are free of multiplicity problems. Thus all SU(N) representations are made as simple as SU(2).

I. INTRODUCTION

The Schwinger construction of SU(2) Lie group and all its representations in terms of two simple harmonic oscillators or equivalently Schwinger bosons [1] is well known. Due to its simplicity, this construction has been widely used in various branches of physics like nuclear physics [2], strongly correlated systems [3], supersymmetry and supergravity

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algebras [4], lattice gauge theories [5], loop quantum gravity [6] etc. The three novel features of SU(2) Schwinger boson construction are its completeness, economy and simplicity. More precisely, the Hilbert space created by the two Schwinger oscillators is isomorphic to the representation space of SU(2) group (see section II). Thus the Schwinger boson representation of SU(2) group is complete (all SU(2) representations occur) as well as economical (every representation occurs once). Further, this construction is also simple as all SU(2) representations are given in terms of monomials (not polynomials) of Schwinger bosons. It is well known that all the above desired features of economy, completeness and elegance associated with SU(2) group are lost when we consider mixed representations of SU(3) or higher SU(N) groups. There has been considerable work in the past in these directions [7–15]. In fact, following the work of Gelfand [8], an explicit realization of a group G leading to its representations which are complete and without multiplicities is known as a model or Gelfand model of G in mathematics literature and is a subject of considerable interest [12]. The reason for multiplicities in SU(N) representations is that SU(N) group requires at least \((N - 1)\) fundamental representations to get its all other irreducible representations. This immediately implies existence of certain non-trivial SU(N) invariant operators (see the references section \[\text{III}, \text{IV} \text{ and V} \text{ and references therein}\). Any two states which differ by an overall presence of such invariants will transform in the same way under SU(N). This leads to the problem of multiplicity which in turn makes the representation theory of SU(N) \((N \geq 3)\) much more involved than SU(2). The standard way to project out these invariants is by using SU(N) Young tableauxes. The symmetrization and anti symmetrization of SU(N) indices along the rows and columns of SU(N) Young tableauxes remove these invariants leading to all SU(N) irreducible representa-
tions. However, the symmetrization and anti-symmetrization operations, in turn, make the representations of even the simplest SU(3) group extremely complicated (see section III). As \( N \) increases the representations become more and more complicated. This renders them useless for any practical application. Note that this is unlike SU(2) case where Schwinger bosons creation operators commute amongst themselves and hence have built in permutation symmetry of SU(2) Young tableaux along its row (see section III). In this work, we define SU(N) irreducible Schwinger bosons (henceforth we call them SU(N) ISB) with built in symmetries of SU(N) Young tableaux. As a result in terms of these SU(N) ISB all SU(N) representations are monomials like in simple SU(2) case. Further, we show that all SU(N) invariant operators constructed out of SU(N) ISB trivially annihilate the above SU(N) representation states. Hence, like SU(2) case, SU(N) ISB representations are multiplicity free. Another feature of the construction is that it iterative in nature. The results obtained for creation operators for SU(N) get carried over to SU(N+1) without any change (see section IV and V). In fact, the present work is SU(N) generalization of SU(3) work [15]. The plan of the paper is as follows. It contains four sections on SU(2), SU(3), SU(4) and SU(N) ISB respectively so that the presentation remains transparent and self contained. In section III we start with SU(2) Schwinger boson construction briefly to highlight all its special and simple features. In section III we describe the earlier work [15] on SU(3) irreducible Schwinger bosons and then reformulate the problem in a language which can be directly generalized to SU(N). Before proceeding to general case, we note that unlike SU(2) and SU(3) the higher SU(N) \((N \geq 4)\) groups have fundamental representations which are not \(N\)-plets leading to different types of group invariants. In fact, this is the reason why the SU(3) irreducible Schwinger boson construction in [15] can
not be directly generalized to higher SU(N) and requires change of language. Therefore in section (IV) we discuss the SU(4) representations explicitly. As we will see this SU(4) section makes the transition from SU(2), SU(3) to SU(N) easy and smooth. In section [V] we construct all the \((N-1)\) SU(N) ISB and show that all SU(N) representations in terms of ISB are monomials without multiplicities. We conclude the work with a brief discussion on parallels between the constraints in the present work and Gauss law constraints in quantum gauge theories [11]. In fact, the constraints leading to the Hilbert spaces containing SU(N) representations without multiplicities are analogous to the Gauss law constraints leading to the physical Hilbert space containing states without gauge multiplicities.

II. SU(2) SCHWINGER BOSON REPRESENTATIONS

The three generators \(\{J_1, J_2, J_3\}\) of the group SU(2) satisfy the following commutation relation among themselves:

\[
[J_a, J_b] = i \epsilon_{abc} J_c; \quad a, b, c = 1, 2, 3.
\]  

This algebra can be realized in terms of a doublet of Harmonic oscillator creation and annihilation operators given by \((a^\dagger_1, a^\dagger_2)\) satisfying the algebra,

\[
[a_\alpha, a^\dagger_\beta] = \delta_\alpha^\beta \quad , \quad [a^\dagger_\alpha, a^\dagger_\beta] = 0 \quad , \quad [a_\alpha, a_\beta] = 0 .
\]  

In terms of these operators,

\[
J^a = \sum_{a,\beta=1}^2 a^{\dagger a} \left( \frac{\sigma^a}{2} \right)^\beta a_\beta .
\]
where $\sigma^a$ denote the Pauli matrices. It is easy to check that these operators satisfy the $SU(2)$ Lie algebra with the $SU(2)$ Casimir:

$$J^2 \equiv \frac{a^\dagger \cdot a}{2} \left( \frac{a^\dagger \cdot a}{2} + 1 \right),$$

(4)

where $a^\dagger \cdot a (= a^\dagger_1 a_1 + a^\dagger_2 a_2)$ is the total number operator.

Thus the representations of $SU(2)$ can be characterized by the eigenvalues of the total occupation number operator with the angular momentum satisfying,

$$j = \frac{(n_1 + n_2)}{2} \equiv \frac{n}{2},$$

(5)

where $n_1$ and $n_2$ are the eigenvalues of $a^\dagger_1 a_1$ and $a^\dagger_2 a_2$ respectively.

An arbitrary $SU(2)$ representation characterized by angular momentum $j = \frac{n}{2}$ is given by the $SU(2)$ Young tableau shown in Figure 1. As harmonic oscillator creation operators $a^\dagger_\alpha (\alpha = 1, 2)$ commute amongst themselves, the simple monomial state:

$$|\psi\rangle^{\alpha_1 \alpha_2 \cdots \alpha_n} = a^{\dagger \alpha_1} a^{\dagger \alpha_2} \cdots a^{\dagger \alpha_n} |0\rangle \equiv O^{\alpha_1 \alpha_2 \cdots \alpha_n} |0\rangle.$$  

(6)

is symmetric under all possible $n!$ permutations of spin indices $(\alpha_1, \alpha_2, \cdots \alpha_n)$.

Above we have defined $O^{\alpha_1 \alpha_2 \cdots \alpha_n} = a^{\dagger \alpha_1} a^{\dagger \alpha_2} \cdots a^{\dagger \alpha_n}$ for later convenience. Thus the symme-

![SU(2) Young tableau for the representation $n = 2j$. Each SU(2) Schwinger boson $a^{\dagger \alpha}$ in (6) creates a Young tableau box.](image)

FIG. 1: $SU(2)$ Young tableau for the representation $n = 2j$. Each $SU(2)$ Schwinger boson $a^{\dagger \alpha}$ in (6) creates a Young tableau box.
states $|\psi^{a_1, a_2, \ldots, a_n}\rangle$ in (6) form an SU(2) irrep with $j = \frac{n}{2}$. From now on we shall say that SU(2) Schwinger bosons are SU(2) irreducible as they have built in symmetry of SU(2) Young tableau. In other words, no explicit symmetrization of spin half indices is required to obtain SU(2) irreducible representations. As mentioned in section I, the aim of the present work is to define SU(N) irreducible Schwinger bosons which have the built in symmetries of SU(N) Young tableaux. Thus all SU(N) representations in terms of SU(N) ISB retain the simplicity and elegance of SU(2) representations in (6).

### III. SU(3) SCHWINGER BOSON REPRESENTATIONS

In this section we briefly review the work in [15] and then recast it in a new framework which is generalizable to SU(N). The details of first part can be found in [15]. The rank of the SU(3) group is two. Therefore, to cover all SU(3) irreducible representations we need two independent harmonic oscillator triplets. Let’s denote them by $\{a^{\dagger}_{\alpha}\} \in 3$ and $\{b^{\dagger}_{\alpha}\} \in 3^*$ with $\alpha = 1, 2, 3$. Now the generators of SU(3) group are written as [17]:

$$Q^a = a^{\dagger} \lambda^a \frac{1}{2} a - b^{\dagger} \bar{\lambda}^a \frac{1}{2} b, \quad a = 1, 2, \cdots, 8. \tag{7}$$

In (7) $\lambda^a$ are the Gell Mann matrices for triplet representation, $-\bar{\lambda}^a$ are the corresponding matrices for the $3^*$ representation where $\bar{\lambda}$ denotes the transpose of $\lambda$. The defining relation (7) implies that under SU(3) $(a^{\dagger})_a$ and $(b^{\dagger})^a$ transform according to 3 and $3^*$ representations. As $Q^a, (a = 1, 2, .., 8)$ in (7) involve both creation and annihilation operators, the SU(3) Casimirs are the total occupation number operators of $a$ and $b$ type oscillators:

$$N_a = a^{\dagger} \cdot a, \quad N_b = b^{\dagger} \cdot b. \tag{8}$$
FIG. 2: SU(3) Young tableau for the representation \((n, m)\). Each SU(3) irreducible Schwinger bosons \(A^a_i(B^b_i)\) creates a single and double Young tableau box.

We represent their eigenvalues by \(n\) and \(m\) respectively and the SU(3) vacuum state \((n = 0, m = 0)\) by \(|0\rangle\). The representations \((n, m)\) are associated with Young tableau shown in Figure 2 with \(n\) single and \(m\) double boxes. At this stage, we define six dimensional Hilbert space \(\mathcal{H}^6_{HO}\) which is created by six oscillators \((a^+_\alpha, b^+_\alpha)\) for \(\alpha = 1, 2, 3\). The basis vectors can be written as:

\[
|n_1, n_2, n_3\rangle \equiv \left(\begin{array}{c} \alpha_1^1 \alpha_2^1 \alpha_3^1 \\ \beta_1^1 \beta_2^1 \beta_3^1 \end{array}\right) |0\rangle ,
\]

or equivalently by:

\[
|\alpha_1^1 \alpha_2^1 \alpha_3^1 \rangle \equiv \mathcal{O}^\dagger_{\beta_1^1 \beta_2^1 \beta_3^1} |0\rangle \equiv \left(\begin{array}{c} \alpha_1^1 \alpha_2^1 \alpha_3^1 \\ \beta_1^1 \beta_2^1 \beta_3^1 \end{array}\right) |\alpha_1^1 \alpha_2^1 \alpha_3^1 \rangle ,
\]

with \(\mathcal{O}^\dagger_{\beta_1^1 \beta_2^1 \beta_3^1} \equiv (a^+_\alpha^1)(a^+_\alpha^2)\ldots(a^+_\alpha^m)(b^+_\beta^1)(b^+_\beta^2)\ldots(b^+_\beta^m)\). The irreducible SU(3) representation states are the states in \(\mathcal{H}^6_{HO}\) which are traceless in any pair of upper and lower indices [16,17]. These SU(3) representation states are explicitly constructed in [13,15] and given by the following polynomial of SU(3) Schwinger bosons:

\[
|\psi\rangle^{\alpha_1 \alpha_2 \ldots \alpha_m}_{\beta_1 \beta_2 \ldots \beta_m} \equiv \left[ O^{\alpha_1 \alpha_2 \ldots \alpha_m}_{\beta_1 \beta_2 \ldots \beta_m} + L_1 \sum_{l_1=1}^{n_1} \sum_{k_1=1}^{n_2} \delta_{\beta_1^1 \beta_1^2} O^{\alpha_1 \alpha_2 \ldots \alpha_m}_{\beta_1 \beta_2 \ldots \beta_m} + L_2 \sum_{l_1=1}^{n_2} \sum_{k_1=1}^{n_3} \delta_{\beta_1^1 \beta_1^2} \right] + L_3 \sum_{l_1=1}^{n_2} \sum_{k_1=1}^{n_3} \sum_{l_2=1}^{n_3} \delta_{\beta_1^1 \beta_2^1} \delta_{\beta_1^2 \beta_2^2} \delta_{\beta_1^3 \beta_2^3} O^{\alpha_1 \alpha_2 \ldots \alpha_m}_{\beta_1 \beta_2 \ldots \beta_m}
\]
The coefficients $L_r$ are given by \[ L_r \equiv \frac{(-1)^r (a^+ \cdot b^+)^r}{(n + m + 1)(n + m)(n + m - 1)\ldots(n + m + 2 - r)} \quad r = 1, 2, \ldots q \equiv \min(n, m), \]
leading to tracelessness conditions:
\[
\sum_{l, k=1}^{3} \delta_{\beta_l \beta_k}^{\alpha_1 \alpha_2 \ldots \alpha_n} |\psi_{\rho}^{\alpha_1 \alpha_2 \ldots \alpha_n} \rangle = 0, \quad \text{for all } l = 1, 2, \ldots n, \text{ and } k = 1, 2, \ldots m.
\]
Note that the conditions (12) represent a single constraint as the SU(3) irreducible states $|\psi_{\rho}^{\alpha_1 \alpha_2 \ldots \alpha_n} \rangle$ are symmetric in upper ($\alpha$) and lower ($\beta$) indices. In fact, in the case of SU(3) the trace zero constraint is exactly equivalent to the vertical anti-symmetry of SU(3) Young tableau in Figure 2. Apart from the complicated and involved representations (10), the Schwinger boson construction of SU(3) suffers from the multiplicity problem. This problem arises because the states which differs from (10) by factors of SU(3) invariant operator $a^+ \cdot b^+$ transform exactly the same way [14, 15]. Therefore, the infinite tower of states:
\[
|\psi_{\rho}^{\alpha_1 \alpha_2 \ldots \alpha_n} \rangle \equiv (a^+ \cdot b^+)^\rho |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n} \rangle,
\]
transform exactly like $(n, m)$ representation in (10). In the SU(2) case we do not face the multiplicity problem because the only invariant operator is the total number operator in (4) which, also being the Casimir simply multiplies the states in (6) by its eigenvalue $n = 2j$. Another equivalent and compact way to obtain and understand SU(3) irreducible representations and its multiplicity problem is through the use of SU(3) invariant Sp(2,R) algebra [14]. As in [14] we define the following SU(3) invariant operators:
\[
k_+ \equiv a^+ \cdot b^+, \quad k_- \equiv a \cdot b, \quad k_0 \equiv \frac{1}{2} (N_a + N_b + 3).
\]
It is easy to check that they satisfy $\text{Sp}(2,\mathbb{R})$ or $\text{SU}(1,1)$ algebra:

\[
[k_-,k_+] = 2k_0, \quad [k_0,k_+] = k_+, \quad [k_0,k_-] = -k_-. \tag{15}
\]

The two mutually commuting $\text{SU}(3)$ and $\text{Sp}(2,\mathbb{R})$ algebras in (7) and (15) were exploited in [14] to label the infinite tower of states in (13) by the additional ‘magnetic quantum number’ of the $\text{Sp}(2,\mathbb{R})$ group. In particular the states in (10) carry the lowest magnetic quantum number and therefore trivially satisfy:

\[
k_- |\psi\rangle_{\beta_1,\beta_2,...,\beta_m}^{a_1,a_2,...,a_n} \equiv a \cdot b |\psi\rangle_{\beta_1,\beta_2,...,\beta_m}^{a_1,a_2,...,a_n} = 0. \tag{16}
\]

In fact, the $\text{SU}(3)$ tracelessness constraint (12) and $\text{Sp}(2,\mathbb{R})$ constraint (16) are exactly equivalent [14, 15]. The constraint (16) also reduces the 6 dimensional harmonic oscillator Hilbert space $\mathcal{H}_{\text{HO}}^6$ to the 5 dimensional $\text{SU}(3)$ Hilbert space $\mathcal{H}_{\text{SU}(3)}^{(5)}$. The $\text{SU}(3)$ Hilbert space $\mathcal{H}_{\text{SU}(3)}^{(5)}$ is generally characterized by $|N_a,N_b,I,M,Y\rangle$ where $N_a,N_b$ are the $\text{SU}(3)$ Casimirs fixing the representation. The other three quantum numbers are usually the total isospin $I$, its third component $M$ and hyper charge $Y$ which are the Casimirs of the chain of canonical subgroup $\text{SU}(2) \otimes \text{U}(1) \in \text{SU}(3)$ and $\text{U}(1) \in \text{SU}(2)$ respectively.

**A. The Irreducible SU(3) Schwinger Bosons**

In recent work [15] we have defined and constructed $\text{SU}(3)$ irreducible creation and annihilation operators which directly create the $\text{SU}(3)$ irreducible Hilbert space spanned by (10) from the vacuum. We define [15]:

\[
A^{\dagger}a = a^{\dagger}a - \frac{1}{N_a + N_b + 1}(a^\dagger \cdot b^\dagger)b^a, \quad A_a = a_a - b_a^\dagger(a \cdot b) \frac{1}{N_a + N_b + 1} \approx a_a
\]

\[
B^{\dagger}a = b^{\dagger}a - \frac{1}{N_a + N_b + 1}(a^\dagger \cdot b^\dagger)a^a, \quad B^a = b^a - a^{\dagger a}(a \cdot b) \frac{1}{N_a + N_b + 1} \approx b^a \tag{17}
\]
In (17) \( \approx \) means that that these identities are weakly satisfied. More explicitly, \( A_\alpha \approx a_\alpha \) means that the actions of \( A_\alpha \) and \( a_\alpha \) are same on SU(3) irreducible Hilbert space satisfying the constraint (16). It is easy to check that the irreducible Schwinger boson creation operators commute amongst themselves:

\[
\left[ A^{\dagger \alpha}, A^{\dagger \beta} \right] = 0, \quad \left[ B^{\dagger \alpha}, B^{\dagger \beta} \right] = 0, \quad \left[ A^{\dagger \alpha}, B^{\dagger \beta} \right] = 0.
\] (18)

Hence the general SU(3) irreducible representations (10) which are extremely complicated polynomials of ordinary SU(3) Schwinger bosons can now be simply written as monomials of SU(3) irreducible Schwinger bosons (15):

\[
|\psi\rangle^\alpha_{\mu_1 \mu_2 \ldots \mu_m} = A^{\dagger \alpha_1} A^{\dagger \alpha_2} \ldots A^{\dagger \alpha_n} B^{\dagger \beta_1} B^{\dagger \beta_2} \ldots B^{\dagger \beta_m} |0\rangle.
\] (19)

The permutation symmetries of the upper and lower indices are inbuilt because of the commutation relations (18). The tracelessness of any mixed state \((n, m)\) is also obvious and immediately follows from the tracelessness of the octet state. To see this, we consider:

\[
|\psi\rangle^\alpha_{\mu_1 \mu_2 \ldots \mu_m} = A^{\dagger \alpha_1} B^{\dagger \beta_1} |\psi\rangle^\alpha_{\mu_2 \ldots \mu_m} = A^{\dagger \alpha_2} \ldots A^{\dagger \alpha_n} B^{\dagger \beta_2} \ldots B^{\dagger \beta_m} |\psi\rangle^\alpha_{\mu_1} = 0.
\] (20)

In (20), we have used the fact that all the \( A^{\dagger} \)'s and \( B^{\dagger} \)'s commute amongst themselves (18) and the octet state \( |\psi\rangle^\alpha_{\mu} \) is traceless. We further note that:

\[
A \cdot B |\psi\rangle^\alpha_{\beta_1 \beta_2 \ldots \beta_m} \approx a \cdot b |\psi\rangle^\alpha_{\alpha_1 \alpha_2 \ldots \alpha_n} = 0,
\]

\[
A^{\dagger} \cdot B^{\dagger} |\psi\rangle^\alpha_{\beta_1 \beta_2 \ldots \beta_m} = \sum_{\gamma=1}^{3} |\psi\rangle^\alpha_{\gamma \beta_1 \beta_2 \ldots \beta_m} = 0.
\] (21)

The only other SU(3) invariant operators which can be constructed out of these irreducible operators are \( A^{\dagger} \cdot A \) and \( B^{\dagger} \cdot B \) which following (16) and (17) are simply the number operators \( a^{\dagger} \cdot a \) and \( b^{\dagger} \cdot b \) respectively. As they are also SU(3) Casimirs, they do not lead to
any multiplicity. Thus this construction in terms of SU(3) irreducible Schwinger bosons also solves the SU(3) multiplicity problem. Further, like in SU(2) case in Figure 1, each Young tableau single (double) box $\in 3 (3^{*})$ representation in Figure 2 corresponds to the irreducible Schwinger boson creation operator $A^{ta} (B^{ta})$. In other words the defining equations of SU(3) irreducible Schwinger bosons (17) already take care of all the symmetries of SU(3) Young tableaux through the constraint (15). No explicit symmetrization or anti-symmetrization is needed. At this stage, in order to formulate SU(N) problem, it is convenient to recast the above results in terms of two harmonic oscillators triplets instead of a triplet and an antitriplet in (7):

$$Q^{a} = a^{\dagger}[1] \frac{\lambda^{a}}{2} a[1] + a^{\dagger}[2] \frac{\lambda^{a}}{2} a[2].$$  \hspace{1cm} (22)

In (22) both $a^{\dagger}[1]$ and $a^{\dagger}[2]$ transform like triplets of SU(3):

$$[Q^{a}, a^{ta}[i]] = \left( a^{\dagger}[i] \frac{\lambda^{a}}{2} \right)^{a}, \quad i = 1, 2.$$  \hspace{1cm} (23)

One can get the anti-triplet $3^{*}$ representation by taking the anti-symmetric combination of the two triplets: $b^{t}_{a} = \epsilon_{a\bar{p}j} a^{\dagger}[1] a^{\dagger}[2]$. In the present representation the four SU(3) invariant operators are [7]:

$$\hat{L}_{ij} = a^{\dagger}[i] \cdot a[j], \quad i, j = 1, 2.$$  \hspace{1cm} (24)

They satisfy U(2) algebra:

$$[\hat{L}_{ij}, \hat{L}_{kl}] = \delta_{jk} \hat{L}_{il} - \delta_{il} \hat{L}_{kj}. $$  \hspace{1cm} (25)

Note that $\hat{L}_{11}$ and $\hat{L}_{22}$ are the two number operators $N_{1}$ and $N_{2}$ respectively which are also the two Casimirs of the SU(3) algebra in (22). We denote their eigenvalues by $n_{1}$
FIG. 3: SU(3) Young table in the representation \([n_1, n_2]\) with two triplets. The same Young tableau with a triplet and an anti-triplet in Figure 2 is characterized by \((n, m)\) with \(n = n_1 - n_2\) and \(m = n_2\) and \(n_2\) respectively. The corresponding SU(3) Young tableau is shown in Figure (3). The advantage of the change of language from (7) to (22) is that the constraint analogous to (16) is now obvious and can be easily generalized to SU(N). The antisymmetrization along the \(n_2\) columns of the SU(3) Young tableau in Figure 3 immediately implies the constraint [7]:

\[
\hat{L}_{12} \equiv a_{[1]}^{\dagger} \cdot a_{[2]} \approx 0.
\]  

(26)

Note that \(\hat{L}_{12}\) is an exchange operator. It replaces type [2] Schwinger boson by type [1] Schwinger boson. Therefore, the only solutions of the constraint (26) are the states which are anti-symmetric in color indices along the \(n_2\) columns of the SU(3) Young tableau in Figure 3. The most general forms of \(A_{[1]}^{\dagger}\) and \(A_{[2]}^{\dagger}\) which transform like triplets and also increase \(N[1]\) and \(N[2]\) by 1 respectively are:

\[
A_{[1]}^{\dagger} = a^{\dagger}[1]
\]  

(27)

\[
A_{[2]}^{\dagger} = a^{\dagger}[2] + F^2(N_1, N_2) \hat{L}_{21} a^{\dagger}[1].
\]
The unknown coefficient $F_1^2(N_1, N_2)$ is fixed by demanding:

$$ (a^+[1] \cdot a[2])A^{ta} \approx 0 \Rightarrow F_1^2(N_1, N_2) = -\frac{1}{N_1 - N_2 + 2}. \quad (28) $$

In (28) we have used $[\hat{L}_{12}, \hat{L}_{21}] = N_1 - N_2$. Note that the coefficient $F_1^2(N_1, N_2)$ is always well defined as the eigenvalues of $N_1$ and $N_2$ corresponding to the SU(3) Young tableau in Figure (3) satisfy $n_1 \geq n_2$. The operator $a^{ta}[1]$ already commutes with the constraint operator $\hat{L}_{12}$ in (26) and therefore remains unchanged in (27). In other words, the first triplet of SU(3) ISB retains the form of SU(2) Schwinger bosons. For later convenience we formally write the first equation in (27) as:

$$ A^{ta}[1]_{SU(3)} = A^{ta}[1]_{SU(2)} = a^{ta}[1]. \quad (29) $$

The above equation emphasizes the form invariance and iterative nature of the construction to be used later in section IV and V. It is easy to check that,

$$ [A^{ta}[1], A^{tp}[1]] = 0, \quad [A^{ta}[2], A^{tp}[2]] = 0. \quad (30) $$

Therefore a general $(n, m)$ irrep of SU(3) is obtained by these new Schwinger bosons as:

$$ |\psi\rangle^{(\beta_1 \cdots \beta_n)(\alpha_1 \cdots \alpha_m)} = A^{tp_1}[2]A^{tp_2}[2] \cdots A^{tp_n}[2]A^{ta_1}[1]A^{ta_2}[1] \cdots A^{ta_m}[1]|0\rangle. \quad (31) $$

The simple monomial construction (31) is equivalent to the SU(3) young tableaux with appropriate symmetries. To see this we construct the simplest mixed octet representation with $n_1 = 2$ and $n_2 = 1$ in (31):

$$ |\psi\rangle^{(\beta)(\alpha_1 \alpha_2)} = A^{tp_1}[2]A^{ta_1}[1]A^{ta_2}[1]|0\rangle = A^{tp_1}[2]A^{ta_1}[1]A^{ta_2}[1]|0\rangle \\ = \frac{1}{3} \left\{ (a^{tp_1}[2]a^{ta_1}[1] - a^{ta_1}[2]a^{tp_1}[1])a^{ta_2}[1] + (a^{tp_1}[2]a^{ta_2}[1] - a^{ta_2}[2]a^{tp_1}[1])a^{ta_1}[1] \right\}|0\rangle. \quad (32) $$
The expression in (32) is first antisymmetrized amongst the column indices $\alpha_1$ and $\beta$, then symmetrized amongst the row indices $\alpha_1 & \alpha_2$. Therefore it has the symmetries of SU(3) Young tableaux in figure 3. This simplest but non-trivial example illustrates the usefulness of the procedure to compute SU(3) representations in terms of SU(3) irreducible Schwinger bosons. *The simple monomial in (37) captures all the symmetries of the SU(3) Young tableau diagram in Figure (3).* Destruction operators corresponding to (27) can be constructed by canonical conjugation of $A^\dagger[1]$ and $A^\dagger[2]$. However, these operators will not commute with the constraint (26) and will take us out of $H_{SU(3)}$. On the other hand, we can also construct the SU(3) irreducible destruction operators weakly commuting with $\hat{L}_{12}$. The construction in exactly same as that of $A^{\dagger \alpha}[2]$ and one obtains:

$$A_{\alpha}[1] = a_{\alpha}[1] + \frac{1}{N_1 - N_2 + 2} \hat{L}_{21} a_{\alpha}[2],$$
$$A_{\alpha}[2] = a_{\alpha}[2]. \quad (33)$$

The irreducible Schwinger boson representations are free from any kind of multiplicity problems as all the non-trivial SU(3) invariant operators are weakly zero:

$$A^\dagger[1] \cdot A[2] \equiv a^\dagger[1] \cdot a[2] \equiv \hat{L}_{12} \approx 0, \quad (34)$$
$$A^\dagger[2] \cdot A[1] = \left( a^{\dagger \alpha}[2] - \frac{1}{N_1 - N_2 + 2} \hat{L}_{21} a^{\dagger \alpha}[1] \right) \left( a_{\alpha}[1] + \frac{1}{N_1 - N_2 + 2} \hat{L}_{21} a_{\alpha}[2] \right)$$
$$= -\frac{1}{(N_1 - N_2 + 2)(N_1 - N_2 + 3)} \left( \hat{L}_{21} \right)^2 \hat{L}_{12} \approx 0, \quad (35)$$
$$A[1] \cdot A^\dagger[2] = \left( a_{\alpha}[1] + \frac{1}{N_1 - N_2 + 2} \hat{L}_{21} a_{\alpha}[2] \right) \left( a^{\dagger \alpha}[2] - \frac{1}{N_1 - N_2 + 2} \hat{L}_{21} a^{\dagger \alpha}[1] \right)$$
$$= -\frac{1}{(N_1 - N_2 + 2)(N_1 - N_2 + 3)} \left( \hat{L}_{21} \right)^2 \hat{L}_{12} \approx 0. \quad (36)$$

Note that in calculating $A^\dagger[2] \cdot A[1]$ and $A[1] \cdot A^\dagger[2]$ in (35) and (36) respectively all the linear terms in $\hat{L}_{21}$ cancel out exactly and the quadratic term $\left( \hat{L}_{21} \right)^2$ is proportional to the
constraint $\hat{L}_{12} \approx 0$ in (26). Note that the total ISB number operators are trivial invariant operators as:

$$A^\dagger[1] \cdot A[1] \approx a^\dagger[1] \cdot a[1] \quad \& \quad A^\dagger[2] \cdot A[2] \approx a^\dagger[2] \cdot a[2] .$$

(37)

As mentioned in the introduction at this stage it is illustrative to give explicit construction of SU(4) irreducible Schwinger bosons before dealing with SU(N) in section (V).

**IV. SU(4) IRREDUCIBLE REPRESENTATIONS**

The rank of SU(4) group is 3. Therefore, as shown in Figure (4), we need three 4-plets $a^\dagger \alpha[i], i = 1, 2, 3$ to construct any irreducible representation of SU(4). The SU(4) generators in terms of these Schwinger bosons are:

$$Q^a = a^\dagger[1] \frac{\Lambda^a}{2} a[1] + a^\dagger[2] \frac{\Lambda^a}{2} a[2] + a^\dagger[3] \frac{\Lambda^a}{2} a[3], \quad a = 1, 2, \cdots, 15.$$  

(38)

In (38) $\Lambda^a$ are the $4 \times 4$ representations of SU(4) Lie algebra. The 12 harmonic oscillators in (38) create a 12 dimensional Hilbert space $H_{HO}^{12}$. The SU(4) invariant group is now U(3) consisting of 9 generators $\hat{L}_{ij}$ in (24) and (25) with $i, j = 1, 2, 3$. Like in SU(3) case, we obtain the vertical anti-symmetry of SU(4) Young tableau in Figure (4) by demanding [7]:

$$\hat{L}_{12} = a^\dagger[1] \cdot a[2] \approx 0$$

(39)

$$\hat{L}_{13} = a^\dagger[1] \cdot a[3] \approx 0$$

(40)

$$\hat{L}_{23} = a^\dagger[2] \cdot a[3] \approx 0.$$  

(41)

The three constraints (39), (40) & (41) impose the vertical anti symmetries in the first-second, first-third & second-third rows respectively of the SU(4) Young tableau shown
Therefore the null space of \( [\hat{L}_{12}, \hat{L}_{13}, \hat{L}_{23}] \) within \( \mathcal{H}_{HO}^{12} \) is the space of SU(4) irreducible representation. This space is \( 9(=12-3) \) dimensional and will be denoted by \( \mathcal{H}_{SU(4)}^9 \). Amongst the nine quantum numbers labeling \( \mathcal{H}_{SU(4)}^9 \), three are the eigenvalues of the three SU(4) Casimir number operators: \( \hat{L}_{ii} = a_\dagger[i] \cdot a[i] \) with \( i = 1, 2, 3 \). The remaining 6 magnetic quantum numbers characterizing a specific state within the above representation are usually chosen to be the eigenvalues of the Casimirs of the canonical subgroup chain: \( SU(3) \otimes U(1) \in SU(4); SU(2) \otimes U(1) \in SU(3) \) and \( U(1) \in SU(2) \). Thus, like in SU(3) case, the dimension of SU(4) representation space computed through canonical way matches with the constraint analysis above (also see [7]). We now come to the explicit construction.

We would like to construct three types of SU(4) irreducible Schwinger bosons \( \hat{A}_i \) with \( i = 1, 2, 3 \) so that the monomial states:

\[
\langle \{ \alpha_1 \ldots \alpha_{n_1} \} \{ \beta_1 \ldots \beta_{n_2} \} \{ \gamma_1 \ldots \gamma_{n_3} \} \rangle_{n_1 \geq n_2 \geq n_3} = \left( A_\dagger^{\gamma_1} [3] \ldots A_\dagger^{\gamma_{n_3}} [3] \right) \left( A^{\beta_1} [2] \ldots A^{\beta_{n_2}} [2] \right) \left( A^{\alpha_1} [1] \ldots A^{\alpha_{n_1}} [1] \right) \langle 0 \rangle
\]

(42)

carry the symmetries of SU(4) Young tableau shown in Figure 4. Note that the ordering is important in (42). Like in SU(3) case, the irreducible Schwinger bosons that increase
$N_1, N_2, N_3$ by one respectively are constructed as:

$$A^{\dagger a}[1] = a^{\dagger a}[1]$$  \hspace{1cm} (43)

$$A^{\dagger a}[2] = a^{\dagger a}[2] + F_1^2(N_1, N_2, N_3) \hat{L}_{21} a^{\dagger a}[1]$$

$$A^{\dagger a}[3] = a^{\dagger a}[3] + F_2^3(N_1, N_2, N_3) \hat{L}_{32} a^{\dagger a}[2] + F_1^3(N_1, N_2, N_3) \hat{L}_{31} a^{\dagger a}[1]$$

$$+ F_2^3(N_1, N_2, N_3) (\hat{L}_{32} \hat{L}_{21}) a^{\dagger a}[1].$$

Note that the three constraints are not independent:

$$[\hat{L}_{12}, \hat{L}_{23}] = \hat{L}_{13}.$$  

Hence implementation of the two constraints (39) and (41) should enable us to compute all the four structure functions $F$. It is clear from (43) that $A^{\dagger a}[1]$ commutes with all the constraint given in (39, 40, 41). The form of $A^{\dagger a}[2]$ in (44) is exactly same as in the case of SU(3) in (27) except that $a$ runs from 1 to 4 and the number operators correspond to that of SU(4). Thus following the same method with the constraint (39) we obtain the same solution (28) for $F_1^2(N_1, N_2, N_3)$:

$$F_1^2(N_1, N_2, N_3) = F_1^2(N_1, N_2) = -\frac{1}{N_1 - N_2 + 2},$$  \hspace{1cm} (44)

leading to,

$$A^{\dagger a}[2] = a^{\dagger a}[2] + \frac{1}{N_1 - N_2 + 2} \hat{L}_{21} a^{\dagger a}[1].$$  \hspace{1cm} (45)

Note that the other two constraints $\hat{L}_{13}$ and $\hat{L}_{23}$ are already satisfied by $A^{\dagger a}[2]$ in (42), i.e.:

$$\hat{L}_{13} A^{\dagger a}[2] \approx \left[ \hat{L}_{13}, A^{\dagger a}[2] \right] = \frac{1}{N_1 - N_2 + 1} \hat{L}_{23} a^{\dagger a}[1] \approx 0$$

$$\hat{L}_{23} A^{\dagger a}[2] \approx 0.$$  \hspace{1cm} (46)
Imposing (39) and (41) and after some algebra we get:

\[ F_2^3 = -\frac{1}{(N_2 - N_3 + 2)}; \quad F_1^3 = -\frac{1}{(N_1 - N_3 + 3)}; \quad F_{21}^3 = \frac{1}{(N_2 - N_3 + 2)(N_1 - N_3 + 3)} \equiv F_2^3 F_1^3. \quad (47) \]

Like in SU(3) case, the construction of the SU(4) irreducible destruction operators is similar and one obtains:

\[ A_{\alpha}[3] = a_{\alpha}[3] \quad (48) \]

\[ A_{\alpha}[2] = a_{\alpha}[2] + \frac{1}{N_2 - N_3 + 2} \hat{L}_{23} \hat{a}_{\alpha}[3] \]

\[ A_{\alpha}[1] = a_{\alpha}[1] + \frac{1}{N_1 - N_2 + 2} \hat{L}_{12} \hat{a}_{\alpha}[2] + \frac{1}{N_1 - N_3 + 3} \hat{L}_{13} \hat{a}_{\alpha}[3] \]

\[ + \frac{1}{(N_1 - N_2 + 2)(N_1 - N_3 + 3)} \hat{L}_{12} \hat{L}_{23} \hat{a}_{\alpha}[3]. \]

One can easily check the commutation relations:

\[ [A^{\dagger \alpha[i]}, A^{\dagger \beta[i]}] = 0; \quad i = 1, 2, 3. \quad (49) \]

In fact, the above identity is trivial for i = 1 as \( A^{\dagger \alpha}[1] \equiv a^{\dagger \alpha}[1]. \) Thus the antisymmetrizations amongst the SU(4) Young tableau column indices in (Figure 4) are implemented by imposing constraints (39), (40) and (41) on the SU(4) irreducible Schwinger bosons.

Having antisymmetrized this way, the commutation relations (49) ensure the horizontal permutation symmetries amongst the indices belong to each of the three rows of Figure 4. As a result the resultant ISB monomial states in (42) carry all the symmetries of SU(4) Young tableau. Further these representations are also multiplicity free as:

\[ A[1] \cdot A^\dagger[2] \approx 0, \quad A[2] \cdot A^\dagger[1] \approx 0 \quad (50) \]

\[ A[1] \cdot A^\dagger[3] \approx 0, \quad A[3] \cdot A^\dagger[1] \approx 0 \quad (51) \]

\[ A[2] \cdot A^\dagger[3] \approx 0, \quad A[3] \cdot A^\dagger[2] \approx 0. \quad (52) \]
The above SU(4) results are analogues of SU(3) results in (34), (35) and (36). An alternative irreducible Schwinger boson construction procedure is to exploit the iterative nature of the solutions. In the present SU(4) case we note that \( A^\alpha \{1 \} \) and \( A^\alpha \{2 \} \) retain the same form as SU(3) and only \( A^\alpha \{3 \} \) has to be constructed to satisfy all the three constraints. In the construction of \( A^\alpha \{3 \} \) also one can use SU(3) ISB so that the first fundamental constraint \( (\hat{L}_{12} \approx 0) \) becomes redundant and only the last constraint \( (\hat{L}_{23} \approx 0) \) has to be implemented by hand. Like in SU(3) case (see eqn. 29), we stress on this form invariance by rewriting the three equations in (43) as:

\[
\begin{align*}
A^\alpha \{1 \}^{[\text{SU}(4)]} & = A^\alpha \{1 \}^{[\text{SU}(3)]} = A^\alpha \{1 \}^{[\text{SU}(2)]} \\
A^\alpha \{2 \}^{[\text{SU}(4)]} & = A^\alpha \{2 \}^{[\text{SU}(3)]} \\
A^\alpha \{3 \}^{[\text{SU}(4)]} & = a^\alpha \{3 \} + G_2^3 \left( a^\dagger \{3 \} \cdot A^\alpha \{2 \}^{[\text{SU}(3)]} \right) A^\alpha \{2 \}^{[\text{SU}(3)]} \\
& + G_1^3 \left( a^\dagger \{3 \} \cdot A^\alpha \{1 \}^{[\text{SU}(3)]} \right) A^\alpha \{1 \}^{[\text{SU}(3)]}.
\end{align*}
\]  

(53)

Note that all the constraints in (39, 40 and 41) are trivially satisfied by \( A^\alpha \{1 \}^{[\text{SU}(4)]} \) and \( A^\alpha \{2 \}^{[\text{SU}(4)]} \) by construction. Now \( \hat{L}_{23} = a^\dagger \{2 \} \cdot a \{3 \} \approx 0 \) can be solved for \( G_2^3 \) and \( G_1^3 \) as:

\[
\begin{align*}
G_2^3 & = -\frac{1}{N_2 - N_3 + 2} \\
G_1^3 & = -\frac{N_1 - N_2 + 2}{(N_1 - N_2 + 1)(N_1 - N_3 + 3)}.
\end{align*}
\]  

(54)

One can check explicitly that with these coefficients the construction in (53), with the coefficients in (44) and (47), is exactly same as (43). We will use this iterative construction for SU(N) in the next section.

V. SU(N) IRREDUCIBLE SCHWINGER BOSONS

The fundamental constituents required to construct any arbitrary irrep of SU(N) are \( N - 1 \) independent Schwinger boson \( N \)-plets given by \( a^\alpha \{1 \}, a^\alpha \{2 \}, a^\alpha \{3 \}, \ldots, a^\alpha \{N - 1 \}, \)
FIG. 5: SU(N) Young tableau for the representation $[n_1, n_2, \cdots, n_{N-1}]$.

with $\alpha = 1, 2, 3, \ldots, N$ since the rank of the group SU(N) is $N - 1$. The SU(N) generators in terms of these Schwinger bosons are:

$$Q^a = \sum_{i=1}^{N-1} a^\dagger[i] \frac{\Lambda^a}{2} a[i], \quad (55)$$

where, $\Lambda^a$'s are the generalization of Gell-Mann matrices for SU(N). The $N(N - 1)$ Harmonic oscillators present in (55) creates a $N(N - 1)$ dimensional Hilbert space $\mathcal{H}_{\text{HO}}^{N(N-1)}$.

There are $(N - 1)$ Casimirs associated with SU(N) group. The SU(N) invariant group is now U(N-1) with $(N - 1)^2$ generators given by $\hat{L}_{ij}, \ i, j = 1, 2, \ldots, N - 1$. In the representation (55) the $(N - 1)$ Casimirs are the number operators $\hat{L}_{ii} \equiv \hat{N}[i] = a^\dagger[i] \cdot a[i]$ with $i = 1, 2, \ldots, N - 1$. Their eigenvalues, specifying a particular representation are denoted by: $(n_1, n_2, \cdots, n_{N-1})$ respectively. For arbitrary SU(N) we obtain the vertical antisymmetry of an Young tableaux by imposing the constraints [7]:

$$\hat{L}_{ij} = a^\dagger[i] \cdot a[j] \approx 0, \quad \text{for } i < j \text{ and } i, j = 1, 2, \ldots, N - 1 \text{ for SU(N).} \quad (56)$$

Therefore the null space of $\hat{L}_{ij}$ for $i < j$ within $\mathcal{H}_{\text{HO}}^{N(N-1)}$ is the space of SU(N) irreducible
representations. There are \( \frac{1}{2}(N-1)(N-2) \) constraints in (56). Therefore the dimension of the Hilbert space containing SU(N) representations in Figure 5 is \( N(N-1)-\frac{1}{2}(N-1)(N-2) = \frac{1}{2}(N-1)(N+2) \). The remaining \( \frac{1}{2}N(N-1) \) \( \left(= \frac{1}{2}(N-1)(N+2) - (N-1) \right) \) SU(N) ‘magnetic quantum numbers’ specify a particular state within the above representation. These magnetic quantum numbers are usually taken as the eigenvalues of the \( \frac{1}{2}N(N-1) \) Casimirs of the canonical subgroup chain:

\[
\text{SU}(N-1) \otimes \text{U}(1) \in \text{SU}(N), \text{SU}(N-2) \otimes \text{U}(1) \in \text{SU}(N-1), \ldots, \text{U}(1) \in \text{SU}(2).
\]

We now come to the explicit construction. We would like to construct \( N-1 \) types of SU(N) irreducible Schwinger bosons \( A^\dagger[i] \) with \( i = 1, 2, \ldots, N-1 \) so that the states:

\[
\left\{ a_{1}^{[1]} a_{2}^{[1]} \ldots a_{n}^{[1]} \right\} \left\{ a_{1}^{[2]} a_{2}^{[2]} \ldots a_{n}^{[2]} \right\} \ldots \left\{ a_{1}^{[N-1]} a_{2}^{[N-1]} \ldots a_{n}^{[N-1]} \right\} \equiv \langle 0 \left| A^{[1]}_{N-1} [N-1] \cdots A^{[N-1]}_{N-1} [N-1] \right| A^{[1]}_{N} [1] \cdots A^{[1]}_{N} [1] \rangle (57)
\]

carry all the symmetries of SU(N) Young tableau. As discussed in the SU(4) section, \( A^\dagger [k], k = 1, 2, \ldots, N-2 \) for SU(N) have exactly the same form as SU(N-1). We only need to construct \( A^\dagger [N-1] \). We construct this in terms of SU(N-1) ISB so that we have to implement only the last fundamental constraint:

\[
\hat{L}_{(N-2)(N-1)} = a^\dagger [N-2] \cdot a [N-1] \approx 0.
\]

Thus the SU(N) irreducible Schwinger bosons are given by,

\[
A^{a[1]}_{SU(N)} = A^{a[1]}_{SU(N-1)} = A^{a[1]}_{SU(N-2)} = \cdots = A^{a[1]}_{SU(2)}
\]

\[
A^{a[2]}_{SU(N)} = A^{a[2]}_{SU(N-1)} = A^{a[2]}_{SU(N-2)} = \cdots = A^{a[2]}_{SU(3)}
\]

\[
A^{a[3]}_{SU(N)} = A^{a[3]}_{SU(N-1)} = A^{a[3]}_{SU(N-2)} = \cdots = A^{a[3]}_{SU(4)}
\]
\[ A^{\dagger a}[N-2]^{SU(N)} = A^{\dagger a}[N-2]^{SU(N-1)}, \]
\[ A^{\dagger a}[N-1]^{SU(N)} = a^{\dagger a}[N-1] + \sum_{i=1}^{N-2} G_i^{N-1} \left( a^+[N-1] \cdot A[i]^{SU(N-1)} \right), \]
\[ A^{\dagger a}[i]^{SU(N-1)}. \]

Note that in (58) \( \alpha = 1, 2, \cdots, N \) and all the number operators are of \( n \)-plets of \( SU(N) \). We again emphasize that the first \( (N-3) \) fundamental constraints

\[ \hat{L}_{i,i+1} = a^+[i] \cdot a[i+1] \approx 0, \quad i = 1, 2, \cdots N-3 \]

are already satisfied by the \( SU(N) \) ISB in (58). Unfolding all the ISB constructions from \( SU(N-1) \) to \( SU(2) \) one obtains the most general form of \( k \)-th irreducible Schwinger bosons:

\[ A^{\dagger a}[k] = a^{\dagger a}[k] + \sum_{r=1}^{k-1} \sum_{(i_1, \ldots, i_r)=1} F^k_{i_1} F^k_{i_2} \cdots F^k_{i_r} \hat{L}_{i_1i_2} \cdots \hat{L}_{i_{r-1}i_r} a^{\dagger a}[i_r]. \] (59)

In (59) \( k = 1, 2, \cdots (N-1) \) and the prime over the second summation \((\sum')\) implies that the ordering \( k > i_1 > i_2 > \ldots > i_r \) has to be maintained. To find out the general form of \( F^k_i(N_1, \ldots, N_{N-1}) \) it is sufficient to apply only the consecutive \((k-1)\) fundamental constraints \( \hat{L}_{p(p+1)} \approx 0 \), where \( p = 1, 2, \ldots, k-1 \) as all other constraints can be obtained as the commutators of these fundamental constraints, e.g.: \( \hat{L}_{13} = [\hat{L}_{12}, \hat{L}_{23}], \hat{L}_{14} = [\hat{L}_{13}, \hat{L}_{34}] = [[\hat{L}_{12}, \hat{L}_{23}], \hat{L}_{34}] \) etc.. It is quite straightforward to show that the constraint \( \hat{L}_{(N-1)N} \approx 0 \) gives

\[ F^k_{k-1}(N_1, \ldots, N_{N-1}) = F^k_{k-1}(N_k, N_{k-1}) = -\frac{1}{N_{k-1} - N_k + 2}. \] (60)

After some algebra, the general constraints \( a^+[i] \cdot a[i+1] \approx 0 \) gives the recurrence relation:

\[ F^k_p(N_1, \ldots, N_{N-1}) = \frac{F^k_{p+1}(N_1, \ldots, N_{N-1})}{1 - (N_p - N_{p+1} + 1) F^k_{p+1}(N_1, \ldots, N_{N-1})}. \] (61)

The solution of (61) with (60) as the boundary condition is:

\[ F^k_i = -\frac{1}{N_i - N_k + 1 + k - i}. \] (62)
Note that these SU(N) solutions for $F^k_i$ reduce to (28) and (47) for N=3 and 4 respectively. Similarly all the $N-1$ fundamental irreducible annihilation operators for SU(N) can also be constructed using the irreducible creation and annihilation operators for SU(N-1). The general $k^{th}$ annihilation operator for SU(N) is given by,

$$ A_a[k]^{SU(N)} = a_a[k] + \sum_{i=k+1}^{N-1} G^K_i \left(a_K[H] \cdot A^\dagger[i]^{SU(N-1)}\right) A_a[i]^{SU(N-1)} , \quad (63) $$
or equivalently,

$$ A_a[k] \equiv a_a[k] + \sum_{r=1}^{N-1} \sum_{\{i_1,i_2,\ldots,i_r=k+1\}} H^i_k H^j_k \cdots H^r_k \hat{L}_{i_1} \hat{L}_{i_2} \cdots \hat{L}_{i_r} a_a[i_r] . \quad (64) $$

In (64) $k = 1, 2, \ldots, (N-1)$ and the prime over the second summation ($\sum'$) implies that the ordering $k < i_1 < i_2 < \ldots < i_r < N - 1$ has to be maintained. The similar algebra as done for creation operators gives:

$$ H^i_k = \frac{1}{N_i - N_k + 1 + k - i} \equiv -F^k_i . \quad (65) $$

The Hilbert space created by SU(N) irreducible Schwinger bosons contains all SU(N) representations and every representation appears once as:

$$ A^\dagger[i] \cdot A[j] \approx 0, \quad \forall \ i \neq j . $$

The only remaining SU(N) invariant operators in terms of SU(N) ISB are $A^\dagger[i] \cdot A[i], \ i = 1, 2, \ldots, (N-1)$. These operators, being weakly related to the SU(N) number operator Casimirs, do not lead to multiplicity.

VI. SUMMARY AND DISCUSSIONS

We conclude that SU(N) representations constructed in terms of SU(N) ISB are complete as well as economical. Further, like in SU(2) case, all representations are monomials
of SU(N) ISB. This is because the SU(N) ISB are defined and constructed such that they
carry the symmetries of SU(N) Young tableaux making explicit symmetrization, antisym-
metrizations redundant. Thus the SU(N) irreducible Schwinger bosons \( N \geq 3 \) provide
a model for SU(N) just like SU(2) Schwinger bosons provide a model for SU(2). At this
stage, following the discussion in \[11\], it is interesting to mention the parallels with quant-
ization of gauge theories. In gauge theories there are many spurious gauge degrees
of freedom. Therefore, all states connected by gauge transformations represent a single
physical state. This multiplicity is removed by imposing the Gauss law constraint on the
physical Hilbert space. In the case of pure electrodynamics the Gauss law constraint is:

\[
\nabla \cdot E |\Psi\rangle_{\text{physical}} = 0 \quad \text{or} \quad \nabla \cdot E \approx 0.
\]

In (66) \( \nabla \cdot E \) represents the divergence of electric field. This is analogous to the constraints
(16,56) which can be interpreted as ‘group theory Gauss law’ constraints in \( \mathcal{H}_{\text{HO}}^{N(N-1)} \).
The representation redundancy in \( \mathcal{H}_{\text{HO}}^{N(N-1)} \) is generated by the generators of invariant
\( U(N-1) \) group. In the case of SU(3), discussed in detail in \[14\] the six dimensional
harmonic oscillator Hilbert space \( \mathcal{H}_{\text{HO}}^{6} \) was completely spanned by vectors labeled with
the 6 quantum numbers belonging to \( SU(3) \otimes Sp(2,R) \) group. Similarly, the harmonic
oscillator Hilbert space \( \mathcal{H}_{\text{HO}}^{N(N-1)} \) can also be completely spanned by vectors labeled by
the \( N(N-1) \) quantum numbers belonging to \( SU(N) \otimes U(N-1) \) or equivalently \( SU(N) \otimes
SU(N-1) \otimes U(1) \). The work in this direction is in progress and will be reported elsewhere.

An important application of SU(N) ISB is in lattice gauge theories \[19\]. The SU(N) ISB
enable us to remove the redundant gauge as well as loop degrees of freedom from lattice
gauge theories \[3,19\] leading to a formulation in terms of loops and strings without any
spurious gauge or loop degrees of freedom. In fact, this has been the starting point and
the motivation behind the present work and the work in [15, 19].

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