Dense binary $PG(t - 1, 2)$-free matroids have critical number $t - 1$ or $t$

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Abstract

The critical threshold of a (simple binary) matroid $N$ is the infimum over all $\rho$ such that any $N$-free matroid $M$ with $|M| > \rho 2^{r(M)}$ has bounded critical number. In this paper, we resolve two conjectures of Geelen and Nelson, showing that the critical threshold of the projective geometry $PG(t - 1, 2)$ is $1 - 3 \cdot 2^{-t}$. We do so by proving the following stronger statement: if $M$ is $PG(t - 1, 2)$-free with $|M| > (1 - 3 \cdot 2^{-t})2^{r(M)}$, then the critical number of $M$ is $t - 1$ or $t$. Together with earlier results of Geelen and Nelson [9] and Govaerts and Storme [11], this completes the classification of dense $PG(t - 1, 2)$-free matroids.

1 Introduction

In this paper, the term matroid refers to a simple binary matroid. We represent such a matroid $M$ as a set of edges, $E(M)$, which is a full-rank subset $\mathbb{F}_2^{r(M)} \setminus \{0\}$ where $r(M)$ is the rank of the matroid. The cardinality of a matroid $M$, denoted $|M|$, is simply the cardinality of its edge set and the critical number, $\chi(M)$, is the smallest $k$ such that there exists a codimension $k$ subspace of $\mathbb{F}_2^{r(M)}$ that is disjoint from $E(M)$. A matroid $M$ contains another matroid $N$ if there is a linear injection $\iota : \mathbb{F}_2^{r(N)} \rightarrow \mathbb{F}_2^{r(M)}$ such that $\iota(E(N)) \subset E(M)$. The projective geometry $PG(t - 1, 2)$ is regarded as a matroid with $r(M) = t$ and $E(M) = \mathbb{F}_2^t \setminus \{0\}$.

Our goal is to understand the relationship between density and critical number for matroids avoiding a fixed matroid $N$. In particular, we are interested in the critical threshold of $N$ - the infimum over all $\rho$ such that any $N$-free matroid $M$ with $|M| > \rho 2^{r(M)}$ has bounded critical number. In this paper we determine the critical threshold of the projective geometries $PG(t - 1, 2)$ and characterize the critical number of $PG(t - 1, 2)$-free matroids with density above that threshold. This question was posed by Geelen and Nelson [9] as a generalization of the following problem in graph theory.

An example of Hajnal shows that there exist triangle-free graphs $G$ with arbitrarily large chromatic number and minimum degree arbitrarily close to

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\(1/2|V(G)|\) (see \([7]\)). Based on this construction, Erdős and Simonovits \([7]\) asked whether triangle-free graphs \(G\) with \(\delta(G) > 1/2|V(G)|\) have bounded chromatic number. This question was solved by Thomassen \([14]\), showing that the chromatic threshold of the triangle is \(1/3\).

In fact, much more is known. It is easy to see that no triangle-free graphs \(G\) exist with \(\delta(G) > 1/2|V(G)|\). Further bounds have been derived \([2, 12, 4]\) and are listed below. All of these bounds are tight.

\[
\begin{array}{c|c|c|c|c}
\delta(G)/|V(G)| & > \frac{1}{2} & > \frac{2}{3} & > \frac{4}{5} & > \frac{5}{6} \\
\chi(G) & \text{no graphs} & \leq 2 & \leq 3 & \leq 4 \\
\end{array}
\]

Goddard and Lyle \([10]\) showed that the chromatic threshold of the complete graph \(K_r\) is \(\frac{2r-2}{2r-3}\). Furthermore, their result allowed them to generalize the above table to \(K_r\)-free graphs. Finally, Allen et al. \([1]\) compute the chromatic threshold of an arbitrary graph in terms of its chromatic number. Namely, for \(c \geq 3\), the chromatic threshold of a graph \(H\) with \(\chi(H) = c\) is one of \(\{\frac{c-1}{c-2}, \frac{3c-2}{4c-3}, \frac{6c-5}{7c-6}\}\).

The analogous question for triangle-free matroids was first investigated by Davydov and Tombak in the context of linear binary codes \([6]\). They essentially showed that the critical threshold of the triangle is at most \(1/4\). Geelen and Nelson \([9]\) proved a lower bound on the critical threshold of all matroids, which resolves the question for the triangle.

The following is known about the relationship between density and critical number for \(PG(t-1, 2)\)-free matroids.

**Theorem 1.1** (\([9]\)). For \(t \geq 2\) and \(\epsilon > 0\), there exist \(PG(t-1, 2)\)-free matroids \(M\) with \(|M| > (1 - 3 \cdot 2^{-t} - \epsilon)2^{r(M)}\) and arbitrarily large \(\chi(M)\).

**Theorem 1.2** (Theorem 3.16 in \([5]\), based on \([6]\)). Any \(PG(1, 2)\)-free matroid \(M\) with \(|M| > \frac{1}{4}2^{r(M)}\) satisfies \(\chi(M) \leq 2\).

**Theorem 1.3** (\([11]\)). For \(t \geq 2\), any \(PG(t-1, 2)\)-free matroid \(M\) with \(|M| > (1 - \frac{1}{4t} \cdot 2^{-t})2^{r(M)}\) satisfies \(\chi(M) = t - 1\).

We use similar ideas to Govaerts and Storme’s proof of Theorem \([13]\) to show that \(1 - 3 \cdot 2^{-t}\) is the critical threshold of \(PG(t-1, 2)\) and to extend Theorem \([12]\) to higher values of \(t\). The following is the main result of this paper.

**Theorem 1.4.** For \(t \geq 2\), any \(PG(t-1, 2)\)-free matroid \(M\) with \(|M| > (1 - 3 \cdot 2^{-t})2^{r(M)}\) satisfies \(\chi(M) \in \{t-1, t\}\).

**Remark 1.5.** Note that any matroid \(M\) with \(\chi(M) = k\) satisfies \(|M| \leq (1 - 2^{-k})2^{r(M)}\). Thus \(\chi(M) \geq t - 1\) for any matroid \(M\) satisfying the hypotheses of Theorems \([13]\) or \([14]\).

The critical threshold of other matroids is still unknown, though Geelen and Nelson have shown the following upper bound.

**Theorem 1.6** (\([8]\)). Any matroid \(N\) has critical threshold at most \(1 - 2 \cdot 2^{-\chi(N)}\).
They also conjecture the following, of which the lower bound is known [9].

**Conjecture 1.7.** A matroid $N$ with $\chi(N) \geq 2$ has critical threshold $1 - i \cdot 2^{-\chi(N)}$ for some $i \in \{2, 3, 4\}$.

Our approach to prove Theorem 1.4 is to induct on $t$, using Theorem 1.2 as the starting point. Our main argument in Section 2 shows that the $t - 1$ case of the main theorem implies the $t$ case for $t \geq 6$. In Section 3 we use some modification of the same methods to show that the $t = 4$ case implies the $t = 5$ case. Finally, in Section 4 we use an alternative argument to show that Theorem 1.2 implies the $t = 3$ and $t = 4$ cases.

## 2 Proof of main theorem for $t \geq 6$

Assuming the main theorem is true for some $t - 1$, we suppose there exists a matroid $M$ which is a counterexample to the main theorem at this value of $t$. In Subsections 2.1 and 2.2 we consider two auxiliary matroids and apply the $t - 1$ case of the main theorem to them in order to derive several properties that any counterexample $M$ must satisfy. Finally, in Subsection 2.3 we derive a contradiction from these properties.

### 2.1 Hyperplane intersection

**Proposition 2.1.** Fix $t \geq 3$. Assuming Theorem 1.4 for $t - 1$, say $M$ is a $PG(t - 1, 2)$-free matroid with $|M| > (1 - 3 \cdot 2^{-t})2^{r(M)}$ and $\chi(M) > t$. Set $Y = F_2^{r(M)} \setminus E(M)$. Then for any hyperplane $H < F_2^{r(M)}$, we have

$$|Y \setminus H| \geq 2^{r(M) - t}.$$  

*Proof. We know that $|E(M) \cap H| > (1 - 3 \cdot 2^{1-t})2^{r(M) - 1}$. Therefore, by assumption, either $E(M) \cap H$ contains $PG(t - 2, 2)$ or $\chi(E(M) \cap H) \leq t - 1$. The latter is not possible, since a codimension $\leq t - 1$ subspace of $H$ which is disjoint from $E(M) \cap H$ is a codimension $\leq t$ subspace of the whole space that is still disjoint from $E(M)$.

Therefore there is some copy of $PG(t - 2, 2)$ in $E(M) \cap H$. Let $G$ be the dimension $t - 1$ subspace of $H$ that contains this projective geometry. We use $G + v$ to denote the coset $\{g + v : g \in G\}$. Suppose there is some $v$ such that $G + v \subset E(M)$. Then there is a copy of $PG(t - 1, 2)$ in $E(M)$ — just take our original copy of $PG(t - 2, 2)$ and all of $G + v$. This is impossible since $M$ is $PG(t - 1, 2)$-free, so every coset of $G$ must intersect $Y$. There are $2^{r(M) - t}$ cosets of $G$ off of $H$, so the desired inequality follows.

We can use the above proposition to give an upper bound on $|Y \cap H|$ for $H$ of any fixed codimension. We will use the following three bounds later in the argument.
Corollary 2.2. Fix $t \geq 3$. Assuming Theorem 1.4 for $t - 1$, say $M$ is a $PG(t - 1, 2)$-free matroid with $|M| > (1 - 3 \cdot 2^{-t})2^r(M)$ and $\chi(M) > t$. Set $Y = E_2^{r(M)} \setminus E(M)$. For any subspace $H \subseteq E_2^{r(M)}$, we have

(i) if codim $H = 1$, then $|Y \cap H| < 2 \cdot 2^r(M) - t$,
(ii) if codim $H = 2$, then $|Y \cap H| < \frac{3}{2} \cdot 2^r(M) - t$,
(iii) if codim $H = 3$, then $|Y \cap H| < \frac{5}{4} \cdot 2^r(M) - t$.

Proof. The first inequality follows directly from Proposition 2.1 and the other two follow by the pigeonhole principle. Namely, for the sake of contradiction, suppose $H$ is a codimension 2 subspace with $|Y \cap H| < \frac{3}{2} \cdot 2^r(M) - t$. Then we can find a coset $H + v$ with $v \notin H$ such that $|Y \cap (H + v)| \geq \frac{1}{2}|Y \setminus H|$. Setting $H' = H \cup (H + v)$, we have

$$|Y \setminus H'| < \frac{2}{3}|Y \setminus H| < 2^r(M) - t,$$

which contradicts Proposition 2.1. This proves (ii).

Similarly, for the sake of contradiction, let $H$ be a codimension 3 subspace with $|Y \setminus H| < \frac{3}{4} \cdot 2^r(M) - t$. Then we can find a coset $H + v$ with $v \notin H$ such that $|Y \cap (H + v)| \geq \frac{1}{4}|Y \setminus H|$. Then setting $H' = H \cup (H + v)$, we have

$$|Y \setminus H'| < \frac{6}{7}|Y \setminus H| < \frac{3}{4} \cdot 2^r(M) - t,$$

which contradicts what was showed above. This proves (iii). 

\[\square\]

2.2 Doubling construction

Definition 2.3. Let $M$ be a matroid. For $v \in E_2^{r(M)}$, define the matroid $M_v$ by $r(M_v) = r(M)$ and $x \in E(M_v)$ if and only if $x, x + v \in E(M)$.

It is easy to compute the size of $M_v$ in terms of our original matroid.

Lemma 2.4. Let $M$ be a matroid. Set $Y = E_2^{r(M)} \setminus E(M)$. For any $v \in E_2^{r(M)}$, we have

$$|M_v| = 2^r(M) - 2|Y| + |Y \cap (Y + v)|,$$

where $Y + v$ denotes the set $\{y + v : y \in Y\}$. In particular, if $|M| > (1 - 3 \cdot 2^{-t})2^r(M)$, then $|M_v| > (1 - 3 \cdot 2^{-t})2^r(M)$.

Proof. We have $E_2^{r(M)} \setminus E(M_v) = Y \cup (Y + v)$. Thus $|E_2^{r(M)} \setminus E(M_v)| = |Y| + |Y + v| - |Y \cap (Y + v)| = 2|Y| - |Y \cap (Y + v)|$, as desired. \[\square\]

The matroid $M_v$ satisfies the following two useful properties.

Proposition 2.5. Fix $t \geq 3$. Let $M$ be a matroid and pick $v \in E(M)$. If $M_v$ contains $PG(t - 2, 2)$, then $M$ contains $PG(t - 1, 2)$. 

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Proposition 2.6. Let $M$ be a matroid and pick $v \in \mathbb{F}_2^r(M)$. If $\chi(M_v) = k$, then there exists a codimension $k$ subspace $H < \mathbb{F}_2^r(M)$ with $v \in H$ and $|E(M) \cap H| \leq 2^{r(M)-k-1}$.

Proof. Let $H$ be a maximal subspace disjoint from $E(M_v)$. By assumption, codim $H = k$. Now if $v \not\in H$, then $H$ and $H + v$ are disjoint so $H \cup (H + v)$ is a codimension $k - 1$ subspace which is also disjoint from $E(M_v)$. This contradicts the maximality of $H$, so $v \in H$.

Now we claim that $|E(M) \cap H| \leq \frac{1}{2}|H|$. We know that for any $x \in E(M) \cap H$, it is not the case that $x + v \in E(M) \cap H$ since then $x, x + v \in E(M_v)$. Therefore at most half of the elements of $H$ can be in $E(M)$.

Combining the properties we derived above with the $t - 1$ case of the main theorem and our bounds from Subsection 2.4 we obtain the following useful properties of any counterexample to our main theorem.

Proposition 2.7. Fix $t \geq 3$. Assuming Theorem 1.4 for $t - 1$, say $M$ is a PG$(t - 1, 2)$-free matroid with $|M| > (1 - 3 \cdot 2^{-t})2^{r(M)}$ and $\chi(M) > t$. Set $Y = \mathbb{F}_2^r(M) \setminus E(M)$. We have the following:

(i) There exist distinct codimension $t - 1$ subspaces $H_1, H_2, \ldots, H_s < \mathbb{F}_2^r(M)$ such that $|Y \cap H_i| \geq 2^{r(M) - t}$ for $1 \leq i \leq s$ and $E(M) \subset \bigcup_{i=1}^s H_i$. Furthermore, $s > 2^t - 3$.

(ii) For all $v \in E(M)$, we have $|Y \cap (Y + v)| \leq 2^{r(M) - t - 1}$.

Proof. By Lemma 2.4 for $v \in E(M)$ we have that $|E(M_v)| > (1 - 3 \cdot 2^{1-t})2^{r(M)}$.

By assumption, either $E(M_v)$ contains PG$(t - 2, 2)$ or $\chi(M_v) \in \{t - 2, t - 1\}$.

By Proposition 2.6 there is a subspace $H$ with codim $H = \chi(M_v) \in \{t - 2, t - 1\}$ and $|Y \cap H| \geq 2^{r(M) - \chi(M_v) - 1}$.

We claim that $\chi(M_v) = t - 1$. If not, then $H$ has codimension $t - 2$ and $|Y \cap H| \geq 2^{r(M) - t + 1}$. Since $t \geq 3$, the subspace $H$ is contained in some hyperplane $H'$ that satisfies $|Y \cap H'| \geq |Y \cap H| \geq 2 \cdot 2^{r(M) - t}$, which contradicts Corollary 2.2 (i).

This implies the first part of (i) since by Proposition 2.6 we can find some $H$ for each $v \in E(M)$ that satisfies $v \in H$. To bound $s$, note that since $E(M) \subset \bigcup_{i=1}^s H_i$, we have

$$\sum_{i=1}^s |E(M) \cap H_i| \geq |E(M)| > (1 - 3 \cdot 2^{-t})2^{r(M)}.$$

Now since $|E(M) \cap H_i| \leq 2^{r(M) - t}$ for all $i$, we conclude that $s > 2^t - 3$. 

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We now can deduce (ii) from the above and Theorem 1.3. We know that \( \chi(M_v) = t - 1 \) and \( M_v \) is \( \text{PG}(t-2,2) \)-free. Therefore \( M_v \) cannot satisfy the hypotheses of Theorem 1.3 since this would imply that \( \chi(M_v) = t - 2 \). Thus we have \( |M_v| \leq (1 - \frac{1}{3}) \cdot 2^{1-t} 2^{r(M)} \). Applying Lemma 2.4 we conclude that 
\[ |Y \cap (Y + v)| < 2^{r(M) - t-1}. \]

\[\boxed{}\]

2.3 Proof of main theorem

Proposition 2.7 (i) is almost strong enough to prove the main theorem – it says that any counterexample \( M \) must have many small subspaces that each contain more than a third of the elements of \( \mathbb{F}_2^{r(M)} \setminus E(M) \).

To finish the proof, we first show that there must be a pair of these subspaces with large intersection.

**Proposition 2.8.** Fix \( t \geq 3 \). Assuming Theorem 1.3 for \( t-1 \), say \( M \) is a \( \text{PG}(t-1,2) \)-free matroid with \( |M| > (1 - 3 \cdot 2^{-t}) 2^{r(M)} \) and \( \chi(M) > t \). Set \( Y = \mathbb{F}_2^{r(M)} \setminus E(M) \). Then there exist distinct codimension \( t-1 \) subspaces \( H_1, H_2 < \mathbb{F}_2^{r(M)} \) such that \( |Y \cap H_1|, |Y \cap H_2| \geq 2^{r(M) - t} \) and \( \text{codim} H_1 \cap H_2 \leq 2t + 2 \).

**Proof.** Applying Proposition 2.7 (i), we choose codimension \( t-1 \) subspaces \( H_1, H_2, \ldots, H_s \) with \( |Y \cap H_i| \geq 2^{r(M) - t} \) for each \( 1 \leq i \leq s \). Furthermore, we have that \( s \geq 4 \) for \( t \geq 3 \).

Now we count
\[ |Y| \geq \sum_{1 \leq i \leq 4} |Y \cap H_i| - \sum_{1 \leq i < j \leq 4} |Y \cap H_i \cap H_j|. \]

If we assume for the sake of contradiction that \( \text{codim} H_i \cap H_j > t + 2 \) for each \( 1 \leq i < j \leq 4 \), we have \( |Y \cap H_i \cap H_j| \leq |H_i \cap H_j| \leq 2^{r(M) - t - 3} \). This implies
\[ 3 \cdot 2^{r(M) - t} > |Y| \geq 4 \cdot 2^{r(M) - t} - 6 \cdot 2^{r(M) - t - 3} = \frac{13}{4} 2^{r(M) - t}, \]
a contradiction.

In fact, something much stronger than the above holds, which we will use in Section 3.

To complete the proof, we use Proposition 2.7 (ii) to show that the intersection of these subspaces cannot contain too many elements of \( \mathbb{F}_2^{r(M)} \setminus E(M) \). Finally, we show that this information contradicts the bounds that were derived in Subsection 2.1.

**Proposition 2.9.** Fix \( t \geq 3 \). Assuming Theorem 1.3 for \( t-1 \), say \( M \) is a \( \text{PG}(t-1,2) \)-free matroid with \( |M| > (1 - 3 \cdot 2^{-t}) 2^{r(M)} \) and \( \chi(M) > t \). Set \( Y = \mathbb{F}_2^{r(M)} \setminus E(M) \). For any codimension \( t \) subspace \( H < \mathbb{F}_2^{r(M)} \), we have
\[ |Y \cap H| < \frac{3}{4} 2^{r(M) - t}. \]
Proof. Since $\chi(M) > t$, we know that there must be some $v \in E(M) \cap H$. Now note that $(Y \cap H) + v \subset H$. Therefore, we have

$$|H| \geq |(Y \cap H) \cup (Y \cap H + v)|$$

$$= 2|Y \cap H| - |Y \cap (Y + v) \cap H|$$

$$\geq 2|Y \cap H| - |Y \cap (Y + v)|.$$ 

Now $|H| = 2^{r(M) - t}$ and $|Y \cap (Y + v)| < 2^{r(M) - t - 1}$ by Proposition 2.7(ii), so the desired inequality follows.

Theorem 2.10. Fix $t \geq 6$. Assuming Theorem 1.4 for $t - 1$, then there exist no $PG(t - 1, 2)$-free matroids $M$ with $|M| > (1 - 3 \cdot 2^{-t})2^{r(M)}$ and $\chi(M) > t$.

Proof. Suppose for the sake of contradiction that $M$ is a counterexample. Set $Y = F^r(M) \setminus E(M)$. By Proposition 2.8, we can find distinct codimension $t - 1$ subspaces $H_1, H_2 \subset F^r(M)$ such that $t \leq \text{codim} H_1 \cap H_2 \leq t + 2$ and $|Y \cap H_1|, |Y \cap H_2| \geq 2^{r(M) - t}$. We have three cases.

Case 1: $\text{codim} H_1 \cap H_2 = t$

Let $H_1 + H_2$ refer to the smallest subspace of $F^r(M)$ containing both $H_1$ and $H_2$. We have $\text{codim} H_1 + H_2 = t - 2$ and

$$|Y \cap (H_1 + H_2)| \geq |Y \cap (H_1 \cup H_2)|$$

$$= |Y \cap H_1| + |Y \cap H_2| - |Y \cap (H_1 \cap H_2)|$$

$$> \frac{5}{4}2^{r(M) - t},$$

where the last line follows from Proposition 2.8.

If $t \geq 5$, this is a contradiction since $H_1 + H_2$ has codimension $t - 2 \geq 3$. Therefore $H_1 + H_2$ is contained in some codimension 3 subspace of $F^r(M)$. However, Corollary 2.2 (iii) says that any codimension 3 subspace of $F^r(M)$ contains strictly fewer than $\frac{5}{4}2^{r(M) - t}$ elements of $Y$.

Case 2: $\text{codim} H_1 \cap H_2 = t + 1$

In this case we have $\text{codim} H_1 + H_2 = t - 3$ and

$$|Y \cap (H_1 + H_2)| \geq |Y \cap H_1| + |Y \cap H_2| - |Y \cap (H_1 \cap H_2)|$$

$$\geq |Y \cap H_1| + |Y \cap H_2| - |H_1 \cap H_2|$$

$$\geq \frac{3}{2}2^{r(M) - t}.$$

Again for $t \geq 5$ we have that $H_1 + H_2$ is contained in a codimension 2 subspace of $F^r(M)$, which contradicts Corollary 2.2 (ii).

Case 3: $\text{codim} H_1 \cap H_2 = t + 2$

In this case, $\text{codim} H_1 + H_2 = t - 4$ and $|Y \cap (H_1 + H_2)| \geq \frac{7}{4}2^{r(M) - t}$. This contradicts Corollary 2.2 (ii) for $t \geq 6$ since then $H_1 + H_2$ is contained in a codimension 2 subspace.
3 Proof of main theorem for $t = 5$

In this section we extend the results of the previous section to the $t = 5$ case. Theorem 2.10 does not apply here since case 3 of that proof requires that $t \geq 6$. However, we can still use that proof to gives us some information in the $t = 5$ case.

**Corollary 3.1.** Assuming Theorem 1.4 for $t = 4$, say $M$ is a $PG(4, 2)$-free matroid with $|M| > (1 - 3 \cdot 2^{-5})2^r(M)$ and $\chi(M) > 5$. Set $Y = F_2^r(M) \setminus E(M)$. If $H_1, H_2 < F_2^r(M)$ are distinct codimension 4 subspaces satisfying $|Y \cap H_1|, |Y \cap H_2| \geq 2^r(M)^{-5}$, then $\text{codim} H_1 \cap H_2 \in \{7, 8\}$.

**Proof.** For $H_1, H_2$ of codimension 4 and distinct, we have that $5 \leq \text{codim} H_1 \cap H_2 \leq 8$. The proof of Theorem 2.10 shows that if $5 \leq \text{codim} H_1 \cap H_2 \leq 7$, then we must have $\text{codim} H_1 \cap H_2 = 7$, as desired. □

We now prove a strengthened version of Proposition 2.8.

**Proposition 3.2.** Assuming Theorem 1.4 for $t = 4$, say $M$ is a $PG(4, 2)$-free matroid with $|M| > (1 - 3 \cdot 2^{-5})2^r(M)$ and $\chi(M) > 5$. Set $Y = F_2^r(M) \setminus E(M)$. If $H_1, H_2, H_3, H_4 < F_2^r(M)$ are pairwise distinct codimension 4 subspaces satisfying $|Y \cap H_i| \geq 2^r(M)^{-5}$ for $1 \leq i \leq 4$, then there exist three pairs $1 \leq i < j \leq 4$ with $\text{codim} H_i \cap H_j = 7$.

**Proof.** This follows from the same argument used to prove Proposition 2.8. By Corollary 3.1, all of the intersections have codimension 7 or 8. Suppose for the sake of contradiction that at most two of the intersections have codimension 7. Then we have

$$3 \cdot 2^r(M)^{-5} > |Y| \geq \sum_{1 \leq i \leq 4} |Y \cap H_i| - \sum_{1 \leq i < j \leq 4} |Y \cap H_i \cap H_j|$$

$$\geq 4 \cdot 2^r(M)^{-5} - 4 \cdot 2^r(M)^{-8} - 2 \cdot 2^r(M)^{-7}$$

$$= 3 \cdot 2^r(M)^{-5},$$

which is a contradiction. □

To complete the proof, we need the following graph-theoretic lemma.

**Lemma 3.3.** Let $G$ be a graph on 30 vertices such that any four vertices of $G$ have at least three edges among them. Then there exists a vertex of $G$ with degree at least 16.

**Proof.** Say vertex $v$ has degree $\delta$. If $\delta \geq 16$ we are done. Otherwise, note that $G$ contains a copy of the complete graph $K_{29-\delta}$. This is because for any three vertices $u_1, u_2, u_3$ that are not neighbors of $v$, all of $u_1, u_2, u_3$ must be pairwise connected.

In particular, $G$ contains a $K_{14}$. Pick two vertices $v_1, v_2$ in the $K_{14}$. Label the vertices not in our $K_{14}$ by $x_1, x_2, \ldots, x_8$ and $y_1, y_2, \ldots, y_8$. Considering the
set of vertices \( x_i, y_i, v_1, v_2 \) for \( 1 \leq i \leq 8 \), there must be at least one edge between \( \{x_i, y_i\} \) and \( \{v_1, v_2\} \). Thus there are at least 8 edges from \( \{v_1, v_2\} \) to outside of the \( K_{14} \). Then one of \( v_1, v_2 \) is connected to at least 4 vertices outside of the \( K_{14} \) and thus has degree at least 17.

**Theorem 3.4.** Assuming Theorem 1.4 for \( t = 4 \), then there exist no \( PG(4, 2) \)-free matroids \( M \) with \( |M| > (1 - 3 \cdot 2^{-5})2^{r(M)} \) and \( \chi(M) > 5 \).

**Proof.** Suppose for the sake of contradiction that \( M \) is a counterexample. Let \( Y \) refer to \( F_2^r(M) \setminus E(M) \). By Proposition 2.7 (i), we can find codimension 4 subspaces \( H_1, H_2, \ldots, H_{30} \) such that \( |Y \cap H_i| \geq 2^{r(M)} - 5 \) for \( 1 \leq i \leq 30 \). Now Proposition 3.2 and Lemma 3.3 imply that we can order the \( H_i \)'s such that \( \text{codim} \, H_i \cap H_{i+1} = 7 \) for \( 1 \leq i \leq 16 \).

We claim that we can find some \( 1 \leq i < j \leq 16 \) such that \( H_i + H_j + H_{17} \) is a hyperplane. Consider \( H_i^+, H_j^+, \ldots, H_{17}^+ \). These are dimension 4 subspaces of \( F_2^r(M) \). We know that \( H_{17}^+ \) intersects each \( H_i^+ \) non-trivially, say at \( \{0, v_i\} \).

Since \( v_i \in H_{17}^+ \setminus \{0\} \) and \( |H_{17}^+ \setminus \{0\}| = 15 \), by the pigeonhole principle there exist some \( 1 \leq i < j \leq 16 \) with \( v_i = v_j \). Thus \( H_i^+ \cap H_j^+ \cap H_{17}^+ \neq \{0\} \), or equivalently, \( H_i + H_j + H_{17} \neq F_2^r(M) \), as desired.

Now note that \( \text{codim} \, H_i \cap H_j = 7 \) since \( H_i + H_j \leq H_i + H_j + H_{17} \leq F_2^r(M) \). Finally, we have

\[
|Y \cap (H_i + H_j + H_{17})| \geq |Y \cap H_i| + |Y \cap H_j| + |Y \cap H_{17}|
- |Y \cap H_i \cap H_j| - |Y \cap H_i \cap H_{17}| - |Y \cap H_j \cap H_{17}|
\geq 3 \cdot 2^{r(M) - 5} - |H_i \cap H_j| - |H_i \cap H_{17}| - |H_j \cap H_{17}|
\geq 3 \cdot 2^{r(M) - 5} - 3 \cdot 2^{r(M) - 7}
= \frac{9}{4} \cdot 2^{r(M) - 5}.
\]

As \( H_i + H_j + H_{17} \) is a hyperplane, this contradicts Corollary 3.2 (i). \( \blacksquare \)

### 4 Proof of main theorem for \( t = 3, 4 \)

To finish the last two cases of the main theorem, we turn to Fourier-analytic techniques. Together with some bounds from Section 2 and below, standard techniques from Fourier analysis imply the \( t = 3 \) case of the main theorem. With some more work, we use similar ideas to prove the \( t = 4 \) case.

We start by using the classical Bose-Burton theorem to derive a bound complementary to Proposition 2.1.

**Theorem 4.1** (Bose-Burton, [3]). Fix \( t \geq 2 \). Let \( M \) be a matroid. If \( |M| > (1 - 2 \cdot 2^{-t})2^{r(M)} \), then \( M \) contains \( PG(t - 1, 2) \).
Corollary 4.2. Fix \( t \geq 3 \). Assuming Theorem [1.4] for \( t - 1 \), say \( M \) is a \( \text{PG}(t - 1, 2) \)-free matroid with \( |M| > (1 - 3 \cdot 2^{-t})2^{r(M)} \) and \( \chi(M) > t \). Set \( Y = \F_2^{r(M)} \setminus E(M) \). For any hyperplane \( H < \F_2^{r(M)} \), we have

\[
|Y \cap H| \geq 2^{r(M) - t}.
\]

Proof. If not, \( |E(M) \cap H| > (1 - 2 \cdot 2^{-t})2^{r(M) - 1} \). Then by Theorem [1.1] there is a copy of \( \text{PG}(t - 1, 2) \) in \( E(M) \cap H \).

The above corollary and Proposition 2.1 serve to give a lower bound on \( \frac{1}{|M|} \sum_{v \in E(M)} |Y \cap (Y + v)| \) using Fourier-analytic techniques. The following proof is based on one found in Tao and Vu [13].

Proposition 4.3. Fix \( t \geq 3 \). Assuming Theorem [1.4] for \( t - 1 \), say \( M \) is a \( \text{PG}(t - 1, 2) \)-free matroid with \( |M| > (1 - 3 \cdot 2^{-t})2^{r(M)} \) and \( \chi(M) > t \). Then letting \( Y = \F_2^{r(M)} \setminus E(M) \), we have

\[
\frac{1}{|M|} \sum_{v \in E(M)} |Y \cap (Y + v)| > \frac{2}{3} |Y|^2.
\]

Proof. Proposition 2.1 and Proposition 4.2 together imply that for any hyperplane \( H < \F_2^{r(M)} \), we have \( |Y \cap H| - |Y \setminus H| < \frac{1}{3} |Y| \). This is saying that the Fourier bias of \( Y \) and of \( E(M) \) are less than \( \frac{1}{3} |Y|^2 \). The rest of the proof is a slightly strengthened version of Lemma 4.13 of [13] and uses much of the same argument.

Set \( Z = \F_2^{r(M)} \). For \( f : Z \to \mathbb{R} \), we define the Fourier transform \( \hat{f} : \hat{Z} \to \mathbb{R} \) by \( \hat{f}(\xi) = \frac{1}{|Z|} \sum_{x \in Z} f(x)\xi(x) \) where \( \hat{Z} \) denotes the Pontryagin dual of \( Z \). We define the convolution \( f * g : Z \to \mathbb{R} \) by \( f * g(x) = \frac{1}{|Z|} \sum_{y \in Z} f(y)g(x - y) \).

To derive the desired result, first note that

\[
\sum_{v \in E(M)} |Y \cap (Y + v)| = |\{(y_1, y_2, v) \in Y \times Y \times E(M) : y_1 + y_2 + v = 0\}|.
\]

Set \( |Y| = \alpha 2^{r(M)} \). We can write the latter quantity as \( 2^{2r(M)} \cdot 1_Y * 1_Y * 1_{E(M)}(0) \).

Then we have

\[
1_Y * 1_Y * 1_{E(M)}(0) = \sum_{\xi \in \hat{Z}} \hat{1}_Y(\xi)\hat{1}_Y(\xi)\hat{1}_{E(M)}(\xi)
\]

\[
\geq \alpha^2(1 - \alpha) \sum_{\xi \in \hat{Z} \setminus \{0\}} \hat{1}_Y(\xi)^2 \cdot |\hat{1}_{E(M)}(\xi)|
\]

\[
> \alpha^2(1 - \alpha) \cdot \frac{1}{3} \alpha \sum_{\xi \in \hat{Z} \setminus \{0\}} \hat{1}_Y(\xi)^2
\]

\[
= \alpha^2(1 - \alpha) \cdot \frac{1}{3} \alpha(\alpha - \alpha^2)
\]

\[
= \frac{2}{3} \alpha^2(1 - \alpha).
\]
The third line follows since the Fourier bias of $1_{E(M)}$ is less than $\frac{1}{4} \alpha$, which means that for $\xi \in \hat{Z} \setminus \{0\}$, we have $|1_{E(M)}(\xi)| < \frac{1}{4} \alpha$. The fourth line follows since $\sum_{\xi \in \hat{Z}} \hat{1}_{A}(\xi)^2 = \frac{|A|^2}{|Z|^2}$ for all $A \subseteq Z$.

Therefore we have

$$
\frac{1}{|M|} \sum_{v \in E(M)} |Y \cap (Y + v)| > \frac{1}{(1 - \alpha)2^{r(M)}} \frac{2}{3} \alpha^2 (1 - \alpha)2^{2r(M)} = \frac{2}{3}\frac{|Y|^2}{2^{r(M)}},
$$

as desired. \hfill \Box

Now for $t = 3$, this lower bound on $\frac{1}{|M|} \sum_{v \in E(M)} |Y \cap (Y + v)|$ contradicts the upper bound on $|Y \cap (Y + v)|$ from Proposition 2.7 (ii).

**Corollary 4.4.** There exist no $PG(2, 2)$-free matroids $M$ with $|M| > (1 - 3 \cdot 2^{-3})2^{r(M)}$ and $\chi(M) > 3$.

**Proof.** For the sake of contradiction, suppose $M$ is a counterexample. Set $Y = \mathbb{F}_2^r(M) \setminus E(M)$. Proposition 2.7 (ii) says that for all $v \in E(M)$, it is the case that $|Y \cap (Y + v)| < 2^{r(M) - 4}$ while Theorem 1.3 gives us that $\frac{|Y|^2}{2^{r(M)}} \geq \frac{11}{32}$. With Proposition 4.3 these bounds give

$$
\frac{1}{16} > \frac{1}{|M|} \sum_{v \in E(M)} \frac{|Y \cap (Y + v)|}{2^{r(M)}} > \frac{2}{3} \frac{|Y|^2}{2^{2r(M)}} \geq \frac{121}{1536},
$$

a contradiction. \hfill \Box

This concludes the $t = 3$ case. To prove the $t = 4$ case, we need to improve the upper bound on $|Y \cap (Y + v)|$ given by Proposition 2.7 (ii). To do so, we use the doubling construction twice.

**Proposition 4.5.** Say $M$ is a $PG(3, 2)$-free matroid with $|M| > (1 - 3 \cdot 2^{-4})2^{r(M)}$ and $\chi(M) > 4$. Set $Y = \mathbb{F}_2^r(M) \setminus E(M)$. For any $v_1, v_2 \in E(M)$ with $v_1 + v_2 \in E(M)$, we have

$$
|Y \cap (Y + v_1)| + |Y \cap (Y + v_2)| < \frac{8}{3}|Y| - \frac{11}{24}2^{r(M)}.
$$

**Proof.** Consider the matroid $(M_{v_1})_{v_2}$. Since $v_2, v_1 + v_2 \in E(M)$, it is also the case that $v_2 \in E(M_{v_1})$. Proposition 2.5 then implies that $(M_{v_1})_{v_2}$ is $PG(1, 2)$-free. We have

$$
Y \cup (Y + v_1) \cup (Y + v_2) \cup (Y + v_1 + v_2) = \mathbb{F}_2^r(M) \setminus E((M_{v_1})_{v_2}).
$$

Therefore, by Theorem 1.3 we know that

$$
|Y \cup (Y + v_1) \cup (Y + v_2) \cup (Y + v_1 + v_2)| \geq \frac{11}{16}2^{r(M)}.
$$

If this was not true, there would be a hyperplane $H$ disjoint from $E((M_{v_1})_{v_2})$. For the same reason as in Proposition 2.6 we have that $v_1, v_2 \in H$, so this
hyperplane then also satisfies $|Y \cap H| \geq \frac{1}{4} |H| = 2^{r(M) - 3}$ which contradicts Corollary 2.2 (i).

Let $c_i$ refer to the number of cosets of $\{0, v_1, v_2, v_1 + v_2\}$ that contain $i$ elements of $Y$. We have $|Y| = c_1 + 2c_2 + 3c_3 + 4c_4$. Say $|Y| = \alpha 2^{r(M)}$ where $\alpha < \frac{4}{16}$. Furthermore, $|Y \cup (Y + v_1) \cup (Y + v_2) | \cup (Y + v_1 + v_2) | = 4c_1 + 4c_2 + 4c_3 + 4c_4 \geq \frac{11}{16} 2^{r(M)}$. Together these imply $c_2 + 2c_3 + 3c_4 < (\alpha - \frac{11}{16}) 2^{r(M)}$.

We claim that

$$|Y \cap (Y + v_1)| + |Y \cap (Y + v_2)| \leq \frac{8}{3} (c_2 + 2c_3 + 3c_4),$$

which is exactly what we want to show. To prove this, we calculate how much each coset of $\{0, v_1, v_2, v_1 + v_2\}$ contributes to the left- and right-hand sides of this equation.

Suppose $G$ is a coset of $\{0, v_1, v_2, v_1 + v_2\}$. Set $Y_G = Y \cap G$. We have four cases.

**Case 1:** $|Y_G| \leq 1$. Such a coset contributes 0 to $c_2 + 2c_3 + 3c_4$. Furthermore, clearly $|Y_G \cap (Y_G + v_1)| = |Y_G \cap (Y_G + v_2)| = 0$ in this case.

**Case 2:** $|Y_G| = 2$. Such a coset contributes 1 to $c_2 + 2c_3 + 3c_4$. Furthermore, exactly one of $Y_G \cap (Y_G + v_1)$, $Y_G \cap (Y_G + v_2)$, $Y_G \cap (Y_G + v_1 + v_2)$ is equal to $Y_G$ and the other two are $\emptyset$. Thus $|Y_G \cap (Y_G + v_1)| + |Y_G \cap (Y_G + v_2)| \in \{0, 2\}$.

**Case 3:** $|Y_G| = 3$. Such a coset contributes 2 to $c_2 + 2c_3 + 3c_4$. Furthermore, it is easy to see that $|Y_G \cap (Y_G + v_1)| = |Y_G \cap (Y_G + v_2)| = 2$, so $|Y_G \cap (Y_G + v_1)| + |Y_G \cap (Y_G + v_2)| = 4$.

**Case 4:** $|Y_G| = 4$. Such a coset contributes 3 to $c_2 + 2c_3 + 3c_4$. In this case, $Y_G \cap (Y_G + v_1) = Y_G \cap (Y_G + v_2) = Y_G$, so $|Y_G \cap (Y_G + v_1)| + |Y_G \cap (Y_G + v_2)| = 8$.

Finally, it is easy to check that in each of the cases above, the contribution to $|Y \cap (Y + v_1)| + |Y \cap (Y + v_2)|$ is at most $8/3$ the contribution to $c_2 + 2c_3 + 3c_4$.

**Theorem 4.6.** There exist no $PG(3, 2)$-free matroids $M$ with $|M| > (1 - 3 \cdot 2^{-4}) 2^{r(M)}$ and $\chi(M) > 4$.

**Proof.** For the sake of contradiction, assume that $M$ is a counterexample. Set $Y = F(\mathcal{M}) \setminus E(M)$. We wish to find a contradiction between Proposition 4.3 and Proposition 4.4. For $|Y| < \frac{3}{4} 2^{r(M)}$, we can check that

$$\frac{2}{3} \frac{|Y|^2}{2^{2r(M)}} \geq \frac{4}{3} \frac{|Y|}{2^{r(M)}} - \frac{11}{48}.$$

It is sufficient to show that we can pair up the elements of $E(M)$ such that if $v_1, v_2$ is a pair, $v_1 + v_2 \in E(M)$. This is because then we have

$$\frac{4}{3} \frac{|Y|}{2^{r(M)}} - \frac{11}{48} \geq \frac{1}{|M|} \sum_{v \in E(M)} \frac{|Y \cap (Y + v)|}{2^{r(M)}} > \frac{2}{3} \frac{|Y|^2}{2^{2r(M)}}.$$
In fact, if we can only pair up all but one of the elements of $E(M)$, we still find a contradiction since

$$
\frac{1}{|M|} \sum_{v \in E(M)} \frac{|Y \cap (Y + v)|}{2^{r(M)}} < \frac{|M| - 1}{|M|} \left( \frac{4}{3} \frac{|Y|}{2^{r(M)}} - \frac{11}{48} \right) + \frac{1}{|M|} \left( \frac{1}{32} \right),
$$

where the bound on the left-over element comes from Proposition 2.7 (ii). Then we can check that the inequality

$$
\frac{2}{3} \frac{|Y|^2}{2^{2r(M)}} > 4 \frac{|Y|}{3} 2^{r(M)} - \frac{11}{48} + \frac{1}{27} \cdot \frac{1}{32}
$$

still holds for $|Y| < \frac{4}{3} 2^{r(M)}$. Since the inequality $4 < \chi(M) \leq r(M)$ implies that $|M| > \frac{13}{16} 2^{r(M)} > 26$, this is sufficient.

To prove that we can pair up all or all but one elements of $E(M)$, we use the following graph-theoretic argument. Construct a graph whose vertices are elements of $E(M)$ and where two vertices are connected by an edge if their sum is also in $E(M)$. Note that a vertex $v$ has degree $|M_v| > \frac{5}{8} 2^{r(M)} > |E(M)|/2$. By Dirac’s theorem, $E(M)$ has a Hamiltonian cycle. Picking alternating edges from this cycle gives the desired matching.

Acknowledgments

This research was conducted at the University of Minnesota Duluth REU and was supported by NSF grant 1358695 and NSA grant H98230-13-1-0273. The author thanks Joe Gallian for suggesting the problem and Levent Alpoge and Noah Arbesfeld for helpful comments on the manuscript.

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