Bayesian inference of the skewness parameter of supra-dense nuclear matter from energetic heavy-ion reactions

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Abstract

Within the Bayesian framework using available constraining bands on the pressure in symmetric nuclear matter (SNM) derived earlier by others in the density range of \(1.3\rho_0\) to \(4.5\rho_0\) from kaon production and nuclear collective flow data in energetic heavy-ion collisions, we infer the posterior probability distribution functions (PDFs) of SNM incompressibility \(K_0\) and skewness \(J_0\) using uniform prior PDFs for them in the ranges of \(220 \leq K_0 \leq 260\) MeV and \(-800 \leq J_0 \leq 400\) MeV. The 68% posterior credible boundaries around the most probable values of \(K_0\) and \(J_0\) are found to be \(222\pm2\) MeV and \(-215\pm20\) MeV, respectively, much narrower than their prior ranges widely used currently in the literature and are consistent with the results of a recent Bayesian analysis of neutron star properties constrained by available X-ray and gravitational wave observations.

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Introduction: Constraining the Equation of State (EOS) of super-dense neutron-rich nuclear matter has been a longstanding and shared goal of both nuclear physics and astrophysics, see, e.g., Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. It is also a major science driver of the new radioactive beam facilities being built around the world, see, e.g., Refs. [19, 20, 21, 22, 23, 24, 25, 26] as well as several existing and planned X-ray observatories and gravitational wave detectors, see, e.g. Refs. [27, 28, 29, 30, 31, 32, 33, 34, 35]. Regardless how one may choose to describe theoretically the EOS of dense neutron-rich matter encountered in either heavy-ion reactions or neutron stars, e.g., in terms of the pressure $P$ as a function of baryon density $\rho$ for cold nucleonic matter in neutron stars at $\beta$ equilibrium, the most fundamental input is the energy per nucleon $E(\rho, \delta)$ in cold nucleonic matter of isospin asymmetry $\delta = (\rho_n - \rho_p)/\rho$ where $\rho_n$ and $\rho_p$ are the densities of neutrons and protons, respectively. The $E(\rho, \delta)$ can be well approximated by [36]

$$E(\rho, \delta) = E_0(\rho) + E_{\text{sym}}(\rho) \cdot \delta^2 + O(\delta^4),$$

(1)

where $E_0(\rho) \equiv E_0(\rho, \delta = 0)$ represents the energy per nucleon in symmetric nuclear matter (SNM) and $E_{\text{sym}}(\rho)$ is the nuclear symmetry energy. There are many interesting and challenging questions about both the $E_0(\rho)$ and $E_{\text{sym}}(\rho)$ in super-dense nucleonic matter. Of course, in both energetic heavy-ion collisions and the core of neutron stars, new phases of matter and new particles may appear at high densities and/or temperatures, raising many new and important questions. Nevertheless, more precise knowledge about major characteristics of high-density SNM EOS $E_0(\rho)$ is useful for understanding the basic features of heavy-ion collisions and/or neutron stars within their minimum models. It is also useful and sometimes a prerequisite for pinning down the even more poorly known nuclear symmetry energy especially at high densities using terrestrial experiments and/or astrophysical observations [37].

Thanks to the great efforts over the last 4 decades by many people, see, e.g., Ref. [38] for an earlier review, the incompressibility $K_0 = 9\rho_0^2[d^2 E_0(\rho)/d\rho^2]_{\rho_0}$ of SNM at its saturation density $\rho_0$ has been relatively well determined to be about
Figure 1: (color online) Constraining bands on the pressure in symmetric nuclear matter as a function of reduced density derived earlier from analyzing kaon production and nuclear collective flow in energetic heavy-ion collisions in Refs. [1, 75, 76].

240±20 MeV [4, 12, 39, 40, 41, 42] or 230±40 MeV [43, 44, 45] while there is a report of somewhat high values in the range of 250 ≤ K_0 ≤ 315 MeV [46] mostly based on systematic studies of the available Giant Monopole Resonance (GMR) data of some heavy nuclei. It has been pointed out by several groups that the main sources of the remaining uncertainties and model dependences in pinning down the K_0 further is its correlations with the uncertain high-order density dependence of both the symmetry energy and E_0(ρ) [12, 43, 44, 45, 47, 48, 49]. Unfortunately, the stiffness of SNM at supra-saturation densities characterized by the skewness parameter J_0 = 27ρ_0^2[d^3E_0(ρ)/dρ^3]_{ρ_0} is hardly known. In fact, even its sign is not determined firmly. For the latest and most comprehensive review of model predictions for J_0 in the range of -369 MeV to 1488 MeV, we refer the reader to Ref. [12]. In particular, negative values of J_0 were suggested by some non-relativistic Skyrme and/or Gogny Hartree-Fock calculations [12, 50, 51, 52, 53], relativistic mean-field models [54], as well as several analyses of some neutron-star observations [55, 56, 57, 58, 59, 60].
considering the constraints on the pressure of SNM imposed by both the flow data in heavy-ion collisions \cite{1} and the mass of PSR J0348+0432 \cite{61}, a range of $-494 \text{ MeV} \leq J_0 \leq -10 \text{ MeV}$ was inferred within a nonlinear relativistic mean field model \cite{54}. While $J_0 = -190^{+40}_{-40} \text{ MeV}$ at 68% confidence level was found in our very recent Bayesian analysis \cite{58} of neutron star radii from X-ray observations and the tidal deformability of GW170817 under the constraints of causality and reproducing the maximum mass of neutron stars at least as high as $M=2.17^{+0.11}_{-0.10} \text{ M}_\odot$ as indicated by the first report \cite{62} of the mass of PSR J0740+6620 \cite{63}. On the other hand, positive values of $J_0$ were predicted by some other relativistic mean field models \cite{12,64}. For example, within a relativistic density functional theory constrained by both terrestrial experiments and astrophysical observations as well as predictions of chiral effective field theories at low densities, hugely positive values of $J_0$ in the range of 300 to 800 MeV were predicted \cite{65}, going beyond the already large range of approximately $-800 \text{ MeV} \leq J_0 \leq 400 \text{ MeV}$ previously known from surveying earlier analyses of terrestrial experiments and astrophysical observations as well as predictions of over 500 nuclear energy density functionals \cite{66,67}. Therefore, much more investigations on the physics associated with $J_0$ are obviously necessary. Indeed, it is very encouraging to note that more efforts are constantly being made by the nuclear physics community to both understand why the $J_0$ is so poorly known and how to better determine it. For example, a recent study in the framework of the Landau-Migdal theory shows that three-particle correlations play a crucial role in determining the value of $J_0$ \cite{68}, consistent with earlier findings within Skyrme/Gogny Hartree-Fock calculations that the $t_3$ term charactering effectively density dependence of many-body interactions/correlations are important but poorly understood for determining the $K_0$ and $J_0$ as well as their correlations \cite{12,43,44,45}. In this regard, it is also interesting to note that the latest and state-of-the-art Quantum Monte Carlo calculations using local interactions derived from chiral effective field theory up to next-to-next-to-leading order found a value of $252 \leq J_0 \leq 1491 \text{ MeV}$ depending on the parametrization of the three-body force used within the statistical Monte Carlo errors and the
uncertainties coming from the truncation of the chiral expansion \[69\].

As pointed out already by Margueron et al. \[12\], there were only few estimates of the poorly known \(J_0\) from analyzing experimental data. Moreover, most of the gross properties and GMR of finite nuclei are only sensitive to the EOS near the so-called crossing-density of about \(0.10\) fm\(^{-3}\) \[43, 44, 70, 71, 72, 73, 74\]. In this work, using the Bayesian statistical approach and the two constraining bands on the SNM pressure shown in Fig. 1 that were derived individually earlier in the density range of \(1.3\rho_0\) to \(2.2\rho_0\) from kaon production \[75, 76\] and \(2.0\rho_0\) to \(4.5\rho_0\) from nuclear collective flow \[1\] in energetic heavy-ion collisions, we infer the posterior probability distribution functions (PDFs) of \(K_0\) and \(J_0\) as well as their correlation with uniform prior PDFs for them in the ranges of \(220 \leq K_0 \leq 260\) MeV and \(-800 \leq J_0 \leq 400\) MeV, respectively. The 68% posterior credible boundaries around the most probable values of \(K_0\) and \(J_0\) are found to be \(222 \pm 2\) MeV and \(-215 \pm 20\) MeV, respectively, representing a significant refinement compared to their prior ranges and may be used as a benchmark for future studies on the EOS of super-dense nuclear matter.

**Approach:** In this section, we provide some details of our approach. First of all, it is necessary to discuss briefly how the constraining bands on the SNM pressure were obtained. Essentially, they were synthesized from systematic transport model analyses of kaon multiplicities and nuclear collective flows in heavy-ion collisions at intermediate and/or relativistic energies. The upper and lower boundaries in different density regions were set by employing different EOSs with and/or without the momentum dependence of single-nucleon mean-field potentials sometimes within different transport codes \[1, 73, 76\]. The underlying values of \(K_0\) used in the original data analyses range from about 170 MeV to 380 MeV depending on if/what kinds of the momentum dependence of single-nucleon potentials were used, and also depending on if/what kinds of in-medium nucleon-nucleon cross sections were used. While the underlying \(J_0\) values of these models were generally not given. It is well known within the Boltzmann transport theory, see, e.g., Ref. \[77\], there is an intrinsic degeneracy between
the single-nucleon mean-field potential and the in-medium nucleon-nucleon cross section in governing the evolution of nucleon phase space distribution function. Consequently, different combinations of nuclear mean-field potentials related to the \( E_0(\rho) \) and in-medium nucleon-nucleon cross sections related to the kinetic pressure built during heavy-ion collisions may reproduce the same observables in heavy-ion collisions. Nevertheless, the underlying zero-temperature pressure of SNM is uniquely determined by the \( E_0(\rho) \) with little influence from the uncertain in-medium nucleon-nucleon cross sections. The nuclear mean-field potential in cold nuclear matter corresponding to the \( E_0(\rho) \) is a direct input in transport model simulations of heavy-ion collisions. Thus, comparing transport model simulations with experimental observations enabled the reliable extraction of SNM pressure at zero temperature over a large density range, of course, under some reasonable and justified assumptions [1, 75, 76, 78, 79, 80, 81, 82, 83, 84, 85, 86]. While the error bands reflecting the remaining uncertainties are still large, the obtained pressure bands shown in Fig. I have been used widely in the literature to test predictions of various nuclear many-body theories.

In this work, we consider the two constraining bands on the cold SNM pressure as “data” with a \( 3\sigma \) error bar (99.7% confidence interval) as the upper and lower limits were given approximately as the absolute boundaries. Moreover, to obtain general constrains on the \( K_0 \) and \( J_0 \) from the data independent of any particular nuclear many-body theory, we adopt the parameterization of \( E_0(\rho) \) as

\[
E_0(\rho) = E_0(\rho_0) + \frac{K_0}{2} \left( \frac{\rho - \rho_0}{3\rho_0} \right)^2 + \frac{J_0}{6} \left( \frac{\rho - \rho_0}{3\rho_0} \right)^3
\]

with \( E_0(\rho_0) = -15.9 \) MeV. It has been widely used in the literature in studying properties of nuclei, neutron stars and heavy-ion collisions, see, e.g., Refs. 2, 4, 12, 56, 58, 67, 87, 88, 89, 90, 91, 85, 86. The corresponding pressure in cold SNM is then

\[
P(\rho) = \rho^2 \frac{dE_0(\rho)}{d\rho} = \frac{\rho^2}{\rho - \rho_0} \left[ K_0 \left( \frac{\rho - \rho_0}{3\rho_0} \right)^2 + \frac{J_0}{2} \left( \frac{\rho - \rho_0}{3\rho_0} \right)^3 \right].
\]

Normally, one performs Taylor expansions of energy density functionals \( e(\rho) \) based on some nuclear many-body theories. The third-order derivative of \( e(\rho) \)
at \( \rho_0 \), i.e., \( 27 \rho_0^3 \frac{d^3 e(\rho)}{d \rho^3} \), is defined as the skewness of SNM EOS. It is necessary to take the value of the derivative at \( \rho_0 \) so that contributions from high-order terms in \( (\rho - \rho_0)/3\rho_0 \) in the Taylor expansion of \( e(\rho) \) vanish. As already discussed in detail in Refs. [56, 58, 88], by design the parameterization of Eq. (2) has the form of a Taylor expansion near \( \rho_0 \) up to the third-order term. While the parameterization itself can be considered as a phenomenological energy density functional, we use it purely as a parameterization in our Bayesian analysis. Effectively, all higher order terms in the Taylor expansion of \( e(\rho) \) have been absorbed in the \( K_0 \) and \( J_0 \) terms of the parameterization. Moreover, the third-order derivative of the parameterized \( E_0(\rho) \) in Eq. (2) is independent of density without having to take its value at \( \rho_0 \) to calculate the value of \( J_0 \).

To calculate the posterior PDFs of \( K_0 \) and \( J_0 \) as well as their correlation function within the standard Bayesian approach, we use the Metropolis-Hastings algorithm [92, 93] in our Markov-Chain Monte Carlo (MCMC) sampling. The posterior probability \( P(\mathcal{M}(K_0, J_0)|D) \) that a realization \( \mathcal{M}(K_0, J_0) \) of our parametric SNM EOS describes correctly the data set denoted by \( D \) can be formulated as

\[
P(\mathcal{M}(K_0, J_0)|D) = C P(D|\mathcal{M}(K_0, J_0)) P(\mathcal{M}(K_0, J_0)),
\]

where \( C \) is a normalization constant and \( P(\mathcal{M}(K_0, J_0)) \) stands for the prior probability distribution function of the model parameters \( K_0 \) and \( J_0 \). We sample the latter uniformly between their minimum and maximum values given in Table 1 according to

\[
p = p_{\text{min}} + (p_{\text{max}} - p_{\text{min}}) x,
\]

where \( p \) denotes \( K_0 \) or \( J_0 \), \( p_{\text{min}} \) and \( p_{\text{max}} \) respectively represent the minimum and maximum values of \( K_0 \) or \( J_0 \), and \( x \) is a random number between 0 and 1. \( P(D|\mathcal{M}(K_0, J_0)) \) is the likelihood function of reproducing the data \( D \) given the model \( \mathcal{M}(K_0, J_0) \). It can be expressed as

\[
P[D|\mathcal{M}(K_0, J_0)] = \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi} \sigma_{D,j}} \exp\left[- \frac{(P_{\text{th},j} - P_{D,j})^2}{2\sigma_{D,j}^2}\right],
\]

where \( N \) is the number of data points. In digitizing the pressures shown in Fig
from both the kaon and flow data, we use 0.1 as the bin size for the reduced density. We have thus $N=26$ (10) for the pressure from the flow (kaon) data set since the relevant density ranges from $2.0\rho_0$ to $4.5\rho_0$ ($1.3\rho_0$ to $2.2\rho_0$). When combining the two data sets (named the combined data), we take the points from the flow data in their overlapping region, which implies that $N=33$ for the combined data set. $\sigma_{D,j}$ represents the $1\sigma$ error bar of the $j$th data point.

Table 1: Prior ranges of the uniformly distributed $K_0$ and $J_0$ parameters, their most probable posterior values and 68%, 90% boundaries inferred from using the pressure band constrained by the kaon, flow as well as both kaon & flow data, respectively. All quantities are in MeV.

| Parameters | 68% posterior boundaries | 90% posterior boundaries |
|------------|--------------------------|--------------------------|
| $K_0$      | $220^{+18}_{-0}$, $222^{+10}_{-2}$, $222^{+2}_{-2}$ | $222^{+30}_{-0}$, $222^{+16}_{-2}$, $222^{+6}_{-2}$ |
| $J_0$      | $-505^{+100}_{-140}$, $-210^{+15}_{-35}, -215^{+20}_{-2}$ | $-505^{+175}_{-225}$, $-210^{+35}_{-50}, -215^{+35}_{-55}$ |

By using the randomly generated parameters $K_0$ and $J_0$ as well as the expression (3) for SNM pressure, one can construct the model $M(K_0, J_0)$, i.e. the theoretical value $P_{th,j}$ for the cold SNM pressure. Subsequently, one can calculate the likelihood of this set of parameters according to Eq. (6). The posterior PDF of each parameter is then determined by the marginal estimation, e.g., the PDF for the parameter $K_0$ is given by

$$P(K_0|D) = \frac{\int P(D|M)dJ_0}{\int P(D|M)P(M)dK_0dJ_0}.$$  

(7)

It is well known that some initial samples in the so-call burn-in period may have to be discarded because the MCMC process does not normally sample from the equilibrium (target) distribution in the beginning, see, e.g., Ref. [94] for more detailed discussions. The length of the burn-in period can be determined by checking the trace plot, i.e., the evolution of either the log posterior or the mean values of the parameters as a function of the step number in the MCMC chain. When the chain has reached stationarity, it starts sampling from its equilibrium (target) distribution, then both the mean and variance of the trace plot should keep relatively constant [95]. Shown in Fig. 2 are the log posterior $-\ln(P)$ (upper window), the mean values of $K_0$ (middle window) and
Figure 2: (Color online) The log-posterior (upper window), $-\ln P(M(K_0, J_0) | D)$, and the mean values of $K_0$ (middle window) and $J_0$ (bottom window) as functions of the step number in the Markov-Chain Monte Carlo sampling of the posterior probability distribution function.

$J_0$ (bottom window) varying with the MCMC steps. It is seen that, after about 10,000 burn-in steps, the log posterior flattens and can traverse the posterior space rapidly, namely, jumping from a remote region of the posterior to another quickly, while the corresponding means of the two parameters become approximately constants. In this work, we discard the 50,000 burn-in steps and use 20 million steps afterwards in calculating the PDFs of $K_0$ and $J_0$.

**Results and Discussions:** The 68% (90%) credible region for the PDF of each parameter, i.e., the so-called the highest posterior density (HPD) interval
where $p_{iL}$ ($p_{iU}$) is the lower (upper) limit of the corresponding HPD interval of the parameter $p_i$. The most probable values of $K_0$ and $J_0$ together with their 68%, 90% credible boundaries are listed in Table using the kaon, flow and their combined data, respectively. The corresponding posterior PDFs of $K_0$ and $J_0$ as well as their correlation are shown in the upper left (kaon only), upper right (flow only) and lower (both kaon and flow) window of Fig. respectively.
red (blue) shadows show the 68% (90%) credible region for the PDFs of each parameter. Several interesting observations can be made:

- Smaller values of $K_0$ are preferred in all cases considered here, while the most probable value of $J_0$ depends strongly on whether the high-density constraint from the flow experiments is used. There is a weak, inverse correlation between the $K_0$ and $J_0$ as the parameters $K_0$ and $J_0$ compensate each other in reproducing the same pressure data under the same condition. Thus, a better knowledge on one of the two parameters will help improve the determination of the other [12, 49].

- The constraining band on the SNM pressure in the density range of $1.3\rho_0$ to $2.2\rho_0$ alone from the kaon data, as shown in the upper left window of Fig. 3, constrain significantly but not very tightly the $K_0$ and $J_0$ parameters relative to their uniform prior PDFs. More quantitatively, they are only loosely constrained to $K_0 = 220^{+18}_{-10}$ MeV and $J_0 = -505^{+105}_{-140}$ MeV at 68% confidence level. Moreover, at the upper limit of $K_0$ (260 MeV) and lower limit of $J_0$ (-800 MeV) of their prior ranges, the PDFs of both parameters are finite, meaning that combinations of these two parameters beyond their prior ranges would give pressures falling into the same constraining band. This is understandable since the $K_0$, as a low-order bulk parameter of SNM EOS, characterizes properties of the SNM around the saturation density, whereas the parameter $J_0$ characterizes the high-density behavior of SNM EOS. The constraining band on the SNM pressure in the density range of $1.3\rho_0$ to $2.2\rho_0$ can put a strong limit on $K_0$ but a weak one on $J_0$.

- As shown in the right window of Fig. 3, the constraining band on the SNM pressure at densities from $2\rho_0$ to $4.5\rho_0$ alone from the flow experiments can constrain the $J_0$ parameter reasonably tightly to $J_0 = 210^{+15}_{-35}$ MeV at 68% confidence level. Simultaneously, due to the tighter constraint on $J_0$, the 68% credible interval of $K_0 = 222^{+10}_{-2}$ MeV is also narrower than that filtered only by the pressure from the kaon experiments. Of course,
as shown in the lower window of Fig. 3, combining the constraining bands on the SNM pressure from both kaon production and flow experiments in the whole density range from $1.3\rho_0$ to $4.5\rho_0$ leads to even more tighter constrains on both parameters, i.e., $K_0 = 222\pm2$ MeV and $J_0 = -215\pm20$ MeV at 68% confidence level. We notice that these constraints from heavy-ion collisions are very consistent while slightly more tight compared to the results of $K_0 = 222^{+26}_{-20}$ MeV and $J_0 = -190^{+40}_{-40}$ MeV at 68% confidence level from our very recent Bayesian analysis [58] of neutron star properties from X-ray and gravitational wave observations using the same prior PDFs for both $K_0$ and $J_0$. Thus, the PDFs obtained here for $K_0$ and $J_0$ and their characteristics summarized in Table 1 may be used as the prior information in future Bayesian inferences of other parameters, such as those characterizing the symmetry energy of neutron-rich matter especially at high densities, from observables of neutron stars and their mergers as well as collisions of high energy radioactive beams.

Having obtained the credible intervals of the parameters $K_0$ and $J_0$, one can easily get the corresponding credible bands for the nucleon energy $E_0(\rho)$ in cold SNM according to Eq. (2). Shown in Fig. 4 are the boundaries of $E_0(\rho)$ at 68% and 90% credible levels. For comparisons, the $E_0(\rho)$ values using the prior limits of $K_0$ and $J_0$ are also shown in the left window. Obviously, the pressures derived from the kaon and flow experiments together impose a tight constraint on the $E_0(\rho)$ up to about $4.5\rho_0$. Moreover, as shown in the right panel of Fig. 4, the heavy-ion constraints on the SNM pressure are more effective in constraining the $E_0(\rho)$ than the radii data of canonical neutron stars used in the recent Bayesian analysis in Ref. [58]. Of course, the constraining bands on the SNM pressure we adopted from heavy-ion collisions is a direct constraint while properties of neutron stars provide some indirect constraints on the $E_0(\rho)$. In particular, the average density in canonical neutron stars is about $2.5\rho_0$ and the radii of these neutron stars are mostly sensitive to the nuclear pressure.
Figure 4: (Color online) The energy per nucleon in symmetric nuclear matter as a function of the reduced density $\rho/\rho_0$. For comparisons, the 68% credible boundaries from heavy-ion reactions are compared with those from Bayesian analyses of neutron star properties from Ref. [58].

around this average density [2]. Consequently, most of the neutron-star observables, such as the radii and maximum mass, are insensitive to the parameter $K_0$, mainly characterizing properties of SNM near $\rho_0$ as shown explicitly in Ref. [58].

**Summary:** In summary, adopting the constraining bands on the cold SNM pressure in the density range from $1.3\rho_0$ to $4.5\rho_0$ from energetic heavy-ion collisions we inferred the PDFs of the underlying incompressibility $K_0$ and skewness $J_0$ parameters of super-dense nuclear matter within the Bayesian framework using a parameterized EOS and uniform prior PDFs in the ranges of $220 \leq K_0 \leq 260$ MeV and $-800 \leq J_0 \leq 400$ MeV. The 68% posterior credible boundaries around the most probable values of $K_0$ and $J_0$ are found to be $222\pm2$ MeV and $-215\pm20$ MeV, representing significant refinements compared to their prior ranges widely used presently in the literature.
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