Symbiotic Solitons in Heteronuclear Multicomponent Bose-Einstein condensates

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We show that bright solitons exist in quasi-one dimensional heteronuclear multicomponent Bose-Einstein condensates with repulsive self-interaction and attractive inter-species interaction. They are remarkably robust to perturbations of initial data and collisions and can be generated by the mechanism of modulational instability. Some possibilities for control and the behavior of the system in three dimensions are also discussed.

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I. INTRODUCTION

Symbiosis is an assemblage of distinct organisms living together. Although the original definition of symbiosis by De Bary did not include a judgment on whether the partners benefit or harm each other, currently, most people use the term symbiosis to describe interactions from which both partners benefit.

In Physics, waves in dispersive linear media tend to expand due to the different velocities at which the wave components propagate. This is not the case in many nonlinear media, in which certain wavepackets, called solitons are able to propagate undistorted due to the balance between dispersion and nonlinearity.

Stable solitons of different subsystems are sometimes able to “live together” and form stable complexes called vector solitons as it happens with Manakov optical solitons or stabilized vector solitons. In some cases, a (large) robust soliton can be used to stabilize a (small) weakly unstable wave.

Multicomponent solitary waves also appear in Bose-Einstein condensates (BECs). In fact, multicomponent BECs support nonlinear waves which do not exist in single component BECs such as domain wall solitons, dark-bright solitons, etc. Most of the previous analyses correspond to homonuclear multicomponent condensates for which the atom-atom interactions are repulsive. However, heteronuclear condensates offer a wider range of possibilities, the main one being the possibility of having a negative inter-species scattering length. This possibility has been theoretically explored in the context of Feshbach resonance management and realized experimentally for boson-fermion mixtures.

In this paper we study the existence and properties of bright solitons in heteronuclear two-component BECs with scattering lengths \(a_{11}, a_{22} > 0\) and \(a_{12} < 0\). We would like to stress the fact that these coefficient combinations do not arise in other systems where similar model equations are used. For instance in nonlinear optics, where the nonlinear Schrödinger equations used to describe the propagation of laser beams in nonlinear media are similar to the mean field equations used to describe Bose-Einstein condensates, the nonlinear coefficients are always of the same sign. The closest analogy could happen in the so-called QPM (quasi-phase-matched) quadratically nonlinear media, where an effective cubic nonlinearity could be “engineered” which could have similar properties but we do not know of any systematic studies of those systems.

Our analysis will show novel features with respect to those already found in single species BECs. For instance, even when solitons do not exist for each of the species, the coupling leads to robust vector solitons. Since the mutual cooperation between these structures is essential for their existence we will refer to these solitons hereafter as symbiotic solitons. We also show how they appear by modulational instability and study some features of their collisions. We also comment on the possibility of obtaining these structures in multidimensional configurations.

II. THE MODEL AND ITS BASIC PROPERTIES

In this paper we will study two-component BECs in the limit of strong transverse confinement ruled by optical lattices. In this limit, Eq. (1), which are of the form

\[
\frac{\partial u_j}{\partial t} = -\frac{1}{2} \frac{\partial^2 u_j}{\partial x^2} + (g_{11}|u_1|^2 + g_{12}|u_2|^2) u_1, \quad (1a)
\]

\[
\frac{\partial u_2}{\partial t} = -\alpha \frac{\partial^2 u_2}{\partial x^2} + (g_{21}|u_1|^2 + g_{22}|u_2|^2) u_2, \quad (1b)
\]

where \(x\) is the adimensional longitudinal spatial variable measured in units of \(a_0 = \sqrt{\hbar/m_1 \omega_\perp}\), \(t\) is the time measured in terms of \(1/\omega_\perp\), and \(u_j(x,t) \equiv u_j(r, \tau) \sqrt[3]{a_0}\). The dimensional reduction leads to

\[
g_{ij} = 2a_{ij} \alpha^{\gamma_{ij}-2}/a_0,
\]

with \(\alpha = m_1/m_2\) and \(a_{ij}\) being the \(s\)-wave scattering length. The normalization for \(u_j\) is \(\int |u_j|^2 d^3 x = N_j\) where \(N_j\) is the number of particles of each species.

Let us first consider constant amplitude solutions of Eq. (1), which are of the form

\[
\phi_j(z,t) = A_j e^{i\beta_j t}, \quad (2a)
\]

\[
\beta_j = g_{jj}|A_j|^2 + g_{j,3-j}|A_{3-j}|^2, \quad (2b)
\]

for \(j = 1, 2\). We will study the evolution of small perturbations of \(\phi_j\) of the form

\[
u_j(z,t) = (A_j + \delta A_j(z,t)) e^{i(\beta_j t + \delta \beta_j(z,t))} \quad (3)
\]
Using Eq. 1 and retaining the first order terms we get partial differential equations for $\delta A_1$, $\delta \beta_1$, $\delta A_2$, $\delta \beta_2$ which can be transformed to Fourier space to obtain

$$\delta A_j(z,t) = \int_R a_0(k) e^{ikx} e^{i\Omega(k)t} dk$$  \hfill (4)

$a_0(k)$ being the Fourier transform of the initial perturbation. Perturbations remain bounded if $\text{Re} \{\Omega(k)\} \leq 0$. Some algebra leads to

$$\Omega^2 = \frac{1}{2} \left(f_1 + f_2 \pm \sqrt{(f_1 - f_2)^2 + 4C^2}\right)$$ \hfill (5)

where $f_j = -(g_{jj} A_j^2 + k^2/4)k^2$, $C^2 = A_1^2 A_2^2 g_{12} k^4$.

The so-called modulational instability (MI) occurs when $\Omega(k)^2 > 0$ for any $k$. For small wavenumbers (worst situation) we get

$$g_{12}^2 > g_{11}g_{22},$$ \hfill (6)

which is analogous to the miscibility criterion for two-component condensates. However, the physical meaning of Eq. 6 is very different since now this instability is a signature of the tendency to form coupled objects between both atomic species. The role of MI in the formation of soliton trains and domains in BEC has been recognized in previous papers [14, 15, 16].

III. VECTOR SOLITONS

Eqs. 1 have sech-type solutions

$$u_j(x,t) = \left(\frac{N_j}{2\omega}\right)^{1/2} \text{sech}\left(\frac{x}{\omega}\right) e^{i\lambda_j t},$$ \hfill (7)

with $\lambda_1 = 1/(2\omega^2)$, $\lambda_2 = \alpha/(2\omega^2)$, and $\omega = 2/(-g_{11} N_1 - g_{12} N_2)$, provided the restriction

$$g_{12}(m_1 N_2 - m_2 N_1) = m_2 g_{22} N_2 - m_1 g_{11} N_1,$$ \hfill (8)

and the MI condition (6) are satisfied. Eq. 3 implies that, given the number of particles in one component, the other is fixed.

Since the self-interaction coefficients are positive, these solitons are supported only by the mutual attractive interaction between both components. This type of vector soliton thus differs from others described for Nonlinear Schrödinger equations of the form Eq. 1, such as the Manakov solitons 3, where all the nonlinear coefficients cooperate to form the solitonic solution. The MI condition 6 implies that the formation of these solitons has a threshold in $g_{12}$ and means that the cross-interaction must be strong enough to be able to overcome the self-repulsion of each atomic cloud. There are no analogues to this condition in single component systems since solitons exist for any value of the self-interaction coefficient $g < 0$. To fix ideas, taking a $^{87}\text{Rb} - ^{41}\text{K}$ mixture with $a_{11} = 69a_0, a_{22} = 99a_0$ the MI condition implies that $a_{12} < -83a_0$ in order to obtain solitons. In Fig. 1 (a) it can be seen how the ratio $N_2/N_1$ is close to 0.4 in the range of values of $-83a_0 > a_{12} > -150a_0$. An hypothetical $^{7}\text{Li} - ^{23}\text{Na}$ mixture with $a_{11} = 5a_0$ and $a_{22} = 52a_0$ (in appropriate quantum states) leads to the curve in Fig. 1 (b), which shows a much larger range of variation.

IV. SOLITON STABILITY

We can use the Vakhitov-Kolokov (VK) criterion to study the stability of solitons given by Eq. 1. To do this, one must study the sign of $\partial \lambda_j/\partial N_j$. For soliton solutions this can be done from the explicit form of $\lambda_j$. After some algebra we find $\lambda_1(N_1)$ and $\lambda_2(N_2)$ and obtain that $\partial \lambda_1/\partial N_1 > 0$, and $\partial \lambda_2/\partial N_2 > 0$ in all their range of existence, which proves the linear stability of the solitons for small perturbations and contradicts the naive intuition that the self-repulsion would lead to intrinsically unstable wavepackets.

We have studied numerically the robustness of symbiotic solitons to finite amplitude perturbations. First we have perturbed both solutions with small amplitude noise and found that, in agreement with the predictions of the VK criterion, they survive after the emission of the noise in the form of radiation. Next we have applied
a stronger perturbation consisting of displacing mutually their centers and observe that a soliton is formed even for relative displacements of the order of the soliton size [Fig. 2]. Finally we have started with sech-type initial data which are not solitons and observe that after the emission of some radiation solitons are formed.

V. GENERATION OF SYMBIOTIC SOLITONS BY MI

To study the generation of these solitons by MI in realistic systems we have considered a multicomponent Bose-Einstein condensate of $^{87}$Rb and $^{41}$K atoms for which the inter-species scattering length $a_{12}$ is controlled by the use of Feschbach resonances as proposed in [10]. To simplify the problem here we do not consider the effect of gravity.

We start by constructing the ground state of the system for an elongated trap typical of the LENS setup with $\omega_\perp = 215$ Hz, $\omega = 16.3$ Hz. For these atomic species $a_{11} = 69a_0$ and $a_{22} = 99a_0$. We adjust the interspecies scattering length to $a_{12} = 95a_0$ during the condensation process. The ground state of this system for $N_1 = 25000, N_2 = 20000$, shown in Fig. 3(a), agrees well with the theoretical predictions for these systems [20].

After the condensate is formed we change instantaneously this quantity to a negative value and at the same time switch off the longitudinal trapping potential and simultaneously this quantity to a negative value and at the same time a coherent phenomenon because of the incoherent nature of interaction between different types of atoms. A related subject of recent interest in Optics is that of partially coherent ones because of the incoherent nature of the vector solitons which arise after the decay of the initial configuration. The many degrees of freedom present in these system open many possibilities for controlling the number and sizes of solitons by appropriately choosing the values of $a_{12}$ before and after the condensate is released and the initial number of particles $N_1, N_2$.

VI. COLLISIONS OF SYMBIOTIC SOLITONS

The robustness of symbiotic solitons manifests also in their collisional behavior and their internal structure makes the interaction of these vector solitons very rich. Since each soliton is a compound object the collisions are at the same time a coherent phenomenon because of the direct overlapping of the same type of atoms and an incoherent one because of the incoherent nature of interaction between different types of atoms. A related subject of recent interest in Optics is that of partially coherent solitons [23].

We have simulated head-on collisions of equal symbiotic solitons of opposite velocities given by

$$u_j = \sqrt{\frac{N_j}{2w}} \text{sech} \left( \frac{x - x_0}{w} \right) e^{i\sqrt{\mu_j}x + i\alpha_{j,+}}$$

$$+ \sqrt{\frac{N_j}{2w}} \text{sech} \left( \frac{x - x_0}{w} \right) e^{-i\sqrt{\mu_j}x + i\alpha_{j,-}}$$

for $j = 1, 2$. $\alpha_{j,\pm}$ are the relative phases and $N_2$ is given by Eq. 8. In Fig. 4 we show some examples units [Fig. 4(c)]. Other smaller and wider solitons exit our integration region traveling at a faster speed.

The final number of solitons depends on the value of $a_{12}$ chosen during the condensation process (which controls the overlapping of the species) and the number of particles $N_1, N_2$ and the negative scattering length $a_{12}$ chosen to destabilize the system. For instance, choosing $a_{12} = -70a_0$, which is below the theoretical limit for MI the evolution of the wavepacket is purely dispersive [see Fig. 4(a)]. Choosing $a_{12} = -87a_0$, above the MI limit but below the choice of Fig. 3 leads to the formation of a single soliton [Fig. 4(b)]. It seems that the larger the scattering length, the larger the number of solitons which arise after the decay of the initial configuration.
of these collisions. Slow [Fig. 5(a-d)] or moderate speed collisions [Fig. 5(e-f)] lead to bound solitons while for larger speeds the picture is not so clear. The specific outcome of the collision depends on the relative soliton phases with the phase difference between the larger components in the symbiotic soliton (in this case Rb) being the dominant ones. For instance collisions with phases \( \theta \equiv (\alpha_{1,1}+, \alpha_{1,-}, \alpha_{2,+}, \alpha_{2,-}) \) (0, 0, \( \pi \), 0) [Fig. 5(e)] and \( \theta = (0, 0, \pi, 0) \) (not shown) both lead to mutual repulsion but the outgoing speeds are different due to the different interactions between the internal components of the soliton. Collisions with higher but still moderate speeds [Fig. 5(e-f)] give independent vector solitons. The outcome of the collisions with zero phases is a bound state of two vector solitons which has internal oscillations, i.e. some sort of multicomponent higher order soliton.

VII. PROSPECTS FOR MULTIDIMENSIONAL SYMBIOTIC SOLITONS

A very interesting question arising naturally is: do these symbiotic solitons exist in multidimensional scenarios? In principle the answer is not evident since the only effect acting against stabilization of multidimensional soliton structures would be collapse, but one could think that in this case collapse could be inhibited because of the repulsive self-interaction, thus a deeper analysis is in order.

The adimensional model equations in two and three dimensions take the form

\[
i \partial u_j / \partial t = \left( - \frac{1}{2m_j} \Delta + V_j + g_{j,j} |u_j|^2 + g_{j,k} |u_k|^2 \right) u_j, \quad (10)
\]

with \( j = 1, 2 \) and \( k = 2, 1 \) correspondingly.

Let us first consider this problem in two spatial dimensions. To study collapse rigorously one usually tries to compute the exact evolution of the wavepacket widths rigorously \[24\]. For the multicomponent case and \( m_1 = m_2 = m \), this was studied by group-theoretical methods by \[25\]. In our case, from the general formulae obtained by Gosh we get a sufficient condition for collapse, which is

\[
\mathcal{H} = \int_{\mathbb{R}^2} \left[ \sum_{j=1,2} \left( |\nabla u_j|^2 / (2m) + V_j |u_j|^2 + g_{j,j} |u_j|^4 / 2 \right) + g_{12} |u_1|^2 |u_2|^2 \right] < 0. \quad (11)
\]

In principle, this is a bad result for obtaining localized structures since it means that arbitrarily close to any stationary solution (for which \( \mathcal{H} = 0 \)), there would be collapsing solutions and thus stationary solutions, if they exist, would be unstable. As it is usual in the framework of collapse problems the situation would be even worse in three spatial dimensions with solutions of arbitrary small number of particles undergoing collapse provided they are initially sufficiently localized.

This means that in principle symbiotic solitons could only be obtained in quasi-1D geometries because of the transverse stabilization effect provided by the trap in a similar way as ordinary bright solitons do.

VIII. CONCLUSIONS AND EXTENSIONS

In this paper we have studied vector solitons in heteronuclear two-component BECs which are supported by their attractive mutual interaction. These symbiotic solitons are linearly stable and remarkably robust and can be generated through modulational instability phenomenon with many possibilities for control. Collisions of these vector solitons show their robustness and open different ways for their manipulation and the design of novel quantum states such as breather-like states. We have also considered multidimensional configurations and shown that collapse may avoid the formation of fully multidimensional symbiotic solitons.

We think that the conceptual ideas behind our work can also be used to understand boson-fermion mixtures. For instance, \( a_{12} \) is known to be negative and large for
quantum degenerate mixtures of $^{87}$Rb and $^{40}$K. In those systems numerical simulations have proven the formation of localized wavepackets which could share the same essential mechanisms for the formation of solitary waves.

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[1] H. A. de Bary, *Die Erscheinung der Symbiose*, Karl J Tübner, Strassburg (1879).
[2] A. Scott, *Nonlinear science*, Oxford University Press (Oxford, 1999).
[3] S.V. Manakov, Zh. Eksp. Teor. Fiz. 65, 505 (1973) [Sov. Phys. JETP 38, 248 (1974)].
[4] G. P. Agrawall, and Y. Kivshar, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, 2003).
[5] G. D. Montesinos, V. M. Pérez-García, and H. Michinel, Phys. Rev. Lett. 92, 133901 (2004).
[6] Z. Musslimani, M. Segev, D. Christodoulides, and M. Soljacic, Phys. Rev. Lett. 84, 1164 (2000); J. Malmberg, A. Carlsson, D. Anderson, M. Lisak, E. Ostrovskaya, and Y. Kivshar, Opt. Lett. 25, 643 (2000); J.J. García-Ripoll, V.M. Pérez-García, E. Ostrovskaya, and Y. Kivshar, Phys. Rev. Lett. 85, 82 (2000); J Yang and D. E. Pelinovsky Phys. Rev. E 67, 016608 (2003).
[7] S. Cohen, and M. Haelterman, Phys. Rev. Lett. 87, 140401 (2001).
[8] K. Kasamatsu, and M. Tsubota, Phys. Rev. Lett. 93, 100402 (2004).
[9] B. P. Anderson, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, Phys. Rev. Lett. 86, 2926 (2001); Th. Busch and J. Anglin, Phys. Rev. Lett. 87, 010401 (2001).
[10] A. Simoni, F. Ferlaino, G. Roati, G. Modugno, and M. Inguscio, Phys. Rev. Lett. 90, 163202 (2003).
[11] S. Inouye, J. Goldwin, M. L. Olsen, C. Ticknor, J. L. Bohn, and D. S. Jin, cond-mat/0406208
[12] C. A. Stan, M. W. Zwierlein, C. H. Schunck, S. M. F. Raupach, and W. Ketterle, Phys. Rev. Lett. 93, 143001 (2004).
[13] G.B. Partridge, A.G. Truscott, and R.G. Hulet, Nature 417, 150 (2002); L.Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Science 296, 1290 (2002).
[14] V. M. Pérez-García, H. Michinel, and H. Herrero, Phys. Rev. A 57, 3837 (1998).
[15] C. Sulem and P. Sulem, “The nonlinear Schrödinger equation: Self-focusing and wave collapse”, Springer, Berlin (2000).
[16] G. Theocharis, Z. Rapti, P. G. Kevrekidis, D. J. Frantzeskakis, and V. V. Konotop, Phys. Rev. A 67, 063610 (2003); L. D. Carr, and J. Brand, Phys. Rev. A 70, 033607 (2004); L. D. Carr, and J. Brand, Phys. Rev. Lett. 92, 040401 (2004); P. G. Kevrekidis, G. Theocharis, D. J. Frantzeskakis and A. Trombettoni, Phys. Rev. A 70, 023602 (2004).
[17] G. Roati, F. Riboli, G. Modugno, and M. Inguscio, Phys. Rev. Lett. 89, 150403 (2002); J. Goldwin, S. Inouye, M. L. Olsen, B. Newman, B. D. DePaola, and D. S. Jin, Phys. Rev. A 70, 021601(R) (2004).
[18] T. Karpinski, M. Breuerzy, S. Ospelkaus-Schwarzer, K. Bongs, M. Gajda, and K. Rzazewski, Phys. Rev. Lett. 93, 100401 (2004).
[19] G. Modugno, M. Modugno, F. Riboli, G. Roati, and M. Inguscio, Phys. Rev. Lett. 89, 190404 (2002).
[20] M. Trippenbach, K. Goral, K. Rzazewski, B. Malomed, and Y. B. Band, J. Phys. B: At. Mol. Opt. Phys. 33 (2000) 4917.
[21] A. M. Kamchatnov, R. A. Kraenkel, B. A. Umarov, Phys. Rev. E 66, 036609 (2002).
[22] F. Riboli and M. Modugno, Phys. Rev. A 65, 063614 (2002).
[23] T.-S. Ku, M.-F. Shih, A. Sukhorukov, and Y. Kivshar, Phys. Rev. Lett. 94, 063904 (2005).
[24] J. J. García-Ripoll, V. M. Pérez-García, P. Torres, Phys. Rev. Lett. 83, 1715 (1999).
[25] P. K. Gosh, Phys. Rev. A 65, 053601 (2002).
[26] V. M. Pérez-García, H. Michinel, J. I. Cirac, M. Lewenstein, P. Zoller, Phys. Rev. Lett. 77, 5320 (1996).