On Fully Split Lacunary Polynomials in Finite Fields

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Abstract

We estimate the number of possible types degree patterns of k-lacunary polynomials of degree t < p which split completely modulo p. The result is based on a combination of a bound on the number of zeros of lacunary polynomials with some graph theory arguments.

1 Introduction

Zeros and factorisations of lacunary polynomials, that is, polynomials of high degree with relatively small number of non-zero coefficients, has always been a subject of active investigation, see [2, 4, 7, 9, 10] and references therein. We say that a polynomial f over a field \( \mathbb{K} \) is k-lacunary if it has at most \( k + 1 \)
non-zero coefficients, including a non-zero constant term, that is, if \( f(0) \neq 0 \) and
\[
f(X) = a_0 + a_1X^{t_1} + \ldots + a_kX^{t_k} \in \mathbb{K}[X] \tag{1}
\]
for some positive integers \( t_1 < \ldots < t_k \).

For example, a classical result of Descartes asserts that a \( k \)-lacunary polynomial \( f \in \mathbb{R}[X] \) may have at most \( 2k \) real roots. Furthermore, Lenstra \([8]\) has shown that for an algebraic number field \( \mathbb{K} \) of degree \( m \) over \( \mathbb{Q} \) and a \( k \)-lacunary polynomial \( f \in \mathbb{K}[X] \), the product \( g \) of all irreducible divisors \( h \mid f \) of degree at most \( \deg h \leq d \) is of degree
\[
\deg g = O \left( k^2 2^{md} md \log(2^{mdk}) \right).
\]

Schinzel \([10]\) has obtained a series of statistical results about the number of \( k \)-lacunary irreducible polynomials with prescribed coefficients. In particular, by \([10] \) Corollary 2, for any algebraic numbers \( a_0, \ldots, a_k \) there are at most \( O \left( T^{(k+1)/2} \right) \) \( k \)-tuples of integers
\[
t = (t_1, \ldots, t_k), \quad 1 \leq t_1 < \ldots < t_k, \tag{2}
\]
with \( t_k \leq T \) and such that the largest non-cyclotomic factor (that is, a factor which does not have roots that are roots of unity) of the \( k \)-lacunary polynomial (1) is reducible over \( \mathbb{K} = \mathbb{Q}(a_1/a_0, \ldots, a_k/a_0) \).

Here we consider a related question about estimating the number \( N_k(p, t) \) of \( k \)-tuples (2) such that there is a \( k \)-lacunary polynomial of the form (1) of degree \( t_k = t \) over the finite field \( \mathbb{K} = \mathbb{F}_p \) of \( p \) elements, where \( p \) is a prime, that fully splits over \( \mathbb{F}_p \).

**Theorem 1.** If a positive integer \( k \) is fixed then for any prime \( p \) and positive integer \( t < p \), we have,
\[
N_k(p, t) \leq t^{k-k[(k-3)/2]-1}p^{(k-1)(k-3)/2}+o(1)
\]
as \( p \to \infty \).

Clearly, Theorem 1 is nontrivial only for \( k > 3 \) and for
\[
t > p^{1-1/k+\varepsilon}, \tag{3}
\]
with some fixed \( \varepsilon > 0 \). Furthermore, for \( t \gg p \) we obtain the bound
\[
N_k(p, t) \leq t^{[k/2]+1+o(1)}.
\]
Our result is based on a rather unusual combination of two techniques: a bound on the number of zeros of lacunary polynomials (see Section 2) and a bound on the so-called domination number of a graph (see Section 3).

Throughout the paper, the implied constants in the symbols ‘O’, ‘≪’ and ‘≫’ may depend on k (we recall that the notations $U \ll V$ and $V \gg U$ is equivalent to $U = O(V)$).

2 Zeros of Lacunary Polynomials

We need the following estimate from [1] on the number of zeros of lacunary polynomials over $\mathbb{F}_p$.

**Lemma 2.** For $k+1 \geq 2$ elements $a_0, a_1, \ldots, a_k \in \mathbb{F}_p^*$ and integers $0 = t_0 < t_1 < \ldots < t_k < p$, the number of solutions $Q$ to the equation

$$\sum_{i=0}^{k} a_i x^{t_i} = 0, \quad x \in \mathbb{F}_p^*,$$

with $t_0 = 0$, satisfies

$$Q \leq 2p^{1-1/k}D^{1/k} + O(p^{1-2/k}D^{2/k}),$$

where

$$D = \min_{0 \leq i \leq k} \max_{j \neq i} \gcd(t_j - t_i, p - 1).$$

**Lemma 3.** For $k+1 \geq 2$ elements $a_0, a_1, \ldots, a_k \in \mathbb{F}_p^*$ and integers $0 = t_0 < t_1 < \ldots < t_k < p$, the multiplicity of any root $\rho$ of the polynomial

$$\sum_{i=0}^{k} a_i x^{t_i} \in \mathbb{F}_p[X]$$

is at most $k$.

**Proof.** Let

$$F(X) = \sum_{i=0}^{k} a_i X^{t_i}.$$
Then for the \( j \) derivative \( F^{(j)}(X) \) we have

\[
F^{(j)}(X)X^j = \sum_{i=0}^{k} \prod_{h=0}^{j-1} (t_i - h)a_iX^{t_i}
\]

(where as usual, we set \( F^{(0)}(X) = F(X) \)). Thus, if \( r \neq 0 \) is a root of multiplicity at least \( k + 1 \leq t_k < p \) in the algebraic closure of \( \mathbb{F}_p \), then

\[
F^{(j)}(r) = 0, \quad j = 0, \ldots, k.
\]

Therefore, the homogeneous system of equations

\[
\sum_{i=0}^{k} \prod_{h=0}^{j-1} (t_i - h)x_i = 0, \quad j = 0, \ldots, k,
\]

has a non-zero solution \( x_i = a_ir^{t_i}, \quad i = 0, \ldots, k \). This implies

\[
\det \left[ \left( \prod_{h=0}^{j-1} (t_i - h) \right)_{i,j=0,\ldots,k} \right] = 0,
\]

which is impossible for \( 0 = t_0 < t_1 < \ldots < t_k < p \) as an easy calculation shows that

\[
\det \left[ \left( \prod_{h=0}^{j-1} (t_i - h) \right)_{i,j=0,\ldots,k} \right] = \prod_{0 \leq i < j \leq k} (t_j - t_i) \neq 0.
\]

The above contradiction implies the desired result. \( \square \)

### 3 Domination Number of a Graph

Let \( G = (V, E) \) be a simple undirected graph of order \( n \). A dominating set \( S \) of \( G \) is a vertex subset such that any vertex of \( V \setminus S \) has a neighbour in \( S \). Intuitively, a dominating set of a graph is a vertex subset whose neighbours, along with themselves, make up the vertex set of the graph.

The minimum cardinality of a dominating set of \( G \) is called the domination number \( \gamma(G) \) of \( G \). In other words,

\[
\gamma(G) = \min_{S \subseteq V(G)} \left\{ |S| : V(G) \subseteq \bigcup_{v \in S} \hat{N}(v) \right\},
\]
where $\hat{N}(v)$ denotes the closed neighbourhood of a vertex $v$.

We denote by $\delta(G)$ the minimum degree of $G$.

When $\delta(G)$ is big enough, there are very good upper bounds for the domination number of the graph $G$ in terms of $\delta(G)$ and $n$ (see, for example, [3, 6]). However, for small values of $\delta(G)$ the classical result of Ore [9] is stronger and provides an upper bound for the domination number of a graph with no isolated vertices:

**Lemma 4.** If $G$ is a graph of order $n$ with $\delta(G) \geq 1$, then

$$
\gamma(G) \leq \frac{n}{2}.
$$

### 4 Proof of Theorem [1]

Since $p > t_k$, by Lemma [3] the multiplicity of each non-zero root of a polynomial of the form (1) does not exceed $k$. Hence, if a polynomial $F(X) \in \mathbb{F}_p[X]$ of the form (1) splits completely over $\mathbb{F}_p$ then the equation

$$a_0 + a_1 x^{t_1} + \ldots + a_n x^{t_k} = 0, \quad x \in \mathbb{F}_p^*,
$$

with $1 \leq t_1 < \ldots < t_k$ has at least $t_k/k$ solutions. Then, from Lemma [2] we have

$$
t_k/k = O \left( p^{1-1/k} D_t^{1/k} \right),
$$

where

$$D_t = \min \max_{0\leq i \leq n} \gcd(t_j - t_i, p - 1).
$$

Thus $D_t t \mid p - 1$ and, since $k$ is fixed,

$$t \geq D_t \gg t_k p^{-(k-1)} = t_k p^{-(k-1)}.
$$

(4)

We now fix $D \mid p - 1$, and for each $t = (t_1, \ldots, t_k)$ construct a graph $G_t(D)$ on $k + 1$ vertices $0, \ldots, k$, connecting $i$ and $j$ if and only if $\gcd(t_i - t_j, p - 1) \geq D$ (where, as before $t_0 = 0$).

Clearly, if $D_t = D$ and $G_t(D) = G$ then $\delta(G) \geq 1$.

Now, for a fixed positive integer $D \leq t < p$ and a graph $G$ with $k + 1$ vertices and $\delta(G) \geq 1$, we estimate the number $M_p(D, G, t)$ of vectors $t =$...
\((t_1, \ldots, t_k) \in \mathbb{Z}^k\) with \(1 \leq t_1 < \ldots < t_k\) and \(t_k = t\) such that \(G_t(D) = G\). Summing over all graphs \(G\) (since \(k\) is fixed there are only finitely many graphs) and admissible values of \(D\), that is, with \(t \geq D \gg t^k p^{-(k-1)}\), see \((4)\), leads to the desired estimate.

Given a graph \(G\) with \(k + 1\) vertices and \(\delta(G) \geq 1\), we now fix a dominating set \(S\) in \(G\) of cardinality \(#S = \lfloor (k+1)/2 \rfloor\), which exists by Lemma \([4]\) (obviously, we can always add more vertices to \(S\) if necessary to guarantee \(#S = \lfloor (k+1)/2 \rfloor\)). So for each \(j \notin S\) with \(j \neq 0, k\), there is \(i \in S\) such that \(\gcd(t_i - t_j, q-1) \geq D\). So if \(t_i\) is fixed, then \(t_j\) can take at most

\[
\sum_{d | p-1} \frac{t}{d} \ll \frac{t}{D} \sum_{d | p-1} 1 = \frac{t}{D} p^{o(1)} \tag{5}
\]

values, where we have used the known bound on the divisor function, (see \([5, \text{Theorem 320}]\)). Finally, when \(t_k = t\) is fixed, each \(t_i, i \in S\), can take at most \(t\) values.

Furthermore, if both \(0, k \in S\) then there are only \(#S - 2 \leq \lfloor (k+1)/2 \rfloor - 2 = \lfloor (k-3)/2 \rfloor\) elements \(t_i\) with \(i \in S \setminus \{0, k\}\) to be chosen. After all values of \(t_i\) with \(i \in S\) are fixed, we see from \((5)\) that the remaining \(k + 1 - #S = \lfloor (k+1)/2 \rfloor\) elements \(t_j, j \notin S\), can be chosen in at most \((tp^{o(1)}/D)^{(k+1)/2}\) ways. So in this case

\[
M_p(D, G, t) \leq t^{(k-3)/2}(t/D)^{(k+1)/2} p^{o(1)} = t^{k-1} D^{-\lfloor (k+1)/2 \rfloor} p^{o(1)}. \tag{6}
\]

If \(0 \in S\) but \(k \notin S\), or \(0 \notin S\) but \(k \in S\), then the same argument implies:

\[
M_p(D, G, t) \leq t^{(k-1)/2}(t/D)^{(k-1)/2} p^{o(1)} = t^{k-1} D^{-\lfloor (k-1)/2 \rfloor} p^{o(1)}. \tag{7}
\]

Finally, if both \(0, k \notin S\) then we get

\[
M_p(D, G, t) \leq t^{(k+1)/2}(t/D)^{(k-3)/2} p^{o(1)} = t^{k-1} D^{-\lfloor (k-3)/2 \rfloor} p^{o(1)}. \tag{8}
\]
Clearly, bound (8) dominates the bounds (6) and (7). In particular, for \( t \geq D \gg t^{k-p-(k-1)} \) we obtain

\[
M_p(D, G, t) \leq t^{k-1-k[(k-3)/2]}p^{(k-1)[(k-3)/2]}+o(1).
\]

Since, as we have mentioned, there are only finitely many possibilities for the graphs \( G_t(D) \), recalling (4) and the bound on the divisor function (see [5, Theorem 320]), we obtain the desired result.

5 Comments

A slight modification of our approach can easily produce a nontrivial bound for \( 1 \leq k \leq 3 \) as well, however we do not know how to relax the condition (3).

It is certainly an interesting question to show that almost all \( k \)-lacunary polynomials of a large degree are irreducible over \( \mathbb{F}_p \). In fact, as a first step one can try to get a lower bound on the degree over \( \mathbb{F}_p \) of the splitting field of a “random” \( k \)-lacunary polynomial.

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