Strong Duality for Generalized Trust Region Subproblem: S-Lemma with Interval Bounds

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Abstract With the help of the newly developed S-lemma with interval bounds, we show that strong duality holds for the interval bounded generalized trust region subproblem under some mild assumptions, which answers an open problem raised by Pong and Wolkowicz [Comput. Optim. Appl. 58(2), 273-322, 2014].

Keywords S-lemma · trust region subproblem · strong duality

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1 Introduction

Consider the interval bounded generalized trust region subproblem:

$$\text{(GTRS)} \quad \inf f(x) \quad \text{s.t. } \alpha \le h(x) \le \beta,$$

where $\alpha \le \beta \in \mathbb{R}$, $f(x)$ and $h(x)$ are quadratic functions, i.e.,

$$f(x) := x^T Ax + 2a^T x + c,$$
$$h(x) := x^T Bx + 2b^T x + d,$$

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$A, B \in \mathbb{R}^{n \times n}$ are symmetric matrices, $a, b \in \mathbb{R}^n$, $c, d \in \mathbb{R}$.

When $B = I$, $b = 0$, and $\alpha \leq 0$, (GTRS) is known as the classical trust region subproblem (TRS), which arises in trust region methods for nonlinear programming [2]. Though (TRS) is explicitly non-convex as $A$ is not necessarily positive semidefinite, the necessary and sufficient optimality condition has been derived, see [3,9]. This makes sense as actually (TRS) enjoys the strong duality [4,5,14].

When $\alpha = -\infty$, (GTRS) reduces to the quadratic programming with a single inequality quadratic constraint (QPIQC), see [8,19] and references therein. Under the primal Slater condition that there is an $\hat{x}$ such that $h(\hat{x}) < \beta$, the necessary and sufficient optimality conditions was derived in [10] and the strong duality for (QPIQC) is actually due to the well-known S-lemma, see the survey paper [11].

When $\alpha = \beta$, (GTRS) is the quadratic programming with a single equality quadratic constraint (QP1EQC). Under the primal Slater condition that there are $x'$ and $x''$ such that $h(x') < \beta < h(x'')$, the necessary and sufficient optimality condition was established in [10]. Suppose $B$ is definite, (QP1EQC) admits the exact semi-definite programming relaxation [19]. Very recently, the strong duality for (QP1EQC) is guaranteed by the new developed S-lemma with equality [16].

The two-sided constrained problem (GTRS) was first introduced in [15], where $b = 0$ is assumed. Under the further assumption that $A$ and $B$ are simultaneously diagonalizable via congruence (SDC) [7], the hidden convexity of (GTRS) was observed [1]. Very recently, (GTRS) have been extensively and deeply studied [13]. In particular, strong duality for (GTRS) was established under the following assumptions:

Assumption 1 ([13])

1. $B \neq 0$.
2. (GTRS) is feasible.
3. The following relative interior constraint qualification holds

   \[(\text{RICQ}) \quad \alpha < B \bullet \hat{X} + 2b^T \hat{x} + d < \beta, \text{ for some } \hat{X} \succ \hat{x}\hat{x}^T.\]

4. (GTRS) is bounded below.
5. (D-GTRS) is feasible.

Assumption [1] is reasonable due to the following facts.

Theorem 1 ([13]) The following holds for the Items in Assumption [1]

(i) If one of the Items 1, 2, 3 in Assumption [1] fails, then an explicit solution of (GTRS) can easily be obtained.
(ii) If Items 1, 2, 3 in Assumption [1] hold and $b = 0$, then Item 4 implies Item 5.
(iii) Item 5 in Assumption [1] implies Item 4.
However, it is still unknown whether Item 4 implies Item 5 when $b \neq 0$, see Remark 2.2 \cite{13}.

Before presenting the strong duality result, we need some definitions. First, introducing one free Lagrange multiplier $\mu$ yields the following Lagrange function:

$$L(x, \mu_+, \mu-) = f(x) + \mu_-(h(x) - \beta) + \mu_+(\alpha - b(x)),$$

where $\mu_+ = \max\{\mu, 0\}, \mu_- = -\min\{\mu, 0\}$. Then, we can write down the Lagrangian dual problem of (GTRS):

$$(D\text{-GTRS}) \sup_{\mu} \left\{ \inf_{x} L(x, \mu_+, \mu-) \right\} = \sup_{\mu} c + \mu d - \mu_\beta + \mu_+ \alpha - s$$

s.t. $\begin{bmatrix} A + \mu B & a + \mu b \\ a^T + \mu b^T & s \end{bmatrix} \succeq 0$,

which is viewed as the dual semidefinite programming (SDP) relaxation for (GTRS). The primal form of SDP relaxation for (GTRS) can be obtained by lifting $x \in \mathbb{R}^n$ to $X := xx^T \in \mathbb{R}^{n \times n}$. Relaxing $X = xx^T$ to $X \succeq xx^T$ yields the following primal SDP relaxation problem:

$$(SDP\text{-GTRS}) \inf_{X} A \cdot X + 2a^T x + c$$

s.t. $\alpha \leq B \cdot X + 2b^T x + d \leq \beta$,

$X \succeq xx^T$,

where the final inequality is equivalent to the linear matrix inequality (LMI)

$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0,$$

according to Schur complement argument. One can verify that (SDP-GTRS) is also the conic dual of (D-GTRS).

Let $v(\cdot)$ denote the optimal value of the problem $\cdot$. We have the following strong duality result.

**Theorem 2 \cite{13}** Under Assumption $\mathbb{1}$, strong duality holds for both (GTRS) and (SDP-GTRS), i.e.,

$$v(GTRS) = v(D\text{-GTRS}) = v(SDP\text{-GTRS}).$$

Moreover, $v(SDP\text{-GTRS})$ is attained.

In this paper, Theorems $\mathbb{1}$ and $\mathbb{2}$ are both extended. More precisely, we prove that Item 4 implies Item 5 when $b \neq 0$, which answers the open question remained in Theorem $\mathbb{1}$. For Theorem $\mathbb{2}$, we show that Items 1 and 3 in Assumption $\mathbb{1}$ are actually sufficient to guarantee the strong duality for (GTRS). As a by-product, Item 2 is redundant since it can be implied by Item 3. The above new results are presented in Section 3. Actually, they are applications of the newly developed S-lemma with interval bounds, which is completely characterized in Section 2. Conclusions are made in Section 4.
Throughout the paper, the notations \( \mathbb{R}^n \) and \( S^n_+ \) denote the \( n \)-dimensional vector space and \( n \times n \) positive semidefinite symmetric matrix space, respectively. Denote by \( A \succ (\succeq) 0 \) the matrix \( A \) is positive (semi)definite. The inner product of two matrices \( A, B \) is denoted by \( A \cdot B = \sum_{i,j=1}^n a_{ij} b_{ij} \). Denote by \( \mathcal{N}(B) \) the null space of \( B \).

### 2 S-Lemma and Generalization

The fundamental S-Lemma was first proved by Yakubovich [17,18] in 1971, see recent surveys [3,11].

**Theorem 3 ([17,18])** Under the Slater assumption that there is an \( x \in \mathbb{R}^n \) such that \( h(x) < 0 \), the system

\[
\begin{align*}
    f(x) &< 0, \ h(x) \leq 0
\end{align*}
\]

is unsolvable if and only if there is a nonnegative number \( \mu \geq 0 \) such that

\[
\begin{align*}
    f(x) + \mu h(x) &\geq 0, \ \forall x \in \mathbb{R}^n.
\end{align*}
\]

Very recently, the S-lemma with equality, known as a long-standing open problem, has been proved by Xia et al. [16].

**Theorem 4 ([16])** Suppose the Slater assumption for equality holds, that is, there are \( x', x'' \in \mathbb{R}^n \) such that \( h(x') < 0 < h(x'') \). Then, except for the case that \( A \) has exactly one negative eigenvalue, \( B = 0, b \neq 0 \) and

\[
\begin{align*}
    \begin{bmatrix}
    V^T AV & V^T (Ax_0 + a) \\
    (x_0^T A + a^T)V & f(x_0)
    \end{bmatrix} \succeq 0,
\end{align*}
\]

where \( x_0 = -\frac{d}{2b^T b}, V \in \mathbb{R}^{n \times (n-1)} \) is the matrix basis of \( \mathcal{N}(b) := \{ x : b^T x = 0 \} \), the system

\[
\begin{align*}
    f(x) &< 0, \ h(x) = 0
\end{align*}
\]

is unsolvable if and only if there is a number \( \mu \) such that

\[
\begin{align*}
    f(x) + \mu h(x) &\geq 0, \ \forall x \in \mathbb{R}^n.
\end{align*}
\]

In this section, as a further extension of Theorems [3] and [4], we characterize the S-lemma with interval bounds, which asks when the following two statements are equivalent:

1. **(S1)** The system

\[
\begin{align*}
    f(x) &< 0, \ \alpha \leq h(x) \leq \beta
\end{align*}
\]

is unsolvable;

2. **(S2)** There is a number \( \mu \in \mathbb{R} \) such that

\[
\begin{align*}
    f(x) + \mu_-(h(x) - \beta) + \mu_+(\alpha - h(x)) &\geq 0, \ \forall x \in \mathbb{R}^n.
\end{align*}
\]

where \( \mu_+ = \max\{\mu, 0\}, \mu_- = -\min\{\mu, 0\} \).
Since the special cases \( \alpha = -\infty \) (or \( \beta = +\infty \)) and \( \alpha = \beta \) have been settled in Theorems 3 and 4, respectively, throughout this paper, we can always make the following assumption:

**Assumption 2** \( -\infty < \alpha < \beta < +\infty \).

The above S-lemma with interval bounds can be regarded as a special case of the general S-procedure [3]. Actually, Polyak [12] succeeded in proving a version of S-procedure involving two quadratic functions in the constraint set:

**Theorem 5** ([12]) Suppose \( n \geq 3 \), \( f_i(x) = x^T A_i x, i = 0, 1, 2 \), real numbers \( \alpha_i, i = 0, 1, 2 \) and there exist \( \mu \in \mathbb{R}^2, x^0 \in \mathbb{R}^n \) such that

\[
\begin{align*}
\mu_1 A_1 + \mu_2 A_2 & > 0, \\
f_1(x^0) & < \alpha_1, f_2(x^0) < \alpha_2.
\end{align*}
\]

Then the system

\[
\begin{align*}
f_0(x) & < \alpha_0, \\
f_1(x) & \leq \alpha_1, f_2(x) \leq \alpha_2
\end{align*}
\]

has no solution if and only if there exist \( \tau_1 \geq 0, \tau_2 \geq 0 \):

\[
\begin{align*}
A_0 + \tau_1 A_1 + \tau_2 A_2 & \geq 0, \\
\alpha_0 + \tau_1 \alpha_1 + \tau_2 \alpha_2 & \leq 0.
\end{align*}
\]

It should be noted that Theorem 5 only implies a special case of the S-lemma with interval bounds where \( a = b = 0 \), and \( B \) is definite.

Now we can establish the general S-lemma with interval bounds. Without loss of generality, we make the following assumption:

**Assumption 3** There exists an \( x \in \mathbb{R}^n \) such that \( \alpha < h(x) < \beta \).

**Theorem 6** Under Assumptions 2 and 3, S-lemma with interval bounds holds except that \( A \) has exactly one negative eigenvalue, \( B = 0 \), \( b \neq 0 \) and there exists a \( \nu \geq 0 \) such that

\[
\begin{bmatrix}
V^T AV & \frac{1}{2\nu} V^T Ab \\
\frac{1}{2\nu} V^T Ab & V^T a
\end{bmatrix}
+ \nu \begin{bmatrix}
\frac{\nu b}{2\nu b} - \frac{\nu}{2}(\alpha + \beta - 2d) \\
\frac{\nu b}{2\nu b} - \frac{\nu}{2}(\alpha + \beta - 2d) & c + \nu(\alpha - d)(\beta - d)
\end{bmatrix}
\geq 0,
\]

(2)

where \( V \in \mathbb{R}^{n \times (n-1)} \) is the matrix basis of \( N(b) \).

Proof. Note that it is trivial to verify that (S2) always implies (S1). It is sufficient to assume (S1) holds and then show (S2) is also true.

We first assume

\[
\alpha \leq \inf_{x \in \mathbb{R}^n} h(x) \leq \sup_{x \in \mathbb{R}^n} h(x) \leq \beta.
\]

Then, (S1) becomes that \( f(x) < 0 \) is unsolvable. It certainly implies (S2) holds with the setting \( \mu = 0 \).
Next, we assume exactly one of the following case occurs:

\[ \alpha \leq \inf_{x \in \mathbb{R}^n} h(x) < \beta < \sup_{x \in \mathbb{R}^n} h(x), \]
\[ \inf_{x \in \mathbb{R}^n} h(x) < \alpha < \sup_{x \in \mathbb{R}^n} h(x) \leq \beta. \]

Without loss of generality, we assume the first case holds. Consequently, the system (1) in (S1) is equivalent to

\[ f(x) < 0, \; h(x) \leq \beta \]

and there is an \( \hat{x} \in \mathbb{R}^n \) such that \( h(\hat{x}) < \beta \), i.e., Slater condition holds.

According to the S-lemma with inequality (i.e., Theorem 3), (S1) holds if and only if there is a number \( \nu \geq 0 \) such that

\[ f(x) + \nu(\alpha - h(x)) \geq 0, \; \forall x \in \mathbb{R}^n. \]

It follows that (S2) holds with \( \mu = -\nu \), which finishes the proof.

Now, under Assumption 3, it is sufficient to assume

\[ \inf_{x \in \mathbb{R}^n} h(x) < \alpha < \beta < \sup_{x \in \mathbb{R}^n} h(x). \]  (3)

Firstly, we further assume either \( A \succeq 0 \) or \( B \neq 0 \). Suppose (S1) holds. Then, for any \( s \in [\alpha, \beta] \), the system

\[ f(x) < 0, \; h(x) - s = 0, \]

is unsolvable. Assumption (3) implies that there are \( x', x'' \in \mathbb{R}^n \) such that \( h(x') < \alpha < \beta < h(x'') \). It follows that

\[ h(x') - s < \alpha - s \leq 0 \leq \beta - s < h(x'') - s. \]

According to Theorem 4, there is a number \( \mu(s) \) such that

\[ f(x) + \mu(s)(h(x) - s) \geq 0, \; \forall x \in \mathbb{R}^n. \]  (4)

(a) Suppose \( \mu(\beta) > 0 \). Let \( \mu = -\mu(\beta) \). Then \( \mu_+ = \mu(\beta) \) and

\[ f(x) + \mu_+(\alpha - h(x)) \geq 0, \; \forall x \in \mathbb{R}^n. \]

(b) Suppose \( \mu(\alpha) < 0 \). Let \( \mu = -\mu(\alpha) \). Then \( \mu_+ = -\mu(\alpha) \) and

\[ f(x) + \mu_+(\alpha - h(x)) \geq 0, \; \forall x \in \mathbb{R}^n. \]

(c) Suppose \( \mu(\alpha) \geq 0 \geq \mu(\beta) \). (4) implies that

\[ f(x) + \mu(\alpha)(h(x) - \alpha) \geq 0, \; \forall x \in \mathbb{R}^n, \]
\[ f(x) + \mu(\beta)(h(x) - \beta) \geq 0, \; \forall x \in \mathbb{R}^n. \]

According to Theorem 3, both the system

\[ f(x) < 0, \; h(x) \leq \alpha, \]
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and the system

\[ f(x) < 0, \ h(x) \geq \beta, \]

are unsolvable. Since \((S_1)\) holds, we have

\[ f(x) \geq 0, \ \forall x \in \mathbb{R}^n, \]

\((S_2)\) holds with \(\mu = 0\).

Therefore, S-lemma with interval bounds holds under the assumption either \(A \succeq 0\) or \(B \neq 0\).

Now we assume \(A \not\succeq 0\) and \(B = 0\). Then, \((S_2)\) cannot hold true. According to Assumption \((4)\), we have \(b \neq 0\). Notice that

\[ \{ x \in \mathbb{R}^n : \alpha \leq h(x) \leq \beta \} = \left\{ \frac{z}{2b^Tb} - b + Vy : z \in [\alpha - d, \beta - d], \ y \in \mathbb{R}^{n-1} \right\} \]

where \(V\) is a matrix basis of \(\mathcal{N}(b)\). Trivially, \((S_1)\) holds if and only if

\[ \inf_{\tilde{h}(z) \leq 0, \ y \in \mathbb{R}^{n-1}} \left\{ f \left( \frac{z}{2b^Tb} + Vy \right) \right\} \geq 0, \]

or equivalently,

\[ \inf_{\tilde{h}(z) \leq 0, \ y \in \mathbb{R}^{n-1}} \left\{ f \left( \frac{z}{2b^Tb} + Vy \right) \right\} \geq 0, \]

where

\[ \tilde{h}(z) := (z - (\alpha - d))(z - (\beta - d)) = z^2 - (\alpha + \beta - 2d)z + (\alpha - d)(\beta - d). \]

Therefore, for any given \(y \in \mathbb{R}^{n-1}\), the system

\[ f \left( \frac{z}{2b^Tb} + Vy \right) < 0, \ \tilde{h}(z) \leq 0 \]

is unsolvable. Since \(\alpha < \beta\), Slater assumption holds for \(\tilde{h}(z) \leq 0\). According to Theorem \ref{thm:slater}, there exists a \(\nu \geq 0\) such that

\[ f \left( \frac{z}{2b^Tb} + Vy \right) + \nu \tilde{h}(z) \geq 0. \] (5)

Notice that

\[ f \left( \frac{z}{2b^Tb} + Vy \right) = \frac{b^TAb}{(2b^Tb)^2} z^2 + \frac{a^Tb}{b^Tb} z + \frac{z}{b^Tb} b^TAVy + 2a^TVy + y^TV^TAVy + c. \]

\ref{eq:slater} can be rewritten as

\[
\begin{bmatrix}
  y \\
  z \\
  1
\end{bmatrix}^T \begin{bmatrix}
  V^TAV & \frac{1}{2b^Tb} V^TAb & V^Ta \\
  \frac{1}{2b^Tb} V^TAb & \frac{a^Tb}{(2b^Tb)^2} + \nu & \frac{a^Tb}{2b^Tb} - \frac{\nu}{2}(\alpha + \beta - 2d) \\
  V^TAv & \frac{a^Tb}{2b^Tb} - \frac{\nu}{2}(\alpha + \beta - 2d) & c + \nu(\alpha - d)(\beta - d)
\end{bmatrix} \begin{bmatrix}
  y \\
  z \\
  1
\end{bmatrix} \geq 0.
\]

Therefore, under the assumption \(A \not\succeq 0\) and \(B = 0\), \((S_1)\) holds if and only if \ref{eq:slater} holds. Since \(A \not\succeq 0\) and \(V^TAV \succeq 0\), it must hold that \(A\) has exactly one negative eigenvalue. \qed
3 Strong Duality for (GTRS)

In this section, we apply the S-lemma with interval bounds to establish strong duality for (GTRS).

We first study the relation between Assumptions 1 and 3.

Lemma 1 Assumption (3) is equivalent to Item 3 in Assumption (1).

Proof. Suppose Assumption (3) is violated, we have either \( \inf_{x \in \mathbb{R}^n} h(x) \geq \beta \) or \( \sup_{x \in \mathbb{R}^n} h(x) \leq \alpha \). We first assume \( \inf_{x \in \mathbb{R}^n} h(x) \geq \beta \). It follows that \( B \succeq 0 \).

For any \( \hat{X} \succeq \hat{x} \hat{x}^T \), we have \( B \cdot (\hat{X} - \hat{x} \hat{x}^T) \geq 0 \). If Item 3 in Assumption (1) holds, we obtain the following contradiction:

\[
\begin{align*}
    h(\hat{x}) &= B \cdot (\hat{x} \hat{x}^T) + 2b^T \hat{x} + d \\
    &\leq B \cdot \hat{X} + 2b^T \hat{x} + d < \beta.
\end{align*}
\]

The other case \( \sup_{x \in \mathbb{R}^n} h(x) \leq \alpha \) can be similarly discussed. Consequently, Item 3 of Assumption (1) implies Assumption (3).

Now we assume Assumption (3) holds, i.e., there is an \( \hat{x} \) such that \( h(\hat{x}) \in (\alpha, \beta) \). Define

\[
\hat{X}(\epsilon) = \hat{x} \hat{x}^T + \epsilon I,
\]

where \( I \) is the identity matrix. Then, we have \( \hat{X}(\epsilon) \succ \hat{x} \hat{x}^T \) for all \( \epsilon > 0 \), and

\[
\lim_{\epsilon \to 0} \left\{ B \cdot \left( \hat{X}(\epsilon) \right) + 2b^T \hat{x} + d \right\} = h(\hat{x}) \in (\alpha, \beta).
\]

Therefore, there is an \( \epsilon_0 > 0 \) such that \( \hat{X}(\epsilon_0) \succ \hat{x} \hat{x}^T \) and

\[
\alpha < B \cdot \hat{X}(\epsilon_0) + 2b^T \hat{x} + d < \beta.
\]

That is, Items 3 of Assumption (1) hold. The proof is complete. \( \square \)

As pointed out by one referee, Item 2 in Assumption (1) is unnecessary as it can be implied by Item 3 according to Lemma (1).

Now, as a main result of this paper, we extend Theorem (2).

Theorem 7 Under Items 1 and 3 in Assumption (1), strong duality holds for both (GTRS) and (SDP-GTRS), i.e.,

\[
v(GTRS) = v(D-GTRS) = v(SDP-GTRS).
\]

Additionally, suppose Item 4 in Assumption (1) holds, \( v(D-GTRS) \) is attained.

Proof. According to Lemma (1), Items 1 and 3 in Assumption (1) imply that \( B \neq 0 \) and Assumption (3) holds. It follows from Theorem (2) that S-lemma with
interval bounds holds. Then, we have

\[ v(GTRS) = \sup_{s \in \mathbb{R}} \left\{ s \mid \{ x \in \mathbb{R}^n \mid f(x) - s < 0, \alpha \leq h(x) \leq \beta \} = \emptyset \right\} \]

\[ = \sup_{s, \mu \in \mathbb{R}} \left\{ s \mid \left[ A + \mu B \right] \begin{bmatrix} a & \mu b \\ a^T & c + \mu d \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \right\} \quad \text{(6)} \]

\[ \leq \inf_{X \in S_n^{n+1}} \left\{ \begin{bmatrix} A & a \\ a^T & c \end{bmatrix} \begin{bmatrix} B \\ b \\ b^T \\ d \end{bmatrix} \begin{bmatrix} X \\ X \end{bmatrix} \in [\alpha, \beta], X_{n+1,n+1} = 1 \right\} \quad \text{(7)} \]

\[ \leq \inf_{x \in \mathbb{R}^n} \left\{ \begin{bmatrix} A & a \\ a^T & c \end{bmatrix} \begin{bmatrix} B \\ b \\ b^T \\ d \end{bmatrix} \begin{bmatrix} X \\ X \end{bmatrix} \in [\alpha, \beta], X = \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \right\} \]

\[ = v(GTRS). \]

It is not difficult to verify that (6) and (7) are exactly the dual SDP (D-GTRS) and primal SDP (SDP-GTRS), respectively. Thus, the strong duality holds for both (GTRS) and (SDP-GTRS).

Now, suppose Item 4 in Assumption 1 also holds, i.e., \( v(GTRS) > -\infty \). Then, we have \( v(SDP-GTRS) = v(GTRS) > -\infty \). Note that, according to Item 3 in Assumption 1 (SDP-GTRS) has a strictly feasible solution. It follows from the standard strong duality theory for SDP that \( v(D-GTRS) \) is attained.

As an immediate corollary of Theorem 7, we improve Item (ii) in Theorem 1 which answers the open question raised in [13] whether Item 4 implies Item 5 when \( b \neq 0 \).

**Corollary 1** Under Items 1 and 3 in Assumption 1, Items 4 and 5 are equivalent.

**Proof.** According to Theorem 7, under Items 1 and 3 in Assumption 1, \( v(GTRS) = -\infty \) if and only if \( v(D-GTRS) = -\infty \), i.e., (D-GTRS) is infeasible.

Finally, Theorem 6 implies that Item 1 in Assumption 1 is necessary for strong duality. Actually, when \( A \) has exactly one negative eigenvalue, \( B = 0, b \neq 0 \) and there is a real number \( \nu \geq 0 \) satisfying (2), according to the proof of Theorem 6, we have

\[ v(GTRS) \geq 0, \quad v(D-GTRS) = -\infty. \]

That is, the duality gap is \( +\infty \).

However, in the case \( B = 0 \), duality gap can be closed by reformulating the constraint \( \alpha \leq h(x) \leq \beta \) as \((h(x) - \alpha)(h(x) - \beta) \leq 0\), which corresponds to a special case of Theorem 6 where \( \alpha = -\infty \).
4 Conclusion

In this paper, we have extended the classical S-lemma to the interval bounded S-lemma. As an application, we establish strong duality for the interval bounded generalized trust region subproblem (GTRS) under some mild assumptions. Our assumptions are much weaker than that in [13]. As a by-product, we answer an open question posted in [13]. The future work includes further extensions and/or applications of our S-lemma with interval bounds.

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