DIMITROV’S QUESTION FOR THE POLYNOMIALS OF DEGREE 1, 2, 3, 4, 5, 6

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Abstract. D. Dimitrov [4] has posted the problem of finding the optimal polynomials that provide the sharpness of Koebe Quarter Theorem for polynomials and asked whether Suffridge polynomials [7] are optimal ones. We disproved Dimitrov’s conjecture for polynomials of degree 3, 4, 5 and 6. For polynomials of degree 1 and 2 the conjecture is valid.

1. Introduction

One of the fundamental results in the geometric complex analysis is the famous Koebe Quarter Theorem. It states that for any function \( f \in \mathcal{U}_n \) the image \( f(\mathbb{D}) \) contains a disc of radius 1/4, whether \( \mathbb{D} = \{|z| < 1\} \) and \( \mathcal{U}_n = \{f(z) : f(0) = 0, f'(0) = 1, f(z) \text{ is univalent in } \mathbb{D}\} \). The 1/4 bound is sharp as it is indicated by the Koebe function \( K(z) = z/(1 - z)^2 \). A natural question is whether the constant 1/4 can be improved for polynomial of specific degree. Say, for polynomials of the first degree it is trivially 1; a simple computation demonstrates that for polynomials of degree 2 it is 1/2. The task was formalized by Dimitrov [4, Problem 5] who posted the following problem

For any \( n \in \mathbb{Z}_+ \) find the polynomial \( p_n(z) \in \mathcal{U}_n \), for which the infimum \( \inf \{|p_n(z)| : z = e^{it}, 0 \leq t \leq 2\pi \} \) is attained.

By the Koebe Quarter Theorem the above infimums are bounded from below by 1/4.

Córdova and Ruscheweyh [3] considered the Suffridge polynomials [7]

\[
S_{n,j}(z) = \sum_{k=1}^{n} \left( 1 - \frac{k - 1}{n} \right) \frac{\sin(\pi j k/(n + 1))}{\sin(\pi j/(n + 1))} z^k.
\]

Note that \( S_{n,j}(z) \in \mathcal{U}_n \) and \( |S_{n,1}(-1)| = \frac{1}{4} \sum_{k=0}^{n-1} \frac{\pi k}{n} \sec^2 \frac{\pi k}{2(n+1)} \to 1/4 \). Hence these polynomials solve the latter problem at least asymptotically.

Note that the value \( \frac{1}{4} \sum_{k=0}^{n-1} \frac{\pi k}{n} \sec^2 \frac{\pi k}{2(n+1)} \) is the Koebe radius only for polynomials \( S_{n,1}(z) \) of even degree. For the polynomials of odd degree the quantity \( \inf \{|S_{n,1}(z)| : |z| = 1\} \) is not achieved at the point \( z = -1 \), rather a different point \( \xi \), such that \( S'_{n,1}(\xi) = 0 \). (see Fig 1).
Fig 1: The image and fragment for $S_{3,1}(\mathbb{D})$.

Note that for $n = 3$, $|S_{3,1}(-1)| \approx 0.3905$ while the Koebe radius is $r_3 \approx 0.3849$. For $n = 5$, $|S_{5,1}(-1)| \approx 0.3215$ while the Koebe radius is $r_5 \approx 0.3196$. Note that $r_2 = 0.5$, $r_4 \approx 0.3455$, $r_6 \approx 0.3069$.

Dimitrov [4, p.15] asked a specific question about the Suffridge polynomial: *Is it the extremal one for every fixed $N$?* Note that they are indeed extremal for $N = 1, 2$. Below we prove that the answer is negative for $N = 3, 4, 5, 6$.

### 2. New extremal polynomials

Univalent polynomials are classical objects of complex analysis. Perhaps, the first systematic approach was suggested by Alexander [1] who proved that the truncated sums of the Taylor series of the function $f(z) = \log(1/(1 - z))$ are univalent in $\mathbb{D}$ polynomials. Note that Alexander’s paper contains many ideas that were not properly estimated at that time, c.f. [5]. The subtlety of the situation well illustrates the fact that a necessary condition of univalency - the derivative does not vanish in $\mathbb{D}$ - implies that the $n$-th coefficient of the polynomial of degree $n$ cannot exceed $1/n$ in absolute value. This is perfectly fine with the logarithm function and awfully wrong with the Koebe function. Thus, Suffridge polynomials can be treated as reasonable substitutions for the function $K(z)$. These polynomials are extremal in a way that they have the $n$-th coefficient exactly $1/n$.

Thus, so far we have two families of extremal univalent polynomials in play - Alexander polynomials and Suffridge polynomials. The main discovery of the current paper is a new extremal family of polynomials that seem to be univalent in $\mathbb{D}$ and might be as important as the two mentioned above series. Namely, the following polynomials were introduced in [8].

$$P_N(z) = \frac{1}{U'_N\left(\cos \frac{\pi}{N+2}\right)} \sum_{k=1}^{N} U'_{N-k+1}\left(\cos \frac{\pi}{N + 2}\right) U_{k-1}\left(\cos \frac{\pi}{N + 2}\right) z^k,$$

where $U_k(x)$ is a family of Chebyshev polynomials of the second kind and $U'_k(x)$ is a derivative.

One given below some examples:

$$P_1(z) = z, \quad P_2(z) = z + \frac{1}{2}z^2,$$
First, let us write

\[ P_3(z) = z + \frac{2}{\sqrt{5}} z^2 + \frac{1}{2} \left(1 - \frac{1}{\sqrt{5}}\right) z^3, \quad P_4(z) = z + \frac{7}{6} z^2 + \frac{2}{3} z^3 + \frac{1}{6} z^4, \]

\[ P_5(z) = z + \frac{8 - 40 (\cos(\pi/7))^2 + 32 (\cos(\pi/7))^3 - 24 \cos(\pi/7))^2}{40 (\cos(\pi/7))^3 - 30 \cos(\pi/7) - 32 (\cos(\pi/7))^2 + 7} \]

\[ \frac{24 (\cos(\pi/7))^3 - 28 (\cos(\pi/7))^2 - 18 \cos(\pi/7) + 4 z^3}{40 (\cos(\pi/7))^3 - 30 \cos(\pi/7) - 32 (\cos(\pi/7))^2 + 7} \]

\[ \frac{16 (\cos(\pi/7))^3 - 16 (\cos(\pi/7))^2 - 12 \cos(\pi/7) + 4 z^4}{40 (\cos(\pi/7))^3 - 30 \cos(\pi/7) - 32 (\cos(\pi/7))^2 + 7} \]

\[ \frac{8 (\cos(\pi/7))^3 - 4 (\cos(\pi/7))^2 - 6 \cos(\pi/7) + 1 z^5}{40 (\cos(\pi/7))^3 - 30 \cos(\pi/7) - 32 (\cos(\pi/7))^2 + 7} \]

\[ P_6(z) = z + \frac{9 + 8 \sqrt{2}}{4 \sqrt{2} + 8} z^2 + \frac{6 \sqrt{2} + 10}{4 \sqrt{2} + 8} z^3 + \frac{4 \sqrt{2} + 6}{4 \sqrt{2} + 8} z^4 + \frac{2 \sqrt{2} + 2}{4 \sqrt{2} + 8} z^5 + \frac{1}{4 \sqrt{2} + 8} z^6 \]

**Theorem 1.** The following presentation is valid for \( t \in (0, \pi) \), \( t \neq \frac{2\pi}{N+2} \)

\[ P_N(e^{it}) = \frac{1}{2 \left(\cos t - \cos \frac{2\pi}{N+2}\right)} + \frac{1 - \cos \frac{2\pi}{N+2}}{(N+2)(1-\cos t)} \sin t \sin \frac{N+2}{2} \frac{\sin \frac{N+2}{2}}{\left(\cos t - \cos \frac{2\pi}{N+2}\right)^2} e^{\frac{N+2}{2}it}. \]

**Proof.** First, let us write \( P_N(z) \) in terms of trigonometric expressions [8]

\[ P_N(z) = \frac{1}{(N+2) \sin \frac{2\pi}{N+2}} \sum_{k=1}^{N} \left( (N-k+3) \sin \frac{(k+1)\pi}{N+2} - \frac{(N-k+1) \sin \frac{(k-1)\pi}{N+2}}{\sin \frac{2\pi}{N+2}} \right) \sin \frac{k\pi}{N+2} z^k \]

Having in mind that

\[ 2 \sin(\pi) - 0 \cdot \sin \frac{N\pi}{N+2} \sin \frac{(N+1)\pi}{N+2} \cdot z^{N+1} = 0 \]

we can change the upper bound for the range in the sum from \( N \) to \( N+1 \). Further modification produces

\[ P_N(z) = \frac{1}{(N+2) \sin \frac{2\pi}{N+2}} \sum_{k=1}^{N+1} \left( (N-k+2) \sin \frac{2k\pi}{N+2} + 2 \frac{\cos \frac{\pi}{N+2}}{\sin \frac{2\pi}{N+2}} \sin^2 \frac{k\pi}{N+2} \right) z^k. \]

An important observation is that

\[ \frac{N+1}{N+2} \cdot S_{N+1,2}(z) = \frac{1}{(N+2) \sin \frac{2\pi}{N+2}} \sum_{k=1}^{N+1} (N-k+2) \sin \frac{2k\pi}{N+2} \cdot z^k, \]

where \( S_{N+1,2}(z) \) is the second Suffridge polynomial of order \( N+1 \). By formula (5) in [7] p. 496], for \( n = N+1 \) and \( j = 2 \) we get
Theorem 2. The following presentation is valid for $t \in (0, \pi)$, $t \neq \frac{2\pi}{N+2}$.

\[
4|P_N(e^{it})|^2 = \left( \frac{\cos \frac{N+2t}{2} - \cos \frac{2\pi}{N+2}}{\cos t - \cos \frac{2\pi}{N+2}} \right)^2 + 2\frac{1 - \cos \frac{2\pi}{N+2}}{N + 2} \left( \frac{\sin t}{\cos t - \cos \frac{2\pi}{N+2}} \right)^2 \left( \frac{\sin \frac{N+2t}{2} - \sin \frac{2\pi}{N+2}}{\cos t - \cos \frac{2\pi}{N+2}} \right)^2 .
\]

We use the Chebyshev polynomials of the first kind $T_n(x)$ and put $t = \frac{2\pi}{N+2}$.

By combining both formulas, we get the formula in the theorem. □

Note that the right hand side has removable singularities, thus in fact it is a trigonometric polynomial.

Let us fix a positive integer $N$ and let $R_N(e^{it}) = |P_N(e^{it})|^2$. The following theorem can be directly verified by tedious standard computations.

**Theorem 2.** The following presentation is valid for $t \in (0, \pi)$, $t \neq \frac{2\pi}{N+2}$.

\[
4|P_N(e^{it})|^2 = \left( \frac{\cos \frac{N+2t}{2} - \cos \frac{2\pi}{N+2}}{\cos t - \cos \frac{2\pi}{N+2}} \right)^2 + 2\frac{1 - \cos \frac{2\pi}{N+2}}{N + 2} \left( \frac{\sin t}{\cos t - \cos \frac{2\pi}{N+2}} \right)^2 \left( \frac{\sin \frac{N+2t}{2} - \sin \frac{2\pi}{N+2}}{\cos t - \cos \frac{2\pi}{N+2}} \right)^2 .
\]

Because the real coefficients symmetry of $P_N(e^{it})$ (the real part is an even function and the imaginary is an odd function of $t$), we denote $|P_N(e^{it})|^2 = R_N(x)$ as a polynomial of $x = \cos(t)$. Let $b = \cos \frac{2\pi}{N+2}$ and $T_N$ be the Chebyshev polynomial of the first kind. From Theorem 2 one can get the following formulas by straightforward computations:

\[
4R_N(x) = \frac{1}{(x-b)^2} + 2\frac{1-b}{(N+2)(x-b)^3} + 2\frac{(1-b)^2}{(N+2)^2}(1-x),
\]

\[
4(R_N(x))' = \frac{2}{(b-x)^3} \left( 1 - \frac{1-b}{1-x} (1-T_{N+2}(x)) \left( 1 - \frac{4(1-b)(1+x)}{(N+2)^2(b-x)^2} - \frac{2(1-b)}{(N+2)^2(1-x)(b-x)} \right) + \frac{1-b}{1-x} \frac{1-b}{1-x} U_{N+1}(x) \frac{1-bx + 3(1-x^2)}{b-x} \right).
\]

**Theorem 3.** If $(R_N(x))' > 0$ for $x \in (-1, 1)$ then the polynomial $P_N(z)$ is univalent in $\mathbb{D}$ and the Koebe radius of this polynomial is $\sqrt{R_N(-1)}$.

It is proved in [8] that the polynomial $P_N(z)$ is typically real and thus the image of the unit circle has no self intersections, the theorem is proved.

Note, that $\sqrt{R_N(-1)} = \frac{1}{4} \sec^2 \frac{\pi}{N+2}$.
3. The case N=1.

In this case $R_1(x) = 1$, thus the Koebe radius is 1.

4. The case N=2.

In this case $R_2(x) = 5/4 + x$, thus the Koebe radius is $\sqrt{R_2(-1)} = 1/2$.

5. The case N=3.

In this case the polynomial $P_3(z)$ is univalent that can be verified using Brennan’s criteria [2]. Also

$$\frac{4R_3(x)}{25} = -2 \frac{37 \cos \left(\frac{1}{5} \pi \right) - 69 + 56 \left(\cos \left(\frac{1}{5} \pi \right)\right)^2}{\left(\cos \left(\frac{1}{5} \pi \right)\right)^2 + 1 - 2 \cos \left(\frac{1}{5} \pi \right)} - \frac{32}{25} \frac{\left(23 \cos \left(\frac{1}{5} \pi \right) - 51 + 49 \left(\cos \left(\frac{1}{5} \pi \right)\right)^2\right)x}{-9 - 5 \cos \left(\frac{1}{5} \pi \right) + 20 \left(\cos \left(\frac{1}{5} \pi \right)\right)^2}$$

$$4R_3'(x) = -32 \frac{23 \cos \left(\frac{1}{5} \pi \right) - 51 + 49 \left(\cos \left(\frac{1}{5} \pi \right)\right)^2}{-9 - 5 \cos \left(\frac{1}{5} \pi \right) + 20 \left(\cos \left(\frac{1}{5} \pi \right)\right)^2} \frac{64}{5} \frac{\left(10 \cos \left(\frac{1}{5} \pi \right) - 14 + 9 \left(\cos \left(\frac{1}{5} \pi \right)\right)^2\right)x}{14 \left(\cos \left(\frac{1}{5} \pi \right)\right)^2 + 3 - 15 \cos \left(\frac{1}{5} \pi \right)}$$

One can check that $R_3'(x)$ is positive on [-1,1], which implies the estimate from above on Koebe radius $|P_3(-1)| = 3 \sqrt{5} \approx 0.382$.

6. The case N=4.

In this case the polynomial $P_4(z)$ is univalent, c.f. [6].

$$4R_4(x) = 40/9 + (112/9)x + (124/9)x^2 + (16/3)x^3$$

and

$$4R_4'(x) = 112/9 + (248/9)x + (48/3)x^2$$

The discriminant is $-37.13...$ therefore the smallest value for $R_4(x)$ is at -1, which implies the estimate from above on Koebe radius $|P_4(-1)| = 1/3$. 

7. The case \( N = 5 \).

In the particular case \( N = 5 \) we get

\[ 4R_5'(x) = \frac{16}{49} \left( 42 \cos (\frac{1}{7} \pi)^3 - 31 \cos (\frac{1}{7} \pi) + 9 - 47 \cos (\frac{1}{7} \pi)^2 \right) + \]

\[ \frac{64}{49} \left( \frac{762 \cos (\frac{1}{7} \pi)^3 - 618 \cos (\frac{1}{7} \pi) - 3 - 323 (\sin (\frac{1}{7} \pi)^2) x}{-11 + 2 (\sin (\frac{1}{7} \pi))^2 - 39 \cos (\frac{1}{7} \pi) + 60 (\cos (\frac{1}{7} \pi))^3} \right) + \]

\[ \frac{192}{49} \left( \frac{940 \cos (\frac{1}{7} \pi)^3 - 761 \cos (\frac{1}{7} \pi) + 81 - 536 (\sin (\frac{1}{7} \pi)^2) x^2}{-18 + 14 (\cos (\frac{1}{7} \pi))^3 + 35 (\sin (\frac{1}{7} \pi))^2} \right) + \]

\[ \frac{128}{7} \left( \frac{380 \cos (\frac{1}{7} \pi)^3 - 293 \cos (\frac{1}{7} \pi) + 17 - 176 (\sin (\frac{1}{7} \pi)^2) x^3}{-22 + 9 (\sin (\frac{1}{7} \pi))^2 - 70 \cos (\frac{1}{7} \pi) + 112 (\cos (\frac{1}{7} \pi))^3} \right). \]

![Fig 2: The graphs \( R_5(x) \) and \( R_5'(x) \)](image)

By decomposing into Taylor polynomial centered at \(-1\) we get

\[ 4R_5'(x) = -\frac{8}{49} \left( 778868087 \cos (\frac{1}{7} \pi)^2 - 791395834 + 2270258054 \cos (\frac{1}{7} \pi)^3 - 1666223113 \cos (\frac{1}{7} \pi) \right) + \]

\[ \frac{32}{49} \left( -58704325 + 88578183 (\cos (\frac{1}{7} \pi))^2 + 57393568 (\cos (\frac{1}{7} \pi))^3 - 61237982 \cos (\frac{1}{7} \pi) \right) (1 + x) + \]

\[ \frac{192}{49} \left( -212691 + 312494 (\cos (\frac{1}{7} \pi))^2 - 238981 \cos (\frac{1}{7} \pi) + 238432 (\cos (\frac{1}{7} \pi))^3 \right) (1 + x)^2 + \]

\[ \frac{128}{7} \left( 17 - 176 (\sin (\frac{1}{7} \pi))^2 + 380 (\cos (\frac{1}{7} \pi))^3 - 293 \cos (\frac{1}{7} \pi) \right) (1 + x)^3 \right). \]
Thus, $R_5'(x) = A_0 + (x + 1)(A_1 + A_2(x + 1) + A_3(x + 1)^2)$ with the obvious choice of $A_j$. Since for $|x| \leq 1$ the value $x + 1$ is positive and $A_i \geq 0$ for $i = 0, 1$ then the inequality

$$A_2^2 - 4A_1A_3 < 0$$

implies that

$$R'(x) > 0; \quad x \in [-1, 1].$$

The verification of (1) is an elementary issue based on approximations of $\cos \pi/7$ and $\sin \pi/7$ from above and below with sufficiently large number of digits.

This proves that the derivative does not intersect the interval and that $R_5'(z) \geq 0$. Thus, $R_5(z)$ is not decreasing on $[-1, 1]$ therefore $P_5(z)$ is univalent by Theorem 3. This gives us an estimate on the Koebe radius $|P_5(-1)| \approx 0.3080$.

8. The case $N = 6$.

In this case

$$4R_6(x) = 2 + \left(8\sqrt{2} - 4\right)x + \left(38 - 12\sqrt{2}\right)x^2 + \left(28\sqrt{2} - 4\right)x^3 + \left(28\sqrt{2} - 10\right)x^4 + \left(-16\sqrt{2} + 32\right)x^5.$$ 

$$4R_6'(x) = 8\sqrt{2} - 4 + 2\left(38 - 12\sqrt{2}\right)x + 3\left(28\sqrt{2} - 4\right)x^2 + 4\left(28\sqrt{2} - 10\right)x^3 + 5\left(-16\sqrt{2} + 32\right)x^4$$

$$= -76\sqrt{2} + 108 + \left(464\sqrt{2} - 660\right)(1 + x) + \left(-732\sqrt{2} + 1068\right)(1 + x)^2 + \left(432\sqrt{2} - 680\right)(1 + x)^3 + \left(-80\sqrt{2} + 160\right)(1 + x)^4$$

$$= \left(108 - 76\sqrt{2} + \frac{-660 + 464\sqrt{2}}{2\sqrt{108 - 76\sqrt{2}}}(x + 1)\right)^2 + \left[\frac{-660 + 464\sqrt{2}}{4(108 - 76\sqrt{2})} - 732\sqrt{2} + 1068 + \frac{432\sqrt{2} - 680}{2\sqrt{108 - 76\sqrt{2}}}(x + 1) + \left(-80\sqrt{2} + 160\right)(1 + x)^2\right](x + 1)^2$$

Applying on argument similar to the formula (1) we get formula (2) which implies the estimate for the Koebe radius $|P_5(-1)| \approx 0.2929$. We conjecture that the obtained estimates in fact are true values.

9. Conclusion

In [8] a new class of polynomials was introduced

$$P_N(z) = \frac{1}{U'_N\left(\cos \frac{\pi}{N+2}\right)} \sum_{k=1}^{N} U'_{N-k+1} \left(\cos \frac{\pi}{N+2}\right) U_{k-1} \left(\cos \frac{\pi}{N+2}\right) z^k,$$
and the extremal property of these polynomials was mentioned

$$\sup_{p_N(z)=z+\sum_{k=2}^{N}a_k z^k} \min \{\Re(p_N(e^{it})) : \Im(p_N(e^{it}) = 0\} = P_N(-1).$$

It was conjectured that these polynomials are univalent and solves Dimitrov problem.

In the present article the first conjecture is proved for $N = 1, ..., 6$ thus for those $N$ the estimates from below on the radius Koebe of polynomials from $U_N$ are obtained. It is shown that those values are smaller then the corresponding ones for Suffridge polynomials $S_{N,1}(z)$.

To prove the case $N > 6$ one needs to verify the criteria given by Theorem 3, which is a not trivial task. Currently we are working on this subject.

Also, let us mention that the polynomials $P_N(z), S_{N,1}(z)$ and their generalizations turnes out to be very helpful in the problem of stabilization of of cycles in nonlinear discrete systems [10, 9].

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