Geometric Reductions, Dynamics and Controls 
for Hamiltonian System with Symmetry

Hong Wang
School of Mathematical Sciences and LPMC, 
Nankai University, Tianjin 300071, P.R.China
E-mail: hongwang@nankai.edu.cn

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Abstract: This is a survey article, from the viewpoint of the completeness of the Marsden-Weinstein reduction, to introduce briefly some recent developments of the symmetric reductions and Hamilton-Jacobi theory of the regular controlled Hamiltonian systems, the nonholonomic controlled Hamiltonian systems and the controlled magnetic Hamiltonian systems. These research reveal the deeply internal relationships of the geometrical structures of phase spaces, the nonholonomic constraint, the dynamical vector fields and the controls of these systems.

Keywords: cotangent bundle, Marsden-Weinstein reduction, Hamilton-Jacobi equation, RCH system, nonholonomic constraint, CMH system.

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1 Marsden-Weinstein Reduction on a Cotangent Bundle

It is well-known that the reduction theory for the mechanical system with symmetry is an important subject and that it is widely studied in the theory of mathematics and mechanics, as well as applications; see Abraham and Marsden [1], Arnold [3], Libermann and Marle [19], Marsden [21], Marsden et al. [22, 27], Marsden and Perlmutter [24], Marsden and Ratiu [26], Marsden and Weinstein [29], Meyer [30], Nijmeijer and Van der Schaft [31] and Ortega and Ratiu [32] and so on., for more details and development.

In mechanics, the phase space of a Hamiltonian system is often the cotangent bundle $T^*Q$ of a configuration manifold $Q$, and the reduction theory on the cotangent bundle of a configuration manifold is a very important special case of general symplectic reduction theory. In the following we first give the Marsden-Weinstein reduction for a Hamiltonian system with symmetry on the cotangent bundle of a smooth configuration manifold with a canonical symplectic form; see Abraham and Marsden [1] and Marsden and Weinstein [29].

Let $Q$ be a smooth manifold and $TQ$ the tangent bundle, $T^*Q$ is the cotangent bundle with a canonical symplectic form $\omega$. Assume that $\Phi : G \times Q \to Q$ is a left smooth action of a Lie group $G$ on the manifold $Q$. The cotangent lift is the action of $G$ on $T^*Q$, $\Phi^* : G \times T^*Q \to T^*Q$ given by $g \cdot \alpha_q = (T\Phi_g)^* \alpha_q$, $\forall \alpha_q \in T^*_q Q$, $q \in Q$. The cotangent lift of any proper (resp. free) $G$-action is proper (resp. free). Moreover, assume that the cotangent lift action is symplectic with respect to the canonical symplectic form $\omega$, and that it has an $\text{Ad}^*$-equivariant momentum map $J : T^*Q \to g^*$ given by $< J(\alpha_q), \xi > = \alpha_q(\xi_Q(q))$, where $\xi \in g$, $\xi_Q(q)$ is the value of the infinitesimal generator $\xi_Q$ of the $G$-action at $q \in Q$, $<, > : g^* \times g \to \mathbb{R}$ is the duality pairing between the dual $g^*$ and $g$. Assume that $\mu \in g^*$ is a regular value of the momentum map $J$, and $G_\mu = \{ g \in G | \text{Ad}_g^* \mu = \mu \}$ is the isotropy subgroup of the coadjoint $G$-action at the point $\mu$. From the Marsden-Weinstein reduction, we know that the reduced space $((T^*Q)_\mu, \omega_\mu)$ is a symplectic manifold.

In the following we give further a precise analysis of the geometrical structure of the symplectic reduced space of $T^*Q$. From Marsden and Perlmutter [24] and Marsden et al. [22], we know that the classification of symplectic reduced space of the cotangent bundle $T^*Q$ as follows: (1) If $\mu = 0$, the symplectic reduced space of cotangent bundle $T^*Q$ at $\mu = 0$ is given by $((T^*Q)_\mu, \omega_\mu) = (T^*(Q/G), \omega_0)$, where $\omega_0$ is the canonical symplectic form of the reduced cotangent bundle $T^*(Q/G)$. Thus, the symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ at $\mu = 0$ is a symplectic vector bundle; (2) If $\mu \neq 0$, and $G$ is Abelian, then $G_\mu = G$, in this case the Marsden-Weinstein symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to symplectic vector bundle $(T^*(Q/G), \omega_0 - B_\mu)$, where $B_\mu$ is a magnetic term; (3) If $\mu \neq 0$, and $G$ is not Abelian and $G_\mu \neq G$, in this case the Marsden-Weinstein symplectic reduced space $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to a symplectic fiber bundle over $T^*(Q/G_\mu)$ with fiber to be the coadjoint orbit $O_\mu$, see the cotangent bundle reduction theorem—bundle version, also see Marsden and Perlmutter [24]...
and Marsden et al. [22].

Thus, from the above discussion, we know that the Marsden-Weinstein symplectic reduced space of a Hamiltonian system defined on the cotangent bundle of a configuration manifold may not be a cotangent bundle. Therefore, the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle $T^*Q$ may not be a Hamiltonian system on a cotangent bundle; that is, the set of Hamiltonian systems with symmetries on the cotangent bundle is not complete under the Marsden-Weinstein reduction.

2 Hamilton-Jacobi Equations

The Hamilton-Jacobi theory is an important research subject in mathematics and analytical mechanics. On the one hand, it provides a characterization of the generating functions of certain time-dependent canonical transformations, such that for a given Hamiltonian system in such a form, its solutions are extremely easy to find by reduction to the equilibrium. On the other hand, it is possible in many cases that the Hamilton-Jacobi equation provides an immediate way to integrate the equation of the motion of a system, even when the problem of the Hamiltonian system itself has not been or cannot be solved completely; see Abraham et al. [1, 2], Arnold [3] and Marsden and Ratiu [26]. In addition, the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it plays an important role in the study of stochastic dynamical systems; see Woodhouse [46], Ge and Marsden [11], and Lázaro-Camí and Ortega [15]. For these reasons, the equation is described as a useful tool in the study of Hamiltonian system theory, which has been extensively developed in recent years and become one of the most active subjects in the study of modern applied mathematics and analytical mechanics.

The Hamilton-Jacobi theory, from the variational point of view, was originally developed by Jacobi in 1866, and it states that the integral of the Lagrangian of a mechanical system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem from the generating function and the geometrical point of view was given by Abraham and Marsden in [1] as follows: letting $Q$ be a smooth manifold and $TQ$ the tangent bundle, $T^*Q$ is the cotangent bundle with a canonical symplectic form $\omega$, and the projection $\pi_Q : T^*Q \rightarrow Q$ induces the map $T\pi_Q : TT^*Q \rightarrow TQ$.

**Theorem 2.1** Assume that the triple $(T^*Q, \omega, H)$ is a Hamiltonian system with the Hamiltonian vector field $X_H$, and $W : Q \rightarrow \mathbb{R}$ is a given generating function. Then the following two assertions are equivalent:

(i) For every curve $\sigma : \mathbb{R} \rightarrow Q$ satisfying that $\dot{\sigma}(t) = T\pi_Q(X_H(dW(\sigma(t))))$, $\forall t \in \mathbb{R}$, $dW \cdot \sigma$ is an integral curve of the Hamiltonian vector field $X_H$.

(ii) $W$ satisfies the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where $E$ is a constant.

From the proof of the Theorem 2.1 given in Abraham and Marsden [1], we know that the assertion (i), with equivalent to Hamilton-Jacobi equation (ii) by the generating function, gives a geometric constraint condition of the canonical symplectic form on the cotangent bundle $T^*Q$ for the Hamiltonian vector field of the system. Thus, the Hamilton-Jacobi equation reveals the deeply internal relationships of the generating function, the canonical symplectic form, and the dynamical vector field of a Hamiltonian system.

However, from the discussion in §1, we know that the set of Hamiltonian systems with symmetries on a cotangent bundle is not complete under the Marsden-Weinstein reduction, and the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle...
It is worthy of noting that if we take that \( \gamma = dW \) in Theorem 2.1, then \( \gamma \) is a closed one-form on \( Q \), and the equation \( d(H \cdot dW) = 0 \) is equivalent to the Hamilton-Jacobi equation \( H(q^i, \frac{\partial W}{\partial q^i}) = E \), where \( E \) is a constant, which was called the classical Hamilton-Jacobi equation. This result was used the formulation of a geometric version of the Hamilton-Jacobi theorem for Hamiltonian systems, see Carinena et al. [6, 7]. Moreover, noting that Theorem 2.1 was also generalized in the context of a time-dependent Hamiltonian system by Marsden and Ratiu in [26], and the Hamilton-Jacobi equation may be regarded as a nonlinear partial differential equation for some generating function \( S \). Thus, the problem becomes how to choose a time-dependent canonical transformation \( \Psi : T^*Q \times \mathbb{R} \to T^*Q \times \mathbb{R} \) which transforms the dynamical vector field of a time-dependent Hamiltonian system to equilibrium such that the generating function \( S \) of \( \Psi \) satisfies the time-dependent Hamilton-Jacobi equation. In particular, for the time-independent Hamiltonian system, one may look for a symplectic map as the canonical transformation. This work offers an important idea: that one can use the dynamical vector field of a Hamiltonian system to describe the Hamilton-Jacobi equation. As a consequence, if we assume that \( \gamma : Q \to T^*Q \) is a closed one-form on \( Q \), and define that \( \gamma^\gamma = T\pi_Q \cdot X_H \cdot \gamma \), where \( X_H \) is the dynamical vector field of the Hamiltonian system \( (T^*Q, \omega, H) \), then we have that \( \gamma^\gamma \) is \( \gamma \)-related; that is, \( T\gamma \cdot X_H = X_H \cdot \gamma \), is equivalent to that \( d(H \cdot \gamma) = 0 \), which was given in Carinena et al. [6, 7]. Motivated by the above research, Wang in [35] gave an important notion and proved a key lemma, which is a modification of the corresponding result of Abraham and Marsden in [1].

**Definition 2.2** The one-form \( \gamma \) is said to be closed with respect to \( T\pi_Q : TT^*Q \to TQ \) if, for any \( v, w \in TT^*Q \), we have that \( d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0 \).

From the above definition we know that if \( \gamma \) is a closed one-form, then it must be closed with respect to \( T\pi_Q : TT^*Q \to TQ \). Conversely, if \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \to TQ \), then it may not be closed. The following lemma is very important for our research, its proof was given in Wang [35].

**Lemma 2.3** Assume that \( \gamma : Q \to T^*Q \) is a one-form on \( Q \), and that \( \lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q \). Then we have that the following two assertions hold:

(i) For any \( x, y \in TQ \), \( \gamma^* \omega(x, y) = -d\gamma(x, y) \), and for any \( v, w \in TT^*Q \), \( \lambda^* \omega(v, w) = -d\gamma(T\pi_Q(v), T\pi_Q(w)) \), since \( \omega \) is the canonical symplectic form on \( T^*Q \).

(ii) For any \( v, w \in TT^*Q \), \( \omega(T\lambda \cdot v, w) = \omega(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)) \).

By using the above Lemma 2.3, we can derive precisely the geometric constraint conditions of the Marsden-Weinstein reduced symplectic form for the dynamical vector field of the Marsden-Weinstein reducible Hamiltonian system; these are called the Type I and Type II Hamilton-Jacobi equations. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-1.

\[
\begin{align*}
\text{Diagram-1}
\end{align*}
\]
Theorem 2.4 (Hamilton-Jacobi Theorem for the Marsden-Weinstein Reduced Hamiltonian System) For the Marsden-Weinstein reducible Hamiltonian system \((T^*Q, G, \omega, H)\) with the canonical symplectic form \(\omega\) on \(T^*Q\) and with the reduced Hamiltonian system \(((T^*Q)_\mu, \omega_\mu, h_\mu)\), assume that \(\gamma : Q \to T^*Q\) is a one-form on \(Q\), and that \(\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q\), and that \(\varepsilon : T^*Q \to T^*Q\) is a \(G_\mu\)-invariant symplectic map, where \(G_\mu\) is the isotropy subgroup of the co-adjoint action at the point \(\mu\). Denote that \(X^\gamma_H = T\pi_Q \cdot X_H \cdot \gamma\), and that \(X^\varepsilon_H = T\pi_Q \cdot X_H \cdot \varepsilon\), where \(X_H\) is the dynamical vector field of the Marsden-Weinstein reducible Hamiltonian system \((T^*Q, G, \omega, H)\). Moreover, assume that \(\mu \in \mathfrak{g}^*\) is a regular value of the momentum map \(J\), and that \(\text{Im}(\gamma) \subset J^{-1}(\mu)\), and that \(\gamma\) is \(G_\mu\)-invariant, and that \(\varepsilon(J^{-1}(\mu)) \subset J^{-1}(\mu)\). Denote that \(\bar{\gamma} = \pi_\mu(\gamma) : Q \to (T^*Q)_\mu\), and that \(\bar{\lambda} = \pi_\mu(\lambda) : J^{-1}(\mu) \subset (T^*Q) \to (T^*Q)_\mu\), and that \(\bar{\varepsilon} = \pi_\mu(\varepsilon) : J^{-1}(\mu) \subset (T^*Q) \to (T^*Q)_\mu\). Then the following two assertions hold:

(i) If the one-form \(\gamma : Q \to T^*Q\) is closed with respect to \(T\pi_Q : T(T^*Q) \to TQ\), then \(\bar{\gamma}\) is a solution of the Type I Hamilton-Jacobi equation \(T\bar{\gamma} \cdot X^\bar{\gamma}_H = X_{h_\mu} \cdot \bar{\gamma}\) for the Marsden-Weinstein reduced Hamiltonian system \(((T^*Q)_\mu, \omega_\mu, h_\mu)\).

(ii) The \(\varepsilon\) and \(\bar{\varepsilon}\) satisfy the Type II Hamilton-Jacobi equation \(T\bar{\varepsilon} \cdot X^\bar{\varepsilon}_H = X_{h_\mu} \cdot \bar{\varepsilon}\), if and only if they satisfy the equation \(T\bar{\varepsilon} \cdot (X_{h_\mu} \cdot \bar{\varepsilon}) = T\lambda \cdot X_H \cdot \varepsilon\), where \(X_{h_\mu}\) and \(X_{h_\mu} \cdot \bar{\varepsilon}\) are Hamiltonian vector fields of the Marsden-Weinstein reduced Hamiltonian functions \(h_{(\mu, a)}\) and \(h_\mu \cdot \bar{\varepsilon} : T^*Q \to \mathbb{R}\), respectively.

See the proof and the more details in Wang [35]. It is worth noting that the Type I Hamilton-Jacobi equation \(T\bar{\gamma} \cdot X^\bar{\gamma}_H = X_{h_\mu} \cdot \bar{\gamma}\) is the equation of the Marsden-Weinstein reduced differential one-form \(\bar{\gamma}\), and that the Type II Hamilton-Jacobi equation \(T\bar{\varepsilon} \cdot X^\bar{\varepsilon}_H = X_{h_\mu} \cdot \bar{\varepsilon}\) is the equation of the symplectic diffeomorphism map \(\bar{\varepsilon}\) and the Marsden-Weinstein reduced symplectic diffeomorphism map \(\bar{\varepsilon}\). The reason that these are called the Type I and Type II Hamilton-Jacobi equations, is that they are the development of the classical Hamilton-Jacobi equation given by Theorem 2.1; (see Abraham and Marsden [1] and Wang [35]).

We know that the orbit reduction for a Hamiltonian system with symmetry is an alternative approach to symplectic reduction given by Marle [20], and Kazhdan, Kostant and Sternberg [13], and that it is different from the Marsden-Weinstein reduction. We also derive precisely the geometric constraint conditions of the regular orbit reduced symplectic form for the dynamical vector field of the regular orbit reducible Hamiltonian system; that is, the Type I and Type II Hamilton-Jacobi equations for the regular orbit reduced Hamiltonian system. See the more details in Wang [35].

3 Regular Controlled Hamiltonian System and Its Dynamics

Since the symplectic reduced space of a Hamiltonian system defined on the cotangent bundle of a configuration manifold may not be a cotangent bundle, hence, the set of Hamiltonian systems with symmetries on the cotangent bundle is not complete under the Marsden-Weinstein reduction. This is a serious problem. If we define directly a controlled Hamiltonian system with symmetry on a cotangent bundle, then it is possible that the Marsden-Weinstein reduced controlled Hamiltonian system may not have a definition. However, from the classification of symplectic reduced space of the cotangent bundle \(T^*Q\), we know that the Marsden-Weinstein symplectic reduced space \(((T^*Q)_\mu, \omega_\mu)\) is symplectically diffeomorphic to a symplectic fiber bundle over \(T^*(Q/G_\mu)\) with fiber to be the coadjoint orbit \(O_\mu\). Thus, if we may define an RCH system on a symplectic fiber bundle, then it is possible to describe uniquely the RCH system on \(T^*Q\) and its regular reduced RCH systems on the associated reduced spaces, and we can study regular reduction theory of the RCH systems with symplectic structures and symmetries, as an extension of the Marsden-Weinstein reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions. This is why the authors in Marsden et al. [28] set up the regular reduction theory of the
RCH system on a symplectic fiber bundle, by using momentum map and the associated reduced symplectic form and from the viewpoint of completeness of regular symplectic reduction.

3.1 Regular Controlled Hamiltonian System

First, our idea in Marsden et al. [28], was that we first define a controlled Hamiltonian system on $T^*Q$ by using the symplectic form, and the such system is called a regular controlled Hamiltonian (RCH) system. Next, we regard a Hamiltonian system on $T^*Q$ as a spacial case of an RCH system without external force and control, and hence, the set of Hamiltonian systems on $T^*Q$ is a subset of the set of RCH systems on $T^*Q$. On the other hand, note that the symplectic reduced space on a cotangent bundle is not complete under the Marsden-Weinstein reduction, and hence, the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle $T^*Q$ may not be a Hamiltonian system on a cotangent bundle. In order to describe, uniformly, the RCH systems defined on a cotangent bundle and on the regular reduced spaces, we first define an RCH system on a symplectic fiber bundle, then we can obtain the RCH system on the cotangent bundle of a configuration manifold as a special case.

Let $(E, M, \pi)$ be a fiber bundle, and for each point $x \in M$, assume that the fiber $E_x = \pi^{-1}(x)$ is a smooth submanifold of $E$ with a symplectic form $\omega_E(x)$; that is, $(E, \omega_E)$ is a symplectic fiber bundle. If, for any Hamiltonian function $H : E \to \mathbb{R}$, we have a Hamiltonian vector field $X_H$ which satisfies the Hamilton's equation; that is, $i_{X_H} \omega_E = dH$, then $(E, \omega_E, H)$ is a Hamiltonian system. Moreover, considering the external force and the control, we can define a kind of regular controlled Hamiltonian (RCH) system on the symplectic fiber bundle $E$ as follows:

**Definition 3.1** (RCH system) An RCH system on $E$ is a 5-tuple $(E, \omega_E, H, F, W)$, where $(E, \omega_E, H)$ is a Hamiltonian system. and the function $H : E \to \mathbb{R}$ is called the Hamiltonian, a fiber-preserving map $F : E \to E$ is called the (external) force map, and a fiber sub-manifold $W$ of $E$ is called the control subset.

Sometimes, $W$ is also denoted as the set of fiber-preserving maps from $E$ to $W$. When a feedback control law $u : E \to W$ is chosen, the 5-tuple $(E, \omega_E, H, F, u)$ is a closed-loop dynamical system. In particular, when $Q$ is a smooth manifold, and $T^*Q$ is its cotangent bundle with a symplectic form $\omega$ (not necessarily canonical symplectic form), then $(T^*Q, \omega)$ is a symplectic vector bundle. If we take that $E = T^*Q$, from the above definition we can obtain an RCH system on the cotangent bundle $T^*Q$; that is, 5-tuple $(T^*Q, \omega, H, F, W)$. Because the fiber-preserving map $F : T^*Q \to T^*Q$ is the (external) force map, which is the reason that the fiber-preserving map $F : E \to E$ is called an (external) force map in above definition. Here for convenience, we assume that all controls appearing in this paper are the admissible controls.

3.2 The Dynamics of an RCH System

In order to describe the dynamics of an RCH system, we have to give a good expression of the dynamical vector field of the RCH system, by using the notation of the vertical lifted map of a vector along a fiber; see Marsden et al. [28].

First, for the notations of the vertical lifts along fiber, we need to consider three case: (1) $\pi : E \to M$ is a fiber bundle; (2) $\pi : E \to M$ is a vector bundle; (3) $\pi : E \to M, E = T^*Q, M = Q$, is a cotangent bundle, which is a special vector bundle. For the cases (2) and (3), we can use the standard definition of the vertical lift operator given in Marsden and Ratiu [26]. But for the case (1), the above operator cannot be used. This question was found by one of referees who give his opinion in a review report of our manuscript. In order to deal with uniformly the three cases, we
have to give a new definition of the vertical lifted maps of a vector along a fiber, and make it to be not conflict with that given in Marsden and Ratiu [26], and it is not and cannot be an extension of the definition of Marsden and Ratiu.

It is worthy of noting that there are two aspects in our new definition. First, for two different points, \(a_x, b_x\) in the fiber \(E_x, x \in M\), how define the moving vertical part of a vector in one point \(b_x\) to another point \(a_x\); Second, for a fiber-preserving map \(F : E \to E\), we know that \(a_x\) and \(F_x(a_x)\) are the two points in \(E_x\), how define the moving vertical part of a tangent vector in image point \(F_x(a_x)\) to \(a_x\). The eventual goal is to give a good expression of the dynamical vector field of RCH system by using the notation of the vertical lift map of a vector along a fiber. Our definitions are reasonable and clear, and should be stated explicitly.

We first consider that \(E\) and \(M\) are smooth manifolds, their tangent bundles \(TE\) and \(TM\) are vector bundles, and for the fiber bundle \(\pi : E \to M\), we consider the tangent mapping \(T\pi : TE \to TM\) and its kernel \(ker(T\pi) = \{\rho \in TE \mid T\pi(\rho) = 0\}\), which is a vector subbundle of \(TE\). Denote that \(VE := ker(T\pi)\), which is called a vertical bundle of \(E\). Assume that there is a metric on \(E\), and that we take a Levi-Civita connection \(A\) on \(TE\), and denote that \(HE := ker(A)\), which is called a horizontal bundle of \(E\), such that \(TE = HE \oplus VE\). Hence, for any \(x \in M\), \(a_x, b_x \in E_x\), any tangent vector \(\rho(b_x) \in Tf_xE\) can be split into horizontal and vertical parts; that is, \(\rho(b_x) = \rho^h(b_x) \oplus \rho^v(b_x)\), where \(\rho^h(b_x) \in Hf_xE\) and \(\rho^v(b_x) \in Vf_xE\). Let \(\gamma\) be a geodesic in \(E_x\) connecting \(a_x\) and \(b_x\), and denote by \(\rho^v_\gamma(a_x)\) a tangent vector at \(a_x\), which is a parallel displacement of the vertical vector \(\rho^v(b_x)\) along the geodesic \(\gamma\) from \(b_x\) to \(a_x\). Since the angle between two vectors is invariant under a parallel displacement along a geodesic, then \(T\pi(\rho^v_\gamma(a_x)) = 0\) and hence, \(\rho^v_\gamma(a_x) \in Vf_xE\). Now, for \(a_x, b_x \in E_x\) and tangent vector \(\rho(b_x) \in Tf_xE\), we can define the vertical lift map of a vector along a fiber by

\[\text{vlift}(TE_x) \times E_x \to TE_x; \quad \text{vlift}(\rho(b_x), a_x) = \rho^v_\gamma(a_x).\]

It is easy to check, from the basic fact in differential geometry, that this map does not depend on the choice of the geodesic \(\gamma\).

If \(F : E \to E\) is a fiber-preserving map, for any \(x \in M\), we have that \(F_x : E_x \to E_x\) and \(TF_x : TE_x \to TE_x\), then for any \(a_x \in E_x\) and \(\rho \in TE_x\), the vertical lift of \(\rho\) under the action of \(F\) along a fiber is defined by

\[\text{vlift}(F_x\rho)(a_x) = \text{vlift}((TF_x\rho)(F_x(a_x)), a_x) = (TF_x\rho)^v_\gamma(a_x),\]

where \(\gamma\) is a geodesic in \(E_x\) connecting \(F_x(a_x)\) and \(a_x\).

In particular, when \(\pi : E \to M\) is a vector bundle, for any \(x \in M\), the fiber \(E_x\) is a vector space. In this case, we can choose the geodesic \(\gamma\) to be a straight line, and the vertical vector is invariant under a parallel displacement along a straight line; that is, \(\rho^v_\gamma(a_x) = \rho^v(b_x)\). Moreover, when \(E = T^*Q\), by using the local trivialization of \(TT^*Q\), we have that \(TT^*Q \cong TQ \times T^*Q\), locally. Note that \(\pi : T^*Q \rightarrow Q\), and \(T\pi : TT^*Q \rightarrow TQ\), then in this case, for any \(\alpha_x, \beta_x \in T^*_xQ, x \in Q\), we know that \((0, \beta_x) \in Vf_xT^*_xQ\), and hence, we can get that

\[\text{vlift}((0, \beta_x)(\beta_x), \alpha_x) = (0, \beta_x)(\alpha_x) = \frac{d}{ds}
|_{s=0} (\alpha_x + s\beta_x),\]

which is consistent with the definition of vertical lift operator along a fiber given in Marsden and Ratiu [26].
For a given RCH System \((T^*Q, \omega, H, F, W)\), the dynamical vector field of the associated Hamiltonian system \((T^*Q, \omega, H)\) is \(X_H\), which satisfies the equation \(i_{X_H}\omega = dH\). Considering the external force \(F : T^*Q \to T^*Q\), which is a fiber-preserving map, by using the above notation of the vertical lift map of a vector along a fiber, the change of \(X_H\) under the action of \(F\) is such that

\[
\text{vlift}(F)X_H(\alpha_x) = \text{vlift}((TFX_H)(F(\alpha_x)), \alpha_x) = (TFX_H)_{T^*}\gamma(\alpha_x),
\]

where \(\alpha_x \in T^*_xQ\), \(x \in Q\) and \(\gamma\) is a straight line in \(T^*_xQ\) connecting \(F_x(\alpha_x)\) and \(\alpha_x\). In the same way, when a feedback control law \(u : T^*Q \to W\), which also is a fiber-preserving map, is chosen, the change of \(X_H\) under the action of \(u\) is such that

\[
\text{vlift}(u)X_H(\alpha_x) = \text{vlift}((TuX_H)(F(\alpha_x)), \alpha_x) = (TuX_H)^{\gamma}(\alpha_x).
\]

As a consequence, we can give an expression of the dynamical vector field of the RCH system as follows:

**Theorem 3.2** The dynamical vector field of an RCH system \((T^*Q, \omega, H, F, W)\) with a control law \(u\) is the synthesis of Hamiltonian vector field \(X_H\) and its changes under the actions of the external force \(F\) and the control \(u\); that is,

\[
X_{(T^*Q, \omega, H, F, u)}(\alpha_x) = X_H(\alpha_x) + \text{vlift}(F)X_H(\alpha_x) + \text{vlift}(u)X_H(\alpha_x),
\]

for any \(\alpha_x \in T^*_xQ\), \(x \in Q\). For convenience, that is simply written as

\[
X_{(T^*Q, \omega, H, F, u)} = X_H + \text{vlift}(F) + \text{vlift}(u),
\]

(3.1)

where \(\text{vlift}(F) = \text{vlift}(F)X_H\), and \(\text{vlift}(u) = \text{vlift}(u)X_H\) are the changes of \(X_H\) under the actions of \(F\) and \(u\). We also denote that \(\text{vlift}(W) = \{\text{vlift}(u)X_H | u \in W\}\). It is worthy of noting that in order to deduce and calculate easily, we always use the simple expressions of the dynamical vector field \(X_{(T^*Q, \omega, H, F, u)}\).

From the expression (3.1) of the dynamical vector field of an RCH system, we know that under the actions of the external force \(F\) and the control \(u\), in general, the dynamical vector field is not Hamiltonian, and hence the RCH system is not yet a Hamiltonian system. However, it is a dynamical system closed relative to a Hamiltonian system, and it can be explored and studied by the extending methods for the external force and the control in the study of Hamiltonian system.

### 3.3 Controlled Hamiltonian Equivalence

For two given Hamiltonian systems \((T^*Q_i, \omega_i, H_i), i = 1, 2\), we say them to be equivalent, if there exists a diffeomorphism \(\varphi : Q_1 \to Q_2\) such that their Hamiltonian vector fields \(X_{H_i}, i = 1, 2\) satisfy the condition \(X_{H_1} \cdot \varphi^* = T(\varphi^*)X_{H_2}\), where the map \(\varphi^* = T^*\varphi : T^*Q_2 \to T^*Q_1\) is the cotangent lifted map of \(\varphi\), and the map \(T(\varphi^*) : TT^*Q_2 \to TT^*Q_1\) is the tangent map of \(\varphi^*\). From Marsden and Ratiu [26], we know that the condition \(X_{H_1} \cdot \varphi^* = T(\varphi^*)X_{H_2}\) is equivalent to the fact that the map \(\varphi^* : T^*Q_2 \to T^*Q_1\) is symplectic with respect to the canonical symplectic forms on \(T^*Q_i, i = 1, 2\).

For two given RCH systems \((T^*Q_i, \omega_i, H_i, F_i, W_i), i = 1, 2\), we also want to define their equivalence; that is, to look for a diffeomorphism \(\varphi : Q_1 \to Q_2\) such that the condition \(X_{(T^*Q_i, \omega_i, H_i, F_i, W_i)} \cdot \varphi^* = T(\varphi^*)X_{(T^*Q_2, \omega_2, H_2, F_2, W_2)}\) holds. However, note that when an RCH system is given, the force map \(F\) is determined, but the feedback control law \(u : T^*Q \to W\) could be chosen. In order to emphasize explicitly the impact of the external force and the control in the study of the RCH systems, by using the above expression (3.1) of the dynamical vector field of the RCH system, we can describe the feedback control law to modify the structure of the RCH system, and the controlled Hamiltonian matching conditions and RCH-equivalence are induced as follows:
**Definition 3.3 (RCH-equivalence)** Suppose that we have two RCH systems \((T^*Q_i, \omega_i, H_i, F_i, W_i)\), \(i = 1, 2\), we say that they are RCH-equivalent, or simply, that \((T^*Q_1, \omega_1, H_1, F_1, W_1) \overset{RCH}{\sim} (T^*Q_2, \omega_2, H_2, F_2, W_2)\), if there exists a diffeomorphism \(\varphi : Q_1 \to Q_2\) such that the following controlled Hamiltonian matching conditions hold:

**RCH-1:** The control subsets \(W_i, i = 1, 2\) satisfy the condition \(W_1 = \varphi(W_2)\), where the map \(\varphi^* = T^*\varphi : T^*Q_2 \to T^*Q_1\) is cotangent lifted map of \(\varphi\).

**RCH-2:** For each control law \(u_i : T^*Q_i \to W_i\), there exists the control law \(u_2 : T^*Q_2 \to W_2\) such that the two closed-loop dynamical systems produce the same dynamical vector fields; that is, \(X(T^*Q_1, \omega_1, H_1, F_1, u_1) \cdot \varphi^* = T(\varphi^*)X(T^*Q_2, \omega_2, H_2, F_2, u_2)\), where the map \(T(\varphi^*) : TT^*Q_2 \to TT^*Q_1\) is the tangent map of \(\varphi^*\).

From the expression (3.1) of the dynamical vector field of the RCH system and the condition \(X(T^*Q_1, \omega_1, H_1, F_1, u_1) \cdot \varphi^* = T(\varphi^*)X(T^*Q_2, \omega_2, H_2, F_2, u_2)\), we have that

\[
(X_{H_1} + \text{vlift}(F_1)X_{H_1} + \text{vlift}(u_1)X_{H_1}) \cdot \varphi^* = T(\varphi^*)[X_{H_2} + \text{vlift}(F_2)X_{H_2} + \text{vlift}(u_2)X_{H_2}].
\]

By using the notation of vertical lift map of a vector along a fiber, for \(\alpha_x \in T_x^*Q_2, x \in Q_2\), we have that

\[
T(\varphi^*)\text{vlift}(F_2)X_{H_2}(\alpha_x) = T(\varphi^*)\text{vlift}((TF_2X_{H_2})(F_2(\alpha_x)), \alpha_x)
\]

\[
= \text{vlift}(T(\varphi^*) \cdot TF_2 \cdot T(\varphi_*)(X_{H_2}(\varphi^*F_2\varphi_*(\varphi^*\alpha_x)), \varphi^*\alpha)
\]

\[
= \text{vlift}(T(\varphi^*F_2\varphi_*)X_{H_2}(\varphi^*F_2\varphi_*(\varphi^*\alpha_x)), \varphi^*\alpha)
\]

\[
= \text{vlift}(\varphi^*F_2\varphi_*)X_{H_2}(\varphi^*\alpha_x),
\]

where the map \(\varphi_* = (\varphi^{-1})^* : T^*Q_1 \to T^*Q_2\). In the same way, we have that \(T(\varphi^*)\text{vlift}(u_2)X_{H_2} = \text{vlift}(\varphi^*u_2\varphi_*)X_{H_2} \cdot \varphi^*\). Thus, the explicit relation between the two control laws \(u_i \in W_i, i = 1, 2\) in **RCH-2** is given by

\[
(\text{vlift}(u_1) - \text{vlift}(\varphi^*u_2\varphi_*)) \cdot \varphi^* = -X_{H_1} \cdot \varphi^* + T\varphi^*(X_{H_2}) + (\text{vlift}(F_1) + \text{vlift}(\varphi^*F_2\varphi_*)) \cdot \varphi^*.
\]

From the above relation (3.2) we know that, when two RCH systems \((T^*Q_i, \omega_i, H_i, F_i, W_i), i = 1, 2\), are RCH-equivalent with respect to \(\varphi^*\), the corresponding Hamiltonian systems \((T^*Q_i, \omega_i, H_i), i = 1, 2\), may not be equivalent with respect to \(\varphi^*\). If two corresponding Hamiltonian systems are also equivalent with respect to \(\varphi^*\), then the control laws \(u_i : T^*Q \to W_i, i = 1, 2\) and the external forces \(F_i : T^*Q_i \to T^*Q_i, i = 1, 2\) in **RCH-2** must satisfy the following condition

\[
\text{vlift}(u_1) - \text{vlift}(\varphi^*u_2\varphi_*) = -\text{vlift}(F_1) + \text{vlift}(\varphi^*F_2\varphi_*).
\]
4.1 Regular Point Reduction of an RCH System

In the subsection, we consider the RCH system with symmetry and a momentum map, and give the regular point reduced RCH system and the RpCH-equivalence for the regular point reducible RCH system, and prove the regular point reduction theorem.

At first, we consider the regular point reducible RCH system. Let $Q$ be a smooth manifold and $T^*Q$ is its cotangent bundle with the symplectic form $\omega$. Let $\Phi : G \times Q \to Q$ be a smooth left action of a Lie group $G$ on $Q$, which is free and proper, so the cotangent lifted left action $\Phi^* : G \times T^*Q \to T^*Q$ is also free and proper. Assume that the cotangent lifted action is symplectic with respect to the symplectic form $\omega$, and that it admits an $Ad^*$-equivariant momentum map $J : T^*Q \to g^*$, where $g$ is the Lie algebra of $G$ and $g^*$ is the dual of $g$. Let $\mu \in g^*$ be a regular value of $J$ and denote that $G_\mu$ is the isotropy subgroup of the coadjoint $G$-action at the point $\mu \in g^*$, and that it is defined by $G_\mu = \{ g \in G | Ad_g^* \mu = \mu \}$. Since $G_\mu(\subset G)$ acts freely and properly on $Q$ and on $T^*Q$, then $Q_\mu = Q/G_\mu$ is a smooth manifold and that the canonical projection $\rho_\mu : Q \to Q_\mu$ is a surjective submersion. It follows that $G_\mu$ acts also freely and properly on $J^{-1}(\mu)$, so that the space $(T^*Q)_\mu = J^{-1}(\mu)/G_\mu$ is a symplectic manifold with the symplectic form $\omega_\mu$ uniquely characterized by the relation that

$$\pi^*_\mu \omega_\mu = i^*_\mu \omega. \quad (4.1)$$

The map $i_\mu : J^{-1}(\mu) \to T^*Q$ is the inclusion and the map $\pi_\mu : J^{-1}(\mu) \to (T^*Q)_\mu$ is the projection. The pair $((T^*Q)_\mu, \omega_\mu)$ is called the Marsden-Weinstein reduced space of $(T^*Q, \omega)$ at $\mu$.

Assume that $H : T^*Q \to \mathbb{R}$ is a $G$-invariant Hamiltonian, the flow $F_t$ of the Hamiltonian vector field $X_H$ leaves the connected components of $J^{-1}(\mu)$ invariant and commutes with the $G$-action, so it induces a flow $f_t^\mu$ on $(T^*Q)_\mu$ defined by $f_t^\mu \cdot \pi_\mu = \pi_\mu \cdot F_t \cdot i_\mu$, and the vector field $X_{h_\mu}$ generated by the flow $f_t^\mu$ on $((T^*Q)_\mu, \omega_\mu)$ is Hamiltonian with the associated regular point reduced Hamiltonian function $h_\mu : (T^*Q)_\mu \to \mathbb{R}$ defined by $h_\mu \cdot \pi_\mu = H \cdot i_\mu$, and the Hamiltonian vector fields $X_H$ and $X_{h_\mu}$ are $\pi_\mu$-related. Moreover, assume that the fiber-preserving map $F : T^*Q \to T^*Q$ and the control subset $W$ of $T^*Q$ are both $G$-invariant. In order to get the $R_p$-reduced RCH system, we also assume that $F(J^{-1}(\mu)) \subset J^{-1}(\mu)$ and that $W \cap J^{-1}(\mu) \neq \emptyset$. Thus, we can introduce a regular point reducible RCH system as follows (see Marsden et al. [28], Wang [36] and Wang [?]):

Definition 4.1 (Regular Point Reducible RCH System) A 6-tuple $(T^*Q, G, \omega, H, F, W)$ with the canonical symplectic form $\omega$ on $T^*Q$, where the Hamiltonian $H : T^*Q \to \mathbb{R}$, the fiber-preserving

map $F : T^*Q \to T^*Q$ and the fiber submanifold $W$ of $T^*Q$ are all $G$-invariant, is called a regular point reducible RCH system if there exists a point $\mu \in g^*$ which is a regular value of the momentum map $J$ such that the regular point reduced system; that is, the 5-tuple $((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, W_\mu)$, where $(T^*Q)_\mu = J^{-1}(\mu)/G_\mu$, $\pi^*_\mu \omega_\mu = i^*_\mu \omega$, $h_\mu \cdot \pi_\mu = H \cdot i_\mu$, $F(J^{-1}(\mu)) \subset J^{-1}(\mu)$, $f_\mu \cdot \pi_\mu = \pi_\mu \cdot F \cdot i_\mu$, $W_\mu = \pi_\mu(W \cap J^{-1}(\mu))$, is an RCH system, which is simply written as the $R_p$-reduced RCH system. Where $((T^*Q)_\mu, \omega_\mu)$ is the $R_p$-reduced space, the function $h_\mu : (T^*Q)_\mu \to \mathbb{R}$ is called the $R_p$-reduced Hamiltonian, the fiber-preserving map $f_\mu : (T^*Q)_\mu \to (T^*Q)_\mu$ is called the $R_p$-reduced (external) force map, and $W_\mu$ is a fiber submanifold of $(T^*Q)_\mu$ and is called the $R_p$-reduced control subset.

It is worthy of noting that for the regular point reducible RCH system $(T^*Q, G, \omega, H, F, W)$, the $G$-invariant external force map $F : T^*Q \to T^*Q$ has to satisfy the conditions $F(J^{-1}(\mu)) \subset J^{-1}(\mu)$, and $f_\mu \cdot \pi_\mu = \pi_\mu \cdot F \cdot i_\mu$ such that we can define the $R_p$-reduced external force map $f_\mu : (T^*Q)_\mu \to (T^*Q)_\mu$. The condition $W \cap J^{-1}(\mu) \neq \emptyset$ in above definition makes that the $G$-invariant control subset $W \cap J^{-1}(\mu)$ can be reduced and the $R_p$-reduced control subset is $W_\mu = \pi_\mu(W \cap J^{-1}(\mu))$. 


In the following we consider the RCH system with symmetry and a momentum map, and give the RpCH-equivalence for the regular point reducible RCH systems, and prove the regular point reduction theorem. Denote by $X_{(T^*Q,G,\omega,H,F,u)}$ the dynamical vector field of the regular point reducible RCH system $(T^*Q,G,\omega,H,F,W)$ with a control law $u$. Assume that it can be expressed by

$$X_{(T^*Q,G,\omega,H,F,u)} = X_H + \text{vlift}(F) + \text{vlift}(u). \quad (4.2)$$

By using the above expression $(4.2)$, for the regular point reducible RCH system we can introduce the regular point reducible controlled Hamiltonian equivalence (RpCH-equivalence) as follows:

**Definition 4.2 (RpCH-equivalence)** Suppose that we have two regular point reducible RCH systems $(T^*Q_i,G_i,\omega_i,H_i,F_i,W_i)$, $i = 1, 2$, we say that they are RpCH-equivalent, or simply, that $(T^*Q_1,G_1,\omega_1,H_1,F_1,W_1) \sim_{RpCH} (T^*Q_2,G_2,\omega_2,H_2,F_2,W_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that the following regular point reducible controlled Hamiltonian matching conditions hold:

**RpCH-1:** For $\mu_i \in \mathfrak{g}^{*}_i$, the regular reducible points of the RCH systems $(T^*Q_i,G_i,\omega_i,H_i,F_i,W_i)$, $i = 1, 2$, the map $\varphi^{\mu}_i = i^{\mu}_i \cdot \varphi \cdot i^{\mu}_i : J_2^{-1}(\mu_2) \to J_1^{-1}(i^{\mu}_1(m_1))$ is $(G_{2\mu_2},G_{1\mu_1})$-equivariant, and $W_1 \cap J_1^{-1}(\mu_1) = \varphi^{\mu}_i(W_2 \cap J_2^{-1}(\mu_2))$, where $\mu = (\mu_1, \mu_2)$, and denote by $i^{-1}_\mu(S)$ the pre-image of a subset $S \subset T^*Q_1$ for the map $i^{\mu}_i : J_1^{-1}(\mu_1) \to T^*Q_1$.

**RpCH-2:** For each control law $u_1 : T^*Q_1 \to W_1$, there exists the control law $u_2 : T^*Q_2 \to W_2$ such that the two closed-loop dynamical systems produce the same dynamical vector fields; that is, $X_{(T^*Q_i,G_i,\omega_i,H_i,F_i,u_i)} \cdot \varphi = T(\varphi^* X_{(T^*Q_2,G_2,\omega_2,H_2,F_2,u_2)})$.

It is worthy of noting that for the regular point reducible RCH system, the induced equivalent map $\varphi^*$ keeps the equivariance of the $G$-action at the regular reducible point. If an $R_p$-reduced feedback control law $u_\mu : (T^*Q)_\mu \to W_\mu$ is chosen, the $R_p$-reduced RCH system $((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)$ is a closed-loop regular dynamic system with a control law $u_\mu$. Assume that its vector field $X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)}$ can be expressed by

$$X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)} = X_{h_\mu} + \text{vlift}(f_\mu) + \text{vlift}(u_\mu), \quad (4.3)$$

where $X_{h_\mu}$ is the Hamiltonian vector field of the $R_p$-reduced Hamiltonian $h_\mu$, and $\text{vlift}(f_\mu) = \text{vlift}(f_\mu)X_{h_\mu}$, $\text{vlift}(u_\mu) = \text{vlift}(u_\mu)X_{h_\mu}$ are the changes of $X_{h_\mu}$ under the actions of the $R_p$-reduced external force $f_\mu$ and the $R_p$-reduced control law $u_\mu$. The dynamical vector fields of the regular point reducible RCH system $(T^*Q,G,\omega,H,F,u)$ and the $R_p$-reduced RCH system $((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)$ satisfy the condition that

$$X_{((T^*Q)_\mu,\omega_\mu,h_\mu,f_\mu,u_\mu)} \cdot \pi_\mu = T(\pi_\mu \cdot X_{(T^*Q,G,\omega,H,F,u)} \cdot i_\mu). \quad (4.4)$$

Thus, we can prove the following regular point reduction theorem for the RCH system, which explains the relationship between the RpCH-equivalence for the regular point reducible RCH systems with symmetries and momentum maps and the RCH-equivalence for the associated $R_p$-reduced RCH systems, its proof was given in Marsden et al. [28] and Wang [36]. This theorem can be regarded as an extension of the regular point reduction theorem of Hamiltonian system under the regular controlled Hamiltonian equivalence conditions.

**Theorem 4.3** Two regular point reducible RCH systems $(T^*Q_i,G_i,\omega_i,H_i,F_i,W_i)$, $i = 1, 2$, are RpCH-equivalent if and only if the associated $R_p$-reduced RCH systems $((T^*Q)_\mu_i,\omega_{\mu_i},h_{\mu_i},f_{\mu_i},W_{\mu_i})$, $i = 1, 2$, are RCH-equivalent.

It is worthy of noting that when the external force and the control of a regular point reducible RCH system $(T^*Q,G,\omega,H,F,W)$ are both zeros; that is, $F = 0$ and $W = \emptyset$, in this case the
RCH system is just a regular point reducible Hamiltonian system \((T^*Q, G, \omega, H)\) itself. Then the following theorem explains the relationship between the equivalence for the regular point reducible Hamiltonian systems with symmetries and the equivalence for the associated \(R_p\)-reduced Hamiltonian systems.

**Theorem 4.4** Two regular point reducible Hamiltonian systems \((T^*Q_i, G_i, \omega_i, H_i), i = 1, 2,\) are equivalent if and only if the associated \(R_p\)-reduced Hamiltonian systems \(((T^*Q_i)\mu_i, \omega_{\mu_i}, h_{\mu_i}), i = 1, 2,\) are equivalent.

See the proof and the more details in Wang [36].

### 4.2 Regular Orbit Reduction of the RCH System

Since the orbit reduction of a Hamiltonian system is an alternative approach to symplectic reduction, which is different from the Marsden-Weinstein reduction, in the subsection, we consider the RCH system with symmetry and a momentum map, and give the regular orbit reduced RCH system and the RoCH-equivalence for the regular orbit reducible RCH system, and prove the regular orbit reduction theorem.

First, we consider the regular orbit reducible RCH system. Assume that the cotangent lifted left action \(\Phi^{T^*} : G \times T^*Q \to T^*Q\) is symplectic, free and proper, and that the action admits an \(Ad^*\)-equivariant momentum map \(J : T^*Q \to g^*\). Let \(\mu \in g^*\) be a regular value of the momentum map \(J\) and \(O_\mu = G \cdot \mu \subset g^*\) be the \(G\)-orbit of the coadjoint \(G\)-action through the point \(\mu\). Since \(G\) acts freely, properly and symplectically on \(T^*Q\), the quotient space \((T^*Q)\sigma_\mu = J^{-1}(O_\mu)/G\) is a regular quotient symplectic manifold with the symplectic form \(\omega_{\sigma_\mu}\) uniquely characterized by the relation that

\[
i_{\sigma_\mu}^* \omega = \pi_{\sigma_\mu}^* \omega_{\sigma_\mu} + J_{\sigma_\mu}^* \omega_{\sigma_\mu},
\]

(4.5)

where \(J_{\sigma_\mu}\) is the restriction of the momentum map \(J\) to \(J^{-1}(O_\mu)\); that is, \(J_{\sigma_\mu} = J \cdot i_{\sigma_\mu}\) and \(\omega_{\sigma_\mu}^+\) is the \(+\)-symplectic structure on the orbit \(O_\mu\) given by

\[
\omega_{\sigma_\mu}^+(\nu)(\xi, \eta) = \langle \nu, [\xi, \eta] \rangle, \quad \forall \nu \in O_\mu, \xi, \eta \in g.
\]

(4.6)

The maps \(i_{\sigma_\mu} : J^{-1}(O_\mu) \to T^*Q\) and \(\pi_{\sigma_\mu} : J^{-1}(O_\mu) \to (T^*Q)\sigma_\mu\) are the natural injection and the projection, respectively. The pair \(((T^*Q)\sigma_\mu, \omega_{\sigma_\mu})\) is called the symplectic orbit reduced space of \((T^*Q, \omega)\) at \(\mu\).

Assume that \(H : T^*Q \to \mathbb{R}\) is a \(G\)-invariant Hamiltonian, the flow \(F_t\) of the Hamiltonian vector field \(X_H\) leaves the connected components of \(J^{-1}(O_\mu)\) invariant and commutes with the \(G\)-action, so it induces a flow \(f^{O_\mu}_t\) on \((T^*Q)\sigma_\mu\), defined by \(f^{O_\mu}_t \cdot \pi_{O_\mu} = \pi_{O_\mu} \cdot f_t \cdot i_{\sigma_\mu}\), and the vector field \(X_{h\sigma_\mu}\) generated by the flow \(f^{O_\mu}_t\) on \(((T^*Q)\sigma_\mu, \omega_{\sigma_\mu})\) is Hamiltonian with the associated regular orbit reduced Hamiltonian function \(h_{\sigma_\mu} : (T^*Q)\sigma_\mu \to \mathbb{R}\) defined by \(h_{\sigma_\mu} \cdot \pi_{O_\mu} = H \cdot i_{\sigma_\mu}\), and the Hamiltonian vector fields \(X_H\) and \(X_{h\sigma_\mu}\) are \(\pi_{O_\mu}\)-related. In general, we maybe thought that the structure of the symplectic orbit reduced space \(((T^*Q)\sigma_\mu, \omega_{\sigma_\mu})\) is more complex than that of the symplectic point reduced space \(((T^*Q)_\mu, \omega_\mu)\), but, from the regular reduction diagram (see Ortega and Ratiu [32]), we know that the regular orbit reduced space \(((T^*Q)\sigma_\mu, \omega_{\sigma_\mu})\) is symplectically diffeomorphic to the regular point reduced space \(((T^*Q)_\mu, \omega_\mu)\), and hence, is also symplectically diffeomorphic to a symplectic fiber bundle. Moreover, assume that the fiber-preserving map \(F : T^*Q \to T^*Q\) and the control subset \(W\) of \(T^*Q\) are both \(G\)-invariant. In order to get the \(R_p\)-reduced RCH system, we also assume that \(F(J^{-1}(O_\mu)) \subset J^{-1}(O_\mu)\) and that \(W \cap J^{-1}(O_\mu) \neq \emptyset\). Thus, we can introduce a regular orbit reducible RCH system as follows (see Marsden et al. [28], Wang [36]):
Definition 4.5 (Regular Orbit Reducible RCH System) A 6-tuple $(T^*Q, G, \omega, H, F, W)$, where the Hamiltonian $H : T^*Q \to \mathbb{R}$, the fiber-preserving map $F : T^*Q \to T^*Q$ and the fiber submanifold $W$ of $T^*Q$ are all $G$-invariant, is called a regular orbit reducible RCH system, if there exists an orbit $O_\mu$, $\mu \in g^*$, where $g$ is a regular value of the momentum map $\mathbf{J}$, such that the regular orbit reduced system; that is, the 5-tuple $(T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, F_{O_\mu}, W_{O_\mu})$, where $(T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G$, $\pi_{O_\mu}^* \omega_{O_\mu} = i_{O_\mu}^{-1} \omega - \mathbf{J}_{O_\mu}^* \omega_{O_\mu}$, $h_{O_\mu} : \pi_{O_\mu} \to H \cdot i_{O_\mu}, F_{O_\mu} : \pi_{O_\mu} \to \pi_{O_\mu} \cdot F \cdot i_{O_\mu}$, and $W_{O_\mu} = \pi_{O_\mu}^{-1}(W \cap J^{-1}(O_\mu))$, is an RCH system, that is simply written as the $O_\mu$-reduced RCH system. Where $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is the $O_\mu$-reduced space, the function $h_{O_\mu} : (T^*Q)_{O_\mu} \to \mathbb{R}$ is called the $O_\mu$-reduced Hamiltonian, the fiber-preserving map $f_{O_\mu} : (T^*Q)_{O_\mu} \to (T^*Q)_{O_\mu}$ is called the $O_\mu$-reduced (external) force map, and $W_{O_\mu}$ is a fiber submanifold of $(T^*Q)_{O_\mu}$, and is called the $O_\mu$-reduced control subset.

It is worthy of noting that for the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$, the $G$-invariant external force map $F : T^*Q \to T^*Q$ has to satisfy the conditions that $F(J^{-1}(O_\mu)) \subset J^{-1}(O_\mu)$, and $f_{O_\mu} : \pi_{O_\mu} \to \pi_{O_\mu} \cdot F \cdot i_{O_\mu}$, such that we can define the reduced external force map $f_{O_\mu} : (T^*Q)_{O_\mu} \to (T^*Q)_{O_\mu}$. The condition that $W \cap J^{-1}(O_\mu) \neq \emptyset$ in above definition makes that the $G$-invariant control subset $W \cap J^{-1}(O_\mu)$ can be reduced and that the reduced control subset is $W_{O_\mu} = \pi_{O_\mu}^{-1}(W \cap J^{-1}(O_\mu))$.

In the following we consider the RCH system with symmetry and a momentum map, and give the RoCh-equivalence for the regular orbit reducible RCH systems, and prove the regular orbit reduction theorem. Denote by $X_{(T^*Q,G,\omega,H,F,u)}$ the dynamical vector field of the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$ with a control law $u$. Assume that it can be expressed by

$$X_{(T^*Q,G,\omega,H,F,u)} = X_H + \text{vlift}(F) + \text{vlift}(u).$$

By using the above expression (4.7), for the regular orbit reducible RCH system we can introduce the regular orbit reducible controlled Hamiltonian equivalence (RoCh-equivalence) as follows.

Definition 4.6 (RoCh-equivalence) Suppose that we have two regular orbit reducible RCH systems $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1)$, $i = 1, 2$, we say that they are RoCh-equivalent, or simply, that $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1) \sim (T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that the controlled Hamiltonian matching conditions hold:

**RoCH-1:** For $O_{\mu_i}$, $\mu_i \in g^*_i$, the regular reducible orbits of the RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, the map $\varphi_{O_{\mu_i}} = i_{O_{\mu_i}}^{-1} \cdot \varphi^* \cdot i_{O_{\mu_2}} : J_2^{-1}(O_{\mu_2}) \to J_1^{-1}(O_{\mu_1})$ is $(G_2, G_1)$-equivariant, and $W_1 \cap J_1^{-1}(O_{\mu_1}) = \varphi_{O_{\mu_1}}^{-1}(W_2 \cap J_2^{-1}(O_{\mu_2}))$, and $J^*_{2O_{\mu_2}} \omega_{2O_{\mu_2}} = (\varphi_{O_{\mu_1}})^* \cdot J^*_{1O_{\mu_1}} \omega_{1O_{\mu_1}}$, where $\mu = (\mu_1, \mu_2)$, and denote by $i_{O_{\mu_1}}^{-1}(S)$ the pre-image of a subset $S \subset T^*Q_1$ for the map $i_{O_{\mu_1}} : J_1^{-1}(O_{\mu_1}) \to T^*Q_1$.

**RoCH-2:** For each control law $u_1 : T^*Q_1 \to W_1$, there exists the control law $u_2 : T^*Q_2 \to W_2$ such that the two closed-loop dynamical systems produce the same dynamical vector fields; that is,$$X_{(T^*Q_1,G_1,\omega_1,H_1,F_1,u_1)} : \varphi^* = T(\varphi^*)X_{(T^*Q_2,G_2,\omega_2,H_2,F_2,u_2)}.$$

It is worthy of noting that for the regular orbit reducible RCH system, the induced equivalent map $\varphi^*$ not only keeps the restriction of the $(+)$-symplectic structure on the regular reducible orbit to $J^{-1}(O_{\mu_1})$, but also keeps the equivariance of $G$-action on the regular reducible orbit. If an $O_{\mu}$-reduced feedback control law $u_{O_{\mu}} : (T^*Q)_{O_{\mu}} \to W_{O_{\mu}}$ is chosen, the $O_{\mu}$-reduced RCH system $((T^*Q)_{O_{\mu}}, \omega_{O_{\mu}}, h_{O_{\mu}}, f_{O_{\mu}}, u_{O_{\mu}})$ is a closed-loop regular dynamic system with the control law $u_{O_{\mu}}$. Assume that its vector field $X_{((T^*Q)_{O_{\mu}}, \omega_{O_{\mu}}, h_{O_{\mu}}, f_{O_{\mu}}, u_{O_{\mu}})}$ can be expressed by

$$X_{((T^*Q)_{O_{\mu}}, \omega_{O_{\mu}}, h_{O_{\mu}}, f_{O_{\mu}}, u_{O_{\mu}})} = X_{h_{O_{\mu}}} + \text{vlift}(f_{O_{\mu}}) + \text{vlift}(u_{O_{\mu}}),$$

where $X_{h_{O_{\mu}}}$ is the dynamical vector field of the $O_{\mu}$-reduced Hamiltonian $h_{O_{\mu}}$, and $\text{vlift}(f_{O_{\mu}}) = \text{vlift}(f_{O_{\mu}})X_{h_{O_{\mu}}}$, $\text{vlift}(u_{O_{\mu}}) = (v_{O_{\mu}})X_{h_{O_{\mu}}}$, are the changes of $X_{h_{O_{\mu}}}$ under the actions of the
$R_o$-reduced external force $f_{O_\mu}$ and the $R_o$-reduced control law $u_{O_\mu}$. The dynamical vector fields of the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, u)$ and the $R_o$-reduced RCH system ($(T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu}$) satisfy the condition that

$$X((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu}) = \pi_{O_\mu} \cdot X(T^*Q, G, \omega, H, F, u) \cdot i_{O_\mu}.$$  \hfill (4.9)

Thus, we can prove the following regular orbit reduction theorem for the RCH system, which explains the relationship between the RoCH-equivalence for the regular orbit reducible RCH systems with symmetries and momentum maps and the RCH-equivalence for the associated $R_o$-reduced RCH systems, its proof was given in Marsden et al. [28] and Wang [36]. This theorem can be regarded as an extension of the regular orbit reduction theorem of Hamiltonian system under the regular controlled Hamiltonian equivalence conditions.

**Theorem 4.7** Two regular orbit reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i), \ i=1,2,$ are RoCH-equivalent if and only if the associated $R_o$-reduced RCH systems $((T^*Q)_{O_\mu_i}, \omega_{O_\mu_i}, h_{O_\mu_i}, f_{O_\mu_i}, W_i)_{O_\mu_i}), \ i=1,2,$ are RCH-equivalent.

It is worthy of noting that when the external force and the control of a regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$ are both zeros; that is, $F = 0$ and $W = \emptyset$, in this case the RCH system is just a regular orbit reducible Hamiltonian system $(T^*Q, G, \omega, H)$ itself. Then the following theorem explains the relationship between the equivalence for the regular orbit reducible Hamiltonian systems with symmetries and the equivalence for the associated $R_o$-reduced Hamiltonian systems.

**Theorem 4.8** If two regular orbit reducible Hamiltonian systems $(T^*Q_i, G_i, \omega_i, H_i), \ i=1,2,$ are equivalent, then their associated $R_o$-reduced Hamiltonian systems $((T^*Q)_{O_\mu_i}, \omega_{O_\mu_i}, h_{O_\mu_i}), \ i=1,2,$ must be equivalent. Conversely, if the $R_o$-reduced Hamiltonian systems $((T^*Q)_{O_{\mu i}}, \omega_{O_{\mu i}}, h_{O_{\mu i}}), \ i=1,2,$ are equivalent and the induced map $\varphi_{O_\mu} : J_2^{-1}(O_{\mu_2}) \rightarrow J_1^{-1}(O_{\mu_1})$ such that $J_{2\mu_2} J_{1\mu_1} = (\varphi_{O_\mu})^* J_{1O_{\mu_1}} \omega_{1O_{\mu_1}}^+$, then the regular orbit reducible Hamiltonian systems $(T^*Q_i, G_i, \omega_i, H_i), \ i=1,2,$ are equivalent.

See the proof and the more details in Wang [36].

**Remark 4.9** If $(T^*Q, \omega)$ is a connected symplectic manifold, and $J : T^*Q \rightarrow g^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma : G \rightarrow g^*$, which is defined by $\sigma(y) := J(g \cdot z) - Ad^*_g J(z)$, where $g \in G$ and $z \in T^*Q$. Then we know that $\sigma$ produces a new affine action $\Theta : G \times g^* \rightarrow g^*$ defined by $\Theta(g, \mu) := Ad^*_g \mu + \sigma(g)$, where $\mu \in g^*$, with respect to which the given momentum map $J$ is equivariant. Assume that the $G$ acts freely and properly on $T^*Q$, and that $G_\mu$ is the isotropy subgroup of $\mu \in g^*$ relative to this affine action $\Theta$, and that $O_\mu = G \cdot \mu \subset g^*$ is the $G$-orbit of the point $\mu$ with respect to the action $\Theta$, and that $\mu$ is a regular value of $J$. Then the quotient space $(T^*Q)_{\mu} = J^{-1}(\mu)/G_\mu$ is a symplectic manifold with the symplectic form $\omega_{\mu}$ uniquely characterized by (4.1); and the quotient space $(T^*Q)_{O_\mu} = J^{-1}(O_{\mu})/G$ is also a symplectic manifold with the symplectic form $\omega_{O_\mu}$ uniquely characterized by (4.5); see Ortega and Ratiu [32]. Moreover, in this case, for the given regular point or regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$, we can also prove the regular point reduction theorem or the regular orbit reduction theorem, by using the above similar ways.

## 5 Hamilton-Jacobi Equations for the RCH System and Its Reduced Systems

We know that under the actions of the external force $F$ and the control $u$, in general, the RCH system $(T^*Q, \omega, H, F, u)$, the $R_o$-reduced RCH system $((T^*Q)_\mu, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu})$ and the $R_o$-reduced
RCH system \(((T^*Q)_{\omega_p}, \omega_{\omega_p}, h_{\omega_p}, f_{\omega_p}, u_{\omega_p})\), all of them are not Hamiltonian systems, and hence, we cannot describe the Hamilton-Jacobi equations for the RCH system, the \(R_p\)-reduced RCH system and the \(R_o\)-reduced RCH system, from the viewpoint of a generating function the same as in Theorem 2.1 (given by Abraham and Marsden in [1]). However, for a given RCH system \((T^*Q, \omega, H, F, W)\) in which \(\omega\) is the canonical symplectic form on \(T^*Q\), by using Lemma 2.3 and the expression (3.1) of the dynamical vector field of the RCH system, we can derive precisely two types of geometric constraint conditions of the canonical symplectic form for the dynamical vector field of the RCH system; that is, the two types of Hamilton-Jacobi equations for the RCH system. Moreover, we generalize the above results for a regular reducible RCH system with symmetry and a momentum map, and derive precisely the two types of Hamilton-Jacobi equations for the \(R_p\)-reduced RCH system and the \(R_o\)-reduced RCH system, and we prove that the RCH-equivalence for the RCH system, and the \(R_p\)-CH-equivalence and the \(R_o\)-CH-equivalence for the regular reducible RCH systems with symmetries and momentum maps, leave the solutions of the corresponding Hamilton-Jacobi equations invariant. These research reveal the deeply internal relationships of the geometrical structures of phase spaces, the dynamical vector fields and the controls of the RCH system and the reduced RCH system.

5.1 Hamilton-Jacobi Equations for an RCH System

For a given RCH system \((T^*Q, \omega, H, F, W)\) in which \(\omega\) is the canonical symplectic form on \(T^*Q\), by using Lemma 2.3 and the expression (3.1) of the dynamical vector field \(X_{(T^*Q,\omega,H,F,W)}\), we can derive precisely two types of geometric constraint conditions of the canonical symplectic form for the dynamical vector field of the RCH system; that is, the two types of Hamilton-Jacobi equations for the RCH system. For convenience, the maps involved in the theorem and its proof are shown in Diagram-2.

![Diagram-2](image)

**Theorem 5.1** (*Hamilton-Jacobi Theorem for an RCH system*) For the RCH system \((T^*Q, \omega, H, F, W)\) with the canonical symplectic form \(\omega\) on \(T^*Q\), assume that \(\gamma : Q \to T^*Q\) is a one-form on \(Q\), and that \(\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q\), and that for any symplectic map \(\varepsilon : T^*Q \to T^*Q\), denote that \(\tilde{X}^\gamma = T\pi_Q \cdot \tilde{X} \cdot \gamma\) and \(\tilde{X}^\varepsilon = T\pi_Q \cdot \tilde{X} \cdot \varepsilon\), where \(\tilde{X} = X_{(T^*Q,\omega,H,F,W)}\) is the dynamical vector field of the RCH system \((T^*Q, \omega, H, F, W)\) with a control law \(u\). Then the following two assertions hold:

(i) If the one-form \(\gamma : Q \to T^*Q\) is closed with respect to \(T\pi_Q : TT^*Q \to TQ\), then \(\gamma\) is a solution of the Type I Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}^\gamma = X_H \cdot \gamma\), where \(X_H\) is the Hamiltonian vector field of the corresponding Hamiltonian system \((T^*Q, \omega, H)\).

(ii) The \(\varepsilon\) is a solution of the Type II Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}^\varepsilon = X_H \cdot \varepsilon\) if and only if it is a solution of the equation \(T\varepsilon \cdot X_H = T\lambda \cdot \tilde{X} \cdot \varepsilon\), where \(X_H\) and \(X_H \cdot \varepsilon \in TT^*Q\) are the Hamiltonian vector fields of the functions \(H\) and \(H \cdot \varepsilon : T^*Q \to \mathbb{R}\), respectively.

See the proof and the more details in Wang [39]. It is worth noting that the Type I Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}^\gamma = X_H \cdot \gamma\), is the equation of the differential one-form \(\gamma\), and that the Type II Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}^\varepsilon = X_H \cdot \varepsilon\), is the equation of the symplectic diffeomorphism map \(\varepsilon\).

Moreover, considering the RCH-equivalence of the RCH systems, we can prove the following theorem, which states that the solutions of two types of Hamilton-Jacobi equations for the RCH...
systems remain invariant under the conditions of RCH-equivalence if the corresponding Hamiltonian systems are equivalent:

**Theorem 5.2** Suppose that two RCH systems, \((T^*Q_i, \omega_i, H_i, F_i, W_i), i = 1, 2\) are RCH-equivalent with an equivalent map \(\varphi : Q_1 \to Q_2\), and that the corresponding Hamiltonian systems, \((T^*Q_i, \omega_i, H_i), i = 1, 2\) are also equivalent. Under the hypotheses and notations of Theorems 5.1, we have that the following two assertions hold:

(i) If the one-form \(\gamma_2 : Q_2 \to T^*Q_2\) is closed with respect to \(T\pi_{Q_2} : TT^*Q_2 \to TQ_2\), then \(\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi : Q_1 \to T^*Q_1\) is also closed with respect to \(T\pi_{Q_1} : TT^*Q_1 \to TQ_1\), and hence, \(\gamma_1\) is a solution of the Type I Hamilton-Jacobi equation for the RCH system \((T^*Q_1, \omega_1, H_1, F_1, W_1)\).

(ii) If the symplectic map \(\varepsilon_2 : T^*Q_2 \to T^*Q_2\) is a solution of the Type II Hamilton-Jacobi equation for the RCH system \((T^*Q_2, \omega_2, H_2, F_2, W_2)\), then \(\varepsilon_1 = \varphi^* \cdot \varepsilon_2 \cdot \varphi : T^*Q_1 \to T^*Q_1\) is a symplectic map, and hence, \(\varepsilon_1\) is a solution of the Type II Hamilton-Jacobi equation for the RCH system \((T^*Q_1, \omega_1, H_1, F_1, W_1)\).

See the proof and the more details in Wang [39].

### 5.2 Hamilton-Jacobi Equations for an \(R_p\)-reduced RCH System

For a given regular point reducible RCH system \((T^*Q, G, \omega, H, F, W)\) with the canonical symplectic form \(\omega\) on \(T^*Q\) and with an \(R_p\)-reduced RCH system \((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)\), by using Lemma 2.3, we can give precisely the geometric constraint conditions of the \(R_p\)-reduced symplectic form for the dynamical vector field of the regular point reducible RCH system; that is, the Type I and Type II Hamilton-Jacobi equations for the \(R_p\)-reduced RCH system \((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)\). First, by using the fact that the one-form \(\gamma : Q \to T^*Q\) is closed with respect to \(T\pi_Q : TT^*Q \to TQ\), and that \(\operatorname{Im}(\gamma) \subseteq J^{-1}(\mu)\), and that \(\gamma\) is \(G_\mu\)-invariant, we can prove the Type I Hamilton-Jacobi theorem for the \(R_p\)-reduced RCH system \((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)\). For convenience, the maps involved in the theorem and its proof are shown in Diagram-3.

![Diagram-3](image)

**Theorem 5.3** (Type I Hamilton-Jacobi Theorem for an \(R_p\)-reduced RCH system) For the regular point reducible RCH system \((T^*Q, G, \omega, H, F, W)\) with the canonical symplectic form \(\omega\) on \(T^*Q\) and with an \(R_p\)-reduced RCH system \((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)\), assume that \(\gamma : Q \to T^*Q\) is a one-form on \(Q\), and that \(\tilde{X}_\gamma = T\pi_Q \cdot \tilde{X} \cdot \gamma\), where \(\tilde{X} = X_{T^*Q, G, \omega, H, F, W}\) is the dynamical vector field of the regular point reducible RCH system \((T^*Q, G, \omega, H, F, W)\) with a control law \(u\). Moreover, assume that \(\mu \in \mathfrak{g}^*\) is a regular value of the momentum map \(J\), and that \(\operatorname{Im}(\gamma) \subseteq J^{-1}(\mu)\), and that \(\gamma\) is \(G_\mu\)-invariant, and that \(\tilde{\gamma} = \pi_\mu(\gamma) : Q \to (T^*Q)_\mu\). If the one-form \(\gamma : Q \to T^*Q\) is closed with respect to \(T\pi_Q : TT^*Q \to TQ\), then \(\tilde{\gamma}\) is a solution of the equation \(T\tilde{\gamma} \cdot \tilde{X} \cdot \gamma = X_{h_\mu} \cdot \tilde{\gamma}\), where \(X_{h_\mu}\) is the Hamiltonian vector field of the \(R_p\)-reduced Hamiltonian function \(h_\mu : (T^*Q)_\mu \to \mathbb{R}\), and the equation is called the Type I Hamilton-Jacobi equation for the \(R_p\)-reduced RCH System \((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)\).

See the proof and the more details in Wang [39]. Moreover, for any \(G_\mu\)-invariant symplectic map \(\varepsilon : T^*Q \to T^*Q\), we can prove the Type II Hamilton-Jacobi theorem for the \(R_p\)-reduced RCH system \((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)\); see Wang [39].
Remark 5.4 It is worth noting that the Type I Hamilton-Jacobi equation \( T\dot{\gamma} \cdot X_{\gamma} = X_{h_{\mu}} \cdot \dot{\gamma} \) is the equation of the \( R_p \)-reduced differential one-form \( \gamma \), and that the Type II Hamilton-Jacobi equation \( T\dot{\gamma} \cdot X_{\dot{\varepsilon}} = X_{h_{\mu}} \cdot \dot{\varepsilon} \) is the equation of the symplectic diffeomorphism map \( \varepsilon \) and the \( R_p \)-reduced symplectic diffeomorphism map \( \dot{\varepsilon} \). If both the external force and the control of the regular point reducible RCH system \((T^*Q, G, \omega, H, F, W)\) are zero; that is, \( F = 0 \) and \( W = 0 \), in this case, the RCH system is just a regular point reducible Hamiltonian system \((T^*Q, G, \omega, H)\) itself. From the proofs of the Theorem 5.3 above, we can also get the Theorem 2.4; that is, the Type I Hamilton-Jacobi equation for the associated Marsden-Weinstein reduced Hamiltonian system (given in Wang [35]). Thus, Theorem 5.3 can be regarded as an extension of the Type I Hamilton-Jacobi equation for the Marsden-Weinstein reduced Hamiltonian system to that for the \( R_p \)-reduced RCH system.

Moreover, considering the \( RpCH \)-equivalence of the regular point reducible RCH systems and using Theorems 4.3, 4.4 and 5.2, we can obtain the Theorem 5.5, which states that the solutions of the two types of Hamilton-Jacobi equations for the regular point reducible RCH systems remain invariant under the conditions of \( RpCH \)-equivalence if the corresponding Hamiltonian systems are equivalent.

\[ \text{Theorem 5.5} \quad \text{Suppose that two regular point reducible RCH systems, } (T^*Q_1, G_1, \omega_1, H_1, F_1, W_1), \quad i = 1, 2 \text{ are } RpCH-\text{equivalent with an equivalent map } \varphi : Q_1 \rightarrow Q_2, \text{ and that the associated } \text{R}_p- \text{reduced RCH systems are } ((T^*Q_1)_{\mu_1}, \omega_{i1}, h_{i1}, f_{i1}, u_{i1}), \quad i = 1, 2. \text{ Assume that the corresponding Hamiltonian systems, } (T^*Q_1, G_1, \omega_1, H_1), \quad \text{and } (T^*Q_2, G_2, \omega_2, H_2), \quad \text{are also equivalent. Then, under the hypotheses and the notations of Theorems 4.3, 4.4 and 5.3, we have that the following two assertions hold:} \]

(i) If the one-form \( \gamma_2 : Q_2 \rightarrow T^*Q_2 \) is closed with respect to \( T\pi Q_2 : TT^*Q_2 \rightarrow TQ_2 \), and \( \gamma_2 = \pi_{2\mu_2}(\gamma_2) : Q_2 \rightarrow (T^*Q_2)_{\mu_2} \) is a solution of the Type I Hamilton-Jacobi equation for the \( \text{R}_p \)-reduced RCH system \(((T^*Q_2)_{\mu_2}, \omega_{2\mu_2}, h_{2\mu_2}, f_{2\mu_2}, u_{2\mu_2})\), then \( \gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi : Q_1 \rightarrow T^*Q_1 \) is a solution of the Type I Hamilton-Jacobi equation for the RCH system \(((T^*Q_1)_{\mu_1}, \omega_{1\mu_1}, h_{1\mu_1}, f_{1\mu_1}, u_{1\mu_1})\), and \( \gamma_1 = \pi_{1\mu_1}(\gamma_1) : Q_1 \rightarrow (T^*Q_1)_{\mu_1} \) is a solution of the Type I Hamilton-Jacobi equation for the \( \text{R}_p \)-reduced RCH system \(((T^*Q_1)_{\mu_1}, \omega_{1\mu_1}, h_{1\mu_1}, f_{1\mu_1}, u_{1\mu_1})\).

(ii) If the \( G_{2\mu_2} \)-invariant symplectic map \( \varepsilon_2 : T^*Q_2 \rightarrow T^*Q_2 \) and \( \varepsilon_2 = \pi_{2\mu_2}(\varepsilon_2) : J_{2^{-1}}(\mu_2)(C T^*Q_2) \rightarrow (T^*Q_2)_{\mu_2} \) satisfy the Type II Hamilton-Jacobi equation for the \( \text{R}_p \)-reduced RCH system \(((T^*Q_2)_{\mu_2}, \omega_{2\mu_2}, h_{2\mu_2}, f_{2\mu_2}, u_{2\mu_2})\), then \( \varepsilon_1 = \varphi^* \cdot \varepsilon_2 \cdot \varphi : T^*Q_1 \rightarrow T^*Q_1 \) and \( \varepsilon_1 = \pi_{1\mu_1}(\varepsilon_1) : J_{1^{-1}}(\mu_1)(C T^*Q_1) \rightarrow (T^*Q_1)_{\mu_1} \) satisfy the Type II Hamilton-Jacobi equation for the \( \text{R}_p \)-reduced RCH system \(((T^*Q_1)_{\mu_1}, \omega_{1\mu_1}, h_{1\mu_1}, f_{1\mu_1}, u_{1\mu_1})\).

See the proof and the more details in Wang [39].

5.3 Hamilton-Jacobi Equations for an \( R_o \)-reduced RCH System

For a given regular orbit reducible RCH system \((T^*Q, G, \omega, H, F, W)\) with the canonical symplectic form \( \omega \) on \( T^*Q \) and with an \( R_o \)-reduced RCH system \(((T^*Q)_{O}, \omega_{O}, h_{O}, f_{O}, u_{O})\), by using Lemma 2.3, we can give precisely the geometric constraint conditions of the \( R_o \)-reduced symplectic form for the dynamical vector field of the regular orbit reducible RCH system; that is, the Type I and Type II Hamilton-Jacobi equations for the \( R_o \)-reduced RCH system \(((T^*Q)_{O}, \omega_{O}, h_{O}, f_{O}, u_{O})\).

In the following, by using Lemma 2.3 and the \( R_o \)-reduced symplectic form \( \omega_{O, \mu} \), for any \( G \)-invariant symplectic map \( \varepsilon : T^*Q \rightarrow T^*Q \), we can derive precisely the Type II Hamilton-Jacobi equation for the \( R_o \)-reduced RCH system \(((T^*Q)_{O}, \omega_{O}, h_{O}, f_{O}, u_{O})\) as follows:

\[ \text{Theorem 5.6} \quad (\text{Type II Hamilton-Jacobi Theorem for an } R_o\text{-reduced RCH system}) \quad \text{For a given regular orbit reducible RCH system } (T^*Q, G, \omega, H, F, W) \text{ with the canonical symplectic form } \omega \text{ on } T^*Q \text{ and with an } R_o\text{-reduced RCH system } ((T^*Q)_{O}, \omega_{O}, h_{O}, f_{O}, u_{O}) \text{, assume that } \gamma : Q \rightarrow T^*Q \]
is a one-form on $Q$, and that $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$. For any $G$-invariant symplectic map $\varepsilon : T^*Q \to T^*Q$, denote that $\tilde{X}^\varepsilon = T\pi_Q \cdot \tilde{X} \cdot \varepsilon$, where $\tilde{X} = X_{(T^*Q,G,\omega,H,F,W)}$ is the dynamical vector field of the regular orbit reducible RCH system $(T^*Q,G,\omega,H,F,W)$ with a control law $u$. Moreover, assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $J$, and that $O_\mu$, $(\mu \in \mathfrak{g}^*)$ is the regular orbit reducible orbit of the corresponding Hamiltonian system $(T^*Q,G,\omega,H)$, and that Im($\gamma$) $\subset$ $J^{-1}(\mu)$, and that $\gamma$ is $G$-invariant, and that $\varepsilon(J^{-1}(O_\mu)) \subset J^{-1}(O_\mu)$. Denote that $\bar{\gamma} = \pi_{O_\mu}(\gamma) : Q \to (T^*Q)_{O_\mu}$, $\lambda = \pi_{O_\mu}(\lambda) : J^{-1}(O_\mu)(C \cdot T^*Q) \to (T^*Q)_{O_\mu}$, and $\bar{\varepsilon} = \pi_{O_\mu}(\varepsilon) : J^{-1}(O_\mu)(C \cdot T^*Q) \to (T^*Q)_{O_\mu}$. Then $\varepsilon$ and $\bar{\varepsilon}$ satisfy the equation $T\bar{\varepsilon} \cdot X_{h_{O_\mu} \cdot \bar{\varepsilon}} = T\bar{\lambda} \cdot \tilde{X} \cdot \varepsilon$ if and only if they satisfy the equation $T\tilde{\gamma} \cdot \tilde{X}^\varepsilon = X_{h_{O_\mu} \cdot \varepsilon}$, where $X_{h_{O_\mu} \cdot \varepsilon}$ and $h_{O_\mu} \cdot \varepsilon \in TT^*Q$ are the Hamiltonian vector fields of the $R_0$-reduced Hamiltonian functions $h_{O_\mu}$ and $h_{O_\mu} \cdot \varepsilon : T^*Q \to \mathbb{R}$, respectively. The equation is a solution of the Type II Hamilton-Jacobi equation for the $R_0$-reduced RCH system $((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})$. Here the maps involved in the theorem are shown in Diagram 4.

See the proof and the more details in Wang [39].

Moreover, for the regular orbit reducible RCH system $(T^*Q,G,\omega,H,F,W)$ with an $R_0$-reduced RCH system $((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})$, we know that the Hamiltonian vector fields $X_H$ and $X_{h_{O_\mu}}$ for the corresponding Hamiltonian system $(T^*Q,G,\omega,H)$ and its $R_0$-reduced system $((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu})$ are $\pi_{O_\mu}$-related; that is, $X_{h_{O_\mu}} \cdot \pi_{O_\mu} = T\pi_{O_\mu} \cdot X_H \cdot i_{O_\mu}$. Then we can prove the Theorem 5.7, which states the relationship between the solutions of Type II Hamilton-Jacobi equations and the regular orbit reduction:

**Theorem 5.7** For the regular orbit reducible RCH system $(T^*Q,G,\omega,H,F,W)$ with the canonical symplectic form $\omega$ on $T^*Q$ and with an $R_0$-reduced RCH system $((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})$, assume that $\gamma : Q \to T^*Q$ is a one-form on $Q$, and that $\varepsilon : T^*Q \to T^*Q$ is a $G$-invariant symplectic map, $\bar{\varepsilon} = \pi_{O_\mu}(\varepsilon) : J^{-1}(O_\mu)(C \cdot T^*Q) \to (T^*Q)_{O_\mu}$. Under the hypotheses and notations of Theorem 5.6, then we have that $\varepsilon$ is a solution of the Type II Hamilton-Jacobi equation $T\tilde{\gamma} \cdot \tilde{X}^\varepsilon = X_{h_{O_\mu} \cdot \varepsilon}$ for the regular orbit reducible RCH system $(T^*Q,G,\omega,H,F,W)$ if and only if $\varepsilon$ and $\bar{\varepsilon}$ satisfy the Type II Hamilton-Jacobi equation $T\tilde{\gamma} \cdot \tilde{X}^\varepsilon = X_{h_{O_\mu} \cdot \varepsilon}$ for the $R_0$-reduced RCH system $((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})$.

See the proof and the more details in Wang [39].

It is worth noting that the different symplectic forms determine the different regular reduced RCH systems. From (4.5) we know that, for the regular orbit reduced symplectic space $(T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G \equiv J^{-1}(\mu)/G \times O_\mu$, if we give a stronger assumption condition; that is, for the one-form $\gamma : Q \to T^*Q$ on $Q$, we assume that Im($\gamma$) $\subset$ $J^{-1}(\mu)$, (note that it is not Im($\gamma$) $\subset$ $J^{-1}(O_\mu)$), and that $\gamma$ is $G$-invariant, then for any $V \in TQ$ and $w \in TT^*Q$, we have that $J^*_\mu \cdot \omega_{O_\mu}(T\gamma \cdot V, w) = 0$, and hence, from (4.5), $i_{O_\mu} \cdot \omega = \pi_{O_\mu} \cdot \omega_{O_\mu} + J^*_\mu \cdot \omega_{O_\mu}$, we have that $\pi_{O_\mu} \cdot \omega_{O_\mu} = i_{O_\mu} \cdot \omega = \omega$ along Im($\gamma$). Thus, we can use Lemma 2.3 for the regular orbit reduced symplectic form $\omega_{O_\mu}$ in the proofs of Theorems 5.6 and 5.7. It is easy to give the wrong results without the precise analysis for the regular orbit reduction case.
6 Controlled Hamiltonian Systems with Nonholonomic Constraints

In mechanics, it happens very often that systems have constraints. A nonholonomic Hamiltonian system is a Hamiltonian system with nonholonomic constraint, and a nonholonomic RCH system is also a RCH system with nonholonomic constraint. Usually, under the restrictions given by the nonholonomic constraints, in general, the dynamical vector fields of the nonholonomic Hamiltonian system and the nonholonomic RCH system may not be Hamiltonian. Thus, we cannot describe the Hamilton-Jacobi equations for the nonholonomic Hamiltonian system and the nonholonomic RCH system from the viewpoint of a generating function as in Theorem 2.1. Since the Hamilton-Jacobi theory is developed based on the Hamiltonian picture of the dynamics, it is a natural idea to extend the Hamilton-Jacobi theory to the nonholonomic Hamiltonian system and the nonholonomic RCH system, and to do so with symmetries and momentum maps; (see León and Wang [17], and Wang [42]).

In order to describe the nonholonomic Hamiltonian system and the nonholonomic RCH system, in the following we first give the completeness and the regularity conditions for nonholonomic constraint of a mechanical system; see León and Wang in [17]. In fact, in order to describe the dynamics of a nonholonomic mechanical system, we need some restriction conditions for the nonholonomic constraints of the system. First, we note that the set of Hamiltonian vector fields forms a Lie algebra with respect to the Lie bracket, since the restriction of a nonholonomic constraint is completely nonholonomic. Usually, under the restrictions given by the nonholonomic constraints of the system. First, we note that the set of Hamiltonian vector fields forms a Lie algebra with respect to the Lie bracket, since the restriction of a nonholonomic constraint is completely nonholonomic. Then the nonholonomic system is said to be completely nonholonomic, if the distribution \( D \subset TQ \) determined by the nonholonomic constraint is completely nonholonomic.

**D-completeness** Let \( Q \) be a smooth manifold and \( TQ \) its tangent bundle. A distribution \( D \subset TQ \) is said to be completely nonholonomic (or bracket-generating) if \( D \) along with all of its iterated Lie brackets that \([D,D], [D,[D,D]], \ldots\), spans the tangent bundle \( TQ \). Moreover, we consider a nonholonomic mechanical system on \( Q \), which is given by a Lagrangian function \( L : TQ \to \mathbb{R} \) subjects to constraint determined by a nonholonomic distribution \( D \subset TQ \) on the configuration manifold \( Q \). Then the nonholonomic system is said to be completely nonholonomic, if the distribution \( D \subset TQ \) determined by the nonholonomic constraint is completely nonholonomic.

**D-regularity** In the following we always assume that \( Q \) is an \( n \)-dimensional smooth manifold with coordinates \( (q^i) \), and that \( TQ \) its tangent bundle with coordinates \( (q^i, \dot{q}^i) \), and that \( T^*Q \) its cotangent bundle with coordinates \( (q^i, p_i) \), which are the canonical cotangent coordinates of \( T^*Q \), and that \( \omega = dq^i \wedge dp_i \) is canonical symplectic form on \( T^*Q \). If the Lagrangian \( L : TQ \to \mathbb{R} \) is hyperregular; that is, the Hessian matrix \( (\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j) \) is nondegenerate everywhere, then the Legendre transformation \( FL : TQ \to T^*Q \) is a diffeomorphism. In this case the Hamiltonian \( H : T^*Q \to \mathbb{R} \) is given by \( H(q, p) = \dot{q} \cdot p - L(q, \dot{q}) \) with Hamiltonian vector field \( X_H \), which is defined by the Hamilton’s equation \( i_{X_H} \omega = dH \), and \( M = FL(D) \) is a constraint submanifold in \( T^*Q \). In particular, for the nonholonomic constraint \( D \subset TQ \), the Lagrangian \( L \) is said to be \( D \)-regular, if the restriction of Hessian matrix \( (\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j) \) on \( D \) is nondegenerate everywhere. Moreover, a nonholonomic system is said to be \( D \)-regular, if its Lagrangian \( L \) is \( D \)-regular. Note that the restriction of a positive definite symmetric bilinear form to a subspace is also positive definite, and hence nondegenerate. Thus, for a simple nonholonomic mechanical system; that is, whose Lagrangian is the total kinetic energy minus potential energy, it is \( D \)-regular automatically.

### 6.1 Nonholonomic Hamiltonian System and Hamilton-Jacobi Equations

A nonholonomic Hamiltonian system is a 4-tuple \((T^*Q, \omega, D, H)\), which is a Hamiltonian system with a \( D \)-completely and \( D \)-regularly nonholonomic constraint \( D \subset TQ \). Under the restriction
given by constraints, in general, the dynamical vector field of a nonholonomic Hamiltonian system may not be Hamiltonian. However, the nonholonomic Hamiltonian system is a dynamical system closely related to a Hamiltonian system. In the following we can derive a distributional Hamiltonian system of the nonholonomic Hamiltonian system \( (T^*Q, \omega, D, H) \), by analyzing carefully the structure for the nonholonomic dynamical vector field and by using the method similar to that used in León and Wang [17] and Bates and Śniatycki [4]. The distributional Hamiltonian system is very important and it is also called a semi-Hamiltonian system in Patrick [33].

Assume that \( L : TQ \to \mathbb{R} \) is a hyperregular Lagrangian, and that the Legendre transformation \( FL : TQ \to T^*Q \) is a diffeomorphism. We consider that the constraint submanifold \( M = FL(D) \subset T^*Q \) and \( i_M : M \to T^*Q \) is the inclusion, and that the symplectic form \( \omega_M = i_M^* \omega \), is induced from the canonical symplectic form \( \omega \) on \( T^*Q \). We define the distribution \( F \) as the pre-image of the nonholonomic constraint \( D \) for the map \( T\pi_Q : TT^*Q \to TQ \); that is, \( F = (T\pi_Q)^{-1}(D) \subset TT^*Q \), which is a distribution along \( M \), and \( F^\circ := \{ \alpha \in T^*T^*Q | \alpha, v \geq 0, \forall v \in TT^*Q \} \) is the annihilator of \( F \) in \( T^*T^*Q|_M \). We consider the following nonholonomic constraints condition that

\[
(i_X \omega - dH) \in F^\circ, \quad X \in TM.
\]

From Cantrijn et al. [5], we know that there exists a unique nonholonomic vector field \( X_n \) satisfying the above condition (6.1) if the admissibility condition \( \text{dim} M = \text{rank} F \) and the compatibility condition \( TM \cap F^\perp = \{0\} \) hold, where \( F^\perp \) denotes the symplectic orthogonal of \( F \) with respect to the canonical symplectic form \( \omega \) on \( T^*Q \). In particular, when we consider the Whitney sum decomposition \( T(T^*Q)|_M = TM \oplus F^\perp \) and the canonical projection \( P : T(T^*Q)|_M \to TM \), then we have that \( X_n = P(X_H) \). See [12] for the more details.

From the condition (6.1) we know that the nonholonomic vector field, in general, may not be Hamiltonian, because of the restriction of nonholonomic constraint. However, we hope to study the dynamical vector field of nonholonomic Hamiltonian system by using the similar method of studying Hamiltonian vector field. From León and Wang [17] and Bates and Śniatycki [4], we can define the distribution \( K = F \cap TM \), and \( K^\perp = F^\perp \cap TM \), where \( K^\perp \) denotes the symplectic orthogonal of \( K \) with respect to the canonical symplectic form \( \omega \). If the admissibility condition \( \text{dim} M = \text{rank} F \) and the compatibility condition \( TM \cap F^\perp = \{0\} \) hold, then we know that the restriction of the symplectic form \( \omega_M \) on \( T^*M \) fibrewise to the distribution \( K \); that is, \( \omega_K = \tau_K^* \omega_M \) is non-degenerate, where \( \tau_K \) is the restriction map to distribution \( K \). It is worthy of noting that \( \omega_K \) is not a true two-form on a manifold, so it does not make sense to speak about it being closed. We call \( \omega_K \) as a distributional two-form to avoid any confusion. Because \( \omega_K \) is non-degenerate as a bilinear form on each fibre of \( K \), there exists a vector field \( X_K \) on \( M \) which takes values in the constraint distribution \( K \), such that the distributional Hamiltonian equation holds; that is,

\[
i_{X_K} \omega_K = dH_K
\]

where \( dH_K \) is the restriction of \( dH_M \) to \( K \), and the function \( H_K \) satisfies \( dH_K = \tau_K^* dH_M \), and \( H_M = \tau_M^* H \) is the restriction of \( H \) to \( M \). Moreover, from the distributional Hamiltonian equation (6.2), we have that \( X_K = \tau_K \cdot X_H \). Thus, the geometric formulation of a distributional Hamiltonian system may be summarized as follows.

**Definition 6.1** (Distributional Hamiltonian System) Assume that the 4-tuple \( (T^*Q, \omega, D, H) \) is a nonholonomic Hamiltonian system, where \( \omega \) is the canonical symplectic form on \( T^*Q \), and \( D \subset TQ \) is a \( D \)-completely and \( D \)-regularly nonholonomic constraint of the system. If there exist a distribution \( K \), an associated non-degenerate distributional two-form \( \omega_K \) induced by the canonical symplectic form and a vector field \( X_K \) on the constraint submanifold \( M = FL(D) \subset T^*Q \), such that the distributional Hamiltonian equation holds; that is, \( i_{X_K} \omega_K = dH_K \), where \( dH_K \) is the restriction of \( dH_M \).
to $\mathcal{K}$, and the function $H_{\mathcal{K}}$ satisfies $dH_{\mathcal{K}} = \tau_{\mathcal{K}} \cdot dH_{\mathcal{M}}$ as defined above, then the triple $(\mathcal{K}, \omega_{\mathcal{K}}, H_{\mathcal{K}})$ is called a distributional Hamiltonian system of the nonholonomic Hamiltonian system $(T^*Q, \omega, D, H)$, and $X_{\mathcal{K}}$ is called a nonholonomic dynamical vector field of the distributional Hamiltonian system $(\mathcal{K}, \omega_{\mathcal{K}}, H_{\mathcal{K}})$. Under the above circumstances, we refer to $(T^*Q, \omega, D, H)$ as a nonholonomic Hamiltonian system with an associated distributional Hamiltonian system $(\mathcal{K}, \omega_{\mathcal{K}}, H_{\mathcal{K}})$.

Since the non-degenerate distributional two-form $\omega_{\mathcal{K}}$ is not symplectic, and the distributional Hamiltonian system $(\mathcal{K}, \omega_{\mathcal{K}}, H_{\mathcal{K}})$ is not yet a Hamiltonian system, and hence, we can not describe the Hamilton-Jacobi equation for a distributional Hamiltonian system the same as in Theorem 2.1. However, for a given nonholonomic Hamiltonian system $(T^*Q, \omega, D, H)$ with an associated distributional Hamiltonian system $(\mathcal{K}, \omega_{\mathcal{K}}, H_{\mathcal{K}})$, we can derive precisely the geometric constraint conditions of the non-degenerate distributional two-form $\omega_{\mathcal{K}}$ for the nonholonomic dynamical vector field $X_{\mathcal{K}}$; that is, the two types of Hamilton-Jacobi equations for the distributional Hamiltonian system $(\mathcal{K}, \omega_{\mathcal{K}}, H_{\mathcal{K}})$. In order to do this, we need first give an important notions and a key lemma, (see also León and Wang [17]). This lemma and the Lemma 2.3 offer an important tool for the proofs of the two types of Hamilton-Jacobi theorems for the distributional Hamiltonian system and the nonholonomic reduced distributional Hamiltonian system.

Assume that $\omega$ is the canonical symplectic form on $T^*Q$, and $D \subset TQ$ is a $D$-regularly nonholonomic constraint, and the projection $\pi_Q : T^*Q \rightarrow Q$ induces the map $T\pi_Q : TT^*Q \rightarrow TQ$. If the one-form $\gamma$ is closed, then $d\gamma(x,y) = 0$, $\forall x, y \in TQ$. Note that for any $v, w \in TT^*Q$, we have that $d\gamma(T\pi_Q(v), T\pi_Q(w)) = \pi^*(d\gamma)(v, w)$ is a two-form on the cotangent bundle $T^*Q$, where $\pi^* : T^*Q \rightarrow T^*T^*Q$. Thus, in the following we can introduce a weaker notion which is an extension of the Definition 2.2 to the nonholonomic context.

**Definition 6.2** The one-form $\gamma$ is called to be closed on $D$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$ if, for any $v, w \in TT^*Q$ and $T\pi_Q(v), T\pi_Q(w) \in D$, we have that $d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0$.

The notion that $\gamma$ is closed on $D$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, is weaker than that $\gamma$ is closed on $D$; that is, $d\gamma(x,y) = 0$, $\forall x, y \in D$. In fact, if $\gamma$ is a closed one-form on $D$, then it must be closed on $D$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$. Conversely, if $\gamma$ is closed on $D$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, then it may not be closed on $D$. Now, we give the important lemma as follows:

**Lemma 6.3** Assume that $\gamma : Q \rightarrow T^*Q$ is a one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \rightarrow T^*Q$. If the Lagrangian $L$ is $D$-regular, and $\text{Im}(\gamma) \subset M = F \mathcal{L}(D)$, then we have that $X_H \cdot \gamma \in F$ along $\gamma$ and $X_H \cdot \lambda \in F$ along $\lambda$; that is, $T\pi_Q(X_H \cdot \gamma(q)) \in \mathcal{D}_q$, $\forall q \in Q$, and $T\pi_Q(X_H \cdot \lambda(q, p)) \in \mathcal{D}_q$, $\forall q \in Q$, $(q, p) \in T^*Q$. Moreover, if the symplectic map $\varepsilon : T^*Q \rightarrow T^*Q$ with respect to the canonical symplectic form $\omega$ on $T^*Q$, satisfies the condition $\varepsilon(M) \subset M$, then we have that $X_{\mathcal{K}} - \varepsilon \in F$ along $\varepsilon$.

See the proof and the more details in León and Wang [17].

By using the Lemma 2.3 and the above Lemma 6.3, we can derive precisely the geometric constraint conditions of the non-degenerate distributional two-form $\omega_{\mathcal{K}}$ for the nonholonomic dynamical vector field $X_{\mathcal{K}}$; that is, the two types of Hamilton-Jacobi equations for the distributional Hamiltonian system $(\mathcal{K}, \omega_{\mathcal{K}}, H_{\mathcal{K}})$. For convenience, the maps involved in the theorem and its proof are shown in Diagram-5.

```
\begin{align*}
\mathcal{M} & \xrightarrow{i_M} T^*Q & \pi_Q \quad & \gamma \quad & T^*Q \\
X_{\mathcal{K}} & \quad \xrightarrow{X_H} & \quad & \quad & \quad \\
\mathcal{K} & \quad \xrightarrow{\pi_K} T(T^*Q) & \xrightarrow{T\pi_Q} & TQ & \xrightarrow{T\pi_Q} T(T^*Q)
\end{align*}
```
Theorem 6.4 (Hamilton-Jacobi Theorem for a Distributional Hamiltonian System) For the nonholonomic Hamiltonian system \((T^*Q, \omega, D, H)\) with an associated distributional Hamiltonian system \((K, \omega_K, H_K)\), assume that \(\gamma : Q \to T^*Q\) is a one-form on \(Q\), and that \(\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q\), and that for any symplectic map \(\varepsilon : T^*Q \to T^*Q\), denote that \(X^\gamma = T\pi_Q \cdot X_H \cdot \gamma\) and \(X^\varepsilon = T\pi_Q \cdot X_H \cdot \varepsilon\), where \(X_H\) is the dynamical vector field of the corresponding unconstrained Hamiltonian system \((T^*Q, \omega, H)\). Moreover, assume that \(\text{Im}(\gamma) \subset M = FL(D)\), and that \(\varepsilon(M) \subset M\), and that \(\text{Im}(T\gamma) \subset K\). Then the following two assertions hold:

(i) If the one-form \(\gamma : Q \to T^*Q\) is closed on \(D\) with respect to \(T\pi_Q : TT^*Q \to TQ\), then \(\gamma\) is a solution of the Type I Hamilton-Jacobi equation \(T\gamma \cdot X^\gamma = X_K \cdot \gamma\) for the distributional Hamiltonian system \((K, \omega_K, H_K)\), where \(X_K\) is the nonholonomic dynamical vector field of the distributional Hamiltonian system.

(ii) The \(\varepsilon\) is a solution of the Type II Hamilton-Jacobi equation \(T\gamma \cdot X^\varepsilon = X_K \cdot \varepsilon\) if and only if it is a solution of the equation \(T\gamma \cdot X^\varepsilon = X_K \cdot \varepsilon\), where \(X_K \cdot \varepsilon\) is the Hamiltonian vector field of the function \(H \cdot \varepsilon : T^*Q \to \mathbb{R}\).

See the proof and the more details in León and Wang [17].

6.2 Nonholonomic Controlled Hamiltonian System and Its Reductions

A nonholonomic RCH system is a 6-tuple \((T^*Q, \omega, D, H, F, W)\), which is an RCH system with a \(D\)-completely and \(D\)-regularly nonholonomic constraint \(D \subset TQ\). Under the restriction given by the constraints, in general, the dynamical vector field of a nonholonomic RCH system may not be Hamiltonian vector field. However the nonholonomic RCH system is a dynamical system closely related to a Hamiltonian system. In the following we can derive a distributional RCH system of the nonholonomic RCH system \((T^*Q, \omega, D, H, F, W)\), by analyzing carefully the structure for the nonholonomic dynamical vector field and by using the method similar to that used in León and Wang [17].

From the above discussion in §6.1, we know that there exist a distribution \(K\), an associated non-degenerate distributional two-form \(\omega_K\) induced by the canonical symplectic form and a vector field \(X_K\) on the constraint submanifold \(M = FL(D) \subset T^*Q\), such that the distributional Hamiltonian equation (6.2) holds, and we have that \(X_K = \tau_K \cdot X_H\).

Moreover, if considering the external force \(F\) and the control subset \(W\), and define that \(F_K = \tau_K \cdot \text{vlift}(F_M)X_H\), and that, for the control law \(u \in W\), \(u_K = \tau_K \cdot \text{vlift}(u_M)X_H\), where \(F_M = \tau_M \cdot F\) and \(u_M = \tau_M \cdot u\) are the restrictions of \(F\) and \(u\) to \(M\); that is, \(F_K\) and \(u_K\) are the restrictions of the changes of the Hamiltonian vector field \(X_H\) under the actions of \(F_M\) and \(u_M\) to \(K\). Then the 5-tuple \((K, \omega_K, H_K, F_K, u_K)\) is a distributional RCH system of the nonholonomic RCH system \((T^*Q, \omega, D, H, F, W)\) with the control law \(u \in W\). Thus, the geometric formulation of the distributional RCH system may be summarized as follows:

**Definition 6.5** (Distributional RCH System) Assume that the 6-tuple \((T^*Q, \omega, D, H, F, W)\) is a nonholonomic RCH system, where \(\omega\) is the canonical symplectic form on \(T^*Q\), and \(D \subset TQ\) is a \(D\)-completely and \(D\)-regularly nonholonomic constraint of the system, and the external force \(F : T^*Q \to T^*Q\) is the fiber-preserving map, and the control subset \(W \subset T^*Q\) is a fiber submanifold of \(T^*Q\). For a control law \(u \in W\), if there exist a distribution \(K\), an associated non-degenerate distributional two-form \(\omega_K\) induced by the canonical symplectic form and a vector field \(X_K\) on the constraint submanifold \(M = FL(D) \subset T^*Q\), such that the distributional Hamiltonian equation holds; that is, \(i_{X_K}\omega_K = dH_K\), where \(dH_K\) is the restriction of \(dH_M\) to \(K\), and the function \(H_K\)
satisfies \( dH_K = \tau_K \cdot dH_M \), and \( F_K = \tau_K \cdot \text{vlift}(F_M)X_H \), and \( u_K = \tau_K \cdot \text{vlift}(u_M)X_H \) as defined above, then the 5-tuple \((K, \omega_K, H_K, F_K, u_K)\) is called a distributional RCH system of the nonholonomic RCH system \((T^*Q, \omega, \mathcal{D}, H, F, u)\), and \( X_K \) is called a nonholonomic dynamical vector field. Denote that

\[
\dot{X} = X_{(K, \omega_K, H_K, F_K, u_K)} = X_K + F_K + u_K
\]

is the dynamical vector field of the distributional RCH system \((K, \omega_K, H_K, F_K, u_K)\), which is the synthesis of the nonholonomic dynamical vector field \(X_K\) and the vector fields \(F_K\) and \(u_K\). Under the above circumstances, we refer to \((T^*Q, \omega, \mathcal{D}, H, F, u)\) as a nonholonomic RCH system with an associated distributional RCH system \((K, \omega_K, H_K, F_K, u_K)\).

It is worthy of noting that if the external force and control of a distributional RCH system \((K, \omega_K, H_K, F_K, u_K)\) are both zeros; that is, \(F_K = 0\) and \(u_K = 0\), in this case, the distributional RCH system is just a distributional Hamiltonian system \((K, \omega_K, H_K)\); see León and Wang [17] for more details. Thus, the distributional RCH system can be regarded as an extension of the distributional Hamiltonian system to the system with the external force and the control.

We know that the reduction of the nonholonomically constrained mechanical systems is also a very important subject in the study of the geometric mechanics, which is regarded as a useful tool for simplifying and studying concrete nonholonomic systems; see León and Wang [17], Bates and Śniatycki [4], Cantrijn et al. [5], Cendra et al. [8], Cushman et al. [9] and [10], Koiller [14], León and Rodrigues [16], and so on. In the following we consider the nonholonomic RCH system with symmetry, as well as with a momentum map. By using the similar method for the nonholonomic reduction (given in Bates and Śniatycki [4], León and Wang [17] and Wang [41]), and analyzing carefully the structures for the nonholonomic reduced dynamical vector fields, we also give the geometric formulations of the nonholonomic reduced distributional RCH systems.

In the following we consider that a nonholonomic RCH system with symmetry and a momentum map is 8-tuple \((T^*Q, G, \omega, J, \mathcal{D}, H, F, W)\), where \(\omega\) is the canonical symplectic form on \(T^*Q\), and the Lie group \(G\), which may not be Abelian, acts smoothly by the left on \(Q\), its tangent lifted action on \(TQ\) and its cotangent lifted action on \(T^*Q\), and \(\mathcal{D} \subset TQ\) is a \(\mathcal{D}\)-completely and \(\mathcal{D}\)-regularly nonholonomic constraint of the system, and \(\mathcal{D}, H, F\) and \(W\) are all \(G\)-invariant. Thus, the nonholonomic RCH system with symmetry and a momentum map is a regular point reducible RCH system with \(G\)-invariant nonholonomic constraint \(\mathcal{D}\). Moreover, in the following we shall give carefully a geometric formulation of the \(J\)-nonholonomic \(R_p\)-reduced distributional RCH system, by using the momentum map and the nonholonomic reduction compatible with regular point reduction.

Note that the Legendre transformation \(\mathcal{F}L : TQ \to T^*Q\) is a fiber-preserving map, and that \(\mathcal{D} \subset TQ\) is \(G\)-invariant for the tangent lifted left action \(\Phi^T : G \times TQ \to TQ\), then the constraint submanifold \(M = \mathcal{F}L(\mathcal{D}) \subset T^*Q\) is \(G\)-invariant for the cotangent lifted left action \(\Phi^{T^*} : G \times T^*Q \to T^*Q\). For the nonholonomic RCH system with symmetry and a momentum map \((T^*Q, G, \omega, J, \mathcal{D}, H, F, W)\), in the same way, we define the distribution \(\mathcal{F}\), which is the pre-image of the nonholonomic constraints \(\mathcal{D}\) for the map \(T\pi_Q : TT^*Q \to TQ\); that is, \(\mathcal{F} = (T\pi_Q)^{-1}(\mathcal{D})\), and the distribution \(K = \mathcal{F} \cap TM\). Moreover, we can also define the distribution two-form \(\omega_K\), which is induced from the canonical symplectic form \(\omega\) on \(T^*Q\); that is, \(\omega_K = \tau_K \cdot \omega_M\), and \(\omega_M = i_M^*\omega\). If the admissibility condition \(\text{dim}M = \text{rank}\mathcal{F}\) and the compatibility condition \(TM \cap F^\perp = \{0\}\) hold, then \(\omega_K\) is non-degenerate as a bilinear form on each fibre of \(K\), there exists a vector field \(X_K\) on \(M\) which takes values in the constraint distribution \(K\), such that for the function \(H_K\), the following distributional Hamiltonian equation holds, that is,

\[
i_{X_K} \omega_K = dH_K,
\]

(6.4)
where the function \( H_K \) satisfies \( dH_K = \tau_K \cdot dH_M \), and \( H_M = \tau_M \cdot H \) is the restriction of \( H \) to \( M \), and from the equation (6.4), we have that \( X_K = \tau_K \cdot X_H \).

Since the nonholonomic RCH system with symmetry and a momentum map is a regular point reducible RCH system with \( G \)-invariant nonholonomic constraint \( D \), for a regular value \( \mu \in g^* \) of the momentum map \( J : T^*Q \to g^* \), we assume that the constraint submanifold \( M \) is clean intersection with \( J^{-1}(\mu) \); that is, \( M \cap J^{-1}(\mu) \neq \emptyset \). Note that \( M \) is also \( G_\mu(\subset G) \) action invariant, and so is \( J^{-1}(\mu) \), because \( J \) is Ad\(^*\)-equivariant. It follows that the quotient space \( M_\mu = (M \cap J^{-1}(\mu))/G_\mu \subset (T^*Q)_\mu \) of the \( G_\mu \)-orbit in \( M \cap J^{-1}(\mu) \), is a smooth manifold with the projection \( \pi_\mu : M \cap J^{-1}(\mu) \to M_\mu \), which is a surjective submersion. Denote that \( i_{M_\mu} : M_\mu \to (T^*Q)_\mu \), and that \( \omega_{M_\mu} = i_{M_\mu}^* \omega_\mu \); that is, the symplectic form \( \omega_{M_\mu} \) is induced from the \( R_p \)-reduced symplectic form \( \omega_\mu \) on \( (T^*Q)_\mu \) given in (4.1). Moreover, the distribution \( F \) is pushed down to a distribution \( F_\mu = T\pi_\mu \cdot F \) on \( (T^*Q)_\mu \), and we define \( K_\mu = \mathcal{F}_\mu \cap TM_\mu \). Assume that \( \omega_{K_\mu} = \tau_{K_\mu} \cdot \omega_{M_\mu} \) is the restriction of the symplectic form \( \omega_{M_\mu} \) on \( (T^*Q)_\mu \) fibrewise to the distribution \( K_\mu \), where \( \tau_{K_\mu} \) is the restriction map to distribution \( K_\mu \). The distributional two-form \( \omega_{K_\mu} \) is not a ”true two-form” on a manifold, which is called as a \( J \)-nonholonomic \( \text{R}_p \)-reduced distributional two-form to avoid any confusion.

From the above construction we know that, if the admissibility condition \( \dim M_\mu = \text{rank} \mathcal{F}_\mu \) and the compatibility condition \( TM_\mu \cap \mathcal{F}_\mu^\perp = \{0\} \) hold, where \( \mathcal{F}_\mu^\perp \) denotes the symplectic orthogonal of \( \mathcal{F}_\mu \) with respect to the \( R_p \)-reduced symplectic form \( \omega_\mu \), then \( \omega_{K_\mu} \) is non-degenerate as a bilinear form on each fibre of \( K_\mu \), and hence, there exists a vector field \( X_{K_\mu} \) on \( M_\mu \), which takes values in the constraint distribution \( K_\mu \), such that for the function \( h_{K_\mu} \), the \( J \)-nonholonomic \( R_p \)-reduced distributional Hamiltonian equation holds, that is,

\[
i_{X_{K_\mu}} \omega_{K_\mu} = dh_{K_\mu}, \tag{6.5}\]

where \( dh_{K_\mu} \) is the restriction of \( dh_{M_\mu} \) to \( K_\mu \), and the function \( h_{K_\mu} \) satisfies \( dh_{K_\mu} = \tau_{K_\mu} \cdot dh_{M_\mu} \), and \( h_{M_\mu} = \tau_{M_\mu} \cdot h_\mu \) is the restriction of \( h_\mu \) to \( M_\mu \), and \( h_\mu \) is the \( R_p \)-reduced Hamiltonian function \( h_\mu : (T^*Q)_\mu \to \mathbb{R} \) defined by \( h_\mu \cdot \pi_\mu = H \cdot i_\mu \). In addition, from the distributional Hamiltonian equation (6.4), \( i_{X_{K_\mu}} \omega_K = dh_K \), we have that \( X_K = \tau_K \cdot X_H \), and from the \( J \)-nonholonomic \( R_p \)-reduced distributional Hamiltonian equation (6.5), \( i_{X_{K_\mu}} \omega_{K_\mu} = dh_{K_\mu} \), we have that \( X_{K_\mu} = \tau_{K_\mu} \cdot X_{h_{K_\mu}} \), where \( X_{h_{K_\mu}} \) is the Hamiltonian vector field of the function \( h_{K_\mu} \), and the vector fields \( X_K \) and \( X_{K_\mu} \) are \( \pi_{\mu} \)-related, that is, \( X_{K_\mu} \cdot \pi_{\mu} = T\pi_\mu \cdot X_K \).

Moreover, if considering the external force \( F \) and the control subset \( W \), and we define the vector fields \( F_K = \tau_K \cdot \text{vlift}(F_M)X_H \), and for a control law \( u \in W \), define that \( u_K = \tau_K \cdot \text{vlift}(u_M)X_H \), where \( F_M = \tau_M \cdot F \) and \( u_M = \tau_M \cdot u \) are the restrictions of \( F \) and \( u_M \) to \( M \); that is, \( F_K \) and \( u_K \) are the restrictions of the changes of Hamiltonian vector field \( X_H \) under the actions of \( F_M \) and \( u_M \) to \( K \), then the 5-tuple \((K, \omega_K, H_K, F_K, u_K)\) is a distributional RCH system corresponding to the nonholonomic RCH system with symmetry and a momentum map \((T^*Q, G, \omega, J, D, H, F, u)\), and the dynamical vector field of the distributional RCH system can be expressed by

\[
\tilde{X} = X(K, \omega_K, H_K, F_K, u_K) = X_K + F_K + u_K, \tag{6.6}
\]

which is the synthesis of the nonholonomic dynamical vector field \( X_K \) and the vector fields \( F_K \) and \( u_K \). Assume that the vector fields \( F_K \) and \( u_K \) on \( M \) are pushed down to the vector fields \( f_{M_\mu} = T\pi_\mu \cdot F_K \) and \( u_{M_\mu} = T\pi_\mu \cdot u_K \) on \( M_\mu \). Then we define that \( f_{K_\mu} = T\pi_{K_\mu} \cdot f_{M_\mu} \), and that \( u_{K_\mu} = T\pi_{K_\mu} \cdot u_{M_\mu} \); that is, \( f_{K_\mu} \) and \( u_{K_\mu} \) are the restrictions of \( f_{M_\mu} \) and \( u_{M_\mu} \) to \( K_\mu \). As a consequence, the 5-tuple \((K_\mu, \omega_{K_\mu}, h_{K_\mu}, f_{K_\mu}, u_{K_\mu})\) is a \( J \)-nonholonomic \( \text{R}_p \)-reduced distributional RCH system of the nonholonomic RCH system with symmetry and a momentum map \((T^*Q, G, \omega, J, D, H, F, W)\),
as well as with a control law \( u \in W \). Thus, the geometrical formulation of the \( J \)-nonholonomic \( R_p \)-reduced distributional RCH system may be summarized as follows.

**Definition 6.6** (\( J \)-Nonholonomic \( R_p \)-reduced Distributional RCH System) Assume that the 8-tuple \((T^*Q,G,\omega,J,D,H,F,W)\) is a nonholonomic RCH system with symmetry and a momentum map, where \( \omega \) is the canonical symplectic form on \( T^*Q \), and \( D \subset TQ \) is a \( D \)-completely and \( D \)-regularly nonholonomic constraint of the system, and \( D \), \( H \), \( F \), and \( W \) are all \( G \)-invariant. For a regular value \( \mu \in g^* \) of the momentum map \( J : T^*Q \to g^* \), assume that there exists a \( J \)-nonholonomic \( R_p \)-reduced distribution \( K_\mu \), an associated non-degenerate and \( J \)-nonholonomic \( R_p \)-reduced distributional two-form \( \omega_{K_\mu} \) and a vector field \( X_{K_\mu} \) on the \( J \)-nonholonomic \( R_p \)-reduced constraint submanifold \( M_\mu = (\mathcal{M} \cap J^{-1}(\mu))/G_\mu \), where \( \mathcal{M} = FL(D) \), and \( \mathcal{M} \cap J^{-1}(\mu) \neq \emptyset \), and \( G_\mu = \{ g \in G \mid Ad_g^* \mu = \mu \} \), such that the \( J \)-nonholonomic \( R_p \)-reduced distributional Hamiltonian equation (6.5) holds, that is, \( i_{X_{K_\mu}} \omega_{K_\mu} = dh_{K_\mu} \), where \( dh_{K_\mu} \) is the restriction of \( dh_{M_\mu} \) to \( K_\mu \), and the function \( h_{K_\mu} \), and the vector fields \( f_{K_\mu} \) and \( u_{K_\mu} \) are defined above. Then the 5-tuple \((K_\mu, \omega_{K_\mu}, h_{K_\mu}, f_{K_\mu}, u_{K_\mu})\) is called a \( J \)-nonholonomic \( R_p \)-reduced distributional RCH system of the nonholonomic RCH system with symmetry and a momentum map \((T^*Q,G,\omega,J,D,H,F,W,u)\) with a control law \( u \in W \), and \( X_{K_\mu} \) is called the \( J \)-nonholonomic \( R_p \)-reduced dynamical vector field. Denote that

\[
\dot{X}_\mu = X\left(\{K_\mu, \omega_{K_\mu}, h_{K_\mu}, f_{K_\mu}, u_{K_\mu}\}\right) = X_{K_\mu} + f_{K_\mu} + u_{K_\mu}
\]

(6.7)

is the dynamical vector field of the \( J \)-nonholonomic \( R_p \)-reduced distributional RCH system \((K_\mu, \omega_{K_\mu}, h_{K_\mu}, f_{K_\mu}, u_{K_\mu})\), which is the synthesis of the \( J \)-nonholonomic \( R_p \)-reduced dynamical vector field \( X_{K_\mu} \) and the vector fields \( f_{K_\mu} \) and \( u_{K_\mu} \). Under the above circumstances, we refer to \((T^*Q,G,\omega,J,D,H,F,W,u)\) as a \( J \)-nonholonomic point reducible RCH system with the associated distributional RCH system \((K, \omega_K, H_K, F_K, u_K)\) and the \( J \)-nonholonomic \( R_p \)-reduced distributional RCH system \((K_\mu, \omega_{K_\mu}, h_{K_\mu}, f_{K_\mu}, u_{K_\mu})\).

Since the non-degenerate and \( J \)-nonholonomic \( R_p \)-reduced distributional two-form \( \omega_{K_\mu} \) is not a "true two-form" on a manifold, and it is not symplectic, and hence the \( J \)-nonholonomic \( R_p \)-reduced distributional RCH system \((K_\mu, \omega_{K_\mu}, h_{K_\mu}, f_{K_\mu}, u_{K_\mu})\) is not a Hamiltonian system, and has no yet generating function, and hence, we can not describe the Hamilton-Jacobi equation for a \( J \)-nonholonomic \( R_p \)-reduced distributional RCH system the same as in Theorem 2.1. However, for a given \( J \)-nonholonomic regular point reducible RCH system \((T^*Q,G,\omega,J,D,H,F,W,u)\) with the associated distributional RCH system \((K, \omega_K, H_K, F_K, u_K)\) and the \( J \)-nonholonomic \( R_p \)-reduced distributional RCH system \((K_\mu, \omega_{K_\mu}, h_{K_\mu}, f_{K_\mu}, u_{K_\mu})\), by using Lemma 2.3 and 6.3, we can derive precisely the geometric constraint conditions of the \( J \)-nonholonomic \( R_p \)-reduced distributional two-form \( \omega_{K_\mu} \) for the \( J \)-nonholonomic regular point reducible dynamical vector field; that is, the two types of Hamilton-Jacobi equations for the \( J \)-nonholonomic \( R_p \)-reduced distributional RCH system \((K_\mu, \omega_{K_\mu}, h_{K_\mu}, f_{K_\mu}, u_{K_\mu})\). At first, by using the fact that the one-form \( \gamma : Q \to T^*Q \) is closed on \( D \) with respect to \( T\pi_Q : T^*Q \to TQ \), and that \( \text{Im}(\gamma) \subset M \cap J^{-1}(\mu) \), and that \( \gamma \) is \( G_\mu \)-invariant, as well as that \( \text{Im}(\hat{T}_\gamma) \subset K_\mu \), we can prove the Type I Hamilton-Jacobi theorem for the \( J \)-nonholonomic \( R_p \)-reduced distributional RCH system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-6.

![Diagram-6](image-url)
Theorem 6.7 (Type I Hamilton-Jacobi Theorem for a $J$-Nonholonomic $R_p$-reduced Distributional RCH System) For a given $J$-nonholonomic regular point reducible RCH system $(T^*Q, G, \omega, J, D, H, F, u)$ with the associated distributional RCH system $(\mathcal{K}, \omega_\mathcal{K}, H_\mathcal{K}, F_\mathcal{K}, u_\mathcal{K})$ and the $J$-nonholonomic $R_p$-reduced distributional RCH system $(\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu}, f_{\mathcal{K}_\mu}, u_{\mathcal{K}_\mu})$, assume that $\gamma : Q \to T^*Q$ is a one-form on $Q$, and that $\bar{X}_\gamma = T\pi_Q \cdot \bar{X} \cdot \gamma$, where $\bar{X} = X_{(\mathcal{K}, \omega_\mathcal{K}, H_\mathcal{K}, F_\mathcal{K}, u_\mathcal{K})} = X_K + F_K + u_K$ is the dynamical vector field of the distributional RCH system $(\mathcal{K}, \omega_\mathcal{K}, H_\mathcal{K}, F_\mathcal{K}, u_\mathcal{K})$ corresponding to the $J$-nonholonomic regular point reducible RCH system with symmetry and a momentum map $(T^*Q, G, \omega, J, D, H, F, u)$. Moreover, assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $J$, and that $\text{Im}(\gamma) \subset M \cap J^{-1}(\mu)$, that $\gamma$ is $G_\mu$-invariant, and that $\tilde{\gamma}_j = \pi_\mu(\gamma) : Q \to M_\mu$, and that $\text{Im}(T\tilde{\gamma}_j) \subset \mathcal{K}_\mu$. If the one-form $\gamma : Q \to T^*Q$ is closed on $D$ with respect to $T\pi_Q : TT^*Q \to TQ$, then $\tilde{\gamma}_j$ is a solution of the equation $T\tilde{\gamma}_j \cdot X_\gamma = X_{\mathcal{K}_\mu} \cdot \tilde{\gamma}_j$. Here $X_{\mathcal{K}_\mu}$ is the $J$-nonholonomic $R_p$-reduced dynamical vector field. The equation $T\tilde{\gamma}_j \cdot X_\gamma = X_{\mathcal{K}_\mu} \cdot \tilde{\gamma}_j$, is called the Type I Hamilton-Jacobi equation for the $J$-nonholonomic $R_p$-reduced distributional RCH system $(\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu}, f_{\mathcal{K}_\mu}, u_{\mathcal{K}_\mu})$.

See the proof and the more details in Wang [42]. Moreover, for any $G_\mu$-invariant symplectic map $\varepsilon : T^*Q \to T^*Q$, we can prove the Type II Hamilton-Jacobi theorem for the $J$-nonholonomic $R_p$-reduced distributional RCH system.

In the following we consider that a nonholonomic RCH system with symmetry and momentum map is 8-tuple $(T^*Q, G, \omega, J, D, H, F, W)$, which is a regular orbit reducible RCH system with $G$-invariant nonholonomic constraint $D$. We can give the geometric formulation of the $J$-nonholonomic $R_o$-reduced distributional RCH system, by using the momentum map and the nonholonomic reduction compatible with the regular orbit reduction.

For a regular value $\mu \in \mathfrak{g}^*$ of the momentum map $J : T^*Q \to \mathfrak{g}^*$, $\mathcal{O}_\mu = G \cdot \mu \subset \mathfrak{g}^*$ is the $G$-orbit of the coadjoint $G$-action through the point $\mu$, we assume that the constraint submanifold $M$ is clean intersection with $J^{-1}(\mathcal{O}_\mu)$; that is, $M \cap J^{-1}(\mathcal{O}_\mu) \neq \emptyset$. It follows that the quotient space $\mathcal{M}_{O_\mu} = (M \cap J^{-1}(\mathcal{O}_\mu))/G \subset (T^*Q)_{O_\mu}$ of the $G$-orbit in $M \cap J^{-1}(\mathcal{O}_\mu)$, is a smooth manifold with projection $\pi_{O_\mu} : M \cap J^{-1}(\mathcal{O}_\mu) \to \mathcal{M}_{O_\mu}$ which is a surjective submersion. Denote that $i_{\mathcal{M}_{O_\mu}} : \mathcal{M}_{O_\mu} \to (T^*Q)_{O_\mu}$, and that $\omega_{\mathcal{M}_{O_\mu}} = i_{\mathcal{M}_{O_\mu}}^* \omega_{O_\mu}$; that is, the symplectic form $\omega_{\mathcal{M}_{O_\mu}}$ is induced from the $R_o$-reduced symplectic form $\omega_{O_\mu}$ on $(T^*Q)_{O_\mu}$ given in (4.5), where $i_{\mathcal{M}_{O_\mu}} : T^*Q_{O_\mu} \to T^*\mathcal{M}_{O_\mu}$. Moreover, the distribution $F$ is pushed down to a distribution $\mathcal{F}_{O_\mu} = T\pi_{O_\mu} \cdot F$ on $(T^*Q)_{O_\mu}$, and we define $\mathcal{K}_{O_\mu} = \mathcal{F}_{O_\mu} \cap T\mathcal{M}_{O_\mu}$. Assume that $\omega_{\mathcal{K}_{O_\mu}} = \tau_{\mathcal{K}_{O_\mu}} \cdot \omega_{\mathcal{M}_{O_\mu}}$ is the restriction of the symplectic form $\omega_{\mathcal{M}_{O_\mu}}$ on $T^*\mathcal{M}_{O_\mu}$ fibrewise to the distribution $\mathcal{K}_{O_\mu}$, where $\tau_{\mathcal{K}_{O_\mu}}$ is the restriction map to distribution $\mathcal{K}_{O_\mu}$. The distributional two-form $\omega_{\mathcal{K}_{O_\mu}}$ is not a "true two-form" on a manifold, which is called as a $J$-nonholonomic $R_o$-reduced distributional two-form to avoid any confusion.

From the above construction we know that, if the admissibility condition $\dim \mathcal{M}_{O_\mu} = \text{rank} \mathcal{F}_{O_\mu}$ and the compatibility condition $T\mathcal{M}_{O_\mu} \cap \mathcal{F}_{O_\mu} = \{0\}$ hold, where $\mathcal{F}_{O_\mu}$ denotes the symplectic orthogonal of $\mathcal{F}_{O_\mu}$ with respect to the $R_o$-reduced symplectic form $\omega_{O_\mu}$, then $\omega_{\mathcal{K}_{O_\mu}}$ is non-degenerate as a bilinear form on each fibre of $\mathcal{K}_{O_\mu}$, and hence there exists a vector field $X_{\mathcal{K}_{O_\mu}}$ on $\mathcal{M}_{O_\mu}$, which takes values in the constraint distribution $\mathcal{K}_{O_\mu}$, such that for the function $h_{\mathcal{K}_{O_\mu}}$, the $J$-nonholonomic $R_o$-reduced distributional Hamiltonian equation holds, that is,

$$i_{X_{\mathcal{K}_{O_\mu}}} \omega_{\mathcal{K}_{O_\mu}} = dh_{\mathcal{K}_{O_\mu}}, \quad (6.8)$$

where $dh_{\mathcal{K}_{O_\mu}}$ is the restriction of $dh_{\mathcal{M}_{O_\mu}}$ to $\mathcal{K}_{O_\mu}$, and the function $h_{\mathcal{K}_{O_\mu}}$ satisfies $dh_{\mathcal{K}_{O_\mu}} = \tau_{\mathcal{K}_{O_\mu}} \cdot dh_{\mathcal{M}_{O_\mu}}$, and $h_{\mathcal{M}_{O_\mu}} = \tau_{\mathcal{M}_{O_\mu}} \cdot h_{O_\mu}$ is the restriction of $h_{O_\mu}$ to $\mathcal{M}_{O_\mu}$, and $h_{O_\mu}$ is the $R_o$-reduced
Hamiltonian function \( h_{O_{\mu}} : (T^*Q)_{O_{\mu}} \to \mathbb{R} \) defined by \( h_{O_{\mu}} \cdot \pi_{O_{\mu}} = H \cdot i_{O_{\mu}} \). In addition, from the distributional Hamiltonian equation; that is, \( i_{X_K} \omega_K = dH_K \), we have that \( X_K = \tau_K \cdot X_H \), and from the \( J \)-nonholonomic \( R_o \)-reduced distributional Hamiltonian equation (6.8) \( i_{X_{K,O_{\mu}}} \omega_{K,O_{\mu}} = dh_{K,O_{\mu}} \), we have that \( X_{K,O_{\mu}} = \tau_{K,O_{\mu}} \cdot X_{h_{K,O_{\mu}}} \), where \( X_{h_{K,O_{\mu}}} \) is the Hamiltonian vector field of the function \( h_{K,O_{\mu}} \), and the vector fields \( X_K \) and \( X_{K,O_{\mu}} \) are \( \pi_{O_{\mu}} \)-related, that is, \( X_{K,O_{\mu}} \cdot \pi_{O_{\mu}} = T \pi_{O_{\mu}} \cdot X_K \).

Moreover, assume that the vector fields \( F_K \) and \( u_K \) on \( M \) are pushed down to the vector fields \( f_{M,O_{\mu}} = T \pi_{O_{\mu}} \cdot F_K \) and \( u_{M,O_{\mu}} = T \pi_{O_{\mu}} \cdot u_K \) on \( M_{O_{\mu}} \). Then we can define that \( f_{K,O_{\mu}} = T \pi_{K,O_{\mu}} \cdot f_{M,O_{\mu}} \) and \( u_{K,O_{\mu}} = T \pi_{K,O_{\mu}} \cdot u_{M,O_{\mu}} \); that is, \( f_{K,O_{\mu}} \) and \( u_{K,O_{\mu}} \) are the restrictions of \( f_{M,O_{\mu}} \) and \( u_{M,O_{\mu}} \) to \( K,O_{\mu} \). In consequence, the 5-tuple \((K,O_{\mu}, h_{K,O_{\mu}}, f_{K,O_{\mu}}, u_{K,O_{\mu}})\) is a \( J \)-nonholonomic \( R_o \)-reduced distributional RCH system of the nonholonomic RCH system with symmetry and momentum map \((T^*Q,G,\omega, J, D, H, F, W)\), as well as with a control law \( u \in W \). Thus, the geometrical formulation of the \( J \)-nonholonomic \( R_o \)-reduced distributional RCH system may be summarized as follows:

**Definition 6.8** (\( J \)-Nonholonomic \( R_o \)-reduced Distributional RCH System) Assume that the 8-tuple \((T^*Q,G,\omega, J, D, H, F, W)\) is a nonholonomic RCH system with symmetry and momentum map, where \( \omega \) is the canonical symplectic form on \( T^*Q \), and \( D \subset TQ \) is a \( D \)-completely and \( D \)-regularly nonholonomic constraint of the system, and \( D, H, F \) and \( W \) are all \( G \)-invariant. For a regular value \( \mu \in \mathfrak{g}^* \) of the momentum map \( J, \mu_O = G \cdot \mu \subset \mathfrak{g}^* \) is the \( G \)-orbit of the coadjoint action through the point \( \mu \), assume that there exists a \( J \)-nonholonomic \( R_o \)-reduced distribution \( K,O_{\mu} \), an associated non-degenerate and \( J \)-nonholonomic \( R_o \)-reduced distributional two-form \( \omega_{K,O_{\mu}} \) and a vector field \( X_{K,O_{\mu}} \) on the \( J \)-nonholonomic \( R_o \)-reduced constraint submanifold \( M_{O_{\mu}} = (M \cap J^{-1}(\mu_O))/G \), where \( M = FL(D) \), and \( M \cap J^{-1}(\mu_O) \neq \emptyset \), such that the \( J \)-nonholonomic \( R_o \)-reduced distributional Hamiltonian equation (6.8) holds, that is, \( i_{X_{K,O_{\mu}}} \omega_{K,O_{\mu}} = dh_{K,O_{\mu}} \), where \( dh_{K,O_{\mu}} \) is the restriction of \( dh_{M,O_{\mu}} \) to \( K,O_{\mu} \) and the function \( h_{K,O_{\mu}} \), and the vector fields \( f_{K,O_{\mu}} \) and \( u_{K,O_{\mu}} \) are defined above. Then the 5-tuple \((K,O_{\mu}, h_{K,O_{\mu}}, f_{K,O_{\mu}}, u_{K,O_{\mu}})\) is called a \( J \)-nonholonomic \( R_o \)-reduced distributional RCH system of the nonholonomic RCH system with symmetry and momentum map \((T^*Q,G,\omega, J, D, H, F, u)\) with a control law \( u \in W \), and \( X_{K,O_{\mu}} \) is called the \( J \)-nonholonomic \( R_o \)-reduced dynamical vector field. Denote that

\[
\dot{X}_{O_{\mu}} = X_{(K,O_{\mu}, h_{K,O_{\mu}}, f_{K,O_{\mu}}, u_{K,O_{\mu}})} = X_{K,O_{\mu}} + f_{K,O_{\mu}} + u_{K,O_{\mu}} \tag{6.9}
\]

is the dynamical vector field of the \( J \)-nonholonomic \( R_o \)-reduced distributional RCH system \((K,O_{\mu}, h_{K,O_{\mu}}, f_{K,O_{\mu}}, u_{K,O_{\mu}})\), which is the synthesis of the \( J \)-nonholonomic \( R_o \)-reduced dynamical vector field \( X_{K,O_{\mu}} \) and the vector fields \( f_{K,O_{\mu}} \) and \( u_{K,O_{\mu}} \). Under the above circumstances, we refer to \((T^*Q,G,\omega, J, D, H, F, u)\) as a \( J \)-nonholonomic regular orbit reducible RCH system with the associated distributional RCH system \((K,O_{\mu}, h_{K,O_{\mu}}, f_{K,O_{\mu}}, u_{K,O_{\mu}})\) and the \( J \)-nonholonomic \( R_o \)-reduced distributional RCH system \((K,O_{\mu}, h_{K,O_{\mu}}, f_{K,O_{\mu}}, u_{K,O_{\mu}})\).

For a given \( J \)-nonholonomic regular orbit reducible RCH system \((T^*Q,G,\omega, J, D, H, F, u)\) with the associated distributional RCH system \((K,O_{\mu}, h_{K,O_{\mu}}, f_{K,O_{\mu}}, u_{K,O_{\mu}})\) and the \( J \)-nonholonomic \( R_o \)-reduced distributional RCH system \((K,O_{\mu}, h_{K,O_{\mu}}, f_{K,O_{\mu}}, u_{K,O_{\mu}})\), by using Lemma 2.3 and 6.3, we can derive precisely the geometric constraint conditions of the \( J \)-nonholonomic \( R_o \)-reduced distributional two-form \( \omega_{K,O_{\mu}} \) for the nonholonomic reducible dynamical vector field; that is, the two types of Hamilton-Jacobi equations for the \( J \)-nonholonomic \( R_o \)-reduced distributional RCH system \((K,O_{\mu}, h_{K,O_{\mu}}, f_{K,O_{\mu}}, u_{K,O_{\mu}})\).

In the following for any \( G \)-invariant symplectic map \( \varepsilon : T^*Q \to T^*Q \), we can give the Type II Hamilton-Jacobi theorem for the \( J \)-nonholonomic \( R_o \)-reduced distributional RCH system. For convenience, the maps involved in the following theorem are shown in Diagram-7.
It is worthy of noting that the Type I Hamilton-Jacobi equation for the J-nonholonomic R_0-reduced distributional RCH system is a one-form on Q, and that λ = γ · π_Ω : T^*Q → T^*Q, and that for any symplectic map ε : T^*Q → T^*Q, denote that $\tilde{X}^\varepsilon = T\pi_Q \cdot \tilde{X} \cdot \varepsilon$, where $\tilde{X} = X_{(K_\omega K_{\pi} H_{F}, u_K)} = X_K + F_K + u_K$ is the dynamical vector field of the distributional RCH system $(K_\omega K_{\pi} H_{F}, F_K, u_K)$ corresponding to the J-nonholonomic regular orbit reducible RCH system with symmetry and a momentum map $(T^*Q, G, \omega, J, D, H, F, u)$. Moreover, assume that $\mu \in g^*$ is a regular value of the momentum map J, and that $\text{Im}(\gamma) \subset M \cap J^{-1}(\Omega_\mu)$, and that $\gamma$ and $\varepsilon$ are G-invariant, and that $\varepsilon(M \cap J^{-1}(\Omega_\mu)) \subset M \cap J^{-1}(\Omega_\mu)$. Then $\bar{\tau}_{\Omega_\mu} \cdot \tilde{X}^\varepsilon = X_{\bar{\Omega}_\mu} \cdot \bar{\varepsilon}_{\Omega_\mu}$. Here $X_{\bar{\Omega}_\mu} \cdot \bar{\varepsilon}_{\Omega_\mu}$ is the Hamiltonian vector field of the function $\bar{h}_{\bar{\Omega}_\mu} \cdot \bar{\varepsilon}_{\Omega_\mu} : T^*Q \rightarrow \mathbb{R}$, and $X_{\bar{\Omega}_\mu}$ is the J-nonholonomic R_0-reduced dynamical vector field. The equation $T\gamma^\varepsilon = X_{\bar{\Omega}_\mu} \cdot \bar{\varepsilon}_{\Omega_\mu}$, is called the Type II Hamilton-Jacobi equation for the J-nonholonomic R_0-reduced distributional RCH system $(K_\mu, \omega_{\bar{\Omega}_\mu}, \bar{h}_{\bar{\Omega}_\mu}, \bar{f}_{\bar{\Omega}_\mu}, u_{\bar{\Omega}_\mu})$.

See the proof and the more details in Wang [42].

For a given J-nonholonomic regular orbit reducible RCH system $(T^*Q, G, \omega, J, D, H, F, W)$ with the associated distributional RCH system $(\bar{K}, \omega_{\bar{K}}, H_{\bar{K}}, F_{\bar{K}}, u_{\bar{K}})$ and the J-nonholonomic R_0-reduced distributional RCH system $(\bar{K}_\mu, \omega_{\bar{\Omega}_\mu}, \bar{h}_{\bar{\Omega}_\mu}, \bar{f}_{\bar{\Omega}_\mu}, u_{\bar{\Omega}_\mu})$, we know that the nonholonomic dynamical vector field $X_K$ and the J-nonholonomic R_0-reduced dynamical vector field $X_{\bar{K}_\mu}$ are $\pi_{\Omega_\mu}$-related; that is, $X_{\bar{K}_\mu} \cdot \pi_{\Omega_\mu} = T\pi_{\Omega_\mu} \cdot X_K \cdot i_{\Omega_\mu}$. Then we can prove the following Theorem 6.10, which states the relationship between the solutions of the Type II Hamilton-Jacobi equations and the J-nonholonomic regular orbit reduction.

**Theorem 6.10** For a given J-nonholonomic regular orbit reducible RCH system $(T^*Q, G, \omega, J, D, H, F, W)$ with the associated distributional RCH system $(\bar{K}, \omega_{\bar{K}}, H_{\bar{K}}, F_{\bar{K}}, u_{\bar{K}})$ and the J-nonholonomic R_0-reduced distributional RCH system $(\bar{K}_\mu, \omega_{\bar{\Omega}_\mu}, \bar{h}_{\bar{\Omega}_\mu}, \bar{f}_{\bar{\Omega}_\mu}, u_{\bar{\Omega}_\mu})$, assume that $\gamma : T^*Q \rightarrow T^*Q$ is a one-form on Q, and that $\varepsilon : T^*Q \rightarrow T^*Q$ is a symplectic map, and that $\bar{\tau}_{\Omega_\mu} = \pi_{\Omega_\mu}(\gamma) : Q \rightarrow \mathcal{M}_{\Omega_\mu}$, and that $\bar{\varepsilon}_{\Omega_\mu} = \pi_{\Omega_\mu}(\varepsilon) : M \cap J^{-1}(\Omega_\mu) \subset T^*Q \rightarrow T^*Q$. Under the hypotheses and the notations of Theorem 6.9, then we have that $\varepsilon$ is a solution of the Type II Hamilton-Jacobi equation $T\gamma \cdot \tilde{X}^\varepsilon = X_K \cdot \varepsilon$ for the distributional RCH system $(\bar{K}, \omega_{\bar{K}}, H_{\bar{K}}, F_{\bar{K}}, u_{\bar{K}})$ if and only if $\varepsilon$ and $\bar{\varepsilon}_{\Omega_\mu}$ satisfy the Type II Hamilton-Jacobi equation $T\bar{\tau}_{\bar{\Omega}_\mu} \cdot \tilde{X}^\bar{\varepsilon} = X_{\bar{K}_\mu} \cdot \bar{\varepsilon}_{\bar{\Omega}_\mu}$ for the J-nonholonomic R_0-reduced distributional RCH system $(\bar{K}_\mu, \omega_{\bar{\Omega}_\mu}, \bar{h}_{\bar{\Omega}_\mu}, \bar{f}_{\bar{\Omega}_mu}, u_{\bar{\Omega}_\mu})$.

See the proof and the more details in Wang [42].

**Remark 6.11** It is worthy of noting that the Type I Hamilton-Jacobi equation $T\bar{\tau}_{\bar{\Omega}_\mu} \cdot \tilde{X}^\bar{\varepsilon} = X_{\bar{K}_\mu} \cdot \bar{\varepsilon}_{\bar{\Omega}_\mu}$ is the equation of the J-nonholonomic R_0-reduced differential one-form $\bar{\tau}_{\bar{\Omega}_\mu}$; and that
the Type II Hamilton-Jacobi equation $T\gamma_{\Omega_\mu} \cdot \hat{X} = X_{K_{\Omega_\mu}} \cdot \tilde{\varepsilon}_{\Omega_\mu}$ is the equation of the symplectic diffeomorphism map $\varepsilon$ and the $J$-nonholonomic $R_\omega$-reduced symplectic diffeomorphism map $\tilde{\varepsilon}_{\Omega_\mu}$. If a $J$-nonholonomic regular point and regular orbit reducible $RCH$ systems we considered $(T^*Q, G, \omega, J, D, H, F, u)$ have not any constrains, in this case, the $J$-nonholonomic $R_\rho$-reduced and $R_\omega$-reduced distributional $RCH$ systems are just the $R_\rho$-reduced $RCH$ system and the $R_\omega$-reduced $RCH$ system. From the Type I and Type II Hamilton-Jacobi theorems; that is, Theorem 6.7 and Theorem 6.9, we can get the Theorem 5.3 and Theorem 5.6, which were given in Wang [39]. It shows that Theorem 6.7 and Theorem 6.9 can be regarded as an extension of the two types of Hamilton-Jacobi theorems for the $R_\rho$-reduced $RCH$ system and the $R_\omega$-reduced $RCH$ system to that for the systems with the nonholonomic context. If the $J$-nonholonomic regular orbit reducible $RCH$ system we considered $(T^*Q, G, \omega, J, D, H, F, u)$ has not any the external force and the control; that is, $F = 0$ and $u = 0$, in this case, the $J$-nonholonomic regular orbit reducible $RCH$ system is just the $J$-nonholonomic regular orbit reducible Hamiltonian system $(T^*Q, G, \omega, J, D, H)$, and with the canonical symplectic form $\omega$ on $T^*Q$. From the Type II Hamilton-Jacobi theorem; that is, Theorem 6.9, we can get the Theorem 5.10 in León and Wang [17]. It shows that Theorem 6.9 can be regarded as an extension of the Type II Hamilton-Jacobi theorem for the $J$-nonholonomic regular orbit reducible Hamiltonian system to that for the system with the external force and the control. In particular, in this case, if the $J$-nonholonomic regular orbit reducible $RCH$ system we considered has not any constrains; that is, $F = 0$, $u = 0$ and $D = \emptyset$, then the $J$-nonholonomic regular orbit reducible $RCH$ system is just a regular orbit reducible Hamiltonian system $(T^*Q, G, \omega, J, D, H)$ with the canonical symplectic form $\omega$ on $T^*Q$, we can obtain the Type II Hamilton-Jacobi equation for the associated $R_\omega$-reduced Hamiltonian system, which is given in Wang [35]. Thus, Theorem 6.9 can be regarded as an extension of the Type II Hamilton-Jacobi theorem for a regular orbit reducible Hamiltonian system to that for the system with the external force, the control and the nonholonomic constrain.

7 Controlled Magnetic Hamiltonian System and Hamilton-Jacobi Equations

A magnetic Hamiltonian system is a canonical Hamiltonian system coupling the action of a magnetic field, so we can introduce a magnetic symplectic form and use it to define a magnetic Hamilton’s equations to describe the magnetic Hamiltonian system, and we drive precisely the geometric constraint conditions of the magnetic symplectic form for the magnetic Hamiltonian vector field; that is, the Type I and Type II Hamilton-Jacobi equations. Moreover, for the magnetic Hamiltonian system with a nonholonomic constraint, we can give a distributional magnetic Hamiltonian system, then derive its two types of Hamilton-Jacobi equations, and we generalize the above results to nonholonomic reducible magnetic Hamiltonian system with symmetry (see Wang [41]).

A controlled magnetic Hamiltonian (CMH) system is a regular controlled Hamiltonian (RCH) system coupling the action of a magnetic field, which is regarded as a magnetic Hamiltonian system with the external force and the control. By using the notation of vertical lift map of a vector along a fiber (see Marsden et al. [28] and Wang [40]), we give a good expression of the dynamical vector field of the CMH system, such that we can describe the magnetic vanishing condition and the CMH-equivalence, and derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of the CMH system; that is, the Type I and Type II Hamilton-Jacobi equations, which are an extension of the two types of Hamilton-Jacobi equations for the magnetic Hamiltonian system to that for the system with the external force and the control, and we prove that the CMH-equivalence for the CMH systems leaves the solutions of corresponding to Hamilton-Jacobi equations invariant, if the associated magnetic Hamiltonian
systems are equivalent. Moreover, we consider the CMH system with the nonholonomic constraint, and derive a distributional CMH system, which is determined by a non-degenerate distributional two-form induced from the magnetic symplectic form. Then we generalize the above results for the nonholonomic reducible CMH system with symmetry, and prove two types of Hamilton-Jacobi theorems for the nonholonomic reduced distributional CMH system (see Wang [40]). These research works reveal the deeply internal relationships of the magnetic symplectic forms, the nonholonomic constraints, the dynamical vector fields and the controls of the CMH systems.

7.1 Magnetic Hamiltonian System

Let $Q$ be an $n$-dimensional smooth manifold and $TQ$ the tangent bundle, $T^*Q$ the cotangent bundle with a canonical symplectic form $\omega$ and the projection $\pi_Q : T^*Q \to Q$ induces the map $\pi_Q^* : T^*Q \to T^*Q$. We consider the magnetic symplectic form $\omega^B = \omega - \pi_Q^*B$, where $\omega$ is the canonical symplectic form on $T^*Q$, and $B$ is the closed two-form on $Q$, and the $\pi_Q^*B$ is called a magnetic term on $T^*Q$. A magnetic Hamiltonian system is a triple $(T^*Q, \omega^B, H)$, which is a Hamiltonian system defined by the magnetic symplectic form $\omega^B$, that is, a canonical Hamiltonian system coupling the action of a magnetic field $B$. For a given Hamiltonian $H$, the dynamical vector field $X^B_H$, which is called the magnetic Hamiltonian vector field, satisfies the magnetic Hamilton’s equation, that is,

$$i_{X^B_H} \omega^B = dH.$$ 

In canonical cotangent bundle coordinates, for any $q \in Q$, $(q, p) \in T^*Q$, we have that

$$\omega = \sum_{i=1}^n dq^i \wedge dp_i, \quad B = \sum_{i,j=1}^n B_{ij} dq^i \wedge dq^j, \quad dB = 0,$$

$$\omega^B = \omega - \pi_Q^*B = \sum_{i=1}^n dq^i \wedge dp_i - \sum_{i,j=1}^n B_{ij} dq^i \wedge dq^j,$$

and the magnetic Hamiltonian vector field $X^B_H$ with respect to the magnetic symplectic form $\omega^B$ can be expressed that

$$X^B_H = \sum_{i=1}^n \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) - \sum_{i,j=1}^n B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i}.$$

See Marsden et al. [22].

In the following we can derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of a magnetic Hamiltonian system; that is, the Type I and Type II Hamilton-Jacobi equations for the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$. In order to do this, in the following we first give an important notion and prove a key lemma, which is an important tool for the proofs of two types of Hamilton-Jacobi theorem for the magnetic Hamiltonian system.

For the one-form $\gamma : Q \to T^*Q$, $d\gamma$ is a two-form on $Q$. Assume that $B$ is a closed two-form on $Q$, then we say that the $\gamma$ satisfies condition $d\gamma = -B$ if, for any $x, y \in TQ$, we have that $(d\gamma + B)(x, y) = 0$. In the following we can give a new notion.

**Definition 7.1** Assume that $\gamma : Q \to T^*Q$ is a one-form on $Q$, we say that the $\gamma$ satisfies condition that $d\gamma = -B$ with respect to $T\pi_Q : TT^*Q \to TQ$ if, for any $v, w \in TT^*Q$, we have that $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$.

From the above definition we know that if $\gamma$ satisfies condition $d\gamma = -B$, then it must satisfy condition $d\gamma = -B$ with respect to $T\pi_Q : TT^*Q \to TQ$. Conversely, if $\gamma$ satisfies condition
\(d\gamma = -B\) with respect to \(T\pi_Q: TT^*Q \to TQ\), then it may not satisfy condition \(d\gamma = -B\).

Now, we can give the following lemma, which can be regarded as an extension of the Lemma 2.3 (given by Wang [35]) to it with the nonholonomic context, and the lemma is a very important tool for our research.

**Lemma 7.2** Assume that \(\gamma: Q \to T^*Q\) is a one-form on \(Q\), and that \(\lambda = \gamma \cdot \pi_Q: T^*Q \to T^*Q\). For the magnetic symplectic form \(\omega^B = \omega - \pi_Q^*B\) on \(T^*Q\), where \(\omega\) is the canonical symplectic form on \(T^*Q\), and \(B\) is a closed two-form on \(Q\), then we have that the following two assertions hold.

(i) For any \(v, w \in TT^*Q\), we have that \(\lambda^*\omega^B(v, w) = -(d\gamma + B)(T\pi_Q(v), T\pi_Q(w))\).

(ii) For any \(v, w \in TT^*Q\), we have that \(\omega^B(T\lambda \cdot v, w) = \omega^B(v, w - T\lambda \cdot w) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w))\).

See the proof and the more details in Wang [40, 41].

Usually, under the impact of the magnetic term \(\pi_Q^*B\), the magnetic symplectic form \(\omega^B = \omega - \pi_Q^*B\), in general, is not the canonical symplectic form \(\omega\) on \(T^*Q\), we cannot prove the Hamilton-Jacobi theorem for a magnetic Hamiltonian system the same as in Theorem 2.1. However, using Lemma 7.2 we can derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of the magnetic Hamiltonian system; that is, the Type I and Type II Hamilton-Jacobi equations for the magnetic Hamiltonian system. For convenience, the maps involved in the theorem are shown in Diagram-8.

**Theorem 7.3** *(Hamilton-Jacobi Theorem for a Magnetic Hamiltonian System)* For a given magnetic Hamiltonian system \((T^*Q, \omega^B, H)\) with a magnetic symplectic form \(\omega^B = \omega - \pi_Q^*B\) on \(T^*Q\), where \(\omega\) is the canonical symplectic form on \(T^*Q\) and \(B\) is a closed two-form on \(Q\), assume that \(\gamma: Q \to T^*Q\) is a one-form on \(Q\), and that \(\lambda = \gamma \cdot \pi_Q: T^*Q \to T^*Q\), and that for any symplectic map \(\varepsilon: T^*Q \to T^*Q\) with respect to \(\omega^B\), denote that \(X^\gamma = T\pi_Q \cdot X^B_H \cdot \gamma\) and \(X^\varepsilon = T\pi_Q \cdot X^B_H \cdot \varepsilon\), where \(X^B_H\) is the dynamical vector field of the magnetic Hamiltonian system \((T^*Q, \omega^B, H)\); that is, the magnetic Hamiltonian vector field. Then the following two assertions hold:

(i) If the one-form \(\gamma: Q \to T^*Q\) satisfies the condition that \(d\gamma = -B\) with respect to \(T\pi_Q: TT^*Q \to TQ\), then \(\gamma\) is a solution of the Type I Hamilton-Jacobi equation \(T\gamma \cdot X^\gamma = X^B_H \cdot \gamma\) for the magnetic Hamiltonian system \((T^*Q, \omega^B, H)\).

(ii) The \(\varepsilon\) is a solution of the Type II Hamilton-Jacobi equation \(T\gamma \cdot X^\varepsilon = X^B_H \cdot \varepsilon\) if and only if it is a solution of the equation \(T\varepsilon \cdot X^B_H = T\lambda \cdot X^B_H \cdot \varepsilon\), where \(X^B_H \cdot \varepsilon \in TT^*Q\) is the magnetic Hamiltonian vector field of the function \(H \cdot \varepsilon : T^*Q \to \mathbb{R}\).

See the proof and the more details in Wang [41]. It is worthy of noting that the Type I Hamilton-Jacobi equation \(T\gamma \cdot X^\gamma = X^B_H \cdot \gamma\) is the equation of the differential one-form \(\gamma\), and that the Type II Hamilton-Jacobi equation \(T\gamma \cdot X^\varepsilon = X^B_H \cdot \varepsilon\) is the equation of the symplectic diffeomorphism map \(\varepsilon\) with respect to \(\omega^B\). If \(B = 0\), in this case the magnetic symplectic form \(\omega^B\) is just the canonical symplectic form \(\omega\) on \(T^*Q\), and the condition that the one-form \(\gamma: Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) with respect to \(T\pi_Q: TT^*Q \to TQ\), becomes that \(\gamma\) is closed with respect to \(T\pi_Q: TT^*Q \to TQ\). Thus, from above Theorem 7.3, we can obtain Theorem 2.5 and Theorem
In order to describe the impact of the different geometric structures and the constraints for the dynamics of a Hamiltonian system, for the magnetic Hamiltonian system with a nonholonomic constraint, we can define a distributional magnetic Hamiltonian system by analyzing carefully the structure of the nonholonomic dynamical vector field, and derive precisely its two types of Hamilton-Jacobi equations. Moreover, we generalize the above results to the nonholonomic reducible magnetic Hamiltonian system with symmetry. We give the definition of a nonholonomic reduced distributional magnetic Hamiltonian system, and prove the two types of Hamilton-Jacobi theorems for the system, which are the development of the Type I and Type II Hamilton-Jacobi theorems for the nonholonomic reduced distributional Hamiltonian system given in León and Wang \cite{17}; See Wang \cite{41}for more details.

### 7.2 Controlled Magnetic Hamiltonian System

In order to describe the impact of the different geometric structures for the dynamics and the Hamilton-Jacobi equations of an RCH system, considering the external force and the control, we can define a kind of controlled magnetic Hamiltonian (CMH) system on $T^*Q$ as follows.

**Definition 7.4 (CMH System)** A controlled magnetic Hamiltonian (CMH) system on $T^*Q$ is a 5-tuple $(T^*Q, \omega^B, H, F, W)$, which is a magnetic Hamiltonian system $(T^*Q, \omega^B, H)$ with the external force $F$ and the control set $W$, where $F : T^*Q \to T^*Q$ is the fiber-preserving map, and $W \subset T^*Q$ is a fiber submanifold, which is called the control subset.

From the above Definition 2.2 and Definition 7.4 we know that a CMH system on $T^*Q$ is also an RCH system on $T^*Q$, however, its symplectic structure is given by a magnetic symplectic form, and the set of the CMH systems on $T^*Q$ is a subset of the set of the RCH systems on $T^*Q$. However, the subset of CMH systems is not closed under the actions of the external force and the control, because the magnetic may be vanishing.

When a feedback control law $u : T^*Q \to W$ is chosen, the 5-tuple $(T^*Q, \omega^B, H, F, u)$ is a regular closed-loop dynamic system. In order to describe the dynamics of the CMH system $(T^*Q, \omega^B, H, F, W)$ with a control law $u$, by using the notation of vertical lift map of a vector along a fiber; (also see Marsden et al. \cite{28}), we can give a good expression of the dynamical vector field of the CMH system as follows: for a given CMH system $(T^*Q, \omega^B, H, F, W)$, the dynamical vector field of the associated magnetic Hamiltonian system $(T^*Q, \omega^B, H)$ is $X^B_H$, which satisfies the equation $1_X^B_H \omega^B = dH$. If considering the external force $F : T^*Q \to T^*Q$, by using the notation of vertical lift map of a vector along a fiber, the change of $X^B_H$ under the action of $F$ is such that

$$\text{vlift}(F)X^B_H(\alpha_x) = \text{vlift}((TFX^B_H)(F(\alpha_x)), \alpha_x) = (TFX^B_H)^\gamma(\alpha_x),$$

where $\alpha_x \in T^*_xQ$, $x \in Q$ and $\gamma$ is a straight line in $T^*_xQ$ connecting $F_x(\alpha_x)$ and $\alpha_x$. In the same way, when a feedback control law $u : T^*Q \to W$ is chosen, the change of $X^B_H$ under the action of $u$ is such that

$$\text{vlift}(u)X^B_H(\alpha_x) = \text{vlift}((TuX^B_H)(u(\alpha_x)), \alpha_x) = (TuX^B_H)^\gamma(\alpha_x).$$

As a consequence, we can give an expression of the dynamical vector field of the CMH system as follows:

**Theorem 7.5** The dynamical vector field of the CMH system $(T^*Q, \omega^B, H, F, W)$ with a control law $u$ is the synthesis of the magnetic Hamiltonian vector field $X^B_H$ and its changes under the actions...
of the external force $F$ and the control law $u$; that is,

$$X_{(T^*Q,\omega^B,H,F,u)}(\alpha_x) = X^B_H(\alpha_x) + \text{vlift}(F)X^B_H(\alpha_x) + \text{vlift}(u)X^B_H(\alpha_x),$$

(7.1)

for any $\alpha_x \in T^*_xQ$, $x \in Q$. For convenience, that is simply written as

$$X_{(T^*Q,\omega^B,H,F,u)} = X^B_H + \text{vlift}(F)^B + \text{vlift}(u)^B.$$  

(7.2)

Where $\text{vlift}(F)^B = \text{vlift}(F)X^B_H$, and $\text{vlift}(u)^B = \text{vlift}(u)X^B_H$ are the changes of $X^B_H$ under the actions of $F$ and $u$. We also denote that $\text{vlift}(W)^B = \bigcup\{\text{vlift}(u)X^B_H | u \in W\}$.

From the expression (7.2) of the dynamical vector field of a CMH system, we know that under the actions of the external force $F$ and the control law $u$, in general, the dynamical vector field may not be magnetic Hamiltonian, and hence the CMH system may not be yet a magnetic Hamiltonian system. However, it is a dynamical system closed relative to a magnetic Hamiltonian system, and it can be explored and studied by the extending methods for the external force and the control in the study of the magnetic Hamiltonian system.

For the magnetic Hamiltonian system $(T^*Q,\omega^B,H)$, its magnetic Hamiltonian vector field $X^B_H$ satisfies the equation $i_{X^B_H}\omega = dH$, and for the associated canonical Hamiltonian system $(T^*Q,\omega,H)$, its canonical Hamiltonian vector field $X_H$ satisfies the equation $i_{X_H}\omega = dH$. Denote that the vector field $X^0 = X^B_H - X_H$, and, from the magnetic symplectic form $\omega^B = \omega - \pi^*_Q B$, we have that

$$i_{X^0}\omega = i_{(X^B_H - X_H)}\omega = i_{X^B_H}\omega - i_{X_H}\omega = i_{X^B_H}(\omega^B + \pi^*_Q B) - i_{X_H}\omega = i_{X^B_H}(\pi^*_Q B).$$

Thus, $X^0$ is called the magnetic vector field and that $i_{X^0}\omega = i_{X^B_H}(\pi^*_Q B)$ is called the magnetic equation, which is determined by the magnetic term $\pi^*_Q B$ on $T^*Q$. When $B = 0$, then $X^0 = 0$, the magnetic equation holds trivially. For the CMH system $(T^*Q,\omega^B,H,F,W)$, from the expression (7.2) of its dynamical vector field, we have that

$$X_{(T^*Q,\omega^B,H,F,u)} = X_H + X^0 + \text{vlift}(F)^B + \text{vlift}(u)^B.$$  

(7.3)

If we choose the external force $F$ and the control law $u$, such that

$$X^0 + \text{vlift}(F)^B + \text{vlift}(u)^B = 0,$$  

(7.4)

then from (7.3) we have that $X_{(T^*Q,\omega^B,H,F,u)} = X_H$; that is, in this case the dynamical vector field of the CMH system is just the canonical Hamiltonian vector field itself, and the motion of the CMH system is just the same as the motion of the canonical Hamiltonian system without the actions of the magnetic, the external force and the control. Thus, the condition (7.4) is called the magnetic vanishing condition for the CMH system $(T^*Q,\omega^B,H,F,W)$, and we have the following theorem:

**Theorem 7.6** If the external force $F$ and the control law $u$ for the CMH system $(T^*Q,\omega^B,H,F,u)$ satisfy the magnetic vanishing condition (7.4), then the dynamical vector field $X_{(T^*Q,\omega^B,H,F,u)}$ of the CMH system is just the canonical Hamiltonian vector field $X_H$ for the associated canonical Hamiltonian system $(T^*Q,\omega,H)$.

For a given CMH system $(T^*Q,\omega^B,H,F,W)$ on $T^*Q$, by using the above Lemma 7.2, we can derive precisely the geometric constraint conditions of the magnetic symplectic form $\omega^B$ for the dynamical vector field $X_{(T^*Q,\omega^B,H,F,u)}$ of the CMH system with a control law $u$; that is, the Type I and Type II Hamilton-Jacobi equations for the CMH system. For convenience, the maps involved in the theorem are shown in Diagram-9.
**Theorem 7.7** (Hamilton-Jacobi Theorem for a CMH System) For a given CMH system \((T^*Q, \omega^B, H)\) with a magnetic symplectic form \(\omega^B = \omega - \pi_\gamma^2 \beta\) on \(T^*Q\), where \(\omega\) is the canonical symplectic form on \(T^*Q\) and \(B\) is a closed two-form on \(Q\), assume that \(\gamma : Q \to T^*Q\) is a one-form on \(Q\), and that \(\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q\), and that for any symplectic map \(\varepsilon : T^*Q \to T^*Q\) with respect to \(\omega^B\), denote that \(\tilde{X}_\gamma = T\pi_Q \cdot \tilde{X} \cdot \gamma\) and \(\tilde{X}_\varepsilon = T\pi_Q \cdot \tilde{X} \cdot \varepsilon\), where \(\tilde{X} = X_\gamma(T^*Q, \omega^B, H, F, W)\) is the dynamical vector field of the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \(u\). Then the following two assertions hold:

(i) If the one-form \(\gamma : Q \to T^*Q\) satisfies the condition that \(d\gamma = -B\) with respect to \(T\pi_Q : TT^*Q \to TQ\), then \(\gamma\) is a solution of the Type I Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}_\gamma = X^B_H \cdot \gamma\), where \(X^B_H\) is the magnetic Hamiltonian vector field of the associated magnetic Hamiltonian system \((T^*Q, \omega^B, H)\).

(ii) The \(\varepsilon\) is a solution of the Type II Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}_\varepsilon = X^B_H \cdot \varepsilon\) if and only if it is a solution of the equation \(T\varepsilon \cdot X^B_H = T\lambda \cdot \tilde{X} \cdot \varepsilon\), where \(X^B_H\) and \(X^B_H \cdot \varepsilon \in TT^*Q\) are the magnetic Hamiltonian vector fields of the functions \(H\) and \(H \cdot \varepsilon : T^*Q \to \mathbb{R}\), respectively.

See the proof and the more details in Wang [40].

**Remark 7.8** It is worthy of noting that the Type I Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}_\gamma = X^B_H \cdot \gamma\) is the equation of the differential one-form \(\gamma\), and that the Type II Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}_\varepsilon = X^B_H \cdot \varepsilon\) is the equation of the symplectic diffeomorphism map \(\varepsilon\) with respect to \(\omega^B\). If the external force and control of the CMH system \((T^*Q, \omega^B, H, F, W)\) are both zeros, that is, \(F = 0\) and \(W = \emptyset\), in this case the CMH system is just a magnetic Hamiltonian system \((T^*Q, \omega^B, H)\), and from the Theorem 7.7, we can obtain the two types of Hamilton-Jacobi equations for the associated magnetic Hamiltonian system; that is, Theorem 7.3 (see Wang [41]). Thus, Theorem 7.7 can be regarded as an extension of the two types of Hamilton-Jacobi equations for a magnetic Hamiltonian system to that for the system with the external force and the control. If \(B = 0\), in this case the magnetic symplectic form \(\omega^B\) is just the canonical symplectic form \(\omega\) on \(T^*Q\), and the condition that the one-form \(\gamma : Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) with respect to \(T\pi_Q : TT^*Q \to TQ\) becomes that \(\gamma\) is closed with respect to \(T\pi_Q : TT^*Q \to TQ\). Thus, from the Theorem 7.7, we can obtain Theorem 5.1 (see Wang [39]). Moreover, if \(B = 0\), \(F = 0\) and \(W = \emptyset\), then, from the Theorem 7.7, we can obtain Theorem 2.5 and Theorem 2.6 in Wang [35]. Thus, Theorem 7.7 can be regarded as an extension of two types of Hamilton-Jacobi equations for a canonical Hamiltonian system to that for the system with the magnetic, the external force and the control.

In the same way, for two given magnetic Hamiltonian systems \((T^*Q_i, \omega^B_i, H_i), i = 1, 2\), we say them to be equivalent, if there exists a diffeomorphism \(\varphi : Q_1 \to Q_2\), which is symplectic with respect to their magnetic symplectic forms, such that their magnetic Hamiltonian vector fields \(X^B_{H_i}, i = 1, 2\) satisfy the condition \(X^B_{H_1} \cdot \varphi^* = T(\varphi^*)X^B_{H_2}\).

For two given CMH systems \((T^*Q_i, \omega^B_i, H_i, F_i, W_i), i = 1, 2\), we also want to define their equivalence, that is, to look for a diffeomorphism \(\varphi : Q_1 \to Q_2\), such that \(X_\gamma(T^*Q_1, \omega^B_{H_1, F_1, W_1}) \cdot \varphi^* = T(\varphi^*)X_\gamma(T^*Q_2, \omega^B_{H_2, F_2, W_2})\). However, it is worthy of noting that because the magnetic may be vanishing and the set of CMH systems, as a subset of the set of RCH systems, is not closed under the
actions of the external force and the control, so we cannot use the RCH-equivalence for the CMH systems. On the other hand, when a CMH system is given, the force map $F$ is determined, but the feedback control law $u : T^*Q \rightarrow W$ could be chosen. In order to describe the feedback control law to modify the structures of the CMH systems, the controlled magnetic Hamiltonian matching conditions and CMH-equivalence are induced as follows:

**Definition 7.9 (CMH-equivalence)** Suppose that we have two CMH systems $(T^*Q_1, \omega^B_1, H_1, F_1, W_1)$, $(T^*Q_2, \omega^B_2, H_2, F_2, W_2)$, if there exists a diffeomorphism $\varphi : Q_1 \rightarrow Q_2$, such that the following CMH-equivalence hold:

**CMH-1:** The control subsets $W_i$, $i = 1, 2$ satisfy the condition $W_1 = \varphi^*(W_2)$, where the map $\varphi^* = T^*\varphi : T^*Q_2 \rightarrow T^*Q_1$ is cotangent lifted map of $\varphi$.

**CMH-2:** For each control law $u_1 : T^*Q_1 \rightarrow W_1$, there exists the control law $u_2 : T^*Q_2 \rightarrow W_2$, such that the two closed-loop dynamical systems produce the same dynamical vector fields; that is, $X(T^*Q_1, \omega^B_1, H_1, F_1, u_1)$ and $X(T^*Q_2, \omega^B_2, H_2, F_2, u_2)$, where the map $T(\varphi^*) : TT^*Q_2 \rightarrow TT^*Q_1$ is the tangent map of $\varphi^*$.

From the expression (7.1) of the dynamical vector field of the CMH system and the condition $X(T^*Q_1, \omega^B_1, H_1, F_1, u_1) \cdot \varphi^* = T(\varphi^*)X(T^*Q_2, \omega^B_2, H_2, F_2, u_2)$, we have that

$$X_B = vlift((F_1)X_B + vlift(u_1)X_B) \cdot \varphi^* = T(\varphi^*)[X_B + vlift(F_2)X_B + vlift(u_2)X_B].$$

By using the notation of vertical lift map of a vector along a fiber, for $\alpha_x \in T^*_xQ_2$, $x \in Q_2$, we have that

$$T(\varphi^*)vlift(F_2)X_B^H(\alpha_x) = T(\varphi^*)vlift(TF_2X_B^H)(F_2(\alpha_x)), \alpha_x)$$

$$= vlift(T(\varphi^*) \cdot TF_2 \cdot (\varphi^*)X_B^H(\varphi^*F_2\varphi_x(\varphi^*\alpha_x)), \varphi^*\alpha)$$

$$= vlift(T(\varphi^*F_2\varphi_x)X_B^H(\varphi^*F_2\varphi_x(\varphi^*\alpha_x)), \varphi^*\alpha)$$

$$= vlift(\varphi^*F_2\varphi_x)X_B^H(\varphi^*\alpha_x),$$

where the map $\varphi_x = (\varphi^{-1})^* : T^*Q_1 \rightarrow T^*Q_2$. In the same way, we have that $T(\varphi^*)vlift(u_2)X_B^H = vlift(\varphi^*u_2\varphi_x)X_B^H \cdot \varphi^*$. Note that $vlift(F_1)^B = vlift(F)X_B^B$, and $vlift(u)^B = vlift(u)X_B^B$, and hence we have that the explicit relation between the two control laws $u_i \in W_i$, $i = 1, 2$ in CMH-2 is given by

$$(vlift(u_1)^B - vlift(\varphi^*u_2\varphi_x)^B) \cdot \varphi^*$$

$$= -X_B^H \cdot \varphi^* + T(\varphi^*)(X_B^H) + (vlift(F_1)^B + vlift(\varphi^*F_2\varphi_x)^B) \cdot \varphi^*.$$  \hspace{1cm} (7.5)

From the above relation (7.5) we know that when two CMH systems $(T^*Q_i, \omega^B_i, H_i, F_i, W_i)$, $i = 1, 2$, are CMH-equivalent with respect to $\varphi^*$, the associated magnetic Hamiltonian systems $(T^*Q_i, \omega^B_i, H_i)$, $i = 1, 2$, may not be equivalent with respect to $\varphi^*$.

On the other hand, note that the magnetic vector field $X^0 = X_B^H - X_H$, from (7.3) and (7.5) we have that

$$(vlift(u_1)^B - vlift(\varphi^*u_2\varphi_x)^B) \cdot \varphi^*$$

$$= -(X_B^H + X_H^B \cdot \varphi^* + T(\varphi^*)(X_H^H + X_0^B) + (vlift(F_1)^B + vlift(\varphi^*F_2\varphi_x)^B) \cdot \varphi^*.$$  

and hence, we have that

$$(X_0^B + vlift(F_1)^B + vlift(u_1)^B) \cdot \varphi^*$$

$$= -X_H^H \cdot \varphi^* + T(\varphi^*)(X_H^H + vlift(F_2)^B + vlift(u_2)^B).$$  \hspace{1cm} (7.6)
If the associated canonical Hamiltonian systems \((T^*Q_i, \omega_i, H_i), i = 1, 2\), are also equivalent with respect to \(\varphi^*\); that is, \(T(\varphi^*) \cdot X_{H_2} = X_{H_1} \cdot \varphi^*\). In this case, from (7.4) and (7.6), we have the following theorem.

**Theorem 7.10** Suppose that two CMH systems \((T^*Q_i, \omega^B_i, H_i, F_i, W_i), i = 1, 2\), are CMH-equivalent with respect to \(\varphi^*\), and the associated canonical Hamiltonian systems \((T^*Q_i, \omega_i, H_i), i = 1, 2\), are also equivalent with respect to \(\varphi^*\). Then we have the following fact that if one system satisfies the magnetic vanishing condition, then another CMH-equivalent system must satisfy the associated magnetic vanishing condition.

Moreover, considering the CMH-equivalence of the CMH systems, we can obtain the following Theorem 7.11, which states that the solutions of the two types of Hamilton-Jacobi equations for the CMH systems leave invariant under the conditions of CMH-equivalence, if the associated magnetic Hamiltonian systems are equivalent.

**Theorem 7.11** Suppose that two CMH systems \((T^*Q_i, \omega^B_i, H_i, F_i, W_i), i = 1, 2\), are CMH-equivalent with an equivalent map \(\varphi : Q_1 \rightarrow Q_2\), and the associated magnetic Hamiltonian systems \((T^*Q_i, \omega^B_i, H_i), i = 1, 2\), are also equivalent with respect to \(\varphi^*\), under the hypotheses and the notations of Theorem 7.7, we have that the following two assertions hold:

(i) If the one-form \(\gamma_2 : Q_2 \rightarrow T^*Q_2\) satisfies the condition that \(d\gamma_2 = -B_2\) with respect to \(T\pi_{Q_2} : TT^*Q_2 \rightarrow TQ_2\), then \(\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi : Q_1 \rightarrow T^*Q_1\) satisfies also the condition that \(d\gamma_1 = -B_1\) with respect to \(T\pi_{Q_1} : TT^*Q_1 \rightarrow TQ_1\), and hence, it is a solution of the Type I Hamilton-Jacobi equation for the CMH system \((T^*Q_1, \omega^B_1, H_1, F_1, W_1)\). Vice versa.

(ii) If the symplectic map \(\varepsilon_2 : T^*Q_2 \rightarrow T^*Q_2\) with respect to \(\omega^B_2\) is a solution of the Type II Hamilton-Jacobi equation for the CMH system \((T^*Q_2, \omega^B_2, H_2, F_2, W_2)\), then \(\varepsilon_1 = \varphi^* \cdot \varepsilon_2 \cdot \varphi^* : T^*Q_1 \rightarrow T^*Q_1\) is a symplectic map with respect to \(\omega^B_1\), and it is a solution of the Type II Hamilton-Jacobi equation for the CMH system \((T^*Q_1, \omega^B_1, H_1, F_1, W_1)\). Vice versa.

See the proof and the more details in Wang [40].

In order to describe the impact of the different group structure of symmetry for the dynamics of a reducible RCH system, we consider the regular point reduction of the CMH system \((T^*Q, \mathcal{H}, \omega^B, H, F, W)\) with symmetry of the Heisenberg group \(\mathcal{H}\). Here the configuration space \(Q = \mathcal{H} \times V, \mathcal{H} = \mathbb{R}^2 \oplus \mathbb{R}\), and \(V\) is a \(k\)-dimensional vector space, and the cotangent bundle \(T^*Q\) with the magnetic symplectic form \(\omega^B = \omega_0 - \pi_1^*B\), where \(\omega_0\) is the usual canonical symplectic form on \(T^*Q\), and \(B = \pi_1^*B\) is the closed two-form on \(Q\), \(B\) is a closed two-form on \(\mathcal{H}\) and the projection \(\pi_1 : Q = \mathcal{H} \times V \rightarrow \mathcal{H}\) induces the map \(\pi_1^* : T^*\mathcal{H} \rightarrow T^*Q\). We note that there is a magnetic term on the cotangent bundle of the Heisenberg group \(\mathcal{H}\), which is related to a curvature two-form of a mechanical connection determined by the reduction of center action of the Heisenberg group \(\mathcal{H}\) (see Marsden et al. [23]), such that we can define a controlled magnetic Hamiltonian system with symmetry of the Heisenberg group \(\mathcal{H}\), and study the regular point reduction of this system. Since a CMH system is also a RCH system, but its symplectic structure is given by a magnetic symplectic form. Thus, the set of the CMH systems with symmetries is a subset of the set of the RCH systems with symmetries, and the subset is not complete under the regular point reduction of the RCH system, because the magnetic may be vanishing for the reduced RCH system. It is worthy of noting that it is different from the regular point reduction of an RCH system defined on a cotangent bundle with the canonical structure, the regular point reduction of a CMH system reveals the deeper relationship of the intrinsic geometrical structures of the RCH systems on a cotangent bundle. See Wang [37] for more details.
7.3 Nonholonomic Controlled Magnetic Hamiltonian System

A nonholonomic CMH system is the 6-tuple \((T^*Q, \omega^B, D, H, F, W)\), which is a CMH system with a \(D\)-completely and \(D\)-regularly nonholonomic constraint \(D \subset TQ\). Under the restriction given by constraint, in general, the dynamical vector field of a nonholonomic CMH system may not be magnetic Hamiltonian, however the system is a dynamical system closely related to a magnetic Hamiltonian system. By analyzing carefully the structure for the nonholonomic dynamical vector field, we give a geometric formulation of the distributional CMH system, which is determined by a non-degenerate distributional two-form induced from the magnetic symplectic form. Moreover, for the nonholonomic CMH system \((T^*Q, \omega^B, D, H, F, u)\) with a control law \(u\) and with an associated distributional CMH system \((K, \omega^B, H_K, F_K^B, u^B_K)\), we can derive precisely the geometric constraint conditions of the non-degenerate, and nonholonomic reduced distributional two-form for the dynamical vector field \(X^B_{(K, \omega^B, H_K, F_K^B, u^B_K)}\); that is, the two types of Hamilton-Jacobi equations for the distributional CMH system \((K, \omega^B, H_K, F_K^B, u^B_K)\); see Wang [?] for more details.

In the following we consider the nonholonomic CMH system with symmetry. By analyzing carefully the structure of the reducible dynamical vector field of the nonholonomic CMH system, we give a geometric formulation of the nonholonomic reduced distributional CMH system. Moreover, we derive precisely the geometric constraint conditions of the non-degenerate, and nonholonomic reduced distributional two-form for the nonholonomic reducible dynamical vector field, that is, the two types of Hamilton-Jacobi equations for the nonholonomic reduced distributional CMH system, which are an extension of the two types of Hamilton-Jacobi equations for the nonholonomic reduced distributional Hamiltonian system given in León and Wang [17].

Assume that the Lie group \(G\) acts smoothly on the manifold \(Q\) by the left, and that we also consider the natural lifted actions on \(TQ\) and \(T^*Q\), and assume that the cotangent lifted left action \(\Phi^T : G \times T^*Q \to T^*Q\) is free, proper and symplectic with respect to the magnetic symplectic form \(\omega^B\) on \(T^*Q\). Then the orbit space \(T^*Q/G\) is a smooth manifold and the canonical projection \(\pi_G : T^*Q \to T^*Q/G\) is a surjective submersion. For the cotangent lifted left action \(\Phi^T : G \times T^*Q \to T^*Q\), assume that \(H : T^*Q \to \mathbb{R}\) is a \(G\)-invariant Hamiltonian, and that the fiber-preserving map \(F : T^*Q \to T^*Q\) and the control subset \(W\) of \(T^*Q\) are both \(G\)-invariant, and that the \(D\)-completely and \(D\)-regularly nonholonomic constraint \(D \subset TQ\) is a \(G\)-invariant distribution for the tangent lifted left action \(\Phi^L : G \times TQ \to TQ\); that is, the tangent of group action maps \(D_q\) to \(D_{qg}\) for any \(q \in Q\). A nonholonomic CMH system with symmetry is 7-tuple \((T^*Q, G, \omega^B, D, H, F, W)\), which is an CMH system with symmetry and \(G\)-invariant nonholonomic constraint \(D\).

In the following we first consider the nonholonomic reduction of a nonholonomic CMH system with symmetry \((T^*Q, G, \omega^B, D, H, F, W)\). Note that the Legendre transformation \(FL : TQ \to T^*Q\) is a fiber-preserving map, and that \(D \subset TQ\) is \(G\)-invariant for the tangent lifted left action \(\Phi^T : G \times TQ \to TQ\), then the constraint submanifold \(M = FL(D) \subset T^*Q\) is \(G\)-invariant for the cotangent lifted left action \(\Phi^L : G \times T^*Q \to T^*Q\). For the nonholonomic CMH system with symmetry \((T^*Q, G, \omega^B, D, H, F, W)\), in the same way, we define the distribution \(M\), which is the pre-image of the nonholonomic constraints \(D\) for the map \(\pi_Q : TT^*Q \to TQ\); that is, \(\pi = (T\pi_Q)^{-1}(D)\), and the distribution \(K = FL\cap TM\). Moreover, we can also define the distribution two-form \(\omega^B_K\), which is induced from the magnetic symplectic form \(\omega^B\) on \(T^*Q\); that is, \(\omega^B_K = T_\pi \omega^B_M\), and \(\omega^B_M = i_M^*\omega^B\). If the admissibility condition \(\text{dim}M \neq \text{rank}F\) and the compatibility condition \(TM \cap F^\perp = \{0\}\) hold, then \(\omega^B_K\) is non-degenerate as a bilinear form on each fibre of \(K\), there exists a vector field \(X^B_K\) on \(M\) which takes values in the constraint distribution \(K\), such that for
the function \( H_K \), the following distributional magnetic Hamiltonian equation holds, that is,

\[
i_{X_K^B} \omega_K^B = dH_K, \tag{7.7}
\]

where the function \( H_K \) satisfies \( dH_K = \tau_K \cdot dH_M \), and \( H_M = \tau_M \cdot H \) is the restriction of \( H \) to \( M \), and from the equation (7.7), we have that \( X_K^B = \tau_K \cdot X_H^B \).

In the following we define that the quotient space \( \tilde{M} = M/G \) of the \( G \)-orbit in \( M \) is a smooth manifold with projection \( \pi_{/G} : M \rightarrow \tilde{M}(\subset T^*Q/G) \), which is a surjective submersion. The reduced magnetic symplectic form \( \omega_{M}^B = \pi_{/G}^* \omega_{\tilde{M}}^B \) on \( \tilde{M} \) is induced from the magnetic symplectic form \( \omega_{\tilde{M}}^B = i^*_M \omega^B \) on \( M \). Since \( G \) is the symmetry group of the system \((T^*Q,G,\omega^B,\mathcal{D},H,F,W)\), all intrinsically defined vector fields and distributions are pushed down to \( \tilde{M} \). In particular, the vector field \( X_M^B \) on \( M \) is pushed down to a vector field \( \bar{X}_\tilde{M}^B = T\pi_{/G} \cdot X_M^B \), and the distribution \( K \) is pushed down to \( \tilde{K} = T\pi_{/G} \cdot K \), and the Hamiltonian \( H \) is pushed down to \( h_{M} \), such that \( h_M \cdot \pi_{/G} = \tau_M \cdot H \). However, \( \omega_K^B \) need not to be pushed down to a distributional two-form defined on \( T\pi_{/G} \cdot K \), despite of the fact that \( \omega_K^B \) is \( G \)-invariant. This is because there may be infinitesimal symmetry \( \eta_K \) that lies in \( M \), such that \( i_{\eta_K} \omega_K^B \neq 0 \). From Bates and Šniatycki [4], we know that in order to eliminate this difficulty, \( \omega_K^B \) is restricted to a sub-distribution \( \mathcal{U} \) of \( K \) defined by

\[
\mathcal{U} = \{ u \in K \mid \omega_K^B(u,v) = 0, \quad \forall v \in \mathcal{V} \cap K \},
\]

where \( \mathcal{V} \) is the distribution on \( M \) tangent to the orbits of \( G \) in \( M \) and it is spanned by the infinitesimal symmetries. Clearly, \( \mathcal{U} \) and \( \mathcal{V} \) are both \( G \)-invariant, project down to \( \tilde{M} \) and \( T\pi_{/G} \cdot \mathcal{V} = 0 \), and define the distribution \( \tilde{K} \) by \( \tilde{K} = T\pi_{/G} \cdot \mathcal{U} \). Moreover, we take that \( \pi_{/G}^* \omega_{\tilde{M}}^B = \tau_{\tilde{M}} \cdot \omega_{\tilde{M}}^B \) is the restriction of the induced magnetic symplectic form \( \omega_{\tilde{M}}^B \) on \( T\tilde{M} \) fibrewise to the distribution \( \mathcal{U} \), where \( \tau_{\tilde{M}} \) is the restriction map to distribution \( \mathcal{U} \), and the \( \omega_{\tilde{M}}^B \) is pushed down to a distributional two-form \( \omega_{\tilde{K}}^B \) on \( \tilde{K} \), such that \( \tau_{\tilde{K}} \omega_{\tilde{K}}^B = \omega_{\mathcal{U}}^B \). We know that distributional two-form \( \omega_{\tilde{K}}^B \) is not a "true two-form" on a manifold, which is called the nonholonomic reduced distributional two-form to avoid any confusion.

From the above construction we know that, if the admissibility condition \( \dim \tilde{M} = \text{rank} \mathcal{F} \) and the compatibility condition \( TM \cap \mathcal{F}^\perp = \{0\} \) hold, where \( \mathcal{F}^\perp \) denotes the symplectic orthogonal of \( \mathcal{F} \) with respect to the reduced magnetic symplectic form \( \omega_{\tilde{M}}^B \), then the nonholonomic reduced distributional two-form \( \omega_{\tilde{K}}^B \) is non-degenerate as a bilinear form on each fibre of \( \tilde{K} \), and hence there exists a vector field \( X_{\tilde{K}}^B \) on \( \tilde{M} \) which takes values in the constraint distribution \( \tilde{K} \), such that the reduced distributional magnetic Hamiltonian equation holds, that is,

\[
i_{X_{\tilde{K}}^B} \omega_{\tilde{K}}^B = dh_{\tilde{K}}, \tag{7.8}
\]

where \( dh_{\tilde{K}} \) is the restriction of \( dh_M \) to \( \tilde{K} \) and the function \( h_{\tilde{K}} : \tilde{M}(\subset T^*Q/G) \rightarrow \mathbb{R} \) satisfies \( dh_{\tilde{K}} = \tau_{\tilde{K}} \cdot dh_M \), and \( h_{\tilde{M}} \cdot \pi_{/G} = H_{\tilde{M}} \) and \( H_{\tilde{M}} \) is the restriction of the Hamiltonian function \( H \) to \( \tilde{M} \), and the function \( h_{\tilde{M}} : \tilde{M}(\subset T^*Q/G) \rightarrow \mathbb{R} \). In addition, from the distributional magnetic Hamiltonian equation (7.7), \( \pi_{/G} \cdot X_{\tilde{K}}^B \), we have that \( X_{\tilde{K}}^B = \tau_{\tilde{K}} \cdot X_{H}^B \), and from the reduced distributional magnetic Hamiltonian equation (7.8), \( \pi_{/G} \cdot X_{\tilde{K}}^B = dh_{\tilde{K}} \), we have that \( X_{\tilde{K}}^B = \tau_{\tilde{K}} \cdot X_{h_{\tilde{K}}}^B \), where \( X_{h_{\tilde{K}}}^B \) is the magnetic Hamiltonian vector field of the function \( h_{\tilde{K}} \) with respect to the reduced magnetic symplectic form \( \omega_{\tilde{M}}^B \), and the vector fields \( X_{\tilde{K}}^B \) and \( X_{\tilde{K}}^B \) are \( \pi_{/G} \)-related, that is, \( X_{\tilde{K}}^B \cdot \pi_{/G} = T\pi_{/G} \cdot X_{\tilde{K}}^B \).

Moreover, considering the external force \( F \) and control subset \( W \), and we define the vector fields \( F_{\tilde{K}}^B = \tau_{\tilde{K}} \cdot \text{vlift}(F_M)X_{H}^B \), and for a control law \( u \in W \), \( u_{\tilde{K}}^B = \tau_{\tilde{K}} \cdot \text{vlift}(u_M)X_{H}^B \), where \( F_M = \tau_M \cdot F \) and \( u_M = \tau_M \cdot u \) are the restrictions of \( F \) and \( u \) to \( M \), that is, \( F_{\tilde{K}}^B \) and \( u_{\tilde{K}}^B \) are the
restrictions of the changes of magnetic Hamiltonian vector field $X^B_H$ under the actions of $F_M$ and $u_M$ to $K$, then the 5-tuple $(K, \omega^K_B, H_K, F^K_B, u^K_B)$ is a distributional CMH system corresponding to the nonholonomic CMH system with symmetry $(T^*Q, G, \omega^B, D, H, F, u)$, and the dynamical vector field of the distributional CMH system can be expressed by

$$\tilde{X} = X^B_{(K, \omega^K_B, H_K, F^K_B, u^K_B)} = X^K_B + F^K_B + u^K_B,$$

(7.9)

which is the synthesis of the nonholonomic dynamical vector field $X^K_B$ and the vector fields $F^K_B$ and $u^K_B$. Assume that the vector fields $F^K_B$ and $u^K_B$ on $\mathcal{M}$ are pushed down to the vector fields $f^B_M = T\pi/G \cdot F^K_B$ and $u^B_M = T\pi/G \cdot u^K_B$ on $\mathcal{M}$. Then we define that $f^K_B = T\tau_K \cdot f^B_M$ and $u^K_B = T\tau_K \cdot u^B_M$, that is, $f^K_B$ and $u^K_B$ are the restrictions of $f^B_M$ and $u^B_M$ to $\tilde{K}$, where $\tau_K$ is the restriction map to distribution $\tilde{K}$, and $T\tau_K$ is the tangent map of $\tau_K$. Then the 5-tuple $(\tilde{K}, \omega^K_B, h^K_B, f^K_B, u^K_B)$ is a nonholonomic reduced distributional CMH system of the nonholonomic reducible CMH system with symmetry $(T^*Q, G, \omega^B, D, H, F, W)$, as well as with a control law $u \in W$. Thus, the geometrical formulation of a nonholonomic reduced distributional CMH system may be summarized as follows.

**Definition 7.12** (Nonholonomic Reduced Distributional CMH System) Assume that the 7-tuple $(T^*Q, G, \omega^B, D, H, F, W)$ is a nonholonomic reduced distributional CMH system with symmetry, where $\omega^B$ is the magnetic symplectic form on $T^*Q$, and $D \subset TQ$ is a $D$-completely and $D$-regularly nonholonomic constraint of the system, and $D$, $H$, $F$ and $W$ are all $G$-invariant. If there exists a nonholonomic reduced distribution $\tilde{K}$, an associated non-degenerate and nonholonomic reduced distributional two-form $\omega^K_B$ and a vector field $X^K_B$ on the reduced constrained submanifold $\tilde{M} = M/G$, where $M = FL(D) \subset T^*Q$, such that the nonholonomic reduced distributional magnetic Hamiltonian equation holds: that is, $i_{X^K_B} \omega^K_B = dh^K_B$, where $dh^K_B$ is the restriction of $dh_M$ to $\tilde{K}$, and the function $h^K_B$ satisfies $dh^K_B = T\tau_K \cdot dh_M$ and $h_M \cdot \pi_G = H_M$, and the vector fields $f^K_B = T\tau_K \cdot f^B_M$ and $u^K_B = T\tau_K \cdot u^B_M$ as defined above. Then the 5-tuple $(\tilde{K}, \omega^K_B, h^K_B, f^K_B, u^K_B)$ is called a nonholonomic reduced distributional CMH system of the nonholonomic reducible CMH system $(T^*Q, G, \omega^B, D, H, F, W)$ with a control law $u \in W$, and $X^K_B$ is called a nonholonomic reduced dynamical vector field. Denote that

$$\tilde{X} = X^B_{(\tilde{K}, \omega^K_B, h^K_B, f^K_B, u^K_B)} = X^K_B + F^K_B + u^K_B$$

(7.10)

is the dynamical vector field of the nonholonomic reduced distributional CMH system $(\tilde{K}, \omega^K_B, h^K_B, f^K_B, u^K_B)$, which is the synthesis of the nonholonomic reduced dynamical vector field $X^K_B$ and the vector fields $F^K_B$ and $u^K_B$. Under the above circumstances, we refer to $(T^*Q, G, \omega^B, D, H, F, u)$ as a nonholonomic reducible CMH system with the associated distributional CMH system $(K, \omega^K_B, H_K, F^K_B, u^K_B)$ and the nonholonomic reduced distributional CMH system $(\tilde{K}, \omega^K_B, h^K_B, f^K_B, u^K_B)$. The dynamical vector fields $\tilde{X} = X^B_{(K, \omega^K_B, h^K_B, f^K_B, u^K_B)}$ and $\tilde{X} = X^B_{(\tilde{K}, \omega^K_B, h^K_B, f^K_B, u^K_B)}$ are $\pi_G$-related; that is, $\tilde{X} \cdot \pi_G = T\tau_K \cdot \tilde{X}$.

For a given nonholonomic reducible CMH system $(T^*Q, G, \omega^B, D, H, F, u)$ with the associated distributional CMH system $(K, \omega^K_B, H_K, F^K_B, u^K_B)$ and the nonholonomic reduced distributional CMH system $(\tilde{K}, \omega^K_B, h^K_B, f^K_B, u^K_B)$, the magnetic vector field $X^0 = X^0_H - X_H$, which is determined by the magnetic equation $i_{X^0} \omega = i_{X^0_H} (\pi_G^* B)$ on $T^*Q$. Note that the vector fields $X^0_H$, $X^0_B$, $X_H$ and the distribution $K$ are pushed down to $\tilde{M}$; that is, $X^0_M = T\pi/G \cdot X^0_M$, $X^0_B = T\pi/G \cdot X^0_B$, and $X_M = T\pi/G \cdot X_M$, where the vector field $X^0_M$, $X^0_B$ and $X_M$ are the restrictions of $X^0_H$, $X^0_B$ and $X_H$ on $\tilde{M}$ and the distribution $K$ is pushed down to a distribution $T\pi/G \cdot K$ on $\tilde{M}$, and define the distribution $\tilde{K}$ by $\tilde{K} = T\pi/G \cdot U$. Denote that $X^0_K = \tau_K(X^0_M) = \tau_K(X^0_B) - \tau_K(X_M) = X^0_B - X^K_B$, from the expression (7.10) of the dynamical vector field of the nonholonomic reduced distributional
CMH system \((\bar{K}, \omega^B_\bar{K}, h_\bar{K}, f^B_\bar{K}, u^B_\bar{K})\), we have that
\[
\dot{X} = X^B_\bar{K} + F^B_\bar{K} + u^B_\bar{K} = X^{\bar{K}} + X^0_\bar{K} + F^B_\bar{K} + u^B_\bar{K}.
\] (7.11)

If the vector fields \(F^B_\bar{K}\) and \(u^B_\bar{K}\) satisfy the following condition
\[
X^0_\bar{K} + F^B_\bar{K} + u^B_\bar{K} = 0,
\] (7.12)
then from (7.11) we have that \(X^B_{(\bar{K}, \omega^B_\bar{K}, h_\bar{K}, f^B_\bar{K}, u^B_\bar{K})} = X^{\bar{K}}\), that is, in this case the dynamical vector field of the nonholonomic reduced distributional CMH system is just the dynamical vector field of the nonholonomic reduced canonical distributional Hamiltonian system without the actions of the magnetic, the external force and the control. Thus, the condition (7.12) is called the magnetic vanishing condition for the nonholonomic reduced distributional CMH system \((\bar{K}, \omega^B_\bar{K}, h_\bar{K}, f^B_\bar{K}, u^B_\bar{K})\).

Since the non-degenerate and nonholonomic reduced distributional two-form \(\omega^B_\bar{K}\) is not a “true two-form” on a manifold, and it is not symplectic, and hence the nonholonomic reduced distributional CMH system \((\bar{K}, \omega^B_\bar{K}, h_\bar{K}, f^B_\bar{K}, u^B_\bar{K})\) is not a Hamiltonian system, and it has no yet generating function, and hence we can not describe the Hamilton-Jacobi equation for the nonholonomic reduced distributional CMH system the same as in Theorem 2.1. However, for a given nonholonomic reducible CMH system \((T^*Q, G, \omega^B, D, H, F, u)\) with the associated distributional CMH system \((\bar{K}, \omega^B_\bar{K}, H_\bar{K}, F^B_\bar{K}, u^B_\bar{K})\) and the nonholonomic reduced distributional CMH system \((\tilde{K}, \omega^B_{\tilde{K}}, h_{\tilde{K}}, f^B_{\tilde{K}}, u^B_{\tilde{K}})\), by using Lemma 7.2 and the following Lemma 7.13, we can derive precisely the geometric constraint conditions of the nonholonomic reduced distributional two-form \(\omega^B_\bar{K}\) for the nonholonomic reducible dynamical vector field \(\tilde{X} = X^B_{(\tilde{K}, \omega^B_{\tilde{K}}, h_{\tilde{K}}, f^B_{\tilde{K}}, u^B_{\tilde{K}})}\); that is, the two types of Hamilton-Jacobi equations for the nonholonomic reduced distributional CMH system \((\bar{K}, \omega^B_\bar{K}, h_\bar{K}, f^B_\bar{K}, u^B_\bar{K})\).

The following Lemma 7.13 can be regarded as an extension of the Lemma 6.3 (given in León and Wang [17]) to it with the nonholonomic magnetic context, and the lemma is a very important tool for the proof of the Hamilton-Jacobi theorem for the nonholonomic reduced distributional CMH system.

**Lemma 7.13** Assume that \(\gamma : Q \rightarrow T^*Q\) is a one-form on \(Q\), and that \(\lambda = \gamma \cdot \pi_Q : T^*Q \rightarrow T^*Q\), and that \(\omega\) is the canonical symplectic form on \(T^*Q\), and that \(\omega^B = \omega - \pi^\gamma_{B} B\) is the magnetic symplectic form on \(T^*Q\). If the Lagrangian \(L\) is \(D\)-regular, and assume that \(\text{Im}(\gamma) \subset M\), where \(M = \mathcal{F}L(D)\), then we have that \(X^B_H \cdot \gamma \in \mathcal{F}\) along \(\gamma\), and that \(X^B_H \cdot \lambda \in \mathcal{F}\) along \(\lambda\); that is, \(T\pi_Q(X^B_H \cdot \gamma(q)) \in \mathcal{D}_q\), \(\forall q \in Q\) and \(T\pi_Q(X^B_H \cdot \lambda(q, p)) \in \mathcal{D}_q\), \(\forall q \in Q, (q, p) \in T^*Q\). Moreover, if a symplectic map \(\varepsilon : T^*Q \rightarrow T^*Q\) with respect to the magnetic symplectic form \(\omega^B\) satisfies the condition that \(\varepsilon(M) \subset M\), then we have that \(X^B_H \cdot \varepsilon \in \mathcal{F}\) along \(\varepsilon\).

See the proof and the more details in Wang [40].

For convenience, the maps involved in the following theorem are shown in Diagram-10.

![Diagram-10](image-url)
Theorem 7.14 (Hamilton-Jacobi Theorem for a Nonholonomic Reduced Distributional CMH System) For a given nonholonomic reducible CMH system \((T^*Q, G, \omega^B, D, H, F, u)\) with the associated distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\) and the nonholonomic reduced distributional CMH system \((\bar{K}, \omega^B_{\bar{K}}, \bar{h}_K, f^B_{\bar{K}}, u^B_{\bar{K}})\), assume that \(\gamma : Q \to T^*Q\) is a one-form on \(Q\), and that \(\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q\), and that for any \(G\)-invariant symplectic map \(\varepsilon : T^*Q \to T^*Q\) with respect to \(\omega^B\), denote that \(\bar{X}^\gamma = T\pi_Q \cdot \bar{X}_\gamma\) and \(\bar{X}^\varepsilon = T\pi_Q \cdot \bar{X}_\varepsilon\), where \(\bar{X}^B_{K(\bar{K}, \omega^B_{\bar{K}}, H^B_{\bar{K}}, u^B_{\bar{K}})} = X^B_K + F^B_K + u^B_K\) is the dynamical vector field of the distributional CMH system corresponding to the nonholonomic reducible CMH system with symmetry \((T^*Q, G, \omega^B, D, H, F, u)\). Moreover, assume that \(\text{Im}(\bar{\gamma}) \subset M\), and that \(\bar{\gamma}\) is \(G\)-invariant, and that \(\varepsilon(M) \subset \mathcal{M}\), and that \(\text{Im}(T\bar{\gamma}) \subset \mathcal{K}\). Denote that \(\bar{\gamma} = \pi_{\gamma/G}(\gamma) : Q \to T^*Q/G\), and that \(\bar{\lambda} = \pi_{\gamma/G}(\lambda) : T^*Q \to T^*Q/G\), and that \(\bar{\varepsilon} = \pi_{\gamma/G}(\varepsilon) : T^*Q \to T^*Q/G\). Then the following two assertions hold:

(i) If the one-form \(\gamma : Q \to T^*Q\) satisfies the condition, \(d\gamma = -B\) on \(D\) with respect to \(T\pi_Q : TT^*Q \to TQ\), then \(\bar{\gamma}\) is a solution of the Type I Hamilton-Jacobi equation \(T\bar{\gamma} \cdot \bar{X}^\gamma = X^B_K \cdot \bar{\gamma}\) for the nonholonomic reduced distributional CMH system \((\bar{K}, \omega^B_{\bar{K}}, \bar{h}_K, f^B_{\bar{K}}, u^B_{\bar{K}})\), where \(X^B_K\) is the nonholonomic reduced dynamical vector field.

(ii) The \(\varepsilon\) and \(\bar{\varepsilon}\) satisfy the Type II Hamilton-Jacobi equation \(T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^B_K \cdot \bar{\varepsilon}\) if and only if they satisfy the equation \(\tau_K \cdot T\bar{\varepsilon} \cdot X^B_K \cdot \bar{\varepsilon} = \bar{T} \cdot \bar{X}_\varepsilon\), where \(X^B_{h^B_K} \cdot \bar{\varepsilon}\) is the magnetic Hamiltonian vector field of the function \(h^B_K \cdot \bar{\varepsilon} : T^*Q \to \mathbb{R}\).

See the proof and the more details in Wang [40].

For a given nonholonomic reducible CMH system \((T^*Q, G, \omega^B, D, H, F, u)\) with the associated distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\) and the nonholonomic reduced distributional CMH system \((\bar{K}, \omega^B_{\bar{K}}, \bar{h}_K, f^B_{\bar{K}}, u^B_{\bar{K}})\), we know that the nonholonomic dynamical vector field \(X^B_K\) and the nonholonomic reduced dynamical vector field \(X^B_{\bar{K}}\) are \(\pi_{\gamma/G}\)-related; that is, \(X^B_K \cdot \pi_{\gamma/G} = T\pi_{\gamma/G} \cdot X^B_{\bar{K}}\)

Then we can prove the following Theorem 7.15, which states the relationship between the solutions of the Type II Hamilton-Jacobi equations and nonholonomic reduction.

Theorem 7.15 For a given nonholonomic reducible CMH system \((T^*Q, G, \omega^B, D, H, F, u)\) with the associated distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\) and the nonholonomic reduced distributional CMH system \((\bar{K}, \omega^B_{\bar{K}}, \bar{h}_K, f^B_{\bar{K}}, u^B_{\bar{K}})\), assume that \(\gamma : Q \to T^*Q\) is a one-form on \(Q\), and that \(\varepsilon : T^*Q \to T^*Q\) is a \(G\)-invariant symplectic map with respect to \(\omega^B\). Denote that \(\bar{\gamma} = \pi_{\gamma/G}(\gamma) : Q \to T^*Q/G\), and that \(\bar{\varepsilon} = \pi_{\gamma/G}(\varepsilon) : T^*Q \to T^*Q/G\). Under the hypotheses and notations of Theorem 7.14, then we have that \(\varepsilon\) is a solution of the Type II Hamilton-Jacobi equation \(T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^B_K \cdot \bar{\varepsilon}\) for the distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\) if and only if \(\varepsilon\) and \(\bar{\varepsilon}\) satisfy the Type II Hamilton-Jacobi equation \(T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^B_K \cdot \bar{\varepsilon}\) for the nonholonomic reduced distributional CMH system \((\bar{K}, \omega^B_{\bar{K}}, \bar{h}_K, f^B_{\bar{K}}, u^B_{\bar{K}})\).

See the proof and the more details in Wang [40].

Remark 7.16 It is worthy of noting that the Type I Hamilton-Jacobi equation \(T\bar{\gamma} \cdot \bar{X}^\gamma = X^B_K \cdot \bar{\gamma}\) is the equation of the nonholonomic reduced differential one-form \(\bar{\gamma}\), and that the Type II Hamilton-Jacobi equation \(T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^B_K \cdot \bar{\varepsilon}\) is the equation of the symplectic diffeomorphism map \(\varepsilon\) and the nonholonomic reduced symplectic diffeomorphism map \(\bar{\varepsilon}\). If the nonholonomic CMH system with symmetry we considered has not any the external force and the control; that is, \(F = 0\) and \(W = \emptyset\), in this case, the nonholonomic CMH system with symmetry \((T^*Q, G, \omega^B, D, H, F, W)\) is just the nonholonomic magnetic Hamiltonian system with symmetry \((T^*Q, G, \omega^B, D, H)\), and with the magnetic symplectic form \(\omega^B\) on \(T^*Q\). From the Hamilton-Jacobi theorem; that is, Theorem 7.14, we can get the Theorems 5.2 and 5.3 given in Wang [41]. It shows that Theorem 7.14 can be regarded as an extension of the two types of Hamilton-Jacobi theorem for the nonholonomic
magnetic Hamiltonian system with symmetry to that for the system with the external force and the control. In particular, in this case, if $B = 0$, then the magnetic symplectic form $\omega^B$ is just the canonical symplectic form $\omega$ on $T^*Q$, from Theorem 7.14, we can also get the Theorems 4.2 and 4.3 given in León and Wang [17]. It shows that the Theorem 7.14 can be regarded as an extension of the two types of Hamilton-Jacobi theorem for the nonholonomic Hamiltonian system with symmetry to that for the system with the magnetic, the external force and the control.

In addition, we also give some generalizations of the above results from the viewpoint of change of the geometrical structures. A natural problem is what and how we could do, if we define a controlled Hamiltonian system on the cotangent bundle $T^*Q$ by using a Poisson structure, and if the symplectic reduction procedure given by Marsden et al. [28] does not work or is not efficient enough. In Wang and Zhang [45], we study the optimal reduction theory of a CH system with Poisson structure and symmetry, by using the optimal momentum map and the reduced Poisson tensor (resp. the reduced symplectic form). We prove the optimal point reduction, the optimal orbit reduction, and the regular Poisson reduction theorems for the CH system, and explain the relationships between the OpCH-equivalence, the OoCH-equivalence, the RPR-CH-equivalence for the optimal reducible CH systems with symmetries and the CH-equivalence for the associated optimal reduced CH systems; see Wang and Zhang [45] for the more details.

It is worthy of noting that if there is no momentum map of the Lie group action for our considered system, then the reduction procedures given in Marsden et al. [28] and Wang and Zhang [45] can not work. One must look for a new way. On the other hand, motivated by the work of Poisson reductions by distribution for Poisson manifolds, see Marsden and Ratiu [25], we note that the phase space $T^*Q$ of the CH system is also a Poisson manifold, and its control subset $W \subset T^*Q$ is a fiber submanifold. If we assume that $D \subset TT^*Q|_W$ is a controllability distribution of the CH system, then we can study naturally the Poisson reduction by controllability distribution for the CH system. For a symmetric CH system, and its control subset $W \subset T^*Q$ is a $G$-invariant fiber submanifold, if we assume that $D \subset TT^*Q|_W$ is a $G$-invariant controllability distribution of the symmetric CH system, then we can give the Poisson reducible conditions by controllability distribution for this CH system, and prove the Poisson reducible property for the CH system and it is kept invariant under the CH-equivalence. We also study the relationship between Poisson reduction by $G$-invariant controllability distribution for the regular (resp. singular) Poisson reducible CH system and Poisson reduction by the reduced controllability distribution for the associated reduced CH system. In addition, we can also develop the singular Poisson reduction and SPR-CH-equivalence for the CH system with symmetry, and prove the singular Poisson reduction theorem. See Ratiu and Wang [34] for more details.

8 Some Applications

Now, it is a natural problem if there is a practical RCH system and how to show the effect on the controls in regular point reduction and Hamilton-Jacobi theory of the system. In the following we shall give two examples of the application for the RCH system; that is, the controlled rigid spacecraft -rotor system and the controlled underwater vehicle-rotor system, and give the Type I and Type II Hamilton-Jacobi equations for the two systems. See Wang [43,44] for more details.

8.1 The Controlled Rigid Spacecraft -Rotor System

We first describe a rigid spacecraft carrying an internal "non-mass" rotor, which is called a carrier body, where "non-mass" means that the mass of a rotor is very very small comparing with the mass of the rigid spacecraft. We first assume that the external forces and the torques acting on
the rigid spacecraft.rotor system are due to buoyancy and gravity, and that the rigid spacecraft.-
rotor system with non-coincident centers of buoyancy and gravity. We put a rotor within the rigid
spacecraft, and assume that the rotor spins under the influence of a control torque \( u \) acting on
the rotor; see Marsden \[21\] and Leonard and Marsden \[18\]. In this case, the configuration space is
\( Q = \text{SO}(3) \otimes \mathbb{R}^3 \times S^1 \cong \text{SE}(3) \times S^1 \), with the first factor being the attitude of the rigid spacecraft
and the drift of the rigid spacecraft in the rotational process and the second factor being the angle
of rotor. For convenience, using the local left trivialization, we denote uniformly that, locally,
\( Q = \text{SE}(3) \times \mathbb{R} \), and \( T^* Q = T^*\text{SE}(3) \times T^* \mathbb{R} \cong \text{SE}(3) \times \text{se}^*(3) \times \mathbb{R} \times \mathbb{R}^* \), and the Hamiltonian
\( H(A, c, \Pi, \Gamma, \alpha, l) : T^* Q \cong \text{SE}(3) \times \text{se}^*(3) \times \mathbb{R} \times \mathbb{R}^* \rightarrow \mathbb{R} \) is given by
\[
H(A, c, \Pi, \Gamma, \alpha, l) = \Omega \cdot \Pi + \alpha \cdot l - L(A, c, \Omega, \alpha, \dot{\alpha})
\]
\[
= \frac{1}{2} [\Pi_1^2 + \Pi_2^2 + (\Pi_3 - l)^2] + \frac{l^2}{I_3} + gh\Gamma \cdot \chi.
\]
(8.1)

In the following as the application of the theoretical result, we shall regard the rigid spacecraft.
rotor system with the control torque \( u \) acting on the rotor and with the non-coincident centers of
the buoyancy and the gravity as a regular point reducible RCH system on the generalization of
the Euclidean group \( Q = \text{SE}(3) \times \mathbb{R} \), in this case, we can give the regular point reduction and the
two types of Hamilton-Jacobi equations for the controlled rigid spacecraft.rotor system with the
non-coincident centers of the buoyancy and the gravity.

Assume that Lie group \( G = \text{SE}(3) \) acts freely and properly on \( Q = \text{SE}(3) \times \mathbb{R} \) by the left
translation on the first factor SE(3) and the trivial action on the second factor \( \mathbb{R} \), and that the
action of SE(3) on phase space \( T^* Q = T^*\text{SE}(3) \times T^* \mathbb{R} \) is by the cotangent lift of the left
translation on SE(3) at the identity, and that the orbit space \( (T^* Q)/\text{SE}(3) \) is a smooth manifold
and \( \pi : T^* Q \rightarrow (T^* Q)/\text{SE}(3) \) is a smooth submersion. Since SE(3) acts trivially on \( \text{se}^*(3) \) and \( \mathbb{R} \times \mathbb{R}^* \), it follows that \( (T^* Q)/\text{SE}(3) \) is diffeomorphic to \( \text{se}^*(3) \times \mathbb{R} \times \mathbb{R}^* \). For \((\mu, a) \in \text{se}^*(3)\), the co-
adjoint orbit \( O_{(\mu, a)} \subset \text{se}^*(3) \) has the induced orbit symplectic form \( \omega_{O_{(\mu, a)}} \) which coincides with
the restriction of the heavy top Lie-Poisson bracket on \( \text{se}^*(3) \) to the co-adjoint orbit \( O_{(\mu, a)} \). From
Abraham and Marsden \[1\], we know that \((T^* Q)_{(\mu, a)}, \omega_{(\mu, a)}) \) is symplectically diffeomorphic to \( (O_{(\mu, a)}, \omega_{O_{(\mu, a)}}) \), and hence we have that the reduced space \((T^* Q)_{(\mu, a)}, \omega_{(\mu, a)}) \) is symplectically
diffeomorphic to \( (O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^*, \omega_{O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^*}) \).

From the expression (8.1) of the Hamiltonian, we know that \( H(A, c, \Pi, \Gamma, \alpha, l) \) is invariant
under the cotangent lift of the left SE(3)-action \( \Phi^T : \text{SE}(3) \times T^* Q \rightarrow T^* Q \). Assume that the
control torque \( u : T^* Q \rightarrow W \) is invariant under the cotangent lift \( \Phi^T \) of the left SE(3)-action,
and the dynamical vector field of the regular point reducible controlled spacecraft.rotor system
\( (T^* Q, \text{SE}(3), \omega_Q, H, u) \) can be expressed by
\[
\tilde{X} = X_{(T^* Q, \text{SE}(3), \omega_Q, H, u)} = X_H + \text{vlift}(u),
\]
(8.2)
where \( \text{vlift}(u) = \text{vlift}(u) \cdot X_H \) is the change of \( X_H \) under the action of the control torque \( u \). For
the point \((\mu, a) \in \text{se}^*(3)\) is the regular value of the momentum map \( J_Q \), we have the \( R_p \)-reduced
Hamiltonian \( h_{(\mu, a)}(\Pi, \Gamma, \alpha, l) : O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^* \subset \text{se}^*(3) \times \mathbb{R} \times \mathbb{R}^* \rightarrow \mathbb{R} \) given by
\[
h_{(\mu, a)}(\Pi, \Gamma, \alpha, l) - \pi_{(\mu, a)} = H(A, c, \Pi, \Gamma, \alpha, l)|_{O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^*},
\]
and the \( R_p \)-reduced control torque \( u_{(\mu, a)} : O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^* \rightarrow W_{(\mu, a)} \subset O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^* \) is given by
\[
u_{(\mu, a)}(\Pi, \Gamma, \alpha, l) - \pi_{(\mu, a)} = u(A, c, \Pi, \Gamma, \alpha, l)|_{O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^*}.
The $R_p$-reduced controlled spacecraft-rotor system is the 4-tuple $(O_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}^*, \widetilde{\omega}_{O_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}^*}, h_{(\mu,a)}, u_{(\mu,a)})$.

For convenience, the maps involved in the following theorem are shown in Diagram-11.

![Diagram-11](image)

**Theorem 8.1** In the case of non-coincident centers of buoyancy and gravity, if the 5-tuple $(T^*Q, SE(3), \omega_Q, H, u)$, where $Q = SE(3) \times \mathbb{R}$, is a regular point reducible rigid spacecraft-rotor system with the control torque $u$ acting on the rotor, then for a point $(\mu,a) \in se^*(3)$, the regular value of the momentum map $Q : SE(3) \times se^*(3) \times \mathbb{R} \times \mathbb{R}^* \to se^*(3)$, the $R_p$-reduced controlled rigid spacecraft-rotor system is the 4-tuple $(O_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}^*, \widetilde{\omega}_{O_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}^*}, h_{(\mu,a)}, u_{(\mu,a)})$. Assume that $\gamma : SE(3) \times \mathbb{R} \to T^*(SE(3) \times \mathbb{R})$ is a one-form on $SE(3) \times \mathbb{R}$, and that $\lambda = \gamma \cdot \pi_{SE(3) \times \mathbb{R}} : T^*(SE(3) \times \mathbb{R}) \to T^*(SE(3) \times \mathbb{R})$, and that $\varepsilon : T^*(SE(3) \times \mathbb{R}) \to T^*(SE(3) \times \mathbb{R})$ is an $SE(3)_{(\mu,a)}$-invariant symplectic map. Denote that $\tilde{X} = T\pi_{SE(3) \times \mathbb{R}} \cdot \tilde{\gamma} \cdot X$, and that $\tilde{X} = T\pi_{SE(3) \times \mathbb{R}} \cdot \tilde{\gamma} \cdot X \cdot \varepsilon$, where $\tilde{X} = X_{T^*Q,SE(3),\omega_Q,H,u}$ is the dynamical vector field of the controlled rigid spacecraft-rotor system $(T^*Q, SE(3), \omega_Q, H, u)$. Moreover, assume that $Im(\gamma) \subset J_{Q1}^{-1}(\mu,a)$, and that $\gamma$ is $SE(3)_{(\mu,a)}$-invariant, and that $\varepsilon(J_{Q1}^{-1}(\mu,a)) \subset J_{Q1}^{-1}(\mu,a)$. Denote that $\tilde{X} = \pi_{O_{(\mu,a)}}(\lambda) : T^*(SE(3) \times \mathbb{R}) \to O_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}^*$, and that $\tilde{X} = \pi_{O_{(\mu,a)}}(\varepsilon) : J_{Q1}^{-1}(\mu,a) \to O_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}^*$. Then the following two assertions hold:

(i) If the one-form $\gamma : SE(3) \times \mathbb{R} \to T^*(SE(3) \times \mathbb{R})$ is closed with respect to $T\pi_{SE(3) \times \mathbb{R}} : TT^*(SE(3) \times \mathbb{R}) \to T(SE(3) \times \mathbb{R})$, then $\tilde{X}$ is a solution of the Type I Hamilton-Jacobi equation $T\tilde{\gamma} \cdot \tilde{X} = X_{h_{(\mu,a)}} \cdot \tilde{\gamma}$.

(ii) The $\varepsilon$ and $\tilde{X}$ satisfy the Type II Hamilton-Jacobi equation $T\tilde{\gamma} \cdot \tilde{X} = X_{h_{(\mu,a)}} \cdot \tilde{\gamma}$ if and only if they satisfy the equation $T\varepsilon \cdot (X_{h_{(\mu,a)}} \cdot \varepsilon) = T\tilde{\gamma} \cdot \tilde{X} \cdot \varepsilon$.

See Wang [43] for more details.

### 8.2 The Controlled Underwater Vehicle-Rotor System

We first describe a underwater vehicle carrying two internal "non-mass" rotors; that is called a carrier body, moving in a given fluid, where "non-mass" means that the mass of a rotor is very very small compared to the mass of the underwater vehicle. We consider that the underwater vehicle-rotor system has a neutrally buoyant, and that rigid body (often ellipsoidal) submerged in an infinitely large volume of the incompressible, the inviscid, the irrotational fluid which is at rest at infinity. The dynamics of the carrier body-fluid system are described by the Kirchhoff’s equations. We first assume that the external forces and the torques acting on the underwater vehicle-rotor system are due to the buoyancy and the gravity, and that the underwater vehicle-rotor system with non-coincident centers of the buoyancy and the gravity. We put two rotors within the underwater vehicle, and assume that the rotors spin under the influence of a control torque $u$ acting on the rotors; see Leonard and Marsden [18]. In this case, the configuration space is $Q = W \times V$, where $W = SE(3) \otimes \mathbb{R}^3 = (SO(3) \otimes \mathbb{R}^3) \otimes \mathbb{R}^3$ is a double semidirect product Lie group and $V = S^1 \times S^1$, with the first factor being the attitude and the position of the underwater vehicle as well as the drift of the underwater vehicle-rotor in the rotational and the translational process, and the second factor being the angles of the rotors. For convenience, using the local left
trivialization, we denote uniformly that, locally, \( Q = \text{SE}(3) \otimes \mathbb{R}^3 \times \mathbb{R}^2 \), and \( T^* Q = T^*(\text{SE}(3) \otimes \mathbb{R}^3 \times \mathbb{R}^2) \cong \text{SE}(3) \otimes \mathbb{R}^3 \times \mathfrak{se}^*(3) \otimes \mathbb{R}^{3*} \times \mathbb{R}^2 \times \mathbb{R}^{2*} \), and the Hamiltonian \( H(A,c,b,\Pi,\Gamma, P, \theta, l) : T^* Q \cong (\text{SE}(3) \otimes \mathbb{R}^3) \times \mathfrak{se}^*(3) \otimes \mathbb{R}^{3*} \times \mathbb{R}^2 \times \mathbb{R}^{2*} \to \mathbb{R} \) is given by
\[
H(A,c,b,\Pi,\Gamma, P, \theta, l) = \Omega \cdot \Pi + v \cdot P + \dot{\theta} \cdot l - L(A,c,b,\Omega,\Gamma, v, \theta, \dot{\theta})
\]
\[
= \frac{1}{2} \left[ \left( \Pi_1 - l_1 \right)^2 + \frac{\Pi_1^2}{I_1} \right] + \frac{P_1^2}{m_1} + \frac{P_2^2}{m_2} + \frac{P_3^2}{m_3} + \frac{l_1^2}{J_1} + \frac{l_2^2}{J_2} + gh \Gamma \cdot \chi.
\]
In the following as the application of the theoretical result, we shall regard the controlled underwater vehicle-rotor system with the control torque \( u \) acting on the rotors and with the non-coincident centers of the buoyancy and the gravity as a regular point reducible RCH system on the generalization of the semidirect product Lie group \( Q = W \times V \), in this case, we can give the regular point reduction and the two types of Hamilton-Jacobi equations for the controlled underwater vehicle-rotor system with the non-coincident centers of the buoyancy and the gravity.

Assume that semidirect product Lie group \( W = \text{SE}(3) \otimes \mathbb{R}^3 \) acts freely and properly on \( Q = (\text{SE}(3) \otimes \mathbb{R}^3) \times \mathbb{R}^2 \) by the left translation on \( \text{SE}(3) \otimes \mathbb{R}^3 \), and that the action of \( W \) on the phase space \( T^* Q \) is by cotangent lift of left translation on \( Q \) at the identity. If \( (\Pi, w_1, w_2) \in \mathfrak{se}^*(3) \otimes \mathbb{R}^{3*} \) is a regular value of the momentum map \( J_Q \), then the \( R_p \)-reduced space
\[
(T^* Q)_{(\Pi, w_1, w_2)} = J_Q^{-1}(\Pi, w_1, w_2) / (\text{SE}(3) \otimes \mathbb{R}^3)_{(\Pi, w_1, w_2)}
\]
is symplectically diffeomorphic to the orbit space \( O_{(\Pi, w_1, w_2)} \times \mathbb{R}^2 \times \mathbb{R}^{2*} \subset \mathfrak{se}^*(3) \otimes \mathbb{R}^{3*} \times \mathbb{R}^2 \times \mathbb{R}^{2*} \) with the induced orbit symplectic form \( \tilde{\omega}_{O_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^{2*}} \).

From the expression (8.3) of the Hamiltonian, we know that \( H(A,c,b,\Pi,\Gamma, P, \theta, l) \) is invariant under the cotangent lift of the left \( \text{SE}(3) \otimes \mathbb{R}^3 \)-action \( \Phi^T : (\text{SE}(3) \otimes \mathbb{R}^3) \times T^* Q \to T^* Q \). Assume that the control torque \( u : T^* Q \to W \) acting on the rotors is invariant under the cotangent lift of the left \( \text{SE}(3) \otimes \mathbb{R}^3 \)-action, and that the dynamical vector field of the regular point reducible controlled underwater vehicle-rotor system \( (T^* Q, \text{SE}(3) \otimes \mathbb{R}^3, \omega_Q, H, u) \) can be expressed by
\[
\dot{X} = X(T^* Q, \text{SE}(3) \otimes \mathbb{R}^3, \omega_Q, H, u) = X_H + \text{vlift}(u),
\]
where \( \text{vlift}(u) = \text{vlift}(u) \cdot X_H \) is the change of \( X_H \) under the action of the control torque \( u \). For the point \( (\mu, a_1, a_2) \in \mathfrak{se}^*(3) \otimes \mathbb{R}^{3*} \), the regular value of the momentum map \( J_Q \), we have the \( R_p \)-reduced Hamiltonian \( h_{(\mu, a_1, a_2)}(\Pi, \Gamma, P, \theta, l) : O_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^{2*} \subset \mathfrak{se}^*(3) \otimes \mathbb{R}^{3*} \times \mathbb{R}^2 \times \mathbb{R}^{2*} \to \mathbb{R} \) given by
\[
h_{(\mu, a_1, a_2)}(\Pi, \Gamma, P, \theta, l) \cdot \pi_{(\mu, a_1, a_2)} = H(A,c,b,\Pi,\Gamma, P, \theta, l)_{\big| O_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^{2*}},
\]
and that the \( R_p \)-reduced control torque \( u_{(\mu, a_1, a_2)} : O_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^{2*} \to C_{(\mu, a_1, a_2)}(\subset O_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^{2*}) \) is given by
\[
u_{(\mu, a_1, a_2)}(\Pi, \Gamma, P, \theta, l) \cdot \pi_{(\mu, a_1, a_2)} = u(A,c,b,\Pi,\Gamma, P, \theta, l)_{\big| O_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^{2*}}.
\]
The \( R_p \)-reduced controlled underwater vehicle-rotor system is the 4-tuple \( (O_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^{2*}, h_{(\mu, a_1, a_2)}, u_{(\mu, a_1, a_2)} \).

For convenience, the maps involved in the following theorem are shown in Diagram-12.
**Theorem 8.2** In the case of non-coincident centers of the buoyancy and the gravity, if the 5-tuple $\left(T^*Q, SE(3)\otimes\mathbb{R}^3, \omega_Q, H, u \right)$, where $Q = SE(3)\otimes\mathbb{R}^3 \times \mathbb{R}^2$, is a regular point reducible underwater vehicle-rotor system with the control torque $u$ acting on the rotors, then, for a point $(\mu, a_1, a_2) \in se^*(3)\otimes\mathbb{R}^3$, the regular value of the momentum map $J_Q : T^*Q \cong SE(3)\otimes\mathbb{R}^3 \times se^*(3)\otimes\mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow se^*(3)\otimes\mathbb{R}^3$, the $R_p$-reduced controlled underwater vehicle-rotor system is the 4-tuple $(O_{(\mu,a_1,a_2)} \times \mathbb{R}^2 \times \mathbb{R}^2, \tilde{\omega}_{O_{(\mu,a_1,a_2)} \times \mathbb{R}^2 \times \mathbb{R}^2, h_{(\mu,a_1,a_2)}, u_{(\mu,a_1,a_2)}}, \text{where } O_{(\mu,a_1,a_2)} \subset se^*(3)\otimes\mathbb{R}^3$ is the co-adjoint orbit of the semidirect product Lie group $SE(3)\otimes\mathbb{R}^3$. Assume that $\gamma : Q \rightarrow T^*Q$ is a one-form on $Q = SE(3)\otimes\mathbb{R}^3 \times \mathbb{R}^2$, and that $\lambda = \gamma \cdot \pi_Q : T^*Q \rightarrow T^*Q$, and that $\varepsilon : T^*Q \rightarrow T^*Q$ is an $(SE(3)\otimes\mathbb{R}^3)_{(\mu,a_1,a_2)}$-invariant symplectic map, where $(SE(3)\otimes\mathbb{R}^3)_{(\mu,a_1,a_2)}$ is the isotropy subgroup of the co-adjoint $SE(3)\otimes\mathbb{R}^3$-action at the point $(\mu,a_1,a_2)$. Denote that $\tilde{X}^\gamma = T\pi_Q \cdot \tilde{X} \cdot \gamma$, and that $\tilde{X}^\varepsilon = T\pi_Q \cdot \tilde{X} \cdot \varepsilon$, where $\tilde{X} = X_{(T^*Q, SE(3)\otimes\mathbb{R}^3, \omega_Q, H, u)}$ is the dynamical vector field of the controlled underwater vehicle-rotor system $(T^*Q, SE(3)\otimes\mathbb{R}^3, \omega_Q, H, u)$. Moreover, assume that $\text{Im}(\gamma) \subset J_Q^{-1}(\{(\mu, a_1, a_2)\})$, and that $\gamma$ is $(SE(3)\otimes\mathbb{R}^3)_{(\mu,a_1,a_2)}$-invariant, and that $\varepsilon(J_Q^{-1}(\{(\mu, a_1, a_2)\}) \subset J_Q^{-1}(\{(\mu, a_1, a_2)\})$. Denote that $\tilde{\gamma} = \pi_{(\mu,a_1,a_2)}(\gamma) : Q = SE(3)\otimes\mathbb{R}^3 \times \mathbb{R}^2 \rightarrow \mathcal{O}$, and that $\tilde{\lambda} = \pi_{(\mu,a_1,a_2)}(\lambda) : T^*Q \rightarrow \mathcal{O}, \tilde{\varepsilon} = \pi_{(\mu,a_1,a_2)}(\varepsilon) : J_Q^{-1}(\{(\mu, a_1, a_2)\}) \rightarrow \mathcal{O}$. Then the following two assertions hold: 

(i) If the one-form $\gamma : Q \rightarrow T^*Q$ is closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, then $\tilde{\gamma}$ is a solution of the Type I Hamilton-Jacobi equation $T\tilde{\gamma} \cdot \tilde{X}^\gamma = X_{h_{(\mu,a_1,a_2)}} \cdot \tilde{\gamma}$. 

(ii) The $\varepsilon$ and the $\tilde{\varepsilon}$ satisfy the Type II Hamilton-Jacobi equation $T\tilde{\gamma} \cdot \tilde{X}^\varepsilon = X_{h_{(\mu,a_1,a_2)}} \cdot \tilde{\varepsilon}$ if and only if they satisfy the equation $T\tilde{\varepsilon} \cdot X_{h_{(\mu,a_1,a_2)}} = T\tilde{\lambda} \cdot \tilde{X}$.

See Wang [44] for more details.

It is worthy of noting that the motions of the controlled rigid spacecraft-rotor system and the controlled underwater vehicle-rotor system are different, and the configuration spaces, the Hamiltonian functions, the actions of Lie groups, the $R_p$-reduced symplectic forms and the $R_p$-reduced systems of the controlled rigid spacecraft-rotor system and the controlled underwater vehicle-rotor system, all of them are also different. However, the two types of Hamilton-Jacobi equations given by calculation in detail are same, that is, the internal rules are same by comparing Theorems 8.1 and 8.2. This is very important! In particular, the method of calculations for the regular point reductions and the two types for the rigid spacecraft with internal rotor and the underwater vehicle with internal rotors is very important and efficient, and it is generalized and used to the more practical controlled Hamiltonian systems. Finally, we also note that there have been a lot of beautiful results of reduction theory of Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. So, it is an important topic to study the application of symmetric reduction and Hamilton-Jacobi theory for the regular controlled Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. These are our goals in future research.

It is well-known that the symmetric reduction and Hamilton-Jacobi theory for Hamiltonian systems are very important research subjects in mathematics and analytical mechanics. Following the theoretical development of geometric mechanics, a lot of important problems involving the subjects are being explored and studied; see Wang and Zhang [45], Ratiu and Wang [34], and Wang [36-38]. In addition, it is very important for a rigorous theoretical work to offer uniformly a composition of the research results from a global view point. In particular, it is the key thought of the researches of geometrical mechanics of the Professor Jerrold E. Marsden to explore and reveal the deeply internal relationship between the geometrical structure of phase space and the dynamical vector field of a mechanical system. It is also our goal of pursuing and inheriting.
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References

[1] Abraham R. and Marsden J.E., Foundations of Mechanics, 2nd edition, Addison-Wesley, 1978.
[2] Abraham R., Marsden J.E. and Ratiu T.S., Manifolds, Tensor Analysis and Applications, Applied Mathematical Science, 75, Springer-Verlag, 1988.
[3] Arnold V.I., Mathematical Methods of Classical Mechanics, 2nd edition, Graduate Texts in Mathematics, 60, Springer-Verlag, 1989.
[4] Bates L. and Śniatycki J., Nonholonomic reduction, Rep. Math. Phys. 32, 99-115(1993).
[5] Cantrijn F., de León M., Marrero J.C. and Martin de Diego D., Reduction of constrained systems with symmetries, J. Math. Phys., 40(2), 795-820(1999).
[6] Cariñena J.F., Gracia X., Marmo G., Martínez E., Muñoz-Lecanda M. and Román-Roy N., Geometric Hamilton-Jacobi theory, Int. J. Geom. Methods Mod. Phys. 3, 1417-1458(2006).
[7] Cariñena J.F., Gracia X., Marmo G., Martínez E., Muñoz-Lecanda M. and Román-Roy N., Geometric Hamilton-Jacobi theory for nonholonomic dynamical systems, Int. J. Geom. Methods Mod. Phys. 7, 431-454(2010).
[8] Cendra H., Marsden J.E. and Ratiu T.S., Geometric mechanics, Lagrangian reduction and nonholonomic systems, In "Mathematics Unlimited 2001 and Beyond" (eds. B. Engquist and W. Schmid), Springer-Verlag, New York, 221-273(2001).
[9] Cushman R., Duistermaat H. and Śniatycki J., Geometry of Nonholonomic Constrained Systems, Advanced series in nonlinear dynamics, 26, (2010).
[10] Cushman R., Kemppainen D., Śniatycki J. and Bates L., Geometry of nonholonomic constraints, Rep. Math. Phys. 36(2/3), 275-286(1995).
[11] Ge Z. and Marsden J.E., Lie-Poisson integrators and Lie-Poisson Hamilton-Jacobi theory, Phys. Lett. A, 133, 133-139(1988).
[12] Gotay M.J. and Nester J.M., Presymplectic Lagrangian systems I: The constraint algorithm and the equivalence theorem, Ann. Inst. Henri Poincaré, Sect. A, 30, 129-142(1979).
[13] Kazhdan D., Kostant B. and Sternberg S.: Hamiltonian group actions dynamical systems of Calogero type, Comm. Pure Appl. Math. 31(1978), 481-508.
[14] Koiller J., Reduction of some classical non-holonomic systems with symmetry, Arch. Rational Mech. Anal. 118, 113-148(1992).
[15] Lázaro-Camí J-A and Ortega J-P, The stochastic Hamilton-Jacobi equation, J. Geom. Mech. 1, 295-315(2009).
[16] de León M. and Rodrigues P.R., Methods of Differential Geometry in Analytical Mechanics, North-Holland, Amsterdam, (1989).
[17] León M. and Wang H., Hamilton-Jacobi theorems for nonholonomic reducible Hamiltonian systems on a cotangent bundle, (arXiv: 1508.07548v3, a revised version).

[18] Leonard N.E, Marsden J.E, Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry, Physica D, 105, 130–162(1997).

[19] Libermann P. and Marle C.M., Symplectic Geometry and Analytical Mechanics, Kluwer Academic Publishers, 1987.

[20] Marle C.M., Symplectic manifolds, dynamical groups and Hamiltonian mechanics, In: Differential Geometry and Relativity, (M. Cahen and M. Flato, eds.), D. Reidel, Boston, 249-269(1976).

[21] Marsden J.E., Lectures on Mechanics, London Mathematical Society Lecture Notes Series, 174, Cambridge University Press, 1992.

[22] Marsden J.E., Misiolek G., Ortega J.P., Perlmutter M. and Ratiu T.S., Hamiltonian Reduction by Stages, Lecture Notes in Mathematics, 1913, Springer, 2007.

[23] Marsden J.E., Misiolek G., Perlmutter M. and Ratiu T.S., Symplectic reduction for semidirect products and central extensions, Diff. Geom. Appl., 9, 173-212 (1998).

[24] Marsden J.E. and Perlmutter M., The orbit bundle picture of cotangent bundle reduction, C. R. Math. Acad. Sci. Soc. R. Can., 22, 33-54 (2000).

[25] Marsden J.E. and Ratiu T.S.: Reduction of Poisson manifolds, Lett. Math. Phys., 11(2), 161-169, (1986).

[26] Marsden J.E. and Ratiu T.S., Introduction to Mechanics and Symmetry, 2nd edition, Texts in Applied Mathematics, 17, Springer-Verlag, 1999.

[27] Marsden J.E., Montgomery R. and Ratiu T.S., Reduction, Symmetry and Phases in Mechanics, In: Memoirs of the American Mathematical Society, 88, American Mathematical Society, Providence, Rhode Island, 1990.

[28] Marsden J.E., Wang H., and Zhang Z.X., Regular reduction of controlled Hamiltonian system with symplectic structure and symmetry, Diff. Geom. Appl., 33(3), 13-45 (2014), (arXiv: 1202.3564v3, a revised version).

[29] Marsden J.E. and Weinstein A., Reduction of symplectic manifolds with symmetry, Rep. Math. Phys., 5, 121–130 (1974).

[30] Meyer K.R., Symmetries and integrals in mechanics, In Peixoto M. (eds), Dynamical Systems, Academic Press, 259–273 (1973).

[31] Nijmeijer H. and Van der Schaft A.J., Nonlinear Dynamical Control Systems, Springer-Verlag, 1990.

[32] Ortega J.P. and Ratiu T.S., Momentum Maps and Hamiltonian Reduction, Progress in Mathematics, 222, Birkhäuser, 2004.

[33] Patrick G.W., Variational development of the semi-symplectic geometry of nonholonomic mechanics, Rep. Math. Phys. 59, 145-184(2007).

[34] Ratiu T.S. and Wang H., Poisson reduction by controllability distribution for a controlled Hamiltonian system, (arXiv: 1312.7047v2, a revised version).
[35] Wang H., Hamilton-Jacobi theorems for regular reducible Hamiltonian systems on a cotangent bundle, Jour. Geom. Phys., 119(2017), 82-102.

[36] Wang H., The geometrical structure of phase space of the controlled Hamiltonian system with symmetry, (arXiv: 1802.01988v3, a revised version).

[37] Wang H., Regular reduction of a controlled magnetic Hamiltonian system with symmetry of the Heisenberg group, (arXiv: 1506.03640v3, a revised version).

[38] Wang H., Symmetric Reduction of Regular Controlled Lagrangian System with Momentum Map, (arXiv: 2103.06563).

[39] Wang H., Hamilton-Jacobi theorems for a regular controlled Hamiltonian system and its reduced systems, (arXiv: 1305.3457, a revised version). To appear in Acta Mathematica Scientia, English Series, 2022.

[40] Wang H., Hamilton-Jacobi equations for the controlled magnetic Hamiltonian systems with nonholonomic constraints, (arXiv: 2110.14175v2, a revised version).

[41] Wang H., Hamilton-Jacobi equations for a nonholonomic magnetic Hamiltonian systems, (arXiv: 2112.00961v2, a revised version).

[42] Wang H., Nonholonomic controlled Hamiltonian system : symmetric reduction and Hamilton-Jacobi equations, (arXiv: 2205.01998).

[43] Wang H., Dynamical equations of the controlled rigid spacecraft with a rotor, (arXiv: 2005.02221).

[44] Wang H., Symmetric reduction and Hamilton-Jacobi equations for the controlled underwater vehicle-rotor system, (arXiv: 1310.3014, a revised version).

[45] Wang H. and Zhang Z.X., Optimal reduction of controlled Hamiltonian system with Poisson structure and symmetry, Jour. Geom. Phys., 62(5), 953-975 (2012).

[46] Woodhouse N.M.J. Geometric Quantization, second ed., Clarendon Press, Oxford, 1992.