LOCAL COHOMOLOGY OF MODULE OF DIFFERENTIALS OF INTEGRAL EXTENSIONS II

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Abstract. In this note \((R, m)\) denotes a complete regular local ring and \(B\) mostly denotes its absolute integral closure. The four objectives of this paper are the following: i) to determine the highest non-vanishing local cohomology of \(\Omega_{B/R}\) in equicharacteristic 0, ii) to establish a connection between each of \(\Omega_{B/R}\) and \(\Omega_{B/V}\) and pull-back of \(\Omega_{A/V}\) via a short exact sequence together with new observations on corresponding local cohomologies in mixed characteristic where \(V\) is the coefficient ring of \(R\) and \(A\) is its absolute integral closure, iii) to demonstrate that \(\Omega_{B/R}\) can be mapped onto a cohomologically Cohen-Macaulay module and iv) to study torsion-free property for \(\Omega_{C/V}\) and \(\Omega_{C/k}\) along with their respective completions where \(C\) is an integral domain and a module finite extension of \(R\). In this connection an extension of Suzuki’s theorem on normality of complete intersections to the formal set-up in all characteristics is accomplished.

1. Introduction

This note is a sequel to our previous work in ([3]). Here \((R, m)\) denotes a complete regular local ring of dimension \(n\), \(B\) denotes its absolute integral closure unless otherwise stated. In ([3]) it was pointed out that \(H^n_m(\Omega_{B/R}) = 0\) and in mixed characteristic \(H^{n-1}_m(\Omega_{B/R}) \neq 0 \iff\) the direct summand property for integral extensions of regular local rings is valid and hence by André’s work in ([1]) it follows that \(H^{n-1}_m(\Omega_{B/R}) \neq 0\) (theorem 4.1, [3]). However, we were not sure that the above observation on non-vanishing of local cohomology ([6]) is valid in equicharacteristic 0 and we raised it as a question along with a couple of other questions in ([3]). In this paper we are mainly concerned with the following: i) to provide affirmative answers to some of the questions raised in ([3]) on highest non-vanishing local cohomology of \(\Omega_{B/R}\) with respect to \(m\) and with respect to any ideal generated by a part of

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regular system of parameters of length \(n - 1\) in equicharacteristic 0, ii) to establish a connection between each of \(\Omega_{B/R}\) and \(\Omega_{B/V}\) and pull back of \(\Omega_{A/V}\) via a short exact sequence together with several new observations on corresponding local cohomologies in mixed characteristic where \(V\) is the coefficient ring of \(R\) and \(A\) is the absolute integral closure of \(V\), iii) to demonstrate that in mixed characteristic \(\Omega_{B/R}\) can be mapped onto cohomologically Cohen-Macaulay modules \(N\) such that for every \(t > 0\), \((0 : p^t)N\) are finitely generated Cohen-Macaulay free \(B/p^tB\)-modules of rank \(n - 1\) and dimension \(n - 1\) and iv) to study torsion free property for standard generators of \(\Omega_{B/V}\), \(\Omega_{B/k}\) and to study the same property for \(\Omega_{C/V}\), \(\Omega_{C/k}\) along with their respective completions where \(C\) is an integral domain and a module finite extension of \(R\). In this connection an extension of Suzuki’s theorem on normality of complete intersection equicharacteristic affine domains to the formal set-up in all characteristics is accomplished. Several new results are also discussed. A brief overview of most of the results of this paper is given below.

In section 2.1 first we derive the following corollary from theorem 3.5 in ([3]):

Let \(R\) be a normal domain containing a field \(k\) of characteristic 0 with a non-null derivation \(D \in \text{Der}_k(R)\). Let \(C\) denote the integral closure of \(R\) in a finite extension of \(Q(R)\) and \(\tilde{C} = \{x \in Q(C)|\text{Tr}(xC) \subset R\}\) ([13]). Let \(\tilde{D} : \Omega_{C/k} \to Q(C)\) denote the \(C\)-linear map induced by the extension of \(D\) to \(Q(C)\). Then \(\text{Im} \tilde{D} \subset \tilde{C}\).

We use the above result to prove the following theorem.

**Theorem 2.2.** Let \((R, m)\) be a complete regular local ring of dimension \(n\) in equicharacteristic 0 and let \(B\) denote its absolute integral closure. Let \(x_2, ..., x_n\) denote a part of a regular system of parameters of \(R\) and \(\underline{x}\) denote the corresponding ideal. Then \(H^{n-1}_m(\Omega_{B/R}) \neq 0\).

As a corollary (2.3) we derive that \(H^{n-1}_m(\Omega_{B/R}) \neq 0\).

The above theorem settles affirmatively question 2a) in equicharacteristic 0 and the corollary settles affirmatively question 1) in (4.4, [3]).

Using the above corollary we are now able to prove the following result in mixed characteristic,

**Proposition 2.5.** The surjective map \(H^{n-1}_m(\Omega_{B/R}) \to H^{n-1}_\underline{x}(\Omega_{B/R})\) is not an isomorphism (here \(p, x_2, ..., x_n\) form a regular system of parameters of \(R\)).

In section 3 first we point out in lemma 3.1 that if \(R_i = V[[X_1, ..., X_i]]\), \(i \leq n - 1, R = R_{n-1}, R_0 = V, B_i = \text{absolute integral closure of } R_i\),
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A = B_0 and B = B_{n-1}, then each of Ω_{B/V} and Ω_{B/R}, is a B_r-summand of Ω_{B/V} and Ω_{B/R} respectively. Similar statement is also valid in the equicharacteristic case.

For lemma 3.2 and corollaries of theorem 3.3 in 3.4, we assume that
\[ [Q(V): Q(\hat{\mathbb{Z}}_p\mathbb{Z})] < \infty \]

Let \( X \) denote the ideal generated by \( X_1, ..., X_{n-1} \) in \( R \).

**Lemma 3.2** points out the following:
\[ H^{n-1}_X(Ω_{A/V} \otimes B) \cong H^{n-1}_m(Ω_{A/V} \otimes B) \cong H^n_m(B). \]
Thus the validity of the direct summand property for integral extensions ([7]) is equivalent to the non-null property for \( H^{n-1}_X(Ω_{A/V} \otimes B) \).

In our next theorem we prove the following.

**Theorem 3.3.** The following sequence is exact:
\[ 0 \to Ω_{A/V} \otimes B \to Ω_{B/V} \to Ω_{B/A} \to 0 \]

In one of the two corollaries of this theorem we point out in 3.4 that
\[ H^1_p(B) \text{ imbeds into both } Ω_{B/V} \text{ and } Ω_{B/R}. \]

This implies (Remark 3.4) that both \( Ω_{B/V} \) and \( Ω_{B/R} \) contain isomorphic copies of a Cohen-Macaulay torsion \( B \)-module. It follows from an exact sequence in proposition 2.5 or in theorem 4.1, ([3]) that both \( Ω_{B/V} \) and \( Ω_{B/k} \) also contain cohomologically Cohen-Macaulay flat \( B \)-modules. For the later observation no restriction on \( Q(V) \) is necessary.

The final theorem of this section demonstrates the following.

**Theorem 3.5.** \( Ω_{B/R} \) can be mapped onto a \( B \)-module \( M \) such that
i) \( M \) is cohomologically Cohen-Macaulay of cohomological dimension \( n-1 \), ii) for every \( t > 0 \), \( (0 : p^t)M \) is finitely generated Cohen-Macaulay free \( B/p^tB \)-module of rank \( n-1 \) and dimension \( n-1 \) and iii) \( (0 : p^t)M \) is a summand of \( (0 : p^t)Ω_{B/R} \).

As a corollary we derive that, for every \( t > 0 \), \( Ω_{B/R} \) contains a finitely generated Cohen-Macaulay free \( B/p^tB \)-module of rank \( n-1 \) and dimension \( n-1 \). Moreover, if \( [Q(V): Q(\hat{\mathbb{Z}}_p\mathbb{Z})] < \infty \), the corresponding rank of this submodule is \( n \).

In section 4 we study torsion-free property for standard generators of each of \( Ω_{B/V} \), \( Ω_{B/k} \) and for each of \( Ω_{C/V} \), \( Ω_{C/k} \) where \( C \) is the integral closure of \( R \) in a finite extension of \( Q(R) \). First we prove the following proposition.

**Proposition 4.1.** Let \( b(\neq 0) \in B \); consider an element \( w \) of the form \( cdb \in Ω_{B/V}, c \in B \). If \( w \) is a torsion element in \( Ω_{B/V} \), then \( b \in A \).

If \( R \) is equicharacteristic i.e. \( R = k[[X_1, ..., X_n]] \), then for every \( b \in B - k \), \( db \in Ω_{B/k} \) is torsion free over \( B \) where \( k \) is the algebraic closure of \( k \). Same arguments work in the affine case.
The above proposition induced me to inquire about conditions under which each of $\Omega_{B/V}$ and $\Omega_{B/k}$ would become torsion free as $B$-modules. This led me to Suzuki’s theorem ([10]) (statement in (4.3)). Theorem 4.3 extends Suzuki’s work on the connection between normality of complete intersection affine domains over a field $k$ and torsion-freeness of the corresponding module of differentials to the formal set-up in all characteristics. To this end we first extend Grothendieck’s result: Proposition 22.7.2 in ([5]) to mixed characteristic in the following way:

**Theorem 4.2.** Let $V$ be a complete discrete valuation ring in mixed characteristic $p$ with maximal ideal $pV$ such that the residue field $k$ of $V$ is perfect. Let $R = V[[X_1, ..., X_{n-1}]]$ and $C = R/I$ be an integral domain such that $p \neq 0$ in $C$. Let $P$ be a prime ideal of $R$ containing $I$ and let $q = P/I$. The following conditions are equivalent.

i) $C_q$ is formally smooth over $V$ for the $q$-adic topology.

ii) There exists derivations $D_i, 1 \leq i \leq r, \in \text{Der}_V(R)$ and elements $\{f_i\}_{1 \leq i \leq r}$ of $I$ such that $\{f_i\}_{1 \leq i \leq r}$ generate $IR_P$ and $\det(D_if_j) \notin P$.

iii) $C_q$ is an unramified regular local ring.

Next we prove the following extension of Suzuki’s theorem.

**Theorem 4.3.** Let $C = R/I$ where $(R, m)$ is either unramified or equicharacteristic complete regular local ring with coefficient ring $V$ (field $k$) such that $V/pV(k)$ is perfect and $I$ is a complete intersection prime ideal in $R$. Let $\hat{\Omega}_{C/V}, \hat{\Omega}_{C/k}$ denote the $m$-adic completions of $\Omega_{C/V}$ and $\Omega_{C/k}$ respectively. We have the following

1) Suppose that if $p \neq 0 \in C$, every prime ideal $P$ of height 1 in $C$ containing $p$ is such that $p \notin P^2C_P$. If $C$ is normal then both $\hat{\Omega}_{C/V}$ and $\hat{\Omega}_{C/k}$ are torsion-free over $C$.

2) If $\hat{\Omega}_{C/V}$ or $\hat{\Omega}_{C/k}$ is torsion free then $C$ is normal.

As a corollary we point out that, with $R$ as above, if $C = R[[Y_1, ..., Y_r]]/I$ is a module finite extension of $R$ and is a complete intersection domain such that $Q(C)$ is a separable extension of $Q(R)$, then part 1) of the above theorem implies that $\Omega_{C/V}$ and $\Omega_{C/k}$ are torsion-free over $C$.

Our final theorem (4.5) points out the connection between derivations of $R$ and the torsion sub-module $T$ of $\hat{\Omega}_{C/V}$, where $C$ is as mentioned in the statement of theorem 4.2. We prove the following:

for any $\omega \neq 0$ in $\hat{\Omega}_{C/V}$, $\omega \in T \iff \tilde{D}_i(\omega) = 0, \tilde{D}_i$ induced by the extension $D_i$ of the standard derivation $\partial/\partial X_i : R \to R$ for $1 \leq i \leq n - 1$ to $Q(C)$. 
For similar observation in the equicharacteristic case see remark 4) in 4.7.

In Corollary 3.4.6, we again point out that \(\Omega_{B/R}\) has an image \(N\) such that \(N\) is cohomologically Cohen-Macaulay i.e. \(H^i_m(N) = 0\) for \(i < n - 1\), \(H^{n-1}_m(N) \neq 0\) and \(\Omega_{B/V}\) can be mapped on to a \(B\)-module \(W\) such that \(H^i_m(W) = 0\) for \(i \geq 0\). \(N\) shares the same properties with \(M\) as in theorem 3.5 and its corollary. Moreover, for any \(x \neq 0 \in m, (0 : x)\) \(N\) can be imbedded into a cohomologically Cohen-Macaulay flat \(B/xB\)-module.

We provide construction of another such module in remark 5), 4.7.

We are thankful to B. Bhatt for his note (about a couple months back) through which we became aware of his awesome piece of work in [2]. We used some of his main results to draw our conclusions on Cohen-Macaulayness in several places within this paper.

**Notations.** Given an extension \(A \rightarrow B\) of commutative rings, the corresponding module of differentials is denoted by \(\Omega_{B/A}\) (instead of \(\Omega^1_{B/A}\)); d.v.r. stands for discrete valuation ring; for any integral domain (ring) \(C, Q(C)\) denotes the field of fractions (quotient ring) of \(C\); for any local ring \((R, m)\), \(E\) denotes the injective hull of \(R/m\) over \(R\), \(E(k(P))\) denotes the injective hull of \(R/P\) over \(R\) and for any \(R\)-module \(M\), \(M^\vee\) stands \(\text{Hom}_R(M, E)\), \(M^*\) stands for \(\text{Hom}_R(M, R)\). Unless stated otherwise, in a local ring by local cohomology of a module we mean the same with respect to the maximal ideal.

**Section 2.**

2.1. For the convenience of the reader we state theorem 3.5, ([3]) and add a sentence or two for indicating its proof.

**Theorem.** (Theorem 3.5, [3]) Let \(R\) be a normal domain of dimension \(n\) containing a field \(k\) of characteristic 0 with a non-null derivation \(D \in \text{Der}_k(R)\) and let \(B\) be its integral closure in an algebraic extension \(F\) of \(K\). Then \(\text{Hom}_R(\Omega_{B/k}, R) \neq 0\). If \((R, m)\) is a complete local normal domain of dimension \(n\) with maximal ideal \(m\) and a non-null derivation \(D \in \text{Der}_k(R)\), then \(H^i_m(\Omega_{B/k}) \neq 0\).

The proof of this theorem is obtained by proving the following assertion: Let \(K\) denote the field of fractions of \(R\). Since characteristic of \(k = 0\), \(D\) can be extended to a derivation \(D\) (same notation): \(B \rightarrow F\) and hence to a derivation: \(B_P \rightarrow F\) for any prime ideal \(P\) of \(R\). Let \(\text{Tr} : F \rightarrow K\) denote the trace map. We define an \(R\)-linear map \(L : \Omega_{B/k} \rightarrow K\) in the following way:
For any \( \omega \in \Omega_{B/k} \), \( L(\omega) = \text{Tr}(\tilde{D}(\omega)) \) where \( \tilde{D} : \Omega_{B/k} \to F \) is the \( B \)-linear map induced by \( D \); the same prescription works for \( \Omega_{B_P/k} \to F \) for any prime ideal \( P \) of \( R \). Then \( \text{im}L \subset R \).

See ([3]) for details.

Next we derive the following corollary.

**Corollary.** Let \( R \) be a normal domain containing a field \( k \) of characteristic 0 with a non-null derivation \( D \in \text{Der}_k(R) \). Let \( C \) denote the integral closure of \( R \) in a finite extension \( Q(C) \) of \( Q(R) \). Let \( \tilde{C} = \{ x \in Q(C) | \text{Tr}(xC) \subset R \} \) ([13]). Then \( \tilde{D}(\Omega_{C/k}) \subset \tilde{C} \), where the \( C \)-linear map \( \tilde{D} : \Omega_{C/k} \to Q(C) \) is induced by the extension of \( D \) to \( Q(C) \).

**Proof.** Recall in the proof of the above theorem we have shown that if \( L = \text{Tr} \cdot \tilde{D} : \Omega_{C/k} \to Q(R) \), then \( \text{im}L \subset R \). Let \( \lambda \in \text{im} \tilde{D} \), i.e. \( \lambda = \tilde{D}(w) \) for some \( w \in \Omega_{C/k} \). For any \( b \in C \), \( b\lambda = b\tilde{D}(w) = \tilde{D}(bw) \).

Hence by above the theorem, it follows that \( \text{Tr}(b\lambda) = \text{Tr}(\tilde{D}(bw)) \in R \); this implies that \( \lambda \in \tilde{C} \). Thus the corollary follows.

2.2. In the following theorem and its corollary we are going to answer affirmatively a question that was raised in ([3]).

**Theorem.** Let \( R = k[[X_1, ..., X_n]] \), characteristic of \( k \) = 0. Let \( B \) be the integral closure of \( R \) in an algebraic extension of \( Q(R) \) such that \( B \) contains an element \( y \) with the property that \( y^2 = X_i \), for some \( i \), \( 1 \leq i \leq n \). Let \( \underline{X} \) denote the ideal \((X_1, ..., \hat{X}_i, ..., X_n)(\hat{\cdot} \equiv \text{omitted}) \). Then \( H^{n-1}_{\underline{X}}(\Omega_{B/R}) \neq 0 \).

**Proof.** Can assume \( i = 1 \). Let \( K = Q(R) \) and \( F = Q(B) \). Let \( D : B \to F \) denote the extension of \( \partial/\partial X_1 : R \to R \), \( \partial/\partial X_1(X_i) = 0 \) for \( i \neq 1 \) and \( = 1 \) for \( i = 1 \). Let \( \tilde{D} : \Omega_{B/k} \to F \) denote the \( B \)-linear map induced by \( D \). By hypothesis,

\[
y^2 = X_1 = 2y dy = dX_1 = \tilde{D}(dy) = 1/2y \text{ and } X_1 dy = 0 \in \Omega_{B/R}. \tag{1}\]

To prove our assertion it will be enough to show that for \( t > 0 \), \( dy(X_2, ..., X_n)^{t-1} \notin (X_2, ..., X_n)\Omega_{B/R} \). If possible, let

\[
dy(X_2, ..., X_n)^{t-1} = \sum_{i=2}^{n} X_i^t w_i, w_i \in \Omega_{B/R}, 2 \leq i \leq n. \tag{2}\]

Consider the short exact sequence (theorem 2.3, lemma 2.4, [3]).

\[
0 \to \Omega_{R/k} \otimes B \to \Omega_{B/k} \xrightarrow{\text{pr}_B} \Omega_{B/R} \to 0 \tag{3}\]

Due to lemma in theorem 3.6, [3] we have a split exact sequence

\[
0 \to \bigcap_{s>0} m^s\Omega_{R/k} \to \Omega_{R/k} \to \bigoplus_{i=1}^{n} R dX_i \to 0 \tag{4}\]

Tensoring with \( B \) we obtain from above the following split exact sequence.
0 \rightarrow \cap_{s>0} m^s\Omega_{R/k} \otimes B \rightarrow \Omega_{R/k} \otimes B \rightarrow \bigoplus_{i=1}^n BdX_i \rightarrow 0.

Let \( \tilde{w}_i \in \Omega_{B/k} \) be such that \( \eta(\tilde{w}_i) = w_i \). The following equality then follows from (2):

\[
dy(X_2, ..., X_n)^{t-1} = \sum_{i=2} X_i^t \tilde{w}_i + \sum b_i dx_i + \lambda, b_i \in B,
\]

\( \lambda \in \cap_{s>0} m^s\Omega_{R/k} \otimes B \)

Note that \( \tilde{D}(\lambda) = 0 \). Applying \( \tilde{D} : \Omega_{B/k} \rightarrow F \) to (3) we obtain:

\[
\tilde{D}(dy)(X_2, ..., X_n)^{t-1} = \sum X_i^t \tilde{D}(\tilde{w}_i) + b_1
\]

From (1), it follows that

\[
(X_2, ..., X_n)^{t-1} = 2y\sum X_i^t \tilde{D}(\tilde{w}_i) + 2yb_1
\]

Let \( S = k[[X_1, ..., X_n, Y]]/(Y^2 - X_1) \approx k[[X_2, ..., X_n, Y]] \).

For any finite extension \( C \) of \( S \) contained in \( B \) such that \( b_i \in C \) and \( \tilde{w}_i \in \Omega_{C/k}, 2 \leq i \leq n \), we have \( \text{Tr}_{Q(C)/Q(R)}(\mu \tilde{D}(\tilde{w}_i)) \mu \in C \subset R \)

(lemma 2.1). By corollary in 2.1 we obtain \( \tilde{D}(\tilde{w}_i) \subset C, 2 \leq i \leq n \).

Since \( \text{Tr}_{Q(C)/Q(R)} = \text{Tr}_{Q(S)/Q(R)} \bullet \text{Tr}_{Q(C)/Q(S)} \), it follows by the above lemma that \( \text{Tr}_{Q(C)/Q(S)}(\tilde{D}(\tilde{w}_i)) \subset S = R\{1/2y, y/2y\}, 2 \leq i \leq n \), where \( S = \{z \in Q(S) | \text{Tr}(zS) \subset R\} \); \( S \) is a free \( R \)-module generated by \( \{1/2y, y/2y\} \) ([9], [13]).

Applying \( \text{Tr}_{Q(C)/Q(S)} \) to (4) we obtain (ignoring the units):

\[
(X_2, ..., X_n)^{t-1} = \sum X_i^t 2y \text{Tr}_{Q(C)/Q(S)}(\tilde{D}(\tilde{w}_i)) + 2yb_1 \text{Tr}_{Q(C)/Q(S)} b_1.
\]

It follows from above that

\[
(X_2, ..., X_n)^{t-1} = \sum X_i^t s_i + 2yc_1, s_i, c_1 \in S
\]

This contradicts the monomial property over \( S \) ([6]). Hence

\[
H_X^{n-1}(\Omega_{B/R}) \neq 0.
\]

2.3. Corollary. Let \((R, m)\) and \( B \) be as above. Then \( H_m^{n-1}(\Omega_{B/R}) \neq 0 \).

Proof. We continue with our notations from the above theorem.

Let us recall that localization commutes with module of differentials even in non-noetherian situations. By corollary 2.1 in ([3]) we have \( H_X^{n-1}(\Omega_{B/R}[1/X_1]) = 0 \). This implies, due to the above theorem and the following exact sequence

\[
H_m^{n-1}(\Omega_{B/R}) \rightarrow H_X^{n-1}(\Omega_{B/R}) \rightarrow H_X^{n-1}(\Omega_{B/R}[1/X_1]) \rightarrow 0
\]

that \( H_m^{n-1}(\Omega_{B/R}) \neq 0 \).

Remark. Since \( H_m^{n}(\Omega_{B/R}) = 0 \) (corollary, 2.1, [3]), the above corollary implies that the highest non-vanishing local cohomology of \( \Omega_{B/R} \)
occurs at the (n-1)-th stage. A proof of this observation can also be obtained without applying corollary 2.1.

2.4. Due to corollary 2 lemma 2.1 in [3] we know if \((R, m)\) is a complete local normal domain of dimension \(n\) of equicharacteristic 0 and \(B\) is an integral domain integral over \(R\), then \(H^a_m(\Omega_{B/R}) = 0\). In this perspective we have the following proposition.

**Proposition.** Let \((R, m)\) be a complete regular local ring of dimension \(n\) of equicharacteristic 0 and \(B\) be an integral domain integral over \(R\). Suppose \(B\) contains a square root of a regular parameter of \(R\). Then for any system of parameters \(y_1, \ldots, y_n\) of \(R\),

\[
\Omega_{B/R} \otimes B/yB = \Omega_{(B/yB)/(R/y)} \neq 0;
\]

here \(y\) denotes the ideal generated by \(y_1, \ldots, y_n\).

**Proof.** Let \(R = k[[X_1, \ldots, X_n]]\) and let \(\underline{X} = (X_1, \ldots, X_n)\). It is enough to show that \(\Omega_{(B/\underline{X}B)/k} \neq 0\). Let \(y \in B\) be such that \(y^2 = X_1\); then

\[
2ydy = dX_1 \quad \text{i.e.} \quad 2X_1dy = ydX_1 \quad \text{in} \quad \Omega_{B/k}.
\]

If possible let \(\text{im} \; dy = 0 \quad \text{in} \quad \Omega_{(B/\underline{X}B)/k}\).

Consider the exact sequence (theorem 2.3, lemma 2.4, [3]):

\[
0 \rightarrow \Omega_{R/k} \otimes B \rightarrow \Omega_{B/k} \rightarrow \Omega_{B/R} \rightarrow 0
\]

Tensoring the sequence with \(B/\underline{X}B\) we obtain the following exact sequence:

\[
\rightarrow \Omega_{R/k} \otimes B/\underline{X}B \rightarrow \Omega_{B/k} \otimes B/\underline{X}B \rightarrow \Omega_{(B/\underline{X}B)/k} \rightarrow 0
\]

im\(dy = 0 \quad \text{in} \quad \Omega_{(B/\underline{X}B)/k} \Rightarrow \)

\[
dy = \sum X_iw_i + \sum b_idX_i + \lambda \quad \text{in} \quad \Omega_{B/k}, \quad b_i \in B, \quad w_i \in \Omega_{B/k} \quad \text{for} \quad 1 \leq i \leq n\]

and \(\lambda \in \bigcap m^i \Omega_{R/k} \otimes B\).

Let \(\tilde{D} : \Omega_{B/k} \rightarrow Q(B)\) denote the \(B\)-linear map induced by the extension of \(\partial/\partial X_1 : R \rightarrow R\). Applying \(\tilde{D}\) to the above equation, we obtain

\[
\frac{1}{2}y = \sum X_iD(w_i) + b_1.
\]

This implies

\[
1 - 2yb_1 = \sum X_i2y\tilde{D}(w_i).
\]

Since \(R\) is complete, \(B\) is local and \(1 - 2yb_1\) is unit, say \(u \in B\). Thus

\[
1 = \sum X_i2iu^{-1}\tilde{D}(w_i).
\]

Applying the trace map we obtain via theorem 2.1.

\[
1 = \sum X_ir_i, \quad r_i \in R \quad \text{— a contradiction.}
\]

**Remark.** This result is valid even if \(R\) is not complete provided \(B\) is local.
2.5. Here we will use theorem 2.3 to prove an observation in the mixed characteristic.

**Proposition 2.5.** Let $R = V[[X_1, ..., X_{n-1}]]$, $m = (p, X_1,..., X_{n-1})$ and $(V, pV)$ is a complete discrete valuation ring. Let $B$ denote the absolute integral closure of $R$ and let $X$ denote the ideal generated by $X_1, ..., X_{n-1}$ in $R$. Then the map $\alpha : \overset{\wedge}{H}^{n-2}(\Omega_{B/R}) \to \overset{\wedge}{H}^{n-1}(\Omega_{B/R})$ is non-null.

**Proof.** Let us recall that the above map is part of a long exact sequence:

$$H^{n-2}_X(\Omega_{B/R}) \to H^{n-2}_X(\Omega_{B/R}[1/p]) \to H^{n-1}_X(\Omega_{B/R}) \to H^{n-1}_X(\Omega_{B/R})$$

(5)

Let $y \in B$ be such that $y^p = X_1 \Rightarrow py^{p-1}dy = dX_1 \Rightarrow pX_1dy = ydX_1 \in \Omega_{B/V} \Rightarrow pyX_1dy = 0 \in \Omega_{B/R}$.

Consider the element $\lambda = (dy(X_2, ..., X_{n-1})^t-1, 0, 0) \in \oplus_{1}^{n-1} \Omega_{B/R}[1/p]$.

Note that $X_1^t X_1^t = X_1^t X_1^t X_1^t = 0 \in \Omega_{B/R}[1/p]$. Thus $\lambda$ represents a non-null element in $H^{n-1}(\overset{\wedge}{\mathcal{X}}, \Omega_{B/R}[1/p])$ (proof of theorem 2.2). Since $\Omega_{B/R}(\Omega_{B/V}) = p\Omega_{B/R}(\Omega_{B/V})$ (theorem 4.1, [3]), we have an exact sequence

$$0 \to H^i_p(\Omega_{B/R}) \to \Omega_{B/R} \to \Omega_{B/R}[1/p] \to 0$$

(6)

It follows that $H^i_X(\Omega^i_{B/R}) \simeq H^i_X(\Omega_{B/R})$ for $i \geq 0$. Then $\alpha$ (class of $\lambda$) = class of $X_1^t X_1^t W_i, \ W_i \in (0 : p^i) \Omega_{B/R}, 2 \leq i \leq n - 1, .... (6)$

We have an exact sequence (theorem 2.3, lemma 2.4, [3]):

$$0 \to \Omega_{B/V} \otimes B \to \Omega_{B/V} \to \Omega_{B/R} \to 0$$

Applying $\text{Hom}_B(B/p^iB, -)$ we obtain the following short exact sequence:

$$0 \to (0 : p^i) \Omega_{B/V} \to (0 : p^i) \Omega_{B/R} \to \Omega_{B/V} \otimes B/p^iB \to 0$$

(7)

Applying $\beta$ to (6) we obtain the following equality from equ (7):

$$p^{i-1} X_1^t X_1^t X_1^t - \sum_{i} X_1^t \mu_i \in \Omega_{B/V} \otimes B/p^iB, \mu_i \in \Omega_{B/V} \otimes B, 1 \leq i \leq n.$$  

Hence,

$$p^{i-1} X_1^t X_1^t X_1^t - \sum_{i} X_1^t \mu_i + p^{i} \nu, \nu \in \Omega_{B/V} \otimes B$$

(8)

Let $D_1$ be the extension of $\partial/\partial X_1 : R \to R$ to $B \to Q(B)$ and let $D_1 : Q_{B/k} \to Q(B)$ denote the corresponding $B$-linear map.

Applying $D_1$ to (8) we obtain
\[p^{t-1}y(X_1, \ldots, X_{n-1})^{t-1} = \Sigma X_i^t b_i + p^t c \text{ where } b_i, c \in B, 1 \leq i \leq n.\]

Since \(y^p = X_1\), the above equality contradicts the monomial property in \(B\). Hence \(im \alpha \neq 0\).

**Remark.** Since \(H_{\Delta}^{n-1}(\Omega_{B/R}[1/p]) = 0\) (corollary 2.1, [3]), the above proposition shows that the surjective map \(H_{\Delta}^{n-1}(\Omega_{B/R}) \to H_{\Delta}^{n-1}(\Omega_{B/R})\) is not an isomorphism. Let us mention that we are still not able to prove that \(H_{\Delta}^{n-1}(\Omega_{B/R}) \neq 0\) in mixed characteristic (raised as a question in [3]).

**Section 3.**

**3.1. Lemma.** Let \(R_i = V[[X_1, \ldots, X_i]], 1 \leq i \leq n - 1, \ R_0 = V\) and \(R_{n-1} = R\). Let \(B, B_i\) denote the absolute integral closure of \(R, R_i\) respectively for \(0 \leq i \leq n - 1\), \(B_0 = A\). Then each of \(\Omega_{B_i/V} \) and \(\Omega_{B_i/R_i}\) is \(B_i\)-summand of \(\Omega_{B/V}\) and \(\Omega_{B/R}\) respectively for \(0 \leq i < n - 1\).

Similar result is also valid in equicharacteristic zero.

**Proof.** Let \(Q_i\) be a prime ideal of \(B\) lying over \((X_{i+1}, \ldots, X_{n-1})\) in \(R\). Since \(B\) is absolutely integrally closed, we have \(Q_i = Q_i^2\), \(B/Q_i\) is the absolute integral closure of \(R_i\) and hence \(B_i \simeq B/Q_i\). Thus \(B_i \xrightarrow{\tilde{j}_i} B \xrightarrow{\tilde{\eta}_i} B/Q_i\) is an \(R_i\)-algebra isomorphism. We have the following exact sequences:

\[Q_i/Q_i^2 \to \Omega_{B/V} \otimes B/Q_i \to \Omega_{(B/Q_i)/V} \to 0\]

and

\[Q_i/Q_i^2 \to \Omega_{B/R} \otimes B/Q_i \to \Omega_{(B/Q_i)/R_i} \to 0.\]

It follows from above that

\[\Omega_{B/V} \otimes B/Q_i \simeq \Omega_{B_i/V}\]

and

\[\Omega_{B/R} \otimes B/Q_i \simeq \Omega_{B_i/R_i}.\]

Thus the composite maps:

\[\Omega_{B_i/V} \xrightarrow{\tilde{j}_i} \Omega_{B/V} \xrightarrow{\tilde{\eta}_i} \Omega_{(B/Q_i)/V}\]

and

\[\Omega_{B_i/R} \xrightarrow{\tilde{j}_i} \Omega_{B/R} \xrightarrow{\tilde{\eta}_i} \Omega_{(B/Q_i)/R_i}\]

are \(B_i\)-module isomorphism. Hence the result.

**3.2.** For the rest of section 3 we assume \(R = V[[X_1, \ldots, X_{n-1}]], V\)-a complete discrete valuation ring in mixed characteristic \(p\) such that \(V/pV\) is perfect; let \(A, B\) denote the absolute integral closures of \(V, R\) respectively. For lemma 3.2 and corollary 2 in 3.4 we assume that \([Q(V) : Q_p] < \infty, Q_p = Q(\hat{Z}_{pZ})\).
Since $\Omega_{Q(A)/Q(V)} = 0, \Omega_{A/V}$ is a $p$-torsion module. In a more general setup Fontaine [4] showed that $\Omega_{A/V}$ is generated by $\{d\xi_t\}_{t \geq 1}$, where $\xi_t$ is a $p^t$-th root of unity such that $\xi_t^p = \xi_t$ for $t > 0$. We have the following lemma.

**Lemma.** $\Omega_{A/V} \otimes B \simeq H^1_p(B)$ and $H^{n-1}_m(\Omega_{A/V} \otimes B) \simeq H^n_m(B)$.

**Proof.** Since $\Omega_{A/V}$ is a $p$-torsion module, $\Omega_{A/V} \simeq \lim_{\to} (0 : p^t)A/V$. In ([12]) it was derived from Fontaine’s work that $(0 : p^t)\Omega_{A/V} \cong A/p^tA$. We have the following commutative diagram:

$$(0 : p^t)\Omega_{A/V} \longrightarrow A/p^tA$$

$$(0 : p^{t+1})\Omega_{A/V} \longrightarrow A/p^{t+1}A$$

Where the horizontal arrows are isomorphisms, the right vertical arrow is multiplication by $p$ and the left vertical arrow is the inclusion map.

It follows from above that $\Omega_{A/V} \simeq \lim_{\to} A/p^tA = H^1_p(A)$.

Hence $\Omega_{A/V} \otimes B \simeq H^1_p(B)$.

Moreover $H^{n-1}_m(\Omega_{A/V} \otimes B) \simeq H^{n-1}_m(H^1_p(B)) \simeq H^n_m(B)$.

**Corollary.** The validity of the monomial property for local rings is equivalent to the condition that $H^{n-1}_m(\Omega_{A/V} \otimes B) \neq 0$, i.e. $H^{n-1}_m(\Omega_{A/V} \otimes B) \neq 0$

3.3. Our next observation is the following.

**Theorem.** Let $AR = A \otimes V$. With notations as above the following sequence is exact:

$$0 \rightarrow \Omega_{A/V} \otimes_B A \xrightarrow{\alpha} \Omega_{B/R} \rightarrow \Omega_{B/AR} \rightarrow 0$$

**Proof.** The following sequence is exact

$$\Omega_{A/V} \otimes_B A \xrightarrow{\alpha} \Omega_{B/R} \rightarrow \Omega_{B/AR} \rightarrow 0.$$
\[ \Omega_{B/R} = \lim_{\to} \Omega_{C/R}. \quad A_t R = A_t \otimes R \simeq A_t[[X_1, \ldots, X_{n-1}]] \simeq R[Y]/(f_t(Y)) \]

\( A_t R \) is a complete regular local ring.

\[ \Omega_{A_t/V} \otimes B \simeq \Omega_{A_t/V} \otimes A_t R \otimes B; \]

\[ \Omega_{A_t/V} \otimes A_t R \simeq \Omega_{A_t/V} \otimes V \simeq \Omega_{A_tR/R}. \]

Write \( f = f_t(Y) \), \( f' = f'_t(Y) \) and \( S = A_t R = R[Y]/(f) \). We need to show \( \Omega_{S/R} \otimes B \to \Omega_{B/R} \) is injective.

It is enough to show that \( \Omega_{S/R} \otimes C \to \Omega_{C/R} \) is injective where \( C \) is any normal domain module finite extension of \( S \) (hence a module finite extension of \( R \)).

\[ \Omega_{S/R} = S/f' S; \text{ hence } \Omega_{S/R} \otimes C = C/f'C. \]

We have an exact sequence ([11]):

\[ \Gamma_{C/S} \beta \to \Omega_{S/R} \otimes C \xrightarrow{v} \Omega_{C/R} \to \Omega_{C/S} \to 0 \]

we need to show \( v \) is injective i.e. to show that \( \text{im} \beta = 0 \).

Since \( S \) and \( C \) are complete local normal domains and \( C \) is a module finite extension of \( S \), for every prime ideal \( P \) of height1 in \( S \), \( C_P \) is a Dedekind domain module finite extension of \( S_P \). Hence, by lemma 2.2 in [3] \( \Gamma_{C/S} P = \Gamma_{C_P/S_P} = 0 \). This implies that \( \text{im} \beta \) has at least codimension 1 in \( \Omega_{S/R} \otimes C = C/f'C \). Since \( C \) is a complete local normal domain, \( C/f'C \) can’t have any sub-module of codimension \( \geq 1 \) in \( C/f'C \). Thus \( \text{im} \beta = 0 \) and the assertion follows.

3.4. Corollaries.

1. There exists a commutative diagram of short exact sequences

```
0 \quad 0
\downarrow \quad \downarrow
\Omega_{A/V} \otimes B \quad \Omega_{A/V} \otimes B
\downarrow \quad \downarrow
0 \quad \Omega_{R/V} \otimes B \quad \quad \Omega_{B/V} \quad \quad \Omega_{B/R} \quad \quad 0
\downarrow \quad \downarrow \quad \downarrow
0 \quad \Omega_{R/V} \otimes B \quad \quad \Omega_{B/A} \quad \quad \Omega_{B/AR} \quad \quad 0
```

we need to show \( v \) is injective i.e. to show that \( \text{im} \beta = 0 \).
Proof. The exactness of the middle row follows from theorem 2.3 and lemma 2.4 in [3]. Since $\Omega_{A/V} \otimes B$ is a $p$-torsion module and $\Omega_{R/V} \otimes B$ is $B$-flat and hence torsion free over $B$, the assertion follows. For local cohomological implications see Remarks 2 in (4.7).

2. There exist exact sequences:

$$0 \to H^1_p(B) \to H^0_p(\Omega_{B/V}) \to H^0_p(\Omega_{B/A}) \to 0$$

and

$$0 \to H^1_p(B) \to H^0_p(\Omega_{B/R}) \to H^0_p(\Omega_{B/AR}) \to 0.$$ 

Proof. By the above corollary we have an exact sequence:

$$0 \to \Omega_{A/V} \otimes_A B \to \Omega_{B/V} \to \Omega_{B/A} \to 0.$$ 

This implies, due to lemma (3.2), and the fact that $\Omega_{A/V} = p\Omega_{A/V}$, the exactness of the first exact sequence.

The second exact sequence is also proved by similar arguments.

Remark. Corollary 2 above implies, due to corollary 5.10 [2], that both $\Omega_{B/V}$ and $\Omega_{B/R}$ contains a Cohen-Macaulay torsion $B$-module. From middle row of the diagram in corollary 1 above it follows, due to theorem 5.1 [2] and flatness of $\Omega_{R/V}$ over $R$, that $\Omega_{B/V}$ also contains a cohomologically Cohen-Macaulay flat $B$-module. For the second assertion no restriction on $Q(V)$ is necessary.

3.5. In our final theorem of this section we prove the following.

Theorem. With $V, R, B$ as in 3.2, $\Omega_{B/R}$ can be mapped onto a $B$-module $M$ such that i) $M$ is cohomologically Cohen-Macaulay of cohomological dimension $n - 1$, ii) for every $t > 0, (0 : p^t)M$ is finitely generated Cohen-Macaulay free $B/p^tB$-module of rank $n - 1$ and dimension $n - 1$ and iii) $(0 : p^t)M$ is a summand of $(0 : p^t)\Omega_{B/R}$.

Proof. We have a short exact sequence:

$$0 \to H^0_p(\Omega_{B/V}) \to \Omega_{B/V} \to \Omega_{B/V}[1/p] \to 0.$$ 

(the exactness follows from the fact that $\Omega_{B/V} \otimes B/pB = 0$(theorem 4.1, [3]))

Let $M$ be the cokernel of $(H^0_p(\Omega_{B/V}) \to \Omega_{B/R})$

Consider the following commutative diagram of short exact sequences:
The exactness of the left column follows from the fact that $\Omega_{R/V}$ being a flat $R$-module ([3]), $\Omega_{R/V} \otimes B$ is a flat $B$-module and hence torsion-free over $B$.

Since $H^i_m(\Omega_{B/V}[1/p]) = 0$ for $i \geq 0$ and $\Omega_{R/V}$ is $R$-flat (theorem 2.3, [3]), it follows from the last row of the above diagram that $H^i_m(M) \cong \Omega_{R/V} \otimes H^{i+1}_m(B) \simeq \bigoplus \nolimits_1^{n-1} H^{i+1}_m(B)$ (lemma, theorem 3.6, [3]) for $i \geq 0$...............................

in particular $H^{n-1}_m(M) \cong \bigoplus \nolimits_1^{n-1} H^{n}_m(B)$.

By theorem 5.1, [2] it follows that $H^i_m(B) = 0$ for $i < n$; hence $H^i_m(M) = 0$ for $i < n - 1$ and $H^{n-1}_m(M) \neq 0$ ([1], [7]). Thus $M$ is cohomologically Cohen-Macaulay of cohomological dimension $n - 1$.

From the last row of the above commutative diagram we obtain $(0 : p^t)M \cong \Omega_{R/V} \otimes B/p^tB \cong \bigoplus \nolimits_1^{n-1} B/p^tB$ (lemma within theorem 3.6, [3]).

Since, by corollary 5.10 [2], $B/p^tB$ is Cohen-Macaulay, our assertion follows.

**Corollary.** For every $t > 0$, $\Omega_{B/R}$ contains a finitely generated Cohen-Macaulay free $B/p^tB$-module of rank $n - 1$ and dimension $n - 1$.

If $[Q(V) : Q(\hat{Z}_{p^t})] < \infty$, then the corresponding rank is $n$.

Proof follows from: a) the above commutative diagram by applying $\text{Hom}(B/p^tB, -)$ to the last row and the last column of the above diagram, b) corollary 2, 3.4 and c) corollary 5.10, [2].

**Remark.** Since $H^{n-1}_m(\Omega_{B/R}) \rightarrow H^{n-1}_m(M) \rightarrow 0$ is exact, the observation in (*) also leads to a new proof of the fact (theorem 4.1, [3]) that
$H^m_{\mathfrak{m}}(\Omega_{B/R}) \neq 0$ if and only if the direct summand property is valid over $R$.

**Section 4.**

**4.1.** Our first proposition deals with torsion-free property of standard generators of $\Omega_{B/V}$.

**Proposition.** Let $R = V[[X_1, \ldots, X_{n-1}]]$ or $k[[X_1, \ldots, X_n]]$ or $k[X_1, \ldots, X_n]$ where characteristic of $k$ is 0. Let $B$ denote an integral extension of $R$ in an algebraic extension of $Q(R)$. If for some $c \in B, b \in B - V, cdb \in \Omega_{B/V}$ is a torsion element, then $b \in A \cap B$ where $A$ is the absolute integral closure of $V$.

If $R$, as above, contains a field $k$ of characteristic 0 then for $b \in B - \bar{k}, db$ is torsion-free, where $\bar{k}$ is the algebraic closure of $k$ in $Q(B)$.

**Proof.** Let $R = V[[X_1, \ldots, X_{n-1}]]$. Let $b \in B$ be such that $db$ is a non-null torsion element in $\Omega_{B/V}$. We have an exact sequence

$$0 \rightarrow \Omega_{R/V} \otimes B \xrightarrow{\tilde{\beta}} \Omega_{B/V} \rightarrow \Omega_{B/R} \rightarrow 0$$

Let $\tilde{c} \in B - \{0\}$ be such that $\tilde{c}'cdb = 0$. Let $\tilde{c} = c'c$; $\tilde{c}db = 0$. $\tilde{c}$ satisfies an integral equation over $R$: $\tilde{c}' + r_1'\tilde{c}'^{-1} + \ldots + r_s' = 0$, $r_i' \in R, 1 \leq i \leq s$ and $r_s' \neq 0$. This implies that $r_s'db = 0$. We write $r$ for $r_s'$ i.e. $rdb = 0, r \neq 0 \in R$.

Since $Q(B)$ is separable algebraic over $Q(R)$, if $f(X)$ denotes the minimal polynomial for $b \over R$ then $f'(b) \neq 0$. Hence

$$f'(b)db = 0...............................................................(10)$$

Let $f(X) = X^t + r_1X^{t-1} + \ldots + r_t$. Then

$$f(b) = b^t + r_1b^{t-1} + \cdots + r_t = 0..................(11)$$

Hence $f'(b)db + \Sigma d(r_i)b^{t-i} = 0$ in $\Omega_{B/V}$. This implies, due to (10)

$$r\Sigma d(r_i)b^{t-i} = 0..............................................(12)$$

Since $\Omega_{R/V}$ is flat over $R$ and $B$ is an integral domain, $\Omega_{R/V} \otimes_R B$ is torsion-free $B$-module. From the above exact sequence we obtain, by (12),

$\Sigma d(r_i) \otimes b^{t-i} = 0$ in $\Omega_{R/V} \otimes_R B$, i.e. $\Sigma d(r_i)b^{t-i} = 0 \in \Omega_{B/V}$.

$R[b]$ is a free $R$-module of rank $t$ (due to (10)) with basis $1, b \ldots b^{t-1}$. We have an isomorphism

$$\eta : \oplus^t_i R \rightarrow R[b], \eta(e_i) = b^{t-i}..................(13)$$
Tensoring (13) with $\Omega_{R/V}$ we get an isomorphism,

$$\tilde{\eta} : \oplus_t^1 \Omega_{R/V} \to \Omega_{R/V} \otimes_R R[b], \quad \tilde{\eta}(d(s_1), \ldots, d(s_t)) = \Sigma d(s_i) \otimes b^{t-i}.$$  

Since $\Omega_{R/V}$ is $R$-flat, $\alpha : \Omega_{R/V} \otimes_R R \to \Omega_{R/V}$ is injective.

Composing this with the injection $\beta : \Omega_{R/V} \otimes_R B \to \Omega_{B/V}$ we have an injective map $\beta \cdot \alpha : \Omega_{R/V} \otimes_R R[b] \to \Omega_{B/V}$.

Since $\beta \cdot \alpha(\Sigma d(r_i) \otimes b^{t-i}) = \Sigma d(r_i) b^{t-i} = 0$, and $\tilde{\eta}$ is an isomorphism we have $d(r_i) = 0$, $1 \leq i \leq t$. This implies that $r_i \in V$, $1 \leq i \leq t$.

Hence from (11) it follows that $b$ is integral over $V$ and thus $b \in A$.

If $R$ is of equicharacteristic 0, the above arguments show that for every $b \in B - \bar{k}$, $db$ is torsion-free over $B$.

### 4.2
As mentioned in the introduction proposition 4.1 led us to look for sufficient conditions for torsion-free property of the module of differentials as a whole. And in this respect we were guided by Suzuki’s work ([10]). Our main goal in 4.2 and 4.3 is to prove a formal version in all characteristics of Suzuki’s theorem on the connection between normality of complete intersection affine domains over a field $k$ and torsion-freeness of the corresponding module of differentials. For this purpose we first extend Grothendieck’s proposition 22.7.2 in ([5]) to mixed characteristic. Grothendieck’s arguments work in this case with several minor modifications.

**Theorem.** Let $V$ be a complete discrete valuation ring in mixed characteristic $p$ with maximal ideal $pV$ such that the residue field $k$ of $V$ is perfect. Let $R = V[[X_1, \ldots, X_{n-1}]]$ and $C = R/I$ be an integral domain such that $p \neq 0$ in $C$. Let $P$ be a prime ideal of $R$ containing $I$ and let $q = P/I$. The following conditions are equivalent.

i) $C_q$ is formally smooth over $V$ for the q-adic topology.

ii) There exists derivations $D_i$, $1 \leq i \leq r$, $\in \text{Der}_V(R)$ and elements $\{f_i\}_{1 \leq i \leq r}$ of $I$ such that $\{f_i\}_{1 \leq i \leq r}$ generate $IR_P$ and $\det(D_if_j) \notin P$.

iii) $C_q$ is an unramified regular local ring.

**Proof.** i) $\iff$ iii). Since $V$ is a d.v.r. and $C_q$ is a domain, $V \to C_q$ is formally smooth if and only if $V/pV \to C_q/pC_q$ is formally smooth. Since $V/pV$ is perfect this is equivalent to $C_q/pC_q$ being a regular local ring, i.e. $C_q$ is an unramified regular local ring.

The proof of i) $\iff$ ii) becomes evident by following Grothendieck’s arguments in his proof of 22.7.2 in ([5]) with relevant minor modifications needed for the mixed characteristic case. Grothendieck’s assumption for ensuring separability is assured here by assuming that the residue field $k$ of $V$ is perfect.

First we recall two basic facts that will be needed in our proof.


a) If \( p \) is in \( J \) for any prime ideal \( J \) of \( R \) then \( p \notin J^2R \) ([8]).

b) \( \widehat{\Omega}_{R/V} \cong \bigoplus_i^1 R dx_i, 1 \leq i \leq n - 1 \), where \( \widehat{\Omega}_{R/V} \) is the m-adic completion of \( \Omega_{R/V}, m = \text{the maximal ideal of } R \) ([3], [5]).

Let \( L = k(p) = R_P/PR_P = C_q/qC_q \). By (22.6.2, (ii), [5]) the condition i) is equivalent to the condition

\[
d_0 : I/I^2 \otimes L \to \Omega_{R/V} \otimes L \to \widehat{\Omega}_{R/V} \otimes L \quad \text{(14)}
\]

and show that condition ii) is equivalent to the assertion that \( \hat{d} \) is injective. Note that \( I/I^2 \otimes L = IR_P/I^2R_P \otimes L \). Condition ii) implies that, if \( F_i = \hat{d}(\bar{f}_i \otimes 1) \), where \( \bar{f}_i \) is the image of \( f_i \) in \( IR_P/I^2R_P \), the matrix \( (F_i, D_j \otimes 1) \) is invertible. This implies \( \{F_i\} \) are linearly independent, and hence, so are \( \{f_i \otimes 1\} \) which generate \( IR_P/I^2R_P \otimes L \).

Thus \( \hat{d} \) is injective. Inversely suppose \( \hat{d} \) is injective and let \( \{f_i \in I\}_{1 \leq i \leq r} \) be such that \( \{\bar{f}_i \otimes 1\} \) form a basis of \( IR_P/I^2R_P \otimes L \). Then \( \{F_i\} \) form a basis of \( \text{Im}(\hat{d}) \). By fact b), \( \widehat{\Omega}_{R/V} \) is a free \( R \)-module of rank \( (n - 1) \) and the \( V \)-derivations of \( R \) generate its dual. The fact that \( \{F_i\} \) are linearly independent implies the existence of \( V \)-derivations \( D_i \) of \( R \) into itself such that the matrix \( (F_i, D_j \otimes 1) \) is invertible as stated in ii).

\( i) \Rightarrow ii) \). We need to show that i) implies that \( \hat{d} \) is injective.

Note that \( \hat{d} \) is the composite of

\[
I/I^2 \otimes L \xrightarrow{\alpha} P/(P^2 + pR) \otimes L \xrightarrow{\beta} \widehat{\Omega}_{R/V} \otimes L,
\]

where \( \beta \) is induced by the standard differential map ”\( d \)”. Since \( R_P \) is a regular local ring and \( IR_P, p \) form a part of a regular system of parameters of \( R_P \), it follows that the natural injection \( \alpha \) is injective. Hence it is enough to show that

\[
P/(P^2 + pR) \otimes L \xrightarrow{\beta} \widehat{\Omega}_{R/V} \otimes L \quad \text{(15)}
\]

is injective.

We have an exact sequence

\[
P/(P^2 + pR) \to \Omega_{R/V} \otimes R/P \to \Omega_{(R/P)/V} \to 0.
\]

Since \( (R, m) \) is a Zariski ring, it follows from (20.7.20, [5]) that

\[
P/(P^2 + pR) \to \widehat{\Omega}_{R/V} \otimes R/P \to \widehat{\Omega}_{(R/P)/V} \to 0
\]

is exact. And hence

\[
P/(P^2 + pR) \otimes L \xrightarrow{\beta} \widehat{\Omega}_{R/V} \otimes L \to \widehat{\Omega}_{(R/P)/V} \otimes L \to 0 \quad \text{(16)}
\]
is exact.

It follows from (Theorem 3.6, [3]) that, since \( p \in P \), \( \text{rank}(\widehat{\Omega}_{(R/P)\otimes L}) = \dim R/P \) and \( \widehat{\Omega}_{R/V} \otimes L \) is of rank \((n-1)\). Moreover, since \( \dim P/(P^2+pR) \otimes L = \text{height} P - 1 \), it follows from (16) that

\[
0 \to P/(P^2+pR) \otimes L \xrightarrow{\partial} \widehat{\Omega}_{R/V} \otimes L \to \widehat{\Omega}_{(R/P)/V} \otimes L \to 0
\]

is exact.

Hence our proof is complete.

4.3. Next we extend Suzuki’s theorem to the formal set-up ([10]).

**Suzuki’s theorem.** Let \( R \) be a complete intersection locality over a field \( k \). Then \( R \) is normal if and only if \( \Omega_{R/k}^* \) is torsion-free, where \( k^* = k \) if characteristic of \( k = 0 \) or a differential constant field of \( R \) if the characteristic is positive.

Our theorem is the following.

**Theorem.** Let \( C = R/I \), where \((R, m)\) is either unramified or equicharacteristic complete regular local ring with coefficient ring \( V \) (field \( k_\ast \)) such that \( V/pV(k_\ast) \) is perfect and \( I \) is a complete intersection prime ideal of \( R \). Let \( \widehat{\Omega}_{C/V}, \widehat{\Omega}_{C/k} \) denote the \( m \)-adic completions of \( \Omega_{C/V}, \Omega_{C/k} \) respectively. We have the following.

i) For mixed-characteristic assume that if \( p \neq 0 \) in \( C \), then every prime ideal \( P \) of height 1 in \( C \) containing \( p \) is such that \( p \notin P^2 C_P \). If \( C \) is normal then \( \widehat{\Omega}_{C/V} \) and \( \widehat{\Omega}_{C/k} \) are torsion-free over \( C \).

ii) If \( \widehat{\Omega}_{C/V} \) or \( \widehat{\Omega}_{C/k} \) is torsion-free then \( C \) is normal.

**Proof.** If \( p \in I \), our assertion reduces to the equicharacteristic case. We here prove the mixed characteristic case; the equicharacteristic case would follow by similar arguments. Our goal is to reduce this formal case to a situation where arguments similar to those given by Suzuki ([10]) would work. Let \( R = V[[X_1, \ldots, X_{n-1}]] \) and let \( I = (f_1, \ldots, f_r) \). In order to attain such reduction, for part i) our main tool is theorem 4.2, in particular the observation (14) in the proof; and for part ii) several observations from ([3]) and ([5]).

i) Let \( d : R \to \Omega_{R/V} \) be the usual derivation. Let \( g(\neq 0) \in R \); then \( g = \lim g_t \) where \( g_t \) is the sum of terms of \( g \) of degree \( \leq t \). We have

\[
dg = \lim d g_t; \quad dg_t = \sum_{i=1}^{n-1} (\partial g_t/\partial X_i)dX_i, \quad \lim dg_t = \sum_{i=1}^{n-1} \lim(\partial g_t/\partial X_i)dX_i = \sum_{i=1}^{n-1}(\partial g/\partial X_i)dX_i \quad \text{(by continuity)}.
\]

Since \( \Omega_{R/V} \) is not Hausdorff, \( dg = \sum(\partial g/\partial X_i)dX_i + \mu \) where \( \mu \in \cap m^t \Omega_{R/V} \). Let us recall that the following
sequence is split exact (lemma in theorem 3.6, [3]):
\[ 0 \to \cap m^t \Omega_{R/V} \to \Omega_{R/V} \xrightarrow{d} \hat{\Omega}_{R/V} (= \bigoplus_{i=1}^{n-1} RdX_i) \to 0 \]

We also denote by \( d \) the composite of \( R \to \Omega_{R/V} \to \hat{\Omega}_{R/V} \); then, for any \( f \in R, df = \Sigma(\partial f/\partial X_i)dX_i \). Since \((R, m)\) is a Zariski ring, it can be checked easily that the exact sequence
\[ I/I^2 \to \Omega_{R/V} \otimes C \xrightarrow{\hat{\alpha}} \hat{\Omega}_{C/V} \to 0 \]
gives rise to the following exact sequence.
\[ I/I^2 \to \hat{\Omega}_{R/V} \otimes C \xrightarrow{\hat{\alpha}} \hat{\Omega}_{C/V} \to 0 \ ..........(17) \]

First we assume that \( C \) is normal. If possible, let \( w \in \hat{\Omega}_{C/V} \) and \( c(\neq 0) \in C \) be such that \( cw = 0 \). Then it follows from (17) that
\[ c \sum_{i=1}^{n-1} \lambda_i \tilde{d}X_i = \sum_{j=1}^{r} \nu_j \hat{df}_j \ ..........(18) \]
where \( \hat{\alpha}(\Sigma \lambda_i \tilde{d}X_i) = w, \tilde{d} = d \otimes Id_C, \lambda_i, \nu_j \in C \); hence
\[ c \Sigma \lambda_i \tilde{d}X_i = \frac{d}{j=1} \nu_j \sum_{i=1}^{n-1} (\partial f_j/\partial X_i) \tilde{d}X_i \ ..........(19) \]

Since \( \tilde{d}X_1, ..., \tilde{d}X_{n-1} \) are linearly independent in \( \hat{\Omega}_{R/V} \otimes C \), we have
\[ \lambda_i = \frac{1}{c} \sum_{j=1}^{d} \nu_j (\partial f_j/\partial X_i) \ ..........(20) \]

. We need to show that \( \nu_j/c \in C, 1 \leq j \leq r \). Then from (18) it would follow that \( w = 0 \in \hat{\Omega}_{C/V} \).

Let \( P \) be a prime ideal of height 1 of \( C \). Then \( C_P \) is a regular local ring. By our assumption \( C_P \) is formally smooth over \( V \) Let \( L = k(P) \)-the field of fractions of \( C_P \). By the proof of proposition 4.2, \( C_P \) is formally smooth over \( V \) if and only if
\[ I/I^2 \otimes L \xrightarrow{d_L} \hat{\Omega}_{R/V} \otimes L \]
is injective, i.e.
\[ I/I^2 \otimes L \xrightarrow{d_L} \bigoplus_{i=1}^{n-1} LdX_i \ ..........(21) \]
is injective.

Due to equation (20), this implies that certain \( r \times r \) minor of \( (\partial f_j/\partial X_i) \) \( 1 \leq j \leq r, 1 \leq i \leq n - 1 \) is invertible and hence \( \nu_j/c \in C_P \) for \( 1 \leq j \leq r \).
d. Since $C$ is normal, $C = \cap C_P$ and thus $\nu_j/c \in C$ for $1 \leq j \leq d$.

ii) $R$ is geometrically regular over $V$. Hence $\Gamma_{R/V} = 0$ (theorem. 2.3 [3]). We have an exact sequence ([11])

$$0 \rightarrow \Gamma_{C/V} \rightarrow I/I^2 \rightarrow \Omega_{R/V} \otimes C \rightarrow \Omega_{C/V} \rightarrow 0$$

(22)

Since $R(C)$ is complete and $I$ is a complete intersection ideal of codimension $r$, $I/I^2$ is a finitely generated free complete $C$-module of rank $r$. Hence, $\Gamma_{C/V}$ and $im \bar{d}$ are also finitely generated complete $C$-modules. Taking the m-adic completion we obtain an exact sequence (20.7.20, [5])

$$0 \rightarrow N \rightarrow I/I^2 \rightarrow \hat{\Omega}_{R/V} \otimes C \rightarrow \hat{\Omega}_{C/V} \rightarrow 0$$

(23)

where $\Gamma_{C/V} \subset N$ (we are not changing the notation for $\bar{d}$).

Since the rank $\hat{\Omega}_{R/V} = \dim R - 1$ and rank $\hat{\Omega}_{C/V} = \dim C - 1$ (21.9.5 [5], theorem 3.6, [3]), via rank calculation it follows that $N = 0$ and hence $\Gamma_{C/V} = 0$. Thus

$$0 \rightarrow I/I^2 \rightarrow \hat{\Omega}_{R/V} \otimes C \rightarrow \hat{\Omega}_{C/V} \rightarrow 0$$

(24)

is exact.

Let us assume that $C$ is not normal. Let $\bar{C}$ denote the normalization of $C$ in $Q(C)$ and let $J = (C : \bar{C}) = \text{ann}_C \bar{C}/C$. Let $P$ be a prime ideal of height 1 in $C$ containing $J$. Then $C_P$ is not regular. By proposition 4.2, rank $\text{jac} (f_1, ..., f_r)$ mod $P < r$. The arguments for the rest of the proof is similar to that given by Suzuki for his proof of the affine case. There exist $b_1, ..., b_r$ of $C$, at least one of which, say $b_1$, is not in $P$ such that $\sum\limits_{i=1}^{r} b_i \bar{d}_i f_i = 0$ in $\hat{\Omega}_{R/V} \otimes C/P$.

Let $\tilde{J} = J : P$. There exist an element $j \in \tilde{J} \bar{C}$ such that $j \notin \bar{C}_P$...........................(25)

Let $j = c/g$, $c, g \in C$. We have

$$j \Sigma b_i \bar{d}_i f_i \in jP \bar{\Omega}_{R/V} \otimes C \subset \tilde{J} \bar{C} \hat{\Omega}_{R/V} \otimes C \subset \hat{\Omega}_{R/V} \otimes C$$

It follows from above and (24) that $j \Sigma b_i \bar{d}_i f_i = \Sigma c_i \bar{d}X_i$, $c_i \in C$

i.e. $c \Sigma b_i \bar{d}_i f_i = g \Sigma c_i \bar{d}X_i$ in $\hat{\Omega}_{R/V} \otimes C$.....................(26)

From (24) it follows that $im(g \Sigma c_i \bar{d}X_i) = 0$ in $\hat{\Omega}_{C/V}$. If $im(\Sigma c_i \bar{d}X_i) \neq 0$ in $\hat{\Omega}_{C/V}$, then this is a torsion element in $\hat{\Omega}_{C/V}$. If $im(\Sigma c_i \bar{d}X_i) = 0$ in
\( \hat{\Omega}_{C/V} \), then it follows from (24) that there are \( r \) elements \( s_1, \ldots, s_r \in C \) such that \( \Sigma c_i \bar{d}X_i = \Sigma s_i \bar{d}f_i \).

Hence it follows from (26) that 
\[ \Sigma (g s_i - c b_i) \bar{d}f_i = 0. \]

Due to (24) \( \bar{d}f_1, \ldots, \bar{d}f_r \) are linearly independent in \( \hat{\Omega}_{R/V} \otimes C \). Hence \( g s_i = c b_i, 1 \leq i \leq r \), i.e. \( j = c/g = s_i/b_i, 1 \leq i \leq r \). Since \( b_1 \notin P \), it follows that \( j \in C_P \) - a contradiction due to (25).

**Corollary.** Let \((R, m)\) be an unramified or an equicharacteristic complete regular local ring and let \( C = R[[X_1, \ldots, X_n]]/I \) be a module finite integral extension of \( R \) such that \( I \) is a complete intersection prime ideal. Suppose that a) \( Q(C)/Q(R) \) is separably algebraic over \( Q(R) \), b) \( \dim R > 1 \) if \( R \) is mixed characteristic, \( \dim R > 0 \) if \( R \) is equicharacteristic and c) in mixed-characteristic for any prime ideal \( P \) of height1 in \( C \) containing \( p \), \( p \notin P^2C/P \). If \( C \) is normal, then \( \Omega_{C/V} \) and \( \Omega_{C/k} \) are torsion-free over \( C \).

**Proof.** We have a commutative diagram of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \to & \Omega_{R/V} \otimes C & \to & \Omega_{C/V} & \to & \Omega_{C/R} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \parallel & & (27) \\
\cap m^t \Omega_{R/V} \otimes C & \to & \cap m^t \hat{\Omega}_{R/V} \otimes C & \to & \cap m^t \Omega_{C/V} & \to & \cap m^t \Omega_{C/R} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

\( \Omega_{C/R} \), being a finitely generated \( C \)-module, is complete. Since \( \Omega_{R/V} \) is a flat \( R \)-module and \( 0 \to \cap m^t \Omega_{R/V} \to \Omega_{R/V} \to \hat{\Omega}_{R/V} \to 0 \) splits, \( \cap m^t \Omega_{R/V} \otimes C \) is a flat \( C \)-module. Since \( C \) is an integral domain the assertion follows immediately.

**4.4. Remarks.** 1. The above diagram shows that for any module-finite extension \( C \) of a power series ring \( R \) such that \( Q(C) \) is separably algebraic over \( Q(R) \), the torsion sub-modules of \( \Omega_{C/V} \) and \( \Omega_{C/k} \) imbed
into \( \hat{\Omega}_{C/V} \) and \( \hat{\Omega}_{C/k} \) respectively and the corresponding local cohomology modules of \( \Omega_{C/V} \) and \( \Omega_{C/k} \) are isomorphic to the same of \( \hat{\Omega}_{C/V} \) and \( \hat{\Omega}_{C/k} \) respectively.

2. With notations as above let \( B \) denote the absolute integral closure of \( C(R) \). Let \( \Omega_{B/V} = \lim \hat{\Omega}_{C/V} \), \( R \subset C \subset B \), \( C \) as in Remark 1 above, in mixed characteristic/equicharacteristic 0. Then the torsion sub-module of \( \Omega_{B/V} \) imbeds into \( \hat{\Omega}_{B/V} \) and the corresponding local cohomology modules of \( \Omega_{B/V} \) and \( \hat{\Omega}_{B/V} \) are isomorphic. This follows from (27) by taking direct limit and the fact that \( \cap m^t \Omega_{R/V} \) is \( R \)-flat and \( \cap m^t \Omega_{R/V} \otimes R/pR = 0 \)

4.5. Our final theorem points out a connection between torsion submodule of \( \hat{\Omega}_{C/V} \) and derivations \( \partial/\partial X_i, 1 \leq i \leq n - 1 \), of \( R \).

**Theorem.** Let \( C \) be a complete local domain in mixed characteristic and \( R = V[[X_1, ..., X_{n-1}]] \hookrightarrow C \) be such that \( C \) is a module finite extension of \( R \). Let \( m \) denote the maximal ideal of \( R \) and let \( \hat{\Omega}_{C/V} \) denote the \( m \)-adic completion of \( \Omega_{C/V} \). Let \( T = \) the torsion sub-module of \( \hat{\Omega}_{C/V} \). Then, for any \( \omega \in \hat{\Omega}_{C/V}, \omega \in T \iff \tilde{D}_i(\omega) = 0 \) for \( 1 \leq i \leq n - 1 \), where \( \tilde{D}_i : \Omega_{C/V} \) or \( \hat{\Omega}_{C/V} \to Q(C) \) is a \( C \)-linear map induced by the extension of the derivation \( \partial/\partial X_i : R \to R, \partial/\partial X_i(X_j) = \delta_{ij} \), to \( D_i : C \to Q(C) \).

**Proof.** \( \iff \): is obvious.

\( \iff \): let \( \hat{d} \) denote the composite of \( C \xrightarrow{d} \Omega_{C/V} \to \hat{\Omega}_{C/V} \) or \( \hat{\Omega}_{C/V} \to \hat{\Omega}_{R/V} \). It follows from the middle column of (27) that \( (\cap m^t \Omega_{R/V} \otimes C) \simeq \cap m^t \Omega_{C/V} \).

It has been pointed out in the beginning of proof of part i), theorem 4.3 that for \( r \in R, \ dr = \Sigma(\partial r/\partial X_i)dX_i + \mu \) where \( \mu \in \cap m^t \Omega_{R/V} \) and \( \hat{d} r = \Sigma(\partial r/\partial X_i)dX_i \).

Let \( \alpha(\neq 0) \in C \). First we want to show that

\[ f'(\alpha)d\alpha = f'(\alpha)\Sigma D_j(\alpha)dX_j + \nu, \nu \in \cap m^t \Omega_{C/V}, \] where \( f(X) \in R[X] \) denotes the minimal polynomial for \( \alpha \).

Let \( f(X) = X^h + r_1X^{h-1} + ... + r_h, \ f(\alpha) = 0. \)

Then \( f'(\alpha)d\alpha + \Sigma_{i=1}^h d(r_i)\alpha^{h-i} = 0. \)
For $1 \leq j \leq n - 1$, \( f'(\alpha)D_j(\alpha) + \sum_{i=1}^{h_i} D_j(r_i)\alpha^{h_i} = 0 \).

Hence
\[
\begin{align*}
&f'(\alpha)d\alpha - \sum_{j=1}^{n-1} f'(\alpha)D_j(\alpha)dX_j \\
&= \sum_{i=1}^{h_i} [d(r_i) - \sum_{j=1}^{n-1} D_j(r_i)dX_j]\alpha^{h_i} \\
&= \sum \nu_i \alpha^{h_i}, \nu_i = [d(r_i) - \sum_{j=1}^{n-1} D_j(r_i)dX_j] \in m^i \Omega_{R/V}.
\end{align*}
\]

Hence
\[
f'(\alpha)d\alpha - \sum_{j=1}^{n-1} f'(\alpha)D_j(\alpha)dX_j \in \cap m^i \Omega_{R/V} \otimes C = \cap m^i \Omega_{C/V}....(28)
\]

Now let \( \omega \in \Omega_{C/V} \); can write \( \omega = \sum_{i=1}^{t} \lambda_i d\alpha_i, \lambda_i, \alpha_i \in C \). Let \( f_i(X) \in R[X] \) denote the minimal polynomial of \( \alpha_i \) and let \( f = \pi f'_i(\alpha_i) \). We have \( \widetilde{D}_j(\omega) = \sum_{i=1}^{t} \lambda_i D_j^\alpha_i \).

It follows from (28) that
\[
f\omega - \sum_{j=1}^{n-1} f\widetilde{D}_j(\omega)dX_j \in \cap m^i \Omega_{C/V}....(29)
\]

Hence it follows from (29) that if \( \widetilde{D}_j(\omega) = 0 \) for \( 1 \leq j \leq n - 1 \), then
\[
f\omega \in \cap m^i \Omega_{C/V} \implies f\text{im} \omega = 0 \in \widehat{\Omega}_{C/V}, \text{i.e. im} \omega \in \widehat{\Omega}_{C/V} \text{ is a torsion element (middle vertical exact sequence in (27))}.
\]

4.6. Corollaries. Notations are as in Remark 2, 4.4.

1. \( \omega \in T(\widehat{\Omega}_{B/V}) \iff \widetilde{D}_i(\omega) = 0, 1 \leq i \leq n - 1 \), where \( T(\widehat{\Omega}_{B/V}) \) denotes the torsion sub-module of \( \widehat{\Omega}_{B/V} \) and \( \widetilde{D}_i \) denotes the corresponding extension: \( \Omega_{B/V} \text{ or } \widehat{\Omega}_{B/V} \to Q(B) \) obtained via the extension of the derivation \( \partial/\partial X_i, 1 \leq i \leq n - 1 \), as mentioned earlier.

This occurs due to the observation that
\[
0 \to \cap m^i \widehat{\Omega}_{R/V} \otimes B \to \Omega_{B/V} \to \widehat{\Omega}_{B/V} \to 0,
\]

obtained by taking direct limit over \( C \) of the middle vertical column in (27), is exact.

2. we have a commutative diagram of short exact sequences:
\[
\begin{array}{cccc}
0 & 0 & \quad & \\
\downarrow & & \downarrow & \\
T(\Omega_{B/V}) & \longrightarrow & T(\bar{\Omega}_{B/V}) & \\
\downarrow & & \downarrow & \\
0 & \longrightarrow & \hat{\Omega}_{R/V} \otimes B & \longrightarrow \Omega_{B/V} \longrightarrow \Omega_{B/R} \longrightarrow 0 \\
\| & & \downarrow & \\
0 & \longrightarrow & \hat{\Omega}_{R/V} \otimes B & \longrightarrow W \longrightarrow N \longrightarrow 0 \\
\downarrow & & \downarrow & \\
0 & 0 & 0 & 
\end{array}
\]

where \( W \) is a \( B \)-submodule of \( \bigoplus_{i=1}^{n-1} Q(B) \) and the vertical map \( \bar{D} : \hat{\Omega}_{B/V} \rightarrow W \) is given by \( \bar{D} = (\bar{D}_1, \ldots, \bar{D}_{n-1}) \); moreover, for \( i \geq 0 \), \( H^i_m(W) = 0 \) and \( H^i_m(N) \simeq \bigoplus_{i=1}^{n-1} H^{i+1}_m(B) \).

The proof follows easily from the facts: i) \( \hat{\Omega}_{B/V} = \lim_{\longrightarrow} \hat{\Omega}_{C/V} \) and the bottom row of (30) is exact

ii) \( W \xrightarrow{p} W \), multiplication by \( p \), is an isomorphism, and iii) \( \hat{\Omega}_{R/V} \otimes B \simeq \bigoplus_{i=1}^{n-1} B \).

Note that \( W \) is torsion-free and \( W/pW = 0 \). Thus \( W \xrightarrow{p} W \) is an isomorphism and hence \( H^i_m(W) = 0 \) for \( i \geq 0 \).

3. \( \Omega_{B/R} \) has an image \( N \) such that i) \( N \) is cohomologically Cohen-Macaulay i.e. \( H^i_m(N) = 0 \) for \( i < n - 1 \), \( H^{n-1}_m(N) \neq 0 \), ii) \( (0 : p^i)N \) is finitely generated Cohen-Macaulay free \( B/p^iB \)-module of rank \( n - 1 \) and of dimension \( n - 1 \), iii) \( (0 : p^i) \) is a summand of \( (0 : p^i) \) for any \( x \neq 0 \in R \), \( (0 : x)N \) can be imbedded into a cohomologically Cohen-Macaulay flat \( B/xB \)-module. Also \( \Omega_{B/V} \) can be mapped onto a \( B \)-module \( W \) such that \( H^i_m(W) = 0 \) for \( i \geq 0 \).

**Proof.** Since \( \Omega_{B/V} \rightarrow \hat{\Omega}_{B/V} \) is onto, the second assertion is obvious from corollary 2, (diagram 30). The last row of (30) implies that \( H^i_m(N) \simeq H^{i+1}_m(\bigoplus_{i=1}^{n-1} B) \), \( (0 : p^i)N \simeq \bigoplus_{i=1}^{n-1} B/p^iB \) and the last column of (30) implies that \( H^{n-1}_m(\Omega_{B/R}) \rightarrow H^{n-1}_m(N) \rightarrow 0 \) is exact. Due to: a) theorem 5.1 and corollary 5.10 in ([2]), b) \( x \) is a non-zero divisor on \( W \) and c) \( \Omega_{R/V} \) is \( R \)-flat, our assertion follows from the last row of (30).
4.7. Remarks. 1) Corollary 2 above provides a new proof of the fact that $H_{n-1}^m(\Omega_{B/R}) \neq 0$ (Theorem 4.1, [3]). Last row of (30) implies that $H_{n-1}^m(N) \simeq \bigoplus_{i=1}^{n-1} H_{m}^i(B)$ and last column of (30) implies that $H_{n-1}^m(\Omega_{B/R}) \rightarrow H_{n-1}^m(N)$ is onto. Hence, $H_{n-1}^m(\Omega_{B/R}) \neq 0$ ([1], [3]).

2) Using theorem 5.1 in ([2]) it follows from proposition 4.2 in ([3]) that $H_i^m(\Omega_{B/V}) \simeq H_i^m(\Omega_{B/R})$ for $i \leq n-2$, and from corollary 1 (3.4) that a) $H_i^m(\Omega_{B/A}) \simeq H_i^m(\Omega_{B/AR})$, for $i \leq n-2$; b) $H_i^m(\Omega_{B/V}) \simeq H_i^m(\Omega_{B/A})$, for $i \leq n-3$ and injectivity of maps at the next level. The crucial question is whether $H_{n-1}^m(\Omega_{A/V} \otimes B) \simeq H_{n}^m(B) \rightarrow H_{n-1}^m(\Omega_{B/V})$ is non-null.

3) B. Bhatt kindly shared with me his approach for providing another new proof of the fact that $H_{n-1}^m(\Omega_{B/R}) \neq 0$ (theorem 4.1, [3]). He used Cohen-Macaulayness of the $m$-adic completion $\hat{B}$ along with facts from derived categories and perfectoids to accomplish his proof.

4) In equicharacteristic 0 same proof works for corresponding statements for theorem 4.5, corollary 1 and corollary 2 in 4.6 without the assertion on local cohomology modules. In positive characteristic the same arguments work for corresponding statement for theorem 4.5 with the assumption that $Q(C)$ is separably algebraic over $Q(R)$.

5) Let $T$ denote the torsion sub-module of $\Omega_{B/V}$, let $U = \text{Coker } (T \rightarrow \Omega_{B/V})$ and $L = \text{Coker } (T \rightarrow \Omega_{B/R})$. Then, as in corollary 2 and corollary 3 in 4.6 it follows that $H_i^m(U) = 0$ for $i \geq 0; H_i^m(L) = 0$ for $i \neq n-1$ and $H_{n-1}^m(L) \simeq H_{n-1}^m(\bigoplus_{i=1}^{n-1} B)$ and for $t > 0$, $(0 : p^t)L \simeq \bigoplus_{i=1}^{n-1} B/p^t B$.

The above isomorphisms are obtained from the exact sequence:

$$0 \rightarrow \Omega_{R/V} \otimes B \rightarrow U \rightarrow L \rightarrow 0.$$ 

Thus $L$ shares the same properties with $N$ in corollary 3, 4.6 above. Moreover, with $N$ as in corollary 2 above, there exists an onto map $\eta : L \rightarrow N$ such that $H_i^m(\ker \eta) = 0$ for $i \geq 0$.

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