ALGEBRAIC K-THÉORY AND PROPERLY INFINITE C*-ALGEBRAS

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Abstract. We show that several known results about the algebraic K-theory of tensor products of algebras with the C*-algebra of compact operators in Hilbert space remain valid for tensor products with any properly infinite C*-algebra.

1. Introduction

Let C* and Ab be the categories of C*-algebras and of abelian groups. Let O be a properly infinite C*-algebra, let K = K(ℓ²(Z≥0)) be the C*-algebra of compact operators, let (e_{j,k})_{j,k∈Z≥0} be the standard system of matrix units for K, and let ⊗ be the spatial tensor product. Consider the corner embedding j: O → O⊗K, given by j(a) = a⊗e_{0,0} for a ∈ O. We show in Proposition 2.2 that if E: C* → Ab is an M2-stable functor then E(j) is an isomorphism

(1) E(j): E(O) ∼→ E(O⊗K).

We use this to show that several known results concerning the algebraic K-theory of tensor products of algebras with K remain valid for tensor products with O. In these results, and throughout the paper, all topological vector spaces (in particular, locally convex algebras) are assumed complete. Also, K_* denotes algebraic K-theory, and K_*^top denotes a suitable version of topological K-theory (the usual one when restricted to Banach algebras). For example we prove in Theorem 3.2 that if A is a C*-algebra then the comparison map is an isomorphism

(2) K_*(A⊗O) ∼→ K_*^top(A⊗O).

In fact this is immediate from (1) and the Karoubi-Suslin-Wodzicki theorem ([12], [15]), according to which K_*(A⊗K) → K_*^top(A⊗K) is an isomorphism. We also prove that if L is a locally multiplicatively convex Fréchet algebra with a uniformly bounded one-sided approximate identity and ⊙ is the projective tensor product, then the comparison map

K_*(L⊗O) → K_*^top(L⊗O)

is an isomorphism. The analogous result for L⊗K is [4, Theorem 8.3.3] (see also [2, Theorem 12.1.1]).

We prove in Theorem 3.4 that if L is a locally convex algebra then the various possible definitions of topological K-theory for locally convex algebras all agree on L⊗K, and moreover they coincide with Weibel’s homotopy algebraic K-theory:

(3) KH_*(L⊗O) ∼ K_*^top(L⊗O).

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The analog of (3) for $L\hat{\otimes}K$ follows from [4, Theorem 6.2.1]. Further, let $HC_*$ be algebraic cyclic homology of $\mathbb{Q}$-algebras. Then we show in Theorem 3.8 that there is a six term exact sequence

$$
K_1^{\text{top}}(L\hat{\otimes}O) \xrightarrow{\cong} HC_{2n-1}(L\hat{\otimes}O) \xrightarrow{\cong} K_2n(L\hat{\otimes}O).
$$

The corresponding statement for $L\hat{\otimes}K$ is a particular case of [4, Theorem 6.3.1].

Finally we show, in Theorem 3.9, that if $A$ is any $\mathbb{C}$-algebra and $\otimes_\mathbb{C}$ denotes the algebraic tensor product over $\mathbb{C}$, then

$$
KH(n)(A \otimes_\mathbb{C} O) = \begin{cases} 
K_0(A \otimes_\mathbb{C} O) & n \text{ even} \\
K_{-1}(A \otimes_\mathbb{C} O) & n \text{ odd}, 
\end{cases}
$$

and there is a six term exact sequence

$$
K_{-1}(A \otimes_\mathbb{C} O) \xrightarrow{\cong} HC_{2n-1}(A \otimes_\mathbb{C} O) \xrightarrow{\cong} K_{2n}(A \otimes_\mathbb{C} O)
$$

and

$$
K_{2n-1}(A \otimes_\mathbb{C} O) \xleftarrow{\cong} HC_{2n-2}(A \otimes_\mathbb{C} O) \xleftarrow{\cong} K_{0}(A \otimes_\mathbb{C} O).
$$

The analogous statement for $A \otimes_\mathbb{C} K$ is a particular case of [4, Theorem 7.1.1].

The first version of this paper and all its results date back to 2007. Although we have both lectured on these results since then, we had not until now distributed the manuscript. Motivated by the recent article [10], where a particular case of (2) is proved, we decided to make our paper publicly available.

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2. Properly infinite algebras and stability

If $A$ is any ring, we write

$$
t_A : A \to M_2(A)
$$

for the canonical inclusion into the upper left corner, given by $t_A(a) = \text{diag}(a, 0)$ for $a \in A$. When $A$ is understood, we just write $t$.

**Definition 2.1.** We say that a functor $E : \mathcal{E} \to \text{Ab}$ from the category of $C^*$-algebras to the category of abelian groups is $M_2$-stable if $E(t)$ is an isomorphism for every $C^*$-algebra $A$.

Note that this definition makes sense for other categories of rings and algebras, such as the category $\mathcal{EC}$ of complete locally convex algebras.

**Proposition 2.2.** Let $E : \mathcal{E} \to \text{Ab}$ be a functor. Assume that $E$ is $M_2$-stable. Let $O$ be a properly infinite $C^*$-algebra and let $K = K(\ell^2(\mathbb{Z}_{>0}))$ be the $C^*$-algebra of compact operators. Then $E$ maps the inclusion $j : O \to O\otimes K$, defined by $j(a) = a\otimes e_{0,0}$ for $a \in O$, to an isomorphism.

**Proof.** Because $O$ is properly infinite, it contains a sequence $(u_n)_{n \in \mathbb{Z}_{>0}}$ of isometries with orthogonal range idempotents; write $u = (u_1, u_2, \ldots) \in O^{1 \times \infty}$ for the corresponding infinite row. Represent an element $a \in O\otimes K$ as an infinite matrix $a = (a_{j,k})_{j,k \in \mathbb{Z}_{>0}}$. For such an element $a$, set

$$
a_{0,+} = (a_{0,k})_{k \in \mathbb{Z}_{>0}}, \quad a_{+,0} = (a_{j,0})_{j \in \mathbb{Z}_{>0}}, \quad \text{and} \quad a_{+,+} = (a_{j,k})_{j,k \in \mathbb{Z}_{>0}}.
$$

Thus

$$
a = \begin{bmatrix} a_{0,0} & a_{0,+} \\ a_{+,0} & a_{+,+} \end{bmatrix}.
$$
Define a \(*\)-homomorphism \(\phi: \mathcal{O} \otimes \mathcal{K} \to M_2(\mathcal{O})\) by

\[
\phi(a) = \begin{bmatrix} a_{0,0} & a_{0,+}u^* \\ ua_{+},0 & ua_{+},+u^* \end{bmatrix}
\]

for \(a \in \mathcal{O} \otimes \mathcal{K}\). By construction, the diagram

\[
\begin{array}{c}
\mathcal{O} \otimes \mathcal{K} \\
\downarrow \phi \\
M_2(\mathcal{O}) \\
\end{array}
\]

commutes. Since \(E(\iota)\) is an isomorphism by hypothesis, it follows that \(E(j)\) is injective and that \(E(\phi)\) is surjective; it remains to show that \(E(j)\) is surjective, or equivalently that \(E(\phi)\) is injective.

For any \(C^*\)-algebra \(A\), we denote by \(M(A)\) its multiplier algebra. Consider the partial isometry

\[
v = \begin{bmatrix}
1 & 0 & 0 & \cdots \\
0 & u_1 & u_2 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & u \\ 0 & 0 \end{bmatrix} \in M(\mathcal{O} \otimes \mathcal{K}).
\]

Define a homomorphism

\[
\psi: \mathcal{O} \otimes \mathcal{K} \to \mathcal{O} \otimes \mathcal{K}
\]

by \(\psi(a) = vav^*\) for \(a \in \mathcal{O}\). We have \(E(\psi) = \text{id}_{E(\mathcal{O} \otimes \mathcal{K})}\) (for example, by [2, Proposition 2.2.6], taking \(B\) there to be \(M(\mathcal{O} \otimes \mathcal{K})\)). Let \(\kappa: M_2(\mathcal{O}) \to \mathcal{O} \otimes \mathcal{K}\) by the \(*\)-homomorphism defined by

\[
\kappa \left( \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} \right) = a_{0,0} \otimes e_{0,0} + a_{0,1} \otimes e_{0,1} + a_{1,0} \otimes e_{1,0} + a_{1,1} \otimes e_{1,1}.
\]

Then the following diagram commutes:

\[
\begin{array}{c}
\mathcal{O} \otimes \mathcal{K} \\
\downarrow \psi \\
M_2(\mathcal{O}) \\
\downarrow \kappa \\
\mathcal{O} \otimes \mathcal{K}. \\
\end{array}
\]

It follows that \(E(\phi)\) is injective; this concludes the proof.

Recall that a functor of \(C^*\)-algebras is \emph{homotopy invariant} if it sends two homotopic homomorphisms to the same homomorphism.

**Corollary 2.3.** Let \(E\) and \(\mathcal{O}\) be as in Proposition 2.2, and further assume that \(E\) is split exact. Then \(E(- \otimes \mathcal{O})\) is homotopy invariant.

**Proof.** The proposition shows that \(E(- \otimes \mathcal{O})\) is \(\mathcal{K}\)-stable; since it is also split exact by hypothesis, it is homotopy invariant by Higson’s homotopy invariance theorem [9, Theorem 3.2.2]. (Note the misprint there: Definition 2.1.8 should be Definition 2.1.10.) □

In the next corollary, we use the notion of \emph{diffotopy invariance} (sometimes called differeotopy invariance) for functors of locally convex algebras, taken from the first part of [7, Definition 4.1.1]; note that \(A[0, 1]\) there is defined to be the space of smooth functions from \([0, 1]\) to \(A\). The definition is essentially the same as for homotopy invariance, replacing continuous homotopies with \(C^\infty\) homotopies (diffotopies; in some papers called diffeotopies). Recall that \(\mathfrak{L}\mathfrak{C}\) is the category of complete locally convex algebras. We require that the multiplication be jointly continuous (as in [7, Definition 2.1]), but we do not require the existence of submultiplicative seminorms.

**Corollary 2.4.** Let \(E: \mathfrak{L}\mathfrak{C} \to \text{Ab}\) be an \(M_2\)-stable split exact functor, and let \(\mathcal{O}\) be a properly infinite \(C^*\)-algebra. Then the functor \(L \mapsto E(L \otimes \mathcal{O})\) is diffotopy invariant.
Proof. Fix \( L \in \mathfrak{L} \mathfrak{C} \) and consider the functor \( F : \mathfrak{C}^* \to \text{Ab} \) given by \( F(A) = E(L \hat{\otimes} (A \otimes \mathcal{O})) \). By Corollary 2.3, \( F \) is homotopy invariant. Hence \( E \) sends the evaluation maps \( L \hat{\otimes} C([0, 1], \mathcal{O}) \to L \hat{\otimes} \mathcal{O} \) at both endpoints to the same map. It follows that the same is true of the evaluation maps \( C^\infty([0, 1], L \hat{\otimes} \mathcal{O}) \to L \hat{\otimes} \mathcal{O} \), since the latter factor through the former. \( \square \)

We will also need the following variant of Proposition 2.2.

**Corollary 2.5.** Let \( E : \mathfrak{L} \mathfrak{C} \to \text{Ab} \) be an \( M_2 \)-stable functor. Let \( k : \mathcal{O} \to \hat{\mathcal{O}} \otimes K \) be the usual inclusion, given by \( a \mapsto a \hat{\otimes} e_{0,0} \) for \( a \in \mathcal{O} \). Then \( E(k) \) is a naturally split monomorphism.

**Proof.** The map \( j \) of Proposition 2.2 factors through \( k \). \( \square \)

### 3. Applications

In this section we apply the results of the previous one to study the algebraic \( K \)-theory of tensor products of different classes of algebras with properly infinite \( C^* \)-algebras. We essentially show that the results of [4], [12], [15], and [17] remain valid if we stabilize with properly infinite \( C^* \)-algebras instead of the compact operators.

The first application concerns the comparison between Quillen’s algebraic \( K \)-theory and the usual topological \( K \)-theory of \( C^* \)-algebras. It is the properly infinite variant of Karoubi’s conjecture for \( C^* \)-algebras, proved in [15] for the \( K \)-stable case.

We need a lemma. See [2, Remark 2.1.13] for why something needs to be done here. We will use the reasoning of this lemma for other categories later.

**Lemma 3.1.** Algebraic \( K \)-theory of \( C^* \)-algebras is \( M_2 \)-stable in the sense of Definition 2.1.

**Proof.** Let \( A \) be a \( C^* \)-algebra. Let \( \bar{A} \) be its unitization as an algebra over \( \mathbb{Z} \). We then get a commutative diagram with split exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \bar{A} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow{\iota_A} & & \downarrow{\iota_{\bar{A}}} & & \downarrow{\iota_{\mathbb{Z}}} & & & & \\
0 & \longrightarrow & M_2(A) & \longrightarrow & M_2(\bar{A}) & \longrightarrow & M_2(\mathbb{Z}) & \longrightarrow & 0.
\end{array}
\]

Apply \( K_n \), getting:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K_n(A) & \longrightarrow & K_n(\bar{A}) & \longrightarrow & K_n(\mathbb{Z}) & \longrightarrow & 0 \\
\downarrow{\iota(A)_*} & & \downarrow{\iota_{\bar{A}})_*} & & \downarrow{\iota_{\mathbb{Z}})_*} & & & & \\
0 & \longrightarrow & K_n(M_2(A)) & \longrightarrow & K_n(M_2(\bar{A})) & \longrightarrow & K_n(M_2(\mathbb{Z})) & \longrightarrow & 0.
\end{array}
\]

Since \( A \) and \( M_2(A) \) are \( C^* \)-algebras, it follows from [15, Corollary 10.4] that the rows are exact. The maps \( (\iota_{\bar{A}})_* \) and \( (\iota_{\mathbb{Z}})_* \) are isomorphisms because \( \bar{A} \) and \( \mathbb{Z} \) are unital. So \( (\iota_A)_* \) is an isomorphism by the Five Lemma. \( \square \)

**Theorem 3.2.** Let \( A \) and \( \mathcal{O} \) be \( C^* \)-algebras, with \( \mathcal{O} \) properly infinite. Then the comparison map \( K_n(A \otimes \mathcal{O}) \to K_n^\text{top}(A \otimes \mathcal{O}) \) from algebraic to topological \( K \)-theory is an isomorphism for all \( n \in \mathbb{Z} \).

**Proof.** Apply the functors \( K_* \) and \( K_*^\text{top} \) to the map \( j : A \otimes \mathcal{O} \to A \otimes \mathcal{O} \otimes K \). Since the comparison map is natural, we obtain a commutative diagram:

\[
\begin{array}{ccccccccc}
K_*(A \otimes \mathcal{O}) & \longrightarrow & K_*^\text{top}(A \otimes \mathcal{O}) \\
\downarrow & & \downarrow \\
K_*(A \otimes \mathcal{O} \otimes K) & \longrightarrow & K_*^\text{top}(A \otimes \mathcal{O} \otimes K).
\end{array}
\]
The right vertical map is well known to be an isomorphism. By Proposition 2.2, applied to the functor \( E = K_n(A \square -) \), and Lemma 3.1, the left vertical map is an isomorphism. By [12, Théorème on page 254] and [15, Theorem 10.9], the same is true of the horizontal map at the bottom. It follows that the top horizontal map is an isomorphism. \( \square \)

Our second application is the properly infinite variant of [4, Theorem 6.2.1 (iii)]. It concerns projective tensor products of locally convex algebras with properly infinite \( C^* \)-algebras, and establishes that for such products, all variants of topological \( K \)-theory coincide with each other and with Weibel’s homotopy algebraic \( K \)-theory. We recall that there are several topological \( K \)-theory groups one can associate to a locally convex algebra \( L \):

- The Bott-periodic Cuntz groups \( kk_n^{\text{top}}(C, L) \). (See [6, Definition 4.2], and see Remark 3.3 below on conflicting notation.)
- The difftopy \( K \)-groups \( KD_n(L) \) for \( n \in \mathbb{Z} \) ([2, §8.2]).
- The difftopy Karoubi-Villamayor style groups \( KV_n^\text{diff}(L) \) for \( n \geq 1 \) ([2, §8.1]).

On the other hand, \( L \), just like any other ring, has associated several algebraic \( K \)-groups:

- Quillen’s \( K \)-groups \( K_n(L) \) for \( n \in \mathbb{Z} \) (defined by Bass for \( n < 0 \)).
- Weibel’s homotopy algebraic \( K \)-groups \( KH_n(L) \) for \( n \in \mathbb{Z} \). (See [16] and [2, §5]. The ring \( \Sigma L \) is defined after [2, Example 2.3.2].)
- The Karoubi-Villamayor algebraic \( K \)-groups \( KV_n(L) \) for \( n \geq 1 \). (In [13], use the discrete norm \( p(0) = 0 \) and \( p(a) = 1 \) for \( a \neq 0 \). The group \( KV_n(L) \) is called \( K^{-n}(L) \) in [13]. Also see [2, §4].)

**Remark 3.3.** The group we are calling here \( kk_n^{\text{top}}(A, B) \) is called \( kk_n^{\text{alg}}(A, B) \) in [6, Definition 4.2], and also in [7]. (Compare the introduction to [7, §6.1] with [6, §4].) It is called \( kk_n^{\text{top}}(A, B) \) in [4]. It is not the same as the groups \( kk_n(A, B) \) in [6, Definition 15.4]. It is also not the same as the algebraic bivariant group \( kk_n(A, B) \) of [3].

**Theorem 3.4.** Let \( L \) be a locally convex algebra and let \( \mathcal{O} \) be a properly infinite \( C^* \)-algebra; put \( \mathcal{M} = L \square \mathcal{O} \). Then there are natural isomorphisms

\[
KH_n(\mathcal{M}) \cong KD_n(\mathcal{M}) \cong kk_n^{\text{top}}(C, \mathcal{M})
\]

for \( n \in \mathbb{Z} \),

\[
KH_n(\mathcal{M}) \cong KV_n^{\text{diff}}(\mathcal{M}) \cong KV_n(\mathcal{M})
\]

for \( n \geq 1 \), and

\[
KH_n(\mathcal{M}) \cong K_n(\mathcal{M})
\]

for \( n \leq 0 \).

**Proof.** All the functors appearing in the theorem are \( M_2 \)-stable. Moreover, there are natural transformations as follows: \( KV_n \to KH_n \) for \( n \geq 1 \) (coming from (44) on page 127 of [2]), \( KV_n^{\text{diff}} \to KD_n \) for \( n \geq 1 \) (coming from the formula before [2, Proposition 8.2.1]), and \( K_n \to KH_n \) for \( n \in \mathbb{Z} \) (see the beginning of [16, §1]: that paper, in contrast to our convention, takes \( KV_n \) to be defined for all \( n \in \mathbb{Z} \)). For any locally convex algebra \( L \), we have algebras \( \Omega L \) as in (31) on page 127 of [2] and \( \Omega^{\text{diff}} L \) as in (64) on page 145 of [2], and there is an obvious map \( \Omega L \to \Omega^{\text{diff}}(L) \). This map induces natural transformations \( KV_n \to KV_n^{\text{diff}} \) for \( n \geq 1 \) and (by taking direct limits) \( KH_n \to KD_n \) for \( n \in \mathbb{Z} \).

By [4, Theorem 6.2.1 (iii)], except for the map \( K_n \to KH_n \), all these natural maps, when applied to \( \mathcal{M} \square \mathcal{K} \), become isomorphisms for all those \( n \) for which they are defined. By [4, Lemma 3.2.1 (ii)], [4, Theorem 6.2.1 (ii)], and [16, Proposition 1.5 (i)], the map \( K_n(\mathcal{M} \square \mathcal{K}) \to KH_n(\mathcal{M} \square \mathcal{K}) \) is an isomorphism for \( n \leq 0 \). (Our theorem says nothing about this map for \( n \geq 1 \).) Next use Corollary 2.5 (including naturality of the splitting), and the fact that a retract of an isomorphism is an isomorphism, to show that all these natural maps (again, except \( K_n \to KH_n \) for \( n \geq 0 \)) are also isomorphisms when evaluated at \( \mathcal{M} \). We have verified all the isomorphisms in the statement of the theorem not involving \( kk_n^{\text{top}}(C, \mathcal{M}) \).
From [7, Theorem 6.2.1 and Definition 2.3.2] (see Remark 3.3 for the discrepancy in notation), we get a natural isomorphism $kk_0^{\text{top}}(\mathcal{C}, \mathcal{M} \hat{\otimes} \mathcal{K}) \cong K_0(\mathcal{M} \hat{\otimes} \mathcal{K})$. Appealing to Corollary 2.5 in the same way as above, we then get a natural isomorphism

$$
kk_0^{\text{top}}(\mathcal{C}) \cong K_0(\mathcal{M}).
$$

The natural isomorphism $KH_n(\mathcal{M}) \cong KD_n(\mathcal{M})$ for $n \in \mathbb{Z}$, which we already proved, implies that the functor $KH_n(- \hat{\otimes} \mathcal{O})$ is difftopy invariant. This functor is $M_2$-stable because $KH_*$ is. It is shown in the proof of [7, Proposition 6.1.2] that the inclusion $M_2(\mathbb{C}) \rightarrow K_{\infty}$ into the smooth compact operators induces a difftopy equivalence $\mathcal{K} \hat{\otimes} M_2(\mathbb{C}) \rightarrow \mathcal{K} \hat{\otimes} K_{\infty}$. Therefore

$$
KH_*(\mathcal{O}) \rightarrow KH_*(\mathcal{K} \hat{\otimes} K_{\infty})
$$

is an isomorphism. Applying Corollary 2.5 in the same way as before, we conclude that $KH_*(\mathcal{O})$ is $K_{\infty}$-stable. Summing up, $KH_*(\mathcal{O})$ satisfies excision ([16, Theorem 2.1]) and is difftopy invariant and $K_{\infty}$-stable.

The functor $kk_0^{\text{top}}(\mathcal{C}, - \hat{\otimes} \mathcal{O})$ also has these properties. For a manifold $\mathcal{M}$, possibly with boundary, and a locally convex algebra $A$, let $C^\infty(\mathcal{M}, A)$ be the algebra of all infinitely often differentiable $A$-valued functions on $\mathcal{M}$, with the topology of uniform convergence of all derivatives in all seminorms on $A$. Define locally convex algebras

$$
C_\infty A = \{f \in C^\infty([0,1], A) : f^{(n)}(0) = 0 \text{ for } n = 1, 2, \ldots \text{ and } f^{(n)}(1) = 0 \text{ for } n = 0, 1, 2, \ldots\},
$$

$$
S_\infty A = \{f \in C_\infty A : f(0) = 0\}, \quad \text{and} \quad S_0 A = \{f \in C^\infty(S^1, A) : f(1) = 0\}.
$$

Now consider the exact sequence

$$
0 \rightarrow S_\infty A \rightarrow C_\infty A \rightarrow A \rightarrow 0.
$$

(This is the same sequence as in [5, §2.1].) The algebra $C_\infty A$ is difftopy equivalent to 0, so $KH_*(C_\infty A) = kk_*^{\text{top}}(\mathcal{C}, A) = 0$. Using excision, we thus get for every $n \in \mathbb{Z}$ a commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \rightarrow & S_\infty A & \rightarrow & C_\infty A & \rightarrow & A & \rightarrow & 0.
\end{array}
$$

Since $KH_0(\mathcal{M} \hat{\otimes} \mathcal{O})) \rightarrow kk_0^{\text{top}}(\mathcal{C}, M \hat{\otimes} \mathcal{O})$ is an isomorphism for all locally convex algebras $M$, and since $S_\infty(M \hat{\otimes} \mathcal{O}) \cong S_\infty(M) \hat{\otimes} \mathcal{O}$, an induction argument shows that the map $KH_n(L \hat{\otimes} \mathcal{O}) \rightarrow kk_n^{\text{top}}(\mathcal{C}, L \hat{\otimes} \mathcal{O})$ is an isomorphism for all $n \geq 0$.

Forming the projective tensor product of a locally convex algebra $A$ with the second extension in [5, §2.3], we further get the exact sequence:

$$
0 \rightarrow K_{\infty} A \rightarrow T_0 \hat{\otimes} A \rightarrow S_0 A \rightarrow 0.
$$

By [5, Theorem 6.4], we have $KH_*(T_0 \hat{\otimes} A) = kk_*^{\text{top}}(\mathcal{C}, T_0 \hat{\otimes} A) = 0$ for every locally multiplicatively convex algebra $A$, that is, every locally convex algebra whose topology is given by submultiplicative seminorms. We claim that the same argument works even without local multiplicative convexity. Indeed, since the theories are defined for locally convex algebras, the proof only needs to be modified at the end by tensoring with the identity map on a general locally convex algebra, and checking that allowing general locally convex algebras does not affect the proof of [5, Lemma 6.3]. (It is shown in [6, Proposition 8.2] that $kk_*^{\text{top}}(\mathcal{C}, T_0 \hat{\otimes} A) = 0$ without local multiplicative convexity, but the analog of [5, Theorem 6.4] is not present in [6].) It follows that we get a commutative diagram
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with exact rows:

$$
\begin{array}{c}
0 \\
\downarrow
\end{array}
\begin{array}{c}
KH_{n+1}(S_0(L\hat{\otimes}O)) \\
\downarrow
\end{array}
\begin{array}{c}
KH_n(K_{\infty}\otimes L\hat{\otimes}O) \\
\downarrow
\end{array}
\begin{array}{c}
0
\end{array}
$$

An argument similar to the one above, now also using $K_{\infty}$-stability, shows that $KH_n(L\hat{\otimes}O) \rightarrow kk^{\text{top}}_n(C, L\hat{\otimes}O)$ is an isomorphism for all $n \leq 0$. □

In view of the theorem above, the topological $K$-groups of a locally convex algebra of the form $L\hat{\otimes}O$ are unambiguously defined. To unify notation, we write

$$
K^*_n(L\hat{\otimes}O) = kk^{\text{top}}_n(C, L\hat{\otimes}O) = KD_*(L\hat{\otimes}O) = KH_*(L\hat{\otimes}O).
$$

Note in particular that these topological $K$-groups are Bott periodic, by [6, Theorem 8.3].

A Fréchet algebra is a locally convex algebra which is metrizable; we call it locally multiplicatively convex if its topology is determined by a sequence $(p_n)_{n \in \mathbb{Z}_{>0}}$ of submultiplicative seminorms. A uniformly bounded left approximate identity in $L$ is a net $(e_\lambda)_{\lambda \in \Lambda}$ of elements of $L$ such that $e_\lambda a \rightarrow a$ for all $a \in L$ and such that $\sup_{n \in \mathbb{Z}_{>0}} \sup_{\lambda \in \Lambda} p_n(e_\lambda) < \infty$. A uniformly bounded right approximate identity is defined similarly.

**Theorem 3.5.** Let $L$ be a locally multiplicatively convex Fréchet algebra with uniformly bounded left or right approximate identity. Then there is a natural isomorphism $K_*(L\hat{\otimes}O) \cong K^{\text{top}}_*(L\hat{\otimes}O)$.

**Proof.** By [2, Theorem 6.6.6], $K$-theory satisfies excision in the category $\mathcal{C}$ of locally multiplicatively convex Fréchet algebras with uniformly bounded left approximate identities. It follows from the same reasoning as in the proof of Lemma 3.1 that $K$-theory is $M_2$-stable on $\mathcal{C}$. Moreover $\mathcal{C}$ is closed under $\otimes$, and any $C^*$-algebra is in $\mathcal{C}$. By [2, Theorem 12.1.1], the map $K_*(L\hat{\otimes}O \hat{\otimes}K) \rightarrow K^{\text{top}}_*(L\hat{\otimes}O \hat{\otimes}K)$ is an isomorphism. The proof is completed by using Corollary 2.5 (including naturality of the splitting), and the fact that a retract of an isomorphism is an isomorphism. □

**Remark 3.6.** For $L$ as in Theorem 3.5, $K_*(L\hat{\otimes}O)$ agrees also with the topological $K$-theory of locally multiplicatively convex Fréchet algebras defined in [14]. To see this, it suffices, by the argument of [2, Remark 12.1.4], to check that $K_0(L\hat{\otimes}O) \cong K_0(L\hat{\otimes}O \hat{\otimes}K_{\infty})$. This follows from Theorem 3.4, using $K_{\infty}$-stability of $kk^{\text{top}}$.

In Theorem 3.8 we consider algebraic cyclic homology of $\mathbb{Q}$-algebras, as in [2, §11.1]. If $A$ is an algebra, we write $HC_*(A) = H_*(C^*(A/\mathbb{Q}))$ for the homology of Connes’ complex defined using the algebraic tensor product, taken over $\mathbb{Q}$:

$$
C^A_*(A/\mathbb{Q}) = (A^{\otimes(n+1)}_{\mathbb{Q}/(n+1)\mathbb{Z}}).
$$

We will need some properties of infinitesimal $K$-theory $K^{\text{inf}}_*$ (as defined, for example, in [2, §11.2]).

**Theorem 3.7** ([1, Remark 4.2], [2, Theorem 11.2.1 (ii)]). On $\mathbb{Q}$-algebras, the functor $K^{\text{inf}}_*$ satisfies excision and is $M_2$-stable.

**Proof.** Once one has excision, $M_2$-stability follows from the same argument as in the proof of Lemma 3.1. As observed in [1, Remark 4.2] and cited in [2, Theorem 11.2.1 (ii)], excision follows from [1, Theorem 0.1]; since the proof is not made explicit in the above references, we give a short argument below.

We will need birelative groups for $K_*$, $K^{\text{inf}}_*$, and $HN_*$ as in [1]. These are defined for rings $A$ and $B$, an ideal $I$ in $A$, and a homomorphism $f : A \rightarrow B$ such that $f|_I$ is injective and $f(I)$ is an ideal in $B$. For $K_*$, see [8, §0.1]; in the other two cases, the definitions are analogous. By construction, there is a long exact sequence

$$
K_{n+1}(B : I) \rightarrow K_n(A, B : I) \rightarrow K_n(A : I) \rightarrow K_n(B : I) \rightarrow K_{n-1}(A, B : I),
$$
and similarly for \( K^\text{inf}_n \) and \( HN_n \). (See [2, §11.2].) Moreover, the fibration of [2, §11.2] involving these gives rise to a long exact sequence

\[
HN_{n+1}(A, B : I) \rightarrow K^\text{inf}_n(A, B : I) \rightarrow K_n(A, B : I) \rightarrow HN_n(A, B : I) \rightarrow K^\text{inf}_{n-1}(A, B : I).
\]

When \( A \) and \( B \) are \( \mathbb{Q} \)-algebras, [1, Theorem 0.1] states that the map \( K_n(A, B : I) \rightarrow HN_n(A, B : I) \) is an isomorphism, so that \( K^\text{inf}_n(A, B : I) = 0 \). Thus \( K^\text{inf}_n(A : I) \rightarrow K^\text{inf}_n(B : I) \) is an isomorphism, which is excision.

We will also need the long exact sequence, for a \( \mathbb{Q} \)-algebra \( A \), in the top row of [2, Diagram (86)] (before Remark 11.3.4 there):

\[
\begin{array}{c}
K^\text{nil}_{n+1}(A) \\
K^\text{nil}_n(A) \\
K_n(A) \\
K^\text{nil}_{n-1}(A)
\end{array}
\]

\begin{align}
&KA \xrightarrow{K^\text{top}_1(M)} HC_{2n-1}(M) \xrightarrow{K^\text{top}_n(M)} K^\text{nil}_{2n}(M) \\
&\downarrow \\
&K^\text{nil}_{2n-1}(M) \xrightarrow{HC_{2n-2}(M)} K^\text{top}_0(M).
\end{align}

**Theorem 3.8.** Let \( L \) be a locally convex algebra and let \( O \) be a properly infinite \( C^* \)-algebra; put \( M = L \otimes O \). For each \( n \in \mathbb{Z} \), there is a natural six term exact sequence of abelian groups as follows:

\[
\begin{array}{c}
K^\text{top}_1(M) \\
HC_{2n-1}(M) \\
K^\text{nil}_{2n-1}(M)
\end{array}
\]
It now follows from [4, Lemma 3.2.1 (ii)] and [16, Proposition 1.5 (i)] that $K_n(A \otimes \mathcal{O}) \to KH_n(A \otimes \mathcal{O})$ is an isomorphism for $n \leq 0$. In view of the sequence (4), it only remains to show that $KH_n(A \otimes \mathcal{O})$ satisfies Bott periodicity in $A$. By Proposition 2.2, there is a natural isomorphism $KH_n(A \otimes \mathcal{O}) \cong KH_n(A \otimes (\mathcal{O} \otimes \mathcal{K}))$. So it suffices to show that $KH_n(A \otimes (\mathcal{O} \otimes \mathcal{K}))$ satisfies Bott periodicity in $A$. The tensor product of operators gives an isomorphism

$$
\mu : \mathcal{K}(t^2(\mathbb{Z}_{\geq 0})) \otimes \mathcal{K}(t^2(\mathbb{Z}_{\leq 0})) \to \mathcal{K}(t^2(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0})).
$$

It makes $KH_*(\mathcal{K}) \cong K_*(\mathcal{K}) \cong K_{\text{top}}^*(\mathcal{K})$ into a graded ring isomorphic to the Laurent polynomials $\mathbb{Z}[t, t^{-1}]$, in which $t$ has degree 2. Next, $1_A \otimes \mathcal{O} (\mathcal{O} \otimes \mu)$ makes $KH_*(A \otimes \mathcal{O} \otimes \mathcal{K})$ into a graded $KH_*(\mathcal{K}) = \mathbb{Z}[t, t^{-1}]$-module. This implies Bott periodicity for $KH_*(A \otimes \mathcal{O} \otimes \mathcal{K})$. □

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