Learning Mixtures of Random Utility Models with Features from Incomplete Preferences

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Abstract

Random Utility Models (RUMs), which subsume Plackett-Luce model (PL) as a special case, are among the most popular models for preference learning. In this paper, we consider RUMs with features and their mixtures, where each alternative has a vector of features, possibly different across agents. Such models significantly generalize the standard PL and RUMs, but are not as well investigated in the literature. We extend mixtures of RUMs with features to models that generate incomplete preferences and characterize their identifiability. For PL, we prove that when PL with features is identifiable, its MLE is consistent with a strictly concave objective function under mild assumptions, by characterizing a bound on root-mean-square-error (RMSE), which naturally leads to a sample complexity bound. We also characterize identifiability of more general RUMs with features and propose a generalized RBCML to learn them. Our experiments on synthetic data demonstrate the effectiveness of MLE on PL with features with tradeoffs between statistical efficiency and computational efficiency. Our experiments on real-world data show the prediction power of PL with features and its mixtures.

1 Introduction

Preference learning is a fundamental machine learning problem in a wide range of applications such as discrete choice analysis [McFadden, 1973], marketing [Berry et al., 1995], meta-search engines [Dwork et al., 2001], information retrieval [Liu, 2009], recommender systems [Baltrunas et al., 2010], crowdsourcing [Chen et al., 2013; Mao et al., 2013], social choice [Azari Soufani et al., 2012], among many others. Plackett-Luce model (PL) [Plackett, 1975; Luce, 1959] is one of the most popular statistical models for preference learning due to its interpretability and computational tractability. In the standard PL, each alternative is parameterized by a real number $\theta$ such that $e^{\theta i}$ represents the alternative’s “quality”. The higher the quality is, the more likely the alternative is ranked higher by an agent.

Extensions of the standard PL has been proposed and studied mostly in three dimensions. The first dimension is PL with features, where features of the agents and/or the alternatives are given, and the model is parameterized by the relationship (often linear, see Definition 2) between the features and the quality of the alternatives. Examples include the conditional logit model [McFadden, 1973], the BLP model [Berry et al., 1995], and bilinear models [Azari Soufani et al., 2013; Schäfer and Hübnermeier, 2018; Zhao et al., 2018a].

The second dimension is PL for partial preferences, where the data consist of partial preferences, often represented by partial orders over the alternatives. Due to the hardness of tackling general partial orders [Liu et al., 2019], most previous work focused on natural sub-cases, including choice data [Train, 2009], top-ranked orders (top-$l$) [Mollica and Tardella, 2017], and pairwise preferences [Hübnermeier et al., 2008]. In particular, in a top-$l$ order, the agent reports a linear order over her most-preferred $l$ alternatives. Top-$l$ orders generalize standard PL ($l = m - 1$, where $m$ is the number of alternatives) and choice data ($l = 1$).

The third dimension is PL mixture models, where $k \geq 1$ PL models are combined via a mixing coefficients $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ and each $\alpha_i$ represents the probability that the data is generated from the $i$-th PL. Mixtures of PLs provide better fitness to data [Tkachenko and Lauw, 2016] and are a popular method for clustering [Gormley and Murphy, 2008], but they are generally hard to compute and are prone to criticisms on interpretability and trustworthiness due to their (non-)identifiability [Zhao et al., 2016].

While there is a large literature on standard PL and its extensions in each of the three dimensions, little is known about the generalization of PL in all three dimensions simultaneously, i.e. mixtures of PL models with features for partial preferences. The literature on general RUMs and their extensions is far more limited. The problem is already highly challenging for top-$l$ orders—to the best of our knowledge, only one previous work studied mixtures of PL models with features for top-$l$ orders [Tkachenko and Lauw, 2016], where an EM algorithm was proposed yet no theoretical guarantees on the model or the algorithm were given.

Motivated by the lack of theoretical understandings of the general PL extensions and learning algorithms, we ask the fol-
Our Contributions. In this paper, we provide the first theoretical characterizations of identifiability for mixtures of $k \geq 1$ PLs with features for top-$l$ orders, denoted by $k$-PL$_X$-TO, where $X$ is the feature matrix. We note that in $k$-PL$_X$-TO, each agent submits a top-$l$ order for possibly different $l$.

Identifiability. We provide necessary and sufficient conditions for PL$_X$-TO and its special case called the bilinear model, denoted by PL$_{X,Z}$-TO, to be identifiable in Theorem 1 and Corollary 1, respectively. Even though $k$-PL$_X$-TO is not identifiable for any $k \geq 2$, we provide a sufficient condition in Theorem 2 for a parameter $k$-PL$_X$-TO to be identifiable, which leads to the generically identifiability of 2-PL$_X$-TO in Corollary 2. It suggests that identifiability may not be a practical concern if the condition is satisfied. We also characterize identifiability of RUMs with features in the appendix.

Strict concavity and sample complexity of MLE. We provide a necessary and sufficient condition for the MLE of PL$_X$-TO to be strictly concave in Theorem 3 and bound on the RMSE of MLE of PL$_X$-TO in Theorem 4, which implies the consistency of MLE and a sample complexity bound.

Table 1 summarizes related works on PL and its extensions that are close to ours. As discussed above, there is a large literature on PL and its extensions in each of the three dimensions, yet no theoretical result is known even for the special non-mixture case PL$_X$-TO for top-$l$ orders in general. [Tkachenko and Lauw, 2016] is the only previous work we are aware of that tackles $k$-PL$_X$-TO, which does not provide theoretical guarantees. Even for PL$_X$-TO, we are only aware of another recent paper [Yildiz et al., 2020], which proposed an ADMM-based algorithm for computing the MLE, but it is unclear whether their algorithm converges to the ground truth because the consistency of MLE was unknown, which is a direct corollary of Theorem 4. To the best of our knowledge, our identifiability results (Theorems 1, 2, Corollaries 1, 2) and the RMSE bound (Theorem 4) are the first for ($k$-)PL$_X$-TO even for linear orders ($l = m - 1$).

Mixtures of PLs with features for choice data (top-1) are well studied and can be dated back to the classical conditional logit model [McFadden, 1973] and the BLP model [Berry et al., 1995]. PL with bilinear features, which is a special case of PL$_X$-TO, has been studied in the literature [Azari Soufiani et al., 2013; Schäfer and Hüllermeier, 2018; Zhao et al., 2018a]. There is a large literature on standard PL (without features) and its mixture models [Hunter, 2004; Soufiani et al., 2013; Negahban et al., 2017; Maystre and Grossglauser, 2015; Khetan and Oh, 2016; Zhao and Xia, 2018; Gormley and Murphy, 2008; Oh and Shah, 2014; Zhao et al., 2016; Chierichetti et al., 2018; Zhao and Xia, 2019; Liu et al., 2019]. Our setting is more general.

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Table 1: The model, identifiability, and consistency of this work compared with previous work.

| Source             | Features | Top-$l$ | Mixtures | Identifiability | Consistency |
|--------------------|----------|---------|----------|----------------|-------------|
| Tkachenko and Lauw, 2016 | ✓        | ✓       | ✓        | ✓ (Thm. 1, 2, Coro. 1, 2) | ✓ (Thm. 4) |
| Yildiz et al., 2020      | ✓        | ✓       | ✓        | ✓              | ✓           |
| McFadden, 1973, Berry et al., 1995 | ✓        | top-1   | ✓        | ✓              | ✓           |
| Grün and Leisch, 2008    | ✓        | top-1   | ✓        | ✓              | ✓           |
| Schäfer and Hüllermeier, 2018 | ✓        | linear orders | ✓        | ✓              | ✓           |
| Zhao et al., 2016, Zhao et al., 2018b, Zhao and Xia, 2019 | ✓        | ✓       | ✓        | ✓              | ✓           |
| Chierichetti et al., 2018 | ✓        | ✓       | ✓        | ✓              | ✓           |
ple 3 for more details. Our Corollary 1 provides a necessary and sufficient condition for the identifiability of more general models. Zhao et al. (2016) characterized the conditions on $k$ (the number of components in the mixture model) and $m$ (the number of alternatives) for mixtures of standard PLs to be (non-)identifiable. Zhao and Xia (2019) characterized (non-)identifiability of mixtures of PLs (without features) for structured partial orders. Grün and Leisch (2008) characterized conditions for mixtures of multinomial logit models to be identifiable. These results do not subsume our results because the model studied in this paper is more general.

Consistency of MLE was proven for the conditional logit model [McFadden, 1973] assuming that each agent provides multiple choice data. Or equivalently, agents with the same features are repeatedly observed. This assumption may not hold in the big data era, where the feature space can be extremely large and it is unlikely that agents would have identical features. Our proof of the RME bound on MLE, which implies consistency, tackles exactly this case and is inspired by the proof of [Khetan and Oh, 2016, Theorem 8] for standard PL, where the Hessian matrix of the model in the main paper and defer results on RUMs to the appendix.

Due to the space constraint, we focus on Plackett-Luce model in the main paper and defer results on RUMs to the appendix.

2 Preliminaries

Let $A = \{a_1, \ldots, a_m\}$ denote the set of $m$ alternatives. Let $\{1, \ldots, n\}$ denote the set of $n$ agents. Given an agent $j$, each alternative is characterized by a column vector of $d \geq 1$ features $\vec{x}_{ji} \in \mathbb{R}^d$. For any $r = 1, \ldots, d$, let $x_{jir}$ denote the $r$th feature of $\vec{x}_{ji}$. A linear order, which is a transitive, antisymmetric, and total binary relation, is denoted by $R = a_1 \succ \cdots \succ a_m$, where $a_i \succ a_j$ means that the agent prefers $a_i$ over $a_j$. Let $\mathcal{L}(A)$ denote the set of all linear orders. A ranked top-$l$ (top-$l$ for short) order has the form $O = a_1 \succ a_2 \cdots \succ a_l \succ \cdots \succ a_m$. It is easy to see that a linear order is a special top-$l$ order with $l = m - 1$. Let $\mathcal{T}(A)$ denote the set of all top-$l$ orders for all $l \in \{1, 2, \ldots, m - 1\}$. An $l$-way order has the form $O = a_{i_1} \succ a_{i_2} \cdots \succ a_{i_l}$. Let $\mathcal{L}(A)$ denote the set of all $l$-way orders for all $l \in \{2, \ldots, m\}$.

Definition 1 (Plackett-Luce model (PL)). The parameter space is $\Theta = \mathbb{R}^m$. The sample space is $\mathcal{L}(A)^n$. Given a parameter $\vec{\theta} \in \Theta$, the probability of any linear order $R = a_{i_1} \succ a_{i_2} \cdots \succ a_{i_m}$ is $\text{Pr}_{PL}(R|\vec{\theta}) = \prod_{p=1}^{m-1} \frac{\exp(\theta_{a_{i_p}})}{\sum_{q=p+1}^{m} \exp(\theta_{a_q})}$.

It follows that the marginal probability for any top-$l$ order $R = a_{i_1} \succ a_{i_2} \cdots \succ a_{i_l} \succ \cdots \succ a_{i_m}$ is $\text{Pr}_{PL}(R|\vec{\theta}) = \prod_{p=1}^{l-1} \frac{\exp(\theta_{a_{i_p}})}{\sum_{q=p+1}^{l} \exp(\theta_{a_{i_q}})}$. In the literature, a normalization constraint on $\vec{\theta}$, e.g. $\sum_{i} \theta_i = 0$, is often required to make the model identifiable. In this paper we do not put such a constraint since it is more convenient to extend the current definition to PL with features.

Let $X_j = [\vec{x}_{1j}, \ldots, \vec{x}_{mj}] \in \mathbb{R}^{d \times m}$ denote the feature matrix for agent $j$, and let $X = [X_1, \ldots, X_n] \in \mathbb{R}^{d \times m \times n}$ denote the feature matrix that concatenates the features for all agents.

Definition 2 (Plackett-Luce model with features (PLX)). Let $X \in \mathbb{R}^{d \times m \times n}$ denote a feature matrix. The parameter space is $\Theta = \mathbb{R}^d$. The sample space is $\mathcal{L}(A)^n$. For any parameter $\vec{\beta} \in \Theta$, the probability of any linear order $R_j = a_{i_1} \succ a_{i_2} \cdots \succ a_{i_m}$ given by agent $j$ is

$$\text{Pr}_{PLX}(R_j|\vec{\beta}) = \prod_{p=1}^{m-1} \frac{\exp(\beta\vec{x}_{ji,p})}{\sum_{q=p+1}^{m} \exp(\beta\vec{x}_{ji,q})}.$$

We note that all feature matrices are assumed given, i.e., not part of the parameter of any model in this paper. PL with bilinear features [Azari Soufiani et al., 2013] is a special case of PLX, where each agent $j \in \{1, \ldots, n\}$ is characterized by a column feature vector $\vec{y}_j \in \mathbb{R}^l$ and each alternative $a_i \in A$ is characterized by a column feature vector $\vec{z}_i \in \mathbb{R}^k$. We note that for any $i \in \{1, \ldots, m\}$, $\vec{z}_i$ is the same across all agents. Let $Y = [\vec{y}_1, \ldots, \vec{y}_n] \in \mathbb{R}^{l \times n}$ denote the agent feature matrix and $Z = [\vec{z}_1, \ldots, \vec{z}_m] \in \mathbb{R}^{k \times m}$ denote the alternative feature matrix.

Definition 3 (Plackett-Luce model with bilinear features (PLX)). Let $Y \in \mathbb{R}^{l \times n}$ denote an agent feature matrix and let $Z \in \mathbb{R}^{k \times m}$ denote an alternative feature matrix. The parameter space consists of matrices $\Theta = \mathbb{R}^{k \times l}$. The sample space is $\mathcal{L}(A)^n$. Given a parameter $B \in \Theta$, the probability of any linear order $R_j = a_{i_1} \succ a_{i_2} \cdots \succ a_{i_m}$ given by agent $j$ is

$$\text{Pr}_{PLX}(R_j|B) = \prod_{p=1}^{m-1} \frac{\exp(B_{ji,p})}{\sum_{q=p+1}^{m} \exp(B_{ji,q})}.$$
The following example shows that non-identifiability of a model can lead to unavoidable untrustworthy interpretations and decisions.

**Example 1.** Suppose an automobile manufacturer is using PL\_Y,Z to learn consumers’ preferences over car models. For simplicity suppose there are two agents \{1, 2\} and two alternatives (car models) \{a_1, a_2\}. Y = \{y_1, y_2\} = \{0.5, 1\}, where each agent is represented by her normalized income (0.5 for the first agent and 1 for the second agent). Z = \[z_1, z_2\] = \[0.6, 1\] and each car is represented by its normalized price (0.6 for the first car and 1 for the second car) and its normalized miles per gallon (0.2 for the first car and 0.5 for the second car).

Let B = \[-1, 8/3\]^T and B’ = \[1, 0\]^T. We show that B and B’ correspond to exactly the same distribution over the two agents’ preferences. In fact, it is not hard to verify that \(z_1^2 B y_1 - z_2^2 B y_1 = z_1^2 B’ y_1 - z_2^2 B’ y_1 = -0.2\) and \(z_1^2 B y_2 - z_2^2 B y_2 = z_1^2 B’ y_2 - z_2^2 B’ y_2 = -0.4\). Therefore, \(Pr_{PL_{Y,Z}}(R_1 = a_1 \succ a_2 | B) = \frac{1 + \exp(z_1^2 B y_1 - z_2^2 B y_1)}{1 + \exp(z_2 B’ y_1 - z_2 B’ y_1)} = Pr_{PL_{Y,Z}}(R_1 = a_1 \succ a_2 | B’).\) Other probabilities can be calculated similarly.

Therefore, it is impossible for any statistical method to distinguish B from B’. This may not be a big concern if the company uses the learned model to predict the preferences of new customers, as both B and B’ would give the same prediction. However, the first components of B and B’ have opposite interpretations. The first component of B being positive is often interpreted as the existence of a negative correlation between an agent’s income and the car’s price, i.e. richer people prefer cheaper cars. The interpretation of the first component of B’ is the opposite. This makes the interpretation and any decision based on it untrustworthy. □

3 Models and Their Identifiability for Top-l Orders

Modeling and learning from different partial orders is desirable in the scenarios where users provide different structures of partial orders [Zhao and Xia, 2019]. Following Zhao and Xia (2019), we extend PL\_X to a model that generates top-l partial orders by introducing a new parameter \(\bar{\phi} = (\bar{\phi}_1, \ldots, \bar{\phi}_{m-1})\), where \(\sum_{i=1}^{m-1} \bar{\phi}_i = 0\). \(\bar{\phi}\) can be viewed as a distribution over \(\{1, 2, \ldots, m - 1\}\) that represents the probability for the agent to report a top-l order, where \(1 \leq l \leq m - 1\). Any model defined in the previous section can be extended to a model that generates a top-l order R with probability \(Pr_{M,\bar{\phi}}(R|\bar{\phi}) = \bar{\phi}_l Pr_{M}(R|\bar{\theta})\), as is illustrated in Figure 1. For example, PL\_X can be extended to PL\_X-TO, formally defined as follows.

**Definition 7 (PL\_X-TO).** Let \(X \in \mathbb{R}^{d \times m_0}\) denote the feature matrix. The parameter space is \(\Theta = \mathbb{R}^d \times \{\phi \in \mathbb{R}_{\geq 0}^{m-1} : \bar{\phi} \cdot 1 = 1\}\). The sample space is \(T(A)\). Given a parameter \((\beta, \bar{\phi})\), the probability of any top-l order \(O_j = a_{i_1} \succ a_{i_2} \succ \ldots \succ a_{i_l} \succ \text{ others}\) given by agent j is \(Pr_{PL_{X,TO}}(O_j | \beta, \bar{\phi}) = \bar{\phi}_l Pr_{PL_X}(O_j | \beta)\), where \(Pr_{PL_X}(O_j | \beta)\) is the marginal probability of \(O_j\) under PL\_X given \(\beta\).

Again, \(X\) is assumed given and not part of the parameter of PL\_X-TO. PL\_X is a submodel of PL\_X-TO (where \(\bar{\phi}_{m-1} = 1\)). PL\_Y,Z-TO and k-PL\_X-TO can be defined similarly, see Appendix A for their formal definitions. The relations between different models mentioned in this paper are shown in the Venn diagram in Figure 2 (PL\_Y,Z and PL\_Y,Z-TO are omitted for simplicity).

As was illustrated in Example 1, identifiability is important if one wants to interpret the learned parameter. For the rest of this section, we focus on identifiability of PL\_X-TO and k-PL\_X-TO.

To characterize the identifiability of PL extensions with features, for each \(j \leq n\), we first define agent j’s normalized feature matrix, denoted by \(\text{norm}(X_j)\).

\[
\text{norm}(X_j) = \begin{bmatrix} \bar{x}_{j2} - \bar{x}_{j1}, \bar{x}_{j3} - \bar{x}_{j1}, \ldots, \bar{x}_{jm} - \bar{x}_{j1} \end{bmatrix}.
\]

We then define \(\text{Norm}(X)\).

\[
\text{Norm}(X) = \{\text{norm}(X_1), \text{norm}(X_2), \ldots, \text{norm}(X_n)\}
\]

In words, \(X\) is normalized by using the feature vector of \(a_1\) as the baseline. Our results still hold if another alternative is used as the baseline. We now present our first identifiability theorem.

**Theorem 1.** For any \(X\), PL\_X-TO is identifiable if and only if \(\text{Norm}(X)\) has full row rank.

The full proof can be found in Appendix G.2.
Example 2. Consider a PLX (a special case of PLX-TO), whose feature matrix \( X \) has three rows \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_3 \), where \( \vec{v}_1 + \vec{v}_2 = \vec{v}_3 \). Therefore, Norm(X) does not have full row rank. Let \( \vec{\beta} = [\beta_1, \beta_2, \beta_3]^\top \) be the ground truth parameter. We construct \( \vec{\beta}' = [\beta_1 + \beta_2 + \beta_3, 0]^\top \). Then it is easy to see \( \vec{\beta}' \cdot X = \vec{\beta} \cdot X \), which further means for any order \( R \), we have \( \Pr_{\text{PLX}}(R | \vec{\beta}') = \Pr_{\text{PLX}}(R | \vec{\beta}) \) by Definition 2. This means this PLX is not identifiable.

Theorem 1 can be applied to characterize the identifiability for PL with bilinear features as in the following corollary.

Corollary 1. For any model PLXZ-TO, where Y \( \in \mathbb{R}^{k \times n} \) and Z \( \in \mathbb{R}^{k \times m} \), PLXZ-TO is identifiable if and only if both Y and norm(Z) have full row rank.

The formal proof can be found in Appendix G.3.

The full row rank condition in Theorem 1 and Corollary 1 is mild as long as \( n \) is not too small, which is the case in many real-world applications. For example, the full row rank condition holds on the sushi dataset used in our real-world experiment. In Appendix F.4, we show that the probability of violating the full row rank condition becomes very small when sampling \( n = 10 \) agents from the sushi dataset, and this probability decays exponentially as \( n \) increases.

In the next example, we show that the sufficient conditions for PLXZ-TO to be identifiable by Schäfer and Hüllemeyer (2018, Proposition 1) is unfortunately flawed.

Example 3. Continuing Example 1, it is not hard to verify that no agent feature or agent feature feature is a constant, which implies that PLXZ-TO is identifiable according to [Schäfer and Hüllemeyer, 2018, Proposition 1]. However, as we showed in Example 1, PLXZ-TO is not identifiable. It’s easy to see that \( \text{norm}(Z) \) has two rows but only one column, which does not have full row rank.

Identifiability of mixtures of PLs with features is at least as challenging as the identifiability of mixtures of standard PLs, which is still an open problem for any \( k \geq 3 \). In the following theorem, we provide a sufficient condition for a parameter in k-PLX-TO to be identifiable.

Theorem 2. If k-PL is identifiable, then for any \( \beta \) such that Norm(\( \beta \)) has full row rank, any parameter \( (\vec{\alpha}, \vec{\beta}, \vec{\phi}) \) with \( \phi_{m-1} > 0 \) is identifiable in k-PLX-TO.

The proof is done by contradiction. The full proof is provided in Appendix G.4.

Corollary 2. For any \( \beta \) such that Norm(\( \beta \)) has full row rank, 2-PLX-TO over four or more features is identifiable modulo label switching.

4 MLE of PLX-TO and Its Consistency

Let \( P = (O_1, \ldots, O_n) \) denote the input data, where for each \( j \leq n \), \( O_j \) is a top-\( l_j \) order. Let \( X \in \mathbb{R}^{d \times mn} \) denote the feature matrix. MLE of PLX-TO computes the parameter that maximize the following log likelihood function:

\[
LL(P|\vec{\phi}, \vec{\beta}) = \sum_{j=1}^{n} \ln \Pr_{\text{PLX-TO}}(O_j|\vec{\beta})
\]

Note that \( \vec{\phi} \) and \( \vec{\beta} \) parameter are separated in the log likelihood function, we can compute them separately as follows.

\[
\vec{\phi}^* = \arg \max_{\vec{\phi}} \sum_{j=1}^{n} \ln \phi_j, \text{ s.t., } \sum_{l=1}^{m-1} \phi_l = 1 \quad (3)
\]

\[
\vec{\beta}^* = \arg \max_{\vec{\beta}} \sum_{j=1}^{n} \ln \Pr_{\text{PLX}}(O_j|\vec{\beta}) \quad (4)
\]

\( \vec{\phi} \) can be easily computed by counting the frequencies of each top-\( l \) order. The main challenge is to accurately estimate the \( \vec{\beta} \) part, which is the main focus of the rest of this section.

The following theorem provides a necessary and sufficient condition for the objective function in (4) to be strictly concave, which turns out to be the same condition for the identifiability of PLX-TO. Strict concavity is desirable because, combined with boundedness, it guarantees the convergence of MLE.

Theorem 3. For any PLX-TO and any data \( P = (O_1, \ldots, O_n) \), the log likelihood function in (4) is strictly concave if and only if \( \text{Norm}(\beta) \) has full row rank.

The full proof can be found in Appendix G.5.

We now introduce an assumption to guarantee the boundedness of MLE for given data \( P \). Boundedness is important for consistency because a strictly concave function may not converge if it is unbounded. For any \( j \leq n \), \( i_1 \leq m \), and \( i_2 \leq m \), we define \( \xi_{j,i_1,i_2} \) as follows.

\[
\xi_{j,i_1,i_2} = \begin{cases} 
1 & \text{if agent } j \text{ prefers } a_{i_1} \text{ over } a_{i_2} \\
-1 & \text{if agent } j \text{ prefers } a_{i_2} \text{ over } a_{i_1} \\
0 & \text{if agent } j \text{’s preference between } a_{i_1} \text{ and } a_{i_2} \text{ is not available}
\end{cases}
\]

Assumption 1. Let \( X \in \mathbb{R}^{d \times mn} \) denote a feature matrix and let \( P \) denote the data. For any \( r \leq d \), there exist \( j_1, j_2 \in \{1, \ldots, n\} \) with \( j_1 \neq j_2 \) and \( i_1, i_2 \in \{1, \ldots, m\} \) with \( i_1 \neq i_2 \) such that \( \xi_{j_1,i_1,i_2} \xi_{j_2,i_1,i_2} (x_{j_1,i_1,r} - x_{j_1,i_2,r})(x_{j_2,i_2,r} - x_{j_2,i_1,r}) < 0 \).

At a high level, Assumption 1 is a mild condition that requires sufficient diversity in agents’ preferences, which mirrors Hunter’s assumption for PL [Hunter, 2004, Assumption 1].

The following lemma shows that Assumption 1 is sufficient for MLE to be bounded.

Lemma 1. For any PLX-TO and data \( P \), if Assumption 1 holds then the MLE in (4) is bounded.

The proof is provided in Appendix G.6. Finally, the following theorem provides a bound on the RMSE of MLE for PLX-TO given that Norm(\( \beta \)) has full row rank.
1. Consistency: We show experiments on synthetic data in this section and way of computing the RMSE bound when the ground truth is unknown. This is obtained by letting $\lambda = 1/n \sum_{j=1}^{n} \ln P_{PLX}(O_j | \beta)$, which is $1/n$ of the objective function in (4). Let $H(\beta)$ denote the Hessian matrix of $L(\beta)$ and $\lambda_1(\beta)$ be the smallest eigenvalue of $-H(\beta)$. Let $\beta_0$ denote the ground truth parameter and $\beta^*$ denote the estimated parameter that is computed using (4). Define $\lambda_{\min} = \min_{0 \leq \sigma \leq 1} \lambda_1(\sigma \beta^* + (1-\sigma) \beta_0)$.

If $\text{Norm}(X)$ has full row rank and Assumption 1 holds, then for any $0 < \delta < 1$, with probability $1 - \delta$,  

$$||\beta^* - \beta_0||_2 \leq \frac{\sqrt{8(m-1)^2 c^2 d \ln(\frac{2d}{\delta})}}{\lambda_{\min} \sqrt{n}},$$  

(5)

where $c$ is the difference between the largest and the smallest entries of $X$.

The full proof is in Appendix G.7.

The RMSE (root-mean-square-error) bound given by Theorem 4 is not tight since we make no assumptions on the distribution of features. While Theorem 4 does provide insights on convergence of MLE:

1. Consistency: as $n$ increases, RMSE $||\beta^* - \beta_0||_2$ decreases at the rate of $1/\sqrt{n}$. When $n$ approaches infinity, RMSE approaches 0.
2. Sample complexity: for any $\epsilon > 0$, $0 < \delta < 1$, $\Pr(||\beta^* - \beta_0||_2 \leq \epsilon) \geq 1 - \delta$ when $n \geq \frac{8(m-1)^2 c^2 d \ln(\frac{2d}{\delta})}{\lambda_{\min}^2 \epsilon^2}$.

This is obtained by letting $\epsilon \geq \frac{\sqrt{8(m-1)^2 c^2 d \ln(\frac{2d}{\delta})}}{\lambda_{\min} \sqrt{n}}$.

3. Approximation of $\lambda_{\min}$: in practice, when the size of data is not too small, $\lambda_{\min}$ can be approximated by $\lambda_1(\beta^*)$ because $\beta^*$ approaches $\beta_0$ as $n$ increases. This gives a practical way of computing the RMSE bound when the ground truth is unknown.

5 Experiments

We show experiments on synthetic data in this section and provide additional experiments on mixture models and on real-world data in the appendix.

Theorem 4. Given any $PL_{X,TO}$ over $m$ alternatives and $n$ agents with the feature matrix $X \in \mathbb{R}^{d \times m}$. Define $L(\beta) = \frac{1}{n} \sum_{j=1}^{n} \ln P_{PLX}(O_j | \beta)$, which is $\frac{1}{n}$ of the objective function in (4). Let $H(\beta)$ denote the Hessian matrix of $L(\beta)$ and $\lambda_1(\beta)$ be the smallest eigenvalue of $-H(\beta)$. Let $\beta_0$ denote the ground truth parameter and $\beta^*$ denote the estimated parameter that is computed using (4). Define $\lambda_{\min} = \min_{0 \leq \sigma \leq 1} \lambda_1(\sigma \beta^* + (1-\sigma) \beta_0)$.

If $\text{Norm}(X)$ has full row rank and Assumption 1 holds, then for any $0 < \delta < 1$, with probability $1 - \delta$,  

$$||\beta^* - \beta_0||_2 \leq \frac{\sqrt{8(m-1)^2 c^2 d \ln(\frac{2d}{\delta})}}{\lambda_{\min} \sqrt{n}},$$  

(5)

where $c$ is the difference between the largest and the smallest entries of $X$.

The full proof is in Appendix G.7.

The RMSE (root-mean-square-error) bound given by Theorem 4 is not tight since we make no assumptions on the distribution of features. While Theorem 4 does provide insights on convergence of MLE:

1. Consistency: as $n$ increases, RMSE $||\beta^* - \beta_0||_2$ decreases at the rate of $1/\sqrt{n}$. When $n$ approaches infinity, RMSE approaches 0.
2. Sample complexity: for any $\epsilon > 0$, $0 < \delta < 1$, $\Pr(||\beta^* - \beta_0||_2 \leq \epsilon) \geq 1 - \delta$ when $n \geq \frac{8(m-1)^2 c^2 d \ln(\frac{2d}{\delta})}{\lambda_{\min}^2 \epsilon^2}$.

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3. Approximation of $\lambda_{\min}$: in practice, when the size of data is not too small, $\lambda_{\min}$ can be approximated by $\lambda_1(\beta^*)$ because $\beta^*$ approaches $\beta_0$ as $n$ increases. This gives a practical way of computing the RMSE bound when the ground truth is unknown.

5 Experiments

We show experiments on synthetic data in this section and provide additional experiments on mixture models and on real-world data in the appendix.

Setup. Fix $m = 10$ and $d = 10$. For each agent and each alternative, the feature vector is generated in $[-1, 1]$ uniformly at random. Each component in $\beta$ is generated uniformly at random in $[-2, 2]$. MLE for $PL_{X,TO}$ was implemented in MATLAB with the built-in fminunc function and tested on a Ubuntu Linux server with Intel Xeon E5 v3 CPUs each clocked at 3.50 GHz. We use mean squared error (MSE) and runtime to measure the statistical efficiency and computational efficiency of algorithms, respectively. Results are shown in Figure 3. All values are averaged over 2000 trials.

Observations. Figure 3 shows the performance of MLE for $PL_{X,TO}$. We observe that MLE decreases as the number of agents increases, which demonstrates consistency of MLE for PL-X,TO. Moreover, learning from top-$l$ preferences with different $l$ values provides tradeoffs between statistical efficiency and computational efficiency.

6 Summary and Future Work

We provide the first set of theoretical results on the identifiability of mixtures of PL with features for top-$l$ preferences. We also identify conditions for the MLE of PL-X,TO to be consistent, and propose an EM algorithm to handle general $k$-PL-X,TO. In the full version of this paper [Zhao et al., 2022], we provide a generalized Rank-Breaking then Composite Marginal Likelihood algorithm for learning RUMs beyond PL from incomplete preferences and show its performance on synthetic data. [Zhao et al., 2022] also includes additional experiments on mixture models as well as missing proofs. Generic identifiability and efficient algorithms for $k$-PL-X,TO are natural questions for future work.

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