Abstract

We present a strong-weak coupling duality for quantum mechanical potentials. Similarly to what happens in quantum field theory, it relates two problems with inverse couplings, leading to a mapping of the strong coupling regime into the weak one, giving information from the nonperturbative region of the parameters space. It can be used to solve exactly power-type potentials and to extract deep information about the energy spectra of polynomial ones.

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Although the solutions of the Schroedinger equation has been studied since its origin, some of its important features were discovered along the time as for example the Aharonov-Bohm effect [1], the study of quantum nondemolition measurements [2], coherent states for general potentials [3], and the geometrical Berry’s phase [4]. Another interesting feature, is the existence of a number of analytical solutions for certain polynomial potentials, when some relations among the potential parameters hold [5]. Moreover some very important applications demand for solutions of anharmonic oscillators, as happens with problems related to tunneling phenomena [6]. Furthermore they can also be used in order to study models analogous to some quantum field theory ones [7].

In this letter we present a very simple and, at same time, very powerful duality for potentials in quantum mechanics. It is capable of defining the analytical dependence of the energy spectrum on the potential parameters, for a large class of potentials. In fact it is just a matter of experimentation to confirm the validity of this feature for potentials like the harmonic oscillator, the Coulomb potential and other exactly solvable potentials. The principal goal of this letter is to show that potentials which can not be treated through perturbation theory are connected with others which can, letting one to use this connection to get the energy in an strong coupling regime through perturbative calculations in the dual potential. This is very similar to the recently studied electric-magnetic duality in \( N = 2 \) supersymmetric Yang-Mills theory [8]. In addition we get the exact behavior of the energy spectra of the general power potential \( V(x) = a x^N \) [9]. We also discuss its application in polynomial potentials [10], [11], [14].

So let us begin by setting the problem in a general way. We start with the one-dimensional time-independent Schroedinger equation

\[
-\frac{\hbar^2}{2\mu} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x),
\]

perform a dilatation in the spatial coordinate \( x \to gx \), so that

\[
-\frac{\hbar^2}{2\mu} \frac{d^2\tilde{\psi}(x)}{dx^2} + U(x)\psi(x) = \mathcal{E}\psi(x),
\]

with \( U(x) \equiv g^2V(gx) \) and \( \mathcal{E} \equiv g^2E \).

As an example we consider the polynomial potential \( V(x) = a x^6 + b x^4 + c x^2 \) [10]. In this case one has:

\[
U(x) = \tilde{a} x^6 + \tilde{b} x^4 + \tilde{c} x^2, \quad \left( \tilde{a} \equiv a g^8, \tilde{b} \equiv b g^6, \tilde{c} \equiv c g^4 \right),
\]

so that, if \( a \) is great compared with \( c \), it is not possible to apply the usual perturbation theory, and even if one use the asymptotic approach like that in [10], valid for great values of the principal quantum number, the problem for the low values of \( n \) is still
maintained. If however one chooses a conveniently small \( g \), it is possible to invert the
situation, obtaining a perturbative potential where \( \tilde{c} \) is much greater than \( \tilde{a} \). In fact, for a
given set of potential parameters, a family of potentials have their energy fixed due to this
symmetry. As an example one can compare two numerical results: \( (a = 100, b = 1, c = 1) \)
with \( E_0 = 13.042036330 \), and \( (a = 10^{-6}, b = 10^{-6}, c = 10^{-4}) \), obtained from the first
setting \( g = 10^{-1} \), with \( E_0 = 0.1304203633 \), which is in accordance with the respective
numerical result. Other important situations can be treated in a quite similar way, as for
instance the problem of atoms in strong magnetic fields \([12]\), interaction of anyons \([13]\,
etc. In all these cases, one can make a connection between the weak and the strong
coupling regime.

Let us now see that this quite simple idea is almost powerful enough to determine by
itself the energy spectra for the power type potentials. In this case \( V(x) = ax^N \), and after
making the above transformations and imposing the covariance of the energy, one gets

\[
E_n = k_n^{(N)} \left( \frac{\hbar^2}{\mu} \right)^{\frac{N}{2}} a^{\frac{N}{2}}, \tag{4}
\]

where \( k_n^{(N)} \) is a factor which can not be determined by the duality, and that for \( N = 2 \)
(harmonic oscillator) will be equal to \( n + \frac{1}{2} \). For the other potentials it is only necessary
determine it for a single value of \( a \) (\( a = 1 \) for example) and the remaining will stay fixed.
One can take for instance the cases with \( N = 4 \) and \( N = 6 \). In these cases the first five
levels are such that one get the results in table I.

Now if one wishes to obtain the value of the energy of any level, with other values
of \( a \) and \( \mu \) (and using the correct value of \( \hbar \)) is just a question of substitution and
evaluation of Eq.(4), using the constants obtained in table I. Furthermore one can also
obtain exactly the wave function by solving the Schroedinger equation, getting an infinite
series depending on the energy. For \( N = 6 \), one has

\[
\Psi(x) \equiv \sum_{n=0}^{\infty} a_n x^n e^{-\frac{\gamma x^4}{4}}, \tag{5}
\]

where \( \gamma \equiv \sqrt{2a} \) and the coefficients \( a_n \) should be computed through the recurrence equa-
tion \( a_{n+2} = \frac{2Ea_n-(2n-1)\gamma a_{n-2}}{(n+1)(n+2)} \). So, substituting the exact energy obtained above, one
gets the corresponding exact wave function.

One can do a comparison between the energy obtained with the method developed
above and that calculated through numerical calculations for some different values of the
potential parameter \( a \) where, for sake of simplicity, we used \( \hbar = \mu = 1 \). In doing so one
obtain the figures appearing in table II. Note that the only difference, when it exists, is in
the last decimal due to the necessary approximation. Nevertheless, if one calculates with
even more precision one sees that this is only a consequence of the approximation.
As our next application of this duality, we consider a polynomial potential like $V(x) = a \ x^N + b \ x^M + c \ x^L$ ($M$ and $L$ lesser than $N$). As for large values of $x$, the $N$-power term dominates, the energy will have its dependence factored and the dependence on the remaining parameters is such that one can write down a series invariant under this symmetry. So we will have something like

$$E_n = a^{2+\frac{2}{N}} \sum_{m=0}^{\infty} b_m x^m,$$

(6)

where $b_0 = k_n^{(N)} \left( \frac{k}{n} \right)^{\frac{N}{N+2}}$, and $x$ must be invariant under this symmetry, so we conclude that $x = a^\alpha b^\beta c^\gamma$, with the coefficients $\alpha, \beta, \gamma$ obeying the equation $\alpha (N + 2) + \beta (M + 2) + \gamma (L + 2) = 0$, in order to keep the symmetry. In fact one can easily verify that the polynomial potentials treated in [10] and [14] have a series where the corresponding expanding variable obeys this equation, so preserving the symmetry outlined here. However in both cases the series diverges when $a$ vanishes, but this feature can be an artifact of the series expansion. One could for example in the potential with $N = 6$, $M = 2$ $L = 0$, try two non-singular functions: $E_n = \left[ \alpha_n^2 a^\alpha + \beta_n^2 b^\beta \right]^{\frac{1}{2}}$ or $E_n = \left[ \alpha_n^4 a + \beta_n^2 b^2 \right]^{\frac{1}{4}}$, which have as expansion series, at a given order, singular expressions like that in (6). Furthermore, it can be verified that they represent upper and lower approximations for the exact levels respectively, with errors below ten percent. Now we are looking for more precise analytically approximated energies.

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TABLE I: $k_n^N$ coefficients for $N$ equal to 4 and 6, for the first five energy levels.

| n | $k_n^{(4)}$   | $k_n^{(6)}$   |
|---|---------------|---------------|
| 0 | 0.66798625    | 0.68070361    |
| 1 | 2.39038996    | 2.56845109    |
| 2 | 4.69361420    | 5.38898250    |
| 3 | 7.34950780    | 8.88132184    |
| 4 | 10.24505604   | 12.80383861   |

TABLE II: Comparison among the energies obtained from equation (4) and from numerical calculations for $N = 6$, $a = 2$ and $a = \pi$.

| n | $E_n(a = 2)$   | $E_{\text{numerical}}(a = 2)$ | $E_n(a = \pi)$ | $E_{\text{numerical}}(a = \pi)$ |
|---|----------------|-------------------------------|---------------|-------------------------------|
| 0 | 0.80949758     | 0.80949758                    | 0.90624478    | 0.90624478                    |
| 1 | 3.05442031     | 3.05442030                    | 3.41946977    | 3.41946976                    |
| 2 | 6.40861633     | 6.408621632                  | 7.17454298    | 7.17454297                    |
| 3 | 10.56173112    | 10.56173111                  | 11.82401784   | 11.82401785                   |
| 4 | 15.22641597    | 15.22641598                  | 17.04620314   | 17.04620315                   |