FLAT SURFACES IN HYPERBOLIC 3-SPACE WHOSE HYPERBOLIC GAUSS MAPS ARE BOUNDED

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Abstract. We construct a weakly complete flat surface in hyperbolic 3-space \( H^3 \) having a pair of hyperbolic Gauss maps both of whose images are contained in an arbitrarily given open disc in the ideal boundary of \( H^3 \). This construction is accomplished as an application of the minimal surface theory. This looks an interesting phenomenon if one comparing the fact that there are no complete minimal (resp. constant mean curvature one) surfaces in \( \mathbb{R}^3 \) (resp. \( H^3 \)) having bounded Gauss maps (resp. bounded hyperbolic Gauss maps).

1. Introduction

It is a classical fact that any complete immersed flat surface in the hyperbolic 3-space \( H^3 \) must be a horosphere or a hyperbolic cylinder, where ‘flat’ means that the Gaussian curvature vanishes identically. However, this does not imply the lack of an interesting global theory for flat surfaces. Gálvez, Martínez and Milán [6] established a Weierstrass-type representation formula for such surfaces. In this paper, we discuss on flat surfaces with admissible singularities. (A singular (i.e., degenerate) point is called admissible if the corresponding points on nearby parallel surfaces are regularly immersed. See [9].) Flat surfaces with admissible singularities in \( H^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2) \) are called flat fronts which can be characterized as the projections of holomorphic immersed Legendrian curves in \( \text{SL}(2, \mathbb{C}) \). Here, a holomorphic map \( L : \mathbb{D}_1 \rightarrow \text{SL}(2, \mathbb{C}) \) is called Legendrian if the pull-back \( L^*\Omega_{\text{SL}} \) vanishes on \( \mathbb{D}_1 \), where \( \Omega_{\text{SL}} \) is the complex contact form on \( \text{SL}(2, \mathbb{C}) \) defined as

\[
\Omega_{\text{SL}} := x_{11} dx_{22} - x_{21} dx_{12} \quad \text{for } (x_{ij})_{i,j=1,2} \in \text{SL}(2, \mathbb{C}).
\]

A flat front in \( H^3 \) induces a pair of hyperbolic Gauss maps \((G_+, G_-)\), both of which are holomorphic mappings into \( \mathbb{C} \cup \{\infty\} \) as follows: Let \( f : M^2 \rightarrow H^3 \) be a flat front defined on a Riemann surface \( M^2 \). The normal geodesic passing through \( f(p) \) meets the ideal boundary \( \partial H^3 \) of the hyperbolic 3-space \( H^3 \) at \( G_+(p) \) if one identify \( \partial H^3 \) by \( \mathbb{C} \cup \{\infty\} \) via the Poincaré half-space model of the hyperbolic 3-space. In [9], two hyperbolic Gauss maps \( G_+ \) and \( G_- \) are indicated by \( G \) and \( G_\ast \), respectively. We are interested in the behavior of hyperbolic Gauss maps of flat surfaces in \( H^3 \). For flat fronts in \( H^3 \), the completeness and the weak completeness are defined (cf. Date: May 22, 2012.

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Completeness implies weak completeness. There are many complete or weakly complete flat fronts in $H^3$ as shown in [9, 10, 11].

It is well-known that the Gauss map (resp. hyperbolic Gauss map) of a complete immersed minimal surface in $\mathbb{R}^3$ (resp. a complete immersed constant mean curvature one surface in $H^3$) can omit at most four points (cf. Fujimoto [4], [5] and Yu [16]). It is well-known that conformal minimal immersions are obtained by taking the real part of null holomorphic immersions. Recently, as an improvement of Nadirashvili’s discovery of complete bounded minimal surfaces in $\mathbb{R}^3$, the existence of complete bounded null holomorphic immersion

$$F : \mathbb{D}_1 \longrightarrow \mathbb{C}^3,$$

of the unit disk $\mathbb{D}_1 \subset \mathbb{C}$ is shown (cf. [1]), where null means that $F_z \cdot F_z$ vanishes identically, here $F_z := dF/dz$ is the derivative of $F$ with respect to the complex coordinate $z$ of $\mathbb{D}_1$ and the dot denotes the canonical complex bilinear form. In fact, properly immersed null holomorphic curves in $\mathbb{C}^3$ of arbitrary topology are constructed in Alarcón and López [1].

It is known that null curves in $\mathbb{C}^3$ are closely related to Legendrian curves in $\mathbb{C}^3$ (cf. Bryant [2] and also Ejiri-Takahashi [3] for the corresponding SL(2, $\mathbb{C}$)-case). It can be easily checked a holomorphic immersion $L : \mathbb{D}_1 \to \text{SL}(2, \mathbb{C})$ is Legendrian if $L^{-1} dL$ is off-diagonal, namely, there exist two holomorphic 1-forms $\omega$ and $\theta$ on $\mathbb{D}_1$ such that

$$L^{-1} dL = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}.$$

As pointed out in the above paragraph, the projection of $L$ into the hyperbolic 3-space gives a flat front in $H^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2)$. Then the singular set of this flat surface in $H^3$ is given by

$$\{ z \in \mathbb{D}_1 : |\rho(z)| = 1 \},$$

where $\rho$ is the meromorphic function defined by $\rho := \theta/\omega$ called the ratio of canonical forms (cf. [9, 10]). By Darboux’s theorem, the contact structure of SL(2, $\mathbb{C}$) is locally equivalent to that of $\mathbb{C}^3$. Moreover, the following explicit transformation

$$T : \mathbb{C}^3 \ni (X, Y, Z) \mapsto \begin{pmatrix} e^{-Z} & Ye^Z \\ Xe^{-Z} & (1 + XY)e^Z \end{pmatrix} \in \text{SL}(2, \mathbb{C})$$

maps holomorphic Legendrian curves in $\mathbb{C}^3$ with contact form

$$\Omega_C := dZ + YdX$$

to those in $\text{SL}(2, \mathbb{C})$. Using this transformation, we prove the following assertion.

**Theorem.** There exists a weakly complete flat front in hyperbolic 3-space whose induced pair of hyperbolic Gauss maps are contained in an arbitrarily given open disk in the ideal boundary of $H^3$.

It should be remarked that there are no compact flat fronts in $H^3$ (cf. [9, Proposition 3.6]). Also, the assumption of weak completeness in the theorem is crucial, since two hyperbolic Gauss maps can omit at most finite points if the given flat front is complete (see Remark 2.5). In contrast to this theorem, Kawakami [7] showed the ratio of canonical forms $\rho$ of weakly complete flat fronts can omit at most three exceptional values.
2. Proof of the theorem

This section is devoted to prove the theorem as we stated in the introduction. Recall that
\[
H^3 := \text{SL}(2, \mathbb{C})/\text{SU}(2) = \{aa^* : a \in \text{SL}(2, \mathbb{C})\} \quad (a^* = \bar{a}).
\]
is the hyperbolic 3-space of constant curvature $-1$. A smooth map $f: \mathbb{D}_1 \to H^3$ is called a (wave) front if there exists a Legendrian immersion $L_f: \mathbb{D}_1 \to T^*_1 H^3$ with respect to the canonical contact structure of the unit cotangent bundle $\pi: T^*_1 H^3 \to H^3$ such that $\pi \circ L_f = f$. For a holomorphic Legendrian immersion $L: \mathbb{D}_1 \to \text{SL}(2, \mathbb{C})$, the projection
\[
f := LL^*: \mathbb{D}_1 \to H^3
\]
gives a flat front in $H^3$ (see [9, 10] for the definition of flat fronts). In particular, the Gaussian curvature vanishes at each point where $f$ is an immersion. We call $L$ in (2.2) the holomorphic lift of $f$. A flat front $f$ is called weakly complete if its holomorphic lift is complete with respect to the pull-back metric
\[
ds^2_L = |\omega|^2 + |\theta|^2 \quad (|\omega|^2 := \omega \bar{\omega}, \ |\theta|^2 := \theta \bar{\theta})
\]
of the canonical Hermitian metric of $\text{SL}(2, \mathbb{C})$ by $L$, where $\omega$ and $\theta$ are holomorphic 1-forms satisfying (1.3) defined on $\mathbb{D}_1$ (cf. [9, 10]). The first fundamental form of $f$ is written as
\[
ds^2_f := |\omega|^2 + |\theta|^2 + \omega \bar{\omega} + \theta \bar{\theta} = |\omega + \bar{\theta}|^2.
\]
On the other hand, the pair of hyperbolic Gauss maps of $f$ is given by
\[
G_+ := \frac{A}{C}, \quad G_- := \frac{B}{D},
\]
where
\[
L := \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

We now prove the theorem in the introduction: Let
\[
F = (X, Y, Z): \mathbb{D}_1 \to \mathbb{C}^3
\]
be a bounded null holomorphic immersion whose induced metric is complete. Without loss of generality, we may assume that
\[
1 < |X| < 2, \quad |Y| < \frac{1}{3}
\]
hold on $\mathbb{D}_1$. Let $(g, \eta dz)$ be the Weierstrass data of $F$, that is,
\[
F_z \left( = \frac{dF}{dz} \right) = \frac{1}{2}(1 - g^2, \sqrt{-1}(1 + g^2), 2g)\eta.
\]
The metric induced by $F$ is written as
\[
\frac{1}{2}(1 + |g|^2)^2 |\eta dz|^2.
\]
By (2.6), the projection $\tilde{F} := (X, Y): \mathbb{D}_1 \to \mathbb{C}^2$ of $F$ is a bounded holomorphic map. Moreover, the following assertion holds:
Lemma 2.1. The metric
\[(2.7) \quad d\sigma^2 := |dX|^2 + |dY|^2,\]
induced by \(\hat{F}\) is a complete Riemannian metric on \(\mathbb{D}_1\). In particular, \(\hat{F}\) is a holomorphic immersion.

Proof. We have that
\[
2d\sigma^2 = 2|\hat{F}_z|^2|dz|^2 = \frac{1}{2}(|1 - g^2| + |1 + g^2|)|\eta dz|^2 = (1 + |g|^4)|\eta dz|^2
\]
\[
\geq \frac{1}{2}(1 + |g|^2)|\eta dz|^2 = |F_z|^2|dz|^2 = |dX|^2 + |dY|^2 + |dZ|^2.
\]
Since \(F\) is a complete immersion, \(d\sigma^2\) is a complete Riemannian metric. \(\square\)

We now consider a new holomorphic immersion defined by
\[
\tilde{F} := (X, Y, W) : \mathbb{D}_1 \longrightarrow \mathbb{C}^3
\]
where
\[
W := -\int_0^z YdX.
\]
Then \(\tilde{F}\) gives a holomorphic Legendrian immersion with respect to the contact form \(\Omega_C\) as in (1.5). Then the induced map (see (1.4) for the definition of \(T\))
\[
L := T \circ \tilde{F} : \mathbb{D}_1 \longrightarrow \text{SL}(2, \mathbb{C})
\]
can be written by
\[(2.8) \quad L = \begin{pmatrix} e^{-W} & Ye^W \\ Xe^{-W} & (1 + XY)e^W \end{pmatrix},
\]
and the mapping \(f : \mathbb{D}_1 \rightarrow H^3\) given by (2.2) is a flat front. In fact, by a straightforward calculation, we have that
\[(2.9) \quad L^{-1}dL = \begin{pmatrix} -(YdX + dW) & e^{2W}(dY - Y^2dX) \\ e^{-2W}dX & YdX + dW \end{pmatrix}.
\]
Since \(YdX + dW = \tilde{F}^*\Omega_C\) vanishes identically, the \(sl(2, \mathbb{C})\)-valued 1-form \(L^{-1}dL\) is off-diagonal (i.e. we just checked that \(L\) is Legendrian). The following assertion holds:

**Proposition 2.2.** The images of two hyperbolic Gauss maps \(G_+\) and \(G_-\) associated to \(f\) lie in the unit disk \(\{\xi \in \mathbb{C} : |\xi| < 1\}\).

Proof. By (2.5) and (2.8), we have that
\[
G_+ = \frac{1}{X}, \quad G_- = \frac{Y}{1 + XY}.
\]
Then the inequalities (2.6) yield the assertion. \(\square\)

To prove the completeness of the metric \(ds_L^2\), we prepare the following assertion:

**Lemma 2.3.** The metric \(ds_L^2\) is positive definite and satisfies the inequality
\[(2.10) \quad ds_L^2 \leq 2ds_L^2.
\]
Proof. Since $T$ is a local diffeomorphism, $L$ is an immersion, and $ds_L^2$ is positive definite. The inequality (2.10) is obtained as follows (cf. (2.3) and (2.4))

$$ds_f^2 = \omega^2 + |\theta|^2 + \omega \theta + \bar{\omega} \bar{\theta} \leq 2|\omega||\theta| + |\omega|^2 + |\theta|^2 \leq 2(|\omega|^2 + |\theta|^2) = 2ds_L^2.$$  

□

The following assertion is a key to prove the theorem.

**Proposition 2.4.** The metric $ds_L^2$ is complete.

**Proof.** We fix a piecewise smooth divergent path $\gamma : [0, \infty) \to \mathbb{D}_1$ arbitrarily. It is sufficient to show that the image of $\gamma$ has infinite length with respect to $ds_L^2$. If $f(\gamma([0, \infty)))$ is unbounded in $H^3$, then the path $\gamma$ must have infinite length with respect to $ds_f^2$ because of the completeness of the hyperbolic space $H^3$. Then the inequality (2.10) implies that $\gamma$ has infinite length with respect to $ds_L^2$. So we assume that $f(\gamma([0, \infty)))$ is bounded in $H^3$. Since SU(2) is compact, (2.1) yields that the image $L(\gamma([0, \infty)))$ is bounded in SL(2, $\mathbb{C}$). By (2.8), there exists a positive constant $m$ such that

$$|e^{-W}| < m, \quad |(1 + XY)e^W| < \frac{m}{3}$$

holds on $\gamma$. Using (2.6), we have that

$$(2.11) \quad \frac{1}{m} \leq |e^{W(\gamma(t))}| \leq m \quad (t \geq 0).$$

On the other hand, it holds that (cf. (2.9))

$$L^{-1}dL = \begin{pmatrix} 0 & e^{2W}dY - Y^2dX \\ e^{-2W}dX & 0 \end{pmatrix}.$$  

By (2.3), it holds along $\gamma$ that

$$ds_L^2 = |e^{-4W}|dX|^2 + |e^{4W}|dY - Y^2dX|^2$$

$$\geq \frac{1}{m^4} \left( |dX|^2 + |dY - Y^2dX|^2 \right)$$

$$\geq \frac{1}{m^4} \left( |dX|^2 + |dY|^2 - 2|Y^2dX||\frac{dY}{2} + |Y|^4|dX|^2 \right)$$

$$\geq \frac{1}{m^4} \left( |dX|^2 + |dY|^2 - (4|Y^2dX|^2 + \frac{|dY|^2}{4}) + |Y|^4|dX|^2 \right)$$

$$\geq \frac{1}{m^4} \left( (1 - 3|Y|^4)|dX|^2 + 3|dY|^2 \right).$$

Again using (2.6),

$$ds_L^2 \geq \frac{1}{m^4} \left( \frac{26|dX|^2}{27} + 3\frac{|dY|^2}{4} \right) \geq \frac{3}{4m^4} \left( |dX|^2 + |dY|^2 \right) = \frac{3}{4m^2}d\sigma^2$$

hold on $\gamma$. Since $d\sigma^2$ is a complete metric (cf. Lemma 2.1), we get the conclusion.  

□

**Proof of the theorem.** Let $r$ be an arbitrarily sufficiently small positive number. We set

$$L_r := \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}L.$$
Then $f_r := L_r^* L_r$ is a desired flat front. In fact, $f_r$ is weakly complete since the induced metric of $L_r$ is complete (cf. Proposition 2.4). On the other hand, the hyperbolic Gauss maps of $f_r$ are equal to $r^2 G_+$ and $r^2 G_-$, and their images are contained in a disk of radius $r^2$ (cf. Proposition 2.2).

**Remark 2.5.** Let $(G_+, G_-)$ be the pair of hyperbolic Gauss maps induced by a complete flat front $f : M^2 \to H^3$, where $M^2$ is a Riemann surface. By [9, Lemma 3.3], there exists a closed Riemann surface $\bar{M}^2$ and finite points $p_1, \ldots, p_n$ on $\bar{M}^2$ such that $M^2$ is bi-holomorphic to $\bar{M}^2 \setminus \{p_1, \ldots, p_n\}$. If $G_+$ (resp. $G_-$) has an essential singularity at some $p_j$ (i.e. $p_j$ is an irregular end), Picard’s theorem implies that $G_+$ (resp. $G_-$) omits at most two points in $\mathbb{C} \cup \{\infty\}$. Otherwise, both $G_+$ and $G_-$ are meromorphic functions on a compact Riemann surface $\bar{M}^2$. Hence both $G_+$ and $G_-$ are surjective maps onto $\mathbb{C} \cup \{\infty\}$.

We now remark that the following problem seems interesting as an analogue of Calabi-Yau problem in minimal surface theory:

**Question.** Are there complete bounded holomorphic Legendrian curves immersed in $\mathbb{C}^3$?

As seen in our previous arguments, this problem is related to our main result. Moreover, it is also closely related to the existence of bounded weakly complete improper affine fronts in the affine space $\mathbb{R}^3$: A notion of IA-maps in the affine 3-space has been introduced by A. Martínez [12]. IA-maps are improper affine spheres with a certain kind of singularities. Since all of IA-maps are wave fronts (see [14, 15]), we call them improper affine fronts (the terminology ‘improper affine fronts’ has been already used in Kawakami-Nakajo [8]). The precise definition of improper affine fronts is given in [15, Remark 4.3]. In [15], weak completeness of improper affine fronts is introduced. Then we have

**Proposition 2.6.** The existence of a complete bounded immersed Legendrian curve as in the above question would imply the existence of a weakly complete improper affine front whose image is bounded.

**Proof.** Let $F = (X, Y, Z)$ be a complete bounded Legendrian immersion into $\mathbb{C}^3$. Since $F$ is Legendrian, (1.5) yields that $dZ = -Y dX$. Here, by completeness of $F$, the induced metric
ds_{F_r}^2 = |dX|^2 + |dY|^2 + |dZ|^2 = |dX|^2 + |dY|^2 + |Y dX|^2 = (|Y|^2 + 1)|dX|^2 + |dY|^2
is complete. Moreover, since the image of $F$ is bounded, we have
ds_{F_r}^2 \leq C(|dX|^2 + |dY|^2) \quad (C > 0 \text{ is a constant}).
Thus, the metric
\begin{equation}
d\tau^2 := |dX|^2 + |dY|^2
\end{equation}
is complete. Hence, we have the following improper affine front $f$ substituting the pair of holomorphic functions $(X, Y)$ into Martinez’ representation formula [12, Theorem 3]

\begin{equation}
f = \left( X + Y, \frac{1}{2}(|X|^2 - |Y|^2) + \Re \left( XY - 2 \int Y dX \right) \right) = \left( X + Y, \frac{1}{2}(|X|^2 - |Y|^2) + \Re (XY + 2Z) \right) : \mathbb{D}_1 \to \mathbb{R}^3.
\end{equation}
Since $dt^2$ in (2.12) is complete, $f$ is weakly complete, by definition of weak completeness given in [15]. The boundedness of $f$ follows from that of $F$. □

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