Integrability of the Manakov–Santini hierarchy

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Abstract

The first example of the so-called “coupled” integrable hydrodynamic chain is presented. Infinitely many commuting flows are derived. Compatibility conditions of the first two of them lead to the remarkable Manakov–Santini system. Integrability of this four component three dimensional quasilinear system of the first order as well as the coupled hydrodynamic chain is proved by the method of hydrodynamic reductions. In comparison with a general case considered by E.V. Ferapontov and K.R. Khusnutdinova, in this degenerate case $N$ component hydrodynamic reductions are parameterized by $N + M$ arbitrary functions of a single variable, where $M$ is a number of branch points of corresponding Riemann surface. These hydrodynamic reductions also are written as symmetric hydrodynamic type systems. New classes of particular solutions are found.
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1 Introduction

This paper is dedicated to a new type of integrable systems, i.e. vector hydrodynamic chains. The simplest example of scalar integrable hydrodynamic chains is the famous Benney hydrodynamic chain (see [2])

\[ A^k_t = A^{k+1}_x + k A^{k-1} A^0_x, \quad k = 0, 1, 2, ... \]  \hspace{1cm} (1)

This hydrodynamic chain can be written in the conservative form (see, for instance, [31])

\[ \partial_t H_k = \partial_x \left( H_{k+1} - \frac{1}{2} \sum_{m=0}^{k-1} H_m H_{k-1-m} \right), \quad k = 0, 1, 2, ..., \] \hspace{1cm} (2)

where all conservation law densities \( H_k \) are polynomial functions with respect to moments \( A^n \). For instance (see [2]), \( H_0 = A^0, H_1 = A^1, H_2 = A^2 + (A^0)^2, H_3 = A^3 + 3A^0A^1, ... \)

In this paper, we consider a simplest generalization of the Benney hydrodynamic chain to a vector case

\[ \begin{pmatrix} q_k \\ f_k \end{pmatrix} = \left( \begin{pmatrix} q_{k+1} \\ f_{k+1} \end{pmatrix} \right)_{x} - f_0 \left( \begin{pmatrix} q_k \\ f_k \end{pmatrix} \right)_{x} - \sum_{m=0}^{k-1} q_{k-1-m} \left( \begin{pmatrix} q_m \\ f_m \end{pmatrix} \right)_{x}, \quad k = 0, 1, 2, ..., \] \hspace{1cm} (3)

where the corresponding vector function contains two species of field variables \( (q_k, f_k) \) only. We believe that much more complicated integrable vector hydrodynamic chains

\[ \vec{A}^k_t = \hat{B}_0(\vec{A}) \vec{A}^{k+1}_x + \hat{B}_1(\vec{A}) \vec{A}^k_x + \ldots + \hat{B}_{k+1}(\vec{A}) \vec{A}^0_x, \quad k = 0, 1, 2, ... \] \hspace{1cm} (4)

can be discovered in coming future. In such a general construction, matrix functions \( \hat{B}_k(\vec{A}) \) can depend on \( \vec{A}^0, \vec{A}^1, ..., \vec{A}^k \) for each index \( k \) only, and vector \( \vec{A}^k \) can depend on an arbitrary number of components.

This vector hydrodynamic chain (3) possesses two obvious reductions. If \( f_k \) vanish, then \( q_k \to H_k \), and (3) reduces to (2); if \( g_k \) vanish, then (3) reduces to the well-known linearly degenerate hydrodynamic chain (see formula (37) in [26])

\[ (f_k)_t = (f_{k+1})_x - f_0 (f_k)_x, \quad k = 0, 1, 2, ... \] \hspace{1cm} (5)

\[ ^1 \text{as usual, a summation for negative values of upper indices should be ignored in all cases below.} \]
Thus, (3) is a natural generalization of two remarkable integrable hydrodynamic chains on “two species” case. This paper is particularly inspired by the first example in physical literature such a Taranov’s bi-chain (describing high frequency electron-positron plasma waves, see detail in [34])

\[
A_k^t = A_{x}^{k+1} + kE A^{k-1}, \quad B_k^t = B_{x}^{k+1} - kEB^{k-1}, \quad k = 0, 1, 2, ..., \]

where

\[
dE = (A^0 - B^0)dx + (A^1 - B^1)dt.
\]

However, this bi-chain has no a “hydrodynamical” origin, i.e. this bi-chain does not possess hydrodynamic reductions (see [11]). In a contrary with this bi-chain, hydrodynamic bi-chain (3) possesses infinitely many hydrodynamic reductions. Moreover, hydrodynamic bi-chain (3) is connected with the three dimensional quasilinear Manakov–Santini system determined by a commutativity condition of two vector fields (see [21]). In this paper, these vector fields are utilized for a derivation of hydrodynamic bi-chain (3). Method of symmetric hydrodynamic reductions (see [30]) is extended on this hydrodynamic bi-chain. A construction of higher symmetries (see [3]) is reformulated for a description of hydrodynamic reductions, higher commuting flows and for an application to the generalized (Tsarev’s) hodograph method (see [35]). Thus, in this paper, we prove an integrability of hydrodynamic bi-chain (3) by the method of hydrodynamic reductions adopted on two species of moments $A^k$ and $B^k$. Moreover, extracting symmetric hydrodynamic reductions (see [30]), we present a list of infinitely many explicit hydrodynamic reductions, i.e. one can construct infinitely many particular solutions for the aforementioned Manakov–Santini system as well as for hydrodynamic bi-chain (3). First examples of three-component hydrodynamic reductions of the Manakov–Santini system were found in [6]. In comparison with the previous preliminary investigation (see [6]), we been able to prove a consistency of all described hydrodynamic reductions with hydrodynamic bi-chain (3) as well as with a whole Manakov–Santini hierarchy.
This paper is organized in the following way. In Section 2, a four component three
dimensional hydrodynamic type system (equivalent to the Manakov–Santini system) is
presented. Due to its dispersionless Lax representation, a first example of integrable
vector hydrodynamic chains is derived. Simple reductions to well-known hydrodynamic
chains are found. In Section 3, a generating function of conservation laws and commuting
flows is described. In Section 4, the Manakov–Santini system is obtained from a new hy-
drodynamic bi-chain. In Section 5, semi-Hamiltonian reductions of the Manakov–Santini
system are considered. It is proved that a corresponding dispersion relation is degener-
ated. In Section 6, two species of Riemann invariants are introduced. Full extended
Gibbons–Tsarev system is derived. In Section 7, all symmetric hydrodynamic reductions
are described, and some of them are explicitly found. In Section 8, the generalized hodo-
graph method is applied for a construction of infinitely many particular solutions for the
Manakov–Santini hierarchy. In Conclusion, a generalization of the approach presented in
this paper is discussed.

2 Lax representation

The function $\lambda(x, t, y; q)$ determined by the pair of linear equations

$$\lambda_t = (q - a) \lambda_x - u_x \lambda_y, \quad \lambda_y = (q^2 - aq - c) \lambda_x - (qu_x + u_t) \lambda_y$$

exists, because a compatibility condition $(\lambda_t)_y = (\lambda_y)_t$ implies to the three dimensional
hydrodynamic type system

$$a_t = (c + u)_x, \quad c_t = ca_x - ac_x - b_x + a_y, \quad u_t = b_x - au_x, \quad b_t = cu_x + u_y,$$

which is equivalent to the well-known Manakov–Santini system (see [21])

$$w_{xy} = w_{tt} + w_x w_{xt} + (u - w_t) w_{xx}, \quad u_{xy} = u_{tt} + u_x^2 + w_x u_{xt} + (u - w_t) u_{xx}.$$
Indeed, the first two equations in (7) reduce to the first equation in (8) by the potential substitution \( a = w_x \) and \( c = w_t - u \), while the second equation in (8) can be obtained from the compatibility condition \((b_t)_x = (b_x)_t\) of other two equations in (7).

An inverse transformation \( \lambda(x, t, y; q) \to q(x, t, y; \lambda) \) leads to the pair of compatible quasilinear equations

\[
q_t = (q - a)q_x + u_x, \quad q_y = (q^2 - aq - c)q_x + qu_x + u_t. \tag{9}
\]

Let us introduce another compatible pair of auxiliary quasilinear equations (formally, replacing \( q_x, q_t, q_y \) and \( u \) by \( f_x, f_t, f_y \) and \( a \), respectively)

\[
f_t = (q - a)f_x + a_x, \quad f_y = (q^2 - aq - c)f_x + qa_x + a_t, \tag{10}
\]

where \( f(x, t, y; q) \) depends on the parameter \( \lambda \) implicitly via a dependence \( q(x, t, y; \lambda) \).

The compatibility conditions \((q_t)_y = (q_y)_t\) and \((f_t)_y = (f_y)_t\) imply to (7) in both cases.

The reduction \( a = 0, c = -u \) reduces (7) to the famous Khokhlov–Zabolotskaya equation (also well known as a dispersionless limit of the Kadomtsev–Petviashvili equation)

\[
\begin{align*}
    u_t &= b_x, \\
    b_t &= u_y - uu_x 
\end{align*} \tag{11}
\]

determined by compatibility condition (9) under the reduction \( f = 0 \); the reduction \( u = 0, b = 0 \) reduces (7) to the remarkable equation (recognized as an anti-self-duality reduction of the famous Yang-Mills equation, see [8])

\[
\begin{align*}
    a_t &= c_x, \\
    c_t &= ca_x - ac_x + a_y, 
\end{align*} \tag{12}
\]

determined by compatibility condition (10) under the reduction \( q = 0 \).

Let us substitute an expansion \( \lambda \to \infty, q \to \infty \), according to a general theory of a relationship between dKP equation (11) and Benney hydrodynamic chain (2); see, for instance, [31])

\[
q = \lambda - \frac{q_0}{\lambda} - \frac{q_1}{\lambda^2} - \frac{q_2}{\lambda^3} - \ldots \tag{13}
\]
in the first equation of (9). Then the first part of hydrodynamic bi-chain (3) is derived, where \( u = q_0 \) and \( b = q_1 \). A substitution of an expansion \((\lambda \to \infty, f \to \infty)\), according to a general theory of a relationship between quasilinear system (12) and hydrodynamic chain (5); see formula (39) in [26]

\[
f = -\frac{f_0}{\lambda} - \frac{f_1}{\lambda^2} - \frac{f_2}{\lambda^3} - \ldots
\]  

(14)
in the first equation of (10) implies to the second part of hydrodynamic bi-chain (3), where \( a = f_0 \) and \( c = f_1 - q_0 - f_0^2/2 \). Thus, whole hydrodynamic bi-chain (3) cannot be derived just from the first equation of (9) as well as from the first linear equation of (6) by an inverse transformation \((\lambda \to \infty, q \to \infty)\), see (13)

\[
\lambda = q + \frac{A^0}{q} + \frac{A^1}{q^2} + \frac{A^2}{q^3} + \ldots
\]  

(15)
which is useful in a derivation of Benney hydrodynamic chain (1) under a corresponding reduction \( f = 0 \) to the dKP case (see, for instance, [31]). In such a case, another formal expansion \((\lambda \to \infty, M \to \infty)\), the so-called Orlov function; see [3]

\[
M = \sum_{n=1}^{\infty} nt^{n-1} \lambda^{n-1} + \sum_{n=1}^{\infty} v_n \lambda^{-n}
\]

for the first equation\(^2\) of (6) can be utilized in a derivation of the second part (see [6]) of hydrodynamic bi-chain (3), where \( p_n = (v_n)_x \) are nothing else but conservation law densities (see below). Here \( t^0 = x, t^1 = t, t^2 = y \). All other higher “times” \( t^k \) are associated with higher commuting hydrodynamic bi-chains (see below). Instead this Orlov function \( M \), in this paper, we use an alternative approach (see (9) and (10)) to derive a generating function of conservation laws \( p \equiv M_x \), where \( \lambda \) is a free parameter; see detail in [6]) as well as a generating function of commuting flows (see below).

A substitution (15) to (14) implies to the expansion \((f \to \infty, q \to \infty)\)

\[
f = -\frac{B^0}{q} - \frac{B^1}{q^2} - \frac{B^2}{q^3} - \ldots
\]  

(16)

\(^2\)It means that the function \( M(\lambda, t, v) \) satisfies (6) as well as the function \( \lambda(q, A) \).
Let us rewrite (10) splitting dynamics with respect to $x, t, y$ and $q$. It means the dynamics (9) is excluded from (10), and $q$ becomes an extra independent variable as well as $x, t, y$. Corresponding equations reduce to the form (cf. (6))

$$f_t = (q - a)f_x - u_x f_q + a_x, \quad f_y = (q^2 - aq - c)f_x - (qu_x + u_t)f_q + qa_x + a_t. \quad (17)$$

A substitution (15) to the first equation in (6) and (16) to the first equation in (17) leads to a generalization of Benney hydrodynamic chain (1) to the vector case (cf. (3))

$$\begin{pmatrix} A^k \\ B^k \end{pmatrix}_t = \begin{pmatrix} A^{k+1} \\ B^{k+1} \end{pmatrix}_x - B^0 \begin{pmatrix} A^k \\ B^k \end{pmatrix}_x + k \begin{pmatrix} A^{k-1} \\ B^{k-1} \end{pmatrix} A^0_x, \quad k = 0, 1, 2, \ldots, \quad (18)$$

where $a = B^0$ and $u = A^0$.

This hydrodynamic bi-chain has three obvious reductions. If $A^k = 0$, then (18) reduces to (5); if $B^k = 0$, then (18) reduces to (1); if $B^k = A^k$, then (18) reduces to the “deformed” Benney hydrodynamic chain

$$A^k_t = A^{k+1}_x - A^0 A_x^k + k A^{k-1} A^0_x, \quad k = 0, 1, 2, \ldots,$$

derived in [13], whose integrability was investigated in [32]. This hydrodynamic chain is connected (see [29]) with the so-called “interpolating system” (see [7]).

Moreover, Manakov–Santini system (8) possesses a natural reduction $u = w_t$ (see [6]) to the continuum limit of the discrete KP hierarchy (see, for instance, [20])

$$\Omega_{xy} = \Omega_{tt} + \frac{1}{2} \Omega_{xt}^2, \quad (19)$$

where $w = \Omega_t$. In such a case (7) reduces (i.e. $c = 0$) to

$$a_t = u_x, \quad a_y = b_x, \quad b_t = u_y, \quad u_t = b_x - au_x.$$

A comparison of the first conservation law $a_t = u_x$ (i.e. $B^0_t = A^0_x$, see the notation in (18)) with the first conservation law of (18)

$$B^0_t = \left( B^1 - \frac{1}{2} (B^0)^2 \right)_x.$$
implies to the first constraint
\[ A^0 = B^1 - \frac{1}{2}(B^0)^2. \]
It means, that hydrodynamic bi-chain (18) reduces to the hydrodynamic chain
\[ B_t^k = B_{x}^{k+1} - B^0B_x^k + kB^{k-1}\left(B^1 - \frac{1}{2}(B^0)^2\right)_x, \tag{20} \]
which is a particular case of a more general class of integrable hydrodynamic chains \((\alpha, \beta, \gamma)\) are arbitrary constants)\n\[ B_t^k = B_{x}^{k+1} - B^0B_x^k + (\alpha k + \beta)B^kB^0 + \gamma kB^{k-1}\left(B^1 + \frac{\beta - \alpha - \gamma(B^0)^2}{2}\right)_x, \]
investigated in [18] and then in [32]. A computation of all other constraints \(A^k(B^0, B^1, ..., B^{k+1})\) is a more complicated task. Nevertheless, the second constraint can be found due to comparison of the fourth equation \(u_t = b_x - au_x\) (see (19)) with the first equation \(A_t^0 = A_x^1 - B^0A_x^0\) from the \(A\)-part of (18). It means that \(b = A^1\) and a substitution of the first constraint \(A^0 = B^1 - (B^0)^2/2\) leads to the conservation law \(B_t^1 = A_x^1\). Taking into account the second equation from (20), the second constraint \(A^1 = B^2 - (B^0)^3/3\) can be easily obtained.

In aforementioned case (19), the first equation in (9) reduces to
\[ p_t = \partial_x(e^p + B^0), \]
where \(p = \ln(q - B^0)\). This is a generating function of conservation laws for the remarkable hydrodynamic chain (see, for instance, [15] and [20])
\[ C_t^k = C_{x}^{k+1} + kC^kC^0_x, \quad k = 0, 1, 2, ..., \]
where \(C^0 = B^0\) and all other moments \(C^k\) are connected with \(B^n\) by inverse triangular point transformations (see detail in [32]). Since this hydrodynamic chain belongs to the Boyer–Finley (a continuum limit of 2D Toda Lattice) hierarchy (see [5]), a corresponding algebraic mapping is given by the expansion \((\lambda \rightarrow \infty, q \rightarrow \infty, p \rightarrow \infty)\)
\[ \lambda = e^p + C^0 + e^{-p}C^1 + e^{-2p}C^2 + ... = q + \frac{C^1}{q - C^0} + \frac{C^2}{(q - C^0)^2} + \frac{C^3}{(q - C^0)^3} + ..., \]
which must coincide with two other expansions ($\lambda \to \infty$, $q \to \infty$; see (15))

\[
\lambda = q + \frac{A^0}{q} + \frac{A^1}{q^2} + \frac{A^2}{q^3} ... = (q - B^0) \exp \left( \frac{B^0}{q} + \frac{B^1}{q^2} + \frac{B^2}{q^3} ... \right).
\]

(21)

Thus, we see that hydrodynamic chain (20) is embedded in hydrodynamic bi-chain (18) by very complicated reductions $A^k(B^0, B^1, ..., B^{k+1})$, while (8) reduces to (19) by the simple constraint $c = 0$ (see (7)). Moreover, in this case, both linear systems (9) and (17) coincide due to aforementioned link (see (16) and (21))

\[
\lambda = (q - B^0)e^{-f}.
\]

The general problem of all possible compatible reductions $A^k(B^0, B^1, ..., B^{k+n})$ should be considered in a separate publication (for each fixed non-negative integer $n$).

### 3 Generating functions

Let us rewrite the first equation of (10) in the form

\[
s_t = (q - a)s_x + sa_x,
\]

where (see (14))

\[
s = \exp f = 1 - \frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \frac{s_2}{\lambda^3} - ...
\]

(22)

Then hydrodynamic bi-chain (3) reduces to the form

\[
(s_k)_t = (s_{k+1})_x - s_0(s_k)_x - \sum_{m=0}^{k-1} q_{k-1-m}(s_m)_x + s_k(s_0)_x, \quad (q_k)_t = (q_{k+1})_x - s_0(q_k)_x - \sum_{m=0}^{k-1} q_{k-1-m}(q_m)_x,
\]

where $s_0 \equiv f_0$ (all other higher unknown functions $s_k$ are polynomial functions of $f_0, q_0, f_1, q_1, ..., f_k, q_k$ for each index $k$). If $q_k$ vanish, this hydrodynamic bi-chain reduces precisely to the form derived in [1].

Let us introduce the so-called vertex operator ($\zeta \to \infty$)

\[
\partial_{\tau(\zeta)} = -\frac{1}{\zeta} \partial_{t^0} - \frac{1}{\zeta^2} \partial_{t^1} - \frac{1}{\zeta^3} \partial_{t^2} - ...
\]

(23)
**Theorem:** Higher commuting hydrodynamic bi-chains are determined by the pair of vertex equations

\[
\partial_{\tau(\zeta)} s(\lambda) = \frac{s(\zeta)\partial_x s(\lambda) - s(\lambda)\partial_x s(\zeta)}{q(\lambda) - q(\zeta)}, \quad \partial_{\tau(\zeta)} q(\lambda) = s(\zeta)\partial_x \ln(q(\lambda) - q(\zeta))
\]  

(24)

by a substitution (13), (22),

\[
q(\zeta) = \zeta - \frac{q_0}{\zeta} - \frac{q_1}{\zeta^2} - \frac{q_2}{\zeta^3} - ..., \quad s(\zeta) = 1 - \frac{s_0}{\zeta} - \frac{s_1}{\zeta^2} - \frac{s_2}{\zeta^3} - ...
\]

and (23).

**Proof:** A direct substitution (13), (22) and (23) in (24) leads to a family of infinitely many hydrodynamic bi-chains. Their commutativity follows from the compatibility conditions \(\partial_{\tau(\zeta)}(\partial_t s(\lambda)) = \partial_t(\partial_{\tau(\zeta)} s(\lambda)), \partial_{\tau(\zeta)}(\partial_y s(\lambda)) = \partial_y(\partial_{\tau(\zeta)} s(\lambda)), \partial_{\tau(\zeta)}(\partial_t q(\lambda)) = \partial_t(\partial_{\tau(\zeta)} q(\lambda)), \partial_{\tau(\zeta)}(\partial_y q(\lambda)) = \partial_y(\partial_{\tau(\zeta)} q(\lambda))\) leading to the extra vertex dynamics (see (7))

\[
\partial_{\tau(\zeta)} a = \partial_x s(\zeta), \quad \partial_{\tau(\zeta)} c = (q(\zeta) - a)\partial_x s(\zeta) - s(\zeta)\partial_x (q(\zeta) - a),
\]

(25)

\[
\partial_{\tau(\zeta)} u = s(\zeta)\partial_x q(\zeta), \quad \partial_{\tau(\zeta)} b = s(\zeta)\partial_x \left(\frac{q^2(\zeta)}{2} + u\right),
\]

which are compatible (i.e. \(\partial_{\tau(\zeta)}(\partial_t a) = \partial_t(\partial_{\tau(\zeta)} a), \partial_{\tau(\zeta)}(\partial_t b) = \partial_t(\partial_{\tau(\zeta)} b), \partial_{\tau(\zeta)}(\partial_t c) = \partial_t(\partial_{\tau(\zeta)} c), \partial_{\tau(\zeta)}(\partial_t u) = \partial_t(\partial_{\tau(\zeta)} u)\)) with (7). Thus, the theorem is proved.

**Remark:** This generating function of commuting flows (25) can be obtained by a substitution (13), (22) in (24) only.

Taking into account the first equation in (25), expansions (22) and (23), the sole function \(S(t^0, t^1, t^2, ...\) can be introduced such that

\[
s(\zeta) = \partial_{\tau(\zeta)} S + 1, \quad s_k = \frac{\partial S}{\partial t^k}.
\]

Then the first equation in (24) implies to

\[
q'(\lambda) = \frac{s(\lambda)\partial_x s'(\lambda) - s'(\lambda)\partial_x s(\lambda)}{\partial_{\tau(\lambda)} s(\lambda)}
\]

\(^3\text{here and below, the sign “prime” means a derivative with respect to a corresponding parameter.}\)
in the limit $\zeta \to \lambda$. Thus, all functions $q_k$ can be expressed via first and second derivatives of the sole function $S$ with respect to corresponding “time” variables $t^n$. A substitution of these dependencies to the second equation in (24) leads to an infinite set of three dimensional quasilinear equations of the third order. From the other hand, three copies of the first equation in (24) imply to a sole vertex equation

$$\frac{[\partial_{\tau(\zeta)} S + 1] \partial_x \partial_{\tau(\lambda)} S - [\partial_{\tau(\lambda)} S + 1] \partial_x \partial_{\tau(\zeta)} S}{\partial_{\tau(\zeta)} \partial_{\tau(\lambda)} S} + \frac{[\partial_{\tau(\eta)} S + 1] \partial_x \partial_{\tau(\lambda)} S - [\partial_{\tau(\lambda)} S + 1] \partial_x \partial_{\tau(\eta)} S}{\partial_{\tau(\eta)} \partial_{\tau(\lambda)} S} + [\partial_{\tau(\lambda)} S + 1] \partial_x \partial_{\tau(\lambda)} S \partial_{\tau(\lambda)} \partial_{\tau(\eta)} S = 0.$$  

An expansion of this generating function with respect to three parameters $\lambda, \zeta, \eta$ at infinity leads to an infinite set of three dimensional quasilinear equations of the second order.

**Remark:** The second equation in (24) implies to

$$s(\lambda) = \frac{\partial_{\tau(\lambda)} q(\lambda)}{\partial_x \ln q'(\lambda)}.$$  

in the limit $\zeta \to \lambda$. Thus, all functions $s_k$ can be expressed via functions $q_0, q_1, ..., q_k$ and their first derivatives with respect to corresponding “time” variables $t^n$. A substitution of these dependencies to the first equation in (24) leads to an infinite set of three dimensional quasi-linear equations of the second order.

$$[q(\lambda) - q(\zeta)] \partial_{\tau(\zeta)} \frac{\partial_{\tau(\lambda)} q(\lambda)}{\partial_x \ln q'(\lambda)} = \partial_{\tau(\zeta)} q(\zeta) \partial_x \frac{\partial_{\tau(\lambda)} q(\lambda)}{\partial_x \ln q'(\lambda)} - \partial_{\tau(\lambda)} q(\zeta) \partial_x \frac{\partial_{\tau(\lambda)} q(\zeta)}{\partial_x \ln q'(\lambda)}.$$  

Generating functions (24) have an obvious two reductions. The choice $s(\lambda) = 1$ means reduction to the generating function associated with the dKP hierarchy (11) and with the Benney hydrodynamic chain (2); the choice $q(\lambda) = \lambda$ means reduction to the generating function associated with hydrodynamic chain (5), quasi-linear system (12) and the so-called “universal” hierarchy (see detail in [1] and [26]).

**Theorem:** Hydrodynamic bi-chain (3) possesses a generating function of conservation laws

$$\partial_t p = \partial_x [(q - a)p],$$  

(26)
where the generating function of conservation law densities is given by

\[ p(\lambda) = \frac{q'(\lambda)}{s(\lambda)}. \]  \hfill (27)

**Proof**: Indeed, taking into account the first equation in (9) and the first equation in (10), a direct substitution (27) in (26) implies to an identity.

**Remark**: A substitution (13) and (22) in (27) leads to an expansion

\[ p = 1 + \frac{p_0}{\lambda} + \frac{p_1}{\lambda^2} + \frac{p_2}{\lambda^3} + ..., \]

where all coefficients \( p_k \) can be expressed polynomially via field variables \( q_0, f_0, q_1, f_1, ..., q_k, f_k \).

Corresponding conservation laws can be written in the form

\[ p_{k,t} = \partial_x \left( p_{k+1} - p_0 p_k - q_k - \sum_{n=0}^{k-1} p_{k-n-1} q_n \right), \quad k = 0, 1, 2, ... \]

Of course, all conservation laws can be written via moments \( A^k, B^k \). For instance, first three of them are

\[ \partial_t B^0 = \partial_x \left( B^1 - \frac{(B^0)^2}{2} \right), \quad \partial_t \left( B^1 + \frac{(B^0)^2}{2} + A^0 \right) = \partial_x \left( B^2 - \frac{(B^0)^3}{3} + A^1 \right), \]

\[ \partial_t \left( B^2 + B^0 B^1 + \frac{(B^0)^3}{6} + 2A^1 + 2A^0 B^0 \right) \]

\[ = \partial_x \left( B^3 + \frac{(B^1)^2}{2} - \frac{(B^0)^2 B^1}{3} - \frac{(B^0)^4}{8} + 2A^2 + 2A^0 B^1 - (B^0)^2 A^0 + (A^0)^2 \right). \]

**Corollary**: A generating function of conservation laws and commuting flows is given by an auxiliary vertex equation

\[ \partial_{\tau(\zeta)} p(\lambda) = \partial_x \left( \frac{s(\zeta) p(\lambda)}{q(\lambda) - q(\zeta)} \right). \]  \hfill (28)

Indeed, a substitution (24) and (27) in the above equation implies to an identity. Moreover, this generating function can be written via the sole function \( S \)

\[ \partial_{\tau(\zeta)} \frac{\partial_x \ln s(\lambda)}{\partial_{\tau(\lambda)} \ln s(\lambda)} = \partial_x \left[ s(\lambda) \partial_x s'(\lambda) - s'(\lambda) \partial_x s(\lambda) \right] s(\zeta) \partial_{\tau(\zeta)} s(\lambda) \]

\[ = \left[ s(\zeta) \partial_x s(\lambda) - s(\lambda) \partial_x s(\zeta) \right] s(\lambda) \partial_{\tau(\lambda)} s(\lambda), \]

or via infinitely many field variables \( q_k \)

\[ \partial_{\tau(\zeta)} \frac{\partial_x q'(\lambda)}{\partial_{\tau(\lambda)} q(\lambda)} = \partial_x \frac{q'(\zeta) \partial_x q'(\lambda) \partial_{\tau(\zeta)} q(\lambda)}{q(\lambda) - q(\zeta) \partial_x q'(\zeta) \partial_{\tau(\lambda)} q(\lambda)}. \]
4 Derivation of the Manakov–Santini system

A substitution of expansions (15) and (16) in the second equations of (6) and (17) implies to the first commuting flow of (18)

\[
\begin{pmatrix}
A^k \\
B^k
\end{pmatrix}_y = \begin{pmatrix}
A^{k+2} \\
B^{k+2}
\end{pmatrix}_x - B^0 \begin{pmatrix}
A^{k+1} \\
B^{k+1}
\end{pmatrix}_x - \left( B^1 - \frac{(B^0)^2}{2} - A^0 \right) \begin{pmatrix}
A^k \\
B^k
\end{pmatrix}_x
\]

\[+ (k + 1) \begin{pmatrix}
A^k \\
B^k
\end{pmatrix}_t \begin{pmatrix}
A^0 \\
B^0
\end{pmatrix} \]

\[
\begin{pmatrix}
A^{k-1} \\
B^{k-1}
\end{pmatrix}_x = (A^1_x - B^0 A^0_x), \ k = 0, 1, 2, ..., \quad (29)
\]

where \( c = B^1 - \frac{(B^0)^2}{2} - A^0 \).

Let us take first two equations from (18) and the first equation from (29)

\[
\begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_t = \begin{pmatrix}
A^1 \\
B^1
\end{pmatrix}_x - B^0 \begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_x,
\]

\[
\begin{pmatrix}
A^1 \\
B^1
\end{pmatrix}_t = \begin{pmatrix}
A^2 \\
B^2
\end{pmatrix}_x - B^0 \begin{pmatrix}
A^1 \\
B^1
\end{pmatrix}_x + \begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_x A^0_x,
\]

\[
\begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_y = \begin{pmatrix}
A^2 \\
B^2
\end{pmatrix}_x - B^0 \begin{pmatrix}
A^1 \\
B^1
\end{pmatrix}_x - \left( B^1 - \frac{(B^0)^2}{2} - A^0 \right) \begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_x + \begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_x A^0_x.
\]

An elimination \( A^2 \) and \( B^2 \) leads to three dimensional four component quasilinear system of the first order

\[
\begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_t = \begin{pmatrix}
A^1 \\
B^1
\end{pmatrix}_x - B^0 \begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_x,
\]

\[
\begin{pmatrix}
A^1 \\
B^1
\end{pmatrix}_t = \begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_y + \left( B^1 - \frac{(B^0)^2}{2} - A^0 \right) \begin{pmatrix}
A^0 \\
B^0
\end{pmatrix}_x.
\]

A substitution \( A^0 = u, B^0 = a, A^1 = b \) and \( B^1 = c + a^2/2 + u \) implies to (7).

Thus, we have proved that three dimensional four component quasilinear system of the first order (7) is equivalent to commuting hydrodynamic bi-chains (18) and (29).
It means that any solution (obtained by the method of hydrodynamic reductions, see below) of (7) can be utilized for a construction of a corresponding solution of two linear systems (6) and (10); then such a solution can be used for construction of a solution of commuting hydrodynamic bi-chains (18) and (29). And vice versa, any solution of commuting hydrodynamic bi-chains (18) and (29) implies immediately to a solution of (7).

5 Semi-Hamiltonian hydrodynamic reductions

Following approach established in [11], all field variables $a, b, c, u$ in (7) are considered as functions of $N$ Riemann invariants $\lambda^k(x, t, y)$, where hydrodynamic reductions are commuting semi-Hamiltonian hydrodynamic type systems

$$
\lambda_i^i = v^i(\lambda)\lambda_x^i, \quad \lambda_y^i = w^i(\lambda)\lambda_x^i, \quad i = 1, 2, ..., N
$$

(30)

for any $N$. Characteristic velocities must satisfy the systems (see [35])

$$
\partial_k \frac{\partial_j v^i}{v^j - v^i} = \partial_j \frac{\partial_k v^i}{v^k - v^i}, \quad i \neq j \neq k;
$$

(31)

$$
\partial_k v^i \frac{v^k - v^i}{w^k - w^i} = \partial_k w^i \frac{w^k - w^i}{w^k - w^i}, \quad i \neq k,
$$

(32)

where $\partial_k \equiv \partial/\partial \lambda^k$. A substitution (30) in (7) implies to the so-called “dispersion relation”

$$
w^i = v^i + av^i - c
$$

(33)

and two differential relations

$$
\partial_i (c + u) = v^i \partial_i a, \quad \partial_i b = (v^i + a) \partial_i u.
$$

(34)

A substitution (33) in (32) together with the first equation from (34) implies to

$$
\partial_i q^k = \frac{\partial_i u}{q^i - q^k}, \quad i \neq k,
$$

(35)
where we introduce \( q^i = v^i + a \) instead \( v^i \). Compatibility conditions \( \partial_k(\partial_i(c + u)) = \partial_i(\partial_k(c + u)) \) and \( \partial_k(\partial_i b) = \partial_i(\partial_k b) \) for each pair of distinct indices lead to

\[
\partial_{ik}u = 2 \frac{\partial_i u \cdot \partial_k u}{(q^i - q^k)^2},
\]

(36)

\[
\partial_{ik}a = \frac{\partial_i u \cdot \partial_k a + \partial_k u \cdot \partial_i a}{(q^i - q^k)^2}.
\]

(37)

System (35)–(37) is an analogue of the Gibbons–Tsarev system for the dKP equation (see [9]). Equations (37) is a linear system whose variable coefficients are determined by solutions of sub-system (35), (36) which is precisely the aforementioned Gibbons–Tsarev system. Original Gibbons–Tsarev system (35), (36) has a general solution parameterized by \( N \) arbitrary functions of a single variable. Linear system (37) also has a general solution parameterized by \( N \) arbitrary functions of a single variable. Thus, whole system (35)–(37) has a general solution parameterized by \( 2N \) arbitrary functions of a single variable.

Thus, this is a first example (in a literature), where number of arbitrary functions twice bigger than in an ordinary theory (see [11]). Let us give an explanation of this deviation here. Three dimensional four component hydrodynamic type system (7) should be written in the form

\[
\begin{pmatrix}
  a \\
  b \\
  c \\
  u
\end{pmatrix}_t = \begin{pmatrix}
  0 & 0 & 1 & 1 \\
  0 & 0 & 0 & c \\
  c & -1 & -a & 0 \\
  0 & 1 & 0 & -a
\end{pmatrix}
\begin{pmatrix}
  a \\
  b \\
  c \\
  u
\end{pmatrix}_x + \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  a \\
  b \\
  c \\
  u
\end{pmatrix}_y.
\]

(38)

A dispersion relation (see [11]) for any three dimensional hydrodynamic type system

\[
\vec{u}_t = \hat{A}\vec{u}_x + \hat{B}\vec{u}_y
\]

is determined by

\[
\det[\hat{A} + \xi\hat{B} - \mu\hat{E}] = 0,
\]

16
where $\xi$ and $\mu$ are constants, and $\hat{E}$ is an identity matrix. In our case,

$$
\begin{vmatrix}
-\mu & 0 & 1 & 1 \\
0 & -\mu & 0 & c+\xi \\
c+\xi & -1 & -\mu-a & 0 \\
0 & 1 & 0 & -\mu-a
\end{vmatrix} = 0
$$

(39)

implies to the reducible quartic

$$(\mu^2 + a\mu - c - \xi)^2 = 0.$$ 

Thus, instead a full algebraic equation of the fourth order (with distinct roots), hydrodynamic type system (7) possesses the degenerated dispersion relation (conic) (cf. (33))

$$\xi = \mu^2 + a\mu - c.$$ 

An existence of hydrodynamic reductions (30) (integrability conditions (31) and (32) are not necessary!) reduces (38) to the form (cf. (39))

$$0 = \begin{vmatrix}
-v^i & 0 & 1 & 1 \\
0 & -v^i & 0 & c+w^i \\
c+w^i & -1 & -v^i-a & 0 \\
0 & 1 & 0 & -v^i-a
\end{vmatrix} \partial_i \begin{pmatrix} a \\ b \\ c \\ u \end{pmatrix}.$$ 

Thus, indeed (see (33)) $\xi$ has a meaning of the characteristic velocity $\vec{w}$, and $\mu$ has a meaning of the characteristic velocity $\vec{v}$. Such an existence of this double conic leads to a double freedom of solutions for generalized Gibbons–Tsarev system (35)–(37).

An existence of such a double conic leads to the following phenomenon. The number $N$ of Riemann invariants $\lambda^k$ can be split on two parts. First $M$ Riemann invariants are branch points of the algebraic curve $\lambda = \Lambda(\lambda; q)$ (see below); all other $N-M$ Riemann invariants are just mark points $r^n$ on this algebraic curve $\lambda = \Lambda(\lambda; q)$. $M$ can run all values from 1 up to $N$. Another origin of this phenomenon is connected with an observation that
whole hydrodynamic bi-chain (18) contains two parts, where each of them is determined by a different linear system (see (6) and (10)). Moreover, first $M$ Riemann invariants are connected with the $A$–part of (18), while other $N−M$ Riemann invariants are connected with the $B$–part of (18).

Since integrable three dimensional four component hydrodynamic type system (7) determined by linear system (6) is associated with commuting hydrodynamic bi-chains (18) and (29), to avoid a lot of unnecessary computations without loss of generality and for simplicity we shall consider just first hydrodynamic bi-chain (18) below.

6 Two species of Riemann invariants

Following the approach presented in [9], let us suppose that all moments $A^k$ are functions of $M$ Riemann invariants $\lambda^n$. In such a case, the first part of (18) reduces to an infinite series of recursive relations

$$\partial_i A^{k+1} = q^i \partial_i A^k - k A^{k-1} \partial_i A^0, \quad k = 0, 1, 2, ..., $$

whose compatibility conditions $\partial_i(\partial_j A^k) = \partial_j(\partial_i A^k)$ are nothing else but the original Gibbons–Tsarev system (35), (36). Moreover, these Riemann invariants $\lambda^i$ are nothing else but branch points $\lambda^i$ of the Riemann surface $\lambda = \Lambda(\lambda; q)$, i.e. $\lambda^i = \Lambda(\lambda; q^i)$, where $q^i(\lambda)$ are solutions of an algebraic system (see [9] and [22])

$$\partial_q \Lambda |_{q=q^i} = 0, \quad i = 1, 2, ..., N.$$ 

Indeed, linear system (6) reduces to the so-called Löwner equation (see [9])

$$\partial_i \Lambda = \frac{\partial_i u}{q - q^i} \partial_q \Lambda,$$ (40)

whose compatibility conditions $\partial_i(\partial_k \Lambda) = \partial_k(\partial_i \Lambda)$ imply to original Gibbons–Tsarev system (35), (36). Thus, hydrodynamic bi-chain (18) reduces to a composition of the inte-
grable hydrodynamic chain

\[ B^k_t = B^{k+1}_x - B^0 B^k_x + k B^{k-1} u_x, \quad k = 0, 1, 2, \ldots \] (41)

and the semi-Hamiltonian hydrodynamic type system (see the first equation in (30))

\[ \lambda^i_t = (q^i(\lambda) - B^0) \lambda^i_x, \quad i = 1, 2, \ldots, M, \] (42)

where functions \( u \) and \( q^i(\lambda) \) satisfy original Gibbons–Tsarev system (35), (36). Nevertheless, linear system (37) cannot be derived from (6), because \( B^0 \) is not a function of Riemann invariants \( \lambda^i \). However, if we consider hydrodynamic reductions of hydrodynamic chain (41), i.e. if we suppose that \( B^0 \) is a function of Riemann invariants \( \lambda^i \), then all higher moments \( B^k \) must be functions of Riemann invariants too. Then compatibility conditions \( \partial_i (\partial_j B^k) = \partial_j (\partial_i B^k) \) lead to (37), where

\[ \partial_i B^{k+1} = q^i \partial_i B^k - k B^{k-1} \partial_i u. \]

A most interesting consequence of aforementioned degeneracy described in the previous Section is an existence of extra Riemann invariants which are just mark points on already determined Riemann surface \( \lambda = \Lambda(\lambda; q) \). Indeed, let us introduce \( N - M \) mark points \( r^k = \Lambda(\lambda; \tilde{q}^k) \) such that each inverse function \( \tilde{q}^j(\lambda, r^j) \) is a solution of the Löwner equation (see (40), where we consider an inverse function \( q(\lambda; \lambda) \))

\[ \partial_i q = \frac{\partial_i u}{q^i - q}, \quad i = 1, 2, \ldots, M. \] (43)

A general solution of this overdetermined system (i.e. the compatibility conditions \( \partial_i (\partial_k \tilde{q}^j) = \partial_k (\partial_i \tilde{q}^j) \)) must be fulfilled due to original Gibbons–Tsarev system (35), (36))

\[ \partial_i \tilde{q}^j = \frac{\partial_i u}{q^i - q^j}, \quad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, N - M \] (44)

depends on \( M \) arbitrary functions of a single variable (because this is a system of the first order in partial derivatives with respect to \( M \) independent variables \( \lambda^i \)) and other \( N - M \)
arbitrary functions of a single variable (because each Riemann invariant \( r^k \) as a mark point on the Riemann surface \( \lambda = \Lambda(\lambda; q) \) is determined up to an arbitrary transformation \( r^k \to R_k(r^k) \). Let us remind that a solution \( \lambda \) of linear system (6) is determined up to an arbitrary transformation \( \lambda \to \tilde{\lambda}(\lambda) \). Then let us introduce the functions \( b_j(\lambda, r^j) \) determined by their first derivatives (the compatibility conditions \( \partial_i(\partial_k b_j) = \partial_k(\partial_i b_j) \) are fulfilled due to original Gibbons–Tsarev system (35), (36))

\[
\partial_i c_j(\lambda, r^j) = \frac{\partial_i u}{(q^i - \tilde{q}^i)^2},
\]

such that

\[
B^0(\lambda, r) = \sum_{j=1}^{N-M} \int \exp c_j(\lambda, r^j) dr^j
\]
satisfies overdetermined system (37). Whole overdetermined system (37), (44), (45) can be rewritten as a quasilinear system of the first order on unknown functions \( \tilde{q}^j, \partial_i B^0 \) and \( \tilde{\partial}_j B^0 = \exp c_j \) (here and below in this Section \( \tilde{\partial}_j = \partial/\partial r^j \)). Thus, its general solution is parameterized by \( N \) arbitrary functions of a single variable. It means, a general solution of complete overdetermined system (35)–(37), (44), (45) is parameterized by \( N + M \) arbitrary functions of a single variable.

Let us introduce \( N \) component hydrodynamic type system (see (42))

\[
\lambda^i_t = (q^i(\lambda) - B^0)\lambda^i_x, \quad r^j_t = (\tilde{q}^j(\lambda, r^j) - B^0) r^j_x.
\]

An integrability condition (i.e. semi-Hamiltonian property (31)) is fulfilled automatically due to (35)–(37), (44), (45).

**Theorem:** Hydrodynamic bi-chain (18) possesses infinitely many \( N \) component hydrodynamic reductions (47) determined by the extended Gibbons–Tsarev system (35)–(37), (44), (45). Then all higher moments \( B^k \) can be reconstructed iteratively in quadratures

\[
dB^{k+1} = \sum_{i=1}^{M} [q^i \partial_i B^k - kB^{k-1} \partial_i u] d\lambda^i + \sum_{j=1}^{N-M} \tilde{q}^j \tilde{\partial}_j B^k dr^j, \quad k = 0, 1, 2, ...
\]

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Proof: Since all moments $B^k$ depend simultaneously on both species of Riemann invariants $\lambda^i$ and $r^j$, then the $B$-part of (18) reduces to the second equation in (47), where (see (48))

$$\partial_i B^{k+1} = q^i \partial_i B^k - kB^{k-1} \partial_i u, \quad \tilde{\partial}_j B^{k+1} = \tilde{q}^j \tilde{\partial}_j B^k. \quad (49)$$

The compatibility conditions $\tilde{\partial}_j (q^i \tilde{\partial}_i B^k) = \tilde{\partial}_i (q^j \tilde{\partial}_j B^k)$, $\partial_i (q^i \partial_i B^k) = \tilde{\partial}_j (q^i \partial_i B^k - kB^{k-1} \partial_i u)$, $\partial_j (q^j \partial_j B^k - kB^{k-1} \partial_i u) = \partial_i (q^j \partial_j B^k - kB^{k-1} \partial_j u)$ lead to the system ($i \neq j$ in the first and third equations below)

$$\tilde{\partial}_i \tilde{\partial}_j B^k = 0, \quad (q^i - \tilde{q}^i) \partial_i \tilde{\partial}_j B^k = \partial_i \tilde{q}^i \cdot \tilde{\partial}_j B^k + k\partial_i u \cdot \tilde{\partial}_i B^{k-1}, \quad (q^j - \tilde{q}^j) \partial_j \tilde{\partial}_i B^k = \partial_j \tilde{q}^j \cdot \tilde{\partial}_i B^k + k(q^i - q^j) (\partial_i u \cdot \tilde{\partial}_j B^{k-1} - \partial_j u \cdot \partial_i B^{k-1}). \quad (50)$$

The second equation in (49) can be written in the common form

$$\tilde{\partial}_j B^k = (\tilde{q}^j)^k \tilde{\partial}_j B^0, \quad k = 0, 1, 2, \ldots \quad (51)$$

A substitution (51) in the first two equations of (50) leads to (44), (45) and (46), where each function $\tilde{q}^i$ depends on all Riemann invariants $\lambda^i$ and just one Riemann invariant $r^j$.

The third equation in (50) coincides with (37) for $k = 0$. The compatibility conditions $\partial_i (\partial_i B^k) = \partial_j (\partial_j B^k)$ can be verified by the induction principle (taking into account the first equation in (49)). Precisely, such a computation was made for $A$-part of hydrodynamic bi-chain (18), i.e. for Benney hydrodynamic chain (1) in [9].

Correspondingly, linear problem (10) determining the $B$-parts of hydrodynamic bi-chains (18) and (29) reduces to (cf. (40))

$$\partial_i F = \frac{\partial_i a}{q^i - q}, \quad \tilde{\partial}_j F = \frac{\tilde{\partial}_j a}{\tilde{q}^j - q}, \quad (52)$$

where $F(\lambda, r)$ is used instead of $f(x, t, y, q)$ for hydrodynamic reductions (30). The dependence $\tilde{q}^i (\lambda, r^j)$ and (46) follow from the compatibility condition $\tilde{\partial}_i (\tilde{\partial}_j F) = \tilde{\partial}_j (\tilde{\partial}_i F)$. The
compatibility condition $\partial_i(\tilde{\partial}_j F) = \tilde{\partial}_j(\partial_i F)$ leads to (44) and (45). The compatibility condition $\partial_i(\partial_j F) = \partial_j(\partial_i F)$ satisfies automatically due to original Gibbons–Tsarev system (35), (36). Similar computations can be repeated for generation function of conservation laws (26). Then the generating function of conservation law densities $p$ can be found in quadratures

$$d\ln p = \sum_{i=1}^{M} \left( \frac{\partial_i u}{(q - q^i)^2} + \frac{\partial_i a}{q - q^i} \right) d\lambda^i + \sum_{j=1}^{N-M} \tilde{\partial}_j a d\tilde{r}^j. \quad (53)$$

It is well known that generating function of conservation laws (26) is associated with the so-called “linearly degenerate” hydrodynamic chains and their linearly degenerate hydrodynamic reductions and (see [26] and Section 9 in [30]). First such an example was the Whitham equations of the Korteweg de Vries equation (see [16]). More general theory was suggested in [17]. Theory of linearly degenerate hydrodynamic type systems is presented in [10], [12], [25]. Following [16] and [17], let us introduce the so-called “quasi-momentum” differential $dP = pd\lambda$ and the so-called “quasi-energy” differential $dQ = (q - a)pd\lambda$. Then characteristic velocities are given by

$$\tilde{q}^i(\lambda, r^j) - B^0(\lambda, r) = \frac{dQ}{dP}|_{\lambda = r^j},$$

where the differentials of “quasi-energy” and “quasi-momentum” possess similar singularities on the Riemann surface $\lambda = \Lambda(\lambda; q)$ at vicinities of mark points $\lambda = r^i$.

Generating function of conservation laws and commuting flows (28) can be written in similar form. In such a case, the generating function of “quasi-energy” differentials is given by

$$d\tilde{Q}(\lambda, \zeta) = W(\lambda, \zeta)p(\lambda)d\lambda \equiv \frac{\exp F(\zeta)}{q(\lambda) - q(\zeta)} p(\lambda)d\lambda.$$

Corresponding generating function of commuting hydrodynamic type systems (see (47))

$$\lambda^i_{\tau(\zeta)} = W^i(\lambda; \zeta)\lambda^i_{x}, \quad r^j_{\tau(\zeta)} = \tilde{W}^j(\lambda; \zeta)\tilde{r}^j$$

is given by (replacing $q(\lambda)$ in the expression for $W(\lambda, \zeta)$ by $q^i$ and $\tilde{q}^j$, respectively)

$$\lambda^i_{\tau(\zeta)} = \frac{\exp F(\zeta)}{q^i - q(\zeta)} \lambda^i_{x}, \quad r^j_{\tau(\zeta)} = \frac{\exp F(\zeta)}{\tilde{q}^j - q(\zeta)} \tilde{r}^j. \quad (54)$$
This hydrodynamic type system possesses generating function (28). In such a case (cf. Section 9 in [30]),

\[ d\ln p = \sum_{i=1}^{M} \frac{\partial_i W}{W_i - W} d\lambda^i + \sum_{j=1}^{N-M} \frac{\tilde{\partial}_j W}{W_j - W} dr^j \]

must coincide with (53). Indeed, a direct substitution of expressions \( W(\lambda, \zeta), W^i(\lambda; \zeta) \) and \( \tilde{W}^j(\lambda, r; \zeta) \) implies to (53). Moreover, hydrodynamic type systems (47) and (54) commute to each other (see Tsarev’s condition (32)). It means, that any hydrodynamic reduction (47) satisfying extended Gibbons–Tsarev system (35)–(37), (44), (45) is compatible with the whole Manakov–Santini hierarchy described in Section 3.

**Remark:** Semi-Hamiltonian property (31) and commutativity condition (32) for hydrodynamic type systems which are hydrodynamic reductions of linearly degenerate hydrodynamic chains can be simplified (see Section 9 in [30]) to the form, respectively

\[ \partial_i \frac{\partial_k V}{V^k - V} = \partial_k \frac{\partial_i V}{V_i - V}, \quad i \neq k; \quad \partial_i \frac{\partial_k V}{V^k - V} = \partial_k \frac{\partial_i V}{W^k - W}, \]

where \( dP = p(\lambda) d\lambda \) is a differential of “quasi-momentum”, \( dQ = V(\lambda) p(\lambda) d\lambda \) is a differential of “quasi-energy” and \( d\tilde{Q} = W(\lambda, \zeta) p(\lambda) d\lambda \) is a generating function of “quasi-energy” differentials. In the case of two species of Riemann invariants (see, for instance, (47) and (54)), the above formulas must be written in the form

\[ \partial_i \frac{\partial_k V}{V^k - V} = \partial_k \frac{\partial_i V}{V^k - V}, \quad \partial_i \frac{\partial_k V}{V^k - V} = \partial_k \frac{\partial_i V}{V^k - V}, \quad i \neq k; \quad \partial_i \frac{\partial_k V}{V^k - V} = \partial_k \frac{\partial_i V}{V^k - V}, \]

\[ \partial_k V = \frac{\partial_k W}{W^k - W}, \quad \partial_k V = \frac{\partial_k W}{W^k - W}, \quad \frac{\partial_k V}{V^k - V} = \frac{\partial_k V}{V^k - V}, \quad \frac{\partial_k W}{V^k - V} = \frac{\partial_k W}{W^k - W}. \]

7 Symmetric hydrodynamic reductions

Benney hydrodynamic chain (1) possesses the special hydrodynamic reduction (see [32])

\[ a_i^j = \left( \frac{(a^i_j)^2}{2} + A^0 \right)_x, \]
where all moments are determined by the choice ($\epsilon_i$ are constants)

$$A^k = \frac{1}{k+1} \sum_{i=1}^{M} \epsilon_i (a^i)^{k+1}, \quad k = 0, 1, 2, ..., \quad \sum_{i=1}^{M} \epsilon_i = 0.$$ 

Let us extend this decomposition on both species of moments ($\epsilon_i$ and $\gamma_j$ are constants)

$$A^k = \frac{1}{k+1} \sum_{i=1}^{M} \epsilon_i (a^i)^{k+1}, \quad B^k = \frac{1}{k+1} \sum_{j=1}^{N-M} \gamma_j (b^j)^{k+1}, \quad k = 0, 1, 2, ... \quad (55)$$

Then bi-chain (18) reduces to the hydrodynamic type system

$$a^i_t = (a^i - B^0)a^i_x + A^0_x, \quad b^j_t = (b^j - B^0)b^j_x + A^0_x, \quad (56)$$

where

$$\sum_{i=1}^{M} \epsilon_i = \sum_{j=1}^{N-M} \gamma_j = 0.$$

In a contrary with semi-Hamiltonian hydrodynamic reductions (see (41) and (42)) of the $A$-part of bi-chain (18), the moment decomposition (see (55))

$$B^k = \frac{1}{k+1} \sum_{j=1}^{N-M} \gamma_j (b^j)^{k+1}, \quad k = 0, 1, 2, ...$$

for the $B$-part of (18) leads to the hydrodynamic chain

$$A^k_t = A^{k+1}_x - B^0 A^k_x + k A^{k-1} A^0_x, \quad k = 0, 1, 2, ...$$

equipped by the non-diagonalizable (i.e. non semi-Hamiltonian!) hydrodynamic type system (cf. (56))

$$b^j_t = (b^j - B^0)b^j_x + A^0_x, \quad j = 1, 2, ..., N - M. \quad (57)$$

Indeed, Riemann invariants $r^k(b)$ cannot exist, because the last term $A^0_x$ cannot be eliminated by any point transformations. Moreover, in a general case, the $A$-part of (18) reduces to the $M$ component hydrodynamic type system

$$a^i_t = (a^i - B^0)a^i_x + A^0_x, \quad i = 1, 2, ..., M, \quad (58)$$
where all moments $A^k$ depend on field variables $a^n$ only, and the function $u = A^0$ satisfies the original Gibbons–Tsarev system (see [30])

$$(a^i - a^k)u_{ik} + u_i\delta u_k - u_k\delta u_i = 0, \quad i \neq k,$$

(59)

where $\delta = \Sigma \partial/\partial a^m$ is a shift operator, $u_i = \partial u/\partial a^i$, $u_{ik} = \partial^2 u/\partial a^i \partial a^k$, all higher moments $A^k$ can be reconstructed iteratively (here and below $\partial_i A^k = \partial A^k/\partial a^i$) in quadratures

$$dA^{k+1} = \sum_{i=1}^{M} [a^i \partial_i A^k + (\delta A^k - kA^{k-1})\partial_i A^0]da^i, \quad k = 0, 1, 2, ...$$

Linear system (6) reduces to the Löwner equation (cf. (40); more detail in [30])

$$\partial_i \Lambda + \frac{\partial_i u}{a^i - q} \left(1 + \sum_{m=1}^{M} \frac{\partial_m u}{a^m - q} \right)^{-1} \partial_q \Lambda = 0,$$

(60)

written via field variables $a^k$, where the equation of Riemann surface $\lambda = \Lambda(a; q)$. The compatibility conditions $\partial_i(\partial_j \Lambda) = \partial_j(\partial_i \Lambda)$ as well as the compatibility conditions $\partial_i[a^i \partial_j A^k + (\delta A^k - kA^{k-1})\partial_j A^0] = \partial_j[a^i \partial_i A^k + (\delta A^k - kA^{k-1})\partial_i A^0]$ lead to (59).

In a contrary, the $B$-part of (18) reduces to (57) if all moments $B^k$ depend on field variables $b^n$ only. It means, that in a general case, moments $B^k$ must depend on two species of field variables $a^n$ and $b^m$ simultaneously (as well as on two species of Riemann invariants $\lambda^i$ and $r^j$). Just in such a case, hydrodynamic type system (57), (58) can be semi-Hamiltonian. Indeed, the $B$-part of (18)

$$B_t^k = B_x^{k+1} - B_x^0 B_x^k + kB_x^{k-1} A_x^0, \quad k = 0, 1, 2, ...$$

reduces to (57) and (58), if (here and below $\partial_i = \partial/\partial b^i$, $\delta = \Sigma \partial/\partial b^m$)

$$\partial_i B^{k+1} = a^i \partial_i B^k + (\delta B^k + \delta B^k - kB^{k-1})\partial_i A^0, \quad \partial_j B^{k+1} = b^j \partial_j B^k.$$

(61)

The last equation can be written in the common form (cf. (51))

$$\partial_j B^k = (b^j)^k \partial_j B^0, \quad k = 0, 1, 2, ...$$

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The compatibility conditions \( \tilde{\partial}_j [a^i \partial_i B^k + (\delta B^k + \tilde{\delta} B^k - k B^{k-1}) \partial_i A^0] = \partial_i[\tilde{b}^i \tilde{\partial}_j B^k], \)
\( \partial_j [a^i \partial_i B^k + (\delta B^k + \tilde{\delta} B^k - k B^{k-1}) \partial_i A^0] = \partial_i[a^i \partial_j B^k + (\delta B^k + \tilde{\delta} B^k - k B^{k-1}) \partial_j A^0] \) and
\( \tilde{\partial}_j(\tilde{b}^i \tilde{\partial}_j B^k) = \tilde{\partial}_j(\tilde{b}^i \tilde{\partial}_i B^k) \) lead to the system (cf. (50); \( i \neq j \) in the first and third equations below)
\[
\tilde{\partial}_i \tilde{\partial}_j B^0 = 0, \quad (a^i - \tilde{b}^i) \partial_i \tilde{\partial}_j B^0 + \partial_i A^0 \cdot \tilde{\partial}_j(\delta B^0 + \tilde{\delta} B^0) = 0, \tag{62}
\]
\[
(a^i - a^j) \partial_i \partial_j B^0 + \partial_i A^0 \cdot \partial_j(\delta B^0 + \tilde{\delta} B^0) - \partial_j A^0 \cdot \partial_i(\delta B^0 + \tilde{\delta} B^0) = 0. \tag{65}
\]

The second equation in (62) reduces to
\[
\partial_i \tilde{\lambda}_j + \frac{\partial_i u}{a^i - b^i} \left( 1 + \sum_{m=1}^{M} \frac{\partial_m u}{a^m - b^j} \right)^{-1} \tilde{\partial}_j \tilde{\lambda}_j = 0, \tag{63}
\]
which can be also obtained from (60) formally replacing \( \Lambda \) on \( N - M \) functions \( \tilde{\lambda}_j \) and correspondingly \( q \) on \( N - M \) field variables \( b^j \). In such a case, a solution of the first equation in (62) is given by (cf. (46))
\[
B^0 = \sum_{j=1}^{N-M} \int \tilde{\lambda}_j(a; b^j) db^j, \tag{64}
\]
A dependence of \( B^0 \) with respect to field variables \( a^k \) is given by the last equation in (62)
\[
(a^i - a^j) \partial_i \partial_j B^0 + \partial_i A^0 \cdot \partial_j(\delta B^0 + \tilde{\delta} B^0) - \partial_j A^0 \cdot \partial_i(\delta B^0 + \tilde{\delta} B^0) = 0, \quad i \neq j, \tag{65}
\]
while all other higher expressions \( B^k(a, b) \) can be found iteratively (see (61))
\[
dB^{k+1} = \sum_{i=1}^{M} [a^i \partial_i B^k + (\delta B^k + \tilde{\delta} B^k - k B^{k-1}) \partial_i A^0] da^i + \sum_{j=1}^{N-M} \tilde{b}^j \tilde{\partial}_j B^k db^j, \quad k = 0, 1, 2, ...
\]

The Löwner equation (60) under the inverse transformation \( \Lambda(a; q) \rightarrow q(a; \lambda) \) (cf. (40)) and (43)) reduces to the form
\[
\partial_i q = \frac{\partial_i u}{a^i - q} \left( 1 + \sum_{m=1}^{M} \frac{\partial_m u}{a^m - q} \right)^{-1} \tag{66}
\]
and the generating function of conservation law densities \( p \) (see (26)) can be found in quadratures (cf. (53))
\[
d \ln p = \sum_{m=1}^{M} \partial_m (q - a) \cdot d \ln (q - a^m) - \sum_{n=1}^{N-M} \tilde{\partial}_n a \cdot d \ln (q - b^n),
\]
26
while (cf. (52)) the function $F(a, b)$ is given by another quadrature (see (22) and (27))
\[ dF = \sum_{m=1}^{M} \partial_m a \cdot d \ln(q - a^m) + \sum_{n=1}^{N-M} \partial_n b \cdot d \ln(q - b^n). \]

**Remark:** System (65) automatically satisfies (for any solution $A^0(a)$ of system (59)) if
\[ B^0 = \sum_{m=1}^{M} \gamma_m a^m + \sum_{n=1}^{N-M} \beta_n b^n, \tag{67} \]
where $\gamma_m$ and $\beta_n$ are arbitrary constants. In such a case
\[ F = \sum_{m=1}^{M} \gamma_m \ln(q - a^m) + \sum_{n=1}^{N-M} \beta_n \ln(q - b^n), \quad p = \frac{1}{\lambda^q \prod_{m=1}^{M} (q - a^m)^{-\gamma_m} \prod_{n=1}^{N-M} (q - b^n)^{-\beta_n}}. \]

All moments $A^k$ can be expressed via field variables $a^i$ as well as Riemann invariants $\lambda^j$. Thus, each field variable $a^i$ is a function of all Riemann invariants $\lambda^n$, and vice versa each Riemann invariant $\lambda^i$ is a function of all Riemann invariants $a^n$. Since all moments $B^k$ are functions of both species of field variables $a^i$ and $b^j$ as well as both species of Riemann invariants $\lambda^i$ and $r^j$, finally, we need to find a dependence of field variables $b^k$ via Riemann invariants $\lambda^i$ and $r^j$. Taking into account (47), (57) and (58) imply to
\[ (\tilde{q}^j - b^n) \cdot \tilde{\partial}_j b^n = 0, \quad j, n = 1, 2, ..., N - M \tag{68} \]
and $(\partial_i = \partial/\partial \lambda^i, \tilde{\partial}_j = \partial/\partial r^j$ in (68) and below in this Section)
\[ \partial_i a^k = \frac{\partial_i u}{q^i - a^k}, \quad \partial_i b^n = \frac{\partial_i u}{q^i - b^n}, \quad i, k = 1, 2, ..., M, \]
which can be obtained directly from (43) replacing $q$ by $a^k$ and $b^n$, respectively. Relations (68) imply to the simple dependence $b^j = \tilde{q}^j(\lambda, r^j)$. Since each Riemann invariant $r^i$ is a mark point on the Riemann surface $\lambda = \Lambda(\lambda; q)$, i.e. $r^j = \Lambda(\lambda; \tilde{q}^j)$, we conclude that equations (58) are connected with (42), while equations (57) reduce to the diagonal form
\[ r^j_t = (b^j - a^j)r^j_x, \]
where Riemann invariants $r^j = \Lambda(\lambda; b^j)$, $j = 1, 2, ..., N - M$. Thus (see (63)), $\tilde{\lambda}_j(a; b^j) = R_j(r^j)$, where $R_j(r^j)$ are arbitrary functions, because Riemann invariants are determined up to an arbitrary transformation $r^j \rightarrow R_j(r^j)$. 

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In the next Section, explicit hydrodynamic reductions are considered. A relationship between field variables $a^k, b^n$ and Riemann invariants $\lambda^i, r^j$ is investigated in detail.

8 Explicit hydrodynamic reductions

A general case contains an arbitrary number of field variables of two species $a^k$ and $b^n$ in $N$ component hydrodynamic type system (56). The most general reduction of $\lambda(p)$ known recently (see [4], [17], [14]) is a combination of rational and logarithmic functions with respect to $p$, i.e.

$$\lambda = \frac{p^{N+1}}{N+1} + a^0_b p^{N-1} + a^0_1 p^{N-2} + \ldots + a^0_{N-1}$$

$$+ \sum_{k=1}^{K} \left[ \sum_{n=1}^{N_k} \frac{a^{(1)}_k}{(p - a_k^{(2)})^n} + \epsilon_k \ln \left( p - a_k^{(2)} \right) \right] + \sum_{m=1}^{M} \delta_k \ln \left( p - a_k^{(3)} \right),$$

where $\epsilon_k$ and $\delta_n$ are constants, $a_k^{(n)}$ are functions. The approach allowing to extract more complicated reductions is presented in [33]. In this Section a simplest case is considered.

$N$ parametric solution of original Gibbons–Tsarev system (59) given by (see [32]; here all constants $\epsilon_i$ are independent, no such a constraint $\sum \epsilon_m = 0$ in general)

$$u = \sum_{m=1}^{M} \epsilon_m a^m$$

(69)

possesses to reconstruct an equation of the Riemann surface $\lambda = \Lambda(a; q)$. A substitution (69) in (60) leads to the so-called waterbag reduction (see [5], [9] and [32])

$$\lambda = q - \sum_{m=1}^{M} \epsilon_m \ln(q - a^m).$$

(70)

The Riemann invariants $\lambda^i(a)$ are branch points of the Riemann surface determined by (70), i.e. the condition ($\lambda_q = 0$)

$$1 = \sum_{m=1}^{M} \frac{\epsilon_m}{q^i - a^m}$$

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leads to $M$ expressions $q^i(a)$ as well as to $M$ inverse expressions $a^i(q)$. Corresponding Riemann invariants $\lambda^i(a)$ are given by (70)

$$
\lambda^i = q^i - \sum_{m=1}^{M} \epsilon_m \ln(q^i - a^m). \quad (71)
$$

Other $N-M$ Riemann invariants $r^j(a, b)$ (see (63)) are mark points on the aforementioned Riemann surface

$$
r^j \equiv \lambda|_{q=b} = b^j - \sum_{m=1}^{M} \epsilon_m \ln(b^j - a^m).
$$

Since $\tilde{\lambda}_j$ are arbitrary functions of corresponding Riemann invariants (i.e. $\tilde{\lambda}_j = Q_j(r^j)$), the function $a$ (see (65)) can be found from the system

$$
\partial_i \partial_j a + \frac{\epsilon_i}{a^i - a^j} \partial_j a - \frac{\epsilon_j}{a^i - a^j} \partial_i a = \epsilon_i \epsilon_j \sum_{m=1}^{N-M} \frac{Q'_m}{(a^i - b^m)(a^j - b^m)}, \quad i \neq j, \quad (72)
$$

where (see (64))

$$
\tilde{\partial}_j a = \tilde{\lambda}_j = Q_j \left( b^j - \sum_{m=1}^{M} \epsilon_m \ln(b^j - a^m) \right). \quad (73)
$$

To avoid a complexity of further computations (an integration of arbitrary functions $Q_j(r^j)$ in (73)), let us restrict our consideration on a first nontrivial case $Q_j(r^j) = \beta_j r^j$, where $\beta_j$ are arbitrary constants. Then (73) leads to

$$
a = B(a) + \frac{1}{2} \sum_{n=1}^{N-M} \beta_n (b^n)^2 + \sum_{m=1}^{M} \sum_{n=1}^{N-M} \epsilon_m \beta_n (a^m - b^n)[\ln(a^m - b^n) - 1], \quad (74)
$$

where the function $B(a)$ satisfies (see (72))

$$
\partial_i \partial_j B + \frac{\epsilon_i}{a^i - a^j} \partial_j B - \frac{\epsilon_j}{a^i - a^j} \partial_i B = 0, \quad i \neq j.
$$

This linear system possesses a general solution parameterized by $N-M$ arbitrary functions of a single variable. Its first nontrivial solution is given by

$$
B = \frac{\delta}{2} \sum_{m=1}^{M} \epsilon_m (a^m)^2 + \sum_{m=1}^{M} \gamma_m a^m, \quad (75)
$$

where $\delta$ and $\gamma_k$ are new arbitrary constants.
Thus, the first nontrivial solution of extended Gibbons–Tsarev system (35)–(37), (44), (45) given by (69), (74) and (75) leads to the first nontrivial \( N \) component hydrodynamic reduction (57), (58) of the Manakov–Santini system. In such a case, the generating function of conservation law densities can be found explicitly

\[
\ln p = -\frac{\delta}{2}q^2 - \delta \sum_{m=1}^{M} \epsilon_m (a^m - q) - \sum_{m=1}^{M} \gamma_m \ln(a^m - q) - q \sum_{n=1}^{N-M} \beta_n \ln(b^n - q) - \sum_{n=1}^{N-M} \beta_n (b^n - q) \\
+ \frac{1}{4} \sum_{m=1}^{M} \sum_{n=1}^{N-M} \beta_n \epsilon_m \ln^2(b^n - q) + 2 \ln(a^m - q) \cdot \ln(b^n - q) - \ln^2(a^m - q)]
\]

(76)

\(-\ln \left(1 + \sum_{m=1}^{M} \frac{\epsilon_m}{a^m - q}\right) + \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{N-M} \beta_n \epsilon_m \left(\text{Li}_2 \frac{a^m - q}{b^n - q} - \text{Li}_2 \frac{b^n - q}{a^m - q}\right),
\]

where the bi-logarithm

\[\text{Li}_2 z = \sum_{k=1}^{\infty} \frac{z^k}{k^2}.\]

A corresponding expression for the function \( F \) is given by

\[
F = \frac{\delta}{2}q^2 + \delta \sum_{m=1}^{M} \epsilon_m (a^m - q) + \sum_{m=1}^{M} \gamma_m \ln(a^m - q) + q \sum_{n=1}^{N-M} \beta_n \ln(b^n - q) + \sum_{n=1}^{N-M} \beta_n (b^n - q) \\
+ \frac{1}{4} \sum_{m=1}^{M} \sum_{n=1}^{N-M} \beta_n \epsilon_m \ln^2(a^m - q) - 2 \ln(a^m - q) \cdot \ln(b^n - q) - \ln^2(b^n - q)]
\]

(77)

\[+ \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{N-M} \beta_n \epsilon_m \left(\text{Li}_2 \frac{b^n - q}{a^m - q} - \text{Li}_2 \frac{a^m - q}{b^n - q}\right).
\]

9 **Generalized hodograph method**

In comparison with integrable hydrodynamic chains their semi-Hamiltonian hydrodynamic reductions possess \( N \) infinite series of conservation laws and commuting flows (cf. (24) and (28)). Let us remind, that hydrodynamic type system (57), (58) is associated with the Riemann surface \( \lambda = \Lambda(a; q) \), whose \( M \) branch points \( \lambda^i = \Lambda(a; q^i) \) determined by \( M \) solutions \( q^i(a) \) of the algebraic equation \( \partial \Lambda(a; q)/\partial q = 0 \) are first \( M \) Riemann invariants; all other \( N - M \) Riemann invariants \( r^j = \Lambda(\lambda; b^j) \) are nothing else but just mark points on this Riemann surface.
Since a dependence $p(q, a, b)$ can be inverted to $q(p, a, b)$ for any hydrodynamic reduction (57), (58), generating function of conservation laws (26)

$$\partial_t p = \partial_x [(q(p) - a)p]$$

leads to $N$ infinite series of conservation laws (see [30]) by virtue of formal expansions

$$p^{(i)} = p_0^i + p_{(1)}^i \lambda^{(i)} + p_{(2)}^i (\lambda^{(i)})^2 + p_{(3)}^i (\lambda^{(i)})^3 + ..., \quad i = 1, 2, ..., M,$$

(78)

$$\tilde{p}^{(j)} = \tilde{p}_0^j + \tilde{p}_{(1)}^j \tilde{\lambda}^{(j)} + \tilde{p}_{(2)}^j (\tilde{\lambda}^{(j)})^2 + \tilde{p}_{(3)}^j (\tilde{\lambda}^{(j)})^3 + ..., \quad j = 1, 2, ..., N - M,$$

where $\lambda^{(i)}$ and $\tilde{\lambda}^{(j)}$ are local parameters at the vicinities of $p_0^i$ and $\tilde{p}_0^j$, respectively. However, in such a case, dependencies $q(p_0^i)$ as well as $q(\tilde{p}_0^j)$ are highly complicated, because the dependence $p(q)$ is much more simpler than $q(p)$ in all known examples (see the previous Section). Thus, instead (78), we utilize another formal expansions given by

$$q^{(i)} = a^i + a_{(1)}^i \lambda^{(i)} + a_{(2)}^i (\lambda^{(i)})^2 + a_{(3)}^i (\lambda^{(i)})^3 + ..., \quad i = 1, 2, ..., M,$$

(79)

$$\tilde{q}^{(j)} = b^j + b_{(1)}^j \tilde{\lambda}^{(j)} + b_{(2)}^j (\tilde{\lambda}^{(j)})^2 + b_{(3)}^j (\tilde{\lambda}^{(j)})^3 + ..., \quad j = 1, 2, ..., N - M,$$

where $\lambda^{(i)}$ and $\tilde{\lambda}^{(j)}$ are local parameters at the vicinities of $a^i$ and $b^j$, respectively. Since, the dependence $p(q)$ possesses $M$ singularities in points $a^i$ and $N - M$ singularities in points $b^j$, formal expansions (78) cannot exist in a general case. Nevertheless, in some special cases, first $N$ conservation laws can be found in the form

$$\partial_t p_0^i = \partial_x [(a^i - a)p_0^i], \quad \partial_t \tilde{p}_0^j = \partial_x [(b^j - a)\tilde{p}_0^j].$$

(80)

Below we restrict our consideration on a simplest sub-case (67), which is compatible with any reduction of $\lambda(p)$ (see the previous Section). In this case, $N$ infinite series of conservation laws can be found utilizing (79). It means that first $N$ conservation laws are determined by (80).
Without loss of generality, let us choose the equation for Riemann surface (70). In such a case (see (67) and (69)), three dimensional four component hydrodynamic type system (7) reduces to the pair of commuting $N$ component hydrodynamic type systems (see (9) and (57), (58))

\[ a_i^i = (a^i - a)a_x^i + u_x, \quad a^i_y = ((a^i)^2 - aa^i - c)a_x^i + a^i u_x + u_t, \quad i = 1, 2, ..., M, \]

\[ b_i^j = (b^j - a)b_x^j + u_x, \quad b^j_x = ((b^j)^2 - ab^j - c)b_x^j + b^j u_x + u_t, \quad j = 1, 2, ..., N - M, \]

where \( \epsilon = \Sigma \gamma_m, \beta = \Sigma \gamma_n, \gamma = \Sigma \gamma_m \)

\[ b = \frac{1}{2} \sum_{m=1}^{M} \epsilon_m (a^m)^2 + \epsilon u, \quad c = \frac{1}{2} \sum_{m=1}^{M} \gamma_m (a^m)^2 + \frac{1}{2} \sum_{n=1}^{N-M} \beta_n (b^n)^2 - \frac{1}{2} a^2 + (\gamma + \beta - 1)u. \]

Then all coefficients $p_i^j(k)$ and $\tilde{p}_j^i(n)$ can be found iteratively, while coefficients $a_i^j(k)(a, b)$ and $b_j^i(n)(a, b)$ are determined by formal expansions of equation $\lambda = \Lambda(a; q)$ of a Riemann surface. A substitution of these expansions $q^i(i)$ and $\tilde{q}_j^i(j)$ together with other four expansions

\[ \partial_{\tau^{(i)}} = \partial_{\psi^i} + \lambda^{(i)} \partial_{\psi^{i+1}} + (\lambda^{(i)})^2 \partial_{\psi^{i+2}} + (\lambda^{(i)})^3 \partial_{\psi^{i+3}} + ..., \]

\[ s^{(i)} = s_{(0)}^{i} + s_{(1)}^{i} \lambda^{(i)} + s_{(2)}^{i} (\lambda^{(i)})^2 + s_{(3)}^{i} (\lambda^{(i)})^3 + ..., \]

\[ \partial_{\tilde{\tau}^{(i)}} = \partial_{\rho^j} + \tilde{\lambda}^{(j)} \partial_{\rho^{j+1}} + (\tilde{\lambda}^{(j)})^2 \partial_{\rho^{j+2}} + (\tilde{\lambda}^{(j)})^3 \partial_{\rho^{j+3}} + ..., \]

\[ \tilde{s}^{(j)} = \tilde{s}_{(0)}^{j} + \tilde{s}_{(1)}^{j} \tilde{\lambda}^{(j)} + \tilde{s}_{(2)}^{j} (\tilde{\lambda}^{(j)})^2 + \tilde{s}_{(3)}^{j} (\tilde{\lambda}^{(j)})^3 + ... \]

to (24) leads to $N$ infinite series of commuting flows. Then the generalized hodograph method admits to construct infinitely many particular solutions in an implicit form (see [35]).
10 Conclusion

Manakov–Santini system (7) is the first example in the theory of three dimensional hydrodynamic type systems, which is naturally equipped by the two component pseudo-differentials \( q, f \), determined by systems in partial derivatives of the first order

\[
\begin{bmatrix}
    q \\
    f
\end{bmatrix}_t = (q - a) \begin{bmatrix}
    q \\
    f
\end{bmatrix}_x + \begin{bmatrix}
    u \\
    a
\end{bmatrix}_x,
\]

\[
\begin{bmatrix}
    q \\
    f
\end{bmatrix}_y = (q^2 - aq - c) \begin{bmatrix}
    q \\
    f
\end{bmatrix}_x + q \begin{bmatrix}
    u \\
    a
\end{bmatrix}_x + \begin{bmatrix}
    u \\
    a
\end{bmatrix}_t.
\]

We believe that more complicated vector pseudo-differentials

\[
\tilde{f}_t = \hat{G}(f, u)\tilde{f}_x + \hat{F}(f, u)\tilde{u}_x
\]

are associated with vector hydrodynamic chains (4).

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