On the static metric extension conjecture, and mass-minimising extensions in fixed conformal classes

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Abstract

The static metric extension conjecture roughly states that the infimum of the ADM energy, over an appropriate space of extensions to a given compact 3-manifolds with boundary, is realised by a static metric [2]. It was shown by Corvino that if the infimum is indeed achieved, then it is achieved by a static metric [7]; however, the more difficult question of whether or not the infumum is achieved, is still an open problem. Later, Bartnik proved that critical points of the ADM mass, over the space of solutions to the Einstein constraints on an asymptotically flat manifold without boundary, correspond to stationary solutions [3]. He further stated that similar ideas should provide a more natural proof of Corvino’s result.

In the first part of this note, we discuss elementary modifications to Bartnik’s argument to adapt it to include a boundary. The conclusion is that critical points of the mass, over the space of extensions satisfying the constraints, correspond to stationary solutions. We then give a sketch of how the proof would be modified to consider the simpler case of scalar-flat extensions and obtain a version of Corvino’s result.

We also consider the space of extensions in a fixed conformal class. Sufficient conditions are given to ensure that the infimum is realised within this class.
1 Introduction

The Barnik mass is said to be the gold standard definition of a quasilocal mass, if only it were possible to actually compute in general. The mass of a domain $\Omega$ in some initial data 3-manifold is as taken to be the infimum of the ADM mass over a space of admissible extensions to $\Omega$, satisfying the Einstein constraints. In [2], where Bartnik first defined the quasilocal mass now bearing his name, the related static metric extension conjecture was posed; it is conjectured that this infimum is achieved by a static extension to $\Omega$.

In 2000, Corvino proved part of this conjecture (Theorem 8 of [7]); he proved that if a minimal ADM energy extension exists then it must be static. Note that we differentiate between the energy and the mass – the latter being the absolute value of the energy-momentum four-vector, while the former refers to the component that is orthogonal to the Cauchy surface. Later, Bartnik suggested that a variational proof of Corvino’s result, based on extending his work on the phase space [3] to manifolds with boundary would be more natural. In [11], the author considered such a variational argument for the Einstein-Yang-Mills constraints on a manifold with boundary, however the boundary data was free to vary and therefore a different conclusion was drawn. In the first part of this note, we give a sketch of how Bartnik’s analysis is modified to the case where the data is fixed on the boundary and provides an alternate proof of Corvino’s result. The extensions considered here fix the first derivative of the metric on the boundary, which is not a requirement of the Bartnik mass; it would be interesting to conduct the full analysis with boundary conditions more suited to the definition of the Bartnik mass.

In Section 2 we introduce the Hilbert manifold of extensions to be considered, which is essentially Bartnik’s phase space with boundary conditions imposed. In Section 3, we introduce energy, momentum and mass definitions, and demonstrate how Corvino’s result on static extensions can be obtained. Finally, in Section 4 we consider a space of extensions in a prescribed conformal class. We give sufficient conditions to ensure that the infimum is realised within the fixed conformal class.

2 The phase space

Let $\mathcal{M}$ be a smooth, connected, oriented, paracompact, asymptotically flat 3-manifold with smooth boundary, $\Sigma$. We also assume that $\mathcal{M}$ has only a single asymptotic end; that is, there exists a compact set $K \supset \Sigma$ such that $\mathcal{M} \setminus K$ is diffeomorphic to $\mathbb{R}^3$ minus the closed unit ball, $\phi : \mathcal{M} \setminus K \to \mathbb{R}^3 \setminus \overline{B_0(1)}$. On $\mathcal{M} \setminus K$ we define $\hat{g}$ to be the pullback of $\hat{g}$ to $\mathcal{M} \setminus K$.

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1Hubert Bray, quoted in a press release from Duke University, refers to the Bartnik mass as the “gold standard for what the correct answer really is” [8].
of the Euclidean metric via $\phi$, and let $r$ be the Euclidean radial coordinate function composed with $\phi$. On $K$, we fix $\hat{g}$ to be smooth, bounded and positive definite, while $r$ is smooth and bounded between $\frac{1}{2}$ and $2$. Unless otherwise stated, $\circ$ will indicate quantities defined with respect to the background metric $\hat{g}$. In order to include the asymptotics and prescribe the data on the boundary, we define the trace-zero weighted Sobolev spaces, which are equipped with the following norms:

$$
\|u\|_{p,\delta} = \left\{ \begin{array}{ll}
\left( \int_M |u|^p r^{-\delta p-3}d\mu_o \right)^{1/p}, & p < \infty, \\
\text{ess sup}_M (r^{-\delta}|u|), & p = \infty,
\end{array} \right.
$$

(2.1)

$$
\|u\|_{k,p,\delta} = \sum_{j=0}^k \|\hat{\nabla}^j u\|_{p,\delta-j}.
$$

(2.2)

The spaces $L^p_\delta$ and $W^{k,p}_\delta$ as the completion of smooth, compactly supported functions on $\mathcal{M} \setminus \Sigma$. Spaces of sections of bundles are defined as usual and we use the standard notation $W^{k,2}_\delta = \mathcal{F}_\delta$. We also make use of the spaces $W^{k,p}_\delta$ and $H^k_\delta$, defined as the completion of smooth functions with bounded support on $\mathcal{M}$. That is, the overline indicates spaces of functions that vanish on the boundary, in the trace sense.

In light of the Bartnik mass, we are interested in the space of possible extensions to a region $\Omega$ in a given initial data set $(\tilde{\mathcal{M}}, \tilde{g}, \tilde{\pi})$. Given such a domain, we let $\mathcal{M}$ be such that $\Sigma$ can be identified with $\partial\Omega$ and extend $(\tilde{g}, \tilde{\pi})$ to a neighbourhood of $\Sigma$. Our space of extensions is then the space of asymptotically flat initial data on $\mathcal{M}$ that agrees with $(\tilde{g}, \tilde{\pi})$ on $\Sigma$. By introducing a background symmetric contravariant 2-tensor density $\hat{\pi}$, supported near $\Sigma$, we can omit reference to $\Omega$ by simply considering $\hat{g}$ and $\hat{\pi}$ on $\Sigma$. Define the spaces

$$
\mathcal{G} := \{g \in S_2 : g > 0, g - \hat{g} \in L^2_{-1/2}\}, \quad \mathcal{K} := \{\pi \in S^2 \otimes \Lambda^3 : \pi - \hat{\pi} \in L^1_{-3/2}\},
$$

$$
\mathcal{N} := L^2_{-5/2}(\Lambda^3 \times T^*\mathcal{M} \otimes \Lambda^3),
$$

where $\Lambda^k$ is the space of $k$-forms on $\mathcal{M}$, and $S_2$ and $S^2$ are symmetric co- and contravariant tensors on $\mathcal{M}$ respectively. The space of potential extensions to $\Omega$ that we consider here will be denoted by $\mathcal{F} = \mathcal{G} \times \mathcal{K}$. Note that $g$, $\hat{\nabla}g$ and $\pi$ are fixed on the boundary in the trace sense.
The constraint map, $\Phi : \mathcal{F} \to \mathcal{N}$, is given by

$$
\Phi_0(g, \pi) = R(g)\sqrt{g} - (\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^k_k)^2)g^{-1/2},
$$

(2.3)

$$
\Phi_i(g, \pi) = 2\nabla_k\pi^k_{i}.
$$

(2.4)

The constraint equations are then given by $\Phi(g, \pi) = (\epsilon, S)$, where $\epsilon$ and $S$ are the source energy and momentum densities respectively, as viewed by a Gaussian normal set of observers.

Bartnik’s work on the phase space makes extensive use of weighted Sobolev-type inequalities, most of which remain valid on an asymptotically flat manifold with boundary (see Theorem 1.2 of [1]), although some care should be taken with the use of the weighted Poincaré inequality. It is straightforward to verify that the majority of Bartnik’s proof of the theorem below is valid in the case discussed here, where $\mathcal{M}$ has a boundary and the initial data is in the trace-zero Sobolev spaces. The only detail that is not obvious, is in establishing that the linearised constraint map has closed range; however, we expect this to be the case and the following result to hold.

**Theorem 2.1** (cf. Theorem 3.12 of [3]). For $(\epsilon, S_i) \in \mathcal{N}$, the set

$$
\mathcal{C} := \{(g, \pi) \in \mathcal{F} : \Phi(\epsilon, S_i)\}
$$

is a Hilbert submanifold of $\mathcal{F}$.

That is, the space of possible extensions to a given domain $\Omega$ is a Hilbert manifold; we refer to this as the constraint submanifold.

3 Static metric extensions

The total ADM energy-momentum covector $\mathbb{P}(g, \pi) = \mathbb{P}_\alpha = (E, p_i)$ is given by

$$
16\pi E := \int_\infty g^{ik}(\nabla_k g_{ij} - \nabla_j g_{ik})dS^j,
$$

(3.1)

$$
16\pi p_i := 2\int_\infty \pi_{ij}dS^j.
$$

(3.2)

Often the quantity $E$ is called the mass, however we reserve the term mass for the quantity, $m = \sqrt{E^2 - |p|^2}$; we assume the dominant energy condition here to ensure this is real. We refer to $E$ and $p_i$ as the energy and momentum respectively.
We are now in a position to discuss critical points of the mass/energy exterior to \( \Sigma \), and in particular show how Bartnik’s work is easily adapted to give another proof of Corvino’s result on static metric extensions. Previously, the author considered evolution exterior to a 2-surface \([11]\); however, the data was not fixed on the boundary so the conclusion is somewhat different. In the context of the static metric extension conjecture, it is more interesting to consider the data as fixed on the boundary. Bartnik’s proof of Corollary 6.2 of Ref. \([3]\), as with the results mentioned above, can easily be verified to hold in the case where \( \mathcal{M} \) has a boundary when the initial data is in the trace-zero weighted Sobolev spaces. This results in the following statement.

**Theorem 3.1** (cf. Corollary 6.2 of \([3]\)). Fix \((g, \pi) \in \mathcal{C}(\varepsilon, S)\), where \((\varepsilon, S) \in L^1\). If \(Dm(g, \pi)[h, p] = 0\) for all \((h, p) \in T_{(g, \pi)}\mathcal{C}(\varepsilon, S)\), then \((g, \pi)\) is a stationary initial data set.

Note that this differs from Corvino’s static extension result, which in our framework, is essentially the following.

**Theorem 3.2.** Fix \(g \in \hat{\mathcal{C}}(s) = \{g \in \mathcal{G} : R(g) = s\}\), where \(s \in L^1\). If \(DE(g)[h] = 0\) for all \(h \in T_g\hat{\mathcal{C}}(s)\), then \(g\) is a static initial data set.

Theorem \([3,2]\) considers extensions to \(\Omega\) that are time-symmetric, so the momentum constraint is not required and the Hamiltonian constraint reduces to the prescribed scalar curvature equation. Bartnik’s proof doesn’t directly give this result, however a variational proof of Theorem \([3,2]\) can be obtained using the same techniques, and is in fact simpler. As such, we simply present a sketch of the proof to illustrate the differences.

**Sketch of proof of Theorem \([3,2]\).** It is straightforward to demonstrate that the level sets \(\hat{\mathcal{C}}(s)\) are indeed Hilbert submanifolds of \(\mathcal{G}\) by following the arguments in Section 3 of \([3]\) with \(\pi = 0\). We consider the following Lagrange function for the Hamiltonian constraint:

\[
L(g; N) = N_\infty E(g) - \int_\mathcal{M} NR(g),
\]

where \(N_\infty\) is a constant, acting as the Lagrange multiplier, and \((N - N_\infty) \in L^2_{-1/2}\). It can be shown that for \(N\) satisfying \((N - N_\infty) \in W^{2,2}_{-1/2}\), we have

\[
DL(g; N)[h] = -\int_\mathcal{M} h \cdot DR(g)^*[N],
\]

for all \(h \in T_g\mathcal{G}\), where \(DR(g)^*\) is the formal adjoint of the linearised scalar curvature map. Let \(\hat{g}\) be a critical point of \(E\) on \(\hat{\mathcal{C}}(s)\), which implies \(DL(\hat{g}; N)[h] = 0\) for all
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$h \in T_{\tilde{g}} \hat{\mathcal{C}}$. A Lagrange multiplier argument then implies the existence of $\hat{N}$ such that

$$DL(g; N)[h] = \int_{\mathcal{M}} \hat{N} DR(g)[h], \quad (3.5)$$

for all $h \in T_{\tilde{g}} \mathcal{G}$. From (3.4) and (3.5), it follows that $DR(g)^*[N + \hat{N}] = 0$, which then implies $g$ is static (cf. Proposition 2.7 of [7]).

To see that $DR(g)^*[N + \hat{N}] = 0$ implies staticity, note that $\xi = (N + \hat{N}, 0, 0, 0)$ corresponds to a hypersurface orthogonal, timelike, and satisfies $D\Phi(g, 0)^*[\xi] = 0$; recall the evolution equations can be expression as

$$\frac{\partial}{\partial t} \begin{bmatrix} g \\ \pi \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \circ D\Phi(g, \pi)^*[\xi]. \quad (3.6)$$

As mentioned in the introduction, fixing the first derivative of the metric is likely too restrictive, and indeed more restrictive than Corvino’s proof requires. It would be interesting to conduct this kind of variational analysis in the case where the metric and the induced mean curvature is fixed on the boundary; a result of Miao demonstrates that if a minimal mass extension exists then it is in fact static, scalar-flat and the mean curvature induced on the boundary by both the interior and exterior metrics matches [12].

4 Energy minimisers in a fixed conformal class

A popular approach to simplify the constraint equations is to look for solutions within a fixed conformal class (see [4] and references therein). In this case, the Hamiltonian constraint becomes elliptic and is therefore far easier to deal with. In this section, we make use of this simplification by considering the space of extensions to $\Omega$ within a given conformal class. Specifically, we consider a fixed metric $\tilde{g} \in \mathcal{G}$ and consider extensions of the form $g(\phi) = e^{4\phi} \tilde{g}$. We also assume that $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B_0(1)}$; that is, we consider the most natural extensions to $\Omega$. This affords us the use of the weighted Poincaré inequality (see, for example, Lemma 3.10 of [3]).

The scalar curvature of $g$ is given by the well-known formula,

$$R(g) = e^{-4\phi}(\bar{R} - 8|\nabla \phi|^2 - 8\bar{\Delta} \phi),$$

where $\sim$ indicates quantities defined with respect to $\tilde{g}$. This allows us to write the
conformal constraint map, \( \Phi : \mathcal{M} \times K \rightarrow \mathcal{N} \), as

\[
\Phi_0(\phi, \pi) = e^{2\phi} \left[ (\tilde{R} - 8|\nabla \phi|^2 - 8\Delta \phi) \sqrt{\tilde{g}} - \tilde{g}_{ik} \tilde{g}_{lj} (\pi^{ij} \pi^{kl} - \frac{1}{2} \pi^{ik} \pi^{jl}) \tilde{g}^{-1/2} \right], \quad (4.1)
\]

\[
\Phi_1(\phi, \pi) = 2e^{4\phi} \left( \tilde{g}_{ip} \tilde{\nabla}_k \pi^{kp} + 4\tilde{g}_{ip} \pi^{kp} \tilde{\nabla}_k \phi - 2\tilde{g}_{jp} \pi^{jp} \tilde{\nabla}_i \phi \right). \quad (4.2)
\]

From this point on, we will raise and lower indices with \( \tilde{g} \) rather than \( g \). We also will consider the weighted Sobolev norms with respect to \( \tilde{g} \) rather than \( \tilde{g} \). Note that the domain of \( \Phi \) enforces the boundary conditions on \((g, \pi)\): in particular, the conformal metric \( \tilde{g} \) must itself be an extension of \( \Omega \) although it need not necessarily satisfy the constraints.

**Proposition 4.1.** For any \( \phi \in \mathcal{H}^{-1/2}_1 \), we have \( g = e^{4\phi} \tilde{g} \in \mathcal{G} \).

**Proof.** It is clear that \( e^{4\phi} \tilde{g} \) is positive-definite, and using the standard weighted Sobolev-type inequalities (see [1] for the weighted Sobolev-type inequalities) we have,

\[
\| e^{4\phi} \tilde{g} - \tilde{g} \|_{2, -1/2} \leq c\| \tilde{g} \|_{\infty, 0} (\| e^{4\phi} - 1 \|_{2, -1/2} + \| e^{4\phi} \tilde{\nabla} \phi \|_{2, -3/2} + \| e^{4\phi} \tilde{\nabla}^2 \phi \|_{2, -5/2})
\]

\[
\leq c\| \tilde{g} \|_{\infty, 0} (\| e^{4\phi} - 1 \|_{2, -1/2} + \| e^{4\phi} \|_{\infty, 0} \| \tilde{\nabla} \phi \|_{1, 2, -3/2}).
\]

Note that \( \phi \) is continuous by the Morrey embedding and \( e^{4\phi} - 1 < 4|\phi| \) near infinity, so \( \| e^{4\phi} - 1 \|_{2, -1/2} < \infty \). \( \square \)

**Proposition 4.2.** Let \((\phi, \pi) \in \mathcal{H}^{-1/2}_1 \times K\) satisfy \( \Phi_0 = 16\pi\rho \), where \( \rho \) is the source energy density. The ADM energy can then be expressed as,

\[
16\pi E = 16\pi \tilde{E} + \int_{\mathcal{M}} (\langle 8|\nabla \phi|^2 - \tilde{R} \rangle \sqrt{\tilde{g}} + (\pi^{ij} \pi_{ij} - \frac{1}{2} (\pi^k_k)^2) / \sqrt{\tilde{g}} + 16\pi e^{-2\phi} \rho) dx^3, \quad (4.3)
\]

where \( \tilde{E} \) is the ADM energy of \( \tilde{g} \).

**Proof.** First we write \( E \) in terms of \( \phi \) and \( \tilde{g} \),

\[
16\pi E = \int_{\tilde{M}} \tilde{g}^{ik} e^{4\phi} \left( 4\tilde{\nabla}_k (\phi) \tilde{g}_{ij} + \tilde{\nabla}_k \tilde{g}_{ij} - 4\tilde{\nabla}_j (\phi) \tilde{g}_{ik} - \tilde{\nabla}_j \tilde{g}_{ik} \right) dS^j
\]

\[
= \int_{\tilde{M}} \tilde{g}^{ik} \left( 4\tilde{\nabla}_k (\phi) \tilde{g}_{ij} + \tilde{\nabla}_k \tilde{g}_{ij} - 4\tilde{\nabla}_j (\phi) \tilde{g}_{ik} - \tilde{\nabla}_j \tilde{g}_{ik} \right) dS^j
\]

\[
+ \int_{\tilde{M}} \tilde{g}^{ik} (e^{4\phi} - 1) \left( 4\tilde{\nabla}_k (\phi) \tilde{g}_{ij} + \tilde{\nabla}_k \tilde{g}_{ij} - 4\tilde{\nabla}_j (\phi) \tilde{g}_{ik} - \tilde{\nabla}_j \tilde{g}_{ik} \right) dS^j \quad (4.4)
\]

We now make use of the following estimate.
Lemma 4.3 (Lemma 4.4 of [3]). Let $S_R$ be the Euclidean sphere of radius $R$, $E_R$ be the exterior region to $S_R$ – the connected component of $M \setminus S_R$ containing infinity – and $A_R$ be the annular region between $S_R$ and $S_2R$. Suppose $u \in H^{1-3/2}(E_{R_0})$, then for every $R \geq R_0$,
\[
\oint_{S_R} |u|dS \leq cR^{1/2}\|u\|_{1,2,-3/2;A_R},
\]
where $c$ is independent of $R$.

This Lemma, along with the H"{o}lder inequality, can now be used to control the second integrand in Eq. (4.4),
\[
\left| \oint_{S_R} \tilde{g}^{ik}(e^{4\phi} - 1) \left( 4\tilde{\nabla}_k(\phi)\tilde{g}_{ij} + \tilde{\nabla}_i\tilde{g}_{kj} - 4\tilde{\nabla}_j(\phi)\tilde{g}_{ik} - \tilde{\nabla}_j\tilde{g}_{ik} \right) dS^j \right|
\leq c\|e^{4\phi} - 1\|_{\infty:S_R}\|\tilde{\nabla}\phi\|_{1:S_R} + \|\tilde{\nabla}\tilde{g}\|_{1:S_R}.
\leq o(R^{1/2})\|e^{4\phi} - 1\|_{\infty:S_R}\|\tilde{\nabla}\phi\|_{1,2,-3/2} + \|\tilde{\nabla}\tilde{g}\|_{1,2,-3/2}
\]
Now making use of the continuity and asymptotics of $e^{4\phi}$ and $\tilde{g}$, the right-hand-side simply becomes $o(1)$ and therefore vanishes as $R$ tends to infinity. Eq. (4.4) now becomes
\[
16\pi E = 16\pi E_0 + 4 \oint_{\infty} \tilde{g}^{ik} \left( \tilde{\nabla}_k(\phi)\tilde{g}_{ij} + \tilde{\nabla}_i\tilde{g}_{kj} - 4\tilde{\nabla}_j(\phi)\tilde{g}_{ik} - \tilde{\nabla}_j\tilde{g}_{ik} \right) dS^j,
\]
which can be expressed in terms of the energy, $E_0$, of $\tilde{g}$,
\[
16\pi E = 16\pi E_0 + 4 \oint_{\infty} \tilde{g}^{ik} \left( \tilde{\nabla}_k(\phi)\tilde{g}_{ij} + \tilde{\nabla}_i\tilde{g}_{kj} - 4\tilde{\nabla}_j(\phi)\tilde{g}_{ik} - \tilde{\nabla}_j\tilde{g}_{ik} \right) dS^j.
\]
Since $(\tilde{g} - \tilde{g}) \in \overline{H}^{-1/2}$ and $\tilde{\nabla}\phi = \partial\phi = \tilde{\nabla}\phi$, Lemma 4.3 can again be used to conclude
\[
16\pi E = 16\pi E_0 + 4 \oint_{\infty} \tilde{g}^{ik} \left( \tilde{\nabla}_k(\phi)\tilde{g}_{ij} + \tilde{\nabla}_i\tilde{g}_{kj} - 4\tilde{\nabla}_j(\phi)\tilde{g}_{ik} - \tilde{\nabla}_j\tilde{g}_{ik} \right) dS^j
\leq 16\pi E - 8 \oint_{\infty} \tilde{\nabla}_j\phi dS^j.
\]
It is now simply a matter of applying the divergence theorem and making use of the Hamiltonian constraint (4.1) to complete the proof. □

When we write $E(g, \pi)$, we mean to take (4.3) to be the definition of the energy, which is well-defined off-shell provided $\tilde{R}$ and the source are integrable.

In the vacuum case ($\rho = 0$), if $\tilde{R} = 0$ then it is clear from (4.3) that the energy of
any solution $g$ in the conformal class of $\hat{g}$ has energy greater than $\hat{E}$, with equality only if $g = \hat{g}$. That is, if there exists a metric $\hat{g}$ in the conformal class of $\tilde{g}$ with $R(\hat{g}) = 0$ then the infimum of the energy is attained by $\hat{g}$. Generically such a scalar-flat extension does not exist though, as our boundary conditions on $\nabla \phi$ are too strong. An argument of Cantor and Brill [5] proves the existence of scalar-flat metrics when no boundary is present, and it is likely straightforward to obtain the same result when Dirichlet or Neumann boundary conditions are imposed; however, this argument does not hold for the (stronger) boundary conditions here and in fact, the result is almost certainly not true in this case.

We will need the following estimate in the proof of the main result of this section.

**Proposition 4.4.** For $u \in W^{1,2}_\delta$ and $\epsilon > 0$, it holds that

$$
\|u\|_{4,\delta} \leq c(\epsilon)\|u\|_{2,\delta} + \epsilon\|u\|_{1,2,\delta}.
$$

**Proof.** This follows from the weighted Hölder and Sobolev inequalities, the definition of the weighted norms, and Young’s inequality:

$$
\|u\|_{4,\delta} = \|u^{1/4}u^{3/4}\|_{4,\delta} \leq \|u^{1/4}\|_{8,\delta/4}\|u^{3/2}\|_{8,3\delta/4}
\leq \|u\|_{2,\delta}^{1/4}\|u\|_{6,\delta}^{3/4}
\leq c(\epsilon)\|u\|_{2,\delta} + \epsilon\|u\|_{6,\delta}
\leq c(\epsilon)\|u\|_{2,\delta} + \epsilon\|u\|_{1,2,\delta}.
$$

$\square$

The main theorem of this section can be divided into the two following cases:

**Theorem 4.5.** Let $S_\alpha^+$ be the set of $(\phi, \pi) \in H^{1/2}_{-1/2}(\mathcal{M}) \times \mathcal{K}$, satisfying the following conditions:

(i) $\tilde{\Phi}_0(\phi, \pi) \geq 0$,

(ii) $\phi \geq -\alpha$,

(iii) $\Phi_0(\phi, \pi) \in L^1_{-3}$.

Then either $\max\{\|\Phi(\phi_{(n)}, \pi_{(n)})\|_{2, -5/2}, \|\nabla_k \pi_{(n)}^{ij}\|_{2, -5/2}\} \to \infty$ for all minimising sequences of $E(\phi, \pi)$, or the infimum is achieved over $S_\alpha^+$. 

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Theorem 4.6. Let $S^0$ be the set of $(\phi, \pi) \in \mathbb{T}^2_{-1/2}(\mathcal{M}) \times \mathcal{K}$, satisfying $\overline{\Phi}(\phi, \pi) = 0$. Then either $\|\nabla_j \pi^i (n)\|_{2,-5/2} \rightarrow 0$ for all minimising sequences of $E(\phi, \pi)$, or the infimum is achieved over $S^0$. 

Furthermore, the infimum is always realised on $S^0$ for any $\beta \in \mathbb{R}^+$. 

$S^0 = \{ (\phi, \pi) \in S^0 : \|\nabla_k \pi^j\|_{2,-5/2} \leq \beta \|\nabla_j \pi^i\|_{2,-5/2} \}$. 

Since the two theorems are so similar, we prove them simultaneously, noting the relevant differences.

Proof. From Proposition 4.2 we have

$$\|\nabla \phi\|_{2,-3/2}^2 + \|\pi\|_{2,-3/2}^2 \leq 32\pi(E - \tilde{E}) + 2 \int_{\mathcal{M}} \tilde{R} \sqrt{g} \, dx^3 - 16\pi \int_{\mathcal{M}} e^{-2\phi} \Phi(\phi, \pi)$$

$$\leq 32\pi E + \tilde{C}. \quad (4.7)$$

This implies that if the initial data is sufficiently large then we can guarantee that the energy is large. Let $S$ be any of the sets $S^0$, $S^0$, $S^0$, and define $E_0 = \inf_{(\phi, \pi) \in S} E(\phi, \pi)$. Now let $(\phi_n, \pi_n)$ be a sequence in the constraint set such that $E(\phi_n, \pi_n) \rightarrow \tilde{n}_B$. Note that Proposition 4.7 and the Poincaré inequality implies that there exists a constant $K$ such that for $\|(\phi, \pi)\|_{H^1_{-1/2} \times L^2_{-3/2}} > K$, we have $E(\phi, \pi) > E_0 + 1$. That is, truncating the beginning of the sequence if necessary, $\|(\phi_n, \pi_n)\|_{H^1_{-1/2} \times L^2_{-3/2}} < K$. In particular, extracting a subsequence if required, $(\phi_n, \pi_n)$ converges weakly in $H^1_{-1/2} \times L^2_{-3/2}$ to a limit, $(\phi_\infty, \pi_\infty)$. It remains to be shown that $(\phi_\infty, \pi_\infty) \in S$.

We assume $\max\{\|\Phi(\phi_n, \pi_n)\|_{2,-5/2}, \|\nabla_j \pi^i (n)\|_{2,-5/2}, \|\nabla_j \pi^i (n)\|_{2,-5/2} \} < C$, and prove below that the infimum is realised in $S$.

For any $\delta_0 \in (-\frac{1}{4}, 0]$, Proposition 4.1 and the definition of $\tilde{\Phi}_0$ give

$$\|\nabla \phi_k\|_{2,-5/2+2\delta_0} \leq c\|\tilde{R}\|_{2,-5/2+2\delta_0} + \|\nabla \phi_k\|_{1,-5/4+\delta_0} + \|\pi_k\|_{1,-5/4+\delta_0} + \|e^{-2\phi_k} \Phi(\phi_k, \pi_k)\|_{2,-5/2+2\delta_0}$$

$$\leq c(\epsilon) (1 + \|\pi_k\|_{2,-5/4+\delta_0} + \|\nabla \phi_k\|_{2,-5/4+\delta_0} + \|\nabla \phi_k\|_{1,-5/4+\delta_0} + \|\nabla \phi_k\|_{1,-5/4+\delta_0} + \|\nabla \phi_k\|_{1,-5/4+\delta_0}), \quad (4.8)$$

which follows from the assumption $\|\Phi(\phi_k, \pi_k)\|_{2,-5/2} < C$ and condition (ii) for the proof of Theorem 4.4 and from $\Phi(\phi_k, \pi_k) = 0$ for the proof of Theorem 4.6.
Similarly, the assumption \( \| \nabla_k \pi_k \|_{2,-5/2} < C \) and the definition of \( \Phi_i \) gives

\[
\| \nabla \pi_k \|_{2,-5/2+2\delta_0} \leq C(\| \nabla \phi_k \|_{4,-5/4+\delta_0} \| \pi_k \|_{4,-5/4+\delta_0} + \| e^{-4\phi_k} \Phi_i(\phi_k, \pi_k) \|_{2,-5/2+2\delta_0}) \leq C(\| \nabla \phi_k \|_{4,-5/4+\delta_0} + \| \pi_k \|_{4,-5/4+\delta_0} + 1) \leq C(\epsilon)(\| \nabla \phi_k \|_{4,-5/4+\delta_0} + \| \pi_k \|_{2,-5/4+\delta_0} + 1) + \epsilon(\| \nabla \phi_k \|_{4,1/2,-5/4+\delta_0} + \| \pi_k \|_{1,2,-5/4+\delta_0}).
\]

(4.9)

We now recall the scale-broken estimate (cf. Theorem 1.10 of [1], Proposition 4.13 of [10]):

\[
\| u \|_{2,2,\delta} \leq C \left( \| \Delta u \|_{2,\delta-2} + \| u \|_{2,0} \right). \tag{4.10}
\]

Combining (4.10) with (1.5), applying the weighted Poincaré inequality to (1.10), and choosing \( \epsilon \) sufficiently small gives

\[
\| \phi_k \|_{2,2,-1/2+2\delta_0} + \| \pi_k \|_{1,2,-3/2+2\delta_0} \leq C \left( 1 + \| \phi_k \|_{1,2,-1/4+\delta_0} + \| \pi_k \|_{2,-5/4+\delta_0} \right).
\]

Choosing \( \delta_0 = -\frac{1}{8} \), this then gives weak convergence in \( H^2_{3/4} \times H^1_{7/4} \), which by the weighted Rellich compactness theorem (Lemma 2.1 of [3]), gives strong convergence in \( H^1_{1/2} \times L^2_{3/2} \).

At this point we consider each case separately. When \( S = S^0 \), the Hamiltonian constraint can be expressed as \( \Delta \phi_k = Q(\nabla \phi_k, \pi_k) \), a quadratic in \( \nabla \phi_k \) and \( \pi_k \). Since bounded polynomial functions are smooth (see, for example, Chapter 26 of [9]), and the map \( \phi \mapsto \nabla \phi \) is bounded linear, \( \Delta \phi_k \rightarrow Q(\nabla \phi_\infty, \pi_\infty) \) in \( L^2_{-5/2} \). By considering the dual pairing \( (\Delta \phi_k, \psi) \) with an arbitrary \( \psi \in L^2_{-1/2} \) and approximating by smooth compactly supported functions, it is simple to show \( \Delta \phi_k \) converges weakly to \( \Delta \phi_\infty \) in \( L^2_{-5/2} \). By uniqueness of limits, it follows that \( \Delta \phi_\infty = Q(\nabla \phi_\infty, \pi_\infty) \); that is, \( (\phi_\infty, \pi_\infty) \) does indeed satisfy the Hamiltonian constraint. The same reasoning implies that \( (\phi, \pi) \) also satisfies the momentum constraint, and therefore \( (\phi_\infty, \pi_\infty) \in S^0 \), and the infimum is achieved over \( S^0_\alpha \). It is clear the above argument demonstrates that the infimum over \( S^0_\beta \) is also achieved.

We now need something akin to the order limit theorem for weak limits to demonstrate that if \( (\phi_k, \pi_k) \in S^0_+ \) then \( (\phi_\infty, \pi_\infty) \) also satisfies conditions (i) and (ii). Consider

\[
F_k = (\bar{R} - 8|\nabla \phi_k|^2 - 8\Delta \phi_k) \sqrt{g} - (\pi_k^2 - \frac{1}{2}(\text{tr}_g \pi_k)^2) \bar{g}^{-1/2},
\]

and
which converges weakly in $L^2_{5/2}$ to

$$F_\infty = (\tilde{R} - 8|\tilde{\nabla}\phi_\infty|^2 - 8\tilde{\Delta}\phi_\infty)\sqrt{\tilde{g}} - (\pi_\infty^2 - \frac{1}{2}(\text{tr}_{\tilde{g}}\pi_\infty)^2)\tilde{g}^{-1/2}.$$ 

We prove $F_\infty \geq 0$ by contradiction: assume there is a bounded set $\Omega \in \mathcal{M}$ such that $F_\infty < 0$ on $\Omega$. Let $\chi_\Omega$ be the characteristic function of $\Omega$, then by the weak convergence of $F_k$ we have

$$\int_\Omega F_k = \int_\mathcal{M} F_k\chi_\Omega \to \int_\mathcal{M} F\chi_\Omega = \int_\Omega F_\infty.$$ 

Since $F_k \geq 0$ by assumption, we have a contradiction and it follows that $\Phi_0(\phi_\infty, \pi_\infty) \geq 0$. An identical albeit simpler argument shows $\phi_\infty \geq -\alpha$, and from the definition of $E$ it is obvious that $\int \Phi(\phi_\infty, \pi_\infty) < \infty$. We therefore conclude $(\phi_\infty, \pi_\infty) \in S^+_\alpha$. \hfill $\blacksquare$

**Remark 4.7.** Theorem 4.5 still holds without the assumption of condition (i), however it is more interesting to impose the weak energy condition.

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