Projected BCS states and spin Hamiltonians for the SO($n_1$) WZW model

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We propose a class of projected BCS wave functions and derive their parent spin Hamiltonians. These wave functions can be formulated as infinite Matrix Product States constructed by chiral correlators of Majorana fermions. In 1D, the spin Hamiltonians can be viewed as SO($n$) generalizations of Haldane-Shastry models. We numerically compute the spin-spin correlation functions and Rényi entropies for $n = 5$ and 6. Together with the results for $n = 3$ and 4, we conclude that these states are critical and their low-energy effective theory is the SO($n_1$) Wess-Zumino-Witten model. In 2D, we show that the projected BCS states are chiral spin liquids, which support non-Abelian anyons for odd $n$ and Abelian anyons for even $n$.

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Introduction. – An efficient description of quantum many-body systems is a challenging problem in modern physics, as the dimension of the Hilbert space scales exponentially with the number of particles. For strongly interacting many-body systems, much of our understanding of their properties comes from physically motivated trial wave functions and/or exact solutions of specific models. A great success of the trial wave function approach is the celebrated Laughlin wave function for the fractional quantum Hall effect at 1/2 filling [1]. Toward exact results, Bethe’s solution of the spin-1/2 Heisenberg chain [2] and integrability of the spin-1/2 Haldane-Shastry model [3] provide invaluable insight for critical spin-1/2 chains.

The justification of trial wave functions is usually a difficult task. For example, the relevance of Anderson’s resonating valence bond (RVB) state [4] for the mechanism of high-$T_c$ superconductivity is still a controversial issue. A useful technique for justifying trial wave functions is to numerically calculate the spin-spin correlation functions and Rényi entropies [5] for the Laughlin wave functions and/or exact solutions of specific models. A great success of the trial wave function approach is the celebrated Laughlin wave function for the fractional quantum Hall effect at 1/2 filling [1]. Toward exact results, Bethe’s solution of the spin-1/2 Heisenberg chain [2] and integrability of the spin-1/2 Haldane-Shastry model [3] provide invaluable insight for critical spin-1/2 chains.

In this work, we propose a class of projected BCS states and derive their parent Hamiltonians. These states can also be represented as infinite Matrix Product States (MPS) [8] constructed from chiral correlators of Majorana fermions. In 1D, the spin Hamiltonians are SO($n$) generalizations of Haldane-Shastry models. We numerically calculate the spin-spin correlation functions and the Rényi entropies for $n = 5$ and 6 and compare the numerical results with field theory predictions from SO($n_1$) criticality. Together with the known results for $n = 3$ and 4, we expect that for general $n$ these states are critical and belong to the SO($n_1$) Wess-Zumino-Witten (WZW) universality class. We also show that the projected BCS states with modified Cooper pair wave functions provide a good description for Ising ordered and disordered phases close to SO($n_1$) criticality. In 2D, the projected BCS states are chiral spin liquids with $p + ip$ pairing symmetry. We find that these topological states support non-Abelian Ising anyons for odd $n$ and Abelian anyons for even $n$, respectively.

Projected BCS wave function. – Constructing the projected BCS wave functions relies on a slave-particle representation of the SO($n$) algebra. Let us start from a 1D periodic chain with even $N$ sites, where the $n$ vectors in each site are represented by using singly occupied fermions, $|n^a⟩ = c^a_0|0⟩$ ($a = 1, \ldots, n$). In terms of fermions, the SO($n$) generators are written as $L^{ab} = i(c^b_0c^0_a - c^b_ac^0_a)$, where $1 \leq a < b \leq n$. To remove unphysical states in this fermionic representation, a single-occupancy constraint is required, $\sum_{a=1}^{n} c^a_j|c^a_j⟩ = 1 \forall j = 1, \ldots, N$, which defines a Gutzwiller projector $P_G$. Then, the projected BCS wave function of our interest is defined by

$$|Ψ⟩ = P_G \exp \left( \sum_{i<j} \frac{1}{z_i - z_j} \sum_{a=1}^{n} c^i_{a,j,a} c^j_{i,a,a} \right) |0⟩,$$

(1)

where $\sum_{a=1}^{n} c^i_{a,j,a} c^j_{i,a,a}$ creates an SO($n$) singlet between sites $i$ and $j$. Note that $|Ψ⟩$ is a coherent superposition of valence-bond singlets of arbitrary range (see Fig. II and hence can be viewed as an RVB state [II]. If we choose $z_j = \exp(i \frac{2\pi}{n} j)$, the amplitude of the Cooper pair wave function $1/|z_i - z_j|$ is the inverse of the chord distance between the sites. Under this choice, $|Ψ⟩$ is both real and translationally invariant, which is the uniform case that we will consider in the following.

Before discussing the properties of $|Ψ⟩$ for general $n$, we establish the relation between [II] and some known results. For $n = 3$, after switching to the standard spin-1/2 Wess-Zumino-Witten (WZW) model, we numerically calculate the spin-spin correlation functions and Rényi entropies for $n = 5$ and 6 and compare the numerical results with field theory predictions from SO($n_1$) criticality. Together with the known results for $n = 3$ and 4, we expect that for general $n$ these states are critical and belong to the SO($n_1$) Wess-Zumino-Witten (WZW) universality class. We also show that the projected BCS states with modified Cooper pair wave functions provide a good description for Ising ordered and disordered phases close to SO($n_1$) criticality. In 2D, the projected BCS states are chiral spin liquids with $p + ip$ pairing symmetry. We find that these topological states support non-Abelian Ising anyons for odd $n$ and Abelian anyons for even $n$, respectively.

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A detailed proof of the equivalence of the projected BCS jorana fields \([16]\) of the primary field allows us to rewrite the projected BCS state (1) as the following infinite MPS:

This Majorana representation belongs to the SU(2) WZW universality class [14]. For \(n=2\) in the vector representation [15], the former has a pure Jastrow form, while the latter includes a Pfaffian factor. To see this difference, let us consider SO(2l) and SO(2l+1) (\(l\) integer) and choose mutually commuting Cartan generators as \(L^{12}, L^{34}, \ldots L^{2l-1,2l}\). For SO(2l), a convenient choice of the Cartan basis is defined by \(0, \ldots, m_\alpha = \pm 1, \ldots, 0\) \(= (n^{2\alpha} \pm i)n^{2\alpha-1}) / \sqrt{2} (\alpha = 1, \ldots, l)\), where \(m_\alpha\) is the eigenvalue of \(L^{2\alpha-1,2\alpha}\). For the vectors \(0, \ldots, m_\alpha = \pm 1, \ldots, 0\), we label their positions in the spin chain by \(x^{(1)}_1 < \cdots < x^{(1)}_{N_1}\). In this basis, the wave function (1) for even \(n = 2l\) takes the form [17]

\[
\Psi(\{m\}) = \rho_m \prod_{\alpha=1}^l \prod_{i<j} (z_i - z_j)^{m_{\alpha,i} m_{\alpha,j}},
\]
where \( \rho_m = \text{sgn}(x_1^{(1)} \ldots x_N^{(1)} \ldots x_1^{(l)} \ldots x_N^{(l)}) \) (sgn: signature of a permutation) if \( \sum_i m_{\alpha,i} = 0 \) \( \forall \alpha \) and \( \rho_m = 0 \) otherwise.

For SO\((2l + 1)\), apart from the vectors \( |0, \ldots, m_\alpha = \pm 1, \ldots, 0\rangle\), there exists an additional vector \( |0, \ldots, 0\rangle = |n^{l+1}\rangle\), which is annihilated by all Cartan generators. Labeling their positions by \( x_1^{(0)} < \ldots < x_N^{(0)}\), the wave function \( \Psi \) for odd \( n = 2l + 1 \) is written as \[ \Psi(|m\rangle) = \rho_m \text{Pf}_0 \left( \frac{1}{z_i - z_j} \right) \prod_{\alpha=1}^l \prod_{i<j}|z_i - z_j|^{m_{\alpha,i}m_{\alpha,j}}, \tag{6} \]

where \( \rho_m = \text{sgn}(x_1^{(0)} \ldots x_N^{(0)}, x_1^{(1)} \ldots x_N^{(1)} \ldots x_1^{(l)} \ldots x_N^{(l)}) \) if \( \sum_i m_{\alpha,i} = 0 \) \( \forall \alpha \) and \( \rho_m = 0 \) otherwise, and the Pfaffian factor \( \text{Pf}_0 \left( \frac{1}{z_i - z_j} \right) \) is restricted to the positions for the extra vector \( |0, \ldots, 0\rangle\).

**Numerical results.**—The power-law decaying correlation functions and the universal scaling of entanglement entropy \( \text{Ref. [19]} \) are characteristic behaviors of conformal critical points in 1D. Even though these quantities are difficult to compute analytically for \( n \), the Jastrow and Pfaffian forms \( \{10\} \) and \( \{7\} \) of the wave functions are very suitable for determining them numerically via the Metropolis Monte Carlo (MC) method \( \{20\} \).

Below we focus on the projected BCS state \( \{1\} \) with \( n = 5 \) and \( 6 \) and provide numerical evidence that they belong to SO\((5,1)\) and SO\((6,1)\) WZW models, respectively.

For critical spin chains in the SO\((n)_1\) WZW universality class, field theory predicts that for \( n < 8 \) the spin-spin correlation function behaves as \( \langle L_1^{ab} L_2^{cd} \rangle \sim (-1)^x/n^\eta \) with \( \eta = n/4 \) \( \{21\} \). For the projected BCS state \( \{1\} \) with \( n = 5 \) and \( 6 \), we have computed the two-point spin correlation \( \langle L_1^{ab} L_2^{cd} \rangle \). Figure 2 shows the numerical results for \( N = 200 \). The critical exponents that best fit with our numerical data are \( \eta = 1.22 \) for SO\((5)\) and \( \eta = 1.42 \) for SO\((6)\) (solid lines in Fig. 2), which agree very well with the field theory predictions (dotted lines).

The entanglement entropy that is easily accessible via the MC method is the Rényi entropy \( S_L^{(2)} = -\ln \text{Tr} \rho_L^2 \) (see Refs. \[ \{8\}, \{22\}, \{24\} \), where \( \rho_L \) is the reduced density matrix of the state in a subsystem of length \( L \). For the SO\((n)_1\) WZW model with \( c = n/2 \) we expect \( S_L^{(2)} = c \ln[\sin(\pi L/N)]/4 + c_2^{(2)} \) \( \{19\} \), where \( c_2^{(2)} \) is a constant. For \( n = 5 \) and \( 6 \), we plot \( S_L^{(2)} \) as a function of \( \ln[\sin(\pi L/N)]/4 \) for \( N = 200 \) in Fig. 3. From our MC data, the estimates of the central charge are \( c = 2.31 \) for SO\((5)\) and \( c = 2.76 \) for SO\((6)\) (solid lines in Fig. 3), which are close to the predicted \( c = n/2 \) (dotted lines) but show some deviations.

The origin of the small deviations of the numerical results and the SO\((n)_1\) predictions may be due to the presence of marginally irrelevant terms in the SO\((n)_1\) WZW model for \( \{1\} \) and its parent Hamiltonian \( \{5\} \), unlike the SU\((n)\) Haldane-Shastry models \( \{25\}, \{26\} \) (including the spin-1/2 Haldane-Shastry model for \( n = 2 \)) being the fixed points of the SU\((n)_1\) WZW model. For \( n = 3 \), the presence of marginal term in the spin-1 Haldane-Shastry model has been confirmed numerically \( \{8\}, \{10\} \). If this is also the case for \( n \geq 5 \), an interesting open question is whether there exist a modified version of \( \{1\} \) and its parent Hamiltonian that represent the fixed point of the SO\((n)_1\) WZW model.

**Away from SO\((n)_1\) criticality.**—After showing that the projected BCS state \( \{1\} \) captures the physics of the SO\((n)_1\) WZW model, it is natural to ask whether similar projected wave functions are relevant for gapped spin chains away from (but close to) SO\((n)_1\) criticality. Let us restrict ourselves to SO\((n)\) symmetric models for sim-
plicity. According to the well-known result by Witten 27, the SO(n) WZW model is equivalent to n massless Majorana fermions, i.e., n Ising models at criticality. For this critical theory, the only relevant perturbation allowed by SO(n) symmetry is the mass term of Majorana fermions. Thus, the low-energy effective theory has the Hamiltonian density $\mathcal{H} = -\frac{1}{2} \sum_{m} (\xi_{R}^{m} \partial_{x} \xi_{R}^{m} - \xi_{L}^{m} \partial_{x} \xi_{L}^{m}) - im \sum_{m} g_{R/L}^{m}$, where $\xi_{R/L}^{m}$ are right (left) moving Majorana fermions, $v$ and $m$ are their velocity and mass. Here we have assumed four-fermion interactions are weak and can be neglected, since they are marginal and only renormalize the mass of Majorana fermions at low-energy limit 28.

The SO(n) criticality corresponds to $m = 0$. The two gapped phases adjacent to the SO(n) criticality are (i) the Ising ordered phase ($m < 0$) and (ii) the Ising disordered phase ($m > 0$). For these two phases, we note that they can be well described by modified projected BCS states. Actually, these two gapped phases and an SO(n)1 critical point (Reshetikhin model 29) are realized in the SO(n) bilinear-biquadratic chain 21, 30. The ideal example that belongs to the Ising ordered phase is the SO(n) AKLT model 31, 32, whose ground state can be represented as a projected BCS state, by replacing $g_{ij} = 1/(z_{i} - z_{j})$ in (11) with $g_{ij} = 1$ 12. For the Ising disordered phase, the ground state of the spin chain is dimerized 21 and hence the valence bonds are short ranged. In this case, a proper Cooper pair wave function for the projected BCS state can be chosen as $g_{ij} \sim \exp(-|z_{i} - z_{j}|/\xi)$, where $\xi$ is the length scale of the valence bonds. In the extreme case, a Cooper pair wave function that is nonvanishing only between neighboring sites yields a Majumdar-Ghosh-like state, corresponding to perfect dimerization. These results imply that both Ising ordered and disordered phases close to SO(n)1 criticality are well described by projected BCS states with properly chosen $g_{ij}$. Indeed, for $n = 3$, it was shown 12 that the projected BCS states with Cooper pair wave functions generated from Kitaev’s Majorana chains 33 are good variational wave functions for the Haldane (Ising ordered) and the dimerized (Ising disordered) phases.

2D chiral spin liquids.— After establishing the relevance of projected BCS states 44 for SO(n)1 criticality in 1D, we move on and discuss their properties in a 2D square lattice, where the $z$’s in (11) are now complex coordinates of the lattice sites. In an analogy with fractional quantum Hall (FQH) states constructed by conformal blocks of their gapless edge CFTs 34, 35, the chiral correlator 36 from the SO(n) WZW model (n massless Majorana fermions) yields chiral spin liquids, which break time reversal symmetry and are spin counterparts of FQH states 36. From the projected BCS form 11, the Cooper pair wave function $1/(z_{i} - z_{j})$ now corresponds to the topological phase of $p+ip$ superconductors 37, supporting chiral gapless Majorana edge modes, which justifies the above bulk-edge correspondence. Below we focus on the anyonic quasiparticle excitations in these 2D states, which have intriguing properties depending on $n \mod 16$, i.e., a 16-fold way.

For odd $n$, the quasiparticles built upon the SO(n) states support non-Abelian statistics. Let us adapt the CFT approach of creating quasihole excitations in FQH states 34 to our spin system. For odd $n$, the SO(n) WZW model has three primary fields: identity field $I$, vector field $v$, and spinor field $s$. Following the CFT approach, creating quasiparticles in the SO(n) state is achieved by adding spinor fields $s$ in the chiral correlator 39. Then, the statistics of quasiparticles are encoded in the fusion rules of the primary fields. In fact, the spinor fields have a nontrivial fusion rule $s \times s = I + v$, together with $s \times v = s$ and $v \times v = I$. These fusion rules resemble those in Ising CFT ($\sigma \times \sigma = I + \varepsilon$, $\sigma \times \varepsilon = \sigma$, and $\varepsilon \times \varepsilon = I$), which are responsible for the non-Abelian statistics of Ising anyons 38. Indeed, the Majorana free field representation of SO(n)1 WZW model allows us to identify the spinor fields $s$ with conformal weight $h_{s} = n/16$ as a product of $n$ Ising $\sigma$ fields ($h_{\sigma} = 1/16$). Thus, we conclude that the SO(n) states support non-Abelian Ising anyons for odd $n$. Note that the case with $n = 3$ recovers the physics of the Moore-Read states 34, 39, while for odd $n \geq 5$ they are natural generalizations of the Moore-Read states.

Now we show that the SO(n) states only support Abelian anyons for even $n$. This subtle difference roots in the fusion rules of the SO(n)1 primary fields. In contrast to odd $n$ case, the SO(n)1 WZW model with even $n$ has two spinor primary fields $s_{+}$ and $s_{-}$ with conformal weight $h_{s_{+}} = h_{s_{-}} = n/16$ 12, apart from the usual identity and vector fields. The fusion rules of spinor and vector fields are $s_{+} \times v = s_{-}$ and $s_{-} \times v = s_{+}$. Depending on the parity of $n/2$, the fusion rules involving two spinor fields are $s_{+} \times s_{+} = s_{-} \times s_{-} = I$, $s_{+} \times s_{-} = v$ for even $n/2$ and $s_{+} \times s_{+} = s_{-} \times s_{-} = v$, $s_{+} \times s_{-} = I$ for odd $n/2$ 40. However, due to the absence of multiplicity in the fusion outcome, only Abelian anyons can exist in the SO(n) states with even $n$.

More precisely, the anyonic properties of the SO(n) states depend on $n \mod 16$ (16-fold way) 41. The topological spin of the quasiparticles generated by SO(n) spinor primary fields is $\theta_{s} = e^{i2\pi h_{s}} = e^{i\pi n/8}$ (for both odd and even $n$), which is a clear signature of the 16-fold way. For example, the quasiparticles $s_{+}$ and $s_{-}$ for SO(8) have $\theta_{s_{+}} = \theta_{s_{-}} = -1$, which are both fermions. Actually, this 16-fold way of the anyonic properties has been analyzed in detail by Kitaev. In Ref. 38, he considered a theory with $Z_{2}$ vortices and free Majorana fermions whose energy band has Chern number $\nu$ and showed that the anyonic properties of the unpaired Majorana modes in the vortex core depends on $\nu \mod 16$. Thus, our present work shows that the SO(n)1 CFT is responsible for this 16-fold way and provides a class of
microscopic Hamiltonians which realize this interesting physics.

**Conclusion and perspective.** To conclude, we have proposed a class of projected BCS states and derived their parent Hamiltonians. These states also have an infinite MPS form generated by chiral correlators of Majorana fermions. In 1D, they can be viewed as SO(n) generalizations of Haldane-Shastry models and capture the physics of the SO(n)_1 WZW model. These results indicate that modified projected BCS states are good variational ansatz for describing Ising ordered and disordered phases close to SO(n)_1 criticality. In 2D, the SO(n) states are chiral spin liquid states, which support non-Abelian Ising anyons for odd n and Abelian anyons for even n. An open question that deserves further investigation is whether these 2D chiral spin liquids are relevant for physical models and materials.

Moreover, our 2D toy models may also shed light on another interesting open question: Can p+i\,p pairing states arise after doping these antiferromagnets?

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Supplemental material

Equivalence of the projected BCS state and the infinite MPS

In this Section, we prove the equivalence of the projected BCS state and the infinite MPS.

Projected BCS state

Let us first expand the projected BCS state

\[ |\Psi\rangle = \mathcal{P}_G \exp \left( \sum_{i<j} \frac{1}{z_i - z_j} \sum_{a=1}^{n} c_{i,a} \dagger c_{j,a} \right) |0\rangle \]

\[ = \mathcal{P}_G \prod_{a=1}^{n} \prod_{i<j} \left( 1 + \frac{1}{z_i - z_j} c_{i,a} \dagger c_{j,a} \right) |0\rangle \]

\[ = \mathcal{P}_G \prod_{a=1}^{n} \left[ \sum_{N_a=0}^{N} \sum_{\text{all allowed } x_1^{(a)}, \ldots, x_N^{(a)}} \text{Pf}_a \left( \frac{1}{z_i - z_j} \right) c_{x_1^{(a)},a} \dagger \cdots c_{x_N^{(a)},a} \right] |0\rangle \]

where \( N_a = (a = 1, \ldots, n) \) is the number of \( c_a^\dagger \) fermions (i.e. \( |n^a\rangle \) vector in the configuration), \( x_1^{(a)} < \cdots < x_n^{(a)} \) are the positions of \( c_a^\dagger \) fermions in the lattice, and the Pfaffian factor \( \text{Pf}_a \left( \frac{1}{z_i - z_j} \right) \) is restricted to the positions for the \( c_a^\dagger \) fermions.

The next step is to implement the Gutzwiller projection. Note that the Gutzwiller projector \( \mathcal{P}_G \) requires single occupancy. Thus, the positions of fermions \( x_1^{(1)} < \cdots < x_N^{(1)}, x_2^{(2)} < \cdots < x_N^{(2)}, \ldots, x_1^{(n)} < \cdots < x_N^{(n)} \) must be all different from each other, so that each site has exactly one fermion. As a result, we have \( \sum_{a=1}^{n} N_a = N \). After implementing the Gutzwiller projector, we obtain

\[ |\Psi\rangle = \sum_{N_1, N_2, \ldots, N_N = 0}^{n} \sum_{\text{all allowed } x_1^{(a)}, \ldots, x_N^{(a)}} \prod_{a=1}^{n} \text{Pf}_a \left( \frac{1}{z_i - z_j} \right) \]

\[ \times (c_{x_1^{(1)},1} \dagger \cdots c_{x_N^{(1)},1} \dagger) (c_{x_1^{(2)},2} \dagger \cdots c_{x_N^{(2)},2} \dagger) \cdots (c_{x_1^{(n)},n} \dagger \cdots c_{x_N^{(n)},n} \dagger) |0\rangle \]

The final step is to rearrange the positions of fermionic operators so that they can be identified as a spin state. This rearrangement only results in a sign, depending on the positions of fermions

\[ |\Psi\rangle = \sum_{N_1, N_2, \ldots, N_N = 0}^{n} \sum_{\text{all allowed } x_1^{(a)}, \ldots, x_N^{(a)}} \prod_{a=1}^{n} \text{Pf}_a \left( \frac{1}{z_i - z_j} \right) \]

\[ \times \text{sgn}(x_1^{(1)}, \ldots, x_N^{(1)}, x_1^{(2)}, \ldots, x_N^{(2)}, \ldots, x_1^{(n)}, \ldots, x_N^{(n)}) |x_1^{(1)}, \ldots, x_N^{(1)}, x_1^{(2)}, \ldots, x_N^{(2)}, \ldots, x_1^{(n)}, \ldots, x_N^{(n)}\rangle \]

where \( |x_1^{(1)}, \ldots, x_N^{(1)}, x_1^{(2)}, \ldots, x_N^{(2)}, \ldots, x_1^{(n)}, \ldots, x_N^{(n)}\rangle \) is a spin configuration labeled by the positions of the vector \( |n^a\rangle \) and \( \text{sgn}(x_1^{(1)}, \ldots, x_N^{(1)}, x_1^{(2)}, \ldots, x_N^{(2)}, \ldots, x_1^{(n)}, \ldots, x_N^{(n)}) \) is the signature of permutation due to the sign factor coming from fermionic anticommutation relations.

Thus, the projected BCS wave function can be written as

\[ \Psi(|x_1^{(1)}>, |x_2^{(2)}>, \ldots, |x_n^{(n)}>) = \text{sgn}(x_1^{(1)}, \ldots, x_N^{(1)}, x_1^{(2)}, \ldots, x_N^{(2)}, \ldots, x_1^{(n)}, \ldots, x_N^{(n)}) \prod_{a=1}^{n} \text{Pf}_a \left( \frac{1}{z_i - z_j} \right) \]

(1)
where \( \{x^{(a)}\} \) is the set of positions satisfying \( x_1^{(a)} < \cdots < x_{N_a}^{(a)} \) (\( N_a \) even and \( \sum_{a=1}^{n} N_a = N \)).

### Infinite MPS

Let us now consider the infinite MPS

\[
|\Psi\rangle = \sum_{a_1, \ldots, a_N=1}^{n} \Psi(a_1, \ldots, a_N)|n^{a_1}, n^{a_2}, \ldots, n^{a_N}\rangle
\]

where \( \Psi(a_1, \ldots, a_N) \) are given by the chiral correlators of Majorana fermion fields \( \chi^a (a = 1, \ldots, n) \)

\[
\Psi(a_1, \ldots, a_N) = \langle \chi^{a_1}(z_1) \chi^{a_2}(z_2) \cdots \chi^{a_N}(z_N) \rangle
\]

To evaluate \( \Psi(a_1, \ldots, a_N) \), we use the two-point correlator of Majorana fermions

\[
\langle \chi^a(z) \chi^b(w) \rangle = \frac{\delta_{ab}}{z-w}
\]

The multipoint correlators of Majorana fermions are obtained by Wick’s theorem

\[
\langle \chi^{a_1}(z_1) \chi^{a_2}(z_2) \cdots \chi^{a_N}(z_N) \rangle = \begin{cases} 
\text{Pf}_a \left( \frac{1}{z_i-z_j} \right) & \text{if } N_a \text{ even} \\
0 & \text{if } N_a \text{ odd}
\end{cases}
\]

Therefore, to obtain a nonvanishing \( \Psi(a_1, \ldots, a_N) \), we must have even \( N_a \). Additionally, \( N_a \), the number of vectors \( |n^a\rangle \) in the spin configuration, must also be even for all \( a = 1, \ldots, n \).

To compare with the projected BCS wave function, let us evaluate the superposition coefficient of the infinite MPS for a spin configuration, which has \( N_a \) vector \( |n^a\rangle \) at positions \( x_1^{(a)} < \cdots < x_{N_a}^{(a)} \) (\( N_a \) even and \( \sum_{a=1}^{n} N_a = N \)). Taking into account the anticommuting nature of Majorana fermion fields, we first pick up the vectors \( |n^b\rangle \) in the spin configuration and rewrite the infinite MPS as

\[
\Psi(\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n)}\}) = \text{sgn}(x_1^{(1)}, \ldots, x_{N_1}^{(1)}, y_1, \ldots, y_{N-L_1}^{(1)})(\chi^{a_1=1}(z^{(1)}_{x_{1}^{(1)}}) \chi^{a_2=1}(z^{(1)}_{x_{2}^{(2)}}) \cdots \chi^{a_N=1}(z^{(1)}_{x_{N_1}^{(N_1)}}))
\]

\[
\times \langle \chi^b(z_{y_1}) \chi^b(z_{y_2}) \cdots \chi^b(z_{y_{N-N_1}}) \rangle \quad (b \neq 1)
\]

\[
= \text{sgn}(x_1^{(1)}, \ldots, x_{N_1}^{(1)}, y_1, \ldots, y_{N-L_1}^{(1)}) \text{Pf}_{a=1} \left( \frac{1}{z_i-z_j} \right)
\]

\[
\times \langle \chi^b(z_{y_1}) \chi^b(z_{y_2}) \cdots \chi^b(z_{y_{N-N_1}}) \rangle
\]

where the positions \( y_1 < \cdots < y_{N-N_1} \) correspond to the vectors \( |n^b\rangle \) with \( b \neq 1 \). The above steps can be repeated from \( b = 2 \) to \( n \). In the end, we obtain

\[
\Psi(\{x^{(1)}\}, \{x^{(2)}\}, \ldots, \{x^{(n)}\}) = \text{sgn}(x_1^{(1)}, \ldots, x_{N_1}^{(1)}, x_1^{(2)}, \ldots, x_{N_2}^{(2)}, \ldots, x_1^{(n)}, \ldots, x_{N_n}^{(n)}) \prod_{a=1}^{n} \text{Pf}_a \left( \frac{1}{z_i-z_j} \right)
\]

Comparing with Eq. (1), we conclude that the infinite MPS and the projected BCS state are equivalent.

### Derivation of the parent Hamiltonian

In this Section, we derive the parent Hamiltonian for the infinite MPS.

### Brief summary of the \( SO(n) \) WZW model

For infinite MPS associated to WZW models, the derivation of the parent Hamiltonian relies on the existence of null vectors in the representation spaces of Kac-Moody algebra. For \( SO(n) \) WZW model, the Kac-Moody algebra is defined by

\[
[J_m^{ab}, J_n^{cd}] = i f^{ab, cd, ef} J_{n+m}^{ef} + n \delta_{ab, cd} \delta_{n+m, 0} \quad m, n \in \mathbb{Z}
\]
where repeated indices are summed over and the SO($n$) structure constant $f^{ab,cd,ef}$ is given by

$$f^{ab,cd,ef} = \delta_{ad} \delta_{be} \delta_{cf} + \delta_{bc} \delta_{ae} \delta_{df} - \delta_{ac} \delta_{be} \delta_{df} - \delta_{bd} \delta_{ae} \delta_{cf}$$

For odd $n$ ($n \geq 3$), the SO($n$)$_1$ WZW model has three primary fields respectively in singlet (denoted by $I$), vector ($v$) and spinor representation ($s$), whose conformal weights are $h_I = 0$, $h_v = 1/2$ and $h_s = n/16$, respectively. For even $n$ ($n \geq 4$), apart from the primary fields in singlet and vector representations ($h_I = 0$ and $h_v = 1/2$), the SO($n$)$_1$ WZW model has two primary fields in spinor representations (denoted by $s_+$ and $s_-$), whose conformal weights are $h_{s_+} = h_{s_-} = n/16$. The SO($n$)$_1$ WZW model has central charge $c = n/2$ and can be constructed by combining $n$ Ising models ($c = n \times \frac{1}{2}$).

For both odd and even $n$, the primary fields in the vector representation have conformal weight $h_v = 1/2$ and are naturally interpreted as Majorana fermions, which are the key ingredients for us to construct the infinite MPS.

For each Majorana fermion $\chi^a$ ($a = 1, \ldots, n$), a primary state $|\chi^a\rangle$ can be defined by

$$|\chi^a\rangle = \chi^a |0\rangle$$

where $|0\rangle$ is the vacuum of the WZW model and satisfies $J^ab |0\rangle = 0$. When acting on the Kac-Moody currents, the primary states satisfy

$$J^ab |\chi^c\rangle = -\sum_{d=1}^n (L^ab)_{cd} |\chi^d\rangle$$

where $L^ab$ are given by

$$(L^ab)_{cd} = i(\delta_{ac} \delta_{bd} - \delta_{bc} \delta_{ad})$$

Note that $L^ab$ form a closed SO($n$) algebra

$$[L^ab, L^{cd}] = i(\delta_{ad} L^{bc} + \delta_{bc} L^{ad} - \delta_{ac} L^{bd} - \delta_{bd} L^{ac}) = i f^{ab,cd,ef} L^{ef}$$

**Null vectors and parent Hamiltonian**

To derive the parent Hamiltonian, one has to find the null vectors in the SO($n$)$_1$ Kac-Moody algebra. The null vectors are descendant states satisfying

$$J^ab |\phi\rangle = 0 \quad (n > 0)$$

For our purpose, we look for null vectors with the following form:

$$|\phi^d\rangle = \sum_{a<b,c} W^d_{abc} J^-1^ab |\chi^c\rangle$$

where $W^d_{abc}$ are the coefficients that have to be determined. They satisfy the orthonormal condition

$$\sum_{a<b,c} (W^d_{abc})^* W^d_{abc} = \delta_d^*d$$

In general, the tensor $W^d_{abc}$ corresponds to a Clebsch-Gordan decomposition. For the SU(2)$_k$ WZW model, the SU(2) Clebsch-Gordan coefficients are known [9]. However, we are not aware of a closed form for the SO($n$) Clebsch-Gordan coefficients. To overcome this difficulty, let us consider the norm of $|\phi^d\rangle$

$$\langle \phi^d | \phi^d \rangle = \sum_{a'<b',c'} \sum_{a<b,c} (W^d_{a'b'c'})^* W^d_{abc} |\chi^{c'}\rangle \langle J^a'b' J^-1^ab | \chi^c\rangle = \sum_{a'<b',c'} \sum_{a<b,c} W^d_{a'b'c'} M_{a'b'c',abc} W^d_{abc} = (W^d)^\dagger MW^d$$
where $W^d$ is viewed as a column vector and $M$ is a matrix defined by

$$M_{a'b',ab',c} = \langle \chi^c | J_1^{a'b'} J_2^{ab'} | \chi^c \rangle$$

If $|\phi^d\rangle$ is a null state, $\langle \phi^d | \phi^d \rangle = (W^d)^\dagger MW^d = 0$. Since $M$ comes from the norm of two descendent states, it is a positive-semidefinite matrix satisfying $(W^d)^\dagger MW^d \geq 0$. Therefore, identifying the orthonormal vectors $W^d$ that belong to the kernel of $M$ gives us all null states $|\phi^d\rangle$. For our SO($n_1$) WZW model, we can write down the explicit form of $M$

$$M_{a'b',ab',c} = \langle \chi^c | J_1^{a'b'} J_2^{ab'} | \chi^c \rangle = \langle \chi^c | J_1^{a'b'} J_2^{ab'} | \chi^c \rangle - \sum_{ef} f^{a'b',ab,ef} J_0^{ef} + \delta_{a'b',ab} \langle \chi^c \rangle$$

$$= \langle \chi^c | \left( -i \sum_{ef} f^{a'b',ab,ef} \sum_g (L_{ef}^g)_{cg} | \chi^g \rangle + \delta_{a'b',ab} \langle \chi^c \rangle \right)$$

$$= -i \sum_{ef} f^{a'b',ab,ef} (L_{ef}^{ab})_{cc'} + \delta_{a'b',ab} \delta_{c',c}$$

where we used Kac-Moody algebra [3] and the properties of the primary state [4]. In this way, the null vectors for the SO($n_1$) WZW model are obtained.

Let us mention that the above approach is a systematic way of finding null vectors and can be easily generalized to other WZW models. The role of the positive-semidefinite matrix $M$ is similar to the Gram matrix for defining the Kac determinant [12] in conformal field theory.

After obtaining $W^d_{abc}$ for all the null vectors, we define the following $K$ tensor [9]

$$K_{a'b',c'}^{ab,c} = \sum_d (W^d_{a'b',c'})^* W^d_{abc} \quad (a < b \text{ and } a' < b')$$

Let us write these $K$ tensors as $n \times n$ matrices, $K_{a'b',c'}^{ab,c} = (K_{a'b',c'}^{ab,c})_{i,j}$. Then, $K_{a'b'}^{ab}$ have the following compact form

$$K_{a'b'}^{ab} = \frac{2}{3} \delta_{a,a'} - \frac{n + 2}{6(n - 1)} i f^{ab,a'b',cd} L^{cd} + \frac{n - 4}{6(n - 1)} (L^{ab} L^{a'b'} + L^{a'b'} L^{ab})$$

Following Ref. [9], we define an operator $\Lambda_i^{ab}$ ($1 \leq a < b \leq n$)

$$\Lambda_i^{ab} = \sum_{j=1}^N \sum_{c<d} \langle K^{(i)} \rangle_{cd} L_{ij}^{cd}$$

$$= \sum_{j=1}^N \sum_{c<d} \left[ \frac{2}{3} L_{ij}^{ab} - \frac{n + 2}{6(n - 1)} i f^{ab,cd,ef} L_i^{ef} L_{ij}^{cd} + \frac{n - 4}{6(n - 1)} (L_i^{ab} L_{ij}^{cd} + L_{ij}^{cd} L_i^{ab}) \right]$$

$$= \sum_{j=1}^N \sum_{c<d} \left[ \frac{2}{3} L_{ij}^{ab} - \frac{1}{n - 1} L_i^{ab} (\tilde{L}_i \cdot \tilde{L}_j) + \frac{1}{3} (\tilde{L}_i \cdot \tilde{L}_j) L_i^{ab} \right]$$

where $w_{ij} \equiv (z_i + z_j)/(z_i - z_j)$ and $\tilde{L}_i \cdot \tilde{L}_j \equiv \sum_{a<b} L_i^{ab} L_j^{ab}$. These operators annihilate the infinite MPS, $\Lambda_i^{ab} |\Psi\rangle = 0 \forall i, a, b$. Moreover, the infinite MPS is an SO($n$) singlet and therefore $\sum_i L_i^{ab} |\Psi\rangle = 0 \forall a, b$. Then, an SO($n$) symmetric parent Hamiltonian can be defined by

$$H = \sum_{i,a<b} (\Lambda_i^{ab})^\dagger \Lambda_i^{ab} + J \sum_{a<b} (\sum_i L_i^{ab})^2 + E_0 \quad (J \geq 0)$$

whose ground state is the infinite MPS with energy $E_0$. 
In 1D, we use \( z_j = \exp(i \frac{2\pi}{9}) \) to ensure translational invariance. Choosing \( J = 2(N - 2)/3 \) and \( E_0 = -2(n - 1)N(N^2 - 4)/9 \), we arrive at the following explicit form of \( H \):

\[
H = - \sum_{i \neq j} w_{ij} \left[ \frac{n+2}{3} (\vec{L}_i \cdot \vec{L}_j) + \frac{n-4}{3(n-1)} (\vec{L}_i \cdot \vec{L}_j)^2 \right] - \frac{n-4}{3(n-1)} \sum_{i \neq j \neq k} w_{ij} w_{jk} (\vec{L}_i \cdot \vec{L}_j)(\vec{L}_i \cdot \vec{L}_k).
\]

(5)

To obtain the above form, the following identities are quite useful

\[
\sum_{a < b} L_i^{ab} (\vec{L}_i \cdot \vec{L}_j) L_i^{ab} = \vec{L}_i \cdot \vec{L}_j \quad (i \neq j)
\]

\[
\sum_{a < b} L_i^{ab} (\vec{L}_i \cdot \vec{L}_j)^2 L_i^{ab} = 2(n-1) - (n-2)(\vec{L}_i \cdot \vec{L}_j) - (\vec{L}_i \cdot \vec{L}_j)^2 \quad (i \neq j)
\]

\[
\sum_{a < b} L_i^{ab} (\vec{L}_i \cdot \vec{L}_j)(\vec{L}_i \cdot \vec{L}_k) L_i^{ab} = 2(\vec{L}_j \cdot \vec{L}_k) - (\vec{L}_i \cdot \vec{L}_k)(\vec{L}_i \cdot \vec{L}_j) \quad (i \neq j \neq k)
\]

Jastrow and Pfaffian wave functions in Cartan basis

In this Section, we derive the explicit Jastrow and Pfaffian forms of the wave functions in Cartan basis.

Cartan basis

Let us first define the Cartan basis. The \( \text{SO}(n) \) algebra is defined by

\[
[L^{ab}, L^{cd}] = i(\delta_{ad} L^{bc} + \delta_{bc} L^{ad} - \delta_{ac} L^{bd} - \delta_{bd} L^{ac})
\]

For \( n = 2l \) and \( 2l + 1 \), we can choose at most \( l \) (rank of the algebra) mutually commuting generators as \( L^{12}, L^{34}, \ldots, L^{2l-1,2l} \). In the vector basis, the \( \text{SO}(n) \) generators are defined by \( L^{ab} = i(\langle n^b \rangle \langle n^a \rangle - \langle n^a \rangle \langle n^b \rangle) \) \((1 \leq a < b \leq n)\). Diagonalizing the Cartan generators gives us the following Cartan basis:

\[
|1,0,\ldots,0\rangle = \frac{1}{\sqrt{2}}(|n^2\rangle + i|n^1\rangle)
\]

\[
|-1,0,\ldots,0\rangle = \frac{1}{\sqrt{2}}(|n^2\rangle - i|n^1\rangle)
\]

\[
|0,1,0,\ldots,0\rangle = \frac{1}{\sqrt{2}}(|n^4\rangle + i|n^3\rangle)
\]

\[
|0,-1,0,\ldots,0\rangle = \frac{1}{\sqrt{2}}(|n^4\rangle - i|n^3\rangle)
\]

\[
\vdots
\]

\[
|0,0,\ldots,1\rangle = \frac{1}{\sqrt{2}}(|n^{2l}\rangle + i|n^{2l-1}\rangle)
\]

\[
|0,0,\ldots,-1\rangle = \frac{1}{\sqrt{2}}(|n^{2l}\rangle - i|n^{2l-1}\rangle)
\]

For \( \text{SO}(2l) \), the above basis is already complete. For \( \text{SO}(2l + 1) \), we have an additional vector \(|n^{2l+1}\rangle\), which is annihilated by all Cartan generators. Thus, we have the following extra vector for \( \text{SO}(2l + 1) \):

\[
|0,0,\ldots,0\rangle = |n^{2l+1}\rangle
\]

Thus, the Cartan basis for \( \text{SO}(2l) \) and \( \text{SO}(2l + 1) \) can be compactly written as

\[
|0,\ldots,m_\alpha = \pm 1,\ldots,0\rangle = \frac{1}{\sqrt{2}}(|n^{2\alpha}\rangle \pm i|n^{2\alpha-1}\rangle) \quad (\alpha = 1,\ldots,l)
\]

\[
|0,0,\ldots,0\rangle = |n^{2l+1}\rangle
\]

Note that \( m_\alpha \) is the eigenvalue of the Cartan generator \( L^{2\alpha-1,2\alpha} \).
After changing the basis, the SO(2) valence-bond singlet operator in the projected BCS state is rewritten as

$$\sum_{a=1}^{2l} c_{i,a}^\dagger c_{j,a}^\dagger = \sum_{a=1}^{l} (c_{i,m_a=1}^\dagger c_{j,m_a=-1} + c_{i,m_a=-1}^\dagger c_{j,m_a=1})$$

Using the above form, the SO(2l) projected BCS state is rewritten as

$$|\Psi_{SO(2l)}\rangle = P_G \exp \left( \sum_{i<j} \frac{1}{z_i - z_j} \sum_{a=1}^{2l} c_{i,a}^\dagger c_{j,a}^\dagger \right) |0\rangle$$

$$= P_G \exp \left( \sum_{i \neq j} \frac{1}{z_i - z_j} \sum_{a=1}^{l} c_{i,m_a=1}^\dagger c_{j,m_a=-1}^\dagger \right) |0\rangle$$

$$= P_G \prod_{a=1}^{l} \prod_{i \neq j} \left( 1 + \frac{1}{z_i - z_j} c_{i,m_a=1}^\dagger c_{j,m_a=-1}^\dagger \right) |0\rangle$$

$$= P_G \prod_{a=1}^{l} \left[ \sum_{N/2}^{N/2} \sum_{N_a=0}^{\infty} \sum_{p_1^{(a)} < \cdots < p_{N_a}^{(a)}} \sum_{q_1^{(a)} < \cdots < q_{N_a}^{(a)}} \det \left( \frac{1}{z_i - z_j} \right) \prod_{p_1^{(a)},m=1}^{p_{N_a}^{(a)},m_a=-1} \prod_{q_1^{(a)},m=1}^{q_{N_a}^{(a)},m_a=-1} \right] |0\rangle$$

where \( \det \left( \frac{1}{z_i - z_j} \right)_{(p_1^{(a)}, \ldots, p_{N_a}^{(a)}), (q_1^{(a)}, \ldots, q_{N_a}^{(a)})} \) is the determinant of the \( N_a \times N_a \) Cauchy matrix restricted to the positions of \( c_{m_a=1}^\dagger \) and \( c_{m_a=-1}^\dagger \) fermions. The following useful identity reduces the Cauchy determinant to a product of Jastrow factors:

$$\det \left( \frac{1}{z_i - z_j} \right)_{(p_1^{(a)}, \ldots, p_{N_a}^{(a)}), (q_1^{(a)}, \ldots, q_{N_a}^{(a)})} = (-1)^{\frac{1}{2} N_a(N_a-1)} \prod_{1 \leq i < j \leq N_a} \frac{(z_{p_1^{(a)}} - z_{p_j^{(a)}})(z_{q_1^{(a)}} - z_{q_j^{(a)}})}{\prod_{1 \leq i < j \leq N_a} (z_{p_i^{(a)}} - z_{q_j^{(a)}})}$$

Note that the sign factor in the Cauchy determinant can be absorbed by rearranging the fermionic operators

$$c_{p_1^{(a)},m=1}^\dagger c_{q_1^{(a)},m=-1}^\dagger \cdots c_{p_{N_a}^{(a)},m=-1}^\dagger c_{q_{N_a}^{(a)},m=1}^\dagger (-1)^{\frac{1}{2} N_a(N_a-1)}$$

$$= (-1)^{\frac{1}{2} N_a(N_a-1)} (c_{p_1^{(a)},m=1}^\dagger \cdots c_{p_{N_a}^{(a)},m=-1}^\dagger) (c_{q_1^{(a)},m=-1}^\dagger \cdots c_{q_{N_a}^{(a)},m=1}^\dagger)$$
Therefore, we obtain

\[ |\Psi_{\text{SO}(2l)}\rangle = P_G \prod_{\alpha=1}^{l} \left[ \sum_{N_\alpha=0}^{N/2} \sum_{p_1^{(\alpha)} < \ldots < p_{N_\alpha}^{(\alpha)}} \sum_{q_1^{(\alpha)} < \ldots < q_{N_\alpha}^{(\alpha)}} \prod_{1 \leq i < j \leq N_\alpha} (z_{p_i^{(\alpha)}} - z_{p_j^{(\alpha)}}) (z_{q_i^{(\alpha)}} - z_{q_j^{(\alpha)}}) \prod_{1 \leq i < j \leq N_\alpha} \frac{(c_{p_i^{(\alpha)},m_\alpha}^\dagger, c_{p_j^{(\alpha)},m_\alpha}^\dagger, \ldots, c_{q_i^{(\alpha)},m_\alpha}^\dagger, c_{q_j^{(\alpha)},m_\alpha}^\dagger)}{c_{p_i^{(\alpha)},m_\alpha}, c_{p_j^{(\alpha)},m_\alpha}, \ldots, c_{q_i^{(\alpha)},m_\alpha}, c_{q_j^{(\alpha)},m_\alpha}} \right] |0\rangle \]

In the next step, we collect the positions \( p_1^{(\alpha)} < \ldots < p_{N_\alpha}^{(\alpha)} \) and \( q_1^{(\alpha)} < \ldots < q_{N_\alpha}^{(\alpha)} \) into a single set \( \{x^{(\alpha)}\} \) with \( x_1^{(\alpha)} < \ldots < x_{2N_\alpha}^{(\alpha)} \). Then, the Jastrow factors can be written as

\[ \frac{\prod_{1 \leq i < j \leq N_\alpha} (z_{p_i^{(\alpha)}} - z_{p_j^{(\alpha)}}) (z_{q_i^{(\alpha)}} - z_{q_j^{(\alpha)}})}{\prod_{1 \leq i < j \leq N_\alpha} (z_{x_i^{(\alpha)}} - z_{x_j^{(\alpha)}})^{m_{\alpha,i}^{\prime} m_{\alpha,j}}} \rightarrow \prod_{1 \leq i < j \leq 2N_\alpha} (z_{x_i^{(\alpha)}} - z_{x_j^{(\alpha)}})^{m_{\alpha,i}^{\prime} m_{\alpha,j}} \]

up to a sign factor. However, the sign factor can again be compensated by rearranging the fermionic operators to the correct order according to \( x_1^{(\alpha)} < \ldots < x_{2N_\alpha}^{(\alpha)} \).

The last step is to implement the Gutzwiller projection and switch to the spin basis. As a result, we obtain the \( \text{SO}(2l) \) wave function in Cartan basis

\[ \Psi(\{m\}) = \rho_m \prod_{\alpha=1}^{l} \prod_{i<j} (z_i - z_j)^{m_{\alpha,i}^{\prime} m_{\alpha,j}} \tag{6} \]

where \( \rho_m = \text{sgn}(x_1^{(1)}, \ldots, x_N^{(1)}, \ldots, x_1^{(l)}, \ldots, x_N^{(l)}) \) if \( \sum m_{\alpha,i} = 0 \ \forall \alpha \) and \( \rho_m = 0 \) otherwise.

For \( n = 4 \), one can further simplify Eq. (6) and show that it is equivalent to a product of two spin-1/2 Haldane-Shastry states, if the four vectors are interpreted as two spin-1/2 states.

**Odd** \( n = 2l + 1 \)

Comparing to \( \text{SO}(2l) \), we have an additional slave-fermion operators in the Cartan basis for \( \text{SO}(2l + 1) \)

\[ c_{m=0}^\dagger = c_{2l+1}^\dagger \]

Then, the valence-bond operator for \( \text{SO}(2l + 1) \) is expressed as

\[ \sum_{a=1}^{2l+1} c_{i,a}^\dagger c_{j,a}^\dagger = c_{i,m=0}^\dagger c_{j,m=0} + \sum_{a=1}^{l} (c_{i,m=1}^\dagger c_{j,m=-1}^\dagger + c_{i,m=-1}^\dagger c_{j,m=1}^\dagger) \]

The expansion of the \( \text{SO}(2l + 1) \) projected BCS state is given by

\[ |\Psi_{\text{SO}(2l+1)}\rangle = P_G \exp \left( \sum_{i<j} \frac{1}{z_i - z_j} \sum_{a=1}^{2l+1} c_{i,a}^\dagger c_{j,a}^\dagger \right) |0\rangle \]

\[ = P_G \exp \left( \sum_{i<j} \frac{1}{z_i - z_j} c_{i,m=0}^\dagger c_{j,m=0} \right) \exp \left( \sum_{i \neq j} \frac{1}{z_i - z_j} \sum_{a=1}^{l} c_{i,m=1}^\dagger c_{j,m=-1}^\dagger \right) |0\rangle \]

\[ = P_G \left[ \sum_{N_0=0}^{N} \prod_{x_i^{(0)} \neq x_j^{(0)} \in x^{(0)}_{N_0}} \text{Pf}_0 \left( \frac{1}{z_i - z_j} c_{x_i^{(0)},m=0}^\dagger \cdots c_{x_j^{(0)},m=0}^\dagger \right) \right] \times \exp \left( \sum_{i \neq j} \frac{1}{z_i - z_j} \sum_{a=1}^{l} c_{i,m=1}^\dagger c_{j,m=-1}^\dagger \right) |0\rangle \]
where the positions of the extra fermion $c_{m=0}^\dagger$ are labeled by $x_1^{(0)} < \cdots < x_N^{(0)}$. The rest of the calculation is similar to the SO$(2l)$ case, except for the presence of a Pfaffian factor due to the extra fermionic mode $c_{m=0}^\dagger$. After some algebra, we obtain the SO$(2l+1)$ wave function in Cartan basis

$$
\Psi(\{m\}) = \rho_m \text{Pf}_0 \left( \frac{1}{z_i - z_j} \right) \prod_{\alpha=1}^{l} \prod_{i<j} (z_i - z_j)^{m_{\alpha,i} m_{\alpha,j}}
$$

(7)

where $\rho_m = \text{sgn}(x_1^{(0)}, \ldots, x_N^{(0)}, x_1^{(1)}, \ldots, x_1^{(l)}, \ldots, x_1^{(l)})$ if $\sum_i m_{\alpha,i} = 0 \ \forall \alpha$ and $\rho_m = 0$ otherwise.

For $n = 3$, the signature of the permutation in $\rho_m$ is reduced to the Marshall sign in the spin-1 Haldane-Shastry state [9].