Inequalities for the fractional convolution operator on differential forms

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Abstract
The purpose of this paper is to derive some Coifman type inequalities for the fractional convolution operator applied to differential forms. The Lipschitz norm and BMO norm estimates for this integral type operator acting on differential forms are also obtained.

Keywords: Fractional convolution operator; Coifman type inequality; BMO norm; Lipschitz norm; Differential form

1 Introduction
As a generalization of functions, the differential form can be regarded as a special kind of vector-valued function. So, if some operators in function spaces are generalized to that in differential forms, similar properties could be obtained as in the function space. In recent years, the research on the generalization of operators from functional spaces to differential forms seems to become a new highlight in the inequalities with differential forms, see [1–6]. In this paper, we mainly consider the following convolution type fractional integrals operator acting on differential forms and develop some norm inequalities for the fractional convolution operator. Given a nonnegative, locally integrable function $K_\alpha$ and $h_\ell(y)$ is a bounded function with a compactly supported set on $\mathbb{R}^n$, write $h_\ell(y) \in L^\infty_\ell$. The fractional convolution operator $F_\alpha$ is defined by a convolution integral

$$F_\alpha h(x) = \sum_I \left( \int_{\mathbb{R}^n} K_\alpha(x - y) h_\ell(y) \, dy \right) \, dx_I,$$

provided this integral exists for almost all $\mathbb{R}^n$, where $h(x) = \sum_I h_\ell(y) \, dx_I$ is a $\ell$-form defined on $\mathbb{R}^n$, the summation is over all ordered $\ell$-tuples $I$. The function $K_\alpha$ is also assumed to be a wide class of kernels satisfying the following growth condition (see [7]):

1. $K_\alpha \in S_\alpha$ if there exists a constant $C > 0$ such that

$$\int_{|x| = a} |K_\alpha(x)| \, dx \leq C a^\alpha; \quad (1.2)$$
(2) $K_\alpha$ is said to satisfy the $L^{\alpha,p}$-Hörmander condition, and write $K_\alpha \in H_{\alpha,p}$. If there exist $c \geq 1$ and $C > 0$ (only dependent on $\varphi$) such that, for all $y \in \mathbb{R}^n$ and $R > c|y|$, 

$$
\sum_{m=1}^{\infty} (2^m R)^{-\alpha} \| K_\alpha(\cdot - y) - K_\alpha(\cdot) \|_{\varphi,|x|^{-2m} R} \leq C,
$$

(1.3)

where $\varphi$ is a Young function defined on $[0, +\infty)$, $|x| \sim s$ stands for the set $\{s < |x| \leq 2s\}$, $O(0,s)$ is a ball with the center at the origin and the radius equal to $s$, and the $\varphi$-mean Luxemburg norm of a function $f$ on a cube (or a ball) $O$ in $\mathbb{R}^n$ is given by 

$$
\| f \|_{\varphi(O)} = \inf \left\{ \lambda > 0 : \frac{1}{|O|} \int_O \varphi\left( \frac{|f|}{\lambda} \right) \, dx \leq 1 \right\}.
$$

(1.4)

Differential forms can be viewed as an extension of functions. When $h(x)$ is a 0-form, the above-mentioned notations are in accord with those of function spaces, and the fractional convolution operator $F_\alpha$ we study in this paper degenerates into the operator which Bernardis discussed in [7]. Namely, for any Lebesgue measurable function $f \in L^\infty_\ell$, $F_\alpha$ is given as follows:

$$
F_\alpha f(x) = \int_{\mathbb{R}^n} K_\alpha(x-y)f(y) \, dy.
$$

(1.5)

This degenerated fractional convolution operator was also introduced by Riveros in [8], who presented weighted Coifman type estimates, two weight estimates of strong and weak type for general fractional operators and gave applications to fractional operators produced by a homogeneous function and a Fourier multiplier.

Now we introduce some notations and definitions. Let $\Theta$ be an open subset of $\mathbb{R}^n$ ($n \geq 2$) and $O$ be a ball in $\mathbb{R}^n$. Let $\rho O$ denote the ball with the same center as $O$ and $\text{diam}(\rho O) = \rho \text{diam}(O)(\rho > 0)$. $|\Theta|$ is used to denote the Lebesgue measure of a set $\Theta \subset \mathbb{R}^n$. Let $\wedge^\ell = \wedge^\ell(\mathbb{R}^n), \ell = 0, 1, \ldots, n$, be the linear space of all $\ell$-forms $h(x) = \sum_I h_I(x) \, dx_I = \sum_I h_{i_1i_2\ldots i_\ell}(x) \, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_\ell}$ in $\mathbb{R}^n$, where $I = (i_1, i_2, \ldots, i_\ell), 1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, are the ordered $\ell$-tuples. Moreover, if each of the coefficients $h_I(x)$ of $h(x)$ is differential on $\Theta$, then we call $h(x)$ a differential $\ell$-form on $\Theta$ and use $D(\Theta, \wedge^\ell)$ to denote the space of all differential $\ell$-forms on $\Theta$. $C^\infty(\Theta, \wedge^\ell)$ denotes the space of smooth $\ell$-forms on $\Theta$. The exterior derivative $d : D(\Theta, \wedge^\ell) \to D(\Theta, \wedge^{\ell+1}), \ell = 0, 1, \ldots, n-1$, is given by 

$$
dh(x) = \sum_{I} \sum_{j=1}^{n} \frac{\partial h_{i_1i_2\ldots i_\ell}(x)}{\partial x_j} \, dx_j \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_\ell}
$$

(1.6)

for all $h \in D(\Theta, \wedge^\ell)$. $L^p(\Theta, \wedge^\ell)(1 \leq p < \infty)$ is a Banach space with the norm $\|h\|_{p,\Theta} = (\int_{\Theta} |h(x)|^p \, dx)^{1/p} = (\int_{\Theta} \sum_I |h_I(x)|^p \, dx)^{1/p} < \infty$. Similarly, the notations $L^p_{\text{loc}}(\Theta, \wedge^\ell)$ and $W^1_{\text{loc}}^p(\Theta, \wedge^\ell)$ are self-explanatory.

From [9], $h$ is a differential form in a bounded convex domain $\Theta$, then there is a decomposition 

$$
h = d(Th) + T(dh),
$$

(1.7)
where $T$ is called a homotopy operator. For the homotopy operator $T$, we know that

$$
\|Th\|_{\ell^p, O} \leq C|O| \text{diam}(O) \|h\|_{\ell^p, O}
$$

(1.8)

holds for any differential form $h \in L^p_{loc}(\Theta, \wedge^\ell)$, $\ell = 1, 2, \ldots, n$, $1 < p < \infty$. Furthermore, we can define the $\ell$-form $h_\Theta \in D'(\Theta, \wedge^\ell)$ by

$$
h_\Theta = \begin{cases} 
|\Theta|^{-1} \int_\Theta h(x) \, dx, & \ell = 0, \\
\partial T(h), & \ell = 1, 2, \ldots, n,
\end{cases}
$$

(1.9)

for all $h \in L^p(\Theta, \wedge^\ell)$, $1 < p < \infty$.

A non-negative function $w \in L^1_{loc}(dx)$ is called a weight. We recall the definitions of the Muckenhoupt weights and the reverse Hölder condition (see [10]). For $1 < p < \infty$, we say that $w \in A_p$ if there exists a constant $C > 0$ such that, for every ball $O \subset \mathbb{R}^n$,

$$
\left( \frac{1}{|O|} \int_O w \, dx \right) \left( \frac{1}{|O|} \int_O w^{-\frac{1}{p'}} \, dx \right)^{p'-1} \leq C.
$$

(1.10)

For the case $p = 1$, $w \in A_1$ if there exists a constant $C > 0$ such that, for every ball $O \subset \mathbb{R}^n$,

$$
\frac{1}{|O|} \int_O w \, dx \leq C \text{ess inf}_{x \in O} w(x).
$$

(1.11)

Also $A_\infty = \bigcup_{p \geq 1} A_p$. It is well known that $A_p \subset A_q$ for all $1 \leq p \leq q \leq \infty$, and also that for $1 < p \leq \infty$, if $w \in A_p$, then there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$.

A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a Young function if it is continuous, convex, increasing and satisfies $\varphi(0) = 0$ and $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Each Young function $\varphi$ has an associated complementary Young function $\bar{\varphi}$ satisfying

$$
t \leq \varphi^{-1}(t)\bar{\varphi}^{-1}(t) \leq 2t
$$

(1.12)

for all $t > 0$, where $\varphi^{-1}(t)$ is the inverse function of $\varphi(t)$ (see [11]).

For each locally integrable function $f$ and $0 \leq \alpha < n$, the fractional maximal operator associated with the Young function $\varphi$ is defined by

$$
M_{\varphi}(x) = \sup_{x \in O} \|f\|_{\varphi(O)}.
$$

(1.13)

For $\alpha = 0$, we write $M_{\varphi}$ instead of $M_{0,\varphi}$. When $\varphi(t) = t$, then $M_{\varphi} = M_0$ is the classical fractional maximal operator. For $\alpha = 0$ and $\varphi(t) = t$, we obtain $M_{0,\varphi} = M$ is the Hardy–Littlewood maximal operator (see [8]).

2 The Coifman type inequalities for the fractional convolution operator

In [7], the inequality for the fractional convolution operator in function with the fractional maximal operator, that is, the Coifman type inequality, is proved.

Lemma 2.1 Let $\varphi$ be a Young function on $[0, +\infty)$ and $f$ be any $n$-tuple function on $\mathbb{R}^n$ with $f \in L^\infty$. Suppose that the fractional convolution operator $F_\alpha = K_\alpha \ast f$ and its kernel satisfies

...
\( K_\alpha \in S_\alpha \cap H_{\alpha,q} \), where \( 0 < \alpha < n \). Then there exists a constant \( C \) such that

\[
\int_{\mathbb{R}^n} |F_\alpha f(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} \left[ M_{\alpha,q} f(x) \right]^p w(x) \, dx
\]  

(2.1)

for any \( 0 < p < \infty \) and \( w \in A_\infty \).

**Theorem 2.1** Let \( \varphi \) be a Young function on \([0, +\infty)\) and \( f, g \) be two functions defined on \( \mathbb{R}^n \) with \(|f(x)| \leq |g(x)|\) for all \( x \in \mathbb{R}^n \). Then for all cubes \( O \) and the Young functions \( \varphi \),

\[
\|f\|_{\varphi(O)} \leq \|g\|_{\varphi(O)}. 
\]  

(2.2)

**Proof** Since \( \varphi \) is a Young function, it follows that

\[
\frac{1}{|O|} \int_O \varphi \left( \frac{|f|}{\|f\|_{\varphi(O)}} \right) \, dx \leq \frac{1}{|O|} \int_O \varphi \left( \frac{|g|}{\|g\|_{\varphi(O)}} \right) \, dx \leq 1
\]

\[
\Rightarrow \|g\|_{\varphi(O)} \in E \cup \left\{ \lambda > 0 : \frac{1}{|O|} \int_O \varphi \left( \frac{|f|}{\lambda} \right) \, dx \leq 1 \right\}
\]

\[
\Rightarrow \|f\|_{\varphi(O)} = \inf E \leq \|g\|_{\varphi(O)}. 
\]  

(2.3)

According to Theorem 2.1, we can get a similar conclusion to Lemma 2.1.

**Theorem 2.2** Let \( \varphi \) be a Young function on \([0, +\infty)\), \( h = \sum I h_I \, dx \) be a differential form on \( \Theta \subset \mathbb{R}^n \), and let all the ordered \( \ell \)-tuples \( I \) satisfy \( h_I \in L^\infty_\text{c} \). Suppose that \( F_\alpha \) is a fractional convolution operator applied to differential forms and its kernel function \( K_\alpha \) satisfies \( K_\alpha \in S_\alpha \cap H_{\alpha,q} \), where \( 0 < \alpha < n \). Then there exists a constant \( C \) such that

\[
\int_{\mathbb{R}^n} |F_\alpha h(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} \left[ M_{\alpha,q} h(x) \right]^p w(x) \, dx
\]  

(2.4)

for any \( 0 < p < \infty \) and \( w \in A_\infty \).

**Proof** By Lemma 2.1 and the following basic inequality

\[
\sum_{i=1}^n |a_i|^s \leq n \left( \sum_{i=1}^n |a_i| \right)^s \leq n^{s+1} \sum_{i=1}^n |a_i|^s,
\]  

(2.5)

where \( s > 0 \) is any constant, it follows that

\[
\|F_\alpha h\|_{p,w,\mathbb{R}^n}^p = \int_{\mathbb{R}^n} |F_\alpha h(x)|^p w(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} \left( \sum_I \left( \int_{\mathbb{R}^n} K_\alpha(x-y) h_I(y) \, dy \right)^2 \right)^{p/2} w(x) \, dx
\]

\[
\leq \int_{\mathbb{R}^n} C_1 \sum_I \left( \int_{\mathbb{R}^n} K_\alpha(x-y) h_I(y) \, dy \right)^p w(x) \, dx
\]

\[
= C_1 \sum_I \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} K_\alpha(x-y) h_I(y) \, dy \right)^p w(x) \, dx
\]
\[ \begin{align*} &\leq C_2 \sum_{I} \int_{\mathbb{R}^n} [M_{a,\bar{h}} h_I(x)]^p w(x) \, dx \\
&= C_2 \int_{\mathbb{R}^n} \sum_{I} [M_{a,\bar{h}} h_I(x)]^p w(x) \, dx \\
&\leq C_3 \int_{\mathbb{R}^n} \left( \sum_{I} M_{a,\bar{h}} h_I(x) \right)^p w(x) \, dx. \tag{2.6} \end{align*} \]

Then, by the definition of the fractional maximal operator, notice that for any $I$ such that $|h_I| \leq |h|$, we obtain that

\[ \sum_{I} M_{a,\bar{h}} h_I(x) = \sum_{I} \sup_{x \in O} |O|^{\frac{\alpha}{n}} \|h_I\|_{\bar{\psi}(O)} \]

\[ \leq C_4 \sum_{I} |O|^{\frac{\alpha}{n}} \sum_{I} \|h_I\|_{\bar{\psi}(O)} \]

\[ \leq C_4 \sum_{I} |O|^{\frac{\alpha}{n}} C_{n} \|h\|_{\bar{\psi}(O)} \]

\[ \leq C_4 M_{a,\bar{h}} h(x). \tag{2.7} \]

Combining (2.6) and (2.7), we have

\[ \|F_{a,h}\|_{p,w,\mathbb{R}^n}^p \leq C_3 \int_{\mathbb{R}^n} \left( \sum_{I} M_{a,\bar{h}} h_I(x) \right)^p w(x) \, dx \]

\[ \leq C_3 \int_{\mathbb{R}^n} \left( C_4 M_{a,\bar{h}} h(x) \right)^p w(x) \, dx \]

\[ \leq C_5 \int_{\mathbb{R}^n} \left( M_{a,\bar{h}} h(x) \right)^p w(x) \, dx. \tag{2.8} \]

**Theorem 2.3** Let $\phi$ be a Young function on $[0, \infty)$, $h = \sum_{i} h_i \, dx_i$ be a differential form on $\Theta \subset \mathbb{R}^n$, and for all the ordered $\ell$-tuples, let $I$ satisfy $h_I \in L_{\infty}^\alpha$. Suppose that $F_{a}$ is a fractional convolution operator on differential forms and its kernel function $K_a$ satisfies $K_a \in S_\alpha \cap H_{a,v}$, where $0 < \alpha < n$. Then there exists a constant $C$ such that

\[ \int_{O} |F_{a,h}(x)|^p \, dx \leq C \int_{O} [M_{a,\bar{h}} h(x)]^p \, dx \tag{2.9} \]

for any $0 < p < \infty$ and all the balls $O \subset \mathbb{R}^n$.

**Proof** By the definition of the $A_{\infty}$-weight, there exist $r_0 \geq 1$ and a constant $C < \infty$ such that, for all the balls $O \subset \mathbb{R}^n$, it follows that

\[ \left( \frac{1}{|O|} \int_{O} w(x) \, dx \right) \left( \frac{1}{|O|} \int_{O} w(x) \, \frac{1}{|O|} \, dx \right)^{r_0-1} \leq C. \tag{2.10} \]
With the arbitrariness of the condition \( w \in A_\infty \) of Theorem 2.2, now get any ball \( O_0 \subset \mathbb{R}^n \) and let

\[
w(x) = \chi_{O_0}(x) = \begin{cases} 
1, & x \in O_0; \\
0, & x \notin O_0.
\end{cases}
\]

It is easy to check that \( w(x) = \chi_{O_0}(x) \) satisfies (2.10). In fact, we have

\[
\left( \frac{1}{|O|} \int_O \chi_{O_0}(x) \, dx \right) \left( \frac{1}{|O|} \int_O \chi_{O_0}(x) \, \frac{1}{|x|} \, dx \right)^{\alpha - 1} \\
= \left( \frac{1}{|O|} |O \cap O_0| \right) \left( \frac{1}{|O|} |O \cap O_0| \right)^{\alpha - 1} \\
= \left( \frac{1}{|O|} |O \cap O_0| \right)^\alpha \leq 1.
\]
(2.11)

Thus

\[
\int_{O_0} |F_\alpha h(x)|^p \, dx \\
= \int_{\mathbb{R}^n} |F_\alpha h(x)|^p \chi_{O_0}(x) \, dx \\
\leq C \int_{\mathbb{R}^n} [M_{\alpha, \varphi} h(x)]^p \chi_{O_0}(x) \, dx \\
\leq C \int_{O_0} [M_{\alpha, \varphi} h(x)]^p \, dx.
\]
(2.12)

\[
\square
\]

If the kernel function \( K_\alpha \) and the coefficient functions \( h_I \) of differential forms are subject to some conditions, the following more important conclusion will be obtained.

**Theorem 2.4** Let \( \varphi \) be a Young function on \([0, +\infty)\), \( h = \sum_I h_I \, dx_I \) be a differential form on \( \Theta \subset \mathbb{R}^n \), and let all the ordered \( \ell \)-tuples \( I \) satisfy \( h_I \in L_{\infty}^\infty \). Suppose that \( F_\alpha \) is a fractional convolution operator on differential forms and its kernel function \( K_\alpha \) satisfies \( K_\alpha \in S_\alpha \cap H_{\alpha, \varphi} \) and \( K_\alpha \in C_0^\infty(\Theta) \), where \( C_0^\infty(\Theta) \) stands for all the \( C^\infty \) functions with compactly supported sets in \( \Theta \) and \( 0 < \alpha < n \). Then there exists a constant \( C \) such that

\[
\|F_\alpha h - (F_\alpha h)_O\|_{p, O} \leq C \text{diam}(O)|O| \|M_{\alpha, \varphi}(dh)\|_{p, O}
\]
(2.13)

for any \( 1 < p < \infty \) and all the balls with \( O \subset \mathbb{R}^n \).

**Proof** By the exterior derivative operator \( d \) and the fractional convolution operator \( F_\alpha \), we obtain that

\[
dh = \sum_{k=1}^n \sum_{I} \frac{\partial h_I(x)}{\partial x_k} \, dx_k \wedge dx_I, \\
F_\alpha(dh) = \sum_{k=1}^n \sum_{I} \left( \int_{\mathbb{R}^n} K_\alpha(x - y) \frac{\partial h_I(y)}{\partial y_k} \, dy \right) \, dx_k \wedge dx_I
\]
(2.14)
and

\[ dF_\alpha(h) = \sum_\ell \sum_{k=1}^n \frac{\partial h_\ell(x)}{\partial x_k} \, dx_k \wedge dx_\ell, \]  

(2.15)

where

\[ h_\ell(x) = \int_{\mathbb{R}^n} K_\alpha(x-y) h_\ell(y) \, dy. \]  

(2.16)

According to (1.7)–(1.9), it follows that

\[ \|F_\alpha h - (F_\alpha h)_0\|_{p,O} = \|T(d(F_\alpha h))\|_{p,O} \leq C_1 \text{diam}(O)|O| \|d(F_\alpha h)\|_{p,O}. \]  

(2.17)

Now we will give the \( L^p \)-norm estimation of \( d(F_\alpha h) \). With \( K_\alpha \in C_0^\infty(\Theta) \) and considering the definition of the general partial derivative (see [12]), we obtain

\[
\|d(F_\alpha h)\|_{p,O}^p = \int_O \left( \sum_\ell \sum_{k=1}^n \left| \frac{\partial h_\ell(x)}{\partial x_k} \right|^2 \right)^{p/2} \, dx \\
= \int_O \left( \sum_\ell \sum_{k=1}^n \left| \int_{\mathbb{R}^n} \frac{\partial K_\alpha(x-y) h_\ell(y)}{\partial x_k} \, dy \right|^2 \right)^{p/2} \, dx \\
= \int_O \left( \sum_\ell \sum_{k=1}^n \left| \int_{\mathbb{R}^n} \frac{\partial K_\alpha(x-y)}{\partial x_k} h_\ell(y) \, dy \right|^2 \right)^{p/2} \, dx \\
= \int_O \left( \sum_\ell \sum_{k=1}^n \left| \int_{\mathbb{R}^n} \frac{\partial h_\ell(y) K_\alpha(x-y)}{\partial y_k} \, dy \right|^2 \right)^{p/2} \, dx \\
= \left\| F_\alpha(dh) \right\|_{p,O}^p, \tag{2.18}
\]

that is

\[ \left\| d(F_\alpha h) \right\|_{p,O}^p = \left\| F_\alpha(dh) \right\|_{p,O}^p. \]  

(2.19)

Combining (2.17) and (2.19), we obtain that

\[
\left\| F_\alpha h - (F_\alpha h)_0 \right\|_{p,O} \leq C_1 \text{diam}(O)|O| \left\| d(F_\alpha h) \right\|_{p,O} \leq C_1 \text{diam}(O)|O| \left\| M_{\alpha,\phi}(dh) \right\|_{p,O}. \tag{2.20}
\]

□
Since a new function is obtained when the differential form is taken as a model, we can get a global inequality in the \( L^p(m) \) domain with Theorem 2.4. Now recall the definition of the \( L^p(m) \) domain introduced by Staples (see [13]).

**Definition 2.1** Let \( \Theta \) be a real subdomain in \( \mathbb{R}^n \). If, for all the functions \( f \in L^p_{\text{loc}}(\Theta) \), there exists a constant \( C \) such that

\[
|\O|^{-1/p} \| f - f_{O_0} \|_{p,\Theta} \leq C \sup_{O \subset \Theta} |\O|^{-1/p} \| f - f_{O} \|_{p,O},
\]

then \( \Theta \) is called an \( L^p(m) \)-average domain, where \( O_0 \) is a fixed ball of \( \Theta \) and \( p \geq 1 \).

**Theorem 2.5** Let \( \varphi \) be a Young function on \( [0, +\infty) \), \( h = \sum h_i \, dx_i \) be a differential form on \( \Theta \subset \mathbb{R}^n \), and let all the ordered \( \ell \)-tuples \( I \) satisfy \( h_I \in L_{\infty}^{\infty} \). Suppose that \( F_a \) is a fractional convolution operator on differential forms and its kernel function \( K_a \) satisfies \( K_a \in S_a \cap H_{a,p} \) and \( K_a \in C_{\infty}^{\alpha}(\Theta) \), where \( 0 < \alpha < n \). Then there exists a constant \( C \) such that

\[
\| F_a h - (F_a h)_{O_0} \|_{p,\Theta} \leq C(\Theta) \| \text{diam}(\Theta) \| M_{a,\varphi} dh(\alpha) \|_{p,\Theta}
\]

for any \( 1 < p < \infty \) and \( O_0 \) is a fixed ball in \( \Theta \).

**Proof** By the definition of the \( L^p(m) \)-average domain and noticing that \( 1 - 1/p \geq 0 \), we have

\[
\begin{align*}
\| F_a h - (F_a h)_{O_0} \|_{p,\Theta} & \leq C_1(\Theta) \| \text{diam}(\Theta) \| M_{a,\varphi} dh(\alpha) \|_{p,\Theta} \\
& \leq C_1(\Theta) \| \text{diam}(\Theta) \| M_{a,\varphi} dh(\alpha) \|_{p,\Theta} \\
& \leq C_2(\Theta) \| \text{diam}(\Theta) \| M_{a,\varphi} dh(\alpha) \|_{p,\Theta} \\
& = C_3(\Theta) \| \text{diam}(\Theta) \| M_{a,\varphi} dh(\alpha) \|_{p,\Theta}.
\end{align*}
\]

\( \square \)

### 3 The Lipschitz and BMO norm inequalities for the fractional convolution operator

It is well known that Lipschitz and BMO norms are two kinds of important norms in differential forms, which can be found in [14]. Now we recall these definitions as follows. Let \( h \in L^1_{\text{loc}}(\Theta, \mathbb{A}^\ell) \), \( \ell = 0, 1, \ldots, n \). We write \( h \in \text{locLip}_k(\Theta, \mathbb{A}^\ell) \), \( 0 \leq k \leq 1 \), if

\[
\| h \|_{\text{locLip}_k, \Theta} = \sup_{\rho \subset \Theta} |O|^{-\rho(k)/n} \| h \|_{1,0} < \infty
\]

(3.1)

for some \( \rho \geq 1 \).

Further, we write \( \text{Lip}_k(\Theta, \mathbb{A}^\ell) \) for those forms whose coefficients are in the usual Lipschitz space with exponent \( k \) and write \( \| h \|_{\text{Lip}_k, \Theta} \) for this norm. Similarly, for \( h \in \text{locLip}_k(\Theta, \mathbb{A}^\ell) \)
\[ L^1_{\text{loc}}(\Theta, \wedge^\ell), \ell = 0, 1, \ldots, n, \text{ we write } h \in BMO(\Theta, \wedge^\ell) \text{ if} \]
\[ \| h \|_{\ast, \Theta} = \sup_{\rho \Theta \subset \Theta} |\rho \Theta|^{-1} \| h - h_{\rho \Theta} \|_{1, \rho \Theta} < \infty \]  
(3.2)

for some \( \rho \geq 1 \).

When \( h \) is a 0-form, Eq. (3.2) reduces to the classical definition of \( BMO(\Theta) \).

**Lemma 3.1** (see [10]) Let \( 0 < p, q < \infty \) and \( 1/s = 1/p + 1/q \). If \( f \) and \( g \) are two measurable functions on \( \mathbb{R}^n \), then
\[ \| fg \|_{s, \Theta} \leq \| f \|_{p, \Theta} \| g \|_{q, \Theta} \]  
(3.3)

for any \( \Theta \subset \mathbb{R}^n \).

**Theorem 3.1** Let \( \varphi \) be a Young function on \( [0, +\infty) \), \( h = \sum_i h_i \, dx_i \) be a differential form on \( \Theta \subset \mathbb{R}^n \), and let all the ordered \( \ell \)-tuples \( I \) satisfy \( \| h_I \|_{L^\infty} \). Suppose that \( F_\alpha \) is a fractional convolution operator on differential forms and its kernel function \( K_\alpha \) satisfies \( K_\alpha \in S_\alpha \cap H_\alpha \), \( \varphi \) and \( K_\alpha \in C_\infty^0(\Theta) \), where \( 0 < \alpha < n \). Then, for any \( 1 < p < \infty \), there exists a constant \( C \) such that
\[ \| F_\alpha h \|_{\text{loc}L^p(\Theta, \Theta)} \leq C \| M_{\varphi, \bar{\Theta}}(dh) \|_{p, \Theta}, \]  
(3.4)

where \( k \) is a constant with \( 0 \leq k \leq 1 \).

**Proof** By Theorem 2.4, we obtain
\[ \| F_\alpha h - (F_\alpha h)_\Theta \|_{p, \Theta} \leq C |O| \text{diam}(O) \| M_{\varphi, \bar{\Theta}} dh(x) \|_{p, \Theta}. \]  
(3.5)

By Lemma 3.1 with \( 1 = 1/p + (p - 1)/p \), for any ball with \( O(\Theta) \), we have
\[ \| F_\alpha h - (F_\alpha h)_\Theta \|_{1, \Theta} \]
\[ = \int_O \| F_\alpha h - (F_\alpha h)_\Theta \| \, dx \]
\[ \leq \left( \int_O \| F_\alpha h - (F_\alpha h)_\Theta \|^p \, dx \right)^{1/p} \left( \int_O \frac{1}{\rho^p} \, dx \right)^{(p-1)/p} \]
\[ = |O|^{(p-1)/p} \| F_\alpha h - (F_\alpha h)_\Theta \|_{p, \Theta} \]
\[ = |O|^{1-1/p} \| F_\alpha h - (F_\alpha h)_\Theta \|_{p, \Theta} \]
\[ \leq |O|^{1-1/p} \| M_{\varphi, \bar{\Theta}} dh(x) \|_{p, \Theta} \]
\[ \leq C_2 |O|^{2-1/p + 1/n} \| M_{\varphi, \bar{\Theta}} dh(x) \|_{p, \Theta}. \]  
(3.6)

By the definition of the Lipschitz norm and \( 2 - 1/p + 1/n - 1 - k/n = 1 - 1/p + 1/n - k/n > 0 \), we obtain
\[ \| F_\alpha h \|_{\text{loc}L^p(\Theta, \Theta)} \]
Let $K$ be defined in a bounded convex domain $\Theta$, then $h \in \text{BMO}(\Theta, \Lambda^i)$ and there exists a constant $C$ such that
\[ \|h\|_{\alpha, \ell} \leq C\|h\|_{\text{locLip}_x, \Theta}. \] (3.8)

By Theorem 3.1 and Lemma 3.2, we get the following conclusion.

**Theorem 4.1** Let $\varphi$ be a Young function on $[0, +\infty)$, $h = \sum_i h_i \, dx_i$ be a differential form on $\Theta \subset \mathbb{R}^n$, and let all the ordered $\ell$-tuples $I$ satisfy $h_i \in L^\infty_{\text{loc}}$. Suppose that $F_\alpha$ is a fractional convolution operator on differential forms and its kernel function $K_\alpha$ satisfies $K_\alpha \in S_\alpha \cap H_{\alpha, \varphi}$ and $K_\alpha \in C_0^\infty(\Theta)$, where $0 < \alpha < n$. Then, for any $1 < p < \infty$, there exists a constant $C$ such that
\[ \|F_\alpha h\|_{\alpha, \ell} \leq C\|M_{\alpha, \varphi}(dh)\|_{p, \ell}. \] (3.9)

**4 Applications**

With regard to the applications of the fractional convolution operator, we will point out that Theorem 2.2 has different expression forms.

**Definition 4.1** (see [7]) Let $K_\alpha(x)$ be a function defined on $\mathbb{R}^n$, if there exist two constants $c \geq 1$ and $C > 0$ such that
\[ |K_\alpha(x - y) - K_\alpha(x)| \leq C \frac{|y|}{|x|^{n+1-\alpha}} \quad |x| > c|y|, \] (4.1)
then the kernel function $K_\alpha$ is said to satisfy the $H_{\alpha, \infty}^*$-condition.

**Lemma 4.1** (see [7]) Let $\varphi$ be any Young function defined on $[0, +\infty)$, then $H_{\alpha, \infty}^* \subset H_{\alpha, \varphi}$.

**Theorem 4.1** Let $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$ and $0 \leq \alpha < n$, then $K_\alpha \in S_\alpha \cap H_{\alpha, \varphi}.$
Proof. Firstly prove that $K_\alpha \in S_\alpha$. By the definition of $K_\alpha$, we have
\[
\int_{|x|<s} |K_\alpha(x)| \, dx \leq \int_{O(0,2s)} \frac{1}{|x|^{n-\alpha}} \, dx \leq \sigma_n (2s)^n \cdot (s)^{n-\alpha} = 2^n \sigma_n s^{\alpha}, \tag{4.2}
\]
where $\sigma_n$ is the volume of a unit sphere $n$ in $\mathbb{R}^n$. Thus $K_\alpha \in S_\alpha$.

Secondly prove that $K_\alpha \in H_\alpha$, $\phi$. According to Lemma 4.1, we only need to prove that $K_\alpha \in H^\ast_{\alpha, \infty}$. If we choose $|x| > 2|y|(c = 2 \geq 1)$ and $y \neq O = (0, \ldots, 0)$ (for $y = O$, it is clearly established), it follows that
\[
|x - y|/|x| \in \begin{cases} 
(\frac{1}{2}, 1), & x, y 	ext{ each component has the same sign;} \\
(1, \frac{3}{2}), & x, y 	ext{ each component has the different sign},
\end{cases}
\]
where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$. Considering that each component of $x, y$ is greater than zero, other cases may be considered similarly. By Lagrange’s mean value theorem
\[
|K_\alpha(x - y) - K_\alpha(x)| \\
= |x - y|^{n-\alpha} - |x|^{n-\alpha} \\
= |x|^{\alpha-n} \left(\frac{|x - y|}{|x|}\right)^{\alpha-n} - 1 \\
\leq (n - \alpha)|x|^{\alpha-n} \left(\frac{|y|}{|x|}\right)^{\alpha-n} |\xi|^{n-1} \\
\leq 2^{n+1-\alpha} (n - \alpha) \frac{|y|}{|x|^{n+1-\alpha}}, \tag{4.3}
\]
where
\[
|x| = \sqrt{\sum_{i=1}^{n} x_i^2}, \xi = (\xi_1, \ldots, \xi_n), \\
\frac{1}{2} < \min \left\{ \frac{|x - y|}{|x|}, 1 \right\} < |\xi| < \max \left\{ \frac{|x - y|}{|x|}, 1 \right\} < \frac{3}{2}.
\]
Thus $K_\alpha \in H^\ast_{\alpha, \infty}$.

Theorems 2.2 and 4.1 yield the following.

**Theorem 4.2** Let $\phi$ be any Young function defined on $[0, +\infty)$ and $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$, then the fractional convolution operator $F_\alpha$ in (1.1) becomes the classical Riesz potential operator
\[
I_\alpha h(x) = \sum_{\ell} \left( \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha}} h_\ell(y) \, dy \right) dx, \tag{4.4}
\]
where $h = \sum h_\ell \, dx_\ell$ is a differential form in $\mathbb{R}^n$ and such that $h_\ell \in L^\infty_{\ell}$ for all the ordered $\ell$-tuples $I$. Then there exists a constant $C$ such that
\[
\int_{\mathbb{R}^n} \left| I_\alpha h(x) \right|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} \left[ M_{\alpha, \phi} h(x) \right]^p w(x) \, dx \tag{4.5}
\]
for any $0 < p < \infty$ and $w \in A_{\infty}$.
Lemma 4.2 (see [7]) Denote by $S^{n-1}$ the unit sphere of $\mathbb{R}^n$, $\Omega$ is a homogeneous function defined on $S^{n-1}$ with $\Omega(x) = \Omega(x')$ and the kernel function $K_\alpha(x) = \Omega(x)/|x|^{n-\alpha}(x \neq 0)$, where $x' = x/|x|(x \neq 0)$. Given a Young function $\varphi$, we define the $L^\varphi$-modulus of continuity of $\Omega$ as

$$\varpi_\varphi(t) = \sup_{|y| \leq t} \|\Omega(\cdot + y) - \Omega(\cdot)\|_{L^\varphi(S^{n-1})} \tag{4.6}$$

and write $\Omega \in L^\varphi(S^{n-1})$. If

$$\int_0^1 \varpi_\varphi(t) \frac{dt}{t} < \infty, \tag{4.7}$$

then $K_\alpha \in S_\varphi \cap H_{\varphi,\alpha}$.

By Theorem 2.2 and Lemma 4.2, we have the following.

Theorem 4.3 Let $\varphi$ be a Young function, $\Omega$ be a homogeneous function in $S^{n-1}$ with $\Omega(x) = \Omega(x')$ and $\Omega \in L^\varphi(S^{n-1})$. Suppose that $F_\alpha$ is the fractional convolution operator with its kernel function $K_\alpha(x) = \Omega(x)/|x|^{n-\alpha}$. Let $h = \sum h_I dx_I$ be a differential form in $\mathbb{R}^n$ with $h_I \in L^\infty$ for all the ordered $\ell$-tuples $I$. If $\int_0^1 \varpi_\varphi(t) \frac{dt}{t} < \infty$, then there exists a constant $C$ such that

$$\int_{\mathbb{R}^n} |F_\alpha h(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} [M_{\varphi,\alpha} h(x)]^p w(x) \, dx \tag{4.8}$$

for any $0 < p < \infty$ and $w \in A_\infty$.

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Competing interests

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Authors’ contributions

All results and investigations of this manuscript were due to the joint efforts of all authors. ZD finished the proof and the writing work. HL gave ZD some excellent advices in the proof and writing. QL took part in the original conceiving and discussion, and carefully checked the second amendment of the English writing and the proof in this article. All authors read and approved the final manuscript.

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