THERMODYNAMIC STABILITY OF A
MULTI-BUBBLE COSMOLOGICAL
MODEL*

GERALD HORWITZ† and OLEG FONAREV
Racah Institute of Physics, The Hebrew University
Jerusalem 91904, Israel

Abstract

Multibubble solutions for a cosmological model which lead to thermal inflationary states due to a semi-classical tunneling of gravity are calculated.

I. One domain solutions

In earlier work done by several different approaches [1, 2], an inflationary solution was found as semiclassical tunneling using a simple model of quantum matter and semiclassical gravity. The model comprised massless noninteracting scalar bosons conformally coupled to gravity in a RW universe with $k = +1$ and a positive phenomenological cosmological constant $\Lambda$. For such model there is a classically forbidden region connecting two classically allowed regions. The inner one of these two is inappropriate for our model since the inner region necessarily involves quantum gravity. The model was evaluated by two different approaches, first by an explicit statistical mechanics approach and then by a wave function of the universe approach. In both of these approaches the initial method was to calculate the solutions in the

*talk at Seventh Marcel Grossman Meeting, Stanford University, July 1994
†Supported by BSF research grant No. 89-00244

1
outer classically allowed region. The result of tunneling is to produce an inflationary thermal solution in the outer region. This tunneling yields both a definition of physical time and entropy appropriate to an inflationary solution.

We use a RW metric written in conformal time coordinates:

$$ds^2 = a^2(u)[du^2 - d\chi^2 - \sin^2 \chi d\Omega^2] = a^2(u)\gamma_{\mu\nu}dx^\mu dx^\nu$$

(1)

The action obtained after going over to a Euclideanized metric as appropriate for a tunneling region is

$$I = \int dx^4 [\frac{3}{8\pi}(\dot{a}^2 + a^2 - \Lambda/3a^4) + \frac{1}{2}(\gamma_{\mu\nu}\phi_{\mu}\phi_{\nu} - \phi^2)] = I_G + I_M$$

(2)

after carrying out a conformal transformation $g = a^2\gamma$ and $\phi = a\varphi$, whence $R/6 = 1$. One then evaluated the norm of the wave function of the universe and showed that

$$P(a) = \int d\phi |\Psi(\phi, a)|^2 = \int \frac{d\beta'}{2\pi i} \int_{W_{KB}} D[a] \int D[\phi] e^{-I(a, \phi, \beta'; a, \phi, 0)}$$

(3)

where $\phi$ and $a$ are periodic functions of $\beta'$. The $\beta'$ integral fixes the total energy to be zero and its saddle point value is the reciprocal temperature when we have exchanged the order of integration as in the second part of equation (3). The saddle point evaluation of $a$ involves replacing $a$ in the integrand by its saddle point value $\bar{a}$. The $\phi$ functional integral gives

$$\int D[\phi] e^{-I(\phi, \beta)} = e^{-\beta F(\beta)}$$

(4)

where $F(\beta)$ is the Helmholtz free energy. Then $P(\bar{a}) = e^S = e^{S_G + S_M}$, where $S$ is the total entropy and $S_G$ and $S_M$ are respectively the entropies of the gravity and of the matter. The common saddle point solutions of the $a$ and $\beta'$ integrations leads to the equation

$$\beta = 2 \int_{x_0}^{x_1} dx \frac{1}{\sqrt{-e + x^2}}$$

(5)

where $x = (\frac{\Lambda}{3}a)^{1/2}a$ and $e$ is the thermal average energy multiplied by $\frac{\Lambda}{3}$; $e = \frac{\Lambda}{3}\pi k_B\beta^{-1} = \alpha\beta^{-4}$. 

2
Seeking the self consistent solution in the tunneling region one finds two solutions in a certain range of the values of $x$ which can most clearly be seen by the following graph, where the dashed curve is the locus of solutions. Each point of the dashed line is the end point of a straight line from $x_-$ at that energy. Thus we see of all there are no solutions for $x$ less than some minimum value $x_0$ and that there are two branches of the solutions for $x_0 < x < x_M = 1/\sqrt{2}$. Both branches correspond to an increasing entropy as a function of $x$ and both are maximum entropy states of nearly equal weight near the maximum value of the upper solution. The upper solution joins the real time domain where it corresponds to a static Einstein solution. The lower one emerges at higher entropy and corresponds to a thermal inflationary solution. Note that in this case there is no need for time to establish the thermal solutions, since it begins in thermal equilibrium when it emerges in the real time domain.

In previous work we discarded the upper solution since it is of lower entropy and is also unstable. Subsequently we began to consider whether it is justified to ignore alternative saddle point solutions which cannot immediately be discarded as being of much less weight. The present paper is an effort to consider another alternative.
II. Multidomain solutions

In the standard functional integral approach, if there is more than one saddle point the wave function in lowest order is a superposition of the several contributions. This method is not viable in our case since we are considering thermally mixed states. We apply instead another approach in which the superposition consists of the different solutions coexisting in spatially separated regions. We worked out in detail the case of two domains; the multibubble solution were treated only approximately. Let us first consider the two bubble case. The two states are a Robertson-Walker (RW) inflationary universe and a static Einstein universe (SE). The two different solutions are matched on a singular surface, which is taken to be spherical. The Euclidean line elements in the two regions take the form:

\[ ds_{RW}^2 = \frac{3}{\Lambda} x^2(u)[du^2 + d\chi_{RW}^2 + \sin^2 \chi_{RW}d\Omega^2] \]  
\[ ds_{SE}^2 = \frac{3}{2\Lambda} [N_{SE}^2(u)du^2 + d\chi_{SE}^2 + \sin^2 \chi_{SE}d\Omega^2] \]  

The singular hypersurface is given either by the equation \( \chi_{RW} = f_{RW}(u) \), or by the equation \( \chi_{SE} = f_{SE}(u) \). The matching conditions on the hypersurface lead to the following relations:

\[ x(u) \sin f_{RW}(u) = \frac{1}{\sqrt{2}} \sin f_{SE}(u) \]  
\[ x^2(u)(1 + \dot{f}_{RW}^2(u)) = \frac{1}{2}(N_{SE}^2(u) + \dot{f}_{SE}^2(u)) \]  

Another constraint follows from the junction conditions that relate the jump in the Einstein tensor to the surface energy \[ 3 \]. As a toy model we assume that the surface energy is purely dynamic and it comes from the surface term in the action. We showed elsewhere \[ 4 \] that the variation of the action with respect to the induced metric on the hypersurface yields, in our case the following constraint:

\[ f_{RW} = \text{constant.} \]  

The thermodynamic entropy we obtained is the contribution from the two states

\[ S = S_{RW} + S_{SE} \]
each contribution being the sum of the gravitational and the matter entropies.

\[ S_{RW} = \frac{9}{2\Lambda} V_{RW} \left[ 4 \int_{x_{-}(e)}^{x_{+}(e)} dx \sqrt{x^2 - x^4 - e} + \frac{4}{3} \alpha^{1/4} e^{3/4} \right] \]  

(12)

\[ S_{SE}(\epsilon) = \frac{9}{2\Lambda} \left[ \frac{1}{4} \int_0^\beta d\bar{u} N_2(\bar{x}) V_{SE}(\bar{x}) + \frac{1}{3} \bar{V}_2(\bar{x}_+) \alpha \beta^{-3}(\bar{e}, \beta) \right] \]  

(13)

Here \( V_{RW} = \int_0^{f_{RW}} \sin^2 \chi \, d\chi \) and \( V_{SE} = \int_{f_{SE}(u)}^\pi \sin^2 \chi \, d\chi \) are the volumes occupied by the RW and SE bubbles respectively. The function \( \bar{x}(u) \) is the saddle point solution of the variational equations for the RW region:

\[ \bar{x}^2 = \bar{x}^2 - \bar{e} \]  

(14)

with \( \bar{e} \) being found from the equation

\[ \left( \frac{\alpha}{\bar{e}} \right)^{\frac{1}{4}} = 2 \int_{x_{-}(\bar{e})}^{x_{+}(\bar{e})} \frac{dx'}{\sqrt{x'^2 - x^4 - \bar{e}}} \]  

(15)

The (proper) temperature in the SE region is related to \( \beta \) with the lapse function:

\[ \beta_2(\bar{e}, \beta) = \int_0^\beta d\bar{u} N_2(\bar{x}) . \]  

(16)

The condition for the two bubbles to be in thermal equilibrium is

\[ \beta_2(\bar{e}, (\alpha/\bar{e})^{\frac{1}{4}}) = (4\alpha)^{\frac{1}{4}} \]  

(17)

Equations (8)–(10), (15)–(17) constitute the conditions to be satisfied in equilibrium. Having found \( \bar{e}, f_{RW}, \bar{x}(u), \bar{f}_{SE}(u) \) and \( \bar{N}_{SE}(u) \) from these equations, we can find the value of the entropy at equilibrium and check the stability from eqs. (11)–(13). Analysis of eqs. (15)–(17) shows that an equilibrium can only exist for the range of parameter \( \alpha \), \( \pi^4/4 < \alpha < \pi^4 \). In this range there is a critical value \( \alpha_c \), such that the equilibrium is stable against global fluctuations when \( \alpha < \alpha_c \) and is quasi-stable when \( \alpha > \alpha_c \). In either case the entropy of the two bubble state is greater than the entropy of the single RW universe.

We now turn to the case of multiple bubbles. There are two possibilities: many RW bubbles in a SE matrix or many SE bubbles in a RW matrix. It can be shown that conditions for equilibrium remain the same as in the case of two bubbles. In our simple model the number of bubbles cannot exceed 25. Moreover, the constraints limit the size of the RW region, so that inflation cannot last long. We are currently exploring possible extensions of the model to overcome these problems.
Acknowledgements

G.H. acknowledges some useful and illuminating discussions with J. Bekenstein.

References

[1] G. Horwitz and D. Weil, *Phys. Rev. Lett.* **48** (1982) 219, R. Brout, G. Horwitz and D. Weil, *Phys. Lett.* **192B** (1987) 318. G. Horwitz and F. Haddad, (in preparation).

[2] R. Brout, *Found. Phys.* **17** (1987) 603, R. Brout, *Phys. Rev.* **D39** (1989) 2436, R. Brout and P. Spindel, *Nucl. Phys.* **B348** (1989) 405, P. Spindel and R. Brout, *Phys. Lett. B320* (1994) 241.

[3] Israel, W., *Nuovo Cim.* **44B** (1966) 1.

[4] O. Fonarev and G. Horwitz, (in preparation).