QUANTUM RANDOM WALKS AND TIME-REVERSAL

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ABSTRACT Classical random walks and Markov processes are easily described by Hopf algebras. It is also known that groups and Hopf algebras (quantum groups) lead to classical and quantum diffusions. We study here the more primitive notion of a quantum random walk associated to a general Hopf algebra and show that it has a simple physical interpretation in quantum mechanics. This is by means of a representation theorem motivated from the theory of Kac algebras: If \( H \) is any Hopf algebra, it may be realised in \( \text{Lin}(H) \) in such a way that \( \Delta h = W(h \otimes 1)W^{-1} \) for an operator \( W \). This \( W \) is interpreted as the time evolution operator for the system at time \( t \) coupled quantum-mechanically to the system at time \( t + \delta \). Finally, for every Hopf algebra there is a dual one, leading us to a duality operation for quantum random walks and quantum diffusions and a notion of the coentropy of an observable. The dual system has its time reversed with respect to the original system, leading us to a CTP-type theorem.

1 Introduction

A Hopf algebra is an algebra \( H \) over a field \( k \), equipped with algebra homomorphisms \( \Delta : H \to H \otimes H \) (the comultiplication) and \( \epsilon : H \to k \) (the counit) such that

\[
(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta, \quad (\epsilon \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \epsilon)\Delta.
\]

In addition, there is a linear map \( S : H \to H \) (the antipode) obeying \( (S \otimes \text{id})\Delta = 1\epsilon = (\text{id} \otimes S)\Delta \).

The axioms are obtained by writing out the axioms of an algebra and reversing the arrows. Thus the algebra multiplication \( \cdot : H \otimes H \to H \) is supplemented by a ‘reversed-multiplication’ \( \Delta \). This means that the algebra axioms are supplemented in such a way as to restore some

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kind of time-reversal or input-output symmetry. Hopf algebras are perhaps among the simplest systems where this kind of input-output symmetry can be formulated.

The goal of this paper is to use Hopf algebras to explore this general notion of input-output symmetry in a precise setting, namely that of random walks and Markov processes. Thus our algebras $H$ will always be thought of as algebras of observables (i.e. as if obtained from boolean algebras or from probability spaces, or quantum algebras of observables in the non-commutative case). In this context, asking that the algebra is a Hopf algebra really does restore an input-output symmetry to the system. An element $h \in H$ is viewed as a random variable and $\Delta h \in H \otimes H$ expresses a linear combination of composite random variables whose joint evaluation would lead to an evaluation of $h$. Thus, while algebras are traditionally used in formulations of (intuitionistic) deductive logic, with the multiplication expressing necessity of a conclusion, the presence of a Hopf algebra puts us into a framework of modal logic, with the comultiplication expressing (in some sense) the notion of ‘possibility’ to supplement the notion of ‘necessity’. We will examine these issues in a probabilistic setting rather than a logical one, based on classical and quantum random processes.

The idea of thinking about Hopf algebras in this context is not a new one. At least in the context of stochastic calculus, it is well known that Hopf algebras provide a tool for formulating random walks and generalizing them. Indeed, this was one of the motivations behind the development of Hopf algebras some years ago, and in more recent times has lead to quantum diffusions and other processes based on Hopf algebras\cite{1,10,9}. From a mathematical point of view there is no particular obstruction to going ahead and formulating the same constructions when the Hopf algebra is an arbitrary (possibly non-commutative) one. In particular, random walks make sense even when the Hopf algebra is non-commutative. On the other hand, it is not at all clear \textit{a priori} if such generalized random walks are anything more than mathematical deformations, i.e. if they are physical quantum processes. It is this question that we address in the present paper.

Our main result in this direction is in Section 3 and is an algebraic operator realization
theorem for arbitrary Hopf algebras. Some of the ideas behind the realization theorem come from the theory of Kac algebras, see\cite{13}\cite{12}\cite{17}, but a feature of our new treatment is that neither the $\ast$-structure nor a Hilbert space need be built in from the start. Thus, our construction should be of interest to Hopf algebraists and in the theory of discrete random processes, as well as in the quantum context. In a quantum mechanical context it means that every Hopf algebra $H$ leads in a reasonably convincing way to a quantum random walk based on a quantum evolution operator $W$ (which we build from $H$). This is such that a step of the random walk on $H$, considered actively as an operation on $H$, consists of the following. First, we embed $h \in H$ as $h \otimes 1 \in H \otimes H$. Here the first copy of $H$ is the algebra of observables at time $t$ and the second copy is the algebra of observables at time $t + \delta$ (i.e. one step further in time). Then we evolve $h \otimes 1$ by the quantum evolution operator $W$ of the joint system $H \otimes H$. Finally, we take an expectation value in the first copy of $H$ to leave us in the second copy at $t + \delta$. This is the quantum step of a quantum random walk. Our theorem asserts that a random walk on any Hopf algebra can always be put into this form.

Section 4 then proceeds to study the impact of Hopf algebra duality in this context. Every Hopf algebra has a dual one, and hence every such quantum random walk has a dual one. We find that it is truly time-reversed with respect to the original. The ideas here are based on the notion of observable-state symmetry (quantum ‘Mach principle’) developed in the quantum-gravitational context in \cite{13}\cite{12}\cite{17}.

We begin in Section 2 with a brief introduction to how Hopf algebra methods can be used to do classical random walks. Nothing in Section 2.1 should be new to experts, however we have not been able to find a suitable treatment elsewhere. We show how to use the Hopf algebra of the real line to easily derive the diffusion equation for Brownian motion from the limit of a random walk given by stepping to the left or right with probability $p, (1 - p)$. Section 2.2 shows how straightforward $q$-deformation techniques lead immediately to a $q$-deformed random walk on $\mathbb{R}$ and a limiting $q$-Brownian process. The latter result can be compared with the quantum stochastic process related to the Azema martingale and based on a similar (but different) Hopf
algebra in [13]. Our more primitive treatment as the limit of a discrete walk seems to be new.

Let us note that the question of when a (quantum) algebra of observables is a Hopf algebra has already been studied by the author in a very different context through a series of papers [13] [14] [16] [12]. We studied particles moving on curved spaces and found that not every background metric admits this possibility of a Hopf algebra structure for the quantization of a particle in it. For example, for a differentiable 1+1 dimensional quantum dynamical system we looked for Hopf algebras of self-dual type and found that the only possibility was for the background to be that of a black-hole type metric [12]. Mathematically, the existence of a non-cocommutative Hopf algebra structure on the (non-commutative) quantum algebra of observables, corresponds to a non-Abelian group-structure on ‘phase-space’. Here we regard the quantum algebra of observables in the fashion of non-commutative geometry as if it is the ring of functions on some space. The phase space does not exist in an ordinary sense for in this case the functions on it would be commutative. Physically, the reason is that the position and co-ordinate functions in phase space can no longer be determined simultaneously because of Heisenberg’s uncertainty principle. Thus, the Hopf algebra structure corresponds to a non-Abelian group structure on phase space but in a language general enough to allow the space to be a quantum space rather than a classical one. The non-Abelianess means that the phase-space as a (quantum) space is curved, and hence led to models exhibiting a unification of quantum effects with gravitational ones. These models in [13] [12] also exhibited a remarkable duality phenomenon, based on Hopf algebra duality and interchanging microscopic (quantum) physics with macroscopic (gravitational physics) [17]. As a byproduct, we were led to a large class of non-commutative and non-cocommutative Hopf algebras which were quite different from the more popular quasitriangular Hopf algebras (quantum groups) of Drinfeld and Jimbo etc. related to Yang-Baxter equations and such topics.

Thus, it is the same mathematical structure, namely an algebra of observables that is a Hopf algebra, which has these two interpretations, (i) in terms of gravitational physics as in the authors previous work and (ii) in terms of probability theory as here. Either application alone would lead us to study Hopf algebras and the fact that the same structure serves both indicates
the possibility of a remarkable unification of the two topics. Roughly speaking, generalized
(quantum) Riemannian geometry can be reformulated in terms of the language of random walks,
and vice-versa. This is one of the long-term motivations behind the present work, and will
be further developed elsewhere. Here we concentrate only on the probabilistic interpretation.
Both (i) and (ii) are quite different again from the standard interpretation of quasitriangular
Hopf algebras in the theory of inverse scattering - where the quantum group is not at all a
quantum algebra of observables but rather a quantum symmetry. This role of quantum symmetry
represents a third potential unification made possible by Hopf algebras, but not something that
we shall develop here. On the other hand, there is a tangential connection between related
quantum spin chains and non-commutative Markov processes which can be mentioned.

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2 Random Walks and Markov Processes via Hopf Algebras

In this preliminary section we explain in elementary terms how Hopf algebras can be used to
describe random walks and Markov processes. Even in simple and well-known cases they encode
the required computation in a very direct way. The reason for this is quite fundamental as
explained in the introduction.

2.1 Brownian Motion

We begin by recalling some standard ideas from classical probability. To be concrete we focus
on \( \mathbb{R} \) (the real line) as our probability space. A probability density function for us is simply a
pointwise positive function \( \rho \in L^1(\mathbb{R}) \) such that \( \int dx \rho(x) = 1 \). We also allow atomic measures
(\( \delta \)-functions). We can use such a \( \rho \) to define a random walk as follows: at \( x_0 = 0 \), \( x_{i+1} = x_i + X \n where \( X \) is chosen randomly with probability distribution \( \rho \). The routine question to ask is:
what is the probability distribution after \( n \) steps, i.e. of \( x_n \)? Clearly it is given by a probability
density $\rho^n$ (say) defined by

$$\rho^n(x) = \int_{y_1+\ldots+y_n=x} dy_1 \cdots dy_n \rho(y_1) \cdots \rho(y_n). \quad (2)$$

Note that in asking after the system at various steps, we are always asking the same mathematical question, namely the distribution of a random variable on the real line. Thus we take the point of view that the question, which we denote $X$, is not changing with each step. Rather, it is the probability distribution of $X$ which is changing with each step. This $X$ is the abstract position observable of the random walk, i.e. ‘where the particle is at’ (statistically speaking). Its expectation value and moments etc are changing with each time step. The distribution (2) immediately gives

$$<1>_{\rho^n} = 1, \quad <X>_{\rho^n} = n <X>_{\rho}, \quad <X^2>_{\rho^n} = n(<X^2>_{\rho} - <X>^2_{\rho}) + n^2 <X>^2_{\rho}$$

where $<>$ denotes expectation value in the marked state. From this we see that the mean position is increasing with each step, as would be expected. Also, if we define $Y^{(n)} = \frac{1}{n}X$ then $<Y^{(n)}>_{\rho^n} = <X>_{\rho}$ is independent of $n$ while its variance $<Y^{(n)2}>_{\rho^n} - <Y^{(n)}>_{\rho}^2 = \frac{1}{n}(<X^2>_{\rho} - <X>^2_{\rho})$ decreases. Similarly the higher variances decrease, so that the rescaled variables $Y^{(n)}$ become more and more sharply peaked. This is the basis of the central limit theorem.

This elementary computation can be done by means of Hopf algebras as follows. Firstly, we regard the abstract question $X$ (roughly speaking) as an element of $L^\infty(\mathbb{R})$ (the Hopf-von Neumann algebra of functions on the real line). It is defined as the coordinate function $X(x) = x$.

Now, a probability distribution assigns to any power $X^n$, or more precisely any bounded function $f(X)$ an expectation value $<f(X)>_{\rho} = \int dx \rho(x)f(x)$. This gives a positive linear functional $\phi$ on $L^\infty(\mathbb{R})$, corresponding to $\rho$ via $\phi(f) = <f(X)>_{\rho}$. Positive means $\phi(f^*f) > 0$ for all $f \neq 0$ and corresponds to $\rho$ real and positive. The normalization corresponds to $\phi(1) = 1$. Thus we can work directly with the Hopf algebra $L^\infty(\mathbb{R})$ in place of random variables, and $\phi$ a (normal, unital) state on $L^\infty(\mathbb{R})$ in place of $\rho$. This is a familiar point of view in the context of classical
mechanics (and quantum mechanics) and $L^\infty(\mathbb{R})$ here is the algebra of observables of the system.

The Hopf algebra structure on $L^\infty(\mathbb{R})$ is $(\Delta f)(x, y) = f(x + y)$, where $L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R}) = L^\infty(\mathbb{R} \times \mathbb{R})$. It expresses the group law on $\mathbb{R}$ in algebraic terms. The co-ordinate function, for example, has the additive comultiplication $(\Delta X)(x, y) = X(x+y) = x+y = (X \otimes 1 + 1 \otimes X)(x, y)$ or as elements of the Hopf algebra,

$$\Delta X = X \otimes 1 + 1 \otimes X.$$  \hfill (4)

The group law played a key role in (2) and likewise, $\Delta$ plays the corresponding role in defining a new state $\phi^n$. Indeed, the comultiplication on any Hopf algebra defines a multiplication in its dual, and it is this multiplication (the convolution product) that we need. Explicitly,

$$(\phi \psi)(f) = (\phi \otimes \psi)(\Delta f), \quad e.g. \quad \phi^n(f) = (\phi^{\otimes n})(\Delta^{n-1} f).$$  \hfill (5)

where $\Delta^{n-1}$ denotes $\Delta$ applied $n - 1$ times. We will come to the formal definition of a Hopf $*$-algebra in Section 2.3, but in our example (and in general) the properties of the $*$-structure are such as to ensure that if $\phi$ is positive then $\phi^n$ is also positive etc, i.e. also a state. Associativity of this convolution algebra is ensured by the first of (1), while $\epsilon$ in (1) is the identity state in the algebra.

In these terms, the computation of (3) looks like

$$< 1 >_{\rho^n} = \phi^n(1) = (\phi^{\otimes n})\Delta^{n-1}1 = (\phi \otimes \cdots \otimes \phi)(1 \otimes \cdots \otimes 1) = \phi(1)^n = 1$$

$$< X >_{\rho^n} = (\phi^{\otimes n})\Delta^{n-1}X = (\phi \otimes \cdots \otimes \phi)(X \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes X) = n\phi(X) = n < X >_{\rho}$$

$$< X^m >_{\rho^n} = (\phi^{\otimes n})\Delta^{n-1}X^m = (\phi \otimes \cdots \otimes \phi)(\sum_{i_1 + \cdots + i_n = m} \binom{m}{i_1 \cdots i_n} X^{i_1} \otimes \cdots \otimes X^{i_n})$$

$$= \sum_{i_1 \cdots i_n} \binom{m}{i_1 \cdots i_n} \phi(X^{i_1}) \cdots \phi(X^{i_n}) = \sum (m)_{i_1 \cdots i_n} < X^{i_1} >_{\rho} \cdots < X^{i_n} >_{\rho}$$

Thus we reproduce the results above in this new language. The computation in this form uses nothing other than the two equations (4) and (5). Not only does it bypass unpleasant convolution integrals as in (2), but it is conceptually rather cleaner also. The conceptual picture is that expectation value of $X$ after $n$ steps is simply the expectation value of $\Delta^{n-1}X$ in the tensor
product system $L^\infty(\mathbb{R})^\otimes n, \phi^\otimes n$. The tensor product system is the system for $n$ independent random variables, each with the same distribution $\phi$, and the iterated comultiplication embeds the algebra of observables for our one particle into this system. If we denote these co-ordinate functions that generate our $n$-fold tensor product by

$$X_i = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1$$  \hfill (6)

(embedded in the $i$'th position), then the random variable $X$ of our one particle embeds as $X_1 + X_2 + \cdots + X_n$. Regarding $\Delta^{n-1}$ as understood, we can simply write $X = X_1 + \cdots + X_n$. Here the $X_i$ are $n$ independent random variables.

To demonstrate the power of this formalism further, let us compute a specific example. In terms of $\rho$ we take an atomic measure with $\rho = p\delta_a + (1-p)\delta_{-a}$, i.e. peaked as $\delta$-functions at $a, -a$. In terms of $\phi$ we have

$$\phi(f) = (p\phi_a + (1-p)\phi_{-a})(f) = pf(a) + (1-p)f(-a)$$  \hfill (7)

where $\phi_a(f) = f(a)$ is the linear map on $L^\infty(\mathbb{R})$ given by evaluation at $a$. We can also introduce the linear map $D(f) = f'(0)$. It is not a state but we can still view it as a (densely defined) map on $L^\infty(\mathbb{R})$ and make the convolution product etc as above, i.e. we formally view it in the convolution algebra. Then the form of $\Delta$ in (5) gives

$$D^n(f) = (D \otimes \cdots \otimes D)(\Delta^{n-1} f) = \frac{\partial}{\partial x_1}|_0 \cdots \frac{\partial}{\partial x_n}|_0 f(x_1 + \cdots + x_n) = f^{(n)}(0).$$  \hfill (8)

Hence in this convolution algebra we have the expansion

$$\phi_a = \epsilon + aD + \frac{a^2}{2!}D^2 + \cdots$$  \hfill (9)

(Taylors theorem). We are now ready to compute the system after $n$ steps (and its limit) as described by the states

$$\phi^n = (\epsilon + 2a(p - \frac{1}{2})D + \frac{a^2}{2!}D^2 + \cdots)^n = (\epsilon + \frac{ct}{n}D + \frac{t\alpha}{n}D^2 + \cdots)^n \rightarrow \phi^\infty = e^{t\alpha D^2 + ctD}$$  \hfill (10)

where $t, c, \alpha$ are defined by $t = n\delta$, $\frac{a^2}{2} = \frac{t\alpha}{n} = \alpha \delta$ and $2a(p - \frac{1}{2}) = \frac{tc}{n} = \delta c$ and we send $a \rightarrow 0$ and $n \rightarrow \infty$ (or $\delta \rightarrow 0$) with $t, c, \alpha$ fixed.
This limiting state describes our random walk after an infinite number of steps, with the step viewed as a step in time of size $\delta$, which tends to zero, i.e. it is the continuous limit of the random walk. We can still evaluate our observables in this limit,

$$\phi^\infty(f) = (e^{t\alpha D^2 + ctD})(f) = (e^{t\alpha \frac{d^2}{dx^2} + c \frac{d}{dx}} f)(0)$$

(11)

and we can also compute the distribution $\rho^\infty$ corresponding to $\phi^\infty$. Formally, if $\phi$ is any state we can formally get back the corresponding density by $\rho(x) = \phi(\delta x)$ (the $\delta$-function at $x$, which of course has to be approximated to lie in $L^\infty(\mathbb{R})$). We have

$$\rho^\infty(y) = \phi^\infty(\delta y) = (e^{t\alpha \frac{d^2}{dx^2} + c \frac{d}{dx}} \delta y)|_{x=0} = (4\pi t)^{-\frac{1}{2}} e^{-\frac{|y-ct|^2}{4\alpha t}}$$

(12)

where the right hand side is the unique solution $G(y, t)$ of the diffusion equation $\frac{\partial}{\partial t} G(y, t) = \alpha \frac{\partial^2}{\partial y^2} G(y, t) - c \frac{\partial}{\partial y} G(y, t)$. Another way to see that this must coincide with $\rho^\infty$ is that the latter is characterized by $\int \rho^\infty(x) f(x) = e^{t\alpha D^2 + ctD}(f)$ for all $f$: Differentiating this $\frac{\partial}{\partial t}$ we have

$$\int \left( \frac{\partial}{\partial t} \rho^\infty \right)(x) f(x) dx = \left( (\alpha D^2 + cD)e^{t\alpha D^2 + ctD} \right)(f)$$

$$= \int \rho^\infty(x)(\alpha \frac{d^2}{dx^2} + c \frac{d}{dx})f(x)dx$$

$$= \int \left( (\alpha \frac{d^2}{dx^2} - c \frac{d}{dx})\rho^\infty(x) \right)f(x)dx$$

(13)

Where $(D \otimes \text{id})\Delta f(X) = f'(X)$. Our derivation of (12) can be compared with more standard derivations based on Stirling’s formula etc. In our case, we used only a limit of the form $(1 + \frac{Z}{n})^n \to e^Z$ in deriving (10).

These elementary computations demonstrate the usefulness and directness of this approach based on Hopf algebras. A remarkable fact can now be observed: in these computations we do not need to assume that the Hopf algebra is commutative! Thus, if $H$ is any Hopf algebra, the convolution product (5) for elements $\phi, \psi$ of $H^*$ make perfectly good sense. We can also take continuum limits by expanding $\phi$ and using the same limiting procedure as in (10) without assuming commutativity. These remarks fit well into the overall context of ‘non-commutative’ or ‘quantum’ probability, where we replace $L^\infty(\Omega)$ for a classical probability space $\Omega$ by a non-commutative von Neumann algebra: we can do random walks by working with not-necessarily
commutative Hopf-von Neumann algebras. Such generalized random walks can be called ‘non-commutative’ or ‘quantum’ random walks.

This situation can be viewed as some of the underlying motivation behind, for example, the notion of a quantum stochastic process. Examples of the latter are indeed known in connection with Hopf algebras[19], though how they might be obtained as the limit of random walks seems to be less well studied, but see for example [11]. Moreover, whether such generalized non-commutative random walks can be viewed as actual quantum mechanical processes is also an open question. These are two of the issues to be addressed in Sections 3,4.

2.2 $q$-Brownian Motion

We now give a non-commutative example. The above random walk leads to Brownian motion (given by the diffusion equation): the non-commutative example will constitute a $q$-deformation of it, and can be called $q$-Brownian motion. A related example has been studied as a quantum stochastic process in [18] under the heading of the Azema process. Our more primitive treatment as the limit of a random walk seems to be new and in any case is based on a different Hopf algebra. Note that a $*$-structure was implicit in the above (the states are taken positive) and this continues to be so in our setting now even though the algebra is non-commutative. As in quantum mechanics, the observables of most interest are the self-adjoint ones and states should be positive in the sense $\phi(h^*h) \geq 0$ for all $h$ in the algebra. We assume they are normalised as $\phi(1) = 1$.

For our Hopf algebra $H$ we take generators $X, g, g^{-1}$ and relations, comultiplication, counit, antipode and $*$-structure

\begin{align}
  gX &= qXg, \quad gg^{-1} = 1 = g^{-1}g, \quad \Delta g = g \otimes g, \quad \Delta X = X \otimes g^{-1} + g \otimes X, \quad \epsilon g = 1, \quad \epsilon X = 0. \\
  SX &= -q^{-1}X, \quad Sg = g^{-1}, \quad X^* = X, \quad g^* = g, \quad q^* = q^{-1}.
\end{align}

This is a standard Hopf algebra [21] but note that $q$ is a parameter of modulus 1 for the $*$-structure that we need. The elements $X^m g^i$ for $m \in \mathbb{Z}_+, i \in \mathbb{Z}$ are a basis. The comultiplication
should be compared with (14). If $\phi$ is any state as above, the distribution $\phi^n$ after $n$ steps is computed from (5) using the comultiplication $\Delta$. It is given by embedding $X$ in $H^\otimes n$ via $\Delta^{n-1}$ and applying $\phi$ to each $H$. From (14) we have

$$\Delta^{n-1} g = g \otimes \cdots \otimes g, \quad \Delta^{n-1} X = X \otimes g^{-1} \otimes \cdots \otimes g^{-1} + g \otimes X \otimes g^{-1} \otimes \cdots \otimes g^{-1} + \cdots + g \otimes \cdots \otimes g \otimes X. \tag{16}$$

If we write the right hand expression as $\Delta^{n-1} X = X_1 + X_2 + \cdots + X_n$ where $X_1 = X \otimes g^{-1} \otimes \cdots \otimes g^{-1}$, $X_2 = g \otimes X \otimes g^{-1} \otimes \cdots \otimes g^{-1}$ etc, we see that the random variable describing the position after $n$ steps is the sum of $n$ random variables $X_i$ embedded in $H^\otimes n$. They are not, however, independent in a usual sense. Instead,

$$X_i X_j = q^2 X_j X_i, \quad i > j \tag{17}$$

from the relations in (14). This shows up when we look at higher moments, where we need to compute $\Delta^{n-1} X^m$. This is a standard computation in connection with the above Hopf algebra.

We have

$$\Delta^{n-1} X^m = \sum_{i_1 + \cdots + i_n = m} \frac{[m]_{q^2}!}{[i_1]_{q^2}! \cdots [i_n]_{q^2}!} X_1^{i_1} \cdots X_n^{i_n}, \quad [i]_q = \frac{1 - q^n}{1 - q} \tag{18}$$

using the relations (17).

**Proposition 2.1** The expectation values after $n$ steps are

$$< g >_\phi^n = < g >_\phi^n, \quad < X >_\phi^n = < X >_\phi, \quad < g >_\phi^n - < g^{-1} >_\phi^n$$

$$< X^2 >_\phi^n = < X^2 >_\phi, \quad < g^2 >_\phi - < g^{-2} >_\phi + [2]_q < Xg >_\phi < g^{-1} X >_\phi \frac{[n]_{q^2} >_\phi - [n]_{q^2^{-2}} >_\phi}{< g^2 >_\phi - < g^{-2} >_\phi}$$

These moments reduce to the usual results (3) for a random walk on $\mathbb{R}$ if we set $q \to 1$ and $g \to 1$ in a strong sense with $< g^{\pm 1} >_\phi < g >_\phi^{\pm 1}$ and $< g^{\pm 2} >_\phi < g >_\phi^{\pm 2}$.

**Proof** The expectation of $g$ and $X$ follows at once from (18). For the expectation of $X^2$ we have to write out $\Delta X^2 = \sum_{i=1}^n X_i^2 + \sum_{i<j}[2]_{q^2} X_i X_j$ from (18) explicitly in terms of the $X_i$. For the first sum we have $X^2 \otimes g^{-2} \otimes \cdots \otimes g^{-2} + g^2 \otimes X \otimes g^{-2} \otimes \cdots \otimes g^{-2} + \cdots$ which gives the first...
term shown. For the second term we note

\[ X_i X_j = g^2 \otimes \cdots \otimes g^2 \otimes X g \otimes 1 \otimes \cdots \otimes 1 \otimes g^{-1} X \otimes g^{-2} \otimes \cdots \otimes g^{-1} \]

where \( X g \) is in the \( i \)'th position and \( g^{-1} X \) is in the \( j \)'th. Applying \( \phi^\otimes n \) gives a factor \( < g^2 >_\phi^{i-1} < g^{-2} >_\phi^{n-j} \). Summing over \( j \), we obtain for the second term in \( < X^2 >_\phi^n \) the result

\[ [2]_{q^2} < X g >_\phi < g^{-1} X >_\phi \times \text{the expression} \]

\[
\sum_{i=1}^{n-1} < g^2 >_\phi^{i-1} \frac{1-<g^{-2}>_\phi^{n-i}}{1-<g^{-2}>_\phi} - \frac{<g^{-2}>_\phi}{(1-<g^{-2}>_\phi)(1-<g^{-2}>_\phi)} \left( <g^2 >_\phi^{n-i} - <g^{-2}>_\phi^{n-i} \right) \\
= \left( <g^2 >_\phi - <g^{-2}>_\phi \right)^{-1} \left( \frac{1-<g^2 >_\phi^{n-i}}{1-<g^2 >_\phi} - \frac{1-<g^{-2}>_\phi^{n-i}}{1-<g^{-2}>_\phi} \right)
\]

after suitable reorganization of the partial fractions. Note that if we write \( < g^{\pm 2} >_\phi = 1 \pm 2 \delta + O(\delta) \) and take \( \delta \to 0 \) then this expression tends to \( \frac{n(n-1)}{2} \) as in Section 2.1 \( \square \)

We can also compute the continuous limit of a random walk on this Hopf algebra, as follows. For simplicity we assume that \( q \) is not a root of unity. For our state \( \phi \) we take

\[ \phi(g^i f(X) g^j) = pf(q^{\frac{i-j}{2}} a) + (1-p)f(-q^{\frac{i-j}{2}} a) \]  \hspace{1cm} (19)

for \( a \in \mathbb{R} \) and \( 0 \leq p \leq 1 \). This reduces to the choice in Section 2.1 when \( i = j \). Note that \( \phi((f(X) g^i)^* f(X) g^j) = \phi(g^i \overline{f}(X) f(X) g^j) = p|f(a)|^2 + (1-p)|f(-a)|^2 \) so that \( \phi \) is positive. Here, and below, we can concentrate on \( f \) given by polynomials or suitable power-series in the \( X \). As in Section 2.1, this algebraic approach is largely for convenience: underlying it is a Hopf-von Neumann algebra similar to an extension of \( L^\infty(\mathbb{R}) \) above (this is needed to make sense of some of the exponentials below).

In order to approximate \( \phi \) we make use of a well known operator of ‘\( q \)-differentiation’ \( \partial_q \) cf[3] to define an operator \( \mathcal{D}_q : H \to H \) by

\[
\mathcal{D}_q g^i f(X) g^j = g^{\frac{i-j}{2}} \frac{f(qX) - f(q^{-1} X)}{(q - q^{-1}) X} g^{\frac{i-j}{2}} q^{-\frac{i-j}{2}} = g^{\frac{i-j}{2}} (\partial_q f) g^{\frac{i-j}{2}} q^{-\frac{i-j}{2}}
\]  \hspace{1cm} (20)
where we have extended the action to \( g \) in such a way that \( \mathcal{D}_q \) is a \( * \)-preserving and (formally) a completely positive operator. Here the \( g^{-\frac{1}{2}} \) is for notational convenience: it should be understood on one side or the other as a factor \( g^{-1} \) (in view of the relations (14)). In particular, we have

\[
\mathcal{D}_q g^i X^m g^j = \frac{[m]_q}{q-q^{-1}} g^{1-i} X^{m-1} g^{j-i} q^{\frac{m-1}{2}} q^{\frac{m-1}{2}}, \quad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{21}
\]

Motivated by this we take a linear functional \( \mathcal{D}_q : H \to \mathbb{C} \) given by evaluation of \( \mathcal{D}_q \) at \( X = 0, g = 1 \) and consider its iterated convolution product according to (5). We have

\[
\mathcal{D}_q (g^i X^m g^j) = \delta_{m,n} g^{i-j} q^{i-j} \tag{22}
\]

\[
\mathcal{D}_q^n (g^i X^m g^j) = (\mathcal{D}_q \otimes \cdots \otimes \mathcal{D}_q) \left( (g^i \otimes \cdots \otimes g^j)(\Delta^{n-1} X^m)(g^j \otimes \cdots \otimes g^j) \right) = \delta_{n,m} [m]_q q^{\frac{i-(n-1)+j}{2}} q^{\frac{i-(n-2)+j}{2}} \cdots q^{\frac{i-(n-1)-j}{2}} = \delta_{n,m} [m]_q q^{i-j} \tag{23}
\]

using (16) to obtain (22) and then (15), (22) to obtain (23).

We also need a (well known) \( q \)-exponential \( e_q \) defined with coefficients \( 1/[i]_q \) in place of the usual \( 1/i! \). From (23) we conclude in the convolution algebra that

\[
(e_q^{\alpha D_q})(g^i f(X) g^j) = f(q^{i-j} a). \tag{24}
\]

Hence we can approximate \( \phi \) just as before, by

\[
\phi^n = (\epsilon + 2a(p - \frac{1}{2}) D_q + \frac{a^2}{[[2]]_q} D_q^2 + \cdots)^n = (\epsilon + \frac{ct}{n} D_q + \frac{t\alpha}{n} D_q^2 + \cdots)^n, \quad \phi^\infty = e^{t\alpha D_q + c t D_q} \tag{25}
\]

where \( t, c, \alpha \) are defined by \( t = n\delta \), \( \frac{a^2}{[[2]]_q} = \frac{t\alpha}{n} \) and \( 2a(p - \frac{1}{2}) = \frac{tc}{n} \) and we send \( a \to 0 \) and \( n \to \infty \) (or \( \delta \to 0 \)) with \( t, c, \alpha \) fixed as before.

We can then evaluate \( \phi^\infty \) on elements of our algebra to compute expectation values. We can also try to introduce a corresponding probability density \( \rho^\infty \) defined via

\[
\phi^\infty(f(X)) = \int_{-\infty}^{\infty} \rho^\infty(x) f(x) dx, \quad \forall f \tag{26}
\]

with respect to ordinary integration (say). This means that we are visualizing our deformed random walk with respect to the usual picture of \( \mathbb{R} \), which is embedded in \( H \) as a subalgebra.
This is not the only possibility (one could, for example, use here a $q$-integration). Note also that

$$\phi^\infty(g^i f(X)g^i) = \phi^\infty(f(X))$$  \(27\)

from (23), so (26) determines $\phi^\infty$ completely. Proceeding with (26) and differentiating with respect to $t$ we have

$$\int (\frac{\partial}{\partial t}\rho^\infty)(x)f(x)dx = ((\alpha D_q^2 + c D_q)e^{t\alpha D_q^2 + ctD_q})(f(X))$$

$$= \phi^\infty((\alpha D_q^2 + c D_q)f(X))$$

$$= \int (\rho^\infty(x)\partial^2_q + c \rho^\infty(x)\partial_q)f(x)dx$$

$$= \int (\alpha\partial^2_q \rho^\infty(x) - c \partial_q \rho^\infty(x))f(x)dx. \quad (28)$$

where we used

$$(D_q \otimes \text{id})\Delta = D_q. \quad (29)$$

This is easily proven from (16) and (22). We also used (27) and the identity

$$\int h(x)f(qx) - f(q^{-1}x) \frac{dx}{(q - q^{-1})x} = \int -h(q^{-1}x')f(x')dx' - \int h(qx') f(x') dx' = - \int h(qx) - h(q^{-1}x) \frac{dx}{(q - q^{-1})x} f(x)dx$$  \(30\)

for all analytic functions $h, f$ such that the contours can be rotated for the changes of variable (so that $\partial^\dagger_q = -\partial_q$ with respect to this $L^2$ inner product on suitable test functions). Thus, $\rho^\infty$ is characterized as the solution of

$$\frac{\partial}{\partial t} \rho^\infty(x,t) = \alpha \partial^2_q \rho^\infty(x,t) - c \partial_q \rho^\infty(x,t). \quad (31)$$

This is a $q$-deformed diffusion equation. Note that unlike the undeformed case, however, our $\phi^\infty$ does not involve ordinary differentiation but rather the non-local operator of $q$-differentiation, and hence there is no reason to think that a smooth solution $\rho^\infty$ to (31) should exist along familiar Gaussian lines. It can perhaps be interpreted stochastically. Our elementary derivation of $q$-Brownian motion can be compared with the treatment of the Azema martingale\[18\] and its quantum stochastic process in \[19\].
2.3 Transition Operators

The last example should convince the reader that it is interesting to consider random walks on non-commutative Hopf algebras, even if only as deformations of standard ones. Note also that in the above random walks, we are not required to use the same state \( \phi \) for the distribution of each step of the walk. We could just as easily have a collection of states \( \{\phi_1, \phi_2, \cdots, \phi_n\} \) to be used successively in each step of the random walk. The distribution after \( n \) steps is given by the convolution product \( \phi_1 \phi_2 \cdots \phi_n \), i.e. the tensor product \( \phi_1 \otimes \phi_2 \cdots \otimes \phi_n \) applied to the image via \( \Delta^{n-1} \) in \( H^\otimes n \). Such a random walk with distinct steps is said to be non-stationary.

Closely related to random walks are Markov processes, and these too can be built on Hopf algebras. In fact, for our purposes they are the same structure from an equivalent ‘active’ point of view. We briefly explain this now. Thus, given a linear functional \( \phi \) on \( H \), we define \( T_{\phi} : H \to H \) by

\[
T_{\phi} = (\phi \otimes \text{id})\Delta
\]

This is equivalent to \( \phi \), and also recovers the convolution multiplication (32) as composition of operators

\[
\epsilon \circ T_{\phi} = \phi, \quad \phi \psi = \epsilon \circ T_{\psi} T_{\phi}
\]

on using the counity and coassociativity axioms (3) of \( \Delta, \epsilon \). Thus, our all-important convolution product is just composition of a corresponding operator \( T_{\phi} : H \to H \). It can be called the Markov transition operator corresponding to \( \phi \).

In terms of the Markov transition operator, the system evolves actively by \( T_{\phi} : H \to H \) and the final expectation values after \( n \) steps are obtained by applying \( \epsilon \) to the evolved observables. Thus, if we have a random walk with varying states \( \phi_1, \cdots, \phi_n \) at each step, the corresponding expectations after \( n \) steps are

\[
\langle f \rangle_{\phi_1 \phi_2 \cdots \phi_n} = \epsilon \circ T_{\phi_n} \cdots T_{\phi_1}(f).
\]

An algebra \( H \) equipped with a step-evolution operator, or operators \( T : H \to H \) is a (possibly non-commutative) Markov process. We see that this is just an ‘active’ way of thinking about
random walks, in which the observables rather than the states evolve. We will say more about this in the next section.

In the above example, the role of the counit was in the evaluation at \( X = 0, g = 1 \) (see (14)). Thus, \( D_q = T_{D_q} \) is the operator \((q\text{-differentiation})\) corresponding to the functional \( D_q = \epsilon \circ D_q \) that was used, see (24). Likewise, the transition operator \( T_{\phi} \) corresponding to the limiting state is

\[
T_{\phi} = e^{\alpha D_q^2 + c t D_q},
\]

i.e. the \( q \)-deformed time evolution.

This formalism of random walks and transition operators works for general Hopf algebras over a field \( k \). Finally, we formalize the situation regarding the \(*\)-structure needed in a quantum mechanical setting. Recall that a \(*\)-algebra means an algebra over \( k = \mathbb{C} \) equipped with an antilinear involutive anti-algebra homomorphism \(*\). Then a Hopf \(*\)-algebra is a \(*\)-algebra \( H \) such that

\[
\Delta h^* = (\Delta h)^* \otimes^*, \quad \epsilon(h^*) = \overline{\epsilon(h)}, \quad (S \circ *)^2 = \text{id}. \tag{35}
\]

Such \(*\)-structures on Hopf algebras have been emphasised by [22], among others. In this case (for \( \phi \) positive), the transition operator \( T_{\phi} \) is a completely positive map [1]. Our example in Section 2.2 fits into this setting and \( D_q \) is completely positive, at least formally.

Also, if \( H \) is a Hopf algebra then \( H^* \) (defined in a suitable way) is also a Hopf algebra and the comultiplication in one is determined by the multiplication in the other. This is just the origin of the convolution product [5], but is also works the other way with \( < \Delta \phi, h \otimes g > = < \phi, hg > \) and \( < S \phi, h > = < \phi, Sh > \) with the comultiplication and antipode of \( H^* \) coming from \( H \) (for this reason we write the evaluation symmetrically as \(< , >\) and say that \( H, H^* \) are dually paired Hopf algebras [13, Sec. 1]). In the Hopf \(*\)-algebra case we have in addition

\[
< \phi^*, h > = \overline{< \phi, (Sh)^* >}. \tag{36}
\]

It follows at once from this and (35) that

\[
T_{\phi}(h)^* = T_{(S\phi)^*}(h^*) \tag{37}
\]
so that $T_\phi$ is $\ast$-preserving iff $\phi$ is *anti-self-adjoint* in the sense

$$\phi^* = S\phi$$  \hspace{1cm} (38)

in $H^\ast$. For our example of Section 2.2, the state $\mathcal{D}_q$ is anti-self-adjoint in this way so that $\mathcal{D}_q$ was necessarily $\ast$-preserving, as easily seen by direct computation. In summary, positivity of $\phi$ corresponds to complete positivity of the corresponding transition operator, while anti-self-adjointness corresponds to the transition operator being $\ast$-preserving.

Let us note finally that Markov processes (like random walks and stochastic processes) can be defined more generally than those based on Hopf algebras, for example in the context of spin chains in [4]. Also, their continuum limit is related to a process of dilation of operators [10].

### 3 Operator Realization of Hopf Algebras

We have seen in the last section how any Hopf algebra $H$ equipped with a linear functional $\phi$ can be interpreted formally as leading to a generalized random walk or Markov process. In the random walk interpretation the distribution of an observable $h \in H$ after $n$ steps is given by embedding $h$ into $H^\otimes n$ as $\Delta^{n-1}h$ and applying the expectation value $\phi$ to each factor. The factors are the steps of the random walk and $\Delta^{n-1}h$ is understood literally as the linear combination of all the ways to arrive at $h$ via the elements in $H^\otimes n$ viewed as successive steps. In this section we explore this interpretation further by proving a general result about the coproduct of an arbitrary Hopf algebra.

Although we are primarily interested in the Hopf $\ast$-algebra situation needed for quantum mechanics, our first result is more general and works over an arbitrary field or commutative ring $k$. Perhaps this more general case will be needed in discrete applications as well as in attempts to unify the present considerations with Planck-scale physics as mentioned in the Introduction. If we have to drop classical geometry at the Planck scale, then it is likely that we will have to drop classical functional analysis too (based on $\mathbb{R}^n$) at some point, and do everything algebraically. While the result is known in some form at a Hopf $\ast$-algebra level [3], our purely algebraic level seems to be more novel.
Theorem 3.1 Let $H$ be a Hopf algebra and view $H \subset \text{Lin}(H)$ by $h \mapsto L_h$ (the left regular representation $L_h(g) = hg$). In the algebra $\text{Lin}(H \otimes H) \supset \text{Lin}(H) \otimes \text{Lin}(H)$ there is an invertible element $W$ such that
\[
\Delta h = W(h \otimes 1)W^{-1}, \quad S h = (\epsilon \otimes \text{id}) \circ W^{-1}(h \otimes ()), \quad W_{12}W_{13}W_{23} = W_{23}W_{12}.
\]
Here $W_{12} = W \otimes 1$, $W_{23} = 1 \otimes W$ in $\text{Lin}((H \otimes 3) \supset \text{Lin}(H)^{\otimes 3}$ (similarly for $W_{13}$). Let $H^* \subset \text{Lin}(H)$ by $\phi \mapsto L^*_\phi$ (the left coregular representation $R^*_\phi = (\text{id} \otimes \phi)\Delta$). Then viewed in this algebra we also have, where defined,
\[
\Delta \phi = W^{-1}(1 \otimes \phi)W, \quad S \phi = (\text{id} \otimes \phi) \circ W^{-1}((\ ) \otimes 1)).
\]
In the finite-dimensional case the subalgebras $H \subset \text{Lin}(H)$ and $H^* \subset \text{Lin}(H)$ together generate all of $\text{Lin}(H)$.

Proof Note that the set of linear maps $\text{Lin}(H \otimes H)$ contains $\text{Lin}(H) \otimes \text{Lin}(H)$ in the standard way but may be larger in the infinite-dimensional case. Explicitly, $W, W^{-1}$ are defined as such linear maps $H \otimes H \to H \otimes H$ by
\[
W(g \otimes h) = \sum g(1) \otimes g(2)h, \quad W^{-1}(g \otimes h) = \sum g(1) \otimes (Sg(2))h.
\] (39)
Here $\Delta g = \sum g(1) \otimes g(2)$ is a standard notation. For brevity, we will omit the $\sum$ signs. We first verify the identities
\[
(W(h \otimes 1)W^{-1})(g \otimes g') = W(h \otimes 1)\rangle (g(1) \otimes (Sg(2))g) = W(hg(1) \otimes (Sg(2))g') = h(1)g(1) \otimes h(2)g(1)(2)(Sg(2))g' = h(1)g \otimes h(2)g' = (\Delta h)\langle (g \otimes g'),
\]
\[
(W^{-1}(1 \otimes \phi)W)(g \otimes g') = W^{-1}(1 \otimes \phi)\rangle (g(1) \otimes g(2)g') = W^{-1}(g(1) \otimes g(2)g'(1)) < \phi, g(2)(2)g'(2) > = g(1) \otimes (Sg(1)(2))g(2)(1)(1) < \phi, g(2)(2)g'(2) > = g(1) \otimes g'(1) < \phi, g(2)(2)g'(2) > = \phi(1) \langle g \otimes \phi(2)g',
\]
Similarly for the antipodes. As for the equations satisfied by $W$ itself, we evaluate on $H \otimes H \otimes H$ as
\[
W_{12}W_{13}W_{23}(g \otimes h \otimes f) = W_{12}W_{13}(g \otimes h(1) \otimes h(2)f) = W_{12}(g(1) \otimes h(1) \otimes g(2)h(2)f)
\]
\[ = g(1)(2)h(1)(2)f = g(1)(2)h(1)(2)f = W_{23}(g(1)(2)h \otimes f) = W_{23}(g \otimes h \otimes f). \]

For the last part we show that every operator \( H \rightarrow H \) arises by actions \( \triangleright \) of \( H \) and \( H^* \) on \( H \), at least when \( H \) is finite-dimensional. In this case every linear operator can be viewed as an element of \( H \otimes H^* \) acting on \( H \) in the usual way by evaluation, namely \( (h \otimes \phi)(g) = \langle \phi, g \rangle \).

We have to represent this by elements of \( H, H^* \) acting via \( \triangleright \). Indeed, as operators in \( \text{Lin}(H) \) we find \( (h \otimes \phi) = h(S^{-1}e_{a(1)})\triangleright(\langle \phi, e_{a(2)} \rangle \otimes f^a) \) where \( e_a \) is a basis of \( H \) and \( f^a \) a dual basis. Thus \( h(S^{-1}e_{a(1)})\triangleright(\langle \phi, e_{a(2)} \rangle \otimes f^a) = h(S^{-1}e_{a(1)})g(1) \otimes \langle \phi, e_{a(2)} \rangle f^a, g(2) \rangle = h(S^{-1}g(2a(1))g(1) \otimes \langle \phi, g(2a(2)) \rangle = h \otimes \phi, g \rangle \) as required. We used coassociativity and the properties of \( S^{-1} \) as skew-antipode. \( \Box \)

If \( W \) is any invertible operator obeying the pentagon identity \( W_{12}W_{13}W_{23} = W_{23}W_{12} \) then it is easy to see that \( \Delta(h) = W(h \otimes 1)W^{-1} \) for any operator \( h \) will always be coassociative and an algebra homomorphism, cf. [9]. If we can arrange also for a counit (typically by restricting our operators to some subalgebra) then this gives a Hopf algebra. A Hopf algebra of this type is very concrete, being realised as operators on some vector space, and hence very suitable for physical applications. On the other hand, it might be thought that only very special Hopf algebras could be obtained concretely in this way, but our theorem says that every Hopf algebra can be realized concretely by acting on itself. Even more, it says that both the Hopf algebra and its dual can be realized concretely at the same time as subalgebras of operators on the same space.

For a quantum-mechanical setting, as well as to make contact with the existing theory of Hopf-von Neumann and Kac algebras, we consider now the situation when \( H \) is a Hopf \( * \)-algebra. To do this, it is helpful to cast the last part of the preceding theorem as a statement about Weyl algebras. If \( H, H^* \) are dually paired Hopf algebras, we define \( w(H) \) (the Weyl algebra of \( H \)) to be the semidirect product \( H \rtimes H^* \) where \( H^* \) acts on \( H \) by the left action \( \triangleright \) above. The multiplication is defined in the linear space \( H \otimes H^* \) by

\[
(h \otimes \phi)(g \otimes \psi) = \sum h g(1) \otimes \phi(1) \psi < \phi(2), g(2) > .
\]
Both $H, H^*$ are subalgebras. Incidentally, this is symmetric in a certain sense between $H, H^*$; one has $w(H^*) \cong w(H)^{op}$ at least if $H$ has a bijective antipode (which we assume). Now, if $H$ is a Hopf $*$-algebra then so is $H^*$ and $w(H)$ becomes a $*$-algebra by

$$(h \otimes \phi)^* = (1 \otimes \phi^*)(h^* \otimes 1).$$  \hspace{1cm} (41)$$

This is the only possibility when we bear in mind that $(h \otimes \phi) = (h \otimes 1)(1 \otimes \phi)$. It is easy to see that it indeed gives a $*$-algebra and that $H, H^*$ are $*$-subalgebras.

**Corollary 3.2** Let $H$ be a finite-dimensional Hopf algebra. Then $\text{Lin}(H) \cong w(H)$ as algebras. The left hand side has the usual composition of linear maps. Hence if $H$ is a Hopf $*$-algebra, $\text{Lin}(H)$ becomes a $*$-algebra.

**Proof** The first part follows at once from the last part of the theorem, where $h \otimes \phi$ in $w(H)$ built on $H \otimes H^*$ corresponds to $h \triangleright (\phi \triangleleft )$ in $\text{Lin}(H)$. Conversely, from the proof there, the element corresponding to $h \otimes \phi \in \text{Lin}(H)$ (acting in the usual way) is $h(S^{-1}e_{a(1)}) \otimes < \phi, e_{a(2)} > f^a \in w(H)$ built on $H \otimes H^*$. Hence $\text{Lin}(H)$ is a $*$-algebra inherited from the $*$-algebra structure of $w(H)$. Since $H, H^*$ are subalgebras, this is such that $(h \triangleright )^* = h^* \triangleright$ and $(\phi \triangleleft )^* = \phi^* \triangleright$ as operators in $\text{Lin}(H)$. \hfill $\Box$

**Proposition 3.3** In the setting of Theorem 3.1, suppose that $H$ is a finite-dimensional Hopf $*$-algebra and let $\text{Lin}(H)$ have the $*$-structure generated by $H, H^*$ (i.e. from $w(H)$). With respect to this $*$-structure, the fundamental operator $W$ is unitary,

$$W^* \otimes ^* = W^{-1}.$$  \hspace{1cm}$W^*$

The right-invariant integral $\int$ on $H$ defines a sesquilinear form $(g, h) = \int g^* h = (\overline{h, g})$ which is compatible with the $*$-structure on $\text{Lin}(H)$ induced by $H, H^*$ in the sense

$$\int (h \triangleright g)^* g' = \int g^* (h^{*} \triangleright g'), \quad \int (\phi \triangleleft g)^* g' = \int g^* (\phi^{*} \triangleright g'), \quad g, g', h \in H, \phi \in H^*.$$  \hspace{1cm}$\int$
Proof For the first part we begin by writing $W$ in terms of such elements from $H, H^*$. Indeed, $W(g \otimes g') = g(1) \otimes g(2) g' = g(1) < f^a, g(2) > \otimes e_a g' = f^{a \triangleright} g \otimes e_a \triangleright g'$, where $e_a$ is a basis of $H$ and $f^a$ is a dual basis. So $W = f^{a \triangleright} \otimes e_a \triangleright$ and hence, by definition, $W^* \otimes ^* = f^{a^* \triangleright} \otimes e_a^* \triangleright = f^{a^* \triangleright} \otimes (S e_a)^* \triangleright$ where $e_a^* = S^{-1} e_a^*, f^{a^*} = f^{a^*}$ are a new mutually dual basis. Dropping the primes, the action of the result is $f^{a \triangleright} g \otimes (S e_a)^* \triangleright g' = g(1) < f^a, g(2) > \otimes (S e_a) g' = g(1) \otimes (S g(2)) g' = W^{-1}(g \otimes g')$.

For the second part we show that this $\ast$-structure on Lin$(H)$ really is an adjoint operation with respect to $(\ , \ )$. The right integral is characterized by $(\text{id} \otimes f) \Delta = 1 f$ and is unique up to scale. This means at once that $(S f)^* = f$ since $(S f)^*$ is also a right integral and has the same normalization, and this implies that $(\ , \ )$ is hermitian as stated. For the operators $h \triangleright$ we have

$(h \triangleright g, g') = \int (h \triangleright g)^* g' = \int (h g)^* g' = \int g^* h^* g' = \int g^* (h \triangleright g') = (g, (h \triangleright) g')$ is automatic. Rather harder is for the operators $\phi \triangleright$,

$$(\phi \triangleright g, h) = \int (\phi \triangleright g)^* h = \int (g(1))^* < \phi, g(2) > h = \int g^* (1) < \phi, (g^* (2))^* > h$$

$$= \int g^* (1) < S^{-1}(\phi^*), g^* (2) > h = \int g^* (1) h(1) < S^{-1}\phi^*(3), g^* (2) > < (S^{-1}\phi^* (2))\phi^*(1), h(2) >$$

$$= \int g^* (1) h(1) < (S^{-1}\phi^* (2)) (1), g^* (2) > < (S^{-1}\phi^* (2)) (2), h(2) > < \phi^*(1), h(3) >$$

$$= \int g^* (1) h(1) < S^{-1}\phi^* (2), g^* (2) h(2) > < \phi^*(1), h(3) >$$

$$= \int (g^* h(1) (1) < S^{-1}\phi^* (2), (g^* h(1) (2) > < \phi^*(1), h(2) >$$

$$= \int g^* h(1) < \phi^*, h(2) > = \int g^* (\phi \triangleright h) = (g, (\phi \triangleright) h).$$

Here the second equality is from the definition of $\phi \triangleright g = g(1) < \phi, g(2) >$. The third equality is from the definition of a Hopf $\ast$-algebra with regard to $\Delta \circ \ast$. The fourth equality is from the relation between the $\ast$ structures in $H, H^*$ and $(S \circ \ast)^2 = \text{id}$. The fifth equality inserts some factors that collapse to $\epsilon(h(2))$ via the fact that $S^{-1}$ is a skew-antipode for $H$. The sixth equality writes the multiplication in $H^*$ in terms of the comultiplication in $H$, and uses coassociativity and that $S^{-1}$ is an anticoalgebra map. The seventh equality writes the comultiplication in $H^*$ in terms of $H$ to combine some factors, while the eighth equality uses that $\Delta$ is an algebra homomorphism. By these manipulations we are ready, in the ninth equality, to use that $f$ is a
right integral, leading to the required result. □

In a functional-analytic context we can take for $H$ a von-Neumann algebra, and in place of $\text{Lin}(H)$ we take $B(\mathcal{H}_\phi)$, the bounded operators on the Hilbert space $\mathcal{H}_\phi$ determined by a state or weight $\phi$. $H$ is embedded in this by the GNS construction. With $\phi = \int$ (the integral on $H$), we have that the *-algebra structure on $\text{Lin}(H)$ becomes the usual adjoint operation $\dagger$ on $B(\mathcal{H}_\phi)$, so that $W^\dagger = W^{-1}$. The structure in Theorem 3.1 and Proposition 3.3 is then characteristic of a Kac algebra $[9]$. This was historically limited to $S^2 = \text{id}$, but this is not needed when formulated along the lines above, cf $[3]$. Although we will work algebraically, we keep in mind this Hopf-von Neumann or Kac algebra setting when we consider quantum mechanical examples. One complication in the Hopf-von Neumann or Kac algebra setting is that the counit is typically unbounded.

As explained in the introduction, the significance of these algebraic results for our present purposes is that they give a direct ‘quantum mechanical’ interpretation of the random walk associated to a Hopf algebra $H$ and a linear functional $\phi$. In terms of the Markov transition operator we see that

$$T_\phi(h) = (\phi \otimes \text{id}) \left( W(h \otimes 1)W^{-1} \right).$$

(42)

Here $\phi$ should be thought of as giving the expectation value over the first copy of $H$ in $H \otimes H \subset \text{Lin}(H) \otimes \text{Lin}(H)$. Thus our random walk, considered actively as a Markov process consist of the following. First, we embed $h \in H$ as $h \otimes 1 \in H \otimes H$. Here the first copy of $H$ is the algebra of observables at time $t$ and the second copy is the algebra of observables at time $t + \delta$ (i.e. one step further in time). Then we evolve $h \otimes 1$ by the quantum evolution operator $W$ of the joint system $H \otimes H$. Finally, we take a conditional expectation value in the first copy of $H$ to leave us in the second copy at $t + \delta$. This represents ‘forgetting’ the details of where the system might have been at the now unobserved time $t$. Thus $T_\phi$ is the quantum step of a quantum random walk in a reasonably physical way.
4 Duality, Coentropy and Time-Reversal

In this section we explore the implications of Hopf algebra duality for the interpretation of quantum random walks on Hopf algebras. A parallel observable-state symmetry in the context of quantum-gravity was explored in [12]. The point of [12] was that when an algebra of observables is a Hopf ∗-algebra, then the dual $H^*$ is also a (Hopf) ∗-algebra. Thus we can, in principle, regard $H^*$ instead as the algebra of observables of some dual quantum system. The original observables are now regarded as elements of $H^{**}$ i.e. linear combinations of states from the dual point of view. Roughly speaking (and not worrying about positivity) the same expectation value $\phi(h)$ of observable $h$ in state $\phi$ is regarded from the dual point of view as $h(\phi)$, the expectation of $\phi$ in state $h$. We gave several examples in [12][16] based on quantum particles on homogeneous spaces. The classical data was a pair of groups $(G_1, G_2)$ (the momentum and position groups respectively) acting on each other in a compatible way. The dual Hopf (von Neumann-Kac) algebra was of the same type with the roles of $G_1, G_2$ interchanged. In the dual picture the quantum particle moves along orbits of $G_2$ in $G_1$ instead of orbits of $G_1$ in $G_2$. Thus, interesting models certainly exist in which the dual system has just as good a physical interpretation as the original system and for which the implications of this observable-state symmetry can be explored. Our considerations will be at a formal algebraic level (there exist algebraic examples too[14]) but the most natural setting that we have in mind is this Hopf-von Neumann or Kac algebra one.

To explore this duality in the present context we fix $H$ a Hopf ∗-algebra and $\phi$ a state on it. We have seen that $\phi$ generates a quantum random walk with transition operator $T_\phi$. Note that there is clearly an arrow of time built into this interpretation as we successively take each step of the walk. It can be expressed by the fact that the completely positive operator $T_\phi$ necessarily increases the entropy of $\phi^n$, as follows.

Firstly, the relative entropy $S(\phi, \psi) \leq 0$ is defined between two states $\phi, \psi$ in a standard way. For the algebra of bounded operators on a Hilbert space, states are of the form $\phi = \sum_i s_i <$
\( \phi_i \| \phi_i >, \psi = \sum_j r_j < \psi_j | \psi_j > \) (convex linear combinations of pure states) and

\[
S(\phi, \psi) = \sum_{i,j} (-s_i \log s_i + s_i \log r_j) | < \phi_i | \psi_j > |^2.
\]

(43)

This definition extends to abstract von Neumann and \( C^* \) algebras as well as into certain algebraic situations. For recent work see [8] and elsewhere. For our purposes below we need only the standard abstract properties of the entropy without worrying too much about its detailed definition. Its heuristic interpretation, as discussed in [3] is that \( e^{S(\phi, \psi)} \) is the probability per unit measurement of the system appearing to be in state \( \phi \) when it is in state \( \psi \). In very general terms it measures the ratio of the impurity of \( \phi \) to that of \( \psi \). For the entropy of a single state an obvious choice for us is to take the entropy relative to \( \epsilon \) or the entropy of \( \epsilon \) relative to the state. For our random walk we have

\[
0 \geq S(\epsilon, \phi^n) = S(\epsilon \otimes^n \circ \Delta^{n-1}, \phi \otimes^n \circ \Delta^{n-1}) \geq S(\epsilon \otimes^n, \phi \otimes^n)
\]

(44)

since \( \Delta^{n-1} \) is a completely positive map (being a *-algebra homomorphism) and using [1]. There is a similar identity \( S(\phi^n, \epsilon) \geq S(\phi \otimes^n, \epsilon \otimes^n) \) although this is less interesting because \( S(\phi, \epsilon) \) etc tend to be \(-\infty \) or 0 (this is the case classically). These inequalities give some information about the relative entropy between \( \epsilon \) and \( \phi^n \). We can also get incremental information in the form

\[
0 \geq S(\phi^n, \phi^{n+1}) = S(\phi^{n-1} \circ T_\phi, \phi^n \circ T_\phi) \geq S(\phi^{n-1}, \phi^n) \geq \cdots \geq S(\epsilon, \phi)
\]

(45)

since \( T_\phi \) is completely positive. This says that as \( n \) grows, the states \( \phi^n \) of our random walk change more and more slowly in the sense that the probability for the system to still appear in state \( \phi^n \) when it is in the next \( \phi^{n+1} \), increases. It also says that the impurity of \( \phi^n \), which is generally less than that of \( \phi^{n+1} \), tends towards the latter. In fact, the reason for this is that as the walk evolves, the states are becoming more and more dissipated or impure, but this however, is in some sense bounded by the right integral \( \int \) from Proposition 3.3 (which we assume is positive). To see this we note that

\[
\int \circ T_\phi = (\phi \otimes \int) \Delta = \phi(1) \int = \int \]

(46)
by the right invariance of $f$. Hence

$$0 \geq S(\phi^{n+1}, \int) = S(\phi^n \circ T_\phi, \int) = S(\phi^n \circ T_\phi, \int \circ T_\phi) \geq S(\phi^n, \int) \geq \cdots \geq S(\epsilon, \int) \quad (47)$$

Here the entropy of $\phi^n$ relative to $f$ indeed increases with each step and in this sense the state becomes more and more similar (before any rescaling) to $f$. It also becomes more and more impure as its degree of impurity approaches the degree of impurity of $f$ as claimed. This integral $f$ represents a kind of ‘maximal entropy’ or ‘maximally impure’ state, and our random walk evolves towards it. In the classical case of functions on a finite group, the integral is precisely the maximally impure state consisting of the average of all the pure states (which are given by evaluation at the various group elements), while $\epsilon$ is a pure state (given by evaluation at the group identity element). In this case it is easy to compute that $S(\epsilon, f) = -\log |\Omega|$ where $|\Omega|$ is the order of the group $\Omega$. There is a similar formula for $S(\epsilon, f)$ for general Hopf algebras and we see that it provides a lower bound for $S(\phi^n, f)$.

The origin of this increase of entropy relative to $f$ is that whereas the evolution of the joint system by $W$ in (42) is invertible, the conditional expectation given by evaluation in the state $\phi$ represents forgetting information. This is the reason that repeated application of $T_\phi$ is entropy increasing in this way. Moreover, we demonstrated in Section 2.2 precisely how this discrete arrow of time leads in a limit to a more familiar continuous arrow of time.

Now for a fixed $H$ we can also consider $a \in H$ as an element of $H^{**}$, and hence this induces a random walk on $H^*$ (a Hopf algebra dually paired with $H$). We need to assume that this $a$ is positive as a linear functional on $H^*$,

**Definition 4.1** An element $a$ in a Hopf $*$-algebra $H$ is copositive if $\langle a, \phi^* \phi \rangle \geq 0$ for all $\phi \in H^*$. It is normalised if $\epsilon(a) = 1$.

For example, if $a$ is grouplike and anti-self-adjoint in the sense $Sa = a^*$, then it is copositive. Convex linear combinations

$$a = \sum_i s_i a_i, \quad Sa_i = a_i^*, \quad \Delta a_i = a_i \otimes a_i$$
of such elements $a_i$ are also copositive. We say that a copositive observable is pure if it is not the convex linear combination of copositives. Now, if $a$ is a normalized copositive element then the random walk induced on $H^*$ has transition operator $T_a : H^* \to H^*$. This random walk when viewed actively as an irreversible evolution among the observables of the dual system (i.e. among elements of $H^*$) can be referred to $H$ where it appears passively as evolution of the states of the dual system. This is given by the adjoint operators $T_a^* : H \to H$. We compute

$$<\phi, T_a^*(h) >=< T_a(\phi), h >= \sum <\phi(1), a > <\phi(2), h >=< \phi, ah >,$$

i.e. $T_a^* = L_a : H \to H$.  

(48)

Thus, the transition operator of the dual random walk corresponds to left multiplication by $a$ when viewed in terms of the original system. Note that information is still being lost with each step even in this picture. An immediate corollary of the remarks in Section 2.3 (at least in the finite-dimensional case) is

**Corollary 4.2** An element $a \in H$ is copositive iff the operator of left-multiplication $L_a : H \to H$ is the adjoint of a completely positive map.

This interpretation of a copositive element $a \in H$ as leading to a quantum random walk in the dual, leads us to define the notion of coentropy of an element of $H$.

**Definition 4.3** Let $a, b \in H$ be copositive elements. The relative coentropy $S^*(a, b)$ is the relative entropy of $a, b$ as linear functionals on $H^*$.

The familiar properties of entropy now appear in dual form for the coentropy. For example, $a \circ T_b = ba$ as linear functionals on $H^*$ (see (48)) means that

$$S^*(ba, bc) \geq S^*(a, c)$$

(49)

for three copositive elements. Likewise, the heuristic interpretation of $S^*(a, b)$ is that $e^{S^*(a, b)}$ should be thought of as the probability per unit trial of mistaking the observable or random variable $b$ for $a$ when examined on random states of the system. In very general terms it is the
ratio of the impurity of $a$ to that of $b$. An example of a copositive element is provided by the left integral element in our Hopf $*$-algebra. This is an element $\Lambda \in H$ such that $h\Lambda = \epsilon(h)\Lambda$ for all $h$, and at least in the finite-dimensional case this exists and is unique up to scale. It is anti-self-adjoint according to $SA = \Lambda^*$, and in nice cases is copositive in the above sense. It corresponds to the right integral $\int$ on $H^*$ and is maximally impure in the way that we have seen above for $\int$.

We now consider $H$ equipped with a copositive element $a$. It induces a random walk in $H^*$, but when referred to $H$ this process is the left multiplication operator. Thus, from the point of view where $H$ is regarded as a quantum algebra of observables, the second process appears not as a random walk on $H$ but as what we will call a creation process. Recall that in any GNS representation of $H$, the element 1 in $H$ leads to a vacuum vector $|0>$ and $L_a$ turns this into another vector $a|0>$, created by $a$. Repeating this for a sequence $a_1, a_2, \cdots a_n$ gives the vector

$$a_n a_{n-1} \cdots a_1 |0>.$$

To be concrete here, one can keep in mind the GNS representation given by the integral $\int$ as in Proposition 3.3. In summary, a geometrical process involving the group structure (comultiplication) of $H^*$ corresponds from the point of view of $H$ to a quantum process involving multiplication in the quantum algebra of observables. This was the point of view of [12] also.

This quantum ‘creation process’ also has a simple classical meaning, at least if we do not worry about the copositivity or normalization requirements. In the classical case the algebra of observables is the algebra of functions on a space $\Omega$. Among them, the characteristic functions of subsets of $\Omega$ correspond to the observables or random variables that specify membership of the subsets. A sequence $a_1, a_2, \cdots$ of such characteristic functions is then a creation process that successively specifies more and more conjunctions as we successively multiply more and more characteristic functions. The first step of the creation process is 1 (the identity function), the next step $a_1.1$ specifies membership of one subset, $a_2.a_1.1$ further specifies an additional membership, etc. This models then the way that classical concepts or observables are created.
as a series of specifications. We have seen that this elementary process corresponds to a random walk in the dual.

Finally, our comments above about the entropy of a random walk now become comments about coentropy in a creation process. Using (49), the analogue of (47) is

\[
0 \geq S^*(a^n+1, \Lambda) \geq S^*(a^n, \Lambda) \geq \cdots \geq S^*(1, \Lambda).
\] (50)

This suggests that for a typical copositive element a the sequence \(1, a, a^2, \cdots\) typically tends to \(\Lambda\) (and perhaps converges to something sensible after rescaling according to a dual central limit theorem). In general terms it becomes more and more impure as it approaches \(\Lambda\) in this coentropic sense. We can easily see this phenomenon in the classical case of the algebra of functions on a (say discrete) group \(\Omega\), as follows. In this case the copositives are convex linear combinations of normalised traces (characters) of representations of \(\Omega\), with the pure copositives being the traces of the irreducible representations. The normalization is \(\epsilon(a) = a(e) = 1\) where \(e\) is the group identity. The integral \(\Lambda\) is given by the Kronecker \(\delta\)-function at \(e\). For Abelian groups we have \(a = \sum \epsilon_i a_i\) where the \(a_i\) are the one-dimensional representations of the group, i.e. the ‘plane waves’. If we proceed (with care) to the continuous case \(\Omega = \mathbb{R}\) then the pure copositives are just the pure frequency waves \(a_\omega = e^{i\omega(\cdot)}\). Our assertion that typical copositive elements give creation processes tending to the integral is then something familiar. For example, we can take the convex linear combination

\[
a = \frac{1}{2}e^{i\omega(\cdot)} + \frac{1}{2}e^{-i\omega(\cdot)} = \cos \omega(\cdot)
\]

(this just corresponds to Brownian motion for the dual random walk as in (49)). It is well-known that squaring such cosine waves introduces higher harmonics. Moreover, raising to higher and higher powers \(a^n\) makes the wave more and more impure as further harmonics are introduced until we have something resembling the integral \(\Lambda\). This is the maximally impure wave containing all frequencies, i.e. the \(\delta\)-function. This is a simple physical interpretation of (50) in the classical case.
If we do not worry about the copositivity condition, we can also give another picture of this general phenomenon in terms of membership of subsets as mentioned above. For this we consider a non-stationary creation process given by a family of subsets containing \( e \) (so that their characteristic functions \( a_1, a_2, \cdots \) are correctly normalised). These indeed typically tend to \( \{ e \} \) (corresponding to \( \Lambda \)) as the only point definitely in their joint intersection. This is a second ‘boolean’ interpretation of (50).

We are now ready to consider a system \((H, \phi, a)\) consisting of a Hopf \(*\)-algebra, a state \( \phi \) and a normalized copositive element \( a \) in \( H \). The first element induces a random walk in \( H \). The second induces a creation process in \( H \). In terms of these elementary considerations, we can express our observable-state duality in this context as a ‘CTP’-type proposition.

**Proposition 4.4** Let \((H, \phi, a)\) be a Hopf \(*\)-algebra equipped with state \( \phi \) (giving a random walk) and a normalised copositive element \( a \) (giving a creation process). Then \((H^*, a, \phi)\) has the same interpretation but with the algebra of observables given by \( H^* \) rather than \( H \) and the roles of \( a, \phi \) interchanged.

More generally, let \((H, \phi_1, \phi_2, \cdots, \phi_n, a_1, a_2, \cdots, a_m)\) be a non-stationary random walk with successive distributions \( \phi_1, \phi_2, \cdots \) for each step, and a non-stationary creation process successively creating the vectors \( a_1|0>, a_2 a_1|0>, \cdots \). Then \((H^*, a_m, a_{m-1}, \cdots, a_1, \phi_n, \phi_{n-1}, \cdots, \phi_1)\) has the same interpretation with the roles of \( H, H^* \), \( a, \phi \) interchanged and the order reversed, i.e. successively applying steps \( a_m, a_{m-1}, \cdots \) and creating vectors \( \phi_n|e>, \phi_{n-1}\phi_n|e>, \cdots \). Here the vacuum vector \( |e> \) in \( H^* \) corresponds to the counit \( \epsilon \).

**Proof** The first part summarises the discussion above. We saw in (38) that the random walk on \( H^* \), when referred to \( H \) is the left regular representation. Similarly with the roles of \( H, H^* \) reversed. We are of course expressing nothing other than the self-duality of the axioms of a Hopf \(*\)-algebra. For the second part we have to note carefully the order in (34). The operator \( T \) as a map \( T : H^* \to \text{Lin}(H) \) is an anti-representation. Its adjoint therefore leads to a map \( L : H^* \to \text{Lin}(H^*) \) (the creation process on \( H^* \)) which is a representation rather than
an antirepresentation (since taking adjoints reverses the order). Note that this reversal is not a feature of our conventions but a genuine aspect of the interpretation of $T_\phi$ as explained in Section 2.3 and of the (quantum) creation process as discussed above. □

Thus, the combined random-walk-creation process in $H$ can be viewed equally well as a combined process in $H^*$, but with the time direction built into the processes on $H^*$ reversed relative to the processes on $H$. Thus the symmetry between observables and states when the quantum algebra of observables is a Hopf algebra\cite{12} also involves a time reversal. Another way to see this reversal is to return to Theorem 3.1. There we see also that the comultiplications of both $H$ and $H^*$ can be understood as conjugation by a joint evolution operator. For the former (which leads to the random walk in $H$) this is by embedding $h$ as $h \otimes 1$ and conjugation by $W(\cdot)W^{-1}$. For the latter (which leads to a random walk in $H^*$) it is by embedding $\phi$ as $1 \otimes \phi$ (hence with a left-right or parity reversal) and conjugation by $W^{-1}(\cdot)W$ (hence with the inverse joint evolution operator). We note that this time-reversal phenomenon associated with Hopf algebra duality is reminiscent of CTP-invariance in particle physics. It suggests a relationship between Hopf algebra duality and particle-antiparticle duality which will be developed further elsewhere.

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