Hyperscaling for Oriented Percolation in 1 + 1 Space–Time Dimensions

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Abstract Consider nearest-neighbor oriented percolation in $d + 1$ space–time dimensions. Let $\rho, \eta, \nu$ be the critical exponents for the survival probability up to time $t$, the expected number of vertices at time $t$ connected from the space–time origin, and the gyration radius of those vertices, respectively. We prove that the hyperscaling inequality $d \nu \geq \eta + 2\rho$, which holds for all $d \geq 1$ and is a strict inequality above the upper-critical dimension 4, becomes an equality for $d = 1$, i.e., $\nu = \eta + 2\rho$, provided existence of at least two among $\rho, \eta, \nu$. The key to the proof is the recent result on the critical box-crossing property by Duminil-Copin et al. [6].

Keywords Oriented percolation · Critical behavior · Critical exponent · Hyperscaling · Box-crossing property

1 Introduction and the Main Results

Oriented percolation is a time-oriented model of percolation. It is also considered as a discrete-time model for the spread of an infectious disease, known as the contact process or the SIS model. Since it became known to exhibit a phase transition and critical behavior, there have been intensive researches in both theory and applications in various fields. Recently, a possible association to the laminar-turbulent flow transition was reported in [20].

Consider the following nearest-neighbor bond oriented percolation on the space–time lattice $\mathbb{L}^d \equiv \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ : \|x\|_1 + t \text{ is even}\}$. A pair of vertices $[(x, s), (y, t)]$ is called a bond if $\|x - y\|_1 = 1$ and $t = s + 1$. Each bond $[(x, t), (y, t + 1)]$ is either occupied with probability $p \in [0, 1]$ or vacant with probability $1 - p$, independently of the other bonds. Let $\mathbb{P}_p$ be the associated probability measure. We say that $(x, s) \in \mathbb{L}^d$ is connected to $(y, t) \in \mathbb{L}^d$, denoted by $(x, s) \rightarrow (y, t)$, if either $(x, s) = (y, t)$ or there is a sequence of occupied bonds
\[
((v_j, j), (v_{j+1}, j+1))_{j=1}^{l-1}
\]
from \(v_s = x \) to \(v_t = y\). We simply write \((x, s) \rightarrow t\) for the event \(\bigcup_y \{ (x, s) \rightarrow (y, t) \}\), and \(s \rightarrow (y, t)\) for the event \(\bigcup_x \{ (x, s) \rightarrow (y, t) \}\).

The major quantities we are interested in are the following. The first quantity is the survival probability up to time \(t\), defined as
\[
\theta_t = \mathbb{P}_{\theta}(\emptyset, 0 \rightarrow t),
\]
where, and in the rest of the paper, the \(p\)-dependence is suppressed for lighter notation. Since \(\{\theta_t\}_{t \in \mathbb{N}}\) is a decreasing sequence of increasing and continuous functions in \(p\), the limit \(\theta_\infty \equiv \lim_{t \uparrow \infty} \theta_t\) is nondecreasing and right-continuous in \(p\). Let
\[
p_c = \inf\{ p \in [0,1] : \theta_\infty > 0 \}.
\]
It is proven in [8] that \(\theta_\infty\) is also left-continuous in \(p\). In particular, \(\theta_\infty = 0\) at \(p = p_c\), which has not been proven yet for unoriented percolation in full generality.

The second and third quantities are the expected number of vertices at time \(t\) connected from the origin \((o, 0)\) and the gyration radius of those vertices, defined as
\[
\chi_t = \sum_x \tau(x, t),
\]
\[
\xi_t = \left( \frac{1}{\chi_t} \sum_x |x|^2 \tau(x, t) \right)^{1/2},
\]
where \(\tau(x, t)\) is the two-point function:
\[
\tau(x, t) = \mathbb{P}_{(o, 0)}((o, 0) \rightarrow (x, t)).
\]
It is first proven in \([1, 12]\), and recently reproved in a much simpler way in \([5]\), that the critical point is unique in the sense that
\[
p_c = \sup \left\{ p \in [0,1] : \sum_{t=0}^{\infty} \chi_t < \infty \right\}.
\]
The sum \(\sum_t \chi_t\) is often called the susceptibility.

Now we briefly summarize the basic properties of those quantities readily obtained from the definition. First we note that, by the Markov property and translation invariance,
\[
\theta_{s+t} \geq \theta_s \theta_t, \quad \chi_{s+t} \leq \chi_s \chi_t.
\]
With the help of the trivial inequality \(\theta_t \leq \chi_t \leq (2t+1)^d \theta_t\), we can conclude that there is a common relaxation time \(\zeta \in [0, \infty]\) such that
\[
\zeta = \lim_{t \uparrow \infty} \frac{-t}{\log \theta_t} = \sup_{t \in \mathbb{N}} \frac{-t}{\log \theta_t} = \lim_{t \uparrow \infty} \frac{-t}{\log \chi_t} = \inf_{t \in \mathbb{N}} \frac{-t}{\log \chi_t}.
\]
Using the second and forth equalities, we can say that \(\zeta\) is bounded away from zero and infinity when \(p < p_c\), implying exponential decay of \(\theta_t\) and \(\chi_t\) in \(t\) in the subcritical regime. This is not the case at the critical point. Moreover, \(\chi_t\) is nondecreasing in \(t\) at \(p = p_c\), because, otherwise, there must be a \(t_0 \in \mathbb{N}\) such that \(\chi_{t_0} < 1\), which together with submultiplicativity implies exponential decay of \(\chi_t\) and convergence of the susceptibility \(\sum_t \chi_t\) at \(p = p_c\), such as
\[
\sum_{t=0}^{\infty} \chi_t = \sum_{n=0}^{\infty} \sum_{k=0}^{t_0-1} \chi_{n0+k} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{t_0-1} \chi_k < \infty,
\]
which is a contradiction to the result in [2]: \(\sum_t \chi_t = \infty\) at \(p = p_c\).
Table 1  Predicted values of the critical exponents in various dimensions (e.g., [15])

|   | $d = 1$ | $d = 2$ | $d = 3$ | $d = 4 - \varepsilon$ | $d \geq 4$ |
|---|---------|---------|---------|------------------------|----------|
| $\rho$ | 0.159464 | 0.451 | 0.73 | $1 - \frac{1}{4} \varepsilon - 0.01283 \varepsilon^2$ | 1 |
| $\eta$ | 0.313686 | 0.230 | 0.12 | $\frac{1}{12} \varepsilon + 0.03751 \varepsilon^2$ | 0 |
| $\nu$ | 0.632613 | 0.568 | 0.526 | $\frac{1}{2} + \frac{1}{12} \varepsilon + 0.008171 \varepsilon^2$ | $\frac{1}{2}$ |
| $\gamma$ | 2.277730 | 1.60 | 1.25 | $1 + \frac{1}{5} \varepsilon + 0.06683 \varepsilon^2$ | 1 |
| $\mu$ | 1.733847 | 1.295 | 1.105 | $1 + \frac{1}{12} \varepsilon + 0.02238 \varepsilon^2$ | 1 |

Let $\rho, \eta, \nu$ be the critical exponents for the above quantities at $p = p_c$: as $t \uparrow \infty$,

$$\theta_t \approx t^{-\rho}, \quad \chi_t \approx t^\eta, \quad \xi_t \approx t^\nu,$$

(1.9)

where $f \approx g$ means that $(\log f)/ \log g$ goes to 1 in the prescribed limit, allowing corrections of slowly varying functions. In higher dimensions $d \gg 4$ ($d > 4$ is enough for sufficiently spread-out models), the lace expansion converges and the above critical exponents take on their mean-field values $\rho = 1, \eta = 0$ and $\nu = 1/2$: the values for branching random walk [3,4,9,10,13,14,18]. In lower dimensions, on the other hand, only numerical values and predictions due to non-rigorous renormalization-group methods are available (see Table 1).

In this paper, we prove the following theorem.

**Theorem 1.1** (i) For any $d \geq 1$, $p \in [0, 1]$ and $t \in \mathbb{N}$, we have

$$\chi_t \leq \frac{4}{3} (4 \xi_t + 1)^d \theta_{t/2}^2,$$

(1.10)

which implies the hyperscaling inequality (assuming existence of $\rho, \eta, \nu$)

$$d \nu \geq \eta + 2 \rho.$$  

(1.11)

(ii) Let $d = 1$ and $p = p_c$. Then, there is a $K > 0$ such that, for any $t \in \mathbb{N}$,

$$\chi_t \geq K \xi_t \theta_t^2,$$

(1.12)

which implies the hyperscaling equality (assuming existence of at least two among $\rho, \eta, \nu$)

$$\nu = \eta + 2 \rho.$$  

(1.13)

**Remarks** 1. The inequality (1.10) was first derived in [19]. Since its proof is easy and short, we will show it again for convenience. It was used in [19] to prove two other hyperscaling inequalities that also involve critical exponents defined in the off-critical regime. For example, if the susceptibility $\sum \chi_t$ and the relaxation time $\zeta$ diverge as $p \uparrow p_c$ as $(p_c - p)^{-\gamma}$ and $(p_c - p)^{-\mu}$ respectively, then, for any $d \geq 1$, we have

$$(d \nu - 2 \rho + 1) \mu \geq \gamma.$$  

(1.14)

If we replace those critical exponents in (1.11) and (1.14) by their mean-field values, then we obtain $d \geq 4$, which is a complement to the aforementioned lace-expansion results. Therefore, the upper-critical dimension $d_c$ for oriented percolation is 4.
2. In general, hyperscaling inequalities are believed to be equalities below and at the model-dependent upper-critical dimension. The values in Table 1 seem to support this belief. The identity (1.13) proves that it is indeed the case for at least \( d = 1 \). For unoriented percolation, for which \( d_c = 6 \), similar results are proven in 2 dimensions by Kesten [11] using the Russo–Seymour–Welsh theorem on the critical box-crossing property [16,17,21]. Since the known critical exponents for 2-dimensional unoriented percolation are rational numbers (e.g., \( \beta = 5/36 \) and \( \gamma = 43/18 \)), it is natural to believe that there must be some balance (i.e., hyperscaling equalities) among those critical exponents. On the other hand, since the values in Table 1 do not seem to be rational numbers, the hyperscaling equality (1.13) is even more surprising.

3. The main reason why the right-hand side of (1.10) is bigger than its left-hand side is due to the inequality

\[
\tau(x, t) = \mathbb{P}_p((o, 0) \rightarrow (x, t)) \\
\leq \mathbb{P}_p((o, 0) \rightarrow t/2, t/2 \rightarrow (x, t)) = \theta_{t/2}^2,
\]

(1.15)

where, and in the rest of the paper, we do not care much about possibilities of, e.g., \( t/2 \) not being an integer, since it is easy (but cumbersome) to make the argument rigorous if we introduce floor functions, etc. The last equality in (1.15) is due to reversibility: if we change the direction of each bond and redefine the connectivity in the time-decreasing direction, then we have the identity \( \mathbb{P}_p(t/2 \rightarrow (x, t)) = \theta_{t/2} \).

4. The following theorem on the critical box-crossing property is the key to show the opposite inequality to (1.15):

**Theorem 1.2** [6, Theorem 1.3] Let

\[
\begin{align*}
V_p(w, t) &= \mathbb{P}_p([0, w] \times [0, t] \text{ is crossed vertically}), \\
H_p(w, t) &= \mathbb{P}_p([0, w] \times [0, t] \text{ is crossed from left to right}).
\end{align*}
\]

(1.16) (1.17)

There exist a constant \( \varepsilon \in (0, 1) \) and an increasing sequence of integers \( \{w_t\}_{t \in \mathbb{N}} \) such that, for all \( t \in \mathbb{N} \),

\[
\begin{align*}
\varepsilon \leq V_{pc}(w_t, 3t) &\leq V_{pc}(3w_t, t) \leq 1 - \varepsilon, \\
\varepsilon \leq H_{pc}(3w_t, t) &\leq H_{pc}(w_t, 3t) \leq 1 - \varepsilon.
\end{align*}
\]

(1.18) (1.19)

We will also use (1.18)–(1.19) to control an upper bound on \( \tau(x, t) \) for \( x > jw_t \) that decays exponentially in \( j \in \mathbb{N} \) (see Lemma 2.1 below). This is a key element to show that \( w_t \) is bounded below by an \( \varepsilon \)-dependent positive multiple of \( \xi_t \).

5. Applying (1.10) and (1.12) to [19, (5.1)] and its reverse, respectively, we can readily show that the hyperscaling inequality (1.14) also becomes an equality for \( d = 1 \), i.e.,

\[
(v - 2\rho + 1)\mu = \gamma.
\]

(1.20)

6. It is easy to show that the hyperscaling inequality (1.11) holds for other finite-range models of oriented percolation and the contact process. It should not be so difficult to prove Theorem 1.2 for the nearest-neighbor models of oriented site percolation and the contact process, hence the hyperscaling equality (1.13) for \( d = 1 \). However, it is not so obvious to prove a similar statement to Theorem 1.2 for longer-range models. This may be worth further investigation.
2 Proof of Theorem 1.1

Proof of Theorem 1.1(i) It suffices to prove the inequality (1.10), as the hyperscaling inequality (1.11) immediately follows by using (1.10) at \( p = p_c \) (and assuming existence of the three critical exponents). First we note that

\[
\chi_t = \frac{1}{\xi_t^2} \sum_x |x|^2 \tau(x, t) \geq 4 \sum_{x:|x| \geq 2\xi_t} \tau(x, t),
\]

hence

\[
\frac{3}{4} \chi_t \leq \sum_{x:|x| \leq 2\xi_t} \tau(x, t).
\]

By (1.15), the right-hand side is further bounded by \((4\xi_t + 1)d\theta_t^2\). This completes the proof of (1.10).

To prove Theorem 1.1(ii), we first assume the following key lemma:

Lemma 2.1 Let \( d = 1 \) and \( p = p_c \). Let \( \varepsilon \in (0, 1) \) and \( w_t \) be the same as in Theorem 1.2.

(i) For any \( t \in \mathbb{N} \) and any \( x \in [-\frac{1}{2} w_t, \frac{1}{2} w_t] \),

\[
\tau(x, t) \geq \varepsilon^6 \theta_t^2.
\]

(ii) For any \( j, t, x \in \mathbb{N} \) with \( j \geq 2 \) and \( j w_t < x \leq (j + 1) w_t \),

\[
\tau(x, t) \leq \varepsilon^{-4} \theta_t^2 (1 - \varepsilon)^{j-2}.
\]

Proof of Theorem 1.1(ii) assuming Lemma 2.1 Again, it suffices to prove the inequality (1.12), as the equality (1.13) is a result of the hyperscaling inequality (1.11) for \( d = 1 \) and the opposite inequality \( \nu \leq \eta + 2\rho \) that immediately follows from (1.12).

To prove (1.12), we first note that, by (2.3),

\[
\chi_t \geq \frac{1}{2} \sum_{x=1}^{\frac{1}{2} w_t} \tau(x, t) \geq \varepsilon^6 w_t \theta_t^2.
\]

To complete the proof, it suffices to show that \( w_t \) is bounded below by a positive multiple of \( \xi_t \). However, by definition,

\[
\xi_t^2 = 2 \sum_{x=1}^{\infty} x^2 \frac{\tau(x, t)}{\chi_t} = 2 \left( \sum_{x=1}^{\frac{1}{2} w_t} x^2 \frac{\tau(x, t)}{\chi_t} + \sum_{j=2}^{\infty} \sum_{x=j w_t}^{(j+1)w_t} x^2 \frac{\tau(x, t)}{\chi_t} \right)
\]

\[
\leq 2 w_t^2 \left( 4 + \sum_{j=2}^{\infty} (j + 1)^2 \frac{\tau(x, t)}{\chi_t} \right).
\]

Then, by using (2.4)–(2.5), we obtain

\[
\xi_t^2 \leq 2 w_t^2 \left( 4 + \frac{1}{\varepsilon^6 \theta_t^2} \sum_{j=2}^{\infty} (j + 1)^2 \max_{j w_t < x \leq (j + 1) w_t} \tau(x, t) \right)
\]

\[
\leq 2 w_t^2 \left( 4 + \varepsilon^{-10} \sum_{j=2}^{\infty} (j + 1)^2 (1 - \varepsilon)^{j-2} \right).
\]
As a result,
\[
\chi_t \geq \frac{\varepsilon^6}{\sqrt{2}} \left( 4 + \varepsilon^{-10} \sum_{j=2}^{\infty} (j + 1)^2 (1 - \varepsilon)^j \right)^{-1/2} \nu_t \theta_t^2. \tag{2.8}
\]
This completes the proof of (1.12). \hfill \Box

The rest of the paper is devoted to showing Lemma 2.1.

**Proof of Lemma 2.1(i).** First we note that, for \(1 \leq x \leq \frac{1}{2} w_t\), the event \((o, 0) \rightarrow (x, t)\) occurs if the following four increasing events occur:

- \((o, 0) \rightarrow t\) in \([-w_t, w_t] \times [0, t]\),
- \(0 \rightarrow (x, t)\) in \([x - w_t, x + w_t] \times [0, t]\),
- \([-\frac{3}{2} w_t, \frac{3}{2} w_t] \times [0, t]\) is crossed from left to right,
- \([-\frac{3}{2} w_t, \frac{3}{2} w_t] \times [0, t]\) is crossed from right to left.

The last two events take care of the possibility that the forward cluster from the origin \((o, 0)\) and the backward cluster from \((x, t)\) do not collide. Using the FKG inequality (see, e.g., [7]), translation invariance and the reversibility explained below (1.15), we obtain

\[
\tau(x, t) \geq \mathbb{P}_p\left((o, 0) \rightarrow t \text{ in } [-w_t, w_t] \times [0, t]\right)^2 \mathbb{H}_p(3w_t, t)^2. \tag{2.9}
\]

We further note that the event \((o, 0) \rightarrow t\) in \([-w_t, w_t] \times [0, t]\) occurs if the following three increasing events occur:

- \((o, 0) \rightarrow t\),
- \([0, w_t] \times [0, t]\) is crossed vertically,
- \([-w_t, 0] \times [0, t]\) is crossed vertically.

Again, by the FKG inequality, translation invariance and the monotonicity \(V_p(w_t, t) \geq V_p(w_t, 3t)\), we obtain

\[
\mathbb{P}_p\left((o, 0) \rightarrow t \text{ in } [-w_t, w_t] \times [0, t]\right) \geq \theta_t V_p(w_t, 3t)^2, \tag{2.10}
\]

hence

\[
\tau(x, t) \geq \theta_t^2 V_p(w_t, 3t)^4 \mathbb{H}_p(3w_t, t)^2. \tag{2.11}
\]

The inequality (2.3) follows from the above inequality at \(p = p_c\) and (1.18)–(1.19). \hfill \Box

**Proof of Lemma 2.1(ii).** Recall that \(j \geq 2\) and \(j w_t < x \leq (j + 1) w_t\). If \((o, 0) \rightarrow (x, t)\), then the following three independent events occur:

- \((o, 0)\) is connected to the boundary \(\partial B_o\) of the box \(B_o \equiv [-w_t, w_t] \times [0, t]\),
- \([w_t, (j - 1) w_t] \times [0, t]\) is crossed from left to right,
- \((x, t)\) is connected from the boundary \(\partial B_x\) of the box \(B_x = [(j - 1) w_t, (j + 2) w_t] \times [0, t]\).

By this observation and using \(\mathbb{H}_p((j - 2) w_t, t) \leq \mathbb{H}_p(w_t, t)^{j-2} \leq \mathbb{H}_p(w_t, 3t)^{j-2}\), we obtain

\[
\tau(x, t) \leq \mathbb{P}_p\left((o, 0) \rightarrow \partial B_o\right) \mathbb{H}_p(w_t, 3t)^{j-2} \mathbb{P}_p\left(\partial B_x \rightarrow (x, t)\right). \tag{2.12}
\]
However, by reversibility and monotonicity, we have
\[
P_p\left(\partial B_x \rightarrow (x, t)\right) \leq P_p\left((o, 0) \rightarrow \partial B_o\right).
\]
(2.13)
Therefore,
\[
\tau(x, t) \leq \frac{\theta_t}{V_{p_c}(w_t, 3t)^2} \leq \epsilon^{-4} \theta_t^2 (1 - \epsilon)^{j-2}.
\]
(2.14)

To bound the probability on the right-hand side by \(\theta_t\), we borrow the idea in the proof of [6, (4.7)]. First, we note that \((o, 0) \rightarrow t\) if the following three increasing events occur:

- \((o, 0) \rightarrow \partial B_o\),
- \([0, w_t] \times [0, t]\) is crossed vertically,
- \([-w_t, 0] \times [0, t]\) is crossed vertically.

By the FKG inequality, translation invariance and the monotonicity \(V_{p_c}(w_t, t) \geq V_{p_c}(w_t, 3t)\), we obtain
\[
\theta_t \geq P_p\left((o, 0) \rightarrow \partial B_o\right) V_{p_c}(w_t, 3t)^2.
\]
(2.15)

To summarize the above computations at \(p = p_c\), we arrived at
\[
\tau(x, t) \leq \left(\frac{\theta_t}{V_{p_c}(w_t, 3t)^2}\right)^2 H_{p_c}(w_t, 3t)^{j-2} \leq \epsilon^{-4} \theta_t^2 (1 - \epsilon)^{j-2},
\]
(2.16)
as required.

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