ON MAXIMUM INDUCED MATCHING NUMBERS OF SPECIAL GRIDS

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Abstract. A subset $M$ of the edge set of a graph $G$ is an induced matching of $G$ if given any two $e_1, e_2 \in M$, none of the vertices on $e_1$ is adjacent to any of the vertices on $e_2$. Suppose that $MIM_G$, a positive integer, is the largest possible size of $M$ in $G$, then, $M$ is the maximum induced matching, $MIM$, of $G$ and $MIM_G$ is the maximum induced matching number of $G$. We obtain some upper bounds for the maximum induced matching numbers of some specific grids.

1. Introduction

For a graph $G$, let $V(G), E(G)$ be vertex and edge sets respectively and let $e \in E(G)$, we define $e = uv$, where $u, v \in V(G)$. Also, the respective order and size of $V(G)$ and $E(G)$ are $|V(G)|$ and $|E(G)|$. For some $M \subseteq E(G)$, $M$ is an induced matching of $G$ if for all $e_1 = u_iu_j$ and $e_2 = v_iv_j$ in $M$, $u_kv_l \notin M$, where $k$ and $l$ are from $\{i, j\}$. Induced matching, a variant of the matching problem, was introduced in 1982 by Stockmeyer and Vazirani[8] and has also been studied under the names strong matchings, ”risk free” marriage problem. It has found theoretical and practical applications in a lot of areas including network problems and cryptology[3]. For more on induced matching and its applications, see [2],[3],[4],[5],[9].

The size of an induced matching is the number of edges in the induced matching and induced matching $M$ of $G$ with the largest possible size is known as the maximum induced matching which is denoted by $MIM$, its size, $MIM_G$, is called the maximum induced matching number (or the strong matching number) of $G$. Obtaining $MIM_G$ is $NP$—hard, even for regular bipartite graphs[4]. However, $MIM_G$ of some graphs have been found in polynomial time ([3], [6]).

A grid $G_{n,m}$ results from the Cartesian product of two paths $P_n$ and $P_m$, resulting in $n$-rows and $m$-columns. Marinescu-Ghemaci in [7], obtained the

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that for any odd grid $G$ by combining the $MIM$ that the $MIM$ Marinescu-Ghemaci also gave useful lower and upper bounds and conjectured that the $MIM$ numbers of grids can be found in polynomial time. Furthermore, by combining the $MIM$ numbers of certain partitions of odd grids, it was shown that for any odd grid $G \equiv G_{n,m}$, $MIM_G \leq \left\lfloor \frac{nm+1}{4} \right\rfloor$. This bound was improved on in [1] for the case where $n \geq 9$ and $m \equiv 1 \mod 4$. In this paper, the Marinescu-Ghemaci bound for the case where $n \geq 9$ and $m \equiv 3 \mod 4$ is considered and more compact values are obtained. The results in this work, combined with some of the results in [7], confirm the $MIM$ numbers of certain graphs, whose $MIM$ numbers’ lower bounds were established in [7].

Section 2, of this work, is a review of definitions and preliminary results needed in this work, while in section 3, we obtain the maximum induced matching number of odd grids.

2. Definitions and Preliminary Results

Grid, $G_{n,m}$, as defined in this work, is the Cartesian product of paths $P_n$ and $P_m$ with $V(P_n) = \{u_1, u_2, \cdots, u_n\}$ and $V(P_m) = \{v_1, v_2, \cdots, v_m\}$. We adapt the following notations from [1]: $V_i = \{u_iv_i, u_2v_i, \cdots, u_nv_i\} \subset V(G_{n,m})$, $i \in [1, m]$ and $U_i = \{u_iv_1, u_2v_2, \cdots, u_nv_m\} \subset V(G_{n,m})$, $i \in [1, n]$. For edge set $E(G_{n,m})$ of $G_{n,m}$, if $u_iv_ju_kv_j \in E(G_{n,m})$ and $u_iv_ju_kv_k \in E(G_{n,m})$, we write $u_{(i,j)}v_j \in E(G_{n,m})$ and $u_{(i,j)}v_k \in E(G_{n,m})$ respectively.

A saturated vertex $v$ is any vertex on an edge in $M$, otherwise, $v$ is unsaturated. We define $v$ as saturable if it can be saturated relative to the nearest saturated vertex. Any vertex that is at least distant-2 from any saturated vertex is saturable. The set of all saturated and saturable vertices on a graph $G$ is denoted by $V_{st}(G)$ and $V_{sb}(G)$ respectively. Clearly, $|V_{st}(G)|$ is even and $V_{st}(G) \subseteq V_{sb}(G)$. Free saturable vertices (FSV) are saturable vertices that can not be on any member of $M$, $FSV = V_{sb} \setminus V_{st}$. Let $G$ be a $G_{n,m}$ grid, we define $G^{[k]}$ as a $G_{n,k}$ subgrid of $G$ induced by $V_{i+1}, V_{i+2}, \cdots, V_{i+k}$.

The following results from [7] on $G$, a $G_{n,m}$ grid, are useful in this work:

**Lemma 2.1.** Let $m, n \geq 2$ be two positive integers and let $G$ be a $G_{n,m}$ grid. Then,

(a) If $m \equiv 2 \mod 4$ and $n$ odd then $|V_{sb}(G)| = \frac{mn+2}{2}$; and $|V_{sb}(G)| = \frac{mn}{2}$ otherwise;

(b) for $m \geq 3$, $m$ odd, $|V_{sb}(G)| = \frac{nm+1}{2}$, for $n \in \{3, 5\}$.

**Theorem 2.2.** Let $G$ be a $G_{n,m}$ grid with $2 \leq n \leq m$. Then,
(a) if \( n \) even and \( m \) even or odd, then \( \text{MIM}_G = \left\lfloor \frac{mn}{4} \right\rfloor \); 
(b) if \( n \in \{3, 5\} \) then for
   \[ \begin{align*}
   (i) \quad & m \equiv 1 \mod 4, \quad \text{MIM}_G = \frac{n(m-1)}{4} + 1 \\
   (ii) \quad & m \equiv 3 \mod 4, \quad \text{MIM}_G = \frac{n(m-1)+2}{4}
   \end{align*} \]

The following theorem is the statement of the bound given by Marinescu-Ghemaci \cite{7}.

**Theorem 2.3.** Let \( G \) be a \( G_{n,m} \) grid, \( m, n \geq 2, \) \( mn \) odd. Then \( \text{MIM}_G \leq \left\lfloor \frac{mn+1}{4} \right\rfloor \).

3. **Maximum induced matching number of odd grids**

The following result and the remark describe the importance of the saturation status of certain vertices in \( G_{5,p} \) grid, where \( p \equiv 2 \mod 4 \).

**Lemma 3.1.** Let \( G \) be a \( G_{n,m} \) grid and let \( \{V_i; V_{i+1}, \ldots, V_{i+p}\} \subset G \) induce \( G^{[p]} \), a \( G_{5,p} \) subgrid of \( G \), where \( p \equiv 2 \mod 4 \). Suppose that \( M_1 \), is an induced matching of \( G^{[p]} \) and that for \( u_3v_j \in V(G^{[p]}), u_3v_j \notin V_{st}(G^{[p]}) \). Then, \( V_{st}(G^{[p]}) \leq 10k + 4 \) and \( M_1 \) is not an MIM of \( G^{[p]} \).

**Proof.** Let \( p = 4k + 2, G^{[2]} \) and \( G^{[p-2]} \) be partitions of \( G \), induced by \( \{V_i; V_{i+1}\} \) and \( \{V_{i+2}, V_{i+3}, \ldots, V_{i+p}\} \), respectively. Since \( u_3v_j \) is not saturated in \( G^{[2]} \), it easy to check that \( |V_{st}(G^{[2]})| = 5 \). From \cite{7}, \( |V_{st}(G^{[p-2]})| = 10k \). Thus \( |V_{st}(G^{[p]})| \leq |V_{st}(G^{[2]})| + 5 \) and therefore, \( |V_{st}(G^{[p]})| \leq 10k + 5 \). This is a contradiction since by \cite{7} \( |V_{st}(G^{[p]})| = 10k + 6 \). \( \square \)

**Remark 3.1.** It should be noted that \( M \) in Lemma 3.1 will still not be MIM of \( G \) if for the vertex set \( A = \{u_1v_1, u_3v_1, u_3v_m, u_5v_m\} \subset V(G) \), any member of \( A \) is unsaturated.

**Lemma 3.2.** Suppose \( u_{(j)}v_i, u_5v_{i-i-1} \in M \) or \( u_{(j)}v_i, u_5v_{i-i-1} \in M \) where \( M \) is an induced matching of \( G \), a \( G_{5,m} \) grid, \( m \equiv 3 \mod 4 \), \( m \geq 23 \) and \( 1 < i < m \), \( i \notin \{4, m-3\} \). Then \( M \) is not a MIM of \( G \).

**Proof.** Let \( G \) be partitioned into \( G^{[m(1)]} \) and \( G^{[m(2)]} \), which are respectively induced by \( A = \{V_1, V_2, \ldots, V_i\} \) and \( B = \{V_{i+1}, V_{i+2}, \ldots, V_m\} \). Suppose that \( M \) is an MIM of \( G \).

Case 1: \( i \equiv 1 \mod 4 \). Let \( m = 4k + 3 \) and set \( i = 4t + 1 \), where \( k \geq 5 \) and \( t > 0 \). Then, \( |m(1)| \equiv 1 \mod 4 \) and \( |m(2)| \equiv 2 \mod 4 \). Since \( u_1v_i, u_2v_i, u_5v_i \) and \( u_5v_{i-1} \) are saturated vertices in \( V_i \), then the only FSV on \( V_{i-1} \) is \( u_3v_{i-1} \).
Suppose that $u_3v_{i-1}$ remains unsaturated. Let $G^{m(3)} \subset G^{m(1)}$ induced by {$V_1, V_2, \cdots, V_{i-2}$}, where $|m(3)| \equiv 3 \mod 4$. By [7], $|V_{st}(G^{m(3)})| = 10t - 4$. Thus, $|V_{st}(G^{m(1)})| \leq 10t$. Suppose that $u_3v_{i-1}$ is saturated, then, $u_5v_{i+1} \in M$. Thus, $u_3v_{i-3} \in V_{i-3} \subset G^{m(4)}$, unsaturated, where $G^{m(4)}$ is $G^{m(3)} \setminus V_{i-2}$. Note that $|m(4)| \equiv 2 \mod 4$. From Lemma [6,1] therefore, $|V_{st}(G^{m(4)})| \leq 10t - 6$ and thus, $|V_{st}G^{m(1)}| \leq 10t - 6 + 6 = 10t$. Now, since $u_1v_i, u_2v_i$ and $u_5v_i$ are saturated vertices in $V_i$, then, $u_3v_{i+1}, u_4v_{i+1} \in V(G^{m(2)})$ are saturated vertices in $G^{m(2)}$.

Claim: Edge $u_3v_{i+1}$ belongs to $M$.

Reason: Suppose that both $u_3v_{i+1}$ and $u_4v_{i+1}$ are not saturated, then $V_{i+1}$ contains no saturated vertices. Let $G^{m(2)} \setminus \{V_{i+1}\} = G^{m(5)}$, where $|m(5)| \equiv 1 \mod 4$. Thus, $|V_{st}(G)| \leq |V_{st}(G^{m(5)})| + |V_{st}(G^{m(3)})| = 10k + 2$. This implies that $M$ requires at least four more saturated vertices to be $MIM$ of $G$. However, $|V_{sb}(G^{m(5)})| = 10(k - t) + 3$ and suppose $u_3v_{i+1}, u_4v_{i+1} \in V_{st}(G)$, then $|V_{st}(G)| \leq 10k + 5$, which in fact, is $|V_{st}(G)| = 10k + 4$. Thus if $u_3v_i, u_5v_{i+1} \in M$, then $M$ is not an $MIM$ of $G$.

Suppose that $u_3v_i, u_5v_{i+1} \in M$. Let $G^{m(1)} = G^{m(1)} \setminus \{V_i\}$ and $G^{m(2)} = G^{m(2)} + \{V_i\}$. Now, $|n(1)| \equiv 0 \mod 4$ and $|n(2)| \equiv 3 \mod 4$. We can see that $|V_{st}(G^{m(2)})| = 10(k - t) + 6$. Now, on $V_{i+1} \subset G^{m(1)}$, only vertices $u_3v_{i-1}$ and $u_4v_{i-1}$ are saturated. Suppose they are both not saturated after all. Let $G^{m(3)} \subset G^{m(1)}$ be induced by {$V_1, V_2, \cdots, V_{i-2}$}, where $|m(3)| \equiv 3 \mod 4$. $|V_{st}(G^{m(3)})| = 10t - 4$. Thus $|V_{st}(G)| = 10k + 2$. Therefore, $M$ requires four saturated vertices to be $MIM$ of $G$. Now, $|V_{sb}(G^{m(3)})| = 10t - 2$, and thus, $V(G^{m(3)})$ contains two extra $FSV$, say, $v_1, v_2$ which are not adjacent. Thus, the maximum number of saturated vertices from the vertex set $v_1, v_2, u_3v_{i-1}, u_4v_{i-1}$ is 2. Therefore, $|V_{st}(G)| \leq 10k + 4$, which is a contradiction.

Case 2. For $i \equiv 2 \mod 4$. Let $G^{p(1)}$ and $G^{p(2)}$ be partitions of $G$ induced by {$V_1, V_2, \cdots, V_i\}$ and {$V_{i+1}, V_{i+2}, \cdots, V_m\}$, with $m = 4k + 3$ and $i = 4t + 2$. Let $u_3v_i$ and $u_5v_{i+1} \in M$. Since $u_3v_i$ belongs $M$ of $G$, then $u_3v_i$ cannot be saturated. Thus, $|V_{st}(G^{p(2)})| \geq 10(k - t) + 2$ for $M$ to be maximal. It can be seen that $|p(2)| \equiv 1 \mod 4$. Now, $u_3v_{i+1}$ and $u_4v_{i+1}$ are saturated vertices in $V_{i+1}$. Suppose both of them are not saturated, then for $G^{p(3)}$ induced by {$V_{i+2}, V_{i+3}, \cdots, V_m\}$, where $|p(3)| \equiv 0 \mod 4$, $|V_{st}(G^{p(3)})| = 10(k - t)$. Thus $u_3v_{i+1}$ and $u_4v_{i+1}$ are saturated vertices and in fact, $u_3v_{i+1} \in M$. On $V_{i+2}$, therefore, there exists three saturated vertices $u_1v_{i+1}, u_2v_{i+2}$ and $u_5v_{i+5}$. Suppose none of these three vertices are saturated. Then, $|V_{st}(G^{p(3)})| \leq |V_{st}(G^{p(4)})| + 2$, with $G^{p(4)}$ induced by {$V_{i+3}, \cdots, V_m\}$ and $|p(4)| \equiv 3 \mod 4$. 
and thus, $|V_{st}(G^{[p(2)]})| \leq 10(t-k) - 2$. Therefore it requires extra four saturated vertices for $M$ to be maximal. There exist two other saturable vertices, $v_1, v_2 \in V(G^{[p(4)]})$ (since $V_{st}(G^{[p(4)]}) = 10(k-t) - 4$ and $V_{sb}(G^{[p(4)]}) = 10(k-t) - 2$). Clearly, $v_1, v_2$ are not adjacent, else they would have formed an edge in $M$. Suppose $v_1, v_2 \in V_{i+3}$. For $v_1$ and $v_2$ to be saturated, they have to be $u_1v_{i+3}$ and one of $u_2v_{i+3}$ and $u_2v_{i+3}$. Thus, $u_5v_{i+3} \in M$ and one of $u_1v_{i+3}, u_2v_{i+3}$ or $u_3v_{i+2}$ belongs to $M$. Let $G^{[p(5)]}$ be induced by $\{V_{i+4}, \ldots V_m\}$, where $|p(5)| \equiv 2 \mod 4$. Now, since $v_3v_{i+2} \in M$, then $u_5v_{i+5} \in V_{i+4}$ is unsaturable and therefore, by Remark 3.1 $|V_{st}(G^{[p(5)]})| = 10(k-t) - 1 + 4$ and thus, $|V_{st}(G^{[p(2)]})| = 10(k-t)$, which is less than required.

The case of $u_5v_{i+1} \in M$ is the same as the case of $u_5v_{i-1} \in M$ for $i \equiv 2 \mod 4$.

Case 3: $i \equiv 0 \mod 4, i \geq 6$ or $i \leq m - 5$, with $u_{(1)}v_i$, $u_5v_{i-1} \in M$. Let $G^{[r(1)]}$ and $G^{[r(2)]}$ be partitions of $G$ which are induced respectively by $\{V_1, V_2, \ldots V_i\}$ and $\{V_{i+1}, V_{i+2}, \ldots, V_m\}$. Since $i \equiv 0 \mod 4$, then $|r(1)| \equiv 0 \mod 4$, while $|r(2)| \equiv 3 \mod 4$. Also, $u_5v_{i-1} \in M$, implies $u_5v_{i+1}$ is unsaturable. Since $i - 2 \equiv 2 \mod 4$, then by Lemma 3.1 and Remark 3.1 $|V_{st}(G^{[r(1)]})| \leq 10t - 2$, implying that for $M$ to be maximal, $|V_{st}(G^{[r(2)]})| \geq 10(k-t) + 8$. It can be seen that $V_{i+1}$ has two only saturable vertices $u_3v_{i+1}$, $u_4v_{i+2}$ left. It should also be noted that if any of $u_3v_{i+1}$ and $u_4v_{i+2}$ is saturated, then $u_3v_{i+3}$ can not be saturated in $G^{[r(3)]}$, a subgrid of $G^{[r(2)]}$ induced by $\{V_{i+2}, V_{i+3}, \ldots, V_m\}$, with $|r(3)| \equiv 2 \mod 4$. Thus suppose $u_3v_{i+1}, u_4v_{i+2} \in V_{st}(G)$, then $|V_{st}(G)| \leq 10(k-t) + 4$. Likewise, if $u_3v_{i+1}, u_4v_{i+2} \notin V_{st}(G)$, $|V_{st}(G)| \leq 10t - 2 + 10(k-t) + 6$.

The case of $u_5v_{i+1} \in M$ follows the same argument as the case of $u_5v_{i-1} \in M$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{A Grid $G \equiv G_{5,23}$ with $MIM_G = 28, u_2v_1, u_4v_4 \in MIM$ of $G$}
\end{figure}
Remark 3.2. (a) In the case of \( i \equiv 0 \mod m \) in Remark 3.2, \( M \) remains a MIM when \( i = 4 \) or when \( i = m - 3 \) as seen in Figure 1 of \(|MIM| = 28\) of \( G_{5,23} \).

(b) It should be noted that the case of \( i \equiv 3 \mod 4 \) has been taken care of by the case of \( i \equiv 1 \mod 4 \) by 'flipping' the grid from right to left or vice versa.

(c) From Lemma 3.2 we see that if for some induced matching \( M \) of \( G_{5,m} \), \( m \equiv 3 \mod 4 \), \( u_{(i)}v_i \) and \( u_{5}v_{(i-1)} \) (or \( u_{5}v_{(i+2)} \)) \( \in M \), then \( M \) is not a maximal induced matching of \( G \) for any \( 1 < i < m \).

Next we investigate some \( M \) of \( G_{5,m} \) if it contains \( u_{(i)}v_i \) and \( u_{5}v_i \).

Lemma 3.3. Suppose \( G = G_{5,m} \), where \( m \geq 23 \) and \( m \equiv 3 \mod 4 \). Let \( u_{(i)}v_i \), \( u_{(i)}v_i \in M \), an induced matching of \( G \) and \( 1 < i < m \), \( i \not\equiv 0 \mod 4 \) then \( M \) is not a MIM of \( G \).

Proof. Suppose that \( i \equiv 2 \mod 4 \). Let \( G^{[n(1)]} \) and \( G^{[n(2)]} \) be partitions of \( G \) induced by \( \{V_1, V_2, \ldots, V_i\} \) and \( \{V_{i+1}, V_{i+2}, \ldots, V_m\} \). Since \( u_{(i)}v_i \) and \( u_{3}v_i \) \( \in M \), then, \( u_{3}v_i \) is unsaturated. Let \( i = 4t + 2 \), for some positive integer \( t \), by Lemma 3.2 then \(|V_{st}(G^{[n(1)]})| = 10t + 4 \). Now, only \( u_{3}v_{i+1} \) is saturable on \( V_{i+1} \). Let \( G^{[m(3)]} \subset G^{[m(2)]} \), induced by \( \{V_{i+2}, \ldots, V_m\} \). Clearly \(|m(3)| = |m(2)| - 1 = 4(k - t)\). Therefore, \(|V_{st}(G^{[m(3)]} + u_{3}v_i)| \leq 10(k - t) + 1 \), which in fact is \( 10(k - t) \). Thus, \(|V_{st}(G)| = 10k - 4 \).

Now, suppose \( i \equiv 1 \mod 4 \). Let \( G^{[n(1)]} \) be induced by \( \{V_1, V_2, \ldots, V_i\} \) and let \( G^{[n(2)]} \) be induced by \( \{V_{i+1}, V_{i+2}, \ldots, V_m\} \). Since \(|n(1)| = 4t + 1 \), by Lemma 3.2 then \(|V_{st}(G^{[n(2)]})| = 10t - 4 \), and thus, \(|G^{[m(1)]}| = 10t\). Also, \( G^{[n(4)]} \) be induced by \( \{V_{i+2}, V_{i+3}, \ldots, V_{m}\} \). Since \(|n(4)| = 4(k - t) + 1 \), then for \( G^{[n(4)]} + u_{5}v_{i+1}, |V_{st}(G^{[n(4)]} + u_{3}v_{i+1})| = 10(k - t) + 4 \). Therefore, \(|V_{st}(G)| \leq 10k + 4 \). Now suppose \( u_{3}v_{(i-2)} \in M \). Then, given \( G^{[n(5)]} \), induced by \( \{V_1, V_2, \ldots, V_{i-3}\} \). We can see that \(|n(5)| \equiv 2 \mod 4 \). By Lemma 3.1 \(|V_{st}(G^{[n(5)]})| = 10t - 6 \). Thus, \(|V_{st}(G^{[n(1)]})| = 10t\) and therefore, \(|V_{st}(G)| \leq 10k + 4 \).

Remark 3.3. Like in Remark 3.2 for \( i \equiv 0 \mod 4 \), it can be seen that \( u_{(i)}v_i, u_{(i)}v_4 \) or \( u_{(i)}v_{m-3}, u_{(i)}v_m \) can be in \( M \) if \( M \) is MIM of \( G \). Also given \( i \equiv 0 \mod 4 \) and \( 4 < i < m - 3 \), for at most only one \( i \), from 1 to \( m \), \( u_{(i)}v_i \) can be a member of maximal \( M \).
Next we investigate the maximality of the induced matching of $G = G_{5,m}$, $m \equiv 3 \mod 4$.

**Lemma 3.4.** Let $u_{\frac{i}{2}}v_i, u_4v_{(i-1)} \in M$ or $u_{\frac{i}{2}}v_i, u_4v_{(i+1)} \in M$ where $M$ is an induced matching of $G$, a $G_{5,m}$ grid, $m \equiv 3 \mod 4$, $m \geq 23$ and $1 < i < m$, $i \not\equiv 0 \mod 4$. Then $M$ is not a MIM of $G$.

**Proof.** Case 1: Let $i \equiv 1 \mod 4$. Suppose that $m = 4k + 3$ and $i = 4t + 1$, $t \geq 1$. Let $G^{[m(1)]}$ and $G^{[m(2)]}$ be two partitions of $G$, induced by $\{V_1, V_2, \ldots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \ldots, V_m\}$ respectively. Since $u_{\frac{i}{2}}v_i, u_4v_{(i-1)} \in M$, then there is no other saturated vertex on both of $V_{i-1}$ and $V_i$. Let $G^{[m(3)]} \subseteq G^{[m(1)]}$ be a grid induced by $\{V_1, V_2, \ldots, V_{i-2}\}$. Now, $n(3) \equiv 3 \mod 4$. Therefore, $|V_{st}(G^{[m(3)]})| = 10t - 4$ and hence, $|V_{st}(G^{[m(1)]})| = 10t$. Now, $|m(2)| \equiv 2 \mod 4$, since $u_{\frac{i}{2}}v_i \in M$, then $u_1v_{i+1} \in V_{i+1}$ is unsaturable. From a previous result, $|V_{st}(G^{[m(2)]})| = 10(k - t) + 4$ and thus, $|V_{st}(G)| = 10k + 4$. For $u_4v_{(i+1)} \in M$. Let $G^{[m(1)]}$ and $G^{[m(2)]}$ be induced by $G^{[m(1)]} \setminus V_i$ and $G^{[m(2)]} + V_i$. Then, $|n(1)| \equiv 0 \mod 4$ and $|n(2)| = 4(k - t) + 3$. It can be seen that on $V_{i-1}$, only $u_3v_{i-1}$ and $u_5v_{i-1}$ are saturable vertices.

Claim: Vertices $u_3v_{i-1}$ and $u_5v_{i-1}$ are not saturable for $M$ to be maximal.

Reason: Suppose without loss of generality, that any of $u_3v_{i-1}$ and $u_5v_{i-1}$ is saturate, say $u_5v_{i-1}$. Then $u_5v_{(i-1)} \in M$. This implies that $v_5v_{i-3}$ is not saturate in $V_{i-3}$. Now $\{V_1, V_2, \ldots, V_{i-3}\}$ induces a grid $G^{[n(4)]}$ and $|n(4)| \equiv 2 \mod 4$. Then, $|V_{st}(G^{[n(4)]})| = 10t - 6$ and thus, $|V_{st}(G^{[n(1)]})| = 10t - 4$. Now, since $|n(2)| = 4(k - t) + 3$, $|V_{st}(G^{[n(2)]})| = 10(k - t) + 6$ and therefore, $|V_{st}(G)| = 10k + 2$.

Case 2: For $i \equiv 2 \mod 4$. Let $G^{[n(1)]}$ and $G^{[n(2)]}$ be two partitions of $G$, induced by $\{V_1, V_2, \ldots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \ldots, V_m\}$ respectively. Since $u_{\frac{i}{2}}v_i$ and $u_4v_{(i-1)} \in M$, vertex $u_5v_i \in V_{st}(G^{[n(1)]})$, and therefore, $|V_{st}(G^{[n(1)]})| = 10t + 4$, where $|n(1)| = 4t + 2$. Also, only $u_3v_{i+1}$ and $u_5v_{i+1}$ are saturable on $V_{i+1}$. Suppose without loss of generality, that both $u_3v_{i+1}$ and $u_5v_{i+1}$ are saturated and thus, $u_3v_{(i+2)}, u_5v_{(i+2)} \in M$. Now, suppose that $G^{[n(4)]}$ is induced by $\{V_{i+3}, V_{i+4}, \ldots, V_m\}$, with $|n(4)| = 4(k - t - 1) + 3$. By following the techniques employed earlier, it can be shown that $|V_{st}(G)| \leq |V_{st}(G^{[n(1)]})| + |V_{st}(G^{[n(2)]})| \leq 10k + 4$. The $u_4v_{(i+4)}$ case, has the same proof as the $u_4v_{(i-1)}$ case. \qed

**Remark 3.4.** There can be only one edge $u_{\frac{i}{2}}v_i \in M$ for which $M$ is MIM of $G_{5,m}$, if $M$ contains $u_{\frac{i}{2}}v_i$ and $u_4v_{(i-1)}$ (or $u_4v_{(i+1)}$), and in this case, $i \equiv 0 \mod 4$ as shown in Figure 2.

**Remark 3.5.** It should be noted that the proof of the $i \equiv 1 \mod 4$ in Lemma 3.4 will hold for $i \equiv 3 \mod 4$ by flipping the grid from right to left.
The previous results and remarks yield the following conclusion.

**Corollary 3.5.** Suppose that \( m \geq 23 \) and \( M \) is the MIM of \( G \), some \( G_{5,m} \) grid. Then, if for at most some positive integer \( i, 1 < i < m \), \( u_{(\ell)}v_i \in M \), then, \( i \equiv 0 \mod 4 \).

**Lemma 3.6.** Let \( M \) be a matching of \( G_{5,m} \) with \( m \equiv 3 \mod 4 \) and let \( u_{(\ell)}v_i, u_{(\ell)}v_j \in M \), \( 1 < i < j < m \), such that \( i \equiv 0 \mod 4 \) and \( j \equiv 0 \mod 4 \), then \( M \) is not an MIM of \( G \).

The claim in Lemma 3.6 can easily be proved using earlier techniques and Lemma 3.1 and Remark 3.1.

**Remark 3.6.** It should be noted from the previous results and from Corollary 3.5 that if \( M \) is the MIM of \( G_{5,m} \), \( m \equiv 3 \mod 4 \), then at most, \( M \) contains two edges of the form \( u_{(\ell)}v_i, u_{(\ell)}v_j \) and \( j \) can only be 4 when \( i = 1 \) or \( i \) can only be \( m - 3 \) when \( j = m \).

**Theorem 3.7.** Let \( M \) be the MIM of \( G \), a \( G_{5,m} \) grid and let \( M \) contain \( u_{(\ell)}v_1 \) and \( u_{(\ell)}v_4 \) (or \( u_{(\ell)}v_{m-3} \) and \( u_{(\ell)}v_m \)). Then there are at least \( 2k + 2 \) saturated vertices on \( U_1 \subset G \).

**Proof.** For \( u_{(\ell)}v_1 \) and \( u_{(\ell)}v_4 \) to be in \( M \), either \( u_{(\ell)}v_4 \in M \) or \( u_{5(\ell)}v_4 \in M \). Now, let \( \{V_6, V_7, \ldots, V_m\} \) induce \( G_{m(1)} \subset G \). Clearly, \( |m(1)| \equiv 2 \mod 4 \) and \( |V_{st}(G_{m(1)})| = 10k - 4 \). Let \( G_{m(1)} \setminus \{u_1v_6, u_1v_7, \ldots, u_1v_m\} \) induce \( G_{m(2)} \subset G_{m(1)} \). Then, \( G_{m(2)} \) is a \( G_{4,m-5} \) subgrid of \( G_{m(1)} \). Now, \( |V_{st}(G_{m(2)})| \leq 8k - 4 \). Thus for \( V(U_1) \subset V(G_{m(1)}) \), \( |V(U)| \geq 2k \). Thus, \( U_1 \) contains at least \( 2k + 2 \) (i.e., \( \frac{m-1}{2} \)) saturated vertices.

Next we investigate \( G_{3,m} \), where \( m \equiv 3 \mod 4 \).
Lemma 3.8. Suppose that $G$ is a $G_{3,m}$ grid with $m \equiv 3 \mod 4$ and $M$ is an induced matching of $G_{3,m}$, with $\{u_{(i,j)}v_i, u_{(i,j)}v_{i+2}, u_{(i,j)}v_j, u_{(i,j)}v_{j+2}\} \in M$ and $i + 2 \geq j$. Then $M$ is not a MIM of $G$.

Proof. Suppose $i + 2 \geq j$. Since $m = 4k + 3$, $|V_{sb}(G)| = 6k + 5$ and $|V_{st}(G)| = 6k + 4$. Thus, $G$ contain at most one FSV. Now from the conditions in the hypothesis, it is clear that $u_3v_i + 1$ and $u_3v_j + 1$ are FSVs in $G$, which is a contradiction. Same argument hold if $i + 2 = j$ since both $u_3v_i + 1$ and $u_3v_i + 3$ are FSVs in $G$.

Remark 3.7. Suppose that $G_n$ is $G_{3,n}$, a subgrid of $G_{3,m}$ and induced by $\{V_{i+1}, V_{i+2}, \ldots, V_{i+n}\}$ and $G'$ is a subgraph of $G$, with $G' = G_n + \{u_3v_i, u_3v_{i+n+1}\}$, then the following are easy to verify. For

(a) $n \equiv 0 \mod 4$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 2$
(b) $n \equiv 1 \mod 4$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 2$
(c) $n \equiv 2 \mod 4$, $|V_{st}(G')| = |V_{sb}(G_n)|$
(d) $n \equiv 3 \mod 4$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 1$

Lemma 3.9. Let $u_{(i,j)}v_j, u_{(i,j)}v_{j+3}, u_{(i,j)}v_k, u_{(i,j)}v_{k+3}, u_{(i,j)}v_i, u_{(i,j)}v_{i+3}$ be in $M$ in induced matching of $G$ a $G_{3,m}$ grid and $m \equiv 3 \mod 4$. Then $M$ is not MIM of $G$.

Proof. Case 1: Let $m = 4p + 3$, $j + 3 = k$ and $l = k + 3$. Suppose $G^{m(1)}$ is a subgraph of $G$, induced by $\{V_{j-1}, V_j, \ldots, V_{i+4}\}$. Then $|m(1)| = 12$, with $u_3v_{j-1}$ and $u_3v_{i+4}$ as FSVs. For one of $u_3v_{j-1}$ and $u_3v_{i+4}$ to be relevant for $M$ to be MIM of $G$, say $u_3v_{j-1}$, then for $G^{m(2)}$, induced by $\{V_1, V_2, \ldots, V_{j-2}\}$, $|V_{sb}(G^{m(2)})|$ must be odd, which can only be if $j - 2 \equiv 3 \mod 4$. So, suppose $j - 2 \equiv 3 \mod 4$, then $|V_{st}(G^{m(2)})| + u_3v_{j-1} \leq |V_{sb}(G^{m(2)})| + 1 = 6q + 6$, where $|m(2)| = 4q + 3$, for $q \geq 1$, since $|m(1)| = 12$ and $|n(2)| = 3 \mod 4$ now let $G^{m(3)} = G^{m(1)} \cup G^{m(2)}$, where $|m(3)| = |m(1)| + |m(3)| = 3 \mod 4$ and $G^{m(4)} \subset G$ be defined as a subgrid of $G$ induced by $\{V_{i+5}, V_{i+6}, \ldots, V_m\}$. Clearly, $|m(4)| \equiv 0 \mod 4$. Since $|V_{sb}(G^{m(4)})| = |V_{st}(G^{m(4)})|$, which is even, then $|V_{st}(G^{m(4)})| + u_3v_{i+4} = |V_{st}(G^{m(4)})| = 6p - 6q - 18$. Now, it can be seen that $|V_{st}(G^{m(1)})| \{u_3v_{j-1}, u_3v_{i+4}\} = 14$. Therefore, $|V_{st}(G)| \leq 6p + 2$ instead of $6p + 4$, and hence a contradiction.

Case 2: Suppose that $j + 3 < k$ and $k + 3 < l$. As in Case 1 and without loss of generality, let $j - 2 \equiv 3 \mod 4$ and let $G^{m(2)}$ still be induced by $\{V_1, V_2, \ldots, V_{j-2}\}$. Also, let $G^{m(4)}$ be induced by $\{V_{i+5}, V_{i+6}, \ldots, V_m\}$ and set $|m(4)| \equiv 3 \mod 4$. Thus, $u_3v_{j-1}$ and $u_3v_{i+4}$ are both relevant for $M$ to be a MIM of $G$ and $|V_{st}(G^{m(2)}) + v_{j-1}| = |V_{sb}(G^{m(2)})| + 1$ and $|V_{st}(G^{m(4)})| + v_{i+4}| = |V_{sb}(G^{m(4)})| + 2$. Therefore, $|V_{st}(G)| \leq 6p + 2$ instead of $6p + 4$, and hence a contradiction.
and thus, \( V_{-4} = |V_{-4b}(G^{m(4)})| + 1 \). Set \( G^{m(2)} = V_{-1} = G^{m(2)} \) and set \( G^{m(2)} = V_{-1} = G^{m(2)} \) and let \( \{ V_1, V_{1+2}, V_{1+3} \} \) induce \( G^{m(5)} \) while \( \{ V_i, V_{i+1}, V_{i+2}, V_{i+3} \} \) induces \( G^{m(6)} \). Furthermore, let \( G^{m(5)} = G^{m(5)} + V_{-4} \) and \( G^{m(6)} \) contain, say, \( h \) columns of \( V_i \) in all, where \( h \equiv 2 \) mod 4. Therefore, for \( G^{m(7)} = G \backslash \left\{ G^{m(2)} \cup G^{m(4)} \cup G^{m(5)} \cup G^{m(6)} \right\} \), \( |m(7)| = m - h = b \equiv 1 \) mod 4.

Let \( b = 4a + 1 \), for some positive integer \( a \) and let \( G^{m(4)} \subset G^{m(7)} \), where \( G^{m(7)} \) is induced by \( \{ V_i, V_{i+1}, V_{i+2}, V_{i+3} \} \) and \( \{ V_{i+4}, V_{i+6}, \cdots, V_{i+2} \} \) with \( m(8) = m(9) = b \). Suppose thus, that \( |m(8)| \equiv 0 \) mod 4, then \( |m(9)| \equiv 3 \) mod 4 and suppose \( |m(8)| \equiv 1 \) mod 4, then \( |m(9)| \equiv 2 \) mod 4. For \( |m(8)| \equiv 0 \) mod 4, let \( G^{m(10)} = G^{m(2)} \cup G^{m(5)} \) be \( G^{m(2)} \cup G^{m(5)} \) and \( G^{m(11)} = G^{m(6)} \cup G^{m(4)} \) be \( G^{m(6)} \cup G^{m(4)} \), where \( |m(2)| + |m(5)| = 4q + 9 \) and \( |m(4)| + |m(6)| = 4r + 9 \), where \( |m(4)| = 4r + 3 \). Therefore, as defined, \( b = |m(7)| = 4p - 4q - 4r - 15 \) and thus \( b = 4(p - q - r - 6) + 3 \). Set \( p - q - r - 6 = f \). Now, for \( |m(8)| \) and \( |m(9)| \), if \( |m(8)| = 4g \), for some positive integer \( g \), then \( |m(9)| = 4(f - g) + 3 \). Next we sum the maximal values of the subgrid of \( G \) as follows: \( |V_{st}(G)| \leq |V_{st}(G^{m(2)} \cup G^{m(5)})| + |V_{st}(G^{m(8)}) + u_{3v_{j+4} + u_{3v_{k-1}}}) + |V_{st}(G^{m(4)}) + |V_{st}(G^{m(9)}) + u_{3v_{k+4} + u_{3v_{l-1}}})| + |V_{st}(G^{m(6)}) \cup G^{m(4)})| \leq 6p + 2 \), which is less than \( 6p + 4 \) and hence a contradiction. For \( |m(8)| \equiv 1 \) mod 4, and \( |m(9)| \equiv 2 \) mod 4, we have \( |m(8)| = 4g + 1 \) and hence \( |m(9)| = 4(f - g) + 2 \) and \( |V_{st}(G^{m(9)}) + u_{3v_{k+4} + u_{3v_{l-1}}})| = 6(f - g) + 4 \) and thus, \( |V_{st}(G)| \leq 6p + 2 \).

Case 3: Suppose \( j + 3 = k \) or \( k + 3 = i \). Without loss of generality, let \( j + 3 = k \). Suppose as in Case 2, \( j - 2 \equiv 3 \) mod 4 and \( m - (i + 4) \equiv 3 \) mod 4. Let \( G^{m(1)} \subset G \), a \( G_{3,9} \) subgrid of \( G \) be induced by \( \{ V_{-1}, v_j, \cdots, v_{j+2} \} \). Then for \( G^{m(2)} = G^{m(2)} \cup G^{m(1)}, \ |n(2)| = m(2) + |n(1)|, \ |n(2)| \equiv 0 \) mod 4. Likewise, suppose \( \{ V_{i-1}, V_i, \cdots, V_m \} \) induces \( G^{m(3)} \), for which \( |n(3)| \equiv 1 \) mod 4. If \( |n(2)| \) and \( |n(3)| \) are \( 4q \) and \( 4r + 1 \) respectively, then \( |n(4)| \equiv 2 \) mod 4. So far, \( G^{m(4)} \), is induced by \( \{ V_{i+4}, V_{i+9}, \cdots, V_{i+2} \} \) and by Remark 3.7 \( |V_{st}(G^{m(4)})| + \{ u_{3v_{j+7} + u_{3v_{l-1}}})| = |V_{sb}(G^{m(4)})| \). By a summation similar to the one at the end of case 2, \( |V_{st}(G)| \leq |V_{st}(G^{m(2)})| + |V_{st}(G^{m(4)})| + |V_{st}(G^{m(3)})| \leq 6p + 2 \).

Remark 3.8. By following the technique employed in Lemma 3.9 it can be established that given \( u_{(i+1)v_j} \), \( u_{(i+1)v_{j+2}} \in M \) and \( u_{(i+1)v_j}, u_{(i+1)v_{j+2}} \in M \) of \( G \), a \( G_{3,m} \) grid, \( m \equiv 3 \) mod 4, \( i + 2 \leq j \), then \( M \) is not a \( MIM \) of \( G \).
Likewise, suppose $v_{(i)}$ is contained in $M$, for all non-negative integer $i$ for which $1 \leq i + 8(n) \leq m$. Let $M$ be the maximum induced matching of $G$. Then,

(a) if $i > 1$, then $i - 1$ is either $2, 3, 4$ or $6$

(b) if $i + 8(n) < m$, for the maximum value of $n$, then $m - (i + 8(n))$ is either $2, 3, 4$ or $6$.

Based on the results so far, we note that if $M$ is the $MIM$ of $G$, a $G_{3,m}$ grid, $m \equiv 3 \mod 4, m \geq 11$, the maximum number of edges of the type $u_{(i)}v_{(i+8)}$ that is contained in $M$, $k$, a positive integer, is $k + 2$ when $m = 8k + 3$ and $k + 3$ when $m = 8k + 7$.

It can be easily established that for $H$ that is a $G_{k,m}$ grid, with $k \equiv 0 \mod 4$ and $m \equiv 3 \mod 4$, which is induced by $U_1, U_2, \ldots, U_k$, if $M_1$ is the $MIM$ of $H$, then, the least saturated vertices in $U_k$ is $\frac{m-1}{2}$. The next result describes the positions of the members of $M_1$ in $E(H)$ if $U_k$ contains $\frac{m-1}{2}$ saturated vertices.

**Lemma 3.10.** Let $H$ be a $G_{k,m}$ grid with $k \equiv 0 \mod 4$ and $m \equiv 3 \mod 4$ and let $U_k$ contain the least possible, $\frac{m-1}{2}$, saturated vertices for which $N$ remains $MIM$ of $H$. Then, for any adjacent vertices $v', v'' \in U_k$, edge $v'v'' \notin M$.

**Proof.** Induced by $\{U_1, U_2, \ldots, U_{k-2}\}$ and $\{U_{k-1}U_k\}$ respectively, let $G_1^{[m]}$ and $G_2^{[m]}$ be partitions of $H$ with $k - 2 \equiv 2 \mod 4$. It can be seen that $|V_{st}(G_1^{[m]})| = |V_{st}(G_1^{[m]})| = \frac{km-2m+2}{2}$. Since $|V_{st}H| = \frac{km}{2}$, then $|V_{st}(G_2^{[m]})| \leq m - 1$. Now, let $G_3^{[m]}$ be a $G_{1,m}$ subgrid (a $P_m$ path) of $H$, induced by $U_k$. By the hypothesis, $U_k$ contains maximum of $\frac{m-1}{2}$ saturated vertices. Now, let $u_kv_i, u_kv_{i+1}$ be adjacent and saturated vertices of $G_3^{[m]}$. Then there are $\frac{m-5}{2}$ other saturated vertices on $G_3^{[m]}$. Without loss of generality, suppose that each of the remaining $\frac{m-5}{2}$ saturated vertices in $G_3^{[m]}$ is adjacent to some saturated vertex in $U_{k-1}$. Now, suppose $u_{k-1}v_j$ is a saturable vertex in $U_{k-1}$ and that $v \in V(H)$, such that $u_{k-1}v_jv \in M_1$. Now, $v \notin U_k$, since all the saturable vertices in $U_k$ is saturated. Likewise, suppose $v \in U_{k-1}$ and then $u_{k-1}v_jv \in E(G_4^{[m]})$, where $G_4^{[m]}$ is a $G_{1,m}$ subgrid of $H$ induced by $U_{k-1}$. Then, clearly, at least one of $u_{k-1}v_j$ and $v$ is adjacent to a saturated vertex in $V_{st}(G_1^{[m]})$. Also, suppose that $v \in U_{k-2}$, since $|V_{st}(G_1^{[m]})| = |V_{st}(G_1^{[m]})|$, then $|V_{st}(G_1^{[m]})| = |V_{st}(G_1^{[m]+u_{k-1}u_j})|$. Hence, $v$ is a FSV in $G_1^{[m]}$. Therefore, $|V_{st}H| \leq |V_{st}G_1^{[m]}| + |V_{st}G_2^{[m]}| \leq \frac{km-4}{2}$, which is a contradiction since $|V_{st}(H)| = \frac{km}{2}$, by [7].

$\square$
Remark 3.10. The implication of Lemma 3.10 is that for a grid $H' \subset H$, which is induced by $\{U_1, U_2, \ldots, U_{k-2}\} \subset V(H)$, $k - 2 \equiv 2 \mod 4$, suppose $U_k$ contains the least possible saturated vertices, $\frac{m-1}{2}$, then $u_kv_2, u_kv_4, \ldots, u_kv_{m-1}$ are saturated as shown in the example in Figure 3, for which $k = 4$ and $m = 7$.

Figure 3. A $G_{4,7}$ Grid with $MIM_G = 11$

Lemma 3.11. Let $G$ be a $G_{3,m}$ with an induced matching $M$ and $G^{(9)}$, induced by $\{V_i, V_{i+2}, \ldots, V_{i+8}\}$ be a $G_{3,9}$ subgrid of $G$. Suppose that $M' \subset M$ is an induced matching of $G^{(9)}$ such that $u_{i+4}v_i, u_{i+8}v_{i+8} \in M'$. No other edge $u_tv_i$, $1 < t < i + 7$ is contained in $M'$. Then for $G^{9(9)} \subset G^{(9)}$, defined as $G^{(9)} \setminus U_1$, $|V_{sb}(G^{9(9)})|$ ≤ 8.

Proof. Let $G^{(7)} = G^{(9)} \setminus \{u_1v_{i+1}, u_{i+2}v_i, \ldots, u_{i+7}v_i\}, V_i, V_{i+8}\}$. It can be seen that $G^{(7)}$ is a $G_{2,7}$ subgrid of $G^{(9)}$. Clearly also, $G^{(7)} \subset G^{9(9)}$. Since $u_2v_i, u_{i+8}v_{i+8} \in M'$, then $u_2v_{i+1}$ and $u_2v_{i+7}$ can not be saturated. Let $G_y \subset G^{(7)}$ be subgraph of $G^{(7)}$, defined as $G^{(7)} \setminus \{u_2v_{i+1}, u_2v_{i+7}\}$. Now, $|V(G_y)| = 12$ and $|V_{sb}(G_y)|$ can be seen to be at most 6. Thus $|V_{sb}(G^{9(9)})| = |V_{sb}(G_y)| + 2 = 8$, since $u_2v_i$ and $u_2v_{i+8}$ are saturated in $M'$.

Remark 3.11. For $U_1 \subset G^{(9)}$ as defined in Lemma 3.11, $U_1$ contains at least 6 saturated vertices, implying that $M'$ contains two edges whose four vertices are from $U_1$.

Corollary 3.12. Let $G$ be a $G_{3,m}$ grid with $m \geq 11$ and $m \equiv 3 \mod 4$. If $M$ is a $MIM$ of $G$. Then $M'$ contains at least $2k'$ edges from $U_1$, where $m = 8k' + 3$ or $m = 8k' + 7$.

Figure 4. A $G \equiv G_{3,23}$ Grid with $MIM_G = 17$
Theorem 3.13. Let $G$ be a $G_{n,m}$ grid, with $m \geq 23$. Then for $n \equiv 1 \mod 4$, $\text{MIM}_G \leq \left\lfloor \frac{2mn-m-3}{8} \right\rfloor$.

Proof. For $n \equiv 1 \mod 4$, $n - 5 \equiv 0 \mod 4$. Let $G_1$ and $G_2$ be partitions of $G$ with $G_1$ induced by $\{U_i, U_2, \cdots, V_{n-5}\}$ and $\{V_{n-4}, V_{n-3}, V_{n-2}, V_{n-1}V_n\}$ respectively. Also, let $M', M''$ be MIM of $G_1$ and $G_2$ respectively. Suppose, $U_{n-5}$ contains at least $\frac{m-1}{2}$ saturated vertices, the least $U_{n-5}$ can contain for $M'$ to remain MIM of $G_1$. By Theorem 3.7, $U_1 \subset G_2$ (the $U_{n-4}$ of $G$) contains at least $2k+2$ saturated vertices with $k = \frac{m-3}{4}$. Following the proof of Theorem 3.7 it is shown that $M''$ contains $\frac{m-3}{4}$ edges of $U_1 \subset G_2$ and either of $u(\frac{1}{2})v_4$ and $u(\frac{1}{2})v_{m-3}$. Now, with $G = G' \cup G''$ and hence, $|M| \leq |M'| + |M''|$, it is obvious therefore, that for each edge $u_\alpha u_\beta \in U_{n-4}$ contained in $M''$, either $u_\alpha$ or $u_\beta$ is adjacent to a saturated vertex on $U_{n-5}$ and also, $u_{n-4}v_4$ (or $u_{n-4}v_{m-3}$) is is adjacent to saturated $u_{n-5}v_4$ (or to saturated $u_{n-4}v_{m-3}$). Hence, $|V_s(G)| \leq \frac{2mn-m-7}{4}$ and thus, $\text{MIM}_G \leq \left\lfloor \frac{2mn-m-7}{8} \right\rfloor$. \qed

Theorem 3.14. Let $G$ be a $G_{n,m}$ grid with $n \equiv 3 \mod 4$ and $m \equiv 3 \mod 4$, $m \geq 11$. Then $\text{MIM}_G \leq \left\lfloor \frac{2mn-m+1}{8} \right\rfloor$ and $\text{MIM}_G \leq \left\lfloor \frac{2mn-m+5}{8} \right\rfloor$ for $m = 8k' + 3$ and $m = 8k' + 7$ respectively.

Proof. The proof follows similar technique as in Theorem 3.13. \qed

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