On the complexity of Commuting Local Hamiltonians, and tight conditions for Topological Order in such systems

Dorit Aharonov∗  Lior Eldar†

January 11, 2013

Abstract

The local Hamiltonian problem plays the equivalent role of SAT in quantum complexity theory. Understanding the complexity of the intermediate case in which the constraints are quantum but all local terms in the Hamiltonian commute, is of importance for conceptual, physical and computational complexity reasons. Bravyi and Vyalyi showed in 2003 [8], using clever applications of the representation theory of C*-algebras, that if the terms in the Hamiltonian are all two-local, the problem is in NP, and the entanglement in the ground states is local. The general case remained open since then. In this paper we extend the results of Bravyi and Vyalyi beyond the two-local case, to the case of three-qubit interactions. We then extend our results even further, and show that NP verification is possible for three-wise interaction between qutrits as well, as long as the interaction graph is embedded on a planar lattice, or more generally, “Nearly Euclidean” (NE). The proofs imply that in all such systems, the entanglement in the ground states is local. These extensions imply an intriguing sharp transition phenomenon in commuting Hamiltonian systems: 3-local NE systems based on qubits and qutrits cannot be used to construct Topological order, as their entanglement is local, whereas for higher dimensional qudits, or for interactions of at least 4 qudits, Topological Order is already possible, via Kitaev’s Toric Code construction. We thus conclude that Kitaev’s Toric Code construction is optimal for deriving topological order based on commuting Hamiltonians.

1 Introduction

The problem of approximating the ground energy of a local Hamiltonian describing a physical system is one of the major problems in condensed matter physics; in the area of quantum computation this problem is called the local Hamiltonian problem [13]. Formally, in the k-local Hamiltonian problem, we are given a Hamiltonian $H$ which is a sum of positive semidefinite terms, each acting on a set of at most $k$ out of $n$ qubits, where $k$ is of order 1, and each term is of bounded norm. Moreover, we are given two numbers, $b > a$ such that $b - a \geq \frac{1}{\text{poly}(n)}$. We are asked whether $H$ has an eigenvalue below $a$ or all its eigenvalues are above $b$, and we are promised that the instance belongs to one of the two cases.

∗School of Computer Science and Engineering, The Hebrew University, Jerusalem, Israel
†School of Computer Science and Engineering, The Hebrew University, Jerusalem, Israel.
It turns out that the problem of understanding ground states and ground values of local Hamiltonians, central to condensed matter physics, is the quantum generalization of one of the most important problems in classical computational complexity, namely, SAT. Indeed, in a seminal work, Kitaev has shown that in parallel to the important of the SAT problem in NP theory, the local Hamiltonian problem is complete for the quantum analogue of NP (denoted QMA) in which both witness and verifier are quantum rather than classical. The analogy between the quantum and the classical problems is derived by viewing the terms of the Hamiltonians as generalizing the notion of classical constraints; energies are viewed as a penalty for a constraint violation. For example, to view the local constraints for the classical SAT as a special instance of local Hamiltonians, we assign for each clause a projection on the assignment forbidden by this clause. The projections we derive are of course all projections in the computational basis; in the general quantum case, the terms need not be diagonal in any particular basis, and the ground state can be highly entangled. This connection linking the physics and the computational complexity problems has drawn much attention over the past few years, and has led to many exciting results and insights (eg., [13, 11, 1, 14, 4, 8]).

The computational view of the local Hamiltonian problem and its connection to classical NP problems, has led Bravyi and Vyalyi in [8] to the following very natural question: what would happen if we only generalize from classical to quantum “half way”: we allow the terms in the Hamiltonian to be projections in any basis, but we restrict them in that all the projections pairwise commute. We are asked to decide whether the ground energy is 0 (namely, there exists a state which is in the ground space of all projections) or it is larger than 0 (for pairwise-commuting projections, the overall energy, namely eigenvalue, of such a state must be at least 1). This problem is the commuting local Hamiltonian problem.\[1\]

The interest in the commuting Hamiltonian problem is related to several important issues in quantum computational complexity. The first is conceptual: a common intuition is that the counter intuitive phenomena in quantum mechanics stems from the fact that non-commuting operators are involved (cf the Heisenberg’s uncertainty principle). One might conjecture, using this intuition, that the commuting local Hamiltonian problem is far weaker than the general local Hamiltonian problem, and might be of the same complexity as SAT, namely, lie in NP. However, a counter intuition exists: The intriguing strictly quantum phenomenon of topological order, which is exhibited for example in Toric codes [12], can be achieved by ground states of commuting Hamiltonians. It is thus natural to ask where does the computational complexity of the commuting Hamiltonian problem lie: is it in NP, is it perhaps quantum-NP complete (where here the relevant quantum analogue of NP is in fact, QMA_1, where there is only one sided error) or maybe the commuting local Hamiltonian problem defines an intermediate computational class of its own?

The study of this problem can also be viewed as tightly related to an exciting major open problem in quantum Hamiltonian complexity: the question of whether a PCP-like theorem holds in the quantum setting or not [2]. Embarrassingly, this problem is still open even for the seemingly much easier case of commuting local Hamiltonians. Clearly, a PCP-type theorem would follow trivially if the commuting local Hamiltonian problem were in NP, but even if this were not true, one might

---

1 We note that this problem is equivalent to the more general case when the terms can be taken as positive-semidefinite operators, since for such an input one can replace each local term with a projection on the non-zero eigenspaces of that term.
still hope to prove a PCP-type theorem for the restricted problem before proceeding to the more
general case. We recall that several results in quantum Hamiltonian complexity, such as the area
law in 1Dim [9] the decay of correlations in gapped Hamiltonians [10], and quantum gap ampli-
fication [2] were all proven exactly in this way, by starting from the easier commuting case, and
generalizing from there; it seems reasonable to hope that better understanding of the commuting
case would help clarify the quantum PCP conjecture in general. More generally, understanding
the complexity of the commuting local Hamiltonian problem might not only shed light on the
role of commutativity in quantum Hamiltonian complexity, and possibly lead to progress on open
problems in quantum complexity theory, but also, it seems that an answer to this question would
necessarily require new insights regarding the nature of multi-particle entanglement.

In [8] an important step was made towards resolving the computational complexity of the
commuting local Hamiltonian problem. Bravyi and Vyalyi showed that for \( k = 2 \), namely for two-
body interactions, regardless of the dimensionality \( d \) of the particles involved, the problem lies in
\( \text{NP} \). The method they use is interesting by itself; They cleverly apply the theory of representations
of \( C^* \)-algebras to the problem. However, their methods break down for three-wise interactions.
The general problem was thus left open by [8], and no progress was noted on this problem since
its inception in 2003.

1.1 Results: The Complexity of Commuting Hamiltonians

In this paper we extend the results of [8] to three-local interactions with the following two results.
We prove:

**Theorem 1.1** The problem of 3-local commuting Hamiltonian on qubits is in \( \text{NP} \).

**Theorem 1.2** (Roughly stated) The problem of 3-local commuting Hamiltonian on qutrits is in \( \text{NP} \), as long
as the interaction graph is planar, and moreover, nearly Euclidean.

The notion of Nearly Euclidean will be defined later (see Definition 5.4); roughly, it formalizes
the requirement that the embedding makes sense physically: no area on the plane can have a par-
ticularly high density of particles, and only close-by particles can interact. This of course includes
also the interesting special case of periodic lattices, or small perturbations of those.

Unlike what might be expected, the extension does not seem to follow easily from the result
of [8]. When attempting to apply the proof of [8] to the case in which three local interactions
are involved, (even when the particles are assumed to be qubits) severe problems occur. We will
provide the overview of the proof of Bravyi and Vyalyi, why it breaks down for three-wise interac-
tions and how we overcome this, later in the introduction. Let us first describe some implications
of our results to the seemingly unrelated topic of conditions for topological order.

1.2 Results: Implications for tight conditions on Topological Order

Topological order is a purely quantum phenomenon; Roughly, a state exhibits a topological order
if there exists a state orthogonal to it, and the two cannot be distinguished or connected by a local
operator. Such characteristics are extremely valuable in the context of fault tolerance, and topo-
logical order has attracted much attention for theoretical and implementation purposes for that
reason. A celebrated example of a system exhibiting Topological Order in the context of quantum computation is the Toric Code, due to Kitaev \cite{12}; it can be defined as the ground space of a set of 4-local commuting operators on qubits arranged on the two dimensional grid, or alternatively, by 3-local commuting interactions between 4-dimensional particles. Topological order defined via commuting local Hamiltonians is particularly interesting: for example, recently it has been shown \cite{5} that such systems are also resilient to local perturbations. It is therefore natural to ask whether it is possible to achieve topological order in ground states of local commuting Hamiltonians, with smaller dimensionality or with less particles interacting. Using the above results, we can resolve this problem to the negative. We show that Kitaev’s construction is optimal in a well defined sense.

Theorem 1.3 Tight conditions for Topological order (Roughly) Consider a system of particles with commuting interactions which are either 2-local, or they are 3-local and the dimensionality of the particles is at most 3. Moreover, assume the interaction graph is Nearly Euclidean planar. Then this system cannot exhibit Topological Order, and moreover, in a well defined sense, the entanglement in the ground space is local. On the other hand, there exist nearly Euclidean planar systems of 3-local interactions with particles of dimensionality 4 that exhibit Topological order, hence we derive a tight boundary between local entanglement and Topological order.

To explain how our results are related to conditions on topological order, observe the following. A key property of topological order states is that their entanglement is non-local. In particular, Bravyi, Hastings and Verstraete showed in \cite{6} (see Theorem (6.2), that if a nearest neighbor quantum circuit generates a state with Topological Order on the $n \times n$ grid, the circuit has to be of depth $\Omega(\sqrt{n})$. The methods we use, as well as those of \cite{8}, however, imply that the ground space has a basis of states with localized entanglement. More precisely, the proofs of Theorems 1.1 and 1.2 also imply that any nearly Euclidean planar commuting Hamiltonian system, that is either 2-local, or 3-local with particle dimensionality $d \leq 3$, has a basis of eigenstates all of which can be generated by a constant depth quantum circuit whose gates act on nearest neighbor particles on the plane (where by nearest neighbor, we mean neighbors in the interaction graph of the Hamiltonian). This means that such systems cannot exhibit topological order in all the states in their groundspace.

The Toric Codes, however, can be easily seen as a the ground space of an instance of the commuting 3-local Hamiltonian problem on qudits of dimension 4 (by gluing pairs of qubits together - see Section 6 and in particular Figure 17). Thus, this is an instance of CLH(3, 4) in which any state in the groundspace exhibits topological order.

Our results thus imply that in the context of “physical” planar systems, there exists a tight boundary between Topological Order systems and constant-depth systems: For $k > 3$ and all $d \geq 2$ or $k = 3$ and $d \geq 4$ there exist nearly Euclidean planar systems which exhibit Topological order, whereas for $k = 3$ and $d < 4$, or for $k = 2$ and any $d$, all nearly Euclidean planar systems have a constant-depth diagonalizing circuit and cannot exhibit Topological order. We deduce that Kitaev’s construction cannot be simplified either in terms of particle dimensionality or number of particles in each interaction, and so it is optimal for commuting Hamiltonians.
1.3 Overview of the proofs of Theorems 1.1 and 1.2

1.3.1 Bird’s eye view on the proof of Bravyi and Vyalyi for two-local case

Let us embark on trying to explain the proof, by first explaining the main idea in the proof of Bravyi and Vyalyi of the two-local case, for any particle dimensionality. To do that, we consider the hypergraph describing the interactions in the Hamiltonian. We observe that in the two-local case, every particle is the center of a “star” of interactions - the interactions acting on $q$ intersect only on $q$. This is not true when interactions are three-local, as one can see in Figure (1).

![Figure 1](image1.png)

**Figure 1:** An example of a 2-local interaction hypergraph (left) and 3-local interaction hypergraph (right). The particles which are star centers are unfilled, and those that are not, are filled in.

Bravyi and Vyalyi prove a lemma (restated here as Lemma 3.3), which shows that particles which are centers of “stars”, are what we call “separable”. This means that if $q$ is such a center of a star, its Hilbert space $\mathcal{H}_q$ can be decomposed to a direct sum of subspaces, which are all preserved by all interactions involving $q$:

$$\mathcal{H}_q = \bigoplus_{\alpha} \mathcal{H}_q^\alpha.$$

Moreover, each subspace $\mathcal{H}_q^\alpha$ can be written as a tensor product of sub-particles,

$$\mathcal{H}_q^\alpha = \bigotimes_k \mathcal{H}_q^\alpha \mathcal{H}_k^\alpha,$$

where $k$ runs over all particles that interact with $q$, and the interaction between $q$ and $k$ is non-trivial only on the relevant sub-particle, $\mathcal{H}_q^\alpha \mathcal{H}_k^\alpha$. This way, the restriction to one of the subspaces implies a decoupling of the interactions involving $q$ to interactions that act on separate sub-particles! When all particles are center of stars as is the case for the two-local, after each particle is restricted to one of its subspaces the restricted Hamiltonian is a set of disjoint edges.

From this derive a proof that the two-local problem lies in NP - essentially, the witness is the specification of the choice $\alpha$ of the correct subspace of each particle, in which the groundstate lies.

The above proof also implies that in the two-local case, there is an eigenbasis of the Hamiltonian in which any eigenstate (and in particular any ground state) has a very limited and local structure of entanglement - the state can be generated by a depth-two quantum circuit which uses only two-local gates. Of course, a natural question is whether these techniques can be applied for the more general case, namely, for higher values of $k$. 

5
1.3.2 What fails when trying to apply [8] to three-wise interactions

Trivially, when generalizing from 2-local interactions to 3-local interactions we immediately loose the star topology which was a crucial component in ([8]). See, for example, Figure 2.

Figure 2: In the example both $H_1$ and $H_2$ share a single qubit $q$ with $H_3$. By Lemma (3.3) $H_1$, and $H_3$ agree on some decomposition of $q$, and so do $H_2$ and $H_3$. Yet, because $H_1$ and $H_2$ share two qubits $p$ and $q$, they do not agree necessarily on the same decomposition of $q$.

However, this example is not truly a problem. Because we restrict our attention to qubits, the low dimensionality implies that one cannot "block-diagonalize" an operator on a qubit $q$ in more than one way. Thus it turns out that in the example above, there is indeed a "consensus" decomposition of $q$ preserved by all 3 operators on $q$. However, consider the example of 4 operators on 4 qubits in Figure 3.

Figure 3: An example of a topology of interactions which can be defined in such a way that, say, for $q_1$, no decomposition exists, which is preserved by all operators acting on it.

Since any pair of operators share 2 qubits, it may be that for none of the qubits does there exist a direct-sum decomposition which is preserved by all operators on that qubit. This in fact emanates
from the nature of the commutativity relation for 3-local terms: it can be generated using not just one particle as in [8] but may involve more complex relations involving two particles.

1.3.3 General idea of the proof

The way we overcome this obstacle is by showing that truly complex structures, such as the example of Figure 3, can only be of local nature, after we remove all separable qubits from the system. In other words, any attempt to expand such examples by adding more interactions with additional qubits inevitably makes are least one qubit separable.

The proofs of the two theorems turn out to require quite different tools to achieve this goal; Below we provide overviews of those proofs.

1.3.4 Proof sketch for the case of three-wise interactions of qubits

Let us start with explaining the proof of Theorem 1.1, namely the case of qubits. We build upon the analysis of [8], identifying qubits which behave "classically". Those are qubits for which there exists an orthogonal basis, such that all operators are block-diagonal w.r.t. this basis. We call those qubits “separable”. If the original system has a zero eigenstate, so does a restricted version of the system when each separable qubit is restricted to one of those subspaces. This restriction of each of the qubits can be provided by the prover, which implies that those qubits can be removed from the problem altogether. Unlike in the case of two-body interactions, however, not all qubits can be removed this way. Most of the proof of Theorem 1.1 is focused on handling the residual problem.

The main point is that in the residual problem, configurations such as that in Figure 3 may exist, however, when the qubits are non-separable, they can only grow to some bounded constant size. More precisely, we prove that the fact that the remaining qubits are not separable, implies severe restrictions on the geometry of the problem. We show that any connected component of the residual problem after removing the separable qubit contains a certain one dimensional structure in the interaction graph, which we call a backbone. The backbone is a long sequence of local Hamiltonian terms, as in Figure 4.

**Figure 4:** After removing all classically-behaving qubits, the residual Hamiltonian contains a certain 1-dimensional structure, denoted as the "backbone". We prove that any term in the Hamiltonian must act on at least two “close” backbone qubits.

The key property is that all Hamiltonian terms in one connected component in the residual problem must act on at least two qubits in the backbone of that component. Moreover, the backbone qubits that a term acts on, must be within a constant distance, in terms of edges of the backbone. Thus, Hamiltonian terms cannot connect distant particles in the backbone; Moreover, there cannot be other “shortcuts” connecting distant parts of the backbone through interactions.
with qubits outside of the backbone. We call this an **almost one dimensional structure**. We use this structure to prove that we can combine sets of constantly many nearest neighbor particles in the backbone into large particles, which are still of constant dimension, but in which interactions only act on nearest neighbors large particles. The methods of [8] can be used here again to show that the large particles can then be sliced, in such a way that all interactions become two-local.

1.3.5 **Proof sketch for the planar three-wise interactions of qutrits**

The proof of Theorem 1.2 which proves a similar result with qutrits on the plane is quite different. The first step is similar: we identify qutrits that behave “classically” (i.e., separable), which enables us to remove them, or at least reduce their dimensionality to 2; We then proceed to prove strong geometric restrictions on the residual problem, in which no particle can be further reduced.

To deduce the geometrical constraints, we would like to make use of the fact that the remaining particles are not separable. However, the phenomenon of separability of a qutrit is somewhat more involved than that of a qubit. We say a qutrit is separable if there exists a non-trivial decomposition of its space to orthogonal subspaces, and each of these subspaces is preserved by all operators acting on the qutrit. But whereas for qubits, due to the low dimensionality, such a decomposition, if exists, is unique, for qutrits this is no longer the case, as there is freedom in the choice of basis of a two dimensional space. This prevents the proof for the case of qubits from going through.

We can overcome this difficulty in the case of nearly Euclidean planar interactions. For such instances, we show that the fact that all qudits are inseparable, implies that each qudit can be acted upon by at most a constant number of operators; More precisely, the degree of each vertex in the interaction graph is at most 5. This is a completely geometrical constraint on the interaction graph of the residual problem, and it is in fact a very strong constraint which suffices for our purposes. We show that if one attempts to cover large regions of the plane with faces with only three edges (corresponding to the three-local interactions) and such that no vertex has degree 6 or more, then “holes” must be formed, i.e., faces which are not distorted-triangles but have more edges, and thus do not correspond to any term in the Hamiltonian. The point is that those holes have a constant density; in other words, each vertex in the graph has a hole within constant distance (measured in number of faces separating the vertex from the hole). We use those holes in order to cut the interaction graph to smaller (constant size) pieces which when combined to particles, induce two-local interactions. The problem thus lies in NP by [8].

We provide more detailed overviews of the two proofs in Subsections (4) and (5), this will be easier after we provide some of the necessary definitions and notations in Section 2.

1.4 **Conclusions and Further work**

The results in this paper focus on two aspects of commuting Hamiltonians: the first is extending the containment in NP also for three body interactions, where a fundamental barrier is encountered exactly when topological order can be present in the ground space. Three body interactions seemed before as the barrier standing between [8] and the extension towards a proof of containment in NP of the general case; here we show that the barrier is far more intriguing, and has to do with the appearance of topological order.
The second aspect is the proof that Kitaev’s celebrated construction of topological order using Toric codes is optimal, a statement which is of interest in various contexts, such as physical implementations of topological order states, the understanding of topological quantum codes, and our general understanding of quantum multiparticle entanglement and quantum states.

The barrier exposed in this paper by no means implies that we should neglect the hope to prove that the general commuting Hamiltonian problem is in NP. In fact, we hope that the barrier encountered here would clarify as to how we need to proceed in order to prove (or possibly disprove) the conjecture that the general problem lies in NP.

One possible direction to explore in order to attack the general commuting case is the following. As is well known, topological order states such as Toric codes do have short classical descriptions, which are in fact classical descriptions of small depth quantum circuits, except those circuits are non-local (i.e., not nearest neighbor on the grid). These are called MERA [16, 3]. Those descriptions allow computing local observables efficiently using a classical computer. From the point of view of NP verification, this is clearly sufficient. If one can show that such MERA descriptions exist for any ground state of commuting Hamiltonian, this would imply that the problem lies in NP. It is possible that the algebraic methods of [8] can be used in an innovative way (perhaps by recursion or by other means) to imply that there exist such MERA-type poly-size classical descriptions of eigenstates for any k-local commuting Hamiltonian on a grid.

Our proofs are quite involved. An indication to this complexity is that though our results imply that the ground states in the systems we study can be generated by constant depth quantum circuits, and thus can only create local entanglement, the locality we derived is quite large - the scale of entanglement involves a number of qubits or qutrits which is of the order of a few tens. An intriguing open question is whether the complexity of our results is essential to the problem, or it can be removed. If not, the proofs indicate that in the three-local systems we study, though the entanglement structure is restricted to being local, still quite complicated (though local) structures of entanglement can emerge, which span large though constant sets of particles, with the constant much larger than the natural scales in the system (say, 2 or 3). It would be interesting to understand this aspect further.

Finally, a technicality in the proof of the qutrit case is that the graph is required to be NE, rather than just planar. We speculate that in fact the NE restriction is not necessary; it is used only in the proof of Claim [5,27] and we believe the claim holds for the general planar case. Indeed, this restriction does not have strong implications for the main message of the paper (namely, the tight boundary between TO and local entanglement), since TO is in any case studied only in such NE systems; still it would be nice to close that corner and it would make our statements somewhat cleaner.

**Organization of Paper:** The structure of this paper is as follows: In Section [2] we lay out some notations and definitions, and in section [3] we restate an important lemma in the representation theory of C*-algebra that was at the center of [8], and use it to reprove their result that the 2-local Hamiltonian problem is in NP (perhaps providing a slightly simpler representation of the proof). Then, in section [4] we prove that CLH for 3-local operators on qubits has a classical verification protocol. In section [5] we extend this result for qutrits, in cases where the interaction graph is a nearly Euclidean planar graph. We then use these two proofs, and the result of [6] to derive tight
2 Background, Notations and Definitions

2.1 Hamiltonians and Hilbert Spaces

We use the following standard notation:

- We denote Hilbert spaces by graphical symbols: $\mathcal{H}, \mathcal{H}_i$, etc. The set of linear operators over the complex numbers, acting on a given Hilbert space $\mathcal{H}$, is denoted by $L(\mathcal{H})$.

- Unless otherwise noted, we denote by $\mathcal{H}$ the Hilbert space of $n$ qudits:

$$\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n.$$

The dimensionality of the qudits is denoted by $d$.

- A $k$-local operator $h$ is an operator which acts on a subset of size $k$ of the $n$ qudits $S \subseteq \{1, \ldots, n\}$, hence $|S| = k$, and we have $H \in L(\otimes_{j \in S} \mathcal{H}_j) \otimes (\otimes_{j \notin S} I_j)$.

Less standard notation includes:

- To specify that an operator $H$ acts non-trivially on some specific qubit $q$, we write $H(q)$. We say an operator acts non-trivially on a particle $q$ if the operator cannot be written as a scalar on $q$ tensor some operator on the remaining particles.

- The set of qudits examined non-trivially by an operator $H_i$ is denoted by $A_i$; The set of particles examined non-trivially by a set of operators $B$, is denoted by $A_B$.

2.2 The Local Hamiltonian Problem and its interaction graph

**Definition 2.1** The $(k,d)$ local Hamiltonian problem for commuting operators, CLH$(k,d)$

The $(k,d)$ local commuting Hamiltonian problem on $n$ qudits of dimension $d$, denoted by CLH$(k,d)$, is defined as follows. We are given a set $S$ of $\text{poly}(n)$ $k$-local projections, $H_i$, acting on $n$ particles each of dimension $d$, such that all terms in $S$ pairwise commute. We are asked whether there exists an eigenstate of

$$H = \sum_{i \in S} H_i$$

with eigenvalue $0$.

**Definition 2.2** $G_S$: Interaction graph of a CLH instance

The interaction graph of an instance $S$ of CLH$(k,d)$ is the graph $G_S = (V, E)$, where $V$ is a set of $n$ nodes, each corresponding to a qudit, and an edge connecting nodes $i$ and $j$ is in $E$ (namely, $(i,j) \in E$) if there exists some $H_m \in S$ such that both $i$ and $j$ belong to $A_m$.

**Definition 2.3** Neighborhood of a qudit in an instance $S$ of CLH$(k,d)$

We denote by $N_S(q)$ the neighboring set of a qudit $q$ w.r.t. $S$, namely, the set of all qudits $p$ adjacent to $q$ in $G_S$.
2.3 Operators Preserving Subspaces

An operator $A$ is said to preserve a subspace $S$ if $A(S) \subseteq S$. The following facts are trivial to prove:

**Fact 2.4** If $A$ is Hermitian, if it preserves a subspace $S$ it also preserves the orthogonal complement of $S$.

**Fact 2.5** If a linear operator $A$ commutes with a projection on a subspace $S$, then $A$ preserves $S$.

2.4 Algebras

In this paper we consider finite dimensional $c^*$-algebras. For the purposes of this paper, these are complex algebras (denoted $\mathcal{A}$) of linear operators (described by matrices) with the additional restriction that $\mathcal{A}$ is closed under the operation of taking adjoints of operators (i.e., the dagger, $\dagger$ operation).

We often refer to the algebra generated by a given set of linear operators, referred to as generators. The generators are always a set of matrices of the same dimensionality; and the algebra generated by them is defined either as the minimal algebra that contains the linear subspaces spanned by the generators, or equivalently, the algebra generated by the set of generators union with the identity matrix.

2.5 Algebras induced by operators

**Definition 2.6** **Algebra induced by an operator**

Let $H = H(q)$ be an operator on $q$, and let us write

$$H = \sum_{\alpha} A_\alpha \otimes B_\alpha$$

such that $A_\alpha$ acts on $q$, and $B_\alpha$ acts on the rest of the environment, and the set $\{B_\alpha\}$ is linearly independent. Then the algebra induced by $H$ on $q$ is the algebra inside $\mathcal{L}(\mathcal{H}_q)$ generated by $\{A_\alpha\}_{\alpha} \cup \{1\}$.

**Fact 2.7** Given an operator $H(q)$, the induced algebra on $q$, $\mathcal{A}^H_q$ is independent of our choice of how to write $H$ as a sum as in Equation 1 so long as the $B_\alpha$ operators are linearly independent.

**Proof:** Let us decompose $H(q)$ in two different ways:

$$H = \sum_{\alpha} A_\alpha \otimes B_\alpha = \sum_{\beta} \hat{A}_\beta \otimes \hat{B}_\beta$$

where the sets $\{B_\alpha\}_{\alpha}$ and $\{\hat{B}_\beta\}_{\beta}$ are each linearly independent. Let $\mathcal{A}, \hat{\mathcal{A}}$ denote the $C^*$-algebra of $H$ on $q$ induced by the first and second decompositions, respectively. We show that $\mathcal{A} \subseteq \hat{\mathcal{A}}$, by symmetry this suffices to show that the two algebras are equal. Let us complete the set $\hat{B}$ into a basis of the second subsystem, by the matrices $\{B_{\alpha'}\}_{\alpha'}$. We can thus write the $\hat{B}$ matrices in terms of this basis:

$$\hat{B}_\beta = \sum_{\alpha} \gamma_{\beta}^{\alpha} B_\alpha + \sum_{\alpha'} \gamma_{\beta}^{\alpha'} B_{\alpha'}$$
with $\gamma_{\beta}$, $\gamma'_{\beta}$ complex numbers. Setting the equation above, in the second decomposition, we get:

$$H = \sum_{\beta} \hat{A}_{\beta} \otimes \left( \sum_{\alpha} \gamma_{\beta}^\alpha B_{\alpha} + \sum_{\alpha'} \gamma'_{\beta}^\alpha B_{\alpha'} \right)$$

which means that

$$H = \sum_{\alpha} \left( \sum_{\beta} \gamma_{\beta}^\alpha \hat{A}_{\beta} \right) \otimes B_{\alpha} + \sum_{\alpha'} \left( \sum_{\beta} \gamma'_{\beta}^\alpha \hat{A}_{\beta} \right) \otimes B_{\alpha'}.$$

We now recall that the decomposition in terms of a basis of linearly independent matrices is unique. Comparing the last equation with

$$H = \sum_{\alpha} A_{\alpha} \otimes B_{\alpha} + \sum_{\alpha'} A_{\alpha'} \otimes B_{\alpha'},$$

the matrix in front of each $B_{\alpha}$ must be the same. Hence, $A_{\alpha}$ is contained in the algebra $\mathcal{A}$ spanned by $\hat{A}_{\beta}$.

**Fact 2.8** Given a Hermitian operator $H(q)$, the induced algebra on $q$ is closed under the adjoint operator.

**Proof:** We write $H = \sum_{\alpha} A_{\alpha} \otimes B_{\alpha}$ with $B_{\alpha}$ linearly independent; then the induced algebra is the one generated by $\{A_{\alpha}\}_{\alpha} \cup \{I\}$. But since $H$ is Hermitian, $H = \sum_{\alpha} A_{\alpha}^\dagger \otimes B_{\alpha}^\dagger$ and so the induced algebra is also the algebra generated by $\{A_{\alpha}^\dagger\}_{\alpha} \cup \{I\}$ by Fact 2.7. This means that the induced algebra also contains the adjoint of the generators, and hence is closed under the adjoint.

A simple but crucial fact to this paper is that if two terms that intersect only on a qubit commute, then the two algebras they induce on $q$ commute:

**Fact 2.9** Consider two Hamiltonian terms $H_{j,k}$ intersecting only on the qudit $j$. Then the algebras $\mathcal{A}_{j,k}$ induced by these operators on $j$ commute with each other.

**Proof:** Let $H_{j,1}$ and $H_{j,2}$ be two commuting operators that share only qudit $j$, and let their decomposition be as follows:

$$H_{j,1} = \sum_{\alpha} A_{\alpha}^1 \otimes B_{\alpha}^1, H_{j,2} = \sum_{\alpha} A_{\alpha}^2 \otimes B_{\alpha}^2$$

where $A_{\alpha}^1$ and $A_{\alpha}^2$ are operators on qudit $j$, and the sets $\{B_{\alpha}^1\}$ and $\{B_{\alpha}^2\}$ are each linearly independent, and act on different qudits. We get:

$$[H_{j,1}, H_{j,2}] = \sum_{\alpha, \beta} [A_{\alpha}^1, A_{\beta}^2] \otimes B_{\alpha}^1 \otimes B_{\beta}^2 = 0.$$

By the linear independence of $\{B_{\alpha}^1\}$ and $\{B_{\alpha}^2\}$ we have that the set $\{B_{\alpha}^1 \otimes B_{\beta}^2\}_{\alpha, \beta}$ is also linearly independent, thus $[A_{\alpha}^1, A_{\beta}^2] = 0$ for all $\alpha, \beta$, meaning that the algebras $\mathcal{A}_{j,1}$ and $\mathcal{A}_{j,2}$ commute.
2.6 Representation theory of algebras

The proof of [8], as well as our proofs, rely on fundamental facts from the representation theory of $C^*$ algebra. For the purposes of this paper, we restrict attention to algebras $A \subseteq L(H)$, such that $A$ is closed under the $^\dagger$ operation. All the algebras we will consider are of this form.

**Definition 2.10** Center of a $C^*$-algebra, $Z(A)$.

The center of a sub-algebra $A$ is defined to be the set of all operators in $A$ which commute with all the elements in $A$. It is denoted by $Z(A)$.

**Definition 2.11** A reducible / irreducible $C^*$-algebra

An algebra $A$ is said to be irreducible if its center is trivial, i.e. $Z(A) = c \cdot I$, and otherwise it is reducible.

**Fact 2.12** Let $A$ be an irreducible subalgebra of $L(H)$, i.e. $Z(A) = c \cdot I$. Then $H$ can be written as a tensor product of two subsystems $H_1 \otimes H_2$ such that

$$A \approx L(H_1) \otimes I(H_2).$$

A generalization of the previous fact is a well-known decomposition theorem from the representation theory of $C^*$-algebras:

**Fact 2.13** Let $A$ be a $C^*$-algebra on some Hilbert space $H$. Then, there exists a decomposition of $H$ into a direct sum of orthogonal subspaces $H_\alpha$, where each $H_\alpha$ is a tensor product of two Hilbert spaces, $H_\alpha = H^1_\alpha \otimes H^2_\alpha$ such that

$$A \approx \bigoplus \alpha L(H^1_\alpha) \otimes I(H^2_\alpha).$$

The projections on the subspaces $H_\alpha$ generate $Z(A)$, and for each subspace $H_\alpha$ the algebra $A_\alpha$ (which is defined to be the algebra $A$ restricted to $H_\alpha$), is irreducible.

**Fact 2.14** Let $A_1$ and $A_2$ be two commuting algebras on a Hilbert space $H$, and let $A_1$ be decomposed by its center as in fact (2.13) - i.e. a decomposition $H = \bigoplus \alpha H_\alpha$ such that $A_1 \approx \bigoplus \alpha L(H^1_\alpha) \otimes I(H^2_\alpha)$. Then $A_2$ preserves each subspace $H_\alpha$ in the decomposition above.

**Proof:** For each subspace $H_\alpha$ there exists a projection $\Pi_\alpha \in Z(A_1)$ whose image is $H_\alpha$. Since $A_1$ and $A_2$ commute, then each projection $\Pi_\alpha$ commutes with all operators in $A_2$. Thus, by Fact 2.5 $A_2$ preserves $H_\alpha$ for all $\alpha$.

3 2-local CLH is in NP (Revised from [8])

The basic facts from the theory of representations of algebras presented in Section 2 can be used to prove an important lemma, which is the basis of the proof of [8]. We start with the following definition and a preliminary claim:

**Definition 3.1** Let $\{A_j\}_{j=1}^k$ be $k$ mutually commuting algebras on some Hilbert space $H$. A separating decomposition is a direct-sum decomposition of $H$:

$$H = \bigoplus \alpha H_\alpha.$$
that is preserved by all algebras, such that inside each subspace $\mathcal{H}_\alpha$ there appears a tensor product structure

$$\mathcal{H}_\alpha = \mathcal{H}_\alpha^0 \otimes \mathcal{H}_\alpha^1 \otimes \mathcal{H}_\alpha^2 \otimes \cdots \otimes \mathcal{H}_\alpha^k$$

such that

$$A_j|_{\mathcal{H}_\alpha} \approx I_{\mathcal{H}_\alpha^0} \otimes I_{\mathcal{H}_\alpha^1} \otimes I_{\mathcal{H}_\alpha^2} \cdots \otimes L(\mathcal{H}_\alpha^j) \otimes I_{\mathcal{H}_\alpha^{j+1}} \cdots \otimes I_{\mathcal{H}_\alpha^k}.$$

**Claim 3.2** Let $\{A_j\}_{j=1}^k$ be $k$ mutually commuting algebras on some Hilbert space $\mathcal{H}$. There exists a separating decomposition of $\mathcal{H}$.

**Proof:** Suppose that the algebra $A_j$ are all irreducible algebras - i.e. have trivial centers. Using fact (2.12) we have that each $A_j$ is isomorphic to the full set of linear operators on some subsystem of $\mathcal{H}$. In other words, $\mathcal{H} = \mathcal{H}^l \otimes \mathcal{H}^{\text{rest}}$ and

$$A_j \approx \bigoplus_{\alpha} L(\mathcal{H}_\alpha^j) \otimes I(\mathcal{H}^{\text{rest}}_\alpha).$$

Consider first $A_1$ and $\mathcal{H}_\alpha^1$. Since the algebras commute, each $A_j$ for $j > 1$ must act as the identity on the $\mathcal{H}_\alpha^1$, and hence acts not trivially only on $\mathcal{H}^{\text{rest}}_\alpha$. We proceed by induction, to derive that each $A_j$ is isomorphic to the full set of linear operators on a separate sub-particle and thus the lemma follows in this case.

Now we generalize to the case where at least one algebra is reducible. Let us examine the algebra $\mathcal{A}$ generated by the set of all operators on the particle $\mathcal{H}$, that commute with any $A \in A_j$ for all $j$. It is easy to check that since the $A_j$ are closed under adjoint (by Fact 2.8) then so is $\mathcal{A}$. By fact (2.13) algebra $\mathcal{A}$ admits a decomposition, such that inside each subspace, it is isomorphic to the full set of operators on some subsystem, tensor with identity. By fact (2.14) this decomposition is preserved by all $A_j$ since they commute with $\mathcal{A}$.

We consider the algebras $A_j$ restricted to these subspaces. We want to show that these restricted algebras are all irreducible. This follows since it turns out that the center of the algebra $\mathcal{A}$ in fact contains the centers of the algebras $A_j$. To show this, first notice that $Z(A_j) \subseteq A$ since any element of $Z(A_j)$ commutes with any element of $A_j$ by definition of a center of an algebra, and also commutes with any element of all the other algebras $A_{j'}$ for $j \neq j'$ since it consists of elements from $A_j$ and the algebras $A_j$ and $A_{j'}$ commute. In fact, since $Z(A_j)$ commutes with all the generators of $\mathcal{A}$, it is also contained in the center of $\mathcal{A}$. So $Z(A_j) \subseteq Z(\mathcal{A})$.

Therefore, since $\mathcal{A}$ is irreducible inside each of the subspaces, so are $A_j$. So the decomposition of the algebra $\mathcal{A}$ decomposes each algebra $A_j$, into irreducible commuting algebras, which by the first paragraph must act on separate subsystems inside each subspace.

We are now ready to prove the following crucial fact, which is the basis for the result of Bravyi and Vyalyi [8] as well as the current paper:

**Lemma 3.3 Decomposition of the center of a star (adapted from [8])**

Let $S$ be an instance of $\text{CLH}(2,d)$ whose interaction graph is a star: this means that there is a particle $j$, and each 2-local $H_{j,k}$ examines $j$ and another particle $k$ (where different terms act on different $k$'s). Then there exists a direct sum decomposition

$$\mathcal{H}^l = \bigoplus_{\alpha} \mathcal{H}_\alpha^l$$

(2)
such that inside each subspace $H^l_{i\alpha}$ there appears a tensor product structure

$$H^l_{j\alpha} = \bigotimes_{(j,k)\in E} H^{l,k}_{i\alpha}$$

where $k$ runs over all other particles, such that all operators $H_{j,k}$ preserve the subspaces $H^l_{i\alpha}$, and moreover,

$$H_{j,k} |_{H^l_{i\alpha}} \in \bigotimes_{l\neq k} I_{H^{l}_{i\alpha}} \otimes \mathcal{L} \left( H^{l,k}_{i\alpha} \otimes H^k \right)$$

**Proof:** We write each Hamiltonian as a sum of tensor product terms.

$$H_{j,k} = \sum_{\alpha} A^k_{i\alpha} \otimes B^k_{i\alpha},$$

where $A^k_{i\alpha}$ acts on $H^l_{i\alpha}$ and $B^k_{i\alpha}$ acts on $H^k_{i\alpha}$, and the operators $\{ B^k_{i\alpha} \}_{\alpha}$ are linearly independent. We consider the $C^*$-algebra generated by $\{ A^k_{i\alpha} \}_\alpha \cup \{ I \}$, and denote it $A_{i,j,k}$. The key point is that any pair of $A_{i,j,k}$ algebras commute, due to Fact (2.9). We can therefore apply claim (3.2) and this implies the result.

The above lemma implies the following. When the interaction graph of the commuting Hamiltonian is a star (which is indeed the case for every qudit in the two-local case), the Hilbert space of the center particle $H^l_{i\alpha}$ can be decomposed into a direct sum of spaces $H^{l,\alpha}_{i\alpha}$, such that each of the terms in the Hamiltonian preserves those subspaces $H^{l,\alpha}_{i\alpha}$. This means that the original system has a zero eigenstate, if and only if for every particle there exists some subspace index $\alpha_0$ such that the restriction of the terms in the Hamiltonians to the subspaces $H^{l,\alpha_0}_{i\alpha}$ is an instance which contains a zero eigenstate. Moreover, under this restriction, the terms in the Hamiltonian become disjoint, namely, they can be described as working on separate particles, described by the subspaces $H^{l,\alpha}_{i\alpha}$. Thus, the interaction graph of the restricted Hamiltonian is simply a set of disjoint edges.

### 3.1 A proof of the two local case

We can now prove the result of [8] stating that $CLH(2,d)$ is in $NP$ for any $d$. We present a slightly modified proof here, since a similar approach will be useful when we generalize the result to $CLH(3,2)$ and $CLH(3,3)$.

The main point is that in the two local case, for any qudit, the interactions involving that qudit form a star. Lemma 3.3 can therefore be applied. Merlin helps Arthur find the ground state by providing him with the correct index $\alpha$ in the decomposition of each particle.

To present the proof in more detail, we use an interactive picture: a communication protocol between Merlin and Arthur. In our protocol, only Merlin sends messages to Arthur, so in fact he can send all messages at once and this concatenation of messages can be viewed as the witness; but the protocol point of view is more convenient for our purposes.

**Algorithm 3.4**

*Input: $S$, an instance of $CLH(2,d)$.***
Repeat until there is no node in the interaction graph whose degree is 2 or more.

1. Merlin picks a vertex $q$, and sends Arthur the description of one subspace $\mathcal{H}_q^\alpha$ from the direct sum decomposition of $\mathcal{H}_q$ given by Lemma 3.3. Both Merlin and Arthur can generate the decomposition so Merlin only needs to send Arthur the index $\alpha$.

2. Both Arthur and Merlin slice the qudit $q$ after restricting it to $\mathcal{H}_q^\alpha$, according to Lemma 3.3. Accordingly, the node $q$ is replaced by at most $N_S(q) + 1$ new nodes, each with a degree 1 in the new interaction graph.

After all particles with degree more than 1 have been removed, Arthur verifies that the Hamiltonian, which is now a set of non-intersecting terms (each term corresponds to a disjoint edge) has a nonzero kernel.

4 The three-local case for qubits

In this section we prove Theorem 1.1. We start with a more detailed overview of the proof.

4.1 Proof overview

As mentioned before, the first step in the proof is to use the tools of [8] to identify and remove qubits that are “separable”, namely, for which there is a decomposition to a direct sum of subspaces, such that all operators acting on the qubit preserve those subspaces. In the case of qubits, when a non-trivial decomposition exists, it must be into two subspaces of dimension one each; when restricting to one such subspace, the state of the qubit becomes some tensor product state with the rest of the system. This means that those qubits can in fact be removed from the system since Merlin can provide their state separately. We have thus reduced the problem to a problem in which all qubits are non-separable; This is done in Subsection 4.2.

We now embark on the most important component in the proof, which is the characterization of the geometric properties of the interaction graph, after the removal of separable qubits. We treat each connected component separately, so we may assume the graph is connected. We ask, how can we constrain the interactions of a qubit $q$, for which we know that no single decomposition exists which all operators on $q$ agree on.

We define a $\triangleright \triangleleft$ (butterfly) relation between two operators acting on the same qubit $q$, $H_1(q)$, $H_2(q)$ if $A_1 \cap A_2 = \{q\}$. We denote this by $H_1 \triangleright \triangleleft H_2$. Each $\triangleright \triangleleft$ relations yields a direct-sum decomposition by (3.3) which is preserved by both operators. As mentioned in the introduction, we notice as a first step, that if there are two butterflies with respect to $q$, $H_1(q) \triangleright \triangleleft H_2(q)$ and $H_1(q) \triangleright \triangleleft H_3(q)$, then due to the low dimensionality, the decompositions induced by both $\triangleright \triangleleft$ relations are the same (see Claim 4.8).

This yields an important transitivity conclusion: i.e., if $H_1(q)$ and $H_2(q)$ agree on some decomposition of $q$, and $H_1(q)$ and $H_3(q)$ agree on some decomposition of $q$, then $H_2(q)$, and $H_3(q)$ agree on the same decomposition. We can now talk about two operators on $q$ which are connected by a path of such butterflies: two operators are said to be $\triangleright \triangleleft$ connected (read this “butterfly-connected”) if there is a sequence of $\triangleright \triangleleft$ relations that connects them, i.e. $H_1 \triangleright \triangleleft H_{i,1} \ldots \triangleright \triangleleft H_{i,m} \triangleright \triangleleft H_2$.

A basic tool in this paper is theorem (4.10) proved in Section (4.3.1):
Theorem 4.1 If any two operators $H_1(q), H_2(q)$ acting on a qubit $q$ are butterfly connected, then $q$ is separable.

This implies that an interaction graph made of non-separable qubits is severely limited, since its operators cannot be all connected by butterfly paths. The operators on any qubit thus cannot “fan-out” too much, as this would induce pairwise $\Rightarrow\Leftrightarrow$ paths and would make this qubit separable.

In Sections (4.3.3) and (4.3.2) we make this intuition more tangible, and show two important conclusions from Theorem (4.10). First, we define an “operator crown” on $q$, which is a set of three operators acting on $q$ organized as in Figure 5.

![Figure 5: An operator crown on qubit $q$.](image)

We show in Claim (4.16) that operator crowns, act as “qubit traps”; This means that if $q$ has such an operator crown, then any operator on $q$ must also act on at least one other qubit of the crown. Second, we show in Claim (4.17) that any two operators on $q$ must either intersect on one other qubit than $q$, or they are connected through another operator $H_x$, which intersects each of them with $q$ and another qubit. Another claim which we call the Bridge claim (Claim 4.18) is a slightly strengthened version of Claim (4.17). These latter properties impose severe restrictions on the geometry of the interaction graph.

Having characterized the local geometric behavior of each individual nonseparable qubit in the residual graph, we are ready to take one step further, and make some claims w.r.t. the global structure of the residual graph. To this end, we define the ”backbone“ of the graph: this is the longest “path of operators” in the residual graph.

![Residual graph diagram](image)

Intuitively, the backbone constitutes the longest possible stretch of ”operator crowns” that are attached back to back, without revisiting qubits that have already been visited. Recall that by Claim (4.16) an operator crown on qubit $q$ essentially ”traps” at least one other qubit of any operator acting on $q$. This means that essentially any operator that acts on a backbone qubit, must act on at least one more backbone qubit which is not very ”far” in terms of backbone edges; this is captured by Lemma (4.21) below.

This ”qubit” entrapment property alone is not sufficient for our purposes, as it does not handle operators that do not act on backbone qubits. We want to show that all operators (if we started with an instance whose interaction graph is connected) must examine at least two backbone qubits, and
therefore these qubits must be close by the above Lemma 4.21. Moreover, we want to show that there are no “shortcuts” between far away qubits in the backbone, through interactions with qubits outside the backbone: consider any qubit outside the backbone which interacts with two backbone qubits, through two different terms in the Hamiltonian. We want to show that even these two qubits cannot be too distant in terms of number of backbone edges. Those two properties are proved in claims (4.23) and (4.24) using the geometric claims above.

The result of all this is the following. Consider a coarse-graining of the backbone, in which say consecutive sets of 20 qubits are aggregated together and are considered as one particle of constant dimension; denote those by $Q_i$. By the above arguments, all interactions inside the backbone are two-local, namely, interact only $Q_i$ and $Q_{i+1}$; and moreover, any qubit outside the backbone may interact only with a specific pair of consecutive large particles $Q_i, Q_{i+1}$.

![Figure 6: The interactions with the backbone](image)

Figure 6: The interactions with the backbone: a backbone of sets $Q_i$ of constantly many qubits, such that each operator acts on $V_i$ and its associated pair of qudits $Q_i, Q_{i+1}$. We note that while the size of $Q_i$ is constant, the size of $V_i$ can be a function of $n$.

We examine this structured problem more closely. Consider the operators interacting $Q_i$ with the qudit to its left, $Q_{i-1}$. Consider also the operators that interact $Q_i$ with the qudit to its right, $Q_{i+1}$. We have a $\triangleright \triangleleft$ relation between any operator acting on $Q_i$ from the left and any operator acting on $Q_i$ from the right. We can then show, very similarly to Lemma (3.3), that there exists a decomposition of the Hilbert space of $Q_i$, such that when we restrict all operators on $Q_i$ to a specific subspace in this decomposition, then $Q_i$ can be written as a tensor product of two subparticles, the left subparticle $Q_{i, \text{left}}$ and the right subparticle $Q_{i, \text{right}}$, and the operators acting on $Q_i$ from the left (right) interact only with the left (right) subparticle of $Q_i$. This paves the way for achieving two-locality: After partitioning each $Q_i$ into those two separate subparticles $Q_{i, \text{left}}$ and $Q_{i, \text{right}}$, we can fuse the right side of one particle with the left side of the next: $Q_{i, \text{right}}$ with $Q_{i+1, \text{left}}$.

The resulting problem is two-local: all interactions are of the form in which one fused particle and one particle out of the backbone interact, or they are 1-local; hence, we get that each fused particle is a center of a star, and the stars are non-intersecting. This is already a problem in NP by Lemma 3.3, namely, by the methods of Bravyi and Vyalyi [8].

We now provide the details.
4.2 Removing Separable Qubits

We start by defining separable qubits, and explaining why they can be removed from the graph, using classical witness. By the end of this section, we will have removed all those qubits and remain with the residual problem in which all qubits are non-separable. We start by defining:

**Definition 4.2 Separable qubit**

A qubit \( q \) is said to be separable if there exists a direct-sum decomposition of its Hilbert space to two one dimensional spaces,

\[
\mathcal{H}_q = \bigoplus_{\alpha \in \{0,1\}} \mathcal{H}_q^\alpha
\]

such that any operator \( H(q) \) which acts on \( q \) preserves this decomposition:

\[
H(q) = \bigoplus_{\alpha} H(q)|_{\mathcal{H}_q^\alpha}
\]

where \( H(q)|_{\mathcal{H}_q^\alpha} \) is the restricted projector. Observe that the restricted projection in this case is also a projection.

This definition is aimed to capture a similar situation to what happens in the case of \( \text{CLH}(2,d) \) when a particle is handled, according to lemma 3.3. Note that the fact that the above direct sum decomposition is into subspaces of dimension 1, meaning that knowing the index \( \alpha \), a separable qubit has a well defined quantum state; Since the verifier will receive the index \( \alpha \) from the prover, this qubit can simply be removed from the graph.

So, the first step in the protocol between Merlin and Arthur is as follows:

**Algorithm 4.3 Restrict Graph \((S)\)**

**Input:** \( S \), an instance of \( \text{CLH}(3,2) \).

Iteratively change \( G_S \) until there is no separable qubit in \( G_S \):

1. Merlin picks a separable qubit \( q \) in \( S \) and sends Arthur an index \( \alpha \) whose corresponding subspace contains a common zero eigenspace of \( S \).

2. Arthur restricts all terms in the Hamiltonian to this subspace, and removes the vertex \( q \) and all its incident edges in \( G_S \). If after this restriction, some term acts trivially on one or more of its particles, Both Arthur and Merlin replace this term (or those terms if there are more than one) by the appropriate 1-qubit terms in the Hamiltonian.

The resulting instance is one without separable qubits, and where all terms act non-trivially on all their particles.

**Claim 4.4** Given a \( \text{CLH}(3,2) \) instance \( S \), algorithm (4.3) generates a \( \text{CLH}(3,2) \) instance \( S_{\text{nosep}} \) with no separable qubits; If \( S \) has a non-trivial common groundspace then Merlin can choose his restrictions so that \( S_{\text{nosep}} \) has such a subspace. If \( S \) has no zero eigenspace, for any choice of subspace restriction the ground energy of \( S_{\text{nosep}} \) is at least 1.
Proof: Let $q$ be a separable qubit which Merlin removes. There exists a basis of $\mathcal{H}_q$, such that any Hamiltonian term $H_i$ is block diagonal in this basis. $S$ has a ground energy zero if and only if one of those subspaces contains such a state. It remains to show that the new instance is still a legal $CLH(3,2)$ instance. Indeed, if two operators $H_1 \in L(H)$ and $H_2 \in L(H)$ commute and are block diagonal in some basis of $q$, then the restricted operators to any 1-dimensional subspace spanned by this basis also commute. This logic is preserved, at every iteration, leading to finer and finer slicing of the original Hilbert space, until we exhaust all separable qubits.

We note that during the process of “slicing” separable qubits, qubits which were previously nonseparable may become separable, yet a qubit cannot be sliced twice.

4.3 Geometric Constraints on the Residual Graph

Here we define Butterflies, operator crowns, operator paths, and provide all sorts of geometric restrictions on the interactions between separable qubits using those notions.

4.3.1 Butterflies and Separability

Definition 4.5 Butterfly
Consider two operators $H_1 = H_1(A,B)$ and $H_2 = H_2(B,C)$, where $B$ is a qubit, and $A$ and $C$ are sets of qubits, such that $A$ does not intersect $C$. We say that $H_1$ and $H_2$ constitute a butterfly and denote $H_1 \triangleright \triangleleft H_2$. A butterfly is always with respect to the particle in the intersection, here $B$; most of the time the identity of the qubit $B$ will be clear from the context and we will omit specifying it.

Claim 4.6 For any pair of operators acting non trivially on $q$, $H_1$ and $H_2$, with $H_1 \triangleright \triangleleft H_2$ with respect to $q$, there exists a non-trivial decomposition of $\mathcal{H}_q$ into a sum of two one dimensional subspaces, each of dimension one, which are preserved by both operators.

Proof: Denote by $A$ the set of qubits which $H_1$ acts upon, excluding $q$. Likewise, denote by $B$ the set of qubits which $H_2$ acts on, excluding $q$. By the definition of the $\triangleright \triangleleft$ relation, we have $A \cap B = \emptyset$. We can consider all operators in $A$ as one qudit. Similarly, we can consider all qubits in $B$ as another qudit. We can then apply lemma 3.3 and conclude that there exists a direct-sum decomposition of $q$ that is preserved by both operators. The reason the decomposition of lemma 3.3 must be non-trivial is that otherwise (namely, a decomposition to a sum of zero and two-dimensional spaces), since $\dim(q) = 2$, it means that one of the operators acts trivially on $q$, contradicting to our assumption.

Definition 4.7 Decomposition induced by the butterfly By Claim 4.6 a butterfly induces a well defined decomposition on its center qubit, which is called the decomposition induced by the butterfly.

Now, what if there are several butterflies in which one qubit participates? Could it be that the decompositions induced on $q$ by different butterflies are different? The following simple claim says that the answer is negative. This follows from the limited dimensionality of the qubit. This clearly leads to strong transitivity relations, as we will soon see, but let us first state and prove the uniqueness of induced decomposition:
Claim 4.8 Unique butterfly induced decomposition of $q$ Consider two butterflies $H_1 \triangleright H_2$, $H_1 \triangleright H_3$, both with respect to $q$, where all three operators act non-trivially on $q$. Then the decompositions induced on $q$ from both butterflies, using Claim 4.6 must be the same.

Proof: As in Claim 4.6, $q$ can be decomposed into a direct sum of two one dimensional subspaces, based on the first butterfly $H_1 \triangleright H_2$. Let $\Pi_q^0, \Pi_q^1$ be the projections on those subspaces of $\mathcal{H}_q$, so $\Pi_q^0 + \Pi_q^1 = I$. We can write

$$H_1 = \Pi_q^0 \otimes I_A + I_q \otimes \Pi_A^1$$

(5)

where $\Pi_A^1$ are some projections on $A$. The operators on $A$ are projections by lemma (3.3).

We claim that if another $\triangleright \triangleleft$ relation with $H_1$ results in a different decomposition of $H_1$

$$H_1 = \Pi_q^0 \otimes \Pi_A^0 + \Pi_q^1 \otimes \Pi_A^1$$

(6)

then this would imply a contradiction. To see this, suppose WLOG that $\Pi_q^0 (\Pi_q^1)$ projects on the state $|0\rangle (|1\rangle)$. And suppose $\Pi_q^0$ projects on the state $a|0\rangle + \beta|1\rangle$ with neither $a$ not $\beta$ equal to 0. Since $q$ is a qubit, we can write WLOG that $\Pi_q^1$ projects on the state $\beta^*|0\rangle - \alpha^*|1\rangle$.

Then we write:

$$\Pi_q^0 = |a|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| + a\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0|$$

and

$$\Pi_q^1 = |\beta|^2|0\rangle\langle 0| + |a|^2|1\rangle\langle 1| - a\beta^*|0\rangle\langle 1| - \alpha^*\beta|1\rangle\langle 0|.$$

Plugging these terms in Equation (6) for $H_1$, and using the fact that $H_1$ is block diagonal in the computational basis, by Equation (5), we set to 0 the terms in tensor with $|0\rangle\langle 1|$ and $|1\rangle\langle 0|$, and get

$$\Pi_A^0 = \Pi_A^1$$

which by Equation (6) means that $H_1$ can be written as $H_1 = I_q \otimes \Pi_A$. Thus, $H_1$ acts trivially on $q$, contrary to our assumption. We would like now to deduce various properties from transitivity. We first define:

Definition 4.9 Butterfly path, Butterfly-connectedness

Consider two operators both acting on a qubit $q$, denoted $H_a$ and $H_b$. We say there is a butterfly path between $H_a$ and $H_b$ if there is a sequence of operators $H_{a_1}, H_{a_2}, ..., H_{a_m}, H_b$ such that $H_a \triangleright H_{a_1} \triangleright H_{a_2} \triangleright H_{a_3} \triangleright \ldots \triangleright H_{a_m} \triangleright H_b$. (Where all butterflies are with respect to $q$). We say that these two operators are butterfly-connected.

Theorem 4.10 Butterfly connectedness of operators implies separability If all pairs of operators $H_1(q), H_2(q)$ acting on a qubit $q$, are connected by a butterfly path on $q$, then $q$ is separable.

Proof: Pick one operator acting on $q$, and now use claim (4.8) along the path connecting it to any other operator on $q$, to show that by transitivity all butterflies along the path induce the same decomposition on $q$, and thus by transitivity all operators on $q$ preserve this decomposition, hence by definition $q$ is separable.
Corollary 4.11 “left-right” Partition implies Separability: If there is a partition of the operator set $S$ into two disjoint non-empty sets $S_{q,\text{left}}$ and $S_{q,\text{right}}$ such that for each $H_i \in S_{q,\text{left}}$ and $H_j \in S_{q,\text{right}}$ we have $A_i \cap A_j \subseteq \{q\}$ (we call this a “left-right” partition), then $q$ is separable.

Proof: For any pair of operators $H_1$ and $H_2$ one can construct a chain of $\triangleright\triangleleft$ relations $H_1 \triangleright H_{j_1} \triangleright \ldots \triangleright H_{j_m} \triangleright H_2$ that goes back and forth between $S_{q,\text{left}}$ and $S_{q,\text{right}}$ as both sets are nonempty. Then by theorem (4.10) we get that $q$ is indeed separable.

What structures then can be present in a graph which contains only non-separable qubits? For that, we need to define one more notion of connectivity, namely operator-path connectivity.

4.3.2 Operator Paths

Definition 4.12 Open and Cyclic Operator Paths
An open operator path is an ordered set of $L$ distinct operators $H_1, \ldots, H_L$ such that for any pair of indices $(i, k)$ where $i \in [L], k \in [L]$ we have:

1. $|A_i \cap A_k| = 2$ for $|i - k| = 1$
2. $|A_i \cap A_k| = 1$ for $|i - k| = 2$.
3. $|A_i \cap A_k| = 0$ for $|i - k| > 2$

Similarly, a closed operator path, is an ordered set of operators that are distinct, except $H_1 = H_L$, and the index additions above are taken modulo $L$. The length of an operator path is defined as the number of its distinct operators.

Claim 4.13 Graph Connectivity implies operator path connectivity
If $S$ is a set of operators such that no qubit in $A_S$ is separable, and $G_S$ is connected, then $A_S$ is also operator-path-connected, i.e., any pair of qubits $q, v \in A_S$, are connected by an operator path which starts with an operator which acts on $q$ and ends with an operator which acts on $v$.

Proof: Let $q \in A_S$. Consider the set $S_q$ built as follows. Start with all operators acting on $q$. Add to $S_q$ any operator which intersects an operator in $S_q$ with an intersection of size 2. Continue until it is impossible to add operators this way. $S_q$ is the final set of operators we get. Now, if $A_{S_q}$ contains $v$, we are done (since clearly we can construct an operator path as desired). If not, consider a path from $q$ to $v$ is $G_S$. Let $w$ be the last qubit that belongs to $A_{S_q}$ along that path. We claim that $w$ is separable by corollary (4.11) as follows: Let us partition the operators acting on $w$ to two non-empty sets: The set $A$ which contains all operators in $S_q$, and $B$ of all other operators. We know that any operator in $B$ cannot intersect any operator in $A$ by 2 particles, since otherwise it would have been in $S_q$. Also, we know that $B$ is non-empty since it contains the operator inducing the edge from $w$ to the next qubit on the path to $v$ in $G_S$, and $A$ is also not empty, since $w$ belongs to $A_{S_q}$. Hence, the conditions of corollary (4.11) apply.
4.3.3 Operator Crowns and Geometrically Constrained Connectivity

We are now ready to deduce various restrictions on the connectivity of the operators acting on a nonseparable qubit \( q \). We will often restrict attention to operator paths all of whose operators act on one particular qubit:

**Definition 4.14 Operator Path on a qubit**

An operator path on \( q \) (where \( q \) is a qubit) is an operator path all of whose operators act on \( q \).

A certain structure which will appear useful is the length-3 operator path on a qubit \( q \):

**Definition 4.15 Operator Crown**

For a qubit \( q \), an operator Crown on \( q \) is a set of 3 operators that act on \( q \), and 4 other distinct qubits, which we call the crown qubits \( a_1, a_2, a_3, a_4 \), as follows: \( H_1(a_1, a_2, q), H_2(a_2, q, a_3), H_3(a_3, q, a_4) \).

See Figure 5 for an illustration. This structure acts as an “operator-trap”: any operator that acts on \( q \) must act also on some qubit which participates in the crown:

**Claim 4.16 Operator Crown as an Operator Trap:** Let \( q \) be a nonseparable qubit, and let \( C \) be some operator crown on \( q \). Then any operator on \( q \) acts on some crown qubit of \( C \).

**Proof:** Suppose on the negative, that there exists some crown \( C = (H_1, H_2, H_3) \) and an operator \( H \), such that \( C \) and \( H \) intersect only on \( q \). Then there exists a subgraph on \( q \) made of 3 operators, namely \( H_1, H_3, H \) such that each pair intersects only on \( q \). Thus \( q \) is separable by the following argument: we prove (\( \ast \)), that any operator on \( q \) has a \( \triangleright \triangleleft \)-path to \( H_1 \). This implies that any two operators are \( \triangleright \triangleleft \)-path connected, so by theorem (4.10) \( q \) is separable. We now show that indeed (\( \ast \)) is true: the operators \( H_1 \) and \( H_3 \) are \( \triangleright \triangleleft \)-path connected to \( H_1 \) as \( H \triangleright \triangleleft H_1, H_3 \triangleright \triangleleft H_1 \). Regarding \( H_2 \), we have \( H_2 \triangleright \triangleleft H_2 \triangleright \triangleleft H_1 \). Any other operator on \( q \) can share two qubits with at most 2 of the operators: \( H_1, H, H_3 \), so it has a \( \triangleright \triangleleft \) path to at least one of them, which implies it is also \( \triangleright \triangleleft \) connected to \( H_1 \).

![Figure 7: A sketch of the proof of Claim (4.16)](image)

We now use Theorem (4.10) and Claim (4.13) to prove that the operators acting on one nonseparable qubit \( q \) must be close in terms of the shortest operator path connecting them:
Claim 4.17 Operators on $q$ are connected by length-three operator paths: Let $q$ be some nonseparable qubit. Then any 2 operators on $q$ are operator-path connected by an operator path on $q$ of length at most 3.

Proof: First, consider the connected component in $G_S$ which contains $q$. By Claim (4.13) it is also operator path-connected. Assume on the negative, that there exist two operators $H_1(q), H_4(q)$ such that any operator path on $q$ that connects them is of length at least 4. Since $H_1 \neq H_4$ this must be an open operator path. We know that any open operator path on $q$ must be of length exactly 4, since if the shortest open path is of length at least 5, we get a structure as in Figure 7 and we contradict Claim (4.16). Let us choose such an operator path on $q$ and denote its operators as follows: $H_1(q, a_1, a_2, a_3)$, $H_2(q, a_2, a_3)$, $H_3(q, a_3, a_4)$, $H_4(q, a_4, a_5)$. Then any operator $H$ that examines $q$ and some qubit of the set $\{a_1, a_2\}$ cannot examine any qubit in the set $\{a_4, a_5\}$ as this would shorten the path between $H_1$ and $H_4$ to $(H_1, H, H_4)$. So let $H(q)$ be some operator on $q$. We claim that it has a $\triangleright \triangleleft$ relation with $H_1$. Indeed, if $H$ does not examine any qubit in the set $\{a_1, a_2\}$, then it has a $\triangleright \triangleleft$ relation with $H_1$. Otherwise, it does not examine any qubit in the set $\{a_4, a_5\}$, so it has a $\triangleright \triangleleft$ relation with $H_1$. Any of the operators $H_2, H_3, H_4$ is also $\triangleright \triangleleft$ connected to $H_1$, thus by Theorem (4.10) $q$ is separable.

We also give a slightly strengthened version of the above. We examine a case where there are two operators $H_1(q), H_2(q)$ on a nonseparable qubit $q$ connected by an operator path $P$ of length 4 on $q$. By Claim (4.17) this is not the minimal length operator path connecting these two operators. We show that in that case, not only is there a shorter path of length 3 between $H_1(q)$ and $H_2(q)$, but there exists a path of length 3 that shortcuts $P$ itself as follows:

Claim 4.18 (Bridge Claim) Let $H_1(q, a_1, a_2)$, $H_2(q, a_2, a_3)$, $H_3(q, a_3, a_4)$, $H_4(q, a_4, a_5)$, be a length 4 open operator path on a nonseparable qubit $q$. Then there exists an operator $H(q, a_2, a_4)$.

Proof: We assume on the negative that no such operator exists, and show that $q$ is separable. All operators of the path are $\triangleright \triangleleft$ connected to $H_1$. Any operator that shares only $q$ with $H_1, \ldots, H_4$ is $\triangleright \triangleleft$ connected to $H_1$. Any operator that shares $q$ and just one other qubit out of $a_1, \ldots, a_4$ is also $\triangleright \triangleleft$ connected to $H_1$. Let $H$ be an operator that acts on $q$ and two other qubits out of $a_1, \ldots, a_4$. These qubits cannot be adjacent so the possible pairs (excluding $(a_2, a_4)$) are $(a_1, a_3)$, $(a_1, a_4)$, $(a_1, a_5)$, $(a_2, a_5)$. Each such pair of qubits examined by $H$, corresponds to a $\triangleright \triangleleft$ relation of $H$ with the operator $H_4, H_2, H_3, H_1$, respectively, so $H$ is $\triangleright \triangleleft$ connected to $H_1$. Thus $q$ is separable by Theorem (4.10).

4.4 The Backbone

Having arrived from the initial input $S$ to one with no separable qubits, using input from Merlin, we now show that with the help of Merlin, the new instance can be viewed as a 2-local problem. This is formalized by the following theorem:

Theorem 4.19 For any instance $S$ of $\text{CLH}(3,2)$ with no separable qubits there exists a partition of the set of vertices of $G_S$ into disjoint sets:

$$V_G = \bigcup_i Q_i$$

where each $Q_i$ is of constant size, such that any operator $H \in S$, acts on qudits from at most 2 such sets $Q_i$. 

24
As previously discussed, the way this theorem is proved, is by identifying a special 1-D structure called the backbone whose properties allow to coarse-grain the set of qubits into constant-dimension sets, such that each operator is 2-local w.r.t. these sets.

### 4.4.1 Properties of the backbone

Here we define and identify the backbone, and prove its desired properties.

**Definition 4.20 Backbone**

For an instance \(S\) of CLH(3,2) we define the backbone to be a maximal length operator path \(B\) in the connectivity graph \(G_S\). If there are several such maximal length paths, we take one of them arbitrarily.

We start by proving a lemma which states that for any backbone qubit, any operator that acts on that qubit must also act on some “nearby” backbone qubit.

**Lemma 4.21 Short range connectivity in the backbone** Let \(B = \{H_1, H_2, ..., H_L\}\) be a backbone of a connected-graph instance \(S\) with no separable qubits, such that \(L > 100\). Let \(q \in A_B\) and let \(H_i(q)\) be some backbone operator. Then for any \(H = H(q) \in S\) that acts on \(q\), there exists another qubit \(p \in A_B\) examined by \(H\), and a backbone operator \(H_j(p) \in B\) such that \(|i - j| \leq 4\) where addition is modulo \(L\) for a closed operator path.

**Proof:** We say \(q \in A_B\) is a “middle” qubit if it is acted upon by 3 consecutive backbone operators \(H_{i-1}, H_i, H_{i+1}\); If \(B\) is a closed path, then all qubits in \(A_B\) are middle qubits. If \(B\) is an open path, all qubits are middle qubits except for the two qubits at the beginning and the two qubits and the end of the path; we call those edge qubits.

The claim easily follows for a middle qubit \(q\): Let \(q\) be such a qubit, acted upon by 3 backbone operators \(H_{i-1}, H_i, H_{i+1}\). These constitute an operator crown on \(q\), so any \(H(q)\) acts on some other qubit \(p\) in \(A_{i-1} \cup A_i \cup A_{i+1}\); \(p\) and \(q\) are thus acted upon by \(H_i(q) \in B, H_j(p) \in B\) where \(|i - j| \leq 2\).

This proves the claim for the case in which the backbone is a closed path. For the case of an open path, we only need to prove the claim for its edge qubits. Let us consider an edge qubit which belongs to the “left” edge, namely, the qubit belongs to \(H_1\); the proof for the other side is essentially identical. Let \(H = H(q,a,b)\) be some operator in \(S\). We discuss separately 3 cases: at least one of \(a, b\) is a middle qubit, at least one of \(a, b\) is an edge qubit, or both of them are outside of \(A_B\).

1. **Either \(a\) or \(b\) is a middle qubit** Assume WLOG that \(a\) is a middle qubit. Suppose on the negative, that the minimal value \(m\) for which there exists some \(H_m \in B\) such that \(a \in A_m\) or \(b \in A_m\) is 6. Let \(C(a) \subseteq A_B\) be the set of the crown qubits of \(a\). Since \(m \geq 6\) then \(q \notin C(a)\), so by Claim 4.16 it must be that \(b \in C(a)\). In addition, since \(m \geq 6\) we know that \(b\) is a middle qubit, which is adjacent to \(a\). \(b\) cannot be in \(A_m\) for \(m \leq 6\) by our assumption; Hence we know that both \(a\) and \(b\) are adjacent “middle qubits” that are distant from \(q\). Consider then the terms \(H\) and \(H_1\). Since both \(a\) and \(b\) are distant from \(q\), we have that \(H \triangleright \triangleleft H_1\) w.r.t qubit \(q\). By claim 4.17 there exists an operator path of length exactly 3 between \(H\) and \(H_1\), which implies the existence of an operator \(H_x(q)\) that shares two qubits with \(H_1\), and hence acts on at most one particle from \(C(a) \cup C(b)\). However, since \(H_1\) shares two particles with \(H\), it must act on either \(a\) or \(b\); this contradicts claim 4.16 w.r.t either qubit \(a\) or qubit \(b\).
2. **a or b is an edge qubit** Suppose at least one of \(a\) or \(b\) are edge qubits, say qubit \(a\). Hence, either \(a \in H_1\) or \(a \in H_L\), in both cases we are done as \(q \in H_1\).

3. **a and b are not in the backbone** In this last case we assume that neither \(a\) nor \(b\) are in \(A_B\). We show that this case is impossible, since it implies that the backbone can be extended, contradictory to the definition of the backbone (Definition 4.20).

Denote by \(H_1(a_1, b_1, a_2), H_2(b_1, a_2, b_2), \) and \(H_3(a_2, b_2, a_3)\) as the first 3 operators. So we assume that either \(q = a_1\) or \(q = b_1\).

If \(q = a_1\), there exists by claim (4.17) an operator path of length at most 3 between \(H\) and \(H_1\). It must be of length exactly 3 by the assumption that \(a, b\) are not in the backbone, and so \(H\) cannot intersect \(H_1\) with two qubits to make a path of length 2. This implies the existence of \(H_x\) that acts on \(a_1\), and shares two qubits with \(H_1\). So if \(H_x\) acts on \(b_1\) (as in Figure 8) then \(H\) and \(H_x\) can be appended to \(B\) to form a \(L + 2\)-length operator path.

![Figure 8:](image)

Otherwise, \(H_x\) acts on \(a_2\) (as in Figure 9). Let us examine now \(H_x\) and \(H_3\) which share qubit \(a_2\). There exists an open operator path on qubit \(a_2\) comprised of the following 4 operators: \(H_x, H_1, H_2, H_3\). Thus by claim (4.18) there exists an operator \(H_y(a_1, a_2, b_2)\) By removing \(H_1, H_2\) from \(B\), and appending \(H, H_x, H_y\) to \(B\), we end up with a backbone of size \(L + 1\).

![Figure 9:](image)

If \(q = b_1\) then \(H\) is connected to \(H_1\) by some operator \(H_x(b_1)\), which shares two qubits with \(H_1\). If \(H_x\) acts on \(a_1\) we can append \(H_x\) to \(B\), and increase its length to \(L + 1\) (Notice that we cannot append \(H\) as well, since \(H, H_2\) share qubit \(b_1\) despite their index difference being
greater than 2 in the new path, and so this is not a legal operator path). Otherwise, $H_x$ acts on $a_2$, as in Figure 10. In this case we can add $H_x$ to $B$, and remove $H_1$, thereby increasing the length of $B$ yet again by 1.

![Figure 10: $q = b_1$ and $H_x$ acts on $a_2$.](image)

Given a backbone $B \subseteq G_S$, we now define the combined particles.

**Definition 4.22 The combined particles** $Q_i$ Let $Q_1, \ldots, Q_M$ be a partition of the qubits in $A_B$,

$$A_B = \bigcup_{i=1}^{M} Q_i$$

generated by grouping together contiguous sets of 20 qubits in the most natural way, from left to right along the backbone. If the number of qubits in $A_B$ does not divide by 20 (but is larger than 20), we simply make the last $Q_M$ larger; in any case each $Q_i$ contains at most 39 qubits. If $A_B$ contains less than 60 particles, than there is just one particles, $Q_1$ containing all of them.

**Claim 4.23 Any operator touches the backbone in at least two close qubits**

Assume that $G_S$ is connected. Then, for any operator $H(a, b, c) \in S$, at least two of the three qubits $a, b, c$, must be contained in $A_B$, and moreover, $\{a, b, c\} \cap A_B$ is contained in two adjacent combined particles $Q_i, Q_{i+1}$ with addition modulo $M$ (or in one combined particle).

**Proof:** We first show that at least two qubits of $\{a, b, c\}$ are in $A_B$. It cannot be that exactly one of its particles is in $A_B$ since this is in contradiction to lemma (4.21). So it is left to rule out the case in which none of its particles are in $A_B$. We will show that this case is impossible. Since $G_S$ is connected, Claim (4.13) applies. Hence, for any qubit of $\{a, b, c\}$ there is an operator path starting from an operator acting on that qubit, to some operator $H_m \in B$. Let $H_l$ denote the first operator on the path for which $A_l$ intersects $A_B$. By Lemma (4.21) $H_l$ examines at least two qubits in $A_B$. So let us now consider $H_{l-1}$. Since $H_{l-1}$ intersects $H_l$ by two qubits, at least one of the intersection qubits must be in $A_B$. So $H_{l-1}$ examines at least one qubit in $A_B$. This is a contradiction to the fact that $H_l$ is the first operator on the path that intersects $A_B$.

Now we show that $\{a, b, c\} \cap A_B$ is distributed among two adjacent $Q_i$’s. We divide to two cases: $\{a, b, c\} \subseteq A_B$ (note that this doesn’t mean that $H_k$ is in the backbone) or only two qubits are in $A_B$.  

27
We start with the first case. Let \( q \in \{a, b, c\} \). Let \( H_i(q) \in B \). Then there exists some \( p \in \{a, b, c\} \), and some \( H_j(p) \in B \) s.t. \( \|i - j\| \leq 4 \) by Lemma \ref{lem:distance-four}. By applying this argument either once or twice we get that for any two qubits in \( \{a, b, c\} \), there are two backbone operators acting on them with indices which are at most 8 apart. This means that these qubits must be contained in two adjacent \( Q_i \)'s (or just in one).

Otherwise, two of the qubits \( \{a, b, c\} \) are in \( A_B \). In this case, consider the first qubit of these 2, which belongs to some operator in the backbone. It follows from lemma \ref{lem:distance-four} that the second qubit belongs to an operator which is at most 4 operators away; Thus, these two qubits belong to two adjacent \( Q_i \)'s (or just in one).

\textbf{Claim 4.24} \textit{No shortcuts outside of backbone}

\textit{Let} \( q \notin A_B \) \textit{be some qubit outside the backbone. Then there exists} \( m \) \textit{such that for all} \( H_i(q) \in S \), \( A_i \cap A_B \subseteq \{Q_m, Q_{m+1}\} \), \textit{with index addition modulo} \( M \).

\textbf{Proof:} Given a qubit \( q \), let \( H_k(q), H_l(q) \) be two operators on \( q \). We first show that \( (A_k \cup A_l) \cap A_B \) are grouped into at most two adjacent combined particles \( Q_m, Q_{m+1} \) for some \( m \).

Since \( q \) is not separable, there exists by claim \ref{lem:distance-three} an operator path on \( q \) of length at most 3 between \( H_k \) and \( H_l \). So there are two cases: either \( |A_k \cup A_l| = 2 \) or there exists an operator \( H_x(q) \) that shares two qubits with both \( H_k \) and \( H_l \).

We will make the claim first for the second case. Since \( q \notin A_B \) the two other qubits examined by \( H_x \) are in \( A_B \) by Claim \ref{lem:close-open}. Let us consider one of the two qubits in \( A_x \cap A_B \), say \( p_1 \), and let \( H_i \in B \) act on \( p_1 \). By lemma \ref{lem:distance-four}, there exists another qubit in \( A_x \cap A_B \), say \( p_2 \), and an operator \( H_j \in B \) acting on \( p_2 \), s.t., \( |i - j| \leq 4 \). Thus the two qubits other than \( q \) which \( H_x \) acts on have backbone operators acting on them which are of distance 4 apart. The same property holds for the qubits of \( A_k \cap A_B \), and \( A_l \cap A_B \). Thus any pair of qubits in \( (A_k \cup A_l) \cap A_B \) belong to operators of index difference at most 12. Thus, \( (A_k \cup A_l) \cap A_B \) are grouped into at most two adjacent combined particles \( Q_m, Q_{m+1} \) for some \( m \).

Since the above property holds for all pairs of operators on \( q \), we have that \( \bigcup_{H_i(q)} A_i \cap A_B \subseteq Q_m \cup Q_{m+1} \) for some \( m \).

\textbf{4.4.2 The Structure of interactions with the Backbone}

By Claims \ref{lem:close-open}, \ref{lem:distance-four}, we realize by now that the input with no separable qubits \( S \) and a connected interaction graph \( G_S \) has, in fact, a very constrained structure. To describe this structure more precisely, we now define a partition of all qubits outside of the backbone into sets denoted by \( V_i \) as follows.

\textbf{Definition 4.25 The partition of the outer qubits} \textit{We define the set} \( V = \bigcup_{i=1}^{M-1} V_i \) \textit{which is comprised of} \( M \) \textit{or} \( M - 1 \) \textit{(depending on whether the backbone is closed or open) disjoint sets of qubits} \( V_1, V_2, \ldots \), \textit{as follows:} \( V \) \textit{is the set of all qubits not in} \( A_B \). The sets} \( V_i \) \textit{are defined iteratively:} \( V_1 \) \textit{is defined as the set of all qubits} \( q \in V \) \textit{satisfying that: for any operator} \( H_k(q) \) \( A_k \cap A_B \subseteq Q_1 \cup Q_2 \). \( V_2 \) \textit{is defined as the set of all qubits} \( q \in V \setminus V_1 \) \textit{satisfying that: for any operator} \( H_k(q) \) \( A_k \cap A_B \subseteq Q_2 \cup Q_3 \). \textit{And so on.}

Note that the above definition indeed defines a partition of all qubits outside the backbone. We thus deduce that the picture of interactions is severely restricted:
Definition 4.26 Almost one dimensional structure: We consider a tuple \((Q, V, S)\) consisting of a set of distinct qudits \(Q = \{Q_i\}\), a set of non-intersecting qubit sets \(V = \{V_i\}\) (non intersecting with the \(Q_i\)s as well) and a set of operators \(S\) on those particles. We say that a tuple \((Q, V, S)\) has an “almost one dimension structure” if each \(H_k \in S\) has \(A_k \subseteq \{Q_i \cap Q_{i+1} \cap V_i\}\) for some \(i\).

See Figure 4 for a schematic example of such a one dimensional structure. By Definition (4.25) and by claims 4.23 and 4.24, we have that the resulting interaction after the partition into \(Q_i\) is indeed almost one dimensional:

Claim 4.27 We are given \(S\) with no separable qubits s.t. \(G_S\) is connected. Consider the backbone \(B\) on \(S\), with \(Q_i\) as defined in Definition 4.22 and \(V, V_i\) as defined in Definition 4.25. Then the tuple \(Q, V, S\) is an almost one dimensional structure, namely, for any interaction \(H_k\) in the instance \(S\), \(A_k\) contained in \(V_i \cup Q_i \cup Q_{i+1}\) for some particular \(i\).

We can now use this to derive two-locality.

4.5 Decomposing the Backbone to get 2-locality

We would now like to discover the two locality by making use of the structure of interactions with the backbone, which was revealed in the previous section. For this, we prove a lemma which is very similar to lemma (3.3), regarding the existence of a separating decomposition as defined in Definition (3.1).

Lemma 4.28 Let \((Q, V, S)\) be a tuple which is an almost one dimensional structure, as in Definition (4.26). For any \(Q_i \in Q\), there exists a direct-sum decomposition of \(\mathcal{H}_{Q_i}\) preserved by all operators in \(S\) which is a separating decomposition w.r.t. any pair of operators \(H_l, H_r\) where \(A_l \subseteq (Q_{i-1}, Q_i, V_{i-1})\) and \(A_r \subseteq (Q_i, Q_{i+1}, V_i)\).

Proof: Fix a backbone qudit \(Q_i\). Denote by \(H_{l,j} \in S\) the \(j\)-th operator such that \(A_{l,j} \subseteq (Q_{i-1}, Q_i, V_{i-1})\) and \(H_{r,k} \in S\) the \(k\)-th operator such that \(A_{r,k} \subseteq (Q_i, Q_{i+1}, V_i)\). Denote by \(A_{l,j}^{Q_i}\) the algebra of \(H_{l,i}\) on \(Q_i\) and by \(A_{r,k}^{Q_i}\) the algebra of \(H_{r,k}\) on \(Q_i\). Finally, denote by \(A_{l,j}^{Q_i}\) the algebra spanned by \(\cup_{j} A_{l,j}^{Q_i}\) and similarly, by \(A_{r,k}^{Q_i}\) the algebra spanned by \(\cup_{k} A_{r,k}^{Q_i}\).

If there is just one qudit \(Q_i\) the lemma follows trivially from lemma (3.3). Otherwise, by definition (4.22) we have \(Q_{i-1} \neq Q_{i+1}\). Then \(H_{l,j} \triangleright H_{r,k}\) w.r.t. \(Q_i\) so by fact (2.9) the algebras \(A_{l,j}^{Q_i}\) and \(A_{r,k}^{Q_i}\) commute for all \(j, k\). As a result \(A_{l,j}^{Q_i}\) and \(A_{r,k}^{Q_i}\) commute, because their generating sets commute. Let us apply fact (3.2): there exists a separating decomposition of \(\mathcal{H}_{Q_i}\) w.r.t. \(A_{l,j}^{Q_i}\) and \(A_{r,k}^{Q_i}\). In each subspace of this decomposition, the restricted algebras act on separate subsystems. Since \(A_{l,j}^{Q_i} \subseteq A_{l,j}^{Q_i}\) for all \(j\), then \(A_{l,j}^{Q_i}\) preserves this decomposition, and acts only on the left subsystem inside each subspace. The same holds true for \(A_{r,k}^{Q_i}\) for all \(k\) w.r.t. the right subsystem. Hence this decomposition is separating w.r.t. any pair of local algebras \(A_{l,j}^{Q_i}\) and \(A_{r,k}^{Q_i}\). Since \(H_{l,j}\) and \(H_{r,k}\) share only \(Q_i\) to begin with, this is a separating decomposition w.r.t. these operators as well.

■
Reducing to a 2-local problem  Merlin sends Arthur for each qudit $Q_i$ a subspace index $\alpha_i$ which by lemma (4.28) partitions all the operators on $Q_i$ into two disjoint left and right subsystems in the $\alpha_i$ subspace of $Q_i$. Since by lemma (4.28) all operators preserve this tensor product subspace, then the tuple $(Q, V, S)$ is satisfiable if and only if there exists such a tensor product subspace in which the restricted instance is satisfiable.

Let $\Pi_{\alpha_i}$ denote the projection on the $\alpha_i$ subspace of backbone qudit $Q_i$. After restricting all operators $H_k \in S$ to the tensor product subspace $\otimes_i \Pi_{\alpha_i}$, each restricted operator $H_k|_{\otimes_i \Pi_{\alpha_i}}$ acts on $H_{Q_i}^{\alpha_i,\text{right}} \otimes H_{Q_i+1}^{\alpha_i+1,\text{left}}$ for some $i$, and possibly some qubit $q \in V_i$. So let us now fuse each pair $H_{Q_i}^{\alpha_i,\text{right}}, H_{Q_i+1}^{\alpha_i+1,\text{left}}$ into a single qudit. We now regard the tuple $(Q, V, S)$ as the modified instance where the qudits $Q_i$ are the fused qudits, and $S$ is the restricted operator set. It can be easily checked that all $H_k \in S$ are now at most 2-local, acting on a single fused qudit and possibly some $q \in V_i$. This problem can then be verified using lemma (3.3). Let $\Pi_{\beta_i}$ denote the projection on the $\beta_i$ subspace of the fused qudit $H_{Q_i}^{\alpha_i,\text{right}} \otimes H_{Q_i+1}^{\alpha_i+1,\text{left}}$. Then Merlin sends Arthur a tensor-product subspace $\otimes_i \Pi_{\beta_i}$ - i.e. a subspace index for each fused qudit.

4.6 Putting together the proof of Theorem 4.19 Containment in NP

To summarize our proof of theorem 4.19, we describe the protocol of the verifier, and recall why it is complete and sound. The witness that Merlin sends Arthur is:

- An index $\alpha$ of a subspace for each separable qubit.
- A description of constant-size sets $Q_i$ and the (possibly non-constant size) sets $V_i$, for all $i$.
- An index $\alpha_i$ for each $Q_i$, as well as the description of the actual subspace $H_{\alpha_i}$, and its tensor product structure, $H_{Q_i}^{\alpha_i,\text{right}} \otimes H_{Q_i+1}^{\alpha_i+1,\text{left}}$. The subspaces are provided by providing the basis vectors. Note that since the dimension of each $Q_i$ is constant, this description is efficient as long as the accuracy required is at most to within inverse exponential in the number of qubits.\(^2\)
- A witness following lemma (3.3) for the two-local problem following the merging of the constant size sets.

The verification procedure is performed as follows: First, Arthur restricts all the operators in $S$ to the subspaces of the separable qubits, provided by Merlin, and derives an instance $S_{\text{nosep}}$ with supposedly no separable qubits. Arthur then verifies that after unifying some qubits of the system into constant size sets $Q_i$ following a recipe by Merlin, the instance $S_{\text{nosep}}$ has an almost 1-dim. structure $(Q, V, S_{\text{nosep}})$. Then, Arthur restricts all operators on $H_{Q_i}$ to the subspace $H_{Q_i}^{\alpha_i}$ as provided by Merlin for each $i$. This is a tensor product of subspaces, one for each $i$. He then fuses right and left remnants of the qudits (i.e. $H_{Q_i}^{\alpha_i,\text{right}}$ with $H_{Q_i+1}^{\alpha_i+1,\text{left}}$ for each $i$) and achieves a 2-local problem. Finally, Arthur uses a witness provided by Merlin for the 2-local problem as in Lemma (3.3).

\(^2\)Throughout the paper we ignore the issue of accuracy and assume that the subspaces are provided with infinite accuracy. The reason is that this issue is not a problem in the context of the CLH problem, since we are only trying to separate a zero eigenvalue from 1 or more. In fact, even inverse polynomial accuracy in each such component will suffice, since there are only polynomially many terms contributing to the error.
We now argue for completeness and soundness: By Claim (4.4) the first step results in an equivalent instance $S_{\text{nosep}}$, which is a positive instance if and only if $S$ is a positive instance. The completeness and soundness of the third step follows exactly the proof of lemma (4.28): i.e. Merlin can find a subspace index $\alpha$ for each $Q_i$, that separates the interaction on $Q_i$ into left-right components, such that the restricted Hamiltonian is a positive instance, if and only if the original Hamiltonian is a positive instance. The correctness of the last step of the protocol, namely, for the two local residual problems, follows from the correctness of [8].

We end with the following corollary:

**Corollary 4.29 Constant Depth Diagonalizing Circuit** For any CLH($3, 2$) Hamiltonian $H$ there exists a quantum circuit of depth 3, where each gate acts on at most 100 qubits that diagonalizes $H$.

**Proof:** A depth 3 circuit is induced in a straightforward manner from the protocol above. All separable qubits can be diagonalized separately from all other particles using 1-local qubit gates. Let us now consider just the nonseparable qubits. We work our way from the end of the protocol in (4.5) backwards: the first two layers handle the 2-local star-topology instance, and the third layer embeds this instance into the backbone. We recall again the following notation: the backbone qubits are denoted by $H_{Q_i}$, and restricted to some subspace $\alpha$ we have $H_{Q_i}^{\alpha,\text{left}}$ and $H_{Q_i}^{\alpha,\text{right}}$ for each $i$. Then when Merlin proves to Arthur the satisfiability of the star-topology instance, he provides a subspace $\Pi_{\beta_i}$ for each fused qudit $H_{Q_i}^{\alpha,\text{right}} \otimes H_{Q_i}^{\alpha,\text{left}}$. To write the circuit, we actually go in reverse order as follows:

1. **Layer 1: Disjoint Hamiltonian Diagonalization**
   A diagonalizing gate for each disjoint Hamiltonian interaction $(q, p)$ where $q \in V_i$ and $p$ is some fused qudit $\left( H_{Q_i}^{\alpha,\text{right}} \otimes H_{Q_i}^{\alpha,\text{left}} \right) |_{\beta_i}$ restricted to some subspace $\beta_i$. Each such gate is a 2-local unitary acting on the qubit $q$ and some qudit of dimensionality less than the product of dimensions of two $Q_i$’s, hence at most a constant.

2. **Layer 2: Embedding of disjoint Hamiltonians into Star Topology** An isometric embedding $V$ of the individual Hilbert spaces $p$ from the item above (which may require adding some ancilla qubits), into a fused qudit that is now still restricted to some subspace $\alpha$. We have
   $$V_2 : \left( H_{Q_i}^{\alpha,\text{right}} \otimes H_{Q_{i+1}}^{\alpha,\text{left}} \right) |_{\beta_i} \mapsto H_{Q_i}^{\alpha,\text{right}} \otimes H_{Q_{i+1}}^{\alpha,\text{left}}.$$
   The isometry $V_2$ acts on possibly numerous individual particles such that the product of their sizes is at most 100.

3. **Layer 3: Embedding of Star Topology instance into backbone** An isometric embedding of each pair of adjacent fused subparticles:
   $$V_3 : \left( H_{Q_i}^{\alpha,\text{left}} \otimes H_{Q_i}^{\alpha,\text{right}} \right) \mapsto H_{Q_i}.$$
   This may also require adding some ancillary qubits.

**Remark:** We remark that in fact, a depth 2 quantum circuit suffices. We will defer the proof to a later version of the paper.
5 The 3-local case for qutrits

In this section we extend the result of the previous section for qutrits, presuming the interaction graph can be embedded on the plane in a special way.

5.1 Formal Definitions

For a planar connected interaction graph $G_S$ of a $CLH(3,3)$ instance $S$, embedded in $R^2$ we define the following:

**Definition 5.1 Op/Noop Faces**

Let $f$ be some face of $G$. $f$ is said to be an “op” face if it has 3 vertices, and there exists an operator $H \in S$ that acts on its 3 vertices (qudits). If there is no operator that acts on its vertices it is called a “noop” face. In this respect, the face extending from the boundary of $G$ to infinity is a “noop” face.

We now define a notion of planar $CLH$:

**Definition 5.2 Planar CLH**

An instance $S$ of $CLH(k,d)$ is said to be “planar CLH” if its interaction graph is planar, connected, and moreover, can be associated with an embedding in the plane, such that every operator in $S$ is associated with an “op” face. In other words, the three vertices corresponding to the particles on which the operator acts, are connected by edges in the graph, such that the area surrounded by those edges does not contain any other vertex.

In Figure 11 we provide examples which explain the difference between the mere requirement that the interaction graph is planar, and our definition of planarity. So far, these are very mild and

![Figure 11](image.png)

Figure 11: Examples of instances of $CLH(k,d)$ whose interaction graphs are embedded on the plane, yet are not instances of planar $CLH$ as defined above:

Fig. a: Suppose that in addition to the operators $H_1, H_2, H_3, H_4$ there also exists an operator $H_5(q_1, q_2, q_3)$. Then there is no “op” face corresponding to $H_5$.

Fig. b: Suppose there are only 3 operators $H_1, H_2, H_3$. There exists a face corresponding to each operator, yet the one corresponding to $H_1$ is not an “op” face since it is not triangular. Note however, that there exists a different embedding of the interaction graph of $H_1, H_2, H_3$ so that it is a legal planar $CLH$ instance.
natural restrictions on top of planarity. We now add one additional “natural” requirement. For this, let us define another notion:

**Definition 5.3 Nearly-Euclidean Triangulation of a Polygon**

A finite planar graph is said to be a Nearly Euclidean (NE) triangulation of a polygon if every face except the infinite face has three edges, the edges are straight lines, and moreover, the ratio between the shortest and longest edge is bounded from above by some overall constant, and the angle between any two incident edges is bounded from below by some overall constant angle.

Notice that this definition ensures that there cannot be areas in which the density of vertices is much higher than in other places, and that the length of a path in the graph is not too far from that of the Euclidean distance between the two end points. In short; propagation along paths in the graph behaves more or less like it would on a periodic lattice, or in Euclidean space.

We now define NE CLH instances:

**Definition 5.4 Nearly Euclidean CLH**

An instance $S$ of planar CLH $(k,d)$ is said to be Nearly Euclidean (NE) if there exists a NE planar triangulation $T_S$ of a convex polygon in the plane, such that the interaction graph of $S$, $G_S$, is a subgraph of this triangulation, i.e. $G_S \subseteq T_S$ and the number of vertices of $T_S$ is at most polynomial in the number of particles (equivalently, vertices) in $S$.

Our main result in this section is:

**Theorem 5.5** The problem of CLH(3,3) restricted to NE planar instances is in NP.

From our proof, we also derive the following corollary:

**Corollary 5.6** Given an instance of NE CLH(3,3), there exists a quantum circuit of depth 3, involving 2-local gates acting on nearest neighbor qutrits in the interaction graph of the original instance, which diagonalizes the Hamiltonian (and thus generates a basis of eigenstates from a basis of input computational basis states).

### 5.2 Proof Overview

Similar to the proof for qubits, our proof of theorem (5.5) above is along the following strategy: identify “classically”-behaving qutrits, and eliminate them using input from Merlin. Then, show that the residual instance can be coarse-grained into a CLH(2,d) instance for a constant $d$. The proof that this coarse graining can be done, however, is entirely different, from the one in the previous chapter.

The first difference is in identifying classically-behaving qutrits; this is slightly more involved than for qubits. The larger qudit dimension allows qutrits to interact in certain configurations that generate qubit separability, but do not generate qutrit separability. Consider for example the following set of 4 operators: $H_1(q,a,b), H_2(q,b,c), H_3(q,c,d), H_4(q,d,e)$ where

$$H_1 = |0\rangle\langle 0|_q \otimes \Pi_a \otimes \Pi_b$$

$$H_2 = |0+1\rangle\langle 0+1|_q \otimes (I - \Pi_b) \otimes \Pi_c$$
\[ H_3 = |1+2\rangle \langle 1+2|_q \otimes (I - \Pi_c) \otimes \Pi_d \\
H_4 = |2\rangle \langle 2|_q \otimes (I - \Pi_d) \otimes \Pi_r. \]

These operators commute yet do not agree on any single decomposition of \( q \), despite the fact that they are all \( \triangleright\triangleleft \) connected. The fact that \( \triangleright\triangleleft \)-connectedness does not imply separability (i.e., no equivalent of Theorem 4.10 holds) destroys the basis for most of the geometric structure we managed to prove in the case of qubits.

From an algebraic perspective the above phenomenon occurs because an operator on a qubit can be written in block-diagonal form w.r.t. only one basis of that qubit (see lemma 4.8). Alternatively stated, any two decompositions of \( H(q) \) w.r.t. the qubit \( q \) preserve each other (i.e., the projections on the subspaces in the decomposition preserve the subspaces in the other decomposition) and are hence identical. Yet for qutrits, this is obviously not the case. Consider for example the following operator:

\[ H = |0\rangle \langle 0|^q \otimes \Pi_{0}^F + |1\rangle \langle 1|^q \otimes \Pi_{1}^F + |2\rangle \langle 2|^q \otimes \Pi_{2}^F \]

where the first term of each summand acts on a qutrit \( q \), and the second term acts on the rest of the system. If the 3 projections \( \Pi_{i}^F \) are linearly independent, then any non-trivial direct-sum decomposition of \( H \) w.r.t. \( q \) must preserve the subspaces corresponding to the decomposition above \( (|0\rangle \langle 0|, |1\rangle \langle 1|, |2\rangle \langle 2|) \). If, however, \( \Pi_{0}^F = \Pi_{1}^F \) then \( H \) may also be written as

\[ H = |+\rangle \langle +|^q \otimes \Pi_{0}^F + |−\rangle \langle −|^q \otimes \Pi_{1}^F + |2\rangle \langle 2|^q \otimes \Pi_{2}^F \]

where \(|+\rangle \langle +|\) projects on the vector \( \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \) and \(|−\rangle \langle −|\) projects on the state \( \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \).

So this decomposition of \( H \) does not preserve the first decomposition.

Nevertheless, notice in the above example that every decomposition must preserve the subspace spanned by \( |2\rangle \), and its orthogonal complement. We call this one dimensional subspace a Critical Subspace. More formally, a critical subspace is a rank 1 projection in the center of the induced algebra of the operator on the target qutrit. For the above example, if \( \Pi_{0}^F, \Pi_{1}^F, \Pi_{2}^F \) are all linearly independent, \( H \) has 3 critical subspaces spanned by the states \( |0\rangle, |1\rangle, |2\rangle \), whereas if \( \Pi_{0}^F = \Pi_{1}^F \neq \Pi_{2}^F \) \( H \)'s only critical subspace is spanned by \( |2\rangle \). We show (facts 5.9 and 5.10) that a \( \triangleright\triangleleft \) relation between a pair of operators acting non-trivially on a qutrit \( q \) implies that each of these operators has a critical subspace on \( q \), and these subspaces are either identical or orthogonal, and moreover each of the operators preserves each other’s critical subspace.

This behavior of critical subspaces is what replaces the notion of unique decomposition in qubits, though it is weaker. We can prove the following, when restricting the interactions to act on the plane. Consider all operators on a qutrit \( q \); each operator has its own critical subspace in the Hilbert space of \( q \). We prove that if the number of operators is large enough, any assignment of critical subspaces to the operators on \( q \) forces all of the operators to preserve at least one of the assigned critical subspaces, and so the qutrit becomes separable. This means that there cannot be more than a small number of operators acting on \( q \); this is proved in Lemma 5.15.

An easy implication of Lemma 5.15 is that in an instance with no separable qutrits, all vertices in the interaction graph must be of degree at most 5. (we state this in Corollary 5.16). This is a crucial point. We show that planar-embedded Hamiltonians in which all the vertices are of degree at most 5, must exhibit an intriguing characteristic, which is in fact entirely geometrical.
Consider a planar embedding of a graph, whose faces are colored black and white. Moreover, only 3-vertex faces can be colored black (black regions correspond to terms in the Hamiltonian). Then there must be a constant density of white “holes”; i.e., any point in the plane is within a constant distance (in terms of number of faces) from such a white hole - i.e., a region where no interaction acts. The proof of this geometric fact uses the Euler Characteristic but is fairly involved and we delay its overview to Subsection (5.6) where it is proven.

The main point is that the existence of those regularly spaced holes allows us, in the case the interaction graph is NE (and this is the only place where we use the NE property in the proof) to coarse grain the set of particles, and by this derive a 2-local instance. This is done in Claim (5.27): the rough idea is to lay down on the plane a “net” that partitions the plane in such a way that in each region, there are only constantly many particles, while making sure that the junctions of the net fall precisely inside those white “holes”. If we combine the particles in each region together, then each term in the Hamiltonian acts on at most 2 of the combined particles. We now proceed to the details.

5.3 Removing Separability

We begin by defining qutrit separability in exactly the same way as qubit separability:

**Definition 5.7** A qutrit q is called separable in an instance S of CLH(3,3) if there exists a non-trivial direct-sum decomposition of \( \mathcal{H}_q \)

\[
\mathcal{H}_q = \bigoplus_{\alpha} \mathcal{H}^\alpha_q
\]

such that any operator \( H(q) \in S \) preserves this decomposition, i.e. \( H(q) = \bigoplus_{\alpha} H(q)|_\alpha \).

Similar to handling CLH(3,2) the NP protocol begins by Merlin helping Arthur remove all separable qutrits from the system. Let \( S \) be some instance of a NE CLH(3,3), and q be some qutrit. If q is separable then following input from Merlin, its dimension is reduced to at most 2, using restriction to some subspace. After the restriction, if the dimension is 1, the particle essentially “vanishes” from the input, so suppose we are left with a 2-dimensional particle, a qubit. This qubit may now be either separable or non-separable. If it is separable, by additional input from Merlin, it “vanishes” from the input. Thus, after all qubit/qutrit separability has been exhausted, we are left with a Hamiltonian acting on qutrits and qubits, all of which are non-separable. We note that the remaining interaction graph is still NE, even though some “op” faces may have now turned into “noop” faces.

5.4 Critical Subspaces and Separability for general CLH(3,3)

We would now like to develop tools that will allow us to prove geometrical restrictions on the residual problem after removing all separable qudits.

**Definition 5.8 Critical Subspace**

Let \( H(q) \) be an operator acting on q. Denote by \( A^H_q \) the algebra induced by H on q. Any rank-1 projection in the center of this algebra \( Z(A^H_q) \), induces a a critical subspace of H on q, which is the one dimensional subspace which is the image of the rank-1 projection.
We now prove several useful facts regarding critical subspaces. The following facts hold for all qudits of dimension \( d \leq 3 \).

**Fact 5.9** Let \( H_1(q), H_2(q) \) be two operators that act non-trivially on \( q \) such that \( H_2(q) \triangleright H_1(q) \). We claim that each of operator has at least one critical subspace on \( q \). Also, each critical subspace of \( H_1 \) is preserved by \( H_2 \) and vice versa.

**Proof:** Let \( A^1_q, A^2_q \) denote the algebras induced by \( H_1(q), H_2(q) \) on the qudit \( q \), respectively. First, we show that both algebras \( A^1_q, A^2_q \) are reducible. Suppose on the negative that it is not the case, and that say, \( A^1_q \) is irreducible. By Fact (2.12), it is isomorphic to \( L(H^1_q) \otimes I_{H^2_q} \). Since \( d = 3 \) or \( d = 2 \), then one of the Hilbert spaces \( H^1_q \) or \( H^2_q \) is of dimension 1. It cannot be that \( H^1_q \) is of dimension 1, since this would imply that the algebra \( A^1_q \) is trivial, and this means that the operator \( H_1 \) acts trivially on \( q \), contrary to our assumption. Thus, it must be that \( H^2_q \) is of dimension 1. Since \( A^1_q, A^2_q \) commute, this implies that \( A^1_q \) is trivial, and so \( H_2(q) \) acts trivially on \( q \) contrary to our assumption.

Thus by fact (2.13), each of the two algebras can be non-trivially decomposed into a direct-sum of algebras, following an orthogonal decomposition of \( H_q \) into a direct-sum of subspaces \( H^a_q \). Since \( d \leq 3 \) at least one such subspace is of dimension 1, so the center of each algebra has a rank 1 projection, and so both operators have a critical subspace on \( q \).

We now show that the critical subspaces of one operator are preserved by the other. Let \( S_c \subseteq H_q \) be a critical subspace of \( H_1(q) \). Let \( \Pi_c \) be the projection on \( S_c \); By Definition (5.8), \( \Pi_c \in Z(A^1_q) \) and thus it is contained in \( A^1_q \). Since \( A^1_q \) commutes with \( A^2_q \) by Fact 2.9, then \( \Pi_c \) commutes with any operator \( A \in A^2_q \), thus by Fact 2.5, \( A^2_q \) preserves \( S_c \), and thus so does \( H_2(q) \).

**Fact 5.10** Let \( H_1(q) \triangleright H_2(q) \) be two operators on \( q \). Let \( S^1_c \) be a critical subspace of \( H_1(q) \) on \( q \). Then any critical subspace of \( H_2(q) \) on \( q \) is either \( S^1_c \) or a subspace orthogonal to it.

**Proof:** Let \( A^1_q, A^2_q \) denote the algebras of \( H_1(q), H_2(q) \) on qubit \( q \). The algebra \( A^2_q \) commutes with \( A^1_q \), and so it commutes with any element in the center of \( A^1_q \), in particular, the projection on \( S^1_c \). By Fact 2.5, any operator in \( A^2_q \) preserves \( S^1_c \). Let \( S^2_c \) be a critical subspace of \( H_2(q) \), and let \( \Pi^2_c \in Z(A^2_q) \) be the projection on it. Since this projection is contained in \( A^2_q \), it too preserves \( S^1_c \). We have that a projection on a one-dim subspace \( S^2_c \) preserves a one dimensional subspace \( S^1_c \) and so either the two are equal or they are orthogonal.

**Fact 5.11** Let \( H_1(q) \) be an operator with a critical subspace on \( q \), and suppose that \( H_1 \) preserves two 1-dim. subspaces of \( q \): \( S_0, S_1 \) such that \( S_0 \) and \( S_1 \) are neither equal nor orthogonal. Then \( S_c = S^0_0 \cap S^1_1 \) is a critical subspace of \( H_1 \).

**Proof:** Since \( H_1 \) preserves both \( S_0 \) and \( S_1 \), it also preserves their two dimensional span, which we shall denote by \( S_{0,1} \), and by Fact 2.4, it also preserves \( S_{0,1}^0 \) which we denote by \( S_2 \).

Let us examine the operator \( H_1 \) restricted to \( S_{0,1}^1 \): \( H_1|_{S_{0,1}^1} \). This operator too preserves \( S_0 \) and \( S_1 \), and so since \( S_0 \) and \( S_1 \) are neither equal nor orthogonal, we get that this operator is block-diagonal w.r.t. two different orthogonal bases: \( S_0, S_0^\delta \) and \( S_1, S_1^\delta \). Since \( S_{0,1} \) is a two dimensional subspace,
we can proceed in a similar way to the proof of Claim (4.8), and conclude that \( H_1 \) restricted to \( S_{0,1} \) is trivial on \( q \), and can be written as

\[
H_1|_{S_{0,1}} = I^q \otimes \Pi_{0,1}^E
\]

where \( \Pi_{0,1}^E \) is a projection on the system not including \( q \). All in all we have:

\[
H_1 = \Pi_{0,1}^q \otimes \Pi_{0,1}^E + \Pi_2^q \otimes \Pi_2^E
\]

where \( \Pi_{0,1}^q \) projects on \( S_{0,1} \) and \( \Pi_2^q \) projects on its orthogonal complement \( S_2 \).

It cannot be that \( \Pi_{0,1}^E \) and \( \Pi_2^E \) are linearly dependent, since this would mean that they are in fact equal, which would imply that \( H_1 \) is trivial, and thus does not have a critical subspace, contradicting the assumption of the statement.

Hence, \( \Pi_{0,1}^E \) and \( \Pi_2^E \) are linearly independent, and so by Definition 2.6 the algebra induced by \( H_1 \) on \( q \), \( A_q \), is spanned by \( \Pi_{0,1}^q, \Pi_2^q \). Its center thus included exactly one rank-1 projection, the projection on \( S_2 \).

---

**Fact 5.12** Let \( H(q) \) act non-trivially on a qudit \( q \), and have an induced algebra \( A_q \) on \( q \) which is reducible. By the notation of definition 5.7, denote the subspaces in the decomposition of the algebra by \( \mathcal{H}_q^a \) and the corresponding projections \( \Pi_a \in Z(A_q) \). Consider an unrelated one-dimensional subspace of \( \mathcal{H}_q \) denoted \( S_0 \). Then if there exists an \( a \) for which \( \Pi_a \) does not preserve \( S_0 \), then \( H(q) \) has a critical subspace on \( q \), denoted \( S_+ \), and \( S_0 \) and \( S_+ \) are neither orthogonal nor equal.

**Proof:** If \( \Pi_a \) does not preserve \( S_0 \), then this means that also its complement \( I - \Pi_a \) does not preserve \( S_0 \). Since \( Z(A_q) \) is trivially closed under complement with \( I \), also \( I - \Pi_a \in Z(A_q) \). Then either \( \Pi_a \) or \( I - \Pi_a \) is a rank-1 projection in \( Z(A_q) \) not preserving \( S_0 \). The image of this rank-1 projection is a critical subspace by definition 5.8. Let us denote it by \( S_+ \). Then \( S_0 \) and \( S_+ \) are neither orthogonal nor equal.

---

### 5.5 All vertices in a planar CLH(3,3) instance are of degree at most 5

Using the results of the previous subsection, we now show that all vertices are of degree at most 5. We prove the following claim for that purpose.

**Claim 5.13** Left-Right Partition implies consensus on decomposition Consider a set of operators on a qudit \( q \) with \( d \leq 3 \), which are separated into two non-empty sets, such that any two operators from these two sets, one from each side, share only \( q \). Then there is a non-trivial direct-sum decomposition of \( q \) which is preserved by all those operators on \( q \).

**Proof:** For a qubit \( q \) the lemma is the same as corollary 4.11, so now we focus on \( q \) being a qutrit. Let us denote the sides of the division as side 1 and side 2. We would like to show that there exists a direct-sum decomposition, preserved by all operators on both sides. First, for any pair of operators \( H_1(q) \) from side 1 and \( H_2(q) \) from side 2 we have \( H_1 \triangleright \triangleleft H_2 \) so by Fact 5.9 both \( H_1 \) and \( H_2 \) have critical subspaces. Let us take some operator on side 1, \( H_1(q) \) whose critical subspace is \( S_0 \). We consider three possible cases and show that in each case, a "consensus" decomposition emerges.

37
1. Case 1: All operators on side 2 have the same critical subspace of $q$, denoted $S_c$. Then all operators on side 1 must preserve this subspace by Fact (5.9), and so $S_c$ is preserved by all operators on $q$.

2. Case 2: There exists an operator in side 2 which has a critical subspace equal to $S_0$. In this case on each side of the division $S_0$ is a critical subspace, so all operators (on both sides) must preserve this subspace by Fact (5.9).

3. Case 3: Neither of the two first cases hold. Then there are at least two distinct critical subspaces on side 2, namely $S_1$ and $S_2$ and none of them is equal to $S_0$. Since $S_0$ is a critical subspace of an operator in side 1 then by fact (5.10) $S_1, S_2$ are both orthogonal to $S_0$. Since they are not the same, they span the entire orthogonal subspace to $S_0$. Any operator on side 1 must preserve both $S_1$ and $S_2$, by Fact (5.9) and thus it preserves the span of $S_1$ and $S_2$, and so must also preserve the orthogonal subspace of this two dimensional subspace, namely $S_0$. By Fact (5.9), any operator on side 2 is $\triangleright\triangleleft$ with $H_1$ and thus must preserve $S_0$ too.

**Corollary 5.14** Let $q$ be a qubit of dimension $d \leq 3$. If all operators on $q$ can be divided to two non-empty sets, such that any operator from one set intersects any operator in the other set in $q$ alone, then $q$ is separable.

**Lemma 5.15** No open operator paths of length $> 4$ Let $S$ be an instance of planar CLH(3, 3), and let $q$ with $d \leq 3$ be a qudit in that instance. Assume there exists a subset of the operators on $q$ that constitute an open operator path on $q$ of length at least 5, then $q$ is separable.

**Proof:** Let $H_1, H_2, H_3, H_4, H_5$ denote a length 5 open operator path on $q$, so $H_1$ and $H_5$ intersect only on $q$. If $q$ is a qubit then by claim (4.16) $q$ is separable, since two qubits of $H_1$ are not part of the operator crown $\{H_3, H_4, H_5\}$. So we now treat the case where $q$ is a qutrit. We first show that all these operators agree on some non-trivial decomposition. The subset $H_1, H_2, H_4, H_5$ yields a division $S_1 = \{H_1, H_2\}$ and $S_2 = \{H_4, H_5\}$ such that any pair of operators, one from each set, intersect only on $q$. Thus, by Claim (5.13) all these 4 operators agree on some non-trivial decomposition of $q$, denoted by $S_0$ and its orthogonal complement $S_0^\perp$.

Let $A_0^\perp$ be the algebra of $H_3(q)$ on $q$. Let $H_q = \bigoplus A_q^\perp$ be the direct-sum decomposition of $H_q$ corresponding to the algebra $A_q^\perp$ whose existence is promised by (2.13). Since $H_3 \triangleright \triangleleft H_1$, then by fact (5.9) $H_3$ has a reducible algebra on $q$. So the decomposition $H_q$ is non-trivial.

Let us examine the behavior of the projections $\Pi_\alpha$ on the subspaces $H_q$ w.r.t. $S_0$. If $S_0$ is preserved by $\Pi_\alpha$ for all $\alpha$ we are done, since by Fact (2.13) $H_3$ preserves $S_0$, and so all 5 operators preserve the decomposition $S_0, S_0^\perp$. So suppose this is not the case.

Then there exists a subspace $H_q$ whose corresponding projection $\Pi_\alpha \in Z(A_q^\perp)$ does not preserve $S_0$. We are now in the situation handled by Fact (5.12) we conclude that either $\Pi_\alpha$ or $I - \Pi_\alpha$ is a rank-1 projection in $Z(A_q^\perp)$ not preserving $S_0$. The image of this rank-1 projection is a critical subspace denoted by $S_+$. Since $S_0$ and $S_+$ are neither orthogonal nor equal.

Since $H_1 \triangleright \triangleleft H_3$ and $H_1 \triangleright \triangleleft H_5$ then by fact (5.9) $H_1$ and $H_5$ preserve $S_+$ and so by our assumption they preserve both $S_0$ and $S_+$. Since their algebras on $q$ are reducible by Fact (5.9), we conclude by Fact (5.11) that they must both have the subspace $S_2 = S_0^\perp \cap S_+^\perp$ as a critical subspace. We get...
that $H_2, H_3, H_4$ all of which intersect either $H_1$ or $H_5$ only on $q$, must preserve $S_2$ by fact (5.9). Therefore, all 5 operators preserve $S_2$.

To complete the proof we now need to show not only $H_1, ..., H_5$ but all operators on $q$ can agree on some preserved non-trivial subspace. We will prove this by induction, "adding back" the other operators on $q$ one by one. Let $H_6, ..., H_L$ be all other operators on $q$. We will assume that all operators $H_1, ..., H_j$ (for $j \geq 5$) agree on a non-trivial decomposition (namely, preserve its subspaces), and in particular, preserve a one dimensional subspace $S_0$, and prove that $H_1, ..., H_{j+1}$ must also agree on one such decomposition (which might be different).

We divide to two cases. Either $H_{j+1}$ preserves $S_0$, in which case we are done, or it doesn’t. Let us therefore assume it doesn’t. Since $S$ is planar, it must be that all operators $H_6, ..., H_L$ satisfy $\triangleright \triangleleft H_2, H_3, H_4$, and so $H_{j+1}$ has at least one $\triangleright \triangleleft$ relation. By Fact (5.9) its algebra on $q$ $A_q^{j+1}$ is reducible, and so it has a non-trivial decomposition of this algebra; using the notion of Fact (2.13) we denote the subspaces in the decomposition by $H_q^a$. Since $H_{j+1}$ does not preserve $S_0$, this means that there exists a subspace $H_q^a$ whose corresponding projection does not preserve $S_0$. We are now again in the situation of Fact (5.12) and we conclude that the center of $A_q^{j+1}$ has a rank-1 projection on a one dimensional space which is neither equal nor orthogonal to $S_0$. This is a critical subspace of $H_{j+1}$ denoted by $S_{j+1}$.

By assumption, $H_2, H_3, H_4$ preserve $S_0$, and by Fact (5.9) $H_2, H_3, H_4$ preserve $S_{j+1}$. We can now apply Fact (5.11) to deduce that the critical subspace of all 3 operators must be orthogonal to the space spanned by $S_0$ and $S_{j+1}$. Let us denote this space by $S_c$. All other operators we mentioned have a $\triangleright \triangleleft$ relation to at least one of these 3 operators, thus by fact (5.9) all operators $H_1, ..., H_{j+1}$ preserve $S_c$.

Corollary 5.16 If an instance of planar CLH(3,3) has no separable qudits, then all the vertices of its interaction graph are of degree at most 5.

Proof: On the negative, let $q$ be a nonseparable qudit in an instance $S$ of NE planar CLH(3,3), with degree is at least 6. Since $q$ is nonseparable then by corollary (5.14) all operators on $q$ are operator-path connected. Since $S$ is a planar instance, it means that there exists an operator path on $q$ comprised of all operators on $q$. If the degree of $q$ is at least 6, this means that $q$ is acted upon by an open operator path of length at least 5 so by lemma (5.15) it is separable, contrary to our assumption.

5.6 Regularly-spaced holes in triangle tilings with no high degree vertices

This subsection is entirely geometrical, and considers tiling of the plane with triangles. We show that if the tiling avoids vertices of degree 6 and above, it must be that the tiling must contain regularly spaced holes, with constant density.

5.6.1 Background and definitions

Definition 5.17 Dual Graph

Given a planar graph $G$, the dual graph of $G$ as follows: The set of vertices of $\hat{G}$ are comprised of all faces in $G$. An edge between vertices of $\hat{G}$ exists, if and only if the respective faces share an edge in $G$.

39
**Definition 5.18 Tessellation**
Given an instance $S$ planar CLH, a tessellation is a subset of operators $T \subseteq S$, whose induced embedding, i.e., the embedding of $G_T$ in $R^2$, is a triangulation of a polygon, which contains no “noop” faces. For a tessellation $T$ we define an external vertex as one which belongs to the infinite face, and otherwise it is an internal vertex.

**Average Degree and the Euler Characteristic** A well known formula connects the average degree of a vertex in a planar graph, with the average number of edges per face. Given a tessellation $T$, and its interaction graph $G_T$ with its embedding in $R^2$, let $a$ be the average number of edges per face (including the infinite face), and $b$ be the average number of edges incident on a vertex. Let $F$ denote the number of faces of $T$ (including the infinite face), $V$ the number of vertices, and $E$ the number of edges. Recall the following definition:

**Definition 5.19 Euler Number** The Euler number $\chi$ is defined to be $\chi = V - E + F$.

Assuming that $G_T$ is a connected planar graph, then by Euler’s theorem we have $\chi = 2$.

One can easily check that the following relations hold:

$$a \cdot F = 2 \cdot E, \quad b \cdot V = 2 \cdot E,$$

since counting the number of edges of each face (including the infinite face) counts twice the number of edges, and similarly, counting the number of edges incident on each vertex also counts twice the number of edges. Using these two, together with the expression for $\chi$ yields the following formula:

$$(a - 2) \cdot (b - 2) = 4 \left(1 - \frac{\chi}{F}\right) \left(1 - \frac{\chi}{V}\right) = 4 \left(1 - \frac{2}{F}\right) \left(1 - \frac{2}{V}\right). \quad (7)$$

where we have used the fact that $\chi = 2$.

**5.6.2 Constant density of holes: Statement and Overview of Proof**
We prove the following claim:

**Claim 5.20** Let $\eta \geq 500$ be some constant. We are given a tessellation $T$, such that there exists an operator $w \in T$ for which the following holds: a) any face in $G_T$ which is at $\tilde{\hat{G}}_T$-distance at most $\eta$ from $w$ is an “op” face, and b) all “op” faces in $G_T$ are at $\tilde{\hat{G}}_T$-distance at most $\eta$ from $w$. Then there exists a vertex $q$ in $T$ whose degree is at least 6.

We will assume that all vertices in $T$ have degree at most 5, and arrive at a contradiction. To do that, we will lower bound $b$, the average number of edges incident on a vertex, by a number larger than 5. This would imply that there must be a vertex of degree 6 or more.

In order to extract interesting information about $b$, we in fact turn to $a$, the average number of edges per face, and prove it is very close to 3. In other words, we would like to show that $a$ is dominated by the “internal” faces which by our assumption are all “op” faces (i.e., their number of edges is 3), whereas the outside multi-edge face is negligible in determining the average number of edges of a face. We can then use Equation (7) we will be able to deduce that $b$ must be strictly larger than 5, since $F, V$ are large.

Our main effort is thus to bound $a$ from above. To this end,
Definition 5.21 Let $\hat{a}$ be the number of edges of the outside face of $T$ (there is only one such face since $T$ is a tessellation).

and we can write

$$a = \frac{1}{F}(3 \cdot (F - 1) + \hat{a}) \quad (8)$$

We will show (and that’s the main effort in the proof) that $\hat{a}$ is bounded by a small constant, 13, so when $F$ is large enough, we will get an upper bound on $a$ which is very close to 3.

The proof that $\hat{a}$ is bounded by a constant goes as follows. We devise a procedure that generates a sequence of tessellations, starting from the tessellation $T_1 = w$, namely the single operator given in the statement of the claim. The sequence of operator sets is denoted $T_1 = w, \ldots, T_i, \ldots, T_m$, for $m = \eta/5$, the first one being $w$ itself, where for each $i$, $T_i \subseteq T_{i+1}$, and for all $i$, $T_i \subseteq T$. Each $T_i$ includes at most 4 additional operators from $T$, compared with $T_{i-1}$. We show that despite the growth in number of faces of the tessellation, the number of external edges cannot cross a certain bound.

To this end, the choice of how to construct the tessellation $T_{i+1}$ from $T_i$ is of particular importance. Suppose that given a tessellation one would construct the $T_i$’s by adding the operators corresponding to some arbitrary path in the dual graph, one by one. Then the number of edges of $\hat{a}$ will grow proportionally to the number of faces $F$; that would be a bad choice. Our scheme thus relies on constructing the $T_i$’s in a spiral fashion, by choosing a “special” external vertex of $T_i$ at each step, and then adding all operators acting on it to “close” an operator path on that vertex. We can show that such a process can continue for a large number of steps ($\eta/5$) without failing, i.e., a “special vertex” is well defined for $i < \eta/5$, and thus the $T_i$ are strictly increasing, while $\hat{a}$ cannot increase beyond some constant value.

We now proceed to the details.

5.6.3 Detailed proof of Claim 5.20

As mentioned, we assume by contradiction that all vertices in the tessellation $T$ have degree at most 5.

Definition 5.22 Consider an external vertex $v$ in $T$. We denote by $n_T(v)^k$ the vertex located $k$ vertices away from $v$ on the path that traverses the external vertices of $T$ in a counterclockwise direction.

The procedure “Tessellate” is given as input $T$ and an operator in $T$, $w$ and returns a sequence of tessellations $T_i$ as follows.

Algorithm 5.23 Tessellate($T, w$)

1. Init ($i = 1$): set $T_1$ to be $w$.

2. ($i = 2$) Choose arbitrarily an external vertex $v_1$ of $T_1$ (all vertices are external at this point) and close an operator path on it, by adding to $T_1$ the necessary operators from $T$ that close the operator path on $v_1$. Denote the new set of operators as $T_2$. 41
3. For \(2 < i \leq \eta/5\), set \(v_i\) as the vertex \(n_{T_{i-1}}(v_{i-1})^k\) that is external in \(T_i\) with minimal \(k\), i.e. the closest left neighbor of \(v_{i-1}\) in the external path of \(T_{i-1}\), which remained external following the closing of the operator path on \(v_{i-1}\). If all external vertices of \(T_{i-1}\) were closed following the appending of operators during the creation of \(T_i\), then set an arbitrary external vertex in \(T_i\) as \(v_i\). Then close an operator path on \(v_i\), again, by adding to \(T_i\) the operators from \(T\) that close an operator path on \(v_i\) (namely, all operators \(H(v_i) \in T \setminus T_i\)). Denote the resulting set as \(T_{i+1}\), and so on.

We claim

**Fact 5.24** Given \(T\) and \(w\), the steps of the procedure “Tessellate” above are well defined for all \(i \in [1, \eta/5]\), and the procedure generates strictly increasing tessellations \(T_1 \subset T_2 \subset \cdots \subset T_{\eta/5} \subset T\).

**Proof:** We show that at each step \(i\) for \(i \in [1, \eta/5]\) there exists a closed operator path in \(T\) on the vertex selected as \(v_i\): By our assumption, all vertices have degree at most 5. This means that each \(T_i\) contains at most 4 more operators than \(T_{i-1}\), and so \(T_i\) contains at most \(4(i-1)+1\) operators. This means that the furthest operator in \(T_i\) from \(w\) (where distance is measured in number of edges in the dual graph) is within distance \(4(i-1)\). Therefore, for \(i \leq \eta/5\), all operators are within distance at most \(4\eta/5\) from \(w\), and thus are all “op” operators by assumption, and also are not adjacent to “noop” faces. Thus, for any vertex \(v\) in \(T_i\), for \(i \in [1, \ldots, \eta/5]\), the operators \(H(v) \in T\) form a closed operator path on \(v\).

We would now like to show the important property for which the procedure was created. Denote by \(\hat{a}_i\) the number of external edges of \(T_i\), and by \(\hat{a}_{\max} = \max_{i=1}^{\eta/5}(\hat{a}_i)\)

**Claim 5.25** \(\hat{a}_{\max} \leq 13\).

**Proof:** We begin by showing the following properties:

1. If \(\hat{a}_i \geq 10\), then there exists some \(k\) such that \(n_{T_i}(v_i)^k\) is external in \(T_{i+1}\).

We want to show that if \(\hat{a}_i \geq 10\), then closing an operator path on any external vertex of \(T_i\) cannot close simultaneously an operator path on all other external vertices of \(T_i\). Take some external vertex \(q\), and close an operator path on it by appending to \(T_i\) at most 4 additional operators from \(T\). Each of these operators examines, in addition to \(q\), two additional qutrits so at most 8 external qutrits of \(T_i\), aside from \(q\), are examined by these operators. Therefore, closing the operator path on \(q\) in \(T_i\) can close additional operator paths on at most 8 other external qutrits of \(T_i\). Thus, if \(\hat{a}_i \geq 10\), then since \(\hat{a}_i\) is equal to the number of external qutrits, there will always be an external vertex of \(T_i\) that remains without a closed operator path following the closing of the operator path on \(q\).

2. For each \(i \in [1, \eta/5]\) if \(\hat{a}_i \geq 10\) then closing the operator path on \(v_i\) increases the number of operators acting on \(v_{i+1}\) by at least 1:

Denote by \(P = v_i, n_{T_i}(v_i)^1, n_{T_i}(v_i)^2, \ldots n_{T_i}(v_i)^m\) the path traversing the external vertices of of \(T_i\) counterclockwise, starting from \(v_i\). Since \(\hat{a}_i \geq 10\), then by item (1), \(v_{i+1}\) is selected by “Tessellate(T,w)” as \(v_{i+1} = n_{T_i}(v_i)^k\) for some \(k\). Then for all \(j < k\) the vertex \(n_{T_i}(v_i)^j\) is an internal qutrit in \(T_{i+1}\) i.e. a vertex such that closing the operator path on \(v_i\) has also closed its operator path. Specifically, for \(j = k-1\) the operator path on \(n_{T_i}(v_i)^j\) was closed at step \(i\). Yet,
Figure 12: The figure shows the first 4 steps of a possible run of the algorithm. The unfilled qutrit, is the one chosen to close the operator path on.

prior to closing, \( n_{T_i}(v_i)^{k-1} \) and \( n_{T_i}(v_i)^k \) are two external neighboring qutrits in \( T_i \), so they share an edge which is part of the external face in \( T_i \), and which must be part of a newly-added operator on \( n_{T_i}(v_i)^{k-1} \). Thus, closing the operator path on \( n_{T_i}(v_i)^{k-1} \) must entail adding at least one operator which acts on both \( n_{T_i}(v_i)^{k-1} \) and \( n_{T_i}(v_i)^k \). Thus the operator count on \( v_{i+1} = n_{T_i}(v_i)^k \) increases by at least 1 at the end of step \( i \).

3. For each \( i \in [4, \eta/5] \) if \( \hat{a}_i \geq 10 \), then the number of operators in \( T_{i+1} \) acting on the selected vertex \( v_{i+1} \), before closing its operator path, is at least 3:

We close the operator paths on the 3 qudits of the input \( w \) after at most 3 steps. After that, all \( v_i \)'s, are ones which are added during previous "closure" processes. When a new vertex is added to \( T_i \), it is added as part of a closed path; hence, when such a \( v_i \) was added, it was added together with two operators acting on it. By item (2), when \( \hat{a}_i \geq 10 \) closing the operator path on \( v_i \) increases number of operators acting on \( v_{i+1} \) by at least 1. Hence, when \( v_{i+1} \) is chosen at step \( i + 1 \), prior to closing its operator path, it is acted upon by at least 3 Hamiltonians.

Now we are ready to show that \( \hat{a}_{\text{max}} \leq 13 \).
We will first show that for all steps $i$ for which $i \geq 5$ and $\hat{a}_{i-1} \geq 10$ we have $\hat{a}_i \geq \hat{a}_{i+1}$.

Recall that we assume that the degree of each vertex in $T$ is at most 5, hence, since the graph is planar, there can be at most 5 operators acting on each vertex. By item (3) above, before closing the path on $v_i$ for $i \geq 5$ and $\hat{a}_{i-1} \geq 10$ there were at least 3 operators acting on $v_i$; Hence, the closing could have added either one or two operators that act on $v_i$. Let us see how each of the cases affects $\hat{a}_i$:

1. Add one face/operator to act on $v_i$ to close the path on $v_i$: In this case, in order to close an operator path we must connect existing vertices, and not add any new ones. Therefore, $\hat{a}_i$ "looses" two edges which are replaced by 1, and decreases overall by 1.

2. Add two faces/operators to act on $v_i$ to close the path on $v_i$: In this case $\hat{a}_i$ gains at most 2 new edges in exchange for at least 2 previous edges so it does not increase.

Thus we have (*) for all steps $i$ for which $i \geq 5$ and $\hat{a}_{i-1} \geq 10$ we have $\hat{a}_i \geq \hat{a}_{i+1}$. On the other hand, at each step $i$ the value of $\hat{a}_i$ can increase by at most 2: exactly two edges are removed from the external path, and at most 4 are added; the latter case occurs if 4 operators are added together with 2 new vertices.

We now claim that for all $i$, $\hat{a}_i \leq 13$. We know that $\hat{a}_1 = 3$, and at each one of the first 4 steps, closing the operator path on each qutrit can increase $\hat{a}$ by at most 2. Hence, $\hat{a}_i \leq 3 + 4 \cdot 2 = 11$ for $i \in \{1, ..., 5\}$. Now, assume by contradiction that $\hat{a}_{j_0} \geq 14$ for some index $j_0 \geq 6$. WLOG let $j_0$ be the first such index for which $\hat{a}$ is strictly larger than 13. Then $\hat{a}_{j_0-1}$ must be either 13 or 12; This means that one step before that, $\hat{a}_{j_0-2}$ must have been $\geq 10$, and notice that $j_0 - 2 \geq 4$. This means that we have an index we have an index $i = j_0 - 1 \geq 5$, such that $\hat{a}_i \geq 10$ and $\hat{a}_{i+1} > a_i$, in contradiction to (*).

We can now prove Claim 5.20

**Proof:** Consider the tessellation $T_{\eta/5}$ generated by the algorithm Tessellate. It is comprised of at least $\eta/5 \geq 100$ faces since each step increases the number of operators by at least one, and so $F \geq 100$. We also have that since $V - E + F = 2$ and $E \leq 5V$, then $F - 2 \leq 4V$, and so $V \geq 98/4$. Plugging this into Equation 7, we have that

$$(a - 2) \cdot (b - 2) \geq 3.6$$

and since $\hat{a}_{\text{max}} \leq 13$, we have $a \leq 3.1$ by Equation 8. This implies that $b > 5$, which means there must be a vertex of degree $> 5$.

**5.7 Proof of regularly spaced holes**

We can now deduce that the interaction graph of a planar CLH$(3,3)$ instance, with no separable qudits, which is a graph whose vertices are all of degree smaller than 6, must have constant “density” of “holes” in it, namely, “noop” faces.

**Claim 5.26** Consider the interaction graph $G_S$ of a planar instance $S$ of CLH$(3,3)$ with no separable qudits. Then for any operator $w \in T$ there exists a “noop” face within distance (in the dual graph) at most $\eta$. 

44
**Proof:** We assume on the negative that there exists a vertex \( w \) with no "noop" faces within distance \( \eta \) from \( w \). This means that any face \( u \) whose distance from face \( w \) is at most \( \eta \) is an "op" face. Consider the set of all operators of distance at most \( \eta \) to \( w \). This set has no gaps (otherwise there would be a "noop" face of distance less than \( \eta \) to \( w \), and so it is a tessellation, which we can denote \( T \). The conditions of Claim (5.20) now hold; therefore there exists a vertex \( v \) in \( T \) whose degree is at least 6 contrary to the fact that \( S \) has no separable qudits. ■

5.8 From regularly spaced holes to two-locality

We thus have a constant density of "noop" faces. Here (and only here) we use the fact that the planar embedding is NE, and devise an NP protocol to reduce the \( CLH(3, 3) \) instance into a \( CLH(2, d) \) instance for constant \( d \). This is formalized in the following claim:

**Claim 5.27** For an instance \( S \) of NE planar \( CLH(3, 3) \) with no separable qudits, there exists additional classical input from Merlin that allows to reduce \( S \) into an instance of \( CLH(2, d) \) for constant \( d \).

5.8.1 Overall approach

We now shift our attention to the NE planar triangulation of a convex polygon, which contains the interaction graph of \( G_S \) as a subgraph. The existence of such a triangulation is guaranteed by the definition of NE planarity (Definition 5.4). The triangles in this triangulation are also associated with "noop" or "op" faces, which inherent their nature ("noop" or "op") from the faces they are contained in, in the original embedding of the graph \( G_S \) in the plane. From now on, we refer to this triangulation \( T \) as our graph. We note that the density of "noop" faces in this graph (namely, Claim 5.26) is still as before, since only "noop" faces were partitioned to triangles - "op" faces were already triangles.

The basic idea is to partition the polygon to constant size sections, using "cuts" along paths in the dual graph, which are non-intersecting, except at "noop" faces. The reader can convince herself that if this can be done, this will ensure that the interactions between the different sets of vertices corresponding to the different sections are two-local, since each interaction term involves only particles from at most two different such areas.

The difficulty in the proof is that due to the lack of regularity of the graph, it is non-trivial to construct explicitly such a net of paths between "noop" faces. Here is where we use the fact that NE planar triangulations of polygons obey nice characteristics, which resemble those of a periodic lattice. The NE requirements imply that the noop faces are scattered in a more of less periodic fashion in terms of Euclidean distance between the noops. We note that no periodicity is assumed here regarding \( T \), and the NE restriction is significantly more general, and cover, essentially, all "reasonable" physical scenarios.

The overall idea is to lay down on the plane a "Brick-Wall", (namely, a square lattice in which the even rows are shifted by a half-square w.r.t. the odd rows - see Figure 13), and find near each vertex of the brick wall a noop face, which we know exists due to Claim 5.20. We essentially perturb the brick wall junctions so that the junctions fall inside those "noop" faces.

A problem arises though from the fact that even if all the junctions of the partition are located at noop faces, locally there may still be qudit triplets that manage to interact across 3 subdivisions,
without intersecting the “noop” itself; this can happen since the sizes of the edges are not all the same.

We thus employ a trick - we in fact find two noops close to each vertex of the brick wall, rather than one. The two noops are required to be not too close to each other. We use the area between the two noops to create what we call a “noop zone”. The advantage is that this noop-zone is larger in diameter than any edge in the graph, and so it prevents the local problems mentioned before; more explicitly, no “op” face can cross and connect all three bricks adjacent to this junction.

To create the noop-zone, we consider a path connecting the two noops, and contract all edges along a path between the two noops, until the two noops intersect in a vertex, which is now larger in dimension but still a constant. This enables us to apply Lemma 3.3 and decompose the new (constant size) vertex to two, using input from Merlin. Separating these two vertices to two, we create a “bridge” between the two noop faces; this causes the two noops to merge into a larger noop face, which is the “noop-zone”. The junction of the brick wall is moved to a location inside this noop-zone.

The proof is somewhat technical, but the above “noop-zone” idea is the only non-trivial idea used in it. We will now proceed to the details.

5.8.2 Useful Characteristics of NE triangulations

Let us denote by $l_{max}, l_{min}$ the sizes of the maximal length and minimal length edges in the planar triangulation. Let us denote by $\theta_{min}$ the minimal angle between any two adjacent edges in this embedding. By our assumptions, both the ratio between $l_{max}, l_{min}$ and the value of $\theta_{min}$ are constants bounded away from 0. For a NE CLH(3,3) instance $S$, set

$$c_0 = l_{max} \cdot 2\eta$$

(9)

where $\eta$ is the constant from claim (5.20), i.e., the bound on the distance from any op face to a noop face (distance is measured in terms of number of edges in the dual graph). We note that $c_0$ upper bounds the Euclidean distance between any face and the closest “noop” face to it, since edges in the dual graph (except those connecting the infinite face) are of length at most $2l_{max}$. Here, we define the distance between two faces as the minimal distance of any pair of points contained in each of them.

Claim 5.28 Given is $T$, a NE planar triangulation of a polygon. Consider a rectangle $R$ in the plane. Then the number of vertices of $T$ that lie inside the rectangle $R$ is at most a constant times the area of the rectangle, where the constant depends on $l_{min}$ and $\theta_{min}$. 

46
Proof: Let \( q \) be some vertex of \( T \). Let \( P(q) \) be the polygon induced by the union of faces on \( q \), and \( C(q) \) be the intersection of half-planes generated by the edges of \( P(q) \). Let \( B(q) \subseteq C(q) \) be the largest disk, centered around \( q \) that is contained in \( C(q) \). Since \( B(q) \subseteq C(q) \) then \( B(q) \) contains a single vertex \(- q \) at its center: on the negative, if \( B(q) \) contains another vertex \( p_1 \) then it also intersects an edge \((q, p_1) \) of some face \((q, p_1, p_2) \), and so it intersects both half-planes of the line \((p_1, p_2) \), contrary to definition.  

Also, by definition, there exists some edge \( e \) in a face containing \( q \), such that \( e \) is opposite to \( q \), and \( B(q) \) is tangent to the line containing \( e \). Thus the radius of \( B(q) \) over all \( q \) is at least \( r_{\min} \) where \( r_{\min} = l_{\min} \sin(\theta_{\min}) \). Since every vertex contained in \( R \) must have at least \( 1/4 \)-th of \( B(q) \) contained in \( R \), then the number of vertices in \( R \) is constant.  

Claim 5.29  Let \( A \) and \( B \) be two faces in a NE triangulation \( T_S \), which are at least \( 6l_{\max} \) distance apart. Then there exist two vertices, \( a \in A, b \in B \), such that the following holds. First, there exists a straight line \( \ell \) crossing both \( A \) and \( B \), such that \( a \in A, b \in B \) are the two vertices whose projections on \( \ell \) are the closest among all pairs of points in \( A \) and \( B \). Denote the distance between the projections of those two points by \( |a − b|_\ell \). There exists a path in \( G \) between \( a, b \), denoted by \( P_{ab} \), which is fully contained in a rectangle \( R_{ab} \) of area \( l_0 \times w_0 \), where \( l_0 = |a − b|_\ell \), \( w_0 = 4l_{\max} + 4l_{\max}/\sin(\theta_{\min})l_{\min} \), and this rectangle contains no other vertex of \( A \) or \( B \). Moreover, the number of vertices in the path is a bounded function of \( |a − b|_\ell \) and the NE parameters \( l_{\min}, l_{\max}, \theta_{\min} \).  

Proof: Consider a line which crosses both faces \( A, B \). Consider the projections of the vertices of those faces on this line. If there is more than one pair which is the closest, then perturb \( \ell \) until \( a, b \) are unique and all other projections are further away. Now, define the rectangle \( R_{ab} \) to be the rectangle two of whose edges are parallel to \( \ell \), and are of length which is the same as the distance between the projection of \( a, b \) on the line (see Figure 14);  

![Figure 14: A sketch of \( R_{ab} \) and \( P_{ab} \).](image)

The other two edges of \( R_{ab} \) are perpendicular, and are taken to be \( w_0 \) as defined in the claim. By construction, \( R_{ab} \) contains no other vertices of \( A \) or \( B \).  

We now construct the path. WLOG, let us assume that \( \ell \) is parallel to the \( y \) axis for ease of presentation. Starting from vertex \( a \) we find a neighboring vertex to \( a \) whose \( y \) coordinate is at least \( \cos(\pi/2 − \theta_{\min})l_{\min} = \sin(\theta_{\min})l_{\min} \). We proceed this way until we reach the first vertex \( a' \) whose \( y \) coordinate is larger than \( 2l_{\max} \). This will be the first segment of the path; denote it by \( P_a \). We claim that this path \( P_a \) is fully contained in the rectangle \( R_{ab} \); This is because the path contains at most \( 2l_{\max}/\sin(\theta_{\min})l_{\min} \) vertices, and so in terms of the \( x \)-axis, we are still within the
$2l_{\text{max}}$ distance from the edge of the rectangle; in terms of the $y$ axis, the largest $y$ coordinate is smaller than $3l_{\text{max}}$ by construction. Notice that the final point of $P_a$ is more than $2l_{\text{max}}$ away from the boundary of the rectangle. We generate in a similar way a path from $b, P_b$, that is contained in $R_{ab}$ to a vertex $b'$ with a $y$ coordinate at most that of $b$ minus $2l_{\text{max}}$. We notice that if we connect $a'$ and $b'$ by a straight line $l_{a'b'}$, the line $l_{a'b'}$ are not only contained inside $R_{ab}$ but every point of it is also at least $2l_{\text{max}}$ from the boundary of the rectangle. We now finish the construction of the path by connecting $a', b'$ by a path inside the rectangle, as follows.

Consider all faces crossed by this line. Find a simple path $P_{a'b'} \in T_S$ between $a'$ and $b'$ that uses only edges of these faces - There exists such a path $P_{a'b'}$ because the graph $T_S$ restricted to the faces crossed by $l_{a'b'}$ is a connected graph. Also, by $\text{NE}$ restrictions, the path $P_{a'b'}$ cannot visit vertices whose Euclidean distance from $l_{a'b'}$ is more than $2l_{\text{max}}$. Hence $P_{a'b'}$ is contained in $R_{ab}$.

Connecting the paths $P_a, P_b, P_{a'b'}$ and removing edges as necessary to derive a simple path, we arrive at the desired simple path $P_{ab}$. Since by claim (5.28), $R_{ab}$ contains at most a number of vertices proportional to its area, the length of $P_{ab}$ is a bounded function of $|a - b|_{\ell}$ and the $\text{NE}$ parameters.

5.8.3 Creating the partition

We are now ready for the proof of the claim.

Proof: (of claim 5.27)

Defining the Grid  We are given an instance $S$ of $\text{NE CLH}(3, 3)$ with an interaction graph $G_S$. Let $T_S$ denote the $\text{NE}$ triangulation of the convex polygon of which $G_S$ is a subgraph: $G_S \subseteq T_S$. Let $\alpha = 7c_0 + 2w_0$ where $w_0$ is the constant from claim (5.29). Let us lay down on the plane a brick-wall $B$ with block-size of constant length $10\alpha$. With small perturbations (shifting and rotating) we can make sure that no edge is parallel to the edges of the planar embedding of $T_S$, and no vertex of the brick wall lands on an edge or vertex of $G$ in this embedding. By Claim (5.28) we know that each cell contains a constant number of vertices of $T_S$, and thus of $G_S$. We note that interactions that connect vertices from different cells are obviously two-local (in terms of number of cells participating in one such interaction) as long as they are far away from the junctions of the brick-wall. Unfortunately, we can still have three-local interactions among those close to the brick-wall junctions. We thus make small modifications of the interaction graph close to every brick-wall junction; we will make sure that those modifications are confined to small disks around those junctions, which do not intersect each other. From now on we focus our attention on one such junction, and explain how to modify the interactions close to it to eliminate the 3-local interactions.

Creating a “noop-zone” near a brick-wall junction  For each face $F_i \in G_S$ that contains the $i$-th junction of $\mathcal{B}$, the point $p_i$, let us denote by $N_i \in T_S$ the closest noop operator to the junction.

Given $N_i$, choose a noop $N'_i$ within Euclidean distance $\in [c_0, 6c_0]$ to it (Euclidean distance between faces is the minimum such distance between any two points in those faces). $N'_i$ exists since we can consider a face whose Euclidean distance to $N_i$ is $\in [3c_0, 4c_0]$, (if no such face exists, the problem is of constant size anyway) and apply Claim (5.20) to find a noop close to this face
within Euclidean distance $c_0$. Since the distance between $N_i$ and $N_i'$ is at least $c_0 > 6l_{\text{max}}$, Claim (5.29) holds, and we can find $a \in N_i, b \in N_i'$, and $R_{ab}$ be a rectangle containing a path $P_{ab}$, as is guaranteed by the claim.

Consider the half-infinite line starting from $a$, and going to infinity in a direction normal to the edge of the rectangle containing $a$. Consider also the half-infinite line starting from $b$ and going to infinity in the parallel direction, normal to the edge of the rectangle containing $b$. These half-infinite lines, augmented with the path $P_{ab}$, partition the plane to two infinite regions, one on each side of $P_{ab}$; we call the sides arbitrarily “left” and “right” (these names may be completely unrelated to true “left” and “right”.) Denote $O_{\text{left}}$ as the set of faces containing at least one vertex of $P_{ab}$, possibly vertices on the left of $P_{ab}$, but no vertex to its right. Denote $O_{\text{right}}$ to be the set of all other faces containing vertices of $P_{ab}$ (by definition, those triangles contain at least one vertex to the right of $P_{ab}$). By the planarity of $T_S$ we have that $O_r \in O_{\text{right}}$ cannot contain any “left” vertex: a face $O_r$ supported on both sides of the divide must have an edge that crosses at least one edge of the path $P_{ab}$ or one of the edges of $N_i$ or $N_i'$. Thus any two operators, one from $O_{\text{right}}$ and one from $O_{\text{left}}$ can only share vertices of $P_{ab}$.

We would now like to modify the interaction graph, and “merge” all vertices along the path $P_{ab}$, to one vertex, whose Hilbert space is the tensor product of all those particles, creating a large though constant dimensional new particle, such that $O_{\text{left}}$ and $O_{\text{right}}$ intersects only through this single particle. This is done as follows. First, we erase all nodes on $P_{ab}$ except $a$. A new large particle, of the dimensionality which is the tensor of all particles in $P_{ab}$, is embedded where previously $a$ was embedded on the plane. For any vertex $v$ that was merged into $a$, and $v'$ a vertex that was previously connected to $v$ by an edge, replace the original edge $(v, v')$ by a straight edge connecting $v'$ to $a$. Also, replace each of the interactions that contained a particle on the path, by the corresponding interaction on the appropriate subparticle in the new merged particle $a$. The remaining particles in that interaction are not changed. We note that due to the change of geometry, some edges may now cross others, but this non-planarity is confined only to an area close to the path, and as we shall see, will not matter to our argument.

The partition of the operators on the vertices of $P_{ab}$ into the sets $O_{\text{left}}$ and $O_{\text{right}}$ now induces a partition of all the operators on $a$, which only intersect on $a$. Hence, the operators on the merged qudit $a$ can be separated, following Claim (3.2), into two qudits, $a'$ and $a''$, using an isometry sent by Merlin. Note that this isometry acts on a constant dimensional particle $a$, since the length of $P_{ab}$ is constant following claim (5.29). We embed $a'$ and $a''$ at some tiny nonzero distance, each to one side of the previous location of $a$, and slightly modify all edges connected to $a$ to fit this small perturbation, keeping them straight for simplicity.

We have arrived at an equivalent instance, except now the two noop faces $N_i$ and $N_i'$ are connected into one region; we denote the unified noop face as $\bar{N}_i$ and call it the “noop-zone”. We note that if initially, either the left or the right side (or both) of the operators on $a$ contains no operator - then we can trivially unite $N_i$ and $N_i'$ into $\bar{N}_i$, by removing edges of $T_S$, and no merging of particles is required.

Let $a_i$ be the middle point on the face not containing $a$ of $N_i$, and $b_i$ the same for $N_i'$ respectively. Since we have chosen the distance between $N_i$ and $N_i'$ to be at least $c_0$, and since the edges that

---

3 we need to add a length of an edge to account for the fact that those distances are taken from different points in the intermediate face, which is why we take loose bounds
contain \( a_i \) and \( b_i \) are not changed during the merging process, then

\[
|a_i - b_i| > 2l_{\text{max}}.
\]

(10)

We denote \( l_i \) as the straight line connecting \( a_i \) and \( b_i \).

Figure 15: An example of a merging step: Figure (a) is the embedding prior to merging - the path \( P_{ab} \) (dashed line) induces a left-right separation of the plane, such that any operator on vertices of the path is supported on only one of these divisions. Figure (b) is the embedding after merging - note that not all faces are drawn for clarity reasons. Note also that some edges cross others - like the edges of \( H_5 \) and \( H_2 \). Figure (c) is after application of the separating isometry on vertex \( a \). It is separated into two vertices, and a large noop one is generated.

We explain now that the graph has not changed except for in a constant size disk around the noops. Let \( B_\alpha(p) \) denote a disk around a point \( p \) with radius \( \alpha \). Since for all \( i \) \( N_i \) is within distance \( c_0 \) from \( F_i \) by (5.20), and since by Claim (5.29) the merging described above of the vertices on the path between \( N_i \) and \( N_i' \), affect edges which are confined to a radius \( \max \{w_0, l_0\} + 2l_{\text{max}} < 6c_0 + 2w_0 \) around \( N_i \), then all edges changed by this merging are contained within \( B_\alpha(p_i) \) for all \( i \), so they are decoupled from each other and the disks do not intersect.
**Rerouting the grid to prevent 3-locality**  Let us reroute the junction so that the “horizontal” line of the T junction when restricted to $B_{a}(p_i)$ is tilted to the straight line going through $b_i$ that is perpendicular to $l_i$. The “vertical” line of T is tilted so that when restricted to $B_{a}(p_i)$ it is the line starting from $b_i$, going to a point in the middle of the short line connecting $a_i$ and $a'$, and then connecting to $a_i$ by a straight line, from which we continue upwards in the same direction as $l_i$, as in Figure 16. Outside of the disk, these lines are connected directly to the edges of the brick-wall. This can be easily achieved using the detours as in Figure 16. Once again, we make sure using small rotations and shifts of the lines that none of these lines pass through any vertex.

![Figure 16: Rerouting the divisions of $B$ to pass through “noop” faces: the dotted lines denote the rerouting of the division so that the intersection $p_i$ is located at a noop $N_i$.](image)

Finally, we show that no local term can act on three-bricks: Let $O$ be an operator in $O_{left}$, then it does not act on the brick to the right of the partition line between $a_i$ and $b_i$. Hence it is at most 2-local. A similar claim holds for any operator in $O_{right}$. Any other operator has edges of length at most $l_{max}$ that cannot cross $N_i$, since it was not modified by the merging process. Since we have that $|a_i - b_i| > 2l_{max}$ by Equation 10, no such operator can act on the three bricks.

6 Tight conditions on Topological Order

Let us now describe more formally the implications of our results to Topological order. We define a Topological Order system as follows:

**Definition 6.1** Topological Order

A quantum state $|\psi_1\rangle$ is said to exhibit Topological Order w.r.t. a lattice $L$, (with one particle on each of its edges) if there exists another state $|\psi_2\rangle$ orthogonal to $|\psi_1\rangle$ with the following properties, for any locally confined observable $O$ on $L$:

1. $\langle \psi_1 | O | \psi_2 \rangle = 0$
2. $\langle \psi_1 | O | \psi_1 \rangle = \langle \psi_2 | O | \psi_2 \rangle$

In this paper, we will say that a quantum system exhibits TO if all states in its groundspace do.
Note that this definition can also be extended to NE graphs - since the notion of locality makes sense in those cases too.

It is shown in [6] that Topological order states cannot be generated by small depth nearest neighbor circuits:

**Theorem 6.2** Let \( U = U_r \cdots U_1 \) be a quantum circuit which is a product of local unitaries \( U_i \) that generates a TO state \( |\psi\rangle \) from the all-zero state, i.e. \( |\psi_1\rangle = U|0\rangle^\otimes n \) on a 2D lattice \( L \) of \( n \) qubits. Then \( r = \Omega(\sqrt{n}) \).

In order to conclude that there exist NE commuting Hamiltonians with \( d \geq 4 \) for which no constant-depth diagonalizing circuit exists, we recall the Toric Code due to Kitaev [12]. The Toric Code is comprised of a finite square 2-dimensional lattice of qubits (edges), such that the top and bottom rows are identified as one, and so are the left and right columns. The system is stabilized by two types of 4-local operators, one which acts on a "plaquette" of 4 edges of the unit square of the grid (denoted as the set \( P \)), and one which acts on a "vertex" of 4 edges at each crossing at the grid (denote as the set \( V \)). The "vertex" operators are \( A = X^\otimes 4 \) and the "plaquette" operators are \( B = Z^\otimes 4 \). The complete Hamiltonian is given by

\[
H = -\sum_{v \in V} A(v) - \sum_{p \in P} B(p).
\]

Let us now consider grouping together the qubits of the Toric Code lattice, by grouping together the two top-right qubits (edges) of every "plaquette".

Figure 17: The figure shows the structure of the Toric Code and the pairing of qubit edges (using the dashed circles) to arrive at a \( CLH(3, 4) \) instance.

It can be easily seen that, up to a global energy-level shift, we arrive at an instance of \( CLH(3, 4) \). Thus, the Toric Code is a special case of \( CLH(3, 4) \). Since all states in the Toric Code groundspace are of Topological Order ([12]) no state in the code can be generated by any constant-depth circuit; Hence, in this case there cannot be a constant-depth nearest-neighbor quantum circuit which diagonalizes the Toric code Hamiltonian, and our conclusion follows.

This conclusion settles an intriguing gap regarding commuting Hamiltonian systems on lattices (more generally, on NE graphs). On the one hand for \( CLH(k, d) \) with \( k \geq 3 \) and \( d \geq 4 \), or \( k > 3 \) no constant-depth diagonalization exists, and Topological order is possible, whereas for
$CLH(k,d)$ with $k < 3$ for all $d$, or $k = 3$ and $d < 4$ constant-depth diagonalization exists, so Topological Order is impossible. Thus, the Toric Code construction is optimal in terms of commuting Hamiltonian dimensionality and size of interactions. This implies Theorem 1.3

A clarifying remark is in place. Theorems 1.1 and 1.2 hold for planar instances, whereas the Toric codes we have just mentioned are defined on a Torus. How can we compare the two? We recall that the Toric codes can actually be generalized to a lattice with a boundary embedded on the plane, as was shown by Bravyi and Kitaev [7], and so this proves the tight boundary provided by Theorem 1.3 between TO and local entanglement, for NE planar instances. In fact, Theorems 1.1 and 1.2 can be extended to handle higher genus surfaces, as long as the graphs are "locally NE planar", i.e., large disks around any vertex are NE planar. From the other side, the proof of Theorem 6.2 holds for such instances too, since locally NE planar embeddings support the Lieb-Robinson bound used in the proof of Theorem 6.2. Hence our implications regarding the tight conditions on TO hold for higher genus surfaces too; we will not provide here the details of this generalization.

7 Acknowledgements

We would like to thank Sergey Bravyi for several important contributions to this paper. First, for pointing us to that our early results on the complexity of the commuting Hamiltonian problem, may have interesting implications on the conditions for the existence of topological order. Second, for showing us the beautiful argument that any circuit generating the Toric Code groundstate is of non-constant depth, and finally for pointing to us to a mistake in an early draft of this paper. We also thank Zeph Landau for useful remarks on an early draft of this paper.

References

[1] D. Aharonov, D. Gottesman, S. Irani, J. Kempe. The power of quantum systems on a line Comm. Math. Physics, vol. 287, no. 1, pp. 41-65 (2009)

[2] D. Aharonov, I. Arad, Z. Landau, U. Vazirani The Detectability Lemma and Quantum Gap Amplification quant-ph/0811.3412, 2008.

[3] M. Aguado, G. Vidal. Entanglement Renormalization and Topological Order Phys. Rev. Lett. 100, 070404 (2008).

[4] S. Bravyi. Efficient algorithm for a quantum analogue of 2-SAT quant-ph/0602108, 2006.

[5] S. Bravyi, M. B. Hastings. A short proof of stability of topological order under local perturbations arXiv:1001.4363v1, 2010.

[6] S. Bravyi, M. B. Hastings, F. Verstraete. Lieb-Robinson Bounds and the Generation of Correlations and Topological Quantum Order Physical Review Letters 97.050401, 2006.

[7] S. Bravyi, A. Kitaev Quantum codes on a lattice with boundary quant-ph/9811052, 1998.
[8] S. Bravyi, M. Vyalyi. Commutative version of the k-local Hamiltonian problem and common eigenspace problem quant-ph/0308021, 2004.

[9] M. B. Hastings. An area law for one-dimensional quantum systems J. Stat. Mech. (2007) P08024

[10] M. B. Hastings. Lieb-Schultz-mattis in higher dimensions Phys. Rev. B, vol. 69, p. 104431, Mar 2004.

[11] J. Kempe, A. Kitaev, O. Regev. The Complexity of the Local Hamiltonian Problem SIAM Journal of Computing, Vol. 35(5), p. 1070-1097 (2006), conference version in Proc. 24th FSTTCS, p. 372-383 (2004)

[12] A. Kitaev. Fault-tolerant quantum computation by anyons Annals Phys. 303 (2003) 2-30

[13] A. Yu. Kitaev, A. H. Shen, M. N. Vyalyi. Classical and Quantum Computation (Graduate Studies in Mathematics) American Mathematical Society, (2002)

[14] R. Oliveira, B. M. Terhal. The complexity of quantum spin systems on a two-dimensional square lattice Quant. Inf, Comp. Vol. 8, No. 10, pp. 0900-0924 (2008)

[15] B. M. Terhal, D. P. DiVincenzo. Adaptive Quantum Computation, Constant Depth Quantum Circuits and Arthur-Merlin Games arXiv:quant-ph/0205133v6

[16] G. Vidal. Entanglement Renormalization Phys. Rev. Lett. 99, 220405 (2007).