Equivariant knots and knot Floer homology

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Abstract
We define several equivariant concordance invariants using knot Floer homology. We show that our invariants provide a lower bound for the equivariant slice genus and use this to give a family of strongly invertible slice knots whose equivariant slice genus grows arbitrarily large, answering a question of Boyle and Issa. We also apply our formalism to several seemingly nonequivariant questions. In particular, we show that knot Floer homology can be used to detect exotic pairs of slice disks, recovering an example due to Hayden, and extend a result due to Miller and Powell regarding stabilization distance. Our formalism suggests a possible route toward establishing the noncommutativity of the equivariant concordance group.

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1 \quad INTRODUCTION

Equivariant knots and concordance have been well-studied historically; see, for example, \cite{5,10,28,29,33}. Recently, there has been a renewed interest in this topic from the viewpoint of more modern invariants, as evidenced by the works of Watson \cite{36}, Lobb–Watson \cite{24}, and Boyle–Issa \cite{3}. The aim of the present article is to investigate the theory of equivariant knots through the lens of knot Floer homology, an extensive package of invariants introduced independently by Ozsváth–Szabó \cite{30} and Rasmussen \cite{32}. Our underlying approach is straightforward: given a strongly invertible knot $(K, \tau)$, we show that $\tau$ induces an appropriately well-defined automorphism of the knot Floer complex $\mathcal{CFK}(K)$. Using the induced action of $\tau$, we construct the following suite of numerical invariants:

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**Theorem 1.1.** Let \((K, \tau)\) be a strongly invertible knot in \(S^3\). Associated to \((K, \tau)\), we have four integer-valued equivariant concordance invariants

\[
\overline{V}_0^\circ(K) \leq V_0^\circ(K) \quad \text{and} \quad \overline{V}_0^{\iota\tau}(K) \leq V_0^{\iota\tau}(K).
\]

In fact, \(\overline{V}_0^\circ\) and \(V_0^\circ\) (where \(\circ \in \{\tau, \iota\}\)) are invariant under the more general relation of isotopy-equivariant homology concordance.

Note that \(\overline{V}_0^\circ\) and \(V_0^\circ\) vanish if \(K\) is equivariantly slice. See Definition 2.9 for the definition of isotopy-equivariant homology concordance.

Obstructions to equivariant sliceness have been investigated by several authors, including Sakuma [33], Cha–Ko [5], and Naik–Davis [10]. However, understanding the equivariant slice genus has only more recently been studied by Boyle–Issa [3]. One of the main results of this paper will be to show that \(\overline{V}_0^\circ\) and \(V_0^\circ\) provide lower bounds for the equivariant slice genus \(\bar{g}_4(K)\) of \((K, \tau)\). In fact, we show that they bound the isotopy-equivariant slice genus; see Definition 2.8. Using this, we provide a family of strongly invertible slice knots \((K_n, \tau_n)\) whose equivariant slice genus grows arbitrarily large, answering a question posed by Boyle–Issa. Prior to the current article, there were no known examples of strongly invertible knots with \(\bar{g}_4(K) - g_4(K)\) provably greater than one.

Surprisingly, our invariants also have applications to several (seemingly) nonequivariant questions. We first show that our formalism can be used to detect exotic pairs of slice disks, recovering an example originally due to Hayden [13]. Note that while knot Floer homology has previously been used to detect exotic higher genus surfaces (see the work of Juhász–Miller–Zemke [19]), the current work represents the first such application of knot Floer homology in the genus-zero case. We also consider the question of bounding the stabilization distance between pairs of disks. Using the work of Juhász–Zemke [21], we show that our examples \(K_n\) give a Floer-theoretic re-proof and extension of a result by Miller–Powell [26], which states that for each integer \(m\), there is a knot \(J_m\) with a pair of slice disks that require at least \(m\) stabilizations to become isotopic.

The invariants of Theorem 1.1 are correction terms derived from the action of \(\tau\) on \(CFK(K)\) following the general algebraic program of Hendricks–Manolescu [16]. Instead of working with numerical invariants, it is also possible to define a local equivalence group in the style of Hendricks–Manolescu–Zemke [17] or Zemke [39]. This follows the approach taken in Dai–Hedden–Mallick in [7] to study cork involutions; and, indeed, the current article is closely related to [7]. In this paper, we define the local equivalence group \(\mathcal{R}_{\tau, d}\) of \((\tau_{K, \iota K})\)-complexes and show that there is a homomorphism from the equivariant concordance group \(\overline{C}\) (defined by Sakuma in [33]) to \(\mathcal{R}_{\tau, d}\):

\[
h_{\tau, d} : \overline{C} \to \mathcal{R}_{\tau, d}.
\]

Interestingly, it turns out that \(\mathcal{R}_{\tau, d}\) is not a priori abelian. It is an open problem whether \(\overline{C}\) is abelian; in principle, our invariants can thus be used to provide a negative answer to this question.† As far as the authors are aware, this is the first example of a (possibly) non-abelian group arising in the setting of local equivalence. See Section 2 for background and further discussion.

Although all of the examples in this paper will be strongly invertible, we also establish several analogous results for 2-periodic knots. We discuss these in Section 8.

† Recently, Di Prisa has shown that \(\overline{C}\) is indeed non-abelian [11]; see Remark 1.13.
1.1 Equivariant slice genus bounds

Our first application will be to show that the invariants of Theorem 1.1 bound the equivariant slice genus $\bar{g}_4(K)$ of $K$ (see Definition 2.1). In fact, we give a bound for a rather more general quantity, defined as follows.

Let $(K, \tau)$ be a strongly invertible knot. Let $W$ be a (smooth) homology ball with boundary $S^3$, and consider any (smooth) self-diffeomorphism $\tau_W$ on $W$ that restricts to $\tau$ on $\partial W$. Note that we do not require $\tau_W$ itself to be an involution. We say that a slice surface $\Sigma$ in $W$ with $\partial \Sigma = K$ is an isotopy-equivariant slice surface (for the given data) if $\tau_W(\Sigma)$ is isotopic to $\Sigma$ rel $K$. Define the isotopy-equivariant slice genus of $(K, \tau)$ by:

$$\tilde{g}_4(K) = \min_{\text{all possible choices of } W \text{ and } \tau_W} \{g(\Sigma)\}.$$  

Here $\tilde{g}_4(K)$ depends on $\tau$, but we suppress this from the notation. The quantity $\tilde{g}_4(K)$ generalizes the obvious notion of equivariant slice genus in several ways. First, we allow ourselves to consider any homology ball $W$ and any diffeomorphism that extends $\tau$, rather than restricting ourselves to $B^4$. Second, we do not require that $\Sigma$ be invariant under the extension of $\tau$, but instead only isotopic to its image. Obviously,

$$\tilde{g}_4(K) \leq \bar{g}_4(K).$$

Although the authors do not have an example in which $\tilde{g}_4(K)$ is distinct from $\bar{g}_4(K)$, this more general quantity will turn out to be critical for several applications. There is also an obvious accompanying notion of isotopy-equivariant homology concordance; see Definition 2.9.

Although the notion of isotopy equivariance may initially seem rather contrived, a slight shift in perspective demonstrates its usefulness. To see this explicitly, let $(K, \tau)$ be a strongly invertible knot in $S^3$. Let $W$ be any (smooth) homology ball with boundary $S^3$ and $\tau_W$ be any extension of $\tau$ over $W$. If $\Sigma \subset W$ is any slice surface for $K$ with $g(\Sigma) < \tilde{g}_4(K)$, then we may immediately conclude that the two surfaces $\Sigma$ and $\tau_W(\Sigma)$ are not isotopic rel $K$. The calculation of $\tilde{g}_4(K)$ thus provides an easy method for generating nonisotopic slice surfaces in the presence of a symmetry on $K$. For example, if $K$ is an equivariant slice knot with $\tilde{g}_4(K) > 0$, then we may take any slice disk $\Sigma$ for $K$ and form its image under any extension $\tau_W$ of $\tau$ (in any homology ball $W$); the resulting pair of slice disks are then automatically nonisotopic rel $K$. We often refer to $\Sigma$ and $\tau_W(\Sigma)$ as a symmetric pair of slice disks. This is in marked contrast to the usual approach taken in the literature, where in order to deploy various invariants, one (naturally) has in mind a specific family of slice disks (or surfaces) that are conjectured to be nonisotopic. The situation here is analogous to the notion of a strong cork introduced by Lin–Ruberman–Saveliev in [23] and studied in [7].

Following the work of Juhász–Zemke [22], we bound $\tilde{g}_4(K)$ in terms of $V_0^\circ$ and $V_\circ^\circ$:

**Theorem 1.2.** Let $(K, \tau)$ be a strongly invertible knot in $S^3$. Then for $\circ \in \{\tau, \tau\tau\}$,

$$\left[1 + \frac{\tilde{g}_4(K)}{2}\right] \leq V_0^\circ(K) \leq V_\circ^\circ(K) \leq \left[1 + \frac{\tilde{g}_4(K)}{2}\right].$$

The computation of $V_0^\circ(K)$ and $V_\circ^\circ(K)$ can thus be used to help construct exotic pairs of slice surfaces for $K$, via the discussion above. In the current paper, we only give the most archetypal
instance of this phenomenon; the authors plan to return to the task of finding a systematic range of examples in future work. Note that by Theorem 1.1, if \( \tilde{g}_4(K) = 0 \) then \( V_0 \) and \( V_0^\circ \) vanish. In the genus-zero case, Theorem 1.1 thus gives a slightly stronger bound than that of Theorem 1.2. This discrepancy is explained in Remark 5.1.

### 1.2 Applications

We now give several computations and applications. Our main class of examples is quite straightforward. Let \( T_{2n,2n+1} \) be the right-handed torus knot and select any strong inversion \( \tau \) on \( T_{2n,2n+1} \) (in fact, this is unique up to conjugation by [33, Proposition 3.1]). As in Figure 1, there are two obvious strong inversions on \( T_{2n,2n+1} \). On one hand, we may take the equivariant connected sum \( \tau# = \tau# \tau \) to obtain an inversion with one fixed point on each summand. On the other, we may consider the strong inversion \( \tau_{sw} \) that interchanges the two factors. Strictly speaking, the latter is a strong inversion on \( T_{2n,2n+1} \); however, as \( T_{2n,2n+1} \) admits an orientation-reversing symmetry, we will occasionally conflate this with \( T_{2n,2n+1} \). We then consider the further equivariant connected sum

\[
K_n = (T_{2n,2n+1} \# T_{2n,2n+1}) \# - (T_{2n,2n+1} \# T_{2n,2n+1})
\]

equipped with the strong inversion

\[
\tau_n = \tau# \# - \tau_{sw}.
\]
FIGURE 2  An equivariant slice knot $J$ with symmetry $\tau$ given by reflection across the obvious vertical axis. The slice disks $D$ and $D'$ are obtained by compressing along the red and blue curves, respectively.

That is, we consider the strong inversion $\tau_{\#}$ on the first copy of $T_{2n,2n+1}#T_{2n,2n+1}$ and take the equivariant connected sum of this with the (orientation-reversed mirror of the) inversion $\tau_{\text{sw}}$ on $T_{2n,2n+1}#T_{2n,2n+1}$. For a discussion of the equivariant connected sum of two strong inversions, see Subsection 2.1. In general, defining the equivariant connected sum requires some additional data, but the application we have in mind will be insensitive to this subtlety; see Remark 6.15. In Figure 1, we perform the equivariant connected sum by (roughly speaking) stacking successive axes end-to-end. Note that $K_n$ is slice.

In Section 6, we establish the following fundamental calculation:

**Theorem 1.3.** For $n$ odd, the pair $(K_n, \tau_n)$ has $V_{\tau_n}^\tau(K_n) \geq n$.

Similar knots were investigated by Hendricks–Hom–Stoffregen–Zemke in [15] and the proof of Theorem 1.3 relies on the computations of [15]. In fact, we also establish that $\bar{V}_{\tau}^\tau(K_n) > 0$, although this is of limited use, and conjecture that the inequality appearing in Theorem 1.3 is an equality. However, as we do not need this for any application, we leave the more detailed computation to the reader.

In [3, Question 1.1], Boyle–Issa asked whether there exists a family of strongly invertible knots for which $\bar{g}_4(K) - g_4(K)$ becomes arbitrarily large. Applying Theorem 1.2, we immediately obtain:

**Theorem 1.4.** For $n$ odd, the pair $(K_n, \tau_n)$ has

$$2n - 2 \leq \bar{g}_4(K_n) \leq \bar{g}_4(K_n).$$

As each $K_n$ is slice, this answers [3, Question 1.1] in the affirmative. The topological intuition behind these examples is quite straightforward: the involutions $\tau_{\#}$ and $\tau_{\text{sw}}$ on $T_{2n,2n+1}#T_{2n,2n+1}$ are very different, so one should expect the equivariant slice genus of $(K_n, \tau_n)$ to be large.

We also consider a particular knot $J$ due to Hayden [13], displayed in Figure 2. In [13], Hayden presents a certain pair of slice disks $D$ and $D'$ for $J$, each with complement having fundamental group $\mathbb{Z}$. By a result of Conway–Powell [6, Theorem 1.2], this implies that $D$ and $D'$ are topolog-
ially isotopic. However, in [13, section 2.1], it is shown that $D$ and $D'$ are not smoothly isotopic (or even diffeomorphic) rel boundary. (See also [18, Theorem 3.2].) Note that $J$ admits a strong inversion $\tau$; a crucial part of the argument in [13] relies on the fact that $D$ and $D'$ are related by the obvious extension of $\tau$ over $B^4$.

In [13, section 2.1], it is noted that $J$ has a close connection to the positron cork $W_0$ of Akbulut–Matveyev [2]. In [7, Theorem 1.15], the action of the cork involution on the Heegaard Floer homology of $\partial W_0$ was investigated. Re-casting these computations in the formalism of the current paper yields:

**Theorem 1.5.** Let $J$ be as in Figure 2. Then $\tilde{\iota}_{g_A}(J) > 0$. In particular, no pair of symmetric slice disks $\Sigma$ and $\tau_W(\Sigma)$ are (smoothly) isotopic rel $J$. This holds for any (smooth) homology ball $W$ with $\partial W = S^3$ and any extension $\tau_W$ of $\tau$ over $W$.

Given the connection between $J$ and $W_0$, it is actually possible to use [7, Theorem 1.15] to provide an immediate proof of Theorem 1.5, as we explain in Remark 7.8. However, going through the proof in the current context explicitly gives:

**Theorem 1.6.** Let $J$ be as in Figure 2 and let $\Sigma$ and $\tau_W(\Sigma)$ be any pair of symmetric slice disks for $J$. Then

$$[F_{W,\Sigma}(1)] \neq [F_{W,\tau_W(\Sigma)}(1)]$$

as elements in either $H_*(CFK(J))$ or $HF^*(J)$. This holds for any (smooth) homology ball $W$ with $\partial W = S^3$ and any extension $\tau_W$ of $\tau$ over $W$.

Here, $F_{W,\Sigma}$ and $F_{W,\tau_W(\Sigma)}$ are the knot Floer cobordism maps associated to (punctured copies of) $\Sigma$ and $\tau_W(\Sigma)$, respectively. We thus explicitly see that $\Sigma$ and $\tau_W(\Sigma)$ are distinguished by their maps on knot Floer homology. Specializing to $\Sigma = D$, this provides a knot Floer-theoretic analogue of the proof of [18, Theorem 3.2], in which $D$ and $D'$ are distinguished using their induced maps on Khovanov homology. Note that Juhász–Miller–Zemke have used knot Floer homology to detect exotic higher genus surfaces [19]. (The fact that the surfaces have genus greater than zero is essential to their argument.) However, the current work represents the first instance of knot Floer homology being applied to detect an exotic pair of disks.

By taking the $n$-fold connected sum $\#_n J$, it is also straightforward to construct an example of a slice knot with $2^n$ different exotic slice disks, which are distinguished by their concordance maps on $HF^*$. We establish this in Theorem 7.10; see [34, Corollary 6.6] for a similar construction. In Theorem 7.11, we extend Theorem 1.5 to an infinite family of knots with exotic pairs of slice disks, which were likewise considered by Hayden in [13].

### 1.3 Algebraic formalism

As discussed previously, our underlying goal will be to show that a strong inversion $\tau$ induces a well-defined action on the knot Floer complex of $K$. We also incorporate the involutive knot Floer automorphism of Hendricks–Manolescu [16] into our formalism, which will allow us to define the invariants $V^\tau_0$ and $\nu^\tau_0$. To construct the action of $\tau$, we first fix an orientation on $K$ and an
ordered pair of basepoints \((w, z)\) that are interchanged by \(\tau\). We refer to these data as a *decoration* on \((K, \tau)\). In Subsection 3.2, we define the action of \(\tau\) associated to a decorated strongly invertible knot:

**Theorem 1.7.** Let \((K, \tau)\) be a decorated strongly invertible knot. Let \(H\) be any choice of Heegaard data compatible with \((K, w, z)\). Then \(\tau\) induces an automorphism

\[
\tau_H : CFK(H) \to CFK(H)
\]

with the following properties:

1. \(\tau_H\) is skew-graded and \(\mathbb{F}[\mathcal{W}, \mathcal{Y}]\)-skew-equivariant;
2. \(\tau_H^2 \simeq \text{id}\);
3. \(\tau_H \circ \iota_H \simeq \zeta_H \circ \iota_H \circ \tau_H\).

Here, \(\iota_H\) is the Hendricks–Manolescu knot Floer involution on \(CFK(H)\) and \(\zeta_H\) is the Sarkar map. Moreover, the homotopy type of the triple \((CFK(H), \tau_H, \iota_H)\) is independent of the choice of Heegaard data \(H\) for the doubly based knot \((K, w, z)\).

This action was originally considered by the second author in the context of establishing a large surgery formula; see the forthcoming paper. Note that \(\tau_H\) and \(\iota_H\) do not in general commute. This is in contrast to the 3-manifold setting; see [7, Lemma 4.4].

In view of the last part of Theorem 1.7, we may suppress writing \(H\) and unambiguously refer to the homotopy type of \((CFK(K), \tau_K, \iota_K)\) as an invariant of the decorated knot \((K, \tau)\). In Subsection 2.2, we formalize these algebraic data by defining the notion of an abstract \((\tau_K, \iota_K)\)-complex. We define an appropriate notion of local equivalence and form the quotient

\[
\mathfrak{R}_{\tau, \iota} = \{\text{abstract \((\tau_K, \iota_K)\)-complexes}\} \slash \text{local equivalence}.
\]

See Subsection 2.2.

The role of the decoration on \((K, \tau)\) turns out to be quite subtle. As we will see, this extra choice of data is needed to form the knot Floer complex of \(K\) and is critical for discussing the invariance properties of \((CFK(K), \tau_K, \iota_K)\). In Subsection 3.4, we introduce the notion of a *decorated* isotopy-equivariant homology concordance and show that in the decorated category, we obtain a map

\[
h_{\tau, \iota} : \{(\text{decorated) strongly invertible knots}\} \slash \text{homology concordance} \to \mathfrak{R}_{\tau, \iota}.
\]

However, this is (in principle) not quite true if the decorations are discarded: in the undecorated setting, an equivariant knot only defines a \((\tau_K, \iota_K)\)-complex up to a certain ambiguity that we refer to as a *twist by \(\zeta_K)\); see Definition 2.20. Nevertheless, we show that \(V_0^\circ\) and \(V_0^\circ\) remain invariants in the undecorated setting.

In Subsection 2.2, we further define a product operation on \(\mathfrak{R}_{\tau, \iota}\) that makes it into a group. We establish an equivariant connected sum formula in Theorem 4.1; this will allow us to prove that \(h_{\tau, \iota}\) constitutes a homomorphism from the equivariant concordance group \(\tilde{C}\) to \(\mathfrak{R}_{\tau, \iota}\).
Theorem 1.8. We have a homomorphism

\[ h_{\tau,\iota} : \tilde{C} \to \mathfrak{K}_{\tau,\iota}. \]

The equivariant concordance group \( \tilde{C} \) consists of the set of directed strongly invertible knots; see Definition 2.5. In Subsection 3.5, we discuss the connection between \( \tilde{C} \) and decorated isotopy-equivariant concordance.

Somewhat surprisingly, it turns out that \( \mathfrak{K}_{\tau,\iota} \) is not \textit{a priori} abelian, although the authors have no explicit example of this. As we discuss in Subsection 2.1, it is currently unknown whether \( \tilde{C} \) is abelian. Hence, in principle Theorem 1.8 can be used to provide examples demonstrating this claim; we plan to return to this question in future work. As far as the authors are aware, this is the first example of a (possibly) non-abelian group arising in the setting of local equivalence. Note that the \( \iota_K \)-local equivalence group of Zemke [39] is abelian.

1.4 Relation to 3-manifold invariants

If \( K \) is an equivariant knot, then any 3-manifold obtained by surgery on \( K \) inherits an involution from the symmetry on \( K \) (see, for example, [7, Lemma 5.2]). In the forthcoming paper, the second author established a large surgery formula relating the action of \( \tau_K \) to the corresponding Heegaard Floer action of the 3-manifold involution. This latter action was defined and studied in [7] in the context of the theory of corks. It follows immediately from the large surgery formula that (with appropriate normalization) the invariants \( \widetilde{V}_0^o \) and \( \underline{V}_0^o \) are none other than the numerical involutive correction terms referenced in [7, Remark 4.5]. Explicitly, for \( p \geq g_3(K) \), we have

\[ -2\widetilde{V}_0^o(K) + \frac{p-1}{4} = \bar{d}_s(S^3_p(K),[0]) \]

\[ -2\underline{V}_0^o(K) + \frac{p-1}{4} = d_s(S^3_p(K),[0]) \]  

(1)

for \( o \in \{\tau, \pi\} \). See [16, Theorem 1.6] for the analogous statements concerning the usual involutive numerical invariants \( \bar{d} \) and \( d \).

The results of [7] easily imply that \( \widetilde{V}_0^o \) and \( \underline{V}_0^o \) are invariant under equivariant concordance (essentially by surgering along the concordance annulus). Hence, it is actually immediate that \( \widetilde{V}_0^o \) and \( \underline{V}_0^o \) obstruct equivariant sliceness. Indeed, [7] already gives several examples of slice knots that are not equivalently slice, as pointed out in [3]. The main import of the present paper is thus to show that \( \widetilde{V}_0^o \) and \( \underline{V}_0^o \) can be used to study higher genus examples, which were not previously accessible.

In [7, Theorem 1.5], it was shown that the invariants of [7, Remark 4.5] satisfy certain inequalities in the presence of negative-definite equivariant cobordisms. In our context, this specializes to inequalities of \( \widetilde{V}_0^o \) and \( \underline{V}_0^o \) involving equivariant crossing changes. In Section 7, we consider several kinds of equivariant crossing changes. We prove:

Theorem 1.9. Let \( K \) be strongly invertible knot. Let \( K' \) be obtained from \( K \) via an equivariant positive-to-negative crossing change (or an equivariant pair of such crossing changes). Then,
if the crossing change is of Type Ia, we have
\[ \bar{V}_0^\tau(K) \geq \bar{V}_0^\tau(K') \text{ and } V_0^\tau(K) \geq V_0^\tau(K'); \]

(2) if the crossing change is of Type Ib, we have
\[ \bar{V}_0^{\tau\tau}(K) \geq \bar{V}_0^{\tau\tau}(K') \text{ and } V_0^{\tau\tau}(K) \geq V_0^{\tau\tau}(K'); \]

(3) if we have an equivariant pair of crossing changes (Type II), we have both
\[ \bar{V}_0^\tau(K) \geq \bar{V}_0^\tau(K') \text{ and } V_0^\tau(K) \geq V_0^\tau(K'); \]
and
\[ \bar{V}_0^{\tau\tau}(K) \geq \bar{V}_0^{\tau\tau}(K') \text{ and } V_0^{\tau\tau}(K) \geq V_0^{\tau\tau}(K'). \]

See Definition 7.6 for a definition of these terms.

A generalization of these ideas will be used to establish Theorem 1.5.

### 1.5 Relation to secondary invariants

In [21], Juhász and Zemke construct several secondary invariants associated to a pair of slice surfaces \( \Sigma \) and \( \Sigma' \) for the same knot. These are shown to give lower bounds for various quantities such as the stabilization distance between \( \Sigma \) and \( \Sigma' \) (see below). Here, we focus on the invariant \( V_0(\Sigma, \Sigma') \) of [21, section 4.5]. It is easy to show:

**Theorem 1.10.** Let \((K, \tau)\) be a strongly invertible knot in \( S^3 \). Let \( W \) be any (smooth) homology ball with boundary \( S^3 \), and let \( \tau_W \) be any extension of \( \tau \) over \( W \). If \( \Sigma \) is any slice disk for \( K \) in \( W \), then
\[
\max\{V_0^\tau(K), V_0^{\tau\tau}(K)\} \leq V_0(\Sigma, \tau_W(\Sigma)).
\]

In [21], \( V_0(\Sigma, \Sigma') \) is defined for surfaces in \( B^4 \), but the extension to general integer homology balls is straightforward. The authors expect further connections with the results of [21], which we intend to investigate in future work.

Let \( W \) be a homology ball with \( \partial W = S^3 \), and let \( \Sigma, \Sigma' \subseteq W \) be two slice surfaces for \( K \). Recall that the stabilization distance \( \mu_{st}(\Sigma, \Sigma') \) is defined to be the minimum of
\[
\max\{g(\Sigma_1), \ldots, g(\Sigma_n)\}
\]
over sequences of slice surfaces \( \Sigma_i \subseteq W \) from \( \Sigma \) to \( \Sigma' \) such that consecutive surfaces are related by either a stabilization/destabilization or an isotopy rel \( K \). (We take the 4-manifold \( W \) as being implicit in the setup and suppress it from the notation.) In [21, Theorem 1.1], Juhász and Zemke show that if \( \Sigma, \Sigma' \subseteq W \) are two slice disks for the same knot, then
\[
V_0(\Sigma, \Sigma') \leq \left\lceil \frac{\mu_{st}(\Sigma, \Sigma')}{2} \right\rceil.
\]
It follows from this that $\mathcal{V}_{\mathcal{O}}^{\circ}$ and $\mathcal{V}_{\mathcal{O}}^0$ can be used to construct pairs of disks with large stabilization distance. Applying Theorem 1.10 and [21, Theorem 1.1] to the examples $(K_n, \tau_n)$ of Subsection 1.2, we immediately obtain:

**Theorem 1.11.** Let $n$ be odd. Let $W$ be any (smooth) homology ball with boundary $S^3$, and let $\tau_W$ be any extension of $\tau_n$ over $W$. Suppose $K_n$ is slice in $W$. Then for any slice disk $\Sigma$,

$$2n - 1 \leq \mu_{\text{st}}(\Sigma, \tau_W(\Sigma)).$$

As $K_n$ is slice in $B^4$, this shows that for any integer $m$, there is some knot with a pair of slice disks that require at least $m$ stabilizations to become isotopic. This provides an alternate proof of a result of Miller-Powell [26, Theorem B]. In fact, Theorem 1.11 is slightly stronger, as the stabilization distance between two surfaces can be strictly less than the number of stabilizations needed to make them isotopic. Moreover, the examples of Theorem 1.11 are inherent to the knots $K_n$, rather than the actual disks: we may start with any slice disk for $K_n$ (in $B^4$ or otherwise) and compute its stabilization distance to its reflection (again, under any extension of $\tau_n$).

**Remark 1.12.** During the completion of this project, Ian Zemke informed us of another family of examples, now independently included in [21, section 10.3]. Our examples use a similar family of knots as in [21, section 10.3], but the slice disks in question are rather different. (In particular, our approach de-emphasizes the construction of the actual disks, and instead requires only that the pair of disks are related by $\tau_W$.)

**Remark 1.13.** Recently, several related results have emerged that have a strong bearing on the work presented here. Although these appeared some time after the initial posting of this paper, we describe them briefly to provide some context.

1. Di Prisa [11] has shown that the equivariant concordance group is indeed non-abelian. The authors do not believe that knot Floer homology detects these examples; it is still unclear whether $\mathcal{K}_{\tau,2}$ is abelian.
2. Building on the Floer-theoretic formalism of the present work (in particular, Theorem 4.3), the authors of this paper (in joint work with Kang and Park) have recently shown that the $(2,1)$-cable of the figure-eight is not slice [9]. This was previously an open question, and as such may provide some motivation for the extensive framework we establish here.
3. Miller and Powell [27] have recently provided a second (more topological) proof of [3, Question 1.1] by utilizing Blanchfield forms. Curiously, the two approaches do not overlap: the examples presented here are not accessible via the methods of [27]; conversely, Floer homology does not give growing genus bounds for the examples of [27].

**Organization**

In Section 2, we give a brief introduction to equivariant concordance and introduce the topological objects that we study in this paper. We then establish the algebraic formalism of local equivalence and define the local equivalence group $\mathcal{K}_{\tau,2}$. In Section 3, we construct the Floer-theoretic action associated to a strong inversion and prove Theorem 1.7. We then define $\mathcal{V}_{\mathcal{O}}^\circ$ and $\mathcal{V}_{\mathcal{O}}^0$ and prove Theorem 1.1. In Section 4, we establish several computational tools involving the action of $\tau$, including...
a connected sum formula. This leads to the proof of Theorem 1.8. We establish the equivariant slice genus bound of Theorem 1.2 in Section 5. Then, in Section 6, we carry out the calculation of Theorem 1.3 regarding the examples \((K_n, \tau_n)\). Finally, in Section 7, we explicitly discuss the relation between our invariants and the work of Dai–Hedden–Mallick [7] and Juhász–Zemke [21]; we prove Theorems 1.5, 1.6, 1.9, and 1.10. Section 8 gives an overview of several analogous results for 2-periodic knots.

2 | BACKGROUND AND ALGEBRAIC FORMALISM

In this section, we give a brief review of equivariant knots. We then establish the algebraic formalism of local equivalence and define and discuss the group \(\mathcal{K}_{\tau, \zeta}\). Throughout, we assume a general familiarity with the knot Floer and involutive knot Floer packages; see [16] and [39].

2.1 | Equivariant knots

Let \(K\) be a knot in \(S^3\) and let \(\tau\) be an orientation-preserving involution on \(S^3\) that sends \(K\) to itself and has nonempty fixed-point set. By the resolution of the Smith Conjecture, we may assume that the fixed-point set of \(\tau\) is an unknot and that \(\tau\) is rotation about this axis [25, 35]. If \(\tau\) preserves orientation on \(K\), then we say that \((K, \tau)\) is \(2\)-periodic (or often just \(periodic\)) and refer to \(\tau\) as a \(periodic\) involution. If \(\tau\) reverses orientation on \(K\), then we say that \((K, \tau)\) is \(strongly\) invertible and refer to \(\tau\) as a \(strong\) inversion. These correspond to the situations where \(\tau\) has zero or two fixed points on \(K\), respectively. In this paper we focus on strongly invertible knots; the periodic case is discussed in Section 8. Although an \(equivariant\) knot may refer to either a strongly invertible or a periodic knot \((K, \tau)\), we will often use this terminology with the strongly invertible case in mind.

We say that \((K_1, \tau_1)\) and \((K_2, \tau_2)\) are \(equivariantly\) \(diffeomorphic\) if there is an orientation-preserving diffeomorphism \(f: S^3 \to S^3\) that sends \(K_1\) to \(K_2\) and has \(\tau_2 \circ f = f \circ \tau_1\).

**Definition 2.1.** Let \((K, \tau)\) be an equivariant knot. A slice surface \(\Sigma \subseteq B^4\) for \(K\) is \(equivariant\) if there exists an involution \(\tau_{B^4}: B^4 \to B^4\) that extends \(\tau\) and has \(\tau_{B^4}(\Sigma) = \Sigma\). The \(equivariant\) \(slice\) \(genus\) of \((K, \tau)\) is defined to be the minimum genus over all equivariant slice surfaces for \((K, \tau)\) in \(B^4\). We denote this quantity by \(\tilde{g}_4(K)\), suppressing the involution \(\tau\).

Equivariant sliceness has been studied by several authors; see, for example, [5, 33], and [10]. Recently, Boyle–Issa [3] studied the equivariant slice genus and were able to present several methods for bounding the equivariant slice genus from below. They moreover construct a family of periodic knots for which \(\tilde{g}_4(K) - g_4(K)\) becomes arbitrarily large. Prior to the current article, there were no known examples of strongly invertible knots with \(\tilde{g}_4(K) - g_4(K)\) provably greater than one.

There is also an obvious notion of equivariant concordance, given by:

**Definition 2.2.** Let \((K_1, \tau_1)\) and \((K_2, \tau_2)\) be two equivariant knots. We say that a concordance \(\Sigma \subseteq S^3 \times I\) from \(K_1\) to \(K_2\) is \(equivariant\) if there exists an involution \(\tau_{S^3 \times I}: S^3 \times I \to S^3 \times I\) that extends \(\tau_1\) and \(\tau_2\) and has \(\tau_{S^3 \times I}(\Sigma) = \Sigma\).
In the strongly invertible setting, it turns out that it is useful to have a refinement of Definition 2.1, which we now explain. If \((K, \tau)\) is a strongly invertible knot, note that \(K\) separates the fixed-point axis of \(\tau\) into two halves.

**Definition 2.3.** A direction on \((K, \tau)\) is a choice of half-axis, together with an orientation on this half-axis.

**Definition 2.4.** Given two directed strongly invertible knots \((K_1, \tau_1)\) and \((K_2, \tau_2)\), we may form their equivariant connected sum, as defined in [33]. This is another directed strongly invertible knot, constructed as follows. Place \(K_1\) and \(K_2\) along the same oriented axis, such that the oriented half-axis for \(K_1\) occurs before the oriented half-axis for \(K_2\). Attach an equivariant band with one foot at the head of the half-axis for \(K_1\) and the other foot at the tail of the half-axis for \(K_2\), as in Figure 3. Define the oriented half-axis for \(K_1 \# K_2\) to run from the tail of the half-axis for \(K_1\) to the head of the half-axis for \(K_2\).

We stress that a choice of direction is necessary to define the equivariant connected sum. Moreover, the equivariant connected sum is not a commutative operation: the strongly invertible knots \((K_1 \# K_2, \tau_1 \# \tau_2)\) and \((K_2 \# K_1, \tau_2 \# \tau_1)\) are not usually equivariantly diffeomorphic (even forgetting about the data of the direction). For further discussion, see [33] or [3, section 2].

To construct an equivariant concordance group, we consider the set of directed strongly invertible knots. This necessitates a refinement of Definition 2.2 that takes into account the extra data of the direction.

**Definition 2.5** [3, Definition 2.4]. Let \((K_1, \tau_1)\) and \((K_2, \tau_2)\) be two directed strongly invertible knots. We say that an equivariant concordance \((\Sigma, \tau_{S^3 \times I})\) between \((K_1, \tau_1)\) and \((K_2, \tau_2)\) is equivariant in a directed sense if the orientations of the half-axes induce the same orientation on the fixed-point annulus \(F\) of \(\tau_{S^3 \times I}\) and the half-axes are contained in the same component of \(F - \Sigma\).

**Definition 2.6** [33]. The equivariant concordance group is formed by quotienting out the set of directed strongly invertible knots by (directed) equivariant concordance. The group operation is given by equivariant connected sum. The inverse of \((K, \tau)\) is constructed by mirroring \(K\) and reversing orientation on the (mirrored) half-axis. We denote this group by \(\tilde{C}\).

It is currently an open question whether \(\tilde{C}\) is abelian. The equivariant concordance group was studied at length by Sakuma [33], who constructed a homomorphism from \(\tilde{C}\) to the additive group \(\mathbb{Z}[t]\) in the form of the \(\eta\)-polynomial.
Remark 2.7. As discussed in [33] and [3], when studying $\tilde{C}$, it is often the convention not to fix an orientation on $K$, due to the fact that the action of $\tau$ reverses orientation on $K$. To follow this convention, we will similarly not require a fixed orientation. However, as most invariants derived from knot Floer homology implicitly do require $K$ to be oriented, in each case we will be careful to check whether the choice of orientation is important.

Finally, we note (as discussed in the introduction) that we bound a rather more general notion than the equivariant slice genus. For completeness, we formally record:

Definition 2.8. Let $(K, \tau)$ be an equivariant knot. Let $W$ be a (smooth) homology ball with boundary $S^3$, and consider any (smooth) self-diffeomorphism $\tau_W$ on $W$ that extends $\tau$. Note that we do not require $\tau_W$ itself to be an involution. We say that a slice surface $\Sigma$ in $W$ with $\partial \Sigma = K$ is an isotopy-equivariant slice surface (for the given data) if $\tau_W(\Sigma)$ is isotopic to $\Sigma$ rel $K$. Define the isotopy-equivariant slice genus of $(K, \tau)$ by:

$$\tilde{ig}_4(K) = \min_{\text{all possible choices of } W \text{ and } \tau_W} \{ g(\Sigma) \}.$$ 

Here $\tilde{ig}_4(K)$ depends on $\tau$, but we suppress this from the notation.

There is also an accompanying notion in the setting of concordance:

Definition 2.9. Let $(K_1, \tau_1)$ and $(K_2, \tau_2)$ be two equivariant knots. An isotopy-equivariant homology concordance between $(K_1, \tau_1)$ and $(K_2, \tau_2)$ consists of a homology cobordism $W$ from $S^3$ to itself, a (smooth) self-diffeomorphism $\tau_W : W \to W$ that extends $\tau_1$ and $\tau_2$, and a concordance $\Sigma \subseteq W$ between $K_1$ and $K_2$ such that $\tau_W(\Sigma)$ is isotopic to $\Sigma$ rel boundary.

2.2 Local equivalence and $R_{\tau, \delta}$

We now give an overview of the framework of local equivalence and define $R_{\tau, \delta}$. We assume the reader has a general familiarity with the ideas of [16] and [39]. Let $C$ be a bigraded, free, finitely generated chain complex over $R = F[\mathcal{U}, \mathcal{V}]$ such that

1. $gr(\delta) = (-1, -1), gr(\mathcal{U}) = (-2, 0), \text{and } gr(\mathcal{V}) = (0, -2)$;
2. $C \otimes F[\mathcal{U}, \mathcal{V}, \mathcal{U}^{-1}, \mathcal{V}^{-1}]$ is homotopy equivalent to $F[\mathcal{U}, \mathcal{V}, \mathcal{U}^{-1}, \mathcal{V}^{-1}]$.

We refer to $C$ as an abstract knot complex, occasionally denoting the two components of the grading by $gr_U$ and $gr_V$. As explained in [38, section 3], given an abstract knot complex, we can formally differentiate the matrix of $\delta$ with respect to $\mathcal{U}$ and $\mathcal{V}$ to obtain $R$-equivariant maps $\Phi$ and $\Psi$. We define the Sarkar map in this context to be $\varsigma_K = \text{id} + \Phi \circ \Psi \simeq \text{id} + \Psi \circ \Phi$. It is a standard fact that $\varsigma_K^2 \simeq \text{id}$.

Definition 2.10. A abstract $(\tau_K, i_K)$-complex is a triple $(C, \tau_K, i_K)$ such that:

1 Technically, $\Phi$ and $\Psi$ are only defined after fixing a basis for $C$. However, the homotopy equivalence classes of $\Phi$ and $\Psi$ are well-defined without a choice of basis; see [39, Corollary 2.9].
(1) $C$ is an abstract knot complex;
(2) $t_K : C \to C$ is a skew-graded, $R$-skew-equivariant chain map such that
\[ t_K^2 \simeq \zeta_K ; \]
(3) $\tau_K$ is a skew-graded, $R$-skew-equivariant chain map such that
\[ \tau_K^2 \simeq \text{id} \quad \text{and} \quad \tau_K \circ t_K \simeq \zeta_K \circ t_K \circ \tau_K . \]

Recall that a map $f : C \to C$ is skew-graded if $\text{gr}(f(x)) = (\text{gr}_V(x), \text{gr}_U(x))$ and is $R$-skew-equivariant if $f(U^i V^j x) = V^j U^i x$.

Definition 2.10 simply says that the pair $(C, t_K)$ is an abstract $t_K$-complex in the sense of Zemke [39, Definition 2.2]. The conditions on $\tau_K$ are from Theorem 1.7. We also note the following extremely important consequence of the commutation condition:

**Lemma 2.11.** Let $(C, \tau_K, t_K)$ be an abstract $(\tau_K, t_K)$-complex. Then $\zeta_K$ commutes with both $\tau_K$ and $t_K$ up to homotopy.

**Proof.** The proof is immediate from the commutation relation between $\tau_K$ and $t_K$, the fact that $t_K^2 \simeq \zeta_K$, and the fact that $\zeta_K^2 \simeq \text{id}$. (In fact, it is possible to show that $\zeta_K$ commutes with any chain map from $C$ to itself, using the equality $\zeta_K = \text{id} + \Phi \circ \Psi$.)

There is a natural notion of homotopy equivalence:

**Definition 2.12.** Two $(\tau_K, t_K)$-complexes $(C_1, \tau_{K_1}, t_{K_1})$ and $(C_2, \tau_{K_2}, t_{K_2})$ are homotopy equivalent if there exist graded, $R$-equivariant homotopy inverses $f$ and $g$ between $C_1$ and $C_2$ such that
\[ f \circ \tau_{K_1} \simeq \tau_{K_2} \circ f \quad \text{and} \quad g \circ \tau_{K_2} \simeq \tau_{K_1} \circ g \]

and
\[ f \circ t_{K_1} \simeq t_{K_2} \circ f \quad \text{and} \quad g \circ t_{K_2} \simeq t_{K_1} \circ g . \]

In this case, we write $(C_1, \tau_{K_1}, t_{K_1}) \simeq (C_2, \tau_{K_2}, t_{K_2})$.

We also have the analogue of local equivalence from [39, Definition 2.4]:

**Definition 2.13.** Two $(\tau_K, t_K)$-complexes $(C_1, \tau_{K_1}, t_{K_1})$ and $(C_2, \tau_{K_2}, t_{K_2})$ are locally equivalent if there exist graded, $R$-equivariant chain maps $f : C_1 \to C_2$ and $g : C_2 \to C_1$ such that
\[ f \circ \tau_{K_1} \simeq \tau_{K_2} \circ f \quad \text{and} \quad g \circ \tau_{K_2} \simeq \tau_{K_1} \circ g . \]

\[ \text{If } f \text{ and } g \text{ are graded, } R \text{-equivariant chain maps, we write } f \simeq g \text{ to mean that } f \text{ and } g \text{ are homotopic via an } R \text{-equivariant homotopy. This means that } f + g = \delta H + H \delta \text{ for some } R \text{-equivariant } H . \text{ If } f \text{ and } g \text{ are skew-graded, skew-} R \text{-equivariant chain maps, we again write } f \simeq g \text{ to mean that } f \text{ and } g \text{ are homotopic via a skew-} R \text{-equivariant homotopy. This means that } f + g = \delta H + H \delta \text{ for some } R \text{-skew-equivariant } H . \text{ Our notation differs slightly from [39], where the convention is to write } \simeq \text{ in the latter case.} \]
and
\[ f \circ t_{K_1} \simeq t_{K_2} \circ f \] and \[ g \circ t_{K_2} \simeq t_{K_1} \circ g. \]
and \( f \) and \( g \) induce homotopy equivalences \( C_1 \otimes \mathbb{F}[U, V, U^{-1}, V^{-1}] \simeq C_2 \otimes \mathbb{F}[U, V, U^{-1}, V^{-1}] \). In this case we write \((C_1, \tau_{K_1}, t_{K_1}) \sim (C_2, \tau_{K_2}, t_{K_2})\). We refer to the maps \( f \) and \( g \) as local maps.

Using the notion of local equivalence, we now define:

**Definition 2.14.** We define the \((\tau_K, t_K)\)-local equivalence group to be

\[ \mathcal{K}_{\tau, t} = \{ \text{abstract } (\tau_K, t_K)\text{-complexes} \} / \text{local equivalence}. \]

The group operation is defined as follows. Given \((C_1, \tau_{K_1}, t_{K_1})\) and \((C_2, \tau_{K_2}, t_{K_2})\), define automorphisms \( \tau \otimes \) and \( t \otimes \) on \( C_1 \otimes C_2 \) as follows. Let

\[ \tau \otimes = \tau_{K_1} \otimes \tau_{K_2} \]
and

\[ t \otimes = (\text{id} \otimes \text{id} + \Phi \otimes \Psi) \circ (t_{K_1} \otimes t_{K_2}). \]

We define the product of two abstract \((\tau_K, t_K)\)-complexes \((C_1, \tau_{K_1}, t_{K_1})\) and \((C_2, \tau_{K_2}, t_{K_2})\) to be \((C_1 \otimes C_2, \tau \otimes, t \otimes)\). This operation gives another abstract \((\tau_K, t_K)\)-complex and is well-defined with respect to local equivalence. The identity is given by the trivial complex \((\mathcal{R}, \tau_{\mathcal{R}}, t_{\mathcal{R}})\), where \( \tau_{\mathcal{R}} = t_{\mathcal{R}} \) is the map on \( \mathcal{R} \) that interchanges \( U \) and \( V \). Inverses are given by dualizing with respect to \( \mathcal{R} \); that is, \((C, \tau_K, t_K)^\vee = (C^\vee, \tau_{K}^\vee, t_{K}^\vee)\). See Lemmas 2.15 and 2.16.

It will also be convenient for us to consider the Sarkar map \( \varsigma \otimes \) on the product of two complexes. Note that

\[ \varsigma \otimes \simeq \text{id} \otimes + \Phi \otimes \Psi \otimes \]
\[ \simeq \text{id} \otimes \text{id} + (\text{id} \otimes \Phi + \Phi \otimes \text{id})(\text{id} \otimes \Psi + \Psi \otimes \text{id}) \]
\[ \simeq \text{id} \otimes \text{id} + \text{id} \otimes \Phi \Psi + \Phi \Psi \otimes \text{id} + \Phi \otimes \Psi + \Psi \otimes \Phi. \]

**Lemma 2.15.** The tensor product induces an associative binary operation on \( \mathcal{K}_{\tau, t} \).

**Proof.** We will be brief, as the majority of the claim is immediate from [39, section 2.3]. We first verify that the product complex is an abstract \((\tau_K, t_K)\)-complex. The only condition that is not either obvious or contained in [39] is the commutation relation

\[ \tau \otimes \circ t \otimes \simeq \varsigma \otimes \circ t \otimes \circ \tau \otimes. \]
To see this, let us expand the left-hand side. Suppressing the subscripts on $\tau$ and $\iota$, we obtain

\[(\tau \otimes \tau)(\text{id} \otimes \text{id} + \Phi \otimes \Psi)(\iota \otimes \iota)\]
\[\simeq (\text{id} \otimes \text{id} + \Psi \otimes \Phi)(\tau \otimes \tau)(\iota \otimes \iota)\]
\[\simeq (\text{id} \otimes \text{id} + \Psi \otimes \Phi)(\xi \tau \otimes \xi \iota)\]
\[\simeq (\text{id} \otimes \text{id} + \Psi \otimes \Phi)(\text{id} \otimes \text{id} + \Phi \Psi \otimes \text{id} + \Phi \Psi \otimes \Phi)(\text{id} \otimes \text{id})(\tau \otimes \tau)\]
\[\simeq (\text{id} \otimes \text{id} + \Phi \Psi + \Psi \Phi \otimes \text{id} + \Phi \Psi \otimes \Phi)(\text{id} \otimes \text{id})(\tau \otimes \tau)\].

Here, in the second line we have used [39, Lemma 2.8], which states that a skew-equivariant map intertwines $\Phi$ and $\Psi$ up to homotopy. In the third line, we have used the commutation property of $\tau$ and $\iota$ in each factor. In the fourth line, we have used the fact that $\xi = \text{id} + \Phi \Psi$. Finally, in the last line we have used the fact that $\Phi$ and $\Psi$ homotopy commute and that $\Phi^2 \simeq \Psi^2 \simeq 0$; see [39, Lemma 2.10] and [39, Lemma 2.11].

On the other hand, the right-hand side is given by

\[(\text{id} \otimes + \Phi \otimes \Psi)(\text{id} \otimes \text{id} + \Phi \otimes \Psi)(\iota \otimes \iota)(\tau \otimes \tau)\]
\[\simeq (\text{id} \otimes \text{id} + \text{id} \otimes \Phi \Psi + \Phi \Psi \otimes \text{id} + \Phi \Psi \otimes \Psi \otimes \Phi)(\text{id} \otimes \text{id} + \Phi \Psi \otimes \text{id} + \Phi \Psi \otimes \Phi)(\iota \otimes \iota)(\tau \otimes \tau)\]
\[\simeq (\text{id} \otimes \text{id} + \text{id} \otimes \Phi \Psi + \Phi \Psi \otimes \text{id} + \Phi \Psi \otimes \Phi)(\iota \otimes \iota)(\tau \otimes \tau)\].

Here, in the second line we have used the fact that $\Phi \otimes = \text{id} \otimes \Phi + \Phi \otimes \text{id}$ (and similarly for $\Psi \otimes$). In the fourth line, we again use the fact that $\Phi$ and $\Psi$ homotopy commute and that $\Phi^2 \simeq \Psi^2 \simeq 0$. The resulting expression is homotopic to the previous.

Checking associativity is straightforward. Indeed, in [39, section 2.3] it is shown that the obvious identity map from $(C_1 \otimes C_2) \otimes C_3$ to $C_1 \otimes (C_2 \otimes C_3)$ intertwines the $t_K$-actions up to homotopy; this clearly intertwines the $\tau_K$-actions. Checking that the tensor product respects local equivalence is likewise immediate. \(\square\)

**Lemma 2.16.** The tensor product operation above gives $\mathfrak{R}_{\tau,\iota}$ the structure of a group.

**Proof.** Again, the majority of the claim is immediate from [39, section 2.3]. The only nontrivial claim is to establish that inverses are given by dualizing, which follows the proof of [39, Lemma 2.18]. Zemke shows that the cotrace and trace maps

\[F : \mathcal{R} \to C \otimes C^\vee \quad \text{and} \quad G : C \otimes C^\vee \to \mathcal{R}\]

have the requisite behavior with respect to localizing and intertwine the $t_K$-actions. We check that $F$ intertwines the actions of $\tau_e$ and $\tau_K \otimes \tau_K^\vee$. As $\tau_e$ squares to the identity, it suffices to show

\[\tau_K \otimes \tau_K^\vee) \circ F \circ \tau_e \simeq F.\]

But [39, eq. (14)] implies

\[(\tau_K \otimes \tau_K^\vee) \circ F \circ \tau_e \simeq (\tau_K^2 \otimes \text{id}) \circ F \simeq (\text{id} \otimes \text{id}) \circ F = F,\]
establishing the claim. The proof for $G$ is similar. Hence, $(C, \tau_K, t_K) \otimes (C, \tau_K, t_K)^\vee$ is locally equivalent to the trivial $(t_K, t_K)$-complex. Checking triviality of the product in the opposite order is similar; use the obvious maps

$$F^r : \mathcal{R} \to C^\vee \otimes C \quad \text{and} \quad G^r : C^\vee \otimes C \to \mathcal{R}.$$ 

This completes the proof. □

Note that instead of considering triples $(C, \tau_K, t_K)$, we may forget $t_K$ or $\tau_K$ and consider pairs $(C, \tau_K)$ or $(C, t_K)$, respectively. The reader will have no trouble in defining appropriate notions of local equivalence for these and forming the analogous local equivalence groups.

**Definition 2.17.** We denote the local equivalence group of $\tau_K$-complexes by $\mathcal{K}_\tau$; this consists of pairs $(C, \tau_K)$ such that $\tau_K$ is a skew-graded, $\mathcal{R}$-equivariant chain map with $\tau_K^2 \simeq \text{id}$. We denote the local equivalence group of $t_K$-complexes by $\mathcal{K}_t$; this consists of pairs $(C, t_K)$ such that $t_K$ is a skew-graded, $\mathcal{R}$-equivariant chain map with $t_K^2 \simeq \xi$. The latter is just the usual local equivalence group of $[39$, Proposition 2.6$]$. We obtain forgetful maps from $\mathcal{K}_{\tau, t}$ to $\mathcal{K}_\tau$ and $\mathcal{K}_t$ by discarding $t_K$ and $\tau_K$, respectively.

**Remark 2.18.** It is possible to have triples $(C_1, \tau_{K_1}, t_{K_1})$ and $(C_2, \tau_{K_2}, t_{K_2})$ such that $(C_1, \tau_{K_1}) \sim (C_2, \tau_{K_2})$ and $(C_1, t_{K_1}) \sim (C_2, t_{K_2})$, but still $(C_1, \tau_{K_1}, t_{K_1}) \not\sim (C_2, \tau_{K_2}, t_{K_2})$. This is because in Definition 2.13, we require $\tau_{K_i}$ and $t_{K_i}$ to be simultaneously intertwined by $f$ and $g$. This will be an important distinction that leads to a great deal of (possible) subtlety in the structure of $\mathcal{K}_{\tau, t}$. For an explicit example of the above phenomenon, see Example 2.27. Compare [7, Example 4.7], which establishes a similar phenomenon in the 3-manifold setting.

### 2.3  |  (Possible) noncommutativity of $\mathcal{K}_{\tau, t}$

We now discuss some subtleties of $\mathcal{K}_{\tau, t}$. The first of these involves a seeming asymmetry in the product operation. Recall that we defined the product $t_K$-action on $C_1 \otimes C_2$ to be $t_{\otimes} = t_{\otimes, A}$, where

$$t_{\otimes, A} = (\text{id} \otimes \text{id} + \Phi \otimes \Psi) \circ (t_{K_1} \otimes t_{K_2}).$$

There is of course a slightly different $t_K$-action on $C_1 \otimes C_2$, given by

$$t_{\otimes, B} = (\text{id} \otimes \text{id} + \Psi \otimes \Phi) \circ (t_{K_1} \otimes t_{K_2}).$$

It is straightforward to check that using $t_{\otimes, B}$ in Definition 2.14 also gives a well-defined operation on $\mathcal{K}_{\tau, t}$. Rather surprisingly, it turns out that these operations are not a priori the same.

**Remark 2.19.** In [39, Lemma 2.14], Zemke considers the map

$$F = \text{id} \otimes \text{id} + \Psi \otimes \Phi.$$ 

This is a homotopy equivalence from $C_1 \otimes C_2$ to itself such that $F \circ t_{\otimes, A} \simeq t_{\otimes, B} \circ F$. Hence, $F$ mediates a homotopy equivalence of pairs $(C_1 \otimes C_2, t_{\otimes, A}) \simeq (C_1 \otimes C_2, t_{\otimes, B})$. For this reason, $t_{\otimes, A}$
and $\iota_{\otimes, B}$ both give the same product structure on $\mathcal{R}_l$. However, the map $F$ above does not provide a homotopy equivalence between the triples $(C_1 \otimes C_2, \tau_{K_1} \otimes \tau_{K_2}, \iota_{\otimes, A})$ and $(C_1 \otimes C_2, \tau_{K_1} \otimes \tau_{K_2}, \iota_{\otimes, B})$. Indeed, $F$ is not $\tau_{K_1} \otimes \tau_{K_2}$-equivariant. We have

$$F \circ (\tau_{K_1} \otimes \tau_{K_2}) = (\text{id} \otimes \text{id} + \Psi \otimes \Phi)(\tau_{K_1} \otimes \tau_{K_2})$$

while

$$(\tau_{K_1} \otimes \tau_{K_2}) \circ F = (\tau_{K_1} \otimes \tau_{K_2})(\text{id} \otimes \text{id} + \Psi \otimes \Phi) \simeq (\text{id} \otimes \text{id} + \Phi \otimes \Psi)(\tau_{K_1} \otimes \tau_{K_2}).$$

In general, it is not true that these are chain homotopic maps.

This discrepancy is closely related to the possible noncommutativity of $\mathcal{R}_{\tau, l}$. Indeed, consider the two products $C_1 \otimes C_2$ and $C_2 \otimes C_1$. There is an obvious isomorphism from $C_1 \otimes C_2$ to $C_2 \otimes C_1$ given by transposition of factors; this clearly intertwines the two $\tau_K$-actions $\tau_{K_1} \otimes \tau_{K_2}$ and $\tau_{K_2} \otimes \tau_{K_1}$. However, it does not intertwine the $\iota_K$-actions: instead, it sends $\iota_{\otimes, A}$ on $C_1 \otimes C_2$ to $\iota_{\otimes, B}$ on $C_2 \otimes C_1$. Hence, $\mathcal{R}_{\tau, l}$ is not necessarily abelian, and in fact this question is equivalent to whether the operations on $\mathcal{R}_{\tau, l}$ induced by $\iota_{\otimes, A}$ and $\iota_{\otimes, B}$ are the same (up to local equivalence). In Theorem 4.1, we establish a connected sum formula showing that using $\iota_{\otimes, A}$ corresponds to taking the equivariant connected sum as in Definition 2.4. Using $\iota_{\otimes, B}$ thus corresponds to modifying Definition 2.4 by placing the half-axis of the first knot above the half-axis of the second.

Unfortunately, the authors do not have an explicit example demonstrating that $\mathcal{R}_{\tau, l}$ is not abelian. Indeed, in all of the examples that the authors have tried, it is possible to find an ad hoc construction of a local equivalence (in fact, even a homotopy equivalence) between $(C_1 \otimes C_2, \tau_{K_1} \otimes \tau_{K_2}, \iota_{\otimes, A})$ and $(C_1 \otimes C_2, \tau_{K_1} \otimes \tau_{K_2}, \iota_{\otimes, B})$. Note that $\mathcal{R}_{\tau, l}$ admits forgetful maps to both $\mathcal{R}_{\tau}$ and $\mathcal{R}_l$, which are both abelian.

### 2.4 Twisting by $\zeta_K$

As discussed in the introduction, our goal will be to associate to a strongly invertible knot a $(\tau_K, \iota_K)$-complex that is well-defined up to homotopy equivalence. Moreover, we wish to show that this local equivalence class is invariant under isotopy-equivariant homology concordance. Unfortunately, it turns out that both of these statements are technically only true if we pass to the decorated category (see Subsections 3.3 and 3.4). To capture this subtlety, we introduce the following notion:

**Definition 2.20.** Let $(C, \tau_K, \iota_K)$ be an abstract $(\tau_K, \iota_K)$-complex. Compose $\tau_K, \iota_K$, or both with any number of copies of $\zeta_K$. By Lemma 2.11, this produces another $(\tau_K, \iota_K)$-complex. We refer to this new complex as being obtained from $(C, \tau_K, \iota_K)$ via a twist by $\zeta_K$.

**Lemma 2.21.** Let $(C, \tau_K, \iota_K)$ be an abstract $(\tau_K, \iota_K)$-complex. Then

$$(C, \zeta_K \circ \tau_K, \iota_K) \simeq (C, \tau_K, \zeta_K \circ \iota_K)$$

and

$$(C, \tau_K, \iota_K) \simeq (C, \zeta_K \circ \tau_K, \zeta_K \circ \iota_K).$$
Proof. To prove the second claim, we use the graded, \( R \)-equivariant automorphism \( \tau_K \circ t_K \). A quick computation using the relation \( \tau_K \circ t_K \simeq \xi_K \circ t_K \circ \tau_K \) shows that this constitutes a homotopy equivalence between \((C, \tau_K, t_K)\) and \((C, \xi_K \circ \tau_K, \xi_K \circ t_K)\). The first claim follows immediately from the second, noting that \( \xi_K^2 \simeq \text{id} \).

Up to homotopy equivalence, a \((\tau_K, t_K)\)-complex thus has only one twist, which is represented by \((C, \xi_K \circ \tau_K, t_K) \simeq (C, \tau_K, \xi_K \circ t_K)\). In general, the authors know of no reason this should be homotopy (or even locally) equivalent to its original, and it is possible that the requisite computation of \( t_K \) does not currently exist in the literature. Note that Lemma 2.21 also implies \((C, \tau_K) \simeq (C, \xi_K \circ \tau_K)\) and \((C, t_K) \simeq (C, \xi_K \circ t_K)\) as pairs. Hence, the distinction between a complex and its twist is a phenomenon that is only present when considering \( \tau_K \) and \( t_K \) simultaneously.

As we will see in Subsections 3.3 and 3.4, twisted complexes will play an important role when we move from the decorated to the undecorated setting. Roughly speaking, if \((K, \tau)\) is an strongly invertible knot without a choice of decoration, then we can only define the homotopy equivalence class of \((C, \tau_K, t_K)\) up to a twist by \( \xi_K \). Nevertheless, we show presently that the numerical invariants \( \mathcal{V}_0^0 \) and \( \mathcal{V}_0^0 \) are unchanged by twisting.

The distinction between a complex and its twist also has an interpretation in terms of the direction on a strongly invertible knot (see Definition 2.3). In Subsection 3.5, we show that a choice of direction on \((K, \tau)\) can be used to determine a homotopy equivalence class of \((\tau_K, t_K)\)-complex. Reversing the direction on \((K, \tau)\) corresponds to applying a twist by \( \xi_K \).

As we will see in Subsections 3.3 and 3.4, twisted complexes will play an important role when we move from the decorated to the undecorated setting. Roughly speaking, if \((K, \tau)\) is an strongly invertible knot without a choice of decoration, then we can only define the homotopy equivalence class of \((C, \tau_K, t_K)\) up to a twist by \( \xi_K \). Nevertheless, we show presently that the numerical invariants \( \mathcal{V}_0^0 \) and \( \mathcal{V}_0^0 \) are unchanged by twisting.

The distinction between a complex and its twist also has an interpretation in terms of the direction on a strongly invertible knot (see Definition 2.3). In Subsection 3.5, we show that a choice of direction on \((K, \tau)\) can be used to determine a homotopy equivalence class of \((\tau_K, t_K)\)-complex. Reversing the direction on \((K, \tau)\) corresponds to applying a twist by \( \xi_K \). In general, reversing the direction on \((K, \tau)\) alters its class in \( \tilde{C} \). Boyle–Issa [3] and Alfieri–Boyle [1] show that several invariants are sensitive to this operation; \( \mathcal{R}_{\uparrow, \tau} \) is thus (in principle) similar, although this fails for the simple examples at our disposal.

### 2.5 Extracting numerical invariants

We now give a brief review of extracting numerical invariants from the local equivalence class of \((C, \tau_K, t_K)\). Recall that given an abstract knot complex \( C \), we may form the large surgery subcomplex, which we denote by \( C_0 \).

**Definition 2.22.** Let \((C, \tau_K, t_K)\) be a \((\tau_K, t_K)\)-complex. The large surgery subcomplex of \( C \) is the subset \( C_0 \) of \( C \) lying in Alexander grading zero; that is, the set of elements \( x \) with \( \text{gr}_U(x) = \text{gr}_V(x) \).

(This is often denoted by by \( A_0 \) elsewhere in the literature.) Strictly speaking, this is not a subcomplex of \( C \); although \( C_0 \) is preserved by \( \partial \), it is not a submodule over \( R \). Instead, we view \( C_0 \) as a singly graded complex over the ring \( \mathbb{F}[U] \), where

\[
U = \mathcal{U} \mathcal{V}.
\]

The Maslov grading of an element is given by \( \text{gr}_U = \text{gr}_V \). When we write \( C_0 \), we will mean this singly graded complex over \( \mathbb{F}[U] \).

Note that although \( \tau_K \) and \( t_K \) are skew-graded, the condition \( \text{gr}_U = \text{gr}_V \) means that \( \tau_K \) and \( t_K \) induce grading-preserving automorphisms on \( C_0 \), which we also denote by \( \tau_K \) and \( t_K \). Moreover, although \( \tau_K \) and \( t_K \) are \( R \)-skew-equivariant, their actions on \( C_0 \) are equivariant with respect to \( U = \mathcal{U} \mathcal{V} \). It follows from [14, Lemma 3.16] that as an automorphism of \( C_0 \), the Sarkar map \( \xi_K \) is
homotopic to the identity. It is then easily checked that

$$(C_0, \tau_K) \quad \text{and} \quad (C_0, t_K \circ \tau_K)$$

are $\iota$-complexes in the sense of [17, Definition 8.1].

We now follow the construction of the involutive numerical invariants $\overline{d}$ and $\underline{d}$ from [16], except that we replace the Heegaard Floer $\iota$-action with the action of $\tau_K$ on $C_0$, where $C_0$ is viewed as a singly graded complex over $\mathbb{F}[U]$. Explicitly, let

$$CFI^\iota(C_0) = \text{Cone} \left( C_0 \xrightarrow{Q(1+\tau_K)} Q \cdot C_0 \right),$$

where $Q$ is a formal variable of grading $-1$. Define

$$d_{\iota}(C_0) = \max \{ r \mid \exists x \in CFI^\iota_r(C_0) \text{ such that } U^n x \neq 0 \text{ and } U^n x \notin \text{im}(Q) \text{ for all } n \}$$

and

$$\overline{d}_{\iota}(C_0) = \max \{ r \mid \exists x \in CFI^\iota_r(C_0) \text{ such that } U^n x \neq 0 \text{ for all } n \text{ and } U^m x \in \text{im}(Q) \text{ for some } m \} + 1.$$

We define the mapping cone $CFI^{\iota}(C_0)$ by replacing $\tau_K$ with $t_K \circ \tau_K$, and define the numerical invariants $d_{\iota}(C_0)$ and $\overline{d}_{\iota}(C_0)$ similarly. Our conventions here are such that if $C$ is the trivial complex $\mathbb{F}[U]$, then $d_{\iota} = \overline{d}_{\iota} = 0$. We now have:

**Definition 2.23.** Let $(C, \tau_K, t_K)$ be an abstract $(\tau_K, t_K)$-complex. Define

$$\overline{V}^\iota_\tau(C) = -\frac{1}{2} \overline{d}_{\iota}(C_0) \quad \text{and} \quad \underline{V}^\iota_\tau(C) = -\frac{1}{2} d_{\iota}(C_0)$$

and

$$\overline{V}^\iota_0(C) = -\frac{1}{2} \overline{d}_{\iota}(C_0) \quad \text{and} \quad \underline{V}^\iota_0(C) = -\frac{1}{2} d_{\iota}(C_0).$$

**Lemma 2.24.** The invariants $\overline{V}^\iota_\tau(C)$ and $\underline{V}^\iota_\tau(C)$ are local equivalence invariants; that is, they factor through $\mathcal{R}_{\tau, \iota}$.

**Proof.** Let $(C_1, \tau_{K_1}, t_{K_1})$ and $(C_2, \tau_{K_2}, t_{K_2})$ be two $(\tau_K, t_K)$-complexes and let $f$ and $g$ be local equivalences between them. As $f$ and $g$ are graded and $R$-equivariant, they induce graded, $\mathbb{F}[U]$-equivariant chain maps between $(C_1)_0$ and $(C_2)_0$, which are easily checked to be local in the sense of [17, Definition 8.5]. The claim follows. \qed

**Lemma 2.25.** The invariants $\overline{V}^\iota_\tau_0(C)$ and $\underline{V}^\iota_\tau_0(C)$ are insensitive to twisting by $\xi_K$. 
Proof. This follows immediately from the fact that $\zeta_K$ is homotopic to the identity as a map on $C_0$; see the proof of \cite[Lemma 3.16]{14}.

Note that the same argument indicates that no additional numerical invariants can be defined by considering (for example) $\tau_K \circ t_K$ in place of $t_K \circ \tau_K$.

2.6 Examples

We now list the $(\tau_K, t_K)$-complexes corresponding to different strong inversions on the figure-eight and the stevedore. These may be derived from the results of \cite[section 4.2]{7} in the following manner. Fix a basis in which the action of $t_K$ is standard, as in \cite[section 8]{16}. For the pairs $(K, \tau)$ at hand, the 3-manifold action of $\tau$ on $HF^-(S^3_+ (K))$ was calculated in \cite[section 4.2]{7}. By the discussion of Subsection 7.1, this determines the action of $\tau_K$ on the homology of $CFK(K)_0$. We then list all automorphisms $\tau_K$ of $CFK(K)$ that induce this action and satisfy the axioms of Definition 2.10. (In particular, note that $\tau_K$ is required to satisfy $\tau_K \circ t_K \simeq \zeta_K \circ t_K \circ \tau_K$.) It turns out that in each example, the resulting automorphism is unique up to $t_K$-equivariant basis change. The proof of this is left to the reader and is an exercise in tedium. Compare \cite[section 4.2]{7}.

Example 2.26. There are two strong inversions $\tau$ and $\sigma$ on the figure-eight, which are displayed in Figure 4. In Figure 5, we display their corresponding actions $\tau_K$ and $\sigma_K$ on $C = CFK(4_1)$ (calculated in the basis with the indicated action of $t_K$). See \cite[Example 4.6]{7}.

Example 2.27. There are two strong inversions $\tau$ and $\sigma$ on the stevedore, which are displayed in Figure 6. In Figure 7, we display their corresponding actions $\tau_K$ and $\sigma_K$ on $C = CFK(6_1)$ (calculated in the basis with the indicated action of $t_K$). See \cite[Example 4.7]{7}. Note that the pairs $(C, t_K)$ and $(C, \sigma_K)$ are individually trivial. In both cases, the local map to the trivial complex is given by sending all generators except for $x_0$ to zero. The local map in the other direction has image $x_0$ in the former case but image $x_0 + e_1$ in the latter. However, there is no local map from the trivial complex that simultaneously commutes with both $\sigma_K$ and $t_K$ (up to homotopy), so the triple $(C, \sigma_K, t_K)$ is nontrivial. This can be checked via exhaustive casework, or by computing $V_{0i}^{cf}(K) = 1$. 

\begin{figure}[h]  
\centering  
\includegraphics[width=\textwidth]{figure4.png}  
\caption{The figure-eight $4_1$ with two strong inversions $\tau$ and $\sigma$.}  
\end{figure}
Remark 2.28. The reader may verify that in each of the above examples, performing a twist by $\xi_K$ does not change the homotopy equivalence class of the relevant triple. We thus suppress writing a choice of decoration or direction in both Examples 2.26 and 2.27.

3 | CONSTRUCTION OF $\tau_K$ AND EQUIVARIANT CONCORDANCE

In this section, we construct the action $\tau_K : CFK(K) \to CFK(K)$ of $\tau$ on the knot Floer complex of $K$. To do this, we first equip $K$ with an orientation and a symmetric pair of basepoints, which
we collectively refer to as a *decoration* on \( K \). We then explain in what sense \( \tau_K \) is independent of the choice of decoration. This turns out to be rather subtle, and will require an extended discussion about identifying different knot Floer complexes for the same knot in the case that the orientation or basepoints are changed. In particular, we show that if \((K, \tau)\) is a decorated strongly invertible knot, then the triple \((\text{CF} \mathcal{K}(K), \tau_K, t_K)\) is well-defined up to homotopy equivalence of \((\tau_K, t_K)\)-complexes. If \((K, \tau)\) does not come with a decoration, then the homotopy equivalence class of \((\text{CF} \mathcal{K}(K), \tau_K, t_K)\) is only defined up to a twist by \( \xi_K \), although the homotopy equivalence class of the pair \((\text{CF} \mathcal{K}(K), \tau_K)\) is still well-defined. See Theorems 3.11 and 3.12.

We then turn to the behavior of \( \tau_K \) under equivariant concordance. Here, we similarly modify the notion of an isotopy-equivariant homology concordance to hold in the decorated setting. We show that a decorated equivariant concordance induces a local equivalence of \((\tau_K, t_K)\)-complexes. In the undecorated setting, this only holds up to a twist applied to one end of the concordance, although we still obtain a local equivalence of \( \tau_K \)-complexes. See Theorems 3.14 and 3.15.

Finally, we discuss the connection between the decorated and directed categories. We show that a choice of direction similarly determines a homotopy equivalence class of \((\tau_K, t_K)\)-complex, and that a concordance in the directed category again induces a local equivalence. We then put everything together and establish Theorem 1.1.

### 3.1 Preliminaries

Defining the action of \( \tau \) will rely on a large number of auxiliary maps. To establish notation, we collect these below. We assume that the reader has some familiarity with the ideas of [37] and [39].

**Definition 3.1.** Let \((K, w, z)\) be an oriented, doubly based knot.

1. Let \( f \) be a diffeomorphism moving \((K, w, z)\) into \((f(K), f(w), f(z))\). If \( \mathcal{H} \) is any choice of Heegaard data for \((K, w, z)\), then we obtain a pushforward set of Heegaard data \( f\mathcal{H} \) for \((f(K), f(w), f(z))\). Moreover, \( f \) induces a tautological chain isomorphism

   \[
   f : \text{CF} \mathcal{K}(\mathcal{H}) \to \text{CF} \mathcal{K}(f\mathcal{H}),
   \]

   which by abuse of notation we also denote by \( f \). We call this the tautological pushforward.

2. If \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are two choices of Heegaard data for \((K, w, z)\), then there is a preferred homotopy equivalence

   \[
   \Phi(\mathcal{H}_1, \mathcal{H}_2) : \text{CF} \mathcal{K}(\mathcal{H}_1) \to \text{CF} \mathcal{K}(\mathcal{H}_2).
   \]

   This is unique (up to homotopy). We refer to \( \Phi(\mathcal{H}_1, \mathcal{H}_2) \) as the naturality map. The set of \( \Phi \) form a transitive system.

3. Let \( \mathcal{H} = ((\Sigma, \alpha, \beta, w, z), J) \) be a choice of Heegaard data for \((K, w, z)\). Then

   \[
   \mathcal{H}' = ((\Sigma, \alpha, \beta, z, w), J)
   \]

   is a choice of Heegaard data for \((K', z, w)\). Note that we interchange the roles of the basepoints \( w \) and \( z \), but we do not reverse orientation on \( \Sigma \) or interchange the roles of \( \alpha \) and \( \beta \). The resulting diagram describes the knot \( K \) with reversed orientation. There is a tautological skew-
graded isomorphism

\[ sw : CFK(H) \to CFK(H') \]

with \( \varpi^i \varpi^j x \mapsto \varpi^j \varpi^i x \), given by mapping each intersection tuple to itself and interchanging the roles of \( \varpi \) and \( \varpi' \). We call \( sw \) the switch map.

(4) Let \( H = ((\Sigma, \alpha, \beta, w, z), J) \) a choice of Heegaard data for \((K, w, z)\). Then

\[ \tilde{H} = ((-\Sigma, \beta, \alpha, z, w), \tilde{J}) \]

is a choice of Heegaard data for \((K, z, w)\). There is a tautological skew-graded isomorphism

\[ \eta : CFK(H) \to CFK(\tilde{H}) \]

with \( \varpi^i \varpi^j x \mapsto \varpi^j \varpi^i x \), given by mapping each intersection tuple to itself and interchanging the roles of \( \varpi \) and \( \varpi' \). We call \( \eta \) the involutive conjugation map. We stress that although \( sw \) and \( \eta \) might appear to be the same map, their codomains are different: the former represents \((K', z, w)\), while the latter represents \((K, z, w)\).

With the exception of the naturality map, we will usually suppress the data of \( H \) and thus the domain of the map in question.

**Lemma 3.2.** The maps \( f, \Phi, sw, \) and \( \eta \) all commute up to homotopy. Moreover, if \( f \) and \( g \) are two diffeomorphisms that commute, then their pushforwards commute up to homotopy.

**Proof.** Follows from naturality results established by Juhász–Thurston–Zemke [20] and Zemke [37].

The maps in Lemma 3.2 should be interpreted as having the proper domain(s). For example, when we say that \( f \) and \( \Phi \) commute, we mean that we have a (homotopy) commutative square

\[
\begin{array}{ccc}
CFK(H_1) & \xrightarrow{\Phi(H_1, H_2)} & CFK(H_2) \\
f \downarrow & & \downarrow f \\
CFK(fH_1) & \xrightarrow{\Phi(fH_1, fH_2)} & CFK(fH_2)
\end{array}
\]

and similarly for the other maps. We thus write (for instance) \( f \circ \Phi(H_1, H_2) \simeq \Phi(fH_1, fH_2) \circ f \) with the understanding that the two instances of \( f \) have different domains. Note that implicitly, we are also claiming these operations commute when applied to Heegaard diagrams. For example, when we write \( f \circ sw \simeq sw \circ f \), we are necessarily claiming that \( fH' = (fH)' \), so that the codomains of both sides may be identified. There are two other important maps that are derived from those in Definition 3.1:

**Definition 3.3.** Let \((K, w, z)\) be an oriented, doubly based knot.
(1) Let $\mathcal{H}$ be any choice of Heegaard data for $(K, w, z)$. Then

$$\eta \circ sw = sw \circ \eta : CFK(\mathcal{H}) \to CFK(\tilde{\mathcal{H}}')$$

provides a filtered isomorphism between $CFK(\mathcal{H})$ and $CFK(\tilde{\mathcal{H}}')$. Note that $CFK(\tilde{\mathcal{H}}')$ is a choice of Heegaard data for $(K', w, z)$; this has the reversed orientation but the same pair of basepoints. We call $\eta \circ sw$ the orientation-reversal map.

(2) Let $\mathcal{H}$ be any choice of Heegaard data for $(K, w, z)$. Let $\rho$ be the half Dehn twist along the orientation of $K$ that moves $w$ into $z$ and $z$ into $w$. This induces a tautological pushforward

$$\rho : CFK(\mathcal{H}) \to CFK(\rho \mathcal{H}).$$

Note that $\rho \mathcal{H}$ represents the doubly based knot $(K, z, w)$. We denote the half Dehn twist against the orientation of $K$ by $\tilde{\rho}$, and denote the induced pushforward similarly.

As the definition of $\rho$ depends on the choice of orientation on $K$, the commutation relations for $\rho$ are slightly more subtle than those in Lemma 3.2. In particular, as $sw$ reverses orientation on $K$, we have the following:

**Lemma 3.4.** The map $\rho$ commutes with $\Phi$ and $\eta$ up to homotopy. However, the maps $\rho$ and $sw$ do not (in general) commute. Instead, we have

$$(\rho \mathcal{H})' = \tilde{\rho} \mathcal{H}' \quad \text{and} \quad (\tilde{\rho} \mathcal{H})' = \rho \mathcal{H}'$$

and

$$sw \circ \rho \simeq \tilde{\rho} \circ sw \quad \text{and} \quad sw \circ \tilde{\rho} \simeq \rho \circ sw.$$  

**Proof.** Follows from naturality results established by Juhász–Thurston–Zemke [20] and Zemke [37].

Finally, we will often employ the following:

**Lemma 3.5.** Let $(K, w, z)$ be a doubly based knot. Let $f$ and $g$ be two diffeomorphisms of $S^3$ such that $f(w) = g(w)$ and $f(z) = g(z)$, and suppose that $f$ and $g$ are isotopic rel $\{w, z\}$,† Let $\mathcal{H}$ be any Heegaard data for $(K, w, z)$ and let $\mathcal{H}'$ be any choice of Heegaard data for $(f(K), f(w), f(z)) \simeq (g(K), g(w), g(z))$. Then

$$\Phi(f \mathcal{H}, \mathcal{H}') \circ f \simeq \Phi(g \mathcal{H}, \mathcal{H}') \circ g.$$  

**Proof.** Follows from naturality results established by Juhász–Thurston–Zemke [20] and Zemke [37].

†That is, there is an isotopy $H_t$ that sends $w$ to $f(w) = g(w)$ and $z$ to $f(z) = g(z)$ for all $t$.  

---
3.2 Construction of $\tau_K$

We now construct $\tau_K$. As usual, we begin by defining $\tau_K$ with respect to a fixed choice of Heegaard data for $K$.

**Definition 3.6.** Let $(K, \tau)$ be a strongly invertible knot. A *decoration* on $(K, \tau)$ is a choice of orientation for $K$, together with an ordered pair of distinct basepoints $(w, z)$ on $K$ that are interchanged by $\tau$. Following the usual notation for a doubly based knot, we denote these data by $(K, w, z)$. Here we introduce a slight abuse of notation, in that $K$ is not considered to have a fixed orientation as part of $(K, \tau)$, but is considered to have a fixed orientation as part of the data $(K, w, z)$.

A decorated knot is just an oriented, doubly based knot in the usual sense, with the caveat that $w$ and $z$ are symmetric under the action of $\tau$. However, because the choice of extra data will be important, we formally emphasize this in Definition 3.6.

Once a decoration for $(K, \tau)$ is chosen, we may select any set of Heegaard data $\mathcal{H}$ for $(K, w, z)$. Define an automorphism

$$\tau_{\mathcal{H}} : CF_K(\mathcal{H}) \rightarrow CF_K(\mathcal{H})$$

as follows. We first apply the tautological pushforward

$$t : CF_K(\mathcal{H}) \rightarrow CF_K(\tau \mathcal{H}).$$

Here, we denote the pushforward by $t$ so as to not create confusion with the overall action $\tau_K$. Note that $\tau \mathcal{H}$ represents $(K', z, w)$, as $\tau$ is an orientation-reversing involution on $K$ and interchanges $w$ and $z$. As $\mathcal{H}'$ also represents $(K', z, w)$, we have a naturality map

$$\Phi(\tau \mathcal{H}, \mathcal{H}') : CF_K(\tau \mathcal{H}) \rightarrow CF_K(\mathcal{H}').$$

Finally, we apply the switch map

$$sw : CF_K(\mathcal{H}') \rightarrow CF_K(\mathcal{H}).$$

**Definition 3.7.** The action $\tau_{\mathcal{H}} : CF_K(\mathcal{H}) \rightarrow CF_K(\mathcal{H})$ is given by the composition

$$\tau_{\mathcal{H}} : CF_K(\mathcal{H}) \xrightarrow{t} CF_K(\tau \mathcal{H}) \xrightarrow{\Phi} CF_K(\mathcal{H}') \xrightarrow{sw} CF_K(\mathcal{H}).$$

Technically, the middle map $\Phi$ is only defined up to homotopy, but this clearly does not affect the homotopy class of $\tau_{\mathcal{H}}$.

**Theorem 1.7** summarizes the salient features of $\tau_{\mathcal{H}}$:

**Proof of Theorem 1.7.** For (1), applying Lemma 3.2 and keeping track of the appropriate domains gives the following chain of homotopies:

$$\tau_{\mathcal{H}}^2 = (sw \circ \Phi(\tau \mathcal{H}, \mathcal{H}') \circ t) \circ (sw \circ \Phi(\tau \mathcal{H}, \mathcal{H}') \circ t)$$

$$= \Phi(\tau \mathcal{H}', \mathcal{H}) \circ \Phi(\mathcal{H}, \tau \mathcal{H}') \circ sw \circ t \circ sw \circ t$$
\[ = \Phi(\tau H', H) \circ \Phi(H, \tau H') \circ sw \circ t \circ t \]
\[ \simeq \text{id}. \]

Here, in the last line we have used the fact that \( t^2 = sw^2 = \text{id} \), together with the fact that the set of \( \Phi \) form a transitive system. Claim (2) follows from observing that the pushforward map \( t \) and the naturality map \( \Phi(\tau H', H') \) are graded and \( \mathcal{R} \)-equivariant, whereas the map \( sw \) is skew-graded and \( \mathcal{R} \)-skew-equivariant. For (3), we apply Lemmas 3.2 and 3.4 to \( \tau_H \circ t_H \). We move all of the naturality maps to the left, simplify, and then collect the pushforward maps together:

\[ \tau_H \circ t_H = (sw \circ \Phi(\tau H, H') \circ t) \circ (\Phi(\rho \bar{H}, H) \circ \rho \circ \eta) \]
\[ \simeq \Phi(\tau H', H) \circ \Phi((\tau \rho \bar{H})^\tau, \tau H') \circ sw \circ t \circ \rho \circ \eta \]
\[ \simeq \Phi(\tau H', H) \circ \Phi(\tau \bar{\rho} \bar{H}', \tau H') \circ t \circ \bar{\rho} \circ sw \circ \eta \]
\[ \simeq \Phi(\tau \bar{\rho} \bar{H}', H) \circ t \circ \bar{\rho} \circ sw \circ \eta. \]

See [16, section 6.1] for the definition of \( t_H \). It will be convenient for us to replace \( s'H \) with \( s'H^{-1} \); this is allowed because \( s'H \simeq \text{id} \). Note that \( s'H^{-1} \) is represented by the basepoint-moving map against the orientation of \( K \). Doing this, we obtain

\[ s'H^{-1} \circ t_H \circ \tau_H = (\Phi(\bar{\rho}^2 H, H) \circ \bar{\rho}^2) \circ (\Phi(\rho \bar{H}, H) \circ \rho \circ \eta) \circ (sw \circ \Phi(\tau H, H') \circ t) \]
\[ \simeq \Phi(\bar{\rho}^2 H, H) \circ \Phi(\bar{\rho}^2 \rho \bar{H}, \bar{\rho}^2 H) \circ \Phi(\bar{\rho}^2 \rho t \bar{H}', \bar{\rho}^2 \rho \bar{H}) \circ \bar{\rho}^2 \circ \rho \circ \eta \circ sw \circ t \]
\[ \simeq \Phi(\bar{\rho}^2 \rho t \bar{H}', H) \circ \bar{\rho}^2 \circ \rho \circ \eta \circ sw \circ t \]
\[ \simeq \Phi(\tau \bar{\rho}^2 \rho \bar{H}', H) \circ t \circ \bar{\rho}^2 \circ \rho \circ sw \circ \eta. \]

The claim then follows from Lemma 3.5 and the fact that \( t \circ \bar{\rho}^2 \circ \rho \simeq t \circ \bar{\rho} \). Finally, the last part of the theorem follows from the fact that the naturality maps commute with each of the factors used in the definitions of \( \tau_H \) and \( t_H \).

\[ \Box \]

### 3.3 Naturality of \( \tau_K \)

Theorem 1.7 shows that \((CFK(K), \tau_H, t_H)\) is a \((\tau_K, t_K)\)-complex whose homotopy type is independent of the choice of Heegaard data for the oriented, doubly based knot \((K, \omega, \zeta)\). Moreover, the homotopy equivalences between such triples are precisely the naturality maps of Definition 3.1. It thus remains to show that \( \tau_H \) is independent of the choice of decoration on \( K \). The reason we have separated this from the claim of Theorem 1.7 is that in general, there is no canonical identification between two knot Floer complexes for \( K \) in the case that the orientation on \( K \) is reversed or the basepoints are changed. For example, although one can write down complexes for \( K \) and \( K' \) that are isomorphic, such an isomorphism is not via a naturality map \( \Phi \).

We begin with the choice of orientation on \( K \). Let \( H \) be any choice of Heegaard data for \((K, \omega, \zeta)\). As discussed previously, we have the orientation-reversing isomorphism

\[ \eta \circ sw : CFK(H) \to CFK(\bar{H}'). \]

Note that the right-hand side represents \((K', \omega, \zeta)\). We now have:
Lemma 3.8. Let $\mathcal{H}$ be any choice of Heegaard data for $(K, w, z)$ and $\mathcal{H}'$ be the corresponding Heegaard data for $(K', w, z)$. Then:

1. $(\eta \circ sw) \circ \tau_\mathcal{H} \simeq \tau_{\mathcal{H}'} \circ (\eta \circ sw)$;
2. $(\eta \circ sw) \circ t_\mathcal{H} \simeq \zeta_{\mathcal{H}'} \circ t_{\mathcal{H}'} \circ (\eta \circ sw)$.

That is, $\eta \circ sw$ provides a homotopy equivalence

$$(CFK(\mathcal{H}), \tau_\mathcal{H}, t_\mathcal{H}) \simeq (CFK(\mathcal{H}'), \tau_{\mathcal{H}'}, \zeta_{\mathcal{H}'} \circ t_{\mathcal{H}'}).$$

Proof. Claim (1) follows immediately from Lemma 3.2, as both $\eta$ and $sw$ commute with all of the individual factors of $\tau_\mathcal{H}$. To see Claim (2), it is more convenient to replace $\zeta_{\mathcal{H}'}$ with $\zeta_{\mathcal{H}'}^{-1}$. Applying Lemmas 3.2 and 3.4, we obtain

$$(\eta \circ sw) \circ \zeta_{\mathcal{H}'} \circ (\eta \circ sw) = (\Phi(\rho^2 \mathcal{H}', \mathcal{H}') \circ \rho^2) \circ (\Phi(\rho \mathcal{H}', \mathcal{H}') \circ \rho \circ \eta) \circ (\eta \circ sw)$$

and

$$\zeta_{\mathcal{H}'}^{-1} \circ t_{\mathcal{H}'} \circ (\eta \circ sw) = (\Phi(\rho^2 \mathcal{H}', \mathcal{H}') \circ \rho^2) \circ (\Phi(\rho \mathcal{H}', \mathcal{H}') \circ \rho \circ \eta) \circ (\eta \circ sw)$$

The desired naturality statement in this case is then subsumed by a more general claim regarding equivariant diffeomorphisms of $S^3$. In general, if $f : S^3 \rightarrow S^3$ is an equivariant diffeomorphism, then the image $(f(K), \tau)$ of $(K, \tau)$ is another strongly invertible knot. We have:

Lemma 3.9. Let $f : S^3 \rightarrow S^3$ be an equivariant diffeomorphism. Let $\mathcal{H}$ be any choice of Heegaard data for $(K, w, z)$ and $f\mathcal{H}$ be the corresponding pushforward data for $(f(K), f(w), f(z))$. Then

1. $f \circ \tau_\mathcal{H} \simeq \tau_{f\mathcal{H}} \circ f$;
2. $f \circ t_\mathcal{H} \simeq t_{f\mathcal{H}} \circ f$. 

That is, \( f \) provides a homotopy equivalence of triples

\[
(CF\mathcal{K}(H), \tau_H, t_H) \simeq (CF\mathcal{K}(fH), \tau_{fH}, t_{fH}).
\]

**Proof.** This follows from the fact that \( f \) commutes with each of the components of \( \tau_H \) and \( t_H \). \( \square \)

Lemma 3.9 says that up to homotopy equivalence, the triple \((CF\mathcal{K}(K), \tau_H, t_H)\) is a well-defined invariant up to equivariant diffeomorphism (in the decorated setting). In particular, by using Figure 8 we may move \((w, z)\) to any other symmetric pair \((w', z')\) so long as \(w\) and \(w'\) lie in the same subarc of \( K \).

Now consider the case in which \( w' \) is chosen to lie in the opposite subarc from \( w \). Due to our analysis of the previous case, we may in fact assume that \( w' = z \) and \( z' = w \). There is then an obvious diffeomorphism that moves \((K, w, z)\) into \((K, z, w)\); namely, the half Dehn twist \( \rho \) along the oriented knot \( K \). However, \( \rho \) does not commute with all the components of \( \tau_H \). We instead have:

**Lemma 3.10.** Let \( H \) be any choice of Heegaard data for \((K, w, z)\) and \( \rho H \) be the corresponding pushforward data for \((K, z, w)\) under the half Dehn twist \( \rho \). Then

1. \( \rho \circ \tau_H \simeq \xi_{\rho H} \circ \tau_{\rho H} \circ \rho \);
2. \( \rho \circ t_H \simeq t_{\rho H} \circ \rho \).

That is, \( \rho \) provides a homotopy equivalence of triples

\[
(CF\mathcal{K}(H), \tau_H, t_H) \simeq (CF\mathcal{K}(\rho H), \xi_{\rho H} \circ \tau_{\rho H}, t_{\rho H}).
\]

**Proof.** To prove the first claim, we compute

\[
\rho \circ \tau_H = \rho \circ (sw \circ \Phi(\tau H, H') \circ t)
\]

\[
\simeq \Phi(\rho \tau H', \rho H) \circ \rho \circ sw \circ t
\]

\[
\simeq \Phi(\rho \tau H', \rho H) \circ \rho \circ t \circ sw
\]
and
\[
\zeta_{\rho H} \circ \tau_{\rho H} \circ \rho = (\Phi(\rho^2 \rho H, \rho H) \circ \rho^2) \circ (sw \circ \Phi(\tau \rho H, (\rho H)' \circ t) \circ \rho
\]
\[
\simeq \Phi(\rho^2 \rho H, \rho H) \circ \Phi(\rho^2(\tau \rho H)' , \rho^2 \rho H) \circ \rho^2 \circ sw \circ t \circ \rho
\]
\[
\simeq \Phi(\rho^2(\tau \rho H)' , \rho H) \circ \rho^2 \circ sw \circ t \circ \rho
\]
\[
\simeq \Phi(\rho^2(\tau \rho H)' , \rho H) \circ \rho^2 \circ \tilde{\rho} \circ t \circ sw.
\]

The claim then follows from Lemma 3.5 and the fact that \(\rho^2 \circ \tilde{\rho} \circ \tau \simeq \rho \circ \tau\). The second assertion of the lemma follows from the fact that \(\rho\) commutes with all the individual components of \(t_H\).

Lemma 3.10 might seem to imply that the homotopy equivalence class of \(\tau_H\) is dependent on the order of the basepoints \(w\) and \(z\). Indeed, without a choice of decoration, it initially appears that \(\tau_H\) is only well-defined up to composition with the Sarkar map. This is a reasonable heuristic, but not quite correct: it is important to stress that there is no canonical way to compare two knot Floer complexes for \(K\) with different pairs of basepoints. Lemma 3.10 should thus be interpreted as a statement specifically regarding the choice of homotopy equivalence \(\rho\) between a choice of Heegaard data for \((K, w, z)\) and a choice of Heegaard data for \((K, z, w)\). A priori, it is possible that a different choice of homotopy equivalence might intertwine \(\tau_H\) and \(\tau_{\rho H}\). Indeed, recall from Lemma 2.21 that \(\zeta_H \circ \tau_H\) and \(\tau_H\) are conjugate up to homotopy. More precisely,

\[
(CFK(H), \tau_H, \zeta_H) \simeq (CFK(H), \zeta_H \circ \tau_H, \zeta_H \circ t_H).
\]

Hence, Lemma 3.10 combined with Lemma 2.21 shows that the homotopy equivalence class of \(\tau_H\) is invariant under exchanging the roles of \(w\) and \(z\), while the homotopy equivalence class of the triple \((CFK(H), \tau_H, \zeta_H)\) is not, at least a priori. Instead, we see that \((CFK(H), \tau_H, \zeta_H)\) is homotopy equivalent to either of the classes

\[
(CFK(\rho H), \zeta_{\rho H} \circ \tau_{\rho H}, \zeta_{\rho H} \circ t_{\rho H}) \simeq (CFK(\rho H), \tau_{\rho H}, \zeta_{\rho H} \circ t_{\rho H}).
\]

The situation is summarized in the following pair of theorems:

**Theorem 3.11.** Let \((K, \tau)\) be a decorated strongly invertible knot. The triple \((CFK(H), \tau_H, \zeta_H)\) is independent, up to homotopy equivalence, of the choice of \(H\) so long as \(H\) is compatible with the chosen decoration; moreover, it is an invariant of \((K, \tau)\) up to equivariant diffeomorphism, interpreted in the decorated setting.

**Proof.** Follows from Theorem 1.7 and Lemma 3.9. \(\square\)

In the decorated setting, we thus suppress the choice of Heegaard data and refer to the homotopy equivalence class of the triple \((CFK(K), \tau_K, t_K)\) unambiguously. In the undecorated setting, we instead have the following:

**Theorem 3.12.** The homotopy equivalence class of \((CFK(H), \tau_H)\) is independent of the choice of decoration on \((K, \tau)\). Reversing orientation or interchanging the basepoints each alters the homotopy equivalence class of \((CFK(H), \tau_H, t_H)\) by a twist.
Proof. Follows from Lemmas 3.8 and 3.10.

In the undecorated setting, we thus refer to \((\mathcal{CFK}(K), \tau_K)\) unambiguously, although this is not entirely natural. However, we must take care when discussing \((\mathcal{CFK}(K), \tau_K, \iota_K)\) in the undecorated setting. Explicitly, we have constructed homotopy equivalences.

\begin{itemize}
  \item \((\mathcal{CFK}(K, w, z), \tau_K, \iota_K) \simeq (\mathcal{CFK}(K_{r}, w, z), \tau_{K_{r}}, \iota_{K_{r}})\) via Lemma 3.8.
  \item \((\mathcal{CFK}(K, w, z), \tau_K, \iota_K) \simeq (\mathcal{CFK}(K, z, w), \iota_{\tau_K}, \iota_K)\) via Lemma 3.10.
  \item \((\mathcal{CFK}(K, w, z), \tau_K, \iota_K) \simeq (\mathcal{CFK}(K, w, z), \iota_{\tau_K}, \iota_{K})\) via Lemma 2.21.
\end{itemize}

Again, however, note that these should not be treated as canonical.

### 3.4 Equivariant concordance

We now turn to the behavior of \(\tau_K\) under equivariant concordance. As in the previous section, we first need to define a notion of equivariant concordance in the decorated setting.

**Definition 3.13.** Let \((K_1, \tau_1)\) and \((K_2, \tau_2)\) be two decorated strongly invertible knots and let \((W, \tau_W, \Sigma)\) be an isotopy-equivariant homology concordance between them. We say that \((W, \tau_W, \Sigma)\) respects the decorations (alternatively, is equivariant in the decorated sense) if:

1. \(\Sigma\) is an oriented knot concordance; and,
2. we can find a pair of properly embedded arcs \(\gamma_1, \gamma_2 \subseteq \Sigma\) such that:
   a. each \(\gamma_i\) has one endpoint on \(K_1\) and one endpoint on \(K_2\), and these endpoints are fixed by \(\tau_1\) and \(\tau_2\), respectively;
   b. we have an isotopy (rel boundary) moving \((\tau_W(\Sigma), \tau_W(\gamma_1), \tau_W(\gamma_2))\) into \((\Sigma, \gamma_1, \gamma_2)\);
   c. the arcs divide \(\Sigma\) into two rectangular regions, one of which contains both \(w_1\) and \(w_2\) (we call this the black region), and the other of which contains both \(z_1\) and \(z_2\) (we call this the white region).

When the context is clear, we refer to such a \(\Sigma\) as a decorated isotopy-equivariant concordance. Note that \(\Sigma\) is just an isotopy-equivariant cobordism for which we can find an appropriate set of isotopy-equivariant dividing curves, in the sense of [37].

**Theorem 3.14.** Let \((K_1, \tau_1)\) and \((K_2, \tau_2)\) be two decorated strongly invertible knots. A decorated isotopy-equivariant concordance between \((K_1, \tau_1)\) and \((K_2, \tau_2)\) induces a local equivalence

\[(\mathcal{CFK}(K_1), \tau_{K_1}, \iota_{K_1}) \sim (\mathcal{CFK}(K_2), \tau_{K_2}, \iota_{K_2}).\]

**Proof.** By work of Zemke [37], we obtain a concordance map

\[F_{W, F} : \mathcal{CFK}(K_1) \to \mathcal{CFK}(K_2).\]

Here, \(F\) represents the concordance \(\Sigma\) with the choice of dividing curves \(\gamma_1\) and \(\gamma_2\). It is standard that \(F_{W, F}\) is grading-preserving and has the requisite behavior under localization. In [16, section 4.5] and [39, Theorem 1.5], it is shown that \(F_{W, F}\) is \(\iota_K\)-equivariant (up to homotopy). It thus suffices to show that it is \(\tau_K\)-equivariant.
Consider the diagram:

\[
\begin{array}{c}
\text{CFK}(K_1, w_1, z_1) \xrightarrow{F_{W,F}} \text{CFK}(K_2, w_2, z_2) \\
\downarrow t \quad \downarrow t \\
\text{CFK}(K_1', z_1, w_1) \xrightarrow{F_{W,W}(F)} \text{CFK}(K_2', z_2, w_2) \\
\downarrow sw \quad \downarrow sw \\
\text{CFK}(K_1, w_1, z_1) \xrightarrow{F_{W,sw(W)(F)}} \text{CFK}(K_2, w_2, z_2)
\end{array}
\]

Here, by \( \text{CFK}(K_1, w_1, z_1) \), we mean any representative for the complex of \((K_1, w_1, z_1)\) in the transitive system of complexes for doubly basepointed knots. (Similarly for the other entries in the diagram; we have thus suppressed writing the naturality maps \( \Phi \) as part of the vertical arrows.)

The first square of this diagram commutes due to the diffeomorphism invariance of link cobordisms \([37, \text{section 1.1}]\). By \( \tau_W(F) \), we mean the image of the decoration of \( F \) under \( \tau_W \). The second square of the diagram also tautologically commutes; here, \( sw(\tau_W(F)) \) is obtained from \( \tau_W(F) \) by interchanging the roles of the black and white regions on \( \tau_W(F) \) and reversing orientation. The fact that our concordance is equivariant in the decorated sense shows that \( sw(\tau_W(F)) \) is isotopic to \( F \) rel boundary, including the dividing curves and coloring of regions on \( \Sigma \). The isotopy invariance of link cobordisms then implies that

\[
F_{W,sw(\tau_W(F))} \simeq F_{W,F}.
\]

This shows that \( F_{W,F} \) homotopy commutes with \( \tau_K \) and hence constitutes a local map from \((\text{CFK}(K_1), \tau_{K_1}, t_{K_1})\) to \((\text{CFK}(K_2), \tau_{K_2}, t_{K_2})\). Turning the concordance around gives the local map in the other direction and completes the proof. \( \square \)

In the decorated setting, the local equivalence class of the triple \((\text{CFK}(K), \tau_K, t_K)\) is thus an invariant of isotopy-equivariant concordance. If \((K_1, \tau_1)\) and \((K_2, \tau_2)\) do not come equipped with decorations, then (according to Theorem 3.12) we may still unambiguously speak of the homotopy equivalence classes of \((\text{CFK}(K_1), \tau_{K_1})\) and \((\text{CFK}(K_2), \tau_{K_2})\). We claim that in the presence of an (undeckored) isotopy-equivariant concordance (as in Definition 2.9), these are again guaranteed to be locally equivalent:

**Theorem 3.15.** Let \((K_1, \tau_1)\) and \((K_2, \tau_2)\) be two strongly invertible knots. An isotopy-equivariant concordance between \((K_1, \tau_1)\) and \((K_2, \tau_2)\) gives a local equivalence of pairs

\[
(\text{CFK}(K_1), \tau_{K_1}) \simeq (\text{CFK}(K_2), \tau_{K_2}).
\]

Moreover, suppose we equip \((K_1, \tau_1)\) and \((K_2, \tau_2)\) with decorations, so that the homotopy equivalence classes of their associated \((\tau_K, t_K)\)-complexes are defined. Then \((\text{CFK}(K_1), \tau_{K_1}, t_{K_1})\) is locally equivalent to either \((\text{CFK}(K_2), \tau_{K_2}, t_{K_2})\) or the twist of \((\text{CFK}(K_2), \tau_{K_2}, t_{K_2})\).
Proof. Because of the discussion following Lemma 2.21, the first claim follows from the second. Thus, let \((K_1, \tau_1)\) and \((K_2, \tau_2)\) be two decorated strongly invertible knots. Let \((W, \tau_W, \Sigma)\) be an equivariant concordance between them that may not be equivariant in the decorated sense. Due to Lemma 3.8, up to twisting the \((\tau_K, t_K)\)-complexes at either end, we may assume that \(\Sigma\) is an oriented concordance.

Now choose any pair of properly embedded arcs \(\gamma_1, \gamma_2 \subseteq \Sigma\) satisfying (a) and (c) of Definition 3.13(2). That is, each \(\gamma_i\) has one end point on \(K_1\) and one endpoint on \(K_2\), and these endpoints are fixed by \(\tau_1\) and \(\tau_2\), respectively. Moreover, the curves \(\gamma_1\) and \(\gamma_2\) divide \(\Sigma\) into two rectangular regions, one of which contains the \(w_i\) basepoints and the other of which contains the \(z_i\) basepoints. Let \(F\) denote this concordance with the choice of dividing arcs \(\gamma_1\) and \(\gamma_2\). As usual, \(F_{W,F}\) commutes with \(t_K\) (up to homotopy). Following the proof of Theorem 3.14, we see that as \(\Sigma\) may not be isotopy equivariant in the decorated sense, we no longer have that \(sw(\tau_W(F))\) is isotopic to \(F\). Instead, \(sw(\tau_W(F))\) is necessarily isotopic to a decorated concordance obtained by applying some number of Dehn twists to \(F\).

The concordance map associated to this altered decoration is given by precomposing the concordance map for \(sw(\tau_W(F))\) with a power of the Sarkar map. Following the proof of Theorem 3.14, we thus see that

\[ \tau_{K_2} \circ F_{W,F} \simeq F_{W,F} \circ (\zeta^n_{K_1} \circ \tau_{K_1}). \]

Hence, \(F_{W,F}\) intertwines \(\tau_{K_1}\) and \(\tau_{K_2}\) up to composition with some power of the Sarkar map. As the Sarkar map is a homotopy involution, the claim follows.

In the undecorated setting, an equivariant concordance thus only induces a local equivalence of \((\tau_K, t_K)\)-triples up to twist. (Of course, note that if our knots are not decorated, then these triples are only defined up to twist anyway.) However, we still obtain a local equivalence between their \(\tau_K\)-complexes.

Having established the necessary naturality results, we now conclude with the construction of the numerical invariants of Theorem 1.1:

**Definition 3.16.** Let \((K, \tau)\) be a strongly invertible knot, which may be neither directed nor decorated. Fix any decoration on \((K, \tau)\) and consider the resulting \((\tau_K, t_K)\)-complex \((CFk(K), \tau_K, t_K)\). Following Subsection 2.5, define:

\[ V_0^{\tau}(K) = V_0^{\tau}(CFk(K), \tau_K, t_K) \quad \text{and} \quad V_0^{\tau}(K) = V_0^{\tau}(CFk(K), \tau_K, t_K) \]

and

\[ V_0^{\tau}(K) = V_0^{\tau}(CFk(K), \tau_K, t_K) \quad \text{and} \quad V_0^{\tau}(K) = V_0^{\tau}(CFk(K), \tau_K, t_K). \]

This is independent of the choice of decoration. Indeed, according to Theorem 3.12, changing the decoration on \((K, \tau)\) corresponds to applying a twist by \(\zeta_K\). However, due to Lemma 2.25, our numerical invariants are not altered by this operation.

Putting everything together, we obtain:
By Theorem 3.15, an isotopy-equivariant homology concordance (in the undecorated category) induces a local equivalence of \((\tau_K, \iota_K)\)-complexes, up to a twist by \(\zeta_K\). By Lemma 2.25, this leaves our numerical invariants unchanged. □

### 3.5 Directed knots

We now turn to the connection between the decorated and directed settings.

**Definition 3.17.** Let \((K, \tau)\) be a directed strongly invertible knot, in the sense of Definition 2.3. We say that a decoration \((K, w, z)\) is compatible with this choice of direction if the oriented subarc of \(K\) containing the \(z\)-basepoint induces the same orientation on its boundary as the chosen half-axis. See Figure 9.

If \((K, \tau)\) is a directed strongly invertible knot, then (up to orientation-preserving equivariant diffeomorphism) there are two compatible decorations on \((K, \tau)\), which are related to each other by simultaneously reversing orientation on \(K\) and interchanging the roles of \(w\) and \(z\). See Figure 9. Note that as discussed in Remark 2.7, a directed strongly invertible knot does not generally come with a specified orientation. Conversely, suppose that \((K, \tau)\) is a decorated strongly invertible knot. Then there are two possible choices of direction that are compatible with this decoration; they are related by simultaneously switching the half-axis and reversing the axis orientation.

**Theorem 3.18.** We have a well-defined set map from the directed equivariant concordance group to the local equivalence group of \((\tau_K, \iota_K)\)-complexes.

\[ h : \tilde{\mathcal{C}} \to \mathfrak{S}_{\tau, d} \]

**Proof.** Let \((K, \tau)\) be a directed strongly invertible knot. As explained above, the choice of direction determines two compatible decorations on \((K, \tau)\), which are related to each other by simultaneously reversing orientation on \(K\) and interchanging the roles of \(w\) and \(z\). By Theorem 3.12, applying both of these operations in succession does not change the homotopy type of the associated \((\tau_K, \iota_K)\)-complex. Hence, using the convention of Definition 3.17, we may unambiguously talk of the \((\tau_K, \iota_K)\)-complex of a directed knot.

Moreover, suppose that we have a directed equivariant concordance \((\Sigma, \tau_{S^1 \times I})\) from \((K_1, \tau_1)\) to \((K_2, \tau_2)\). Definition 2.5 implies that we can find a pair of arcs \(\gamma_1\) and \(\gamma_2\) that run along the length

---

**FIGURE 9** Two decorations compatible with a fixed choice of direction.
of $\Sigma$ and are fixed by $\tau_{S^3 \times I}$. We may choose our compatible decorations on $K_1$ and $K_2$ such that $\Sigma$ is an oriented concordance. Then $\Sigma$ (with the arcs $\gamma_1$ and $\gamma_2$) forms a decorated concordance in the sense of Definition 3.13.

\section{CONNECTED SUMS}

In this section, we establish further fundamental results regarding the action of $\tau_K$ and show that the map $h$ from Theorem 3.18 is a group homomorphism.

\subsection{Connected sums}

We begin with the connected sum formula. Let $(K_1, \tau_1)$ and $(K_2, \tau_2)$ be two directed strongly invertible knots. As discussed in Subsection 2.1, we may form the equivariant connected sum $(K_1 \# K_2, \tau_1 \# \tau_2)$, which is another directed strongly invertible knot. Note that according to Theorem 3.18, we have a well-defined (up to homotopy equivalence) $(\tau_K, \nu_K)$-complex for each of the directed pairs $(K_1, \tau_1)$, $(K_2, \tau_2)$, and $(K_1 \# K_2, \tau_1 \# \tau_2)$, obtained by choosing a compatible decoration in each case.

\begin{theorem}
Let $(K_1, \tau_1)$ and $(K_2, \tau_2)$ be directed strongly invertible knots and $K_1 \# K_2$ be their equivariant connected sum. Then

\[ (CPK(K_1 \# K_2), \tau_{K_1 \# K_2}, \nu_{K_1 \# K_2}) \text{ and } (CPK(K_1) \otimes CPK(K_2), \tau_{\otimes}, \nu_{\otimes}) \]

are homotopy equivalent, where

\[ \tau_{\otimes} = \tau_{K_1} \otimes \tau_{K_2} \]

and

\[ \nu_{\otimes} = (id \otimes id + \Phi \otimes \Psi)(\nu_{K_1} \otimes \nu_{K_2}). \]

\end{theorem}

\begin{proof}
Define an equivariant cobordism from $(S^3, K_1, \tau_1) \sqcup (S^3, K_2, \tau_2)$ to $(S^3, K_1 \# K_2, \tau_1 \# \tau_2)$ by attaching a 1-handle and then a band in the obvious manner. Denote this by $(W, \tau_W, \Sigma)$; the surface $\Sigma$ is schematically depicted in Figure 10. The knots $K_1$ and $K_2$ are represented by the two inner circles and have half-axes given by their respective horizontal diameters (oriented from left-to-right). Their connected sum $K_1 \# K_2$ is represented by the outer ellipse and has half-axis defined similarly. We place $w$- and $z$-basepoints on $K_1$, $K_2$, and $K_1 \# K_2$ as indicated in Figure 10; note that these are compatible with each of the chosen directions. Let $F$ be the set of dividing arcs on $\Sigma$ consisting of the three indicated horizontal arcs. This makes $\Sigma$ into a cobordism that is equivariant in the decorated sense.

In [39, Theorem 1.1], Zemke shows that the map

\[ F_{W, F} : CPK(K_1) \otimes CPK(K_2) \to CPK(K_1 \# K_2) \]

defined by the link cobordism with decoration $F$ is a homotopy equivalence, together with the map in the other direction constructed by turning the cobordism around. (Indeed, Figure 10 is
FIGURE 10 Decorated equivariant cobordism from $K_1 \sqcup K_2$ to $K_1 \# K_2$. Black dots represent $w$-basepoints; white dots represent $z$-basepoints. The action of $\tau_w$ is given by reflection across the horizontal axis. See [39, fig. 6.1].

just [39, fig. 6.1], which corresponds to the map $G_1$ in [39, Theorem 1.1].) Moreover, according to [39, Theorem 1.1], this homotopy equivalence intertwines $(id \otimes id + \Phi \otimes \Psi)(\iota_{K_1} \otimes \iota_{K_2})$ on the incoming end with the connected sum involution $\iota_{K_1 \# K_2}$ on the outgoing end. We thus simply need to show that $F_{W,F}$ intertwines $\tau_{K_1} \otimes \tau_{K_2}$ with $\tau_{K_1 \# K_2}$. This follows from the same argument as in Theorem 3.14. We have the commutative diagram:

\[
\begin{array}{ccc}
CF\mathcal{K}(K_1) \otimes CF\mathcal{K}(K_2) & \xrightarrow{F_{W,F}} & CF\mathcal{K}(K_1 \# K_2) \\
\downarrow t \otimes t & & \downarrow t \\
CF\mathcal{K}(K'_1) \otimes CF\mathcal{K}(K'_2) & \xrightarrow{F_{W,F}(w)} & CF\mathcal{K}(K'_1 \# K'_2) \\
\downarrow sw \otimes sw & & \downarrow sw \\
CF\mathcal{K}(K_1) \otimes CF\mathcal{K}(K_2) & \xrightarrow{F_{W,F}(sw(F))} & CF\mathcal{K}(K_1 \# K_2)
\end{array}
\]

Each of the two squares commutes tautologically. It is clear from Figure 10 that $sw(\tau_w(F))$ coincides with $F$; hence $F_{W,F}$ intertwines $\tau_{K_1} \otimes \tau_{K_2}$ with $\tau_{K_1 \# K_2}$. The proof for the reversed cobordism map is similar. □

Remark 4.2. Note that the above proof does not allow us to use $\Psi \otimes \Phi$ in place of $\Phi \otimes \Psi$ in the statement of Theorem 4.1, unless the conventions of Definition 2.4 are also changed. This asymmetry is due to the fact that we have specifically used the map $G_1$ in [39, Theorem 1.1]. The map $G_2$ in [39, Theorem 1.1] intertwines $(id \otimes id + \Psi \otimes \Phi)(\iota_{K_1} \otimes \iota_{K_2})$ with $\iota_{K_1 \# K_2}$. However, $G_2$ does not correspond to a decoration that is geometrically equivariant; see [39, fig. 5.1]. See the discussion in Subsection 2.3.

This completes the proof of Theorem 1.8:

Proof of Theorem 1.8. Follows from Theorems 3.18 and 4.1. □
4.2 The swapping involution

We now compute the action of the swapping involution described in Subsection 1.2. In general, given a knot $K$ in $S^3$, we can form the connected sum $K \# K'$ as in Figure 11. As discussed in [3, section 2], this admits an obvious strong inversion. In fact, as discussed in [3, section 2], we obtain a homomorphism from the usual concordance group to $\tilde{\mathcal{C}}$ by choosing the half-axis depicted in Figure 11. We call this the \textit{swapping involution} on $K \# K'$ and denote it by $\tau_{sw}$. Our goal will be to calculate the $(\tau_K, t_K)$-complex of $(K \# K', \tau_{sw})$ (with this choice of direction).

To compute the action of $\tau_{sw}$, we need to discuss the construction of $K \# K'$ more precisely. Assume that $(K, w, z)$ is an oriented, doubly based knot in $S^3$. We think of the projection of $K$ as lying entirely to the left of a vertical axis. Denote $180^\circ$ rotation about this axis by $\tau$. We obtain another doubly based knot $(\tau K, \tau w, \tau z)$ by applying $\tau$ to $K$. Although this can of course be identified with $K$, it will be helpful for us to emphasize the second copy of $K$ as being the image of the first under $\tau$; we thus henceforth write $\tau K$ rather than $K$. We moreover modify the decoration on $\tau K$ by applying $sw$; this gives $(\tau K', \tau z, \tau w)$.

As in Figure 11, we now attach a $\tau$-equivariant band to form the connected sum of $K$ and $\tau K'$. It will be convenient for us to assume that this band has a particular arrangement with respect to the basepoints on $K$ and $\tau K'$. Specifically, we require the foot of our band on $K$ to lie on the oriented subarc of $K$ running from $z$ to $w$, and the foot of our band on $\tau K'$ to lie on the oriented subarc running from $\tau w$ to $\tau z$. We furthermore place a pair of symmetric basepoints $w'$ and $z'$ on $K \# \tau K'$ in such a way so that $w'$ lies on $K$ and $z'$ lies on $\tau K'$. See Figure 12. Note that this
makes \((K\#\tau K', w', z')\) into a decorated strongly invertible knot, and this choice of decoration is compatible with the direction chosen in Figure 11.

Before proceeding further, we first construct the induced action of \(\tau\) on the disjoint union \((K, w, z) \sqcup (\tau K', \tau z, \tau w)\). Define a chain map
\[
\tau_{\text{exch}} : C\mathcal{F}K(K, w, z) \otimes C\mathcal{F}K(\tau K', \tau z, \tau w) \to C\mathcal{F}K(K, w, z) \otimes C\mathcal{F}K(\tau K', \tau z, \tau w)
\]
as follows. First apply the tautological pushforward associated to \(\tau\). This induces an isomorphism from \(C\mathcal{F}K(K, w, z)\) to \(C\mathcal{F}K(\tau K, \tau w, \tau z)\), and also an isomorphism from \(C\mathcal{F}K(\tau K', \tau z, \tau w)\) to \(C\mathcal{F}K(K', z, w)\). We thus obtain a map
\[
t : C\mathcal{F}K(K, w, z) \otimes C\mathcal{F}K(\tau K', \tau z, \tau w) \to C\mathcal{F}K(K', z, w) \otimes C\mathcal{F}K(\tau K, \tau w, \tau z),
\]
which sends the first factor on the left isomorphically onto the second factor on the right, and the second factor on the left isomorphically onto the first factor on the right. We then apply the map \(sw\) from Definition 3.1 in each factor:
\[
sw \otimes sw : C\mathcal{F}K(K', z, w) \otimes C\mathcal{F}K(\tau K, \tau w, \tau z) \to C\mathcal{F}K(K, w, z) \otimes C\mathcal{F}K(\tau K', \tau z, \tau w).
\]
The action of \(\tau_{\text{exch}}\) is thus defined by the composition
\[
\tau_{\text{exch}} = (sw \otimes sw) \circ t.
\]
Note that this is just the action of \(\tau_K\) defined in Subsection 3.2, generalized to the symmetric link \((K, w, z) \sqcup (\tau K', \tau z, \tau w)\).

We now establish the main theorem of this subsection:

**Theorem 4.3.** Denote the induced action of \(\tau_{sw}\) also by \(\tau_{sw}\). Then
\[
(C\mathcal{F}K(K \# \tau K'), \tau_{sw}, t_{K \# \tau K'}) \text{ and } (C\mathcal{F}K(K) \otimes C\mathcal{F}K(\tau K'), \tau_\otimes, t_\otimes)
\]
are homotopy equivalent, where
\[
\tau_\otimes = (\text{id} \otimes \text{id} + \Psi \otimes \Phi) \circ \tau_{\text{exch}}
\]
and
\[
t_\otimes = \zeta_\otimes \circ (\text{id} \otimes \text{id} + \Psi \otimes \Phi) \circ (t_K \otimes t_{\tau K'}).
\]

**Proof.** As in the proof of Theorem 4.1, we consider the pair-of-pants cobordism \((W, \tau_W, \Sigma)\) from \(K \sqcup \tau K'\) to \(K \# \tau K'\). Decorate \(\Sigma\) with the set \(P\) of dividing curves depicted in Figure 13. The involution \(\tau_W\) on this cobordism restricts to \(\tau\) on the incoming end and \(\tau_{sw}\) on the outgoing end. As in Figure 12, this is given by reflection across the vertical axis.
Now, the decoration $F$ is not equivariant with respect to $\tau_W$. Nevertheless, we have the following homotopy-commutative diagram:

$$
\begin{array}{ccc}
\CF K(K, w, z) \otimes \CF K(\tau K', \tau z, \tau w) & \xrightarrow{F_{W,F}} & \CF K(K \# \tau K', w', z') \\
\downarrow{t} & & \downarrow{t} \\
\CF K(K', z, w) \otimes \CF K(\tau K, \tau w, \tau z) & \xrightarrow{F_{W,sw(\tau)}} & \CF K(K' \# \tau K, z', w') \\
\downarrow{sw \otimes sw} & & \downarrow{sw} \\
\CF K(K, w, z) \otimes \CF K(\tau K', \tau z, \tau w) & \xrightarrow{F_{W,sw(\tau)}} & \CF K(K \# \tau K', w', z')
\end{array}
$$

Here, the decoration $sw(\tau_W(F))$ is obtained by switching the designation of white and black regions in $\tau_W(F)$ and reversing orientation. Hence, we obtain

$$
\tau_{sw} \circ F_{W,F} \simeq F_{W,sw(\tau_W(F))} \circ \tau_{exch}.
$$

We now claim that

$$
F_{W,sw(\tau_W(F))} \simeq F_{W,F} \circ (\id \otimes \id + \Psi \otimes \Phi).
$$

To see this, we use the bypass relation for link cobordism maps established in [39, Lemma 1.4]. A schematic outline of the bypass relation is given in Figure 14.

In our case, we apply the bypass relation to the dotted disk in the top-left of Figure 15. The effect of applying the bypass relation is also depicted in Figure 15 and yields the claim. It follows that $F_{W,F}$ intertwines $\tau_{sw}$ with $(\id \otimes \id + \Psi \otimes \Phi) \circ \tau_{exch}$.
We now consider the behavior of $F_{W,F}$ with respect to $t_K$. Note that $F_{W,F}$ is not the same as the map used in the connected sum formula of Theorem 4.1, and thus does not necessarily intertwine $t_K \# t_K'$ and $(\text{id} \otimes \text{id} + \Phi \otimes \Psi) \circ (t_K \otimes t_{K'})$. Instead, we have the following commutative diagram:

Here, $\eta(F)$ is obtained from $F$ by interchanging the roles of the white and black regions of $F$, but not reversing orientation. Hence, we obtain

$$t_K \# t_K' \circ F_{W,F} \simeq (\rho \circ F_{W,\eta(F)} \circ (\tilde{\rho} \otimes \tilde{\rho})) \circ (t_K \otimes t_{K'}) .$$

We now claim that

$$\rho \circ F_{W,\eta(F)} \circ (\tilde{\rho} \otimes \tilde{\rho}) \simeq F_{W,F} \circ \zeta \otimes (\text{id} \otimes \text{id} + \Psi \otimes \Phi).$$

Indeed, the reader should check that applying an oppositely oriented half-Dehn twist to each end of $\eta(F)$ gives a decoration isotopic to $sw(\tau_W(F))$. Hence,

$$\tilde{\rho} \circ F_{W,\eta(F)} \circ (\tilde{\rho} \otimes \tilde{\rho}) \simeq F_{W,sw(\tau_W(F))}.$$
Applying formula (2) for $F_{\mathcal{W},sw(\tau_{\mathcal{W}}(P'))}$ and using the fact that $\rho$ and $\tilde{\rho}$ differ by an application of the Sarkar map, we obtain

$$\rho \circ F_{\mathcal{W},\eta(P')} \circ (\tilde{\rho} \otimes \tilde{\rho}) \simeq \zeta_# \circ F_{\mathcal{W},P} \circ (\text{id} \otimes \text{id} + \Psi \otimes \Phi).$$

Here, $\zeta_#$ is the Sarkar map on the connected sum $K\#\tau K'$. The fact that the Sarkar map can be computed algebraically shows that $\zeta_# \circ F_{\mathcal{W},P} \simeq F_{\mathcal{W},P} \circ \zeta_\otimes$, as $F_{\mathcal{W},P}$ is an explicit homotopy equivalence that identifies $CF\mathcal{K}(K) \otimes CF\mathcal{K}(\tau K')$. For completeness, however, we include a more concrete topological proof in Lemma 4.4. The desired claim follows.

Finally, we show that turning $F_{\mathcal{W},P}$ around constitutes a homotopy inverse to $F_{\mathcal{W},P'}$. To see that these are homotopy inverses, note that

$$F_{\mathcal{W},P} \simeq q_K \circ F_{\mathcal{W},P'} \circ (\text{id} \otimes \rho).$$

Here, $q$ is a quarter-Dehn twist and $P'$ is the decoration in Figure 16.

Writing $F_{\mathcal{W},P'}$ for the reverse of $F_{\mathcal{W},P}$, we thus have

$$F_{\mathcal{W},P'} \simeq (\text{id} \otimes \tilde{\rho}) \circ F_{\mathcal{W},P'} \circ \tilde{q}_K.$$

As in the proof of Theorem 4.1, $F_{\mathcal{W},P'}$ and $F_{\mathcal{W},P'}$ are homotopy inverses. The claim follows. \[\square\]

**Lemma 4.4.** With $\mathcal{F}$ as in the proof of Theorem 4.3, we have $\zeta_# \circ F_{\mathcal{W},P} \simeq F_{\mathcal{W},P'} \circ \zeta_\otimes$.

**Proof.** As in the proof of Theorem 4.3, write $F_{\mathcal{W},P} \simeq q \circ F_{\mathcal{W},P'} \circ (\text{id} \otimes \rho)$. Using the fact that $q$ and $\rho$ are induced by orientation-preserving diffeomorphisms, it is straightforward to check that $q \circ \zeta_# \simeq \zeta_# \circ q$ and $(\text{id} \otimes \rho) \circ \zeta_\otimes \simeq \zeta_\otimes \circ (\text{id} \otimes \rho)$. (For the latter, simply note that $\Phi$ and $\Psi$ commute with all such pushforward maps.) It thus suffices to prove the lemma with the decoration $P'$ in place of $\mathcal{F}$. Applying the definition of $\zeta_\otimes$, this reduces to showing

$$\Psi \Phi \circ F_{\mathcal{W},P'} \simeq F_{\mathcal{W},P'} \circ (\text{id} \otimes \Phi \Psi + \Phi \Psi \otimes \text{id} + \Phi \otimes \Psi + \Psi \otimes \Phi).$$

We repeatedly apply suitable bypass relations. First note that $\Psi \Phi \circ F_{\mathcal{W},P'}$ is the map associated to the decoration on the left in Figure 17. Applying the bypass relation to the disk on the left-hand side gives the two decorations shown on the right. We denote these by $F_1$ and $F_2$, respectively.

We then further apply a bypass relation to $F_1$. Doing this for the disk on the right-hand side of Figure 17 gives the two decorations shown in Figure 18, which we denote by $F_3$ and $F_4$. Note that $F_{\mathcal{W},P} \simeq F_{\mathcal{W},P'} \circ (\Psi \Phi \otimes \text{id})$. 

![Figure 16](image-url) Writing $\mathcal{F}$ in terms of $\mathcal{F}'$. 

\[\text{FIGURE 16} \quad \text{Writing } \mathcal{F} \text{ in terms of } \mathcal{F}'.\]
We now apply a final bypass relation to $\mathcal{F}_3$. Doing this for the disk indicated in Figure 18 gives the two decorations shown in Figure 19, which we denote by $\mathcal{F}_5$ and $\mathcal{F}_6$. Note that $F_{W,F_3} \simeq F_{W,F_3} \circ (\Psi \otimes \Phi)$, while $F_6$ is just $F'$.

Putting the results of Figures 18 and 19 together, we have that

$$F_{W,F_1} \simeq F_{W,F_1} \circ (\Psi \otimes \Phi) + F_{W,F'} + F_{W,F'} \circ (\Psi \otimes \Phi).$$

Now, note that in Figure 17, the decorations $F_{W,F_1}$ and $F_{W,F_2}$ are related by reflection across the vertical axis. By applying similar bypass relations to $F_2$ (using the reflections of the disks for $F_1$) we obtain

$$F_{W,F_2} \simeq F_{W,F_2} \circ (\Phi \otimes \Psi) + F_{W,F'} + F_{W,F'} \circ (\Phi \otimes \psi).$$

Adding these two relations together and using the fact that $\Psi$ and $\Phi$ homotopy commute gives the desired result. □
We may assume that $\Sigma$ is exactly equivariant near $\partial \Sigma = K \subseteq \partial W$. We have chosen an arc of fixed points lying on $\Sigma$; this is represented by the dotted line. Puncturing $W$ at a point on this dotted line gives an isotopy-equivariant knot cobordism from the unknot to $K$. This is schematically represented by cutting out the sphere indicated in the figure. Stabilizing $\Sigma$ is represented by the 1-handle with feet near the dotted line.

5 | EQUIVARIANT SLICE GENUS BOUNDS

We now prove Theorem 1.2. This closely follows [22, Theorem 1.7].

Proof of Theorem 1.2. Let $(K, \tau)$ be a strongly invertible knot, which may be neither directed nor decorated. Let $\Sigma$ be an isotopy-equivariant slice surface for $(K, \tau)$ in some homology ball $(W, \tau_W)$. We may assume $\tau_W$ acts as $\tau \times \text{id}$ on some collar neighborhood $(\partial W) \times I$. We may furthermore assume that the isotopy from $\tau_W(\Sigma)$ to $\Sigma$ does not move this collar neighborhood of $\partial W$ and that $\Sigma$ is exactly equivariant near $\partial W$. Hence, we can puncture $\Sigma$ at some fixed point of $\tau_W$ near $\partial W$ and treat $\Sigma$ and $\tau_W(\Sigma)$ as isotopy-equivariant knot cobordisms from the unknot (with the obvious strong inversion) to $(K, \tau)$. See Figure 20.

For reasons that will be clear presently, it will be convenient for us to stabilize $\Sigma$ a certain number of times. If the genus of $\Sigma$ is even, then we stabilize $\Sigma$ twice; if the genus of $\Sigma$ is odd, then we stabilize $\Sigma$ once. We denote the stabilized surface by $\Sigma'$; note that the genus of $\Sigma'$ is even. We carry out the stabilization equivariantly near $\partial W$, so that $\Sigma'$ is still isotopic to $\tau_W(\Sigma')$ rel $K$. See Figure 20.

Now fix any pair of dividing arcs $\mathcal{F}$ on $\Sigma'$ such that the resulting black and white regions have equal genus. Note that this implicitly fixes a decoration on the ends of $\Sigma'$, but due to Lemma 2.25 this choice of decoration on $K$ does not affect the statement of the theorem. Consider the knot cobordism map $F_{W,F}$. We have the usual commutative diagram

$$
\begin{array}{c}
\text{CFK}(U) \\
\downarrow t \\
\text{CFK}(U') \\
\downarrow s_{W} \\
\text{CFK}(U)
\end{array} \xrightarrow{F_{W,F}} 
\begin{array}{c}
\text{CFK}(K) \\
\downarrow t \\
\text{CFK}(K') \\
\downarrow s_{W} \\
\text{CFK}(K)
\end{array}
$$

where we have suppressed the choice of basepoints. Importantly, we have not assumed that $\Sigma$ (or $\Sigma'$) is isotopy-equivariant in the decorated sense. Hence, although $\tau_W(\Sigma')$ is isotopic to $\Sigma'$,
it is not true that the image of the decoration $\tau_W(F)$ under this isotopy must coincide with the decoration $sw(F)$. Indeed, in general we might obtain a completely different decoration on $\Sigma'$. We thus instead invoke [22, Proposition 5.5]. This states that if $\Sigma'$ is any stabilized surface and $F^A$ and $F^B$ are any two sets of dividing curves (each consisting of a pair of dividing arcs) on $\Sigma'$ with $\chi(F^A_w) = \chi(F^B_w)$ and $\chi(F^A_z) = \chi(F^B_z)$, then

$$[F_{W,F^A}(1)] = [F_{W,F^B}(1)].$$

Hence, in our case $F_{W,F^A}$ and $F_{W,sw(\tau_W(F))}$ are chain homotopic. This shows that $F_{W,F^A}$ induces a $\tau_K$-equivariant map from the trivial complex of the unknot to $CF\mathcal{K}(K)$. Our argument here is almost identical to that of [22, Theorem 1.7]; there, the authors show that $F_{W,F^A}$ is $\zeta_K$-equivariant (up to homotopy).

The map $F_{W,F^A}$ has grading shift $(-g(\Sigma'), -g(\Sigma'))$. We thus obtain a map from the trivial complex $F[U]$ to the large surgery complex $C_0$ of $CF\mathcal{K}(K)$. This lowers grading by $g(\Sigma')$ and is a homotopy equivalence after inverting $U$. It follows that

$$d_{\tau}(C_0) \geq -g(\Sigma').$$

This gives the inequality

$$V^\tau_0(K) \leq \left\lfloor \frac{1 + g(\Sigma)}{2} \right\rfloor$$

keeping in mind the number of stabilizations relating $\Sigma$ to $\Sigma'$. The same argument, together with the fact that $F_{W,F^A}$ also homotopy commutes with $\tau_K$, gives the desired inequality for $\tilde{V}^\tau_0(K)$. The other claims of the theorem are obtained by turning the cobordism around and reversing orientation.

**Remark 5.1.** The reader may be confused as to why Theorem 1.2 is weaker than Theorem 1.1 in the genus-zero case. This is because in the proof of Theorem 1.2, we stabilize in order to deal with the possible nonequivariance of the dividing curves. However, in the genus-zero case, no stabilizations are actually necessary. Following the more specialized proof of Theorem 1.2 in this situation gives the same conclusion as from Theorem 1.1.

## 6 TORUS KNOTS

We now bound the invariants $\tilde{V}^\tau_0$ and $V^\tau_0$ for the strongly invertible knots $(K_n, \tau_n)$ from the introduction. Recall that $K_n$ is constructed by taking the connected sum of $(T_{2n,2n+1} \# T_{2n,2n+1}, \tau_#)$ with the mirror of $(T_{2n,2n+1} \# T_{2n,2n+1}, \tau_{sw})$. We deal with each one of these two factors in turn. Throughout, let $n$ be odd.

### 6.1 The connected sum involution

Using the connected sum formula, we first compute the $\tau_K$-complex of $(T_{2n,2n+1} \# T_{2n,2n+1}, \tau_#)$. For this, we need to know the $\tau_K$-complex of $(T_{2n,2n+1}, \tau)$, where $\tau$ is the unique strong inversion on $T_{2n,2n+1}$. 

Definition 6.1. Let $C_n$ be the staircase complex associated to the parameter sequence

$$(c_{-2n+1}, c_{-2n+2}, ..., c_{2n-2}, c_{2n-1}) = (1, 2n - 1, 2, 2n - 2, ..., 2n - 2, 2, 2n - 1, 1).$$

This is displayed in Figure 21. Explicitly, $C_n$ is generated by the elements

$x_k$ for $-2n + 2 \leq k \leq 2n - 2$ and $k$ even; and

$y_{\ell}$ for $-2n + 1 \leq \ell \leq 2n - 1$ and $\ell$ odd

and has nonzero differentials given by

$$\partial x_k = \gamma^{c_k - 1} y_k - 1 + \gamma^{c_k + 1} y_{k+1}.$$ 

Together with the definition of $\partial$, the convention that $\text{gr}_V(y_{-2n+1}) = \text{gr}_U(y_{2n-1}) = 0$ determines the gradings of all of the generators of $C_n$. It will be helpful for us to explicitly record:

$$\text{gr}_U(y_{2n-1-2i}) = -2(1 + 2 + \cdots + i)$$

$$\text{gr}_V(y_{-2n+1+2i}) = -2(1 + 2 + \cdots + i).$$

(3)

There is a unique skew-graded homotopy involution on $C_n$, which is given by

$$\tau(x_k) = x_{-k}$$

$$\tau(y_{\ell}) = y_{-\ell}.$$ 

In [15, Proposition 3.1], it is shown that the knot Floer complex of $T_{2n,2n+1}$ is homotopy equivalent to $C_n$. By Theorem 1.7, we know that $\tau_K$ is a skew-graded homotopy involution on $CFK(T_{2n,2n+1})$. Thus, the $\tau_K$-complex of $(T_{2n,2n+1}, \tau)$ is given by $(C_n, \tau)$, with $\tau$ as above. Applying Theorem 4.1, we conclude that the $\tau_K$-complex of $(T_{2n,2n+1}\# T_{2n,2n+1}, \tau_{\#})$ is homotopy equivalent to $(C_n \otimes C_n, \tau \otimes \tau)$. The goal for this subsection will be to extract a usable representative of this local equivalence class. Our computations here are similar to those of [15, section 3].

Definition 6.2. Let $D_n$ be the staircase complex associated to the parameter sequence

$$(d_{-4n+2}, d_{-4n+3}, ..., d_{4n-3}, d_{4n-2}) =$$

$$(1, 2n - 1, 1, 2n - 1, 2, 2n - 2, 2, 2n - 2, 3, ..., 2n - 2, 2, 2n - 2, 2, 2n - 1, 1, 2n - 1, 1).$$
This is displayed in the upper half of Figure 22. Explicitly, $D_n$ is generated by the elements

$$w_k \text{ for } -4n+3 \leq k \leq 4n-3 \text{ and } k \text{ odd;}$$

$$z_\ell \text{ for } -4n+2 \leq \ell \leq 4n-2 \text{ and } \ell \text{ even}$$

and has nonzero differentials given by

$$\partial w_k = \gamma^{d_k-1} z_{k-1} + \delta^{d_k+1} z_{k+1}.$$ 

Like $C_n$, the complex $D_n$ has a unique skew-graded homotopy involution, which we again denote by $\tau$. It will also be useful to consider the square complex $S_n$, which is displayed in the lower half of Figure 22.

This is generated by $\{r_0, r_{-1}, r_1, t\}$, with differential

$$\partial r_0 = \gamma^n r_{-1} + \delta^n r_1, \quad \partial r_{-1} = \delta^n t, \quad \partial r_1 = \gamma^n t, \quad \text{and} \quad \partial t = 0.$$ 

Using the fact that $\text{gr}_V(z_{-4n+2}) = \text{gr}_U(z_{4n-2}) = 0$, we again have (for example)

$$\text{gr}_U(z_{4n-2-2i}) = \begin{cases} -4(1 + 2 + \cdots + i/2) & \text{for } i \text{ even} \\ -4(1 + 2 + \cdots + (i-1)/2) - (i + 1) & \text{for } i \text{ odd} \end{cases}$$

$$\text{gr}_V(z_{-4n+2+2i}) = \begin{cases} -4(1 + 2 + \cdots + i/2) & \text{for } i \text{ even} \\ -4(1 + 2 + \cdots + (i-1)/2) - (i + 1) & \text{for } i \text{ odd} \end{cases}$$

The grading on $S_n$ is such that $\text{gr}(t) = \text{gr}(z_0)$. Note that $\text{gr}(r_{-1}) = \text{gr}(w_{-1})$ and $\text{gr}(r_1) = \text{gr}(w_1)$.
Definition 6.3. Let $\mathcal{E}_n = D_n \oplus S_n$. Define an involution $\tau$ on $\mathcal{E}_n$ as follows. On $D_n$, we define $\tau$ to be almost the same as in Definition 6.2, but slightly different on $w_{-1}$, $w_1$, and $z_0$. On $S_n$, we define $\tau$ to be the obvious reflection map.

\[ \tau(\omega_k) = \omega_{-k} \quad \text{for } k \neq -1, 1 \]
\[ \tau(w_{-1}) = w_1 + r_1 \]
\[ \tau(w_1) = w_{-1} + r_{-1} \]
\[ \tau(z_{\ell}) = z_{-\ell} \quad \text{for } \ell \neq 0 \]
\[ \tau(z_0) = z_0 + t. \]

Roughly speaking, $\tau$ acts as reflection on the staircase and the square, but additionally maps some of the staircase generators to (sums of staircase generators with) square generators. Unlike the action of $\psi_K$ in [15, section 3.2.1], however, none of the square generators are mapped to staircase generators.

The main claim of this subsection is that $(\mathcal{E}_n, \tau)$ is locally equivalent to $(\mathcal{C}_n \otimes \mathcal{C}_n, \tau \otimes \tau)$. We show this by constructing local maps in both directions. The forward direction is straightforward from the work of [15]. In what follows, we write $\leq$ to indicate the presence of a local map from one $\tau_K$-complex to another; see also the discussion of Subsection 7.2.

Lemma 6.4. We have $(\mathcal{E}_n, \tau) \leq (\mathcal{C}_n \otimes \mathcal{C}_n, \tau \otimes \tau)$.

Proof. In [15, section 3.2.1], it is shown that $\mathcal{C}_n \otimes \mathcal{C}_n$ admits the subcomplex $\mathcal{V}_n$ displayed in Figure 23. Explicitly, $\mathcal{V}_n$ is spanned by

\[ \{y_1y_i\} \cup \{y_iy_{i+2}\}_{i \leq -3} \cup \{y_iy_{i+2}\}_{i \geq -1}, \]
together with
\[
\{y_i x_{i+1}\}_{i \leq -3} \cup \{x_i y_{i+1}\}_{i \leq -2} \cup \{x_i y_{i-1}\}_{i \geq 0} \cup \{y_i x_{i-1}\}_{i \geq 1},
\]
and
\[
y_1 y_{-1} + y_{-1} y_1, \quad y_{-1} x_0 + x_0 y_{-1}, \quad y_1 x_0 + x_0 y_1, \quad \text{and} \quad x_0 x_0.
\]
The first two collections of generators span a staircase complex, while the last four generators span a square complex.

There is an obvious map \(\varphi : \mathcal{E}_n \to \mathcal{C}_n \otimes \mathcal{C}_n\) given by mapping \(\mathcal{E}_n\) isomorphically onto \(\mathcal{Y}_n\); compare Figures 22 and 23. It is straightforward to check that this has the requisite behavior under localization: observe that \(y_{1-2n} y_{-1-2n}\) is nontorsion. To check equivariance, recall that \(\tau(x_k) = x_{-k}\) and \(\tau(y_\ell) = y_{-\ell}\). An examination of Figure 23 shows that \(\tau \otimes \tau\) acts as reflection on \(\mathcal{Y}_n\), except at the generators
\[
\begin{align*}
(\tau \otimes \tau)(x_0 y_{-1}) &= x_0 y_1 = y_1 x_0 + (y_1 x_0 + x_0 y_1) \\
(\tau \otimes \tau)(y_1 x_0) &= y_{-1} x_0 = x_0 y_{-1} + (y_{-1} x_0 + x_0 y_{-1}) \\
(\tau \otimes \tau)(y_1 y_{-1}) &= y_{-1} y_1 = y_1 y_{-1} + (y_1 y_{-1} + y_{-1} y_1).
\end{align*}
\]
This coincides exactly with the action of \(\tau\) on \(\mathcal{E}_n\).

We now construct a map from \(\mathcal{E}_n^\vee \) to \(\mathcal{C}_n^\vee \otimes \mathcal{C}_n^\vee\). It will be helpful for us to first discuss some auxiliary lemmas regarding the dual staircase complexes \(\mathcal{C}_n^\vee\) and \(\mathcal{D}_n^\vee\). Our first lemma concerns elements in \(\mathcal{C}_n^\vee \otimes \mathcal{C}_n^\vee\) of the form \(x_p^\vee \otimes y_q^\vee\). Roughly speaking, we claim that if the value of \(p + q\) is fixed, then the grading of \(x_p^\vee \otimes y_q^\vee\) is minimized when the difference \(|p - q|\) is minimized. Similar statements hold for elements of the form \(y_p^\vee \otimes y_q^\vee\). We make this more explicit by introducing the following terminology:

**Definition 6.5.** Let \(k\) be odd and let \(p + q = k\) with \(p\) even and \(q\) odd. We call \((p, q)\) difference-minimizing in the following situations.

1. \(k \equiv 1\) mod 4: we require \(p = (k - 1)/2\) and \(q = (k + 1)/2\).
2. \(k \equiv 3\) mod 4: we require \(p = (k + 1)/2\) and \(q = (k - 1)/2\).

Let \(\ell\) be even and let \(p + q = \ell\) with \(p\) and \(q\) both odd. We call \((p, q)\) difference-minimizing in the following situations.

1. \(\ell \equiv 2\) mod 4: we require \(p = q = \ell/2\).
2. \(\ell \equiv 0\) mod 4: we require \(p = (\ell - 1)/2\) and \(q = (\ell + 1)/2\), or vice versa.

In each case, note that the difference \(|p - q|\) is minimized, subject to the constraints on the parity of \(p\) and \(q\) and the condition that the value of \(p + q\) is fixed. The distinction between \(k\) and \(\ell\) is due to our choice of notation for the generators of \(\mathcal{D}_n\), and will become clear presently.

**Lemma 6.6.** Let \(k\) be odd. Then
\[
\min_{\substack{p+q=k \\quad \text{if} \quad p \quad \text{even,} \quad q \quad \text{odd}}} \{\text{gr}_U(x_p^\vee \otimes y_q^\vee)\} \quad \text{and} \quad \min_{\substack{p+q=k \\quad \text{if} \quad p \quad \text{even,} \quad q \quad \text{odd}}} \{\text{gr}_V(x_p^\vee \otimes y_q^\vee)\}
\]
both occur when \((p, q)\) is difference-minimizing. Similarly, let \(\ell\) be even. Then
\[
\min_{\substack{p+q=\ell \quad \text{p odd, q odd}}} \{\text{gr}_U(y^\vee_p \otimes y^\vee_q)\} \quad \text{and} \quad \min_{\substack{p+q=\ell \quad \text{p odd, q odd}}} \{\text{gr}_V(y^\vee_p \otimes y^\vee_q)\}
\]
both occur when \((p, q)\) is difference-minimizing.

Proof. First note that for any \(i\), we have:
\[
\begin{align*}
\text{gr}_U(x^\vee_{i+1}) - \text{gr}_U(x^\vee_{i-1}) &= -2n - 1 + i \\
\text{gr}_V(x^\vee_{i+1}) - \text{gr}_V(x^\vee_{i-1}) &= 2n + 1 + i,
\end{align*}
\]
and
\[
\begin{align*}
\text{gr}_U(y^\vee_{i+2}) - \text{gr}_U(y^\vee_{i}) &= -2n + 1 + i \\
\text{gr}_V(y^\vee_{i+2}) - \text{gr}_V(y^\vee_{i}) &= 2n + 1 + i.
\end{align*}
\]
These claims are verified using the differentials in the definition of \(C_n\); the reader may find it helpful to consult Figure 21.

Consider the first claim of the lemma. Observe
\[
\begin{align*}
\text{gr}_U(x^\vee_{p+2} \otimes y^\vee_{q-2}) &= \text{gr}_U(x^\vee_p \otimes y^\vee_q) + (-2n - 1 + (p + 1)) - (-2n + 1 + (q - 2)) \\
&= \text{gr}_U(x^\vee_p \otimes y^\vee_q) + p - q + 1.
\end{align*}
\]
Similarly, we have
\[
\begin{align*}
\text{gr}_V(x^\vee_{p+2} \otimes y^\vee_{q-2}) &= \text{gr}_V(x^\vee_p \otimes y^\vee_q) + (2n + 1 + (p + 1)) - (2n + 1 + (q - 2)) \\
&= \text{gr}_U(x^\vee_p \otimes y^\vee_q) + p - q + 3.
\end{align*}
\]
Note that due to the parity constraints on \(p\) and \(q\) and the fact that \(p + q = k\), the value of \(p - q\) is fixed modulo 4. Treating both of the above as finite-difference equations, it is clear that to minimize both \(\text{gr}_U(x^\vee_p \otimes y^\vee_q)\) and \(\text{gr}_V(x^\vee_p \otimes y^\vee_q)\) we are searching for \((p, q)\) such that \(p - q + 1\) and \(p - q + 3\) are both in \([0, 4]\). An examination of Definition 6.5 gives the claim for \(k\) odd. The claim for \(\ell\) even is established in an analogous manner. 

The following lemma relates the gradings of elements of \(D_n^\vee\) and elements of \(C_n^\vee \otimes C_n^\vee\), and will be important for constructing a map from the former into the latter.

Lemma 6.7. Let \(k\) be odd and let \(p + q = k\) with \(p\) even and \(q\) odd. If \((p, q)\) is difference-minimizing, then
\[
\text{gr}(w^{\vee}_k) = \text{gr}(x^{\vee}_p \otimes y^{\vee}_q) = \text{gr}(y^{\vee}_q \otimes x^{\vee}_p).
\]
Let \(\ell\) be even and let \(p + q = \ell\) with \(p\) and \(q\) both odd. If \((p, q)\) is difference-minimizing, then
\[
\text{gr}(z^{\vee}_\ell) = \text{gr}(y^{\vee}_p \otimes y^{\vee}_q) = \text{gr}(y^{\vee}_q \otimes y^{\vee}_p).
\]
Proof. We prove the second claim and leave the first to the reader. Assume \((p, q)\) is difference-minimizing. Write \(\ell = 4n - 2 - 2i, p = 2n - 1 - 2r,\) and \(q = 2n - 1 - 2s;\) note that \(r + s = i.\) From (3) and (4), we have
\[
gr_U(z_\ell^\vee) = -\text{gr}_U(z_\ell) \begin{cases} 4(1 + 2 + \cdots + i/2) & \text{for } i \text{ even} \\ 4(1 + 2 + \cdots + (i - 1)/2) + (i + 1) & \text{for } i \text{ odd} \end{cases}
\]
and
\[
gr_U(y_p^\vee) = -\text{gr}_U(y_p) = 2(1 + 2 + \cdots + r) \quad \text{and} \quad \text{gr}_U(y_q^\vee) = -\text{gr}_U(y_q) = 2(1 + 2 + \cdots + s).
\]
Suppose \(\ell \equiv 2 \mod 4.\) Then \(i\) is even, and an examination of Definition 6.5 shows \(r = s = i/2.\) In this case, we clearly have \(\text{gr}_U(z_\ell^\vee) = \text{gr}_U(y_p^\vee) + \text{gr}_U(y_q^\vee) = \text{gr}_U(y_p^\vee \otimes y_q^\vee).\) Suppose \(\ell \equiv 0 \mod 4.\) Then \(i\) is odd, and we have \(r = (i + 1)/2\) and \(s = (i - 1)/2\) (or vice versa). An inspection of the equalities above once again gives the claim. An analogous argument for \(\text{gr}_V\) completes the proof. □

We now establish the major claim of this subsection:

**Lemma 6.8.** We have \((E_n^\vee, \tau^\vee) \leq (C_n^\vee \otimes C_n^\vee, \tau^\vee \otimes \tau^\vee)\).

Proof. We define a grading-preserving map \(\psi : E_n^\vee \to C_n^\vee \otimes C_n^\vee\) as follows. For any \(\ell\) even, let
\[
\psi(z_\ell^\vee) = \sum_{i+j=\ell} U^* \psi x_0^\vee \otimes y_j^\vee.
\]
Here, the right-hand side is formed by considering all possible products \(y_i^\vee \otimes y_j^\vee\) with \(i\) and \(j\) odd and \(i + j = \ell.\) Each term is multiplied by powers of \(U\) and \(V\) so that the resulting grading is equal to that of \(z_\ell^\vee.\) Note that this is possible due to Lemmas 6.6 and 6.7. Indeed, by Lemma 6.6, \(\text{gr}(y_i^\vee \otimes y_j^\vee)\) is minimized when \((i, j)\) is difference-minimizing. We then multiply every other term on the right-hand side by powers of \(U\) and \(V\) so as to have grading equal to this minimal grading. But by Lemma 6.7, the minimal grading is none other than \(\text{gr}(z_\ell^\vee).\)

We similarly define:
\[
\psi(w_k^\vee) = \sum_{i+j=k} U^* \psi x_i^\vee \otimes y_j^\vee + y_j^\vee \otimes x_i^\vee
\]
\[
\psi(r_{-1}^\vee) = \sum_{i+j=0, i<j} U^* \psi x_{i-1}^\vee \otimes y_j^\vee + U^* \psi y_i^\vee \otimes x_{j-1}^\vee
\]
\[
\psi(r_1^\vee) = \sum_{i+j=0, i<j} U^* \psi x_{i+1}^\vee \otimes y_j^\vee + U^* \psi y_i^\vee \otimes x_{j+1}^\vee
\]
\[
\psi(r_0^\vee) = x_0^\vee \otimes x_0^\vee
\]
\[
\psi(t^\vee) = \sum_{i+j=0, i<j} U^* \psi y_i^\vee \otimes y_j^\vee.
\]
As before, Lemmas 6.6 and 6.7 guarantee that in each of the above equations, there exist unique powers of \(U\) and \(V\) that make \(\psi\) grading-preserving. Note that \(\text{gr}(t^\vee) = \text{gr}(z_0^\vee),\) while \(\text{gr}(r_{-1}^\vee) = \text{gr}(w_{-1}^\vee)\) and \(\text{gr}(r_1^\vee) = \text{gr}(w_1^\vee).\)
We claim that \( \psi \) is a chain map. Because both sides of the equation \( \partial \psi = \psi \partial \) are homogenous, it suffices to prove this in the quotient where we set \( \mathcal U = \mathcal V = 1 \). (The reader who is unconvinced of this fact may consult [12, section 2.4], in which an analogous situation is discussed.) The claim is then straightforward from the definitions; the only subtle cases are to verify \( \partial \psi = \psi \partial \) on \( r_1^\vee \) and \( r_{-1}^\vee \). For the former, we have

\[
\partial \psi(r_{-1}^\vee) = \sum_{i+j=0, i<j} \partial(x_i^\vee \otimes y_j^\vee) + \partial(y_i^\vee \otimes x_j^\vee) \\
= \sum_{i+j=0, i<j} x_i^\vee \otimes x_j^\vee - x_i^\vee \otimes x_{j+1}^\vee + \ldots + x_{i-1}^\vee \otimes x_{j+1}^\vee \\
= \sum_{i+j=0, i<j} x_{i-1}^\vee \otimes x_{j+1}^\vee + x_{i+1}^\vee \otimes x_{j-1}^\vee.
\]

Identifying this as a telescoping series shows that it is equal to \( x_0^\vee \otimes x_0^\vee \) (note that \( x_i^\vee = 0 \) for \( i < -2n + 1 \)). A similar computation holds for \( \partial \psi(r_1^\vee) \).

Checking that \( \psi \) has the requisite behavior under localization and is \( \tau \)-equivariant is straightforward; the only subtle cases are (again setting \( \mathcal U = \mathcal V = 1 \)):

\[
\tau \psi(t^\vee) = \sum_{i+j=0, i<j} y_i^\vee \otimes y_j^\vee = \psi(t^\vee) + \psi(z_0^\vee)
\]

together with

\[
\tau \psi(r_{-1}^\vee) = \sum_{i+j=0, i<j} x_{i+1}^\vee \otimes y_j^\vee + y_{i-1}^\vee \otimes x_{j+1}^\vee = \psi(r_{-1}^\vee) + \psi(w_{-1}^\vee)
\]

and

\[
\tau \psi(r_1^\vee) = \sum_{i+j=0, i<j} x_{i-1}^\vee \otimes y_j^\vee + y_{i-1}^\vee \otimes x_{j-1}^\vee = \psi(r_{1}^\vee) + \psi(w_{1}^\vee).
\]

This completes the proof.

We thus obtain the overall computation:

**Lemma 6.9.** For \( n \) odd, we have \((C\mathcal PK(T_{2n,2n+1} \# T_{2n,2n+1}), \tau_\#) \sim (\mathcal E_n, \tau)\).

**Proof.** Follows from Lemmas 6.4 and 6.8. □

**Remark 6.10.** In [15, section 3.2.1], the local equivalence class of \((T_{2n,2n+1} \# T_{2n,2n+1}, \iota_{\#})\) was similarly identified with \((\mathcal E_n, \iota_K)\) for a certain involution \( \iota_K \) on \( \mathcal E_n \). In fact, the map of Lemma 6.4 is both \( \tau \)- and \( \iota_K \)-equivariant. However, the map of Lemma 6.8 is not \( \iota_K \)-equivariant. We thus do not determine the \((\tau_K, \iota_K)\)-class of \( T_{2n,2n+1} \# T_{2n,2n+1} \) in this paper; only the \( \tau_K \)-class.

### 6.2 The swapping involution

We now turn to the \( \tau_K \)-class of \((T_{2n,2n+1} \# T_{2n,2n+1}, \tau_{\#})\). Although the full local equivalence class turns out to be difficult to compute, for our purposes it will suffice to establish an inequality. Let
FIGURE 24 The subcomplex $\mathcal{W}_n$. Note that the top two rows form a staircase complex, such that $\partial(x_0y_{-1}) = y^ny_{-1}y_{-1} + U^ny_1y_{-1}$.

FIGURE 25 The box complex $B_n$ and its dual $B_n^\vee$. See [15, fig. 3.4].

Let $D_n$ be the staircase complex equipped with the unique skew-graded involution of Definition 6.2. Then we claim:

**Lemma 6.11.** We have $(D_n, \tau) \leq (CFK(T_{2n,2n+1} \# T_{2n,2n+1}), \tau_{sw})$.

**Proof.** Consider the subcomplex $\mathcal{W}_n$ of $CFK(T_{2n,2n+1} \# T_{2n,2n+1}) \cong C_n \otimes C_n$ displayed in Figure 24. This is similar to the upper half of Figure 23, but it is not quite the same: the second of the two rows has many of the tensor products occurring with transposed factors.

We claim that $\tau_{sw}$ preserves this subcomplex and acts as the obvious reflection map. To see this, consider the exchange involution $\tau_{exch}$ defined in Subsection 4.2. This sends $\tau_{exch}(x_iy_j) = y_{-j}x_{-i}, \tau_{exch}(y_iy_j) = y_{-j}y_{-i}$.

Moreover, it is clear from the definition of $\partial$ on $C_n$ that $\Psi \otimes \Phi$ vanishes on generators of the form $x_iy_j, y_iy_j$, and $y_iy_j$. Applying Theorem 4.3 then gives the desired computation of $\tau_{sw}$. Mapping $(D_n, \tau)$ isomorphically onto $(\mathcal{W}_n, \tau_{sw})$ completes the proof. \hfill $\Box$

### 6.3 The involution on $K_n$

We now finally turn to the $\tau_K$-class of $(K_n, \tau_n)$. Our first step will be to understand the complex $E_n \otimes D_n^\vee$. This follows [15, section 3.2.2].

**Definition 6.12.** Let $B_n$ be the box complex displayed on the left in Figure 25. This has five generators $v, r_0, r_{-1}, r_1, t$, with differential

$$
\partial v = 0, \quad \partial r_0 = y^n r_{-1} + U^n r_1, \quad \partial r_{-1} = U^n t, \quad \partial r_1 = y^n t, \quad \text{and} \quad \partial t = 0.
$$
The gradings of these generators are such that $\text{gr}(v) = \text{gr}(t) = (0, 0)$. Define an involution $\tau$ on $B_n$ by setting

$$
\begin{align*}
\tau(v) &= v + t \\
\tau(r_0) &= r_0 \\
\tau(r_{-1}) &= r_1 \\
\tau(r_1) &= r_{-1} \\
\tau(t) &= t.
\end{align*}
$$

Note that the action of $\tau$ sends the singleton generator $v$ to (the sum of $v$ with) a square complex generator. However, unlike in [15, section 3.2.2], $\tau$ does not send the opposite corner of the square back to $v$. The reader should compare the complexes $B_n$ and $S_n$.

The utility of $B_n$ is given by the following lemma:

**Lemma 6.13.** We have $(B_n, \tau) \sim (E_n, \tau) \otimes (D_n, \tau)^\vee$.

**Proof.** This is similar to [15, Proposition 3.5]. We construct maps

$$
f : E_n \to D_n \otimes B_n
$$

and

$$
g : E_n^\vee \to D_n^\vee \otimes B_n^\vee
$$

as follows. The map $f$ is given by

$$
\begin{align*}
f(w_i) &= w_i v \text{ for } i \leq -1 \\
f(w_1) &= w_1 (v + t) + z_0 r_1 \\
f(w_i) &= w_i (v + t) \text{ for } i \geq 3 \\
f(z_i) &= z_i v \text{ for } i \leq 0 \\
f(z_i) &= z_i (v + t) \text{ for } i \geq 2
\end{align*}
$$

$$
f(r_0) = z_0 r_0 \\
f(r_{-1}) = z_0 r_{-1} \\
f(r_1) = z_0 r_1 \\
f(t) = z_0 t.
$$

It is easily checked that $f$ is a grading-preserving chain map; the only subtlety is checking that $\partial f = f \partial$ on $w_1$. We have:

$$
\begin{align*}
\partial f(w_1) &= \partial (w_1 (v + t) + z_0 r_1) = \gamma^n z_0 + \omega^n z_2 (v + t) + \gamma^n z_0 t \\
f(\partial w_1) &= f(\gamma^n z_0 + \omega^n z_2) = \gamma^n z_0 v + \omega^n z_2 (v + t),
\end{align*}
$$

which are equal to each other. Checking $\tau$-equivariance is likewise straightforward; the only subtle cases are for $w_{-1}$, $w_1$, and $z_0$. For these, we have

$$
\begin{align*}
\tau f(w_{-1}) &= \tau(w_{-1} v) = w_1 (v + t) \\
f(\tau w_{-1}) &= f(w_1 + r_1) = w_1 (v + t) + z_0 r_1 + z_0 r_1
\end{align*}
$$

$$
and
\[ \tau f(w_1) = \tau((w_1 + t) + z_0 r_1^\vee) = w_{-1}^\vee v + z_0 r_{-1} \]
\[ f(\tau w_1) = f(w_{-1} + r_{-1}) = w_{-1}^\vee v + z_0 r_{-1} \]

and
\[ \tau f(z_0) = \tau(z_0 v) = z_0(v + t) \]
\[ f(\tau z_0) = f(z_0 + t) = z_0 v + z_0 t. \]

This completes the verification of \( f \).

The map \( g \) is given by
\[ g(r_0^\vee) = z_0^\vee r_0^\vee + w_{-1}^\vee r_1^\vee + w_1^\vee r_{-1}^\vee \]
\[ g(w_1^\vee) = w_1^\vee v^\vee \]
\[ g(r_{-1}^\vee) = w_{-1}^\vee t^\vee + z_0^\vee r_{-1}^\vee \]
\[ g(z_1^\vee) = z_1^\vee v^\vee \]
\[ g(r_1^\vee) = w_1^\vee t^\vee + z_0^\vee r_1^\vee \]
\[ g(t^\vee) = z_1^\vee t^\vee. \]

An examination of the right-hand side of Figure 25 shows that \( g \) is a grading-preserving chain map. Checking \( \tau \)-equivariance is likewise straightforward; the only subtle cases are for \( r_{-1}^\vee, r_1^\vee \), and \( t^\vee \). For these, we have
\[ \tau g(r_{-1}^\vee) = \tau((w_{-1}^\vee t^\vee + z_0^\vee r_{-1}^\vee)) = w_{-1}^\vee (t^\vee + v^\vee) + z_0^\vee r_{-1}^\vee \]
\[ g(\tau r_{-1}^\vee) = g(r_1^\vee + w_1^\vee) = w_1^\vee t^\vee + z_0^\vee r_1^\vee + w_1^\vee v^\vee \]

and
\[ \tau g(r_1^\vee) = \tau((w_1^\vee t^\vee + z_0^\vee r_1^\vee)) = w_1^\vee (t^\vee + v^\vee) + z_0^\vee r_{-1}^\vee \]
\[ g(\tau r_1^\vee) = g(r_{-1}^\vee + w_{-1}^\vee) = w_{-1}^\vee t^\vee + z_0^\vee r_{-1}^\vee + w_{-1}^\vee v^\vee \]

and
\[ \tau g(t^\vee) = \tau(z_0^\vee t^\vee) = z_0^\vee (t^\vee + v^\vee) \]
\[ g(\tau t^\vee) = g(t^\vee + z_0^\vee) = z_0^\vee t^\vee + z_0^\vee v^\vee. \]

This completes the verification for \( g \). \( \square \)

We are now finally in a position to state our fundamental computation:

**Lemma 6.14.** We have \( (\text{CFK}(K_n), \tau_n) \trianglelefteq (B_n, \tau) \).

**Proof.** By Lemma 6.9, we have
\[ (\text{CFK}(T_{2n,2n+1} \# T_{2n,2n+1}), \tau_\#) \sim (E_n, \tau). \]
By Lemma 6.11, we have

\[(\text{CF} K(T_{2n,2n+1} \# T_{2n,2n+1}), \tau_s \psi) \leq (D_n, \tau)^\psi.\]

Tensoring these together and utilizing Theorem 4.1, we thus have that

\[(\text{CF} K(K_n), \tau_n) \leq (E_n, \tau) \otimes (D_n, \tau)^\psi \sim (B_n, \tau),\]

where the final local equivalence follows from Lemma 6.13.

\[\square\]

This immediately yields the proof of Theorem 1.3:

**Proof of Theorem 1.3.** It is straightforward to check that an inequality as in Lemma 6.14 implies an inequality of the large-surgery numerical invariants defined in Subsection 2.5:

\[d_{\tau}(\text{CF} K(K_n)_0) \leq d_{\tau}((B_n)_0) \quad \text{and} \quad \bar{d}_\tau(\text{CF} K(K_n)_0) \leq \bar{d}_\tau((B_n)_0).\]

See Subsection 7.2 for further discussion. A direct computation shows that

\[d_{\tau}((B_n)_0) = -2n \quad \text{and} \quad \bar{d}_\tau((B_n)_0) = 0.\]

Applying Definition 2.23 completes the proof.

\[\square\]

Remark 6.15. Throughout this section, we have only worked with the \(\tau_K\)-complexes of our knots. These are insensitive to the choice of direction. Moreover, as discussed in Subsection 2.3, the (possible) non-abelian nature of \(\mathfrak{K}_{\tau,\psi}\) does not arise unless the action of \(\tau_K\) is considered simultaneously. Thus, the computations of this section hold regardless of the way the equivariant connected sum is performed, not just following the conventions of Figure 1.

## 7 | RELATION TO OTHER INVARIANTS

We now relate the present paper to the results of [7] and [22].

### 7.1 | Equivariant large surgery

We begin with a brief review of [7]. Let \(Y\) be a \(\mathbb{Z}/2\mathbb{Z}\)-homology sphere and let \(\tau\) be an involution on \(Y\). Note that \(Y\) has a single spin structure \(\mathfrak{s}\) that is necessarily sent to itself by \(\tau\). In [7, section 4], it is shown that \(\tau\) induces a well-defined automorphism of \(\text{CF}^-(Y, \mathfrak{s})\), which we also denote by \(\tau\).

Moreover, in [7, section 4] it is shown that \(\tau\) is a homotopy involution. The pair \((\text{CF}^-(Y, \mathfrak{s}), \tau)\) thus constitutes an abstract \(\iota\)-complex in the sense of [17, Definition 8.1]. Taking the local equivalence class of this \(\iota\)-complex gives an element

\[h_{\iota}(Y) = [(\text{CF}^-(Y, \mathfrak{s})[-2], \tau)].\]
in the local equivalence group $\mathfrak{F}$ of [17, Proposition 8.8]. (The grading shift is a convention due to the definition of the grading on $CF^-$.) This is an invariant of equivariant $\mathbb{Z}/2\mathbb{Z}$-homology bordism. In fact, one can construct a $\mathbb{Z}/2\mathbb{Z}$-homology bordism group of involutions and show that $h_\tau$ constitutes a homomorphism from this group into $\mathfrak{F}$; see [7, Theorem 1.2] and [7, section 2]. We may also consider the map $\iota \circ \tau$ in place of $\tau$, which is similarly a homotopy involution. This gives another $\iota$-complex whose local equivalence class

$$h_{\iota \circ \tau}(Y) = [(CF^-(Y, \mathfrak{F})[-2], \iota \circ \tau)]$$

is another (generally different) element of $\mathfrak{F}$. For each of these elements, one can extract the numerical invariants $\bar{d}$ and $d$ following the procedure described by Hendricks and Manolescu [16]. This yields numerical invariants $\bar{d}_\tau$ and $d_\tau$ associated to $h_\tau$, as well as invariants $\bar{d}_{\iota \tau}$ and $\bar{d}_{\iota \tau}$ associated to $h_{\iota \circ \tau}$.

Remark 7.1. The discussion of [7] is phrased in terms of integer homology spheres, but the extension to $\mathbb{Z}/2\mathbb{Z}$-homology spheres is straightforward. Note that in this more general situation, the gradings of our complexes take values in $\mathbb{Q}$, and so $\bar{d}_{\iota}$ and $\bar{d}_{\iota}$ may be $\mathbb{Q}$-valued.

If $(K, \tau)$ is an equivariant knot, then any surgery on $K$ inherits an involution; see, for example, [7, section 5]. In the forthcoming paper, the second author showed the following:

**Theorem 7.2** (Theorem 1.1 in a forthcoming paper by Mallick). Let $(K, \tau)$ be an equivariant knot and let $p \geq g_3(K)$. Then there is an absolutely graded isomorphism

$$\left( CF^-(S^3_p(K), [0]) \left[ \frac{p-1}{4} - 2 \right], \tau \right) \cong (CF(K)_0, \tau_K).$$

On the left-hand side, $S^3_p(K)$ is large surgery on $K$ and $\tau$ is the automorphism on $CF^-(S^3_p(K), [0])$ induced by the inherited 3-manifold involution. On the right-hand side, $CF(K)_0$ is the large surgery subcomplex of $CF(K)$ and $\tau_K$ is the restriction of the action defined in Subsection 3.2. A similar statement holds replacing $\tau$ with $\iota \circ \tau$ and $\tau_K$ with $\iota_K \circ \tau_K$.

Remark 7.3. There are several confusing conventions regarding absolute gradings. For the sake of being explicit, we give an explanation of these for the reader.

(1) Note that the “trivial complex” $CF^-(S^3)$ consists of a single $\mathbb{F}[U]$-tower starting in Maslov grading $-2$. However, when discussing Floer complexes in the abstract, it is generally preferable to treat this complex as starting in Maslov grading zero. This explains the shift by $-2$ in the definition of $h_\tau$ and $h_{\iota \circ \tau}$.

(2) Similarly, if $K$ is the unknot, then the large surgery complex $C_0$ (as defined in Subsection 2.5) consists of a single $\mathbb{F}[U]$-tower starting in Maslov grading zero. This explains the extra $-2$ in the isomorphism of Theorem 7.2.

(3) In Subsection 2.5, we have defined $\bar{d}_{\iota}(C_0)$ and $\bar{d}_{\iota}(C_0)$ in such a way so that the shift by $-2$ is already taken into account. Indeed, note that $\bar{d}_{\iota}(C_0) = \bar{d}_{\iota}(C_0) = 0$ for the large surgery complex of the unknot. However, when defining $\bar{d}_{\iota}$ and $\bar{d}_{\iota}$ in terms of an actual 3-manifold complex $CF^-(Y)$, it is necessary to add two to each of the definitions.
Taking into account the grading shift, Definition 2.23 and Theorem 7.2 immediately imply the relations referenced in Subsection 1.4:

\[-2\tilde{V}_0^\circ(K) + \frac{p - 1}{4} = \tilde{d}_\circ(S^3_p(K), [0])\]

and

\[-2V_0^\circ(K) + \frac{p - 1}{4} = d_\circ(S^3_p(K), [0]).\]

for \(\circ \in \{\tau, \iota \tau\}\).

### 7.2 Inequalities

A key property of the local equivalence group \(\mathfrak{E}\) is that it is partially ordered; see, for example, [7, Definition 3.6]. This partial order is consistent with the numerical invariants \(\tilde{d}\) and \(d\), in the sense that if \([(C_1, \tau_1)] \leq [(C_2, \tau_2)]\), then

\[d(C_1) \leq d(C_2)\quad \text{and}\quad \tilde{d}(C_1) \leq \tilde{d}(C_2).\]

In [7, Theorem 1.5], it was shown that if \((Y_1, \tau_1)\) and \((Y_2, \tau_2)\) are two homology spheres with involutions and \(W\) is an equivariant negative-definite cobordism from \(Y_1\) to \(Y_2\), then under certain circumstances we obtain inequalities

\[[(CF^-(Y_1), \tau_1)] \leq [(CF^-(Y_2), \tau_2)]\]

and/or

\[[(CF^-(Y_1), \iota \circ \tau_1)] \leq [(CF^-(Y_2), \iota \circ \tau_2)].\]

It is thus possible to bound the numerical invariants of \((Y_1, \tau_1)\) by topologically constructing equivariant negative-definite cobordisms into other manifolds \((Y_2, \tau_2)\). See [7, section 5] and [7, section 7] for further discussion and examples.

For convenience, we briefly review these results here, generalizing them slightly in the case of \(\mathbb{Z}/2\mathbb{Z}\)-homology spheres. Let \(Y_1\) and \(Y_2\) be two \(\mathbb{Z}/2\mathbb{Z}\)-homology spheres equipped with involutions \(\tau_1\) and \(\tau_2\). Let \(W\) be a cobordism from \(Y_1\) to \(Y_2\) equipped with a self-diffeomorphism \(f : W \to W\) that restricts to \(\tau_i\) on \(Y_i\). In what follows, we will be interested in spin\(^c\)-structures \(\mathfrak{s}\) on \(W\) such that the Heegaard Floer grading shift

\[\Delta(W, \mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}\]

is zero.

**Theorem 7.4** [7, Proposition 4.10]. Let \(W\) be a negative-definite cobordism as above with \(b_1(W) = 0\). Suppose \(W\) admits a spin\(^c\)-structure \(\mathfrak{s}\) such that \(\Delta(W, \mathfrak{s}) = 0\) and \(\mathfrak{s}\) restricts to the unique spin structure on \(\partial W\). Then:

1. if \(f_* \mathfrak{s} = \mathfrak{s}\), we have \(h_{\iota \tau_1}(Y_1) \leq h_{\iota \tau_2}(Y_2)\);
2. if \(f_* \mathfrak{s} = \hat{\mathfrak{s}}\), we have \(h_{\iota \circ \tau_1}(Y_1) \leq h_{\iota \circ \tau_2}(Y_2)\).
The proof is the same as that of [7, Proposition 4.10] and proceeds by considering the Heegaard Floer cobordism map associated to $(W, \mathfrak{s})$.

We consider a particularly important family of such cobordisms, constructed as follows. Let $(K, \tau)$ be an equivariant knot and let $Y_1 = S_3^3(K)$ be large, odd surgery on $K$. Define an equivariant cobordism $W$ by symmetrically attaching $(-1)$-framed 2-handles along unknots that have linking number zero with $K$ (as well as with each other):

**Definition 7.5.** Three important instances of this construction are given in the top row of Figure 26. We categorize these as follows.

1. **Type Ia:** Attach a single $(-1)$-framed 2-handle along an equivariant unknot that has no fixed points along the axis of $\tau$.
2. **Type Ib:** Attach a single $(-1)$-framed 2-handle along an equivariant unknot that has two fixed points along the axis of $\tau$.
3. **Type II:** Attach a pair of $(-1)$-framed 2-handles that are interchanged by $\tau$.

Consider a handle attachment of Type Ia. Let $x$ be the element of $H_2(W, \partial W; \mathbb{Z})$ corresponding to the core of the attached 2-handle and let $\mathfrak{s}$ be the spin$^c$-structure on $W$ corresponding to the dual of $x$. This restricts to the unique spin structure on the ends of $W$, as can be seen from the fact that the map $H^2(W; \mathbb{Z}) \rightarrow H^2(\partial W; \mathbb{Z})$ corresponds to the map $H_2(W, \partial W; \mathbb{Z}) \rightarrow H_1(\partial W; \mathbb{Z})$ under Poincaré duality. As $x$ has self-intersection $-1$, we moreover have $\Delta(\mathfrak{s}) = 0$. Finally, it is easily checked that $f_*\mathfrak{s} = \mathfrak{s}$. Handle attachments of Type Ib are similar, except that $f_*\mathfrak{s} = \tilde{\mathfrak{s}}$. To understand handle attachments of Type II, let $x$ and $y$ be the elements of $H_2(W, \partial W; \mathbb{Z})$ represented by the cores of the attached 2-handles. Then $W$ admits both a spin$^c$-structure with $f_*\mathfrak{s} = \mathfrak{s}$ (corresponding to the dual of $x + y$) and a spin$^c$-structure with $f_*\mathfrak{s} = \tilde{\mathfrak{s}}$ (corresponding to the
dual of $x - y$). Once again, these both restrict to the unique spin structure on the ends of $W$ and have $\Delta(\mathfrak{s}) = 0$. See [7, section 5.2] for further discussion.

In the context of equivariant knots, this immediately gives a set of crossing change inequalities for $\overline{V}^0_0$ and $V^0_0$. As in Figure 26, define:

**Definition 7.6.** Let $K$ be an equivariant knot. We categorize equivariant positive-to-negative crossing changes as follows.

1. **Type Ia:** The crossing change occurs along the axis of symmetry and the two strands of the crossing point in opposite directions along the axis (Figure 26, top left).
2. **Type Ib:** The crossing change occurs along the axis of symmetry and the two strands of the crossing point in the same direction along the axis (Figure 26, top middle).
3. **Type II:** We perform a symmetric pair of crossing changes (Figure 26, top right).

**Proof of Theorem 1.9.** Passing to large surgery, each of the possible crossing changes is mediated by a 2-handle attachment of the corresponding type. By Theorem 7.4, we thus have

$$d_{\circ}(S^3_p(K), [0]) \leq d_{\circ}(S^3_p(K'), [0]) \quad \text{and} \quad d_{\circ}(S^3_p(K), [0]) \leq d_{\circ}(S^3_p(K'), [0])$$

for $\circ \in \{\tau, \iota \tau\}$. Applying the relation (1) immediately gives the desired conclusion. \[\square\]

**Remark 7.7.** In Definition 7.5, attaching our 2-handles along unknots is not essential. However, we emphasize the unknot case due to its ease of use and connection with the current paper.

### 7.3 Exotic slice disks

We now turn to the proofs of Theorems 1.5 and 1.6. We establish the former by showing that at least one of $V^\tau_0(J)$ and $V^{\iota \tau}_0(J)$ is greater than zero.

**Proof of Theorem 1.5.** We exhibit an equivariant cobordism (of Type II) from $(+1)$-surgery on $J$ to $(-1)$-surgery on the knot $6_2$, where the latter is equipped with a certain involution $\tau$. This is done in Figures 27 and 28; compare [7, section 7.5]. Note that the first picture in Figure 27 certainly constitutes an equivariant cobordism from $S^3_{+1}(J)$ to some homology sphere with involution; we identify this homology sphere as $S^3_{-1}(6_2)$. However, we will not bother to make this identification equivariant, although this can be done. That is, it turns out (rather surprisingly) that we will not need to explicitly identify the involution $\tau$ on $S^3_{-1}(6_2)$.

As the cobordism is of Type II, by Theorem 7.4 it suffices to show that at least one of

$$d_{\tau}(S^3_{-1}(6_2)) \quad \text{and} \quad d_{\iota \tau}(S^3_{-1}(6_2))$$

is strictly less than zero. Indeed, we would then have that at least one of $d_{\tau}(S^3_{+1}(J))$ and $d_{\iota \tau}(S^3_{+1}(J))$ is strictly less than zero. As $J$ is genus one, $(+1)$-surgery constitutes large surgery, so this implies that one of $V^\tau_0(J)$ and $V^{\iota \tau}_0(J)$ is strictly greater than zero. (Note that the grading shift $(p - 1)/4$ in the relation (1) in this case is zero.)

The desired claim is established in [7, section 7.5], but for the sake of completeness we outline the argument here. First, the knot Floer homology of $6_2$ is easily calculated from the Alexander
FIGURE 27 Attaching a pair of equivariant 2-handles to (+1)-surgery on $J$. In (a), we perform the equivariant handle attachment. In (b), we blow down the right-hand unknot and slide the left-hand unknot partway along its band. In (c), we untwist the left-hand band. In (d), we slide the right-hand band over the unknot. In (e), we untwist the right-hand band. Manipulations are continued in Figure 28.

polynomial of $6_2$. The Floer homology $HF^-(S^3_{-1}(6_2))$ can then be calculated using the surgery formula; the action of the Hendricks–Manolescu involution $\iota$ on $HF^-(S^3_{-1}(6_2))$ can also be calculated (see, for example, [7, section 7.5]). The result is displayed in Figure 29.

Now, either $\tau$ acts on $HF^-(S^3_{-1}(6_2))$ by fixing the central $Y$-shape, or it acts on the central $Y$-shape by reflection. By direct calculation, in the former case we have

$$d_\tau(S^3_{-1}(6_2)) = 0 \quad \text{and} \quad d_{\iota\tau}(S^3_{-1}(6_2)) = -2$$

while in the latter, we have

$$d_\tau(S^3_{-1}(6_2)) = -2 \quad \text{and} \quad d_{\iota\tau}(S^3_{-1}(6_2)) = 0.$$

This completes the proof.

Remark 7.8. It possible to provide an immediate proof of Theorem 1.5 as a topological corollary to [7, Theorem 1.15], as follows. Let $\Sigma$ and $\tau_W(\Sigma)$ be any pair of symmetric slice disks for $J$ in some homology ball $W$. One can show that (+1)-surgery on $J$ is diffeomorphic to $Y = \partial W_0$, where $W_0$ is the positron cork of Akbulut–Matveyev [2]. Moreover, under this diffeomorphism, the induced action of $\tau$ on $S^3_{+1}(J)$ is the usual cork involution on $Y$. Extend the (+1)-surgery on $J$ along the disks $\Sigma$ and $\tau_W(\Sigma)$ to obtain two homology balls $B_1$ and $B_2$, each with boundary $Y$. (Here, $\partial B_1$ and $\partial B_2$ are identified via the obvious identity map.) Using $\tau_W$, we obtain a diffeomorphism $f : B_1 \rightarrow B_2$ that restricts to the cork involution on $Y$. (Note that $f$ restricts to $\tau_W$ on the complement of a tubular neighborhood of $\Sigma$.) If $\Sigma$ and $\tau_W(\Sigma)$ were isotopic rel boundary, then we would have that $(W, \Sigma)$ and $(W, \tau_W(\Sigma))$ were diffeomorphic rel boundary. This would imply the existence of a diffeomorphism $g : B_1 \rightarrow B_2$ restricting to the identity on $Y$. Then $g^{-1} \circ f$ is a self-diffeomorphism of $B_1$ restricting to the cork involution on $Y = \partial B_1$. However, in [7, Theorem 1.15], it is shown that no such extension exists.

We now explain why the proof of Theorem 1.5 implies Theorem 1.6. We begin with the following:
FIGURE 28 Identifying the result of the equivariant handle attachment. Figure (a) is a copy of (e) from Figure 27; note that both knots in the figure are unknots. In (b), we retract the right-hand band of the (+1)-curve along itself to clearly make it into an unknot. In (c), we move a strand of the (−1)-curve slightly to make the blowdown more apparent. In (d) we blow down the (+1)-curve. In (e) through (g), we isotope the result to look like the end result of [7, fig. 39]. The reader may check that this is 62.

Lemma 7.9. Let $(K, \tau)$ be a strongly invertible knot in $S^3$. Let $W$ be any (smooth) homology ball with $\partial W = S^3$ and let $\tau_W$ be any extension of $\tau$ over $W$. Let $\Sigma$ and $\tau_W(\Sigma)$ be a pair of symmetric slice disks for $K$. Then $\tau_K([F_W, \Sigma(1)]) = [F_W, \tau_W(\Sigma)(1)]$ as elements of $H_*(\mathcal{CFK}(K))$.

Proof. The proof is the same as that of Theorem 1.2. Let $F$ be a decoration on $\Sigma$. We again have the commutative diagram
Note that the decoration $sw(\tau_W(F))$ is associated to the disk $\tau_W(\Sigma)$, which is not necessarily isotopic to $\Sigma$. As $\Sigma$ and $\tau_W(\Sigma)$ are disks, no additional subtlety involving the decoration arises, and we may instead write $F_{W,\Sigma}$ and $F_{W,\tau_W(\Sigma)}$ along the top and bottom rows in the above diagram, respectively, to represent that these maps are unique up to chain homotopy. Using the fact that $\tau_K$ acts trivially on $\mathcal{H}(U)$ immediately gives the claim. 

The nontriviality of our numerical invariants then easily obstructs $F_{W,\Sigma}(1)$ and $F_{W,\tau_W(\Sigma)}(1)$ from being homologous:

**Proof of Theorem 1.6.** Let $\Sigma$ and $\tau_W(\Sigma)$ be a pair of symmetric slice disks for $J$ and suppose that $[F_{W,\Sigma}(1)] = [F_{W,\tau_W(\Sigma)}(1)]$ as elements of $H_*(\mathcal{CFK}(J))$. Lemma 7.9 then implies that $[F_{W,\Sigma}(1)]$ is a $\tau_K$-invariant element in $H_*(\mathcal{CFK}(J))$. Using this (and the fact that $F_{W,\Sigma}$ has zero grading shift), it is straightforward to construct an absolutely graded, $\tau_K$-equivariant local map from the trivial complex into $\mathcal{CFK}(J)$. This shows that $0 \leq d_\tau(\mathcal{CFK}(J)_0)$ and thus that $V_{\tau_0}(J) \leq 0$. Moreover, as $F_{W,\Sigma}$ homotopy commutes with $\tau_K$, we also know that $[F_{W,\Sigma}(1)]$ is $\tau_K$-equivariant. Hence, $[F_{W,\Sigma}(1)]$ is in fact $\tau_K \circ \tau_K$-equivariant. This likewise shows that $0 \leq d_\tau(\mathcal{CFK}(J)_0)$ and thus that $V_{\tau_0}(J) \leq 0$, contradicting the proof of Theorem 1.5.

We now verify that $[F_{W,\Sigma}(1)] \neq [F_{W,\tau_W(\Sigma)}(1)]$ as elements of $\mathcal{HFK}(J) \cong H_*(\mathcal{CFK}(J)/(U, V))$. This follows algebraically from the previous paragraph and an analysis of $\mathcal{CFK}(J)$. We first calculate the ranks of $\mathcal{HFK}(J)$ in each Alexander and Maslov grading. This can be done using the knot Floer calculator implemented in SnapPy [4]; the results are displayed in Figure 30.

Note that this computation of $\mathcal{HFK}(J)$ uses the conventions of Ozsváth-Szabó. Although $J$ is not thin, a similar analysis as in (for example) [31] allows us to determine the full knot Floer complex from the hat version. Translating into the conventions used by Zemke gives the complex displayed in Figure 31. (For a discussion of this procedure, see, for example, [8, section 2.]) This consists of a singleton generator $v$, together with four squares. Two of these squares are spanned by $\mathcal{H}$- or $\mathcal{V}$-powers of $\{a_i, b_i, c_i, d_i\} (i = 1, 2)$ and have a corner in $(\text{gr}_U, \text{gr}_V)$-bigrading $(0, 0)$. The other two are spanned by by $\mathcal{H}$- or $\mathcal{V}$-powers of $\{e_i, f_i, g_i, h_i\} (i = 1, 2)$ and have a corner in $(\text{gr}_U, \text{gr}_V)$-bigrading $(-1, -1)$.

Now, $x = F_{W,\Sigma}(1)$ and $\tau_K x = F_{W,\tau_W(\Sigma)}(1)$ are cycles in $\mathcal{CFK}(J)$ that we know are not homologous. In the quotient $\mathcal{CFK}(J)/(U, V)$, the images of $x$ and $\tau_K x$ remain cycles. The only way...
| Alexander | Maslov | Rank of $\widehat{HFK}(J)$ |
|-----------|--------|-----------------------------|
| -1        | -2     | 2                           |
| -1        | -1     | 2                           |
| 0         | -1     | 4                           |
| 0         | 0      | 5                           |
| 1         | 0      | 2                           |
| 1         | 1      | 2                           |

**Figure 30** Rank of $\widehat{HFK}(J)$ in each Alexander and Maslov grading.

**Figure 31** The complex $CFK(J)$, spanned by $v$ together with $\{a_i, b_i, c_i, d_i\}$ and $\{e_i, f_i, g_i, h_i\}$ for $i = 1, 2$. Bigradings of generators are given on the right.

for these images to become homologous in $CFK(J)/(U, V)$ would be for $x - \tau_\epsilon x$ to be (homological to) a nonzero element of $CFK(J)$ lying in the image of $(\mathcal{V}, \mathcal{V})$. However, an examination of Figure 31 shows that there are no elements of $CFK(J)$ with $\text{gr}_U = \text{gr}_V = 0$ that lie in the image of $(\mathcal{V}, \mathcal{V})$, a contradiction.

We now show that taking the $n$-fold connected sum of $J$ with itself gives a slice knot with $2^n$ distinct exotic slice disks, distinguished by their concordance maps on $\widehat{HFK}$. For a similar construction, see [34, Corollary 6.6].

**Theorem 7.10.** The (equivariant) connected sum $\#_n J$ admits $2^n$ distinct exotic slice disks, distinguished by their concordance maps on $\widehat{HFK}$.

**Proof.** As in Figure 2, let $D$ and $D'$ be the pair of exotic slice disks for $J$ from [13, section 2.1]. For each binary string $s$ of length $n$, there is an obvious slice disk $D_s$ for $\#_n J$ constructed by taking the boundary sum of copies of $D$ and $D'$, Explicitly, each index in $s$ with a 0 contributes a copy of $D$, while each index with a 1 contributes a copy of $D'$; see Figure 32. The fact that $D$ and $D'$ are topologically isotopic easily shows that the $2^n$ disks constructed in this manner are topologically isotopic rel boundary.

Now, we may identify

$$\widehat{HFK}(\#_n J) = \bigotimes_n \widehat{HFK}(J).$$
Under this identification, $[F_{B^4,D^4}(1)]$ is the tensor product of copies of $[F_{B^4,D^4}(1)]$ and $[F_{B^4,D^4'}(1)]$, each in the appropriate index. But $[F_{B^4,D^4}(1)] \neq [F_{B^4,D^4'}(1)]$ as elements of the vector space $\tilde{HF}(J)$. It follows that the $[F_{B^4,D^4}(1)]$ are different for different strings $s$. This completes the proof. □

Finally, we generalize Theorem 1.5 to an infinite family of knots $J_n$ with exotic pairs of slice disks, considered by Hayden in [13, section 2.3]. These are displayed on the left in Figure 33 and are obtained from the knot $J$ of Theorem 1.5 by adding pairs of (negative) full twists, as indicated. We have an obvious pair of slice disks for $J_n$ given by compressing along the displayed red and blue curves. In [13, fig. 9], Hayden constructs a handle diagram for the complement of these disks; it is immediate from [13, fig. 9] that the disk exteriors have fundamental group $\mathbb{Z}$ and thus that the disks are topologically isotopic. Here, we show that knot Floer homology obstructs any two symmetric pair of disks for $J_n$ from being smoothly isotopic.

**Theorem 7.11.** Let $J_n$ (for $n \geq 0$) be as in Figure 33. Then $\tilde{g}_{4}(J_n) > 0$. In particular, no pair of symmetric slice disks $\Sigma$ and $\tau_W(\Sigma)$ are (smoothly) isotopic rel $J_n$. This holds for any (smooth) homology ball $W$ with $\partial W = S^3$ and any extension $\tau_W$ of $\tau$ over $W$.

**Proof.** Clearly, (+1)-surgery on $J_n$ admits a negative-definite equivariant cobordism to (+1)-surgery on $J$, given by attaching (−1)-framed 2-handles along the green curves indicated on the right in Figure 33. Noting that each $J_n$ has Seifert genus one, it follows from Theorem 7.4 that

$$\tilde{V}_0^\ast(J_n) \geq \tilde{V}_0^\ast(J) \quad \text{and} \quad V_0^\ast(J_n) \geq V_0^\ast(J)$$
and
\[ \overline{V}^{\tau}_{0}(J) \geq \overline{V}^{\tau}_{0}(J_n) \quad \text{and} \quad \overline{V}^{\tau}_{0}(J) \geq \overline{V}^{\tau}_{0}(J). \]

The claim then immediately follows from our bounds on the invariants of \( J \).

\[ \square \]

### 7.4 Secondary invariants

We now discuss the secondary invariant \( V_0(\Sigma, \Sigma') \) of [21, section 4.5]. This is defined as follows. Let \( K \) be a knot in \( S^3 \). For simplicity, let \( W \) be a homology ball with boundary \( S^3 \) and let \( \Sigma \) and \( \Sigma' \) be two slice disks for \( K \) in \( W \). Consider the elements \( F_{W,\Sigma}(1) \) and \( F_{W,\Sigma'}(1) \) in \( CFK(K) \). (Note that as \( \Sigma \) and \( \Sigma' \) are disks, no choice of decoration is needed; the more general definition of \( V_0(\Sigma, \Sigma') \) in [21, section 4.5] requires a discussion of a specific set of dividing curves.) Viewing \( F_{W,\Sigma}(1) \) and \( F_{W,\Sigma'}(1) \) as elements of the large surgery subcomplex \( CFK(K)_0 \), define
\[
V_0(\Sigma, \Sigma') = \min\{ n \in \mathbb{Z} \geq 0 \mid U^n \cdot [F_{W,\Sigma}(1)] = U^n \cdot [F_{W,\Sigma'}(1)] \text{ in } H_*(CFK(K)_0) \}.
\]

In [21, Theorem 1.1], it is shown that \( V_0(\Sigma, \Sigma') \) bounds the stabilization distance between \( \Sigma \) and \( \Sigma' \) from below:
\[
V_0(\Sigma, \Sigma') \leq \left\lceil \frac{\mu_{st}(\Sigma, \Sigma')}{2} \right\rceil.
\]

The following is straightforward:

**Proof of Theorem 1.10.** Let \( \Sigma \) be any slice disk for \( K \) in \( W \), and suppose that \( V_0(\Sigma, \tau_{W}(\Sigma)) = n \). By Lemma 7.9, we have
\[
\tau_K([F_{W,\Sigma}(1)]) = [F_{W,\tau_{W}(\Sigma)}(1)].
\]

Multiplying both sides by \( U^n \), we obtain a \( \tau_K \)-invariant element in the homology of \( CFK(K)_0 \) with grading \(-2n\). This implies \( \delta_{\tau}(CFK(K)_0) \geq -2n \) and thus that \( V^{\tau}_{0}(K) \leq n \). Moreover, as \( F_{W,\Sigma} \) homotopy commutes with \( \tau_K \), we also know that \( [F_{W,\Sigma}(1)] \) is \( \tau_K \)-equivariant. Hence, we obtain an \( \tau_K \circ \tau_K \)-invariant element in the homology of \( CFK(K)_0 \) with grading \(-2n\). This similarly shows that \( V^{\tau}_{0}(K) \leq n \).

\[ \square \]

### 8 PERIODIC KNOTS

We close this paper by discussing a similar family of results in the periodic setting. As in the strongly invertible case, it is possible to define an action of \( \tau_K \) associated to a 2-periodic knot \( K \) and consider the notion of a periodic \( (\tau_K, \iota_K) \)-complex. The same subtlety as in Subsection 3.4 arises, in that this is only an invariant of equivariant concordance in the decorated category. Nevertheless, once again we may define numerical invariants \( \overline{V}^0_0 \) and \( \overline{V}^0_0 \), and these give bounds for the equivariant slice genus. Although much of the formalism is thus the same, the authors have not yet been able to find many interesting calculations of periodic invariants. One key difference
is that in the periodic case, there is no natural notion of equivariant connected sum. Correspondingly, it turns out that the set of periodic $(τ_K, t_K)$-complexes (up to local equivalence) does not seem to admit a natural group structure.

### 8.1 Construction of $τ_K$

We begin with the construction of $τ_K$. Let $(K, τ)$ be a 2-periodic knot. In contrast to the strongly invertible case, it is natural to assume that $K$ is oriented (as $K$ may not come with an orientation-reversing diffeomorphism). Let $w$ and $z$ be a pair of basepoints on $K$ that are interchanged by $τ$, and let $H$ be any choice of compatible Heegaard data for $(K, w, z)$. Taking the pushforward under $τ$ gives a tautological isomorphism

$$t : CFK(H) \to CFK(τH).$$

The latter complex represents the same knot, but now the roles of the basepoints have been interchanged. We now apply a half-Dehn twist that moves $w$ into $z$ and $z$ into $w$:

$$ρ : CFK(∥H) \to CFK(ρτH).$$

Finally, we have the naturality map

$$Φ(ρτH, H) : CFK(ρτH) \to CFK(H).$$

We thus define $τ_H$ to be

$$τ_H : CFK(H) \xrightarrow{t} CFK(τH) \xrightarrow{ρ} CFK(ρτH) \xrightarrow{Φ} CFK(H).$$

As before, $τ_H$ is independent of the choice of Heegaard data for $(K, w, z)$.

The proof of the following is analogous to that of Theorem 1.7:

**Theorem 8.1.** Let $(K, τ)$ be an (oriented) 2-periodic knot and fix a pair of symmetric basepoints $(w, z)$ on $K$. Let $H$ be any choice of Heegaard data compatible with $(K, w, z)$. Then $τ$ induces an automorphism

$$τ_H : CFK(H) \to CFK(H)$$

with the following properties:

1. $τ_H$ is filtered and $𝔽[.Weight, V]$-equivariant;
2. $τ_H^2 ≃ ζ_H$;
3. $τ_H ∘ t_H ≃ t_H ∘ τ_H$.

Here, $t_H$ is the Hendricks–Manolescu knot Floer involution on $CFK(H)$ and $ζ_H$ is the Sarkar map. Moreover, the homotopy type of the triple $(CFK(H), τ_H, t_H)$ is independent of the choice of Heegaard data $H$ for the doubly based knot $(K, w, z)$. 
Proof. Left to the reader; analogous to Theorem 1.7.

Note the difference in all three properties with Theorem 1.7.

Remark 8.2. It turns out that the analogous subtlety to Subsection 3.3 does not arise at this stage: the homotopy class of \((CFK(H), \tau_H, \iota_H)\) is independent of the choice of symmetric basepoints \(w\) and \(z\). This is because any two pairs \((w, z)\) and \((w', z')\) on \(K\) are related by a \(\tau\)-equivariant basepoint-pushing diffeomorphism along \(K\). Unlike in the strongly invertible case, the associated pushforward map commutes with all the components of \(\tau_H\). Combined with the fact that \(K\) comes with an orientation, this shows that we may unambiguously refer to the \((\tau_K, \iota_K)\)-complex of \((K, \tau)\), without specifying any additional data.

8.2 Periodic \((\tau_K, \iota_K)\)-complexes

Given Theorem 8.1, it is natural to define a \((\tau_K, \iota_K)\)-complex formalism in the periodic setting:

Definition 8.3. A periodic \((\tau_K, \iota_K)\)-complex is a triple \((C, \tau_K, \iota_K)\) such that:

1. \(C\) is an abstract knot complex;
2. \(\iota_K : C \to C\) is a skew-graded, \(R\)-skew-equivariant chain map such that \(\iota_K^2 \simeq \zeta_K\);

3. \(\tau_K\) is a graded, \(R\)-equivariant chain map such that

\[\tau_K^2 \simeq \zeta_K\quad\text{and}\quad \tau_K \circ \iota_K \simeq \iota_K \circ \tau_K.\]

The notions of homotopy equivalence and local equivalence carry over without change. We may also define the notion of a twist by \(\zeta_K\) as before. However, it should be noted that there is no analogue of Lemma 2.21 in the periodic setting.

Remark 8.4. The principal difference between local equivalence in the periodic and strongly invertible settings is the absence of a natural group structure in the former. Indeed, the reader can check that trying a product law such as

\[\tau_\otimes = \tau_1 \otimes \tau_2\]

or even

\[\tau_\otimes = (\text{id} \otimes \text{id} + \Phi \otimes \Psi) \circ (\tau_1 \otimes \tau_2)\]

does not satisfy \(\tau_\otimes^2 \simeq \zeta_\otimes\).

The algebraic procedure of Subsection 2.5 also carries over without change to define numerical invariants \(V^\tau_0(K), V^\iota_0(K), \tilde{V}^\iota_0(K),\) and \(\tilde{V}^{\iota\iota}_0(K)\). In the forthcoming paper, the second author estab-
lished a large surgery formula for periodic knots. Thus, $\overline{V}_0^\circ$ and $\underline{V}_0^\circ$ again have the interpretation as invariants associated to large surgeries.

### 8.3 Equivariant concordance and cobordism

As in Subsection 2.1, we may define the notion of an isotopy-equivariant homology concordance between two periodic knots $(K_1, \tau_1)$ and $(K_2, \tau_2)$. The subtlety of Subsection 3.4 again arises: even if $\Sigma$ is equivariant or isotopy-equivariant, it is unclear whether an equivariant or isotopy-equivariant pair of arcs on $\Sigma$ can be chosen. We thus have:

**Theorem 8.5.** Let $(K_1, \tau_1)$ and $(K_2, \tau_2)$ be two periodic knots in $S^3$. Suppose that we have an isotopy-equivariant homology concordance between $(K_1, \tau_1)$ and $(K_2, \tau_2)$. Then $(CF\mathcal{K}(K_1), \tau_{K_1}, t_{K_1})$ is locally equivalent to either $(CF\mathcal{K}(K_2), \tau_{K_2}, t_{K_2})$ or $(CF\mathcal{K}(K_2), \tau_{K_2}, t_{K_2})$. Hence, $\overline{V}_0^\circ(K)$ and $\underline{V}_0^\circ(K)$ are invariant under isotopy-equivariant homology concordance.

**Proof.** Left to the reader; analogous to Theorem 3.15.

Finally, we formally record that the results of Theorems 1.2 and 1.10 hold in the periodic setting:

**Theorem 8.6.** Let $(K, \tau)$ be a 2-periodic knot in $S^3$. Then for $\circ \in \{\tau, \iota\tau\}$,

$$-\left[\frac{1 + \tilde{g}_4(K)}{2}\right] \leq \overline{V}_0^\circ(K) \leq \underline{V}_0^\circ(K) \leq \left[\frac{1 + \tilde{g}_4(K)}{2}\right].$$

**Proof.** Left to the reader; analogous to Theorem 1.2.

**Theorem 8.7.** Let $(K, \tau)$ be any 2-periodic knot in $S^3$. Let $W$ be any (smooth) homology ball with boundary $S^3$, and let $\tau_W$ be any extension of $\tau$ over $W$. If $\Sigma$ is any slice disk for $K$ in $W$, then

$$\max\{V_0^\tau(K), V_0^{\iota \tau}(K)\} \leq V_0(\Sigma, \tau_W(\Sigma)).$$

**Proof.** Left to the reader; analogous to Theorem 1.10.

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