A voter model with time dependent flip rates

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Abstract. We introduce time variation in the flip rates of the voter model. This type of generalization may be applied to other diffusion-like models in which interaction rates at the microscopic level may change with time, for example in models of language change, allowing the representation of changes in speakers’ learning rates over their lifetime. The mean time taken to reach consensus varies in a nontrivial way with the rate of change of the flip rates, varying between bounds given by the mean consensus times for static homogeneous (the original voter model) and static heterogeneous flip rates. By considering the mean time between interactions for each agent, we derive excellent estimates of the mean consensus times and exit probabilities for any timescale of flip rate variation. The scaling of consensus times with population size on complex networks is correctly predicted, and is as would be expected for the ordinary voter model. Heterogeneity in the initial distribution of opinions has a strong effect, considerably reducing the mean time to consensus, while increasing the probability of survival of the opinion which initially occupies the most slowly changing agents. The mean times taken to reach consensus for different states are very different. An opinion originally held by the fastest changing agents has a smaller chance of succeeding, and takes much longer to do so than an evenly distributed opinion.

Keywords: population dynamics (theory), interacting agent models, stochastic processes

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1. Introduction

Neutral diffusion-like or copying process models have been applied in a very broad range of fields, from social phenomena [1] and language change [2,3], to ecology [4]–[6] and population genetics [7] among many others. In all such models, different alternative items—species, opinions or language variants for example—are copied between neighbouring sites until one finally dominates the whole system. Here we consider the effect of time variation in the rate of update at each site in such models. This has particular relevance to language change, in which the rate at which speakers adapt to their language environment changes with age. The items copied are alternative variants of a language element; different ways of ‘saying the same thing’. Young speakers adapt very quickly, but once they reach adulthood many speakers barely change their language use [8,9]. We show that the time taken to reach consensus depends in a nontrivial way on the timescale of the variations of the agent update rate; see figure 1. By considering the mean time between interactions for each agent, we are able to obtain excellent estimates of the mean time for reaching consensus, even in the intermediate regime where the timescale of consensus formation and the timescale of flip rate change are of the same order, and interact in a nontrivial way.

The generalization and the methods that we will describe could be applied to any of the models mentioned, but for simplicity we concentrate on the voter model [10], in which agents in a population possess one of two discrete opinions. At each step an agent is chosen and imports the opinion of a randomly selected neighbour. The rate at which a particular agent is chosen for update is their flip rate, and it is the time variance of these flip rates that we consider in this paper. The voter model has become an emblematic opinion spreading model due to its simplicity and tractability, as well as its distinction from other coarsening phenomena such as in the Ising model [11]. The original voter model has been extended to include network structure [12]–[14] and the effect of changes in the microscopic interactions [15,16]. Masuda et al investigated the effect of heterogeneity in
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Figure 1. Mean consensus time $T$ as a function of the period $S$ of the change of flip rates for three different flip rate functions $r(t/S)$. Numerical simulations are shown as open symbols for the sinusoidal function $r_{\text{sin}}$ (circles), linear ‘sawtooth’ $r_{\text{lin}}$ (triangles) and quadratic sawtooth $r_{\text{qdr}}$ (squares). Solid curves show analytic calculations using equation (20). The dashed line shows mean consensus time $T_{\text{hom}}$ for homogeneous flip rates, and dotted lines those for heterogeneous but static flip rates, $T_{\text{het}}$. Simulation results are averages over $10^4$ runs with homogeneous initial conditions and population size $N = 500$.

the flip rates of agents [17]. While they have some similarity in spirit, the present model differs from other time dependent models including such effects as latency or ageing of states [18]–[20] in that the change in flip rate does not depend on the opinion or the time of adoption of the opinion. That is, we are not interested in ageing of the opinions but that of the agents themselves. The present work is probably most similar to the ‘exogenous update’ rule work of [20]; however in that case the update rule is the same for all agents and is reset when an agent becomes available for update, rather than changing independently.

In the heterogeneous voter model [17], a population of $N$ agents (labelled $i = 1, 2, \ldots, N$) possess opinions $x_i$ which take values either 1 or 0. Agents are selected for update asynchronously with frequency proportional to flip rates $r_i$, which may be different for each agent. Update consists of an agent importing the opinion of a neighbour, selected uniformly at random. In the present study, we extend this model to consider the flip rates to be time dependent, and thus $r_i(t)$. The method described here is perfectly amenable to use in considering complex network structure. For simplicity, we first consider a well mixed population (that is, a fully connected network) before examining more complex structures.

The mean time to consensus for homogeneous flip rates $r_i = r, \forall i$, that is, the standard voter model, is well established [12, 21, 22], and is found by assuming that the
population of agents relaxes quickly to a quasi-stationary state (QSS), followed by a much longer period characterized by collective motion. The mean time to consensus can be calculated for this second stage by writing a Fokker–Planck equation for a conserved centre-of-mass variable. This gives a lower bound and a good approximation to the total mean time to consensus. When heterogeneity is introduced in the flip rates, the consensus time is always increased, and can be predicted from the moments of the flip rate distribution [17]. For very slowly changing flip rates, the consensus time is essentially the same as for the static heterogeneous case. For very quickly changing flip rates, the consensus time may be reduced to the value found for homogeneous flip rates with the same mean. For intermediate periods, the time to consensus varies smoothly between these two extremes, as can be seen in figure 1. Using the distribution of flip rates existing in the population simply returns the static heterogeneous result, which does not vary with the period of variation of the flip rates. Instead, we calculate effective flip rates, found by considering the mean time between interactions for each agent. Using the moments of the distribution of these effective flip rates returns the correct qualitative behaviour, and is in excellent quantitative agreement with numerical results (figure 1).

2. Analysis

The mean time to consensus can be calculated through a Fokker–Planck equation (FPE) formalism. See for example [21, 12] or the rigorous treatment in [22]. The essential idea is that after an initial, rapid, period of mixing, the individual opinions settle into a long lived meta-stable distribution. This quasi-stationary state (QSS) is characterized by a weighted mean opinion which is conserved by the dynamics. The mean time to consensus can be calculated by considering the evolution of only this central variable.

2.1. Mean consensus time for static flip rates

For orientation, we first calculate the mean consensus times for static homogeneous (i.e. the basic voter model) and static heterogeneous flip rates. We define a weighted mean opinion $\xi(t)$ as follows:

$$\xi(t) = \frac{\sum_i x_i(t)/r_i}{\sum_i 1/r_i} = \sum_i Q_i x_i(t)$$

where

$$Q_i = \frac{1}{N(\langle 1/r \rangle r_i)}$$

and $\langle \cdots \rangle$ signifies a population average. We choose $\xi$ because it is conserved by the dynamics [23, 24, 15, 21]:

$$\frac{d}{dt} \langle \xi(t) \rangle = 0.$$  \hfill (3)

This also means that the probability that the population eventually reaches consensus in the state $\mathbf{1}$ is simply

$$P(x \to \mathbf{1}) = \xi(0) = \xi_0.$$ \hfill (4)
Choosing the time increment $\delta t$ This agrees with the result obtained previously in [17].

As in the homogeneous case, the moments $\mu_k$ give the standard result

$$ T_{\text{hom}} \approx N \left[ x_0 \ln x_0 + (1 - x_0) \ln(1 - x_0) \right]. $$

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2.2. Time dependent flip rates

Now we are ready to consider the case where the flip rates \( r_i(t) \) can vary with time. We assume that the flip rates \( r_i(t) \) follow some periodic function with the period \( S \) acting as a control variable. Initial values for \( r_i \) are chosen by selecting \( s_i \) uniformly at random from \([0, 1]\) and setting \( r_i(0) = f(s_i) \) for some \( f(s) \). It is convenient to ensure a stable distribution of \( r_i \) values in the population over time. In the language change application, this corresponds to a stable distribution of speaker ages in a population, with old speakers periodically replaced by young ones, and whose members’ learning rates all follow the same function of age (i.e. \( r(s) \)). This is obviously a very crude model, and our aim here is simply to demonstrate the general effect of time variation in such interactions. To achieve this, we define a periodic version of \( f(s) \): let \( r(s) \equiv f(s - [s]) \). For an agent with initial flip rate \( r_i(0) = r(s_i) \), we set

\[
r_i(t) = r \left( s_i + \frac{t}{S} \right).
\]  

This ensures that the period is equal to \( S \) and also that at any time, the overall distribution of \( r_i(t) \) values in a large population follows \( dr = df(s)/ds \).

We postulate that the change in observed consensus time is due to an interaction between the timescales of flip rate change and of opinion change (or consensus formation) of the population. When the flip rates change extremely slowly, consensus will be reached with essentially no change in flip rates; hence the static heterogeneous result equation (10) applies. If the flip rates change extremely quickly, agent \( i \) will cycle through all the possible values of \( r_i \) in a very short time compared with the rate at which she interacts. We can therefore calculate an approximate consensus time by replacing \( r_i \) with \( \langle r_i \rangle_t = r_0 \). We see, then, that in the limit of very quickly changing flip rates \( T \) is given by equation (11), i.e. the consensus time in the standard homogeneous voter model. The heterogeneous flip rate consensus time (10) and the fast change limit (11) provide approximate upper and lower bounds for the consensus time. As can be seen in figure 1, these bounds agree very well with numerical results in the two limits, and consensus times for intermediate regimes lie between the two.

We can calculate a more rigorous interpolation between the results (10) and (11) by observing that in both cases, the weight for each agent’s state in the sum for \( \xi(t) \) in equation (1) is proportional to \( 1/r_i \), which is the expected time interval between interactions for agent \( i \). For intermediate \( S \), let us generalize by defining \( \tau_i(t) \) to be the expected interval between interactions for agent \( i \), where the dependence on \( t \) indicates that this update interval varies because \( r_i(t) \) varies with time. Consider a sequence of short intervals of length \( \Delta t \), beginning at time \( t \). The probability that \( i \) is selected in the first interval is \( r_i(t) \Delta t \). The probability that \( i \) is selected in the second (having not been selected in the first) is \( [1 - r_i(t) \Delta t]r_i(t + \Delta t) \Delta t \), and so on. The probability that \( i \) is selected in the \((k + 1)\)th such interval is thus

\[
r_i(t + k \Delta t) \Delta t \prod_{l=0}^{k-1} [1 - r_i(t + l \Delta t) \Delta t].
\]

Taking \( \Delta t \to 0 \) we can write the terms in the product as exponentials, i.e. \( 1 - r_i(t + l \Delta t) \Delta t \to \exp\{-r_i(t') \Delta t\} \), where we have rewritten \( t + k \Delta t \to t' \) and \( t + l \Delta t \to t'' \). The
product then becomes an integral in the argument of the exponential, so the probability that \( i \) is selected in the interval \([t', t' + dt]\) becomes

\[
  r_i(t') \, dt \exp \left\{ - \int_t^{t'} r_i(t'') \, dt'' \right\}.
\]  \tag{14}

Multiplying by the waiting times \((t' - t)\) and integrating gives the expected waiting time

\[
  \tau_i(t) = \int_t^\infty (t' - t) r_i(t') \exp \left\{ - \int_t^{t'} r_i(t'') \, dt'' \right\} \, dt'.
\]  \tag{15}

We can then define an effective flip rate

\[
  \tilde{r}_i(t) \equiv \frac{1}{\tau_i(t)}.
\]  \tag{16}

For the static case, we recover \( \tilde{r}_i(t) = r_i \). For extremely quickly varying \( r_i(t) \), \( \tilde{r}_i(t) = \bar{r} \), which is the same value as was obtained in the original homogeneous voter model. To see this, notice that fluctuations in \( \int_t^{t'} r(t'') \, dt'' \) die out very quickly, so after some small time \( \sigma \), \( \exp \left\{ - \int_t^{t'} r_i(t'') \, dt'' \right\} \approx \exp \{-(t' - t)\bar{r}\} \), while for times less than \( \sigma \), the exponential term is close to 1 and \( \int_t^{t+\sigma} (t'-t) r_i(t') \, dt' \approx \sigma^2 \bar{r} \), giving \( \tau_i(t) \approx \sigma^2 \bar{r}_0 + e^{-\sigma \bar{r}} (1 + \sigma \bar{r})/\bar{r} \approx 1/\bar{r} \).

These limits agree with the two limits obtained through the qualitative arguments above.

For the reduction to a single variable, we then define

\[
  \tilde{\xi}(t) \equiv \frac{\sum_i \tau_i(t) x_i(t)}{\sum_i \tau_i(t)}.
\]  \tag{17}

The argument proceeds as before, so we in effect replace \( 1/\bar{r} \) by \( \tau_i(0) \) and \( \bar{r} \) by \( \tilde{r}_0 = \tilde{r}_i(0) \) in equation (9), to give

\[
  -1 = \frac{1}{N \tau_i(0) \tilde{r}_0} \tilde{\xi}_0 (1 - \tilde{\xi}_0) \frac{d^2}{d\tilde{\xi}_0} T^*.
\]  \tag{18}

For homogeneous initial conditions, \( \tilde{\xi}_0 = x_0 \). We can utilize the fact that all agents follow the same flip rate function \( r(s) \), but starting at different initial values of \( s \), to estimate \( \tau_i(0) \). In the large-\( N \) limit, the agent’s initial values of \( s_i \) evenly populate the interval \([0, 1] \). For large \( N \), then, we can replace the average over agents by an average over \( s \), giving

\[
  \bar{\tau}_i(0) \approx \int_0^1 \tau(s) \, ds \equiv \tau_0,
\]  \tag{19}

where \( \tau(s) \equiv \tau_i(0) \) for \( i \) such that \( r_i(0) = r(s) \). Finally then we can write

\[
  T \approx N \tau_0 \left[ x_0 \ln x_0 + (1 - x_0) \ln(1 - x_0) \right]
\]  \tag{20}

for homogeneous initial conditions. This analytic calculation is in excellent agreement with numerical results for various \( r(s) \) distributions over the whole range of \( S \), as can be seen in figure 1.
Higher moments can be calculated iteratively using equations [25]

\[-nT_{n-1} = \frac{1}{N\tau_0} \tilde{\xi}_0(1 - \tilde{\xi}_0) \frac{d^2}{d\xi_0^2} T_n,\]  

where \(T_n\) is the \(n\)th moment of the consensus time distribution, leading to expressions in terms of polylogarithm functions. The variance of consensus times calculated in this way is in excellent agreement with numerical simulations (not shown). In fact equation (21) only differs from the ordinary voter model by a factor \(\tau_0\), so the whole distribution of consensus times has the same shape as that for the ordinary voter model, once time is rescaled by \(\tau_0\). This is borne out in simulations; see figure 3.

2.3. Network structure

In [15] it was shown that a similar mean-field approach is sufficient to reproduce the population size scaling of the mean consensus time on heterogeneous networks for the ordinary voter model. By assuming flip rates to be independent of degree, a similar treatment can be performed here. We now show that the population size scaling for time varying flip rates (and hence also for the heterogeneous flip rate model of [17]) depends on the network degree distribution in the same way as in the ordinary voter model.

We define the weighted mean to be

\[\xi(t) = \frac{\sum_i x_i(t) q_i/r_i}{\sum_i q_i/r_i}\]  

where \(q_i\) is the degree of voter \(i\). Because the probability that voter \(i\) is chosen for update is proportional to \(1/q_i\), we again find that \(\xi\) is conserved by the dynamics. Carrying out averages over \(q_i\) and \(r_i\) separately (they are chosen independently) and, as before, replacing population averages with distribution moments we find, for large populations,

\[\langle \delta \xi^2 \rangle \approx \frac{1}{r_0 \mu \langle q \rangle^2 N^3} \sum_q n_q q^2 \left[ x(1 - x_q) + x_q(1 - x) \right].\]

Here \(n_q\) is the number of voters having degree \(q\), and \(x_q\) is the mean opinion of such voters. Finally, in the QSS we set \(x_q = x = \xi\) giving

\[\langle \delta \xi^2 \rangle \approx \frac{\langle q^2 \rangle}{r_0 \mu \langle q \rangle^2 N^2} \xi (1 - \xi).\]

This differs from the fully connected result only by a factor \(\langle q^2 \rangle / \langle q \rangle^2\), and hence the mean consensus time for heterogeneous flip rates on a network goes as

\[T \propto N \mu - \frac{\langle q \rangle^2}{\langle q^2 \rangle}.\]

It follows that for time varying flip rates and homogeneous initial conditions, the mean consensus time goes as (compare equation (20))

\[T \propto N \tau_0 \mu - \frac{\langle q \rangle^2}{\langle q^2 \rangle}.\]

As can be seen in figure 2, this agrees with numerical simulation for Erdős–Rényi networks and for networks with power-law degree distributions.
Figure 2. Mean consensus time $T$ versus population size $N$ on different networks, plotted on logarithmic axes. From top to bottom: Erdős–Rényi graph with mean degree 7 (\(\square\)), scale-free graph with exponent $\gamma = 3.5$ (\(\bigcirc\)) and scale-free graph with $\gamma = 2.5$ for $S = 2048, 32$ and $0.5$ (triangles). Solid lines are analytic predictions based on equation (26) which predicts $T \propto N$ for the Erdős–Rényi and scale-free cases with $\gamma = 3.5$ and $T \propto N^{2/3}$ for $\gamma = 2.5$. All data are for $r_{\text{lin}}(s)$ with $x_0 = 0.5$ and homogeneous initial conditions.

Figure 3. Distribution $p_T(t/\tau_0)$ of consensus times rescaled by $\tau_0$ for $r_{\text{qdr}}(s)$ and $S = 0.5$ (\(\square\)), $S = 32$ (\(\bigodot\)) and $S = 2048$ (\(\bigtriangleup\)), and for $r_{\text{lin}}(s)$ with $S = 128$ (\(\times\)). Frequencies were binned in rescaled time intervals of 50. Under this time rescaling, all distributions collapse onto the same curve. Inset: the same data plotted with a logarithmic vertical axis, compared with a decaying exponential function (line).
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Figure 4. Probability of reaching consensus at the state 1 as a function of S for inhomogeneous initial conditions. Open squares are simulation results for $r_{\sin}(s)$. Half the agents have $x_i(0) = 1$ and initial flip rate $r_{\sin}(s_i)$ with $s_i \in [1/2, 1)$. The remaining agents have $x_i(0) = 0$ and $s_i \in [0, 1/2)$. Data points are from $2 \times 10^4$ runs with $N = 500$. The solid curve shows $\tilde{\xi}_0$ calculated using equations (15)–(17).

2.4. Inhomogeneous initial conditions

We now consider the effect of heterogeneity in the initial opinions of agents. For the static heterogeneous case, the probability $P(x \to 1)$ of reaching consensus state 1 also depends on the flip rates of the agents. Opinion 1 is more likely to achieve consensus if it is initially that of the more slowly changing agents, and vice versa.

Returning to equation (17), $\langle \tilde{\xi} \rangle$ is a conserved quantity to the extent that the replacement of $r_i(t)$ by $\tilde{r}_i(t)$ in the Fokker–Planck equation (8) is a valid approximation. The probability of reaching consensus in the state 1 is then simply $\tilde{\xi}_0$. This probability varies with the period of change $S$ of the flip rates. When the flip rates change very fast, the agents are essentially identical (having effective flip rate $\bar{r}$) and there is no initial configuration dependence. Conversely, the effect of initial inhomogeneity in opinion (that is, correlation between the initial flip rate and initial opinion) will be strongest for extremely slowly varying flip rates. This can be seen in figure 4.

The mean consensus time again obeys equation (18), but the presence of inhomogeneous initial conditions is felt through the fact that now $\tilde{\xi}_0 \neq x_0$, leading to

$$T \approx N\tau_0[\tilde{\xi}_0 \ln \tilde{\xi}_0 + (1 - \tilde{\xi}_0) \ln(1 - \tilde{\xi}_0)].$$

This means that we must calculate $\tilde{\xi}_0$ using the initial distribution of $x_i$. In the examples shown here this was done by integrating equation (15) numerically and then computing equation (17) as an integral over $s$ (compare equation (19)) to find $\tilde{\xi}_0$. The value of $\tau_0$ depends only on $r(s)$ and $S$, and so is independent of the initial conditions. As for the homogeneous case, $\tau_0$ increases with $S$, and because the fastest changes in $\tau_0(S)$ and $\tilde{\xi}_0(S)$ occur in different ranges of $S$ (compare figure 1 with 4) the final curve for $T(S)$ has the complicated shape seen in figure 5, as borne out in simulation.

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Figure 5. (a) Mean consensus time as a function of $S$ for inhomogeneous initial conditions. Open circles are simulation results for $r_{\text{sin}}(s)$. The solid curve shows the consensus time calculated using equation (27). (b) Mean time for reaching each of the two possible consensus states: ($\triangle$) mean time for reaching final state 1; ($\nabla$) mean time for reaching final state 0. Analytical predictions of equations (28) and (29) are shown as solid lines (blue and red respectively). For comparison the analytic predictions for the overall mean time to consensus (solid, green) from equation (27)—the same as shown in panel (a)—and for the homogeneous case (dashed) from equation (20) are also shown. Simulation conditions are the same as for figure 4.

To calculate the mean time to consensus at a particular state, we again solve equation (18) but the appropriate boundary conditions are different. This gives

$$T_0 \approx N\tau_0 \frac{\tilde{\xi}_0}{(1 - \tilde{\xi}_0)} \ln \tilde{\xi}_0,$$

(28)

$$T_1 \approx N\tau_0 \frac{(1 - \tilde{\xi}_0)}{\tilde{\xi}_0} \ln(1 - \tilde{\xi}_0).$$

(29)

Combining these two times in proportion to the probability for reaching each state returns equation (27).

3. Numerical simulations

We performed simulations of the model with different distributions of $r_i$. We considered the following functional forms for $r(s)$, chosen to give some variety of interesting functional forms:

$$r_{\text{qhr}}(s) = r_0 - r_d + 3r_d(1 - s + \lfloor s \rfloor)^2,$$

(30)

$$r_{\text{sin}}(s) = r_0 + r_d \sin(2\pi s),$$

(31)

$$r_{\text{lin}}(s) = r_0 - r_d + 2r_d(s - \lfloor s \rfloor).$$

(32)

The mean of each function is $\mu_1 = r_0$, which we generally choose to be equal to 1. The parameter $r_d$ controls the amplitude of the variations. To ensure that the values
of $r_i$ are always strictly positive, we require $r_d < r_0$. The mean time to consensus and probability of reaching a certain final state depend only on the distribution of $r_i$ values in the population and on the period $S$. Careful consideration of equations (15) and (19) which involve integration over every possible interval of $r(s)$ leads to the conclusion that a reordering of the function $r(s)$ would lead to the same mean consensus time. For example, time reversed versions of any of the $r(s)$ functions would yield the same results.

The mean time to consensus for different flip rate periods $S$ is presented in figure 1 for the three $r(s)$ functions tried. All move from a time similar to that found for homogeneous $r_i$ (the original voter model) for small $S$ to a value close to that found for static $r_i$ with the same distribution (heterogeneous voter model) for large $S$. See equations (10) and (11) in section 2. The transition occurs over a similar range of $S$ for each model, though the shape differs a little.

The distribution of consensus times rises rapidly to a peak value at small times, and this is followed by a long tail very closely approximated by an exponential decay. In equation (20) we see that the mean consensus times for different $r$ functions or values of $S$ differ only by the factor $\tau_0$. In fact, the whole distribution of consensus times has the same shape, so if we rescale by $\tau_0$, the distributions collapse onto the same curve, as shown in figure 3. In the inset the data are plotted against a logarithmic scale, showing the exponential tail clearly. Interestingly the decay $p_T \propto e^{-\lambda t}$ generally does not have $\lambda = 1/T$ as might be naively expected.

We also carried out simulations with inhomogeneous initial conditions. Because we have so far dealt only with a simple fully connected (or well mixed) population, the inhomogeneity is in the correlation between initial $r_i$ values and initial opinions. Shown in figures 4 and 5 are results for $r_{\sin}(s)$. The agents were divided into two equal groups. Agents in the first group had $x_i(0)$ set to 0, and $s_i(0)$ chosen uniformly in $[0, 1/2)$. The agents in the second group had $x_i(0)$ set to 1, and $s_i(0)$ chosen uniformly in $[1/2, 1)$. In this way $x_0 = 1/2$ but the agents with initial opinion 1 all had initial values of $r_i$ below $r_0$, while those with initial opinion 0 all had $r_i(0) > r_0$. This means that the probability of reaching final state 1 is greater than $x_0$. As can be seen in figure 4, the effect is largest for large $S$.

As $S$ decreases, the timescale of change of $r_i$ eventually becomes shorter than the timescale of consensus, and the effect of the initial inhomogeneity is lost. The time for reaching consensus also changes with $S$, as for the homogeneous case, but now in a more complicated way; see figure 5(a). The inhomogeneity reduces the overall time for reaching consensus from that found in the heterogeneous case, as can be seen by comparing the dotted and central solid curves in figure 5(b), due to the inertia of the slowly changing agents, who now initially share a common opinion, combined with the speed with which the agents holding the weaker opinion may be changed.

It is also interesting to compare the mean times taken for reaching each of the final states, 0 and 1. The mean time for reaching 1 is almost as short as (but not shorter than) that for homogeneous $r$. The agents originally holding opinion 1 have a lot of weight, so the majority of agents in the mixed QSS will have opinion 1 and quickly kill off opinion 0. In the minority of cases (see figure 4) the final state is 0. In this case, the time taken for reaching consensus is very long, significantly longer then the time taken for heterogeneous $r$ with homogeneous initial conditions, as can be seen in figure 5(b). This has ramifications for language change, as we expect to see similar effects in neutral
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Figure 6. (a) Mean consensus time versus population size for different values of $S$, using $r_{\text{lin}}(s)$. Symbols show numerical results; lines show predictions of equation (20) which grow linearly with $N$. (b) The difference between numerical and analytic values for $T/N$ as a function of $N$ at several values of $S$. For small $S$, the difference is large for small $N$ but converges rapidly to 0. For larger $S$ the difference also falls rapidly, but converges to a small yet nonzero value.

models of language change, such as that described in [2]. Inclusion of age related variation in flip rates delays consensus in two ways. The mean time for reaching consensus, which corresponds to a language variant becoming established as the convention in a population, is always longer for heterogeneous flip rates than for a perfectly homogeneous population, so any changes of learning with age will delay the establishment of a convention. The effect is exacerbated by the fact that new variants tend to originate in the youngest members of the population [26], corresponding to opinion 0 in the present model, meaning that the time taken for a new variant to overtake the population is even longer. For example, in [3] the mean time for reaching consensus among British and Irish immigrants to New Zealand in a feasible neutral model was found to be much longer than the observed time. As just described, generational effects only make this situation worse, suggesting that some kind of selection effect (preference for one variant over another) must have been at work in this situation.

To establish the effect of system size, we repeated numerical simulations for a range of values of $N$, from 50 up to 1000. In figure 6(a) we plot $T$ as a function of $N$ for several values of $S$ across a broad range. We see that the mean consensus times grow linearly with $N$, confirming that the calculated scaling $T \propto N$ is correct. Numerical results are in excellent agreement with analytic predictions for larger $N$, given by equation (20). In figure 6(b) we plot the absolute difference between the numerical and analytic results as a function of $N$, for several $S$ values. We see that the results for small $N$ do not agree at all well with analytic predictions, which are based on a large $N$ approximation. For larger $N$, however, the numerical results quickly converge to the analytic curve, coinciding for $N \geq 400$. For small $S$, the difference is consistent with zero for $N$ from approximately $N = 400$. For larger $S$ values, the numerical results never exactly converge to the prediction, but the difference achieves its minimum value from $N$ around 400 and remains the same as $N$ increases (aside from statistical fluctuations). All of the results presented above are for $N = 500$. Above this value we do not expect to see any improvement in the results.

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Finally, we extended the simulations to a population of voters on a network. We carried out simulations of the model for Erdős–Rényi networks and networks whose degree distributions follow a decaying power law of the form \( P(q) \propto q^{-\gamma} \) for large degree \( q \).

The qualitative behaviour is exactly the same, with mean consensus time rising with a logistic-shaped curve from a minimum at small \( S \) to a maximum at large \( S \). Equation (26) predicts that consensus times should depend on the degree distribution of an uncorrelated network through a factor \( \langle q^2 \rangle / \langle q^2 \rangle \). In general, this factor does not depend on network size, meaning that consensus times will grow linearly with \( N \). If \( \langle q^2 \rangle \) diverges with \( N \), a different scaling of mean consensus time with \( N \) will emerge. For networks with power-law degree distributions with \( \gamma < 3 \), \( \langle q^2 \rangle \propto N^{(3-\gamma)/(\gamma-1)} \) meaning that we expect to find \( T \propto N^{(2\gamma-4)/(\gamma-1)} \). In figure 2 we plot mean consensus time for various \( N \) for an Erdős–Rényi network with mean degree 7, and power-law degree distributed networks with \( \gamma = 3.5 \) and \( \gamma = 2.5 \). As expected, \( T \) grows linearly with \( N \) for the Erdős–Rényi network and power-law network with \( \gamma = 3.5 \). For \( \gamma = 2.5 \) the mean consensus time grows sublinearly with \( N \), with an exponent close to the expected value of 2/3.

4. Discussion

In this paper we have introduced a generalization of the voter model in which the flip rates of agents vary with time. Interaction between the timescale of consensus formation and the timescale of flip rate change leads to nontrivial dependence of mean consensus time on the period of flip rate change. For very rapidly changing flip rates, the mean time to consensus agrees with that found for the voter model with fixed homogeneous flip rates. As the period of change of the flip rates lengthens, so does the consensus time, until it saturates at the time found for static heterogeneous flip rates. An analytic estimate of the mean consensus time can be found by calculating the expected interval between interactions for each agent, then using the usual method of assuming a quickly reached quasi-stationary state followed by a slow escape to consensus. The mean consensus time is calculated for this second stage through a Fokker–Planck equation for a single conserved centre-of-mass variable. The results obtained by this method are in excellent agreement with numerical simulation. The overall mean time to consensus, the mean time for reaching a particular final state and the rescaling of the distribution of consensus times are all correctly predicted. We also found that complex population structures such as networks with power-law degree distributions affect the scaling of the mean consensus time with population size in the same way as they do for static flip rates.

For simplicity here we used periodic flip rates, but the method used is readily applicable to more complex variations in flip rates—for example, if each agent’s flip rate varies with a different period, or indeed if each followed a different function entirely. More generally, time dependent interactions in any copying process may be modelled and analysed in a similar way, to consider for example seasonal variation in invasion rates in ecological models, or time variation in the strength of synaptic interactions in neuronal models. This is particularly relevant to language change. Speakers learn more quickly when they are young and more slowly or almost not at all once they reach adulthood. The effect of such ageing can be modelled by exactly the kind of time varying interactions

\footnote{We did not restrict multiple edges in this simulation.}
described here. The results presented here for this very simple model suggest that in a more realistic language change model, heterogeneity in learning rates will increase the mean time taken to reach consensus, and this will be further exacerbated as a new language variant is more likely to appear in the faster learning (i.e. the younger) members of the population. The inertia of the (older) slowly adapting speakers will contribute both to enhanced survival probability for the existing convention, and in the event that a new variant does take over, to increasing the time required for this to happen. Extension of time variation of update rates to such more realistic models is therefore a natural avenue for future investigation. Another possible extension would be to consider flip rates that depend on local dynamical processes, which is relevant for example in the case of neuronal models in which a neuron’s response depends on recent activity.

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