Optimization on fractal sets

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Abstract
In the sequel, we outline necessary and sufficient condition to the existence of extrema of a function on a self-similar set, and we describe discrete gradient algorithm to find the extrema.

Keywords Extrema · Fractal · Laplacian · Discrete gradient · Dynamic programming

Introduction
In 1636, in a written correspondence with Martin Mersenne, Pierre de Fermat established a necessary condition for the existence of the minimum and the maximum of a function [7]:

“When a quantity, for example the ordinate of a curve, reached its maximum or its minimum, in a situation infinitely close, his increase or decrease is null.”

Since then, many results have been set, ranging from free and constraint conditions, giving birth to numerical algorithms that enable one to find the extremas of a function. Until now, optimization has mainly concerned regular domains, without really suitable results for fractal sets. As said by G. Hardy [10] on a close subject, we might say that it was “in consequence of the methods employed”.

The birth of analysis on fractals, especially, the work of J. Kigami [11–15], bridged this gap, by introducing a new kind of differential operators, whose intrinsic properties are analogous to differential ones. For instance, the Laplacian of a function \( u \) defined on a specific fractal set \( \mathcal{F} \), at a given point \( X \in \mathcal{F} \), is “equal to the limit, in a...
suitable renormalized sense, of the difference between an average value of the function in a neighborhood of \( X \) and \( u(X) \)” [18]. If related numerical methods have been developed, for instance, by R. Strichartz [16,18,19], the field of analysis on fractals remains not completely explored, in particular as regards optimization.

In the sequel, in the spirit of Fermat paper, we try to extend results of smooth analysis on extremas in the case of fractals. The novelty of our work lays in the use of the aforementioned differential operators, specifically designed for fractal sets. To begin with, we examine existence conditions, then, we present a numerical algorithm that enable us to find local extrema of a continuous function defined on a fractal set.

1 Framework of the study

In the following, we place ourselves in the euclidian space of dimension \( d \in \{1, 2, 3\} \).

Notation We classically denote by \( \mathbb{N}^* \) the set of strictly positive integers.

1.1 Self-similar sets

Notation In the sequel, \( \mathcal{F} \) denotes a fractal domain of Hausdorff dimension \( D_H (\mathcal{F}) \); \( N \) is a strictly positive integer, and \( \{ f_1, \ldots, f_N \} \) is a set of contractive maps, where, for any integer \( i \) of \( \{1, \ldots, N\} \), \( R_i \in ]0, 1[ \) is the contraction ratio of \( f_i \), and \( P_i \in \mathbb{R}^d \) the fixed point of \( f_i \).

Theorem 1.1 (Gluing Lemma [3])

Given a complete metric space \((E, \delta)\), a strictly positive integer \( N \), and a set \( \{ f_i \}_{1 \leq i \leq N} \) of contractions on \( E \) with respect to the metric \( \delta \), there exists a unique non-empty compact subset \( K \subset E \) such that:

\[
K = \bigcup_{i=1}^{N} f_i (K)
\]

The set \( K \) is said to be self-similar with respect to the family \( \{ f_1, \ldots, f_N \} \), and called attractor of the iterated function system (IFS) \( \{ f_1, \ldots, f_N \} \).

Definition 1.1 (Boundary (or initial) graph) We denote by \( V_0 \) the ordered set of the (boundary) points:

\[
\{ P_1, \ldots, P_{N_0} \}
\]

The set of points \( V_0 \), where, for any \( i \) of \( \{1, \ldots, N_0\} \), the point \( P_i \) is linked to the other points \( P_j \), constitutes an complete graph, that we denote by \( \mathcal{F}_0 \).

\( V_0 \) is called the set of vertices of the graph \( \mathcal{F}_0 \).
Definition 1.2 \( m \)th order graph, \( m \in \mathbb{N}^* \)

For any strictly positive integer \( m \), we set:

\[
V_m = \bigcup_{i=1}^{N} f_i (V_{m-1}).
\]

The set of points \( V_m \), where the points of an \( m \)th-order cell are linked in the same way as \( F_0 \), is an oriented graph, which we denote by \( F_m \).

\( V_m \) is called the set of vertices of the graph \( F_m \).

By extension, we write:

\[
F_m = \bigcup_{i=1}^{N} f_i (F_{m-1}).
\]

Property 1.2 For any natural integer \( m \):

\[
V_m \subset V_{m+1}.
\]

Property 1.3 The set \( \bigcup_{m \in \mathbb{N}} V_m \) is dense in \( F \).

Definition 1.3 (Word) Given a strictly positive integer \( m \), we call number-letter any integer \( W_i \) of \( \{1, \ldots, N\} \), and word of length \( |W| = m \), on the graph \( F_m \), any set of number-letters of the form:

\[
W = (W_1, \ldots, W_m).
\]

We write:

\[
f_W = f_{W_1} \circ \cdots \circ f_{W_m}.
\]

Definition 1.4 (Vertices) Two points \( X \) and \( Y \) of \( F \) are called vertices of the graph \( F \) if there exists a natural integer \( m \) such that:

\[
(X, Y) \in V_m^2.
\]

Definition 1.5 (Edge relation) Given a natural integer \( m \), two points \( X \) and \( Y \) of \( F_m \) are called adjacent if and only if \( X \) and \( Y \) are neighbors in \( F_m \). We write:

\[
X \sim_m Y
\]

This edge relation ensures the existence of a word \( W = (W_1, \ldots, W_m) \) of length \( m \), such that \( X \) and \( Y \) both belong to the iterate:

\[
f_W V_0 = (f_{W_1} \circ \cdots \circ f_{W_m}) V_0
\]
Given two points $X$ and $Y$ of $\mathcal{F}$, we say that $X$ and $Y$ are adjacent if and only if there exists a natural integer $m$ such that:

$$X \sim_Y m$$

**Definition 1.6** (*Addresses*) Given a natural integer $m$, and a vertex $X$ of $\mathcal{F}_m$, we call address of the vertex $X$ an expression of the form

$$X = f_{\mathcal{W}}(P_i)$$

where $\mathcal{W}$ is a word of length $m$, and $i$ a natural integer in $\{1, \ldots, N\}$.

**Property 1.4** Subcell–Junction points *Given a natural integer $m$, the graph $\mathcal{F}_m$ can be written as the finite union of $N^m$ subgraphs:*

$$\mathcal{F}_m = \bigcup_{|\mathcal{W}|=m} f_{\mathcal{W}}(\mathcal{F}_0)$$

*For any word $\mathcal{W}$ of length $m$, $f_{\mathcal{W}}(\mathcal{F}_0)$ is called $m$th-order cell, or subcell.*

**Notation**  

$i$. Given a strictly positive integer $m$, we denote by $\Sigma_m$ the set of words $\mathcal{W} \in \{1, \ldots, N\}^m$ of length $m$.

$ii$. We then set:

$$\Sigma = \bigcup_{m \in \mathbb{N}^*} \Sigma_m$$

**Notation** For the sake of clarity, we refer, from now on, to a self-similar set either by $\mathcal{F}$ or by:

$$(\mathcal{F}, S, (f_i)_{i \in S})$$

with $S = \{1, \ldots, N\}$.

**Notation** We denote by:

$i$. $\sigma$, the shift map from $\Sigma$ to $\Sigma$ which, for any word $\mathcal{W}$, deletes the first “letter” i.e.:

$$\sigma(12233\ldots) = 2233\ldots$$

$ii$. $\pi$, the surjective map from $\Sigma$ to $\mathcal{F}$ defined for every infinite “word” $\mathcal{W} = \mathcal{W}_1\mathcal{W}_2\ldots \in \Sigma$, by:

$$\pi(W) = \bigcap_{m \in \mathbb{N}^*} f_{\mathcal{W}_1\ldots\mathcal{W}_m}(\mathcal{F})$$

where $f_{\mathcal{W}_1\ldots\mathcal{W}_m} = f_{\mathcal{W}_1} \circ \cdots \circ f_{\mathcal{W}_m}$. 

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\[ C_{L,F} = \bigcup_{(i,j) \in \Sigma^2, i \neq j} (F_i \cap F_j) \]

**Definition 1.7** We define:

i. The critical set:

\[ C_L = \pi^{-1}(C_{L,F}) \]

ii. The post-critical set:

\[ P = \bigcup_{m \in \mathbb{N}^*} \sigma^m(C_L) \]

**Property 1.5**

\[ V_0 = \pi \left( \bigcup_{m \in \mathbb{N}^*} \sigma^m(C_L) \right) \]

**Notation** Given a natural integer \( m \), we denote by \( N_m \) the number of vertices of the graph \( F_m \), \( A_m \) the edge set of \( F_m \), and by \( C \) the cardinal of the set \( C_{L,F} \).

**Proposition 1.6** One has:

\[ N_0 = N_0 \]

and, for any strictly positive integer \( m \):

\[ N_m = N \times N_{m-1} - C \quad , \quad A_m = N \times A_{m-1} \]

where \( C = \#C_{L,F} \).

**Proof** The graph \( F_m \) is the union of \( N \) copies of the graph \( F_m \). Each copy has \( A_{m-1} \) edges, and shares vertices with others copies. One can thus consider the copies as the vertices of a complete graph which has a number of edges equal to \( C \), so there are \( C \) vertices to discount.

\( \Box \)

**Remark 1.1** One may check that:

\[ A_m = N^m A_0 = N^m (N_0(N_0 - 1)) \quad , \quad N_m = N^m N_0 - \left( \frac{1-N^m}{1-N} \right) C = O(N^m). \]
**Definition 1.8** *(Neighborhood system [14])* Let \((\mathcal{F}, S, (f_i)_{i \in S})\) be a self-similar structure. For any \(X \in \mathcal{F}\), and any natural integer \(m\), we set:

\[
\mathcal{F}_{m,X} = \bigcup_{W \in \Sigma_m, X \in f_W(\mathcal{F})} f_W(\mathcal{F})
\]

which is called system of neighborhood of \(X\).

**Definition 1.9** *(Self-similar measure on \(\mathcal{F}\) [19])* A measure \(\mu\) with full support on \(\mathbb{R}^d\) is called **self-similar measure** on \(\mathcal{F} = \bigcup_{i=1}^{N} f_i(\mathcal{F})\) if, given a family of strictly positive weights \((\mu_i)_{1 \leq i \leq N}\) such that:

\[
\sum_{i=1}^{N} \mu_i = 1
\]

one has:

\[
\mu = \sum_{i=1}^{N} \mu_i \circ f_i^{-1}
\]

**Property 1.7** *(Building of a self-similar measure on \(\mathcal{F}\)*) We set, for any integer \(i\) belonging to \(\{1, \ldots, N\}\):

\[
\mu_i = R^D_{i}(\mathcal{F})
\]

One has then:

\[
\sum_{i=1}^{N} R^D_{i}(\mathcal{F}) = 1.
\]

which enables us to define a self-similar measure \(\mu\) on \(\mathcal{F}\) through:

\[
\mu = \sum_{i=1}^{N} \mu_i \circ f_i^{-1}
\]

One may note that the measure \(\mu\) corresponds to the normalized \(D_H(\mathcal{F})\)-dimensional Hausdorff measure \(\mathcal{H}^{D_H(\mathcal{F})}\) (we refer to [8]):

\[
\mu(E) = \frac{\mathcal{H}^{D_H(\mathcal{F})}(E \cap \mathcal{F})}{\mathcal{H}^{D_H(\mathcal{F})}(\mathcal{F})}
\]

for any subset \(E \subset \mathbb{R}^d\).
1.2 Laplacians, on self-similar sets

Definition 1.10 (Energy, on the graph $\mathcal{F}_m$, $m \in \mathbb{N}$, of a pair of functions) Given a natural integer $m$, and two real valued functions $u$ and $v$, defined on the set $V_m$ of the vertices of $\mathcal{F}_m$, we introduce the energy, on the graph $\mathcal{F}_m$, of the pair of functions $(u, v)$, as:

$$E_{\mathcal{F}_m}(u, v) = \sum_{(X, Y) \in V_m^2, X \sim Y} (u(X) - u(Y)) (v(X) - v(Y))$$

For the sake of simplicity, we write:

$$E_{\mathcal{F}_m}(u, v) = \sum_{X \sim Y} (u(X) - u(Y)) (v(X) - v(Y))$$

Definition 1.11 (Dirichlet form on a measured space) (we refer to the paper [4], or the book [9]) Given a measured space $(E, \mu)$, a Dirichlet form on $E$ is a bilinear symmetric form, that we denote by $\mathcal{E}$, defined on a vectorial subspace $D$ dense in $L^2(E, \mu)$, such that:

1. For any real-valued function $u$ defined on $D$: $\mathcal{E}(u, u) \geq 0$.
2. $D$, equipped with the inner product which, to any pair $(u, v)$ of $D \times D$, associates:

$$(u, v)_{\mathcal{E}} = (u, v)_{L^2(E, \mu)} + \mathcal{E}(u, v)$$

is a Hilbert space.
3. For any real-valued function $u$ defined on $D$, if:

$$u_* = \min \{ \max(u, 0), 1 \} \in D$$

then: $\mathcal{E}(u_*, u_*) \leq \mathcal{E}(u, u)$ (Markov property, or lack of memory property).

Definition 1.12 (Dirichlet form, on a finite set (see [15])) Let $V$ denote a finite set $V$, equipped with the usual inner product which, to any pair $(u, v)$ of functions defined on $V$, associates:

$$(u, v) = \sum_{p \in V} u(p) v(p)$$

A Dirichlet form on $V$ is a symmetric bilinear form $\mathcal{E}$, such that:

1. For any real valued function $u$ defined on $V$: $\mathcal{E}(u, u) \geq 0$.
2. $\mathcal{E}(u, u) = 0$ if and only if $u$ is constant on $V$.
3. For any real-valued function $u$ defined on $V$, if: $u_* = \min \{ \max(u, 0), 1 \}$, i.e.:

$$\forall p \in V : u_*(p) = \begin{cases} 1 & \text{if } u(p) \geq 1 \\ u(p) & \text{if } 0 < u(p) < 1 \\ 0 & \text{if } u(p) \leq 0 \end{cases}$$
then: \( \mathcal{E}(u_*, u_*) \leq \mathcal{E}(u, u) \) (Markov property).

**Property 1.8** Given a natural integer \( m \), the map, which, to any pair of real-valued functions \((u, v)\) defined on \( V_m \), associates:

\[
\mathcal{E}_{\mathcal{F}_m}(u, v) = \sum_{X \sim Y_m} (u(X) - u(Y)) (v(X) - v(Y))
\]

is a Dirichlet form on \( \mathcal{F}_m \). Moreover:

\[
\mathcal{E}_{\mathcal{F}_m}(u, u) = 0 \iff u \text{ is constant}
\]

**Proposition 1.9** For any strictly positive integer \( m \), if \( u \) is a real-valued function defined on \( V_{m-1} \), its harmonic extension, denoted by \( \tilde{u} \), is obtained as the extension of \( u \) to \( V_m \) which minimizes the energy:

\[
\mathcal{E}_{\mathcal{F}_m}(\tilde{u}, \tilde{u}) = \sum_{X \sim Y_m} (\tilde{u}(X) - \tilde{u}(Y))^2
\]

**Remark 1.2** Concretely: The link between \( \mathcal{E}_{\mathcal{F}_m} \) and \( \mathcal{E}_{\mathcal{F}_{m-1}} \) is obtained through the introduction of two strictly positive constants \( r_m \) and \( r_{m-1} \) such that:

\[
rm \sum_{X \sim Y_m} (\tilde{u}(X) - \tilde{u}(Y))^2 = rm-1 \sum_{X \sim Y_m} (u(X) - u(Y))^2
\]

In particular:

\[
r_1 \sum_{X \sim Y_1} (\tilde{u}(X) - \tilde{u}(Y))^2 = r_0 \sum_{X \sim Y_0} (u(X) - u(Y))^2
\]

For the sake of simplicity, one fixes the value of the initial constant: \( r_0 = 1 \). Then:

\[
\mathcal{E}_{\mathcal{F}_1}(\tilde{u}, \tilde{u}) = \frac{1}{r_1} \mathcal{E}_{\mathcal{F}_0}(\tilde{u}, \tilde{u})
\]

We set:

\[
r = \frac{1}{r_1}
\]

and:

\[
\mathcal{E}_m(u) = rm \sum_{X \sim Y_m} (\tilde{u}(X) - \tilde{u}(Y))^2
\]
Since the determination of the harmonic extension of a function appears to be a local problem, on the graph $F_{m-1}$, which is linked to the graph $F_m$ by a similar process as the one that links $F_1$ to $F_0$, one deduces, for any strictly positive integer $m$:

$$\mathcal{E}_{F_m}(\tilde{u}, \tilde{u}) = \frac{1}{r_1} \mathcal{E}_{F_{m-1}}(\tilde{u}, \tilde{u})$$

By induction, one gets:

$$r_m = r_1^m = r^{-m}$$

If $v$ is a real-valued function, defined on $V_{m-1}$, of harmonic extension $\tilde{v}$, we write:

$$\mathcal{E}_m(u, v) = r^{-m} \sum_{X \sim Y} (\tilde{u}(X) - \tilde{u}(Y)) (\tilde{v}(X) - \tilde{v}(Y))$$

For further precision on the construction and existence of harmonic extensions, we refer to [17].

**Definition 1.13** (Renormalized energy, for a continuous function $u$, defined on $F_m$, $m \in \mathbb{N}$) Given a natural integer $m$, one defines the normalized energy, for a continuous function $u$, defined on $F_m$, by:

$$\mathcal{E}_m(u) = \sum_{X \sim Y} r^{-m} (u(X) - u(Y))^2$$

**Definition 1.14** (Normalized energy, for a continuous function $u$, defined on $F$) Given a function $u$ defined on $V_\ast = \bigcup_{i \in \mathbb{N}} V_i$ one defines the normalized energy:

$$\mathcal{E}(u) = \lim_{m \to +\infty} \sum_{X \sim Y} r^{-m} (u(X) - u(Y))^2$$

**Definition 1.15** (Dirichlet form, for a pair of continuous functions defined on $F$) We define the Dirichlet form $\mathcal{E}$ which, to any pair of real-valued, continuous functions $(u, v)$ defined on the graph $F$, associates, subject to its existence:

$$\mathcal{E}(u, v) = \lim_{m \to +\infty} \sum_{X \sim Y} r^{-m} (u|_{V_m}(X) - u|_{V_m}(Y)) (v|_{V_m}(X) - v|_{V_m}(Y))$$
**Notation** We denote by:

i. \( \text{dom } \mathcal{E} \) the subspace of continuous functions defined on \( \mathcal{F} \), such that:

\[
\mathcal{E}(u) < +\infty
\]

ii. \( \text{dom}_0 \mathcal{E} \) the subspace of continuous functions defined on \( \mathcal{F} \), which take the value zero on \( V_0 \), and such that:

\[
\mathcal{E}(u) < +\infty
\]

**Lemma 1.10** The map:

\[
\text{dom } \mathcal{E} / \text{Constants} \times \text{dom } \mathcal{E} / \text{Constants} \rightarrow \mathbb{R},
\]

\[
(u, v) \mapsto \mathcal{E}(u, v)
\]

defines an inner product on \( \text{dom } \mathcal{E} / \text{Constants} \).

**Theorem 1.11** \((\text{dom } \mathcal{E} / \text{Constants}, \mathcal{E}(\cdot, \cdot))\) is a complete Hilbert space.

**Definition 1.16** (Harmonic function) A real-valued function \( u \), defined on \( V_* = \bigcup_{i \in \mathbb{N}} V_i \), is said to be **harmonic** if, for any natural integer \( m \), its restriction \( u|_{V_m} \) is harmonic:

\[
\forall m \in \mathbb{N}, \forall X \in V_m \setminus V_0 : \Delta_m u|_{V_m}(X) = 0
\]

**Notation** We denote by \( \text{dom } \Delta \) the existence domain of the Laplacian, on \( \mathcal{F} \), as the set of functions \( u \) of \( \text{dom } \mathcal{E} \) such that there exists a continuous function on \( \mathcal{F} \), denoted by \( \Delta u \), that we call **Laplacian of** \( u \), such that, for any \( v \in \text{dom } \mathcal{E}, v|_{\mathcal{F}_0} = 0 \):

\[
\mathcal{E}(u) = \lim_{m \to +\infty} \sum_{X \sim Y} r_m^{-m} (u|_{V_m}(X) - u|_{V_m}(Y))^2 < +\infty
\]

**Definition 1.17** We denote by \( \text{dom } \Delta \) the existence domain of the Laplacian, on \( \mathcal{F} \), as the set of functions \( u \) of \( \text{dom } \mathcal{E} \) such that there exists a continuous function on \( \mathcal{F} \), denoted by \( \Delta u \), that we call **Laplacian of** \( u \), such that, for any \( v \in \text{dom } \mathcal{E}, v|_{\mathcal{F}_0} = 0 \):

\[
\mathcal{E}(u, v) = \lim_{m \to +\infty} \sum_{X \sim Y} r_m^{-m} (u|_{V_m}(X) - u|_{V_m}(Y)) (v|_{V_m}(X) - v|_{V_m}(Y)) = -\int_{\mathcal{F}} v \Delta u \, d\mu
\]

**Theorem 1.12**

\((u \in \text{dom } \Delta \text{ and } \Delta u = 0) \text{ if and only if } u \text{ is harmonic}\)
Notation

i. Given a natural integer $m$, $S(\mathcal{H}_0, V_m)$ denotes the space of spline functions “of level $m$, $u$, defined on $\mathcal{F}$, continuous, such that, for any word $W$ of length $m$, $u \circ T_W$ is harmonic, i.e.:

$$\Delta_m (u \circ T_W) = 0$$

ii. $\mathcal{H}_0 \subset \text{dom } \Delta$ denotes the space of harmonic functions, i.e. the space of functions $u \in \text{dom } \Delta$ such that:

$$\Delta u = 0$$

Property 1.13 For any natural integer $m$: $S(\mathcal{H}_0, V_m) \subset \text{dom } E$.

Theorem 1.14 (Pointwise formula) Let $m$ be a strictly positive integer, $X \in V_* \setminus V_0$, and $\psi^m_X \in S(\mathcal{H}_0, V_m)$ a spline function such that:

$$\psi^m_X(Y) = \begin{cases} 
\delta_{XY} & \forall Y \in V_m \\
0 & \forall Y \notin V_m
\end{cases}, \text{ where } \delta_{XY} = \begin{cases} 
1 & \text{if } X = Y \\
0 & \text{else}
\end{cases}$$

i. For any function $u$ of $\text{dom } \Delta$, such that its Laplacian exists, the sequence

$$\left( r^{-m} \left\{ \int_\mathcal{F} \psi^m_X \, d\mu \right\}^{-1} \Delta_m u(X) \right)_{m \in \mathbb{N}}$$

converges uniformly towards

$$\Delta u(X)$$

ii. Conversely, given a continuous function $u$ on $\mathcal{F}$ such that the sequence

$$\left( r^{-m} \left\{ \int_\mathcal{F} \psi^m_X \, d\mu \right\}^{-1} \Delta_m u(X) \right)_{m \in \mathbb{N}}$$

converges uniformly towards a continuous function on $V_* \setminus V_0$, one has:

$$u \in \text{dom } \Delta \text{ and } \Delta u(X) = \lim_{m \to +\infty} r^{-m} \left\{ \int_\mathcal{F} \psi^m_X \, d\mu \right\}^{-1} \Delta_m u(X)$$

1.3 Existence of extrema

Definition 1.18 (Extrema) Given a continuous function $u$ defined on the fractal set $\mathcal{F}$, and $X \in \mathcal{F}$, we say that $u$:

i. has a global minimum (resp. a global maximum) at $X$ if:

$$\forall Y \in \mathcal{F} : \ u(X) \leq u(Y) \quad (\text{resp. } u(X) \geq u(Y))$$
ii. a local minimum (resp. a local maximum) at $X$ if there exists a neighborhood $V$ of $X$ such that

$$\forall Y \in V : \ u(X) \leq u(Y) \quad \text{(resp. } u(X) \geq u(Y)) .$$

**Theorem 1.15** Given a continuous function $u$ defined on the compact fractal set

$$\mathcal{F} = \bigcup_{i=1}^{N} f_i (\mathcal{F})$$

the Weierstrass extreme value theorem ensures the existence of:

$$\min_{X \in \mathcal{F}} u \quad \text{and} \quad \max_{X \in \mathcal{F}} u$$

**Theorem 1.16** (Laplacian test for fractals [14]) Given a continuous, real-valued function $u$ defined on $\mathcal{F}$, and belonging to $\text{dom}\Delta$:

i. If $u$ admits a local maximum at $X_0 \in \mathcal{F}$, then:

$$\Delta u(X_0) \leq 0$$

ii. If $u$ admits a local minimum at $X_0 \in \mathcal{F}$, then:

$$\Delta u(X_0) \geq 0$$

**Proof** i. If $u$ admits a local maximum at $X_0$, then for sufficiently large values of the integer $m$:

$$\Delta_m u(X_0) = \sum_{X_0 \sim m Y} (u(Y) - u(X_0)) \leq 0 .$$

This enables us to conclude that:

$$\Delta_\mu u(X) = \lim_{m \to \infty} r^{-m} \left( \int_{K} \psi_{X_m}^{(m)} d\mu \right)^{-1} \Delta_m u(X_m) \leq 0$$

ii. can be proved in a similar way.

## 2 Numerical algorithm and dynamic programming

In the following, we present a numerical algorithm, based on discrete gradient, to find a local maximizer (resp. local minimizer) of a continuous function $u$ on $K$. 
2.1 The algorithm

We recall that $F_m = (V_m, A_m)$ is the oriented graph approximation of $F$ of order $m$, where $V_m$ is the vertices set and $A_m$ is the edge set. We can check that the distance between two connected vertices is of order $2^{-m}$.

In order to find the local maximizer, we provide every edge $\{XY\}$ with the weight $D_{XY} = u(Y) - u(X)$. In order to find an appropriate approximation of the maximizer $X^*$, we fix a degree of tolerance $\varepsilon > 0$, and the graph $F_m$ of order $m$ such that:

$$2^{-m} \leq \varepsilon .$$

Starting at an arbitrary point $X_0$ in $V_m$, we follow the direction of the maximal positive gradient at $X_0$, i.e.

$$\max_{X_0 \sim m Y} \{D_{X_0 Y} \mid D_{X_0 Y} > 0\} .$$

We replace the initial point by $\arg \max_{X_0 \sim m Y} \{D_{X_0 Y} \mid D_{X_0 Y} > 0\}$, and we do the same operation until

$$\max_{X_0 \sim m Y} \{D_{X_0 Y} \mid D_{X_0 Y} \geq 0\} = 0 .$$

In this case, the algorithm stop and we have the approximation of $X^*$. We can can summarize the algorithm in the following steps:

**Discrete gradient algorithm**

1. Fix a degree of tolerance $\varepsilon > 0$.
2. Build the graph $F_m$, for $m$ such that:

$$2^{-m} \leq \varepsilon .$$

3. Fix $X = X_0$ for $X_0 \in V_m$.
4. While $\max_{X \sim Y m} \{D_{XY} \mid D_{XY} \geq 0\} > 0$:

   Update $X = \arg \max_{X \sim Y m} \{D_{XY} \mid D_{XY} > 0\}$.

5. Return $X$.

2.2 Numerical analysis and dynamic programming

The algorithm presented above can be viewed as a dynamical programming algorithm on the directed graph $F_m$, $m \in \mathbb{N}^*$ (we refer to [5]). Values of the function can be
calculated recursively:

\[ v^0_X = 0, \quad \forall X \in V_m \quad v^m_X = B \left( v^{m-1}_X \right) = \sup_{Y \in V_m} \left( D_{XY} + v^{n-1}_X \right) \]

where \( B \) denotes the Bellman operator:

\[ B : \mathbb{R}^{V_m} \rightarrow \mathbb{R}^{V_m} \]

\[ (B(v))_X = \sup_{Y \in V_m} (D_{XY} + v_Y) \]

where \( D_{XY} \) is the weight associated to the edge \( XY \in A_m \) and \( D_{XY} = -\infty \) if \( XY \notin A_m \).

As shown in the first section, \( \#A_m = \mathcal{O}(N^m) \) and \( \#V_m = \mathcal{O}(N^m) \). The number of possible transitions is at most of order \( \#A_m = \mathcal{O}(N^m) \). Since \( \#V_m = \mathcal{O}(N^m) \)

one can deduce that the calculation time of the maximum is of order \( \mathcal{O}(N^{2m}) \).

### 2.2.1 Sierpiński simplices

Sierpiński simplices are sparse graph. Thus, the calculation time required for the gradient algorithm can be optimized: each vertex has a finite number of neighbors, which ensures a calculation time at step \( m \in \mathbb{N}^* \) which is of order \( \mathcal{O}(N^m) \).

Moreover, computations can be simplified using the fact that every vertex \( X \) has two addresses:

\[ X = f_{\mathcal{W}_1}(P_i) = f_{\mathcal{W}_2}(P_j) \]

where \((\mathcal{W}_1, \mathcal{W}_2) \in \Sigma^2 \) and \((P_i, P_j) \in V^2_0\), with \( i \neq j \). Thus, the neighbors of \( X \) are given by:

\[
\left\{ \bigcup_{k \neq i} f_{\mathcal{W}_2}(P_k) \right\} \cup \left\{ \bigcup_{\ell \neq j} f_{\mathcal{W}_2}(P_\ell) \right\}
\]

### The Sierpiński Gasket

In the case of Sierpiński Gasket, one can optimize the calculation time of the gradient algorithm. Since, for any natural integer \( m \):

\[
\#V_m = \frac{3^{m+1} + 3}{2} \quad \text{and} \quad \#A_m = 2 \times 3^{m+1}
\]

and given the fact that every vertex \( X \in V_m \setminus V_0 \) has four neighbors, it follows that the calculation time of the maximum at step \( m \in \mathbb{N}^* \) is thus of order \( \mathcal{O}(3^m) \).
In the sequel (see Figs. 1, 2, 3 and 4), we present results of our algorithm in the case of Sierpiński Gasket with vertices:

$$P_0 = (0, 0), \quad P_1 = (1, 0), \quad P_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

for the value $m = 6$.

The color function is related to the gradient of the one at stake, high values ranging from red to blue.

For this first example, the algorithm starts with $X_0 = \left(\frac{5}{8}, \frac{\sqrt{3}}{8}\right)$, following the largest gradient (red points), the algorithm converges to the local maximum 1 at $(1, 0)$. 
Fig. 3  isovalues of the function

\[ X \mapsto g(X) = -\|X - \left( \frac{1}{2}, \frac{\sqrt{3}}{4} \right) \|^2 \]

Fig. 4  The algorithm path from

\[ X_0 = \left( \frac{3}{4}, \frac{\sqrt{3}}{4} \right) \] to \( X^* \)

In the second example, the algorithm starts with \( X_0 = \left( \frac{3}{4}, \frac{\sqrt{3}}{4} \right) \), and converges to the global maximum 0 at \( \left( \frac{1}{2}, \frac{\sqrt{3}}{4} \right) \).

**The Sierpiński Tetrahedron**

The Sierpiński Tetrahedron requires a calculation time for the maximum at step \( m \in \mathbb{N}^* \) which is of order \( O(4^m) \).

In the sequel (see Figs. 5 and 6), we present results of our algorithm in the case of Sierpiński Tetrahedron, with vertices:

- \( P_0 = (0, 0, 0) \)
- \( P_1 = (1, 0, 0) \)
- \( P_2 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right) \)
- \( P_3 = \left( \frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}} \right) \)
Optimization on fractal sets

Fig. 5 Isovalues of the function $X \mapsto f(X) = \| X \|^2$

Fig. 6 The algorithm path from $X_0 = \left( \frac{41}{64}, \frac{1}{64} + \frac{\sqrt{3}}{8}, \frac{1}{64} \right)$ to $X^*$ for the value $m = 6$.

The color function is related to the gradient of the one at stake, high values ranging from red to blue.

For this first example, the algorithm start with $X_0 = \left( \frac{41}{64}, \frac{1}{64} + \frac{\sqrt{3}}{8}, \frac{1}{64} \right)$, following the largest gradient (red points), the algorithm converges to the local maximum 1 at $(1, 0, 0)$.

2.2.2 Self-similar curves

Self-similar curves require a calculation time which is of order $O(N^m)$, due to the fact that every vertex has only two neighbors. In such cases: $V_0 = \{ P_0, P_1 \}$. Thus, every
Fig. 7 Isovalues of the function $X \mapsto f(X) = \|X\|^2$

Fig. 8 The algorithm path from $X_0 = (0, 0)$ to $X^*$

vertex $X$ has exactly two addresses:

$$X = f_{\mathcal{W}_i}(P_0) = f_{\mathcal{W}_{i+1}}(P_1)$$

where $(\mathcal{W}_i, \mathcal{W}_j) \in \Sigma^2$. The neighbors of $X$ are thus given by:

$$f_{\mathcal{W}_{i-1}}(P_0) \quad \text{and} \quad f_{\mathcal{W}_{j+1}}(P_1)$$

where $\mathcal{W}_{i-1}$ (resp. $\mathcal{W}_{j+1}$) is the next (resp. the past) address of $\mathcal{W}_i$ (resp. $\mathcal{W}_j$) in the lexicographical order.

In the sequel (see Figs. 7 and 8), we present results of our algorithm in the case of the Minkowski curve, with $V_0 = \{(0, 0); (1, 0)\}$, for the value $m = 3$.

The color function is related to the gradient of the one at stake, high values ranging from red to blue.
For this example, the algorithm starts with \( X_0 = (0, 0) \); following the largest gradient (red points), the algorithm converges to the local maximum \( \left( \frac{5}{4096} \right) \) at \( \left( \frac{1}{32}, \frac{1}{64} \right) \).

**Conclusion**

Our algorithm enables one to identify the extremal value of a function defined on a fractal. It is all the more important, for instance in the perspective of further applications to computer aided design. We either think of structures with fractal components, or ones that take into account the irregularities of surfaces encountered during manufacturing processes (rapid prototyping in particular).

One can find other ways of dealing with fractals, for instance, a very original and interesting one uses non-Newtonian calculus in the case of the Sierpiński Gasket [2]. It is based on Burgin’s non-Diophantine arithmetic. One considers a one-to-one map \( f \) from a fractal set \( \mathcal{F} \) into \( \mathbb{R} \). This leads to the definition of arithmetic rules as follows:

\[
\forall (X, Y) \in \mathcal{F}^2 : \begin{cases} 
X \oplus Y = f^{-1}(f(X) + f(Y)) \\
X \ominus Y = f^{-1}(f(X) - f(Y)) \\
X \otimes Y = f^{-1}(f(X) \times f(Y)) \\
X \odot Y = f^{-1}(f(X) / f(Y))
\end{cases}
\]

Arithmetic rules (existence of a unit element, commutativity, associativity, distributivity) are preserved.

Concretely, one considers two sets \( X \) and \( Y \) equipped with bijections

\[
f_X : X \to \mathbb{R}, \quad f_Y : Y \to \mathbb{R}
\]

and related arithmetics

\[
\{ \oplus_X, \ominus_X : X \times X \to X \}, \quad \{ \oplus_Y, \otimes_Y : Y \times Y \to Y \}
\]

respectively defined by \( f_X \) and \( f_Y \).

The bijection

\[
f = f_Y^{-1} \circ f_X : X \to Y
\]

makes it possible to consider derivatives of functions

\[
A : X \to Y
\]
in the following way [1]:

\[
\frac{dA(X)}{dX} = \lim_{h \to 0} \left\{ A \left( X \oplus_X f_X^{-1}(h) \right) \ominus_Y A(X) \right\} \ominus_Y f_Y^{-1}(h) \\
= \lim_{h \to 0} \left\{ A \left( X \oplus_X h \ominus_Y A(X) \right) \ominus_Y h_Y \right\} \\
= \lim_{h \to 0} \left\{ A \left( X \oplus_X h \right) \ominus_Y A(X) \right\} \ominus_Y f(h).
\]

Of course, such an approach is not equivalent to ours. The interest is that it enables one to define Fourier transforms on Cantor sets for instance (see [1], or to formulate differential equations on Koch curves (see [6]). Comparison of both approaches, especially on a numerical point of view, could be very interesting.

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