Variant Domination Types for a Complete \( h \)-ary Tree

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Abstract

Graph \( G = (V, E) \) is a tool that can be used to simplify and solve network problems. Domination is a typical network problem that graph theory is well suited for. A subset of nodes in any network is called dominating if every node is contained in this subset, or is connected to a node in it via an edge. Because of the importance of domination in different areas, variant types of domination have been introduced according to the purpose they are used for. In this paper, two domination parameters the first is the restrained and the second is secure domination have been chosen. The secure domination, and some types of restrained domination in one type of trees is called complete \( h \)-ary tree \( (T_{c,h,r}) \) are determined.

Keywords: Complete \( h \)-ary trees, Domination, Restrained domination, Secure domination Mathematical subject classification: 05C69

Introduction:

Let \( G = (V, E) \) be a finite undirected simple graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). \( N(v) = \{u \in V; uv \in E(G)\} \) and \( N[v] = N(v) \cup \{v\} \), are open neighborhood and closed neighborhood sets of \( v \) respectively. The induced subgraph \( G[D] \) is a graph of vertices set \( D \subseteq V(G) \) of a graph \( G \) together with any edges whose endpoints are both in this subset. Any notion or definition which is not found here could be found in (1,2). An independent vertex set of a graph \( G \) is a subset of the vertices such that no two vertices in the subset represent an edge of \( G \), (2).

In any graph \( G \) the set \( D \subseteq V(G) \) is a dominating set if every vertex \( v \in V \) is either an element of \( D \) or is adjacent to an element of \( D \). The domination number of \( G \), denoted by \( \gamma(G) \), is the cardinality of a minimum dominating set of \( G \), (3). Because of the importance of domination in different areas, variant types of domination have been introduced according to the purpose they are used for. The domination parameters have been formed either by putting a condition on the vertices of a dominating set \( D \), or by putting a condition on the vertices in \( V - D \) or on both. A total dominating set \( D \) of a graph \( G \) is a dominating such that \( G[D] \) has no isolated vertices.st. An independent dominating set is a vertex subset which is both independent and dominating.

Various types of domination of a graph \( G \) have been defined and studied by several authors, they are listed in the appendix of Haynes (4,5). For more details about parameters of domination number that depend on vertex dominating with condition on dominating set \( D \), see (6,7,8,9), and for condition on vertices in set \( V - D \), see (10,11,12,13,14).

Here, variation types of the domination theme, namely that of restrained domination are studied. In a graph \( G \) with a dominating set \( D \), and the following condition put on the vertices of set \( V - D \), such that the open neighborhood of every vertex in \( G[V - D] \) is not an empty set. If this condition is verified then \( D \) is a restrained dominating set in \( G \). The number \( \gamma_{r}(G) \), is the minimum cardinality of a restrained dominating set of \( G \), (15). When, \( D \) is a restrained and independent then it is called an independent restrained dominating set, (16). The independent restrained domination number of \( G \), denoted by \( \gamma_{r}(G) \), is the smallest cardinality of an independent restrained dominating set of \( G \).
The total restrained domination number of $G$, denoted by $\gamma_r(G)$, is the smallest cardinality of a total restrained dominating set of $G$ (when the dominating set $D$ is restrained and has no isolated vertices in $G[D]$). (17). For the following condition: each $u \in V - D$, there exists a vertex $v \in D$ such that $uv \in E$ and $(D - \{v\}) \cup \{u\}$ is a dominating set, if the this condition is verified then $D$ is a secure dominating set of $G$. The minimum cardinality of a secure dominating set in $G$ is the secure domination number denoted by $\gamma_s(G)$, (18,19).

In this paper, the domination (restrained domination, total restrained domination, independence restrained domination, and secure domination) number of a complete $h$-ary root, $h \geq 2$, $r \geq 3$ are determined.

**Lemma 1** (15) If $D$ is a minimum restrained dominating set of a tree $T$; then every pendent of $T$ belongs to $D$.

**Restricted domination in a complete h-ary tree**
In this section, restrained, independent restrained and total restrained domination for a complete $h$-ary tree $T_{c,h,r}$, are determined:

**Theorem 2** If $G = T_{c,h,r}$, is a complete h-ary tree; then for $r \geq 3$,

$$\gamma_r(G) = \begin{cases} 
\frac{h^{r+3}}{h^3 - 1} - 1 & \text{if } r \equiv 0 \pmod{3}, \\
\frac{h^{r+3}}{h^3 - 1} & \text{if } r \equiv 1 \pmod{3}, \\
\frac{h^{r+3}}{h^3 - 1} + h & \text{if } r \equiv 2 \pmod{3}.
\end{cases}$$

**Proof.**
Looking for a set such that, this set contains as possible a minimum restrained dominating set. So, let’s consider $D = \bigcup_{i=0}^{\left\lfloor \frac{r}{3} \right\rfloor} D_i$, where $D_i = \{v; v \text{ is a vertex of depth } r - 3i \text{ in } G\}$ for all $i = 0, 1, \ldots, \left\lfloor \frac{r}{3} \right\rfloor$ and $|D| = \sum_{i=0}^{\left\lfloor \frac{r}{3} \right\rfloor} h^{r-3i}$. Three cases are obtained.

(i) If $r \equiv 0 \pmod{3}$, then set $D$ dominates the vertices of $G$ where, induce subgraph $G[V - D]$ has no isolated vertices, since between any two sets $D_i$ and $D_{i+1}$, $i = 0, 1, \ldots, \left\lfloor \frac{r}{3} \right\rfloor - 1$, there are adjacent vertices of depths $r - 3i - 1$ and $r - 3i - 2$, $i = 0, 1, \ldots, \left\lfloor \frac{r}{3} \right\rfloor - 1$. Therefore, in this case set $D$ is a restrained dominating set in $G$. Thus, $\gamma_r(G) \leq \sum_{i=0}^{\left\lfloor \frac{r}{3} \right\rfloor} h^{r-3i}$. If there is a dominating set $A$ of $|A|$ vertices with $|A| < \sum_{i=0}^{\left\lfloor \frac{r}{3} \right\rfloor} h^{r-3i}$ then $A$ will definitely not contain the pendant vertices and according to Lemma 1.1, every leaf (pendant) of $T$ belong to the minimum restrained dominating set in $T$, then $A$ cannot be restrained. Thus, set $D$ is the minimum set. So that in this case $\gamma_r(G)$ equals to $\gamma_r(G) = |D| = \sum_{i=0}^{\left\lfloor \frac{r}{3} \right\rfloor} h^{r-3i}$.

(ii) If $r \equiv 1 \pmod{3}$, then the vertices which are not dominated by the set $D$ is only the root vertex, so $D \cup \{v_0\}$ is the minimum restrained dominating set in $G$. (as an example, see Fig. 2).

Therefore in this case,
\[ y_r(G) = 1 + \sum_{i=0}^{r+1} h^{r-3i} = 1 + \frac{(h^{r+3} - h)}{h^3 - 1} \]

(iii) If \( r \equiv 2 \) (mod 3), then the set \( D \) can dominate the graph as a minimum restrained domination except the vertices of depth one plus the root vertex \( v_0 \).

.1+ Thus \( y_r(G) = 1 + h + \sum_{i=0}^{\lfloor \frac{r}{3} \rfloor} h^{r-3i} = \frac{h^{r+3}}{h^3 - 1} \)

Figure 2: Restrained dominating set in \( T_{c,2A} \)

Theorem 3 If \( G = T_{c,h,r} \) is a complete h-ary tree \( r \) restrained dominating set, then \( y_r(G) = y_r(G) \) for \( r \equiv 0 \) (mod 3).

Proof.

Looking for a set such that, this set contains as possible a minimum total restrained dominating set. So, let’s consider \( D = \bigcup_{i=0}^{\lfloor \frac{r}{4} \rfloor} D_i \), where \( D_i = \{ v: v \) is a vertex of depth \( r = 4i \) and \( r = 4i - 1 \) in \( G \} \) and \( |D| = \sum_{i=0}^{\lfloor \frac{r}{4} \rfloor} (h^{r-4i} + h^{r-4i-1}) \), so there are four cases:

(i) if \( r \equiv 1 \) (mod 4), then the vertices in set \( D \) dominate \( G \) and \( G[V - D] \) is a graph with no isolated vertices, since between any two sets \( D_i \) and \( D_{i+1} \), \( i = 0, 1, ..., \frac{r-1}{4} - 1 \), there are adjacent vertices of depths \( r - 4i - 2 \) and \( r - 4i - 3, i = 0, 1, ..., \frac{r-1}{4} - 1 \). Therefore, \( D \) is the restrained dominating set in \( G \).

Thus \( y_r(G) \leq \sum_{i=0}^{\frac{r-1}{4}} (h^{r-4i} + h^{r-4i-1}) \), (as an example, see Fig. 3).

Assuming there exists a set \( A \) such that this set is a dominating set in \( G \) with \( |A| < \sum_{i=0}^{\frac{r-1}{4}} (h^{r-4i} + h^{r-4i-1}) \), then set \( A \) will definitely not contain the pendant vertices and according to Lemma 1.1 every leaf (pendant) of \( T \) belongs to the minimum restrained dominating set in \( T \). Therefore, dominating set \( A \) in \( G \) can not be restrained. Thus, set \( D \) is restrained dominating and is the minimum set with

\[ y_r(G) = |D| = \sum_{i=0}^{\frac{r-1}{4}} (h^{r-4i} + h^{r-4i+1}) \left( \frac{1 - h^{(\frac{r-1}{4})+1}}{h^4-1} \right) = (h + 1) \left( \frac{h^{r-3} - 1}{h^{4} - 1} \right) \]

(ii) if \( r \equiv 2 \) (mod 4) then set \( D \) dominates the graph \( G \), but in \( G[V - D] \) root vertex \( v_0 \) is an isolated vertex, so \( D \cup \{ v_0 \} \) is a total restrained dominating set in \( G \) and it’s the minimum. Therefore, in this case

\[ y_r(G) = 1 + \sum_{i=0}^{\frac{r-1}{4}} (h^{r-4i} + h^{r-4i-1}) = 1 + \sum_{i=0}^{\frac{r-2}{4}} (h^{r-4i} + h^{r-4i+1}) \]

\[ = 1 + \frac{(h^{r+3} + h^{r+4}) - h^{(\frac{r-2}{4})+1}}{h^4-1} = 1 + \frac{h^3 + h^2}{h^{4} - 1} \]

Theorem 4 If \( G = T_{c,h,r} \) is a complete h-ary tree, then for \( r \geq 5 \),

\[ y_r(G) = \begin{cases} (h + 1) \left( \frac{h^{r+3} - 1}{h^4 - 1} \right), & \text{if } r \equiv 1 \text{ (mod 4)}, \\ 1 + (h^2 + h) \left( \frac{h^{r+2} - 1}{h^4 - 1} \right), & \text{if } r \equiv 2 \text{ (mod 4)}, \\ 1 + (h^3 + h^2) \left( \frac{h^{r+1} - 1}{h^4 - 1} \right), & \text{if } r \equiv 3 \text{ (mod 4)}, \\ 1 + 3h + (h^3 + h^2) \left( \frac{h^{r} - 1}{h^4 - 1} \right), & \text{if } r \equiv 0 \text{ (mod 4)}. \end{cases} \]
(iii) If \( r \equiv 3(\text{mod } 4) \), then the set \( D \) dominates \( G \) except the root vertex. Let \( D \cup \{u\} \) be the dominating set of \( G \) where, \( u \) is any vertex of depth one. So, \( D \cup \{u\} \) is a minimum total restrained dominating set in \( G \). Therefore the result is gotten.

\[
\begin{align*}
\gamma_{rt}(G) &= 1 + \sum_{i=0}^{\left\lfloor \frac{r-2}{4} \right\rfloor} (h^{r-4i} + h^{-4i-1}) \\
&= 1 + \left( h^3 + h^2 \right) \left( 1 - h^{-4} h^{-4-1} \right).
\end{align*}
\]

(iv) If \( r \equiv 0(\text{mod } 4) \), then the set \( D \) is the minimum total restrained dominating set in \( G \) except the vertices of depth two and one plus the root vertex. Thus , \( \gamma_{rt}(G) = 1 + h + 2h + \sum_{i=0}^{\left\lfloor \frac{r-2}{4} \right\rfloor} (h^{r-4i} + h^{-4i-1}) = 1 + h + 2h + (h^3 + h^2) \left( h^{r-1} - \frac{1}{h^4-1} \right) \).

![Figure 3: Total restrained dominating set in \( T_{c,2,5} \)](image)

**Secure domination in a complete h-ary tree**

**Theorem 5** If \( G = T_{c,h,r} \) is a complete \( h \)-ary tree; then for \( r \geq 3 \),

\[
\gamma_S(G) = \left[ \frac{h^{r+1} - 1}{h^2 - 1} \right] + h^{r-1} (h - 1).
\]

**Proof.**

Case 1. If \( r \) is odd.

Let's consider \( D = A_1 \cup A_2 \), where \( A_1 = \bigcup_{i=0}^{\left\lfloor \frac{r}{2} \right\rfloor} D_i \) where \( D_i = \{v: v \text{ is a vertex of depth } r - 2i - 1\} \) in \( G \), \( |A_1| = \sum_{i=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} h^{r-2i-1}, \) and \( A_2 = \{v: v \text{ is a vertex of depth } r\}, \) such that \( |A_2| = h^r - h^{r-1}, |D| = |A_1| + |A_2| \)

\[
= \sum_{i=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} h^{2i} + h^r - h^{r-1} = \frac{h^{r+1} - 1}{h^2 - 1} + h^{r-1} (h - 1).
\]

Thus, all vertices of \( G \) are dominated by set \( D \).

Every vertex in \( D_i \) is adjacent to some vertices in row \( r_{i+1} \). Therefore, for each vertex \( v \) in \( D_i \) there is a vertex \( u \in V - D \) such that, the swap \( ((D - \{v\}) \cup \{u\}) \) is a dominating set since the vertices in \( r_{i+1} \) are dominated by the vertices of \( D_i \) in row \( i \) and the vertices of \( D_{i+2} \) in row \( r_{i+2} \). Therefore, \( D \) is the secure dominating set in \( G \). Thus, \( \gamma_S(G) = \frac{h^{r+1} - 1}{h^{2-1}} + h^{r-1} (h - 1) \). If there is another dominating set say \( F \), and \( |F| < |D| \), then there are at least \( h \) vertices of \( G \) which are not dominated by set \( F \). Thus, \( D \) is the minimum and it is a secure dominating set. So that, the secure domination number in this case of \( T_{c,h,r} \) is

\[
\gamma_S(G) = |D| = \left[ \frac{h^{r+1} - 1}{h^2 - 1} \right] + h^{r-1} (h - 1) = \left[ \frac{h^{r+1} - 1}{h^{2-1}} \right] + h^{r-1} (h - 1).
\]

Case 2. If \( r \) is even, then \( = A_1 \cup A_2 \), where \( D \) is a dominating set such that \( = \bigcup_{i=0}^{\left\lfloor \frac{r}{2} \right\rfloor} D_i \) where \( D_i = \{v: v \text{ is a vertex of depth } r - 2i\} \) in \( G \), and \( |A_1| = \sum_{i=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} h^{r-2i-1} \). While,

\[
A_2 \text{ is the same set in Case1.}
\]

Thus, \( |D| = |A_1| + |A_2| \), and \( D \) is the minimum secure dominating set as same proof in Case 1(Fig.4).

\[
\gamma_S(G) = |D| = \sum_{i=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} h^{r-2i-1} + h^r - h^{r-1} = \frac{h^{r+1} - 1}{h^{2-1}} + h^{r-1} (h - 1) = \left[ \frac{h^{r+1} - 1}{h^{2-1}} \right] + h^{r-1} (h - 1).
\]
Conclusion:

Domination number for some types of graph domination is calculated for a complete $h$-ary root tree $T_{c,h,r}$. A restrained domination number for complete $h$-ary root, $h \geq 2$, can be determined form $r \geq 3$, while the total restrained domination can be determined from $r \geq 5$. When, $r = 3$ and $4$ the complete $h$-ary tree has no total restrained domination. The independence restrained domination number equals to restrained domination number for the complete $h$-ary tree, when $r \equiv 0 (mod\ 3)$. When $(r \equiv 1, 2 (mod\ 3))$, the graph has no independence restrained dominating set. The secure domination in a complete $h$-ary tree can be determined from $r \geq 3$.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
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أنواع مختلفة من الهيمنة في بيان شجرة شعاع الجذر المتكامل

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الخلاصة:
يعتبر البيان $G = (V, E)$ أداة جيدة لحلول بعض مشاكل شبكات. من هذه المشاكل هي مسألة الهيمنة في الشبكات، والتي تدرس عن طريق نظرية البيانات بعد تحويل الشبكة إلى بيان الذي هو مجموعة من الرؤوس مع مجموعة من الحافات التي تربط بين هذه الرؤوس.

اي مجموعة جزءية من رؤوس البيان هي مجموعة هيمنة في البيان إذا كان أي رأس في البيان ينتمي لمجموعة الهيمنة أو له جوار في هذه المجموعة. رقم الهيمنة هو قياس أصغر مجموعة تهيمن على البيان. لأهمية هذا الموضوع في مختلف المجالات، انواع مختلفة من الهيمنة في البيانات تم استخدامها في هذا البحث.

في هذا البحث تم اختيار نوعين من الهيمنة. الأول هو الهيمنة المميزة حيث الشرط وضع على مجموعة الرؤوس خارج المجموعة المميزة. أما النوع الثاني وهو الهيمنة الأمينة وهي المجموعة المهيمنة مع وضع شرط على المجموعة المهيمنة. تم دراسة الهيمنة الأمينة مع أنواع مختلفة من الهيمنة المميزة على عائلة من الأشجار وهي شجرة شعاع الجذر المتكامل ذو العمق $h$ والفرع $r$.

الكلمات المفتاحية: شجرة شعاع الجذر المتكامل ذو العمق $h$ والفرع $r$, الهيمنة، الهيمنة المميزة، الهيمنة الأمينة.