 SU(n) AND U(n) REPRESENTATIONS OF THREE-MANIFOLDS WITH BOUNDARY

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Abstract
Let $M^3$ be a compact, connected, oriented three-dimensional manifold with non-empty boundary, $\partial M$. This paper obtains results on extending on extending flat vector bundles or equivalently representations of the fundamental group from $S$, part of the boundary, to the whole manifold $M^3$. The proof uses the introduction, investigation, and complete computation up to sign of new numerical invariants $\lambda_{U(n)}(M^3, S)$ (respectively, $\lambda_{SU(n)}(M^3, S)$), where $S$ is a proper connected subsurface of $\partial M$ which is connected. These numerical invariants 'count with multiplicities and signs' the number of representations up to conjugacy of the fundamental group of $M$, $\pi_1(M)$, to the unitary group $U(n)$ (resp., the special unitary group $SU(n)$) which when restricted to $S$ are conjugate to a specified irreducible representation of $\pi_1(S)$. These invariants are inspired by the work of Casson on $SU(2)$ representations of closed manifolds [1]. All the invariants treated here are seen to be independent of the choice of irreducible representation, $\phi$, on the surface $S$ in the boundary of $M^3$, $\partial M^3$.

If the difference of the Euler characteristics $T = \chi(S) - \chi(M)$ is non-negative, a $(\dim U(n)) \times T$-dimensional cycle is produced that carries information about the space of such $U(n)$ representations. For $T = 0$, the above integer invariant results. For $T > 0$, under the assumption that $\phi$ sends each boundary component of $S$ to the identity, a list of invariants results which are expressed as a homogeneous polynomial in many variables, $\lambda_{U(n)}(M, S, o)$. Here $o$ records the relevant choices of orientations needed. (Such a pattern of polynomial invariants is reminiscent of the work of Donaldson on 4-manifolds [21].) The $SU(n)$ case is treated in a parallel fashion, giving other polynomial invariants for $T > 0$.

For $T = 0$, the resulting integer invariants for $U(n)$ (resp., $SU(n)$) are explicitly computed up to sign in all cases. For $T > 0$, an example is given to show that the $U(n)$ (resp., $SU(n)$) invariants are non-zero in some cases.

Some applications to problems of extending $U(n)$ (resp., $SU(n)$) representations to the fundamental groups of three-manifolds, in particular to the case of rational homology cobordisms, are given in §1. After a historical review in §2, §3 explains and states the main results on the invariants, §4 defines the invariants, §5 proves the theorems, while §6 treats stabilization issues. §7 summarizes that, for $T = 0$, the case of the numerical invariant, there is a $U(n)$ (resp., $SU(n)$) gauge theoretic reformulation, analogous to Taube’s $SU(2)$ gauge theoretic reformulation [39] of Casson’s invariant but here using a Fredholm operator $Z/2$ spectral flow.

1. Some Applications to Extending Representations

Flat vector bundles over a manifold correspond to linear representations of the fundamental group, with the correspondence given by holonomy. Extending representations from the fundamental groups of (part of) the boundary of a manifold to the fundamental group of the whole manifold, or equivalently, extending flat bundles, are important in defining and investigating invariants in many contexts in both high and low dimensional topology and geometry. This section states results on this problem of extending representations of the fundamental groups from (part of) the boundary of a three-manifold. These results are obtained using the invariants defined and investigated in later sections. Theorem [14] treats the case of connected boundary and theorem [1,2] the case when the boundary consists of two boundary components and the three-manifold is a rational homology cobordism between them.

Theorem 1.1. Let $M^3$ be an oriented, connected, compact three-manifold with connected boundary and $S$ a connected subsurface of its boundary for which the inclusion of $S$ in $M^3$, $i : S \subset M^3$, induces an isomorphism of first rational homology groups, i.e.,

$$i_* : H_1(S, Q) \xrightarrow{\cong} H_1(M^3, Q),$$

Then picking a base point $p$ in $S$:
1. Case $U(n)$: For any $U(n)$ representation $\phi : \pi_1(S, p) \to U(n)$ there exists a representation $\phi' : \pi_1(M^3, p) \to U(n)$ which restricts on $S$ to $\phi$.

2. Case $SU(n)$: If moreover, $\phi$ is irreducible and has image in $SU(n)$, then its extension $\phi'$ may be chosen to enjoy the same property.

This result covers many topologically quite different possible choices of the surface $S$ which may have one or many boundary circles. For example, if $M^3$ is a three-manifold for which the inclusion of the boundary, a Riemann surface of genus $g$, $\partial M \subset M^3$ induces a surjection $H_1(\partial M, Q) \to H_1(M^3, Q)$ of rational homology, $S$ may be chosen to be a thickening in $\partial M$ of a wedge of circles suitably chosen, or if $g$ is even, a suitable once-punctured Riemann surface of genus $g/2$, or many other possibilities satisfying the homological condition of theorem 1.1.

A three-dimensional, oriented, rational homology cobordism consists of a compact, connected, oriented 3-manifold $W$ with two boundary components, denoted $N_1, N_2$, such that the maps induced by inclusion on first and second rational homology

$$H_1(N_1, Q) \to H_1(W, Q) \text{ and } H_2(N_1, Q) \to H_2(W, Q)$$

are isomorphisms. By Poincaré duality, this is equivalent to the same condition on the other boundary component,

$$H_1(N_2, Q) \to H_1(W, Q) \text{ and } H_2(N_2, Q) \to H_2(W, Q)$$

are isomorphisms; or again equivalently, to the condition that both $H_1(N_1, Q) \to H_1(W, Q)$ and $H_1(N_2, Q) \to H_1(W, Q)$ are isomorphisms.

It is of natural topological interest to understand to what extent a homology cobordism behaves like a cylinder. A theorem of this nature is proved in this paper:

**Theorem 1.2.** Let $W^3$ be a compact, connected, oriented 3-manifold with boundary consisting of two connected components, denoted $N_1, N_2$, such that the inclusion $i : N_1 \subset W^3$ induces an isomorphism of rational first homology,

$$i_* : H_1(N_1, Q) \cong H_1(W^3, Q).$$

Then $W^3$ is a rational homology cobordism and, picking a base point $p$ in $S$:

1. Case $U(n)$: For any $U(n)$ representation,

$$\phi : \pi_1(N_1, p) \to U(n),$$

there exists a representation $\phi' : \pi_1(W^3, p) \to U(n)$ which restricts on $N_1$ to $\phi$.

2. Case $SU(n)$: If moreover, $\phi$ is irreducible and has image in $SU(n)$, then its extension $\phi'$ may be chosen to enjoy the same property.

2. Historical background on representation counting

Let $M^3$ be a closed, connected, oriented 3-manifold which is a homology sphere, that is, its integral homology is the same as that of the standard 3-sphere, $H_*(M^3, Z) \cong H_*(S^3, Z)$. Casson introduced an invariant which `counts with signs and multiplicities' the number of representations up to conjugacy of the fundamental group of $M^3$, $\pi_1(M^3)$, to the special unitary group $SU(2)$ up to conjugacy [possibly after suitable perturbation] [11 36 37]. He further derived surgery formulae for effectively computing his invariant and established relations to the Rohlin invariant, and other subjects. Allied invariants have since been studied by Boyer and Nicas [14], Walker [40] [rational homology spheres], Lescop [32] [general closed oriented three-manifolds], Lin [33] [knot invariants], Herald [30] [knot invariants], Frohman and Long [26] [knot invariants], Ruberman and Saveliev [35] [four dimensional invariants].
In a different direction, Casson’s invariant was reinterpreted in an analytical gauge theoretic fashion by Taubes [39], forging an important further bond between geometric topology and mathematical physics. The topological sign assigned by Casson to a representation that is transverse is in Taubes’ setting expressed in terms of a spectral flow. The works of Casson and Taubes have been the inspiration for much subsequent work; a notable example was the work of Floer [22, 23, 24] who gave an instanton theory refinement and a $\mathbb{Z}/8$ graded homology theory with the grading given by a spectral flow modulo 8. The Euler characteristic of this Floer homology is the numerical $SU(2)$ invariant of Casson and Taubes for three-dimensional homology spheres.

Because of difficulties in treating the singular strata which correspond to reducible representations in the spaces of representations (see [27, 28]) to $SU(n)$ or $U(n)$, Casson’s original invariant for closed 3-manifolds was restricted to the group $SU(2)$ and to integral homology 3-spheres. In the decades since, substantial efforts expanded the class of 3-manifolds which can be treated, but have only very partially lifted the restriction on the Lie group. The difficulties escalate as $n$ increases, as the number and kinds of singular strata increase in the associated symplectic varieties of $SU(n)$ (resp., $U(n)$) representations.

For $G = SU(2)$, a definitive picture emerged from a series of extensions. The work of Boyer and Lines [13] extended the invariant to the case where the fundamental group equals $\mathbb{Z}/n$ by adding rational corrections for each reducible representation. The work of Walker [40] completely extended the invariant to rational homology spheres, $H_*(M^3, Q) \cong H_*(S^3, Q)$, defining an invariant valued in the rationals $Q$ and proving all the analogues of Casson’s original treatment. The calculations of Boyer and Lines involve Dedekind sums associated to the reducible representations of lens spaces, and in Walker’s work their generalizations. The work of Lescop [32] employing a topological-combinatorial approach treated in generality oriented closed 3-manifolds, still for $G = SU(2)$. The work of Boden and Nicas treats an $SU(n)$ knot invariant for $n \geq 2$ [11], as does the work of Frohman [25].

Various extensions to the case of $SU(3)$ have been pursued by several authors in the standard topological Casson setting and in the gauge setting of Taubes. For example, there is an $SU(3)$ invariant of Boden and Herald [8] for integral homology 3-spheres with explicit computations carried out by Boden, Herald, Kirk, Klassen [9, 10]. A different $SU(3)$ Casson type invariant was introduced by Ronnie Lee and the present authors; this invariant is perturbative [16]. However, to date the increasing difficulties with the stratification of the representation spaces have precluded a general definition for $SU(n)$ representations, for $n > 3$, for closed 3-manifolds.

As seen in this paper, by considering representations to $U(n)$ (respectively, $SU(n)$) which restrict to a specified irreducible representation to $U(n)$ (resp., $SU(n)$) on part of the boundary of a non-closed manifold, the, in general, as yet unresolved, technical difficulties caused by the need for the closed case to treat the contributions of the singular strata are absent. As seen below, the $U(n)$ (resp., $SU(n)$) numerical analogue of the Casson invariant in the present context can then be computed homologically for all $n$, at least up to sign.

3. Results on numerical and polynomial invariants

The present section deals with Casson type invariants in the context of compact, connected, oriented 3-manifolds $M^3$ with non-empty but connected boundary $\partial M^3$ together with a specified embedding

$$S_1 \subset \partial M^3$$

of a connected [non-closed] subsurface $S_1$ into $\partial M^3$ with base point $p$, and a choice of irreducible representation of the fundamental group of $S_1$,

$$\phi : \pi_1(S_1, p) \to U(n),$$
to the unitary group $U(n)$. The underlying idea is to construct invariants that measure the number of representations of the fundamental group of $M^3$, $\pi_1(M^3)$, to $U(n)$ which restrict on $S_1$ to the chosen representation, $\phi$, up to conjugation. It is then shown that the resulting invariants are independent of the choice of such irreducible $\phi$.

An entirely parallel analysis for the special unitary group $SU(n)$ proceeds in a like manner with minor changes and is indicated below. More generally, there are also definitions and results analogous to the results of this section for counting irreducible representations to any compact, connected Lie group $G$. For such general Lie groups irreducible means, as usual, that the only elements of $G$ which commute with all elements of the image of the given representation are those in the center of the group $G$.

The initial restriction to irreducible representations $\phi$ is crucial in this section; however, since any representation to $U(n)$ is a sum of irreducible representations, general results may often be treated by first studying this special case, as in the first statement of theorems 1.1 and 1.2. Moreover, the case of manifolds with several boundary components may often be reduced to the case of one, by connecting the boundary components by disjointly embedded paths and deleting a tubular neighborhood of them. By these means the results on this special case have general implications, for example theorem 1.2.

The aim is to “count appropriately with signs and multiplicities” the number of representations, up to conjugacy, of the fundamental group of $M^3$, $\pi_1(M^3)$, to $U(n)$ (respectively, $SU(n)$) which restrict to $\phi$, up to conjugacy. To deal with issues of global sign, additionally a choice of orientations of $M^3$ and of the real cohomology groups $H^1(M, R), H^2(M, R), H^1(S_1, R)$ is needed, see [8]. These orientation choices are recorded via the symbol $o$.

Firstly, it will be shown that this can be carried out in the case that there is equality of Euler characteristics,

$$\chi(S_1) = \chi(M^3),$$

to yield integer valued invariants,

$$\lambda_{U(n)}(M^3, S_1) \in \mathbb{Z},$$
$$\lambda_{SU(n)}(M^3, S_1) \in \mathbb{Z},$$

which are shown to be independent of the irreducible representation $\phi$ chosen on $S_1$. The absolute values $|\lambda_{U(n)}(M^3, S_1)|, |\lambda_{SU(n)}(M^3, S_1)|$ are independent of the orientation choices recorded by $o$. Let $p$ be the base point of $S_1$ chosen to lie in the boundary of $S_1$.

In contradistinction to the original $SU(2)$ Casson invariant, these invariants $\lambda_{U(n)}(M^3, S_1) \in \mathbb{Z}$ and $\lambda_{SU(n)}(M^3, S_1) \in \mathbb{Z}$ will be shown to be of a homological character, as seen from the following theorem which, in particular, completely determines them up to sign.

**Theorem 3.1.** Case $U(n)$: In the case that $\chi(S_1) = \chi(M^3)$ and $\partial M^3$ connected:

1. If $\lambda_{U(n)}(M^3, S_1) \neq 0$, then there is at least one representation $\phi' : \pi_1(M^3, p) \to U(n)$ restricting to $\phi : \pi_1(S_1, p) \to U(n)$.

2. The value of the invariant $\lambda_{U(n)}(M^3, S_1)$ is independent of the choice of irreducible representation $\phi$.

3. If the rational homology $H_2(M^3, Q)$ is non-vanishing or the induced mapping on rational homology

$$H_1(S_1, Q) \to H_1(M^3, Q)$$

There are also definitions and results analogous to the results of this section for counting irreducible representations to any compact, connected Lie group $G$. For such general Lie groups irreducible means, as usual, that the only elements of $G$ which commute with all elements of the image of the given representation are those in the center of the group $G$.
is not an isomorphism, then \( \lambda_{U(n)}(M^3, S_1) = 0 \).

(4) If the induced mapping on rational cohomology

\[ H_1(S_1, Q) \cong H_1(M^3, Q) \]

is an isomorphism, then \( H_2(M^3, Q) = 0 \) and the absolute value \( |\lambda_{U(n)}(M^3, S_1)| \) equals \( K^n \), where \( K \) is the order of the finite abelian group \( H^2(M^3, S_1, Z) \), the second cohomology. In particular, \( \lambda_{U(n)}(M^3, S_1) \neq 0 \) in this case.

(5) \( \lambda_{U(n)}(M^3, S_1) = (-1)^n \lambda_{U(n)}(-M^3, S_1) \) where \(-M^3\) is \( M^3 \) with the opposite orientation.

**Theorem 3.2.** Case \( SU(n) \): The same results hold replacing \( U(n) \) by \( SU(n) \) except that in statement (4), \( K^n \) is replaced by \( K^{n-1} \) and in statement (5), \((-1)^n \) is replaced by \((-1)^{n-1}\).

These changes from \( K^n \) and \((-1)^n \) to \( K^{n-1} \) and \((-1)^{n-1} \) reflect that the rank of \( U(n) \) as a Lie group is \( n \) and the rank of \( SU(n) \) is \( n - 1 \).

Recall that for an arbitrary compact, oriented, odd dimensional manifold \( N \) with boundary \( \partial N \), their Euler characteristics are related by:

\[ \chi(N) = (1/2) \chi(\partial N). \]

This is readily verified by forming the double of \( N \), called \( \tilde{N} \), obtained by taking two disjoint copies of \( N \) with the tautological identification on their boundaries. By Poincaré duality for the closed odd dimensional manifold \( \tilde{N} \), the Euler characteristic of \( \tilde{N} \) vanishes and so, by additivity of the Euler characteristic, one obtains \( 0 = \chi(\tilde{N}) = 2 \chi(N) - \chi(\partial N) \).

In the present context, if the connected surface \( \partial M^3 \) has genus \( g \), the Euler characteristic of \( M^3 \) is just

\[ \chi(M^3) = (1/2) \chi(\partial M^3) = (1/2)(2 - 2g) = 1 - g. \]

So the condition that \( \chi(M^3) = \chi(S_1) \) of theorem 5.1 is just that \( \chi(S_1) = 1 - g \). That is, \( H_1(S_1, Z) \cong Z^g \). For example, \( S_1 \) could be under the Euler characteristic condition be chosen to be a regular neighborhood of a bouquet of \( g \) circles imbedded into \( \partial M^3 \) which has genus \( g \); but as noted in the discussion after theorem 11, there are in general many other topologically distinct possible choices for \( S_1 \).

Secondly and correspondingly, if \( \chi(S_1) - \chi(M^3) = T > 0 \) is even, [note, \( T = (g - k) \) with \( \chi(S_1) = 1 - g_1 \)], then (in analogue to Donaldson’s invariants of four dimensional manifolds [21]) additional invariants will be defined, under the mild assumption that \( \phi \) sends each homotopy class representing a boundary component of \( S_1 \) to the identity in \( U(n) \).

Note : This additional assumption excludes a few low genus cases. More precisely, let the connected surface \( S_1 \) be obtained from a closed Riemann surface of genus \( k \) by deleting \( l \) 2-disks. In particular, \( \chi(S_1) = 1 - g_1 = (2 - 2k - l - 1 = 1 - (2k + l - 1) \) or \( g_1 = 2k + l - 1 \) and \( S_1 \) has \( l \) boundary components. A representation with the above property will factor through the fundamental group of the closed surface. In particular, it will be reducible if \( k = 0, 1 \) for \( n \geq 2 \). Thus the discussion of polynomial invariants below implicitly uses the condition that \( g_1 \) is greater than the number of components of the boundary of \( S_1 \) plus one. With this assumption, any irreducible representation of \( \pi_1(S_1) \) can be deformed through irreducible representations to an irreducible representation that sends each boundary component to the identity.

Explicitly, let \( I, J \) (possibly vacuous) be two multi-indices of pairs of non-negative integers, \( I = [(i_1, r_1), (i_2, r_2), \ldots, (i_a, r_a)] \), \( J = [(j_1, s_1), (j_2, s_2), \ldots, (j_b, s_b)] \) such that \( i_1 < i_2 < \cdots < i_a \) and \( j_1 < j_2 < \cdots < j_b \) with

\[ T = \chi(S_1) - \chi(M^3) = (\sum_{p=1}^{a} (2i_p)r_p) + (\sum_{p=1}^{b} (4j_p - 2)s_p). \]
Then one defines a rational invariant 
\[ \lambda_{I,J,U(n)}(M^3, S_1, o) \]
which “counts with signs and multiplicities” the number of the \((\dim U(n)) \times T\) parameter families of representations which extend to \(M^3\) the irreducible representation \(\phi\), where \(\phi\) is chosen to send each boundary component of \(S_1\) to the identity element \(Id \in U(N)\). As explained in §4 \(o\) records a choice of the relevant orientations those of \(M^3\), and \(H^2(M, R), H^1(M, R), H^1(S_1, R)\).

All these may be recorded by a single homogeneous polynomial invariant, 
\[ \Lambda_{U(n)}(M, S_1, o) = \Sigma_{I,J} \lambda_{I,J,U(n)}(M, S_1, o) X_{I,J}, \]
with the sum over multi-indices \(I, J\), as above and \(X_{I,J}\) denoting the monomial:
\[ X_{I,J} = (x_{i_1}^1 x_{i_2}^2 \cdots x_{i_r}^r)(y_{j_1}^1 y_{j_2}^2 \cdots y_{j_s}^s) \]
Here one should regard \(x_i\) to be in degree \(2i\) and \(y_j\) to be in degree \(4j - 2\).

The corresponding multi-index invariants for \(SU(n)\), \(\lambda_{I,J,SU(n)}(M^3, S_1, o)\), are indexed by multi-indices (possibly vacuous), \(I = [(i_1, r_1), (i_2, r_2), \cdots, (i_a, r_a)]\), \(J = [(j_1, s_1), (j_2, s_2), \cdots, (j_b, s_b)]\), of non-negative indices with \(i_1 < i_2 < \cdots < i_a\) and \(j_1 < j_2 < \cdots < j_b\) subject to the added constraints \(i_1 > 1\), if \(I\) is present, and \(j_1 > 1\), if \(J\) is present. All these may be recorded by a single homogeneous polynomial invariant
\[ \Lambda_{SU(n)}(M, S_1, o) = \Sigma_{I,J} \lambda_{I,J,U(n)}(M, S_1, o) X_{I,J} \]
with the sum over multi-indices \(I, J\), as above with the additional constraints \(i_1 > 1\), if \(I\) present and \(j_1 > 1\), if \(J\) is present.

**Theorem 3.3.** Case \(U(n)\): Let \(\phi\) send each homotopy classes representing a boundary component of \(S_1\) to the identity in \(U(n)\). Then with \(T = \chi(S_1) - \chi(M^3)\) one has:

1. If \(\Lambda_{U(n)}(M^3, S_1, o) \neq 0\), then there is a \((\dim U(n)) \times T\) dimensional family of representations \(\{\phi^\lambda\} : \pi(M^3) \to U(n)\), each restricting to \(\phi : \pi(S_1) \to U(n)\).
2. The value of the invariant \(\Lambda_{U(n)}(M^3, S_1, o)\) is independent of the choice of irreducible representation \(\phi\) sending the homotopy class of each boundary component of \(S\) to the identity.
3. The invariant \(\Lambda_{U(n)}(M^3, S_1, o)\) changes by the sign \((-1)^n\) under a change of orientation of any of \(M^3\), \(H^1(M, R), H^2(M, R), H^1(S_1, R)\). In particular, the absolute value of the coefficient of any monomial term in the polynomial \(\Lambda_{U(n)}(M, S_1, o)\) is independent of the orientation choices \(o\).

**Theorem 3.4.** Case \(SU(n)\): The same results, but with the added constraint on indices \(i_1 > 1\), if \(I\) present, and \(j_1 > 1\), if \(J\) is present, hold replacing \(U(n)\) by \(SU(n)\) except that in statement (4), \(K^n\) is replaced by \(K^{n-1}\) and in statement (5), \((-1)^n\) is replaced by \((-1)^{n-1}\).

It would be interesting to have formulae for the polynomial invariants \(\Lambda_{U(n)}(M, S_1, o)\), \(\Lambda_{SU(n)}(M, S_1, o)\), analogous to theorems §3.1 §3.2 parts 3 and 4. An example is given at the end of section §6 to show that these invariants do not always vanish.

There are for general compact, connected, Lie groups \(G\) definitions and results analogous to the results of this section for ‘counting with signs and multiplicities’ the number of representations which restrict to a conjugate of a prescribed, irreducible representation on part of the boundary; and more generally, there are polynomial invariants associated to \(G\) when \(T > 0\). These are all again independent of the choice of the irreducible representation prescribed on part of the boundary.

The present treatment for 3-manifolds with boundary of some representation-theoretic invariants which had encountered symplectic and topological obstacles for closed three-manifolds, may suggest a paradigm in some other settings, e.g., for other invariants of three-manifolds with boundary.
4. Definition of the Invariants

Let $M^3$ be a compact, connected, oriented 3-manifold with non-empty, connected boundary, $\partial M^3$, of genus $g$. Let

$$\partial M^3 = S_1 \cup S_2 \text{ with } \partial S_1 = \partial S_2 = S_1 \cap S_2$$

be a decomposition of the boundary of $M^3$ into two non-empty connected subsurfaces, $S_1, S_2$. Pick a point $p \in \partial S_1 = S_1 \cap S_2$ to serve as the base point for all fundamental groups.

It is assumed that there is chosen a fixed irreducible representation, called $\phi$,

$$\phi : \pi_1(S_1) \to U(n)$$

of the fundamental group of $S_1$ to $U(n)$. The treatment of $SU(n)$ is entirely parallel, replacing $U(n)$ by $SU(n)$, except where noted in the computations.

Following Casson’s approach [11], choose handlebodies $H_1, H_2$ of genus $h_1, h_2$, respectively, such that $M^3$ is a union of $H_1, H_2$ adapted to the decomposition of $\partial M^3$. That is, firstly $H_1$ is obtained from the regular neighborhood $S_1 \times [0,1]$ in $M^3$ of $S_1 = S_1 \times 0$ by adding one handle; secondly $H_2$ is obtained from the regular neighborhood $S_2 \times [0,1]$ in $M^3$ of $S_2 = S_2 \times 0$ by adding one handles.

Hence, the fundamental groups of $\pi_1(H_1)$, $\pi_1(H_2)$ are free groups obtained from the fundamental groups $\pi_1(S_1), \pi_2(S_2)$, also free groups, by adding added generators, $h_1 - g_1, h_2 - g_2$, of them, respectively. These handlebodies, $H_1, H_2$, are chosen to intersect in a subsurface $U \subset M^3$ with the added properties:

$$M^3 = H_1 \cup H_2 \text{ with } S_1 = H_1 \cap (\partial M^3) \text{ and } S_2 = H_2 \cap (\partial M^3)$$

$$U = H_1 \cap H_2 \text{ with } \partial H_1 = S_1 \cup U \text{ and } \partial H_2 = S_2 \cup U$$

$$U \text{ meets } \partial M^3 \text{ transversally along } S_1 \cap S_2$$

$$U \text{ is connected}$$

Recall that a handlebody of genus $h$ is a regular neighborhood of a bouquet of $h$ circles embedded in $R^3$; equivalently, it is the union of a closed 3-ball, $D^3$, together with $h$ 1-handles, $D^2 \times [0,1]$ suitably attached along the boundaries pieces $D^2 \times 0, D^2 \times 1$ to disjoint parts of the boundary $\partial D^3$.

In particular, if the handlebodies, $H_1, H_2$ have genus $h_1, h_2$, then $\chi(H_1) = 1 - h_1, \chi(H_2) = 1 - h_2$ and the fundamental groups, $\pi_1(H_1), \pi_1(H_2)$ are free groups on $h_1, h_2$ generators, respectively.

The following notation is used: For a space $X$ and subspace $S \subset X$ with base point $p \in S$, let $R^\#(X)$ denote the space of representations of the fundamental group, $\pi_1(X, p)$, to $U(n)$ and $R(X)$ be the quotient of $R^\#(X)$ by the conjugation action of $U(n)$. For a specified representation $\phi : \pi_1(S, p) \to U(n)$, define $R^\#(X, S, \phi)$ to be the space of representations of $\pi_1(X, p)$ to $U(n)$ which when restricted to $\pi_1(S, p)$ give the specified representation $\phi$. Let $R(X, S, [\phi])$ be the image in $R(X)$ of $R^\#(X, S, \phi)$ under the natural projection. Alternatively, $R(X, S, [\phi])$ may be described as the space of representations of $\pi_1(X, p)$ to $U(n)$ which on $S$ equals $\phi$ up to conjugation, divided out by the conjugation quotient action. In particular, it only depends on $\phi$ up to conjugation.

Note that since $\phi : \pi_1(S_1, p) \to U(n)$ is assumed to be irreducible, the natural quotient mapping $R^\#(M^3, S_1, \phi) \to R(M^3, S_1, [\phi])$ sending a representation $\rho \in R^\#(M^3)$ to its image in $R(M^3) = R^\#(M^3)/\text{conjugation}$ is a bijection:

$$R^\#(M^3, S_1, \phi) \cong R(M^3, S_1, [\phi])$$

This holds since the irreducibility of $\phi$ means that the conjugation action of $U(n)$ is free modulo the action of the center $Z(U(n)) \cong U(1)$ of $U(n)$ which conjugates trivially. Otherwise expressed, the projective unitary group $PU(n) = U(n)/Z(U(n))$ acts freely on the irreducible representations.
Hence, by $\phi$ irreducible, to suitably count with signs and multiplicities the number of points in $R(M^3, S_1, [\phi])$, it suffices to consider the space of representations $R^\#(M^3, S_1, \phi)$ instead, a useful simplification.

To give an explicit model of $H_1, H_2, U = H_1 \cap H_2$ take a triangulation, say $K$ of $M^3$ for which $S_1 = |L_1|, S_2 = |L_2|$, for subcomplexes $L_1, L_2$ of $K$. Now in the second baricentric subdivision, $K''$, let $H_a$ be the union of closed simplices of $K''$ which contain as a vertices of $K''$ the baricenter of a vertex of $K$, i.e., that vertex again, or the baricenter of an edge of $K$. Correspondingly, let $H_b$ be the subcomplex of $K''$ which is the union of the closed simplices of $K''$ which contain as a vertex of $K''$ a baricenter of a 3-simplex of $K$ or a baricenter of a 2-simplex of $K$. It is a standard fact that $H_a, H_b$ are handlebodies with $M^3 = H_a \cup H_b$ and $H_a \cap H_b$ a separating surface for $M^3$[38]. Such a decomposition is called a Heegaard decomposition. Now let $H'_1$ obtained by adjoining to $H_a$ those closed simplices of $K''$ which have a vertex which is a baricenter of a 2-simplex of the subcomplex $S_1$. Correspondingly, let $H'_2$ be obtained by adjoining to $H_a$ those closed simplices of $K''$ which have a vertex which is a baricenter of a 2-simplex of the subcomplex $S_2$. This gives a handlebody decomposition of $M^3$ with the property that there is an isotopy of $M^3$ carrying the intersections $H'_1 \cap \partial M^3, H'_2 \cap \partial M^3$ to $S_1, S_2$, respectively. Let $H_1, H_2$ be the respective images of $H'_1, H'_2$. This constructs, as desired, handlebodies adapted to the decomposition $\partial M^3 = S_1 \cup S_2$.

In this model, the pieces, $H_1, H_2, S_1, U = H_1 \cap H_2$, are all connected, each is homotopy equivalent to a bouquet of circles, and their fundamental groups are free groups. Also, the fundamental group of $H_1$ is the free group on $h_1$ generators obtained by adding $h_1 - g_1$ free generators to the fundamental group of $\pi_1(S_1)$; similarly, the fundamental group of $H_2$ is the free group on $h_2$ generators obtained by adding $h_1 - g_1$ free generators to the fundamental group of $\pi_1(S_1)$.

In this model, $H_1$ is the union of a regular neighborhood, $S_1 \times [0, 1]$, of $S_1$ in $H_1$ and added 1-handles, say $j$ of them. By isotopying these added 1-handles, they may be arranged to attach to a fixed 2-disk, say $D^2 \subset S_1 \times 1 \subset M^3$, That is, $H_1$ is exhibited as the boundary connected sum of $S_1 \times [0, 1]$, a handlebody of genus $g_1$, and a handlebody $\tilde{H}$ of genus $j$, with common intersection the 2-disk, $D^2$. Thus, $h_1 = g_1 + j$. In particular, there is an isomorphism $R^\#(S_1) \cong U(n)^{g_1}, R^\#(H_1) \cong U(n)^{h_1}, and R^\#(H_1, S_1, \phi) \cong U(n)^{h_1-g_1}$ and in a parallel manner $R^\#(S_2) = U(n)^{g_2}, R^\#(H_2) \cong U(n)^{h_2},$ since the inclusion $\pi_1(S_1) \subset \pi_1(H_1)$ merely adds $j = (h_1 - g_1)$ new free generators. A representation in $R^\#(H_1, \phi)$ will be specified on the $g_1$ free generators of $\pi_1(S_1)$ by $\phi$, but arbitrary on the remaining $j = h_1 - g_1$ free generators. With these identifications $R^\#(H_1) = U(n)^{g_1} \times U(n)^{(h_1-g_1)}$, also.

Expressed in another fashion, there are isomorphisms $R^\#(H_1) \cong U(n)^{h_1}, R^\#(H_2) \cong U(n)^{h_2}, R^\#(S_1) \cong U(n)^{g_1}, R^\#(S_2) \cong U(n)^{g_2}, and R^\#(U) \cong U(n)^u$ where $\chi(H_1) = 1 - h_1, \chi(H_2) = 1 - h_2$ and $\chi(S_1) = 1 - g_1, \chi(U) = 1 - u$.

Moreover, the inclusions $S_1 \subset H_1, S_2 \subset H_2$ induce the surjections $R^\#(H_1) = U(n)^{h_1} \twoheadrightarrow R^\#(S_1) = U(n)^{g_1}$ and $R^\#(H_2) = U(n)^{h_2} \twoheadrightarrow R^\#(S_2) = U(n)^{g_2}$ defined by projecting to the first $g_1, g_2$ factors, respectively.

Next, note that $M^3 = H_1 \cup H_2$ with $U = H_1 \cap H_2$, so the fundamental group $\pi_1(M^3, p)$ is, by van Kampen’s theorem, the amalgamated free product of $\pi_1(H_1, p)$ and $\pi_1(H_2, p)$ along $\pi_1(U, p)$. Hence, a representation $f : \pi_1(M^3, p) \rightarrow U(n)$ is uniquely specified by its restrictions to $\pi_1(H_1, p)$ and
\[ \pi_1(H_2, p) \]; and given a pair of such representations, they arise from a representation of \( \pi_1(M^3, p) \) if and only if their restrictions to \( \pi_1(U, p) \) are equal. That is, as Casson had observed in the analogous closed manifold setting, \( R^\#(M^3) \) is precisely the intersection of the images of \( R^\#(H_1) \) and \( R^\#(H_2) \) in \( R^\#(U) \).

\[ R^\#(M^3) = [\text{Image } R^\#(H_1)] \cap [\text{Image } R^\#(H_2)] \text{ in } R^\#(U) \]

In particular,

\[ R^\#(M^3, S_1, \phi) = [\text{Image } R^\#(H_1, S_1, \phi)] \cap [\text{Image } R^\#(H_2)] \text{ in } R^\#(U) \]

while

\[ R^\#(M^3, S_1, \phi) \cong R(M^3, S_1, [\phi]) \]

Let \( K : R^\#(H_1) \to R^\#(U) \), \( L : R^\#(H_2) \to R^\#(U) \) be the mappings defined by the homomorphisms of fundamental groups \( \pi_1(U) \to \pi_1(H_1) \), \( \pi_1(U) \to \pi_1(H_2) \) induced by the inclusions \( U \subset H_1, U \subset H_2 \). Let \( J : R^\#(H_1, \phi) \subset R^\#(H_1) \) be the inclusion. In this notation, by \( \phi \) irreducible, the set

\[ R(M^3, S_1, [\phi]) \cong R^\#(M^3, S_1, \phi) \]

to be counted “with signs and multiplicities” is identified with the image via \( (a, b) \mapsto L(b) \) of the set of pairs

\[ \{ (a, b) \mid a \in R^\#(H_1, \phi), b \in R^\#(H_2) \text{ with } K \cdot J(a) = L(b) \}. \]

Now, the fundamental group of the connected oriented surface \( U \) is a free group on, say, \( u \) generators, so picking standard generators gives an identification

\[ R^\#(U) \cong U(n)^u \]

describing the mapping space with the group \( U(n)^u \).

Let \( F \) be the mapping defined by

\[ F : R^\#(U) \times R^\#(U) \cong U(n)^u \times U(n)^u \to U(n)^u \cong R^\#(U) \]

\[ (r, s) \mapsto r \cdot s^{-1} \]

In these terms, \( R^\#(M^3, S_1, \phi) \) is identified as the image of the pairs \( (a, b) \in R^\#(H_1, \phi) \times R^\#(H_2) \) for which \( F(K \cdot J(a), L(b)) = Id \in U(n)^u \). In other words, for the composite mapping

\[ G = F(K \cdot J(\ast), L(\ast)) : ( R^\#(H_1, \phi) \times R^\#(H_2) ) \to R^\#(U) \cong U(n)^u \]

the inverse of the point \( Id \in U(n)^u \) is precisely the set of representations \( R^\#(M^3, S_1, \phi) \cong R(M^3, S_1, [\phi]) \).

Since the spaces \( R^\#(U), R^\#(H_1, \phi), R^\#(H_2) \) are each of the type \( U(n)^x \) for various \( x \)'s, they are also smooth manifolds. So once oriented, making the mappings \( K \cdot J, L \) transverse [by a small perturbation, if needed], as mappings of compact, oriented, smooth manifolds, defines an intersection cycle and class

\[ [\text{cyc}] := [\text{intersection}] \in H_W(R^\#(U), Z) \]

of \( (K \cdot J)(R^\#(H_1, S_1, \phi)) \) and \( L(R^\#(H_2)) \) in \( R^\#(U) \), when \( W = \text{dim } R^\#(H_1, \phi) + \text{dim } R^\#(H_2) - \text{dim } R^\#(U) \) is non-negative. This intersection is a well defined cycle up to boundaries. It is defined by moving the images of the mappings \( K \cdot J, L \) to be transverse and taking the inverse image of \( Id \) under \( G \). As will be seen, this oriented intersection cycle regarded as a class in \( R^\#(U) \) has a homological nature.

It is claimed that

\[ W = (\text{dim } U(n)) \cdot (\chi(S_1) - \chi(M^3)) = (\text{dim } U(n)) \cdot T \]
By definition,

\[ T = W / (\dim U(n)) \]

\[ = (\dim R^#(H_1, \phi) + \dim R^#(H_2) - \dim R^#(U)) / (\dim U(n)) \]

\[ = (h_1 - g_1) + h_2 - u = (h_1 + h_2 - u - g_1) \]

while by \( M^3 = H_1 \cup H_2 \) with \( H_1 \cap H_2 = U \),

\[ \chi(S_1) - \chi(M^3) \]

\[ = \chi(S_1) - (\chi(H_1) + \chi(H_2) - \chi(U)) \]

\[ = (1 - g_1) - ((1 - h_1) + (1 - h_2) - (1 - u)) = (h_1 + h_2 - u - g_1) \]

Hence, \( W \) is as claimed.

**Orientations:**

To specify the orientations of the above smooth manifolds, it suffices to orient the tangent spaces at the trivial representation of the spaces \( R^#(U), R^#(H_1), R^#(H_2) \) and the tangent space at \( \phi \) of \( R^#(H_1, S_1, \phi) \). This last is the kernel of the restriction of the tangent space of \( R^#(H_1) \) at \( \phi \) to the tangent space of \( R^#(S_1) \) at \( \phi \), which is surjective. Hence, by definition, it suffices to orient the tangent spaces at the identity of \( R^#(U), R^#(H_1), R^#(H_2), R^#(S_1) \). Since each of the fundamental groups is free, these tangent spaces are precisely, \( H^1(U, R) \oplus u(n), H^1(H_1, R) \oplus u(n), H^1(H_2, R) \oplus u(n), H^1(S_1, R) \oplus u(n) \), where \( u(n) \) denotes the Lie algebra of \( U(n) \).

Note the long exact Mayer-Vitesoris sequence for reduced real cohomology:

\[ 0 \rightarrow H^1(M, R) \rightarrow H^1(H_1, R) \oplus H^1(H_2, R) \rightarrow H^1(U, R) \]

\[ \rightarrow H^2(M, R) \rightarrow H^2(H_1, R) \oplus H^2(H_2, R) = 0 \oplus 0 \rightarrow H^2(U, R) = 0 \]

since \( H_1, H_2 \) are handlebodies, so have the homotopy type of bouquets of circles and, in particular, have vanishing second homology groups. By \( U \) connected, \( H^0(U, Q) = 0 \). Hence, specifying orientations for \( H^2(M, R), H^1(M, R) \) determines compatible choices of orientations for the triple \( H^1(U, R) \oplus u(n), H^1(H_1, R) \oplus u(n), H^1(H_2, R) \oplus u(n) \). In this way, the choices of orientations of \( M^3 \) and of \( H^2(M, R), H^1(M, R), H^1(S_1, R) \) determine the orientation for the cycle

\[ [\text{cyc}] \in H_W(R^#(U), Z). \]

The choice of orientation for \( M^3 \) determines an orientation on \( S_1 \subset \partial M^3 \) and so orientations for each of the boundary circles \( \partial S_1 = \partial U \). Also, the induced orientation from \( M^3 \) on \( H_1 \) containing \( S_1 \) give an orientation on \( U \subset H_1 \) which together with the given orientations of the boundary circles of \( \partial U \) specify an compatible orientation of \( H^2(U, \partial U, R) = R \). By this means the class \([\text{cyc}]\) is well defined, given the four orientations of \( M^3 \) and of \( H^2(M, R), H^1(M, R), H^1(S_1, R) \) which are together recorded here by the letter o. Since \((-1)^{\dim u(n)} = (-1)^n\), changing the orientation of any one of these four terms introduces a change of sign by \((-1)^n\) in the value of the cycle \([\text{cyc}]\).

Alternatively and quite generally, by facts about cap products and Poincaré duality, employing the mapping

\[ G = F(K \cdot J(*), L(*)) : R^#(H_1, \phi) \times R^#(H_2) \rightarrow R^#(U), \]

the intersection class \([\text{cyc}] \in H_W(R^#(U), Z)\) may be described as follows:

Let \( \{R^#(U)\} \) denote the specified oriented generator of the top cohomology, \( H_{\dim} R^#(U)(R^#(U), Z) \cong Z \). Let \([R^#(H_1, \phi)], [R^#(H_2)]\) denote the specified oriented generators in top dimensional homology of \( H_{\dim} R^#(H_1, \phi)(R^#(H_1, \phi), Z) \cong Z \) and \( H_{\dim} R^#(H_2)(R^#(H_2), Z) \cong Z \), respectively; then there is the equality of homology classes

\[ [\text{cyc}] = G_*([R^#(H_1, \phi)] \times [R^#(H_2)]) \cap G^*\{ \{R^#(U)\} \} \in H_W(R^#(U), Z) \]
Here $\cap$ is the cap product pairing from (homology, cohomology) to homology. This demonstrates the intrinsically homological (in the representation spaces) nature of these invariants.

Additionally, any two irreducible representations $\phi'$s can be continuously deformed to each other through irreducible representations; so the homology class $[cyc]$ is left unchanged. Hence, invariants based on $[cyc]$ are independent of the choice of $\phi$.

In the case of equality, $\chi(M^3) = \chi(S_1)$, the associated cycle is a 0-cycle. Adding up the intersection points according to their signs, after a generic perturbation, gives a sum of points with signs and multiplicities which is the desired integer, assuming that the individual pieces, $R^\#(U), R^\#(H_1,S_1,\phi), R^\#(H_2)$ which are all of the form $U(n)^\ast$ for various $\chi'$s, have been oriented compatibly given $o$. Let $\hat{U}$ denote the closed Riemann surface obtained from $U$ by collapsing individually each boundary circle to a point. Let $u$ be the genus of $\hat{U}$. In order to get a result independent of the choice of Heegaard decomposition an additional sign is needed, $(-1)^{\dim U(n)-u} = (-1)^{n-u}$, see [33] where the question of stabilization is explicitly addressed.

The reader will note that the symbol $o$ is dispensed with in the case $T = 0$, corresponding to the integer invariant. The reason is that for $\lambda_{U(n)}(M^3,S_1)$, (resp., $\lambda_{SU(n)}(M^3,S_1)$), to not vanish, necessarily $H_2(M^3,\mathbb{R}) = 0$ and $H_1(S_1,\mathbb{R}) \to H_1(M^3,\mathbb{R})$ is an isomorphism [see below]. Thus in that case, the only one where signs are an issue, one can just chose $o$ so that the orientation of $H_1(S_1,\mathbb{R})$ and $H_1(M^3,\mathbb{R})$ correspond. With this orientation convention, the associated numerical invariant depends only on that of $M^3$, whence the notion $\lambda_{U(n)}(M^3,S_1)$, (resp., $\lambda_{SU(n)}(M^3,S_1)$).

That is, with these orientation conventions, the definition of the required integer-valued invariant in the case $T = \chi(S_1) - \chi(M^3) = 0$ is:

$$\lambda_{U(n)}(M^3,S_1) = (-1)^{n-u} \text{[intersection number of } R^\#(H_1,S_1,\phi) \text{ and } R^\#(H_2) \text{ in } R^\#(U)\]$$

If one takes absolute values, then the choice of orientation classes $o$ of these manifolds will have no effect on the result. Similar remarks apply to the class $[cyc]$ generally.

By the above, for $\chi(S_1) - \chi(M^3) = T = 0$, the integer invariant $\lambda_{U(n)}(M^3,S_1)$ has absolute value:

$$|\lambda_{U(n)}(M^3,S_1)| = |(\text{degree of the mapping } G : ( R^\#(H_1,\phi) \times R^\#(H_2) ) \to R^\#(U))|$$

where $|\ast|_1$ denotes the absolute value.

The arguments of this section, by replacing $U(n)$ by $SU(n)$ and $(-1)^{\dim u(n)-u} = (-1)^{n-u}$ by $(-1)^{\dim SU(n)-u} = (-1)^{(n-1)-u}$ defines $\lambda_{SU(n)}(M^3,S_1)$ for the case $T = 0$ and the cycle $[cyc]$ as well.

5. **Proof of theorems 3.1 (resp., 3.2), 1.1 and 1.2**

Homological methods of extending representations from finite groups were studied in a classical short paper of Gerstenhaber and Rothaus [29]. As in their work, the degrees of maps between products of unitary groups will be utilized.

Assume $\chi(M^3) = \chi(S_1)$, i.e., $g = g_1$, or equivalently, $(h_1 + h_2) - g_1 = u$.

In order to compute the degree of the mapping

$$G : U(n)^{h_1-g_1} \times U(n)^{h_2} \to U(n)^u$$

$$R^\#(M^3,S_1,\phi) \times R^\#(H_2) \to R^\#(U),$$

it is helpful to review facts about the cohomology of $U(n)$. A computation of this degree will prove theorem 3.1 parts 2,3,4. This detailed discussion of the $U(n)$ case will be followed by discussion of the corresponding $SU(n)$ case.
Note that the required mapping 
\[ G : U(n)^N \to U(n)^N, \]
with \( N = (h_1 + h_2) - g_1 = u \) arises in a very special manner. There is a homomorphism, say \( \rho : F_N \to F_N \) of free groups and under the natural identification
\[ \text{Hom}(F_N, U(n)) \cong U(n)^N, \]
the induced mapping is precisely \( G \). Here \( \text{Hom}(F_N, U(n)) \) is the space of homomorphisms of \( F_N \) to \( U(n) \). It is identified with \( U(n)^N \) by sending \( \rho \) to its evaluation, \( (\rho(y_1), \rho(y_2), \cdots, \rho(y_N)) \), for free generators \( y_1, \cdots, y_N \) of \( F_N \).

This special character is clear for the mappings \( J : R^\#(H_1) \cong U(n)^{h_1} \to R^\#(U) \cong U(n)^u \) and \( K : R^\#(H_1) \cong U(n)^{h_1} \to R^\#(U) \cong U(n)^u \) since they are induced by homomorphisms of the free groups \( \pi_1(U) \subset \pi_1(H_1), \pi_1(U) \subset \pi_1(H_2) \). It also holds with the prescribed identifications for \( R^\#(H_1, \phi) \subset R^\#(U) \). Hence, equally so for the composite \( G = F(I \cdot J(\star), K(\star)) \), as \( F \) is defined by \( (r, s) \mapsto r \cdot s^{-1}. \)

The Lie group \( U(n) \) is known to have integral cohomology an exterior algebra on generators \( x[j] \in H^{2j+1}(U(n), Z) \) for \( j = 0, \cdots, n - 1 \), see A. Borel [15]. Since \( U(n) \) is a group, this cohomology has a Hopf algebra structure. As is well known, the classes \( x[j] \) are primitive, that is, \( m^*(x[j]) = x[j \otimes 1 + 1 \otimes x[j]] \) where \( m : U(n) \times U(n) \to U(n) \) is the group multiplication mapping, \( m(a, b) = a \cdot b \).

In particular, the generator of the top cohomology group \( H^{\dim U(n)}(U(n), Z) = Z \) is represented by the product \( \prod_{j=0}^{n-1} x[j] \).

Correspondingly, the product of \( N \) copies of \( U(n) \) has cohomology \( H^*(U(n)^N, Z) \) equal to the exterior algebra on generators \( \{ \pi^*_k(x[j]) \mid j = 0, \cdots, (n - 1), k = 1 \cdots N \} \) where \( \pi_k : U(n)^N \to U(n) \) is the projection onto the \( k^{th} \) factor. Hence, to compute the degree of the mapping \( G : U(n)^N \to U(n)^N \) it will suffice to find the pull backs of the \( \pi^*_k(x[j]) \) in \( H^{2j+1}(U(n)^N, Z) \); that is, the pull backs of \( x[j] \in H^{2j+1}(U(n), Z) \) under the mapping
\[ \pi_k \cdot G : U(n)^N \to U(n) \]
where the last sends \( (A[1], A[2], \cdots, A[N]) \mapsto w[k] \) where \( w[k] \) is a finite product of these matrices to various powers.

For example, the mapping \( f : U(n) \to U(n) \) sending \( A \mapsto A^r \) will, by primitivity, send \( x[j] \) to \( r \cdot x[j] \) and hence have degree \( r^n \) as a mapping of \( U(n) \to U(n) \).

More generally, for \( f : U(n)^N \to U(n) \) obtained by taking products of the matrices \( (A[1], \cdots, A[N]) \) and their inverses, if \( A[k] \) appears in the word \( w \) in the \( A[i]'s \) with total multiplicity \( m[k] \), the sum of powers of its occurrences, then
\[ f^*(x[j]) = \Sigma_{k=1}^{N} m[k] \pi_k^*(x[j]). \]

Even more generally, let \( G : U(n)^N \to U(n)^N \) be the mapping induced by a homomorphism of free groups \( f : F_N \to F_N \) with \( F_N \) a free group on generators \( y_1, \cdots, y_N \), under the identification
\[ \text{Hom}(F_N, U(n)) \cong U(n)^N. \]
sends the entry \( (A[1], \cdots, A[N]) \in U(n)^N \) to \( (W[1], W[2], \cdots, W[N]) \) with each \( W[j] \) a product of the matrices \( \{ A[i] \} \) and their inverses. Let \( m[i, k] \) denote the sum of powers to which \( A[k] \) is raised in toto in the product giving the entry \( W[i] \). By the above
\[ G^*(\pi_i^* x[j]) = \Sigma_{k=1}^{N} m[i, k] \pi_k^*(x[j]) \]
independent of \( j = 0, \cdots, (n - 1) \).

That is, the primitive classes, the span of the \( \pi_k^*(x[j]), k = 1, \cdots, N \), in dimension \( 2j + 1 \), each transform via the matrix \( M = \{ m[i, k] \} \). Since the generator of \( H^{\dim U(n)}(U(n)^N, Z) = \)
isomorphism it follows that $H_1(F_N, Z) = Z[y_i, i = 1 \cdots N] \to H_1(F_N, Z) = Z[y_i, i = 1 \cdots N]$. Hence, $|\det(G)| = |(\det f)|^n$.

For the example at hand, the abelianization may be described as follows: The inclusions $S_1 \subset H_1, U \subset H_1, U \subset H_2$ induce maps of vector spaces of the same rank $H$ on homology $H$. Let the composite mapping of free abelian groups $F \to F$ be called $G$. As above, the Mayer Vietoris sequence of the triple $(M^3, H_1, H_2)$ with $U = H_1 \cap H_2$ gives a long exact sequence in reduced rational cohomology:

$$
\begin{array}{c}
0 \to H^1(M, Q) \to H^1(H_1, Q) \oplus H^1(H_2, Q) \\
\to H^2(M, Q) \to H^2(H_1, Q) \oplus H^2(H_2, Q) = 0 \oplus 0 \to H^2(U, Q) = 0
\end{array}
$$

Since the image by $h \otimes Q$ of $(L \otimes Q) \oplus H^1(H_2, Q)$ in $H^1(U, Q)$ is contained in the image of $H^1(H_1, Q) \oplus H^1(H_2, Q)$ in $H^1(U, Q)$, the map induced by $h$ tensor the rationals necessarily is not onto if $H^2(M^3, Q)$ is non-zero. Hence, $H^2(M^3, Q) \neq 0$ implies that $\lambda_{U(n)}(M^3, S_1) = 0$.

If $H^2(M^3, Q) = 0$, then $\chi(M^3) = 1 - \dim H^1(M^3, Q) = 1 - g = 1 - g_1$, so $H^1(M^3, Q) \to H^1(S_1, Q)$ is a map of vector spaces of the same rank $g$. If an non-zero element of $H^1(M^3, Q)$ were in the kernel of this mapping, then its image lies in $L \otimes Q$ and also maps to a non-zero element of $(L \otimes Q) \oplus H^1(H_2, Q)$ which goes to zero in $H^1(U, Q)$. Hence, unless $H^1(M^3, Q) \to H^1(S_1, Q)$ is an isomorphism it follows that $h$ tensor the rationals is not an isomorphism and so the degree of $h$ is zero and $\lambda_{U(n)}(M^3, S_1) = 0$ again. This gives the proof of theorem 3.1 part 3.

In the case, $H^1(M^3, Q) \to H^1(S_1, Q)$ is an isomorphism, one proves $H^2(M^3, Q) = 0$ as follows: By Poincaré duality $H^2(M^3, Q) \cong H_1(M^3, \partial M, Q)$ which fits into the long exact sequence on homology $H_1(\partial M, Q) \to H_1(M^3, Q) \to H_1(M^3, \partial M, Q) \to H_0(\partial M, Q) \to H_0(M^3, Q)$. The last map is an isomorphism by $M^3, \partial M^3$ connected. Since $i_* : H_1(S_1, Q) \to H_1(M^3, Q)$ is an isomorphism, a fortiori, $H_1(\partial M, Q) \to H_1(M^3, Q)$ is onto. Consequently, by this long exact sequence $H_1(M^3, \partial M, Q) \cong H_2(M^3, Q)$ vanishes.

There is again a long exact sequence for reduced integral cohomology:

$$
\begin{array}{c}
0 \to H^1(M, Z) \to H^1(H_1, Z) \oplus H^1(H_2, Z) \to H^2(M, Z) \\
\to H^2(H_1, Z) \oplus H^2(H_2, Z) = 0 \oplus 0 \to H^2(U, Z) = 0
\end{array}
$$

By exactness and the above, $H^1(M^3, Z) \cong Z^g$ and $H^1(S_1, Z) \cong Z^g$ with the induced mapping $Z^g \to Z^g$ rationally an isomorphism. Let $B$ be the kernel of the mapping $H^1(U, Z) \to H^2(M^3, Z)$. 


The abelian group $H^2(M^3, Z)$ by the above is of finite order. Since $h$ factors as $L \oplus H^1(H_2, Z) \to B \subset H^1(U, Z)$, the absolute values of its determinant is the product of the order of the finite abelian group $H^2(M^3, Z)$ and the absolute value of the determinant of the truncated mapping

$$\hat{h} : L \oplus H^1(H_2, Z) \xrightarrow{i} H^1(H_1, Z) \oplus H^1(H_2, Z) \xrightarrow{b-c} B$$

of free abelian groups which is an isomorphism when tensored with the rationals. The required determinant is just the order of $B/[\text{Image } \hat{h}]$ times the order $|H^2(M^3, Z)|$.

Additionally, there is the short exact sequence $0 \to L \xrightarrow{k} H^1(H_1, Z) \xrightarrow{a} H^1(S_1, Z) \to 0$ which fits together to form the two short exact sequences in the tableau: with $m = a \oplus 0$.

\[
\begin{array}{ccc}
0 & \to & L \oplus H^1(H_2, Z) \\
& \downarrow & \xrightarrow{k \oplus \text{id}} \\
0 & \to & H^1(H_1, Z) \oplus H^1(H_2, Z) \xrightarrow{m} H^1(S_1, Z) \to 0 \\
& \downarrow & (b-c) \\
& & B \\
& \downarrow & \\
& & 0
\end{array}
\]

Define mappings $R : H^1(S_1, Z)/[\text{Image } m \cdot i] \to B/[\text{Image } (b-c)(k \oplus \text{Id})]$ and $S : B/[\text{Image } (b-c)(k \oplus \text{Id})] \to H^1(S_1, Z)/[\text{Image } m \cdot i]$ by $R(m(x)) = (b-c)(x)$ and $S((b-c)(y)) = m(y)$. By exactness, they are well defined and are inverses.

Hence, $|\text{det}(h)| = |H^2(M^3, Z)| \cdot |\text{det}(\hat{h})| = |H^2(M^3, Z)| \cdot |H^1(S_1, Z)/[m \cdot i]|H^1(M^3, Z)|$ where $m \cdot i$ appears in the long exact sequence of the pair $(M, S_1)$:

$$H^1(M^3, Z) \xrightarrow{m \cdot i} H^1(S_1, Z) \to H^2(M^3, S_1, Z) \to H^2(M^3, Z) \to H^2(S_1, Z) = 0$$

Now the last exact sequence shows that $|H^2(M^3, S_1, Z)| = |H^2(M^3, Z)| \cdot |H^1(S_1, Z)/[m \cdot i]|H^1(M^3, Z)|$ so $\text{det}(h) = |H^2(M^3, S_1, Z)|$, as desired. This then proves that $|\lambda_{U(n)}(M^3, S_1)| = K^n$ with $K = |H^2(M^3, S_1, Z)|$, as claimed.

By definition if there were no representations extending to $\pi_1(M^3)$ the given representation $\phi$, then the invariant $\lambda_{U(n)}(M^3, S_1)$ would be zero. This observation completes the proof of theorem 3.1.

The $SU(n)$ case proceeds in a parallel fashion with minor changes, proving theorem 3.2.

**Proof of theorem 1.1.** In the case that $i_* : H_1(S, Q) \to H_1(M^3, Q)$ is an isomorphism, by the above argument $H_2(M^3, Q) = 0$ so $i_* : H_j(S, Q) \to H_j(M^3, Q)$ is an isomorphism for $j = 0, 1, 2, 3$. In particular, $\chi(S) = \chi(M^3)$.

Hence, theorem 3.1 applies to $(M^3, S)$ to extend any irreducible representation into $U(n)$. Since any irreducible $U(n)$ representation is a sum of irreducibles ones, assembling these extensions proves theorem 1.1 in the $U(n)$ case. The $SU(n)$ case proceeds similarly using theorem 3.2 instead.

**Proof of theorem 1.2.** To show that $H_2(N_2, Q) \to H_2(W^3, Q)$ is a rational isomorphism consider the long exact sequence for the pair $(W, N_1)$. By Poincaré duality $0 = H_0(W, N_2, Q) \cong H_3(W, N_1, Q)^*$, so by the exact sequence $H_3(W, N_1, Q) \to H_2(N_1, Q) \to H_2(W, Q)$, the induced map $i_* : H_2(N_1, Q) \to H_2(W, Q)$ is one to one. Since $H_2(N_1, Q) = Q$ to show that this mapping is an isomorphism it suffices to show that $H_2(W, Q)$ is of rank one. Now since $H_1(N_1, Q) \to H_1(W, Q)$ is an isomorphism, a fortiori, $H_1(\partial W, Q) \to H_1(W, Q)$ is onto. But by the long exact sequence of the pair $(W, \partial W)$,
$H_1(\partial W, Q) \to H_1(W, Q) \to H_1(W, N_1, Q) \to H_0(\partial W, Q) \to H_0(W, Q)$, this surjection implies that $H_1(W, N_1, Q)$ maps isomorphically to the kernel of $H_0(\partial N, Q) \to H_0(W, Q)$ which has rank one. Again, by Poincaré duality $H_2(W, Q) \cong H_1(W, N_1, Q)^* \cong Q$. In toto, this proves $i_* : H_2(N_1, Q) \cong H_2(W, Q)$. That is, $W$ is a rational homology cobordism.

Take a properly embedded path $\gamma$ from a point, say $p$, of the boundary component $N_1$ to a point $q$ of the other boundary component $N_2$ of $W^3$. Let $M^3$ be the result of deleting a small tubular neighborhood of the path $\gamma$. It has one boundary component, the connected sum of $N_1$ and $N_2$. Suppose that its boundary, $\partial M^3$, is of genus $g$. Let this tubular neighborhood intersect $N_1$ in a 2-disk $D^2$ about $p$.

If $S$ is not all of $N_1$, chose $\gamma$ so that $D^2$ is disjoint from $S \subset N_1$. The manifold $W^3$ can be recovered from $M^3$ by adding in the tubular neighborhood of $\gamma$. Note that $W^3$ has the homotopy type of $M^3$ with the 2-disk $D^2$ added, $W^3 \cong M^3 \cup D^2$.

Note that by assumption, the restriction mapping $i_* : H^2(W, Q) \to H^2(N_1, Q) = Q$ is an isomorphism.

Let $N_1'$ be the closure of $N_1 - D^2$. Now $W^3$ has $M^3 \cup D^2$ as a deformation retract which keeps $M$ fixed. Now identify $N_1$ with $N_1' \cup D^2$, then the Mayer-Vietoris sequence gives the commutative diagram with exact rows for rational coefficients:

\[
\begin{array}{ccccccc}
H^1(\partial D^2) = Q & \to & H^2(W') & \to & H^2(M) & \to & H^2(\partial D^2) = 0 \\
\downarrow \cong & & \downarrow \cong i & & \downarrow & & \downarrow \cong \\
H^1(\partial D^2) = Q & \cong & H^2(N_1) & \to & H^2(N_1') & \to & H^2(\partial D^2) = 0
\end{array}
\]

This shows that $H^2(M^3, Q) = 0$.

Similarly, the restriction mapping $H^1(M^3, Q) \to H^1(N_1', Q)$ is an isomorphism from the commutative diagram of exact sequences for reduced rational cohomology:

\[
\begin{array}{ccccccc}
0 & \to & H^1(W) & \to & H^1(M^3 \cup D^2) & \to & H^1(M) \oplus H^1(D^2) = (H^1(M) \oplus 0) & \to & H^1(\partial D^2) = Q \\
\downarrow & & \downarrow \cong i & & \downarrow & & \downarrow \cong \\
0 & \to & H^1(N_1') & \to & H^1(N_1' \cup D^2) & \cong & H^1(N_1') \oplus H^1(D^2) & \to & H^1(\partial D^2) = Q
\end{array}
\]

Now let $\phi : \pi_2(N_1) \to U(n)$ be any representation. Since any such is a direct sum of irreducibles, we may assume $\phi$ is irreducible. Then theorem 3.1 applies to the manifold $M^3$ and subspace $N_1' \subset \partial M^3$, for the restriction, say $\phi'$ of $\phi$ to $\pi_1(N_1')$. By that theorem there is a representation $\Phi'$ of $\pi_1(M^3)$ to $U(n)$ which extends $\phi'$ to all of $\pi_1(M^3)$. But since $\phi'(\partial D^2) = \phi(\partial D^2) = Id$, the representation $\Phi'$ of $\pi_1(M^3)$ extends to a representation, say $\Phi$ of $\pi_1(W^3) = \pi_1(M^3 \cup D^2)$ which restricts precisely to $\phi$.

The $SU(n)$ case proceeds in a parallel fashion. The integral cohomology of $SU(n)$ is a Hopf-algebra generated by primitive classes in dimensions $3, 5, 7, \cdots, 2n-1$ with cup product the generator of the top cohomology $H^{n^2-1}(SU(n), Z)$. For this reason, for $T = 0$ the associated degree in this $SU(n)$ case, computing the invariant $\lambda_{SU(n)}(M^3, S_1)$, is $K^{n-1}$ in view of the $n-1$ primitive generators. In a similar fashion, the dependence on the change of orientation goes as $(-1)^{n-1}$ and the definition of the invariant has the stabilizing term $(-1)^{(n-1)-u}$ with $u$ the genus of $\hat{U}$, the closed Riemann surface obtained from $U$ by collapsing each individual boundary circle of $U$ to a point.

6. Stabilization and invariants for $T > 0$.

The classical results of J. Singer \[38\] imply that any two handlebody decompositions adapted as here to the boundary decomposition $\partial M_3 = S_1 \cup S_2$ are related by isotopies and a standard stabilization process. This applies equally well to the special decomposition considered here which arise by adding one handles to the regular neighborhoods of $S_1, S_2$ in $\partial M^3$. The stabilization
process has an explicit description: Given $H_1, H_2$ one chooses a 3-ball, say $D^3$ in the interior of $M^3$ which intersects $U$ in a 2-disk, say $D^2$ and $H_1, H_2$ in two 3-balls, say $B_1, B_2$ respectively. Then this decomposition into $H_1, H_2$ is modified by replacing $D^3 \cap H_1$ by $B_1$ union an additional handle obtained by connecting two interior points of $D^2$ by and an arc in $B_2$ and thickening it. If this new handlebody is called $H'_1$ and the closure of $M^3 - H'_1$ is called $H'_2$, then $(H'_1, H'_2)$ form a handlebody decomposition adapted to the pair $(S_1, S_2)$. Let $U' = H'_1 \cap H'_2$. The genus of $U'$ is that of $U$ increased by 1.

Since the union $D^3 \cup H'_j$ is isotopic to $H_j$, $j=1,2$, and the union $U' \cup D^3$ has $U$ as a deformation retract, the inclusions $H'_j \cup D^3 \cup H'_j \cong H_j$ and $U' \subset U' \cup D^3 \sim U$ define standard inclusions:

$$R^\#(H_1) \subset R^\#(H_1), \ R^\#(H_2) \subset R^\#(H_2), \ R^\#(U) \subset R^\#(U'), \ R^\#(H_1, \phi) \subset R^\#(H'_1, \phi)$$

A simple check shows that the pointwise intersection

$$[\text{Image } R^\#(H_1, S_1, \phi)] \cap [\text{Image } R^\#(H_2)] \text{ in } R^\#(U)$$

equals the intersection

$$[\text{Image } R^\#(H'_1, S_1, \phi)] \cap [\text{Image } R^\#(H'_2)] \text{ in } R^\#(U')$$

Correspondingly, making these transverse in $U$ and extending this to $U'$ shows that the intersection cycle $[\text{cyc}] \in H_W(R^\#(U), Z)$ maps under the inclusion $R^\#(U) \subset R^\#(U')$ to the intersection cycle $[\text{cyc}'] \in H_W(R^\#(U'), Z)$ However, a close check of signs shows that with the standard orientations for the choice $o = \text{orientation of } (H^1(M, R), H^2(M, R), H_1(S_1, R))$ the stabilization process gives a sign change of $(-1)^{\dim U(n)}$ under the process of one stabilization.

Again, let $U$ be the closed surface obtained by collapsing each component of the boundary of $U$ individually to a point and $u$ denote the genus of $U$.

Hence, defining

$$\text{proper}[\text{cyc}] = (-1)^{\dim U(n) - u} [\text{cyc}] \in H_W(R^\#(U), Z)$$

yields a cycle which is natural for the stabilization process. This cycle is in dimension

$$W = (\dim U(n))T = (\dim U(n)) (\chi(S_1) - \chi(M^3)).$$

In order to have universal cohomology classes on which to evaluate these cycles, one turns to the work of Atiyah and Bott on Yang-Mills gauge theory in the context of Riemann surfaces. There they consider the space of $U(n)$ representations,

$$R^\#_{U(n)}(\Sigma) = \text{Hom}(\pi_1(\Sigma), U(n)),$$

of the fundamental group of a closed Riemann surface of genus $g$ and compute the rational homology of the quotient by conjugation,

$$\mathcal{R}_{U(n)}(\Sigma) = R^\#_{U(n)}(\Sigma)/\text{conjugation}.$$

The classifying space for flat $U(n)$ connections is the classifying space $BG_{U(n)}$, with $\mathcal{G}_{U(n)}$ the gauge group of continuous mappings,

$$\mathcal{G}_{U(n)} = \text{Map}(\Sigma, U(n));$$

so there are natural mappings

$$\mathcal{R}_{U(n)}(\Sigma) \to B\text{Map}(\Sigma, U(n)) = BG_{U(n)}$$

which stabilize well under increasing genus.

As it happens, there are universal classes for $H^\ast(B\text{Map}(\Sigma, U(n)), Q) : 2g$ exterior generators each in dimensions $1, 3, \cdots, 2n-1$ [corresponding to the first homology]; one polynomial generator each in
dimensions 2, 4, · · · , 2n and one polynomial generator in dimensions 2, 4, · · · , 2n − 2 [corresponding to second homology]; one polynomial generator each in dimensions 2, 4, · · · , 2n [corresponding to the zero-th homology]. Altogether this is a product of free exterior and polynomial generators. The generators are labeled by the chosen basis for \( H^1(\Sigma, Z), H_2(\Sigma, Z) = Z, H^0(\Sigma, Z) = Z \).

Under increasing genus these generators map by restriction to corresponding generators [or to zero if that part of the homology dies in the restriction]. Hence, one may potentially form universal classes which when evaluated on the proper cycles, \([\text{cyc}], \text{give invariants independent of the handlebody chosen in computing them.} \]

Naturally, there are corresponding statements for the case \( SU(n) \).

Since Atiyah and Bott’s work \([3]\) dealt with representations of fundamental groups of closed Riemann surfaces, we must restrict the chosen representation \( \phi : \pi_1(\Sigma, p) \to U(n) \) to have the property that it sends each boundary component of the subsurface \( S_1 \) to the identity. Then the intersection cycle will lie in the smaller space of representations which are restrictions of the fundamental group of \( \hat{S}_1 \), the closed Riemann surface obtained by filling in the boundary disks.

In this manner, for each of Atiyah and Bott’s universal characteristic classes \([3]\), there is an associated invariant. This is parallel to the 4-manifold invariants of Donaldson \([21]\).

Two significant issues remain in the \( U(n) \) case. The cycles defined here lie in \( R^\#(\hat{U}) \) for the associated closed surface \( \hat{U} \) once the above restriction on \( \phi \) is adopted. However, in the paper of Atiyah and Bott \([3]\) the classes factor through \( R(\hat{U}) \). Hence, in the universal example, the gauge group \( G_{U(n)} \) is to be replaced by the smaller group \( G'_{U(n)} = \text{Map}'(\Sigma, U(n)) \) of maps which are pointed, i.e., mappings which send a fixed base point to the identity. Atiyah and Bott show that the integral cohomology of \( BG_{U(n)} \) is the tensor product of the integral cohomology of \( BU(n) \) with the integral cohomology of \( G'_{U(n)} \).

The cohomology of the classifying space \( BG'_{U(n)} \) lacks the generators one each in dimensions 2, 4, · · · , 2n corresponding to the zero-dimensional generator \( H^0(\Sigma, Z) = Z \).

Secondly, since the mapping to the universal space utilizes a choice of basis for the homology \( H^1(\hat{U}, Z) \) to identify the generators, in order to produce true invariants independent of these choices, it is necessary to look for elements of the algebra invariant under the automorphisms of \( H^1(\Sigma, Z) \) endowed with its cup pairing, i.e., invariant under the natural action of the discrete symplectic group \( Sp(2g, Z) \).

Consequently, each of the 2g exterior generators in dimension \( 2k − 1 \) contribute but one invariant class, while all the polynomial generators survive. Passing to the limit, the universal symplectically invariant classes form an algebra with: polynomial generators in dimensions 2, 6, 10, 2(2n − 1) [from the symplectic form on the odd generators], two polynomial generators in each even dimension \( j \) from the polynomial classes corresponding to \( H^2(\Sigma, Z) \).

These classes stabilize to give two polynomial generators \( x_i, y_i \) with \( x_i \) in dimension \( 2i \) and \( y_i \) in dimension \( 4j − 2 \), the second from the symplectically invariant classes.

In consequence, for any monomial

\[
X_{I,J} = (x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_a}^{r_a})(y_{j_1}^{s_1} y_{j_2}^{s_2} \cdots y_{j_b}^{s_b})
\]

with

\[
T = \chi(S_1) - \chi(M^3) = \left( \Sigma_{p=1}^{a}(2i_p)(r_p) + \Sigma_{p=1}^{b}(4j_p - 2)(s_p) \right)
\]
there is an invariant \( \lambda_{U(n),I,J}(M^3, S_1, \phi) \) counting with signs and multiplicities the number of \((\dim U(n)) \times T\) parameter families of representations which extend to \(M^3\) the irreducible representation \(\phi\), where \(\phi\) is chosen to send each boundary component of \(S_1\) to \(Id\).

These may be codified as a single homogeneous polynomial

\[
\Lambda_{U(n)}(M, S_1, o) = \Sigma_{I,J} \lambda_{U(n),I,J}(M, S_1, o) \cdot X_{I,J}
\]

in a manner reminiscent of the work of Donaldson on 4-manifolds [21].

For the \(SU(n)\) case, the cohomology of the classifying space for \(BG_{SU(n)}\) is a exterior algebra on classes in dimensions 3, 5, \(\cdots\), 2n-1 and polynomial algebras on generators in dimensions 4, 6, \(\cdots\), 2n and also 4, 6, \(\cdots\), 2n-2.

Hence, deleting the polynomial invariant arising from \(H^0(\hat{U}, Z)\) and replacing the exterior generators by the symplectically invariant ones, we arrive at polynomial generators \(x_i\) in dimension 2i for \(i > 1\) and polynomial generators \(y_i\) in dimensions 4j-2 for \(j > 1\).

The corresponding multi-index invariants for \(SU(n)\), \(\lambda_{SU(n),I,J}(M^3, S_1, \phi)\), are indexed by multi-indices (possibly vacuous) \(I = [(i_1, r_1), (i_2, r_2), \cdots, (i_a, r_a)], J = [(j_1, s_1), (j_2, s_2), \cdots, (j_b, s_b)]\) of non-negative indices with \(i_1 < i_2 < \cdots < i_a\) and \(j_1 < j_2 < \cdots < j_b\) subject to the added constraints \(i_1 > 1\) if \(I\) is present and \(j_1 > 1\) if \(J\) is present. Evaluating these on \([\text{cyc}]\) yields the desired \(SU(n)\) invariants: For

\[
T = \chi(S_1) - \chi(M^3) = ( \Sigma_{p=1}^a (2i_p)(r_p) + \Sigma_{p=1}^b (4j_p - 2)(s_p) )
\]

there is an invariant \(\lambda_{SU(n),I,J}(M^3, S_1, \phi)\) counting with signs and multiplicities the number of \((\dim SU(n)) \times T\) parameter families of representations which extend to \(M^3\) the irreducible representation \(\phi\), where \(\phi\) is chosen to send each boundary component of \(S_1\) to \(Id\).

All these may be recorded by a single homogeneous polynomial invariant

\[
\Lambda_{SU(n)}(M, S_1, o) = \Sigma_{I,J} \lambda_{SU(n),I,J}(M, S_1, o) \cdot X_{I,J}
\]

with the sum over multi-indices \(I, J\) as above with the added constraints \(i_1 > 1\) if \(I\) present and if \(j_1 > 1\) if \(J\) present.

With these choices theorems [33] [34] holds.

**Example 6.1.** Non-trivial homogeneous polynomial invariants.

A simple example in which these polynomial invariants are non-vanishing is provided as follows:

Let \(S\) be a Riemann surface of genus \(g\) with one boundary component and \(M^3 = S \times [0, 1]\). Now choose a simple closed curve, say \(\gamma\), which separates \(S \times 0\) into a genus \(h\) subsurface, say \(S_1\), with single boundary component \(\gamma\) and remaining genus \(g-h\) subsurface, say \(T\). Now suppose that the genus of \(S_1\) is at least two, and an irreducible representation \(\phi : \pi_1(S_1) \to U(n)\) is chosen, as may be done, so that \(\phi\) sends the boundary component \(\gamma\) to the identity. In the standard way, \(\pi_1(S_1)\) is a free group on generators, say \(\{a_j, b_j \mid j = 1, \cdots, h\}\), with \(\gamma\) represented by the product of commutators \(\prod_{j=1}^h [a_j, b_j]\). Then \(\pi_1(S \times 0)\) may be presented as a free group of these \(2h\) generators plus \(2(g-h)\) more, say \(\{a_j, b_j \mid j = h+1, \cdots, g\}\), where the boundary of \(S \times 0\) is represented by the product of commutators \(\prod_{j=1}^g [a_j, b_j]\). Since the fundamental groups of \(S \times 0\) and \(M^3 = S \times [0, 1]\) are isomorphic, the space of representations of \(\pi_1(M^3)\) into \(U(n)\) is just the product \(R^\#(M) = \times_{k=1}^{2g} U(n)\) and the space of of representations into \(U(n)\) restricting to \(\phi\) is just the product \(R^\#(M, S_1, \phi) = \times_{k=2h+1}^{2g} U(n) = U(n)^{2(g-h)}\). The cohomology class \(y_j\) pulls back to this product as a sum of terms \(\Sigma_{k=1}^{g-h} \pi_j^*(a_j \otimes b_j)\) where \(\pi_j\) is the projection to the \(j^{th}\) pair \(U(n) \times U(n)\) in \(U(n)^{2(g-h)} = (U(n) \times U(n))^{g-h}\) and \(a_j \otimes b_j\) is the class in \(H^{2j-1}(U(n), z) \otimes H^{2j-1}(U(n), Z)\).
representing the product of the primitive classes, here \(a_j, b_j\), in these dimensions and factors. The product of the \(a_j\)'s over \(j = 1, \cdots, n\) evaluates the fundamental class of \(U(n)\) to one as does the product of the \(b_j\)'s. Consequently, the characteristic class \((\prod_{j=1}^n y_j)^{g-h}\) in dimension \(2(g - h) \cdot (\dim U(n))\) evaluates the cycle \(U(n)^{2(g-h)}\) giving \(((g - h)!)^n\) up to sign.

Similarly for \(SU(n)\), evaluating the characteristic class \((\prod_{j=2}^n y_j)^{g-h}\) in dimension \(2(g - h) \cdot (\dim SU(n))\) on the corresponding cycle \(SU(n)^{h-g}\) yields up to sign \(((g - h)!)^{n-1}\).

Questions about extending various kinds of representations arise naturally in low dimensional topology and in group theory, e.g., \cite{29, 17, 18, 19, 20, 6}. It is tempting to wonder if the present results can be profitably generalized to the combinatorial group theory setting.

7. Gauge Theoretic Reformulation

It’s natural to reformulate the present \(U(n)\) (respectively, \(SU(n)\)) representation-theoretic numerical invariants in terms of gauge theory of flat \(U(n)\) (resp., \(SU(n)\)) connections, in analogy to the work of Taubes on \(SU(2)\) flat connections \cite{39}. This can be carried out by utilizing the \(\mathbb{Z}/2\mathbb{Z}\) valued spectral flow invariant in the setting of real bounded Fredholm operators of index 0 acting on a separable Hilbert space, see \cite{39} page 571 or generally the ‘real K theory’, denoted \(KR(X)\), of Atiyah, Patodi, and Singer, or the direct analysis of Koschorke \cite{4, 2, 31}, in place of the \(Z\) valued spectral flow invariant of Atiyah, Patodi, and Singer \cite{5, 12} in the setting of self-adjoint operators utilized by Taubes. This will be discussed in a future paper.

Such a reformulation offers the possibility of defining Floer type homology groups \cite{22, 23, 24} in these \(U(n)\) and \(SU(n)\) settings for manifolds with boundary. However, carrying out a gauge-theoretic reformulation for the more general homogeneous polynomial invariants introduced here would be more challenging.

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