ON COHOMOLOGICAL DECOMPOSABILITY OF ALMOST–KÄHLER STRUCTURES

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Abstract. We study the $J$-invariant and $J$-anti-invariant cohomological subgroups of the de Rham cohomology of a compact manifold $M$ endowed with an almost-Kähler structure $(J, \omega, g)$. In particular, almost-Kähler manifolds satisfying a Lefschetz type property and solvmanifolds endowed with left-invariant almost-complex structures are investigated.

INTRODUCTION

Cohomological properties of compact complex, and, more generally, almost-complex, manifolds have been recently studied by many authors; see, e.g., [3], respectively [11,12], and the references therein. The study of the cohomology of almost-complex manifolds is motivated, in particular, by a question of Donaldson, [10, Question 2], relating the tamed and compatible symplectic cones of a compact 4-dimensional almost-complex manifold (see, e.g., [20]), and by the analogous question arising for compact higher-dimensional complex manifolds (see [20, page 678] and [26, Question 1.7]). (We recall that a symplectic structure $\omega$ on a manifold $M$ is said to tame an almost-complex structure $J$ if $\omega_x(u_x, J_x u_x) > 0$ for any $x \in M$ and for any $u \in T_x M \setminus \{0\}$, and it is said to be compatible with $J$ if $g := \omega(\cdot, J \cdot)$ is a $J$-Hermitian metric; in the latter case, the triple $(J, \omega, g)$ is called an almost-Kähler structure on $M$.)

Following T.-J. Li and the third author, [20], an almost-complex structure $J$ on a $2n$-dimensional manifold $M$ is called $C^\infty$-pure-and-full if

$$H^2_{dR}(M; \mathbb{R}) = H^{(1,1)}_J(M)_{\mathbb{R}} \oplus H^{(2,0),(0,2)}_J(M)_{\mathbb{R}},$$

where $H^{(1,1)}_J(M)_{\mathbb{R}}$ and $H^{(2,0),(0,2)}_J(M)_{\mathbb{R}}$ denote the subgroups of $H^2_{dR}(M; \mathbb{R})$ whose elements can be represented by forms of types $(1,1)$ and $(2,0)+(0,2)$ respectively. In the notation of T. Drăghici, T.-J. Li, and the third author, [11], $H^{(1,1)}_J(M)_{\mathbb{R}} =: H^+_J(M)$ and $H^{(2,0),(0,2)}_J(M)_{\mathbb{R}} =: H^-_J(M)$ are the $J$-invariant and $J$-anti-invariant cohomology subgroups respectively.

In [11, Theorem 2.3], Drăghici, Li, and the third author proved that every almost-complex structure on a compact 4-dimensional manifold is $C^\infty$-pure-and-full. This
is no longer true in dimensions greater than four; see, e.g., \([15, \text{Example 3.3}]\); see also \([12]\).

The groups \(H^j_j(M)\) and \(H^{2,0} (M)\) appear as a natural generalization of the Dolbeault cohomology to the non-integrable case; see, e.g., \([20, \text{Proposition 2.1}]\). In fact, compact Kähler manifolds are \(C^\infty\)-pure-and-full, and, in this case, \(H^j_j(M) \approx H^j_j(M) \cap H^{2} (M)\) and \(H^{2,0} (M) \approx \left( H^{2,0} (M) \oplus H^{0,2} (M) \right) \cap H^{2} (M)\).

We remark that on a compact complex manifold, other cohomologies can be defined, namely, the Bott-Chern and Aeppli cohomologies. In \([3]\), the problem of cohomology decomposition in terms of the Bott-Chern cohomology groups is investigated, providing in particular a characterization of compact complex manifolds satisfying the \(\partial \bar{\partial}\)-Lemma.

With compact Kähler manifolds being \(C^\infty\)-pure-and-full, in this paper we are interested in the study of the cohomological subgroups \(H^j_j(M)\) and \(H^{2,0} (M)\) for almost-Kähler manifolds.

On the one hand, A. Fino and the second author, \([15, \text{Proposition 3.2}]\), as well as Drăghici, Li, and the third author, \([11, \text{Proposition 2.8}]\), proved that the almost-complex structure of a compact almost-Kähler manifold is \(C^\infty\)-pure. On the other hand, we prove the following result, therefore showing a difference between the integrable and the non-integrable cases.

**Proposition 4.1** Let \(X := \mathbb{Z}[i]^3 \setminus (\mathbb{C}^3, \ast)\) be the real manifold underlying the Iwasawa manifold. Then there exists an almost-Kähler structure \((J, \omega, g)\) on \(X\) which is \(C^\infty\)-pure and \(C^\infty\)-full. Furthermore, the Lefschetz type operator \(\mathcal{L}_\omega : \omega \wedge \cdot : \wedge^2 M \to \wedge^4 M\) of the almost-Kähler structure \((J, \omega, g)\) does not take \(g\)-harmonic 2-forms to \(g\)-harmonic 4-forms.

In studying cohomological decomposition of the de Rham cohomology of almost-Kähler manifolds, the third author introduced a *Lefschetz type property* for 2-forms; see Definition \([2, \text{Definition 2.2}]\). Such a property is stronger than the Hard Lefschetz Condition on 2-classes, namely, the property that \(\omega^n - 2 \cdot \cdot : H^2_d (M) \to H^{2n} (M)\) is an isomorphism, where \(2n := \dim M\).

We study such a Lefschetz type property on almost-Kähler manifolds \((M, J, \omega, g)\) in relation to the existence of a cohomological decomposition of \(H^2_d (M)\). More precisely, we prove the following result.

**Theorem 2.3.** Let \((M, J, \omega, g)\) be a compact almost-Kähler manifold. Suppose that there exists a basis of \(H^2_d (X)\) represented by \(g\)-harmonic 2-forms which are of pure type with respect to \(J\). Then the Lefschetz type property on 2-forms is satisfied.

Note that by the hypothesis, it follows, in particular, that \(J\) is \(C^\infty\)-pure-and-full and \(C^\infty\)-pure-and-full, \([15, \text{Theorem 3.7}]\). Note also that Fino and the second author provided in \([15]\) several examples of compact non-Kähler solvmanifolds admitting a basis of harmonic representatives of pure type with respect to the almost-complex structure. In \([13, \text{[2]}]\), Drăghici, Li, and the third author asked whether such a Lefschetz type property on 2-forms is actually equivalent to \(C^\infty\)-fullness for every almost-Kähler nilmanifold and solvmanifold, without any further assumption. Theorem \([2,3]\) and Proposition \([11]\) provide results and examples in favour of a possibly positive answer to their question.
In [12] Theorem 1.1, starting with a compact complex surface \((M, J)\), it is shown that the dimension \(h^J\) of the \(J\)-anti-invariant cohomology subgroup \(H^J(M)\) of any metric related almost-complex structure \(\tilde{J}\) on \(M\) (namely, an almost-complex structure \(\tilde{J}\) on \(M\) inducing the same orientation as the one induced by \(J\) and with a common compatible metric), such that \(\tilde{J} \neq \pm J\), can be 0, 1, or 2, and a description of such almost-complex structures \(\tilde{J}\) having \(h^\tilde{J} \in \{1, 2\}\) is provided. Furthermore, it is conjectured that \(h^\tilde{J} = 0\) for a generic almost-complex structure \(J\) on a compact 4-dimensional manifold, and that if \(h^\tilde{J} \geq 3\), then \(J\) is integrable, [12] Conjecture 2.4, Conjecture 2.5]. One could set a similar question for higher-dimensional manifolds, asking Question 5.2: are there examples of non-integrable almost-complex structures \(J\) on a compact \(2n\)-dimensional manifold with \(h^J > n(n - 1)\)?

Finally, we prove a Nomizu type result for the subgroups \(H^J(M)\) of completely-solvable solvmanifolds \(M = \Gamma\backslash G\) endowed with left-invariant almost-complex structures \(J\). More precisely, denote the Lie algebra associated to \(G\) by \(\mathfrak{g}\), and consider

\[
H^J_{\{p,q\}}(\mathfrak{g}; \mathbb{R}) := \{ a = [\alpha] \in H^\bullet(\wedge^\bullet \mathfrak{g}^*; d) : \alpha \in \wedge^\{p,q\}(\mathfrak{g}; \mathbb{R}) \subseteq H^\bullet_{dR}(M; \mathbb{R}) \}
\]

the subgroup of \(H^\bullet_{dR}(M; \mathbb{R})\) that consists of classes admitting a left-invariant representative of type \((p, q) + (q, p)\), where

\[
\wedge^\{p,q\}(\mathfrak{g}; \mathbb{R}) := (\wedge^{p,q}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^* \oplus \wedge^{q,p}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*) \cap \wedge^\bullet \mathfrak{g}^*.
\]

Then the following result holds.

**Theorem 5.4.** Let \(M = \Gamma\backslash G\) be a solvmanifold endowed with a left-invariant almost-complex structure \(J\), and denote the Lie algebra naturally associated to \(G\) by \(\mathfrak{g}\). For any \(p, q \in \mathbb{N}\), the map \(j : H^\{p,q\}(\mathfrak{g}; \mathbb{R}) \to H^\{p,q\}(\mathfrak{g}; \mathbb{R}) \mathbb{R}\) induced by left-translations is injective, and if \(H^\bullet_{dR}(\wedge^\bullet \mathfrak{g}^*; d) \simeq H^\bullet_{dR}(M; \mathbb{R})\) (for instance, if \(M\) is a completely-solvable solvmanifold), then \(j : H^\{p,q\}(\mathfrak{g}; \mathbb{R}) \to H^\{p,q\}(\mathfrak{g}; \mathbb{R}) \mathbb{R}\) is in fact an isomorphism.

In particular, it follows that \(\dim_{\mathbb{R}} H^J(M) \leq n(n - 1)\) for every left-invariant almost-complex structure on a completely-solvable solvmanifold.

1. **\(C^\infty\)-pure-and-full almost-complex structures**

1.1. **Subgroups of the de Rham cohomology of almost-complex manifolds.** We start by fixing some notation and recalling some recent results on cohomological properties of almost-complex manifolds; for more details see, e.g., [12][11][13][15][20] and the references therein.

Let \(J\) be a smooth almost-complex structure on a compact \(2n\)-dimensional manifold \(M\). Denote by \(\wedge^r M\) the bundle of \(r\)-forms on \(M\); we denote with the same symbol \(\wedge^r M := \Gamma(M, \wedge^r M)\) the space of smooth global sections of the bundle \(\wedge^r M\). Then \(J\) extends to a complex automorphism of \(T^CM = TM \otimes \mathbb{C}\) such that \(T^CM = T^{1,0}J \oplus T^{0,1}M\), where \(T^{1,0}J\) and \(T^{0,1}M\) are the \((\pm 1)\)-eigenbundles. The action of \(J\) can be extended to the space \(\wedge^r(M; \mathbb{C})\) of smooth global sections of the bundle \(\wedge^r(M; \mathbb{C}) := \wedge^r M \otimes \mathbb{C}\), getting the following decomposition:

\[
\wedge^r(M; \mathbb{C}) = \bigoplus_{p+q=r} \wedge^{p,q}_J M.
\]
Then the space $\wedge^r M$ of real smooth differential $r$-forms decomposes as

$$\wedge^r M = \bigoplus_{p+q=r, \ p \leq q} \wedge^r_{p,q} (M)_{\mathbb{R}},$$

where, for $p < q$ (later on, we do not distinguish the cases $p < q$ and $p = q$),

$$\wedge^r_{p,q} (M)_{\mathbb{R}} := \{ \alpha \in \wedge^p_{p,q} M \oplus \wedge^q_{p,q} M : \alpha = \alpha \},$$

$$\wedge^r_{p,p} (M)_{\mathbb{R}} := \{ \alpha \in \wedge^p_{p,p} M : \alpha = \alpha \}.$$

In particular, for $r = 2$, we will adopt the following notation:

$$\wedge^1_{1,1} (M)_{\mathbb{R}} := \wedge_+^1 M, \quad \wedge^1_{2,0},(0,2) (M)_{\mathbb{R}} := \wedge^-_M;$$

this is consistent with the decomposition in invariant and anti-invariant parts of $\wedge^2 M$ under the natural action of $J$ on $\wedge^2 M$, given by $J \alpha (\cdot, \cdot) := \alpha (J \cdot, J \cdot)$.

We will refer to forms in $\wedge^1_{1,1} (M)_{\mathbb{R}}$, respectively $\wedge^1_{2,0},(0,2) (M)_{\mathbb{R}}$, as forms of pure type with respect to $J$.

For a finite set $S$ of pairs of integers, let

$$Z^S_j := \bigoplus_{(p,q) \in S, \ p \leq q} Z^j_{p,q}, \quad B^S_j := \bigoplus_{(p,q) \in S, \ p \leq q} B^j_{p,q},$$

where

$$Z^j_{p,q} := \{ \alpha \in \wedge^j_{p,q} (M)_{\mathbb{R}} : d\alpha = 0 \},$$

$$B^j_{p,q} := \{ \beta \in \wedge^j_{p,q} (M)_{\mathbb{R}} : \text{there exists } \gamma \text{ such that } d\gamma = \beta \}.$$

Define

$$H^S_j (M)_{\mathbb{R}} := \frac{Z^S_j}{B^S_j}.$$

Let $B$ be the space of $d$-exact forms. Since $\frac{Z^S_j}{B^S_j} = \frac{Z^S_j}{B^S_j}$, a natural inclusion $\rho_S : \frac{Z^S_j}{B^S_j} \to \frac{Z^S_j}{B^S_j}$ is defined. As in [20], we will write $\rho_S (\frac{Z^S_j}{B^S_j})$ simply as $\frac{Z^S_j}{B^S_j}$, and consequently the cohomology spaces $H^S_j (M)_{\mathbb{R}}$ can be identified as

$$H^S_j (M)_{\mathbb{R}} = \{ [\alpha] \in H^j_{dR} (M; \mathbb{R}) : \alpha \in Z^S_j \} = \frac{Z^S_j}{B^S_j}.$$

Therefore, there is a natural inclusion

$$H^1_{j,1}(M)_{\mathbb{R}} + H^1_{j,2},(0,2) (M)_{\mathbb{R}} \subseteq H^2_{dR}(M; \mathbb{R}).$$

1.2. $C^\infty$-pure-and-full and pure-and-full almost-complex structures. As in [20], we set the following definition.

**Definition 1.1** ([20] Definition 2.2, Definition 2.3, Lemma 2.2]). An almost-complex structure $J$ on a manifold $M$ is said to be

- $C^\infty$-pure if $H^1_{j,1}(M)_{\mathbb{R}} \cap H^1_{j,2},(0,2) (M)_{\mathbb{R}} = \{ 0 \}$,
- $C^\infty$-full if $H^2_{dR}(M; \mathbb{R}) = H^1_{j,1}(M)_{\mathbb{R}} + H^1_{j,2},(0,2) (M)_{\mathbb{R}}$,
- $C^\infty$-pure-and-full if

$$H^2_{dR}(M; \mathbb{R}) = H^1_{j,1}(M)_{\mathbb{R}} \oplus H^1_{j,2},(0,2) (M)_{\mathbb{R}}.$$
According to the previous notation, we will write\[H^+_J(M) := H^{(1,1)}_J(M)_\mathbb{R}, \quad H^-_J(M) := H^{(2,0),(0,2)}_J(M)_\mathbb{R}.\]

Similar definitions in terms of currents can be given, introducing the notion of pure-and-full almost-complex structure; we refer to [20] §2.2.2 for further details and results. More precisely, on an almost complex manifold \((M, J)\), the space \(\mathcal{E}_k(M)_\mathbb{R}\) of real \(k\)-currents has a decomposition \(\mathcal{E}_k(M)_\mathbb{R} = \bigoplus_{p+q=k} \mathcal{E}^J_{(p,q),(q,p)}(M)_\mathbb{R}\), where \(\mathcal{E}^J_{(p,q),(q,p)}(M)_\mathbb{R}\) denotes the space of real \(k\)-currents of bi-dimension \((p, q) + (q, p)\). 

Let \(Z^J_{(2,0),(0,2)}\) and \(Z^J_{(1,1)}\) denote the spaces of real \(d\)-closed currents of bi-dimension \((2, 0) + (0, 2)\), respectively \((1, 1)\), and \(B^J_{(2,0),(0,2)}\) and \(B^J_{(1,1)}\) denote the spaces of real \(d\)-exact currents of bi-dimension \((2, 0) + (0, 2)\), respectively \((1, 1)\). Denote by \(B\) the space of boundaries. Let, as in [20],

\[
    H^J_{(1,1)}(M)_\mathbb{R} := \left\{ \alpha \in H_2(M; \mathbb{R}) : \alpha \in Z^J_{(1,1)} \right\} = \frac{Z^J_{(1,1)}}{B}, \\
    H^J_{(2,0),(0,2)}(M)_\mathbb{R} := \left\{ \alpha \in H_2(M; \mathbb{R}) : \alpha \in Z^J_{(2,0),(0,2)} \right\} = \frac{Z^J_{(2,0),(0,2)}}{B}.
\]

We recall the following definition.

**Definition 1.2** ([20] Definition 2.15, Definition 2.16]). An almost complex structure \(J\) on a manifold \(M\) is said to be pure if \(H^J_{(1,1)}(M)_\mathbb{R} \cap H^J_{(2,0),(0,2)}(M)_\mathbb{R} = \{0\}\). It is said to be full if \(H_2(M; \mathbb{R}) = H^J_{(1,1)}(M)_\mathbb{R} + H^J_{(2,0),(0,2)}(M)_\mathbb{R}\). Therefore, an almost complex structure \(J\) is pure-and-full if and only if

\[
    H_2(M; \mathbb{R}) = H^J_{(1,1)}(M)_\mathbb{R} \oplus H^J_{(2,0),(0,2)}(M)_\mathbb{R}.
\]

In [20] Proposition 2.1] it is shown that, given a compact complex manifold \((M, J)\) of complex dimension \(n\), if \(n = 2\) or \(J\) is Kähler, then \(J\) is \(C^\infty\)-pure-and-full, and

\[
    H^J_{(1,1)}(M)_\mathbb{R} \simeq H^1_{\partial\bar{\partial}}(M) \cap H^2_{dR}(M; \mathbb{R})
\]

and

\[
    H^J_{(2,0),(0,2)}(M)_\mathbb{R} \simeq \left( H^2_{\partial\bar{\partial}}(M) \oplus H^0_{\partial\bar{\partial}}(M) \right) \cap H^2_{dR}(M; \mathbb{R}).
\]

In view of this result the subgroups \(H^J_{(1,1)}(M)_\mathbb{R}\) and \(H^J_{(2,0),(0,2)}(M)_\mathbb{R}\) of the de Rham cohomology can be viewed as an analogue of the Dolbeault cohomology groups for non-integrable almost-complex structures.

In [11] Theorem 2.3] the following result is proven.

**Theorem 1.3** ([11] Theorem 2.3]). If \(M\) is a compact manifold of dimension 4, then any almost-complex structure \(J\) on \(M\) is \(C^\infty\)-pure-and-full.

This is no longer true in dimensions higher than 4: in [15] Example 3.3], a compact non-\(C^\infty\)-pure almost-complex structure on a 6-dimensional nilmanifold is constructed. Therefore, the previous theorem can be considered as a sort of Hodge decomposition theorem in the non-Kähler case.
2. Cohomological properties of almost-Kähler manifolds

2.1. Lefschetz type property on almost-Kähler manifolds with pure type harmonic representatives. Given a compact $2n$-dimensional almost-Kähler manifold $(M, J, \omega, g)$, we are interested in studying the property of being $C^\infty$-pure-and-full.

First we recall the following result.

Proposition 2.1 ([11, Proposition 2.8], [15, Proposition 3.2]). If $J$ is an almost-complex structure on a compact manifold $M$ and $J$ admits a compatible symplectic structure, then $J$ is $C^\infty$-pure.

Furthermore, Fino and the second author proved that an almost-Kähler manifold admitting a basis of harmonic 2-forms whose elements are of pure type with respect to the almost-complex structure is $C^\infty$-pure-and-full and pure-and-full, [15, Theorem 3.7]. They also provided several examples of compact non-Kähler solvmanifolds satisfying the above assumption in [15].

For the purpose of studying the property of being $C^\infty$-pure-and-full on almost-Kähler manifolds, we recall the following definition.

Definition 2.2. Given a compact $2n$-dimensional symplectic manifold $(M, \omega)$, denote by $L_\omega: \wedge^2 M \to \wedge^{2n-2} M$, $L_\omega(\alpha) := \omega^{n-2} \wedge \alpha$ the Lefschetz type operator (on 2-forms) associated with $\omega$.

Then one says that the compact $2n$-dimensional almost-Kähler manifold $(M, J, \omega, g)$ satisfies the Lefschetz type property (on 2-forms) if $L_\omega$ takes $g$-harmonic 2-forms to $g$-harmonic $(2n-2)$-forms.

Furthermore, we recall some notions and results from [6, 22, 27]; see also [7, 23]. Let $(M, \omega)$ be a compact $2n$-dimensional symplectic manifold. Extend $\omega^{-1}: T^*M \to TM$ to the whole exterior algebra of $T^*M$. For any $k \in \mathbb{N}$, the symplectic $\star_\omega$ operator is defined as

$$\star_\omega: \wedge^k M \to \wedge^{2n-k} M, \quad \beta \wedge \star_\omega \alpha = \omega^{-1}(\alpha, \beta) \frac{\omega^n}{n!}, \quad \forall \alpha, \beta \in \wedge^k M.$$ 

One can prove that $\star_\omega^2 = \text{id}_{\wedge \bullet M}$ [6, Lemma 2.1.2].

For any $k \in \mathbb{N}$, define the symplectic co-differential operator

$$\delta_\omega: \wedge^k M \to \wedge^{k-1} M, \quad \delta_\omega(\wedge^k M) := (-1)^{k+1} \star_\omega d \star_\omega;$$

this operator has been studied by J.-L. Brylinski in [6] for Poisson manifolds. In the context of generalized complex geometry (see, e.g., [16]), it can be interpreted as the symplectic counterpart of the operator $d^c := -i (\partial - \overline{\partial})$ in complex geometry; see [7].

By definition, $(M, \omega)$ satisfies the Hard Lefschetz Condition if, for each $k \in \mathbb{N}$, the map

$$[\omega]^k \hookrightarrow: H^{n-k}_{\text{dR}}(M; \mathbb{R}) \to H^{n+k}_{\text{dR}}(M; \mathbb{R})$$

is an isomorphism. O. Mathieu [22, Corollary 2] and, independently, D. Yan [27, Theorem 0.1] proved that, given a compact symplectic manifold $(M, \omega)$, any de Rham cohomology class has a (possibly non-unique) $\omega$-symplectically harmonic representative (that is, a $d$-closed $\delta_\omega$-closed representative) if and only if the Hard Lefschetz Condition holds.

We can now prove the following result.
Theorem 2.3. Let \((M, J, \omega, g)\) be a compact almost-Kähler manifold. Suppose that there exists a basis of \(H^2_{dR}(X; \mathbb{R})\) represented by \(g\)-harmonic 2-forms which are of pure type with respect to \(J\). Then the Lefschetz type property on 2-forms is satisfied.

Proof. Recall that on a \(2n\)-dimensional almost-Kähler manifold \((M, J, \omega, g)\) the Hodge \(*_g\) operator and the symplectic \(*_\omega\) operator are related by \(*_\omega = *_g J\) [6 Theorem 2.4.1, Remark 2.4.4]. Therefore, for forms of pure type with respect to \(J\), the properties of being \(g\)-harmonic and of being \(\omega\)-symplectically harmonic are equivalent. The theorem follows noting that [27 Lemma 1.2] \([\mathcal{L}_\omega, d] = 0\) and \([\mathcal{L}_\omega, \delta_\omega] = d\); hence \(\mathcal{L}_\omega\) sends \(\omega\)-symplectically harmonic 2-forms (of pure type with respect to \(J\)) to \(\omega\)-symplectically harmonic \((2n-2)\)-forms (of pure type with respect to \(J\)).

Remark 2.4. We note that if \((M, J, \omega, g)\) is a compact \(2n\)-dimensional almost-Kähler manifold satisfying the Lefschetz type property on 2-forms and \(J\) is \(C^\infty\)-full, then \(J\) is \(C^\infty\)-pure-and-full and \(C^\infty\)-full-and-full.

Indeed, we have already remarked that \(J\) is \(C^\infty\)-pure; see Proposition [2.1]. Moreover, since \(J\) is \(C^\infty\)-full, \(J\) is also pure by [20] Proposition 2.5. We now recall the argument in [15] to prove that \(J\) is also full.

Firstly, note that if the Lefschetz type property on 2-forms holds, then

\[
[\omega^{n-2}] = : H^2_{dR}(M; \mathbb{R}) \to H^{2n-2}_{dR}(M; \mathbb{R})
\]

is an isomorphism. Therefore, we get that

\[
H^{2n-2}_{dR}(M; \mathbb{R}) = H^{(n,n-2),(n-2,n)}_J(M) + H^{(n-1,n-1)}_J(M) \mathbb{R} ;
\]

indeed (following the argument in [15 Theorem 4.1]), since \([\omega^{n-2}] = : H^2_{dR}(M; \mathbb{R}) \to H^{2n-2}_{dR}(M; \mathbb{R})\) is in particular surjective, we have

\[
H^{2n-2}_{dR}(M; \mathbb{R}) = [\omega^{n-2}] \dashv H^2_{dR}(M; \mathbb{R})
\]

\[
= [\omega^{n-2}] \dashv \left( H^{(2,0),(0,2)}_J(M) \mathbb{R} \oplus H^{(1,1)}_J(M) \mathbb{R} \right)
\]

\[
\subseteq H^{(n,n-2),(n-2,n)}_J(M) \mathbb{R} + H^{(n-1,n-1)}_J(M) \mathbb{R} ,
\]

yielding the above decomposition of \(H^{2n-2}_{dR}(M; \mathbb{R})\). Then, it follows that \(J\) is also full; see, for example, [1 Theorem 2.1].

2.2. A family of almost-Kähler manifolds satisfying the Lefschetz type property on 2-forms. Let \(n\) be the 6-dimensional nilpotent Lie algebra whose structure equations, with respect to a basis \(\{e^j\}_{j \in \{1, \ldots, 6\}}\) of \(n^*\), are given by

\[
d e^1 = d e^2 = d e^3 = 0 , \quad d e^4 = e^{23} , \quad d e^5 = e^{13} , \quad d e^6 = e^{12}
\]

(we write \(e^{jk}\) instead of \(e^j \wedge e^k\)). Using a result of Mal’tsev [21 Theorem 7], the connected simply-connected Lie group \(G\) associated with \(n\) admits a discrete co-compact subgroup \(\Gamma\); let \(N := \Gamma \backslash G\) be the (compact) nilmanifold obtained as a quotient of \(G\) by \(\Gamma\). Note that \(N\) is not formal by a theorem of K. Hasegawa [17 Theorem 1, Corollary].
Fix $\alpha > 1$ and take
$$
\omega_\alpha := e^{14} + \alpha \cdot e^{25} + (\alpha - 1) \cdot e^{36};
$$
since $d\omega_\alpha = 0$ and $\omega_\alpha^3 \neq 0$, we get that $\omega_\alpha$ is a left-invariant symplectic form on $N$. Set
$$
J_\alpha e_1 := e_4, \quad J_\alpha e_2 := \alpha e_5, \quad J_\alpha e_3 := (\alpha - 1) e_6, \quad J_\alpha e_4 := -e_1, \quad J_\alpha e_5 := -\frac{1}{\alpha} e_2, \quad J_\alpha e_6 := -\frac{1}{\alpha - 1} e_3,
$$
where $\{e_1, \ldots, e_6\}$ denotes the global dual frame of $\{e^1, \ldots, e^6\}$ on $N$. It is immediate to check that

- setting $g_\alpha(\cdot, \cdot) := \omega_\alpha(\cdot, J_\alpha \cdot)$, the triple $(J_\alpha, \omega_\alpha, g_\alpha)$ gives rise to a family of left-invariant almost-Kähler structures on $N$;
- denoting by
  $$
  E^1_\alpha := e^1, \quad E^2_\alpha := \alpha e^2, \quad E^3_\alpha := (\alpha - 1) e^3, \\
  E^4_\alpha := e^4, \quad E^5_\alpha := e^5, \quad E^6_\alpha := e^6,
  $$
then $\{E^1_\alpha, \ldots, E^6_\alpha\}$ is a $g_\alpha$-orthonormal co-frame on $N$; with respect to this new co-frame, we easily obtain the following structure equations:
$$
d E^1_\alpha = d E^2_\alpha = d E^3_\alpha = 0, \quad d E^4_\alpha = \frac{1}{\alpha(\alpha - 1)} E^2_\alpha^{13}, \\
d E^5_\alpha = \frac{1}{\alpha - 1} E^3_\alpha^{13}, \quad d E^6_\alpha = \frac{1}{\alpha} E^2_\alpha^{12}.
$$

Then
$$
\varphi^1_\alpha := E^1_\alpha + i E^4_\alpha, \quad \varphi^2_\alpha := E^2_\alpha + i E^5_\alpha, \quad \varphi^3_\alpha := E^3_\alpha + i E^6_\alpha
$$
are $(1, 0)$-forms with respect to the almost-complex structure $J_\alpha$, and
$$
\omega_\alpha = E^1_\alpha^{14} + E^2_\alpha^{25} + E^3_\alpha^{36}.
$$

By a result of K. Nomizu [25, Theorem 1] (see Theorem 5.3), the de Rham cohomology of $N$ is straightforwardly computed:
$$
H^2_{dR}(N; \mathbb{R}) \simeq \mathbb{R} \left( E^{15}_\alpha, E^{16}_\alpha, E^{24}_\alpha, E^{26}_\alpha, E^{34}_\alpha, E^{35}_\alpha, E^{14}_\alpha, \frac{1}{\alpha} E^{25}_\alpha, \frac{1}{\alpha} E^{25}_\alpha, \frac{1}{\alpha - 1} E^{36}_\alpha \right)
$$
(where we have listed the $g_\alpha$-harmonic representatives instead of their classes).

Note that the listed $g_\alpha$-harmonic representatives of $H^2_{dR}(N; \mathbb{R})$ are of pure type with respect to $J_\alpha$: hence, the almost-complex structure $J_\alpha$ is $C^\infty$-pure-and-full by [15, Theorem 3.7]. In particular, note that
$$
H^2_{dR}(N; \mathbb{R}) \simeq \mathbb{R} \left( i \alpha \varphi^1_\alpha + i \varphi^2_\alpha, i(\alpha - 1) \varphi^2_\alpha + i \alpha \varphi^3_\alpha, \exists \varphi^1_\alpha, \exists \varphi^2_\alpha, \exists \varphi^3_\alpha \right)
$$
$$
\oplus \left( \exists \varphi^2_\alpha, \exists \varphi^3_\alpha, \exists \varphi^{23}_\alpha \right);
$$
hence $h^+_\alpha(N) = 5$ and $h^-_{J_\alpha}(N) = 3$.

Moreover, one explicitly notes that
$$
\mathcal{L}_{\omega_\alpha} E^{15}_\alpha = E^{1536}_\alpha = *_{g_\alpha} E^{24}_\alpha, \quad \mathcal{L}_{\omega_\alpha} E^{16}_\alpha = E^{1625}_\alpha = *_{g_\alpha} E^{34}_\alpha, \\
\mathcal{L}_{\omega_\alpha} E^{24}_\alpha = E^{2436}_\alpha = *_{g_\alpha} E^{15}_\alpha, \quad \mathcal{L}_{\omega_\alpha} E^{26}_\alpha = E^{2614}_\alpha = *_{g_\alpha} E^{35}_\alpha, \\
\mathcal{L}_{\omega_\alpha} E^{34}_\alpha = E^{3425}_\alpha = *_{g_\alpha} E^{16}_\alpha, \quad \mathcal{L}_{\omega_\alpha} E^{35}_\alpha = E^{3514}_\alpha = *_{g_\alpha} E^{26}_\alpha.
$$
while

\[ \mathcal{L}_{\omega_\alpha} \left( E_\alpha^{14} + \frac{1}{\alpha} E_\alpha^{25} \right) = -\frac{\alpha + 1}{\alpha} E_\alpha^{145} - \frac{1}{\alpha} E_\alpha^{2356} - E_\alpha^{1346} \]

where

\[ d \ast g_\alpha \mathcal{L}_{\omega_\alpha} \left( E_\alpha^{14} + \frac{1}{\alpha} E_\alpha^{25} \right) = d \left( -\frac{\alpha + 1}{\alpha} E_\alpha^{36} - E_\alpha^{25} - \frac{1}{\alpha} E_\alpha^{14} \right) = 0 , \]

and, by a similar computation, \( d \ast g_\alpha \mathcal{L}_{\omega_\alpha} \left( e^{25} + e^{36} \right) = 0 \). This proves explicitly that \( \omega_\alpha \) satisfies the Lefschetz type property on 2-forms.

The nilmanifold \( N \) is not formal by a theorem of Hasegawa [17, Theorem 1, Corollary]. The non-formality of \( M \) can also be proved by giving a non-zero triple Massey product on \( N \); see [19]. Since

\[ [E_\alpha^1] \sim [E_\alpha^3] = (\alpha - 1) \left[ dE_\alpha^5 \right] = 0 , \quad [E_\alpha^3] \sim [E_\alpha^2] = -\alpha \left( \alpha - 1 \right) \left[ dE_\alpha^4 \right] = 0 , \]

we get that the triple Massey product

\[ \langle [E_\alpha^1], [E_\alpha^3], [E_\alpha^2] \rangle = -\left( \alpha - 1 \right) \left[ E_\alpha^{25} + \alpha E_\alpha^{14} \right] \]

does not vanish, and hence \( N \) is not formal.

In summary, we have proven the following result.

**Proposition 2.5.** There is a non-formal 6-dimensional nilmanifold \( N \) endowed with a 1-parameter family \( \{(J_\alpha, \omega_\alpha, g_\alpha)\}_{\alpha > 1} \) of left-invariant almost-Kähler structures being \( C^\infty \)-pure-and-full and pure-and-full and satisfying the Lefschetz type property on 2-forms.

**Remark 2.6.** It has to be noted that \( \omega_\alpha \wedge \cdot : \Lambda^2 N^6 \to \Lambda^4 N^6 \) induces an isomorphism in cohomology \( [\omega_\alpha] \sim : H^2_{dR}(N, \mathbb{R}) \to H^4_{dR}(N, \mathbb{R}) \), while according to [3, Theorem A], \( [\omega_\alpha]^2 \rightsquigarrow : H^1_{dR}(N, \mathbb{R}) \to H^5_{dR}(N, \mathbb{R}) \) is not an isomorphism.

3. Almost-Kähler \( C^\infty \)-pure-and-full structures

3.1. The Nakamura manifold of completely solvable type. Take \( A \in \text{SL}(2; \mathbb{Z}) \) with two different real eigenvalues \( e^\lambda \) and \( e^{-\lambda} \) with \( \lambda > 0 \), and fix \( P \in \text{GL}(2; \mathbb{R}) \) such that \( PAP^{-1} = \text{diag}(e^\lambda, e^{-\lambda}) \). For example, take

\[ A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad P := \begin{pmatrix} \frac{1-\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}-1}{2} & 1 \end{pmatrix} , \]

and consequently \( \lambda = \log \frac{3+\sqrt{5}}{2} \). Let \( M^6 := M^6(\lambda) \) be the compact complex manifold

\[ M^6 := \mathbb{S}^1 \times \mathbb{R}^{2} \times \mathbb{T}^2_{\mathbb{C}}, (x^1, x^3, x^5, x^6) \]

where \( \mathbb{T}^2_{\mathbb{C}} \) is the 2-dimensional complex torus \( \mathbb{T}^2_{\mathbb{C}} := \frac{\mathbb{C}^2}{\mathbb{Z}^2} \) and \( T_1 \) acts on \( \mathbb{R} \times \mathbb{T}^2_{\mathbb{C}} \) as \( T_1 (x^1, x^3, x^4, x^5, x^6) := (x^1 + \lambda, e^{-\lambda} x^3, e^\lambda x^4, e^{-\lambda} x^5, e^\lambda x^6) \). The manifold \( M^6 \) can be seen as a compact quotient of a completely-solvable Lie group by a discrete co-compact subgroup [13, Example 3.1] (denote the Lie algebra naturally associated to the completely-solvable Lie group of \( M^6 \) by \( g \)). Using coordinates \( x^1 \) on \( \mathbb{S}^1 \), \( x^1 \) on \( \mathbb{R} \) and \( (x^3, x^4, x^5, x^6) \) on \( \mathbb{T}^2_{\mathbb{C}} \), we set

\[ e^1 := dx^1, \quad e^2 := dx^2, \quad e^3 := e^{x^1} dx^3, \quad e^4 := e^{-x^1} dx^4, \quad e^5 := e^{x^1} dx^5, \quad e^6 := e^{-x^1} dx^6 \]
as a basis for \( \mathfrak{g}^* \); therefore, with respect to \( \{ e^i \}_{i \in \{1, \ldots, 6\}} \), the structure equations are the following:

\[
d e^1 = d e^2 = 0, \quad d e^3 = e^{13}, \quad d e^4 = -e^{14}, \quad d e^5 = e^{15}, \quad d e^6 = -e^{16}.
\]

### 3.2. The de Rham cohomology of the Nakamura manifold.

Let \( J \) be the almost-complex structure on \( M^6 \) defined by the complex \((1,0)\)-forms given by

\[
\varphi^1 := \frac{1}{2}(e^1 + i e^2), \quad \varphi^2 := e^3 + i e^5, \quad \varphi^3 := e^4 + i e^6.
\]

It is straightforward to check that \( J \) is integrable.

With \( M^6 \) being a compact quotient of a completely-solvable Lie group, one computes the de Rham cohomology of \( M^6 \) easily by A. Hattori’s theorem [19, Corollary 4.2]; see Theorem 5.3:

\[
\begin{align*}
H^1_{dR}(M^6; \mathbb{C}) & \cong \mathbb{C} \langle \varphi^1, \bar{\varphi}^1 \rangle, \\
H^2_{dR}(M^6; \mathbb{C}) & \cong \mathbb{C} \langle \varphi^{11}, \varphi^{23}, \varphi^{32}, \varphi^{23}, \varphi^{23} \rangle, \\
H^3_{dR}(M^6; \mathbb{C}) & \cong \mathbb{C} \langle \varphi^{123}, \varphi^{132}, \varphi^{123}, \varphi^{213}, \varphi^{312}, \varphi^{231}, \varphi^{123} \rangle
\end{align*}
\]

(for the sake of clearness, we write, for example, \( \varphi^{AB} \) in place of \( \varphi^A \wedge \bar{\varphi}^B \) and we list the harmonic representatives with respect to the metric \( g := \sum_{j=1}^3 \varphi^j \circ \bar{\varphi}^j \) instead of their classes). Therefore, \( M^6 \) is geometrically formal; i.e., the product of \( g \)-harmonic forms is still \( g \)-harmonic, and therefore it is formal. That is, the de Rham complex of \( M \) is formal as a differential graded algebra; see, e.g., [9].

Furthermore, it can be easily checked that

\[
\omega := e^{12} + e^{34} + e^{56}
\]
gives rise to a symplectic structure on \( M^6 \) satisfying the Hard Lefschetz Condition. We obtain the following result.

**Proposition 3.1** ([14, Proposition 3.2]). The manifold \( M^6 \) is formal and admits a symplectic form \( \omega \) satisfying the Hard Lefschetz Condition.

Note also that \( \tilde{\omega} := \frac{1}{2} (\varphi^{11} + \varphi^{22} + \varphi^{33}) \) is not \( d \)-closed but \( d \tilde{\omega}^2 = 0 \), from which it follows that the manifold \( M^6 \) admits a balanced metric.

Moreover, since \( M^6 \) is a compact quotient of a completely-solvable Lie group, by the Hasegawa theorem (Main Theorem) we have the following result; see also [14, Theorem 3.3]. (We recall that a compact complex manifold is said to belong to class \( C \) of Fujiki if it admits a proper modification from a Kähler manifold.)

**Theorem 3.2** (Main Theorem). The manifold \( M^6 \) admits no Kähler structure and it is not in class \( C \) of Fujiki.

### 3.3. An almost-Kähler structure on the Nakamura manifold.

According to Hasegawa’s theorem (Main Theorem), any integrable complex structure on \( M^6 \) (for example, the \( J \) defined in §3.2) does not admit any symplectic structure compatible with it. Therefore, we consider the almost-complex structure \( J' \) defined by

\[
J' e^1 := -e^2, \quad J' e^3 := -e^4, \quad J' e^5 := -e^6.
\]

Considering

\[
\psi^1 := \frac{1}{2} (e^1 + i e^2), \quad \psi^2 := e^3 + i e^4, \quad \psi^3 := e^5 + i e^6
\]
as a co-frame for the space of $(1,0)$-forms on $(M^6, J')$, one can compute
\[ d\psi^1 = 0, \quad d\psi^2 = \psi^{12} + \psi^{12}, \quad d\psi^3 = \psi^{13} + \psi^{13}, \]
from which it is clear that $J'$ is not integrable. Note that the $J'$-compatible 2-form
\[ \omega' := e^{12} + e^{34} + e^{56} \]
is $d$-closed. Hence, $(M^6, J', \omega')$ is an almost-Kähler manifold.

Moreover, since any cohomology class in $\mathfrak{h}(\mathfrak{g})$, where we have listed the harmonic representatives with respect to the metric $g' := \sum_{j=1}^6 e^j \otimes e^j$ instead of their classes; note that the listed $g'$-harmonic representatives are of pure type with respect to $J'$. Therefore, $J'$ is obviously $C^\infty$-full; it is also $C^\infty$-pure by [15, Proposition 3.2] or [11, Proposition 2.8] (see Proposition 2.1). Moreover, since any cohomology class in $H^2_{\partial R}(M^6; \mathbb{R})$ (respectively, in $H^2_{\partial R}(M^6; \mathbb{R})$) has a $g'$-harmonic representative in $\mathcal{Z}^{(1,1)}_{J'}$ (respectively, in $\mathcal{Z}^{(2,0),(0,2)}_{J'}$), by [15, Theorem 3.7] we have that $J'$ is also pure-and-full. One can explicitly check that the Lefschetz type operator $\mathcal{L}_{g'}: \wedge^2 M^6 \to \wedge^4 M^6$ introduced in [2] takes $g'$-harmonic 2-forms to $g'$-harmonic 4-forms, since
\[
\mathcal{L}_{g'} e^{12} = e^{1234} + e^{1256} = *g' (e^{34} + e^{56}), \quad \mathcal{L}_{g'} e^{36} = e^{1236} = *g' e^{45}, \\
\mathcal{L}_{g'} e^{34} = e^{1234} + e^{3456} = *g' (e^{12} + e^{56}), \quad \mathcal{L}_{g'} e^{45} = e^{1245} = *g' e^{36}, \\
\mathcal{L}_{g'} e^{56} = e^{1256} + e^{3456} = *g' (e^{12} + e^{34}).
\]

Resuming, we have shown the following result.

**Proposition 3.3.** Let $M^6$ be the Nakamura manifold. Then there exist a complex structure $J$ and an almost-Kähler structure $(J', \omega', g')$, both of which are $C^\infty$-pure-and-full and pure-and-full.

Furthermore, the Lefschetz type operator of the almost-Kähler structure $(J', \omega', g')$ takes $g'$-harmonic 2-forms to $g'$-harmonic 4-forms.

Inspired by the argument of the proof of [11, Theorem 2.3] (see Theorem 1.3), one can ask the following question (compare also [13, §2]). We provide in Proposition 4.1 an example of a non-$C^\infty$-full almost-Kähler structure for which the Lefschetz type property on 2-forms does not hold.

**Question 3.4.** Let $(M, J, \omega, g)$ be a compact $2n$-dimensional almost-Kähler manifold satisfying the Lefschetz type property on 2-forms. Is $J$ $C^\infty$-full?

### 4. An almost-Kähler non-$C^\infty$-full structure

Let $X := \mathbb{Z}[i]^3 \setminus (\mathbb{C}^3, *)$ be the Iwasawa manifold, where the group structure on $\mathbb{C}^3$ is defined by
\[(z_1, z_2, z_3) * (w_1, w_2, w_3) := (z_1 + w_1, z_2 + w_2, z_3 + z_1 w_2 + w_3).
\]
Considering the standard complex structure induced by the one on $\mathbb{C}^3$ and setting $\{\varphi^1, \varphi^2, \varphi^3\}$ as a global co-frame for the $(1,0)$-forms on $X$, by Hattori’s theorem
Corollary 4.2] (see Theorem 5.3) one gets that
\[
H^2_{dR}(X; \mathbb{C}) \cong \mathbb{R} \langle \varphi^{13} + \varphi^{13}, i(\varphi^{13} - \varphi^{13}), \varphi^{23} + \varphi^{23}, i(\varphi^{23} - \varphi^{23}), \varphi^{12} - \varphi^{21}, i(\varphi^{12} + \varphi^{21}), i\varphi^{11}, i\varphi^{22} \rangle \otimes_{\mathbb{R}} \mathbb{C},
\]
where we have listed the harmonic representatives with respect to the metric \( g := \sum_{h=1}^3 \varphi^h \circ \varphi^h \) instead of their classes. Set
\[
\varphi^1 = e^1 + ie^2, \quad \varphi^2 = e^3 + ie^4, \quad \varphi^3 = : e^5 + ie^6;
\]
then
\[
d e^5 = -e^{13} + e^{24}, \quad d e^6 = -e^{14} - e^{23},
\]
the other differentials being zero. Therefore,
\[
H^2_{dR}(X; \mathbb{R}) \cong \mathbb{R} \langle e^{15} - e^{26}, e^{16} + e^{25}, e^{35} - e^{46}, e^{36} + e^{45}, e^{13} + e^{24}, e^{23} - e^{14}, e^{12}, e^{34} \rangle.
\]
Set
\[
v_1 := e^{15} - e^{26}, \quad v_2 := e^{16} + e^{25}, \quad v_3 := e^{35} - e^{46}, \quad v_4 := e^{36} + e^{45},
\]
\[
v_5 := e^{13} + e^{24}, \quad v_6 := e^{23} - e^{14}, \quad v_7 := e^{12}, \quad v_8 := e^{34}.
\]
Consider the almost-Kähler structure \((J, \omega, g)\) on \(X\) defined by
\[
J e^1 := -e^6, \quad J e^2 := -e^5, \quad J e^3 := -e^4, \quad \omega := e^{16} + e^{25} + e^{34}.
\]
We easily get that
\[
\mathbb{R} \langle v_2, v_3 + v_5, v_4 - v_6, v_8 \rangle \subseteq H^+_j(X), \quad \mathbb{R} \langle v_1, v_3 - v_5, v_4 + v_6 \rangle \subseteq H^-_j(X).
\]

We claim that the previous inclusions are actually equalities, and in particular that \(J\) is a non-\(C^\infty\)-full almost-Kähler structure on \(X\). Indeed, we first note that by [15, Proposition 3.2] or [11, Proposition 2.8] (see Proposition 2.1), \(J\) is \(C^\infty\)-pure, since it admits a symplectic structure compatible with it. Moreover, we recall that a \(C^\infty\)-full almost-complex structure is also pure by [20, Proposition 2.30], and therefore it also satisfies that
\[
H_{3.1,1.3}(X) \cap H_{2.2}(X) = \{0\};
\]
see [11, Theorem 2.4]. Therefore, our claim reduces to proving that \(J\) does not satisfy [11]. Note that
\[
e^{3456} = [e^{3456} - d e^{135}] = [e^{3456} + e^{1234}]
\]
and that \(e^{3456} + e^{1234} \notin \Lambda_{3.1,1.3}(X)\) while \(e^{3456} - e^{1234} \notin \Lambda_{2.2}(X)\), and so \(H_{3.1,1.3}(X) \cap H_{2.2}(X) \ni e^{3456};\) therefore [11] does not hold, and hence \(J\) is not \(C^\infty\)-full.

Let \(L_\omega\) be the Lefschetz type operator of the almost-Kähler structure \((J, \omega, g)\). Then, we have \(L_\omega(e^{12}) = e^{1234} = d(e^{245});\) i.e., \(L_\omega\) does not take \(g\)-harmonic 2-forms in \(g\)-harmonic 4-forms.

Hence, we have proved the following result.
Proposition 4.1. Let $X := \mathbb{Z}[i]^{3} \setminus (\mathbb{C}^{3}, \ast)$ be the real manifold underlying the Iwasawa manifold. Then there exists an almost-Kähler structure $(J, \omega, g)$ on $X$ which is $C^\infty$-pure and non-$C^\infty$-full.

Furthermore, the Lefschetz type operator of the almost-Kähler structure $(J, \omega, g)$ does not take $g$-harmonic 2-forms to $g$-harmonic 4-forms.

5. ALMOST-COMPLEX MANIFOLDS WITH LARGE ANTI-INARIANT COHOMOLOGY

Given an almost-complex structure $J$ on a compact manifold $M$, it is natural to ask how large the cohomology subgroup $H^{(2,0),(0,2)}_{J}(M; \mathbb{R})$ can be. In this direction, Drăghici, Li, and the third author raised the following question in [12].

Question 5.1 ([12] Conjecture 2.5). Are there compact 4-dimensional manifolds $M$ endowed with non-integrable almost-complex structures $J$ such that $\dim_{\mathbb{R}} H^{−}_J(M) \geq 3$?

Here we present a 1-parameter family $\{J_t\}_t$ of (non-integrable) almost-complex structures on the 6-dimensional torus $\mathbb{T}^6$ having $\dim_{\mathbb{R}} H^{−}_J(\mathbb{T}^6; \mathbb{R})$ greater than 3; see also [11 §4]. For $t$ small enough, set $\alpha_t := \alpha_t(x^3) \in C^\infty(\mathbb{T}^6)$ such that $\alpha_0(x^3) \equiv 1$ and set

$$\varphi_t^1 := dx^1 + i \alpha_t \, dx^4, \quad \varphi_t^2 := dx^2 + i \, dx^5, \quad \varphi_t^3 := dx^3 + i \, dx^6;$$

therefore, the structure equations are

$$d \varphi_t^1 = i \, d \alpha_t \wedge dx^4, \quad d \varphi_t^2 = 0, \quad d \varphi_t^3 = 0.$$

Straightforward computations give that the $J$-anti-invariant $d$-closed 2-forms are of the type

$$\psi = \frac{C}{\alpha_t} \left( dx^{13} - \alpha_t \, dx^{46} \right) + D \left( dx^{16} - \alpha_t \, dx^{34} \right) + E \left( dx^{23} - dx^{56} \right)$$

$$+ F \left( dx^{26} - dx^{35} \right),$$

where $C, D, E, F \in \mathbb{R}$ (we shorten $dx^j \wedge dx^k$ by $dx^{jk}$). Moreover, the forms $dx^{23} - dx^{56}$ and $dx^{26} - dx^{35}$ are clearly harmonic with respect to the standard flat metric $\sum_{j=1}^6 dx^j \otimes dx^j$, while the classes of $dx^{16} - \alpha_t \, dx^{34}$ and $dx^{13} - \alpha_t \, dx^{46}$ are non-zero, their harmonic parts being non-zero. Hence, we get that $h_{J_0} = 6$ and $h_{J_t} = 4$ for small $t \neq 0$.

In the general case, we ask the following natural question.

Question 5.2. Are there examples of non-integrable almost-complex structures $J$ on a compact $2n$-dimensional manifold with $\dim_{\mathbb{R}} H^{−}_J(M) > n(n-1)$?

Now consider a solvmanifold $M = \Gamma \setminus G$, namely, a compact quotient of a connected simply-connected solvable Lie group $G$ by a co-compact discrete subgroup $\Gamma$. Denote the Lie algebra naturally associated to $G$ by $\mathfrak{g}$, and consider $(\wedge^\bullet \mathfrak{g}^*, d)$ to be the subcomplex of the de Rham complex $(\wedge^\bullet M, d)$ given by the left-invariant differential forms. The following result by Nomizu [25] and Hattori [19] holds.

Theorem 5.3 ([25] Theorem 1, [19] Theorem 4.2]). Let $M$ be a nilmanifold or, more in general, a completely-solvable solvmanifold. Then $H^\bullet(\wedge^\bullet \mathfrak{g}^*, d) \simeq H_{dR}^\bullet(M; \mathbb{R})$. 
Let $J$ be a left-invariant almost-complex structure on $M$, namely, an almost-complex structure on $M$ induced by an almost-complex structure on $G$ that is invariant under the action of $G$ on itself given by left-translations. Given $p,q \in \mathbb{N}$, denote by

$$H^\ast_{J}(p,q)\mathbb{R}(g) := \left\{ a = [\alpha] \in H^\ast \left( \wedge^p g^\ast, d \right) : \alpha \in \wedge^p J(p,q) \mathbb{R}(g^\ast) \right\} \subseteq H^\ast_{\mathbb{R}}(M; \mathbb{R})$$

the subgroup (see, e.g., [8] Lemma 9) of $H^\ast_{\mathbb{R}}(M; \mathbb{R})$ that consists of classes admitting a left-invariant representative of type $(p,q) + (q,p)$, where $\wedge^p J(p,q) \mathbb{R}(g^\ast) := \left\langle \wedge^p \left( g \otimes_\mathbb{R} C \right) \ast \otimes \wedge^q \left( g \otimes_\mathbb{R} C \right) \ast \right\rangle \cap \wedge^q g^\ast$.

Using Belgun’s symmetrization trick [4 Theorem 7], one can prove the following Nomizu type result, which relates the subgroups $H^\ast_{J}(p,q) \mathbb{R}(M)$ with their left-invariant part $H^\ast_{J}(p,q) \mathbb{R}(g)$.

**Theorem 5.4.** Let $M = \Gamma \backslash G$ be a solvmanifold endowed with a left-invariant almost-complex structure $J$, and denote the Lie algebra naturally associated to $G$ by $g$. For any $p,q \in \mathbb{N}$, the map

$$j: H^\ast_{J}(p,q) \mathbb{R}(g) \rightarrow H^\ast_{J}(p,q) \mathbb{R}(M)$$

induced by left-translations is injective, and if $H^\ast_{\mathbb{R}}(\wedge^q g^\ast, d) \simeq H^\ast_{\mathbb{R}}(M; \mathbb{R})$ (for instance, if $M$ is a completely-solvable solvmanifold), then $j: H^\ast_{J}(p,q) \mathbb{R}(g) \rightarrow H^\ast_{J}(p,q) \mathbb{R}(M)$ is in fact an isomorphism.

**Proof.** Since $J$ is left-invariant, left-translations induce the map $j: H^\ast_{J}(p,q) \mathbb{R}(g) \rightarrow H^\ast_{J}(p,q) \mathbb{R}(M)$.

Since by J. Milnor’s Lemma [24 Lemma 6.2] $G$ is unimodular, one can take in particular a bi-invariant volume form $\eta$ on $M$ such that $\int_M \eta = 1$. Consider the Belgun symmetrization map in [4 Theorem 7], namely,

$$\mu: \wedge^\ast M \rightarrow \wedge^\ast g^\ast, \quad \mu(\alpha) := \int_M \alpha \mid_m \eta(m).$$

Since $\mu$ commutes with $d$ by [4 Theorem 7], it induces the map $\mu: H^\ast_{\mathbb{R}}(M; \mathbb{R}) \rightarrow H^\ast(\wedge^q g^\ast, d)$, and since $\mu$ commutes with $J$, it preserves the bi-graduation; therefore it induces the map $\mu: H^\ast_{J}(p,q) \mathbb{R}(M) \rightarrow H^\ast_{J}(p,q) \mathbb{R}(g)$. Moreover, since $\mu$ is the identity on the space of left-invariant forms by [4 Theorem 7], we get the commutative diagram

$$\begin{array}{c}
H^\ast_{J}(p,q) \mathbb{R}(g) \xrightarrow{j} H^\ast_{J}(p,q) \mathbb{R}(M) \xrightarrow{\mu} H^\ast_{J}(p,q) \mathbb{R}(g) \\
\text{id} \end{array}$$

hence $j: H^\ast_{J}(p,q) \mathbb{R}(g) \rightarrow H^\ast_{J}(p,q) \mathbb{R}(M)$ is injective and $\mu: H^\ast_{J}(p,q) \mathbb{R}(M) \rightarrow H^\ast_{J}(p,q) \mathbb{R}(g)$ is surjective.

Furthermore, when $H^\ast(\wedge^q g^\ast, d) \simeq H^\ast_{\mathbb{R}}(M; \mathbb{R})$ (for instance, when $M$ is a completely-solvable solvmanifold, by Hattori’s theorem [19 Theorem 4.2]; see Theorem 5.3), since $\mu(\wedge^\ast g^\ast) = \text{id}(\wedge^\ast g^\ast)$ by [4 Theorem 7], we get that $\mu: H^\ast_{\mathbb{R}}(M; \mathbb{R}) \rightarrow H^\ast(\wedge^q g^\ast, d)$ is the identity map, and hence $\mu: H^\ast_{J}(p,q) \mathbb{R}(M) \rightarrow H^\ast_{J}(p,q) \mathbb{R}(g)$ is also injective and hence an isomorphism. \hfill \square
In particular, if $M = \Gamma \backslash G$ is a $2n$-dimensional completely-solvable solvmanifold endowed with a left-invariant almost-complex structure $J$, then
\[
\dim_{\mathbb{R}} H^+_{J}(M) \leq n(n - 1) \quad \text{and} \quad \dim_{\mathbb{R}} H^{-}_{J}(M) \leq n^2;
\]
this provides a partial negative answer to Question 5.2.

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