Finite sequences representing expected order statistics

by

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Abstract

Characterizations of finite sequences $\beta_1 < \cdots < \beta_n$ representing expected values of order statistics from a random sample of size $n$ are given. As a by-product, a characterization of binomial mixtures, when the mixing random variable is supported in the open interval $(0, 1)$, is presented; this enables the exact description of the convex hull of the open binomial curve, as well as the open moment curve.

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1 Introduction

In the present note we consider the following problem: Given $n$ real numbers

$\beta_1 < \cdots < \beta_n$,

under what conditions on $\beta$’s is there an integrable random variable (r.v.) $X$ such that

$\mathbb{E} X_{i:n} = \beta_i, \quad 1 \leq i \leq n$?

[Here, $X_{1:n} \leq \cdots \leq X_{n:n}$ are the order statistics of independent, identically distributed r.v.’s $X_1, \ldots, X_n$, each with distribution like $X$.] Notice that the number $n$ is held fixed; the question for infinite sequences is closely connected to the Hausdorff (1921) moment problem, and its answer is well-known from the works of Huang (1998), Kadane (1971, 1974), Kolodynski (2000) and Papadatos (2017). Some relative results for the finite case can be found in Mallows (1973).
2 A characterization of finite sequences of expected order statistics

Without loss of generality we may consider the numbers
\[ \tilde{\beta}_i = \frac{\beta_i - c}{\lambda}, \quad c \in \mathbb{R}, \quad \lambda > 0, \]
instead of \( \beta_i \). Clearly these numbers will be the expected order statistics (=EOS) from \( (X - c)/\lambda \) if and only if the \( \beta \)'s are the EOS from \( X \).

First, we seek for a necessary condition. Assume that \( X \) is a non-degenerate random variable with distribution function (d.f.) \( F \) and \( \mathbb{E}|X| < \infty \). Let \( X_1, \ldots, X_n \) be independent, identically distributed (i.i.d.) random variables with d.f. \( F \), and denote by \( X_{1:n} \leq \cdots \leq X_{n:n} \) the corresponding order statistics. It is known that
\[ \mathbb{E}X_{k:k} := \mu_k = \binom{n}{k}^{-1} \sum_{j=k}^{n} \binom{j-1}{k-1} \mu_{j:n}, \quad k = 1, \ldots, n, \tag{1} \]
where \( \mu_{j:n} := \mathbb{E}X_{j:n}, \quad j = 1, \ldots, n \); this follows by a trivial application of Newton’s formula to the expression \( \mu_k = k \int_0^1 u^{k-1} F^{-1}(u) \left[ u + (1 - u) \right]^{n-k} du \), where \( F^{-1}(u) := \inf\{ x : F(x) \geq u \}, \quad 0 < u < 1 \), is the left-continuous inverse of \( F \). From (1) with \( k = 1, 2 \),
\[ \mu_1 = \frac{\mu_{1:n} + \cdots + \mu_{n:n}}{n}, \quad \mu_2 = \frac{2}{n(n-1)} \sum_{j=2}^{n} (j-1) \mu_{j:n}. \tag{2} \]

On the other hand, it is well-known (see Jones and Balakrishnan (2002)) that
\[ \mu_{j+1:n} - \mu_{j:n} = \binom{n}{j} \int_0^\alpha F(x)^j (1 - F(x))^{n-j} dx > 0, \quad j = 1, \ldots, n - 1, \tag{3} \]
where \( \alpha < \omega \) are the endpoints of the support of \( X \); actually this formula goes back to Karl Pearson (1902). Notice that \( -\infty \leq \alpha < \omega \leq \infty, \alpha < \omega \) because \( F \) is non-degenerate, and the integral in (3) is finite since \( X \) is integrable. From (1) and (3) (applied to \( n = 2 \)),
\[ \mu_2 - \mu_1 = \int_\alpha^\omega F(x)(1 - F(x)) dx, \]
while (2) yields
\[ \mu_2 - \mu_1 = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} i(n-i)(\mu_{i+1:n} - \mu_{i:n}). \tag{4} \]

Choosing \( c = n^{-1} \sum_{j=1}^{n} \mu_{j:n}, \lambda = (n(n-1))^{-1} \sum_{i=1}^{n-1} i(n-i)(\mu_{i+1:n} - \mu_{i:n}) > 0 \), the numbers \( \mu_{j:n} = (\mu_{j:n} - c)/\lambda \) are the EOS from \( (X - c)/\lambda \). Therefore, by considering \( (X - c)/\lambda \) in place of \( X \), we may further assume that \( \mu_2 - \mu_1 = 1 \). Then,
\[ \int_\alpha^\omega F(x)(1 - F(x)) dx = 1. \]
Since \( F(y)(1 - F(y)) > 0 \) for \( y \in (\alpha, \omega) \), and zero outside \([\alpha, \omega]\), it follows that \( f_Y(y) := F(y)(1 - F(y)) \) defines a Lebesgue density of a random variable, say \( Y \), supported in the (finite or infinite) interval \((\alpha, \omega)\). From (4), the numbers \( p_j := j(n-j)(\mu_{j+1:n} - \mu_{j:n})/(n(n-1)) \), \( j = 1, \ldots, n-1 \), are strictly positive probabilities (summing to 1). Also, the integral in (3) can be rewritten as

\[
\int_\alpha^\omega \left( F(y)^{j-1}(1 - F(y))^{n-j-1} \right) f_Y(y) dy = \mathbb{E} \left\{ T^{j-1}(1 - T)^{n-j-1} \right\},
\]

where \( T := F(Y) \) is a random variable taking values in the interval \((0, 1)\) w.p. 1, because, by definition, \( \Pr(\alpha < Y < \omega) = 1 \). Hence, (3) reads as

\[
p_j = \frac{j(n-j)}{n(n-1)} \binom{n}{j} \mathbb{E} \left\{ T^{j-1}(1 - T)^{n-j-1} \right\}, \quad j = 1, \ldots, n-1,
\]

and we have shown the following

**Proposition 1** If \( X_1, \ldots, X_n \) are i.i.d. integrable non-degenerate r.v.’s, then there exists an r.v. \( T \), with \( \Pr(0 < T < 1) = 1 \), such that

\[
\frac{(j+1)(n-j-1)(\mu_{j+2:n} - \mu_{j+1:n})}{\sum_{i=1}^{n-1} i(n-i)(\mu_{i+1:n} - \mu_{i:n})} = \mathbb{E} \left\{ \binom{n-2}{j} T^j (1 - T)^{n-2-j} \right\},
\]

\[
\quad j = 0, \ldots, n-2. \tag{5}
\]

It is of interest to observe that the binomial moments of \( T \) appear in the r.h.s. of (4). Clearly, the r.v. \( T \) in this representation need not be unique; any other r.v. \( T' \) with \( \Pr(0 < T' < 1) = 1 \), possessing identical moments up to order \( n-2 \) with \( T \), will fulfill the same relationship.

**Remark 1** For any integrable non-degenerate r.v. \( X \) with d.f. \( F \) we may define the r.v. \( T \) as in the proof of Proposition [4] that is, \( T = F(Y) \) where \( Y \) has density \( f_Y(y) = F(y)(1 - F(y))/\lambda \) with \( \lambda = \int F(1 - F) \). It can be shown, using Lemma 4.1 in Papadatos (2001), that the d.f. of \( T \) is specified by

\[
\Pr(T < t) = \frac{1}{\lambda} \left[ t(1 - t)F^{-1}(t) - \int_0^t (1 - 2u)F^{-1}(u) du \right], \quad 0 < t < 1. \tag{6}
\]

Notice that \( \lambda = \int_0^1 (2t-1)F^{-1}(t)dt \) and, hence, the function \( F^{-1} \) determines uniquely the d.f. of \( T \). Moreover, (4) shows that the entire location-scale family of \( X \), \( \{c + \lambda X : c \in \mathbb{R}, \lambda > 0\} \), is mapped to a single r.v. \( T \in (0, 1) \). Provided that \( X \) has (finite or infinite) interval support, non-vanishing density \( f \) and differentiable inverse d.f. \( F^{-1} \), we conclude from (6) that a density of \( T \) is given by

\[
f_T(t) = \frac{t(1 - t)}{\lambda f(F^{-1}(t))}, \quad 0 < t < 1. \tag{7}
\]
Next, we proceed to verify that the preceding procedure can be inverted, showing
sufficiency of (5). To this end, we shall make use of the following lemma, which is of
independent interest in itself. A detailed proof is postponed to the appendix.

**Lemma 1** Let $T$ be an r.v. with d.f. $F_T$ such that $\mathbb{P}(0 < T < 1) = 1$. Then, there
exists a unique, non-degenerate, integrable, r.v. $X$, satisfying

$$
\mathbb{E}X_1 = 0 \quad \text{and} \quad \mathbb{E}X_{k+2:k+2} - \mathbb{E}X_{k+1:k+1} = \mathbb{E}T^k, \quad k = 0, 1, \ldots,
$$

where $X_{k:k} = \max\{X_1, \ldots, X_k\}$ with $X_1, X_2, \ldots$ being i.i.d. copies of $X$. The inverse
distribution function of $X$ is given by

$$
F_0^{-1}(t) := \frac{F_T(t -)}{t(1 - t)} - 4F_T(\frac{1}{2} -) - \int_{1/2}^{t} \frac{2u - 1}{u^2(1 - u)^2} F_T(u) du - c_T,
$$

$0 < t < 1$, where $F_T(t -) = \mathbb{P}(T < t), f_{1/2}^1 du = -f_t^{1/2} du$ for $t < 1/2$,

$$
c_T := \mathbb{E}\left[\frac{1}{T} I(T \geq \frac{1}{2})\right] - \mathbb{E}\left[\frac{1}{1 - T} I(T < \frac{1}{2})\right],
$$

and $I$ denotes an indicator function.

**Remark 2** Any r.v. $T \in (0, 1)$ can be viewed as the *expected order statistics generator*
of its corresponding r.v. $X$ with inverse d.f. $F_0^{-1}$ as in (9). This is so because the map
$T \to X$ (i.e., $F_T \to F_0 \equiv F_X$), defined implicitly by Lemma 1, is one to one and onto
from the space $T = \{T : \mathbb{P}(0 < T < 1) = 1\}$ to $\mathcal{H} = \{X : \mathbb{E}X = 0, \mathbb{E}X_{2:2} = 1\}$, where
identically distributed r.v.’s are considered as equal. Its inverse is given by
Remark 1 (with $\lambda = 1$, since $X \in \mathcal{H}$). In view of (13), below, it is the suitable
(and unique) transformation that quantifies the characterization of Hoeffding (1953),
stating that the sequence of expected order statistics characterizes the corresponding
distribution. It also provides an explicit connection of the (infinite) sequence of
expected order statistics to the Hausdorff (1921) moment problem; see Kadane (1971,
1974), Huang (1998), Kolodynski (2000), Papadatos (2017).

**Remark 3** Suppose that the r.v. $T$ of Lemma 1 is absolutely continuous with density
$f_T$. Assume also that the corresponding r.v. $X$ (with $\mathbb{E}X = 0, \mathbb{E}X_{2:2} = 1$, inverse
d.f. $F_0^{-1}$ as in (9)) is absolutely continuous, admitting a non-vanishing density $f_0$ in the
(finite or infinite) interval support of $X$, and that $F_0^{-1}$ is differentiable. Then
(see Remark 1),

$$
f_T(t) = \frac{t(1 - t)}{f_0(F_0^{-1}(t))}, \quad F_0^{-1}(t) = \int_{1/2}^{t} \frac{f_T(u)}{u(1 - u)} du - c_T, \quad 0 < t < 1.
$$

For example, if $T$ is Beta($2, 2$) then $X$ is uniform in $(-3, 3)$; if $T$ is Beta($2, 1$) then
$X = 2 \mathcal{E} - 2$ where $\mathcal{E}$ is standard exponential; if $T$ is Beta($1, 2$) then $X = 2 - 2 \mathcal{E}$; if $T$
is standard uniform then $X$ is standard logistic with density $f_0(x) = e^{-x}/(1 + e^{-x})^2,
x \in \mathbb{R}$; if $T$ is degenerate with $\mathbb{P}(T = \rho) = 1$ then (9) shows that $X$ is a two-valued
r.v. with $\mathbb{P}(X = -1/\rho) = \rho, \mathbb{P}(X = 1/(1 - \rho)) = 1 - \rho$. 

4
The characterization for finite $n$ reads as follows.

**Theorem 1** Given $n$ real numbers $\beta_1 < \cdots < \beta_n$, the following are equivalent.

(i) The $\beta$’s are EOS, that is, there exist i.i.d. integrable non-degenerate r.v.’s $X_1, \ldots, X_n$ such that $\mathbb{E} X_{j:n} = \beta_j$, $j = 1, \ldots, n$.

(ii) There exists an r.v. $T$, with $\mathbb{P}(0 < T < 1) = 1$, such that

$$\frac{(j+1)(n-j-1)(\beta_{j+2} - \beta_{j+1})}{\sum_{i=1}^{n-1} i(n-i)(\beta_{i+1} - \beta_i)} = \mathbb{E} \left\{ \binom{n-2}{j} T^j (1-T)^{n-2-j} \right\},$$

for $j = 0, \ldots, n-2$. (11)

(iii) There exists an r.v. $T$, with $\mathbb{P}(0 < T < 1) = 1$, such that

$$\frac{n-1}{\binom{n-1}{k+1}} \sum_{i=1}^{n-1} i(n-i)(\beta_{i+1} - \beta_i) \sum_{j=k+1}^{n-1} (n-j) \binom{j}{k+1} (\beta_{j+1} - \beta_j) = \mathbb{E} T^k,$$

for $k = 0, \ldots, n-2$. (12)

**Proof:** The equivalence of (11) and (12) follows by a straightforward computation, while the implication (i) ⇒ (ii) is proved in Proposition 1. In order to verify (ii) ⇒ (i), assume that (11) is satisfied for some $T$ with $\mathbb{P}(0 < T < 1) = 1$, and consider the r.v. $X$ as defined in Lemma 1. Let $\mu_{j:n} = \mathbb{E} X_{j:n}$ and $\mu_k = \mathbb{E} X_{k:k}$. Then,

$$\mu_{j:n} = n \binom{n-1}{j-1} \sum_{i=j}^{n} (-1)^{i-j} \binom{n-j}{i-j} \frac{\mu_i}{i}, \quad j = 1, \ldots, n;$$

(13)

see Mallows (1973), Arnold et al. (1992), David and Nagaraja (2003). It follows that

$$\mu_{j+2:n} - \mu_{j+1:n} = \binom{n}{j+1} \sum_{i=j+1}^{n} (-1)^{i-j} \binom{n-j-1}{i-j} \mu_i, \quad j = 0, \ldots, n-2.$$

By a trivial application of the binomial theorem to $(1-T)^{n-2-j}$, and since $\mathbb{E} T^k = \mu_{k+2} - \mu_{k+1}$, see (3), we obtain

$$\mathbb{E} \left\{ \binom{n-2}{j} T^j (1-T)^{n-2-j} \right\} = \binom{n-2}{j} \sum_{i=j+1}^{n} (-1)^{i-j} \binom{n-j-1}{i-j} \mu_i,$$

for $j = 0, \ldots, n-2$.

Hence, for $j = 0, \ldots, n-2$,

$$\frac{(j+1)(n-j-1)}{n(n-1)} (\mu_{j+2:n} - \mu_{j+1:n}) = \mathbb{E} \left\{ \binom{n-2}{j} T^j (1-T)^{n-2-j} \right\},$$

and (11) implies that for some $\lambda > 0$,

$$\beta_{j+1} - \beta_j = \lambda (\mu_{j+1:n} - \mu_{j:n}), \quad j = 1, \ldots, n-1.$$

It follows by induction on $j$ that $\beta_j = \beta_1 + \lambda (\mu_{j:n} - \mu_{1:n})$, and therefore, $(\beta_j - c)/\lambda = \mu_{j:n}$, with $c = \beta_1 - \lambda \mu_{1:n}$. Hence, the numbers $((\beta_j - c)/\lambda)^n_{j=1}$ are expected order statistics, and thus, the same is true for $\beta$’s.
Remark 4 The r.h.s. of (11) corresponds to a Binomial Mixture (of a particular form, since $P(T = 0) = P(T = 1) = 0$). Obviously, the necessary and sufficient condition (12) is always satisfied for $n = 2$ and $n = 3$. Hence, the true problem begins at $n = 4$.

3 The truncated moment problem for finite open intervals

In this section, we obtain a precise characterization by invoking results from the truncated moment problem for finite intervals. The existing results are limited to compact intervals and are not applicable to our case, since, according to the characterization of Theorem 1, a suitable $T$ lies in the open interval $(0, 1)$ w.p. 1.

Definition 1 Given $n \geq 4$ numbers $\beta_1 < \cdots < \beta_n$, let $\beta = (\beta_1, \ldots, \beta_n)$ and define the vector $(\nu_k)_{k=0}^{n-2} = \nu = \nu(\beta)$ by

$$
\nu_k := \frac{n-1}{\lambda(n-1)} \sum_{j=k+1}^{n-1} (n-j) \binom{j}{k+1} (\beta_{j+1} - \beta_j), \quad k = 0, \ldots, n-2,
$$

where $\lambda = \lambda(\beta) := \sum_{i=1}^{n-1} i(n-i)(\beta_{i+1} - \beta_i) > 0$.

It is easily checked that $1 = \nu_0 > \nu_1 > \cdots > \nu_{n-2} > 0$, and that the vector $\nu$ is invariant under location-scale transformations on the $\beta$’s.

According to Theorem 1, the $\beta$’s are EOS if and only if the $\nu$’s fulfill the truncated moment problem in the interval $(0, 1)$. However, for the truncated moment problem, well-known results exist for a compact interval $[a, b]$; see, e.g., Theorem IV.1.1 of Karlin and Studden (1966) or Theorems 10.1, 10.2 in Schmüdgen (2017). In order to obtain the corresponding necessary and sufficient conditions for open intervals, we shall make use of the following

Theorem 2 (Richter-Tchakaloff Theorem; see Schmüdgen (2017), Theorem 1.24). Let $(X, \mathcal{F}, \mu)$ be a measure space and $V$ a finite dimensional linear subspace of $L^1(\mathbb{R}, \mu)$. Define the linear functional $L_\mu$ by $L_\mu(f) := \int f d\mu$, $f \in V$. Then, there exists a measure $\mu_0$ in $(X, \mathcal{F})$, supported on $k \leq \dim V$ points of $X$, such that $L_{\mu_0} \equiv L_\mu$ on $V$, that is, $\int f d\mu_0 = \int f d\mu$ for all $f \in V$.

A symmetric $n \times n$ matrix $A$ with real entries is positive definite (denoted by $A \succ 0$) if $x^T A x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where $x^T$ denotes the transpose of a column vector $x \in \mathbb{R}^n$. Similarly, $A$ is positive semi-definite (or nonnegative definite) if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$, and this is denoted by $A \succeq 0$.

Definition 2 (Hankel matrices). Let $n \in \{4, 5, \ldots\}$, $0 \leq \varepsilon < 1/2$, and consider the numbers $\nu_k$ as in Definition 1.
(i) Case $n = 2m + 2$: We define

$$A_0(\varepsilon) := (\nu_{i+j})_{i,j=0}^m, \quad B_0(\varepsilon) := (\nu_{i+j+1} - \nu_{i+j+2} - \varepsilon(1 - \varepsilon)\nu_{i+j})_{i,j=0}^{m-1}$$

and $A_0 := A_0(0)$, $B_0 := B_0(0)$.

(ii) Case $n = 2m + 3$: We define

$$A_1(\varepsilon) := (\nu_{i+j+1} - \varepsilon\nu_{i+j})_{i,j=0}^m, \quad B_1(\varepsilon) := ((1 - \varepsilon)\nu_{i+j} - \nu_{i+j+1})_{i,j=0}^m$$

and $A_1 := A_1(0)$, $B_1 := B_1(0)$.

Notice that the matrices $A_0(\varepsilon), A_1(\varepsilon), B_1(\varepsilon)$ are of order $m + 1$, while $B_0(\varepsilon)$ is of order $m$. The following theorem contains our main result; compare with Mallows (1973).

**Theorem 3** Let $n \in \{4, 5, \ldots\}$, $\beta_1 < \cdots < \beta_n$, and $(\nu_0, \ldots, \nu_{n-2})$ as in Definition 11.

(i) If $n = 2m + 2$, then the $\beta$’s are EOS if and only if $A_0(\varepsilon) \succeq 0$ and $B_0(\varepsilon) \succeq 0$ for some $\varepsilon \in (0, 1/2)$, where $A_0(\varepsilon)$ and $B_0(\varepsilon)$ are given by Definition 21.

(ii) If $n = 2m + 3$, then the $\beta$’s are EOS if and only if $A_1(\varepsilon) \succeq 0$ and $B_1(\varepsilon) \succeq 0$ for some $\varepsilon \in (0, 1/2)$, where $A_1(\varepsilon)$ and $B_1(\varepsilon)$ are given by Definition 22.

(iii) If $n = 2m + 2$, the condition $A_0 \succ 0$ and $B_0 \succ 0$ is sufficient, but not necessary, for the $\beta$’s to be EOS. Similarly, if $n = 2m + 3$, the condition $A_1 \succ 0$ and $B_1 \succ 0$ is sufficient, but not necessary, for the $\beta$’s to be EOS.

**Proof:** (i) and (ii): According to Theorems 10.1, 10.2 in Schmüddgen (2017), or Theorem IV.1.1 of Karlin and Studden (1966), the condition $A_i(\varepsilon) \succeq 0$ and $B_i(\varepsilon) \succeq 0$ $(i = 0$ or 1) is necessary and sufficient for $(\nu_k)_{k=0}^{n-2}$ to be a truncated moment sequence in the interval $[\varepsilon, 1 - \varepsilon]$.

Assume first that $A_i(\varepsilon) \succeq 0$ and $B_i(\varepsilon) \succeq 0$ for some $\varepsilon \in (0, 1/2)$. Since $\nu_0 = 1$, any solution (=representing measure) $\mu$ will be a probability measure. Equivalently, the r.v. $T$ with d.f. $F_T(x) = \mu((0, \infty, x])$ takes values in $[\varepsilon, 1 - \varepsilon] \subseteq (0, 1)$ and satisfies $\mathbb{E}T^k = \nu_k$, $k = 0, \ldots, n - 2$. From Theorem 1(iii) it follows that the $\beta$’s are EOS.

To prove necessity, assume that the $\beta$’s are EOS. From Theorem 1(iii) we can find an r.v. $T$ with $\mathbb{E}T^k = \nu_k$, $k = 0, \ldots, n - 2$, and $\mathbb{P}(0 < T < 1) = 1$. Let $\mu_T$ be the probability measure of $T$ and consider the probability space $(\mathcal{X}, \mathcal{F}, \mu) := ((0, 1), \mathcal{B}, \mu_T)$, where $\mathcal{B}$ is the Borel $\sigma$–field on $(0, 1)$. Define the space $V$ of real polynomials $f : (0, 1) \to \mathbb{R}$ of degree $\leq n - 2$; obviously, $V$ is a linear subspace of $L^1(\mathbb{R}, \mu_T)$ of dimension $n - 1$ (finite). Consider also the Riesz functional $L_{\mu_T} : V \to \mathbb{R}$ defined by $L_{\mu_T}(f) := \int f d\mu_T = \sum_{k=0}^{n-2} a_k x^k$ for $f(x) = \sum_{k=0}^{n-2} a_k x^k \in V$. Form Richter-Tchakaloff Theorem (see Theorem 2 above), there exists a measure $\mu_0$, supported in at most $n - 1$ points of $\mathcal{X} = (0, 1)$, such that $L_{\mu_0} \equiv L_{\mu_T}$ on $V$; in particular, $\nu_k = \int_{(0,1)} x^k d\mu_T(x) = \int_{(0,1)} x^k d\mu_0(x)$, $k = 0, \ldots, n - 2$. Thus, $\mu_0$ is a probability
measure \( (\nu_0 = 1) \) supported on a finite number of points in \((0, 1)\), possessing the same initial \( n - 2 \) moments as \( \mu_T \). This means that \( \mu_0 \) solves the truncated moment problem for \((\nu_k)_{k=0}^{n-2} \) in the interval \([t_1, t_2] \), where \( t_1 \in (0, 1) \) is the minimum supporting point of \( \mu_0 \) and \( t_2 \in (0, 1) \) the maximum one. Choose \( \varepsilon > 0 \) such that \( \varepsilon < \min\{t_1, 1 - t_2\} \).

Then, the sequence \((\nu_k)_{k=0}^{n-2} \) is the moment sequence of \( \mu_0 \), supported in the interval \([\varepsilon, 1 - \varepsilon] \), and Theorems 10.1, 10.2 in Schmüdgen (2017) imply that \( A_i(\varepsilon) \geq 0 \) and \( B_i(\varepsilon) \geq 0 \) (\( i = 0 \) or \( 1 \)).

(iii) First we prove sufficiency. Denote by \( \lambda_{\min}(M) \) (resp. \( \lambda_{\max}(M) \)) the smallest (resp. the largest) eigenvalue of a real symmetric matrix \( M \). For the case \( n = 2m + 2 \), the matrix \( A_0(\varepsilon) \) is independent of \( \varepsilon \), hence, \( A_0(\varepsilon) = A_0 > 0 \) by hypothesis. Moreover, \( B_0(\varepsilon) = B_0 - \varepsilon(1 - \varepsilon)M_0 \) for some real symmetric matrix \( M_0 \); see (13). Since \( \lambda_{\min}(B_0) > 0 \) by assumption, it follows that for any \( x = (x_0, \ldots, x_m) \in \mathbb{R}^m \),

\[
x^T B_0(\varepsilon)x = x^T B_0 x - \varepsilon(1 - \varepsilon)x^T M_0 x \geq [\lambda_{\min}(B_0) - \varepsilon(1 - \varepsilon)\lambda_{\max}(M_0)]x^T x \geq 0,
\]

if \( \varepsilon > 0 \) is sufficiently small. Hence, the sufficient condition (i), namely, \( A_0(\varepsilon) \geq 0 \) and \( B_0(\varepsilon) \geq 0 \) for some small \( \varepsilon > 0 \), is satisfied. Similarly, when \( n = 2m + 3 \) we have \( A_1(\varepsilon) = A_1 - \varepsilon M_1 \) and \( B_1(\varepsilon) = B_1 - \varepsilon M_1 \) for some real symmetric matrix \( M_1 \); see (14). From \( \lambda_{\min}(A_1) > 0 \), \( \lambda_{\min}(B_1) > 0 \), it follows that for any \( x \in \mathbb{R}^{m+1} \),

\[
x^T A_1(\varepsilon)x = x^T A_1 x - \varepsilon x^T M_1 x \geq [\lambda_{\min}(A_1) - \varepsilon\lambda_{\max}(M_1)]x^T x \geq 0,
\]

and \( x^T B_1(\varepsilon)x \geq [\lambda_{\min}(B_1) - \varepsilon\lambda_{\max}(M_1)]x^T x \geq 0 \), provided \( \varepsilon > 0 \) is sufficiently small. Hence, the sufficient condition (ii), \( A_1(\varepsilon) \geq 0 \) and \( B_1(\varepsilon) \geq 0 \) for some small \( \varepsilon > 0 \), is satisfied. Therefore, in both cases, the condition (iii) is sufficient for the \( \beta \)'s to represent EOS.

Finally, we show that the condition (iii), namely \( A_i > 0 \) and \( B_i > 0 \) (\( i = 0 \) or \( 1 \)), is not necessary. To this end, consider the sequence \( \beta_j := \sum_{k=n+1-j}^{n} \binom{n}{k} \), \( j = 1, \ldots, n \). Then, \( \beta_{j+1} - \beta_j = \binom{n}{j} \) (\( j = 1, \ldots, n - 1 \)) and a straightforward computation yields \( \nu_k = 2^{-k} \), \( k = 0, \ldots, n - 2 \); see (14). Suppose first that \( n = 2m + 2 \) and let \( x^T = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \). Then, \( x^TA_0 x = (\sum_{k=0}^{m} x_k/2^k)^2 \), and since \( m \geq 1 \), the matrix \( A_0 \) is singular (hence, not positive definite). Similarly, for \( x^T = (x_0, \ldots, x_{m-1}) \in \mathbb{R}^m \), \( x^T B_0 x = (1/4)(\sum_{k=0}^{m-1} x_k/2^k)^2 \), which is positive definite if and only if \( m = 1 \) (\( n = 4 \)). On the other hand, \( A_0(\varepsilon) = A_0 \geq 0 \) for all \( \varepsilon \in (0, 1/2) \), while \( x^T B_0(\varepsilon)x = (1/4 - \varepsilon(1 - \varepsilon))(\sum_{k=0}^{m-1} x_k/2^k)^2 \geq 0 \) for small enough \( \varepsilon > 0 \). According to characterization (i), the given \( \beta \)'s are EOS, although the numbers \( \nu_k(\beta) \) \( (k = 0, \ldots, n - 2) \) do not satisfy the condition \( A_0 > 0 \) and \( B_0 > 0 \).

Next, suppose that \( n = 2m + 3 \). Then, \( A_1 = B_1 \) and it follows that \( x^T A_1 x = x^T B_1 x = (1/2)(\sum_{k=0}^{m} x_k/2^k)^2 \), showing that \( A_1 \) (and \( B_1 \)) is singular and positive semi-definite. On the other hand, \( x^T A_1(\varepsilon)x = x^T B_1(\varepsilon)x = (1/2 - \varepsilon)(\sum_{k=0}^{m} x_k/2^k)^2 \geq 0 \), and (ii) shows that the \( \beta \)'s are EOS. Hence, although the numbers \( \nu_k(\beta) \) \( (k = 0, \ldots, n - 2) \) do not satisfy the condition \( A_1 > 0 \) and \( B_1 > 0 \), the corresponding \( \beta \)'s are EOS.

It can be checked that the given \( \beta \)'s are the EOS from the two-valued r.v. \( X \) with \( P(X = 0) = P(X = 2^n) = 1/2 \).

Remark 5 (a) Assume that for \( i = 0 \) or \( 1 \), \( A_i \geq 0 \), \( B_i \geq 0 \), and either \( \det A_i = 0 \) or \( \det B_i = 0 \) (or both). Then, the measure \( \mu = \mu_0 \) is \([0, 1]-determinate \) from its
Consequently, from Lemma 1 we conclude that the corresponding r.v. \( X \) of the given expected order statistics, is also unique, namely, \( I \mathbb{P}(T \lor (12) \text{ is uniquely determined (in fact, Example 2})

The case \( \beta \equiv 4 \) we have \( \mu \equiv 0 \) and \( \lambda \equiv A \) in (17) we immediately deduce that, e.g., the numbers \( (0, 2, 11, 13) \equiv \) positive semi-definite, and by Sylvester’s criterion, \( A_0 = A_0(\varepsilon) \) is positive semi-definite if and only if

\[
(b_2 - b_1)(b_4 - b_3) \geq \left( \frac{2}{3}(b_3 - b_2) \right)^2.
\]

According to Theorem 3(i), the \( \beta \)'s are EOS if and only if (17) is satisfied. Based on (17) we immediately deduce that, e.g., the numbers \( (0, 2, 5, 7) \equiv \) EOS, while the numbers \( (0, 2, 11, 13) \equiv \) not. For the first set of numbers, the r.v. \( T \equiv (11) \) or \( (12) \equiv \) uniquely determined (in fact, \( T \equiv 1/2 \), because \( \nu(\beta) \equiv (1, 1/2, 1/4) = (1, \mathbb{E}T, \mathbb{E}T^2) \), showing that \( \text{Var}(T) = 0 \); see Remark 5(a) and (14) of Definition 1. Consequently, from Lemma 1 we conclude that the corresponding r.v. \( X \), assuming the given expected order statistics, is also unique, namely, \( \mathbb{P}(X = -1/2) = \mathbb{P}(X = 15/2) = 1/2; \) see Remarks 3 2.

Example 2 (The case \( n = 5 \)). It can be checked that for \( n = 5 \), the \( 2 \times 2 \) matrices \( A_1, B_1 \), see (16), (14), are positive semi-definite if and only if \( (b_3 - b_2)(b_5 - b_4) \geq \frac{1}{2}(b_4 - b_3)^2 \) and \( (b_2 - b_1)(b_4 - b_3) \geq \frac{1}{2}(b_3 - b_2)^2 \). Moreover, if both inequalities are strict (case \( ++ \)) then \( A_1 \succ 0 \), \( B_1 \succ 0 \), and Theorem 3(iii) shows that the \( \beta \)'s are EOS. If, however, one (or both) of the inequalities reduces to an equality, one has to check the condition (ii) of Theorem 3 in detail. For instance, if both matrices are singular (case \( 0,0 \)), then \( A_1(\varepsilon) \geq 0 \) and \( B_1(\varepsilon) \geq 0 \) for \( 0 < \varepsilon < \min\{b_3 - b_2, b_4 - b_3\}/(b_4 - b_2) \), and the \( \beta \)'s are again EOS. As an example of the \( 0,0 \)-case consider the numbers \( (0, 1, 5, 13, 21) \), representing the EOS from the (uniquely defined) r.v. \( X \) with \( \mathbb{P}(X = -1/10) = 2/3, \mathbb{P}(X = 121/5) = 1/3. \) However, both cases \( 0,++ \) (e.g., \( (0, 9, 11, 13, 14) \)) and \( ++,0 \) (e.g., \( (0, 1, 3, 5, 14) \)) imply that the \( \beta \)'s are not EOS. To see this, assume that \( 2(b_3 - b_2)(b_5 - b_4) = (b_4 - b_3)^2 \) and \( 2(b_2 - b_1)(b_4 - b_3) > (b_3 - b_2)^2 \); case \( 0,++). Then, \( B_1 \succ 0 \) (hence, \( B_1(\varepsilon) \geq 0 \) for small \( \varepsilon > 0 \)) and \( A_1 \geq 0 \) with \( \text{det} A_1 = 0 \). It can be verified that for \( x^T = (x_0, x_1) := (b_4 - b_3, -(b_4 - b_2)) \), \( x^TA_1(\varepsilon)x = -\varepsilon \Delta \), where \( \Delta > 0 \) depends only on \( \beta \)'s, and thus, according to Theorem 3(ii), the \( \beta \)'s cannot be EOS. By the same reasoning, this is also true for the \( (+,0) \)-case. Therefore, the complete characterization for \( n = 5 \) says for the \( \beta \)'s to be EOS it is necessary and sufficient that either \( 2(b_2 - b_1)(b_4 - b_3) = (b_3 - b_2)^2 \) and
\[2(\beta_3 - \beta_2)(\beta_5 - \beta_4) = (\beta_4 - \beta_3)^2, \text{ or } 2(\beta_2 - \beta_1)(\beta_4 - \beta_3) > (\beta_3 - \beta_2)^2 \text{ and } 2(\beta_3 - \beta_2)(\beta_5 - \beta_4) > (\beta_4 - \beta_3)^2; \text{ that is, either both matrices } A_1, B_1 \text{ are positive definite, or both are positive semi-definite and singular. We do not know if the situation is similar for odd values of } n \geq 7.\]

Our final result characterizes the binomial mixtures for which the mixing distribution is supported in the open interval \((0, 1)\) (cf. Wood, 1992, 1999). The proof, being an immediate application of Theorems 1, 3, is omitted.

**Theorem 4** Let \(p = (p_0, \ldots, p_n) \ (n \geq 2)\) be a probability vector \((p_i \geq 0, \sum_{i=0}^n p_i = 1)\) and \(u = u(p) = (u_0, \ldots, u_n)\), where

\[u_k := \binom{n}{k}^{-1} \sum_{j=k}^n \binom{j}{k} p_j, \quad k = 0, 1, \ldots, n.\]

If \(n = 2m\) set

\[A(\varepsilon) = (u_{i+j})_{i,j=0}^{m}, \quad B(\varepsilon) = (u_{i+j+1} - u_{i+j+2} - \varepsilon(1 - \varepsilon)u_{i+j})_{i,j=0}^{m-1},\]

and if \(n = 2m + 1\) set

\[A(\varepsilon) = (u_{i+j+1} - \varepsilon u_{i+j})_{i,j=0}^{m}, \quad B(\varepsilon) = ((1 - \varepsilon)u_{i+j} - u_{i+j+1})_{i,j=0}^{m}.\]

Then, the following are equivalent.

(i) \(A(\varepsilon) \succeq 0\) and \(B(\varepsilon) \succeq 0\) for some \(\varepsilon\) with \(0 < \varepsilon < 1/2\).

(ii) \(p \in \text{Conv}[B_0]\), where

\[B_0 = \left\{(\binom{n}{j} p^j (1 - p)^{n-j})_{j=0}^n, \quad 0 < p < 1\right\}\]

is the open binomial probability curve (without its endpoints) and \(\text{Conv}[X]\) denotes the convex hull of \(X \subseteq \mathbb{R}^{n+1}\).

(iii) There exists an r.v. \(V\) with \(\mathbb{P}(0 < V < 1) = 1\) such that

\[p_j = \mathbb{E} \left\{\binom{n}{j} V^j (1 - V)^{n-j}\right\}, \quad j = 0, 1, \ldots, n.\]

(iv) \(u \in \text{Conv}[M_0]\), where \(M_0 = \{(1, t, t^2, \ldots, t^n), \ 0 < t < 1\}\) is the open moment curve (without its endpoints).

(v) There exists an r.v. \(V\) with \(\mathbb{P}(0 < V < 1) = 1\) such that

\[u_k = \mathbb{E} V^k, \quad k = 0, 1, \ldots, n.\]

Let \(x(t) = (1, t, t^2, \ldots, t^n), \ 0 \leq t \leq 1\). A simple application of Theorem 3 shows that for \(n \geq 3\) (in contrast to the case \(n = 2\)), the line segment \((1 - \lambda)x(0) + \lambda x(t_0), \ 0 \leq \lambda < 1\), lies outside \(\text{Conv}[M_0]\). Given \(\lambda_0, t_0 \in (0, 1)\), it follows from Farkas’ Lemma (see, e.g., Bertsimas and Tsitsiklis (1997), Theorem 4.6) that for any \(m\), and any given collection \(\{t_0, \ldots, t_m\} \subseteq (0, 1)\), we can find a polynomial \(p\) with \(\text{deg}(p) \leq n\), such that \((1 - \lambda_0)p(0) + \lambda_0 p(t_0) < 0\) and \(p(t_i) \geq 0, \ i = 0, 1, \ldots, m.\)
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easily seen that $I_P(X)$ follows from the classical result of Hoeffding (1953), since the sequence $\mathbb{E}X_{k:k}$ determines the triangular array $\mathbb{E}X_{j:n}$, and vice-versa; see [13], [14]. We proceed to verify that $F_0^{-1}$ is indeed the distribution inverse of an integrable r.v. $X$ satisfying (5). Left continuity of $F_0^{-1}$ follows automatically from its definition. Moreover, the function $g(t) := F_0^{-1}(t) + c_T$ can be written as

$$g(t) = \begin{cases} 
-4[F_T(t-1) - F_T(t)] - \int_t^{1/2} \frac{1-2u}{u^2(1-u)^2} [F_T(u) - F_T(t-)]du, & 0 < t \leq \frac{1}{2}, \\
4[F_T(t) - F_T(\frac{1}{2})] + \int_{1/2}^{t} \frac{2u-1}{u^2(1-u)^2} [F_T(t) - F_T(u)]du, & \frac{1}{2} \leq t < 1,
\end{cases}$$

showing that $g(t) \geq 0$ for $t \geq 1/2$ and $g(t) \leq 0$ for $t \leq 1/2$. For $t_1 < t_2$ with $1/2 \leq t_1 < t_2 < 1$ we have

$$g(t_2) - g(t_1) = \frac{F_T(t_2) - F_T(t_1)}{t_1(1-t_1)} + \int_{t_1}^{t_2} \frac{2u-1}{u^2(1-u)^2} [F_T(t_2) - F_T(u)]du \geq 0.$$ 

Similarly, for $t_1 < t_2$ with $0 < t_1 < t_2 \leq 1/2$,

$$g(t_2) - g(t_1) = \frac{F_T(t_2) - F_T(t_1)}{t_2(1-t_2)} + \int_{t_1}^{t_2} \frac{1-2u}{u^2(1-u)^2} [F_T(u) - F_T(t_1)]du \geq 0.$$ 

Therefore, $g$, and hence $F_0^{-1}$, is nondecreasing. In order to verify that $F_0^{-1} \in L^1$, we shall calculate the integrals

$$J_1 = \int_0^{1/2} |g(t)|dt = \int_0^{1/2} g(t)dt, \quad J_2 = \int_{1/2}^1 |g(t)|dt = \int_{1/2}^1 g(t)dt.$$ 

In the following we shall make repeatedly use of the fact that for a nonnegative r.v. $Y$, $\mathbb{E}Y^2 = \int_0^\infty \mathbb{P}(Y > t)dt$ or $\mathbb{E}Y = \int_0^\infty \mathbb{P}(Y \geq t)dt$, followed by subsequent applications of Tonelli’s theorem. Using (18) and Tonelli’s theorem we have

$$J_1 = 2F_T(\frac{1}{2}) - 4 \int_0^{1/2} F_T(t)dt + \int_0^{1/2} \frac{1-2u}{u^2(1-u)^2} \int_0^u [F_T(u) - F_T(t)]dtdu,$$

on noting that $F_T(t) = F_T(t-)$ a.e. Considering the nonnegative r.v. $Y = h(T) = TI(T \leq u)$ it is easily seen that $\mathbb{P}(Y > t) = F_T(u) - F_T(t)$ for $t < u$, and the probability is zero for $t \geq u$. Hence,
\[ \mathbb{E}Y = \int_0^u [F_T(u) - F_T(t)]\,dt, \] and also, \[ \mathbb{E}h(T) = \int_{[0,u]} t\,dF_T(t). \] Since these expectations are equal, we obtain
\[ \int_0^u [F_T(u) - F_T(t)]\,dt = \int_{[0,u]} t\,dF_T(t). \]

Substituting this equality to the double integral in \( J_1 \) and interchanging once again the order of integration (since the integrand is nonnegative), we obtain
\[ \int_0^{1/2} \int_0^u \frac{1 - 2u}{u^2(1 - u)^2} [F_T(u) - F_T(t)]\,dtdu = \int_{[0,1/2]} \left( \frac{1}{1 - t} - 4t \right) dF_T(t) < \infty. \]

In order to evaluate the exact value of \( J_1 \), it remains to express the integral \( \int_0^{1/2} F_T(t)\,dt \) in terms of integrals w.r.t. \( dF_T \). For \( u \in (0,1) \), consider the nonnegative r.v. \( Y = h(T) = T I(T < u) \), for which \( \mathbb{P}(Y > t) = F_T(u) - F_T(t) \) for \( t < u \), and zero otherwise. Then, \( \mathbb{E}Y = \int_0^u [F_T(u) - F_T(t)]\,dt = uF_T(u) - \int_0^u F_T(t)\,dt, \) and \( \mathbb{E}h(T) = \int_{[0,u]} t\,dF_T(t); \) thus,
\[ \int_0^u F_T(t)\,dt = uF_T(u) - \int_0^u [F_T(u) - F_T(t)]\,dt = uF_T(u) - \int_{[0,u]} t\,dF_T(t). \]

Setting \( u = 1/2 \) we find
\[ 2F_T(\frac{1}{2}) - 4 \int_0^{1/2} F_T(t)\,dt = \int_{(0,1/2)} 4tdF_T(t), \]
and finally, since \( \int_{(0,1/2)} \left( \frac{1}{1 - t} - 4t \right) dF_T(t) = \int_{(0,1/2)} \left( \frac{1}{1 - t} - 4t \right) dF_T(t) \) (because the integrand vanish for \( t = 1/2 \)), we conclude that
\[ J_1 = \int_{(0,1/2)} \frac{1}{1 - t} dF_T(t) = \mathbb{E} \left[ \frac{1}{1 - T} I(T < 1/2) \right]. \]

Using (18) we rewrite \( J_2 \) as
\[ J_2 = -2F_T(\frac{3}{2}) + 4 \int_{1/2}^1 F_T(t)\,dt + \int_{1/2}^1 \frac{2u - 1}{u^2(1 - u)^2} \int_u^1 [F_T(t) - F_T(u)]\,dtdu, \]
by Tonelli’s theorem. Substituting
\[ \int_u^1 [F_T(t) - F_T(u)]\,dt = \int_{[u,1]} (1 - t)\,dF_T(t) \]
in the inner integral (noting that both integrals represent the expectation of \( Y = (1 - T)I(T > u) \)), and changing the order of integration, we arrive at
\[ J_2 = -2F_T(\frac{3}{2}) + 4 \int_{1/2}^1 F_T(t)\,dt + \int_{(1/2,1)} (1 - 4(1 - t))\,dF_T(t) < \infty. \]

For \( u \in (0,1) \) consider the r.v. \( Y = (1 - T)I(T \geq u) \), so that, \( \mathbb{P}(Y \geq y) = F_T(1 - y) - F_T(u) \) for \( y \leq 1 - u \), and zero otherwise. Then,
\[ \mathbb{E}Y = \int_0^{1-u} [F_T(1 - y) - F_T(u - y)]\,dy = \int_u^1 [F_T(t) - F_T(u - y)]\,dt, \]
and this expectation is also equal to \( \int_{[u,1]} (1 - t)\,dF_T(t) \). Hence,
\[ \int_u^1 F_T(t)\,dt = (1 - u)F_T(u - y) + \int_{[u,1]} (1 - t)\,dF_T(t). \]
Substituting $u = 1/2$ we obtain
\[-2F_T(t,1/2^+) - 4 \int_{1/2}^1 F_T(t) dt = \int_{[1/2,1]} 4(1-t) dF_T(t),\]
and since \( \int_{(1/2,1)} (1/2 - 4(1-t)) dF_T(t) = \int_{(1/2,1)} (1/2 - 4(1-t)) dF_T(t) \), we conclude that
\[J_2 = \int_{[1/2,1]} \frac{1}{2} dF_T(t) = E \left[ \frac{1}{2} I(T \geq 1/2) \right].\]
The preceding argument not only shows that \( F_0^{-1} \in L^1 \), but also proves that
\[\int_0^1 g(t) dt = J_2 - J_1 = c_T,\]
with \( c_T \) as in (10), and therefore, the expectation of the r.v. \( X \) with inverse d.f. \( F_0^{-1} = g - c_T \) is zero.

Next, set \( \mu_k = EX_{k,k} = k \int_0^1 t^{k-1} F_0^{-1}(t) dt, k = 1, 2, \ldots \), and let \( R = F_T(t,1/2^-) \). In view of (18), write \( \mu_k + c_T = \int_0^1 k t^{k-1} g(t) dt = I_2 - I_1 \) where
\[I_1 = \int_0^{1/2} 4k t^{k-1} [R - F_T(t)] dt + \int_0^{1/2} \int_t^{1/2} \frac{1 - 2u}{u^2 (1-u)^2} [F_T(u) - F_T(t)] dudt,\]
\[I_2 = \int_{1/2}^1 4k t^{k-1} [F_T(t) - R] dt + \int_{1/2}^1 \int_{1/2}^t \frac{2u - 1}{u^2 (1-u)^2} [F_T(t) - F_T(u)] dudt,\]
noting that the integrands may be different form the original ones (suggested from (18)) at sets of measure zero. Changing the order of integration, according to Tonelli’s theorem, we see that
\[I_1 = \int_0^{1/2} 4k t^{k-1} [R - F_T(t)] dt + \int_0^{1/2} \frac{1 - 2u}{u^2 (1-u)^2} \int_0^u k t^{k-1} F_T(u) - F_T(t) dtdu,\]
\[I_2 = \int_{1/2}^1 4k t^{k-1} [F_T(t) - R] dt + \int_{1/2}^1 \frac{2u - 1}{u^2 (1-u)^2} \int_u^1 k t^{k-1} [F_T(t) - F_T(u)] dtdu.\]
Consider the expectation of \( Y = T^k I(T \leq u) \). Since \( P(Y > y) = F_T(u) - F_T(y^{1/k}) \) for \( y < u^k \), and zero otherwise, we obtain
\[\int_{[0,u]} t^k dF_T(t) = E Y = \int_0^u \left[ F_T(u) - F_T(y^{1/k}) \right] dy = \int_0^u k t^{k-1} [F_T(u) - F_T(t)] dt.\]
Thus,
\[I_1 = \int_0^{1/2} 4k t^{k-1} [F_T(1/2^+) - F_T(t)] dt + \int_0^{1/2} \frac{1 - 2u}{u^2 (1-u)^2} \int_{(0,u]} t^k dF_T(t) du = \int_0^{1/2} 4k t^{k-1} [F_T(1/2^+) - F_T(t)] dt + \int_{(0,1/2]} k \left( \frac{1}{(1 - t)^2} - 4 \right) dF_T(t).\]
Next, set \( Y = T^k I(T < u) \) (for \( 0 < u < 1 \)) with \( EY = \int_{(0,u]} t^k dF_T(t) \), and observe that \( P(Y > y) = F_T(u-) - F_T(y^{1/k}) \) for \( y < u^k \) (and zero otherwise), to conclude the identity
\[\int_{(0,u]} t^k dF_T(t) = \int_0^u \left[ F_T(u-) - F_T(y^{1/k}) \right] dy = \int_0^u k t^{k-1} [F_T(u-) - F_T(t)] dt.\]
Applying this with \( u = 1/2 \) we obtain an explicit simple formula for \( I_1 \):

\[
I_1 = \int_{(0,1/2]} 4t^k dF_T(t) + \int_{(0,1/2]} t^k \left( \frac{1}{t(1-t)} - 4 \right) dF_T(t) = \int_{(0,1/2]} t^k dF_T(t).
\]

Finally, in order to calculate \( I_2 \), consider the auxiliary variable \( Y = (1 - T^k)I(T > u) \) (with \( 0 < u < 1 \)), for which \( \mathbb{P}(Y \geq y) = F_T((1 - y)^{1/k}) - F_T(u) \) for \( y < 1 - u^k \), and zero otherwise. The alternative expressions for its expectation yield

\[
\int_{(u,1)} (1-t^k)dF_T(t) = \int_0^{1-u^k} [F_T((1-y)^{1/k}) - F_T(u)] dy = \int_u^{kt^{-1}} [F_T(t) - F_T(u)] dt.
\]

Hence,

\[
I_2 = \int_{1/2}^1 4kt^{-1} [F_T(t) - F_T(\frac{1}{2})] dt + \int_{1/2}^1 \frac{2u-1}{u^2(1-u)^2} \int_{(u,1)} (1-t^k)dF_T(t) du,
\]

and applying once again Tonelli’s theorem, we get

\[
I_2 = \int_{1/2}^1 4kt^{-1} [F_T(t) - F_T(\frac{1}{2})] dt + \int_{(1/2,1)} (1-t^k)\left( \frac{1}{t(1-t)} - 4 \right) dF_T(t).
\]

If we set \( Y = (1 - T^k)I(T \geq u) \), we see that \( \mathbb{P}(Y \geq y) = F_T((1 - y)^{1/k}) - F_T(u-) \) for \( y \leq 1 - u^k \), and zero otherwise, obtaining

\[
\int_{[u,1]} (1-t^k)dF_T(t) = \int_0^{u^{-1}} [F_T((1-y)^{1/k}) - F_T(u-)] dy = \int_u^{kt^{-1}} [F_T(t) - F_T(u-)] dt.
\]

Applying this identity with \( u = 1/2 \) we conclude that

\[
I_2 = \int_{(1/2,1)} 4(1-t^k)dF_T(t) + \int_{(1/2,1)} (1-t^k)\left( \frac{1}{t(1-t)} - 4 \right) dF_T(t) = \int_{(1/2,1)} \frac{1-t^k}{t(1-t)} dF_T(t).
\]

By the preceding calculations,

\[
\mu_k + c_T = I_2 - I_1 = \int_{[1/2,1]} \frac{1-t^k}{t(1-t)} dF_T(t) - \int_{(0,1/2]} \frac{t^k}{t(1-t)} dF_T(t)
\]

(observe that the r.h.s. equals to \( c_T \) for \( k = 1 \), showing once again that \( \mu_1 = 0 \)). Therefore, for \( k = 0, 1, \ldots \),

\[
\mu_{k+2} - \mu_{k+1} = \int_{[1/2,1]} \frac{(1-t^{k+2})-(1-t^{k+1})}{t(1-t)} dF_T(t) - \int_{(0,1/2]} \frac{t^{k+2}-t^{k+1}}{t(1-t)} dF_T(t)
\]

\[
= \int_{(0,1)} t^k dF_T(t) = \mathbb{E}T^k,
\]

and the proof is complete.