Syracuse Random Variables and the Periodic Points of Collatz-type maps

Maxwell C. Siegel *

Monday, July 13, 2020

Abstract

Let \( p \) be an odd prime, and consider the map \( H_p \) which sends an integer \( x \) to either \( \frac{x}{2} \) or \( \frac{px+1}{2} \) depending on whether \( x \) is even or odd. The values at \( x = 0 \) of arbitrary composition sequences of the maps \( \frac{x}{2} \) and \( \frac{px+1}{2} \) can be parameterized over the 2-adic integers (\( \mathbb{Z}_2 \)) leading to a continuous function from \( \mathbb{Z}_2 \) to \( \mathbb{Z}_p \) which the author calls the “numen” of \( H_p \), denoted \( \chi_p \); the \( p = 3 \) case turns out to be an alternative version of the “Syracuse Random Variables” constructed by Tao[10]. This paper establishes the “Correspondence Theorem”, which shows that an odd integer \( \omega \) is a periodic point of \( H_p \) if and only if \( \omega = \chi_p(\ n) / (1 - r_p(\ n)) \) for some integer \( n \geq 1 \), where \( r_p(\ n) = p^\#_1(\ n) / 2^\lambda(\ n) \), where \( \#_1(\ n) \) is the number of 1s digits in the binary expansion of \( n \) and \( \lambda(\ n) \) is the number of digits in the binary expansion of \( n \). Using this fact, the bulk of the paper is devoted to examining the Dirichlet series associated to \( \chi_p \) and \( r_p \), which are used along with Perron’s Formula to reformulate the Correspondence Theorem in terms of contour integrals, to which Residue calculus is applied so as to obtain asymptotic formulae for the quantities therein. A sampling of other minor results on \( \chi_p \) are also discussed in the paper’s final section.

Keywords: Collatz Conjecture, \( p \)-adic analysis, 2-adic integers, Mellin transform, parity vector, contour integration, Pontryagin duality, \( px+1 \) map, \( 3x+1 \) map, \( 5x+1 \) map, Bayesian inference.

Contents

1 The Numen \( \chi_p \) and the Correspondence Theorem 8
   1.1 Construction and Characterization of \( \chi_p \) 8
   1.2 The Correspondence Theorem 15

---

*University of Southern California, Dornsife School of Letters & Sciences, Department of Mathematics. Department of Mathematics 3620 S. Vermont Ave., KAP 104 Los Angeles, CA 90089-2532. E-mail: maxwelcs@usc.edu Declarations of interest: none
Our Toolbox

2.1 Complex-Analytic Methods ........................................ 23
2.2 Rising-continuity and functions from $\mathbb{Z}_p$ to $\mathbb{Z}_q$ .... 34

3 Asymptotics, Blancmanges, and Contour Integrals 40

3.1 Blancmange Curves, $\sum_{n=1}^N r_p(n)$, & $\sum_{n=1}^N \chi_p(n)$ .... 40
3.1.1 Pictures at an Exhibition ...................................... 40
3.1.2 The Function $\zeta_p$ ........................................... 47
3.1.3 The summatory function of $\chi_p$ ................................ 54
3.2 Integral Criteria for Periodic Points ............................ 60

4 Miscellaneous Results 69

4.1 A Lipschitz-type estimate for $\chi_p$ ............................ 69
4.2 The Probabilistic Approach and Bayesian Inference ............ 72
4.3 An Archimedean Upper Bound for $\chi_p$ on certain subsets of $\mathbb{Z}_2$ . 80

List of Figures

1 The blancmange curve $T_{1/2}(x)$ on $[-1,1]$ ....................... 41
2 $\Bl(x)$, the $\#_1$-summatory blancmange ............................ 42
3 $\Bl_3(x)$, the $r_3$-summatory blancmange ............................. 46
4 Graph of a segment of $\Bl_3(x)$, with blue regions showing the values of $x$ for which $r_3(x) < 1$ .............................. 46
5 Log-scale graphs of $|\Bl_3(x)|$ (red), $|\Bl_5(x)|$ (blue), and $|\Bl_7(x)|$ (green) ........................................ 47

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Introduction

Let $p$ be an odd prime, and consider the map $H_p : \mathbb{N}_0 \to \mathbb{N}_0$ (where $\mathbb{N}_k \overset{\text{def}}{=} \{n \in \mathbb{Z} : x \geq k\}$) defined by:

$$H_p(x) \overset{\text{def}}{=} \begin{cases} \frac{x}{2} & \text{if } n \equiv 0 \\ \frac{nx+1}{2} & \text{if } n \equiv 1 \end{cases}$$

(1)

where $a \equiv b \pmod{r}$ denotes the congruence of $a$ and $b$ mod $r$. In 1978, Böhm and Sontacchi [11] proved what has since become a well-known result in the Collatz literature: the fact that every integer periodic point of the Collatz map must be of the form:

$$\sum_{k=1}^n \frac{2^{m-b_k-1}3^{k-1}}{2^m-3^n}$$

(2)
for some integers \( m, n \geq 1 \) and some strictly increasing sequence of positive integers \( b_1 < b_2 < \cdots < b_n \). The use of probabilistic techniques to study the Collatz map’s dynamics is by no means new. Tao’s 2019 paper on the subject, for examples, attack 2 by treating the \( b_k \)s as random variables. A second probabilistic approach is to study the parity vector of a given input \( x \) under an \( H_p \) map; this is the sequence \( [x], [H_p(x)], [H_p^2(x)], \ldots \) of the values mod 2 of the iterates of \( x \) under \( H_p \). [10]

The present paper came about as a result of the author’s idea to subordinate the input \( x \) to its parity vector. In other words, rather than asking “given \( x \), what is \( x \)’s parity vector under \( H_p \)”, we consider the question “given a parity vector, what \( x \) could have generated it?” Doing so quickly leads to a method of parameterizing \( x \) over the space of all possible parity vectors, which we identify with \( \mathbb{Z}_2 \), the space of 2-adic integers.

Our method is as follows. First, we define the maps:

\[
\begin{align*}
h_0(x) & \overset{\text{def}}{=} \frac{x}{2} \\
h_1(x) & \overset{\text{def}}{=} \frac{px + 1}{2}
\end{align*}
\]

(the branches of \( H_p \)). Next, for each \( n \in \mathbb{N}_1 \), let \( \{0, 1\}^n \) denote the set of all \( n \)-tuples whose entries are 0s and 1s. We denote an arbitrary such tuple as \( j = (j_1, \ldots, j_n) \), and we say \( j \) is non-zero if \( j \) contains at least one 1. Given any such tuple, we write \( |j| \) to denote the length of \( j \), so that any tuple can be written as \( j = (j_1, \ldots, j_{|j|}) \). For any integer \( m \geq 1 \), we write \( j^m \) to denote the tuple obtained by concatenating \( m \) copies of \( j \).

Writing:

\[
\mathcal{J} \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} \{0, 1\}^n
\]

(3)
to denote the set of all tuples of 0s and 1s of finite length, we then consider the composition sequence \( h_j \) defined by:

\[
h_j = h_{j_1} \circ \cdots \circ h_{j_{|j|}}, \quad \forall j \in \mathcal{J}
\]

(4)
Note the near-equivalence of \( j \) with parity vectors. Indeed, given any \( x, n \in \mathbb{N}_1 \), there exists a unique \( j \in \mathcal{J} \) of length \( n \) so that \( H_p^m(x) = h_j(x) \). Said \( j \) is the parity vector for the first \( n \) iterates of \( x \) under \( H_p \), albeit written in reverse order.

Since the \( h_j \)s are affine linear maps (that is, of the form \( ax + b \)), so any composition sequence \( h_j \). As such, for any \( j \in \mathcal{J} \), we have:

\[
h_j(x) = h_j^*(0) x + h_j(0) = \frac{p^{\#_1(j)} \#_1(j)}{2^{|j|}} x + h_j(0), \quad \forall j \in \mathcal{J}
\]

(5)
where:

\[
\#_1(j) \overset{\text{def}}{=} \text{number of 1s in } j
\]

(6)
Observing that the quantity \( p^{\#_1(j)/2^{|j|}} \) has a \( p \)-adic magnitude of \( p^{-\#_1(j)} \)—which tends to zero in \( \mathbb{Z}_p \) whenever \( \#_1(j) \to \infty \), we then have that:

\[
\lim_{\#_1(j) \to \infty} h_1(x) \overset{Z_p}{=} \lim_{\#_1(j) \to \infty} h_1(0), \quad \forall x \in \mathbb{Z}
\]

In letting the number of 1s in \( j \) tends to infinity, we can identify \( j \) with the 2-adic integer \( z \) whose 2-adic digits are precisely the entries of \( j \). Since the above limit is independent of the value of \( x \), this allows us to realize the map \( j \mapsto h_1(0) \) as a function \( \chi_p : \mathbb{Z}_2 \to \mathbb{Z}_p \) by letting the length of \( j \) tend to infinity; the author calls \( \chi_p \) the **numen** of \( H_p \).

It is worth noting that this approach has the peculiar distinction of being neither novel nor well-known. While the author independently formulated this approach in October of 2019, at around the same time, Tao published a paper \[14\] on the Collatz Conjecture, focusing on the analysis of what he called “Syracuse random variables”. Tao’s Syracuse random variables and the author’s numen are essentially one and the same. Indeed, equation 1.3 on page 5 of Tao’s paper is equivalent to \[5\]. Tao mentions the fact that his Syracuse random variables can be unified into a single 3-adic-valued random variable—the author’s \( \chi_3 \)—but his paper chooses not to adopt that point of view. Another noteworthy difference in our approaches is that whereas Tao builds the Syracuse random variables using a purely probabilistic approach, treating them as collections of geometrically distributed random variables, the present paper treats the unified Syracuse random variable (and its corresponding generalizations for the other \( px + 1 \) maps) explicitly parameterizes these functions in terms of the properties of the binary representations of integers. There are other similarities in the two approaches, to be discussed in Section 4.

While Tao’s work already establishes the importance of the Syracuse random variables—and hence, of \( \chi_p \)—to understanding the Collatz Conjecture and its relatives, the driving force behind the present paper’s analysis is based off the following relation which, to all appearances, seems to have gone unnoticed. \( \chi_p \) is intimately related to Böhm and Sontacchi’s formula \[2\] for the periodic points of \( H_3 \). As established in Section 1 of this paper (**Lemma 3**), \[2\] turns out to be equivalent to the functional equation \( \chi_3 \):

\[
\chi_p(B(t)) \overset{Z_p}{=} \frac{\chi_p(t)}{1 - r_p(t)}, \quad \forall t \in \mathbb{N}_1
\]

where:

\[
B(t) = \frac{t}{1 - 2^{\lambda(t)}}
\]

\[
r_p(t) = \frac{p^{\#_1(t)}}{2^{\lambda(t)}}
\]

\[
\#_1(t) = \text{number of 1s in the binary expansion of } t \in \mathbb{N}_1
\]

\[
\lambda(t) = \text{total number of digits in the binary expansion of } t \in \mathbb{N}_1
\]
The principal result of Section 1 (which the author terms the **Correspondence Theorem**) is that:

\[ \chi_p(\mathbb{Z}_2) \cap \mathbb{Z} = \{ \chi_p(B(t)) : t \in \mathbb{N}_0 \} \cap \mathbb{Z} = \{ \text{periodic points of } H_p \text{ in } \mathbb{Z} \} \quad (12) \]

As such, \( \chi_p \) presents us with a means of attacking the problem of determining the periodic points of the \( H_p \) maps. Beyond this fundamental result, the numen itself (along with related objects, especially \( r_p \)) has the advantage of providing a unified foundation—both archimedean (\( \mathbb{N}_0, \mathbb{Z} \), viewed as subsets of \( \mathbb{R} \)) and non (\( \chi_p(B(\mathbb{N}_0)) \subseteq \mathbb{Q} \) viewed as a subset of \( \mathbb{Z}_p \)) for studying both the \( H_p \) maps’ dynamics and the more general number-theoretic properties implicated therein.

While the author has obtained a smorgasbord of various interesting minor results (which shall be detailed in Section 4), the principal aim of this paper—Section 3—is to use classical complex-analytic Mellin transform methods to obtain an “exact formula” (a.k.a., generalized Fourier series) for \( F \). The essence of this approach is to use the Dirichlet series associated to the functions \( \chi_3 \) and \( r_3 \) to restate the **Correspondence Theorem** in terms of a contour integral

\[ \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\kappa_n(s)C_{3,\omega}(s)}{s(s+1)^2} \, ds = \frac{1}{2} \]

where:

\[ C_{3,\omega}(s) = B_3(s) - \frac{1}{\omega} \Xi_3(s) \]

\[ B_3(s) = \sum_{n=0}^{\infty} \frac{\frac{3}{2} - r_3(n)}{(n+1)^s} \]

\[ \Xi_3(s) = \sum_{n=0}^{\infty} \frac{\chi_3(n)}{(n+1)^s} \]

and where:

\[ \kappa_n(s) = n \left( (n+2)^{s+1} - 3(n+1)^{s+1} + 3n^{s+1} - (n-1)^{s+1} \right) \]

\[ + (n+2)^{s+1} - (n+1)^{s+1} - \left( n^{s+1} - (n-1)^{s+1} \right) \]

By shifting the contour of integration in [135] from \( \text{Re}(s) = 2 \) to \( \text{Re}(s) = -1/4 \), we obtain an expression \( R_3(\omega, n) \) [143] so that:

\[ \frac{1}{2\pi i} \int_{-2-i\infty}^{-2+i\infty} \frac{\kappa_n(s)C_{3,\omega}(s)}{s(s+1)^2} \, ds = R_3(\omega, n) + O \left( n^{3/4} \right) \]

**Note:** here, we adopt the admitted abuse of notation by writing \( j \) and \( j^m \) in place of \( \beta(j) \) and \( \beta(j^m) \); so, \( \chi_3(j) \), for instance, means \( \chi_3 \) evaluated at the integer \( \beta(j) \) whose sequence of binary digits are exactly the entries of \( j \).
Definition: Let $X^+_3$ denote the set of all positive integers $n$ for which $r_3(n) > 1$; equivalently, we say that a tuple $j \in J$ is in $X^+_3$ if and only if $r_3(j) \overset{\text{def}}{=} r_3(\beta(j)) > 1$.

Theorem I (Contour integral criteria for periodic points of the Collatz Map): An odd integer $\omega$ is a periodic point of the Collatz map if and only if there is a non-zero $j \in J$ so that:

$$\frac{1}{2} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\kappa_{jm}(s) C_{3,\omega}(s)}{s(s+1)^2} ds, \forall m \in \mathbb{N}_1$$

Shifting the integral from $\text{Re}(s) = 2$ to $\text{Re}(s) = -1/4$ and taking residues into account gives the equivalent condition:

$$\frac{1}{2} = R_3(\omega,j^m) + \frac{1}{2\pi i} \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \frac{\kappa_{jm}(s) C_{3,\omega}(s)}{s(s+1)^2} ds, \forall m \in \mathbb{N}_1$$

Theorem II (Asymptotic and limit formulae): For all $\omega \in \mathbb{C} \setminus \{0\}$, all $j \in J$, and all $m \in \mathbb{N}_1$:

$$\chi_3(B(j)) = \frac{3\omega}{2} + (\chi_3(B(j)) - \omega) r_3^m(j) - \frac{\omega}{2\pi i} \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \frac{\kappa_{jm}(s) C_{3,\omega}(s)}{s(s+1)^2} ds$$

Shifting the integral from $\text{Re}(s) = 2$ to $\text{Re}(s) = -1/4$ and taking residues into account gives:

$$\chi_3(B(j)) = \frac{3\omega}{2} + (\chi_3(B(j)) - \omega) r_3^m(j) - \omega R_3(\omega,j^m) - \frac{\omega}{2\pi i} \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \frac{\kappa_{jm}(s) C_{3,\omega}(s)}{s(s+1)^2} ds$$

In particular, in both these formulae, note that since $r_3(j) < 1$ if and only if $j \in X^+_3$, for such $j$, we can take the limit as $m \to \infty$ to get:

$$\chi_3(B(j)) = \frac{3\omega}{2} - \omega \lim_{m \to \infty} \left( R_3(\omega,j^m) + \frac{1}{2\pi i} \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \frac{\kappa_{jm}(s) C_{3,\omega}(s)}{s(s+1)^2} ds \right)$$

In all of these formulae, we have the asymptotic estimate:

$$\left| \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \frac{\kappa_{jm}(s) C_{3,\omega}(s)}{s(s+1)^2} ds \right| \ll (\beta(j^m))^{3/4}$$

where the constant depends only on $\omega$. 


Theorem III (Explicit Formula for $R_3$): The following formula holds for all $\omega \in \mathbb{C} \setminus \{0\}$ for all $n \in \mathbb{N}_2$:

$$R_3(\omega, n) = -\frac{1}{\omega \ln 2} \sum_{k \neq 0} \frac{\zeta \left(1 + \frac{2k\pi i}{\ln 2}\right) + F \left(1 + \frac{2k\pi i}{\ln 2}\right)}{(1 + \frac{2k\pi i}{\ln 2})^2} \kappa_n \left(1 + \frac{2k\pi i}{\ln 2}\right) \Big(1 + \frac{2k\pi i}{\ln 2}\right) O(n)$$

$$- \frac{1}{4\ln 2} \sum_{k \neq 0} \frac{G \left(1 + \frac{2k\pi i}{\ln 2}\right)}{(1 + \frac{2k\pi i}{\ln 2})^2} \kappa_n \left(1 + \frac{2k\pi i}{\ln 2}\right) O(n)$$

$$+ \frac{3}{2} - \frac{G(1)}{4\ln 2} - \frac{1}{\omega \ln 2} \left(2 - \gamma + \frac{\ln 2}{2} + \frac{4 + \ln 2}{2} F(1) - F'(1)\right)$$

$$- \frac{1}{\omega} F(1) \frac{\kappa'_n(1) + \frac{2}{\omega \ln 2 \kappa_n(1)}}{O(1/n)}$$

$$+ \frac{2}{\omega \ln 2} \sum_{k \neq 0} \frac{\zeta \left(1 + \frac{2k\pi i}{\ln 2}\right) + F \left(1 + \frac{2k\pi i}{\ln 2}\right)}{(1 + \frac{2k\pi i}{\ln 2})^2} \kappa_n \left(1 + \frac{2k\pi i}{\ln 2}\right) \Big(1 + \frac{2k\pi i}{\ln 2}\right) O(1)$$

Here:

$$F(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \binom{s + n - 1}{n} \Xi_3 (s + n)$$

$$G(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \binom{s + n - 1}{n} \zeta_3 (s + n)$$

Outline of the Paper:

Section 1 constructs the numen $\chi_p$ and proves a variety of its properties, culminating with the Correspondence Theorem.

Section 2 consists of a review of the pertinent tools from analytical number theory (Perron’s formula, etc.) as well as a handful of technical results, all to be utilized in Sections 3 and 4. This includes some bits of $p$-adic analysis and $p$-adic Fourier analysis, particularly as regards the van der Put basis for continuous functions from $\mathbb{Z}_p$ to $\mathbb{Z}_q$ where $p$ and $q$ are distinct primes.

Section 3 analyzes the summatory functions of $r_p$ and $\chi_p$, as well as the ensuing Dirichlet series ($\zeta_3$ and $\Xi_3$, respectively). Asymptotics for the summatory functions are established, functional equations and analytic continuations are derived for the aforementioned Dirichlet series, and the Correspondence Theorem from Section 1 is reformulated as a contour integral. Using standard estimates on the growth rate of the Riemann Zeta Function (RZF) along vertical lines in the complex plane, similar growth rates are established for $\zeta_3$ and $\Xi_3$. 

7
Using these asymptotics, we can justify shifting the contour of integration in our reformulation of the Correspondence Theorem and use residues to compute an asymptotic formula for the integral in question, which we give in full.

Section 4 features an array of different minor results obtained by the author. First, a Lipschitz-type estimate for the $p$-adic absolute value of $\chi_p$. Second, the author gives his version of the probabilistic formulation given by Tao in [14]. This is accompanied by a crude but hopefully interesting inequality providing a necessary condition for an odd integer $\omega$ to be a periodic point of $H_p$; namely, roughly speaking, such a periodic point has the property that the probability that $\chi_p$ is congruent to $\omega$ modulo $p^n$ cannot decrease too quickly as $n \to \infty$.

Finally, using a sorcerous intermixing of the topologies of $\mathbb{C}$ and $\mathbb{C}_p$, $p$-adic Fourier analysis is used to provide a method (“the $L^1$-method”) of bounding the archimedean absolute values of $\mathbb{C}_p$-valued functions when those functions are restricted to appropriate subsets of their domains on which they take values in $\mathbb{Q} \subseteq \mathbb{C}_p$. This is then used to obtain upper bounds on the absolute value of $\chi_p$ over subsets of $\mathbb{Z}_2$ consisting of all those $2$-adic integers in whose $2$-adic expansions there are at least $\lceil \log_2 p \rceil$ 0s between any two consecutive 1s.

1 The Numen $\chi_p$ and the Correspondence Theorem

1.1 Construction and Characterization of $\chi_p$

As Lagarias and others have done, we begin by reinterpreting $j$ in terms of binary expansions of integers [10, 5]. In writing a tuple $j = (j_1, \ldots, j_{|j|}) \in J$, we say a tuple $i \in J$ was obtained by “adding $m$ terminal zeroes to $j$” whenever:

$$i = \left( j_1, \ldots, j_{|j|}, 0, \ldots, 0 \right)_{m \text{ times}}$$

Definition 1.1.1:

I. We define an equivalence relation $\sim$ on $J$ by the property that, for all $i, j \in J$, $i \sim j$ if and only if one of $i$ or $j$ can be obtained by adding finitely many terminal zeroes to the other.

Observe that the value of $h_i(0)$ is independent of the number of terminal zeroes in $j$:

$$h_i(0) = h_j(0), \ \forall i, j \in J \text{ such that } i \sim j$$

II. Let $J/\sim$ be the set of equivalence classes of $J$ under the equivalence relation $\sim$ defined above. Then, we define the map $\beta : J/\sim \to \mathbb{N}_0$ by:

$$\beta(j) \stackrel{\text{def}}{=} \sum_{\ell=1}^{\lfloor j \rfloor} j_\ell 2^{\ell-1} \quad (18)$$
Note:

i. $\beta(j)$ is well-defined with respect to $\sim$, since adding terminal zeroes to $j$ does not change the value of $\beta(j)$.

ii. Observe that $\beta(i) = \beta(j)$ occurs if and only if $i \sim j$.

iii. The map $\beta$ is well-known in Collatz literature, where—following Lagarias’ convention—it is usually denoted by $Q$. [10]

III. Our first definition of the **numen** of $H_p$ is the function $\chi_p : \mathbb{N}_0 \to \mathbb{Q}$ defined by the rule:

$$\chi_p(t) \overset{\text{def}}{=} h_{\beta^{-1}(t)}(0), \ \forall t \in \mathbb{N}_0$$

where $\beta^{-1}(t)$ is any element $j$ of $\mathcal{J}$ so that $\beta(j) = t$.

$\chi_p$ is well-defined, since the choice of a pre-image of $t$ under $\beta$ does not affect the value of $\chi_p(t)$.

**Definition 1.1.2:**

I. For all $t \in \mathbb{N}_0$, let $\#_1(t)$ denote the number of 1s in the binary/2-adic expansion of $t$. We extend $\#_1$ to $\mathbb{Z}_2$ by defining $\#_1(\bar{z}) = \infty$ for all $\bar{z} \in \mathbb{Z}_2 \setminus \mathbb{N}_0$.

II. For all $t \in \mathbb{N}_0$, let $\lambda(t)$ denote the number of digits (0s and 1s) in the binary/2-adic expansion of $t$. Using the convention that $\lambda(0) = 0$, it follows that:

$$\lambda(t) = \lceil \log_2 (t + 1) \rceil, \ \forall t \in \mathbb{N}_0$$

We extend $\lambda$ to $\mathbb{Z}_2$ by defining $\lambda(\bar{z}) = \infty$ for all $\bar{z} \in \mathbb{Z}_2 \setminus \mathbb{N}_0$. Note also that $\lambda(t) = \lceil \log_2 t \rceil + 1$ for all $t \in \mathbb{N}_1$.

Note that $\#_1$ and $\lambda$ satisfy the functional equations:

$$\#_1(2^mt + k) = \#_1(t) + \#_1(k), \ \forall t \in \mathbb{N}_0, \ \forall m \in \mathbb{N}_1, \ \forall k \in \{0, \ldots, 2^m - 1\}$$

$$\lambda(2^mt + k) = \lambda(t) + m, \ \forall t \in \mathbb{N}_1, \ \forall m \in \mathbb{N}_1, \ \forall k \in \{0, \ldots, 2^m - 1\}$$

In particular:

$$\#_1(2t) = \#_1(t)$$

$$\#_1(2t + 1) = \#_1(t) + 1$$

for all $t \in \mathbb{N}_0$, and:

$$\lambda(2t) = \lambda(2t + 1) = \lambda(t) + 1$$

for all $t \in \mathbb{N}_1$.

Letting $t \in \mathbb{N}_1$, and writing $\beta^{-1}(t)$ to denote the shortest tuple $j$ for which $\beta(t) = j$, we then have the identities:

$$\#_1(\beta^{-1}(t)) = \#_1(t)$$

$$|\beta^{-1}(t)| = \lambda(t)$$
The first identity tells us that, given a positive integer \( t \), the number of 1s in the string \( j = \beta^{-1} (t) (\#_1 (\beta^{-1} (t))) \) is equal to the number of 1s in the 2-adic expansion of \( t (\#_1 (t)) \). The second identity tells us that, given a positive integer \( t \), the length of the shortest tuple \( j \) for which \( \beta (j) = t (|\beta^{-1} (t)|) \) is equal to the number of digits in the 2-adic expansion of \( t (\lambda (t)) \).

Letting \( t \in \mathbb{N}_1 \), we can write \( 5 \) using \( \beta_1 (t) \) in lieu of \( j \). This gives us:

\[
h_{\beta^{-1}(t)} (x) = \frac{p^{\#_1 (\beta^{-1}(t))}}{2^{\|\beta^{-1}(t)\|}} x + h_{\beta^{-1}(t)} (0), \quad \forall x \in \mathbb{Z}, \forall t \in \mathbb{N}_1
\]  

(23)

Applying Remark 1.1.5, using \( \chi_p (t) \) to denote \( h_{\beta^{-1}(t)} (0) \), we then obtain:

\[
h_{\beta^{-1}(t)} (x) = \frac{p^{\#_1 (t)}}{2^{\lambda (t)}} x + \chi_p (t), \quad \forall x \in \mathbb{Z}, \forall t \in \mathbb{N}_1
\]  

(24)

Next, as is well known, every positive integer \( t \) has a unique binary expansion of the form:

\[
t = \sum_{k=1}^{\#_1 (t)} 2^{n_k}
\]

for some strictly increasing sequence of non-negative integers \( 0 \leq n_1 < n_2 < \cdots < n_K < \infty \). We can extend this notation to include 0 by defining the sum to be 0 when \( t = 0 \). Using this notation, we can give our first somewhat closed-form expression for \( \chi_p (t) \).

**Proposition 1.1.1:** Let \( t \in \mathbb{N}_0 \), and write the 2-adic expansion of \( t \) as \( \sum_{k=1}^{\#_1 (t)} 2^{n_k} \). Then:

\[
\chi_p \left( \sum_{k=1}^{\#_1 (t)} 2^{n_k} \right) = \sum_{k=1}^{\#_1 (t)} \frac{p^{k-1}}{2^{n_k+1}}
\]

(25)

where the sum on the right is defined to be 0 whenever \( t = 0 \).

**Remark 1.1.6:** This formula is well-known in Collatz literature; see [14] for but one example. Indeed, as remarked by Tao, it follows by induction that, as defined, \( \chi_p \) is injective (Ibid). However, this injectivity does not necessarily hold when we consider the value of \( \chi_p \) modulo \( p^N \) for some \( N \in \mathbb{N}_1 \). This line of thinking leads to the aforementioned Lipschitz-type estimates that instigated the researches detailed in this paper.

**Proof:** Since \( \chi_p (0) \) is defined to be 0, formula (25) holds for \( t = 0 \). So, fix \( K \in \mathbb{N}_1 \) and let \( t \) be any positive integer for which \( \#_1 (t) = K \). We write \( t = \sum_{k=1}^{K} 2^{n_k} \), and proceed by induction on \( K \).

When \( K = 1 \), \( \chi (2^{n_1}) = h_{\beta^{-1}(2^{n_1})} (0) \). Note that the \( n_1 + 1 \) tuple \( j = \)
\[
\left(0, \ldots, 0, 1\right)_{n_1 \text{ times}} \text{ then satisfies:}
\]
\[
\beta(j) = \sum_{t=1}^{n_1+1} j2^{t-1} = 0 \cdot 2^{1-1} + 0 \cdot 2^{2-1} + \cdots + 0 \cdot 2^{n_1-1} + 1 \cdot 2^{(n_1+1)-1} = 2^{n_1}
\]

and so:
\[
\chi_p(2^{n_1}) = h_{\beta^{-1}(2^{n_1})}(0) = h_{0^{n_1}}(h_1(0)) = \frac{1}{2^{n_1}} \cdot \frac{p \cdot 0 + 1}{2} = \frac{1}{2^{n_1+1}} = \sum_{k=1}^{1} p^{k-1}
\]

Thus, the formula holds for \( K = 1 \).

Now, proceeding to the inductive step, suppose the formula \( 25 \) holds for an arbitrary \( K \geq 1 \). So, let \( t \) be any non-negative integer with at most \( K \) non-zero binary digits. Without loss of generality, suppose \( t \) has \( K \) non-zero binary digits, with \( t = \sum_{k=1}^{K} 2^{n_k} \) for some \( n_k \). Note that \( \lambda(t) = n_K + 1 \) (that is, \( 2^{\lambda(t)-1} = 2^{\left\lfloor \log_2 t \right\rfloor} \) is the largest power of 2 which is less than or equal to \( 2 \)).

Next, let \( n_{K+1} \) be any integer greater than \( n_K \). Then:
\[
\sum_{k=1}^{K+1} 2^{n_k} = t + 2^{n_{K+1}} = t + 0 \cdot 2^{n_{K+1}} + \cdots + 0 \cdot 2^{n_{K+1}+(n_{K+1}-n_K-1)} + 2^{n_{K+1}}
\]

Hence:
\[
\chi_p\left( \sum_{k=1}^{K+1} 2^{n_k} \right) = \chi_p\left( t + 2^{n_{K+1}} \right) = \left( h_{\beta^{-1}(t)} \circ h_{0^{n_K+1-n_K-1}} \circ h_1 \right)(0) = h_{\beta^{-1}(t)}\left( \frac{1}{2 \cdot 2^{n_K+1-n_K-1}} \right) = h_{\beta^{-1}(t)}\left( \frac{1}{2^{n_{K+1}-n_K}} \right)
\]

\( \text{use } 24 \):
\[
\chi_p(t) = \sum_{k=1}^{K} p^{k-1} \cdot 2^{n_k+1}; \#1(t) = K
\]

\[
\begin{align*}
\left( n_K = \lambda(t) - 1; \chi_p(t) = \sum_{k=1}^{K} p^{k-1} \cdot 2^{n_k+1}; \#1(t) = K \right) & = \frac{p^K}{2^{n_{K+1}+1}} + \sum_{k=1}^{K} \frac{p^{k-1}}{2^{n_k+1}} \\
& = \sum_{k=1}^{K+1} \frac{p^{k-1}}{2^{n_k+1}}
\end{align*}
\]
Thus, the $K$th case implies the $(K+1)$th case. As such, by induction, the formula holds for all $K \in \mathbb{N}_0$.

Q.E.D.

Next, we $2$-adically interpolate $\chi_p$ into a function $\chi_p : \mathbb{Z}_2 \to \mathbb{Z}_p$. To do this, we note that, in $25$, as $\#_1 (t) \to \infty$, the series $\sum_{k=1}^{\#_1(t)} \frac{p^{k-1}}{2^{n_k + 1}}$ converges to a limit in $\mathbb{Z}_p$. As such:

**Definition 1.1.3:** We $2$-adically interpolate $\chi_p$ to a function $\chi_p : \mathbb{Z}_2 \to \mathbb{Z}_p$ by way of the rule:

$$\chi_p \left( \sum_{k=1}^{\infty} a_k 2^{n_k} \right) = \sum_{k=1}^{\infty} \frac{p^{k-1}}{2^{n_k + 1}}$$

(26)

where $\{n_k\}_{k \in \mathbb{N}_1}$ is any strictly increasing sequence of non-negative integers. We write $\equiv$ to emphasize that, in the above equality, the series’ convergence occurs in $\mathbb{Z}_p$, rather than in $\mathbb{R}$.

While this gives us a valid $2$-adic interpolation of $\chi_p$, this interpolation is not continuous on all of $\mathbb{Z}_2$. As an example, the sequence $\{2^n\}_{n \in \mathbb{N}_1}$ converges to 0 in $\mathbb{Z}_2$, yet, using $25$, we have that $\chi_p (2^n) = 2^{-n-1}$. Were $\chi_p : \mathbb{Z}_2 \to \mathbb{Z}_p$ continuous, $\chi_p (2^n) = 2^{-n-1}$ would converge in $\mathbb{Z}_p$ to $\chi_p (0) = 0$, but $2^{-n-1}$ has unit magnitude in $\mathbb{Z}_p$ for all $n \in \mathbb{N}_1$. Nevertheless, we do have the following:

**Lemma 1.1.1:** $\chi_p : \mathbb{Z}_2 \to \mathbb{Z}_p$ is continuous as a function from $\mathbb{Z}_2 \setminus \mathbb{N}_0$ to $\mathbb{Z}_p$.

(That is to say, in the terminology of Section 2.2, $\chi_p$ is “rising-continuous”.) In particular:

$$\chi_p (\bar{3}) \equiv \lim_{n \to \infty} \chi_p (\bar{3}2^n)$$

(27)

Proof: Let $\bar{3} \in \mathbb{Z}_2 \setminus \mathbb{N}_0$ be arbitrary, and write $\bar{3} = \sum_{n=0}^{\infty} a_n 2^n$, where the $a_n$s are the $2$-adic digits of $\bar{3}$. Then, let $\{m\}_{m \in \mathbb{N}_1} \subseteq \mathbb{Z}_2 \setminus \mathbb{N}_0$ be any sequence converging to $\bar{3}$ in $2$-adic absolute value. or each $m$, we write $\bar{3}_m = \sum_{n=0}^{\infty} a_{m,n} 2^n$. This convergence is equivalent to saying that, for every $N \in \mathbb{N}_1$, there is an $m_N \in \mathbb{N}_1$ so that $\bar{3}_m \equiv \bar{3}$ for all $m \geq m_N$, and hence, that $a_{m,n} = a_n$ for all $n \in \{0, \ldots, N - 1\}$ whenever $m \geq m_N$. As such:

$$\bar{3}_m \equiv \sum_{n=0}^{N-1} a_n 2^n + \sum_{n=N}^{\infty} a_{m,n} 2^n, \forall m \geq m_N$$

Next, we can write the rational integer $\sum_{n=0}^{N-1} a_n 2^n$ as:

$$\sum_{n=0}^{N-1} a_n 2^n = \sum_{k=1}^{K_N} 2^{n_k}$$

for some $K_N \in \mathbb{N}_0$ (where $\sum_{n=0}^{N-1} a_n 2^n = 0$ whenever $K_N = 0$). In terms of $N$, $K_N$ is the number of $1$s present in the $2$-adic expansion of $[\bar{3}_{2N}] = \sum_{n=0}^{N-1} a_n 2^n$. 

12
Anyhow, using \[26\] we then have that:

\[
\chi_p(\z) \equiv \chi_p \left( \sum_{k=1}^{K_N} 2^{n_k} + \sum_{n=N}^{\infty} a_{m,n} 2^n \right) \equiv \sum_{k=1}^{K_N} p^{k-1} 2^{n_k+1} + O(p^{K_N})
\]

Indeed, here, \(\sum_{n=N}^{\infty} a_{m,n} 2^n\) —the error between \(\z\) and \(\z_m\) —contains any and all 1s in the 2-adic expansion of \(\z_m\) that occur after the \(K_N\)th 1 in said 2-adic expansion. \[26\] tells us that \(2^{n_k}\)—the term of the 2-adic expansion of \(\z_m\) corresponding to the \(n_k\)th 1 in said 2-adic expansion—yields the term \(\frac{p^{k-1}}{2^{n_k+1}}\) in \(\chi_p(\z_m)\). As such, since \(\sum_{n=N}^{\infty} a_{m,n} 2^n\) contains all the 1s after the \(K_N\)th 1, it must be that all of the corresponding terms of \(\chi_p(\z)\) have powers of \(p\) which are \(\geq p^{K_N+1-1} = p^{K_N}\); hence the \(O(p^{K_N})\) term. Similarly:

\[
\chi_p(\z) \equiv \chi_p \left( \sum_{k=1}^{K_N} 2^{n_k} + \sum_{n=N}^{\infty} a_{m,n} 2^n \right) \equiv \sum_{k=1}^{K_N} p^{k-1} 2^{n_k+1} + O(p^{K_N})
\]

As such, for any \(N \in \mathbb{N}_1\), \(|\z - \z_m|_2 \leq p^N\) implies that there are positive integers \(K_N\) and \(m_N\) so that:

\[
|\chi_p(\z) - \chi_p(\z_m)|_p = O(p^{-K_N}), \quad \forall m \geq m_N.
\]

Now, the clincher: since \(\z\) is a 2-adic integer which is \textit{not} in \(\mathbb{N}_0\), the 2-adic expansion of \(\z\) contains infinitely many 1s. Thus, as \(N \to \infty\), the number of 1s digits in the 2-adic expansion of the integer \(\z|_{2^N} = \sum_{n=0}^{N-1} a_n 2^n\) will tend to \(\infty\). “The number of 1s digits in the 2-adic expansion of the integer \(\z|_{2^N}\)” is, of course, exactly the quantity we denoted by \(K_N\). Thus, \(\z\) being an element of \(\mathbb{Z}_2 \setminus \mathbb{N}_0\) guarantees that the magnitude of \(|\chi_p(\z) - \chi_p(\z_m)|_p\) tends to \(0\) in \(\mathbb{R}\) as \(N \to \infty\). As such, we have that \(\z_m \to \z\) in \(\mathbb{Z}_2\) forces \(\chi_p(\z_m)\) to converge to \(\chi_p(\z)\) in \(\mathbb{Z}_p\) whenever \(\z \in \mathbb{Z}_2 \setminus \mathbb{N}_0\), which proves the continuity of \(\chi_p : \mathbb{Z}_2 \setminus \mathbb{N}_0 \to \mathbb{Z}_p\). Q.E.D.

Using some extra bits of terminology, we can give \(\chi_p\) a more closed-form expression.

**Definition 1.1.4:** Let \(\z \in \mathbb{Z}_2\). Then, we write \(\{\beta_k(\z)\}_{k \in \{1, \ldots, \#(\z)\}}\) to denote the unique, strictly increasing sequence of non-negative integers so that the 2-adic expansion of \(\z\) is given by:

\[
\z = \sum_{k=1}^{\#(\z)} 2^{\beta_k(\z)}
\]  

(28)

The above sum is defined to be 0 whenever \(\z = 0\).

**Theorem 1.1.1** (\(\chi_p\) functional equations): \(\chi_p\) is the unique function from
\(Z_2\) to \(Z_p\) satisfying the functional equations:

\[\begin{align*}
\chi_p(2^k z) &= \frac{1}{2} \chi_p(z) \\
\chi_p(2^k z + 1) &= p \chi_p(z) + 1
\end{align*}\]

(29)

for all \(z \in Z_2\).

**Proof:** Let \(t = \sum_{k=1}^{K} 2^{n_k} \in \mathbb{N}_0\). Then (25) implies:

\[\chi_p \left( 2 \sum_{k=1}^{K} 2^{n_k} \right) = \chi_p \left( \sum_{k=1}^{K} 2^{n_k+1} \right) = \sum_{k=1}^{K} p^{k-1} 2^{n_k+2} = \frac{1}{2} \sum_{k=1}^{K} p^{k-1} 2^{n_k+2} = \chi_p \left( \sum_{k=1}^{K} 2^{n_k} \right)\]

Similarly:

\[\chi_p \left( 2 \left( \sum_{k=1}^{K} 2^{n_k} \right) + 1 \right) = \chi_p \left( 1 + 2^{n_1+1} + 2^{n_2+1} + \cdots + 2^{n_K+1} \right)\]

\[= \frac{1}{2} \left( 1 + p^{n_1+1} + \frac{p}{2^{n_2+2}} + \cdots + \frac{p^K}{2^{n_K+2}} \right)\]

\[= \frac{1}{2} \left( 1 + p \chi_p \left( \sum_{k=1}^{K} 2^{n_k} \right) \right)\]

Hence (29) hold true for all \(z \in \mathbb{N}_0\). By the 2-adic interpolability of \(\chi_p\), we have that \(\sum_{k=1}^{K} 2^{n_k}\) tends to an arbitrary element of \(Z_2 \setminus \mathbb{N}_0\) as \(K \to \infty\), which then shows that \(\chi_p\) satisfies the functional equations for all \(z \in Z_2 \setminus \mathbb{N}_0\). Since \(\chi_p\) also satisfies these functional equations for all \(z \in \mathbb{N}_0\), it then follows that it satisfies them for all \(z \in Z_2\).

Finally, let \(f : Z_2 \to Z_p\) be any function satisfying:

\[f(2^k z) = \frac{1}{2} f(z)\]

\[f(2^k z + 1) = \frac{p f(z) + 1}{2}\]

for all \(z \in Z_2\). Setting \(t = 0\) gives \(f(0) = \frac{1}{2} f(0)\), which forces \(f(0) = 0\). Consequently, \(f(1) = f(2 \cdot 0 + 1) = \frac{p f(0) + 1}{2} = \frac{1}{2}\), and so \(f(2^{n_1}) = \frac{1}{2^{n_1+r}} f \left( \frac{1}{2} \right) = \frac{1}{2^{n_1+r}}\).
Then, for any $n_2 > n_1$:

$$f(2^{n_1} + 2^{n_2}) = f\left(2^{n_1} \left(2^{n_2-n_1} + 1\right)\right)$$

$$= \frac{1}{2^{n_1}} f\left(2^{n_2-n_1} + 1\right)$$

$$= \frac{1}{2^{n_1}} \frac{p f\left(2^{n_2-n_1-1} + 1\right)}{2}$$

$$= \frac{1}{2^{n_1+1}} + \frac{p}{2^{n_1+1}} f\left(2^{n_2-n_1-1}\right)$$

$$= \frac{1}{2^{n_1+1}} + \frac{p}{2^{n_1+1}} f\left(1\right)$$

$$= \frac{1}{2^{n_1+1}} + \frac{p}{2^{n_2+1}}$$

Continuing in this manner, by induction, it then follows that:

$$f\left(\sum_{k=1}^{K} 2^{n_k}\right) = \sum_{k=1}^{K} \frac{p^{k-1}}{2^{n_k+1}} = \chi_p \left(\sum_{k=1}^{K} 2^{n_k}\right)$$

and thus, that $f(t) = \chi_p(t)$ holds for all $t \in \mathbb{N}_0$. Applying the argument from part (I) then shows that $f(t) = \chi_p(t)$ on $\mathbb{N}_0$ then determines the values of $f$ at every other 2-adic integer, forcing $f(\beta) = \chi_p(\beta)$ for all $t \in \mathbb{Z}_2$.

Q.E.D.

### 1.2 The Correspondence Theorem

In this subsection we prove the titular Correspondence Theorem (Theorem 1.2.1): an odd integer $\omega$ is a periodic point of $H_p$ if and only if $\omega = \chi_p(\beta)$ for some $\beta \in \mathbb{Z}_2$. To do this, we begin with a nearly tautological observation.

**Proposition 1.2.1:** Let $\omega \in \mathbb{Z}$ be a periodic point of $H_p$ (i.e., $H_p^n(\omega) = \omega$ for some $n \in \mathbb{N}_1$). There exists a $t \in \mathbb{N}_0$ so that:

$$\chi_p(t) \equiv \left(1 - \frac{p^{\#_1(t)}}{2^{\lambda(t)}}\right) \omega$$

(30)

Additionally, for any $\beta \in \mathcal{J}$ for which $h_\beta(\omega) = \omega$, it must be that:

$$\chi_p(\beta) \equiv \left(1 - \frac{p^{\#_1(\beta)}}{2^{\lambda(\beta)}}\right) \omega$$

(31)

**Proof:** If $\omega \in \mathbb{N}_1$ is a periodic point of $H_p$ (say, of period $n$), there is then a unique $\beta \in \mathcal{J}$ of length $n$ so that $h_\beta(\omega) = H_p^n(\omega) = \omega$. As such, for $t = \beta$:

$$\omega = h_\beta(\omega) h_{\beta^{-1}}(\omega) = \frac{p^{\#_1(t)} \omega}{2^{\lambda(t)}} + \chi_p(t)$$

$$\Downarrow$$

$$\left(1 - \frac{p^{\#_1(t)}}{2^{\lambda(t)}}\right) \omega = \chi_p(t)$$
By the same reasoning, for any \( j \in J \) for which \( h_j(\omega) = \omega \), it must be that:

\[
\chi_p(\beta(j)) = \left(1 - \frac{p_{\#1(\beta(j))}}{2\lambda(\beta(j))}\right)\omega
\]

holds true as an equality in \( \mathbb{R} \), as desired.

Q.E.D.

**Lemma 1.2.1:** Every periodic point of \( H_p \) in \( \mathbb{N}_1 \) is of the form:

\[
\chi_p \left( \frac{t}{1 - 2^\lambda(t)} \right)
\]

for some \( t \in \mathbb{N}_1 \). In particular, if \( \omega \) is the periodic point of \( H_p \), then:

\[
\omega = \chi_p \left( \frac{\beta(j)}{1 - 2^\lambda(\beta(j))} \right)
\]

where \( j \) is any element of \( J \) such that \( h_j(\omega) = \omega \).

Proof: For a periodic point \( \omega \), let \( j \in J \) be the shortest tuple for which \( h_j(\omega) = \omega \), and let \( j^n \) denote the concatenation of \( n \) copies of \( j \). No matter how many copies of \( j \) we concatenate, the associated composition sequence leaves \( \omega \) fixed:

\[
h_{j^n}(\omega) = h_j^{\circ n}(\omega) = h_j^{\circ n-1}(h_j(\omega)) = h_j^{\circ n-1}(\omega) = \cdots = \omega
\]

As such, \( \text{31} \) yields:

\[
\chi_p(\beta(j^n)) = \left(1 - \frac{p_{\#1(\beta(j^n))}}{2\lambda(\beta(j^n))}\right)\omega
\]

Now, observe that:

\[
\beta(j^n) = \sum_{m=0}^{n-1} 2^{j_{2m}} \left(j_{1} + j_{2} \cdot 2 + \cdots + j_{2^{m}} \cdot 2^{j_{2^{m}} - 1}\right) = \frac{1 - 2^{j_{2n}}}{1 - 2^{j_{1}}}} \beta(j)
\]

Letting \( t \) denote \( \beta(j) \), we write:

\[
t_n \overset{\text{def}}{=} \beta(j^n), \quad \forall n \in \mathbb{N}_1
\]

Since \( |j| = \lambda(t) \), we can then write:

\[
t_n = \frac{1 - 2^n\lambda(t)}{1 - 2^{\lambda(t)}t}
\]

Additionally, since \( j^n \) is the concatenation of \( n \) copies of \( j \), we have that:

\[
\#_1(\beta(j^n)) = \#_1(j^n) = \#_1(j)n = \#_1(t)n
\]

\[
\lambda(\beta(j^n)) = |j^n| - 1 = n|j| - 1 = n\lambda(t) - 1
\]

16
As such:

$$\chi_p (\beta (j^n)) = \left(1 - \frac{p^\#_1 (\beta (j^n))}{2^{\lambda (\beta (j^n))}}\right) \omega$$

$$\downarrow$$

$$\chi_p \left(\frac{1 - 2^n \lambda (t)}{1 - 2^{\lambda (t)}} t\right) = \left(1 - \left(\frac{p^\#_1 (t)}{2^{\lambda (t)}}\right)^n\right) \omega$$ (36)

Finally, since:

$$\lim_{n \to \infty} \frac{1 - 2^n \lambda (t)}{1 - 2^{\lambda (t)}} t \overset{z_2}{=} \frac{t}{1 - 2^{\lambda (t)}}, \quad \forall t \in \mathbb{N}_0$$

and since \( \frac{t}{1 - 2^{\lambda (t)}} \in \mathbb{N}_0 \) if and only if \( t = 0 \), the rising continuity of \( \chi_p : \mathbb{Z}_2 \to \mathbb{Z}_p \) guarantees that:

$$\chi_p \left(\frac{t}{1 - 2^{\lambda (t)}}\right) \overset{z_2}{=} \lim_{n \to \infty} \chi_p \left(\frac{t}{1 - 2^{\lambda (t)}} 2^n \lambda (t)\right)$$

$$\overset{z_2}{=} \lim_{n \to \infty} \chi_p \left(\frac{1 - 2^n \lambda (t)}{1 - 2^{\lambda (t)}} t\right) \overset{z_2}{=} \lim_{n \to \infty} \left(1 - \left(\frac{p^\#_1 (t)}{2^{\lambda (t)}}\right)^n\right) \omega$$

$$\overset{z_2}{=} \omega$$

This proves every periodic point of \( H_p \) in \( \mathbb{Z} \) is of the form \( 32 \) for some \( t \in \mathbb{N}_1 \); in particular, \( t = \beta (j) \), where \( j \) is any element of \( J \) satisfying \( h_j (\omega) = \omega \), which gives us (33)

Q.E.D.

**Definition 1.2.1**: Let \( B : \mathbb{N}_0 \to \mathbb{Z}_2 \) be defined by:

$$B (t) \overset{\text{def}}{=} \frac{t}{1 - 2^{\lambda (t)}} = \frac{t}{1 - 2^{\log_2 (t+1)}} \quad \forall t \in \mathbb{N}_0$$ (37)

**Proposition 1.2.2**: \( B \) extends to a continuous function from \( \mathbb{Z}_2 \) to \( \mathbb{Z}_2 \). In particular:

I. The restriction of \( B \) to \( \mathbb{Z}_2 \setminus \mathbb{N}_0 \) is the identity map: \( B (t) = t \) for all \( t \in \mathbb{Z}_2 \setminus \mathbb{N}_0 \). (Hence, with respect to the Haar probability measure of \( \mathbb{Z}_2 \), \( B \) is equal to the identity map almost everywhere on \( \mathbb{Z}_2 \).)

II. \( B (2^n - 1) = -1 \) for all \( n \in \mathbb{N}_1 \).

Proof: As a sequence \( \{t_n\}_{n \in \mathbb{N}_1} \) of positive integers converges 2-adically to a limit \( t \in \mathbb{Z}_2 \), one of two things can happen. If \( \lambda (t_n) \) remains bounded as \( n \to \infty \), then \( t \) must be an element of \( \mathbb{N}_0 \), and so, \( \lim_{n \to \infty} B (t_n) = B (t) \). The other possibility is that \( \lambda (t_n) \to \infty \) as \( n \to \infty \), in which case, \( 2^{\lambda (t_n)} \) tends to 0 in \( \mathbb{Z}_2 \), and hence:

$$\lim_{n \to \infty} B (t_n) = \lim_{n \to \infty} \frac{t_n}{1 - 2^{\lambda (t_n)}} \overset{z_2}{=} \lim_{n \to \infty} \frac{t_n}{1 - 0} = t$$

17
which shows that $B(t) = t$ for all $t \in \mathbb{Z}_2 \setminus \mathbb{N}_0$. Thus, we can continuously (and, therefore, uniquely) extend $B$ to a function on $\mathbb{Z}_2$, and this extension equals the identity map for all $t \in \mathbb{Z}_2 \setminus \mathbb{N}_0$, a set of full measure in $\mathbb{Z}_2$.

Finally, for (II), note that $2^n - 1 = \frac{2^n - 1}{2 - 1} = \sum_{k=0}^{n-1} 2^k$, and so, $\lambda(2^n - 1) = n$, and so:

$$B(2^n - 1) = \frac{2^n - 1}{1 - 2^n} = \frac{2^n - 1}{1 - 2^n} = \frac{2^n - 1}{1 - 2^n} = -1$$

as claimed.

Q.E.D.

Using $B(t)$, we can re-interpret as a functional equation relating $\chi_p(B(t))$ and $\chi_p(t)$.

**Lemma 1.2.2** ($\chi_p \circ B$ functional equation):

$$\chi_p(B(t)) \equiv \frac{\chi_p(t)}{1 - r_p(t)}, \quad \forall t \in \mathbb{N}_1 \tag{38}$$

**Proof:** Letting $t \in \mathbb{N}_1$, we have that:

$$\chi_p(B(t)) \equiv \chi_p \left( \sum_{m=0}^{\infty} \sum_{k=1}^{\#_1(t)} 2^{m\lambda(t) + \beta_k(t)+1} \right)$$

$$= \sum_{k=1}^{\#_1(t)} \frac{p^{k-1}}{2^{\beta_k(t)+1}} + \sum_{k=1}^{\#_1(t)} \frac{p^{k+\#_1(t)-1}}{2^{\lambda(t)+\beta_k(t)+1}} + \sum_{k=1}^{\#_1(t)} \frac{p^{k+2\#_1(t)-1}}{2^{2\lambda(t)+\beta_k(t)+1}} + \cdots$$

$$= \sum_{m=0}^{\infty} \sum_{k=1}^{m\#_1(t)} \frac{p^{k+m\#_1(t)-1}}{2^{m\lambda(t)+\beta_k(t)+1}}$$

$$= \sum_{m=0}^{\infty} \left( \frac{p^{\#_1(t)}}{2^{\lambda(t)}} \right)^m \sum_{k=1}^{\#_1(t)} \frac{p^{k-1}}{2^{\beta_k(t)+1}} \chi_p(t)$$

(if $t \neq 0$; $\equiv \chi_p(t) \frac{1 - p^\#_1(t)}{1 - p^\#_1(t)}$)

$$= \frac{\chi_p(t)}{1 - r_p(t)}$$

Thus, the identity holds for all $t \in \mathbb{N}_1$.

Q.E.D.

**Lemma 1.2.2** shows that $B(\mathbb{N}_1)$ is a subset of $\mathbb{Q} \cap \mathbb{Z}_2$ mapped into $\mathbb{Q}$ by $\chi_p$.

It would be of interest to know if the “converse” holds true; that is, is $B(\mathbb{N}_1)$ the pre-image of $\mathbb{Q} \cap \mathbb{Z}_p$ under $\chi_p$?

**Proposition 1.2.3**: allows us to view $\chi_p \circ B$ as a function from $\mathbb{N}_1$ to $\mathbb{R}$ defined by the rule:
\[ \chi_p(B(t)) = \frac{\chi_p(t)}{1 - r_p(t)} \]  

(39)

Proof: Because the series defining \( \chi_p(B(t)) \) simplifies in \( \mathbb{Z}_p \) to the product of a rational number (\( \chi_p(t) \)) and a \( p \)-adically convergent geometric series whose common ratio is a rational number, the universality of the geometric series formula allows us to conclude that for every \( t \in \mathbb{N}_0 \), the value of \( \chi_p(B(t)) \) is an honest-to-goodness rational number. Consequently, using the right-hand side of (38) we can define \( \chi_p \circ B \) as a function from \( \mathbb{N}_1 \) to \( \mathbb{R} \).

Q.E.D.

Finally, we have the main result of this section, which shows why the numen is of interest:

**Theorem 1.2.1 (The Correspondence Theorem):** An odd integer \( \omega \) is a periodic point of \( H_p \) if and only if:

\[ \omega = \chi_p(B(t)) = \frac{\chi_p(t)}{1 - \frac{p^{\#_1(t)}}{2^{\lambda(t)}}} = \sum_{k=1}^{\#_1(t)} \frac{2^{\lambda(t)-\beta_k(t)-1}p^{-k-1}}{2^{\lambda(t)} - p^{\#_1(t)}} \]  

(40)

for some \( t \in \mathbb{N}_0 \).

Proof: Let \( \omega \) be an odd non-zero periodic point of \( H_p \). Then, by Lemma 1.2.1, there is a \( t \in \mathbb{N}_1 \) so that \( \chi_p(t) = \left(1 - \frac{p^{\#_1(t)}}{2^{\lambda(t)}}\right)\omega \). Hence:

\[ \omega = \frac{\chi_p(t)}{1 - \frac{p^{\#_1(t)}}{2^{\lambda(t)}}} = \chi_p(B(t)) \]

as desired.

II. Let \( t \in \mathbb{N}_1 \), and suppose that \( \chi_p(B(t)) \in \mathbb{Z} \). Letting \( \omega \) denote \( \chi_p(B(t)) \)—and noting that we then have \( \omega = \frac{\chi_p(t)}{1 - \frac{p^{\#_1(t)}}{2^{\lambda(t)}}} \)—by 23 we can write:

\[ h_{\beta(t)}(\omega) = \frac{p^{\#_1(t)}\omega}{2^{\lambda(t)}} + \chi_p(t) = \frac{p^{\#_1(t)}\omega}{2^{\lambda(t)}} + \omega \left(1 - \frac{p^{\#_1(t)}}{2^{\lambda(t)}}\right) = \omega \]

Letting \( j \in \beta^{-1}(t) \), it then follows that \( h_j(\omega) = \omega \).

Claim: Let \( t \in \mathbb{N}_1 \), and suppose there is a \( j \in \beta^{-1}(t) \) so that \( h_j(\omega) = \omega \). Then, \( \omega \) is a periodic point of \( H_p \).

Proof of Claim: Let \( t \) and \( j \) be as given. By way of contradiction, suppose that \( \omega \) was not a periodic point of \( H_p \). Since the composition sequence \( h_j \) fixes \( \omega \), our counterfactual assumption on \( \omega \) forces there to be an \( n \in \{1, \ldots, |j|\} \) so that:

\[ H^n(\omega) = h_{j_1, \ldots, j_n}(\omega) \]

but for which:

\[ H^{n+1}(\omega) \neq h_{j_1, \ldots, j_n, j_{n+1}}(\omega) \]
(Note that \( t \in \mathbb{N}_1 \) then guarantees that \(|j| \geq 1\). So, at some point in applying to \( \omega \) the maps in the composition sequence defining \( h_j \), we must have applied the “wrong” map; that is, \( h_{j_1,\ldots,j_n} (\omega) \) was odd (resp. even), but \( j_{n+1} = 0 \) (resp. \( j_{n+1} = 1 \)), causing the composition sequence \( h_j \) to diverge from the “natural” path corresponding to the motions of \( \omega \) under iterations of \( H \).

Noting that for any \( m \in \mathbb{N}_0 \):

\[
h_0 (2m + 1) = \frac{2m + 1}{2} \in \mathbb{Z} \left[ \frac{1}{2} \right] \backslash \mathbb{Z}
\]

\[
h_1 (2m) = \frac{2pm + 1}{2} \in \mathbb{Z} \left[ \frac{1}{2} \right] \backslash \mathbb{Z}
\]

we see that applying the “wrong” map at any step in the composition sequence sends us from \( \mathbb{Z} \) to an element of \( \mathbb{Z} \left[ \frac{1}{2} \right] \) (the dyadic rational numbers) of the form \( \frac{a}{2} \), where \( a \) is an odd integer. So, letting \( m \in \mathbb{N}_1 \) and \( a \in 2\mathbb{Z} + 1 \) be arbitrary, we have that:

\[
|h_0 \left( \frac{a}{2^m} \right)|_2 = \left| \frac{a}{2^m+1} \right|_2 = 2 \left| \frac{a}{2^m} \right|_2
\]

\[
|\frac{a}{2^m+1}|_2 = \left| \frac{pa + 2m+1}{2^{m+1}} \right|_2 = \text{St}_\Delta \left| \frac{pa}{2^{m+1}} \right|_2 = 2 \left| \frac{a}{2^m} \right|_2
\]

where \( \text{St}_\Delta \) denotes an application of the strong triangle inequality, using the facts that \( a \) and \( p \) are odd and \( m \geq 1 \).

Since a dyadic rational number \( q \) is a non-integer if and only if \( q = \frac{a}{2^m} \) for some \( a \in 2\mathbb{Z} + 1 \) and \( m \in \mathbb{N}_1 \), this shows that, for any such \( q \), \( |h_0 (q)|_2 = |h_1 (q)|_2 = 2|q|_2 \). So, \( |h_i (q)|_2 = 2^{\lfloor \frac{i}{2} \rfloor} |q|_2 \) for all such \( q \) and any \( i \in J \). Hence:

\[
h_i (q) \notin \mathbb{Z}, \forall i \in J, \forall q \in \mathbb{Z} \left[ \frac{1}{2} \right] \backslash \mathbb{Z}
\]

In other words, once a composition sequence of \( h_0 \) and \( h_1 \) outputs a non-integer dyadic rational, there is no way to apply composition sequences of \( h_0 \) and \( h_1 \) to arrive back at an integer output.

In summary, our assumption that \( \omega \in \mathbb{Z} \) was not a periodic point of \( H_p \) forces \( h_{j_1,\ldots,j_n,j_{n+1}} (\omega) \neq H^{n+1} (\omega) \), and hence, forces \( h_{j_1,\ldots,j_n,j_{n+1}} (\omega) \) to be a non-integer dyadic rational number. But then, \( \mathbb{Z} \) contains:

\[
\omega = h_j (\omega) = h_{j_{n+2},\ldots,j_j} (h_{j_1,\ldots,j_n,j_{n+1}} (\omega)) \notin \mathbb{Z}
\]

which is not an integer. This is impossible. Consequently, for our \( j \), \( h_j (\omega) = \omega \) forces \( \omega \) to be a periodic point of \( H_p \). This proves the Claim. \( \checkmark \)

Since \( t \) was an arbitrary positive-integer for which \( \chi_p (B (t)) \in \mathbb{Z} \), this shows that \( \chi_p (B (t)) \) is a periodic point of \( H_p \) in \( \mathbb{Z} \) for any \( t \in \mathbb{N}_1 \) for which \( \chi_p (B (t)) \in \mathbb{Z} \). Finally, we need to deal with the fact that our periodic points must be odd.
To see why this must be so, taking the 2-adic absolute value of the expression in Lemma 1.2.2 for any \( t \in \mathbb{N}_1 \), we have that:

\[
\left| \chi_p(B(t)) \right|_2 = \frac{\chi_p(t)}{1 - \frac{2^{\lambda(t)}}{2^{\#_1(t)}}} = \left| \frac{2^{\lambda(t)} \chi_p(t)}{2^{\lambda(t)} - p^{\#_1(t)}} \right|_2 = \left| 2^{\lambda(t)} \chi_p(t) \right|_2
\]

Since \( t \in \mathbb{N}_1 \) implies \( \chi_p(t) \in \mathbb{Q} \), and since \( \lambda(t) = \beta_{\#_1(t)}(t) + 1 \), we have that:

\[
\left| 2^{\lambda(t)} \chi_p(t) \right|_2 = \left| \sum_{k=1}^{\#_1(t)} 2^{\lambda(t)-\beta_k(t)-1} p^{k-1} \right|_2 = \left| 2C + p^{\#_1(t)-1} \right|_2
\]

where \( C \) is some non-negative integer depending on \( p \) and \( t \). Since \( p^{\#_1(t)-1} \) is always an odd integer, this shows that:

\[
\left| 2^{\lambda(t)} \chi_p(t) \right|_2 = \left| 2C + p^{\#_1(t)-1} \right|_2 = 1
\]

and hence:

\[
\left| \chi_p(B(t)) \right|_2 = \left| \frac{2^{\lambda(t)} \chi_p(t)}{2^{\lambda(t)} - p^{\#_1(t)}} \right|_2 = \left| 2^{\lambda(t)} \chi_p(t) \right|_2 = 1
\]

which shows that, for any \( t \in \mathbb{N}_1 \), both the numerator and denominator of the rational number \( \chi_p(B(t)) \) are co-prime to 2. Consequently, when \( \chi_p(B(t)) \) is an integer for \( t \in \mathbb{N}_1 \), it is necessarily an odd integer, as desired.

This proves the **Correspondence Theorem**.

Q.E.D.

The following table displays the first few values of the various functions considered thus far. As per Proposition 1.2.3, note that the integer values attained by \( \chi_3(B(t)) \) and \( \chi_5(B(t)) \) are all periodic points of the shortened collatz map \( H_3 \)—this includes fixed points at negative integers, as well.

By looking at \( \chi_p(B(t)) \), we can see definite patterns emerge, such as the fact that:

\[
\chi_p(B(2^n - 1)) = \frac{1}{2^n}, \quad \forall n \in \mathbb{N}_1
\]

and, more significant, that \( \chi_3(B(t)) \) is more often positive than negative, but that the opposite holds for \( p = 5 \) (and, heuristically, for all odd integers \( p \geq 5 \)). Most significant, in the author’s opinion, however, is the example of:

\[
\chi_p(B(10)) = \frac{4 + p}{16 - p^2} = \frac{1}{4 - p}
\]
In showing that every integer periodic point \( x \) of \( H_p \) is of the form:

\[
\omega = \chi_p (B(t)) = \frac{\chi_p(t)}{1 - \frac{\lambda(t)}{2^{\mu(t)}}} = \frac{\#_1(t)}{1 - \frac{\lambda(t)}{2^{\lambda(t)}}} = \sum_{k=1}^{\#_1(t)} \frac{2^{\lambda(t)} - \beta_k(t) - 1}{2^{\lambda(t)} - p^{\#_1(t)}} p^{k-1}
\]

for some \( t \in \mathbb{N}_0 \), Proposition 1.2.3 demonstrates that \( |2^{\lambda(t)} - p^{\#_1(t)}| = 1 \) is a sufficient condition for \( \chi_p (B(t)) \) to be a periodic point of \( H_p \). However, as the \( t = 10 \) case shows, this is not a necessary condition, because there could be values of \( t \) where the numerator and denominator of \( \chi_p (B(t)) \) share a common divisor that, when cancelled out, reduce \( \chi_p (B(t)) \) to an integer, despite the potentially large absolute value of \( 2^{\lambda(t)} - p^{\#_1(t)} \) for that value of \( t \). In fact, thanks to P. Mihăilescu’s resolution of Catalan’s Conjecture, it would seem that estimates on the archimedean size of \( 2^{\lambda(t)} - p^{\#_1(t)} \) do not help us understand \( \chi_p (B(t)) \) a whit better!

**Mihăilescu’s Theorem** [2]: The only choice of \( x, y \in \mathbb{N}_1 \) and \( m, n \in \mathbb{N}_2 \) for which:

\[
x^m - y^n = 1
\]

are \( x = 3, m = 2, y = 2, n = 3 \) (that is, \( 3^2 - 2^3 = 1 \)).

With Mihăilescu’s Theorem, it is easy to see that, for any odd integer \( p \geq 3 \), \( |2^{\lambda(t)} - p^{\#_1(t)}| \) will never be equal to 1 for any \( t \geq 3 \). Consequently,
for all prime $p \geq 3$, any rational integer value of $\chi_p(B(t))$ (and hence, any periodic point of $H_p$) of the form $\chi_p(B(t))$, where $t \geq 3$, must occur as a result of $\sum_{k=1}^{\#_1(t)} 2^{\lambda(t)-\beta_k(t)} - 1 p^{k-1}$ being a multiple of $2^{\lambda(t)} - p^{\#_1(t)} \in \mathbb{Z} \setminus \{-1, 1\}$. This would suggest that classical techniques of transcendence theory (ex: Baker’s Method) or additive number theory (ex: the Circle Method) need not bother to show up if and when the call is ever sent out to gather the miraculous mathematical army needed to conquer the Collatz Conjecture. Instead, it seems that the key—at least for proving the (non-)existence of periodic points—is to understand the roots of the polynomials:

$$\sum_{k=1}^{\#_1(t)} 2^{\lambda(t)-\beta_k(t)} - 1 x^{k-1}, \sum_{k=1}^{\#_1(t)} p^{k-1} x^{\lambda(t)-\beta_k(t)} - 1, 2^{\lambda(t)} - x^{\#_1(t)}, x^{\lambda(t)} - y^{\#_1(t)}, \text{ etc.}$$

over the finite fields ($\mathbb{Z}/q\mathbb{Z}$, for prime $q$).

2 Our Toolbox

This section is the paper’s principal mathematical toolbox, consisting of all the lemmata and other assorted knick-knacks we shall need.

2.1 Complex-Analytic Methods

We need a slight generalization of Perron’s Formula:

**Theorem 2.1.1 Generalized Perron Formula**: Consider a sequence $\{a_k\}_{k \in \mathbb{N}_1} \subseteq \mathbb{C}$, and let:

$$D_A(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Writing $\sigma_a(D_A)$ to denote the abscissa of absolute convergence of $D_A$, for any $m \in \mathbb{N}_0$ and any $b > \max\{0, \sigma_a(D_A)\}$, we have that:

$$\sum_{k=1}^{\lfloor x \rfloor - 1} a_k \left(1 - \frac{k}{\lfloor x \rfloor} \right)^m = \frac{m!}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{x^s}{s(s+1) \cdots (s+m)} D_A(s) \, ds, \forall n \in \mathbb{N}_2$$

**Proposition 2.1.1**: Letting $D_A$ and $b$ be as given in [13] define $A : [1, \infty) \to \mathbb{C}$ by:

$$A(x) \overset{\text{def}}{=} \sum_{k=1}^{\lfloor x \rfloor} a_k$$

Then:
\[ \sum_{k=1}^{\lfloor x \rfloor} A(k) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\lfloor x \rfloor + 1}{s(s+1)} \frac{(x+1)^s}{D_A(s)} \, ds, \quad \forall x \in \mathbb{R} \geq 1 \] (45)

\[ a_{\lfloor x \rfloor} = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{(x+1)^s}{s(s+1)} \left[ 2|\lfloor x \rfloor| x^s + 2|\lfloor x \rfloor - 1| (x-1)^s \right] D_A(s) \, ds, \quad \forall x \in \mathbb{R} \geq 1 \] (46)

**Proof:** We apply summation by parts:

\[ \sum_{k=1}^{\lfloor x \rfloor} a_k \left( 1 - \frac{k}{\lfloor x \rfloor} \right) = \left( 1 - \frac{\lfloor x \rfloor - 1}{\lfloor x \rfloor} \right) \sum_{k=1}^{\lfloor x \rfloor - 1} a_k - \sum_{k=1}^{\lfloor x \rfloor - 2} \left( \left( 1 - \frac{k + 1}{\lfloor x \rfloor} \right) - \left( 1 - \frac{k}{\lfloor x \rfloor} \right) \right) \sum_{j=1}^{\lfloor x \rfloor} a_j \]

\[ = \frac{1}{\lfloor x \rfloor} A(\lfloor x \rfloor - 1) - \sum_{k=1}^{\lfloor x \rfloor - 2} \left( -\frac{1}{\lfloor x \rfloor} \right) A(k) \]

\[ = \frac{1}{\lfloor x \rfloor} \sum_{k=1}^{\lfloor x \rfloor - 1} A(k) \]

Setting \( m = 1 \) in (43), replace the left-hand side with \( \frac{1}{\lfloor x \rfloor} \sum_{k=1}^{\lfloor x \rfloor - 1} A(k) \), multiply both sides by \( \lfloor x \rfloor \), and then replace \( x \) with \( x + 1 \); this gives (45).

Applying the backward difference operator:

\[ \nabla \{ f \}(x) \overset{\text{def}}{=} f(x) - f(x - 1) \]

to (45) twice with respect to \( x \) then yields (46) Q.E.D.

While the formula for \( a_{\lfloor x \rfloor} \) is good, it can be made better. At present, the denominator of the integrand introduces a decay of \( O\left(|s|^{-2}\right) \). Our purposes, however, will require \( O\left(|s|^{-3}\right) \) decay. This can be done by integrating with respect to \( x \); the fact that \( a_{\lfloor x \rfloor} \) is constant over the interval \( x \in (n, n+1) \) will then accrue one more backward difference into our integral.

**Proposition 2.1.2:** Letting \( D_A \) and \( b \) be as given in (43) we have that:

\[ a_n = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\kappa_n(s) D_A(s)}{s(s+1)^2} \, ds, \quad \forall n \in \mathbb{N}_1 \] (47)

where:

\[ \kappa_n(s) = n \left( (n+2)^{s+1} - 3(n+1)^{s+1} + 3n^{s+1} - (n-1)^{s+1} \right) \]

+ \( (n+2)^{s+1} - (n+1)^{s+1} - (n+1)^{s+1} - (n-1)^{s+1} \)

\[ = n\nabla^3 \left( (n+2)^{s+1} \right) + \nabla \left( (n+2)^{s+1} \right) - \nabla \left( n^{s+1} \right) \] (48)
where:
\[
\nabla^k (f(n)) \overset{\text{def}}{=} \nabla^k \{f\} (n)
\]

Proof: We note that:
\[
\int_n^{n+1} a_{i[x]} dx = a_n \int_n^{n+1} dx = a_n
\]
\[
\int_n^{n+1} ([x] + c) (x + c)^s dx = (n + c) \int_n^{n+1} (x + c)^s dx
\]
\[
= \frac{n + c}{s + 1} \left( (n + c + 1)^{s+1} - (n + c)^{s+1} \right)
\]

As such:
\[
\int_n^{n+1} \left( ([x] + 1) (x + 1)^s - 2 [x] x^s + [x - 1] (x - 1)^s \right) dx
\]
becomes:
\[
\frac{n}{s + 1} \left( (n + 2)^{s+1} - 3 (n + 1)^{s+1} + 3 n^{s+1} - (n - 1)^{s+1} \right)
\]
\[
+ \frac{1}{s + 1} \left( (n + 2)^{s+1} - (n + 1)^{s+1} - n^{s+1} + (n - 1)^{s+1} \right)
\]

which is:
\[
\frac{n \nabla^3 \left( (n + 2)^{s+1} \right)}{s + 1} + \frac{\nabla \left( (n + 2)^{s+1} \right) - \nabla \left( n^{s+1} \right)}{s + 1}
\]

Consequently:
\[
a_n = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{n \nabla^3 \left( (n + 2)^{s+1} \right)}{s (s + 1)^2} + \frac{\nabla \left( (n + 2)^{s+1} \right) - \nabla \left( n^{s+1} \right)}{s (s + 1)^2} D_A(s) ds
\]
\[
= \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\kappa_n(s) D_A(s)}{s (s + 1)^2} ds
\]
as desired.

**Proposition 2.1.3** (Behavior and Asymptotics of \(\kappa_n(s)\)):

I. Let \(n \in \mathbb{N}_1\) be arbitrary. Then, as a function of the complex variable \(s\), \(\kappa_n(s)\) is holomorphic for \(\text{Re}(s) > -1\) (and is an entire function whenever \(n \geq 2\)) with a simple zero at \(s = 0\); that is, \(\kappa_n(0) = 0\) for all \(n \in \mathbb{N}_1\). Also, \(\kappa_n(1) = 4\), for all \(n \in \mathbb{N}_1\).

II. For all \(s \in \mathbb{C}\), with \(\text{Re}(s) = \sigma\):
\[
|\kappa_n(s)| \leq |s + 1| \left( \frac{\sigma (\sigma - 1)}{3} + 2 \right) (n + 2)^\sigma, \quad \forall n \in \mathbb{N}_1
\]  
(49)
III.

\[ \kappa_n'(s) = n \nabla^3 \left( (n + 2)^{s+1} \ln (n + 2) \right) + \nabla \left( (n + 2)^{s+1} \ln (n + 2) \right) - \nabla \left( n^{s+1} \ln n \right) \]  

\[ \text{(50)} \]

with:

\[ |\kappa_n'(s)| \leq n \sup_{x \in [n-1,n+2]} x^{s-2} |3s^2 - 1 + s(s^2 - 1) \ln x| 
+ 2 \sup_{x \in [n,n+2]} |(s+1) \nabla (x^s \ln x) + \nabla (x^s)| \]

\[ \text{(51)} \]

for all \( n \geq 2 \). In particular:

\[ |\kappa_n'(0)| \leq \frac{n}{(n-1)^2} + 2 \ln \left( 1 + \frac{1}{n-1} \right) = O \left( \frac{1}{n} \right) \]  

\[ \text{(52)} \]

\[ |\kappa_n'(1)| \leq 4 \ln \left( n + 3 + \frac{1}{n+1} \right) + 4n \ln \left( 1 + \frac{1}{n+1} \right) + 4 + \frac{2}{n-1} = O \left( \ln n \right) \]

\[ \text{(53)} \]

Proof: Using:

\[ \kappa_n(s) = n \left( (n + 2)^{s+1} - 3(n + 1)^{s+1} + 3n^{s+1} - (n-1)^{s+1} \right) 
+ \left( (n + 2)^{s+1} - (n + 1)^{s+1} + n^{s+1} - (n-1)^{s+1} \right) \]

we see that for \( n \geq 2 \), \( \kappa_n(s) \) is a linear combination of functions of the form \( a \cdot b^s \) for real numbers \( a \neq 0 \) and \( b > 0 \), forcing \( \kappa_n(s) \) to be entire. When \( n = 1 \), we have \( 0^{s+1} \), which is zero for all \( \Re(s) > -1 \) but is otherwise undefined or singular for \( \Re(s) \leq -1 \), showing that for all \( n \geq 1 \), \( \kappa_n(s) \) is holomorphic for \( \Re(s) > -1 \).

Setting \( s = 1 \), the formula for \( \kappa_n(1) \) ends up simplifying to 4. Setting \( s = 0 \), the formula for \( \kappa_n(0) \) ends up simplifying to 0, indicating that \( \kappa_n(s) \) has a zero at \( s = 0 \). Differentiating with respect to \( s \) gives:

\[ \kappa_n'(s) = n \nabla^3 \left( (n + 2)^{s+1} \ln (n + 2) \right) + \nabla \left( (n + 2)^{s+1} \ln (n + 2) \right) - \nabla \left( n^{s+1} \ln n \right) \]

(which proves \[ \text{(50)} \] for (III)). Setting \( s = 0 \), we obtain:

\[ \kappa_n'(0) = n \nabla^3 \left( (n + 2) \ln (n + 2) \right) + \nabla \left( (n + 2) \ln (n + 2) \right) - \nabla \left( n \ln n \right) = n(n + 2) \ln (n + 2) - 3n(n + 1) \ln (n + 1) + 3n^2 \ln n - n(n - 1) \ln (n - 1) + (n + 2) \ln (n + 2) - (n + 1) \ln (n + 1) - n \ln n - (n - 1) \ln (n - 1) = (n + 1)(n + 2) \ln (n + 2) - (n + 1)(n + 1) \ln (n + 1) + n(3n - 1) \ln n - (n - 1)^2 \ln (n - 1) \]
Now, $\kappa_n(s)$ vanishes at $s = 0$ with order 1 if and only if $\kappa'_n(0) \neq 0$. Setting the above expression equal to 0 gives:

$$1 = \frac{(n + 2)^{(n+1)(n+2)} n^{3n-1}}{(n + 1)^{(3n+1)(n+1)} (n - 1)^{2(n-1)}}$$

Clearly $n$ cannot be 1. Next, note that if $n \geq 2$ is even (resp. odd), the fraction on the right is of the form even/odd (resp. odd/even), and such a fraction can never be equal to 1; any fraction $a/b$ which reduces to 1 must be of the form even/even or odd/odd. Thus, $\kappa'_n(0) \neq 0$ for any $n \in \mathbb{N}_1$, which shows that $\kappa_n$’s zero at $s = 0$ is indeed of order 1.

For the estimates on $\kappa_n$, we note that:

$$x^s - (x - 1)^s = s \int_{x-1}^{x} t^{s+1} dt$$

implies:

$$|\nabla (x^s)| = |x^s - (x - 1)^s| \leq |s| x^{\sigma-1}, \ \forall x \in \mathbb{R} \geq 1, \ \forall s \in \mathbb{C} \setminus \{0\} \quad (54)$$

Thus, by induction:

$$|\nabla^k (x^s)| \leq |s| \left( \frac{\sigma - 1)!}{(\sigma - k)!} \right) x^{\sigma-k}, \ \forall x \in \mathbb{R} \geq 1, \ \forall s \in \mathbb{C} \setminus \{0\}, \ \forall k \in \mathbb{N}_1 \quad (55)$$

where in both formulae, $\sigma = \text{Re} (s)$.

So:

$$\kappa_n(s) = n \nabla^3 \left( (n + 2)^{s+1} \right) + \nabla \left( (n + 2)^{s+1} \right) - \nabla (n^{s+1})$$

implies:

$$|\kappa_n(s)| \leq \sigma (\sigma - 1) |s + 1| n (n + 2)^{\sigma-2} + |s + 1| (n + 2)^\sigma + |s + 1| n^\sigma$$

$$\leq |s + 1| \left( \frac{\sigma (\sigma - 1)}{n + 2} + 2 \right) (n + 2)^\sigma$$

$$(n \geq 1) ; \leq |s + 1| \left( \frac{\sigma (\sigma - 1)}{3} + 2 \right) (n + 2)^\sigma$$

which proves (49).

Finally, using:

$$\kappa'_n(s) = n \nabla^3 \left( (n + 2)^{s+1} \ln (n + 2) \right) + \nabla \left( (n + 2)^{s+1} \ln (n + 2) \right) - \nabla (n^{s+1} \ln n)$$

we have:

$$\nabla \left( (n + 2)^{s+1} \ln (n + 2) \right) - \nabla (n^{s+1} \ln n) = \int_n^{n+2} \frac{d}{dx} \{\nabla \{ x^{s+1} \ln x \} \{x\} \} \ dx$$

$$= \int_n^{n+2} ((s + 1) \nabla (x^s \ln x) + \nabla (x^s)) \ dx$$

27
and so:

\[
\left| \nabla \left( (n + 2)^{s+1} \ln (n + 2) \right) - \nabla (n^{s+1} \ln n) \right| \leq 2 \sup_{x \in [n, n+2]} |(s + 1) \nabla (x^s \ln x) + \nabla (x^s)|
\]

\[
= 2 \sup_{x \in [n, n+2]} |(s + 1) \nabla (x^s \ln x) + \nabla (x^s)|
\]

For the third-order difference, we proceed inductively:

\[
\nabla (n^{s+1} \ln n) = \int_{n-1}^{n} t^s (1 + (s + 1) \ln t) \, dt
\]

\[
\nabla^2 (n^{s+1} \ln n) = \int_{n-1}^{n} t^s (1 + (s + 1) \ln t) \, dt - \int_{n-2}^{n-1} t^s (1 + (s + 1) \ln (t - 1)) \, dt
\]

\[
= \int_{n-1}^{n} \left( \int_{t-1}^{t} \frac{d}{dx} \{ x^s (1 + (s + 1) \ln x) \} \right) dx \, dt
\]

\[
= \int_{n-1}^{n} \int_{t-1}^{t} x^{s-1} (2s + 1 + s (s + 1) \ln x) \, dx \, dt
\]

\[
\nabla^3 (n^{s+1} \ln n) = \int_{n-1}^{n} \int_{t-1}^{t} \int_{x-1}^{x} y^{s-2} (3s^2 - 1 + s (s^2 - 1) \ln y) \, dy \, dx \, dt
\]

\[
\nabla^3 (n^{s+1} \ln n) \leq \sup_{y \in [n-3, n]} \int_{x-1}^{x} y^{s-2} |(s^2 - 1 + s (s^2 - 1) \ln y|
\]

and so:

\[
\left| \nabla^2 \left( (n + 2)^{s+1} \ln (n + 2) \right) \right| \leq \sup_{x \in [n-1, n+2]} \left| x^{s-2} (3s^2 - 1 + s (s^2 - 1) \ln x) \right|
\]

Consequently:

\[
|\kappa_n' (s)| \leq n \sup_{x \in [n-1, n+2]} x^{s-2} (3s^2 - 1 + s (s^2 - 1) \ln x) + 2 \sup_{x \in [n, n+2]} |(s + 1) \nabla (x^s \ln x) + \nabla (x^s)|
\]

which is plugged in $s = 0$ and $s = 1$ gives 52 and 53 respectively.

Q.E.D.

**Proposition 2.1.4:** Let $\{a_n\}_{n \geq 1}$ be a sequence of complex numbers, and suppose that there is a real number $c > 0$ so that the $a_n$'s summatory function, $A(N)$, satisfies:

\[
|A(N)| \ll N^c \ln N \quad \text{as} \quad N \to \infty
\]
Then:

\[ \sup_{t \in \mathbb{R}} |D_A(\sigma + it)| < \infty, \ \forall \sigma > c \]

Proof: Writing \( s = \sigma + it \), summing by parts gives:

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \lim_{N \to \infty} \left( \frac{A(N)}{N^s} + \sum_{n=1}^{N-1} \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) A(n) \right)
\]

(if \( \sigma > c \)):

\[
= \sum_{n=1}^{\infty} \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) A(n)
\]

\[
= (1 - 2^{-s}) a_1 + \sum_{n=2}^{\infty} \frac{1}{n^s} \left( 1 - \frac{1}{(1 + \frac{1}{n})^s} \right) A(n)
\]

\( (k-\Sigma \text{ conv. unif. in } n) \):

\[
= (1 - 2^{-s}) a_1 + \sum_{k=1}^{\infty} \left( -\frac{s}{k} \right) \sum_{n=2}^{\infty} \frac{A(n)}{n^{\sigma + it + k}}
\]

Since \( (1 - 2^{-s}) a_1 \) is uniformly bounded for \( \text{Re}(s) = \sigma \), we have that \( \sigma > c \) implies:

\[
|D_A(\sigma + it)| \leq C_\sigma + \sum_{k=1}^{\infty} \left| \left( \frac{-s}{k} \right) \sum_{n=2}^{\infty} \frac{A(n)}{n^{\sigma + it + k}} \right|
\]

\[
\ll C_{\sigma, A} \sum_{k=1}^{\infty} \left| \left( -\frac{s}{k} \right) \sum_{n=2}^{\infty} \frac{\ln n}{n^{\sigma - c + k}} \right|
\]

\[
= C_{\sigma, A} \sum_{k=1}^{\infty} \left| \left( -\frac{s}{k} \right) \zeta'(\sigma - c + k) \right|
\]

Since:

\[
\lim_{k \to \infty} \left| \frac{(k+1)^{\zeta'}(\sigma - c + k + 1)}{(k)^{\zeta'}(\sigma - c + k)} \right| = \lim_{k \to \infty} \left| \frac{s+k}{k+1} \frac{\zeta'(\sigma - c + k + 1)}{\zeta'(\sigma - c + k)} \right|
\]

\[
= \lim_{k \to \infty} \frac{\zeta'(\sigma - c + k + 1)}{\zeta'(\sigma - c + k)}
\]

\[
= \lim_{k \to \infty} \frac{\zeta'(k+1)}{\zeta'(k)}
\]

\[
= \lim_{k \to \infty} \frac{\ln 2}{2^k} + \frac{\ln 3}{3^k} + \cdots
\]

\[
= \lim_{k \to \infty} \frac{\ln 2}{2^k} + \left( \frac{2}{3} \right)^k \frac{\ln 3}{3^k} + \left( \frac{2}{4} \right)^k \frac{\ln 4}{4^k} + \cdots
\]

\[
= \frac{\ln 2}{2} \ln 2
\]

= \frac{1}{2}
we have that, by the ratio test, when \( \sigma > c \), the series:

\[
\sum_{k=1}^{\infty} \left( -\frac{s}{k} \right) \zeta' (\sigma - c + k)
\]

converges absolutely for each \( s \in \mathbb{C} \); note that this convergence occurs compactly with respect to \( s \). Consequently, when \( \sigma > c \), \( \sum_{k=1}^{\infty} (-s) \zeta' (\sigma - c + k) \) defines an entire function of the complex variable \( s \). Moreover, it can be shown that:

\[
\sigma > c \Rightarrow \sup_{t \in \mathbb{R}} \left| \sum_{k=1}^{\infty} \left( -\frac{\sigma - it}{k} \right) \zeta' (\sigma - c + k) \right| < \infty
\]

from which it follows that:

\[
\sup_{t \in \mathbb{R}} |D_A (\sigma + it)| \ll C_{\sigma,A} \sup_{k=1}^{\infty} \left| \left( -\frac{\sigma - it}{k} \right) \zeta' (\sigma - c + k) \right| < \infty
\]

as desired. Q.E.D.

**Lemma 2.1.1**: Let \( \{a_n\}_{n \geq 1} \) be a sequence of complex numbers, and suppose that there is a real number \( c > 0 \) so that the \( a_n s^i \) summatory function, \( A(N) \), satisfies:

\[
|A(N)| \ll N^c \ln N \text{ as } N \to \infty
\]

Then, for all \( b > c \), the integral:

\[
\int_{b-i\infty}^{b+i\infty} \frac{k_n(s) D_A(s)}{s(s+1)^2} ds
\]

is absolutely convergent for all \( n \in \mathbb{N}_1 \).

Proof: For brevity, let us write:

\[
K(x,s) \overset{\text{def}}{=} (|x| + 1) (x + 1)^s - 2 |x| x^s + |x - 1| (x - 1)^s
\]

so that \([46]\) becomes:

\[
a_{[x]} = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{K(x,s)}{s(s+1)} D_A(s) ds
\]

The growth condition on \( |A(N)| \) guarantees that \( C_{A,b} \overset{\text{def}}{=} \sup_{t \in \mathbb{R}} |D_A(b + it)| < \infty \) for all \( b > c \), and hence, for all \( x > 1 \):

\[
\int_{b-i\infty}^{b+i\infty} \frac{K(x,s)}{s(s+1)} D_A(s) ds \leq ([x] + 1) \int_{b-i\infty}^{b+i\infty} \left| \frac{(x+1)^s}{s(s+1)} D_A(s) \right| ds
\]

\[
+ 2 |x| \int_{b-i\infty}^{b+i\infty} \left| \frac{x^s}{s(s+1)} D_A(s) \right| ds
\]

\[
+ |x - 1| \int_{b-i\infty}^{b+i\infty} \left| \frac{(x-1)^s}{s(s+1)} D_A(s) \right| ds
\]

\[
\leq C_{A,b}' \left( ([x] + 1) (x + 1)^b + 2 |x| x^b + |x - 1| (x - 1)^b \right)
\]
where:

\[ C'_{A,b} = C_{A,b} \int_{b-i\infty}^{b+i\infty} \frac{ds}{s(s+1)} \]

Thus, \[ 58 \] converges absolutely for all \( x > 1 \), with:

\[ a_{\lfloor x \rfloor} \ll (\lfloor x \rfloor + 1)(x + 1)^b + 2x|x|^b + \lfloor x - 1 \rfloor (x - 1)^b \ll (x + 1)^{b+1} \]

Since we obtained \[ 47 \] by integrating \[ 46 \] with respect to \( x \in (n, n+1) \) we have that:

\[
\left| \kappa_n (s) D_A(s) \right| \leq \left| \kappa_n (b + i\infty) \right| \left| \kappa_n (s) D_A(s) \right| \]

\[
\leq \int_{b-i\infty}^{b+i\infty} \left| K(x, s) D_A(s) \right| ds
\]

\[
\leq \int_{n}^{n+1} \left( \int_{b-i\infty}^{b+i\infty} \left| K(x, s) D_A(s) \right| ds \right) dx
\]

\[
\leq \sup_{x \in [n, n+1]} (x + 1)^{b+1}
\]

which shows that our integral is indeed absolutely convergent for all \( n \geq 1 \).

Q.E.D.

While we have shown that the integral will converge, we need stronger estimates on \( \kappa_n (s) \) to be able to get what we want.

**Lemma 2.1.2**: For all \( s = \sigma + it \in \mathbb{C} \) and all \( n \in \mathbb{N}_2 \):

\[ |\kappa_n (s)| \ll |s|^n \text{ as } |\text{Im}(s)| \to \infty \] (59)

where the constant of proportionality is independent of both \( \text{Im}(s) \) and \( n \).

Proof: Let:

\[ M_n (s) \overset{\text{def}}{=} \frac{\kappa_n (s)}{n^s} \] (60)

Noting that for any real number \( r \):

\[ \int_{1}^{\infty} \frac{x^r}{x^{s+1}} dx = \frac{1}{s - r}, \ \forall \text{Re}(s) > r \] (61)

we have for:

\[ \sigma_{n,b} \overset{\text{def}}{=} \inf \left\{ \sigma \in \mathbb{R} : \int_{1}^{\infty} \frac{M_n (b + ix)}{x^{s+1}} dx < \infty \right\} \] (62)
$M_n$ satisfies

$$|M_n(b + it)| \ll |t|^\sigma \quad \text{as } |t| \to \infty, \ \forall \sigma \geq \sigma_{n,b} \quad (63)$$

Now:

$$\left| \int_1^\infty \frac{\kappa_n(b + ix)}{n^{b+ix}} \frac{1}{x^{s+1}} dx \right| = \left| \frac{1}{n^b} \int_1^\infty \kappa_n(b + ix) e^{-i(ln n)x} x^{s+1} dx \right|$$

(use 49):

$$\leq \frac{1}{n^b} \int_1^\infty \frac{|b + 1 + ix|}{x^{\sigma+1}} \left( \frac{|b(b-1)|}{3} + 2 \right) (n + 2)^b \frac{(n + 2)^b}{n^b} \int_1^\infty \sqrt{x^2 + (b + 1)^2} \frac{x}{x^{\sigma+1}} dx$$

$$\ll \left( \frac{|b(b-1)|}{3} + 2 \right) (n + 2)^b \frac{(n + 2)^b}{n^b} O(1)$$

Thus, $\sigma = 1$ must be greater than or equal to $\sigma_{n,b}$, and so:

$$|M_n(b + it)| \ll |t| \quad \text{as } |t| \to \infty$$

where the constant of proportionality is independent of $n$. Consequently:

$$|\kappa_n(\sigma + it)| = |n^\sigma M_n(s)| \ll |t|^\sigma \ll |s|^\sigma \quad \text{as } |t| = |\text{Im}(s)| \to \infty$$

as desired.

Q.E.D.

Our final proposition is a computation of an upper bound on an integral which occurs when estimating the integral in Lemma 2.1.1.

**Proposition 2.1.5:**

$$\left| \int_{b + i\infty}^{b - i\infty} \frac{dz}{z \left| z^2 + 1 \right|^s} \right| = \frac{2}{b^2} \arccosh \left( \frac{b^2 + 4b + 2}{|b|^2} \right), \ \forall b \in \mathbb{R} \setminus \{0, -1\}$$

$^1$Technically, 61 only implies $|M_n(b + it)| \ll |t|^\sigma_{n,b}$ as $t \to \infty$. However, by the Reflection Principle:

$$M_n(b - it) = \overline{M_n(b + it)}$$

and hence, the bound for $t \to \infty$ also holds as $t \to -\infty$. 

32
Proof: Letting \( x = \frac{z - b}{i} \), \( dx = \frac{1}{i} \, dz \)

\[
\left| \frac{1}{i} \int_{b-i\infty}^{b+i\infty} \frac{dz}{|z| \, |z+1|^2} \right| = \int_{-\infty}^{\infty} \frac{dx}{|z+1+i|^2} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2 + b^2 \left( x^2 + (b+1)^2 \right)}} = 2 \int_{0}^{\infty} \frac{dx}{\sqrt{x^2 + b^2 \left( x^2 + (b+1)^2 \right)}}
\]

Letting:

\[
x = \sqrt{u - (b+1)^2}
\]

\[
dx = \frac{du}{2\sqrt{u - (b+1)^2}}
\]

we have that:

\[
\frac{1}{i} \int_{b-i\infty}^{b+i\infty} \frac{dz}{|z| \, |z+1|^2} = 2 \int_{0}^{\infty} \frac{dx}{\sqrt{x^2 + b^2 \left( x^2 + (b+1)^2 \right)}} = \int_{(b+1)^2}^{\infty} \frac{dx}{u \sqrt{(u+b^2-(b+1)^2) \left( u-(b+1)^2 \right)}} = \int_{(b+1)^2}^{\infty} \frac{du}{u \sqrt{u-(b^2+4b+2) u + (2b+1)(b+1)^2}} = \int_{(b+1)^2}^{\infty} \frac{du}{u \sqrt{u-(b^2+4b+2)^2-b^4}} = \int_{\frac{b^2}{2}}^{\infty} \frac{du}{u \sqrt{(u+b^2+4b+2) \sqrt{u^2-b^4}}} = \int_{\frac{b^2}{2}}^{\infty} \frac{(2/b^2) du}{\sqrt{(u+b^2+4b+2) \left( 2u/b^2 \right)^2-1}}
\]

\((t = 2u/b^2)\):

\[
= \frac{2}{b^2} \int_{1}^{\infty} \frac{1}{t \cdot \frac{b^2+4b+2}{b^2} \sqrt{t^2-1}} \, dt = \frac{2}{b^2} \arccosh \left( \frac{b^2+4b+2}{b^2} \right) \frac{1}{\sqrt{(b^2+4b+2)^2-1}}
\]

(Wolfram Alpha)

Q.E.D.
Finally, because it will appear everywhere, we adopt the notation:

\[
\sigma_p \overset{\text{def}}{=} \log_2 \left( \frac{p + 1}{2} \right) \tag{64}
\]

### 2.2 Rising-continuity and functions from \( \mathbb{Z}_p \) to \( \mathbb{Z}_q \)

In dealing with \( \chi_p \), we will vary the particular domain and range/co-domain being used to make sense of the function. Sometimes we will work \( p \)-adically; other times, we will work in \( \mathbb{C} \). Intermixing archimedean and non-archimedean convergence in this manner is infamously dangerous. In general, if a series of rational numbers \( \sum_{n=0}^{\infty} a_n \) converges to a sum \( s \in \mathbb{R} \) and to a sum \( s_p \) in \( \mathbb{Z}_p \), it cannot be assumed that \( s = s_p \). This mistake is as old as the \( p \)-adic numbers themselves, going all the way back to Kurt Hensel himself.[16] However, there are two special cases where \( s \) and \( s_p \) must be equal, and these are all the cases we need:

Case 1: All but finitely many of the \( a_n \)'s are 0. Here, the sum \( \sum_{n=0}^{\infty} a_n \) reduces to a sum of finitely many non-zero rational numbers, which is necessarily convergent in any completion of \( \mathbb{Q} \).

Case 2: All but finitely many of the \( a_n \)'s form a geometric sequence. That is, there is a rational number \( r \) and an \( n_0 \in \mathbb{N}_0 \) so that \( a_{n_0+k} = a_{n_0} r^k \) for all \( k \in \mathbb{N}_0 \). The reason for this is due to the universality of the geometric series formula:

\[
\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}
\]

which is to say, if a geometric series of rational numbers converges in both \( \mathbb{R} \) and \( \mathbb{Z}_p \), then the sum of the series in \( \mathbb{R} \) is the same as the sum of the series in \( \mathbb{Z}_p \).

We start with van der Put’s basis for continuous functions on \( \mathbb{Z}_p \).

**Theorem 2.2.1 (The van der Put Theorem (vdPT))[2]:** Letting \( p \) be a prime, number, define the following\(^2\)

I. Let \( \lambda_p : \mathbb{N}_0 \to \mathbb{N}_0 \) be the function so that \( \lambda_p (n) \) denotes the number of digits in the base \( p \) expansion of \( n \); we define \( \lambda_p (0) \) to be 0. (Note that \( \lambda_p (n) = \lceil \log_p n \rceil + 1 \) for all \( n \in \mathbb{N}_0 \), and \( \lambda_p (n) = \lfloor \log_p (n+1) \rfloor \) for all \( n \in \mathbb{N}_0 \).)

II. We call the indicator functions \( \left\{ p^{\lambda_p (j)} \overset{j \in \mathbb{N}_0}{\equiv} j \right\} \) the \textbf{van der Put functions} and refer to \( \left\{ p^{\lambda_p (j)} \overset{j \in \mathbb{N}_0}{\equiv} j \right\} \) as the \textbf{van der Put basis}.

III. Given an integer \( n \geq 1 \), let \( n_- \) denote the integer obtained by deleting the right-most/terminal non-zero \( p \)-adic digit of \( n \). That is:

\[
n_- = n - n_{\lambda_p (n)} p^{\lambda_p (n)-1} \tag{65}
\]

---

\(^2\)The author writes \( \mathbb{N}_\kappa \) to denote the set of all integers \( \geq \kappa \).
where \( n_{\lambda_p(n)} \) is the coefficient of the largest power of \( p \) present in the \( p \)-adic expansion of \( n \).

With this terminology, let \( K \) be any complete field extension of \( \mathbb{Z}_p \), and let \( C(\mathbb{Z}_p, K) \) be the \( K \)-vector space of continuous functions \( f : \mathbb{Z}_p \to K \). Then:

I. For any choice of constants \( \{c_n\}_{n \in \mathbb{N}_0} \subseteq K \), the van der Put series:

\[
\sum_{n=0}^{\infty} c_n \left[ \lambda_p(n) \equiv n \right]
\]

converges in \( K \) uniformly for \( \lambda \in \mathbb{Z}_p \) if and only if \( |c_n|_p \to 0 \) as \( n \to \infty \).

II. Let \( f : \mathbb{Z}_p \to K \) be arbitrary, and set:

\[
c_n(f) \overset{\text{def}}{=} \begin{cases} f(0) & \text{if } n = 0 \\ f(n) - f(n-) & \text{if } n \geq 1, \forall n \in \mathbb{N}_0 \end{cases}
\]  

We call \( c_n(f) \) the \( n \)th van der Put coefficient of \( f \). Then, \( |c_n(f)|_p \to 0 \) as \( n \to \infty \) if and only if \( f \in C(\mathbb{Z}_p, K) \). When \( f \in C(\mathbb{Z}_p, K) \), \( f \) has the van der Put series representation:

\[
f(\lambda) \overset{K}{=} c_0(f) + \sum_{n=0}^{\infty} c_n(f) \left[ \lambda^p(n) \equiv n \right], \forall \lambda \in \mathbb{Z}_p
\]

which is uniformly convergent\(^3\) over \( \mathbb{Z}_p \).

III. Every \( f \in C(\mathbb{Z}_p, K) \) is uniquely determined by its the van der Put coefficients, \( \{c_n(f)\}_{n \in \mathbb{N}_0} \).

This construction extends to a more general class of functions, such as \( C(\mathbb{Z}_p, \mathbb{Z}_q) \), the space of continuous \( q \)-adic integer valued functions of a \( p \)-adic integer variable. In particular, the author found it useful to coin the following bits of terminology:

**Definition 2.2.1:** Let \( p \) and \( q \) be primes, possibly distinct.

I. We write \( \mathbb{Z}_p' \) to denote the set:

\[
\mathbb{Z}_p' \overset{\text{def}}{=} \mathbb{Z}_p \setminus \mathbb{N}_0 = \{ \lambda \in \mathbb{Z}_p : \lambda \text{ is not a non-negative real integer} \}
\]  

II. A sequence \( \{\lambda_n\}_{n \in \mathbb{N}_0} \subseteq \mathbb{Z}_p \) is said to be \((\text{p-adically}) \text{ rising}\) (or is called a \((\text{p-adically}) \text{ rising sequence}\)) whenever it is convergent and its limit is an element of \( \mathbb{Z}_p' \).

**Remark:** The author chose the term “rising” to refer to such sequences to reflect the fact that a sequence \( \{n_j\}_{j \in \mathbb{N}_0} \subseteq \mathbb{N}_0 \) of non-negative integers is \( p \)-adically rising if and only if the number of non-zero \( p \)-adic digits in each \( n_j \) “rises” to \( \infty \) as \( j \to \infty \).

\(^3\)We write \( K \) to make clear that the equality in question occurs in \( K \) and that all notions of convergence utilized on either side of the equality are those belonging to \( K \) and its \( p \)-adic absolute value.
III. Let $K$ be a complete ring extension of $\mathbb{Z}_q$, and let $f : \mathbb{Z}_p \to K$ be a function. We say $f$ is rising-continuous whenever $\lim_{n \to \infty} f(\{z_n\}) \equiv f(\lim_{n \to \infty} z_n)$ holds for every rising sequence $\{z_n\}_{n \in \mathbb{N}_0} \subseteq \mathbb{Z}_p$.

**Proposition 2.2.1:** Let $K$ be a complete ring extension of $\mathbb{Z}_q$, and let $f : \mathbb{Z}_p \to K$ be a function. Then, $f$ is rising-continuous if and only if the following conditions hold true:

I. $f$ is continuous on $\mathbb{Z}'_p$.

II. $\lim_{j \to \infty} f(n_j) \equiv f(\lim_{j \to \infty} n_j)$ holds true for all sequences of non-negative integers $\{n_j\}$ which are $p$-adically rising.

**Remark:** As the author discovered when reading through [1], exercise 62.B on page 192 of that text contains half of the characterization of rising-continuity; the requirement that (II) occur for all rising sequences in $\mathbb{Z}_p$ (rather than arbitrary convergent sequences in $\mathbb{Z}_p$) is what is needed to guarantee to make things work.

The utility of the notion of “rising continuity” occurs in the light of the interpolation of a function $f : \mathbb{N}_0 \to \mathbb{Q}$ to a function $\tilde{f} : \mathbb{Z}_p \to \mathbb{Z}_q$. First, however, a useful identity:

**Proposition 2.2.2:** Consider a function $f : \mathbb{Z}_p \to F$, where $F$ is a valued field of characteristic zero which is complete as a metric space with respect to the absolute value induced by $F$’s valuation. Then, we have the identity:

$$\sum_{n=0}^{\infty} c_n(f) \left[ \frac{n}{p^{v_\mathfrak{p}(n)}} \equiv n \right] = \lim_{k \to \infty} f(\{[\mathfrak{z}]_{p^k}\}), \quad \forall \mathfrak{z} \in \mathbb{Z}_p$$

(69)

where $[\mathfrak{z}]_{p^k}$ denotes the unique integer in $\{0, \ldots, p^k - 1\}$ congruent to $\mathfrak{z}$ modulo $p^k$. We call this the van der Put identity.

**Definition 2.2.2:** Let $p$ and $q$ be primes, let $K$ be a complete ring extension of $\mathbb{Z}_q$, and let $f : \mathbb{N}_0 \to K$ be a function. We say $f$ has/admits a rising-continuation to $\mathbb{Z}_p$ if (p-adically) rising-continuable to $\mathbb{Z}_p$ whenever there exists a rising-continuous function $g : \mathbb{Z}_p \to K$ so that $g(n) = f(n)$ for all $n \in \mathbb{N}_0$. We call such a $g$ a (p-adic) rising-continuation of $f$ (to $\mathbb{Z}_p$).

**Lemma 2.2.1:** Let $f : \mathbb{N}_0 \to K$ be a function admitting a rising-continuation to $\mathbb{Z}_p$. Then:

I. The rising-continuation of $f$ is unique.

II. Letting $\tilde{f}$ denote the rising-continuation of $f$, we have that:

$$\tilde{f}(\mathfrak{z}) = \lim_{j \to \infty} f(\{[\mathfrak{z}]_{p^j}\}), \quad \forall \mathfrak{z} \in \mathbb{Z}_p$$

that is, we make sense of $f(\mathfrak{z})$ for $\mathfrak{z} \in \mathbb{Z}'_p$ by using the above limit procedure (the convergence here is point-wise with respect to $\mathfrak{z}$). In fact, more generally, we have that:

$$\tilde{f}(\mathfrak{z}) = \lim_{j \to \infty} f(n_j), \quad \forall \mathfrak{z} \in \mathbb{Z}'_p$$

36
for any rising sequence \( \{n_j\}_{j \in \mathbb{N}} \subseteq \mathbb{N}_0 \) converging to \( z \in \mathbb{Z}_p' \).

Rising-continuations always admit van der Put series representations.

**Theorem 2.2.1:** Let \( p \) and \( q \) be primes, let \( K \) be a complete ring extension of \( \mathbb{Z}_q \), let \( f : \mathbb{N}_0 \to K \) be a rising-continuable function, and let \( \tilde{f} \) be the rising-continuation of \( f \) to \( \mathbb{Z}_p \). Then, \( \tilde{f} \) is given by:

\[
\tilde{f} (\bar{z}) = \sum_{n=0}^{\infty} c_n (f) \left[ \bar{z}^{p^{\lambda_p(n)}} \equiv n \right]
\]

(70)

where the series converges point-wise (to \( \tilde{f} \)) for all \( z \in \mathbb{Z}_p \).

**Proof:** Use (69) and **Lemma 2.2.1**.

Q.E.D.

Next, we show that, for certain values of \( z \), under the appropriate circumstances, it is possible for the van der Put series of \( \tilde{f} \) to converge to \( \tilde{f} (z) \) not only in the non-archimedean topology of \( K \), but in the archimedean topology of \( \mathbb{R} \) and/or \( \mathbb{C} \), as well! This result is best described as an interpolation theorem for real or complex-valued functions on the non-negative integers that happen to admit a rising-continuation to a rising-continuous function from a \( p \)-adic space to a \( q \)-adic one.

**Theorem 2.2.2:** Let \( p \) and \( q \) be a primes, let \( K \) be a complete ring extension of \( \mathbb{Z}_q \), suppose that \( K' = \mathbb{C} \cap K \) is non-empty, and let \( f : \mathbb{N}_0 \to K \) be a function whose image (that is, \( f(\mathbb{N}_0) \)) lies in \( K' \). If \( f \) has a rising-continuation \( \tilde{f} : \mathbb{Z}_p \to K \), then:

\[
\tilde{f} (\bar{z}) = \sum_{n=0}^{\infty} c_n (f) \left[ \bar{z}^{p^{\lambda_p(n)}} \equiv n \right]
\]

(71)

(meaning that the van der Put series converges in both \( \mathbb{C} \) and \( K \) to the true value of \( \tilde{f} (\bar{z}) \)) occurs for all \( z \in \mathbb{Z}_p \) for which either of the following conditions hold true:

I. \( c_n (f) \left[ \bar{z}^{p^{\lambda_p(n)}} \equiv n \right] = 0 \) for all but at most finitely many \( n \in \mathbb{N}_0 \).

II. There exist constants \( a, r \in K' \) so that:

\[
\sum_{n=0}^{\infty} c_n (f) \left[ \bar{z}^{p^{\lambda_p(n)}} \equiv n \right] \equiv a \sum_{n=0}^{\infty} r^n
\]

where \( \max \left\{ |r|_\mathbb{C}, |r|_q \right\} < 1 \), where \( |\cdot|_\mathbb{C} \) is the absolute value on \( \mathbb{C} \). In particular, for such a \( z \), we have that:

\[
\tilde{f} (\bar{z}) = \frac{a}{1 - r}
\]
We write $D_{p,q}(f)$ or $D_{p,q}(\tilde{f})$ to denote the set of all $\mathfrak{z} \in \mathbb{Z}_p$ for which (71) holds true. We call $D_{p,q}(f)$ or $D_{p,q}(\tilde{f})$'s (or $\tilde{f}$'s) domain of double-convergence.

Proof: Let everything be as given.

I. Suppose $c_n(f)[\mathfrak{z}^{p^{\lambda_p(n)}} \equiv n] = 0$ for all but at most finitely many $n \in \mathbb{N}_0$. Since $c_n(f) \in K' \subseteq \mathbb{C}$ for all $n \in \mathbb{N}_0$, we then have that (71) reduces to a sum of finitely many terms in $K'$. Being finite, this sum necessarily converges, and does so in both $K$ and $\mathbb{C}$, and its sum is an element of $K'$. By Theorem 2.2.1, the convergence of the finite sum in $K' \subseteq K$ tells us that the value of the sum is equal to the true value of $\tilde{f}(\mathfrak{z})$. On the other hand, since $K'$ is also a subring of $\mathbb{C}$, this tells us that $\tilde{f}(\mathfrak{z}) \in \mathbb{C}$, and hence, that the value to which the series converges in $\mathbb{C}$ is the same as the value to which it converges in $K$.

II. Suppose there are $a,r \in K'$, with $\max\{|r|_C, |r|_q\} < 1$, so that

$$\sum_{n=0}^{\infty} c_n(f)[\mathfrak{z}^{p^{\lambda_p(n)}} \equiv n] K' = a \sum_{n=0}^{\infty} r^n$$

Since $|r|_q < 1$ and $|r|_C < 1$ we have that:

$$\sum_{n=0}^{\infty} r^n \equiv \frac{1}{1-r}$$

as such, by the universality of the geometric series formula, it follows that the sum of this geometric series in $K$ is equal to (as an element of $K'$) its sum in $\mathbb{C}$. Consequently:

$$\sum_{n=0}^{\infty} c_n(f)[\mathfrak{z}^{p^{\lambda_p(n)}} \equiv n] K' = a \sum_{n=0}^{\infty} r^n \equiv a \times \frac{1}{1-r}$$

Since $a$ and $\frac{1}{1-r}$ are both elements of $K'$, their product is also in $K'$, and is thus simultaneously in $K$ and in $\mathbb{C}$. Thus:

$$\tilde{f}(\mathfrak{z}) = \sum_{n=0}^{\infty} c_n(f)[\mathfrak{z}^{p^{\lambda_p(n)}} \equiv n] \equiv a \times \frac{1}{1-r} \equiv \frac{a}{1-r}$$

as desired.

Q.E.D.

Using the fact that for each $\mathfrak{z} \in \mathbb{N}_0$, all but finitely many terms of $\tilde{f}(\mathfrak{z})$'s van der Put series vanish, we can truncate the van der Put series to obtain a
closed-form expression for \( f : \mathbb{N}_0 \rightarrow \mathbb{Q} \) which converges for all \( z \in \mathbb{N}_0 \). Similarly, we can truncate \( \tilde{f} (z) \) to obtain a closed-form expression for \( f \), albeit over \( \mathbb{Z}_p \), which converges to the true value of \( f \) for all sufficiently small \( x \in \mathbb{N}_0 \).

Corollary 2.2.1: Let \( p \) and \( q \) be a primes, let \( K \) be a complete ring extension of \( \mathbb{Z}_q \), suppose that \( K' \) is non-empty, and let \( f : \mathbb{N}_0 \rightarrow K \) be a \( p \)-adicly rising-continuable function whose image (that is, \( f (\mathbb{N}_0) \)) lies in \( K' \). Then:

I. \[
f (x) \equiv c_x (f) + \sum_{n=0}^{p \cdot \lambda_p (x) - 1} c_n (f) \left[ n \equiv x \pmod{p} \right], \quad \forall x \in \mathbb{N}_0
\] (72)

where the sum is defined to be 0 when \( x = 0 \). This can also be written as:

\[
f (x) \equiv c_0 (f) + c_x (f) + \sum_{m=1}^{\lambda_p (x) - 1} \sum_{n=p^{-1}}^{p^{m-1}} c_n (f) \left[ n \equiv x \pmod{p} \right], \quad \forall x \in \mathbb{N}_1
\] (73)

II. If we define \( \hat{f}_N : \mathbb{Z}_p \rightarrow \mathbb{C} \) by:

\[
\hat{f}_N (\hat{z}) \triangleq \sum_{n=0}^{\hat{z}^N - 1} \left[ \frac{\hat{z}^n}{\hat{z}^{p^m} - n} \right] f (n)
\]

Then \( \hat{f}_N \) is continuous (in particular, locally-constant), and satisfies:

\[
\hat{f}_N (n) = f (n), \quad \forall n \in \{0, \ldots, p^N - 1\}
\]

Finally, we need to note the formula for the Fourier Coefficients of a \( q \)-adic complex-valued function of a \( p \)-adic integer variable. We can formally derive this by using the identity:

\[
\left[ \frac{\hat{z}^n}{\hat{z}^{p^m} - n} \right] = \frac{1}{p^m} \sum_{k=0}^{p^{m-1}} e^{2\pi i \frac{k \cdot \frac{\hat{z}^n}{\hat{z}^{p^m}}}{p^m}}
\] (74)

to obtain a formula for a formal \textit{q-adic van der Put series} \( f (\hat{z}) \), which is an expression of the form:

\[
f (\hat{z}) = \sum_{n=0}^{\infty} c_n (f) \left[ \frac{\hat{z}^n}{\hat{z}^{p^m} - n} \right]
\] (75)

where the \( c_n (f) \)s are constants in \( \mathbb{C}_q \). In [74], \( \{ \cdot \}_p \) is the \( p \)-adic fractional part. [74] holds both over \( \mathbb{C} \) and \( \mathbb{C}_q \) for any prime \( q \) (including \( q = p \)). In the non-archimedean case, we fix an embedding of the complex roots of unity \( \{ e^{2\pi i / p^n} \}_{n \in \mathbb{N}_1} \) in \( \mathbb{C}_q \) and, for all \( \eta \in \mathbb{Q}_p \), writing \( \eta = \frac{\hat{z}}{p^n} \), where \( \hat{z} \in \mathbb{Z}_p \) and
\( n \in \mathbb{N}_1 \), we abuse notation slightly to write \( e^{2\pi i/\rho} \) to denote the \( \lfloor \rho \rfloor \)th power of the image of \( e^{2\pi i/\rho} \) under our chosen embedding. Applying 74 to 75, we obtain the formal identities:

**Theorem 2.2.3:** Let \( p \) and \( q \) be distinct primes, and let \( f \) be a formal van der Put series of the form 75. Then, the formal \((p,q)\)-adic Fourier coefficient of \( f \), denoted \( \hat{f}(t) \), are the formal sums so that:

\[
f(z) = \sum_{t \in \hat{\mathbb{Z}}_2} \hat{f}(t) e^{2\pi i \{t\}_p}
\]

and are given by:

\[
\hat{f}(t) = \sum_{n = 1}^{\infty} \frac{c_n(f)}{p^{\lambda_p(n)}} e^{-2\pi n i t}, \forall t \in \hat{\mathbb{Z}}_2
\]

In particular, the following are equivalent:

i. \( \hat{f}(t) \) converges in \( \mathbb{C}_q \) for all \( t \in \hat{\mathbb{Z}}_2 \).

ii. \( \lim_{n \to \infty} c_n(f) = 0 \)

iii. \( f \) is a continuous \( \mathbb{C}_q \)-valued function on \( \mathbb{Z}_p \).

iv. 76 converges uniformly in \( \mathbb{C}_q \) for all \( t \in \hat{\mathbb{Z}}_2 \).

**Remark:** If \( f : \mathbb{Z}_p \to \mathbb{C} \) has a van der Put series expression so that 77 converges in \( \mathbb{C} \) for all \( t \in \hat{\mathbb{Z}}_p \), and \( \hat{f} \in L^1\left(\hat{\mathbb{Z}}_p\right) \), then 76 converges absolutely to \( f \) in \( \mathbb{C} \) for all \( \hat{t} \in \hat{\mathbb{Z}}_p \). If \( \hat{f} \in L^2\left(\hat{\mathbb{Z}}_p\right) \), then 76 converges to \( f \) in \( \mathbb{C} \) for almost every \( \hat{t} \in \hat{\mathbb{Z}}_p \).

### 3 Asymptotics, Blancmanges, and Contour Integrals

#### 3.1 Blancmange Curves, \( \sum_{n=1}^{N} r_p(n) \), \& \( \sum_{n=1}^{N} \chi_p(n) \)

##### 3.1.1 Pictures at an Exhibition

As it turns out, like in the episode of Monty Python’s Flying Circus where people everywhere are being mysteriously transformed into Scotsmen, behind the asymptotics of \( \chi_p \) and \( r_p \) lurks a blancmange—only of the mathematical type, rather than an earthly dessert from outer space. The mathematical kind of blancmanges are **Blancmange curves**, named after the dessert they so happen to resemble, also known as the Takagi Function, after their discoverer. These are a family of fractal curves created by repeated midpoint subdivision. There is a great deal to be said about these curves; the exposition here is liberally borrowed from Lagarias’ excellent article [16] on the subject. The
Figure 1: The blancmange curve $T_{1/2} (x)$ on $[-1, 1]$
one obtains:

\[
\sum_{k=1}^{2^n-1} \#_1(k) a^{\#_1(k)-1} b^k = \sum_{k=0}^{n-1} \frac{b^{2^k}}{1 + a \cdot b^{2^k}} \prod_{m=0}^{n-1} \left(1 + a \cdot b^{2^m}\right) \tag{82}
\]

which, after setting \(a = b = 1\) gives:

\[
\sum_{k=1}^{2^n-1} \#_1(k) = \sum_{k=0}^{n-1} \frac{1}{2} \prod_{m=0}^{n-1} 2 = \sum_{k=0}^{n-1} 2^{n-1} = n \cdot 2^{n-1}
\]

Replacing \(n\) with \(\log_2 (n + 1)\) suggests that:

\[
\sum_{m=1}^{n} \#_1(m) \approx \frac{n + 1}{2} \log_2 (n + 1) \text{ as } n \to \infty \tag{83}
\]

Plotting:

\[
\text{Bl}(x) \overset{\text{def}}{=} \frac{|x| + 1}{2} \log_2 (|x| + 1) - \sum_{m=1}^{x} \#_1(m) \tag{84}
\]

reveals the blancmange:

In essence, our naïve substitution of \(\log_2 (n + 1)\) into the summatory function of \(\#_1(n)\) produced the dominant growth term; subtracting off that term reveals the blancmange batrachion hidden beneath it. Delightfully, this naïveté is fully and rigorously justified. In 1968, Trollope provided a closed-form expression demonstrating exactly the relation between \(\sum_{m=1}^{n} \#_1(m)\) and the Takagi function \(T(x)\):
Theorem 3.1.1.1 (Trollope 1968) [16]:

\[
\sum_{m=1}^{n} \#_1(m) = \frac{(n+1)(\lambda(n)+1) - 2\lambda(n)}{2} - 2^{\lambda(n)} - 2T\left(\frac{n+1}{2\lambda(n)-1} - 1\right), \quad \forall n \in \mathbb{N}_1
\]  

(85)

Tellingly, this formula can also be obtained using the Perron’s Formula Mellin transform methods detailed in Section 2.2 [3]. The methods used there are what inspired the present approach.

Using the identity [97] we can do exactly the same for \( r_p \) and \( \chi_p \). For \( r_p \), the resulting formula is:

\[
\sum_{m=1}^{2^n-1} r_p(m) = \frac{p}{p-1} \left( \left( \frac{p+1}{2} \right)^n - 1 \right)
\]  

(86)

Replacing \( n \) with \( \log_2(n+1) \) gives the suggestion:

\[
\sum_{m=1}^{n} r_p(m) \approx \frac{p}{p-1} \left( (n+1)^{\sigma_p} - 1 \right) = O(n^{\sigma_p})
\]  

(87)

which we now make rigorous.

Lemma 3.1.1.1 (Replacing \( n \) with \( \log_2(n+1) \) to obtain dominant asymptotics is justified): Let \( \{a_m\}_{m \in \mathbb{N}_1} \) be a sequence of non-negative real numbers for which there is a continuously differentiable \( A : (0, \infty) \to [0, \infty) \), with \( A(0) = 0 \) and with \( \sup_{x>0} A'(x) < \infty \) so that:

\[
\sum_{m=1}^{2^n-1} a_m = A(n), \quad \forall n \in \mathbb{N}_1
\]

Then:

\[
\sum_{m=1}^{n} a_m \leq A(\log_2(n+1)) + \sup_{x>0} |A'(x)|, \quad \forall n \in \mathbb{N}_1
\]

Proof: Let \( A(n) = \sum_{m=1}^{n} a_m \). Then:

\[
A(n) = \sum_{m=1}^{2^n-1} a_m = A(2^n - 1)
\]
Hence:

\[ \sum_{m=1}^{2^\lceil x \rceil - 1} a_m = A \left( 2^\lceil x \rceil - 1 \right) = A \left( \lceil x \rceil \right) \]

\[ \Downarrow \]

\[ A \left( 2^{\lceil \log_2 (x+1) \rceil} - 1 \right) = A \left( \lceil \log_2 (x+1) \rceil \right) \]

\[ \Downarrow \]

\[ A \left( 2^\lambda (x) - 1 \right) = A \left( \lceil \log_2 (x+1) \rceil \right) = A \left( [\log_2 (x+1)] \right) \]

Since \( A \) is continuously differentiable, and since \( A(0) = 0 \):

\[ A \left( \lceil \log_2 (x+1) \rceil \right) = A \left( \lceil \log_2 (x+1) \rceil \right) - A(0) \]

\[ = \int_0^{\lceil \log_2 (x+1) \rceil} A'(t) \, dt \]

\[ = \int_0^{\log_2 (x+1)} A'(t) \, dt + \int_{\log_2 (x+1)}^{\lceil \log_2 (x+1) \rceil} A'(t) \, dt \]

\[ = A \left( \log_2 (x+1) \right) - A(0) + \int_{\log_2 (x+1)}^{\lceil \log_2 (x+1) \rceil} A'(t) \, dt \]

\[ = A \left( \log_2 (x+1) \right) + \int_{\log_2 (x+1)}^{\lceil \log_2 (x+1) \rceil} A'(t) \, dt \]

By the mean value theorem, for all \( x > 0 \), there is an \( \eta_x \in [\log_2 (x+1), \lceil \log_2 (x+1) \rceil] \) so that:

\[ \int_{\log_2 (x+1)}^{\lceil \log_2 (x+1) \rceil} A'(t) \, dt = A'(\eta_x) \leq \sup_{t>0} A'(t) < \infty \]

and so, for all \( n \in \mathbb{N}_1 \):

\[ \sum_{m=1}^{n} a_m = \sum_{m=1}^{2^{\lceil \log_2 (n+1) \rceil} - 1} a_m \]

\[ \leq \sum_{m=1}^{2^{\lceil \log_2 (n+1) \rceil} - 1} a_m \]

\[ = A(\lceil \log_2 (n+1) \rceil) \]

\[ = A(\log_2 (n+1)) + A'(\eta_n) \]

\[ \leq A(\log_2 (n+1)) + \sup_{x>0} A'(x) \]

as desired.

Q.E.D.
Applying this to $86$ gives:

**Proposition 3.1.1.1** (Dominant asymptotics of $\sum_{m=1}^{n} r_p (m)$):

$$\sum_{m=1}^{n} r_p (m) \leq \frac{p}{p-1} ((n+1)^{\sigma_p} - 1) \tag{88}$$

Since $\sigma_3 = 1$, setting $p = 3$ yields $\frac{3n}{2}$. Thus, the case of the Collatz map ($p = 3$) is distinguished as the only one where the summatory function of $r_p$ grows *linearly* with $n$ when $p = 3$. For all larger values of $p$, however, we instead get power-law type growth, with polynomial growth occurring precisely when $p + 1$ is a power of $2$.

Now, like with $\#_1$, let us subtract off our asymptotic estimate $87$ and define:

**Definition 3.1.1.1:** We write $Bl_p : [0, \infty) \to \mathbb{R}$ to denote the function:

$$Bl_p (x) \overset{\text{def}}{=} \sum_{m=1}^{\lfloor x \rfloor} \left(1 - r_p (m)\right) \tag{92}$$

As constructed, note that:

$$\tilde{Bl}_p (x + 1) = 1 - r_p (\lfloor x \rfloor) + \tilde{Bl}_p (x)$$

and thus, that $r_p (x) < 1$ if and only if $\tilde{Bl}_p (x + 1) > \tilde{Bl}_p (x)$. Figure 4 shows a section of the graph of $Bl_3 (x)$, with the blue-shaded regions denoting the values of $x$ for which $r_3 (x) < 1$. 

45
Figure 3: $Bl_3(x)$, the $r_3$-summatory blancmange

Figure 4: Graph of a segment of $Bl_3(x)$, with blue regions showing the values of $x$ for which $r_3(x) < 1$. 
When we plot $\tilde{B}_p(x)$ for values of $p$ other than 3, a key difference emerges: when $p > 3$, there is significantly less “wiggle” in the curve. Figure 5 shows $|\tilde{B}_p(x)|$ graphed logarithmically; the places where the curves decrease correspond to the values of $x$ where $r_p(x) < 1$.

Defining:

$$X_p^+ = \{ n \in \mathbb{N}_1 : r_p(n) < 1 \}$$

$$X_p^- = \{ n \in \mathbb{N}_1 : r_p(n) > 1 \}$$

conjecturally, it appears that for odd $p \geq 3$:

$$\limsup_{N \to \infty} \frac{|X_p^+ \cap \{1, \ldots, N\}|}{N} > \limsup_{N \to \infty} \frac{|X_p^- \cap \{1, \ldots, N\}|}{N} \iff p = 3$$

with the preponderance of local minima in the $|\tilde{B}_3(x)|$ curve in Figure 5 and the absence thereof in the $|\tilde{B}_5(x)|$ and $|\tilde{B}_7(x)|$ curves being computational evidence in support of this conjecture.

### 3.1.2 The Function $\zeta_p$

We begin with some useful notations, as well as a generating function identity:
Definition 3.1.2.1: Let \( p \) be an odd integer \( \geq 3 \).
I. We define \( \zeta_p(s) \) as the Dirichlet series associated to \( r_p \):
\[
\zeta_p(s) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{r_p(n)}{(n+1)^s}
\]
(93)

II. We define \( B_p(s) \) as the Dirichlet Series associated to \( \nabla \{ B_l_p \} \):
\[
B_p(s) = \sum_{n=0}^{\infty} \frac{\Delta (n^{\sigma_p}) - r_p(n)}{(n+1)^s}
\]
(94)

Remark: expanding \( \Delta (n^{\sigma_p}) \) (which, note, is 1 when \( p = 3 \)), we get:
\[
B_p(s) = \begin{cases} 
\frac{3}{2} \zeta(s) - \zeta_3(s) & \text{if } p = 3 \\
\frac{p}{p-1} \zeta(s - \sigma_p) - \sum_{n=1}^{\infty} \frac{n^{\sigma_p}}{(n+1)^{s-1}} - \zeta_p(s) & \text{if } p \geq 5 
\end{cases}
\]
(95)

Proposition 3.1.2.1: Let \( F \) be any valued field of characteristic 0, and let \( a \) and \( z \) be any elements of \( F \). Then, we have the identities:
\[
\sum_{n=0}^{2^{m-1}-1} a^{\#_1(n)} z^n = \prod_{j=0}^{m-1} \left( 1 + az^{2^j} \right), \ \forall m \in \mathbb{N}_0
\]
(96)
\[
\sum_{n=2^{m-1}}^{2^m-1} a^{\#_1(n)} z^n = az^{2^{m-1}} \prod_{j=0}^{m-2} \left( 1 + az^{2^j} \right), \ \forall m \in \mathbb{N}_1
\]
(97)
where the products are defined to be 1 whenever \( m \) is 0 and 1, respectively.

Proof:
I. Letting:
\[
C_a(z) \overset{\text{def}}{=} \sum_{n=0}^{\infty} a^{\#_1(n)} z^n
\]
(98)
Note that:
\[
\frac{C_a(z) + C_a(-z)}{2} = \sum_{n=0}^{\infty} a^{\#_1(2n)} z^{2n} = \sum_{n=0}^{\infty} a^{\#_1(n)} z^{2n} = C_a(z^2)
\]
and:
\[
\frac{C_a(z) - C_a(-z)}{2} = \sum_{n=0}^{\infty} a^{\#_1(2n+1)} z^{2n+1} = az \sum_{n=0}^{\infty} a^{\#_1(n)} z^{2n} = azC_a(z^2)
\]
Adding these two identities yields:
\[
C_a(z) = (1 + az) C_a(z^2)
\]
(99)
Hence:

\[ C_a(z) = (1 + az) C_a(z^2) \]

\[ = (1 + az) (1 + az^2) C_a(z^2^2) \]

\[ \vdots \]

\[ = \lim_{N \to \infty} C_a \left( z^{2^N} \right) \prod_{n=0}^{N-1} \left( 1 + az^{2^n} \right) \]

\((\forall |z| < 1) \); \( = C_a(0) \prod_{n=0}^{\infty} \left( 1 + az^{2^n} \right) \)

\( (C_a(0) = a^{n_1(0)} = a^0 = 1) ; = \prod_{n=0}^{\infty} \left( 1 + az^{2^n} \right) \)

and so:

\[ C_a(z) = \prod_{n=0}^{\infty} \left( 1 + az^{2^n} \right) ; \forall |z| < 1 \]  \hspace{1cm} (100)

Since the partial product:

\[ \prod_{j=0}^{m-1} \left( 1 + az^{2^j} \right) \]

contains all terms of \(C_a\)’s power series of degree: \( \leq \sum_{j=0}^{m-1} 2^j = 2^m - 1 \), it then follows that:

\[ \prod_{j=0}^{m-1} \left( 1 + az^{2^j} \right) = \sum_{n=0}^{2^m-1} a^{n_1(n)} z^n \]

as desired. \( \checkmark \)

II. Using (I), we note that:

\[ \sum_{n=2^m-1}^{2^{m-1}} a^{n_1(n)} x z^n = \prod_{j=0}^{m-1} \left( 1 + az^{2^j} \right) - \prod_{j=0}^{m-2} \left( 1 + az^{2^j} \right) \]

\[ = \left( 1 + az^{2^{m-1}} \right) \prod_{j=0}^{m-2} \left( 1 + az^{2^j} \right) - \prod_{j=0}^{m-2} \left( 1 + az^{2^j} \right) \]

\[ = az^{2^{m-1}} \prod_{j=0}^{m-2} \left( 1 + az^{2^j} \right) \]

which is the desired identity.

Q.E.D.

The convergence details of \( \zeta_p(s) \), as well as the functional equation it satisfies and nature and location of its singularities are detailed in the following Lemma:
**Proposition 3.1.2.2:**
I. The defining Dirichlet series for $\zeta_3(s)$ converges for $\text{Re}(s) > 1$.
II. $\zeta_p(s)$ satisfies the recursive functional equation:

$$
\zeta_p(s) = \frac{(p - 1)^{-1}}{2^{s-\sigma_p} - 1} \sum_{n=1}^{\infty} \frac{1}{2^n} \binom{s + n - 1}{n} \zeta_p(s + n), \ \forall \text{Re}(s) \in (\sigma_p - 1, \infty) \setminus \{\sigma_p\}
$$

where:

$$
\binom{s + n - 1}{n} = \frac{\Gamma(s + n)}{n!\Gamma(s)}
$$

Using this equation, one can analytically continue $\zeta_p(s)$ to meromorphic function on $\mathbb{C}$.

**Proof:**
I. Writing $\sigma = \text{Re}(s)$:

$$
|\zeta_p(s)| \leq \sum_{n=0}^{\infty} \frac{r_p(n)}{(n + 1)^\sigma}
$$

$$
= 1 + \sum_{n=1}^{\infty} \frac{r_p(n)}{(n + 1)^\sigma}
$$

$$
(r_p(n) = \frac{p^\#(n)}{2^k}, \forall n \in \{2^{k-1}, \ldots, 2^k - 1\})
$$

$$
= 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{n=2^{k-1}}^{2^k-1} \frac{p^\#(n)}{(n + 1)^\sigma}
$$

$$
< 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{(2^{k-1})^\sigma} \sum_{n=2^{k-1}}^{2^k-1} \frac{p^\#(n)}{p(p+1)^{k-1}}
$$

$$
= 1 + \frac{2\sigma p}{p + 1} \sum_{k=1}^{\infty} \left(\frac{p + 1}{2^{\sigma + 1}}\right)^k
$$

Since geometric series converges for all $\sigma > \log_2 \left(\frac{p+1}{2}\right) = \sigma_p$, we have that the Dirichlet series defining $\zeta_p(s)$ converges for $\text{Re}(s) > \sigma_p$. ✓

II. Using the functional equations:

$$
r_p(2n) = \frac{1}{2} r_p(n)
$$

$$
r_p(2n + 1) = \frac{p}{2} r_p(n)
$$

50
we have that:

\[ \zeta_p(s) = \sum_{n=0}^{\infty} \frac{r_p(n)}{(n+1)^s} \]

(split \(n \mod 2\):

\[ = \sum_{n=0}^{\infty} \frac{r_p(2n)}{(2n+1)^s} + \sum_{n=0}^{\infty} \frac{r_p(2n+1)}{(2n+2)^s} \]

\[ = \frac{1}{2^{s+1}} \sum_{n=0}^{\infty} \frac{r_3(n)}{(n+1)\frac{1}{2}^s} + \frac{p}{2^{s+1}} \sum_{n=0}^{\infty} \frac{r_p(n)}{(n+1)^s} \]

\[ = \frac{1}{2^{s+1}} \sum_{n=0}^{\infty} \frac{r_p(n)}{(n+1)\frac{1}{2}^s} + \frac{p}{2^{s+1}} \zeta_p(s) \]

Solving for \(\zeta_p(s)\) gives:

\[ \zeta_p(s) = \frac{1}{2^{s+1} - p} \sum_{n=0}^{\infty} \frac{r_p(n)}{(n+1)^s} \] (102)

Now, letting:

\[ g_p(x) \overset{\text{def}}{=} \sum_{n=0}^{\infty} r_p(n) e^{-(n+1)x} = e^{-x} + \frac{p}{2} e^{-2x} + \frac{p^2}{2^2} e^{-3x} + \cdots \] (103)

we note that:

\[ \int_{0}^{\infty} x^{s-1} g_p(n) \, dx = \Gamma(s) \sum_{n=0}^{\infty} \frac{r_p(n)}{(n+1)^s} \] (104)

and hence:

\[ \zeta_p(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} x^{s-1} g_p(x) \, dx \] (105)
Thus, manipulating \( \zeta_p(s) \) we find that:

\[
\zeta_p(s) = \frac{1}{2^{s+1} - p} \sum_{n=0}^{\infty} \frac{r_p(n)}{(n + \frac{1}{2})^s} = \frac{1}{2^{s+1} - p} \Gamma(s) \int_0^\infty x^{s-1} \sum_{n=0}^{\infty} r_p(n) e^{-(n+\frac{1}{2})x} \, dx
\]

\[
= \frac{1}{2^{s+1} - p} \Gamma(s) \int_0^\infty x^{s-1} r_p(n) e^{-(n+1)x} \, dx
\]

\[
= \frac{1}{2^{s+1} - p} \Gamma(s) \int_0^\infty x^{s-1} \sum_{n=0}^{\infty} r_p(n) e^{-(n+1)x} \, dx
\]

\[
= \frac{1}{2^{s+1} - p} \Gamma(s) \int_0^\infty x^{s-1} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^n}{2^n} \, g_p(x) \, dx
\]

\[
= \frac{1}{2^{s+1} - p} \Gamma(s) \int_0^\infty x^{s-n-1} \, g_p(x) \, dx
\]

\[
= \frac{1}{2^{s+1} - p} \Gamma(s) \int_0^\infty x^{s-n-1} \, g_p(x) \, dx
\]

Pulling out the \( n = 0 \) term gives us:

\[
\zeta_p(s) = \frac{1}{2^{s+1} - p} \left( \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^n}{2^n} \right) \Gamma(s + n) \frac{1}{\zeta_p(s+n)} \int_0^\infty x^{s-n-1} \, g_p(x) \, dx
\]

\[
\zeta_p(s) = \frac{1}{2^{s+1} - p} \sum_{n=0}^{\infty} \left( \frac{1}{2^n} \right) (s + n - 1) \zeta_p(s+n)
\]

As for the convergence of the series, we first note that for fixed \( n \):

\[
\left( \frac{s + n - 1}{n} \right) = \frac{1}{n!} \prod_{k=0}^{n-1} (s + k)
\]

is a polynomial in \( s \) of degree \( n \). As such, upon writing \( \sigma = \text{Re}(s) \), the ratio
test gives:

\[
\lim_{n \to \infty} \left| \frac{1}{\pi i} \binom{s+n}{n+1} \zeta_p (s + n + 1) \right| = \frac{1}{2} \lim_{n \to \infty} \left| \frac{(s+n)!}{(s+n-1)(n+1)!} \right| \approx 1
\]

\[
\zeta_p (s + n) = \frac{1}{2} \lim_{n \to \infty} \left| \frac{1}{n+1} \right|
\]

Since this is less than 1, the series will be absolutely convergent for all values of \( s \) for which its individual terms are finite-valued. Since \( \zeta_p (s + n) \) is finite valued for all \( \Re(s) > \sigma_p \), we have that the terms of the series will be finite valued for all \( n \geq 1 \) and all \( \Re(s) > \sigma_p - 1 \), except for \( s \in \sigma_p + 2\pi i \mathbb{Z} \), where \( \frac{1}{2} \frac{1}{2^{s-1}} \frac{1}{2^{s-1} - 1} \) has its simple poles. Pulling out terms from the sum for \( n \in \{1, \ldots, N\} \) and working recursively then allows us to determine the singularities of \( \zeta_p (s) \) with \( \Re(s) > \sigma_p - N - 1 \), thereby furnishing an analytic continuation of \( \zeta_p (s) \) to a meromorphic function on \( \mathbb{C} \).

Q.E.D.

**Lemma 3.1.2.1:** For all \( \epsilon \in (0, 1) \) and every \( n \in \mathbb{N}_0 \), there is a polynomial \( P_{\epsilon,n} (s) \) of degree \( n \) with rational-coefficients (depending only on \( n \) and \( \epsilon \)) so that:

\[
|\zeta_3 (s)| \leq \frac{P_{\epsilon,n} (|s|)}{|2s^2 - 1|}, \quad \forall \Re(s) \geq \epsilon - n
\]

That is to say we have that for each \( n \in \mathbb{N}_0 \):

\[
|\zeta_3 (\sigma + it)| \ll |t|^n, \quad \forall \sigma \in (-n, n+1)
\]

Proof:

Using the functional equation for \( \zeta_3 \), we have that:

\[
(2^{s-1} - 1) \zeta_3 (s) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{2^n} \binom{s+n-1}{n} \zeta_3 (s + n)
\]

where the right-hand side converges for all \( \Re(s) > 0 \). As such, for every \( b > 0 \), there is a constant \( C_b > 0 \) which uniformly bounds the right-hand side in magnitude for all \( \Re(s) > b \). So, letting \( \epsilon \in (0, 1) \) be arbitrary, we have that:

\[
|\zeta_3 (s)| \leq \frac{C_\epsilon}{|2s^2 - 1|}, \quad \forall \Re(s) \geq \epsilon, \quad \forall \epsilon > 0
\]

Since:

\[
(2^{s-1} - 1) \zeta_3 (s) = \frac{8}{8} \zeta_3 (s + 1) + \frac{1}{4} \sum_{n=2}^{\infty} \frac{1}{2^n} \binom{s+n-1}{n} \zeta_3 (s + n)
\]

53
we have that the series on the right is bounded uniformly in magnitude by some constant $C'$ for all $\Re (s) \geq \epsilon - 1$. Meanwhile, for such $s$, $\zeta_3 (s + 1)$ is bounded in magnitude by $C$. As such:

$$|\zeta_3 (s)| \leq \frac{C}{2^{s-1} - 1}, \ \forall \Re (s) \geq \epsilon - 1, \ \forall \epsilon > 0$$

Continuing by induction, it follows that for every $\epsilon \in (0, 1)$ and every $n \in \mathbb{N}$, there is a rational-coefficient polynomial $P_{\epsilon,n} (s)$ of degree $n$ so that:

$$|\zeta_3 (s)| \leq P_{\epsilon,n} (|s|) |2\lambda - 1|, \ \forall \Re (s) \geq \epsilon - n,$$

Q.E.D.

### 3.1.3 The summatory function of $\chi_p$

As shown in Lemma 1.1.1, $\chi_p : \mathbb{Z}_2 \to \mathbb{Z}_p$ is rising-continuous—continuous on $\mathbb{Z}_2 \setminus \mathbb{N}_0$. Consequently, we can use the work of Section 2.2 to obtain formulae for $\chi_p$ over $\mathbb{Z}_p$ and $\mathbb{C}$.

To begin with, using our formula:

$$\chi_p (n) = \sum_{k=1}^{\#_1 (n)} p^k \beta (n) + 1 \ \forall n \in \mathbb{N}_0$$

the $n \geq 1$th van der Put coefficients of $\chi_p$:

$$c_n (\chi_p) \equiv \chi_p (n) - \chi_p \left( n - 2^\lambda (n) - 1 \right)$$

are immediately computed by noting that since $n - 2^\lambda (n) - 1$ is but $n$ with the right-most 1 in its 2-adic digit expansion removed, we have that:

$$\chi_p \left( n - 2^\lambda (n) - 1 \right) = \sum_{k=1}^{\#_1 (n) - 1} p^k \frac{\beta (n)}{2^\lambda (n) + 1}$$

and as such (since $c_0 (\chi_p) = \chi_p (0) = 0$), that:

$$c_n (\chi_p) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{p^{\#_1 (n) - 1}}{2^\lambda (n)} & \text{if } n \geq 1 \end{cases} \quad (107)$$

Consequently, $\chi_p : \mathbb{Z}_2 \to \mathbb{Z}_p$ admits the van der Put series:

$$\chi_p (\mathfrak{z}) \equiv \sum_{n=1}^{\infty} p^{\#_1 (n) - 1} \left[ n \equiv \mathfrak{z} \pmod{2^\lambda (n)} \right], \ \forall \mathfrak{z} \in \mathbb{Z}_2 \quad (108)$$

where, as indicated, the series converges in $\mathbb{Z}_p$; recall that the convergence here is strictly point-wise. Moreover, letting $D_{2,p}$ denote the set of double-convergence
for \[108\] we have that said series will converge simultaneously in both \(\mathbb{Z}_p\) and \(\mathbb{R}\) to the true value of \(\chi_p (\tilde{x}) \in \mathbb{Q} \cap \mathbb{Z}_p\) for all \(\tilde{x} \in D_{2,p}\):

\[
\chi_p (\tilde{x}) = \sum_{n=1}^{\infty} \frac{\beta^{\#_1(n)-1}}{2^{\lambda(n)}} \left( n \equiv \tilde{x} \mod 2^{\lambda(n)} \right), \quad \forall \tilde{x} \in D_{2,p}
\]

Restricting the above series to the case where \(\tilde{x} = x \in \mathbb{N}_0\), the terms with \(n \geq 2^\lambda(x)\) vanish, and we obtain:

\[
\chi_p (x) = \sum_{n=1}^{2^\lambda(x)-1} \frac{\beta^{\#_1(n)-1}}{2^{\lambda(n)}} \left( n \equiv x \mod 2^{\lambda(n)} \right), \quad \forall x \in \mathbb{N}_1
\]

By \[111\] the Fourier coefficients of \(\chi_p\) formally compute to be:

\[
\hat{\chi}_p (t) = \sum_{n=\frac{|t|}{2}}^{\infty} \frac{c_n (\chi_p)}{2^{\lambda(n)}} e^{-2\pi n t} = \sum_{n=\max\{\frac{|t|}{2},1\}}^{\infty} \frac{\beta^{\#_1(n)-1}}{2^{\lambda(n)}} e^{-2\pi n t}
\]

where \(t \in \hat{\mathbb{Z}}_2\) (the Pontryagin dual of \(\mathbb{Z}_2\), viewed here as \(\mathbb{Z} [\frac{1}{2}] / \mathbb{Z}\) embedded in the unit interval). By Lemma 1.1.1., since \(\chi_p\) is not continuous on all of \(\mathbb{Z}_2\), these formulae fail to converge for all \(t \in \hat{\mathbb{Z}}_2\); in fact, it can be shown that they fail to converge in either \(\mathbb{C}_p\) or \(\mathbb{C}\) for any \(t \in \hat{\mathbb{Z}}_2\). To circumvent this difficulty, we begin by noting that, by restricting the value of \(x\), we can eliminate the dependence of the upper limit of \[110\]s \(n\) sum. Specifically, fixing \(N \in \mathbb{N}_1\), we have that:

\[
\chi_p (x) = \sum_{n=1}^{2^N-1} \frac{\beta^{\#_1(n)-1}}{2^{\lambda(n)}} \left( n \equiv x \mod 2^{\lambda(n)} \right), \quad \forall x \in \{0, \ldots, 2^N-1\}
\]

simply because the \(n\)th Iverson brackets vanish for all \(n > x\). When expanding the Iverson brackets in this series into \(e^{2\pi n t}_2\), the values of \(t\)s present will all be less than or equal to \(2^N\) in \(2\)-adic magnitude. Consequently, the Fourier coefficients of the right-hand side will have compact (in fact, finite) support on \(\hat{\mathbb{Z}}_2\), guaranteeing that the \(L^1(\hat{\mathbb{Z}}_2)\) condition will hold for this truncation. To that end, we institute the following definition:

**Definition 3.1.3.1:** For each \(N \in \mathbb{N}_1\), define \(\chi_{p,N} : \mathbb{Z}_2 \to \mathbb{R}\) by:

\[
\chi_{p,N} (\tilde{x}) = \sum_{n=0}^{2^N-1} \chi_p (n) \left[ \tilde{x} \equiv n \mod 2^N \right], \quad \forall \tilde{x} \in \mathbb{Z}_2
\]

By \[111\] we obtain the following formula for the Fourier coefficients of \(\chi_{p,N}:

\[
\hat{\chi}_{p,N} (t) = \begin{cases} 
\sum_{n=\max\{\frac{|t|}{2},1\}}^{2^N-1} \frac{\beta^{\#_1(n)-1}}{2^{\lambda(n)}} e^{-2\pi n t_2} & \text{if } 0 \leq |t|_2 \leq 2^N \\
0 & \text{if } |t|_2 > 2^N 
\end{cases}, \quad \forall t \in \hat{\mathbb{Z}}_2
\]

55
Consequently, we obtain:

\[
\chi_{p,N}(j) = \sum_{k=0}^{2^N-1} \left( \frac{1}{2^N} \sum_{n=1}^{2^N-1} \chi_p(n) e^{-2\pi ikn/2^N} \right) \bar{\chi}_{p,N}(k/2^N) e^{2\pi i \{k/2^N\}} \tag{114}
\]

Noting that:

\[
\hat{\chi}_{p,N}(0) = \frac{1}{2^N} \sum_{n=1}^{2^N-1} \chi_p(n) \tag{115}
\]

we can then compute the summatory function of \(\chi_p\) and obtain its dominant asymptotics by our replacement method.

**Proposition 3.1.3.1:**

\[
\sum_{n=0}^{2^N-1} \chi_p(n) = \begin{cases} 
N - \frac{1}{4} & \text{if } p = 3 \\
\frac{2^{N+3}}{p^3} \left( \frac{p+1}{4} \right)^{N-1} - 1 & \text{if } p \neq 3
\end{cases} \tag{116}
\]

Thus, replacing \(N\) with \(\log_2(N+1)\) we get dominant asymptotics for the summatory function of \(\chi_p\):

\[
\sum_{n=0}^{N} \chi_p(n) \ll \begin{cases} 
\frac{N+1}{4} \log_2 \left( \frac{N+1}{4} \right)^p - \frac{N+1}{p-3} & \text{if } p = 3 \\
\frac{1}{4^m} \sum_{n=2^m}^{2^{m+1}-1} \chi_p(n) - 1 & \text{if } p \neq 3
\end{cases} \tag{117}
\]

Proof: Using (115) and (113) we have that

\[
\frac{1}{2^N} \sum_{n=0}^{2^N-1} \chi_p(n) = \hat{\chi}_{p,N}(0)
\]

\[
= \sum_{n=\max(9,1)}^{2^N-1} \frac{p\#_1(n)-1}{4^{\lambda(n)}}
\]

\[
= \sum_{n=1}^{2^N-1} \frac{p\#_1(n)-1}{4^{\lambda(n)}}
\]

\[
= \frac{1}{4^m} \sum_{m=1}^{N-1} \frac{1}{4^m} \sum_{n=2^m}^{2^{m+1}-1} p\#_1(n)-1
\]

(use 97): \[
= \frac{1}{p} \sum_{m=1}^{N-1} \frac{1}{4^m} \left( \frac{p+1}{4} \right)^{m-1}
\]

\[
= \begin{cases} 
\frac{N-1}{4} & \text{if } p = 3 \\
\frac{1}{p-3} \left( \frac{p+1}{4} \right)^{N-1} - 1 & \text{if } p \neq 3
\end{cases}
\]
Multiplying by $2^N$ then yields \[116\] Q.E.D.

We pause here to note that computer calculations imply the inequality
\[
\frac{n + 1}{4} \log_2 \left(\frac{n + 1}{2}\right) \leq \sum_{k=1}^{n} \chi_3(k) \leq \frac{n + 1}{4} + \frac{n + 1}{4} \log_2 \left(\frac{n + 1}{2}\right), \quad \forall n \in \mathbb{N}
\]
(118)
although, admittedly, the author does not yet have a proof for the lower bound.

Now, we proceed with Dirichlet series.

**Definition 3.1.3.2:**
\[
\Xi_p(s) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{\chi_p(n)}{(n+1)^s}
\]
(119)

Using \[117\]

**Proposition 3.1.3.2:**
\[
\sigma > \sigma_p \Rightarrow \sup_{t \in \mathbb{R}} |\Xi_p(\sigma + it)| < \infty
\]
(120)

Proof: Using \[117\] we have that:
\[
\sum_{n=0}^{N} \chi_p(n) = \begin{cases} 
\frac{N+1}{4} \log_2 \left(\frac{N+1}{2}\right) & \text{if } p = 3 \\
\frac{N+1}{4(N+1)^2} - \frac{N+1}{p-3} & \text{if } p \neq 3
\end{cases}
\]
and hence:
\[
\sum_{n=0}^{N-1} \chi_p(n) \ll N^{\sigma_p} \ln N
\]
Consequently, by **Proposition 2.1.4**, this bound on the summatory function of $\chi_p(n)$ implies \[120\] Q.E.D.

**Theorem 3.1.3.1** (Functional equation for analytic continuation of $\Xi_p(s)$): $\Xi_p(s)$ is analytically continuable to a meromorphic function on $\mathbb{C}$ by way of the functional equation:
\[
\Xi_p(s) = \frac{\zeta(s)}{2^{s-\sigma_p} - 1} + \frac{1}{2^{s-\sigma_p} - 1} \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{s + n - 1}{n}\right) \Xi_p(s + n)
\]
(121)

Proof:
\[ \Xi_p(s) = \sum_{n=0}^{\infty} \chi_p(2n) \frac{(2n)^s}{(2n+1)^s} + \sum_{n=0}^{\infty} \chi_p(2n+1) \frac{(2n+2)^s}{(2n+1)^s} \]

(use 29):

\[ \Xi_p(s) = \frac{1}{2s+1} \sum_{n=0}^{\infty} \chi_p(n) \frac{(n+\frac{1}{2})^s}{\Gamma(s+1)} + \frac{1}{2s+1} \sum_{n=0}^{\infty} \frac{p \chi_p(n) + 1}{\Gamma(s+1)} \]

Next:

\[ \sum_{n=0}^{\infty} \frac{\chi_p(n)}{(n+\frac{1}{2})^s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \chi_p(n) \int_0^{\infty} t^{s-1} e^{-(n+\frac{1}{2})t} dt \]

\[ (\chi_p(0) = 0); \quad = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{t/2} \left( \sum_{n=0}^{\infty} \chi_p(n) e^{-(n+1)t} \right) dt \]

\[ = \frac{1}{(s-1)!} \sum_{m=0}^{\infty} \frac{(1/2)^m}{m!} \int_0^{\infty} t^{s+m-1} \left( \sum_{n=0}^{\infty} \chi_p(n) e^{-(n+1)t} \right) dt \]

\[ = \sum_{m=0}^{\infty} \left( \frac{1}{2} \right)^m \frac{(s+m-1)!}{m!(s-1)!} \int_0^{\infty} t^{s+m-1} \left( \sum_{n=0}^{\infty} \chi_p(n) \right) dt \]

\[ = \sum_{m=0}^{\infty} \left( \frac{1}{2} \right)^m \frac{(s+m-1)!}{m!(s-1)!} \frac{t^{s+m-1}}{\Gamma(s+m)} \Xi_p(s+m) \]

So, pulling out the \( n = 0 \) term:

\[ \Xi_p(s) = \frac{\zeta(s)}{2s+1-p} + \frac{1}{2s+1-p} \sum_{n=0}^{\infty} \chi_p(n) \frac{(n+\frac{1}{2})^s}{\Gamma(s+1)} \]

\[ = \frac{\zeta(s)}{2s+1-p} + \frac{1}{2s+1-p} \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \left( \frac{s+n-1}{n} \right) \Xi_p(s+n) \]

\[ = \frac{\zeta(s)}{2s+1-p} + \Xi_p(s) + \frac{1}{2s+1-p} \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n \left( \frac{s+n-1}{n} \right) \Xi_p(s+n) \]

\[ \uparrow \]

\[ \Xi_p(s) = \frac{\zeta(s)}{2s+1-p} + \frac{1}{2s+1-p} \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n \left( \frac{s+n-1}{n} \right) \Xi_p(s+n) \]

\[ = \frac{\zeta(s)}{2s-\log_2(\frac{p+1}{2})} + \frac{1}{2s-\log_2(\frac{p+1}{2})} - 1 \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n \left( \frac{s+n-1}{n} \right) \Xi_p(s+n) \]
Next, like with $\zeta_3$, we need to establish asymptotics for the growth of $\Xi_3(s)$ along vertical lines in $\mathbb{C}$. For our purposes, it will suffice to characterize the behavior of $\Xi_3(s)$ along the line $\text{Re}(s) = -1/4$.

**Proposition 3.1.3.3** (Growth of $\Xi_3(s)$ along $\text{Re}(s) = -1/4$):

\[
\left| \Xi_3\left(-\frac{1}{4} + it\right) \right| \ll |t|^{9/8} \quad (123)
\]

Proof: When $p = 3$, the Dirichlet series defining $\Xi_3(s)$ converges absolutely for all $\text{Re}(s) > \sigma_3 = 1$. Since the series in (121) starts at $n = 1$, this shows that, when $p = 3$, said series will converge for all $\text{Re}(s) > 0$. Pulling the $n = 1$ term out from the series gives:

\[
\Xi_3(s) = \frac{2\zeta(s) + s\Xi_3(s + 1)}{2^s - 2} + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n \binom{s + n - 1}{n} \Xi_3(s + n) \quad (124)
\]

where, as indicated, for any $\delta > 0$, the remaining series converges absolutely for all $\text{Re}(s) \geq -1 + \delta$. As such, setting $s = -1/4 + it$, we get:

\[
\Xi_3\left(-\frac{1}{4} + it\right) = \frac{2\zeta\left(-\frac{1}{4} + it\right) + \left(-\frac{1}{4} + it\right) \Xi_3\left(\frac{3}{4} + it\right)}{2^{-\frac{1}{4} + it} - 2} + O(1) \quad (125)
\]

where the constant of proportionality implied by the $O(1)$ is independent of $t$.

The growth of $\zeta(s)$ along lines of the form $\text{Re}(s) = \sigma$ is a much-studied topic. From Titchmarsh’s classic text on the Zeta function [14], we have the estimates:

\[
\left| \zeta\left(-\frac{1}{4} + it\right) \right| \ll |t|^{3/4} \quad (126)
\]

\[
\left| \zeta\left(\frac{3}{4} + it\right) \right| \ll |t|^{1/8} \quad (127)
\]

As such, (125) implies:

\[
\left| \Xi_3\left(-\frac{1}{4} + it\right) \right| \ll |t|^{3/4} + |t| \left| \Xi_3\left(\frac{3}{4} + it\right) \right| \quad (128)
\]

To get asymptotics for $\Xi_3\left(\frac{3}{4} + it\right)$, we return to (121):

\[
\Xi_3(s) = \frac{2\zeta(s)}{2^s - 2} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \binom{s + n - 1}{n} \Xi_3(s + n) \quad (129)
\]

Q.E.D.
Setting $s = 3/4 + it$ gives:

$$
Ξ_3 \left( \frac{3}{4} + it \right) = \frac{2\zeta(\frac{3}{4} + it)}{2^{\frac{1}{4}} + it - 2} + O(1)
$$

\[ \downarrow \]

$$
\left| Ξ_3 \left( \frac{3}{4} + it \right) \right| \ll \left| \zeta \left( \frac{3}{4} + it \right) \right| \ll |t|^{1/8}
$$

Hence, $128$ becomes:

$$
\left| Ξ_3 \left( -\frac{1}{4} + it \right) \right| \ll |t|^{3/4} + |t| \left| Ξ_3 \left( \frac{3}{4} + it \right) \right| \ll |t|^{3/4} + |t| |t|^{1/8} = |t|^{3/4} + |t|^{9/8} \ll |t|^{9/8}
$$

as desired.

Q.E.D.

### 3.2 Integral Criteria for Periodic Points

We know that an odd integer $\omega \in 2\mathbb{Z} + 1$ is a periodic point of $H_p$ if and only if there is an integer $n \geq 1$ so that:

$$
\omega = \frac{\chi_p(n)}{1 - r_p(n)}
$$

Due to the erratic behavior of $r_p(n)$ (being greater than 1 for some values of $n$ and less than 1 for others), the fraction on the right is difficult to analyze directly. As such, instead, we will consider the equivalent statement:

$$
(1 - r_p(n)) \omega - \chi_p(n) = 0 \quad (130)
$$

Now:

**Definition 3.2.1:** Let $\omega \in \mathbb{C}$. Then, write $C_{p,\omega}(s)$ to denote the function:

$$
C_{p,\omega}(s) \overset{\text{def}}{=} B_p(s) - \frac{1}{\omega} \Xi_p(s) \quad (131)
$$

**Proposition 3.2.1:**

I. $\sup_{t \in \mathbb{R}} |C_{p,\omega}(\sigma + it)| < \infty$ for all $\sigma > \sigma_p$.

II. $\left| C_{3,\omega} \left( -\frac{1}{4} + it \right) \right| \ll |t|^{9/8} \quad (132)$
Proof: (1) follows immediately from the fact that $C_{p,\omega}$ is a linear combination of two Dirichlet series with $\sigma_p$ as their abscissa of convergence. Likewise, since:

$$\left|B_{3,\omega} \left( -\frac{1}{4} + it \right) \right| = \left| \frac{3}{2} \zeta \left( -\frac{1}{4} + it \right) - \zeta_3 \left( -\frac{1}{4} + it \right) \right|$$

(use ?? & ??) $\ll |t|^{3/4} + |t|$$

$$\ll |t|$$

we have that, by [123]

$$\left|C_{3,\omega} \left( -\frac{1}{4} + it \right) \right| \leq \left| B_{3,\omega} \left( -\frac{1}{4} + it \right) - \frac{1}{\omega} \Xi_3 \left( -\frac{1}{4} + it \right) \right|$$

$$\ll |t| + |t|^{9/8}$$

$$\ll |t|^{9/8}$$

as desired.

Q.E.D.

Since $C_{p,\omega} (s)$ is a Dirichlet series with an abscissa of absolute convergence of $\sigma_p$, [17] and Lemma 2.1.1 tell us that:

$$\frac{1}{2\pi i} \int_{\sigma_p + 1 - i\infty}^{\sigma_p + 1 + i\infty} \frac{k_n(s) C_{p,\omega}(s)}{s (s+1)^2} ds = \frac{p}{p-1} \Delta \left( n^{\sigma_p} - r_p(n) \right) - \frac{1}{\omega} \chi_p(n) \quad (133)$$

Adding and subtracting 1 to the right-hand side, we have that $1-r_p(n)-\frac{1}{\omega} \chi_p(n)$ vanishes if and only if $\omega$ is an odd integer periodic point of $H_p$ associated to $n \geq 1$. Consequently, we have:

**Lemma 3.2.1:** An odd integer $\omega$ is a periodic point of $H_p$ if and only if there is an $n \in \mathbb{N}_1$ for which:

$$\frac{1}{2\pi i} \int_{\sigma_p + 1 - i\infty}^{\sigma_p + 1 + i\infty} \frac{k_n(s) C_{3,\omega}(s)}{s (s+1)^2} ds = \frac{p}{p-1} \left( (n+1)^{\sigma_p} - n^{\sigma_r} \right) - 1 \quad (134)$$

In the case of the Collatz map ($p = 3$), this reduces to:

$$\frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \frac{k_n(s) C_{3,\omega}(s)}{s (s+1)^2} ds = \frac{1}{2} \forall b > 1 \quad (135)$$

Our aim is to use Cauchy’s Theorem, integrating over the boundary of the vertical strip $-1/4 < \text{Re} (s) < 2$ to compute the contour integral [133] using the residues of the integrand:

$$\left( \int_{2 - i\infty}^{2 + i\infty} + \int_{-\frac{1}{4} + i\infty}^{-\frac{1}{4} - i\infty} \right) \frac{k_n(s) C_{3,\omega}(s)}{s (s+1)^2} ds = \frac{2\pi i}{\alpha_{\lambda_3}} \sum_{\alpha \in \lambda_3} \text{Res}_{\alpha} \left[ \frac{k_n(s) C_{3,\omega}(s)}{s (s+1)^2} \right]$$

61
where $\Lambda_3$ is the set of all poles of the integrand in the strip $-1/4 < \text{Re} (s) < 2$.

The problem is, our estimates [59] and [132] do not appear sufficient to guarantee the convergence of the integral along $\text{Re} (s) = -\frac{1}{4}$:

$$\int_{-\frac{1}{4} - i\infty}^{-\frac{1}{4} + i\infty} \left| \kappa_n (s) C_{3,\omega} (s) \right| \frac{ds}{s (s + 1)^2} \ll \int_{-\frac{1}{4} - i\infty}^{-\frac{1}{4} + i\infty} \left| \frac{sn^{-1/4} \cdot s^{9/8}}{s (s + 1)^2} \right| \frac{ds}{s (s + 1)^2}$$

$$= \frac{1}{n^{3/4}} \int_{-\frac{1}{4} - i\infty}^{-\frac{1}{4} + i\infty} \left| s \right|^{-7/8} ds$$

$$= \infty$$

Fortunately, there is a way around this obstacle, and it is detailed in the following proposition:

**Proposition 3.2.2:**

$$\int_{-\frac{1}{4} - i\infty}^{-\frac{1}{4} + i\infty} \left| \kappa_n (s) C_{3,\omega} (s) \right| \frac{ds}{s (s + 1)^2} \ll n^{3/4}, \forall n \in \mathbb{N}_1, \forall \omega \in \mathbb{C} \setminus \{0\}$$

Proof: Recall that $\kappa_n (s)$ is a linear combination of terms of the form $x^{n+1}$, where $x$ is either $n + 2$, $n + 1$, $n$, or $n - 1$. Letting $x \in \mathbb{R} > 0$ be arbitrary, we note that our estimate [132] implies:

$$\int_{-\frac{1}{4} - i\infty}^{-\frac{1}{4} + i\infty} \left| \frac{x^{n+1} C_{3,\omega} (s)}{s (s + 1)^2} \right| \frac{ds}{s (s + 1)^2} \ll x^{3/4} \int_{-\frac{1}{4} - i\infty}^{-\frac{1}{4} + i\infty} \left| \frac{s^{9/8}}{s (s + 1)^2} \right| \frac{ds}{s (s + 1)^2} = x^{3/4} \int_{-\frac{1}{4} - i\infty}^{-\frac{1}{4} + i\infty} \frac{ds}{s (s + 1)^2}$$

Since:

$$\int_{-\frac{1}{4} - i\infty}^{-\frac{1}{4} + i\infty} \left| \kappa_n (s) C_{3,\omega} (s) \right| \frac{ds}{s (s + 1)^2}$$

is a linear combination of finitely many integrals of the form:

$$\int_{-\frac{1}{4} - i\infty}^{-\frac{1}{4} + i\infty} \left| \frac{x^{n+1} C_{3,\omega} (s)}{s (s + 1)^2} \right| \frac{ds}{s (s + 1)^2} \ll x^{3/4}$$

we have that it must be absolutely convergent, due to the well-known fact that the sum of finitely many functions in $L^1 (U)$ (where $U$ is some set) is again a function in $L^1 (U)$, and that it satisfies the $O \left( n^{3/4} \right)$ upper bound.

Q.E.D.

We have refrained from further exploring [134] for $p > 3$ for simplicity’s sake seeing as—at a glance—it appears that the growth of $C_{p,\omega}$ along vertical lines in the Re$(s) < 0$ half-plane will be larger than $|s|^2$ for all odd integers $p \geq 7$, thereby preventing the argument given in the above proposition from applying.
Given that $\sigma_5 = \log_2 3 \in (1, 2)$, it seems probable that the above argument might manage to extend to the $p = 5$ case, though we will not explore that here.

Now, we shift our contour integral and use the residue theorem.

**Definition 3.2.1:** Letting $\Lambda_3$ denote the set of all poles of the function:

$$\frac{\kappa_n(s) C_3,\omega(s)}{s(s+1)^2}$$

with $-1/4 < \text{Re}(s) < 2$ (note that this set is independent of both $n$ and $\omega$), define:

$$R_3(\omega, n) \overset{\text{def}}{=} \sum_{\alpha \in \Lambda_3} \text{Res}_\alpha \left[ \frac{\kappa_n(s) C_{p,\omega}(s)}{s(s+1)^2} \right]$$  \hspace{1cm} (136)

In keeping with our aforementioned abuse of notation, given $j \in J$, we shall write $R_3(\omega, j)$ to denote $R_3(\omega, \beta(j))$, where—recall—$\beta(j)$ is the integer whose sequence of binary digits is $j$.

**Theorem 3.2.1:** An odd integer $\omega$ is a periodic point of $H_3$ if and only if there is an integer $n \geq 1$ so that:

$$R_3(\omega, n) + \frac{1}{2\pi i} \int_{-\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \kappa_n(s) C_{3,\omega}(s) \frac{ds}{s(s+1)^2} = \frac{1}{2}$$  \hspace{1cm} (137)

Proof: Since our integrand is absolutely integrable over $-1/4+i\mathbb{R}$, **Cauchy’s Theorem** gives:

$$\left( \int_{2-i\infty}^{2+i\infty} - \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \right) \frac{\kappa_n(s) C_{3,\omega}(s)}{s(s+1)^2} ds = 2\pi i R_3(\omega, n)$$  \hspace{1cm} (138)

Using (133) for $p = 3$ to evaluate the $\text{Re}(s) = 2$ integral, we then obtain:

$$R_3(\omega, n) + \frac{1}{2\pi i} \int_{-\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \kappa_n(s) C_{3,\omega}(s) \frac{ds}{s(s+1)^2} = \frac{1}{2} R_3(\omega, n) + \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \kappa_n(s) C_{3,\omega}(s) \frac{ds}{s(s+1)^2}$$

$$= \frac{3}{2} - r_3(n) - \frac{1}{\omega} \chi_3(n)$$

which is the desired identity.

Q.E.D.

**Corollary 3.2.1:**

I. An odd integer $\omega$ is a periodic point of $H_3$ if and only if there is an integer $n \geq 1$ so that:

$$R_3(\omega, n) + \frac{1}{2\pi i} \int_{-\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \kappa_n(s) C_{3,\omega}(s) \frac{ds}{s(s+1)^2} = \frac{1}{2}$$  \hspace{1cm} (139)
II. (Asymptotic formula for $\chi_3 (B (j)))$: For all $j \in J$, all $\omega \in \mathbb{C} \setminus \{0\}$, and all $m \in \mathbb{N}_1$:

$$\chi_3 (B (j)) = \frac{3\omega}{2} + (\chi_3 (B (j)) - \omega) r_3^m (j) - \omega R_3 (\omega, j^m) - \frac{\omega}{2\pi i} \int_{-\frac{1}{4} - i\infty}^{\frac{1}{4} + i\infty} \frac{\kappa_{3m} (s) C_{3, \omega} (s)}{s (s + 1)^2} ds$$

(140)

III. For all $j \in X_3^+$ and all $\omega \in \mathbb{C} \setminus \{0\}$:

$$\chi_3 (B (j)) = \frac{3\omega}{2} - \omega \lim_{m \to \infty} \left( R_3 (\omega, j^m) + \frac{1}{2\pi i} \int_{-\frac{1}{4} - i\infty}^{\frac{1}{4} + i\infty} \frac{\kappa_{3m} (s) C_{3, \omega} (s)}{s (s + 1)^2} ds \right)$$

(141)

IV. A positive odd integer $\omega$ is a periodic point of $H_3$ if and only if there is a $j \in X_3^+$ such that:

$$R_3 (\omega, j^m) = \frac{1}{2} + (1 - r_3 (j^m)) - \frac{1}{\omega} \chi_3 (j^m)$$

(142)

holds for all $m \in \mathbb{N}_1$.

Proof:

I. Since an odd integer $\omega$ is a periodic point of the Collatz map if and only if $$(1 - r_3 (n)) - \frac{1}{\omega} \chi_3 (n) = 0,$$ we have that, by (137), $\omega$ is a periodic point if and only if (139) holds true.

II. Starting with (133) (with a 1 added and subtracted in) and $p = 3$, we have

$$\frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \frac{\kappa_n (s) C_{3, \omega} (s)}{s (s + 1)^2} ds = \frac{1}{2} + (1 - r_3 (n)) - \frac{1}{\omega} \chi_3 (n)$$

Now, using our abuse of notation, let us replace $n$ with the tuple $j$ whose entries are $n$'s binary digits. This gives:

$$\frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \frac{\kappa_j (s) C_{3, \omega} (s)}{s (s + 1)^2} ds = \frac{1}{2} + (1 - r_3 (j)) - \frac{1}{\omega} \chi_3 (j)$$

Next, observe the identity:

$$(1 - r_p (j^m)) - \frac{1}{\omega} \chi_p (j^m) = (1 - r_p^m (j)) \left( 1 - \frac{1}{\omega} \frac{\chi_p (j)}{1 - r_p (j)} \right), \forall m \in \mathbb{N}_1$$

Consequently:

$$\frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \frac{\kappa_{3m} (s) C_{3, \omega} (s)}{s (s + 1)^2} ds = \frac{1}{2} + (1 - r_3^m (j)) \left( 1 - \frac{1}{\omega} \chi_3 (B (j)) \right)$$
and so:

\[ \chi_3 (B(j)) = \frac{3\omega}{2} + (\chi_3 (B(j)) - \omega) r_3^m (j) - \frac{\omega}{2\pi i} \int_{-\infty}^{\infty} \frac{\kappa_j (s) C_{3,\omega} (s)}{s(s+1)^2} ds \]

Using (138) we obtain:

\[ \chi_3 (B(j)) = \frac{3\omega}{2} + (\chi_3 (B(j)) - \omega) r_3^m (j) - \omega R_3 (\omega, j^m) - \frac{\omega}{2\pi i} \int_{-\infty}^{\infty} \frac{\kappa_j (s) C_{3,\omega} (s)}{s(s+1)^2} ds \]

as desired.

III. \( r_3 (j) < 1 \) if and only if \( j \in X_3^+ \). For such a \( j \), \( r_3^m (j) \to 0 \) as \( m \to \infty \), and (140) becomes:

\[ \chi_3 (B(j)) = \frac{3\omega}{2} - \omega \lim_{m \to \infty} \left( R_3 (\omega, j^m) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\kappa_j (s) C_{3,\omega} (s)}{s(s+1)^2} ds \right) \]

which is (140).

IV. By the Correspondence Theorem, an odd integer \( \omega \) is a periodic point of the Collatz map if and only if there is a non-zero \( j \) so that \( \chi_3 (B(j)) = \omega \). For such a pair of \( \omega \) and \( j \), (140) reduces to:

\[ \omega = \frac{3\omega}{2} + (\omega - \omega) r_3^m (j) - \omega R_3 (\omega, j^m) - \frac{\omega}{2\pi i} \int_{-\infty}^{\infty} \frac{\kappa_j (s) C_{3,\omega} (s)}{s(s+1)^2} ds \]

and hence, since the odd integer \( \omega \) is necessarily non-zero:

\[ \frac{1}{2} = R_3 (\omega, j^m) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\kappa_j (s) C_{3,\omega} (s)}{s(s+1)^2} ds \]

as desired.

Q.E.D

Now, we proceed to evaluate the residues. Doing so is simple but extremely tedious. As such, it is best to examine all the requisite information and “chunking”; we leave the base computational details to be worked out by the reader.

Theorem 3.2.2 (Explicit Formula for \( R_3 \)): The following formula holds for
all \( \omega \in \mathbb{Z} \) for all \( n \in \mathbb{N}_2 \):

\[
R_3(\omega, n) = -\frac{1}{\omega \ln 2} \sum_{k \neq 0} \frac{\zeta \left( 1 + \frac{2k\pi i}{\ln 2} \right) + F \left( 1 + \frac{2k\pi i}{\ln 2} \right)}{\left( 1 + \frac{2k\pi i}{\ln 2} \right) \left( 2 + \frac{2k\pi i}{\ln 2} \right)^2} \kappa_n \left( 1 + \frac{2k\pi i}{\ln 2} \right) O(n)
\]

\[
- \frac{1}{4 \ln 2} \sum_{k \neq 0} \frac{G \left( 1 + \frac{2k\pi i}{\ln 2} \right)}{\left( 1 + \frac{2k\pi i}{\ln 2} \right) \left( 2 + \frac{2k\pi i}{\ln 2} \right)^2} \kappa_n \left( 1 + \frac{2k\pi i}{\ln 2} \right) O(n)
\]

\[
+ \frac{3}{2} - G(1) + \frac{1}{\omega \ln 2} \left( 2 - \gamma + \ln 2 + 4 \frac{\ln 2}{2} F(1) - F'(1) \right)
\]

\[
- \frac{1}{\omega \ln 2} \kappa_n'(1) + \frac{1}{\omega \ln 2} \kappa_n'(0)
\]

\[
+ \frac{1}{\omega \ln 2} \sum_{k \neq 0} \frac{\zeta \left( 1 + \frac{2k\pi i}{\ln 2} \right) + F \left( 1 + \frac{2k\pi i}{\ln 2} \right)}{\left( 1 + \frac{2k\pi i}{\ln 2} \right) \left( 2 + \frac{2k\pi i}{\ln 2} \right)^2} \kappa_n \left( 2k\pi i \right) O(1)
\]

\[
+ \frac{1}{16 \ln 2} \sum_{k \neq 0} \frac{G \left( 1 + \frac{2k\pi i}{\ln 2} \right)}{\left( 1 + \frac{2k\pi i}{\ln 2} \right) \left( 2 + \frac{2k\pi i}{\ln 2} \right)^2} \kappa_n \left( 2k\pi i \right) O(1)
\]

To compute \( R_3(\omega, n) \), since \( \sigma_3 = -1/4 \) we need only compute the residues of:

\[
\kappa_n(s) C_{3,\omega}(s) = \frac{\kappa_n(s)}{s(s+1)^2} \left( \frac{3}{2} \zeta(s) - \zeta_3(s) - \frac{1}{\omega} \Xi_3(s) \right)
\]

at all of its poles in the strip \(-\frac{1}{2} < \text{Re}(s) < 2\). Here are the relevant singular expansions, notations, and noteworthy details involved in computing the residues of this function.

- As was shown in [49]

\[
|\kappa_n(s)| \ll O(n^\sigma)
\]

This asymptotic is very useful to keep in mind when making sense of the formula for \( R_3 \) to be given below. It should be noted that \( \kappa_n(0) \) and \( \kappa_n(1) \) are something of an exception: for all \( n \), \( \kappa_n(1) = 4 \); meanwhile, for any \( n \in \mathbb{N}_1 \), \( \kappa_n(s) \) (as a holomorphic function of the complex variable \( s \)) has a simple zero at \( s = 0 \).

- \( \Xi_3 \) For \( \Xi_3 \), writing out its functional equation gives:

\[
\Xi_3(s) = \frac{\zeta(s)}{2s-1 - 1} + \frac{1}{2s-1 - 1} \sum_{n=1}^\infty \left( \frac{1}{2} \right)^n \frac{(s+n-1)^n}{n} \Xi_3(s+n) O(1) \text{ for } \text{Re}(s) > 0
\]

66
Thus, we have that $\Xi_3(s)$ has a double pole at $s = 1$, thanks to $\frac{\zeta(s)}{2^{s-1} - 1}$, and simple poles at $s = 1 + \frac{2k\pi i}{\ln 2}$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Pulling out the $n = 1$ term as well gives:

$$
\Xi_3(s) = \frac{\zeta(s)}{2^{s-1} - 1} + \frac{1}{2} s \Xi_3(s + 1) + \frac{1}{2^{s-1} - 1} \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n \binom{s + n - 1}{n} \Xi_3(s + n)
$$

The singular part of $\Xi_3(s)$ as $\operatorname{Re}(s) \to 0$ is $s \Xi_3(s + 1)$. Since $\Xi_3(s + 1)$ has a double pole at $s = 0$, the extra factor of $s$ forces $s \Xi_3(s + 1)$—and hence, $\Xi_3(s)$—to have a simple pole at $s = 0$. Meanwhile, for $k \in \mathbb{Z} \setminus \{0\}$, we have that $s \Xi_3(s + 1)$ (and hence, $\Xi_3(s)$) has a simple pole at $s = \frac{2k\pi i}{\ln 2}$. These singularities remain unchanged when we move to consider the function:

$$
\kappa_n(s) \Xi_3(s)
$$

when we compute $R_3$.

In particular, note that since $\kappa_n(s)$ has a simple zero at $s = 0$:

$$
\frac{\kappa_n(s) \Xi_3(s)}{s(s + 1)^2} \approx \kappa_n(s) \Xi_3(s + 1) = \frac{\kappa_n(s)}{s} \times s \Xi_3(s + 1) = \kappa_n'(0) s \Xi_3(s + 1) \text{ as } s \to 0
$$

where “$\approx$” means “has the same singular behavior as”. Hence, in computing the residue at $s = 0$, we will accrue a factor of $\kappa_n'(0)$.

For brevity, in our formula, we shall use the abbreviation:

$$
F(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \binom{s + n - 1}{n} \Xi_3(s + n)
$$

so that:

$$
\Xi_3(s) = \frac{\zeta(s) + F(s)}{2^{s-1} - 1}
$$

• (ζ3) Writing out the functional equation for $\zeta_3$ gives:

$$
\zeta_3(s) = \frac{1/4}{2^{s-1} - 1} \sum_{n=1}^{\infty} \frac{1}{2^n} \binom{s + n - 1}{n} \zeta_3(s + n)
$$

Hence, $\zeta_3(s)$ have simple poles for $s \in 1 + \frac{2n\pi i}{\ln 2} \mathbb{Z}$. Pulling out the $n = 1$ term gives:

$$
\zeta_3(s) = \frac{1}{8} \frac{s \zeta_3(s + 1)}{2^{s-1} - 1} + \frac{1/4}{2^{s-1} - 1} \sum_{n=2}^{\infty} \frac{1}{2^n} \binom{s + n - 1}{n} \zeta_3(s + n)
$$
and so, $\zeta_3(s) \sim s\zeta_3(s+1)$ as $\text{Re}(s) \to 0$. Since $\zeta_3(s+1)$ has a simple pole at $s = 0$, the extra factor of $s$ makes $\zeta_3(s)$ holomorphic at $s = 0$. However, all of the simple poles of $\zeta_3(s)$ at $s = 1 + \frac{2k\pi i}{\ln 2}$ for $k \in \mathbb{Z}\setminus\{0\}$ then induce simple poles of $\zeta_3(s)$ at $s = \frac{2k\pi i}{\ln 2}$ for $k \in \mathbb{Z}\setminus\{0\}$. These singularities remain unchanged when we move to consider the function:

$$\frac{\kappa_n(s)\zeta_3(s)}{s(s+1)^2}$$

when we compute $R_3$.

For brevity, in our formula, we shall use the abbreviation:

$$G(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \binom{s+n-1}{n} \zeta_3(s+n)$$  \hspace{1cm} (154)

so that:

$$\zeta_3(s) = \frac{G(s)}{4^{2s-1}-1}$$

- ($\zeta$) Since $\kappa_n(s)/s$ has a removable singularity at $s = 0$, the only pole of the function $\frac{\kappa_n(s)\zeta(s)}{s(s+1)^2}$ with $-1/4 < \text{Re}(s) < 2$ is the simple pole of $\zeta(s)$ at $s = 1$.

Performing the computations, we find that:

$$\text{Res}_{\frac{2k\pi i}{\ln 2}} \left[ \frac{\kappa_n(s)\zeta_3(s)}{s(s+1)^2} \right] = \left\{ \begin{array}{ll}
\left(\frac{\zeta_3'(1)}{2\ln 2} - \frac{1}{\ln 2} - 1\right) & \text{if } k = 0 \\
\frac{1}{\ln 2} \frac{\zeta(1+\frac{2k\pi i}{\ln 2}) + F(1+\frac{2k\pi i}{\ln 2})}{(1+\frac{2k\pi i}{\ln 2})^2} \kappa_n(1+\frac{2k\pi i}{\ln 2}) & \text{else}
\end{array} \right.$$  \hspace{1cm} (146)

$$\text{Res}_{\frac{2k\pi i}{\ln 2}} \left[ \frac{\kappa_n(s)\zeta_3(s)}{s(s+1)^2} \right] = \left\{ \begin{array}{ll}
\frac{-2\zeta'(0)}{\ln 2} & \text{if } k = 0 \\
\frac{-2}{\ln 2} \frac{\zeta(1+\frac{2k\pi i}{\ln 2}) + F(1+\frac{2k\pi i}{\ln 2})}{(1+\frac{2k\pi i}{\ln 2})^2} \kappa_n(1+\frac{2k\pi i}{\ln 2}) & \text{else}
\end{array} \right.$$  \hspace{1cm} (147)

$$\text{Res}_{1+\frac{2k\pi i}{\ln 2}} \left[ \frac{\kappa_n(s)\zeta_3(s)}{s(s+1)^2} \right] = \frac{1}{4\ln 2} \frac{G(1+\frac{2k\pi i}{\ln 2})}{(1+\frac{2k\pi i}{\ln 2})^2} \kappa_n \left(1+\frac{2k\pi i}{\ln 2}\right)$$  \hspace{1cm} (148)

$$\text{Res}_{1+\frac{2k\pi i}{\ln 2}} \left[ \frac{\kappa_n(s)\zeta_3(s)}{s(s+1)^2} \right] = -\frac{1}{16\ln 2} \frac{G(1+\frac{2k\pi i}{\ln 2})}{(1+\frac{2k\pi i}{\ln 2})^2} \kappa_n \left(\frac{2k\pi i}{\ln 2}\right), \forall k \in \mathbb{Z}\setminus\{0\}$$  \hspace{1cm} (149)

Putting everything together yields 143

Q.E.D.
4 Miscellaneous Results

4.1 A Lipschitz-type estimate for $\chi_p$

First, to address the observation which originally motivated the author to pursue this paper’s numen-based line of inquiry, we prove an explicit Lipschitz estimate for $\chi_p$. A good rule of thumb for the $H_p$ maps is that their “2-ness” is generally regular and well-behaved; it is in their “$p$-ness” that their peculiar irregularities show themselves. Case in point, as mentioned previously, $\chi_p$ is not, in general, injective modulo $p^N$ for arbitrary $N$. Not only that, but, the values of $\chi_p$ are not equally distributed across the residue classes mod any given $p^N$. Indeed, as [26] shows, $\chi_p(t)$ is a unit of is $\mathbb{Z}_p$ (and hence, its value mod $p^N$ is always co-prime to $p$) for all non-zero $t \in \mathbb{Z}_2$.

As such, it would be desirable to establish control over the distribution of $\chi_p(t)$ mod $p^m$ as $t$ varies over $\{0, \ldots, 2^n - 1\}$, for any given integers $n \geq 1$ and $m \in \{1, \ldots, n\}$. Our next result consists of exactly such a relation, stated here as a system of congruences between the locations of the binary digits of $s$ and $t$.

First, however, a definition:

**Definition 4.1**: Let $a$ be an integer. We say an odd prime $p$ is a friend of $a$ whenever $a$ is a primitive root of unity modulo $p^k$ for every $k \in \mathbb{N}_1$.

The next proposition gives a sufficient condition for $p$ to be a friend of $a$.

**Proposition 4.1**: Let $p$ be an odd prime, and let $a$ be an integer which is a primitive root of unity modulo $p$ and modulo $p^2$. Then, $a$ is a primitive root of unity modulo $p^k$ for all $k \in \mathbb{N}_1$.

Proof: See [17].

Q.E.D.

Remark: That is to say, $p$ is a friend of $a$ whenever $a$ is a primitive root of unity modulo $p$ and modulo $p^2$.

**Theorem 4.1**: Let the prime $p$ be a friend of 2. Then, given any integers $m, s, t \in \mathbb{N}_0$, the congruence $\chi_p(s)^{p^m} \equiv \chi_p(t)$ is satisfied whenever the congruences

\[
\beta_k(s) \equiv \beta_k(t) \\
\beta_k(s) \equiv \beta_k(t)
\]

Note that $\beta_k(s) \equiv \beta_k(t)$ always holds true for $k = m$, because the integers $\beta_m(s)$ and $\beta_m(t)$, being integers, are necessarily congruent to one another mod $p^{m+m} = 1$. 

69
hold for all \( k \in \{1, \ldots, m\} \). More generally:

\[
|\chi_p(s) - \chi_p(t)|_p \leq \max_{k \geq 1} \left[ \beta_k(s)^{p^{-1}} \beta_k(t)^{(1 + \nu_p(\beta_k(s) - \beta_k(t))) - k + 1} \right], \forall s, t \in \mathbb{Z}_2
\]

(152)

Here \( \nu_p(m) \) is the \( p \)-adic valuation of \( m \), and \( \beta_k(s)^{p^{-1}} \beta_k(t)^{(1 + \nu_p(\beta_k(s) - \beta_k(t))) - k + 1} \) is an Iverson bracket, which evaluates to 1 whenever the enclosed statement is true and evaluates to 0 whenever the enclosed statement is false.

To give the proof of Theorem 4.1, we need a formula for the \( p \)-adic magnitude of \( 2^m - 1 \). For that, we need to know the multiplicative order of 2 modulo \( p^k \), for \( k \in \mathbb{N}_1 \).

**Proposition 4.2:** Let \( p \) be a friend of 2. Then:

\[
|2^m - 1|_p = p^{-\left[m \equiv 0 \pmod{\nu_p(m)+1}\right]}, \forall m \in \mathbb{N}_0
\]

(153)

Proof: We need to determine the largest power of \( p \) that divides \( 2^m - 1 \). To do this, let \( k \in \mathbb{N}_1 \) so that \( p^k \mid (2^m - 1) \). By elementary group theory, this forces \( m \) to be a multiple of \( \text{ord}_{p^k}(2) \). By Proposition 4.1, since \( p \) is a friend of 2, we know that:

\[\text{ord}_{p^k}(2) = \phi(p) = (p - 1)p^{k-1}\]

where \( \phi \) is Euler’s totient function. So, in order that \( p^k \mid (2^m - 1) \) occur, we need for \( m \equiv (p-1)p^{k-1} \pmod{0} \). As such, \( |2^m - 1|_p = p^{-\kappa} \), where \( \kappa \) is the largest positive integer for which \( m \equiv (p-1)p^{k-1} \pmod{0} \). If no such \( \kappa \) exists, then \( |2^m - 1|_p = 1 \).

To that end, note that the congruence \( m \equiv (p-1)p^{k-1} \pmod{0} \) implies \( m \equiv 0 \pmod{0} \). As such, \( m \) being a multiple of \( p - 1 \) is a necessary condition for the existence of a \( \kappa \) so that \( m \equiv (p-1)p^{k-1} \pmod{0} \). Additionally, if \( m \) is a multiple of \( p - 1 \), then \( k = 1 \) satisfies \( m \equiv (p-1)p^{k-1} \pmod{0} \). As such, if \( m \) is a multiple of \( p - 1 \), then \( \kappa \) necessarily exists and is \( \geq 1 \). Consequently, for \( m \geq 1 \):

\[
|2^m - 1|_p \begin{cases} 1 & \text{if } (p - 1) \nmid m \\ p^{-\max\left\{ k \in \mathbb{N}_1 : (p-1)p^{k-1} \mid m \right\}} & \text{if } (p - 1) \mid m \end{cases}
\]

\[
= \begin{cases} 1 & \text{if } (p - 1) \nmid m \\ p^{-\nu_p\left(\frac{p^m}{p} \right)} & \text{if } (p - 1) \mid m \end{cases}
\]

\[
= p^{-\left[(p-1)\min\nu_p\left(\frac{p^m}{p} \right)\right]} \quad \nu_p(p - 1) = 0;
\]

\[
= p^{-\left[m \equiv 0 \pmod{\nu_p(m)+1}\right]}
\]

70
Q.E.D.

**Proof of Theorem 4.1:** Using 26 it follows that, for all \( s, t \in \mathbb{Z}_2 \):

\[
|\chi_p(s) - \chi_p(t)|_p = \left[ \sum_{k=1}^{\infty} p^{k-1} \left( \frac{1}{2^{\beta_k(t)+1}} - \frac{1}{2^{\beta_k(s)+1}} \right) \right]_p
\]

(Strong \( \Delta \)-Ineq.):

\[
|f^{k-1} \left( \frac{1}{2^{\beta_k(t)+1}} - \frac{1}{2^{\beta_k(s)+1}} \right) |_p
\]

\[
= \max_{k \geq 1} \left[ p^{k-1} \left( \frac{2^{\beta_k(s)} - 2^{\beta_k(t)}}{2^{\beta_k(t)+\beta_k(s)+1}} \right) \right]_p
= \max_{k \geq 1} \left[ \frac{2^{\beta_k(t)}}{p^{k-1}} \| 2^{\beta_k(s)} - 2^{\beta_k(t)} \|_p \right]
\]

\[
(\text{p is odd}) = \max_{k \geq 1} \frac{1}{p^{k-1}} \left[ 2^{\beta_k(t)} \right]_p |2^{\beta_k(s)} - 2^{\beta_k(t)} - 1|_p
\]

\[
= \max_{k \geq 1} \left[ \frac{1}{p^{k-1}} \left[ 2^{\beta_k(t)} \right]_p \right] \left[ 2^{\beta_k(s)} - 2^{\beta_k(t)} - 1 \right]_p
\]

\[
\beta_k(s) - \beta_k(t) \in \mathbb{N}_0, \ \forall k ; \quad \frac{p}{p^{k-1}} \left[ \beta_k(s) - \beta_k(t) \equiv_{p-1} 0 \right] \left( \nu_p(\beta_k(s) - \beta_k(t)) + 1 \right)
\]

\[
\frac{p}{p^{k-1}} \left[ \beta_k(s) - \beta_k(t) \equiv_{p-1} 1 \right] \left( 1 + \nu_p(\beta_k(s) - \beta_k(t)) \right)
\]

This gives the sought-after Lipschitz-type estimate 152.

To prove the congruences condition, fix \( m, s, t \in \mathbb{N}_0 \). Then, in order for \( \chi(s) \) and \( \chi(t) \) to be congruent mod \( p^m \) as elements of \( \mathbb{Z}_p \), it suffices that:

\[
|\chi_p(s) - \chi_p(t)|_p \leq p^{-m}, \ \text{and thus, that:}
\]

\[
- \left\lfloor \beta_k(s) \equiv_{p-1} \beta_k(t) \right\rfloor \left( 1 + \nu_p(\beta_k(s) - \beta_k(t)) \right) \leq p^{-m}, \ \forall k \in \mathbb{N}_1
\]

and thus, that:

\[
\left\lfloor \beta_k(s) \equiv_{p-1} \beta_k(t) \right\rfloor \left( 1 + \nu_p(\beta_k(s) - \beta_k(t)) \right) \geq m, \ \forall k \in \mathbb{N}_1
\]

Now, if \( k \geq m + 1 \), then:

\[
\left\lfloor \beta_k(s) \equiv_{p-1} \beta_k(t) \right\rfloor \left( 1 + \nu_p(\beta_k(s) - \beta_k(t)) \right) + k - 1 \geq 0 + (m + 1) - 1 = m
\]

and so, the inequality automatically holds for \( k \geq m + 1 \). Thus, to have \( \chi_p(s) \equiv \chi_p(t) \), it suffices that:

\[
\left\lfloor \beta_k(s) \equiv_{p-1} \beta_k(t) \right\rfloor \left( 1 + \nu_p(\beta_k(s) - \beta_k(t)) \right) + k - 1 \geq m
\]

71
holds for $k \in \{1, \ldots, m\}$. Adding $-k + 1$ to both sides then gives:

$$\left[ \beta_k(s)^{p-1} \beta_k(t) \right] (1 + \nu_p (\beta_k(s) - \beta_k(t))) \geq m - k + 1$$

Finally, suppose there is a $k \in \{1, \ldots, m\}$ for which $\beta_k(s)$ is not congruent to $\beta_k(t)$ mod $p - 1$. Then:

$$\left[ \beta_k(s)^{p-1} \beta_k(t) \right] (1 + \nu_p (\beta_k(s) - \beta_k(t))) \geq m - k + 1$$

$$\downarrow$$

$$k \geq m + 1$$

which is impossible. As such, it must be that $\beta_k(s)^{p-1} \beta_k(t)$ holds for all $k \in \{1, \ldots, m\}$ whenever $\chi_p(t)^{p^m} \equiv \chi_p(s)$. Thus, we have that if $1 + \nu_p (\beta_k(s) - \beta_k(t)) \geq m - k + 1$ and $\beta_k(s)^{p-1} \beta_k(t)$ hold true for all $k \in \{1, \ldots, m\}$, it must be that $\chi(t)^{p^m} \equiv \chi(s)$. Since:

$$1 + \nu_p (\beta_k(s) - \beta_k(t)) \geq m - k + 1$$

$$\downarrow$$

$$\frac{\beta_k(s) - \beta_k(t)}{p^{m-k}} \in \mathbb{Z}$$

$$\downarrow$$

$$\beta_k(s)^{p^{m-k}} \equiv \beta_k(t)$$

we then have that the validity of the congruences [154] for all $k \in \{1, \ldots, m\}$ is then sufficient to guarantee $\chi_p(t)^{p^m} \equiv \chi_p(s)$.

Q.E.D.

4.2 The Probabilistic Approach and Bayesian Inference

Here, we elaborate slightly on the common ground between the present work and Tao’s paper.

**Definition 4.2.1**: The characteristic function of $\chi_p$ is the function $\varphi_p : \hat{\mathbb{Z}}_p \to \mathbb{C}$ defined by:

$$\varphi_p(t) \overset{\text{def}}{=} \int_{\mathbb{Z}_p} e^{2\pi i \{tx \oplus (3)\}_p} d\delta$$

(154)

Since $\chi_p$ is co-prime to $p$ except at 0, we can view the probability mass function for the values of $\chi_p$ modulo powers of $p$ as a function $f_p : \hat{\mathbb{Z}}_p \to [0, 1]$ defined by:

$$f_p\left(\frac{k}{p^n}\right) \overset{\text{def}}{=} P\left(\chi_p^{p^n} \equiv k\right) = \int_{\mathbb{Z}_p} \left[ \chi_p^{p^n}(\delta) \equiv k \right] d\delta$$

72
which holds for all \( n \geq 1 \) and all \( k \) co-prime to \( p \). We define \( f_p (0) \) to be 0. Then:

\[
f_p (t) = \frac{1}{|t|_p} \sum_{|s|_p \leq |t|_p} \varphi_p (s) e^{-2\pi i st|t|_p}, \quad \forall t \in \hat{\mathbb{Z}}_p \setminus \{0\}
\]

where the sum is taken over all \( s \in \hat{\mathbb{Z}}_p \) whose \( p \)-adic magnitude is less than or equal to \( |t|_p \). In this terminology, Tao’s Syrac \( (\mathbb{Z}/3^n\mathbb{Z}) \) is the projection of \( \chi_3 \mod 3^n \). In this—the author’s terminology—the most difficult result of Tao’s paper (the Fourier decay estimate) can then be stated as:

**Theorem 4.2.1** [10]: For any real constant \( A > 0 \), there exists a real constant \( C_A > 0 \) (depending only on \( A \)) so that:

\[
\left| \sum_{|t|_3=3^n} f_3 (t) e^{2\pi itx} \right| \leq \frac{C_A}{n^A}, \quad \forall x \in \mathbb{Z} : \gcd (x, 3) = 1, \quad \forall n \in \mathbb{N}_1 \quad (155)
\]

The author is curious to see if Tao’s techniques could be applied to obtain comparable estimates for the \( f_p \)s. That being said, working independently, the author derived Tao’s Lemma 1.12 using \( \varphi_p \):

\[
\varphi_p (t) = \frac{1}{2^{\omega_{|t|_p}(2) - 1}} \sum_{j=0}^{\omega_{|t|_p}(2)-1} 2^j e^{2\pi i j t} \varphi_p (2^j pt), \quad \forall t \in \hat{\mathbb{Z}}_p \setminus \{0\} \quad (156)
\]

where:

**Definition 4.2.2**: For integers \( a, b \), with \( a \geq 2 \) and \( \gcd (a, b) = 1 \), \( \omega_a (b) \) denotes the multiplicative order of the integer \( b \) in \( (\mathbb{Z}/a\mathbb{Z})^\times \).

Equation (156) can be obtained by first establishing the functional equation:

\[
\varphi_p (2t) = \frac{1}{2} \varphi_p (t) + \frac{1}{2} e^{2\pi it} \varphi_p (pt), \quad \forall t \in \hat{\mathbb{Z}}_p \quad (157)
\]

which uniquely characterizes \( \varphi_p \). Equation (156) can be solved to obtain:

\[
\varphi_p (t) = \sum_{j_1=0}^{\omega_{|t|_p}(2)-1} \sum_{j_2=0}^{\omega_{|t|_p/p^2}(2)-1} \cdots \sum_{j_{-\nu_p(t)}=0}^{\omega_{|t|_p/p^{\nu_p(t)}(2)-1}} \frac{2^{2j_1+\cdots+j_{-\nu_p(t)}} e^{2\pi it} \prod_{t=0}^{\nu_p(t)-1} (2^{\omega_{|t|_p/p^{(2)}}(2) - 1})}{\prod_{t=0}^{\nu_p(t)-1} (2^{\omega_{|t|_p/p^2}(2) - 1})}, \quad \forall t \in \hat{\mathbb{Z}}_p \setminus \{0\} \quad (158)
\]

Here, \( \nu_p (t) \) is the \( p \)-adic valuation of \( t \).

In the case where \( 2 \) is a friend of \( p \), this simplifies to:

\[
\varphi_p (t) = \sum_{j_1=0}^{\nu_p(t)-1} \sum_{j_2=0}^{\nu_p(t)-1} \cdots \sum_{j_{-\nu_p(t)}=0}^{\nu_p(t)-1} \frac{2^{2j_1+\cdots+j_{-\nu_p(t)}} e^{2\pi it} \prod_{t=0}^{\nu_p(t)-1} (2^{\omega_{|t|_p/p^{(2)}}(2) - 1})}{\prod_{t=0}^{\nu_p(t)-1} (2^{\omega_{|t|_p/p^2}(2) - 1})}, \quad \forall t \in \hat{\mathbb{Z}}_p \setminus \{0\} \quad (159)
\]
Here, note that for non-zero $t$, $-\nu_p(t)$ is the value of $n$ so that $t$’s irreducible fraction representation is of the form $t = k/p^n$.

We can also use this to obtain a formula for $f_p$.

**Definition 4.2.3**: We adopt the convention of using **bold** letters to denote finite-length tuples of integers. In what follows, $u$ and $j$ are tuples of finite length.

I. We define the **empty tuple** to be the unique tuple of length 0 (which contains nothing at all).

II. We write $|j|$ to denote the **length** of $j$—that is, the number of entries of $j$.

III. We write $\Sigma(j)$ to denote the **sum** of the entries of $j$.

IV. Given a tuple $j = (j_1, \ldots, j_n)$, we write:

$$\Sigma_\ell(j) \overset{\text{def}}{=} \sum_{h=1}^{\ell} j_h, \quad \forall \ell \in \mathbb{N}_1$$

(160)

In particular, note that $\Sigma_\ell(j) = \Sigma(j)$ for all $\ell \geq |j|$.

V. 

$$\alpha_p(j) \overset{\text{def}}{=} \sum_{\ell=1}^{\frac{|j|}{2}} 2^{\Sigma_\ell(j)} p^{\ell-1}$$

(161)

$$\alpha_p(u; j) \overset{\text{def}}{=} \sum_{\ell=1}^{\frac{|j|}{2}} 2^{(p-1)\Sigma_\ell(u)} 2^{\Sigma_\ell(j)} p^{\ell-1}$$

(162)

with:

$$\alpha_p(u; j) = \alpha_p(j) \text{ if } |u| = 0$$

(163)

VI. We write $(\mathbb{Z}/k\mathbb{Z})^n$ to denote the set of all $n$-tuples whose entries are taken from the set $\{0, 1, \ldots, k-1\}$.

**Theorem 4.2.2**: Let $p$ be an odd prime such that 2 is a primitive root of unity mod $p$ and mod $p^2$. Then, for all $n \in \mathbb{N}_1$ and all $k \in \mathbb{Z}$:

$$P\left(x_p \equiv_k \right) = \sum_{u_1=0}^{p^n-1-1} \sum_{u_2=0}^{p^n-2-1} \cdots \sum_{u_{n-1}=0}^{p-1} \sum_{j \in (\mathbb{Z}/(p-1)\mathbb{Z})^n} 2^{(p-1)\Sigma(u)} 2^{\Sigma(j)} \frac{\alpha_p(u; j)}{\prod_{\ell=0}^{p^n-1} (2^{(p-1)\ell} - 1)}$$

(164)

Finally, an elementary application of Bayes’ Theorem establishes a relationship between the decay of $f_p$ and periodic points. The motivation for constructing the numen was to be able to perform analysis on the space of all parity vectors by treating said vectors as the sequences of digits of 2-adic integers. In
do so, however, we had to allow the parity vectors to vary independently of the input $x$ that generated them. Indeed, formula \[24\]

$$h_{\beta^{-1}(t)}(x) = \frac{p^{#_1(t)}}{2^{\lambda(t)}} x + \chi_p(t), \quad \forall x \in \mathbb{Z}, \quad \forall t \in \mathbb{N}_1$$

holds regardless of the value of $x$. To figure out how to re-establish a connection between $x$ and $t$ (i.e., between $x$ and the reverse parity-vector $j = \beta^{-1}(t)$), we begin with the observation that, in terms of subsets of $\mathbb{Z}_2$ and $\mathbb{Z}_p$, $\chi_p$ satisfies the identity:

$$\chi_p(j + 2^m \mathbb{Z}_2) = h_{\beta^{-1}([j]_{2^m})}(x) + p^{#_1([j]_{2^m})} \mathbb{Z}_p, \quad \forall m \in \mathbb{N}_1, \quad \forall j \in \mathbb{Z}_2, \quad \forall x \in \mathbb{Z}$$

(165)

By the **Correspondence Theorem**, we know that a non-zero $x$ is a periodic point of $H_p$ if and only if there is a $k \in \mathbb{N}_1$ so that $h_{\beta^{-1}(k)}(x) = x$, and hence, if and only if:

$$\chi_p(k + 2^m \mathbb{Z}_2) = x + p^{#_1(k)} \mathbb{Z}_p, \quad \forall m \geq \lambda(k)$$

Note here that we have two pieces of information: the value of $\chi_p(j)$ modulo a power of $p$, and the value of $j$ modulo a power of 2. Using conditional probability, we can link these two pieces of information.

**Proposition 4.2.1:** Let $x$ be a non-zero periodic point of $H_p$. Then, $x$ is co-prime to $p$.

**Proof:** Suppose $x$ is a non-zero periodic point of $H_p$ which is a multiple of $p$. Then, there is a $y \in \mathbb{Z}$ so that $H_p(y) = x$. Since $H_p(2n) = n$ and $H_p(2n + 1) = pn + \frac{p+1}{2}$, the only way that $x = H_p(y)$ can be a multiple of $p$ is if $x = H_p(y) = y/2$. That is, if $x$ is a multiple of $p$, then: $H_p^{-1}(\{x\}) = 2x$. Consequently, the $m$th pre-image of $x$ under $H_p$ is $2^m x$, which is equal to $x$ for some $m \in \mathbb{N}_1$ if and only if $x = 0$. Since $x$ being a periodic point of $H_p$ forces $x$ to be in the $m$th pre-image of $x$ under $H_p$ for some $m \in \mathbb{N}_1$, this forces $x = 0$, which is a contradiction.

Q.E.D.

**Proposition 4.2.2:** Let $x \in \mathbb{Z}$, let $m \in \mathbb{N}_1$, and let $a \in \mathbb{Z}_2$. Then:

$$P\left(\chi_p^{p^{#_1([a]_{2^m})}} \equiv x \mid 2^m \equiv a\right) = \left[ h_{\beta^{-1}([a]_{2^m})}(x) \equiv 2^m \right]$$

(166)

**Remark:** Letting $j \in \mathcal{J}$ be non-zero, and letting $a = \beta(j)$, formula 166 can be written as:

$$P\left(\chi_p^{p^{#_1(j)}} \equiv x \mid 2^{2^j} \equiv \beta(j)\right) = \left[ h_j(x) \equiv x \right], \quad \forall x \in \mathbb{Z}, \forall j \in \mathcal{J} \setminus \{0\}$$

(167)
Proof: Let \( x \in \mathbb{Z} \), let \( m \in \mathbb{N}_1 \), and let \( a \in \mathbb{Z}_2 \). The two “events” we are concerned with are \( \chi_p \) taking the value of \( x \) modulo \( p^{\#_1([a]_{2^m})} \) and \( \tilde{z} \) (the parity vector) taking the value \( k \) mod \( 2^m \). That is to say, we are comparing the probability that the number \( t = \beta(\tilde{z}) \) produced by an arbitrary parity vector is congruent to \( a \) mod \( 2^m \) with the probability that the output of \( \chi_p \) along that parity vector (that is, at \( t = \beta(\tilde{z}) \)) is congruent to \( x \) mod \( p^{\#_1([a]_{2^m})} \).

The probability, here, is the measure of the set of all \( z \in \mathbb{Z}_2 \) congruent to \( a \) mod \( 2^m \); this is the measure of the support of the indicator function \( \left[ z_{2^m} \equiv a \right] \).

Likewise, the probability for \( \chi_p \) is the measure of the support of the indicator function \( \left[ \chi_p(\tilde{z}) \equiv x \right] \). Consequently, using the definition of conditional probability in terms of joint probabilities, we have that:

\[
P \left( \chi_p \equiv x \mid z_{2^m} \equiv a \right) = \frac{P \left( \chi_p \equiv x \& z_{2^m} \equiv a \right)}{P \left( z_{2^m} \equiv a \right)}
\]

\[
= \frac{1}{P \left( z_{2^m} \equiv a \right)} \int_{\mathbb{Z}_2} \left[ z_{2^m} \equiv a \right] \left[ \chi_p(\tilde{z}) \equiv x \right] d\tilde{z}
\]

\[
= 2^m \int_{a+2^m\mathbb{Z}_2} \left[ \chi_p(\tilde{z}) \equiv x \right] d\tilde{z}
\]

\[
= 2^m \int_{1/2^m} h_{\beta^{-1}([a]_{2^m})}(x) p^{\#_1([a]_{2^m})} \equiv x d\tilde{z}
\]

as desired.

Q.E.D.

**Lemma 4.2.1**: For any odd prime \( p \), any integer \( x \) which is co-prime to \( p \), and any \( j \in J \):

\[
\left[ h_j(x) \equiv x \right] \leq 2|j| f_p \left( \frac{x}{p^{\#_1(j)}} \right)
\]

(168)

**Remark**: Since the Iverson bracket can only take the values 0 and 1, the bracket must vanish whenever the upper bound is less than 1.
Proof: Let \( p, x, m, \) and \( j \) be as given. Setting \( m = |j| \) and \( a = \beta(j) \), with the help of Proposition 4.2.2, we use Bayes' Theorem, we obtain:

\[
P\left(\chi_p^{p\equiv_1(j)} \equiv x \mid \beta(j) \right) = \frac{P\left(\chi_p^{p\equiv_1(j)} \equiv x \right) P\left(\beta(j) \mid \chi_p^{p\equiv_1(j)} \equiv x \right)}{P\left(\beta(j) \right)}\]

The co-primality of \( x \) to \( p \) justifies us writing:

\[
P\left(\chi_p^{p\equiv_1(j)} \equiv x \right) = f_p \left( \frac{x}{p^{\#_1(j)}} \right)\]

Finally, using the well-known fact that probabilities are numbers in \([0,1]\), we obtain the upper bound:

\[
\left[ h_j(x)^{p\equiv_1(j)} \equiv x \right] = 2^{2|j|} f_p \left( \frac{x}{p^{\#_1(j)}} \right) \frac{P\left(\beta(j) \mid \chi_p^{p\equiv_1(j)} \equiv x \right)}{\leq 1} \leq 2^{2|j|} f_p \left( \frac{x}{p^{\#_1(j)}} \right)
\]

as desired.

Q.E.D.

**Theorem 4.2.3** (Necessary inequality for \( h_j(x)^{p\equiv_1(j)} \equiv x \)): Let \( p \) be an odd prime. Then, for any \( x \in \mathbb{Z} \) with \( \gcd(x,p) = 1 \), if there is a \( j \in \mathcal{J} \) so that \( h_j(x)^{p\equiv_1(j)} \equiv x \), it must be that:

\[
|j| \geq -\log_2 f_p \left( \frac{x}{p^{\#_1(j)}} \right) \quad (169)
\]

**Remark:** Equivalently, for such an \( x \), if \( 0 < |j| < -\log_2 f_p \left( \frac{x}{p^{\#_1(j)}} \right) \), then \( h_j(x)^{p\equiv_1(j)} \not\equiv x \).

Proof: Letting \( j \in \mathcal{J} \) be arbitrary, the fact that \( f_p \leq 1 \) implies that upon taking \( \log_2 \) of \([168]\) and multiplying by \(-1\), we obtain:

\[
|j| - \log_2 \left[ h_j(x)^{p\equiv_1(j)} \equiv x \right] \geq -\log_2 f_p \left( \frac{x}{p^{\#_1(j)}} \right)
\]

Thus, given \( x \), if \( j \) is such that \( h_j(x)^{p\equiv_1(j)} \equiv x \), the logarithm of the bracket vanishes, and we obtain \([169]\).
Corollary 4.2.1 (Periodic Point Inequality): Let \( x \in \mathbb{Z} \) be co-prime to \( p \). If, for every non-zero \( j \in J \), there is an \( n \in \mathbb{N}_1 \) so that:

\[
 f_p \left( \frac{x}{p^{n \#_1(j)}} \right) < \frac{1}{2^n |j|} \tag{170}
\]

then \( x \) is not a periodic point of \( H_p \).

Remark: Equivalently, if \( x \) is a non-zero periodic point of \( H_p \), then there is a \( j \in J \) for which:

\[
 f_p \left( \frac{x}{p^{n \#_1(j)}} \right) \geq \frac{1}{2^n |j|}, \quad \forall n \in \mathbb{N}_1 \tag{171}
\]

In particular, we can take \( j \) to be the shortest non-zero string for which \( h_j(x) = x \).

Proof: Let \( j \in J \) be non-zero (i.e., \( \#_1(j) \geq 1 \)). Taking:

\[
 x^{p^{n \#_1(j)}} \equiv h_j^n(x) = r_p(j)x + \chi_p(j)
\]

replace \( j \) with \( j^n \), the \( n |j| \)-tuple consisting of \( n \) concatenated copies of \( j \). This yields

\[
 x^{p^{n \#_1(j)}} \equiv h_j^n(x) = r_p(j)x + \frac{1 - r_p^n(j)}{1 - r_p(j)} \chi_p(j)
\]

which is equivalent to:

\[
 x^{p^{n \#_1(j)}} \equiv \frac{\chi_p(j)}{1 - r_p(j)} = \chi_p(B(j))
\]

So:

\[
 \chi_p(B(j))^{p^{n \#_1(j)}} \equiv x \Rightarrow f_p \left( \frac{x}{p^{n \#_1(j)}} \right) \geq \frac{1}{2^n |j|}
\]

Hence:

\[
 f_p \left( \frac{x}{p^{n \#_1(j)}} \right) < \frac{1}{2^n |j|} \Rightarrow \chi_p(B(j))^{p^{n \#_1(j)}} \not\equiv x \Rightarrow \chi_p(B(j)) \neq x
\]

By the **Correspondence Theorem**, every non-zero integer periodic point of \( H_p \) is of the form \( \chi_p(B(j)) \) for some non-zero \( j \in J \). Consequently if for every non-zero \( j \in J \), there is an \( n \in \mathbb{N}_1 \) so that [170] holds true, then there is no non-zero \( j \) for which \( \chi_p(B(j)) = x \), and hence, \( x \) is not a non-zero periodic point of \( H_p \).

Q.E.D.

As a sample application, we prove:
Proposition 4.2.3 (Lower bound on $f_3(1/3^n)$)

\[ f_3\left(\frac{1}{3^n}\right) \geq \frac{1}{4^n}, \quad \forall n \in \mathbb{N} \quad (172) \]

Proof: Let $(0,1)^n$ denote the 2m-tuple obtained by concatenating m copies of the 2-tuple $(0,1)$ (ex: $(0,1)^2 = (0,1,0,1)$, etc.).

Claim: If $m \in \mathbb{N}$, then the only integer $x$ satisfying $h_{(0,1)^m}(x) = x$ is $x = 1$.

Proof of Claim: Noting that:

\[ h_{0,1}(x) = \frac{3x + 1}{4} \]

we have that:

\[ h_{(0,1)^m}(x) = h_{0,1}^{om}(x) \]

\[ = \left(\frac{3}{4}\right)^m x + \frac{1}{4} \sum_{k=0}^{m-1} \left(\frac{3}{4}\right)^k \]

\[ = \left(\frac{3}{4}\right)^m (x - 1) + 1 \]

So, if $h_{(0,1)^m}(x) = x$, then:

\[ x = \left(\frac{3}{4}\right)^m (x - 1) + 1 \]

\[ \downarrow \]

\[ 0 = \left(\left(\frac{3}{4}\right)^m - 1\right) (x - 1) \]

which forces either $m = 0$ or $x = 1$. Since $m$ must be positive, this forces $x = 1$, which proves the claim. ✓

Consequently, since $x = 1$ is an odd periodic point of $H_3$ with minimal string $\mathbf{j} = (0,1)$, \textbf{171} forces \textbf{172} to be true.

Q.E.D.

While it is tantalizing to try to combine \textbf{171} or \textbf{170} with \textbf{155} this does not lead to anything useful, not even if we assume Tao’s conjecture on the refinability of his $O\left(\frac{n}{\log n}\right)$ bound in \textbf{155} to an $O\left(e^{-An}\right)$ bound; our crude application of “probability is a number between 0 and 1” to obtain \textbf{171} and \textbf{170} makes for an estimate too weak to yield more desirable results. As such, the author wonders if it might be possible to refine these Bayesian inequalities to something more useful, as well as to explore their converses.
4.3 An Archimedean Upper Bound for $\chi_p$ on certain subsets of $\mathbb{Z}_2$

By carefully exploiting the universality of the geometric series, we can nearly cheat at $p$-adic analysis to obtain archimedean bounds for restrictions of non-archimedean-valued functions to appropriately chosen domains, enabling us to obtain upper bounds on the rising-continuation of a function $f : \mathbb{N}_0 \to \mathbb{Q}$. The author calls this approach the “$L^1$-method”.

**Theorem 4.3.1 (The $L^1$-Method):** Let $f : \mathbb{N}_0 \to \mathbb{Q}$ admit a rising-continuation $\tilde{f} : \mathbb{Z}_p \to \mathbb{Z}_q$, and let $U \subseteq \mathbb{Z}_p$ be a set on which the van der Put series:

$$S \{f\} (\mathfrak{z}) \overset{\text{def}}{=} \sum_{n=0}^{\infty} c_n (f) \left[ \mathfrak{z} p^\chi_p(n) \equiv n \right]$$

converges in $\mathbb{R}$ (though *not* necessarily to the correct/true value of $f$). Write $f' : U \to \mathbb{R}$ to denote the function defined by:

$$f'(\mathfrak{z}) = S \{f\} (\mathfrak{z}), \quad \forall \mathfrak{z} \in U$$

that is, $f'$ is defined by the sum of $S \{f\} (\mathfrak{z})$ in $\mathbb{R}$. Finally, let $\eta : \mathbb{Z}_p \to U$ be a measurable function, and let $g : \mathbb{Z}_p \to \mathbb{R}$ be defined by $g \overset{\text{def}}{=} f' \circ \eta$. With these definitions:

I. If $g \in L^1(\mathbb{Z}_p)$, letting $\hat{g} : \hat{\mathbb{Z}}_p \to \mathbb{C}$ denote the Fourier coefficients of $g$, we have that $g(\mathfrak{z})$’s complex-valued Fourier series is uniformly convergent in $\mathbb{C}$ over $\mathbb{Z}_p$, and that:

$$\hat{f}(\eta(\mathfrak{z})) \overset{\mathbb{C}}{=} \sum_{t \in \hat{\mathbb{Z}}_p} \hat{g}(t) e^{2\pi i (t \eta)(\mathfrak{z})}, \quad \forall \mathfrak{z} \in \eta^{-1}(D_{p,q}(f))$$

where, recall, $D_{p,q}(f)$ is the set of $\mathfrak{z} \in \mathbb{Z}_p$ for which $S \{f\} (\mathfrak{z})$ reduces to either a finite sum or to a geometric series which is convergent in both $\mathbb{R}$ and $\mathbb{Z}_q$ (a.k.a. $f$’s domain of “double convergence”).

**Warning:** For values of $\mathfrak{z} \notin \eta^{-1}(D_{p,q}(f))$, the sum of $g$’s Fourier series is spurious—it need not have any bearing on the true value of $\tilde{f}$ at $\mathfrak{z}$.

II. (Archimedean bound on $\tilde{f}$)

$$\sup_{\mathfrak{z} \in \eta(\mathbb{Z}_p) \cap D_{p,q}(f)} \left| \tilde{f}(\mathfrak{z}) \right| \leq \| \hat{g} \|_1$$

(173)

**Proof:** Let $f$, $\tilde{f}$, $U$, $f'$, $\eta$, and $g$ be as given. Since the real-sum of the van der Put series converges point-wise on $U$ to $f'$, $f'$ is then a measurable real-valued function on $U$. Since $\eta : \mathbb{Z}_p \to U$ is measurable, the composition $g = f' \circ \eta$ is a real-valued measurable function on $\mathbb{Z}_p$. So, we can proceed using Fourier analysis of a real-valued function of a $p$-adic variable.
Next, suppose \( g \) is integrable. Since \( g \) is integrable, it has an absolutely convergent complex-valued Fourier series:

\[
f'(\eta(z)) \overset{\text{def}}{=} g(z) = \sum_{t \in \mathbb{Z}_p} \hat{g}(t) e^{2\pi i tz}, \quad \forall z \in \mathbb{Z}_p
\]

The key thing to note is that when \( z \in D_{p,q}(f) \), by Theorem ?, the sum of \( S^k(f) \) converges in both \( \mathbb{C} \) and \( \mathbb{Z}_q \), and to the same quantity, necessarily a rational number:

\[
f'(z) = \tilde{f}(z), \quad \forall z \in \eta^{-1}(D_{p,q}(f))
\]

Consequently:

\[
g(z) \overset{\text{def}}{=} f'(\eta(z)) = \tilde{f}(\eta(z)), \quad \forall z \in \eta^{-1}(D_{p,q}(f))
\]

which proves (II).

Finally, applying the triangle inequality gives the elementary estimate:

\[
\sup_{z \in \mathbb{Z}_p} |f'(\eta(z))| \leq \sum_{t \in \mathbb{Z}_p} |\hat{g}(t)| = \|\hat{g}\|_1
\]

Thus:

\[
\|\hat{g}\|_1 \geq \sup_{z \in \mathbb{Z}_p} |f'(\eta(z))|
\]

\[
\geq \sup_{z \in \eta^{-1}(D_{p,q}(f))} |f'(\eta(z))|
\]

\[
= \sup_{z \in \eta(\mathbb{Z}_p) \cap D_{p,q}(f)} |f'(z)|
\]

(174)

which yields \([173]\)

Q.E.D.

The moral of the \( L^1 \)-method is that even though the values of the Fourier series of \( g \) are not meaningful at every \( z \in \mathbb{Z}_p \), we can still obtain information about true values of \( \tilde{f} \) at some \( z \in \mathbb{Z}_p \).

As an application, we can use the \( L^1 \)-method to obtain bounds on \( \chi_p(z) \) on appropriately chosen subsets of \( \mathbb{Z}_2 \).

**Definition 4.3.1**: Recall that, for any \( z \in \mathbb{Z}_2 \setminus \mathbb{N}_0 \), we can compute \( \chi_p(z) \) by writing:

\[
\chi_p(z) \overset{\text{def}}{=} \sum_{k=1}^{#_1(z)} \frac{p^{k-1}}{2^m(z)+1}
\]

As such:

81
I. We write $U_p$ to denote the set of 2-adic integers for which for the series on the right hand side converges to a (finite) sum in $\mathbb{R}$:

$$U_p \overset{\text{def}}{=} \left\{ z \in \mathbb{Z}_2 : \sum_{k=1}^{\#_1(z)} \frac{p^{k-1}}{2^{\beta_k(z)+1}} \in \mathbb{R} \right\}$$

As defined, note that $U_p$ contains $\mathbb{N}_0$ and $B(X_p^+)$, but not $B(X_p^-)$.

II. We then define $S_p$ as the set of all $z \in U_p$ for which the sum of:

$$\sum_{k=1}^{\#_1(z)} \frac{p^{k-1}}{2^{\beta_k(z)+1}}$$

in $\mathbb{R}$ is not equal to its sum in $\mathbb{Q}_p$. We call the elements of $S_p$ spurious points of $\chi_p$. Additionally, when $z$ is spurious, the value of $\sum_{k=1}^{\#_1(z)} \frac{p^{k-1}}{2^{\beta_k(z)+1}}$ in $\mathbb{R}$ is called a spurious value of $\chi_p$.

III. More generally, given any formula for $\chi_p(z)$ at some $z \in \mathbb{Z}_2$ (note, such a formula could be $p$-adic valued (a series in $\mathbb{Q}_p$) or complex-valued (a series in $\mathbb{C}$)), we say that the quantity specified by said formula at the indicated value of $z$ is spurious whenever it is not equal to the true value of $\chi_p(z) \in \mathbb{Z}_p$. On the other hand, we say the specified by the formula at the indicated value of $z$ is correct whenever that value is equal to the true value of $\chi_p(z) \in \mathbb{Z}_p$.

The importance of the spurious values of $z$ is in their form, rather than their value. As we saw in Section 1, the form of the series $\sum_{k=1}^{\#_1(z)} \frac{p^{k-1}}{2^{\beta_k(z)+1}}$ which gives the $p$-adic value of $\chi_p(z)$ is completely determined by the system of functional equations in $[29]$. The spurious values of $z$ can thus be thought of as those 2-adic integers for which the series $\sum_{k=1}^{\#_1(z)} \frac{p^{k-1}}{2^{\beta_k(z)+1}}$ generated by the 1s and 0s in $z$’s 2-adic expansion is well-behaved enough to converge to some value in $\mathbb{R}$. As such, the functional equations $[29]$ hold true for all $z \in S_p$. That is to say:

**Proposition 4.3.1:** Let $\chi'_p : U_p \to \mathbb{R}$ be the function defined by:

$$\chi'_p(n) \overset{\text{def}}{=} \chi_p(n), \forall n \in \mathbb{N}_0 \tag{175}$$

$$\chi'_p(z) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \frac{p^{k-1}}{2^{\beta_k(z)+1}}, \forall z \in U_p \setminus \mathbb{N}_0 \tag{176}$$

Thus, $\chi'_p(z)$ give the correct value of $\chi_p(z)$ whenever $z$ is a non-spurious element of $D_p$, and takes potentially spurious values everywhere else on its domain. In particular, note that $\chi'_p(z)$ necessarily takes only non-negative real values, and vanishes if and only if $z = 0$. 

82
Then, $\chi'_p$ satisfies the same functional equations as $\chi_p$:

$$\chi'_p(2z) = \frac{1}{2} \chi'_p(z)$$
$$\chi'_p(2z+1) = \frac{p\chi'_p(z) + 1}{2}$$
for all $z \in D_p$. In other words, we can think of $\chi'_p$ is a real-valued solution to this system (subject to the initial condition $\chi'_p(0) = 0$, whereas $\chi_p$ is the $p$-adic valued solution to this initial value problem.)

Proof: Since, for each $z \in U_p$, $\chi'_p(z)$ is given by the sum of the series in $\mathbb{R}$, the functional-equation properties of that series hold in $\mathbb{R}$. Consequently, we have that the functional equations satisfied by $\chi_p$ must also be satisfied by $\chi'_p$.

Q.E.D.

Note that this tells us that, as a subset of $\mathbb{Z}_2$, $U_p$ is mapped into itself by the maps $z \mapsto 2z$ and $z \mapsto 2z + 1$.

While it might be overly optimistic to hope that an approach as simple as the one detailed in Theorem 4.3.1 could be of any use, the author has found at least one tractable application of this method, to be detailed in this subsection.

With respect to future work, what remains to be seen is the extent to which the method and its conclusions can be refined (particularly with regards to the choice of $\eta$ and $V$).

The case we shall deal with relies on the following elementary characterization of $U_p$ in terms of the spacing of the 1s in the 2-adic digits of its elements.

**Proposition 4.3.2**: Let $\bar{z} \in Z_2 \setminus N_0$, and recall that for each $n \in N_1$ we write $\beta_n(\bar{z})$ to denote the non-negative integer so that $2^{\beta_n(\bar{z})}$ is the $n$th smallest term in the 2-adic expansion of $\bar{z}$. Then:

I. A necessary condition for $\bar{z}$ to be an element of $U_p$ is:

$$\lim_{n \to \infty} \sup (\beta_{n+1}(\bar{z}) - \beta_n(\bar{z})) \geq \frac{\ln p}{\ln 2}$$

II. A sufficient condition for $\bar{z}$ to be an element of $U_p$ is:

$$\lim_{n \to \infty} \inf (\beta_{n+1}(\bar{z}) - \beta_n(\bar{z})) > \frac{\ln p}{\ln 2}$$

That is, a necessary condition that a 2-adic integer $\bar{z}$ be an element of $U_p$ is that there are infinitely many 1s in the 2-adic digits of $\bar{z}$ which are separated from the next 1 by at least $\lceil \frac{\ln p}{\ln 2} \rceil - 1$ consecutive 0s. Likewise, a sufficient condition for $\bar{z}$ to be an element of $U_p$ is that all but finitely many 1s in the 2-adic digits of $\bar{z}$ are separated from the next 1 by at least $\lceil \frac{\ln p}{\ln 2} \rceil - 1$ consecutive 0s.
Proof: Fix $\mathfrak{z} \in \mathbb{Z}_2 \setminus \mathbb{N}_0$. As a short-hand, we write $\beta_n$ in lieu of $\beta_n (\mathfrak{z})$. Then, we have that:

$$\chi'_p (\mathfrak{z}) \equiv \sum_{n=1}^{\infty} \frac{p^{n-1}}{2^{\beta_n + 1}}$$

Now, let $z_n = [\mathfrak{z}]_{2^{\beta_n + 1}}$. Then, observe that $\#_1 (z_n) = n$ and $\lambda (n) = \beta_n + 1$. Consequently:

$$r_p (z_n) = \frac{p^{\#_1(z_n)}}{2^{\lambda(z_n)}} = \frac{p^n}{2^{\beta_n + 1}}$$

As such:

$$\chi'_p (\mathfrak{z}) \equiv \sum_{n=1}^{\infty} r_p (z_n)$$

Noting that:

$$r_p (z_n + 1) = \frac{p^{n+1}}{2^{\beta_{n+1} + 1}} = \frac{p}{2^{\beta_{n+1} - \beta_n}} \frac{p^n}{2^{\beta_n + 1}} = \frac{p}{2^{\beta_{n+1} - \beta_n}} r_p (z_n)$$

we then apply the Ratio test: $\sum_{n=0}^{\infty} r_p (z_n)$ will diverge whenever:

$$1 < \liminf_{n \to \infty} \frac{r_p (z_n + 1)}{r_p (z_n)} = \liminf_{n \to \infty} \frac{p}{2^{\beta_{n+1} - \beta_n}}$$

Taking natural logarithms gives:

$$\limsup_{n \to \infty} (\beta_{n+1} - \beta_n) < \frac{\ln p}{\ln 2}$$

Hence:

$$\limsup_{n \to \infty} (\beta_{n+1} - \beta_n) \geq \frac{\ln p}{\ln 2}$$

is a necessary condition for $\mathfrak{z}$ to be in $U_p$. On the other hand, the series will converge whenever:

$$1 > \limsup_{n \to \infty} \frac{r_p (z_n + 1)}{r_p (z_n)} = \limsup_{n \to \infty} \frac{p}{2^{\beta_{n+1} - \beta_n}}$$

which yields:

$$\liminf_{n \to \infty} (\beta_{n+1} - \beta_n) > \frac{\ln p}{\ln 2}$$

and thus gives us a sufficient condition for $\mathfrak{z}$ to be in $U_p$.

Q.E.D.

As mentioned previously, these convergence criteria are rather weak, and so, improving them (tightening the necessary condition and/or loosening the sufficient condition) is likely the first step toward refining the results to be given below. That being said, the sufficient condition $\liminf_{n \to \infty} (\beta_{n+1} (\mathfrak{z}) - \beta_n (\mathfrak{z})) > \frac{\ln p}{\ln 2}$ is of particular interest to us, because the set of all $\mathfrak{z} \in \mathbb{Z}_2$ for which it holds can, in fact, be continuously parameterized over $\mathbb{Z}_2$. 

84
The trick to using the $L^1$ method appears to be cleverly constructing subsets of $U_p$ which can be continuously parameterized over $\mathbb{Z}_2$ by some function $\eta$, where $\eta$ satisfies functional equations that are compatible with the functional equations satisfied by $\chi'_p$. As it turns out, the set of all $\mathbf{z} \in \mathbb{Z}_2$ in which all but finitely many 1s are separated from the next 1 by at least $\lceil \ln p/\ln 2 \rceil - 1$ consecutive 0s is homeomorphic to $\mathbb{Z}_2$, and the homeomorphism satisfies exactly the kind of functional equations needed to make the Fourier coefficients of $\chi'_p$'s pre-composition with that homeomorphism explicitly computable.

**Definition 4.3.2:**
I. For every integer $\kappa \geq 2$, define the function $\tau_\kappa : \mathbb{Z}_2 \to \mathbb{Z}_2$ by:

$$
\tau_\kappa \left( \sum_{n=0}^{\infty} a_n 2^{\kappa n} \right) \text{ def } = \sum_{n=0}^{\infty} a_n 2^{\kappa n}
$$

That is, $\tau$ inserts $\kappa - 1$ zeros between every two 1s in the 2-adic digits of its input.

As pointed out by Alain Robert [7], note that the functions $\tau_\kappa$ are injective and continuous, satisfying the $\kappa$-Hölder condition:

$$
|\tau_\kappa (x) - \tau_\kappa (y)|_2 = |x - y|_2^\kappa, \; \forall x, y \in \mathbb{Z}_2
$$

II. Let $D_\kappa \text{ def } = \tau_\kappa (\mathbb{Z}_2)$ denote the image of $\mathbb{Z}_2$ under $\tau_\kappa$. That is, $D_\kappa$ is the set of all 2-adic integers whose 2-adic digits contain at least $\kappa - 1$ zeroes between any two 1s. With these definitions, using the $L^1$ method, we can then prove:

**Theorem 4.3.2:** Let $p$ be an odd integer $\geq 3$, and let $\kappa$ be any positive integer for which the following statements all hold true:

$$
\kappa \geq \lceil \log_2 p \rceil
$$

$$
2^\kappa - 1 > p
$$

Then, $H_p$ can have only finitely many periodic points in $\mathbb{N}_1$ that are routed through $D_\kappa$. In particular, any such periodic point $\omega$ satisfies:

$$
\omega \leq \frac{2^{\kappa - 1} (2^{\kappa + 1} - 2 - p)}{(2^\kappa - p - 1) (2^{\kappa + 1} - p - 1)}
$$

**Proposition 4.3.3:** If $\kappa \geq \lceil \log_2 p \rceil$, then $D_\kappa \subseteq U_p$.

Proof: Fix $\kappa \geq \lceil \log_2 p \rceil$, and let $\mathbf{z} \in D_\kappa$ be arbitrary. Then, there is a $\mathbf{\eta} \in \mathbb{Z}_2$ so that

$$
\mathbf{z} = \tau_\kappa (\mathbf{\eta}) = \sum_{n=1}^{\infty} 2^{\kappa \beta_n(\mathbf{\eta})}
$$
Since $\kappa \geq \lceil \log_2 p \rceil$, we have that any two 1s in the 2-adic digital expansion of $\mathfrak{z} = \tau_\kappa (\mathfrak{y})$ have at least $\lceil \log_2 p \rceil > \frac{\ln p}{\ln 2}$ zeroes between them; that is:

$$\beta_{n+1} (\mathfrak{z}) - \beta_n (\mathfrak{z}) = \beta_{n+1} (\tau_\kappa (\mathfrak{y})) - \beta_n (\tau_\kappa (\mathfrak{y})) \geq \kappa = \lceil \log_2 p \rceil > \frac{\ln p}{\ln 2}, \ \forall n \in \mathbb{N}_1$$

Consequently, by Proposition 4.3.2, $\mathfrak{z} = \tau_\kappa (\mathfrak{y})$ satisfies the sufficient condition to be an element of $U_p$, and is therefore contained in $U_p$.

Q.E.D.

**Definition 4.3.3:** Let:

$$\chi_{p;\kappa} \overset{\text{def}}{=} \chi'_p \circ \tau_\kappa \quad (181)$$

$$\hat{\chi}_{p;\kappa} (t) \overset{\text{def}}{=} \int_{\mathbb{Z}_2} \chi'_{p} (\tau_\kappa (\mathfrak{z})) e^{-2\pi i t \mathfrak{z}} \quad (182)$$

By the $L^1$ method, to prove Theorem 4.3.2, it suffices to show that $\hat{\chi}_{p;\kappa} \in L^1 (\mathbb{Z}_2)$ whenever $\kappa$ satisfies [179] (184) for $\chi_{p;\kappa}$. Then, we show that [184] has a unique solution in $L^1 (\mathbb{Z}_2)$, and that the Fourier coefficients of this solution must be given by [187] and [188]. This then completely determines $\chi_{p;\kappa}$ and $\hat{\chi}_{p;\kappa}$. We can then directly show that $\hat{\chi}_{p;\kappa} \in L^1 (\mathbb{Z}_2)$.

**Proposition 4.3.4:** For all $\kappa \in \mathbb{N}_1$, the following functional equations hold true for all $\mathfrak{z} \in \mathbb{Z}_2$:

$$\tau_\kappa (2 \mathfrak{z}) = 2^\kappa \tau_\kappa (\mathfrak{z})$$

$$\tau_\kappa (2 \mathfrak{z} + 1) = 2^\kappa \tau_\kappa (\mathfrak{z}) + 1 \quad (183)$$

Proof: Let $\kappa \in \mathbb{N}_1$ be arbitrary, and write $\mathfrak{z} = \sum_{n=0}^{\infty} \delta_n 2^n \in \mathbb{Z}_2$. Then:

I. $\tau_\kappa (2 \mathfrak{z}) = \tau_\kappa (\sum_{n=1}^{\infty} \delta_{n-1} 2^n) = \sum_{n=1}^{\infty} \delta_{n-1} 2^{\kappa n} = \sum_{n=0}^{\infty} \delta_n 2^{\kappa n + \kappa} = 2^\kappa \tau_\kappa (\mathfrak{z})$

II. $\tau_\kappa (2 \mathfrak{z} + 1)$

$$\begin{align*}
\tau_\kappa (2 \mathfrak{z} + 1) &= \tau_\kappa \left( 1 + \sum_{n=1}^{\infty} \delta_{n-1} 2^n \right) \\
&= 1 + \sum_{n=1}^{\infty} \delta_{n-1} 2^{\kappa n} \\
&= 1 + \sum_{n=0}^{\infty} \delta_n 2^{\kappa n + \kappa} \\
&= 1 + 2^\kappa \tau_\kappa (\mathfrak{z})
\end{align*}$$

Q.E.D.
Proposition 4.3.5: Let $\kappa$ be arbitrary. Then, for all $\mathfrak{f} \in \mathbb{Z}_2$:

$$\chi_{p,\kappa} (2\mathfrak{f}) = \frac{\chi_{p,\kappa} (\mathfrak{f})}{2^{\kappa}}$$  \hspace{1cm} \text{(184)}

$$\chi_{p,\kappa} (2\mathfrak{f} + 1) = \frac{p\chi_{p,\kappa} (\mathfrak{f}) + 2^{\kappa - 1}}{2^{\kappa}}$$

Proof:

I. $\chi_{p,\kappa} (2\mathfrak{f}) = \chi'_{p} (\tau_{\kappa} (2\mathfrak{f})) = \chi'_{p} (2^{\kappa} \tau_{\kappa} (\mathfrak{f})) = 2^{-\kappa} \chi'_{p} (\tau_{\kappa} (\mathfrak{f})) = 2^{-\kappa} \chi_{p,\kappa} (\mathfrak{f})$

II. $\chi_{p,\kappa} (2\mathfrak{f} + 1) = \chi'_{p} (\tau_{\kappa} (2\mathfrak{f} + 1)) = \chi'_{p} (2^{\kappa} \tau_{\kappa} (\mathfrak{f}) + 1) = \frac{p\chi'_{p} (2^{\kappa - 1} \tau_{\kappa} (\mathfrak{f})) + 1}{2} = \frac{p2^{1-\kappa} \chi'_{p} (\tau_{\kappa} (\mathfrak{f})) + 1}{2} = \frac{p\chi'_{p} (\tau_{\kappa} (\mathfrak{f})) + 2^{\kappa - 1}}{2^{\kappa}} = \frac{p\chi_{p,\kappa} (\mathfrak{f}) + 2^{\kappa - 1}}{2^{\kappa}}$

Q.E.D.

Lemma 4.3.1: Let $p$ be an odd integer $\geq 3$, and let $\kappa \geq 1$ be arbitrary. Let $f \in L^1 (\mathbb{Z}_2)$ satisfy the functional equations:

$$f (2\mathfrak{f}) = \frac{f (\mathfrak{f})}{2^{\kappa}}$$  \hspace{1cm} \text{(185)}

$$f (2\mathfrak{f} + 1) = \frac{pf (\mathfrak{f}) + 2^{\kappa - 1}}{2^{\kappa}}$$

for all $\mathfrak{f} \in \mathbb{Z}_2$, and let:

$$\hat{f} (t) = \int_{\mathbb{Z}_2} f (\mathfrak{f}) e^{-2\pi i \mathfrak{f} t} d\mathfrak{f}$$

denote the Fourier transform of $f$. Then:

I. $\hat{f} (t) = \frac{1 + pe^{-2\pi i t}}{2^{\kappa + 1}} \hat{f} (2t) + \frac{e^{-2\pi i t}}{4} \mathbbm{1}_0 (2t), \forall t \in \hat{\mathbb{Z}}_2$  \hspace{1cm} \text{(186)}

II. The values of $\hat{f} (t)$ are given by:

$$\hat{f} (0) = \frac{2^{\kappa - 1}}{2^{\kappa + 1} - 1 - p}$$

$$\hat{f} \left( \frac{1}{2} \right) = \frac{1 - 2^{\kappa}}{2^{\kappa + 1} - 1 - p}$$
\[ \hat{f}(t) = \hat{f}(1) \prod_{n=0}^{\log_2 \left( \frac{|t|}{4} \right)} \frac{1 + p \left( e^{-2\pi it} \right)^{2^n}}{2^{n+1}}, \ \forall t \in \hat{Z}_2 \setminus \{0\} \]

where the product over \( n \) is defined to be 1 when \( t = \frac{1}{2} \) (i.e., when the upper limit of the product is \( \log_2 \left( \frac{1/2}{4} \right) = \log_2 \frac{2}{4} = -1 \)).

This shows that if there exists a solution \( f \in L^1(Z_2) \) to the system of functional equations, then it must be unique, and is given by the function with the Fourier coefficients described in (II).

Proof:

I. Let \( f \) be as given. Then, for all \( t \in \hat{Z}_2 \):

\[
\hat{f}(t) = \int_{Z_2} f(z) e^{-2\pi i \{t\} z} dz
\]

\[
= \int_{2Z_2} f(z) e^{-2\pi i \{t\} z} dz + \int_{2Z_2+1} f(z) e^{-2\pi i \{t\} z} dz
\]

\[
\left( \ell = \frac{k}{2}, \ \eta = \frac{k-1}{2} \right)
\]

\[
= \frac{1}{2^{\kappa+1}} \int_{Z_2} f(x) e^{-2\pi i \{2t\} x} dx + \frac{1}{2} \int_{Z_2} f(2n + 1) e^{-2\pi i \{2n+1\} x} dx
\]

\[
+ \frac{e^{-2\pi it}}{4} \int_{Z_2} e^{-2\pi i \{2\eta\} x} dx
\]

\[
= \frac{1 + pe^{-2\pi it}}{2^{\kappa+1}} \hat{f}(2t) + \frac{e^{-2\pi it}}{4^\kappa} \hat{f}(0)
\]

and so:

\[
\hat{f}(t) = \frac{1 + pe^{-2\pi it}}{2^{\kappa+1}} \hat{f}(2t) + \frac{e^{-2\pi it}}{4^\kappa} \hat{f}(0)
\]

which is \[180\]

II. Letting \( t = 0 \) in \[180\] gives:

\[
\hat{f}(0) = \frac{1 + p}{2^{\kappa+1}} \hat{f}(0) + \frac{1}{4}
\]

\[
\hat{f}(0) = \frac{2^{\kappa-1}}{2^{\kappa+1} - 1 - p}
\]

88
Next, letting \( t = \frac{1}{2} \) in \[186\] and using the fact that \( \hat{f}(0) = \frac{2^{\kappa - 1}}{2^{\kappa + 1} - 1 - p} \) and that \( \hat{f} \) is periodic with period 1 gives:

\[
\hat{f}\left(\frac{1}{2}\right) = \frac{1 - p}{2^{\kappa + 1}} \hat{f}(0) - \frac{1}{4} = \frac{1 - p}{2^{\kappa + 1}} \frac{2^{\kappa - 1}}{2^{\kappa + 1} - 1 - p} - \frac{1}{4} = \frac{1}{2} \frac{1 - 2^{\kappa}}{2^{\kappa + 1} - 1 - p}
\]

Now, note that in \[186\]

\[
\hat{f}(t) = \frac{1 + pe^{-2\pi it}}{2^{\kappa + 1}} \hat{f}(2t) + \frac{e^{-2\pi it}}{4} \mathbf{1}_0(2t)
\]

the indicator function \( \mathbf{1}_0(2t) \) vanishes for all \( t \in \hat{Z}_2 \) with \( |t|_2 \geq 4 \) (i.e., for \( t \neq 0, \frac{1}{2} \)). Consequently, we have:

\[
\hat{f}(t) = \frac{1 + pe^{-2\pi it}}{2^{\kappa + 1}} \hat{f}(2t), \ \forall |t|_2 \geq 4
\]

Using this functional equation, it follows that for any \( t \) with \( |t|_2 = 2^m \geq 2^2 = 4 \), there will be \( m - 2 \) terms in the product defining \( \hat{f}(t) \), ending with a constant term of \( \hat{f}\left(\frac{1}{2}\right) \). Since \( m - 2 = \log_2 |t|_2 - 2 = \log_2 \frac{|t|_2}{4} \) this can be written as:

\[
\hat{f}(t) = \hat{f}\left(\frac{1}{2}\right) \prod_{n=0}^{\log_2 \left|\frac{|t|_2}{4}\right|} \frac{1 + p \left(e^{-2\pi it}\right)^{2^n}}{2^{\kappa + 1}}, \ \forall |t|_2 \geq 4
\]

When \( t = \frac{1}{2} \), \( \log_2 \left|\frac{|t|_2}{4}\right| = -1 \). As such, if we adopt the convention that the product over \( n \) is defined to be 1 when \( t = \frac{1}{2} \), we then have that:

\[
\hat{f}(t) = \hat{f}\left(\frac{1}{2}\right) \prod_{n=0}^{\log_2 \left|\frac{|t|_2}{4}\right|} \frac{1 + p \left(e^{-2\pi it}\right)^{2^n}}{2^{\kappa + 1}}, \ \forall t \in \hat{Z}_2 \setminus \{0\}
\]

as desired.

Q.E.D.

**Lemma 4.3.2:** Let \( p \) be an odd integer \( \geq 3 \), let \( \kappa \) be a positive integer, and let \( g : \hat{Z}_2 \to \mathbb{C} \) be the function defined by:

\[
g(0) = \frac{2^{\kappa - 1}}{2^{\kappa + 1} - 1 - p}
\]

\[
g\left(\frac{1}{2}\right) = \frac{1}{2} \frac{1 - 2^{\kappa}}{2^{\kappa + 1} - 1 - p}
\]

\[
g(t) = g\left(\frac{1}{2}\right) \prod_{n=0}^{\log_2 \left|\frac{|t|_2}{4}\right|} \frac{1 + p \left(e^{-2\pi it}\right)^{2^n}}{2^{\kappa + 1}}, \ \forall t \in \hat{Z}_2 \setminus \{0\}
\]

where the product is 1 when \( t = \frac{1}{2} \).
With these hypotheses, if \( p < 2^{\kappa} - 1 \), then \( g \in L^1(\hat{\mathbb{Z}}_2) \). In particular:

\[
\|g\|_{L^1(\hat{\mathbb{Z}}_2)} \leq \frac{2^{\kappa-1} (2^{\kappa+1} - 2 - p)}{(2^{\kappa} - p - 1) (2^{\kappa+1} - p - 1)}
\]

Proof: We have that:

\[
\|g\|_{L^1(\hat{\mathbb{Z}}_2)} = |g(0)| + \left| g \left( \frac{1}{2} \right) \right| + \left| g \left( \frac{1}{2} \right) \right| \sum_{t \in \hat{\mathbb{Z}}_2} \prod_{n=0}^{\log_2 \|t\|_2} \left| \frac{1 + p (e^{-2\pi i t})^{2^n}}{2^{\kappa+1}} \right|
\]

Now:

\[
\sum_{t \in \hat{\mathbb{Z}}_2} \prod_{n=0}^{\log_2 \|t\|_2} \left| \frac{1 + p (e^{-2\pi i t})^{2^n}}{2^{\kappa+1}} \right| \leq \sum_{t \in \hat{\mathbb{Z}}_2} \left( \frac{p + 1}{2^{\kappa+1}} \right)^{-1+\log_2 \|t\|_2}
\]

We evaluate the sum over \( t \) by writing \( \{t \in \hat{\mathbb{Z}}_2 : \|t\|_2 \geq 4 \} \) as the union of \( \{t \in \hat{\mathbb{Z}}_2 : \|t\|_2 = 2^m \} \) over \( m \in \mathbb{N}_2 \):

\[
\sum_{t \in \hat{\mathbb{Z}}_2, \|t\|_2 \geq 4} \left( \frac{p + 1}{2^{\kappa+1}} \right)^{-1+\log_2 \|t\|_2} = \sum_{m=2}^{\infty} \sum_{t \in \hat{\mathbb{Z}}_2, \|t\|_2 = 2^m} \left( \frac{p + 1}{2^{\kappa+1}} \right)^{m-1} = \sum_{m=2}^{\infty} \left( \frac{p + 1}{2^{\kappa+1}} \right)^{m-1} \sum_{t \in \hat{\mathbb{Z}}_2, \|t\|_2 = 2^m} 1
\]

The \( t \)-sum on the far right is simply the number of elements of \( \{t \in \hat{\mathbb{Z}}_2 : \|t\|_2 = 2^m \} \), which is \( 2^{m-1} \) (the number of irreducible fractions in \((0, 1)\) with a denominator of \(2^m\)). As such:

\[
\sum_{t \in \hat{\mathbb{Z}}_2, \|t\|_2 \geq 4} \prod_{n=0}^{\log_2 \|t\|_2} \left| \frac{1 + p (e^{-2\pi i t})^{2^n}}{2^{\kappa+1}} \right| \leq \sum_{m=2}^{\infty} \left( \frac{p + 1}{2^{\kappa+1}} \right)^{m-1} 2^{m-1} = \sum_{m=1}^{\infty} \left( \frac{p + 1}{2^{\kappa}} \right)^m
\]

Thus, the condition \( 2^\kappa - 1 > p \) is sufficient to guarantee:

\[
\|g\|_{L^1(\hat{\mathbb{Z}}_2)} \leq |g(0)| + \left| g \left( \frac{1}{2} \right) \right| + \left| g \left( \frac{1}{2} \right) \right| \sum_{m=1}^{\infty} \left( \frac{p + 1}{2^{\kappa}} \right)^m < \infty
\]

Using \( g(0) = \frac{2^{\kappa-1}}{2^{\kappa+1} - 1 - p}, \ g \left( \frac{1}{2} \right) = \frac{1}{2} \frac{2^{\kappa-1} - 2^{\kappa}}{2^{\kappa+1} - 1 - p} \), and summing the geometric series then yields:

\[
\|g\|_{L^1(\hat{\mathbb{Z}}_2)} \leq \frac{2^{\kappa-1} (2^{\kappa+1} - 2 - p)}{(2^{\kappa} - p - 1) (2^{\kappa+1} - p - 1)}
\]
as desired.
Q.E.D.

In summary all that we have shown:

**Theorem 4.3.3**: Let \( p \) be an odd integer \( \geq 3 \), let \( \kappa \) be an integer \( \geq \lceil \log_2 p \rceil \)
so that \( 2^{\kappa} - 1 > p \). Then:

I. The Fourier coefficients of \( \chi_{p,\kappa} \) are given by:

\[
\hat{\chi}_{p,\kappa}(0) = \frac{2^{\kappa-1}}{2^{\kappa+1} - 1 - p} \quad (187)
\]

\[
\hat{\chi}_{p,\kappa}(t) = \hat{\chi}_{p,\kappa}(1/2) \prod_{n=0}^{\log_2 \frac{t}{|t|}} \frac{1 + p(e^{-2\pi i t})^{2^n}}{2^{\kappa+1}}, \quad \forall t \in \hat{\mathbb{Z}}_2 \setminus \{0\} \quad (188)
\]

where the product is defined to be 1 when \( t = \frac{1}{2} \).

II. \( \hat{\chi}_{p,\kappa} \in L^1(\hat{\mathbb{Z}}_2) \), with:

\[
\|\hat{\chi}_{p,\kappa}\|_{L^1(\hat{\mathbb{Z}}_2)} \leq \frac{2^{\kappa-1}(2^{\kappa+1} - 2 - p)}{(2^\kappa - p - 1)(2^{\kappa+1} - p - 1)} \]

Proof: Use Lemmata 4.3.1 & 4.3.2 along with the Fourier Inversion Theorem
for the Fourier Transform over \( \mathbb{Z}_2 \) and \( \hat{\mathbb{Z}}_2 \) to compute the formulae
for the Fourier coefficients of \( \chi_{p,\kappa} \). The \( p < 2^\kappa - 1 \) condition then guarantees
the \( L^1 \) summability of \( \hat{\chi}_{p,\kappa} \) and the specified upper bound on its \( L^1 \) norm.
Q.E.D.

Proof of Theorem 4.3.2: Letting \( \kappa \) satisfy \ref{condition:179}, since \( \omega_{p,\kappa} \) is in \( L^1(\hat{\mathbb{Z}}_2) \),
by the \( L^1 \) method (here, \( \eta \) is \( \tau_\kappa \)) and it must be that any periodic point of \( H_p \)
in routed through \( D_\kappa \) is \( \leq \|\hat{\chi}_{p,\kappa}\|_{L^1(\hat{\mathbb{Z}}_2)} \leq \frac{2^{\kappa-1}(2^{\kappa+1} - 2 - p)}{(2^\kappa - p - 1)(2^{\kappa+1} - p - 1)} \),
which then forces there to be only finitely many such periodic points. Since \( D_\kappa \subseteq U_p \), this shows
that there can be only finitely many periodic points of \( H_p \) in \( \mathbb{N}_0 \) which are routed
through \( D_\kappa \), because any such periodic point must be \( \leq \frac{2^{\kappa-1}(2^{\kappa+1} - 2 - p)}{(2^\kappa - p - 1)(2^{\kappa+1} - p - 1)} \).
Q.E.D.

**References**

[1] Schikhof, W. (1985). *Ultrametric Calculus: An Introduction to p-Adic Analysis* (Cambridge Studies in Advanced Mathematics). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511623844

[2] Cohen, H. (2008). *Number theory: Volume II: Analytic and modern tools* (Vol. 240). Springer Science & Business Media
[3] Flajolet, P., Grabner, P., Kirschenhofer, P., Prodinger, H., & Tichy, R. F. (1994). “Mellin transforms and asymptotics: digital sums”. Theoretical Computer Science, 123(2), 291-314.

[4] Flajolet, P., Gourdon, X., & Dumas, P. (1995). “Mellin transforms and asymptotics: Harmonic sums”. Theoretical computer science, 144(1), 3-58.

[5] Lagarias, J. C. The $3x + 1$ problem and its generalizations, Amer. Math. Monthly 92 (1985), 3–23.

[6] Radyna, Y. (2004). “Change of variable in integrals over p-adic and adelic domains”. In North-Holland Mathematics Studies (Vol. 197, pp. 267-272). North-Holland.

[7] Robert, A. M. (2013). A course in $p$-adic analysis (Vol. 198). Springer Science & Business Media. Chicago

[8] Rozier, O. (2018). “Parity sequences of the $3x+1$ map on the $2$-adic integers and Euclidean embedding”. arXiv preprint arXiv:1805.00133.

[9] Siegel, M. C. (2019). “Conservation of Singularities in Functional Equations Associated to Collatz-Type Dynamical Systems; or, Dreamcatchers for Hydra Maps”. arXiv preprint arXiv:1909.09733.

[10] Tao, Terence. (2019) “Almost all orbits of the Collatz map attain almost bounded values.” arXiv: Probability : n. pag.

[11] Folland, Gerald B. A course in abstract harmonic analysis. Vol. 29. CRC press, 2016.

[12] Washington, Lawrence C. Introduction to Cyclotomic Fields. Second Edition. Vol. 83. New York: Springer New York, 1997.

[13] Corrado Böhm and Giovanna Sontacchi (1978), “On the existence of cycles of given length in integer sequences like $x_n + 1 = x_n/2$ if $x_n$ even, and $x_{n+1} = 3x_n + 1$ otherwise”, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 64 (1978), 260–264. (MR 83h:10030)

[14] Titchmarsh, E. C., and D. R. Heath-Brown. (1986) The theory of the Riemann zeta-function. Oxford [Oxfordshire]: Clarendon Press.

[15] Conrad, K. “Infinite Series in $p$-Adic” Fields. <https://kconrad.math.uconn.edu/blurbs/gradnumthy/infseriespadic.pdf>

[16] Lagarias, J.C. “The Takagi function and its properties”, Functions in number theory and their probabilistic aspects, RIMS Kôkyûroku Bessatsu, B34, Res. Inst. Math. Sci. (RIMS), Kyoto, 2012, pp. 153–189. MR 301484

[17] EuYu (https://math.stackexchange.com/users/9246/euyu), 2 is a primitive root mod $3^h$ for any positive integer $h$, URL (version: 2017-04-13): https://math.stackexchange.com/q/594800