A result on s-semipermutable subgroups of finite groups and some applications

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ABSTRACT
Let \( p \) be a prime number, \( G \) be a \( p \)-solvable finite group and \( P \) be a Sylow \( p \)-subgroup of \( G \). We prove that \( G \) is \( p \)-supersolvable if \( NG(P) \) is \( p \)-supersolvable and if there is a subgroup \( H \) of \( P \) with \( P' \leq H \leq \Phi(P) \) such that \( H \) is s-semipermutable in \( G \). As applications, we simplify the proofs of some known results and also generalize some known results.

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1. Introduction
In this paper, all groups are assumed to be finite. We use standard notation and terminology, see for example [8] or [12]. Throughout, let \( p \) be an arbitrary but fixed prime number.

Recall that two subgroups \( H \) and \( K \) of a group \( G \) are said to permute if \( HK = KH \), or equivalently if \( HK \) is a subgroup of \( G \). A subgroup \( H \) of a group \( G \) is said to be s-permutable in \( G \) if \( H \) permutes with every Sylow subgroup of \( G \). A subgroup \( H \) of a group \( G \) is called s-semipermutable in \( G \) if \( H \) permutes with every Sylow \( q \)-subgroup \( Q \) of \( G \) for all primes \( q \) not dividing \( |H| \).

By a result of Wielandt [12, Kapitel IV, Satz 8.1], a group \( G \) with Sylow \( p \)-subgroup \( P \) is \( p \)-nilpotent if \( P \) is regular and \( N_G(P) \) is \( p \)-nilpotent. By [12, Kapitel III, Satz 10.2 (a)], a \( p \)-group \( P \) is regular if the nilpotency class of \( P \) is less than \( p \), i.e. if \( Z_{p-1}(P) = P \). Consequently, a group \( G \) with Sylow \( p \)-subgroup \( P \) is \( p \)-nilpotent if \( Z_{p-1}(P) = P \) and \( N_G(P) \) is \( p \)-nilpotent. Recently, Xu and Li [16] generalized this result as follows.

Theorem 1.1. ([16, Theorem 1.3]) Let \( G \) be a group of order divisible by \( p \), and let \( P \) be a Sylow \( p \)-subgroup of \( G \). Suppose that there is a normal subgroup \( H \) of \( P \) with \( H \leq \Phi(P) \) and \( P/H = Z_{p-1}(P)/H \) such that \( H \) is s-semipermutable in \( G \). Suppose moreover that \( N_G(P) \) is \( p \)-nilpotent. Then \( G \) is \( p \)-nilpotent.

The main result of this paper is a version of Theorem 1.1 for \( p \)-supersolvable groups. Our point of departure is the observation that a \( p \)-solvable group \( G \) with Sylow \( p \)-subgroup \( P \) is \( p \)-supersolvable if \( P \) is abelian and \( N_G(P) \) is \( p \)-supersolvable. To see this, it is enough to consider the case \( O_{p'}(G) = 1 \). Then we have \( CG(O_p(G)) \leq O_p(G) \) by [10, Chapter 6, Theorem 3.2], whence \( P = O_p(G) \) as \( P \) is abelian. So \( G = N_G(O_p(G)) = N_G(P) \) is \( p \)-supersolvable, as claimed.
Our main result generalizes this observation as follows.

**Theorem 1.2.** Let $G$ be a $p$-solvable group, and let $P$ be a Sylow $p$-subgroup of $G$. Suppose that there is a subgroup $H$ of $P$ such that $P' \leq H \leq \Phi(P)$ and such that $H$ is $s$-semipermutable in $G$. Suppose moreover that $N_G(P)$ is $p$-supersolvable. Then $G$ is $p$-supersolvable.

In view of Theorems 1.1 and 1.2, one might wonder whether the condition $P' \leq H \leq \Phi(P)$ in Theorem 1.2 can be replaced by the condition that $H \trianglelefteq P$, $H \leq \Phi(P)$ and $P/H = Z_{p-1}(P/H)$. The following example demonstrates that this is not the case.

**Example 1.3.** Let $G$ be the group indexed in GAP [9] as SmallGroup(216,153), and let $P$ be a Sylow 3-subgroup of $G$. Then $G$ is solvable and hence 3-solvable. Also, $P$ has nilpotency class 2. So, with $H = 1$, we have that $H \trianglelefteq P$, $H \leq \Phi(P)$ and $P/H = Z_2(P/H) = Z_3$. Moreover, $H$ is $s$-semipermutable in $G$, and $N_G(P)$ is 3-supersolvable. However, $G$ is not 3-supersolvable.

The paper is organized as follows. After collecting some preliminary results in Section 2, we will prove Theorem 1.2 in Section 3. After that, in Section 4, we will apply Theorem 1.2 to give a new proof of a special case of Theorem 1.1. Namely, we use Theorem 1.2 to show that a group $G$ with Sylow $p$-subgroup $P$ is $p$-nilpotent provided that $N_G(P)$ is $p$-nilpotent and that there is a subgroup $H$ of $P$ with $P' \leq H \leq \Phi(P)$ such that $H$ is $s$-semipermutable in $G$. Finally, in Section 5, we will use Theorems 1.1 and 1.2 to generalize a result of Liu and Yu [15] on pronormal subgroups and a result of Chen et al. [7] on weakly $\mathcal{H}$-subgroups and to simplify the proofs of these results.

### 2. Preliminaries

In this section, we collect some results needed for the proof of Theorem 1.2 and for our applications of Theorems 1.1 and 1.2.

**Lemma 2.1.** Let $H$ be an $s$-semipermutable subgroup of a group $G$.

1. ([5, Lemma 3.1]) If $H$ is a $p$-subgroup of $G$ and $N$ is a normal subgroup of $G$, then $HN/N$ is $s$-semipermutable in $G/N$.
2. ([5, Lemma 3.2]) If $H$ is a $p$-subgroup of $G$ and if $N$ is a normal $p$-subgroup of $G$, then $H \cap N$ is normalized by $O^p(G)$.
3. ([5, Lemma 3.3]) If $H \trianglelefteq K \leq G$, then $H$ is $s$-semipermutable in $K$.
4. ([13, Theorem A]) If $H$ is a $p$-group, then $H^G$ is solvable, where $H^G$ denotes the normal closure of $H$ in $G$.

**Lemma 2.2.** Let $G$ be a group of order divisible by $p$, and let $P$ be a Sylow $p$-subgroup of $G$. Suppose that $P$ has a subgroup $D$ with $1 < |D| < |P|$ such that any subgroup of $P$ with order $|D|$ is $s$-permutable in $G$. If $P$ is a nonabelian 2-group and $|D| = 2$, suppose moreover that any cyclic subgroup of $P$ with order 4 is $s$-permutable in $G$. Then $G$ is $p$-supersolvable.

**Proof.** This follows from [17, Theorem 1.1].

Following Ballester-Bolinches et al. [3], we say that a subgroup $H$ of a group $G$ is $c$-supplemented in $G$ if there is a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_G$, where $H_G$ is the core of $H$ in $G$.

**Lemma 2.3.** ([1, Theorem 3.2 and Corollary 3.4]) Let $P$ be a nontrivial normal $p$-subgroup of a group $G$. Suppose that there is a subgroup $D$ of $P$ with $1 < |D| < |P|$ such that every subgroup of $P$ with order $|D|$ or $p|D|$ is $c$-supplemented in $G$. If $P$ is a nonabelian 2-group and $|D| = 1$, suppose moreover that any cyclic
proof of Theorem 1.2

Proof of Theorem 1.2. Suppose that the theorem is not true, and let $G$ be a counterexample with minimal order. We will derive a contradiction in several steps.

1. $G/N$ is $p$-supersolvable for any nontrivial normal subgroup $N$ of $G$.

Let $1 \neq N \leq G$ and $\overline{G} := G/N$. Then $\overline{G}$ is $p$-solvable, $\overline{P}$ is a Sylow $p$-subgroup of $\overline{G}$, we have $\overline{P}^p = \overline{P} \leq \overline{H} \leq \overline{\Phi(P)} = \Phi(\overline{P})$, and $\overline{H}$ is $s$-semipermutable in $\overline{G}$ by Lemma 2.1 (1). Also $N_G(\overline{P}) = \overline{N_G(P)}$ is $p$-supersolvable. The minimality of $G$ implies that $\overline{G}$ is $p$-supersolvable.

2. $O_p'(G) = 1$.

Assume that $O_p'(G) \neq 1$. Then $G/O_p'(G)$ is $p$-supersolvable by (1), which implies that $G$ is $p$-supersolvable. This contradiction shows that $O_p'(G) = 1$.

3. We have $\Phi(G) = 1$, and $O_p(G)$ is the unique minimal normal subgroup of $G$.

Assume that $\Phi(G) \neq 1$. Then $G/\Phi(G)$ is $p$-supersolvable by (1). Applying [12, Kapitel VI, Satz 8.6 a)], we conclude that $G$ is $p$-supersolvable. This contradiction shows that $\Phi(G) = 1$.

Let $N_1, \ldots, N_t$ be the distinct minimal normal subgroups of $G$. From [11, Theorem 1.8.17], we see that $O_p(G) = N_1 \times \cdots \times N_t$. To finish the proof of (3), it is enough to show that $t = 1$.

Assume that $t > 1$. By [11, Theorem 1.3.7], $G \cong G/(N_1 \cap N_2)$ is isomorphic to a subgroup of $(G/N_1) \times (G/N_2)$. Since $G/N_1$ and $G/N_2$ are $p$-supersolvable by (1), we have that $(G/N_1) \times (G/N_2)$ is $p$-supersolvable. It follows that $G$ is $p$-supersolvable. This contradiction yields $t = 1$.

4. $H \cap O_p(G) = 1$.

Set $U := H \cap O_p(G)$. As $P' \leq H$, we have $U \leq P$. Hence, $U$ is the intersection of two normal subgroups of $P$, so $U \leq P$ and thus $P \leq N_G(U)$. By Lemma 2.1 (2), we also have $O_p(G) \leq N_G(U)$. Thus $G = PO_p'(G) \leq N_G(U)$ and hence $U \leq G$.

Since $O_p(G)$ is minimal normal in $G$ by (3), we either have $U = 1$ or $U = O_p(G)$. Assume that $U = O_p(G)$. Then $O_p(G) \leq H \leq \Phi(P)$, and [12, Kapitel III, Hilfssatz 3.3] implies that $O_p(G) \leq \Phi(G)$, which contradicts (3). So we have $U = 1$, as required.

5. Each maximal subgroup of $G$ contains $O_p(G)$ or $P'$.

Let $M$ be a maximal subgroup of $G$ such that $O_p(G) \not\leq M$. We have to show that $P' \leq M$.

By (4), we have $(P \cap M)' \cap O_p(G) \leq P' \cap O_p(G) \leq H \cap O_p(G) = 1$. Thus

$$(P \cap M)' \cap O_p(G) = P' \cap O_p(G) = 1.$$

As $O_p(G) \not\leq M$, we have $G = M O_p(G)$. So $P = P \cap G = P \cap M O_p(G) = (P \cap M) O_p(G)$. This implies that

$$(P/O_p(G))' = (P \cap M)' O_p(G)/O_p(G) \cong (P \cap M)'/(P \cap M)' \cap O_p(G) \cong (P \cap M)' .$$

On the other hand, we have

$$(P/O_p(G))' = P' O_p(G)/O_p(G) \cong P'/(P' \cap O_p(G)) \cong P'.$$

So we have $(P \cap M)' \cong (P/O_p(G))' \cong P'$. As $(P \cap M)' \leq P'$, it follows that $P' = (P \cap M)' \leq M$. This completes the proof of (5).
5. Applications

In this section, we use Theorems 1.1 and 1.2 to generalize some recent results on pronormal subgroups and weakly $\mathcal{H}$-subgroups obtained in [15] and [7] and to shorten their proofs.

Recall that a subgroup $H$ of a group $G$ is said to be pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for each $g \in G$. Examples of pronormal subgroups are normal subgroups, Sylow subgroups and Hall subgroups of soluble groups.
Recently, Liu and Yu [15] proved the following theorem.

**Theorem 5.1.** ([15, Theorem 1.3]) Let $G$ be a group of order divisible by $p$ such that $(|G|, p - 1) = 1$, and let $P$ be a Sylow $p$-subgroup of $G$. Suppose that $P' \leq G$ and that $P$ has a subgroup $D$ with $1 < |D| < |P|$ such that any subgroup of $P$ with order $|D|$ is pronormal in $N_G(P)$. If $P$ is a nonabelian 2-group and $|D| = 2$, suppose moreover that any cyclic subgroup of $P$ with order 4 is pronormal in $N_G(P)$. Then $G$ is $p$-nilpotent.

By [8, Chapter I, Lemma 6.3 (d)], a subgroup $H$ of a group $G$ is normal in $G$ provided that $H$ is both subnormal and pronormal in $G$. If $P$ is a Sylow $p$-subgroup of a group $G$, then any subgroup of $P$ is subnormal in $N_G(P)$ since any subgroup of $P$ is subnormal in $P$ and $P \leq N_G(P)$. Consequently, a subgroup of $P$ is pronormal in $N_G(P)$ if and only if it is normal in $N_G(P)$. Therefore, when we replace the word “pronormal” by the word “normal” in Theorem 5.1, then what we obtain is just a reformulation of Theorem 5.1.

As is well-known, any normal subgroup of a group $G$ is $s$-permutable in $G$, but the converse is not true. In the next theorem, we take the hypotheses of Theorem 5.1, but we replace pronormality by $s$-permutability, and we also weaken the condition $P' \leq G$.

**Theorem 5.2.** Let $G$ be a group of order divisible by $p$ such that $(|G|, p - 1) = 1$, and let $P$ be a Sylow $p$-subgroup of $G$. Assume that there is a normal subgroup $H$ of $P$ with $H \leq \Phi(P)$ and $P/H = Z_{p-1}(P/H)$ such that $H$ is $s$-semipermutable in $G$. Suppose that $P$ has a subgroup $D$ with $1 < |D| < |P|$ such that any subgroup of $P$ with order $|D|$ is $s$-permutable in $N_G(P)$. If $P$ is a nonabelian 2-group and $|D| = 2$, suppose moreover that any cyclic subgroup of $P$ with order 4 is $s$-permutable in $N_G(P)$. Then $G$ is $p$-nilpotent.

**Proof.** By Theorem 1.1, it suffices to show that $N_G(P)$ is $p$-nilpotent. Lemma 2.2 shows that $N_G(P)$ is $p$-supersolvable. We have $(|N_G(P)|, p - 1) = (|G|, p - 1) = 1$, and so [14, Lemma 2.4 (4)] implies that $N_G(P)$ is $p$-nilpotent, as required. 

Note that Theorem 5.1 is covered by Theorem 5.2. In the next theorem, we again take the hypotheses of Theorem 5.1, but we remove the condition that $(|G|, p - 1) = 1$ and add the condition that $G$ is $p$-solvable. We also weaken the condition that $P' \leq G$, and we replace pronormality by $s$-permutability.

**Theorem 5.3.** Let $G$ be a $p$-solvable group, and let $P$ be a Sylow $p$-subgroup of $G$. Assume that there is a subgroup $H$ of $P$ such that $P' \leq H \leq \Phi(P)$ and such that $H$ is $s$-semipermutable in $G$. Suppose that $P$ has a subgroup $D$ with $1 < |D| < |P|$ such that any subgroup of $P$ with order $|D|$ is $s$-permutable in $N_G(P)$. If $P$ is a nonabelian 2-group and $|D| = 2$, suppose moreover that any cyclic subgroup of $P$ with order 4 is $s$-permutable in $N_G(P)$, then $G$ is $p$-supersolvable.

**Proof.** By Theorem 1.2, it suffices to show that $N_G(P)$ is $p$-supersolvable. But this follows immediately from Lemma 2.2.

Following Bianchi et al. [6], we say that a subgroup $H$ of a group $G$ is an $H$-subgroup of $G$ if $N_G(H) \cap H^g \leq H$ for all $g \in G$. Following Asaad et al. [2], we say that a subgroup $H$ of a group $G$ is a weakly $H$-subgroup of $G$ if there is a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K$ is an $H$-subgroup of $G$.

Recently, Chen et al. [7] proved the following theorem.

**Theorem 5.4.** ([7, Theorem A]) Let $p$ be the smallest prime dividing the order of a group $G$, and let $P$ be a Sylow $p$-subgroup of $G$. Suppose that $P' \leq G$ and that every maximal subgroup of $P$ is a weakly $H$-subgroup of $N_G(P)$. Then $G$ is $p$-nilpotent.
By [6, Theorem 6 (2)], a subgroup \( H \) of a group \( G \) is normal in \( G \) provided that \( H \) is both subnormal and an \( \mathcal{H} \)-subgroup of \( G \). In particular, if \( P \) is a Sylow \( p \)-subgroup of a group \( G \), then a subgroup of \( P \) is an \( \mathcal{H} \)-subgroup of \( N_G(P) \) if and only if it is normal in \( N_G(P) \). Consequently, a subgroup of \( P \) is \( c \)-supplemented in \( N_G(P) \) if it is a weakly \( \mathcal{H} \)-subgroup of \( N_G(P) \).

In the next two results, we take the hypotheses of Theorem 5.4, but we replace “weakly \( \mathcal{H} \)-subgroup” by “\( c \)-supplemented”, and we also modify some other conditions.

**Theorem 5.5.** Let \( p \) be the smallest prime dividing the order of a group \( G \), and let \( P \) be a Sylow \( p \)-subgroup of \( G \). Assume that there is a normal subgroup \( H \) of \( P \) with \( H \leq \Phi(P) \) and \( P/H = Z_{p-1}(P/H) \) such that \( H \) is \( s \)-semipermutable in \( G \). Suppose that there is a subgroup \( D \) of \( P \) with \( 1 \leq |D| < |P| \) such that every subgroup of \( P \) with order \( |D| \) or \( p|D| \) is \( c \)-supplemented in \( N_G(P) \). If \( P \) is a nonabelian \( 2 \)-group and \( |D| = 1 \), suppose moreover that any cyclic subgroup of \( P \) with order 4 is \( c \)-supplemented in \( N_G(P) \). Then \( G \) is \( p \)-nilpotent.

**Proof.** By Theorem 1.1, it suffices to show that \( N_G(P) \) is \( p \)-nilpotent. Lemma 2.4 shows that \( N_G(P) \) is \( p \)-nilpotent.

**Theorem 5.6.** Let \( P \) be a Sylow \( p \)-subgroup of a \( p \)-solvable group \( G \). Assume that there is a subgroup \( H \) of \( P \) such that \( P' \leq H \leq \Phi(P) \) and such that \( H \) is \( s \)-semipermutable in \( G \). Suppose that there is a subgroup \( D \) of \( P \) with \( 1 \leq |D| < |P| \) such that every subgroup of \( P \) with order \( |D| \) or \( p|D| \) is \( c \)-supplemented in \( N_G(P) \). If \( P \) is a nonabelian \( 2 \)-group and \( |D| = 1 \), suppose moreover that any cyclic subgroup of \( P \) with order 4 is \( c \)-supplemented in \( N_G(P) \). Then \( G \) is \( p \)-supersolvable.

**Proof.** As a consequence of Lemma 2.3, every chief factor of \( N_G(P) \) below \( P \) is cyclic. Consequently, \( N_G(P) \) is \( p \)-supersolvable. So \( G \) is \( p \)-supersolvable by Theorem 1.2.

**Disclosure Statement**

The authors report there are no competing interests to declare.

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