An extremal problem in coloring of hypergraphs

Tapas Kumar Mishra  Sudebkumar Prasant Pal
Dept. of Computer Science and Engineering
IIT Kharagpur 721302, India

Friday 10th July, 2015

Abstract

Let $G(V,E)$ be a $k$-uniform hypergraph. A hyperedge $e \in E$ is said to be properly $(r,p)$ colored by an $r$-coloring of vertices in $V$ if $e$ contains vertices of at least $p$ distinct colors in the $r$-coloring. An $r$-coloring of vertices in $V$ is called a strong $(r,p)$ coloring if every hyperedge $e \in E$ is properly $(r,p)$ colored by the $r$-coloring. We study the maximum number of hyperedges that can be properly $(r,p)$ colored by a single $r$-coloring and the structures that maximizes number of properly $(r,p)$ colored hyperedges.

Keywords: Hypergraph, coloring, Strong coloring, extremal problem

1 Introduction

Let $G(V,E)$ be a $n$-vertex $k$-uniform hypergraph. A hyperedge $e \in E$ is properly $(r,p)$ colored by an $r$-coloring of vertices if $e$ consists of at least $p$ distinctly colored vertices. A strong $(r,p)$ coloring of $G$ is an $r$-coloring of the vertices of $V$, such that $\forall e \in E, e$ consists of at least $p$ distinctly colored vertices. We note that for a fixed $r$ and $p$, $G$ may not have any strong $(r,p)$ coloring. Moreover, its not too hard to see that the decision problem is also NP-complete, since (i) the decision problem of bicolorability of hypergraphs is NP-complete ([7]), and (ii) proper $(2,2)$ coloring of $G$ is equivalent to proper bicoloring of $G$. Given $n,k,r,$ and $p$, we study the maximum number of hyperedges $M(n,k,r,p)$ of any $n$-vertex $k$-uniform hypergraph $G(V,E)$ that can be properly $(r,p)$ colored by a single $r$-coloring.

This problem has an equivalent counterpart in graphs. A proper coloring of a edge in graphs denotes the vertices of the edge getting different colors. A graph is properly colored if its every edges is properly colored. Consider an $r$-coloring of a $n$-vertex graph $H(V,E')$. For any $K_k$ in $H$, $k \in \mathbb{N}$, a rainbow of size $x$ exists if there exists a $K_k$ which is a subgraph of the $K_k$, $x \leq k$, and is properly colored. Consider the problem of finding the maximum number of distinct $K_k$’s in an $r$-coloring such that each $K_k$ has a rainbow of size $p$. It is easy to see that this problem is equivalent to the problem of finding the maximum number of hyperedges in an $n$-vertex $k$-uniform hypergraph $G(V,E)$ that can be properly $(r,p)$ colored by a single $r$-coloring: each
"K_k is replaced by a k-uniform hyperedge and a rainbow of size p denotes p distinctly colored vertices in the hyperedge.

This problem has been motivated by the separation problems in graphs.

Definition 1. Let \( [n] \) denote the set 1, 2, ..., n. A set \( S \subseteq [n] \) separates i from j if \( i \in S \) and \( j \notin S \). A set \( S \) of subsets of \( [n] \) is a separator if, for each \( i,j \in [n] \) with \( i \neq j \), there is a set \( S \) in \( S \) that separates \( i \) from \( j \). If, for each \( (i,j) \in [n] \times [n] \) with \( i \neq j \), there is a set \( S \in S \) that separates \( i \) from \( j \) and a set \( T \in S \) that separates \( j \) from \( i \), then \( S \) is called a complete separator. Moreover, with the additional constraint that the sets \( S \) and \( T \) that separate \( i \) and \( j \) are required to be disjoint, then \( S \) is called a total separator.

We refer the reader to [11, 4, 5, 17, 2, 14, 8] for discussions and results on separating families for graphs. The notion of separation for hyperedges is introduced in [9]. A family \( S = \{S_1,\ldots,S_t\} \) is called a separator for a \( k \)-uniform hypergraph \( G(V,E) \), \( S_i \subseteq V \) for \( 1 \leq i \leq t \), such that every hyperedge \( e \in E \) has a nonempty intersection with at least one \( S_i \) and \( V \setminus S_i \). We consider the following problem of separation for \( k \)-uniform hypergraphs. Let \( G(V,E) \) be a \( k \)-uniform hypergraph. A set \( S_1 = \{S_{11},\ldots,S_{1r}\} \) \( (r,p) \)-separates a hyperedge \( e \in E \) if (i) \( S_{1j} \subseteq V \), \( S_{1j} \neq \emptyset \), \( 1 \leq j \leq r \), (ii) \( \cup_j S_{1j} = V \), and, (ii) \( e \) has nonempty intersection with at least \( min\{\#e,p\} \) elements of \( S_1 \). Observe that the maximum number of hyperedges that can be \( (r,p) \)-separated by a single family \( S_1 \) is \( M(n,k,r,p) \).

Consider the problem of maximizing profit between a player \( P \) and an adversary \( A \). Adversary \( A \) provides \( n, k, r \) and \( p \) to the player \( P \). \( P \) performs some calculation on those parameters and finds out a number \#e. Now, \( A \) constructs a \( n \)-vertex \( k \)-uniform hypergraph with \#e hyperedges and colors the vertices with \( r \)-colors. If \( A \) can properly \( (r,p) \) color at least \#e hyperedges in a hypergraph, then \( A \) wins. If \( A \) cannot properly \( (r,p) \) color at least \#e hyperedges in a hypergraph, then \( P \) wins. However, the profit of \( P \) is given by \( \binom{n}{k} - \#e \). So, given a fixed \( n, k, r \) and \( p \), what value of \#e should \( P \) use so that he is guaranteed a win and his profit is maximized. Observe that if \( P \) chooses \#e to be \( M(n,k,r,p) + 1 \), then he is guaranteed a win with maximum profit.

The problem has many applications in resource allocation and scheduling. Consider the problem where there are total \( n \) resources \( \{v_1,\ldots,v_n\} \), \( m \) processes \( \{e_1,\ldots,e_m\} \). Each process has a distinct wish-list of \( k \) resources. There are \( r \) time slots. A process can execute if it gets at least \( p \) distinct resources in different time slots. The problem is to maximize the number of processes that can be executed within \( r \) time slots. The solution to the above problem is equivalent to the maximum number of hyperedges that can be properly \( (r,p) \) colored by a single \( r \)-coloring in an \( n \)-vertex \( k \)-uniform hypergraph \( G(V,E) \), where \( V = \{v_1,\ldots,v_n\} \), \( E = \{e_1,\ldots,e_m\} \). Throughout the paper, \( G \) denotes a \( k \)-uniform hypergraph with vertex set \( V \) and hyperedge set \( E \), unless otherwise stated.

1.1 Motivation

Turán’s theorem is a fundamental result in graph theory that gives the maximum number of edges that can be present in a \( K_{r+1} \) free graph. The problem was first stated by Mantel [18, 1] for the special case of triangle free graphs. He proved that the maximum number of edges in an \( n \)-vertex
any

In order to estimate exactly coloring of a Turán density number of hyperedges in an n-vertex k-uniform hypergraph not containing a copy of F. The number of distinct hyperedges that consists of exactly \( k \)-uniform hypergraph \( F \) the Turán number \( \text{ex}(n, F) \) is the maximum number of hyperedges in a graph \( G(V, E) \) that does not contain some arbitrary subgraph \( F \).

**Definition 2.** Given an \( k \)-uniform hypergraph \( F \) the Turán number \( \text{ex}(n, F) \) is the maximum number of hyperedges in an \( n \)-vertex \( k \)-uniform hypergraph not containing a copy of \( F \). The Turán density \( \pi(F) \) of \( F \) is

\[
\pi(F) = \lim_{n \to \infty} \frac{\text{ex}(n, F)}{\binom{n}{k}}.
\]

They showed that for a arbitrary graph \( F \) and a fixed \( \epsilon > 0 \), there exists a \( n_0 \) such that for any \( n > n_0 \),

\[
(1 - \frac{1}{\chi(F) - 1} - \epsilon) \frac{n^2}{2} \leq \text{ex}(n, F) \leq (1 - \frac{1}{\chi(F) - 1} + \epsilon) \frac{n^2}{2},
\]

where \( \chi(F) \) denotes the chromatic number of graph \( F \). For complete graph \( K_{t+1} \), the chromatic number is \( t+1 \); so, the result due to Erdős et al. for \( \text{ex}(n, F) \) reduces to an approximate version of Turán’s theorem. If \( F \) is bipartite, \( \text{ex}(n, H) \leq \epsilon n^2 \), for \( \epsilon > 0 \). This also implies that for a graph \( G, \pi(F) = (1 - \frac{1}{\chi(F)} \frac{n}{n}) \).

Having solved the problem for \( F = K_1 \), Turán [16] posed the natural generalization of the problem for determining \( \text{ex}(n, F) \) where \( F = K^k_t \) is a complete \( k \)-uniform hypergraph on \( t \) vertices. The minimum number of hyperedges in an \( k \)-uniform hypergraph \( G \) on \( n \) vertices such that any subset of \( r \) vertices contains at least one hyperedge of \( G \) is the Turán number \( T(n, r, k) \). Note that \( G \) has this property if and only if the complementary \( k \)-uniform hypergraph \( G' \) is \( K^k_{n-r} \)-free; thus \( T(n, r, k) + \text{ex}(n, K^k_{n-r}) = \binom{n}{k} \). There is extensive study of both \( T(n, r, k) \) and \( \text{ex}(n, K^k_{n-r}) \) and we refer the reader to two surveys [13, 6] for details. All the above results assumes that the host graph or hypergraph is arbitrary. Mubayi and Talbot [10], and Talbot [15] introduced Turán problems with coloring conditions, which could also be viewed from the perspective of a constrained host graph. They considered a new type of extremal hypergraph problem: given an \( k \)-uniform hypergraph \( F \) and an integer \( r \geq 2 \), determine the maximum number of hyperedges in an \( F \)-free, \( r \)-colorable \( r \)-graph on \( n \) vertices. In similar direction, we pose the following problem: maximize the number of hyperedges in a \( r \)-coloring of a \( n \)-vertex \( k \)-uniform hypergraph \( G \), such that no hyperedge of \( G \) consists of less than \( p \) colors.

**1.2 Our Results**

In order to estimate \( M(n, k, r, p) \), we first consider the case when \( r \) divides \( n \) and compute the number of distinct hyperedges that consists of exactly \( p \) distinct colors under any balanced \( r \) coloring of a \( K^k_n \). Let \( m(n, k, r, p) \) denote the number of distinct hyperedges that consists of exactly \( p \) distinct colors under any balanced \( r \) coloring of a \( K^k_n \). We prove the following lemma.
Lemma 1. For a fixed value of $n$, $k$, $r$ and $p$, $m(n,k,r,p) = \binom{n}{p} \left( \binom{r}{k}^p - p \binom{r-1}{k} + \binom{r-2}{k} \right)\cdots(-1)^p \binom{r}{k}$, where $c$ is the smallest integer such that $\frac{r}{c} \geq k$.

Observe that summing over all the hyperedges with exactly $i$ distinct colors, $1 \leq i \leq p-1$, we get the number of hyperedges that are colored with at most $p-1$ colors by any balanced $r$-coloring, provided $r$ divides $n$. In Section 3, we show that the number of distinct hyperedges that consists of at least $p$ distinct colors is maximized when the $r$-coloring is balanced. Therefore, we conclude the following theorem.

Theorem 1. The maximum number of properly $(r,p)$ colored hyperedges of a $K_r^n$ in any $r$-coloring $(i)$ is $M(n,k,r,p) = \binom{n}{r} - \sum_{i=1}^{p-1} m(n,k,r,i)$, where $m(n,k,r,i) = \binom{n}{i} \left( \binom{r}{k}^i - i \binom{r-1}{k} + \binom{r-2}{k} \right)\cdots(-1)^p \binom{r}{k}$, and, $c$ is the smallest integer such that $\frac{r}{c} \geq k$, and, $(ii)$ the $r$-coloring that maximizes the number of properly colored hyperedges splits the vertex set into equal sized parts, provided $r$ divides $n$.

Furthermore, we generalize the above theorem for arbitrary $n$ i.e. to cases where $n$ does not divide $r$ and derive a upper and lower bound for $M(n,k,r,p)$ as given by the following theorem.

Theorem 2. For a fixed $n$, $k$, $r$ and $p$, the maximum number of properly $(r,p)$ colored $k$-uniform hyperedges $M(n,k,r,p)$ on any $n$-vertex hypergraph $G$ is at most $\binom{n}{r} - \sum_{i=1}^{p-1} m(n,k,r,i)$ and at least $\binom{n}{r} - \sum_{i=1}^{p-1} m(n_1,k,r,i)$, where $n_1 = \lfloor \frac{n}{r} \rfloor \cdot r$, $n_2 = \lfloor \frac{n}{r} \rfloor \cdot r$, and $m(n',k,r,i) = \binom{n'}{i} \left( \binom{r}{k}^i - i \binom{r-1}{k} + \binom{r-2}{k} \right)\cdots(-1)^p \binom{r}{k}$, and, $c$ is the smallest integer such that $\frac{r}{c} \geq k$. Moreover, the number of properly $(r,p)$ colored hyperedges is maximized when the $r$-coloring is balanced.

1.3 Notations

1. For a set $A$, $\binom{|A|}{r}$ denotes the set of all the distinct $r$-element subsets of $A$. For instance, $\binom{n}{r}$ denotes the set of all the distinct $r$-element subsets of $\{1,\ldots,n\}$, $|\binom{n}{r}| = \binom{n}{r}$.

2. For a set $S = \{S_1,\ldots,S_l\}$, for any fixed $l$, $U(S)$ denotes the union of the elements, i.e $U(S) = S_1 \cup \ldots \cup S_l$.

3. Lexicographic ordering. Consider a $n$-element set $V = \{v_1,\ldots,v_n\}$ and a set of $k$-element subsets $E = \{e_1,\ldots,e_m\}$ of $V$, where $e_i \subset V$, for $1 \leq i \leq m$. For any $v_q, v_r \in V$, $v_q \prec v_r$ if $q \leq r$. Let $e_i, e_j \in E$, where $e_i = \{v_{i1},\ldots,v_{ik}\}$ and $e_j = \{v_{j1},\ldots,v_{jk}\}$. Then, $e_i \prec e_j$ if there exists an index $l$ such that $v_{il} = v_{jl}, v_{il-1} = v_{jl-1}$ and $v_{il} \prec v_{jl}$. An ordering $O$ of subsets of $E$ is a lexicographic ordering if for every $e_i, e_j \in O$, $e_i$ precedes $e_j$ in $O$ if and only if $e_i \prec e_j$.

2 Exact Number of properly $(r,p)$ colored hyperedges in a balanced partition

Let $G(V,E)$ be a $n$-vertex $k$-uniform hypergraph, where $V$ denotes the vertex set and $E$ denotes the set of hyperedges. An $r$-coloring $X$ of vertices in $V$ partitions the vertex set into $r$ color
classes $A = \{A_1, ..., A_r\}$, where $A_j \subseteq V$, $1 \leq j \leq r$ and every vertex $v \in A_j$ receives the same color under $X$. An $r$-coloring of vertices is called balanced if every color class is of almost same size, i.e. for all $A_j \in A$, $|A_j| = \lfloor \frac{n}{r} \rfloor$ or $|A_j| = \lceil \frac{n}{r} \rceil$. Let $p$ be some fixed integer, $1 < p \leq r$ and $p \leq k$. In this section, we study the number of distinct hyperedges that consists of exactly $p$ distinct colors under any balanced $r$ coloring of $G$. Throughout the section, we assume that $n$ is divisible by $r$, such that for all $A_j \in A$, $|A_j| = \frac{n}{r}$.

Consider a balanced $r$ coloring $X$ of vertices a $K_n^r$. Let $A = \{A_1, ..., A_r\}$ denote the corresponding color partition. Let $m(n,k,r,p)$ denote the number of distinct hyperedges that consists of exactly $p$ distinct colors under $X$. Let $B$ be the set of all the $p$-element subsets $B_i$ of $A$, $1 \leq i \leq \binom{r}{p}$ i.e. $B = \{B_i|B_i$ is the $i$th $p$-element subset of $\binom{[A]}{p}\}$. Consider the $i$th $p$-element subset $B_i \in B$. Let $m_i(n,k,r,p)$ denote the number of distinct hyperedges $e \in E$ that consists of exactly $p$ distinct colors under $X$ and $e \subseteq U(B_i)$. Due to the balanced nature of the $r$-coloring $X$, note that $m_i(n,k,r,p) = m_i(n,k,r,p)$, for any $B_i, B_j \in B$. Observe that

$$m(n,k,r,p) = \sum_{B \in B} m_i(n,k,r,p) = \binom{r}{p} m_i(n,k,r,p).$$

(1)

So, we focus our attention on computing $m_i(n,k,r,p)$ for a fixed $p$-element subset $B_i \in B$. Without loss of generality, we consider $B_1 = \{A_1, ..., A_r\}$ as the fixed $p$-element subset of $B$ and compute $m_1(n,k,r,p)$.

There are exactly $p$ subsets of size $p-1$ of $B_1$. Let these sets be $P_1, ..., P_p$, in the lexicographic order. Let $N(P_j)$ denote the number of hyperedges $e \in E$ such that $e \subseteq U(P_j)$, $P_j \in B_1$, and let $N(P_j)$ denote the number of hyperedges $e \in E$ such that $e \subseteq U(P_j) \cap ... \cap U(P_i)$ and, $P_j, ..., P_i \in B_1$. Observe that $N(P_j) = \binom{\frac{n}{r}(p-1)}{k}$, $1 \leq j \leq p$. So,

$$\sum_{1 \leq j \leq p} N(P_j) = p \binom{\frac{n}{r}(p-1)}{k}. \tag{2}$$

Note that if $e \subseteq U(P_{j_1})$ and $e \subseteq U(P_{j_2})$, then $e \subseteq U(P_{j_1}) \cap U(P_{j_2})$. Observe that $P_{j_1}$ and $P_{j_2}$ can have at most $p-2$ parts in common; $e \subseteq U(P_{j_1} \cap P_{j_2})$ implies that $e$ lies in a fixed $p-2$ parts of $P_{j_1}$, that is also a subset of $P_{j_2}$. So, number of hyperedges $e$ that lie in a fixed $p-2$ parts $P_{j_1} \cap P_{j_2}$ is $N(P_{j_1}P_{j_2}) = \binom{\frac{n}{r}(p-2)}{k}$. Since there are exactly $\binom{r}{2}$ distinct pairs of the form $\{P_{j_1}, P_{j_2}\}$, total number of hyperedges $e$ that are subsets of $p-2$ sized subsets of $B_1$ is

$$\sum_{1 \leq j_1 < j_2 \leq p} N(P_{j_1}P_{j_2}) = \binom{p}{2} \binom{\frac{n}{r}(p-2)}{k}. \tag{3}$$

Let $c$ be the smallest integer such that $\frac{4}{r}c \geq k$. Then, $\frac{4}{r}(c-1) < k$. Consider any fixed $c-1$ parts $A_{j_1}, ..., A_{j(c-1)}$. Observe that for any hyperedge $e \in E$, $e \not\subseteq A_{j_1} \cup ... \cup A_{j(c-1)}$. So, we compute all the summations of the form $\sum_{1 \leq j_1 < j_2 < ... < j_c \leq p} N(P_{j_1}P_{j_2}...P_{j_c})$ till $\frac{4}{r}c \geq k$, where

$$\sum_{1 \leq j_1 < j_2 < ... < j_c \leq p} N(P_{j_1}P_{j_2}...P_{j_c}) = \binom{p}{c} \binom{\frac{n}{r}c}{k}. \tag{4}$$
Observe that if a hyperedge $e$ is a subset of $B_1$ and is not a subset of any of the $P_j$, $P_j \in \{P_1, ..., P_p\}$, then $e$ consists of exactly $p$ colors in the $r$-coloring. The total number of hyperedges $e \subseteq B_1$ is $N(B_1) = \binom{n}{k}^p$. So, by definition, $N(P'_1, ..., P'_p)$ denotes all the hyperedges $e \subseteq B_1$ such that $e$ consist of exactly $p$ colors, i.e. $m_1(n, k, r, p) = N(P'_1, ..., P'_p)$. In order to compute $m_1(n, k, r, p)$, we use the fundamental result of inclusion exclusion stated below.

**Theorem 3.** [12] Let $A$ be any $n$-element set, and let $P_1, ..., P_m$ denote $m$ properties of elements of $A$. Let $A_i \subset A$ be the subset of elements of $A$ with property $P_i$. Let $N(P_i)$ denote the number of elements of $A$ with property $P_i$, i.e. $N(P_i) = |A_i|$, for $1 \leq i \leq m$. Let $N(P_1P_2...P_l) = |A_i \cap A_j \cap ... \cap A_l|$. Let $N(P'_i)$ denote the number of elements of $A$ that does not satisfy property $P_i$ and the number of elements with none of the properties $P_1, P_2, ..., P_l$ is denoted by $N(P'_1P'_2...P'_l)$. Then,

$$N(P'_1P'_2...P'_m) = n - \sum_{1 \leq i \leq m} N(P_i) + \sum_{1 \leq i < j \leq m} N(P_iP_j) - ... + (-1)^m N(P_1P_2...P_m).$$

So, using principle of inclusion exclusion [3] we have,

$$N(P'_1, ..., P'_p) = N(B_1) - \sum_{1 \leq j < p} N(P_j) + \sum_{1 \leq j < 2 \leq p} N(P_jP_2) - ... (-1)^c \sum_{1 \leq j_1 < j_2 < ... < j_c \leq p} N(P_{j_1}P_{j_2}...P_{j_c})$$

$$= \binom{n}{k}^p - p \binom{n}{k}^p \binom{p-1}{k} + \binom{n}{k}^p \binom{p}{2} \binom{p-2}{k} ... (-1)^c \binom{p}{c} \binom{p-c}{k}.$$ (6)

Now, using Equation(1) we get, $m(n, k, r, p) = \binom{n}{k}^p \left( \binom{p}{k} \binom{p-1}{k} + \binom{p}{2} \binom{p-2}{k} ... (-1)^c \binom{p}{c} \binom{p-c}{k} \right),$ where $c$ is the smallest integer such that $\frac{n}{c} \geq k$. This concludes the proof of Lemma.

Observe that summing over all the hyperedges with exactly $i$ distinct colors, $1 \leq i \leq p - 1$, we get the number of hyperedges that are colored with at most $p$ colors by any balanced $r$-coloring, provided $r$ divides $n$. Therefore, the exact number of properly $(r, p)$ colored hyperedges in a balanced partition is

$$M(n, k, r, p) = \binom{n}{k} - \sum_{i=1}^{p-1} m(n, k, r, i).$$

(7)

Consider the case when $r = p = 2$, i.e., when we are performing a bicoloring on $n$ vertices and proper coloring of a hyperedge $e$ denote $e$ becoming non-monochromatic under the bicoloring. Observe that $M(n, k, 2, 2) = \binom{n}{k} - m(n, k, 2, 1)$, and $m(n, k, 2, 1) = 2 * \binom{\frac{n}{k}}{k}$. Therefore, $M(n, k, 2, 2) = \binom{n}{k} - 2 * \binom{\frac{n}{k}}{k}$, which agrees with the existing results. Note that $M(n, k, r, p)$ is a non-decreasing function of $n$. So, $M(n-1, k, r, p) \leq M(n, k, r, p) \leq M(n+1, k, r, p)$.

Let $x(i, j, n, k, r) = \binom{\frac{n}{k}}{k} - \frac{r-1}{i-1} x(i, j-1, n, k, r)$. $x(i, j, n, k, r)$ denotes the number of hyperedges that are colored with less than or equal to $j$ colors by an $r$-coloring, when counted with respect to color classes of size $i$, $i \geq j$. Here, the term $\binom{\frac{n}{k}}{k}$ accounts for every hyperedge $e \in E$, that is a subset of some fixed $i$ color parts of the $r$-coloring. Any $(j - 1)$-sized color parts are repeated $r - j + 1$ times when counted over all $j$-sized color classes; however, we need to count it exactly once. Each hyperedge inside some fixed $i$-sized set is counted $i - j$ times over all the $j - 1$ sized sets. So, $\binom{\frac{n}{k}}{k} - \frac{r-1}{i-1} x(i, j-1, n, k, r) + \frac{1}{i-j+1} x(i, j, n, k, r)$ counts the number of hyperedges that are colored with less than or equal to $j$ colors by an $r$-coloring,
when counted with respect to color classes of size \( i, i \geq j \). \( \frac{1}{j-1} x(i, j-1, n, k, r) \) term is added in order to include the hyperedges colored with less than or equal to \( j-1 \) colors. Observe that \( x(p-1, p-1, n, k, r) \) denotes the number of hyperedges colored with less than or equal to \( p-1 \) colors by a balanced \( r \)-coloring. Therefore,

\[
M(n, k, r, p) = \binom{n}{k} - x(p-1, p-1, n, k, r). \tag{8}
\]

### 3 Maximizing the number of properly \((r, p)\) colored hyperedges

In this section, we show that the number of properly \((r, p)\) colored hyperedges is maximized when the \( r \)-coloring is balanced. We show that the number of hyperedges colored with less than or equal to \( p-1 \) colors is minimized for a balanced \( r \)-coloring, thereby proving the above claim.

Consider an \( r \)-coloring \( X \) of vertices a \( K_n^k \). Let \( A = \{A_1, \ldots, A_r\} \) denote the corresponding color partition and let \( |A_i| = n_i \), for \( 1 \leq i \leq r \). Let \( m_X(n, k, r, p) \) denote the number of distinct hyperedges that consists of at most \( p \) distinct colors under \( X \). Let \( n_1 \geq n_2 + 2 \). Then we have the following lemma.

**Lemma 2.** The number of hyperedges colored with at most \( p \) colors is reduced by moving a vertex \( v \in A_1 \) from \( A_1 \) to \( A_2 \), i.e. switching the color of \( v \) from 1 to 2 produces an \( r \)-coloring \( X' \) such that \( m_{X'}(n, k, r, p) < m_X(n, k, r, p) \).

**Proof** In order to prove that \( m_{X'}(n, k, r, p) < m_X(n, k, r, p) \), we analyze: (i) the gain \( g \): the number of hyperedges \( e \in E \) such that \( e \) is colored with greater than \( p \) colors under \( X \) and \( e \) receives at most \( p \) colors under \( X' \), and (ii) the loss \( l \): the number of hyperedges \( e \in E \) such that \( e \) is colored with at most \( p \) colors under \( X \) and \( e \) receives at least \( p+1 \) colors under \( X' \). Note that a hyperedge \( e \in E \) contributes to \( g \) or \( l \) if and only if \( v \in e \). Since \( m_{X'}(n, k, r, p) = m_X(n, k, r, p) + g - l \), in order to prove Lemma 2, we need to show that \( l > g \).

Let \( y(n, k, r, p) \) denote the minimum number of \( k \)-uniform hyperedges on \( n \) labeled vertices that are colored with exactly \( p \) colors by any \( r \) coloring. Observe that a hyperedge \( e \in E \) contributes to \( g \) if and only if it consists of exactly \( p+1 \) colors in \( X \), \( v \in e \) and includes no other vertex from \( A_1 \), i.e., \( e \cap A_1 = v \), and includes at least one vertex from \( A_2 \), i.e., \( e \cap A_2 \geq 1 \). So, gain due to switching \( v \) from \( A_1 \) to \( A_2 \) is

\[
g = \sum_{i=1}^{c} \binom{n_2}{i} y(n-n_1-n_2, k-i-1, r-2, p-1), \tag{9}
\]

where \( c \) be the smallest integer such that \( \frac{4}{c} \geq k \). In each of the \( c \) terms in the summation, \( \binom{n_2}{i} \) denotes the number of ways to choose exactly \( i \) vertices from \( A_2 \) (of color 2), \( y(n-n_1-n_2, k-i-1, r-2, p-1) \) denotes the minimum number of hyperedges that can be formed consisting of exactly \( k-\left(\frac{i+1}{i} \right) \) vertices from \( A \setminus \{A_1 \cup A_2\} \) and exactly \( p-1 \) distinct colors. The \( k-(i+1) \) vertices from \( A \setminus \{A_1 \cup A_2\} \) with \( p-1 \) distinct colors combined with \( i \) vertices from \( A_2 \) and \( v \) from \( A_1 \) forms the hyperedges \( e \) consisting of exactly \( p+1 \) colors under coloring \( X \) including \( v \), \( e \cap A_1 = v \), and \( |e \cap A_2| = i \).
Similarly, a hyperedge $e \in E$ contributes to $l$ if and only if it consists of exactly $p$ colors in $X$, includes no other vertex from $A_2$, i.e., $e \cap A_2 = \emptyset$, and $v \in e$ and includes at least one vertex other than $v$ from $A_1$, i.e., $|e \cap A_1| \geq 2$. So, loss due to switching $v$ from $A_1$ to $A_2$ is

$$l = \sum_{i=1}^{r} \binom{n_1 - 1}{i} y(n - n_1 - n_2, k - i - 1, r - 2, p - 1).$$

(10)

Since $n_1 \geq n_2 + 2$, $n_1 - 1 > n_2$. So, comparing $l$ and $g$ term-wise, we get $l > g$ as desired. \qed

Lemma \[2\] implies that the number of hyperedges colored with less than $p$ colors can be minimized until the color partition $\{A_1, \ldots, A_r\}$ is balanced, i.e. for every $i$, $1 \leq i \leq r$, $\frac{n}{p} \leq |A_i| \leq \frac{n}{g}$. Therefore, the number of properly $(r, p)$ colored hyperedges is maximized when the $r$-coloring is balanced. So, using Equation \[7\] Theorem \[1\] follows.

Observe that even if $r$ does not divide $n$, the $r$-coloring that maximizes the number of properly colored hyperedges splits the vertex set into almost equal sized parts (from Lemma \[2\]) of either $\left\lceil \frac{n}{r} \right\rceil$ or $\left\lfloor \frac{n}{r} \right\rfloor$ size. Therefore, we can get an upper bound on $M(n,k,r,p)$ by computing the minimum number of hyperedges including vertices of at most $p - 1$ distinct colors with $\left\lceil \frac{n}{r} \right\rceil \cdot r$ vertices and subtracting from \( \binom{n}{r} \). Furthermore, we can get a lower bound on $M(n,k,r,p)$ by computing the minimum number of hyperedges including vertices of at most $p - 1$ distinct colors with $\left\lfloor \frac{n}{r} \right\rfloor \cdot r$ vertices and subtracting from \( \binom{n}{r} \). This observation combined with Theorem \[1\] proves Theorem \[2\].

For the special case when $r = p = k$, we can compute $M(n,k,r,p)$ much easily. Observe that any hyperedge must contain one vertex each from each of the color classes $\{A_1, \ldots, A_r\}$ in order to be properly $(r, p)$ colored. So, the number of properly colored hyperedges under any $r$-coloring is $|A_1||A_2|\ldots|A_r|$. Using the second part of Theorem \[2\] $M(n,k,r,p) = |A_1||A_2|\ldots|A_r|$, where $\{A_1, \ldots, A_r\}$ is a balanced partition. So, we have the following corollary.

**Corollary 1.** The number of properly $(r, p)$ colored hyperedges of a $K^r_n$ in any $r$-coloring is $|A_1||A_2|\ldots|A_r|$ when $r = p = k$. Moreover, the $r$-coloring that maximizes the number of properly colored hyperedges splits the vertex set into almost equal sized parts.

**References**

[1] Martin Aigner and Günter M. Ziegler. *Proofs from THE BOOK*, pages 235–240. Springer-Verlag, New York, 1998.

[2] T.J. Dickson. On a problem concerning separating systems of a finite set. *J. Combin. Theory*, 7:191–196, 1969.

[3] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[4] G. Katona. On separating systems of a finite set. *J. Combin. Theory I*, pages 174–194, 1966.

[5] G. Katona. Chapter 23 - combinatorial search problems. In Jagdish N. Srivastava, editor, *A Survey of Combinatorial Theory*, pages 285 – 308. North-Holland, 1973.
[6] Peter Keevash. Hypergraph turan problems. *Surveys in Combinatorics*, pages 83–140, 2011.

[7] László Lovász. Coverings and colorings of hypergraphs. In *Proc. 4th Southeastern Conference on Combinatorics, Graph Theory, and Computing*. Utilitas Mathematica Publishing, Winnipeg, pages 3–12, 1973.

[8] CAI Mao-cheng. Solutions to Edmond’s and Katona’s problems on families of separating subsets. *Discrete Mathematics*, 47:13–21, 1983.

[9] Tapas Mishra and Sudebkumar Prasant Pal. Bicoloring covers for graphs and hypergraphs. *arXiv preprint arXiv:1501.00343*, 2015.

[10] D. Mubayi and J. Talbot. Extremal problems for t-partite and t-colorable hypergraphs. *Electronic J. Combin.*, 15, 2008.

[11] A. Rényi. On random generating elements of a finite boolean algebra. *Acta Sci. Math. (Szeged)*, 22(1-2):75–81, 1961.

[12] Kenneth H. Rosen. *Discrete Mathematics and Its Applications*. McGraw-Hill Higher Education, 5th edition, 2002.

[13] Alexander Sidorenko. What we know and what we do not know about turán numbers. *Graphs and Combinatorics*, 11(2):179–199, 1995.

[14] J. Spencer. Minimal completely separating systems. *J. Combin. Theory*, 8:446–447, 1970.

[15] John Talbot. Chromatic turán problems and a new upper bound for the turán density of. *European Journal of Combinatorics*, 28(8):2125 – 2142, 2007. EuroComb ’05 - Combinatorics, Graph Theory and Applications EuroComb ’05 - Combinatorics, Graph Theory and Applications.

[16] P. Turán. Research problems. *MTA Mat.Kutató Int.KözL.*, 6:417–423, 1961.

[17] I. Wegener. On separating systems whose elements are sets of at most k elements. *Discrete Math.*, 28:219–222, 1979.

[18] Douglas Brent West. *Introduction to Graph Theory*, pages 41–42. Prentice Hall, 2nd edition, 2001.