Good Representations and Solvable Groups

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Dedicated to William Fulton on his 60th birthday

1. Introduction

The purpose of this paper is to provide a characterization of solvable linear algebraic groups in terms of a geometric property of representations. Representations with a related property played an important role in the proof of the equivariant Riemann–Roch theorem [EG2]. In that paper, we constructed representations with that property (which we call freely good) for the group of upper triangular matrices in $\text{GL}_n$. We noted that it seemed unlikely that such representations exist for arbitrary groups; the main result of this paper implies that they do not.

To state our results, we need some definitions. A representation $V$ of a linear algebraic group $G$ is said to be good (resp. freely good) if there exists a nonempty $G$-invariant open subset $U$ such that

(i) $G$ acts properly (resp. freely) on $U$;

(ii) $V \setminus U$ is the union of a finite number of $G$-invariant linear subspaces.

Note that freely good representations were called “good” in [EG2].

The main result of the paper is the following theorem.

**Theorem 1.1.** Let $G$ be a connected algebraic group over a field $k$ of characteristic not equal to 2. Then $G$ is solvable if and only if $G$ has a good representation. Moreover, if $G$ is solvable and $k$ is perfect then $G$ has a freely good representation.

In characteristic 2, a solvable group still has good representations, and a partial converse holds (Corollary 4.1). A key step in the proof of the main result is Theorem 4.1, which is inspired by an example of Mumford [MFK, Ex. 0.4].

In characteristic 0, solvable groups are characterized by a weaker property that does not require the action to be proper. (In general, if $G$ acts properly on $X$ then $G$ acts with finite stabilizers on $X$, but the converse need not hold.)

**Theorem 1.2.** Let $G$ be a connected algebraic group over a field of characteristic 0. Suppose that $G$ has a representation $V$ that contains a nonempty open set $U$ such that:

1. the complement of $U$ is a finite union of invariant linear subspaces; and
2. $G$ acts with finite stabilizers on $U$.

Then $G$ is solvable.
Examples (see Section 6) show that this weaker property does not characterize solvability in positive characteristic.

2. Preliminaries

Groups and Representations. We let $k$ denote a field with algebraic closure $\bar{k}$ and separable closure $k_s$. If $Z$ is a $k$-variety and $k' \supset k$ is any extension of $k$, then $Z(k')$ denotes the $k'$-valued points of $Z$ and $Z_k$ denotes the $k$-variety $Z \times_k k'$.

All groups in this paper are assumed to be linear algebraic groups over a field $k$. We assume that such a group $G$ is geometrically reduced (i.e., that $G_k$ is reduced). The identity component of a group $G$ is denoted $G_0$.

Unless otherwise stated, a representation $V$ of a group $G$ is assumed to be $k$-rational; that is, $V$ is a $k$-vector space and the action map $G \times V \to V$ is a morphism of $k$-varieties.

If $k' \supset k$ is a field extension then we call a $k'$-rational representation $V$ of $G_{k'}$ a $k'$-representation of $G$. We say that $V$ is defined over $k$ if it is obtained by base change from a $k$-rational representation.

If $k' \supset k$ is a Galois field extension, then $\text{Gal}(k'/k)$ acts on $k'$-representations of $G$: Indeed, let $V$ be a $k'$-representation of $G$ corresponding to a $k'$-morphism $\rho: G_{k'} \to \text{GL}(V)$. For $g \in G(k_s)$ and $\sigma$ in $\text{Gal}(k'/k)$, we define $\sigma \rho(g)$ as follows (cf. [B, AG 14.3, 24.5]). Because $\rho$ is defined over $k'$, for any $\tau \in \text{Gal}(k_s/k')$ and any $g \in G(k_s)$ we have $\tau(\rho(\tau^{-1}(g))) = \rho(g)$. Thus, if $\sigma \in \text{Gal}(k'/k) = \text{Gal}(k_s/k)/\text{Gal}(k'/k)$, then

$$\sigma'(\rho((\sigma')^{-1}g)) \in \text{GL}(V)(k_s)$$

is independent of the lift of $\sigma$ to an element $\sigma' \in \text{Gal}(k_s/k)$. We will call this point $\sigma \rho(g)$ and set $\sigma \rho(g) = \sigma(\rho(\sigma^{-1}g))$.

The $k'$-representation $V$ is obtained by base change from a representation defined over $k$ if and only if $\sigma \rho = \rho$ for all $\sigma \in \text{Gal}(k'/k)$.

Free and Proper Actions. The action of a group $G$ on a scheme $X$ is said to be free if the action map $G \times X \to X \times X$ is a closed embedding. The action is said to be proper if the map $G \times X \to X \times X$ is proper. If the action is proper then the stabilizer of every point is finite. If the stabilizer of every geometric point is a trivial group scheme then we say that the action is set-theoretically free. An action that is set-theoretically free and proper is free [EG1].

Let $H \to G$ be a finite morphism of algebraic groups. If $G$ acts properly on a scheme $X$ then $H$ also acts properly on $X$. Thus, if $V$ is a good representation of $G$ then $V$ is also a good representation of $H$ via the action induced by the map $H \to G$. Moreover, if $H$ is a closed subgroup and $V$ is a freely good representation of $G$, then $V$ is a freely good representation of $H$.

Example 2.1. Let $B$ be the group of upper triangular matrices in $\text{GL}(n)$. The group $B$ acts by left multiplication on the vector space $V$ of upper triangular matrices; it acts with trivial stabilizers on the open subset $U$ of invertible upper triangular matrices. Since the matrices are upper triangular, $V \setminus U$ is the union of the...
invariant subspaces \( L_i = \{ A \in V \mid A_{ii} = 0 \} \). This representation is freely good because the action of \( B \) on \( U \) is identified with \( B \) acting itself by left multiplication. The map \( B \times B \to B \times B \) given by \( (A, A') \mapsto (A, AA') \) is an isomorphism, so the action of \( B \) on \( U \) is free.

By contrast, the action of \( \text{GL}(n) \) by left multiplication on the vector space \( M_n \) of \( n \times n \) matrices is not good.

3. Existence of Good Representations

In this section we show that every connected solvable group \( G \) has good representations and, if \( k \) is perfect, freely good representations.

By the Lie–Kolchin theorem, \( G \) is trigonalizable; that is, it can be embedded in the group \( B \subset \text{GL}_n \) of upper triangular matrices.

Let \( V \) be the vector space of upper triangular \( n \times n \) matrices. The group \( B \) acts on \( V \) by left multiplication, and we have seen that this representation is freely good. By restriction, \( V \) is a good representation of \( G \).

Consider the morphism

\[
G \to \text{GL}(V)
\]

corresponding to the action of \( G \) on \( V \).

Since \( G \) is a morphism of schemes of finite type, it is defined over a field extension \( k' \) of finite degree. Write \( V = V \otimes k' \) for the corresponding \( k' \)-representation; then we have \( G \to \text{GL}(V) \).

Case I: \( k' \) is Separable over \( k \)

(This will occur when \( k \) is perfect.) In this case we will use Galois descent to construct a freely good representation of \( G \).

Replacing \( k' \) by a possibly bigger field extension, we may assume that \( k' \supset k \) is Galois. Enumerate the elements of \( \text{Gal}(k'/k) \) as \( \{ \sigma_1, \sigma_2, \ldots, \sigma_d \} \) and consider the representation \( \Phi: G \to \text{GL}(V \otimes k) \) where \( G \) acts on the \( j \)-th factor by the representation \( \sigma_j: G \to \text{GL}(V) \).

We define \( U = V \otimes d \) to be the open set whose \( k\)-rational points are the \( d \)-tuples \( (A_1, \ldots, A_d) \), where some \( A_i \) is invertible. We realize \( U \) as a complement of \( G \)-invariant linear subspaces as follows. Let \( L_i = \{ A \in V \mid A_{ij} = 0 \} \), a \( G \)-invariant subspace of \( V \). Given a \( d \)-tuple \( (j_1, \ldots, j_d) \), define

\[
L_{(j_1, \ldots, j_d)} = L_{j_1} \oplus \cdots \oplus L_{j_d}.
\]

This is a \( G \)-invariant subspace of \( V \otimes d \) and \( U = V \setminus \bigcup L_{(j_1, \ldots, j_d)} \).

Lemma 3.1 (cf. [EG2, Thm. 2.2]). \( G \) acts freely on \( U \).

Proof. Since \( G \) is a closed subgroup of \( B \) and the open set \( U \) is \( B \)-invariant, it suffices to show that \( B \) acts freely on \( U \). To do this, we must show that the map \( B \times U \to U \times U \) given on \( k \)-points by

\[
(A, A_1, A_2, \ldots, A_d) \mapsto (AA_1, \sigma_1(AA_2^{-1}(A_2)), \ldots, \sigma_d(AA_d^{-1}(A_d)))
\]

is a closed embedding.
First we show that the image $Z$ of $B^0 \times U_d$ is closed in $U_d \times U_d$. Let $(A_1, A_2, \ldots, A_d, C_1, \ldots, C_d)$ be matrix coordinates on $U_d \times U_d$. Expanding the inverse out in terms of the adjoint, we see that the image is contained in the subvariety defined by the matrix equations

$$\sigma_j \sigma_i^{-1}(\det A_i)C_j = \sigma_j \sigma_i^{-1}(\text{Adj } A_i)A_j.$$ 

Suppose that a $2d$-tuple of matrices $(A_1, A_2, \ldots, A_d, C_1, C_2, \ldots, C_d) \in U_d \times U_d$ satisfies the matrix equations above. At least one of the $A_i$ and one of the $C_j$ is invertible because we are in $U_d \times U_d$. Let $A = \sigma_i^{-1}(C_i A_i^{-1})$. Substituting into our equations we see that $C_j$ is invertible and $C_j = \sigma_j(A \sigma_i^{-1}(A_j))$. Hence every point satisfying the matrix equations is in the image $Z$ of $B^0 \times U_d$, so $Z$ is closed.

The variety $Z$ is covered by open sets of the form

$$\{(A_1, A_2, \ldots, A_j, \ldots, A_d, AA_1, \ldots, AA_j, \ldots, AA_d) \mid \det A_j \neq 0\}.$$ 

These open sets are isomorphic to $V^{d-1} \times B^k$, where $V^{d-1}$ is the $(d-1)$-fold Cartesian product of $V$. Hence the image is smooth—in particular, normal. The action of $G^k$ on $U_d$ is set-theoretically free, so $G^k \times U_d \to Z$ is a birational bijection. By Zariski’s main theorem (cf. [B, AG18]), a birational bijection of normal varieties is an isomorphism; hence $G^k \times U_d \to Z$ is an isomorphism. Therefore, $G^k \times U_d \to U_d \times U_d$ is a closed embedding. \hfill $\square$

**Remark.** The proof of [EG2, Thm. 2.2] is incomplete; the last paragraph of the preceding argument is needed.

For any basis of $V$, there is a natural choice of basis so that, with respect to this basis, if $g \in G(k_s)$ then $\Phi(g)$ is represented by the block diagonal matrix

$$\begin{bmatrix}
\rho(g) & \sigma_2 \rho(g) & \cdots & \sigma_d \rho(g) \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{bmatrix}.$$ 

This representation is not defined over $k$ because the Galois group acts by permuting the blocks. More precisely, we have the following. Given a $d \times d$ matrix $M$, let $M[n]$ denote the $nd \times nd$ matrix whose $ij$ block is $M_{ij} \cdot I_n$, where $I_n$ is the $n \times n$ identity matrix. If $\sigma \in \text{Gal}(k'/k)$, let $J_\sigma$ denote the permutation matrix corresponding to the permutation $\sigma_i \mapsto \sigma \sigma_i$. In matrix form, for $g \in G(k_s)$ we have

$$\sigma \Phi(g) = J_\sigma[n]^{-1} \Phi(g) J_\sigma[n].$$

We will show that $\Phi$ is $k'$-isomorphic to a freely good representation defined over $k$. Choose a primitive element $\alpha$ for the extension $k' \supset k$, and let $A$ be the $d \times d$ matrix with $A_{ij} = \sigma_j(\alpha^i)$. The matrix $A$ is invertible since $\alpha, \sigma_2(\alpha), \ldots, \sigma_d(\alpha)$ are exactly the roots of the irreducible polynomial $f \in k[x]$ of $\alpha$ over $k$, so $\det A = \prod_{1 \leq j \leq d}(-1)^{i_j}(\sigma_{i_j}(\alpha) - \sigma_{j}(\alpha)) \neq 0$. The Galois group acts by $\sigma(A) = AJ_\sigma$. Consider the morphism $\Psi : G_{k'} \to GL(V^{\otimes d})$ defined by $\Psi(g) = A[n] \Phi(g) A[n]^{-1}$ for $g \in k_s$. Then $\sigma \Psi(g) = \Psi(g)$ for any $\sigma \in \text{Gal}(k'/k)$; hence $\Psi$ is defined over $k$. 


Each of the subspaces \( L_{(j_1, \ldots, j_d)} \) is \( G_{k'} \)-invariant under the action \( \psi \). Moreover, because each \( L_{(j_1, \ldots, j_d)} \) is a vector subspace of \( V^{\oplus d} \) defined over \( k \), the corresponding subrepresentations \( G_{k'} \to \text{GL}(L_{(j_1, \ldots, j_d)}) \) are also defined over \( k \). Therefore \( \psi \) is obtained by base change from a freely good representation of \( G \).

**Case II: The General Case**

Here we may assume that there is a freely good \( k' \)-rational representation \( \rho : G_{k'} \to \text{GL}(V) \) defined over a finite normal extension \( k' \) of \( k \). Then \( k' \supset k \) factors as \( k' = k_0k \) with \( k_0 \) purely inseparable of degree \( p^n \) and \( k'/k \) Galois. The Frobenius endomorphism on \( V \) induces a group homomorphism of \( \text{GL}(V) \). Composing \( \rho \) with the \( n \)th power of Frobenius on \( \text{GL}(V) \), we obtain a representation defined over \( k'' \). Because the Frobenius has finite kernel, this representation will no longer be faithful. However, the action of Frobenius is trivial on geometric points, so \( G \) will act properly on an open set whose complement is a union of linear subspaces. We can now use the Galois descent argument of Case I to obtain a good \( k \)-rational representation of \( G \).

**4. Characterization of Solvable Groups by Good Representations**

In this section we show that if \( \text{char } k \neq 2 \) then every group with a good representation is solvable. However, many of the results of this section are valid in arbitrary characteristic. In particular, we prove that every reductive group with a good representation is a torus. We only need that \( \text{char } k \neq 2 \) in part of the proof of Theorem 1.1. We will explicitly say when we start assuming this; until then, \( \text{char } k \) is arbitrary.

Let \( T \) be the diagonal torus in \( \text{SL}_2 \) and let \( N(T) \) be the normalizer of \( T \). We will first show that \( N(T) \) has no good representations. We begin by recalling some facts about \( N(T) \). First, let

\[
J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix};
\]

we will also write

\[
H(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.
\]

The group \( N(T) \) is generated by \( T \) and \( J \); it has two components, \( T \) and \( J(T) \). The action of \( \text{SL}_2 \) on its 2-dimensional standard representation \( V \) induces an action on \( S(V^*) \cong k[x, y] \) given by

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}; y \mapsto ay - cx, \quad x \mapsto -by + dx.
\]

Let \( W_i \) denote the subspace of \( S(V^*) \) spanned by \( x^i \) and \( y^i \); this is an irreducible representation of \( N(T) \) of dimension 2 (if \( i > 0 \)). Let \( W_0 \) denote the 1-dimensional irreducible representation of \( N(T) \) on which \( T \) acts trivially and \( J \) acts by multiplication by \( -1 \).
If $\text{char } k \neq 2$, the group $N(T)$ is linearly reductive; that is, its action on any representation is completely reducible [MFK, p. 191]. The next lemma shows that much of this survives in arbitrary characteristic.

**Lemma 4.1.** Let $V$ be a representation of $N(T)$.

1. As a representation of $N(T)$, $V$ splits as a direct sum of $N(T)$-submodules:
   \[ V = V_0 \oplus \bigoplus_{i>0} V_{\pm i}. \]
   Here $V_{\pm i}$ is the sum of the $i$ and $-i$ weight spaces of $T$ on $V$, and $V_j$ is the $j$-weight space.

2. The action of $N(T)$ on $V_{\pm i}$ ($i > 0$) is completely reducible, and $V_{\pm i}$ is isomorphic as $N(T)$-module to a direct sum of copies of $W_i$.

3. If $\text{char } k \neq 2$, then $V_0$ is isomorphic to a direct sum of copies of $W_0$ and $W'_0$.

**Proof.**

1. Because the action of $T$ on $V$ is completely reducible, we can decompose $V = \bigoplus V_i$ as a $T$-module. As $JH(t)J^{-1} = H(t^{-1})$, we have $JV_i = V_{-i}$. Hence $V_{\pm i}$ is an $N(T)$-submodule and so we have the desired direct sum decomposition of $V$.

2. Let $v_1, \ldots, v_d$ be a basis for $V_i$ ($i > 0$). The map $v_i \mapsto y$, $Jv_i \mapsto -x$ defines an isomorphism of the span of $v_i$, $Jv_i$ (denoted $\langle v_i, Jv_i \rangle$) with $W_i$, and the map $W_i^d \rightarrow V_{\pm i}$, taking the $r$th component to $\langle v_i, Jv_i \rangle$, is an $N(T)$-module isomorphism.

3. Decompose the 0-weight space of $V$ into the $+1$ and $-1$ eigenspaces of $J$; these are isomorphic to sums of copies of $W_0$ and $W'_0$, respectively.

The proof of the following result was motivated by [MFK, Ex. 0.4].

**Theorem 4.1.** The group $N(T)$ has no good representations.

**Proof.** If a group $G$ has good representations, then so does $G_\mathbb{C}$, so we may assume that $k$ is algebraically closed. Suppose that $V$ is a representation of $N(T)$, and let $U \subset V$ be the complement of a finite set of invariant linear subspaces $S$. We will show that $N(T)$ does not act properly on $U = V \setminus \bigcup_{L \in S} L$. The strategy of the proof is as follows. Consider the action map $\Phi : N(T) \times U \rightarrow U \times U$. We will find a closed subvariety $Z$ of $N(T) \times U$ whose closed points are of the form

\[
\left( \begin{array}{cc}
0 & -\lambda^{-1} \\
\lambda & 0
\end{array} \right), v_k
\]

and whose image is not closed in $U \times U$. Hence $\Phi$ is not proper, so the representation is not good.

We now carry out the proof. Decompose $V = \bigoplus V_i$, where $V_i$ is the $i$-weight space of $V$ for $T$. Pick $u \in U$ and write $u = \sum u_i$, where $u_i \in V$. Some of the $u_i$ may be 0; let $d$ be the dimension of the space spanned by the nonzero $u_i$.

**Step 1.** If $a_i \neq 0$ for all $i$ with $u_i \neq 0$, then $w = \sum a_i u_i \in U$. Indeed, suppose not; then $w \in L$ for some $L \in S$. For almost all choices $t_1, \ldots, t_d$ of $d$ elements of $k^*$, the vectors...
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\[ H(t_q)w = \sum_q t_q^{i_q} a_{i_q} u_{i_q} \in L \]

are linearly independent. (Here \(i_1, \ldots, i_d\) are the indices \(i_p\) with \(u_{i_p} \neq 0\).) This follows because the \(d \times d\) matrix \(A\) with entries

\[ A_{pq} = t_q^{i_p} \]

is nonsingular for almost all \(t_1, \ldots, t_d\). (This is because \(\det A\) is a sum of monomials, where each monomial is a product of one term from each row and each column; each monomial has different multi-degree, so \(\det A\) is not the zero polynomial.) Therefore, the vectors \(H(t_q)w\) span the same space as the \(u_i\) and so

\[ u_2 \not\in U; \]

contradicting our assumption that \(u_2 \in U\). We conclude that \(w \in U\), as claimed.

A similar argument shows that \(Ju_0 + \sum_{i \neq j} u_i \in U\).

Step 2. There exists an element \(u' = \sum u_i' \in U\) with \(Ju_i' = u_{-i}'\) for all \(i > 0\). To see this, suppose \(u_{-i} \neq Ju_j\) for some \(j > 0\). Let \(W_j \subset V_j \oplus V_{-j}\) be the subspace of vectors of the form \(v_j + Ju_j\) \((v_j \in V_j)\). Note that \(W_j\) generates \(V_j \oplus V_{-j}\) as an \(N(T)\)-module. Consider the affine linear subspace

\[ B = \sum_{i \neq j} u_i + W_j. \]

We claim that \(B \cap U\) is nonempty. If it is empty then, because \(B\) is affine linear and is contained in a finite union of the subspaces in \(S\), we see that \(B \subset L\) for some \(L \in S\). But then the span of \(B\) is contained in \(L\), so \(\sum_{i \neq j} u_i \in L\) and \(W_j \subset L\). Because \(L\) is \(N(T)\)-stable, \(V_j \oplus V_{-j} \subset L\) as well; hence \(u \in \sum_{i \neq j} u_i + (V_j \oplus V_{-j}) \subset L\), contradicting \(u \in U\). We conclude that \(B \cap U\) is nonempty. Replacing \(u\) by an element of \(B \cap U\), which we again call \(u\), we do not change \(u_i\) for \(i \neq j\) but we do obtain \(u_{-j} = Ju_j\). Iterating this process, we obtain \(u'\) of the desired form.

Replacing \(u\) by \(u'\), we will assume that \(Ju_i = u_{-i}\) for all \(i > 0\). From the \(N(T)\)-module isomorphism of \(\langle u_i, Ju_i \rangle\) with \(\langle x', y' \rangle\) we see that, for \(i > 0\),

\[ \begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix} : u_i \mapsto \lambda^i u_{-i}, \quad u_{-i} \mapsto (-1)^i \lambda^{-i} u_i. \]

Note also that

\[ \begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix} u_0 = Ju_0. \]

Step 3. Define

\[ v_\lambda = u_0 + \sum_{i > 0} (u_{-i} + \lambda^{-i} u_i), \]

\[ v_\lambda' = Ju_0 + \sum_{i > 0} (u_{-i} + (-1)^i \lambda^{-i} u_i). \]

For all \(\lambda \neq 0\), both \(v_\lambda\) and \(v_\lambda'\) are in \(U\) (by Step 1). Define \(Z\) to be the closed subvariety of \(N(T) \times U\) whose points are the pairs
Then
\[
\begin{pmatrix}
0 & -\lambda^{-1} \\
\lambda & 0
\end{pmatrix}
\begin{pmatrix}
v_k
\end{pmatrix}.
\]

Consider the point
\[
(v, v') = \left( u_0 + \sum_{i > 0} u_{-i}, J u_0 + \sum_{i > 0} u_{-i} \right).
\]

Reasoning as in Step 1 shows that \( u \) is in the \( N(T) \)-module generated by \( v \) or \( v' \), so if either \( v \) or \( v' \) were in \( L \) then \( u \) would be; but this is impossible since \( u \in U \). Hence \( v \) and \( v' \) are in \( U \), so \( (v, v') \in U \times U \). Also, \( (v, v') \) is not in \( \Phi(Z) \) but is in the closure of \( \Phi(Z) \) in \( U \times U \). We conclude that \( \Phi \) is not proper, so the representation is not good.

*Proof of Theorem 1.1.* Let \( G \) be a connected nonsolvable linear algebraic group. Consider the surjective map \( \pi : G \rightarrow G_1 = G/\mathcal{R}_u G \), where \( \mathcal{R}_u G \) is the unipotent radical of \( G \) and where \( G_1 \) is reductive. Because \( G \) is not solvable, \( G_1 \) is neither trivial nor a torus. Let \( T \) be a maximal torus of \( G \). Then \( T_1 = \pi(T) \) is a maximal torus of \( G_1 \), and \( \pi \) induces an isomorphism of Weyl groups \( W(T, G) \rightarrow W(T_1, G_1) \) [B, 11.20]. (Here \( W(T, G) = N_G(T)/Z_G(T) \), where \( N_G \) and \( Z_G \) denote normalizer and centralizer of \( T \) in \( G \), and similarly for \( G_1 \).) Because \( \ker \pi \) is a unipotent group, \( \pi|_T : T \rightarrow T_1 \) is an isomorphism. Note that \( Z_G(T) = T \cdot (\mathcal{R}_u G)^T \) [B, 13.17]. Because \( G_1 \) is reductive, this fact (applied to \( G_1 \)) implies that \( Z_{G_1}(T_1) = T_1 \). Moreover, any \( g_1 \in N_{G_1}(T_1) \) can be lifted to \( g \in N_G(T) \). This follows because the isomorphism of Weyl groups just described, together with the structure of the centralizers, implies that each component of \( N_{G_1}(T_1) \) is the image of a surjective map of a component of \( N_G(T) \).

Since \( G_1 \) is not a torus, there is a root \( \alpha \) and a homomorphism \( \phi_\alpha : \text{SL}_2 \rightarrow G_1 \) with kernel either trivial or the set of matrices \( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \) with \( a^2 = 1 \). Moreover (using the subscript \( \text{SL}_2 \) to denote terms for \( \text{SL}_2 \) defined in the previous subsection), \( \phi_\alpha(T_{\text{SL}_2}) \subset T \) and \( J_1 := \phi_\alpha(J_{\text{SL}_2}) \in N_{G_1}(T_1) \). (See [J, p. 176] for these facts.) Let \( H_1 = \phi_\alpha(N(T_{\text{SL}_2})) \); its identity component \( H_1^0 = \phi_\alpha(T_{\text{SL}_2}) \subset T_1 \). Because \( H_1 \) is a finite image of \( N(T_{\text{SL}_2}) \), it has no good representations (and hence neither does \( G_1 \)).

**Up to this point, char \( k \) has been arbitrary; now we assume that char \( k \neq 2 \).**

Because \( \pi|_T \) is an isomorphism, there is a unique subgroup \( H^0 \subset T \) projecting isomorphically to \( H_1^0 \). As noted previously, we can choose a lift \( J \in N_G(T) \) of \( J_1 \in N_{G_1}(T_1) \). Write \( J = J_1 J_a \) for the Jordan decomposition of \( J \). Because \( \text{char} \ k \neq 2 \), \( J_1 \) is semisimple and so \( \pi(J_a) = 1 \). Therefore we can replace \( J \) by \( J_a \) and assume that \( J \) is semisimple. Now \( J^2 \) corresponds to the identity element in the Weyl group (as \( J_1^2 \) does), so \( J^2 \in Z_G(T) = T \cdot (\mathcal{R}_u G)^T \). Since \( J \) is semisimple, we conclude that \( J^2 \in T \). Because \( J_1^2 \) is in the subgroup \( H_1^0 \) of \( T_1 \) and \( T \) maps isomorphically to \( T_1 \), we conclude that \( J^2 \in H^0 \). Therefore, the group \( H \) generated by \( H^0 \) and \( J \) maps isomorphically to \( H_1 \) and thus has no good representations; hence \( G \) has no good representations. This proves Theorem 1.1. \( \square \)
The proof of Theorem 1.1 yields the following weaker statement in characteristic 2. Note that Levi decompositions need not exist in positive characteristic [B, 11.22].

Corollary 4.1. Suppose \( \text{char } k = 2 \). If the connected algebraic group \( G \) has a Levi decomposition and if \( G \) has a good representation, then \( G \) is solvable. In particular, any connected reductive group with a good representation is diagonalizable.

Proof. Suppose that \( G = LN \), where \( L \) is reductive and \( N \) unipotent. If \( G \) has a good representation then so does \( L \). As we have shown, this implies that \( L \) is a torus, so \( G \) is solvable. \( \square \)

5. Proof of Theorem 1.2

If \( G \) has a representation \( V \) that contains an open set \( U \) whose complement is a finite union of invariant subspaces such that \( G \) acts with finite stabilizers on \( U \), then \( G \) also has such a representation. Thus we can assume that \( k \) is algebraically closed.

Assume that \( G \) is not solvable, and let \( V \) be a representation of \( G \). Since the characteristic is 0, \( G \) has a Levi subgroup \( L \). Since \( G \) is assumed to be nonsolvable, \( L \) contains a Borel subgroup that is not a torus. Hence \( L \) contains a nontrivial unipotent subgroup \( N \).

Since the characteristic is 0 and \( L \) is reductive, \( V \) decomposes as a direct sum \( V = V_1 \oplus V_2 \oplus \cdots \oplus V_p \) of irreducible \( L \)-modules. Every vector in the subspace \( V^N = V_1^N \oplus V_2^N \oplus \cdots \oplus V_p^N \) has a positive dimensional stabilizer. Since \( N \) is unipotent, \( V_i^N \neq 0 \) for each \( i \) and so \( L(V_i^N) \) spans all of \( V_i \). Hence the subset \( LV^N = L(V_1^N) \oplus \cdots \oplus L(V_p^N) \) that consists of vectors with positive dimensional stabilizers cannot be contained in any proper \( L \)-invariant subspace. Since \( L \) is a subgroup of \( G \), this means that \( LV^N \) is not contained in any proper \( G \)-invariant subspace. Hence \( V \) does not have properties (1) and (2). \( \square \)

6. Examples and Complements

In this section we discuss set-theoretic versions of the conditions “freely good” and “good”. We will say a representation \( V \) is set-theoretically freely good (resp. set-theoretically good) if it contains a nonempty open subset \( U \) whose complement is a union of invariant subspaces such that \( G \) acts with trivial stabilizers (resp. finite stabilizers) on \( U \) (cf. Theorem 1.2). Surprisingly, these conditions are not enough to characterize solvability in arbitrary characteristic.

Example 6.1. Let \( V \) be the standard representation of \( SL_2 \) and let \( V_d = S(V^*) \) be the vector space of homogeneous forms of degree \( d \). As in Section 4, \( SL_2 \) acts on \( V_d \). If \( p = \text{char } k \) is an odd prime then \( W_p = V_{2p-2} \oplus V_1 \) is a set-theoretically freely good representation of \( SL_2 \). The reason is as follows. The stabilizer of any pair of forms \( (f(x, y), l(x, y)) \) is trivial as long as \( l(x, y) \neq 0 \) and the coefficient of \( x^{p-1}y^{p-1} \) in \( f \) is nonzero. Since the characteristic is \( p \), the subspace
$L_{2p-2} \subseteq V_{2p-2}$ of forms with no $x^{p-1}y^{p-1}$ term is an $SL_2$ invariant subspace (cf. [J, II2.16]). Thus, $SL_2$ acts with trivial stabilizers on the open set $U_p = W_p \setminus (W_{2p-2} \oplus V_1 \cup V_{2p-2} \oplus 0)$.

In characteristic 2, the representation $W_2 = V_2 \oplus V_1$ is not set-theoretically freely good because the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ stabilizes the pair $(x^2y^2, x+y)$. However, $W_2$ is set-theoretically good.

In positive characteristic, we do not know if the group $SL_n$ admits set-theoretically good representations for $n \geq 3$.

**Example 6.2.** Assume that $k$ is algebraically closed and that $\text{char} \; k \neq 2$. Then $G = \text{PGL}_2$ has no representation that is set-theoretically freely good. Indeed, let

$$g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad H = \{1, g\}. $$

If $V$ is any representation of $G$, then $V^H$ generates $V$ as a representation of $G$. Indeed, this holds if $V$ is irreducible because, for any vector $v$, the vector $v + gv$ is a nonzero $H$-invariant. Since $\text{char} \; k \neq 2$, the action of $H$ is completely reducible; hence, if

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$$

is an exact sequence of $G$-modules then the corresponding sequence of $H$-invariants is also exact. By induction, we may assume that $V_1^H$ and $V_3^H$ generate $V_1$ and $V_3$ as $G$-modules, and a diagram-chase then shows that $V_2^H$ generates $V_2$ as a $G$-module.

It follows that if $V$ is any representation then there is no proper invariant linear subspace of $V$ containing $V^H$. Therefore, $V$ is not set-theoretically freely good. A similar argument shows that $\text{PGL}_n$ and $\text{GL}_n$ do not have set-theoretically freely good representations.

We conclude with a proposition about the inductive construction of good representations.

**Proposition 6.1.** Let $G$ be a connected linear algebraic group and $H$ a normal subgroup. Assume that $k$ is algebraically closed. If $H$ and $G/H$ have set-theoretically freely good representations, then so does $G$.

**Proof.** For this proof only, we will use “good” to mean “set-theoretically freely good”. Let $W$ be a good representation of $H$, with $M_i$ a finite set of proper invariant subspaces containing the vectors with nontrivial stabilizers. Because $G/H$ is affine [B, Thm. 6.8], the vector bundle $G \times^H W$ is generated by a finite-dimensional space of global sections $\Gamma$. We will view sections of the vector bundle as regular functions $\gamma : G \rightarrow W$ satisfying $\gamma(gh) = h^{-1} \cdot \gamma(g)$, where on the right side we are using the action of $H$ on $W$. The action of $G$ on the space of sections of the vector bundle corresponds to the left action of $G$ on regular functions: $(g \cdot \gamma)(g_0) = \gamma(g^{-1}g_0)$. Because the action of $G$ on regular functions is locally finite, by enlarging the space $\Gamma$ we may assume that $\Gamma$ is stable under the $G$-action.
Define $L_i$ to be the subspace of $\Gamma$ consisting of those elements of $\Gamma$ that are sections of $G \times^HM_i$. Each $L_i$ is a $G$-stable subspace of $\Gamma$. Let $\Gamma^0$ denote the complement of the $L_i$ in $\Gamma$.

Let $V$ be a good representation of $G/H$, viewed as a representation of $G$ via the map $G \rightarrow G/H$. We claim that $V \oplus \Gamma$ is a set-theoretically good representation of $G$. Indeed, let $V_j$ be a finite set of invariant subspaces of $V$ containing the vectors with nontrivial stabilizer. It suffices to show that the vectors with nontrivial stabilizer in $V \oplus \Gamma$ are contained in the union of the subspaces $V_j \oplus \Gamma$ and $V \oplus L_i$. To see this, let $(v, \gamma)$ be in the complement of these subspaces; hence $v \notin V_j$ and $\gamma \notin L_i$ for any $i, j$. We must show that $\text{stab}_G(v, \gamma) = H$. Let $h \in \text{stab}_G(v, \gamma)$. As before, we will view $\gamma$ as a function $G \rightarrow W$. Because $\gamma$ is not in any $L_i$, we have that $\gamma$ is not a section of $G \times^HM_i$ for any $i$. In other words, the open subsets $W \setminus M_i$ of $G$ are nonempty. Choose $g_0$ in the intersection of these sets, so $s(g_0) \notin M_i$ for any $i$. Our hypothesis implies that $h \cdot \gamma = \gamma$. By definition, we have

$$(h \cdot \gamma)(g_0) = \gamma(h^{-1}g_0) = \gamma(g_0(g_0^{-1}h^{-1}g_0)) = (g_0^{-1}hg_0)\gamma(g_0).$$

But $\text{stab}_H \gamma(g_0) = \{1\}$, so we conclude that $h = 1$, as desired. 

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