Horizontal diameter of unit spheres with polar foliations

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Abstract: For a singular Riemannian foliation $\mathcal{F}$ on a Riemannian manifold $M$, a curve is called horizontal if it meets the leaves of $\mathcal{F}$ perpendicularly. For a polar foliation $\mathcal{F}$ on a unit sphere $S^n$, we show that the horizontal diameter of $S^n$ is $\pi$, i.e., any two points in $S^n$ can be connected by a horizontal curve of length $\leq \pi$.

A singular Riemannian foliation $\mathcal{F}$ on a Riemannian manifold $M$ is a decomposition of $M$ into smooth injectively immersed submanifolds $L(x)$, called leaves, such that it is a singular foliation and any geodesic starting orthogonally to a leaf remains orthogonal to all leaves it intersects. A leaf $L$ of $\mathcal{F}$ (and each point in $L$) is called regular if the dimension of $L$ is maximal, otherwise $L$ is called singular; see [7, 10, 14].

A singular Riemannian foliation is called a polar foliation if, for each regular point $p$, there is a totally geodesic complete immersed submanifold $\Sigma_p$, called section, that passes through $p$ and that meets each leaf orthogonally. A typical example of a polar foliation is the partition of a Riemannian manifold into parallel submanifolds to an isoparametric submanifold $L$ in a Euclidean space. Recall that a submanifold $L$ of a Euclidean space is called isoparametric if its normal bundle is flat and the principal curvatures along any parallel normal vector field are constant; see [1, 6, 16, 17, 18].

For a singular Riemannian foliation $\mathcal{F}$ on a Riemannian manifold $M$, a curve is called horizontal if it meets the leaves of $\mathcal{F}$ perpendicularly. When $M$ has positive curvature, Wilking [19] proved that any two points in $M$ can be connected by a piecewise smooth horizontal curve. Thus Wilking introduced the horizontal metric $g_H$ on $M$ by defining the horizontal distance of two points as the infimum over the length of all horizontal curves connecting these two points. Besides the intrinsic interest in such object, one reason to study $g_H$ is its connection with Sub-Riemannian geometry; see [11, 15]. It is natural to define the horizontal diameter $\text{diam}_H M$ of $M$ by

\[ \text{diam}_H M = \sup \{ d_H(p, q) \mid p, q \in M \}, \]

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where \(d_H(p, q)\) is the horizontal distance of \(p\) and \(q\). Notice that \(diam_H M \geq diam(M)\), where \(diam(M)\) is the diameter of \(M\) defined by its Riemannian metric.

Recently a lot of progress has been made in the singular Riemannian foliations of round spheres \([3, 4, 5, 8, 9, 12, 13]\). In \([15]\) we studied the horizontal diameter rigidity of unit sphere \(S^n\). We proved that for many classes of singular Riemannian foliations on \(S^n\), the horizontal diameter of \(S^n\) is \(\pi\), i.e., any two points in \(S^n\) can be connected by a horizontal curve of length \(\leq \pi\). Based on our results, we also proposed the following rigidity conjecture:

**Rigidity Conjecture.** For any singular Riemannian foliation on a unit sphere \(S^n\), we have \(diam_H S^n = \pi\).

We call this conjecture “rigidity” since it asserts that the inequality \(diam_H S^n \geq diam(S^n) = \pi\) should be an equality. In this paper we prove this conjecture for the case of polar foliation:

**Theorem A.** For any polar foliation on a unit sphere \(S^n\), we have \(diam_H S^n = \pi\).

Notice that Theorem A is not new. In fact, by \([17]\) polar foliations of unit sphere \(S^n\) with codimension \(\geq 2\) give rise to spherical buildings, so the horizontal diameter of \(S^n\) must be \(\pi\). On the other hand, the case of codimension 1 had been proved by \([2]\). However, in this paper we will give a short and uniform proof.

Since a polar foliation on a unit sphere \(S^n\) is also an isoparametric foliation on \(S^n\) \([1]\), the main references of this paper are \([6, 16, 17, 18]\).

**Proof of Theorem A.** For any \(p, q \in S^n\), we will show that \(d_H(p, q) \leq \pi\). Since the union of all regular leaves are open and dense in \(S^n\), we can assume that both \(L(p)\) and \(L(q)\) are regular leaves. Let \(\Sigma\) be the section passing \(p\), then the Coxeter group \(W\) on \(\Sigma\) determines a chamber complex \(C(\Sigma, W)\) on \(\Sigma\) \([16]\). Let \(\tilde{p}\) be the antipodal point of \(p\), then \(L(\tilde{p})\) is also a regular leaf. Since both \(L(\tilde{p})\) and \(L(q)\) are regular leaves, there is a unique point \(\tilde{q} \in L(q) \cap \Sigma\) such that \(\tilde{q}\) and \(\tilde{p}\) are in the interior of a same chamber. Choose a minimal horizontal geodesic \(\gamma\) from \(p := \gamma(0)\) to \(\tilde{q} := \gamma(t_0)\), and assume that \(\gamma'(0) = \xi\). Extend \(\xi\) to a parallel normal vector field \(\xi\) of \(L(p)\). We set \(M := L(p)\). Since \(M\) is a regular leaf of a polar foliation on unit sphere, we get that \(M\) is an isoparametric submanifold of \(S^n\) \([1] [15]\). Thus the shape operator \(A_\xi\) has constant eigenvalues. Assume that \(A_\xi\) has \(g\) distinct (constant) eigenvalues, which we label by

\[
\lambda_i = \cot \theta_i, \ 0 < \theta_i < \pi, \ 1 \leq i \leq g,
\]

where the \(\theta_i\) form an increasing sequence, and \(\lambda_i\) has constant multiplicity \(m_i\) on \(M\). We denote the corresponding eigendistributions of \(A_\xi\) by

\[
E_i(x) = \{ X \in T_xM \mid A_\xi X = \lambda_i X \}.
\]

If we denote by \(S_i(x)\) the leaf of \(E_i\) through \(x \in M\), then \(S_i(x)\) is a \(m_i\)-dimensional metric sphere.

We consider the parallel leaf \(f_t : M \to L(\gamma(t))\) defined by

\[
f_t(x) = \cos t \ x + \sin t \ \xi(x).
\]
Then for any \( x \in M \) and \( 1 \leq i \leq g \), \( f_{\theta_i}(x) \) is a focal point of \( M \) since we have
\[
f_{\theta_i}(S_i(x)) = f_{\theta_i}(x).
\] (1)

Consider now the family \( \mathcal{B} \) of broken horizontal geodesics from \( p \), whose projection is the same as \( \gamma \), that are allowed to change directions at the singular leaves. For any time \( t \), define \( \mathcal{B}(t) := \{ c(t) \mid c \in \mathcal{B} \} \), which is a subset of \( L(\gamma(t)) \). Then
\[
\mathcal{B}(t) = f_t(p) = \gamma(t) \text{ for } t \in [0, \theta_1).
\]

By (1) we get that
\[
\mathcal{B}(t) = f_t(S_i(p)) \text{ for } t \in [\theta_1, \theta_2).
\]

Define \( S_2 \circ S_1(p) := \{ S_2(x) \mid x \in S_1(p) \} \). Since
\[
f_{\theta_2}(S_2 \circ S_1(p)) = f_{\theta_2}(S_1(p)),
\]
we get
\[
\mathcal{B}(t) = f_t(S_2 \circ S_1(p)) \text{ for } t \in [\theta_2, \theta_3).
\]

Hence, after \( g \) steps, we get that
\[
\mathcal{B}(t) = f_t(M) = L(\gamma(t)) \text{ for } t \geq \theta_g.
\] (2)

By the proof of Proposition 2.3 in [17] we get that
\[
S_g \circ S_{g-1} \circ \cdots \circ S_1(p) = M.
\] (3)

In fact, Follow [6] we define
\[
N_g = \{ (y_1, \cdots, y_g) \mid y_1 \in S_1(p), y_2 \in S_2(y_1), \cdots, y_g \in S_g(y_{g-1}) \},
\]
then \( N_g \) is an iterated sphere bundle of dimension \( m := \sum_{1 \leq i \leq g} m_i = \dim M \), so \( N_g \) is a compact manifold of the same dimension as \( M \).

Now consider the map \( u_g : N_g \to M \) defined by
\[
u_g(y_1, \cdots, y_g) = y_g.
\]

Thus
\[
u_g(N_g) = S_g \circ S_{g-1} \circ \cdots \circ S_1(p).
\]

By Theorem 5.2 in [6] \( (N_g, u_g) \) is an \( \mathbb{Z}_2 \)-orientable Bott-Samelson cycle at \( p \). Now by the surjectivity of the top Bott-Samelson cycle we get that \( u_g : N_g \to M \) is surjective. This proves (3).

By (2), (3) we have
\[
\mathcal{B}(t) = f_t(M) = L(\gamma(t)) \text{ for } t \geq \theta_g.
\]

Since \( \bar{q} = \gamma(t_0) \) and \( \bar{p} = \gamma(\pi) \) are in the interior of a same chamber, and the focal points can only occur in the boundary of the chambers that \( \gamma \) meets, we get that there is no focal point on \( \gamma \mid_{[t_0, \pi]} \). Thus \( t_0 > \theta_g \). It follows that
\[
\mathcal{B}(t_0) = L(\gamma(t_0)) = L(\bar{q}) = L(q),
\]
which means that \( p \) and \( q \) can be joined by a broken horizontal geodesic of length \( t_0 \). Thus \( \text{diam}_H S^n = \pi \) since \( d_H(p, q) \leq t_0 \leq \pi \). \( \square \)

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