Functional evolution of quantum cylindrical waves

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Abstract.
Kuchař showed that the quantum dynamics of (1 polarization) cylindrical wave solutions to vacuum general relativity is determined by that of a free axially symmetric scalar field along arbitrary axially symmetric foliations of a fixed flat 2+1 dimensional spacetime. We investigate if such a dynamics can be defined unitarily within the standard Fock space quantization of the scalar field. Classical scalar field evolution from an initial flat slice to an arbitrary (in general, curved) slice of the flat spacetime can be decomposed into (i) ‘time’ evolution in which the spatial Minkowskian coordinates of the initial slice serve as spatial coordinates of the final slice, followed by (ii) the action of a spatial diffeomorphism of the final slice on the data obtained from (i). We show that although the functional evolution of (i) is unitarily implemented in the quantum theory, generic spatial diffeomorphisms of (ii) are not. Our results imply that a Tomanaga-Schwinger type functional evolution of quantum cylindrical waves is not a viable concept even though, remarkably, the more limited notion of functional evolution in Kuchař’s ‘half parametrized formalism’ is well-defined.

1. Introduction
Given the difficulties of constructing a quantum theory of gravity, it is of use to build intuition from the study of simpler toy models. While cosmological mini-superspaces [1, 2] have yielded valuable insights, their finite-dimensional nature precludes the occurrence of field-theoretic aspects of quantum gravity. In this regard, vacuum cylindrically symmetric gravitational fields with 2 hypersurface-orthogonal, commuting Killing vectors (one generating translations along ‘z’ and the other, rotations along ‘φ’) constitute a useful infinite-dimensional midi-superspace. Quantization of this cylindrical wave midi-superspace was initiated by Kuchař [3] and further studied by a number of authors [4, 5, 6, 7, 8]. While the latter works deal with quantization after fixing a gauge, our focus here is on aspects of Dirac quantization of cylindrical waves.

Every cylindrical wave solution is determined by a corresponding axially-symmetric solution to the free scalar wave equation on a fixed 3 dimensional flat spacetime. Specifically, the cylindrical wave line element is

$$ds^2 = e^{-\phi}(-dT^2 + (dR)^2) + R^2 e^{-\phi}(d\Phi)^2 + e^{\phi}(dZ)^2$$

(1)

and that of the flat spacetime is

$$ds^2 = -(dT)^2 + (dR)^2 + R^2 (d\Phi)^2.$$ 

(2)
$\phi(R, T)$ is the free scalar field and $\gamma(R, T)$ is its energy in a box of size $R$ at time $T$. Kuchař defined a canonical transformation from the Arnowitt-Deser-Misner (ADM) phase space of cylindrical waves to that of an axially-symmetric parametrised field theory on 2+1 dimensional flat spacetime and studied the formal Dirac quantization of the latter system [3].

Parametrised field theory (PFT) is free field theory on flat spacetime in a diffeomorphism invariant disguise [11]. It describes field evolution on arbitrary (and in general, curved) foliations of the flat spacetime instead of only the usual flat foliations, by treating the 'embedding variables' which describe the foliation as dynamical variables to be varied in the action in addition to the scalar field. In the present context the coordinates $X_\alpha := (T, R)$ are parametrised by a new set of arbitrary coordinates $x_\alpha := (t, r)$ such that for fixed $t$, $X_\alpha(t, r)$ define a spacelike slice in Minkowski spacetime with radial coordinate $r$. General covariance of PFT ensues from the arbitrary choice of $x_\alpha$ and implies that in its canonical description, evolution from one slice of an axisymmetric foliation to another is generated by constraints.

Indeed, under Kuchař’s canonical transformation, appropriate combinations of the Hamiltonian and diffeomorphism constraints of the midi-superspace take the form of the constraints appropriate to PFT, namely,

$$ C_\alpha(x) := P_\alpha(x) + h_\alpha[\phi, \pi, X_\alpha](x) = 0 , $$

where $P_\alpha$ and $\pi$ are the momenta conjugate to $X_\alpha$ and $\phi$ and $h_\alpha$ is related to the stress-energy of the scalar field. In the Dirac quantization, a formal operator version of $C_\alpha$ acting on a physical state $|\Psi\rangle$ of the theory is given by

$$ \left( \frac{1}{i} \frac{\delta}{\delta X_\alpha} + \hat{h}_\alpha \right) |\Psi\rangle = 0 . $$

Eq.(4) takes the form of a functional Schrödinger equation which represents infinitesimal evolution of the quantum state $|\Psi\rangle$ from one Cauchy slice to another.

2. The main question and Results

In this work we investigate if the formal equation (4) is well defined in the context of the standard Fock space quantization of the free scalar field on flat spacetime. More precisely, we investigate if there exists a unitary transformation on the standard Fock space which implements evolution between any two Cauchy slices of the flat spacetime and whose infinitesimal version is equation (4).

What do we mean by evolution between two slices? Since the classical theory is that of a free scalar field we explicitly (and uniquely) know the evolved data $(\phi, \pi)$ on the slice $t = t_1$ in terms of initial data at $t = t_0$. Since the equations for scalar field evolution are linear, we also know how the corresponding quantum operators are related. This is what we mean by evolution. Since this notion of evolution is independent of the choice of interpolating foliation between the two slices, our subsequent considerations are phrased solely in terms of data on the two slices.

We inquire if the operators at the initial and final slices are unitarily related. If they are, then in analogy to the definition of the Schrödinger picture from the Heisenberg picture in usual quantum mechanics, we can define the Schrödinger state at any time $t$ as the unitary image of the state at time $t_0$. Techniques appropriate to our investigation here have been

1 Kuchař’s canonical transformation identifies $T$ with the spatial integral of one of the extrinsic curvature components of the cylindrical wave metric (1) on a $t=$ constant slice.

2 For a beautiful exposition of the Schrödinger and Heisenberg pictures in a canonical treatment of PFT on (n+1)-dimensional flat spacetime (in the absence of axisymmetry), we refer the reader to [11, 12].

3 Note that if we restrict attention to only flat slices, the standard Hamiltonian is the generator of the desired unitary transformation. The non-triviality enters solely due to the possibility of evolution along directions which are not isometries of the spacetime.
developed in [9, 10] for the case of PFT on (n+1)-dimensional flat spacetime in the absence of axisymmetry. That work arose as an effort to rigorously implement the considerations of Kuchař in [11, 12]. The strategy pursued in [9, 10] is as follows. Classical scalar field evolution from an initial flat slice to any final slice is a linear canonical transformation on the scalar field phase space. Instead of phrasing this canonical transformation in terms of the scalar field and its momentum, one can instead use appropriate linear functionals of these fields. The linear functionals used are such that when they are evaluated on the initial slice, they reduce to the familiar mode coefficients (i.e. the classical correspondents of the standard annhiliation and creation operators). It then follows (from the fact that evolution is a canonical transformation) that these functionals evaluated on the final slice are related by a Bogolubov transformation to their initial values. The criterion for unitary implementability of a Bogolubov transformation is that the 'β' matrix be Hilbert-Schmidt (i.e. the absolute value of its diagonal elements should be summable) [13, 14].

The results of [9, 10] are as follows. In 1+1 dimensions operator evolution is unitary, the functional Schrödinger picture exists as the unitary image of the standard Heisenberg-picture-based Fock quantization and the infinitesimal version of the unitary transformation on Heisenberg states implies that the Schrödinger states satisfy an equation of the form (4) (albeit, with an extra ‘anomaly potential’ term [12, 9]). In n + 1 dimensions (n > 1), the β coefficients for evolution along generic curved-foliations are not Hilbert-Schmidt. Hence the functional Schrödinger picture does not exist (at least in the form usually envisioned, as the unitary image of the Heisenberg picture) and the functional Schrödinger equation cannot be given any obvious meaning. The case of interest here, namely 2+1 dimensions with fewer degrees of freedom (only the axisymmetric ones), is a tantalizing intermediate case and as we shall demonstrate we get a suitably ‘in between’ answer.

Here, our interest is in evolution between two axisymmetric, but otherwise arbitrary, Cauchy slices Σ_i and Σ_f in the flat spacetime (2). Note that the restriction of the spacetime coordinate R to any axisymmetric spatial slice defines a natural coordinate system on the slice. We refer to this natural coordinate by the same symbol R. In order to accommodate arbitrary reparameterizations (R, T) → (r, t), it is necessary to admit arbitrary radial coordinates r_f(R) on Σ_f and r_i(R) on Σ_i. Despite this, it suffices for our purposes, to consider evolution from a fixed initial slice Σ_0, chosen to be the T = 0 slice coordinatized by the natural radial coordinate R, to an arbitrary final slice Σ, coordinatized by an arbitrary radial coordinate r = r(R). This entails no loss of generality since, if in the quantum theory all evolution is unitary, quantum evolution from any Σ_i to any Σ_f may be constructed as the (inverse) unitary evolution from Σ_i to Σ_0 followed by unitary evolution from Σ_0 to Σ_f.

Now, consider the restricted evolution from Σ_0 to Σ such that the radial coordinate r on Σ is chosen to be the natural radial coordinate R i.e. r is identified with the restriction of the spacetime coordinate R to Σ. We shall refer to such evolution as the ‘half parametrized’ evolution. Next, we reconsider the general case of interest, namely that of evolution from Σ_0 with coordinate R to Σ with coordinate r = r(R). The uniqueness of evolution from initial data implies that (φ(r), π(r)) on Σ can be obtained by first evolving the initial data from Σ_0 to Σ by a ‘half parametrized’ evolution and then subjecting the result to the spatial diffeomorphism defined by r(R).

Now we state the main result. 4 First, we construct the Bogolubov transformation corresponding to the most general evolution from Σ_0 to Σ. The resulting β matrix is,

$$ \beta(k_1, k_2) = \frac{k_2}{2} \int_0^\infty dr \left( J_1(k_1 r) J_1(k_2 R) - \left( \frac{R}{r} \right) J_0(k_1 r) J_0(k_2 R) \right) R^r $$

4 For technical detail readers should consult [15].
\[
- i \left( \frac{R}{r} J_0(k_1r)J_1(k_2R) + J_1(k_1r)J_0(k_2R) \right) T^2 \right) e^{ik_2T}.
\]

Then, we particularize the Bogolubov transformation to the case of ‘half parametrized’ evolution defined above and show that the $\beta$ matrix is Hilbert-Schmidt. This implies that the finite functional evolution corresponding to Kuchař’s ‘half parameterized’ formalism [3] is well defined. The action of a generic spatial diffeomorphism on data on any slice $\Sigma$, however, is shown to be not unitarily implementable in quantum theory. This implies that the most general Tomonaga-Schwinger type functional evolution of quantum cylindrical waves is not a viable concept. We reinforce this result by showing that the operator corresponding to the generator of generic spatial diffeomorphisms of $\Sigma_0$ does not have a well defined action on the Fock vacuum. The action of the normal-ordered smeared infinitesimal generator of generic spatial diffeomorphism on the Fock vacuum is

\[
\| \int_0^\infty dr rf(r) : \phi(r,0)\phi'(r,0) : |0\rangle \|^2 = \frac{1}{2} \int_0^\infty dk_1 \int_0^\infty dk_2 F(k_1k_2) ,
\]

where

\[
F(k_1k_2) = k_1k_2 \int_0^\infty dr rf(r) \left[ J_0(k_1r)J_1(k_2r) + J_1(k_1r)J_0(k_2r) \right].
\]

Here $f(r)$ is a smearing function with a compact support. The normal-ordered smeared infinitesimal generator of generic spatial diffeomorphism exists if the integral on the right hand side of the Eq. (6) does, which is shown not to be the case.

3. Discussion

The work of Kuchař [3] maps the dynamics of cylindrical waves to that of an axisymmetric free scalar field along arbitrary axisymmetric foliations of the fixed (2+1)-dimensional flat spacetime (2). In this work we studied quantum evolution of this free field operator from the initial flat $T = 0$ slice, $\Sigma_0$, with radial coordinate $R$, to an arbitrary slice $\Sigma$ with radial coordinate $r$. We showed that operator evolution is unitarily implemented in the standard Fock representation when $r$ is chosen to coincide with $R$ on $\Sigma$. The transition to a general radial coordinate $r(R)$ from $R$ is obtained by the action of a corresponding spatial diffeomorphism on the field variables on $\Sigma$. We showed that for generic choices of $r(R)$, this diffeomorphism is not unitarily implemented in the standard Fock space representation.

In (1+1)-dimensional parametrized field theory (PFT) where such operator evolution is unitary, a Tomonaga-Schwinger type of functional evolution can be defined as the unitary evolution of Fock states from the initial $T = 0$ surface to an arbitrary one [9]. Our results in this work indicate that a similar notion of functional evolution for quantum cylindrical waves is not a viable concept for generic choices of foliations starting from the slice $\Sigma_0$. One may inquire if such functional evolution can be defined in the standard Fock representation by choosing some other slice than $\Sigma_0$ as the initial slice (this would entail expressing the standard mode coefficients $a(k)$ as functionals of data on the new initial slice and analysing the Bogolubov transformation corresponding to evolution of these mode coefficients). Though we have not investigated this question, we suspect that no such choice of initial slice can render generic evolution of operators as unitary.

Our results do indicate, however, that it should be possible to define functional evolution of quantum states along the restricted class of foliations wherein the radial coordinate on each slice is $R$. Such evolution is formally described by the ‘half parametrized’ formalism of [3]. It would be of interest to construct the Schrödinger picture states as the unitary image of the Heisenberg-picture Fock states and to show that they satisfy a functional Schrödinger equation along the lines of [9], for this restricted evolution.
Our results have been obtained for foliations satisfying the specific boundary conditions requiring that the embeddings be ‘Minkowskian’ near the axis of symmetry, \( r = 0 \), and at spatial infinity. Specifically, we require that \( R(r) - r \) and \( T'(r) \) be compactly supported away from \( r = 0 \) and \( r \to \infty \). The fact that \( T'(r) \) and \( R(r) - r \) vanish at the axis and at spatial infinity are direct consequences of the boundary conditions for cylindrical waves in their ADM description (see the appendix of [16]). Our conditions of compact support for these quantities are more restrictive than the conditions in [16] and it would be of interest to see if our proofs could be generalised for those weaker boundary conditions. In any case, what is required is a set of boundary conditions for the ADM description of cylindrical waves which allows the definition of a consistent Hamiltonian framework and which induce conditions on \( T(r) \), \( R(r) \) such that our proofs still go through. We suspect that requiring fall offs of various phase space variables to be faster than any power of \( r^{-1} \) at spatial infinity and any power of \( r \) at the axis should suffice. It would be good to check this.

In section 2 we showed that a generic infinitesimal generator of spatial diffeomorphisms on \( \Sigma_0 \) does not have a well defined action on the Fock vacuum. This ties in neatly with the lack of unitary implementability of finite spatial diffeomorphisms. What about generators of infinitesimal ‘half parametrized’ evolution on \( \Sigma_0 \)? From [3] the relevant Hamiltonian density on any slice \( \Sigma \) with radial coordinate \( R \) is given by

\[
\mathcal{H} = \frac{1}{2} (1 - T_R^2)^{-1} [(R^{-\frac{3}{2}} - R^\frac{3}{2} T_R \phi_R)^2 + \frac{1}{2} R \phi_R^2] .
\] (8)

On \( \Sigma_0 \), \( T_R = 0 \), \( \pi = \phi_T \) and \( \mathcal{H} \) equals the standard flat spacetime energy density. \( H(f) := \int dR f(R) \mathcal{H} \) is a generator of ‘half parametrized’ evolution on \( \Sigma_0 \). In [5] it was shown that for smooth \( f \) vanishing fast enough at the axis and at spatial infinity, the normal-ordered operator \( \hat{H}(f) \) is densely defined and has a well-defined action on the Fock vacuum. Since we know that the standard flat space Hamiltonian, \( \hat{H}_0 \), is also a well-defined operator, we conclude from [5] that \( \hat{H}(f + 1) = \hat{H}(f) + \hat{H}_0 \) which generates nontrivial evolution at infinity and along the axis is also well-defined. This ties in neatly with the existence of finite ‘half parametrized’ evolution.

It can be checked that the Poisson bracket between two such generators of ‘half parametrized’ evolution on \( \Sigma_0 \), \( H(f_1) \) and \( H(f_2) \), is the generator of a spatial diffeomorphism. It seems puzzling that despite \( \hat{H}(f_1), \hat{H}(f_2) \) being densely defined operators, the generators of generic spatial diffeomorphisms do not have a well-defined action on the Fock vacuum. A possible resolution is that the relevant shift vector does not satisfy our requirements of genericity since it is built out of \( f_1, f_2 \) in a specific way. However, we suspect that the resolution lies in the well-known problem of domains for unbounded operators: although the vacuum is in the domain of both \( \hat{H}(f_1) \) and \( \hat{H}(f_2) \), it need not be so for their product and/or commutator. It would be of interest to confirm this.

We conclude with some remarks regarding the relevance of our results to a Dirac quantization of PFT and of gravity. At first glance, our results seem to indicate that a straightforward Dirac quantization of axisymmetric PFT in 2+1 dimensions (and hence of quantum cylindrical waves) which is equivalent to the standard Fock quantization, does not exist. \(^5\) Indeed it seems that the lack of unitary implementability of spatial diffeomorphisms for quantum cylindrical waves has an adverse lesson for Loop Quantum Gravity (see for example [17]) since the entire formalism is based on a unitary representation of spatial diffeomorphisms. However, a more careful Loop Quantum Gravity type treatment of PFT merits the construction of a kinematic Hilbert space

\(^5\) Note that a classical Hamilton- Jacobi type of canonical transformation along the lines of [11] should result in the classical Heisenberg picture constraints which directly yield, upon Dirac quantization, the standard Fock space. However the PFT constraints which are naturally available take the form appropriate to the classical Schrödinger picture [11, 12]. Our results imply that a direct Dirac quantization of the latter, without further canonical transformations, is not equivalent to the standard Fock representation.
representation for both the matter and the embedding variables rather than formally setting the action of $\hat{X}^\alpha(x)$ to be multiplication by the embedding variable $X^\alpha(x)$ and that of the conjugate momentum variable to be $\hat{P}_\alpha(x) = -i\frac{\delta}{\delta X^\alpha(x)}$. Such a treatment is currently in progress [18] and suggests that when the embedding sector is properly defined in the quantum theory along with an appropriate notion of embedding dependent Hilbert spaces for the matter sector, a satisfactory Dirac quantization which is equivalent to the standard Fock quantization can be constructed notwithstanding the results of this work and [10].

Acknowledgments
We thank Charles Torre for useful comments on this work.

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