ON THE $\mathcal{W}$-GRAVITY SPECTRUM AND ITS $G$-STRUCTURE

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Abstract: We present results for the BRST cohomology of $\mathcal{W}[g]$ minimal models coupled to $\mathcal{W}[g]$ gravity, as well as scalar fields coupled to $\mathcal{W}[g]$ gravity. In the latter case we explore an intricate relation to the (twisted) $g$ cohomology of a product of two twisted Fock modules.
1 Introduction

The BRST quantization of two dimensional $\mathcal{W}[\mathfrak{g}]$ gravity coupled to $\mathcal{W}[\mathfrak{g}]$ matter poses the interesting mathematical problem of computing the semi-infinite cohomology of a $\mathcal{W}$-algebra with values in a tensor product of two (positive energy) $\mathcal{W}$-modules. In this note we study this cohomology both for free scalar fields as well as for $\mathcal{W}$ minimal models coupled to $\mathcal{W}$-gravity, i.e. we study the cohomology of the tensor products of two Fock spaces at irrational $\alpha^2$, and of an irreducible $\mathcal{W}$-module with a Fock space. In these cases we give the complete results for the cohomologies. The work described in this paper is an extension of [1, 2], where we presented results for the case in which the ‘Liouville’ momentum takes values in one specific Weyl chamber. We refer to [1] for further references on the subject.

Strictly speaking, the relevant BRST operator has only been shown to exist, by explicit construction, for $\mathcal{W}_3 \equiv \mathcal{W}[sl(3)]$ [3, 4]. However, since our analysis is insensitive to the specific form of the BRST operator, pending the existence proof we have formulated our results for arbitrary simple, simply-laced Lie algebras $\mathfrak{g}$.

For irrational $\alpha^2$, it turns out that there is an intimate connection between the $\mathcal{W}[\mathfrak{g}]$ cohomology of a tensor product of two $\mathcal{W}$ Fock spaces and the (twisted) $\mathfrak{g}$ cohomology of the product of two twisted $\mathfrak{g}$ Fock spaces, which is the finite-dimensional analogue of a $G/G$ coset model. Closely related observations have been made in [5, 6].

This note is organized as follows. In section 2 we discuss the (twisted) $\mathfrak{g}$ cohomology of the product of two twisted Fock spaces. In appendix A we give the complete result for $\mathfrak{g} \cong sl(2)$ and $sl(3)$. Our results in this section mainly serve the purpose to formulate the results for the $\mathcal{W}$-cohomology through the correspondence alluded to above, but we believe they are also interesting in their own right. In section 3 we consider the $\mathcal{W}[\mathfrak{g}]$ cohomology of a tensor product of two Fock spaces and explains the correspondence with the (twisted) $\mathfrak{g}$ cohomology. Finally, in section 4, we present a complete result for the $\mathcal{W}[\mathfrak{g}]$ minimal models coupled to $\mathcal{W}[\mathfrak{g}]$ gravity. At the end we included a table of some of the states for explicitness. We compare our results to previously obtained results, in particular those of [4, 8], and find complete agreement.

2 $G$-cohomology of a product of two twisted Fock spaces

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Fix a triangular decomposition $\mathfrak{g} \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and a corresponding Chevalley basis $\{e_{-\alpha}, h_i, e_\alpha\}, \alpha \in \Delta_+, i = 1, \ldots, \text{rank} \mathfrak{g}$. For any $\mathfrak{g}$-module $V$ in the BGG-category $\mathcal{O}$ [2] (loosely speaking, the category of modules with weights bounded from above), we can consider its ‘twisted’
cohomology $H_{tw}^{i}(g, V)$, which is the finite-dimensional analogue of the so-called ‘semi-infinite’ cohomology introduced by Feigin [10]. This cohomology is defined as follows (see [11, 12] for more details): Introduce a ghost system $(b_{A}, c^{A})$ for each generator $e_{A}$ of $g$, with (anti-)commutators $\{b_{A}, c^{B}\} = \delta_{A}^{B}$, and denote the corresponding ghost Fock space $F^{gh}$. The (physical) ghost vacuum $|gh\rangle$ satisfies
\[ b_{i}|gh\rangle = c^{-\alpha}|gh\rangle = 0, \quad \alpha \in \Delta_{+}, \quad i = 1, \ldots, \text{rank } g. \tag{2.1} \]
The ghost Fock space is graded by ghost number, $\text{gh}(c^{A}) = 1$, $\text{gh}(b_{A}) = -1$, and is a $g$-module under the action
\[ \pi^{gh}(e_{A}) = - \sum_{B,C} f_{AB}^{C} c^{B} b_{C}. \tag{2.2} \]
Note that the highest weight of $F^{gh}$ equals $2\rho$, where $\rho$ is the principal vector of $g$ ($\langle \rho, \alpha \rangle = 1$, $\forall \alpha \in \Delta_{+}$), as is easily computed via
\[ \pi^{gh}(h_{i})|gh\rangle = - \sum_{\alpha \in \Delta_{+}} f_{-\alpha}^{i} c^{-\alpha} b_{-\alpha}|gh\rangle = \sum_{\alpha \in \Delta_{+}} (\alpha_{i}^{\vee}, \alpha)|gh\rangle = (\alpha_{i}^{\vee}, 2\rho)|gh\rangle. \tag{2.3} \]
The (twisted) cohomology $H_{tw}^{i}(g, V)$ is defined as the cohomology of the (BRST) operator
\[ d = \sum_{A} c^{A} \left( \pi(e_{A}) + \frac{1}{2} \pi^{gh}(e_{A}) \right), \tag{2.4} \]
acting on the (graded) complex $V \otimes F^{gh}$.

The twisted cohomology of a subalgebra of $g$ is defined similarly by restricting to the appropriate subset of generators. In particular we are interested in the cohomologies of (twisted) nilpotent subalgebras $n_{w}^{+} \equiv w \cdot n_{+} \cdot w^{-1}$, corresponding to Weyl group elements $w \in W$. In this case the sums run over $\alpha \in w(\Delta_{+})$.

To orient the discussion it is worth noting in the “untwisted” case ($w' = 1$) that the computation of $H_{tw}^{i}(n_{+}, V)_{\lambda} \cong H^{i}(n_{+}, V)_{\lambda} \cong \text{Ext}^{i}_{\mathcal{O}}(M_{\lambda}, V)$ for various modules $V \in \mathcal{O}$ is a classical problem in mathematics. In the case when $V$ is a finite-dimensional irreducible module $L_{\lambda}$ the result is well-known [13] (see (2.7) below, where this result is derived as an illustration of standard techniques). For many other interesting modules, such as Verma modules, the problem has only been solved partially.

Besides Verma modules $M_{\lambda}$ and contragredient Verma modules $\overline{M}_{\lambda}$ there exists a class of modules in $\mathcal{O}$, the so-called twisted Fock spaces $F_{\lambda}^{w}$ (labelled by elements $w \in W$), that interpolate between $M_{\lambda}$ and $\overline{M}_{\lambda}$. [Our conventions are such that $F_{\lambda}^{1} \cong \overline{M}_{\lambda}$ and $F_{\lambda}^{w_{0}} \cong M_{\lambda}$.] These modules are, in a sense, finite-dimensional analogues of Wakimoto modules [14] and were introduced in [11] (see [13] for explicit realizations). They are uniquely characterized by the property that they are free over $\mathcal{U}(n_{+}^{w} \cap n_{-})$, cofree over $\mathcal{U}(n_{+}^{w} \cap n_{+})$ and have a unique highest weight vector (of weight $\lambda$).
The cohomology of a twisted Fock space \( F^w_\lambda \) with respect to the nilpotent subalgebra \( n_+^w \) with the same twist \( w \), is given by \([\ref{1}], \ref{2}\)

\[
H^i_{\text{tw}}(n_+^w, F^w_\lambda) \cong \delta^{i,0} \mathcal{C}_{\lambda^+ - w\rho}.
\] (2.5)

In fact, \((2.3)\) uniquely characterizes the module \( F^w_\lambda \) in the category \( O \).

For \( \Lambda \) an integral dominant weight, \( \text{i.e.} \ \Lambda \in P_+ \), there exist resolutions of the irreducible module \( L_\Lambda \) in terms of twisted Fock spaces \( F^w_\lambda \) (for any \( w \in W \)) with terms

\[
C^n_w L_\Lambda \cong \bigoplus_{\{\sigma \in W|\ell_w(\sigma) = i\}} F^w_{\sigma \Lambda},
\] (2.6)

where \( \ell_w(\sigma) \) is the twisted length of \( \sigma \in W \), which can be expressed in terms of the usual length \( \ell \) through \( \ell_w(\sigma) = \ell(w^{-1}\sigma) - \ell(w^{-1}) \) and \( \sigma \ast \Lambda = \sigma(\Lambda + \rho) - \rho \) denotes a shifted action of the Weyl group.

To illustrate an application of \((2.6)\) we reproduce the known result for \( H^i_{\text{tw}}(n_+^w, L_\Lambda) \) as alluded to above. Simply take a resolution of \( L_\Lambda \) in terms of Fock spaces twisted by the same \( w \in W \) and apply \((2.3)\) to the resulting double complex. We find

\[
H^i_{\text{tw}}(n_+^w, L_\Lambda) \cong \bigoplus_{\{\sigma \in W|\ell_w(\sigma) = i\}} \mathcal{C}_{\sigma \ast \Lambda + \rho - w\rho}.
\] (2.7)

We are interested in computing the twisted cohomology \( H^i_{\text{tw}}(g, F^w_\lambda \otimes F^w_{\mu \rho}) \), or rather the cohomology relative to the Cartan subalgebra \( t \), which we will denote by \( H^i_{\text{tw}}(g, t; F^w_\lambda \otimes F^w_{\mu \rho}) \). [The Weyl group element \( w_0 \) denotes the unique element of longest length in \( W \).] This cohomology corresponds to the physical states of the finite-dimensional analogue of the so-called \( G/G \)-model. Using the fact that \( g \cong n_+^w \oplus t \oplus n_+^{w'w_0} \), it can be related to a generalization of \((2.3)\) by invoking a reduction theorem (see \( \text{e.g.} \ \ref{12}\))

\[
H^i_{\text{tw}}(g, t; F^w_\lambda \otimes F^w_{\mu \rho}) \cong \bigoplus_{p+q=i} \left( H^p_{\text{tw}}(n_+^w, F^w_\lambda) \otimes H^q_{\text{tw}}(n_+^{w'w_0}, F^w_{\mu \rho}) \right)_t
\]

\[
\cong H^i_{\text{tw}}(n_+^{w'}, F^w_\lambda)_{-\mu - \rho - w\rho}.
\] (2.8)

One can show that nonzero cohomology can only arise if \( \lambda \) and \( \mu \) can be parametrized as

\[
\lambda = \sigma \ast \Lambda, \quad \mu = -\sigma' \ast \Lambda - 2\rho,
\] (2.9)

for some dominant weight \( \Lambda \) and \( \sigma \preceq \sigma' \), \( \sigma, \sigma' \in W \). [We take the usual Bruhat ordering “\( \preceq \)” on \( W \) \([\ref{16}]\). In particular for \( sl(2) \) and \( sl(3) \), this is just the ordering of \( W \) by the length \( \ell(\cdot) \).] Moreover, if one restricts the discussion to dominant integral weights, \( \text{i.e.} \ \Lambda \in P_+ \), then one can show that the cohomology does not depend on the particular \( \Lambda \in P_+ \). We will henceforth restrict the discussion to \( \Lambda \in P_+ \) and adopt the shorthand notations \( F^w_\sigma = F^w_{\sigma \Lambda} \) and \( F^w_{\sigma'w_0^\prime} = F^w_{\sigma'\Lambda - 2\rho} \).
In order to summarize the computation of the dimensions of these cohomology groups, we introduce a set of polynomials by

$$
P_{w,w'}^{\sigma,\sigma'}(q) = \sum_i (-1)^{\ell_w(\sigma) + \ell_w(\sigma') + i} q^i \dim H^i_{tw}(g, h; F^w_{\sigma} \otimes F^{-w'w_0}_{-\sigma'}) . \quad (2.10)$$

They satisfy the following basic relations:

1. $P_{\sigma,\sigma'}^{w,w'}(q) = 0$ for $\sigma > \sigma'$.
2. (i) $P_{\sigma,\sigma'}^{w,w'}(q) = 1$, (ii) $P_{\sigma,\sigma'}^{w,w}(q) = \delta_{\sigma,\sigma'}$, (iii) $P_{\sigma,\sigma'}^{w,w'}(1) = \delta_{\sigma,\sigma'}$.
3. $P_{\sigma,\sigma'}^{w,w'}(q) = P_{\sigma,\sigma'}^{ww_0,w_0w}(q)$ (reflection symmetry).
4. $P_{\sigma,\sigma'}^{w,w'}(q) = P_{\sigma,\sigma'}^{ww_0,w'w_0}(q^{-1})$ (Poincaré duality).

The identities in 2) follow from the fact that (i) for $\sigma = \sigma'$ there is only one state in the complex, (ii) for $w = w'$ the cohomology is given by $(2.5)$, and (iii) by applying the Lefschetz principle. Identity 3) follows from the observation that in $(2.9)$ we could equally well have swapped the order of the two Fock spaces and chosen the weight $\Lambda' = -w_0\Lambda$ to parametrize $(\lambda, \mu)$. Finally, identity 4) follows from the fact that the module contragredient to $F^w_\lambda$ is $F^{ww_0}_\lambda$. In Appendix A we list all polynomials for $g \cong sl(2)$ and $sl(3)$.

In the particular case $(w, w') = (w_0, 1)$, where the polynomials correspond to the $g$ cohomology of a product of two Verma modules $M_\sigma \otimes M_{-\sigma'}$, it is known that $P_{\sigma,\sigma'}^{w_0,1}(q) = R_{\sigma,\sigma'}(q)$ for $\ell(\sigma') - \ell(\sigma) \leq 3$ [14]. Here, $R_{\sigma,\sigma'}(q)$ denote the Kazhdan-Lusztig $R$-polynomials [18, 16].

The above discussion has a straightforward generalization to the affine Lie algebras, in which case one is interested in computing the relative semi-infinite cohomology $H^{\infty/2+i}(\hat{g}, \hat{t}; F^w_\lambda \otimes F^{w_0w_0}_\mu)$ of the tensor product of two Wakimoto modules, twisted by finite-dimensional Weyl group elements $w, w' \in \hat{W}$ [12]. Once more the cohomology can arise only for the weights satisfying the affine analogue of $(2.9)$, with $\sigma, \sigma'$ in the affine Weyl group $\hat{W}$.

It is known that for $w = w'$ (see, e.g. sections 4 and 5 in [12])

$$H^{\infty/2+i}(\hat{g}, \hat{t}; F^w_\sigma \otimes F^{w_0w_0}_\sigma) \cong \delta_{\sigma,\sigma'} \delta^{i,0} \mathcal{C}' , \quad (2.11)$$

which is the analogue of $(2.5)$. We expect that for general $w$ and $w'$ the set of polynomials as in $(2.10)$ with $\sigma, \sigma' \in \hat{W}$ will in fact be the same as in the finite-dimensional case. For $\hat{sl}(2)$ this can be verified explicitly from the results of [6]. The general case appears to be an open problem.
3 \( \mathcal{W} \)-cohomology of Fock spaces at irrational \( \alpha^2_+ \)

Let \( \mathcal{W}[g] \) be the \( \mathcal{W} \)-algebra associated to some simply-laced simple Lie algebra \( g \) (see [19] for a review and a list of notations). In the remainder of this paper we will present some new results for the semi-infinite cohomology \( H^i(\mathcal{W}[g], V^M \otimes V^L) \) of \( \mathcal{W}[g] \) on the product of two positive energy \( \mathcal{W}[g] \) modules \( V^M \) and \( V^L \). Specifically, for the ‘Liouville’ module \( V^L \), representing the \( \mathcal{W}[g] \) gravity sector, we will take the Fock space of an appropriate set of free scalar fields, while for the matter module \( V^M \) we will take either a Fock space (section 3) or an irreducible module, i.e. \( \mathcal{W}[g] \) minimal model (section 4). Although, in general, the Cartan subalgebra of \( \mathcal{W}[g] \) will not be diagonalizable on the modules \( V \), there is still an analogue of the relative cohomology. One can show that the cohomology possesses a multiplet structure of \( 2^\ell \) states (where \( \ell = \text{rank} \, g \)), which is essentially due to the ghost zero modes [8, 1]. The lowest ghost number state in each multiplet will be called a prime state. Throughout this paper we will only formulate the results for the prime states in the cohomology.

To be precise, our results are only valid for \( g \cong \mathfrak{sl}(2) \) and \( \mathfrak{sl}(3) \), for these are the only cases for which the differential (BRST operator) has been constructed explicitly (see [3, 4] for the latter, and also [20] for some higher rank results). However, one expects that such a differential exists for the other \( \mathcal{W}[g] \) algebras. In the discussion below we use only very generic properties of the differential, and our results should therefore be valid for the other \( \mathcal{W}[g] \) algebras as well.

Let \( F(\Lambda, \alpha_0) \) denote the Fock space of \( \ell \equiv \text{rank} \, g \) scalar fields \( \phi^k(z) \), normalized such that \( \phi^k(z)\phi^l(w) = -\delta^{kl} \ln(z - w) \), coupled to a background charge \( \alpha_0 \rho \). The Fock space vacuum \( |\Lambda\rangle \) is labelled by a vector \( \Lambda \) in the weight space of \( g \) such that \( p^k|\Lambda\rangle = k|\Lambda\rangle \).

A realization of \( \mathcal{W}[g] \) on the Fock space \( F(\Lambda, \alpha_0) \) can be constructed by means of the Drinfel’d-Sokolov reduction. In particular, the stress energy tensor is given by

\[
T(z) = -\frac{1}{2}(\partial \phi(z) \cdot \partial \phi(z)) - i\alpha_0 \rho \cdot \partial^2 \phi(z).
\]

It generates the Virasoro subalgebra of \( \mathcal{W}[g] \). The central charge and conformal dimension of \( F(\Lambda, \alpha_0) \) are given by

\[
c = \ell - 12\alpha_0^2|\rho|^2, \quad h(\Lambda) = \frac{1}{2}(\Lambda, \Lambda + 2\alpha_0 \rho).
\]

Let us first consider the cohomology \( H^i(\mathcal{W}[g], F(\Lambda^M, \alpha_0^M) \otimes F(\Lambda^L, \alpha_0^L)) \). Imposing the condition that the total central charge vanishes in order for the differential to be nilpotent, leads to the following parametrization of \( \alpha_0^M \) and \( \alpha_0^L \)

\[
\alpha_0^M = \alpha_+ + \alpha_-, \quad -i\alpha_0^L = \alpha_+ - \alpha_-, \quad \alpha_+ \alpha_- = -1.
\]

By standard arguments, based on the composition series of a Fock space, one can show that the cohomology is trivial (i.e. contains at most tachyonic states) if either...
\( F(\Lambda^M, \alpha_0^M) \) or \( F(\Lambda^L, \alpha_0^L) \) is irreducible (see e.g. [22]). Recall that a Fock space \( F(\Lambda, \alpha_0) \) is reducible if and only if there exists a root \( \alpha \in \Delta_+ \) such that (see e.g. [22])

\[
(\Lambda + \alpha_0 \rho, \alpha) \in \pm (N\alpha_+ + N\alpha_-),
\]

and ‘completely degenerate’ if \([3, 4]\) holds for all roots \( \alpha \in \Delta_+ \). Therefore, in the reducible case, it is convenient to parametrize \( \Lambda \) by \( \Lambda + \alpha \rho = w^{-1}(\alpha_{+}\sigma(\Lambda^{(+)}) + \alpha_{-}(\Lambda^{(-)}) + \rho)) \equiv \Lambda(w, \sigma) + \alpha_0 \rho, \) (3.5)

for some weights \( \Lambda^{(+)}, \Lambda^{(-)} \) and \( w, \sigma \in W \). If \( F(\Lambda, \alpha_0) \) is reducible in the direction of \( \alpha \in \Delta_+ \), then we can choose \( \Lambda^{(+)}, \Lambda^{(-)} \) such that \( (\Lambda^{(+)}, \alpha) \in \mathbb{N}, (\Lambda^{(-)}, \alpha) \in \mathbb{N} \). If \( F(\Lambda, \alpha_0) \) is completely degenerate then one can choose \( \Lambda^{(+)}, \Lambda^{(-)} \in P_+ \) as well as \( \sigma = 1 \), and if moreover \( \alpha_0^2 \) is irrational then this parametrization of \( \Lambda \) in terms of \( \Lambda^{(+)}, \Lambda^{(-)} \in P_+ \) and \( w \in W \) is unique. In this paper we will only consider the completely degenerate case. In the case that \( F(\Lambda, \alpha_0) \) is reducible only in certain directions the cohomology essentially reduces to the one of a smaller \( \mathcal{W} \)-algebra. It can be analysed similarly.

The highest weight vectors of all \( F(\Lambda(w, 1)), w \in W \), have the same eigenvalues with respect to the \( \mathcal{W}[g] \) generators because these are invariant under \( \Lambda + \alpha_0 \rho \rightarrow w(\Lambda + \alpha_0 \rho) \). Therefore, all \( F(\Lambda(w, 1)) \) contain one and the same irreducible \( \mathcal{W}[g] \) module \( L(\Lambda) \). Thus, by arguing in the usual way, we obtain a family of resolutions \( C^i_wL(\Lambda) \) of \( L(\Lambda) \), parametrized by \( w \in W \), that are based on \( F(\Lambda(w, 1)) \). For \( \alpha_0^2 \notin \mathcal{Q} \) these resolutions have a finite number of terms (see section 4 for \( \alpha_0^2 \in \mathcal{Q} \)). Specifically, for \( \Lambda = \alpha_{+}\Lambda^{(+)}, \alpha_{-}\Lambda^{(-)} \) we have

\[
C^i_wL(\Lambda^{(+)}, \Lambda^{(-)}) \cong \bigoplus_{\{\sigma \in W|w(\sigma) = i\}} F(\Lambda(w, \sigma)). \tag{3.6}
\]

where \( \Lambda(w, \sigma) \) is defined in (3.3). For \( w = 1 \) these resolutions were already constructed in [22, 23].

Let us now turn to the discussion of the cohomology \( \text{H}^i(\mathcal{W}[g], F(\Lambda^M, \alpha_0^M) \otimes F(\Lambda^L, \alpha_0^L)) \). By analysis of the Kac-determinants for \( \mathcal{W}[g] \) [1] and standard reasoning using composition series for \( F \), one can argue that:

For \( \Lambda^M \) of the form (3.3), i.e.

\[
\Lambda^M(w, \sigma) + \alpha_0^M \rho = w^{-1}\left(\alpha_{+}\sigma(\Lambda^{(+)}) + \alpha_{-}(\Lambda^{(-)}) + \rho\right) \tag{3.7}
\]

the cohomology \( \text{H}^i(\mathcal{W}[g], F(\Lambda^M, \alpha_0^M) \otimes F(\Lambda^L, \alpha_0^L)) \) can only be nontrivial if \( \Lambda^L = \Lambda^L(w', \sigma') \) for some \( w', \sigma' \in W \), where

\[
i(\Lambda^L(w', \sigma') + \alpha_0^L \rho) = w'^{-1}\left(\alpha_{+}\sigma'(\Lambda^{(+)}) + \alpha_{-}(\Lambda^{(-)}) + \rho\right) \tag{3.8}
\]
At this point we note a remarkable similarity between the resolutions of \((3.6)\) and \((2.6)\), which suggests that the matter module \(F(\Lambda^M(w, \sigma))\) behaves like the twisted Wakimoto module \(F^\sigma_w\), while the Liouville module \(F(\Lambda^L(w', \sigma'))\) behaves like the dual of the matter module, \(i.e.\) like \(F_{w_0}^{-w_0}\). For irrational \(\alpha^2_+\) we therefore expect a close correspondence between the \(\mathcal{W}[g]\) cohomology of a tensor product of two Fock spaces and the (twisted) \(g\) cohomology of a tensor product of two twisted Wakimoto modules. More precisely, we assert that

\[
\sum_i (-1)^{\ell_w(\sigma) + \ell_{w'}(\sigma') + i} q^i \dim H^i(\mathcal{W}[g], F(\Lambda^M(w, \sigma)) \otimes F(\Lambda^L(w', \sigma'))) = \mathcal{P}^{w, w'}_{\sigma, \sigma'}(q), \quad (3.9)
\]

where the polynomials \(\mathcal{P}^{w, w'}_{\sigma, \sigma'}(q)\) were defined in \((2.10)\) (see Appendix A for an explicit list of all polynomials for \(g \cong sl(2)\) and \(sl(3)\)).

For \(g \cong sl(2)\) the result agrees with \([21, 24]\). For \(sl(3)\) the result \((3.9)\) is in complete agreement with the states explicitly constructed in \([7]\) (in \([7]\) only states corresponding to, in our conventions, \(w = \sigma, w' = \sigma'\) of ghost number \(\ell_{w'}(w') = -\ell(w')\) were considered). We have explicitly constructed some additional physical states. The results are consistent with \((3.9)\).

Let us now examine the consequences of \((3.9)\) for the (chiral) ground ring at irrational \(\alpha^2_+\) in the \(\mathcal{W}[sl(3)]\) case. From the explicit results in Appendix A we conclude that for every \(\Lambda^{(+)}, \Lambda^{(-)} \in P_+\) we have a ground ring element iff \((w, w') = (1, w_0), \ (\sigma, \sigma') = (1, \nu_0)\). Moreover, this element is unique. Let us denote it by \(\phi_{(\Lambda^{(+)}, \Lambda^{(-)})}\). The ground ring is freely generated by \(\phi_{(\Lambda_1, 0)}, \phi_{(\Lambda_2, 0)}, \phi_{(0, \Lambda_1)}\) and \(\phi_{(0, \Lambda_2)}\) (in the notation of \([7]\) these correspond to \(\gamma^0_2, \gamma^0_1, x_1\) and \(x_2\), respectively). [Note that our present conventions differ from those in \([7]\).]

It is reasonable to believe that an equation similar to \((3.9)\) holds for rational \(\alpha^2_+\) if one replaces the \(g\) Fock space \(F^w\) with its corresponding affinization \(i.e.\) a (twisted) Wakimoto module of \(\tilde{g}\), and let \(\sigma\) run over the affine Weyl group. In particular we claim (see \((2.11)\)):

\[
H^i(\mathcal{W}[g], F(\Lambda^M(w, \sigma)) \otimes F(\Lambda^L(w, \sigma'))) (i.e. w = w') is nontrivial only if \sigma = \sigma' in which case it is one-dimensional and concentrated in dimension \(i = 0\) (tachyonic state).
\]

For \(\tilde{sl}(2)\) this claim is known to be correct, while for other \(\tilde{g}\) it is consistent with sample calculations. In particular, this assertion leads to the cohomology of \(\mathcal{W}\) minimal models (section 4) where it does not contradict previous results. [In the affine Lie algebra case it corresponds to \((2.11)\). For the analogous statement for (contragredient-) Verma modules see \([1]\).]

For \(g \cong sl(2)\) it is well-known that the dimensions of the cohomology groups \(H^i(\mathcal{W}[g], F(\Lambda^M(w, \sigma)) \otimes F(\Lambda^L(w', \sigma')))\) are insensitive as to whether \(\alpha^2_+\) is rational
or irrational, as a consequence of an $SO(2,\mathbb{D})$ symmetry relating different values of $\alpha_+$. This $SO(2,\mathbb{D})$ symmetry does not persist to higher rank $\mathcal{W}$-algebras, and in fact for $\mathfrak{g} = sl(3)$ it has been shown that the cohomology at $c^M = 2$ (corresponding to $\alpha_+ = \pm 1$) is considerably larger than for irrational $\alpha_+^2$ (see [2] for some results for $\mathcal{W}_3$ at $c^M = 2$). There are two, possibly related, reasons for this phenomenon. Firstly, for $\alpha_+^2 = \pm 1$ the parametrization (3.5) in terms of $\Lambda^{(+)}$, $\Lambda^{(-)}$ is considerably larger than for irrational $\alpha_+^2$. Secondly, at $\alpha_+ = \pm 1$, both the complex and the cohomology carry a representation of $\mathfrak{g}$, where the $\mathfrak{g}$-generators are given by the (zero modes of a) Frenkel-Kac-Segal vertex operator construction in the matter sector.

4 \ $\mathcal{W}$-cohomology of minimal models

In this section we will give a complete classification of physical states for a $\mathcal{W}[\mathfrak{g}]$ minimal model coupled to $\mathcal{W}[\mathfrak{g}]$ gravity.

The $\mathcal{W}[\mathfrak{g}]$ minimal models arise for $\alpha_+^2 = p/p' \in \mathbb{Q}$ ($p$ and $p'$ relatively prime integers) and are labelled by two integrable weights $\Lambda^{(\pm)} \in P_{\mp}^{h^-}$ and $\Lambda^{(-)} \in P_{\pm}^{h^+}$ such that the highest weight is given by $\Lambda = \alpha_+ \Lambda^{(+) \pm} \alpha_- \Lambda^{(-)}$. Here, $P_{\pm}$ denotes the set of integrable weights of $\mathfrak{g}$ at level $k$ and $h^\vee$ is the dual Coxeter number.

We have a set of resolutions, parametrized by $w \in W$, of the minimal model $L(\Lambda^{(+)}, \Lambda^{(-)})$ in terms of twisted Fock spaces similar to (3.6)

$$C_w^i L(\Lambda^{(+)}, \Lambda^{(-)}) \cong \bigoplus_{\{\sigma \in \hat{W} | \ell_w(\sigma) = i\}} F(\Lambda(w, \sigma)),$$

where $\Lambda(w, \sigma)$ is defined as in (3.5), but now the sum over $\sigma$ runs over the affine Weyl group $\hat{W}$. [We recall that any $w \in \hat{W}$ can be written as $w = t_\beta \bar{w}$ for some $\bar{w} \in W$ and translation $t_\beta$ such that $w \lambda = t_\beta \bar{w} \lambda = \bar{w} \lambda + k \beta$ for affine weights $\lambda$ of level $k$. In the following $\rho$ is regarded as an element of $P_{\pm}^{h^\vee}$.] The twisted length $\ell_w$ on the affine Weyl group $\hat{W}$ is defined by (see [14, 15])

$$\ell_w(\sigma) = \lim_{N \to \infty} \left( \ell(t_{-Nw^p} \sigma) - \ell(t_{-Nw^p}) \right).$$

(4.2)

Since, for each $\sigma \in \hat{W}$ there are only a finite number of possible cancellations between $t_{-Nw^p}$ and $\sigma$, the limit in (4.2) is in fact reached at a finite value of $N$. Note furthermore that for $\sigma \in W$ the length defined by (4.2) reduces to the usual twisted length $\ell_w(\sigma) = \ell(w^{-1} \sigma) - \ell(w^{-1})$.

Now, consider the cohomology $H^i(\mathcal{W}[\mathfrak{g}], L(\Lambda^{(+)\pm}, \Lambda^{(-)}) \otimes F(\Lambda^L, \alpha_0^L))$. By taking an arbitrary resolution $C_w^i L(\Lambda^{(+)\pm}, \Lambda^{(-)})$ of $L(\Lambda^{(+)\pm}, \Lambda^{(-)})$ one finds that the cohomology is nontrivial if and only if $\Lambda^L = \Lambda^L(w, \sigma)$ for some $w \in W$ and $\sigma \in \hat{W}$ where

$$-i(\Lambda^L(w, \sigma) + \alpha_0^L \rho) = w^{-1} \left( \alpha_+ \sigma(\Lambda^{(+)\pm} + \rho) + \alpha_- (\Lambda^{(-)} + \rho) \right).$$

(4.3)
Now, as discussed in section 3, $H^i(\mathcal{W}[g], F(\Lambda^M(w, \sigma)) \otimes F(\Lambda^L(w, \sigma'))) \cong \delta_{\sigma, \sigma'} \delta^{i(q)}$. Thus by taking, for any $\Lambda^L = \Lambda^L(w, \sigma)$, a resolution $C_w L(\Lambda^{(+)}, \Lambda^{(-)})$ of $L(\Lambda^{(+)}, \Lambda^{(-)})$ with the same twist $w \in W$, the same argument as in e.g. [24] immediately yields

1. The cohomology $H^i(\mathcal{W}[g], L(\Lambda^{(+)}, \Lambda^{(-)}) \otimes F(\Lambda^L, \alpha^L_0))$ is nontrivial iff $\Lambda^L = \Lambda^L(w, \sigma)$ for some $w \in W$ and $\sigma \in \hat{W}$.

2. For $\Lambda^L = \Lambda^L(w, \sigma)$ there is precisely one (prime) state in the cohomology. Its ghost number is given by $\ell_w(\sigma)$ and its energy level by $E = \frac{1}{2} |\alpha_+(\Lambda^{(+)} + \rho) + \alpha_- (\Lambda^{(-)} + \rho)|^2 - \frac{1}{2} |\Lambda^L(w, \sigma) + \alpha^L_0\rho|^2$.

In [1] we obtained this result for the particular case that $-i(\Lambda^L + \alpha^L_0\rho)$ is in the fundamental Weyl chamber $D_+$. This corresponds to those $\Lambda^L(w, \sigma)$ where for any $\sigma \in \hat{W}$ the Weyl group element $w = w_\sigma$ is determined in such a way that $-i(\Lambda^L(w_\sigma, \sigma) + \alpha^L_0\rho) \in D_+$. The above result extends our previous work to all Weyl chambers. For $g \cong sl(2)$ it agrees with [21, 24]. For $g \cong sl(3)$ and the trivial module $L(\Lambda^{(+)}, \Lambda^{(-)}) \cong \mathcal{C}$ (i.e. $p = 3, p' = 4$ and $\Lambda^{(+)} = \Lambda^{(-)} = 0$) the results can be compared to those for the ‘two-scalar $W_3$ string’ [8]. We find a complete agreement. For illustrative purposes we provide, in this case, a table of physical states for low ghost numbers. In each row the table lists, for given $w \in \hat{W}$ ($\ell(w) \leq 4$), the values of the (energy-)level $E$ and the ghost number $\ell_w(\sigma)$, $w \in W$ (see (4.2)), of the corresponding (prime) state in $H^i(\mathcal{W}_3, \mathcal{C} \otimes F(\Lambda^L(w, \sigma)))$. The table can be compared to the results of [8].

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A Appendix

In this appendix we give some explicit polynomials $P_{\sigma, \sigma'}^{w, w'}(q)$. In particular we present an exhaustive list for $g \cong sl(2)$ and $sl(3)$. The results have been verified by explicit computation.

$$P_{\sigma, \sigma'}^{w, w'}(q) = \begin{cases} 1 & \text{if } \sigma = \sigma' \\ 0 & \text{otherwise} \end{cases} \quad (A.1)$$

$$P_{\sigma, \sigma'}^{w, w'}(q) = \begin{cases} q^{\pm 1} - 1 & \text{if } \sigma \prec r_{w\alpha_i} \sigma = \sigma', \ w\alpha_i \in \Delta_+ \\ 0 & \text{otherwise} \end{cases} \quad (A.2)$$
For $i \neq j$

$$\mathcal{P}_{\sigma,\sigma'}^{w,w_{\gamma},r_{i}}(q) = \begin{cases} 1 & \text{if } \sigma = \sigma' \\ q^{\pm 1} - 1 & \text{if } \sigma < r_{w_{\alpha},\sigma} = \sigma', \ w_{\alpha} \in \Delta_{\pm} \\
(q \pm 1 - 1)^{2} & \text{or } \sigma < r_{w_{\alpha},\sigma} = \sigma', \ w_{\alpha} \in \Delta_{\pm} \\
(q - 1)(q^{-1} - 1) & \text{if } \sigma < r_{w_{\alpha},\sigma} < r_{w_{\alpha},r_{w_{\alpha},\sigma}} = \sigma', \ w_{\alpha} \in \Delta_{\pm} \end{cases}$$

Note that (A.3) can be summarized as

$$\mathcal{P}_{\sigma,\sigma'}^{w,w_{\gamma},r_{i}}(q) = \sum_{\sigma_{0} \leq \sigma_{0} \leq \sigma_{\gamma}} \mathcal{P}_{\sigma,\sigma_{0}'}^{w_{\gamma},r_{i}}(q) \mathcal{P}_{\sigma_{0}',\sigma'}^{w_{\gamma},r_{i}}(q).$$

(A.4)

The right hand side of (A.4) is in fact the polynomial associated to the first term in the spectral sequence of $H^{i}_{w}(g; F^{w}_{\sigma} \otimes F_{-\sigma}^{w_{\gamma}r_{i}w_{\alpha}})$ with respect to the decomposition $g \cong n_{+}^{w_{\gamma}} \oplus t \oplus n_{+}^{w_{\gamma}w_{\alpha}}$. In this particular case this spectral sequence collapses at the first term, hence the result (A.4).

For $sl(3)$ the above determines all but the six polynomials $\mathcal{P}_{\sigma,\sigma'}^{w_{\gamma}w_{\alpha}}(q)$, which are listed below

$$\mathcal{P}_{\sigma,\sigma'}^{w_{\gamma}w_{\alpha}}(q) = \mathcal{P}_{\sigma,\sigma'}^{1,w_{\gamma}}(q^{-1}) = \begin{cases} 1 & \text{if } \sigma = \sigma' \\ q - 1 & \text{if } \ell(\sigma') - \ell(\sigma) = 1 \\
(q - 1)^{2} & \text{if } \ell(\sigma') - \ell(\sigma) = 2 \\
(q^{3} - 2q^{2} + 2q - 1) & \text{if } \ell(\sigma') - \ell(\sigma) = 3 \\
\end{cases}$$

(A.5)

For $i, j \in \{1, 2\}, i \neq j$

$$\mathcal{P}_{\sigma,\sigma'}^{r_{i},r_{j},r_{i}}(q) = \mathcal{P}_{\sigma,\sigma'}^{r_{i},r_{j},r_{i}}(q^{-1}) = \begin{cases} 1 & \text{if } \sigma = \sigma' \\ q - 1 & \text{if } \sigma < r_{i} \sigma = \sigma' \\
q^{-1} - 1 & \text{or } \sigma < r_{j} \sigma = \sigma' \\
(q - 1)(q^{-1} - 1) & \text{if } \sigma < r_{i} \sigma < r_{i} r_{j} = \sigma' \\
\end{cases}$$

(A.6)

References

[1] P. Bouwknegt, J. McCarthy and K. Pilch, USC-93/11, hep-th/9302086, Lett. Math. Phys. 29 (1993), to be published.

[2] P. Bouwknegt, J. McCarthy and K. Pilch, On the BRST structure of $W_{3}$ gravity coupled to $c = 2$ matter, USC-93/14, hep-th/9303164.

[3] J. Thierry-Mieg, Phys. Lett. B197 (1987) 368.
[4] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Phys. Lett. B292 (1992) 35.

[5] V. Sadov, On the spectra of $\hat{sl}(N)_k/\hat{sl}(N)_k$-cosets and $W_N$ gravities, HUTP-92/A055, hep-th/9302060; The hamiltonian reduction of the BRST complex and $N = 2$ SUSY, HUTP-93/A006, hep-th/9304049.

[6] O. Aharony, O. Ganor, J. Sonnenschein and S. Yankielowicz, Phys. Lett. B305 (1993) 35; and references therein.

[7] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Nucl. Phys. B401 (1993) 304.

[8] H. Lu, C.N. Pope, X.J. Wang and K.W. Wu, The complete spectrum of the $W_3$ string, CTP TAMU 50/93, hep-th/9309041, and references therein.

[9] I.N. Berstein, I.M. Gel’fand and S.I. Gel’fand, in: Proc. Summer School of the Bolyai János Math. Soc., ed. I.M. Gel’fand (New York, 1975).

[10] B.L. Feigin, Usp. Mat. Nauk 39 (1984) 195.

[11] B.L. Feigin and E.V. Frenkel, Comm. Math. Phys. 128 (1990) 161.

[12] P. Bouwknegt, J. McCarthy and K. Pilch, J. Geom. Phys. 11 (1993) 225.

[13] R. Bott, Ann. Math. 66 (1957) 203; B. Kostant, Ann. Math. 74 (1961) 329.

[14] M. Wakimoto, Comm. Math. Phys. 104 (1986) 605.

[15] P. Bouwknegt, J. McCarthy and K. Pilch, Prog. Theor. Phys. Suppl. 102 (1990) 67.

[16] J.E. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press (1990).

[17] O. Gabber and A. Joseph, Ann. Sci. Ec. Norm. Sup. 14 (1981) 261; B.D. Boa, Contemp. Math. 139 (1991) 1; K.J. Karlin, Trans. Amer. Math. Soc. 249 (1986) 29.

[18] D. Kazhdan and G. Lustig, Inv. Math. 53 (1979) 165.

[19] P. Bouwknegt and K. Schoutens, Phys. Rep. 223 (1993) 183.
[20] H. Lu, C.N. Pope and X.J. Wang, *On higher-spin generalizations of string theory*, CTP TAMU-22/93, hep-th/9304115; K. Hornfeck, *Explicit construction of the BRST charge for $W_4$*, DFTT-25/93, hep-th/9306019; C.-J. Zhu, *The BRST quantization of the nonlinear $WB_2$ and $W_4$ algebras*, SISSA/77/93/EP, hep-th/9306026.

[21] B.H. Lian and G.J. Zuckerman, Phys. Lett. **B254** (1991) 417; Phys. Lett. **B266** (1991) 21; Comm. Math. Phys. **145** (1992) 561.

[22] E. Frenkel, *$W$-algebras and Langlands-Drinfel’d correspondence*, in Proc. of the 1991 Cargèse workshop on “New Symmetry Principles in Quantum Field Theory,” eds. J. Fröhlich et. al., Plenum Press (1992).

[23] M. Niedermaier, Comm. Math. Phys. **148** (1992) 249.

[24] P. Bouwknegt, J. McCarthy and K. Pilch, Comm. Math. Phys. **145** (1992) 541.
| $\sigma \backslash w$ | E  | $r_1$ | $r_2$ | $r_{2r_1}$ | $r_{1r_2}$ | $r_{1r_2r_1}$ |
|----------------|-----|-------|-------|-------------|-------------|---------------|
| 1              | 0   | 0     | 0     | 0           | 0           | 0             |
| $r_1$          | 1   | 1     | −1    | 1           | −1          | −1            |
| $r_2$          | 1   | 1     | 1     | −1          | 1           | −1            |
| $r_0$          | 2   | −1    | −1    | 1           | 1           | 1             |
| $r_{1r_2}$     | 3   | 2     | 0     | 2           | 0           | −2            |
| $r_{2r_1}$     | 3   | 2     | 2     | 0           | −2          | 0             |
| $r_{1r_0}$     | 4   | 0     | −2    | 2           | 2           | −2            |
| $r_{2r_0}$     | 4   | 0     | 2     | −2          | −2          | 2             |
| $r_{0r_1}$     | 5   | −2    | −2    | 0           | 2           | 0             |
| $r_{0r_2}$     | 5   | −2    | 0     | −2          | 0           | 2             |
| $r_{1r_2r_1}$  | 4   | 3     | 1     | 1           | −1          | −1            |
| $r_{0r_1r_0}$  | 6   | −1    | −3    | 1           | 3           | −1            |
| $r_{0r_2r_0}$  | 6   | −1    | 1     | −3          | −1          | 3             |
| $r_{1r_2r_0}$  | 8   | 3     | −1    | 3           | 1           | −3            |
| $r_{2r_1r_0}$  | 8   | 3     | 3     | −1          | −3          | 1             |
| $r_{1r_0r_2}$  | 9   | 1     | −3    | 3           | 3           | −3            |
| $r_{2r_0r_1}$  | 9   | 3     | 3     | −3          | −3          | 3             |
| $r_{0r_1r_2}$  | 11  | −3    | −3    | −1          | 3           | 1             |
| $r_{0r_2r_1}$  | 11  | −3    | −1    | −3          | 1           | 3             |
| $r_{1r_2r_1r_0}$ | 10 | 4   | 2     | 2           | −2          | −2            |
| $r_{1r_0r_2r_0}$ | 11 | 2   | −2    | 4           | 2           | −4            |
| $r_{2r_0r_1r_0}$ | 11 | 2   | 4     | −2          | −4          | 2             |
| $r_{0r_1r_0r_2}$ | 13 | −2   | −4    | 2           | 4           | −2            |
| $r_{0r_2r_0r_1}$ | 13 | −2   | 2     | −4          | −2          | 4             |
| $r_{1r_2r_0r_1}$ | 14 | 4   | 0     | 4           | 0           | −4            |
| $r_{2r_1r_0r_2}$ | 14 | 4   | 4     | 0           | −4          | 0             |
| $r_{0r_1r_2r_1}$ | 14 | −4   | −2    | −2          | 2           | 2             |
| $r_{1r_0r_2r_1}$ | 16 | 0   | −4    | 4           | 4           | −4            |
| $r_{2r_0r_1r_2}$ | 16 | 0   | 4     | −4          | −4          | 4             |
| $r_{0r_1r_2r_0}$ | 18 | −4   | −4    | 0           | 4           | 0             |
| $r_{0r_2r_1r_0}$ | 18 | −4   | 0     | −4          | 0           | 4             |

Table 1: Physical states for the two-scalar $W_3$ string