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Diophantine Properties Associated to the Equilibrium Configurations of an Isochronous N-Body Problem

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Recently a solvable N-body problem featuring several free parameters has been investigated, and conditions on these parameters have been identified which guarantee that this system is isochronous (all its solutions are periodic with a fixed period) and that it possesses equilibria. The N coordinates $\tilde{z}_n$ characterizing the equilibrium configurations are in some cases explicitly known, in others coincide with the N zeros of certain para-Jacobi polynomials or are arbitrary numbers. In the present paper the behavior of this N-body system in the immediate vicinity of its equilibria is studied, and Diophantine relations satisfied by the N coordinates $\tilde{z}_n$ are thereby identified.

1. Introduction

Recently a solvable N-body problem featuring several free parameters has been investigated, and conditions on these parameters have been identified which guarantee that this system is isochronous (all its solutions are periodic with the same fixed period, independent of the initial data) and that it possesses equilibria [1]. In the present paper Diophantine findings are derived from these results.

The idea to obtain Diophantine relations for the coordinates $\tilde{z}_n$ of the equilibria of an (autonomous) isochronous N-body model is rather simple. One focuses on the behavior of that system in the immediate vicinity of its equilibria. The standard solution of the linearized equations of motion characterizing this behavior entails that these motions are a linear superposition of periodic motions, the frequencies of which coincide with the N eigenvalues of certain specific $N \times N$ matrices constructed with the coordinates $\tilde{z}_n$ of the equilibria. But if the system is isochronous, all its motions—including of course those in the immediate vicinity of its equilibria—are completely periodic with a fixed period. Hence the eigenvalues associated with the motions in the immediate vicinity of the equilibria must all be integer multiples of a common frequency. Thus, the outcome of this approach is to identify specific $N \times N$ matrices all eigenvalues of which must be integer multiples of a common factor: a Diophantine finding. This approach has been extensively used to arrive at Diophantine findings: see for instance Appendix C, entitled “Diophantine findings and conjectures”, of monograph [2]. Let us also note that, while the fact that the N-body model providing the starting point for this approach is isochronous guarantees that Diophantine findings emerge via this
approach (of course, provided equilibria exist and can be found), such findings can also emerge from models which are not altogether isochronous: see below.

The present paper presents some new results of this kind. It is organized as follows. In the next Section 2 the findings of [1] are reviewed—largely verbatim, but merely to the extent needed to make the present paper understandable to readers who prefer not to go preliminarily through [1] (although such neglect is not recommended); and the strategy to arrive at the results of the present paper is detailed. This section can be omitted in a first reading by whoever is primarily interested in the new findings reported in the present paper, who might therefore immediately jump (albeit at the risk of missing some notational indications) to Section 3 where these results are reported. These findings reveal Diophantine properties associated with the \( N \) coordinates \( \bar{z}_n \) characterizing the equilibrium configurations of the solvable \( N \)-body problem treated in [1], which in some cases (i.e. for certain assignments of the parameters of this model) are explicitly known, in other cases coincide with the \( N \) zeros of certain para-Jacobi polynomials [8] or are just a set of arbitrary numbers. These results are then proven in the subsequent Section 4 by investigating the behavior in the immediate vicinity of its equilibria of the \( N \)-body problem treated in [1]. A very terse Section 5, entitled “Outlook”, concludes the paper by mentioning possible future developments. A number of useful identities are collected in Appendix A.

2. Preliminaries

The solvable \( N \)-body problem discussed in [1] is characterized by the Newtonian equations of motion

\[
\bar{z}_n = -E \bar{z}_n - (N - 1) A_1 + B \bar{z}_n - 2 (N - 1) A_3 \bar{z}_n^2 + \sum_{\ell = 1, \ell \neq n}^N \left\{ (\bar{z}_n - \bar{z}_\ell)^{-1} \left[ 2 \bar{z}_n \dot{\bar{z}}_\ell - (D_0 + D_1 \bar{z}_n) (\dot{\bar{z}}_n + \dot{\bar{z}}_\ell) -D_2 \bar{z}_n (\bar{z}_n \bar{z}_\ell + \bar{z}_\ell \bar{z}_n) + 2 \left( A_1 + A_2 \bar{z}_n + A_3 \bar{z}_n^2 \right) \bar{z}_n \right] \right\} .
\]

(2.1)

Notation 2.1. Here and hereafter \( N \) is an arbitrary positive integer (generally \( N \geq 2 \); the \( N \) coordinates \( \bar{z}_n(t) \) are, generally complex, variables depending on the real (independent) variable \( t \) (“time”); superimposed dots denote time-differentiations; the indices \( n, m, k, \ell \) take all integer values from 1 to \( N \), unless otherwise indicated; the 8 (generally complex) constants \( A_1, A_2, A_3, B, D_0, D_1, D_2, E \) are a priori arbitrary (but see below). The \( N \) coordinates \( \bar{z}_n(t) \) are of course moving in the complex \( \bar{z} \)-plane as the time \( t \) evolves. But they can be identified with the coordinates \( \bar{r}_n(t) \) of \( N \) (unit mass, pointlike) particles moving in the “physical”, horizontal plane spanned by the real 2-vector \( \bar{r} \equiv (x, y) \) via the relation \( \bar{r}_n \equiv (x_n, y_n) \) with the \( 2N \) Cartesian coordinates \( x_n \) and \( y_n \) corresponding to the real and imaginary parts of the complex number \( \bar{z}_n = x_n + i y_n \) (see for instance Chapter 4, entitled “Solvable and/or integrable many-body problems in the plane, obtained by complexification”, of Ref. [4]). Here and hereafter \( i \) is the imaginary unit, \( i^2 = -1 \). In the following we generally work with complex variables, but we feel free to refer to the evolution of the \( N \) coordinates \( \bar{z}_n(t) \) as describing an \( N \)-body problem. ■
The \( N \) coordinates \( z_n(t) \) coincide with the \( N \) zeros of a time-dependent polynomial of degree \( N \) in \( z \) which evolves in time according to the \textit{linear, autonomous}, partial differential equation (PDE)

\[
\psi_t + \left( D_0 + D_1 z + D_2 z^2 \right) \psi_t + \left[ E - (N-1) D_2 z \right] \psi_t + \left( A_1 + A_2 z + A_3 z^2 \right) \psi_{zz} - \left[ (N-1) A_1 B z + 2 (N-1) A_3 z^3 \right] \psi = 0; \tag{2.2a}
\]

\[
\psi (z,t) = \prod_{n=1}^{N} \left[ z - z_n(t) \right] = z^N + \sum_{m=1}^{N} \left[ c_m(t) z^{N-m} \right]. \tag{2.2b}
\]

\textbf{Notation 2.2.} In (2.2a) appended variables denote partial differentiations; while (2.2b), besides displaying the identification of the \( N \) coordinates \( z_n(t) \) evolving according to the Newtonian equations of motion (2.1) with the \( N \) zeros of the time-dependent (monic) polynomial \( \psi (z,t) \) evolving according to the PDE (2.2a), introduces the \( N \) coefficients \( c_m(t) \) of the (monic) polynomial \( \psi (z,t) \).

The \( N \) coefficients \( c_m(t) \) evolve according to the system of \( N \) linear autonomous ordinary differential equations (ODEs)

\[
\begin{align*}
\dot{c}_m + (N+1-m) D_0 \dot{c}_{m-1} + [(N-m) D_1 + E] \dot{c}_m - m D_2 \dot{c}_{m+1} \\
- (N+1-m) (m-1) A_1 c_{m-1} - m [(2N-m-1) A_2 + B] c_m + m (m+1) A_3 c_{m+1} = 0.
\end{align*} \tag{2.3}
\]

Hence the \textit{solution} of this system is detailed by the following

\textbf{Proposition 2.1:}

\[
c_m(t) = \sum_{j=1}^{N} \gamma_j \hat{c}_m^{(j)} \exp (\lambda_j t), \tag{2.4}
\]

where the \( 2N \) numbers \( \gamma_j \) are \textit{a priori} arbitrary (or can be \textit{a posteriori} fixed, in the context of the initial-value problem, by imposing consistency with the \( 2N \) initial data \( c_m (0) \), \( \dot{c}_m (0) \)); while the \( 2N \) quantities \( \lambda_j \), respectively the \( 2N^2 \) components \( \hat{c}_m^{(j)} \) of the \( 2N \) vectors \( \hat{c}^{(j)} \equiv \left( \hat{c}_1^{(j)}, \hat{c}_2^{(j)}, \ldots , \hat{c}_N^{(j)} \right) \), are the \( 2N \) \textit{eigenvalues}, respectively the \( 2N \) \textit{eigenvectors}, of the generalized eigenvalue problem

\[
(\lambda^2 I + \lambda \ D + A) \hat{c}^{(j)} = 0, \tag{2.5a}
\]

with the two \( (N \times N) \)-matrices \( A \) and \( D \) defined componentwise as follows:

\[
A_{m,n} = - (N+1-m) \ (m-1) A_1 \delta_{m,n+1} - m \ [(2N-m-1) A_2 + B] \delta_{m,n} + m (m+1) A_3 \delta_{m,n-1}, \tag{2.5b}
\]

\[
D_{m,n} = (N+1-m) D_0 \delta_{m,n+1} + [(N-m) D_1 + E] \delta_{m,n} - m D_2 \delta_{m,n-1}. \tag{2.5c}
\]

\textbf{Notation 2.3.} \( N \)-vectors respectively \( (N \times N) \)-matrices are denoted by underlined lower-case respectively upper-case Latin letters; their components respectively their elements are denoted by the corresponding (not underlined) letters. \( I \) is the \textit{unit} \( (N \times N) \)-matrix, \( I_{nn} = \delta_{nn} \); in the following we will generally omit to write out this matrix. Of course above and hereafter \( \delta_{nm} \equiv \delta_{m,n} \) is the standard Kronecker symbol, \( \delta_{nn} = 1 \) if \( n = n \), \( \delta_{mn} = 0 \) if \( m \neq n \).
Remark 2.1. Clearly formula (2.4) holds as written only if all the $2N$ eigenvalues $\lambda_j$ are different among themselves. It continues to hold in a limiting sense if two or more eigenvalues coincide, in which case clearly the time evolution in the right-hand side of (2.4) also features terms depending on integer powers of the time variable $t$. ■

Remark 2.2. If the real parts of the $2N$ eigenvalues $\lambda_j$ vanish and they are all different among themselves, it is plain from (2.2b) with (2.4) that, for all time, the coordinates $z_n(t)$ remain confined to a finite region of the complex $z$-plane. ■

Remark 2.3. If

$$A_1 = D_0 = 0 \quad \text{or} \quad A_3 = D_2 = 0 \quad (2.6)$$

the two matrices $A$ and $D$ are triangular; the $2N$ eigenvalues $\lambda_j$ are then given by the explicit formulas

$$\lambda_m^{(\pm)} = -\frac{1}{2} \left[ (N - m) \, D_1 + E \pm \Delta_m \right], \quad (2.7a)$$

$$\Delta_m = \left\{ (N - m) \, D_1 + E \right\}^2 + 4 \, m \, [(2N - m - 1) \, A_2 + B]^{1/2} \quad (2.7b)$$

where, say, $\lambda_{2m-1} = \lambda_m^{(+)}$, $\lambda_{2m} = \lambda_m^{(-)}$, $m = 1, \ldots, N$. Clearly in this case all these eigenvalues $\lambda_j$ are imaginary if there hold the following two conditions on the parameters: (i) $D_1$ and $E$ are imaginary (or vanishing); (ii) $A_2$ and $B$ are real and have values which entail that $\Delta_m$ is imaginary (for $m = 1, 2, \ldots, N$). But no assignment of the parameters—other than the special one described in the following Proposition 2.2—yields $2N$ eigenvalues $\lambda_j$ which are all different among themselves and are all integer multiples of the same nonvanishing imaginary number $i \, \omega$. ■

In the following we shall focus on models satisfying one of the two conditions (2.6), in order to take advantage of the explicit expressions (2.7) of the eigenvalues characterizing the time evolution of the dynamical system (2.3). Additional conditions which guarantee that this time evolution is actually isochronous are detailed by the following

**Proposition 2.2.** If one of the two conditions (2.6) holds (see Remark 2.3), and there moreover hold the following 4 additional conditions on the parameters: (i) the real parts of $D_1$ and $E$ vanish,

$$D_1 = i \delta, \quad E = i \eta \quad (2.8a)$$

with $\delta$ and $\eta$ two real parameters; (ii) the remaining parameters are all real; (iii) $A_2 > -\delta^2/4$ so that the quantity

$$\omega = \sqrt{A_2 + \delta^2/4} \quad (2.8b)$$

is as well real (and, by convention, positive, $\omega > 0$); (iv) there holds the following relation among the 5 parameters $N$, $\delta$, $\eta$, $A_2$, $B$:

$$(N \, \delta + \eta) \, (\delta + 2 \, \omega) + 2 \, [(2N - 1) \, A_2 + B] = 0 \quad (2.8c)$$

then the expression (2.7) of the $2N$ eigenvalues becomes

$$\lambda_m^{(\pm)} = i \left[ (N \, \delta + \eta) \left( \pm \frac{1}{2} \right) + m \left( \frac{\delta}{2} \pm \omega \right) \right]. \quad (2.9)$$
If, moreover,

$$\delta = \frac{p_1}{q_1} \omega , \quad \eta = \frac{p_2}{q_2} \omega ,$$

(2.10)

with $q_1, q_2$ two arbitrary positive integers, $p_1, p_2$ two arbitrary integers, and $p_1 \neq \pm 2q_1$ (so that $\delta \pm 2\omega \neq 0$), and the eigenvalues (which then read)\(\lambda_m^{(\pm)} = i \left[(N p_1 q_2 + p_2 q_1) (\pm 1 - 1) + m (p_1 \pm 2q_1) q_2\right] \Omega,\)

(2.11a)

$$\Omega = \frac{\omega}{2 q_1 q_2}$$

(2.11b)

are all different among themselves, then clearly all the coefficients $c_m(t)$, hence as well the polynomial (2.2b), evolve in time periodically with the same period,

$$c_m(t \pm T) = c_m(t) , \quad \psi(z, t \pm T) = \psi(z, t) , \quad T = \frac{2\pi}{\Omega} .$$

(2.12)

Hence (see (2.2b)) in this special case all the coordinates $z_n(t)$ also evolve in time periodically with the same period $T$ (or possibly, some of them, with a period which is a generally small, integer multiple of $T$, due to the possibility that over the time evolution some of the zeros of the polynomial $\psi(z, t)$ exchange their roles; for a discussion of this possibility, including a justification of the assertion that the integer multiple in question is “generally small”, see [5]). Anyway, in this special case the $N$-body problem (2.1) is isochronous. ■

Remark 2.4. Note, however, that the phenomenon described at the end of Proposition 2.2 cannot happen for motions sufficiently close (in particular, infinitesimally close) to equilibrium configurations, such as those considered below. ■

The next development is to consider the $N$-body problem (2.1) in the immediate vicinity of its equilibrium configurations, which are clearly characterized by $N$ coordinates $\bar{z}_n$ satisfying the $N$ nonlinear algebraic equations

$$(N - 1) A_1 - B \bar{z}_n + 2 (N - 1) A_3 \bar{z}_n^2$$

$$= 2 \left(A_1 + A_2 \bar{z}_n + A_3 \bar{z}_n^2\right) \bar{z}_n \sum_{\ell=1, \ell\neq n}^N (\bar{z}_n - \bar{z}_\ell)^{-1} .$$

(2.13)

Clearly the $N$ equilibrium coordinates $\bar{z}_n$ can be equivalently characterized—see (2.2)—as the $N$ zeros of the polynomial

$$\bar{\Psi}(z) = \prod_{n=1}^N (z - \bar{z}_n) = z^N + \sum_{m=1}^N \bar{c}_m z^{N-m} ,$$

(2.14a)

satisfying the time-independent version of the PDE (2.2a), i.e. the ODE

$$\left(A_1 + A_2 z + A_3 z^2\right) z \bar{\Psi}' - \left[(N - 1) A_1 - B z + 2 (N - 1) A_3 z^2\right] \bar{\Psi}'$$

$$-N \left[(N - 1) A_2 - A_3 z\right] \bar{\Psi} = 0 .$$

(2.14b)

Notation 2.4. Here and hereafter appended primes denote derivatives with respect to the argument of the function they are appended to. ■
To investigate the behavior of the \(N\)-body problem (2.1) in the immediate vicinity of its equilibria one of course sets

\[ z_n(t) = \bar{z}_n + \varepsilon \zeta_n(t) , \tag{2.15} \]

with the \(N\) time-independent coordinates \(\bar{z}_n\) characterizing the equilibria and \(\varepsilon\) infinitesimal. The equations of motion (2.1) get thereby linearized:

\[ \ddot{\zeta} + D \dot{\zeta} + A \zeta = 0 , \tag{2.16} \]

with the two \(N \times N\) matrices \(D\) and \(A\) obtained—in terms of the equilibrium coordinates \(\bar{z}_n\)—in a standard manner (see below) from the \(N\)-body problem (2.1).

The general solution of this system of \(N\) linear equations of motion—characterizing the behavior of the \(N\)-body problem (2.1) in the immediate neighborhood of its equilibria, see (2.15)—reads

\[ \zeta_n(t) = \sum_{j=1}^{2N} \eta_j \tilde{\phi}_n^{(j)} \exp(\bar{\lambda}_j t) \tag{2.17a} \]

where the \(2N\) numbers \(\eta_j\) are \textit{a priori} arbitrary (or can be \textit{a posteriori} fixed, in the context of the initial-value problem, by imposing consistency with the \(2N\) initial data \(\zeta_n(0), \dot{\zeta}_n(0)\)), while the \(2N\) quantities \(\bar{\lambda}_j\), respectively the \(2N^2\) components \(\tilde{\phi}_n^{(j)}\) of the \(2N\) \(N\)-vectors \(\tilde{\phi}_n^{(j)} \equiv \left( \tilde{\phi}_1^{(j)}, \tilde{\phi}_2^{(j)}, \ldots, \tilde{\phi}_N^{(j)} \right)\), are the \(2N\) eigenvalues, respectively the \(2N\) eigenvectors, of the generalized eigenvalue problem

\[ (\bar{\lambda}^2 + D \bar{\lambda} + A) \tilde{\phi}_n^{(j)} = 0 \tag{2.17b} \]

(with analogous remarks to those made above after (2.5), see in particular Remark 2.1).

It is then plain that these \(2N\) eigenvalues \(\bar{\lambda}_j\)—which characterize the behavior of the \(N\)-body problem (2.1) in the immediate neighborhood of its equilibria—coincide with the \(2N\) eigenvalues \(\lambda_j\) of the generalized eigenvalue problem (2.5a), which, via (2.2b) and (2.4), characterize the general behavior of the solutions \(z_n(t)\) of the \(N\)-body problem (2.1):

\[ \bar{\lambda}_j = \lambda_j , \quad j = 1, 2, \ldots, 2N . \tag{2.18} \]

Hence—in all cases when the \(2N\) eigenvalues \(\bar{\lambda}_j\) can be explicitly computed, see above—one can similarly assert that the eigenvalues \(\bar{\lambda}_j\) are known; and in the special cases when the \(N\)-body problem (2.1) is isochronous—hence the \(2N\) eigenvalues \(\bar{\lambda}_j\) are, up to a common integer factor, integer numbers, see (2.11)—the same Diophantine assertion can be made for the \(2N\) eigenvalues \(\bar{\lambda}_j\) of the generalized eigenvalue problem (2.17b).

These shall be the main findings of the present paper. To obtain them we must still find the two \(N \times N\) matrices \(\bar{D}\) and \(\bar{A}\). To this end we now take again advantage of the results of [1], in particular those concerning the equilibria of the \(N\)-body problem (2.1).

We already saw that the numbers \(\bar{z}_n\) providing the \(N\) coordinates characterizing the equilibria of the \(N\)-body problem (2.1) coincide with the \(N\) zeros of the (monic) polynomial \(\Psi(z)\), see (2.14a), satisfying the ODE (2.14b). This ODE—featuring, in addition to the arbitrary positive integer \(N\), the \(4 \text{ a priori} \) arbitrary parameters \(A_1, A_2, A_3, B\) (any single one of which can of course be divided away, unless it vanishes)—is generally not of hypergeometric type hence not reducible to the ODEs characterizing the classical polynomials (except in special cases, see below). Yet it can be rather
Diophantine Properties Associated to the Equilibrium Configurations of an Isochronous N-Body Problem

explicitly solved in terms of quadratures over elementary functions (see [1]), as detailed by the following

**Proposition 2.3.** The general solution of the ODE (2.14b) is given by the formula

\[
\psi(z) = z^N \varphi(z),
\]

(2.19)

with

\[
\varphi(z) = \rho + \gamma \int z^v (x - z_+)^{v_+} (x - z_-)^{v_-} dz,
\]

(2.20a)

or equivalently

\[
\varphi(z) = \rho + \gamma \int_{1/z} \frac{dy}{z} (1 - z_+ y)^{v_+} (1 - z_- y)^{v_-}.
\]

(2.20b)

The parameters \(v, v_+\) and \(z_+\) are related to the 5 parameters \(N, A_1, A_2, A_3, B\) by the following formulas:

\[
v = -(N + 1), \quad v_+ + v_- = N - 1,
\]

(2.21a)

\[
b = -\frac{1}{2} \left[ 3(N-1) a_2 + (v_+ - v_-) \Delta \right],
\]

(2.21b)

\[
\Delta^2 = a_2^2 - 4 a_1,
\]

(2.21c)

\[
z_\pm = \frac{-a_2 \pm \Delta}{2},
\]

(2.21d)

where

\[
a_1 = \frac{A_1}{A_3}, \quad a_2 = \frac{A_2}{A_3}, \quad b = \frac{B}{A_3}.
\]

(2.21e)

Of course this solution holds as written for generic values of the 5 parameters \(N, A_1, A_2, A_3, B\); special cases are treated separately below. ■

**Notation 2.5.** In (2.20) and throughout \(\rho\) and \(\gamma\) denote two arbitrary (integration) constants, without implying that they have the same values in the different equations where they appear. ■

These findings imply that there holds (see [1]) the following

**Proposition 2.4.** In the generic case (implying in particular that \(A_3 \neq 0, A_1 \neq 0\) and \(A_2^2 - 4A_1A_3 \neq 0\)) the single (Diophantine) restriction on the 5 parameters \(N, A_1, A_2, A_3, B\) reading

\[
B = -\frac{1}{2} \left[ 3(N-1) A_2 + (N + 1 - 2k) \left( A_2^2 - 4A_1A_3 \right)^{1/2} \right]
\]

(2.22)

with \(k\) an arbitrary integer in the range \(1 \leq k \leq N\), is necessary and sufficient to guarantee that the general solution of the ODE (2.14b) be polynomial, indeed (up to an arbitrary overall multiplicative
constant, assigned here so as to make the polynomial $\bar{\psi}(z)$ monic this general solution reads

$$\bar{\psi}(z) = z^N + \gamma \sum_{m=1}^{N} (\bar{c}_m z^{N-m}) ,$$

(2.23a)

$$\bar{c}_m = \frac{(-1)^m}{m} \sum_{j=\max(0,m-k)}^{\min(N-k,m-1)} \binom{N-k}{j} \binom{k-1}{m-1-j} z^{j} z^{m-1-j} ,$$

(2.23b)

where $\gamma$ is an arbitrary (integration) constant. Here notation (2.21) is employed, and the standard notation for the binomial coefficient,

$$\binom{p}{q} \equiv \frac{p!}{q!(p-q)!} ,$$

(2.23c)

where $p$ is an arbitrary nonnegative integer and $q$ is an integer in the range $0 \leq q \leq p$. ■

Remark 2.5. Note that the generic case considered in this Proposition 2.4 violates the conditions (2.6) allowing an explicit evaluation of the eigenvalues $\lambda_j$, see above Remark 2.3. ■

We therefore hereafter focus on other, less generic, cases in which the general solution of the ODE (2.14b) is polynomial: see the following Proposition 2.5, reporting the findings of [1] concerning all the equilibria of the $N$-body problem (2.1), hence the solutions $\bar{z}_n$ of the set of $N$ algebraic equations (2.13) which coincide with the $N$ zeros $\bar{z}_n$ of the polynomial solutions, see (2.14a), of the ODE (2.14b).

Proposition 2.5. The following conditions on the 5 parameters $N, A_1, A_2, A_3, B$ are necessary and sufficient to imply that the general solution of the ODE (2.14b) be a polynomial of degree $N$, hence they are as well necessary and sufficient to imply that the $N$-body problem (2.1) feature a (time-independent) equilibrium solution, the configuration of which is indeed provided by the $N$ zeros $\bar{z}_n$ of this polynomial (and it is a genuine equilibrium configuration, featuring $N$ coordinates $\bar{z}_n$ all different among themselves).

Case (i) (the most generic):

$$A_3 \neq 0, \quad A_1 \neq 0, \quad \text{and} \quad A_2^2 - 4 A_1 A_3 \neq 0 ,$$

(2.24a)

$$B = -\frac{1}{2} \left[ 3 (N-1) A_2 + (N-1-2k) (A_2^2 - 4 A_1 A_3)^{1/2} \right] ,$$

(2.24b)

with $k$ an arbitrary integer in the range $1 \leq k \leq N$ (see (2.22)). Then the equilibrium configuration is identified by the $N$ zeros $\bar{z}_n$ of the polynomial (2.23), see Proposition 2.4 (of course, for generic values of the arbitrary parameter $\gamma$; excluding exceptional values—such as $\gamma = 0$—which are incompatible with the assumption that the $N$ zeros $\bar{z}_n$ of the polynomial (2.23) be all different among themselves). Note that this case violates the conditions (2.6) allowing to evaluate explicitly the eigenvalues $\lambda_j$ (see Remark 2.3).

Case (ii):

$$A_3 \neq 0, \quad A_2 \neq 0, \quad A_1 = 0 .$$

(2.25)

There are then two subcases.
Diophantine Properties Associated to the Equilibrium Configurations of an Isochronous N-Body Problem

Case (ii.a):

\[ B = -(N - 1) A_2 \ , \tag{2.26a} \]

with the explicit expression of the \( N \) equilibrium coordinates \( \bar{z}_n \) reading

\[ \bar{z}_n = -\frac{A_2}{A_3} \left[ 1 + \gamma \exp \left( \frac{2 \pi i n}{N} \right) \right]^{-1} . \tag{2.26b} \]

Case (ii.b):

\[ B = -N A_2 , \tag{2.27a} \]

with the explicit expression of the \( N \) equilibrium coordinates \( \bar{z}_n \) reading

\[ \bar{z}_n = -\frac{A_2}{A_3} \left[ 1 + \gamma \exp \left( \frac{2 \pi i n}{N-1} \right) \right]^{-1} , \quad n = 1,2,\ldots,N-1 ; \quad \bar{z}_n = 0 . \tag{2.27b} \]

Let us repeat that, here and throughout, \( i \) is of course the imaginary unit, \( i^2 = -1 \); while here \( \gamma \) is a constant parameter, arbitrary except for the following restrictions: \( \gamma \neq 0, \gamma \neq -1, \) and moreover in Case (ii.a) \( \gamma \neq 1 \) if \( N \) is even, in Case (ii.b) \( \gamma \neq 1 \) if \( N \) is odd.

Case (iii):

\[ A_3 \neq 0, \quad A_2 \neq 0, \quad A_1 = \frac{A_2^2}{4 A_3} \neq 0 , \tag{2.28a} \]

\[ B = -\frac{3 (N - 1) A_2}{2} , \tag{2.28b} \]

with the explicit expression of the \( N \) equilibrium coordinates \( \bar{z}_n \) reading

\[ \bar{z}_n = -\frac{A_2}{2 A_3} \left[ 1 + \gamma \exp \left( \frac{2 \pi i n}{N} \right) \right]^{-1} , \quad n = 1,2,\ldots,N , \tag{2.28c} \]

where \( i \) is again the imaginary unit and \( \gamma \) a constant parameter, arbitrary except for the following restrictions: \( \gamma \neq 0, \gamma \neq -1, \) and \( \gamma \neq 1 \) if \( N \) is even. Note that this case violates the conditions (2.6) allowing to evaluate explicitly the eigenvalues \( \lambda_j \) (see Remark 2.3).

Case (iv):

\[ A_3 = 0, \quad A_2 \neq 0, \quad A_1 \neq 0 . \tag{2.29a} \]

In this case the parameter \( B \) must satisfy the Diophantine condition

\[ B = (-2 N + 1 + k) A_2 . \tag{2.29b} \]

Here \( k \) is an integer in the range \( 1 \leq k \leq N \), and the \( N \) equilibrium coordinates \( \bar{z}_n \) are the \( N \) zeros of the monic polynomial

\[ \Psi(z) = z^N + \gamma \sum_{m=k}^{N} (\bar{e}_m z^{N-m}) , \tag{2.30a} \]
with $\gamma$ an arbitrary (nonvanishing) constant parameter and
\begin{equation}
\tilde{\epsilon}_m = \left( N - k \right) \frac{a^{m-k}}{m}, \quad a = \frac{A_1}{A_2}, \quad m = k, k+1, \ldots, N.
\end{equation}
(2.30b)

Equivalently, these $N$ zeros are those of the following function:
\begin{equation}
\varphi(z) = 1 + \gamma P_n^{(1-k, -N-1+k)} \left( \frac{z+2a}{z} \right), \quad a = \frac{A_1}{A_2},
\end{equation}
(2.31)
with $\gamma$ an arbitrary (nonvanishing) constant, $P_n^{(\alpha, \beta)}(x)$ the standard (conveniently renormalized, to make it monic) Jacobi polynomial (see for instance [6]) and $k$ again an arbitrary integer in the range $1 \leq k \leq N$. Note that this is in fact a para-Jacobi polynomial [8]; and it also easily seen that the polynomial $\bar{\psi}(z)$ coincides itself with the following para-Jacobi polynomial [8]:
\begin{equation}
\bar{\psi}(z) = a_2^n N p_n \left( 0, k - 1; \gamma, 1 + \frac{2z}{a} \right).
\end{equation}
(2.32)

Case (v):
\begin{equation}
A_3 = A_2 = 0, \quad A_1 \neq 0.
\end{equation}
(2.33)

Then if $B$ does not vanish, $B \neq 0$, the basic assumption that $\bar{\psi}(z)$, see (2.14a), be a polynomial of degree $N$ in $z$ is not compatible with the ODE (2.14b), hence in this case there is no equilibrium configuration of the $N$-body problem (2.1); while if $B$ does vanish, $B = 0$, then (up to an arbitrary overall multiplicative constant, which has been fixed here to make the polynomial $\bar{\psi}(z)$ monic)
\begin{equation}
\bar{\psi}(z) = z^N + \gamma
\end{equation}
(2.34a)
with $\gamma$ an arbitrary (nonvanishing) constant, entailing for its $N$ zeros the simple formula
\begin{equation}
\tilde{z}_n = \gamma \exp \left( \frac{2\pi i n}{N} \right).
\end{equation}
(2.34b)

The arbitrary constants $\gamma$ in (2.34a) and (2.34b) are of course different, being related to each other in an obvious manner.

Case (vi):
\begin{equation}
A_3 = A_2 = A_1 = 0.
\end{equation}
(2.35)

In this case if $B$ does not vanish, $B \neq 0$, the $N$-body problem (2.1) has no genuine equilibrium configuration; while, if instead also $B$ vanishes, $B = 0$, then any configuration $\tilde{z}_n$ with the $N$ coordinates $\tilde{z}_n$ all different among themselves is a genuine equilibrium configuration of the $N$-body problem (2.1) (but it is easily seen that in this case the $N$-body problem (2.1) is not altogether isochronous, because $N$ of the $2N$ eigenvalues $\lambda_j$ vanish). ■

Remark 2.6. Let us re-emphasize that we characterize an equilibrium configuration as genuine if the $N$ coordinates $\tilde{z}_n$ are all different among themselves, $\tilde{z}_n \neq \bar{z}_m$ if $n \neq m$; and that in Proposition 2.5—indeed, throughout this paper—we focus on such equilibria, to avoid the ambiguities otherwise caused by the denominators present in the right-hand sides of the equations (2.1) and (2.13). Readers interested in cases when this restriction does not hold are referred to Appendix A of [1]. ■
Diophantine Properties Associated to the Equilibrium Configurations of an Isochronous $N$-Body Problem

Let us conclude this Section 2 by listing the cases of interest in the present paper. They are those such that the $N$-body problem (2.1) possesses a genuine equilibrium configuration (see Proposition 2.5) and moreover allows to compute explicitly the eigenvalues $\lambda_j$ (see Remark 2.3). We also highlight the conditions which are sufficient to guarantee the isochrony of the $N$-body problem (see Proposition 2.2, with $q = q_1$, $p = p_1$, $p \neq 2q$, $\bar{\omega} = \omega/(2q)$).

Case 1.

\begin{align}
A_1 &= D_0 = 0, \quad A_2 = (4q^2 - p^2) \bar{\omega}^2, \quad A_3 \neq 0, \\
B &= -(N-1)A_2, \quad D_1 = 2i p \bar{\omega}, \quad E = -N(2q + p) i \bar{\omega}.
\end{align}

(2.36a)

Note that $D_2$ remains unrestricted. Here $p$, $q$ and $\bar{\omega}$ are 3 a priori arbitrary parameters. The corresponding eigenvalues $\lambda_m^{(\pm)}$ are

$$
\lambda_m^{(\pm)} = i \left[ N(p - 2q) \left( \frac{\pm 1 - 1}{2} \right) + m(p \pm 2q) \right] \bar{\omega}.
$$

(2.36b)

And the corresponding equilibrium coordinates read (see (6.57))

$$
\bar{z}_n = - \frac{A_2}{A_3 (1 + \gamma u_n)} = - \frac{A_2 w_n}{A_3}.
$$

(2.36c)

These results follow via elementary algebra from the relevant results of Proposition 2.2 and of Case (ii.a) of Proposition 2.5. The corresponding $N$-body model (2.1) is isochronous iff $p$ and $q$ are integers and $p \pm 2q \neq 0$.

Case 2.

\begin{align}
A_1 &= D_0 = 0, \quad A_2 = (4q^2 - p^2) \bar{\omega}^2, \quad A_3 \neq 0, \\
B &= -NA_2, \quad D_1 = 2i p \bar{\omega}, \quad E = -[(N-1)2q + (N+1)p]i \bar{\omega}.
\end{align}

(2.37a)

Again, $D_2$ remains unrestricted and $p$, $q$, $\bar{\omega}$ are 3 a priori arbitrary parameters. The corresponding eigenvalues $\lambda_m^{(\pm)}$ now read

$$
\lambda_m^{(\pm)} = i \left[ N(p - 2q + 2) \left( \frac{\pm 1 - 1}{2} \right) + m(p \pm 2q) \right] \bar{\omega}.
$$

(2.37b)

And the corresponding equilibrium coordinates read (see (6.57) with $u_n = \exp \left( \frac{2\pi i n}{N-1} \right)$)

$$
\bar{z}_n = - \frac{A_2}{A_3 (1 + \gamma u_n)} = - \frac{A_2 w_n}{A_3}, \quad n = 1, \ldots, N-1; \quad \bar{z}_n = 0.
$$

(2.37c)

These results follow via elementary algebra from the relevant results of Proposition 2.2 and of Case (ii.b) of Proposition 2.5. Again, the corresponding $N$-body model (2.1) is isochronous iff $p$ and $q$ are integers and $p \pm 2q \neq 0$. 

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Case 3.

\[ A_1 \neq 0 , \quad A_2 = (4q^2 - p^2) \tilde{\omega}^2 , \quad A_3 = D_2 = 0 , \quad D_1 = 2i p \tilde{\omega} , \]
\[ B = (-2N + k + 1) A_2 , \quad E = -[k (2q - p) + 2N p] i \tilde{\omega} . \]  

(2.38a)

Here \( k \) is any integer in the range from 1 to \( N \), \( k = 1, 2, ..., N \), while \( D_0 \) is unrestricted and \( p, q, \tilde{\omega} \) are \( a \ priori \) arbitrary parameters. The corresponding eigenvalues \( \lambda_m^{(\pm)} \) now read

\[ \lambda_m^{(\pm)} = i \left[ k (p - 2q) \left( \frac{\pm 1 - 1}{2} \right) + m (p \pm 2q) \right] \tilde{\omega} . \]

(2.38b)

And the corresponding equilibrium coordinates \( \tilde{z}_n \) are the \( N \) zeros of the para-Jacobi polynomial \( p_n (0, k - 1; \gamma; 1 + \frac{2 + k}{\omega}) \), see (2.32) and, more explicitly, (2.30). These results follow via elementary algebra from the relevant results of Proposition 2.2 and of Case (iv) of Proposition 2.5. Again, the corresponding \( N \)-body model (2.1) is isochronous iff \( p \) and \( q \) are integers and \( p \pm 2q \neq 0 \).

Case 4.1.

\[ A_1 \neq 0 , \quad A_2 = A_3 = B = D_2 = 0 , \]
\[ D_1 = 2i \omega , \quad E = -2Ni \omega . \]  

(2.39a)

and \( D_0 \) unrestricted. The corresponding eigenvalues \( \lambda_m^{(\pm)} \) now read

\[ \lambda_m^{(+)} = 2i m \omega , \quad \lambda_m^{(-)} = 0 . \]

(2.39b)

Case 4.2

\[ A_1 \neq 0 , \quad A_2 = A_3 = B = D_2 = 0 , \]
\[ D_1 = -2i \omega , E = i p \omega . \]  

(2.40a)

Here \( D_0 \) is unrestricted and \( p \) is an arbitrary rational number. The corresponding eigenvalues \( \lambda_m^{(\pm)} \) now read

\[ \lambda_m^{(+)} = 0 , \quad \lambda_m^{(-)} = i (2N - p - 2m) \omega . \]

(2.40b)

As for the corresponding equilibrium coordinates, they read, in both these two Cases 4.1 and 4.2, as follows (see (6.57)):

\[ \tilde{z}_n = \gamma u_n . \]

(2.41)

Clearly in these two cases the corresponding \( N \)-body model is not isochronous. These results follow via elementary algebra from the relevant results of Proposition 2.2 and of Case (v) of Proposition 2.5.
Case 5.1.

\[ A_1 = A_2 = A_3 = B = 0, \quad D_1 = 2 i \omega, \quad E = -2 N i \omega, \]  
(2.42a)

here \( D_0 = 0 \) or \( D_2 = 0 \). The corresponding eigenvalues \( \lambda_{m}^{(\pm)} \) now read

\[ \lambda_{m}^{(+)} = 2 i m \omega, \quad \lambda_{m}^{(-)} = 0. \]  
(2.42b)

Case 5.2.

\[ A_1 = A_2 = A_3 = B = 0, \quad D_1 = -2 i \omega, \quad E = i p \omega, \]  
(2.43a)

here \( D_0 = 0 \) or \( D_2 = 0 \) and \( p \) is an arbitrary rational number. The corresponding eigenvalues \( \lambda_{m}^{(\pm)} \) now read

\[ \lambda_{m}^{(+)} = 0, \quad \lambda_{m}^{(-)} = i (2 N - p - 2 m) \omega. \]  
(2.43b)

And the corresponding equilibrium coordinates \( \bar{z}_n \) are in both cases \( N \), quite arbitrary numbers (all different among themselves, for a genuine equilibrium).

These results follow via elementary algebra from the relevant results of Proposition 2.2 and of Case (vi) of Proposition 2.5. Again, the corresponding \( N \)-body model (2.1) is not isochronous.

Finally let us recall that we excluded here the Cases (i) and (iii) of Proposition 2.5 because they violate the conditions allowing to compute explicitly the eigenvalues \( \lambda_j \) (see Remark 2.3).

3. Main findings

In this section the main findings of the present paper are reported; and tersely commented upon at the end.

Proposition 3.1. Let the \( N \times N \) matrices \( \bar{A} \) and \( \bar{D} \) be defined, componentwise, as follows:

\[ \bar{A}_{nn} = -\frac{(4 q^2 - p^2) \bar{\omega}^2}{6} (N^2 - 1) \left( 1 + \frac{1}{\bar{u}_n} \right), \]  
(3.44a)

\[ \bar{A}_{nm} = -2 (4 q^2 - p^2) \bar{\omega}^2 \frac{\bar{u}_n}{1 + \bar{u}_n} \left( \frac{1 + \bar{u}_m}{\bar{u}_n - \bar{u}_m} \right)^2, \quad n \neq m; \]  
(3.44b)

\[ \bar{D}_{nn} = \frac{(N - 1) (-2 i p \bar{\omega} + \alpha)}{2 \bar{u}_n} - (2 N q + p) i \bar{\omega}, \]  
(3.44c)

\[ \bar{D}_{nm} = \left( -2 i p \bar{\omega} + \frac{\alpha}{1 + \bar{u}_n} \right) \frac{1 + \bar{u}_m}{\bar{u}_n - \bar{u}_m}, \quad n \neq m. \]  
(3.44d)

Here \( \bar{u}_n = \gamma u_n = \gamma \exp(2 \pi i n/N) \) (see (6.57a)), with \( \gamma, \bar{\omega} \) (see (2.36b)) and \( \alpha = D_2 a_2 \) three arbitrary parameters and \( p \) and \( q \) two arbitrary integers (see (2.10)). Then the \( 2N \) roots of the determinantal
polynomial $P_{2N}(\lambda) \equiv \det (\lambda^2 + \lambda \overline{D} + \overline{A})$ are $\lambda_j = i\tilde{\omega}s_j$ with the $2N$ numbers $s_j$ all integers, indeed

$$P_{2N}(\lambda) \equiv \det(\lambda^2 + \lambda \overline{D} + \overline{A}) = \prod_{m=1}^{N} (\lambda - \lambda_m^{(+)}) (\lambda - \lambda_m^{(-)})$$  \hspace{1cm} (3.45a)

with

$$\lambda_m^{(\pm)} = i \left[ N (p - 2) \left( \frac{\pm 1 - 1}{2} \right) + m (p \pm 2) \right] \tilde{\omega}.$$  \hspace{1cm} (3.45b)

Of course these formulas, (3.44) and (3.45), also hold for arbitrary (not integer) values of the two parameters $p$ and $q$. ■

These results follow from Case 1 (as detailed in the last part of the preceding Section 2), as shown in the following Section 4.

The results which follow from Case 2 coincide with those of Proposition 3.1, up to a shift of the parameter $N$; hence they are not reported.

**Proposition 3.2.** Let the $N \times N$ matrices $\tilde{A}$ and $\tilde{D}$ be defined, componentwise, as follows:

$$\tilde{A}_{nm} = -\frac{N^2 - 1}{6} A_{11}; \quad \tilde{A}_{nm} = -2 A_1 \frac{u_n}{(u_n - u_m)^2}, \quad n \neq m;$$  \hspace{1cm} (3.46a)

$$D_{nm} = E + \frac{N - 1}{2} \left( D_1 + \frac{D_0}{u_n} \right)$$  \hspace{1cm} (3.46b)

$$D_{nm} = \frac{D_0 + D_1}{u_n - u_m}, \quad n \neq m.$$  \hspace{1cm} (3.46c)

Here again $u_n = \gamma u_n = \gamma \exp(2 \pi i n/N)$ (see (6.57a)), with $\gamma$, $D_0$ and $A_1$ three arbitrary parameters. As for $D_1$ and $E$, either $D_1 = 2 i \omega$, $E = -2 N i \omega$ (corresponding to Case 4.1 of Section 2), or $D_1 = -2 i \omega$, $E = i p \omega$, with $\omega$ an arbitrary parameter and $p$ an arbitrary integer (corresponding to Case 4.2 of Section 2). Then (in both cases) the $2N$ roots of the determinantal polynomial $P_{2N}(\lambda) \equiv \det (\lambda^2 + \lambda \overline{D} + \overline{A})$ are $\lambda_j = i\tilde{\omega}s_j$ with the $2N$ numbers $s_j$ all integers (actually, $N$ of them vanishing), indeed

$$P_{2N}(\lambda) \equiv \det(\lambda^2 + \lambda \overline{D} + \overline{A}) = \prod_{m=1}^{N} (\lambda - \lambda_m)$$  \hspace{1cm} (3.47)

with

$$\lambda_m = \lambda_m^{(+)} = 2 i m \omega,$$  \hspace{1cm} (3.48a)

when

$$D_1 = 2 i \omega, \quad E = -2 N i \omega,$$  \hspace{1cm} (3.48b)

and

$$\lambda_m = \lambda_m^{(-)} = i (2 N - p - 2m) \omega$$  \hspace{1cm} (3.48c)

when

$$D_1 = -2 i \omega, \quad E = i p \omega.$$  \hspace{1cm} (3.48d)
Of course the last two formulas, (3.48c) and (3.48d), also hold for arbitrary (not integer) values of the parameter \( p \). ■

These results follow from Case 4 (as detailed in the last part of the preceding Section 2), as shown in the following Section 4.

**Proposition 3.3**. Let the \( N \times N \) matrices \( \tilde{A} \) and \( \tilde{D} \) be defined, componentwise, as follows:

\[
\tilde{A}_{nm} = \frac{(4q^2 - p^2)}{6} \tilde{\omega}^2 \left( (k^2 - 1) x_n^2 + 2 (k - 1) (2N - k + 1) x_n \right. \\
+ 6 (2N - k - 1) + 4 (N - 1) (N - k - 1) + (k - 1)^2) \left( 1 - x_n^2 \right)^{-1}, \tag{3.49a}
\]

\[
\tilde{A}_{nm} = 2 \left( 4q^2 - p^2 \right) \tilde{\omega}^2 \frac{1 - x_n^2}{(x_n - x_m)^2}, \quad n \neq m; \tag{3.49b}
\]

\[
\tilde{D}_{nn} = - \left[ k (2q - p) + 2N p \right] i \tilde{\omega} \\
- \left[ \alpha + 2 i p \tilde{\omega} (x_n - 1) \right] \frac{k - 1 + (2N - k - 1) x_n}{2 (1 - x_n^2)}, \tag{3.49c}
\]

\[
\tilde{D}_{nm} = [\alpha + 2 i p \tilde{\omega} (x_n - 1)] (x_n - x_m)^{-1}, \quad n \neq m. \tag{3.49d}
\]

Here \( x_n \) are the \( N \) zeros of the para-Jacobi polynomial \( p_n(0, k; \gamma, x) \) [8], with \( k \) an arbitrary integer in the range \( 1 \leq k \leq N \), and the parameter \( \gamma \) an arbitrary number; and as well arbitrary are the two parameters \( \tilde{\omega} \) (see (2.36b)) and \( \alpha = 2D_0/\tilde{\omega} \), and the two integers \( p \) and \( q \). Then the \( 2N \) roots of the determinantal polynomial \( P_{2N}(\tilde{\lambda}) \equiv \det(\tilde{\lambda}^2 + \lambda \tilde{D} + \tilde{A}) \) are \( \tilde{\lambda}_j = i\tilde{\omega} s_j \) with the \( 2N \) numbers \( s_j \) all integers, indeed

\[
P_{2N}(\tilde{\lambda}) \equiv \det(\tilde{\lambda}^2 + \lambda \tilde{D} + \tilde{A}) = \prod_{m=1}^{N} (\lambda - \lambda_m^{(+)})(\lambda - \lambda_m^{(-)}) \tag{3.50a}
\]

with

\[
\lambda_m^{(\pm)} = i \left[ k (p - 2q) \left( \frac{\pm 1 - 1}{2} \right) + m (p \pm 2q) \right] \tilde{\omega}. \tag{3.50b}
\]

Of course these formulas, (3.49) and (3.50), also hold for arbitrary (not integer) values of the two parameters \( p \) and \( q \). ■

These results follow from Case 3 (as detailed in the last part of the preceding Section 2), as shown in the following Section 4.

**Proposition 3.4**. Let the \( N \times N \) matrix \( \tilde{D} \) be defined, componentwise, as follows:

\[
\tilde{D}_{nm} = E + \sum_{\ell=1, \ell \neq n}^{N} \left[ (\tilde{z}_n - \tilde{z}_\ell)^{-1} (D_0 + D_1 \tilde{z}_n + D_2 \tilde{z}_n \tilde{z}_\ell) \right], \tag{3.51a}
\]

\[
\tilde{D}_{nm} = [(\tilde{z}_n - \tilde{z}_m)]^{-1} (D_0 + D_1 \tilde{z}_n + D_2 \tilde{z}_n \tilde{z}_m), \quad n \neq m. \tag{3.52a}
\]

Here \( D_0 = 0 \) or \( D_2 = 0 \) and the \( N \) coordinates \( \tilde{z}_n \) are now as well arbitrary (but different among themselves). As for \( D_1 \) and \( E \), either \( D_1 = 2 i \tilde{\omega} \), \( E = -2 N i \tilde{\omega} \) (corresponding to Case 5.1 of
Section 2), or \( D_1 = -2i\omega, E = ip\omega \), with \( \omega \) an arbitrary parameter and \( p \) an arbitrary integer (corresponding to Case 5.2 of Section 2). Then the determinantal polynomial \( P_n(\lambda) \equiv \det(\lambda + \bar{D}) \) reads

\[
P_n(\lambda) \equiv \det(\lambda + \bar{D}) = \prod_{m=1}^{N} (\lambda - \lambda_m),
\]

(3.53)

with

\[
\lambda_m = \lambda_m^{(+)} = 2im\omega,
\]

(3.54a)

when

\[
D_1 = 2i\omega, E = -2NI\omega,
\]

(3.54b)

and

\[
\lambda_m = \lambda_m^{(-)} = i(2N - p - 2m)\omega
\]

(3.54c)

when

\[
D_1 = -2i\omega, E = ip\omega.
\]

(3.54d)

Of course the last two formulas, (3.54c) and (3.54d), also hold for arbitrary (not integer) values of the parameter \( p \). ■

These results follow from Case 5 (as detailed in the last part of the preceding Section 2), as shown in the following Section 4.

Let us complete this section with the following comments on the results reported herein.

These findings are somewhat analogous to previous results: see in particular those reviewed in [7], in Appendix D (entitled “Remarkable matrices and related identities”) of [4] and in Appendix C (entitled “Diophantine findings and conjectures”) of [2], as well as those reported in the relevant papers quoted in the Preface of the updated paperback version of [2] and in the very recent papers [8] and [9]. The novelty is that the \( N \times N \) matrices identified herein—whose eigenvalues are exhibited above and have a Diophantine character—depend on quite a few arbitrary parameters; more than the somewhat analogous \( N \times N \) matrices previously identified as featuring analogous properties. On the other hand we expect that these results could also be proven by techniques analogous to those previously employed (see in particular [7] and [4]): indeed, it is generally the case—in the field of special functions and related topics—that alternative demonstrations are easily produced after some findings have been identified and proven to begin with...

4. Behavior near equilibria and consequential Diophantine findings

In this section we pursue the investigation of the behavior of the \( N \)-body problem (2.1) in the immediate vicinity of its equilibria (when they exist), see (2.15) and (2.16) yielding (2.17). Our task here is to compute explicitly the two \( N \times N \) matrices \( \bar{A} \) and \( \bar{D} \), see (2.17b). This is a standard task.

We report here the result, which the diligent reader will easily verify by inserting the ansatz (2.15) in the equations of motion (2.1). Componentwise, these two \( N \times N \) matrices read as follows:
\[ \tilde{A}_{nn} = - \left\{ B - 4 \left( N - 1 \right) A_3 \tilde{z}_n \right. \]
\[ + 2 \left( A_1 + 2 A_2 \tilde{z}_n + 3 A_3 \tilde{z}_n^2 \right) \sum_{\ell=1, \ell \neq n}^{N} \left( \tilde{z}_n - \tilde{z}_\ell \right)^{-1} \]
\[ - 2 \left( A_1 + A_2 \tilde{z}_n + A_3 \tilde{z}_n^2 \right) \tilde{z}_n \sum_{\ell=1, \ell \neq n}^{N} \left( \tilde{z}_n - \tilde{z}_\ell \right)^{-2} \right\}, \quad (4.55a) \]

\[ \tilde{A}_{nm} = -2 \left( A_1 + A_2 \tilde{z}_n + A_3 \tilde{z}_n^2 \right) \tilde{z}_n \left( \tilde{z}_n - \tilde{z}_m \right)^{-2}, \quad n \neq m ; \quad (4.55b) \]

\[ \tilde{D}_{nn} = E + \sum_{\ell=1, \ell \neq n}^{N} \left[ \left( \tilde{z}_n - \tilde{z}_\ell \right)^{-1} \left( D_0 + D_1 \tilde{z}_n + D_2 \tilde{z}_n^2 \tilde{z}_\ell \right) \right], \quad (4.56a) \]

\[ \tilde{D}_{nm} = \left( D_0 + D_1 \tilde{z}_n + D_2 \tilde{z}_n^2 \right) \left( \tilde{z}_n - \tilde{z}_m \right)^{-1}. \quad (4.56b) \]

The next task is to evaluate these two \( N \times N \) matrices \( \tilde{A} \) and \( \tilde{D} \) by inserting in their definitions (4.55) and (4.56) the explicit expressions of the \( N \) coordinates \( \tilde{z}_n \) which correspond to the various equilibria of the dynamical system (2.1). The relevant Cases 1-5 are reviewed at the end of Section 2.

We consider firstly the assignments corresponding to Case 1. Then one gets (via the notation and the identities reported in the Appendix) the explicit expressions (3.44) of the two \( N \times N \) matrices \( \tilde{A} \) and \( \tilde{D} \). Proposition 3.1 is thereby proven.

Secondly, we consider the assignments corresponding to Case 2, proceeding as in the preceding case. But at the end of the relevant computations we conclude that this case yields the same findings as the previous one (up to appropriate redefinitions of the parameter \( N \)).

Thirdly, we consider the assignments corresponding to the two Cases 4. Then one gets (again, via the notation and the identities reported in the Appendix) the explicit expressions (3.46) of the two \( N \times N \) matrices \( \tilde{A} \) and \( \tilde{D} \). Proposition 3.2 is thereby proven.

Fourthly, we consider the assignments corresponding to Case 3. Then one gets the explicit expressions (3.49) of the two \( N \times N \) matrices \( \tilde{A} \) and \( \tilde{D} \). Proposition 3.3 is thereby proven.

Finally, we consider the assignments corresponding to Case 5. Then one finds that the matrix \( \tilde{A} \) vanishes identically, \( \tilde{A} = 0 \), while the matrix \( \tilde{D} \) has the explicit expression (3.51). Proposition 3.4 is thereby proven.

5. Outlook

We believe that the search for new Diophantine findings of the kind reported in this paper is far from over; as well as the search for new \( N \)-body problems amenable to exact treatments. Hence this search constitutes an open—and, we opine, an interesting—research task, that we plan ourselves, and hope others, will be able and willing to pursue.
6. Appendix A: Identities

In this Appendix we display a number of identities which are presumably known and can in any case be easily proven by going through them sequentially. The protagonists of these identities are the $N$-th roots of unity,

$$u_n = \exp\left(\frac{2\pi i n}{N}\right),$$

(6.57a)

and the related quantities

$$w_n = (1 + \gamma u_n)^{-1}$$

(6.57b)

with $\gamma$ an arbitrary number (itself, however, not an $N$-th root of unity, so that the $N$ quantities $w_n$ are all finite).

$$(1 - u_n) + (1 - u_n^{-1}) = (1 - u_n) (1 - u_n^{-1}) = 2 \left[1 - \cos\left(\frac{2\pi n}{N}\right)\right],$$

(6.58a)

$$(1 - u_n)^{-1} + (1 - u_n^{-1})^{-1} = \frac{(1 - u_n) + (1 - u_n^{-1})}{(1 - u_n) (1 - u_n^{-1})} = 1.$$  

(6.58b)

$$\sum_{\ell = 1, \ell \neq n}^{N} \left[ f \left( \frac{u_n}{u_\ell} \right) \right] = \sum_{\ell = 1}^{N-1} \left[ f \left( u_\ell \right) \right] = \sum_{\ell = 1}^{N-1} \left[ f \left( u_\ell^{-1} \right) \right]$$

$$= -f(1) + \sum_{\ell = 1}^{N} \left[ f \left( u_\ell \right) \right] = -f(1) + \sum_{\ell = 1}^{N} \left[ f \left( u_\ell^{-1} \right) \right].$$

(6.59)

In these formulas, (6.59), $f(u)$ is an arbitrary function.

$$\sum_{\ell = 1}^{N} \left( \frac{u_n}{u_\ell} \right)^{p} = \sum_{\ell = 1}^{N} u_\ell^{p} = \sum_{\ell = 1}^{N} u_\ell^{-p} = 0, \quad p = 1, 2, ..., N - 1,$$

(6.60a)

$$\sum_{\ell = 1, \ell \neq n}^{N} \left( \frac{u_n}{u_\ell} \right)^{p} = \sum_{\ell = 1}^{N-1} u_\ell^{p} = \sum_{\ell = 1}^{N-1} u_\ell^{-p} = -u_n^{p} = -1, \quad p = 1, 2, ..., N - 1.$$  

(6.60b)

$$\frac{1}{2} \sum_{\ell = 1}^{N-1} \left[ (1 - u_\ell)^{-1} + (1 - u_\ell^{-1})^{-1} \right] = \frac{1}{2} \sum_{\ell = 1}^{N-1} (1) = \frac{N - 1}{2},$$

(6.61a)

$$\sum_{\ell = 1, \ell \neq n}^{N} \left[ \frac{1}{(1 - u_\ell) (1 - u_\ell^{-1})} \right] = \sum_{\ell = 1}^{N-1} \left[ (1 - u_\ell)^{-1} u_\ell \right] = \sum_{\ell = 1}^{N-1} \left[ (u_\ell^{-1} - 1)^{-1} \right]$$

$$= \sum_{\ell = 1}^{N-1} (u_\ell - 1)^{-1} = -\sum_{\ell = 1}^{N-1} (1 - u_\ell)^{-1} = -\frac{N - 1}{2},$$

(6.61b)
Diophantine Properties Associated to the Equilibrium Configurations of an Isochronous N-Body Problem

\[ \sum_{\ell=1, \ell \neq n}^N \left( 1 - \frac{u_n}{u_\ell} \right)^{-1} u_\ell^{-1} = u_n^{-1} \sum_{\ell=1, \ell \neq n}^N \left( 1 - \frac{u_n}{u_\ell} \right)^{-1} \frac{u_\ell}{u_n} = -\frac{N-1}{2 u_n}; \]  \hspace{1cm} (6.61c)

\[ \sum_{\ell=1, \ell \neq n}^N \frac{1}{\left( 1 - \frac{u_n}{u_\ell} \right)} = u_n \sum_{\ell=1, \ell \neq n}^N \frac{1}{\left( 1 - \frac{u_n}{u_\ell} \right)} = \frac{N-3}{2 u_n}; \]  \hspace{1cm} (6.62a)

\[ \sum_{\ell=1, \ell \neq n}^N \left( 1 - \frac{u_n}{u_\ell} \right)^{-1} u_\ell^2 = u_n^2 \sum_{\ell=1, \ell \neq n}^N \left( 1 - \frac{u_n}{u_\ell} \right)^{-1} = \frac{N-3}{2 u_n}; \]  \hspace{1cm} (6.62b)

\[ \sum_{\ell=1, \ell \neq n}^N \frac{1}{\left( 1 - \frac{u_n}{u_\ell} \right)} = u_n \sum_{\ell=1, \ell \neq n}^N \frac{1}{\left( 1 - \frac{u_n}{u_\ell} \right)} = \frac{N-1}{2}; \]  \hspace{1cm} (6.64a)

\[ \sum_{\ell=1, \ell \neq n}^N \left( 1 - \frac{u_n}{u_\ell} \right)^{-1} = 1 - u_n \sum_{\ell=1, \ell \neq n}^N \left( 1 - \frac{u_n}{u_\ell} \right)^{-1} = \frac{N-1}{2}; \]  \hspace{1cm} (6.64b)

\[ \sum_{\ell=1, \ell \neq n}^N \frac{1}{\left( 1 - \frac{u_n}{u_\ell} \right)} = u_n \sum_{\ell=1, \ell \neq n}^N \frac{1}{\left( 1 - \frac{u_n}{u_\ell} \right)} = \frac{N-3}{2 u_n}; \]  \hspace{1cm} (6.64c)

\[ \sum_{\ell=1, \ell \neq n}^N \left( 1 - \frac{u_n}{u_\ell} \right)^{-1} = -u_n \sum_{\ell=1, \ell \neq n}^N \left( 1 - \frac{u_n}{u_\ell} \right)^{-1} = -\frac{N-1}{2 u_n}; \]  \hspace{1cm} (6.64d)

\[ \sum_{\ell=1, \ell \neq n}^N \frac{1}{\left( 1 - \frac{u_n}{u_\ell} \right)} = u_n \sum_{\ell=1, \ell \neq n}^N \frac{1}{\left( 1 - \frac{u_n}{u_\ell} \right)} = -\frac{N-3}{2 u_n}; \]  \hspace{1cm} (6.64e)

\[ w_n - w_\ell = -\gamma w_n w_\ell \left( u_n - u_\ell \right) \]
\[ = -\gamma \left( u_n^{-1} + \gamma \frac{u_\ell}{u_n} + \gamma^2 u_\ell^2 \right)^{-1} \left( 1 - \frac{u_\ell}{u_n} \right), \]  \hspace{1cm} (6.65a)
\[(w_n - w_\ell)^{-1} = -\gamma^{-1} \left( u_n^{-1} + \gamma + \frac{u_\ell}{u_n} + \gamma^2 u_\ell \right) \left( 1 - \frac{u_\ell}{u_n} \right)^{-1} \] ; \quad (6.65b)

\[\sigma^{(1)}_n (N; \gamma) \equiv \sum_{\ell=1, \ell \neq n}^{N} (w_n - w_\ell)^{-1} = -\frac{N-1}{2} \gamma u_n \left( 1 - \gamma^2 u_n^2 \right) . \quad (6.66)\]

\[\sigma^{(2)}_n (N; \gamma) \equiv \sum_{\ell=1, \ell \neq n}^{N} (w_n - w_\ell)^{-2} . \quad (6.67)\]

\[
\left[ \sum_{\ell=1, \ell \neq n}^{N} (w_n - w_\ell)^{-1} \right]^2 = \left[ \sigma^{(1)}_n (N; \gamma) \right]^2 \\
= \sum_{\ell=1, \ell \neq n}^{N} (w_n - w_\ell)^{-1} \sum_{\ell' = 1, \ell' \neq n}^{N} (w_n - w_{\ell'})^{-1} \\
= \sigma^{(2)}_n (N; \gamma) + \sum_{\ell=1, \ell' = 1, \ell \neq n, \ell' \neq n}^{N} \left[ (w_n - w_\ell)^{-1} - (w_n - w_{\ell'})^{-1} \right] (w_\ell - w_{\ell'})^{-1} \\
= \sigma^{(2)}_n (N; \gamma) + 2 \sum_{\ell=1, \ell' = 1, \ell \neq n, \ell' \neq n}^{N} \left[ (w_n - w_\ell)^{-1} (w_\ell - w_{\ell'})^{-1} \right] \\
= 3 \sigma^{(2)}_n (N; \gamma) + 2 \sum_{\ell=1, \ell' = 1, \ell \neq n, \ell' \neq n}^{N} \left[ (w_n - w_\ell)^{-1} (w_\ell - w_{\ell'})^{-1} \right]; \quad (6.68a)
\]

\[\sigma^{(2)}_n (N; \gamma) = \frac{1}{3} \left[ \sigma^{(1)}_n (N; \gamma) \right]^2 - \frac{2}{3} \sum_{\ell=1, \ell \neq n}^{N} (w_n - w_\ell)^{-1} \sigma^{(1)}_n (N; \gamma) ; \quad (6.68b)\]

\[\sigma^{(2)}_n (N; \gamma) = \frac{1}{3} \left[ \sigma^{(1)}_n (N; \gamma) \right]^2 + \frac{N-1}{3} \gamma \sum_{\ell=1, \ell \neq n}^{N} (w_n - w_\ell)^{-1} \left( u_\ell^{-1} - \gamma^2 u_\ell \right) \] ; \quad (6.68c)

\[(w_n - w_\ell)^{-1} \left( u_\ell^{-1} - \gamma^2 u_\ell \right) \\
= -\gamma^{-1} \left( u_n^{-1} + \gamma + \frac{u_\ell}{u_n} + \gamma^2 u_\ell \right) \left( u_\ell^{-1} - \gamma^2 u_\ell \right) \left( 1 - \frac{u_\ell}{u_n} \right)^{-1} \\
= \left( u_n^{-1} + \gamma \right) \left[ 1 - \gamma (u_\ell)^{-1} + \gamma u_\ell (1 + \gamma u_\ell) \left( 1 - \frac{u_\ell}{u_n} \right)^{-1} \right] \left( 1 - \frac{u_\ell}{u_n} \right)^{-1} \quad (6.69)
Diophantine Properties Associated to the Equilibrium Configurations of an Isochronous N-Body Problem

\[
\sigma_n^{(2)}(N; \gamma) = -\frac{(N-1) \left(1 + \gamma u_n\right)^2}{12 \left(\gamma u_n\right)^2} \cdot \left[ (N-5) \left(1 + \gamma^2 u_n^2\right) + 2(N+1) \gamma u_n \right].
\]

(6.70)

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