THE WEAVE PRODUCT OF TWO CONICS

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ABSTRACT

Tensor diagrams are a handy way to depict complicated relationships between objects in projective geometry. One of the simpler ones takes two copies of a $3 \times 3$ matrix and computes its adjugate. In this paper, we give a geometric interpretation of this construction when two different matrices are used. To do so, we interpret them as coefficient matrices of conic sections in the projective plane and relate the diagram construction to their common tangents.

Keywords Projective Geometry · Tensor Diagrams · Multi-linear Algebra

In this paper, we introduce the weave product $\varpi(A, B)$ of two $3 \times 3$-matrices $A, B$ over any field $K$ and see how it can be interpreted when the matrices represent conic sections in the projective plane. For this, fix a field $K$.

For an introduction to projective planes and basic properties of tensor diagrams, see [1]. Here, we will only state the very minimal definitions and properties necessary for the subsequent discussion. Anyone familiar with these concepts can directly start at Definition [1.1]

1 Tensor diagrams

A square matrix $T = (T_{i,j})$ over $K$ is an array of field elements which are identified by two indices, where each index runs from 1 to $d$. For $V := K^d$, it can represent both a linear function $f : V \to V$ and a bilinear function $g : V \times V \to K$. To differentiate these cases, we write $T = T^i_j$ in the first case and $T = T^{i,j}$ in the second. For vectors $u, v \in V$, linear maps are thusly evaluated as $f(u)_i = \sum_{j=1}^{d} T^j_i u_j$ and bilinear functions as $g(u, v) = \sum_{i,j=1}^{d} T^{i,j} u_i v_j$. The indices that iterate over the input vectors are at the top and the indices iterating over the result are at the bottom. Moreover, when multiplying two matrices $T, S$ that both represent linear transformations $V \to V$, the resulting matrix has entries $(AB)^i_j = \sum_{k=1}^{d} T^k_i S^j_k$.

Note that the running indices in each sum appear exactly once as an upper input index and once as a lower output index.

In this system, a column vector $v$ is written as $v = v_i$ as it serves as an input for functions as the ones shown above. Dually, a row vector $l$, which represents a linear functional $V \to K$, is written as $l = l^j$ indicating that it takes a single column vector as its input. And the dot product of $l$ and $v$—or the evaluation of $l$ at $v$—is written as $\sum_{i=1}^{d} l^i v_i$.

All of this now generalizes to arrays of field elements with an arbitrary number of indices, which we call tensors. I.e., a tensor $A$ is an array of $M \cdot N$ elements in $K$ indexed by $M$ input and $N$ output indices

$$A^1_{j_1} \ldots ^M_{j_M} \in K$$

where each index runs from 1 to $d$. We multiply two or more tensors the same way as the vectors and matrices above: As sums over products of the entries of tensors such that every running index appears exactly once as an upper and a lower index.

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In more traditional tensor algebra, this is called a contraction and not a multiplication. But here we stick with the term multiplication as it is used in [1].
Since those expressions will get very unwieldy quickly, a diagram technique is introduced: Any tensor will be represented as a diagram in which the tensor itself is a node, the output indices are arrows leaving the node and the input indices are arrows entering it. This gives us

- matrices representing linear transformations
- matrices representing bilinear forms
- column vectors
- and row vectors/linear functionals

Note that we only label arrows if multiple ones are going in or multiples are going out, since only those could be confused with each other.

When there are several tensors present in a diagram, this diagram represents a product of them. Any loose arrows in this diagram represent in- or output indices of the product, depending on how they are attached to a node. Apart from that, arrows are allowed to connect two nodes—representing a running index in the sum defining the product. With this, the diagram

\[
\begin{array}{c}
  v \\
  T \\
\end{array}
\]

depicts the matrix-vector product \( f(v) = Tv \) with which we started and

\[
\begin{array}{c}
  u \\
  T \\
  v \\
\end{array}
\]

is the bilinear map \( g(u, v) \). Similarly,

\[
\begin{array}{c}
  v \\
  l \\
\end{array}
\]

is \( lv \) for a row vector \( l \) (or \( l(v) \) if interpreted as a linear functional \( l \)).

From now on, we will limit the tensors we use to \( d = 3 \). In particular, all matrices from here on out are \( 3 \times 3 \) matrices. We define the \( \delta \)-tensor via

\[
\delta^i_j := \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
\]

Interpreted as a matrix, this is the unit matrix, and we represent it via an arrow that is not attached to any node:

\[
\begin{array}{c}
i \\
\end{array}
\]

We define the \( \varepsilon \)-tensors via

\[
\varepsilon_{i,j,k} := \varepsilon^{i,j,k} := \begin{cases} 
1, & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3, \\
-1, & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3, \\
0, & \text{if two of } i, j, k \text{ are equal}.
\end{cases}
\]

We represent them via the diagrams
The importance of the $\varepsilon$-tensor lies in the following fact: Let $u, v, w$ be three column vectors. Then,

$$
\begin{array}{c}
\varepsilon_{i,j,k} \\
\end{array}
$$

is the determinant of these three vectors. An equivalent statement holds for the other $\varepsilon$-tensor and row vectors. Moreover, we can simplify a product of two $\varepsilon$-tensors into a sum of products of $\delta$-tensors:

$$
\sum_{k=1}^{3} \varepsilon_{i,j,k} \varepsilon_{k,l,m} = \delta_{i}^{l} \delta_{j}^{m} - \delta_{i}^{m} \delta_{j}^{l}
$$
or in tensor diagrams

$$
\begin{array}{c}
\varepsilon^{i,j,k} \varepsilon_{k,l,m} = \delta_{i}^{l} \delta_{j}^{m} - \delta_{i}^{m} \delta_{j}^{l}
\end{array}
$$

This is known as the $\varepsilon$-$\delta$-rule. These last two statements are straightforward to verify, but can also be found in Section 13.7 of [1].

With the $\varepsilon$-tensor we can now define the object of interest of this paper.

**Definition 1.1.** For two symmetric matrices $A, B$ we define their **weave product** as either

$$
\varepsilon(A, B) :=
\begin{array}{c}
A
\end{array}
$$
or

$$
\varepsilon(A, B) :=
\begin{array}{c}
A
\end{array}
$$

depending on whether they both represent quadratic forms on $V$ or its dual space $V^*$. 

It is well-known that $\varepsilon(A, A) = -2 \cdot \adj(A)$ (the adjugate matrix of $A$, see e.g. Section 14.4 in [1]). In particular, $\varepsilon(A, A)$ is, up to a scalar factor, the inverse of $A$ if it exists. Until now, there is no interpretation of $\varepsilon(A, B)$ for $A \neq B$. Recall that the diagram depicts a concrete formula for the entries of this matrix:

$$
\varepsilon(A, B)^{i,j} = \sum_{k,l,m,n=1}^{3} \varepsilon^{i,k,l} \varepsilon^{j,m,n} \cdot A_{l,m} \cdot B_{k,n}
$$
Due to the two $\varepsilon$-tensors, only four of the 81 summands are non-zero, and so it’s straightforward to expand the sum if necessary. The goal, however, is to find a geometric relation between $A$, $B$ and $\varpi(A, B)$. This will be done in the next section. Here, we finish the section with some algebraic identities that will be useful for that.

**Theorem 1.2.** Let $A, B$ be symmetric matrices that represent quadratic forms on $V$. Then the following hold:

(a) $\varpi(A, B) = \varpi(B, A)$.

(b) If $A$ is invertible, $\varpi(A, B) = \det(A) \cdot \left( A^{-1} BA^{-1} - \text{tr} \left( A^{-1} B \right) A^{-1} \right)$.

(c) If $B$ is invertible, $\varpi(A, B) = \det(B) \cdot \left( B^{-1} AB^{-1} - \text{tr} \left( B^{-1} A \right) B^{-1} \right)$.

The same statements hold if $A$ and $B$ represent quadratic forms on the dual space $V^*$.

**Proof.**

(a) If we interchange $A$ and $B$ in the diagram of $\varpi(A, B)$ we effectively swap two arrows of each $\varepsilon$-tensor. Each such swap will introduce a factor of $-1$ by the definition of the $\varepsilon$-tensor; and since this happens twice, nothing changes.

(b) Due to the equality

\[
\varpi(A^{-1}, A^{-1}) = -2 \cdot \det(A^{-1}) \cdot (A^{-1})^{-1} = -2 \cdot \det(A^{-1}) \cdot A,
\]

we can replace $A$ by $-\frac{1}{2} \det(A) \cdot \varpi(A^{-1}, A^{-1})$ in the diagram definition of $\varpi(A, B)$.

We remember the scalar factor $-\frac{1}{2} \det(A)$ for later and focus on the diagram on the right-hand side. We can apply the $\varepsilon$-$\delta$-rule on the left in the diagram:
In the second line, we transformed the first summand to the negative of the second one by swapping two arrows of the upper-right $\varepsilon$-tensor. Then, we can add them. As argued in the proof of part (a), this introduces a factor of $-1$. Now we cancel the $-2$ with the $-\frac{1}{2} \cdot \det(A)$ from the beginning to get the determinant as the sole scalar prefactor; which we again keep for later. We apply the $\varepsilon$-$\delta$-rule on the right in the diagram:

\[
\begin{array}{c}
A^{-1} \\
A^{-1} \quad \quad \\
B
\end{array}
\quad = \quad
\begin{array}{c}
A^{-1} \\
A^{-1} \\
B
\end{array}
\quad - \\
\begin{array}{c}
A^{-1} \\
A^{-1} \\
B
\end{array}
\]

The first summand represents just the ordinary matrix product $A^{-1} BA^{-1}$. Moreover, the diagram

\[
\begin{array}{c}
A^{-1} \\
B
\end{array}
\quad = \quad
\begin{array}{c}
A^{-1} \\
A^{-1} B
\end{array}
\]

represents the expression $\sum_{i=1}^{3} (A^{-1} B)_{i,i}$ which is the trace of $A^{-1} B$. So, the second summand is $- \text{tr}(A^{-1} B) \cdot A^{-1}$. Together with the scalar prefactor $\det(A)$ from above, this gives the desired result.

(c) The third statement follows directly from parts (a) and (b). Alternatively, an analogous proof as the one for part (b) can be applied.

When $A$ and $B$ are quadratic forms on $V^*$, we can simply flip every arrow around in the diagrams above and get the same results.

\[\Box\]

2 The weave product of two conics

The projective plane over $K$, which we denote by $KP^2$, can be seen as an extension of the affine plane $K^2$. This is (usually) achieved by embedding

\[K^2 \hookrightarrow K^3 \setminus \{0\}, \quad (x, y) \mapsto \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}\]
and then considering every vector only up to non-zero scalar multiples. I.e., all points in $\mathbb{K}P^2$ are equivalence classes of vectors $p \in \mathbb{K}^3 \setminus \{0\}$:

$$[p] := \{ \lambda p \mid \lambda \in \mathbb{K}^\times \}.$$ 

We usually write $p$ instead of $[p]$ as no confusion can arise.

Lines in $\mathbb{K}P^2$ are given as solution spaces to homogeneous linear equations. I.e., a row vector $l$ represents a line and a point $p$ lies on it iff $lp = 0$. Similarly, curves of degree 2—which we call conics—are solutions to homogeneous polynomials of degree 2. I.e., they are given as quadratic forms which are in turn represented by a (w.l.o.g. symmetric) matrix $A = A^{i,j}$ containing the coefficients of the polynomial. Thus, a point $p$ lies on a conic iff, in classical notation, $p^T A p = 0$ or, in tensor diagram notation:

$$\begin{array}{c}
p \\ \text{A} \\ p
\end{array} = 0$$

A conic is **degenerate** if its polynomial factors into two linear terms.

Every conic has a dual $B = B_{i,j}$ representing a quadratic equation which describes when a line is tangent to the conic. This dual is given by the adjugate matrix $\text{adj}(A)$ which is a multiple of $A^{-1}$ in case $A$ is non-degenerate.

With this, we can now give a geometric interpretation of the weave product.

**Theorem 2.1.** Given two non-degenerate conics in $\mathbb{K}P^2$ represented by the symmetric matrices $A$ and $B$, respectively. Then, the following statement holds:

There are four lines which are tangent to both conics. They touch the given conics in four points each; so there are eight points in total. Then there exists another conic, which runs through these eight points and this conic is given by $\varpi (A^{-1}, B^{-1})$.

![Figure 1: The weave product of the given conics.](image)

The result is insofar surprising, as five points uniquely define a conic. The existence of a conic running through eight given points is not imminently clear. Furthermore, note that the tangents and the points in which they touch the conics might only exist over the algebraic closure of $\mathbb{K}$. They can still fulfil the quadratic equation given by $\varpi (A^{-1}, B^{-1})$, though.

**Proof.** Let $p$ be a point on $A$. The tangent $l$ in $p$ to $A$ is given by $l = (Ap)^T$. It is also tangent to $B$ if and only if its pole $q$ with respect to $B$ lies on $l$ itself. This pole can be computed as $q = B^{-1} l^T$. To check whether it lies on $l$, it has to hold that

$$0 = lq = (Ap)^T B^{-1} (Ap) = p^T (AB^{-1} A) p.$$ 

See Theorems 9.1 and 9.2 in [1] for the calculations of tangents and poles. Note, that this only works for symmetric conic matrices.
The equation above means that the points at which the common tangents touch $A$ also have to lie on the conic given by $AB^{-1} A$. Applying the same arguments to $B$, we get that the tangent touching points on $B$ have to lie on $BA^{-1} B$. So, we get the eight tangent touching points we are interested in as the intersection of $A$ and $AB^{-1} A$, and $B$ and $BA^{-1} B$.

![Figure 2: The auxiliary conics $AB^{-1} A$ and $BA^{-1} B$.](image)

Any conic that runs through the intersection points of two conics can be written as a linear combination of the given conics. If a conic exists that runs through all eight tangent touching points we have here, the above arguments show that it must be a linear combination of $A$ and $AB^{-1} A$ as well as a linear combination of $B$ and $BA^{-1} B$.

Now, applying Theorem 1.2 parts (b) and (c) to $A^{-1}$ and $B^{-1}$ instead of $A$ and $B$ immediately tell us how to find such linear combinations:

$$\det(A^{-1}) \cdot (AB^{-1} A + \text{tr}(AB^{-1}) \cdot A) = \det(B^{-1}) \cdot (BA^{-1} B + \text{tr}(BA^{-1}) \cdot B)$$

In particular, either linear combination is realized by the weave product $\varpi(A^{-1}, B^{-1})$.

3 The importance of Theorem 2.1

To end this paper, we want to shortly talk about the qualitative relation between the conic from Theorem 2.1 and tensor diagrams.

Firstly, there is Hatton’s proof of Article 133 in [2]. He uses cross ratios and harmonic point constructions to show that this conic exists. So, the existence itself of the conic in question is one of dozens—maybe hundreds—of relatively basic statements about conic sections in projective geometry that have been known for quite some time.

Secondly, observe that most arguments in the proof of Theorem 2.1 have nothing to do with tensor diagrams. In particular, the tangent touching points in questions are the intersections of $A$ with $AB^{-1} A$, and of $B$ with $BA^{-1} B$, respectively. So, even without Theorem 1.2 we know that there have to be $\lambda, \mu, \nu, \kappa \in K$ such that

$$\lambda AB^{-1} A + \mu A - \nu BA^{-1} B - \kappa B = 0.$$ 

Looking at the individual entries of these matrices, these are "just" nine linear equations in the four indeterminants $\lambda, \mu, \nu, \kappa$. These coefficients are still relatively complex terms in the entries of $A$ and $B$, but solving this linear system is probably nothing a CAS or any human with a day or two of spare time cannot do. It is unclear, though, that the coefficients will be immediately recognized as $\lambda = \det(A^{-1})$, $\mu = \det(A^{-1}) \text{tr}(AB^{-1})$, etc. Which is something we get for free when working with tensor diagrams as described above.

Consequently, the importance of Theorem 2.1 does not lie in the existence of this peculiar conic that runs through eight special points, and not even in the fact that it makes the final step in the proof easier. But it gives the weave product diagram a concrete geometric interpretation, which is remarkable: The weave product is a very small and simple diagram that combines two matrices, and using the same matrix twice gives the adjugate. While it is not surprising that some small diagrams result in well-known constructions from linear algebra—the inverse, the determinant, the trace etc.—it is very much not obvious that something like the geometric statement from Theorem 2.1 is what the

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2 And that it is equivalent to an entirely different characterization.

3 The fact that the problem was reduced to a linear system is noteworthy in itself: Starting with actually computing the common tangents to two conics would immediately mean solving a cubic equation.
weave product diagram should represent.

Lastly, it shall be noticed that the author is still unaware of a concrete, geometric interpretation of the weave product if the matrices are seen as projective transformations instead of conics. I.e., how does the following diagram relate to the actions of $T, S \in \text{PGL}_3(K)$ as automorphism on $\mathbb{P}^2$?

![Diagram](image)

The identities from Theorem 1.2 still hold here—albeit via slightly different tensor diagram calculations. But since linear combinations of projective transformation have less direct interpretations than the linear combination of conics, it is not even clear whether these identities can help clarify this variant.

**References**

[1] Richter-Gebert, Jürgen. 2011. *Perspectives on Projective Geometry*. Springer.

[2] Hatton, John L. S. 1913. *The Principles of Projective Geometry Applied to the Straight Line and Conic*. Cambridge University Press.