The Heun operator as a Hamiltonian

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Abstract

It is shown that the celebrated Heun operator

\[ H = -(a_0 x^3 + a_1 x^2 + a_2 x) \frac{d^2}{dx^2} + (b_0 x^2 + b_1 x + b_2) \frac{d}{dx} + c_0 x \]

is the Hamiltonian of the $sl(2, R)$-quantum Euler–Arnold top of spin $\nu$ in a constant magnetic field. For $a_0 = 0$ it is canonically equivalent to $BC\{A\}$–Calogero–Moser–Sutherland quantum models; if $a_0 = 0$, ten known one-dimensional quasi-exactly-solvable problems are reproduced, and if, in addition, $b_0 = c_0 = 0$, then four well-known one-dimensional quantal exactly-solvable problems are reproduced. If spin $\nu$ of the top takes a (half)-integer value the Hamiltonian possesses a finite-dimensional invariant subspace and a number of polynomial eigenfunctions occur. Discrete systems on uniform and exponential lattices are introduced which are canonically equivalent to the one described by the Heun operator.

Keywords: Heun operator, Calogero–Moser–Sutherland model, quasi-exactly-solvable problem, Euler–Arnold quantum top, uniform lattice, exponential lattice, magnetic field

1. The Heun operator

It is evident that the Heun operator $P_3 \partial_x^2 + P_2 \partial_x + P_1$ plays an exceptionally important role in different physical sciences and mathematics, see e.g. the recent papers [1], [2], and some references therein, respectively. It is characterized by four regular singular points and has the form

\[ H = -(a_0 x^3 + a_1 x^2 + a_2 x) \partial_x^2 + (b_0 x^2 + b_1 x + b_2) \partial_x + c_0 x, \quad \partial_x \equiv \frac{d}{dx}. \]  

It depends on seven free parameters ($a_{0,1,2}$, $b_{0,1,2}$, $c_0$): we do not fix normalization (overall factor) but choose the reference point for $x$ in such a way that the coefficient $P_3$ in front of the second derivative vanishes at $x = 0$; for the sake of future convenience, we do not factorize the coefficient $(a_0 x^2 + a_1 x + a_2) x$ further to a product of monomials. The operator (1) is
defined up to additive constant $c_1$—it is the reference point for the spectral parameter and coincides with the accessory parameter in the Heun equation.

On the basis of monomials the operator $H_e$ has the form of a tridiagonal matrix. Note that the celebrated Heun equation (3) (see [4]) for a general discussion and also [1, 2, 5]

$$\frac{1}{p_3} (H_e - c_1) u(x) = 0,$$

occurs, where $c_1$ is the accessory parameter.

2. The Euler–Arnold quantum top

The Hamiltonian of the $sl(2, R)$-quantum Euler–Arnold top of spin $\nu$ in a constant magnetic field $B$ is given by

$$H = t^0 J_0 J_0 + t^+ J_+ J_+ + t^0 J_0 J_0 + t^0 J_0 J_0 + B^+ J_+ + B^0 J_0 + B^- J_-,$$

(2)

where $t^0, t^+, t^0, B^+, B^0, B^-$ are constants. Here the $J$'s span the $sl(2, R)$-Lie algebra; they obey the $sl(2, R)$ commutation relations: making linear transformations the general $sl(2, R)$ quantum top in a constant magnetic field can be reduced to the form (2). It is evident that this top is integrable: the second Casimir operator $C_2$ is the integral, it always commutes with $H$. The top (2) can be constrained by imposing a condition that $C_2$ is a constant,

$$C_2 \equiv \frac{1}{2} \{ J_+, J_- \} - J_0 J_0 = -\nu(\nu + 1),$$

(3)

where $\{ J_+, J_- \} \equiv J_+ J_- + J_- J_+$ denotes the anti-commutator. Here $t^{\alpha, \beta}$ with $\alpha, \beta = \pm, 0$ defines the tensor of inertia, $B = (B^+, B^0, B^-)$ defines the magnetic field. Thus, the Hamiltonian (2) with constraint (3) depends on seven free parameters as well as the operator (1).

3. $sl(2, R)$-Lie algebra

The $sl(2, R)$-algebra can be naturally spanned by differential operators, by difference (shift) operators and by difference (dilation) operators.

3.1. $sl(2, R)$-Lie algebra and differential operators

In first-order differential operators the $sl(2, R)$-Lie algebra is realized as follows,

$$J_+ = \partial_+, \quad J_0 = x \partial_x - \nu, \quad J_- = x (x \partial_x - 2 \nu),$$

(4)

where the parameter $\nu$ is spin. It is easy to see that in representation (4) the Casimir operator $C_2$ is the constant, see (3), thus, the constraint is fulfilled.

By substituting (4) into (2) we get the Heun operator (1), hence, $H = H_e$. In particular,

$$-a_0 = t^0, \quad b_0 = t^0(1 - 3 \nu) + B^+, \quad c_0 = 2 \nu(\nu^+ - B^+).$$

(5)

These parameters satisfy a remarkable condition

$$-2 \nu(2 \nu - 1) a_0 + 2 \nu b_0 + c_0 = 0,$$

(6)

which implies that for any $a_0, b_0, c_0$ one can find $\nu$ such that this condition is fulfilled. This $\nu$ corresponds to the $sl(2, R)$ spin of representation (4). One can reparameterize (1) by replacing...
\[ c_0 = 2 \nu (2 \nu - 1) a_0 - 2 \nu b_0, \]
to have the condition (6) fulfilled automatically. It is evident that the Heun operator acts in infinite-dimensional space of monomials \( \{ \ldots x^{-\nu}, \ldots x^{1+\nu}, x^\nu \} \).

### 3.2. sl(2, R)-Lie algebra and difference (shift) operators

Let us introduce the shift operator,

\[ T_\delta f(x) = f(x + \delta), \quad T_\delta = e^{\partial \delta}, \]
and construct a canonical pair of shift operators (see e.g. [6])

\[ D_\delta = \frac{T_\delta - 1}{\delta}, \quad x_\delta = x T_\delta = x(1 - \delta D_\delta), \]

where the operator \( D_\delta \),

\[ D_\delta f(x) = \frac{f(x + \delta) - f(x)}{\delta}, \]
is the so-called Norlund derivative. It is easy to check that \([D_\delta, x_\delta] = 1\), hence, forming the canonical pair.

In first-order difference (shift) operators the sl(2, R)-Lie algebra is realized as follows,

\[ J_\nu = D_\nu, \quad J_0 = x_\delta D_\delta - \nu, \quad J_\nu = x_\delta (x_\delta D_\delta - 2 \nu), \]

where the parameter \( \nu \) is spin, while \( \delta \) is an arbitrary parameter. It is easy to see that in representation (8) the Casimir operator \( C_2 \) is the constant, see (3); it does not depend on \( \delta \), thus, the constraint is fulfilled. By substituting (2) into (8) we get the finite-difference (shift) Heun operator

\[ H^{(s)}_\nu = - (a_0 x_\delta^3 + a_1 x_\delta^2 + a_2 x_\delta) D_\delta^2 + (b_0 x_\delta^2 + b_1 x_\delta + b_2) D_\delta + c_0 x_\delta, \]
see (1). This operator acts on uniform lattice space with spacing \( \delta \)

\[ \{ \ldots, x - 2\delta, x - \delta, x, x + \delta, x + 2\delta, \ldots \} \]
marked by \( x \in \mathbb{R} \)—a position of a central (or reference) point of the lattice. Explicitly, it is a five-point lattice operator,

\[ H^{(s)}_\nu f(x) = A_{-3}(x)f(x - 3\delta) + A_{-2}(x)f(x - 2\delta) + A_{-1}(x)f(x - \delta) + A_0(x)f(x) + A_1(x)f(x + \delta), \]
where \( A_\nu \) are polynomials. In the limit \( \delta \to 0 \), the operator \( H^{(s)}_\nu \) becomes the Heun operator \( H_\nu \) (1).

### 3.3. sl(2, R)-Lie algebra and difference (dilation) operators

Let us introduce the dilation operator,

\[ T_q f(x) = f(qx), \quad T_q = q^A, A \equiv x \partial_x, \]
and construct a canonical pair of dilation operators [7]

\[ D_q = x^{-1} \frac{T_q - 1}{q - 1}, \quad x_q = \frac{A(q - 1)}{T_q - 1} x, \]

where \( x_q D_q = A \) and \( D_q x_q = \partial_x A = A + 1 \). The operator \( D_q \) is the so-called Jackson symbol (or the Jackson derivative). Both operators \( x_q, D_q \) are pseudodifferential operators with action on monomials as follows,
\[ D_q x^n = \{n\}_q x^{n-1}, x_q x^n = \frac{n + 1}{(n + 1)_q} x^{n+1}, \]

where \(\{n\}_q = \frac{1 - q^n}{1 - q}\) is the so-called q-number.

In first-order difference (dilation) operators the \(sl(2, \mathbb{R})\)-Lie algebra is realized in the following way,

\[ J_\pm = D_q, J_0 = x \partial_x - \nu, J_x = x_q (x \partial_x - 2 \nu), \] (11)

see (4), where the parameter \(\nu\) is spin. It is easy to see that in representation (11) the Casimir operator \(C_2\) is the constant, see (3); it does not depend on \(q\), thus, the constraint is fulfilled. By substituting (2) into (11) we get the finite-difference (dilation) Heun operator

\[ H^{(d)}_\nu = -(a_0 x_q^2 + a_1 x_q + a_2 x_q \partial x) D_q^2 + (b_0 x_q^2 + b_1 x_q + b_2) D_q + c_0 x_q, \] (12)

see (1) and (9). In fact, it is the differential-difference operator

\[ H^{(d)}_\nu = -(a_0 x_q^2 x \partial x + a_1 x_q x \partial_x + a_2 x \partial x) D_q + (b_0 x_q x \partial x + b_1 x \partial x + b_2 D_q) + c_0 x_q, \]

acting on the exponential lattice \(\{x^n\}_q\), \(n = 0, 1, 2, \ldots\) with the reference point marked by \(x \in \mathbb{R}\).

4. Properties

- Operators \(H_\nu, H^{(ς)}_\nu, H^{(d)}_\nu\) are canonically equivalent—see (7) and (10).
- If \(2 \nu = n\) is an integer, the Heun operator \(H_\nu(H^{(ς)}_\nu, H^{(d)}_\nu)\) (1) ((9) and (12)) has \((n + 1)\)-dimensional invariant subspace in polynomials \(\mathcal{P}_n\), hence, \([H_\nu, \partial_\nu^{n+1}] : \mathcal{P}_n \to \{0\}\) and \(\partial_\nu^{n+1} = (J_\nu)^{n+1}\) is a particular integral [8]; also \([H^{(d)}_\nu, (J_\nu)^{n+1}] : \mathcal{P}_n \to \{0\}\)—see (8) and (11).
- ‘Polynomial’ isospectrality: let us assume the Heun operator \(H_\nu(x)\) has a formal polynomial eigenfunction,

\[ \phi(x) = \sum_{k=0}^{n} a_k x^k, \]

with eigenvalue \(\epsilon\), then the discrete (shift) Heun operator \(H^{(ς)}_\nu(x)\) has a formal polynomial eigenfunction

\[ \phi^{(ς)}_\nu(x) = \sum_{k=0}^{n} a_k x^{(k)}, \]

where \(x^{(k)} = x(x - \delta) \ldots (x - k\delta)\) is the Pochhammer symbol or quasi-monomial, with eigenvalue \(\epsilon\) and the discrete (dilation) Heun operator \(H^{(d)}_\nu(x)\) has a formal polynomial eigenfunction

\[ \phi^{(d)}_\nu(x) = \sum_{k=0}^{n} a_k \frac{k!}{(k)_q!} x^k, \]

where \(k! = 1 \cdot 2 \cdot \ldots \cdot k\) is factorial and \((k)_q! = [1]_q \cdot [2]_q \cdot \ldots \cdot [k]_q\) is the so-called q-factorial, with eigenvalue \(\epsilon\).
- Making a canonical (gauge rotation) transformation

\[ \partial_x \to \partial_x + \mathcal{A} = e^{-\phi} \partial_x e^\phi, x \to x, \] (13)

where \(\mathcal{A} = \phi(x)\), the Heun operator (1) can be transformed to the Schrödinger operator with rational potential.
where $D_{\alpha}(x)$ is a one-dimensional Laplace–Beltrami operator with contravariant metric 
\( g^{ij} = P_{ij} \), \( B = \frac{3}{16}a_0 + 23b_0 + c_0 + \frac{h_3^3}{4a_0} \) is the parameter, \( Q_2 \) is a second-degree polynomial. Then making a change of variables—a-another canonical transformation, 
\[ \partial_{x'} = \frac{1}{x''} \partial_{\tau}, x' = x(\tau), \] 
where \((x')^2 = a_0x^3 + a_1x^2 + a_2x\)—we arrive at the Schrödinger operator in the Cartesian coordinate \( \tau \),
\[ H_{\tau}(\tau) = -\partial^2_{\tau} + V(x(\tau)), \]
\[ V(x(\tau)) = Bx(\tau) + \frac{Q_2(x(\tau))}{P_3(x(\tau))}. \]

- ‘Canonical’ covariance. Take \( (x - \alpha)\mu \) as the gauge factor, where \( \alpha \) is root of the cubic equation \( P_3(\alpha) = 0 \). For any parameters \( \{a, b, c\} \) one can indicate \( \mu \) such that the gauge-rotated Heun operator remains the Heun operator,
\[ (x - \alpha)^{-\mu} H_{\tau}(x; \{a, b, c\}) \] 
\( (x - \alpha)\mu = H_{\tau}(x; \{a, \hat{b}, \hat{c}\}) \).

For example, for zero root, \( \alpha = 0 \), the exponent \( \mu = 1 + \frac{h_3}{a_2} \). For the Lame operator \( \mu = \frac{1}{2} \) for any root \( \alpha \).

- If \( a_0 \neq 0 \):
  
  (i) \( H_{\tau} \) with additive constant \( c_1 \) is factorizable in the \( sl(2, R) \)-algebra

\[ H_{\tau} + c_1 = T_a T_b, \]
where \( T_{a,b} = \alpha_{a,b}J_a + \beta_{a,b}J_0 + \gamma_{a,b}J_+ + D_{a,b} \) with \( \alpha_{b} = 0 \) and \( c_1 = D_bD_0 \);

(ii) \( H_{\tau} \) is canonically equivalent to the \( BC_1 \) elliptic Inozemtsev Hamiltonian [9] (see [10]) and to the \( BC_1(\lambda) \) elliptic Calogero–Moser–Sutherland Hamiltonian (see e.g. [11]) written in Cartesian coordinate \( \tau \); if in (1), \( P_2 = -P_3/2, H_{\tau} \) coincides with the Lame operator.

**Example.** \( BC_1 \) elliptic Calogero–Moser–Sutherland model.

Take the Heun operator
\[ h_{BC}(x) = 4(x^3 - 3 \lambda x^2 + 3\delta x)\partial_x^2 + 6(1 + 2\mu)(x - \lambda), \]
where \( \lambda, \delta, \mu, n \) are parameters; here \( a_0 = -4, b_0 = 6(1 + 2\mu), c_0 = -2n(2n + 1 + 6\mu), \) see (1) and \( n = 2s \), see (6). In the space of monomials \( x^k, k = 0, 1, 2, \ldots \) the operator \( h_{BC}(x) \) has the form of a tridiagonal, Jacobi matrix. In terms of \( sl(2, R) \)-generators (4), it reads
\[ h_{BC} = 4 J^- J^0(n) - 12 \lambda J^0(n) J^0(n) + 12 \delta J^0(n) J^+ + 2(4n + 1 + 6\mu) J^- + 12 \lambda(n + 2\mu) J^0(n) + 6 \delta(n + 2 + 2\mu) J^- + \lambda n(n + 2). \]
Making the gauge rotation of (17),
\[ \mathcal{H}_{BC} = -\frac{1}{2} \left( \Psi_0 \right) h_{BC} \Psi_0^{-1}, \Psi_0 = [x^3 - 3 \lambda x^2 + 3 \delta x]^2, \]
and changing \( x \to \tau \),
\[ (x')^2 = 4(x^3 - 3 \lambda x^2 + 3 \delta x), \]
we arrive at \( BC_1 \) elliptic Calogero–Moser–Sutherland Hamiltonian [12],
\[ \mathcal{H}_B = -\frac{1}{2} \partial^2_\tau + 2\mu(\mu - 1) \varphi(2\tau) + (2n + 2\mu)(n + 2\mu) \varphi(\tau), \]
where \( \varphi(\tau) \equiv \varphi(\tau | g_2, g_3) \) is the Weierstrass function with its invariants parametrized as follows,
\[ g_2 = 12(\lambda^2 - \delta), \quad g_3 = 4\lambda(2\lambda^2 - 3\delta). \]
Here \( \lambda, \delta \) are parameters and \( e = -\lambda \) is the root of the \( \varphi \)-Weierstrass function, \( \varphi'(-\lambda) = 0 \).
In such a parametrization the variable \( x \) has the form
\[ x = \varphi(\tau | g_2, g_3) + \lambda. \]

- If \( a_0 = 0 \), but \( b_0 > 0 \), all known ten families of quasi-exactly-solvable (QES) Hamiltonians written in the Cartesian coordinate \( \tau \) are reproduced [13–15].
- If \( a_0 = 0 \), and \( b_0 = c_0 = 0 \), the Riemann (hypergeometrical) operator occurs, thus, all known quantal exactly-solvable problems (the Harmonic oscillator, the Morse and the Pöschl–Teller potentials, the Coulomb problem) are reproduced when written in the Cartesian coordinate \( \tau \).

5. Classical limit

The classical limit appears when in the top Hamiltonian (2) and constraint (3) the generators \( J_s \) span the \( sl(2, R) \) Poisson algebra, thus, the Lie bracket is replaced by the Poisson bracket. In this case we have the classical \( sl(2, R) \) top in a constant magnetic field. Hence, the Hamiltonian is the polynomial in coordinate and momentum of the third and second degrees, respectively. Following the Igor Krichever suggestion, we call this procedure de-quantization.

The classical version of the Hamiltonian (14) based on de-quantization, \( \partial_x \to ip, x \to q \), is characterized by rational potential,
\[ \mathcal{H}_c(q, p) = P_3(q)p^2 + B q + \frac{Q_2(q)}{P_3(q)}, \]
see (14), where \( p \) is the classical momentum. Classical trajectories (in phase space), \( \mathcal{H}_c = E \), are algebraic curves.

6. Conclusions

In this letter we show that the Heun operator is the Hamiltonian of the \( sl(2, R) \)-(non-compact) Lie algebra, quantum Euler–Arnold top in a constant magnetic field. It reveals the links on the level of quantum canonical equivalence between three different quantum systems: tops, (generalized) Calogero–Moser–Sutherland systems and discrete systems, all in one
dimension. It implies that in the Fock space (universal enveloping Heisenberg algebra $U_{H_3}$ with vacuum attached) formalism all three systems are described by a single polynomial in $H_3$ generators. There are clear indications that similar links exist in multi-dimensional cases involving $A_n$, $BC_n$, $Du$ quantum Calogero–Moser–Sutherland models and $sl(n + 1, R)$-algebra quantum Euler–Arnold tops, $G_2$ quantum Calogero–Moser–Sutherland models and $g^{(2)}$-polynomial algebra, quantum tops (see [16] for the $A_2/G_2$ case), and also other Calogero–Moser–Sutherland models and polynomial algebra, quantum tops. These links will be described elsewhere.

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