Oscillations and instabilities in neutron stars with poloidal magnetic fields

S. K. Lander and D. I. Jones

School of Mathematics, University of Southampton, Southampton SO17 1BJ

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ABSTRACT

We study the time evolution of non-axisymmetric linear perturbations of a rotating magnetized neutron star, whose magnetic field is purely poloidal. The background stellar configurations are generated self-consistently, with multipolar field configurations and allowing for distortions to the density distribution from rotational and magnetic forces. The perturbations split into two symmetry classes, with perturbations in one class being dominated by an instability generic to poloidal fields, which is localized around the ‘neutral line’ where the background field vanishes. Rotation acts to reduce the effect of this instability. Perturbations in the other symmetry class do not suffer this instability and in this case we are able to resolve Alfvén oscillations, whose restoring force is the magnetic field; this is the first study of non-axisymmetric Alfvén modes of a star with a poloidal field. We find no evidence that these modes form a continuum.

In a rotating magnetized star, we find that there are no pure Alfvén modes or pure inertial modes, but hybrids of these. We discuss the nature of magnetic instabilities and oscillations in magnetars and pulsars, finding the dominant Alfvén mode from our simulations has a frequency comparable with observed magnetar quasi-periodic oscillations (QPOs).

Key words: instabilities – MHD – stars: magnetic field – stars: neutron – stars: oscillations – stars: rotation.

1 INTRODUCTION

Neutron stars (NSs) are notable for the extreme strength of their magnetic fields, with surface fields reaching $\sim 10^{15}$ G for magnetars and interior fields perhaps being an order of magnitude stronger. We expect many aspects of NS physics to be influenced by their magnetic fields, but we still have limited understanding of the actual structure of these fields; there are still open questions concerning their strength in the stellar interior, the relative proportions of poloidal and toroidal components and the possible effect of superconductivity (among others).

One particular motivation for improving modelling of NS magnetic fields is the observation of quasi-periodic oscillations (QPOs) in the aftermath of giant flares from magnetars; these provide the first direct evidence of NS oscillations and give us a potential probe of the interior physics of these stars.

One way to build up an improved understanding of NS magnetic fields is to explore the equilibria and dynamics of a simplified NS model. We choose to study a non-relativistic fluid star, but allowing for the effects of rotation and a magnetic field. In Lander & Jones (2009) we explored NS structures within this model, generating self-consistent equilibrium solutions for stars with purely poloidal, purely toroidal and mixed poloidal–toroidal magnetic fields. To understand the stability and oscillations of these stars we perform time evolutions of perturbations, using our NS equilibria as background configurations. In Lander, Jones & Passamonti (2010) we studied the oscillation spectrum of NSs with purely toroidal magnetic fields, whilst Lander & Jones (2010) presented results for toroidal-field instabilities. This paper discusses oscillations and instabilities of poloidal-field NSs, whilst in future we hope to complete this study of magnetized NSs by exploring mixed-field configurations.

We begin by discussing the governing equations for our NS model, both for the background equilibria and for the time evolution of the perturbations. We find that perturbations in one symmetry class are unstable in a region where the background field vanishes (consistent with the analytic work of e.g. Wright 1973 and Markey & Tayler 1973), with this unstable behaviour being reduced by the effect of rotation.

We quantify the effect of magnetic field strength and rotation on the instability’s growth rate. Perturbations in the other symmetry class, by contrast, appear to evolve stably; this confirms a prediction from Markey & Tayler (1973) about the nature of the dominant poloidal-field instability. For these stable perturbations we are able to resolve many Alfvén oscillation periods and extract their mode frequencies; this is the
first study of non-axisymmetric Alfvén modes in a poloidal-field star, complementing a number of recent studies on axisymmetric oscillations (Sotani, Kokkotas & Stergioulas 2008; Cerdá-Durán, Stergioulas & Font 2009; Colaiuda, Beyer & Kokkotas 2009; Sotani & Kokkotas 2009). We next turn to the oscillations of rotating magnetized stars, showing that these are a hybrid of inertial and Alfvén modes. We conclude by discussing our work in the context of magnetars and pulsars and discuss the possible proportions of poloidal and toroidal components required for stability.

2 GOVERNING EQUATIONS AND NUMERICS

We begin by describing the equations governing our NS model, both for the axisymmetric stationary background configuration and the non-axisymmetric perturbations. A fuller account of the numerical methods used in the code and its performance may be found in Lander et al. (2010); these are only described briefly here.

We model a NS as a self-gravitating, rotating, magnetized polytropic fluid with perfect conductivity, in Newtonian gravity. We wish to study linear perturbations of this star; for this our governing equations consist of a set of stationary background equations and a set of equations describing the time evolution of the perturbations. The background configuration has a purely poloidal magnetic field \( B_0 \) and may be (rigidly) rotating:

\[
0 = -\nabla P_0 - \rho_0 \nabla \Phi_0 - \rho_0 \Omega \times (\Omega \times r) + \frac{1}{4\pi} (\nabla \times B_0) \times B_0, \tag{1}
\]

\[
\nabla^2 \Phi_0 = 4\pi G \rho_0, \tag{2}
\]

\[
P_0 = k \rho_0^\gamma, \tag{3}
\]

where \( P \) is stellar pressure, \( \rho \) density, \( \Phi \) gravitational potential, \( G \) gravitational constant and \( \Omega \) angular velocity; 0-subscripts denote background quantities. Finally, we will take \( \gamma = 2 \) throughout this study, as a rough approximation to a NS equation of state.

Many studies of poloidal-field oscillations assume a dipolar field configuration, given by some simple analytic expression. In contrast to these, we solve for the field and fluid together, using a non-linear iterative procedure. The result is a self-consistent field configuration composed of a sum of different multipolar contributions. These higher multipoles are more significant in more distorted stars, providing small corrections to non-rotating and highly magnetized stars and becoming comparable with the dipolar field in rapidly rotating stars. Using the magnetic vector potential \( A \) defined through \( B = \nabla \times A \), one may show that the magnetic vector potential (in spherical-polar coordinates) satisfies the equation

\[
\nabla^2 (A_0 \sin \phi) = -\kappa \rho \rho \sin \theta \sin \phi, \tag{4}
\]

where \( \kappa \) is a constant governing the strength of the field; a derivation of this is given in Lander & Jones (2009). From the vector potential we obtain the background magnetic field configuration:

\[
B_0 = \nabla \times (A_0 \phi \hat{e}_\phi); \tag{5}
\]

since \( B \) is poloidal it is described by a toroidal vector potential. These background equations are solved numerically using an iterative procedure to find stationary equilibrium configurations, as detailed in Lander & Jones (2009; see also Tomimura & Eriguchi 2005). Our method allows for distortions of the star due to both rotational and magnetic forces; for the purely poloidal fields considered here, both of these forces will act to make the star oblate.

For the perturbation equations, we work in the rotating frame of the background and write our equations in terms of the perturbed density \( \delta \rho \), the mass flux \( f = \rho_0 \dot{r} \) and a magnetic variable \( \beta = \rho_0 \delta B \). We additionally make the Cowling approximation – neglecting the perturbed gravitational force – to avoid the computational expense of solving the perturbed Poisson equation. Our perturbations are then governed by seven equations:

\[
\rho_0 \frac{\partial f}{\partial t} = -\gamma P_0 \nabla \delta \rho - 2 \Omega \times f + \left( \frac{2 - \gamma}{\gamma} \right) P_0 \nabla \rho_0 - \frac{1}{4\pi} (\nabla \times B_0) \times \delta B + \frac{1}{4\pi} (\nabla \times \beta) \times B_0, \tag{6}
\]

\[
\frac{\partial \delta \rho}{\partial t} = -\nabla \cdot f, \tag{7}
\]

\[
\frac{\partial \delta \beta}{\partial t} = \nabla \times (f \times B_0) - \frac{\nabla \rho_0}{\rho_0} \times (f \times B_0). \tag{8}
\]

The axisymmetry of the background allows us to decompose the perturbation equations in the azimuthal index \( m \), thus reducing the 3D system of equations to a 2D one. It is also beneficial to isolate perturbations of a particular \( m \) when discussing oscillation modes and instabilities.
2.1 Boundary conditions

Instead of working with the spherical-polar radial coordinate \( r \), we employ a coordinate \( x \) fitted to isopycnic surfaces, which for non-spherical stars is a function of \( r \) and \( \theta \). Doing this gives us a very simple set of boundary conditions at the stellar surface:

\[
f(x = R) = \beta(x = R) = 0, \quad \delta P(x = R) = 0.
\]

Since we deal with \( m \neq 0 \) perturbations, we should also enforce a zero-displacement condition at the centre:

\[
f(x = 0) = \beta(x = 0) = 0, \quad \delta P(x = 0) = 0.
\]

Perturbation variables are also zero at the pole, except for \( m = 1 \), where \( f, f_\phi, \beta_\theta \) and \( \beta_\phi \) are non-zero; in the case of these quantities, it is their \( \theta \)-derivatives which vanish.

Finally, it may be shown that perturbations of stars with purely poloidal background fields form two sets based on shared symmetry properties: \( S_1 = \{ f, f_\phi, \delta \beta_\theta \} \) and \( S_2 = \{ f, \beta_\phi, \beta_\phi \} \). All the perturbations within a set have the same equatorial symmetry (say, even/symmetric), whilst all the perturbations in the other set have the opposite symmetry (which will then be odd/antisymmetric). We may enforce these symmetries as an additional set of boundary conditions, reducing our (2D) numerical domain to one quadrant of a disc. There are, therefore, two symmetry classes of solution: one class \( \mathcal{P}^+ \), where elements of \( S_1 \) are even (with those in \( S_2 \) odd) and one class \( \mathcal{P}^- \), where \( S_1 \) elements are odd (with \( S_2 \) elements even).

For more details on the boundary conditions summarized here, see Lander et al. (2010) or Lander & Jones (2010). Finally, we note that whilst the poloidal background field vanishes on a ‘neutral line’ within the star (see Section 3), we do not impose vanishing of the perturbations there.

2.2 Initial data

Depending on the initial data chosen and the symmetry class specified (see above), either ‘axial-led’ or ‘polar-led’ perturbations (using the terminology of Lockitch & Friedman 1999) will be excited. In particular, axial-led perturbations are those in the class \( \mathcal{P}^- \) and polar-led perturbations are in the \( \mathcal{P}^+ \) class. For evolutions of polar-led perturbations, we begin with an initial spherical harmonic perturbation in the density: \( \delta \rho \sim Y_{lm}(\theta, \phi) \), whilst if we wish to excite axial-led perturbations we prescribe a ‘magnetic’ spherical harmonic perturbation in the velocity: \( \mathbf{v} \sim \nabla Y_{lm} \times \mathbf{e}_r \).

Evolving perturbations on a background star with a poloidal field, we find that axial initial data strongly excites an unstable mode. Although some oscillatory behaviour can be seen too, the unstable mode dominates the evolutions and prevents us from extracting information about axial-led modes. With polar initial data, however, there is no evidence of an instability, and in this case we are able to resolve enough Alfvén oscillations to study polar-led mode frequencies. For this reason, all results in the section on instabilities come from evolutions using axial initial data (and the corresponding symmetry class \( \mathcal{P}^- \)), whilst all results in the oscillation section use polar initial data (and the other symmetry class \( \mathcal{P}^+ \)).

2.3 Numerics

As described above, we need only evolve perturbations on one quadrant of a disc, by exploiting symmetries of the original 3D system of equations. We employ a McCormack predictor–corrector algorithm to evolve the perturbation equations in time. We add an artificial viscosity term to our equations to damp out numerical instabilities that can emerge when using finite-difference routines, but we ensure that this is as small as possible – otherwise it could unnecessarily damp the physical hydromagnetic instabilities which we aim to study. We additionally employ a divergence-cleaning routine, since numerical error in the evolution of the perturbed magnetic field can result in an unphysical monopolar term: \( \nabla \cdot \delta \mathbf{B} \neq 0 \) (Dedner et al. 2002). We do not use artificial resistivity in this work.

2.4 Dimensionless and physical quantities

In our code, we non-dimensionalize all quantities by dividing them by the requisite combination of gravitational constant \( G \), equatorial radius \( R \) and maximum stellar density \( \rho_m \). Some results presented in this paper are in terms of dimensionless quantities; this is either because they are frequently used in the literature or because there is no particular benefit to quoting them in physical units. Dimensionless quantities are denoted with a hat; for example \( \hat{\Omega} = \Omega / \sqrt{G \rho_m} \). Note that dimensionless frequencies (i.e., the rotation \( \Omega \) or mode frequency \( \sigma \)) are quoted in terms of \( \text{radians} \) rather than \( \text{cycles} \) per unit code-time.

In all cases where we have redimensionalized our results, this is done to the same ‘canonical’ NS, with a mass of 1.4 solar masses and whose radius would be 10 km if it were non-rotating and unmagnetized (i.e., in hydrostatic equilibrium and spherical). For this canonical star, the maximum density is 2.2 × 10^15 g cm\(^{-3}\). Rotation frequencies may be redimensionalized using the approximate relation

\[
\Omega \ [\text{rad s}^{-1}] \approx 12 \ 100 \hat{\Omega}.
\]
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with the same relation holding for mode frequencies \( \sigma \). The corresponding rotation frequency \( \nu \) in cycles s\(^{-1}\) may be obtained using

\[
\nu [\text{Hz}] = \frac{\Omega \, [\text{rad s}^{-1}]}{2\pi} \approx 1900\Omega,
\]

– again, the same relation may be used to obtain mode frequencies in Hz.

3 INSTABILITIES OF PURELY POLOIDAL FIELDS

The first indications that purely poloidal fields could generically suffer hydromagnetic instabilities appeared in the studies of Wright (1973) and Markey & Tayler (1973). Using coordinates adapted to the magnetic field geometry, these authors found unstable perturbations in the vicinity of the ‘neutral line’, where the magnetic field vanishes (see Fig. 1). Instability was predicted to occur after approximately one Alfvén crossing time – the time taken for a magnetic perturbation to travel around the star.

The existence of unstable perturbations around the neutral line of a poloidal field is analogous to the toroidal-field case, for which Tayler (1973) showed the existence of an \( m = 1 \) instability localized around the magnetic field’s symmetry axis (the region where a toroidal field vanishes). A more general result for poloidal-field stability was given by Van Assche et al. (1982), who showed that any poloidal field with closed field lines is unstable around the neutral line. Markey & Tayler (1973) suggest that the open field lines, in contrast, may help reduce the effect of the instability.

In contrast to the toroidal-field case, there does not seem to be a definite conclusion from these analytic studies about which values of azimuthal index \( m \) correspond to the most unstable modes; this is a consequence of them utilizing field-line adapted coordinates rather than global spherical polars. One might, then, expect instabilities to be present for a variety of \( m \) – and this was indeed found in the numerical studies of Geppert & Rheinhardt (2006) and Braithwaite (2006).

Analytic work (Pitts & Tayler 1985) suggests that rotation may help to stabilize magnetic fields; this conclusion was backed up by the results of numerical evolutions reported in Geppert & Rheinhardt (2006). In contrast, the work of Braithwaite (2006) finds that rotation plays no stabilizing role on poloidal fields; however, since this study models a star as spherical (with no distorting centrifugal force) it may be less relevant for describing fast-rotating, highly distorted stars.

To summarize, the signatures of the poloidal-field instability described in earlier work are localized unstable growth around the neutral line, onset after one Alfvén crossing time, (probable) existence for a variety of \( m \) and (perhaps) stabilized by rotation. With these in mind, we now turn to our results for the behaviour of perturbations of a star with a poloidal field.

To look for a poloidal-field instability, we begin by approximating the Alfvén crossing time \( \tau_\Lambda \) using volume-averaged background quantities:

\[
\tau_\Lambda \approx \frac{2R}{\mathcal{C}_\Lambda} = 2R \sqrt{\frac{4\pi \langle \rho \rangle}{\langle B^2 \rangle}},
\]

where \( R \) is the stellar radius, \( \mathcal{C}_\Lambda \) the Alfvén speed and angular brackets denote volume averages. In dimensionless form, we find that \( \bar{\tau}_\Lambda \approx 77 \) for a star with a (redimensionalized) field strength of \( \bar{B} = 3.0 \times 10^{16} \, \text{G} \). In a plot of perturbed magnetic energy \( \delta M = \int (\delta B^2 / 8\pi) \, dV \) against time, we would therefore expect to see an instability manifest itself around \( \bar{t} \approx 77 \); this is confirmed in Fig. 2. To confirm that the

Figure 1. Schematic field-line geometry for a poloidal magnetic field. Around the star’s symmetry axis the field lines are ‘open’, in the sense that they extend outside the star; near the equatorial surface there is a region of field lines which close within the star. At the centre of these closed field lines is the neutral line (represented by the crosses in 2D, with the dashed line showing its circular form in the 3D star); along this line the magnetic field strength drops to zero.
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Figure 2. \( m = 2 \) instability of a poloidal magnetic field in a non-rotating star. We plot the magnetic energy \( \delta \hat{M} \) against time \( \hat{t} \), both in dimensionless form, for three different grid resolutions. We see that the onset time for the instability is independent of resolution (appearing at around the expected value of \( t_A \approx 77 \)), and its growth rate converges, consistent with it being a physical instability. The results are for a star with average field strength \( \bar{B} = 3.0 \times 10^{16} \) G.

origin of this instability is physical rather than numerical, Fig. 2 also shows that the result is similar for three different grid resolutions: low (16 \( r \)-points \( \times \) 15 \( \theta \)-points), medium (32 \( \times \) 30) and high (64 \( \times \) 60). The three growth rates converge at second order, the intended accuracy of the time-evolution code.

In Fig. 3 we compare the behaviour of \( \delta \hat{M} \) for stars with a fixed field strength (3.0 \( \times \) 10\(^{16} \) G) but differing rotation rates, finding that rotation does act to slow the instability’s growth rate. We will return to quantify the change in growth rate later in the section.

We next turn to the prediction that an instability should be localized around the magnetic field’s neutral line. From a global evolution, it is not straightforward to separate the behaviour of the instability from stable perturbations. We attempt to isolate the unstable perturbation by dividing the value of \( |\delta B| \) at each point by its value close to the start of the evolution, in this way removing the shape of initial stable perturbations. We plot the resulting (dimensionless) quantity with a logarithmic scale. After the onset of instability, all plots we obtain are similar to that shown in Fig. 4, for azimuthal indices \( m = 1, 2 \) and 4. Note that because our analysis is linear, the perturbed magnetic energy can grow indefinitely without diminishing the energy of the background configuration; in reality, non-linear effects would become important after the initial period of unstable growth.

Comparing the shape of the unstable perturbation with the background field configuration – shown in Fig. 5 – we see that the instability is clearly dominant around the neutral point. Furthermore, it appears to remain localized in this closed field-line region, fitting with the suggestion of Markey & Tayler (1973) that open field lines are likely to provide a stabilizing influence.

Having studied qualitative features of poloidal-field instabilities, we conclude this section with some quantitative results. As a measure of instability we define a growth rate \( \xi \) by

\[
\xi = \frac{1}{\Delta t} \Delta \left( \ln \left( \frac{\delta \hat{M}}{\delta \hat{M}_0} \right) \right),
\]

(14)

Figure 3. The stabilizing effect of rotation on purely poloidal magnetic fields (for \( m = 2 \) perturbations). The magnetic energy is plotted against time for three different rotation rates. We see that increasing the rotation rate decreases the growth rate of the instability; the gradient of \( \delta \hat{M} \) is reduced in the regime where the instability dominates. As for the previous plot, each configuration has a field strength of \( \bar{B} = 3.0 \times 10^{16} \) G.

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Figure 4. The magnitude of the perturbed magnetic field after the onset of instability, plotted with a logarithmic scale. The most unstable perturbations are visible around the neutral line, where the background field goes to zero. This plot is for the \( m = 2 \) instability, but very similar results emerge for \( m = 1 \) and 4 perturbations.

Figure 5. The poloidal-field configuration of the background star. The field strength decreases to zero around the neutral line, which is located at a dimensionless radius \( r/R \approx 0.8 \) from the centre.

where \( M_0 \) is the magnetic energy of the background star. This simply measures the degree of exponential growth of the unstable mode and is easy to identify once the instability has come to dominate the behaviour of the system. In Fig. 6 we use \( \xi \) to investigate the dependence of the instability on the background field strength and rotation rate of the star. From the left-hand panel we see that the growth rate scales linearly with field strength, with higher \( m \) instabilities appearing to grow faster. In the right-hand panel of Fig. 6 we plot the \( m = 2 \) instability growth rate as a function of rotation frequency, up to \( \Omega \approx 0.58\Omega_\kappa \). Rotation is seen to reduce the growth of this instability, but with finite-duration evolutions we cannot conclusively say that it will be removed altogether – however, we suggest that it may be stabilized in a real star by a combination of rotation and some additional physics (e.g. viscosity). A similar picture of rotational stabilization seems to emerge for stars with \( \Omega > 0.6\Omega_\kappa \) and for \( m = 1 \) and 4 instabilities, although evolutions in these cases appear to be more prone to late-time (numerical) instability.

As mentioned in Sections 2.1 and 2.2, each evolution of perturbations we perform uses one of two possible symmetry classes, dictating the equatorial symmetry properties of each perturbation variable. Perturbations in the \( P^+ \) symmetry class evolve stably for many Alfvén time-scales and we are able to study the corresponding ‘polar-led’ modes (see the following section). The generic instability of poloidal fields – explored in this section – manifests itself only in the behaviour of \( P^- \) perturbations. We may understand this through a more detailed discussion of the study of Markey & Tayler (1973). These authors work in a system of coordinates fitted to closed field lines, the geometry of which is a torus enclosing the neutral line. They first ‘open up’ this torus into a cylinder to enable comparison with known instabilities in a cylindrical field geometry (see Markey & Tayler 1973; fig. 2). In the simplest case, there are two main instabilities: the varicose instability (also known as the sausage instability) and the kink instability. Within a star, the most unstable perturbations will be non-radial – those that do not involve work against the star’s gravitational potential. Since the varicose instability does involve radial motion, the kink instability is
likely to be the dominant one in stars with poloidal fields. In the spherical-polar coordinates of the star, the kink instability involves motion of fluid elements in the $\theta$-direction across the equatorial plane. Along this plane $v_\theta = 0$ for the symmetry class $\mathcal{P}^+$ but not for the class $\mathcal{P}^-$ – and so unstable behaviour is only seen in the latter symmetry class. We believe this is the first numerical confirmation of this aspect of poloidal-field instabilities.

4 OSCILLATION MODES

As described above, perturbations in the $\mathcal{P}^-$ symmetry class are dominated by unstable behaviour, preventing us from resolving this class of Alfvén mode ($a$-mode) frequencies. In the $\mathcal{P}^+$ class, however, the perturbations behave in a stable oscillatory manner; in this section we study their mode frequencies. All results reported in this section are for $m = 2$ modes – both for brevity and because these are easiest to resolve with our code. To check that these are representative of other azimuthal indices, we performed evolutions for $m = 1$ and 4 oscillations too; the results suggested that there are no qualitative differences, but that $m = 1$ modes are of slightly lower frequency and $m = 4$ modes are of slightly higher frequency than the $m = 2$ case.

Before looking at results for oscillation modes, we discuss some terminology we will use to describe modes – this is based on notation employed by Lockitch & Friedman (1999) to describe inertial modes ($i$-modes). The eigenfunctions of $i$-modes are – with the exception of the $r$-mode – a sum of spherical harmonic contributions $Y_{lm}$ and so cannot be labelled with a single index $l$ (although they do have a single $m$). In all cases, however, the sums are dominated by the $Y_{lm}$ contributions from $l = m$ up to some value $l = l_0$; the contributions beyond $l_0$ are all far smaller. There may be several modes of the same $m$ and $l_0$, so these are enumerated with an additional index $k$ and denoted by $Y_{lm,k}$. Since we focus on $m = 2$ modes, we will suppress the $m$-index. Finally, we recall from Section 2.1 that we will have two classes of oscillation, corresponding to the two symmetry classes: these are axial-led ($\mathcal{P}^+$) and polar-led ($\mathcal{P}^+$) modes. Lockitch & Friedman (1999) showed that for axial-led modes the lowest contributing $Y_{lm}$ to their eigenfunction (i.e., $l = m$) is axial and for polar-led modes the lowest term is polar.

4.1 Modes of a non-rotating magnetized star

In the non-rotating case, we find three peaks at lower frequency than the $f$-mode in the frequency spectrum; we identify these as magnetically restored modes. We label these modes $a_1$, $a_2$, and $a_3$ based on the amplitude of their peaks, from strongest to weakest. We expect $a$-mode frequencies to be proportional to the Alfvén speed $c_A = B / \sqrt{4\pi \rho}$ and hence scale roughly linearly with $B$; this is borne out in our results, shown in Fig. 7.

We can gain some understanding about the eigenfunction structure of these $a$-modes by comparison with the behaviour of the code for inertial modes (in unmagnetized stars). In this case, we find that the lowest $l_0$ modes have the highest amplitude peaks in frequency space – these modes are excited more strongly because of the finite resolution of the numerical grid, combined with the low-$l$ initial data we use. This provides us with a useful rough diagnostic to identify $a$-modes: we suggest that the eigenfunction of the strong peak $a_1$ contains lower $Y_{lm}$ contributions than the $a_2$ or $a_3$ modes.

In a perfectly conducting medium, like the model NS considered in this paper, magnetically restored oscillations can occur in a continuous band of frequencies rather than being discrete global modes. This result was established by analytic work for an incompressible medium (see e.g. Goossens, Poedts & Hermans 1985) and more recent numerical work has suggested that the axisymmetric oscillations of compressible stars may form a continuum too (Sotani et al. 2008; Cerdá-Durán et al. 2009; Colaiuda et al. 2009). It is known, however, that dissipative effects like viscosity and resistivity can act to remove the continuum (e.g. Ireland et al. 1992).
To test for a mode continuum, we look at the oscillation frequencies of perturbed quantities at different points within the star. If our system has discrete global modes, we expect all these local oscillation frequencies to be equal; if there is a continuous mode spectrum, then oscillation frequencies will be position dependent. From our evolutions, we find the former: mode frequencies at different points within the star are equal, within the resolution dictated by the length of our evolutions (of the order of 1 per cent). Whilst this appears to contradict recent studies on magnetar QPOs that have found evidence for a mode continuum (among them Sotani et al. 2008; Cerdá-Durán et al. 2009 and Colaiuda et al. 2009), it should be emphasized that these studies are not quite comparable. For one, we study non-axisymmetric oscillations, rather than the $m = 0$ modes in these earlier studies; in addition we have only looked at polar-led modes, in contrast with the axial oscillations of the other papers. Our work is, however, in agreement with the study of polar Alfvén modes by Sotani & Kokkotas (2009), who also found a discrete oscillation spectrum.

### 4.2 Modes of a rotating magnetized star

Next we consider the mode spectrum of a rotating star with a poloidal magnetic field, but let us first recall our results on toroidal-field oscillation modes from Lander et al. (2010). In this earlier work, we found hybrid magneto-inertial modes, whose character was Alfvén-like for slow rotation and inertial-like in more rapidly rotating stars (Lander et al. 2010), so that their character depended on the ratio of magnetic to kinetic energy $M/T$ – we expect to see this happen in the poloidal-field case too. In the toroidal-field case, we found two polar-led $a$-modes in the non-rotating case. After rotational splitting, one half of each $a$-mode appeared to become a zero-frequency mode in the $M/T \to 0$ limit, whilst the other half of each mode became an inertial mode: $i_1^2$ and $i_2^2$. This led us to classify the $a$-modes as analogues of their respective inertial modes, namely $\tilde{i}_1^\ast$ and $\tilde{i}_2^\ast$.

In some respects, we find that polar-led oscillations of a poloidal-field star appear to differ qualitatively from the results for toroidal fields. In the non-rotating toroidal-field case we found two clear $a$-mode peaks, whereas with a poloidal field we find three – one of which is stronger than the others. Adding rotation splits these three modes into six – i.e., three corotating and three counter-rotating magneto-inertial modes (see Fig. 8). Close to Keplerian velocity $\Omega_K$, these all appear to become known polar-led inertial modes: the two $l_0 = 3$ modes and the four $l_0 = 5$ modes of a rotating unmagnetized star (see, e.g. Lockitch & Friedman 1999).

The large number of avoided crossings (at which the character of two modes changes) present in Fig. 8 makes it difficult for us to make conclusive statements about the eigenfunctions of $a_1$, $a_2$ and $a_3$, because unlike the toroidal case we cannot easily track these modes from the Alfvén-dominated to inertial-dominated regimes. However, since each $a$-mode splits into a corotating and counter-rotating branch, we expect them to become a corresponding pair of corotating and counter-rotating $i$-modes. We also know from Lockitch & Friedman (1999) that $i_1^\ast$, $i_2^\ast$, $i_3^\ast$ corotate with the star, and $i_1^\ast$, $i_2^\ast$, $i_3^\ast$ are counter-rotating. Combining this information with the speculation that the strong peak $a_1$ is a lower-$l_0$ mode than $a_2$ and $a_3$, we suggest that $a_1$ becomes the pair of modes $\{i_1^\ast, i_2^\ast\}$ as $M/T \to 0$, whilst $a_2 \to \{i_1^\ast, i_3^\ast\}$ and $a_3 \to \{i_2^\ast, i_4^\ast\}$.

Finally, we note the slightly anomalous behaviour of the highest frequency hybrid mode shown in Fig. 8, which begins at $\Omega = 0$ as one branch of the $a_3$ mode. This mode does not approach its apparent inertial counterpart $i_5^\ast$ as closely as the other hybrid modes in the high-$\Omega$
Polar-led $m = 2$ hybrid magneto-inertial modes for a NS with field strength $\vec{B} = 6.0 \times 10^{16}$ G; the plot is clearer for this highly magnetized star than at lower values of $\vec{B}$. When $\Omega = 0$, there are three pure Alfvén modes: $a_1$, $a_2$ and $a_3$. Rotation splits each into corotating and counter-rotating modes, which are seen to become known inertial modes for high rotation rates (with the possible exception of the highest frequency mode; see text), although the large number of avoided crossings makes it difficult to track each mode individually. In these dimensionless units, Keplerian velocity $\Omega_K \approx 0.72$.

regime. Since the other five hybrid modes do convincingly become inertial modes, however, we suggest that this discrepancy is due to an avoided crossing with the corotating branch of the $f$-mode (not shown in the figure) at $\Omega \approx 0.6$.

5 POLOIDAL-FIELD EFFECTS IN MAGNETARS AND PULSARS

In the previous two sections, we presented results for instabilities and oscillation mode frequencies at a variety of field strengths and rotation rates. Here we will apply these to two specific cases: a canonical model magnetar with $v = 0$ Hz and $\vec{B} = 10^{16}$ G and a model pulsar NS with $\nu = 1$ Hz (a common, if slightly low value) and $\vec{B} = 10^{14}$ G (assuming a fairly high surface field strength of $10^{13}$ G); in both cases we assume that the volume-averaged magnetic field $\vec{B}$ is an order of magnitude greater than the surface field strength.

We first consider the stability of poloidal fields in each model star. Although the magnetar field strength is 100 times that of the pulsar, a purely poloidal field would be unstable in both – however, the instability growth rate, linear in $\vec{B}$, would be 100 times slower in the pulsar, even if it were non-rotating. The additional effect of its rotation will further slow the growth of the pulsar’s poloidal-field instability. For the magnetar, we see from the left-hand plot of Fig. 6 that its growth rate $\zeta \approx 300$ s$^{-1}$; equivalently its $e$-folding time-scale is around 3 ms. If the pulsar were non-rotating, its $e$-folding time-scale would be 0.3 s; rotation will increase this somewhat, though we cannot quantify this using our results. This is because our rotating background stars are constructed by specifying the oblateness of the star (which depends on the grid spacing) rather than its rotation frequency, so our slowest rotating stellar models still rotate far faster than the 1 Hz of the model pulsar we wish to discuss here.

Although a purely poloidal field is unstable for a number of azimuthal indices, the instability is highly localized; this was predicted by Markey & Tayler (1973) and our results (Figs 4 and 5) are in good agreement with this. As suggested by Wright (1973), a toroidal field which is fairly strong in the neighbourhood of the neutral line could remove any poloidal-field instabilities. Interestingly, the toroidal field would not need to be globally strong to remove such instabilities. Based on this, we suggest that in a stable mixed-field configuration the maximum values of the poloidal and toroidal components may well be comparable, but their respective energies need not be (the toroidal-field energy is likely to be considerably smaller, since the toroidal component need only occupy a small volume of the star). This suggests that the ‘twisted-torus’ magnetic field configurations discussed by (for example) Yoshida, Yoshida & Eriguchi (2006), Lander & Jones (2009) and Ciolfi et al. (2009) may be stable equilibrium solutions, despite having toroidal components whose energy is $\lesssim 10$ per cent of the total magnetic energy.

Since purely poloidal fields are generically unstable, they are not candidates for long-lived magnetic field configurations in magnetars. For this reason, we cannot be confident that our results for polar-led Alfvén oscillations should closely resemble those of a real magnetar. In addition, it is unclear how many of the observed magnetar QPOs (see Watts & Strohmayer 2007 for details of these) originate as Alfvén

A typical rotation frequency for magnetars is $v = 0.1$ Hz (equivalently $\Omega \approx 5 \times 10^{-5}$ in dimensionless units), but we will assume that such a small value is negligible when $\vec{B} \sim 10^{16}$ G – as suggested by Figs 6 (right-hand plot) and 8.
modes of the interior, as opposed to elastic modes of the crust (for example). However, if future observations of magnetar oscillations do include frequencies best fitted by our results, it could indicate that these QPOs represent a-modes of dominantly poloidal-field configurations.

The frequencies of the three polar-led a-modes \(a_1, a_2\) and \(a_3\) exist in the ratio 7:4:11 (with a maximum discrepancy of 2 per cent). For our model magnetar, we would expect a strong Alfvén QPO (corresponding to \(a_1\)) at 83 Hz, with less significant QPOs at 47 Hz (\(a_2\)) and 130 Hz (\(a_3\)). It is interesting to note that our predicted value of 83 Hz is rather close to the two dominant magnetar QPO frequencies observed to date: 84 Hz for SGR 1900+14 and 92 Hz for SGR 1806–20 (Watts & Strohmayer 2007).

For our model pulsar, we know that the oscillations will be hybrid magneto-inertial modes, but predicting whether they will be more Alfvén like or inertial like is difficult – as before, this is because of finite resolution dictating the allowed rotation rates for our background stars. To attempt to gauge the relative influences of \(\Omega\) and \(\tilde{B}\) on the model pulsar’s oscillation modes, let us return to Fig. 8 (for a 6 \(\times 10^{16}\) G star). In this figure, the Alfvén mode \(a_1\) is seen to split into two hybrid modes, with frequencies \(\sim 10\) per cent different from their inertial counterparts at \(\tilde{\Omega} = 0.31 (\nu = 600\) Hz). We also know that hybrid-mode frequencies scale with \(M/T\) (see Lander et al. 2010 for more on this), and that \(M/T \sim \tilde{B}^2/\Omega^2\). For our model pulsar, \(\tilde{B}\) is a factor of 600 smaller than in the plot and \(\Omega\) is also a factor of 600 smaller, so \(M/T\) is roughly equal to its value for the Fig. 8 star at \(\tilde{\Omega} = 0.31\); from this we suggest that the pulsar’s oscillation modes will be dominantly inertial, with a magnetic correction of \(\sim 10\) per cent.

6 DISCUSSION

In this paper we have studied the behaviour of perturbations of a NS with a purely poloidal magnetic field. The background equilibrium configurations are generated self-consistently; the density distribution may be distorted by a combination of magnetic and rotational forces and the magnetic field is multipolar and poloidal, generalizing earlier work for dipole fields with a simple analytic form.

A number of previous studies have shown that poloidal fields generically suffer instabilities localized around the ‘neutral line’ (where the background magnetic field vanishes); we confirm that these are also present in our global evolution of perturbations and for our particular poloidal field. The instability is present for a number of non-axisymmetric perturbations – we have studied it for azimuthal indices \(m = 1, 2\) and \(4\) – but only in one of the two symmetry classes of perturbation. We argue that this confirms a prediction from Markey & Tayler (1973) about which modes are most unstable. Our results suggest that rotation has a stabilizing effect on purely poloidal magnetic fields.

For the stable symmetry class, we are able to evolve perturbations for many Alfvén oscillation periods and hence extract mode frequencies. These non-axisymmetric oscillations appear to be global modes, in contrast with the continuous spectrum of axisymmetric modes discussed recently by various authors (Sotani et al. 2008; Cerdá-Durán et al. 2009; Colaiuda et al. 2009). We find three Alfvén modes whose frequencies scale linearly with the field strength. These are rotationally split into pairs of modes; one of these co-rotates with the star and the other counter-rotates. As the stellar rotation rate is increased (or the field strength reduced), these become inertial modes; so in the case of a rotating magnetized star, pure inertial modes and pure Alfvén modes are replaced by hybrid magneto-inertial modes.

There are a few qualitative differences between the results presented here and our earlier work on toroidal-field oscillations (Lander et al. 2010) and instabilities (Lander & Jones 2010). Poloidal-field instabilities are not predominantly confined to \(m = 1\) as toroidal-field instabilities are, but instead exist for a variety of \(m\). The poloidal-instability growth rates we have studied (\(m = 1, 2, 4\)) are all lower than the \(m = 1\) toroidal-instability growth rate, though of the same order of magnitude. It appears from our results that in the poloidal-field case, the instability is more localized than the toroidal-field instability studied in Lander & Jones (2010). Perhaps the most notable difference is that the poloidal-field instability only exists in one symmetry class of perturbations rather than in both. In terms of oscillations, the character of hybrid magneto-inertial modes appears different if the background field is poloidal rather than toroidal. In the former case, we find that each hybrid mode becomes inertial as the rotation rate is increased; in the latter case, some of the hybrid modes appeared to become zero-frequency modes in this limit.

Our conclusion that purely poloidal fields are generically unstable is hardly surprising; however, it was not guaranteed that these localized instabilities would show up in evolutions of a discretized set of perturbations on a relatively coarse grid. Having shown that our code (at fairly low resolution) is adequate to study these unstable perturbations, we now discuss some other time evolutions of magnetar oscillations which have not found such instabilities and speculate on reasons for this. The work of Colaiuda et al. (2009) uses a coordinate system adapted to magnetic field lines in order to reduce the perturbation equations to a 1D system. In these coordinates, there is no coupling between different closed field lines, which, together with a boundary condition of not evolving perturbations on the neutral line, allows for stable evolutions without numerical viscosity. Sotani et al. (2008) did not employ these field-line coordinates and had to use numerical viscosity to suppress instabilities connected with evolving this intrinsically 1D system in 2D. The related work of Cerdá-Durán et al. (2009) does not mention instabilities and their plots of local QPO amplitudes within the star show that there is little or no growth immediately around the neutral line. Their work uses the anelastic approximation to suppress fluid modes, but it is not clear whether that should also remove poloidal-field instabilities.

All of these studies are based on long-term stable evolutions of perturbations on a magnetized background star; furthermore, they evolve axial perturbations, which our work indicates are particularly unstable for poloidal fields. The important difference between their work and ours appears to be that all three papers we have discussed specialize to axisymmetric (\(m = 0\)) perturbations, whereas we are currently only able to evolve perturbations with \(m \geq 1\). Axisymmetric perturbations may simply be less unstable; evolving these perturbations with (for example) numerical viscosity may be sufficient to remove instabilities altogether.
We have discussed some of the roles a poloidal field may play in both magnetars and pulsars. The unstable nature of purely poloidal fields means that they are unlikely to be good models of any NS magnetic field, although the instability growth rate is considerably slower for pulsars than magnetars. We find that the dominant polar-led Alfvén mode of a poloidal-field magnetar has a frequency of around 83 Hz, whilst pulsar oscillation modes will be hybrid magneto-inertial in nature. Although dominantly inertial, these pulsar modes may have magnetic corrections of \( \sim 10 \) per cent for fairly typical field strengths and rotation rates.

It has long been thought that a magnetic field may be stable with a suitable combination of toroidal and poloidal components (e.g. Tayler 1980). Since the poloidal-field instability only occurs close to the neutral line, we suggest that the addition of a toroidal component in this region alone may provide stabilization. The toroidal component would have to be locally comparable in strength with the poloidal component, but need only occupy a small volume – in this case, its contribution to the total magnetic energy could be quite modest. From this, we have an indication that some recent models of mixed-field equilibria (Yoshida et al. 2006; Ciolfi et al. 2009; Lander & Jones 2009) may be stable, despite having only a small percentage of magnetic energy contained in the toroidal component. We hope to explore the stability and oscillation spectra of these mixed-field configurations in a future study.

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