Discretization of inverse scattering on a half line

Evgeny L. Korotyaev\textsuperscript{1,2}

\textsuperscript{1}Academy for Advance Interdisciplinary Studies, Northeast Normal University, Changchun, Jilin, China
\textsuperscript{2}HSE University, 3A Kantemirovskaya ulitsa, St. Petersburg, Russia

Correspondence
Evgeny L. Korotyaev, Academy for Advance Interdisciplinary Studies, Northeast Normal University, Changchun, 130024 Jilin, China; and HSE University, 3A Kantemirovskaya ulitsa, St. Petersburg, 194100, Russia.
Email: korotyaev@gmail.com

Dedicated to Sergei Kuksin (Paris and Moscow) on the occasion of his 65th birthday.

Abstract
We solve inverse scattering problem for Schrödinger operators with compactly supported potentials on the half line. We discretize S-matrix: We take the value of the S-matrix on some infinite sequence of positive real numbers. Using this sequence obtained from S-matrix, we recover uniquely the potential by a new explicit formula, without the Gelfand–Levitan–Marchenko equation.

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1 | INTRODUCTION AND MAIN RESULTS

1.1 | Introduction

We consider Schrödinger operators $T_q y = -y'' + q(x)y$, $y(0) = 0$ on $L^2(\mathbb{R}^+)$. We assume that the potential $q$ belongs to real classes $L^\alpha$ or $P_\alpha$ defined by

\begin{align*}
L^\alpha &= \{ f \in L^\alpha(\mathbb{R}^+, \mathbb{R}) : \text{supp } f \subset [0,1] \}, \\
P_\alpha &= \{ f \in L^\alpha : \text{supp } f = 1 \}, \alpha \geq 1.
\end{align*}

It is well known (see [6, 10, 33] and references therein) that the operator $T_q$ has only purely absolutely continuous spectrum $[0, \infty)$ plus a finite number $m \geq 0$ of negative eigenvalues $E_1 < \ldots < E_m < 0$. We introduce the Jost solutions $f_+(x, k)$ of the equation

\begin{equation}
-f''_+ + qf_+ = k^2 f_+, \quad (x, k) \in \mathbb{R}^+ \times \mathbb{C} \setminus \{0\},
\end{equation}

under the conditions $f_+(x, k) = e^{i k x}$, $x \geq 1$, and the Jost function $\psi(k) = f_+(0, k)$. The Jost function $\psi$ is entire and satisfies

\begin{equation}
\psi(k) = 1 + O(1/k) \quad \text{as} \quad |k| \to \infty, \quad k \in \mathbb{C}^+,
\end{equation}

uniformly in $\arg k \in [0, \pi]$. The function $\psi(k)$ has exactly $m \geq 0$ simple zeros in $\mathbb{C}^+$ given by

\begin{equation}
k_j = i|E_j|^\frac{1}{2} \in i\mathbb{R}^+, \quad j \in \mathbb{N}_m = \{1, 2, \ldots, m\},
\end{equation}

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possibly one simple zero at 0, and an infinite number of zeros (so-called resonances) \(0 \leq |k_{m+1}| \leq |k_{m+2}| \leq ... \) in \(\mathbb{C}_-\).

The S-matrix is defined by

\[
S(k) = \frac{\overline{\psi}(k)}{\psi(k)} = \frac{\overline{\psi(-k)}}{\psi(k)} = e^{-i2\xi(k)}, \quad k \in \mathbb{R} \setminus \{0\},
\]

where the function \(\xi(k)\) is the phase shift. Recall that the function \(\xi\) is odd and continuous on \(\mathbb{R} \setminus \{0\}\) and satisfies:

\[
\begin{aligned}
\xi(k) &= -\frac{1}{2k} \int_0^1 qdx + o(1) \quad \text{as } k \to +\infty \\
\xi(\pm k) &= \mp\pi \left(m + \frac{n_0(\psi)}{2}\right) + o(1) \quad \text{as } k \to +0
\end{aligned}
\]

Here, \(n_0(f)\) is the multiplicity of 0 as a zero of a function \(f\) and note that \(n_0(\psi) \leq 1\).

There are a lot of results about an inverse scattering problem: To determine the potential \(q\) using the given phase shifts \(\xi(k)\) defined on all \(k > 0\), see [6, 10, 28, 33] and references therein. Marchenko [33] proved that the mapping \(q \to \text{spectral data} = \{\xi + \text{eigenvalues} + \text{norming constants}\}\) is a bijection between a specific class of potentials and spectral data. Similar results were obtained by Krein, see [28–30]. We discuss the case of compactly supported potentials, which associated with resonances. Zworski obtained first results about the characterization problem (Proposition 8, p. 293, Corollary p. 295 in [39]) on the real line. He gave necessary and sufficient conditions for some function to be a transmission coefficient for some compactly supported potential \(q\). But unfortunately, it is shown in [21] that this statement is not correct. Inverse problems (characterization, recovering, uniqueness) for compactly supported potentials in terms of resonances were solved by Korotyaev for a Schrödinger operator with a compactly supported potential on the real line [21] and the half-line [19], see also Zworski [38] and Brown–Knowles–Weikard [5] concerning the uniqueness. Moreover, there are other results about perturbations of the following model (unperturbed) potentials by compactly supported potentials: step potentials [8], periodic potentials [24], and linear potentials (corresponding to one-dimensional Stark operators) [25]. We mention also that inverse resonance scattering (characterization, recovering, uniqueness) were discussed by Korotyaev and Mokeev for a Dirac operator and for canonical systems on the half-line [26] and the real line [27].

We consider the following inverse problem: to recover the compactly supported potential when the phase shifts are given at some increasing sequence of energy. We give the physical motivation. When conducting an experiment (scattering of a particle by a potential), the phase shift is measured in a discrete set of points of energy. By this discrete set of phase shift values (+ proper eigenvalues and norming constants), a potential is restored using numerical methods. Our main goal is to solve the inverse scattering problem on the half-line according to the values of the phase shift in a discrete set of points only.

Introduce a class \(\mathcal{J}_\alpha\) of all possible Jost functions for \(q \in \mathcal{P}_\alpha\).

**Definition J.** By \(\mathcal{J}_\alpha, \alpha \in [1, \infty)\), we mean the class of all entire functions \(f\) having the form

\[
f(k) = 1 + \frac{\tilde{F}(k) - \tilde{F}(0)}{2ik}, \quad k \in \mathbb{C},
\]

where \(\tilde{F}(k) = \int_0^1 F(x)e^{2i\pi kx}dx\) and \(F \in \mathcal{P}_\alpha\). The zeros of \(f\) satisfy:

(i) The set of zeros (counted with multiplicity) of \(f\) is symmetric with respect to the imaginary line, \(f(k) \neq 0\) for any \(k \in \mathbb{R} \setminus \{0\}\) and \(f\) has possibly a simple zero at \(k = 0\).

(ii) All zeros \(k_1, ..., k_m, m \geq 0\) of the function \(f\) in \(\mathbb{C}_+\) are simple, belong to \(i\mathbb{R}_+\) and if they are labeled by \(|k_1| > |k_2| > ... > |k_m| > 0\), then they satisfy

\[
(-1)^j f(-k_j) > 0, \quad \forall \ j \in \mathbb{N}_m := \{1, 2, ..., m\},
\]

**Remark.** The class \(\mathcal{J}_1\) was introduced in [19] to study the case of potentials \(q \in \mathcal{P}_1\). Recall that the condition (1.7) in Definition J is equivalent to the following condition (see [19]):
(ii') \( f(-k_j) \neq 0 \) for all \( j \in \mathbb{N}_m \) and the function \( f \) has an odd number \( \geq 1 \) of zeros on any interval \((-k_j, -k_{j+1}), j \in \mathbb{N}_{m-1}\) and an even number \( \geq 0 \) of zeros on the interval \((-k_m, 0]\).

Introduce the set \( S_\alpha \) of all possible S-matrices:

\[
S_\alpha = \left\{ S(k) = \frac{\psi(-k)}{\psi(k)}, k \in \mathbb{R} : \psi \in J_\alpha \right\}, \quad \alpha \geq 1.
\]

We sometimes write \( \psi(k, q), k_n(q), \ldots \) instead of \( \psi(k), k_n, \ldots \) when several potentials are being dealt with. In [19], it was shown that the mapping \( q \to \psi(k, q) \) is the bijection between \( P_1 \) and \( J_1 \). We improve this result, including the characterization.

**Theorem 1.1.**

(i) Each mapping \( q \to \psi(\cdot, q), \alpha \in [1, 2] \) is a bijection between \( P_\alpha \) and \( J_\alpha \). Moreover, there exists an algorithm to recover the potential \( q \in P_\alpha \) in terms of its eigenvalues and resonances.

(ii) Each mapping \( S : P_\alpha \to S_\alpha, \alpha \in [1, 2] \) given by \( q \to S(\cdot, q) \) is a bijection.

**1.2 Main results**

We consider the following inverse problem: to describe a potential \( q \) when the S-matrix \( S(r, q) \) is given for some increasing sequence of energy. Define the class of such positive sequences

\[
\mathcal{R} = \{(r_n)_{1}^{\infty} : 0 < r_1 < r_2 < r_3 < \ldots, \quad |r_n - \frac{\pi}{2} n| < \frac{\pi}{8} \quad \forall n \geq n_+(r)\}.
\]

We note that if \( r \in \mathcal{R} \), then there exists an integer \( n_+(r) \) such that \( |r_n - \frac{\pi}{2} n| < \frac{\pi}{8} \) for all \( n \geq n_+(r) \). We use \( r \in \mathcal{R} \), since due to the Kadets Theorem [17], the system of functions \( 1, e^{\pm i2r_n}, n \in \mathbb{N} \) is a Riesz basis of \( L^2(-1, 1) \). Note that instead of the Kadets Theorem [17], we can use other results about a Riesz basis, see, for example, Avdonin [3], Avdonin-Ivanov [4], Kacnel’son [16], and references therein.

We write \( H_\mathbb{C} \) for the complexification of the real Hilbert space \( H \). Let \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \) and \( X = \mathbb{N} \) or \( X = \mathbb{Z}_+ \). Introduce the real space \( \ell^2(X) \) of all sequences \( h = (h_n)_{n \in X} \) equipped with the norm \( \|h\| = \sum_{n \in X} |h_n|^2 < \infty \). Let \( \ell^2 = \ell^2(\mathbb{N}) \) for shortness. Define the ball

\[
B_\epsilon(q) = \{ u \in L^2 : \|q - u\| < \epsilon \}, \quad \text{for} \quad (q, \epsilon) \in L^2 \times \mathbb{R}_+.
\]

We need the following properties of S-matrix.

**Proposition 1.2.**

(i) Let \( q \in L^1 \) and \((r_n)^\infty_1 \in \mathcal{R} \). Then,

\[
S(r_n, q) = 1 + \frac{\hat{q}(0) - \hat{q}_c(r_n)}{ir_n} + \frac{O(1)}{n^2} \quad \text{as} \quad n \to \infty,
\]

where \( \hat{q}_c(k) = \int_0^1 q(x) \cos 2kx \, dx \), uniformly on the bounded subsets of \( L^1_\mathbb{C} \). Moreover, there exists a sequence of real numbers \( \sigma = (\sigma_n)^\infty_0 \) such that

\[
S(r_n, q) = e^{\frac{\sigma_n + \sigma_0}{r_n}}, \quad \forall n \in \mathbb{N},
\]

where \( \sigma_0 = \int_0^1 q \, dx \) and \( \sigma_n \to 0 \) as \( n \to \infty \). In particular, if \( q \in L^2 \), then \( \sigma \in \ell^2(\mathbb{Z}_+) \).
(ii) Let $q^0 \in L^2$ and let $\delta > 0$. There exists $\varepsilon > 0$ such that if $q \in B_\varepsilon(q^0)$, then

$$|S(r_n, q) - S(r_n, q^0)| \leq \delta \quad \forall n \geq 1. \quad (1.11)$$

Moreover, let a sequence $(\sigma^0_n)^\infty_{n=0}$ corresponding to $q^0$ be given. Then, there exists a unique sequence of real numbers $\sigma = (\sigma_n)^\infty_{n=0}$ corresponding to $q \in B_\varepsilon(q^0)$ such that all $t_n = \sigma_n - \sigma^0_n$ satisfy

$$t_0 = \int_0^1 (q - q^0)dx, \quad |t_n| \leq \varepsilon + \pi \delta r_n \quad \forall n \geq 1, \quad \text{and} \quad t_n \to 0 \quad \text{as} \quad n \to \infty,$$

and all $t_n$ are small for $\delta, \varepsilon$ small enough.

Remarks.

(1) We see that the coordinates $\sigma_0$ and $\sigma_n$ for $n$ large enough in (1.10) can be defined uniquely from asymptotics, but other coordinate $\sigma_n$ for finite $n$ cannot be defined uniquely. Nonetheless, Theorem 1.3 will show that if $S(r_n, q_1) = S(r_n, q_2)$ for some $q_1, q_2 \in L^1$ and all $n \in \mathbb{N}$, then $q_1 = q_2$.

(2) If we fix $q^0$ and all $\sigma^0_n$, then all coordinates $t_n, n \geq 0$ can be defined uniquely. Below we obtain parameterization of $q \in B_\varepsilon(q^0)$ in terms of the local coordinates $t_n, n \geq 0$, that is, $\sigma_n = \sigma^0_n + t_n$.

We consider the inverse problem in terms of S-matrix. First, we consider the bijection between potentials $q \in P_\alpha$ and meromorphic functions $S \in S_\alpha$ for each $\alpha \in [1, 2]$. Second, we discretize S-matrix: We take the value of the S-matrix on some infinite sequence of positive real numbers and show that this sequence determined the potential uniquely.

Recall some definitions. We write $H_C$ for the complexification of the real Hilbert space $H$. Suppose that $H_1, H$ are real separable Hilbert spaces. The mapping $F : H_1 \to H$ is a local real analytic isomorphism iff for any $y \in H_1$, it has a continuation $\tilde{F}$ into some complex neighborhood $y \in U \subset H_1 \subset H_C$, which is a bijection between $U$ and some open set $\tilde{F}(U) \subset H_C$ and if $\tilde{F}, \tilde{F}^{-1}$ are analytic mappings on $U, \tilde{F}(U)$, respectively. $F$ is a (global) isomorphism if it is both a bijection and a local isomorphism.

Let $q^0 \in L^2$ and let $\sigma^0 = (\sigma^0_n)^\infty_{n=0} \in \ell^2(\mathbb{Z}_+)$ be some sequence corresponding to $q^0$; let $\delta > 0$ be small enough. Due to Proposition 1.2, there exists $\varepsilon > 0$ such that if $q \in B_\varepsilon(q)$, then we have $|S(r_n, q) - S(r_n, q^0)| \leq \delta$ for all $n \geq 1$ and there exists a unique sequence $\sigma = (\sigma_n)^\infty_{n=0} \in \ell^2(\mathbb{Z}_+)$ such that $\sigma_n = \sigma^0_n + t_n$, and $t_n$ is small enough. Thus, we obtain

$$S(r_n, q^0) = e^{\frac{\sigma_0 + \sigma_n}{r_n}}, \quad S(r_n, q) = e^{\frac{\sigma_0 + \sigma_n}{r_n}}, \quad (1.13)$$

where $\sigma_n, n \in \mathbb{Z}_+$ is uniquely defined for small $\varepsilon > 0$. Here, $t_n = \sigma_n - \sigma^0_n, n \geq 0$ are local coordinates. For fixed $q^0 \in L^2$ and $\varepsilon > 0$ small enough, we can define the mapping $\sigma : B_\varepsilon(q^0) \to \ell^2(\mathbb{Z}_+)$ by

$$q \to \sigma = (\sigma_n)^\infty_{n=0}. \quad (1.14)$$

We formulate the first main results.

**Theorem 1.3.**

(i) Let a sequence $r = (r_n)^\infty_{n=0} \in R$. If $S(r_n, q_1) = S(r_n, q_2)$ for some $q_1, q_2 \in L^1$ and all $n \in \mathbb{N}$, then we have $q_1 = q_2$.

(ii) Let $q^0 \in L^2$ and let $\varepsilon > 0$ be small enough. Then, the mapping $\sigma : B_\varepsilon(q^0) \to \ell^2(\mathbb{Z}_+)$ given by $q \to \sigma = (\sigma_n)^\infty_{n=0}$ is a local analytic bijection.

Remarks.

(1) In the inverse theory of Marchenko [33] or Krein [30], one needs the phase shift plus eigenvalues and norming constants. In (i), we need only $S(r_n, q), n \geq 1$. 


2) The proof of (ii) is more complicated and all of Section 3 is devoted to it. Moreover, in Section 3, we have additional results about the mapping \( q \to \sigma(q) \).

In order to discuss other inverse problems, we define sets of potentials:

\[
\mathcal{L}_+^\alpha = \{ q \in \mathcal{L}_+^\alpha : \psi(k, q) \neq 0, \forall k \in \mathbb{C}_+ \}, \quad \mathcal{P}_+^\alpha = \mathcal{L}_+^\alpha \cap \mathcal{P}_\alpha, \quad \alpha \geq 1,
\]

that is, when corresponding Schrödinger operators \(-y'' + qy\) have no eigenvalues.

Let \( q_\star \in \mathcal{L}^\alpha \). Consider a Jost function \( \psi_\star = \psi(k, q_\star) \) with zeros \( k_j \in \mathbb{C}_+, j = 1, \ldots, m \). We define a new function \( F = B(k)\psi_\star (k) \) using the Blaschke product \( B \) given by

\[
B(k) = \prod_{j=1}^m \frac{k + k_j}{k - k_j}, \quad k \in \mathbb{C}_+,
\]

where the function \( F \neq 0 \) in \( \mathbb{C}_+ \). Results from [19] (see Theorem 2.1 below) give that \( F \in \mathcal{L}_1^1 \) and \( F = \psi(k, q) \) is the Jost function for a unique potential \( q \in \mathcal{L}_1^1 \). Note that there are more results about zeros of Jost functions in [19, 20, 35]. Thus, we have two steps to solve the following inverse problem:

1) to recover the potential \( q_\star \), when \( q \in \mathcal{L}_1^1 \) and \( B(k) \) are given and \( \psi(k, q_\star), \psi(k, q) \in \mathcal{L}_1^1 \).
2) the inverse problem for functions \( \psi(\cdot, q) \in \mathcal{L}_1^1 \), when \( q \in \mathcal{L}_1^1 \).

We consider the first step: to recover \( q_\star \), when \( q \in \mathcal{L}_1^1 \) and all zeros \( k_j, j = 1, \ldots, m \) of \( \psi_\star \) in \( \mathbb{C}_+ \) are given. Jost and Kohn [14] adapted the Gelfand–Levitan method [12] to determine a potential \( q_\star \) from the function \( q \) and the zeros of \( \psi(k, q_\star) \) in \( \mathbb{C}_+ \). We describe their results for our case, when a potential \( q \in \mathcal{P}_1^\alpha \) is given, and for simplicity, we assume that \( m = 1 \). For any \( k_\star = ir_\star, r_\star > 0 \), and any constant \( c > 0 \), the function \( \psi_\star (k) = \psi(k, q_\star) \) is a Jost function for a potential \( q_\star \in L^1(\mathbb{R}_+) \) given by

\[
q_\star = q + q_o, \quad q_o = -(\log A)'', \tag{1.16}
\]

where the function \( A(x) = 1 + c \int_0^x \varphi^2(s, k_\star, q)ds \) and \( \varphi(x, k, q) \) is the solution of the equation

\[
-\varphi'' + q\varphi = k^2\varphi, \quad \varphi(0, k, q) = 0, \quad \varphi'(0, k, q) = 1.
\]

Moreover, \( q_o, q_o' \in L^1(\mathbb{R}_+) \) and \( c = \int_0^\infty \varphi^2(x, k_\star, q_\star)dx > 0 \) is the normalizing constant, which can be any positive number. It is important that they determine asymptotics:

\[
q_o = -\frac{2}{c} \left(2r_\star\right)^5 e^{-2r_\star x}(1 + o(1)) \quad \text{as} \quad x \to \infty. \tag{1.17}
\]

Recall results from [19]: Let \( q \in \mathcal{P}_1 \) and let \( \psi_\star = \psi(k, q)^{k-k_\star}/k+k_\star \in \mathcal{J}_1 \) for some \( k_\star \in i\mathbb{R}_+ \). Then, \( \psi_\star = \psi(k, q_\star) \) is the Jost function for a unique potential \( q_\star \in \mathcal{P}_1 \). In this case, the potential \( q_o \) is compactly supported and the constant \( c = \frac{2|k|e^{|k|}}{|i\psi'(-k_\star)\psi(1,k_\star)|} > 0 \) (see more in Theorem 4.2).

Thus, results and asymptotics (1.17) of Jost and Kohn [14] are not correct for specific cases. Then, the identity (1.16) of Jost and Kohn [14] jointly with the remark from [19] solve the first step. Note that if we take \( \psi \in \mathcal{J}_\alpha \) for some \( q \in \mathcal{P}_\alpha, \alpha \in [1, 2] \), and \( \psi_\star \in \mathcal{J}_1 \), then (1.16) gives that \( q_\star \in \mathcal{P}_\alpha \). We describe more the first step in Section 4.

We consider the second step, which is the main and more important step.

Consider two Sturm–Liouville problems for \( q \in L^1(0,1) \):

\[
-y'' + qy = \lambda y, \quad y(0) = y(1) = 0, \tag{1.18}
\]

\[
-y'' + qy = \lambda y, \quad y(0) = y'(1) = 0. \tag{1.19}
\]
Let $\mu_n$ and $\tau_n$, $n \geq 1$ be eigenvalues of the problems (1.18) and (1.19), respectively. It is well known that these eigenvalues are simple, interlace $\tau_1 < \mu_1 < \tau_2 < \mu_2 < \ldots$, and satisfy:

$$
\begin{align*}
\tau_n &= \pi^2 (n - \frac{1}{2})^2 + \sigma_o + \bar{\tau}_n, \quad n \geq 1, \quad \bar{\tau}_n \to 0 \text{ as } n \to \infty, \\
\mu_n &= (\pi n)^2 + \sigma_o + \bar{\mu}_n, \quad n \geq 1, \quad \bar{\mu}_n \to 0 \text{ as } n \to \infty,
\end{align*}
$$

(1.20)

and $\sigma_o = \int_0^1 qdx$, see [34]. Moreover, if $q \in L^2(0, 1)$, then we have $(\tau_n)_{1}^{\infty} \in \ell^2$ and $(\mu_n)_{1}^{\infty} \in \ell^2$. In our consideration, we use the following crucial fact from [22]: $q \in L^2_+$ if and only if the first eigenvalue $\tau_1$ of (1.19) satisfies $\tau_1 \geq 0$, that is,

$$
q \in L^2_+ \iff \tau_1 \geq 0.
$$

(1.21)

Moreover, we have that $\psi(0, q) = 0$ for some $q \in L^2_+$ iff the eigenvalues $\tau_1 = 0$.

We fix model (unperturbed) sequences (corresponding to potential $q = 0$) by

$$
p^o = (p^o_n)_{1}^{\infty}, \quad n^o = (n^o_n)_{1}^{\infty}, \quad \bar{n}^o = (\bar{n}^o_n)^2.
$$

In order to prove Theorem 1.5, we need following preliminary results.

**Proposition 1.4.** Let $q \in L^1_+$. Then, the function $k - \xi(k)$ is strongly increasing in $k \in [0, \infty)$ and $1 - \xi' > 0$. Moreover, for each $n \geq 1$, the equation $k - \xi(k) = p^o_n$ has a unique solution $p_n \geq 0$ such that

$$
p_n = p^o_n + \xi(p_n), \quad \xi(p_n) = \frac{\sigma_o + \sigma_n}{2p^o_n}, \quad \sigma_o = \int_0^1 qdx,
$$

(1.22)

$$
\mu_n = p^2_n, \quad \tau_n = p^2_{2n-1} \quad \forall \ n \geq 1,
$$

(1.23)

where $\mu_n, \tau_n$ are eigenvalues of the problems (1.18) and (1.19), respectively. If in addition $q \in L^2_+$, then we have $(\sigma_n)_{1}^{\infty} \in \ell^2(\mathbb{Z}^+)$.

For a potential $q \in L^2_+$ due to Proposition 1.4, we have an increasing sequence $0 \leq p_1 < p_2 < \ldots$, where $p_j = p^o_j + \frac{\sigma_o + \sigma_j}{2p^o_j}$ and $(\sigma_j)_{1}^{\infty} \in \ell^2$. For $q \in L^2_+$, we define the spectral data $\mathfrak{S}_o$ by

$$
\mathfrak{S}_o = \{ (\sigma_j)_{j=0}^{\infty} \in \ell^2(\mathbb{Z}^+) : 0 \leq p_1 < p_2 < p_3 < \ldots, \quad p_j = p^o_j + \frac{\sigma_o + \sigma_j}{2p^o_j}, \quad j \in \mathbb{N} \}.
$$

Using asymptotics (1.22) with $\xi(p_j) = \frac{\sigma_o + \sigma_j}{2p^o_j}$ we define the **phase mapping** $\phi : L^2_+ \to \mathfrak{S}_o$ by:

$$
q \to \phi = (\sigma_j)_{j=0}^{\infty}, \quad \sigma_o = q_0 := \int_0^1 qdx.
$$

(1.24)

We formulate the second main result about the inverse problem.

**Theorem 1.5.** The phase mapping $\phi : L^2_+ \to \mathfrak{S}_o$ given by (1.24) is a bijection between $L^2_+$ and $\mathfrak{S}_o$. Moreover, for any $\phi \in \mathfrak{S}_o$, the corresponding potential $q \in L^2_+$ has the form:

$$
q(x) = \sigma_0 - 2 \frac{d^2}{dx^2} \log(\Gamma_n \det \Omega(x, p)), \quad x \in (0, 1),
$$

(1.25)

where $\Omega(x, p), x \in (0, 1)$ is the infinite matrix whose elements $\Omega_{n, j}$ are given by

$$
\Omega_{n, j}(x, p) = \frac{n^2_n - n^2_0}{n^2_n - n^2_0} \left\{ \cos \sqrt{n^2_n x} + \frac{(-1)^n - \cos 2\sqrt{n^2_n}}{\sin 2\sqrt{n^2_n}} \sin \sqrt{n^2_n} x, \quad \frac{\sin p^o_j x}{p^o_j} \right\},
$$

(1.26)

$$
n_n = p^2_n - \sigma_o = n^o_n + \bar{n}_n, \quad (\bar{n}_n)_{1}^{\infty} \in \ell^2,
$$

where $\Gamma_n \det \Omega(x, p)$.
the sequence \((p_n)_1^\infty\) is defined in Proposition 1.4, \(\{u, v\}_\omega = uv' - u'v\) and

\[
\Gamma_n = \prod_{j > n \geq 1} \left( \frac{n_n - n_0^0}{n_n - n_j}, \frac{n_n^0 - n_j}{n_n^0 - n_j^0} \right). \tag{1.27}
\]

Remarks.

(1) Here, \(\Omega - I\) is a trace class operator and \(\det \Omega(x, p)\) is well defined. The proof is based on the recovery identity for the Sturm–Liouville problem on the unit interval from [23], which was adopted from the case of even potentials, see p. 117 in [37].

(2) Recovering similar to (1.25) is well known due to Jost and Kohn [14]. In addition, similar formulas are used for reflectionless potentials on the line, see, for examples, [1, 9, 10], where only eigenvalues and norming constants are used. In (1.25), we use only the phase shift.

(3) In order to obtain similar results for matrix-valued potentials for second-order or first-order systems, we need to use [7] or [31, 32].

Example. We compute \(q\) with one parameter, when \(n_n, n \geq 1\) have the following form: The first \(n_1 = p_1^2, 0 < p_1 < \pi\) is free and all other \(n_n = (p_n^o)^2, n \geq 2\), are frozen. Let

\[
v = \sin \frac{\pi}{2}x, \quad u = \cos \sqrt{n_1}x - \frac{1 + \cos 2\sqrt{n_1}}{2\sqrt{n_1}} \sin \sqrt{n_1}x = C \sin \sqrt{n_1}(1 - x),
\]

where \(C = \sin n_1\). They satisfy

\[
u'' = -n_1u, \quad u'' = -(\pi/2)^2v, \quad w = uv' - u'v, \quad w' = w'' - u''v = Eu = n_1 - n_1^o.
\]

Then, from Theorem 1.5, we determine the corresponding potential \(q\) by

\[
q = -2(\ln w)'' = -2\left(\frac{w'}{w}\right)' = 2\left(\frac{w''}{w} - \frac{ww''}{w^2}\right).
\]

In this case, \(\Omega\) is a scalar. If we take the potential \(q\) with \(m\) free parameters \(0 < n_1 < \ldots < n_m < n_{m+1}^o\) and all other \(n_n = n_n^o, n > m\) are frozen, then \(\Omega\) is the \(m \times m\) matrix.

1.3 | Smooth potentials

In order to discuss the case of smooth potentials, we define the Sobolev space \(W_\alpha\) and the class \(W_\alpha^+\) by

\[
W_\alpha = \{q, q^(0) \in L^2(0, 1)\}, \quad W_\alpha^+ = \{q \in L^2_+ : q|_{(0, 1)} \in W_\alpha\}, \quad \alpha \geq 0.
\]

Recall results from [34]: If \(q \in W_\alpha\), then eigenvalues \(\mu_n\) and \(\tau_n\), \(n \geq 1\) of the problems (1.18) and (1.19) have the asymptotics

\[
\sqrt{\mu_n} = \sqrt{\mu_0} + \frac{\sigma_0}{2\sqrt{\mu_0}} + \frac{1}{\sqrt{\mu_0}} \sum_{1 \leq j \leq \alpha+1} \frac{a_j}{(4\mu_0^)^j} + \frac{u_n}{n^{\alpha+1}}, \tag{1.28}
\]

\[
\sqrt{\tau_n} = \sqrt{\tau_0} + \frac{\sigma_0}{2\sqrt{\tau_0}} + \frac{1}{\sqrt{\tau_0}} \sum_{1 \leq j \leq \alpha+1} \frac{b_j}{(4\tau_0^)^j} + \frac{v_n}{n^{\alpha+1}}, \tag{1.29}
\]

for some real \(a = (a_j^1)_{1}^{\infty}, b = (b_j^1)_{1}^{\infty} \in \mathbb{R}^\infty\) and \((u_n)^\infty, (v_n)^\infty \in \ell^2\). If \(q \in W_\alpha^+\), then due to (1.23), the sequences \(\tau_n = p_{2n-1}, \quad \mu_n = p_{2n}^2, n \geq 1\) satisfy (1.28)–(1.29). We define the spectral data \(\Xi_\alpha, \alpha \geq 1\) (similar to the case \(\alpha = 0\)) for
If \( q \in W_\alpha^+ \) for some \( \alpha \geq 1 \), then the sequence \( p_{2n-1} = \sqrt{\tau_n}, p_{2n} = \sqrt{\mu_n}, n \geq 1 \) has the asymptotics (1.28), (1.29). Thus, we have the mapping from \( W_\alpha^+ \) into \( \mathcal{G}_\alpha \) given by

\[
\phi_\alpha : q \to \{a, b, \sigma\}, \quad a = (a_j)_1^\alpha, b = (b_j)_1^\alpha, \quad \sigma = (\sigma_n)_0^\infty.
\]

(1.30)

**Corollary 1.6.** Each mapping \( \phi_\alpha : W_\alpha^+ \to \mathcal{G}_\alpha, \alpha \geq 1 \) is a bijection between \( W_\alpha^+ \) and \( \mathcal{G}_\alpha \).

The plan of this paper is as follows. In Section 2, we prove the main results. In Section 3, we consider the mapping \( q \to S(r_n, q), n \in \mathbb{N} \), for some increasing sequence \( r_n > 0, n \geq 1 \), and finish the proof of Theorem 1.3. In Section 4, we discuss results of Jost and Kohn.

## 2 | PROOF OF MAIN THEOREMS

### 2.1 | Preliminaries

We recall some well-known facts about entire functions (see, e.g., [18]). An entire function \( f(z) \) is said to be of exponential type if there is a constant \( A \) such that \( |f(z)| \leq \text{const.} \cdot e^{A|z|} \) everywhere. The function \( f \) is said to belong to the Cartwright class \( \mathcal{E}_{\text{Cart}} \), if \( f(z) \) is entire, of exponential type, and satisfies:

\[
\int_{\mathbb{R}} \frac{\log(1 + |f(x)|)dx}{1 + x^2} < \infty, \quad \rho_+(f) = 0, \quad \rho_-(f) = 2,
\]

where the types \( \rho_{\pm}(f) \) are given by \( \rho_{\pm}(f) = \limsup_{y \to \infty} \frac{\log |f(z_0 + iy)|}{y} \). Let \( f(z) \) belong to the Cartwright class \( \mathcal{E}_{\text{Cart}} \) and denote by \( (z_n)_1^n \) the sequence of its zeros \( \neq 0 \) (counted with multiplicity), so arranged that \( 0 < |z_1| \leq |z_2| \leq \ldots \). Then, we have the Hadamard factorization:

\[
f(z) = f(0)e^{iz} \lim_{r \to +\infty} \prod_{|z_n| \leq r} \left(1 - \frac{z}{z_n}\right),
\]

uniformly in every bounded disk and

\[
\sum_{z_n \neq 0} \frac{|\text{Im} z_n|}{|z_n|^2} < \infty.
\]

(2.2)

We discuss Jost functions. It is well known that the Jost solution \( f_+(x, k) \) satisfies the equation

\[
f_+(x, k) = e^{ixk} - \int_x^1 \frac{\sin k(x-t)}{k} q(t)f_+(t, k)dt, \quad x \in [0, 1],
\]

(2.3)

for all \( (k, q) \in \mathbb{C} \times P_1 \). Note that the Jost function \( \psi(k) = f_+(0, k) \) is entire. The Jost function \( \psi(k) \) is real on the imaginary line and then satisfies \( \overline{\psi(k)} = \psi(-k) \) for all \( k \in \mathbb{C} \).

We need standard results about the Jost function for \( q \in P_1 \) (see, e.g., [10, 19, 21]).
(1) Introduce the solutions \( \varphi(x,k), \tilde{\varphi}(x,k) \) of the equation \(-y'' + qy = k^2 y\) under the conditions: \( \varphi(0,k) = \tilde{\varphi}'(0,k) = 0 \) and \( \varphi'(0,k) = \tilde{\varphi}(0,k) = 1 \). Note that the function \( \varphi \) has the form

\[
\varphi(x,k) = \frac{f_+(0,k)}{2ik} (f_+(x,k)S(k) - f_+(x,-k)) \quad \forall (x,k) \in \mathbb{R}_+ \times \mathbb{R}.
\]  

(2.4)

From (2.4), we obtain the well-known identity for any \( k > 0, x \geq 1 \):

\[
\varphi(x,k) = \frac{|\psi(k)|}{k} \sin(kx - \xi(k)).
\]  

(2.5)

(2) The Jost function is expressed in terms of the fundamental solutions \( \varphi, \tilde{\varphi} \) for all \( k \in \mathbb{C} \) by

\[
e^{-ikf_+(0,k)} = \varphi'(1,k) - ike^{-ik\varphi(1,k)},
\]

\[
e^{-ikf_+'(0,k)} = ike^{-ik\tilde{\varphi}(1,k)} - \tilde{\varphi}'(1,k).
\]  

(2.6)

(3) Recall that \( \hat{q}(k) = \int_0^1 q(x)e^{2ikx}dx \). The Jost function \( \psi(k) = f_+(0,k), q \in L^1_C \), satisfies

\[
|\psi(k)| \leq e^{|v|-\omega}, \quad |\psi(k) - 1| \leq \omega e^{|v|-\omega}, \quad v = \text{Im } k,
\]  

(2.7)

\[
\psi(t) = 1 + \psi_1 + \psi_2, \quad \psi_1(k) = \frac{\hat{q}(k) - \hat{q}(0)}{2ik}, \quad \psi_2(k) \leq \omega^2 e^{|v|-\omega},
\]  

(2.8)

where \( |k|_1 = \max\{1, |k|\} \) and \( \omega = \min\{|q|, \frac{||q||}{|k|}\} \) and \( ||q|| = \int_0^1 |q(x)|dx \).

(4) The Jost function has the following asymptotics:

\[
\log \psi(k) = -\psi_1(k) + \frac{O(1)}{k^2},
\]  

(2.9)

\[
\xi(k) = \frac{\hat{q}(0) - \hat{q}_c(k)}{2k} + \frac{O(1)}{k^2},
\]  

(2.10)

as \( |k| \to \infty, k \in \overline{\mathbb{C}_+} \), and uniformly with respect to \( \text{arg } k \in [0, \pi] \) and \( \hat{q}_c(k) = \int_0^1 q(x) \cos 2kx dx \).

Below we need the following results.

**Theorem 2.1.**

(i) The mapping \( q \to \psi(\cdot, q) \) acting from \( P_1 \) to \( J_1 \) is a bijection.

(ii) Let \( \tilde{\psi} = B(k)\psi(k, q) \) for some \( q \in P_1 \), where \( B(k) = \prod_{j=1}^{n_0} \frac{k+k_j}{k-k_j} \). Then, \( \tilde{\psi} \) is the Jost function \( \psi(k, \tilde{q}) \) for a unique potential \( \tilde{q} \in L^1_+ \).

(iii) Let \( q \in P_1 \). Then, the types \( \rho_+(\psi) = 0 \) and \( \rho_-(\psi) = 2 \) and the Jost function \( \psi \) is given by

\[
\psi(k) = k^{n_0} \tilde{q}^{(n_0)}(0) e^{ik} \lim_{l \to \infty} \prod_{|k_n| \leq r, k_n \neq 0} \left( 1 - \frac{k}{k_n} \right), \quad k \in \mathbb{C},
\]  

(2.11)

uniformly on compact subsets of \( \mathbb{C} \), where \( n_0 = n_0(\psi) \in \{0, 1\} \). Furthermore, the shift function \( \xi \) has the form

\[
\xi(k) = -\pi \left( m + \frac{n_0}{2} \right) + \int_0^k \xi'(t)dt, \quad \xi'(k) = 1 + \sum_{n=1}^{\infty} \frac{\text{Im } k_n}{|k - k_n|^2}, \quad k \geq 0,
\]  

(2.12)
uniformly on compact subsets of $\mathbb{R} \setminus \{0\}$; and norming constants $\mathcal{C}_j := \int_0^\infty f_j^2(x, k_j) \, dx$ satisfy

$$\mathcal{C}_j = -i \frac{\psi'(k_j)}{\psi(-k_j)} \quad \forall \ j \in \mathbb{N}_m. \quad (2.13)$$

Moreover, we can recover the potential $q$ in terms of its eigenvalues and resonances.

(iv) The following trace formula holds true:

$$-2k \, \text{Tr} (R(k) - R_0(k)) = \frac{n_0}{k} + i + \lim_{r \to \infty} \sum_{k_n \neq 0, |k_n| < r} \frac{1}{k - k_n}.$$  \hspace{1cm} (2.14)

uniformly on compact subsets of $\mathbb{C} \setminus \{k_n, n \in \mathbb{N}\}$.

Proof. Proof of (i)–(iii) was given in [19]. We show (2.14). Define the Fredholm determinant

$$D(k) = \det(I + Y(k)), k \in \mathbb{C}_+,$$  \hspace{1cm} where $Y(k) = |q|^{1/2} R_0(k^2) |q|^{1/2} \text{sign} q.$

It is well known that $Y(k)$ is the trace class operator and thus the determinant is well defined. Recall the known fact, see, for example, [13]. Let an operator-valued function $\Omega : \mathcal{D} \to \mathcal{B}_1$ be analytic for some domain $\mathcal{D} \subset \mathbb{C}$ and $(I + \Omega(z))^{-1}$ be bounded for any $z \in \mathcal{D}$. Then, for the function $F(z) = \det(I + \Omega(z))$, we have

$$F'(z) = F(z) \, \text{Tr}((I + \Omega(z))^{-1} \Omega'(z)). \quad (2.15)$$

Thus, due to (2.15) and the identity $R = R_0 - RqR_0$, we obtain the known identity

$$\frac{D'(k)}{D(k)} = 2k \, \text{Tr}(I + Y(k))^{-1} Y'(k) = -2k \, \text{Tr}(R(k^2) - R_0(k^2)).$$

The Hadamard factorization (2.11) yields the derivative of the Jost function

$$\frac{\psi'(k)}{\psi(k)} = \frac{n_0}{k} + i + \lim_{r \to \infty} \sum_{k_n \neq 0, |k_n| < r} \frac{1}{k - k_n}.$$  \hspace{1cm} (2.16)

Using a known result of Jost–Pais [15] that the Jost function is a Fredholm determinant, $\psi = D$, we obtain (2.14). \hfill \Box

Remark.

(1) Due to (2.1), the series in (2.12) converges absolutely.

(2) It should be noted that there is a paper [2], where the inverse scattering problem for Schrödinger operators with compactly supported potentials is solved. These results are similar to results from [19], obtained 10 years before.

Proof of Theorem 1.1. We omit the proof of (i), since it repeats one for the case $\alpha = 1$ from [19]. We prove only (ii). We will prove that the mapping $S : P_{\alpha} \to S_{\alpha}$ given by $q \to S(k, q)$ is an injection. Let $\psi_j(k)$ be the Jost function for potentials $q_j \in P_{\alpha}, j = 1, 2$ and let $S_j = \psi_j(-k_j) / \psi_j(k)$ be the S-matrix. We assume that $S_1 = S_2$. We show that $q_1 = q_2$. The properties of the class $J_{\alpha}$ give that the functions $\psi_1$ and $\psi_2$ have the same zeros $\neq 0$. Note that $n_0(\psi_2) = n_0(\psi_1) \leq 1$, since the functions $\psi_1, \psi_2$ have the same even number of zeros on the interval $(-k_m, 0]$. This and (2.11) yield $\psi_1 = \psi_2$ and Theorem 1.1 implies $q_1 = q_2$.

We show that the mapping $S : P_{\alpha} \to S_{\alpha}$ is onto (the characterization). Let $S = \frac{f(-k)}{f(k)}$ for some $f \in J_{\alpha}$. Then, due to Theorem 1.1, there exists a unique $q \in P_{\alpha}$ such that $f = \psi(k, q)$ and then we obtain $S = \frac{f(-k)}{f(k)} = \frac{\psi(-k, q)}{\psi(k, q)}$. \hfill \Box

2.2 Proof of main results

We are ready to prove main theorems.
Proof of Theorem 1.3. (i) Let $\psi_j$ be the Jost function for the potential $q_j \in L^1$, $j = 1, 2$. Define the function

$$F(k) = \psi_1(k)\psi_2(-k) - \psi_2(k)\psi_1(-k), \quad k \in \mathbb{C}.$$ 

The function $F$ is entire and has the following properties:

1. $F \in L^2(\mathbb{R})$, since the asymptotics of the functions $\psi_1, \psi_2$ from (1.2) gives $F(k) = O(1/k)$ as $k \to \pm \infty$.
2. $F(x_n) = 0$ for all $x_n, n \in \mathbb{N}$, and $F(0) = 0$.
3. The function $F$ has a type $\leq 2$.

Then, from (1), (3), and the Paley–Wiener Theorem (see p. 30 in [18]), we deduce that

$$F(k) = \int_{-1}^{1} g(x)e^{-2i(kx)}dx, \quad \text{for some } g \in L^2(-1,1).$$

Recall the Kadets Theorem [17]: If a sequence $r = (r_n)_{n=1}^{\infty} \in \mathcal{R}$, then the system of the functions $1, e^{\pm 2\pi ir_n}$ is a Riesz basis of $L^2(-1,1)$. Thus, the Kadets Theorem and the properties (1)–(3) yield $F = 0$ and then $\frac{\psi_1(k)}{\psi_1(-k)} = \frac{\psi_2(k)}{\psi_2(-k)}$ for all $k \in \mathbb{C}$. Then, Theorem 1.1(i) implies that $q_1 = q_2$.

The proof of (ii) will be given in Section 3. \qed

In order to prove main results, we discuss the properties of the sequence $p_n, n \geq 1$.

Proof of Proposition 1.4. Let $q \in L^1_+$. Then, from (2.12), we obtain that $1 - \xi'(k) > 0$ for all $k \geq 0$ and $k - \xi(k) = \frac{\pi}{2} n_o + \int_{0}^{k} (1 - \xi'(t))dt > 0$. Asymptotics (2.10) yields that $k - \xi(k) \to +\infty$ as $k \to +\infty$. Thus, for each $n \geq 1$, the equation $k - \xi(k) = p_o = \frac{\pi}{2} n$ has a unique solution $p_n \geq 0$ such that due to (2.10), we have

$$p_n = p_o + \sigma_0 + \sigma_n \frac{2p_o}{n}, \quad \sigma_0 = q_o = \int_{0}^{1} q dx \quad \text{and} \quad \sigma_n \to 0 \quad \text{as} \quad n \to \infty. \quad (2.16)$$

Due to (2.5), the function $\varphi(1,k) = \frac{|\psi(k)|}{k} \sin(k - \xi(k)), k > 0$ has the following zeros $p_{2n}, n \geq 1$. Thus, each $p_{2n}^2 = \mu_n, n \geq 1$, is the eigenvalues of the problem $-y'' + qy = \lambda y, y(0) = y(1) = 0$.

The function $\varphi'(1,k) = |\psi(k)| \cos(k - \xi(k)), k > 0$ has the following zeros $p_{2n-1}, n \geq 1$. Thus, each $p_{2n-1}^2 = \tau_n, n \geq 1$, is the eigenvalue of the problem $-y'' + qy = \lambda y, y(0) = y'(1) = 0$.

Let $q \in L^2_+$. Then, from (2.10) and (2.16) we have $p_n = p_o + \sigma_0^2 + \sigma_n\frac{2p_o}{n}$, where $(\sigma_n)_{n=0}^{\infty} \in \ell^2(\mathbb{Z}_+)$. \qed

We prove our main results.

Proof of Theorems 1.5. Define sets of spectral data $\mathcal{N}_+ = \{(n_j)_{j=1}^{\infty} \in \mathcal{N} : 0 \leq n_1\}$ and

$$\mathcal{R} = \left\{ n = (n_j)_{j=1}^{\infty} : n_1 < n_2 < n_3 < ..., n_n = \frac{(nn)^2}{4} + \bar{n}_o + \bar{n}_n, \quad \bar{n}_o \in \mathbb{R}, (\bar{n}_n)_{n=1}^{\infty} \in \ell^2 \right\},$$

and a real space $\mathcal{H} = L^2((0,1),\mathbb{R})$. Recall Corollary 1.4 from [23]: The mapping $r : \mathcal{H} \to \mathcal{N}$ given by

$$q \to r = (r_j)_{j=1}^{\infty}, \quad \text{where} \quad r_{2j-1} = \tau_j, \quad r_{2j} = \mu_j, \quad j \geq 1, \quad (2.17)$$

is a real-analytic isomorphism between $\mathcal{H}$ and $\mathcal{N}$. Moreover, the potential $q - q_o$, where $q_o = \int_{0}^{1} q dx$ has the form (1.25), where $\tau_j = \tau_j - q_o, j \in \mathbb{N}$.

Due to Proposition 1.4, for each $q \in L^2_+$, there exist a unique increasing sequence of nonnegative numbers $(p_{n})_{n=1}^{\infty}$ such that

1. $p_{2n}^2 = \mu_n, n \geq 1$ are the eigenvalues of the problem $-y'' + qy = \lambda y, y(0) = y(1) = 0$;
2. $p_{2n-1}^2 = \tau_n > 0, n \geq 1$ are the eigenvalues of the problem $-y'' + qy = \lambda y, y(0) = y'(1) = 0$;
3. the sequence $p_n = p_o + \sigma_0 + \sigma_n\frac{2p_o}{n}$, where $\sigma = (\sigma_n)_{n=0}^{\infty} \in \ell^2$. 

\(\square\)
Thus, the sequence \((p_j^2)_{j=0}^\infty\) belongs to \(\mathcal{N}_+\) and due to the properties of the mapping defined by (2.17), the mapping \(q \to (p_j^2)_{j=0}^\infty\) is a bijection between \(L^2_+\) and \(\mathcal{N}_+\). Consider the phase mapping \(\phi : L^2_+ \to \mathcal{G}_0\) given by \(q \to \phi = (\sigma_j)_{j=0}^\infty\), where \(\sigma_0 = q_0 = \int_0^1 q\,dx\). It is clear that the mapping \((p_j) \to \phi = (\sigma_j)_{j=0}^\infty\) from \(\mathcal{N}_+\) into \(\mathcal{G}_0\) is a bijection between \(\mathcal{N}_+\) and \(\mathcal{G}_0\). Then, we take a shifted potential \(q - q_0\) and a shifted sequence \(\tilde{n} = (\tilde{n}_j)_{j=0}^\infty\in\ell^2\) given by \(\tilde{n}_j = p_j^2 - \sigma_0 = (\pi_j)^2 + \tilde{n}_j\), \((\tilde{n}_j)_{j=0}^\infty\in\ell^2\). Then, using Corollary 1.4 from [23], we obtain the identity (1.25). □

**Proof of Corollary 1.6.** Due to Theorem 1.5, the mapping \(\phi_0 : L^2_+ \to \mathcal{G}_0\) is a bijection. Recall that (1.23) implies \(\tau_n = p_{2n-1}, \mu_n = p_{2n}\), for all \(n \geq 1\). Recall results from [34]:

The potential \(q \in W_\alpha\) for some \(\alpha \geq 0\) if and only if the eigenvalues \(\tau_n\) and \(\mu_n\), \(n \geq 1\) (corresponding to the problems (1.18) and (1.19)) have asymptotic estimates given by (1.28)–(1.29).

Then, from this result, we deduce that the mapping \(\phi_\alpha : W^+_\alpha \to \mathcal{G}_\alpha\) given by (1.30) is a bijection between \(W^+_\alpha\) and \(\mathcal{G}_\alpha\) for any \(\alpha \geq 0\). □

### 2.3 Even potentials

Recall that \(\mu_j = \mu_j(q), n \geq 1\) are the Dirichlet eigenvalues of the problem (1.18) and \(\tau_j = \tau_j(q), n \geq 1\) are the mixed eigenvalues of the problem (1.19) on the unit interval corresponding to \(q\) and they satisfy \(\tau_1 < \mu_1 < \tau_2 < \mu_2 < \ldots\). We need to recall results from [22], which will be crucial for us:

\[
q \in L^2_+ \iff \tau_1(q) \geq 0 \iff \mu_1(q) > 0, \quad \varphi'(1,0,q) \geq 0.
\]  

(2.18)

We discuss the inverse problem for the case of even potentials \(q \in L^2_+\), where the set of even potentials is defined by

\[
L^2_+ = \left\{ q \in L^2_+ : q(x) = q(1-x), \quad \forall x \in (0,1) \right\}.
\]

The condition, when \(q \in L^2_+\), is formulated in terms of \(\tau_1 \geq 0\). In Theorem 1.5, we have used \(\tau_j, \mu_j\) for all \(j \in \mathbb{N}\) and the condition (2.18) was very natural. But for even potentials, we plan to use only the Dirichlet eigenvalues \(\mu_j, j \geq 1\). Thus, we have to rewrite conditions \(q \in L^2_+\) only in terms of Dirichlet eigenvalues (2.21).

**Lemma 2.2.** Let \(q \in L^2_+\). Then, the function \(\varphi'(1,\lambda)\) has the form

\[
\varphi'(1,\lambda) = \cos \sqrt{\lambda} + \sum_{j=1}^{\infty} \frac{(-1)^j - \cos \sqrt{\mu_j}}{\lambda - \mu_j} \varphi(1,\lambda) \varphi(1,0), \quad \lambda \in \mathbb{R},
\]

(2.19)

where \(\dot{u} = \frac{\partial}{\partial \lambda} u\) and the series converges uniformly on compact sets in \(\mathbb{C}\), in particular,

\[
\varphi'(1,0) = 1 - \sum_{j \geq 1} \frac{1 - \cos(\sqrt{\mu_j} - \pi)}{\mu_j(-1)^j} \varphi(1,0)\varphi(1,0),
\]

(2.20)

where \((-1)^j \varphi(1,\mu_j) > 0\) and \(\varphi(1,0) > 0\). Moreover, we have

\[
q \in L^2_+ \iff \mu_1(q) > 0, \quad \sum_{j \geq 1} \frac{1 - \cos(\sqrt{\mu_j} - \pi)}{\mu_j(-1)^j} \varphi(1,0) \varphi(1,0) \leq 1.
\]

(2.21)
Proof. We need the interpolation formula from McKean–Trubowitz’s paper [36] (see Theorem 2, p. 174): Let \( F = \varphi'(1, \lambda) - \cos \sqrt{\lambda}, \) then the following identity,

\[
F(\lambda) = \sum_{n \geq 1} \frac{F(\mu_j)\varphi(1, \lambda)}{(\lambda - \mu_j)\varphi(1, \mu_j)}, \quad \lambda \in \mathbb{C},
\]

(2.22)

holds true, where the series converges uniformly on compact sets in \( \mathbb{C}. \) It is well known that if \( q \in \mathcal{L}^2_{+,e}, \) then \( \varphi'(1, \mu_j) = (-1)^j \) for all \( j \geq 1, \) see, for example, [11], which yields \( F(\mu_j) = (-1)^j - \cos \sqrt{\mu_j} \) and (2.19), (2.20). From (2.20), (2.18), we obtain (2.21).

For \( q \in \mathcal{L}^2_{+,e}, \) due to Proposition 1.4, we have an increasing sequence \( (p_j)^\infty_{j=1} \) such that

\[
0 \leq p_1 < p_2 < \ldots, \quad \text{where} \quad p_j = p^0_j + \xi(p_j), \quad \xi(p_j) = \frac{\sigma_0 + \sigma_j}{2p^0_j}, \quad (\sigma_j)^\infty_{j=0} \in \ell^2(\mathbb{Z}+),
\]

and \( \sigma_0 = \int_0^1 q dx. \) We consider even potentials \( q \in \mathcal{L}^2_{+,e}, \) then we need less spectral data and we take only even sequences \( p_{2j} = \sqrt{\mu_j}, j \geq 1. \) Moreover, in this case, the Dirichlet eigenvalues \( \mu_j \) have to satisfy an additional condition (2.21). Thus, using these conditions, we define the spectral data \( \mathfrak{S}_{o,e} \) for even potentials by

\[
\mathfrak{S}_{o,e} = \left\{ \sigma_e = (\sigma_{2j})^\infty_{j=0} \in \ell^2(\mathbb{Z}+): 0 < p_2 < p_4 < \ldots, \quad p_{2j} = p^0_{2j} + \frac{\sigma_0 + \sigma_j}{2p^0_j}, \quad j \geq 1, \right\}
\]

(2.23)

The additional condition (2.23) for potentials from \( q \in \mathcal{L}^2_{+,e} \) corresponds to the fact that the eigenvalue \( \tau_1 \geq 0, \) see (2.21). Using asymptotics (1.22) with \( \xi(p_j) = \frac{\sigma_j + \sigma_j}{2p^0_j} \) for even \( j = 2n, \) similar to the phase mapping \( \phi: \mathcal{L}^2_{+} \rightarrow \mathfrak{S}_{o}, \) we define the even phase mapping \( \phi_e: \mathcal{L}^2_{+,e} \rightarrow \mathfrak{S}_{o,e} \) by:

\[
q \rightarrow \phi_e = \sigma_e = (\sigma_{2j})^\infty_{j=0}, \quad \sigma_0 = q_0 = \int_0^1 q dx.
\]

(2.24)

We formulate our result about the inverse problem for even potentials. For this case we, use only Dirichlet eigenvalues \( \mu_n = p^2_{2n} \) and do not use eigenvalues \( \tau_n = p^2_{2n-1}, n \in \mathbb{N} \) for mixed boundary conditions. In order to recover the potential \( q \in \mathcal{L}^2_{+,e} \) in (2.25), we need a simple modification of Dirichlet eigenvalues \( \mu_n = p^2_{2n} \) and we define the sequences of shifted Dirichlet eigenvalues \( m_n = \mu_n - \sigma_0, n \geq 1, \) such that \( (m_n - (\pi n)^2)^\infty_{n=0} \in \ell^2. \)

**Theorem 2.3.** The even phase mapping \( \phi_e: \mathcal{L}^2_{+,e} \rightarrow \mathfrak{S}_{o,e} \) is a bijection between \( \mathcal{L}^2_{+,e} \) and \( \mathfrak{S}_{o,e}. \) Moreover, for any \( \phi_e(q) = \sigma_e \in \mathfrak{S}_{o,e}, \) the corresponding potential \( q \in \mathcal{L}^2_{+,e} \) has the form:

\[
q(x) = \sigma_o - 2 \frac{d^2}{dx^2} \log \left( G_m \det \Omega(x, m) \right), \quad x \in (0, 1),
\]

(2.25)

\[
m_n = \mu_n - \sigma_0 = (\pi n)^2 + \tilde{m}_n, \quad \text{where} \quad (\tilde{m}_n)^\infty_{n=0} \in \ell^2,
\]

where \( \Omega(x, m) \) is the infinite matrix whose elements \( \Omega_{n,j} \) are given by

\[
\Omega_{n,j}(x, m) = m_n - (\pi n)^2 \left\{ \cos \sqrt{m_n}x + \frac{(-1)^j - \cos \sqrt{m_{nj}}}{\sin \sqrt{m_n} \sin \sqrt{m_{nj}} \frac{\sin \pi j x}{\pi j}} \right\},
\]

(2.26)

where \( \{u, v\}_w = uv' - u'v \) and \( G_m = \prod_{j \geq 1} \left( \frac{m_n - (\pi n)^2}{m_n - m_j}, \frac{(\pi n)^2 - m_j}{(\pi n)^2 - (\pi j)^2} \right). \)
Proof. The proof for even potentials is similar to the generic case of Theorem 1.5, but the formulas (2.25), (2.26) for even potentials are taken from p. 117 of [37]. There is a small difference between (1.25) and (2.25).

Let \( q \in L^2_{+,e} \). Then, due to Theorem 1.5 and (2.21), we deduce that \( \phi_e(q) = \sigma_e \in S_0,e \).

Recall the results from [37]. Define the space of even potentials

\[
\mathcal{H}_e = \left\{ f \in L^2(0,1) : \int_0^1 f(x)dx = 0, f(x) = f(1-x) \forall x \in (0,1) \right\}.
\]

Following the book of Pöschel and Trubowitz [37], we define the set \( \mathcal{R}(\mu_0) \) of all real, strictly increasing sequences by

\[
\mathcal{R}_0 = \left\{ \mu = (\mu_n)_{n=1}^{\infty} : \mu_1 < \mu_2 < \mu_3 < \ldots, \mu_n = (\pi n)^2 + \tilde{\mu}_n, (\tilde{\mu}_n)_{n=1}^{\infty} \in \ell^2 \right\}.
\]

Let \( \mu_n(q) \), \( n \geq 1 \) be the Dirichlet eigenvalues for \( q \in \mathcal{H}_e \). The mapping \( \mu_n(\mathcal{H}_e) \to \mathcal{R}_0 \) is a bijection between \( \mathcal{H}_e \) and \( \mathcal{R}_0 \). Moreover, the potential \( q \) is recovered by the identity (2.25) \( q = q - \sigma_o \) in terms of its eigenvalues \( (\mu_n(q))_{n=1}^{\infty} \).

Let \( q \in L^2_{+,e} \). Then, due to Theorem 1.5, the sequence \( \mu_n = \frac{p_n^2}{2n} \) are the Dirichlet eigenvalues for \( q \in \mathcal{H}_e \). Due to Lemma 2.2, the sequence \( \sigma_e \in S_{0,e} \) and the above results from [37] give that the mapping \( \phi_e : L^2_{+,e} \to S_{0,e} \) is an injection.

Let us have \( \sigma_e = (\sigma_{2n})_{n=0}^{\infty} \in S_{0,e} \). Then, we have an increasing sequence \( \mu_n = \frac{p_n^2}{2n}, n \in \mathbb{N} \), and the above results from [37] give that \( \mu_n \) are the Dirichlet eigenvalues for \( q \in \mathcal{H}_e \) such that condition (2.23) holds true, which jointly with (2.21) gives \( q \in L^2_{+,e} \). Thus, the even phase mapping \( \phi_e : L^2_{+,e} \to S_{0,e} \) is a bijection between \( L^2_{+,e} \) and \( S_{0,e} \). Moreover, the above results from [37] give (2.25).

\[ \square \]

3 | ANALYTIC MAPPING

3.1 | Analytic properties of S-matrix

We now discuss differentiable maps. Let \( \mathcal{H} \) and \( \mathcal{G} \) be separable complex Hilbert spaces. Let \( U \subset \mathcal{H} \) be open. A map \( f : U \to \mathcal{G} \) is differentiable at \( q \in U \) if there exists a bounded map \( A : \mathcal{H} \to \mathcal{G} \) such that

\[
\| f(q + h) - f(q) - Ah \| = o(\|h\|) \quad h \to 0.
\]

The linear map \( A \) is uniquely determined and is called the derivative of \( f \) at \( q \) and is denoted by \( \frac{\partial}{\partial q} f \). The map \( f \) is differentiable on \( U \), if it is differentiable at each point in \( U \).

We very often consider complex-valued differentiable functions \( f \), which are defined on open subsets of \( L^2_{\mathbb{C}}(0,1) \). In this case, \( \frac{\partial}{\partial q} f \) is a bounded linear functional on \( L^2_{\mathbb{C}}(0,1) \). It follows from the Riesz representation theorem that there is a unique element \( f'_q = \frac{\partial}{\partial q} f \) in \( L^2_{\mathbb{C}}(0,1) \) such that

\[
\frac{\partial}{\partial q} f(v) = \langle v, f'_q \rangle = \int_0^1 v(x)f'_q(x)dx
\]

for all \( v \in L^2_{\mathbb{C}}(0,1) \), where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( L^2(0,1) \). The function \( f'_q = \frac{\partial}{\partial q} f \) is the gradient of \( f \) at \( q \). Our notation imitates the usual finite-dimensional notation.

We discuss analytic maps. By definition, a map \( f : U \to F \) is analytic on \( U \), if it is continuously differentiable on \( U \). It is the natural generalization of Riemann’s notion of analytic function of one complex variable.

Recall that \( \varphi(x,k) \) is the solutions of the equation \( -\varphi'' + q \varphi = k^2 \varphi \) under the conditions: \( \varphi(0,k) = 0, \varphi'(0,k) = 1 \), and \( q \in L^2_{\mathbb{C}}(0,1) \). We need properties of the fundamental solution \( \varphi(x,k) = \varphi(x,k,q) \), see, for example, [37]. First, we recall the following standard estimates:

\[
|\varphi(x,k,q)| \leq \frac{e^{x|\text{Im} k|+\omega}}{|k|_1}, \quad |\varphi(x,k,q) - \varphi_0(x,k)| \leq \frac{\omega}{2|k|_1}e^{x|\text{Im} k|+\omega},
\]

(3.1)

where \( |k|_1 = \max\{1,|k|\}, \omega = \min\{\|q\|,\|q\|/|k|\} \), \( \|q\| = \int_0^1 |q(x)|dx \),

where \( \varphi_0(x,k) = \frac{\sin kx}{k} \). Second, we recall the well-known analytic properties from [37].
Lemma 3.1. The functions $\varphi(1, k, q), \varphi'(1, k, q)$ are entire on $(k, q) \in \mathbb{C} \times L^2_C(0, 1)$. Their gradients are given by

\[
\frac{\partial \varphi(1, k, q)}{\partial q(x)} = (A\varphi)(x, k, q),
\]

\[
\frac{\partial \varphi'(1, k, q)}{\partial q(x)} = (B\varphi)(x, k, q),
\]

where $x \in [0, 1]$ and

\[
A(x, \lambda, q) = \varphi(1)\vartheta(x) - \varphi(x)\vartheta(1), \quad B(x, \lambda, q) = \varphi'(1)\vartheta(x) - \varphi(x)\vartheta'(1),
\]

and here $\varphi(x) = \varphi(x, k, q), \vartheta(x) = \vartheta(x, k, q), ...$, for shortness.

We need properties of the Jost function as a function of a potential from the Hilbert space.

Lemma 3.2.

(i) The function $\psi(k, q)$ is entire on $\mathbb{C} \times L^2_C$ and its gradient is given by

\[
\frac{\partial \psi(k, q)}{\partial q(x)} = \varphi(x, k, q)f_+(x, k, q), \quad x \in [0, 1].
\]

(ii) For each $q \in L^2_C$, there exists $\zeta \in L^2_C$ such that the following identities hold true:

\[
\psi(k, q) = 1 + \hat{g}(k) = 1 + \frac{\hat{\zeta}(k) - \hat{\zeta}(0)}{2ik}, \quad \forall k \in \mathbb{C},
\]

where the function $g(t) = \int_1^t \zeta(x)dx$.

(iii) Let $q^o \in L^2_C$ and let $\delta > 0$. There exists $\varepsilon > 0$ such that if $\|q - q^o\|_{L^2} \leq \varepsilon$, then

\[
|\psi(k, q) - \psi(k, q^o)| \leq \delta \quad \forall k \in \mathbb{R}.
\]

Moreover, let $u > 0$ and let, in addition, $2\delta < C_* := \min_{k \geq u} |\psi(k, q^o)|$ (where $C_* > 0$). Then, for all $q \in B_\varepsilon(q^o)$ and $\varepsilon > 0$ small enough, we have

\[
\min_{k \geq u} |\psi(k, q)| \geq \frac{C_*}{2}.
\]

Proof. (i) Lemma 3.1 and (2.6) imply that the function $\psi(k, q)$ is entire on $\mathbb{C} \times L^2_C$. Let for shortness $f_+(x, k, q) = f_+(x, k, q)$, $\varphi(x) = \varphi(x, k, q), ...$ Using (2.6) and Lemma 3.1, we obtain

\[
\frac{\partial \psi(k)}{\partial q(x)} = e^{ikx} \left( \frac{\partial \varphi'(1)}{\partial q(x)} - ik \frac{\partial \varphi(1)}{\partial q(x)} \right)
\]

\[
= e^{ikx} \varphi(x) \left( \vartheta(x)\varphi'(1) - \varphi(x)\vartheta'(1) \right) - ik \left( \vartheta(x)\varphi(1) - \varphi(x)\vartheta(1) \right)
\]

\[
= e^{ikx} \varphi(x) \left( \vartheta(x)\varphi'(1) - ik\varphi(1) \right) + \varphi(x) \left[ - \vartheta'(1) + ik \vartheta(1) \right]
\]

\[
= \varphi(x) \left( \vartheta(x)\psi(0) + \varphi(x)\psi'(0) \right) = \varphi(x)f_+(x).
\]

The statement (ii) was proved in [19] for real $q$. The proof for the complex case is similar.
(iii) Let \( q = q^o + q \), \( p = q^o + t q \), and let \( F(k, x, t) = \frac{\partial}{\partial p(x)} \psi(k, p) \big|_{p=q^o+tq} \). Then, for real \( k \), we have

\[
\psi(k, q) - \psi(k, q^o) = \int_0^1 \frac{\partial}{\partial t} \psi(k, q^o + t q) dt = \int_0^1 \langle F(k, \cdot, t), q \rangle dt, \tag{3.9}
\]

where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( L^2(0, 1) \). From (i), (2.7), (3.1), we obtain

\[
|F(k, x, t)| = |\varphi(x, k, p) f_+(x, k, p)| \leq e^{2\|p\|} \leq e^{2\|q^o\| + \varepsilon}, \tag{3.10}
\]

since \( \|p\| = \|q^o + t q\| \leq \|q^o\| + \|q\| \leq \|q^o\| + \varepsilon \). Then, substituting the last estimate into (3.9), we get

\[
|\psi(k, q) - \psi(k, q^o)| \leq \int_0^1 e^{2\|q^o\| + \varepsilon} \left| \frac{k}{\|q\|} \right| \|q\| dt = \|q\| e^{2\|q^o\| + \varepsilon} \leq \varepsilon e^{2\|q^o\| + \varepsilon} = \delta. \tag{3.11}
\]

Let \( u > 0 \). Then, \( C_* = \min_{k \geq u} |\psi(k, q^o)| > 0 \), since \( \psi(k, q^o) \to 1 \) as \( k \to \infty \) and \( \psi(k, q^o) \neq 0 \) for all \( k > 0 \). From (3.11), for \( \delta \leq C_*/2 \), we obtain

\[
|\psi(k, q)| = |\psi(k, q^o) + \psi(k, q) - \psi(k, q^o)| \geq |\psi(k, q^o)| - |\psi(k, q^o) - \psi(k, q)|
\]

\[
\geq |\psi(k, q^o)| - \delta \geq C_* - \delta \geq C_*/2,
\]

which yields (3.8).

From these two lemmas, we obtain properties of functions \( S(k, q) \).

**Lemma 3.3.** Each function \( S(k, q) \) for any fixed \( k > 0 \) is real analytic on \( \mathcal{L}^2 \) and its gradient is given by

\[
\frac{\partial S(k, q)}{\partial q(x)} = -2ik \frac{\varphi^2(x, k, q)}{\psi^2(k, q)}. \tag{3.12}
\]

Moreover, let in addition, \( b > 0 \). Then, the following asymptotics holds true:

\[
S(k, q) = 1 + \frac{q_0 - \hat{q}_c(k)}{ik} + O(\|q\|^2), \tag{3.13}
\]

\[
\frac{\partial S(k, q)}{\partial q(x)} = -2i k \left( \sin k x + \frac{O(\|q\|)}{k} \right)^2,
\]

as \( |k| \to \infty \), \( \text{Im} k \in [-b, b] \), uniformly on the bounded subsets of \([0, 1] \times \mathcal{L}^2_C\).

**Proof.** Let for shortness \( f_+(x, k) = f_+(x, k, q) \), \( \varphi(x, k) = \varphi(x, k, q) \). Due to the definition \( S(k) = \frac{\psi(-k)}{\psi(k)} \) and using (2.6) and (3.5) and (2.4), we deduce that

\[
\frac{\partial S(k)}{\partial q(x)} = \frac{1}{\psi(k)} \frac{\partial \psi(-k)}{\partial q(x)} - S(k) \frac{1}{\psi(k)} \frac{\partial \psi(k)}{\partial q(x)}
\]

\[
= \varphi(x, -k) \frac{f_+(x, -k)}{\psi(k)} - S(k) \varphi(x, k) \frac{f_+(x, k)}{\psi(k)}
\]

\[
= \frac{\varphi(x, k)}{\psi(k)} \left( f_+(x, -k) - S(k) f_+(x, k) \right) = -2ik \frac{\varphi^2(x, k)}{\psi^2(k)}.
\]
From (2.7) and (2.8), we obtain

\[
S(k, q) = \frac{\psi(-k)}{\psi(k)} = \left(1 + \frac{q_0 - \hat{q}(-k)}{2ik} + \psi_2(-k)\right) \left(1 - \frac{q_0 - \hat{q}(k)}{2ik} + \psi_2(k)\right)^{-1}
\]

\[
= 1 + \frac{q_0 - \hat{q}(k)}{ik} + O\left(\frac{||q||^2}{k^2}\right).
\]

From (3.12), (3.1) and (2.7), (2.8), we obtain

\[
\frac{\partial S(k)}{\partial q(x)} = -2i k \psi^2(x, k) \psi_2(k) = -2i k \left(\sin kx + O\left(\frac{||q||}{k}\right)\right)^2 \left(1 + O\left(\frac{||q||}{k}\right)\right)^{-2},
\]

which yields (3.13).

**Proof of Proposition 1.2.** (i) From (3.13), we have

\[
S(r_n, q) = 1 + \frac{\sigma_0 + z_n}{ir_n}, \quad \sigma_0 = \int_0^1 q dx, \quad z_n = -\hat{q}_c(r_n) + O\left(\frac{||q||^2}{n}\right) \quad (3.14)
\]
as \(r_n \to \infty\), uniformly on the bounded subsets of \(L^2\). This yields asymptotics (1.9). Using this asymptotics, we obtain (1.10).

(ii) Let \(q^o \in L^2\) and let \(\delta > 0\), and let \(q \in B_\delta(q^o)\) for \(\epsilon > 0\) small enough. Let \(\psi(k) = \psi(k, q)\) and \(\psi^o(k) = \psi(k, q^o)\). Define \(g(k) = \psi(k) - \psi^o(k)\). We have

\[
S(k, q) - S(k, q^o) = \frac{\psi(-k)}{\psi(k)} - \frac{\psi^o(-k)}{\psi^o(k)} = \frac{A(k)}{B(k)}, \quad B(k) = \psi^o(k)\psi(k),
\]

\[
A(k) = \psi(-k)\psi^o(k) - \psi^o(-k)\psi(k) = \psi^o(k)g(-k) - \psi^o(-k)g(k), \quad (3.15)
\]

From (3.7), we obtain that for any \(\bar{\delta} > 0\), there exists \(\epsilon > 0\) such that if \(||q - q^o||_{L^2} \leq \epsilon\), then \(|g(k)| = |\psi(k, q) - \psi(k, q^o)| \leq \bar{\delta}\) for all \(k \in \mathbb{R}\). Using this estimate and (2.7), we obtain

\[
|A(k)| \leq e^{\omega \bar{\delta}} + e^{\omega \bar{\delta}} = 2e^{\omega \bar{\delta}}. \quad (3.16)
\]

Let \(C_* = C_*(u) = \min_{k > u} |\psi^o(k)\psi(k)|\) for some \(u > 0\) and using (3.8), we have

\[
|B(k)| = |\psi^o(k)\psi(k)| \geq C_*^2/2 > 0, \quad \forall k > u. \quad (3.17)
\]

The identity (3.15) and estimates (3.16) and (3.17) imply

\[
|S(k, q) - S(k, q^o)| = \frac{|A(k)|}{|B(k)|} \leq \frac{2e^{\omega \bar{\delta}}}{C_*^2/2} = \bar{\delta} \frac{e^{\omega \bar{\delta}}}{C_*}, \quad \forall k > u. \quad (3.18)
\]

We show (1.12). In (1.10), we have that

\[
S(r_n, q) = e^{\frac{c_0 + z_n}{ir_n}} \quad \forall n \in \mathbb{N}, \quad \sigma_0 = \int_0^1 q dx, \quad \sigma_n \to 0 \text{ as } n \to \infty,
\]

\[
S(r_n, q^o) = e^{\frac{c^o_0 + z^o_n}{ir_n}} \quad \forall n \in \mathbb{N}, \quad \sigma^o_0 = \int_0^1 q^o dx, \quad \sigma^o_n \to 0 \text{ as } n \to \infty, \quad (3.19)
\]

where \((\sigma_n)^o, (\sigma^o_n)^o \in ell^2(\mathbb{Z}_+).\) Let \(t_n = \sigma_n - \sigma^o_n, n \geq 0\). From (3.18) and (3.19) at \(u < r_1\), we obtain

\[
|U_n - 1| \leq \delta, \quad \text{where} \quad U_n := S(r_n, q)/S(r_n, q^o) = e^{\frac{t_n}{ir_n}}, \quad t_0 = \int_0^1 (q - q^o) dx,
\]
and
\[
\frac{|t_0 + t_n|}{\pi r_n} \leq |\sin \left( \frac{t_0 + t_n}{r_n} \right) | \leq \delta, \quad \text{and } t_n \to 0, \quad \text{as } n \to \infty,
\]
which yields (1.12).

### 3.2 Inverse problem

We consider the following inverse problem: to recover the potential when the S-matrix \( S(r, q) \) is given for some increasing sequence of energy. Let \( q^o \in L^2 \). We consider the local inverse problem for \( q \in B_\varepsilon(q^o) = \{ w \in L^2 : ||w - q^o|| \leq \varepsilon \} \) for some \( \varepsilon > 0 \). For a fixed sequence \((r_n)_1^\infty \in R \) and \( q^o \in L^2 \), we define the mappings

\[
q \rightarrow b = (b_n)_0^\infty, \quad b_0 = q_0 - q_0^0, \quad b_n = e^{i \frac{b_n}{r_n}} U_n(q, q^o), \quad U_n = S(r_n, q)/S(r_n, q^o),
\]

for \( n \geq 1 \). It is clear that \( b_0(q) \) is an entire function of \( q \in L^2 \).

**Lemma 3.4.** Let \( r \in R \) and \( q^o \in L^2 \). Then, each \( b_n = e^{i \frac{b_n}{r_n}} S(r_n, q)/S(r_n, q^o), \ n \geq 1, \) is real analytic on \( L^2 \). Moreover, its gradient is given by

\[
\frac{\partial b_n(q)}{\partial q(x)} = i \frac{b_n(q)}{r_n} \left( 1 - \frac{2 r_n^2}{\varphi^2(r_n, q)} \varphi^2(x, r_n, q) \right),
\]

and the following asymptotics hold true:

\[
b_n(q) = \hat{q}c(r_n) + O\left( \frac{\|q\|^2}{n} \right),
\]

\[
\frac{\partial b_n(q)}{\partial q(x)} = \cos 2 r_n x + O\left( \frac{\|q\|}{n} \right),
\]

as \( n \to \infty \), uniformly on the bounded subsets of \([0, 1] \times L^2 \), where \( \hat{q}c(k) = \int_0^1 q(x) \cos 2kx \, dx \).

Moreover, for any \( \delta_1 \in (0, 1) \), there exists some \( \varepsilon = \varepsilon(\delta_1, q^o) > 0 \) such that

\[
|b_n(q) - 1| \leq \delta_1 < 1 \quad \forall \ (n, q) \in \mathbb{N} \times B_\varepsilon(q^o).
\]

**Proof.** Lemma 3.3 gives that each \( b_n(q) \), \( n \geq 0 \) is real analytic on \( L^2 \). Using (3.12) we obtain (3.21). Substituting (3.13) into \( b_n = e^{i \frac{b_n}{r_n}} S(r_n, q)/S(r_n, q^o) \), and into \( \frac{\partial b_n(q)}{\partial q(x)} \) given by (3.21), we obtain (3.22). Let \( e_n = e^{i \frac{b_n}{r_n}} \). Then, \( |e_n - 1| \leq \frac{|b_n|}{r_n} \) and using (1.11), we obtain

\[
|b_n - 1| = |e_n U_n - 1| = |e_n U_n - e_n + e_n - 1| \leq |U_n - 1| + |e_n - 1| \leq \delta + \frac{|b_n|}{r_n} \leq \delta + \varepsilon \frac{\|q\|}{r_n} = \delta_1.
\]

Let \( q^o \in L^2 \) and \( \varepsilon > 0 \) be small enough. Then, due to (3.23), we can define the mapping \( f : B_\varepsilon(q^o) \to \ell^2(\mathbb{Z}_+) \) by \( q \rightarrow f = (f_n)_0^\infty \), where the components are given by

\[
f_0 = b_0 = q_0 - q_0^0, \quad f_n(q) = -ir_n \log b_n(q), \quad n \geq 1,
\]

and the branch of log is defined by \( \log 1 = 0 \). We can do it, since in (3.23) we have obtained that \( |b_n - 1| \leq \delta_1 < 1 \) for all \( (n, q) \in \mathbb{N} \times B_\varepsilon(q^o) \) and some small \( \delta_1 = \delta_1(\varepsilon, q^o) \).

**Theorem 3.5.** Let a sequence \((r_n)_1^\infty \in R \) and \( q^o \in L^2 \). Then, we have the following:

\[\text{□}\]
(i) Each mapping \( \mathfrak{f}_n : q \to \mathfrak{f}_n, n \geq 1 \), defined by \( (3.24) \) acting from \( B_\varepsilon(q^0) \) into \( \mathbb{R} \), is real analytic and its gradient is given by

\[
\frac{\partial \mathfrak{f}_n(q)}{\partial q(x)} = 1 - \frac{(2r_n^2)}{\psi^2(r_n, q)} \varphi^2(x, r_n, q). \tag{3.25}
\]

Moreover, the mapping \( \mathfrak{f} : q \to (\mathfrak{f}_n)_{n=0}^\infty \) acting from \( B_\varepsilon(q^0) \) into \( \ell^2(\mathbb{Z}_+) \) is real analytic and is an injection.

(ii) Let \( q^0 = 0 \). Then, the mapping \( \mathfrak{f}'(0) = \left. \frac{\partial \mathfrak{f}(q)}{\partial q} \right|_{q=0} : \mathcal{L}^2 \to \ell^1_+ \) has the form

\[
\mathfrak{f}'(0) = \Phi : \mathcal{L}^2 \to \ell^2(\mathbb{Z}_+) \tag{3.26}
\]

and is a linear isomorphism between \( \mathcal{L}^2 \) and \( \ell^2(\mathbb{Z}_+) \), where \( \Phi \) is the Fourier transformation \( \Phi : \mathcal{L}^2 \to \ell^2(\mathbb{Z}_+) \) defined by

\[
(\Phi h)_0 = \int_0^1 h(x)dx, \quad (\Phi h)_n = \int_0^1 h(x) \cos 2r_nxdx, \quad n \geq 1. \tag{3.27}
\]

(iii) For each \( q \in \mathcal{L}^2 \), the mapping \( \left. \frac{\partial \mathfrak{f}(q)}{\partial q} \right|_{q} : \mathcal{L}^2 \to \ell^2(\mathbb{Z}_+) \) has an inverse.

Proof. (i) Consider the function \( \mathfrak{f}_n(q) = -ir_n \log b_n(q), n \geq 1 \)

We recall the following result (see, e.g., [37]):

Let \( f : D \to H \) be a mapping from an open subset \( D \) of a complex Hilbert space into a Hilbert space \( H \) with orthogonal basis \( e_n, n \geq 1 \). Then, \( f \) is analytic on \( D \) if and only if \( f \) is locally bounded, and each coordinate function \( f_n = (f, e_n) : D \to \mathbb{C} \) is analytic on \( D \). Moreover, the derivative of \( f \) is given by the derivatives of its coordinate function:

\[
\frac{\partial f}{\partial q}(h) = \sum_{n=1}^{\infty} \frac{\partial f_n(h)}{\partial q} e_n. \tag{3.28}
\]

Due to this result and Lemmas 3.4, the mapping \( q \to \mathfrak{f}(q) \) from \( \mathcal{L}^2 \) to \( \ell^2(\mathbb{Z}_+) \) is real analytic. Theorem 1.3(i) yields that this mapping is an injection.

(ii) Let \( \mathfrak{f}'(q) = \left. \frac{\partial \mathfrak{f}(q)}{\partial q} \right|_{q(x)} \) and let \( \mathfrak{f}'_n(q) = \left. \frac{\partial \mathfrak{f}_n(q)}{\partial q(x)} \right|_{q=0} n \geq 1 \) for shortness. Consider the case \( q^0 = 0 \). From (3.21), we obtain

\[
\mathfrak{f}'_0(q) = 1, \quad \mathfrak{f}'_n(q) \big|_{q=0} = 1 - 2(r_n^2) \frac{\varphi^2(x, r_n, 0)}{\psi^2(r_n, 0)} = \cos 2r_nx. \tag{3.29}
\]

This yields that \( \mathfrak{f}'(0) \) has the form (3.26). The Kadets Theorem (see the proof of Theorem 1.1) implies that \( 1, \cos 2r_nx, \quad n \geq 1 \), forms a Riesz basis in \( L^2(0, 1) \).

(iii) Let \( q \in \mathcal{L}^2 \). Using the asymptotic estimate (3.22), we see that \( \mathfrak{f}'(q) \) is the sum of the Fourier transform \( \mathfrak{f}'(0) = \Phi \) and a compact operator \( K \) for all \( q \in \mathcal{L}^2 \). That is,

\[
\mathfrak{f}'(q)h = (\mathfrak{f}'_n, h)_{n=0}^\infty = \Phi h + Kh, \quad h \in \mathcal{L}^2.
\]

Consequently, \( \mathfrak{f}' \) is a Fredholm operator. We show now that the operator \( \mathfrak{f}'(q) \) is invertible by contradiction. Let \( h \in \mathcal{L}^2, h \neq 0 \), be a solution of the equation

\[
\mathfrak{f}'(q)(h) = 0, \quad \Leftrightarrow \quad \left\{ (\mathfrak{f}'_n(q), h) = 0, n \geq 0 \right\}, \tag{3.30}
\]

for some fixed \( q \in \mathcal{L}^2 \). We rewrite this in the form

\[
(\mathfrak{f}'_0(q), h) = h_0 = \int_0^1 h(x)dx = 0, \quad (\mathfrak{f}'_n(q), h) = \int_0^1 \left( 1 - 2(r_n^2) \frac{\varphi^2(x, r_n, q)}{\psi^2(r_n, q)} \right) h(x)dx = 0, \quad n \geq 1. \tag{3.31}
\]
In order to prove $h = 0$, we introduce the function

$$g(k) = k \int_0^1 h(x) \varphi^2(x, k) dx, \quad k \in \mathbb{C},$$

which is odd and entire, where $\varphi(x, k) = \varphi(x, k, q)$. From (3.30) and (3.21), we deduce that

$$g(\pm r_n) = \pm r_n \int_0^1 h(x) \varphi^2(x, \pm r_n) dx = 0, \quad \forall \ n \geq 1.$$

The entire function $g$ has the following properties:

1. $g \in L^2(\mathbb{R})$, since $\varphi(x, k) = \frac{\sin kx + O(1/k)}{k}$ as $k \to \pm \infty$.
2. $g(\pm r_n) = 0$ for all $r_n, n \in \mathbb{N}$, and $g(0) = 0$.
3. The function $g$ has a type $\leq 2$.

Then, Paley–Wiener Theorem (see p. 30 in [18]) gives $g(k) = \int_{-1}^1 \hat{g}(x)e^{-2ikx} dx$. Thus, properties (1)–(3) and the Kadets Theorem (see the proof of Theorem 1.1) imply that $g = 0$.

We need to show $h = 0$. Due to Lemma 3.6, the sequence $\varphi^2(x, \sqrt{\mu_n})$ and $\varphi^2(x, \sqrt{\tau_n}), n \geq 1$, form a Riesz basis in $L^2(0, 1)$. Then, $\int_0^1 h(x) \varphi^2(x, k) dx = 0$ for all $k \in \mathbb{C}$, we obtain $h = 0$.

**Lemma 3.6.** Let $q \in L^2(0, 1)$. Then, functions $\varphi^2(x, \sqrt{\mu_n})$ and $\varphi^2(x, \sqrt{\tau_n}), n \geq 1$, form the basis in $L^2(0, 1)$.

**Proof.** Define the space $L^2_{ev}(0, 2) = \{v \in L^2(0, 2) : v(x) = v(2-x), x \in (0, 2)\}$ of even functions. For $q \in L^2(0, 1)$, we define an even potential $\tilde{q} \in L^2_{ev}(0, 2)$ by

$$\tilde{q}(x) = \begin{cases} q(x), & x \in (0, 1) \\ q(2-x), & x \in (1, 2). \end{cases}$$

We consider the Schrödinger operator $\tilde{H}_D$ with the Dirichlet boundary conditions on the interval $[0, 2]$ given by $\tilde{H}_D f = -f'' + \tilde{q} f$, $f(0) = f(2) = 0$. Let $\tilde{\mu}_n$ and $\tilde{f}_n, n \geq 1$ be eigenvalues and the corresponding eigenfunctions of $\tilde{H}_D$. The eigenvalues $\tau_n, \mu_n, n \geq 1$, and the corresponding eigenfunctions $g_n = \varphi(x, \sqrt{\mu_n}), f_n = \varphi^2(x, \sqrt{\mu_n})$ for $q$ on $[0, 1]$ satisfy

$$-g_n'' + q g_n = \tau_n(q) g_n, \quad g_n(0) = g_n'(1) = 0,$$

$$-f_n'' + q f_n = \mu_n f_n, \quad f_n(0) = f_n(1) = 0.$$

Direct calculations (see, e.g., [23]) give that the eigenfunctions $\tilde{f}_n$ and the eigenvalues $\tilde{\mu}_n, n \geq 1$, satisfy $\tilde{\mu}_{2n-1} = \tau_n(q)$ and $\tilde{\mu}_{2n} = \mu_n(q)$ and

$$\tilde{f}_{2n-1}(x) = \begin{cases} g_n(x), & x \in (0, 1) \\ g_n(2-x), & x \in (1, 2), \end{cases}, \quad \tilde{f}_{2n}(x) = \begin{cases} f_n(x), & x \in (0, 1) \\ -f_n(2-x), & x \in (1, 2). \end{cases}$$

Recall that $\frac{\partial \tilde{\mu}_n}{\partial q} = f_n^2(x)$ and the sequence of functions $f_n^2(x), n \geq 1$, forms a Riesz basis in $L^2(0, 1)$ (see, e.g., [37]). Thus, functions $\varphi^2(x, \sqrt{\mu_n})$ and $\varphi^2(x, \sqrt{\tau_n}), n \geq 1$, form a Riesz basis in $L^2(0, 1)$.

**Proof of Theorem 1.3(ii).** Proof of (ii) follows from Theorem 3.5.

4 | JOST–KOHN’S IDENTITIES

We discuss the solution of Jost and Kohn [14] for specific cases. Our goal is to show that at some conditions on $k_s, q \in P_1$, there exists a solution to the Gelfand–Levitan equation, which gives the compactly supported function $q_s - q$ and the norming constant $c$ is uniquely determined by $k_s, q$. 


Let \( \#(E) \) be the number of zeros of \( \psi \) (counted with multiplicity) in the set \( E \subset \mathbb{C} \).

**Definition K.** Let \( \psi \in \mathcal{J}_1 \). By \( \mathcal{K}(\psi) \), we mean the class of all \( k_+ \in i\mathbb{R} \) such that

(i) \( \psi(-k_+) = 0 \) and \( \psi(k_+) \neq 0 \).

(ii) If \( k_+ \in (k_1, +i\infty) \), then the number \( \#(-k_+,-k_1) \geq 1 \) is odd.

(iii) If \( k_+ \in (k_j,k_{j+1}) \) for some \( j \in \mathbb{N}, \) then \( \#(-k_+,-k_j) \geq 1 \) is odd.

(iv) If \( k_+ \in (0,k_m) \), then \( \#(-k_-,-k_+) \geq 1 \) is odd.

Note that \( \#(-k_-,-k_+) \geq 1 \) can be any odd number for a Jost function \( \psi(k,q) \) with a specific potential \( q \), see [22]. We describe potentials \( q \), corresponding to the Jost function \( \psi_*(k) = \psi(k) \frac{k-k_+}{k+k_+} \) belonging to \( \mathcal{J}_1 \) and \( i\psi(-k_+)\varphi(1,k_+) > 0 \).

**Lemma 4.1.** Let \( \psi \in \mathcal{J}_1 \) for some \( q \in \mathcal{P}_1 \) and let \( \psi(-k_+) = 0 \) at \( k_+ = ir_+ \), \( r_+ > 0 \). Then,

\[
\varphi(x,k_+) = -\frac{\psi(k_+)}{2r} e^{rx}, \quad \forall x \geq 1, \tag{4.1}
\]

\[
\varphi^2(1,k_+) - 2r \int_0^1 \varphi^2(t,k_+) dt = ie^{i\kappa}\psi'(-k_+)\varphi(1,k_+). \tag{4.2}
\]

There are two cases:

(i) If \( k_+ \in \mathcal{K}(\psi) \), then \( \psi_*(k) = \psi(k) \frac{k-k_+}{k+k_+} \) belongs to \( \mathcal{J}_1 \) and

\[
i\psi'(-k_+)\varphi(1,k_+) > 0. \tag{4.3}
\]

(ii) If \( k_+ \notin \mathcal{K}(\psi) \), then \( \psi_*(k) = \psi(k) \frac{k-k_+}{k+k_+} \notin \mathcal{J}_1 \) and

\[
i\psi'(-k_+)\varphi(1,k_+) < 0. \tag{4.4}
\]

**Proof.** We determine \( \varphi(x,k) \) for \( x > 1, k = ir \in i\mathbb{R} \). The function \( \varphi \) has the form

\[
\varphi(x,k) = c_1 e^{r(x-1)} + c_2 e^{-r(x-1)}, \quad x > 1, \tag{4.5}
\]

for some real \( c_1, c_2 \). Using (2.6), we obtain at \( x = 1 \):

\[
\begin{cases}
\varphi(1,k) = c_1 + c_2, \\
\varphi'(1,k) = -ikc_1 + ic_2,
\end{cases}
\]

\[
\begin{aligned}
c_1 &= \frac{i\kappa \varphi(1,k) - \varphi'(1,k)}{2ik} = -e^{-ik} \psi(k) \frac{2ik}{2ik} \\
c_2 &= \frac{i\kappa \varphi(1,k) + \varphi'(1,k)}{2ik} = e^{ik} \psi(-k) \frac{2ik}{2ik}.
\end{aligned} \tag{4.6}
\]

From (4.6), (4.5), and \( \psi(-k_+) = 0 \), we obtain (4.1).

Differentiating the equation \( -\varphi'' + q \varphi = k^2 \varphi \) with respect to \( k \) yields

\[
-\varphi'' + q \varphi = 2k \varphi + k^2 \varphi.
\]

Multiplying this equation by \( \varphi \) and the equation \( -\varphi'' + q \varphi = \lambda \varphi \) by \( \varphi \) and taking the difference, we obtain

\[
2k \varphi^2 = \varphi'' \varphi - \varphi'' \varphi = \{\varphi,\varphi\}',
\]
where \(\{u, v\} = uv' - u'v\). Then, we have the standard identity
\[
2k \int_0^1 \varphi^2(t, k)dt = \{\dot{\varphi}, \varphi\}(t, k) \bigg|_0^1 = (\varphi \varphi' - \varphi' \varphi)(1, k) = -\varphi(ik\varphi + \varphi')(1, k),
\] (4.7)
since \(\varphi(0, k) = 0\) and \(\varphi'(0, k) = 1\) for all \(k\) and \(\varphi'(x, k) = -ik\varphi(x, k)\) for \(x \geq 1\).

Due to \(\varphi(-k^*) = 0\), we obtain from (2.6) at \(k = -k^*\):
\[
\left(e^{-ik\varphi (k)}\right)'_k = e^{-ik\varphi (k)} = (\varphi' - i\varphi - ik\dot{\varphi})(1, k),
\]
\[
-e^{ik^*\varphi(-k^*)} = (i\varphi - \varphi' - ik^*\dot{\varphi})(1, k),
\]
\[
i\varphi(1, k) + e^{ik\varphi(-k)} = - (ik\varphi + \varphi')(1, k),
\] (4.8)
since \(\varphi(1, k) = \varphi(1, k)\) for all \(k\).

Differentiating \(\dot{\varphi}(k) = \varphi_*(k)k^{-k}k_{k^*}^{-k_{k^*}}\) at \(k = -k^*_s\) we have
\[
i\dot{\varphi}(k) = \frac{i\varphi_*(k)}{k-k^*_s} = \frac{\varphi_*(k)}{-2k^*_s} = \frac{\varphi_*(k)}{-2|k^*_s|} = \frac{(1)^{j+1}\varphi_*(k)}{(1)^{j+1}|k^*_s|},
\]
(4.10)
We need the following fact from [22]: If \(k^*_s \in (k_j, k_{j+1})\) then \(k^*_s \in (\mu_j, \mu_{j+1})\), which yields \((-1)^j\varphi(1, k^*_s) > 0\). Here, \((\mu_n)_{n \in \mathbb{N}}\) is an increasing sequence of eigenvalues of the problem \(-y'' + qy = \lambda y, y(0) = y(1) = 0\) on the unit interval \([0,1]\). This jointly with (4.10) gives (4.3).

Similar arguments imply (ii) and, in particular, (4.4).

We present the main results of this section about Jost–Kohn’s identities. We show when their potential is compactly supported and when it is not.

**Theorem 4.2.** Let \(\psi \in J_1\) for some \(q \in P_1\) and let \(k_s = ir_s \in i\mathbb{R} \setminus \{k_1, \ldots, k_m\}\) and a norming constant \(\epsilon > 0\). Then, \(\psi_\epsilon = \psi(k)k_{k^*}^{-k_{k^*}}\) is the Jost function for a unique potential \(q_\epsilon\) given by
\[
q_\epsilon = q + q_0, \quad q_0 = -(\log A)'',
\] (4.11)
where \(A(x) = 1 + \epsilon \int_0^x \varphi^2(s, k_s, q)ds\) and \(q_0, q'_0 \in L^1(\mathbb{R}_+)\). Moreover, there are two cases:

(i) Let \(k_s \in \mathcal{K}(\psi)\). Then,
\[
\psi_\epsilon \in J_1, \quad \text{and} \quad \epsilon_s = \frac{2|k^*_s|e^{|k^*_s|}}{i\psi(\varphi_\epsilon)(1, k_s)} > 0.
\] (4.12)

Moreover, we have \(q_\epsilon \in P_1\) if \(\epsilon = \epsilon_s\) and \(q_\epsilon\) is not compactly supported if \(\epsilon \neq \epsilon_s\).

(ii) Let \(k_s \notin \mathcal{K}(\psi)\). Then, \(\psi_\epsilon \notin J_1\) and \(q_\epsilon\) is not compactly supported for any \(\epsilon > 0\).

**Proof.** Let \(\psi \in J_1\) for some \(q \in P_1\) and let \(k_s \in i\mathbb{R}_+ \setminus \{k_1, \ldots, k_m\}\). We recall the Jost–Kohn result [14]: For the function \(\psi_\epsilon(k) = \psi(k)k_{k^*}^{-k_{k^*}}\) and any norming constant \(\epsilon > 0\), there exists a unique potential \(q_\epsilon\) given by (4.11). In order to determine
the potential $q_*$, the Gel’fand–Levitan equation is used, here the perturbed operator $T_*$ has the additional eigenvalue $E_* = k_*^2$, see, for example, [10] or [6]. The Gel’fand–Levitan equation in this case is given by

$$G(x, t) = -G_0(x, t) - \int_0^x G_0(t, s) G(x, s) ds, \quad t \geq x,$$

$$G_0(x, t) = c\varphi(x) \varphi(t), \quad \varphi(x) = \varphi(x, k_*).$$

(4.13)

This equation has a degenerate kernel and is easily solved, since we can rewrite $G$ in the form

$$G(x, t) = g(x) \varphi(t),$$

(4.14)

where $g$ is an unknown function. Then, the Gel’fand–Levitan equation becomes

$$\left(g(x) + c\varphi(x) + g(x) \int_0^x \varphi^2(s) ds\right) \varphi(t) = 0,$$

(4.15)

which has the solution $g(x) A(x) = -c \varphi(x), \quad A(x) = 1 + c \int_0^x \varphi^2(s) ds$, and then

$$g(x) = -\frac{c \varphi(x)}{A(x)}, \quad G(x, t) = -\frac{c \varphi(x) \varphi(t)}{A(x)}.$$

(4.16)

Thus, the potential $q_*$ for the perturbed operator $T_{q_*}$ has the form $q_* = q + q_o$, where $q_o$ is given by

$$q_o = -\left(\log A\right)'' = \frac{(A')^2}{A^2} - \frac{A''}{A^2} = \frac{A'}{A} Q, \quad Q = A' - \frac{A'' A}{A'},$$

(4.17)

which yields (4.11) and it is the well-known Jost–Kohn identity (see [14]).

(i) Let $k_* = ir_* \in \mathcal{K}(\psi)$ and let $a = -\frac{\psi(k_*)}{2r_*}$. Then, from (4.1) and (4.17), we obtain for $x > 1$:

$$\varphi(x, k_*) = ae^{r_* x}, \quad A' = c\varphi^2(x), \quad A'' = 2r_* A', \quad Q = A' - 2r_* A, \quad Q' = 0.$$  

(4.18)

The function $Q = A' - 2r_* A$ is continuous on $\mathbb{R}_+$ and is a constant on $x \geq 1$. Then due to (4.2), we have

$$Q|_{x=1} = c\varphi^2(1) - 2r_* A(1) = c\left(\varphi^2(1) - 2r_* \int_0^1 \varphi^2(t) dt\right) - 2r_* = c e^{ik} \psi(-k) \varphi(1) - 2r_*,$$  

(4.19)

since $A(1) = 1 + \epsilon \int_0^1 \varphi^2(t, k) dt$. From (4.19), we deduce that $Q = 0$ if we take $\epsilon = \frac{2r_* e^{r_*}}{i \psi(-k_*) \varphi(1, k_*)}$ and then $q_o(x) = 0$ for all $x > 1$.

(ii) Let $k_* \not\in \mathcal{K}(\psi)$. Then, Lemma 4.1 implies that $\psi_* \not\in \mathcal{J}_1$, since its zeros do not satisfy conditions (i) and (ii) in Definition J. We show that $q_*$ is not compactly supported. Assume that $q_*$ is compactly supported. Then due to Theorem 2.1, the zeros of $\psi_*$ have to satisfy conditions (i) and (ii) in Definition J.

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