On the spread of outerplanar graphs

Daniel Gotshall*  Megan O’Brien*  Michael Tait*

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Abstract

The spread of a graph is the difference between the largest and most negative eigenvalue of its adjacency matrix. We show that for sufficiently large \( n \), the \( n \)-vertex outerplanar graph with maximum spread is a vertex joined to a linear forest with \( \Omega(n) \) edges. We conjecture that the extremal graph is a vertex joined to a path on \( n - 1 \) vertices.

1 Introduction

The spread of a square matrix \( M \) is defined to be

\[
S(M) := \max_{i,j} |\lambda_i - \lambda_j|,
\]

where the maximum is taken over all pairs of eigenvalues of \( M \). That is, \( S(M) \) is the diameter of the spectrum of \( M \). The spread of general matrices has been studied in several papers (e.g. [4, 9, 18, 24, 25, 34, 37]). In this paper, given a graph \( G \) we will study the spread of the adjacency matrix of \( G \), and we will call this quantity the spread of \( G \) and denote it by \( S(G) \). Since the adjacency matrix of an \( n \)-vertex graph is real and symmetric, it has a full set of real eigenvalues which we may order as \( \lambda_1 \geq \cdots \geq \lambda_n \). In this case, the spread of \( G \) is given simply by \( \lambda_1 - \lambda_n \).

The study of the spread of graphs was introduced in a systematic way by Gregory, Hershkowitz, and Kirkland in [15]. Since then, the spread of graphs has been studied extensively. A problem in this area with an extremal flavor is to maximize or minimize the spread over a fixed family of graphs. This problem has been considered for trees [1], graphs with few cycles [13, 27, 38], the family of all \( n \)-vertex graphs [3, 6, 29, 31, 32, 35], bipartite graphs [6], graphs with a given matching number [19] or girth [36] or size [22].

We also note that spreads of other matrices associated with a graph have been considered extensively (e.g. [2, 12, 14, 16, 21, 23, 26, 39, 40, 41]), but in this paper we will focus on

*Department of Mathematics & Statistics, Villanova University, email addresses \{dgotshall, mobrie71, michael.tait\}@villanova.edu. The third author is partially supported by National Science Foundation grant DMS-2011553.
the adjacency matrix. This paper examines the question of maximizing the spread of an
$n$-vertex outerplanar graph. A graph is outerplanar if it can be drawn in the plane with
no crossings and such that all vertices are incident with the unbounded face. Similarly to
Wagner’s theorem characterizing planar graphs, a graph is outerplanar if and only if it does
not contain either $K_{2,3}$ or $K_4$ as a minor. Maximizing the spread of this family of graphs
is motivated by the extensive history on maximizing eigenvalues of planar or outerplanar
graphs, for example [5, 7, 8, 10, 11, 17, 20, 28, 30, 42, 43].

Our main theorem comes close to determining the outerplanar graph of maximum spread.
Let $P_k$ denote the path on $k$ vertices and $G \lor H$ the join of $G$ and $H$. A linear forest is a
disjoint union of paths.

**Theorem 1.1.** For $n$ sufficiently large, any graph which maximizes spread over the family
of outerplanar graphs on $n$ vertices is of the form $K_1 \lor F$ where $F$ is a linear forest with
$\Omega(n)$ edges.

We leave it as an open problem to determine whether or not $F$ should be a path on $n − 1$
vertices, and we conjecture that this is the case.

**Conjecture 1.2.** For $n$ sufficiently large, the unique $n$-vertex outerplanar graph of maximum
spread is $K_1 \lor P_{n-1}$.

### 2 Preliminaries

Let $G$ be an outerplanar graph of maximum spread and let $A$ be its adjacency. We will
frequently assume that $n$ is sufficiently large. We will use the characterization that a graph
is outerplanar if and only if it does not contain $K_{2,3}$ or $K_4$ as a minor. In particular, $G$
does not contain $K_{2,3}$ as a subgraph. Given a vertex $v \in V(G)$, the neighborhood of $v$ will be
denoted by $N(v)$ and its degree by $d_v$. If $f, g : \mathbb{N} \to \mathbb{R}$ we will use $f = \mathcal{O}(g)$ to mean that
there exists a constant $c$ such that $f(n) ≤ cg(n)$ for $n$ sufficiently large. $f = \Omega(g)$ means
that $g = \mathcal{O}(f)$ and $f = \Theta(g)$ means that $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$. We will occasionally have
sequences of inequalities where we will abuse notation and mix inequality symbols with $\mathcal{O}(\cdot)$
and $\Theta(\cdot)$.

Let the eigenvalues of $A$ be represented as $\lambda_1 ≥ \lambda_2 ≥ \cdots ≥ \lambda_n$. For any disconnected
graph, adding an edge between the connected components will not decrease $\lambda_1$ and will also
not decrease $−\lambda_n$. Therefore, without loss of generality, we may assume that $G$ is connected.
By the Perron-Frobenius theorem we may assume that the eigenvector $x$ corresponding to
$\lambda_1$ has $x_u > 0$ for all $u$.

Furthermore, we will normalize $x$ so that it has maximum entry equal to 1, and let $x_w = 1$
where $w$ is a vertex attaining the maximum entry in $x$. Note there may be more than one
such vertex, in which case we can arbitrarily choose and fix $w$ among all such vertices. The
other eigenvector of interest to us corresponds to $\lambda_n$, call it $z$. We will also normalize $z$
so that its largest entry in absolute value has absolute value 1 and let $w'$ correspond to a vertex
with maximum absolute value in $z$ (so $x_{w'}$ equals 1 or $−1$).
We will implement the following known equalities for the largest and smallest eigenvalues:

\[ \lambda_1 = \max_{\mathbf{x}' \neq 0} \frac{(\mathbf{x}')^t \mathbf{A} \mathbf{x}'}{(\mathbf{x}')^t \mathbf{x}'} = \frac{\mathbf{x}^t \mathbf{A} \mathbf{x}}{\mathbf{x}^t \mathbf{x}} \quad (2.1) \]

\[ \lambda_n = \min_{\mathbf{z}' \neq 0} \frac{(\mathbf{z}')^t \mathbf{A} \mathbf{z}'}{(\mathbf{z}')^t \mathbf{z}'} = \frac{\mathbf{z}^t \mathbf{A} \mathbf{z}}{\mathbf{z}^t \mathbf{z}} \quad (2.2) \]

An important result from equations 2.1 and 2.2 and the Perron-Frobenius Theorem is that for any strict subgraph \( H \) of \( G \), we have \( \lambda_1(A(G)) > \lambda_1(A(H)) \). Finally, we will use the following theorem from \cite{33}.

**Theorem 2.3.** For \( n \) large enough, \( K_1 \lor P_{n-1} \) has maximum spectral radius over all \( n \)-vertex outerplanar graphs.

### 3 Vertex of Maximum Degree

The main goal of this section is to prove that \( G \) has a vertex of degree \( n - 1 \). This is stated in the following theorem.

**Theorem 3.1.** For \( n \) large enough, we have \( d_w = n - 1 \).

As a first step, we will get preliminary upper and lower bounds on the largest and smallest eigenvalues of \( G \). First we obtain an upper bound on the spectral radius.

**Lemma 3.2.** \( \lambda_1 \leq \sqrt{n} + 1 \).

*Proof.* We define the graph \( G_1 \) to be the graph \( K_1 \lor P_{n-1} \). By Theorem 2.3 we know that any outerplanar graph on sufficiently many vertices cannot have a spectral radius larger than that of \( G_1 \). Now define \( G_2 \) as \( G_1 \) with another edge joining the endpoints of the path, so \( G_2 = K_1 \lor C_{n-1} \). Clearly \( G_1 \) is a subgraph of \( G_2 \). Putting all this together gives us

\[ \lambda_1(G) \leq \lambda_1(G_1) < \lambda_1(G_2) = \sqrt{n} + 1, \]

where the last equality can be calculated using an equitable partition with two parts (the dominating vertex and the cycle).

Next we bound \( |\lambda_n| \).

**Lemma 3.3.** For \( n \) sufficiently large, \( \sqrt{n - 1} - 2 \leq |\lambda_n| \leq \sqrt{n - 1} + 2 \).

*Proof.* The upper bound on \( |\lambda_n| \) follows immediately from Lemma 3.2 and the well-known fact \( \lambda_1 \geq |\lambda_n| \) for any graph. Now to get the lower bound, since \( G \) is the outerplanar graph on \( n \) vertices that maximizes spread, we have

\[ S(G) \geq S(K_{1,n-1}) \]
\[
\lambda_1(K_{1,n-1}) - \lambda_n(K_{1,n-1}) = \sqrt{n-1} - (-\sqrt{n-1}) = 2\sqrt{n-1}.
\]

So
\[
2\sqrt{n-1} \leq \lambda_1(G) - \lambda_n(G) \leq \sqrt{n+1} - \lambda_n(G) < \sqrt{n-1} + 2 - \lambda_n(G).
\]

Hence we have
\[
-\lambda_n \geq \sqrt{n-1} - 2.
\]

Essentially the same proof also gives a lower bound for \(\lambda_1\).

**Corollary 3.4.** For \(n\) large enough we have \(\lambda_1 \geq \sqrt{n-1} - 2\).

We shall use Lemma 3.3 to obtain a lower bound on the degree of each vertex.

**Lemma 3.5.** Let \(u\) be an arbitrary vertex in \(G\). Then \(d_u > |z_u|n - \mathcal{O}(\sqrt{n})\) and \(d_u > x_u n - \mathcal{O}(\sqrt{n})\).

**Proof.** We will show the first part explicitly. When \(\lambda_n\) is the smallest eigenvalue for our graph, we have
\[
|\lambda_n^2 z_u| = \left| \sum_{y \sim u} \sum_{v \sim y} z_v \right| \leq d_u + \sum_{y \sim u} \sum_{v \sim y, v \neq u} |z_v|.
\]

Recall that an outerplanar graph cannot have a \(K_{2,3}\). This implies every vertex in \(G\) has at most two neighbors in \(N(u)\), meaning the eigenvector entry for each vertex contained in the neighborhood of \(N(u)\) can be counted at most twice. Hence \(\sum_{y \sim u} \sum_{v \sim y} |z_v| \leq 2 \sum_{v \neq u} |z_v|\). Note
\[
|\lambda_n| |z_v| \leq \sum_{v' \sim v} |z_{v'}| \leq \sum_{v' \sim v} 1 = d_v.
\]

So we have
\[
\sum_{y \sim u} \sum_{v \sim y} |z_v| \leq 2 \sum_{v \neq u} |z_v| \leq \frac{2}{|\lambda_n|} \sum_{v \neq u} d_v \leq \frac{4e(G)}{|\lambda_n|} \leq \frac{4(2n-3)}{|\lambda_n|},
\]
as \(e(G) \leq 2n - 3\) by outerplanarity. Combining and using Lemma 3.3, we have
\[
(\sqrt{n-1} - 2)^2 |z_u| \leq |\lambda_n^2 z_u| \leq d_u + \frac{4(2n-3)}{|\lambda_n|} \leq d_u + \frac{8n}{\sqrt{n-1} - 2},
\]
for \(n\) sufficiently large. Isolating \(d_u\) gives the result. A similar proof can be written to justify the lower bound with respect to \(x\) and we omit these details.

\(\square\)
Lemma 3.6. We have $d_w > n - \mathcal{O}(\sqrt{n})$ and $d_{w'} > n - \mathcal{O}(\sqrt{n})$. For every other vertex $u$ we get $|z_u|, x_u = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$, for $n$ sufficiently large.

Proof. The bound on $d_w$ and $d_{w'}$ follows immediately from the previous lemma and the normalization that $|z_{w'}| = x_w = 1$. Now consider any other vertex $u$. We know that $G$ contains no $K_{2,3}$ and hence can have at most 2 common neighbors with $w$. Thus $d_u = \mathcal{O}(\sqrt{n})$. By Lemma 3.5 we have that there are constants $c_1$ and $c_2$ such that

$$c_1\sqrt{n} > d_u x_u - c_2\sqrt{n},$$

and

$$c_1\sqrt{n} > d_u |z_u| - c_2\sqrt{n},$$

for sufficiently large $n$. This implies the result.

If $w$ and $w'$ were distinct vertices, then for sufficiently large $n$ they would share many neighbors, contradicting outerplanarity. Hence we immediately have the following important fact.

Corollary 3.7. For $n$ sufficiently large we have $w = w'$.

Hence from now on, we will denote the vertex in Corollary 3.7 by $w$, and furthermore for the remainder of the paper we will also assume that $z_w = 1$ without loss of generality. Before deriving our next result quantifying the other entries of $z$, we first need to define an important vertex set.

Definition 3.8. Recall that $w$ is the fixed vertex of maximum degree in $G$. Let $B = V(G) \setminus (N(w) \cup \{w\})$.

Now we consider the $z_u$ eigenvector entries of vertices in $B$. At the moment we know that each eigenvector entry for a vertex in $B$ has order at most $1/\sqrt{n}$. The next lemma shows that in fact the sum of all of the eigenvector entries of $B$ has this order.

Lemma 3.9. For $n$ large enough, we have that $\sum_{u \in B} |z_u|$ and $\sum_{u \in B} x_u$ are each $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$.

Proof. Let $u \in B$. Since $u$ is not adjacent to $w$, all of the neighbors of $u$ have eigenvector entry of order at most $1/\sqrt{n}$ and the size of $B$ is also of order at most $1/\sqrt{n}$ by Lemma 3.6.

Hence there is a constant $C$ such that

$$|\lambda_n||z_u| \leq \sum_{v \sim u} |z_v| \leq \frac{C d_u}{\sqrt{n}},$$

and $|B| \leq C\sqrt{n}$. Now

$$\sum_{u \in B} |z_u| \leq \frac{1}{|\lambda_n|} \sum_{u \in B} \left(\frac{C d_u}{\sqrt{n}}\right) \leq \frac{C}{|\lambda_n|\sqrt{n}} (e(B, V(G) \setminus B) + 2e(B)).$$
Each vertex in $B$ is connected to at most two vertices in $N(u)$, so $e(B, V(G) \setminus B) \leq 2|B| \leq 2C\sqrt{n}$. The graph induced on $B$ is outerplanar, so $e(B) \leq 2|B| - 3 < 2C\sqrt{n}$. Finally, using Lemma 3.3 we get the required result. A slightly modified version of this argument proves the bound on $\sum_{u \in B} x_u$.

We will use Lemma 3.9 to show that $B$ is empty, and this will complete the proof of Theorem 3.1. First we define the following alteration of $G$. Let $t$ be an arbitrary vertex in $B$ (if it exists).

**Definition 3.10.** Let $G^*$ be the graph defined such that its adjacency matrix $A^*$ satisfies

$$A^*_{ij} = \begin{cases} 1 & \text{if } i = w, j = t \\ 0 & \text{if } i = t, j \neq w \\ A_{ij} & \text{otherwise} \end{cases}$$

That is, to get $G^*$ from $G$ we add an edge from $t$ to $w$ and remove all other edges incident with $t$. In particular, the only neighbor of the $t$ vertex in $G^*$ is $w$.

**Lemma 3.11.** For large enough $n$, $B$ is empty.

**Proof.** Assume for contradiction that $B$ is nonempty. Then there is a vertex $t$ such that $t \not\sim w$. Define $G^*$ as in Definition 3.10. Furthermore, we define the vector $z^*$ which slightly modifies $z$ as follows.

$$z^*_u = \begin{cases} z_u & \text{if } u \neq t \\ -|z_u| & \text{if } u = t \end{cases}$$

That is, if $z_t < 0$ then $z^*$ is the same vector as $z$ and otherwise we flip the sign of $z_t$. Note that $(z^*)^T z^* = z^T z$. Now

$$S(G^*) - S(G) \geq \left( \frac{x^T A^* x}{x^T x} - \frac{(z^*)^T A^* z^*}{z^T z} \right) - \left( \frac{x^T A x}{x^T x} - \frac{z^T A z}{z^T z} \right)$$

$$= \frac{2x_t}{x^T x} \left( 1 - \sum_{v \sim t} x_v \right) + \frac{2z_t}{z^T z} \left( \text{sgn}(z_t) + \sum_{v \sim t} z_v \right)$$

$$\geq \frac{2x_t}{x^T x} \left( 1 - \sum_{v \sim t} x_v \right) + \frac{2|z_t|}{z^T z} \left( 1 - \left| \sum_{v \sim t} z_v \right| \right),$$

where $\text{sgn}(z_t)$ equals 1 if $z_t > 0$ and $-1$ otherwise.

$$\left| \sum_{v \sim t} z_v \right| \leq \sum_{v \sim t} |z_v| \leq \sum_{v \sim t} |z_v| + \sum_{v \in B} |z_v|.$$

There are at most 2 terms in the first sum, and so by Lemmas 3.6 and 3.9 we have

$$\left| \sum_{v \sim t} z_v \right| = O\left( \frac{1}{\sqrt{n}} \right).$$
Similarly, we have
\[ \sum_{v \sim t} x_v = O \left( \frac{1}{\sqrt{n}} \right). \]

This implies \( 1 - \sum_{v \sim t} x_v > 0 \) and \( 1 - \left| \sum_{v \sim t} z_v \right| > 0 \) for \( n \) large enough, which implies that

\[ \frac{2x_t}{x^T x} \left( 1 - \sum_{v \sim t} x_v \right) + \frac{2|z_t|}{z^T z} \left( 1 - \left| \sum_{v \sim t} z_v \right| \right) > 0 \]

for \( n \) large enough. Hence \( S(G^*) > S(G) \), contradicting the assumption that \( G \) is spread-extremal. \( \square \)

We have finally achieved our goal for this section. Theorem 3.1 follows immediately from the definition of \( B \) and the fact that it is empty, as implied by Lemma 3.11.

### 4 Determining Graph Structure

By Theorem 3.1, the vertex \( w \) has degree \( n - 1 \), or equivalently, \( K_{1,n-1} \) is a subgraph of \( G \). Since \( G \) is \( K_{2,3} \)-free, the graph induced by the neighborhood of \( w \) has maximum degree at most 2. Furthermore, this subgraph cannot contain a cycle, otherwise \( G \) would contain a wheel-graph and this is a \( K_4 \)-minor. Any graph of maximum degree at most 2 that does not contain a cycle is a disjoint union of paths. Therefore, we know that \( G \) is given by a \( K_1 \lor F \) where \( F \) is a disjoint union of paths. Our next task is to study \( F \). To this end, we denote the number of edges in \( F \) by \( m \). Our main theorem, Theorem 1.1 is proved if we can show that \( m = \Omega(n) \). Before doing this, we need a more accurate estimate for the eigenvector entries.

**Lemma 4.1.** For any \( u \neq w \), we have
\[ z_u = \frac{1}{\lambda_n} + \frac{d_u - 1}{\lambda_n^2} + \Theta \left( \frac{1}{n^{3/2}} \right). \]

**Proof.** As we are only considering outerplanar graphs, we have \( d_u \in \{1, 2, 3\} \) for all \( u \neq w \). Hence
\[ \lambda_n z_u = \sum_{y \sim u} z_y \]
\[ = z_w + \sum_{y \sim u \atop y \neq w} z_y \]
\[ = 1 - \Theta \left( \frac{1}{\sqrt{n}} \right) \]
by Lemma 3.0 and our normalization.
Note that as \( \lambda_n z_u = 1 + z_{u_1} + z_{u_2} > 0 \) and \( \lambda_n < 0 \), we must have \( z_u < 0 \). Next we repeat the argument to improve our estimate,

\[
\lambda_n z_u = z_w + \sum_{y \sim u \atop y \neq w} z_y
\]

\[= 1 + (d_u - 1) \left( \frac{1}{\lambda_n} + \Theta \left( \frac{1}{n} \right) \right).\]

Now we apply our bounds on \( \lambda_n \) in Lemma 3.3 to get

\[
z_u = \frac{1}{\lambda_n} + \frac{d_u - 1}{\lambda_n^2} + \Theta \left( \frac{1}{n^{3/2}} \right).
\]

We can use equivalent reasoning to obtain a very similar approximation for the \( x_u \) entries,

**Lemma 4.2.** For any \( u \neq w \), we have \( x_u = \frac{1}{\lambda_1} + \frac{d_u - 1}{\lambda_1^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \)

**Proof.** By the same logic as in the proof of Lemma 4.1, we have

\[
\lambda_1 x_u = \sum_{w \sim u} x_w
\]

\[= x_w + \sum_{y \sim u \atop y \neq w} x_y
\]

\[= 1 + \Theta \left( \frac{1}{\sqrt{n}} \right) \text{ by Corollary 3.6}
\]

So \( x_u = \frac{1}{\lambda_1} + \Theta \left( \frac{1}{n} \right) \). Next we repeat the argument to improve our estimate,

\[
\lambda_1 x_u = x_w + \sum_{y \sim u \atop y \neq w} x_y
\]

\[= 1 + (d_u - 1) \left( \frac{1}{\lambda_1} + \Theta \left( \frac{1}{n} \right) \right).
\]

So \( x_u = \frac{1}{\lambda_1} + \frac{d_u - 1}{\lambda_1^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \) according to our bounds on \( \lambda_1 \) in Lemma 3.2

Using Lemma 4.1 we can now get tighter estimates on \( \lambda_n \).

**Lemma 4.3.** We have that \( \lambda_n = -\sqrt{n - 1} + \frac{m}{n - 1} + \Theta \left( \frac{m}{n^{3/2}} \right) \).

**Proof.** We first define the vector

\[
y_2 = \begin{bmatrix}
\frac{1}{\sqrt{n-1}} \\
-\frac{1}{\sqrt{n-1}} \\
\vdots \\
-\frac{1}{\sqrt{n-1}}
\end{bmatrix}
\]

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where \( y_2 \) has \( n \) entries and \( w \) corresponds to the 1 entry. As \( \lambda_n \) is the minimum Rayleigh quotient we have

\[
\lambda_n \leq \frac{y_2^T A y_2}{y_2^T y_2} \cdot 2 \sum_{i \sim j} (y_2)_i (y_2)_j \]

\[
= \frac{2}{n-1} \left( -\frac{1}{\sqrt{n-1}} \right) + m \left( \frac{1}{n-1} \right) \]

\[
= -\sqrt{n-1} + \frac{m}{n-1}.
\]

In order to show the lower bound on \( \lambda_n \), we need to realize that \( \lambda_n \) is the minimum possible Rayleigh quotient. So

\[
\lambda_n = \frac{z^T A z}{z^T z} = \frac{2 \sum_{i \sim j} z_i z_j}{z^T z} = \frac{2 \sum_{w \sim k} z_w z_k}{z^T z} + \frac{2 \sum_{i \sim j, i,j \neq w} z_i z_j}{z^T z}.
\]

Note the first term is the Rayleigh quotient for the star subgraph centered at the vertex \( w \) of maximum degree. Hence it is bounded from below by \( -\sqrt{n-1} \). More specifically, we have

\[
2 \sum_{w \sim k} z_w z_k \geq z^T A(K_{1,n-1}) z \geq \lambda_n(A(K_{1,n-1})) = -\sqrt{n-1}.
\]

Applying Lemma 4.1 we have

\[
\lambda_n \geq -\sqrt{n-1} + \frac{2 \sum_{i \sim j, i,j \neq w} \left( \frac{1}{\lambda_n} + \frac{d_i-1}{\lambda_n^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \right) \left( \frac{1}{\lambda_n} + \frac{d_j-1}{\lambda_n^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)}{1 + \left( \sum_{\ell \neq w} \frac{1}{\lambda_n} + \frac{d_\ell-1}{\lambda_n^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)^2}.
\]

For \( \ell \neq w \) we have \( 0 \leq d_\ell - 1 \leq 2 \). Note that for \( n \) large enough we have that \( \frac{1}{\lambda_n} + \frac{d_\ell-1}{\lambda_n^2} + \Theta \left( \frac{1}{n^{3/2}} \right) < 0 \), and so we have a lower bound if we replace all \( d_\ell - 1 \) terms in the numerator by 2 and in the denominator by 0, and so we have

\[
\lambda_n \geq -\sqrt{n-1} + \frac{2 \sum_{i \sim j, i,j \neq w} \left( \frac{1}{\lambda_n} + \frac{2}{\lambda_n^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \right) \left( \frac{1}{\lambda_n} + \frac{2}{\lambda_n^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)}{1 + \sum_{\ell \neq w} \left( \frac{1}{\lambda_n} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)^2} \]

\[
= -\sqrt{n-1} + \frac{2m \left( \frac{1}{\lambda_n} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)}{1 + (n-1) \left( \frac{1}{\lambda_n} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)}.
\]
Since $-\sqrt{n-1} - 2 \leq \lambda_n \leq \sqrt{n-1} + 2$, we have that $\frac{1}{\lambda_n^2} = \frac{1}{n-1} + \Theta\left(\frac{1}{n^{3/2}}\right)$. Hence we have

$$\lambda_n \geq -\sqrt{n-1} + \frac{2m\left(\frac{1}{n-1} + \Theta\left(\frac{1}{n^{3/2}}\right)\right)}{1 + (n-1)\left(\frac{1}{n-1} + \Theta\left(\frac{1}{n^{3/2}}\right)\right)}$$

$$= -\sqrt{n-1} + \frac{2m}{n-1} + \Theta\left(\frac{m}{n^{3/2}}\right)$$

$$= -\sqrt{n-1} + \left(\frac{m}{n-1} + \Theta\left(\frac{m}{n^{3/2}}\right)\right)\left(1 + \Theta\left(\frac{1}{\sqrt{n}}\right)\right),$$

completing the proof.

We claim a very similar result for $\lambda_1$.

**Lemma 4.4.** $\lambda_1 = \sqrt{n-1} + \frac{m}{n-1} + \Theta\left(\frac{m}{n^{3/2}}\right)$.

**Proof.** First we prove the lower bound. Define the vector

$$y_3 = \begin{bmatrix} 1 \\ \sqrt{n-1} \\ \sqrt{n-1} \\ \vdots \\ \sqrt{n-1} \end{bmatrix},$$

where $w$ corresponds to the entry 1. As $\lambda_1$ is the maximum possible Rayleigh quotient we have

$$\lambda_1 \geq \frac{y_3^T A y_3}{y_3^T y_3} = \frac{2 \sum_{i \sim j} (y_3)_i (y_3)_j}{y_3^T y_3}$$

$$= \frac{2}{(n-1)} \left(1 + \frac{1}{\sqrt{n-1}}\right) + m \left(1 + \Theta\left(\frac{1}{\sqrt{n}}\right)\right)$$

$$= \sqrt{n-1} + \frac{m}{n-1}.$$

For the upper bound, we use the fact that $\lambda_1$ is the maximum possible Rayleigh quotient similarly to the previous lemma. So

$$\lambda_1 = \frac{x^T A x}{x^T x} = \frac{2 \sum_{i \sim j} x_i x_j}{x^T x}$$

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\[
\begin{align*}
2 \sum_{w \sim k} x_w x_k &= \frac{x^T A(K_{1,n-1}) x}{x^T x} + 2 \sum_{i \sim j, i,j \neq w} x_i x_j.
\end{align*}
\]

As before, the first term is
\[
\frac{x^T A(K_{1,n-1}) x}{x^T x} \leq \lambda_1(A(K_{1,n-1})) = \sqrt{n - 1}.
\]

Applying Lemma 4.2 and using \(0 \leq d_u - 1 \leq 2\) for all \(u \neq w\) we have
\[
\lambda_1 \leq \sqrt{n - 1} + \frac{2 \sum_{i \sim j, i,j \neq w} \left( \frac{1}{\lambda_1} + \frac{d_{x_1} - 1}{\lambda_1^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \right) \left( \frac{1}{\lambda_1} + \frac{d_{x_1} - 1}{\lambda_1^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)}{1 + \left( \sum_{\ell \neq w} \frac{1}{\lambda_1} + \frac{d_{x_1} - 1}{\lambda_1^2} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)^2}
\]
\[
\leq \sqrt{n - 1} + \frac{2 \sum_{i \sim j, i,j \neq w} \left( \frac{1}{\lambda_1} + \frac{2}{\lambda_1} + \Theta \left( \frac{1}{n^{3/2}} \right) \right) \left( \frac{1}{\lambda_1} + \frac{2}{\lambda_1} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)}{1 + \sum_{\ell \neq w} \left( \frac{1}{\lambda_1} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)^2}
\]

Using \(\sqrt{n - 1} - 2 \leq \lambda_1 \leq \sqrt{n - 1} + 2\), we have
\[
\lambda_1 \leq \sqrt{n - 1} + \frac{2m \left( \frac{1}{n-1} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)}{1 + (n-1) \left( \frac{1}{n-1} + \Theta \left( \frac{1}{n^{3/2}} \right) \right)}
\]
\[
= \sqrt{n - 1} + \frac{2m \left( \frac{1}{n-1} + \Theta \left( \frac{m}{n^{3/2}} \right) \right)}{2 + \Theta \left( \frac{1}{n^{1/2}} \right)}
\]
\[
= \sqrt{n - 1} + \left( \frac{m}{n-1} + \Theta \left( \frac{m}{n^{3/2}} \right) \right) \left( 1 + \Theta \left( \frac{1}{\sqrt{n}} \right) \right),
\]
completing the proof.

We are now in a position to prove our main theorem.

**Proof of Theorem 1.1.** As before, let \(G_1\) be the graph \(K_1 \vee P_{n-1}\) and let \(A_1\) be its adjacency matrix with the dominating vertex corresponding to the first row and column. Since \(G\) is spread-extremal we must have \(S(G) \geq S(G_1)\). We lower bound \(S(G_1)\) using the vectors
\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix}
-1 + \sqrt{n} \\
1 \\
1 \\
\vdots \\
1 
\end{bmatrix}
\quad \text{and} \quad
\mathbf{v}_2 &= \begin{bmatrix}
-1 - \sqrt{n} \\
1 \\
1 \\
\vdots \\
1 
\end{bmatrix}.
\end{align*}
\]
Using these vectors and the Rayleigh principle, we have that

\[
S(G_1) \geq \frac{\mathbf{v}_1^T A_1 \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{v}_1} - \frac{\mathbf{v}_2^T A_1 \mathbf{v}_2}{\mathbf{v}_2^T \mathbf{v}_2}
\]

\[
= \frac{2(n-1)(\sqrt{n}-1) + (n-2)(1)}{2n-2\sqrt{n}} - \frac{2(n-1)(-\sqrt{n}-1) + (n-2)(1)}{2n+2\sqrt{n}}
\]

\[
= \frac{2(n-1)(\sqrt{n}-1) + (n-1)(1)}{2n-2\sqrt{n}} - \frac{1}{2n-2\sqrt{n}}
\]

\[
= \frac{n-1}{\sqrt{n}-1} + \frac{n-1}{\sqrt{n}+1} - \frac{\sqrt{n}}{n^2-n} \geq 2\sqrt{n} - \frac{1}{n}.
\]

Combining this with Lemmas 4.3 and 4.4 we have

\[
2\sqrt{n} - \frac{1}{n} \leq \lambda_1(G) - \lambda_n(G) = \left( \sqrt{n}-1 + \frac{m}{n-1} + \Theta \left( \frac{m}{n^{3/2}} \right) \right) - \left( -\sqrt{n}-1 + \frac{m}{n-1} + \Theta \left( \frac{m}{n^{3/2}} \right) \right).
\]

Therefore there exists a constant \(C\) such that for \(n\) large enough we have

\[
2\sqrt{n} - \frac{1}{n} \leq 2\sqrt{n}-1 + C \cdot \frac{m}{n^{3/2}}.
\]

Since \(\sqrt{n} - \sqrt{n-1} = \Omega(n^{-1/2})\), rearranging gives the final result.

\(\Box\)

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