WEYL DENOMINATOR IDENTITY FOR THE AFFINE LIE SUPERALGEBRA $\mathfrak{gl}(2|2)$

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Abstract. We prove the Weyl denominator identity for the affine Lie superalgebra $\mathfrak{gl}(2|2)$ conjectured by V. Kac and M. Wakimoto in [KW]. As it was pointed out in [KW], this gives a new proof of the Jacobi identity for the number of presentations of a given integer as a sum of 8 squares.

0. Introduction

The denominator identities for Lie superalgebras were formulated and partially proven in the paper of V. Kac and M. Wakimoto [KW]. In the same paper it was shown how various classical identities in number theory as the number of representation of a given integer as a sum of $d$ squares can be obtained, for some $d$, by evaluation of certain denominator identities. The following cases are considered in the paper [KW]:

(a) basic Lie superalgebras, i.e. the finite-dimensional simple Lie superalgebras, which have a reductive even part and admit an even non-degenerate invariant bilinear form;

(b) the affinization of basic Lie superalgebras with non-zero dual Coxeter number;

(c) the (twisted) affinization of a strange Lie superalgebras $Q(n)$;

(d) the affinization of $\mathfrak{gl}(2|2)$ (this is the smallest basic Lie superalgebras with zero dual Coxeter number).

Some of the cases (a), (b) are proven in [KW]; the proof is based on combinatorics of root systems and a certain result from representation theory. The rest of (a) was proven in [G1] using only combinatorics of root systems. The rest of (b) was proven in [G2] using (a) and the existence of Casimir operator. The case (c) was proven in [Z] analytically. In the present paper we prove (d), i.e. the identity for the affine Lie superalgebra $\mathfrak{gl}(2|2)$ conjectured in [KW], 7.1. The proof uses the existence of Casimir operator and an idea of [Z].

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In order to write down the identity, we introduce the following infinite products after [DK]: for a parameter $q$ and a formal variable $x$ we set

$$(1 + x)^q := \prod_{n=0}^{\infty} (1 + q^n x), \quad \text{and} \quad (1 - x)^q := \prod_{n=0}^{\infty} (1 - q^n x).$$

These infinite products converge for any $x \in \mathbb{C}$ if the parameter $q$ is a real number $0 < q < 1$. In particular, they are well-defined for $0 < x = q < 1$ and $(1 \pm q)^q := \prod_{n=1}^{\infty} (1 \pm q^n)$.

Take the formal variables $x, y_1, y_2$. The denominator identity for $\mathfrak{gl}(2|2)$ can be written in the following form

$$(1 - x)^\infty (1 - qx^{-1})^\infty (1 - x y_1 y_2)^\infty (1 - q(xy_1 y_2)^{-1})^\infty ((1 - q)^4)^4 = \prod_{i=1}^{2}(1 + y_i)^\infty (1 + q y_i^{-1})^\infty (1 + x y_i)^\infty (1 + x q^{-1} y_i)^\infty$$

$$= \frac{((1 - q)^2)^2}{(1 - qy_1^{-1} y_2)^\infty (1 - q y_1 y_2)^\infty} \cdot \sum_{n=-\infty}^{\infty} \frac{q^n}{(1 + q^n y_1)(1 + q^n y_2)} - \frac{q^n x}{(1 + q^n x y_1)(1 + q^n x y_2)}.$$

Expanding the factor $\frac{((1 - q)^2)^2}{(1 - qy_1^{-1} y_2)^\infty (1 - q y_1 y_2)^\infty}$ in the region $q < |\frac{y_1}{y_2}| < q^{-1}$ we obtain (see Lemma 1.3.1)

$$\frac{((1 - q)^2)^2}{(1 - qy_1^{-1} y_2)^\infty (1 - q y_1 y_2)^\infty} = 1 + \sum_{n=1}^{\infty} f_n \left( \frac{y_1}{y_2} \right),$$

where $f_n(y) := (y^n + y^{-n} - y^{n-1} - y^{1-n}) \sum_{j=0}^{\infty} (-1)^j q^{(j+1)(j+2n)/2}$

and this gives the identity conjectured by V. Kac and M. Wakimoto.

The left-hand side of the identity represents the Weyl denominator $\hat{R}$ for the affine Lie superalgebra $\mathfrak{gl}(2|2)$; the second factor in the right-hand side is the analogue of the right-hand side of the denominator identity for affine Lie superalgebras with non-zero dual Coxeter number. Note that the denominator identity for the affine Lie superalgebra $\mathfrak{sl}(2|2)$ can be obtained from the denominator identity for $\mathfrak{gl}(2|2)$ by taking $y_1 = y_2$; as a result, the denominator identity for $\mathfrak{sl}(2|2)$ is almost similar to the denominator identity for affine Lie superalgebras with non-zero dual Coxeter number with one extra-factor $(1 - q)^\infty$ in the left-hand side (since the dimension of Cartan subalgebra for $\mathfrak{sl}(2|2)$ is less by one than the dimension of Cartan subalgebra for $\mathfrak{gl}(2|2)$).

As it is shown in [KW], the evaluation of this identity gives the following Jacobi identity [J]:

$$(2) \quad \Box(q)^8 = 1 + 16 \sum_{j,k=1}^{\infty} (-1)^{(j+1)k} k^3 q^{jk},$$
where \( \Box(q) = \sum_{j \in \mathbb{Z}} q^{j^2} \) and thus the coefficient of \( q^n \) in the power series expansion of \( \Box(q)^d \) is the number of representation of a given integer as a sum of \( d \) squares (taking into the account the order of summands).

In Section 1 we introduce notation. In Section 2 we prove the identity (1). In Section 3 we recall how to deduce the Jacobi identity from the identity (1).

1. Notation

1.1. Root system. Consider \( V := \mathbb{R}^5 \) endowed by a bilinear form and an orthogonal basis \( \varepsilon_1, \varepsilon_2, \delta_1, \delta_2, \delta \) such that \( (\varepsilon_i, \varepsilon_i) = 1 = (\delta_i, \delta_i) \) for \( i = 1, 2 \) and \( (\delta, \delta) = 0 \). Set

\[
\beta_1 := \delta_1 - \varepsilon_1, \quad \alpha := \varepsilon_1 - \varepsilon_2, \quad \beta_2 := \varepsilon_2 - \delta_2, \quad \gamma := \beta_1 + \alpha + \beta_2 = \delta_1 - \delta_2.
\]

The root system of \( \mathfrak{g}l(2|2) \) is \( \Delta_0 = \{ \pm \alpha, \pm \gamma \}, \Delta_1 = \{ \pm \beta_i; \pm (\alpha + \beta_i) \}_{i=1,2} \). The affine root system is \( \hat{\Delta}_i = \bigcup_{s \in \mathbb{Z}} (\Delta_i + s\delta) \), \( i = 0, 1 \).

We consider the following sets of simple roots for \( \mathfrak{g}l(2|2) \) and \( \mathfrak{g}l(2|2) \) respectively:

\[
\Pi := \{ \beta_1, \alpha, \beta_2 \}, \quad \hat{\Pi} := \{ \beta_1, \alpha, \beta_2, \delta - \gamma \}.
\]

One has

\[
\Delta_+ = \{ \alpha, \gamma; \beta_i, \alpha + \beta_i \}_{i=1,2}, \quad \hat{\Delta}_+ = \Delta_+ \cup \bigcup_{s=1}^{\infty} (\Delta + s\delta), \quad \hat{\rho} = \rho = -\frac{\beta_1 + \beta_2}{2}.
\]

Set

\[
Q^+ := \sum_{\mu \in \Pi} \mathbb{Z}_{\geq 0}\mu, \quad \hat{Q}^+ := \sum_{\mu \in \hat{\Pi}} \mathbb{Z}_{\geq 0}\mu.
\]

1.1.1. For \( \nu \in \Delta_0 \) let \( s_\nu \in \text{Aut}(V) \) be the reflection with respect to \( \nu \), i.e. \( s_\nu(\lambda) = \lambda - \frac{(\lambda, \nu)}{(\nu, \nu)} \nu \). The Weyl group \( W \) of \( \Delta_0 \) takes form \( W = W_{\alpha} \times W_{\gamma} \), where \( W_{\alpha} \) (resp., \( W_{\gamma} \)) is generated by the reflection \( s_{\alpha} \) (resp., \( s_{\gamma} \)).

For \( \nu \in V \) introduce \( t_\nu \in \text{Aut}(V) \) by the formula

\[
t_\mu(\lambda) = \lambda - (\lambda, \mu)\delta.
\]

Then \( t_\mu t_\nu = t_{\mu + \nu} \). For \( \nu \in \Delta_0 \) we denote by \( T_\nu \) the infinite cyclic group generated by \( t_\nu \) and by \( \hat{T}_\nu \) the group generated by \( s_\nu \) and \( t_\nu \). The Weyl group of \( \mathfrak{g}l(2|2) \) is \( \hat{W} = W_{\alpha} \times \hat{W}_{\gamma} \).

Notice that \( \delta \) and \( \beta_1 - \beta_2 \) lie in the kernel of the bilinear form so these vectors are \( \hat{W} \)-stable.

For a subgroup \( G \) of the Weyl group we introduce the following operator:

\[
\mathcal{F}_G := \sum_{w \in G} \text{sgn } w \cdot w.
\]

1.2. Algebra \( \mathcal{R} \). We are going to use notation of [G2], 1.4, which we recall below.
1.2.1. Consider the space \( \hat{h}^* = V \oplus \mathbb{R}\Lambda_0 \) and extend our bilinear form by \((\Lambda_0, \delta) = 1,(\Lambda_0, \Lambda_0) = (\Lambda_0, \epsilon_i) = (\Lambda_0, \delta_i) = 0 \) for \( i = 1, 2 \). The Weyl group \( \hat{W} \) acts on \( \hat{h}^* \) as follows: the reflections act by the same formulas and the action of \( t_\mu \) extends by the standard formula

\[
t_\mu(\lambda) = \lambda + (\lambda, \delta)\mu - ((\lambda, \mu) + (\mu, \mu)/(\lambda, \delta))\delta, \quad \mu \in V, \lambda \in \hat{h}^*
\]

Call a \( \hat{Q}^+\)-cone a set of the form \((\lambda - \hat{Q}^+)\), where \( \lambda \in \hat{h}^* \).

1.2.2. For a formal sum of the form \( \sum_{\nu \in \hat{h}^*} b_\nu e^\nu \), \( b_\nu \in \mathbb{Q} \) define the support of \( Y \) by \( \text{supp}(Y) := \{ \nu \in \hat{h}^* | b_\nu \neq 0 \} \). Let \( \mathcal{R} \) be a vector space over \( \mathbb{Q} \), spanned by the sums of the form \( \sum_{\nu \in \hat{Q}^+} b_\nu e^{\lambda - \nu} \), where \( \lambda \in \hat{h}^*, b_\nu \in \mathbb{Q} \). In other words, \( \mathcal{R} \) consists of the formal sums \( Y = \sum_{\nu \in \hat{h}^*} b_\nu e^\nu \) with the support lying in a finite union of \( \hat{Q}^+\)-cones.

Clearly, \( \mathcal{R} \) has a structure of commutative algebra over \( \mathbb{Q} \). If \( Y \in \mathcal{R} \) is such that \( YY' = 1 \) for some \( Y' \in \mathcal{R} \), we write \( Y^{-1} := Y' \).

1.2.3. Action of the Weyl group. For \( w \in \hat{W} \) set \( w(\sum_{\nu \in \hat{h}^*} b_\nu e^\nu) := \sum_{\nu \in \hat{h}^*} b_\nu e^{w\nu} \). One has \( wY \in \mathcal{R} \) iff \( w(\text{supp} Y) \) is a subset of a finite union of \( \hat{Q}^+\)-cones.

Let \( W' \) be a subgroup of \( \hat{W} \). Let \( \mathcal{R}_{W'} := \{ Y \in \mathcal{R} | wY \in \mathcal{R} \text{ for any } w \in W' \} \). Clearly, \( \mathcal{R}_{W'} \) is a subalgebra of \( \mathcal{R} \).

1.2.4. Infinite products. An infinite product of the form \( Y = \prod_{\nu \in X} (1 + a_\nu e^{-\nu})^{r(\nu)} \), where \( a_\nu \in \mathbb{Q}, r(\nu) \in \mathbb{Z}_{\geq 0} \) and \( X \subset \hat{\Delta} \) is such that the set \( X \setminus \hat{\Delta}_+ \) is finite, can be naturally viewed as an element of \( \mathcal{R} \); clearly, this element does not depend on the order of factors.

Let \( \mathcal{V} \) be the set of such infinite products. For any \( w \in \hat{W} \) the infinite product

\[
wY := \prod_{\nu \in X} (1 + a_\nu e^{-w\nu})^{r(\nu)},
\]

is again an infinite product of the above form, since, as one easily sees ([G2], Lem. 1.2.8), the set \( w\hat{\Delta}_+ \setminus \hat{\Delta}_+ \) is finite. Hence \( \mathcal{V} \) is a \( \hat{W} \)-invariant multiplicative subset of \( \mathcal{R}_{\hat{W}} \).

The elements of \( \mathcal{V} \) are invertible in \( \mathcal{R} \): using the geometric series we can expand \( Y^{-1} \) (for example, \((1 - e^\alpha)^{-1} = -e^{-\alpha}(1 - e^{-\alpha})^{-1} = -\sum_{i=1}^{\infty} e^{-i\alpha}) \).

1.2.5. The subalgebra \( \mathcal{R}' \). Denote by \( \mathcal{R}' \) the localization of \( \mathcal{R}_{\hat{W}} \) by \( \mathcal{V} \). By above, \( \mathcal{R}' \) is a subalgebra of \( \mathcal{R} \). Observe that \( \mathcal{R}' \not\subset \mathcal{R}_{\hat{W}} \): for example, \((1 - e^{-\alpha})^{-1} \in \mathcal{R}' \), but \((1 - e^{-\alpha})^{-1} = \sum_{j=0}^{\infty} e^{-j\alpha} \not\in \mathcal{R}_{\hat{W}} \). We extend the action of \( \hat{W} \) from \( \mathcal{R}_{\hat{W}} \) to \( \mathcal{R}' \) by setting \( w(Y^{-1}Y') := (wY)^{-1}(wY') \) for \( y \in \mathcal{V}, Y' \in \mathcal{R}_{\hat{W}} \).
An infinite product of the form $Y = \prod_{\nu \in X} (1 + a_{\nu} e^{-\nu})^{r(\nu)}$, where $a_{\nu}, X$ are as above and $r(\nu) \in \mathbb{Z}$ lies in $R'$, and $wY = \prod_{\nu \in X} (1 + a_{\nu} e^{-w\nu})^{r(\nu)}$. One has

$$\text{supp}(Y) \subset \lambda' - \mathcal{Q}^+,$$

where $\lambda' = - \sum_{\nu \in X \Delta_{+} : a_{\nu} \neq 0} r_{\nu} \nu$.

**Remark.** Set $q := e^{-\delta}, x := e^{-\alpha}, y_{i} := e^{-\beta_{i}}$, and write elements of $R'$ as power series in these variables. Since $\{e^{-\nu}, \nu \in \Pi\} = \{x, y_{1}, y_{2}, q(xy_{1}y_{2})^{-1}\}$, the support of $Y \in R'$ corresponds to the expansion of $Y$ in the region $|q| < |xy_{1}y_{2}|; |x|, |y_{1}|, |y_{2}| < 1$.

1.2.6. Let $W'$ be a subgroup of $\hat{W}$. For $Y \in R'$ we say that $Y$ is $W'$-invariant (resp., $W'$-anti-invariant) if $wY = Y$ (resp., $wY = sgn(w)Y$) for each $w \in W'$.

Let $Y = \sum a_{w} e_{w} \in R_{W'}$ be $W'$-anti-invariant. Then $a_{w} e_{w} = (-1)^{sgn(w)} a_{w}$ for each $w$ and $w \in W'$. In particular, $W'$ sup$(Y) = \text{supp}(Y)$, and, moreover, for each $w \in \text{supp}(Y)$ one has $\text{Stab}_{W'} w \subset \{ w \in W' | sgn(w) = 1 \}$. The condition $Y \in R_{W'}$ is essential: for example, for $W' = \{ \text{id}, s_{\alpha} \}$, the expressions $Y := e^{\alpha} - e^{-\alpha}, Y^{-1} = e^{-\alpha}(1 - e^{-2\alpha})^{-1}$ are $W'$-anti-invariant, but supp$(Y^{-1}) = -\alpha, -3\alpha, \ldots$ is not $s_{\alpha}$-invariant.

Take $Y = \sum a_{w} e_{w} \in R_{W'}$. The sum $F_{W'}(Y) = \sum_{w \in W'} sgn(w) wY$ is an element of $R$ if for each $w$ the sum $\sum_{w \in W'} sgn(w) a_{w}$ is finite (i.e., $W' \cap \text{supp}(Y)$ is finite). In this case $F_{W'}(Y) \in \mathcal{R}$ and, writing $F_{W'}(Y) = \sum b_{w} e_{w}$, we obtain $b_{w} = \sum_{w \in W'} sgn(w) a_{w}$ so $b_{w} = sgn(w) b_{w}$ for each $w \in W'$. We conclude that

$$Y \in R_{W'} \& F_{W'}(Y) \in \mathcal{R} \implies \begin{cases} F_{W'}(Y) \in R_{W'}; \\ F_{W'}(Y) \text{ is } W'\text{-anti-invariant;} \\ \text{supp}(F_{W'}(Y)) \text{ is } W'\text{-stable.} \end{cases}$$

Let us call a vector $\lambda \in \hat{h}^{*}$ $W'$-regular if $\text{Stab}_{W'} \lambda = \{ \text{id}\}$. Say that the orbit $W' \lambda$ is $W'$-regular if $\lambda$ is $W'$-regular (so the orbit consists of $W'$-regular points). If $W'$ is an affine Weyl group, then for any $\lambda \in \hat{h}^{*}$ the stabilizer $\text{Stab}_{W'} \lambda$ is either trivial or contains a reflection. Thus for $W' = \hat{W}_{\alpha}, \hat{W}_{\gamma}$ one has

$$Y \in R_{W'} \& F_{W'}(Y) \in \mathcal{R} \implies \text{supp}(F_{W'}(Y)) \text{ is a union of } W'\text{-regular orbits.}$$

1.2.7. **Remark.** For $Y \in R'$ the sum $F_{W'}(Y)$ is not always $W'$-anti-invariant: for example, for $W' = \{ \text{id}, s_{\alpha} \}$ one has $F_{W'}((1 - e^{-\alpha})^{-1}) = (1 - e^{-\alpha})^{-1} - (1 - e^{\alpha})^{-1} = 1 + 2e^{-\alpha} + 2e^{-2\alpha} + \ldots$ which is not $W'$-anti-invariant.

1.3. **Another form of denominator identity.** Introduce the following elements of $\mathcal{R}$:

$$R_{0} := \prod_{\nu \in A_{\alpha}^{\bot}} (1 - e^{-\nu}), \quad R_{1} := \prod_{\nu \in A_{1}^{\bot}} (1 + e^{-\nu}), \quad R := \frac{R_{0}}{R_{1}},$$

$$\hat{R}_{0} := \prod_{\nu \in A_{\alpha}^{\bot}} (1 - e^{-\nu}), \quad \hat{R}_{1} := \prod_{\nu \in A_{1}^{\bot}} (1 + e^{-\nu}), \quad \hat{R} := \frac{\hat{R}_{0}}{\hat{R}_{1}}.$$ The products $Re_{\rho}$ and $\hat{R}e_{\rho}$ are $\hat{W}$-anti-invariant elements of $\mathcal{R}'$ (see, for instance, [G2], Lem. 1.5.1).
1.3.1. **Lemma.** In the region $q < |y| < q^{-1}$ one has
\[
\frac{((1-q)\infty)^2}{(1-qy)\infty(1-qy^{-1})\infty} = \sum_{n=-\infty}^{\infty} (y^n + y^{-n} - y_{n-1} - y_{1-n}) \sum_{j=0}^{\infty} (-1)^j q^{(j+1)(j+2n)/2}
\]
and this expression lies in $\mathcal{R}$ for $y = e^{\beta_2 - \beta_1}$.

**Proof.** Consider the root system $\mathfrak{sl}(2|1)$ with the odd simple roots $\beta'_1, \beta'_2$ and the even positive root $\alpha' = \beta'_1 + \beta'_2$. Note that the corresponding element $\rho'$ is equal to zero. Consider the corresponding affine root system, let $\delta'$ be the minimal imaginary root and $\hat{W}'$ be its Weyl group. The affine denominator identity for $\mathfrak{sl}(2|1)$ written for $z := e^{-\delta}$ takes form
\[
\frac{2((1+z)\infty(1-z)\infty)^2}{(1-\xi)\infty(1+\xi^{-1})\infty(1-z\xi^{-1})\infty(1+z\xi)\infty} = \sum_{n=-\infty}^{\infty} (-1)^n z^n \left( \frac{1}{1-z^n} - \frac{1}{1+z^n} \right).
\]
Both sides are well-defined for real $0 < z < 1$ and $\beta'_i$ such that $e^{\beta'_i} \neq z^n$ for $n \in \mathbb{Z}$. Taking $e^{\alpha'} = -1$ and $e^{-\beta'_i} = -\xi$ we obtain $e^{-\beta'_i} = e^{-\alpha} e^{\beta'_i} = \xi^{-1}$ and the evaluation gives
\[
\frac{2((1+z)\infty(1-z)\infty)^2}{(1-\xi)\infty(1+\xi^{-1})\infty(1-z\xi^{-1})\infty(1+z\xi)\infty} = \sum_{n=-\infty}^{\infty} (-1)^n z^n \left( \frac{1}{1-z^n\xi} - \frac{1}{1+z^n\xi} \right).
\]
For $z^2 = q, \xi^2 = y$ we get
\[
\frac{((1-q)\infty)^2}{(1-qy)\infty(1-qy^{-1})\infty(1-y)} = \sum_{m=-\infty}^{\infty} (-1)^m q^m \frac{1}{1-q^my}. \quad \text{For } m > 0 \text{ one has } \frac{1}{1-q^my} = \sum_{k=0}^{\infty} q^{mk} y^k \text{ and } \frac{1}{1-q^{-m}y} = -\sum_{k=0}^{\infty} q^{-mk} y^{-k} \text{ so}
\]
\[
\frac{((1-q)\infty)^2}{(1-qy)\infty(1-qy^{-1})\infty} = 1 + (1-y) \sum_{m=1}^{\infty} (-1)^m \sum_{k=0}^{\infty} (q^{m^2/m+2mk} y^k - q^{m^2/m+2mk} y^{-k})
\]
\[
= 1 + \sum_{m=0}^{\infty} (-1)^m \sum_{k=0}^{\infty} (q^{(m+1)(m+2k)} y^k - q^{(m+1)(m+2k)} y^{-k})
\]
as required. One readily sees that the right-hand side of the above expression lies in $\mathcal{R}$ for $y = e^{\beta_2 - \beta_1}$.

1.3.2. Set $x := e^{-\alpha}, y_i := e^{-\beta_i}$ for $i = 1, 2$ and $q := e^{-\delta}$. Under this substitution, the left-hand side of (1) becomes $\hat{R}$ and, using Lemma 1.3.1, we rewrite (1) in the following form
\[
\hat{R}e^{\rho} = (1 + \sum f_n e^{-\rho} \mathcal{F}_{W_n} \left( \frac{e^{\rho}}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right)).
\]
Denote by $LHS$ (resp., $RHS$) the left-hand (resp., right-hand) side of the identity (3).
The denominator identity for $\mathfrak{gl}(2|2)$ takes the form

\[
\mathcal{F}_{W_\alpha}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \text{Re}^\rho = \mathcal{F}_{W_\gamma}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}).
\]

so (3) can be rewritten as

\[
\hat{\text{Re}}^\rho = (1 + \sum f_n) \mathcal{F}_{T_\alpha}(\text{Re}^\rho).
\]

In the sequel we need the following lemma.

1.3.3. **Lemma.** If $\mathcal{F}_{T_\alpha}(\text{Re}^\rho)$ is well-defined (as an element of $\mathcal{R}$), then

\[
\mathcal{F}_{T_\alpha}(\text{Re}^\rho) = \mathcal{F}_{T_\gamma}(\text{Re}^\rho).
\]

**Proof.** Note that $(\gamma - \alpha, \rho) = (\gamma - \alpha, \beta_i) = 0$ for $i = 1, 2$ so $\frac{e^\rho}{(1 - e^{-\beta_1})(1 - e^{-\beta_2})}$ is invariant with respect to the action of $t_{\gamma - \alpha}$. Therefore

\[
\mathcal{F}_{T_\alpha}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \mathcal{F}_{T_\gamma}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}).
\]

Using the formula (4), we obtain

\[
\mathcal{F}_{T_\alpha}(\text{Re}^\rho) = \mathcal{F}_{T_\alpha} \circ \mathcal{F}_{W_\gamma}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \mathcal{F}_{W_\gamma} \circ \mathcal{F}_{T_\alpha}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \mathcal{F}_{T_\gamma} \circ \mathcal{F}_{W_\gamma}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \mathcal{F}_{T_\gamma}(\text{Re}^\rho),
\]

as required. \(\square\)

As a corollary, (3) can be rewritten as $\hat{\text{Re}}^\rho = (1 + \sum f_n) \mathcal{F}_{T_\gamma}(\text{Re}^\rho)$.

**2. Proof of the denominator identity**

2.1. By \[1.2.3\] LHS is an invertible element of $\mathcal{R}'$. In this subsection we show that RHS is a well-defined element of $\mathcal{R}$.

For $w \in \hat{W}_\alpha$ set

\[
S_w := \text{supp}(w(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})})).
\]

One has

\[
t_{n\alpha}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \frac{e^\rho q^n}{(1 + q^n e^{-\beta_1})(1 + q^n e^{-\beta_2})}.
\]
Then for $n > 0$ one has

\[
S_{id} = \{\rho - k_1 \beta_1 - k_2 \beta_2\}, \quad S_{\alpha} = \{\rho - (1 + k_1 + k_2) \alpha - k_1 \beta_1 - k_2 \beta_2\}, \\
S_{\rho_{\text{na}}} = \{\rho - (1 + k_1 + k_2) \delta - k_1 \beta_1 - k_2 \beta_2\}, \\
S_{\rho_{\text{na}}} = \{\rho - n(1 + k_1 + k_2) \delta - (1 + k_1 + k_2) \alpha - k_1 \beta_1 - k_2 \beta_2\}, \\
S_{\rho_{\text{na}}} = \{\rho - n(1 + k_1 + k_2) \delta - (1 + k_1 + k_2) \alpha + k_1 \beta_1 + k_2 \beta_2\},
\]

where $k_1, k_2 \geq 0$. Observe that the above sets are pairwise disjoint so the sum $F_{W_{\alpha}}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})})$ is well-defined and its support lies in $\rho - \hat{Q}^+$. Clearly, the sum $1 + \sum f_n$ is well-defined.

One readily sees that

\[
n \delta + k \beta_1 - k \beta_2 \in \hat{Q}^+ \iff |k| \leq n
\]

Thus the support of $1 + \sum f_n$ lies in $\{0\} \cup \{-m \delta + k \beta_1 - k \beta_2, m > 0\} \cap -\hat{Q}^+$ (in particular, $(1 + \sum f_n) \in R$). Hence $RHS$ is a well-defined element of $\mathcal{R}$.

2.2. **Lemma.** The expansion of $\frac{RHS \ LHS}{LHS}$ in the region $|q| < |xy_1y_2|, |x|, |y_1|, |y_2| < 1$ is of the form $1 + \sum_{n=1}^{\infty} \sum_{j=-n}^{n} a_n \cdot q^n \cdot (\frac{y_2}{y_1})^j$, where $a_n \in \mathbb{Z}$.

**Proof.** Recall that $LHS = \hat{R} e^\rho$ and that $\hat{R} \in \mathcal{Y}$ (see 1.2.4 for notation). By 2.1 $RHS \in \mathcal{R}$. Therefore the fraction

\[
Y := \frac{RHS}{LHS} = \hat{R}^{-1} e^{-\rho} \cdot RHS
\]

lies in $\mathcal{R}$.

Clearly, $-\rho \in \text{supp}(\hat{R}^{-1} e^{-\rho}) \subset (-\rho - \hat{Q}^+)$. Since $\text{supp}(RHS) \subset \rho - \hat{Q}^+$, we conclude that $\text{supp}(Y) \subset -\hat{Q}^+$. By (5) the coefficient of $e^\rho$ in $RHS$ is 1; clearly, the coefficient of $e^{-\rho}$ in $\hat{R}^{-1} e^{-\rho}$ is also 1, so the coefficient of $e^0 = 1$ in $Y$ is 1. In the light of Remark 1.2.5 the required assertion is equivalent to the inclusion

\[
\text{supp}(Y) \subset \{-n \delta + j(\beta_1 - \beta_2) | n \geq 0, |j| \leq n\}.
\]

Retain notation of 1.2.1. The element $\hat{\rho}_\alpha := 2\Lambda_0 + \frac{a}{2}$ is the standard element for the corresponding copy of $\mathfrak{g}(\mathbb{C}) \subset \mathfrak{g}(\mathbb{C})$. Recall that $\hat{R} = \frac{R_\alpha}{R_1}$ (see 1.3 for notation) so $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho} = \hat{R}_0 e^{\hat{\rho}_\alpha} \cdot (\hat{R} e^{\rho})^{-1}$. By 1.2.4 $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho}$ belongs to $\mathcal{R}_W$. It is a standard fact that $\hat{R}_0 e^{\hat{\rho}_\alpha}$ is $\hat{W}_\alpha$-anti-invariant. Recall that $\hat{R} e^{\rho}$ is $\hat{W}$-anti-invariant. Thus $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho}$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_W$. One has

\[
\hat{R}_0 e^{\hat{\rho}_\alpha} Y = \hat{R}_1 e^{\hat{\rho}_\alpha - \rho} \cdot RHS = (1 + \sum f_n) \cdot \hat{R}_1 e^{\hat{\rho}_\alpha - \rho} \cdot F_{W_{\alpha}}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}).
\]
The $\hat{W}_\alpha$-invariance of $\hat{R}_1 e^{\beta_0 - \rho}$ gives

$$\hat{R}_1 e^{\beta_0 - \rho} \cdot \mathcal{F}_{\hat{W}_\alpha} \left( \frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right) = \mathcal{F}_{\hat{W}_\alpha} \left( \frac{\hat{R}_1 e^{\beta_0}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$$

so

$$\hat{R}_0 e^{\beta_0} Y = (1 + \sum f_n) \cdot \mathcal{F}_{\hat{W}_\alpha} \left( \frac{\hat{R}_1 e^{\beta_0}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right).$$

By 2.1, $\mathcal{F}_{\hat{W}_\alpha} \left( \frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$ lies in $\mathcal{R}$ so $\mathcal{F}_{\hat{W}_\alpha} \left( \frac{\hat{R}_1 e^{\beta_0}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$ lies in $\mathcal{R}$. By 1.2.4, the term

$$\frac{\hat{R}_1 e^{\beta_0}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} = e^{\beta_0} \prod_{\beta \in \Delta_{\mathfrak{sl}_2} \setminus \{\beta_1, \beta_2\}} (1 + e^{-\beta})$$

lies in $\mathcal{R}_\hat{W}$. Therefore, in the light of 1.2.6, $\mathcal{F}_{\hat{W}_\alpha} \left( \frac{\hat{R}_1 e^{\beta_0}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. Observe that $\frac{n}{y_2} = e^{\beta_2 - \beta_1}$ is $\hat{W}$-invariant so $f_n$ is $\hat{W}$-invariant. Thus $(1 + \sum f_n) \mathcal{F}_{\hat{W}_\alpha} \left( \frac{\hat{R}_1 e^{\beta_0}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. As a result, $\hat{R}_0 e^{\beta_0} Y$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$.

Write $Y = Y_1 + Y_2$, where $Y_1 = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} a_{n,j} q^n (\frac{n}{y_2})^j$ and $Y_2$ does not have monomials of the form $q^n (\frac{n}{y_2})^j$, i.e. supp$(Y) = $ supp$(Y_1) \prod$ supp$(Y_2)$. One has $Y_i \in \mathcal{R}$ because supp$(Y_i) \subset$ supp$(Y) \subset -\mathcal{Q}^+$ ($i = 1, 2$).

Since $\frac{n}{y_2} = e^{\beta_2 - \beta_1}$ is $\hat{W}$-invariant, $Y_1$ is a $\hat{W}$-anti-invariant element of $\mathcal{R}_\hat{W}$. Since $\hat{R}_0 e^{\beta_0}$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$, the product $\hat{R}_0 e^{\beta_0} Y_1$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. By above, $\hat{R}_0 e^{\beta_0} Y$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. Hence $\hat{R}_0 e^{\beta_0} Y_2$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$.

Assume that $Y_2 \neq 0$. Recall that supp$(Y_2) \subset -\mathcal{Q}^+$. Let $\mu$ be a maximal element in supp$(Y_2)$ with respect to the standard partial order $\mu \leq \nu$ if $(\nu - \mu) \in \mathcal{Q}^+$. Then $\hat{\rho}_\alpha + \mu$ is a maximal element in the support of $\hat{R}_0 e^{\beta_0} Y_2$. By 1.2.6, this support is the union of $\hat{W}_\alpha$-regular orbits, so $\hat{\rho}_\alpha + \mu$ is a maximal element in a regular $\hat{W}_\alpha$-orbit (regularity means that each element has the trivial stabilizer in $\hat{W}_\alpha$). Since $\mu \in -\mathcal{Q}^+$ one has $\frac{2(\mu, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$. Therefore $\frac{2(\hat{\rho}_\alpha + \mu, \alpha)}{(\alpha, \alpha)} = 1 + (\mu, \alpha), \frac{2(\hat{\rho}_\alpha + \mu, \delta - \alpha)}{(\delta - \alpha, \delta - \alpha)} = 1 + (\mu, \delta - \alpha)$ are positive integers so $(\mu, \alpha), (\mu, \delta - \alpha) \geq 0$. Since $\mu \in -\mathcal{Q}^+$ one has $(\mu, \delta) = 0$ and thus $(\mu, \alpha) = 0$.

The element $\hat{\rho}_\gamma := -2 A_0 + \frac{\gamma}{2}$ is the standard element for the corresponding copy of $\mathfrak{sl}_2$. Using Lemma 1.3.3 we obtain

$$\hat{R}_0 e^{\hat{\rho}_\gamma} Y = (1 + \sum f_n) \mathcal{F}_{\hat{W}_\gamma} \left( \frac{\hat{R}_1 e^{\hat{\rho}_\gamma}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right).$$

Repeating the above reasoning for $\hat{W}_\gamma$ we obtain $(\mu, \gamma) = 0$. Hence $(\mu, \alpha) = (\mu, \gamma) = 0$ and $\mu \in -\mathcal{Q}^+$. This implies $\mu = -m \delta + k(\beta_1 - \beta_2)$, which contradicts to the construction.
of $Y_2$. Hence $Y_2 = 0$ so $Y = Y_1$ that is $\text{supp}(Y) \subset \{-n\delta + j(\beta_1 - \beta_2)\}$. Combining the condition $\text{supp}(Y) \subset -\hat{Q}^+$ and (9), we obtain the required inclusion (7). \hfill \Box

2.3. Evaluation. By Lemma [2,2] $\frac{\text{RHS}}{\text{LHS}}$ is a function of one variable $y := \frac{y_1}{y_2}$. In order to establish the identity $\text{LHS} = \text{RHS}$, it is enough to verify that $\frac{\text{RHS}}{\text{LHS}}(y) = 1$ for a fixed $x$ and some $y_2, y_1$ satisfying $y_1 = yy_2$. We will check this for $x = 1, y_2 = y, y_1 = y^2$, (i.e. $e^{-\alpha} = -1, e^{-\beta_1} = y^2, e^{-\beta_2} = y$).

One has $\frac{\text{RHS}}{\text{LHS}} = \hat{R}^{-1}(\text{RHS} \cdot e^{-p})$. We write $\text{RHS} \cdot e^{-p} = AB$, where

$$A := \frac{((1-q)^{\infty})^2}{(1-qy)^{\infty}(1-qy^{-1})^{\infty}}, \quad B := e^{-p} \cdot \mathcal{F}_{W_\alpha} \left( \frac{e^p}{(1+e^{-\beta})(1+e^{-\beta_2})} \right), \quad y = \frac{y_1}{y_2},$$

2.3.1. Recall that an infinite product $\prod_{i=1}^{\infty} (1+g_i(z))$, where $g_i(z)$ are holomorphic functions in $U \subset \mathbb{C}$ is called normally convergent in $U$ if $\sum g_i(z)$ normally converges in $U$. By [R], a normally convergent infinite product converges to a function $g(z)$, which is holomorphic in $U$; moreover, the set of zeros of $g(z)$ is the union of the sets of zeros of $1 + g_i(z)$ and the order of each zero is the sum of the orders of the corresponding zeros of $1 + g_i(z)$.

The denominator of $A(y)$ normally converges in any $U \subset X$, where $X \subset \mathbb{C}$ is a compact not containing 0. Thus $A(y)$ is a meromorphic function in the region $0 < |y|$ with simple poles at the points $y = q^n$, $n \in \mathbb{Z} \setminus \{0\}$.

2.3.2. The evaluation of $\hat{R}$ takes the form

$$\hat{R}(y) = \frac{2 \prod_{n=0}^{\infty} (1-q^n)(1+q^n)^2(1+q^{n-1}y^3)(1+q^ny^{-3})(1+q^{n+1}y^{-s})(1-q^{n+1}y^{-s})}{\prod_{n=0}^{\infty} \prod_{s=1}^{2} (1+q^n y^s)(1+q^{n+1}y^{-s})(1-q^ny^s)(1-q^{n+1}y^{-s})}.$$  

All infinite product in the above expression normally converge in any $U \subset X$, where $X \subset \mathbb{C}$ is a compact not containing 0. Therefore $\hat{R}(y)$ is a meromorphic function in the region $0 < |y|$, and

$$(8) \quad \frac{A}{\hat{R}}(y) = \frac{(1-y) \prod_{n=0}^{\infty} (1-q^n)^2(1+q^n)^2(1-q^{n+1}y^{-2}) \prod_{s=1}^{2} (1+q^n y^s)(1+q^{n+1}y^{-s})}{2 \prod_{n=0}^{\infty} (1-q^{2n})^2(1+q^{n+1}y^{-3})(1+q^{n+1}y^{-3})}$$

is a meromorphic function in the region $0 < |y|$ with simple poles, the zero of order two at $y = 1$ and all other zeros of order one; the set of poles (resp., zeros) is $P$ (resp., $Z$):

$$P := \{y \mid y^3 = -q^m \& y \neq -q^k\}_{k,m \in \mathbb{Z}}, \quad Z := \{y \mid y^2 = \pm q^m\}_{m \in \mathbb{Z}}.$$

One readily sees from (8) that

$$\lim_{y \to 1} (y - 1) - \frac{2A(1-y)}{\hat{R}(y)} = 2, \quad \frac{A(qy)}{\hat{R}(y)} = \frac{A(y)}{\hat{R}(y)} \cdot \frac{q(1-qy)}{1-y}.$$
2.3.3. Recall that

\[ B = \sum_{n=-\infty}^{\infty} \frac{q^n}{(1 + q^n e^{-\beta_1})(1 + q^n e^{-\beta_2})} - \frac{q^n e^{-\alpha}}{(1 + q^n e^{-\beta_1-\alpha})(1 + q^n e^{-\beta_2-\alpha})} \]

so the evaluation takes the form

\[ B(y) = \sum_{n=-\infty}^{\infty} \frac{y^n}{(1+q^n y)(1+q^n y^2)} + \frac{y^n}{(1+q^n y^2 - 1)(1+q^n y - 1)} \]

Each point \( y \in \mathbb{C} \) such that \( y^2 \neq \pm q^n \) for \( n \in \mathbb{Z} \) has a neighborhood \( U \) such that the above sums converge absolutely and uniformly. Thus \( B(y) \) is a meromorphic function in the region \( 0 < |y| \) with poles at the points \( \{ y \mid y^2 = \pm q^n \} \), where all poles are simple except the pole of order two at \( y = 1 \). Let us verify that \( B(y) = 0 \) for each \( y \in P \). For \( y^3 = -q^k, y \notin \{-q^m\} \) one has

\[ \frac{y}{1 \pm q^n y^2} = \frac{y}{1 \mp q^{n+k} y^{-1}} = \mp \frac{1}{1 \mp q^{-n-k} y} \]

so \( B(y) = 0 \). Hence \( \frac{AB_R}{R}(y) \) is a holomorphic function in the region \( 0 < |y| \).

From the second formula of \( (10) \) one sees that \( B(qy) = q^{-1} \frac{1-y}{1-qy} \); combining with \( (9) \) we get \( \frac{AB_R(qy)}{R} = \frac{AB_R}{R}(y) \). Since \( \frac{AB_R}{R}(y) \) is a holomorphic function in the region \( 0 < |y| \), this function is constant. One has

\[ \lim_{y \to 1} (1-y)^2 \cdot B(y) = \lim_{y \to 1} (1-y)^2 \frac{1}{(1-y)(1-y^2)} = \frac{1}{2}. \]

Using \( (9) \) we obtain \( \frac{AB_R}{R}(1) = 1 \) so \( \frac{AB_R}{R}(y) \equiv 1 \) (for \( 0 < |y| \)). This completes the proof of denominator identity.

3. Application to Jacobi identity \( (2) \)

Recall the Gauss’ identity (which follows easily from the Jacobi triple product)

\[ \Box(-q) = \frac{(1-q)^\infty}{(1+q)^\infty}. \]

The evaluation of the identity \( (11) \) at \( y_1 = y_2 = 1 \) gives

\[ \frac{((1-x)_q^\infty(1-qx^{-1})_q^\infty)^2((1-q)_q^\infty)^4}{4((1+q)_q^\infty)^4((1+x)_q^\infty(1+qx^{-1})_q^\infty)^2} = \sum_{n=-\infty}^{\infty} a_n, \]

where \( a_n := \frac{q^n}{(1+q^n)(1+q^n^*)} - \frac{q^n^*}{(1+q^n^*)(1+q^n)}. \)

We divide both sides of the above identity by \( \frac{1-x^2}{16} \) and take the limit \( x \to 1 \); we get

\[ \frac{(1-q)^\infty}{(1+q)^\infty} = 1 - 16 \sum_{n=1}^{\infty} \frac{q^n(q^{2n} - 4q^n + 1)}{(1+q^n)^4}, \]
since
\[
\lim_{x \to 1} \frac{a_0}{(x - 1)^2} = \frac{1}{16}, \quad \lim_{x \to 1} \frac{a_n + a_{-n}}{(x - 1)^2} = -\frac{q^n (q^{2n} - 4q^n + 1)}{(1 + q^n)^4}.
\]

Using the expansion \((a + 1)^{-4} = \sum_{j=0}^{\infty} (-1)^j \frac{(j+1)(j+2)(j+3)}{6} a^j\), we obtain
\[
\Box(q)^8 = \left(\frac{(1-q)^{\infty}}{(1+q)^{\infty}}\right)^8 = 1 + 16 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^j j^3 q^{nj},
\]
which implies the required identity
\[
\Box(q)^8 = 1 + 16 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j+nj} j^3 q^{nj}.
\]

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