Bounding Optimality Gap in Stochastic Optimization via Bagging: Statistical Efficiency and Stability

Henry Lam and Huajie Qian
Department of Industrial Engineering and Operations Research
Columbia University

Abstract

We study a statistical method to estimate the optimal value, and the optimality gap of a given solution for stochastic optimization as an assessment of the solution quality. Our approach is based on bootstrap aggregating, or bagging, resampled sample average approximation (SAA). We show how this approach leads to valid statistical confidence bounds for non-smooth optimization. We also demonstrate its statistical efficiency and stability that are especially desirable in limited-data situations, and compare these properties with some existing methods. We present our theory that views SAA as a kernel in an infinite-order symmetric statistic, which can be approximated via bagging. We substantiate our theoretical findings with numerical results.

1 Introduction

Consider a stochastic optimization problem

$$Z^* = \min_{x \in \mathcal{X}} \{ Z(x) = E_F[h(x, \xi)] \}$$  (1)

where $\xi \in \Xi$ is generated under some distribution $F$, and $E_F[\cdot]$ denotes its expectation. We focus on the situations where $F$ is not known, but instead a collection of i.i.d. data for $\xi$, say $\xi_{1:n} = (\xi_1, \ldots, \xi_n)$, are available. Obtaining a good solution for (1) under this setting has been under active investigation both from the stochastic and the optimization communities. Common methods include the sample average approximation (SAA) [46, 28], stochastic approximation (SA) or gradient descent [29, 8, 39], and (distributionally) robust optimization [14, 4, 55, 3]. These methods aim to find a solution that is nearly optimal, or in some way provide a safe approximation. Applications of the generic problem (1) and its data-driven solution techniques span from operations research, such as inventory control, revenue management, portfolio selection (see, e.g., [46, 5]) to risk minimization in machine learning (e.g., [23]).

This paper concerns the estimation of $Z^*$ using limited data. Moreover, given a solution, say $\hat{x}$, a closely related problem is to estimate the optimality gap

$$G(\hat{x}) = Z(\hat{x}) - Z^*$$  (2)

This allows us to assess the quality of $\hat{x}$, in the sense that the smaller $G(\hat{x})$ is, the closer is the solution $\hat{x}$ to the true optimum in terms of achieved objective value. More precisely, we will focus

A preliminary conference version [30] of this work appears in the Proceedings of the Winter Simulation Conference 2018.
on inferring a lower confidence bound for $Z^*$, and, correspondingly, an upper bound for $\mathcal{G}(\hat{x})$ - noting that its first term $Z(\hat{x})$ can be treated as a standard population mean of $h(\hat{x}, \xi)$ that is estimable using a sample independent of the given $\hat{x}$, or that $\mathcal{G}(\hat{x})$ can be represented as the max of the expectation of $h(\hat{x}, \xi) - h(x, \xi)$ whose estimation is structurally the same as $Z^*$.

This problem is motivated by the fact that many state-of-the-art solution methods mentioned before are only amenable to crude, worst-case performance bounds. For instance, [48] and [28] provide large deviations bounds on the optimality gap of SAA in terms of the diameter or cardinality of the decision space and the maximal variance of the function $h$. [39] and [24] provide bounds on the expected value and deviation probabilities of the SAA iterates in terms of the strong convexity parameters, space diameter and maximal variance. These bounds can be refined under additional structural information (e.g., [47]). While they are very useful in understanding the behaviors of the optimization procedures, using them as a precise assessment on the quality of an obtained solution may be conservative. Because of this, a stream of work study approaches to validate solution performances by statistically bounding optimality gaps. [37, 1, 35, 45] investigate the use of SAA to estimate these bounds, [32] validate the performances of SAA iterates by using convexity conditions. [49, 42] study approaches like the jackknife and probability metric minimization to reduce the bias in the resulting gap estimates. [2] utilize gap estimates to guide sequential sampling. [37, 6, 34] investigate the use of empirical and profile likelihoods to estimate optimal values. Our investigation in this paper follows the above line of work on solution validation, focusing on the situation when data are limited and hence the statistical efficiency becomes utmost important. We also point out a related series of work that validate feasibility under uncertain constraints (e.g., [36, 40, 54, 12, 11]), though their problem of interest is beyond the scope of this paper, as we focus on deterministically constrained problems and objective value performances.

More precisely, we introduce a bootstrap aggregating, or commonly known as bagging [9], approach to estimate a lower confidence bound for $Z^*$. This comprises repeated resampling of data to construct SAAAs, and ultimately averaging the resampled optimal SAA values. We demonstrate how this approach applies under very general conditions on the cost function $h$ and decision space $X$, while enjoys high statistical efficiency and stability. Compared to procedures based on batching (e.g., [37]), which also have documented benefits in wide applicability and stability, the data recycling in our approach breaks free a tradeoff between the tightness of the resulting bound and the statistical accuracy/correctness exhibited by batching. In cases where sufficient smoothness is present and central limit theorem (CLT) for SAA (e.g., [46, 1]) can be directly applied, we also see that our approach gains stability regarding standard error estimation, thanks to the smoothing effect brought by bagging. Nonetheless, our approach generally requires higher computational load than these previous methods due to the need to solve many resampled programs. While we focus primarily on statistical performances, towards the end of this paper we will discuss some computational implications.

The theoretical justification of our bagging scheme comes from viewing SAA as a kernel in an infinite-order symmetric statistic [22], and an established optimistic bound for SAA as its asymptotic limit. A symmetric statistic is a generalization of sample mean in which each summand consists of a function (i.e., kernel) acting on more than one observation [41]. In particular, the size of the SAA program can be seen as precisely the kernel “order” (or “degree”), which depends on the data size and is consequently of an infinite-order nature. Our bagging scheme serves as a Monte Carlo approximation for this symmetric statistic. As a main methodological contribution, we analyze the asymptotic behaviors of the statistic and the resulting bounds as the SAA size grows, and translate them into efficient performances of our bagging scheme. Finally, we note that the notion of infinite-
order symmetric statistics has been used in analyzing ensemble machine learning predictors like random forests \[52\]; our SAA kernels are, from this view, in parallel to the base learners in the latter context.

Finally, we mention that \[21\] has also studied the resampling of SAA programs to construct confidence intervals for the optimal values of stochastic programs. Our approach connects with, but also differs substantially from \[21\] in several regards. In terms of scope of applicability, \[21\] focuses on mixed-integer linear programs, while we consider cost functions that can be generally non-Donsker. However, we instead require an additional “non-degeneracy” condition that depends on the cost function and the underlying probability distribution. In terms of methodology, \[21\] utilizes the quantiles of the resampled distribution to generate confidence intervals, by observing the same limiting distribution between an original CLT and the bootstrap CLT. The resampling in \[21\] requires a “two-layer” extended bootstrap where each resample is drawn from a new sample of the true distribution (as opposed to some bootstrap methods that allows repeated resample from the same original sample, with the availability of a conditional bootstrap CLT). Thus the approach requires substantial data size or otherwise resorting to subsampling. Our bagging approach, in contrast, is based on a direct use of Gaussian limit and standard error estimation in the CLT for the optimistic bound. Our burden lies on the bootstrap size requirement to obtain consistent standard error estimate, and less on the data size requirement.

We summarize our contributions of this paper as follows:

1. Motivated from the challenges of existing techniques (Section 2), we introduce a bagging procedure to estimate a lower confidence bound for \(Z^*\), correspondingly an upper confidence bound for \(G(\hat{x})\) (Section 3). We present the idea of our procedure that views SAA as a kernel in a symmetric statistic, and an optimistic bound for SAA as its associated limiting quantity (Section 4).

2. We analyze the asymptotic behaviors of the infinite-order symmetric statistic generated from the SAA kernel, under minimal smoothness requirements on the optimization problem. Moreover, when smoothness conditions are introduced, we demonstrate how these behaviors recover the classical CLT on SAA. These results are presented in Section 5. The mathematical developments without smoothness conditions utilize a combination of probabilistic coupling arguments and a new hypergeometric representation associated with the Hajek projection \[50\] (Appendices A.1 and A.2). The developments to recover the classical CLT use another analysis-of-variance (ANOVA) decomposition and a maximal deviation bound for empirical processes (Appendix A.3).

3. Building on the above results, we demonstrate how the bounds generated from our bagging procedure exhibit asymptotically correct coverages, and improve a tradeoff between the bound tightness and the statistical accuracy in existing batching schemes. This efficiency gain can be seen by an asymptotic comparison of the standard error in our estimator and an interpretation using conditional Monte Carlo. These developments are in Sections 6 and 7 with mathematical details in Appendices A.4-A.8.

4. We explain the stability in our generated bounds brought by the smoothing effect of bagging in estimating standard error. This compares favorably with the direct use of CLT in situations where the objective function is smooth. This property is supported by our numerical experiments (Section 8).
2 Existing Challenges and Motivation

We discuss some existing methods and their challenges, to motivate our investigation. We start the discussion with the direct use of asymptotics from sample average approximation (SAA).

2.1 Using Asymptotics of Sample Average Approximation

When the cost function $h$ in (1) is smooth enough, it is known classically that a central limit theorem (CLT) governs the behavior of the estimated optimal value in SAA, namely

$$\hat{Z}_n = \min_{x \in \mathcal{X}} \frac{1}{n} \sum_{i=1}^{n} h(x, \xi_i). \quad (3)$$

We first introduce the following Lipschitz condition:

**Assumption 1** (Lipschitz continuity in the decision). The cost function $h(x, \xi)$ is Lipschitz continuous with respect to $x$, in the sense that

$$|h(x_1, \xi) - h(x_2, \xi)| \leq M(\xi)\|x_1 - x_2\|$$

for any $x_1, x_2 \in \mathcal{X}$, where $M(\xi)$ satisfies $E[M^2(\xi)] < \infty$.

Denote “$\Rightarrow$” as convergence in distribution. The following result is taken from [46]:

**Theorem 1** (Extracted from Theorem 5.7 in [46]). Suppose that Assumption 1 holds, $E[h(\tilde{x}, \xi)^2] < \infty$ for some point $\tilde{x} \in \mathcal{X}$, and $\mathcal{X}$ is compact. Given i.i.d. data $\xi_{1:n} = (\xi_1, \ldots, \xi_n)$, consider the SAA problem (3). The SAA optimal value $\hat{Z}_n$ satisfies

$$\sqrt{n}(\hat{Z}_n - Z^*) \Rightarrow \inf_{x \in \mathcal{X}^*} Y(x) \quad (4)$$

where $\mathcal{X}^*$ is the set of optimal solutions for (1), and $Y(x)$ is a centered Gaussian process on $\mathcal{X}^*$ that has a covariance structure defined by $\text{Cov}(h(x_1, \xi), h(x_2, \xi))$ between any $x_1, x_2 \in \mathcal{X}^*$.

Roughly speaking, Theorem 1 stipulates that, under the depicted conditions, one can use (4) to obtain

$$\hat{Z}_n - \frac{\hat{q}}{\sqrt{n}}$$

as a valid lower confidence bound for $Z^*$ (and analogously for $G(\hat{x})$ given $\hat{x}$), where $\hat{q}$ is some suitable error term that captures the quantile of the limiting distribution in (4). Indeed, in the case of estimating $G(\hat{x})$, [1] provides an elegant argument that shows that, to achieve $1 - \alpha$ confidence, one can take $\hat{q} = z_{1-\alpha}\hat{\sigma}$ where $z_{1-\alpha}$ is the standard normal critical value and $\hat{\sigma}$ is a standard deviation estimate, regardless of whether the limit in (4) is a Gaussian distribution (or in other words the solution is unique). [1] calls this the single-replication procedure. More precisely, $\hat{\sigma}^2$ is obtained from

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left( h(\hat{x}, \xi_i) - h(\hat{x}_n^*, \xi_i) - (\hat{h}(\hat{x}) - \hat{h}(\hat{x}_n^*)) \right)^2$$

where $\hat{x}_n^*$ is the solution from (3), and $\hat{h}(\hat{x}) - \hat{h}(\hat{x}_n^*) = (1/n) \sum_{i=1}^{n} (h(\hat{x}, \xi_i) - h(\hat{x}_n^*, \xi_i))$.

Though Theorem 1 (and other related work, e.g., [15, 28]) is very useful, there are at least two reasons why one would need more general methods:
1. When the decision space contains discrete elements (e.g., combinatorial problems), Assumption 1 does not hold anymore. There is no guarantee in using the bound (5), i.e., it may still be correct but conservative, or it may simply possess incorrect coverages. We note, however, that for some class of problems (e.g., two-stage mixed-integer linear programs), extensions to Theorem 1 and approaches such as quantile-based bootstrapping (e.g., [21]) are useful alternatives.

2. If the SAA solutions have a “jumping” behavior, namely that program (1) has several near-optimal solutions with hugely differing objective variances, then the standard deviation estimate $\hat{\sigma}$ needed in the bound (5) can be unreliable. This is because $\hat{\sigma}$ depends heavily on $\hat{x}_n^*$, which can fall close to any of the possible near-optimal solutions with substantial chance and make the resulting estimation noisy. This issue is illustrated in, e.g., Examples 1 and 2 in [1]. We should also mention that, as an additional issue, the bias in $\hat{Z}_n$ relative to $Z^*$ can be quite large in any given problem, i.e., arbitrarily close to order $1/\sqrt{n}$ described in the CLT, even if all the conditions in Theorem 1 hold [41]. Note that this bias is in the optimistic direction (i.e., the resulting bound is still correct, but conservative), and it also appears in the “optimistic bound” approach that we discuss next. There have been techniques such as jackknife [41, 42] and probability metric minimization [49] in reducing this bias effect.

2.2 Batching Procedures

An alternate approach is to use the optimistic bound [37, 45, 25]

$$E[\hat{Z}_n] \leq Z^*$$

(6)

where $E[\cdot]$ in (6) is taken with respect to the data in constructing the SAA value $\hat{Z}_n$. The bound (6) holds for any $n \geq 1$, as a direct consequence from exchanging the expectation and the minimization operator in the SAA, and holds as long as $\xi_{1:n}$ are i.i.d.

The bound (6) offers a simple way to construct a lower bound for $Z^*$ under great generality. Note that the left hand side of (6) is a mean of SAA. Thus, if one can “sample” a collection of SAA values, then a lower confidence bound for $Z^*$ can be constructed readily by using a standard estimate of population mean. To “sample” SAA values, an approach suggested by [37] is to batch the i.i.d. data set $\xi_{1:n}$ into say $m$ batches, each batch consisting of $k$ observations, so that $mk = n$ (we ignore rounding issues). For each $j = 1, \ldots, m$, solve an SAA using the $k$ observations in the $j$-th batch; call this value $\hat{Z}_j$. Then use

$$\hat{Z}_k - z_{1-\alpha} \frac{\hat{\sigma}}{\sqrt{m}}$$

(7)

where $\hat{Z}_k = (1/m) \sum_{j=1}^{m} \hat{Z}_j$ and $\hat{\sigma}^2 = (1/(m-1)) \sum_{j=1}^{m} (\hat{Z}_j - \hat{Z}_k)^2$ are the sample mean and variance from $\hat{Z}_j$, $j = 1, \ldots, m$, and $z_{1-\alpha}$ is the $(1 - \alpha)$-level standard normal quantile.

The bound (7) does not rely on any continuity of $h$, and $\hat{\sigma}/\sqrt{m}$ is simply the sample standard deviation for a sample mean. In these regards, the bound largely circumvents the two concerns described before.

Nonetheless, there is an intrinsic tradeoff between tightness and statistical accuracy in this batching approach. On one hand, $m$ must be chosen big enough (e.g., roughly $> 30$) so that one can use the CLT to justify the approximation (7). Moreover, the larger is $m$, typically the smaller is the magnitude of the standard error in the second term of (7). On the other hand, the larger
is $k$, the closer is $E[\hat{Z}_k]$ to $Z^*$ in (6), leading to a tighter lower bound for $Z^*$. This is thanks to a monotonicity property in that $E[Z_n]$ is non-decreasing in $n$ [37]. Therefore, there is a tradeoff between the statistical accuracy controlled by $m$ (in terms of the validity of the CLT and the magnitude of the standard error term) and the tightness controlled by $k$ (in terms of the position of $E[\hat{Z}_k]$ in (6)). In the batching or the so-called multiple-replication approach of [37], this tradeoff is confined to the relation $mk = n$. There have been suggestions to improve this tradeoff, e.g., by using overlapping batches [35, 34], but their validity requires uniqueness or exponential convergence of the solution (e.g., in discrete decision space).

2.3 Motivation and Overview of Our Approach

Thus, in general, when the sample size $n$ is small, the batching approach appears to necessarily settle for a conservative bound in order to retain statistical validity/accuracy. The starting motivation for the bagging procedure that we propose next is to break free this tightness-accuracy tradeoff. In particular, we offer a bound roughly in the form

$$Z_{k}^{bag} - \frac{q_{bag}}{\sqrt{n}}$$

where $Z_{k}^{bag}$ is a point estimate obtained from bagging many resampled SAA values, and $k$ signifies the size of the resampled SAA (i.e., the “bags”). The quantity $q_{bag}$ relies on a standard deviation estimate of $Z_{k}^{bag}$. Our method operates at a similar level of generality as batching and handles the two concerns Points 1 and 2 in Section 2.1. The estimate $q_{bag}$ does not succumb to the “jumping” solution behavior, and the bound holds regardless of the continuity to the decision. Moreover, compared to the batching bound (7), our bound has a standard error term shrunk to order $1/\sqrt{n}$ from $1/\sqrt{m}$ (and relies on an asymptotic on $n$, not $m$), thus gaining higher statistical precision. In fact, this term regains the same order of precision level as the bound (5) that uses SAA asymptotics directly.

On the other hand, we will show that the choice of $k$ in (8), which affects the tightness, can be taken as roughly $o(\sqrt{n})$ in general. Compared with the direct-CLT bound (5), our bound appears less tight. However, we have gained estimation stability of $q_{bag}$ and, moreover, we consider conditions more general than when (5) is applicable. We will see that if we re-impose Lipschitz continuity on the decision (i.e., Assumption 1), then $k$ can be set arbitrarily close to the order of $n$. This means that our approach is almost as statistically efficient as the bound (5), with the extra benefit of stability in estimating $q_{bag}$.

Nonetheless, we point out that our approach requires solving resampled SAA programs many times, and is thus computationally more costly than batching and direct-CLT methods. The higher computation cost is the price to pay to elicit our benefits depicted above. Our approach is thus most recommended when statistical performance is of higher concern than computation efficiency, prominently in small-sample situations.

The next section will explain our procedure in more detail. A key insight is to view SAA as a symmetric kernel and the optimistic bound (6) as a limiting quantity of an associated symmetric statistic, which can be estimated by bagging. On a high level, the stability in estimating the standard error $q_{bag}$ can be attributed to the nature of bagging as a smoother [10, 19].
3 Bagging Procedure to Estimate Optimal Values

This section presents our approach. Instead of batching the data, we uniformly resample \( k \) observations from \( \xi_{1:n} \) for many, say \( B \), times. We use each resample to form an SAA problem and solve it. We then average all these resampled SAA optimal values. The resampling can be done with or without replacement (we will discuss some differences between the two). We summarize our procedure in Algorithm 1.

Algorithm 1 Bagging Procedure for Bounding Optimal Values

Given \( n \) observations \( \xi_{1:n} = \{\xi_1, \ldots, \xi_n\} \), select a positive integer \( k \)

\[
\text{for } b = 1 \text{ to } B \text{ do}
\]

Randomly sample \( \xi_k^b = (\xi_1^b, \ldots, \xi_k^b) \) uniformly from \( \xi_{1:n} \) (with or without replacement), and solve

\[
\hat{Z}_k^b = \min_{x \in X} \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i^b)
\]

end for

Compute \( \tilde{Z}^{bag}_k = \frac{1}{B} \sum_{b=1}^{B} \hat{Z}_k^b \) and

\[
\tilde{\sigma}^2_{IJ} = \begin{cases} 
\sum_{i=1}^{n} \text{Cov}_*(N_i^*, \hat{Z}_k^*)^2, & \text{if resampling is with replacement} \\
\left(\frac{n}{n-k}\right)^2 \sum_{i=1}^{n} \text{Cov}_*(N_i^*, \hat{Z}_k^*)^2, & \text{if resampling is without replacement} 
\end{cases}
\]

where

\[
\text{Cov}_*(N_i^*, \hat{Z}_k^*) = \frac{1}{B} \sum_{b=1}^{B} (N_i^b - \frac{k}{n})(\hat{Z}_k^b - \hat{Z}_k^{bag})
\]

end for

Output \( \hat{Z}_k^{bag} - z_{1-\alpha} \tilde{\sigma}_{IJ} \)

In the output of Algorithm 1 the first term \( \hat{Z}_k^{bag} \) is the average of many bootstrap resampled SAA values, which resembles a bagging predictor by viewing each SAA as a “base learner” [9]. The quantity \( \text{Cov}_*(N_i^*, \hat{Z}_k^*) \) in (10) is the covariance between the count of a specific observation \( \xi_i \) in a bootstrap resample, denoted \( N_i^* \), and the resulting resampled SAA value \( \hat{Z}_k^* \). The quantity \( \tilde{\sigma}^2_{IJ} \) is an empirical version of the so-called infinitesimal jackknife (IJ) estimator [19], which has been used to estimate the standard deviation of bagging schemes, including in random forests or tree ensembles [53]. The additional constant factor \( (n/(n-k))^2 \) in the second line of (9) is a finite-sample correction specific to resampling without replacement. Although the IJ variance estimator is not affected by this factor in the asymptotic regime that we will discuss, we find it significantly improves the finite-sample performance of our method.

4 SAA as Symmetric Kernel

We explain how Algorithm 1 arises. In short, the \( \hat{Z}_k^{bag} \) in Algorithm 1 acts as a point estimator for \( E[\hat{Z}_k] \) in the optimistic bound (6), whereas \( \tilde{\sigma}^2_{IJ} \) captures the standard error in using this point estimator.
To be more precise, let us introduce a functional viewpoint and write

\[ W_k(F) = E_{F^k}[H_k(\xi_1, \ldots, \xi_k)] \] (11)

where

\[ H_k(\xi_1, \ldots, \xi_k) = \min_{x \in X} \frac{1}{k} \sum_{i=1}^k h(x, \xi_i) \]

is the SAA value, expressed more explicitly in terms of the underlying data used. Here, the expectation \( E_{F^k}[\cdot] \) is generated with respect to i.i.d. variables \((\xi_1, \ldots, \xi_k)\), i.e., \( F^k \) denotes the product measure of \( k \) \( F \)'s. For convenience, we denote \( E[\cdot] \) as the expectation either with respect to \( F \) or the product measure of \( F \)'s when no confusion arises. Also, we denote \( W_k = W_k(F) \).

With these notations, the optimistic bound (6) can be expressed as

\[ W_k(F) \leq Z^* \]

with the best bound being \( W_\infty = \lim_{k \to \infty} W_k \leq Z^* \) thanks to the monotonicity property of the expected SAA value mentioned before.

Suppose that we have used sampling with replacement in Algorithm 1. Also say we use infinitely many bootstrap replications, i.e., \( B = \infty \). Then, the estimator \( \hat{Z}_k^{bag} \) in Algorithm 1 becomes precisely

\[ \hat{Z}_k^{bag} = W_k(\hat{F}) \]

where \( \hat{F} \) is the empirical distribution formed by \( \xi_{1:n} \), i.e., \( \hat{F}(\cdot) = (1/n) \sum_{i=1}^n \delta_{\xi_i}(\cdot) \) where \( \delta_{\xi_i}(\cdot) \) is the delta measure at \( \xi_i \). If \( W_k(\cdot) \) is “smooth” in some sense, then one would expect \( W_k(\hat{F}) \) to be close to \( W_k(F) \). Indeed, when \( k \) is fixed, \( W_k(F) \), which is expressible as the \( k \)-fold expectation under \( F \) in (11), is multi-linear, i.e.,

\[ W_k(F) = E_{F^k}[H_k(\xi_1, \ldots, \xi_k)] = \int \cdots \int H_k(\xi_1, \ldots, \xi_k) \prod_{j=1}^k dF(\xi_j) \]

and is always differentiable with respect to \( F \) (in the Gateaux sense) from the theory of von Mises statistical functionals [44]. This ensures that \( W_k(\hat{F}) \) is close to \( W_k(F) \) probabilistically, as elicited by a CLT (Theorem 2 below).

Note that \( W_k(F) \) is exactly the average of \( H_k(\xi_{i_1}, \ldots, \xi_{i_k}) \) over all possible combinations of \( \{\xi_{i_1}, \ldots, \xi_{i_k}\} \) drawn with replacement from \( \xi_{1:n} \). This is equivalent to

\[ V_{n,k} = \frac{1}{n^k} \sum_{i_j \in \{1, \ldots, n\}, j = 1, \ldots, k} H_k(\xi_{i_1}, \ldots, \xi_{i_k}) \] (12)

which is the so-called \( V \)-statistic. If we have used sampling without replacement in Algorithm 1 we arrive at the estimator (assuming again \( B = \infty \))

\[ U_{n,k} = \frac{\binom{n}{k}}{n^k} \sum_{(i_1, \ldots, i_k) \in \mathcal{C}_k} H_k(\xi_{i_1}, \ldots, \xi_{i_k}) \] (13)

where \( \mathcal{C}_k \) denotes the collection of all subsets of size \( k \) in \( \{1, \ldots, n\} \). The quantity (13) is known as the \( U \)-statistic. The \( V \) and \( U \) estimators in (12) and (13) both belong to the class of symmetric statistics [44, 50, 13], since the estimator is unchanged against a shuffling of the ordering of the data
ξ_{1:n}. Correspondingly, the $H_k(\cdot)$ function is known as the symmetric kernel. Symmetric statistics generalize the sample mean, the latter corresponding to the case when $k = 1$.

When $B < \infty$, then $V_{n,k}$ and $U_{n,k}$ above are approximated by a random sampling of the summands on the right hand side of (12) and (13). These are known as incomplete $V$- and $U$-statistics [33, 7, 27], and are precisely our $\hat{Z}_{bag}^k$. As $B$ is chosen large enough, $\hat{Z}_{bag}^k$ will well approximate $V_{n,k}$ and $U_{n,k}$.

To discuss further, we make the following assumptions:

**Assumption 2** ($L_2$-boundedness). We have

$$E \sup_{x \in \mathcal{X}} |h(x, \xi)|^2 < \infty$$

Denote $g_k(\xi) = E[H_k(\xi_1, \ldots, \xi_k)|\xi_1 = \xi]$. Denote $Var(\cdot) = Var_F(\cdot)$ as the variance under $F$.

**Assumption 3** (Finite non-zero variance). We have $0 < Var(g_k(\xi)) < \infty$.

We have the following asymptotics of $U_{n,k}$ and $V_{n,k}$:

**Theorem 2.** Suppose $k \geq 1$ is fixed, and Assumptions 2 and 3 hold. Then

$$\sqrt{n}(U_{n,k} - W_k) \Rightarrow N(0, k^2 Var(g_k(\xi))) \quad (14)$$

and

$$\sqrt{n}(V_{n,k} - W_k) \Rightarrow N(0, k^2 Var(g_k(\xi))) \quad (15)$$

as $n \to \infty$, where $N(0, k^2 Var(g_k(\xi)))$ is a normal distribution with mean 0 and variance $k^2 Var(g_k(\xi))$.

**Proof.** Assumption 2 implies that $EH_k(\xi_{i_1}, \ldots, \xi_{i_k})^2 < \infty$ for any (possibly identical) indices $i_1, \ldots, i_k$, since

$$EH_k(\xi_{i_1}, \ldots, \xi_{i_k})^2 \leq \frac{1}{k^2} E \sup_{x \in \mathcal{X}} \left( \sum_{j=1}^k h(x, \xi_{i_j}) \right)^2 \leq E \sup_{x \in \mathcal{X}} |h(x, \xi)|^2 < \infty \quad (16)$$

by the Minkowski inequality. Then, under (16) and Assumption 3, (14) follows from Theorem 12.3 in [50], and (15) follows from Section 5.7.3 in [44].

Theorem 2 is a consequence of the classical CLT for symmetric statistics. The expression $kg_k(\xi)$, as a function defined on the space $\mathcal{X}$, is the so-called influence function of $W_k(F)$, which can be viewed as its functional derivative with respect to $F$ [26]. Alternately, for a $U$-statistic $U_{n,k}$, the expression is the so-called Hajek projection [50], which is the projection of the statistic onto the subspace generated by the linear combinations of $f_i(\xi_i), i = 1, \ldots, n$ and any measurable function $f_i$. It turns out that these two views coincide, and the $U$- and $V$-statistics (whose approximation uses the projection viewpoint and the functional derivative viewpoint respectively) obey the same CLT as depicted in Theorem 2.

The output of Algorithm 1 is now evident given Theorem 2. When $B = \infty$, $\hat{Z}_{bag}^n$ is precisely $U_{n,k}$ under sampling without replacement or $V_{n,k}$ under sampling with replacement. The quantity $\hat{\sigma}_{IJ}^2$ in Algorithm 1, an empirical IJ estimator, can be shown to approximate the asymptotic variance $k^2 Var(g_k(\xi))/n$ as $n, B \to \infty$, by borrowing recent results in bagging [19, 52] (Theorems 11 and
below show stronger results). Then the procedural output is the standard CLT-based lower confidence bound for $W_k$.

The discussion above holds for a fixed $k$, the sample size used in the resampled SAA. It also shows that, at least asymptotically, using with or without replacement does not matter. However, using a fixed $k$ regardless of the size of $n$ is restrictive and leads to conservative bounds. The next subsection will relax this requirement and present results on a growing $k$ against $n$, which in turn allows us to get a tighter $W_k = E[\hat{Z}_k]$ in the optimistic bound \([\text{(6)}]\).

## 5 Asymptotic Behaviors with Growing Resample Size

We first make the following strengthened version of Assumption 2:

**Assumption 4** ($L_2+\delta$-bounded modulus of continuity). We have

$$E \sup_{x \in \mathcal{X}} |h(x, \xi) - h(x, \xi')|^{2+\delta} < \infty$$

where $\xi, \xi'$ are i.i.d. generated from $F$.

Assumption 4 holds quite generally, for instance under the following sufficient conditions:

**Assumption 5** (Uniform boundedness). $h(\cdot, \cdot)$ is uniformly bounded over $\mathcal{X} \times \Xi$.

**Assumption 6** (Uniform Lipschitz condition). $h(x, \xi)$ is Lipschitz continuous with respect to $\xi$, where the Lipschitz constant is uniformly bounded in $x \in \mathcal{X}$, i.e.,

$$|h(x, \xi) - h(x, \xi')| \leq L \|\xi - \xi'\|$$

where $\|\cdot\|$ is some norm in $\Xi$. Moreover, $E\|\xi\|^{2+\delta} < \infty$.

**Assumption 7** (Majorization).

$$|h(x, \xi) - h(x, \xi')| \leq f(\xi) + f(\xi')$$

where $E f(\xi)^{2+\delta} < \infty$.

That Assumption 5 implies Assumption 4 is straightforward. To see how Assumption 6 implies Assumption 4, note that, if the former is satisfied, we have

$$E \sup_{x \in \mathcal{X}} |h(x, \xi) - h(x, \xi')|^{2+\delta} \leq L^{2+\delta} E\|\xi - \xi'\|^{2+\delta} < \infty$$

Similarly, Assumption 7 implies Assumption 4 because the former leads to

$$E \sup_{x \in \mathcal{X}} |h(x, \xi) - h(x, \xi')|^{2+\delta} \leq E(f(\xi) + f(\xi'))^{2+\delta} < \infty$$

Next, we also make the following assumption:

**Assumption 8** (Non-degeneracy). We have

$$P\left(\min_{x \in \mathcal{X}} \{h(x, \xi) - Z(x)\} > 0\right) + P\left(E \left[\min_{x \in \mathcal{X}} \{h(x, \xi) - h(x, \xi')\}|\xi'\right] > 0\right) > 0 \tag{17}$$

where $\xi, \xi' \sim i.i.d. F$. 

Roughly speaking, Assumption 8 means that $\xi$ is sufficiently mixed so that the optimal value of a data-driven optimization problem with only one (or two) data point can deviate away from its mean. This assumption holds, e.g., when $\mathcal{X}$ lies in a positive region in the real space that is bounded away from the origin. The assumption can be further relaxed in practical problems. For example, one can replace $\mathcal{X}$ in (17) by a smaller region that can possibly contain any candidates of optimal solutions. Moreover, if the cost function is Lipschitz (i.e., Assumption 1 holds), it suffices to replace the entire decision space $\mathcal{X}$ in (17) with the set of optimal solutions $\mathcal{X}^*$, namely:

**Assumption 9** (A weaker non-degeneracy condition). We have

$$P\left(\min_{x \in \mathcal{X}^*} \{h(x, \xi) - Z^*\} > 0\right) + P\left(E\left[\min_{x \in \mathcal{X}^*} \{h(x, \xi) - h(x, \xi')\}\mid \xi'\right] > 0\right) > 0 \quad (18)$$

where $\mathcal{X}^*$ is the set of optimal solutions for (11). In particular, when the optimal solution is unique, i.e., $\mathcal{X}^* = \{x^*\}$, this assumption is reduced to $\text{Var}(h(x^*, \xi)) > 0$.

An important implication of the above two assumptions is to ensure that $k^2\text{Var}(g_k(\xi))$ is bounded away from 0 even as $k$ grows, thus leading to a behavior similar to Assumption 3 for the finite $k$ case.

**Lemma 1** (Non-degenerate asymptotic variance). Suppose Assumption 2 holds. Also, suppose either Assumption 8 holds, or that Assumptions 1 and 9 hold jointly and $\mathcal{X}$ is compact. Then $k^2\text{Var}(g_k(\xi)) > \epsilon > 0$ for some constant $\epsilon$, when $k$ is sufficiently large.

The proof of Lemma 1 uses a coupling argument between $g_k(\xi) = E[H_k(\xi_1, \ldots, \xi_k)\mid \xi_1 = \xi]$, which is a conditional expectation on $H_k$, and $E[H_k(\xi_1, \ldots, \xi_k)]$, the full expectation on $H_k$, by assigning the same random variables $\xi_2, \ldots, \xi_k$. This coupling is used to bound the difference $g_k(\xi) - E[H_k(\xi_1, \ldots, \xi_k)]$ used in calculating the variance $\text{Var}(g_k(\xi))$, which then combines with the non-degeneracy condition (Assumption 8 or 9) to get a lower bound for $\text{Var}(g_k(\xi))$. See Appendix A.1 for the detailed proof.

We have the following asymptotics:

**Theorem 3** (CLT for growing resample size under sampling without replacement). Suppose Assumptions 2, 4 and 8 hold. If the resample size $k = o(\sqrt{n})$, then

$$\frac{\sqrt{n}(U_{n,k} - W_k)}{k\sqrt{\text{Var}(g_k(\xi))}} \Rightarrow N(0,1)$$

where $N(0,1)$ is the standard normal variable.

**Theorem 4** (CLT for growing resample size under sampling with replacement). Suppose Assumptions 2, 4 and 8 hold. If the resample size $k = O(n^\gamma)$ for some $\gamma < \frac{1}{2}$, then

$$\frac{\sqrt{n}(V_{n,k} - W_k)}{k\sqrt{\text{Var}(g_k(\xi))}} \Rightarrow N(0,1)$$

where $N(0,1)$ is the standard normal variable.

Theorems 3 and 4 are analogs of Theorem 2 when $k \to \infty$. In both theorems, we see that there is a limit in how large $k$ we can take relative to $n$, which is thresholded at roughly order $\sqrt{n}$. A symmetric statistic with a growing $k$ is known as an infinite-order symmetric statistic.
Theorem 5 (Validity of nearly full resample under Lipschitzness). Suppose Assumptions 1, 2, 4 and 7 hold, and that the decision space $\mathcal{X}$ is compact. Then the conclusion of Theorem 3 holds by choosing $k = o(n)$.

Theorem 6 (Recovery of the classical CLT for SAA under solution uniqueness). In addition to the conditions in Theorem 5, if we further assume that (11) has a unique optimal solution $x^* \in \mathcal{X}$, then the conclusion of Theorem 5 holds for any $k \leq n$. Moreover we have $k^2 \text{Var}(g_k(\xi)) \rightarrow \text{Var}(h(x^*, \xi))$ and $W_k - Z^* = o(1/\sqrt{k})$ as $k \rightarrow \infty$. In particular, if $k \geq cn$ for some constant $\epsilon > 0$, then

$$\sqrt{n}(U_{n,k} - Z^*) \Rightarrow N(0, \text{Var}(h(x^*, \xi)))$$

where $N(0, \text{Var}(h(x^*, \xi)))$ is the normal variable with mean zero and variance $\text{Var}(h(x^*, \xi))$. 

, and has been harnessed in analyzing random forests [22]. Theorems 3 and 4 give the precise conditions under which the SAA kernel results in an asymptotically converging infinite-order symmetric statistic.

The proof of Theorem 3 utilizes a general projection theorem, in which one can translate the convergence of a projected statistic into convergence of the beginning statistic, if the ratio of their variances tends to 1 (Theorem 11.2 in [50]; restated in Theorem 13 in Appendix A). In our case, the considered projection is the Hajek projection of the infinite-order $U$-statistic. To execute this theorem, we approximate the variance ratios between the projection and the remaining orthogonal component. This requires using a further coupling argument among the higher-order conditional expectations, and combining with a representation of the variance ratio in terms of moments of hypergeometric random variables. Then, the CLT for the $U$-statistic follows by verifying the Lyapunov condition of the Hajek-projected $U$-statistic.

From Theorem 3, the conclusion of Theorem 4 follows by using a relation between $U$- and $V$-statistics in the form

$$n^k(U_{n,k} - V_{n,k}) = (n^k - nP_k)(U_{n,k} - R_{n,k})$$

where $nP_k = n(n-1)\cdots(n-k+1)$ and $R_{n,k}$ is the average of all $H_k(\xi_{i_1}, \ldots, \xi_{i_k})$ with at least two of $i_1, \ldots, i_k$ being the same (see, e.g., Section 5.7.3 in [44]). By carefully controlling the difference between $U_{n,k}$ and $V_{n,k}$, one can show an asymptotic for $V_{n,k}$ under a similar growth rate of $k$ as that for $U_{n,k}$. This leads to a slightly less general result for $V_{n,k}$ in Theorem 4. We mention that the growth rates of $k$ in both Theorems 3 and 4 are sufficient conditions. We will also see in the next section that, under further conditions, the growth of $k$ can be allowed bigger.

The proofs of Theorems 3 and 4 are both in Appendix 1.2. These two theorems conclude that $U_{n,k}$ and $V_{n,k}$ continue to well approximate the optimistic bound $W_k$ even as $k \rightarrow \infty$, under the depicted assumptions and bounds on the growth rate.

Taking one step further, the following shows that bagging under sampling without replacement achieves almost the same efficiency as the direct use of CLT for SAA in (5).
Note that, compared with Theorems 3 and 4, the centering quantity in Theorem 6 is changed from $W_k$ to $Z^*$. The asymptotic distribution is Gaussian with variance precisely the objective variance at $x^*$. This recovers Theorem 1 in the special case where $\mathcal{X}^* = \{x^*\}$. If the uniqueness condition does not hold, there could be a discrepancy between the optimistic bound $W_\infty$ and $Z^*$ (This can be hinted by observing the different types of limits between Theorems 3, 4 and Theorem 1, namely Gaussian versus the minimum of a Gaussian process).

We obtain Theorems 5 and 6 from a different path than Theorem 3, in particular by looking at the variance of $U_{n,k}$ via an analysis-of-variance (ANOVA) decomposition \cite{20} of the symmetric kernel $H_k$. Thanks to the uncorrelatedness among the ANOVA terms, we can control the variance of $U_{n,k}$ by using a bound from \cite{52}, which can be shown to depend on the maximal deviation of an empirical process generated by the centered cost function indexed by the decision, i.e., $\mathcal{F} := \{h(x,\cdot) - Z(x) : x \in \mathcal{X}\}$. The Lipschitz assumption allows us to estimate this maximal deviation using empirical process theory. Appendix A.3 shows the proof details.

6 Statistical Properties of Bagging Bounds and Comparisons with Batching

We analyze the properties of our confidence bounds implied from Theorems 3 and 4, namely consisting of a point estimator $U_{n,k}$ or $V_{n,k}$ and a standard error $k \sqrt{\text{Var}(g_k(\xi))/n}$. We first show that the latter is of order $1/\sqrt{n}$, thus reconciling with our claim in (8) and demonstrating an asymptotically higher statistical precision compared to the batching bound in (7).

**Proposition 1** (Magnitude of the standard error). Under Assumption 2, we have $k^2 \text{Var}(g_k(\xi)) \leq C$ for some constant $C > 0$, as $k \to \infty$. Consequently, the asymptotic standard deviation of $U_{n,k}$ or $V_{n,k}$, namely $k \sqrt{\text{Var}(g_k(\xi))/n}$, is of order $O(1/\sqrt{n})$.

Note that Proposition 1 is quite general in that it does not impose any growth rate restriction on $k$. We also note that, under conditions that provide a CLT for the SAA (i.e., Theorem 1), the $\tilde{\sigma}$ in the batching bound (7) can be of order $O(1/\sqrt{k})$ as the data size per batch $k$ grows, and thus the resulting term there can be controlled to be $O(1/\sqrt{n})$ like ours (and also the direct-CLT bound (5)). Nonetheless, Proposition 1 is free of such type of assumptions. Its proof uses the coupling argument in bounding the variance that appears in the proof of Theorem 3. The proof details are in Appendix A.4.

The following shows a more revealing result on the higher statistical efficiency of our bagging procedure compared to batching:

**Theorem 7** (Asymptotic variance reduction). Recall that $\tilde{Z}_k$ is the point estimate in the bound (7) given by the batching procedure. Assume the same conditions and resample sizes of either Theorem 2 or 3 in the case of resampling without replacement, or Theorem 4 in the case of resampling with replacement. With the same batch size and resample size, both denoted by $k$, we define the asymptotic ratios of variance

$$r_U := \limsup_{n,k \to \infty} \frac{\text{Var}(U_{n,k})}{\text{Var}(\tilde{Z}_k)}, \quad r_V := \limsup_{n,k \to \infty} \frac{\text{Var}(V_{n,k})}{\text{Var}(\tilde{Z}_k)}.$$  \hspace{1cm} (20)

We have $r_U = r_V = \limsup_{k \to \infty} k \text{Var}(g_k(\xi))/\text{Var}(H_k) \leq 1$, and in particular

1. $r_U = r_V = 0$ when $\lim_{k \to \infty} k \text{Var}(H_k) = \infty$
2. \( r_U = r_V < 1 \) when the conditions of Theorem 8 hold, \( X^* \) is not a singleton and the covariance \( \text{Cov}(h(x_1, \xi), h(x_2, \xi)) \) for \( x_1, x_2 \in X^* \) is not a constant.

3. \( r_U = r_V = 1 \) when the conditions of Theorem 7 hold and \( X^* \) is a singleton.

The following example shows that in the second case of Theorem 7, i.e. when the cost function is Lipschitz continuous in decision and there are multiple optimal solutions, the asymptotic ratio of variance not only is strictly less than 1 but also can be arbitrarily close to 0.

Example 1. Consider the cost function

\[
h(x, \xi) = \begin{cases} (2 - x)\xi_1 + (x - 1)\xi_2 & \text{if } 1 \leq x \leq 2 \\ \vdots & \vdots \\ (j + 1 - x)\xi_j + (x - j)\xi_{j+1} & \text{if } j < x \leq j + 1 \\ \vdots & \vdots \\ (d - x)\xi_{d-1} + (x - (d - 1))\xi_d & \text{if } d - 1 < x \leq d \end{cases}
\]

for \( x \in [1, d] \) and uncertain quantity \( \xi = (\xi_1, \ldots, \xi_d) \) where \( \xi_j, j = 1, \ldots, d \) are independent standard normal variables. In other words, at \( x = j \) the cost \( h(x, \xi) \) is set to \( \xi_j \) and everywhere else given by a linear interpolation between the two neighboring integer points. In this case, the objective is constantly zero over the entire decision space so \( X^* = [1, d] \). The SAA value \( H_k = \min_{j=1,\ldots,d} \xi_j \) where \( \xi_j \) is the sample mean of the \( j \)-th component \( \xi_j \), hence \( \sqrt{k}H_k \) is the minimum of \( d \) independent standard normal variables. A direct application of Corollary 1.9 in [16] leads to \( k\text{Var}(H_k) \geq C/\log d \) for some universal constant \( C > 0 \). In Appendix A.5 we show that \( \lim_{k \to \infty} k^2\text{Var}(g_k(\xi)) = 1/d \). Therefore \( r_U = r_V \leq \log d/(Cd) \).

Furthermore, the following shows that the point estimator under sampling without replacement always has a smaller variance than the batching estimator, for any \( n \) and \( k \):

Theorem 8 (Variance reduction under any finite sample). Recall that \( \tilde{Z}_k \) is the point estimate in the bound (7) given by the batching procedure. Let \( \xi_{(1)}, \ldots, \xi_{(n)} \) be the order statistic of the data set \( \xi_1, \ldots, \xi_n \). With the same batch size and resample size, both denoted by \( k \), we have

\[
\text{Var}(\tilde{Z}_k) = \text{Var}(U_{n,k}) + E[\text{Var}(\tilde{Z}_k|\xi_{(1)}, \ldots, \xi_{(n)})]
\]

and hence \( \text{Var}(\tilde{Z}_k) \geq \text{Var}(U_{n,k}) \) for any \( k \geq 1 \).

Proof. By the law of total variance we have

\[
\text{Var}(\tilde{Z}_k) = E[\text{Var}(\tilde{Z}_k|\xi_{(1)}, \ldots, \xi_{(n)})] + \text{Var}(E[\tilde{Z}_k|\xi_{(1)}, \ldots, \xi_{(n)})].
\]

The desired conclusion follows from noticing that \( E[\tilde{Z}_k|\xi_{(1)}, \ldots, \xi_{(n)}] = U_{n,k} \). \( \square \)

Theorem 8 reinforces the smaller standard error in bagging compared to batching from asymptotic to any finite sample, provided that we use sampling without replacement. Intuitively, bagging eliminates the additional variability contributed from the ordering of the data, whereas the batching estimator is subject to change if the data are reordered. Alternately, one can also interpret bagging as a conditional Monte Carlo scheme applied on the batching estimator given the data ordering.

Next, the following result concerns the biases of \( U_{n,k} \) and \( V_{n,k} \):
Theorem 9 (Bias). Under the same assumptions and resample sizes as Theorems 3 and 4, the bias of $U_{n,k}$ in estimating $W_k$ is 0, whereas the bias of $V_{n,k}$ in estimating $W_k$ is $O((k^2/n)^l + k/n)$ where $l$ is any fixed positive integer.

The zero-bias property of $U_{n,k}$ is trivial: Each summand in its definition is an SAA value with distinct i.i.d. data, and thus has mean exactly $W_k$. On the other hand, the summands in $V_{n,k}$ are SAA values constructed from potentially repeated observations, which induces bias relative to $W_k$. The proof of the latter again utilizes the relation (19), and is left to Appendix A.6.

From Theorem 9, we see that $U_{n,k}$ outperforms $V_{n,k}$ in terms of bias control. When $k$ is fixed, such an advantage for $U_{n,k}$ is relatively mild, since the bias of $V_{n,k}$ in estimating the optimistic bound $W_k$ is of order 1/n. However, as $k$ grows, this advantage becomes more significant, and the bias of $V_{n,k}$ can be arbitrarily close to $O(1)$ (when $k \approx \sqrt{n}$).

Theorems 5, 8 and 9 together justify that, in terms of both standard error and bias, sampling without replacement, i.e., $U_{n,k}$, seems to be the more recommendable choice for our bagging procedure. However, in our numerical experiments in Section 8, $U_{n,k}$ and $V_{n,k}$ appear to perform quite similarly.

Lastly, we should mention that the biases depicted in Theorem 9 concern the estimators of $W_k$, but do not capture the discrepancy between $W_k$ and $Z^*$. The latter quantity is of separate interest. As discussed at the end of Section 2.1, it can be generally reduced by existing methods like the jackknife or probability metric minimization [42, 49].

7 Error Estimates and Coverages

Finally, we analyze the use of the IJ estimator in approximating the standard error and the error coming from the Monte Carlo noise in running the bootstrap. Together with the results in Section 5 and 6, these will give us an overall CLT on the output from Algorithm 1. First, we have the following consistency of the IJ variance estimator, relative to the magnitude of the target standard error:

Theorem 10 (Relative consistency of IJ estimator under resampling without replacement). Consider resampling without replacement. Under the same conditions and resample sizes of either Theorem 3 or 4, the IJ variance estimator is relatively consistent, i.e.

$$\frac{n^2}{(n-k)^2} \sum_{i=1}^{n} \text{Cov}_p^2(N_i^*, H_k^*) / \frac{k^2}{n} \text{Var}(g_k(\xi)) \overset{p}{\to} 1.$$  

Theorem 11 (Relative consistency of IJ estimator under resampling with replacement). Consider resampling with replacement. Under the same conditions and resample sizes of Theorem 4, the IJ variance estimator is relatively consistent, i.e.

$$\sum_{i=1}^{n} \text{Cov}_p^2(N_i^*, H_k^*) / \frac{k^2}{n} \text{Var}(g_k(\xi)) \overset{p}{\to} 1.$$  

Theorem 10 is justified by adopting the arguments for random forests in [52] and a weak law of large numbers, and Theorem 11 follows from analyzing the difference between $U$- and $V$-statistics as in the proof of Theorem 4. Appendix A.7 shows the details.

When a large enough bootstrap size $B$ is used in Algorithm 1, the Monte Carlo errors in estimating the point estimator and its variance both vanish. This gives an overall CLT for the output of our bagging procedure, as in the next theorem:
Theorem 12 (CLT for Algorithm 1). Under the same conditions and resample sizes of either Theorem 3 or 5 in the case of resampling without replacement, or Theorem 4 in the case of resampling with replacement, if the bootstrap size $B$ in Algorithm 1 is such that $B/(kn) \to \infty$, then the output of Algorithm 1 satisfies
\[ \frac{\tilde{Z}_{bag}^k - W_k}{\tilde{\sigma}_{IJ}} \Rightarrow N(0,1) \]
where $N(0,1)$ is the standard normal variable.

An immediate consequence of Theorem 12 is the correct coverage of the true optimal value:

Corollary 1 (Correct coverage from Algorithm 1). Under the same assumptions, growth rates of the resample size $k$ and the bootstrap size $B$ in Theorem 12, the output of Algorithm 1 satisfies
\[ P\left( \tilde{Z}_{bag}^k - z_{1-\alpha}\tilde{\sigma}_{IJ} \leq Z^* \right) \geq P\left( \tilde{Z}_{bag}^k - z_{1-\alpha}\tilde{\sigma}_{IJ} \leq W_k \right) \to 1 - \alpha \]
where $P$ is generated under the data $\xi_{1:n}$.

Theorem 12 and Corollary 1 thus close our analyses by showing an exact asymptotic coverage of our bagging bound for the optimistic bound $W_k$, and a correct asymptotic coverage for $Z^*$, where the exactness of the later depends on the discrepancy between $W_k$ and $Z^*$. Additionally, Theorem 6 stipulates that this discrepancy vanishes under the same setting as when the classical SAA CLT has a normal limit, and thus hints that our bound for $Z^*$ is close to having exact coverage in this case.

Lastly, note that $B$ needs to be taken to have order greater than $kn$ to wash away the Monte Carlo error under the considered conditions. To achieve the best result regarding the tightness of the bound, in the case of non-Lipschitzness (Theorems 3 and 1) we would choose $k$ to be close to $\sqrt{n}$, which means the need of roughly order $n^{3/2}$ bootstrap size or optimization programs to solve, whereas under Lipschitzness (Theorems 5 and 6) we would choose $k$ to be close to $n$, giving a bootstrap size of order $n^2$. As discussed previously, because of the computational load, our bagging scheme is most recommended for small-sample situation where $n$ is relatively small. If computation is a concern, one can always use a smaller $k$ in our scheme to speed up computation, with the price of generating a more conservative bound.

8 Numerical Experiments

In this section we provide numerical tests to demonstrate the validity of our bagging-based procedures and compare them to the batching procedure given in (7) and the single-replication procedure given in (5).

Four stochastic optimization problems are tested. The first problem we consider is the $(1 - \alpha_1)$-level conditional value at risk (CVaR) of a standard normal variable $\xi$

\[ \min_{x \in \mathbb{R}} x + \frac{1}{\alpha_1} E[(\xi - x)_+] \quad (21) \]

where $(\cdot)_+ := \max\{\cdot, 0\}$ denotes the positive part. We set $\alpha_1 = 0.1$, namely, we are solving for the 90%-level CVaR of the standard normal, whose true value can be calculated to be 1.755.

The second one is a portfolio optimization problem where one seeks to minimize the $(1 - \alpha_2)$-level CVaR risk measure of an investment portfolio subject to that the expected return of the
investment exceeds some target level. Let $\xi = (\xi_1, \ldots, \xi_5)^T$ be the vector of random returns of five different assets whose joint distribution follows a multivariate normal, $x = (x_1, \ldots, x_5)^T$ be the holding proportions of the assets, and $b$ be the target level of expected return. The optimization is described by

$$
\begin{align*}
\min_{c,x} \quad & c + \frac{1}{\alpha_2} E[(-\xi^T x - c)_+] \\
\text{s.t.} \quad & E[\xi^T x] \geq b \\
& \sum_{i=1}^5 x_i = 1 \\
& x_i \geq 0 \text{ for } i = 1, \ldots, 5.
\end{align*}
$$

(22)

In particular, the random return vector $\xi$ follows $N(\mu, \Sigma)$ where the mean $\mu = (1, 2, 3, 4, 5)^T$ and the covariance $\Sigma$ is randomly generated, $\alpha_2 = 0.05$ and $b = 3$. Note that the cost function here, as well as that in (21), is piecewise linear hence Lipschitz continuous, and the optimal solution is unique. Therefore we expect all the methods to perform well for these two problems. Note that, to avoid feasibility complications that divert our focus, in (22) we assume knowledge of the expected return $\mu$ so the constraint becomes $\mu^T x \geq b$.

To describe the third problem, suppose there are ten different items labeled as $\#1$ through $\#10$ each of which incurs a random loss $\xi_i$, and one is required to pick at least one out of the ten items and at most two items among $\#7, \#8, \#9, \#10$ in such a way that the total expected loss is minimized. Mathematically, the problem can be formulated as the following stochastic linear integer program

$$
\begin{align*}
\min_{x} \quad & E[\xi^T x] \\
\text{s.t.} \quad & Ax \leq b \\
& x_i \in \{0, 1\} \text{ for } i = 1, 2, \ldots, 10
\end{align*}
$$

(23)

where $\xi$ follows $N(\mu, \Sigma)$ with mean $\mu = (-1, -7/9, -5/9, \ldots, 7/9, 1)^T \in \mathbb{R}^{10}$ and covariance $\Sigma$ randomly generated, $b = (-1, 2)^T$ and

$$A = \begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}.$$

It is straightforward to see that picking the items with negative expected losses, i.e., $\#1$ through $\#5$, gives the minimum total loss, and hence the unique optimal solution is $x^* = (1, 1, 1, 1, 0, 0, 0, 0, 0, 0)^T$ with a total loss $-2.78$. Because of the integrality requirement the single-replication procedure is not theoretically justified and can exhibit incorrect coverage. When implementing the methods we solve the SAA problems by a direct enumeration (feasible thanks to the relatively low dimensionality).

The fourth optimization problem is the following simple stochastic linear program

$$
\begin{align*}
\min_{x} \quad & E[-0.05x + (3 - 2x)\xi] \\
\text{s.t.} \quad & -1 \leq x \leq 1
\end{align*}
$$

(24)

where the uncertain quantity $\xi$ is a standard normal and the decision $x$ is a scalar. It is clear that the optimal value is $-0.05$ at $x^* = 1$. This problem serves to highlight that, although the optimization is highly smooth, using past methods may give subpar finite-sample performances due to a delicate interplay between the variance and jumping behavior of the estimated solution. It then illustrates how bagging can be a resolution in such a scenario.
8.1 Lower Bounds of Optimal Values

In this subsection we use Algorithm 1 without replacement (\(U\)-statistic), Algorithm 1 with replacement (\(V\)-statistic), the batching procedure (7) and the single-replication procedure (5) to compute lower confidence bounds for the optimal value \(Z^* = \min_{x \in X} Z(x)\). Specifically, we first simulate an i.i.d. data set \(\xi_1, \ldots, \xi_n\) of size \(n\), and then compute a 95% lower bound of the optimal value using each of the four methods. As suggested by Theorem 12, we set \(B\), the number of resamples, in Algorithm 1 to be roughly \(5nk\) to wash out the effect of Monte Carlo error in estimating the covariances. This is in accordance with our focus on statistical efficiencies, under the presumed adequate resources in solving SAA problems. In the batching procedure we use the quantile of \(t\)-distribution with \(m - 1\) degrees of freedom when there are less than 30 batches, so as to enhance finite-sample performances as suggested in [37], whereas in other procedures we use the normal quantile.

Tables 1 and 2 summarize the results for problem (21) when the data size \(n = 50\) and \(n = 300\), whereas Table 3 shows those for problem (24). We compute 1000 confidence bounds from 1000 independently generated data sets, and then average the results to estimate coverage probability (c.p.\(\%\)), mean of the lower bound (mean) and standard deviation of the lower bound (std.). We use \(k\) to denote either batch size in the batching procedure or resample size in our bagging procedures. The “NA” entries in the tables correspond to the cases where \(n/k < 2\) hence the batching procedure is not tested. The “Single-replication” column of each table has only one row because all the \(n\) data are used to form the SAA in the single-replication procedure.

**Table 1:** Problem (21), \(n = 50\). Lower bounds of optimal values.

| \(k\) | Batching | \(U\)-statistic | \(V\)-statistic | Single-replication |
|------|----------|-----------------|-----------------|-------------------|
|      | c.p.(\%) | mean            | std.            | c.p.(\%)         |
| 10   | 99.4     | 1.00            | 0.29            | 99.4             |
| 25   | 97.1     | 0.36            | 0.96            | 99.7             |
| 40   | NA       | NA              | NA              | 98.6             |

**Table 2:** Problem (21), \(n = 300\). Lower bounds of optimal values.

| \(k\) | Batching | \(U\)-statistic | \(V\)-statistic | Single-replication |
|------|----------|-----------------|-----------------|-------------------|
|      | c.p.(\%) | mean            | std.            | c.p.(\%)         |
| 10   | 100      | 1.37            | 0.10            | 100              |
| 30   | 99.1     | 1.48            | 0.11            | 99.8             |
| 50   | 97.8     | 1.50            | 0.12            | 98.7             |
| 100  | 96.9     | 1.44            | 0.19            | 97.9             |
| 150  | 96.1     | 1.20            | 0.42            | 98.0             |
| 250  | NA       | NA              | NA              | 96.9             |

Tables 1-3 show that for a wide range of resample sizes, namely from 10 to more than half of the data size, our bagging procedure generates statistically valid lower bounds in the sense that the coverage probabilities are equal to or above the nominal value 95%. The batching and single-replication procedures also generate valid confidence bounds. The results across different values of
Table 3: Problem (24), n = 100. Lower bounds of optimal values.

| k  | Batching | U-statistic | V-statistic | Single-replication |
|----|----------|-------------|-------------|--------------------|
|    | c.p.(%) | mean  | std. | c.p.(%) | mean  | std. | c.p.(%) | mean  | std. | c.p.(%) | mean  | std. |
| 5  | 100     | −1.23 | 0.38 | 100     | −1.21 | 0.35 | 100     | −1.23 | 0.37 | 95.5    | −0.63 | 0.59 |
| 10 | 98.9    | −1.08 | 0.48 | 99.8    | −0.99 | 0.39 | 100     | −1.01 | 0.37 | 95.5    | −0.63 | 0.59 |
| 25 | 94.0    | −0.93 | 0.63 | 98.9    | −0.79 | 0.44 | 99.0    | −0.82 | 0.40 | 95.5    | −0.63 | 0.59 |
| 50 | 95.0    | −1.57 | 1.55 | 97.1    | −0.67 | 0.47 | 97.5    | −0.72 | 0.44 | 95.5    | −0.63 | 0.59 |
| 70 | NA      | NA    | NA  | 95.1    | −0.62 | 0.51 | 97.5    | −0.69 | 0.43 | 95.5    | −0.63 | 0.59 |

$k$ also verify the relation between the resample size and tightness of the optimistic bound \( \frac{1}{m} \). To be specific, in all the tables, as the resample size $k$ grows, the mean lower bound gets closer to the true optimal value 1.755 in Tables 1 and 2 and −0.05 in Table 3. In particular, in the case of problem (21) and $n = 300$ (Table 2) both $U$-statistic and $V$-statistic provide a lower bound as good as 1.55 with coverage probability 97%-98% by using $k = 100, 150, 250$. It therefore appears that, with the bagging procedures, one can obtain a relatively tight bound for the optimal value and in the meantime retain good statistical accuracy, by using a resample size $k$ that is roughly half the data size.

Although the bounds generated from all considered methods are statistically valid, they differ in tightness and stability. We observe that our bagging procedures appear to output tighter and stabler bounds on the optimal value than batching. In each of Tables 1-3 under the same batch size or resample size $k$, the bounds given by $U$-statistic and $V$-statistic are always larger in terms of the mean, and meanwhile less variable as measured by the standard deviation, than those by batching. The difference in tightness and stability becomes more noticeable as $k$ increases. This is in accordance with benefit of reducing variance in using bagging procedures as illustrated by Theorems 7 and 8.

The results also show the tradeoff between tightness and statistical accuracy in the batching procedure. According to the monotonicity property of the optimistic bound, the confidence bound should exhibit a monotonic trend of becoming tighter as the batch size $k$ increases. However, in all the tables the mean lower bound first gets tighter for relatively small batch size but then becomes looser again as the size further increases. For example, in Table 3 the tightest bound (in terms of the mean) is −0.93 at $k = 25$ and in Table 2 the tightest is 1.50 at $k = 50$. This non-monotonic behavior appears since, as the batch size gets large, too few batches are available for the procedure to maintain the desired statistical accuracy (i.e. a coverage probability above 95%). To mitigate this issue, we resort to using $t$-quantile in place of normal which loosens the bound in exchange for correct coverages. In fact, if we change the $t$-quantile to normal the coverage probability drops to 92% in Table 1 and 86% in Table 2 in our experiment. Note that such kind of tradeoff no longer appears in our bagging procedures as the bound always gets tighter and at the same time has the desired coverage level even for large $k$.

### 8.2 Upper Bounds of Optimality Gaps

Now we test our methods in bounding optimality gaps of solutions. In our experiments we first solve the SAA formed by $n_1$ data points $\xi_1, \ldots, \xi_{n_1}$ to obtain a solution $\hat{x}$. We then generate $n_2$ independent data points $\xi_{n_1+1}, \ldots, \xi_{n_1+n_2}$. These (and possibly the first $n_1$ data points) are then used
Table 4: Problem (22), n = 40, n_1 = 20, n_2 = 20. Upper bounds of optimality gaps by BC.

| k  | Batch | U-statistic | V-statistic | Single-replication |
|----|-------|-------------|-------------|--------------------|
|    | c.p.(%) | mean  | std. | c.p.(%) | mean  | std. | c.p.(%) | mean  | std. | c.p.(%) | mean  | std. |
| 2  | 99.7  | 3.94  | 1.79 | 99.8  | 3.82  | 1.79 | 99.9  | 3.79  | 1.76 | 93.0  | 1.89  | 1.65 |
| 5  | 99.5  | 3.17  | 1.79 | 99.4  | 3.01  | 1.78 | 99.3  | 3.07  | 1.88 |       |       |      |
| 10 | 99.5  | 3.10  | 1.96 | 98.7  | 2.53  | 1.75 | 97.6  | 2.50  | 1.74 |       |       |      |
| 20 | 99.0  | 6.94  | 5.36 | 97.1  | 2.09  | 1.71 | 97.6  | 2.02  | 1.69 |       |       |      |
| 27 | NA    | NA    | NA  | 96.9  | 2.10  | 1.67 | 95.1  | 2.00  | 1.61 |       |       |      |

Table 5: Problem (22), n = 40, n_1 = 20, n_2 = 20. Upper bounds of optimality gaps by CRN.

| k  | Batch | U-statistic | V-statistic | Single-replication |
|----|-------|-------------|-------------|--------------------|
|    | c.p.(%) | mean  | std. | c.p.(%) | mean  | std. | c.p.(%) | mean  | std. | c.p.(%) | mean  | std. |
| 2  | 99.9  | 4.32  | 2.41 | 100   | 4.36  | 2.41 | 100   | 4.24  | 2.35 | 91.4  | 1.20  | 1.31 |
| 5  | 99.3  | 3.66  | 2.71 | 99.1  | 3.18  | 2.14 | 99.5  | 3.24  | 2.28 |       |       |      |
| 10 | 99.3  | 4.92  | 4.59 | 97.1  | 2.43  | 2.02 | 98.7  | 2.73  | 2.21 |       |       |      |
| 15 | NA    | NA    | NA  | 94.7  | 2.13  | 1.87 | 98.2  | 2.53  | 2.22 |       |       |      |
| 20 | NA    | NA    | NA  | NA    | NA    | NA  | 96.7  | 2.09  | 1.81 |       |       |      |

to compute an upper confidence bound for the optimality gap \( \mathcal{G}(\hat{x}) = Z(\hat{x}) - Z^* \). For convenience
we denote \( n = n_1 + n_2 \) as the total sample size in the experiments.

We consider two approaches to bounding the gap, one reusing the first \( n_1 \) data points, and the other not. The first approach is to use the Bonferroni Correction (BC). Specifically, we use the second group of \( n_2 \) data to compute \( U = \hat{x} + z_{0.975} \hat{\sigma}/\sqrt{n_2} \) as a 97.5% upper confidence bound of \( Z(\hat{x}) \), where \( \hat{x}, \hat{\sigma}^2 \) are the sample mean and variance of \( h(\hat{x}, \xi_{n_1+1}), \ldots, h(\hat{x}, \xi_n) \), and compute a 97.5% lower confidence bound \( L \) of the true optimal value \( Z^* \) using all the \( n \) data as in the previous section. In the end we output \( U - L \) as a confidence bound for the gap \( \mathcal{G}(\hat{x}) \). By BC we know
\[
P(U - L \geq Z(\hat{x}) - Z^*) \geq P(U \geq Z(\hat{x})) + P(L \leq Z^*) - 1 \approx 97.5% + 97.5% - 1 = 95%
\]
hence \( U - L \) is an asymptotically valid 95% confidence bound for the gap.

The second approach is a Common Random Numbers (CRN) variance-reduction technique proposed by [37] in this context. Consider minimizing a different objective \( E[h(x, \xi) - h(\hat{x}, \xi)] \) as a whole, where \( \hat{x} \) is viewed as fixed, whose optimal value is exactly \(-\mathcal{G}(\hat{x})\). We use the second group of \( n_2 \) data to compute a 95% lower confidence bound for this new optimization problem, and then negate the lower bound to obtain a valid upper bound for \( \mathcal{G}(\hat{x}) \).

Tables [4][6] and [8] summarize the results for problems (22) (23) (24) using BC, while Tables [5][7] and [9] display those using CRN. Note that, in either approach, in order to guarantee the statistical accuracy of the confidence bound a relatively small number of data (e.g., around 30) would suffice. In view of this, we choose \( n_2 \) around 30 in all the experiments.

We see a few similar observations as in Section [8.1] where we compute lower bounds for optimal values. The two bagging procedures generate statistically valid upper bounds in almost all the cases (mildly undercover in the case \( k = 30 \) of Table [9]). The bounds by batching also possess
Table 6: Problem (23), n = 100, n_1 = 64, n_2 = 36. Upper bounds of optimality gaps by BC.

| k  | Batching |   |   |   | U-statistic |   |   |   | V-statistic |   |   |   | Single-replication |   |   |   |
|----|----------|---|---|---|-------------|---|---|---|-------------|---|---|---|-------------------|---|---|---|
|    | c.p.(%)  | mean | std. |   | c.p.(%)     | mean | std. |   | c.p.(%)     | mean | std. |   | c.p.(%)        | mean | std. |
| 5  | 100      | 5.77 | 1.84 |   | 100         | 5.60 | 1.69 |   | 100         | 5.58 | 1.74 |   | 100            | 4.31 | 1.67 |
| 10 | 100      | 5.44 | 1.94 |   | 100         | 4.93 | 1.61 |   | 100         | 5.10 | 1.74 |   |                  |      |     |
| 25 | 100      | 5.57 | 2.16 |   | 100         | 4.61 | 1.69 |   | 100         | 4.78 | 1.67 |   |                  |      |     |
| 50 | 100      | 11.04| 6.76 |   | 100         | 4.49 | 1.64 |   | 100         | 4.45 | 1.60 |   |                  |      |     |
| 70 | NA       | NA   | NA   |   | 100         | 4.33 | 1.62 |   | 100         | 4.42 | 1.63 |   |                  |      |     |
| 90 | NA       | NA   | NA   |   | 100         | 4.25 | 1.63 |   | 100         | 4.47 | 1.64 |   |                  |      |     |

Table 7: Problem (23), n = 100, n_1 = 64, n_2 = 36. Upper bounds of optimality gaps by CRN.

| k  | Batching |   |   |   | U-statistic |   |   |   | V-statistic |   |   |   | Single-replication |   |   |   |
|----|----------|---|---|---|-------------|---|---|---|-------------|---|---|---|-------------------|---|---|---|
|    | c.p.(%)  | mean | std. |   | c.p.(%)     | mean | std. |   | c.p.(%)     | mean | std. |   | c.p.(%)        | mean | std. |
| 6  | 100      | 2.14 | 1.04 |   | 100         | 1.83 | 0.76 |   | 100         | 1.94 | 0.74 |   | 86.1            | 0.93 | 0.84 |
| 9  | 99.3     | 1.88 | 1.17 |   | 100         | 1.44 | 0.75 |   | 100         | 1.63 | 0.78 |   |                  |      |     |
| 18 | 96.7     | 2.46 | 2.59 |   | 99.4        | 1.17 | 0.83 |   | 99.7        | 1.27 | 0.72 |   |                  |      |     |
| 30 | NA       | NA   | NA   |   | 94.0        | 0.97 | 0.84 |   | 99.6        | 1.23 | 0.82 |   |                  |      |     |

the desired coverage probability in most cases, but are looser (i.e., larger) than those given by U- and V-statistics. It can be seen that the tightest bound by batching can be twice that by bagging (e.g., in Tables 5 and 7). Like in Tables 1-3 the batching bounds are also more variable, as measured by the standard deviation, than the bagging-based bounds under the same resample size k.

Some new observations are as follows. First, we see that the single-replication procedure suffers from severe under-cover issues in problems (23) and (24) (86.1% in Tables 7 and 79.5% in Table 9). In problem (23) this can be attributed to the integrality requirement on the decision. In problem (24) the optimization itself is smooth, and the issue lies in the delicate relation between the variance and the jumping behavior of the estimated solution. We find that with high probability the candidate solution ˆx is −1 (with optimality gap 0.1), and, given ˆx = −1, solving the SAA associated with the new cost function h(x, ξ) − h(−1, ξ) gives the solution −1 again with high probability. However this way the estimated variance ˆσ^2 will be zero (because the new cost function is constantly 0 at x = −1) which causes the under-cover issue. Similar observations have been discussed in Section 6 of [1]. On the contrary, our bagging procedures mitigate this by estimating the variance using all the resampled SAA solutions.

Second, in general the CRN approach enjoys the benefit of generating tighter and stabler confidence bounds than the BC approach thanks to variance reduction. By comparing Table 6 with Table 7 or Table 8 with Table 9 we see that this benefit of CRN becomes more significant when one invests more data in obtaining ˆx, i.e. when n_1 is chosen larger. This is because, the closer the estimated solution ˆx gets to the true optimum x^*, the smaller is the variance of the gap function h(x, ξ) − h(ˆx, ξ) at the optimum (i.e., x^*) due to the continuity of its variance (as a function of x), which in turn leads to a smaller standard error. We also observe that the BC approach tends to over-cover the optimality gap, potentially because of the looseness of the union bound.
Table 8: Problem (24), $n = 100, n_1 = 64, n_2 = 36$. Upper bounds of optimality gaps by BC.

| $k$ | Batching | $U$-statistic | $V$-statistic | Single-replication |
|-----|----------|---------------|---------------|--------------------|
|     | c.p.(%)  | mean          | std.          | c.p.(%)            | mean          | std.          | c.p.(%) | mean          | std. |
| 5   | 100      | 2.25          | 1.06          | 100                | 2.14          | 1.03          | 100     | 2.22          | 1.05 |
| 10  | 100      | 2.06          | 1.11          | 100                | 1.95          | 1.06          | 100     | 2.01          | 1.06 |
| 25  | 100      | 2.02          | 1.30          | 100                | 1.72          | 1.07          | 100     | 1.75          | 1.06 |
| 50  | 100      | 3.75          | 3.15          | 100                | 1.59          | 1.07          | 100     | 1.70          | 1.09 |
| 70  | NA       | NA            | NA            | 100                | 1.54          | 1.11          | 100     | 1.66          | 1.10 |

Table 9: Problem (24), $n = 100, n_1 = 64, n_2 = 36$. Upper bounds of optimality gaps by CRN.

| $k$ | Batching | $U$-statistic | $V$-statistic | Single-replication |
|-----|----------|---------------|---------------|--------------------|
|     | c.p.(%)  | mean          | std.          | c.p.(%)            | mean          | std.          | c.p.(%) | mean          | std. |
| 3   | 100      | 1.61          | 0.55          | 100                | 1.53          | 0.50          | 100     | 1.53          | 0.53 |
| 9   | 97.8     | 1.29          | 0.89          | 99.9               | 1.10          | 0.57          | 99.9    | 1.12          | 0.55 |
| 18  | 90.8     | 2.09          | 2.13          | 98.1               | 0.93          | 0.66          | 99.3    | 0.98          | 0.63 |
| 30  | NA       | NA            | NA            | 92.1               | 0.87          | 0.79          | 97.7    | 0.91          | 0.67 |

9 Conclusion

We have studied a bagging approach to estimate bounds for the optimal value, and consequently the optimality gap for a given solution in stochastic optimization. We demonstrate how our approach works under minimal regularity conditions, including for non-smooth problems, and exhibits competitive statistical efficiency and stability. Compared to batching, our approach generates a new tradeoff between bound tightness and statistical accuracy that is especially beneficial in small-sample situations. Compared to approaches based on direct SAA asymptotics, our approach requires less smoothness conditions on the objectives and gives more stable estimates thanks to the smoothing effect of bagging. These benefits, however, are offset by the price of more computation in repeatedly solving SAA programs. We have developed the theoretical properties of our approach by viewing SAA as a kernel in infinite-order symmetric statistics, and have illustrated our findings with numerical results.

A Proofs

A.1 Proof of Lemma 1

We show the lemma under the two alternate sets of listed assumptions. First is under Assumptions 2 and 3. Second is under Assumptions 1, 2, 9 and that $\mathcal{X}$ is compact.

Proof of Lemma 1 (using Assumption 3). Under Assumption 2 by (16) we have $EH_k(\xi_1, \ldots, \xi_k)^2 < \infty$ and hence $Var(g_k(\xi)) = Var(E[H_k(\xi_1, \ldots, \xi_k)|\xi_1]) \leq Var(H_k(\xi_1, \ldots, \xi_k))$ is well-defined and fi-
nite. By the Chebyshev inequality, we have

\[
\begin{align*}
\kappa^2 \text{Var}(g_k(\xi)) \\
= \kappa^2 \text{Var}(E[H_k(\xi_1, \ldots, \xi_k)|\xi_1]) \\
= \text{Var} \left( E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] \right) \\
\geq \eta^2 \mathbb{P} \left( \left| E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] - E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] \right| > \eta \right)
\end{align*}
\]

since

\[
E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] = E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right]
\]

Thus,

\[
\eta^2 \left[ \mathbb{P} \left( E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] - E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] > \eta \right) \\
+ \mathbb{P} \left( E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] - E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] < -\eta \right) \right]
\]

(25)

Consider the two terms in (25). We use a coupling argument to bound them and show that they lead to the two terms in (17) that are independent of \( k \). For the first term in (25),

\[
P \left( E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] - E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] > \eta \right)
\]

\[
= P \left( E \left[ \min_{x \in \mathcal{X}} \left\{ h(x, \xi'_1) + \sum_{i \neq 1} h(x, \xi_i) \right\} \right] - E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] > \eta \right)
\]

where \( \xi_1, \xi'_1, \xi_2, \ldots, \xi_k \) are all independent

\[
= P \left( E \left[ \min_{x \in \mathcal{X}} \left\{ h(x, \xi'_1) + \sum_{i \neq 1} h(x, \xi_i) \right\} - \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] > \eta \right)
\]

\[
\geq P \left( E \left[ h(x_{\epsilon}(\xi'), \xi'_1) + \sum_{i \neq 1} h(x_{\epsilon}(\xi'), \xi_i) - \sum_{i=1}^{k} h(x_{\epsilon}(\xi'), \xi_i) \right] > \eta + \epsilon \right)
\]

(26)

where \( x_{\epsilon}(\xi') \) is an \( \epsilon \)-optimal solution for the optimization \( \min_{x \in \mathcal{X}} \left\{ h(x, \xi'_1) + \sum_{i \neq 1} h(x, \xi_i) \right\} \) that only depends on \( \xi' = \{\xi'_1, \xi_2, \ldots, \xi_k\} \). The last inequality follows since \( \min_{x \in \mathcal{X}} \left\{ h(x, \xi'_1) + \sum_{i \neq 1} h(x, \xi_i) \right\} \geq h(x_{\epsilon}(\xi'), \xi'_1) + \sum_{i \neq 1} h(x_{\epsilon}(\xi'), \xi_i) - \epsilon \) by the definition of \( x_{\epsilon}(\xi') \) and \( \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \leq \sum_{i=1}^{k} h(x_{\epsilon}(\xi'), \xi_i) \)}
by the definition of minimization. Note that (26) is equal to
\[
P \left( E \left[ h(x_\varepsilon(\xi'), \xi'_1) - h(x_\varepsilon(\xi'), \xi_1) \right] > \eta + \varepsilon \right)
\]

\[
= P \left( E \left[ h(x_\varepsilon(\xi'), \xi'_1) - Z(x_\varepsilon(\xi')) \right] > \eta + \varepsilon \right)
\]
since \(\xi_1\) is independent of \(x_\varepsilon(\xi')\) and \(\xi'_1\)

\[
\geq P \left( E \left[ \min_{x \in X} \{ h(x, \xi'_1) - Z(x) \} \right] > \eta + \varepsilon \right)
\]

\[
= P \left( \min_{x \in X} \{ h(x, \xi'_1) - Z(x) \} > \eta + \varepsilon \right)
\]

Similarly, for the second term in (25), we have

\[
P \left( E \left[ \min_{x \in X} \left\{ \sum_{i=1}^k h(x, \xi_i) \right\} \right] - E \left[ \min_{x \in X} \left\{ \sum_{i=1}^k h(x, \xi_i) \right\} \right] < -\eta \right)
\]

\[
= P \left( E \left[ \min_{x \in X} \left\{ \sum_{i=1}^k h(x, \xi_i) \right\} - E \left[ \min_{x \in X} \left\{ \sum_{i=1}^k h(x, \xi_i) \right\} \right] > \eta \right) \right)
\]

\[
= P \left( E \left[ \min_{x \in X} \left\{ \sum_{i=1}^k h(x, \xi_i) \right\} - E \left[ \min_{x \in X} \left\{ h(x, \xi'_1) + \sum_{i \neq 1} h(x, \xi_i) \right\} \right] > \eta \right) \right)
\]

where \(\xi_1, \xi'_1, \xi_2, \ldots, \xi_k\) are all independent

\[
= P \left( E \left[ \min_{x \in X} \left\{ \sum_{i=1}^k h(x, \xi_i) - \min_{x \in X} \left\{ h(x, \xi'_1) + \sum_{i \neq 1} h(x, \xi_i) \right\} \right\} \xi'_1 > \eta \right) \right)
\]

where \(x_\varepsilon(\xi), \xi = \{\xi_1, \ldots, \xi_n\},\) is an \(\varepsilon\)-optimal solution of \(\min_{x \in X} \sum_{i=1}^n h(x, \xi_i);\)

this follows since \(\min_{x \in X} \sum_{i=1}^k h(x, \xi_i) + \varepsilon \geq \sum_{i=1}^k h(x_\varepsilon(\xi), \xi_i)\) by the definition of \(x_\varepsilon(\xi)\) and

\[
\min_{x \in X} \left\{ \sum_{i \neq 1} h(x, \xi_i) \right\} \leq h(x_\varepsilon(\xi), \xi'_1) + \sum_{i \neq 1} h(x_\varepsilon(\xi), \xi_i)
\]

\[
= P \left( E \left[ h(x_\varepsilon(\xi), \xi'_1) - h(x_\varepsilon(\xi), \xi'_1) \right] > \eta + \varepsilon \right)
\]

\[
\geq P \left( E \left[ \min_{x \in X} \{ h(x, \xi'_1) - h(x, \xi_1) \} \right] > \eta + \varepsilon \right)
\]

Combining (27) and (28) into (25), we get \(17). \]

Proof of Lemma 7 (using Assumption 2). We first argue consistency of the SAA solutions. Since Assumptions 1 and 2 hold, by Theorem 7.48 in [16] we have \(\sup_{x \in X} \left\{ \frac{1}{k} \sum_{i=1}^k h(x, \xi_i) - Z(x) \right\} \rightarrow 0\) almost surely as \(k \rightarrow \infty\). Denote by \(X_k^*\) the set of optimal solutions for the SAA problem formed by
ξ₁, ..., ξₖ. Note that \( \hat{X}_k \neq \emptyset \) because of Lipschitz continuity and compactness of \( \mathcal{X} \). Assumption 4 also implies Lipschitzness of \( Z \), i.e. \( |Z(x_1) - Z(x_2)| \leq EM(\xi)\|x_1 - x_2\| \). With all these ingredients, Theorem 5.3 in [46] then ensures that almost surely \( \sup_{x \in \hat{X}_k} \inf_{x' \in \mathcal{X}^*} \|x - x'\| \to 0 \). Moreover, since \( \sup_{x \in \hat{X}_k} \inf_{x' \in \mathcal{X}^*} \|x - x'\| \leq D_X \), where \( D_X \) is the diameter of \( \mathcal{X} \), we have \( E[\sup_{x \in \hat{X}_k} \inf_{x' \in \mathcal{X}^*} \|x - x'\|^2] \to 0 \) by bounded convergence theorem.

We now follow the line of arguments in the proof that uses Assumption 8. Here, we can work with exact optimal solutions in place of \( \epsilon \)-optimal solutions because \( \hat{X}_k \neq \emptyset \). Following the coupling argument in the previous proof, we have for the first term in (25)

\[
P \left( E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] - E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] > \eta \right)
\]

\[
\geq P \left( E \left[ h(x(\xi'), \xi_1) - Z(x(\xi')) \right] > \eta \right)
\]

\[
\geq P \left( E \left[ h(x'(\xi'), \xi_1) - Z(x'(\xi')) - (M(\xi_1') + EM(\xi))\|x'(\xi') - x'(\xi')\| \xi_1' \right] > \eta \right)
\]

\[
\geq P \left( \min_{x \in \mathcal{X}^*} \{h(x, \xi_1') - Z^*\} - (M(\xi_1') + EM(\xi))E \left[ \|x(\xi') - x'(\xi')\| \xi_1' \right] > \eta \right)
\]

where \( x(\xi') \) is an optimal solution for \( \min_{x \in \mathcal{X}} \{h(x, \xi_1') + \sum_{i=2}^{k} h(x, \xi_i)\} \) and \( x'(\xi') \in \mathcal{X}^* \) minimizes \( \|x(\xi') - x'(\xi')\| \) (minimum is achieved because \( \mathcal{X}^* \) is compact). Since

\[
E \left[ \|x(\xi') - x'(\xi')\| \xi_1' \right] \leq E \left[ \sup_{x \in \hat{X}_k} \inf_{x' \in \mathcal{X}^*} \|x - x'\| \right] \leq \left( E \left[ \sup_{x \in \hat{X}_k} \inf_{x' \in \mathcal{X}^*} \|x - x'\|^2 \right] \right)^{\frac{1}{2}} \to 0
\]

we know \( E[\|x(\xi') - x'(\xi')\| \xi_1'] = o_p(1) \) on one hand. On the other hand, \( M(\xi_1') + EM(\xi) = O_p(1) \), hence \( (M(\xi_1') + EM(\xi))E[\|x(\xi') - x'(\xi')\| \xi_1'] = o_p(1) \). By Slutsky’s theorem

\[
\min_{x \in \mathcal{X}^*} \{h(x, \xi_1') - Z^*\} - (M(\xi_1') + EM(\xi))E \left[ \|x(\xi') - x'(\xi')\| \xi_1' \right] \Rightarrow \min_{x \in \mathcal{X}} \{h(x, \xi_1') - Z^*\}
\]

which leads to

\[
\liminf_{k \to \infty} P \left( \min_{x \in \mathcal{X}} \{h(x, \xi_1') - Z^*\} - (M(\xi_1') + EM(\xi))E \left[ \|x(\xi') - x'(\xi')\| \xi_1' \right] > \eta \right)
\]

\[
\geq P \left( \min_{x \in \mathcal{X}^*} \{h(x, \xi_1') - Z^*\} > \eta \right).
\]

For the second term in (25), we have the following lower bound by a similar argument

\[
P \left( E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] - E \left[ \min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i) \right] < -\eta \right)
\]

\[
\geq P \left( E \left[ h(x, \xi_1) - h(x, \xi_1) \right] > \eta \right)
\]

\[
\geq P \left( E \left[ \min_{x \in \mathcal{X}^*} \{h(x, \xi_1) - h(x, \xi_1)\} \xi_1' \right] - E[M(\xi_1)\|x(\xi') - x'(\xi')\|] - M(\xi_1)E\|x(\xi) - x'(\xi)\| > \eta \right)
\]
where $x(\xi)$ is an optimal solution for $\min_{x \in \mathcal{X}} \sum_{i=1}^{k} h(x, \xi_i)$ and $x'(\xi) \in \mathcal{X}^*$ minimizes $\|x(\xi) - x'(\xi)\|$. Again $E[M(\xi_1)\|x(\xi) - x'(\xi)\|]$ and $M(\xi_1')E[\|x(\xi) - x'(\xi)\|]$ are both $o_p(1)$, and by convergence in distribution we obtain the lower bound

$$P \left( E \left[ \min_{x \in \mathcal{X}^*} \{h(x, \xi_1) - h(x, \xi'_1)\} \right] \xi'_1 > \eta \right)$$

in place of (28). This completes the proof.

\[ \square \]

### A.2 Proofs of Theorems 3 and 4

We need the following result from [50]:

**Theorem 13** (Theorem 11.2 in [50]). Let $L_n$ be a linear space of random variables with finite second moment that contains the constants. Let $T_n$ be a random variable with projection $S_n$ onto $L_n$. If

$$\frac{\text{Var}(T_n)}{\text{Var}(S_n)} \to 1 \text{ as } n \to \infty$$

then

$$\frac{T_n - ET_n}{sd(T_n)} = \frac{S_n - ES_n}{sd(S_n)} \xrightarrow{p} 0 \text{ as } n \to \infty$$

where $sd(\cdot)$ denotes the standard deviation.

For any random variable in the form $T = T(\xi_1, \ldots, \xi_n)$, we also use the notation $\hat{T}$ to denote the Hajek projection, namely, the projection of $T$ onto the space spanned by $\sum_{i=1}^{n} f_i(\xi_i)$ where $f_i$'s are any measurable functions. By [50], we know that, if $\xi_1, \ldots, \xi_n$ are i.i.d. and $T$ has finite second moment,

$$\hat{T} = \sum_{i=1}^{n} E[T|\xi_i] - (n-1)ET$$

To proceed, we also define

$$g_{k,c}(\xi_1, \ldots, \xi_c) = E[H_k(\xi_1, \ldots, \xi_k)|\xi_1 = \xi'_1, \ldots, \xi_c = \xi'_c]$$

as the conditional expectation of $H_k$ given the first $c$ variables. In particular, by our definition before, $g_k(\xi) = g_{k,1}(\xi)$ and $H_k(\xi_1, \ldots, \xi_k) = g_{k,k}(\xi_1, \ldots, \xi_k)$.

We have the following lemma on the estimate of $g_{k,c}(\cdot)$:

**Lemma 2.** Suppose Assumption 2 holds. For $\xi_1, \ldots, \xi_c, \xi'_1, \ldots, \xi'_c \overset{i.i.d.}{\sim} F$, we have

$$|g_{k,c}(\xi'_1, \ldots, \xi'_c) - E[g_{k,c}(\xi_1, \ldots, \xi_c)]| \leq \frac{1}{k} \sum_{i=1}^{c} E \left[ \sup_{x \in \mathcal{X}} |h(x, \xi'_i) - h(x, \xi_i)| \right]$$
Proof. Let $\xi_i, \xi_i'$ be i.i.d. variables generated from $F$. Assumption 2 ensures $E H_k(\xi_1, \ldots, \xi_k)^2 < \infty$ and hence $Var(g_k,c(\xi_1, \ldots, \xi_c)) = Var(E[H_k(\xi_1, \ldots, \xi_k)|\xi_1, \ldots, \xi_c]) \leq Var(H_k(\xi_1, \ldots, \xi_k))$ is well-defined and finite. Consider

\[ g_{k,c}(\xi_1', \ldots, \xi_c') - E[g_{k,c}(\xi_1, \ldots, \xi_c)] = E[H_k(\xi_1', \ldots, \xi_c', \xi_k)\xi_1', \ldots, \xi_c'] - E[H_k(\xi_1, \ldots, \xi_k)] \]

\[ = E \left[ \min_{x \in X} \left\{ \frac{1}{k} \sum_{i=1}^{c} h(x, \xi_i') + \frac{1}{k} \sum_{i=c+1}^{k} h(x, \xi_i) \right\} \right] - E \left[ \min_{x \in X} \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) \right] \]

\[ = E \left[ \min_{x \in X} \left\{ \frac{1}{k} \sum_{i=1}^{c} h(x, \xi_i') + \frac{1}{k} \sum_{i=c+1}^{k} h(x, \xi_i) \right\} - \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) \right] \]

since $\xi_i, \xi_i'$ are all independent

\[ \leq E \left[ \frac{1}{k} \sum_{i=1}^{c} h(x_{c,k}(\xi_i'), \xi_i') + \frac{1}{k} \sum_{i=c+1}^{k} h(x_{c,k}(\xi_i), \xi_i) - \frac{1}{k} \sum_{i=1}^{k} h(x_{c,k}(\xi_i), \xi_i) \right] + \epsilon \quad (29) \]

where $x_{c,k}(\xi)$ is an $\epsilon$-optimal solution of $\min_{x \in X} \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i)$. The last inequality follows since

\[ \min_{x \in X} \left\{ \frac{1}{k} \sum_{i=1}^{c} h(x, \xi_i') + \frac{1}{k} \sum_{i=c+1}^{k} h(x, \xi_i) \right\} \leq \frac{1}{k} \sum_{i=1}^{c} h(x_{c,k}(\xi_i'), \xi_i') + \frac{1}{k} \sum_{i=c+1}^{k} h(x_{c,k}(\xi_i), \xi_i) \]

by the definition of minimization and

\[ \min_{x \in X} \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) \geq \frac{1}{k} \sum_{i=1}^{k} h(x_{c,k}(\xi), \xi_i) - \epsilon \]

by the definition of $x_c(\xi)$. Note that (29) is equal to

\[ E \left[ \frac{1}{k} \sum_{i=1}^{c} (h(x_{c,k}(\xi_i'), \xi_i') - h(x_{c,k}(\xi), \xi_i)) \right] \xi_1', \ldots, \xi_c' \] + \epsilon

\[ \leq E \left[ \frac{1}{k} \sum_{i=1}^{c} \sup_{x \in X} |h(x, \xi_i') - h(x, \xi_i)| \xi_i', \ldots, \xi_c' \right] + \epsilon \quad (30) \]

\[ = \frac{1}{k} \sum_{i=1}^{c} E \left[ \sup_{x \in X} |h(x, \xi_i') - h(x, \xi_i)| \xi_i' \right] + \epsilon \quad (31) \]

since $\xi_1', \ldots, \xi_c, \xi_1', \ldots, \xi_c'$ are independent
Similarly,

\[ E[g_{k,c}(\xi_1, \ldots, \xi_c)] - g_{k,c}(\xi'_1, \ldots, \xi'_c) = E[H_k(\xi_1, \ldots, \xi_k)] - E[H_k(\xi'_1, \ldots, \xi'_k)] \]

\[ = E \left[ \min_{x \in X} \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) \right] - E \left[ \min_{x \in X} \left\{ \frac{1}{k} \sum_{i=1}^{c} h(x, \xi'_i) + \frac{1}{k} \sum_{i=c+1}^{k} h(x, \xi_i) \right\} \right] \]

\[ = E \left[ \min_{x \in X} \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - \min_{x \in X^c} \left\{ \frac{1}{k} \sum_{i=1}^{c} h(x, \xi'_i) + \frac{1}{k} \sum_{i=c+1}^{k} h(x, \xi_i) \right\} \xi'_1, \ldots, \xi'_c \right] \]

since \( \xi_i, \xi'_i \) are all independent

\[ \leq E \left[ \frac{1}{k} \sum_{i=1}^{k} h(x_{e,k}(\xi_i), \xi_i) - \frac{1}{k} \sum_{i=1}^{c} h(x_{e,k}(\xi'_i), \xi'_i) - \frac{1}{k} \sum_{i=c+1}^{k} h(x_{e,k}(\xi'_i), \xi'_i) \right] \]

\[ = E \left[ \frac{1}{k} \sum_{i=1}^{c} h(x_{e,k}(\xi'_i), \xi_i) - h(x_{e,k}(\xi'_i), \xi'_i) \right] \xi'_1, \ldots, \xi'_c \]

\[ \leq \frac{1}{k} \sum_{i=1}^{c} E \left[ \sup_{x \in X} |h(x, \xi_i) - h(x, \xi'_i)| \right] \xi'_1, \ldots, \xi'_c \]

Combining (31) and (32), and noting that \( \epsilon \) is arbitrary, we have

\[ g_{k,c}(\xi'_1, \ldots, \xi'_c) - E[g_{k,c}(\xi_1, \ldots, \xi_c)] \leq \frac{1}{k} \sum_{i=1}^{c} E \left[ \sup_{x \in X} |h(x, \xi'_i) - h(x, \xi_i)| \right] \xi'_i \]

which concludes the lemma.

We are now ready to prove Theorem 3.

Proof of Theorem 3 By Assumption 2 we have \( E H_k(\xi_1, \ldots, \xi_k)^2 < \infty \) by (10) and hence the centered \( U \)-statistic \( U_{n,k} - W_k \) satisfies \( E(U_{n,k} - W_k)^2 < \infty \). Following Section 12.1 in van der Vaart, the Hajek projection of \( U_{n,k} - W_k \) is

\[ (U_{n,k} - W_k) = \sum_{i=1}^{n} E[U_{n,k} - W_k | \xi_i] = \sum_{i=1}^{n} \frac{1}{(k)} \sum_{(i_1, \ldots, i_k) \in C_k} E[H_k(\xi_{i_1}, \ldots, \xi_{i_k}) - W_k | \xi_i] \]

(33)

Note that

\[ E[H_k(\xi_i, \ldots, \xi_k) - W_k | \xi_i] = \begin{cases} E[H_k(\xi_1, \ldots, \xi_k) - W_k | \xi_i] = g_k(\xi_i) - W_k & \text{if } i \in \{i_1, \ldots, i_k\} \\ 0 & \text{otherwise} \end{cases} \]
For each $i$, the number of $E[H_k(\xi_i, \ldots, \xi_{i_k}) - W_k|\xi_i]$ in which the first case above happens, among all summands in the inner summation in the left hand side of (33), is $\binom{n-1}{k-1}$. Therefore, (33) is equal to

$$\binom{n-1}{k-1} \frac{n}{k} \sum_{i=1}^{n} (g_k(\xi_i) - W_k) = \frac{k}{n} \sum_{i=1}^{n} (g_k(\xi_i) - W_k)$$

(34)

Since $\xi_1, \ldots, \xi_n$ are i.i.d., we have

$$\text{Var}(U_{n,k} - W_k) = \frac{k^2}{n} \text{Var}(g_k(\xi))$$

(35)

By Theorem 13, if we can prove that

$$\frac{\text{Var}(U_{n,k})}{\text{Var}((U_{n,k} - W_k))} \rightarrow 1$$

(36)

and

$$\frac{(U_{n,k} - W_k) - E(U_{n,k} - W_k)}{sd((U_{n,k} - W_k))} \Rightarrow N(0,1)$$

(37)

Then

$$\frac{U_{n,k} - W_k}{sd(U_{n,k})} = \left( \frac{U_{n,k} - W_k}{sd(U_{n,k})} - \frac{(U_{n,k} - W_k) - E(U_{n,k} - W_k)}{sd((U_{n,k} - W_k))} \right) + \frac{(U_{n,k} - W_k) - E(U_{n,k} - W_k)}{sd((U_{n,k} - W_k))} \Rightarrow N(0,1)$$

by Slutsky’s Theorem. Furthermore, by (36) and Slutsky’s Theorem again, we have

$$\frac{U_{n,k} - W_k}{sd((U_{n,k} - W_k))} = \frac{U_{n,k} - W_k}{sd(U_{n,k})} \frac{sd(U_{n,k})}{sd((U_{n,k} - W_k))} \Rightarrow N(0,1)$$

Note that, by (35),

$$\frac{U_{n,k} - W_k}{sd((U_{n,k} - W_k))} = \frac{\sqrt{n}}{k} \sqrt{\text{Var}(g_k(\xi))}$$

and hence we conclude the theorem.

By (34) and (35), the left hand side of (37) can be written as

$$\frac{(k/n) \sum_{i=1}^{n} (g_k(\xi_i) - W_k)}{(k/\sqrt{n})sd(g_k(\xi))} = \frac{\sqrt{n}}{k} \frac{(1/n) \sum_{i=1}^{n} (g_k(\xi_i) - W_k)}{sd(g_k(\xi))}$$

Thus (37) is equivalent to

$$\frac{\sqrt{n}((1/n) \sum_{i=1}^{n} (g_k(\xi_i) - W_k))}{sd(g_k(\xi))} \Rightarrow N(0,1)$$

(38)

The rest of the proof focuses on showing (36) and (38).
Proof of (36): Consider

\[
\text{Var}(U_{n,k}) = \frac{1}{\binom{n}{k}^2} \sum_{(i_1, \ldots, i_k; (i'_1, \ldots, i'_k)) \in \mathcal{C}_k} \text{Cov}(H_k(\xi_{i_1}, \ldots, \xi_{i_k}), H_k(\xi_{i'_1}, \ldots, \xi_{i'_k}))
\]

\[
= \frac{1}{\binom{n}{k}^2} \sum_{c=1}^{k} \binom{k}{c} \binom{n-k}{k-c} \text{Var}(g_{k,c}(\xi_1, \ldots, \xi_c))
\]

\[
= \sum_{c=1}^{k} \frac{\binom{k}{c} \binom{n-k}{k-c}}{\binom{n}{k}} \text{Var}(g_{k,c}(\xi_1, \ldots, \xi_c))
\]

(39)

(40)

where the second equality follows by counting the number of combinations of \((i_1, \ldots, i_k)\) and \((i'_1, \ldots, i'_k)\) in the summation that have \(c\) overlapping indices. For each \(c\), this number follows by first picking \(k\) out of \(n\) indices from \(\{1, \ldots, n\}\) to place in \((i_1, \ldots, i_k)\), then choosing \(c\) from these \(k\) numbers to place in \((i'_1, \ldots, i'_k)\) and \(k-c\) from the remaining \(n-k\) numbers to place in the remaining spots in \((i'_1, \ldots, i'_k)\). Note also that if \((i_1, \ldots, i_k)\) and \((i'_1, \ldots, i'_k)\) have \(c \geq 1\) overlapping indices, say by relabeling and the symmetry of \(H_k\) we write the indices as \((i_1, \ldots, i_k)\) and \((i_1, \ldots, i'_c, i_{c+1}', \ldots, i'_k)\), then

\[
\text{Cov}(H_k(\xi_{i_1}, \ldots, \xi_{i_k}), H_k(\xi_{i'_1}, \ldots, \xi_{i'_k}))
\]

\[
= \text{Cov}(E[H_k(\xi_{i_1}, \ldots, \xi_{i_k})|\xi_{i_1}, \ldots, \xi_{i_c}, \xi_{i_{c+1}'}, \ldots, \xi_{i'_k}], E[H_k(\xi_{i_1}, \ldots, \xi_{i_k})|\xi_{i_1}, \ldots, \xi_{i_c}, \xi_{i_{c+1}'}, \ldots, \xi_{i'_k}])
\]

\[
= \text{Var}(g_{k,c}(\xi_1, \ldots, \xi_c))
\]

since \(\xi_{c+1}, \ldots, \xi_k\) and \(\xi_{c+1}', \ldots, \xi'_k\) are independent. Finally, if \((i_1, \ldots, i_k)\) and \((i'_1, \ldots, i'_k)\) have no overlapping index then \(\text{Cov}(H_k(\xi_{i_1}, \ldots, \xi_{i_k}), H_k(\xi_{i'_1}, \ldots, \xi_{i'_k})) = 0\). Therefore the equality in (39) holds.

On the other hand, by (35), we have \(\text{Var}(U_{n,k} - W_k) = \frac{k^2}{n} \text{Var}(g_k(\xi))\). Also, by Lemma 1, \(\text{Var}(g_k(\xi)) > 0\). Combining these with (40) gives

\[
\frac{\text{Var}(U_{n,k})}{\text{Var}(U_{n,k} - W_k)} = \frac{n \sum_{c=1}^{k} \frac{\binom{k}{c} \binom{n-k}{k-c}}{\binom{n}{k}} \text{Var}(g_{k,c}(\xi_1, \ldots, \xi_c))}{k^2 \text{Var}(g_k(\xi))}
\]

\[
= \frac{n}{k^2} \frac{\binom{k}{1} \binom{n-k}{k-1}}{\binom{n}{k}} + \frac{n \sum_{c=2}^{k} \frac{\binom{k}{c} \binom{n-k}{k-c}}{\binom{n}{k}} \text{Var}(g_{k,c}(\xi_1, \ldots, \xi_c))}{k^2 \text{Var}(g_k(\xi))}
\]

(41)

Consider the first term in (41). We have

\[
\frac{n}{k^2} \frac{\binom{k}{1} \binom{n-k}{k-1}}{\binom{n}{k}} = \frac{n}{k^2} \frac{k \cdot (n-k) (n-k-1) \cdots (n-2k+2)/(k-1)!}{n(n-1) \cdots (n-k+1)/k!}
\]

\[
= \frac{n}{k^2} \frac{n-k}{n-1} \frac{n-k-1}{n-2} \cdots \frac{n-2k+2}{n-k+1}
\]

\[
= \left(1 - \frac{k-1}{n-1}\right) \left(1 - \frac{k-1}{n-2}\right) \cdots \left(1 - \frac{k-1}{n-k+1}\right)
\]

(42)
For $k = o(\sqrt{n})$,
\[
\left| \sum_{j=n-k+1}^{n-1} \log \left( 1 - \frac{k-1}{j} \right) \right| \leq C \sum_{j=n-k+1}^{n-1} \frac{k-1}{j} \\
\leq C \int_{n-k}^{n-1} \frac{k-1}{u} du \\
= C(k-1) \log \frac{n-1}{n-k} \\
= -C(k-1) \log \left( 1 - \frac{k-1}{n-1} \right) \\
\leq \hat{C}(k-1) \frac{k-1}{n-1} \\
= o(1)
\]
where $C, \hat{C} > 0$ are some constants. Therefore, from (42), we get
\[
\frac{n}{k^2} \cdot \frac{\binom{k}{c} \binom{n-k}{k-c}}{\binom{n}{k}} \to 1 \quad (43)
\]
as $n \to \infty$.

Now consider the second term in (41). By Lemma 2 for $c \geq 1$,
\[
\begin{align*}
\text{Var}(g_{k,c}(\xi_1, \ldots, \xi_c)) &\leq E \left( \frac{1}{k} \sum_{i=1}^{c} E \left[ \sup_{x \in X} |h(x, \xi'_i) - h(x, \xi_i)| \right] \right)^2 \\
&\leq \frac{1}{k^2} \left( \sum_{i=1}^{c} E \left( \sup_{x \in X} |h(x, \xi'_i) - h(x, \xi_i)| \right) \right)^2 \quad \text{by the Minkowski inequality} \\
&\leq \frac{c^2 M}{k^2} \quad (44)
\end{align*}
\]
for some $M > 0$ by Assumption 4.

Note also that
\[
\frac{\binom{k}{c} \binom{n-k}{k-c}}{\binom{n}{k}}
\]
is the probability mass at $c$ of a hypergeometric variable with parameters $(n, k, k)$. Note that such a variable takes domain $\{\max(2k-n, 0), \ldots, k\}$, which equals $\{0, \ldots, k\}$ for $n$ sufficiently large since $k = o(\sqrt{n})$. This hypergeometric variable has second moment equal to
\[
\frac{k^2(n-k)^2}{n^2(n-1)} + \frac{k^4}{n^2} \quad (45)
\]
Thus, for $n$ large enough, and using Lemma 1, we have

\[
\frac{n \sum_{c=2}^{k} \binom{k}{c} \binom{n-k}{c} Var(g_{k,c}(\xi_1, \ldots, \xi_c))}{k^2 Var(g_k(\xi))} \leq \frac{n \sum_{c=2}^{k} \binom{k}{c} \binom{n-k}{c} \epsilon M}{k^2}
\]

for some constant $\epsilon > 0$

(46)

by using (45) and \( \binom{k}{c} \binom{n-k}{c} = \frac{k^2}{n} (1 + o(1)) \) that we have proven using (45)

\[
= \left( \frac{(n-k)^2}{n(n-1)} + \frac{k^2}{n} - (1 + o(1)) \right) M \epsilon
\]

(47)

since $k = o(\sqrt{n})$.

Combining (45) and (46) into (41), we get

\[
\frac{Var(U_{n,k})}{Var(U_{n,k} - W_k)} \rightarrow 1
\]

Proof of (38): By Lemma 2 denoting $\xi, \xi' \text{i.i.d.} \sim F$, we have

\[
E|g_k(\xi) - W_k|^{2+\delta} \leq E \left( \frac{1}{k} E \left[ \sup_{x \in X} |h(x, \xi') - h(x, \xi)| \right] \right)^{2+\delta} \leq \frac{1}{k^{2+\delta}} E \sup_{x \in X} |h(x, \xi') - h(x, \xi)|^{2+\delta} \leq \frac{\tilde{M}}{k^{2+\delta}}
\]

for some $\tilde{M} > 0$ by Assumption 4. Moreover, by Lemma 1 we have $Var(g_k(\cdot)) \geq \epsilon/k^2$ for some $\epsilon > 0$ for $k$ sufficiently large.

Hence

\[
\frac{nE|g_k(\xi) - W_k|^{2+\delta}}{(nVar(g_k(\xi)))^{1+\delta/2}} \leq \frac{n\tilde{M}/k^{2+\delta}}{(n\epsilon/k^2)^{1+\delta/2}} = \frac{\tilde{M}}{n^{\delta/2} \epsilon^{1+\delta/2}} \rightarrow 0
\]

as $n \rightarrow \infty$. The Lyapunov condition then implies the central limit theorem in (38).

Proof of Theorem 4. Let $c(n, k, s)$ count the number of mappings $\phi : \{1, 2, \ldots, k\} \rightarrow \{1, 2, \ldots, n\}$ such that $|\phi(1, 2, \ldots, k)| = s$, or equivalently, count the number of $\xi_{i_1}, \ldots, \xi_{i_k}$ such that $i_1, \ldots, i_k$ covers $s$ distinct indices, and let $A_{n,s}$ be the average of all $H_k(\xi_{i_1}, \ldots, \xi_{i_k})$ with $s$ distinct indices. In particular, $A_{n,k} = U_{n,k}$. The V-statistic can be expressed for a fixed $l \geq 0$ as

\[
n^k V_{n,k} = \sum_{s=k-l}^{k} c(n, k, s) A_{n,s} + \left( n^k - \sum_{s=k-l}^{k} c(n, k, s) \right) R_{n,l}
\]
where \( R_{n,t} \) is the average of all \( H_k(\xi_{i_1}, \ldots, \xi_{i_k}) \) with at most \( k - l - 1 \) distinct indices. We have

\[
n^k(U_{n,k} - V_{n,k}) = n^kU_{n,k} - \sum_{s=k-l}^{k} c(n,k,s)(U_{n,k} + A_{n,s} - U_{n,k}) - (n^k - \sum_{s=k-l}^{k} c(n,k,s))R_{n,t}
\]

\[
= (n^k - \sum_{s=k-l}^{k} c(n,k,s))(U_{n,k} - R_{n,t}) - \sum_{s=k-l}^{k-1} c(n,k,s)(A_{n,s} - U_{n,k})
\]

\[
= \left( \sum_{s=1}^{k-l-1} c(n,k,s) \right)(U_{n,k} - R_{n,t}) - \sum_{s=k-l}^{k-1} c(n,k,s)(A_{n,s} - U_{n,k}). \tag{48}
\]

We want to show that the two terms in (48) are both \( o_p(n^{k-1/2}) \) so that the desired conclusion follows by Slutsky’s theorem. To this end, we let

\[
l = \left\lfloor \frac{1}{2(1 - 2\gamma)} \right\rfloor \tag{49}
\]

the reason for which shall be clear later.

To bound the first term in (48), note that \( c(n,k,s) \) can be written as

\[
c(n,k,s) = S(k,s)n(n-1)\cdots(n-s+1)
\]

where \( S(k,s) \) is the Stirling number of the second kind with parameters \( k, s \), which is the number of partitions of a set of size \( k \) into \( s \) non-empty subsets. It’s shown in [43] that for \( k \geq 2 \) and \( 1 \leq s \leq k - 1 \)

\[
S(k,s) \leq \frac{1}{2} \binom{k}{s} s^{k-s}. \tag{50}
\]

Hence

\[
\sum_{s=1}^{k-l-1} c(n,k,s) \leq \frac{1}{2} \sum_{s=1}^{k-l-1} \binom{k}{s} s^{k-s} n^s.
\]

Note that the ratio between two neighboring \( \binom{k}{s} s^{k-s} n^s \) is

\[
\left( \frac{k}{s-1} \right) (s-1)^{k-s+1} n^{s-1} / \binom{k}{s} s^{k-s} n^s = \frac{(s-1)^{k-s+1}}{(k-s+1)s^{k-s-1}n} \leq \frac{s^2}{n} \leq \frac{k^2}{n} = o(1),
\]

therefore

\[
\sum_{s=1}^{k-l-1} c(n,k,s) \leq \frac{1}{2} \left( 1 + \sum_{s=1}^{k-l-2} \left( \frac{k^2}{n} \right)^s \right) \binom{k}{l+1} (k - l - 1)^{l+1} n^{k-l-1}
\]

\[
\leq \frac{1}{2(1 - k^2/n)} \binom{k}{l+1} (k - l - 1)^{l+1} n^{k-l-1} = O(k^{2l+2} n^{k-l-1}) = O((\frac{k^2}{n})^{l+1} n^k).
\]

For the particular choice of \( l \) shown in (49), the above bound is \( o(n^{k-1/2}) \). Since both \( U_{n,k} \) and \( R_{n,t} \) are \( O_p(1) \) by Assumption [2], the first term in (48) is \( O_p(n^{k-1}) \).

For the second term in (48), it suffices to show that for each \( k - l \leq s \leq k - 1 \) it holds \( c(n,k,s)(A_{n,s} - U_{n,k}) = o_p(n^{k-1/2}) \) since there are only \( l \) of them. Since \( l \) is now viewed as a
constant, from the upper bound \([50]\) for \(s \geq k - l\) it follows that \(S(k, s) = O(k^{2(k-s)})\).\[4\] If we can argue that \(A_{n,s} - U_{n,k} = O_p(k^{-1})\), then each summand can be bounded as
\[
O_p(k^{2(k-s)-1}n^s) = O_p(n^{2\gamma(k-s)-\gamma+s}) = O_p(n^{k+s-1})
\]
where the last equality holds because \(\gamma < 1/2\) hence \(2\gamma(k-s) - \gamma + s\) increases in \(s\). This implies an upper bound of order \(O_p(n^{k-1/2})\) again because \(\gamma < 1/2\). Now we show \(A_{n,s} - U_{n,k} = O_p(k^{-1})\) by a coupling argument. The value of \(A_{n,s}\) can be computed from the same resamples \(\xi_1, \ldots, \xi_{ik}\) (with \(k\) distinct data points) used to compute \(U_{n,k}\), by removing \(k-s\) of them and fill in with those remaining in the resample. To be specific, we use \(I_k = (I(1), \ldots, I(k))\) to represent a sequence of length \(k\) where \(I(j) \in \{1, \ldots, n\}\) for each \(j \leq k\), define \(|I_k|\) to be the number of distinct indices in \(I_k\). For convenience we denote by \(\hat{I}_k(j_1 : j_2) = (I_k(j_1), \ldots, I_k(j_2))\) the sub-sequence for \(1 \leq j_1 \leq j_2 \leq k\) and \(\hat{\xi}_k = (\xi_{I_k(1)}, \ldots, \xi_{I_k(k)})\).

This leads to
\[
|A_{n,s} - U_{n,k}| \leq \frac{(n-k)!}{n!} \sum_{|I_k| = k} \frac{1}{s^{k-s}} \sum_{|I'_k| = s, I'_k(1:1) = I_k(1:1)} \left| H_k(\hat{\xi}'_k) - H_k(\hat{\xi}_k) \right|
\]
\[
\leq \frac{(n-k)!}{n!} \sum_{|I_k| = k} \frac{1}{s^{k-s}} \sum_{|I'_k| = s, I'_k(1:1) = I_k(1:1)} \sup_{x \in X} \left| \frac{1}{k} \sum_{j=s+1}^k \left( h(x, \xi'_{I_k(j)}) - h(x, \xi_{I_k(j)}) \right) \right|
\]
\[
\leq \frac{(n-k)!}{n!} \sum_{|I_k| = k} \frac{1}{s^{k-s}} \sum_{|I'_k| = s, I'_k(1:1) = I_k(1:1)} \sum_{j=s+1}^k \sup_{x \in X} \left| h(x, \xi'_{I_k(j)}) - h(x, \xi_{I_k(j)}) \right|
\]
\[
\leq \frac{1}{k} \sum_{j=s+1}^k \frac{(n-k)!}{n! s^{k-s}} \sum_{|I_k| = k} \sum_{|I'_k| = s, I'_k(1:1) = I_k(1:1)} \sup_{x \in X} \left| h(x, \xi'_{I_k(j)}) - h(x, \xi_{I_k(j)}) \right|
\]
\[
= \frac{k-s}{k} \frac{2}{n(n-1)} \sum_{1 \leq i_1 < i_2 \leq n} \sup_{x \in X} \left| h(x, \xi_{i_1}) - h(x, \xi_{i_2}) \right|
\]

where the last equality is because \(I'_k(j)\) and \(I_k(j)\) are distinct indices and the gross sum over \(I_k, I'_k\) puts equal weight on each pair \((i_1, i_2)\). Due to Assumption 4, we have

\[
E[|A_{n,s} - U_{n,k}|] \leq \frac{k-s}{k} E[\sup_{x \in X} |h(x, \xi) - h(x, \xi')|] = O\left(\frac{1}{k}\right) = O\left(\frac{1}{k}\right).
\]
A.3 Proofs of Theorems 5 and 6

The Hajek projections of the symmetric kernel $H_k$ and the symmetric statistic $U_{n,k}$ are

$$\hat{H}_k = W_k + \sum_{i=1}^{k} (g_k(\xi_i) - W_k)$$

$$\hat{U}_{n,k} = W_k + \frac{k}{n} \sum_{i=1}^{n} (g_k(\xi_i) - W_k).$$

As discussed in Section 5, we will use the ANOVA decomposition [20] of the symmetric kernel $H_k$ to allow for a larger resample size $k$ in obtaining Theorem 5. We have the following variance bound from [52] in analyzing random forests:

Lemma 3. (Adapted from Lemma 7 of [52]) Under Assumption 2, for any $k \leq n$ it holds

$$E(U_{n,k} - \hat{U}_{n,k})^2 \leq \frac{k^2}{n^2} E(H_k - \hat{H}_k)^2.$$  

Proof. [52] prove this bound in the context of random forests where $H_k$ is a regression tree and $U_{n,k}$ is the random forest obtained from aggregating the resampled trees (without replacement). Although the context they focus on is different from ours, their proof in fact automatically works for general symmetric kernels and U-statistics including the SAA values considered in this paper. Therefore proof for this lemma is omitted. Note that in Lemma 7 of [52] the right hand side is the total variance $\text{Var}(H_k)$ instead of $E(H_k - \hat{H}_k)^2$, however this comes from upper bounding $E(H_k - \hat{H}_k)^2$ by $\text{Var}(H_k)$ in their proof so the bound with $E(H_k - \hat{H}_k)^2$ remains true.

This allows us to derive a CLT under an additional assumption on $k$:

Theorem 14. Under Assumptions 2, 4 and 8, if the resample size $k$ is chosen such that

$$k^2 E(H_k - \hat{H}_k)^2 = o(n)$$  

(51)

then

$$\frac{\sqrt{n}(U_{n,k} - W_k)}{k \sqrt{\text{Var}(g_k(\xi))}} \Rightarrow N(0, 1)$$

where $N(0, 1)$ is the standard normal.

Proof of Theorem 14. According to the proof of Theorem 3 we only need to verify that $E(U_{n,k} - \hat{U}_{n,k})^2/\text{Var}(U_{n,k}) \rightarrow 0$, or equivalently $E(U_{n,k} - \hat{U}_{n,k})^2/\text{Var}(\hat{U}_{n,k}) \rightarrow 0$. Under the choice of $k$ we have $E(U_{n,k} - \hat{U}_{n,k})^2 = o(1/n)$ due to Proposition 3 whereas $\text{Var}(\hat{U}_{n,k}) = k^2 \text{Var}(g_k(\xi))/n \geq \epsilon/n$ for $k$ large enough. This completes the proof.

We state an upper bound for the left hand side of (51) in terms of the maximum deviation of the cost function from its mean.

Lemma 4. We have $|H_k - Z^*| \leq \sup_{x \in \mathcal{X}} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right|$, hence

$$E(H_k - \hat{H}_k)^2 \leq \text{Var}(H_k) \leq E(H_k - Z^*)^2 \leq E\left[ \left( \sup_{x \in \mathcal{X}} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right| \right)^2 \right].$$
Proof. Let $x^*$ be an optimal solution of the original optimization (1), and $x_k^*$ be an optimal solution of the SAA formed by $\xi_1, \ldots, \xi_k$. If $H_k \leq Z^*$, since $Z(x_k^*) \geq Z^*$, we have $|H_k - Z^*| \leq \sup_{x \in X} \left| \frac{1}{k} \sum_{i=1}^k h(x, \xi_i) - Z(x) \right|$. Otherwise, if $H_k > Z^*$, then obviously $Z^* < H_k \leq \frac{1}{k} \sum_{i=1}^k h(x, \xi_i)$, hence again $|H_k - Z^*| \leq \frac{1}{k} \sum_{i=1}^k h(x, \xi_i) - Z(x)$. This proves the first inequality. For the second part of the lemma, the inequality $E(H_k - \bar{H}_k)^2 \leq Var(H_k)$ follows from the projection property of Hajek projection and the other two are obvious.

To proceed, we need to introduce concepts in empirical processes and some notations. Denote by

$$\mathcal{F} := \{ h(x, \cdot) - Z(x) : x \in X \}$$

the family of centered cost functions indexed by the decision $x \in X$. Note that for centered functions the Lipschitz condition holds with a slightly larger constant than $M(\xi)$

$$|h(x_1, \xi) - Z(x_1) - (h(x_2, \xi) - Z(x_2))| \leq (M(\xi) + EM(\xi)) \|x_1 - x_2\|.$$

For a vector $x \in \mathbb{R}^d$, let $\|x\|$ be its $L_2$ norm, and for a random variable $X$ we define $\|X\|_p := (E|X|^p)^{1/p}$ for $p \geq 1$. We equip the function space $\mathcal{F}$ defined above with the norm $\| \cdot \|_2$. We denote by $N(\epsilon, \mathcal{X}, \| \cdot \|)$ the covering number, with ball size $\epsilon$, of the decision space, and by $N_{||}(\epsilon, \mathcal{F}, \| \cdot \|_2)$ the bracketing number, with bracket size $\epsilon$, of the function space $\mathcal{F}$.

We need a few results adapted from [51]. The first result connects the complexity of the function space $\mathcal{F}$ to that of the decision space $X$:

**Lemma 5** (Adapted from Theorem 2.7.11 of [51]). Suppose Assumption 1 holds and the decision space $X$ is compact, then for any $\epsilon > 0$

$$N_{||}(4\epsilon M(\xi) \|_2, \mathcal{F}, \| \cdot \|_2) \leq N(\epsilon, \mathcal{X}, \| \cdot \|) .$$

The second result gives an upper bound of the covering number of the decision space $X$, hence an upper bound of the bracketing number of $\mathcal{F}$ because of the first result.

**Lemma 6.** Let $D_X$ be the diameter of the decision space $X$ with respect to the $L_2$ norm $\| \cdot \|$, then $N(\epsilon, \mathcal{X}, \| \cdot \|) \leq (3D_X/\epsilon)^d$ for all $\epsilon \leq D_X$.

**Proof.** Problem 6 in Section 2.1 of [51] states that the $\epsilon$-packing number of a Euclidean ball of radius $R$ in $\mathbb{R}^d$ is bounded above by $(3R/\epsilon)^d$, and the lemma follows from the fact that the covering number is always no more than the packing number and that $X$ can be contained in a Euclidean ball of radius $D_X$.

The third result concerns the first order moment of the maximum deviation.

**Lemma 7** (Adapted from Theorem 2.14.2 of [51]). Let $\hat{h}(\xi) = \sup_{x \in X} |h(x, \xi) - Z(x)|$. We have for all $k$

$$\sqrt{k} E \left[ \sup_{x \in X} \left| \frac{1}{k} \sum_{i=1}^k h(x, \xi_i) - Z(x) \right| \right] \leq C \|\hat{h}(\xi)\|_2 \int_0^1 \sqrt{1 + \log N_{||}(\epsilon \|\hat{h}(\xi)\|_2, \mathcal{F}, \| \cdot \|_2)} d\epsilon$$

where $C$ is a universal constant.

We also need the following result that translates an upper bound of the first order moment to one for higher order moments:
Lemma 8 (Adapted from Theorem 2.14.5 of [51]). For any \( p \geq 2 \) it holds
\[
\sqrt{k} \left( \mathbb{E} \left[ \sup_{x \in \mathcal{X}} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right|^{p} \right] \right)^{\frac{1}{p}} \leq C \left( \sqrt{k} \mathbb{E} \left[ \sup_{x \in \mathcal{X}} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right| \right] + k^{-\frac{1}{2}} \| \tilde{h}(\xi) \|_p \right)
\]
where \( C \) is a constant depending only on \( p \), and \( \tilde{h} \) is the same as in Lemma 7.

Now we turn to the problem of further bounding the upper bound in Lemma 14, which can be viewed as the maximum deviation of the empirical process generated by the cost function. Specifically, we show that this can be controlled at the canonical rate \( 1/\sqrt{k} \) in the case of Lipschitz continuous cost function. We have:

Theorem 15. Suppose Assumptions 7 and 2 hold, and that the decision space \( \mathcal{X} \) is compact, then we have
\[
\mathbb{E} \left[ \sup_{x \in \mathcal{X}} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right|^{2} \right] = O(\frac{1}{k}).
\]

Proof. First we conclude the following upper bound of the maximum deviation
\[
\sqrt{k} \mathbb{E} \left[ \sup_{x \in \mathcal{X}} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right| \right] \leq C \| \tilde{h}(\xi) \|_2 \int_{0}^{1} \sqrt{1 + \log N \left( \frac{\epsilon \| \tilde{h}(\xi) \|_2 \| \mathcal{X}, \| \cdot \| \right)} \right) \right) \text{ by Lemmas 7 and 5}
\]
\[
\leq C \| \tilde{h}(\xi) \|_2 \left( 1 + \int_{0}^{1} \sqrt{\log N \left( \frac{\epsilon \| \tilde{h}(\xi) \|_2 \| \mathcal{X}, \| \cdot \| \right)} \right) \right) \text{ since } \sqrt{a + b} \leq \sqrt{a} + \sqrt{b}
\]
\[
\leq C \| \tilde{h}(\xi) \|_2 \left( 1 + \int_{0}^{1} \sqrt{\log \frac{12D_{\mathcal{X}} \| M(\xi) \|_2}{\epsilon \| \tilde{h}(\xi) \|_2}} \right) \text{ by Lemma 6 and } N(\epsilon, \mathcal{X}, \| \cdot \|) = 1 \text{ for } \epsilon \geq D_{\mathcal{X}}
\]
\[
= C \| \tilde{h}(\xi) \|_2 + 12C D_{\mathcal{X}} \| M(\xi) \|_2 \int_{0}^{1} \sqrt{\log \frac{1}{\epsilon}} \text{ d} \epsilon
\]
\[
\leq C' \left( \| \tilde{h}(\xi) \|_2 + \sqrt{\log \left( 3 \sqrt{12D_{\mathcal{X}} \| M(\xi) \|_2} \right) (4D_{\mathcal{X}} \| M(\xi) \|_2 \| \tilde{h}(\xi) \|_2) \right) \right) < \infty
\]
where \( C' \) is another universal constant. Then we apply the Lemma 8 with \( p = 2 \) to get
\[
k \mathbb{E} \left[ \sup_{x \in \mathcal{X}} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right|^{2} \right] \leq C^2 \left( \sqrt{k} \mathbb{E} \left[ \sup_{x \in \mathcal{X}} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right| \right] + \| \tilde{h}(\xi) \|_2 \right)^2 < \infty
\]
which concludes Theorem 15.

With all these preparations, Theorem 5 can be readily proved:

Proof of Theorem 5. By Lemma 4 and Theorem 15 we have \( E(H_k - \tilde{H}_k)^2 = O(1/k) \) hence \( k^2 E(H_k - \tilde{H}_k)^2 = O(k) \), which is \( o(n) \) when \( k = o(n) \). The CLT then follows from Theorem 13.

We now prove Theorem 6.
Proof of Theorem 6. If we show \( \text{Var}(H_k)/\text{Var}(\tilde{H}_k) \to 1 \) as \( k \to \infty \), then the conclusion follows from

\[
E(H_k - \tilde{H}_k)^2 = \text{Var}(H_k) - \text{Var}(\tilde{H}_k) = o(\text{Var}(\tilde{H}_k)) = o(k^2 \text{Var}(g_k(\xi))) = o\left(\frac{1}{k}\right)
\]

and Theorem 14, where the last equality is due to Proposition 1. Recall that \( EH_k = E\tilde{H}_k = W_k \).

In fact we will show the stronger results

\[
kE(H_k - W_k)^2 \to \text{Var}(h(x^*, \xi))
\]

\[
kE(\tilde{H}_k - W_k)^2 = k^2 \text{Var}(g_k(\xi)) \to \text{Var}(h(x^*, \xi))
\]

where \( x^* \) is the unique optimal solution. The way we prove these two moment convergence results is to first show that the left hand side weakly converges to some variable that has the desired variance and then use uniform integrability to conclude convergence in moments.

We first prove the \( \leq \) direction of (53). Under the depicted conditions, Theorem 11 entails that

\[
\sqrt{k}(H_k - Z^*) \Rightarrow N(0, \text{Var}(h(x^*, \xi)))
\]

on one hand. On the other hand, from Lemma 4 we have

\[
|H_k - Z^*|^{2+\delta} \leq \sup_{x \in X} |\frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x)|^{2+\delta}
\]

and Lemma 8 with \( p = 2 + \delta \) implies

\[
E \left[ k^{1+\frac{\delta}{2}} \sup_{x \in X} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right|^{2+\delta} \right] \leq C \left( \sqrt{k} E \left[ \sup_{x \in X} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right| \right] + k^{-\frac{\delta}{2(2+\delta)}} \|\tilde{h}(\xi)\|_{2+\delta}^{2+\delta} \right).
\]

Note that the first term on the right hand side is bounded because of (52). Let \( \xi, \xi' \) be i.i.d. copies of the uncertain variable, then the second term

\[
\|\tilde{h}(\xi)\|_{2+\delta}^{2+\delta} = E \sup_{x \in X} |h(x, \xi) - Z(x)|^{2+\delta}
\]

\[
\leq E \sup_{x \in X} E_{\xi'} |h(x, \xi) - h(x, \xi')|^{2+\delta} \quad \text{by Jensen’s inequality}
\]

\[
\leq E \sup_{x \in X} |h(x, \xi) - h(x, \xi')|^{2+\delta} < \infty. \quad \text{by Assumption 4}
\]

This guarantees that

\[
\sup_{k} (\sqrt{k} |H_k - Z^*|)^{2+\delta} \leq \sup_{k} E \left[ k^{1+\frac{\delta}{2}} \sup_{x \in X} \left| \frac{1}{k} \sum_{i=1}^{k} h(x, \xi_i) - Z(x) \right|^{2+\delta} \right] < \infty
\]

therefore the sequence of random variables \( k(H_k - Z^*)^2 \) is uniformly integrable. Since \( \sqrt{k}(H_k - Z^*) \) is asymptotically normal we conclude \( kE(H_k - Z^*)^2 \to \text{Var}(h(x^*, \xi)) \), and hence

\[
\limsup_{k} kE(H_k - W_k)^2 \leq \limsup_{k} kE(H_k - Z^*)^2 = \text{Var}(h(x^*, \xi)).
\]

Next we show (54). Recall from the proof of Lemma 1 that when \( \mathcal{X}^* = \{x^*\} \)

\[
k(g_k(\xi'_1) - W_k) \begin{cases} 
\geq h(x^*, \xi'_1) - Z^* - (M(\xi'_1) + EM(\xi))E \|x(\xi') - x'(\xi')\| \xi'_1 \\
\leq h(x^*, \xi'_1) - Z^* + E[M(\xi'_1)\|x(\xi) - x'(\xi)\|] + M(\xi'_1)E\|x(\xi) - x'(\xi)\|
\end{cases}
\]

where \( x(\xi), x'(\xi), x(\xi'), x'(\xi') \) are the same as those in the proof of Lemma 1. We have shown that the errors are all \( o_p(1) \), hence \( k(g_k(\xi'_1) - W_k) \Rightarrow h(x^*, \xi'_1) - Z^* \). On the other hand, when verifying
Lyapunov condition in proving Theorem \(8\) we have already seen that \(\sup_k k^{2+\delta} |g_k(\xi'_1) - W_k|^{2+\delta} < \infty\). Therefore uniform integrability of \(k^2(g_k(\xi'_1) - W_k)^2\) follows and as \(k \to \infty\)

\[
kE(\tilde{H}_k - W_k)^2 = k^2E(g_k(\xi) - W_k)^2 \to E(h(x^*, \xi) - Z^*)^2 = Var(h(x^*, \xi))
\]

which is exactly (54).

Now (55), (54) and the relation \(E(H_k - W_k)^2 \geq E(\tilde{H}_k - W_k)^2\) together imply (53).

To justify the order of bias, note that (55) and (53) force that as \(k \to \infty\)

\[
k(Z^* - W_k)^2 = kE(H_k - Z^*)^2 - kE(H_k - W_k)^2 \to 0
\]

hence \(Z^* - W_k = o(1/\sqrt{k})\).

The CLT when \(k \geq n\) follows from \(Z^* - W_k = o(1/\sqrt{k}) = o(1/\sqrt{n})\), variance convergence (54) and Slutsky’s theorem. \(\square\)

### A.4 Proof of Proposition 11

**Proof of Proposition 11** Using Lemma 2 with \(c = 1\), we have

\[
Var(g_k(\xi)) = E(E[H_k(\xi_1, \ldots, \xi_k) | \xi_1] - E[H_k(\xi_1, \ldots, \xi_k)])^2
\]

\[
\leq \frac{1}{k^2}E\left(E\left(\sup_{x \in X} |h(x, \xi'_1) - h(x, \xi_1)| \right)^2\right)
\]

\[
\leq \frac{1}{k^2}E\left(\sup_{x \in X} |h(x, \xi'_1) - h(x, \xi_1)| \right)^2 \text{ by Jensen’s inequality}
\]

\[
\leq \frac{1}{k^2}E\left(\sup_{x \in X} |h(x, \xi'_1)| + \sup_{x \in X} |h(x, \xi_1)| \right)^2
\]

\[
\leq \frac{4}{k^2}E\sup_{x \in X} |h(x, \xi)|^2 = O\left(\frac{1}{k^2}\right) \text{ by Minkowski inequality and Assumption 2}
\]

This concludes the proposition. \(\square\)

### A.5 Proofs of Theorem 7 and the Claim in Example 11

**Proof of Theorem 7** From the batching procedure it is clear that \(Var(\tilde{Z}_k) = Var(H_k)/m\) where \(m\) is the number of batches such that \(mk = n\) if rounding errors are ignored. For our U-statistic, note that \(Var(\tilde{U}_{n,k}) = k^2Var(g_k(\xi))/n = k^2Var(H_k)/n,\) and that the resample sizes in Theorems \(8\) and \(10\) satisfy the relation (51) hence by Lemma 3 it holds \(Var(U_{n,k} - \tilde{U}_{n,k}) = o(1/n) = o(Var(U_{n,k})).\)

So the asymptotic ratio

\[
\limsup_{n,k \to \infty} \frac{Var(U_{n,k})}{Var(\tilde{Z}_k)} = \limsup_{n,k \to \infty} \frac{Var(\tilde{U}_{n,k})}{Var(H_k)/m} = \limsup_{n,k \to \infty} \frac{k^2Var(H_k)/n}{Var(H_k)/m} = \limsup_{k \to \infty} \frac{Var(H_k)}{Var(H_k)} = 1
\]

(56)

Then \(r_U = 1\) follows from the fact that \(Var(\tilde{H}_k) \leq Var(H_k).\) Under the conditions and resample sizes of Theorem 1 we have \(E(V_{n,k} - \tilde{U}_{n,k})^2 = o(1/n) = o(Var(U_{n,k})))\) from the proof of Theorem 9 hence \(r_Y = r_U\) follows.

When \(kVar(H_k) \to \infty\), it’s obvious that (56) is equal to 0 since \(kVar(H_k) = O(1)\) by Proposition 1.
Under the conditions of Theorem 3, Theorem 4 states that $\sqrt{k}(H_k - Z^*)$ weakly converges to the infimum of a Gaussian process $\{Y(x)\}_{x \in \mathcal{X}}$ whose covariance structure is defined by $\text{Cov}(h(x_1, \xi), h(x_2, \xi))$. Note that when the covariance is not a constant, the infimum of the Gaussian process does not follow a Gaussian distribution. Now suppose $\limsup_{k \to \infty} \text{Var}(\hat{H}_k)/\text{Var}(H_k) = 1$ then there exists a subsequence $H_{k_s}$ such that

$$\text{Var}(\hat{H}_{k_s})/\text{Var}(H_{k_s}) \to 1 \text{ as } s \to \infty$$

which implies that $\text{Var}(H_{k_s} - \hat{H}_{k_s}) = o(\text{Var}(\hat{H}_{k_s})) = o(1/k_s)$. By Theorem 14 this ensures that $U_{k_s, k_s}$, or equivalently $H_{k_s}$, has a Gaussian limit as $s \to \infty$, however, we already know any subsequence of $H_k$ has a non-Gaussian limit. If the optimal solution is unique, then from (53) and (54) in the proof of Theorem 6 we know (56) is equal to 1.

**Proof of the claim in Example 7.** Like in the proof of Theorem 6, we can follow the coupling argument for Lemma 1 to get

$$k(g_k(\xi'_1) - W_k) = E \left[ \min_{x \in \mathcal{X}} \left\{ h(x, \xi'_1) + \sum_{i \neq 1} h(x, \xi_i) \right\} - \min_{x \in \mathcal{X}} \sum_{i = 1}^k h(x, \xi_i) \right] \xi'_1$$

$$\leq E \left[ h(x(\xi), \xi'_1) - h(x(\xi), \xi_1) \right]$$

where $x(\xi)$ is the optimal solution for $\min_{x \in \mathcal{X}} \sum_{i = 1}^k h(x, \xi_i)$ hence is independent of $\xi'_1$. Note that $x(\xi)$ is uniformly distributed among $\{1, 2, \ldots, d\}$, and that for any fixed $\xi_1$ the solution $x(\xi)$ will weakly converge to the same uniform distribution. Therefore

$$E \left[ h(x(\xi), \xi'_1) - h(x(\xi), \xi_1) \right] = \frac{1}{d} \sum_{j=1}^d \xi'_{1,j} - E[h(x(\xi), \xi_1)]$$

$$= \frac{1}{d} \sum_{j=1}^d \xi'_{1,j} - E \left[ \frac{1}{d} \sum_{j=1}^d \xi_{1,j} + \sum_{j=1}^d (P(x(\xi) = j) - \frac{1}{d}) \xi_{1,j} \right]$$

$$= \frac{1}{d} \sum_{j=1}^d \xi'_{1,j} - E \left[ \sum_{j=1}^d (P(x(\xi) = j) - \frac{1}{d}) \xi_{1,j} | \xi_1 \right]$$

$$= \frac{1}{d} \sum_{j=1}^d \xi'_{1,j} + o(1) \text{ by dominated convergence theorem}$$

where $\xi'_{1,j}$ and $\xi_{1,j}$ are the $j$-th components of $\xi'_1$ and $\xi_1$ respectively. Therefore we have shown $k(g_k(\xi'_1) - W_k) \leq \sum_{j=1}^d \xi'_{1,j}/d + o(1)$. Similarly, denoting by $x(\xi')$ the optimal solution for $\min_{x \in \mathcal{X}} \{ h(x, \xi'_1) + \}$
\[ \sum_{i=2}^{k} h(x, \xi_i) \], the lower bound can be obtained as
\[
k(g_k(\xi'_1) - W_k) \geq E \left[ h(x(\xi'), \xi'_1) - h(x(\xi'), \xi_1) \right] \xi'_1\]\[
= E \left[ h(x(\xi'), \xi'_1) \right] \xi'_1 \text{ by independence between } \xi_1 \text{ and } x(\xi')\]
\[
= \frac{1}{d} \sum_{j=1}^{d} \xi'_{1,j} + \sum_{j=1}^{d} (P(x(\xi') = j|\xi'_1) - \frac{1}{d}) \xi'_{1,j}\]
\[
= \frac{1}{d} \sum_{j=1}^{d} \xi'_{1,j} + o_p(1).\]

The lower and upper bounds agree so
\[
k(g_k(\xi'_1) - W_k) \Rightarrow \frac{1}{d} \sum_{j=1}^{d} \xi_{1,j} = N(0, \frac{1}{d}).\]

On the other hand \(k^2(g_k(\xi'_1) - W_k)^2\) is uniformly integrable as argued in the proof of Theorem 6, hence \(k^2\text{Var}(g_k(\xi)) = E[k^2(g_k(\xi'_1) - W_k)^2] \to 1/d.\)

A.6 Proof of Theorem 9

Proof of Theorem 9. For \(U_{n,k}\), note that each summand in its definition is an SAA value with distinct i.i.d. data, and thus has mean exactly \(W_k\). For \(V_{n,k}\), recall the relation (18)
\[
n^k(U_{n,k} - V_{n,k}) = \left( \sum_{s=1}^{k-1-l} c(n, k, s) (U_{n,k} - R_{n,l}) - \sum_{s=k-l}^{k-1} c(n, k, s) (A_{n,s} - U_{n,k}) \right).\]

Note that \(U_{n,k}\) is unbiased for \(W_k\), and that \(ER_{n,l} = O(1)\) since Assumption 2 implies for any indices \(i_1, \ldots, i_k \in \{1, \ldots, n\}\) that \(|EH_k(\xi_{i_1}, \ldots, \xi_{i_k})| \leq E \sup_{x \in X} |h(x, \xi)|\). In the proof of Theorem 4 we have shown that \(E|A_{n,s} - U_{n,k}| = O(1/k)\), \(\sum_{s=1}^{k-l-1} c(n, k, s) = O((k^2/n)^{l+1}n^k)\) when \(k = o(\sqrt{n})\), and that \(c(n, k, s) = O(k^{2(k-s)}n^s)\) for \(s \geq k-l\). Therefore
\[
n^k \left| EV_{n,k} - W_k \right| \leq O\left( \frac{k^2}{n} \right)^{l+1} n^k + O(1/k) \sum_{s=k-l}^{k-1} O(k^{2(k-s)}n^s),\]

Since \(k^2/n = o(1)\), it holds \(\sum_{s=k-l}^{k-1} O(k^{2(k-s)}n^s) = O(k^2n^{k-1})\), which leads to \(EV_{n,k} - W_k = O((k^2/n)^{l+1} + k/n)\) for any fixed \(l \geq 0.\)

A.7 Proofs of Theorems 10 and 11

Proof of Theorem 10. [52] provides a proof in the context of random forests. Since their proof can be adapted to our optimization context, we shall directly borrow some intermediate results there which hold for general symmetric kernels and U-statistics, and only focus on parts that rely on the particular SAA kernel considered there. Readers are referred to the proof of Theorem 9 in [52] for explanations of the borrowed results.

Note that both Theorems 3 and 5 can be viewed as special cases of Theorem 4 where \(E(H_k - \bar{H}_k)^2 \) is \(O(1)\) and \(O(1/k)\) respectively. So it suffices to show consistency in the more general setting.
under Theorem 14 and, if not implied by (51), the additional requirement \( k = o(n) \). The finite-sample correction \( n^2/(n-k)^2 \to 1 \) hence we can ignore it for proving consistency. The IJ variance estimator now can be expressed as

\[
\sum_{i=1}^{n} \text{Cov}^2(N_i^*, H_k^*) = \sum_{i=1}^{n} \left( E_x[H_k^*] \sum_{j=1}^{k} 1(\xi_{ij} = \xi_i) \right) - E_x[N_i^*]E_x[H_k^*])^2 \\
= \sum_{i=1}^{n} \left( k E_x[H_k^* 1(\xi_{i1} = \xi_i)] - \frac{k}{n} U_{n,k} \right)^2 \\
= \frac{k^2}{n^2} \sum_{i=1}^{n} \left( E_x[H_k^* \xi_{i1} = \xi_i] - U_{n,k} \right)^2 \\
= \frac{k^2}{n^2} \sum_{i=1}^{n} (A_i + R_i)^2
\]

where \( \xi_{i1}, \ldots, \xi_{ik} \) are resampled from \( \xi_1, \ldots, \xi_n \) without replacement, and

\[
A_i = E_x[H_k^* | \xi_{i1} = \xi_i] - E_x[\hat{H}_k^*] \\
R_i = E_x[H_k^* \hat{H}_k^* | \xi_{i1} = \xi_i] - E_x[H_k^* - \hat{H}_k^*].
\]

First we deal with \( R_i \)'s. Lemma 13 in [52] shows that \( ER_i^2 \leq CE(H_k - \hat{H}_k)^2/n \) for some universal constant \( C \). Note that here the expectation is taken with respect to the original data \( \xi_1, \ldots, \xi_n \) rather than the resampled data. In that lemma they only conclude the weaker result \( ER_i^2 \leq C \text{Var}(H_k)/n \), however, it is obvious from their proof that \( \text{Var}(H_k) \) can be replaced by \( E(H_k - \hat{H}_k)^2 \) since \( R_i \) contains only high order ANOVA terms of \( H_k \). This bound implies

\[
E \left( \frac{k^2}{n} \sum_{i=1}^{n} R_i^2 \right) = O \left( \frac{k^2}{n^2} E(H_k - \hat{H}_k)^2 \right) = o \left( \frac{1}{n} \right)
\]

Then we analyze the \( A_i \)'s. Lemma 12 in [52] shows that \( \sum_{i=1}^{n} (A_i - (g_k(\xi_i) - W_k))^2/n = o_p(\text{Var}(g_k(\xi))) \). By Cauchy Schwartz inequality we only have to prove

\[
\frac{1}{n} \sum_{i=1}^{n} (g_k(\xi_i) - W_k)^2/\text{Var}(g_k(\xi)) \overset{p}{\to} 1
\]

in order to justify

\[
\frac{1}{n} \sum_{i=1}^{n} A_i^2/\text{Var}(g_k(\xi)) \overset{p}{\to} 1,
\]

and then consistency follows from (60), (58) and an application of Cauchy Schwartz inequality to the cross term \( \sum_{i=1}^{n} A_i R_i \). To proceed, we need the following weak law of large numbers:

**Lemma 9** (Theorem 2.2.9 from [18]). For each \( n \) let \( Y_{n,i}, 1 \leq i \leq n \) be independent. Let \( b_n > 0 \) with \( b_n \to \infty \), and let \( \hat{Y}_{n,i} = Y_{n,i}1(\text{|Y}_{n,i}| \leq b_n) \). Suppose that, as \( n \to \infty \), \( \sum_{i=1}^{n} P(|Y_{n,i}| > b_n) \to 0 \) and \( b_n^{-2} \sum_{i=1}^{n} EY_{n,i}^2 \to 0 \), then

\[
\frac{\sum_{i=1}^{n} Y_{n,i} - \sum_{i=1}^{n} E\hat{Y}_{n,i}}{b_n} \overset{p}{\to} 0.
\]
We apply the lemma to $Y_{ni} = (g_k(\xi_i) - W_k)^2/\text{Var}(g_k(\xi))$ with $b_n = n$. To verify the conditions

$$nP\left(\frac{(g_k(\xi_i) - W_k)^2}{\text{Var}(g_k(\xi))} > n\right) = nP(|g_k(\xi_i) - W_k|^{2+\delta} > (n\text{Var}(g_k(\xi)))^{1+\frac{\delta}{2}})$$

$$\leq \frac{n}{(n\text{Var}(g_k(\xi)))^{1+\frac{\delta}{2}}} E|g_k(\xi_i) - W_k|^{2+\delta} \text{ by Markov inequality}$$

$$\leq \frac{n}{(n\text{Var}(g_k(\xi)))^{1+\frac{\delta}{2}}} \frac{M}{k^{2+\delta}} \text{ by the proof of Theorem 3}$$

$$= \frac{M}{n^\frac{\delta}{2}(k^2\text{Var}(g_k(\xi)))^{1+\frac{\delta}{2}}} = O(n^{-\frac{\delta}{2}}) \to 0$$

and

$$\frac{1}{n} E \left[\frac{(g_k(\xi_i) - W_k)^4}{\text{Var}(g_k(\xi))^2} \left(\frac{(g_k(\xi_i) - W_k)^2}{\text{Var}(g_k(\xi))} \leq n\right)\right] \leq \frac{1}{n} E \left[\frac{|g_k(\xi_i) - W_k|^{2+\delta}}{(n\text{Var}(g_k(\xi)))^{1+\frac{\delta}{2}}} n^{1-\frac{\delta}{2}} \left(\frac{(g_k(\xi_i) - W_k)^2}{\text{Var}(g_k(\xi))} \leq n\right)\right]$$

$$\leq \frac{1}{n^\frac{\delta}{2}} E \left[\frac{|g_k(\xi_i) - W_k|^{2+\delta}}{(n\text{Var}(g_k(\xi)))^{1+\frac{\delta}{2}}} \left(\text{Var}(g_k(\xi)) > n\right)\right] \leq \frac{\tilde{M}}{n^\frac{\delta}{2}(k^2\text{Var}(g_k(\xi)))^{1+\frac{\delta}{2}}} \to 0.$$

It remains to show that

$$\left|1 - E \left[\frac{(g_k(\xi_i) - W_k)^2}{\text{Var}(g_k(\xi))} \left(\frac{(g_k(\xi_i) - W_k)^2}{\text{Var}(g_k(\xi))} \leq n\right)\right]\right| = \left|E \left[\frac{(g_k(\xi_i) - W_k)^2}{\text{Var}(g_k(\xi))} \left(\frac{(g_k(\xi_i) - W_k)^2}{\text{Var}(g_k(\xi))} > n\right)\right]\right|$$

$$\leq \left(\frac{1}{n} E \left[\frac{|g_k(\xi_i) - W_k|^{2+\delta}}{(n\text{Var}(g_k(\xi)))^{1+\frac{\delta}{2}}} \left(\text{Var}(g_k(\xi)) > n\right)\right]\right) \frac{1}{n} \to 0 \text{ by Markov inequality.}$$

With all these conditions verified, we can conclude (59) from Lemma 9 and complete the proof.  

\textbf{Proof of Theorem 11.} Given Theorem 10 it suffices to show that the IJ variance estimator under resampling with replacement differs by only $o_p(1/n)$ from the one without replacement. Since quantities under both resampling with and without replacement will be involved in this proof, we attach * to quantities under resampling without replacement, and ā to those with replacement. We have

$$\sum_{i=1}^{n} \text{Cov}^2(\bar{N}^*_i, H^*_k) = \frac{k^2}{n^2} \sum_{i=1}^{n} (E^*[H^*_k | \xi_i = \xi_i] - V_{n,k})^2$$

where $\xi_1, \ldots, \xi_k$ are resampled from $\xi_1, \ldots, \xi_n$ with replacement. By comparing (59) and (61) and using Cauchy Schwartz inequality

$$\left|\sum_{i=1}^{n} \text{Cov}^2(\bar{N}^*_i, H^*_k) - \sum_{i=1}^{n} \text{Cov}^2(N^*_i, H^*_k)\right| \leq \frac{k^2}{n^2} \sum_{i=1}^{n} (v_i - u_i)^2 + 2 \sqrt{\sum_{i=1}^{n} \text{Cov}^2(N^*_i, H^*_k) \cdot \frac{k^2}{n^2} \sum_{i=1}^{n} (v_i - u_i)^2}$$
where \( v_i = E_s[H^*_k | \xi_{i_1} = \xi_i] - V_n,k \) and \( u_i = E_s[H^*_k | \xi_{i_1} = \xi_i] - U_n,k \). If we show that \( E(V_n,k - U_n,k)^2 = o(1/n) \) and \( E(E_s[H^*_k | \xi_{i_1} = \xi_i] - E_s[H^*_k | \xi_{i_1} = \xi_i])^2 = o(1/n) \), then \( E[\sum_{i=1}^{n}(v_i - u_i)^2] = o(1) \) and under the condition \( k = O(n^\gamma) \) with \( \gamma < 1/2 \) we have

\[
\sum_{i=1}^{n} \text{Cov}^2_s(N^*_i, H^*_k) - \sum_{i=1}^{n} \text{Cov}^2_s(N^*_i, H^*_k) = \frac{k^2}{n^2} o_p(1) + \sqrt{O_p\left(\frac{1}{n}, \frac{k^2}{n^2}\right)} = o_p\left(\frac{1}{n}\right)
\]

which concludes the theorem.

The first error \( E(V_n,k - U_n,k)^2 = o(1/n) \) can be deduced from (48) in the proof of Theorem 4. We only need to notice that, in the setting of that proof, \( E(U_n,k - R_{n,l})^2 = O(1) \) due to Assumption 2 and that each \( E(A_{n,s} - U_{n,k})^2 = O(1/k^2) \) for \( s \geq k - l \) due to Assumption 4.

The second error \( E(E_s[H^*_k | \xi_{i_1} = \xi_i] - E_s[H^*_k | \xi_{i_1} = \xi_i])^2 = o(1/n) \) needs some further discussion. We study \( E(E_s[H^*_k | \xi_{i_1} = \xi_i] - E_s[H^*_k | \xi_{i_1} = \xi_i])^2 \) without loss of generality. Given that the first resampled data point \( \xi_{i_1} = \xi_1 \), for any fixed integer \( l \geq 0 \) we obtain the following decomposition of \( E_s[H^*_k | \xi_{i_1} = \xi_1] \) similar to that in the proof of Theorem 4

\[
n^{k-1} E_s[H^*_k | \xi_{i_1} = \xi_1] = \sum_{s=k-l}^{k-1} c(n - 1, k - 1, s) A_s + (n^{k-1} - \sum_{s=k-l}^{k-1} c(n - 1, k - 1, s)) R_l
\]

where \( A_s \) is the average of all \( H_k(\xi_1, \xi_{i_2}, \ldots, \xi_{i_k})'s \) where \( \xi_{i_2}, \ldots, \xi_{i_k} \) contain exactly \( s \) distinct data and none of them is \( \xi_1 \), and \( R_l \) is the average of all other \( H_k(\xi_1, \xi_{i_2}, \ldots, \xi_{i_k})'s \). Note that, in particular, \( A_{k-1} = E_s[H^*_k | \xi_{i_1} = \xi_1] \). We have the following analog of (48)

\[
n^{k-1} (E_s[H^*_k | \xi_{i_1} = \xi_1] - E_s[H^*_k | \xi_{i_1} = \xi_1]) = (n^{k-1} - \sum_{s=k-l}^{k-1} c(n - 1, k - 1, s)) (A_{k-1} - R_l) - \sum_{s=k-l}^{k-2} c(n - 1, k - 1, s) (A_s - A_{k-1}).
\]

Note that the coefficient of the first term does not match the form of (48), but we have

\[
n^{k-1} - \sum_{s=k-l}^{k-1} c(n - 1, k - 1, s) = n^{k-1} - (n - 1)^{k-1} + \sum_{s=1}^{k-l-2} c(n - 1, k - 1, s).
\]

Like in the proof of Theorem 4

\[
\sum_{s=1}^{k-l-2} c(n - 1, k - 1, s) = O(\left(\frac{k^2}{n}\right)^{l+1} (n - 1)^{k-1}) \cdot E(A_{k-1} - R_l)^2 = O(1)
\]

\[
c(n - 1, k - 1, s) = O(k^{2(k-1-s)} n^s) \text{ and } E(A_s - A_{k-1})^2 = O\left(\frac{1}{k^2}\right) \text{ for } s \geq k - 1 - l.
\]

Moreover by Bernoulli’s inequality \((1 + x)^r \geq 1 + rx \) for any integer \( r \geq 0 \) and real \( x \geq -1 \)

\[
n^{k-1} - (n - 1)^{k-1} = n^{k-1}(1 - (1 - \frac{1}{n})^{k-1}) \leq n^{k-2}(k - 1).
\]

With all these bounds and Minkowski inequality we get

\[
E(E_s[H^*_k | \xi_{i_1} = \xi_1] - E_s[H^*_k | \xi_{i_1} = \xi_1])^2 = O\left(\left(\frac{k^2}{n}\right)^{l+1} E(A_{k-1} - R_l)^2 + \sum_{s=k-l}^{k-2} (\frac{k^2}{n})^{2(k-1-s)} E(A_s - A_{k-1})^2\right)
\]

\[
= O\left(\left(\frac{k^2}{n}\right)^{l+1} + \frac{k^2}{n^2}\right) = o\left(\frac{1}{n}\right)
\]
when $l$ is chosen according to (49).

A.8 Proofs of Theorem 12 and Corollary 1

Proof of Theorem 12. We have two tasks. One is that $\tilde{Z}_{k_{bag}}^n - U_{k_{bag}} = o_p(1/\sqrt{n})$ when resampling without replacement, or $\tilde{Z}_{k_{bag}}^n - V_{k_{bag}} = o_p(1/\sqrt{n})$ with replacement, so that by Slutsky’s theorem the CLTs still hold with $U_{k_{bag}}$ or $V_{k_{bag}}$ replaced by their estimate $\tilde{Z}_{k_{bag}}^n$. The other thing is that

$$\left| \sum_{i=1}^n \tilde{Cov}_i(N_i^*, \tilde{Z}_k^*) - \sum_{i=1}^n Cov_i(N_i^*, H_k^*) \right| = o_p(1/n)$$

so that the variance estimation is consistent and CLTs remain valid by Slutsky’s theorem.

The first task is relatively easy. Note that $\tilde{Z}_{k_{bag}}^n$ is unbiased in either case, and

$$Var_*(\tilde{Z}_{k_{bag}}^n) = \frac{1}{B} Var_*(H_k^*) \leq \frac{1}{B} E_*H_k^{*2} \leq \frac{1}{Bn} \sum_{i=1}^n \sup_{x \in X} |h(x, \xi_i)|^2$$

where the last inequality follows from the argument used in (16). Due to Assumption 2 and the strong law of large numbers $\sum_{i=1}^n \sup_{x \in X} |h(x, \xi_i)|^2 / n \overset{P}{\to} E \sup_{x \in X} |h(x, \xi)|^2 < \infty$, hence $Var_*(\tilde{Z}_{k_{bag}}^n) = O_p(1/B)$. If $B/(kn) \to \infty$ we have

$$E_*(\tilde{Z}_{k_{bag}}^n - U_{k_{bag}})^2 = o_p(1/kn) \quad E_*(\tilde{Z}_{k_{bag}}^n - V_{k_{bag}})^2 = o_p(1/kn)$$

For a non-negative random variable, if its conditional expectation is of order $o_p(1)$, then itself is also $o_p(1)$. Therefore $(\tilde{Z}_{k_{bag}}^n - U_{k_{bag}})^2 = o_p(1/n)$ and $(\tilde{Z}_{k_{bag}}^n - V_{k_{bag}})^2 = o_p(1/n)$.

For the second task, we first deal with resampling without replacement. By Cauchy Schwartz inequality the Monte Carlo error can be bounded as

$$\left| \sum_{i=1}^n \tilde{Cov}_i(N_i^*, \tilde{Z}_k^*) - \sum_{i=1}^n Cov_i(N_i^*, H_k^*) \right| \leq \sum_{i=1}^n (\tilde{Cov}_i - Cov_i)^2 + 2 \sum_{i=1}^n Cov_i \sum_{i=1}^n (\tilde{Cov}_i - Cov_i)^2$$

where $\tilde{Cov}_i = Cov_*(N_i^*, H_k^*)$ and $\tilde{Cov}_i = \tilde{Cov}_i(N_i^*, \tilde{Z}_k^*)$ for short. Since $\sum_{i=1}^n Cov_i$ is the desired variance of order $1/n$, we only need to show $\sum_{i=1}^n (\tilde{Cov}_i - Cov_i)^2 = o_p(1/n)$. By computing variances of the sample covariances one can get

$$E_*\left[ \sum_{i=1}^n (\tilde{Cov}_i - Cov_i)^2 \right] \leq \sum_{i=1}^n \left( \frac{1}{B} E_*[(H_k^* - E_*H_k^*)^2(N_i^* - k/n)^2] + \frac{1}{B^2} Var_*(H_k^*) Var_*(N_i^*) + \frac{2}{B} Cov_i^2 \right)$$

$$\leq \frac{1}{B} E_*[(H_k^* - E_*H_k^*)^2] \sum_{i=1}^n (N_i^* - k/n)^2 + \frac{1}{B^2} Var_*(H_k^*) \sum_{i=1}^n Var_*(N_i^*) + \frac{2}{B} \sum_{i=1}^n Cov_i^2. \quad (62)$$

Note that $\sum_{i=1}^n Cov_i^2 = O_p(1/n)$, $Var_*(H_k^*) = O_p(1)$, and $\sum_{i=1}^n (N_i^* - k/n)^2 = k(n-k)/n$, $Var_*(N_i^*) = k(n-k)/n^2$ since $N_i^* = 0$ or $1$ and $\sum_{i=1}^n N_i^* = k$. With all these bounds, we have

$$E_*\left[ \sum_{i=1}^n (\tilde{Cov}_i - Cov_i)^2 \right] = O_p\left( \frac{k}{B} + \frac{k}{B^2} + \frac{1}{Bn} \right) = O_p\left( \frac{k}{B} \right).$$
If \( B/(kn) \to \infty \), then \( E_*\left[ \sum_{i=1}^{n} (\text{Cov}_i - \text{Cov}_*)^2 \right] = o_p(1/n) \), which implies \( \sum_{i=1}^{n} (\text{Cov}_i - \text{Cov}_*)^2 = o_p(1/n) \).

In the case of resampling with replacement, we have the same bound (62), where \( \text{Var}_*(N_i^*) = k(n-1)/n^2 \) and \( \text{Var}_*(H_i^*) = O_p(1) \). However, the first term becomes more complicated. We bound the first term by a conditioning argument on \( N_i^* \)

\[
E_*[H_k^2 | N_i^* = s] \leq E_*\left[ (\frac{1}{k} \sum_{j=1}^{k} \sup_{x \in X} |h(x, \xi_j)|)^2 | N_i^* = s \right]
\]

\[= \frac{s}{k} \sup_{x \in X} |h(x, \xi)|^2 + \frac{k-s}{k} \frac{1}{n-1} \sum_{j \neq i} \sup_{x \in X} |h(x, \xi_j)|^2 \quad \text{by Minkowski inequality}
\]

therefore

\[
E_*[H_k^2 (N_i^* - \frac{k}{n})^2] = \sum_{s=0}^{k} E_*[H_k^2 | N_i^* = s] (s - \frac{k}{n})^2 P(N_i^* = s)
\]

\[\leq \sum_{s=0}^{k} \left( \frac{s}{k} \sup_{x \in X} |h(x, \xi)|^2 + \frac{k-s}{k} \frac{1}{n-1} \sum_{j \neq i} \sup_{x \in X} |h(x, \xi_j)|^2 \right) (s - \frac{k}{n})^2 P(N_i^* = s).
\]

Now we have

\[
E_*[H_k^2 \sum_{i=1}^{n} (N_i^* - \frac{k}{n})^2] = \sum_{i=1}^{n} E_*[H_k^2 (N_i^* - \frac{k}{n})^2]
\]

\[\leq \sum_{i=1}^{k} \left( \sum_{i=1}^{n} \sup_{x \in X} |h(x, \xi)|^2 \right) (s - \frac{k}{n})^2 P(N_i^* = s)
\]

\[\leq \sum_{i=1}^{n} \sup_{x \in X} |h(x, \xi)|^2 \text{Var}_*(N_i^*) = O_p(k)
\]

and the first term can be bounded as

\[
E_*[(H_k^* - E_*H_k^*)^2 \sum_{i=1}^{n} (N_i^* - \frac{k}{n})^2] \leq 2E_*[H_k^2 \sum_{i=1}^{n} (N_i^* - \frac{k}{n})^2] + 2(E_*H_k^*)^2 E_*[\sum_{i=1}^{n} (N_i^* - \frac{k}{n})^2] = O_p(k).
\]

With these bounds, \( E_*[\sum_{i=1}^{n} (\text{Cov}_i - \text{Cov}_*)^2] = O_p(k/B) \) and the conclusion follows.

**Proof of Corollary** 4 From Theorem 12 we have

\[
P \left( \frac{Z_{k}^{\text{bag}} - WK}{\hat{\sigma}_{IJ}} \leq Z_{1-\alpha} \right) \to 1 - \alpha \quad (63)
\]

Note that

\[
P \left( \frac{Z_{k}^{\text{bag}} - WK}{\hat{\sigma}_{IJ}} \leq z_{1-\alpha} \right) = P \left( Z_{k}^{\text{bag}} - Z_{1-\alpha} \hat{\sigma}_{IJ} \leq W_K \right) \leq P \left( Z_{k}^{\text{bag}} - Z_{1-\alpha} \hat{\sigma}_{IJ} \leq Z^* \right) \quad (64)
\]

by (63). Combining (63) and (64) gives the conclusion.
Acknowledgements
We gratefully acknowledge support from the National Science Foundation under grants CMMI-1542020, CMMI-1523453 and CAREER CMMI-1653339/1834710.

References
[1] G. Bayraksan and D. P. Morton. Assessing solution quality in stochastic programs. *Mathematical Programming*, 108(2-3):495–514, 2006.
[2] G. Bayraksan and D. P. Morton. A sequential sampling procedure for stochastic programming. *Operations Research*, 59(4):898–913, 2011.
[3] A. Ben-Tal, D. Den Hertog, A. De Waegenaere, B. Melenberg, and G. Rennen. Robust solutions of optimization problems affected by uncertain probabilities. *Management Science*, 59(2):341–357, 2013.
[4] D. Bertsimas, V. Gupta, and N. Kallus. Robust sample average approximation. *Mathematical Programming*, 171(1-2):217–282, 2018.
[5] J. R. Birge and F. Louveaux. *Introduction to Stochastic Programming*. Springer Science & Business Media, 2011.
[6] J. Blanchet, Y. Kang, and K. Murthy. Robust Wasserstein profile inference and applications to machine learning. *arXiv preprint arXiv:1610.05627*, 2016.
[7] G. Blom. Some properties of incomplete U-statistics. *Biometrika*, 63(3):573–580, 1976.
[8] V. S. Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*, volume 48. Springer, 2009.
[9] L. Breiman. Bagging predictors. *Machine learning*, 24(2):123–140, 1996.
[10] P. Bühlmann and B. Yu. Analyzing bagging. *The Annals of Statistics*, 30(4):927–961, 2002.
[11] G. C. Calafiore. Repetitive scenario design. *IEEE Transactions on Automatic Control*, 62(3):1125–1137, 2017.
[12] A. Carè, S. Garatti, and M. C. Campi. FAST – Fast algorithm for the scenario technique. *Operations Research*, 62(3):662–671, 2014.
[13] V. De la Peña and E. Giné. *Decoupling: From Dependence to Independence*. Springer Science & Business Media, 2012.
[14] E. Delage and Y. Ye. Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Operations Research*, 58(3):595–612, 2010.
[15] D. Dentcheva, S. Penev, and A. Ruszczyński. Statistical estimation of composite risk functionals and risk optimization problems. *Annals of the Institute of Statistical Mathematics*, 69(4):737–760, 2017.
[16] J. Ding, R. Eldan, and A. Zhai. On multiple peaks and moderate deviations for the supremum of a gaussian field. *The Annals of Probability*, 43(6):3468–3493, 2015.

[17] J. Duchi, P. Glynn, and H. Namkoong. Statistics of robust optimization: A generalized empirical likelihood approach. *arXiv preprint arXiv:1610.03425*, 2016.

[18] R. Durrett. *Probability: Theory and Examples*. Cambridge university press, 2010.

[19] B. Efron. Estimation and accuracy after model selection. *Journal of the American Statistical Association*, 109(507):991–1007, 2014.

[20] B. Efron and C. Stein. The jackknife estimate of variance. *The Annals of Statistics*, 9(3):586–596, 1981.

[21] A. Eichhorn and W. Römisch. Stochastic integer programming: Limit theorems and confidence intervals. *Mathematics of Operations Research*, 32(1):118–135, 2007.

[22] E. W. Frees. Infinite order U-statistics. *Scandinavian Journal of Statistics*, 16(1):29–45, 1989.

[23] J. Friedman, T. Hastie, and R. Tibshirani. *The Elements of Statistical Learning*, volume 1. Springer series in statistics New York, NY, USA:, 2001.

[24] S. Ghadimi and G. Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013.

[25] P. Glasserman. *Monte Carlo Methods in Financial Engineering*, volume 53. Springer Science & Business Media, 2013.

[26] F. R. Hampel. The influence curve and its role in robust estimation. *Journal of the American Statistical Association*, 69(346):383–393, 1974.

[27] S. Janson. The asymptotic distributions of incomplete U-statistics. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 66(4):495–505, 1984.

[28] A. J. Kleywegt, A. Shapiro, and T. Homem-de-Mello. The sample average approximation method for stochastic discrete optimization. *SIAM Journal on Optimization*, 12(2):479–502, 2002.

[29] H. Kushner and G. G. Yin. *Stochastic Approximation and Recursive Algorithms and Applications*, volume 35. Springer Science & Business Media, 2003.

[30] H. Lam and H. Qian. Assessing solution quality in stochastic optimization via bootstrap aggregating. In *Proceedings of the Winter Simulation Conference (to appear)*, 2018.

[31] H. Lam and E. Zhou. The empirical likelihood approach to quantifying uncertainty in sample average approximation. *Operations Research Letters*, 45(4):301–307, 2017.

[32] G. Lan, A. Nemirovski, and A. Shapiro. Validation analysis of mirror descent stochastic approximation method. *Mathematical programming*, 134(2):425–458, 2012.

[33] J. Lee. *U-Statistics: Theory and Practice*. Marcel Dekker, 1990.
[34] D. Love and G. Bayraksan. Overlapping batches for the assessment of solution quality in stochastic programs. In *Proceedings of the Winter Simulation Conference*, pages 4184–4195, 2011.

[35] D. Love and G. Bayraksan. Overlapping batches for the assessment of solution quality in stochastic programs. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 25(3):20, 2015.

[36] J. Luedtke and S. Ahmed. A sample approximation approach for optimization with probabilistic constraints. *SIAM Journal on Optimization*, 19(2):674–699, 2008.

[37] W.-K. Mak, D. P. Morton, and R. K. Wood. Monte Carlo bounding techniques for determining solution quality in stochastic programs. *Operations Research Letters*, 24(1-2):47–56, 1999.

[38] L. Mentch and G. Hooker. Quantifying uncertainty in random forests via confidence intervals and hypothesis tests. *The Journal of Machine Learning Research*, 17(1):841–881, 2016.

[39] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on Optimization*, 19(4):1574–1609, 2009.

[40] B. Pagnoncelli, S. Ahmed, and A. Shapiro. Sample average approximation method for chance constrained programming: theory and applications. *Journal of Optimization Theory and Applications*, 142(2):399–416, 2009.

[41] A. Partani. *Adaptive Jacknife Estimators for Stochastic Programming*. PhD thesis, 2007.

[42] A. Partani, D. P. Morton, and I. Popova. Jackknife estimators for reducing bias in asset allocation. In *Proceedings of the 38th Winter Simulation Conference*, pages 783–791, 2006.

[43] B. C. Rennie and A. J. Dobson. On Stirling numbers of the second kind. *Journal of Combinatorial Theory*, 7(2):116–121, 1969.

[44] R. J. Serfling. *Approximation Theorems of Mathematical Statistics*, volume 162. John Wiley & Sons, 2009.

[45] A. Shapiro. Monte Carlo sampling methods. *Handbooks in Operations Research and Management Science*, 10:353–425, 2003.

[46] A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on Stochastic Programming: Modeling and Theory*. SIAM, 2009.

[47] A. Shapiro and T. Homem-de-Mello. On the rate of convergence of optimal solutions of Monte Carlo approximations of stochastic programs. *SIAM Journal on Optimization*, 11(1):70–86, 2000.

[48] A. Shapiro and A. Nemirovski. On complexity of stochastic programming problems. In *Continuous Optimization*, pages 111–146. Springer, 2005.

[49] R. Stockbridge and G. Bayraksan. A probability metrics approach for reducing the bias of optimality gap estimators in two-stage stochastic linear programming. *Mathematical Programming*, 142(1-2):107–131, 2013.
[50] A. W. Van der Vaart. *Asymptotic Statistics*, volume 3. Cambridge University Press, 2000.

[51] A. W. Van der Vaart and J. A. Wellner. *Weak Convergence and Empirical Processes with Applications to Statistics*. Springer, 1996.

[52] S. Wager and S. Athey. Estimation and inference of heterogeneous treatment effects using random forests. *Journal of the American Statistical Association*, (just-accepted), 2017.

[53] S. Wager, T. Hastie, and B. Efron. Confidence intervals for random forests: the jackknife and the infinitesimal jackknife. *Journal of Machine Learning Research*, 15(1):1625–1651, 2014.

[54] W. Wang and S. Ahmed. Sample average approximation of expected value constrained stochastic programs. *Operations Research Letters*, 36(5):515–519, 2008.

[55] W. Wiesemann, D. Kuhn, and M. Sim. Distributionally robust convex optimization. *Operations Research*, 62(6):1358–1376, 2014.