The plasma-solid transition

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Abstract: Using a criterion proposed by Salpeter and standard solid-state physics, we have determined conditions for the occurrence of the plasma-solid transition. Possible astrophysical applications are discussed.

Introduction

Phase transitions of various kinds have been, and still are, actively contributing to important changes occurring in the world around us. The aim of the present paper is a contribution to the study of the conditions for the occurrence of the so-called “plasma-solid (PS)” transition.

Simple physical reasoning shows that the PS transition must certainly occur in various astrophysical settings, such as in proto-planetary and protostellar clouds. On the one hand, it is widely known that the universe contains different kinds of plasmas. On the other hand, it is equally widely known that solid objects also exist in the universe. This implies that there must exist a physical region where a transition between the two regimes takes place.

In our preliminary work on the PS transition \cite{1}, we have determined the conditions for its occurrence in two simple idealized systems: a pure Fermi-Dirac and a pure Bose-Einstein gas. The object of the present calculation is again a determination of the conditions for the occurrence of the PS transition, but this time on a more realistic footing. Our starting point is the criterion for the occurrence of the PS transition (although it was not called by that name) proposed by Salpeter \cite{2}.

Salpeter’s paper was devoted to a thorough discussion of a zero-temperature plasma. In its third part, he considered a system of positive ions of given charge and mass, rigidly fixed in the nodes of a perfect crystal lattice, and went on to estimate the zero point energy of the ions. He showed that the behavior of such a system can be described by the ratio

\[ \frac{E_{\text{zero}}}{E_{\text{box}}} \]
where \( E_{z,p} \) denotes the zero point energy of the ions and \( E_C \) is the Coulomb energy. According to the analysis presented in [2], a PS transition occurs for \( f = 1 \). Calculations reported in [2] were estimates, and strictly valid only for \( T = 0 \text{ K} \). Our calculations reported in the following section go beyond those assumptions in at least two aspects: (i) we have not used estimates but exact calculations, and (ii) our results take into account the influence of temperature.

Calculations

Finding a general expression for the energy of a real solid that takes into account most (if not all) of its characteristics is a formidable problem in solid state physics (for example [3], [4]). Various approximations to the complete solution exist; one of them is the Debye model of a solid. Within the Debye model, the energy per mole of a solid is given by the following expression [3]:

\[
E = N \left\{ u(v) + 9nk_B T \left[ \left( \frac{T}{\theta} \right)^3 \int_0^{\theta/T} \left( \frac{1}{2} + \frac{1}{\exp(\xi) - 1} \right) \xi^3 d\xi \right] \right\},
\]

where \( N \) denotes the number of elementary cells in a mole of the material, \( n \) the number of particles per elementary cell, \( u(v) = \frac{U}{N} \) = the static lattice energy per cell, \( \theta \) the Debye temperature, \( k_B \) the Boltzmann constant, and in the following we will use the convention \( k_B = 1 \). The second integral in Eq. (2) can be solved as [5]

\[
I_1 = \int_0^x \frac{t^n dt}{e^t - 1} = x^n \left[ \frac{1}{n} - \frac{x}{2(n + 1)} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k+n)(2k)!} \right]
\]

under the condition \( n \geq 1, |x| < 2\pi \). Here, the symbol \( B_{2k} \) denotes Bernoulli’s numbers. It follows from Eq. (2) that the energy per particle of a solid within the Debye model is given by

\[
E_p = 9T \left( \frac{T}{\theta} \right)^3 I
\]
where
\[ I = \frac{1}{2} \int_0^{\theta/T} \xi^3 d\xi + \int_0^{\theta/T} \frac{\xi^3 d\xi}{e^\xi - 1} \]  \hspace{1cm} (5)

Using Eq. (3) in Eq. (5), one finally gets
\[ \left( \frac{T}{\theta} \right)^3 I = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k + 3)(2k)!} \left( \frac{\theta}{T} \right)^{2k}, \]  \hspace{1cm} (6)

which implies that
\[ E_p = 3T \left[ 1 + 3 \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k + 3)(2k)!} \left( \frac{\theta}{T} \right)^{2k} \right] \]  \hspace{1cm} (7)

or, expanding explicitly
\[ E_p \cong 3T \left[ 1 + \frac{1}{20} \left( \frac{\theta}{T} \right)^2 - \frac{1}{1680} \left( \frac{\theta}{T} \right)^4 + \ldots \right]. \]  \hspace{1cm} (8)

The energy of a Fermi-Dirac gas of particles of mass \( m \) and spin \( s \) enclosed in a volume \( V \) is given by [6]
\[ E_e = \frac{gV m^{3/2}}{2^{1/2} \pi^2 \hbar^3} F_{3/2} (\beta \mu) \]  \hspace{1cm} (9)

All the symbols in this equation have their usual meaning, in particular, \( g = 2s + 1 \), and \( F_{3/2} (\beta \mu) \) is a particular case of a Fermi integral
\[ F_k (\beta \mu) = \int_0^\infty \frac{e^k d\epsilon}{1 + \exp [\beta (\epsilon - \mu)]}. \]  \hspace{1cm} (10)

Eq. (10) can be developed as [7]
\[ F_k (\beta \mu) = \frac{\mu^{k+1}}{k+1} + \frac{1}{T} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \mu^n} \left[ \mu^k \right] \left[ 1 - (-1)^n \right] \left[ 1 - 2^{-n} \right] T^n \Gamma [n+1] \varsigma [n+1], \]  \hspace{1cm} (11)

where \( \Gamma \) and \( \varsigma \) are the gamma function and Riemann zeta function, respectively.
Expanding to fourth order and inserting the temperature dependence of the chemical potential, one gets the following result for the energy per particle of the electron gas

\[
\frac{E_e}{N} = \frac{g m^{3/2} 2^{1/2} T^{5/2}}{5 \mu_0^{5/2}} \left[ 1 + \frac{5\pi^2}{12} \left( \frac{T}{\mu_0} \right)^2 - \frac{\pi^4}{36} \left( \frac{T}{\mu_0} \right)^4 \right]
\]

(12)

The chemical potential of the electron gas at \( T = 0 \) K is denoted here by \( \mu_0 = An^{2/3} \), with \( A = (3\pi^2)^{2/3} h^2/2m \).

Introducing \( B = (2^{1/2}/5) g m^{3/2}/(\pi^4 h^3) \) simplifies the notation in Eq. (12).

Discussion

Salpeter’s ratio defined in Eq. (1) can be formed using Eqs. (8) and (12). The result obtained this way can be expressed as follows

\[
f \simeq \frac{9}{140 BT^3} \frac{\mu_0^{3/2}}{36\mu_0^4} \left[ 1680T^4 + 84(T\theta)^2 - \theta^4 \right] \quad (13)
\]

Imposing the condition \( f = 1 \) on Eq.(13), it becomes possible to determine the values of the parameters of a PS transition. Solving Eq. (13) for the Debye temperature \( \theta \), and introducing in the solution the density dependence of the chemical potential at \( T = 0 \) K one obtains the following expression

\[
\theta = \left( \frac{2}{3} \right)^{1/2} \left[ 63T^2 + \frac{1}{A^2 n^4} (\sqrt{7 An^{2/3}}(1107n^2 A^3 T^4 + 5\sqrt{A n^{2/3}}(ABn^{2/3} \pi^4 T_7 - 36A^5 Bn^{10/3}T^3 - 15 A^3 B \pi^3 n^2 T^5))^{1/3} \right]^{1/2}.
\]

(14)

This equation links the relevant parameters of a plasma (number density and temperature) with those of the solid (Debye temperature) that can condense from it. We have thus obtained the equation of state of a system undergoing a PS transition. Apart from being interesting from the point of view of pure statistical physics, this result can find astrophysical applications in studies of dust and gas clouds. For recent examples of observational studies of such clouds, see for example [8] [9].

The number density and the temperature of an astrophysical cloud can be determined from observations using the methods of plasma physics, such
as the analysis of spectral line broadening, or in some cases, in-situ measurements of interplanetary gas and dust. Once these values are known for a given cloud, the Debye temperature of the solid material that can condense from it is given by Eq. (14). The Debye temperature is a unique characteristic of every solid, which gives hope that the method discussed here could lead to a possibility to determine the chemical composition of a condensing proto-planetary system. Such a determination would of course be only indirect, because after calculating the value of $\theta$ from Eq. (14), one would have to rely on an identification of the material (or a mixture of materials) based on the corresponding calculated value.

Consider Eq. (14) in the limiting case $T \to 0$, $\mu_0 \to 0$. Astronomically, this limit corresponds to an interstellar cloud sufficiently far away from any bright star. Developing Eq. (14) into series in the chemical potential and retaining just the first two terms, one gets the following expression for the Debye temperature

$$\theta = K_1 T^{7/6} \mu_0^{-1/2} + K_2 T^{5/6} \mu_0^{1/2}$$

(15)

The numerical values of the constants in Eq. (15) can be determined by developing Eq. (14) into a series. By doing this, it follows that $K_1 = (2/3)^{1/2} 5^{1/6} \pi^{1/4} 2^{3/2} B^{1/6}$ and $K_2 = 3^{3/2} 7^{3/4} 2^{-1/2} \pi^{-2/3} (5B)^{-1/6}$.

Here, the constant $B$ is the one defined after Eq. (12). For sufficiently small values of the chemical potential, Eq. (15) can be simplified into

$$\theta = K_1 T^{7/6} \mu_0^{-1/2}$$

(16)

Inserting the values of all the constants which occur in Eq. (16), one finally arrives at

$$\theta = 50.8 T^{7/6} n^{-1/3}$$

(17)

This expression has two possible applications. It can be used for the calculation of $\theta$ if the temperature and the number density of the cloud are known. Typical values of the temperature and density of the molecular clouds in our Galaxy are $20 < T [K] < 60$ and $10^3 < n [\text{cm}^{-3}] < 10^5$. Inserting in Eq. (17) $n = 2000 \text{cm}^{-3}$ and $T = 43 \text{ K}$ gives $\theta = 320 \text{ K}$, which is nearly the experimentally known value of the Debye temperature of the chemical element magnesium. This result could be interpreted as indication that in a molecular cloud, grains of Mg can condense under these densities and temperatures.
In a similar way, the pair of values $n = 10^4 \text{cm}^{-3}$, $T = 120 \text{ K}$ would lead to $	heta = 628 \text{ K}$. Such a value is only slightly higher than the experimental value for the element Si. Note that this last result has a direct significance for the interpretation of observations, because emission from silicate grains has indeed been detected [10]. Turning the argument around, if solid particles are observed in a cloud, and if their chemical composition can be determined, it becomes possible to calculate the Debye temperature from the principles of solid state physics. If, in addition, temperature is known from spectroscopy, one can determine the value of the chemical potential, and finally, the number density of a cloud from Eq. (16). Further work along these lines is in progress and will be discussed elsewhere.

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References

[1] V. Celebonovic and W. Däppen: in: Contributed papers of the 20th SPIG Conference, (ed. Z.Lj.Petrovic, M.M.Kuraica, N.Bibic and G.Malovic), p.527, published by the Inst. of Physics, Faculty of Physics and INN. Vinca, Beograd, Yugoslavia (2000) (preprint available at LANL: astro-ph/0007337).

[2] E.E. Salpeter: Astrophys.J., 134, 669 (1961).  
[3] M. Born and K. Huang: Dynamical theory of crystal lattices, OUP, Oxford (1968).  
[4] A. Davydov: Theorie du Solide, Editions ”Mir”, Moscow (1980).  
[5] M. Abramowitz and I. A. Stegun: Handbook of mathematical functions, Dover Publications Inc., New York (1972).  
[6] L. D. Landau and E. M. Lifchitz: Statistical Physics, Vol. 1, Nauka, Moscow (1976) (in Russian).  
[7] V. Celebonovic: Publ. Astron. Obs. Belgrade, 60, 16, (1998). (preprint available at LANL: astro-ph/9802279).  
[8] N. E. Jessop and D. Ward-Thompson: Mon. Not. R. Astr. Soc. (in press) (preprint available at LANL: astro-ph/0012093).  
[9] E. Grun, H. Krueger and M. Landgraf: in: The Heliosphere at Solar Minimum: The Ulysses Perspective, eds. A. Balogh, R. Marsden, E. Smith, Springer-Praxis, Heidelberg (in press) (preprint available at LANL: astro-ph/0012224).  
[10] D. Cesarsky, A. P. Jones, L. Lequeux, L. Verstrate: Astron. Astrophys., 358, 708 (2000).