WEIGHT DISTRIBUTION OF CYCLIC CODES DEFINED BY QUADRATIC FORMS AND RELATED CURVES
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ABSTRACT. We consider cyclic codes \( C \) associated to quadratic trace forms in \( m \) variables \( Q(x) = \text{Tr}_{q^m/q}(xR(x)) \) determined by a family \( \mathcal{L} \) of \( q \)-linearized polynomials \( R \) over \( \mathbb{F}_{q^m} \), and three related codes \( \mathcal{C}_{L,0}, \mathcal{C}_{L,1} \) and \( \mathcal{C}_{L,2} \). We describe the spectra for all these codes when \( \mathcal{L} \) is an even rank family, in terms of the distribution of ranks of the forms \( Qh \) in the family \( \mathcal{L} \), and we also compute the complete weight enumerator for \( \mathcal{C} \). Let \( \text{Tr}_q \) be the trace function from \( \mathbb{F}_{q^n} \) to \( \mathbb{F}_q \). We consider cyclic codes \( Q \) of length \( n \) over \( \mathbb{F}_q \) as a subspace of \( \mathbb{F}_{q^n} \) of dimension \( k \). If \( C \) has minimal distance \( d = \min \{d(c, c') : c, c' \in \mathcal{C}, c \neq c'\} \), where \( d(c, c') \) is the Hamming distance in \( \mathbb{F}_q^n \), then \( C \) is called an \([n, k, d]\)-code. One of the most important families of codes are the cyclic ones. A code is cyclic if given a codeword \( c = (c_1, \ldots, c_n) \in \mathcal{C} \) the cyclic shift \( s(c) = (c_n, c_1, \ldots, c_{n-1}) \) is also in \( \mathcal{C} \). The weight of \( c \in \mathcal{C} \) is \( w(c) = \#\{0 \leq i \leq n : c_i \neq 0\} \); that is, the number of non-zero coordinates of \( c \). For \( 0 \leq i \leq n \) the numbers \( A_i = \#\{c \in \mathcal{C} : w(c) = i\} \) are called the frequencies and the sequence \( \text{Spec}(C) = (A_0, A_1, \ldots, A_n) \) is called the weight distribution or the spectrum of \( C \). A good reference for general coding theory is the book [7].

Fix \( \alpha \) a generator of \( \mathbb{F}_{q^n}^* \). Consider \( h(x) = h_1(x) \cdots h_t(x) \in \mathbb{F}_q[x] \) where \( h_j(x) \) are different irreducible polynomials over \( \mathbb{F}_q \). For each \( j = 1, \ldots, t \), let \( g_j = \alpha^{n_j} \) be a root of \( h_j(x) \), \( n_j \) be the order of \( q_j \) and \( m_j \) be the minimum positive integer such that \( q_j^{m_j} \equiv 1 \pmod{n_j} \). Then, \( \deg(h_j(x)) = m_j \) for all \( j \). Put \( n = \frac{q^m-1}{\delta} \) where \( \delta = \gcd(q^m-1, s_1, \ldots, s_t) \). Then, by Delsarte’s Theorem of trace and duals (2), the \( q \)-ary code \( C = \{c(a_1, \ldots, a_t) : a_j \in \mathbb{F}_{q^{m_j}} \} \) with

\[
(1.1) \quad c(a_1, \ldots, a_t) = \left( \sum_{j=1}^t \text{Tr}_{q^{m_j}/q}(a_j) \text{Tr}_{q^{m_j}/q}(a_j g_j) \cdots \text{Tr}_{q^{m_j}/q}(a_j g_j^{-1}) \right),
\]

where \( \text{Tr}_{q^{m_j}/q} \) is the trace function from \( \mathbb{F}_{q^{m_j}} \) to \( \mathbb{F}_q \), is an \([n, k]\)-cyclic code with check polynomial \( h(x) \) and dimension \( k = m_1 + \cdots + m_t \).

The computation of the spectra of cyclic codes is in general a difficult task. The recent survey [3] of Dinh, Li and Yue shows the progress made on this problem in the last 20 years using different techniques: exponential sums, special nonlinear functions over finite fields, quadratic forms, Hermitian forms graphs, Cayley graphs, Gauss and Kloosterman sums. In [5], Feng and Luo computed the weight distribution of the cyclic code of length \( n = p^m - 1 \) with zeros \( \{\alpha^{-1}, \alpha^{-(p^d+1)}\} \),

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where $\alpha$ is a generator of $\mathbb{F}_p$, $\ell \geq 0$ and $m/(m, \ell)$ odd, by using a perfect nonlinear function. In another work \cite{11}, they used quadratic forms to calculate the weight distribution of the cyclic codes with zeros $\{\alpha^{-2}, \alpha^{-\ell^2+1}\}$ and $\{\alpha^{-\ell}, \alpha^{-\ell^2+1}\}$, respectively, when $p$ is an odd prime and $(m, \ell) = 1$. These methods were used by other authors to calculate the spectra of other cyclic codes over $\mathbb{F}_p$ when $p$ is an odd prime. All these results are summarized in Theorem 2.4 in \cite{3}.

In this paper, we will explicitly compute the weight distributions of some general families of cyclic codes over $\mathbb{F}_q$. In particular, we will compute the spectra of cyclic codes with zeros $\{\alpha^{-\ell^2+1}\}$ and $\{\alpha^{-\ell}, \alpha^{-\ell^2+1}\}$ and $\{1,\alpha^{-\ell}, \alpha^{-\ell^2+1}\}$ in all characteristics, where $\alpha$ is a generator of $\mathbb{F}_{q^m}$ and $m/(m, \ell)$ is even (i.e. new cases not covered in \cite{14} and more general ones), by using quadratic forms and related exponential sums.

We now give a brief summary of the results in the paper. In Section 2 we recall quadratic forms $Q$ in $m$ variables over finite fields and their absolute invariants: the rank and the type. We define certain exponential sums $S_{Q,\beta}$ and compute their values and distributions (Lemma \ref{lemma}). We then consider the particular quadratic form $Q_{\gamma, \ell}(x) = \text{Tr}_{q^m/q}(\gamma x^{\ell+1})$, with $\gamma \in \mathbb{F}_q$, $\ell \in \mathbb{N}$. We recall the distribution of rank and types given by Klapper in \cite{8} and \cite{9}. These facts will be later used (Sections 3–5) to compute the spectra of some families of cyclic codes.

In the next section, we consider cyclic codes defined by general quadratics forms determined by $g$-linearized polynomials and compute their spectra in some cases. More precisely, we consider $\mathcal{L} = \langle x^{q^1}, x^{q^2}, \ldots, x^{q^s} \rangle \subset \mathbb{F}_{q^m}[x]$, the associated code

$$C_{\mathcal{L}} = \{(\text{Tr}_{q^m/q}(xR(x)))_{x \in \mathbb{F}_{q^m}} : R \in \mathcal{L}\}$$

and three related codes $C_{\mathcal{L},0}$, $C_{\mathcal{L},1}$ and $C_{\mathcal{L},2}$ (see \ref{thm}). If $\mathcal{L}$ is an even rank family (see Definition \ref{def}), we give the weight distributions of $C_{\mathcal{L},0}$, $C_{\mathcal{L},1}$ and $C_{\mathcal{L},2}$ (see Theorems \ref{thm} and \ref{thm} and Tables 1 to 4). In Proposition \ref{prop}, we also give the complete weight enumerator of $C_{\mathcal{L}}$.

In the next sections we consider two particular even rank families: $\mathcal{L} = \langle x^{q^s} \rangle$ and $\mathcal{L} = \langle x^{q^1}, x^{q^2} \rangle$, with $\ell \in \mathbb{N}$. In Section 4, we compute the spectrum of the code $C_{\mathcal{L}}$ defined by the family of quadratic forms $Q_{\gamma, \ell}(x) = \text{Tr}_{q^m/q}(\gamma x^{\ell+1})$, $\gamma \in \mathbb{F}_{q^m}$, and the spectra of the related codes $C_{\ell,0}$, $C_{\ell,1}$ and $C_{\ell,2}$ (see Theorems \ref{thm} and \ref{thm} and Tables 5–8). As a consequence, $C_{\mathcal{L}}$ turns out to be a 2-weight code. The complete weight enumerator of $C_{\mathcal{L}}$ is given in Corollary \ref{cor}. In Section 5 we obtain similar results for the codes $C_{\ell,3\ell}$, $C_{\ell,3\ell,0}$, $C_{\ell,3\ell,1}$ and $C_{\ell,3\ell,2}$ (see Theorems \ref{thm} and \ref{thm} and Tables 9 and 10).

In the last section, we consider Artin-Schreier curves of the form

$$C_{R, \beta} : y^p - y = xR(x) + \beta x$$

where $p$ is prime, $\beta \in \mathbb{F}_{p^m}$ and $R$ is a $p$-linearized polynomial over $\mathbb{F}_{p^m}$. In Proposition 6.1 we give simple necessary and sufficient conditions for these curves to be optimal, that is curves attaining the equality in the Hasse-Weil bound, in terms of the degree of $R$ and the rank $r$ of the associated quadratic form $Q_R(x) = \text{Tr}_{p^m/p}(xR(x))$. We then show in Theorem 6.3 that there are several maximal and minimal curves in the family

$$y^p - y = \gamma x^{p^s+1} + \beta x, \quad \gamma \in \mathbb{F}_{p^m}^*, \quad \beta \in \mathbb{F}_{p^m}.$$ 

In the binary case $p = 2$, Van der Geer and Van der Vlugt have found the same curves for $\ell = 1$ and $\beta = 0$ (see \cite{12}). Thus, we extend their result for any $p$, $\ell$ and $\beta$. We also show the existence of optimal curves of the form

$$x^p - y = \gamma_1 x^{p^s+1} + \gamma_3 x^{p^{2s}+1} + \beta x$$

with $\gamma_1, \gamma_3 \in \mathbb{F}_{p^m}^*$, $\beta \in \mathbb{F}_{p^m}$. 

2. Quadratic forms over finite fields and exponential sums

A quadratic form in $\mathbb{F}_q^m$ is an homogeneous polynomial $q(x)$ in $\mathbb{F}_q^m[x]$ of degree 2. We want to consider more general functions. Any function

$$Q : \mathbb{F}_q^m \rightarrow \mathbb{F}_q$$

can be identified with a polynomial of $m$ variables over $\mathbb{F}_q$ via an isomorphism $\mathbb{F}_q^m \simeq \mathbb{F}_q^m$ of $\mathbb{F}_q$-vector spaces. Such $Q$ is said to be a quadratic form if the corresponding polynomial is homogeneous of degree 2. The rank of $Q$, is the minimum number $r$ of variables needed to represent $Q$ as a polynomial in several variables. Alternatively, the rank of $Q$ can be computed as the codimension of the $\mathbb{F}_q$-vector space $V = \{ y \in \mathbb{F}_q^m : Q(y) = 0, Q(x+y) = Q(x), \forall x \in \mathbb{F}_q^m \}$. That is $|V| = q^{m-r}$. Two quadratic forms $Q_1, Q_2$ are equivalent if there is an invertible $\mathbb{F}_q$-linear function $S : \mathbb{F}_q^m \rightarrow \mathbb{F}_q^m$ such that $Q_1(x) = Q_2(S(x))$.

Fix $Q$ a quadratic form from $\mathbb{F}_q^m$ to $\mathbb{F}_q$. It will be convenient to consider, for each $\beta \in \mathbb{F}_q^m$ and $\xi \in \mathbb{F}_q$, the number

$$N_{Q,\beta}(\xi) = \# \{ x \in \mathbb{F}_q^m : Q(x) + \text{Tr}_{q^m/q}(\beta x) = \xi \}.$$  

We will abbreviate $N_Q(\xi) = N_{Q,0}(\xi)$, $N_{Q,\beta} = N_{Q,\beta}(0)$ and $N_Q = N_Q(0) = \# \ker Q$. It is a classic result that quadratic forms over finite fields are classified in three different equivalent classes. This classification depends on the parity of the characteristic (see for instance [10]). But in both characteristics (even or odd), there are 2 classes with even rank (usually called type 1 and 3) and one of odd rank. For even rank, we will use the notation

$$\varepsilon_Q = \begin{cases} +1 & \text{if } Q \text{ is of type 1,} \\ -1 & \text{if } Q \text{ is of type 3,} \end{cases}$$

and call this sign the type of $Q$. The number $N_Q(\xi)$ does not depend on the characteristic and it is given by

$$N_Q(\xi) = q^{m-1} + \varepsilon_Q \nu(\xi) q^{m-\frac{r-1}{2}}$$

where $\nu(0) = q - 1$ and $\nu(z) = 1$ if $z \in \mathbb{F}_q^*$ (see [10]). From the works [8], [9] of Klapper we also know the distribution of these numbers $N_{Q,\beta}(\xi)$, which are given as follows.

**Lemma 2.1.** Let $Q$ be a quadratic form of $m$ variables over $\mathbb{F}_q$ of even rank $r$. Then, for all $\xi \in \mathbb{F}_q$, there are $q^m - q^r$ elements $\beta \in \mathbb{F}_q^m$ such that $N_{Q,\beta}(\xi) = q^{m-1}$ and $q^{r-1} + \varepsilon_Q \nu(c) q^{\frac{r-1}{2}}$ elements $\beta \in \mathbb{F}_q^m$ such that $N_{Q,\beta}(\xi) = q^{m-1} + \varepsilon_Q \nu(\xi+c) q^{m-\frac{r-1}{2}},$ where $c$ runs on $\mathbb{F}_q$.

Given a quadratic form $Q$, we can define the exponential sums

$$S_{Q,\beta}(\beta) = \sum_{a \in \mathbb{F}_q^*} \sum_{x \in \mathbb{F}_q^m} \zeta_q^{\text{Tr}_{q^m/q}(aQ(x) + \text{Tr}_{q^m/q}(\beta x) + b)}$$

where $\zeta_q = e^{\frac{2\pi i}{q}}$ and put $S_Q(\beta) = S_{Q,0}(\beta)$. We now give the values of $S_{Q,\beta}(\beta)$ and their distributions.
Lemma 2.2. Let $Q(x)$ be a quadratic form over $\mathbb{F}_q$ of even rank $r$. Then,

$$S_Q(\beta) = \begin{cases} 0 & q^m - q^r \text{ times}, \\ \varepsilon(q-1)q^{m-\frac{r}{2}} & q^{r-1} + \varepsilon(q-1)q^{\frac{r}{2}-1} \text{ times}, \\ -\varepsilon q^{m-\frac{r}{2}} & (q^{r-1} - \varepsilon q^{\frac{r}{2}-1})(q-1) \text{ times}. \end{cases} $$

$$S_{Q,b}(\beta) = \begin{cases} 0 & q^m - q^r \text{ times}, \\ \varepsilon(q-1)q^{m-\frac{r}{2}} & q^{r-1} - \varepsilon q^{\frac{r}{2}-1} \text{ times}, \\ -\varepsilon q^{m-\frac{r}{2}} & q^{r-1} + \varepsilon q^{\frac{r}{2}-1} \text{ times}. \end{cases} $$

Proof. Notice that

$$S_Q(\beta) = \sum_{a \in \mathbb{F}_q^*} \sum_{x \in \mathbb{F}_{q^m}} \zeta_p^{Tr_{q/p}(a(Q(x)+Tr_{q^m/q}(\beta x)))} = \sum_{x \in \mathbb{F}_{q^m}} \sum_{a \in \mathbb{F}_q} \zeta_p^{Tr_{q/p}(a(Q(x)+Tr_{q^m/q}(\beta x)))} - q^m.$$ 

Hence, $S_Q(\beta) = qN_{Q,\beta}(0) - q^m$. Similarly, we get $S_{Q,b}(\beta) = qN_{Q,b}(-b) - q^m$. The result now follows from Lemma 2.1.

The quadratic form $Tr_{q^m/q}(\gamma x^{d+1})$. A whole family of quadratic forms over $\mathbb{F}_q$ in $m$ variables are given by

$$Q_R(x) = Tr_{q^m/q}(xR(x))$$

where $R(x)$ is a $q$-linearized polynomial over $\mathbb{F}_q$. We are interested in the simplest case, when $R(x)$ is the monomial $R_{\gamma,\ell}(x) = \gamma x^\ell$ with $\ell \in \mathbb{N}$, $\gamma \in \mathbb{F}_q^*$, i.e.

$$Q_{\gamma,\ell}(x) = Tr_{q^m/q}(\gamma x^{\ell+1}).$$

The next theorems, due to Klapper, give the distribution of ranks and types of the family of quadratic forms $\{Q_{\gamma,\ell}(x) = Tr_{q^m/q}(\gamma x^{\ell+1}) : \gamma \in \mathbb{F}_{q^m}, \ell \in \mathbb{N}\}$.

For integers $m, \ell$ we will use the following notations

$$m_\ell = \frac{m}{(m,\ell)} \quad \text{and} \quad \varepsilon_\ell = (-1)^{\frac{1}{2}m_\ell}$$

and denote the set of $(q^\ell + 1)$-th powers in $\mathbb{F}_{q^m}$ by

$$S_{q,m}(\ell) = \{ x^{d+1} : x \in \mathbb{F}_{q^m}^* \}.$$ 

**Theorem 2.3** (even characteristic ([8])). Let $q$ be a power of 2 and $m, \ell \in \mathbb{N}$ such that $m_\ell$ is even. Then $Q_{\gamma,\ell}$ is of even rank and we have:

(a) If $\varepsilon_\ell = \pm 1$ and $\gamma \in S_{q,m}(\ell)$ then $Q_{\gamma,\ell}$ is of type $\mp 1$ and has rank $m - 2(m,\ell)$.

(b) If $\varepsilon_\ell = \pm 1$ and $\gamma \notin S_{q,m}(\ell)$ then $Q_{\gamma,\ell}$ is of type $\pm 1$ and has rank $m$.

For $q$ odd, consider the following sets of integers

$$X_{q,m}(\ell) = \{ 0 \leq t \leq N : t \equiv \ell (L) \} \quad \text{and} \quad Y_{q,m}(\ell) = \{ 0 \leq t \leq N : t \equiv \frac{L}{2} (L) \}$$

where $N = q^m - 1$ and $L = q^{(m,\ell)} + 1$.

**Theorem 2.4** (odd characteristic ([9])). Let $q$ be a power of an odd prime $p$ and let $m, \ell \in \mathbb{N}$ be non negative integers. Put $\gamma = \alpha^t$ with $\alpha$ a primitive element in $\mathbb{F}_{q^m}$. Then, we have:

(a) If $\varepsilon_\ell = 1$ and $t \in X_{q,m}(\ell)$ then $Q_{\gamma,\ell}$ is of type $-1$ and has rank $m - 2(m,\ell)$.

(b) If $\varepsilon_\ell = 1$ and $t \notin X_{q,m}(\ell)$ then $Q_{\gamma,\ell}$ is of type $1$ and has rank $m$.

(c) If $m_\ell$ is even, $\varepsilon_\ell = -1$ and $t \in Y_{q,m}(\ell)$ then $Q_{\gamma,\ell}$ is of type $1$ and has rank $m - 2(m,\ell)$.

(d) If $m_\ell$ is even, $\varepsilon_\ell = -1$ and $t \notin Y_{q,m}(\ell)$ then $Q_{\gamma,\ell}$ is of type $-1$ and has rank $m$. 

We will need the following result whose proof is elementary.

Lemma 2.5. Let $q$ be a prime power and $m, \ell$ integers. If $m_\ell$ is even then $(q^m - 1, q^\ell + 1) = q^{(m, \ell)} + 1$.

Lemma 2.6. Let $M = \#S_{q,m}(\ell)$, $M_1 = \#X_{q,m}(\ell)$ and $M_2 = \#Y_{q,m}(\ell)$ and put $M' = q^m - 1 - M$, $M'_1 = q^m - 1 - M_1$ and $M'_2 = q^m - 1 - M_2$. If $m_\ell$ is even then

$$M = M_1 = M_2 = \frac{q^m - 1}{q^{(m, \ell)} + 1} \quad \text{and} \quad M' = M'_1 = M'_2 = \frac{q^{(m, \ell)} q^m - 1}{q^{(m, \ell)} + 1}.$$ 

Proof. Let $\alpha$ be a primitive element of $\mathbb{F}_{q^m}$, then $S_{q,m}(\ell) = \langle \alpha^q q^1 \rangle$, this implies that $M = \frac{q^m - 1}{q^{(m, \ell)} + 1}$, by Lemma 2.5. On the other hand, if $k, N, s_1, s_2$ are non-negative integers with $k \mid N$ and $0 \leq s_1, s_2 \leq k - 1$, then

$$\# \{ i \in \{1, \ldots, N \} : i \equiv s_1 \mod k \} = \# \{ i \in \{1, \ldots, N \} : i \equiv s_2 \mod k \} = \frac{N}{k}.$$ 

Therefore, if $q^{(m, \ell)} + 1 \mid q^m - 1$, we have that $M_1 = M_2 = \frac{q^m - 1}{q^{(m, \ell)} + 1}$. Clearly, we obtain that $M' = M'_1 = M'_2 = q^{(m, \ell)} M$ as we wanted. \hfill \square

3. Weight distribution of cyclic codes defined by trace forms

Let $\mathcal{L} \subset \mathbb{F}_{q^m}[x]$ be a finite dimensional $\mathbb{F}_{q^m}$-subspace containing $q$-linearized polynomials only, i.e.

$$\mathcal{L} = \langle x^q, x^{q^2}, \ldots, x^{q^s} \rangle \subset \mathbb{F}_{q^m}[x]$$

for some non-negative integers $\ell_1, \ldots, \ell_s$ with $\ell_i \neq \ell_j$ for $i \neq j$. Define the $q$-ary code

$$C_\mathcal{L} = \left\{ c_R = \left( \text{Tr}_{q^m/q}(xR(x)) \right)_{x \in \mathbb{F}_{q^m}} : R \in \mathcal{L} \right\} \subset \mathbb{F}_q^n$$

with length $n = q^m - 1$ and the related codes

$$C_{\mathcal{L},0} = \left\{ c_{R,b} = \left( \text{Tr}_{q^m/q}(xR(x)) + b \right)_{x \in \mathbb{F}_{q^m}} : R \in \mathcal{L}, \ b \in \mathbb{F}_q \right\},$$

$$C_{\mathcal{L},1} = \left\{ c_R(\beta) = \left( \text{Tr}_{q^m/q}(xR(x) + \beta x) \right)_{x \in \mathbb{F}_{q^m}} : R \in \mathcal{L}, \ \beta \in \mathbb{F}_{q^m} \right\},$$

$$C_{\mathcal{L},2} = \left\{ c_{R,b}(\beta) = \left( \text{Tr}_{q^m/q}(xR(x) + \beta x) + b \right)_{x \in \mathbb{F}_{q^m}} : R \in \mathcal{L}, \ \beta \in \mathbb{F}_{q^m}, \ b \in \mathbb{F}_q \right\}.$$ 

Notice that $c_{R,b} = c_R + b$ and $c_{R,b}(\beta) = c_R(\beta) + b$; moreover, we have that $c_{R,0} = c_R(0) = c_R$, $c_{R,b}(0) = c_{R,b}$ and $c_{R,0}(\beta) = c_R(\beta)$. Then, we have

$$C_\mathcal{L} \subset C_{\mathcal{L},0}, \ C_{\mathcal{L},1} \subset C_{\mathcal{L},2}.$$

All of these codes are cyclic since one can check that their words have the form $(1, 1, \ldots, 1)$. In our case, this can be seen directly. If $\alpha$ is a primitive element of $\mathbb{F}_{q^m}$ then

$$c_R = \left( \text{Tr}_{q^m/q}(xR(x)) \right)_{x \in \mathbb{F}_{q^m}} = \left( \text{Tr}_{q^m/q}(\alpha^i R(\alpha^i)) \right)_{i = 0}^{q^m - 2}.$$ 

The cyclic shift $s(c_R) = \left( \text{Tr}_{q^m/q}(\alpha^i R(\alpha^i)) \right)_{i = 0}^{q^m - 2} = c_S$ with $S(x) = \alpha^{-1} R(\alpha^{-1} x) \in \mathcal{L}$, hence $s(c_R)$ is in $C_\mathcal{L}$ and the code is cyclic. Similarly for the other codes.

Definition 3.1. A family $\mathcal{L} \subset \mathbb{F}_{q^m}[x]$ of $q$-linearized polynomials has the even rank property or is an even rank family if the quadratic form $Q_R(x) = \text{Tr}_{q^m/q}(xR(x))$ has even rank for any $R \in \mathcal{L}$.
Let $\mathcal{L}$ be an even rank family of $q$-linearized polynomials. Then, $Q_R(x) = \text{Tr}_{q/p}(xR(x))$ has constant type in the family; that is $Q_R(x)$ is of type 1 or of type $-1$ for every $R \in \mathcal{L}$. Therefore, given $r$ a non-negative integer, we can define

$$K_r = \{ R \in \mathcal{L} : Q_R \text{ has rank } r \},$$

(3.3)

$$K_{r,1} = \{ R \in \mathcal{L} \setminus \{0\} : Q_R \text{ has rank } r \text{ of type } 1 \},$$

(3.4)

$$K_{r,2} = \{ R \in \mathcal{L} \setminus \{0\} : Q_R \text{ has rank } r \text{ of type } 3 \}.$$

We have $K_0 = \{0\}$ and $K_r = K_{r,1} \cup K_{r,2}$ for $r > 0$, and we denote their cardinalities by

$$M_{r,1} = \#K_{r,1}, \quad M_{r,2} = \#K_{r,2}, \quad M_r = \#K_r.$$

Note that $M_0 = 1$ and $M_r = M_{r,1} + M_{r,2}$ for $r > 0$. Finally, we denote the set of ranks in $\mathcal{L}$ by

$$R_L = \{ r \in \mathbb{Z}_{>0} : \exists R \in \mathcal{L} \text{ with } Q_R \text{ of rank } r \}.$$

For any positive integer $r$, we define the set

$$[r]_q := \{ q^\ell + 1 : 0 < \ell < r \} = \{ q + 1, q^2 + 1, \ldots, q^{r-1} + 1 \}.$$

We now restate Lemma 2.1 in [13] in more generality and give a proof for completeness. We will need it to calculate the dimensions of the four families of codes considered in this section.

**Lemma 3.2.** Let $m$ be a positive even integer and let $M = \{1\} \cup [\frac{m}{2}]_q$. If $\alpha$ is a primitive element of $\mathbb{F}_{q^m}$ then we have:

(a) $\alpha^{-u}$ and $\alpha^{-v}$ are not conjugated for all distinct elements $u, v \in M$.

(b) The minimum $m_u \in \{1, \ldots, m\}$ such that $q^{m_u} u \equiv u \pmod{q^m - 1}$ is $m$ for all $u \in M$.

**Proof.** For (a) it is enough to show that $q^s(q^\ell_1 + 1) \not\equiv q^\ell_2 + 1 \pmod{q^m - 1}$ and $q^s \not\equiv q^\ell \pmod{q^m - 1}$ for $1 \leq s \leq m - 1$ and $\ell, \ell_1, \ell_2 < \frac{m}{2}$ with $\ell_1 \neq \ell_2$. We will show the first statement. Suppose that there is some $s \in \{1, \ldots, m\}$ such that $q^s(q^\ell_1 + 1) \equiv q^\ell_2 + 1 \pmod{q^m - 1}$. Then,

$$q^{s+\ell_1} + q^s \equiv q^\ell_2 + 1 \pmod{q^m - 1}.$$

If $s + \ell_1 < m$ then $q^{s+\ell_1} + q^s = q^{\ell_2} + 1$ as integers. The uniqueness of the $q$-ary expansion of integers implies that $s + \ell_1 = \ell_2$ and $s = 0$, which cannot happen. Now, if $s + \ell_1 > m$ then $s > \frac{m}{2}$ since by hypothesis $\ell_1 < \frac{m}{2}$, and hence there exists a positive integer $t < \frac{m}{2}$ such that $s = m - t$. Notice that $\ell_1 > t$ and $0 < \ell_1 - t < \frac{m}{2}$, thus $q^{s+\ell_1} + q^s \equiv q^{\ell_1-t} + q^s \pmod{q^m - 1}$ and hence

$$q^{\ell_1-t} + q^s \equiv q^{\ell_2} + 1 \pmod{q^m - 1}.$$

Since all the powers are less than $m$, we obtain $q^{\ell_1-t} + q^s = q^{\ell_2} + 1$ as integers, by unicity of $q$-ary expansion we obtain that $s = \ell_2$ and $\ell_1 - t = 0$ which cannot occur. Therefore, $\alpha^{-u}$ and $\alpha^{-v}$ are not conjugated for all $u \neq v$ in $[\frac{m}{2}]_q$. In a similar way, it can be shown that $\alpha^{-1}$ and $\alpha^{-u}$ are not conjugated for all $u \in [\frac{m}{2}]_q$.

The item (b) can be proved by a similar argument as in (a).

We are now in a position to give the weight distribution of the four codes considered. We give the spectra in two theorems.

**Theorem 3.3.** Let $q$ be a prime power, $m$ a non-negative integer and $\mathcal{L} = \langle x^{q^{\ell_1}}, x^{q^{\ell_2}}, \ldots, x^{q^{\ell_s}} \rangle$ an ideal in $\mathbb{F}_{q^m}[x]$ such that $1 \leq \ell_1 < \ell_2 < \cdots < \ell_s < \frac{m}{2}$. If $\mathcal{L}$ is an even rank family then the dimensions of the cyclic codes $C_\mathcal{L}$ and $C_{\mathcal{L,0}}$ are $ms$ and $ms + 1$ respectively and their spectra are given by Tables 1 and 2 below.
Table 1. Weight distribution of $C_{L_i} (r \in R_{L_i}, i = 1, 2)$.

| weight | frequency |
|--------|-----------|
| $w_0 = 0$ | 1         |
| $w_1 = q^m - q^{m-1} + (-1)^i(q - 1)q^{m-1} - 1$ | $M_{r,i}$ |

Table 2. Weight distribution of $C_{L_0} (r \in R_{L_i}, i = 1, 2)$.

| weight | frequency |
|--------|-----------|
| $w_0 = 0$ | 1         |
| $w_1 = q^m - 1$ | $q - 1$ |
| $w_2 = q^m - q^{m-1} + (-1)^i(q - 1)q^{m-1} - 1$ | $M_{r,i}$ |
| $w_3 = q^m - q^{m-1} + (-1)^i(q - 1)q^{m-1} - 1$ | $M_{r,i}(q - 1)$ |

Proof. By definition $w(c_R) = \#\{x \in \mathbb{F}_q^m : Q_R(x) \neq 0\}$, then

$$w(c_R) = q^m - 1 - \#\{x \in \mathbb{F}_q^m : Q_R(x) = 0\}.$$  

Analogously, $w(c_{R,h}) = q^m - 1 - \#\{x \in \mathbb{F}_q^m : Q_R(x) = -b\}$. Then we have that

$$w(c_{R,h}) = \begin{cases} 
q^m - N_{Q_R}(0) & \text{if } b = 0, \\
q^m - N_{Q_R}(-b) - 1 & \text{if } b \neq 0.
\end{cases}$$

If $Q_R$ has rank $r$ and type $\varepsilon_R$ then

$$w(c_{R,h}) = \begin{cases} 
q^m - q^{m-1} - \varepsilon_R (q - 1)q^{m-1} - 1 & \text{if } b = 0, \\
q^m - q^{m-1} + \varepsilon_R q^{m-1} - 1 & \text{if } b \neq 0.
\end{cases}$$

From these facts, using the numbers $M_r, M_{r,i}$, and the set $R_L$, we obtain the weights and frequencies given in Tables 1 and 2, and the result thus follows.

Let us consider the polynomials $h(x) = \prod_{j=1}^s h_{t_j}(x)$, where $h_{t_j}(x)$ are the minimal polynomials of $\alpha^{-(q^j + 1)}$ over $\mathbb{F}_q$ with $\alpha$ a primitive element of $\mathbb{F}_{q^m}$ and $j = 1, \ldots, s$. By Delsarte’s Theorem, if $n = \frac{q^m - 1}{\delta}$ with $\delta = \gcd(q^m - 1, q^{t_j} + 1, \ldots, q^{t_s} + 1)$ then $h(x)$ is the check polynomial of the cyclic code

$$C_L^* = \{c(a_1, \ldots, a_s) = \left( \sum_{j=1}^s \text{Tr}_{q^m/q}(a_j g_j^i) \right)^{n} : a_j \in \mathbb{F}_{q^m} \} \subset \mathbb{F}_q^n$$

where $g_j = \alpha^{q^j + 1}$ for $j = 1, \ldots, s$. Since the dimension of a cyclic code is given by the degree of its check polynomial, we have

$$\dim C_L^* = \deg h(x)$$

It is known, by general theory of finite fields, that the degree of the minimal polynomial over $\mathbb{F}_q$ of an element $u \in \mathbb{F}_q$ is given by the size of its cyclotomic coset, and this size coincides with the minimum $1 \leq m_u \leq m$ such that $q^{m_u}u \equiv u \pmod{q^m - 1}$. By Lemma 3.2 all of the elements in $L$ are not conjugated to each other and $\deg h_{t_j}(x) = m$ for $j = 1, \ldots, s$. Hence $\deg h(x) = sm$ and thus $\dim C_L^* = sm$.

On the other hand, if $R(x) = \sum_{j=1}^s a_j x^{q^j} \in L$, by linearity of the trace function we have that

$$c_R = \left( \text{Tr}_{q^m/q}(xR(x)) \right)_{x \in \mathbb{F}_q^m} = \left( \sum_{j=1}^s \text{Tr}_{q^m/q}(a_j \alpha^{(q^j + 1)}) \right)^{q^m-1} = \left( \sum_{j=1}^s \text{Tr}_{q^m/q}(a_j g_j^i) \right)^{q^m-1}.$$

Weight distribution of Table 1.

| weight | frequency |
|--------|-----------|
| $w_0 = 0$ | 1         |
| $w_1 = q^m - q^{m-1} + (-1)^i(q - 1)q^{m-1} - 1$ | $M_{r,i}$ |

Weight distribution of Table 2.

| weight | frequency |
|--------|-----------|
| $w_0 = 0$ | 1         |
| $w_1 = q^m - 1$ | $q - 1$ |
| $w_2 = q^m - q^{m-1} + (-1)^i(q - 1)q^{m-1} - 1$ | $M_{r,i}$ |
| $w_3 = q^m - q^{m-1} + (-1)^i(q - 1)q^{m-1} - 1$ | $M_{r,i}(q - 1)$ |
Notice that if \( n = \frac{q^m - 1}{q} \) as before, by modularity we get
\[
\left( \sum_{j=1}^{s} \text{Tr}_{q^m/q}(a_j g_j^i) \right)^n = \left( \sum_{j=1}^{s} \text{Tr}_{q^m/q}(a_j g_j^i) \right)^{n-1} \quad (3.8)
\]
for every \( 1 \leq t \leq \delta \). Thus, denoting \( c = c(a_1, \ldots, a_s) \in \mathcal{C}_L \), by \( 3.3 \) we have that \( c_R = (c | \cdots | c) \) \( \delta \)-times. Hence all the words in \( \mathcal{C}_L \) are obtained by \( \delta \)-concatenation of the words of the cyclic code \( \mathcal{C}_p \), this implies that the dimension of these codes are the same. Therefore \( \dim \mathcal{C}_L = \dim \mathcal{C}_p = sm \). The same argument shows that \( \dim \mathcal{C}_{L,0} = sm + 1 \). \( \square \)

**Theorem 3.4.** Let \( q \) be a prime power, \( m \) a non-negative integer and \( \mathcal{L} = \langle x^{q_1}, x^{q_2}, \ldots, x^{q_s} \rangle \) an ideal in \( \mathbb{F}_{q^m}[x] \) such that \( 1 \leq \ell_1 \leq \ell_2 < \cdots < \ell_s < \frac{q}{q^m}. \) If \( \mathcal{L} \) is an even rank family, then the dimensions of the cyclic codes \( \mathcal{C}_{L,1} \) and \( \mathcal{C}_{L,2} \) are \( m(s+1) \) and \( m(s+1) + 1 \) respectively and their spectra are given by Tables 3 and 4 below.

**Table 3.** Weight distribution of \( \mathcal{C}_{L,1} \) \( (r \in \mathcal{R}_L, \ i = 1, 2). \)

| weight               | frequency |
|----------------------|-----------|
| \( w_0 = 0 \)        | 1         |
| \( w_1,i = q^m - q^{m-1} \) | \( \sum_{r \in \mathcal{R}_L} M_r(q^m - q^r) \) |
| \( w_2,i = q^m - q^{m-1} + (-1)^i(q - 1)q^{m-1} \) | \( M_{r,i}(q^{r-1} + (-1)^{i+1}(q - 1)q^{m-1}) \) |
| \( w_3,i = q^m - q^{m-1} + (-1)^i+1q^{m-1} \) | \( M_{r,i}(q^{r-1} + (-1)^iq^{m-1})(q - 1) \) |

**Table 4.** Weight distribution of \( \mathcal{C}_{L,2} \) \( (r \in \mathcal{R}_L, \ i = 1, 2). \)

| weight               | frequency |
|----------------------|-----------|
| \( w_0 = 0 \)        | 1         |
| \( w_1 = q^m - q^{m-1} \) | \( \sum_{r \in \mathcal{R}_L} M_r(q^m - q^r) \) |
| \( w_2,i = q^m - q^{m-1} + (-1)^i(q - 1)q^{m-1} \) | \( M_{r,i}(q^{r-1} + (-1)^{i+1}(q - 1)q^{m-1}) \) |
| \( w_3,i = q^m - q^{m-1} + (-1)^i+1q^{m-1} \) | \( M_{r,i}(q^{r-1} + (-1)^iq^{m-1})(q - 1) \) |
| \( w_4 = q^m - 1 \) | \( q - 1 \) |
| \( w_5 = q^m - q^{m-1} - 1 \) | \( (q - 1) \sum_{r \in \mathcal{R}_L} M_r(q^m - q^r) \) |
| \( w_6,i = q^m - q^{m-1} - 1 + (-1)^i(q - 1)q^{m-1} \) | \( M_{r,i}(q^{r-1} + (-1)^i(q^{m-1} - 1))(q - 1) \) |
| \( w_7,i = q^m - q^{m-1} - 1 + (-1)^i+1q^{m-1} \) | \( M_{r,i}(q^{r-1} + (-1)^i(q^{m-1} - 1))(q - 1) \) |

**Proof.** The dimensions of \( \mathcal{C}_{L,1} \) and \( \mathcal{C}_{L,2} \) can be obtained in the same way as in Theorem 3.3 using Lemma 3.2.

Now, let \( R \in \mathcal{L} \) and suppose the quadratic form \( Q_R \) has rank \( r \) and type \( \varepsilon_R \). Let’s see the weights of the words of \( \mathcal{C}_{L,1} \). By the orthogonality property of the characters of \( \mathbb{F}_q \), we have that
\[
w(c_R(\beta)) = q^m - 1 - \# \{ x \in \mathbb{F}_q^m : Q_R(x) + \text{Tr}_{q^m/q}(\beta x) = 0 \}
\]
\[
= q^m - 1 - \frac{1}{q} \sum_{\alpha \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q^m} \text{Tr}_{q^m/q}(\alpha(Q_R(x)+\text{Tr}_{q^m/q}(\beta x)))
\]
\[
= q^m - 1 - \frac{1}{q} \left\{ \sum_{\alpha \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q^m} \text{Tr}_{q^m/q}(\alpha(Q_R(x)+\text{Tr}_{q^m/q}(\beta x))) - q \right\}
\]
\[
= q^m - 1 - \frac{1}{q} \left\{ \sum_{\alpha \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q^m} \text{Tr}_{q^m/q}(\alpha(Q_R(x)+\text{Tr}_{q^m/q}(\beta x))) + q^m - q \right\}.
\]
Therefore
\begin{equation}
(3.9) \quad w(c_R(\beta)) = q^m - q^{m-1} - \frac{1}{q} S_{Q_R}(\beta),
\end{equation}
where \( S_{Q_R} \) is the exponential sum (2.1), with \( b = 0 \). In the same way, when \( b \neq 0 \), we get
\begin{equation}
(3.10) \quad w(c_{R,b}(\beta)) = q^m - q^{m-1} - 1 - \frac{1}{q} S_{Q_R,b}(\beta).
\end{equation}
Notice that if \( R = 0 \), then \( Q_R = 0 \) and, for all \( \beta \neq 0 \), we have \( w(c_0(\beta)) = q^m - q^{m-1} \). If \( R \) and \( \beta \) are zeros, then \( w(c_{0,0}(0)) = q^m - 1 \) if \( b \neq 0 \). When \( b = 0 \) we will denote \( c_R(\beta) = c_{R,0}(\beta) \).

Now, let \( K_{r,1} \) and \( K_{r,2} \) be as in (3.3). Then, \( \varepsilon_R = (-1)^{i+1} \) if \( R \in K_{r,i}, i = 1, 2 \). By Lemma 2.2 we have that
\[
w(c_{R,b}(\beta)) = \begin{cases} 
0 & \text{if } b = 0, R = 0, \\
q^m - q^{m-1} & \text{if } b = 0, R \in K_{r,i}, \text{ for } q^m - q^t \beta^s, \\
q^m - q^{m-1} + (-1)^i(q-1)q^{m-\frac{t}{q} - 1} & \text{if } b = 0, R \in K_{r,i}, \text{ for } q^{r-1} + (-1)^{i+1}(q-1)q^{\frac{t}{q} - 1} \beta^s, \\
q^m - q^{m-1} + (-1)^i(q-1)q^{m-\frac{t}{q} - 1} - 1 & \text{if } b \neq 0, R \in K_{r,i}, \text{ for } q^{r-1} + (-1)^i(q^{\frac{t}{q} - 1})(q-1) \beta^s, \\
q^m - q^{m-1} - 1 & \text{if } b \neq 0, R \in K_{r,i}, \text{ for } q^m - q^t \beta^s, \\
q^m - q^{m-1} + (-1)^i(q-1)q^{m-\frac{t}{q} - 1} - 1 & \text{if } b \neq 0, R \in K_{r,i}, \text{ for } q^{r-1} + (-1)^i(q^{\frac{t}{q} - 1})(q-1) \beta^s, \\
q^m - q^{m-1} + (-1)^i(q-1)q^{m-\frac{t}{q} - 1} - 1 & \text{if } b \neq 0, R \in K_{r,i}, \text{ for } q^{r-1} + (-1)^i(q^{\frac{t}{q} - 1})(q-1) \beta^s,
\end{cases}
\]
with \( i = 1, 2 \). From this, the result readily follows. 

\textbf{Remark 3.5.} (i) A code is \textit{t-divisible} if the weight of every codeword is divisible by \( t \). Note that from Tables 1–4, the code \( C_L \) is \( q^{m-\frac{t}{q} - 1}(q-1) \)-divisible, the code \( C_{L,0} \) is \( (q-1) \)-divisible if and only if \( M_{r,2} = 0 \) and that \( C_{L,1} \) is \( q^{m-\frac{t}{q} - 1} \)-divisible.

(ii) If one perform the sum of the frequencies in each of the Tables 1–4 one checks that the dimensions of the codes are the ones given in Theorems 3.3 and 3.4.

\textit{Complete weight enumerator.} Suppose that the elements of \( \mathbb{F}_q \) are ordered by \( \omega_0, \omega_1, \ldots, \omega_{q-1} \), where \( \omega_0 = 0 \). The composition of the vector \( v = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{F}_q^n \) is defined by
\[
\text{comp}(v) = (t_0, t_1, \ldots, t_{q-1}),
\]
where each \( t_i = t_i(v) = \# \{ 0 \leq j \leq n-1 : v_j = \omega_i \} \). Clearly, we have that \( \sum_{i=0}^{q-1} t_i = n \). Let \( C \) be a linear code of length \( n \) over \( \mathbb{F}_q \) and let
\[
A(t_0, t_1, \ldots, t_{q-1}) = \# \{ c \in C : \text{comp}(c) = (t_0, t_1, \ldots, t_{q-1}) \}.
\]
The \textit{complete weight enumerator} of \( C \) is the polynomial
\begin{equation}
(3.11) \quad W_C(z_0, z_1, \ldots, z_{q-1}) = \sum_{(t_0, \ldots, t_{q-1}) \in B_n} A(t_0, t_1, \ldots, t_{q-1}) z_0^{t_0} z_1^{t_1} \cdots z_{q-1}^{t_{q-1}},
\end{equation}
where \( B_n = \{ (t_0, \ldots, t_{q-1}) : t_i \geq 0, t_0 + \cdots + t_{q-1} = n \} \).

\textbf{Lemma 3.6.} Let \( C \) be a linear code of length \( n \) over \( \mathbb{F}_q \) such that \( t_i(c) = t_j(c) \) for all \( i, j > 0 \) and \( c \in C \). Then, if \( A_\ell = \# \{ c \in C : w(c) = \ell \} \), we have that
\[
W_C(z_0, z_1, \ldots, z_{q-1}) = \sum_{\ell=0}^{n} A_\ell z_0^{n-\ell} z_1^{\frac{t}{q} - 1} \cdots z_{q-1}^{\frac{t}{q} - 1}.
\]

\textit{Proof.} Let \( c = (c_0, \ldots, c_{n-1}) \in C \). Since \( t_i(c) = t_j(c) \) for \( i, j > 0 \) and \( \sum_{i=0}^{q-1} t_i = n \), we have that \( t_i = \frac{n-t_0}{q-1} \). On the other hand, since \( w(c) = n - \# \{ 0 \leq j \leq n-1 : c_j = 0 \} = n - t_0 \), we have that \( t_0 = n - w(c) \), and thus \( t_1 = \frac{w(c)}{q-1} \) (note that \( C \) has to be necessarily \( (q-1) \)-divisible). Therefore,
we have that \( A(t_0, \ldots, t_{q-1}) = A_w(c) \) if \( t_0 = n - w(c) \) for some \( c \in C \) and \( t_i = t_j \) for all \( i, j > 0 \), and \( A(t_0, \ldots, t_{q-1}) = 0 \) otherwise. \( \square \)

As a direct consequence of the lemma, we obtain the complete weight enumerator of \( C_L \).

**Proposition 3.7.** Let \( q \) be a prime power, \( m \) a non-negative integer and \( \mathcal{L} = \langle x^{q^2}, x^{q^4}, \ldots, x^{q^{2s}} \rangle \) an ideal in \( \mathbb{F}_q[x] \) such that \( 1 \leq \ell_1 < \ell_2 < \cdots < \ell_s < \frac{m}{2} \). If \( \mathcal{L} \) is an even rank family then the complete weight enumerator of \( C_L \) is given by

\[
W_{C_L}(z_0, \ldots, z_{q-1}) = z_0^n + \sum_{i=1}^{2} \sum_{r,i} M_{r,i} z_0^{a(r,i)} z_1^{b(r,i)} \cdots z_{q-1}^{c(r,i)}
\]

where \( M_{r,i} \) and \( R_L \) are as in (3.4) and (3.3) and

\[
a(r,i) = q^{m-1} + (-1)^{i+1}(q-1)q^{m-\frac{r}{2}} - 1, \quad b(r,i) = q^{m-1} + (-1)^{i}q^{m-\frac{r}{2}} - 1.
\]

4. The codes associated to \( x^{q^{\ell+1}} \)

Here, we consider the codes \( C_L, C_{L,0}, C_{L,1} \) and \( C_{L,2} \) from the previous section but in the particular case of \( \mathcal{L} = \langle x^{q^2} \rangle \), that we denote by \( C_\ell, C_{\ell,0}, C_{\ell,1} \) and \( C_{\ell,2} \). We will compute the spectra of these codes using Theorems 3.3 and 3.4 and Tables 1–4, but we explicitly compute the rank distribution in \( L \) and their associated numbers \( M_{r,i} \).

**The codes \( C_\ell \) and \( C_{\ell,0} \).** Consider the irreducible cyclic code \( C_\ell \) and the code \( C_{\ell,0} \) over \( \mathbb{F}_q \), with check polynomial \( h_\ell(x) \) and \( h_\ell(x)(x-1) \), respectively, where \( h_\ell \) is the minimal polynomial of \( \alpha^{-(q^{\ell+1})} \), with \( \alpha \) a primitive element. By Delsarte’s Theorem these codes can be described by

\[
C_\ell = \left\{ c(\gamma) = ( \text{Tr}_{q^m/q}(\gamma^{(q^{\ell+1})i}))_{i=0}^{n-1} : \gamma \in \mathbb{F}_{q^m} \right\},
\]

\[
C_{\ell,0} = \left\{ c_b(\gamma) = ( \text{Tr}_{q^m/q}(\gamma^{(q^{\ell+1})i} + b))_{i=0}^{n-1} : \gamma \in \mathbb{F}_{q^m}, b \in \mathbb{F}_q \right\}.
\]

Note that \( c_0(\gamma) = c(\gamma) \) and that \( C_\ell \subset C_{\ell,0} \).

Now we give the parameters and the spectra of these codes.

**Theorem 4.1.** Let \( q \) be a prime power and \( m, \ell \) positive integers such that \( \ell < \frac{m}{2} \) and \( m_\ell = \frac{m}{m, \ell} \) is even. Then, \( C_\ell \) is a \([n, m, d]_{q}\)-code with \( n = \frac{q^{m-1}d'}{d} \) and \( d = \frac{1}{\ell}q^{\frac{m}{2}-1}(q-1)d' \) where \( D = q^{(m, \ell)} + 1 \) and

\[
d' = \begin{cases} q^{\frac{m}{2}} - 1 & \text{if } \frac{1}{2}m_\ell \text{ even,} \\ q^{\frac{m}{2}} - q^{(m, \ell)} & \text{if } \frac{1}{2}m_\ell \text{ odd.} \end{cases}
\]

On the other hand, \( C_{\ell,0} \) is a \([n, m+1, d]_{q}\)-code with \( d = \frac{d}{\ell}q^{m-1}(q-1) - \tilde{d} \) and

\[
\tilde{d} = \begin{cases} q^{\frac{m}{2} + (m, \ell)-1} + 1 & \text{if } \frac{1}{2}m_\ell \text{ even,} \\ q^{\frac{m}{2} + (m, \ell)-1} - 1 & \text{if } \frac{1}{2}m_\ell \text{ odd.} \end{cases}
\]

The weight distributions of \( C_\ell \) and \( C_{\ell,0} \) are given by Tables 5 and 6 below.

**Table 5.** Weight distribution of \( C_\ell \).

| weight | frequency |
|--------|-----------|
| 0      | 1         |
| \( \frac{1}{\ell} \{ q^m - q^{m-1} + (-1)^{\frac{m}{2}}(q-1)q^{\frac{m}{2} + (m, \ell)-1} \} \) | \( n \) |
| \( \frac{1}{\ell} \{ q^m - q^{m-1} + (-1)^{\frac{m}{2}+1}(q-1)q^{\frac{m}{2}} \} \) | \( nq^{(m, \ell)} \) |
Table 6. Weight distribution of $C_{ℓ,0}$.

| weight | frequency |
|--------|-----------|
| $0$    | $1$       |
| $q^{m-1}/q$ | $q-1$ |
| $\frac{1}{q} \left( q^m - q^{m-1} + (-1) \frac{q^m}{q} (q-1) \frac{q^m}{q} (q-1) \right)$ | $n$ |
| $\frac{1}{q} \left( q^m - q^{m-1} + (-1) \frac{q^m+1}{q} (q-1) \frac{q^m+1}{q} (q-1) \right)$ | $nq^{(m,ℓ)}$ |
| $\frac{1}{q} \left( q^m - q^{m-1} + (-1) \frac{q^m+1}{q} (q-1) \frac{q^m+1}{q} (q-1) \right)$ | $n(q-1)$ |
| $\frac{1}{q} \left( q^m - q^{m-1} + (-1) \frac{q^m+1}{q} (q-1) \frac{q^m+1}{q} (q-1) \right)$ | $nq^{(m,ℓ)}(q-1)$ |

Proof. Let us begin by computing the length $n$ of these codes. Since $m_ℓ$ is even, by Lemma 2.5 we have that $n = \frac{q^{m_ℓ}-1}{q^2-1}$. Thus $q^{(m,ℓ)} + 1 | q^m - 1$ and by Lemma 2.6 we have $n = M$ or $n = M_1 = M_2$ in even or odd characteristic respectively, where $M$, $M_1$ and $M_2$ are the cardinalities of the sets $S_{q,m}(ℓ)$, $X_{q,m}(ℓ)$ and $Y_{q,m}(ℓ)$ defined in (2.7) and (2.8). Notice that in this case $q^m - 1 - M = nq^{(m,ℓ)}$ in even characteristic and $q^m - 1 - M_1 = q^m - 1 - M_2 = nq^{(m,ℓ)}$ in odd characteristic.

Let $L_ℓ = \langle x^q \rangle$, then

$$R \in L_ℓ \iff R(x) = γ x^{q_ℓ} = R_γ(x) \text{ for some } γ \in F_{q^m}.$$ 

Thus $Q_R(x) = \text{Tr}_{q^m/q}(γ x^{q+1}) = Q_{γ,ℓ}(x)$.

Notice that the codes $C_{L_ℓ}$ and $C_{L_ℓ,0}$ as in (3.1), are obtained from $(q^m - 1, q^ℓ + 1)$-copies of the codes $C_ℓ$ and $C_{0,0}$ in (4.1), respectively. In terms of weights, this mean that

$$w(c_γ(γ)) = \frac{w(c_{R_γ,0})}{(q^m - 1, q^ℓ + 1)}.$$

By Theorems 2.8 and 2.9, $L_ℓ$ is an even rank family. Furthermore, $R_{L_ℓ} = \{0, m, -m(ℓ, ℓ)\}$. If $q$ is even, by Theorem 2.9 we have that

$$M_{m,2} = M_{m-2(m, ℓ),1} = 0, \quad M_{m,1} = nq^{(m,ℓ)}, \quad M_{m-2(m, ℓ),2} = n, \quad \text{if } \frac{1}{2}m_ℓ \text{ even},$$

$$M_{m,1} = M_{m-2(m, ℓ),1} = n, \quad M_{m,2} = nq^{(m,ℓ)}, \quad \text{if } \frac{1}{2}m_ℓ \text{ odd}.$$ 

Similarly, if $q$ is odd, Theorem 2.4 implies that

$$M_{m,2} = M_{m-2(m, ℓ),1} = 0, \quad M_{m,1} = nq^{(m,ℓ)}, \quad M_{m-2(m, ℓ),2} = n, \quad \text{if } \frac{1}{2}m_ℓ \text{ even},$$

$$M_{m,1} = M_{m-2(m, ℓ),1} = n, \quad M_{m,2} = nq^{(m,ℓ)}, \quad \text{if } \frac{1}{2}m_ℓ \text{ odd}.$$ 

Now, by Theorem 3.3, we obtain the weights and frequencies given in Tables 5 and 6. Finally, by studying the values in the tables, we get the minimal distances for both codes. \hfill $\square$

Corollary 4.2. Under the same hypothesis as before, the complete weight enumerator of $C_ℓ$ is

$$W_{C_ℓ}(z_0, \ldots, z_{q-1}) = z_n^n + n z_0^{a_0} z_1^{a_1} \ldots z_{q-1}^{a_{q-1}} + nq^{(m,ℓ)} z_0^{a_0'} z_1^{a_1'} \ldots z_{q-1}^{a_{q-1}'}$$

where

$$a_0 = n - \frac{(q-1)q^{m_ℓ}}{q^{(m,ℓ)+1}} (1 + (-1)^{\frac{m_ℓ}{2}} q^{(m,ℓ)}), \quad a_1 = \frac{q^{m_ℓ-1}}{q^{(m,ℓ)+1}} (1 + (-1)^{\frac{m_ℓ}{2}} q^{(m,ℓ)}),$$

$$a_0' = n - \frac{(q-1)q^{m_ℓ+1}}{q^{(m,ℓ)+1}} (1 + (-1)^{\frac{m_ℓ}{2}} q^{(m,ℓ)}), \quad a_1' = \frac{q^{m_ℓ-1}}{q^{(m,ℓ)+1}} (1 + (-1)^{\frac{m_ℓ}{2}} q^{(m,ℓ)}).$$

Example 4.3. Let $q = 2$, $m = 8$ and $ℓ = 1$. By Theorems 3.3 and 3.4, the codes $C_ℓ$ and $C_{ℓ,0}$ have parameters [85, 8, 40] and [85, 9, 37] respectively, with weight enumerators given by

$$W_{C_ℓ}(x) = 1 + 170x^{120} + 85x^{144} \quad \text{and} \quad W_{C_{0,0}}(x) = 1 + 85x^{37} + 170x^{40} + 170x^{45} + 85x^{48} + x^{85}.$$
The codes $C_{\ell,1}$ and $C_{\ell,2}$. Consider the codes $C_{\ell,1}$ and $C_{\ell,2}$ over $\mathbb{F}_q$, with check polynomials $h_\ell(x)h_1(x)$ and $h_\ell(x)h_1(x)(x-1)$, respectively. Here, $h_\ell$ and $h_1(x)$ are the minimal polynomials of $\alpha\cdot \alpha_{(q^\ell+1)}$ and $\alpha^{-1}$ respectively, where $\alpha$ is a primitive element of $\mathbb{F}_{q^m}$. By Delsarte’s Theorem, these codes are given by

$$
C_{\ell,1} = \left\{ c(\gamma, \beta) = (\text{Tr}_{q^m/q}(\gamma x^q + \beta x))_{x \in \mathbb{F}_q^*} : \gamma, \beta \in \mathbb{F}_{q^m} \right\},
$$

$$
C_{\ell,2} = \left\{ c_0(\gamma, \beta) = (\text{Tr}_{q^m/q}(\gamma x^q + \beta x) + b)_{x \in \mathbb{F}_q^*} : \gamma, \beta \in \mathbb{F}_{q^m}, b \in \mathbb{F}_q \right\}.
$$

As before, for $m, \ell$ positive integers such that $m/(m, \ell)$ even we denote $n = \frac{q^{m-1}}{q^{(m,\ell)}-1}$. We now give the parameters and the spectra of these codes.

**Theorem 4.4.** Let $q$ be a prime power and $m, \ell$ positive integers such that $m_\ell$ is even. Then, $C_{\ell,1}$ is a $[N, 2m, d]_q$-code with $N = q^m - 1$ and $d = q^m(q - 1) - d'$ with

$$
d' = \begin{cases} 
q^{\frac{m}{2}+(m,\ell)-1} & \text{if } \frac{1}{2}m_\ell \text{ even}, \\
(q-1)q^{\frac{m}{2}+(m,\ell)-1} & \text{if } \frac{1}{2}m_\ell \text{ odd}
\end{cases}
$$

and $C_{\ell,2}$ is a $[N, 2m+1, d-1]_q$-code. The weight distributions of the codes $C_{\ell,1}$ and $C_{\ell,2}$ are given by Tables 7 and 8 below.

**Table 7.** Weight distribution of $C_{\ell,1}$.

| weight | frequency |
|--------|-----------|
| $0$ | $1$ |
| $q^m - q^{m-1}$ | $n(q^m - q^{m-2}(m,\ell)) + q^m - 1$ |
| $q^m - q^{m-1} + (q-1)^{\frac{m}{2}+\ell+1}(q-1)q^\frac{m}{2} + 1$ | $nq^{(m,\ell)}(q^{m-1} + (q-1)^{\frac{m}{2}+\ell+1}(q-1)q^\frac{m}{2} - 1)$ |
| $q^m - q^{m-1} + (q-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2} + (m,\ell)-1$ | $n(q^{m-2}(m,\ell) + (q-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2} - (m,\ell)-1)$ |
| $q^m - q^{m-1} + (-1)^{\frac{m}{2}+\ell}q^\frac{m}{2} + (m,\ell)-1$ | $nq^{(m,\ell)}(q^{m-1} + (q-1)^{\frac{m}{2}+\ell}q^\frac{m}{2} - 1)(q-1)$ |
| $q^m - q^{m-1} + (-1)^{\frac{m}{2}+\ell+1}q^\frac{m}{2} + (m,\ell)-1$ | $n(q^{m-2}(m,\ell) + (q-1)^{\frac{m}{2}+\ell+1}q^\frac{m}{2} - (m,\ell)-1)(q-1)$ |
| $q^m - q^{m-1} - 1$ | $n(q^m - q^{m-2}(m,\ell))(q-1) + (q^m - 1)(q-1)$ |
| $q^m - q^{m-1} + (-1)^{\frac{m}{2}+\ell+1}(q-1)q^\frac{m}{2} - 1$ | $nq^{(m,\ell)}(q^{m-1} + (q-1)^{\frac{m}{2}+\ell+1}q^\frac{m}{2} - 1)(q-1)$ |
| $q^m - q^{m-1} + (q-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2} + (m,\ell)-1$ | $n(q^{m-2}(m,\ell) + (q-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2} - 1)(q-1)$ |
| $q^m - q^{m-1} + (-1)^{\frac{m}{2}+\ell}q^\frac{m}{2} + (m,\ell)-1$ | $nq^{(m,\ell)}(q^m - q^{m-2}(m,\ell) + (1)(q^m - 1)(q-1)$ |
| $q^m - q^{m-1} - 1$ | $n(q^m - q^{m-2}(m,\ell) + q^m - 1)$ |

**Table 8.** Weight distribution of $C_{\ell,2}$.

| weight | frequency |
|--------|-----------|
| $0$ | $1$ |
| $q^m - q^{m-1}$ | $n(q^m - q^{m-2}(m,\ell))(q-1) + (q^m - 1)(q-1)$ |
| $q^m - q^{m-1} + (q-1)^{\frac{m}{2}+\ell+1}(q-1)q^\frac{m}{2} - 1$ | $nq^{(m,\ell)}(q^{m-1} + (q-1)^{\frac{m}{2}+\ell+1}q^\frac{m}{2} - 1)(q-1)$ |
| $q^m - q^{m-1} + (q-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2} + (m,\ell)-1$ | $n(q^{m-2}(m,\ell) + (q-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2} - 1)(q-1)$ |
| $q^m - q^{m-1} + (-1)^{\frac{m}{2}+\ell}q^\frac{m}{2} + (m,\ell)-1$ | $nq^{(m,\ell)}(q^m - q^{m-2}(m,\ell) + (1)(q^m - 1)(q-1)$ |
| $q^m - q^{m-1} - 1$ | $n(q^m - q^{m-2}(m,\ell) + q^m - 1)$ |
| $q^m - q^{m-1} + (q-1)^{\frac{m}{2}+\ell+1}(q-1)q^\frac{m}{2}$ | $nq^{(m,\ell)}(q^{m-1} + (q-1)^{\frac{m}{2}+\ell+1}q^\frac{m}{2} - 1)$ |
| $q^m - q^{m-1} + (q-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2}$ | $n(q^{m-2}(m,\ell) + (q-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2} - 1)$ |
| $q^m - q^{m-1} + (-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2}$ | $nq^{(m,\ell)}(q^{m-1} + (q-1)^{\frac{m}{2}+\ell}(q-1)q^\frac{m}{2} - 1)$ |
| $q^m - q^{m-1} + (-1)^{\frac{m}{2}+\ell+1}(q-1)q^\frac{m}{2} - 1$ | $nq^{(m,\ell)}(q^{m-1} + (q-1)^{\frac{m}{2}+\ell+1}q^\frac{m}{2} - 1)$ |
Proof. Note that \(C_{\ell,1} = C_{\ell,1} \) and \(C_{\ell,2} = C_{\ell,2} \) with \(L_\ell = \langle x^\ell \rangle \) where \(C_{\ell,1}, C_{\ell,2} \) are the codes defined in \([5,3] \). Then, by Theorem \([5,3] \) it is enough to compute the numbers \(M_{r,1}, M_{r,2} \) and the set \(R_{L_\ell} \). They have been calculated in the proof of the Theorem \([4,1] \). Therefore, the Tables 7 and 8 give us the spectra of the codes \(C_{\ell,1} \) and \(C_{\ell,2} \) as we wanted. \(\square\)

Example 4.5. Let \(q = 2, m = 8 \) and \(\ell = 1 \) as in Example \([5,3] \). By Theorem \([5,4] \) the codes \(C_{\ell,1} \) and \(C_{\ell,2} \) have parameters \([255, 16, 112] \) and \([255, 17, 111] \), respectively. Also, we have

\[
W_{C_{\ell,1}}(x) = 1 + 3060 x^{112} + 23120 x^{120} + 16575 x^{128} + 20400 x^{136} + 2380 x^{144} \\
W_{C_{\ell,2}}(x) = 1 + 2380 x^{111} + 3060 x^{112} + 20400 x^{119} + 23120 x^{120} + 16575 x^{127} + 16575 x^{128} + 23120 x^{135} + 20400 x^{136} + 3060 x^{143} + 2380 x^{144} + x^{255}.
\]

Remark 4.6. (i) From Tables 5–8 we see that \(C_\ell \) is a 2-weight code, \(C_{\ell,0} \) and \(C_{\ell,1} \) are 5-weight codes and \(C_{\ell,2} \) is an 11-weight code. Also, one checks that \(C_\ell \) is \(q^{\frac{2m}{\ell}} - 1 \)-divisible and \(C_{\ell,1} \) is \(q^{\frac{2m}{\ell}} - 1 \)-divisible. These facts are in accordance with Klapper’s Theorems \([2,3] \) and \([2,4] \) and Remark \([5,5] \).

(ii) In the binary case (i.e. \(q = 2 \)), the codes \(C_{\ell,0} \) and \(C_{\ell,2} \) have symmetric spectrum, that is \(A_i = A_{n-i} \) for every \(i \), since the word 11\( \cdots \)11 is in these codes (there is a weight \(w = n \)).

Remark 4.7. It can be shown, via Pless power moments, that if \(q = 2 \) and \((m, \ell) = 1 \) the dual code of \(C_\ell \) is optimal in the sense that its minimal distance is maximum in the class of cyclic codes with generator polynomial \(m_\alpha (x)m_{\alpha'} (x) \) over \(F_2 \). This condition of optimality is equivalent to the function \(f(x) = x^2 \) defined over \(F_{2^m} \) being an APN function (see \([1] \)). In our case, \(f_\ell (x) = x^{2^\ell+1} \) with \((m, \ell) = 1 \), is a well-known APN function, namely the Kasami–Gold function.

5. Codes associated to \(L_{\ell,3\delta} \)

In this section we consider the codes \(C_{\ell}, C_{\ell,0}, C_{\ell,1} \) and \(C_{\ell,2} \) associated to the family of \(p\)-linearized polynomials

\[
\mathcal{L} = \mathcal{L}_{\ell,3\delta} = \langle x^{\ell}, x^{\ell^3} \rangle \subset \mathbb{F}_{p^m} [x]
\]

where \(p \) is an odd prime and \(m_\ell = m/(m, \ell) \) even. The next theorem summarizes, in our notation, the results proved in \([13] \), have the following expressions:

(a) If \( \frac{1}{2}m_\ell \) is odd, then \(M_{m,1} = M_{m-2\delta,1} = M_{m-4\delta,1} = M_{m-6\delta,1} = 0 \) and

\[
M_{m,2} = \frac{p^{2m+4\delta} - p^{2m} - p^{2m+\delta} + p^{m+4\delta} + p^{m+\delta} - p^{\delta} + p^{3\delta} - p^{3\delta}}{p^{6\delta} + p^{5\delta} - p^{4\delta} - p^{\delta} - 1},
\]

\[
M_{m-2\delta,1} = \frac{p^{2m-2\delta} (p^{\delta} - p^{3\delta} - 1) + p^{m-2\delta} (p^{\delta} - p^{3\delta} + 1) - p^{3\delta} (p^{3\delta} - p^{\delta} + 1) + (p^{\delta} + p^{2\delta} ) (\sum_{i=0}^{\delta}(1)^{i+1} p^{i\delta})}{p^{6\delta} + p^{5\delta} - p^{4\delta} - p^{\delta} - 1},
\]

\[
M_{m-4\delta,2} = \frac{p^{2m-4\delta} (p^{3\delta} + p^{\delta} - 1) - p^{m-4\delta} (p^{3\delta} + p^{\delta} - 1) + p^{\delta} (p^{2\delta} - p^{\delta} + 1) - (p^{\delta} + p^{2\delta} ) (\sum_{i=0}^{\delta}(1)^{i+1} p^{i\delta})}{p^{6\delta} + p^{5\delta} - p^{4\delta} - p^{\delta} - 1},
\]

\[
M_{m-6\delta,1} = \frac{p^{2m-6\delta} - p^{m-6\delta} + p^{3\delta} - p^{3\delta} + p^{2\delta} - p^{\delta} - p^{\delta} - 1}{p^{6\delta} + p^{5\delta} - p^{4\delta} - p^{\delta} - 1}.
\]
(b) If $\frac{1}{2}m_\ell$ is even, then

$$M_{m,2} = M_{m-2\ell,1} = M_{m-4\ell,2} = M_{m-6\ell,1} = 0$$

and

$$M_{m,2} = \frac{p^{2m-6\delta} - p^{2m+4\delta} - p^{m+\delta} + p^{m+4\delta} - p^{m-\delta} - p^{m+4\delta}}{p^{m} + p^{m+3\delta} - p^{m+7\delta} - p^{m+11\delta} - p^{m+13\delta}},$$

$$M_{m-2\ell,1} = \frac{p^{2m-2\delta}(p^{7\delta} - p^{2\delta} - 1)}{p^{m}-p^{m+\delta} + p^{m+3\delta} - p^{m+7\delta} - p^{m+11\delta} - p^{m+13\delta}},$$

$$M_{m-4\ell,2} = \frac{p^{2m-3\delta}(p^{5\delta} - p^{\delta} - 1) - p^{m-3\delta}(p^{2\delta} - p^{\delta} - 1) + p^{\delta}(p^{2\delta} - p^{\delta} - 1) + (p^{3\delta} - 2\delta - p^{\delta})(\sum_{i=0}^{(m-1)^{1}}(\sum_{i=0}^{(m-1)^{1}}(\sum_{i=0}^{(m-1)^{1}})^{1}))}{p^{m}-p^{m+\delta} + p^{m+3\delta} - p^{m+7\delta} - p^{m+11\delta} - p^{m+13\delta}},$$

$$M_{m-6\ell,1} = \frac{p^{2m-3\delta} - p^{m-3\delta} + 1 - p^{\delta} - p^{2\delta} + p^{2\delta} - p^{\delta} - p^{\delta} - p^{2\delta}}{p^{m}-p^{m+\delta} + p^{m+3\delta} - p^{m+7\delta} - p^{m+11\delta} - p^{m+13\delta}}.$$

In [3], the distribution of ranks and types given in the previous theorem was used to calculate the spectra of the codes $C_L$ and $C_{L,1}$ with $L = L_{3\ell,3\ell}$. Fortunately, this information is enough to calculate the spectra of $C_{L,0}$ and $C_{L,2}$ also, which follows directly from Theorems 3, 3, 3, 4 and 3, 1.

**Theorem 5.2.** Let $p$ be an odd prime and let $m, \ell$ be positive integers such that $m_\ell = m/(m, \ell)$ is even with $m > 6\ell$. Then, $C_{L_{3\ell,3\ell}}$ is a $[n, 2m+1, d]$-code with $n = p^m - 1$ and $d = p^{m-1}(p - 1) - d'$ where

$$d' = \begin{cases} p^{\frac{m}{3} + 3(m, \ell) - 1} + 1 & \text{if } \frac{1}{2}m_\ell \text{ even}, \\ (p - 1)p^{\frac{m}{3} + 3(m, \ell) - 1} & \text{if } \frac{1}{2}m_\ell \text{ odd}, \end{cases}$$

and $C_{L_{3\ell,3\ell}}$ is a $[n, 3m + 1, \tilde{d}]$-code with $\tilde{d} = d$ if $\frac{1}{2}m_\ell$ is even and $\tilde{d} = d - 1$ if $\frac{1}{2}m_\ell$ is odd. The weight distributions of the codes $C_{L_{3\ell,3\ell}}$ and $C_{L_{3\ell,3\ell}}$ are given by Tables 9 and 10 below.

We set these notations for the next two tables:

$$F_0 = \frac{p^{2m-6\delta} - p^{2m+4\delta} - p^{m+\delta} + p^{m+4\delta} - p^{m-\delta} - p^{m+4\delta}}{p^{2m} + p^{m+3\delta} - p^{m+7\delta} - p^{m+11\delta} - p^{m+13\delta}},$$

$$F_1 = \frac{p^{2m-2\delta}(p^{7\delta} - p^{2\delta} - 1)}{p^{m}-p^{m+\delta} + p^{m+3\delta} - p^{m+7\delta} - p^{m+11\delta} - p^{m+13\delta}},$$

$$F_2 = \frac{p^{2m-3\delta}(p^{5\delta} + p^{\delta} - 1) - p^{m-3\delta}(p^{2\delta} - p^{\delta} - 1) + p^{\delta}(p^{2\delta} - p^{\delta} - 1) + (p^{3\delta} - 2\delta - p^{\delta})(\sum_{i=0}^{(m-1)^{1}}(\sum_{i=0}^{(m-1)^{1}}(\sum_{i=0}^{(m-1)^{1}})^{1}))}{p^{m}-p^{m+\delta} + p^{m+3\delta} - p^{m+7\delta} - p^{m+11\delta} - p^{m+13\delta}},$$

$$F_3 = \frac{p^{2m-3\delta} - p^{m-3\delta} + 1 - p^{\delta} - p^{2\delta} + p^{2\delta} - p^{\delta} - p^{2\delta}}{p^{m}-p^{m+\delta} + p^{m+3\delta} - p^{m+7\delta} - p^{m+11\delta} - p^{m+13\delta}}.$$

**Table 9.** Weight distribution of $C_{L_{3\ell,3\ell}}$.

| weight | frequency |
|--------|-----------|
| $p^m - p^{m-1} + (-1)^{\frac{m}{3} + 1}(p - 1)p^{\frac{m}{3} + 1}$ | $F_0$ |
| $p^m - p^{m-1} + (-1)^{\frac{m}{3} + 1}(p - 1)p^{\frac{m}{3} + 1}$ | $F_1$ |
| $p^m - p^{m-1} + (-1)^{\frac{m}{3} + 1}(p - 1)p^{\frac{m}{3} + 1}$ | $F_2$ |
| $p^m - p^{m-1} + (-1)^{\frac{m}{3} + 1}(p - 1)p^{\frac{m}{3} + 1}$ | $F_3$ |
| $p^m - 1$ | $p - 1$ |
| $p^m - p^{m-1} + (-1)^{\frac{m}{3} + 1}p^{\frac{m}{3} + 1}$ | $(p - 1)F_1$ |
| $p^m - p^{m-1} + (-1)^{\frac{m}{3} + 1}p^{\frac{m}{3} + 1}$ | $(p - 1)F_2$ |
| $p^m - p^{m-1} + (-1)^{\frac{m}{3} + 1}p^{\frac{m}{3} + 1}$ | $(p - 1)F_3$ |
Table 10. Weight distribution of $\mathcal{C}_{L, M}$.2

| weight | frequency |
|--------|-----------|
| $p^m - 1$ | $p^m - 1 + \sum_{i=0}^3 F_i (p^m - p^{m-i(m, \ell)})$ |
| $p^m - p^{m-1} + (1)^{m+1}(p-1)p^{p-1}$ | $(p^{m-1} + (1)^{m+1}(p-1)p^{p-1})F_0$ |
| $p^m - p^{m-1} + (1)^{m+1}(p-1)p^{2(m, \ell)-1}$ | $(p^{m-2} + (1)^{m+1}(p-1)p^{2(m, \ell)-1})F_1$ |
| $p^m - p^{m-1} + (1)^{m+1}(p-1)p^{3(m, \ell)-1}$ | $(p^{m-6} + (1)^{m+1}(p-1)p^{3(m, \ell)-1})F_3$ |
| $p^m - p^{m-1} - 1$ | $(p-1)^{m-1 + \sum_{i=0}^3 F_i (p^m - p^{m-i(m, \ell)})}$ |
| $p^m - p^{m-1} - 1 + (1)^{m+1}(p-1)p^{p-1}$ | $(p^{m-2} + (1)^{m+1}(p-1)p^{p-1})F_0$ |
| $p^m - p^{m-1} - 1 + (1)^{m+1}(p-1)p^{2(m, \ell)-1}$ | $(p^{m-4} + (1)^{m+1}(p-1)p^{2(m, \ell)-1})F_1$ |
| $p^m - p^{m-1} - 1 + (1)^{m+1}(p-1)p^{3(m, \ell)-1}$ | $(p^{m-6} + (1)^{m+1}(p-1)p^{3(m, \ell)-1})F_3$ |
| $p^m - p^{m-1} - 1 + (1)^{m+1}(p-1)p^{p-1}$ | $(p^{m-2} + (1)^{m+1}(p-1)p^{2(m, \ell)-1})F_0$ |
| $p^m - p^{m-1} - 1 + (1)^{m+1}(p-1)p^{2(m, \ell)-1}$ | $(p^{m-4} + (1)^{m+1}(p-1)p^{2(m, \ell)-1})F_1$ |
| $p^m - p^{m-1} - 1 + (1)^{m+1}(p-1)p^{3(m, \ell)-1}$ | $(p^{m-6} + (1)^{m+1}(p-1)p^{3(m, \ell)-1})F_3$ |

Remark 5.3. The weight distributions of $\mathcal{C}_{L, M}$ and $\mathcal{C}_{L, M}.1$ are determined by those of $\mathcal{C}_{L, M}.0$ and $\mathcal{C}_{L, M}.2$, respectively. More precisely, the weight distribution of $\mathcal{C}_{L, M}$ is given by the first 5 rows of Table 9, and the spectrum of $\mathcal{C}_{L, M}.1$ is given by the first 10 rows of Table 10. Therefore, $\mathcal{C}_{L, M}$ is a 4-weight code, $\mathcal{C}_{L, M}.0$ and $\mathcal{C}_{L, M}.1$ are 9-weight codes and $\mathcal{C}_{L, M}.2$ is a 19-weight code.

As a direct consequence of Proposition 3.7, we obtain the following.

Corollary 5.4. Under the hypothesis of Theorem 5.2, the complete weight enumerator of $\mathcal{C}_{L, M}$ is given by

$$W_{\mathcal{C}_{L, M}}(z_0, \ldots, z_{p-1}) = z_0^{p^m-1} + \sum_{i=0}^3 F_i z_0^{a_i} z_1^{b_i} \cdots z_{p-1}^{b_i},$$

where, for each $i = 0, \ldots, 3$, the numbers $F_i$ are given in (5.1) and

$$a_i = p^m - 1 + (1)^{i+1} \xi \ell (p-1)p^{p+i(m, \ell)-1} - 1 \quad \text{and} \quad b_i = p^m - 1 + (1)^{i+1} \xi \ell p^{p+i(m, \ell)-1}.$$

Proof. By the previous remark, the weight enumerator of $\mathcal{C}$ is $W_{\mathcal{C}}(x) = 1 + \sum_{i=0}^3 R_i x^{c_i}$ where

$$c_i = (p-1)(p^m - 1 + (1)^{i+1} \xi \ell p^{p+i(m, \ell)+1}).$$

Thus, by Proposition 3.7, we have $W_{\mathcal{C}}(z_0, z_1, \ldots, z_{p-1}) = z_0^{p^m-1} + \sum_{i=0}^3 F_i z_0^{a_i} z_1^{b_i} \cdots z_{p-1}^{b_i}$, where

$$a_i = p^m - 1 - c_i \quad \text{and} \quad b_i = \frac{c_i}{\xi \ell}.$$ From these identities and (5.2) we get the desired expressions for $a_i$ and $b_i$, and thus the result follows.

6. Optimal curves

Fix $q = p^n$ with $p$ prime. In this section we will consider Artin-Schreier curves of the form

$$C_{R, \beta} : \quad y^p - y = xR(x) + \beta x$$
where $R(x)$ is any $p$-linearized polynomial over $\mathbb{F}_q$ and $\beta \in \mathbb{F}_q$. A good treatment of Artin-Schreier curves can be found in Chapter 3 by Gneri-zbudak in [3]. They are associated to the codes $C_{\mathcal{L} ,}$ studied in Sections 3–5 which are defined by quadratic forms $Q_R(x) = \text{Tr}_{p^m/p}(xR(x))$, or similar ones, of Section 2. Given a family $\mathcal{L}$ of $p$-linearized polynomials, we define the family

$$\Gamma_\mathcal{L} = \{C_{\mathcal{R}, \beta} : R \in \mathcal{L}, \beta \in \mathbb{F}_q\}$$

of curves $C_{\mathcal{R}, \beta}$ as in (6.1).

We begin by showing necessary and sufficient conditions for the family $\mathcal{L}$ to contain optimal curves (maximal or minimal); that is, curves attaining equality in the Hasse-Weil bound (see Theorem 5.2.3 in [11])

$$|\#C(\mathbb{F}_q) - (q + 1)| \leq 2q\sqrt{q}.$$

Proposition 6.1. Assume $\mathcal{L}$ is an even rank family of $p$-linearized polynomials over $\mathbb{F}_{p^m}$. Let $R \in \mathcal{L}$ and let $r$ be the rank of the quadratic form $Q_R(x) = \text{Tr}_{p^m/p}(xR(x))$ and $v = v_p(\deg R)$ be the $p$-adic value of $\deg R$. Then, the family $\Gamma_\mathcal{L}$ in (6.2) contains optimal curves, both maximal and minimal, if and only if there is some $R \in \mathcal{L}$ with

$$v = \frac{m-r}{2}.$$

In this case we have:

(i) If $p$ is odd, the curve $C_{\mathcal{R}, \beta} \in \Gamma_\mathcal{L}$ is maximal (resp. minimal) if and only if the codeword $c_R(\beta) = (\text{Tr}_{p^m/p}(xR(x) + \beta x))_{x \in \mathbb{F}_{p^m}}$ in $C_{\mathcal{L},1}$ has weight $w(c_R(\beta)) = w_{2,1}$ (resp. $w_{2,2}$) as in Table 3.

(ii) If $p = 2$, the curve $C_{\mathcal{R}, \beta} \in \Gamma_\mathcal{L}$ is maximal (resp. minimal) if and only if either $w(c_R(\beta)) = w_{2,1}$ (resp. $w_{2,2}$) or else $w(c_R(\beta)) = w_{3,2}$ (resp. $w_{3,1}$) as in Table 3.

Proof. Consider the cyclic code $C_{\mathcal{L},1} = \{c_R(\beta) = (\text{Tr}_{p^m/p}(xR(x) + \beta x))_{x \in \mathbb{F}_{p^m}} : R \in \mathcal{L}, \beta \in \mathbb{F}_{p^m}\}$ as in (6.1). The weight of the codeword $c_R(\beta)$ is related to the number of $\mathbb{F}_{p^m}$-rational points of the curve $C_{\mathcal{R}, \beta}$ given in (6.1). In fact, by Hilbert’s Theorem 90 we have

$$\text{Tr}_{p^m/p}(xR(x) + \beta x) = 0 \iff y^p - y = xR(x) + \beta x \quad \text{for some } y \in \mathbb{F}_{p^m}.$$

Since $C_{\mathcal{R}, \beta}$ is a $p$-covering of $\mathbb{P}^1$, considering the point at infinity, we get

$$\#C_{\mathcal{R}, \beta}(\mathbb{F}_{p^m}) = 1 + p \#\{x \in \mathbb{F}_{p^m} : \text{Tr}_{p^m/p}(xR(x) + \beta x) = 0\} = p^{m+1} + 1 - p w(c_R(\beta))$$

where the values of $w(c_R(\beta))$ are given in Table 3 with $q = p$. On the other hand, as an application of the Riemann-Hurwitz formula, the curve $y^p - y = f(x)$ with $f(x) \in \mathbb{F}_q[x]$, has genus $g = \frac{1}{2}(p-1)(\deg f)$ since the degree of $f$ is coprime with $p$ (see Example 2.4 in [3]). Hence, $C_{\mathcal{R}, \beta}$ has genus

$$g(C_{\mathcal{R}, \beta}) = \frac{1}{2}(p-1)(\deg R) = \frac{1}{2}(p-1)p^{v},$$

since $(\deg xR(x) + \beta x, p) = (p^{v} + 1, p) = 1$, for $R \neq 0$. By the Hasse-Weil bound for curves we have that

$$p^m + 1 - (p-1)p^{v+\frac{m}{2}} \leq \#C_{\mathcal{R}}(\mathbb{F}_{p^m}) \leq p^m + 1 + (p-1)p^{v+\frac{m}{2}}.$$

To find maximal or minimal curves we need to ensure equality in the above inequalities; that is, by (6.3) and (6.4) we want that

$$p^{n+1} - p w(c_R(\beta)) = p^{n} \pm (p-1)p^{v+\frac{m}{2}},$$

where the sign (+ (resp. −)) corresponds to a maximal (resp. minimal) curve. Looking at Table 3 with $q = p$, we check that this could only happen if and only if $v = \frac{m-r}{2}$ and the weight $w(c_R(\beta))$ is $w_{2,1}$ (resp. $w_{2,2}$) for a maximal (resp. minimal) curve. Because of the presence of the factors $p-1$ in the weights, additional curves appear in the case $p = 2$. They correspond to $w(c_R(\beta)) = w_{3,2}$.
(resp. $w_{3,1}$) for a maximal (resp. minimal) curve. Since the type of the quadratic form is fixed, only one of the two kind of maximal (or minimal) curves can appear if $p = 2$. □

Next, as an application of the spectra of cyclic codes, for a fixed number $\ell$ we consider the Artin-Schreier curves

$$
C_{\gamma,\beta} : y^p - y = \gamma x^{p^{\ell+1}} + \beta x, \quad \gamma \in \mathbb{F}_p^*, \beta \in \mathbb{F}_p
$$

(6.5)

$$
C_{\gamma_1,\gamma_2,\beta} : y^p - y = \gamma_1 x^{p^{2\ell+1}} + \gamma_2 x^{p^{\ell+1}} + \beta x, \quad \gamma_1, \gamma_2 \in \mathbb{F}_p^*, \beta \in \mathbb{F}_p
$$

related to the codes $C_{\ell,1}$ and $C_{(\ell,3\ell),1}$ of Section 4 and 5, respectively; and we will show that the families $\{C_{\gamma,\beta}\}$ and $\{C_{\gamma_1,\gamma_2,\beta}\}$ contain several maximal and minimal curves.

We begin by computing the $\mathbb{F}_p^*$-rational points of the curves in the first family $\{C_{\gamma,\beta}\}$.

**Proposition 6.2.** Let $m$ and $\ell$ be positive integers such that $m\ell$ is even and let $p$ a prime number. Consider the curve $C_{\gamma,\beta}$ as in (6.5) with $\gamma \in \mathbb{F}_p^*$ and $\beta \in \mathbb{F}_p$. Fix $\gamma = \alpha^l$ and put $\varepsilon_\ell = (-1)^{\ell^2}p^{\ell}$. Then, we have:

(a) If $p > 2$, with $\frac{1}{2}m\ell$ even and $t \equiv 0 \pmod{p^{(m,\ell)} + 1}$, then

$$
\#C_{\gamma,\beta}(\mathbb{F}_p) = \begin{cases} 
p^m + 1 & \text{for } p^m - p^{m-2(m,\ell)} \beta's, 
p^m + 1 - p^{(m,\ell)}(p-1)p^\frac{m}{2} & \text{for } p^m - 2(p-1)p^\frac{m}{2} - (p-1)p^\frac{m}{2} - (m,\ell) - 1 \beta's, 
p^m + 1 + p^{(m,\ell)}p^\frac{m}{2} & \text{for } (p^m - 2(p-1)p^\frac{m}{2} - (m,\ell) - 1)(p-1) \beta's. 
\end{cases}
$$

(b) If $p > 2$, with $\frac{1}{2}m\ell$ even and $t \not\equiv 0 \pmod{p^{(m,\ell)} + 1}$, then

$$
\#C_{\gamma,\beta}(\mathbb{F}_p) = \begin{cases} 
p^m + 1 + (p-1)p^\frac{m}{2} & \text{for } p^m - 1 + (p-1)p^\frac{m}{2} \beta's, 
p^m + 1 - p^\frac{m}{2} & \text{for } (p-1)(p^{m-1} - p^\frac{m}{2}) \beta's. 
\end{cases}
$$

(c) If $p > 2$, with $\frac{1}{2}m\ell$ odd and $t \equiv \frac{(m,\ell) + 1}{2} \pmod{p^{(m,\ell)} + 1}$, then

$$
\#C_{\gamma,\beta}(\mathbb{F}_p) = \begin{cases} 
p^m + 1 & \text{for } p^m - p^{m-2(m,\ell)} \beta's, 
p^m + 1 - p^{(m,\ell)}(p-1)p^\frac{m}{2} & \text{for } p^m - 2(p-1)p^\frac{m}{2} - (p-1)p^\frac{m}{2} - (m,\ell) - 1 \beta's, 
p^m + 1 - p^{(m,\ell)}p^\frac{m}{2} & \text{for } (p^m - 2(p-1)p^\frac{m}{2} - (m,\ell) - 1)(p-1) \beta's. 
\end{cases}
$$

(d) If $p > 2$, with $\frac{1}{2}m\ell$ odd and $t \not\equiv \frac{(m,\ell) + 1}{2} \pmod{p^{(m,\ell)} + 1}$, then

$$
\#C_{\gamma,\beta}(\mathbb{F}_p) = \begin{cases} 
p^m + 1 - (p-1)p^\frac{m}{2} & \text{for } p^m - 1 - (p-1)p^\frac{m}{2} \beta's, 
p^m + 1 + p^\frac{m}{2} & \text{for } (p-1)(p^{m-1} + p^\frac{m}{2}) \beta's. 
\end{cases}
$$

(e) If $p = 2$ and $\gamma \in S_{2,\ell}(m) = \{x^{2^{\ell+1}} : x \in \mathbb{F}_2\}$ then

$$
\#C_{\gamma,\beta}(\mathbb{F}_2) = \begin{cases} 
2^{m+1} & \text{for } 2^{m} - 2^{m-2(m,\ell)} \beta's, 
2^{m+1} - \varepsilon_\ell 2^{\frac{m}{2} + (m,\ell)} & \text{for } 2^{m-2(m,\ell)} - \varepsilon_\ell 2^{\frac{m}{2} + (m,\ell)} \beta's, 
2^{m+1} + \varepsilon_\ell 2^{\frac{m}{2} + (m,\ell)} & \text{for } 2^{m-2(m,\ell)} + \varepsilon_\ell 2^{\frac{m}{2} + (m,\ell)} \beta's. 
\end{cases}
$$

(f) If $p = 2$ and $\gamma \notin S_{2,\ell}(m)$ then

$$
\#C_{\gamma,\beta}(\mathbb{F}_2) = \begin{cases} 
2^{m+1} - \varepsilon_\ell 2^\frac{m}{2} & \text{for } 2^{m-1} - \varepsilon_\ell 2^\frac{m}{2} \beta's, 
2^{m+1} + \varepsilon_\ell 2^\frac{m}{2} & \text{for } 2^{m-1} + \varepsilon_\ell 2^\frac{m}{2} \beta's. 
\end{cases}
$$

**Proof.** Consider the cyclic code $C_{\ell,1} = \{c_{\gamma,\beta} = (\mathbb{F}_p/\gamma x^{p^{\ell+1}} + \beta x)x \in \mathbb{F}_p^* : \gamma, \beta \in \mathbb{F}_p^*\}$. By the same argument as in the previous proof, we have that

$$
\#C_{\gamma,\beta}(\mathbb{F}_p) = p^{m+1} - 1 - p w(c_{\gamma,\beta}).
$$
Thus, the number of rational points of $C_{\gamma, \beta}$ are obtained from Tables 5–8, by using Theorems 2.3 and 2.4 by straightforward calculations. 

We now show the existence of optimal curves in the family $\{C_{\gamma, \beta}\}$. We will use Proposition 6.1 to prove the existence of optimal curves and Proposition 6.2 to count the number of them.

**Theorem 6.3.** Let $p$ be a prime number. Let $m$ and $\ell$ non-negative integers with $\ell \mid m$ such that $m_\ell = \frac{m}{\ell}$ is even and $\gamma = \alpha^3 \in \mathbb{F}_{p^m}$. Then, we have the following:

(a) Let $p$ be odd. Then, the curve $C_{\gamma, \beta}$ as in (6.3) is

(i) minimal if $\frac{1}{2}m_\ell$ is even and $t \equiv 0 \pmod{p^\ell + 1}$, for $p^{m-2\ell-1} - (p-1)p^{\frac{m}{\ell}-\ell-1}$ elements $\beta$,

(ii) maximal if $\frac{1}{2}m_\ell$ is odd and $t \equiv 2\gamma + 1 \pmod{p^\ell + 1}$, for $p^{m-2\ell-1} + (p-1)p^{\frac{m}{\ell}-\ell-1}$ elements $\beta$.

(b) Let $p = 2$ and $\gamma \in S_{2, m}(\ell) = \{x^{2^\ell+1} : x \in \mathbb{F}_{2^m}^\ast\}$. Then,

(i) there are $2m-2\ell+1 - 2^\ell - \ell - 1$ elements $\beta$ such that $C_{\gamma, \beta}$ is minimal and

(ii) there are $2m-2\ell-1 + 2^\ell - \ell - 1$ elements $\beta$ such that $C_{\gamma, \beta}$ is maximal.

**Proof.** Consider the family $\mathcal{L} = (x^{p^\ell})$ of $p$-linearized polynomials over $\mathbb{F}_{p^m}$, with $p$ prime. By Klapper’s Theorems 2.3 and 2.4, $\mathcal{L}$ is an even rank family. Thus, the family of curves $\Gamma_{\mathcal{L}}$ in (6.1) is in fact the family $\{C_{\gamma, \beta}\}$ in (6.3). Now, applying Proposition 6.1, by using Tables 3 and 7 and Theorems 2.3 and 2.4, we get the existence part of the statement. Finally, invoking Proposition 6.2 we get the number of such optimal curves.

**Example 6.4.** Suppose that $p = 2$, $m = 4$ and $\ell = 1$. Consider the curve

$$C_{\gamma, \beta} : y^2 + y = \gamma x^3 + \beta x, \quad \gamma \in \mathbb{F}_{16}^\ast, \quad \beta \in \mathbb{F}_{16}$$

which is in particular an elliptic curve. Suppose that $\gamma \in S_{2, 4}(1)$. Then, by Theorem 6.3 $C_{\gamma, \beta}$ is minimal for only one element $\beta$ and it is maximal for two elements $\beta$.

If $\gamma = z^3$ for some $z \in \mathbb{F}_{16}$ then, by the affine change of variable $u = zx$, the curve $C_{\gamma, \beta}$ turns out to be isomorphic to the curve $C_{1, \lambda} : y^2 + y = u^3 + \lambda u$, where $\lambda = \beta z^{-1}$. This curve is minimal (9 rational points) only for $\lambda = 0$ and it is maximal (25 rational points) for $\lambda = 1, \alpha^5$ and $\alpha^{10}$. That is,

$$y^2 + y = u^3$$

is a minimal elliptic curve and

$$y^2 + y = u^3 + u, \quad y^2 + y = u^3 + \alpha^5 u, \quad y^2 + y = u^3 + \alpha^{10} u$$

are maximal elliptic curves over $\mathbb{F}_{16}$.

We now show that the family $\{C_{\gamma_1, \gamma_2, \beta}\}$ contains optimal curves.

**Proposition 6.5.** Let $p$ be an odd prime and let $m, \ell$ be non-negative integers such that $\ell \mid m$, $m_\ell = \frac{m}{\ell}$ is even and $m > 6\ell$. If $\frac{1}{2}m_\ell$ is odd (resp. even), the Artin-Schreier curve $C_{\gamma_1, \gamma_2, \beta}$ as in (6.3) is maximal (resp. minimal) for some $\gamma_1, \gamma_2 \in \mathbb{F}_{p^m}^\ast$ and $\beta \in \mathbb{F}_{p^m}$.

**Proof.** The family $\mathcal{L} = (x^{p^\ell}, x^{3p^\ell})$ of $p$-linearized polynomials over $\mathbb{F}_{p^m}$ has the even rank property, by Theorem 5.1. By Table 10 and Theorem 5.2 we have that if $\frac{1}{2}m_\ell$ is odd (resp. even) then there exists $R \in \mathcal{L}$ with $\deg R = p^{3\ell}$ and $Q_R$ of rank $r = m - 6\ell$ and type 1 (resp. 3). Thus, we have that $v = \frac{2m_\ell - 1}{2} = 3\ell$, where $v = v_p(\deg R)$, and the result follows directly from Proposition 6.1.

**Example 6.6.** Take an odd prime, $\ell = 1$ and $m > 6$ even. Then, the Artin-Schreier curve $y^p - y = \gamma_1 x^{3p^\ell} + \gamma_2 x^{p^\ell} + \beta x$ is maximal in $\mathbb{F}_{p^{2k}}$ and minimal in $\mathbb{F}_{p^{2k+2}}$ for any $k \geq 2$, for at least one $\gamma_1, \gamma_2 \in \mathbb{F}_{p^m}^\ast$ and $\beta \in \mathbb{F}_{p^m}$, where $\mathbb{F}_{p^m}$ stands for $\mathbb{F}_{p^{4k}}$ or $\mathbb{F}_{p^{4k+2}}$ depending on the case. For instance,

$$y^3 - y = \gamma_1 x^{28} + \gamma_2 x^4 + \beta x$$
is maximal in $F_{3^8} = F_{6561}$ and minimal in $F_{3^{10}} = F_{59049}$ for at least one $\gamma_1, \gamma_2, \beta$ in the corresponding field. Similarly,
\[ y^5 - y = \gamma_1 x^{125} + \gamma_2 x^6 + \beta x \]
is maximal in $F_{5^8} = F_{390625}$ and minimal in $F_{5^{10}} = F_{9765625}$ for some elements $\gamma_1, \gamma_2, \beta$ in the ground field.

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