OPTIMAL SIZE OF BUSINESS AND DIVIDEND STRATEGY IN A NONLINEAR MODEL WITH REFINANCING AND LIQUIDATION VALUE

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Abstract. This paper investigates the optimal control problem with a nonlinear capital process attributed to internal competition factors. Suppose that the company can control its capital process by paying dividend, refinancing and changing the size of business. The transaction costs generated by the control processes as well as the liquidation value at ruin are considered. We aim at seeking the optimal control strategies for maximizing the company’s value. The results show that the company should expand the business scale when the current capital increases. The refinancing may only happen at the moments when, and only when, the capital is null. The dividends should be paid out according to barrier strategy if the dividend rate is unconstrained or threshold strategy if the dividend rate is bounded, respectively.

1. Introduction. From the view of corporation finance, the company’s value can be measured by the expected discounted sum of dividends until the time of ruin. It depends on the size of business and dividend strategy. On the one hand, company has its own benefits and drawbacks that come from either increasing in size, or remaining small, and these will depend on the market in which the company is in, the current economy, and in some cases the preferences of the manager. Generally speaking, the diminishing of the size of business can reduce the company’s risk as well as its potential profit simultaneously. On the other hand, the company also faces the compromise of distributing too much of the company’s cash reserves as dividends, which is costly due to ruin risk, or postponing distribution too long. The latter option prolongs the company’s lifespan, but it is inefficient because of discounting. In the field of actuarial mathematics research, optimal size of business and dividend strategy has been widely studied in linear risk-return models. Barrier or threshold dividend strategy is usually proven to be optimal. For example, [20].

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However, as pointed out by [7] and [8], the linear relationship between risk and return may be an oversimplification of realistic situations. In particular, an excessive size of business may not be a recipe for a high return. This motivates the quest for how an optimum size may be changed when the relationship of risk and return may no longer be linear. Motivated by a workforce control problem, [7] introduced a new risk model when the drift term was no longer linear and there was no one-to-one correspondence between the drift and the diffusion terms. He incorporated two important scenarios from the real world: an internal competition factor inside the company and a liability that a company had to face. To maximize the expected discounted sum of dividend payments until the time of ruin, [7] suggested an optimum size of business and optimal dividend strategy. [8] and [14] continued studying this optimization problem when the size of business was constrained. For other nonlinear risk models in actuarial science, we refer to [24] and [2].

When the company is on the brink of ruin, it can choose to shed risk by refinancing or withdraw from the market. The former needs refinancing costs including proportional and fixed transaction costs generated by advisory, consulting and issuance of securities; and the latter may lead to ruin and corresponding liquidation (or terminal) value, say $P$. If $P$ is negative, the company is fined for going into ruin. If $P$ is positive, we can interpret it as the value that has accrued from the sale of non-liquid assets. To maximize the company’s value, the decision to refinance or not depends on the relationships among the model’s parameters. The literature on the optimum size of business and dividend strategy with refinancing includes [17], [15], [16], [6] and so on, in which the company’s value was defined as the difference between the expected discounted dividends and the expected discounted refinancing costs until the time of ruin. The literature on the optimum size of business and dividend strategy with liquidation value includes [19], [21], [12], [23] and so on, in which the company’s value was defined as the expected sum of discounted dividends less the expected discounted liquidation value. It is interesting to note that [22] studied the combinational optimization problem of the size of business and dividend distribution with refinancing and liquidation value. They measured the company’s value by the expected sum of the discounted liquidation value and the discounted dividends less the expected discounted refinancing until the time of ruin. By adopting some techniques in optimal control theory, they obtained the explicit solutions for joint optimal control strategies.

Inspired by [22], we extend the optimization problem in [7] by adding refinancing strategy and liquidation value in this paper. Under the objective of maximizing the company’s value, we seek the optimal size of business, dividend and refinancing strategies. The influences of transaction costs and liquidation value on the strategies are considered. The results illustrate that the company should expand its business size within the range $[u_0, \frac{\mu}{\sigma}]$ when the current capital increases. The refinancing can only happen at the moments when, and only when, the capital is null and the future profitability is relatively high. The dividends should be paid out with barrier strategy if the dividend rate is unconstrained and with threshold strategy if the dividend rate is bounded, respectively.

The rest of this paper is organized as follows. In Section 2, we introduce a nonlinear model including internal competition factor as well as debt and formulate the optimal control problem. In Section 3, we consider two suboptimal problems.
when the dividend rate is unrestricted. By comparing the solutions of two suboptimal problems, we identify the closed-form solution to the general optimal problem, which depends on the relationships among the model’s parameters. In Section 4, we solve the analogous optimization problem when the dividend rate is bounded.

2. Model formulations and the optimal control problems. Let \( \{B_t\}_{t \geq 0} \) be a standard Brownian motion on the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) with \( \mathcal{F}_t = \sigma\{B_s : s \leq t\} \). Let’s start with the classical capital process \( \{X_t^\pi\}_{t \geq 0} \) controlled by the strategy \( \pi = (u^*, L^\pi) \) as following

\[
X_t^\pi = x + \int_0^t (\mu u_s^\pi - \delta)\,ds + \int_0^t \sigma u_s^\pi \,dB_s - L_t^\pi,
\]

where \( X_0^\pi = x \) is the initial capital. \( \{u_t^\pi\}_{t \geq 0} \) and \( \{L_t^\pi\}_{t \geq 0} \) are two adapted control processes with respect to \( \mathcal{F}_t \), which represent the size of business unit at time \( t \) and the dividend payoff up to time \( t \), respectively. \( \mu > 0 \) is the expected net profit per unit, \( \sigma > 0 \) reflects the volatility of the business, \( \delta \geq 0 \) is a constant liability payment rate (or transaction costs generated by the control process). The company’s value is evaluated by the following performance function

\[
V(x; \pi) = \mathbb{E}_x \left( \int_0^{T^\pi} e^{-rt} \,dL_t^\pi \right),
\]

where \( \mathbb{E}_x \) represents the expectation conditioned on \( X_0^\pi = x > 0 \), \( r > 0 \) denotes the discount rate, \( T^\pi = \inf\{t : X_t^\pi < 0\} \) is called the time of ruin. The classical optimization problem is to seek the value function

\[
V(x) = \sup_{\pi \in \Pi} V(x; \pi),
\]

and associated optimal strategy \( \pi = (u^*, L^\pi) \) in set of admissible strategies \( \Pi \) such that \( V(x) = V(x; u^*) \).

In the model (2.1), the expected return \( \mu u_t^\pi \) and the risk described by \( \sigma u_t^\pi \) are perfectly correlated with each other. However, it was pointed out in [7] that an excessive taking of risk may not necessarily be the recipe for a high expected return.

This motivates the quest for a risk process beyond the class of linear risk processes. As far as the insurance market is concerned, as [14] said, with the consideration of internal competition factors of reinsurance markets arising from sentiment on risk preferences among reinsurers, it is of practical relevance to consider nonlinear risk processes. Particularly, when a risk-averse reinsurer has a preferred risk level and wishes to impose additional service charges on insurance companies seeking services beyond the target level, other reinsurers may demand additional charges for those seeking services with risk levels below the preferred level so as to gain market shares. Therefore, [7] generalized the risk model (2.1) by taking an internal competition factor inside the company, and described the controlled capital process by

\[
X_t^\pi = x + \int_0^t (\mu u_s^\pi - a(u_s^\pi)^2 - \delta)\,ds + \int_0^t \sigma u_s^\pi \,dB_s - L_t^\pi.
\]

Here, \( a \) is the “internal competition factor” between different units inside the company, the condition \( \mu > \sqrt{4\delta} \) is required such that the maximum \( \zeta := \max_{u \geq 0} (\mu u - a u^2 - \delta) \) is positive. Under the objective function in (2.3), [7] obtained the value function and corresponding optimal control strategy. [8] and [14] further studied this optimization problem when \( u \) was constrained.
In this paper, we incorporate refinancing and liquidation value in the model proposed by [7]. We continue using the process \( \{u_t\}_{t \geq 0} \) to describe the control policy for the size of business and denoting \( L_t \) by the cumulative amount of dividends paid from time 0 to time \( t \). Furthermore, we denote \( G_t = \sum_{n=1}^{\infty} I_{(\tau_n \leq t)} \eta_n \) by the cumulative amount of refinancing by issuing equities from time 0 to time \( t \). It is described by a sequence of increasing stopping times \( \{\tau_n, n = 1, 2, \cdots\} \) and a sequence of random variables \( \{\eta_n, n = 1, 2, \cdots\} \), which represent the times and the amounts of equity issuance, respectively. Given a joint strategy \( \pi = (u^\pi, L^\pi, G^\pi) \), the controlled capital process follows that

\[
X_t^\pi = x + \int_0^t (\mu(s) - au^2(s) - \delta)ds + \int_0^t \sigma u(s)dB_s - L_t^\pi + G_t^\pi.
\] (2.5)

The company selects a strategy \( \pi = (u^\pi, L^\pi, G^\pi) \) at any time \( t \) based on information available up to and including time \( t \), say \( \mathcal{F}_t \). Similar to [7], we also assume that the condition \( \mu > \sqrt{4\delta} \) is true in this paper. Then we will give the definition of admissible strategy.

**Definition 2.1.** A strategy \( \pi = (u^\pi, L^\pi, G^\pi) \) is said to be admissible if

(i) \( u^\pi = \{u_t^\pi\}_{t \geq 0} \) is an \( \mathcal{F}_t \)-adapted process with \( u_t^\pi \geq 0 \) for all \( t \geq 0 \).

(ii) \( L^\pi = \{L_t^\pi\}_{t \geq 0} \) is an increasing, \( \mathcal{F}_t \)-adapted càdlàg process with \( L_0^\pi = 0 \) and satisfies \( \Delta L_t^\pi = L_t^\pi - L_{t-}^\pi \leq X_t^\pi \) for all \( t \geq 0 \).

(iii) \( \{\tau_n^\pi\} \) is a sequence of stopping times w.r.t. \( \mathcal{F}_t \), and \( 0 \leq \tau_1^\pi < \cdots < \tau_n^\pi < \cdots \), a.s.

(iv) \( \eta_n^\pi \geq 0, n = 1, 2, \cdots \) is measurable w.r.t. \( \mathcal{F}_{\tau_n^\pi} \).

(v) \( \text{P}(\lim_{n \to \infty} \tau_n^\pi < T) = 0, \forall T > 0 \).

We write \( \Pi \) for the space of these admissible strategies. For each strategy \( \pi \in \Pi \), the ruin time of the controlled process \( X_t^\pi \) is defined as \( T^\pi = \inf\{t : X_t^\pi < 0\} \). We assume that the shareholders need to pay \( \beta_2 \eta + K \) to meet the capital injection of \( \eta \). The factor \( \beta_2 > 1 \) measures the proportional costs, \( K > 0 \) is the fixed costs. Proportional costs on dividends transaction are taken into account through the value of \( \beta_1 \), with \( 0 < \beta_1 \leq 1 \) representing the net proportion of leakages from the dividends received by shareholders after transaction costs have been paid.

**Problem 1.** We define the company value associated with the strategy \( \pi \in \Pi \) by the following performance function:

\[
V(x; \pi) = \mathbb{E}_x \left( \beta_1 \int_0^{T^\pi} e^{-rs}dL_s^\pi - \sum_{n=1}^{\infty} e^{-rT_n^\pi} (\beta_2 \eta_n^\pi + K) I_{(\tau_n^\pi \leq T^\pi)} + Pe^{-rT^\pi} \right),
\] (2.6)

where \( P \in \mathbb{R} \) is the liquidation value at ruin. We focus on finding the value function

\[
V(x) = \max_{\pi \in \Pi} V(x; \pi)
\] (2.7)

and the associated optimal strategy \( \pi^* \in \Pi \) such that \( V(x) = V(x; \pi^*) \). It is easy to prove that \( V(x) \) is bounded by \( \beta_1 x + PL_{\{r \geq 0\}} + \zeta/r \) with \( \zeta = \max_{u \geq 0} (\mu u - au^2 - \delta) \).

Unlike [8] and [14], this paper does not impose restriction on the size of business \( u \). But we shall discuss another optimization problem when the dividend rate is bounded by a ceiling \( M > 0 \), then the cumulated dividend process can be written as \( L_t = \int_0^t l_s ds \) with \( 0 \leq l_s \leq M \). The optimal control problem with restricted dividend rate can be found in [10], [3], [22] and so on. In what follows, we rewrite the admissible strategies and the optimization problem under new assumption.
Definition 2.2. A strategy \( \bar{\pi} = (\bar{u}, \bar{L}, \bar{G}) \) is said to be admissible if
(i) \( \bar{u} = \{u^\pi_t\}_{t \geq 0} \) is an \( \mathcal{F}_t \)-adapted process with \( u^\pi_t \geq 0 \) for all \( t \geq 0 \).
(ii) \( \bar{L} = \{L^\pi_t\}_{t \geq 0} \) is an increasing, \( \mathcal{F}_t \)-adapted càdlàg process satisfying \( L^\pi_0 = 0 \) and \( \bar{L} = \int_0^T \bar{L} ds \) with \( 0 \leq \bar{L} \leq M \).
(iii) \( \{\tau^\pi_n\} \) is a sequence of stopping times w.r.t. \( \mathcal{F}_t \) and \( 0 \leq \tau^\pi_1 < \cdots < \tau^\pi_n < \cdots \), a.s..
(iv) \( \eta^\pi_n \geq 0, n = 1, 2, \ldots \) is measurable w.r.t. \( \mathcal{F}_{\tau^\pi_n} \).
(v) \( \Pr(\lim_{n \to \infty} \tau^\pi_n < T) = 0, \forall T > 0 \).

The class of admissible strategies is denoted by \( \Pi \). For each strategy \( \bar{\pi} \in \Pi \), the ruin time of the controlled process \( X^\pi \) is defined as \( T^\pi = \inf\{t : X^\pi_t < 0\} \).

Problem 2. When a ceiling \( M > 0 \) is imposed on the dividend rate, we define the company value associated with the strategy \( \bar{\pi} \in \Pi \) by the following performance function:

\[
V(x; \bar{\pi}) = \mathbb{E}_x \left( \beta_1 \int_0^{T^\pi} e^{-r s} \xi^\pi_s ds - \sum_{n=1}^{\infty} e^{-r \tau^\pi_n} (\beta_2 \eta^\pi_n + K) I(\tau^\pi_n \leq T^\pi) + Pe^{-r T^\pi} \right). \tag{2.8}
\]

We aim at finding the value function

\[
\bar{V}(x) = \max_{\bar{\pi} \in \Pi} V(x; \bar{\pi}), \tag{2.9}
\]

and the optimal strategy \( \bar{\pi}^* \in \Pi \) such that \( \bar{V}(x) = V(x; \bar{\pi}^*) \). It is easy to see that \( \bar{V}(x) \) is bounded by \( \beta_1 x + PI_{\{P \geq 0\}} + M/r \).

For future reference, let’s define two notations for each function \( \omega(x) \):

- \( L \omega(x) = \max_{y \geq 0} \{\omega(x + y) - \beta_2 y - K\} \)
- \( \mathcal{L} u \omega(x) = \frac{1}{2} \sigma^2 u^2 \omega''(x) + (\mu u - \alpha u^2 - \delta) \omega'(x) - r \omega(x) \)

3. Unrestricted dividend rate. In this section, we consider the optimization problem without restriction on the dividend rate. By applying the standard control theory (see [4]), we give the HJB equations associated Problem 1:

\[
\max \left\{ \max_{u \geq 0} \{ \mathcal{L} u v(x) \}, \beta_1 - v'(x), \mathcal{M} v(x) - v(x) \right\} = 0, \tag{3.1}
\]

\[
\max \{ \mathcal{M} v(0) - v(0), P - v(0) \} = 0. \tag{3.2}
\]

Certainly, the following verification theorem plays an important role in the solving process.

Theorem 3.1. Let \( v(x) \) be a twice continuously differentiable, increasing and concave solution of HJB equations (3.1) and (3.2), it has \( v(x) \geq V(x) \) for all \( x \geq 0 \). Furthermore, if there exists some strategy \( \pi^* \in \Pi \) such that \( v(x) = V(x; \pi^*) \), then \( v(x) = V(x) \) and \( \pi^* \) is optimal.

Proof. Please see the proof procedure in Appendix A. \( \square \)

Adopting the method in [22], let’s begin by investigating two categories of sub-optimal control problem in the following two subsections, whose solutions play an important role in constructing the overall optimal control solutions.
3.1. The solution to the problem without refinancing. Suppose that it is optimal to withdraw from the market whenever the capital hits 0, then the corresponding boundary conditions are $v(0) = P$ and $Mv(0) - v(0) \leq 0$. Then we know that the candidate solution $f(x)$ for $v(x)$ should satisfy that

$$\max_{u \geq 0} \{u''f(x)\} = 0, \quad 0 < x < b_1,$$  
$$f'(x) - \beta_1 = 0, \quad x \geq b_1,$$  
$$f(0) = P,$$  
$$Mf(0) - f(0) \leq 0,$$

with unknown parameter $b_1 \geq 0$, which is the switch level for dividends policy. Motivated by Theorem 3.1, we shall find a twice continuously differentiable, increasing and concave solution $f(x)$ and then construct associated optimal strategy. Differentiating Eq. (3.3) with respect to $u$ and setting the derivative to zero yield

$$f''(x) = \frac{2au - \mu}{\sigma^2 u}. \quad (3.7)$$

The concavity of $f(x)$ requires

$$u \leq \frac{\mu}{2a}. \quad (3.8)$$

Plugging Eq. (3.7) in Eq. (3.3) yields

$$(\mu u(x) - 2\delta)f'(x) = 2rf(x). \quad (3.9)$$

Taking derivative with respect to $x$ on both sides of Eq. (3.9) and using Eq. (3.7) again, we obtain

$$u'(x) = \frac{(\mu - 2au(x))(\mu u(x) - 2\delta) + 2r\sigma^2 u(x)}{\mu \sigma^2 u(x)} \quad (3.10)$$

Let’s define the following function for convenience:

$$\phi(x) := (\mu - 2ax)(\mu x - 2\delta) + 2r\sigma^2 x. \quad (3.12)$$

Note that

$$(4a\delta + \mu^2 + 2r\sigma^2)^2 > (4a\delta + \mu^2 + 2r\sigma^2)^2 - 16a\delta\mu^2 = (4a\delta - \mu^2 + 2r\sigma^2)^2 + 8r\mu^2\sigma^2 > 0.$$  

Using the properties of quadratic function, we know that the equation $\phi(x) = 0$ has two solutions:

$$\rho_1 = \frac{1}{4a\mu} \left((4a\delta + \mu^2 + 2r\sigma^2) - \sqrt{(4a\delta + \mu^2 + 2r\sigma^2)^2 - 16a\delta\mu^2}\right) \in (0, \frac{2\delta}{\mu}),$$

$$\rho_2 = \frac{1}{4a\mu} \left((4a\delta + \mu^2 + 2r\sigma^2) + \sqrt{(4a\delta + \mu^2 + 2r\sigma^2)^2 - 16a\delta\mu^2}\right) \in (\frac{\mu}{2a}, \infty).$$

Obviously, $\phi(x) > 0$ holds for $x \in (\rho_1, \frac{\mu}{2a})$. Given some number $u_0 \in (\rho_1, \frac{\mu}{2a})$, define a function

$$Q(x) := \int_{u_0}^{x} \frac{\mu \sigma^2 z}{(\mu - 2az)(\mu z - 2\delta) + 2r\sigma^2 z} \, dz, \quad u_0 \leq x \leq \frac{\mu}{2a}. \quad (3.11)$$

It is easy to see that the integrand in Eq. (3.11) is positive, so $Q(x)$ is increasing strictly on the interval $[u_0, \frac{\mu}{2a}]$. Consequently, the inverse $Q^{-1}(x)$ of the function
$Q(x)$ exists. Since the function $Q^{-1}(x)$ satisfies the same first order ordinary differential equation as the function $u(x)$ (see (3.10)) and $Q(u_0) = 0$, we get the optimal size of business as

$$u(x) = Q^{-1}(x) \in [u_0, \frac{\mu}{2a}], \quad 0 \leq x \leq x_0,$$

(3.12)

with $x_0 = Q(\frac{\mu}{2a})$. Obviously, $u(x)$ grows as capital $x$ increases. For $0 \leq x \leq x_0$, it follows from Eq. (3.7) that

$$f'(x) = ke^{\int_{x_0}^{x} \frac{a-2e^{y(x)}}{\sigma u(x)} dz},$$

(3.13)

with undetermined parameter $k > 0$. Then

$$f''(x) = \frac{k(2au(x) - \mu)}{\sigma^2 u(x)} e^{\int_{x_0}^{x} \frac{a-2e^{y(x)}}{\sigma^2 u(x)} dz} < 0, \quad 0 \leq x < x_0,$$

(3.14)

with $f''(x_0) = 0$. Applying Eq. (3.4) and the continuities of $f'(x)$ and $f''(x)$ at $x_0$, we get $x_0 = b_1$ and $k = \beta_1$. Namely, there exists a common switch level for the size of business and dividend policy. Combining with Eqs. (3.5) and (3.13) leads to

$$f(x) = \int_{x}^{b_1} e^{\int_{x}^{y} \frac{a-2e^{z(x)}}{\sigma^2 u(z)} dz} dy + P, \quad 0 \leq x \leq b_1.$$

(3.15)

Next, the key is to determine the minimum size of business $u(0) = u_0$. We can rewrite Eq. (3.9) as

$$\beta_1(\mu u_0 - 2\delta) e^{\int_{0}^{\frac{\mu}{2a}} \frac{a-2e^{y(x)}}{\sigma^2 u(x)} dz} = 2r P.$$

(3.16)

Let’s do variable change of $y = u(z)$, using Eq. (3.10) and $u(b_1) = \frac{\mu}{2a}$, we arrive at

$$\varphi(u_0) = P,$$

(3.17)

where the function $\varphi(y)$ is of the form

$$\varphi(y) = \frac{\beta_1}{2r} (\mu y - 2\delta) \exp \left\{ \int_{0}^{\mu y} \frac{\mu(\mu - 2az)}{(\mu - 2az)(\mu z - 2\delta) + 2r \sigma^2 z} dz \right\}, \quad \rho_1 < y \leq \frac{\mu}{2a}.$$

Some calculations indicate that $\varphi(y)$ is increasing on $(\rho_1, \frac{\mu^2}{4a})$ since

$$\varphi'(y) = \exp \left\{ \int_{\mu y}^{\mu y + \frac{\mu(\mu - 2az)}{(\mu - 2az)(\mu z - 2\delta) + 2r \sigma^2 z} dz} \right\} \frac{\beta_1 \mu \sigma^2 y}{\varphi(y)} > 0, \quad \rho_1 < y \leq \frac{\mu}{2a}.$$

So, $\varphi(y)$ attains its maximum $\frac{\beta_1}{2r} \left( \frac{\mu^2}{4a} \right) - \delta > 0$ at $y = \frac{\mu}{2a}$. Moreover,

$$\lim_{y \to \rho_1^+} \varphi(y) = \frac{\beta_1}{2r} (\mu \rho_1 - 2\delta) \lim_{y \to \rho_1^+} \exp \left\{ \int_{\mu y}^{\mu y + \frac{\mu(\mu - 2az)}{(\mu - 2az)(\mu z - 2\delta) + 2r \sigma^2 z} dz} \right\}$$

$$= \frac{\beta_1}{2r} (\mu \rho_1 - 2\delta) \exp \left\{ \int_{\mu y}^{\mu y + \frac{\mu(\mu - 2az)}{-2a \mu (z - \rho_1)(z - \rho_2) dz} dz} \right\}$$

$$= -\infty.$$

(3.18)

Therefore, Eq. (3.17) has a unique root $u_0 \in (\rho_1, \frac{\mu}{2a})$ in the case of $P < \frac{\beta_1}{2r} \left( \frac{\mu^2}{4a} \right) - \delta$. More specifically, it has $u_0 \in (\rho_1, \frac{2a}{\mu})$ if $P < 0$ and $u_0 \in (\frac{2a}{\mu}, \frac{\mu}{2a})$ if $0 \leq P < \frac{\beta_1}{2r} \left( \frac{\mu^2}{4a} \right) - \delta$.

Moreover, Eq. (3.4) and the continuity of $f(x)$ at $b_1$ result in

$$f(x) = \beta_1(x - b_1) + f(b_1), \quad x \geq b_1.$$

(3.19)
This implies that the excess should be paid out immediately as dividends whenever the capital exceeds the barrier $b_1$. In view of the continuity of $u(x)$ at $b_1$, we set $u(x) = u(b_1) = Q^{-1}(b_1) = \frac{\beta_1}{\mu}$. In this case, we set $b_1 = 0$, which means the optimal strategy is immediately to pay all capital as dividend and claim the liquidation value $P$. Then we can verify that the solution of Eqs. (3.3)-(3.5) takes the form as

$$f(x) = \beta_1 x + P, \quad x \geq 0.$$  \hfill (3.20)

Then, it remains to check inequality (3.6) in following three cases.

(i) $P < \frac{\beta_1}{\mu^2} \left( \frac{1}{4\delta} - \delta \right)$ and $\beta_1 < f'(0+) = \beta_1 e^{b_1} \frac{u - 2au(x)}{\tau(x)} \leq \beta_2$. In this case, the solution $f(x)$ is given by Eqs. (3.15) and (3.19). We know that $f'(x) \leq \beta_2$ holds for any $x > 0$ since $f'(x)$ is decreasing on $[0, \infty)$. Therefore, $\mathcal{A}f(0) - f(0) = \max \{f(y) - \beta_2 y - K\} - f(0) = -K < 0$, inequality (3.6) follows.

(ii) $P < \frac{\beta_1}{\mu^2} \left( \frac{1}{4\delta} - \delta \right)$ and $f'(0+) = \beta_1 e^{b_1} \frac{u - 2au(x)}{\tau(x)} > \beta_2$. In this case, the solution $f(x)$ is also given by Eqs. (3.15) and (3.19). We note that $f'(x)$ is decreasing on $[0, b_1]$ and takes value on $[\beta_1, f'(0+)]$. Then there exists a unique number $\eta_1 \in (0, b_1)$ such that $f'(\eta_1) = \beta_2$. Define the integral

$$I(z) := \int_0^z (f'(x) - \beta_2)dx = f(z) - f(0) - \beta_2 z.$$  \hfill (3.21)

Obviously, inequality (3.6) holds if and only if

$$K \geq I(\eta_1).$$  \hfill (3.22)

(iii) $P \geq \frac{\beta_1}{\mu^2} \left( \frac{1}{4\delta} - \delta \right)$. In this case, the solution $f(x)$ is given by Eq. (3.20). We have $f'(x) \equiv \beta_1 < \beta_2$. Thereby, $\mathcal{A}f(0) - f(0) = -K < 0$.

### 3.2. The solution to the problem with forced refinancing.

Then, it is left to discuss the solution of HJB equations when

$$P < \frac{\beta_1}{\mu^2} \left( \frac{1}{4\delta} - \delta \right), \quad f'(0+) = \beta_1 e^{b_1} \frac{u - 2au(x)}{\tau(x)} > \beta_2 \quad \text{and} \quad I(\eta_1) > K.$$  \hfill (3.23)

Eq. (3.22) tells that $\mathcal{A}f(0) - f(0) \leq 0$ does not hold under the condition (3.23), $f(x)$ can not solve HJB equations in this case. It suggests that it is better to refinance rather than to withdraw from the market when the capital is null. Then the associated boundary conditions correspond to $v(0) \geq P$ and $\mathcal{A}v(0) - v(0) = 0$. Correspondingly, the candidate solution $\tilde{f}(x)$ for $v(x)$ should satisfy that

$$\max_{\tilde{u} \geq 0} \{\mathcal{A} \tilde{u} \tilde{f}(x)\} = 0, \quad 0 < x < b_2,$$  \hfill (3.24)

$$\tilde{f}'(x) - \beta_1 = 0, \quad x \geq b_2,$$  \hfill (3.25)

$$\tilde{f}(0) \geq P,$$  \hfill (3.26)

$$\mathcal{A} \tilde{f}(0) - \tilde{f}(0) = 0,$$  \hfill (3.27)

with some parameter $b_2 \geq 0$, which is the switch level for dividend policy. Comparing Eqs. (3.24)-(3.27) with Eqs. (3.3)-(3.6), we obtain the following solution

$$\tilde{f}(x) = \begin{cases} 
\beta_1 \int_0^{x+b_2} e^{b_1} \frac{u - 2au(y)}{\tau(y)} dy + P, & 0 \leq x \leq b_2, \\
\beta_1 (x - b_2) + \tilde{f}(b_2), & x \geq b_2.
\end{cases}$$  \hfill (3.28)
with $b_2 = b_1 - p^*$ and $p^* > 0$. Namely, $\tilde{f}(x)$ can be obtained by shifting $f(x)$ in (3.15) and (3.19) to the left $p^*$ unit. The associated optimal size of business is

$$\tilde{u}^*(x) = \begin{cases} Q^{-1}(x + p^*), & 0 \leq x \leq b_2, \\ \frac{\mu}{\sigma^2}, & x \geq b_2. \end{cases} \tag{3.29}$$

Apparently, $\tilde{f}(x)$ and $\tilde{u}^*(x)$ satisfy Eqs. (3.24)-(3.26) automatically. Now we proceed to seek the number $p^* > 0$ such that Eq. (3.27) holds. Define a function $\psi(p)$ in $p$ as follows:

$$\psi(p) := f(\eta_1) - f(p) - \beta_2(\eta_1 - p) - K, \quad 0 \leq p \leq \eta_1,$$

where $f(x)$ is shown in Eq. (3.15). It follows from Eq. (3.23) that

$$\psi(0) = f(\eta_1) - f(0) - \beta_2 \eta_1 - K = I(\eta_1) - K > 0. \tag{3.30}$$

In addition, we deduce that

$$\psi(\eta_1) = -K < 0 \quad \text{and} \quad \psi'(p) = \beta_2 - f'(p) < 0. \tag{3.31}$$

Consequently, there exists a unique $p^* \in (0, \eta_1)$ such that $\psi(p^*) = 0$, that is

$$f(\eta_1) - f(p^*) - \beta_2(\eta_1 - p^*) - K = 0, \tag{3.32}$$

or, equivalently,

$$\tilde{f}(\eta_2) - \tilde{f}(0) - \beta_2 \eta_2 - K = 0 \tag{3.33}$$

with $\eta_2 := \eta_1 - p^*$. Note that $\tilde{f}'(\eta_2) = f'(\eta_1) = \beta_2$. Then,

$$\mathcal{M}\tilde{f}(0) = \max_{y \geq 0}\{\tilde{f}(y) - \beta_2 y - K\} = \tilde{f}(\eta_2) - \beta_2 \eta_2 - K = \tilde{f}(0) \tag{3.34}$$

is true, Eq. (3.27) follows. By the way, $p^*$ is decreasing with respect to $K$.

Finally, we should verify that $f(x)$ and $\tilde{f}(x)$ given above satisfy HJB equations (3.1) and (3.2) for all $x \geq 0$ in different cases. We only give the proof for $f(x)$, the method is also applicable to $\tilde{f}(x)$.

**Step 1.** To show $\max_{u \geq 0}\{\mathcal{A}^u\tilde{f}(x)\} \leq 0$ on $[0, \infty)$.

(i) If $0 \leq x \leq b_2$, by construction, $\tilde{f}(x)$ and $\tilde{u}^*(x)$ satisfy Eq. (3.24). That is

$$\max_{u \geq 0}\{\mathcal{A}^u\tilde{f}(x)\} = \mathcal{A}^\tilde{u}^*\tilde{f}(x) = 0. \tag{3.28}$$

(ii) If $x > b_2$, it has $\tilde{f}(x) \geq \tilde{f}(b_2)$, $\tilde{f}'(x) = \tilde{f}'(b_2) = \beta_1$ and $\tilde{f}''(x) = \tilde{f}''(b_2) = 0$. So, for each $\tilde{u} \geq 0$, we derive that

$$\mathcal{A}^\tilde{u}\tilde{f}(x) = \frac{1}{2} \sigma^2 \tilde{u}^2 \tilde{f}''(x) + (\mu \tilde{u} - \tilde{a} \tilde{u}^2 - \delta) \tilde{f}'(x) - r \tilde{f}(x)$$

$$= \frac{1}{2} \sigma^2 \tilde{u}^2 \tilde{f}''(b_2) + (\mu \tilde{u} - \tilde{a} \tilde{u}^2 - \delta) \tilde{f}'(b_2) - r \tilde{f}(b_2)$$

$$\leq \frac{1}{2} \sigma^2 \tilde{u}^2 \tilde{f}''(b_2) + (\mu \tilde{u} - \tilde{a} \tilde{u}^2 - \delta) \tilde{f}'(b_2) - r \tilde{f}(b_2)$$

$$= \mathcal{A}^\tilde{u}\tilde{f}(b_2) \leq 0. \tag{3.29}$$

**Step 2.** To show $\tilde{f}'(x) \geq \beta_1$. It can be established directly from the expression of $\tilde{f}(x)$ in Eq. (3.28).
• **Step 3.** To show \( \mathcal{M} \hat{f}(x) \leq \hat{f}(x) \). We write that

\[
\mathcal{M} \hat{f}(x) - \hat{f}(x) = \max_{\eta \geq 0} \{ \hat{f}(x + \eta) - \beta_2 \eta - K \} - \hat{f}(x)
\]

\[
= \max_{\eta \geq 0} \left\{ \int_x^{x+\eta} (\hat{f}'(s) - \beta_2)ds \right\} - K.
\]

(i) If \( 0 \leq x \leq \eta_2 \), it has \( \hat{f}'(x) - \beta_2 \geq 0 \) if and only if \( 0 \leq x \leq \eta_2 \). Therefore,

\[
\mathcal{M} \hat{f}(x) - \hat{f}(x) = \max_{\eta \geq 0} \left\{ \int_x^{x+\eta} (\hat{f}'(s) - \beta_2)ds \right\} - K
\]

\[
\leq \int_0^{\eta_2} (\hat{f}'(s) - \beta_2)ds - K = 0,
\]

the equality holds if and only if \( x = 0 \).

(ii) If \( \eta_2 < x < \infty \), the inequality \( \hat{f}'(x) - \beta_2 < 0 \) is always true, then

\[
\mathcal{M} \hat{f}(x) - \hat{f}(x) = \max_{\eta \geq 0} \left\{ \int_x^{x+\eta} (\hat{f}'(s) - \beta_2)ds \right\} - K
\]

\[
= -K < 0.
\]

• **Step 4.** Obviously, \( \hat{f}(x) \) satisfies Eq. (3.2).

### 3.3. The value function and associated optimal strategy

We have studied two categories of suboptimal models above, one is the classical model without refinancing, the other never goes bankrupt by refinancing. Then we will identify the value functions and the optimal strategies corresponding to the suboptimal models, depending on the relationships among the parameters.

**Theorem 3.2.** The value function \( V(x) \) and associated optimal control strategy \( \pi^* \) can be identified in following cases, which exhaust all possibilities.

(i) Suppose \( P < \frac{\beta_1}{\sigma} \left( \frac{a^2}{4a} - \delta \right) \) and \( \beta_1 < f'(0+) = \beta_1 e^{\int_0^{\frac{a^2}{4a}} \frac{-2au(u)}{\sigma^2u(u)} du} \leq \beta_2 \) hold, then \( V(x) \) is consistent with \( f(x) \) given in Eqs. (3.15) and (3.19). The associated optimal size of business is

\[
u^{\pi^*} (x) = \begin{cases} Q^{-1}(x), & 0 \leq x \leq b_1, \\ \frac{\sigma}{\pi}, & x \geq b_1. \end{cases}
\]

\( L^{\pi^*} \) is a barrier dividend strategy with level \( b_1 > 0 \) described by

\[
L_t^{\pi^*} = (x - b_1)_+ + \int_0^t I_{(X_s^{\pi^*} = b_1)} dL_s^{\pi^*}.
\]

It is always unprofitable to refinance, so \( G_t^{\pi^*} = 0 \). Thereby, the capital process controlled by optimal strategy \( \pi^* = (u^{\pi^*}, L^{\pi^*}, G^{\pi^*}) \) follows that

\[
\begin{cases} X_t^{\pi^*} = x + \int_0^t (\mu u^{\pi^*}(X_s^{\pi^*}) - a [u^{\pi^*}(X_s^{\pi^*})]^2 - \delta)ds + \int_0^t \sigma u^{\pi^*}(X_s^{\pi^*})dB_s - L_t^{\pi^*}, \\ X_t^{\pi^*} \leq b_1, \quad t > 0. \end{cases}
\]

(ii) Suppose \( P < \frac{\beta_1}{\sigma} \left( \frac{a^2}{4a} - \delta \right), \) \( f'(0+) = \beta_1 e^{\int_0^{\frac{a^2}{4a}} \frac{-2au(u)}{\sigma^2u(u)} du} > \beta_2 \) and \( K \geq I(\eta_1) \) hold, then the value function and associated optimal strategy take the same forms as those in (i).

(iii) Suppose \( P \geq \frac{\beta_1}{\sigma} \left( \frac{a^2}{4a} - \delta \right) \) holds, then \( V(x) \) coincides with \( f(x) \) given in Eq. (3.20). The associated optimal strategy is to pay all the capital as dividends and,
thereafter, claim the salvage value $P$, i.e., $L_t^* = x, G_t^* = 0$ and $u^*(x)$ can be arbitrary on $[0, \infty)$.

(iv) Suppose $P < \frac{\beta_1}{\pi} (\nu^2 - \delta), f'(0^+) = \beta_1 e^{\int_0^{+\pi} \frac{a - 2a(x)}{\sigma^2(x)} \, dx} > \beta_2$ and $K < I(\eta_1)$ hold, then $V(x)$ is consistent with $\tilde{f}(x)$ given in Eq. (3.28). $L^*$ is a barrier dividend strategy with level $b_2 > 0$ described by

$$L_t^* = (x-b_2)_+ + \int_0^t I_{(X^*_t=b_2)} \, dL^*_t. \quad (3.40)$$

It is profitable to refinance when, and only when, the capital is null, the capital immediately jumps to $\eta_2$ once it reaches 0 by issuing equity. Mathematically, $G^*$ is characterized by

$$\left\{ \begin{array}{l}
\int_0^\infty I_{\{t; X_t^* > 0\}} \, dG_t^* = 0, \\
\tau_t^* = \inf\{t \geq 0 : X_t^* = 0\}, \\
\tau_n^* = \inf\{t > \tau_{n-1}^* : X_{t-n}^* = 0\}, n = 2, 3, \cdots, \\
\eta_n^* = \eta_2 = \eta_1 - \beta^*, n = 1, 2, \cdots.
\end{array} \right. \quad (3.41)$$

The associated optimal reinsurance strategy $\tilde{u}^*$ is determined by Eq. (3.29). Thereby, the capital process controlled by $\pi^* = (\tilde{u}^*, \pi^*, G^*)$ follows that

$$\left\{ \begin{array}{l}
X^*_t = x + \int_0^t \left( \mu \tilde{u}^*(X^*_s) - a \tilde{u}^*(X^*_s) \right) \, ds + \int_0^t \sigma \tilde{u}^*(X^*_s) \, dB_s \\
+ \sum_{n=1}^\infty I_{\{\tau_{n-1}^* \leq s \leq \tau_n^*\}} \eta_n^* - L_t^*, \\
0 \leq X^*_t \leq b_2, \quad t > 0.
\end{array} \right. \quad (3.42)$$

Proof. We can easily prove that $V(x)$ given in 4 cases is indeed a twice continuously differentiable, increasing and concave solution to HJB equations (3.1) and (3.2). In addition, $f(x) = V(x; \pi^*)$ and $\tilde{f}(x) = V(x; \pi^*)$ hold in different cases, respectively. Recalling Theorem 3.1, all results in this theorem can be confirmed. We only provide the proof of (iv) in Appendix B as a sample.

Remark 1. The results in Theorem 3.2 illustrate that the optimal strategy does not involve refinancing when the cost factors $K$ or $\beta_2$ for issuing equities is too high relative to the company’s profitability. In this case, the optimization problem reduces to that studied in [7] with $P = 0$.

4. Restricted dividend rate. In this section, we consider the optimization problem when a ceiling $M$ is imposed on the dividend rate. With reference to the theory of optimal control, the HJB equations corresponding to Problem 2 are given by

$$\max \left\{ \begin{array}{l}
\max_{u \geq 0, \lambda \geq 0, \eta \geq 0} \{ \mathcal{A} u \theta(x) + l(\beta_1 - \psi'(x)) \}, \mathcal{A} \theta(x) - \psi(x) \}\right. = 0, \quad (4.1)
\max \{ \mathcal{A} \theta(0) - \psi(0), P - \psi(0) \} = 0. \quad (4.2)$$

Then, similar to Section 3, we will solve HJB equations according to different boundary conditions. By investigating the relationships among the model’s parameters, Problem 2 can be solved completely. Firstly, we state the verification theorem as following.

Theorem 4.1. Let $\theta(x)$ be a twice continuously differentiable, increasing and concave solution of HJB equations (4.1) and (4.2), then $\theta(x) \geq \bar{V}(x)$ for all $x \geq 0$. Furthermore, if there exists some strategy $\pi^* = (u^*, \pi^*, G^*)$ such that $\theta(x) = V(x; \pi^*)$, then $\theta(x) = \bar{V}(x)$ and $\pi^*$ is associated optimal strategy.
4.1. The case without refinancing. Suppose that it is optimal to withdraw from the market whenever the capital reaches 0, then the corresponding boundary conditions are \( \bar{v}(0) = P \) and \( A\bar{v}(0) - \bar{v}(0) \leq 0 \). Following the approach in control theory, the candidate solution \( g(x) \) for \( \bar{v}(x) \) should satisfy that

\[
\max_{u \geq 0, M \geq 0} \{ \alpha^u g(x) + l(\beta_1 - g'(x)) \} = 0, \quad (4.3)
\]

\[
P - g(0) = 0, \quad (4.4)
\]

\[
A g(0) - g(0) \leq 0. \quad (4.5)
\]

We try to find the increasing, concave and twice continuously differentiable solution \( g(x) \), which has a switch level \( d_1 > 0 \) such that \( g'(d_1) = \beta_1 \). Then it has \( g'(x) \geq \beta_1 \) for \( 0 \leq x \leq d_1 \), so

\[
\max_{u \geq 0, M \geq 0} \{ \alpha^u g(x) + l(\beta_1 - g'(x)) \} = \max_{u \geq 0} \{ \alpha^u g(x) \} = 0. \quad (4.6)
\]

Similar to Eq. (3.10), the maximizer \( u(x) \) satisfies the following differential equation

\[
u'(x) = \frac{(\mu - 2au(x)) + \mu u(x) - 2\beta^2 u(x)}{\mu^2 u(x)}. \quad (4.7)
\]

Let’s define a function

\[
S(x) := \int_{u_0}^x \frac{\mu^2 z}{(\mu - 2az)(\mu z - 2\beta^2) + 2r\sigma^2 z} \, dz, \quad u_0 \leq x \leq u^*, \quad (4.8)
\]

where \( u_0 < u^* < \frac{\beta_1}{2a} \) will be determined later. Then \( u(x) \) can be expressed in terms of \( S^{-1}(x) \) as

\[
u(x) = S^{-1}(x) \in [u_0, u^*], \quad 0 \leq x \leq d_1, \quad (4.9)
\]

which is an increasing function with \( u(d_1) := u^* > 0 \) and \( u(0) := u_0 < u^* \). Repeating a similar process in Subsection 3.1, we provide the candidate solution to Eq. (4.6)

\[
g(x) = \beta_1 \int_0^x e^{\frac{d_1}{r} \frac{\mu - 2\beta_1 z}{\mu^2 z}} \, dy + P, \quad 0 \leq x \leq d_1. \quad (4.10)
\]

For \( x > d_1 \), it has \( g'(x) < \beta_1 \). We conjecture that it is optimal to pay dividends at the maximum rate \( M \) and maintain the business scale at a constant size \( u^* \). Accordingly,

\[
\max_{u \geq 0, M \geq 0} \{ \alpha^u g(x) + l(\beta_1 - g'(x)) \} = \alpha^u g(x) + M(\beta_1 - g'(x)) = 0, \quad (4.11)
\]

or equivalently,

\[
\frac{1}{2} \sigma^2(u^*)^2 g''(x) + [\mu u^* - a(u^*)^2 - \delta - M]g'(x) - rg(x) + \beta_1 M = 0. \quad (4.12)
\]

With regard to the properties that \( g(x) \) is bounded and \( g'(d_1) = \beta_1 \), we obtain

\[
g(x) = \frac{\beta_1 M}{r} + \frac{\beta_1}{\gamma} e^{\gamma(x-d_1)}, \quad x \geq d_1, \quad (4.13)
\]

where \( \gamma \) is the negative root of the equation

\[
\frac{1}{2} \sigma^2(u^*)^2 \gamma^2 + [\mu u^* - a(u^*)^2 - \delta - M] \gamma - r = 0. \quad (4.14)
\]

To match the continuity condition

\[
\frac{g''(d_1+)}{g'(d_1+)} = \frac{g''(d_1-)}{g'(d_1-)},
\]
we derive that
\[
\gamma = -\frac{\mu - 2au^*}{\sigma^2 u^*}.
\tag{4.15}
\]
Then, substituting Eq. (4.15) into Eq. (4.14) yields
\[
\chi(u^*):=(\mu - 2au^*)(\mu u^* - 2\delta - 2M) + 2r\sigma^2 u^* = 0.
\tag{4.16}
\]
In view of the definition of \(\rho_1\), we derive that
\[
\chi(\rho_1) = -2M(\mu - 2\rho_1) < 0, \quad \chi(\frac{\mu}{2a}) = 2r\sigma^2 \cdot \frac{\mu}{2a} > 0.
\tag{4.17}
\]
Therefore \(u^* \in (\rho_1, \frac{\mu}{2a})\) is uniquely determined by Eq. (4.16). In addition, the continuity of \(u(x)\) at point \(d_1\) leads to
\[
u(d_1) = S^{-1}(d_1) = u^*,
\tag{4.18}
\]
i.e., \(d_1 = S(u^*) \geq 0\). Finally, we need to determine the value of \(u_0\). Similar to Eq. (3.9), we have
\[
(\mu u(x) - 2\delta)g'(x) = 2rg(x).
\tag{4.19}
\]
Therefore, combining with Eq. (4.10), setting \(x = 0\) yields
\[
\beta_1(\mu u_0 - 2\delta) e^{\int_0^{d_1} \frac{\mu - 2au(z)}{\sigma^2 u(z)} dz} = 2rP.
\tag{4.20}
\]
Doing variable change of \(y = u(z)\) and applying Eq. (4.7) and \(u(d_1) = u^*\), we obtain
\[
\psi(u_0) = P,
\tag{4.21}
\]
where the function \(\psi(y)\) is of the form
\[
\psi(y) = \frac{\beta_1}{2r} (\mu y - 2\delta) \exp \left\{ \int_y^{u^*} \frac{\mu (\mu - 2az)}{(\mu - 2az)(\mu z - 2\delta) + 2r\sigma^2 z} dz \right\}, \quad \rho_1 < y \leq u^*.
\]
Analogous to \(\varphi(y)\), we can prove that the function \(\psi(y)\) is increasing on \((\rho_1, u^*]\), which attains its maximum \(\frac{\beta_1}{2r}(\mu u^* - 2\delta)\) at \(y = u^*\) and \(\lim_{y \to \rho_1} \psi(y) = -\infty\). Therefore, when \(P < \frac{\beta_1}{2r}(\mu u^* - 2\delta)\), Eq. (4.21) has a unique root \(u_0 \in (\rho_1, u^*)\).

However, there does not exist appropriate solution to Eq. (4.21) under the condition \(P \geq \frac{\beta_1}{2r}(\mu u^* - 2\delta)\). In this case, we set \(d_1 = 0\), which means the optimal strategy is immediately to pay dividends at the maximum rate \(M\) until the time of ruin. We conjecture that \(g'(x) \leq \beta_1\) for all \(x \geq 0\). Then
\[
\max_{u \geq 0, M \geq 0} \{ e^{au} g(x) + l(\beta_1 - g'(x)) \} = \max_{u \geq 0} \{ e^{au} g(x) + M(\beta_1 - g'(x)) \},
\tag{4.22}
\]
equivalently,
\[
\max_{u \geq 0} \left\{ \frac{1}{2} \sigma^2 u^2 g''(x) + [\mu u - au^2 - \delta - M]g'(x) - r g(x) + \beta_1 M \right\} = 0.
\tag{4.23}
\]
Taking derivative with respect to \(u\) yields
\[
\frac{g''(x)}{g'(x)} = -\frac{\mu - 2au(x)}{\sigma^2 u(x)}.
\tag{4.24}
\]
We further expect the optimal size of business \(u(x)\) to be some constant, which proves to be \(u^* \in (u_0, \frac{\mu}{2a})\) later. Then the solution to Eq. (4.24) takes the form
\[
g(x) = C + De^{\gamma x}
\tag{4.25}
\]
with
\[ \gamma = -\frac{\mu - 2au^*}{\sigma^2 u^*}. \] (4.26)

By putting Eq. (4.26) into Eq. (4.23), we deduce that \( C = \beta_1 M/r \) and \( u^* \in (u_0, \frac{\mu}{2\sigma}) \) satisfies Eq. (4.16). Moreover, the boundary condition \( g(0) = P \) leads to \( D = P - \beta_1 M/r \). We remain to verify the condition \( g'(x) \leq \beta_1 \) for all \( x \geq 0 \), equivalently,
\[ g'(0+) = D \gamma \leq \beta_1 \iff P \geq \frac{\beta_1}{\gamma} + \frac{\beta_1 M}{r} = -\frac{\beta_1 \sigma^2 u^*}{\mu - 2au^*} + \frac{\beta_1 M}{r} = \frac{\beta_1}{2r}(\mu u^* - 2\delta), \] (4.27)

where the second equality stems from Eq. (4.16).

Finally, we come to check Eq. (4.5) in following 3 cases.

(i) \( P < \frac{\beta_1}{2r}(\mu u^* - 2\delta) \) and \( \beta_1 < g'(0+) = \beta_1 e^{\int_0^{d_1} \frac{\mu - 2au^*}{\sigma^2 u^*(x)} dx} \leq \beta_2 \). In this case, the solution \( g(x) \) is given by Eqs. (4.10) and (4.13). We know \( g'(x) \leq \beta_2 \) holds for each \( x > 0 \) since \( g'(x) \) is decreasing on \((0, \infty)\). Therefore \( \mathcal{M}g(0) - g(0) = \max_{y \geq 0} \{g(y) - \beta_2 y - K\} = g(0) = -K < 0 \) follows.

(ii) \( P < \frac{\beta_1}{2r}(\mu u^* - 2\delta) \) and \( g'(0+) = \beta_1 e^{\int_0^{d_1} \frac{\mu - 2au^*}{\sigma^2 u^*(x)} dx} > \beta_2 \). In this case, the solution \( g(x) \) is also given by Eqs. (4.10) and (4.13). The derivative \( g'(x) \) is strictly decreasing on \((0, d_1] \) with the range of value \([\beta_1, g'(0+)]\). Then there exists a unique number \( \eta_1 \in (0, b_1) \) such that \( g'(\eta_1) = \beta_2 \). Define the integral
\[ J(z) := \int_0^z (g'(x) - \beta_2) dx = g(z) - g(0) - \beta_2 z. \] (4.28)

Apparently, Eq. (4.5) holds if and only if
\[ K \geq J(\eta_1). \] (4.29)

(iii) \( P \geq \frac{\beta_1}{2r}(\mu u^* - 2\delta) \). In this case, the solution \( g(x) \) is given by Eq. (4.25). The inequality \( g'(x) \leq \beta_1 < \beta_2 \) holds for each \( x \geq 0 \). Then, the inequality \( \mathcal{M}g(0) - g(0) = -K < 0 \) is valid.

4.2. The case with forced refinancing. Now, we shall discuss the solution of HJB equations under the following condition
\[ P < \frac{\beta_1}{2r}(\mu u^* - 2\delta), \quad g'(0+) = \beta_1 e^{\int_0^{d_1} \frac{\mu - 2au^*}{\sigma^2 u^*(x)} dx} > \beta_2 \quad \text{and} \quad K < J(\eta_1). \] (4.30)

In the case of (4.30), the function \( g(x) \) can not solve HJB equations since the inequality \( \mathcal{M}g(0) - g(0) \leq 0 \) does not hold. It means that refinancing is profitable when the capital reaches \( 0 \). Correspondingly, the boundary conditions become \( \mathcal{M}P(0) - P = 0 \) and \( P \geq 0 \). The solution \( \tilde{g}(x) \) for \( \tau(x) \) should satisfy that
\[ \max_{\tilde{g} \geq 0, \mu \geq 0} \{x \tilde{g}^x + l(\beta_1 - \tilde{g}(x))\} = 0, \quad x \geq 0, \] (4.31)
\[ \tilde{g}(0) \geq P, \] (4.32)
\[ \mathcal{M} \tilde{g}(0) - \tilde{g}(0) = 0. \] (4.33)

We give the explicit expression as following
\[ \tilde{g}(x) = \begin{cases} \beta_1 \int_0^{x+q^*} e^{\int_y^{d_1} \frac{\mu - 2au^*}{\sigma^2 u^*(s)} ds} dy + P, & 0 \leq x \leq d_2, \\ \frac{\beta_1 M}{r} + (P - \frac{\beta_1 M}{r})e^{\gamma(x-d_2)}, & x \geq d_2, \end{cases} \] (4.34)
where \( u(z) = S^{-1}(z), d_2 = d_1 - q^* \) and \( q^* \in (0, \bar{\eta}_1) \) is the unique solution of the following equation
\[
g(\bar{\eta}_1) - g(q^*) - \beta_2(\bar{\eta}_1 - q^*) - K = 0, \tag{4.35}
\]
or, equivalently,
\[
g(\bar{\eta}_1) - g(0) - \beta_2\bar{\eta}_2 - K = 0, \tag{4.36}
\]
where \( \bar{\eta}_2 := \bar{\eta}_1 - q^* \) and \( g(x) \) is of the form (4.10). The associated optimal size of business is
\[
\bar{u}^\pi^*(x) = \begin{cases} 
S^{-1}(x + q^*), & 0 \leq x \leq d_2, \\
u^*, & x \geq d_2. 
\end{cases} \tag{4.37}
\]

Obviously, \( q^* \) is also decreasing with respect to \( K \). Similar to \( \tilde{f}(x) \), we can verify that \( g(x) \) and \( g_q(x) \) given in Subsection 4.1 and 4.2 satisfy HJB equations for all \( x \geq 0 \) in different cases, respectively. Also, they are twice continuously differentiable, increasing and concave. We omit the verification process.

4.3. Value function and associated optimal strategy. Similar to Section 3.3, the solution of Problem 2 identifies with the corresponding ones in either the first suboptimal problem without refinancing or the second one with forced refinancing, depending on the relationship among the model’s parameters. We present the explicit solution to Problem 2 as following.

**Theorem 4.2.** The value function \( \bar{V}(x) \) and associated optimal strategy \( \bar{\pi}^* \) can be identified in following 4 cases, which exhaust all possibilities.

(i) If \( P < \frac{\beta_1}{2\gamma} (\mu u^* - 2\delta) \) and \( \beta_1 < g'(0^+) = \beta_1 e^{\int_0^{d_1} \frac{\mu - 2\mu u(z)}{\sigma^2 u(z)} dz} \leq \beta_2 \), then \( \bar{V}(x) \) is consistent with \( g(x) \) defined by Eqs. (4.10) and (4.13). \( L^{\pi^*} \) is a threshold dividend strategy with level \( d_1 > 0 \), which is depicted by the following dividend rate
\[
l^{\pi^*}(x) = \begin{cases} 
M, & x > d_1, \\
0, & 0 \leq x \leq d_1. 
\end{cases} \tag{4.38}
\]

It is unprofitable to refinance and ruin is allowed, so \( G^{\pi^*}_t \equiv 0 \). The optimal size of business \( \bar{u}^\pi^*(x) \) is
\[
\bar{u}^\pi^*(x) = \begin{cases} 
S^{-1}(x), & 0 \leq x \leq d_1, \\
u^*, & x \geq d_1. 
\end{cases} \tag{4.39}
\]

Thus, the capital process controlled by \( \bar{\pi}^* = (u^{\pi^*}, L^{\pi^*}, G^{\pi^*}) \) satisfies that
\[
X^\pi^*_t = x + \int_0^t [\mu u^* (X^\pi^*_s) - a(u^* (X^\pi^*_s))^2 - \delta]ds + \int_0^t \sigma u^* (X^\pi^*_s)dB_s - L^{\pi^*}_t. \tag{4.40}
\]

(ii) If \( P < \frac{\beta_1}{2\gamma} (\mu u^* - 2\delta) \), \( g'(0^+) = \beta_1 e^{\int_0^{d_1} \frac{\mu - 2\mu u(z)}{\sigma^2 u(z)} dz} > \beta_2 \) and \( K \geq J(\bar{\eta}_1) \), the value function and associated optimal strategy take the same forms as those in (i).

(iii) If \( P \geq \frac{\beta_1}{2\gamma} (\mu u^* - 2\delta) \), then
\[
V(x) = \frac{\beta_1}{\delta} x + (P - \frac{\beta_1}{\delta}) e^{\gamma x}. \tag{4.41}
\]
The optimal strategy is to pay dividends at the maximum rate \( M \) until the time of ruin, no capital is injected all the time, the size of business remains unchanged, the liquidation value \( P \) is claimed once the ruin occurs. That is to say,

\[
l^\pi_\ast(x) = M, \quad x \geq 0,
\]

\[
u^\pi_\ast(x) = u^\ast, \quad x \geq 0,
\]

\[G^\pi_\ast(t) = 0, \quad t \geq 0.
\]

(iv) If \( P < \frac{\beta_1}{2\sigma}(\mu u^\ast - 2\delta) \), \( g'(0+) = \beta_1 e^{\int_0^{\bar{t}_1} \frac{2xu(x)}{x^2 + 1} \, dz} > \beta_2 \) and \( K < J(\bar{\eta}_1) \), then \( V(x) \) coincides with \( \tilde{g}(x) \) given by Eq. (4.34). \( \tilde{L}^\pi \) is a threshold dividend strategy with level \( d_2 > 0 \), which is depicted by the following dividend rate

\[
l^\pi_\ast(x) = \begin{cases} M, & x > d_2, \\ 0, & 0 \leq x \leq d_2. \end{cases}
\]

It is optimal to refinance when, and only when, the capital is null, the capital immediately jumps to \( \bar{\eta}_2 \) once it reaches 0 by issuing equities. Mathematically, \( \tilde{G}^\pi \) is characterized by

\[
\tilde{L}^\pi_\ast(x) = \begin{cases} S^{-1}(x + q^\ast), & 0 \leq x \leq d_2, \\ u^\ast, & x \geq d_2. \end{cases}
\]

Thereby, the capital process controlled by \( \tilde{\pi} = (\tilde{u}^\pi_\ast, \tilde{L}^\pi_\ast, \tilde{G}^\pi_\ast) \) follows that

\[
X_t^\tilde{\pi} = x + \int_0^t (\mu \tilde{u}(X_s^\tilde{\pi}) - a[\tilde{u}(X_s^\tilde{\pi})]^2 - \delta) \, ds + \int_0^t \sigma \tilde{u}(X_s^\tilde{\pi}) \, dB_s + \sum_{n=1}^{\infty} I_{(\tau^\pi_n \leq t)} \eta_n^\pi - \tilde{L}^\pi_\ast.
\]

Remark 2. The results in Theorem 3.2 and Theorem 4.2 illustrate that the company should postpone refinancing as long as possible, it should choose refinancing when and only when the proportional cost factor \( \beta_2 \) and the fixed cost \( K \) are not too high; it should expand the business size with the increasing of capital; it should pay dividends to shareholders with barrier strategy if the dividend rate is bounded, or pay dividends with threshold strategy if the dividend rate is unbounded. These results justify our intuition. In addition, all results obtained in Section 4 are compatible with those in Section 3, respectively, when the dividend ceiling \( M \) goes to infinity. That is, the optimal control problem without dividend restriction can be seen as the limiting optimal control problem with bounded dividend rate.

Appendix A. Proof of Theorem 3.1.

For each given strategy \( \pi = (u^\pi, L^\pi, G^\pi) \in \Pi \), define \( \Lambda_{L} = \{s : L^\pi_s \neq L^\pi_s\} \), \( \Lambda_{G} = \{s : G^\pi_s \neq G^\pi_s\} \) \( \{\tau^\pi_1, \tau^\pi_2, \ldots, \tau^\pi_n, \ldots\} \). Let \( \hat{L}^\pi_t = \sum_{s \in \Lambda_{L}, s \leq t} (L^\pi_s - L^\pi_s) \) be the discontinuous part of \( L^\pi_t \) and \( \tilde{L}^\pi_t = L^\pi_t - \hat{L}^\pi_t \) be the continuous part of
respectively. Then, applying the generalized Itô formula, we derive that

\[ e^{-r(t\wedge T^\pi)}v(X_{t\wedge T^\pi}^\pi) - v(x) = \int_0^{t\wedge T^\pi} e^{-rs} \varphi'(X_s^\pi)ds + \int_0^{t\wedge T^\pi} (\mu u(X_s^\pi) - au^2(X_s^\pi) - \delta)v'(X_s^\pi)dB_s 
- \int_0^{t\wedge T^\pi} e^{-rs}v'(X_s^\pi)d\hat{L}_s^\pi + \int_0^{t\wedge T^\pi} e^{-rs}v'(X_s^\pi)d\hat{G}_s^\pi 
+ \sum_{s \in \Delta_{\Delta L}^\pi, s \leq t\wedge T^\pi} e^{-rs}\left(v(X_s^\pi) - v(X_{s-}^\pi)\right). \quad (A.1)\]

The last term on the right hand side can be written as

\[ \sum_{s \in \Delta_{\Delta L}^\pi, s \leq t\wedge T^\pi} e^{-rs}\left(v(X_s^\pi) - v(X_{s-}^\pi)\right) \]

\[ = \sum_{s \in \Delta_{\Delta L}^\pi, s \leq t\wedge T^\pi} e^{-rs}\left(v(X_s^\pi) - v(X_{s-}^\pi)\right) + \sum_{s \in \Delta_{\Delta L}^\pi, s \leq t\wedge T^\pi} e^{-rs}\left(v(X_s^\pi) - v(X_{s-}^\pi)\right) \]

\[ \leq -\sum_{s \in \Delta_{\Delta L}^\pi, s \leq t\wedge T^\pi} e^{-rs}\beta_1(L_s^\pi - L_{s-}^\pi) + \sum_{n=1}^{\infty} e^{-r\tau_i^\pi}(\beta_2\eta_n^\pi + K)I_{\{\tau_i^\pi \leq t\wedge T^\pi\}}. \quad (A.2)\]

where the inequality is due to that \( v(x) \) satisfies the HJB equation Eq. (3.1) with \( v'(x) \geq \beta_1 \) and \( \mathcal{M} v(x) \leq v(x) \). Moreover, in view of Eq. (3.1), the first term on the right hand side of (A.1) is non-positive. So substituting (A.2) into (A.1), we obtain

\[ e^{-r(t\wedge T^\pi)}v(X_{t\wedge T^\pi}^\pi) \leq v(x) + \int_0^{t\wedge T^\pi} (\mu u(X_s^\pi) - au^2(X_s^\pi) - \delta)e^{-rs}v'(X_s^\pi)dB_s 
- \beta_1 \int_0^{t\wedge T^\pi} e^{-rs}dL_s^\pi + \sum_{n=1}^{\infty} e^{-r\tau_i^\pi}(\beta_2\eta_n^\pi + K)I_{\{\tau_i^\pi \leq t\wedge T^\pi\}}. \quad (A.3)\]

Since \( v(x) \) is increasing and \( v(0) \geq P \), we have

\[ e^{-r(t\wedge T^\pi)}P \leq v(x) + \int_0^{t\wedge T^\pi} (\mu u(X_s^\pi) - au^2(X_s^\pi) - \delta)e^{-rs}v'(X_s^\pi)dB_s 
- \beta_1 \int_0^{t\wedge T^\pi} e^{-rs}dL_s^\pi + \sum_{n=1}^{\infty} e^{-r\tau_i^\pi}(\beta_2\eta_n^\pi + K)I_{\{\tau_i^\pi \leq t\wedge T^\pi\}}. \quad (A.4)\]

The stochastic integral with respect to the Brownian motion in (A.4) is a uniformly integrable martingale, if \( v'(x) \) is bounded. Taking expectation and limit on both sides of (A.4) yields

\[ v(x) \geq \mathbb{E}_x\left(\beta_1 \int_0^{T^\pi} e^{-rs}dL_s^\pi - \sum_{n=1}^{\infty} e^{-r\tau_i^\pi}(\beta_2\eta_n^\pi + K)I_{\{\tau_i^\pi \leq T^\pi\}} + Pe^{-rT^\pi}\right) \]

\[ = V(x; \pi). \quad (A.5)\]

Consequently, \( v(x) \geq V(x) \) by (2.6). Applying the method in \( \Pi \), we can also prove above results when \( v'(x) \) is unbounded by modifying above proof process, it is omitted here.

**Appendix B. Proof of (iv) in Theorem 3.2.**

In the case of \( P < \frac{\bar{b}}{\bar{r}}(\frac{\mu^2}{4\sigma^2} - \delta) \), \( f'(0+) > \beta_2 \) and \( K < I(\eta_1) \), it proves that \( \hat{f}(x) \) satisfies Eq. (3.1) and (3.2), so it has \( \hat{f}(x) \geq V(x) \) according to Theorem 3.1.
We can deduce that the function $b(u) := \mu u - au^2 - \delta$ and the function $\theta(u) := \sigma u$ are uniformly bounded real-valued functions on $[u_0, \frac{\mu}{2\sigma}]$ satisfying a uniform Lipschitz condition. Similar to [9], using Theorem 3.1 in [13] the strategy $\pi^*$ is uniquely determined by Eqs. (3.29), (3.40)-(3.42). Since $\mathcal{A}''u^* \tilde{f}(X_t^\pi) = 0$ for $0 \leq X_t^\pi \leq b_2$, one has
\[
\int_0^{t \wedge \tau^x} e^{-rs} \mathcal{A}''u^* \tilde{f}(X_s^\pi) ds = \int_0^{t \wedge \tau^x} e^{-rs} \mathcal{A}''u^* \tilde{f}(X_s^\pi) \mathbb{I}_{\{0 \leq X_s^\pi \leq b_2\}} ds = 0. \tag{B.1}
\]
Furthermore, Eqs. (3.25) and (3.27) lead to
\[
\sum_{s \in \Lambda_s^x \cup \Lambda_s^y, \pi \leq t \wedge \tau^x} e^{-rs} \left( f^*(X_s^\pi) - f(X_s^\pi) \right) \\
= \sum_{s \in \Lambda_s^x, \pi \leq t \wedge \tau^x} e^{-rs} \left( f^*(X_s^\pi) - f(X_s^\pi) \right) \mathbb{I}_{\{X_s^\pi = b_2\}} \\
+ \sum_{s \in \Lambda_s^y, \pi \leq t \wedge \tau^x} e^{-rs} \left( f^*(X_s^\pi) - f(X_s^\pi) \right) \mathbb{I}_{\{X_s^\pi = 0\}} \\
= - \sum_{s \in \Lambda_s^x, \pi \leq t \wedge \tau^x} e^{-rs} \beta_1 (L_s^\pi - L_s^{\pi^*}) \\
+ \sum_{n=1}^{\infty} e^{-rs} (\beta_2 \eta_n^\pi + K) \mathbb{I}_{\{\tau_n^\pi \leq t \wedge \tau^x\}}. \tag{B.2}
\]
Replacing $\pi, \tau^x, v$ with $\pi^*, \tau^{\pi^*} = \infty$, $\tilde{f}$ in Itô formula (A.1) and taking expectations, we have
\[
\tilde{f}(x) = \mathbb{E}_x [e^{-rt} \tilde{f}(X_t^\pi)] + \mathbb{E}_x \left( \beta_1 \int_0^t e^{-rs} dL_s^\pi - \sum_{n=1}^{\infty} e^{-rs} (\beta_2 \eta_n^\pi + K) \mathbb{I}_{\{\tau_n^\pi \leq t\}} \right). \tag{B.3}
\]
Letting $t \to \infty$, the first term on the right hand side vanishes, then we obtain
\[
\tilde{f}(x) = \mathbb{E}_x \left( \beta_1 \int_0^\infty e^{-rs} dL_s^\pi - \sum_{n=1}^{\infty} e^{-rs} (\beta_2 \eta_n^\pi + K) \mathbb{I}_{\{\tau_n^\pi < \infty\}} \right) = V(x; \pi^*), \tag{B.4}
\]
which, together with $\tilde{f}(x) \geq V(x)$, establishes that $\tilde{f}(x) = V(x) = V(x; \pi^*)$.

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