NIJENHUIS ALGEBRAS, NS ALGEBRAS AND N-DENDRIFORM ALGEBRAS

PENG LEI AND LI GUO

Abstract. In this paper we study (associative) Nijenhuis algebras, with emphasis on the relationship between the category of Nijenhuis algebras and the categories of NS algebras. This is in analogy to the well-known theory of the adjoint functor from the category of Lie algebras to that of associative algebras, and the more recent results on the adjoint functor from the categories of dendriform and tridendriform algebras to that of Rota-Baxter algebras. We first give an explicit construction of free Nijenhuis algebras and then apply it to obtain the universal enveloping Nijenhuis algebra of an NS algebra. We further apply the construction to determine the binary quadratic nonsymmetric algebra, called the N-dendriform algebra, that is compatible with the Nijenhuis algebra. As it turns out, the N-dendriform algebra has more relations than the NS algebra.

Contents

1. Introduction
2. Free Nijenhuis algebra on an algebra
   2.1. A basis of the free Nijenhuis algebra
   2.2. The product in a free Nijenhuis algebra
   2.3. The proof of Theorem
3. NS algebras and their universal enveloping algebras
4. From Nijenhuis algebras to N-dendriform algebras
   4.1. Background and the statement of Theorem
   4.2. The proof of Theorem
References

1. Introduction

Through the antisymmetry bracket \([x, y] := xy - yx\), an associative algebra \(A\) defines a Lie algebra structure on \(A\). The resulting functor from the category of associative algebras to that of Lie algebras and its adjoint functor have played a fundamental role in the study of these algebraic structures. A similar relationship holds for Rota-Baxter algebras and dendriform algebras.

This paper studies a similar relationship between (associative) Nijenhuis algebras and NS algebras.

A Nijenhuis algebra is a nonunitary associative algebra \(N\) with a linear endomorphism \(P\) satisfying the Nijenhuis equation:

\[
P(x)P(y) = P(P(xy) + P(xy)) - P^2(xy), \quad \forall x, y \in N.
\]

The concept of a Nijenhuis operator on a Lie algebra originated from the important concept of a Nijenhuis tensor that was introduced by Nijenhuis [25] in the study of pseudo-complex manifolds in the 1950s and was related to the well-known concepts of Schouten-Nijenhuis bracket, the
Frölicher-Nijenhuis bracket \([\mathcal{I}]\) and the Nijenhuis-Richardson bracket. Nijenhuis operator operators on a Lie algebra appeared in \([24]\) in a more general study of Poisson-Nijenhuis manifolds and then more recently in \([14, 5]\) in the context of the classical Yang-Baxter equation.

The Nijenhuis operator on an associative algebra was introduced by Carinena and coauthors \([3]\) to study quantum bi-Hamiltonian systems. In \([24]\), Nijenhuis operators are constructed by analogy with Poisson-Nijenhuis geometry, from relative Rota-Baxter operators.

Note the close analogue of the Nijenhuis operator with the more familiar Rota-Baxter operator of weight \(\lambda\) (where \(\lambda\) is a constant) defined to be a linear endomorphism \(P\) on an associative algebra \(R\) satisfying

\[
P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy), \quad \forall x, y \in R.
\]

The latter originated from the probability study of G. Baxter \([3]\), was studied by Cartier and Rota and is closely related to the operator form of the classical Yang-Baxter equation. Its study has experienced a quite remarkable renascence in the last decade with many applications in mathematics and physics, most notably the work of Connes and Kreimer on renormalization of quantum field theory \([3, 14, 11]\). See \([14]\) for further details and references.

The recent theoretic developments of Nijenhuis algebras have largely followed those of Rota-Baxter algebras. Commutative Nijenhuis algebras were constructed in \([7, 23]\) following the construction of free commutative Rota-Baxter algebras \([17]\).

Another development followed the relationship between Rota-Baxter algebras and dendriform algebras. Recall that a dendriform algebra, defined by Loday \([22]\), is a vector space \(D\) with two binary operations \(<\) and \(>\) such that

\[
(x < y) < z = x < (y \star z), \quad (x > y) < z = x > (y > z), \quad (x \star y) > z = x > (y > z), \quad x, y, z \in D,
\]

where \(\star := < + >\). Similarly a tridendriform algebra, defined by Loday and Ronco \([23]\), is a vector space \(T\) with three binary operations \(<, >\) and \(\cdot\) that satisfy seven relations. Aguiar \([1]\) showed that for a Rota-Baxter algebra \((R, P)\) of weight 0, the binary operations

\[
x <_P y := xP(y), \quad x >_P y := P(x)y, \quad \forall x, y \in R,
\]

define a dendriform algebra on \(R\). Similarly, Ebrahimi-Fard \([5]\) showed that, for a Rota-Baxter algebra \((R, P)\) of non-zero weight, the binary operations

\[
x <_P y := xP(y), \quad x >_P y := P(x)y, \quad x \cdot_P y := \lambda xy, \quad \forall x, y \in R,
\]

define a tridendriform algebra on \(R\).

As an analogue of the tridendriform algebra, the concept of an NS algebra was introduced by Leroux \([21]\), to be a vector space \(M\) with three binary operations \(<, >\) and \(\cdot\) that satisfy four relations (see Eq. \((14)\)). As an analogue of the Rota-Baxter algebra case, it was shown \([21]\) that, for a Nijenhuis algebra \((N, P)\), the binary operations

\[
x <_P y := xP(y), \quad x >_P y := P(x)y, \quad x \cdot_P y := -P(xy), \quad \forall x, y \in R,
\]

defines an NS algebra on \(R\).

Considering the adjoint functor of the functor induced by the above mentioned map from Rota-Baxter algebras to (tri-)dendriform algebras, the Rota-Baxter universal enveloping algebra of a (tri-)dendriform algebra was constructed in \([3]\). For this purpose, free Rota-Baxter algebras was first constructed.

In this paper we give a similar approach for Nijenhuis algebras, but we go beyond the case of Rota-Baxter algebras. Our first goal is to give an explicit construction of free Nijenhuis algebras.
in Section 2. We consider both the cases when the free Nijenhuis algebra is generated by a set and by another algebra. Other than its role in the theoretical study of Nijenhuis algebras, this construction allows us to construct the universal enveloping algebra of an NS algebra. We achieve this in Section 3.

Knowing that a Nijenhuis algebra gives an NS algebra, it is natural to ask what other dendriform type algebras that Nijenhuis algebras can give in a similar way. As a second application of our construction of free Nijenhuis algebras, we determine all “quadratic nonsymmetric” relations that can be derived from Nijenhuis algebras and find that one can actually derive more relations than defined by the NS algebra in Eq. (16). This discussion is presented in Section 4.

Notation: In this paper \( k \) is taken to be a field. A \( k \)-algebra is taken to be nonunitary associative unless otherwise stated.

2. Free Nijenhuis algebra on an algebra

We start with the definition of free Nijenhuis algebras.

Definition 2.1. Let \( A \) be a \( k \)-algebra. A free Nijenhuis algebra over \( A \) is a Nijenhuis algebra \( F_N(A) \) with a Nijenhuis operator \( P_A \) and an algebra homomorphism \( j_A : A \to F_N(A) \) such that, for any Nijenhuis algebra \( N \) and any algebra homomorphism \( f : A \to N \), there is a unique Nijenhuis algebra homomorphism \( \bar{f} : F_N(A) \to N \) such that \( \bar{f} \circ j_A = f \):

![Diagram]

For the construction of free Nijenhuis algebras, we follow the construction of free Rota-Baxter algebras [8,16] by bracketed words. Alternatively, one can follow [9] to give the construction by rooted trees that is more in the spirit of operads [24]. One can also follow the approach of Gröbner-Shirshov bases [3]. Because of the lack of a uniform approach (see [18,19] for some recent attempts in this direction) and to be notationally self-contained, we give some details. We first display a \( k \)-basis of the free Nijenhuis algebra in terms of bracketed words in § 2.1. The product on the free Nijenhuis algebra is given in § 2.2 and the universal property of the free Nijenhuis algebra is proved in § 2.3.

2.1. A basis of the free Nijenhuis algebra. Let \( A \) be a \( k \)-algebra with a \( k \)-basis \( X \). We first display a \( k \)-basis \( X_\infty \) of \( F_N(A) \) in terms of bracketed words from the alphabet set \( X \).

Let \( [ \) and \( ] \) be symbols, called brackets, and let \( X' = X \cup \{ [\], ] \}. \) Let \( M(X') \) denote the free semigroup generated by \( X' \).

Definition 2.2. ([3,13]) Let \( Y, Z \) be two subsets of \( M(X') \). Define the alternating product of \( Y \) and \( Z \) to be

\[
\Lambda(Y, Z) = \bigcup_{r \geq 1} (Y[Z])^r \bigcup \bigcup_{r \geq 0} (Y[Z])^r Y \bigcup (Z Y)^r \bigcup \bigcup_{r \geq 1} (Z Y)^r [Z].
\]

We construct a sequence \( \bar{x}_n \) of subsets of \( M(X') \) by the following recursion. Let \( \bar{x}_0 = X \) and, for \( n \geq 0 \), define

\[
\bar{x}_{n+1} = \Lambda(X, \bar{x}_n).
\]
Further, define
\[
\mathcal{X}_\infty = \bigcup_{n \geq 0} \mathcal{X}_n = \lim_{n \to \infty} \mathcal{X}_n.
\]
Here the second equation in Eq. (3) follows since \( \mathcal{X}_1 \supseteq \mathcal{X}_0 \) and, assuming \( \mathcal{X}_n \supseteq \mathcal{X}_{n-1} \), we have
\[
\mathcal{X}_{n+1} = \Lambda(X, \mathcal{X}_n) \supseteq \Lambda(X, \mathcal{X}_{n-1}) \supseteq \mathcal{X}_n.
\]

By [8, 10] we have the disjoint union
\[
\mathcal{X}_\infty = \bigcup_{r \geq 1} \left( \left( \bigcup_{r \geq 0} (X[\mathcal{X}_\infty])^r \right) \bigcup \left( \bigcup_{r \geq 0} (X[\mathcal{X}_\infty])^r X \right) \right).
\]
Further, define
\[
x = x_1 \cdots x_b,
\]
where \( x_i, 1 \leq i \leq b \), is alternatively in \( X \) or in \( [\mathcal{X}_\infty] \). This decomposition will be called the \textbf{standard decomposition} of \( x \).

For \( x \) in \( \mathcal{X}_\infty \) with standard decomposition \( x_1 \cdots x_b \), we define \( b \) to be the \textbf{breadth} \( b(x) \) of \( x \), we define the \textbf{head} \( h(x) \) of \( x \) to be 0 (resp. 1) if \( x_1 \) is in \( X \) (resp. in \( [\mathcal{X}_\infty] \)). Similarly define the \textbf{tail} \( t(x) \) of \( x \) to be 0 (resp. 1) if \( x_b \) is in \( X \) (resp. in \( [\mathcal{X}_\infty] \)).

\[ \text{The product in a free Nijenhuis algebra.} \]
Let
\[
F_N(A) = \bigoplus_{x \in \mathcal{X}_\infty} kx.
\]
We now define a product \( \circ \) on \( F_N(A) \) by defining \( x \circ x' \in F_N(A) \) for \( x, x' \in \mathcal{X}_\infty \) and then extending bilinearly. Roughly speaking, the product of \( x \) and \( x' \) is defined to be the concatenation whenever \( t(x) \neq h(x') \). When \( t(x) = h(x') \), the product is defined by the product in \( A \) or by the Nijenhuis relation in Eq. (4).

To be precise, we use induction on the sum \( n := d(x) + d(x') \) of the depths of \( x \) and \( x' \). Then \( n \geq 0 \). If \( n = 0 \), then \( x, x' \) are in \( X \) and so are in \( A \) and we define \( x \circ x' = x \cdot x' \in A \subseteq F_N(A) \). Here \( \cdot \) is the product in \( A \).

Suppose \( x \circ x' \) have been defined for all \( x, x' \in \mathcal{X}_\infty \) with \( n \geq k \geq 0 \) and let \( x, x' \in \mathcal{X}_\infty \) with \( n = k + 1 \).

First assume the breadth \( b(x) = b(x') = 1 \). Then \( x \) and \( x' \) are in \( X \) or \( [\mathcal{X}_\infty] \). Since \( n = k + 1 \) is at least one, \( x \) and \( x' \) cannot be both in \( X \). We accordingly define
\[
x \circ x' = \begin{cases} xx', & \text{if } x \in X, x' \in [\mathcal{X}_\infty], \\
x'x, & \text{if } x \in [\mathcal{X}_\infty], x' \in X, \\
\lfloor x \rfloor \circ \lfloor x' \rfloor + [\lfloor x \rfloor \circ [\lfloor x' \rfloor]] - \lfloor [\lfloor x \rfloor \circ [\lfloor x' \rfloor]] \rfloor, & \text{if } x = [\lfloor x \rfloor], x' = [\lfloor x' \rfloor] \in [\mathcal{X}_\infty]. \end{cases}
\]

Here the product in the first and second case are by concatenation and in the third case is by the induction hypothesis since for the three products on the right hand side we have
\[
d([\lfloor x \rfloor]) + d([\lfloor x' \rfloor]) = d([\lfloor x \rfloor]) + d([\lfloor x' \rfloor]) - 1 = d(x) + d(x') - 1, \\
d([\overline{x}]) + d([\overline{x}]) = d([\overline{x}]) + d([\overline{x}]) - 1 = d(x) + d(x') - 1, \\
d([\overline{x}]) + d([\overline{x}]) = d([\overline{x}]) - 1 + d([\overline{x}]) - 1 = d(x) + d(x') - 2
\]
which are all less than or equal to \( k \).
Now assume \( b(x) > 1 \) or \( b(x') > 1 \). Let \( x = x_1 \cdots x_b \) and \( x' = x'_1 \cdots x'_{b'} \) be the standard decompositions from Eq. (3). We then define
\[
x \circ x' = x_1 \cdots x_{b-1}(x_b \circ x'_1) x'_2 \cdots x'_{b'},
\]
where \( x_b \circ x'_1 \) is defined by Eq. (3) and the rest is given by concatenation. The concatenation is well-defined since by Eq. (4), we have \( h(x_b) = h(x_b \circ x'_1) \) and \( t(x'_1) = t(x_b \circ x'_1) \). Therefore, \( t(x_{b-1}) \neq h(x_b \circ x'_1) \) and \( h(x'_2) \neq t(x_b \circ x'_1) \).

We have the following simple properties of \( \circ \).

**Lemma 2.3.** Let \( x, x' \in \mathcal{X}_\infty \). We have the following statements.
(a) \( h(x) = h(x \circ x') \) and \( t(x') = t(x \circ x') \).
(b) If \( t(x) \neq h(x') \), then \( x \circ x' = xx' \) (concatenation).
(c) If \( t(x) \neq h(x') \), then for any \( x'' \in \mathcal{X}_\infty \),
\[
(xx') \circ x'' = x(x' \circ x''), \quad x'' \circ (xx') = (x'' \circ x)x'.
\]

Extending \( \circ \) bilinearly, we obtain a binary operation
\[
F_N(A) \otimes F_N(A) \rightarrow F_N(A).
\]
For \( x \in \mathcal{X}_\infty \), define
\[
N_A(x) = [x].
\]
Obviously \([x]\) is again in \( \mathcal{X}_\infty \). Thus \( N_A \) extends to a linear operator \( N_A \) on \( F_N(A) \). Let
\[
j_X : X \rightarrow \mathcal{X}_\infty \rightarrow F_N(A)
\]
be the natural injection which extends to an algebra injection
\[
j_A : A \rightarrow F_N(A).
\]

The following is our first main result which will be proved in the next subsection.

**Theorem 2.4.** Let \( A \) be a \( k \)-algebra with a \( k \)-basis \( X \).
(a) The pair \( (F_N(A), \circ) \) is an algebra.
(b) The triple \( (F_N(A), \circ, N_A) \) is a Nijenhuis algebra.
(c) The quadruple \( (F_N(A), \circ, N_A, j_A) \) is the free Nijenhuis algebra on the algebra \( A \).

The following corollary of the theorem will be used later in the paper.

**Corollary 2.5.** Let \( M \) be a \( k \)-module and let \( T(M) = \bigoplus_{n \geq 1} M^{\otimes n} \) be the reduced tensor algebra over \( M \). Then \( F_N(T(M)) \), together with the natural injection \( i_M : M \rightarrow T(M) \xrightarrow{j_T(M)} F_N(T(M)) \), is a free Nijenhuis algebra over \( M \), in the sense that, for any Nijenhuis algebra \( N \) and \( k \)-module map \( f : M \rightarrow N \) there is a unique Nijenhuis algebra homomorphism \( \hat{f} : F_N(T(M)) \rightarrow N \) such that \( \hat{f} \circ k_M = f \).

**Proof.** This follows immediately from Theorem 2.4 and the fact that the construction of the free algebra on a module (resp. free Nijenhuis algebra on an algebra, resp. free Nijenhuis on a module) is the left adjoint functor of the forgetful functor from algebras to modules (resp. from Nijenhuis algebras to algebras, resp. from Nijenhuis algebras to modules), and the fact that the composition of two left adjoint functors is the left adjoint functor of the composition. \( \square \)
2.3. The proof of Theorem 2.4.

Proof. We just need to verify the associativity. For this we only need to verify
\[ (x' \circ x'') \circ x''' = x' \circ (x'' \circ x''') \]
for \( x', x'', x''' \in \mathfrak{X}_\infty \). We will do this by induction on the sum of the depths \( n := d(x') + d(x'') + d(x''') \). If \( n = 0 \), then all of \( x', x'', x''' \) have depth zero and so are in \( X \). In this case the product \( \circ \) is given by the product \( \cdot \) in \( A \) and so is associative.

Assume the associativity holds for \( n \leq k \) and assume that \( x', x'', x''' \in \mathfrak{X}_\infty \) have \( n = d(x') + d(x'') + d(x''') = k + 1 \).

If \( t(x') = h(x'') \), then by Lemma 2.3,
\[ (x' \circ x'') \circ x''' = (x' \circ x'') \circ x''' = x'(x'' \circ x'''') = x' \circ (x'' \circ x'''') \]
A similar argument holds when \( t(x'') \neq h(x''') \).

Thus we only need to verify the associativity when \( t(x') = h(x'') \) and \( t(x'') = h(x''') \). We next reduce the breadths of the words.

Lemma 2.6. If the associativity
\[ (x' \circ x'') \circ x''' = x' \circ (x'' \circ x'''') \]
holds for all \( x', x'' \) and \( x''' \) in \( \mathfrak{X}_\infty \) of breadth one, then it holds for all \( x', x'' \) and \( x''' \) in \( \mathfrak{X}_\infty \).

Proof. We use induction on the sum of breadths \( m := b(x') + b(x'') + b(x''') \). Then \( m \geq 3 \). The case when \( m = 3 \) is the assumption of the lemma. Assume the associativity holds for \( 3 \leq m \leq j \) and take \( x', x'', x''' \in \mathfrak{X}_\infty \) with \( m = j + 1 \). Then \( j + 1 \geq 4 \). So at least one of \( x', x'', x''' \) have breadth greater than or equal to 2.

First assume \( b(x') \geq 2 \). Then \( x' = x'_1 x'_2 \) with \( x'_1, x'_2 \in \mathfrak{X}_\infty \) and \( t(x'_1) \neq h(x'_2) \). Thus by Lemma 2.3, we obtain
\[ (x' \circ x'') \circ x''' = ((x'_1 x'_2) \circ x'') \circ x''' = (x'_1 (x'_2 \circ x'')) \circ x''' = x'_1 ((x'_2 \circ x'') \circ x'''') \]
Similarly,
\[ x' \circ (x'' \circ x''') = (x'_1 x'_2) \circ (x'' \circ x''') = x'_1 (x'_2 \circ (x'' \circ x'''')) \]
Thus \( (x' \circ x'') \circ x''' = x' \circ (x'' \circ x'''') \) whenever \( (x'_1 \circ x'') \circ x''' = x'_1 \circ (x'' \circ x'''') \). The latter follows from the induction hypothesis. A similar proof works if \( b(x''') \geq 2 \).

Finally if \( b(x') \geq 2 \), then \( x' = x'' x''' \) with \( x'_1, x'_2 \in \mathfrak{X}_\infty \) and \( t(x'_1) \neq h(x'_2) \). By applying Lemma 2.3 repeatedly, we obtain
\[ (x' \circ x'') \circ x''' = (x' \circ (x'' x''')) \circ x''' = ((x' \circ x''') x'''') \circ x''' = (x' \circ x''') (x'' \circ x'''') \]
In the same way, we have
\[ (x' \circ x'_1)(x'_2 \circ x'''') = x' \circ (x'' \circ x'''') \]
This again proves the associativity. \( \square \)

To summarize, our proof of the associativity has been reduced to the special case when \( x', x'', x''' \) in \( \mathfrak{X}_\infty \) are chosen so that
(a) \( n := d(x') + d(x'') + d(x''') = k + 1 \) with the assumption that the associativity holds when \( n \leq k \).
(b) the elements have breadth one and
(c) \( t(x') = h(x'') \) and \( t(x'') = h(x''') \).
By item [b], the head and tail of each of the elements are the same. Therefore by item [c], either all the three elements are in $X$ or they are all in $[\mathbb{X}_\infty]$. If all of $x', x'', x'''$ are in $X$, then as already shown, the associativity follows from the associativity in $A$.

So it remains to consider the case when $x', x'', x'''$ are all in $[\mathbb{X}_\infty]$. Then $x' = [\mathbb{X}'], x'' = [\mathbb{X}''], x''' = [\mathbb{X}''']$ with $\mathbb{X}', \mathbb{X}'', \mathbb{X}''' \in \mathbb{X}_\infty$. Using Eq. (3) and bilinearity of the product $\circ$, we have

$$(x' \circ x'') \circ x''' = ([x] \circ [x']) \circ [x''] - ([x] \circ [x']) \circ [x''']$$

Applying the induction hypothesis in $n$ to the fifth term ($\circ [x'] \circ [x'']$) and the eighth term, and then use Eq. (3) again, we obtain

$$(x' \circ x'') \circ x''' = ([x] \circ [x']) \circ [x''] - ([x] \circ [x']) \circ [x''']$$

By a similar computation, we obtain

$$x' \circ (x'' \circ x''') = ([x] \circ [x']) \circ [x''] - ([x] \circ [x']) \circ [x''']$$

Now by induction, the $i$-th term in the expansion of $(x' \circ x'') \circ x'''$ matches with the $\sigma(i)$-th term in the expansion of $x' \circ (x'' \circ x''')$. Here the permutation $\sigma \in \Sigma_{11}$ is given by

$$\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & 6 & 9 & 2 & 4 & 7 & 11 & 5 & 3 & 8 & 10
\end{pmatrix}.$$ 

This completes the proof of Theorem 2.4 (a).

(b). The proof follows from the definition $N_A(x) = [x]$ and Eq. (3).

(c). Let $(N, \ast, P)$ be a Nijenhuis algebra with multiplication $\ast$. Let $f : A \to N$ be a $k$-algebra homomorphism. We will construct a $k$-linear map $\bar{f} : F_N(A) \to N$ by defining $\bar{f}(x)$ for $x \in \mathbb{X}_n$.

We achieve this by defining $\bar{f}(x)$ for $x \in \mathbb{X}_n$, $n \geq 0$, inductively on $n$. For $x \in \mathbb{X}_0 := X$, define $\bar{f}(x) = f(x)$. Suppose $\bar{f}(x)$ has been defined for $x \in \mathbb{X}_n$ and consider $x$ in $\mathbb{X}_{n+1}$ which is, by definition and Eq. (4),

$$\Lambda(X, \mathbb{X}_n) = \left( \bigcup_{r \geq 1} \left( X[\mathbb{X}_n] \right)^r \right) \bigcup \left( \bigcup_{r \geq 0} \left( X[\mathbb{X}_n] \right)^r \bigcup \mathbb{X}_n \right).$$


Let \( \mathbf{x} \) be in the first union component \( \bigsqcup_{r \geq 1} (X \langle \mathfrak{x}_r \rangle)' \) above. Then
\[
\mathbf{x} = \prod_{i=1}^{r} (x_{2i-1} \langle \mathfrak{x}_{2i} \rangle)
\]
for \( x_{2i-1} \in X \) and \( x_{2i} \in \mathfrak{x}_r \), \( 1 \leq i \leq r \). By the construction of the multiplication \( \circ \) and the Nijenhuis operator \( N_A \), we have
\[
\mathbf{x} = \circ_{i=1}^{r} (x_{2i-1} \circ [x_{2i}]) = \circ_{i=1}^{r} (x_{2i-1} \circ N_A(x_{2i})).
\]
Define
\[
\bar{f}(x) = \circ_{i=1}^{r} (\bar{f}(x_{2i-1}) \ast N(\bar{f}(x_{2i}))).
\]
where the right hand side is well-defined by the induction hypothesis. Similarly define \( \bar{f}(x) \) if \( x \) is in the other union components. For any \( x \in \mathfrak{x}_n \), we have \( P_A(x) = [x] \in \mathfrak{x}_n \), and by the definition of \( \bar{f} \) in (Eq. (12))) we have
\[
\bar{f}([x]) = P(\bar{f}(x)).
\]
So \( \bar{f} \) commutes with the Nijenhuis operators. Combining this equation with Eq. (12) we see that if \( x = x_1 \cdots x_b \) is the standard decomposition of \( x \), then
\[
\bar{f}(x) = \bar{f}(x_1) \cdots \bar{f}(x_b).
\]
Note that this is the only possible way to define \( \bar{f}(x) \) in order for \( \bar{f} \) to be a Nijenhuis algebra homomorphism extending \( f \).

It remains to prove that the map \( \bar{f} \) defined in Eq. (12) is indeed an algebra homomorphism. For this we only need to check the multiplicity
\[
\bar{f}(\mathbf{x} \circ \mathbf{x}') = \bar{f}(\mathbf{x}) \circ \bar{f}(\mathbf{x}')
\]
for all \( \mathbf{x}, \mathbf{x}' \in \mathfrak{x}_n \). For this we use induction on the sum of depths \( n := d(\mathbf{x}) + d(\mathbf{x}') \). Then \( n \geq 0 \). When \( n = 0 \), we have \( \mathbf{x}, \mathbf{x}' \in X \). Then Eq. (13) follows from the multiplicity of \( \bar{f} \). Assume the multiplicity holds for \( \mathbf{x}, \mathbf{x}' \in \mathfrak{x}_n \) with \( n \geq k \) and take \( \mathbf{x}, \mathbf{x}' \in \mathfrak{x}_n \) with \( n = k + 1 \). Let \( \mathbf{x} = x_1 \cdots x_b \) and \( \mathbf{x}' = x_1' \cdots x_{b'}' \) be the standard decompositions. Since \( n = k + 1 \geq 1 \), at least one of \( x_b \) and \( x_{b'}' \) is in \( [\mathfrak{x}_n] \). Then by Eq. (13) we have,
\[
\bar{f}(x_b \circ x_b') = \begin{cases} 
\bar{f}(x_b x_b'), & \text{if } x_b \in X, x_b' \in [\mathfrak{x}_n], \\
\bar{f}(x_b x_b'), & \text{if } x_b \in [\mathfrak{x}_n], x_b' \in X,
\end{cases}
\]
\[
\bar{f}([\mathfrak{x}_b] \circ [\mathfrak{x}_b']) + [\mathfrak{x}_b] \circ [\mathfrak{x}_b'] - [\mathfrak{x}_b] \circ [\mathfrak{x}_b'] = P(\bar{f}(\mathfrak{x}_b)) * \bar{f}(\mathfrak{x}_b').
\]
In the first two cases, the right hand side is \( \bar{f}(x_b) \ast \bar{f}(x_b') \) by the definition of \( \bar{f} \). In the third case, we have, by Eq. (13), the induction hypothesis and the Nijenhuis relation of the operator \( P \) on \( N \),
\[
\bar{f}([\mathfrak{x}_b] \circ [\mathfrak{x}_b']) + [\mathfrak{x}_b] \circ [\mathfrak{x}_b'] - [\mathfrak{x}_b] \circ [\mathfrak{x}_b'] = P(\bar{f}(\mathfrak{x}_b)) \ast \bar{f}(\mathfrak{x}_b') + P(\bar{f}(\mathfrak{x}_b) \ast \mathfrak{x}_b') - P(\bar{f}(\mathfrak{x}_b) \ast \mathfrak{x}_b').
\]
Therefore $\bar{f}(x_b \circ x'_1) = \bar{f}(x_b) \ast \bar{f}(x'_1)$. Then

$$
\bar{f}(x \circ x') = \bar{f}(x_1 \cdots x_{b-1}(x_b \circ x'_1)x'_2 \cdots x'_{b'}) \\
= \bar{f}(x_1) \ast \cdots \ast \bar{f}(x_{b-1}) \ast \bar{f}(x_b \circ x'_1) \ast \bar{f}(x'_2) \cdots \bar{f}(x'_{b'}) \\
= \bar{f}(x_1) \ast \cdots \ast \bar{f}(x_{b-1}) \ast \bar{f}(x_b) \ast \bar{f}(x'_1) \ast \bar{f}(x'_2) \cdots \bar{f}(x'_{b'}) \\
= \bar{f}(x) \ast \bar{f}(x').
$$

This is what we need. \hfill \Box

3. NS algebras and their universal enveloping algebras

The concept of an NS algebra was introduced by Leroux \cite{Leroux} as an analogue of the dendriform algebra of Loday \cite{Loday} and the tridendriform algebra of Loday and Ronco \cite{Loday-Ronco}.

**Definition 3.1.** An NS algebra is a module $M$ with three binary operations $<, >$ and $\cdot$ that satisfy the following four relations

$$
(x < y) < z = x < (y \ast z), \quad (x > y) < z = x > (y < z),
$$

$$(x \ast y) > z = x > (y > z), \quad (x \ast y) \cdot z + (x \cdot y) < z = x > (y \cdot z) + x \cdot (y \ast z).
$$

for $x, y, z \in M$. Here $\ast$ denotes $< + > + \cdot$.

NS algebras share similar properties as dendriform algebras. For example, the operation $\ast$ defines an associative operation. Another similarity is the following theorem which is an analogue of the results of Aguiar \cite{Aguiar} and Ebrahimi-Fard \cite{Ebrahimi-Fard} that a Rota-Baxter algebra gives a dendriform algebra or a tridendriform algebra.

**Theorem 3.2.** \cite{Leroux} A Nijenhuis algebra $(N, P)$ defines an NS algebra $(N, <_P, >_P, \cdot_P)$, where

$$
x <_P y = xP(y), \quad x >_P y = P(x)y, \quad x \cdot_P y = -P(xy).
$$

Let $\mathcal{NA}$ denote the category of Nijenhuis algebras and let $\mathcal{NS}$ denote the category of NS algebras. It is easy to see that the map from $\mathcal{NA}$ to $\mathcal{NS}$ in Theorem 3.2 is compatible with the morphisms in the two categories. Thus we obtain a functor

$$
\mathcal{E} : \mathcal{NA} \to \mathcal{NS}.
$$

We will study its left adjoint functor.

Motivated by the enveloping algebra of a Lie algebra and the Rota-Baxter enveloping algebra of a tridendriform algebra \cite{Ebrahimi-Fard}, we are naturally led to the following definition.

**Definition 3.3.** Let $M$ be an NS-algebra. A **universal enveloping Nijenhuis algebra** of $M$ is a Nijenhuis algebra $U_N(M) \in \mathcal{NA}$ with a homomorphism $\rho : M \to U_N(M)$ in $\mathcal{NS}$ such that for any $N \in \mathcal{NA}$ and homomorphism $f : M \to N$ in $\mathcal{NS}$, there is a unique $\bar{f} : U_N(M) \to N$ in $\mathcal{NA}$ such that $\bar{f} \circ \rho = f$. 

Let \( M := (M, <, >, \bullet) \in \mathbf{NS} \). Let \( T(M) = \bigoplus_{n \geq 1} M^\otimes n \) be the tensor algebra. Then \( T(M) \) is the free algebra generated by the \( k \)-module \( M \). By Corollary 2.5, \( F_N(T(M)) \), with the natural injection \( i_M : M \rightarrow T(M) \rightarrow F_N(T(M)) \), is the free Nijenhuis algebra over the vector space \( M \).

Let \( J_M \) be the Nijenhuis ideal of \( F_N(T(M)) \) generated by the set
\[
\{ x < y - xP(y), \ x > y - P(x)y, \ x \bullet y = P(x \otimes y) \mid x, y \in M \}
\]
Let \( \pi : F_N(T(M)) \rightarrow F_N(T(M))/J_M \) be the quotient map.

**Theorem 3.4.** Let \((M, <, >, \bullet)\) be an \( \mathbf{NS} \) algebra. The quotient Nijenhuis algebra \( F_N(T(M))/J_M \), together with \( \rho := \pi \circ i_M \), is the universal enveloping Nijenhuis algebra of \( M \).

**Proof.** The proof is similar to the case of tridendriform algebras and Rota-Baxter algebras [8]. So we skip some of the details.

Let \((N, P)\) be a Nijenhuis algebra and let \( f : M \rightarrow N \) be a homomorphism in \( \mathbf{NS} \). More precisely, we have \( f : (M, <, >, \bullet) \rightarrow (N, <', >', \bullet') \). We will complete the following commutative diagram, using notations from Corollary 2.5.

\[
\begin{array}{ccc}
T(M) & \xrightarrow{k_M} & M \\
\downarrow f & & \downarrow f \\
F_N(T(M)) & \xrightarrow{j_{T(M)}} & F_N(T(M))/J_M \\
\end{array}
\]

By the universal property of the free algebra \( T(M) \) over \( M \), there is a unique homomorphism \( \hat{f} : T(M) \rightarrow N \) such that \( \hat{f} \circ k_M = f \). So \( \hat{f}(x_1 \otimes \cdots \otimes x_n) = f(x_1) \cdots \cdots f(x_n) \). Here \( \ast \) is the product in \( N \). Then by the universal property of the free Nijenhuis algebra \( F_N(T(M)) \) over \( T(M) \), there is a unique morphism \( \hat{f} : F_N(T(M)) \rightarrow N \) in \( \mathbf{NA} \) such that \( \hat{f} \circ j_{T(M)} = \hat{f} \). By Corollary 2.5, \( \hat{f} = \hat{f} \).

Then
\[
\hat{f} \circ i_M = \hat{f} \circ j_{T(M)} \circ k_M = \hat{f} \circ k_M = f.
\]
So for any \( x, y \in M \), we check that
\[
\hat{f}(x < y - xP(y)) = 0, \ \hat{f}(x > y - P(x)y) = 0, \ \hat{f}(x \bullet y - P(x \otimes y)) = 0.
\]
Thus \( J_M \) is in \( \ker(\hat{f}) \) and there is a morphism \( \hat{f} : F_N(T(M))/J_M \rightarrow N \) in \( \mathbf{NA} \) such that \( \hat{f} = \hat{f} \circ \pi \).

Then by the definition of \( \rho = \pi \circ i_M \) in the theorem and Eq. (21), we have
\[
\hat{f} \circ \rho = \hat{f} \circ \pi \circ i_M = \hat{f} \circ i_M = f.
\]
This proves the existence of \( \hat{f} \).

Suppose \( \hat{f} : F_N(T(M))/J_M \rightarrow N \) is also a homomorphism in \( \mathbf{NA} \) such that \( \hat{f} \circ \rho = f \). Then
\[
(\hat{f} \circ \pi) \circ i_M = f = (\hat{f} \circ \pi) \circ i_M.
\]
By Corollary 2.5, the free Nijenhuis algebra \( F_N(T(M)) \) over the algebra \( T(M) \) is also the free Nijenhuis algebra over the vector space \( M \) with respect the natural injection \( i_M \). So we have \( \hat{f} \circ \pi = \hat{f} \circ \pi \) in \( \mathbf{NA} \). Since \( \pi \) is surjective, we have \( \hat{f} \circ \pi = \hat{f} \). This proves the uniqueness of \( \hat{f} \). \( \square \)
4. From Nijenhuis algebras to N-dendriform algebras

In this section, we consider an inverse of Theorem 3.2 in the following sense. Suppose \((N, P)\) is a Nijenhuis algebra and define binary operations

\[ x \prec_P y = xP(y), \quad x \succ_P y = P(x)y, \quad x \cdot_P y = -P(xy). \]

By Theorem 3.2, the three operations satisfy the NS relations in Eq. (16). Our inverse question is, what other quadratic nonsymmetric relations could \((N, \prec_P, \succ_P, \cdot_P)\) satisfy? We recall some background on binary quadratic nonsymmetric operads in order to make the question precise. We then determine all the quadratic nonsymmetric relations that are consistent with the Nijenhuis operator.

4.1. Background and the statement of Theorem 4.2. For details on binary quadratic nonsymmetric operads, see [16, 24].

**Definition 4.1.** Let \(k\) be a field.

(a) A **graded vector space** is a sequence \(P := \{P_n\}_{n \geq 0}\) of \(k\)-vector spaces \(P_n, n \geq 0\).

(b) A **nonsymmetric (ns) operad** is a graded vector space \(P = \{P_n\}_{n \geq 0}\) equipped with **partial compositions**:

\[(22) \quad \circ_i := \circ_{m,n,i} : P_m \otimes P_n \longrightarrow P_{m+n-1}, \quad 1 \leq i \leq m,\]

such that, for \(\lambda \in P_\ell, \mu \in P_m\) and \(\nu \in P_n\), the following relations hold.

(i) \((\lambda \circ_i \mu) \circ_{i+j} \nu = \lambda \circ_i (\mu \circ_j \nu), \quad 1 \leq i \leq \ell, 1 \leq j \leq m.\)

(ii) \((\lambda \circ_i \mu) \circ_{k+1+m} \nu = (\lambda \circ_k \nu) \circ_i \mu, \quad 1 \leq i < k \leq \ell.\)

(iii) There is an element \(id \in P_1\) such that \(id \circ \mu = \mu\) and \(\mu \circ id = \mu\) for \(\mu \in P_n, n \geq 0\).

An ns operad \(P = \{P_n\}\) is called **binary** if \(P_1 = k.id\) and \(P_n, n \geq 3\) are induced from \(P_2\) by composition. Then in particular, for the free operad, we have

\[(23) \quad P_3 = (P_2 \circ_1 P_2) \oplus (P_2 \circ_2 P_2),\]

which can be identified with \(P_2 \otimes P_2 \oplus P_2 \otimes P_2\). A binary ns operad \(P\) is called **quadratic** if all relations among the binary operations in \(P_2\) are derived from \(P_3\).

Thus a binary, quadratic, ns operad is determined by a pair \((V, R)\) where \(V = P_2\), called the **space of generators**, and \(R\) is a subspace of \(V^\otimes 2 \oplus V^\otimes 2\), called the **space of relations**. So we can denote \(P = P(V)/(R)\).

Note that a typical element of \(V^\otimes 2\) is of the form \(\sum_{i=1}^k \circ_{i}^{(1)} \otimes \circ_{i}^{(2)}\) with \(\circ_{i}^{(1)}, \circ_{i}^{(2)} \in V, 1 \leq i \leq k\).

Thus a typical element of \(V^\otimes 2 \oplus V^\otimes 2\) is of the form

\[
\left( \sum_{i=1}^k \circ_{i}^{(1)} \otimes \circ_{i}^{(2)} \right) \oplus \sum_{j=1}^m \circ_{j}^{(3)} \otimes \circ_{j}^{(4)},\]

\(\circ_{i}^{(1)}, \circ_{i}^{(2)}, \circ_{j}^{(3)}, \circ_{j}^{(4)} \in V, 1 \leq i \leq k, 1 \leq j \leq m, k, m \geq 1.\)

For a given binary quadratic ns operad \(P = P(V)/(R)\), a \(k\)-vector space \(A\) is called a **\(P\)-algebra** if \(A\) has binary operations (indexed by) \(V\) and if, for

\[
\left( \sum_{i=1}^k \circ_{i}^{(1)} \otimes \circ_{i}^{(2)} \right) \oplus \sum_{j=1}^m \circ_{j}^{(3)} \otimes \circ_{j}^{(4)} \in R \subseteq V^\otimes 2 \oplus V^\otimes 2
\]
with $\odot_j^{(1)}, \odot_j^{(2)}, \odot_j^{(3)}, \odot_j^{(4)} \in V, 1 \leq i \leq k, 1 \leq j \leq m$, we have

\begin{equation}
\sum_{i=1}^{k} (x \odot_i^{(1)} y) \odot_i^{(2)} z = \sum_{j=1}^{m} x \odot_j^{(3)} (y \odot_j^{(4)} z), \quad \forall \ x, y, z \in A.
\end{equation}

For example, from Eq. (26) the NS algebras are precisely the $\mathcal{P}$-algebras where $\mathcal{P} = \mathcal{P}(V)/(R)$ with $R$ being the subspace of $V^{\otimes 2} \oplus V^{\otimes 2}$ spanned by the four elements

\begin{align*}
(\ast >, > \otimes >), & \quad (\ast \otimes \ast, \ast \otimes \ast), \\
(\ast >, > \otimes >), & \quad (\ast \otimes \ast, \ast \otimes \ast),
\end{align*}

where $\ast = \prec + > + \ast$. More precisely, any $\mathcal{P}$-algebra $A$ satisfies the relations

\begin{align*}
(x < y) < z & = x < (y < z) + x < (y \ast z), \\
(x > y) < z & = x > (y < z), \\
(x < y) > z + (x > y) > z + (x \ast y) > z & = x > (y > z), \quad \forall x, y, z \in A \\
(x < y) \ast z & = x \ast (y > z), \\
(x > y) \ast z + (x \ast y) < z + (x \ast y) \ast z & = x > (y \ast z) + x \ast (y < z) + x \ast (y \ast z).
\end{align*}

Note that the relations of the NS algebra in Eq. (16) is contained in the space spanned by the relations in Eq. (25). We call $\mathcal{P}$ defined by the relations in Eq. (25) the $\textbf{N-dendriform operad}$ and call a quadruple $(A, \prec, >, \ast)$ satisfying Eq. (26) an $\textbf{N-dendriform algebra}$. Let $\textbf{ND}$ denote the category of $\textbf{N-dendriform algebras}$. Then we have the following immediate corollary of Theorem 4.2.

\textbf{Corollary 4.3.} \quad (a) There is a natural functor

\begin{equation}
\mathcal{F} : \textbf{NA} \rightarrow \textbf{ND}, \quad (N, P) \mapsto (N, \prec, >, \ast).
\end{equation}

(b) There is a natural (inclusion) functor

\begin{equation}
\mathcal{G} : \textbf{ND} \rightarrow \textbf{NS}, \quad (M, \prec, >, \ast) \mapsto (M, \prec, >, \ast).
\end{equation}
(c) The functors $\mathcal{F}$ and $\mathcal{S}$ give a refinement of the functor $\mathcal{E} : \text{NA} \to \text{NS}$ in Eq. (16) in the sense that the following diagram commutes

$$
\begin{array}{ccc}
\text{NA} & \xrightarrow{\mathcal{F}} & \text{ND} \\
\downarrow{\mathcal{E}} & & \downarrow{\mathcal{S}} \\
\text{NS}
\end{array}
$$

(29)

4.2. The proof of Theorem 4.2. With $V = k\{<,>,\cdot\}$, we have

$$V^{\otimes 2} \oplus V^{\otimes 2} = \bigoplus_{\mathcal{G} \in \{\otimes_1, \otimes_2, \otimes_3, \otimes_4\}} k(\otimes_1 \otimes \otimes_2, \otimes_3 \otimes \otimes_4).$$

Thus any element $r$ of $V^{\otimes 2} \oplus V^{\otimes 2}$ is of the form

$$r := a_1(< \otimes <, 0) + a_2(< \otimes >, 0) + a_3(< \otimes <, 0) + b_1(> \otimes <, 0) + b_2(> \otimes >, 0) + b_3(> \otimes <, 0) + c_1(\cdot \otimes <, 0) + c_2(\cdot \otimes >, 0) + c_3(\cdot \otimes <, 0) + d_1(0, < \otimes <) + d_2(0, < \otimes >) + d_3(0, < \otimes <) + e_1(0, > \otimes <) + e_2(0, > \otimes >) + e_3(0, > \otimes <) + f_1(0, \cdot \otimes <) + f_2(0, \cdot \otimes >) + f_3(0, \cdot \otimes <)$$

where the coefficients are in $k$.

Let $\mathcal{P} = \mathcal{P}(V)/(R)$ be an operad satisfying the condition in Item (a). Let $r$ be in $R$ expressed in the above form. Then for any Nijenhuis algebra $(N, P)$, the quadruple $(N, <, >, \cdot)$ is a $\mathcal{P}$-algebra. Thus

$$a_1(x <p y) <p z + a_2(x <p y) >p z + a_3(x <p y) \cdot_p z + b_1(x >p y) <p z + b_2(x >p y) >p z + b_3(x >p y) \cdot_p z + c_1(x \cdot_p y) <p z + c_2(x \cdot_p y) >p z + c_3(x \cdot_p y) \cdot_p z + d_1 x >p (y <p z) + d_2 x >p (y <p z) + d_3 x >p (y \cdot_p z) + e_1 x <p (y >p z) + e_2 x <p (y <p z) + e_3 x <p (y \cdot_p z) + f_1 x \cdot_p (y >p z) + f_2 x \cdot_p (y <p z) + f_3 x \cdot_p (y \cdot_p z) = 0, \forall x, y, z \in N.$$

By the definitions of $<p, >p, \cdot_p$ in Eq. (17), we have

$$a_1 x P(y) P(z) + a_2 P(x P(y)) z - a_3 P(x P(y) z) + b_1 P(x y P(z)$$

$$+ b_2 P(x P(y) z) - b_3 P(x P(y) z) - c_1 P(x y) P(z) - c_2 P(x y) P(z) + c_3 P(x y) z + d_1 P(x P(y) z) + d_2 P(x y) P(z) - d_3 P(x P(y) z) + e_1 x P(y) z + e_2 x P(y P(z)) - e_3 x P(y P(z)) - f_1 x P(y) z - f_2 P(x P(y) z) + f_3 P(x P(y) z) = 0.$$

Since $P$ is a Nijenhuis operator, we further have

$$a_1 x P(y P(z)) + a_1 x P(P(y) z) - a_1 x P^2(y z) + a_2 P(x P(y) z) - a_3 P(x P(y) z)$$

$$+ b_1 P(x y P(z)) + b_2 P(x P(y) z) - b_3 P(x P(y) z)$$
Collecting similar terms, we obtain
\[
(a_1 + e_2)xP(yP(z)) + (a_1 + e_1)xP(yP(z)) - (a_1 + e_3)xP(\{yP(z)\}) + (a_2 + d_1)xP(xP(y))z \\
-(a_3 + f_1)xP(xP(y))z + (b_1 + d_2)xP(xP(y))z + (b_2 + d_1)xP(xP(y))z - (b_3 + d_3)xP(xP(y))z \\
-(c_1 + f_2)xP(\{xyP(z)\}) + (c_3 - c_1)xP(\{xyP(z)\}) + (c_1 + d_3)xP(\{xyP(z)\}) \\
-(c_2 + d_1)xP(\{xyP(z)\}) + (c_3 - c_1)xP(\{xyP(z)\}) = 0.
\]

Now we take the special case when \((N, P)\) is the free Nijenhuis algebra \((F_N(T(M)), P_{T(M)})\) defined in Corollary 2.3 for our choice of \(M = k[x, y, z]\) and \(P_{T(M)}(u) = [u]\). Then the above equation is just
\[
(a_1 + e_2)x[y]\{z\} + (a_1 + e_1)x[y]\{z\} - (a_1 + e_3)x[y]\{z\} + (a_2 + d_1)x[y]\{z\} + (a_3 + f_1)x[y]\{z\} + (b_1 + d_2)x[y]\{z\} + (b_2 + d_1)x[y]\{z\} - (b_3 + d_3)x[y]\{z\} \\
-(c_1 + f_2)x[y]\{z\} + (c_3 - c_1)x[y]\{z\} + (c_1 + d_3)x[y]\{z\} \\
-(c_2 + d_1)x[y]\{z\} + (f_3 - d_3)x[y]\{z\} = 0.
\]

Note that the set of elements
\[
x[y]\{z\}, x[y]\{z\}, x[y]\{z\}, x[y]\{z\}, x[y]\{z\}, x[y]\{z\}, x[y]\{z\}, [x[y]\{z\}]
\]

is a subset of the basis \(\mathcal{X}_\infty\) of the free Nijenhuis algebra \(F_N(T(M))\) and hence is linearly independent. Thus the coefficients must be zero, that is,
\[
a_1 = -e_1 = -e_2 = -e_3, \\
a_2 = b_2 = c_2 = -d_1, \\
a_3 = -f_1, b_1 = -d_2, \\
b_3 = c_1 = c_3 = -f_2 = -f_3 = -d_3.
\]

Substituting these equations into the general relation \(r\), we find that the any relation \(r\) that can be satisfied by \(<_p,>_p, \bullet_p\) for all Nijenhuis algebras \((N, P)\) is of the form
\[
r = a_1\left( (x < y) < z - x < (y < z) - x < (y < z) - x < (y \bullet z) \right) \\
+ b_1\left( (x > y) < z - x > (y < z) \right) \\
+ d_1\left( x > (y > z) - (x < y) > z - (x > y) > z - (x \bullet y) > z \right) \\
+ a_3\left( (x < y) \bullet z - x \bullet (y > z) \right) \\
+ b_3\left( (x > y) \bullet z + (x \bullet y) < z + (x \bullet y) \bullet z - x > (y \bullet z) - x \bullet (y < z) - x \bullet (y \bullet z) \right),
\]
where \( a_1, b_1, d_1, a_3, b_3 \in \mathbb{k} \) can be arbitrary. Thus \( r \) is in the subspace prescribed in Item [b], as needed.

\( [b] \Rightarrow [a] \) We check directly that all the relations in Eq. (24) are satisfied by \((N, <_p, >_p, \bullet_p)\) for every Nijenhuis algebra \((N, P)\). First of all

\[
(x <_p y) <_p z = xP(y)P(z) = xP(yP(z)) + xP(P(y)z) - xP^2(yz) = x <_p (y <_p z) + x <_p (y >_p z) + x <_p (y \bullet_p z),
\]

proving the first equation in Eq. (24). The proofs of the second and third equations are similar.

For the fourth equation, we have

\[
(x <_p y) \bullet_p z = -P((xP(y))z) = -P(xP(y)z)) = x \bullet_p (y >_p z).
\]

Finally for the last equation, we verify

\[
(x >_p y) \bullet_p z + (x \bullet_p y) <_p z + (x \bullet_p y) \bullet_p z
\]

\[
= -P((P(xy)z) - P(xyP(z)) + P(P(xy)z)
\]

\[
= -P(xP(yz)) - P(xyP(z)) + P^2(xy) + P(xyP(z))
\]

\[
= -P(xP(yz)) - P(xyP(z)) + P^2(xy) - P(xyP(z))
\]

So the two sides of the last equation agree.

Thus if the relation space \( R \) of an operad \( \mathcal{P} = \mathcal{P}(V)/(R) \) is contained in the subspace spanned by the vectors in Eq. (25), then the corresponding relations are linear combinations of the equations in Eq. (26) and hence are satisfied by \((N, <_p, >_p, \bullet_p)\) for each Nijenhuis algebra \((N, P)\). Therefore \((N, <_p, >_p, \bullet_p)\) is a \( \mathcal{P} \)-algebra. This completes the proof of Theorem 4.2.

References

[1] M. Aguiar, On the associative analog of Lie bialgebras, Journal of Algebra, 244, (2001), 492-532.
[2] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, Pacific J. Math., 10, (1960), 731-742.
[3] L. A. Bokut, Y. Chen and J. Qiu, Gröbner-Shirshov bases for associative algebras with multiple operators and free Rota-Baxter algebras, J. Pure Appl. Algebra 214 (2010) 89-110.
[4] J. Carriñoa, J. Grabowski and G. Marmo, Quantum bi-Hamiltonian systems, Internat. J. Modern Phys. A, 15, (2000), 4797-4810.
[5] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem., Comm. Math. Phys., 210, (2000), no. 1, 249-273.
[6] K. Ebrahimi-Fard, Loday-type algebras and the Rota-Baxter relation, Lett. Math. Phys. 61 (2002) 139-147.
[7] K. Ebrahimi-Fard, On the associative Nijenhuis relation, The Electronic Journal of Combinatorics, Volume 11(1), R38, (2004).
[8] K. Ebrahimi-Fard and L. Guo, Rota-Baxter algebras and dendriform algebras, Jour. Pure Appl. Algebra 212 (2008), 320-339.
[9] K. Ebrahimi-Fard and L. Guo, Free Rota-Baxter algebras and rooted trees, *J. Algebra and Its Applications* 7 (2008), 167-194.

[10] K. Ebrahimi-Fard, L. Guo and D. Kreimer, Spitzer’s Identity and the Algebraic Birkhoff Decomposition in pQFT, *J. Phys. A: Math. Gen.*, 37, (2004), 11037-11052.

[11] K. Ebrahimi-Fard, L. Guo and D. Manchon, Birkhoff type decompositions and the Baker-Campbell-Hausdorff recursion, *Comm. Math. Phys.*, 267, (2006), 821-845. arXiv:math-ph/0602004.

[12] K. Ebrahimi-Fard and P. Leroux, Generalized shuffles related to Nijenhuis and TD-algebras, *Comm. Algebra* 37 (2009) 3065-3094.

[13] A. Frölicher and A. Nijenhuis, Theory of vector valued differential forms. Part I, *Indag. Math.* 18 (1956) 338-360.

[14] I. Z. Golubchik and V.V. Sokolov, One more type of classical Yang-Baxter equation, *Funct. Anal. Appl.* 34 (2000), 296-298.

[15] I. Z. Golubchik and V.V. Sokolov, Generalized Operator Yang-Baxter Equations, Integrable ODEs and Nonassociative Algebras, *J. of Nonlinear Math. Phys.* 7 (2000), 184-197.

[16] L. Guo, An Introduction to Rota-Baxter Algebras, to be published by Higher Education Press (China) and International Press (US).

[17] L. Guo and W. Keigher, Baxter algebras and shuffle products, *Adv. Math.*, 150, (2000), 117-149.

[18] L. Guo, W. Sit and R. Zhang, On Rota’s problem for linear operators in associative algebras, *Proc. ISSAC 2011*, 147-154.

[19] L. Guo, W. Sit and R. Zhang, Differential type operators and Gröbner-Shirshov bases, preprint.

[20] Y. Kosmann-Schwarzbach and F. Magri, Poisson-Nijenhuis structures, *Ann. Inst. Henri Poincaré* 53 (1990), 35-81.

[21] P. Leroux, Construction of Nijenhuis operators and dendriform trialgebras, *Int. J. Math. Math. Sci.* (2004), no. 40-52, 2595-2615.

[22] J.-L. Loday, Dialgebras, in *Dialgebras and related operads, Lecture Notes in Math.*, 1763, (2001), 7-66.

[23] J.-L. Loday and M. Ronco, Trialgebras and families of polytopes, in “Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory” *Contemp. Math.* 346 (2004) 369-398.

[24] J.-L. Loday and B. Vallette, Algebraic Operads, *Grundlehren Math. Wiss.* 346, Springer, Heidelberg, 2012.

[25] A. Nijenhuis, $X_{n-1}$-forming sets of eigenvectors. *Indag. Math.* 13 (1951) 200-212.

[26] K. Uchino, Twisting on associative algebras and Rota-Baxter type operators, *J. Noncommut. Geom.* 4 (2010) 349-379.

**Department of Mathematics, Lanzhou University, Lanzhou, Gansu 730000, China**

*E-mail address:* leip@lzu.edu.cn

**Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102, USA**

*E-mail address:* liguo@newark.rutgers.edu