1. Introduction

In this note we give two integrability criteria for a representation \( \alpha : \mathfrak{g} \to \text{End}(\mathcal{D}) \) of a Banach–Lie algebra \( \mathfrak{g} \) as skew-symmetric unbounded operators on a dense domain \( \mathcal{D} \) of a Hilbert space, that is, sufficient conditions for the existence of a continuous unitary representation of a simply connected Banach–Lie group with Lie algebra \( \mathfrak{g} \) (when it exists) whose derived representation extends \( \alpha \).

After Nelson's famous criterion [Nel59], new ones (for finite dimensional Lie algebras) appeared in the late sixties and early seventies, also based on analytic vectors (see [FSSS72, Sim72]) or on smooth vectors (see [JM84]). In both cases the key result is the validity of the commutation relation

\[
e^{\alpha(x)} \alpha(y) e^{-\alpha(x)} = \alpha(e^{\text{ad}x} y),
\]

for every \( x \in \text{a Lie-generating subset of } \mathfrak{g} \) and any \( y \in \mathfrak{g} \). It is used to transfer computations in the space of operators to computations in the Lie algebra so that the integrability follows from formulas in \( \mathfrak{g} \).

We consider a Banach–Lie algebra \( \mathfrak{g} \) which decomposes as

\[
\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n,
\]

where \( \mathfrak{a}_j, j = 1, 2, \ldots, n \), are closed subspaces and we prove that the representation integrates if (1) holds for every \( x \in \cup \mathfrak{a}_j \) and \( y \in \mathfrak{g} \). This also leads to a criterion based on analytic vectors and generalising [FSSS72]: the representation integrates if \( \mathcal{D} \) consists of analytic vectors for every \( \alpha(x), x \in \cup \mathfrak{a}_j \) (for a generalisation of Nelson's criterion see [LM99]). The new difficulty in the Banach setting is that we have to differentiate paths of the form

\[
t \mapsto e^{\alpha(x(t))} v,
\]

where \( x(t) \) is a smooth path in \( \mathfrak{a}_j \), but (1) enables us to do that. The derivative of (2) involves the logarithmic derivative of \( x(t) \), therefore a formula relating such derivatives is needed.

These results are particularly well-suited for the study of symmetric or 3-graded Lie algebras. In a forthcoming work [MN10] with Karl-Hermann Neeb we will use them to prove a generalisation of the Lüscher–Mack Theorem [LM75, Appendix C] to the Banach–Lie setting. This is a crucial tool for the study of the representation theory of the automorphisms groups of infinite dimensional real symmetric domains.

The integrability criteria are stated in Section 2. In Section 3 the relation (1) is discussed and used to prove the differentiability of (2) while the relevant formula for logarithmic derivatives is given in Section 4. The proof of the integrability criteria is achieved in Section 5.

2. Main results

Let \( \mathfrak{g} \) be a Banach–Lie algebra with Lie bracket \([\cdot, \cdot]\). A representation \( \alpha \) of \( \mathfrak{g} \) on a (dense) subspace \( \mathcal{D} \) of a Hilbert space \( \mathcal{H} \) is a linear map which associates to any
\(x \in \mathfrak{g}\) a skew-symmetric unbounded operator \(\alpha(x) : \mathcal{D} \to \mathcal{D}\), in such a way that
\[
\alpha([x,y]) = \alpha(x)\alpha(y) - \alpha(y)\alpha(x).
\]
The representation is said to be **strongly continuous** if for every \(v \in \mathcal{D}\) the map
\[
\alpha^v : \mathfrak{g} \to \mathcal{H}, \; x \mapsto \alpha(x)v
\]
is continuous.

Let \(G\) be a Banach–Lie group with Lie algebra \(\mathfrak{g}\) and \(\pi : G \to \text{U}(\mathcal{H})\) be a continuous unitary representation of \(G\) in \(\mathcal{H}\). Each element \(x \in \mathfrak{g}\) gives rise to a one-parameter unitary group \(\pi(\exp tx)\) and hence by Stone’s theorem to a skew-adjoint operator
\[
d\pi(x)v := \frac{d}{dt}\Big|_{t=0} \pi(\exp tx)v
\]
defined on the set of vectors \(v\) for which the limit exists. The derived representation is the (strongly continuous) representation of \(\mathfrak{g}\) defined on the space \(\mathcal{H}^\infty\) of smooth vectors (those vectors for which the orbit map is smooth) by
\[
d\pi(x) := d\pi(x)|_{\mathcal{H}^\infty}.
\]
When \(G\) is finite dimensional \(\mathcal{H}^\infty\) is dense and since it is invariant under the action of \(G\), the operators \(d\pi(x)\) are essentially skew-adjoint, but for an arbitrary Banach-Lie group \(\mathcal{H}^\infty\) may be empty. For issues related to smooth vectors for representations of infinite dimensional Lie groups see [Nee10b].

We say that a strongly continuous representation \(\alpha\) on the dense domain \(\mathcal{D}\) integrates to a continuous unitary representation of \(G\) if there exists such a representation \(\pi\) with \(\mathcal{D} \subseteq \mathcal{H}^\infty\) and \(d\pi|_{\mathcal{D}} = \alpha\).

We below assume that \(\mathfrak{g}\) has a decomposition
\[
\mathfrak{g} = a_1 \oplus a_2 \oplus \cdots \oplus a_n,
\]
where \(a_j, j = 1, 2, \ldots, n\), are closed subspaces.

The main theorem will be stated for a strongly continuous representation \(\alpha\) on a dense domain \(\mathcal{D}\) which satisfies the following assumptions:

\begin{itemize}
  \item[(A1)] For all \(x \in \bigcup a_j\), \(\alpha(x)\) is essentially skew-adjoint, i.e., its closure \(\overline{\alpha(x)}\) is skew-adjoint, hence generates a strongly continuous one-parameter unitary group \(e^{\overline{\alpha(x)}} := e^{\text{ad}x}, t \in \mathbb{R}\).
  \item[(A2)] For all \(x \in \bigcup a_j\), \(e^{\overline{\alpha(x)}} \mathcal{D} \subseteq \mathcal{D}\).
  \item[(A3)] For all \((x,y) \in (\bigcup a_j, \mathfrak{g})\) and \(v \in \mathcal{D}\), the commutation relation
\[
e^{\alpha(x)}\alpha(y)e^{-\alpha(x)}v = \alpha(e^{\text{ad}x}y)v
\]
holds.
\end{itemize}

**Theorem 1.** Let \(G\) be a simply connected Banach–Lie group with Lie algebra \(\mathfrak{g}\). Any strongly continuous representation of \(\mathfrak{g}\) satisfying (A1-3) integrates to a continuous unitary representation of \(G\).

First we give a corollary in which the assumption (A3) is weakened (see [JMS83] Theorem 9.1), where in the case of finite dimensional Lie algebras it is weakened even further).

**Corollary 2.** Let \(G\) be a simply connected Banach–Lie group with Lie algebra \(\mathfrak{g}\). Any strongly continuous representation of \(\mathfrak{g}\) satisfying (A1-2) and
\begin{itemize}
  \item[(A3')] For all \((x,y) \in (\bigcup a_j, \mathfrak{g})\) and \(v \in \mathcal{D}\), the map \(\mathbb{R} \to \mathcal{D}, t \mapsto \alpha(y)e^{\alpha(x)}v\) is continuous.
\end{itemize}
integrates to a continuous unitary representation of \(G\).
The second corollary is an integrability criterion based on analytic vectors. It generalises the Integrability Theorem [Nee10b, 6.8], where \( \mathcal{D} \) is assumed to consists of analytic vectors for every \( \alpha(x), x \in \mathfrak{g} \), but the techniques involved are completely different. For finite dimensional Lie algebras it was proved by M. Flato, J. Simon, H. Snellman, and D. Sternheimer (see [FSSS72]). Note that J. Simon also proved [Sim72] that is it sufficient to assume that \( \mathcal{D} \) consists of analytic vectors for every \( \alpha(x), x \in \mathcal{S} \), where \( \mathcal{S} \) is a Lie-generating subset of \( \mathfrak{g} \).

**Corollary 3.** Let \( G \) be a simply connected Banach–Lie group with Lie algebra \( \mathfrak{g} \). Let \( \alpha \) be a strongly continuous representation of \( \mathfrak{g} \) over a dense domain which consists of analytic vectors for the operators \( \alpha(x), x \in \mathfrak{S} \). Then \( \alpha \) integrates to a continuous unitary representation of \( G \).

In the next section we show how the corollaries follow from Theorem [1], which will be proved in the last section.

### 3. The Commutation Relation

In this section we first prove that the assumptions of Corollary [2] as well as those of Corollary [3] lead to the assumptions (A1-3). Those results can be found for finite dimensional Lie algebras in [JM81, Ch. 3] and in [FSSS72] respectively, but although they extend directly to the Banach setting we give full proofs for the sake of completeness. Then we show that the commutation relation [1] implies the differentiability of a family of operators which is crucial in the proof of the main theorem.

We will need the following product rule.

**Lemma 4.** Let \( E \) and \( F \) be two Banach spaces and \( L_0(E,F) \) denote the space of continuous linear operators from \( E \) to \( F \) endowed with the strong operator topology. Let \( t \mapsto K(t) \in L_0(E,F) \) be a continuous path such that \( t \mapsto K(t)v \) is differentiable for every \( v \) in a subspace \( \mathcal{D} \) of \( E \) and let \( \gamma(t) \) be a differentiable path in \( \mathcal{D} \). We write \( K'(t) : \mathcal{D} \to F \) for the linear operator obtained by \( K'(t)v := \frac{d}{dt} K(t)v, v \in \mathcal{D} \). Then \( t \mapsto K(t) \gamma(t) \) is differentiable with

\[
\frac{d}{dt} K(t) \gamma(t) = K'(t) \gamma(t) + K(t) \gamma'(t).
\]

**Proof.** We write

\[
\frac{1}{h} (K(t + h) \gamma(t + h) - K(t) \gamma(t)) = \frac{1}{h} (K(t + h) \gamma(t + h) - K(t + h) \gamma(t)) + \frac{1}{h} (K(t + h) \gamma(t) - K(t) \gamma(t)).
\]

The second term converges to \( K'(t) \gamma(t) \) and the first one is equal to

\[
K(t + h) \left( \frac{\gamma(t + h) - \gamma(t)}{h} - \gamma'(t) \right) + K(t) \gamma'(t).
\]

which converges to \( K(t) \gamma'(t) \) by the Principle of Uniform Boundedness: If \( \mathcal{C} \) is a compact neighbourhood of \( t \), then for every \( v \in \mathcal{H} \), \( \sup_{s \in \mathcal{C}} K(s)v \) is bounded and hence \( \sup_{s \in \mathcal{C}} ||K(s)|| \) is bounded. \( \square \)

**Lemma 5.** Consider two unbounded operators \( A \) and \( B \) defined on a dense domain \( \mathcal{D} \) of the Hilbert space \( \mathcal{H} \). Assume that \( A \) is essentially skew-adjoint, that \( A \mathcal{D} \subseteq \mathcal{D} \) and \( e^{tA} \mathcal{D} \subseteq \mathcal{D} \), and that \( B \) is closable. Let \( v \in \mathcal{D} \) such that \( t \mapsto B e^{tA} v \) is continuous. Then \( t \mapsto B e^{tA} v \) is differentiable with

\[
\frac{d}{dt} B e^{tA} v = B A e^{tA} v.
\]
Proof. We have
\[ e^{tA}v - v = \int_0^t Ae^{sA}vdv. \]

Let \( \overline{B} \) be the closure of \( B \). Its domain \( \mathcal{D}(\overline{B}) \) is a Banach space when endowed with the graph norm \( ||w||_B := ||w|| + ||Bw|| \), where \( ||\cdot|| \) is the Hilbert norm in \( \mathcal{H} \), and then \( \overline{B} : \mathcal{D}(\overline{B}) \to \mathcal{H} \) is a continuous linear operator. By assumption the map \( s \mapsto Ae^{sA}v \) is continuous for the graph norm and hence the integral \( \int_0^t Ae^{sA}vdv \) exists as a Riemann integral in \( \mathcal{D}(\overline{B}) \). We therefore have
\[ Be^{tA}v - Bv = \int_0^t BAe^{sA}vdv, \]
and the claim follows. \( \square \)

**Proposition 6.** Let \( \alpha \) be a strongly continuous representation of the Banach–Lie algebra \( \mathfrak{g} \) over a dense domain \( \mathcal{D} \) of a Hilbert space \( \mathcal{H} \). Let \( x \in \mathfrak{g} \) such that \( \alpha(x) \) is essentially skew-adjoint and such that the associated one-parameter unitary group leaves \( \mathcal{D} \) invariant. Let \( \mathfrak{a} \) be a closed subspace of \( \mathfrak{g} \) which invariant under \( \text{ad}x \). Assume that for every \( y \in \mathfrak{a} \) the map \( \mathbb{R} \to \mathcal{H}, t \mapsto \alpha(y)e^{\alpha(x)}v \) is continuous. Then we have for every \( y \in \mathfrak{a} \) the commutation relation
\[ e^{\alpha(x)}\alpha(y)e^{-\alpha(x)} = \alpha(e^{\text{ad}x}y). \]

Proof. (See [1M84, 3.2 and 3.3]) Let \( v \in \mathcal{D} \). We want to prove that the map
\[ [0, 1] \to \mathcal{H}, s \mapsto e^{(1-s)\alpha(x)}\alpha(e^{s\text{ad}x}y)e^{(s-1)\alpha(x)}v \]
is constant. We will apply Lemma 4 with
\[ K(s) : \mathfrak{a} \to \mathcal{H}, z \mapsto e^{(1-s)\alpha(x)}\alpha(z)e^{(s-1)\alpha(x)}v \]
and \( \gamma(s) = e^{s\text{ad}x}y \). The continuity of the operator \( K(s) \) follows directly from the strong continuity of \( \alpha \). The continuity of the map \( s \mapsto \alpha(z)e^{(s-1)\alpha(x)}v \) implies by Lemma 5 the differentiability of \( s \mapsto \alpha(z)e^{(s-1)\alpha(x)}v \), and the derivative is
\[ \frac{d}{ds}\alpha(z)e^{(s-1)\alpha(x)}v = \alpha(z)\alpha(x)e^{(s-1)\alpha(x)}v. \]
Hence, by Lemma 4 the map \( s \mapsto K(s)z = e^{(1-s)\alpha(x)}\alpha(z)e^{(s-1)\alpha(x)}v \) is differentiable with derivative
\[ K'(s)z = e^{(1-s)\alpha(x)}[\alpha(z), \alpha(x)]e^{(s-1)\alpha(x)}v. \]
Since \( s \mapsto e^{s\text{ad}x}y \) is analytic and \( \frac{d}{ds}e^{s\text{ad}x}y = [x, e^{s\text{ad}x}y] \), again by Lemma 4 we have
\[ \frac{d}{ds}e^{(1-s)\alpha(x)}[\alpha(e^{s\text{ad}x}y), \alpha(x)]e^{(s-1)\alpha(x)}v + e^{(1-s)\alpha(x)}[\alpha([x, e^{s\text{ad}x}y]), \alpha(x)]e^{(s-1)\alpha(x)}v = 0. \]
\( \square \)

Let \( \alpha : \mathfrak{g} \to \text{End}(\mathcal{D}) \) be a strongly continuous representation of \( \mathfrak{g} \) by skew-symmetric operators and let \( x \in \mathfrak{g} \) such that \( \mathcal{D} \) consists of analytic vectors for \( \alpha(x) \). By Nelson’s Theorem [Nie59, Lemma 5.1] \( \alpha(x) \) then is essentially skew-adjoint. The key result here is [FSSS72, Proposition 1], the proof of which carries over to the Banach case without any change.
Proposition 7 ([FSSS72]). Let \( \alpha \) and \( \alpha' \) be strongly continuous representations of the Banach–Lie algebra \( \mathfrak{g} \) over the domains \( \mathcal{D} \) and \( \mathcal{D}' \) (respectively), dense in \( \mathcal{H} \), with \( \mathcal{D} \subseteq \mathcal{D}' \), and hence for any \( y \in \mathfrak{g} \), \( \alpha(y) \) is the restriction of \( \mathcal{D}' \) of \( \alpha'(y) \). Then, if \( \mathcal{D} \) is a domain of analytic vectors for some \( \alpha(x), x \in \mathfrak{g} \), we have for any \( v \in \mathcal{D}, w \in \mathcal{D}' \), denoting by \( \langle \cdot, \cdot \rangle \) the scalar product in \( \mathcal{H} \),

\[
\langle -e^{\alpha(x)}\alpha(y)v, w \rangle = \langle e^{\alpha(x)}v, \alpha'(\text{ad}_x y)w \rangle.
\]

Remark that if for every \( y \in \mathfrak{g} \) one is given on \( \mathcal{D}' \supseteq \mathcal{D} \) an unbounded skew-symmetric operator \( \alpha'(y) : \mathcal{D}' \to \mathcal{D}' \), so that \( \alpha'(y)|_{\mathcal{D}} = \alpha(y) \), then \( \alpha' \) is automatically a strongly continuous Lie algebra homomorphism. Indeed, for \( v \in \mathcal{D} \) and \( w \in \mathcal{D}' \), we have

\[
\langle -\alpha(y)v, w \rangle = \langle v, \alpha'(y)w \rangle.
\]

Hence

\[
\langle v, \alpha'([x, y])w \rangle = \langle \alpha([y, x])v, w \rangle = \langle \alpha(y)\alpha(x) - \alpha(x)\alpha(y)v, w \rangle = \langle v, \alpha'(y)\alpha'(y) - \alpha'(y)\alpha'(y)w \rangle
\]

show that \( \alpha' \) is a Lie algebra homomorphism. Moreover a representation of Banach–Lie algebra is strongly continuous if and only if it is weakly continuous ([Nee10a, Lemma 4.2]). The domain \( \mathcal{D} \) being dense, the set of functionals \( \{\langle \cdot, v \rangle, v \in \mathcal{D}\} \) separates the points in \( \mathcal{H} \) and hence \( \alpha' \) is strongly continuous.

Assume now that \( \alpha(y) \) is essentially skew-adjoint for every \( y \in \bigcup \mathfrak{a}_j \), and let

\[
\mathcal{D}' := \bigcap_{\ell \in \mathbb{N}, y \in \bigcup \mathfrak{a}_j} \mathcal{D}(\overline{\alpha(y_1) \ldots \alpha(y_\ell)}),
\]

where \( \overline{\alpha(y)} \) denotes the closure of \( \alpha(y) \). Then \( \mathcal{D}' \) contains \( \mathcal{D} \) and is invariant under \( \overline{\alpha(y)} \), \( y \in \bigcup \mathfrak{a}_j \), so we can set for such \( y \),

\[
\alpha'(y) := \overline{\alpha(y)}|_{\mathcal{D}'},
\]

and for \( y = y_1 + \ldots + y_n \in \mathfrak{g}, y_j \in \mathfrak{a}_j \),

\[
\alpha'(y) = \alpha'(y_1) + \ldots + \alpha'(y_n).
\]

By the preceding remark this defines a strongly continuous representation of \( \mathfrak{g} \) extending \( \alpha \). Applying Proposition 7 several times, we see, since \( \alpha(y)^* = \overline{\alpha(y)} \) for \( y \in \bigcup \mathfrak{a}_j \), that for every \( y_1, \ldots, y_\ell \in \bigcup \mathfrak{a}_j \) and \( w \in \mathcal{D}' \), \( e^{\alpha(x)}w \in \mathcal{D}(\overline{\alpha(y_1) \ldots \alpha(y_\ell)}) \), and hence

\[
e^{\alpha(x)}\mathcal{D}' \subseteq \mathcal{D}'.
\]

Then Proposition 7 also shows that the commutation relation holds on \( \mathcal{D}' \):

Corollary 8. Let \( \alpha \) be a strongly continuous representation of the Banach–Lie algebra \( \mathfrak{g} \) over a dense domain \( \mathcal{D} \) such that \( \alpha(y) \) is essentially skew-adjoint for every \( y \in \bigcup \mathfrak{a}_j \). Let \( x \in \mathfrak{g} \) such that \( \mathcal{D} \) consists of analytic vectors for \( \alpha(x) \). Let \( \mathcal{D}' \supseteq \mathcal{D} \) be the domain defined in (3) and \( \alpha' \) the corresponding extension of \( \alpha \). Then \( e^{\alpha(x)} \) leaves \( \mathcal{D}' \) invariant and we have for every \( y \in \mathfrak{g} \) the commutation relation

\[
e^{\alpha(x)}\alpha'(y)e^{-\alpha(x)} = \alpha'(e^{\text{ad}_x y}).
\]

For the next proposition we will need the following lemma:
Lemma 9. Consider two essentially skew-adjoint operators $A$ and $B$ defined on a common dense domain $D$ and assume that for all $s \in \mathbb{R}$, $e^{sA}D \subseteq D$. Let $v \in D$ be such that $s \mapsto Be^{sA}v$ is continuous. Then

$$e^{tB}v - e^{tA}v = \int_0^t e^{sB}(B - A)e^{(t-s)A}vd\alpha.$$ 

Proof. We apply Lemma $\PageIndex{4}$ with $K(s) = e^{sB}$ and $\gamma(s) = e^{(t-s)A}v$ to obtain

$$\frac{d}{ds} e^{sB}e^{(t-s)A}v = e^{sB}(B - A)e^{(t-s)A}v.$$ 

By assumption the right hand side is continuous. Hence the claim follows from the Fundamental Theorem of Calculus. $\Box$

Proposition 10. Let $\alpha$ be a strongly continuous representation of the Banach–Lie algebra $\mathfrak{g}$ on a dense domain $D$. Let $I$ be a real interval and $I \to \mathfrak{g}$, $t \mapsto x(t)$ be a continuous path such that each $\alpha(x(t))$ is essentially skew-adjoint and $e^{\alpha(x(t))}D \subseteq D$, $s \in \mathbb{R}$. If, for every $t$, we have for sufficiently small $h$ and every $s \in \mathbb{R}$ the commutation relation

$$(4) \quad e^{\alpha(x(t))}\alpha(x(t + h)) = e^{\alpha(x(t))}\alpha(e^{s\text{ad}x(t)}x(t + h)),$$

then for every $v \in D$ the map $(s, t) \mapsto e^{\alpha(x(t))}v$ is continuous. If moreover the path $x(t)$ is differentiable, then $t \mapsto e^{\alpha(x(t))}v$ is differentiable with

$$\frac{d}{dt} e^{\alpha(x(t))}v = \alpha\left(\int_0^1 e^{s\text{ad}x(t)x'(t)}ds\right)e^{\alpha(x(t))}v.$$ 

Proof. Let us fix $t \in I$ and let $t + N_t$ be a convex neighbourhood of $t$ in $I$ such that for $h \in N_t$ the relation $(4)$ holds. Let $v \in D$. Rewriting $(4)$ as

$$\alpha(x(t + h))e^{\alpha(x(t))}v = e^{\alpha(x(t))}\alpha(e^{s\text{ad}x(t)}x(t + h))v,$$

we see that $s \mapsto \alpha(x(t+h))e^{\alpha(x(t))}v$ is continuous. We can therefore apply Lemma $\PageIndex{9}$ to obtain

$$e^{\alpha(x(t + h))}v - e^{\alpha(x(t))}v = \int_0^h e^{\alpha(x(t + s))}\alpha(x(t + h) - x(t)) e^{(s-u)\alpha(x(t))}vdu$$

$$= \int_0^h e^{\alpha(x(t + s))}e^{(s-u)\alpha(x(t))}\alpha\left(e^{(u-s)\text{ad}x(t)}x(t + h) - x(t)\right)e^{(s-u)\alpha(x(t))}vdu.$$ 

Thus, writing $||\cdot||$ for the norm in $\mathfrak{g}$,

$$\left|e^{\alpha(x(t + h))}v - e^{\alpha(x(t))}v\right| \leq |s| |\alpha| ||e^{(u-s)\text{ad}x(t)}|| ||x(t + h) - x(t)||$$

and $(s, t) \mapsto e^{\alpha(x(t))}v$ is continuous. Now assume that $t \mapsto x(t)$ is differentiable. Let us write

$$\frac{e^{\alpha(x(t + h))} - e^{\alpha(x(t))}}{h} = \int_0^1 e^{\alpha(x(t + hs + h))}\alpha\left(\frac{x(t + hs + h) - x(t)}{h}\right)e^{(1-s)\alpha(x(t))}vdu$$

and let us define on $N_t$ the function

$$z(h) = \begin{cases} \frac{x(t + hs + h) - x(t)}{h} & \text{for } h \neq 0, \\ x'(t) & \text{for } h = 0. \end{cases}$$

The formula

$$\alpha(z(h))e^{(1-s)\alpha(x(t))}v = e^{(1-s)\alpha(x(t))}\alpha(e^{(s-1)\text{ad}z(t)}z(h))v,$$

which holds for $h \neq 0$ by assumption and for $h = 0$ by continuity, shows that

$$(s, h) \mapsto \alpha(z(h))e^{(1-s)\alpha(x(t))}v$$
is continuous, and hence
\[(s, h) \mapsto e^{s\alpha(x(t+h))}\alpha(z(h))e^{(1-s)\alpha(x(t))}v\]
is continuous. We can therefore pass to the limit under the integral sign to derive that
\[
\frac{d}{dt}e^{\alpha(x(t))} = \int_0^1 e^{s\alpha(x(t))}\alpha(x'(t))e^{(1-s)\alpha(x(t))}vds = \int_0^1 \alpha(e^{s\text{ad} x(t)}x'(t))e^{\alpha(x(t))}vds,
\]
and the claim follows from the linearity and the continuity of \(\alpha\).

\[\square\]

4. THE RIGHT-LOGARITHMIC DERIVATIVE

Let \(\mathfrak{g}\) be a Banach–Lie algebra. Let \(U \subset \mathfrak{g}\) be a symmetric starlike neighbourhood of 0 in \(\mathfrak{g}\) such that the Dynkin series \(x \ast y\) converges in \(U \times U\). Then, for \(x \in U\), the maps
\[
\lambda_x y := x \ast y, \quad \rho_x y := y \ast x \quad \text{and} \quad c_x y := x \ast y \ast (-x)
\]
are local diffeomorphisms at the origin. Moreover the differential at 0 of \(c_x\) is given by
\[
Dc_x(0) = e^{\text{ad} x}.
\]

Let \(I\) denote an interval of the real line. The right logarithmic derivative (see \cite[II.4]{Neeb06}) of a smooth path \(\gamma : I \to U\) is defined by
\[
\delta(\gamma)_t = D\rho_{\gamma(t)}(0)^{-1}\gamma'(t).
\]

**Lemma 11.** Let \(x \in U\) and \(\gamma(t) = tx\). Then
\[
\delta(\gamma)_t = x.
\]

**Proof.** We have \(D\rho_{tx}(0)x = \lim_{h \to 0} \frac{hx + tx - tx}{h} = \lim_{h \to 0} \frac{(h+t)x - tx}{h} = x. \quad \square\)

**Lemma 12.** Let \(\alpha, \beta : I \to U\) two differentiable paths such that \((\alpha \ast \beta)(t) := \alpha(t) \ast \beta(t) \in U\). Then
\[
\delta(\alpha \ast \beta)_t = \delta(\alpha)_t + e^{\text{ad} \alpha(t)}\delta(\beta)_t.
\]

**Proof.** We have
\[
\delta(\alpha \ast \beta)_t = D\rho_{(\alpha \ast \beta)(t)}(0)^{-1}(\alpha \ast \beta)'(t)
\]
\[
= D\rho_{\alpha(t)}(0)^{-1}D\rho_{\beta(t)}(0)^{-1}(D\rho_{\beta(t)}(\alpha(t))\alpha'(t) + D\lambda_{\alpha(t)}(\beta(t))\beta'(t))
\]
\[
= D\rho_{\alpha(t)}(0)^{-1}\alpha'(t) + D\rho_{\alpha(t)}(0)^{-1}D\rho_{\beta(t)}(\alpha(t))^{-1}D\lambda_{\alpha(t)}(\beta(t))\beta'(t)
\]
\[
= D\rho_{\alpha(t)}(0)^{-1}\alpha'(t) + Dc_{\alpha(t)}(0)D\rho_{\beta(t)}(0)^{-1}\beta'(t)
\]
\[
= \delta(\alpha)_t + e^{\text{ad} \alpha(t)}\delta(\beta)_t. \quad \square\]

The next lemma says that the logarithmic derivative is the pull-back of the Maurer-Cartan form on \(\mathfrak{g}\) (and may therefore be defined for any path in \(\mathfrak{g}\)).

**Lemma 13.** Let \(\alpha : I \to U\) be a differentiable path. Then
\[
\delta(\alpha)_t = \int_0^1 e^{s\text{ad} \alpha(t)}\alpha'(t)ds.
\]

**Proof.** Let us fix \(t \in I\) and consider the map
\[
\psi : [0, 1] \to \mathfrak{g}, \quad s \mapsto \delta(s\alpha)_t.
\]

Then
\[
\psi(s+h) = \delta((s+h)\alpha)_t = \delta((s\alpha) \ast (h\alpha))_t.
\]
so we obtain with Lemma 12,
\[
\psi(s + h) = \delta(s\alpha)t + e^{s\text{ad}(t)}\delta(h\alpha)t.
\]
Hence we have
\[
\psi'(s) = \lim_{h \to 0} \frac{\psi(s + h) - \psi(s)}{h} = \lim_{h \to 0} e^{s\text{ad}(t)}\frac{1}{h}\delta(h\alpha)t = e^{s\text{ad}(t)}\alpha'(t),
\]
and the result follows by integration. \qed

Assume now that \( g \) decomposes as
\[
g = a_1 \oplus a_2 \oplus \cdots \oplus a_n,
\]
where \( a_j, j = 1, 2, \ldots, n \) are closed subspaces. Then, for every \( j = 1, 2, \ldots, n \), there exists a 0-neighbourhood \( V_j \) in \( a_j \) such that the map
\[
V_1 \times V_2 \times \cdots \times V_n \to g, \quad (x_1, x_2, \ldots, x_n) \mapsto x_1 * x_2 * \cdots * x_n
\]
is a diffeomorphism onto its image. From now on we assume that we have chosen \( U \) starlike and small enough so that it is contained in this image. So if \( x, y \in U \) then
\[
(tx) * y = x_1(t) * x_2(t) * \cdots * x_n(t)
\]
where \( t \mapsto x_j(t) \in a_j \) is analytic.

**Proposition 14.** We have
\[
x = \sum_{j=1}^{n} e^{\text{ad} x_1(t)} \cdots e^{\text{ad} x_{j-1}(t)} \int_0^1 e^{s \text{ad} x_j(t)} x_j'(t)ds
\]

*Proof.* Let \( \gamma(t) = (tx) * y = x_1(t) * x_2(t) * \cdots * x_n(t) \). The result follows by computing, using the preceding lemmas, the right logarithmic derivative of \( \gamma \) in its two expressions. \qed

5. **Proof of the main theorem**

Let us consider a strongly continuous representation \( \alpha \) of \( g \) on a dense domain \( D \) of the Hilbert space \( H \) which satisfies the assumptions (\( A1-3 \)) and recall the notations of the preceding section.

If
\[
z = z_1 * \cdots * z_n \in V_1 * \cdots * V_n
\]
we set
\[
\pi(z) := e^{\alpha(z_1)}e^{\alpha(z_2)} \cdots e^{\alpha(z_n)}.
\]
Let \( U' \) be a starlike 0-neighbourhood in \( g \) so that \( U' * U' \subseteq U \). Then it suffices to show that for every \( x, y \in U' \),
\[
\pi(x * y) = \pi(x)\pi(y),
\]
see, e.g., [Bou89, Ch. 3, §6, Lemma 1.1].

Let us write
\[
(tx) * y = x_1(t) * x_2(t) * \cdots * x_n(t), \quad x_j(t) \in a_j,
\]
so that
\[
\pi((tx) * y) = e^{\alpha(x_1(t))}e^{\alpha(x_2(t))} \cdots e^{\alpha(x_n(t))}.
\]
Let \( v \in D \) and
\[
\gamma(t) = \pi((tx) * y)v.
\]
Thanks to Proposition 10, we can use Lemma 4 several times to see that the map $t \mapsto \gamma(t)$ is differentiable with

$$
\gamma'(t) = \sum_{j=1}^{n} e^{\alpha(x_1(t))} \ldots e^{\alpha(x_{j-1}(t))} \alpha \left( \int_{0}^{1} e^{s \text{ad} x_j(t)} x'_j(t)ds \right) e^{\alpha(x_j(t))} \ldots e^{\alpha(x_n(t))} v.
$$

Then repeated use of the commutation relation yields

$$
\gamma'(t) = \alpha \left( \sum_{j=1}^{n} e^{\text{ad} x_1(t)} \ldots e^{\text{ad} x_{j-1}(t)} \int_{0}^{1} e^{s \text{ad} x_j(t)} x'_j(t)ds \right) \gamma(t),
$$

which, according to Proposition 14, amounts to

$$(7) \quad \gamma'(t) = \alpha(x) \gamma(t).$$

But is it well known (cf. [FSSS72, P. 431]) that the solution of the initial value problem

$$\gamma(t) \in D, \quad \gamma'(t) = \alpha(x) \gamma(t), \quad \alpha(0) = \pi(y)v,$$

is unique (and given by $e^{\alpha(x)} \pi(y)v$). Therefore we have

$$\pi(tx) \pi(y)v = \pi((tx) \ast y)v,$$

and this equality extends to $\mathcal{H}$ to give (1) by evaluation at $t = 1$. We also derive from (7), with $y = 0$, that $\overline{\text{d} \pi(x)} \cap D = \alpha(x)$, and since by construction $\pi(tx)D \subseteq D$, the operator $\alpha(x)$ is essentially skew-adjoint, i.e. $\alpha(x) = d\pi(x)$. It also follows from (5) that (1) holds for every $x \in g$. Hence for $v \in D$ the map

$$g \rightarrow \mathcal{H}, \quad x \mapsto e^{\alpha(x)} v$$

is continuous (see Proposition 10), and this implies that the representation $\pi$ is continuous. Now we have

$$D \subseteq D_{\infty}^{g} := \bigcap_{\ell \in \mathbb{N}, y_{\ell} \in g} D(\overline{\text{d} \pi(y_{\ell})} \ldots \overline{\text{d} \pi(y_{1})}),$$

but we know by [Nee10a, Lemma 3.4, Remark 8.3] that $D_{\infty}^{g}$ coincides with the space of smooth vectors for $\pi$. This concludes the proof.

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