Convergence and Sample Complexity of Natural Policy Gradient Primal-Dual Methods for Constrained MDPs

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Abstract

We study sequential decision making problems aimed at maximizing the expected total reward while satisfying a constraint on the expected total utility. We employ the natural policy gradient method to solve the discounted infinite-horizon optimal control problem for Constrained Markov Decision Processes (constrained MDPs). Specifically, we propose a new Natural Policy Gradient Primal-Dual (NPG-PD) method that updates the primal variable via natural policy gradient ascent and the dual variable via projected sub-gradient descent. Although the underlying maximization involves a nonconcave objective function and a nonconvex constraint set, under the softmax policy parametrization we prove that our method achieves global convergence with sublinear rates regarding both the optimality gap and the constraint violation. Such convergence is independent of the size of the state-action space, i.e., it is dimension-free. Furthermore, for log-linear and general smooth policy parametrizations, we establish sublinear convergence rates up to a function approximation error caused by restricted policy parametrization. We also provide convergence and finite-sample complexity guarantees for two sample-based NPG-PD algorithms. Finally, we use computational experiments to showcase the merits and the effectiveness of our approach.

Keywords: Constrained Markov decision processes; Natural policy gradient; Constrained nonconvex optimization; Method of Lagrange multipliers; Primal-dual algorithms.

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1. Introduction

Reinforcement learning (RL) studies sequential decision-making problems with the objective of maximizing expected total reward while interacting with an unknown environment (Sutton and Barto, 2018). Markov Decision Processes (MDPs) are typically used to model the dynamics of the environment. However, in many safety-critical applications, e.g., in autonomous driving (Fisac et al., 2018), robotics (Ono et al., 2015), cyber-security (Zhang et al., 2019b), and financial management (Abe et al., 2010), the control system is also subject to constraints on its utilities/costs. In this setting, constrained Markov Decision Processes (constrained MDPs) are used to model the environment dynamics (Altman, 1999) and, in addition to maximizing the expected total reward it is also important to take into account the constraint on the expected total utility/cost as an extra learning objective.

Policy gradient (PG) (Sutton et al., 2000) and natural policy gradient (NPG) (Kakade, 2002) methods have enjoyed substantial empirical success in solving MDPs (Schulman et al., 2015; Lillicrap et al., 2015; Mnih et al. 2016; Schulman et al., 2017; Sutton and Barto, 2018). PG methods, or more generally direct policy search methods, have also been used to solve constrained MDPs (Uchibe and Doya, 2007; Borkar, 2005; Bhatnagar and Lakshmanan, 2012; Chow et al., 2017; Tessler et al., 2019; Liang et al., 2018; Paternain et al., 2022; Achiam et al., 2017; Spooner and Savani, 2020), but most existing theoretical guarantees are asymptotic and/or only provide local convergence guarantees to stationary-point policies. On the other hand, it is desired to show that, for arbitrary initial condition, a solution that enjoys $\epsilon$-optimality gap and $\epsilon$-constraint violation is computed using a finite number of iterations and/or samples. It is thus imperative to establish global convergence guarantees for PG methods when solving constrained MDPs.

In this work, we provide a theoretical foundation for non-asymptotic global convergence of the NPG method in solving optimal control problems for constrained MDPs and answer the following questions:

(i) Can we employ NPG methods to solve optimal control problems for constrained MDPs?

(ii) Do NPG methods converge to the globally optimal solution that satisfies constraints?

(iii) What is the convergence rate of NPG methods and the effect of the function approximation error caused by a restricted policy parametrization?

(iv) What is the sample complexity of model-free NPG methods?

Preview of key contributions

Our key contributions are:

(i) We propose a simple but effective primal-dual algorithm for solving discounted infinite-horizon optimal control problems for constrained MDPs. Our Natural Policy Gradient Primal-Dual (NPG-PD) method employs natural policy gradient ascent to update the primal variable and projected sub-gradient descent to update the dual variable.

(ii) We exploit the structure of softmax policy parametrization to establish global convergence guarantees in spite of the fact that the objective function in maximization
problem is not concave and the constraint set is not convex. In particular, we prove that our NPG-PD method achieves global convergence with rate $O(1/\sqrt{T})$ in both the optimality gap and the constraint violation, where $T$ is the total number of iterations. Our convergence guarantees are dimension-free, i.e., the rate is independent of the size of the state-action space.

(iii) We establish convergence with rate $O(1/\sqrt{T})$ in both the optimality gap and the constraint violation for log-linear and general smooth policy parametrizations, up to a function approximation error caused by restricted policy parametrization.

(iv) We provide convergence and finite-sample complexity guarantees for two sample-based NPG-PD algorithms. We also utilize computational experiments to showcase the merits and the effectiveness of our approach.

At this point it is worth highlighting the main differences between the results of this paper and those of our earlier one (Ding et al., 2020). Although our algorithmic framework utilizes the NPG-PD method of Ding et al. (2020), our practical characterization of function approximation error, which is based on the estimation-transfer error decomposition in both the optimality gap and the constraint violation, facilitates the derivation of convergence and sample complexity results described in (iii) and (iv) above. In contrast, earlier function approximation study of Ding et al. (2020) utilizes the classical notion of compatible function approximation which is often challenging to control in modern machine learning. Moreover, Ding et al. (2020) follows standard drift analysis of constraint violation in online optimization and it yields sub-optimal rates relative to the tabular case.

Our results as summarized in (iii) and (iv) above build on PG methods (Agarwal et al., 2021) and they provide a novel contribution to the constrained MDP setting. In contrast to our earlier work (Ding et al., 2020), we establish the optimal rate for log-linear and general smooth policy parameterizations up to a function approximation error. By providing a new regret-type primal-dual analysis, we show that the derived rate for the function approximation case matches the optimal rate for the tabular case. Furthermore, in contrast to the sample complexity result of Ding et al. (2020), which holds only when the estimates of value functions are bounded, we employ a stochastic gradient method that does not require boundedness. This allows us to improve sample complexity from $O(1/\epsilon^6)$ to $O(1/\epsilon^4)$, where $\epsilon > 0$ is a desired level of accuracy.

In addition to these technical differences, we also characterize zero constraint violation performance of our method in both tabular and function approximation settings and conduct computational experiments on a set of benchmark robotic tasks to demonstrate the merits and the effectiveness of our approach.

Related work

Our work builds on Lagrangian-based constrained MDP algorithms (Altman, 1999; Abad et al., 2002; Abad and Krishnamurthy, 2003; Borkar, 2005). However, convergence guarantees of these algorithms are either local (to stationary-point or locally optimal policies) (Bhatnagar and Lakshmanan, 2012; Chow et al., 2017; Tessler et al., 2019) or asymptotic (Borkar, 2005). In the tabular setting, we compare the convergence rates in Table 1 by assuming the exact evaluation of policy gradients. When function approximation is used for policy
parametrization, Yu et al. (2019) recognized the lack of convexity and showed asymptotic convergence (to a stationary point) of a method based on successive convex relaxations. In contrast, we establish convergence to a globally optimal solution in spite of the lack of convexity. Recent references (Paternain et al., 2019, 2022) are closely related to our work. Paternain et al. (2019) provide duality analysis for constrained MDPs in the policy space and propose a provably convergent dual descent algorithm by assuming access to a nonconvex optimization oracle. However, it is not clear how to obtain the solution to a primal nonconvex optimization problem and the global convergence guarantees are not established. Paternain et al. (2022) propose a primal-dual algorithm and provide computational experiments but do not offer any convergence analysis. In spite of the lack of convexity, our work provides global convergence guarantees for a new primal-dual algorithm without using any optimization oracles. For the function approximation setting, we compare the convergence rates and sample complexities in Table 2. Other related policy optimization methods include CPG (Uchibe and Doya, 2007), accelerated PDPO (Liang et al., 2018), CPO (Achiam et al., 2017; Yang et al., 2020b), FOCOPS (Zhang et al., 2020b), and IPPO (Liu et al., 2020b) but theoretical guarantees for these algorithms are still lacking. Recently, optimism principles have been used for efficient exploration in constrained MDPs (Singh et al., 2020; Zheng and Ratliff, 2020. Ding et al., 2021; Qiu et al., 2020; Efroni et al., 2020; Bai et al., 2020; Yu et al., 2021; Liu et al., 2021a; Wei et al., 2022). In comparison, our work focuses on the optimization landscape within a primal-dual framework both in the model-based and the model-free setup.

Our work is also pertinent to recent global convergence results for PG methods. Fazel et al. (2018); Malik et al. (2020); Mohammadi et al. (2019, 2020, 2021, 2022) provided global convergence guarantees and quantified sample complexity of (natural) PG methods for nonconvex linear quadratic regulator problem of both discrete- and continuous-time systems. Zhang et al. (2019a) showed that locally optimal policies for MDPs are achievable using PG methods with reward reshaping. Wang et al. (2019) demonstrated that (natural) PG methods converge to the globally optimal value when overparametrized neural networks are used. A variant of NPG, trust-region policy optimization (TRPO) (Schulman et al., 2015), converges to the globally optimal policy with overparametrized neural networks (Liu et al., 2019) and for regularized MDPs (Shani et al. 2020). Bhandari and Russo (2019, 2021) studied global optimality and convergence of PG methods from a policy iteration perspective. Agarwal et al. (2021) characterized global convergence properties of (natural) PG methods and studied computational, approximation, and sample size issues. Additional recent advances along these lines include (Mei et al., 2020. Zhang et al., 2020a; Cen et al., 2021. Liu et al. 2020a; Khodadadian et al. 2022). While all these references handle a lack of convexity in the objective function, additional effort is required to deal with nonconvex constraints that arise in optimal control of constrained MDPs. Our paper addresses this challenge.

We also remark on a very recent work on Lagrangian-based policy optimization. Liu et al. (2021b); Li et al. (2021); Ying et al. (2022) examined a two-time scale scheme for updating primal-dual variables by updating policy in an inner loop via an NPG-style subroutine for each dual iterate. In spite of improved convergence that results from the proposed modifications of the Lagrangian and the dual update, the double-loop scheme often increases computational cost and introduces difficulty in parameter tuning. Furthermore, Liu et al. (2021b a); Bai et al. (2022) proposed policy optimization algorithms that offer zero constraint violation in the end of training but they did not consider infinite state spaces.
Natural Policy Gradient Primal-Dual Method for Constrained MDPs

| Algorithm                        | Iteration/Sample complexities |
|----------------------------------|-------------------------------|
| PG-PD (Abad et al., 2002)        | asymptotic                    |
| PG-PD (Borkar, 2005)             | asymptotic                    |
| NPG-PD (Theorem 9, Theorem 27)   | $O\left(1/\epsilon^2\right) / O\left(1/\epsilon^4\right)$ |

Table 1: Complexity comparison of our NPG-PD method with closely-related algorithms for the tabular case with finitely many states/actions. The iteration complexity is determined by the number of gradient-based updates that an algorithm takes to achieve $\epsilon$-optimality gap and $\epsilon$-constraint violation, $\frac{1}{T} \sum_{t=0}^{T-1} (V_t^*(\rho) - V_t^{(t)}(\rho)) \leq \epsilon$ and $\left[\frac{1}{T} \sum_{t=0}^{T-1} (b - V_t^{(t)}(\rho))\right]_+ \leq \epsilon$, and the sample complexity is determined by the number of trajectory rollouts.

| Algorithm                        | Iteration/Sample complexities |
|----------------------------------|-------------------------------|
| PDO (Chow et al., 2017)          | asymptotic                    |
| RCPO (Tessler et al., 2019)      | asymptotic                    |
| CBP (Jain et al., 2022)          | $O\left(1/\epsilon^2\right) + \epsilon_{fa} / -$ |
| C-NPG-PD (Bai et al., 2023)      | $O\left(1/\epsilon^2\right) + \epsilon_{fa} / O\left(1/\epsilon^4\right) + \bar{\epsilon}_{bias}$ |
| NPG-PD (Theorem 15, Theorem 27)  | $O\left(1/\epsilon^2\right) + \epsilon_{fa} / O\left(1/\epsilon^4\right) + \epsilon_{bias}$ |
| NPG-PD (Theorem 22, Theorem 26)  | $O\left(1/\epsilon^2\right) + \epsilon_{fa} / O\left(1/\epsilon^4\right) + \epsilon_{bias}$ |

Table 2: Complexity comparison of our NPG-PD method with closely-related algorithms for the function approximation case with potentially infinitely many states/actions. Up to a function approximation error $\epsilon_{fa}$, the iteration complexity is determined by the number of gradient-based iterations an algorithm takes to ensure $\epsilon$-optimality gap and $\epsilon$-constraint violation, $E\left[\frac{1}{T} \sum_{t=0}^{T-1} (V_t^*(\rho) - V_t^{(t)}(\rho))\right] \leq \epsilon$ and $E\left[\left[\frac{1}{T} \sum_{t=0}^{T-1} (b - V_t^{(t)}(\rho))\right]_+\right] \leq \epsilon$, and the sample complexity is determined by the number of trajectory rollouts. The bias error $\bar{\epsilon}_{bias}$ contains the transfer error $\epsilon_{bias}$, that captures how well the function class represents the true function, and the error of policy representation.
Paper outline
In Section 2, we formulate an optimal control problem for constrained Markov decision processes and provide necessary background material. In Section 3, we describe our natural policy gradient primal-dual method. We provide convergence guarantees for our algorithm under the tabular softmax policy parametrization in Section 4 and under log-linear and general smooth policy parametrizations in Section 5. We establish convergence and finite-sample complexity guarantees for two model-free primal-dual algorithms in Section 6 and provide computational experiments in Section 7. We close the paper with remarks in Section 8.

2. Problem setup
In Section 2.1, we introduce constrained Markov decision processes. In Section 2.2, we present the method of Lagrange multipliers, formulate a saddle-point problem for the constrained policy optimization, and exhibit several problem properties: strong duality, boundedness of the optimal dual variable, and constraint violation. In Section 2.3, we introduce a parametrized formulation of the constrained policy optimization problem, provide an example of a constrained MDP which is not convex, and present several useful policy parametrizations.

2.1 Constrained Markov decision processes
We consider a discounted Constrained Markov Decision Process (Altman, 1999),

$$\text{CMDP}(S, A, P, r, g, b, \gamma, \rho)$$

where $S$ is a finite state space, $A$ is a finite action space, $P$ is a transition probability measure which specifies the transition probability $P(s' | s, a)$ from state $s$ to the next state $s'$ under action $a \in A$, $r: S \times A \to [0, 1]$ is a reward function, $g: S \times A \to [0, 1]$ is a utility function, $b$ is a constraint offset, $\gamma \in [0, 1)$ is a discount factor, and $\rho$ is an initial distribution over $S$.

For any state $s_t$, a stochastic policy $\pi: S \to \Delta_A$ is a function in the probability simplex $\Delta_A$ over action space $A$, i.e., $a_t \sim \pi(\cdot | s_t)$ at time $t$. Let $\Pi$ be a set of all possible policies. A policy $\pi \in \Pi$, together with initial state distribution $\rho$, induces a distribution over trajectories

$$\tau = \{(s_t, a_t, r_t, g_t)\}_{t=0}^\infty,$$

where $s_0 \sim \rho$, $a_t \sim \pi(\cdot | s_t)$ and $s_{t+1} \sim P(\cdot | s_t, a_t)$ for all $t \geq 0$.

Given a policy $\pi$, the value functions $V^*_r, V^*_g: S \to \mathbb{R}$ associated with the reward $r$ or the utility $g$ are determined by the expected values of total discounted rewards or utilities received under policy $\pi$,

$$V^*_r(s) := \mathbb{E} \left[ \sum_{t=0}^\infty \gamma^t r(s_t, a_t) \mid \pi, s_0 = s \right], \quad V^*_g(s) := \mathbb{E} \left[ \sum_{t=0}^\infty \gamma^t g(s_t, a_t) \mid \pi, s_0 = s \right]$$

where the expectation $\mathbb{E}$ is taken over the randomness of the trajectory $\tau$ induced by $\pi$.

Starting from an arbitrary state-action pair $(s, a)$ and following a policy $\pi$, we also introduce the state-action value functions $Q^*_r(s, a), Q^*_g(s, a): S \times A \to \mathbb{R}$ together with their advantage
functions $A^r_\pi, A^g_\pi : S \times A \to \mathbb{R}$,

$$Q^\pi_\omega(s, a) := \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t \diamond (s_t, a_t) \mid \pi, s_0 = s, a_0 = a \right]$$

$$A^\pi_\omega := Q^\pi_\omega(s, a) - V^\pi_\omega(s)$$

where the symbol $\diamond$ represents $r$ or $g$. Since $r, g \in [0, 1]$, we have

$$V^\pi_\omega(s) \in \left[ 0, \frac{1}{1-\gamma} \right]$$

and their expected values under the initial distribution $\rho$ are determined by

$$V^\pi_\omega(\rho) := \mathbb{E}_{s_0 \sim \rho} [V^\pi_\omega(s_0)].$$

Having defined a policy as well as the state-action value functions for the discounted constrained MDP, the objective is to find a policy that maximizes the expected reward value over all policies subject to a constraint on the expected utility value,

$$\max_{\pi \in \Pi} V^r_\omega(\rho)$$

subject to $V^g_\omega(\rho) \geq b$. 

In view of the aforementioned boundedness of $V^r_\omega(s)$ and $V^g_\omega(s)$, we set the constraint offset $b \in (0, 1/(1-\gamma)]$ to make Problem (1) meaningful.

**Remark 1** For notational convenience we consider a single constraint in Problem (1) but our convergence guarantees are readily generalizable to the problems with multiple constraints.

### 2.2 Method of Lagrange multipliers

By dualizing constraints (Luenberger and Ye, 1984; Bertsekas, 2014), we cast Problem (1) into the following max-min problem,

$$\max_{\pi \in \Pi} \min_{\lambda \geq 0} V^\pi_\omega(\rho) + \lambda (V^g_\omega(\rho) - b)$$

where $V^{\pi,\lambda}_L(\rho) := V^r_\omega(\rho) + \lambda (V^g_\omega(\rho) - b)$ is the Lagrangian of Problem (1), $\pi$ is the primal variable, and $\lambda$ is the nonnegative Lagrange multiplier or dual variable. The associated dual function is defined as

$$V^\lambda_D(\rho) := \max_{\pi \in \Pi} V^{\pi,\lambda}_L(\rho).$$

Instead of utilizing the linear program method (Altman, 1999), we employ direct policy search method to solve Problem (2). Direct methods are attractive for three reasons: (i) they allow us to directly optimize/monitor the value functions that we are interested in; (ii) they can deal with large state-action spaces via policy parameterization, e.g., neural nets; and (iii)
they can utilize policy gradient estimates via simulations of the policy. Since Problem (1) is a nonconcave constrained maximization problem with the policy space $\Pi$ that is often infinite-dimensional, Problems (1) and (2) are challenging.

In spite of these challenges, Problem (1) has nice properties in the policy space when it is strictly feasible. We adapt the standard Slater condition (Bertsekas, 2014) and assume strict feasibility of Problem (1) throughout the paper.

**Assumption 2 (Slater condition)** There exists $\xi > 0$ and $\pi \in \Pi$ such that $V^*_g(\rho) - b \geq \xi$.

The Slater condition is mild in practice because we usually have a priori knowledge on a strictly feasible policy, e.g., the minimal utility is achievable by a particular policy so that the constraint becomes loose.

Let $\pi^*$ denote an optimal solution to Problem (1), let $\lambda^*$ be an optimal dual variable

$$\lambda^* \in \text{argmin} \ V^\lambda_D(\rho)$$

and let the set of all optimal dual variables be $\Lambda^*$. We use the shorthand notation $V^\pi^*_r(\rho) = V^*_r(\rho)$ and $V^\pi^*_g(\rho) = V^*_g(\rho)$ whenever it is clear from the context. We recall the strong duality for constrained MDPs and we prove boundedness of optimal dual variable $\lambda^*$.

**Lemma 3 (Strong duality and boundedness of $\lambda^*$)** Let Assumption 2 hold. Then

(i) $V^*_r(\rho) = V^*_g(\rho)$;

(ii) $0 \leq \lambda^* \leq (V^*_r(\rho) - V^*_g(\rho)) / \xi$.

**Proof** The proof of (i) is standard; e.g., see Altman (1999, Theorem 3.6) or Paternain et al. (2019, Theorem 1) or Paternain et al. (2022, Theorem 3). The proof of (ii) builds on the constrained convex optimization (Beck, 2017, Section 8.5). Let $\Lambda_a := \{\lambda \geq 0 | V^\lambda_D(\rho) \leq a\}$ be a sublevel set of the dual function for $a \in \mathbb{R}$. For any $\lambda \in \Lambda_a$, we have

$$a \geq V^\lambda_D(\rho) \geq V^\pi_g(\rho) + \lambda (V^*_g(\rho) - b) \geq V^\pi_g(\rho) + \lambda\xi$$

where $\pi$ is a Slater point. Thus, $\lambda \leq (a - V^*_g(\rho)) / \xi$. If we take $a = V^*_r(\rho) = V^*_D$, then $\Lambda_a = \Lambda^*$ which proves (ii).

Let the value function associated with Problem (1) be determined by

$$v(\tau) := \text{maximize}_{\pi \in \Pi} \{V^\pi_r(\rho) | V^\pi_g(\rho) \geq b + \tau\}.$$  

Using the concavity of $v(\tau)$ (e.g., see Paternain et al. (2019, Proposition 1)), in Lemma 4 we establish a bound on the constraint violation, thereby extending a result from constrained convex optimization (Beck, 2017, Section 8.5) to a constrained non-convex setting.

**Lemma 4 (Constraint violation)** Let Assumption 2 hold. For any $C \geq 2\lambda^*$, if there exists a policy $\pi \in \Pi$ and $\delta > 0$ such that $V^*_r(\rho) - V^*_g(\rho) + C [b - V^*_g(\rho)]_+ \leq \delta$, then $[b - V^*_g(\rho)]_+ \leq 2\delta / C$, where $[x]_+ = \max(x, 0)$.
Proof By the definition of $v(\tau)$, we have $v(0) = V^*(\rho)$. We also note that $v(\tau)$ is concave (see the proof of Paternain et al. (2019, Proposition 1)). First, we show that $-\lambda^* \in \partial v(0)$. By the definition of $V_L^{\pi,\lambda}(\rho)$ and the strong duality in Lemma 3,

$$
V_L^{\pi,\lambda^*}(\rho) \leq \max_{\pi \in \Pi} V_L^{\pi,\lambda^*}(\rho) = V_p^*(\rho) = V_p^*(\rho) = v(0), \text{ for all } \pi \in \Pi.
$$

Hence, for any $\pi \in \{\pi \in \Pi \mid V_g^{\pi}(\rho) \geq b + \tau\}$,

$$
v(0) - \tau\lambda^* \geq V_L^{\pi,\lambda^*}(\rho) - \tau\lambda^* = V_p^*(\rho) + \lambda^* (V_g^*(\rho) - b) - \tau\lambda^* = V_p^*(\rho) + \lambda^* (V_g^*(\rho) - b - \tau) \geq V_p^*(\rho).
$$

Maximizing the right-hand side of this inequality over $\{\pi \in \Pi \mid V_g^{\pi}(\rho) \geq b + \tau\}$ yields

$$
v(0) - \tau\lambda^* \geq v(\tau)
$$

and, thus, $-\lambda^* \in \partial v(0)$.

On the other hand, if we take $\tau = -(b - V_g^*(\rho))_+$, then

$$
V_p^*(\rho) \leq V_p^*(\rho) = v(0) \leq v(\tau).
$$

Combining (3) and (4) yields $V_p^*(\rho) - V_p^*(\rho) \leq -\tau\lambda^*$. Thus,

$$(C - \lambda^*) |\tau| = -\lambda^* |\tau| + C |\tau| = \tau\lambda^* + C |\tau| \leq V_p^*(\rho) - V_p^*(\rho) + C |\tau|$$

which completes the proof by applying the assumed condition on $\pi$. \qed

Aided by the above properties implied by the Slater condition, we target the max-min Problem (2) in a primal-dual domain.

2.3 Policy parametrization

Introduction of a set of parametrized policies $\{\pi_{\theta} \mid \theta \in \Theta\}$ brings Problem (1) into a constrained optimization problem over the finite-dimensional parameter space $\Theta$,

$$
\max_{\theta \in \Theta} V_r^{\pi_{\theta}}(\rho)
$$

subject to $V_g^{\pi_{\theta}}(\rho) \geq b$. 

9
A parametric version of max-min Problem (2) is given by
\[
\begin{align*}
\maximize_{\theta \in \Theta} & \minimize_{\lambda \geq 0} V_{r}^{\pi_{\theta}}(\rho) + \lambda(V_{g}^{\pi_{\theta}}(\rho) - b).
\end{align*}
\] (6)

where \(V_{L}^{r \pi_{\theta}, \lambda}(\rho) := V_{r}^{\pi_{\theta}}(\rho) + \lambda(V_{g}^{\pi_{\theta}}(\rho) - b)\) is the associated Lagrangian and \(\lambda\) is the Lagrange multiplier. The dual function is determined by \(V_{D}^{\pi_{\theta}}(\rho) := \maximize_{\theta} V_{r}^{\pi_{\theta}}(\rho), \lambda\). The primal maximization problem (5) is finite-dimensional but not concave even if in the absence of a constraint (Agarwal et al., 2021). In Lemma 5 we prove that, in general, Problem (5) is not convex because it involves maximization of a non concave objective function over non convex constraint set. The proof is provided in Appendix A and it utilizes an example of a constrained MDP in Figure 1.

**Lemma 5 (Lack of convexity)** There exists a constrained MDP for which the objective function \(V_{r}^{\pi_{\theta}}(s)\) in Problem (5) is not concave and the constraint set \(\{\theta \in \Theta | V_{g}^{\pi_{\theta}}(s) \geq b\}\) is not convex.

![Figure 1: An example of a constrained MDP for which the objective function \(V_{r}^{\pi_{\theta}}(s)\) in Problem (5) is not concave and the constraint set \(\{\theta \in \Theta | V_{g}^{\pi_{\theta}}(s) \geq b\}\) is not convex.](image)

In general, the Lagrangian \(V_{L}^{r \pi_{\theta}, \lambda}(\rho)\) in Problem (6) is convex in \(\lambda\) but not concave in \(\theta\). While many algorithms for solving max-min optimization problems, e.g., those proposed in Lin et al. (2019); Nouiehed et al. (2019); Yang et al. (2020a), require extra assumptions on the max-min structure or only guarantee convergence to a stationary point, we exploit problem structure and propose a new primal-dual method to compute globally optimal solution to Problem (6). Before doing that, we first introduce several useful classes of policies.

**Direct policy parametrization.** A direct parametrization of a policy is the probability distribution,
\[
\pi_{\theta}(a | s) = \theta_{s,a} \text{ for all } \theta \in \Delta_{|S|}
\]
where \(\theta_{s} \in \Delta_{A} \text{ for any } s \in S, \text{i.e.}, \theta_{s,a} \geq 0 \text{ and } \sum_{a \in A} \theta_{s,a} = 1\). This policy class is complete since it directly represents any stochastic policy. Even though it is challenging to deal with
from both theoretical and the computational viewpoints (Mei et al., 2020; Agarwal et al., 2021), it offers a sanity check for many policy search methods.

**Softmax policy parametrization.** This class of policies is parametrized by the softmax function,

\[ \pi_{\theta}(a | s) = \frac{\exp(\theta_{s,a})}{\sum_{a' \in A} \exp(\theta_{s,a'})} \text{ for all } \theta \in \mathbb{R}^{||S|| \cdot ||A||}. \]  

The softmax policy can be used to represent any stochastic policy and its closure contains all stationary policies. It has been utilized to study convergence properties of many RL algorithms (Bhandari and Russo, 2019; Agarwal et al., 2021; Mei et al., 2020; Cen et al., 2021; Khodadadian et al., 2022) and it offers several algorithmic advantages: (i) it equips the policy with a rich structure so that the NPG update works like the classical multiplicative weights update in the online learning literature (e.g., see Cesa-Bianchi and Lugosi (2006)); (ii) it can be used to interpret the function approximation error (Agarwal et al., 2021).

**Log-linear policy parametrization.** A log-linear policy is given by

\[ \pi_{\theta}(a | s) = \frac{\exp(\theta^\top \phi_{s,a})}{\sum_{a' \in A} \exp(\theta^\top \phi_{s,a'})} \text{ for all } \theta \in \mathbb{R}^d \]  

where \( \phi_{s,a} \in \mathbb{R}^d \) is the feature map at a state-action pair (s, a). The log-linear policy builds on the softmax policy by applying the softmax function to a set of linear functions in a given feature space. More importantly, it exactly characterizes the linear function approximation via policy parametrization (Agarwal et al. 2021); see Miryoosefi and Jin (2022); Amani et al. (2021) for linear constrained MDPs.

**General policy parametrization.** A general class of stochastic policies is given by \{\pi_{\theta} | \theta \in \Theta\} with \( \Theta \subset \mathbb{R}^d \) without specifying the structure of \( \pi_{\theta} \). The parameter space has dimension \( d \) and this policy class covers a setting that utilizes nonlinear function approximation, e.g., (deep) neural networks (Liu et al., 2019; Wang et al., 2019).

When we choose \( d \ll ||S|| \cdot ||A|| \) in either the log-linear policy or the general nonlinear policy, the policy class has a limited expressiveness and it may not contain all stochastic policies. Motivated by this observation, the theory that we develop in Section 5 establishes global convergence up to error caused by the restricted policy class.

### 3. Natural policy gradient primal-dual method

In Section 3.1, we provide a brief summary of three basic algorithms that have been used to solve constrained policy optimization problem (5). In Section 3.2, we propose a natural policy gradient primal-dual method which represents an extension of natural policy gradient method to constrained optimization problems.

#### 3.1 Constrained policy optimization methods

We briefly summarize three basic algorithms that can be employed to solve the primal problem (5). We assume that the value function and the policy gradient can be evaluated exactly for any given policy.
We first introduce some useful definitions. The discounted visitation distribution $d^\pi_{s_0}$ of a policy $\pi$ and its expectation over the initial distribution $\rho$ are respectively given by,

$$d^\pi_{s_0}(s) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t P^\pi(s_t = s \mid s_0)$$

$$d^\rho_{s_0}(s) = \mathbb{E}_{s_0 \sim \rho} \left[ d^\pi_{s_0}(s) \right]$$

where $P^\pi(s_t = s \mid s_0)$ is the probability of visiting state $s$ at time $t$ under the policy $\pi$ with an initial state $s_0$. When the use of parametrized policy $\pi_\theta$ is clear from the context, we use $V_\theta r(\rho)$ to denote $V^\pi \pi_\theta r(\rho)$. When $\pi_\theta(\cdot \mid s)$ is differentiable and when it belongs to the probability simplex, i.e., $\pi_\theta \in \Delta_\Theta$ for all $\theta$, the policy gradient (PG) of the Lagrangian (6) is determined by,

$$\nabla_\theta V_\theta^\lambda L(s_0) = \nabla_\theta V_\theta r(s_0) + \lambda \nabla_\theta V_\theta g(s_0)$$

$$= \frac{1}{1 - \gamma} \mathbb{E}_{s_0 \sim d_0^\pi} \mathbb{E}_{a \sim \pi_\theta(\cdot \mid s)} \left[ A_\theta^\lambda(s, a) \nabla_\theta \log \pi_\theta(a \mid s) \right]$$

where $A_\theta^\lambda(s, a) = A_\theta(s, a) + \lambda A_\theta(g(s, a))$.

**Dual method**

When strong duality in Lemma 3 holds, it is convenient to work with the dual formulation of the primal problem (5),

$$\text{minimize } \lambda \geq 0 \quad V_\lambda D(\rho). \quad (10)$$

While the dual function is convex regardless of concavity of the primal maximization problem, it is often non-differentiable (Bertsekas, 2008). Thus, a projected dual subgradient descent can be used to solve the dual problem,

$$\lambda^{(t+1)} = \mathcal{P}_+ \left( \lambda^{(t)} - \eta \partial_\lambda V_\lambda^{\lambda(t)}(\rho) \right)$$

where $\mathcal{P}_+ (\cdot)$ is the projection to the non-negative real axis, $\eta > 0$ is the stepsize, and $\partial_\lambda V_\lambda^{\lambda(t)}(\rho) := \partial_\lambda V_\lambda^{\lambda(t)}(\rho)|_{\lambda = \lambda^{(t)}}$ is the subgradient of the dual function evaluated at $\lambda = \lambda^{(t)}$.

The dual method works in the space of dual variables and it requires efficient evaluation of the subgradient of the dual function. We note that computing the dual function $V_\lambda^D(\rho)$ for a given $\lambda = \lambda^{(t)}$ in each step amounts to a standard unconstrained RL problem (Paternain et al., 2019). In spite of global convergence guarantees for several policy search methods in the tabular setting, it is often challenging to obtain the dual function and/or to compute its sub-gradient, e.g., when the problem dimension is high and/or when the state space is continuous. Although the primal problem can be approximated using the first-order Taylor series expansion (Achiam et al., 2017; Yang et al., 2020b), inverting Hessian matrices becomes the primary computational burden and it is costly to implement the dual method.
Primal method

In the primal method, a policy search strategy works directly on the primal problem (5) by seeking an optimal policy in a feasible region. The key challenge is to ensure the feasibility of the next iterate in the search direction, which is similar to the use of the primal method in nonlinear programming (Luenberger and Ye, 1984).

An intuitive approach is to check the feasibility of each iterate and determine whether the constraint is active (Xu et al., 2021). If the iterate is feasible or the constraint is inactive, we move towards maximizing the single objective function; otherwise, we look for a feasible direction. For the softmax policy parametrization (7), this can be accomplished using a simple first-order gradient method,

\[
\begin{align*}
\theta^{(t+1)}_{s,a} &= \theta^{(t)}_{s,a} + \eta G^{(t)}_{s,a}(\rho) \\
G^{(t)}_{s,a}(\rho) &= \begin{cases} \\
\frac{1}{1-\gamma} A^{(t)}_r(s,a), & \text{when } V^{(t)}_g(\rho) < b - \epsilon_b \\
\frac{1}{1-\gamma} A^{(t)}_g(s,a), & \text{when } V^{(t)}_g(\rho) \geq b - \epsilon_b
\end{cases}
\end{align*}
\]  

(11)

where we use the \( A^{(t)}_r(s,a) \) and \( A^{(t)}_g(s,a) \) to denote \( A^{(t)}_r(s,a) \) and \( A^{(t)}_g(s,a) \), respectively, \( G^{(t)}_{s,a}(\rho) \) is the gradient ascent direction determined by the scaled version of advantage functions, and \( \epsilon_b > 0 \) is the relaxation parameter for the constraint \( V^{\pi_{\theta}}_g(\rho) \geq b \). When the iterate violates the relaxed constraint, \( V^{\pi_{\theta}}_g(\rho) \geq b - \epsilon_b \), it maximizes the constraint function to gain feasibility. More reliable evaluation of the feasibility often demands a more tractable characterization of the constraint, e.g., by utilizing Lyapunov function (Chow et al., 2018), Gaussian process modeling (Sui et al., 2018), backward value function (Satija et al., 2020), and logarithmic penalty function (Liu et al., 2020b). Hence, the primal method offers the adaptability of adjusting a policy to satisfy the constraint, which is desirable in safe training applications. However, global convergence theory is still lacking and recent progress (Xu et al., 2021) requires a careful relaxation of the constraint.

Primal-dual method

The primal-dual method simultaneously updates primal and dual variables (Arrow, 1958). With the direct parametrization \( \pi_{\theta}(a \mid s) = \theta_{s,a} \), a basic primal-dual method performs the following Policy Gradient Primal-Dual (PG-PD) update (Abad and Krishnamurthy, 2003),

\[
\begin{align*}
\theta^{(t+1)} &= \mathcal{P}_\Theta \left( \theta^{(t)} + \eta_1 \nabla_\theta V^{\pi_{\theta}}_L(\rho) \right) \\
\lambda^{(t+1)} &= \mathcal{P}_\Lambda \left( \lambda^{(t)} - \eta_2 (V^{\pi_{\theta}}_g(\rho) - b) \right)
\end{align*}
\]  

(12)

where \( \nabla_\theta V^{\pi_{\theta}}_L(\rho) := \nabla_\theta V^{\pi_{\theta}}_r(\rho) + \lambda^{(t)} \nabla_\theta V^{\pi_{\theta}}_g(\rho) \), \( \eta_1 > 0 \) and \( \eta_2 > 0 \) are the stepsizes, \( \mathcal{P}_\Theta \) is the projection onto probability simplex \( \Theta := \Delta_{S}^{[A]} \), and \( \mathcal{P}_\Lambda \) is the projection that will be specified later. For the max-min formulation (6), PG-PD method (12) directly performs projected gradient ascent in the policy parameter \( \theta \) and descent in the dual variable \( \lambda \), both
over the Lagrangian $V^\pi_\rho \lambda (\rho)$. The primal-dual method overcomes disadvantages of the primal and dual methods either by relaxing the precise calculation of the subgradient of the dual function or by changing the descent direction via tuning of the dual variable. While this simple method provides a foundation for solving constrained MDPs (Chow et al., 2017; Tessler et al., 2019), lack of convexity in (6) makes it challenging to establish convergence theory for the primal-dual method, which is the primary objective of this paper.

We first leverage structure of constrained policy optimization problem (5) to provide a positive result in terms of optimality gap and constraint violation.

Theorem 6 (Restrictive convergence: direct policy parametrization) Let Assumption 2 hold with a policy class $\{ \pi_\theta = \theta \mid \theta \in \Theta \}$ and let $\Lambda = \left[ 0, \frac{2}{((1 - \gamma) \xi)} \right]$, $\rho > 0$, $\lambda^{(0)} = 0$, and $\theta^{(0)}$ be such that $V^*_\rho (\rho) \geq V^*_r (\rho)$. If we choose $\eta_1 = O(1)$ and $\eta_2 = O(1/\sqrt{T})$, then the iterates $\theta^{(t)}$ generated by PG-PD method (12) satisfy

\[
\text{(Optimality gap)} \quad \frac{1}{T} \sum_{t=0}^{T-1} \left( V^*_r (\rho) - V^*_r (\rho) \right) \leq C_1 \frac{|A| |S|}{(1 - \gamma)^6 T^{1/4}} \left\| d^\pi_\rho / \rho \right\|_\infty^2
\]

\[
\text{(Constraint violation)} \quad \frac{1}{T} \sum_{t=0}^{T-1} \left( b - V^*_r (\rho) \right) \leq C_2 \frac{|A| |S|}{(1 - \gamma)^6 T^{1/4}} \left\| d^\pi_\rho / \rho \right\|_\infty^2
\]

where $C_1$ and $C_2$ are absolute constants that are independent of $T$.

For the tabular constrained MDPs with direct policy parametrization, Theorem 6 guarantees that, on average, the optimality gap $V^*_r (\rho) - V^*_r (\rho)$ and the constraint violation $b - V^*_r (\rho)$ decay to zero with the sublinear rate $1/T^{1/4}$. However, this rate explicitly depends on the sizes of state/action spaces $|S|$ and $|A|$, and the distribution shift $\left\| d^\pi_\rho / \rho \right\|_\infty$ that specifies the exploration factor. A careful initialization $\theta^{(0)}$ that satisfies $V^*_r (\rho) \geq V^*_r (\rho)$ is also required. We leave it as future work to prove a less restrictive rate.

The proof of Theorem 6 is provided in Appendix B and it exploits the problem structure that casts the primal problem (5) as a linear program in the occupancy measure (Altman, 1999) and applies the convex optimization analysis. This method is not well-suited for large-scale problems, and projections onto the high-dimensional probability simplex are not desirable in practice. We next introduce a natural policy gradient primal-dual method to overcome these challenges and provide stronger convergence guarantees.

3.2 Natural policy gradient primal-dual (NPG-PD) method

The Fisher information matrix induced by $\pi_\theta$,

\[
F_\rho (\theta) := \mathbb{E}_{s \sim d^\rho_\theta} \mathbb{E}_{a \sim \pi_\theta (\cdot | s)} \left[ \nabla_\theta \log \pi_\theta (a | s) (\nabla_\theta \log \pi_\theta (a | s))^T \right]
\]

is used in the update of the primal variable in our primal-dual algorithm. The expectations are taken over the randomness of the state-action trajectory induced by $\pi_\theta$ and Natural
Policy Gradient Primal-Dual (NPG-PD) method for solving Problem (6) is given by,

\[
\begin{align*}
\theta^{(t+1)} &= \theta^{(t)} + \eta_1 \Lambda_{\theta}^\dagger(\theta^{(t)}) \nabla_{\theta} V_{\theta}^{(t)}(\rho) \\
\lambda^{(t+1)} &= \mathcal{P}_{\Lambda}\left(\lambda^{(t)} - \eta_2 (V_{g}^{(t)}(\rho) - b)\right)
\end{align*}
\] (13)

where \(\dagger\) denotes the Moore-Penrose inverse of a given matrix, \(\mathcal{P}_{\Lambda}(\cdot)\) denotes the projection to the interval \(\Lambda\) that will be specified later, and \((\eta_1, \eta_2)\) are constant positive stepsizes in the updates of primal and dual variables. The primal update \(\theta^{(t+1)}\) is obtained using a pre-conditioned gradient ascent via the natural policy gradient \(\Lambda_{\theta}^\dagger(\theta^{(t)}) \nabla_{\theta} V_{\theta}^{(t)}(\rho)\) and it represents the policy gradient of the Lagrangian \(V_{\theta}^{(t)}(\rho)\) in the geometry induced by the Fisher information matrix \(\Lambda_{\theta}(\theta^{(t)})\). On the other hand, the dual update \(\lambda^{(t+1)}\) is obtained using a projected sub-gradient descent by collecting the constraint violation \(b - V_{g}^{(t)}(\rho)\), where, for brevity, we use \(V_{L}^{(t)}(\rho)\) and \(V_{g}^{(t)}(\rho)\) to denote \(V_{\theta}^{(t)}(\rho)\) and \(V_{g}^{(t)}(\rho)\), respectively.

In Section 4, we establish global convergence of NPG-PD method (13) under the softmax policy parametrization. In Section 5, we examine the general policy parametrization and, in Section 6, we analyze sample complexity of two sample-based implementations of NPG-PD method (13).

**Remark 7** The performance difference lemma (Kakade and Langford, 2002; Agarwal et al., 2021), which quantifies the difference between \(V_{\pi}^{n}(s_0)\) and \(V_{\pi'}^{n}(s_0)\) for any two policies \(\pi\) and \(\pi'\) and any state \(s_0\),

\[
V_{\pi}^{n}(s_0) - V_{\pi'}^{n}(s_0) = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim d_{s_0}^n, a \sim \pi(\cdot | s)} \left[ A_{\pi}^{n}(s, a) \right]
\]

is utilized in our analysis, where the symbol \(\diamond\) denotes \(r\) or \(g\).

4. Tabular softmax parametrization: dimension-free global convergence

We first examine NPG-PD method (13) under softmax policy parametrization (7). Strong duality in Lemma 3 holds on the closure of the softmax policy class, because of completeness of the softmax policy class. Even though maximization problem (5) is not concave, we establish global convergence of our algorithm with dimension-independent convergence rates.

We first exploit the softmax policy structure to show that the primal update in (13) can be expressed in a more compact form; see Appendix C for the proof.

**Lemma 8 (Primal update as MWU)** Let \(\Lambda := [0, 2/((1 - \gamma)\xi)]\) and let \(A_{L}^{(t)}(s, a) := A_{r}^{(t)}(s, a) + \lambda^{(t)} A_{g}^{(t)}(s, a)\). Under softmax parametrized policy (7), NPG-PD algorithm (13) can be brought to the following form,

\[
\begin{align*}
\theta_{s,a}^{(t+1)} &= \theta_{s,a}^{(t)} + \eta_1 \Lambda_{\theta}^\dagger(\theta^{(t)}) \nabla_{\theta} V_{\theta}^{(t)}(\rho) \\
\lambda^{(t+1)} &= \mathcal{P}_{\Lambda}(\lambda^{(t)} - \eta_2 (V_{g}^{(t)}(\rho) - b))
\end{align*}
\] (14a)
Furthermore, the primal update in (14a) can be equivalently expressed as

$$
\pi^{(t+1)}(a | s) = \pi^{(t)}(a | s) \frac{\exp \left( \frac{m}{1-\gamma} A_L^{(t)}(s, a) \right)}{Z^{(t)}(s)}
$$

(14b)

where $Z^{(t)}(s) := \sum_{a \in A} \pi^{(t)}(a | s) \exp \left( \frac{m}{1-\gamma} A_L^{(t)}(s, a) \right)$.

The primal updates in (14a) do not depend on the state distribution $\mathcal{d}_\rho^{(t)}$ that appears in NPG-PD algorithm (13) through the policy gradient. This is because of the Moore-Penrose inverse of the Fisher information matrix in (13). Furthermore, policy update (14b) is given by the multiplicative weights update (MWU) which is commonly used in online linear optimization (Cesa-Bianchi and Lugosi, 2006). In contrast to the online linear optimization, an advantage function appears in the MWU policy update at each iteration in (14b).

In Theorem 9, we establish global convergence of NPG-PD algorithm (14a) with respect to both the optimality gap $V^* - V^{(t)}(\rho)$ and the constraint violation $b - V_g^{(t)}(\rho)$. Even though we set $\theta_{k,a}^{(0)} = 0$ and $\lambda^{(0)} = 0$ in the proof of Theorem 9 in Section 4.1, global convergence can be established for arbitrary initial conditions.

**Theorem 9** (Global convergence: softmax policy parametrization) Let Assumption 2 hold for $\xi > 0$ and let us fix $T > 0$ and $\rho \in \Delta_S$. If we choose $\eta_1 = 2\log |A|$ and $\eta_2 = 2(1-\gamma)/\sqrt{T}$, then the iterates $\pi(t)$ generated by algorithm (14) satisfy,

$$
\begin{align*}
\text{(Optimality gap)} & \quad \frac{1}{T} \sum_{t=0}^{T-1} (V^*(\rho) - V^{(t)}(\rho)) \leq \frac{7}{(1-\gamma)^2} \frac{1}{\sqrt{T}} \\
\text{(Constraint violation)} & \quad \left[ \frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) \right]_+ \leq \frac{2}{\xi} + \frac{4\xi}{(1-\gamma)^2} \frac{1}{\sqrt{T}}.
\end{align*}
$$

Theorem 9 demonstrates that, on average, the reward value function converges to its globally optimal value and that the constraint violation decays to zero. In other words, for a desired accuracy $\epsilon$, it takes $O(1/\epsilon^2)$ iterations to compute the solution which is $\epsilon$ away from the globally optimal one (with respect to both the optimality gap and the constraint violation). We note that the required number of iterations only depends on the desired accuracy $\epsilon$ and is independent of the sizes of the state and action spaces. Although maximization problem (5) is not concave, our rate $(1/\sqrt{T}, 1/\sqrt{T})$ for optimality/constraint violation gap outperforms the classical one $(1/\sqrt{T}, 1/T^{3/4})$ (Mahdavi et al., 2012) and it matches the achievable rate for solving online convex minimization problems with convex constraint sets (Yu et al., 2017). Moreover, in contrast to the bounds established for PG-PD algorithm (12) in Theorem 6, the bounds in Theorem 9 for NPG-PD algorithm (13) under softmax policy parameterization do not depend on the initial distribution $\rho$.

As shown in Lemma 10 in Section 4.1, the reward and utility value functions are coupled and the natural policy gradient method in the unconstrained setting does not provide monotonic improvement to either of them (Agarwal et al., 2021, Section 5.3). To address this challenge, we introduce a new line of non-convex analysis by bridging the online regret analysis in unconstrained MDPs (Even-Dar et al., 2009; Agarwal et al., 2021) and the
Lagrangian methods in constrained optimization (Beck, 2017). To bound the optimality gap, via a drift analysis of the dual update we first establish the bounded average performance in Lemma 11 in Section 4.1. Furthermore, instead of using methods from constrained convex optimization (Mahdavi et al., 2012; Yu et al., 2017; Wei et al., 2020; Yuan and Lamperski, 2018), which either require extra assumptions or have slow convergence rate, under strong duality we establish that the constraint violation for nonconvex Problem (5) converges with the same rate as the optimality gap. To the best of our knowledge, this appears to be the first such result for nonconvex constrained optimization problems. Finally, since local TRPO update can be well approximated by NPG update when the stepsize is small (Schulman et al., 2015), our NPG-based analysis also suggests convergence of constrained policy optimization (Achiam et al., 2017; Yang et al., 2020b).

4.1 Proof of Theorem 9

We first utilize the performance difference lemma to show joint policy improvement per iteration in the reward and utility value functions. We show that neither of them is necessarily monotonic.

**Lemma 10 (Non-monotonic improvement)** For any distribution of the initial state \( \mu \), iterates \((\pi^{(t)}, \lambda^{(t)})\) of algorithm (14) satisfy

\[
V_r^{(t+1)}(\mu) - V_r^{(t)}(\mu) + \lambda^{(t)}(V_g^{(t+1)}(\mu) - V_g^{(t)}(\mu)) \geq \frac{1 - \gamma}{\eta_1} \mathbb{E}_{s \sim \mu} \log Z^{(t)}(s) \geq 0. \tag{15}
\]

**Proof** Let \( d^{(t+1)}_\mu := d^{(t+1)}_\mu \). The performance difference lemma in conjunction with the multiplicative weights update in (14b) yield,

\[
V_r^{(t+1)}(\mu) - V_r^{(t)}(\mu) = \frac{1}{1 - \gamma} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \left[ \sum_{a \in A} \pi^{(t+1)}(a \mid s) A_r^{(t)}(s, a) \right] \\
= \frac{1}{\eta_1} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \left[ \sum_{a \in A} \pi^{(t+1)}(a \mid s) \log \left( \frac{\pi^{(t+1)}(a \mid s)}{\pi^{(t)}(a \mid s)} \right) \right] \\
- \frac{\lambda^{(t)}}{1 - \gamma} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \left[ \sum_{a \in A} \pi^{(t+1)}(a \mid s) A_g^{(t)}(s, a) \right] \\
= \frac{1}{\eta_1} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \left[ D_{KL} \left( \pi^{(t+1)}(a \mid s) \parallel \pi^{(t)}(a \mid s) \right) \right] \\
+ \frac{1}{\eta_1} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \log Z^{(t)}(s) \\
- \frac{\lambda^{(t)}}{1 - \gamma} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \left[ \sum_{a \in A} \pi^{(t+1)}(a \mid s) A_g^{(t)}(s, a) \right]
\]
where the last equality follows from the definition of the Kullback-Leibler divergence or relative entropy between distributions $p$ and $q$, $D_{KL}(p || q) := \mathbb{E}_{x \sim p} \log(p(x)/q(x))$. Furthermore,

\[
\frac{1}{\eta_1} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \left[ D_{KL} \left( \pi^{(t+1)}(a \mid s) \parallel \pi^{(t)}(a \mid s) \right) \right] + \frac{1}{\eta_1} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \log Z^{(t)}(s) - \frac{\lambda^{(t)}}{1 - \gamma} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \left[ \sum_{a \in A} \pi^{(t+1)}(a \mid s) A^{(t)}_g(s, a) \right]
\]

\[
\geq \frac{1}{\eta_1} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \log Z^{(t)}(s) - \frac{\lambda^{(t)}}{1 - \gamma} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \left[ \sum_{a \in A} \pi^{(t+1)}(a \mid s) A^{(t)}_g(s, a) \right]
\]

\[
\equiv \frac{1}{\eta_1} \mathbb{E}_{s \sim d^{(t+1)}_\mu} \log Z^{(t)}(s) - \lambda^{(t)} (V^{(t+1)}_g(\mu) - V^{(t)}_g(\mu))
\]

is a consequence of the performance difference lemma, where we drop a nonnegative term in (a) and (b). The first inequality in (15) follows from a componentwise inequality $d^{(t+1)}_\mu \geq (1 - \gamma)\mu$, which is obtained using (9).

Now we prove that $\log Z^{(t)}(s) \geq 0$. From the definition of $Z^{(t)}(s)$ we have

\[
\log Z^{(t)}(s) = \log \left( \sum_{a \in A} \pi^{(t)}(a \mid s) \exp \left( \frac{\eta_1}{1 - \gamma} \left( A^{(t)}_r(s, a) + \lambda^{(t)} A^{(t)}_g(s, a) \right) \right) \right)
\]

\[
\geq \sum_{a \in A} \pi^{(t)}(a \mid s) \log \left( \exp \left( \frac{\eta_1}{1 - \gamma} \left( A^{(t)}_r(s, a) + \lambda^{(t)} A^{(t)}_g(s, a) \right) \right) \right)
\]

\[
= \frac{\eta_1}{1 - \gamma} \sum_{a \in A} \pi^{(t)}(a \mid s) A^{(t)}_r(s, a) + \frac{\eta_1}{1 - \gamma} \sum_{a \in A} \lambda^{(t)} \pi^{(t)}(a \mid s) A^{(t)}_g(s, a)
\]

\[
\equiv 0
\]

where in (a) we apply the Jensen’s inequality to the concave function $\log(x)$. On the other hand, the last equality follows from the definitions of $A^{(t)}_r(s, a)$ and $A^{(t)}_g(s, a)$, which yield

\[
\sum_{a \in A} \pi^{(t)}(a \mid s) A^{(t)}_r(s, a) = \sum_{a \in A} \pi^{(t)}(a \mid s) (Q^{(t)}_r(s, a) - V^{(t)}_r(s)) = 0
\]

\[
\sum_{a \in A} \pi^{(t)}(a \mid s) A^{(t)}_g(s, a) = 0.
\]

Lemma 10 states that each primal update (14b) improves the Lagrangian-like term $V^{(t)}_r(\mu) + \lambda^{(t)} V^{(t)}_g(\mu)$ to $V^{(t+1)}_r(\mu) + \lambda^{(t)} V^{(t+1)}_g(\mu)$, with improvement depending on the previous primal-dual update $(\pi^{(t)}, \lambda^{(t)})$. This lemma can be viewed as a constrained version
of the policy improvement established for the unconstrained case (Agarwal et al. 2021) resulting from setting $\lambda^{(t)} = 0$. In fact, the dual iterate $\lambda^{(t)}$ captures how the constraint violation of policy improvement affects the reward value function, which is a unique feature of constrained policy optimization. Because of this superimposed effect, there is no monotonic improvement guarantee for either reward or utility value functions.

As pointed out by Beck (2017), in constrained convex optimization the primal iterate cannot reduce the unconstrained objective function monotonically and some averaging scheme has to be imposed. In our non-convex context, we examine the average of value functions, which is similar to regret analysis in online optimization. We next compare the average value functions of policy iterates generated by algorithm (14) with the ones that result from the use of optimal policy.

**Lemma 11 (Bounded average performance)** Let Assumption 2 hold and let us fix $T > 0$ and $\rho \in \Delta_S$. Then the iterates $(\pi^{(t)}(s), \lambda^{(t)})$ generated by algorithm (14) satisfy

$$
\frac{1}{T} \sum_{t=0}^{T-1} \left( (V^{*}_r(\rho) - V^{(t)}_r(\rho)) + \lambda^{(t)}(V^*_g(\rho) - V^{(t)}_g(\rho)) \right) \leq \frac{\log |A|}{\eta_1 T} + \frac{1}{(1 - \gamma)^2T} + \frac{2\eta_2}{(1 - \gamma)^2}. \tag{16}
$$

**Proof** Let $d^* := d^*_p$. The performance difference lemma in conjunction with the multiplicative weights update in (14b) yield,

$$
V^*_r(\rho) - V^{(t)}_r(\rho) = \frac{1}{1 - \gamma} E_{s \sim d^*} \left[ \sum_{a \in A} \pi^*(a \mid s) A^{(t)}_r(s, a) \right] - \frac{\lambda^{(t)}}{1 - \gamma} E_{s \sim d^*} \left[ \sum_{a \in A} \pi^*(a \mid s) A^{(t)}_g(s, a) \right].
$$

Application of the definition of the Kullback–Leibler divergence or relative entropy between distributions $p$ and $q$, $D_{KL}(p \parallel q) := \mathbb{E}_{x \sim p} \log(p(x)/q(x))$, and the performance difference lemma again yield,

$$
V^*_r(\rho) - V^{(t)}_r(\rho) = \frac{1}{\eta_1} E_{s \sim d^*} \left[ D_{KL}(\pi^*(a \mid s) \parallel \pi^{(t)}(a \mid s)) - D_{KL}(\pi^*(a \mid s) \parallel \pi^{(t+1)}(a \mid s)) \right] + \frac{1}{\eta_1} E_{s \sim d^*} \log Z^{(t)}(s) - \frac{\lambda^{(t)}}{1 - \gamma} E_{s \sim d^*} \sum_{a \in A} \pi^*(a \mid s) A^{(t)}_g(s, a) \tag{17}
$$

On the other hand, the first inequality in (15) with $\mu = d^*$ becomes

$$
V^{(t+1)}_r(d^*) - V^{(t)}_r(d^*) + \lambda^{(t)}(V^{(t+1)}_g(d^*) - V^{(t)}_g(d^*)) \geq \frac{1 - \gamma}{\eta_1} E_{s \sim d^*} \log Z^{(t)}(s). \tag{18}
$$
Finally, we use (19) to yield, 

\[
\frac{1}{T} \sum_{t=0}^{T-1} (V^*_r(\rho) - V^{(t)}_r(\rho)) 
\]

\[
= \frac{1}{\eta T} \sum_{t=0}^{T-1} E_s \sim d^* \left[ D_{KL} \left( \pi^*(a \mid s) \| \pi^{(t)}(a \mid s) \right) - D_{KL} \left( \pi^*(a \mid s) \| \pi^{(t+1)}(a \mid s) \right) \right] 
\]

\[
+ \frac{1}{\eta T} \sum_{t=0}^{T-1} E_s \sim d^* \log Z^{(t)}(s) - \frac{1}{T} \sum_{t=0}^{T-1} \lambda^{(t)} (V^*_g(\rho) - V^{(t)}_g(\rho)) 
\]

\[
\leq \frac{1}{\eta T} \sum_{t=0}^{T-1} E_s \sim d^* \left[ D_{KL} \left( \pi^*(a \mid s) \| \pi^{(t)}(a \mid s) \right) - D_{KL} \left( \pi^*(a \mid s) \| \pi^{(t+1)}(a \mid s) \right) \right] 
\]

\[
+ \frac{1}{(1-\gamma)T} \sum_{t=0}^{T-1} (V^{(t+1)}_r(d^*) - V^{(t)}_r(d^*)) 
\]

\[
+ \frac{1}{(1-\gamma)T} \sum_{t=0}^{T-1} \lambda^{(t)} (V^{(t+1)}_g(d^*) - V^{(t)}_g(d^*)) - \frac{1}{T} \sum_{t=0}^{T-1} \lambda^{(t)} (V^*_g(\rho) - V^{(t)}_g(\rho)). 
\]

Hence, application of (18) to the average of (17) over \( t = 0, 1, \ldots, T - 1 \) leads to,

\[
\frac{1}{T} \sum_{t=0}^{T-1} \lambda^{(t)} (V^{(t+1)}_g(\mu) - V^{(t)}_g(\mu)) 
\]

\[
= \frac{1}{T} \sum_{t=0}^{T-1} (\lambda^{(t+1)} V^{(t+1)}_g(\mu) - \lambda^{(t)} V^{(t)}_g(\mu)) + \frac{1}{T} \sum_{t=0}^{T-1} (\lambda^{(t)} - \lambda^{(t+1)}) V^{(t+1)}_g(\mu) 
\]

\[
\leq \frac{1}{T} \lambda^{(T)} V^{(T)}_g(\mu) + \frac{1}{T} \sum_{t=0}^{T-1} |\lambda^{(t)} - \lambda^{(t+1)}| V^{(t+1)}_g(\mu) 
\]

\[
\leq \frac{2\eta_2}{1-\gamma^2} 
\]

where we take a telescoping sum for the first sum in (a) and drop a non-positive term, and in (b) we utilize \(|\lambda^{(T)}| \leq \eta_2 T/(1-\gamma)\) and \(|\lambda^{(t)} - \lambda^{(t+1)}| \leq \eta_2/(1-\gamma)\), which follows from the dual update in (14a), the non-expansiveness of projection \( P_\Lambda \), and boundedness of the value function \( V^{(t)}_g(\mu) \leq 1/(1-\gamma)\). Application of (20) with \( \mu = d^* \) and the use of telescoping sum to (19) yields,

\[
\frac{1}{T} \sum_{t=0}^{T-1} (V^*_r(\rho) - V^{(t)}_r(\rho)) 
\]

\[
\leq \frac{1}{\eta T} E_s \sim d^* D_{KL} \left( \pi^*(a \mid s) \| \pi^{(0)}(a \mid s) \right) + \frac{1}{(1-\gamma)T} V^{(T)}_r(d^*) + \frac{2\eta_2}{(1-\gamma)^3} 
\]

\[
- \frac{1}{T} \sum_{t=0}^{T-1} \lambda^{(t)} (V^*_g(\rho) - V^{(t)}_g(\rho)). 
\]

Finally, we use \( D_{KL}(p \| q) \leq \log |A| \) for \( p \in \Delta_A \) and \( q = \text{Unif}_A \), \( V^{(T)}_r(d^*) \leq 1/(1-\gamma) \), and \( V^*_g(\rho) \geq b \) to complete the proof. \( \blacksquare \)
Lemma 11 shows that the average difference between \((V^*_g(\rho), V^*_g(\rho))\) and \((V^{(t)}_g(\rho), V^{(t)}_g(\rho))\) can be bounded by a \((T, \eta_1, \eta_2)\)-term. As aforementioned, when there is no constraint (e.g., \(\eta_2 = 0\)), it is straightforward to strengthen Lemma 11 as the fast rate result in the unconstrained case (Agarwal et al., 2021). We also note that this average performance analysis generalizes to the function approximation setting in Section 5, with additional characterization of approximation errors.

**Proof** [Proof of Theorem 9]

**Bounding the optimality gap.** From the dual update in (14a) we have

\[
0 \leq (\lambda(T))^2 = \sum_{t=0}^{T-1} ((\lambda(t+1))^2 - (\lambda(t))^2)
\]

\[
= \sum_{t=0}^{T-1} \left( (\mathcal{P}_\lambda (\lambda(t) - \eta_2(V^{(t)}_g(\rho) - b)) - (\lambda(t))^2 \right)
\]

\[
\leq \sum_{t=0}^{T-1} \left( (\lambda(t) - \eta_2(V^{(t)}_g(\rho) - b))^2 - (\lambda(t))^2 \right)
\]

\[
= 2\eta_2 \sum_{t=0}^{T-1} \lambda(t)(b - V^{(t)}_g(\rho)) + \eta_2^2 \sum_{t=0}^{T-1} (V^{(t)}_g(\rho) - b)^2
\]

\[
\leq 2\eta_2 \sum_{t=0}^{T-1} \lambda(t)(V^*_g(\rho) - V^{(t)}_g(\rho)) + \frac{\eta_2^2 T}{(1 - \gamma)^2}
\]

where (a) because of the projection \(\mathcal{P}_\lambda\), (b) is because of the feasibility of the optimal policy \(\pi^*: V^*_g(\rho) \geq b\), and \(|V^{(t)}_g(\rho) - b| \leq 1/(1 - \gamma)\). Hence,

\[
-\frac{1}{T} \sum_{t=0}^{T-1} \lambda(t)(V^*_g(\rho) - V^{(t)}_g(\rho)) \leq \frac{\eta_2}{2(1 - \gamma)^2}.
\]

(21b)

To obtain the optimality gap bound, we now substitute (21b) into (16), apply \(D_{KL}(p \parallel q) \leq \log |A|\) for \(p \in \Delta_A\) and \(q = \text{Unif}_A\), and take \(\eta_1 = 2 \log |A|\) and \(\eta_2 = 2(1 - \gamma)/\sqrt{T}\).

**Bounding the constraint violation.** For any \(\lambda \in \left[0, 2/((1 - \gamma)\xi)\right]\), from the dual update in (14a) we have

\[
|\lambda^{(t+1)} - \lambda|^2 \overset{(a)}{=} |\lambda(t) - \eta_2(V^{(t)}_g(\rho) - b) - \lambda|^2
\]

\[
= |\lambda(t) - \lambda|^2 - 2\eta_2(V^{(t)}_g(\rho) - b)(\lambda(t) - \lambda) + \eta_2^2(V^{(t)}_g(\rho) - b)^2
\]

\[
\overset{(b)}{\leq} |\lambda(t) - \lambda|^2 - 2\eta_2(V^{(t)}_g(\rho) - b)(\lambda(t) - \lambda) + \frac{\eta_2^2}{(1 - \gamma)^2}
\]

where (a) is because of the non-expansiveness of projection \(\mathcal{P}_\lambda\) and (b) is because of \((V^{(t)}_g(\rho) - b)^2 \leq 1/(1 - \gamma)^2\). Averaging the above inequality over \(t = 0, \ldots, T - 1\) yields

\[
0 \leq \frac{1}{T}|\lambda(T) - \lambda|^2 \leq \frac{1}{T}|\lambda(0) - \lambda|^2 - \frac{2\eta_2}{T} \sum_{t=0}^{T-1} (V^{(t)}_g(\rho) - b)(\lambda(t) - \lambda) + \frac{\eta_2^2}{(1 - \gamma)^2},
\]

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which implies,
\[
\frac{1}{T} \sum_{t=0}^{T-1} (V_g^{(t)}(\rho) - b)(\lambda_t - \lambda) \leq \frac{1}{2\eta_2T} |\lambda^{(0)} - \lambda|^2 + \frac{\eta_2}{2(1-\gamma)^2}. \tag{22}
\]
We now add (22) to (16) on both sides of the inequality, and utilize $V_g^*(\rho) \geq b$,
\[
\frac{1}{T} \sum_{t=0}^{T-1} (V_r^*(\rho) - V_r^{(t)}(\rho)) + \frac{\lambda}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) \\
\leq \frac{\log|A|}{\eta_1T} + \frac{1}{(1-\gamma)^2T} + \frac{2\eta_2}{(1-\gamma)^3} + \frac{1}{2\eta_2T} |\lambda^{(0)} - \lambda|^2 + \frac{\eta_2}{2(1-\gamma)^2}. \tag{23}
\]
Taking $\lambda = 2/((1-\gamma)\xi)$ when $\sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) \geq 0$ and $\lambda = 0$ otherwise, we obtain
\[
V_r^*(\rho) - \frac{1}{T} \sum_{t=0}^{T-1} V_r^{(t)}(\rho) + \frac{2}{(1-\gamma)\xi} \left[ b - \frac{1}{T} \sum_{t=0}^{T-1} V_g^{(t)}(\rho) \right] + \frac{2\eta_2}{(1-\gamma)^3} + \frac{2}{\eta_2(1-\gamma)^2\xi^2T} + \frac{\eta_2}{2(1-\gamma)^2}.
\]
Note that both $V_r^{(t)}(\rho)$ and $V_g^{(t)}(\rho)$ can be expressed as linear functions in the same occupancy measure (Altman 1999 Chapter 10) that is induced by policy $\pi^{(t)}$ and transition $P(s'|s,a)$. The convexity of the set of occupancy measures shows that the average of $T$ occupancy measures is an occupancy measure that produces a policy $\pi'$ with value $V_r^{\pi'}$ and $V_g^{\pi'}$. Hence, there exists a policy $\pi'$ such that $V_r^{\pi'}(\rho) = \frac{1}{T} \sum_{t=0}^{T-1} V_r^{(t)}(\rho)$ and $V_g^{\pi'}(\rho) = \frac{1}{T} \sum_{t=0}^{T-1} V_g^{(t)}(\rho)$. Thus,
\[
V_r^*(\rho) - V_r^{\pi'}(\rho) + \frac{2}{(1-\gamma)\xi} \left[ b - V_g^{\pi'}(\rho) \right] + \frac{2\eta_2}{(1-\gamma)^3} + \frac{2}{\eta_2(1-\gamma)^2\xi^2T} + \frac{\eta_2}{2(1-\gamma)^2}.
\]
Applying Lemma 4 with $2/((1-\gamma)\xi) \geq 2\lambda^*$ yields
\[
\left[ b - V_g^{\pi'}(\rho) \right] \leq \frac{\xi\log|A|}{\eta_1T} + \frac{\xi}{(1-\gamma)T} + \frac{2\eta_2\xi}{(1-\gamma)^2} + \frac{2}{\eta_2(1-\gamma)\xi T} + \frac{\eta_2\xi}{2(1-\gamma)}
\]
which leads to our constraint violation bound if we further utilize $\frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) = b - V_g^{\pi'}(\rho), \eta_1 = 2\log|A|$, and $\eta_2 = 2(1-\gamma)/\sqrt{T}$.

4.2 Zero constraint violation

It is natural to employ a conservative constraint $V_g^{\pi\delta}(\rho) \geq b + \delta$ for some $\delta > 0$ in Problem (5). When our desired accuracy $\epsilon$ is small enough, there exists some $\delta$ for algorithm (14) to get zero constraint violation.
Corollary 12 (Zero constraint violation: softmax policy parametrization) Let Assumption 2 hold for $\xi > 0$ and let us fix $\rho \in \Delta_S$ and replace the constraint of Problem (5) by $V^\pi_g(\rho) \geq b$, where $b := b + \delta$ for some $\delta > 0$. For $\epsilon < \xi/2$, there exists $\delta = O(\epsilon)$ such that if we choose $T = \Omega(1/\epsilon^2)$, $\eta_1 = 2 \log |A|$, and $\eta_2 = 2(1 - \gamma)/\sqrt{T}$, the iterates $\pi^{(t)}$ generated by algorithm (14) satisfy,

\[
(Optimality gap) \quad \frac{1}{T} \sum_{t=0}^{T-1} (V^*_{\pi}(\rho) - V^{(t)}_{\pi}(\rho)) = O(\epsilon)
\]

\[
(Constraint violation) \quad \left[ \frac{1}{T} \sum_{t=0}^{T-1} (b - V^{(t)}_g(\rho)) \right]_+ \leq 0.
\]

**Proof** The proof idea is similar to the one used in the proof of Theorem 9. Using the new constraint $V^\pi_g(\rho) \geq b$, Problem (5) satisfies Assumption 2 for $\xi := \xi - \delta$ where $\delta < \xi$, and there exists an optimal policy $\pi^\star$. Without loss of generality, by restricting $\delta < \xi/2$, we can replace $\Lambda$ by $\bar{\Lambda} := [0, 4/((1 - \gamma)\xi)]$ which contains $[0, 4/((1 - \gamma)\xi)]$ for any such $\xi$. Thus, we can apply NPG-PD algorithm (13) to this conservative problem using the projection set $\bar{\Lambda}$. It is straightforward to check that Lemma 11 holds for $V^\pi_{\pi}(\rho)$ and $V^\pi_g(\rho)$. Thus, bounding of the optimality gap in the proof of Theorem 9 proves that after $T = \Omega(1/\epsilon^2)$ iterations,

\[
\frac{1}{T} \sum_{t=0}^{T-1} (V^\pi_{\pi}(\rho) - V^{(t)}_{\pi}(\rho)) = O(\epsilon).
\]

Let $q^\star$ and $\bar{q}^\star$ be the occupancy measures induced by policies $\pi^\star$ and $\bar{\pi}^\star$, respectively. In the occupancy measure space, Problem (5) becomes a linear program and, thus, $V^\pi_{\pi}(\rho) = \langle r, q^\star \rangle$ and $V^\pi_g(\rho) = \langle r, \bar{q}^\star \rangle$. By the continuity of optimal value function in convex optimization (Terazono and Matani, 2015), $|V^\pi_{\pi}(\rho) - V^\pi_g(\rho)| \leq 2\epsilon/((1 - \gamma)\xi)$ for $\delta = \epsilon$. Therefore, we can replace $V^\pi_{\pi}(\rho)$ in (23) by $V^\pi_{\pi}(\rho)$ to bound the optimality gap by the same desired accuracy $\epsilon$ up to some problem-dependent constant.

For establishing bound on the constraint violation, the key change begins from (23). Since we use $b = b + \delta$ and $V^\pi_{\pi}(\rho)$, the right-hand side of (23) contains an extra term $2\epsilon/((1 - \gamma)\xi) - \lambda \delta$. Similarly, there are two options for selecting $\lambda$: $\lambda = 4/((1 - \gamma)\xi)$ when $\sum_{t=0}^{T-1} (b - V^{(t)}_g(\rho)) \geq 0$ and $\lambda = 0$ otherwise. In the first case, if we set $\delta = \epsilon$, then the extra term is $-2\epsilon/((1 - \gamma)\xi)$ which allows us to cancel the rate $O(1/\sqrt{T})$ when we choose $T = \Omega(1/\epsilon^2)$; the second case on the other hand is exactly the zero constraint violation. ■

5. Function approximation: convergence rate and optimality

Let us consider a general form of NPG-PD algorithm (13),

\[
\theta^{(t+1)} = \theta^{(t)} + \frac{\eta_1}{1 - \gamma} w^{(t)}
\]

\[
\lambda^{(t+1)} = \mathcal{P}_\Lambda \left( \lambda^{(t)} - \eta_2 \left( V^{(t)}_g(\rho) - b \right) \right)
\]

\[
23
\]
When the compatible function approximation error is zero, the global convergence follows. The intuition behind this is that any minimizer of $E^\nu(w; \theta, \lambda)$ is a minimizer of $E^\nu(w; \theta, \lambda) + \lambda \nabla_\theta V^{\theta, \lambda}_L(\rho)$ allows us to rewrite it as

\[
(1 - \gamma)F^\dagger_\rho(\theta) \nabla_\theta V^{\theta, \lambda}_L(\rho) \in \arg\min_{w_\nu} E^\nu_\nu(w_\nu; \theta) \tag{27}
\]

where $\dagger$ denotes $r$ or $g$.

Let the minimal error be $E^\nu_\nu, * := \min_{w_\nu} E^\nu_\nu(w_\nu; \theta)$, where the compatible function approximation error $E^\nu_\nu(w_\nu; \theta)$ is given by

\[
E^\nu_\nu(w_\nu; \theta) := \mathbb{E}_{(s,a) \sim \nu} \left[ \left( A^\theta_\nu(s,a) - w_\nu^T \nabla_\theta \log \pi_\theta(a | s) \right)^2 \right]. \tag{28}
\]

When the compatible function approximation error is zero, the global convergence follows from Theorem 9. However, this is not the case for a general policy class because it may not include all possible policies (e.g., if we take $d \ll |S||A|$ for the tabular constrained MDPs). The intuition behind compatibility is that any minimizer of $E^\nu_\nu(w_\nu; \theta)$ can be used as the NPG direction without affecting convergence theory; also see discussions in Kakade (2002); Sutton et al. (2000); Agarwal et al. (2021).

Since the state-action measure $\nu$ of some feasible comparison policy $\pi$ is not known, we introduce an exploratory initial distribution $\nu_0$ over state-action pairs and define a state-action visitation distribution $\nu^\pi_{\nu_0}$ of a policy $\pi$ as

\[
\nu^\pi_{\nu_0}(s,a) = (1 - \gamma) \mathbb{E}_{(s_0,a_0) \sim \nu_0} \left[ \sum_{t=0}^{\infty} \gamma^t P^\pi(s_t=s, a_t=a | s_0, a_0) \right]
\]

\begin{align*}
V^\pi_f(\rho) &= V^\pi_D(\rho) \geq V^\pi_D(\rho) \geq V^\pi_f(\rho) - M \epsilon_\pi
\end{align*}

where $\epsilon_\pi := \max_s \| \pi(\cdot | s) - \pi_\theta(\cdot | s) \|_1$ is the policy approximation error and $M > 0$ is a problem-dependent constant. Application of item (ii) in Lemma 3 to the set of all optimal dual variables $\lambda^*_\nu$ yields $\lambda^*_\nu \in [0, 2/((1 - \gamma)\xi)]$ and, thus, $\Lambda = [0, 2/((1 - \gamma)\xi)]$.

To quantify errors caused by the restricted policy parametrization, let us first generalize NPG. For a distribution over state-action pair $\nu \in \Delta_{S \times A}$, we introduce the compatible function approximation error as the following regression objective (Kakade, 2002),

\[
E^\nu_\nu(w; \theta, \lambda) := \mathbb{E}_{(s,a) \sim \nu} \left[ \left( A^\theta_\nu(s,a) - w^T \nabla_\theta \log \pi_\theta(a | s) \right)^2 \right]
\]

where $A^\theta_\nu(s,a) := A^\theta_\nu(s,a) + \lambda A_\nu^g(s,a)$. We can view NPG in (13) as a minimizer of $E^\nu(w; \theta, \lambda)$ for $\nu(s,a) = d_\rho^\nu(s) \rho_\theta(a | s)$,

\[
(1 - \gamma)F^\dagger_\rho(\theta) \nabla_\theta V^{\theta, \lambda}_L(\rho) \in \arg\min_{w} E^\nu(w; \theta, \lambda). \tag{26}
\]

Expression (26) follows from the first-order optimality condition and the use of $\nabla_\theta V^{\theta, \lambda}_L(\rho) := \nabla_\theta V^\pi_f(\rho) + \lambda \nabla_\theta V^\pi_g(\rho)$ allows us to rewrite it as

\[
(1 - \gamma)F^\dagger_\rho(\theta) \nabla_\theta V^{\theta, \lambda}_L(\rho) \in \arg\min_{w_\nu} E^\nu_\nu(w_\nu; \theta) \tag{27}
\]

where $\dagger$ denotes $r$ or $g$.
When there are no compatible function approximation errors, the log-linear policy up-
well-posed, which is similar to imposing regularization in practice. Let the exact minimizer be
where the bounded domain \( W \) is given by \( W = \{ w | \| w \| \leq W \} \). We next prove convergence of (25) for the log-linear and for the general smooth policy classes.

5.1 Log-linear policy class
We first consider policies \( \pi_\theta \) in the log-linear class (8), with linear feature maps \( \phi_{s,a} \in \mathbb{R}^d \). In this case, the gradient \( \nabla_\theta \log \pi_\theta(a \mid s) \) becomes a shifted version of feature \( \phi_{s,a} \),

\[
\nabla_\theta \log \pi_\theta(a \mid s) = \phi_{s,a} - E_{a' \sim \pi_\theta(\cdot \mid s)}[\phi_{s,a'}] =: \tilde{\phi}_{s,a}.
\]

Thus, the compatible function approximation error (28) captures how well the linear function
in (25) for \( \nu = r \) or \( g \), where \( \nu(t)(s,a) = d_\rho(t)(s)\pi_\theta(t)(a \mid s) \) is an on-policy state-action visitation distribution. This is because the softmax function is invariant to any terms that are independent of the action.

Let us consider an approximate solution,

\[
w_\circ(t) \approx \arg\min_{\|w\|_2 \leq W} \mathcal{E}_\circ(t)(w; \theta(t))
\]

where the bounded domain \( W > 0 \) can be viewed as an \( \ell_2 \)-regularization. We point out that it is possible to remove the domain restriction when some regularity assumptions on the feature maps are made in a sample-based algorithm (Bach and Moulines, 2013). We restrict the domain to make the approximate solution well-defined even when it is not well-posed, which is similar to imposing regularization in practice. Let the exact minimizer be \( w_\circ,\star(t) \in \arg\min_{\|w\|_2 \leq W} \mathcal{E}_\circ(t)(w; \theta(t)) \). Fixing a state-action distribution \( \nu(t) \), the estimation error in \( w_\circ(t) \) arises from the discrepancy between \( w_\circ(t) \) and \( w_\circ,\star(t) \), which comes from the randomness in a sample-based optimization algorithm and the mismatch between the linear function and the true state-action value function. We represent the estimation error as

\[
\mathcal{E}_{\circ,\text{est}}(t) := E \left[ \mathcal{E}_\circ(t)(w_\circ(t); \theta(t)) - \mathcal{E}_\circ(t)(w_\circ,\star(t); \theta(t)) \right]
\]
where the expectation \( \mathbb{E} \) is taken over the randomness of approximate algorithm that is used to solve (31).

Note that the state-action distribution \( \nu^{(t)} \) is on-policy. To characterize the effect of distribution shift on \( w_{\circ,\star}^{(t)} \), let us introduce some notation. We represent a fixed distribution over state-action pairs \((s, a)\) by

\[
\nu^*(s, a) := d_{\rho}^r(s) \circ \text{Unif}_A(a).
\]

(32)

The fixed distribution \( \nu^* \) samples a state from \( d_{\rho}^r(s) \) and an action uniformly from \( \text{Unif}_A(a) \). We characterize the error in \( w_{\circ,\star}^{(t)} \) that arises from the distribution shift using the transfer error,

\[
\mathcal{E}_{o,bias}^{(t)} := \mathbb{E} \left[ \mathcal{E}^{\nu^*} \left( w_{\circ,\star}^{(t)}; \theta^{(t)} \right) \right].
\]

The transfer error characterizes the expressiveness of function approximation that is affected by the feature maps \( \phi_{s,a} \in \mathbb{R}^d \) and the quality of exact minimizer \( w_{\circ,\star}^{(t)} \).

**Assumption 13 (Estimation error and transfer error)** Both the estimation error and the transfer error are bounded, i.e., \( \mathcal{E}_{o,est}^{(t)} \leq \epsilon_{est} \) and \( \mathcal{E}_{o,bias}^{(t)} \leq \epsilon_{bias} \), where \( \circ \) denotes \( r \) or \( g \).

When we apply a sample-based algorithm to (31), it is standard to have \( \epsilon_{est} = O(1/\sqrt{K}) \), where \( K \) is the number of samples; e.g., see Shalev-Shwartz and Ben-David (2014, Theorem 14.8). A special case is the exact tabular softmax policy parametrization for which \( \epsilon_{bias} = \epsilon_{est} = 0 \), since the feature maps \( \phi_{s,a} \in \mathbb{R}^d \) reduce to indicator functions of the state/action spaces.

For any state-action distribution \( \nu \), we define \( \Sigma_{\nu} := \mathbb{E}_{(s,a) \sim \nu} \left[ \phi_{s,a} \phi_{s,a}^\top \right] \) and, to compare \( \nu \) with \( \nu^* \), we introduce the relative condition number,

\[
\kappa := \sup_{w \in \mathbb{R}^d} \frac{w^\top \Sigma_{\nu} w}{w^\top \Sigma_{\nu^*} w}.
\]

**Assumption 14 (Relative condition number)** For an initial state-action distribution \( \nu_0 \) and \( \nu^* \) determined by (32), the relative condition number \( \kappa \) is finite.

With the estimation error \( \epsilon_{est} \), the transfer error \( \epsilon_{bias} \), and the relative condition number \( \kappa \) in place, in Theorem 15 we provide convergence guarantees for algorithm (25) using the approximate update (31). Even though we set \( \theta^{(0)} = 0 \) and \( \lambda^{(0)} = 0 \) in the proof of Theorem 15, global convergence can be established for arbitrary initial conditions.

**Theorem 15 (Convergence and optimality: log-linear policy parametrization)** Let Assumption 2 hold for \( \xi > 0 \) and let us fix a state distribution \( \rho \) and a state-action distribution \( \nu_0 \). If the iterates \( (\theta^{(t)}, \lambda^{(t)}) \) generated by algorithm (25) and (31) with \( \|\phi_{s,a}\| \leq B \) and \( \eta_1 = \eta_2 = 1/\sqrt{T} \) satisfy Assumptions 13 and 14, then,

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (V^*_r(\rho) - V_r^{(t)}(\rho)) \right] \leq \frac{C_3}{(1 - \gamma)^5} \frac{1}{\sqrt{T}} + \frac{2 + 4/\xi}{(1 - \gamma)^2} \left( \sqrt{|A|} \epsilon_{bias} + \sqrt{\kappa |A| \epsilon_{est} / (1 - \gamma)} \right),
\]

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) \right] \leq \frac{C_4}{(1 - \gamma)^4} \frac{1}{\sqrt{T}} + \left( \frac{4 + 2\xi}{1 - \gamma} \right) \left( \sqrt{|A|} \epsilon_{bias} + \sqrt{\kappa |A| \epsilon_{est} / (1 - \gamma)} \right).
\]

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where \( C_3 := 1 + \log |A| + 5B^2W^2/\xi \) and \( C_4 := (1 + \log |A| + B^2W^2)\xi + (2 + 4B^2W^2)/\xi \).

Theorem 15 shows that, on average, the reward value function converges to its globally optimal value and that the constraint violation decays to zero (up to an estimation error \( \epsilon_{\text{est}} \) and a transfer error \( \epsilon_{\text{bias}} \)). When \( \epsilon_{\text{bias}} = \epsilon_{\text{est}} = 0 \), the rate \( (1/\sqrt{T}, 1/\sqrt{T}) \) matches the result in Theorem 9 for the exact tabular softmax case. In contrast to the optimality gap, the lower order of effective horizon \( 1/(1 - \gamma) \) in the constraint violation yields a tighter error bound.

**Remark 16** In the standard error decomposition,

\[
\mathcal{E}_0^{\nu(t)}(w_{\theta}^{(t)}; \theta^{(t)}) = \mathcal{E}_0^{\nu(t)}(w_{\hat{\theta}}^{(t)}; \theta^{(t)}) - \mathcal{E}_0^{\nu(t)}(w_{\tilde{\theta}}^{(t)}; \theta^{(t)}) + \mathcal{E}_0^{\nu(t)}(w_{\hat{\theta}}^{(t)}; \theta^{(t)})
\]

the difference term is the standard estimation error that result from the discrepancy between \( w_{\theta}^{(t)} \) and \( w_{\hat{\theta}}^{(t)} \), and the last term characterizes the approximation error in \( w_{\hat{\theta}}^{(t)} \). In Corollary 17, we repeat Theorem 15 in terms of an upper bound \( \epsilon_{\text{approx}} \) on the approximation error,

\[
\mathcal{E}^{(t)}_{\text{approx}} := \mathbb{E} \left[ \mathcal{E}_0^{\nu(t)}(w_{\tilde{\theta}}^{(t)}; \theta^{(t)}) \right].
\]

Since \( \mathcal{E}^{(t)}_{\text{approx}} \) utilizes on-policy state-action distribution \( \nu^{(t)} \), the error bounds in Corollary 17 depend on the worst-case distribution mismatch coefficient \( \|\nu^*/\nu_0\|_\infty \). In contrast, application of estimation and transfer errors in Theorem 15 does not involve the distribution mismatch coefficient. Therefore, the error bounds in Theorem 15 are tighter than the ones in Corollary 17 that utilizes the standard error decomposition.

**Corollary 17 (Convergence and optimality: log-linear policy parametrization)** Let Assumption 2 hold for \( \xi > 0 \) and let us fix a state distribution \( \rho \) and a state-action distribution \( \nu_0 \). If the iterates \( (\theta^{(t)}, \lambda^{(t)}) \) generated by algorithm (25) and (31) with \( \|\phi_{s,a}\| \leq B \) and \( \eta_1 = \eta_2 = 1/\sqrt{T} \) satisfy Assumption 13 except for \( \mathcal{E}^{(t)}_{\text{bias}} \), Assumption 14, and \( \mathcal{E}^{(t)}_{\text{approx}} \leq \epsilon_{\text{approx}} \), \( \rho = r \) or \( g \), then,

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (V_\gamma^*(\rho) - V_\gamma^{(t)}(\rho)) \right] \leq \frac{C_3}{(1 - \gamma)^5} \frac{1}{\sqrt{T}} + \frac{C_4}{(1 - \gamma)^4} \left( \sqrt{\frac{|A| \epsilon_{\text{approx}}}{1 - \gamma}} \frac{\|\nu^*/\nu_0\|_\infty}{1 - \gamma} + \frac{\kappa |A| \epsilon_{\text{est}}}{1 - \gamma} \right)
\]

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) \right] \leq \frac{C_3'}{(1 - \gamma)^5} \frac{1}{\sqrt{T}} + \frac{C_4'}{(1 - \gamma)^4} \left( \sqrt{\frac{|A| \epsilon_{\text{approx}}}{1 - \gamma}} \frac{\|\nu^*/\nu_0\|_\infty}{1 - \gamma} + \frac{\kappa |A| \epsilon_{\text{est}}}{1 - \gamma} \right)
\]

where \( C_3 := 1 + \log |A| + 5B^2W^2/\xi \), \( C_4 := (1 + \log |A| + B^2W^2)\xi + (2 + 4B^2W^2)/\xi \), \( C_4' := (2 + 4/\xi)/(1 - \gamma)^2 \), and \( C_4' := (4 + 2\xi)/(1 - \gamma) \).

**Proof** From the definitions of \( \mathcal{E}_0^{\nu^*} \) and \( \mathcal{E}_0^{\nu(t)} \) we have

\[
\mathcal{E}_0^{\nu^*}(w_{\hat{\theta}}^{(t)}; \theta^{(t)}) \leq \left\| \frac{\nu^*}{\nu(t)} \right\|_\infty \mathcal{E}_0^{\nu(t)}(w_{\hat{\theta}}^{(t)}; \theta^{(t)}) \leq \frac{1}{1 - \gamma} \left\| \frac{\nu^*}{\nu_0} \right\|_\infty \mathcal{E}_0^{\nu(t)}(w_{\hat{\theta}}^{(t)}; \theta^{(t)})
\]

where the second inequality is because of \( (1 - \gamma)\nu_0 \leq \nu^{(t)} \). Thus,

\[
\mathcal{E}^{(t)}_{\text{approx}} \leq \frac{1}{1 - \gamma} \left\| \frac{\nu^*}{\nu_0} \right\|_\infty \mathcal{E}^{(t)}_{\text{approx}}
\]

which allows us to replace \( \mathcal{E}^{(t)}_{\text{bias}} \) in the proof of Theorem 15 by \( \mathcal{E}^{(t)}_{\text{approx}} \).
5.2 Proof of Theorem 15

We provide a regret-type analysis for general smooth policy class that subsumes the log-linear policy class. Using the policy smoothness, we first generalize Lemma 11 to the function approximation setting. Then, we can utilize the function approximation error to contain possible duality error and characterize the regret and violation performance in Lemma 18.

**Lemma 18 (Regret/violation lemma)** Let Assumption 2 hold for $\xi > 0$, let us fix a state distribution $\rho$ and $T > 0$, and let $\log \pi_\theta(a|s)$ be $\beta$-smooth in $\theta$ for any $(s,a)$. If the iterates $(\theta^t, \lambda^t)$ are generated by algorithm (25) with $\theta^0 = 0$, $\lambda^0 = 0$, $\eta_1 = \eta_2 = 1/\sqrt{T}$, and $\|w^t\| \leq W$, then,

\[
\frac{1}{T} \sum_{t=0}^{T-1} (V^*_r(\rho) - V^r_t(\rho)) \leq \frac{C_3}{(1-\gamma)^5} \frac{1}{\sqrt{T}} + \sum_{t=0}^{T-1} \frac{\text{err}^t_r(\pi^*)}{(1-\gamma)_T} + \sum_{t=0}^{T-1} \frac{2 \times \text{err}^t_g(\pi^*)}{(1-\gamma)_T^2}
\]

\[
\frac{1}{T} \sum_{t=0}^{T-1} (\beta - V^g_t(\rho)) \leq \frac{C_4}{(1-\gamma)^4} \frac{1}{\sqrt{T}} + \sum_{t=0}^{T-1} \frac{\xi \times \text{err}^t_r(\pi^*)}{T} + \sum_{t=0}^{T-1} \frac{2 \times \text{err}^t_g(\pi^*)}{(1-\gamma)_T}
\]

where $C_3 := 1 + \log |A| + 5\beta W^2/\xi$, $C_4 := (1 + \log |A| + \beta W^2)\xi + (2 + 4\beta W^2)/\xi$, and

\[
\text{err}^t_\phi(\pi) := \mathbb{E}_{s \sim d^\rho} \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ A^t_\phi(s,a) - (w^t_\phi)^\top \nabla_{\theta} \log \pi^t(a|s) \right]
\]

where $\phi = r$ or $g$.

**Proof** The smoothness of the log-linear policy in conjunction with an application of Taylor series expansion to $\log \pi^t_\theta(a|s)$ yield

\[
\log \frac{\pi^t_\theta(a|s)}{\pi^t_\theta(a|s)} + (\theta^{t+1} - \theta^t)^\top \nabla_{\theta} \log \pi^t_\theta(a|s) \leq \frac{\beta}{2} \|\theta^{t+1} - \theta^t\|^2 \quad (33)
\]
where \( \theta^{(t+1)} - \theta^{(t)} = \eta_1 w^{(t)}/(1 - \gamma) \). Fixing \( \pi \) and \( \rho \), we use \( d \) to denote \( d^\pi \) to obtain,

\[
\mathbb{E}_s \sim d \left( D_{\text{KL}}(\pi(\cdot | s) \| \pi_\theta^{(t)}(\cdot | s)) - D_{\text{KL}}(\pi(\cdot | s) \| \pi_\theta^{(t+1)}(\cdot | s)) \right) \\
= - \mathbb{E}_s \sim d \mathbb{E}_a \sim \pi(\cdot | s) \log \frac{\pi_\theta^{(t)}(a | s)}{\pi_\theta^{(t+1)}(a | s)} \\
\overset{(a)} \geq \frac{\eta_1}{1 - \gamma} \mathbb{E}_s \sim d \mathbb{E}_a \sim \pi(\cdot | s) \left[ \nabla_\theta \log \pi_\theta^{(t)}(a | s) w^{(t)} \right] - \beta \frac{\eta_1^2}{2(1 - \gamma)^2} \| w^{(t)} \|^2 \\
\overset{(b)} = \frac{\eta_1}{1 - \gamma} \mathbb{E}_s \sim d \mathbb{E}_a \sim \pi(\cdot | s) \left[ \nabla_\theta \log \pi_\theta^{(t)}(a | s) w_g^{(t)} \right] \\
+ \frac{\eta_1}{1 - \gamma} \mathbb{E}_s \sim d \mathbb{E}_a \sim \pi(\cdot | s) \left[ \nabla_\theta \log \pi_\theta^{(t)}(a | s) \left( w_r^{(t)} + \lambda^{(t)} w_g^{(t)} \right) - (A_r^{(t)}(s, a) + \lambda^{(t)} A_g^{(t)}(s, a)) \right] \\
- \beta \frac{\eta_1^2}{2(1 - \gamma)^2} \left( \| w_r^{(t)} \|^2 + (\lambda^{(t)})^2 \| w_g^{(t)} \|^2 \right) \\
\overset{(c)} \geq \frac{1}{1 - \gamma} \left( V^\pi_\rho(\rho) - V^{(t)}_\rho(\rho) \right) + \frac{\eta_1}{1 - \gamma} \lambda^{(t)} \left( V^\pi_\rho(\rho) - V^g_\rho(\rho) \right) \\
- \eta_1 \text{err}_r^{(t)}(\pi) - \eta_1 \lambda^{(t)} \text{err}_g^{(t)}(\pi) - \beta \frac{\eta_1^2 W^2}{(1 - \gamma)^2} - \beta \frac{\eta_1^2 W^2}{(1 - \gamma)^2} (\lambda^{(t)})^2 
\]

where \((a)\) is because of \((33)\). On the other hand, we use the update \( w^{(t)} = w_r^{(t)} + \lambda^{(t)} w_g^{(t)} \) for a given \( \lambda^{(t)} \) in \((b)\) and in \((c)\) we apply the performance difference lemma, definitions of \( \text{err}_r^{(t)}(\pi) \) and \( \text{err}_g^{(t)}(\pi) \), and \( \| w_\rho^{(t)} \| \leq W \). Rearrangement of the above inequality yields

\[
V^\pi_\rho(\rho) - V^{(t)}_\rho(\rho) \\
\leq \frac{1}{1 - \gamma} \left( \frac{1}{\eta_1} \mathbb{E}_s \sim d \left( D_{\text{KL}}(\pi(\cdot | s) \| \pi_\theta^{(t)}(\cdot | s)) - D_{\text{KL}}(\pi(\cdot | s) \| \pi_\theta^{(t+1)}(\cdot | s)) \right) \right) \\
+ \frac{1}{1 - \gamma} \text{err}_r^{(t)}(\pi) + \frac{2}{(1 - \gamma)^2} \text{err}_g^{(t)}(\pi) + \beta \frac{\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4\eta_1 W^2}{(1 - \gamma)^5} \\
- \lambda^{(t)} \left( V^\pi_\rho(\rho) - V^{(t)}_\rho(\rho) \right) 
\]

where we utilize \( 0 \leq \lambda^{(t)} \leq 2/((1 - \gamma)\xi) \) from the dual update in \((25)\).
Averaging the above inequality above over \( t = 0, 1, \ldots, T - 1 \) yields
\[
\frac{1}{T} \sum_{t=0}^{T-1} (V_r^\pi(\rho) - V_r^{(t)}(\rho))
\leq \frac{1}{(1-\gamma)\eta_1 T} \sum_{t=0}^{T-1} (E_s \sim d \left( D_{KL}(\pi(\cdot | s) \| \pi_g^{(t)}(\cdot | s)) - D_{KL}(\pi(\cdot | s) \| \pi_g^{(t+1)}(\cdot | s)) \right) + \frac{1}{(1-\gamma)T} \sum_{t=0}^{T-1} \text{err}^{(t)}(\pi) + \frac{2}{(1-\gamma)^2\xi T} \sum_{t=0}^{T-1} \text{err}^{(t)}(\pi) + \frac{\beta \eta_1 W^2}{(1-\gamma)^3} + \frac{\beta \eta_1 W^2}{(1-\gamma)^5\xi^2} + \frac{1}{T} \sum_{t=0}^{T-1} \lambda^{(t)}(V_g^\pi(\rho) - V_g^{(t)}(\rho))
\]

which implies that,
\[
\frac{1}{T} \sum_{t=0}^{T-1} (V_r^\pi(\rho) - V_r^{(t)}(\rho))
\leq \frac{\log |A|}{(1-\gamma)\eta_1 T} + \frac{1}{(1-\gamma)T} \sum_{t=0}^{T-1} \text{err}^{(t)}(\pi) + \frac{2}{(1-\gamma)^2\xi T} \sum_{t=0}^{T-1} \text{err}^{(t)}(\pi) + \beta \frac{\eta_1 W^2}{(1-\gamma)^3} + \beta \frac{4\eta_1 W^2}{(1-\gamma)^5\xi^2} + \frac{1}{T} \sum_{t=0}^{T-1} \lambda^{(t)}(V_g^\pi(\rho) - V_g^{(t)}(\rho)).
\]

If we choose the comparison policy \( \pi = \pi^* \), then we have
\[
\frac{1}{T} \sum_{t=0}^{T-1} (V_r^\pi(\rho) - V_r^{(t)}(\rho)) + \frac{1}{T} \sum_{t=0}^{T-1} \lambda^{(t)}(V_g^\pi(\rho) - V_g^{(t)}(\rho))
\leq \frac{\log |A|}{(1-\gamma)\eta_1 T} + \frac{1}{(1-\gamma)T} \sum_{t=0}^{T-1} \text{err}^{(t)}(\pi^*) + \frac{2}{(1-\gamma)^2\xi T} \sum_{t=0}^{T-1} \text{err}^{(t)}(\pi^*) + \beta \frac{\eta_1 W^2}{(1-\gamma)^3} + \beta \frac{4\eta_1 W^2}{(1-\gamma)^5\xi^2}.
\]

**Proving the first inequality.** By the same reasoning as in (21a),
\[
0 \leq (\lambda^{(T)})^2 = \sum_{t=0}^{T-1} ((\lambda^{(t+1)})^2 - (\lambda^{(t)})^2)
\leq 2\eta_2 \sum_{t=0}^{T-1} \lambda^{(t)}(b - V_g^{(t)}(\rho)) + \eta_2^2 \sum_{t=0}^{T-1} (V_g^{(t)}(\rho) - b)^2
\leq 2\eta_2 \sum_{t=0}^{T-1} \lambda^{(t)}(V_g^\pi(\rho) - V_g^{(t)}(\rho)) + \frac{\eta_2^2 T}{(1-\gamma)^2}
\]

where \((a)\) is because of feasibility of \( \pi^* \): \( V_g^\pi(\rho) \geq b \), and \( |V_g^{(t)}(\rho) - b| \leq 1/(1-\gamma) \). Hence,
\[
-\frac{1}{T} \sum_{t=0}^{T-1} \lambda^{(t)}(V_g^\pi(\rho) - V_g^{(t)}(\rho)) \leq \frac{\eta_2}{2(1-\gamma)^2}.
\]
By adding the inequality (35b) to (34) on both sides and taking $\eta_1 = \eta_2 = 1/\sqrt{T}$, we obtain the first inequality.

**Proving the second inequality.** Since the dual update in (25) is the same as the one in (14a), we can use the same reasoning to conclude (22). Adding the inequality (22) to (34) on both sides and using $V_g^*(\rho) \geq b$ yield

$$
\frac{1}{T} \sum_{t=0}^{T-1} (V^*_r(\rho) - V_r^{(t)}(\rho)) + \lambda \frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho))
$$

$$
\leq \frac{\log |A|}{(1 - \gamma)\eta_1 T} + \frac{1}{(1 - \gamma)T} \sum_{t=0}^{T-1} \text{err}_r^{(t)}(\pi^*) + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \text{err}_g^{(t)}(\pi^*)
$$

$$
+ \beta \frac{\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4\eta_1 W^2}{(1 - \gamma)^5 \xi^2} + \frac{1}{2 \eta_2 T} \log(0) - \log|A| + \frac{\eta_2}{2(1 - \gamma)^2}.
$$

Taking $\lambda = \frac{2}{(1 - \gamma)\xi}$ when $\sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) \geq 0$ and $\lambda = 0$ otherwise, we obtain

$$
V^*_r(\rho) - \frac{1}{T} \sum_{t=0}^{T-1} V_r^{(t)}(\rho) + \frac{2}{(1 - \gamma)\xi} \left[ b - \frac{1}{T} \sum_{t=0}^{T-1} V_g^{(t)}(\rho) \right]
$$

$$
\leq \frac{\log |A|}{(1 - \gamma)\eta_1 T} + \frac{1}{(1 - \gamma)T} \sum_{t=0}^{T-1} \text{err}_r^{(t)}(\pi^*) + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \text{err}_g^{(t)}(\pi^*)
$$

$$
+ \beta \frac{\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4\eta_1 W^2}{(1 - \gamma)^5 \xi^2} + \frac{2}{\eta_2(1 - \gamma)^2 \xi^2 T} + \frac{\eta_2}{2(1 - \gamma)^2}.
$$

Since $V_r^{(t)}(\rho)$ and $V_g^{(t)}(\rho)$ are linear functions in the occupancy measure (Altman, 1999, Chapter 10), there exists a policy $\pi'$ such that $V_r^{\pi'}(\rho) = \frac{1}{T} \sum_{t=0}^{T-1} V_r^{(t)}(\rho)$ and $V_g^{\pi'}(\rho) = \frac{1}{T} \sum_{t=0}^{T-1} V_g^{(t)}(\rho)$. Hence,

$$
V^*_r(\rho) - V_r^{\pi'}(\rho) + \frac{2}{(1 - \gamma)\xi} \left[ b - V_g^{\pi'}(\rho) \right]
$$

$$
\leq \frac{\log |A|}{(1 - \gamma)\eta_1 T} + \frac{1}{(1 - \gamma)T} \sum_{t=0}^{T-1} \text{err}_r^{(t)}(\pi^*) + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \text{err}_g^{(t)}(\pi^*)
$$

$$
+ \beta \frac{\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4\eta_1 W^2}{(1 - \gamma)^5 \xi^2} + \frac{2}{\eta_2(1 - \gamma)^2 \xi^2 T} + \frac{\eta_2}{2(1 - \gamma)^2}.
$$

Application of Lemma 4 with $2/((1 - \gamma)\xi) \geq 2\lambda^*$ yields

$$
\left[ b - V_g^{\pi'}(\rho) \right] + \leq \frac{\xi \log |A|}{\eta_1 T} + \frac{\xi}{T} \sum_{t=0}^{T-1} \text{err}_r^{(t)}(\pi^*) + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \text{err}_g^{(t)}(\pi^*)
$$

$$
+ \beta \frac{\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4\eta_1 W^2}{(1 - \gamma)^5 \xi^2} + \frac{2}{\eta_2(1 - \gamma)^2 \xi^2 T} + \frac{\eta_2}{2(1 - \gamma)^2}.
$$

which leads to our constraint violation bound if we further utilize $\frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) = b - V_g^{\pi'}(\rho)$ and $\eta_1 = \eta_2 = 1/\sqrt{T}$. 

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The analysis of Lemma 18 is based on the generalization of Lemma 11 to the function approximation setting using policy smoothness. A crucial step is to use the original optimal policy as our comparison policy in hindsight. Although the strong duality may not hold because of insufficient expressiveness of the parametrized policy class, we can characterize the regret and violation performances, up to function approximation errors.

**Proof [Proof of Theorem 15]**

When \( \| \phi_{s,a} \| \leq B \), for the log-linear policy class, \( \log \pi_{\theta}(a \mid s) \) is \( \beta \)-smooth with \( \beta = B^2 \). By Lemma 18, it remains to consider the randomness in sequences of \( w(t) \) and the error bounds for \( \text{err}_r(t) \) and \( \text{err}_g(t) \). Application of the triangle inequality yields

\[
\text{err}_r(t)(\pi^*) \leq \mathbb{E}_{s \sim d^*_r} \mathbb{E}_{a \sim \pi^*(\cdot \mid s)} \left[ A_r(t)(s, a) - (w_{r,*}^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a \mid s) \right] + \mathbb{E}_{s \sim d^*_r} \mathbb{E}_{a \sim \pi^*(\cdot \mid s)} \left[ (w_{r,*}^{(t)} - w_r^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a \mid s) \right].
\]

Application of (30) and \( A_r^{(t)}(s, a) = Q_r^{(t)}(s, a) - \mathbb{E}_{a' \sim \pi((\cdot \mid s))} Q_r^{(t)}(s, a') \) yields

\[
\mathbb{E}_{s \sim d^*_r} \mathbb{E}_{a \sim \pi^*(\cdot \mid s)} \left[ A_r^{(t)}(s, a) - (w_{r,*}^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a \mid s) \right] \\
= \mathbb{E}_{s \sim d^*_r} \mathbb{E}_{a \sim \pi^*(\cdot \mid s)} \left[ Q_r^{(t)}(s, a) - \phi_{s,a}^\top w_{r,*}^{(t)} \right] \\
- \mathbb{E}_{s \sim d^*_r} \mathbb{E}_{a' \sim \pi((\cdot \mid s))} \left[ Q_r^{(t)}(s, a') - \phi_{s,a'}^\top w_{r,*}^{(t)} \right] \\
\leq \sqrt{\mathbb{E}_{s \sim d^*_r} \mathbb{E}_{a \sim \pi^*(\cdot \mid s)} \left[ Q_r^{(t)}(s, a) - \phi_{s,a}^\top w_{r,*}^{(t)} \right]^2} \\
+ \sqrt{\mathbb{E}_{s \sim d^*_r} \mathbb{E}_{a' \sim \pi((\cdot \mid s))} \left[ Q_r^{(t)}(s, a') - \phi_{s,a'}^\top w_{r,*}^{(t)} \right]^2} \\
\leq 2 \sqrt{|A| \mathbb{E}_{s \sim d^*_r} \mathbb{E}_{a \sim \text{Unif}_A} \left[ Q_r^{(t)}(s, a) - \phi_{s,a}^\top w_{r,*}^{(t)} \right]^2} \\
= 2 \sqrt{|A| \mathcal{E}_r^{(t)}(w_{r,*}; \theta^{(t)})}.
\]

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Similarly,

\[
E_{s \sim d^\nu_p} E_{a \sim \pi^* (\cdot | s)} \left[ \left( w_{r, *}^{(t)} - w_r^{(t)} \right)^\top \nabla_{\theta} \log \pi(t)(a | s) \right]
= E_{s \sim d^\nu_p} E_{a \sim \pi^* (\cdot | s)} \left[ \left( w_{r, *}^{(t)} - w_r^{(t)} \right)^\top \phi_{s,a} \right]
- E_{s \sim d^\nu_p} E_{a' \sim \pi(t) (\cdot | s)} \left[ \left( w_{r, *}^{(t)} - w_r^{(t)} \right)^\top \phi_{s,a'} \right]
\leq 2 \sqrt{|A|} \left\| w_{r, *}^{(t)} - w_r^{(t)} \right\|_{\Sigma_{\nu,t}}^2
\]

where \( \Sigma_{\nu,t} := E_{(s,a) \sim \nu^t} \left[ \phi_{s,a} \phi_{s,a}^\top \right] \). From the definition of \( \kappa \) we have

\[
\left\| w_{r, *}^{(t)} - w_r^{(t)} \right\|_{\Sigma_{\nu,t}}^2 \leq \kappa \left\| w_{r, *}^{(t)} - w_r^{(t)} \right\|_{\Sigma_{\nu,0}}^2 \leq \frac{\kappa}{1 - \gamma} \left\| w_{r, *}^{(t)} - w_r^{(t)} \right\|_{\Sigma_{\nu,t}}^2
\]

where we use \((1 - \gamma) \nu_0 \leq \nu^{(t)} := \nu^{(t)}\) in the second inequality. Evaluation of the first-order optimality condition of \( w_{r, *}^{(t)} \in \text{argmin}_{\|w_r\| \leq W} \mathcal{E}_{\nu}^{\nu(t)}(w_r; \theta^{(t)}) \) yields

\[
\left( w_r - w_{r, *}^{(t)} \right)^\top \nabla_{\theta} \mathcal{E}_{\nu}^{\nu(t)}(w_{r, *}^{(t)}; \theta^{(t)}) \geq 0, \text{ for any } w_r \text{ satisfying } \|w_r\| \leq W.
\]

Thus,

\[
\mathcal{E}_{\nu}^{\nu(t)}(w_r; \theta^{(t)}) - \mathcal{E}_{\nu}^{\nu(t)}(w_{r, *}^{(t)}; \theta^{(t)})
= E_{s,a \sim \nu(t)} \left[ \left( Q_r^{(t)}(s,a) - \phi_{s,a}^\top w_{r, *}^{(t)} + \phi_{s,a}^\top w_r^{(t)} - \phi_{s,a}^\top w_{r, *}^{(t)} \right)^2 \right] - \mathcal{E}_{\nu}^{\nu(t)}(w_{r, *}^{(t)}; \theta^{(t)})
= 2 \left( w_{r, *}^{(t)} - w_r^{(t)} \right)^\top E_{s,a \sim \nu(t)} \left[ \left( Q_r^{(t)}(s,a) - \phi_{s,a}^\top w_{r, *}^{(t)} \right) \phi_{s,a} \right]
+ E_{s,a \sim \nu(t)} \left[ \left( \phi_{s,a}^\top w_{r, *}^{(t)} - \phi_{s,a}^\top w_r^{(t)} \right)^2 \right]
= \left( w_r - w_{r, *}^{(t)} \right)^\top \nabla_{\theta} \mathcal{E}_{\nu}^{\nu(t)}(w_{r, *}^{(t)}; \theta^{(t)}) + \left\| w_r - w_{r, *}^{(t)} \right\|_{\Sigma_{\nu,t}}^2
\geq \left\| w_r - w_{r, *}^{(t)} \right\|_{\Sigma_{\nu,t}}^2.
\]
Taking \( w_r = w_r^{(t)} \) in the above inequality and combining it with (38) and (39), yield

\[
\mathbb{E}_{s \sim d_r^T} \mathbb{E}_{a \sim \pi^*} \left[ (w_r^{(t)} - w_r^{(t)}) ^\top \nabla \log \pi^{(t)} (a \mid s) \right] \\
\leq 2 \sqrt{\frac{\kappa |A|}{1 - \gamma}} \left( \mathcal{E}_r^{(t)} (w_r^{(t)}; \theta^{(t)}) - \mathcal{E}_r^{(t)} (w_{r,\lambda}; \theta^{(t)}) \right).
\]

(Substitution of (37) and (40) into the right-hand side of (36) yields)

\[
\mathbb{E} \left[ \text{err}^{(t)}_r (\pi^*) \right] \leq 2 \sqrt{|A|} \mathbb{E} \left[ \mathcal{E}_r^{(t)} (w_r^{(t)}; \theta^{(t)}) \right] + 2 \sqrt{\frac{\kappa |A|}{1 - \gamma}} \mathbb{E} \left[ \mathcal{E}_r^{(t)} (w_r^{(t)}; \theta^{(t)}) - \mathcal{E}_r^{(t)} (w_{r,\lambda}; \theta^{(t)}) \right].
\]

By the same reasoning, we can establish a similar bound on \( \mathbb{E} [\text{err}^{(t)}_r (\pi^*)] \). Finally, our desired results follow by applying Assumption 13 and Lemma 18.

**Remark 19 (Zero constraint violation: log-linear policy parametrization)** As done in Section 4.2, it is straightforward to refine the constraint violation in Theorem 15 and Corollary 17 to be zero by replacing the constraint in Problem (5) by a conservative constraint \( V_g^{\pi_\theta} (\rho) \geq b + \delta \) for some \( \delta > 0 \). In Theorem 15, for instance, an extra term \(-\Omega(\delta)\) caused by the conservatism enters into the right-hand side of the constraint violation. Given a desired accuracy \( \epsilon > 0 \), if \( \epsilon_{\text{est}} = O(\epsilon^2) \) and \( \epsilon_{\text{bias}} = O(\epsilon^2) \) are small, and \( T = \Omega(1/\epsilon^2) \), then \(-\Omega(\delta)\) cancels those positive terms to get zero constraint violation,

\[
\left[ \frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)} (\rho)) \right] \geq 0.
\]

Similarly, we can make an argument for zero constraint violation for Corollary 17. We also note that \( \epsilon_{\text{est}} \) and \( \epsilon_{\text{bias}} \) can be made small in practice.

### 5.3 General smooth policy class

For a general class of smooth policies (Zhang et al., 2019a; Agarwal et al., 2021), we now establish convergence of algorithm (25) with approximate gradient update,

\[
w^{(t)} = w_r^{(t)} + \lambda^{(t)} w_g^{(t)}
\]

\[
w_{\circ}^{(t)} \approx \arg\min_{\|w_0\|_2 \leq W} E_\circ^{(t)} (w_0; \theta^{(t)})
\]

where \( \circ \) denotes \( r \) or \( g \) and the exact minimizer is given by \( w_{\circ,\star}^{(t)} \in \arg\min_{\|w_0\|_2 \leq W} E_\circ^{(t)} (w_0; \theta^{(t)}) \).

**Assumption 20 (Policy smoothness)** For all \( s \in S \) and \( a \in A \), \( \log \pi_\theta (a \mid s) \) is a \( \beta \)-smooth function of \( \theta \),

\[
\| \nabla_\theta \log \pi_\theta (a \mid s) - \nabla_\theta' \log \pi_{\theta'} (a \mid s) \| \leq \beta \| \theta - \theta' \| \text{ for all } \theta, \theta' \in \mathbb{R}^d.
\]
Since both tabular softmax and log-linear policies satisfy Assumption 20 (Agarwal et al., 2021), Assumption 20 covers a broader function class relative to softmax policy parametrization (7).

Given a state-action distribution \( \nu^{(t)} \), we introduce the estimation error as

\[
E^{(t)}_{\nu, \text{est}} := \mathbb{E} \left[ E^{(t)}_\circ \left( w^{(t)}_\circ; \theta^{(t)} \right) - E^{(t)}_\circ \left( w^{(t)}_\circ; \theta^{(t)} | \theta^{(t)} \right) \right].
\]

Furthermore, given a state distribution \( \rho \) and an optimal policy \( \pi^* \), we define a state-action distribution \( \nu^*(s, a) := d^\pi_\rho(s)\pi^*(a | s) \) as a comparator and introduce the transfer error,

\[
E^{(t)}_{\nu, \text{bias}} := \mathbb{E} \left[ E^{\nu^*}_\circ \left( w^{(t)}_\circ; \theta^{(t)} \right) \right].
\]

For any state-action distribution \( \nu \), we define a Fisher information-like matrix induced by \( \pi_\theta \),

\[
\Sigma^{\theta}_{\nu} = \mathbb{E}_{(s, a) \sim \nu} \left[ \nabla_\theta \log \pi_\theta(a | s) \left( \nabla_\theta \log \pi_\theta(a | s) \right)^\top \right]
\]

and use \( \Sigma^{(t)}_{\nu} \) to denote \( \Sigma^{\theta_{(t)}}_{\nu} \).

**Assumption 21 (Estimation/transfer errors and relative condition number)** The estimation and transfer errors as well as the expected relative condition number are bounded, i.e., \( E^{(t)}_{\nu, \text{est}} \leq \epsilon_{\text{est}} \) and \( E^{(t)}_{\nu, \text{bias}} \leq \epsilon_{\text{bias}} \), for \( \diamond = r \) or \( g \), and

\[
\mathbb{E} \left[ \sup_{w \in \mathbb{R}^d} \frac{w^\top \Sigma^{(t)}_{\nu} w}{w^\top \Sigma^{(t)}_{\nu} w} \right] \leq \kappa.
\]

We next provide convergence guarantees for algorithm (25) in Theorem 22 using the approximate update (41). Even though we set \( \theta^{(0)} = 0 \) and \( \lambda^{(0)} = 0 \) in the proof of Theorem 22, convergence can be established for arbitrary initial conditions.

**Theorem 22 (Convergence and optimality: general policy parametrization)** Let Assumptions 2 and 20 hold and let us fix a state distribution \( \rho \), a state-action distribution \( \nu_0 \), and \( T > 0 \). If the iterates \((\theta^{(t)}, \lambda^{(t)})\) generated by algorithm (25) and (41) with \( \eta_1 = \eta_2 = 1/\sqrt{T} \) satisfy Assumption 21 and \( \|w^{(t)}_\circ\| \leq W \), then,

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \left( V^*_\diamond(\rho) - V^{(t)}_{\diamond}(\rho) \right) \right] \leq \frac{C_3}{(1 - \gamma)^5} \frac{1}{\sqrt{T}} + \frac{1 + 2/\xi}{(1 - \gamma)^2} \left( \sqrt{\epsilon_{\text{bias}}} + \sqrt{\frac{\kappa \epsilon_{\text{est}}}{1 - \gamma}} \right)
\]

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (b - V^{(t)}_{\diamond}(\rho)) \right] \leq \frac{C_4}{(1 - \gamma)^4} \frac{1}{\sqrt{T}} + \frac{2 + \xi}{1 - \gamma} \left( \sqrt{\epsilon_{\text{bias}}} + \sqrt{\frac{\kappa \epsilon_{\text{est}}}{1 - \gamma}} \right)
\]

where \( C_3 := 1 + \log |A| + 5bW^2/\xi \) and \( C_4 := (1 + \log |A| + bW^2)\xi + (2 + 4bW^2)/\xi \).

**Proof** Since Lemma 18 holds for any smooth policy class that satisfies Assumption 20, it remains to bound \( \text{err}^{(t)}_{\circ}(\pi^*) \) for \( \diamond = r \) or \( g \). We next separately bound each term on the
right-hand side of (36). For the first term,
\[
\begin{aligned}
E_s \sim d^*_\nu \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ A_r^{(t)}(s, a) - (w_{r,*}^{(t)})^\top \nabla_\theta \log \pi_\theta^{(t)}(a | s) \right] \\
\leq \sqrt{E_s \sim d^*_\nu \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ A_r^{(t)}(s, a) - (w_{r,*}^{(t)})^\top \nabla_\theta \log \pi_\theta^{(t)}(a | s) \right]^2} \\
= \sqrt{E_r^{(t)}(w_{r,*}^{(t)}; \theta^{(t)})}.
\end{aligned}
\] (42)

Similarly,
\[
\begin{aligned}
E_s \sim d^*_\nu \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ (w_{r,*}^{(t)} - w_r^{(t)})^\top \nabla_\theta \log \pi_\theta^{(t)}(a | s) \right] \\
\leq \sqrt{E_s \sim d^*_\nu \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ (w_{r,*}^{(t)} - w_r^{(t)})^\top \nabla_\theta \log \pi_\theta^{(t)}(a | s) \right]^2} \\
= \sqrt{\left\| w_{r,*}^{(t)} - w_r^{(t)} \right\|_{\Sigma_{v_0}^{(t)}}^2}.
\end{aligned}
\] (43a)

Let \( \kappa^{(t)} := \left\| (\Sigma_{v_0}^{(t)})^{-1/2} \Sigma_{\nu(t)}^{(t)} (\Sigma_{v_0}^{(t)})^{-1/2} \right\|_2 \) be the relative condition number at time \( t \). Thus,
\[
\begin{aligned}
\left\| w_{r,*}^{(t)} - w_r^{(t)} \right\|_{\Sigma_{v_0}^{(t)}}^2 &\leq \left\| (\Sigma_{v_0}^{(t)})^{-1/2} \Sigma_{\nu(t)}^{(t)} (\Sigma_{v_0}^{(t)})^{-1/2} \right\| \left\| w_{r,*}^{(t)} - w_r^{(t)} \right\|_{\Sigma_{v_0}^{(t)}}^2 \\
&\overset{(a)}{\leq} \frac{\kappa^{(t)}}{1 - \gamma} \left\| w_{r,*}^{(t)} - w_r^{(t)} \right\|_{\Sigma_{v_0}^{(t)}}^2 \\
&\overset{(b)}{=} \frac{\kappa^{(t)}}{1 - \gamma} \left( E_r^{(t)}(w_r^{(t)}; \theta^{(t)}) - E_r^{(t)}(w_{r,*}^{(t)}; \theta^{(t)}) \right)
\end{aligned}
\]
where we use \((1 - \gamma)\nu_0 \leq \nu_0^{(t)} := \nu^{(t)} \) in (a), and we get (b) by the same reasoning as bounding (39). Taking an expectation over the inequality above from both sides yields
\[
\begin{aligned}
E \left[ \left\| w_{r,*}^{(t)} - w_r^{(t)} \right\|_{\Sigma_{v_0}^{(t)}}^2 \right] &\leq E \left[ \frac{\kappa^{(t)}}{1 - \gamma} E \left[ E_r^{(t)}(w_r^{(t)}; \theta^{(t)}) - E_r^{(t)}(w_{r,*}^{(t)}; \theta^{(t)}) \mid \theta^{(t)} \right] \right] \\
&\leq E \left[ \frac{\kappa^{(t)}}{1 - \gamma} \epsilon_{\text{est}} \right] \\
&\leq \frac{\kappa \epsilon_{\text{est}}}{1 - \gamma}
\end{aligned}
\] (43b)

where the last two inequalities are because of Assumption 21.

Substitution of (42) and (43) to the right-hand side of (36) yields an upper bound on \( E[\text{err}^{(t)}_\pi^*(\pi^*)] \). By the same reasoning, we can establish a similar bound on \( E[\text{err}^{(t)}_\theta^*(\pi^*)] \). Finally, application of these upper bounds to Lemma 18 yields the desired result.
Remark 23 (Zero constraint violation: general policy parametrization) We can refine the constraint violation in Theorem 22 to be zero by replacing the constraint in Problem (5) by a conservative constraint $V^\pi_\theta(\rho) \geq b + \delta$ for some $\delta > 0$, which admits the same reasoning as Remark 19.

6. Sample-based NPG-PD algorithms

We now leverage convergence results established in Theorems 15 and 22 to design two model-free algorithms that utilize sample-based estimates. In particular, we propose a sample-based extension of NPG-PD algorithm (25) with function approximation and $\Lambda = [0, 2/(1-\gamma)\xi]$,.

$$
\theta^{(t+1)} = \theta^{(t)} + \frac{\eta_1}{1-\gamma} \hat{w}^{(t)}
$$

$$
\lambda^{(t+1)} = \mathcal{P}_\Lambda \left( \lambda^{(t)} - \eta_2 \left( \hat{V}_g^{(t)}(\rho) - b \right) \right)
$$

where $\hat{w}^{(t)}$ and $\hat{V}_g^{(t)}(\rho)$ are the sample-based estimates of the gradient and the value function. At each time $t$, we can access constrained MDP environment by executing a policy $\pi$ with terminating probability $1 - \gamma$. For the minimization problem in (41), we can run stochastic gradient descent (SGD) for $K$ rounds, $w_{o,k+1} = \mathcal{P}_{\|w_{o,k}\| \leq W} (w_{o,k} - \alpha_k G_{o,k})$, where $\alpha_k$ is the stepsize. Here, $G_{o,k}$ is a sample-based estimate of the population gradient $\nabla_\theta E^{\nu(t)}_\phi (w_\phi; \theta^{(t)})$.

$$
G_{o,k} = 2 \left( (w_{o,k})^\top \nabla_\theta \log \pi_\theta^{(t)}(a \mid s) - \hat{A}_o^{(t)}(s,a) \right) \nabla_\theta \log \pi_\theta^{(t)}(a \mid s)
$$

$\hat{A}_o^{(t)}(s,a) := \hat{Q}_o^{(t)}(s,a) - \hat{V}_o^{(t)}(s)$, $\hat{Q}_o^{(t)}(s,a)$ and $\hat{V}_o^{(t)}(s)$ are undiscounted sums that are collected in Algorithm 2. In addition, we estimate $\hat{V}_g^{(t)}(\rho)$ using an undiscounted sum in Algorithm 3. As shown in Appendix D, $G_{o,k}$, $\hat{A}_o^{(t)}(s,a)$, and $\hat{V}_g^{(t)}(\rho)$ are unbiased estimates and we approximate gradient using the average of the SGD iterates $\hat{w}^{(t)} = \frac{1}{K(K+1)} \sum_{k=1}^{K} (k+1) (w_{r,k} + \lambda^{(t)} w_{g,k})$, which is an approximate solution for least-squares regression (Lacoste-Julien et al., 2012).

To establish sample complexity of Algorithm 1, we assume that the score function $\nabla_\theta \log \pi(a \mid s)$ has bounded norm and the policy parametrization $\pi_\theta$ has non-degenerate Fisher information matrix (Zhang et al., 2019a; Agarwal et al., 2021; Liu et al., 2020a; Ding et al., 2022).

Assumption 24 (Lipschitz policy) For $0 \leq t < T$, the policy $\pi^{(t)}$ satisfies

$$
\left\| \nabla_\theta \log \pi^{(t)}(a \mid s) \right\| \leq L_\pi, \text{ where } L_\pi > 0.
$$

Assumption 25 (Fisher-non-degenerate policy) There exists $\sigma_F > 0$ such that

$$
\Sigma_{\nu}^{\theta} \succeq \sigma_F I
$$

for all $\nu$ and $\theta \in \mathbb{R}^d$, where $I$ is the identity matrix in $\mathbb{R}^{d \times d}$.
Algorithm 1 Sample-based NPG-PD algorithm with general policy parametrization

1: **Initialization:** Learning rates $\eta_1$ and $\eta_2$, number of SGD iterations $K$, SGD learning rate $\alpha_k = \frac{2}{\sigma^2(k+1)}$ for $k \geq 0$.
2: Initialize $\theta^{(0)} = 0$, $\lambda^{(0)} = 0$.
3: for $t = 0, \ldots, T - 1$ do
4: Initialize $w_{r,0} = w_{g,0} = 0$.
5: for $k = 0, 1, \ldots, K - 1$ do
6: Estimate $\hat{A}_r(s, a)$ and $\hat{A}_g(s, a)$ for some $(s, a) \sim \nu^{(t)}$, using Algorithm 2 with policy $\pi^{(t)}_{\theta'}$.
7: Take a step of SGD,
8: end for
9: Set $\bar{w}^{(t)}_r = \bar{w}^{(t)}_r + \lambda^{(t)}(t)\nu^{(t)}_g$, where
10: Estimate $\hat{V}^{(t)}_g(\rho)$ using Algorithm 3 with policy $\pi^{(t)}_{\theta'}$.
11: Natural policy gradient primal-dual update
12: end for

Algorithm 2 A-Unbiased estimate ($A^\text{est}_{\diamond}$, $\diamond = r$ or $g$)

1: **Input:** Initial state-action distribution $\nu_0$, policy $\pi$, discount factor $\gamma$.
2: Sample $(s_0, a_0) \sim \nu_0$, execute the policy $\pi$ with probability $\gamma$ at each step $h$; otherwise, accept $(s_h, a_h)$ as the sample.
3: Start with $(s_h, a_h)$, execute the policy $\pi$ with the termination probability $1 - \gamma$. Once terminated, add all rewards/utilities from step $h$ onwards as $\hat{Q}^\pi_{\diamond}(s_h, a_h)$ for $\diamond = r$ or $g$, respectively.
4: Start with $s_h$, sample $a'_h \sim \pi(\cdot | s_h)$, and execute the policy $\pi$ with the termination probability $1 - \gamma$. Once terminated, add all rewards/utilities from step $h$ onwards as $\hat{V}^\pi_{\diamond}(s_h)$ for $\diamond = r$ or $g$, respectively.
5: **Output:** $(s_h, a_h)$ and $\hat{A}^\pi_{\diamond}(s_h, a_h) := \hat{Q}^\pi_{\diamond}(s_h, a_h) - \hat{V}^\pi_{\diamond}(s_h), \diamond = r$ or $g$. 

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Algorithm 3  \(V\)-Unbiased estimate (\(V_{\text{est}}\))

1: **Input:** Initial state distribution \(\rho\), policy \(\pi\), discount factor \(\gamma\).
2: Sample \(s_0 \sim \rho\), execute the policy \(\pi\) with the termination probability \(1 - \gamma\). Once terminated, add all utilities up as \(\hat{V}_g^\pi(\rho)\).
3: **Output:** \(\hat{V}_g^\pi(\rho)\).

Assumption 25 holds for the Gaussian policy class (Fatkhullin et al., 2023), neural softmax policy class (Ding et al., 2022), and some neural policies (Liu et al., 2020a). We introduce it in order to tighten sample complexity, although this assumption does not necessarily hold for the tabular softmax policy (Fatkhullin et al., 2023).

In Theorem 26, we establish sample complexity of Algorithm 1.

**Theorem 26 (Sample complexity: general policy parametrization)** Let Assumptions 2, 20, 24, and 25 hold and let us fix a state distribution \(\rho\), a state-action distribution \(\nu_0\), and \(T > 0\). If the iterates \((\theta(t), \lambda(t))\) are generated by the sample-based NPG-PD method described in Algorithm 1 with \(\eta_1 = \eta_2 = 1/\sqrt{T}\) and \(\alpha_k = 2/(\sigma_F(k + 1))\), in which \(K\) rounds of trajectory samples are used at each time \(t\), then,

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (V_g^*(\rho) - V_g^{(t)}(\rho)) \right] \leq C_5 \frac{1}{(1 - \gamma)^3} \sqrt{T} \left( \frac{2\kappa G^2}{\sigma_F(K + 1)} + \sqrt{\epsilon_{\text{bias}}} \right) + \frac{1 + 2/\xi}{(1 - \gamma)^3} \left( \frac{\sqrt{\epsilon_{\text{bias}}}}{\sigma_F} + \frac{2\kappa G^2}{\sigma_F(K + 1)} \right) \frac{1}{(1 - \gamma)^3} \sqrt{T}
\]

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (V_g^*(\rho) - V_g^{(t)}(\rho)) \right] \leq C_6 \frac{1}{(1 - \gamma)^3} \sqrt{T} \left( \frac{2\kappa G^2}{\sigma_F(K + 1)} + \sqrt{\epsilon_{\text{bias}}} \right) + \frac{2 + \xi}{(1 - \gamma)^2} \left( \frac{\sqrt{\epsilon_{\text{bias}}}}{\sigma_F} + \frac{2\kappa G^2}{\sigma_F(K + 1)} \right) \frac{1}{(1 - \gamma)^3} \sqrt{T}
\]

where \(C_5 := 2 + \log |A| + 5\beta W^2/\xi\), \(C_6 := (2 + \log |A| + \beta W^2)\xi + (2 + 4\beta W^2)/\xi\), and \(G^2 := 4(W^2L_\pi^2 + 2/(1 - \gamma)^2)\).

In Theorem 26, the sampling effect appears as an error rate \(1/\sqrt{K}\), where \(K\) is the size of sampled trajectories. This rate follows the standard SGD result (Lacoste-Julien et al., 2012) and it will be reduced to \(1/K^{1/4}\) under less restrictive assumptions on the policy class (Shamir and Zhang, 2013). When \(\epsilon_{\text{bias}} = 0\), it takes \(O(1/\epsilon^4)\) sampled trajectories for Algorithm 1 to output an \(\epsilon\)-optimal policy. The proof of Theorem 26 in Appendix E follows the proof of Theorem 22 except that we use sample-based estimates of gradients in the primal update and sample-based value functions in the dual update.

Algorithm 4 is utilized for log-linear policy parametrization. For the feature \(\phi_{s,a}\) that has bounded norm \(\|\phi_{s,a}\| \leq B\), the sample-based gradient in SGD has the second-order moment bound \(G^2 := 4(W^2B^2 + 2/(1 - \gamma)^2)B^2\). In Theorem 27, we establish sample complexity of Algorithm 4; see Appendix F for proof.

**Theorem 27 (Sample complexity: log-linear policy parametrization)** Let Assumption 2 hold and let us fix a state distribution \(\rho\) and a state-action distribution \(\nu_0\). If the iterates \((\theta(t), \lambda(t))\) generated by the sample-based NPG-PD method described in Algorithm 4 with \(\|\phi_{s,a}\| \leq B\), \(\eta_1 = \eta_2 = 1/\sqrt{T}\), and \(\alpha_k = 2/(\sigma_F(k + 1))\), in which \(K\) rounds of trajectory samples are used at each time \(t\), and there exists \(\sigma_F > 0\) such that \(\mathbb{E}_{(s,a) \sim \nu(t)} \left[ \phi_{s,a} \phi_{s,a}^\top \right] \succeq \sigma_F I\),
Algorithm 4 Sample-based NPG-PD algorithm with log-linear policy parametrization

1: **Input**: Learning rates $\eta_1$ and $\eta_2$, number of SGD iterations $K$, SGD learning rate $\alpha_k = \frac{2}{\sigma_F(k+1)}$ for $k \geq 0$.
2: Initialize $\theta^{(0)} = 0$, $\lambda^{(0)} = 0$,
3: for $t = 0, \ldots, T-1$ do
4: Initialize $w_{r,0} = w_{g,0} = 0$.
5: for $k = 0,1,\ldots,K-1$ do
6: Estimate $\hat{Q}_{r}^{(t)}(s, a)$ and $\hat{Q}_{g}^{(t)}(s, a)$ for some $(s, a) \sim \nu^{(t)}$, using Algorithm 5 with log-linear policy $\pi_{\theta}^{(t)}$.
7: Take a step of SGD,
\[
    w_{r,k+1} = \mathcal{P}[\|w_r\| \leq W] \left( w_{r,k} - 2\alpha_k (\phi_{s,a}^T w_{r,k} - \hat{Q}_{r}^{(t)}(s, a)) \phi_{s,a} \right)
\]
\[
    w_{g,k+1} = \mathcal{P}[\|w_g\| \leq W] \left( w_{g,k} - 2\alpha_k (\phi_{s,a}^T w_{g,k} - \hat{Q}_{g}^{(t)}(s, a)) \phi_{s,a} \right).
\]
8: end for
9: Set $\hat{w}_{r}^{(t)} = \hat{w}_{r}^{(t)} + \lambda^{(t)} \hat{w}_{g}^{(t)}$, where
\[
    \hat{w}_{r}^{(t)} = \frac{2}{K(K+1)} \sum_{k=0}^{K-1} (k+1)w_{r,k} \quad \text{and} \quad \hat{w}_{g}^{(t)} = \frac{2}{K(K+1)} \sum_{k=0}^{K-1} (k+1)w_{g,k}.
\]
10: Estimate $\hat{V}_{g}^{(t)}(\rho)$ using Algorithm 3 with log-linear policy $\pi_{\theta}^{(t)}$.
11: Natural policy gradient primal-dual update
\[
    \theta^{(t+1)} = \theta^{(t)} + \frac{\eta_1}{1-\gamma} \hat{w}_{r}^{(t)} \tag{45}
\]
\[
    \lambda^{(t+1)} = \mathcal{P}_{[0,2/(1-\gamma\xi)]} \left( \lambda^{(t)} - \eta_2 (\hat{V}_{g}^{(t)}(\rho) - b) \right).
\]
12: end for

Algorithm 5 $Q$-Unbiased estimate ($Q_{\phi}^{\text{est}}$, $\phi = r$ or $g$)

1: **Input**: Initial state-action distribution $\nu_0$, policy $\pi$, discount factor $\gamma$.
2: Sample $(s_0, a_0) \sim \nu_0$, execute the policy $\pi$ with probability $\gamma$ at each step $h$; otherwise, accept $(s_h, a_h)$ as the sample.
3: Start with $(s_h, a_h)$, execute the policy $\pi$ with the termination probability $1-\gamma$. Once terminated, add all rewards/utilities from step $h$ onwards as $\hat{Q}_{\phi}^{\pi}(s_h, a_h)$ for $\phi = r$ or $g$, respectively.
4: **Output**: $(s_h, a_h)$ and $\hat{Q}_{\phi}^{\pi}(s_h, a_h)$, $\phi = r$ or $g$. 

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then,

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (V^*_r(\rho) - V^{(t)}_r(\rho)) \right] \leq \frac{C_5}{(1-\gamma)^5} \frac{1}{\sqrt{T}} + \frac{2 + 4/\xi}{(1-\gamma)^3} \left( \sqrt{|A|\epsilon_{bias}} + \sqrt{\frac{2\kappa |A| G^2}{\sigma_F(K+1)}} \right)
\]

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (b - V^{(t)}_g(\rho)) \right] \leq \frac{C_6}{(1-\gamma)^4} \frac{1}{\sqrt{T}} + \frac{4 + 2\xi}{(1-\gamma)^2} \left( \sqrt{|A|\epsilon_{bias}} + \frac{2\kappa |A| G^2}{\sigma_F(K+1)} \right)
\]

where \( C_5 := 2 + \log |A| + 5\beta W^2/\xi \) and \( C_6 := (2 + \log |A| + \beta W^2)\xi + (2 + 4\beta W^2)/\xi \).

When we specialize the log-linear policy to be the softmax policy, Algorithm 4 becomes a sample-based implementation of NPG-PD method (14) that utilizes the state-action value functions. In this case, \( \epsilon_{bias} = 0 \) and \( B = 1 \) in Theorem 27. When there are no sampling effects, i.e., as \( K \to \infty \), our rate \( (1/\sqrt{T}, 1/\sqrt{T}) \) matches the rate in Theorem 9. It takes \( O(1/\epsilon^4) \) sampled trajectories for Algorithm 4 to output an \( \epsilon \)-optimal policy.

7. Computational experiments

We utilize a set of robotic tasks to demonstrate the merits and the effectiveness of our sample-based NPG-PD method described in Algorithm 1. In our computational experiments robotic agents are trained to move along a straight line or in a plane with speed limits for safety (Zhang et al., 2020b). We compare the performance of our NPG-PD algorithm with two classes of representative state-of-the-art methods: (i) two classical primal-dual policy search methods: Trust Region Policy Optimization based Lagrangian (TRPOLag) method and Proximal Policy Optimization based Lagrangian (PPOLag) method (Ray et al., 2019); (ii) two methods that utilize the state-of-the-art policy optimization techniques: Constrained Update Projection (CUP) approach (Yang et al., 2022) and First Order Constrained Optimization in Policy Space (FOCOPS) algorithm (Zhang et al., 2020b). We conduct computational experiments in the OmniSafe framework (Ji et al., 2023) and implement robotic environments using the OpenAI Gym (Brockman et al., 2016) for the MuJoCo physical simulators (Todorov et al., 2012).

We train six MuJoCo robotic agents to walk: Ant-v1, Humanoid-v1, HalfCheetah-v1, Walker2d-v1, Hopper-v1, and Swimmer-v1, while constraining the moving speed to be under a given threshold. Figure 2 shows that, in the first two tasks, our NPG-PD algorithm uniformly outperforms other four methods by reaching higher rewards while maintaining similar constraint satisfaction costs. This outstanding performance of NPG-PD is also demonstrated in HalfCheetah-v1 and Walker2d-v1 tasks in Figure 3; in particular, we note that NPG-PD achieves a performance similar to that of PPOLag and that they both outperform the other three methods in Walker2d-v1 task. On the other hand, PPOLag does not perform well in Hopper-v1 in Figure 4. For the last two tasks, Figure 4 shows a competitive performance of NPG-PD with two state-of-the-art methods: FOCOPS and CUP. Even though early oscillatory behavior slows down convergence of NPG-PD in Hopper-v1, it achieves higher rewards than CUP and FOCOPS. This demonstrates that NPG-PD can not only converge faster than classical Lagrangian-based primal-dual methods but also achieve performance of state-of-the-art policy optimization methods.
8. Concluding remarks

We have proposed a Natural Policy Gradient Primal-Dual algorithm for solving optimal control problems for constrained MDPs. Our algorithm utilizes natural policy gradient ascent to update the primal variable and projected sub-gradient descent to update the dual variable. Although the underlying maximization involves a nonconcave objective function and a nonconvex constraint set, we have established global convergence for either softmax or general smooth policy parametrizations and have provided finite-sample complexity guarantees for two model-free extensions of the NPG-PD algorithm. To the best of our knowledge, our work is the first to offer finite-time performance guarantees for policy-based primal-dual methods in the context of discounted infinite-horizon constrained MDPs.

In future work, we will attempt to address oscillatory behavior which commonly arises in primal-dual methods (Stooke et al., 2020). When the two-time scale scheme for updating primal and dual variables is used, the fast convergence rate can be achieved by incorporating modifications to the objective function or to the update of the dual variable into the algorithm design (Liu et al., 2021b; Li et al., 2021; Ying et al., 2022). It is relevant to examine whether similar techniques can improve convergence of single-time scale primal-dual algorithms and if constraint violation can be reduced to zero (Bai et al., 2022). Other open issues include addressing sample efficiency of policy gradient primal-dual algorithms in the presence of strategic exploration (Agarwal et al., 2020; Zanette et al., 2021; Zeng et al., 2022), reuse of off-policy samples, examining robustness against adversaries, as well as off-line policy optimization for constrained MDPs.

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Figure 2: Learning curves of NPG-PD method (—, blue), CUP (Yang et al., 2022) (—, red), FOCOPS (Zhang et al., 2020b) (—, orange), TRPOLag (Ray et al., 2019) (—, black), and PPOLag (Ray et al., 2019) (—, green) for Ant-v1 and Humanoid-v1 robotic tasks with the speed limit 25. The vertical axes represent the average reward and the average cost (i.e., average speed). The solid lines show the means of 1000 bootstrap samples obtained over 3 random seeds and the shaded regions display the bootstrap 95% confidence intervals.
Figure 3: Learning curves of NPG-PD method (—, blue), CUP (Yang et al., 2022) (—, red), FOCOPS (Zhang et al., 2020b) (—, orange), TRPOLag (Ray et al., 2019) (—, black), and PPOLag (Ray et al., 2019) (—, green) for HalfCheetah-v1 and Walker2d-v1 robotic tasks with the speed limit 25. The vertical axes represent the average reward and the average cost (i.e., average speed). The solid lines show the means of 1000 bootstrap samples obtained over 3 random seeds and the shaded regions display the bootstrap 95% confidence intervals.

Appendix A. Proof of Lemma 5

We prove Lemma 5 by providing a concrete constrained MDP example as shown in Figure 1. States $s_3$, $s_4$, and $s_5$ are terminal states with zero reward and utility. We consider non-trivial state $s_1$ with two actions: $a_1$ moving ‘up’ and $a_2$ going ‘right’, and the associated value
functions are given by

\[ V_T^\pi(s_1) = \pi(a_2 \mid s_1) \pi(a_1 \mid s_2) \]

\[ V_g^\pi(s_1) = \pi(a_1 \mid s_1) + \pi(a_2 \mid s_1) \pi(a_1 \mid s_2). \]
We consider the following two policies $\pi^{(1)}$ and $\pi^{(2)}$ using the softmax parametrization (7),
\[
\begin{align*}
\theta^{(1)} &= (\log 1, \log x, \log x, \log 1) \\
\theta^{(2)} &= (-\log 1, -\log x, -\log x, -\log 1)
\end{align*}
\]
where the parameter takes form of $(\theta_{s_1,a_1}, \theta_{s_1,a_2}, \theta_{s_2,a_1}, \theta_{s_2,a_2})$ with $x > 0$.

First, we show that $V^\pi_r$ is not concave. We compute that
\[
\begin{align*}
\pi^{(1)}(a_1 | s_1) &= \frac{1}{1 + x}, \quad \pi^{(1)}(a_2 | s_1) = \frac{x}{1 + x}, \quad \pi^{(1)}(a_1 | s_2) = \frac{x}{1 + x} \\
V^{(1)}_r(s_1) &= \left( \frac{x}{1 + x} \right)^2, \quad V^{(1)}_g(s_1) = \frac{1 + x + x^2}{(1 + x)^2} \\
\pi^{(2)}(a_1 | s_1) &= \frac{x}{1 + x}, \quad \pi^{(2)}(a_2 | s_1) = \frac{1}{1 + x}, \quad \pi^{(2)}(a_1 | s_2) = \frac{1}{1 + x} \\
V^{(2)}_r(s_1) &= \left( \frac{1}{1 + x} \right)^2, \quad V^{(2)}_g(s_1) = \frac{1 + x + x^2}{(1 + x)^2}.
\end{align*}
\]

Now, we consider policy $\pi^{(\zeta)}$,
\[
\zeta \theta^{(1)} + (1 - \zeta) \theta^{(2)} = (\log 1, \log (x^{2\zeta-1}), \log (x^{2\zeta-1}), \log 1)
\]
for some $\zeta \in [0, 1]$, which is defined on the segment between $\theta^{(1)}$ and $\theta^{(2)}$. Therefore,
\[
\begin{align*}
\pi^{(1)}(a_1 | s_1) &= \frac{1}{1 + x^{2\zeta-1}}, \quad \pi^{(1)}(a_2 | s_1) = \frac{x^{2\zeta-1}}{1 + x^{2\zeta-1}}, \quad \pi^{(1)}(a_1 | s_2) = \frac{x^{2\zeta-1}}{1 + x^{2\zeta-1}} \\
V^{(\zeta)}_r(s_1) &= \left( \frac{x^{2\zeta-1}}{1 + x^{2\zeta-1}} \right)^2, \quad V^{(\zeta)}_g(s_1) = \frac{1 + x^{2\zeta-1} + (x^{2\zeta-1})^2}{(1 + x^{2\zeta-1})^2}.
\end{align*}
\]

When $x = 3$ and $\zeta = \frac{1}{2}$,
\[
\frac{1}{2} V^{(1)}_r(s_1) + \frac{1}{2} V^{(2)}_r(s_1) = \frac{5}{16} > V^{(\frac{1}{2})}_r(s_1) = \frac{4}{16}
\]
which implies that $V^\pi_r$ is not concave.

When $x = 10$ and $\zeta = \frac{1}{2}$,
\[
V^{(1)}_g(s_1) = V^{(2)}_g(s_1) \geq 0.9 \quad \text{and} \quad V^{(\frac{1}{2})}_g(s_1) = 0.75
\]
which shows that if we take constraint offset $b = 0.9$, then $V^{(1)}_g(s_1) = V^{(2)}_g(s_1) \geq b$, and $V^{(\frac{1}{2})}_g(s_1) < b$ in which the policy $\pi^{(\frac{1}{2})}$ is infeasible. Therefore, the set $\{\theta \mid V^\pi_g(s) \geq b\}$ is not convex.
Appendix B. Proof of Theorem 6

Let us first recall the notion of occupancy measure (Altman, 1999). An occupancy measure $q^\pi$ of a policy $\pi$ is defined as a set of distributions generated by executing $\pi$,

$$q^\pi_{s,a} = \sum_{t=0}^{\infty} \gamma^t P(s_t = s, a_t = a \mid \pi, s_0 \sim \rho)$$

(46)

for all $s \in S$, $a \in A$. For brevity, we put all $q^\pi_{s,a}$ together as $q^\pi \in \mathbb{R}^{\left|S\right|\times\left|A\right|}$ and $q^\pi_a = [q^\pi_{1,a}, \cdots, q^\pi_{\left|S\right|,a}]^\top$. For an action $a$, we collect transition probabilities $P(s' \mid s, a)$ for all $s', s \in S$ to have the shorthand notation $P_a \in \mathbb{R}^{\left|S\right|\times\left|S\right|}$. The occupancy measure $q^\pi$ has to satisfy a set of linear constraints given by $\mathcal{Q} := \{q^\pi \in \mathbb{R}^{\left|S\right|\times\left|A\right|} \mid \sum_{a \in A} (I - \gamma P_a^\top) q^\pi_a = \rho$ and $q^\pi \geq 0\}$. With a slight abuse of notation, we write $r \in [0, 1]^{\left|S\right|\times\left|A\right|}$ and $g \in [0, 1]^{\left|S\right|\times\left|A\right|}$. Thus, the value functions $V^\pi_r, V^\pi_g : S \to \mathbb{R}$ under the initial state distribution $\rho$ are linear in $q^\pi$:

$$V^\pi_r(\rho) = \langle q^\pi, r \rangle \quad \text{and} \quad V^\pi_g(\rho) = \langle q^\pi, g \rangle : = F_g(q^\pi).$$

We are now in a position to consider the primal problem (5) as a linear program,

$$\max_{q^\pi \in \mathcal{Q}} F_r(q^\pi) \quad \text{subject to} \quad F_g(q^\pi) \geq b$$

(47)

where the maximization is over all occupancy measures $q^\pi \in \mathcal{Q}$. Once we compute a solution $q^\pi$, the associated policy solution $\pi$ can be recovered via

$$\pi(a \mid s) = q^\pi_{s,a} \sum_{a \in A} q^\pi_{s,a} \quad \text{for all} \ s, a \in A.$$

(48)

Abstractly, we let $\pi^\vartheta : \mathcal{Q} \to \Delta_A^{\left|S\right|}$ be a mapping from an occupancy measure $q^\pi$ to a policy $\pi$. Similarly, as defined by (46) we let $q^\pi : \Delta_A^{\left|S\right|} \to \mathcal{Q}$ be a mapping from a policy $\pi$ to an occupancy measure $q^\pi$. Clearly, $q^\pi = (\pi^\vartheta)^{-1}$.

Despite the non-convexity essence of (5) in policy space, the reformulation (47) reveals underlying convexity in occupancy measure $q^\pi$. In Lemma 28, we exploit this convexity to show the average policy improvement over $T$ steps.

Lemma 28 (Bounded average performance) Let assumptions in Theorem 6 hold. Then, the iterates $(\theta^{(t)}, \lambda^{(t)})$ generated by PG-PD method (12) satisfy

$$\frac{1}{T} \sum_{t=0}^{T-1} Z^{(t)}(F_r(q^{\theta^{(t)}}) - F_r(q^{\theta^{(t)}})) + \frac{1}{T} \sum_{t=0}^{T-1} \lambda^{(t)} (F_g(q^{\theta^{(t)}}) - F_g(q^{\theta^{(t)}})) \leq \frac{D_\theta L_\theta}{T^{3/4}}$$

(49)

where $D_\theta := \frac{8\left|S\right|}{(1-\gamma)^2} \|d^\pi_{\rho}/\rho\|_\infty^2$ and $L_\theta := \frac{2\left|A\right|(1+2/\xi)}{(1-\gamma)^3}$.

Proof From the dual update in (12) we have $0 \leq \lambda^{(t)} \leq 2/(1-\gamma)^2$. From the smooth property of the value functions under the direct policy parametrization (Agarwal et al., 2021, Lemma D.3) we have

$$\left|F_r(q^\theta) - F_r(q^{\theta^{(t)}}) - \langle \nabla \theta F_r(q^{\theta^{(t)}}, \theta - \theta^{(t)}) \right| \leq \frac{\gamma \left|A\right|}{(1-\gamma)^3} \|\theta - \theta^{(t)}\|_2^2.$$
If we fix $\lambda^{(t)} \geq 0$, then
\[
\left| (F_r + \lambda^{(t)} F_g)(q^\theta) - (F_r + \lambda^{(t)} F_g)(q^{\theta(t)}) - \langle \nabla_\theta F_r(q^{\theta(t)}) + \lambda^{(t)} \nabla_\theta F_g(q^{\theta(t)}), \theta - \theta^{(t)} \rangle \right| \leq \frac{L_\theta}{2} \| \theta - \theta^{(t)} \|^2.
\]
Thus,
\[
(F_r + \lambda^{(t)} F_g)(q^\theta) \geq (F_r + \lambda^{(t)} F_g)(q^{\theta(t)}) + \langle \nabla_\theta F_r(q^{\theta(t)}) + \lambda^{(t)} \nabla_\theta F_g(q^{\theta(t)}), \theta - \theta^{(t)} \rangle - \frac{L_\theta}{2} \| \theta - \theta^{(t)} \|^2.
\]
We note that the primal update in (12) is equivalent to
\[
\theta^{(t+1)} = \arg \max_{\theta \in \Theta} \left\{ V_r^{\theta(t)}(\rho) + \lambda^{(t)} V_g^{\theta(t)}(\rho) + \langle \nabla_\theta V_r^{\theta(t)}(\rho) + \lambda^{(t)} \nabla_\theta V_g^{\theta(t)}(\rho), \theta - \theta^{(t)} \rangle - \frac{1}{2 \eta_1} \| \theta - \theta^{(t)} \|^2 \right\}.
\]
By taking $\eta_1 = 1/L_\theta$ and $\theta = \theta^{(t+1)}$ in (50),
\[
(F_r + \lambda^{(t)} F_g)(q^{\theta(t+1)}) \geq \maximize_{\theta \in \Theta} \left\{ (F_r + \lambda^{(t)} F_g)(q^{\theta(t)}) + \langle \nabla_\theta F_r(q^{\theta(t)}) + \lambda^{(t)} \nabla_\theta F_g(q^{\theta(t)}), \theta - \theta^{(t)} \rangle - \frac{L_\theta}{2} \| \theta - \theta^{(t)} \|^2 \right\}
\]
where $\theta_{\alpha} := \pi^{\alpha}(\alpha q^\pi + (1 - \alpha)q^{\theta(t)})$, we apply (50) for the second inequality, and the last inequality is due to $\pi^{\alpha} \circ q^\pi = \text{id}_{\mathcal{S}_A}$ and linearity of $q^\theta$ in $\theta$. Since $F_r$ and $F_g$ are linear in $q^\theta$, we have
\[
(F_r + \lambda^{(t)} F_g)(q^{\theta_{\alpha}}) = \alpha(F_r + \lambda^{(t)} F_g)(q^\pi) + (1 - \alpha)(F_r + \lambda^{(t)} F_g)(q^{\theta(t)}).
\]
By the definition of $\pi^{\alpha}$,
\[
(\pi^{\alpha}(q) - \pi^{\alpha}(q'))_{s_a} = \frac{1}{\sum_{a \in A} q_{s_a}} (q_{s_a} - q'_{s_a}) + \frac{\sum_{a \in A} q'_{s_a} - \sum_{a \in A} q_{s_a}}{\sum_{a \in A} q_{s_a}} q_{s_a} - \frac{\sum_{a \in A} q_{s_a}}{\sum_{a \in A} q_{s_a}} q_{s_a}.
\]
which, together with \( \|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \), gives
\[
\|q^\pi(q) - q^\pi(q')\|^2 \\
\leq 2 \sum_{s \in S} \sum_{a \in A} \frac{(q_{sa} - q'_{sa})^2}{(\sum_{a \in A} q_{sa})^2} + 2 \sum_{s \in S} \sum_{a \in A} \left( \frac{\sum_{a \in A} q'_{sa} - \sum_{a \in A} q_{sa}}{\sum_{a \in A} q_{sa} \sum_{a \in A} q_{sa}} \right)^2 (q'_{sa})^2
\leq 2 \sum_{s \in S} \frac{1}{(\sum_{a \in A} q_{sa})^2} \left( \sum_{a \in A} (q_{sa} - q'_{sa})^2 + \left( \sum_{a \in A} q'_{sa} - \sum_{a \in A} q_{sa} \right)^2 \right).
\]

Therefore,
\[
\|\theta - \theta^{(t)}\|^2 \\
= \left\| q^\pi \left( \alpha q^\pi + (1 - \alpha)q^{(t)} \right) - q^\pi \left( q^{(t)} \right) \right\|^2 \\
\leq \sum_{s \in S} \frac{2\alpha^2}{(\sum_{a \in A} q_{sa}^{(t)})^2} \left( \left( \sum_{a \in A} q_{sa}^* \right)^2 + \left( \sum_{a \in A} q_{sa}^{(t)} \right)^2 \right)
\]
in which the upper bound further can be relaxed into
\[
\sum_{s \in S} \frac{4\alpha^2}{(\sum_{a \in A} q_{sa}^{(t)})^2} \left( \left( \sum_{a \in A} q_{sa}^* \right)^2 + \left( \sum_{a \in A} q_{sa}^{(t)} \right)^2 \right) \\
= 4\alpha^2 \sum_{s \in S} \frac{(d_{r^*}(s))^2 + (d_{r^{(t)}}(s))^2}{(d_{r^{(t)}}(s))^2} \\
\leq 4\alpha^2 |S| + 4\alpha^2 |S| \left\| \frac{d_{r^*}}{d_{r^{(t)}}} \right\|_\infty^2 \\
\leq 4\alpha^2 |S| \left( 1 + \frac{1}{(1 - \gamma)^2} \left\| \frac{d_{r^*}}{\rho} \right\|_\infty \right) \\
\leq \alpha^2 D_\theta
\]
where we apply \( d_{r^{(t)}} \geq (1 - \gamma)\rho \) componentwise in the second inequality.

We now apply (52) and (53) to (51),
\[
(F_r + \lambda^{(t)} F_g)(q^\pi) - (F_r + \lambda^{(t)} F_g)(q^{(t+1)})
\leq \min_{\alpha \in [0,1]} \left\{ L_\theta \left\| \theta - \theta^{(t)} \right\|^2 + (F_r + \lambda^{(t)} F_g)(q^\pi) - (F_r + \lambda^{(t)} F_g)(q^{(t+1)}) \right\}
\leq \min_{\alpha \in [0,1]} \left\{ \alpha^2 D_\theta L_\theta + (1 - \alpha) \left( (F_r + \lambda^{(t)} F_g)(q^\pi) - (F_r + \lambda^{(t)} F_g)(q^{(t+1)}) \right) \right\}
\]

49
which further implies

\[(F_r + \lambda^{(t+1)} F_g)(q^{\theta^*}) - (F_r + \lambda^{(t+1)} F_g)(q^{\theta^{(t+1)}})\]

\[\leq \min_{\alpha \in [0,1]} \left\{ \alpha^2 D_\theta L_\theta + (1 - \alpha)((F_r + \lambda^{(t)} F_g)(q^{\theta^*}) - (F_r + \lambda^{(t)} F_g)(q^{\theta^{(t)}})) \right\} \tag{54}\]

\[- (\lambda^{(t)} - \lambda^{(t+1)})(F_g(q^{\theta^*}) - F_g(q^{\theta^{(t+1)}})).\]

We check the right-hand side of the inequality (54). By the dual update in (12), it is easy to see that

\[- (\lambda^{(t)} - \lambda^{(t+1)})(F_g(q^{\theta^*}) - F_g(q^{\theta^{(t+1)}})) \leq |\lambda^{(t)} - \lambda^{(t+1)}|/(1 - \gamma) \leq \eta_2/(1 - \gamma)^2.\]

We discuss three cases: (i) when \(\alpha^{(t)} < 0\), we set \(\alpha = 0\) for (54),

\[(F_r + \lambda^{(t+1)} F_g)(q^{\theta^*}) - (F_r + \lambda^{(t+1)} F_g)(q^{\theta^{(t+1)}}) \leq \frac{D_\theta L_\theta}{2\sqrt{T}}; \tag{55}\]

(ii) when \(\alpha^{(t)} > 1\), we set \(\alpha = 1\) that leads to

\[(F_r + \lambda^{(t+1)} F_g)(q^{\theta^*}) - (F_r + \lambda^{(t+1)} F_g)(q^{\theta^{(t+1)}}) \leq \frac{3}{2} D_\theta L_\theta, \text{ i.e., } \alpha^{(t+1)} \leq 3/4.\]

Thus, this case reduces to the next case (iii): \(0 \leq \alpha^{(t)} \leq 1\) in which we can express (54) as

\[(F_r + \lambda^{(t+1)} F_g)(q^{\theta^*}) - (F_r + \lambda^{(t+1)} F_g)(q^{\theta^{(t+1)}})\]

\[\leq \left(1 - \left(\frac{(F_r + \lambda^{(t)} F_g)(q^{\theta^*}) - (F_r + \lambda^{(t)} F_g)(q^{\theta^{(t)}})}{4D_\theta L_\theta} \right)\right) \times \left((F_r + \lambda^{(t)} F_g)(q^{\theta^*}) - (F_r + \lambda^{(t)} F_g)(q^{\theta^{(t)}})\right)\]

\[+ \frac{D_\theta L_\theta}{2\sqrt{T}}\]

or equivalently,

\[\alpha^{(t+1)} \leq \left(1 - \frac{\alpha^{(t)}}{2}\right) \alpha^{(t)} + \frac{1}{4\sqrt{T}}. \tag{56}\]

By choosing \(\lambda^{(0)} = 0\) and \(\theta^{(0)}\) such that \(V_r^{\theta^{(0)}}(\rho) \geq V_r^{\theta^*}(\rho)\), we know that \(\alpha^{(0)} \leq 0\). Thus, \(\alpha^{(1)} \leq 1/(4\sqrt{T})\). By (55), the case \(\alpha^{(1)} \leq 0\) is trivial. Without loss of generality, we assume that \(0 \leq \alpha^{(t)} \leq 1/\sqrt{T}\). By induction over \(t\) for (56),

\[\alpha^{(t+1)} \leq \left(1 - \frac{\alpha^{(t)}}{2}\right) \alpha^{(t)} + \frac{1}{4\sqrt{T}} \leq \frac{1}{\sqrt{T}}. \tag{57}\]

By combining (55) and (57), and averaging over \(t = 0, 1, \ldots, T - 1\), we get the desired bound. \[\blacksquare\]

**Proof** [Proof of Theorem 6]
Bounding the optimality gap. By the dual update (12) and $\lambda(0) = 0$, it is convenient to bound $(\lambda(T))^2$ by

$$
(\lambda(T))^2 = \sum_{t=0}^{T-1} ((\lambda(t+1))^2 - (\lambda(t))^2)
$$

$$
= 2\eta_2 \sum_{t=0}^{T-1} \lambda(t) (b - F_g(q^{\theta(t)})) + \frac{\eta_2}{2} \sum_{t=0}^{T-1} (F_g(q^{\theta(t)}) - b)^2
$$

$$
\leq 2\eta_2 \sum_{t=0}^{T-1} \lambda(t) (F_g(q^*) - F_g(q^{\theta(t)})) + \frac{\eta_2^2 T}{(1 - \gamma)^2},
$$

where the inequality is due to the feasibility of the optimal policy $\pi^*$ or the associated occupancy measure $q^* = q^{\theta^*}$: $F_g(q^*) \geq b$, and $|F_g(q^{\theta(t)}) - b| \leq 1/(1 - \gamma)$. The above inequality further implies

$$
-\frac{1}{T} \sum_{t=0}^{T-1} \lambda(t) (F_g(q^*) - F_g(q^{\theta(t)})) \leq \frac{\eta_2}{2(1 - \gamma)^2}.
$$

By substituting the above inequality into (49) in Lemma 28, we show the desired optimality gap bound, where we take $\eta_2 = (1 - \gamma)^2 D_\theta L_\theta / (2\sqrt{T})$.

Bounding the constraint violation. From the dual update in (12) we have for any $\lambda \in [0, 2/(1 - \gamma)\xi]$

$$
|\lambda(t+1) - \lambda|^2 \leq (a) |\lambda(t) - \eta_2 (F_g(q^{\theta(t)}) - b) - \lambda|^2
$$

$$
\leq (b) |\lambda(t) - \lambda|^2 - 2\eta_2 (F_g(q^{\theta(t)}) - b) (\lambda(t) - \lambda) + \frac{\eta_2^2}{(1 - \gamma)^2}
$$

where (a) is due to the non-expansiveness of projection $P_\Lambda$ and (b) is due to $(F_g(q^{\theta(t)}) - b)^2 \leq 1/(1 - \gamma)^2$. Summing it up from $t = 0$ to $t = T - 1$, and dividing it by $T$, yield

$$
\frac{1}{T} |\lambda(T) - \lambda|^2 - \frac{1}{T} |\lambda(0) - \lambda|^2
$$

$$
\leq -\frac{2\eta_2}{T} \sum_{t=0}^{T-1} (F_g(q^{\theta(t)}) - b) (\lambda(t) - \lambda) + \frac{\eta_2^2}{(1 - \gamma)^2}
$$

which further implies,

$$
\frac{1}{T} \sum_{t=0}^{T-1} (F_g(q^{\theta(t)}) - b) (\lambda(t) - \lambda) \leq \frac{|\lambda(0) - \lambda|^2}{2\eta_2 T} + \frac{\eta_2}{2(1 - \gamma)^2}.
$$
We note that \( F_g(q^{\theta^*}) \geq b \). By adding the inequality above to (49) in Lemma 28 from both sides,
\[
\frac{1}{T} \sum_{t=0}^{T-1} (F_r(q^{\theta^*}) - F_r(q^{\theta(t)})) + \frac{\lambda}{T} \sum_{t=0}^{T-1} (b - F_g(q^{\theta(t)})) \leq \frac{D_0 L_\theta}{T^{1/4}} + \frac{1}{2\eta_2 T} |\lambda^{(0)} - \lambda|^2 + \frac{\eta_2}{2(1 - \gamma)^2}.
\]
We choose \( \lambda = \frac{2}{(1 - \gamma)\xi} \) if \( \sum_{t=0}^{T-1} (b - F_g(q^{\theta(t)})) \geq 0 \); otherwise \( \lambda = 0 \). Thus,
\[
F_r(q^{\theta^*}) - F_r(q') + \frac{2}{(1 - \gamma)\xi} [b - F_g(q')] \leq \frac{D_0 L_\theta}{T^{1/4}} + \frac{1}{2\eta_2 (1 - \gamma)^2} \xi^2 T + \frac{\eta_2}{2(1 - \gamma)^2}
\]
where there exists \( q' \) such that \( F_r(q') := \frac{1}{T} \sum_{t=0}^{T-1} F_r(q^{\theta(t)}) \) and \( F_g(q') := \frac{1}{T} \sum_{t=0}^{T-1} F_g(q^{\theta(t)}) \) by the definition of occupancy measure.

Application of Lemma 4 with \( 2/((1 - \gamma)\xi) \geq 2\lambda^* \) yields
\[
[b - F_g(q')] \leq \frac{(1 - \gamma)\xi D_0 L_\theta}{T^{1/4}} + \frac{1}{2\eta_2 (1 - \gamma)\xi T} + \frac{\eta_2 \xi}{2(1 - \gamma)}
\]
which readily leads to the desired constraint violation bound by noting that
\[
\frac{1}{T} \sum_{t=0}^{T-1} (b - F_g(q^{\theta(t)})) = b - F_g(q')
\]
and taking \( \eta_2 = \frac{8|A||S|12(1 + 2/\xi)}{(1 - \gamma)^4 \sqrt{T}} \|d^\pi_r / \rho\|^2_\infty \) and \( \|d^\pi_r / \rho\|^2_\infty \geq (1 - \gamma)^2 \).

\[\blacksquare\]

Appendix C. Proof of Lemma 8

The dual update follows Lemma 3. Since \( \lambda^* \leq (V^{\pi^{\theta^*}}_{r}(\rho) - V^{\theta^*}_{r}(\rho)) / \xi \) with \( 0 \leq V^{\pi^*}_{r}, V^{\theta^*}_{r} \leq 1/(1 - \gamma) \), we take projection interval \( \Lambda = [0, 2/((1 - \gamma)\xi)] \) such that upper bound \( 2/((1 - \gamma)\xi) \) is such that \( 2/((1 - \gamma)\xi) \geq 2\lambda^* \).

We now verify the primal update. We expand the primal update in (13) into the following form,
\[
\theta^{(t+1)} = \theta^{(t)} + \eta_1 F^\dagger_{\rho}(\theta^{(t)}) \nabla_{\theta} V_{r}^{\theta^{(t)}}(\rho) + \eta_1 \lambda^{(t)} F^\dagger_{\rho}(\theta^{(t)}) \nabla_{\theta} V_{g}^{\theta^{(t)}}(\rho).
\]

We now deal with: \( F^\dagger_{\rho}(\theta^{(t)}) \nabla_{\theta} V_{r}^{\theta^{(t)}}(\rho) \) and \( F^\dagger_{\rho}(\theta^{(t)}) \nabla_{\theta} V_{g}^{\theta^{(t)}}(\rho) \). For the first one, the proof begins with solutions to the following approximation error minimization problem:
\[
\text{minimize } E_r(w) := \mathbb{E}_{s \sim a^{\pi^*}_{r}, a \sim \pi_\theta(a \mid s)} \left[ (A_{r}^{\pi^*_\theta}(s, a) - w^\top \nabla_{\theta} \log \pi_\theta(a \mid s))^2 \right].
\]

Using the Moore-Penrose inverse, the optimal solution reads,
\[
w^*_r = F^\dagger_{\rho}(\theta) \mathbb{E}_{s \sim a^{\pi^*}_{r}, a \sim \pi_\theta(a \mid s)} \left[ \nabla_{\theta} \log \pi_\theta(a \mid s) A_{r}^{\pi^*_\theta, \lambda}(s, a) \right] = (1 - \gamma) F^\dagger_{\rho}(\theta) \nabla_{\theta} V^{\pi^*_\theta, \lambda}_{r}(\rho)
\]
where $F_{\rho}(\theta)$ is the Fisher information matrix induced by $\pi_\theta$. One key observation from this solution is that $w_s^*$ is parallel to the NPG direction $F_{\rho}^\top(\theta)\nabla_\theta V_{r,\pi_\theta}(\rho)$.

On the other hand, it is easy to verify that $A_{\pi_\theta}^r$ is a minimizer of $E_r(w)$. The softmax parametrization (7) implies that

$$\frac{\partial \log \pi_\theta(a \mid s)}{\partial \theta_{a',s'}} = \mathbb{I}\{s = s'\} \left( \mathbb{I}\{a = a'\} - \pi_\theta(a' \mid s) \right)$$

where $\mathbb{I}\{E\}$ is the indicator function of event $E$ being true. Thus, we have

$$w^T \nabla_\theta \log \pi_\theta(a \mid s) = w_{s,a} - \sum_{a' \in A} w_{s,a'} \pi_\theta(a' \mid s).$$

The above equality together with the fact: $\sum_{a \in A} \pi_\theta(a \mid s) A_{\pi_\theta}^r(s,a) = 0$, shows that $E_r(A_{\pi_\theta}^r) = 0$. However, $A_{\pi_\theta}^r$ may not be the unique minimizer. We consider the following general form of possible solutions,

$$A_{\pi_\theta}^r + u, \text{ where } u \in \mathbb{R}^{\{|S|\mid |A|\}}.$$

For any state $s$ and action $a$ such that $s$ is reachable under $\rho$, using (59) yields

$$u^T \nabla_\theta \log \pi_\theta(a \mid s) = w_{s,a} - \sum_{a' \in A} w_{s,a'} \pi_\theta(a' \mid s).$$

Here, we make use of the following fact: $\pi_\theta$ is a stochastic policy with $\pi_\theta(a \mid s) > 0$ for all actions $a$ in each state $s$, so that if a state is reachable under $\rho$, then it will also be reachable using $\pi_\theta$. Therefore, we require zero derivative at each reachable state:

$$u^T \nabla_\theta \log \pi_\theta(a \mid s) = 0$$

for all $s, a$ so that $u_{s,a}$ is independent of the action and becomes a constant $c_s$ for each $s$. Therefore, the minimizer of $E_r(w)$ is given up to some state-dependent offset,

$$F_{\rho}^\top(\theta) \nabla_\theta V_{r,\pi_\theta}(\rho) = \frac{A_{\pi_\theta}^r}{1 - \gamma} + u$$

(60)

where $u_{s,a} = c_s$ for some $c_s \in \mathbb{R}$ for each state $s$ and action $a$.

We can repeat the above procedure for $F_{\rho}^\top(\theta^{(t)}) \nabla_\theta V_{g,\pi_\theta}^{(t)}(\rho)$ and show,

$$F_{\rho}^\top(\theta) \nabla_\theta V_{g,\pi_\theta}^{(t)}(\rho) = \frac{A_{\pi_\theta}^{(t)}}{1 - \gamma} + v$$

(61)

where $v_{s,a} = d_s$ for some $d_s \in \mathbb{R}$ for each state $s$ and action $a$.

Substituting (60) and (61) into the primal update (58) yields,

$$\theta^{(t+1)} = \theta^{(t)} + \frac{\eta_1}{1 - \gamma} \left( C_{\pi_\theta}^{(t)} + \lambda^{(t)} A_{\pi_\theta}^{(t)} \right) + \eta_1 \left( u + \lambda^{(t)} v \right)$$

$$\pi^{(t+1)}(a \mid s) = \pi^{(t)}(a \mid s) \frac{\exp \left( \frac{\eta_1}{1 - \gamma} \left( A_{\pi_\theta}^{(t)}(s,a) + \lambda^{(t)} A_{\pi_\theta}^{(t)}(s,a) \right) + \eta_1 \left( c_s + \lambda^{(t)} d_s \right) \right)}{Z^{(t)}(s)}$$

where the second equality also utilizes the normalization term $Z^{(t)}(s)$. Finally, we complete the proof by setting $c_s = d_s = 0$.  

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Appendix D. Sample-based NPG-PD algorithm with function approximation

We describe a sample-based NPG-PD algorithm with function approximation in Algorithm 1. We note the computational complexity of Algorithm 1: each round has expected length $2/(1 - \gamma)$ so the expected number of total samples is $4KT/(1 - \gamma)$; the total number of gradient computations $\nabla \theta \log \pi^{(t)}(a|s)$ is $2KT$; the total number of scalar multiplies, divides, and additions is $O(dKT + KT/(1 - \gamma))$.

The following unbiased estimates that are useful in our analysis.

$$
\mathbb{E} \left[ \hat{V}^{(t)}_{\theta}(s) \right] = \mathbb{E} \left[ \sum_{k=0}^{K' - 1} g(s_k, a_k) | \theta^{(t)}, s_0 = s \right] = \mathbb{E} \left[ \sum_{k=0}^{\infty} \mathbb{I}\{K' - 1 \geq k \geq 0\} g(s_k, a_k) | \theta^{(t)}, s_0 = s \right] = \sum_{k=0}^{\infty} \mathbb{E} \mathbb{E}_{K'} \left[ \mathbb{I}\{K' - 1 \geq k \geq 0\} \right] g(s_k, a_k) | \theta^{(t)}, s_0 = s = \sum_{k=0}^{\infty} \mathbb{E} \mathbb{E}_{K'} \left[ \mathbb{I}\{K' - 1 \geq k \geq 0\} \right] \gamma^k g(s_k, a_k) | \theta^{(t)}, s_0 = s = \sum_{k=0}^{\infty} \gamma^k g(s_k, a_k) | \theta^{(t)}, s_0 = s = V^{(t)}_{\theta}(s)
$$

where we apply the Monotone Convergence Theorem and the Dominated Convergence Theorem for (a) and swap the expectation and the infinite sum in (c), and in (b) we use $\mathbb{E}_{K'} \left[ \mathbb{I}\{K' - 1 \geq k \geq 0\} \right] = 1 - P(K' < k) = \gamma^k$ since $K' \sim \text{Geo}(1 - \gamma)$, a geometric distribution.

By a similar argument as above,

$$
\mathbb{E} \left[ \hat{Q}^{(t)}_{r}(s, a) \right] = \mathbb{E} \left[ \sum_{k=0}^{K' - 1} r(s_k, a_k) | \theta^{(t)}, s_0 = s, a_0 = a \right] = \mathbb{E} \left[ \sum_{k=0}^{\infty} \mathbb{I}\{K' - 1 \geq k \geq 0\} r(s_k, a_k) | \theta^{(t)}, s_0 = s, a_0 = a \right] = \sum_{k=0}^{\infty} \mathbb{E} \mathbb{E}_{K'} \left[ \mathbb{I}\{K' - 1 \geq k \geq 0\} \right] r(s_k, a_k) | \theta^{(t)}, s_0 = s, a_0 = a = \sum_{k=0}^{\infty} \mathbb{E} \mathbb{E}_{K'} \left[ \mathbb{I}\{K' - 1 \geq k \geq 0\} \right] \gamma^k r(s_k, a_k) | \theta^{(t)}, s_0 = s, a_0 = a = \mathbb{E} \sum_{k=0}^{\infty} \gamma^k r(s_k, a_k) | \theta^{(t)}, s_0 = s, a_0 = a = Q^{(t)}_{r}(s, a),
$$

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Therefore,

\[
\mathbb{E} \left[ \hat{A}^{(t)}_r(s, a) \right] = \mathbb{E} \left[ \hat{Q}^{(t)}_r(s, a) \right] - \mathbb{E} \left[ \hat{V}^{(t)}_r(s) \right] = Q^{(t)}_r(s, a) - V^{(t)}_r(s) = A^{(t)}_r(s, a).
\]

We also provide a bound on the variance of \( \hat{V}^{(t)}_g(s) \),

\[
\text{Var} \left[ \hat{V}^{(t)}_g(s) \right] = \mathbb{E} \left[ \left( \hat{V}^{(t)}_g(s) - V^{(t)}_g(s) \right)^2 \mid \theta^{(t)}, s_0 = s \right]
= \mathbb{E} \left[ \left( \sum_{k=0}^{K'-1} g(s_k, a_k) - V^{(t)}_g(s) \right)^2 \mid \theta^{(t)}, s_0 = s \right]
= \mathbb{E}_{K'} \left[ \mathbb{E} \left[ \left( \sum_{k=0}^{K'-1} g(s_k, a_k) - V^{(t)}_g(s) \right)^2 \right] \mid K' \right]
\leq \mathbb{E}_{K'} \left[ (K')^2 \mid K' \right]
= \frac{1}{(1 - \gamma)^2}
\]

where (a) is due to \( 0 \leq g(x_k, a_k) \leq 1 \) and \( V^{(t)}_g(s) \geq 0 \) and (b) is clear from \( K' \sim \text{Geo}(1 - \gamma) \).

Similarly, we have the variance bound \( \text{Var} \left[ \hat{Q}^{(t)}_r(s, a) \right] \leq \frac{1}{(1 - \gamma)^2} \).

By the sampling scheme of Algorithm 2, we can show that \( G_{r,k} \) is an unbiased estimate of the population gradient \( \nabla_{\theta} E^{(t)}_r(w_r; \theta^{(t)}) \),

\[
\mathbb{E}_{(s,a) \sim d^{(t)}} [G_{r,k}] = 2 \mathbb{E} \left[ \left( w_{r,k}^\top \nabla_{\theta} \log \pi^{(t)}_\theta(a \mid s) - \hat{A}^{(t)}_r(s, a) \right) \nabla_{\theta} \log \pi^{(t)}_\theta(a \mid s) \right]
= 2 \mathbb{E} \left[ \left( w_{r,k}^\top \nabla_{\theta} \log \pi^{(t)}_\theta(a \mid s) - \mathbb{E} \left[ \hat{A}^{(t)}_r(s, a) \mid s, a \right] \right) \nabla_{\theta} \log \pi^{(t)}_\theta(a \mid s) \right]
= 2 \mathbb{E} \left[ \left( w_{r,k}^\top \nabla_{\theta} \log \pi^{(t)}_\theta(a \mid s) - \hat{A}^{(t)}_r(s, a) \right) \nabla_{\theta} \log \pi^{(t)}_\theta(a \mid s) \right]
= \nabla_{w_r} E^{(t)}_r(w_r; \theta^{(t)}).
\]

**Appendix E. Proof of Theorem 26**

We first adapt Lemma 18 to the sample-based case as follows.

**Lemma 29 (Sample-based regret/violation lemma)** Let Assumption 2 hold and let us fix a state distribution \( \rho \) and \( T > 0 \). Assume that \( \log \pi_\theta(a \mid s) \) is \( \beta \)-smooth in \( \theta \) for any \((s, a)\). If the iterates \((\pi^{(t)}, \lambda^{(t)})\) generated by the Algorithm 1 with \( \theta^{(0)} = 0, \lambda^{(0)} = 0, \rho \) and \( T \), then

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^T \left( V^{(t)}_r(s, a) - V^*_r(s, a) \right)^2 \right] \leq c \frac{\beta^2}{\gamma^2} \frac{1}{T^2},
\]

where \( c \) is a constant depending on \( \beta, \gamma, \rho \).
\[ \eta_1 = \eta_2 = 1/\sqrt{T}, \text{ and } \| \hat{\omega}_r^{(t)} \|, \| \hat{\omega}_g^{(t)} \| \leq W, \text{ then,} \]

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (V_r^*(\rho) - V_r^{(t)}(\rho)) \right] \leq \frac{C_5}{(1 - \gamma)^5} \sqrt{\frac{1}{T}} + \sum_{t=0}^{T-1} \mathbb{E} \left[ \text{err}_r^{(t)}(\pi^*) \right] + \sum_{t=0}^{T-1} 2\mathbb{E} \left[ \text{err}_g^{(t)}(\pi^*) \right]
\]

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) \right] \leq \frac{C_6}{(1 - \gamma)^4} \sqrt{\frac{1}{T}} + \sum_{t=0}^{T-1} \xi \mathbb{E} \left[ \text{err}_r^{(t)}(\pi^*) \right] + \sum_{t=0}^{T-1} 2\mathbb{E} \left[ \text{err}_g^{(t)}(\pi^*) \right]
\]

where \( C_5 = 2 + \log |A| + 5\beta W^2/\xi \), \( C_6 = (2 + \log |A| + 3\beta W^2)\xi + (2 + 4\beta W^2)/\xi \), and

\[
\widehat{\text{id}}^{(t)}(\pi) = \mathbb{E}_{s \sim d^*} \mathbb{E}_{a \sim \pi(\cdot | s)} \left[ A^{(t)}_r(s,a) - \left( \hat{\omega}_r^{(t)} \right) \nabla \log \pi^{(t)}(a \mid s) \right] , \text{ where } \circ = r \text{ or } g.
\]

**Proof**: The smoothness of log-linear policy in conjunction with an application of Taylor’s theorem to \( \pi^{(t)}_\theta(a \mid s) \) yield

\[
\log \frac{\pi^{(t)}_\theta(a \mid s)}{\pi^{(t+1)}_\theta(a \mid s)} + \left( \theta^{(t+1)} - \theta^{(t)} \right)^\top \nabla \log \pi^{(t)}_\theta(a \mid s) \leq \frac{\beta}{2} \| \theta^{(t+1)} - \theta^{(t)} \|^2
\]

where \( \theta^{(t+1)} - \theta^{(t)} = \frac{\eta_t}{1 - \gamma} \hat{\omega}^{(t)} \). We unfold \( d_{\rho}^{\pi^*} \) as \( d^* \) since \( \pi^* \) and \( \rho \) are fixed. Therefore,

\[
\mathbb{E}_{s \sim d^*} \left( D_{\text{KL}}(\pi^*(\cdot \mid s) \| \pi^{(t)}_\theta(\cdot \mid s)) - D_{\text{KL}}(\pi^*(\cdot \mid s) \| \pi^{(t+1)}_\theta(\cdot \mid s)) \right)
\]

\[
= - \mathbb{E}_{s \sim d^*} \mathbb{E}_{a \sim \pi^*(\cdot \mid s)} \log \frac{\pi^{(t)}_\theta(a \mid s)}{\pi^{(t+1)}_\theta(a \mid s)}
\]

\[
\geq \eta_t \mathbb{E}_{s \sim d^*} \mathbb{E}_{a \sim \pi^*(\cdot \mid s)} \left[ \left( \hat{\omega}^{(t)} \right)^\top \nabla \log \pi^{(t)}_\theta(a \mid s) \right] - \beta \frac{\eta_t^2}{2(1 - \gamma)^2} \| \hat{\omega}^{(t)} \|^2
\]

\[
\geq \eta_t \lambda^{(t)} \mathbb{E}_{s \sim d^*} \mathbb{E}_{a \sim \pi^*(\cdot \mid s)} \left[ \left( \hat{\omega}^{(t)} \right)^\top \nabla \log \pi^{(t)}_\theta(a \mid s) \right] - \beta \frac{\eta_t^2}{2(1 - \gamma)^2} \| \hat{\omega}^{(t)} \|^2
\]

\[
\geq \eta_t \lambda^{(t)} \left( V^{(t)}_r(\rho) - V^{(t)}_r(\rho) \right) + \eta_t \lambda^{(t)} \left( V^{(t)}_g(\rho) - V^{(t)}_g(\rho) \right)
\]

\[
- \eta_t \widehat{\text{id}}^{(t)}(\pi^*) - \eta_t \lambda^{(t)} \widehat{\text{id}}^{(t)}_g(\pi^*) - \beta \frac{\eta_t^2 W^2}{(1 - \gamma)^2} - \beta \frac{\eta_t^2 W^2}{(1 - \gamma)^2} \lambda^{(t)}^2
\]
where \( \hat{w}(t) = \hat{w}_r(t) + \lambda(t) \hat{w}_g(t) \) for a given \( \lambda(t) \), in the last inequality we apply the performance difference lemma, notation of \( \tilde{\text{err}}_r^{(t)}(\pi^*) \) and \( \tilde{\text{err}}_g^{(t)}(\pi^*) \), and \( \|\hat{w}_r(t)\|, \|\hat{w}_g(t)\| \leq W \).

Rearranging the inequality above leads to,

\[
V^*_r(\rho) - V^*_g(\rho) \leq \frac{1}{1 - \gamma} \left( \frac{1}{\eta_1} \mathbb{E}_{s \sim d^*} \left( D_{KL}(\pi^* \mid \mid \pi^{(t)}(\cdot \mid s)) - D_{KL}(\pi^* \mid \mid \pi^{(t+1)}(\cdot \mid s)) \right) \right)
\]

\[
+ \frac{1}{1 - \gamma} \tilde{\text{err}}_r^{(t)}(\pi^*) + \frac{2}{(1 - \gamma)^2 \xi} \tilde{\text{err}}_g^{(t)}(\pi^*) + \beta \frac{\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4 \eta_1 W^2}{(1 - \gamma)^5 \xi^2}
\]

\[
- \lambda(t) \left( V^*_g(\rho) - V^*_g(\rho) \right)
\]

where we utilize \( 0 \leq \lambda(t) \leq 2/((1 - \gamma)\xi) \) from the dual update of Algorithm 1.

Therefore,

\[
\frac{1}{T} \sum_{t=0}^{T-1} (V^*_r(\rho) - V^*_g(\rho)) \leq \frac{1}{(1 - \gamma)\eta_1 T} \sum_{t=0}^{T-1} \left( \mathbb{E}_{s \sim d^*} \left( D_{KL}(\pi^* \mid \mid \pi^{(t)}(\cdot \mid s)) - D_{KL}(\pi^* \mid \mid \pi^{(t+1)}(\cdot \mid s)) \right) \right)
\]

\[
+ \frac{1}{(1 - \gamma)T} \sum_{t=0}^{T-1} \tilde{\text{err}}_r^{(t)}(\pi^*) + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \tilde{\text{err}}_g^{(t)}(\pi^*) + \beta \frac{\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4 \eta_1 W^2}{(1 - \gamma)^5 \xi^2}
\]

\[
- \frac{1}{T} \sum_{t=0}^{T-1} \lambda(t) \left( V^*_g(\rho) - V^*_g(\rho) \right)
\]

\[
\leq \frac{\log |A|}{(1 - \gamma)\eta_1 T} + \frac{1}{(1 - \gamma)T} \sum_{t=0}^{T-1} \tilde{\text{err}}_r^{(t)}(\pi^*) + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \tilde{\text{err}}_g^{(t)}(\pi^*)
\]

\[
+ \beta \frac{\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4 \eta_1 W^2}{(1 - \gamma)^5 \xi^2} + \frac{1}{T} \sum_{t=0}^{T-1} \lambda(t) \left( V^*_g(\rho) - V^*_g(\rho) \right)
\]

where in the last inequality we take a telescoping sum of the first sum and drop a non-positive term. Taking the expectation over the randomness in sampling on both sides of the inequality above yields

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (V^*_r(\rho) - V^*_g(\rho)) \right] + \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \lambda(t) \left( V^*_g(\rho) - V^*_g(\rho) \right) \right]
\]

\[
\leq \frac{\log |A|}{(1 - \gamma)\eta_1 T} + \frac{1}{(1 - \gamma)T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \tilde{\text{err}}_r^{(t)}(\pi^*) \right] + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \tilde{\text{err}}_g^{(t)}(\pi^*) \right] + \beta \frac{\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4 \eta_1 W^2}{(1 - \gamma)^5 \xi^2}
\]

(62)
where in the second inequality we drop a non-positive term and use the fact where the second inequality is due to the feasibility of the policy \( \pi^* \).

\[
0 \leq (\lambda(t))^2 = \sum_{t=0}^{T-1} ((\lambda(t+1))^2 - (\lambda(t))^2)
\]

\[
\leq \sum_{t=0}^{T-1} \left( (\lambda(t) - \eta_2(\hat{V}_g(t)(\rho) - b))^2 - (\lambda(t))^2 \right)
\]

\[
= 2\eta_2 \sum_{t=0}^{T-1} \lambda(t)(b - \hat{V}_g(t)(\rho)) + \eta_2^2 \sum_{t=0}^{T-1} (\hat{V}_g(t)(\rho) - b)^2
\]

\[
\leq 2\eta_2 \sum_{t=0}^{T-1} \lambda(t)(V^*_g(\rho) - \hat{V}_g(t)(\rho)) + 2\eta_2 \sum_{t=0}^{T-1} \lambda(t)(V^*_g(\rho) - \hat{V}_g(t)(\rho))
\]

\[
+ \eta_2^2 \sum_{t=0}^{T-1} (\hat{V}_g(t)(\rho) - b)^2
\]

where the second inequality is due to the feasibility of the policy \( \pi^* \): \( V^*_g(\rho) \geq b \). Since \( V^*_g(\rho) \) is a population quantity and \( \hat{V}_g(t)(\rho) \) is an estimate that is independent of \( \lambda(t) \) given the past history, \( \lambda(t) \) is independent of \( V^*_g(\rho) - \hat{V}_g(t)(\rho) \) at time \( t \) and thus \( \mathbb{E}[\lambda(t)(V^*_g(\rho) - \hat{V}_g(t)(\rho))] = 0 \) due to the fact \( \mathbb{E}[\hat{V}_g(t)(\rho)] = V^*_g(\rho) \); see it in Appendix D. Therefore,

\[
- \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \lambda(t)(V^*_g(\rho) - \hat{V}_g(t)(\rho)) \right] \leq \mathbb{E} \left[ \frac{\eta_2}{2T} \sum_{t=0}^{T-1} (\hat{V}_g(t)(\rho) - b)^2 \right] \leq \frac{2\eta_2}{(1 - \gamma)^2}
\]

where in the second inequality we drop a non-positive term and use the fact

\[
\mathbb{E} \left[ (\hat{V}_g(t)(\rho))^2 \right] = \text{Var} \left[ \hat{V}_g(t)(s) \right] + \left( \mathbb{E} \left[ \hat{V}_g(t)(s) \right] \right)^2 \leq \frac{2}{(1 - \gamma)^2}
\]

where the inequality is due to that \( \text{Var}[\hat{V}_g(t)(s)] \leq 1/(1 - \gamma)^2 \); see it in Appendix D, and \( \mathbb{E} \left[ \hat{V}_g(t)(\rho) \right] = V^*_g(\rho) \), where \( 0 \leq V^*_g(\rho) \leq 1/(1 - \gamma) \).

Adding the inequality (63) to (62) on both sides and taking \( \eta_1 = \eta_2 = 1/\sqrt{T} \) yield the first inequality.

**Proving the second inequality.** From the dual update in Algorithm 1 we have for any \( \lambda \in \Lambda := \left[ 0, 1/(1 - \gamma) \xi \right] \),

\[
\mathbb{E} \left[ |\lambda(t+1) - \lambda|^2 \right]
\]

\[
= \mathbb{E} \left[ |\mathcal{P}_A(\lambda(t) - \eta_2(\hat{V}_g(t)(\rho) - b)) - \mathcal{P}_A(\lambda)|^2 \right]
\]

\[
\overset{(a)}{=} \mathbb{E} \left[ |\lambda(t) - \eta_2(\hat{V}_g(t)(\rho) - b) - \lambda|^2 \right]
\]

\[
= \mathbb{E} \left[ |\lambda(t) - \lambda|^2 \right] - 2\eta_2 \mathbb{E} \left[ (\hat{V}_g(t)(\rho) - b)(\lambda(t) - \lambda) \right] + \eta_2^2 \mathbb{E} \left[ (\hat{V}_g(t)(\rho) - b)^2 \right]
\]

\[
\overset{(b)}{=} \mathbb{E} \left[ |\lambda(t) - \lambda|^2 \right] - 2\eta_2 \mathbb{E} \left[ (\hat{V}_g(t)(\rho) - b)(\lambda(t) - \lambda) \right] + \frac{3\eta_2^2}{(1 - \gamma)^2}
\]
where \((a)\) is due to the non-expansiveness of projection \(P_\lambda\) and \((b)\) is due to \(\mathbb{E}[\left(\hat{V}_g^{(t)}(\rho) - b\right)^2] \leq 2/(1 - \gamma)^2 + 1/(1 - \gamma)^2\). Summing it up from \(t = 0\) to \(t = T - 1\) and dividing it by \(T\) yield

\[
0 \leq \frac{1}{T} \mathbb{E}\left[|\lambda(T) - \lambda|^2\right] \\
\leq \frac{1}{T} \mathbb{E}\left[|\lambda(0) - \lambda|^2\right] - \frac{2\eta_2}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[\left(\hat{V}_g^{(t)}(\rho) - b\right)(\lambda(t) - \lambda)\right] + \frac{3\eta_2^2}{(1 - \gamma)^2}
\]

which further implies that

\[
\mathbb{E}\left[\frac{1}{T} \sum_{t=0}^{T-1} \left(V_r^{(t)}(\rho) - V_g^{(t)}(\rho)\right)\right] + \lambda \mathbb{E}\left[\frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho))\right] \\
\leq \frac{1}{(1 - \gamma)\eta T} \sum_{t=0}^{T-1} \mathbb{E}\left[\text{err}_r^{(t)}(\pi^*)\right] + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \mathbb{E}\left[\text{err}_g^{(t)}(\pi^*)\right] \\
+ \beta \frac{4\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4\eta_1 W^2}{(1 - \gamma)^5 \xi^2} + \frac{1}{2\eta_2 T} \mathbb{E}\left[|\lambda(0) - \lambda|^2\right] + \frac{2\eta_2}{(1 - \gamma)^2}.
\]

By taking \(\lambda = \frac{2}{(1 - \gamma)\xi}\) when \(\sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) \geq 0\); otherwise \(\lambda = 0\), we reach

\[
\mathbb{E}\left[V_r^{*}(\rho) - \frac{1}{T} \sum_{t=0}^{T-1} V_r^{(t)}(\rho)\right] + \frac{2}{(1 - \gamma)\xi} \mathbb{E}\left[b - \frac{1}{T} \sum_{t=0}^{T-1} V_g^{(t)}(\rho)\right] \\
\leq \frac{\log |A|}{(1 - \gamma)\eta T} + \frac{1}{(1 - \gamma)T} \sum_{t=0}^{T-1} \mathbb{E}\left[\text{err}_r^{(t)}(\pi^*)\right] + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \mathbb{E}\left[\text{err}_g^{(t)}(\pi^*)\right] \\
+ \beta \frac{4\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4\eta_1 W^2}{(1 - \gamma)^5 \xi^2} + \frac{2}{\eta_2(1 - \gamma)^2 \xi^2 T} + \frac{2\eta_2}{(1 - \gamma)^2}.
\]

Since \(V_r^{(t)}(\rho)\) and \(V_g^{(t)}(\rho)\) are linear functions in the occupancy measure (Altman, 1999, Chapter 10), there exists a policy \(\pi'\) such that \(V_r^{\pi'}(\rho) = \frac{1}{T} \sum_{t=0}^{T-1} V_r^{(t)}(\rho)\) and \(V_g^{\pi'}(\rho) = \frac{1}{T} \sum_{t=0}^{T-1} V_g^{(t)}(\rho)\). Hence,

\[
\mathbb{E}\left[V_r^{*}(\rho) - V_r^{\pi'}(\rho)\right] + \frac{2}{(1 - \gamma)\xi} \mathbb{E}\left[b - V_g^{\pi'}(\rho)\right] \\
\leq \frac{\log |A|}{(1 - \gamma)\eta T} + \frac{1}{(1 - \gamma)T} \sum_{t=0}^{T-1} \mathbb{E}\left[\text{err}_r^{(t)}(\pi^*)\right] + \frac{2}{(1 - \gamma)^2 \xi T} \sum_{t=0}^{T-1} \mathbb{E}\left[\text{err}_g^{(t)}(\pi^*)\right] \\
+ \beta \frac{4\eta_1 W^2}{(1 - \gamma)^3} + \beta \frac{4\eta_1 W^2}{(1 - \gamma)^5 \xi^2} + \frac{2}{\eta_2(1 - \gamma)^2 \xi^2 T} + \frac{2\eta_2}{(1 - \gamma)^2}.
\]
Application of Lemma 4 with $2/((1 - \gamma)\xi) \geq 2\lambda^*$ yields

$$
\mathbb{E} \left[ b - V_g^{*'}(\rho) \right] + \leq \frac{\xi \log |A|}{\eta T} + \frac{\xi}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \text{err}^{(t)}(\pi^*) \right] + \frac{2}{(1 - \gamma)T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \text{err}^{(t)}(\pi^*) \right] \\
+ \beta \frac{\eta_1 W^2}{(1 - \gamma)^2} + \beta \frac{4\eta_1 W^2}{(1 - \gamma)^4 \xi} + \frac{2}{\eta_2 (1 - \gamma)\xi T} + \frac{2\eta_2 \xi}{(1 - \gamma)}.
$$

which leads to our constraint violation bound if we utilize $\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (b - V_g^{(t)}(\rho)) \right] = \mathbb{E} \left[ b - V_g^{*'}(\rho) \right]$ and taking $\eta_1 = \eta_2 = 1/\sqrt{T}$.

Proof [Proof of Theorem 26]

By Lemma 29, we only need to consider the randomness in sequences of $\hat{w}^{(t)}$ and bound $\mathbb{E} \left[ \text{err}^{(t)}(\pi^*) \right]$ for $\phi = r$ or $g$. Application of the triangle inequality yields

$$
\hat{\text{err}}^{(t)}(\pi^*) \leq \left| \mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ A_{r}^{(t)}(s, a) - (w_{r*,}^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a | s) \right] \right| + \left| \mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ (w_{r*,}^{(t)} - \hat{w}_{r}^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a | s) \right] \right| \tag{64}
$$

where $w_{r*,}^{(t)} \in \arg\min_{\|w_r\|_2 \leq W} \mathbb{E}_{r}^{(t)}(w_r; \theta^{(t)})$. We next bound each term in the right-hand side of (64), separately. For the first term,

$$
\mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ A_{r}^{(t)}(s, a) - (w_{r*,}^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a | s) \right] \\
\leq \sqrt{\mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left( A_{r}^{(t)}(s, a) - (w_{r*,}^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a | s) \right)^2} \tag{65}
$$

$$
= \sqrt{\mathbb{E}_{r}^{(t)}(w_{r*,}^{(t)}; \theta^{(t)})}.
$$

Similarly,

$$
\mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ (w_{r*,}^{(t)} - \hat{w}_{r}^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a | s) \right] \\
\leq \sqrt{\mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ ((w_{r*,}^{(t)} - \hat{w}_{r}^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a | s))^2 \right]} \tag{66}
$$

$$
= \sqrt{\|w_{r*,}^{(t)} - \hat{w}_{r}^{(t)}\|_{L^2}^{(t)}}.
$$
We let $\kappa(t) := \| (\Sigma_v)^{-1/2} \Sigma_{\nu, t} (\Sigma_v)^{-1/2} \|_2$ be the relative condition number at time $t$. Thus,
\[
\left\| w_{r,*}^{(t)} - \hat{w}_r \right\|_{\Sigma_{\nu, t}}^2 \leq \left\| (\Sigma_v)^{-1/2} \Sigma_{\nu, t} (\Sigma_v)^{-1/2} \right\| \left\| w_{r,*}^{(t)} - \hat{w}_r \right\|_{\Sigma_{\nu, t}}^2
\]
\[
\leq \frac{\kappa(t)}{1 - \gamma} \left\| w_{r,*}^{(t)} - \hat{w}_r \right\|_{\Sigma_{\nu, t}}^2 \tag{67}
\]
where we use $(1 - \gamma)\nu_0 \leq \nu_0 \nu_0 := \nu(t)$ in $(a)$, and we get $(b)$ due to that the first-order optimality condition for $w_{r,*}^{(t)}$,
\[
(w_r - w_{r,*}^{(t)})^\top \nabla_{\theta} E_r^{\nu(t)}(w_{r,*}^{(t)}; \theta(t)) \geq 0, \text{ for any } w_r \text{ satisfying } \| w_r \| \leq W.
\]

further implies that
\[
E_r^{\nu(t)}(w_r; \theta(t)) - E_r^{\nu(t)}(w_{r,*}^{(t)}; \theta(t))
\]
\[
= \mathbb{E}_{s,a \sim \nu(t)} \left[ \left( A_r(t)(s,a) - \phi_{s,a}^\top w_{r,*}^{(t)} + \phi_{s,a}^\top \hat{w}_r - \phi_{s,a} \right)^2 \right] - E_r^{\nu(t)}(w_{r,*}^{(t)}; \theta(t))
\]
\[
= 2 \left( w_{r,*}^{(t)} - w_r \right)^\top \mathbb{E}_{s,a \sim \nu(t)} \left[ \left( A_r(t)(s,a) - \phi_{s,a}^\top w_{r,*}^{(t)} \right) \phi_{s,a} \right]
\]
\[
+ \mathbb{E}_{s,a \sim \nu(t)} \left[ \left( \phi_{s,a}^\top w_{r,*}^{(t)} - \phi_{s,a} \right)^2 \right]
\]
\[
= \left( w_r - w_{r,*}^{(t)} \right)^\top \nabla_{\theta} E_r^{\nu(t)}(w_{r,*}^{(t)}; \theta(t)) + \left\| w_r - w_{r,*}^{(t)} \right\|_{\Sigma_{\nu, t}}^2
\]
\[
\geq \left\| w_r - w_{r,*}^{(t)} \right\|_{\Sigma_{\nu, t}}^2.
\]

Taking an expectation over $(67)$ from both sides yields
\[
\mathbb{E} \left[ \left\| w_{r,*}^{(t)} - \hat{w}_r \right\|_{\Sigma_{\nu, t}}^2 \right] \leq \mathbb{E} \left[ \frac{\kappa(t)}{1 - \gamma} \mathbb{E} \left[ E_r^{\nu(t)}(\hat{w}_r; \theta(t)) - E_r^{\nu(t)}(w_{r,*}^{(t)}; \theta(t)) \mid \theta(t) \right] \right]
\]
\[
\leq \mathbb{E} \left[ \frac{\kappa(t)}{1 - \gamma} \frac{2G^2}{\sigma_F(K + 1)} \right] \tag{68}
\]
\[
\leq \frac{2\kappa G^2}{\sigma_F(1 - \gamma)(K + 1)}
\]
where $(a)$ is due to the standard SGD result (Lacoste-Julien et al. 2012): for $\alpha_k = 2/(\sigma_F(k + 1))$, 
\[
E_{r, \text{est}}^{(t)} = \mathbb{E} \left[ E_r^{\nu(t)}(\hat{w}_r; \theta(t)) - E_r^{\nu(t)}(w_{r,*}^{(t)}; \theta(t)) \right] \leq \frac{2G^2}{\sigma_F(K + 1)}.
\]
and (b) follows from Assumption 21. Here, it is straightforward to check the second-order moment of stochastic gradient $G_{\diamond,k}$ using Assumption 24:

$$
\mathbb{E}\left[\|G_{\diamond,k}\|^2\right] \leq 4L_\pi^2 \left(W^2 L_\pi^2 + \frac{2}{(1-\gamma)^2}\right) := G^2.
$$

Substitution of (66) and (68) into the right-hand side of (64) yields an upper bound on $\mathbb{E}[\text{err}_r(t)(\pi^*)]$. By the same reasoning, we can establish a similar bound on $\mathbb{E}[\text{err}_g(t)(\pi^*)]$. Finally, application of these upper bounds to Lemma 29 leads to our desired results.

Appendix F. Proof of Theorem 27

By $\|\phi_{s,a}\| \leq B$, for the log-linear policy class, $\log \pi_\theta(a \mid s)$ is $\beta$-smooth with $\beta = B^2$. By Lemma 29, we only need to consider the randomness in sequences of $\hat{\omega}^{(t)}$ and the error bounds for $\mathbb{E}[\text{err}_r^{(t)}(\pi^*)]$ and $\mathbb{E}[\text{err}_g^{(t)}(\pi^*)]$. We first use (64) and consider the following cases. By (30) and $A_r^{(t)}(s,a) = Q_r^{(t)}(s,a) - \mathbb{E}_{a' \sim \pi(t)\cdot \mid \cdot}Q_r^{(t)}(s,a')$,

\begin{align*}
\mathbb{E}_{s \sim d_p^t} \mathbb{E}_{a \sim \pi^\ast(\cdot \mid s)} & \left[ A_r^{(t)}(s,a) - (w_{r,\ast}^{(t)})^\top \nabla_{\theta} \log \pi^{(t)}(a \mid s) \right] \\
& = \mathbb{E}_{s \sim d_p^t} \mathbb{E}_{a \sim \pi^\ast(\cdot \mid s)} \left[ Q_r^{(t)}(s,a) - \phi_{s,a}^\top w_{r,\ast}^{(t)} \right] \\
& \quad - \mathbb{E}_{s \sim d_p^t} \mathbb{E}_{a' \sim \pi(t)(\cdot \mid s)} \left[ Q_r^{(t)}(s,a') - \phi_{s,a'}^\top w_{r,\ast}^{(t)} \right] \\
& \leq \sqrt{\mathbb{E}_{s \sim d_p^t} \mathbb{E}_{a \sim \pi^\ast(\cdot \mid s)} \left( Q_r^{(t)}(s,a) - \phi_{s,a}^\top w_{r,\ast}^{(t)} \right)^2} + \sqrt{\mathbb{E}_{s \sim d_p^t} \mathbb{E}_{a' \sim \pi(t)(\cdot \mid s)} \left( Q_r^{(t)}(s,a') - \phi_{s,a'}^\top w_{r,\ast}^{(t)} \right)^2} \\
& \leq 2 \sqrt{|A|} \mathbb{E}_{s \sim d_p^t} \mathbb{E}_{a \sim \text{Unif}_A} \left( Q_r^{(t)}(s,a) - \phi_{s,a}^\top w_{r,\ast}^{(t)} \right)^2 \\
& = 2 \sqrt{|A|} \mathbb{E}_r^{(t)}(w_{r,\ast}^{(t)}; \theta^{(t)}).
\end{align*}

(69)
Similarly,
\[
\mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ (w_{r,s}^{(t)} - \hat{w}_r^{(t)})^\top \nabla_\theta \log \pi^{(t)}(a | s) \right] \\
= \mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a \sim \pi^*(\cdot | s)} \left[ (w_{r,s}^{(t)} - \hat{w}_r^{(t)})^\top \phi_{s,a} \right] \\
- \mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a' \sim \pi^{(t)}(\cdot | s)} \left[ (w_{r,s}^{(t)} - \hat{w}_r^{(t)})^\top \phi_{s,a'} \right] \\
\leq 2 \sqrt{|A|} \mathbb{E}_{s \sim d^*_p} \mathbb{E}_{a \sim \text{Unif}_A} \left[ \frac{1}{|A|} \left( (w_{r,s}^{(t)} - \hat{w}_r^{(t)})^\top \phi_{s,a} \right)^2 \right] \\
= 2 \sqrt{|A|} \left\| w_{r,s}^{(t)} - \hat{w}_r^{(t)} \right\|_{\Sigma_{\nu}}^2
\]

where \( \Sigma_{\nu} := \mathbb{E}_{(s,a) \sim \nu} \left[ \phi_{s,a} \phi_{s,a}^\top \right] \). By the definition of \( \kappa \),
\[
\left\| w_{r,s}^{(t)} - \hat{w}_r^{(t)} \right\|_{\Sigma_{\nu}}^2 \leq \kappa \left\| w_{r,s}^{(t)} - \hat{w}_r^{(t)} \right\|_{\Sigma_{\nu_0}}^2 \leq \frac{\kappa}{1 - \gamma} \left\| w_{r,s}^{(t)} - \hat{w}_r^{(t)} \right\|_{\Sigma_{\nu}}^2
\]

where we use \((1 - \gamma)v_0 \leq \nu_0^{\pi^{(t)}} := \nu^{(t)}\) in the second inequality. We note that \( w_{r,s}^{(t)} \in \arg\min_{\|w_r\| \leq W} \mathcal{E}_r^{\nu^{(t)}}(w_r; \theta^{(t)}) \). Application of the first-order optimality condition for \( w_{r,s}^{(t)} \) yields
\[
(w_r - w_{r,s}^{(t)})^\top \nabla_\theta \mathcal{E}_r^{\nu^{(t)}}(w_{r,s}^{(t)}; \theta^{(t)}) \geq 0, \text{ for any } w_r \text{ satisfying } \|w_r\| \leq W.
\]

Thus,
\[
\mathcal{E}_r^{\nu^{(t)}}(w_r; \theta^{(t)}) - \mathcal{E}_r^{\nu^{(t)}}(w_{r,s}^{(t)}; \theta^{(t)}) \\
= \mathbb{E}_{s,a \sim \nu^{(t)}} \left[ \left( Q^{(t)}(s, a) - \phi_{s,a}^\top w_{r,s}^{(t)} + \phi_{s,a}^\top w_{r,s}^{(t)} - \phi_{s,a}^\top w_r \right)^2 \right] - \mathcal{E}_r^{\nu^{(t)}}(w_{r,s}^{(t)}; \theta^{(t)}) \\
= 2 (w_{r,s}^{(t)} - w_r)^\top \mathbb{E}_{s,a \sim \nu^{(t)}} \left[ \left( Q^{(t)}(s, a) - \phi_{s,a}^\top w_{r,s}^{(t)} \right) \phi_{s,a} \right] \\
+ \mathbb{E}_{s,a \sim \nu^{(t)}} \left[ \left( \phi_{s,a}^\top w_{r,s}^{(t)} - \phi_{s,a}^\top w_r \right)^2 \right] \\
= (w_r - w_{r,s}^{(t)})^\top \nabla_\theta \mathcal{E}_r^{\nu^{(t)}}(w_{r,s}^{(t)}; \theta^{(t)}) + \left\| w_r - w_{r,s}^{(t)} \right\|_{\Sigma_{\nu^{(t)}}}^2 \\
\geq \left\| w_r - w_{r,s}^{(t)} \right\|_{\Sigma_{\nu^{(t)}}}^2.
\]

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Taking \( w_r = \hat{w}_r(t) \) in the inequality above and combining it with (71) and (70) yield

\[
\mathbb{E}_{s \sim d^r_s} \mathbb{E}_{a \sim \pi^*_s(\cdot | s)} \left[ (w_{r,*} - \hat{w}_r(t)) \nabla_{\theta} \log \pi(t)(a | s) \right] \\
\leq 2 \sqrt{\frac{|A| \kappa}{1 - \gamma} \left( \mathcal{E}_r^{\mu_r(t)}(\hat{w}_r(t); \theta(t)) - \mathcal{E}_r^{\mu_r(t)}(w_{r,*}; \theta(t)) \right)}.
\]

(72)

We now substitute (69) and (72) into the right-hand side of (64),

\[
\mathbb{E} \left[ \text{err}_r^{(t)}(\pi^*) \right] \leq 2 \sqrt{|A| \mathbb{E} \left[ \mathcal{E}_r^{\mu_r(t)}(w_{r,*}; \theta(t)) \right]} + 2 \sqrt{\frac{|A| \kappa}{1 - \gamma} \mathbb{E} \left[ \mathcal{E}_r^{\mu_r(t)}(\hat{w}_r(t); \theta(t)) - \mathcal{E}_r^{\mu_r(t)}(w_{r,*}; \theta(t)) \right]} \\
\leq 2 \sqrt{|A| \mathbb{E} \left[ \mathcal{E}_r^{\mu_r(t)}(w_{r,*}; \theta(t)) \right]} + 2 \sqrt{\frac{|A| \kappa}{1 - \gamma} \frac{2G^2}{\sigma_F(K + 1)}},
\]

where the second inequality is due to the standard SGD result (Lacoste-Julien et al., 2012): for \( \alpha_k = 2/(\sigma_F(k + 1)) \),

\[
\mathcal{E}_r^{(t)} = \mathbb{E} \left[ \mathcal{E}_r^{\mu_r(t)}(\hat{w}_r(t); \theta(t)) - \mathcal{E}_r^{\mu_r(t)}(w_{r,*}; \theta(t)) \right] \leq \frac{2G^2}{\sigma_F(K + 1)}.
\]

By the same reasoning, we can find a similar bound on \( \mathbb{E} \left[ \text{err}_g^{(t)}(\pi^*) \right] \). Finally, our desired results follow by applying Assumption 13 and Lemma 29.
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