BORDISM OF CONSTRAINED MORSE FUNCTIONS

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Abstract. We call a Morse function $f$ on a closed manifold $k$-constrained if neither $f$ nor $-f$ has critical points of indefinite Morse index $< k$. In this paper we study bordism groups of $k$-constrained Morse functions, and thus interpolate between the case $k = 1$ of bordism groups of Morse functions (computed by Ikegami) and the case $k \gg 1$ of bordism groups of special generic functions (computed by Saeki).

We employ Levine’s elimination of cusps, Stein factorization, the two-index theorem of Hatcher-Wagoner, and a handle extension theorem for fold maps due to Gay-Kirby to show that the notion constrained bordism is strongly related to so-called connective bordism. As an application of our results we show that the oriented bordism group of constrained Morse functions detects exotic Kervaire spheres in certain dimensions.

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1. Introduction

Bordism theory for differentiable maps has been a central issue in the field of global singularity theory ever since it was initiated in the mid-1950s by René Thom [31]. Thom computed bordism groups of embedded manifolds, and his homotopy theoretic approach relies crucially on the Pontrjagin-Thom construction. Later, Rimányi and Szűcs [25] adopted this concept to study bordism groups of smooth
maps with certain prescribed types of singularities, where they considered maps of $m$-manifolds into a fixed $n$-manifold in the case of positive codimension, $n - m > 0$. Kalmár [13] found an analogous construction for smooth maps with prescribed singular fibers in the case of negative codimension.

More explicit methods of geometric topology have recently been applied successfully to compute bordism groups of smooth maps with concrete singularities of the mildest types. For instance, bordism groups of Morse functions, which were originally introduced by Ikegami and Saeki in [12], have been determined entirely by Ikegami [11] by using Levine’s cusp elimination technique [19] and the Kervaire semi-characteristic [14]. Furthermore, employing the technique of Stein factorization [2] as well as Cerf’s pseudo-isotopy theorem [3], Saeki [27] showed that oriented bordism groups of so-called special generic functions, i.e., Morse functions having only minima and maxima as critical points, are isomorphic to groups of homotopy spheres [15]. More generally, Sadykov [26] combined the Pontrjagin-Thom construction with Smale-Hirsch theory [9] to express bordism groups of special generic maps in terms of stable homotopy theory.

In this paper, we develop a geometric-topological approach to study bordism groups of Morse functions whose critical points are subject to the following type of index constraints. For a given integer $k \geq 1$ we call a Morse function on a closed $n$-manifold $k$-constrained if all indefinite Morse indices of its critical points are contained in the interval $\{k, \ldots, n-k\}$. Thus, the notion of a $k$-constrained Morse function interpolates between ordinary Morse functions ($k = 1$) and special generic functions ($k > n/2$). From the viewpoint of Morse theory [22] we make the fundamental observation that a closed manifold of dimension $n \neq 4$ admits a $k$-constrained Morse function if and only if it is $(k - 1)$-connected (where the case $n = 3$ relies on Perelman’s solution to the smooth Poincaré conjecture). This observation suggests that bordism groups of constrained Morse functions should be related strongly to so-called connective bordism groups (see Section 2), which will in fact be manifest in our Theorem [14]. In the context of a generalization of the Madsen-Weiss theorem, Perlmutter [24] has recently imposed Morse index constraints of the above form on Morse functions defined on morphisms of the bordism category.

The central notion of $k$-constrained bordism (see Definition 3.2) involves so-called fold maps of bordisms into the plane. By definition, a fold map on a manifold of dimension $n + 1$ is a smooth map all of whose singular points are determined by map germs of the form

$$(t, x_1, \ldots, x_n) \mapsto (t, -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2).$$

Thus, fold maps into the plane can locally be thought of as one-parameter families of Morse functions, and the absolute index $\max\{i, n-i\}$ of a fold point with the above map germ turns out to be a locally constant invariant of singular points. Depending on whether the absolute index of a component of the singular locus is Therefore, index constraints imposed on Morse functions induce constraints on the absolute index of fold maps in a natural way. In generalization of the notion of (in)definite Morse critical points, fold points of index $n$ are called definite, and indefinite otherwise. In Definition 3.2, we will introduce the correct notion of bordism between constrained Morse functions.
Kitazawa’s construction of so-called round fold maps \cite[p. 339]{16} implies that total spaces of fiber bundles over spheres with fiber a twisted sphere admit fold maps into the plane with connected indefinite fold locus. The general existence problem for fold maps in the presence of index constraints has been posed by Saeki in Problem 5.13 of \cite[p. 200]{28}. Technical difficulties in approaching this problem arise from the fact that Eliashberg’s \(h\)-principle \cite{6} cannot be used when constructing fold maps which are exposed to index constraints. The results of this paper can be considered as a partial solution to Saeki’s problem for the case of fold maps into \(\mathbb{R}^2\) that are subject to our type of index constraints.

In the following discussion of our main results the focus lies on oriented bordism groups; unoriented versions of the results hold in an analogous way, and some details are pointed out in Remark \ref{remark:unoriented} and Remark \ref{remark:unoriented2}.

Besides the oriented \(n\)-dimensional bordism group of \(k\)-constrained Morse functions, which will be denoted by \(M^n_k\) (see Definition \ref{definition:bordism}), our two main results below involve the oriented \(k\)-connective \(n\)-dimensional bordism group \(C^n_k\) as reviewed in Section \ref{section:bordism} as well as the group \(G^n_k\) of oriented \(k\)-constrained generic \(n\)-dimensional bordism (see Definition \ref{definition:connective}). These two bordism groups both interpolate between the smooth oriented bordism group and the group of homotopy spheres (see Remark \ref{remark:interpolation}).

In our first main result we extend Saeki’s result \cite{27} on bordism groups of special generic functions to constrained Morse functions by relating groups of constrained generic bordism to connective bordism groups as follows.

**Theorem 1.1.** Let \(n \geq 6\) and \(1 \leq k < n\) be integers. Then, there exist homomorphisms as follows:

\[(i) \quad \varepsilon^n_k : C^n_k \to G^n_k, \quad [M^n] \mapsto [f], \quad \text{where} \ f : M^n \to \mathbb{R} \ \text{denotes an arbitrarily chosen} \ k\text{-constrained Morse function, and} \]

\[(ii) \quad \delta^n_k : G^n_k \to C^n_{k-1}, \quad [f : M^n \to \mathbb{R}] \mapsto \left[\sharp(M^n)\right], \quad \text{where} \ \sharp(M^n) \ \text{denotes the oriented connected sum of the connected components of} \ M^n. \ (We \ use \ the \ convention \ that} \ \sharp(\emptyset) = S^n,)\]

Moreover, for \(1 < k < n\), the natural homomorphism \(C^n_k \to C^n_{k-1}, \ [M^n] \mapsto [M^n]\), factors as the composition \(\delta^n_k \circ \varepsilon^n_k\), and the natural homomorphism \(G^n_k \to G^n_{k-1}, \ [f : M^n \to \mathbb{R}] \mapsto [f]\), factors as the composition \(\varepsilon^n_{k-1} \circ \delta^n_k\).

The two-index theorem of Hatcher and Wagoner \cite{8} (see Section \ref{section:two-index}) will serve as an essential tool for showing that the homomorphism of part \((i)\) is well-defined in the case \(1 < k < n/2\) (see the proof of Proposition \ref{proposition:two-index}). Furthermore, we will exploit a handle extension theorem for constrained Morse functions that has recently been established by Gay and Kirby \cite{7} in the context of symplectic geometry (see Section \ref{section:handle_extension}). In order to show that the homomorphism of part \((ii)\) is well-defined, we use Stein factorization for generic maps into the plane which are subject to certain fold index constraints (see Section \ref{section:stein_factorization}).

Our second main result (see Theorem \ref{theorem:main_result} below) reveals that bordism groups of constrained Morse functions have a structure similar to that of bordism groups of Morse functions \cite{11}, whereas a somewhat surprising phenomenon arises in dimensions of the form \(n \equiv 3 \mod 4\) (see parts \((iii)\) and \((iv)\)). Namely, the size of the group \(M^n_k\) is governed by an integer \(\kappa_{(n+1)/4}\) that measures the existence of closed \(k\)-constrained bordisms of dimension \(n+1\) with odd Euler characteristic (see...
Theorem 1.2. Let $n \geq 4$ and $1 < k \leq n/2$ be integers. The oriented $n$-dimensional bordism group of $k$-constrained Morse functions $\mathcal{M}_k^n$ fits into a short exact sequence of abelian groups

$$0 \to A_k^n \xrightarrow{\alpha_k^n} \mathcal{M}_k^n \xrightarrow{\beta_k^n} G_k^n \oplus \mathbb{Z}^{[n/2]-k} \to 0,$$

where the homomorphisms $\alpha_k^n$ and $\beta_k^n$ are defined in Lemma 5.3 and Lemma 5.1, respectively. We have either $A_k^n = 0$ or $A_k^n = \mathbb{Z}/2$, depending on the following cases:

(i) If $n$ is even, then $A_k^n = 0$, so $\beta_k^n$ is an isomorphism.

(ii) If $n \equiv 1 \mod 4$, then $A_k^n = \mathbb{Z}/2$, and $\alpha_k^n$ admits a splitting (see Lemma 5.5).

(iii) If $n \equiv 3 \mod 4$ and $k \leq \kappa(n+1)/4$, then $A_k^n = 0$, so $\beta_k^n$ is an isomorphism.

(iv) If $n \equiv 3 \mod 4$ and $k > \kappa(n+1)/4$, then $A_k^n = \mathbb{Z}/2$.

Furthermore, the constants $\kappa(n+1)/4$ and $\gamma(n+1)/4$ (see Definition 5.6 and Definition 2.3, respectively) are related by $\kappa(n+1)/4 \leq \gamma(n+1)/4$. As pointed out in Remark 3.6 of [27, p. 296], index constraints for fold singularities seem to be important for understanding global differential topological phenomena like smooth structures on manifolds. As a new instance of this principle we apply our results in Section 6 to show how bordism groups of constrained Morse functions can detect individual exotic smooth structures on spheres. More specifically, Theorem 6.2 states that, in infinitely many dimensions of the form $n \equiv 1 \mod 4$, the exotic Kervaire $n$-sphere can be characterized among all exotic $n$-spheres by the property that it admits a $(n-1)/2$-constrained Morse function representing $0 \in \mathcal{M}_k^n$. Results of this type are relevant in the context of a concrete positive TFT which has recently been constructed by Banagl (see Section 10 of [1]), and in [33] we use them to compute Banagl’s aggregate invariant of various exotic spheres.

Notation. All manifolds and maps considered in this paper are differentiable of class $C^\infty$. If $M^n$ is a manifold with boundary, then we write $\text{int} M^n = M^n \setminus \partial M^n$ for its interior. For an oriented manifold $M^n$ the manifold equipped with the opposite orientation will be denoted by $-M^n$. Writing $||x||^2 := x_1^2 + \cdots + x_n^2$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote the unit $n$-disc of radius $r$ by $D_r^n = \{x \in \mathbb{R}^n; ||x|| \leq r\}$. The symbol $\cong$ will either mean orientation preserving diffeomorphism of manifolds or isomorphism of groups.

Acknowledgements. Many parts of the present paper originate from the author’s Heidelberg PhD thesis, and the author would like to express his deep gratitude to his supervisor Professor Markus Banagl for inspiring guidance during the creation of this work. Moreover, the author would like to thank Professor Osamu Saeki for helpful discussions.

The author is grateful to the German National Merit Foundation (Studienstiftung des deutschen Volkes) for financial support. Moreover, the author has been supported by JSPS KAKENHI Grant Number JP17H06128.
2. Connective Bordism

Theorem 1.1 relates constrained generic bordism groups (see Definition 3.1) to connective bordism groups defined as follows (compare [30, Example 17, p. 51]).

**Definition 2.1.** Fix integers \( n \geq 2 \) and \( 1 \leq k < n \). Let \( M^n \) and \( N^n \) be non-empty \( k \)-connected oriented closed \( n \)-manifolds. An oriented \( k \)-connective bordism from \( M^n \) to \( N^n \) is a \( k \)-connected oriented compact manifold \( W^{n+1} \) with boundary \( \partial W^{n+1} = M^n \sqcup N^n \).

The oriented \( k \)-connective \( n \)-bordism group \( C^n_k \) is the set of equivalence classes \([M^n]\) of non-empty \( k \)-connected oriented closed \( n \)-manifolds \( M^n \) subject to the equivalence relation of oriented \( k \)-connective bordism.

It can be shown that \( C^n_k \) is for any \( 1 \leq k < n \) an abelian group with group law induced by oriented connected sum, \([M^n] + [N^n] := [\sharp(M^n \sqcup N^n)]\), identity element represented by the standard sphere \( S^n \), and inverses induced by reversing the orientation, \(-[M^n] = [-M^n]\). Moreover, using the characterization of \( h \)-cobordisms given in [22, p. 108], one can show that \( C^n_k \) coincides for \( k > (n-1)/2 \) with the group of homotopy spheres \( \Theta_n \) as defined in [15].

**Proposition 2.2.** Let \( 1 < k \leq (n-1)/2 \). The natural homomorphism \( C^n_k \rightarrow C^n_{k-1} \), \([M^n] \mapsto [M^n]\), is injective for \( k \equiv 3, 5, 6, 7 \mod 8 \).

**Proof.** If \( M^n \) represents \( 0 \in C^n_{k-1} \), then there exists an oriented \((k-1)\)-connected compact manifold \( W^{n+1} \) with boundary \( \partial W^{n+1} = M^n \). Note that any triangulation of \( W^{n+1} \) is \((k-1)\)-parallelizable, that is, \( TW \) is trivial over the \((k-1)\)-skeleton (see [21, Section 5, p. 49]). The obstruction for being \( k \)-parallelizable vanishes since \( \pi_{k-1}(SO(n)) = 0 \) for \( k \equiv 3, 5, 6, 7 \mod 8 \) (see the proof of [15, Theorem 3.1, p. 508]). Therefore, by [21, Theorem 3, p. 49] \( W \) can be made \( k \)-connected by a finite sequence of surgeries without changing \( M^n = \partial W^{n+1} \). Hence, if \( M^n \) happens to be \( k \)-connected, then \( M^n \) represents \( 0 \in C^n_k \). \( \square \)

Poincaré duality implies that orientable closed manifolds with odd Euler characteristic can only exist in dimensions which are a multiple of 4. For instance, \( CP^{2i} \) is for any integer \( i \geq 1 \) a simply connected closed \( 4i \)-manifold with odd Euler characteristic. We define a sequence \( \gamma_1, \gamma_2, \ldots \) of positive integers as follows (compare Problem 2.6 in [5, p. 151]).

**Definition 2.3.** For every integer \( i \geq 1 \) let \( \gamma_i \) be the greatest integer \( k \geq 1 \) for which there exists a \( k \)-connected closed manifold \( V^{4i} \) with odd Euler characteristic (or, equivalently, odd signature).

By an argument analogous to the proof of Proposition 2.2 we can show that \( \gamma_i \neq 2, 4, 5, 6 \mod 8 \) for all \( i \geq 1 \).

**Example 2.4.** Note that \( \gamma_i < 2i \) because \( 2i \)-connected closed \( 4i \)-manifolds are homotopy spheres. For odd \( i \), we always have \( \gamma_i = 1 \) because \( 2 \)-connected closed \( 4i \)-manifolds \( V^{4i} \) are spinable, which implies that their signature is a multiple of 16 according to Ochanine’s generalization of Rochlin’s theorem (see [23, p. 133]). For even \( i \), we have \( \gamma_i \geq 3 \) because the quaternionic projective space \( HP^i \) has odd Euler characteristic. In particular, \( \gamma_3 = 3 \). When \( i = 4j \) is a multiple of 4, then \( \gamma_i \geq 7 \) because the \( j \)-fold product \( \bigcirc P^2 \times \cdots \times \bigcirc P^2 \) of the octonian projective plane...
\(\mathbb{O}P^2\) is a 7-connected closed manifold with odd Euler characteristic. In particular, \(\gamma_4 = 7\).

3. Preliminaries on Generic Maps into the Plane

In this section we collect essential techniques for constructing and studying generic maps from bordisms into the plane.

Fix an integer \(n \geq 1\). Recall that any smooth map of a manifold \(X^{n+1}\) into the plane can be approximated arbitrarily well in the Whitney \(C^\infty\) topology by a smooth map \(G: X^{n+1} \to \mathbb{R}^2\) whose singular locus \(S(G) = \{x \in X^{n+1}; \text{rank} d_xG < 2\}\) consists of those \(x \in X^{n+1}\) admitting coordinate charts centered at \(x\) and \(G(x)\), respectively, in which \(G\) has one of the following normal forms:

1. \((t, x_1, \ldots, x_n) \mapsto (t, tx_1 + x_1^2 \pm x_2^2 \pm \cdots \pm x_n^2)\), i.e., \(x\) is a cusp of \(G\).
2. \((t, x_1, \ldots, x_n) \mapsto (t, \pm x_1^2 \pm \cdots \pm x_n^2)\), i.e., \(x\) is a fold point of \(G\).

**Definition 3.1.** Let \(f: M^n \to \mathbb{R}\) and \(g: N^n \to \mathbb{R}\) be \(k\)-constrained Morse functions on oriented closed \(n\)-manifolds. An oriented \(k\)-constrained generic bordism from \(f\) to \(g\) is an oriented bordism \(W^{n+1}\) from \(M^n\) to \(-N^n\) equipped with a generic map \(G: W^{n+1} \to \mathbb{R}^2\) such that

1. there exist tubular neighborhoods \(M^n \times \{0, \varepsilon\} \subset W^{n+1}\) of \(M^n \times \{0\} = M^n \subset W^{n+1}\) and \(N^n \times (1-\varepsilon, 1] \subset W^{n+1}\) of \(N^n \times \{1\} = N^n \subset W^{n+1}\) such that
   \[G|_{M^n \times \{0, \varepsilon\}} = f \times \text{id}_{\{0, \varepsilon\}}, \quad G|_{N^n \times (1-\varepsilon, 1]} = g \times \text{id}_{\{1-\varepsilon, 1\}}.\]
2. all absolute indices of fold points of \(G\) are contained in \([\lceil n/2 \rceil, \ldots, n-k] \cup \{n\}\).

The oriented \(k\)-constrained generic \(n\)-bordism group \(\mathcal{G}_k^n\) is the set of equivalence classes \([f]\) of \(k\)-constrained Morse functions \(f: M^n \to \mathbb{R}\) on oriented closed \(n\)-manifolds subject to the equivalence relation of oriented \(k\)-constrained generic bordism.

**Definition 3.2.** Let \(f: M^n \to \mathbb{R}\) and \(g: N^n \to \mathbb{R}\) be \(k\)-constrained Morse functions on oriented closed \(n\)-manifolds. An oriented \(k\)-constrained bordism from \(f\) to \(g\) is an oriented \(k\)-constrained generic bordism from \(f\) to \(g\) without cusps.

The oriented \(n\)-bordism group of \(k\)-constrained Morse functions \(\mathcal{M}_k^n\) is the set of equivalence classes \([f]\) of \(k\)-constrained Morse functions \(f: M^n \to \mathbb{R}\) on oriented closed \(n\)-manifolds subject to the equivalence relation of oriented \(k\)-constrained bordism.

Note that \(\mathcal{G}_k^n\) and \(\mathcal{M}_k^n\) are abelian groups. In both cases, the group law is induced by disjoint union, \([f]: M^n \to \mathbb{R}\) and \([g]: N^n \to \mathbb{R}\) := \([f \sqcup g]: M^n \sqcup N^n \to \mathbb{R}\), the identity element is represented by the unique map \(f_0: \emptyset \to \mathbb{R}\), and the inverse of \([f]: M^n \to \mathbb{R}\) is given by \([-f]: -M^n \to \mathbb{R}\).

There are natural homomorphisms \(\mathcal{G}_k^n \to \mathcal{G}_k^l\) and \(\mathcal{M}_k^n \to \mathcal{M}_k^l\) whenever \(l \geq k\). Moreover, there is a natural homomorphism \(\mathcal{M}_k^n \to \mathcal{G}_k^n\) which maps the class \([f]: M^n \to \mathbb{R}\) to the class \([f]: M^n \to \mathbb{R}\) in \(\mathcal{G}_k^n\).

**Remark 3.3.** By definition, the groups \(\mathcal{G}_k^n\) and \(\mathcal{M}_k^n\) coincide for \(k > n/2\) both with the oriented bordism group of special generic functions on \(n\)-manifolds \(\overline{\Gamma}(n,1)\) as defined in [27].
Remark 3.4. Varying $k$, the group $G^k_n$ interpolates between the smooth oriented bordism group $\Omega^{SO}_n$ (an isomorphism $G^k_n \xrightarrow{\cong} \Omega^{SO}_n$ is given by $[f]: M^n \to \mathbb{R} \mapsto [M^n]$) and, by $\mathbb{Z}$, the group of homotopy spheres $\Theta_n \cong \overline{\Gamma(n,1)} = G^k_n$ ($k > n/2$).

3.1. Elimination of Cusps; Cusps and Euler Characteristic. We refer to [11] for a detailed discussion of the material presented in this section.

Recall from [11] Definition 2.2, p. 213] that there exist homomorphisms

$$\varphi_\lambda: \mathcal{M}^n \to \mathbb{Z}, \quad [f] \mapsto C_\lambda(f) - C_{n-\lambda}(f), \quad \lambda \in \{0, \ldots, n\},$$

where $C_\mu(f)$ denotes the number of critical points of $f$ of Morse index $\mu$. For any integer $1 < k \leq n/2$ we use the natural homomorphism $\mathcal{M}^n \to \mathcal{M}^1_n$ to define a homomorphism (compare [11] Definition 2.3, p. 213]

$$\Phi^k_n: \mathcal{M}^n \to \mathbb{Z}^{\lceil n/2 \rceil - k}, \quad [f] \mapsto (\varphi_{\lceil(n+3)/2\rceil}(f), \ldots, \varphi_{n-k}(f)).$$

Levine’s technique [19] for eliminating pairs of cusps of generic maps into the plane (see also [11] Section 3, pp. 215ff) can be used as in [11] to prove the following

Theorem 3.5. Suppose that $n \geq 2$. Let $G: W^{n+1} \to \mathbb{R}^2$ be an oriented $k$-constrained generic bordism from $g_0: M^n_0 \to \mathbb{R}$ to $g_1: M^n_1 \to \mathbb{R}$. Suppose that $\Phi^k_n([g_0]) = \Phi^k_n([g_1])$. Moreover, if $n$ is odd, then suppose that $G$ has an even number of cusps. Then, $[g_0] = [g_1] \in \mathcal{M}^n_k$.

Proof. We make $W^{n+1}$ connected by using the oriented connected sum operation, and modify $G$ accordingly while performing the oriented connected sum along small 2-discs centered at definite fold points of $G$. If $W^{n+1}$ is connected, then $G$ is homotopic rel $\partial W^{n+1}$ to an oriented $k$-constrained bordism from $g_0$ to $g_1$ by means of an iterated elimination of matching pairs of cusps. For details, see [11] proof of Theorem 2.7, p. 220ff].

Remark 3.6. For $k = n/2 > 1$ it can be shown that any oriented $n/2$-constrained generic bordism $G: W^{n+1} \to \mathbb{R}^2$ is already an oriented $n/2$-constrained bordism. Indeed, the map $G$ cannot have cusps because the occuring absolute fold indices $n$ and $n/2$ are not consecutive integers when $n/2 > 1$. Consequently, $\mathcal{M}^{n/2}_k = G^{n/2}_n$.

By an adaption of the proof of [11] Lemma 5.2, p. 226] we have the following

Proposition 3.7. Let $G: W^{n+1} \to \mathbb{R}^2$ be an oriented $1$-constrained generic bordism from $g_0: M^n_0 \to \mathbb{R}$ to $g_1: M^n_1 \to \mathbb{R}$. Let $c$ denote the number of cusps of $G$, and let $\nu$ denote the number of critical points of $g_0 \sqcup g_1$. Then, $\nu$ is even, and $c + \nu/2 \equiv \chi(W^{n+1}) \mod 2$, where $\chi(W^{n+1})$ denotes the Euler characteristic of $W^{n+1}$.

3.2. Two-Index Theorem. The purpose of this section is to discuss the two-index theorem of Hatcher and Waggoner [8]. This theorem is based on a parametrized implementation of the Smale trick, by which one may trade a Morse critical point of index $i$ for one of index $i + 2$ by creating a pair of critical points of successive indices $i + 1$ and $i + 2$, and then cancelling the Morse critical point of index $i$ with that of index $i + 1$. Under stronger assumptions the Smale trick has been used by Cerf in his proof of the pseudo-isotopy theorem (see [3] Lemma 0, p. 101].

Theorem 3.8. Fix integers $n \geq 5$ and $1 < k < n/2$. Suppose that $f_0, f_1: M^n \to \mathbb{R}$ are $k$-constrained Morse functions on a closed manifold $M^n$. Then, there exists an oriented $k$-constrained generic bordism $F: M^n \times [0,1] \to \mathbb{R}^2$ from $f_0$ to $f_1$. 


Proof. Without loss of generality, we may assume that $M^n$ is connected, and that $f_0(M^n) = f_1(M^n) = [0,1]$. For $i = 0,1$ let $c_0^i$ and $c_1^i$ denote the unique critical points of $f_i$ of index 0 and 1, respectively. For $i,j \in \{0,1\}$ and suitable $\varepsilon > 0$ there exist orientation preserving embeddings $\iota_j^i: D^n_{2\varepsilon} \to M$ such that $\iota_j^i(0) = c_j^i$ and

$$(f_i \circ \iota_j^i)(x) = e^j(|x|^2) := j + (-1)^j |x|^2, \quad x \in D^n_{2\varepsilon}. \quad (*)$$

Furthermore, for possibly smaller $\varepsilon > 0$, there exists an isotopy $H: [0,1] \times [0,1] \to M$ of diffeomorphisms $H_t := H(t,-): M \to M$ such that $H_0 = \text{id}_M$ and $H_1 \circ \iota_0^i = \iota_1^i$ for $j = 0,1$. Therefore, after replacing $f_1$ by $f_1 \circ H_1$, we may without loss of generality work with the assumption that $f_0 \circ \iota_0^i = f_1 \circ \iota_1^i$ for $j = 0,1$. Set $V := M \setminus (\iota_0^i(\text{int} D^n_{\varepsilon}) \cup \iota_1^i(\text{int} D^n_{\varepsilon}))$ and $V^j := U \cap V \cong S^{n-1}$ for $j = 0,1$. Then, $f_1$ restricts for $i = 0,1$ to a Morse function $g_i := f_1|\cdot: (V, V^0, V^1) \to ([\varepsilon^2, 1-\varepsilon^2], [\varepsilon^2, 1-\varepsilon^2])$ all of whose critical points have Morse index contained in the set $\{k, \ldots, n-k\}$. Choose a generic 1-parameter family $g_t$, $t \in [0,1]$, as described in [4, Theorem 9.4, pp. 188f] with regular level sets $V_0 = g_t^{-1}(\varepsilon^2)$ and $V_1 = g_t^{-1}(1-\varepsilon^2)$. Since the cardinality of the set $\{k, \ldots, n-k\}$ is at least 2 (recall that $k/n < 2/3$), we can use the two-index theorem of Hatcher and Wagoner (see [3] Chapter V, Proposition 3.5) in the form presented in [4] Section 9.9, pp. 212f to modify the family $g_t$ rel $g_0$ and $g_1$ iteratively in such a way that the resulting generic map $G: V \times [0,1] \to [\varepsilon^2, 1-\varepsilon^2] \times [0,1]$, $(x,t) \mapsto (g_t(x), t)$ is $k$-constrained.

In the following, we sketch the construction of the desired map $F$, which amounts to a careful extension of $G$ over $U^j \times [0,1]$ for $j = 0,1$. (The construction is presented in full detail in [3] Section 8.4 using [5] Appendix B.) Without loss of generality, we may assume for $t \in [0,1]$ that $g_t = g_0$ when $t$ is near 0, and that $g_t = g_1$ when $t$ is near 1. We extend $g: V \times [0,1] \to [\varepsilon^2, 1-\varepsilon^2]$ to a smooth map $\tilde{g}: \tilde{V} \times [0,1] \to \mathbb{R}$ for some open neighborhood $\tilde{V}$ of $V$ in $M$ such that, for $t \in [0,1]$, $\tilde{g}|_{\tilde{V} \times \{t\}} = f_0|_{\tilde{V}}$ when $t$ is near 0, and that $\tilde{g}|_{\tilde{V} \times \{t\}} = f_1|_{\tilde{V}}$ when $t$ is near 1. For $j = 0,1$, we define a tubular neighborhood of $V^j \times [0,1]$ in $\tilde{V} \times [0,1]$ by

$$\alpha^j: (-\delta, \delta) \times V^j \times [0,1] \to \tilde{V} \times [0,1], \quad \alpha^j(u,v,t) = (\iota_0^j(\rho(u) \cdot (\iota_0^j)^{-1}(v)), t),$$

where $\rho: (-1/2, 1/2) \to \mathbb{R}$ is given by $\rho(r) = \sqrt{r+1}$. By construction, we have $\text{pr}_{[0,1]} \circ \alpha^j = \text{pr}_{[0,1]}$ and $(\tilde{g} \circ \alpha^j)(u,v,t) = e^j(\varepsilon^2(u+1))$ when $t \in [0,1]$ is near 0 or near 1. For $j = 0,1$, we use the technique of integral curves of vector fields on manifolds with boundary (see [1] Chapter 6 §2, pp. 149f) to construct another tubular neighborhood of $V^j \times [0,1]$ in $V \times [0,1]$, say

$$\beta^j: (-\delta, \delta) \times V^j \times [0,1] \to \tilde{V} \times [0,1],$$

such that $\text{pr}_{[0,1]} \circ \beta^j = \text{pr}_{[0,1]}$ and $(\tilde{g} \circ \beta^j)(u,v,t) = e^j(\varepsilon^2(u+1))$. By adapting the proof of [1] Theorem 5.3, p. 112] we can construct for some open neighborhood $U \subset \tilde{V} \times [0,1]$ of $\partial V \times [0,1]$ an isotopy rel $U \cap (\tilde{V} \times \{0,1\})$ from the inclusion $U \hookrightarrow \tilde{V} \times [0,1]$ to an embedding $\theta: U \to \tilde{V} \times [0,1]$ such that $\theta \circ \alpha^j = \beta^j$ on a neighborhood of $V^j \times [0,1]$ in $\tilde{V} \times [0,1]$. A version of the isotopy extension theorem (see [10] Theorem 1.4, p. 180] provides an ambient isotopy rel a neighborhood of $M \times [0,1]$ in $M \times [0,1]$ from $\text{id}_{M \times [0,1]}$ to an automorphism $\Theta$ of $M \times [0,1]$ such that $\Theta \circ \alpha^j = \beta^j$ for $j = 0,1$ on a neighborhood of $V^j \times [0,1]$ in $M \times [0,1]$. Finally,
the desired map \( F : M \times [0, 1] \to \mathbb{R}^2 \) can be defined as
\[
F(x, t) = \begin{cases} 
(G \circ \Theta)(x, t), & \text{if } x \in V, \\
(e^j((x_j^0)^{-1}(x))^2), & \text{if } x \in U^j.
\end{cases}
\]

\[\square\]

Remark 3.9. In [27] Lemma 3.1, p. 291], Cerf’s pseudo-isotopy theorem [34 is used to show that the statement of Theorem 3.8 also holds for \( n \geq 6 \) and \( k > n/2 \).

3.3. Handle Extension Theorem. Using the techniques of standard Morse functions (see Theorem 3.13) and forward handles (see Remark 3.11) of Gay and Kirby [7], we prove a handle extension theorem (Theorem 3.10) for constrained Morse functions (see also [33, Chapter 7]).

Let \( W_0^n \) and \( W_1^n \) be oriented closed manifolds of dimension \( n \), and let \( W^{n+1} \) be an oriented compact \((n+1)\)-manifold with oriented boundary \( \partial W = W_0 \sqcup W_1 \). Let us consider the problem of extending a given \( k \)-constrained Morse function \( f_0 : W_0 \to \mathbb{R} \) over \( W \), i.e. to construct a \( k \)-constrained Morse function \( f_1 : W_1 \to \mathbb{R} \), and an oriented \( k \)-constrained bordism \( F : W \to \mathbb{R}^2 \) from \( f_0 \) to \( f_1 \). For \( k = 1 \) this can always be achieved by applying the techniques of Section 3.1 to a suitable generic extension of \( f_0 \) over \( W \). The “handle extension theorem” addresses this problem for \( k > 1 \), provided that \( W \) admits a handle decomposition with only handles of a single index contained in \( \{k+1, \ldots n-k\} \), and that \( f_0 \) is nicely compatible with the attaching maps of the handles of \( W \) (see Section 3.3.1). More specifically, the purpose of Section 3.3.3 is to derive the following result.

Theorem 3.10. Let \( n \geq 5 \) and \( k \in \{2, \ldots, [n/2] - 1\} \). Suppose that \( \tau : W^{n+1} \to [0, 1] \) is a Morse function with regular value sets \( W_0 = \tau^{-1}(0) \) and \( W_1 = \tau^{-1}(1) \) such that all critical points of \( \tau \) have the same Morse index \( \lambda \in \{k+1, \ldots, n-k\} \) and are contained in the slice \( \tau^{-1}(1/2) \). Furthermore, suppose that \( W_0^n \) is \((k-1)\)-connected. Then, there exists a smooth function \( \sigma : W \to \mathbb{R} \) with the following properties:

(i) \( \sigma \) restricts for every \( t \neq 1/2 \) to a \( k \)-constrained Morse function
\[\sigma_t : \tau^{-1}(t) \to \mathbb{R}.
\]

(ii) \( \sigma \) and \( \tau \) form the components of an oriented \( k \)-constrained bordism
\[(\sigma, \tau) : W \to \mathbb{R} \times [0, 1]
\]

from \( \sigma_0 \) to \( \sigma_1 \).

Remark 3.11. Following [7], the main idea behind the construction of the desired function \( \sigma \) in Theorem 3.10 can be illustrated as follows for a bordism \( W \) of dimension \( n + 1 = 2 \) with a single critical point \( c \) of index \( \lambda = 1 \) (see Figure 1).

Choose local coordinates \((x, y) \in \mathbb{R}^\lambda \times \mathbb{R}^{n+1-\lambda} \) centered at \( c \) in which \( \tau \) has the normal form \( (x, y) \mapsto -||x||^2 + ||y||^2 + 1/2 \), and consider the so-called forward \( \lambda \)-handle (see [7, Fig. 29])
\[\mathbb{R}^\lambda \times \mathbb{R}^{n+1-\lambda} \to \mathbb{R}^2, \quad (x, y) \mapsto (||x||^2 + ||y||^2 + 1/2, y_1).\]

By composition with the dihomeomorphism \( \mathbb{R}^2 \to \mathbb{R}^2, \quad (a, b) \mapsto (a - b^2, b) \), this map can be shown to be a fold map with a single fold line (namely, the \( y_1 \)-axis) of absolute index \( \max\{\lambda, n-\lambda\} \). Still working in the local chart around \( c \), the proof of Proposition 3.20 uses a bump function to modify the forward \( \lambda \)-handle outside a
Figure 1. Construction of \( \sigma \) as the height function on a 2-dimensional cobordism \( W \). The fold lines of \((\sigma, \tau)\) are marked as bold lines on \( W \). Note that \( \sigma \) restricts to an excellent Morse function on \( W_0 \) that is standard with respect to the left-hand sphere of the critical point \( c \) of \( \tau \) in \( W_0 \).

compact neighborhood of the origin in a way that allows to extend it to the desired function \((\sigma, \tau)\) on all of \( W \). By means of integral curves of a gradient-like vector field of \( \tau \) we reduce this extension problem for suitable \( t_- \in (0, 1/2) \) to the construction of a \( k \)-constrained Morse function \( \sigma_- : \tau^{-1}(t_-) \to \mathbb{R} \) which is in addition standard (see [7 Section 4]) with respect to the left-hand sphere of the critical point of \( \tau \). Then, \( \sigma \) can be taken to be a height function in Figure 1. Finally, as indicated in Figure 1 the fold lines of \((\sigma, \tau)\) are given by the suspended fold points of \( \sigma_- \), and one additional indefinite fold line of absolute index \( \max\{\lambda, n-\lambda\} \) coming from the forward \( \lambda \)-handle.

Remark 3.12. The original intention of [7] is to navigate between so-called Morse 2-functions, i.e., generic maps from a bordism into a surface. Motivated by the study of Lefschetz fibrations in the context of symplectic geometry, Gay and Kirby focus on Morse 2-functions without definite fold points, and with connected fibers. Our achievement in Theorem 3.10 is to adapt their method to the case that stronger index constraints are imposed on indefinite absolute indices of fold points (compare [7, Remark 1.6, p. 8]).

3.3.1. Standard Morse Functions. In Theorem 3.13 below we recall a result of Gay and Kirby (see [7, Theorem 4.2]) on the existence of so-called indefinite standard Morse functions. For this purpose, let \( Y_0^{n-1} \) and \( Y_1^{n-1} \) denote nonempty closed manifolds of dimension \( n-1 \geq 1 \), and let \( Y^n \) be a connected compact \( n \)-manifold with boundary \( \partial Y = Y_0 \sqcup Y_1 \). For \( i = 1, \ldots, N \) let

\[
\phi_i : L^d_i \times \text{int} \, D^{n-d}_\varepsilon \to Y \setminus \partial Y, \quad \varepsilon > 0,
\]

be pairwise disjoint embeddings, where \( L^d_i \) are closed manifolds of dimension \( d < n/2 \). Fix real numbers \( 0 < z_1 < \cdots < z_N < 1 \).
Theorem 3.13. There exists a Morse function \( g: Y^n \to [0,1] \) with regular level sets \( Y_0 = f^{-1}(0) \) and \( Y_1 = f^{-1}(1) \), and with the following properties:

(i) \( g \) is standard with respect to the pairs \((\phi_i, z_i)\), i.e., there exists \( \varepsilon' \in (0, \varepsilon) \) such that for \( i = 1, \ldots, N \),

\[
g(\phi_i(u, v)) = v_1 + z_i, \quad (u, v) \in L^d \times \text{int } D^{n-d}_{\varepsilon' - d}.
\]

(ii) \( g \) is indefinite, i.e., \( g \) has neither critical points of index 0 nor of index \( n \).

(iii) Every critical point \( c \) of \( g \) of index \( \leq d \) satisfies \( g(c) < z_1 \), whereas every critical point \( c \) of \( g \) of index \( > d \) satisfies \( g(c) > z_N \).

Remark 3.14. By property (i) of Theorem 3.13, the submanifold \( L_i = \phi_i(L_i \times 0) \subset Y \) lies in the fiber \( g^{-1}(z_i) \) of \( g \). Furthermore, the framing of \( L_i \) in \( Y \) induced by \( \phi_i \) is nicely compatible with \( g \) in such a way that \( g \) has no critical points in \( \phi_i(L_i \times \text{int } D^{n-d}_{\varepsilon' - d}) \).

The following corollary is concerned with the existence of constrained Morse functions that are standard with respect to prescribed pairs \((\phi_i, z_i)\).

Corollary 3.15. Let \( n \geq 5 \) and \( k \in \{2, \ldots, [n/2] - 1\} \). Let \( M^n \) be a \((k-1)\)-connected closed \( n \)-manifold. Suppose that \( L^d_1, \ldots, L^d_k \) are closed manifolds of dimension \( d \in \{k, \ldots, [n/2] - 1\} \), and that, for some \( \varepsilon > 0 \), there exist embeddings \( \phi_i: L^d_i \times \text{int } D^{n-d}_{\varepsilon' - d} \to M^n \), \( i = 1, \ldots, N \), whose images have pairwise disjoint closures. Furthermore, let \( 0 < z_1 < \cdots < z_N < 1 \) be real numbers. Then, \( M^n \) admits a \( k \)-constrained Morse function \( f: M^n \to \mathbb{R} \) such that, for some \( C > 0 \),

\[
f(\phi_i(u, v)) = C \cdot u_1 + z_i, \quad (u, v) \in L^d_i \times \text{int } D^{n-d}_{\varepsilon' - d}, \quad i = 1, \ldots, N.
\]

Proof. Choose disjoint embeddings \( \iota_0, \iota_1: D^n \to M^n \) which are also disjoint to the closures of the images of the embeddings \( \phi_i \). We apply Theorem 3.13 to the compact \( n \)-manifold \( Y^n = M^n \setminus (\iota_0(\text{int } D^n) \cup \iota_1(\text{int } D^n)) \) with boundary components \( Y_0^{-1} = \iota_0(S^{n-1}) \) and \( Y_1^{-1} = \iota_1(S^{n-1}) \) to obtain a Morse function \( g: Y^n \to [0,1] \) with regular level sets \( Y_0 = g^{-1}(0) \) and \( Y_1 = g^{-1}(1) \) satisfying properties (i) to (iii). By construction, \( Y^n \), \( Y_0^{-1} \) and \( Y_1^{-1} \) are simply connected, and we have

\[
H_j(Y_0^{-1}; \mathbb{Z}) = H_j(Y_1^{-1}; \mathbb{Z}) = 0, \quad j = 1, \ldots, k - 1.
\]

Hence, in view of properties (ii) and (iii), we can use standard arguments of Morse theory (including Theorem 8.1, page 100) as well as the techniques used in the proof of Theorem 7.8, p. 97) to eliminate all critical points of \( g \) of indices \( 1, \ldots, k - 1 \) by a homotopy with compact support in \( g^{-1}((0, z_1)) \), and all critical points of \( g \) of indices \( n - k + 1, \ldots, n - 1 \) by a homotopy with compact support in \( g^{-1}((z_N, 1)) \). Therefore, we may replace property (ii) by the stronger assumption that all indices of critical points of \( g \) are contained in the set \( \{k, \ldots, n - k\} \).

Set \( C = \varepsilon'/\varepsilon \in (0, 1) \). By means of the isotopy extension lemma [10] Theorem 1.3, p. 180] we construct a diffeomorphism \( \rho: \text{int } D^{n-d}_{2\varepsilon} \to \text{int } D^{n-d}_{\varepsilon' - d} \) such that \( \rho(v) = C \cdot v \) for \( ||v|| < \varepsilon \) and \( \rho(v) = v \) for \( ||v|| > 3\varepsilon/2 \). Let \( \Xi \) denote the automorphism of \( Y^n \) which extends the automorphisms \( \phi_i \circ (\text{id}_{L^d_i} \times \rho) \circ \phi_i^{-1} \) of \( \phi_i(L^d_i \times \text{int } D^{n-d}_{\varepsilon' - d}) \), \( i = 1, \ldots, N \), by the identity map. Then, by property (i) of \( g \),

\[
(g \circ \Xi)(\phi_i(u, v)) = C \cdot v_1 + z_i, \quad (u, v) \in L^d_i \times \text{int } D^{n-d}_{\varepsilon' - d}, \quad i = 1, \ldots, N.
\]
Finally, we use [22] Lemma 3.7, p. 26 to extend \( g \circ \Xi : Y^n \to [0, 1] \) to the desired Morse function \( f : M^n \to \mathbb{R} \) with two additional critical points, namely one at \( \iota_0(0) \) of index 0, and one at \( \iota_1(0) \) of index \( n \).

**Remark 3.16.** Corollary 3.15 will eventually be brought to bear in the proof of Theorem 3.10 (see Section 3.3.3) in the case where \( L_i = S^d \), and the embeddings \( \phi_i \) play the role of attaching maps of \((d+1)\)-handles.

### 3.3.2. Constructing Fold Maps from Local Handles into the Plane

We refer to [33] Section 7.2, pp. 178ff for complete details of the construction presented here.

Fix a pair \((m, \lambda)\) consisting of integers \( m \geq 2 \) and \( \lambda \in \{1, \ldots, m - 1\} \). Define a Morse function \( \mu : \mathbb{R}^m \to \mathbb{R} \) with a single critical point of Morse index \( \lambda \) at the origin of \( \mathbb{R}^m = \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \) by

\[
\mu : \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \to \mathbb{R}, \quad \mu(p,q) = -||p||^2 + ||q||^2.
\]

A gradient-like vector field \( v \) for \( \mu \) is given by

\[
v : \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \to \mathbb{R}^m, \quad v(p,q) = (-p,q).
\]

Note that the flow \( \eta \) of \( v \) is given by

\[
\eta : \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \times \mathbb{R} \to \mathbb{R}^m, \quad \eta(p,q,t) = (e^{-t}p, e^t q).
\]

Indeed, for any point \( x = (p,q) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \), the integral curve

\[
\eta_x : \mathbb{R} \to \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}, \quad \eta_x(t) = \eta(p,q,t) = (e^{-t}p, e^t q),
\]

satisfies \( \eta_x(0) = x \) and \( \eta_x'(t) = (-e^{-t}p, e^t q) = v(\eta_x(t)) \) for all \( t \in \mathbb{R} \).

**Definition 3.17.** Given \( \varepsilon, \delta > 0 \), we define the local \((\varepsilon, \delta)\)-handle by

\[
H^{\varepsilon}_{\delta} = \{(p,q) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} ; -\delta^2 \leq -||p||^2 + ||q||^2 \leq \delta^2, \ |\mu(p,q)| < \varepsilon \cdot \sqrt{\delta^2 + \delta^2}\}. 
\]

The following two lemmas (see [33] Proposition 7.2.4, p. 180) are essentially observed in the proof of [22] Theorem 3.12, page 30]. Note that one part of the boundary of \( H^{\varepsilon}_{\delta} \) is contained in the slice \( \mu^{-1}(-\delta^2) \), and forms a tubular neighborhood of the left-hand sphere of the unique critical point of \( \mu \).

**Lemma 3.18.** The local \((\varepsilon, \delta)\)-handle \( H^{\varepsilon}_{\delta} \) is an \( m \)-dimensional submanifold of \( \mathbb{R}^m = \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \) with boundary \( \partial H^{\varepsilon}_{\delta} = L^{\varepsilon}_{\delta} \cup R^{\varepsilon}_{\delta} \), where \( L^{\varepsilon}_{\delta} := H^{\varepsilon}_{\delta} \cap \mu^{-1}(\delta^2) \) and \( R^{\varepsilon}_{\delta} := H^{\varepsilon}_{\delta} \cap \mu^{-1}(-\delta^2) \). There are diffeomorphisms

\[
\lambda^{\varepsilon}_{\delta} : S^{\lambda-1} \times \text{int } D^{m-\lambda}_{\varepsilon} \overset{\cong}{\to} L^{\varepsilon}_{\delta}, \quad \lambda^{\varepsilon}_{\delta}(u,v) = (\sqrt{||v||^2 + \delta^2} \cdot u, v),
\]

\[
\rho^{\varepsilon}_{\delta} : \text{int } D^{\lambda}_{\varepsilon} \times S^{m-\lambda-1} \overset{\cong}{\to} R^{\varepsilon}_{\delta}, \quad \rho^{\varepsilon}_{\delta}(u,v) = (u, \sqrt{||u||^2 + \delta^2} \cdot v).
\]

Let \( Z := (\mathbb{R}^\lambda \times \{0\}) \cup (\{0\} \times \mathbb{R}^{m-\lambda}) \). The flow \( \eta \) of \( \mu \) is used in the following lemma to express \( H^{\varepsilon}_{\delta} \setminus Z \) in terms of \( L^{\varepsilon}_{\delta} \setminus Z \).

**Lemma 3.19.** Let \( \varepsilon, \delta > 0 \). We define manifolds with boundary by

\[
X^{\varepsilon}_{\delta} := \{(v,t) \in \mathbb{R}^{m-\lambda} \times \mathbb{R} ; 0 < ||v|| < \varepsilon, 0 \leq t \leq 1/2 \cdot \log(1 + \delta^2/||v||^2)\},
\]

\[
Y^{\varepsilon}_{\delta} := \{(x,t) = (p,q,t) \in L^{\varepsilon}_{\delta} \times \mathbb{R} ; q \neq 0, 0 \leq t \leq \log(||p||/||q||)\}.
\]

The diffeomorphism \( \lambda^{\varepsilon}_{\delta} \) (see Lemma 3.18) induces a diffeomorphism

\[
S^{\lambda-1} \times X^{\varepsilon}_{\delta} \overset{\cong}{\to} Y^{\varepsilon}_{\delta}, \quad (u,v,t) \mapsto (\lambda^{\varepsilon}_{\delta}(u,v), t).
\]
The flow $\eta$ of $\mu$ restricts to a diffeomorphism

$$Y_{\delta}^\varepsilon \xrightarrow{\sim} H_{\delta}^\varepsilon \setminus Z, \quad (x, t) = (p, q, t) \mapsto \eta(p, q, t) = (e^{-t}p, e^tq).$$

By composition we obtain a diffeomorphism

$$\Lambda_{\varepsilon} : S^{\lambda-1} \times X_{\delta}^\varepsilon \xrightarrow{\sim} H_{\delta}^\varepsilon \setminus Z, \quad (u, v, t) \mapsto (e^{-t} \cdot \sqrt{||v||^2 + \delta^2} \cdot u, e^t \cdot v).$$

The next result implements the technique of forward handles (see \cite{7} Fig. 29).

**Proposition 3.20.** Let $\varepsilon > 0$. If $\delta > 0$ is sufficiently small, then there exists a smooth function $\nu : H_{\delta}^\varepsilon \to \mathbb{R}$ with the following properties:

(i) The map $(\mu|_{H_{\delta}^\varepsilon}, \nu) : H_{\delta}^\varepsilon \to \mathbb{R}^2$ is a fold map whose singular locus is the fold line $S_{\delta} := \{0\} \times [-\delta, \delta] \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{n-1-\lambda}$ of absolute index $\max\{\lambda, m-1-\lambda\}$.

(ii) For $0 < |s| \leq \delta^2$ the set of critical points of the restriction $\nu|_{H_{\delta,s}^\varepsilon} : H_{\delta,s}^\varepsilon \to \mathbb{R}$ of $\nu$ to the slice $H_{\delta,s}^\varepsilon := H_{\delta}^\varepsilon \cap \mu^{-1}(s)$ is given by

$$H_{\delta,s}^\varepsilon \cap S_{\delta} = \begin{cases} \emptyset, & s < 0, \\ \{(0, \pm \sqrt{s}, 0)\} : = \{x_s^\pm\}, & s > 0. \end{cases}$$

We have $H_{\delta,-\delta}^\varepsilon = L_{\delta}^\varepsilon$, and the restriction $\nu|_{L_{\delta}^\varepsilon} : L_{\delta}^\varepsilon \to \mathbb{R}$ is the projection $(p, q) \mapsto q_1$. Moreover, if $s > 0$, then $x_s^-$ is a non-degenerate critical point of $\nu|_{H_{\delta,s}^\varepsilon}$ of Morse index $\lambda$ at level $-\sqrt{s}$, and $x_s^+$ is a non-degenerate critical point of $\nu|_{H_{\delta,s}^\varepsilon}$ of Morse index $m-\lambda-1$ at level $+\sqrt{s}$.

(iii) There exists $\varepsilon' \in (0, \varepsilon)$ such that $\nu(\eta_\varepsilon(t)) = \nu(\eta_\varepsilon(0))$ for all $(x, t) \in Y_{\delta}^\varepsilon \setminus Y_{\delta}^\varepsilon$.

**Proof.** Let $\varepsilon_0 := \varepsilon/3$. First, we construct a smooth map $\nu_\varepsilon : H_{\delta}^{\varepsilon_0} \to \mathbb{R}$ on the open subset $H_{\delta}^{\varepsilon_0} \subset H_{\delta}^\varepsilon$, and a smooth map $\nu_\varepsilon : H_{\delta}^\varepsilon \setminus Z \to \mathbb{R}$ on the open subset $H_{\delta}^\varepsilon \setminus Z \subset H_{\delta}^\varepsilon$ such that $\nu_\varepsilon$ and $\nu_\varepsilon$ agree on the intersection $H_{\delta}^{\varepsilon_0} \cap (H_{\delta}^\varepsilon \setminus Z) = H_{\delta}^{\varepsilon_0} \setminus Z$.

Let $\nu_\varepsilon$ be the smooth map given by the projection to the $(\lambda+1)$-th coordinate:

$$\nu_\varepsilon : H_{\delta}^{\varepsilon_0} \to \mathbb{R}, \quad x = (p, q) \mapsto q_1.$$

For the construction of $\nu_\varepsilon$, choose a smooth map $\xi : [0, \infty) \to \mathbb{R}$ such that $\xi([0, \infty)) \subset [0, 1]$, $\xi(r) = 1$ for $r < \varepsilon_0^2$, and $\xi(r) = 0$ for $r > (2\varepsilon_0)^2$. Then, use Lemma \ref{Lemma 3.19} to define the smooth map $\nu_\varepsilon : H_{\delta}^\varepsilon \setminus Z \to \mathbb{R}$ by requiring that

$$(\nu_\varepsilon \circ \eta)(x, t) = e^{t \cdot \xi(||x||^2)}q_1$$

for all $(x, t) = (p, q, t) \in Y_{\delta}^\varepsilon$. Since $\nu_\varepsilon$ and $\nu_\varepsilon$ agree at all $H_{\delta}^{\varepsilon_0} \setminus Z$

$$(\nu_\varepsilon \circ \eta)(x, t) = e^{t \cdot \xi(||x||^2)}q_1 = e^tq_1 = (\nu_\varepsilon \circ \eta)(x, t)$$

for all $(x, t) = (p, q, t) \in Y_{\delta}^\varepsilon$ because $||q|| < \varepsilon_0$ whenever $x = (p, q) \in L_{\delta}^{\varepsilon_0}$.

It remains to show that the glued map

$$\nu : H_{\delta}^\varepsilon \to \mathbb{R}, \quad \nu(x) = \begin{cases} \nu_\varepsilon(x), & \text{if } x \in H_{\delta}^{\varepsilon_0}, \\ \nu_\varepsilon(x), & \text{if } x \in H_{\delta}^\varepsilon \setminus Z, \end{cases}$$

satisfies the desired properties when $\delta > 0$ is sufficiently small.

(i) By construction, $(\mu|_{H_{\delta}^\varepsilon}, \nu)$ restricts on $H_{\delta}^{\varepsilon_0}$ to the forward $\lambda$-handle discussed in Remark \ref{Remark 3.11}. Let us check that $(\mu|_{H_{\delta}^\varepsilon}, \nu)$ is a submersion on $H_{\delta}^\varepsilon \setminus Z$. Since
precomposition with the diffeomorphisms $\Phi^\infty_\delta$ of Lemma 3.18 is constant in the variable $u \in S^{\lambda-1}$, we have to show that the following map is a submersion:

\[ F : X^\epsilon_\delta \to \mathbb{R}^2, \quad (v, t) \mapsto (-\delta^2 e^{-2t} + ||v||^2 (e^{2t} - e^{-2t}), e^t \xi(||v||^2) v_1). \]

Writing $r := ||v||^2$, the Jacobian of $F$ at $(v, t)$ is given by the $2 \times (m - \lambda + 1)$-matrix

\[
\begin{pmatrix}
2v_1(e^{2t} - e^{-2t}) & 2v_2(e^{2t} - e^{-2t}) & \ldots & 2v_{m-\lambda}(e^{2t} - e^{-2t}) & 2\delta^2 e^{-2t} + 2r(e^{2t} + e^{-2t}) \\
(1 + 2t\xi'(r)v_1^2)e^t\xi(r) & 2\xi(r)v_1v_2e^t\xi(r) & \ldots & 2\xi'(r)v_1v_{m-\lambda}e^t\xi(r) & e^t\xi(r)v_1v(r)
\end{pmatrix}
\]

For $i \in \{2, \ldots, m - \lambda\}$ the determinant of the $2 \times 2$-submatrix given by the first and the $i$-th column is

\[
det \begin{pmatrix}
2v_1(e^{2t} - e^{-2t}) \\
(1 + 2t\xi'(r)v_1^2)e^t\xi(r)
\end{pmatrix} 
\begin{pmatrix}
2v_i(e^{2t} - e^{-2t}) \\
2\xi'(r)v_1v_i e^t\xi(r)
\end{pmatrix} = -2v_i(e^{2t} - e^{-2t})e^t\xi(r).
\]

This determinant vanishes if and only if $t = 0$ or $v_i = 0$. Thus, the rank of the Jacobian of $F$ at $(v, t)$ remains to be investigated only in the case that $t = 0$ or $v_2 = \cdots = v_{m-\lambda} = 0$. For this purpose, we consider the $2 \times 2$-submatrix of the Jacobian of $F$ at $(v, t)$ consisting of the first and the last column, which is

\[
det \begin{pmatrix}
2v_1(e^{2t} - e^{-2t}) & 2\delta^2 e^{-2t} + 2r(e^{2t} + e^{-2t}) \\
(1 + 2t\xi'(r)v_1^2)e^t\xi(r) & e^t\xi(r)v_1v(r)
\end{pmatrix} = 2v_1^2(e^{2t} - e^{-2t})e^t\xi'(r)(r) - (2\delta^2 e^{-2t} + 2r(e^{2t} + e^{-2t}))(1 + 2t\xi'(r)v_1^2)e^t\xi'(r).
\]

If $t = 0$, then this determinant is further equal to $-(2\delta^2 + 4r)$, which is always negative. In the following, we assume that $v_2 = \cdots = v_{m-\lambda} = 0$. Then, $r = ||v||^2 = v_1^2$, and the determinant vanishes if and only if the following term vanishes

\[ r(e^{2t} - e^{-2t})(\delta^2 e^{-2t} + r(e^{2t} + e^{-2t}))(1 + 2t\xi'(r)r). \]

If $r < \varepsilon_0^2$, then $\xi'(r) = 1$ and $\xi'(r) = 0$, and $(\ast)$ reduces to

\[ r(e^{2t} - e^{-2t}) - (\delta^2 e^{-2t} + r(e^{2t} + e^{-2t})) = -(2r + \delta^2)e^{-2t} < 0. \]

In the following, we will assume that $\varepsilon_0^2 \leq r < \varepsilon^2$. Choose $\delta > 0$ so small that

\[ \log(1 + \delta^2/\varepsilon_0^2) < \min\{1/(2\varepsilon^2 \max_{s \in \mathbb{R}} |\xi'(s)|), \sinh^{-1}(1/4)\}. \]

for all $(v, t) = (v_1, 0, \ldots, 0, t) \in X^\epsilon_\delta$ with $\varepsilon_0^2 \leq r = v_1^2$ that

\[ 0 \leq t < \min\{1/(4\varepsilon^2 \max_{s \in \mathbb{R}} |\xi'(s)|), 1/2 \cdot \sinh^{-1}(1/4)\}. \]

Consequently, $1 + 2t\xi'(r)r > 1/2 > e^{2t} - e^{-2t} \geq 0$. Therefore, we obtain the following estimate for the expression $(\ast)$:

\[ (\ast) \leq r(e^{2t} - e^{-2t}) - r(1 + 2t\xi'(r)r) = r [e^{2t} - e^{-2t}] - (1 + 2t\xi'(r)r) \leq 0. \]

(ii) Let $0 < |s| \leq \delta^2$. Any critical point of $\nu^{H^\delta}_{\epsilon,s}$ is necessarily a critical point of the map $(\mu|_{H^\delta_{\epsilon,s}}, \nu)$, whose singular locus $S_{\delta}$ has been determined in part (i). If $s < 0$, then $H^\delta_{\epsilon,s} \cap S_{\delta} = \emptyset$, and the claim about $\nu^{L^\delta}_{\epsilon,s}$ holds by construction. If $s > 0$, then the points of $H^\delta_{\epsilon,s} \cap S_{\delta} = \{(0, \pm \sqrt{s}, 0)\} = \{x^\pm_s\}$ can be checked to be non-degenerate critical points of $\nu^{H^\delta}_{\epsilon,s}$, with the desired properties by a straightforward computation. For this purpose, note that $x^\pm_s$ is contained in the open subset $R^{\varepsilon_0}_{\sqrt{s}} \subset H^\delta_{\epsilon,s}$ (see Lemma 3.18). On $R^{\varepsilon_0}_{\sqrt{s}} \cap \nu$ agrees with $\nu_{<s}$, and can be studied near $x^\pm_s$ by means of

\[ \rho^{\varepsilon_0}_{\sqrt{s}} : \text{int} D^e_{\varepsilon_0} \times S^{m-\lambda-1} \to R^{\varepsilon_0}_{\sqrt{s}}, \quad \rho^{\varepsilon_0}_{\sqrt{s}}(u, v) = (u, \sqrt{|u|^2 + s \cdot v}), \]
and the inverse of the stereographic projection,
\[
\sigma_\pm: \mathbb{R}^{m-\lambda-1} \overset{\cong}{\longrightarrow} S^{m-\lambda-1} \setminus \{(\mp 1, 0, \ldots, 0)\},
\]
\[
w = (w_1, \ldots, w_{m-\lambda-1}) \mapsto \left( \frac{|w|^2 - 1}{|w|^2 + 1}, \frac{2w_1}{|w|^2 + 1}, \ldots, \frac{2w_{m-\lambda-1}}{|w|^2 + 1} \right).
\]

(iii) Let \( \varepsilon' = 2\varepsilon_0 \) and \((x, t) = (p, q, t) \in Y^\varepsilon \setminus \overline{Y^\varepsilon_1} \). Then, it follows from \( \eta_x(t) \in H^\varepsilon_0 \setminus Z \) and \( ||q||^2 > (2\varepsilon_0)^2 \) that
\[
\nu(\eta_x(t)) = (\nu_\varepsilon \circ \eta)(x, t) = e^{t \cdot \xi(||q||^2)}q_1 = q_1 = \nu(\eta_x(0)).
\]

\[\Box\]

3.3.3. \textbf{Proof of Theorem 3.10.} Without loss of generality we may assume that \( k + 1 \leq \lambda \leq [n/2] \). (In fact, it suffices to show that \( W_1 \) is \((k-1)\)-connected. By an argument similar to [22, Remark 1, p. 70] we can show that \( W_1 \) is simply connected. Then the claim follows because the effect of a \( p \)-surgery on a closed manifold of dimension \( d \geq p + 2 \) does not affect the homology groups in dimensions strictly below \( \min(p, d-p-1) \). In our case, \( p = \lambda - 1 \in \{k, \ldots, n-k-1\} \).

Let \( \xi \) be a gradient-like vector fields of \( \tau \) (see [22, Definition 3.2, p. 20]). Let \( c_1, \ldots, c_N \) be the critical points of \( \tau \). For every \( i = 1, \ldots, N \) there exist open neighborhoods \( U_i \) of \( c_i \) in \( W \) and \( V_i \) of \( 0 \in \mathbb{R}^{n+1} \), and a chart \( \psi_i: U_i \overset{\cong}{\longrightarrow} V_i \) of \( W \) with \( \psi_i(c_i) = 0 \), such that the pair \((\tau, \xi)\) corresponds via \( \psi_i \) to the pair \((\mu, \nu)\) introduced at the beginning of Section 3.3.2, i.e.,
\[
\tau \circ \psi_i^{-1} = \mu|_{V_i} + 1/2,
\]
\[
d\psi_i \circ \xi \circ \psi_i^{-1} = \nu|_{V_i}.
\]

We choose \( \varepsilon, \delta > 0 \) so small that the local \((2\varepsilon, \delta)\)-handle \( H^\varepsilon_{\delta} \) of Definition 3.17 is contained in \( V_i \) for all \( i = 1, \ldots, N \). By choosing \( \delta > 0 \) small enough, we may in addition assume that there exists a smooth map \( \nu: H^\varepsilon_{\delta} \rightarrow \mathbb{R} \) with the properties (i) to (iii) of Proposition 3.20.

We apply Corollary 3.15 to the \((k-1)\)-connected closed \( n \)-manifold \( M^n = \tau^{-1}(1/2 - \delta^2) \), the left-hand spheres \( L_i = S^{\lambda-1} \) of dimension \( d = \lambda - 1 \), some fixed real numbers \( 0 < z_1 < \cdots < z_N < 1 \), and the embeddings
\[
\phi_i: S^{\lambda-1} \times \text{int} D^n_{2\varepsilon} \overset{\lambda-1/2}{\longrightarrow} L^\varepsilon_{\delta} \overset{\psi_i^{-1}}{\longrightarrow} U_i \cap M^n \hookrightarrow M^n, \quad i \in \{1, \ldots, N\},
\]
to obtain a \( k \)-constrained Morse function \( f: M^n \rightarrow \mathbb{R} \) such that, for some \( C > 0 \),
\[
f(\phi_i(u, v)) = C \cdot v_1 + z_i, \quad (u, v) \in S^{\lambda-1} \times \text{int} D^n_{2\varepsilon}^{\lambda-1}, \quad i = 1, \ldots, N.
\]

Note that it suffices to construct the desired map \( \sigma \) in a neighborhood of \( \tau^{-1}(1/2) \) in \( W \). In fact, a smooth function \( \sigma: \tau^{-1}(1/2 - \delta^2, 1/2 + \delta^2) \rightarrow \mathbb{R} \) with the desired properties is given by
\[
\sigma(w) = \begin{cases} 
C \cdot \nu(\psi_i(w)) + z_i, & \text{if } w \in \psi_i^{-1}(H^\varepsilon_{\delta}) \text{ for some } i = 1, \ldots, N, \\
f(M \cap \Gamma_w), & \text{else},
\end{cases}
\]
where \( \Gamma_w \subset W \) denotes the image of the uniquely determined maximally extended integral curve through \( w \) with respect to \( \xi \). To show that \( \sigma \) is a well-defined smooth function, use the behavior of \( f \) on \( \psi_i^{-1}(L^\varepsilon_{\delta}) \) to that of \( \nu \) on \( L^\varepsilon_{\delta} \), and use property (iii) of \( \nu \) in Proposition 3.20. The desired properties of \( \sigma \) follow from the properties (i) and (ii) of \( \nu \) in Proposition 3.20.
This completes the proof of Theorem 3.10.

3.4. Stein Factorization. The importance of Stein factorization for the global study of singularities of smooth maps was first realized when Burlet and de Rham [2] used it as a tool to study special generic maps of 3-manifolds into the plane.

We recall the concept of Stein factorization of an arbitrary continuous map \( f: X \to Y \) between topological spaces. Define an equivalence relation \( \sim_f \) on \( X \) as follows. Two points \( x_1, x_2 \in X \) are called equivalent, \( x_1 \sim_f x_2 \), if they are mapped by \( f \) to the same point \( y := f(x_1) = f(x_2) \in Y \), and lie in the same connected component of \( f^{-1}(y) \). The quotient map \( \pi_f: X \to X/\sim_f \) gives rise to a unique set-theoretic factorization of \( f \) of the form

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\pi_f} & \searrow{\overline{f}} & \\
X/\sim_f & & 
\end{array}
\]

If we equip the quotient space \( X_f := X/\sim_f \) with the quotient topology induced by the surjective map \( \pi_f: X \to X_f \), then it follows that the maps \( \pi_f \) and \( \overline{f} \) are continuous.

In the following, both the above diagram and the quotient space \( X_f = X/\sim_f \) will be referred to as the Stein factorization of \( f \).

In generalization of [27, Lemma 2.1, p. 290], the following result clarifies the structure of Stein factorization for \( k \)-constrained generic bordisms \( W^m \to \mathbb{R}^2 \) for \( m \geq 3 \) and \( k > 1 \).

**Theorem 3.21.** Let \((W^m, M_1, M_2)\) be a smooth manifold triad of dimension \( m := \dim W \geq 3 \). Suppose that \( F: W^m \to \mathbb{R}^2 \) is an oriented \( k \)-constrained generic bordism, and that \( F \) is a stable map.

If \( k > 1 \), then the Stein factorization \( W_F = W/\sim_F \) of \( F \) can be given the structure of a compact smooth manifold of dimension 2 with corners in such a way that \( \pi_F: W \to W_F \) is a generic smooth map and \( \overline{F}: W_F \to \mathbb{R}^2 \) is an immersion. Furthermore, if \( D(F) \) denotes the union of the definite fold lines of \( F \), then the boundary of \( W_F \) decomposes as

\[
\partial W_F = \pi_F(\partial W) \cup \pi_F(D(F)),
\]

where \( \pi_F(\partial W) \cap \pi_F(D(F)) = \pi_F(\partial W \cap D(F)) \) is the set of corners of \( W_F \), and \( \pi_F \) restricts to an embedding \( D(F) \to \partial W_F \).

**Proof.** The claims follow from [17] Theorem 2.2, p. 2609]. Note that the only local neighborhoods of points in \( W_F \) that can occur are those of types \((a), (b_1), (b_3), (c_2)\) and \((d_2)\) in [17] Figure 1, p. 2610] because we have excluded fold points of absolute index \( m - 2 \). This implies that \( W_F \) is a topological 2-manifold. Finally, the desired smooth structure on \( W_F \) is induced by requiring \( \overline{F} \) to be a local diffeomorphism. □

4. Proof of Theorem 1.1

Fix integers \( n \geq 3 \) and \( 1 \leq k < n \).

Lemma 4.1 below is a straightforward generalization of [27, Lemma 3.2, p. 292], and will be used frequently. Note that Lemma 4.1 can be considered as a version...
of Theorem 3.10 for the case that all critical points of \( \tau \) have the same Morse index \( \lambda \in \{1, m - 1\} \).

**Lemma 4.1.** Let \( f : M^n \to \mathbb{R} \) be a \( k \)-constrained Morse function. Then, there exist a \( k \)-constrained Morse function \( g : \sharp(M^n) \to \mathbb{R} \) and an oriented \( k \)-constrained bordism \( G : W^{n+1} \to \mathbb{R}^2 \) from \( g \) to \( f \).

Proposition 4.2 below generalizes [27, Lemma 3.3, p. 293] by showing that the assignment \( \{ f : M^n \to \mathbb{R} \} \mapsto \{ \sharp(M^n) \} \) defines a well-defined map \( \delta_n^k : G_n^k \to C_{n-1}^k \) for any integer \( 1 < k < n \). Then, it follows that \( \delta_n^k : G_n^k \to C_{n-1}^k \) is a homomorphism because the existence of an orientation preserving diffeomorphism \( \sharp(M \sqcup N) \cong \sharp(\sharp(M) \sqcup \sharp(N)) \) implies that \( \delta_n^k([f] + [g]) = [\sharp(M \sqcup N)] = [\sharp(\sharp(M) \sqcup \sharp(N))] = \delta_n^k([f]) + \delta_n^k([g]) \) for any \( k \)-constrained Morse functions \( f : M^n \to \mathbb{R} \) and \( g : N^n \to \mathbb{R} \).

**Proposition 4.2.** Suppose that \( 1 < k < n \). Let \( f : M^n \to \mathbb{R} \) and \( g : N^n \to \mathbb{R} \) be \( k \)-constrained Morse functions which are oriented \( k \)-constrained generic bordant. Then, there exists an oriented \((k - 1)\)-connective bordism \( V^{n+1} \) from \( M^n \) to \( N^n \).

![Figure 2](image-url)  
**Figure 2.** Stein factorization \( \pi_H : U \to U_H \) of \( H : U^{n+1} \to \mathbb{R}^2 \). Exemplarily, only some 1-handle \( T_1 \) is shown explicitly. The line segment \([0, 1] \cong L \subset U_H \) is chosen as indicated by the dotted line.

**Proof.** By assumption there exists an oriented \( k \)-constrained generic bordism from \( f \sqcup -g : M^n \sqcup -N^n \to \mathbb{R} \) to \( \emptyset \to \mathbb{R} \). Lemma 4.1 implies that there exists a \( k \)-constrained Morse function \( h : \sharp(M^n \sqcup -N^n) \to \mathbb{R} \) and an oriented \( k \)-constrained generic bordism \( H : U^{n+1} \to \mathbb{R}^2 \) from \( h \) to \( \emptyset \to \mathbb{R} \). By Theorem 3.21 the Stein factorization \( U_H \) of \( H \) (see Figure 2) is a connected oriented compact 2-manifold with two corners, where we have assumed without loss of generality that \( H \) is stable. (In fact, \( H \) can always be achieved to be stable by first choosing \( h \) to have a single
critical point on each level set, and then perturbing $H$ on the interior of $U^{n+1}$.)

Then, by the classification of compact surfaces, we may think of $U_H$ as a half disc with a finite number $r \geq 0$ of $1$-handles attached, say $T_1, \ldots, T_r$. Let $L_1, \ldots, L_r \subset U^{n+1}$ be the line segments indicated in Figure 2. We choose an oriented compact submanifold $V^{n+1}$ of $U^{n+1}$ with boundary $\partial V = \sharp(M^n \sqcup -N^n) \sqcup -P^n$, where $P^n \subset U^{n+1}$ is an oriented submanifold of codimension $1$ which is constructed as the preimage $P^n = \pi_H^{-1}(L)$ of a smooth line segment $L \subset U_H$ chosen as indicated in Figure 2. Here, we have assumed that $L$ avoids images of cusps of $\pi_H$, and is transverse to $\pi_H(S(\pi_H))$. Then, we have an orientation preserving diffeomorphism $P^n \cong \sharp(P_1 \sqcup (-P_1) \sqcup \cdots \sqcup P_r \sqcup (-P_r))$, where $P_i := \pi_H^{-1}(L_i)$. By construction, the Stein factorization of the restriction $F := H|_{V'}$ is diffeomorphic to a rectangle, $V_F \cong [0,1] \times [0,1]$, in such a way that $\{0,1\} \times [0,1]$ corresponds to $\pi_F(\partial V)$, and $[0,1] \times \{0,1\}$ corresponds to $\pi_F(S)$, where $S$ denotes the definite fold locus of $F$. Let $\rho: V_F \to [0,1]$ denote the projection of $V_F \cong [0,1] \times [0,1]$ to the first factor. Then, by slightly perturbing $\rho$ on the interior of $V_F$, we can achieve that the composition $\rho \circ \pi_F: V^{n+1} \to [0,1]$ is a $k$-constrained Morse function with regular level sets $\sharp(M^n \sqcup N^n)$ and $P^n$. Consequently, $V^{n+1}$ is a $(k-1)$-connective bordism from $\sharp(M^n \sqcup N^n)$ to $P^n$, and the claim follows.

□

From now on, suppose that $n \geq 6$.

Let us show that the choice of $k$-constrained Morse functions $f: M^n \to \mathbb{R}$ on given $k$-connected oriented closed manifolds $M^n$ induces a well-defined map $\varepsilon^n_k: C^n_k \to G^n_k; [M^n] \mapsto [f: M^n \to \mathbb{R}]$. Note that $\varepsilon^n_k$ will be a homomorphism by Lemma 4.1. We distinguish between the cases that $k > n/2$, $k = n/2$ and $k < n/2$.

For $k > n/2$, note that $C^n_k = \Theta_n$, the group of homotopy spheres, and $G^n_k = \tilde{\Gamma}(n,1)$, the bordism group of special generic functions (see Remark 3.3). Then, $\varepsilon^n_k$ coincides with the homomorphism

$$\tilde{\Phi}: \Theta_n \to \tilde{\Gamma}(n,1)$$

introduced in the proof of [27, Theorem 1.1, p. 288] (compare [27, p. 294]), and is well-defined by [27, Lemma 3.1, p. 291] because $n \geq 6$.

For $k = n/2$ we have already noted in Section 3 that $C^n_{n/2} = \Theta_n$, the group of homotopy spheres. All indefinite critical points of an $n/2$-constrained Morse function $f: M^n \to \mathbb{R}$ are of index $n/2$, so there are none if $M^n$ is a homotopy sphere. Hence, the desired homomorphism $\varepsilon^n_{n/2}: C^n_{n/2} \to G^n_{n/2}$ is the composition of the above homomorphism $C^n_{n/2} = \Theta_n \xrightarrow{\tilde{\Phi}} \tilde{\Gamma}(n,1) = G^n_{n/2+1}$ and the homomorphism $G^n_{n/2+1} \to G^n_{n/2}$.

Let $k < n/2$. For $k = 1$, the map $\varepsilon^n_1$ is just the composition of the map $C^n_1 \to \Omega^{SO}_n$, $[M^n] \mapsto [M^n]$, with the inverse of the isomorphism $G^n_1 \xrightarrow{\cong} \Omega^{SO}_n$ described in Remark 3.4. For $1 < k < n/2$, evaluation of the map $\varepsilon^n_k: C^n_k \to G^n_k$ on a given element of $C^n_k$ depends a priori on the choice of a representative $M^n$, and on the choice of a $k$-constrained Morse function $f: M^n \to \mathbb{R}$. Independence of choices follows from the following result (compare [27, Lemma 3.1, p. 291]).

**Proposition 4.3.** Suppose that $1 < k < n/2$. Let $W^{n+1}$ be an oriented $(k-1)$-connective bordism from $M^n_0$ to $M^n_1$, and let $g_0: M^n_0 \to \mathbb{R}$ and $g_1: M^n_1 \to \mathbb{R}$ be $k$-constrained Morse functions. If $W^{n+1}$ is $k$-connected, then there exists an oriented $k$-constrained generic bordism $G: W^{n+1} \to \mathbb{R}^2$ from $g_0$ to $g_1$. 
Proof. By the rearrangement theorem \[22, p. 44\] we can modify an arbitrarily chosen Morse function \( f: W^{n+1} \to [-1/2,n+1+1/2] \) with regular level sets \( M^n_v = f^{-1}(-1/2) \) and \( M^n_1 = f^{-1}(n+1+1/2) \) to be self-indexing, i.e., the Morse index of every critical point \( c \) of \( f \) is \( f(c) \). Following the proof of Smale’s h-cobordism theorem \[22, Theorem 9.1, p. 107\], we then modify \( f \) in such a way that all Morse indices of critical points of \( M^k \) defined in a straightforward way by forgetting about orientations in Definition 2.1 Theorem 1.1 carry over to unoriented bordism groups. Remark 3.4, p. 293]). For \( k \) the argument of the proof of Proposition 4.2 in order to show that the epimorphism \( \Phi \) is surjective, where \( \Phi \) is defined in Remark 3.4, and the analogously defined isomorphism \( \delta \) of lemmas. Hence, \( f: M^n \to R \) is a special generic function, \( \pi(M^n) \) is a homotopy n-sphere, and the claim follows from \[27\] Lemma 3.1, p. 291] because \( n \geq 6 \).

This completes the proof of Theorem 1.1.

Remark 4.4. Unoriented versions \( \mathcal{D}^n_k \) and \( \mathcal{H}^n_k \) of the groups \( \mathcal{C}^n_k \) and \( \mathcal{G}^n_k \), can be defined in a straightforward way by forgetting about orientations in Definition 2.1 and Definition 3.1 respectively. It is easy to show that the natural epimorphisms \( \mathcal{C}^n_k \to \mathcal{D}^n_k \), \( [M^n] \to [M^n] \), has kernel \( 2\mathcal{C}^n_k \). Moreover, for \( k > 1 \) we can modify the argument of the proof of Proposition 4.2 in order to show that the epimorphism \( \mathcal{G}^n_k \to \mathcal{H}^n_k \), \( [f: M^n \to R] \to [f: M^n \to R] \), has kernel \( 2\mathcal{G}^n_k \) (compare \[27\] Remark 3.4, p. 293). For \( k = 1 \) we use the isomorphism \( \mathcal{G}^n_1 \cong \Omega^S_n \) described in Remark 3.4 and the analogously defined isomorphism \( \mathcal{H}^n_1 \cong \Omega^O_n \) to the unoriented smooth bordism group to show that the homomorphism \( \mathcal{G}^n_k \to \mathcal{H}^n_k \), \( [f: M^n \to R] \to [f: M^n \to R] \), has kernel \( 2\mathcal{G}^n_1 \). (In fact, note that the homomorphism \( \Omega^S_n \to \Omega^O_n \), \( [M^n] \to [M^n] \), has kernel \( 2\Omega^S_n \).) Hence, all the statements of Theorem 1.1 carry over to unoriented bordism groups.

5. Proof of Theorem 1.2

Fix integers \( n \geq 4 \) and \( 1 < k \leq n/2 \). We present the proof of Theorem 1.2 in a sequence of lemmas.

Lemma 5.1. The homomorphism
\[
\beta^n_k: \mathcal{M}^n_k \to \mathcal{G}^n_k \oplus \mathbb{Z}^\oplus \mathbb{Z}^\oplus [n/2]^{-k}, \quad [f: M^n \to R] \to ([f], \Phi^n_k([f])),
\]

is surjective, where \( \Phi^n_k: \mathcal{M}^n_k \to \mathbb{Z}^\oplus \mathbb{Z}^\oplus [n/2]^{-k} \) is defined in Section 3.1.
Morse functions with exactly Morse function use the same argument as in [11, p. 220] to modify \( g \) iteratively by introducing pairs of critical points of successive Morse indices in order to produce a \( k \)-constrained Morse function \( f: M^n \to \mathbb{R} \) which satisfies \( \Phi^n_k([f]) = (c_1, \ldots, c_{[n/2]-k}) \). Since \( [f] = [g] \) in \( G^n_k \) by means of a generic homotopy that realizes the sequence of births of critical point pairs, we obtain \( \beta^n_k([f: M^n \to \mathbb{R}]) = ([g], c_1, \ldots, c_{[n/2]-k}) \).

The following lemma completes the proof of part (i).

**Lemma 5.2.** If \( n \) is even, then \( \beta^n_k \) is injective.

**Proof.** Suppose that \( \beta^n_k([f]) = 0 \in G^n_k \oplus \mathbb{Z}^{{[n/2]}-k} \) for some \( [f: M^n \to \mathbb{R}] \in M^n_k \). Then, there exists an oriented \( k \)-constrained generic bordism \( G: W^{n+1} \to \mathbb{R}^2 \) from \( f \) to \( f_0 \). Since \( \Phi^n_k([f]) = 0 \), Theorem 3.5 implies that \( [f] = [f_0] = 0 \in M^n_k \). □

In order to studying the kernel of \( \beta^n_k \) when \( n \) is odd, we introduce a homomorphism \( \alpha^n_k: \mathbb{Z}/2 \to M^n_k \) as follows.

**Lemma 5.3.** Suppose that \( n \) is odd, and let \( l := (n-1)/2 \). If \( f, g: S^n \to \mathbb{R} \) are Morse functions with exactly 4 critical points whose Morse indices form the set \( \{0, l, l+1, n\} \), then \( [f] = [g] \in M^n_k \). Hence, there is a well-defined homomorphism \( \alpha^n_k: \mathbb{Z}/2 \to M^n_k \), \( \bar{T} \mapsto [f_\alpha: S^n \to \mathbb{R}] \), where \( f_\alpha: S^n \to \mathbb{R} \) denotes a fixed Morse function with the above properties.

**Proof.** By Theorem 3.8 there exists an oriented \( l \)-constrained generic bordism \( G: S^n \times [0, 1] \to \mathbb{R}^2 \) from \( f \) to \( g \). Moreover, Proposition 3.7 implies that the number of cusps of \( G \) is even. Hence, the claim that \( [f] = [g] \in M^n_k \) follows from Theorem 3.5. As \( S^n \) admits an orientation reversing automorphism, we may choose \( g = -f \), and obtain \( 2[f] = 0 \). □

**Lemma 5.4.** If \( n \) is odd, then \( \text{im} \alpha^n_k = \ker \beta^n_k \).

**Proof.** Note that \( \Phi^n_k([f_\alpha]) = 0 \in G^n_k \) and \( \Phi^n_k([f_\alpha]) = 0 \in \mathbb{Z}^{{[n/2]}-k} \), and thus \( \text{im} \alpha^n_k \subset \ker \beta^n_k \). Conversely, suppose that \( \beta^n_k([f]) = 0 \in G^n_k \oplus \mathbb{Z}^{{[n/2]}-k} \) for some \( [f: M \to \mathbb{R}] \in M^n_k \). Then, there exists an oriented \( k \)-constrained generic bordism \( G: W^{n+1} \to \mathbb{R}^2 \) from \( f \) to \( f_0 \). Moreover, by means of Theorem 3.8 we can construct an oriented \( k \)-constrained generic bordism \( G_\alpha: D^{n+1} \to \mathbb{R}^2 \) from \( f_\alpha \) to \( f_\alpha \). By Proposition 3.7, exactly one of the oriented \( k \)-constrained generic bordisms \( G \) and \( G \cup G_\alpha \), say \( G_0 \), has an even number of cusps. Hence, Theorem 3.5 implies that \( 0 = [f_0] = [f] + m[f_\alpha] \in M^n_k \) for suitable \( m \in \{0, 1\} \), and we conclude that \( \ker \beta^n_k \subset \text{im} \alpha^n_k \). □

**Lemma 5.5.** Below completes the proof of part (ii). For this purpose, recall from [11] Definition 2.5, p. 214] that there is for \( n \equiv 1 \mod 4 \) a well-defined homomorphism \( \Lambda: M^n_k \to \mathbb{Z}/2 \) which assigns to \( [f] \in M^n_k \) the element \( \Lambda([f]) = \sigma(f) \) where \( \sigma(f) = \sum_{\lambda=0}^{(n-1)/2} C_\lambda(f) \), and \( \sigma(M^n; \mathbb{Q}) \) denotes the Kervaire semi-characteristic of \( M^n \) over \( \mathbb{Q} \) (see [14]). Composition with the natural homomorphism \( M^n_k \to M^n_{2k} \) yields a homomorphism \( \Lambda^n_k: M^n_k \to \mathbb{Z}/2 \) which turns out to be a splitting of \( \alpha^n_k \) for \( n \equiv 1 \mod 4 \).

**Lemma 5.5.** For \( n \equiv 1 \mod 4 \) we have \( \Lambda^n_k \circ \alpha^n_k = \text{id}_{\mathbb{Z}/2} \).
Proof. It suffices to note that $\Lambda_k^n((f_n)) = \mathbb{I} \in \mathbb{Z}/2$, which follows from $\sigma(f_n) = 2$ and $\sigma(S^n; \mathbb{Q}) = 1$.

In Lemma 5.7 below we prove the remaining parts (iii) and (iv) by characterizing injectivity of $\alpha_k^n$ for $n \equiv 3 \mod 4$. For this purpose, we introduce the required sequence $\kappa_1, \kappa_2, \ldots$ of positive integers in Definition 5.6 below. Note that $CP^{2i}$ is for any integer $i \geq 1$ a closed $4i$-manifold with odd Euler characteristic, and any generic map $CP^{2i} \to \mathbb{R}^2$ defines an oriented 1-constrained generic bordism from $f_0: \emptyset \to \mathbb{R}$ to $f_0: \emptyset \to \mathbb{R}$.

Definition 5.6. For every integer $i \geq 1$ let $\kappa_i$ be the greatest integer $k \geq 1$ for which there exists an oriented $k$-constrained generic bordism $V^{4i} \to \mathbb{R}^2$ from $f_0: \emptyset \to \mathbb{R}$ to $f_0: \emptyset \to \mathbb{R}$ such that $V^{4i}$ has odd Euler characteristic (or, equivalently, odd signature).

Lemma 5.7. Suppose that $n \equiv 3 \mod 4$. Then, $\alpha_k^n$ is injective if and only if $k > \kappa_{(n+1)/4}$.

Proof. As in the proof of Lemma 5.4 we can construct an oriented $k$-constrained generic bordism $G_0: D^{n+1} \to \mathbb{R}^2$ from $f_0$ to $f_0$.

"$\Leftarrow$". Let $k > \kappa_{(n+1)/4}$. If we suppose that $[f_0] = 0 \in \mathcal{M}_k^n$, then there exists an oriented $k$-constrained bordism $F: W^{n+1} \to \mathbb{R}^2$ from $f_0$ to $f_0$. From Proposition 3.7 we conclude that $\chi(W)$ is even because $F$ has no cusps and $f_0$ has 4 critical points. Hence, $V^{n+1} := W^{n+1} \cup_{S^n} D^{n+1}$ is an oriented closed manifold with odd Euler characteristic. Now $F$ and $G_0$ glue to an oriented $k$-constrained generic bordism $V^{n+1} \to \mathbb{R}^2$ from $f_0$ to $f_0$, which contradicts the assumption that $k > \kappa_{(n+1)/4}$.

"$\Rightarrow$". Suppose that $k \leq \kappa_{(n+1)/4}$, and let $V^{n+1}$ be a closed manifold with odd Euler characteristic which admits an oriented $k$-constrained generic bordism $G_1: V^{n+1} \to \mathbb{R}^2$ from $f_0$ to $f_0$. Note that both $G_0$ and $G_1$ have an odd number of cusps by Proposition 5.7. Hence, the oriented $k$-constrained generic bordism $G_0 \cup G_1: D^{n+1} \cup V^{n+1} \to \mathbb{R}^2$ from $f_0$ to $f_0$ has an even number of cusps. Therefore, $[f_0] = 0 \in \mathcal{M}_k^n$ by Theorem 3.5.

Remark 5.8. We do not know if the short exact sequence in Theorem 1.2(iv) splits.

Finally, the sequences $\kappa_1, \kappa_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$ are related to each other as follows. The inequality $\gamma_i \leq \kappa_i$ is (for $\gamma_i > 1$) a consequence of Proposition 4.3 Moreover, the inequality $\kappa_i \leq \gamma_i + 1$ follows (for $\kappa_i > 1$) from Proposition 4.2 where one has to take care of the parity of the Euler characteristic.

This completes the proof of Theorem 1.2.

Remark 5.9. Note that $\kappa_i < 2i$ for all $i \geq 1$ (where we have used Remark 3.6 and Proposition 3.7 to exclude the case that $\kappa_i = 2i$). Hence, for $\gamma_i = 2i - 1$ (which happens (at least) for $i = 1, 2, 4$) we have $\kappa_i = \gamma_i$. Moreover, $\gamma_i \neq 2, 4, 5, 6$ mod 8 implies that $\kappa_i \neq 5, 6$ mod 8 because $\gamma_i \in \{\kappa_i - 1, \kappa_i\}$.

Remark 5.10. An unoriented version $\mathcal{N}_k^n$ of the group $\mathcal{M}_k^n$ can be defined in a straightforward way by forgetting orientations in Definition 3.2. Then, one obtains a version of Theorem 1.2 involving the unoriented bordism groups $\mathcal{N}_k^n$ and $\mathcal{H}_k^n$ (see Remark 4.4), where Definition 5.6 and Definition 2.3 have to be modified appropriately.
6. Detecting Exotic Kervaire Spheres

We discuss how bordism groups of constrained Morse functions are capable of detecting exotic Kervaire spheres in certain high dimensions. Recall that Kervaire spheres are a concrete family of homotopy spheres that can be obtained from a plumbing construction as follows (see [18, p. 162]). The unique Kervaire sphere $\Sigma_n$ of dimension $n = 4k + 1$ can be defined as the boundary of the parallelizable $(4k+2)$-manifold given by plumbing together two copies of the tangent disc bundle of $S^{2k+1}$.

In Theorem 6.2 we will need to impose stronger conditions on the dimension $n$. These originate from a result of Stolz [29] on highly connected bordisms that is used in our argument, and we do not know if they can be eliminated.

Let $\Theta_n$ denote the group of homotopy $n$-spheres. Recall that $\Theta_n$ consists of $h$-cobordism classes of oriented differentiable homotopy $n$-spheres, and its group structure is induced by forming the oriented connected sum of homotopy spheres. The group of homotopy spheres has been introduced and studied by Kervaire and Milnor [15], who showed that for $n \geq 5$, $\Theta_n$ is a finite abelian group. As remarked in [15, p. 505], $\Theta_n$ classifies for $n \geq 5$ oriented diffeomorphism classes of oriented closed $n$-manifolds homeomorphic to $S^n$, where non-trivial classes are usually called exotic spheres.

Let $bP_{n+1} \subset \Theta_n$ denote the subgroup of those homotopy $n$-spheres that can be realized as the boundary of a parallelizable compact manifold (see [15, p. 510]). For instance, we have $[\Sigma_K^n] \in bP_{4k+2}$ by construction of the Kervaire spheres, and $bP_{n+1} = 0$ whenever $n$ is even by [15, Theorem 5.1, p. 512]. The following result is part of the classification theorem of homotopy spheres (see [20, Theorem 6.1, p. 123]).

**Theorem 6.1.** Suppose that $n = 4k + 1$ for some integer $k \geq 1$. Then, $bP_{n+1} = \{[S^n], [\Sigma_K^n]\}$, where $\Sigma_K^n$ denotes the unique Kervaire sphere of dimension $n$. Moreover, $bP_{n+1} \cong \mathbb{Z}/2$ whenever $n + 3 \notin \{2^1, 2^2, 2^3, \ldots\}$, whereas $bP_{n+1} = 0$ for $n \in \{5, 13, 29, 61\}$. We have $\Theta_n/bP_{n+1} \cong \text{coker } J_n$, where $J_n: \pi_n(SO) \to \pi_n^n$ denotes the stable $J$-homomorphism.

The application of our main results to Kervaire spheres is as follows.

**Theorem 6.2.** Suppose that $n \geq 237$ and $n \equiv 13 \text{ mod } 16$. Let $l := (n-1)/2$. Then, for any exotic $n$-sphere $\Sigma^n$ the following statements are equivalent:

(i) $\Sigma^n$ is diffeomorphic to the Kervaire $n$-sphere $\Sigma_K^n$.

(ii) $\Sigma^n$ admits an $l$-constrained Morse function which represents $0 \in \mathcal{M}_p^l$.

**Remark 6.3.** There are infinitely many dimensions $n \equiv 13 \text{ mod } 16$ for which $bP_{n+1} \neq \Theta_n$, that is, $\Theta_n$ contains exotic spheres different from the Kervaire sphere. In fact, by Theorem 6.1 it suffices to show that coker $J_n$ is non-trivial for infinitely many such $n$. For this purpose, note that for $n \equiv 5 \text{ mod } 8$ the domain of the stable $J$-homomorphism $J_n: \pi_n(SO) \to \pi_n^n$ satisfies $\pi_n(SO) = 0$, so that coker $J_n = \pi_n^n$ (see the proof of Theorem 3.1 in [15, p. 508]). Now, we claim that $\pi_n^n \neq 0$ whenever $n = 2(p + 1)(p - 1) - 3$ for an odd prime $p$. Indeed, setting $(r, s) = (1, 0)$, we can write any such $n$ in the form $n = 2(rp + s + 1)(p - 1) - 2(r - s) - 1$, where $0 \leq s < r \leq p - 1$ and $r - s \neq p - 1$. Hence, it follows from [32, Theorem B, p. 191] that the $p$-primary component of $\pi_n^n$ is $\mathbb{Z}_p$. 


Proof (of Theorem 6.2). By composition of the natural homomorphism $C^n_{l+1} \to C^n_l$ with the homomorphisms of Theorem 1.1 we can define homomorphisms
\[
c^n_0 : \Theta_n = C^n_{l+1} \to C^n_l, \quad [\Sigma^n] \mapsto [\Sigma^n],
\]
\[
g^n_0 : \Theta_n \xrightarrow{c^n_0} C^n_l \xrightarrow{\varepsilon^n_l} G^n_l, \quad [\Sigma^n] \mapsto [f],
\]
\[
c^n_{l-1} : \Theta_n \xrightarrow{g^n_0} G^n_l \xrightarrow{\delta^n_l} C^n_{l-1}, \quad [\Sigma^n] \mapsto [\Sigma^n],
\]
where $f : \Sigma^n \to \mathbb{R}$ denotes an arbitrarily chosen $l$-constrained Morse function in the definition of $g^n_0$. Denote the kernels of $c^n_l$, $g^n_l$ and $c^n_{l-1}$ by $C^n_l$, $G^n_l$ and $C^n_{l-1}$, respectively. Then, by construction, $C^n_l \subset G^n_l \subset C^n_{l-1}$.

Note that Proposition 2.2 implies that $C^n_l = C^n_{l-1}$ because $l = 6 \mod 8$. Thus, we have shown that $G^n_l = C^n_l$. Furthermore, the inclusion $bP_{n+1} \subset C^n_l$ holds since by [21, Theorem 3, p. 49] any parallelizable compact smooth manifold $W$ of dimension $m = n + 1$ can be made $l = (m/2) - 1$-connected by a finite sequence of surgeries without changing $\partial W$. Conversely, by a theorem of Stolz (see [29, Theorem B(ii), p. XIX]), the inclusion $C^n_l \subset bP_{n+1}$ holds because $m := n + 1$ is by assumption of the form $m = 2k + d$ for $d = 0$ and some odd integer $k \geq 113$. All in all, we have shown that $G^n_l = C^n_l = bP_{n+1} = ([S^n], [\Sigma_K^n])$, where the last equality is taken from Theorem 6.1.

Thus, for an exotic $n$-sphere $\Sigma^n$ statement (i) holds if and only if $[\Sigma^n] \in G^n_l$, that is, $[f] = 0 \in G^n_l$ for some (and hence, any) $l$-constrained Morse function $f : \Sigma^n \to \mathbb{R}$. Equivalently, by Theorem 1.2(ii), $\Sigma^n$ satisfies statement (ii). (In fact, by Theorem 1.2(ii) and Lemma 5.5 we have an isomorphism $M^n_l \cong G^n_l \oplus \mathbb{Z}/2$ given by $[f] : M^n \to \mathbb{R} \mapsto ([f], \Lambda^n_l([f]))$). If $\Sigma^n$ admits an $l$-constrained Morse function $f$ which represents $0 \in M^n_l$, then obviously $[f] = 0 \in G^n_l$. Conversely, suppose that $\Sigma^n$ admits an $l$-constrained Morse function $f$ which represents $0 \in G^n_l$. If necessary, we modify $f$ by introducing an additional pair of Morse critical points of subsequent indices $l$ and $l + 1$ in order to adjust the parity of $\sigma(f) = \sum_{\lambda = 0}^l C_\lambda(f)$ in such a way that $\Lambda^n_l([f]) = \sigma(f) - \sigma(\Sigma^n; \mathbb{Q}) = 0 \in \mathbb{Z}/2$. Then, $[f] = 0 \in M^n_l$. □

Remark 6.4. The groups $C^n_l$ and $G^n_l$ introduced in the proof of Theorem 6.2 are part of natural subgroup filtrations $C^n_{[n/2]} \subset \cdots \subset C^n_1$ and $G^n_{[n/2]} \subset \cdots \subset G^n_1$ of $\Theta_n$ which can be defined for any $n \geq 6$ as follows. Using Theorem 1.1 we define for $1 \leq k \leq [n/2]$ the homomorphisms
\[
c^n_k : \Theta_n = C^n_{[n/2]+1} \to C^n_{k+1} \xrightarrow{c^n_{k+1}} G^n_{k+1} \xrightarrow{\delta^n_{k+1}} C^n_k, \quad [\Sigma^n] \mapsto [\Sigma^n],
\]
\[
g^n_k : \Theta_n \xrightarrow{c^n_k} C^n_k \xrightarrow{\varepsilon^n_k} G^n_k, \quad [\Sigma^n] \mapsto [f],
\]
where $f : \Sigma^n \to \mathbb{R}$ denotes an arbitrarily chosen $k$-constrained Morse function in the definition of $g^n_k$. If $C^n_k$ and $G^n_k$ denote the kernels of $c^n_k$ and $g^n_k$, respectively, then $C^n_k \subset G^n_k$ for $1 \leq k \leq [n/2]$, and $G^n_k \subset C^n_{k-1}$ for $2 \leq k \leq [n/2]$ (compare [33, Theorem 10.1.3, p. 243]). Analogously to the proof of Theorem 6.2 we can use [21, Theorem 3, p. 49] to show that $bP_{n+1} \subset C^n_{[n/2]}$.

In general, we do not know whether the resulting filtrations $C^n_k$ and $G^n_k$ of $\Theta_n$ coincide or not.

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