FROM TAUB NUMBERS TO THE BONDIMASS

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Abstract

Taub numbers are studied on asymptotically 
flat backgrounds with Killing symmetries. When the field equations are solved for a background spacetime and higher order functional derivatives (higher order variational derivatives of the Hilbert Lagrangean) are solved for perturbations from the background, such perturbed spacetimes admit zeroth, first, and second order Taub numbers. Zeroth order Taub numbers are Komar constants (up to numerical factors) or Penrose-Goldberg constants of the background. For a Killing symmetry of the background, first order Taub numbers give the contribution of the linearized perturbation to the associated background quantity, such as the perturbing mass. Second order Taub numbers give the contribution of second order perturbations to the background quantity. The Bondi mass is a sum of first and second order Taubs numbers on a Minkowski background.

1 INTRODUCTION

To define the Bondi mass one needs an asymptotically flat manifold and the notion of future null infinity \( I^+ \). The Bondi metric is axi symmetric and has 4 metric functions \( V; U; \phi; \eta \): 

\[
g_{\text{Bondi}} \, dx \cdot dx = (V e^2 = r^2 (u^2 e^2 ) du^2 + 2e^2 \, dudr + 2r^2 U e^2 \, dud) \]

where \( V = 0 (l=r); \quad U = 0 (l=r^2); \quad \phi = 0 (l=r^2) \). The asymptotic form of metric function \( V \in \mathcal{V} = r \cdot 2M (u); \quad O (l=r); \quad M (u); \) is the Bondi mass 

\[
2M (u) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} : \]

The Bondi mass is the 2-surface integral of the mass aspect over a topological 2-sphere at \( I^+ \)

\[
M_{\text{Bondi}} = \frac{1}{8} \int_{S^2} \left[ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \right] \frac{1}{8} \int_{S^2} \left[ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right] d
\]

2 EARLY CALCULATIONS OF THE BONDIMASS

The first computation of the Bondi mass was done by J.N. Goldberg with use of the Einstein pseudotensor and von Freud superpotential: 

\[
\left( \frac{\Phi_{\text{Einstein}}}{g_{\text{Einstein}}} \right) = \Phi \text{ (superpotential)} + \left( \text{pseudotensor} \right)
\]

*To appear in Class. Quantum Grav., 14, 1899 (1997)

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Calculating in an asymptotically Cartesian frame and transforming to null spherical coordinates, Goldberg integrated the superpotential over an asymptotic 2-sphere at future null infinity and, using the generator of asymptotic time translations, obtained the Bondi mass.

W. McNamara and Tam burial were the first to construct a tensorial calculation by modifying the K on an ar superpotential. For the null surface $u = \text{const}$, they added a term which eliminated all surface derivatives. Using an asymptotic symmetry, the integral of the modified K on an ar superpotential, called a "linkage", yields the Bondi mass at $t^+$. Unfortunately the linkage construction does not arise from a variational principle.

3 Taub Method

The Taub method is fully developed in [1]. We write the Bondi metric in a perturbation expansion as $g^{\text{Bondi}}(u; t) = h^{(1)} + h^{(2)} + \ldots$. To insure a single coordinate system for each tensor in the perturbation expansion, the Minkowski background metric is constructed from the Bondi metric. We solve the Newman-Penrose form of Einstein's equations with initial data $\theta_0(u_0; t) = 0$ ($u_0; t) = \frac{1}{2}(u_0; t) = 0$; $u_0; t) = 0$. We obtain Bondi metric functions $V = r$, $U = \frac{1}{r}$, $i = 0$, i.e., the Minkowski background. The symmetry generator of mass, the background timelike Killing vector, $k_i = (\theta_i; t)$, is covariant constant. The Taub superpotential $U_{\text{Taub}}(k_i; h) = (g) k_i \{ [h]$, $[h + h] \} = 0$ is used to compute Taub numbers where the $n^{th}$ Taub number is $n (k_i; h^{(n)}) = \frac{1}{2} \int_{\partial N} U_{\text{Taub}}(k_i; h^{(n)}) dS$.

Each of the $h^{(n)}$ and hence each of the $n$ derives from a variational derivative of the Hilbert action, the $n^{th}$ number from the $n + 1^{th}$ variation.

The Bondi mass results from the sum of $1$st and second numbers: $M_{\text{Bondi}} = 1 (k_i; h^{(1)}) + 2 (k_i; h^{(2)})$. $1$ has the non-radiative mass contribution and $2$ contains the news function.

4 Perturbation Calculation

To nd $1$ solve D E $\log (h^{(1)}) = 0$ for $h^{(1)}$ using the Newman-Penrose equations on the Minkowski background with tetrad $\{ (1 = du, n = \frac{1}{2}du + dr, m = \sqrt{r^2 + d^2}) \}$. Find the spin coe$cients, m$ e metric components, and Weyl tensor components with initial data $\theta_0(u_0; t); 0 \{ u_0; t) ; \frac{1}{2} (u_0; t); 0 (u; t) \} g.

$h^{(1)} = [\frac{1}{2} + \frac{1}{2} r = 0] 1 \{ 0 = \frac{1}{2} r \} m m \{ 0 = \frac{1}{2} r \} m m \{ 0 = \frac{1}{2} r \} (1 \text{ m } + m \text{ m })$.

Substitute into $U_{\text{Taub}}(k_i; h^{(1)})$ and obtain $1 = \frac{1}{1} R (\frac{1}{2} + \frac{1}{2}) d$: This is $M_{\text{Bondi}}$ when the news $\theta_0 \frac{1}{2} = 0$; Vacuum Schwarzschild has $\frac{1}{2} = 0$; $\frac{1}{2} = m$ and so $M_{\text{Bondi}} = 1 = m$.

To nd $2$ iterate the Newman-Penrose equations. The second order Bianchi identities for the Weyl tensor components have second order quantities as sources whereas the first order equations were source-free.
$$U_{\text{Taub}}(k_t;h^{(2)}) = \frac{1}{g^1_n} \left( 2 \frac{\theta_0}{r^3} + O_3 \right);$$

$$2 \langle k_t; h^{(2)} \rangle = \frac{1}{8} \int_{S^2} R \theta_0(0,0) \, \text{d}S^2;$$

$$M_{\text{Bondi}} = 1 + 2; \text{ Higher orders fall off faster and do not contribute to the Bondi mass.}$$

1. H. Bondi, M. van der Burg, and A. Metzner, Proc. Roy. Soc. (London) A 269, 21 (1962).
2. J. N. Goldberg, Phys. Rev. 131, 1367 (1963).
3. L. Tamburino and J. W. in 't Veld, Phys. Rev. 150, 1039 (1966).
4. E. N. Glass, Phys. Rev. D 47, 474 (1993).
5. M. G. Naber and E. N. Glass, J. Math. Phys. 35, 5969 (1994).
6. E. N. Glass and Mark Naber, J. Math. Phys. 35, 1834 (1994).
7. W. E. Couch, R. J. Torrence, A. I. Janis, and E. T. Newman, J. Math. Phys. 9, 484 (1968).