Asymptotic expansions of Kummer hypergeometric functions with three asymptotic parameters $a$, $b$ and $z$

N. M. Temme*  E. J. M. Veling†

August 23, 2022

Abstract

In a recent paper [11] new asymptotic expansions are given for the Kummer functions $M(a, b, z)$ and $U(a, b + 1, z)$ for large positive values of $a$ and $b$, with $z$ fixed and special attention for the case $a \sim b$. In this paper we extend the approach and also accept large values of $z$. The new expansions are valid when at least one of the parameters $a$, $b$, or $z$ is large. We provide numerical tables to show the performance of the expansions.

Keywords Asymptotic expansions; Kummer functions; Confluent hypergeometric functions
AMS Classification Primary 41A60; Secondary 33C15

1 Introduction

We derive new asymptotic expansions of the Kummer functions $M(a, b, z)$ and $U(a, b + 1, z)$ in which all three positive parameters $a$, $b$, and $z$ are allowed to be large, and they are even valid when at least one of the parameters $a$, $b$, or $z$ is large. The methods of the recent paper [11] require only a minor modification to include the argument $z$ as a large parameter. Again we use a uniform method to derive the asymptotic expansion of a Laplace-type integral of the form

$$F_{\lambda}(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty s^{\lambda-1} e^{-zs} f(s) \, ds,$$

with $z$ as a large positive parameter. If the parameter $\lambda > 0$ is fixed we can use Watson’s lemma. However, when $\lambda$ is allowed to become large, there is a positive saddle point, and the asymptotic approach can be based on Laplace’s method. The uniformity aspect is that we combine both methods in one approach.

The method can also be used for loop integrals of the form

$$G_{\lambda}(z) = \frac{\Gamma(\lambda + 1)}{2\pi i} \int_{-\infty}^{(0+)} s^{-\lambda-1} e^{zs} g(s) \, ds,$$

with $z$ as a large positive parameter.
We show in the next section that \( G_\lambda(z) \) can be interpreted as an analytic continuation with respect to \( \lambda \) of \( F_\lambda(z) \) (after changing a notation), and in this way we can reduce the number of four expansions needed in [11] to only two. The asymptotic analysis of Kummer functions (or confluent hypergeometric functions) has been discussed in great detail in the literature. A simple result is available for \( M(a, b, z) \) when \( b \to \infty \), with \( a = O(1) \) and \( z = O(1) \), because in that case the defining convergent power series has an asymptotic character. In [10, Chapter 10] several results for \( M(a, b, z) \) and \( U(a, b, z) \) are derived for large \( a \) or \( b \), also in combination with large \( z \). The classical results on large \( z \) expansions are considered in [7] and [8], and summarised in [5], where we can also find expansions for large parameters. See also [2] and [6], where the results are derived for the Whittaker functions. In the notation of the Whittaker functions \( M_{\kappa, \mu}(z) \) and \( W_{\kappa, \mu}(z) \), the uniformity aspects considered in the present paper are similar to those with large \( z \), \( \kappa \), and \( \mu \), paying special attention to the case \( \kappa \sim -\mu \), \( \mu > 0 \). In a recent paper [3] expansions are given for the Whittaker functions for large values of \( \mu \), which are uniformly valid for \( 0 \leq \kappa/\mu \leq 1 - \delta \) and \( 0 \leq \arg(z) \leq \pi \).

The asymptotic method We summarise the main steps in the construction of the asymptotic expansions. For details we refer to the Appendix and [10, Chapter 25]. The Kummer functions can be written as integrals of the form

\[
\int e^{-z \phi(t)} \Phi(t) dt,
\]

where \( c = 1 \) for the \( M \)-function and \( c = \infty \) for the \( U \)-function. The functions \( \phi(t) \) and \( \phi(s) \) will be described in a later section, see (2.1). The function \( \phi(t) \) has one saddle point \( t_0 \) (a zero of \( \phi'(t) \)) in \( (0, c) \), and \( t_0 \) moves to the origin under the influence of an extra parameter. A typical example is the function

\[
\psi(s) = s - \mu \ln s, \quad \psi'(s) = s - \mu s, \quad (2.2)
\]

where the saddle point \( s_0 = \mu \) tends to zero when \( \mu \) does. At that same time \( \psi(s) \to s \), and the saddle point vanishes. In the cases covered in this paper, the integral as in (2.1) will become a Laplace-type integral as in (1.1):
Using an integration by parts scheme we can obtain the asymptotic expansion of (1.1)

\[ F_\lambda(z) \sim z^{-\lambda} \sum_{n=0}^{\infty} \frac{f_n(\mu)}{z^n}, \quad z \to \infty, \quad \lambda \geq 0. \quad (2.5) \]

In the Appendix we explain how the coefficients \( f_n(\mu) \) can be obtained. In the same way we can find the expansion of the contour integral in (1.2)

\[ G_\lambda(z) \sim z^\lambda \sum_{n=0}^{\infty} (-1)^n \frac{g_n(\mu)}{z^n}, \quad z \to \infty, \quad \lambda \geq 0. \quad (2.6) \]

In the previous article [11] we considered the cases \( a \leq b \) and \( a \geq b \) for each Kummer function, resulting in four expansions. Here we combine the method for \( a \leq b \) and \( a \geq b \) by exploiting the strong relationship between the functions \( F_\lambda(z) \) and \( G_\lambda(z) \), because \( G_\lambda(z) \) can be seen as the analytic continuation with respect to \( \lambda \) of \( F_\lambda(z) \), which becomes defined for \( \Re \lambda \leq 0 \).

To show this, we first observe that the integral in (1.2) exists for all finite complex values of \( \lambda \neq -1, -2, 3, \ldots \). For a start, let \( \Re \lambda < 0 \). Then we can use \((-\infty, 0]\) as the path of integration and obtain

\[ G_\lambda(z) = \frac{\Gamma(\lambda + 1)}{2\pi i} \left( e^{-\pi i(-\lambda-1)} - e^{\pi i(-\lambda-1)} \right) \int_{-\infty}^{0} |s|^{-\lambda-1} e^{zs} g(s) \, ds = \frac{1}{\Gamma(-\lambda)} \int_{0}^{\infty} s^{-\lambda-1} e^{-zs} g(-s) \, ds. \quad (2.7) \]

This can be used for \( \lambda = -1, -2, 3, \ldots \), and it becomes \( F_\lambda(z) \) when we replace \( \lambda \) by \(-\lambda\) and \( g(s) \) by \( f(-s) \). We conclude this as follows.

**Lemma 2.1.** The function \( G_\lambda(z) \) defined in (1.2) is an analytic function for all complex values of \( \lambda \) and is, with \( g(s) = f(-s) \), the analytic continuation with respect to \( \lambda \) of \( F_\lambda(z) \), which is initially defined for \( \Re \lambda > 0 \).

In the following sections we use this lemma when an asymptotic expansion derived for \( b \geq a \) will also be used for \( b \leq a \). In this way we reduce the four different methods used in [11] to two approaches.

In this paper we use the positive argument \( z \) of the Kummer functions as the principal asymptotic parameter, in the asymptotic analysis we scale the parameters \( a \) and \( b \) with respect to \( z \) by using

\[ \alpha = \frac{a}{z}, \quad \beta = \frac{b}{z}. \quad (2.8) \]

Special values of the Kummer functions are

\[ M(a, a, z) = e^z, \quad U(a, a + 1, z) = z^{-a}, \quad (2.9) \]

and our asymptotic expansions reduce smoothly to these elementary values as \( b \to a \). We prefer giving expansions of \( U(a, b + 1, z) \) for this reason, and not of \( U(a, b, z) \), which becomes the incomplete gamma function \( e^z \Gamma(1 - a, z) \) as \( b \to a \). For more details on the Kummer functions we refer to [5].
In the following two sections the asymptotic expansions have a front term of the form $e^{zA}$, where $A$ can be written in terms of

$$A(\mu) = \mu (\tau - \ln \tau - 1) - \alpha \ln(1 - \mu \tau), \quad (2.10)$$

where $\mu = \pm(\alpha - \beta)$, $\tau = t_0/\mu$, and $t_0$ is the relevant saddle point.

3 The expansion of $M(a, b, z)$

We start with $b \geq a$ and use the notation

$$\lambda = b - a, \quad \mu = \frac{\lambda}{z} = \beta - \alpha = \frac{b - a}{z}. \quad (3.1)$$

The Kummer relation $M(a, b, z) = e^{z}M(b - a, b, -z)$ together with the integral

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a) \Gamma(b - a)} \int_0^1 e^{zt} t^{b-a-1} (1 - t)^{b-a} \, dt, \quad \Re a > 0, \quad \Re(b - a) > 0, \quad (3.2)$$

gives

$$M(a, b, z) = \frac{\Gamma(b) e^z}{\Gamma(a) \Gamma(b - a)} \int_0^1 e^{-zt} t^{b-a-1} (1 - t)^{a-1} \, dt \quad (3.3)$$

where

$$\phi(t) = t - \alpha \ln(1 - t) - \mu \ln t, \quad \phi'(t) = -\frac{t^2 - (\beta + 1)t + \mu}{t(1 - t)}. \quad (3.4)$$

Note that in this case $\Phi(t) = \frac{1}{t(1 - t)}$, the function shown in (2.1).

The saddle point $t_0$ inside the interval $(0, 1)$ is given by

$$t_0 = \frac{1}{2} (\beta + 1) - \frac{1}{2} \sqrt{(\beta + 1)^2 - 4\mu}$$

$$= \frac{2\mu}{\beta + 1 + \sqrt{(\beta + 1)^2 - 4\mu}} = \frac{2\mu}{\beta + 1 + \sqrt{(\beta - 1)^2 + 4\alpha}}. \quad (3.5)$$

with expansion

$$t_0 = \frac{\mu}{\beta + 1} + \frac{\mu^2}{(\beta + 1)^3} + \mathcal{O}(\mu^3), \quad \mu \to 0. \quad (3.6)$$

From this, and from $\phi'(t)$ given in (3.4), we see that the saddle point $t_0$ vanishes, as $\mu \to 0$.

We use the transformation given in (2.3) and obtain

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^{zA} F_{\lambda}(z), \quad F_{\lambda}(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty s^{\lambda - 1} e^{-zs} f(s) \, ds, \quad (3.7)$$

where

$$A = \phi(t_0) - \psi(s_0), \quad f(s) = \frac{s}{t(1 - t)} ds, \quad \frac{dt}{ds} = \frac{\psi'(s)}{\phi'(t)}, \quad f(s) = \frac{s - \mu}{(\beta + 1) t - t^2 - \mu}. \quad (3.8)$$
As in (2.5), with details in the Appendix, we can obtain the expansion

\[ M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^{zA} z^{-\lambda} \sum_{n=0}^{\infty} \frac{f_n(\mu)}{z^n}, \quad z \to \infty. \quad (3.9) \]

To find \( f_0(\mu) \) we need the derivative \( dt/ds \) at \( s = s_0 = \mu \). Using l’Hôpital’s rule we have

\[ \left. \frac{dt}{ds} \right|_{s=s_0} = \frac{\psi''(s_0)}{\phi''(t_0)} = \frac{1}{\mu}, \quad \phi''(t_0) = \frac{\beta t_0^2 - 2\mu t_0 + \mu}{t_0^2(1-t_0)^2}. \quad (3.10) \]

This gives

\[ \left( \left. \frac{dt}{ds} \right|_{s=s_0} \right)^2 = \frac{\psi''(s_0)}{\phi''(t_0)} \implies f_0(\mu) = \sqrt{\frac{\mu}{\beta t_0^2 - 2\mu t_0 + \mu}}. \quad (3.11) \]

We normalize the coefficients of the expansion by writing

\[ M(a, b, z) \sim e^{zA} f_0(\mu) \sum_{n=0}^{\infty} \frac{\bar{f}_n(\mu)}{z^n}, \quad \bar{f}_0(\mu) = f_0(\mu). \quad (3.12) \]

as \( z \to \infty \) and \( b \geq a \).

In applications and numerical testing it may be convenient to use a scaled function. We write

\[ M(a, b, z) = e^{zA} \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} \bar{M}(a, b, z), \quad (3.13) \]

and we summarise the above results in the following theorem.

**Theorem 3.1.** The scaled Kummer function defined in (3.13) has the asymptotic expansion

\[ \bar{M}(a, b, z) \sim e^{-zA} f_0(\mu) \sum_{n=0}^{\infty} \frac{\bar{f}_n(\mu)}{z^n}, \quad z \to \infty, \quad (3.14) \]

uniformly with respect to \( b \geq a \geq a_0 \), where \( a_0 \) is a fixed positive parameter, \( \mu \) is defined in (3.1), \( A = A(\mu) \) is defined in (2.10), \( f_0(\mu) \) is given in (3.11). The first coefficients are \( \bar{f}_0(\mu) = 1 \) and

\[ \bar{f}_1(\mu) = \frac{\mu \tau^2 (\mu^2 \tau^5 - 13\mu^2 \tau^4 - \mu \tau^4 + 21\tau^3 \mu + 4\mu \tau^2 - 9\tau^2 - 2\tau - 1)}{12(\mu \tau^2 - 1)^3(1 - \mu \tau)}, \quad (3.15) \]

where (see (3.5))

\[ \tau = \frac{t_0}{\mu} = \frac{2}{\beta + 1 + \sqrt{(\beta + 1)^2 - 4\mu}}. \quad (3.16) \]

**Proof.** For details of the proof we refer to [9]. The relation between \( s \) and \( t \) is one-to-one and analytic in a domain around the positive \( s \)-axis, where \( f(s) \) of (3.8) is an analytic function. The second saddle point \( t_+ \) that follows from (3.5) by changing the sign in front of the square root, corresponds with a complex point \( s_+ \) that follows from the transformation in (2.3), and this point is a singularity of \( f(s) \).
Remark 3.2. The expansions (3.9) and (3.14) with coefficients \( F \) and \( \tilde{f}_n(\mu) \), suggest the separation of the large parameter \( z \) and the uniformity parameter \( \mu \) as two independent parameters. However, \( \mu \) depends on the three considered parameters \( a, b \) and \( z \), as follows from (3.1), although the explicit form of the scaled coefficient \( \tilde{f}_1(\mu) \) given in (3.15) (and that of the not shown higher coefficients) shows only two parameters \( \tau \) and \( \rho \). We use the current notation in order to stay close to the method described in the cited literature.

Note that \( \tilde{f}_1(0) = 0 \), just like all coefficients \( \tilde{f}_n(\mu) \) with \( n \geq 1 \) that we have computed. Also, if \( \mu \to 0 \), the factor \( \Phi = e^{z^A} \Gamma(\beta) / \Gamma(\alpha) \) becomes \( e^z \), and the asymptotic expansion gives the correct value \( e^z \) when \( \mu \to 0 \).

In numerical calculations with small values of \( \mu \), we should write the front term \( e^{-zA}f_0(\mu) \) with \( t_0 \) replaced by \( \mu \tau \) (see (3.16)). By (3.8) and (3.11) we have (see (2.10))

\[
f_0(\mu) = \frac{1}{\sqrt{\beta \mu \tau^2 - 2 \mu \tau + 1}}, \quad A = A(\mu) = \mu (\tau - \ln \tau - 1) - \alpha \ln(1 - \mu \tau). \tag{3.18}
\]

3.1 The case \( a \geq b \)

We use Lemma 2.1 and write the function \( F_\lambda(z) \) defined for \( \lambda > 0 \) in (3.7) as

\[
F_\lambda(z) = \frac{\Gamma(-\lambda + 1)}{2\pi i} \int_{-\infty}^{0+} e^{zs} f(-s) ds. \tag{3.19}
\]

The right-hand side can be used as the analytic continuation of \( F_\lambda(z) \) with respect to \( \lambda \) into the half-plane \( \lambda \leq 0 \), which gives

\[
F_{-\lambda}(z) = \frac{\Gamma(\lambda + 1)}{2\pi i} \int_{-\infty}^{0+} e^{zs} f(-s) ds, \quad \lambda \neq -1, -2, -3, \ldots . \tag{3.20}
\]

The saddle point analysis of the right-hand side of (3.20) proceeds as for the function \( F_\lambda(z) \) defined in (3.7), and we can use a similar procedure for integration by parts as explained in the Appendix. The expansions in (2.6) shows \((-1)^n\) because of the slightly different integration by parts method compared with the one for obtaining (2.5). On the other hand, the functions in (7.14) have the same structure with \( s \) replaced by \(-s \) and \( \mu \) by \(-\mu \). Thus we find exactly the same expansion as in (3.12) and (3.14).
Corollary 3.3. Using Lemma 2.1 and Theorem 3.1 we conclude that the asymptotic expansions in (3.12) and (3.14) can be used for \( b \geq a \geq a_0 \) as well as for \( b_0 \leq b \leq a \), where \( a_0 \) and \( b_0 \) are fixed positive numbers.

3.2 The range of the three parameters

With the upper bound of the remainder of the expansion, as stated in the proof of Theorem 3.1, global information may become available about the expansion and the range of the parameters. However, it is always useful to look at the coefficients of the expansion to get more detailed information. We discuss two points of interest in which limiting forms of the coefficients are relevant, and both points apply to the expansion in (3.14) as well as to that in (4.12) for the \( U \)-function.

- **The behaviour for large values of \( \mu \).**

The initial interest to derive the expansions in this paper is validity when the parameter \( \mu \) tends to zero. But we also want to find out how the expansion behaves for larger values of \( \mu \).

Inspecting the coefficient \( \tilde{f}_1(\mu) \) in (3.15) we see that for large values of \( \mu \) it behaves like \( O(1/\mu) \), uniformly with respect to the parameter \( \tau \) given in (3.16). For the coefficients we derived for the numerical computations we find \( \tilde{f}_k(\mu) = O(1/\mu^k) \) as \( \mu \to \infty \). From this behaviour we conclude that the expansion in (3.14) has a double asymptotic property: it is valid when \( \mu \) or \( z \) is large or if both are large.

Numerical experiments confirm this property. If we take \( a = 0.5, b = 0.7, z = 100 \), our expansions with terms up to \( n = 4 \) gives a result with relative error \( 2.0407 \times 10^{-11} \) when we use a test based on the Wronski relation for the Kummer functions. When we take \( a = 50.5, b = 100.7, z = 1 \), the Wronski test gives the error \( 6.51 \times 10^{-13} \).

- **The behaviour for \( t_0 \to 1 \).**

The expansion in (3.12) becomes useless when \( t_0 \to 1 \). In that case the factor \( \mu \tau - 1 = t_0 - 1 \) in the denominator of \( \tilde{f}_1(\mu) \) in (3.15) tends to zero. This happens with all coefficients \( \tilde{f}_k(\mu) \) of the expansion, even so that every \( \tilde{f}_k(\mu), k \geq 1 \), has a factor \( (\mu \tau - 1)^k \) in the denominator.

By setting the numerator of \( \phi'(t) \) in (3.4) equal to 0 we find that when \( t_0 \) is replaced by \( \rho \), where \( \rho \in (0, 1) \), the following linear relation between \( \alpha \) and \( \beta \) arises:

\[
\alpha = \rho^2 - \rho + (1 - \rho)\beta. \tag{3.21}
\]

In Figure 1 we show the shaded domain between the line \( \alpha = \beta \) (for these values \( t_0 = 0 \)) and the line that follows from the relation in (3.21) (where \( t_0 = \rho \)). Inside the coloured domain we have \( 0 \leq t_0 \leq \rho \). After selecting \( \rho \) we can use the condition \( \alpha \geq \rho^2 - \rho + (1 - \rho)\beta \) on the parameters \( \alpha \) and \( \beta \) to use the asymptotic expansion with \( t_0 \leq \rho \). In the figure we have taken \( \rho = \frac{4}{5} \).

In the sector above the diagonal \( \alpha = \beta \) we have \( t_0 < 0 \), because in that case \( \mu < 0 \). As we have explained in the previous subsection, we can use the expansion in (3.12) also for \( \mu < 0 \), and we see that for all positive values of \( \alpha \) and \( \beta \) above the line governed by the relation in (3.21) we have \( t_0 \leq \rho \).
Figure 1: In the coloured domain the saddle point $t_0$ satisfies $0 \leq t_0 \leq \rho < 1$. The line from the point $(\rho, 0)$ to the right has the equation given in (3.21). In the figure we have used $\rho = \frac{4}{5}$. For further details we refer to the text.

The other quantity $\mu_\tau^2 - 1 = t_0^2/\mu - 1$ in the denominator of $\tilde{f}_1(\mu)$ vanishes if $(\beta + 1)t_0 = 2\mu$, which implies $(\beta - 1)^3 = -4\alpha$. This cannot happen if $\alpha$ and $\beta$ are positive. In fact, when this happens two saddle points coincide, and we need Airy functions to describe the asymptotic behaviour; see [2].

We conclude that we can use the expansions in (3.12) and (4.12) for a wide range of the parameters $a$, $b$ and $z$.

4 The expansion of $U(a, b, z)$

In this section we again use the notation

$$\lambda = b - a, \quad \mu = \frac{\lambda}{z} = \beta - \alpha = \frac{b - a}{z},$$

and we start the analysis assuming that $b \geq a$. We take the contour integral

$$U(a, b, z) = \frac{\Gamma(1 - a)}{2\pi i} \int_{-\infty}^{(0+)} e^{z s} s^{-\lambda - 1} (1 - s)^{b - a - 1} ds, \quad \Re z > 0,$$

where $a \neq 1, 2, 3, \ldots$. The contour cuts the real axis between 0 and 1. At that point the fractional powers are determined by $\text{ph} (1 - s) = 0$ and $\text{ph} s = 0$. We use the Kummer relation $U(a, b, z) = z^{1-b} U(a - b + 1, 2 - b, z)$ and obtain

$$U(a, b + 1, z) = \frac{z^{1-b} \Gamma(\lambda + 1)}{2\pi i} \int_{-\infty}^{(0+)} e^{z t} t^{-\lambda - 1} (1 - t)^{-a} \, dt.$$  

We write this in the form

$$U(a, b + 1, z) = \frac{z^{-b} \Gamma(\lambda + 1)}{2\pi i} \int_{-\infty}^{(0+)} e^{z \varphi(t)} \frac{dt}{t},$$

where $\varphi(t) = t^{-\lambda} (1 - t)^{-a}$.
where
\[ \phi(t) = t - \alpha \ln(1 - t) - \mu \ln t, \quad \phi'(t) = -t^2 - (\beta + 1)t + \mu. \] (4.5)

The saddle point \( t_0 \) inside the interval \((0, 1)\) is given by
\[ t_0 = \frac{1}{2}(\beta + 1) - \frac{1}{2}\sqrt{(\beta + 1)^2 - 4\mu} = \frac{2\mu}{\beta + 1 + \sqrt{(\beta + 1)^2 - 4\mu}}, \] (4.6)

with expansion
\[ t_0 = \frac{\mu}{\beta + 1} + \frac{\mu^2}{(\beta + 1)^3} + \mathcal{O}(\mu^3), \quad \mu \to 0. \] (4.7)

Again, we see that the saddle point \( t_0 \) vanishes as \( \mu \to 0 \).

We use the transformation shown in (2.3) and obtain
\[ U(a, b + 1, z) = z^{-b} e^{zA} G_\lambda(z), \quad G_\lambda(z) = \Gamma(\lambda + 1) \int_{-\infty}^{(0+)} s^{-\lambda - 1} e^{zs} p(s) ds, \] (4.8)

where, with \( s_0 = \mu \),
\[ A = \phi(t_0) - \psi(s_0), \quad p(s) = \frac{s}{t} \frac{dt}{ds}, \quad \frac{dt}{ds} = \frac{\psi'(s)}{\phi'(t)}, \quad \frac{p(s)}{p_0(\mu)} = \frac{(t - 1)(s - \mu)}{t^2 - (\beta + 1)t + \mu}. \] (4.9)

As in (2.6) we can obtain the expansion
\[ U(a, b + 1, z) \sim e^{zA} \sum_{n=0}^{\infty} (-1)^n \frac{p_n(\mu)}{z^n}, \quad z \to \infty. \] (4.10)

The first coefficient is
\[ p_0(\mu) = \frac{\mu}{t_0} \sqrt{\frac{\psi''(s_0)}{\phi''(t_0)}} = (1 - t_0) \sqrt{\frac{\mu}{\beta t_0^2 - 2\mu t_0 + \mu}} \] (4.11)

When the parameters are large it may be convenient in numerical tests to use a scaled function as we did for the \( M \)-function in (3.13). Here we write
\[ U(a, b + 1, z) = z^{-a} \tilde{U}(a, b + 1, z) \] (4.12)

We summarise the results for the \( U \)-function in the following theorem.

**Theorem 4.1.** The scaled Kummer function defined in (4.12) has the asymptotic expansion
\[ \tilde{U}(a, b + 1, z) \sim e^{zA} p_0(\mu) \sum_{n=0}^{\infty} (-1)^n \frac{\tilde{p}_n(\mu)}{z^n}, \quad \tilde{p}_n(\mu) = \frac{p_n(\mu)}{p_0(\mu)}, \quad z \to \infty, \] (4.13)

uniformly with respect to \( b \geq a \geq a_0 \), where \( a_0 \) is a fixed positive parameter, \( \mu \) is defined in (4.4), \( A = A(\mu) \) is defined in (2.10), \( p_0(\mu) \) is given in (4.11). The first coefficients of the expansion (4.12) are \( \tilde{p}_0(\mu) = 1 \) and
\[ \tilde{p}_1(\mu) = \frac{\mu \tau^2 (1 - \tau) (\mu^2 \tau^4 - \mu \tau^3 + 8 \mu \tau^2 - 9 \tau + 1)}{12 (\mu \tau^2 - 1)^3 (\mu \tau - 1)}, \] (4.14)

where \( \tau = t_0 / \mu \), with \( t_0 \) given (4.6).
Proof. The saddle point contour of the integral in (4.8) is given by $\Im \psi(s) = \Im \psi(\mu) = 0$, and is governed by
\begin{equation}
\rho = \mu \frac{\theta}{\sin \theta}, \quad s = \rho e^{i\theta}, \quad -\pi < \theta < \pi.
\end{equation}
In the $t$-plane a similar contour through the saddle point $t_0$ can be defined. On these contours the relation between $t$ and $s$ is one-to-one, and $p(s)$ is analytic on the contour given in (4.15). For large $s$ and $t$ on the saddle point contours we have $s \sim t$, and from (4.9) we conclude that $p(s) \sim 1$ for large $s$ on the contour given in (4.15), and it can be verified that all derivatives are bounded. It follows, as in the proof of Theorem 3.1, that we can find a bound for the remainder in the finite expansion related to the expansion given in (4.13).

For small values of $\mu$, we write the quantities $p_0(\mu)$ and $A$ in terms of $\tau$. We have
\begin{equation}
p_0(\mu) = \frac{1 - \mu \tau}{\sqrt{\beta \mu \tau^2 - 2 \mu \tau + 1}}, \quad A = A(\mu) = \mu (\tau - \ln \tau - 1) - a \ln(1 - \mu \tau).
\end{equation}
We see that $A \to 0$ and $p_0(\mu) \to 1$ as $\mu \to 0$, that is, as $b \to a$. Also, all coefficients $\tilde{p}_n(\mu)$, $n \geq 1$, tend to zero when $\mu \to 0$ and in that case $\tilde{U}(a,b+1,z) \to 1$. This confirms that the expansion of $U(a,b+1,z)$ tends to the elementary value $U(a,a+1,z) = z^{-a}$ given in (2.9).

4.1 The case $a \geq b$

As in Section 3.1 we have the following result about the case $a \geq b$.

Corollary 4.2. Using Lemma 2.1 and Theorem 4.1 we conclude that the asymptotic expansions in (4.10) and (4.13) can be used for $b \geq a \geq a_0$ as well as for $b_0 \leq b \leq a$, where $a_0$ and $b_0$ are fixed positive numbers.

5 Numerical verifications

For a detailed discussion on the computation of Kummer functions and other hypergeometric functions we refer to the recent paper [4], where arbitrary-precision implementations are considered. Our paper focuses on asymptotic methods for the Kummer functions, and in this section we give information on the performance of our expansions using a limited number of terms.

To avoid comparisons by using other software, the relative errors shown in the tables are computed by verifying recurrence relations written in the stable forms
\begin{equation}
\frac{z M(a+1,b+1,z) + b M(a,b,z)}{b M(a+1,b,z)} = 1, \quad \frac{a U(a+1,b,z) + U(a,b-1,z)}{U(a,b,z)} = 1.
\end{equation}
When we use the scaled functions introduced in (3.13) and (4.12) we can write these relations as
\begin{equation}
\frac{z \tilde{M}(a+1,b+1,z) + a \tilde{M}(a,b,z)}{z \tilde{M}(a+1,b,z)} = 1, \quad \frac{a \tilde{U}(a+1,b,z) + z \tilde{U}(a,b-1,z)}{z \tilde{U}(a,b,z)} = 1.
\end{equation}
We can also use the relation
\begin{equation}
a M(a,b,z) U(a+1,b+1,z) + \frac{a}{b} M(a+1,b+1,z) U(a,b,z) = \frac{e^{z \Gamma(b)}}{z^{\Gamma(a)}},
\end{equation}

\[10\]
which follows from the Wronskian of the Kummer functions (see [5, Section 3.12(vi), 13.3(ii)])

In terms of the scaled functions we can write

\[ \frac{a}{z} \tilde{M}(a, b, z) \tilde{U}(a + 1, b + 1, z) + \tilde{M}(a + 1, b + 1, z) \tilde{U}(a, b, z) = 1. \]  

(5.4)

We give tables showing the relative errors in the computations for a selection of the parameters \( a, b \) and \( z \). Our computations are done with Maple (version 2021.2), with \( \text{Digits} = 16 \), without using the multi-precision possibilities. We have compared our asymptotic results with Maple’s \( KummerM \) and \( KummerU \) function codes, and found that for the \( M \)-functions this comparison is reliable, but for the \( U \)-functions it is not (again, with \( \text{Digits} = 16 \)). For example, when we take \( a = 130.0, b = 25.1 \) and \( z = 100.0 \), Maple gives the value \( U(a, b + 1, z) = -2.033 \ldots \times 10^{-234} \), a negative result! Matlab (version R2021b, \( \text{Digits} = 16 \)) gives \( U(a, b + 1, z) = 3.872389298555866 \times 10^{-293} \), and our asymptotic result gives \( U_{\text{asymp}}(a, b + 1, z) = 3.8723892985556390 \times 10^{-293} \).

We have also done a few other tests with Matlab (version R2021b, \( \text{Digits} = 16 \)) for the parameter values \( a = 130.0, b = 25.1 \) and \( z = 100.0 \), and we conclude that Matlab performs better than Maple in this example, with a recursion test on \( KummerU \) based on (5.1) giving a relative error 0.

In Table 1 we give the relative errors in the computation of the scaled functions \( \tilde{M}(a, b, z) \) and \( \tilde{U}(a, b, z) \) for \( z = 500 \), \( b = 500 \), several values of \( a \) by using expansions (3.14) and (4.12) with terms up to \( n = 4 \). The errors are computed by using the scaled recurrence relations in (5.2). We observe that for these \( a, b \) for \( n = 0 \) (expansions with only one term equal to 1) the approximations give a nice estimate, and that for \( n = 3 \) and \( n = 4 \) the relative errors are nearly the same.

To handle ratios of gamma functions with large arguments, which occur in the expansion in (3.12), we can use

\[ \frac{\Gamma(b)}{\Gamma(a)} = e^{\ln \Gamma(b) - \ln \Gamma(a)}, \]  

(5.5)

or the asymptotic expansions of \( \Gamma(z) \) or \( \ln \Gamma(z) \); see [1, Section 5.11].

Another aspect that requires attention in numerical evaluations of the asymptotic results is the factor \( e^{\pm z A} \) in front of the expansions. \( A \) has always the form \( \pm A(\pm \mu) \), with \( A(\mu) = \mu (\tau - \ln \tau - 1) - \alpha \ln(1 - \mu \tau) \) given in (2.10). Especially when \( \mu \) is small (which we always allow in our asymptotic results), the logarithmic term must be calculated accurately. For small \( x \) we have \( \ln(1 + x) = x + O(x^2) \), and when we first compute \( 1 + x \) information may get lost. For, say

---

1With these parameter values and \( \text{Digits} = 32 \) Maple’s procedure \( KummerU \) gives \( 3.8723892985558665 \times 10^{-293} \) with \( U_{\text{asymp}}(a, b + 1, z) \approx 3.872389298556251 \times 10^{-293} \). Compared with a computation with Matlab, \( \text{Digits} = 32 \), \( KummerU \) gives \( 3.8723892985558665 \times 10^{-293} \), identical with the Maple result.
Table 1: Relative errors in the computation of the scaled functions \( \tilde{M}(a, b, z) \) and \( \tilde{U}(a, b, z) \) for \( z = 500 \), \( b = 500 \), several values of \( a \) by using expansion \( (3.14) \) with terms up to \( n = 4 \). The errors are computed by using the scaled recurrence relations in \( (5.2) \).

| \( a \) | \( n = 0 \) | \( n = 1 \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) |
|---|---|---|---|---|---|
| \( M(a, b, z) \) | | | | | |
| 99 | \( 0.48 \times 10^{-05} \) | \( 0.43 \times 10^{-08} \) | \( 0.16 \times 10^{-09} \) | \( 0.76 \times 10^{-12} \) | \( 0.46 \times 10^{-13} \) |
| 199 | \( 0.16 \times 10^{-05} \) | \( 0.12 \times 10^{-08} \) | \( 0.98 \times 10^{-11} \) | \( 0.10 \times 10^{-14} \) | \( 0.46 \times 10^{-13} \) |
| 299 | \( 0.82 \times 10^{-06} \) | \( 0.55 \times 10^{-09} \) | \( 0.14 \times 10^{-11} \) | \( 0.10 \times 10^{-12} \) | \( 0.11 \times 10^{-12} \) |
| 399 | \( 0.51 \times 10^{-06} \) | \( 0.30 \times 10^{-09} \) | \( 0.30 \times 10^{-12} \) | \( 0.16 \times 10^{-14} \) | \( 0.39 \times 10^{-14} \) |
| 499 | \( 0.35 \times 10^{-06} \) | \( 0.19 \times 10^{-09} \) | \( 0.39 \times 10^{-13} \) | \( 0.10 \times 10^{-14} \) | \( 0.40 \times 10^{-15} \) |
| \( U(a, b, z) \) | | | | | |
| 501 | \( 0.35 \times 10^{-06} \) | \( 0.19 \times 10^{-09} \) | \( 0.37 \times 10^{-13} \) | \( 0.10 \times 10^{-14} \) | \( 0.10 \times 10^{-14} \) |
| 601 | \( 0.26 \times 10^{-06} \) | \( 0.13 \times 10^{-09} \) | \( 0.53 \times 10^{-13} \) | \( 0.25 \times 10^{-13} \) | \( 0.25 \times 10^{-13} \) |
| 701 | \( 0.20 \times 10^{-06} \) | \( 0.89 \times 10^{-10} \) | \( 0.64 \times 10^{-13} \) | \( 0.23 \times 10^{-13} \) | \( 0.23 \times 10^{-13} \) |
| 801 | \( 0.16 \times 10^{-06} \) | \( 0.66 \times 10^{-10} \) | \( 0.74 \times 10^{-13} \) | \( 0.33 \times 10^{-13} \) | \( 0.33 \times 10^{-13} \) |
| 901 | \( 0.13 \times 10^{-06} \) | \( 0.51 \times 10^{-10} \) | \( 0.24 \times 10^{-12} \) | \( 0.20 \times 10^{-12} \) | \( 0.20 \times 10^{-12} \) |
Table 2: Relative errors in the computation of the Wronski relation (5.4) for the scaled functions $\tilde{M}(a, b, z)$ and $\tilde{U}(a, b, z)$ for $z = 500$ and several values of $a$ and $b$ by using expansion (3.14) and (4.12) with terms up to $n = 4$.

| $a$  | $b = 101$ | $b = 301$ | $b = 501$ | $b = 701$ | $b = 901$ |
|------|-----------|-----------|-----------|-----------|-----------|
| 101  | $0.00 \times 10^{-00}$ | $0.46 \times 10^{-12}$ | $0.14 \times 10^{-11}$ | $0.42 \times 10^{-12}$ | $0.71 \times 10^{-12}$ |
| 301  | $0.52 \times 10^{-13}$ | $0.40 \times 10^{-15}$ | $0.32 \times 10^{-13}$ | $0.73 \times 10^{-13}$ | $0.63 \times 10^{-13}$ |
| 501  | $0.13 \times 10^{-12}$ | $0.15 \times 10^{-13}$ | $0.10 \times 10^{-13}$ | $0.50 \times 10^{-14}$ | $0.27 \times 10^{-13}$ |
| 701  | $0.17 \times 10^{-12}$ | $0.89 \times 10^{-13}$ | $0.31 \times 10^{-13}$ | $0.00 \times 10^{-00}$ | $0.67 \times 10^{-13}$ |
| 901  | $0.14 \times 10^{-12}$ | $0.14 \times 10^{-12}$ | $0.18 \times 10^{-12}$ | $0.14 \times 10^{-13}$ | $0.10 \times 10^{-14}$ |

$-\frac{1}{2} \leq x \leq \frac{1}{2}$, we use the relation

$$\text{arctanh } z = \frac{1}{2} \ln \frac{1 + z}{1 - z} \implies \ln(1 + x) = 2 \text{arctanh} \frac{x}{2 + x},$$

(5.6)

and either use a power series expansion of $\text{arctanh } z$ for, say $-\frac{1}{3} \leq z \leq \frac{1}{5}$, or the Maple code for $\text{arctanh } z$.

6 Concluding remarks

We derived new asymptotic expansions for positive values of $a$, $b$ and $z$, at least one of which is large, and in this way we have been able to extend the results of [11], which are only valid for large $a$ and $b$. By proving the relation between an integral on the positive line and a contour integral in the complex plane we have also reduced the number of expansions from four to two. With numerical tables we have demonstrated the performance of the expansions for a small selection of the parameters. More extensive testing is needed to verify the performance of the new expansions with respect of the range of the parameters $a$ and $b$ when given a value of $z$ and the number of terms of the expansions.

7 Appendix: Evaluating the coefficients

In the two Sections 3 and 4, we use the transformation in (2.3), and the first step is to express $t$ as a function of $s$ near $s_0 = \mu$, the saddle point in the $s$-domain where $\psi'(s) = (s - \mu)/s$ vanishes. We explain the procedure considering the transformation for Section 3, where, see (3.4),

$$\phi(t) = t - \alpha \ln(1 - t) - \mu \ln t, \quad \phi'(t) = -\frac{t^2 - (\beta + 1)t + \mu}{t(1 - t)},$$

(7.1)

with saddle point $t_0 \in (0, 1)$ given by

$$t_0 = \frac{1}{2}(\beta + 1) - \frac{1}{2} \sqrt{(\beta + 1)^2 - 4\mu} = \frac{2\mu}{\beta + 1 + \sqrt{(\beta + 1)^2 - 4\mu}}.$$
To find $t$ as function of $s$ near $s_0$ we write the transformation in (2.3) in the form of the local expansions

$$\sum_{k=2}^{\infty} \frac{\phi^{(k)}(t_0)}{k!} (t - t_0)^k = \sum_{k=2}^{\infty} \frac{\psi^{(k)}(s_0)}{k!} (s - s_0)^k,$$

(7.3)

which we can write as

$$(t - t_0) \sqrt{\sum_{k=2}^{\infty} \frac{1}{k!} \phi^{(k)}(t_0)(t - t_0)^{k-2}} = (s - s_0) \sqrt{\sum_{k=2}^{\infty} \frac{1}{k!} \psi^{(k)}(s_0)(s - s_0)^{k-2}},$$

(7.4)

where the square roots are positive for positive values of $s$ and $t$. The relation satisfies the condition imposed on the mapping in (2.3), that is, $\text{sign}(t - t_0) = \text{sign}(s - s_0)$ in the present case for $t \in (0, 1)$ and $s > 0$.

We substitute the expansion $t = t_0 + \sum_{k=1}^{\infty} t_k (s - s_0)^k$ and find for the first coefficient

$$t_1 = \sqrt{\frac{\psi^{(2)}(s_0)}{\phi^{(2)}(t_0)},}$$

(7.5)

where, again, the sign of the square root is positive to satisfy the condition $\text{sign}(s - s_0) = \text{sign}(t - t_0)$. The other coefficients $t_k$ can be found by simple computer algebra methods. The first few are

$$t_2 = \frac{\psi_3 - \phi_3 t_1^2}{6 \phi_2 t_1}, \quad t_3 = \frac{5 \phi_3^2 t_1^6 - 3 \phi_2 \phi_4 t_1^6 - 4 \phi_3 \psi_3 t_1^4 + 3 \phi_2 \psi_4 t_1^4 - \psi_5^2}{72 \phi_2^2 t_1^2},$$

(7.6)

where $\phi_k = \phi^{(k)}(t_0), \psi_k = \psi^{(k)}(s_0), k \geq 2$.

The next step is to find the coefficients $a_k(\mu)$ of the expansion $f(s) = \sum_{k=0}^{\infty} a_k(\mu)(s - s_0)^k$, where in the present example $f(s)$ is given in (3.8). The first coefficients are

$$a_0(\mu) = \frac{\mu t_1}{t_0(1 - t_0)}, \quad a_1(\mu) = \frac{2 \mu t_2 - 2 \mu t_2 t_0 + 2 \mu t_2 t_0 - \mu t_1}{t_0^2 (1 - t_0)^2},$$

(7.7)

When we have enough of these coefficients, we can evaluate the coefficients $f_n(\mu)$ needed in the expansions in (2.5) and (2.6). The first few $f_n(\mu)$ are

$$f_0(\mu) = a_0(\mu), \quad f_1(\mu) = \mu a_2(\mu), \quad f_2(\mu) = \mu (2 a_3(\mu) + 3 \mu a_4(\mu)),$$

$$f_3(\mu) = \mu (6 a_4(\mu) + 20 \mu a_5(\mu) + 15 \mu^2 a_6(\mu)),$$

$$f_4(\mu) = \mu (24 a_5(\mu) + 130 \mu a_6(\mu) + 210 \mu^2 a_7(\mu) + 105 \mu^3 a_8(\mu)),$$

(7.8)

Remark 7.1. The reviewer of [14] observed: It seems that the numerical coefficients are the same as the sequence A269940 in the OEIS. It would be worth investigating this in the future. See also https://oeis.org/A269940. This would imply

$$f_n(\mu) = \sum_{k=1}^{n} T(n, k) \mu^k a_{k+n}(\mu), \quad T(n, k) = \sum_{m=0}^{k} (-1)^{m+k} \binom{n+k}{n+m} s(n + m, m),$$

(7.9)
where \( s(n,m) \) are the Stirling numbers of the first kind. We have proved this relation by using mathematical induction.

Observe that, to avoid square roots in the formulas, we do not substitute \( t_0 \) given in (7.2) and \( t_1 \) in (7.5). A second point is to reduce the number of variables in the formulas. We see that the given scaled coefficient \( \tilde{f}_1(\mu) \) given in (3.15) is a function of two parameters only, namely \( \mu \) and \( \tau = t_0/\mu \). The derivatives of \( \phi(t) \) given in (7.1) at \( t_0 \) are functions of \( t_0, \mu \) and \( \beta \), but we have used the relation \((\beta + 1)t_0 = t_0^2 + \mu \) (see the numerator of \( \phi'(t) \) in (7.1)) to eliminate \( \beta \) from the formulas. This way we can make the final coefficients as simple as possible.

A final point is to have stable representations. When we would use \( \tilde{f}_1(\mu) \) given in (3.15) with \( \tau \) replaced by \( t_0/\mu \), a form arises that is still analytic at \( \mu = 0 \) (a crucial value in our asymptotics), but from a numerical point of view it becomes undefined at \( \mu = 0 \). The representation of \( \tilde{f}_1(\mu) \) in (3.15) in terms of the parameter \( \tau \) is stable for small values of \( \mu \), just as the higher coefficients.

### 7.1 The integration by parts procedure

To explain the relation between the coefficients \( f_n(\mu) \) and \( a_n(\mu) \) as shown in (7.8) we give a few steps in the integration by parts procedure. We write

\[
F_\lambda(z) = z^{-\lambda}f(\mu) - \frac{1}{z\Gamma(\lambda)} \int_0^\infty \frac{f(s) - f(\mu)}{s - \mu} \, ds,
\]

(7.10)

where

\[
f_1(s) = s \frac{d}{ds} \left( f(s) - f(\mu) \right).
\]

(7.11)

Continuing this procedure we obtain for \( K = 0, 1, 2, \ldots \)

\[
z^\lambda F_\lambda(z) = \sum_{k=0}^{K-1} \frac{f_k(\mu)}{z^k} + \frac{1}{z\Gamma(\lambda)} E_K(z, \mu),
\]

\[
f_k(s) = s \frac{d}{ds} \left( f_{k-1}(s) - f_{k-1}(\mu) \right), \quad k = 1, 2, \ldots, \quad f_0(s) = f(s),
\]

(7.12)

\[
E_K(z, \mu) = \frac{1}{\Gamma(\lambda)} \int_0^\infty s^{\lambda-1} e^{-zs} f_K(s) \, ds.
\]

Eventually we obtain the complete asymptotic expansion

\[
F_\lambda(z) \sim z^{-\lambda} \sum_{n=0}^\infty \frac{f_n(\mu)}{z^n}.
\]

(7.13)

As shown in (7.8), the coefficients \( f_n(\mu) \) can be expressed in terms of the coefficients \( a_n(\mu) \). To verify this we write

\[
f(s) = \sum_{m=0}^\infty a_m(\mu)(s - \mu)^m, \quad f_n(s) = \sum_{m=0}^\infty c_m^{(n)}(s - \mu)^m, \quad c_m^{(0)} = a_m(\mu).
\]

(7.14)
We see that $f_n(\mu) = c_0^{(n)}$ and we have from (7.12)

$$f_{n+1}(s) = \sum_{m=0}^{\infty} c_{m+1}^{(n)}(s-\mu)^m = s \sum_{m=1}^{\infty} c_{m}^{(n)}(m-1)(s-\mu)^{m-2}. \quad (7.15)$$

This gives the recursion

$$c_{m}^{(n+1)} = mc_{m+1}^{(n)} + \mu(m+1)c_{m+2}^{(n)}, \quad m, n = 0, 1, 2, \ldots . \quad (7.16)$$

All these coefficients can be expressed in terms of $a_m(\mu)$, and especially $c_0^{(n)} = f_n(\mu)$. This gives the relations between $f_n(\mu)$ and $a_n(\mu)$, the first ones being given in (7.8).

The procedure described here can be applied in exactly the same way to obtain the coefficients $p_n(\mu)$ of the asymptotic expansion in (4.10). A limited number of coefficients are provided in this article, but more of these are available from the authors.

Acknowledgments

The authors thank the reviewers for carefully reading the manuscript and for constructive suggestions.

NMT acknowledges financial support from Ministerio de Ciencia e Innovación, project MTM2012-11686.

NMT thanks CWI, Amsterdam, and the Universidad de Cantabria, Spain, for support.

EJMV thanks TUDelft, Delft, for support.

References

[1] R. A. Askey and R. Roy. Chapter 13, Gamma Function. In NIST Handbook of Mathematical Functions, pages 135–147. U.S. Dept. Commerce, Washington, DC, 2010.

[2] T. M. Dunster. Uniform asymptotic expansions for Whittaker’s confluent hypergeometric functions. SIAM J. Math. Anal., 20(3):744–760, 1989.

[3] T. M. Dunster. Uniform asymptotic expansions for the Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ with $\mu$ large. Proc. A., 477(2252):Paper No. 20210360, 18, 2021.

[4] F. Johansson. Computing hypergeometric functions rigorously. ACM Trans. Math. Software, 45(3):Art. 30, 26, 2019.

[5] A. B. Olde Daalhuis. Chapter 13, Confluent Hypergeometric Functions. In NIST Handbook of Mathematical Functions, pages 321–349. Cambridge University Press, Cambridge, 2010. http://dlmf.nist.gov/13

[6] F. W. J. Olver. Whittaker functions with both parameters large: uniform approximations in terms of parabolic cylinder functions. Proc. Roy. Soc. Edinburgh Sect. A, 86(3-4):213–234, 1980.
[7] F. W. J. Olver. *Asymptotics and Special Functions*. AKP Classics. A K Peters Ltd., Wellesley, MA, 1997. Reprint of the 1974 original [Academic Press, New York].

[8] L. J. Slater. *Confluent Hypergeometric Functions*. Cambridge University Press, New York, 1960.

[9] N. M. Temme. Laplace type integrals: transformation to standard form and uniform asymptotic expansions. *Quart. Appl. Math.*, 43(1):103–123, 1985.

[10] N. M. Temme. *Asymptotic Methods for Integrals*, volume 6 of *Series in Analysis*. World Scientific Publishing Co. Pte. Ltd., Hackensack, 2015.

[11] N. M. Temme. Asymptotic expansions of Kummer hypergeometric functions for large values of the parameters. *Integral Transforms and Special Functions*, February 2021.