Mixed Finite Element Matrix for Plate Bending Analysis Using Weighted-Residual Integral and Weak Formulation

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Abstract—The Conventional Finite Element Method for the analysis of plate bending based on a the principle of stationarity of total potential energy which is well known as Stiffness Formulation, using a variety of conformal and non-conformal elements was widely applicable in early of numerical solutions for such problems, later in order to improve the efficiency of such solutions, a more general and flexible formulation called Mixed Formulation based on variational principles which can be regarded as an extension of the Stiffness principle become important alternative. The explicit stiffness matrix for plate bending was given in the most if it’s not in all literatures so the reader can easy follow the solving procedure for such problems and verifying any published results, however such like-matrix (Augmented Matrix) in case of mixed formulation not given, at least for simple elements in the literatures as well as concerned researches. This quietly leads to some difficulties concerning the verification as well as understanding the published results. The main objective of this paper is to introduce this matrix in abbreviated size followed by a applicability through a some detailed examples for some bending models. The derived matrix will be helpful subject of research work in addition to, reveals a very good feeling with understanding and verification the published results in plus to comparing with the analytical solutions of different plate bending problems as the reader can do that.

Index Terms—Finite Element Method, Plate Bending, Weighted-Residual Integral, Weak Formulation.

I. INTRODUCTION

In the Conventional Finite Element Method (CFEM), which is based on the Stiffness Formulation, yields a very accurate displacement for most cases, while the stresses (or moments), which are derived as post-calculation from the displacement field, are not continuous across the boundaries. Although this drawback of (CFEM) may be considered as the major one for solving various engineering problems up to date, and in order to have stress continuity across the element boundary, the moments should be included as independent variables in the formulation. Such an approach is referred to as Mixed Formulation, which is a more general and flexible formulation utilizing variational principles that can be regarded as an extension of the stiffness principle taking significant place in modern field of engineering analysis. Although this Mixed Formulation is a matter of many researches for Finite Element Method as well as other techniques, for plate bending problems, but the explicit forms of the Augmented Matrix were not given for some reasons [1]. This paper describes the construction of this augmented matrix and presents its explicit form through systematic way in order to overcome the size difficulty for finite element plate bending models. In absence of such explicit matrices for all, the correctness and fineness of the published results so obtained remains in doubt due to many interpretations, such as the mathematical expressions of matrix itself, threatening of boundary conditions, typographical errors, etc. In this work, some of these doubtfully interpretations were showed and thus contributing in comparing and verifying many of results so obtained in many published research, as well as it will help advanced researchers in this field. Hence it can indicate the reason for this summarized detailed work in order to be simple, but a brief recall revision for many of basic concepts may be necessarily convenient for the reader while some detail of proofs was omitted, for the sake of brevity.

II. REVIEW OF SOME BASIC CONCEPTS OF ENGINEERING AND MATHEMATICAL ANALYSIS

In general the finite element method is a technique for constructing approximation functions required in an element-wise application of any variational method, and in order to bring this method in easiness view of understanding, it is necessary to start this review from the method of weighted residuals and weighted-integral statements to arrive at the weak formulation of differential equations, because the weak formulation plays a significance role in the classification of boundary conditions into the so-called natural and essential boundary conditions, which in turn are important in obtaining the interpolation functions as well as the nodal degrees of freedom to be considered in the finite element model. The approximate solution of the Governing Differential Equation (GDEq) such as \( F(u', u, x), x \in [a, b] \) of a problem consists of assuming and substituting a trial function with undetermined coefficients: \( U_N(x) = \sum_{j=1}^{N} c_j \phi_j + \phi_0 \), in which will satisfy neither the (GDEq) in its domain nor the boundary conditions on its boundary produce a so-called by CRANDALL [2]; the equation residual or simply an error: \( R(x) = F(U', U, x) \) which was abbreviated by [3] as the Residual \( R(x) \) in satisfying such equations, classified by this later reference as domain residual and boundary residual.

The determination of such undetermined coefficients to make residual identical to zero everywhere will yield the exact solution of (GDEq), (i.e. this process sometimes called differential formulation), but due the fact that such process.

DOI: http://dx.doi.org/10.24018/ejers.2020.5.8.2041
Vol 5 | Issue 8 | August 2020
does not always result in the required of linearly independent algebraic equations corresponding to those unknowns. CRANDALL [2] introduced a criteria to insure that there are exactly the same number of equations as there unknowns and make this residual stays close to zero throughout the interval [a,b] which in turn provides a first category of techniques for solve such problems which are:

1- Collocation method developed in [4]: This technique depends on choosing as many locations $x_k$ as there are undetermined coefficients and making the residuals $R(x_k)$ vanishes at those locations. Using Dirac delta function $\delta(x-x_k)$, this method can be expressed in integral form as: $\int_a^b \delta(x-x_k)R(x)dx = 0$.

2- Subdomain method [5]: Depending on dividing the desired interval into as many as subdomains as there are undetermined coefficients, by calculating and enforcing the average value of the residual in each subdomain to be zero where mathematically, the average value of any function $R(x)$ in an interval $[a,b]$ defined as $(\int_a^b R(x)dx) / (\int_a^b dx)$ and this vanishes when the numerator is zero. Hence: $\int_a^b R(x)dx = 0$.

3- Galerkin’s method [6]: It requires that the weighted averages of the residual over the desired interval should vanish, that is $(\int_a^b w(x)R(x)dx) / (\int_a^b w(x)dx)$ where $w(x)$ is a weight function and this also vanishes when the numerator is zero, hence $\int_a^b w(x)R(x)dx = 0$ ; which mathematically implies that both of $w(x)$ & $R(x)$ are orthogonal to each other in the interval $[a,b]$.

4- The method of least squares [7]: This method depends on the minimization technique as it is well known in elementary calculus, the necessary condition for a function $f(x)$ to have a maximum or a minimum (extremum) at a certain point is that the first derivative of the function shall vanish at that point, $(df(x)/dx) = 0$, while for a function of several variables is that all its first order partial derivatives should vanish thus to minimize the integral of square of the residual over the interval $[a,b]$, casted as:

$$\frac{\partial}{\partial c_i} \int_a^b R^2(x)dx = \int_a^b R(x) \frac{\partial R(x)}{\partial c_i} dx = 0$$

These integral-statements suggest the so-called in general, integral-residual methods in which the weighed-integral statements for certain specified problem provide a new function in integral form of (GDEq) of the problem such as $I(u) = \int_a^b F(x,u,u')dx$, observing that $I$ depends on $u(x)$. These forms are referred to as Functionals which are functions defined by integrals, whose integrands in general are functions of dependent variables and their derivatives that are themselves functions of other parameters such as a position in one, two or three dimensions, time, etc. Thus the functional is a function of functions. For example, of this terminology, the most well-known functional is the total potential energy functional provided in solid mechanics.

In Similar manner as in elementary calculus the extremum of functionals can be obtained through a process called variation of the functional providing the so-called stationary condition of the functional which is the one for which the variation of integral is zero, i.e. $\delta I(u) = 0 \rightarrow \delta I(u) = \delta \int_a^b F(x,u,u')dx = \int_a^b \delta F(x,u,u')dx = 0$ , where the symbol $\delta$ is called variation operator. The laws for the variation of sums, products, ratios, powers and so forth are completely analogous to differentiation operator. However, it is important to note that the differential of a function $df$ in differential calculus represents a first-order approximation to the change in the function along a particular curve while the variation of the functional $\delta F$ in calculus of variations is the first-order approximation to the change in the functional from curve to curve. This branch of science is termed variational principles, and for detailed reference on this topic, refer to [8].

If the position vector of any point on anticlockwise sense oriented curve in xy-plane is parameterized with arc length $s$ then in Cartesian system is: $r(s) = x(s)i + y(s)j$, where $i$, $j$ & $k$ are standard basis vectors in which the later one is perpendicular to xy-plane and the corresponding unit tangent vector is $dr(ds) = dx/ds i + dy/ds j$. Cross product $(dr/\|dr\| x k)$ gives a unit normal vector $\pm n$ to the curve at that point, that is: $n = dy/\|ds\ i - dx/\|ds\ j$ abbreviated as: $n = n_x i + n_y j$ where $n_x$ & $n_y$ are its horizontal and vertical components respectively and its norm $\|n\| = \sqrt{n_x^2 + n_y^2} = 1$. In general, for any level curve $\theta(x,y) = c$, the unit normal is given in terms of gradient vector $\nabla \theta$ as $n = \pm \nabla \theta / \|\nabla \theta\|$. Since the polar form of any vector in xy-plane is given by $r = (x = \|r\| \cos \theta)i + (y = \|r\| \sin \theta)j$, where $\theta$ is the angle between positive x-axis and the vector $r$, then the unit normal vector: $n = dy/\|ds\ i - dx/\|ds\ j \equiv \cos \theta i - (-\sin \theta)j$, therefore: $n_x = \cos \theta$ & $n_y = \sin \theta$, which called direction cosines of the unit normal vector $r$, implies that $dy/\|ds\ = n_x \Rightarrow dy = n_x ds \equiv \cos \theta ds$, and $-dx/\|ds\ = n_y \Rightarrow dx = -n_y ds \equiv -\sin \theta ds$.

An important theorems and relationships in vector integral calculus may be needed later during formulation process should be recalled here for making basic reference. The double integral $\int_R w(x,y) \frac{\partial^2 f(x,y)}{\partial y^2} dR$, over the two-dimensional region $R$ defined by $R \in \{x_1 = a \leq x \leq x_2 = b, y_1 = p(x) \leq y \leq y_2 = q(x)\}$ bounded by simple closed curve $C$ as shown in Fig. 1, is related to a line integral around $C$ through carried out the integration first over $x$ and then over $y$, yielding a relation (or Green's identities) which serves as branch of the so-called Green-Gauss Theorem:
The Mixed Formulation for orthotropic thin plate bending element based on Kirchhoff plate theory can be introduced according to Hellinger-Reissner functional, but here a technique of weighted-residual methods will be followed, which are based on weighted-integral statements of the governing equations which in turn reduced to the so-called Weak Formulations as it is introduced by [3] in the following sub titles.

### A. Review of Mixed Formulation Principle

The Weighted-residual method is introduced to provide means for obtaining as many independent relations as there are unknown coefficients in approximate solution when it is substituted in a weighted integral statement of a given governing differential equation. These independent relations can be obtained through two main ways as:

The first one by direct substitution of a trial function into a weighted-integral statement of the GDEq. such that the residual in GDEq. be orthogonal to the set of weight functions i.e. \( \int_{a}^{b} w(x) R(x) \, dx = 0 \), which will provide the required number of equations for determining the coefficients of the trial function. The trial function as well as the weight functions must satisfy the continuity requirements by weighted-integral statement and all types of the specified homogenous boundary conditions of the problem, in which it may require higher-order functions.

The second way is exactly as the first one but before substitution is made, the even-order of differentiation of dependent variable in GDEq must be equally distributed among the dependent variable and the weight function. This procedure is well known as a weak form of a weighted-integral statement of GDEq. Hence it is clear that the continuity requirements on trial functions will be reduced, and thus the weak-form name is organized. This weakness of the continuity can be performed using Green-Gauss theorem (generalized integration by parts indicated above), therefore as result of this weakened process the so-called natural boundary conditions will be included in the formulation which in turn the trial functions need not be to satisfying the conditions priorly, rather than it should satisfy only the so-called essential condition. The construction of weak formulation can be summarized in the following three steps organized by [3]:

1. Moving all expressions of the governing differential equation to one side and multiplying the entire equation with a weight function and recasting it in a homogenous weighted integral form over its domain.

2. Weaken the continuity by trading the differentiation from dependent variable to the weight function in the governing differential equation using Green-Gauss theorem.

3. Identifying of the primary and secondary variables of the weak form, with the aid of role suggested by [3] that is: after completing weak process, boundary terms will involve both the weight function and the dependent variable then, the actual dependent variable of the problem which is expressed in the same form as the weight function, is called the Primary variable, while the coefficients of the weight function and its derivatives in the boundary expressions are referred to as the Secondary variables, and their
specifications in case of stiffness formulation, represents the Natural Boundary Conditions, and the Essential Boundary Conditions respectively.

B. Review of the Basic Relations and Governing Equations for Bending of Thin Plate

An element $dx \times dy$ of a plate subjected to uniformly distributed load per unit area $q(x, y)$ as shown in the Fig. 2, where the positive sign convention for moments (i.e. stress resultants) and forces per unit length are indicated:

\[ Q_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y}, Q_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x}, \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q \]

The properties of natural anisotropic as well as structural anisotropy material depending on the direction, and in such material all the strains are coupled to all the stresses [10]. The orthogonally anisotropic is an anisotropic material that has symmetrical three mutually perpendicular planes with respect to its elastic properties [11], such plate material is referred to as the orthotropic plate that has wide practical applications in engineering, where if the principal material axes that are normal to the planes of symmetry of 2D-orthotropy coincide with the $x$ & $y$ coordinate axes then the four elastic constants; young's moduli along the principal material axes $E_x, E_y$ and passion's ratios $v_{xz} & v_{yz}$ that characterize the decrease in a specified-direction during force applied in the other one, since in general the lateral strain is proportional to the longitudinal strain [12], are required for full description of such orthotropic material stress-strain relationships, where due to symmetry the Betti’s reciprocal theorem reveals $E_x v_{yz} = E_y v_{xz}$; therefore $v_{xz} = v_{yz} E_x / E_y$, and for sake of simplifying, the following notations will be used henceforth: $v_{xy} = \sqrt{v_{yz} v_{xz}}$, $E_{xy} = \sqrt{E_x E_y}$. The bending rigidities $D_{xx}, D_{yy}$, twisting rigidity $D_{xy}$ and the shear modulus $G_{xy}$ which characterize changes of angles between principle directions $x$ & $y$, for plate of thickness $t$ are given by [14]:

\[ D_x = E_t t^3/12(1 - v_{xz} v_{yz}), D_y = E_t t^3/12(1 - v_{yz} v_{xy}) \]

\[ D_{xy} = G_{xy} t^3/12 \]

relating bending and twisting moments ($m_x, m_y$ & $m_{xy}$) with deflection $w$ are [14]:

\[ m_x = -D_x(\partial^2 w/\partial x^2 + v_y \partial^2 w/\partial y^2) \]

\[ m_y = -D_y(\partial^2 w/\partial y^2 + v_x \partial^2 w/\partial x^2) \]

\[ m_{xy} = -2 D_{xy} \partial^2 w/\partial x \partial y \]

and the relationship between bending, twisting moments and applied load $q$ is governed by the following differential equation of equilibrium

\[ -\left(\frac{\partial^2 m_x}{\partial x^2} + 2 \frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2}\right) = q \]

Substitution of expressions (3) into (4) yields the governing differential equation for the deflection of the orthotropic plate as [14]:

\[ D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q(x, y) \]

where $H = D_1 + 2D_{xy}$, and $D_1 = v_x D_y = v_y D_x$

C. Weak Form Formulation For Bending Of Orthotropic Thin Plate

Solving relations (3) for $\partial^2 w / \partial x^2, \partial^2 w / \partial y^2$ & $2 \partial^2 w / \partial x \partial y$ yields:

\[ \partial^2 w / \partial x^2 = -\left(D_{11} m_x - D_{12} m_y\right) \]

\[ \partial^2 w / \partial y^2 = -\left(D_{22} m_y - D_{21} m_x\right) \]

\[ 2 \partial^2 w / \partial x \partial y = -D_{33} m_{xy} \]

Remark:
The second equation in the last set (6) was given in [15] in a similar manner as: $\partial^2 w / \partial y^2 = -\left(D_{12} m_{xx} - D_{11} m_{yy}\right)$ and with disregarding the differences between constant notations, we think it involves sign typographical mistake should be corrected as in (6), where:

\[ D_{11} = 1/((1 - v_x v_y)D_x), D_{12} = v_y/(1 - v_x v_y)D_y \]

\[ D_{21} = v_x/(1 - v_x v_y)D_x, D_{22} = 1/(1 - v_x v_y)D_y \]

\[ D_{33} = 1/D_{xy} \]

But since $D_{21} = v_x/(1 - v_x v_y)D_x \equiv 12v_x/E_{xt^3}$ and $D_{12} = 12v_y/E_{yt^3} \equiv 12v_y/E_{yt^3} = D_{21}$, then the last constants conveniently are redefined as:

\[ D_{11} = 12/E_{xt^3}, D_{12} = D_{21} = 12v_x/E_{xt^3} \]

\[ D_{22} = 12/E_{yt^3}, D_{33} = 12/G_{xy} t^3 \]

Applying three steps procedure for the set of equations (4) & (6), where for better understanding view this applying for equation (4) over a region $R$ and the weight function $v$, yields:

\[ \int_R v \left(-\frac{\partial^2 m_x}{\partial x^2} - 2 \frac{\partial^2 m_{xy}}{\partial x \partial y} - \frac{\partial^2 m_y}{\partial y^2} - q\right) dR = 0 \]

with aid of the relations (1 & 2), indicated above, the first term becomes:
\[ - \int_R \nabla^2 \psi \, dR = \int_R \nabla \psi \cdot \nabla \omega \, dR - \oint_C \psi \, n_x \, ds \]

Examination of the boundary term in the light of work done concept shows that the weight function represents a deflection \( v \sim w \) therefore:

\[ - \int_R \nabla^2 \psi \, dR = \int_R \nabla \psi \cdot \nabla \omega \, dR - \oint_C \psi \, n_x \, ds \]

The term \(-2 \int_R \nabla^2 \psi \, dR\) can be extended as:

\[ - \int_R \nabla^2 \psi \, dR - \int_R \nabla^2 \psi \, dR, \text{ then trading the differentiation with respect to } x \text{ in the first part while to } y \text{ in the other one, yields} \]

\[ - \int_R \nabla^2 \psi \, dR = \int_R \nabla \psi \cdot \nabla \omega \, dR - \oint_C \psi \, n_x \, ds \]

Finally, this process step yields

\[ \left\{ \begin{array}{l}
\int_R \left[ \frac{\partial \psi}{\partial x} \frac{\partial m_x}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial m_x}{\partial y} + \frac{\partial \psi}{\partial x} \frac{\partial m_y}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial m_y}{\partial y} - q \, \psi \right] \, dR - \\
\oint_C \left[ \frac{\partial m_x}{\partial x} n_x + \frac{\partial m_y}{\partial y} n_y + \frac{\partial m_{xy}}{\partial x} n_x + \frac{\partial m_{xy}}{\partial y} n_y \right] \, ds = 0 \\
\int_R \left[ \frac{\partial m_{xy}}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial m_{xy}}{\partial y} \frac{\partial w}{\partial y} - D_{33} m_{xy} m_{xy} \right] \, dR - \\
\oint_C \left[ m_{xy} \frac{\partial w}{\partial x} n_x + m_{xy} \frac{\partial w}{\partial y} n_y \right] \, ds = 0
\end{array} \right. \]

where from equilibrium relations mentioned earlier that is \( Q_n = Q_x n_x + Q_y n_y \), and application of the same treatment to the rest of equations (6), yields the following set of weak-form equations as:

1. \[ \left\{ \begin{array}{l}
\int_R \left[ \frac{\partial \psi}{\partial x} \frac{\partial m_x}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial m_x}{\partial y} + \frac{\partial \psi}{\partial x} \frac{\partial m_y}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial m_y}{\partial y} - q \, \psi \right] \, dR - \\
\oint_C \left[ \frac{\partial m_x}{\partial x} n_x + \frac{\partial m_y}{\partial y} n_y + \frac{\partial m_{xy}}{\partial x} n_x + \frac{\partial m_{xy}}{\partial y} n_y \right] \, ds = 0 \\
\int_R \left[ \frac{\partial m_{xy}}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial m_{xy}}{\partial y} \frac{\partial w}{\partial y} - D_{33} m_{xy} m_{xy} \right] \, dR - \\
\oint_C \left[ m_{xy} \frac{\partial w}{\partial x} n_x + m_{xy} \frac{\partial w}{\partial y} n_y \right] \, ds = 0
\end{array} \right. \]

Using appropriate interpolation function for each of the field variables in the problem, which can be written as:

\[ \text{Field Variable} = [N_p][\Delta_p], \text{ where } [N_p] \text{ is the shape function matrix and } \Delta_p \text{ is the corresponding nodal values of field variable } P. \text{ Substituting these interpolation functions for field variables } [m_x, m_y, m_{xy}, w] \text{ into weak form set, the following finite element model can be obtained:} \]

\[ \left[ \begin{array}{cccc}
[K_{11}] & [K_{12}] & [K_{13}] & [K_{14}] \\
[K_{22}] & [K_{23}] & [K_{24}] & [K_{25}] \\
[K_{33}] & [K_{34}] & [K_{35}] & [K_{36}] \\
[K_{44}] & [K_{45}] & [K_{46}] & [K_{47}] \\
\end{array} \right] \left[ \begin{array}{c}
m_1 \\
m_2 \\
m_3 \\
m_4 \\
\end{array} \right] = \left[ \begin{array}{c}
[F_1] \\
[F_2] \\
[F_3] \\
[F_4] \\
\end{array} \right] + \left[ \begin{array}{c}
[0] \\
[0] \\
[0] \\
[0] \\
\end{array} \right] \quad (7) \]

Where, for rectangular plate element \( x \in [0.a] \) \& \( y \in [0.b] \):

\[ [K_{11}] = -D_{11} \int_0^a \int_0^b [N_{mx}]^T [N_{mx}] dxdy \]

\[ [K_{12}] = (D_{12})^* \int_0^a \int_0^b [N_{mx}]^T [N_{my}] dxdy \]

\[ [K_{13}] = \int_0^a \int_0^b [N_{my}]^T [N_{nx}] dxdy \]

Remark: \((D_{12})^* \) this term was given with minus sign in eq. (10.16) in Reference [15], which seems to also be a typographical error.

\[ [K_{44}] = \int_0^a \int_0^b [N_{xy}]^T [N_{xy}] dxdy \]

\[ [K_{33}] = (-D_{33})^* \int_0^a \int_0^b [N_{mx}]^T [N_{mx}] dxdy \]

Remark: in eq. (10.16) in Reference [15], the term \((-D_{33})^* \) was given with positive sign, which seems to also be a typographical error, and it should be as indicated in part 4 above.

\[ [K_{34}] = \int_0^a \int_0^b \left[ \frac{\partial}{\partial y} [N_{mx}]^T \right] \frac{\partial}{\partial x} [N_{w}] \right] dxdy \]

\[ [K_{13}] = [K_{23}] = [K_{34}] = [0] \]

\[ [K_{21}] = [K_{12}]^T, [K_{31}] = [K_{13}]^T, [K_{32}] = [K_{23}]^T \]

\[ [K_{41}] = [K_{42}]^T, [K_{43}] = [K_{24}]^T \]

and the external applied loads vector \([F_i]\) = \( \int_R q \, w dR \)

The other sub-vectors which represent the work done by end actions along the boundaries are:

\[ [F_1] = \oint_C \left[ m_x \frac{\partial w}{\partial x} n_x + m_y \frac{\partial w}{\partial y} n_y \right] ds \]

\[ [F_2] = \oint_C \left[ m_y \frac{\partial w}{\partial y} n_x + m_x \frac{\partial w}{\partial x} n_y \right] ds \]

\[ [F_3] = \oint_C \left[ \frac{\partial m_{xy}}{\partial x} n_x + \frac{\partial m_{xy}}{\partial y} n_y \right] ds \]

Since \( \frac{\partial m_{x}}{\partial x} = V_x \) and \( \frac{\partial m_{y}}{\partial y} = V_y \) are the shear forces along y-direction and x-direction respectively, and if the twisting moment terms in the last \([F_4]\) integrating once more by parts then:
where at end points of each directions $x$ & $y$, the terms $[m_{xy}]_{y=0,b}$ & $[m_{xy}]_{x=0,a}$ if they exist, will produce vertical concentrated forces of magnitude $F_c = -2m_{xy}$ at the corner of the polygonal plates and thus $[m_{xy} \ w \ n_{y}]_{y=0,b} & [m_{xy} \ w \ n_{y}]_{x=0,a}$ are set to zero because they produce no work due to $w = 0$ for all supporting cases. Therefore the original $[F_3]$ is replaced by:

$$[F_3] = \int_C \left( \left( V_x n_x + V_y n_y \right) w + \frac{\partial w}{\partial y} m_{xy} n_x + \frac{\partial w}{\partial x} m_{xy} n_y \right) ds$$

All these vectors will vanish at inter-element boundaries due to equilibrium requirements at the any two adjacent boundaries of any two successive elements, but at outer edges (i.e. boundaries of the plate), these line integrals must be investigated for different support-type edges. For example, by considering a rectangular plate then:

**Simply supported along $x$-axis;** $n_x = 0 , n_y = \pm 1$

$[F_3] = 0$ due to $n_x = 0 , [F_3] = 0$ due to $m_{xy} = 0$

$[F_3] \neq 0$, since $m_{xy} \neq 0$ and $\partial w / \partial x$ is arbitrary

$[F_3] \neq 0$, $w = 0$, but $m_{xy} \neq 0$ and $\partial w / \partial x$ is arbitrary

**Simply supported along $y$-axis;** $n_x = \pm 1, n_y = 0$

$[F_3] = 0$ due to $m_{xy} = 0 , [F_3] = 0$ due to $n_y = 0$

$[F_3] \neq 0$, since $m_{xy} \neq 0$ and $\partial w / \partial y$ is arbitrary

$[F_3] \neq 0$, $w = 0$, but $m_{xy} \neq 0$ and $\partial w / \partial y$ is arbitrary

**Built-in supported along $x$-axis;** $n_x = 0, n_y = \pm 1$

$[F_3] = 0$ due to $n_x = 0 , [F_3] = 0$ due to $\partial w / \partial y = 0$

$[F_3] = [F_3] = 0$ due to $m_{xy} = 0$ and $w = 0$

**Built-in supported along $y$-axis;** $n_x = \pm 1, n_y = 0$

$[F_3] = 0$ due to $\partial w / \partial x = 0 , [F_3] = 0$ due to $n_y = 0$

$[F_3] = [F_3] = 0$ due to $m_{xy} = 0$ and $w = 0$

**Suspended free edge along $x$-axis;** $n_x = 0, n_y = \pm 1$

$[F_3] = 0$ due to $n_x = 0 , [F_3] = 0$ due to $m_{xy} = 0$

$[F_3] \neq 0$, since $m_{xy} \neq 0$ and $\partial w / \partial x$ is arbitrary

$[F_3] \neq 0$, $w = 0$, but $m_{xy} \neq 0$ and $\partial w / \partial x$ is arbitrary

**Suspended free edge along $y$-axis;** $n_x = \pm 1, n_y = 0$

$[F_3] = 0$ due to $m_{xy} = 0 , [F_3] = 0$ due to $n_y = 0$

$[F_3] \neq 0$, since $m_{xy} \neq 0$ and $\partial w / \partial y$ is arbitrary

$[F_3] \neq 0$, $w = 0$, but $m_{xy} \neq 0$ and $\partial w / \partial y$ is arbitrary

while if it is entirely free edge then all these line integrals vanish. This example shows that the line integrals $[F_3]$ and its identical part of $[F_3]$ which are the same as that in Hellinger-Reissner functional (i.e. $\oint m_{mx} \partial w / \partial s \ ds$) should be included in this formulation as:

$$[F_3] = \int_0^a \left\{ \left[ N_{n_{xy}} \right]^T \frac{\partial}{\partial x} [N_{w}] \right\}_{y=b} - \left[ N_{n_{xy}} \right]^T \frac{\partial}{\partial x} [N_{w}] \right\}_{y=0} \right\} dx + \int_0^b \left\{ \left[ N_{n_{xy}} \right]^T \frac{\partial}{\partial y} [N_{w}] \right\}_{x=a} - \left[ N_{n_{xy}} \right]^T \frac{\partial}{\partial y} [N_{w}] \right\}_{x=0} \right\} dy$$

where $[F_3] = [F_3]^T$, $[K_{34}] = [K_{34}]^T$, and the imposing of boundary conditions process will fix the necessary behavior, therefore $[K_{34}]$ becomes:

$$[K_{34}] = \int_0^b \int_0^a \left\{ \left[ N_{n_{xy}} \right]^T \frac{\partial}{\partial x} [N_{w}] \right\}_{y=b} - \left[ N_{n_{xy}} \right]^T \frac{\partial}{\partial x} [N_{w}] \right\}_{y=0} \right\} dx \ dy - \int_0^b \int_0^a \left\{ \left[ N_{n_{xy}} \right]^T \frac{\partial}{\partial y} [N_{w}] \right\}_{x=a} - \left[ N_{n_{xy}} \right]^T \frac{\partial}{\partial y} [N_{w}] \right\}_{x=0} \right\} dy$$

D. Weak Form Formulation for Bending of Isotropic Thin Plate

The same foregoing detailed explanation can be extended to the isotropic plate which is the one that has the same material property in all directions, those are:

$$v_x = v_y = v_{xy} = v, E_x = E_y = E_{xy} = E$$

$$D_x = D_y = E t^3/12 (1 - \nu^2) \equiv D$$

$$G_{xy} = E/2(1 + \nu) \equiv G$$

$$D_{xy} = E t^3/24(1 + \nu) = (1 - \nu)D/2$$

$$D_1 = E t^3/12 (1 - \nu^2) = v D$$

$$B = D_1 + 2D_{xy} = v D + 2 (1 - \nu)/2D = D$$

and thus:

$$D_{11} = 1/(1 - \nu^2)D_1 = 12/E t^3 \equiv D_{22}$$

$$D_{12} = v \sqrt{1 - \nu^2}D_1 = 12v/E t^3 \equiv D_{21}$$

$$D_{33} = 1/D_{xy} = 12/Gt^3 = 24(1 + \nu)/E t^3$$

where $E$ is the modulus of elasticity and $v$ is a passion ratio of an isotropic plate.

E. The Four Node Serendipity Quadrilateral Elements

Zienkiewicz called the elements whose nodes are only on the their boundaries, as ‘Serendip family’ elements, by referring to the famous princess of Serendip noted for chance discoveries [16] and the most known one is the 4-noded (corner-noded) element, and according to the formulation in this article there exist four degrees of freedom DOF at each node $\{m_{x}, m_{y}, m_{xy}, w\}$. Since for each of the field variables of the problem, this element has one unknown nodal value per node point; totally four unknowns, then mathematically four undetermined constants must be employed in the polynomial expression chosen to represent each filled variable ; for example the field variable $m_{x}$ is approximated by bi-linear polynomial as:

$$m_{x}(x, y) = a_0 + a_1 x + a_2 y + a_3 xy$$

$$m_{x}(x, y) = \{1 \ x \ y \ xy\}[a_0 \ a_1 \ a_2 \ a_3]^T = [F][A]$$

Hence such element is termed as linear element, since the variation of shape functions about a boundary is of that order. In this article derivation of shape functions for the linear 4-noded rectangular element (abbreviated as [R4]) with local numbering system as shown in Fig. 3 is reviewed as:
where: \( m_x(0,0) = m_{x1}, m_x(a,0) = m_{x2}, m_x(a,b) = m_{x3}, m_x(0,b) = m_{x4} \) therefore:

\[
\begin{bmatrix}
m_{x1} \\ m_{x2} \\ m_{x3} \\ m_{x4}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & a & b & ab \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a \\ a \\ a \\ a
\end{bmatrix}
\]

\[
[C] = [\Delta_{mxy}]^{-1} =
\begin{bmatrix}
m_{x1} & \ -m_{x1} + m_{x2} \\
-\frac{m_{x1} + m_{x4}}{a} & \ -\frac{m_{x1} + m_{x4}}{b} \\
m_{x1} - m_{x2} + m_{x3} - m_{x4} & \ \frac{m_{x2} - m_{x4}}{ab}
\end{bmatrix}
\]

yield the required interpolation (trial) function for \( m_x \) as:

\[
m_x = [F][C][\Delta_{mxy}]^{\frac{1}{2}} = [N_{mxy}][\Delta_{mxy}],\text{ where}
\]

\[
[N_{mxy}] = \begin{bmatrix}
1 - \frac{x}{a} y + \frac{ab}{a} b & x y & \frac{ab}{a} y & \frac{ab}{b} y
\end{bmatrix}
\]

is called shape functions matrix. Then in general the field variables are:

\[
m_x = [N_{mxy}][\Delta_{mxy}] \text{, } m_y = [N_{mxy}][\Delta_{mxy}], \quad w = [N_{mxy}][\Delta_{mxy}]
\]

Substitution of these interpolation function into sub-matrices (7) and performing the necessary multiplications and integrations yields the final closed (explicit) form of augmented matrix \( [K_e] \).

Since all the field variables are approximated with same interpolation function (i.e. \([N]\) is the same for all of them) and if the elastic constants are valued terms while letter \( v \) standing for non-valued terms then these matrices can be classified as the following four categories:

\[
[A] = \int_0^a \int_0^b \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} dx dy
\]

\[
[B] = \int_0^a \int_0^b \frac{\partial[N]^{T}}{\partial x} \frac{\partial[N]}{\partial x} dx dy
\]

\[
[C] = \int_0^a \int_0^b \frac{\partial[N]^{T}}{\partial y} \frac{\partial[N]}{\partial y} dx dy
\]

\[
[D] = \int_0^a \int_0^b \left( \frac{\partial[N]^{T}}{\partial y} \frac{\partial[N]}{\partial x} + \frac{\partial[N]^{T}}{\partial x} \frac{\partial[N]}{\partial y} \right) dx dy - [R]
\]

\[
[R] = \int_0^a \left( \frac{\partial[N]^{T}}{\partial x} \right)_{y=a} \left( \frac{\partial[N]}{\partial x} \right)_{y=0} dx
\]

\[
\int_0^a \frac{\partial[N]^{T}}{\partial x} \left|_{y=a} \right. dy - \int_0^a \frac{\partial[N]^{T}}{\partial y} \left|_{x=0} \right. dy
\]

Explicit expressions of the elements of these matrices for the 4-node serendipity rectangular elements finite element model are given in the appendix A. Thus the augmented matrix \([K_e]_{16 \times 16}\) for orthotropic plate can be constructed from the so-called Basic Matrix \([k_b]_{ij}\) of this formulation which is defined as:

\[
[k_b]_{ij} = \begin{bmatrix}
-D_{11}a_{ij} & -D_{12}a_{ij} & 0 & b_{ij} \\
-D_{12}a_{ij} & -D_{22}a_{ij} & 0 & c_{ij} \\
0 & 0 & -D_{33}a_{ij} & d_{ij} \\
c_{ij} & a_{ij} & e_{ij} & 0
\end{bmatrix}_{4 \times 4}
\]

where all the notations are as declared above, and the Element Augmented Matrix \([K_e]\) having the following arrangement:

\[
[K_e] = \begin{bmatrix}
[k_b]_{11} & [k_b]_{12} & [k_b]_{13} & [k_b]_{14} \\
[k_b]_{21} & [k_b]_{22} & [k_b]_{23} & [k_b]_{24} \\
[k_b]_{31} & [k_b]_{32} & [k_b]_{33} & [k_b]_{34} \\
[k_b]_{41} & [k_b]_{42} & [k_b]_{43} & [k_b]_{44}
\end{bmatrix}_{16 \times 16}
\]

Corresponding to the nodal values vector of the four nodes 1, 2, 3 & 4 according to local numbering system are:

\[
[m_x,m_y,m_{xy},w_1,...,m_x,m_y,m_{xy},w_4]^T
\]

While the augmented matrix \([K_e]_{16 \times 16}\), for isotropic can be obtained by the same Basic Matrix \([k_b]_{ij}\) just by using isotropic elastic constants indicated above.

As a sample, the shape of Overall Augmented matrix \([K_G]\) which is obtained from the implementation of this procedure where 0 is standing for zero-valued terms while letter \( v \) standing for non-zero valued one, is:

\[
[K_G] =
\]

**F. Element Augmented Matrix Builder Code**

This Element Augmented Matrix can be easily constructed from building block; Basic Matrix through implementation of the earlier discussed procedure in the following simple visual Basic coding:

For \( i = 1 \) To 4

\[
 m = (i - 1) * \text{DOF} + 1
\]

For \( j = 1 \) To 4
n = (j - 1) * DOF + 1

\[ kb(m, n) = -D_{ii} \ a(i,j) \]

\[ kb(m, n + 1) = D_{ij} \ a(i,j) \]

\[ kb(m, n + 3) = c(i,j) \]

\[ kb(m + 1, n) = -D_{ii} \ a(i,j) \]

\[ kb(m + 1, n + 1) = -2D_{ij} \ a(i,j) \]

\[ kb(m + 1, n + 2) = 0 \]

\[ kb(m + 1, n + 3) = c(i,j) \]

\[ kb(m + 2, n) = 0 \]

\[ kb(m + 2, n + 1) = 0 \]

\[ kb(m + 2, n + 3) = wD_{ij} \ a(i,j) \]

\[ kb(m + 3, n) = b(i,j) \]

\[ kb(m + 3, n + 1) = c(i,j) \]

\[ kb(m + 3, n + 2) = e(i,j) \]

\[ kb(m + 3, n + 3) = 0 \]

Next \( j \)

Next \( i \)

IV. IMPLEMENTATIONS OF THE DERIVED BASIC MATRIX AND NUMERICAL RESULTS

The above mixed rectangular element is now used with its Augmented Matrix to solve simple problems of square plates of length \( L \) with different types of edge conditions, under uniformly distributed load and concentrated load at the center of the plate and the results were compared with those from a many published literatures as well as with the available exact solutions.

Table I shows a comparison of displacements, bending moments at center and twisting moment at corner of all simply-supported plate subjected to uniform loading & concentrated load at the middle, with those parameters calculated by [15].

| TABLE I: RESULTS OF (SS-SS) PLATE ANALYSIS |
|---------------------------------------------|
| \( k_{m,n} \) | \( \text{DOF} \) | \( \text{Uniformly Distributed Load} \) | \( \text{Concentrated Load} \) |
| | | \( \alpha_{xx} \) | \( \alpha_{yy} \) | \( \alpha_{xy} \) | \( \alpha_{xx} \) | \( \alpha_{yy} \) | \( \alpha_{xy} \) |
| \( m \times n \) | | | | | | | |
| 1 x 1 | 16 | 461.3 | 719.6 | 719.6 | 290.6 | 184.5 | 184.5 | 287.8 |
| 2 x 2 | 36 | 423.7 | 524.6 | 524.6 | 308.2 | 134.8 | 134.8 | 340.0 |
| 3 x 3 | 64 | 414.0 | 497.6 | 497.6 | 316.0 | 125.2 | 125.2 | 382.8 |
| 4 x 4 | 100 | 410.6 | 489.2 | 489.2 | 319.1 | 121.6 | 121.6 | 411.7 |
| 5 x 5 | 144 | 409.0 | 485.4 | 485.4 | 321.0 | 119.8 | 119.8 | 434.3 |
| 6 x 6 | 196 | 408.2 | 483.4 | 483.4 | 322.0 | 118.7 | 118.7 | 453.0 |
| 7 x 7 | 256 | 407.6 | 482.2 | 482.2 | 322.7 | 118.1 | 118.1 | 468.8 |
| 8 x 8 | 324 | 407.3 | 481.4 | 481.4 | 323.1 | 117.6 | 117.6 | 482.5 |
| 16x16 | 1156 | 407.6 | 479.7 | 479.7 | 323.4 | 116.5 | 116.5 | 554.1 |
| Exact sol. | 406.20 | 478.86 | -324.6 | 116 |

Multiplier | \( 10^4 \text{Dips}^2 \) | \( 10^4 \text{qu}^2 \) | \( 10^4 \text{qu}^2 \) | \( 10^4 \text{qu}^2 \) |

Remark: These results obtained by imposing the boundary conditions: \( w = 0, \ mx = 0 \) at edge \( x = 0 \) and \( my = 0 \) at center line \( y = a/2 \), and \( y = 0 \) and \( my = 0 \) at center line \( y = b/2 \), in which they are exactly agreed with results in Reference [15].

Table II shows a comparison of displacements, bending moments at center of all clamped square plate subjected to uniform loading & concentrated load at the middle, with those parameters calculated by [15].

| TABLE II: RESULTS OF (CC-CC) PLATE ANALYSIS |
|---------------------------------------------|
| \( 1/4^b \) of plate | | \( \text{Uniformly Distributed Load} \) | \( \text{Concentrated Load} \) |
| | | \( \alpha_{xx} \) | \( \alpha_{yy} \) | \( \alpha_{xy} \) |
| Mesh size | \( \text{DOF} \) | BCS 1 | BCS 2 | BCS 1 | BCS 2 |
| 1 x 1 | 16 | 181.7 | 166.4 | 166.4 | 566.9 | 519.3 | 519.3 |

Table III shows a comparison of displacements, bending moment at the center and bending moment at the mid of clamped edge of a square plate with two opposite sides clamped and the other two simply-supported (C-C \( l x \) & S-S\( y \)), and subjected to uniform loading & concentrated load at the middle, with those parameters calculated by [15].

| TABLE III: RESULTS OF (C-C&S-S) PLATE ANALYSIS |
|---------------------------------------------|
| \( 1/4^b \) of plate | | \( \text{Uniformly Distributed Load} \) | \( \text{Concentrated Load} \) |
| | | \( \alpha_{xx} \) | \( \alpha_{yy} \) | \( \alpha_{xy} \) |
| Mesh size | \( \text{DOF} \) | BCS 1 | BCS 2 | BCS 1 | BCS 2 |
| 1 x 1 | 16 | 246.4 | 246.4 | 246.4 | 675.2 | 675.2 | 978.5 | 978.5 | -234.8 | 234.8 |
| 2 x 2 | 36 | 2168.2 | 2168.2 | 2168.2 | 4247.2 | 4247.2 | 846.8 | 846.8 | -150.6 | 150.7 |
| 3 x 3 | 64 | 203.7 | 203.7 | 203.7 | 368.1 | 368.1 | 780.1 | 780.1 | -163.4 | 163.4 |
| 4 x 4 | 100 | 198.7 | 198.7 | 198.7 | 352.6 | 352.6 | 751.4 | 751.4 | -163.8 | 163.8 |
| 5 x 5 | 144 | 196.3 | 196.3 | 196.3 | 345.2 | 345.2 | 736.6 | 736.6 | -164.3 | 164.3 |
| 6 x 6 | 196 | 194.9 | 194.9 | 194.9 | 343.1 | 343.1 | 727.9 | 727.9 | -164.7 | 164.7 |
| 7 x 7 | 256 | 191.4 | 191.4 | 191.4 | 339.0 | 339.0 | 722.3 | 722.3 | -165.0 | 165.0 |
| 8 x 8 | 324 | 193.5 | 193.5 | 193.5 | 337.4 | 337.4 | 718.5 | 718.5 | -165.2 | 165.2 |
| 16x16 | 1156 | 192.2 | 192.2 | 192.2 | 333.7 | 333.7 | 708.3 | 708.3 | -165.8 | 165.8 |

Multiplier | \( 10^4 \text{Dips}^2 \) | \( 10^4 \text{qu}^2 \) | \( 10^4 \text{qu}^2 \) | \( 10^4 \text{qu}^2 \) |

Remark: These results obtained by imposing the boundary conditions: \( w = 0, \ mx = 0 \) at simply side \( x = 0 \) and \( y = 0 \) at clamped side \( x = a/2 \) and \( y = b/2 \) at both center lines \( x = a/2 \) and \( y = b/2 \). But naturally is to impose \( m_{xy} = 0 \) at clamped side \( y = 0 \),The given results for \( 6 \times 8 \) by Reddy [15] is in fact it should be for \( 5 \times 5 \).

Table IV contains the values of deflection, bending moment and twisting moment at the specified locations, for
a square plate with two opposite sides simply-supported along y-direction and the other two opposite sides are clamped and free sides along x-direction (S-Sy / C-Fx).

| Mesh size | DOF | αex | αes | αex | αes |
|-----------|-----|-----|-----|-----|-----|
| x = 2 | 1 | 58.73 | 554.7 | 802.2 | 751.0 | 542.6 | 484.4 |
| x = 4 | 1 | 58.68 | 550.7 | 825.6 | 762.5 | 546.8 | 430.0 |
| x = 8 | 1 | 572.4 | 571.6 | 781.9 | 577.3 | 295.4 | 295.2 |
| x = 12 | 1 | 569.3 | 569.0 | 569.4 | 287.6 | 286.6 |
| x = 16 | 1 | 568.2 | 568.1 | 566.8 | 283.7 | 283.6 |
| x = 20 | 1 | 567.7 | 567.6 | 565.4 | 282.3 | 282.3 |
| x = 24 | 1 | 567.4 | 567.4 | 564.7 | 281.6 | 281.5 |

Exact sol. | N.G | N.G | N.G

Maple | 567.6 | 562.8 | 279.8

Multiplier | $10^4$ $Dqa^2$ | $10^4$ $qa^2$ | $10^4$ $qa^2$

Remark:

$BC`s$: 1: $w=0$, $m_y=0$ at simply side $x=0$ and $w=0$, $m_x=0$ at clamped side $y=0$ and $m_y=0$ at free side and $m_y=0$ along center line $x=a/2$

$BC`s$: 2: $w=0$, $m_y=0$ at simply side $x=0$ and $w=0$, and $m_y=0$ at free side and $m_y=0$ along center line $x=a/2$

In this case the results were compared for two choices of imposing boundary conditions, although the second one, in which $m_y$, not imposing as boundary condition at clamped edge converges in somewhat like the first one but starting with more nearly converged value than the alternative, however utilizing the deflection formula that is given by Timoshenko (set of Eqs. d,e,g,h) [14], with the aid of Maple program package, emphasizing that the twisting moment $m_y$ along the clamped edge must be zero as it’s well known, in other words it should be imposed boundary condition thus the reason for why not imposing in this example as well as table II above which agreed with Reddy results [15], not understandable for the authors. In this context and for more clarifying this situation, the graphs of $m_y$ at clamped edge along $x$-axis and $m_y$ at free edge are showed in Fig. 4 and Fig. 5, where they are not zero-values there, and the value of the first one found from this work as (-0.035459) for mesh 10x20 is in agreement with the given calculated value (-0.0355) in the table (7.2.6) by Reddy [17]. While $m_y$ at clamped edge are in zero-values.

Table V contains the values of deflection, bending moment and twisting moment at the specified locations, for a square plate with two opposite sides simply-supported along y-direction and the other two opposite sides are free sides along x-direction (S-Sy / C-Fx).

| Mesh size | DOF | $\alpha_{ex}$ | $\alpha_{es}$ | $\alpha_{ex}$ | $\alpha_{es}$ |
|-----------|-----|---------------|---------------|---------------|---------------|
| x = 2 | 1 | 0.009830 | 0.11572 | 0.027935 | 0.012296 | 0.1342716 |
| x = 4 | 1 | 0.012323 | 0.121838 | 0.028607 | 0.014339 | 0.1316923 |
| x = 8 | 1 | 0.012755 | 0.122217 | 0.027479 | 0.014715 | 0.1317655 |
| x = 16 | 1 | 0.012904 | 0.122378 | 0.027320 | 0.014845 | 0.1312539 |
| x = 24 | 1 | 0.012973 | 0.122439 | 0.027223 | 0.014905 | 0.1311957 |
| x = 1 | 1 | 0.1184 | 112.4 | 0.1225 | 0.02787 | 0.1311603 |

Exact sol. | 0.1109 | 0.1225 | 0.0271 | 0.01509 | 0.131877 |

Maple | 0.1184 | 112.4 | 0.1225 | 0.02787 | 0.1311603 |

Multiplier $Dqa^2$, $qa^2$, $qa^2$, $qa^2$.

Remark:

$w=0$, $m_y=0$ at simply side $x=0$, and $m_y=0$ at free side $y=0$, $m_x=0$ at center line $y=b/2$, and $x=a/2$ in which these boundary conditions agreed with analytical solutions as checked by Maple

Finally Table VI contains the values of deflection, bending moment and twisting moment, at the specified locations, for a simply-supported square orthotropic (graphite-epoxy) plate having material properties as: $E_x = 7.8 \times 10^9$ psi, $E_y = 2.6 \times 10^6$ psi, $G_{xy} = 1.3 \times 10^6$ psi & $v_{xy} = 0.25$ under uniformly distributed transverse load.

Table VI: Results of all S-S orthotropic plate analysis

| Mesh size | DOF | $\alpha_{ex}$ | $\alpha_{es}$ | $\alpha_{ex}$ | $\alpha_{es}$ |
|-----------|-----|---------------|---------------|---------------|---------------|
| x = 2 | 1 | 2.9738 | 2.9737 | 0.9435 | 0.9435 | 3.6290 | 3.6290 |
| x = 4 | 1 | 3.1562 | 3.1041 | 0.8002 | 0.8078 | 2.7712 | 2.8937 |
| x = 8 | 1 | 3.1255 | N.G | 0.7788 | N.G | 2.7447 | N.G |
| x = 16 | 1 | 3.1109 | 0.0390 | 0.7717 | 0.7737 | 2.7465 | 2.7851 |
| x = 24 | 1 | 3.1033 | N.G | 0.7684 | N.G | 2.7488 | N.G |
| x = 1 | 1 | 3.0989 | 3.0901 | 0.7667 | 0.7676 | 2.7504 | 2.7685 |
| x = 4 | 1 | 3.0960 | N.G | 0.7657 | N.G | 2.7514 | N.G |
| x = 8 | 1 | 3.0942 | 3.0890 | 0.7654 | 0.7654 | 2.7523 | 2.7628 |
| x = 16 | 1 | 3.0895 | 3.0890 | 0.7635 | N.G | 2.7551 | N.G |

Exact sol. | 3.0876 | 0.7628 | 2.7556 |

Multiplier $10^4$ $H/qa^2$, $10^4$ $qa^2$, $10^4$ $qa^2$. 

Remark:
Where these results obtained by imposing the boundary conditions: $w=0$, $m_n=0$ at edge $x=0$ & $m_n=0$ at center line $x=a/2$ and $w=0$, $m_n=0$ at $y=0$ & $m_n=0$ at center line $y=b/2$; these results matching Reddy results [15] only in first line (1 x 1) while the others were not; although the formulation is clear and the our calculated results seems to be very close to the analytical solutions. The reason for such problem is not recognized for the authors.

Finally, the second author recognizing that the deep search carried out by first author in analytical and numerical formulation which lead to the above results and the following conclusion.

V. CONCLUSION

The introduced step by step derivation of explicit augmented finite element matrix using mixed formulation for plate bending, which in turn leads to a relatively small space and size of such an explicit terms of the matrix when are utilized in either by hand or computer programming calculations.

Applying of such augmented matrix in solving plate problems with a necessary discussion on reasonable treatment of imposing boundary conditions as it should be in accordance with actual problem behavior rather than looking for a more attractive values, reveals very good results when compared with either analytical or numerical solutions of plate bending problems in available literature as the reader can possibly verify. This augmented matrix will provide an easy and direct means and a land mark reference in verifying many obtained results and avoiding its incompatibleness wrong interpretations in the past and future research developments in the area of solving plate bending problems.

APPENDIX

| $a(i , j)$ | 1 | 2 | 3 | 4 |
|------------|---|---|---|---|
| 1          | -1| 1 | -1| 1 |
| 2          | -1| 1 | -1| 1 |
| 3          | -1| 1 | -1| 1 |
| 4          | -1| 1 | -1| 1 |

The values of $e(i , j) = e(i/j)/2$

| $e(i/j)$ | 1 | 2 | 3 | 4 |
|----------|---|---|---|---|
| 1        | -1| -1| -1| -1|
| 2        | 1 | 1 | 1 | 1 |
| 3        | -1| -1| -1| -1|
| 4        | 1 | 1 | 1 | 1 |

ACKNOWLEDGMENT

The authors are very grateful to former Honor Professor of civil engineering; Mustafa M. Tawil, university of Tripoli, Libya, for reviewing the manuscript and the technical comments which support this work.

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