QUANTUM SUBGROUPS OF $GL_{\alpha,\beta}(n)$

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Abstract. Let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $\ell \in \mathbb{N}$, odd with $\ell \geq 3$. We determine all Hopf algebra quotients of the quantized coordinate algebra $\mathcal{O}_{\alpha,\beta}(GL_n)$ when $\alpha^{-1}\beta$ is a primitive $\ell$-th root of unity and $\alpha, \beta$ satisfy certain mild conditions, and we characterize all finite-dimensional quotients when $\alpha^{-1}\beta$ is not a root of unity. As a byproduct we give a new family of non-semisimple and non-pointed Hopf algebras with non-pointed duals which are quotients of $\mathcal{O}_{\alpha,\beta}(GL_n)$.

Dedicated to Nicolás Andruskiewitsch in his 50th birthday.

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1. Introduction

One-parameter quantizations of the general linear group \( GL_n \) are well-known objects that arises as dual Hopf algebras of the Drinfeld-Jimbo quantized enveloping algebras \( U_q(\mathfrak{gl}_n) \) and were studied by many authors, for example [FR T], [HH], [Ma], [PW], [TT] and [Tk2]. The problem of constructing families of multiparameter quantum groups was first raised in [Ma2] and subsequently treated in [Tk], [Re], [OY], [Hi], [LS], [AE] among others. Two-parameter deformations of \( \mathcal{O}(GL_n) \) were introduced by M. Takeuchi in [Tk], see also [Ko], and they were studied in detail in [DPW]. In [Tk] a two-parameter deformation of the enveloping algebra \( U(\mathfrak{gl}_n) \) is also defined and it is proven that there exists a Hopf pairing between them. In the case of the one-parameter deformation, one can prove that this pairing is perfect. This is why the Hopf algebras \( \mathcal{O}_q(GL_n) \) are dual to \( U_q(\mathfrak{gl}_n) \). A family depending on more parameters has been constructed independently by Sudbery [S] and Reshetikhin [Re]. It is shown in [AST] that this family can be obtained and characterized by a construction of Manin and they explain how the algebras in the family are twists of \( \mathcal{O}_q(GL_n) \) by 2-cocycles. We have chosen to study the two-parameter deformation given by Takeuchi simply because they consist of a family of objects which cannot be obtained by 2-cocycle deformations of \( \mathcal{O}_q(GL_n) \), see Remark 3.2 (c).

Several authors, among them Benkart and Witherspoon [BW1, BW2], Jing [Ji] and Kulish [Ku], defined also two-parameter deformations of \( U(\mathfrak{gl}_n) \) which are particular cases of the multiparameter deformations, see for example [CM2] and in particular [S], where they are defined using a pairing with a multiparameter deformation of the coordinate ring of \( GL_n \). These two-parameter deformations of \( U(\mathfrak{gl}_n) \) are closely related to each other, for example Takeuchi’s deformation \( U_{\alpha,\beta}(\mathfrak{gl}_n) \) is isomorphic to the deformation \( U_{r,s^{-1}}(\mathfrak{gl}_n) \) given by Benkart and Witherspoon but as coalgebra they have the opposite coproduct. Recently, N. Hu and Y. Pei [HP] defined two-parameter deformations of \( U(\mathfrak{g}) \) for any semisimple Lie algebra and showed that they can be realized as Drinfeld doubles. This generalizes previous results of Benkart and Witherspoon for type A, Bergeron, Gao and Hu for type B, C and D [BGH1, BGH2], Hu and Shi for type G [HS] and Bai and Hu for type E [BH].

In this paper we determine the Hopf algebra quotients of the two parameter quantization \( \mathcal{O}_{\alpha,\beta}(GL_n) \) of the coordinate algebra on \( GL_n \) when \( \alpha^{-1}\beta \) is a primitive \( \ell \)-th root of unity, \( \ell \in \mathbb{N} \), odd with \( \ell \geq 3 \) and \( \alpha, \beta \) satisfy certain mild conditions, and we characterize all finite-dimensional quotients when \( \alpha^{-1}\beta \) is not a root of unity. As a consequence, we give in Thm. 5.32 a new family of non-semisimple and non-pointed Hopf algebras with non-pointed duals, which are quotients of \( \mathcal{O}_{\alpha,\beta}(GL_n) \) and cannot be obtained as quotients of \( \mathcal{O}_\epsilon(G) \), with \( G \) a connected, simply connected, simple complex Lie group, \( \epsilon \) a primitive \( s \)-th root of unity, see [AG2].
Quantum Subgroups of $GL_{\alpha,\beta}(n)$

It is crucial for the determination of the quotients the relation between $U_{\alpha,\beta}(\mathfrak{gl}_n)$ and $\mathcal{O}_{\alpha,\beta}(GL_n)$, as well as some known facts about the pairing between them and the center of $U_{\alpha,\beta}(\mathfrak{gl}_n)$. For this reason, we rely on results of [BKL, BW1, BW2, BW3] and [DPW].

In order to study quantum subgroups of more general quantum groups, we believe that it would be necessary to define first, as in the case of one-parameter deformations [L], the rational form of multiparameter deformations of coordinate rings of reductive or more general algebraic groups. This involves the study of the representation theory of these quantum groups at roots of 1, since the quantized coordinate rings would be generated as algebras by the matrix coefficients of representations of type 1. One may also use for the definition of these type of quantum groups the pairing between the quantized coordinate rings and the quantized enveloping algebras, but in the case where the parameters are roots of unity, the pairing is degenerate, which makes the representation theory more complicated. For multiparameter deformations of other simple Lie algebras, see [BGH], [AE].

The problem of determining the quantum subgroups of a quantum group was first considered by P. Podleś [P] for quantum $SU(2)$ and $SO(3)$. Then, the characterization of all finite-dimensional Hopf algebra quotients of the quantized coordinate algebra $\mathcal{O}_q(SL_N)$ was obtained by Eric Müller [Mu2]. For more general simple groups, all possible quotients of $\mathcal{O}_\epsilon(G)$ were determined in [AG2] in the case where the parameter $\epsilon$ is a root of unity, generalizing the results of Müller. They are parameterized by data $D = (I_+, I_-, N, \Gamma, \sigma, \delta)$ where $I_+$ and $-I_-$ are subsets of the basis of a fixed root system of Lie($G) = \mathfrak{g}$, $N$ is a finite abelian group related to $I = I_+ \cup -I_-$, $\Gamma$ is an algebraic group, $\sigma : \Gamma \to L$ is an injective morphism of algebraic groups, where $L \subseteq G$ is a connected algebraic subgroup associated to $I$ and $\delta : N \to \hat{\Gamma}$ is a group map into the character group of $\Gamma$. The corresponding quotient $A_D$ fits into a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{i} & \mathcal{O}_\epsilon(G) & \xrightarrow{\pi} & \mathfrak{u}_\epsilon(\mathfrak{g})^* & \longrightarrow & 1 \\
 & & \downarrow{\iota_\sigma} & & \downarrow{\eta_D} & & \downarrow{\kappa} & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{i} & A_D & \xrightarrow{\kappa} & \hat{H} & \longrightarrow & 1,
\end{array}
\]

where $\mathfrak{u}_\epsilon(\mathfrak{g})$ is the Frobenius-Lusztig kernel of $\mathfrak{g}$ and $H^*$ is a Hopf subalgebra of $\mathfrak{u}_\epsilon(\mathfrak{g})$ determined by the triple $(I_+, I_-, N)$. In particular, the quotients $A_D$ fits into a central exact sequence of Hopf algebras.

We prove that the quotients of $\mathcal{O}_{\alpha,\beta}(GL_n)$ follow the same pattern when $\alpha^{-1}\beta$ is a primitive $\ell$-th root of unity and $\alpha, \beta$ satisfy certain mild conditions, see Thm. 5.23. If $\alpha^{-1}\beta$ is not a root of unity, the finite-dimensional quotients are the function algebras of finite subgroups of the diagonal torus in $GL_n(k)$. The definition of a subgroup datum for $\mathcal{O}_{\alpha,\beta}(GL_n)$ is the following:

**Definition 1.1.** A subgroup datum of the quantum group $\mathcal{O}_{\alpha,\beta}(GL_n)$ is a collection $D = (I_+, I_-, N, \Gamma, \sigma, \delta)$ where
• \( I_+, I_- \subseteq \{1, \ldots, n-1\} \). These subsets determine an algebraic subgroup \( L \) of \( GL_n \) consisting in block matrices whose nonzero blocks are in the diagonal, see Remark \[5.19].

• \( N \) is a subgroup of \( \hat{T} \), see Remark \[5.20].

• \( \Gamma \) is an algebraic group.

• \( \sigma : \Gamma \rightarrow L \) is an injective homomorphism of algebraic groups.

• \( \delta : N \rightarrow \hat{T} \) is a group homomorphism.

If \( \Gamma \) is finite, we call \( \mathcal{D} \) a finite subgroup datum. The following theorem is the main result of the paper. Part (a) is completely analogous to \[Mu2, Thm. 4.1\] with almost the same proof and part (b) is also analogous to \[AG2, Thm. 2.17\], but its proof is different since the quantum group \( \mathcal{O}_{\alpha,\beta}(GL_n) \) is given by generators and relations. In particular, several technical lemmata will be needed to prove this part of the theorem.

**Theorem 1.** Let \( q : \mathcal{O}_{\alpha,\beta}(GL_n) \rightarrow A \) be a surjective Hopf algebra map.

(a) If \( \alpha^{-1}\beta \) is not a root of unity and \( \dim A \) is finite, then \( A \) is a function algebra of a finite subgroup of the diagonal torus in \( GL_n(k) \).

(b) If \( \alpha^{-1}\beta \) is a primitive \( \ell \)-th root of unity with \( \alpha^\ell = 1 = \beta^\ell \), then there is a bijection between

(i) Hopf algebra quotients \( q : \mathcal{O}_{\alpha,\beta}(GL_n) \rightarrow A \)

(ii) Subgroup data of \( \mathcal{O}_{\alpha,\beta}(GL_n) \) up to equivalence.

In Section 4 we give the proof of part (a) and in Section 5 we give the proof of part (b). Specifically, in Sec. 5.3 we carry out the construction of a quotient \( \mathcal{A}_D \) of \( \mathcal{O}_{\alpha,\beta}(GL_n) \) starting from a subgroup datum \( \mathcal{D} \), see Thm. \[5.23\]. In Section 5.4, we attach a subgroup datum \( \mathcal{D} \) to an arbitrary Hopf algebra quotient \( A \) and prove that \( \mathcal{A}_D \simeq A \) as quotients of \( \mathcal{O}_{\alpha,\beta}(GL_n) \). Finally, in Sec. 5.5 we study the lattice of quotients \( \mathcal{A}_D \). This concludes the proof of the theorem.

As a consequence, any quotient \( \mathcal{A}_D \) also fits into a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
1 & \rightarrow & \mathcal{O}(GL_n) & \rightarrow & \mathcal{O}_{\alpha,\beta}(GL_n) & \rightarrow & \mathcal{u}_{\alpha,\beta}(gl_n)^* & \rightarrow & 1 \\
& & \downarrow{\iota}\sigma & & \downarrow{\pi} & & \downarrow{\pi} & & \\
1 & \rightarrow & \mathcal{O}(\Gamma) & \rightarrow & \mathcal{A}_D & \rightarrow & \mathcal{H} & \rightarrow & 1,
\end{array}
\]

where \( \mathcal{u}_{\alpha,\beta}(gl_n)^* \) is a quotient of the restricted quantum group of \( gl_n \) defined in \[BW1\], and \( \mathcal{H} \) is a Hopf subalgebra of \( \mathcal{u}_{\alpha,\beta}(gl_n) \) determined by the triple \( (I_+, I_-, N) \). In particular, the quotient \( \mathcal{A}_D \) fits into a central exact sequence of Hopf algebras. It is not known if a kind of converse is true, that is, if a Hopf algebra is a central extension and it satisfies some additional but specific properties, then it is a quotient of a quantum group. An example of a specific property, for instance, is being finite-dimensional and generated
by a simple subcoalgebra of dimension 4 stable by the antipode. This fact was proved by Ţ Stefan [S] and it is used with profit in the classification of Hopf algebras of small dimension, see for example [GV], [N].

The paper is organized as follows. We recall in Section 2 some known facts about Hopf algebras, central extensions of Hopf algebras and PI-Hopf triples. In Section 3 we recall the definition of the two-parameter deformation of the coordinate ring of $GL_n$, the universal enveloping algebra of $gl_n$, the pairing between them and some results due to Kharchenko [K] and Benkart and Witherspoon [BW1] on a PBW-basis of $U_{\alpha,\beta}(gl_n)$. As already mentioned, we prove Thm. 1 in Sec. 4 and Sec. 5, and we end the paper by giving some properties and relations between distinct quantum subgroups in the case where the parameters are roots of unity. As a byproduct we obtain a new family of finite-dimensional non-semisimple and non-pointed Hopf algebras with non-pointed duals, see Thm. 5.32.

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2. Preliminaries

2.1. Conventions. We work over an algebraically closed field $k$ of characteristic zero and by $k^\times$ we denote the group of units of $k$. We write $G_\ell$ for the group of $\ell$-th roots of unity. Our references for the theory of Hopf algebras are [Mo] and [Sw], for Lie algebras [Hu] and for quantum groups [J] and [BG]. If $\Gamma$ is a group, we denote by $\hat{\Gamma}$ the character group. The antipode of a Hopf algebra $H$ is denoted by $S$. The Sweedler notation is used for the comultiplication of $H$ but dropping the summation symbol. The set of group-like elements of a coalgebra $C$ is denoted by $G(C)$. We also denote by $C^+ = \text{Ker} \varepsilon$ the augmentation ideal of $C$, where $\varepsilon : C \to k$ is the counit of $C$. Let $A \xrightarrow{\pi} H$ be a Hopf algebra map, then $A^{coH} = A^{co\pi} = \{a \in A \mid (\text{id} \otimes \pi)(\Delta(a)) = a \otimes 1\}$ denotes the subalgebra of right coinvariants and $\leftarrow^{coH} A = \leftarrow^{co\pi} A = \{a \in A \mid (\pi \otimes \text{id})(\Delta(a)) = 1 \otimes a\}$ denotes the subalgebra of left coinvariants.

Let $H$ be a Hopf algebra, $A$ a right $H$-comodule algebra with structure map $\delta : A \to A \otimes H$, $a \mapsto a_{(0)} \otimes a_{(1)}$ and $B = A^{coH}$. The extension $B \subseteq A$ is called a Hopf Galois extension or $H$-Galois if the canonical map $\lambda : A \otimes_B A \to A \otimes H$, $a \otimes b \mapsto ab_{(0)} \otimes b_{(1)}$ is bijective.

Definition 2.1. A Hopf pairing between two Hopf algebras $U$ and $H$ is a bilinear form $(-, -) : H \times U \to \mathcal{R}$ such that, for all $u, v \in U$ and $f, h \in H$, ...
\[
\begin{align*}
(i) \quad (h, uv) &= (h_{(1)}, u)(h_{(2)}, v); \\
(ii) \quad (fh, u) &= (f, u_{(1)})(h, u_{(2)}); \\
(iii) \quad (1, u) &= \varepsilon(u); \\
(iv) \quad (h, 1) &= \varepsilon(h).
\end{align*}
\]

Hence \((h, S(u)) = (S(h), u)\), for all \(u \in U\), \(h \in H\). Given a Hopf pairing, one has Hopf algebra maps \(U \rightarrow H^\circ\) and \(H \rightarrow U^0\), where \(H^\circ\) and \(U^0\) are the Sweedler duals. The pairing is called perfect if these maps are injections.

2.2. Central extensions of Hopf algebras. We recall some results on quotients and extensions of Hopf algebras.

Definition 2.2. A sequence of Hopf algebras maps \(1 \rightarrow B \xrightarrow{\iota} A \xrightarrow{\pi} H \rightarrow 1\), where \(1\) denotes the Hopf algebra \(k\), is exact if \(\iota\) is injective, \(\pi\) is surjective, \(\text{Ker} \ \pi = AB^+\) and \(B = \text{co}^\pi A\).

If the image of \(B\) is central in \(A\), then \(A\) is called a central extension of \(B\). We shall use the following result.

Proposition 2.3. Let \(A\) and \(K\) be Hopf algebras, \(B\) a central Hopf subalgebra of \(A\) such that \(A\) is left or right faithfully flat over \(B\) and \(p : B \rightarrow K\) a Hopf algebra epimorphism. Then \(H = A/AB^+\) is a Hopf algebra and \(A\) fits into the exact sequence \(1 \rightarrow B \xrightarrow{\iota} A \xrightarrow{\pi} H \rightarrow 1\). If we set \(\mathcal{J} = \text{Ker} \ p \subseteq B\), then \((\mathcal{J}) = A\mathcal{J}\) is a Hopf ideal of \(A\) and \(A_p := A/(\mathcal{J})\) is the pushout given by the following diagram:

\[
\begin{array}{c}
B \xrightarrow{\iota} A \\
p \downarrow \quad \downarrow q \\
K \xrightarrow{j} A_p
\end{array}
\]

Moreover, \(K\) can be identified with a central Hopf subalgebra of \(A_p\) and \(A_p\) fits into the exact sequence \(1 \rightarrow K \xrightarrow{\jmath} A_p \xrightarrow{\pi_p} H \rightarrow 1\). \[\square\]

Remark 2.4. Let \(A\) and \(B\) be as in Prop. 2.3 then the following diagram of central exact sequences is commutative.

\[
\begin{array}{c}
1 \rightarrow B \xrightarrow{\iota} A \xrightarrow{\pi} H \xrightarrow{\pi} 1 \\
p \downarrow \quad \downarrow q \\
1 \rightarrow K \xrightarrow{j} A_p \xrightarrow{\pi_p} H \xrightarrow{\pi} 1.
\end{array}
\]

The following general fact is due to Masuoka, see [AG2, Lemma 1.14].

Lemma 2.5. Let \(H\) be a bialgebra over an arbitrary commutative ring, and let \(A, A'\) be right \(H\)-Galois extensions over a common algebra \(B\) of \(H\)-coinvariants. If \(A'\) is right \(B\)-faithfully flat, then any \(H\)-comodule algebra map \(\theta : A \rightarrow A'\) that is identical on \(B\) is an isomorphism. \[\square\]
Recall that a $k$-algebra $A$ is called affine if it is finitely generated as an algebra and a ring $R$ is called a polynomial identity ring or PI-ring for short, if there exists a monic polynomial $f$ in the free algebra $\mathbb{Z}\langle X \rangle$ on a set $X = \{x_1, \ldots, x_m\}$ such that $f(r_1, \ldots, r_m) = 0$ for all $r \in R$.

**Definition 2.6.** [BG, Def. III.4.1] A PI-Hopf triple $(B, H, \overline{H})$ over $k$ consists of three Hopf algebras such that

(i) $H$ is a $k$-affine $k$-Hopf algebra.
(ii) $B$ is a central Hopf subalgebra of $H$ which is a domain and such that $H$ is a finitely-generated $B$-module.
(iii) $\overline{H} := H/B^+H$ is the finite-dimensional Hopf algebra quotient.

We end this section with the following results.

**Lemma 2.7.** Let $H$ be a $k$-affine $k$-Hopf algebra, $B$ a central Hopf subalgebra of $H$ which is a domain and such that $H$ is a finitely-generated $B$-module and denote by $\overline{H} := H/B^+H$ the finite-dimensional Hopf algebra given by the quotient. Then

(i) [BG, Lemma III.4.2] $B$ is an affine $k$-algebra. Thus $H$ is a noetherian PI-algebra and $\mathbb{Z}(H)$ is an affine algebra.
(ii) [BG, Thm. III.4.5] $H$ is a finitely generated projective $B$-module.
(iii) [BG, Lemma III.4.6] $B \subseteq H$ is a faithfully flat $\overline{H}$-Galois extension.

In particular, $B = {}^{\text{co}}H^H = H^{\text{co}\overline{H}}$. □

**Remark 2.8.** By Lemma [27] (iii) and [Mo, Prop. 3.4.3], any PI-Hopf triple $(B, H, \overline{H})$ gives rise to a central extension of Hopf algebras – see Def. 2.2

$$1 \to B \xrightarrow{\iota} H \xrightarrow{\pi} \overline{H} \to 1.$$ 

3. **Two-parameter deformations of classical objects**

In this section we recall the definition and some basic properties of the two-parameter quantization of the coordinate algebra of $GL_n^{\alpha,\beta}$ as well as the two-parameter quantization of $U(gl_n)$ given in [Tk].

**3.1. The quantum group $GL_{\alpha,\beta}(n)$.**

**Definition 3.1.** [Tk, Sec. 2] Let $\alpha, \beta \in k^\times$ and $n \in \mathbb{N}$. The algebra $O_{\alpha,\beta}(M_n)$ is the $k$-algebra generated by the elements $\{x_{ij} : 1 \leq i, j \leq n\}$ satisfying the following relations:

\[
\begin{align*}
x_{ik}x_{ij} &= \alpha^{-1}x_{ij}x_{ik} & \text{if } j < k, \\
x_{jk}x_{ik} &= \beta x_{ik}x_{jk} & \text{if } i < j, \\
x_{jk}x_{il} &= \beta \alpha x_{il}x_{jk} & \text{and} \\
x_{jl}x_{ik} - x_{ik}x_{jl} &= (\beta - \alpha)x_{il}x_{jk} & \text{if } i < j \text{ and } k < l.
\end{align*}
\]

This algebra is a non-commutative polynomial algebra in the variables $x_{ij}$ and has no non-zero divisor. It has a basis $\{\prod_{i,j} e_{ij}^{e_{ij}} \mid e_{ij} \in \mathbb{N}_0\}$, where the
products are formed with respect to a fixed ordering of \( \{x_{ij} : 1 \leq i, j \leq n\} \). It has a bialgebra structure determined by

\[
\Delta(x_{ij}) = \sum_{s=1}^{n} x_{is} \otimes x_{sj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{ij}.
\]

If \( \alpha = 1 = \beta \) this commutative algebra is just \( \mathcal{O}(\mathbb{M}_n(\mathbb{k})) \). Thus \( \mathcal{O}_{\alpha,\beta}(\mathbb{M}_n) \) defines a two-parameter quantization \( M_{\alpha,\beta}(n, \mathbb{k}) \) of the semigroup scheme \( M(n, \mathbb{k}) \).

The quantum determinant \( g = |X| \), where \( X = (x_{ij})_{1 \leq i, j \leq n} \) denotes the \( n \times n \)-matrix with coefficients \( x_{ij} \), is defined by

\[
g = \sum_{\sigma \in S_n} (-\beta)^{-\ell(\sigma)} x_{\sigma(1)} \cdots x_{\sigma(n)} = \sum_{\sigma \in S_n} (-\alpha^{-1})^{-\ell(\sigma)} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)},
\]

It is a group-like element and we have that \( x_{ij}g = (\beta \alpha)^{j-i}gx_{ij} \) for all \( 1 \leq i, j \leq n \). Thus, the powers of \( g \) satisfy the left and right Ore condition. The localization of \( \mathcal{O}_{\alpha,\beta}(\mathbb{M}_n) \) at the powers of \( g \) gives the Hopf algebra \( \mathcal{O}_{\alpha,\beta}(GL_n) := \mathcal{O}_{\alpha,\beta}(\mathbb{M}_n)[g^{-1}] \), which is the Hopf algebra \( A_{\alpha^{-1},\beta} \) in \[TK\]. This Hopf algebra corresponds to the quantum group \( GL_{\alpha,\beta}(n) \). The antipode \( S \) is given by

\[
S(x_{ij}) = (-\beta)^{j-i}g^{-1}|X_{ji}| = (-\alpha^{-1})^{j-i}|X_{ji}|g^{-1},
\]

where \( |X_{ji}| \) denotes the quantum determinant of the \((n-1) \times (n-1)\) minor obtained by removing the \( j \)-th row and the \( i \)-th column. Hence

\[
S^2(x_{ij}) = (\alpha^{-1}\beta)^{j-i}x_{ij}.
\]

**Remark 3.2.** (a) By taking different values of \( \alpha, \beta \), e.g. \( (\alpha, \beta) = (q^{-1}, q) \) or \( (1, q) \) one obtains the well-known one parameter deformations of \( \mathcal{O}(GL_n) \), the standard in the first case and the Dipper-Donkin [DD] deformation in the second. Hence we will assume that \( \alpha^{-1} \neq \beta \) and \( \alpha \neq 1 \neq \beta \).

(b) In [TK3], it is studied the problem of the cocycle deformations of the quantum groups \( \mathcal{O}_{\alpha^{-1},\beta}(GL_n) \). It is proved in Thm. 2.6 that the bialgebra \( \mathcal{O}_{\alpha^{-1},\beta}(\mathbb{M}_n) \) is isomorphic to a cocycle deformation of \( \mathcal{O}_{\alpha^{-1},\beta}(\mathbb{M}_n) \) if and only if \( \alpha^{-1}\beta = \alpha'^{-1}\beta' \) or \( \alpha \beta = (\alpha'^{-1}\beta')^{-1} \) (here we have used Takeuchi’s notation to avoid confusion with the reference). Moreover, one has by Cor. 2.8 that if \( \alpha^{-1}\beta \neq 1 \) then \( \mathcal{O}_{\alpha^{-1},\beta}(GL_n) \) is not a cocycle deformation of commutative Hopf algebras, and \( \mathcal{O}_{\alpha^{-1},\beta}(\mathbb{M}_n) \) is not a cocycle deformation of commutative bialgebras.

(c) It is not difficult to see that \( \mathcal{O}_{\alpha^{-1},\beta}(GL_n) \) and \( \mathcal{O}_{\alpha,\beta^{-1}}(GL_n) \) are isomorphic as Hopf algebras. The isomorphism is determined by defining \( x_{ij} \mapsto y_{n+1-i,n+1-j} \), where the elements \( x_{ij} \) and \( y_{ij} \) denote the canonical generators of \( \mathcal{O}_{\alpha^{-1},\beta}(\mathbb{M}_n) \) and \( \mathcal{O}_{\alpha,\beta^{-1}}(\mathbb{M}_n) \), respectively (see [DPW, Prop. 1.11]). Moreover, in [DPW] Thm. 2.4, Cor. 2.6] it is proved that if \( \alpha^{-1}\beta = \alpha'^{-1}\beta' \) or \( \alpha^{-1}\beta = (\alpha'^{-1}\beta')^{-1} \) then \( \mathcal{O}_{\alpha^{-1},\beta}(GL_n) \) and \( \mathcal{O}_{\alpha'^{-1},\beta'}(GL_n) \)
are isomorphic as coalgebras; in particular, their categories of comodules are equivalent, compare with Remark (b).

3.2. Quantum Borel subgroups of $GL_{\alpha,\beta}(n)$. Let $J_+$ be the ideal of $O_{\alpha,\beta}(GL_n)\langle t \rangle$ generated by the elements $\{x_{ij}\}_{i > j}$. By the relations in Def. 3.1 it is a two-sided ideal and since $\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}$ and $\varepsilon(x_{ij}) = \delta_{ij}$, it follows that $J_+$ is a coideal. Moreover, by equation (4) we know that $S(x_{ij}) = (-\beta)^{j-i} g^{-1} |X_{ji}|$ and $|X_{ji}| \in I_+$, whence $S(J_+) \subseteq J_+$ and $J_+$ is a Hopf ideal. The Hopf algebra quotient $O_{\alpha,\beta}(GL_n)/J_+$ corresponds to the two-parameter deformation of a Borel subalgebra of $GL_n(k)$ and it is denoted by $O_{\alpha,\beta}(B^+)$. Denote by

$$t_+ : O_{\alpha,\beta}(GL_n) \to O_{\alpha,\beta}(B^+)$$

the canonical Hopf algebra quotient and $t_+(x_{ij}) = \tilde{x}_{ij}$ for all $1 \leq i, j \leq n$. Then $O_{\alpha,\beta}(B^+)$ is generated as an algebra by the elements $\{\tilde{x}_{ij} | 1 \leq i \leq j \leq n\}$ satisfying the relations

$$(6)\quad \tilde{x}_{ik}\tilde{x}_{ij} = \alpha^{-1}\tilde{x}_{ij}\tilde{x}_{ik} \quad \text{if} \ j < k,$$

$$\tilde{x}_{jk}\tilde{x}_{ik} = \beta\tilde{x}_{ik}\tilde{x}_{jk} \quad \text{if} \ i < j,$$

$$\tilde{x}_{ik}\tilde{x}_{il} = \beta\tilde{x}_{il}\tilde{x}_{ik} \quad \text{and}$$

$$\tilde{x}_{jl}\tilde{x}_{ik} = \tilde{x}_{ik}\tilde{x}_{jl} \quad \text{if} \ i < j \text{ and } k < l.$$

The elements $\{\tilde{x}_{ii}\}_{1 \leq i \leq n}$ are invertible group-like elements which commute with each other and $t_+(g^{-1}) = \tilde{x}_{11}^{-1} \cdots \tilde{x}_{nn}^{-1}$. Moreover, the set

$$\left\{ \prod_{i<j} \tilde{x}_{ij}^{e_{ij}} \prod_i \tilde{x}_{ii}^{e_{ii}} | e_{ij} \in \mathbb{N}_0, e_i \in \mathbb{Z} \right\}$$

is a linear basis of $O_{\alpha,\beta}(B^+)$, for some fixed ordering of $\{\tilde{x}_{ij} | 1 \leq i < j \leq n\}$. In particular, it has no non-zero divisors.

Analogously, by taking the ideal $J_-$ generated by the elements $\{x_{ij}\}_{i < j}$, one defines the borel subalgebra $O_{\alpha,\beta}(B^-) := O_{\alpha,\beta}(GL_n)/J_-$. Denote by $t_- : O_{\alpha,\beta}(GL_n) \to O_{\alpha,\beta}(B^-)$ the Hopf algebra quotient and $t_-(x_{ij}) = \hat{x}_{ij}$ for all $1 \leq i, j \leq n$. The following lemma is [PW] Thm. 8.1.1 in the case of two-parameter deformations.

Lemma 3.3. The following algebra map is injective

$$\delta = (t_+ \otimes t_-) \circ \Delta : O_{\alpha,\beta}(GL_n) \to O_{\alpha,\beta}(B^+) \otimes O_{\alpha,\beta}(B^-).$$

Proof. Since $\delta(x_{ij}) = \sum_{i,j \leq k} \tilde{x}_{ik} \otimes \tilde{x}_{kj}$, then $\delta(x_{ij}) \neq 0$ for all $1 \leq i, j \leq n$. Moreover, since $\delta$ is an algebra map, $\delta(x) = 0$ if and only if $\delta(xg^t) = \delta(xg^t) = 0$ for all $t \in \mathbb{Z}$. As $\{g^{-t} \prod_{i,j} \tilde{x}_{ij}^{e_{ij}} | t, e_{ij} \in \mathbb{N}_0\}$ is a set of generators of $O_{\alpha,\beta}(GL_n)$, we may assume that if $\delta(x) = 0$, then $x \in O_{\alpha,\beta}(M_n)$.

By Def. 3.1 we know that $O_{\alpha,\beta}(M_n)$ has a linear basis consisting of monomials of the form

$$m_e = \prod_{i<j} x_{ij}^{e_{ij}} \prod_i x_{ii}^{e_{ii}} \prod_{i>j} x_{ij}^{e_{ij}}.$$
where $e = (e_{ij})_{1 \leq i,j \leq n}$ runs over the set of $n \times n$ matrices with coefficients in $\mathbb{Z}_+$, and the product of the $x_{ij}$'s is taken with respect to a fixed order. Define the degree of a monomial $m_e$ to be the matrix $e$. Then the opposite lexicographic order (i.e. $x_{ij} \geq x_{kl}$ if $i < k$ and if $i = k$ then $j \leq l$), induces a partial order in the monomials according to their degree. Thus

$$\delta(m_e) = c \prod_{i<j} x_{ij} \cdot \prod_i x_{ii}^{e_{ii} + \sum_{i>j} e_{ij}} \otimes \prod_j x_{jj}^{e_{jj} + \sum_{i<j} e_{ij}} \cdot \prod_{i>j} x_{ij}^{e_{ij}} + \text{lower terms},$$

with $c \neq 0$, since by Def. 3.3 and equations (6) changing the order of the factors in a monomial only results in a nonzero scalar factor and some lower terms. From this follows that if $e \neq f = (f_{ij})_{1 \leq i,j \leq n}$ then $\delta(m_e)$ and $\delta(m_f)$ have different leading terms. This implies the linear independence of the $\delta(m_e)$'s, which implies that $\delta$ is injective on $\mathcal{O}_{\alpha,\beta}(M_n)$.

3.3. The quantum group $U_{\alpha,\beta}(\mathfrak{gl}_n)$. Now we recall the definition of the two-parameter deformation $U_{\alpha,\beta}(\mathfrak{gl}_n)$ of the enveloping algebra of $\mathfrak{gl}_n$, following [BW3] but changing the notation of the parameters $\alpha = r$ and $\beta = s$ to stress the relation with the two-parameter deformation $\mathcal{O}_{\alpha,\beta}(GL_n)$.

**Definition 3.4.** [BW3] Sec. 1] Assume that $\alpha \neq \beta$. The algebra $U_{\alpha,\beta}(\mathfrak{gl}_n)$ is the $k$-algebra generated by the elements $\{a_i, a_i^{-1}, b_i, b_i^{-1}, e_j, f_j : 1 \leq i \leq n, 1 \leq j < n\}$ satisfying the following relations: for all $1 \leq i, k \leq n$, $1 \leq j, l < n$,

\[
\begin{align*}
    a_i, b_k & \quad \text{commute with each other and} \\
    a_i a_i^{-1} &= a_i^{-1} a_i = b_i b_i^{-1} = b_i^{-1} b_i = 1, \\
    a_i e_j &= \alpha^{\delta_{ij} - \delta_{i,j+1}} e_j a_i, \quad b_i e_j = \beta^{\delta_{ij} - \delta_{i,j+1}} e_j b_i \\
    a_i f_j &= \alpha^{-\delta_{ij} + \delta_{i,j+1}} f_j a_i, \quad b_i f_j = \beta^{-\delta_{ij} + \delta_{i,j+1}} f_j b_i, \\
    [e_j, f_i] &= \frac{\delta_{ij}}{\alpha - \beta} (a_j b_{i+1} - a_{j+1} b_j), \\
    [e_j, e_i] &= \begin{cases} f_j, f_i = 0 & \text{if } |j - i| > 1, \\
    0 & \text{otherwise} \end{cases} \\
    0 &= e_j^2 e_{j+1} - (\alpha + \beta) e_j e_{j+1} e_j + \alpha \beta e_{j+1}^2 e_j, \\
    0 &= e_j e_{j+1}^2 - (\alpha + \beta) e_{j+1} e_j e_{j+1} + \alpha \beta e_{j+1}^2 e_j, \\
    0 &= f_j^2 f_{j+1} - (\alpha^{-1} + \beta^{-1}) f_j f_{j+1} f_j + \alpha^{-1} \beta^{-1} f_{j+1} f_j^2, \\
    0 &= f_j f_{j+1}^2 - (\alpha^{-1} + \beta^{-1}) f_{j+1} f_j f_{j+1} + \alpha^{-1} \beta^{-1} f_{j+1}^2 f_j,
\end{align*}
\]

Let $w_j = a_j b_{j+1}$ and $w'_j = a_{j+1} b_j$ for all $1 \leq j < n$. The algebra $U_{\alpha,\beta}(\mathfrak{gl}_n)$ has a Hopf algebra structure determined by the elements $a_i, b_i$ being grouplikes, the $e_j$ being $(w_j, 1)$-primitives and the $f_j$ being $(1, w'_j)$-primitives, for all $1 \leq i \leq n, 1 \leq j < n$. That is,

\[
\begin{align*}
    \Delta(a_i) &= a_i \otimes a_i, \quad \Delta(b_i) = b_i \otimes b_i, \\
    \Delta(e_j) &= e_j \otimes 1 + w_j \otimes e_j \quad \text{and} \quad \Delta(f_j) = 1 \otimes f_j + f_j \otimes w'_j.
\end{align*}
\]
Similarly as before, by taking different values of the parameters $\alpha, \beta$, one obtains the well-known one parameter deformations of $U(\mathfrak{g}_n)$ as quotients of this one. For example, if $(\alpha, \beta) = (q, q)$, then the group-like elements $a_ib_i^{-1}$ are central and the quotient of $U_{q,q}$ by the Hopf ideal $(a_ib_i^{-1} - 1 : 1 \leq i \leq n)$ can be identified with the Drinfeld-Jimbo Hopf algebra $U_q(\mathfrak{g}_n)$.

There is a canonical triangular decomposition
\begin{equation}
U_{\alpha, \beta}(\mathfrak{g}_n) = U_{\alpha, \beta}^+(\mathfrak{g}_n) \otimes U_{\alpha, \beta}^0(\mathfrak{g}_n) \otimes U_{\alpha, \beta}^-(\mathfrak{g}_n),
\end{equation}
where $U_{\alpha, \beta}^+(\mathfrak{g}_n)$, $U_{\alpha, \beta}^0(\mathfrak{g}_n)$ and $U_{\alpha, \beta}^-(\mathfrak{g}_n)$ are the subalgebras generated by $\{e_i\}_{1 \leq i < n}$, $\{a_j^{\pm 1}, b_j^{\pm 1}\}_{1 \leq j \leq n}$, respectively.

Let $U_{\alpha, \beta}(\mathfrak{b}^+)$ be the Hopf subalgebra of $U_{\alpha, \beta}(\mathfrak{g}_n)$ generated by the elements $\{e_j, w_j^{\pm 1}, a_j^{\pm 1} | 1 \leq j < n\}$ and $U_{\alpha, \beta}(\mathfrak{b}^-)$ be the Hopf subalgebra generated by the elements $\{f_j, (w_j')^{\pm 1}, b_j^{\pm 1} | 1 \leq j < n\}$. Then, by the triangular decomposition (9), the multiplication $m : U_{\alpha, \beta}(\mathfrak{b}^+) \otimes U_{\alpha, \beta}(\mathfrak{b}^-) \to U_{\alpha, \beta}(\mathfrak{g}_n)$, is surjective. Thus, its transpose $m^* : U_{\alpha, \beta}(\mathfrak{g}_n)^* \to [U_{\alpha, \beta}(\mathfrak{b}^+) \otimes U_{\alpha, \beta}(\mathfrak{b}^-)]^*$ defines an injective map. Moreover, by [BW2, Lemma 2.2 and Thm. 2.7] there exists a Hopf algebra pairing between $U_{\alpha, \beta}(\mathfrak{b}^+)$ and $U_{\alpha, \beta}(\mathfrak{b}^-)$, and $U_{\alpha, \beta}(\mathfrak{g}_n) \simeq D(U_{\alpha, \beta}(\mathfrak{b}^+), U_{\alpha, \beta}(\mathfrak{b}^-)^{\text{cop}})$, the double related to the pairing.

3.3.1. A PBW-type basis of $U_{\alpha, \beta}(\mathfrak{g}_n)$. If $\alpha \neq -\beta$, $U_{\alpha, \beta}(\mathfrak{g}_n)$ admits a PBW-type basis because of the next theorem, which is a special case of [K] Thm. $A_n$, see [BKL, Thm. 3.2]. First, let $\{\mathcal{E}_{i,j} : 1 \leq j \leq i < n\}$ be the elements defined by
\begin{equation}
\mathcal{E}_{j,j} = e_j \quad \text{and} \quad \mathcal{E}_{i,j} = e_i \mathcal{E}_{i-1,j} - \alpha^{-1} \mathcal{E}_{i-1,j} e_i = [e_i, \mathcal{E}_{i-1,j}]_{\alpha^{-1}},
\end{equation}
for all $1 \leq j < i < n$. Then, relation (7) can be reformulated by saying
\begin{equation}
e_{i+1} \mathcal{E}_{i+1,i} = \beta^{-1} \mathcal{E}_{i+1,i} e_{i+1}, \quad \mathcal{E}_{i+1,i} e_i = \beta^{-1} e_i \mathcal{E}_{i+1,i}.
\end{equation}
Analogously, define $\{\mathcal{F}_{i,j} : 1 \leq j \leq i < n\}$ by letting $\mathcal{F}_{j,j} = f_j$ and $\mathcal{F}_{i,j} = f_i \mathcal{F}_{i-1,j} - \beta^{-1} \mathcal{F}_{i-1,j} f_i$ for $1 \leq j < i < n$. By [BW1, p. 5] we have
\begin{equation}
w_s \mathcal{E}_{k,l} = \alpha^s \mathcal{E}_{k,l} w_s,
\end{equation}
for all $1 \leq s < n$. For $k \geq l$, denote $w_{k,l} = w_k w_{k-1} \cdots w_l$ and set $\zeta = 1 - \alpha^{-1} \beta$. The following lemma is due to Benkart and Witherspoon.

**Lemma 3.5.** [BW1, Lemma 2.19] For $1 \leq l \leq k < n$,
\begin{equation}
\Delta(\mathcal{E}_{k,l}) = \mathcal{E}_{k,l} \otimes 1 + w_{k,l} \otimes \mathcal{E}_{k,l} + \zeta \sum_{j=l}^{k-1} \mathcal{E}_{k,j+1} w_{j,l} \otimes \mathcal{E}_{j,l}. \quad \square
\end{equation}

The following theorem gives a PBW-basis of $U_{\alpha, \beta}(\mathfrak{g}_n)$.

**Theorem 3.6.** [K] Thm. $A_n$ Assume that $\alpha \neq -\beta$. Then
\begin{enumerate}[(i)]
\item $B_{\alpha}^+ = \{\mathcal{E}_{i_1,j_1} \mathcal{E}_{i_2,j_2} \cdots \mathcal{E}_{i_p,j_p} | (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_p, j_p)\}$ and $B_{\alpha}^- = \{\mathcal{F}_{i_1,j_1} \mathcal{F}_{i_2,j_2} \cdots \mathcal{F}_{i_p,j_p} | (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_p, j_p)\}$ (lexicographical ordering) are linear basis of $U_{\alpha, \beta}^+(\mathfrak{g}_n)$ and $U_{\alpha, \beta}^-(\mathfrak{g}_n)$ respectively.
\end{enumerate}
suppose the formula holds for $k$. We prove it by induction on $\psi$.

$$1 \leq (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_p, j_p)$$

and $B_1^{-} = \{ f_{i_1,j_1} f_{i_2,j_2} \cdots f_{i_p,j_p} | (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_p, j_p) \}$ (lexicographical ordering) are linear basis of $U_{\alpha, \beta}(gl_n)$ and $U_{\alpha, \beta}(gl_n)$ respectively, where $e_{i,j} = e_i e_{i-1} \cdots e_j$ and $f_{i,j} = f_i f_{i-1} \cdots f_j$ for $i \geq j$. □

Since $U_{\alpha, \beta}(gl_n)$ is a group algebra, combining these bases and using the triangular decomposition, one obtains a PBW-basis for $U_{\alpha, \beta}(gl_n)$.

There is also a two-parameter analog of $U_{\alpha, \beta}(gl_n)$ and $U_{\alpha, \beta}(gl_n)$. The Hopf algebras $O_{\alpha, \beta}(GL_n)$ and $U_{\alpha, \beta}(gl_n)$ are associated with each other by a Hopf pairing $\langle - , - \rangle : U_{\alpha, \beta}(gl_n) \times O_{\alpha, \beta}(GL_n) \rightarrow \mathbb{k}$ which is defined on the generators by

$$\langle a_i , x_{st} \rangle = \delta_{st} \alpha^{i_s}, \quad \langle b_i , x_{st} \rangle = \delta_{st} \beta^{i_t},$$

$$\langle e_j , x_{st} \rangle = \delta_{js} \delta_{j+1,t}, \quad \langle f_j , x_{st} \rangle = \delta_{j+1,s} \delta_{jt},$$

where $1 \leq i \leq n$, $1 \leq j < n$, $1 \leq s, t \leq n$, see [5]. Clearly, the Hopf pairing defines a Hopf algebra map

$$\psi : O_{\alpha, \beta}(GL_n) \rightarrow U_{\alpha, \beta}(gl_n)^{\circ},$$

given by $\psi(x_{st})(u) = \langle u, x_{st} \rangle$, for all $1 \leq s, t \leq n$ and for $u = a_i, b_i, e_j, f_j$ with $1 \leq i \leq n$, $1 \leq j < n$. Following [5, Sec. 4], we say that $O_{\alpha, \beta}(GL_n)$ is connected if $\psi$ is injective. In case that $\alpha$ and $\beta$ are roots of unity, $O_{\alpha, \beta}(GL_n)$ is not connected. However, a map induced by $\psi$ is injective on certain quotients of these quantum groups and this fact is needed to do the first step in the construction of the quantum subgroups of $O_{\alpha, \beta}(GL_n)$. The following technical lemmata will be needed in the sequel.

**Lemma 3.7.** For all $1 \leq i, j \leq n$ and $1 \leq l \leq k < n$,

(i) $\langle E_{k,l}, x_{ij} \rangle = (-1)^{k-l} \alpha^{l-k} \delta_{l,i} \delta_{k+1,j},$

(ii) $\langle F_{k,l}, x_{ij} \rangle = \delta_{k+1,i} \delta_{l,j}.$

**Proof.** We prove it by induction on $k \geq l$. By definition we know that $\langle E_{l,l}, x_{ij} \rangle = \delta_{l,i} \delta_{l+1,j}$, hence the formula holds for $k = l$. Now suppose the formula holds for $k = l - 1 \geq l$, then

$$\langle E_{k,l}, x_{ij} \rangle = \langle e_k E_{k-1,l}, x_{ij} \rangle - \alpha^{-1} \langle E_{k-1,l} e_k, x_{ij} \rangle$$

$$= \sum_{m=1}^{n} \langle e_k, x_{im} \rangle \langle E_{k-1,l}, x_{mj} \rangle - \alpha^{-1} \sum_{m=1}^{n} \langle E_{k-1,l}, x_{im} \rangle \langle e_k, x_{mj} \rangle$$

$$= \delta_{k,i} \langle E_{k-1,l}, x_{k+1,j} \rangle - \alpha^{-1} \langle E_{k-1,l}, x_{i,k} \rangle \delta_{k+1,j}$$

$$= \delta_{k,i} (-1)^{k-1-l} \alpha^{l-k+1} \delta_{l,k} \delta_{k+1,j} - \alpha^{-1} (-1)^{k-1-l} \alpha^{l-k+1} \delta_{l,i} \delta_{k+1,l}$$

$$= (-1)^{k-l} \alpha^{l-k} \delta_{l,i} \delta_{k+1,j}.$$
which finishes the proof of part (i). The proof of (ii) is analogous. By
definition we know that \( \langle F_{l,t}, x_{ij} \rangle = \langle f_l, x_{ij} \rangle = \delta_{l+1,i} \delta_{t,j} \), hence the formula
holds for \( k = l \). Now suppose the formula holds for \( k - 1 \geq l \), then
\[
\langle F_{k,t}, x_{ij} \rangle = \langle f_k F_{k-1,t}, x_{ij} \rangle - \beta^{-1} \langle F_{k-1,t} f_k, x_{ij} \rangle \\
= \sum_{m=1}^{n} \langle f_k, x_{im} \rangle \langle F_{k-1,t}, x_{mj} \rangle - \beta^{-1} \sum_{m=1}^{n} \langle F_{k-1,t}, x_{im} \rangle \langle f_k, x_{mj} \rangle \\
= \delta_{k+1,i} \langle F_{k-1,t}, x_{kj} \rangle - \beta^{-1} \langle F_{k-1,t}, x_{i,k+1} \rangle \delta_{k,j} \\
= \delta_{k+1,i} \delta_{k,k} \delta_{t,j} - \beta^{-1} \delta_{k,i} \delta_{t,j} \delta_{k,j} \\
= \delta_{k+1,i} \delta_{t,j},
\]
which finishes the proof of part (ii). \( \square \)

4. Finite quantum subgroups of \( GL_{\alpha,\beta}(n) \), \( \alpha^{-1} \beta \) not a root of 1

In this section we prove Thm. 4.1(a), as in [Mu2 Thm. 4.1].

**Theorem 4.1.** If \( \alpha \) or \( \beta \) is not a root of unity, then the finite-dimensional
quotients of \( O_{\alpha,\beta}(GL_n) \) are just the function algebras of finite subgroups
of the diagonal torus in \( GL_n(k) \).

**Proof.** Let \( q : O_{\alpha,\beta}(GL_n) \to A \) be a surjective Hopf algebra map such that
dim \( A \) is finite. Then by [R], the antipode \( S_A \) of \( A \) has finite (even) order,
say 2\( t \). Then by [AG] it follows that \( q(x_{ij}) = S_A^{2t}(q(x_{ij})) = q(S^{2t}(x_{ij})) =
(\alpha \beta)^{2t(j-1)} q(x_{ij}) \). Since \( \alpha \beta \) is not a root of unity, we have that \( q(x_{ij}) = 0 \)
for all \( 1 \leq i, j \leq n \) with \( i \neq j \). Hence, \( A \) is a finite-dimensional
quotient of \( k[x_{11}, x_{22}, \ldots, x_{nn}] \), the coordinate algebra of the diagonal torus
\( T \subseteq GL_n(k) \). Thus, \( A \) is isomorphic to the algebra of functions of a finite
subgroup of \( T \). \( \square \)

**Remark 4.2.** If \( A \) is infinite-dimensional, it may not be commutative: take
the quotient given by \( q : O_{\alpha,\beta}(GL_n) \to O_{\alpha,\beta}(B^+) \), where \( O_{\alpha,\beta}(B^+) \)
is the Borel quantum subgroup defined in Sec. 4.2. Then by construction,
\( O_{\alpha,\beta}(B^+) \) is not commutative, non-semisimple and \( S^2 \neq \text{id} \).

5. Quantum subgroups of \( GL_{\alpha,\beta}(n) \), \( \alpha^{-1} \beta \) a primitive root of 1

In this section we determine \emph{all} the quotients of \( O_{\alpha,\beta}(GL_n) \) when \( \alpha \) and
\( \beta \) are roots of unity and satisfy certain mild conditions. The proof follows
the ideas of [AG2], but hard computational methods are required.

5.1. Central Hopf subalgebras and PI-Hopf triples. Let \( \ell \in \mathbb{N} \) be
an odd natural number such that \( \alpha^{-1} \beta \) is a primitive \( \ell \)-th root of unity
and \( \alpha^\ell = 1 = \beta^\ell \). In [DPW Thm. 3.1], Du, Parshall and Wang define
a generalization of the quantum Frobenius map
\[
F^\#: O(GL_n) \to O_{\alpha,\beta}(GL_n), \quad X_{ij} \mapsto x_{ij}^\ell, \quad g \mapsto g^\ell,
\]
which is a Hopf algebra monomorphism that corresponds intuitively to a surjective map of quantum groups $F : GL_{\alpha,\beta}(n) \rightarrow GL(n)$.

For this reason, Thm. 3.6 and the definition of $U_{\alpha,\beta}(gl_n)$ we will assume from now on that

(12) $\alpha \neq \pm \beta, \beta^{-1}, \alpha^{-1} \beta$ is a primitive $\ell$-th root of 1 and $\alpha^\ell = 1 = \beta^\ell$,

where $\ell$ is supposed to be odd and $\ell \geq 3$.

Proposition 5.1. [DPW] Prop. 3.8] With the above notation and assumptions and identifying $O(GL_n) = F^#(O(GL_n))$ we have

(i) The Hopf subalgebra $O(GL_n)$ is central in $O_{\alpha,\beta}(GL_n)$.
(ii) $O_{\alpha,\beta}(GL_n)$ is faithfully flat over $O(GL_n)$.
(iii) $\overline{H} := O_{\alpha,\beta}(GL_n)/O(GL_n)^+O_{\alpha,\beta}(GL_n)$ has dimension $\ell^n$. □

Although the construction of the quantized coordinate rings differs from the one given in [DL], the explicit definition given by generators and relations allows us to give as in [AG2] a coalgebra section to $\pi$.

Corollary 5.2. The Hopf algebra surjection $\pi : O_{\alpha,\beta}(GL_n) \rightarrow \overline{H}$ admits a coalgebra section $\gamma$.

Proof. From [DPW] Prop. 3.5] it follows that $\{ \prod_{i,j} \bar{x}_{ij}^{e_{ij}} | 0 \leq e_{ij} < \ell \}$ is a basis of $\overline{H}$ for some fixed ordering of the $\pi(x_{ij}) = \bar{x}_{ij}$. With this in mind, define the linear map

(13) $\gamma : \overline{H} \rightarrow O_{\alpha,\beta}(GL_n), \quad \gamma(\prod_{i,j} \bar{x}_{ij}^{e_{ij}}) = \prod_{i,j} x_{ij}^{e_{ij}}$.

Clearly, $\gamma$ is a linear section of $\pi$. Moreover, a direct calculation shows that $\gamma$ is also a coalgebra map. □

We end this subsection with the following corollary.

Corollary 5.3. $(O(GL_n), O_{\alpha,\beta}(GL_n), \overline{H})$ is a PI-Hopf triple and one has the central extension $1 \rightarrow O(GL_n) \xrightarrow{\iota} O_{\alpha,\beta}(GL_n) \xrightarrow{\pi} \overline{H} \rightarrow 1$. □

5.2. Restricted two-parameter quantum groups. We recall now the definition of the restricted two-parameter quantum groups given in [BW1]. They are finite-dimensional quotients of the two-parameter quantum groups $U_{\alpha,\beta}(gl_n)$ given by Def. 3.4] Since $\alpha$ and $\beta$ are roots of unity, $U_{\alpha,\beta}(gl_n)$ contains central elements which generate a Hopf ideal.

The following theorem is a very small variation of results in [BW1] for $U_{\alpha,\beta}(sl_n)$, which hold with the same proofs, since the only difference relays on central group-like elements: use for example that $w_k - 1 := a_k b_k^{\ell} - 1 = a_k^{\ell} b_k^{\ell} - 1 + (a_k - 1)$ for all $1 \leq k < n$.

Theorem 5.4. (i) [BW1] Thm. 2.6] The elements $e_{k,l}^\ell, f_{k,l}^\ell, a_i^\ell, b_i^\ell$ with $1 \leq l \leq k < n$ and $1 \leq i \leq n$ are central in $U_{\alpha,\beta}(gl_n)$. 

Let $I_n$ be the ideal generated by the elements \( \mathcal{E}_{k,l}^\ell, \mathcal{F}_{k,l}^\ell, a_i^{\ell} - 1, b_i^{\ell} - 1 \) with \( 1 \leq l < k < n \) and \( 1 \leq i \leq n \). Then $I_n$ is a Hopf ideal.

**Remark 5.5.** Let $B$ be the subalgebra of $U_{\alpha,\beta}(\mathfrak{gl}_n)$ generated by $e_i^\ell, f_i^\ell, a_i^{\pm\ell}, b_i^{\pm\ell}$ with \( 1 \leq k < n \) and \( 1 \leq i \leq n \). Clearly, it is central by Thm. 5.4. Denote by $\mathfrak{gl}_n$ the quotient of $U_{\alpha,\beta}(\mathfrak{gl}_n)$ by the ideal $I_{\alpha,\beta}(\mathfrak{gl}_n)$ generated by the elements $\mathcal{E}_{k,j}^\ell, \mathcal{F}_{k,j}^\ell, (a_{k+1}b_k)^\ell - 1$ with \( 1 \leq j \leq k < n \). In this case, $\mathfrak{gl}_n$ is a pointed Hopf algebra which is a Drinfeld double and \( \dim \mathfrak{gl}_n = \ell^{(n-1)(n+2)} \). Moreover, one has the commutative diagram

\[
\begin{array}{ccc}
U_{\alpha,\beta}(\mathfrak{gl}_n) & \overset{\iota}{\longrightarrow} & U_{\alpha,\beta}(\mathfrak{gl}_n) \\
\downarrow & & \downarrow \\
\mathfrak{gl}_n & \overset{j}{\underset{\sim}{\longrightarrow}} & \mathfrak{gl}_n.
\end{array}
\]

Indeed, one can use the PBW-basis of $U_{\alpha,\beta}(\mathfrak{gl}_n)$ and $U_{\alpha,\beta}(\mathfrak{gl}_n)$ – see Thm. 3.6 - to prove that the map $j$ is injective, since $I_n \cap U_{\alpha,\beta}(\mathfrak{gl}_n) = J_n$.

**Lemma 5.8.** $U_{\alpha,\beta}(\mathfrak{gl}_n)$ is a pointed Hopf algebra of dimension $\ell^{n^2+n}$.

**Proof.** Since $U_{\alpha,\beta}(\mathfrak{gl}_n)$ is pointed, $U_{\alpha,\beta}(\mathfrak{gl}_n)$ is pointed by [Mo] Cor. 5.3.5]. By Thm. 5.6 $U_{\alpha,\beta}(\mathfrak{gl}_n)$ has a linear basis consisting of the elements

\[
\mathcal{E}_{i_1,j_1}^{m_1} \mathcal{E}_{i_2,j_2}^{m_2} \cdots \mathcal{E}_{i_p,j_p}^{m_p} a_1^{r_1} a_2^{r_2} \cdots a_n^{r_n} b_1^{s_1} b_2^{s_2} \cdots b_n^{s_n} \mathcal{F}_{k_1,l_1}^{t_1} \mathcal{F}_{k_2,l_2}^{t_2} \cdots \mathcal{F}_{k_q,l_q}^{t_q},
\]

where \( (i_1,j_1) < (i_2,j_2) < \cdots < (i_p,j_p) \) and \( (k_1,l_1) < (k_2,l_2) < \cdots < (k_q,l_q) \) are lexicographically ordered and all powers range between 0 and \( \ell - 1 \). Then \( \dim U_{\alpha,\beta}(\mathfrak{gl}_n) = \ell^{n^2+n} \).

Let $q = \alpha^{-1}\beta$ and denote $h_i = a_i^{-1}b_i$ for all \( 1 \leq i \leq n \). Since $\alpha, \beta \in \mathbb{G}_\ell$, there exist $0 < n_\alpha, n_\beta < \ell$ such that $q^{n_\alpha} = \alpha$ and $q^{n_\beta} = \beta$, by the assumptions in (12). Let $\mathcal{I}_\ell$ be the ideal of $U_{\alpha,\beta}(\mathfrak{gl}_n)$ generated by the central group-like elements \( \{ h_i^{n_\alpha}a_i^{-1} - 1, h_i^{-1}b_i^{n_\beta} - 1 \} \) \( 1 \leq i \leq n \) and define

\[
\mathfrak{u}_{\alpha,\beta}(\mathfrak{gl}_n) := U_{\alpha,\beta}(\mathfrak{gl}_n)/\mathcal{I}_\ell.
\]
Lemma 5.9. \( \hat{u}_{\alpha,\beta}(gl_n) \) is a pointed Hopf algebra of dimension \( \ell^{n^2} \).

Proof. First, \( \hat{u}_{\alpha,\beta}(gl_n) \) is pointed by [Mo] Cor. 5.3.5. Since \( \hat{u}_{\alpha,\beta}(gl_n) \) is the quotient of \( u_{\alpha,\beta}(gl_n) \) by the ideal \( I_\ell \) generated by central group-like elements, from Thm. 3.6 and the proof of Lemma 5.8 – see (14), it follows that \( \hat{u}_{\alpha,\beta}(gl_n) \) has a linear basis consisting of the elements

(15) \[ \mathcal{E}_{i_1,j_1} \mathcal{E}_{i_2,j_2} \cdots \mathcal{E}_{i_p,j_p} \mathcal{H}_{1}^{r_1} \mathcal{H}_{2}^{r_2} \cdots \mathcal{H}_{n}^{r_n} \mathcal{F}_{k_1,l_1} \mathcal{F}_{k_2,l_2} \cdots \mathcal{F}_{k_q,l_q}, \]

where \((i_1,j_1) < \cdots < (i_p,j_p)\) and \((k_1,l_1) < \cdots < (k_q,l_q)\) are lexicographically ordered and \(0 \leq m_i, r_j, k_l \leq \ell - 1\). Then \( \dim \hat{u}_{\alpha,\beta}(gl_n) = \ell^{n^2} \). \( \Box \)

As in the case of one-parameter deformations of classical objects, the restricted quantum group \( \hat{u}_{\alpha,\beta}(gl_n) \) is associated to the Hopf algebra \( \mathcal{H} \) given by Prop. 5.1 in terms of a Hopf pairing, induced from the Hopf pairing between \( U_{\alpha,\beta}(gl_n) \) and \( \mathcal{O}_{\alpha,\beta}(gl_n) \), see [DL] 6.1, [BG] III.7.10.

Lemma 5.10. The Hopf pairing \( \langle -,- \rangle : U_{\alpha,\beta}(gl_n) \times \mathcal{O}_{\alpha,\beta}(GL_n) \rightarrow k \) from Subsection 3.3.2 induce a Hopf pairing \( \langle -,- \rangle' : \hat{u}_{\alpha,\beta}(gl_n) \times \mathcal{H} \rightarrow k \). In particular, there exists a Hopf algebra map \( \overline{\psi} : \mathcal{H} \rightarrow \hat{u}_{\alpha,\beta}(gl_n)^* \) given by \( \overline{\psi}(\pi(x))(r(u)) = \langle r(u), \pi(x) \rangle' \) for all \( x \in \mathcal{O}_{\alpha,\beta}(GL_n), u \in U_{\alpha,\beta}(gl_n) \) such that the following diagram commutes

(16) \[ \begin{array}{ccc} \mathcal{O}_{\alpha,\beta}(GL_n) & \xrightarrow{\psi} & U_{\alpha,\beta}(gl_n) \circ \pi \\
\mathcal{H} & \xrightarrow{\overline{\psi}} & \hat{u}_{\alpha,\beta}(gl_n)^* \end{array} \]

where \( \psi(x)(u) = \langle u, x \rangle \) for all \( u \in U_{\alpha,\beta}(gl_n) \) and \( x \in \mathcal{O}_{\alpha,\beta}(GL_n) \).

Proof. First note that \( \hat{u}_{\alpha,\beta}(gl_n) \) is the quotient of \( U_{\alpha,\beta}(gl_n) \) by the ideal \( I_\ell \) generated by the central elements \( e_{k,j}^{\ell}, \mathcal{F}_{k,j}, h_i^{\ell} - 1, h_i^{n_\alpha} a_i^{1} - 1, h_i^{n_\beta} b_i^{1} - 1 \) with \(1 \leq j \leq k < n, 1 \leq i \leq n\). Thus, to see that the pairing \( \langle -,- \rangle' \) is well-defined, it is enough to prove \( \mathcal{O}(GL(n))^+ \subseteq J_r \) and \( I_\ell \subseteq J_l \), where \( J_r \) and \( J_l \) are the right and left radicals of the pairing. Since by Prop. 5.1 (i), \( \Delta(x_{st}^{\ell}) = \sum_{k=1}^{n} x_{sk}^{\ell} \otimes x_{kt}^{\ell} \), it suffices to show for the first inclusion that \( \langle u, x_{st}^{\ell} \rangle = \delta_{st} \varepsilon(u) \) for all \( u = a_i, b_i, e_j, f_j \), with \(1 \leq i \leq n, 1 \leq j < n\). But this follows from the definition in Section 3.3.2 for example,

\[ \langle a_i, x_{st}^{\ell} \rangle = \langle \Delta^{(\ell-1)}(a_i), x_{st}^{\otimes \ell} \rangle = \langle a_i, x_{st}^{\ell} \rangle = \delta_{st} (\alpha^{\delta_{is}}) = \delta_{st} = \delta_{st} \varepsilon(a_i), \]

and since \( \Delta^{(\ell-1)}(e_i) = \sum_{j=0}^{\ell-1} (a_j b_{i+1})^{\otimes j} \otimes e_i \otimes 1^{\otimes \ell-j-1} \) it follows that...
Finally, we have to show that \( E \). The proof for the other elements is similar. Now, to prove that \( I \) elements. But for all \( 1 \leq h \leq H \),

\[
\langle e_i, x_{st}^\ell \rangle = \langle \Delta^{(\ell-1)}(e_i), x_{st}^{\otimes \ell} \rangle = \sum_{j=0}^{\ell-1} \langle (a_i b_{i+1}^j) \otimes e_i \otimes 1^{\otimes \ell-j-1}, x_{st}^{\otimes \ell} \rangle
\]

\[
= \sum_{j=0}^{\ell-1} \delta_{st}^j \delta_{i+1, t} \delta_{s, t}^{\ell-j-1} \alpha^{(\delta_{i,s} \beta_{i+1,s})^j}
\]

\[
= \delta_{st}^{\ell-1} \delta_{i+1, t} \sum_{j=0}^{\ell-1} (\alpha^{(\delta_{i,s} \beta_{i+1,s})^j}) = 0 = \delta_{st} \varepsilon(e_i).
\]

The proof for the other elements is similar. Now, to prove that \( I_n \subseteq J_l \), it is enough to show it for the generators of the ideal \( I_n \) which are central elements. But for all \( 1 \leq s, t \leq n \) we have

\[
\langle h_i^\ell, x_{st} \rangle = \langle h_i^{\otimes \ell}, x_{st} \rangle = \sum_{1 \leq r_1, \ldots, r_{\ell-1} \leq n} \langle h_i^{\otimes \ell}, x_{s,r_1} \otimes x_{r_1, r_2} \otimes \cdots \otimes x_{r_{\ell-1}, t} \rangle
\]

\[
= \sum_{1 \leq r_m \leq n} \delta_{s,r_1} \delta_{r_1, r_2} \cdots \delta_{r_{\ell-1}, t} \alpha^{-1} (\delta_{i,s} \beta_{i+1,s})^{\ell-1} \beta_{i+1,s}^{r_1} \cdots \beta_{i+1,s}^{r_{\ell-1}} = \delta_{st}^\ell = \delta_{st}.
\]

Hence \( h_i^\ell - 1 \in J_l \) for all \( 1 \leq i \leq n \). Analogously, for all \( 1 \leq s, t \leq n \) we have

\[
\langle h_i^{n a_i^{-1}}, x_{st} \rangle = \sum_{m=1}^{n} \langle h_i^{n a_i}, x_{s,m} \rangle \langle a_i^{-1}, x_{mt} \rangle
\]

\[
= \sum_{m=1}^{n} \delta_{s,m} \alpha^{-n a_i \delta_{i,s} \beta_{a_i \delta_{i,s}}} \delta_{m,t} \alpha^{-\delta_{i,m}} = \delta_{s,t} \alpha^{-n a_i \delta_{i,s} \beta_{a_i \delta_{i,s}}} \alpha^{-\delta_{i,s}}
\]

\[
= \delta_{s,t} \beta_{i,s}^{n a_i} \alpha^{-\delta_{i,s}} = \delta_{s,t} \alpha^{\delta_{i,s} \beta_{i,s}} = \delta_{s,t}.
\]

where the second equality follows from the calculations made for \( h_i^\ell \). Thus, \( h_i^{n a_i^{-1}} - 1 \in J_l \) for all \( 1 \leq i \leq n \). By a similar calculation one can show that \( h_i^{n b_i^{-1}} - 1 \in J_l \) for all \( 1 \leq i \leq n \). Indeed, for all \( 1 \leq s, t \leq n \) we have

\[
\langle h_i^{n b_i^{-1}}, x_{st} \rangle = \sum_{m=1}^{n} \langle h_i^{n b_i}, x_{s,m} \rangle \langle b_i^{-1}, x_{mt} \rangle = \delta_{s,t} \alpha^{-n b_i \delta_{i,s} \beta_{n b_i \delta_{i,s}}} \beta^{-\delta_{i,s}}
\]

\[
= \delta_{s,t} \beta_{i,s}^{n b_i} \beta^{-\delta_{i,s}} = \delta_{s,t} \beta_{i,s} \beta^{-\delta_{i,s}} = \delta_{s,t}.
\]

Finally, we have to show that \( \mathcal{C}_l^{k,j} \) and \( \mathcal{F}_l^{k,j} \) are in \( J_l \) for all \( 1 \leq j \leq k < n \). Let \( 1 \leq s, t \leq n \), then
\[
\langle \mathcal{E}_{k,j}, x_{st} \rangle = \sum_{1 \leq r_1, \ldots, r_{l-1} \leq n} \langle \mathcal{E}_{k,j}, x_{r_1, r_2} \rangle \cdots \langle \mathcal{E}_{k,j}, x_{r_{l-1}, t} \rangle \\
= \sum_{1 \leq r_m \leq n} (-1)^{\ell(k-j)} \alpha^{\ell(k-i)} \delta_{j,s} \delta_{k+1,r_1} \delta_{j,r_1} \cdots \delta_{j,r_{l-1}} \delta_{k+1,t} \\
= (-1)^{\ell(k-j)} \delta_{s,t} \delta_{k+1,j} = 0,
\]
where the second equality follows from Lemma 3.7 (i) and the last one from 1 \leq j \leq k < n. Hence \( \mathcal{E}_{k,j} \in J_l \) for all 1 \leq j \leq k < n. Moreover,
\[
\langle \mathcal{F}_{k,j}, x_{st} \rangle = \sum_{1 \leq r_1, \ldots, r_{l-1} \leq n} \langle \mathcal{F}_{k,j}, x_{r_1, r_2} \rangle \cdots \langle \mathcal{F}_{k,j}, x_{r_{l-1}, t} \rangle \\
= \sum_{1 \leq r_m \leq n} \delta_{k+1,s} \delta_{j,r_1} \delta_{k+1,r_1} \delta_{j,r_2} \cdots \delta_{k+1,r_{l-1}} \delta_{j,t} = \delta_{k+1,s} \delta_{j,k+1,j} = 0,
\]
where the second equality follows from Lemma 3.7 (ii) and the last one from 1 \leq j \leq k < n. Hence \( \mathcal{F}_{k,j} \in J_l \) for all 1 \leq j \leq k < n and thus \( I_n \subseteq J_l \).

Since \( \mathcal{O}(GL(n))^\oplus \subseteq J_r \), there exists a Hopf algebra map \( \overline{\psi} : \overline{\mathcal{F}} \to U_{\alpha,\beta}(\mathfrak{gl}_n) \) such that \( \overline{\psi} \circ \pi = \psi \). Thus, to prove the last assertion, we need to show that \( \text{Im} \overline{\psi} \subseteq {}^t r(\hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \mathfrak{)} \). But since \( I_n \subseteq J_l \), it follows that \( \overline{\psi}(\bar{x}_{st})(h) = (x_{st}, h) = 0 \) for all \( h \in I_n \) and the map \( \overline{\psi}(\bar{x}_{st})(-) \) given by
\[
\overline{\psi}(\bar{x}_{st})(r(h)) = (r(h), \bar{x}_{st})' = (h, x_{st}) = \overline{\psi}(\bar{x}_{st})(h),
\]
defines an element in \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \) for all 1 \leq s, t \leq n. \( \square \)

5.2.1. Connectedness. The following proposition is a key step for the construction of the quotients. In the case of one-parameter deformations, this result is well-known, see [BGIII.7.10]. In the terminology of Takeuchi [Tk2], the proposition says that the Hopf algebra \( \overline{\mathcal{F}} \) is connected. Since it is finite-dimensional this also proves that it is isomorphic to the dual of the restricted (pointed) quantum group \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \). Since the proof of this fact is rather technical, for the sake of clarity, we divide it in several lemmata.

**Proposition 5.11.** \( \overline{\psi} : \overline{\mathcal{F}} \to \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \) is injective and \( \overline{\mathcal{F}} \simeq \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \).

As pointed out in the proof of Lemma 5.10 \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \) is the quotient of \( U_{\alpha,\beta}(\mathfrak{gl}_n) \) by the ideal \( I_n \) generated by the central elements \( \mathcal{E}_{k,j}^\ell, \mathcal{F}_{k,j}, h_i^k - 1, h_i^{n_i - 1} - 1, h_i^{n_j} - 1 \) with 1 \leq j \leq k < n, 1 \leq i \leq n. Hence, the triangular decomposition (17) \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \cong \hat{u}_{\alpha,\beta}^\pm(\mathfrak{gl}_n) \otimes \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \otimes \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \),

where \( \hat{u}_{\alpha,\beta}^\pm(\mathfrak{gl}_n) = r(U_{\alpha,\beta}(\mathfrak{gl}_n)^\pm) \) and \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) = r(U_{\alpha,\beta}(\mathfrak{gl}_n)^\circ) \), and \( r \) is the canonical map, see Lemmata 5.8 and 5.9. In particular, these subalgebras are generated by \( \{ r(e_i) \}_{1 \leq i < n}, \{ r(f_i) \}_{1 \leq i < n} \) and \( \{ r(h_j) \}_{1 \leq j \leq n} \), respectively. Take \( \hat{u}_{\alpha,\beta}(\mathfrak{b}^+) = r(U_{\alpha,\beta}(\mathfrak{b}^+)) \) and \( \hat{u}_{\alpha,\beta}(\mathfrak{b}^-) = r(U_{\alpha,\beta}(\mathfrak{b}^-)) \). These subalgebras of \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \) are generated by the elements \( \{ r(e_j), r(w_{\pm}^1), r(a_n)_{\pm 1} | 1 \leq
"H/π defines an injective map. By the triangular decomposition (17), the map given by the multiplication $t$ by Lemma 3.3, it follows that $π$ maps by the very definition of this quotients, there exist surjective Hopf algebra surjective quotients. For example, $K_+ = T/π(J_+)$ and $K_- = T/π(J_-)$ (see Subsection 3.2), and denote by $t_± : T → K_±$ the Hopf algebra quotients. For example, $K_+$ is generated as an algebra by the elements $\{x_{ij} : 1 ≤ i ≤ j ≤ n\}$ satisfying the relations

$$
\begin{align*}
x_{ij}^\ell &= \delta_{ij} \quad \text{for all } i ≤ j, \\
x_{ik}x_{ij} &= \alpha^{-1}x_{ij}x_{ik} \quad \text{if } j < k, \\
x_{jk}x_{ik} &= \beta x_{ik}x_{jk} \quad \text{if } i < j, \\
x_{jk}x_{jl} &= \beta x_{il}x_{jk} \quad \text{and} \\
x_{ij}x_{ik} &= x_{ik}x_{ij} \quad \text{if } i < j \text{ and } k < l.
\end{align*}
$$

Moreover, the elements $\{x_{ij}\}_{1 ≤ i ≤ n}$ are invertible group-like elements which commute with each other and the set $\{\prod_{i<j} x_{ij}^\ell \prod_{i<j} x_{ii}\} 0 ≤ d_{ij}, d_i < \ell$ is a linear basis of $K_+$, for some fixed ordering of $\{x_{ij} : 1 ≤ i ≤ j ≤ n\}$. Thus, by the very definition of this quotients, there exist surjective Hopf algebra maps $π_± : O_{α,β}(B^±) → K_±$ such that the following diagram commute

$$
\begin{array}{cccc}
O_{α,β}(GL_n) & \xrightarrow{t_±} & O_{α,β}(B^±) \\
\pi \downarrow & & \pi \downarrow \\
T & \xrightarrow{π_±} & K_±.
\end{array}
$$

Now let $δ := (\tilde{t}_+ \otimes \tilde{t}_-) Δ : T → K_+ \otimes K_-$. Then the commutativity of diagram (19) implies that the diagram

$$
\begin{array}{cccc}
O_{α,β}(GL_n) & \xrightarrow{π} & T \\
\delta \downarrow & & \tilde{δ} \downarrow \\
O_{α,β}(B^+) \otimes O_{α,β}(B^-) & \xrightarrow{π_+ \otimes π_-} & K_+ \otimes K_-.
\end{array}
$$

commutes. This implies that $\tilde{δ}$ is injective. Indeed, let $h ∈ T$ and $x ∈ O_{α,β}(GL_n)$ such that $π(x) = h$. If $δ(h) = 0$, then

$$
\begin{align*}
0 &= (\tilde{t}_+ \otimes \tilde{t}_-) Δ(π(x)) = (\tilde{t}_+ \otimes \tilde{t}_-)(π \otimes π)Δ(x) = (\tilde{t}_+ π \otimes \tilde{t}_- π)Δ(x) \\
&= (π_+ π_+ \otimes π_- π_-)Δ(x) = (π_+ \otimes π_-)δ(x),
\end{align*}
$$

where the fourth and the fifth equalities follow from the commutativity of diagrams (19) and (20). Thus $δ(x) ∈ Ker π_+ \otimes π_-$. But since $δ$ is injective by Lemma 3.3, it follows that $x ∈ Ker π$ and whence $h = 0$.\"
Lemma 5.12. There exist Hopf algebra maps $\psi_{\pm} : K_{\pm} \to \hat{u}_{\alpha,\beta}(b^\pm)^*$ such that the following diagram commutes

\[
\begin{array}{ccc}
\hat{K} & \xrightarrow{\psi} & \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \\
\downarrow i_{\pm} & & \downarrow \\
K_{\pm} & \xrightarrow{\psi_{\pm}} & \hat{u}_{\alpha,\beta}(b^\pm)^*.
\end{array}
\]

Proof. We prove it only for $\psi_{\pm}$ since the proof for $\psi_-$ is completely analogous. Consider $\bar{x}_{ij} \in \pi(I_+) \subseteq \hat{K}$, that is, $i > j$. Then $\bar{\psi}(\bar{x}_{ij})(r(u)) = \langle r(u), \bar{x}_{ij} \rangle = \langle u, x_{ij} \rangle = 0$ for all $u \in \hat{u}_{\alpha,\beta}(b^\pm)$, since by the definition of the pairing it holds for the generators of this Hopf subalgebra. This implies that there exists a Hopf algebra map $\psi_{\pm} : K_{\pm} \to \hat{u}_{\alpha,\beta}(b^\pm)^*$ such that the diagram (21) commutes. $\square$

The following lemma will be needed also for the proof of Prop. 5.11. Regrettably, it is quite technical and its proof involves lots of calculations, but we think the computations can not be avoided, which seems to be in general the case in the theory of quantum groups. In order of calculations, let us prove it, we apply the method in [Tk2].

Let $M = (M_{ij})_{1 \leq i, j \leq n}$ and $N = (N_{ij})_{1 \leq i, j \leq n}$ be upper triangular matrices and denote $\mathcal{E}^M = \mathcal{E}^{M_1} \cdots \mathcal{E}^{M_n}$ and $\bar{x}^N = \bar{x}^{N_1} \cdots \bar{x}^{N_n}$, where $\mathcal{E}^{M_i} = \mathcal{E}^{M_{i-1},i} \cdots \mathcal{E}^{M_{i+1},i}$ and $\bar{x}^{N_i} = \bar{x}^{N_{i-1}} \cdots \bar{x}^{N_{i+1}}$. Consider the subalgebras $A$, $B$ of $K_{\pm}$ and $U$, $V$ of $\hat{u}_{\alpha,\beta}(b^\pm)$ given by

\[
A = \mathbb{C}\{\bar{x}_{ij}, \bar{x}_{ji}^{-1} | 1 < j, k \leq n\}, \quad B = \mathbb{C}\{\bar{x}_{1j}, \bar{x}_{11}^{-1} | 1 < j \leq n\},
\]

\[
U = \mathbb{C}\{w_t, E_{ij} | 1 < j, 1 < t\}, \quad V = \mathbb{C}\{E_{i1} | 1 \leq i \leq n-1\},
\]

where the notation means that the elements are generators. Note that $A$ and $U$ are Hopf subalgebras and $\Delta(B) \subseteq B \otimes K_+$, $\Delta(V) \subseteq \hat{u}_{\alpha,\beta}(b^\pm) \otimes V$.

Lemma 5.13. For $1 \leq r, s, \ell$ and $a \in A$, $b \in B$, $u \in U$ and $v \in V$,

1. $\langle \mathcal{E}_{n-1,1}^{r}, \bar{x}_{1,n}^{s} \bar{x}_{1,\ell} \cdots \bar{x}_{1,k_1} \rangle = \langle \mathcal{E}_{n-1,1}^{r}, \bar{x}_{1,n}^{s} \bar{x}_{1,\ell} \cdots \bar{x}_{1,k_1} \rangle \varepsilon(\bar{x}_{1,k_1+1} \cdots \bar{x}_{1,\ell})$, for all $k_1 \neq n$, with $s + 1 \leq l \leq t$ and $0 \leq s$, where $\langle \mathcal{E}_{n-1,1}^{r}, \bar{x}_{1,n}^{s} \rangle = \delta_{r,s} r^{(r-1)/2} \langle \mathcal{E}_{n-1,1}^{r}, \bar{x}_{1,n}^{s} \rangle r \prod_{j=0}^{r-1} \Phi_{r-j}(\alpha^2)$.
2. $\langle v, a \rangle = \varepsilon(v) \varepsilon(a)$ and $\langle u, b \rangle = \varepsilon(u) \varepsilon(b)$.
3. $\langle \mathcal{E}^{M}, \bar{x}^{N} \rangle = \lambda \delta_{M,N}$, where $\lambda$ is a non-zero scalar.

Proof. (i) First we need to do some computations that will be needed in the sequel. For all $1 \leq m_j \leq n$, $1 \leq j \leq t$ we have

\[
\langle \mathcal{E}_{n-1,1}^{r}, \bar{x}_{1,1} \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle = \langle \mathcal{E}_{n-1,1}^{r}, \bar{x}_{1,1} \bar{x}_{1,n} \rangle \sum_{i=1}^{t} \delta_{n,m_i} \alpha^{i-1} \prod_{j \neq i} \delta_{1,m_j},
\]

\[
\langle \mathcal{E}_{n-1,1}^{r}, \bar{x}_{1,1} \bar{x}_{1,n} \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle = \beta^r \langle \mathcal{E}_{n-1,1}^{r}, \bar{x}_{1,1} \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle.
\]
Indeed, for $(a)$ we have

$$
\langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle = \\
\langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \rangle \varepsilon(\bar{x}_{1,m_2} \cdots \bar{x}_{1,m_t}) + \langle w_{n-1,1}, \bar{x}_{1,m_1} \rangle \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_2} \cdots \bar{x}_{1,m_t} \rangle + \\
+ \zeta \sum_{j=1}^{n-2} \langle \varepsilon_{n-1,j+1}, \bar{x}_{1,m_1} \rangle \langle \varepsilon_{j,1}, \bar{x}_{1,m_2} \cdots \bar{x}_{1,m_t} \rangle \\
= \delta_{n,m_1} \delta_{1,m_2} \cdots \delta_{1,m_t} \langle \varepsilon_{n-1,1}, \bar{x}_{1,n} \rangle + \alpha \delta_{1,m_1} \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_2} \cdots \bar{x}_{1,m_t} \rangle + \\
+ \zeta \sum_{j=1}^{n-2} \sum_{k=1}^{n} \langle \varepsilon_{n-1,j+1}, \bar{x}_{1,k} \rangle \langle w_{j,1}, \bar{x}_{k,m_1} \rangle \langle \varepsilon_{j,1}, \bar{x}_{1,m_2} \cdots \bar{x}_{1,m_t} \rangle \\
= \delta_{n,m_1} \delta_{1,m_2} \cdots \delta_{1,m_t} \langle \varepsilon_{n-1,1}, \bar{x}_{1,n} \rangle + \alpha \delta_{1,m_1} \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_2} \cdots \bar{x}_{1,m_t} \rangle + \\
= (\delta_{n,m_1} \delta_{1,m_2} \cdots \delta_{1,m_t} + \alpha \delta_{1,m_1} \delta_{n,m_2} \cdots \delta_{1,m_t} + \\
+ \alpha^2 \delta_{1,m_1} \delta_{1,m_2} \delta_{n,m_3} \delta_{1,m_4} \cdots \delta_{1,m_{t-1}} + \\
+ \alpha^{t-1} \delta_{1,m_1} \cdots \delta_{1,m_{t-1}} \delta_{n,m_t}) \langle \varepsilon_{n-1,1}, \bar{x}_{1,n} \rangle \\
= \langle \varepsilon_{n-1,1}, \bar{x}_{1,n} \rangle \sum_{i=1}^{t} \delta_{n,m_i} \alpha^{i-1} \prod_{j \neq i} \delta_{1,m_j},
$$

where the third equality follows from Lemma 3.7(i) and the fourth equality from the preceeding equalities. For $(b)$ we have for all $r \geq 1$ that

$$
\langle \varepsilon_{n-1,1}, \bar{x}_{n,n}^{r} \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle = \\
\langle \varepsilon_{n-1,1}, \bar{x}_{n,n}^{r} \rangle \varepsilon(\bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t}) + \langle w_{n-1,1}, \bar{x}_{n,n}^{r} \rangle \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle + \\
+ \zeta \sum_{j=1}^{n-2} \langle \varepsilon_{n-1,j+1}, \bar{x}_{n,n}^{r} \rangle \langle \varepsilon_{j,1}, \bar{x}_{1,m_2} \cdots \bar{x}_{1,m_t} \rangle \\
= \langle \varepsilon_{n-1,1}, \bar{x}_{n,n}^{r} \rangle \varepsilon(\bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t}) + \beta^r \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle \\
= \beta^r \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle,
$$

where the second and the third equalities follows from the fact that the element $\bar{x}_{n,n}$ is group-like and the elements $\varepsilon_{i,j}$ are nilpotent in $\mathfrak{u}_{\alpha,\beta}(b^+)$.

Now we proceed with the proof. We prove it by induction on $s$. Suppose first that $s = 0$, then, we have to prove that $\langle \varepsilon_{n-1,1}^{r}, \bar{x}_{1,k_1} \cdots \bar{x}_{1,k_t} \rangle = 0$ for all $r \geq 1$. The case $r = 1$ follows from $(a)$, since $k_j \neq n$ for all $1 \leq j \leq t$. Let $r > 1$, then

$$
\langle \varepsilon_{n-1,1}^{r}, \bar{x}_{1,k_1} \cdots \bar{x}_{1,k_t} \rangle = \\
= \sum_{m_j=1}^{n} \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle \langle \varepsilon_{n-1,1}^{r-1}, \bar{x}_{m_1,k_1} \cdots \bar{x}_{m_t,k_t} \rangle
$$

where the third equality follows from Lemma 3.7(i) and the fourth equality from the preceeding equalities. For $(b)$ we have for all $r \geq 1$ that

$$
\langle \varepsilon_{n-1,1}, \bar{x}_{n,n}^{r} \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle = \\
\langle \varepsilon_{n-1,1}, \bar{x}_{n,n}^{r} \rangle \varepsilon(\bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t}) + \langle w_{n-1,1}, \bar{x}_{n,n}^{r} \rangle \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle + \\
+ \zeta \sum_{j=1}^{n-2} \langle \varepsilon_{n-1,j+1}, \bar{x}_{n,n}^{r} \rangle \langle \varepsilon_{j,1}, \bar{x}_{1,m_2} \cdots \bar{x}_{1,m_t} \rangle \\
= \langle \varepsilon_{n-1,1}, \bar{x}_{n,n}^{r} \rangle \varepsilon(\bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t}) + \beta^r \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle \\
= \beta^r \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle,
$$

where the second and the third equalities follows from the fact that the element $\bar{x}_{n,n}$ is group-like and the elements $\varepsilon_{i,j}$ are nilpotent in $\mathfrak{u}_{\alpha,\beta}(b^+)$.

Now we proceed with the proof. We prove it by induction on $s$. Suppose first that $s = 0$, then, we have to prove that $\langle \varepsilon_{n-1,1}^{r}, \bar{x}_{1,k_1} \cdots \bar{x}_{1,k_t} \rangle = 0$ for all $r \geq 1$. The case $r = 1$ follows from $(a)$, since $k_j \neq n$ for all $1 \leq j \leq t$. Let $r > 1$, then

$$
\langle \varepsilon_{n-1,1}^{r}, \bar{x}_{1,k_1} \cdots \bar{x}_{1,k_t} \rangle = \\
= \sum_{m_j=1}^{n} \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle \langle \varepsilon_{n-1,1}^{r-1}, \bar{x}_{m_1,k_1} \cdots \bar{x}_{m_t,k_t} \rangle
$$

where the third equality follows from Lemma 3.7(i) and the fourth equality from the preceeding equalities. For $(b)$ we have for all $r \geq 1$ that

$$
\langle \varepsilon_{n-1,1}, \bar{x}_{n,n}^{r} \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle = \\
\langle \varepsilon_{n-1,1}, \bar{x}_{n,n}^{r} \rangle \varepsilon(\bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t}) + \langle w_{n-1,1}, \bar{x}_{n,n}^{r} \rangle \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle + \\
+ \zeta \sum_{j=1}^{n-2} \langle \varepsilon_{n-1,j+1}, \bar{x}_{n,n}^{r} \rangle \langle \varepsilon_{j,1}, \bar{x}_{1,m_2} \cdots \bar{x}_{1,m_t} \rangle \\
= \langle \varepsilon_{n-1,1}, \bar{x}_{n,n}^{r} \rangle \varepsilon(\bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t}) + \beta^r \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle \\
= \beta^r \langle \varepsilon_{n-1,1}, \bar{x}_{1,m_1} \cdots \bar{x}_{1,m_t} \rangle,
where the second equality follows from the comultiplication of $E_{n,1}$, the third from Lemma 3.7 (i), the fifth from the fact that $x_{n,k_2} = 0$, because $k_2 \neq n$, the sixth by (a) and the last equality from the fact that $x_{n,k_j} = 0$ because $k_j \neq n$. Now let $s > 0$. For $r = 1$ we have that

$$
\langle E_{n,1}, x_{1,n} \rangle = \langle E_{n,1}, x_{1,n} \rangle \langle x_{1,n}, x_{1,k_1} \rangle + \langle w_{n,1}, x_{1,n} \rangle \langle E_{n,1}, x_{1,n} \rangle \langle x_{1,n}, x_{1,k_1} \rangle + \zeta \sum_{j=1}^{n-1} \langle E_{n,1}, x_{1,n} \rangle \langle E_{n,1}, x_{1,n} \rangle \langle x_{1,n}, x_{1,k_1} \rangle
$$

$$
= \langle E_{n,1}, x_{1,n} \rangle \delta_{s,1} \langle x_{1,k_1} \rangle + \zeta \sum_{j=1}^{n-1} \langle E_{n,1}, x_{1,n} \rangle \langle E_{n,1}, x_{1,n} \rangle \langle x_{1,n}, x_{1,k_1} \rangle
$$

where the last equality follows from part (i). Assume now that $r > 1$ and $s > 1$. Then
\[ \langle \mathcal{E}_{n-1,1}^{r}, \bar{x}_{1,n}^{s} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle = \]
\[ \sum_{m_{j}=1}^{n} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,m_{1}} \cdots \bar{x}_{1,m_{t}} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{m_{1},n} \cdots \bar{x}_{m_{s},n} \bar{x}_{m_{s+1},k_{s+1}} \cdots \bar{x}_{m_{t},k_{t}} \rangle \]
\[ = \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,n} \cdots \bar{x}_{1,n} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle + \]
\[ + \alpha \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,n} \cdots \bar{x}_{1,n} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle + \cdots + \]
\[ + \alpha^{s} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,n} \cdots \bar{x}_{1,n} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle + \cdots + \]
\[ + \alpha^{s-1} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,n} \cdots \bar{x}_{1,n} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle = \]
\[ \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,n}^{s-1} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle + \]
\[ + \alpha^{2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,n}^{s-1} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle + \cdots + \]
\[ + \alpha^{2(s-1)} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,n}^{s-1} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle = \phi_{s} \langle \alpha^{2} \rangle \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,n}^{s-1} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle, \]
where the second equality follows by (a), the third from \( k_{j} \neq n \) for all \( s+1 \leq j \leq t \) and the fourth from the commuting relations of \( \bar{x}_{i,j} \). But

\[ \langle \mathcal{E}_{n-1,1}, \bar{x}_{n,n} \bar{x}_{1,n}^{s} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle = \sum_{m_{j}=1}^{n} \langle \mathcal{E}_{n-1,1}, \bar{x}_{n,m_{0}} \bar{x}_{1,m_{1}} \cdots \bar{x}_{1,m_{t}} \rangle \cdot \]
\[ \cdot \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{m_{0},n} \bar{x}_{1,m_{1}} \cdots \bar{x}_{m_{s},n} \bar{x}_{m_{s+1},k_{s+1}} \cdots \bar{x}_{m_{t},k_{t}} \rangle = \sum_{m_{j}=1}^{n} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,m_{1}} \cdots \bar{x}_{1,m_{t}} \rangle \cdot \]
\[ \cdot \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,m_{1}} \cdots \bar{x}_{m_{s},n} \bar{x}_{m_{s+1},k_{s+1}} \cdots \bar{x}_{m_{t},k_{t}} \rangle \]
\[ = \sum_{m_{j}=1}^{n} \beta \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,m_{1}} \cdots \bar{x}_{1,m_{t}} \rangle \cdot \]
\[ \cdot \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,m_{1}} \cdots \bar{x}_{m_{s},n} \bar{x}_{m_{s+1},k_{s+1}} \cdots \bar{x}_{m_{t},k_{t}} \rangle \]
\[ = \beta \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n}^{s-1} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle + \]
\[ + \alpha^{2} \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n}^{s-1} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle + \cdots + \]
\[ + \alpha^{2(s-1)} \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n}^{s-1} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle = \beta \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \phi_{s} \langle \alpha^{2} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n}^{s-1} \bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_{t}} \rangle, \]
where the fourth equality follows from (a) and the commuting relations of the elements \( \bar{x}_{i,j} \). Following in this way we get for \( m \geq \min(r,s) + 1 \) that
\[ \langle \mathcal{E}_{n-1,1}^r, \bar{x}_{n,n} \bar{x}_{1,n} \bar{x}_{1,k_1} \cdots \bar{x}_{1,k_t} \rangle = \]
\[ \beta^{m(m+1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^m \prod_{j=0}^{m-1} \Phi_{s-j}(\alpha^2) \langle \mathcal{E}_{n-1,1}^{r-m}, \bar{x}_{n,n} \bar{x}_{1,n} \bar{x}_{1,k_s+1} \cdots \bar{x}_{1,k_t} \rangle. \]

If \( r < s \), then taking \( m = r - 1 \) we get by (b), (a) and \( k_j \neq n \) that
\[ \langle \mathcal{E}_{n-1,1}^r, \bar{x}_{n,n} \bar{x}_{1,n} \bar{x}_{1,k_s+1} \cdots \bar{x}_{1,k_t} \rangle = \]
\[ \beta^{(r-1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^{r-1} \prod_{j=0}^{r-2} \Phi_{s-j}(\alpha^2) \langle \mathcal{E}_{n-1,1}^{r-s-1}, \bar{x}_{n,n} \bar{x}_{1,n} \bar{x}_{1,k_s+1} \cdots \bar{x}_{1,k_t} \rangle \]
\[ = \beta^{(r-1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^{r-1} \prod_{j=0}^{r-2} \Phi_{s-j}(\alpha^2) \beta^r \langle \mathcal{E}_{n-1,1}^{r-s-1}, \bar{x}_{1,n} \bar{x}_{1,k_s+1} \cdots \bar{x}_{1,k_t} \rangle \]
\[ = 0. \]

If \( s < r \) then taking \( m = s \) we get again by (b), (a) and \( k_j \neq n \) that
\[ \langle \mathcal{E}_{n-1,1}^r, \bar{x}_{n,n} \bar{x}_{1,n} \bar{x}_{1,k_s+1} \cdots \bar{x}_{1,k_t} \rangle = \]
\[ \beta^{s(s+1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^s \prod_{j=0}^{s-1} \Phi_{s-j}(\alpha^2) \langle \mathcal{E}_{n-1,1}^{r-s}, \bar{x}_{n,n} \bar{x}_{1,n} \bar{x}_{1,k_s+1} \cdots \bar{x}_{1,k_t} \rangle \]
\[ = \beta^{s(s+1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^s \prod_{j=0}^{s-1} \Phi_{s-j}(\alpha^2) \sum_{m_j=1}^{n} \langle \mathcal{E}_{n-1,1}, \bar{x}_{n,n} \bar{x}_{1,n,m_{s+1}} \cdots \bar{x}_{1,m_t} \rangle \cdot \langle \mathcal{E}_{n-1,1}^{r-s-1}, \bar{x}_{n,n} \bar{x}_{m_{s+1},k_{s+1}} \cdots \bar{x}_{m_t,k_t} \rangle \]
\[ = \beta^{s(s+1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^s \prod_{j=0}^{s-1} \Phi_{s-j}(\alpha^2) \sum_{m_j=1}^{n} \beta^s \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,m_{s+1}} \cdots \bar{x}_{1,m_t} \rangle \cdot \langle \mathcal{E}_{n-1,1}^{r-s-1}, \bar{x}_{n,n} \bar{x}_{m_{s+1},k_{s+1}} \cdots \bar{x}_{m_t,k_t} \rangle = 0. \]

Finally, if \( r = s \), taking \( m = r - 1 \) we get by (b), (a) and \( k_j \neq n \) that
\[ \langle \mathcal{E}_{n-1,1}^r, \bar{x}_{n,n} \bar{x}_{1,n} \bar{x}_{1,k_s+1} \cdots \bar{x}_{1,k_t} \rangle = \]
\[ \beta^{(r-1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^{r-1} \prod_{j=0}^{r-2} \Phi_{r-j}(\alpha^2) \langle \mathcal{E}_{n-1,1}, \bar{x}_{n,n} \bar{x}_{1,n} \bar{x}_{1,k_s+1} \cdots \bar{x}_{1,k_t} \rangle \]
\[ = \beta^{(r-1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^{r-1} \prod_{j=0}^{r-2} \Phi_{r-j}(\alpha^2) \beta^r \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \bar{x}_{1,k_s+1} \cdots \bar{x}_{1,k_t} \rangle \]
\[ = \beta^{(r+1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^r \prod_{j=0}^{r-2} \Phi_{r-j}(\alpha^2) \epsilon(\bar{x}_{1,k_{s+1}} \cdots \bar{x}_{1,k_t}). \]
Hence, it follows that

\[
\langle \mathcal{E}_{n-1,1}^r, \bar{x}_{1,n} \bar{x}_{1,k_1+1} \ldots \bar{x}_{1,k_l} \rangle = \\
\Phi_s(\alpha^2) \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle \langle \mathcal{E}_{n-1,1}^{r-1}, \bar{x}_{n,n} \bar{x}_{1,n}^{-1} \bar{x}_{1,k_1+1} \ldots \bar{x}_{1,k_l} \rangle \\
= \delta_{r,s} \beta^{(r-1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^r \prod_{j=0}^{r-1} \Phi_{r-j}(\alpha^2) \varepsilon(\bar{x}_{1,k_1+1} \ldots \bar{x}_{1,k_l}).
\]

Finally, a direct computation shows that

\[
\langle \mathcal{E}_{n-1,1}^r, \bar{x}_{1,n} \rangle = \beta^{(r-1)/2} \langle \mathcal{E}_{n-1,1}, \bar{x}_{1,n} \rangle^r \prod_{j=0}^{r-1} \Phi_{r-j}(\alpha^2).
\]

(ii) Clearly, it is enough to prove it on the generators. By Lemma 3.7 (i) we have for all $1 < i \leq j \leq n$ and $1 \leq k \leq n-1$ that $\langle \mathcal{E}_{k,1}, \bar{x}_{i,j} \rangle = 0 = \varepsilon(\mathcal{E}_{k,1})\varepsilon(\bar{x}_{i,j})$. The proof of the other equality is completely analogous.

(iii) We prove it by induction on $n$. By (ii) and the paragraph before the lemma we have

\[
\langle \mathcal{E}^M, \bar{x}^N \rangle = \langle \mathcal{E}^{M_n} \ldots \mathcal{E}^{M_1}, \bar{x}^{N_n} \ldots \bar{x}^{N_1} \rangle \\
= \langle (\mathcal{E}^{M_n} \ldots \mathcal{E}^{M_1})(1), \bar{x}^{N_1} \rangle \langle (\mathcal{E}^{M_n} \ldots \mathcal{E}^{M_1})(2), \bar{x}^{N_2} \ldots \bar{x}^{N_n} \rangle \\
= \langle (\mathcal{E}^{M_n})(1) \ldots (\mathcal{E}^{M_1})(1), \bar{x}^{N_1} \rangle \langle (\mathcal{E}^{M_n})(2) \ldots (\mathcal{E}^{M_1})(2), \bar{x}^{N_2} \ldots \bar{x}^{N_n} \rangle \\
= \langle (\mathcal{E}^{M_n})(1) \ldots (\mathcal{E}^{M_1})(1), \bar{x}^{N_1} \rangle \langle (\mathcal{E}^{M_n})(2) \ldots (\mathcal{E}^{M_1})(2), \bar{x}^{N_2} \ldots \bar{x}^{N_n} \rangle \\
\cdot \varepsilon((\mathcal{E}^{M_1})(2)) \\
= \langle (\mathcal{E}^{M_n})(1) \ldots (\mathcal{E}^{M_2})(1), \bar{x}^{N_1} \rangle \langle (\mathcal{E}^{M_1})(1), \bar{x}^{N_1} \rangle \langle (\mathcal{E}^{M_2})(2), \bar{x}^{N_2} \ldots \bar{x}^{N_n} \rangle \varepsilon((\mathcal{E}^{M_1})(2)) \\
= \varepsilon((\mathcal{E}^{M_n})(1) \ldots (\mathcal{E}^{M_2})(1)) \langle (\mathcal{E}^{M_1})(1), \bar{x}^{N_1} \rangle \langle (\mathcal{E}^{M_2})(2), \bar{x}^{N_2} \ldots \bar{x}^{N_n} \rangle \\
= \langle \mathcal{E}^{M_1}, \bar{x}^{N_1} \rangle \langle \mathcal{E}^{M_n} \ldots \mathcal{E}^{M_2}, \bar{x}^{N_2} \ldots \bar{x}^{N_n} \rangle.
\]

Thus, by induction we need only to evaluate the first factor

\[
\langle \mathcal{E}^{M_1}, \bar{x}^{N_1} \rangle = \langle \mathcal{E}_{n-1,1}^{M_1,n} \ldots \mathcal{E}_{1,1}^{M_1,2}, \bar{x}_{1,n}^{N_1} \ldots \bar{x}_{1,n}^{N_1} \rangle.
\]

But

\[
\langle \mathcal{E}_{n-1,1}^{M_1,n} \ldots \mathcal{E}_{1,1}^{M_1,2}, \bar{x}_{1,n}^{N_1} \ldots \bar{x}_{1,n}^{N_1} \rangle = \\
\sum_{r<s,N_1,N_2} \langle \mathcal{E}_{n-1,1}^{M_1,n} \ldots \mathcal{E}_{1,1}^{M_1,2}, \bar{x}_{1,n}^{N_1} \ldots \bar{x}_{1,n}^{N_1} \rangle
\]
By the definition of the maps, we have the diagram

\[
\begin{array}{ccc}
\pi & \longrightarrow & K_+ \otimes K_- \\
\psi & \downarrow & \\
\hat{u}_{\alpha,\beta}(g \langle n \rangle) & \overset{t_m}{\longrightarrow} & \hat{u}_{\alpha,\beta}(b^+)^* \otimes \hat{u}_{\alpha,\beta}(b^-)^*,
\end{array}
\]

which is commutative. By Lemma 5.14, \(\overline{\psi}_+\) and \(\overline{\psi}_-\) are injective, then \(\overline{\psi}_+ \otimes \overline{\psi}_-\) is injective. Since \(t_m\) and \(\overline{\pi}\) are injective by (18) and (20), it follows that \(\overline{\psi}\) is also injective. Since both Hopf algebras have the same dimension by Prop. 5.14 (iii) and Lemma 5.9, it follows that \(\overline{\psi}\) is an isomorphism.

Using Cor. 5.3 we get the following corollary.

**Corollary 5.15.** \((O(GL_n), O_{\alpha,\beta}(GL_n), \hat{u}_{\alpha,\beta}(g \langle n \rangle)^*)\) is a PI-Hopf triple and one has the central extension of Hopf algebras

\[
(23) \quad 1 \rightarrow O(GL_n) \xrightarrow{\pi} O_{\alpha,\beta}(GL_n) \xrightarrow{\psi} \hat{u}_{\alpha,\beta}(g \langle n \rangle)^* \rightarrow 1. \]
5.3. Construction of the quotients. In this subsection we perform the construction of the quotients of \( \mathcal{O}_{\alpha,\beta}(GL_n) \), as in [AG2]. By Corollary 5.15 \( \mathcal{O}_{\alpha,\beta}(GL_n) \) fits into a central exact sequence

\[
1 \to \mathcal{O}(GL_n) \xrightarrow{i} \mathcal{O}_{\alpha,\beta}(GL_n) \xrightarrow{\pi} \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \to 1.
\]

Let \( \varphi : \mathcal{O}_{\alpha,\beta}(GL_n) \to A \) be a surjective Hopf algebra map. The Hopf subalgebra \( K = \varphi(\mathcal{O}(GL_n)) \) is central in \( A \) and whence \( A \) is an \( H \)-extension of \( K \), where \( H \) is the Hopf algebra \( H = A/KA^+ \). Indeed, by Lemma 2.17 (i), \( \mathcal{O}_{\alpha,\beta}(GL_n) \) is noetherian and whence \( A \) is also noetherian. Then by [Sch2] Thm. 3.3, \( A \) is faithfully flat over every central Hopf subalgebra; in particular over \( K \), and the claim follows directly from [Mo, Prop. 3.4.3]. Since \( K \) is a quotient of \( \mathcal{O}(GL_n) \), there is an algebraic group \( \Gamma \) and an injective map of algebraic groups \( \sigma : \Gamma \to GL_n \) such that \( K \simeq \mathcal{O}(\Gamma) \). Moreover, since \( \varphi(\mathcal{O}_{\alpha,\beta}(GL_n)\mathcal{O}(GL_n)^+) = KA^+ \), we have \( \mathcal{O}_{\alpha,\beta}(GL_n)\mathcal{O}(GL_n)^+ \subseteq \text{Ker} \pi \varphi \), where \( \pi : A \to H \) is the canonical projection. Since \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \simeq \mathcal{O}_{\alpha,\beta}(GL_n)/[\mathcal{O}_{\alpha,\beta}(GL_n)\mathcal{O}(G)^+] \) by Prop. 5.11 there is a surjective map \( r : \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \to H \) and the following diagram

\[
\begin{array}{ccc}
1 & \to & \mathcal{O}(GL_n) \\
\downarrow{i} & & \downarrow{\pi} \\
\mathcal{O}_{\alpha,\beta}(GL_n) & \xrightarrow{\varphi} & \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \\
\downarrow{r} & & \downarrow{\sigma} \\
1 & \to & H
\end{array}
\]

is commutative. Therefore, every quotient of \( \mathcal{O}_{\alpha,\beta}(GL_n) \) fits into a similar diagram and can be constructed using central extensions. As in [AG2] we construct the quotients in three steps.

5.3.1. First step. The surjective map \( r : \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)^* \to H \) induces an injective Hopf algebra map \( t^r : H^* \to \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \). Denote also by \( H^* \) the Hopf subalgebra of \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \) given by the image of \( t^r \). Since \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \) is finite-dimensional and pointed by Lemma 5.9 by [AG2] Cor. 1.12 – see also [CM, Mul] – the Hopf subalgebras of \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \) are parameterized by triples \( (\Sigma, I_+, I_-) \) where \( \Sigma \) is a subgroup of \( T := G(\hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)) \simeq (\mathbb{Z}/\ell\mathbb{Z})^n \),

\[
I_+ = \{i | e_i \in H^*, 1 \leq i < n\} \text{ and } I_- = \{i | f_i \in H^*, 1 \leq i < n\},
\]

such that \( w_i = h_i^{\alpha_1}h_i^{-\alpha_2} \in \Sigma \) if \( i \in I_+ \) and \( w'_j = h_j^{\alpha_1}h_j^{-\alpha_2} \in \Sigma \) if \( j \in I_- \).

The Hopf subalgebras \( U_{\alpha,\beta}(l) \) of \( U_{\alpha,\beta}(\mathfrak{gl}_n) \) and \( \hat{u}_{\alpha,\beta}(l) \) of \( \hat{u}_{\alpha,\beta}(\mathfrak{gl}_n) \).

Definition 5.16. For every triple \( (\Sigma, I_+, I_-) \) define \( U_{\alpha,\beta}(l) \) to be the subalgebra of \( U_{\alpha,\beta}(\mathfrak{gl}_n) \) generated by the elements

\[
\{a_k, a_k^{-1}, b_k, b_k^{-1}, e_i, f_j | 1 \leq k \leq n, i \in I_+, j \in I_-\}.
\]

Note that \( U_{\alpha,\beta}(l) \) does not depend on \( \Sigma \). With this definition, the following proposition is clear.

Proposition 5.17. \( U_{\alpha,\beta}(l) \) is a Hopf subalgebra of \( U_{\alpha,\beta}(\mathfrak{gl}_n) \).
Let $I_n$ be the Hopf ideal of $U_{\alpha,\beta}(\mathfrak{gl}_n)$ given by Thm. 5.4 (ii) and define $J_n = I_n \cap U_{\alpha,\beta}(l)$. Clearly, $J_n$ is a Hopf ideal of $U_{\alpha,\beta}(l)$ and the quotient defines the Hopf algebra $u_{\alpha,\beta}(l) = U_{\alpha,\beta}(l)/J_n$ such that the diagram

$$
\begin{array}{ccc}
U_{\alpha,\beta}(l) & \longrightarrow & U_{\alpha,\beta}(\mathfrak{gl}_n) \\
\downarrow & & \downarrow \\
u_{\alpha,\beta}(l) & \longrightarrow & u_{\alpha,\beta}(\mathfrak{gl}_n),
\end{array}
$$

commutes, where $j : u_{\alpha,\beta}(l) \rightarrow u_{\alpha,\beta}(\mathfrak{gl}_n)$ is given by $j(a + J_n) = a + I_n$ for all $a \in U_{\alpha,\beta}(l)$. Clearly, $j$ is a well-defined Hopf algebra map that makes the diagram commute. Moreover, it is injective since $j(a + J_n) = 0$ if and only if $a \in I_n \cap U_{\alpha,\beta}(l) = J_n$.

Recall that $\hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)$ is the Hopf algebra given by $u_{\alpha,\beta}(\mathfrak{gl}_n)/\mathcal{I}_\ell$, where $\mathcal{I}_\ell$ is the ideal generated by the central group-likes $\{h_i^{n_a}a_i^{\ell-1}, h_i^{n_b}b_i^{\ell-1} - 1 | 1 \leq i \leq n\}$. Then, $\mathcal{J}_\ell := \mathcal{I}_\ell \cap u_{\alpha,\beta}(l)$ is a Hopf ideal of $u_{\alpha,\beta}(l)$. Thus we define

$$(25) \quad u_{\alpha,\beta}(l) := u_{\alpha,\beta}(l)/\mathcal{J}_\ell,$$

to be the finite-dimensional Hopf algebra given by the quotient.

Recall that all Hopf subalgebras of $\hat{u}_{\alpha,\beta}(\mathfrak{gl}_n)$ are determined by triples $(\Sigma, I_+, I_-)$ where $\Sigma \subseteq \mathbb{T}$ is a subgroup. Since $U_{\alpha,\beta}(l)$ is determined by $I_+$ and $I_-$, then clearly $u_{\alpha,\beta}(l)$ is the Hopf subalgebra of $u_{\alpha,\beta}(\mathfrak{gl}_n)$ that corresponds to the triple $(\mathbb{T}, I_+, I_-)$.

The quantized coordinate algebra $O_{\alpha,\beta}(L)$. In this subsection we construct the quantum groups $O_{\alpha,\beta}(L)$ associated to the triple $(\mathbb{T}, I_+, I_-)$. Since the pairing between $U_{\alpha,\beta}(\mathfrak{gl}_n)$ and $O_{\alpha,\beta}(GL_n)$ is degenerate, we can not follow directly the construction made in [AC2] 2.1.3. The advantage here is that the Hopf algebra $O_{\alpha,\beta}(GL_n)$ is given by generators and relations. This allows us to give an explicit construction of $O_{\alpha,\beta}(L)$. This construction can be already seen in the work of Müller [Mu2, MuI], but without mention of the triple. For every triple $(\mathbb{T}, I_+, I_-)$ define the sets

$$
(26) \quad \mathcal{I}_+ = \{(i, j) | i \leq k < j, \ k \notin I_+\}, \quad \mathcal{I}_- = \{(i, j) | j \leq k < i, \ k \notin I_-\},
$$

and let $\mathcal{I}$ be the two-sided ideal of $O_{\alpha,\beta}(GL_n)$ generated by the elements $\{x_{i,j} | (i, j) \in \mathcal{I}_+ \cup \mathcal{I}_-\}$. Then $\mathcal{I}$ is a Hopf ideal and one has the central sequence of Hopf algebras

$$
(27) \quad 1 \longrightarrow \mathcal{O}(GL_n)/\mathcal{J} \longrightarrow \mathcal{O}_{\alpha,\beta}(GL_n)/\mathcal{I} \longrightarrow \mathcal{H}/\pi(\mathcal{I}) \longrightarrow 1,
$$

where $\mathcal{J} = \mathcal{I} \cap \mathcal{O}(GL_n)$ and $\pi(\mathcal{I})$ are Hopf ideals of $\mathcal{O}(GL_n)$ and $\mathcal{H} \cong \hat{u}_{\alpha,\beta}(l)^*$, respectively.

**Lemma 5.18.**

(a) $\mathcal{J}$ is the ideal of $\mathcal{O}(GL_n)$ generated by the elements $\{x_{i,j} | (i, j) \in \mathcal{I}_+ \cup \mathcal{I}_-\}$.

(b) $\mathcal{H}/\pi(\mathcal{I}) \cong \hat{u}_{\alpha,\beta}(l)^*$.
(c) The sequence (27) is exact and fits into the commutative diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & O(GL_n) & \longrightarrow & O_{\alpha,\beta}(GL_n) & \longrightarrow & u_{\alpha,\beta}(gl_n)^* & \longrightarrow & 1 \\
& & \downarrow{\text{Res}} & & \downarrow{\pi_L} & & \downarrow{\text{res}} & & \\
1 & \longrightarrow & O(GL_n)/J & \longrightarrow & O_{\alpha,\beta}(GL_n)/I & \longrightarrow & u_{\alpha,\beta}(l)^* & \longrightarrow & 1,
\end{array}
\]

(d) The Hopf algebra surjection $\pi_L$ admits a coalgebra section $\gamma_L$.

Proof. (a) By Prop. 5.1, we identify $O(GL_n)$ with its image under $F^#$; that is, it is the Hopf subalgebra generated by the elements $\{x_{ij}^I \mid 1 \leq i, j \leq n\}$. Then $J = I \cap O(GL_n)$ is contained in the image of $F^#$, which is an algebra map and whence $J$ is generated by the elements $\{x_{ij}^I \mid (i, j) \in I_+ \cup I_\cdot\}$. But since $(O(GL_n)/J)^+$ is central in $O_{\alpha,\beta}(GL_n)/I$ and $O_{\alpha,\beta}(GL_n)/I$ is noetherian, by [Sch2] Thm. 3.3 $O_{\alpha,\beta}(GL_n)/I$ is faithfully flat over $O(GL_n)/J$ and by [Mo] Prop. 3.4.3 we have the central exact sequence

\[
1 \longrightarrow O(GL_n)/J \longrightarrow O_{\alpha,\beta}(GL_n)/I \longrightarrow K \longrightarrow 1,
\]

where $K$ is the Hopf algebra given by the quotient

\[
K = (O_{\alpha,\beta}(GL_n)/I)/[(O(GL_n)/J)^+(O_{\alpha,\beta}(GL_n)/I)].
\]

But since $(O(GL_n)/J)^+(O_{\alpha,\beta}(GL_n)/I) = (O(GL_n)^+O_{\alpha,\beta}(gl_n))/I$, it follows that $K \simeq \bar{H}/\pi(I) \simeq u_{\alpha,\beta}(l)^*$. Thus we have the central exact sequence

\[
1 \longrightarrow O(GL_n)/J \longrightarrow O_{\alpha,\beta}(GL_n)/I \longrightarrow \hat{u}_{\alpha,\beta}(l)^* \longrightarrow 1.
\]

To see that the above exact sequence fits into the commutative diagram, we need only to show that $\text{res } \pi = \pi_L \text{ Res }$. But for all $x \in O_{\alpha,\beta}(GL_n)$ we have $\pi_L \text{ Res } (x) = \pi_L(x + I) = \pi(x) + \pi(I) = \text{res } (\pi(x))$.

(d) Using that the set in (29) is a basis of $\bar{H}/\pi(I)$, we define as in Corollary 5.2 the linear map $\hat{\gamma}_L : \bar{H}/\pi(I) \rightarrow O_{\alpha,\beta}(L)$ by $\hat{\gamma}_L(\prod_{i,j} x_{ij}^{e_{ij}}) = \prod_{i,j} x_{ij}^{e_{ij}}$. Clearly, $\hat{\gamma}_L$ is a linear section of $\pi_L$. Again, a direct calculation shows that $\hat{\gamma}_L$ is also a coalgebra map. Finally, since $\bar{H}/\pi(I) \simeq u_{\alpha,\beta}(l)^*$, the morphism
\( \gamma_L \) given by the composition of \( \tilde{\gamma}_L \) and this isomorphism gives the desired coalgebra section. \( \square \)

**Remark 5.19.** Since \( O(GL_n)/J \) is a commutative Hopf algebra which is a quotient of \( O(GL_n) \), there exists an algebraic subgroup \( L \) of \( GL_n \) such that \( O(GL_n)/J \cong O(L) \). Moreover, by the lemma above, we know that this subgroup is given by the set of complex matrices \( M = (m_{ij}) \) of \( GL_n \) such that \( m_{ij} = 0 \) if \((i,j) \in I_+ \cup I_- \).

We set then

\[
O_{\alpha,\beta}(L) := O_{\alpha,\beta}(GL_n)/I.
\]

Consequently, we can re-write the commutative diagram (28) as

\[
1 \longrightarrow O(GL_n) \stackrel{\pi}{\longrightarrow} O_{\alpha,\beta}(GL_n) \longrightarrow \hat{u}_{\alpha,\beta}(gl_n)^* \longrightarrow 1
\]

\[
1 \longrightarrow O(L) \stackrel{\pi_L}{\longrightarrow} O_{\alpha,\beta}(L) \longrightarrow \hat{u}_{\alpha,\beta}(l)^* \longrightarrow 1.
\]

**5.3.2. Second step.** In this subsubsection we apply a previous result of [AG2] on quantum subgroups given by a pushout construction, to perform the second step of the construction.

Fix a triple \((T, I_+, I_-)\) and let \( L \) be the algebraic group associated to this triple as above, see Remark 5.19. Let \( \Gamma \) be an algebraic group and \( \sigma: \Gamma \rightarrow GL_n \) be an injective homomorphism of algebraic groups such that \( \sigma(\Gamma) \subseteq L \). Then we have a surjective Hopf algebra map \( \iota\sigma: O(L) \rightarrow O(\Gamma) \). Applying the pushout construction given by Prop. 2.3, we obtain a Hopf algebra \( A_{l,\sigma} \) which is part of an exact sequence of Hopf algebras and fits into the following commutative diagram

\[
1 \longrightarrow O(G) \stackrel{\iota}{\longrightarrow} O_{\alpha,\beta}(GL_n) \longrightarrow \hat{u}_{\alpha,\beta}(gl_n)^* \longrightarrow 1
\]

\[
1 \longrightarrow O(L) \stackrel{\iota_L}{\longrightarrow} O_{\alpha,\beta}(L) \longrightarrow \hat{u}_{\alpha,\beta}(l)^* \longrightarrow 1
\]

\[
1 \longrightarrow O(\Gamma) \stackrel{\iota\sigma}{\longrightarrow} A_{l,\sigma} \longrightarrow \hat{u}_{\alpha,\beta}(l)^* \longrightarrow 1.
\]

In particular, if \( \Gamma \) is finite then \( \dim A_{l,\sigma} = |\Gamma| \dim \hat{u}_{\alpha,\beta}(l) = |\Gamma|\ell_n^2 - |I_+ \cup I_-| \), see [AG2] Rmk. 2.11.

**5.3.3. Third step.** In this subsection we make the third and last step of the construction. As in [AG2], it consists essentially on taking a quotient by a Hopf ideal generated by differences of central group-like elements of \( A_{l,\sigma} \).

Recall that from the beginning of this section we fixed a surjective Hopf algebra map \( r: \hat{u}_{\alpha,\beta}(l)^* \rightarrow H \) and \( H^* \) is determined by the triple \((\Sigma, I_+, I_-)\).

Since the Hopf subalgebra \( \hat{u}_{\alpha,\beta}(l) \) is determined by the triple \((T, I_+, I_-)\) with
\[ T \supseteq \Sigma, \text{ we have that } H^* \subseteq \mathfrak{u}_{\alpha,\beta}(l) \subseteq \mathfrak{u}_{\alpha,\beta}(gl_n). \] Denote by \( v : \mathfrak{u}_{\alpha,\beta}(l)^* \to H \) the surjective Hopf algebra map induced by this inclusion.

**Remark 5.20.** By definition, \( T \) is the group generated by the group-like elements \( h_i, \ 1 \leq i \leq n \). Let \( \delta_j, \ 1 \leq j \leq n \) denote the characters on \( T \) given by \( \delta_j(h_i) = q^{\delta_{ij}} \), where \( q = \alpha^{-1}\beta \) is a primitive \( \ell \)-th root of unity which is fixed. By Lemma 5.10 and Prop. 5.11 we know that

\[ \langle \bar{x}_{jj}, h_i \rangle = \langle \bar{x}_{jj}, a_i^{-1}b_i \rangle = \langle \bar{x}_{jj}, b_i \rangle = \alpha^{-\delta_{ij} \beta^{\delta_{ij}}} = q^{\delta_{ij}}. \]

Thus, if we restrict \( \langle \bar{x}_{jj}, - \rangle \) to \( T \), we may identify \( \bar{x}_{jj} = \delta_j \) for all \( 1 \leq i \leq n \).

Let \( T_I \) be the subgroup of \( T \) generated by the elements \( \{ w_i, w'_j : \ i \in I_+, \ j \in I_- \} \) and denote by \( \rho : \hat{T} \to \hat{\Sigma}, \rho_I : \hat{T} \to \hat{T}_I \) the group homomorphisms between the character groups induced by the inclusions. Set \( N = \text{Ker} \rho \) and \( M_I = \text{Ker} \rho_I \).

The following lemma is analogous to [AG2, Lemma 2.14].

**Lemma 5.21.** (a) Every \( \chi \) of \( M_I \) defines an element \( \bar{\chi} \) of \( G(\mathfrak{u}_{\alpha,\beta}(l)^*) \) which is central. In particular, since \( N \subseteq M_I \), \( \bar{\chi} \in G(\mathfrak{u}_{\alpha,\beta}(l)^*) \cap Z(\mathfrak{u}_{\alpha,\beta}(l)^*) \) for all \( \chi \in N \).

(b) \( H \simeq \mathfrak{u}_{\alpha,\beta}(l)^*/(\bar{\chi} - 1 | \chi \in N) \).

**Proof.** (a) Let \( \chi \in M_I \) and define \( \bar{\chi} \in G(\mathfrak{u}_{\alpha,\beta}(l)^*) \) on the generators of \( \mathfrak{u}_{\alpha,\beta}(l) \) by

\[ \bar{\chi}(e_i) = 0, \quad \bar{\chi}(f_j) = 0 \quad \text{and} \quad \bar{\chi}(h_k) = \chi(h_k) \quad \text{for all } i \in I_+, \ j \in I_- \]

To see that is well-defined, it is enough to show that

\[ \bar{\chi}(e_i f_j - f_j e_i) = \frac{1}{\alpha - \beta} \bar{\chi}(a_i b_{i+1} - a_{i+1} b_i) \quad \text{for all } i \in I_+ \cap I_- \]

But \( \bar{\chi}(a_i b_{i+1} - a_{i+1} b_i) = \bar{\chi}(a_i b_{i+1}) - \bar{\chi}(a_{i+1} b_i) = \chi(w_i) - \chi(w'_i) = 0 \), since \( w_i \) and \( w'_i \) are in \( T_I \). Let us see now that \( \bar{\chi} \) is central. Since \( \mathfrak{u}_{\alpha,\beta}(l) \) admits a triangular decomposition which is induced by the triangular decomposition of \( \mathfrak{u}_{\alpha,\beta}(gl_n) \), we may assume that an element of \( \mathfrak{u}_{\alpha,\beta}(l) \) is a linear combination of elements of the form \( hm \) with \( h \in T \) and \( m \) is a product of some powers of the elements \( e_i, \ i \in I_+ \) and \( f_j, \ j \in I_- \). Let \( \theta \in \mathfrak{u}_{\alpha,\beta}(l)^* \), then

\[ \bar{\chi}(\theta(hm)) = \bar{\chi}(h(\theta(m_{(1)})\theta(hm_{(2)}))) = \bar{\chi}(h)(\bar{\chi}(\theta(m_{(1)})\theta(hm_{(2)}))) = \chi(h)\bar{\chi}(\theta(hm)) \]

and

\[ \theta(\bar{\chi}(hm)) = \theta(hm_{(1)})\bar{\chi}(hm_{(2)}) = \theta(hm_{(1)})\bar{\chi}(h)(\bar{\chi}(m_{(2)})) = \theta(hm_{(1)})\chi(h)\bar{\chi}(m_{(2)}) \]

which implies that \( \bar{\chi} \) is central.

(b) By (a) we know that \( \bar{\chi} \) is a central group-like element of \( \mathfrak{u}_{\alpha,\beta}(l)^* \) for all \( \chi \in N \). Hence the quotient \( \mathfrak{u}_{\alpha,\beta}(l)^*/(\bar{\chi} - 1 | \chi \in N) \) is a Hopf algebra.

On the other hand, we know that \( H^* \) is determined by the triple \( (\Sigma, I_+, I_-) \) and consequently \( H^* \) is included in \( \mathfrak{u}_{\alpha,\beta}(l) \). If we denote \( v : \mathfrak{u}_{\alpha,\beta}(l)^* \to H \) the surjective map induced by this inclusion, we have that \( \text{Ker} v = \{ f \in \)


\( u_\alpha(1)^\ast : f(h) = 0, \forall h \in H^\ast \). But \( \tilde{\chi} - 1 \in \text{Ker} \nu \) for all \( \chi \in N \), by definition. Hence there exists a surjective Hopf algebra map

\[ \gamma : \hat{u}_{\alpha, \beta}(1)^\ast / (\tilde{\chi} - 1) \mid \chi \in N \to H. \]

But by the proof of Lemma 5.18 and the fact that \( \hat{d} \) central and can be described as \( Y(\chi) \)

\[ \text{Hence there exists a surjective Hopf algebra map } \]

\[ \sum \left( \hat{u}_{\alpha, \beta}(1)^\ast \right) \]

\[ \nu : \chi \rho \text{ and denote } u \text{ surjective, the image of } \]

\[ \gamma \text{ which implies that } \gamma \text{ is an isomorphism.} \]

Before going on with the construction we need the following technical lemma. Let \( X = \{ \tilde{x} \mid \chi \in M_I \} \) be the set of central group-like elements of \( \hat{u}_{\alpha, \beta}(1)^\ast \) given by Lemma 5.21.

**Lemma 5.22.** There exists a subgroup \( Z := \{ \partial^m \mid m \in M_I \} \) of \( G(A_{L, \sigma}) \) isomorphic to \( X \) consisting of central elements.

**Proof.** By Lemma 5.18 \( d \), we know that there exists a coalgebra section

\[ \gamma_L : \hat{u}_{\alpha, \beta}(1)^\ast \to O_{\alpha, \beta}(L). \]

Then, we obtain the group of group-like elements given by \( Y = \{ \gamma_L(\tilde{x}) \mid \chi \in M_I \} \) in \( O_{\alpha, \beta}(L) \). Let \( d = \tilde{x}_{11} \tilde{x}_{22} \cdots \tilde{x}_{nn} \in O_{\alpha, \beta}(L) \) and denote \( \pi_L(d) = D \). By Remark 5.20 we know that each \( \tilde{x} \in M_I \subset T \cong (\mathbb{Z}/\ell\mathbb{Z})^n \) corresponds to an element \( D^m \) with \( m \in (\mathbb{Z}/\ell\mathbb{Z})^n \). By abuse of notation we write \( \tilde{x} = D^m \) with \( m \in M_I \). Then using the definition of \( \gamma_L \) and the elements \( d^m, m \in M_I \), it can be seen that the elements of \( Y \) are central and can be described as \( Y = \{ d^m \mid m \in M_I \} \).

Since the map \( \nu : O_{\alpha, \beta}(L) \to A_{L, \sigma} \) given by the pushout construction is surjective, the image of \( Y \) defines a group of central group-likes in \( A_{L, \sigma} \):

\[ Z = \{ \partial^m = \nu(d^m) \mid m \in M_I \}. \]

Besides, \( |Z| = |Y| = |X| = |M_I| \). Indeed, \( \nu(Z) = \nu(\nu(Y)) = \nu(L(Y)) = \nu(L) = \nu(\gamma_L(X)) = X \) since the diagram (32) is commutative and \( \pi_L \gamma_L = \text{id} \). Hence \( |\nu(Z)| = |X| \), from which the assertion follows.

For the definition of a subgroup datum \( D = (I^+, L^-, N, \Gamma, \sigma, \delta) \), see Def. 1.1. We prove next our first main result, which is needed to prove Thm. 1.1.

**Theorem 5.23.** Let \( D = (I^+, L^-, N, \Gamma, \sigma, \delta) \) be a subgroup datum. Then there exists a Hopf algebra \( A_D \) which is a quotient of \( O_{\alpha, \beta}(GL_n) \) and fits into the central exact sequence

\[ 1 \to O(\Gamma) \xrightarrow{i} A_D \xrightarrow{\hat{r}} H \to 1. \]

Concretely, \( A_D \) is given by the quotient \( A_{L, \sigma}/J_\delta \) where \( J_\delta \) is the two-sided ideal generated by the set \( \{ \partial^z - \delta(z) \mid z \in N \} \) and the following diagram of
exact sequences of Hopf algebras is commutative

\[ \begin{array}{ccccccccc}
1 & \rightarrow & \mathcal{O}(GL_n) & \xrightarrow{\iota} & \mathcal{O}_{\alpha, \beta}(GL_n) & \xrightarrow{\pi} & \check{u}_{\alpha, \beta}(gl_n)^* & \rightarrow & 1 \\
& & \downarrow{\text{res}} & & \downarrow{\text{Res}} & & \downarrow{\rho} & & \\
1 & \rightarrow & \mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_{\alpha, \beta}(L) & \xrightarrow{\pi_L} & \check{u}_{\alpha, \beta}(l)^* & \rightarrow & 1 \\
& & \downarrow{\iota_T} & & \downarrow{\nu} & & \downarrow{\check{\pi}} & & \\
1 & \rightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{l, \sigma} & \xrightarrow{\check{\pi}} & \check{u}_{\alpha, \beta}(l)^* & \rightarrow & 1 \\
& & \downarrow{\iota_r} & & \downarrow{v} & & \downarrow{\check{\pi}} & & \\
1 & \rightarrow & \mathcal{O}(\Gamma) & \xrightarrow{i} & A_D & \xrightarrow{\check{\pi}} & H & \rightarrow & 1.
\end{array} \]

Proof. By Remark 5.20, \( N \) determines a subgroup \( \Sigma \) of \( \mathbb{T} \) and the triple \((\Sigma, I_+, I_-)\) gives rise to a surjective Hopf algebra map \( r : u_\ast(g)^* \rightarrow H \). Since \( \sigma : \Gamma \rightarrow L \subseteq G \) is injective, by the first two steps developed before one can construct a Hopf algebra \( A_{l, \sigma} \) which is a quotient of \( \mathcal{O}_{\alpha, \beta}(GL_n) \) and an extension of \( \mathcal{O}(\Gamma) \) by \( u_{\alpha, \beta}(l)^* \), where \( u_{\alpha, \beta}(l) \) is the Hopf subalgebra of \( u_\ast(g) \) associated to the triple \((\mathbb{T}, I_+, I_-)\). Moreover, by Lemma 5.21 (b) and the proof of Lemma 5.22, \( H \) is the quotient of \( u_{\alpha, \beta}(l)^* \) by the two-sided ideal \((D^m - 1) \mid m \in N\). If \( \delta : N \rightarrow \check{\Gamma} \) is a group map, then the elements \( \delta(m) \) are central group-like elements in \( A_{l, \sigma} \) for all \( m \in N \), and the two-sided ideal \( J_\delta \) of \( A_{l, \sigma} \) generated by the set \( \{\delta^m - \delta(m) \mid m \in N\} \) is a Hopf ideal, because by Lemma 5.22 the elements \( \delta^m \) are central. Hence, by [Mu2] Prop. 3.4 (c) the following sequence is exact \( 1 \rightarrow \mathcal{O}(\Gamma)/J_\delta \rightarrow A_{l, \sigma}/J_\delta \rightarrow u_{\alpha, \beta}(l)^*/\check{\pi}(J_\delta) \rightarrow 1 \), where \( J_\delta = J_\delta \cap \mathcal{O}(\Gamma) \). Since \( \check{\pi}(\delta^m) = D^m \) and \( \check{\pi}(\delta(m)) = 1 \) for all \( m \in N \), we have that \( \check{\pi}(J_\delta) \) is the two-sided ideal of \( u_{\alpha, \beta}(l)^* \) given by \((D^m - 1) \mid m \in N\), which implies by Lemma 5.21 (b) that \( u_{\alpha, \beta}(l)^*/\check{\pi}(J_\delta) = H \). Hence, if we denote \( A_D := A_{l, \sigma}/J_\delta \), we can re-write the exact sequence of above as

\[ 1 \rightarrow \mathcal{O}(\Gamma)/J_\delta \rightarrow A_D \rightarrow H \rightarrow 1. \]

To finish the proof it is enough to see that \( J_\delta = J_\delta \cap \mathcal{O}(\Gamma) = 0 \), and this can be proved exactly as it was proved in [AG2] Thm. 2.17. \( \square \)

5.4. Characterization of the quotients. Let \( \theta : \mathcal{O}_{\alpha, \beta}(GL_n) \rightarrow A \) be a surjective Hopf algebra map. In the following we prove that \( A \simeq A_D \) as Hopf algebras for some subgroup datum \( D \).

By the first paragraph of Subsection 5.3, we know that \( A \) fits into a commutative diagram

\[ \begin{array}{ccccccccc}
1 & \rightarrow & \mathcal{O}(GL_n) & \xrightarrow{\iota} & \mathcal{O}_{\alpha, \beta}(GL_n) & \xrightarrow{\pi} & \check{u}_{\alpha, \beta}(gl_n)^* & \rightarrow & 1 \\
& & \downarrow{\iota_T} & & \downarrow{\rho} & & \downarrow{\check{\pi}} & & \\
1 & \rightarrow & \mathcal{O}(\Gamma) & \xrightarrow{i} & A & \xrightarrow{\check{\pi}} & H & \rightarrow & 1,
\end{array} \]
where $\sigma : \Gamma \to GL_n$ is an injective map of algebraic groups and $H^*$ is a Hopf subalgebra of $\hat{u}_{\alpha,\beta}(gl_n)$ determined by a triple $(\Sigma, I_+, I_-)$. Let $N$ correspond to $\Sigma$ as in the paragraph before Lemma [5.21]. Our aim is to show that there exists $\delta$ such that $A \simeq A_D$ for the subgroup datum $D = (I_+, I_-, N, \Gamma, \sigma, \delta)$. Recall from Subsection [5.3.1] the definition of the Hopf subalgebra $\hat{u}_{\alpha,\beta}(l)$ of $\hat{u}_{\alpha,\beta}(gl_n)$ given by the sets $I_+$ and $I_-$. In particular, $H^* \subseteq \hat{u}_{\alpha,\beta}(l) \subseteq \hat{u}_{\alpha,\beta}(gl_n)$. Denote by $v : \hat{u}_{\alpha,\beta}(l)^* \to H$ the surjective Hopf algebra map induced by the inclusion. The following lemma is crucial for the determination of the quantum subgroups; see [AG2] Lemma 3.1.

**Lemma 5.24.** There exist Hopf algebra maps $u, w$ such that the following diagram with exact rows commutes. In particular, $\sigma(\Gamma) \subseteq L \subseteq GL_n$.

\[
\begin{array}{cccccc}
1 & \to & O(GL_n) & \overset{\iota}{\to} & O_{\alpha,\beta}(GL_n) & \overset{\pi}{\to} & \hat{u}_{\alpha,\beta}(gl_n)^* & \to & 1 \\
\downarrow{\text{res}} & & \downarrow{\iota_L} & & \downarrow{\pi_L} & & \downarrow{\hat{\pi}} & & \downarrow{v} & & \downarrow{\hat{\nu}} & & \downarrow{w} & & \downarrow{\nu} & & \downarrow{\theta} & & \downarrow{u} & & \downarrow{\theta} & & \downarrow{\iota} & & 1 \\
1 & \to & O(L) & \overset{\iota}{\to} & O_{\alpha,\beta}(L) & \overset{\pi}{\to} & \hat{u}_{\alpha,\beta}(l)^* & \to & 1 \\
1 & \to & O(\Gamma) & \overset{\iota}{\to} & A & \overset{\theta}{\to} & H & \to & 1.
\end{array}
\]

**Proof.** To show the existence of the maps $u$ and $w$ it is enough to show that $\text{Ker Res} \subseteq \text{Ker } \theta$, since $u$ is simply $w \iota_L$. This clearly implies that $v \pi_L = \hat{\pi} w$.

By (30) we know that $\text{Ker Res} = \mathcal{I}$, where $\mathcal{I}$ is the two-sided ideal of $O_{\alpha,\beta}(GL_n)$ generated by the elements $\{x_{ij} \mid (i, j) \in \mathcal{I}_+ \cup \mathcal{I}_-\}$ and

\[\mathcal{I}_+ = \{(i, j) \mid i \leq k < j, \ k \notin I_+\}\]  
and \[\mathcal{I}_- = \{(i, j) \mid j \leq k < i, \ k \notin I_-\}\].

Then $\hat{\pi}(\theta(x)) = v \pi(x) = v \pi_L \text{Res}(x) = 0$ for all $x \in \mathcal{I}$, and this implies for all $x \in \mathcal{I}$ that $\theta(x) \in O(\Gamma)^+ A = \theta(O(GL_n)^+ O_{\alpha,\beta}(GL_n))$. Then for all $x \in \text{Ker Res}$, there exist $a \in O(GL_n)^+ O_{\alpha,\beta}(GL_n)$ and $c \in \text{Ker } \theta$ such that $x = c + a$. In particular, this holds for all $x_{ij}$ with $(i, j) \in \mathcal{I}_+ \cup \mathcal{I}_-$, that is $x_{ij} = a_{ij} + c_{ij}$, for some $a_{ij} \in O(GL_n)^+ O_{\alpha,\beta}(GL_n)$ and $c_{ij} \in \text{Ker } \theta$. Comparing degrees in both sides of the equality we have that $a_{ij} = 0$, which implies that each generator of $\mathcal{I}$ must lie in $\text{Ker } \theta$. \hfill $\square$

**Remark 5.25.** Any algebraic group $\Gamma$ appearing in an exact sequence given by a quotient of $O_{\alpha,\beta}(GL_n)$ must be composed by block matrices $M = (m_{ij})_{1 \leq i, j \leq n}$ such that $m_{ij} = 0$ if $(i, j) \in \mathcal{I}_+ \cup \mathcal{I}_-$.

The following lemma shows the convenience of characterizing the quotients $A_{l, \sigma}$ of $O_{\alpha,\beta}(GL_n)$ as pushouts.
Lemma 5.26. A is a quotient of $A_{l,\sigma}$ and the following diagram commutes

\[
\begin{array}{ccccccccc}
1 & \rightarrow & O(GL_n) & \xrightarrow{\iota} & O_{\alpha,\beta}(GL_n) & \xrightarrow{\pi} & \hat{\mathfrak{u}}_{\alpha,\beta}(\mathfrak{g}_n)^* & \rightarrow & 1 \\
\downarrow{\text{res}} & & \downarrow{\text{Res}} & & \downarrow{\pi_L} & & \downarrow{p} & & \downarrow{1} \\
1 & \rightarrow & O(L) & \xrightarrow{\iota_L} & O_{\alpha,\beta}(L) & \xrightarrow{\pi_L} & \hat{\mathfrak{u}}_{\alpha,\beta}(l)^* & \rightarrow & 1 \\
\downarrow{u} & & \downarrow{\nu} & & \downarrow{\nu} & & \downarrow{v} & & \downarrow{1} \\
1 & \rightarrow & O(\Gamma) & \xrightarrow{j} & A_{l,\sigma} & \xrightarrow{\hat{\pi}} & \hat{\mathfrak{u}}_{\alpha,\beta}(l)^* & \rightarrow & 1 \\
\downarrow{i} & & \downarrow{t} & & \downarrow{1} & & \downarrow{1} & & \downarrow{1} \\
1 & \rightarrow & O(\Gamma) & \xrightarrow{\iota} & A & \xrightarrow{\hat{\pi}} & H & \rightarrow & 1.
\end{array}
\]

Proof. Using the maps $u, w$ given by Lemma 5.24, we have that $w\iota_L = \hat{\iota}u$, that is, the following diagram commutes

\[
\begin{array}{ccccccc}
\mathcal{O}(L) & \xrightarrow{\iota_L} & \mathcal{O}_{\epsilon}(L) & \xrightarrow{\nu} & A_{l,\sigma} & \xrightarrow{\hat{\pi}} & H \\
\downarrow{u} & & \downarrow{\nu} & & \downarrow{\nu} & & \downarrow{v} \\
\mathcal{O}(\Gamma) & \xrightarrow{j} & A_{l,\sigma} & \xrightarrow{\hat{\pi}} & H & \rightarrow & 1.
\end{array}
\]

Since $A_{l,\sigma}$ is a pushout, there exists a unique Hopf algebra map $t : A_{l,\sigma} \rightarrow A$ such that $tv = w$ and $tj = \hat{i}$. This implies also that $v\pi = \hat{\pi}t$ and therefore the diagram (35) is commutative. \qed

Let $(\Sigma, I_+, I_-)$ be the triple that determines $H$. Recall that $T_I$ is the subgroup of $T$ generated by the elements $\{w_i, w'_j : i \in I_+, j \in I_-, \rho : \widehat{T} \rightarrow \widehat{\Sigma} \text{ and } \rho_I : \widehat{T} \rightarrow \widehat{T_I} \text{ denote the group homomorphisms between the character groups induced by the inclusions and } N = \ker \rho, M_I = \ker \rho. \]

By Lemmata 5.21 and 5.22 we know that the Hopf algebra $A_{l,\sigma}$ contains a set of central group-like elements $Z = \{\partial^m | m \in M_I\}$ such that $\hat{\pi}(\partial^m) = D^m$ for all $m \in M_I$ and $H = \hat{\mathfrak{u}}_{\alpha,\beta}(l)^*/(D^m - 1) | m \in N)$. To see that $A = A_D$ for a subgroup datum $D = (I_+, I_-, N, \Gamma, \sigma, \delta)$ it remains to find a group map $\delta : N \rightarrow \hat{\Gamma}$ such that $A \simeq A_{l,\sigma}/J_\delta$. This is given by the following lemma which finishes one part of the proof of Thm. The proof will be completed by Thm. 5.29 in the following subsection.

Lemma 5.27. There exists a group homomorphism $\delta : N \rightarrow \hat{\Gamma}$ such that $J_\delta = (\partial^m - \delta(m) | m \in N)$ is a Hopf ideal of $A_{l,\sigma}$ and $A \simeq A_D = A_{l,\sigma}/J_\delta$.

Proof. Let $\partial^m \in Z$. Then $\hat{\pi}t(\partial^m) = v\pi(\partial^m) = 1$ for all $m \in N$, by Lemma 5.21 (b). Since $t(\partial^m)$ is a group-like element, this implies that $t(\partial^m) \in$
objects are surjective Hopf algebra maps \( q \).

By an abuse of notation we write \( A \) for the category of quotients of \( O \) provided by the set of subgroup data. Eventually, this will be the partial order in the set of quotients given by Thm. 5.23, using the data \( D \) of \( O_{\alpha,\beta}(GL_n) \).

We will describe the partial order in the set \( \{q_D\} \), \( D \) a subgroup datum, of quotients \( q_D : O_{\alpha,\beta}(GL_n) \to A_D \) given by Thm. 5.23 using the data provided by the set of subgroup data. Eventually, this will be the partial order in the set of all quotients of \( O_{\alpha,\beta}(GL_n) \). We begin by the following definition. By an abuse of notation we write \([A_D] = [q_D]\).

**Definition 5.28.** Let \( D = (I_+, I_-, N, \Gamma, \sigma, \delta) \) and \( D' = (I'_+, I'_-, N', \Gamma', \sigma', \delta') \) be subgroup data of \( O_{\alpha,\beta}(GL_n) \). We say that \( D \leq D' \) iff

- \( I'_+ \subseteq I_+ \) and \( I'_- \subseteq I_- \).
- \( N \subseteq N' \) or equivalently \( \Sigma' \subseteq \Sigma \).
- There exists a morphism of algebraic groups \( \tau : \Gamma' \to \Gamma \) such that \( \sigma \tau = \sigma' \).
- \( \delta' \eta = \iota \tau \delta \), where \( \eta : N \to N' \) denotes the canonical inclusion.

If we denote \( I = I_+ \cup I_- \) and \( I' = I'_+ \cup I'_- \), the first condition implies that \( I' \subseteq I \). Denote by \( \rho_I : \hat{T} \to \hat{T}_I \) and \( \rho_{I'} : \hat{T} \to \hat{T}_{I'} \) the epimorphisms between the character groups induced by the inclusions and let \( M_I = \text{Ker} \rho_I, M_{I'} = \text{Ker} \rho_{I'} \). Then \( M_I \subseteq M_{I'} \).
We say that $\mathcal{D} \simeq \mathcal{D}'$ iff $\mathcal{D} \leq \mathcal{D}'$ and $\mathcal{D}' \leq \mathcal{D}$. This means that $I_+ = I_+'$ and $I_- = I_-'$, $N = N'$, there exists an isomorphism of algebraic groups $\tau : \Gamma' \rightarrow \Gamma$ such that $\sigma \tau = \sigma'$ and $\delta' = \tau \delta$.

The following theorem completes the proof of Thm. [1] (b) and its proof is completely analogous to [AG2, Thm. 2.20].

**Theorem 5.29.** Let $\mathcal{D}$ and $\mathcal{D}'$ be subgroup data. Then
\begin{itemize}
  \item[(a)] $[A_\mathcal{D}] \leq [A_\mathcal{D}']$ iff $\mathcal{D} \leq \mathcal{D}'$.
  \item[(b)] $[A_\mathcal{D}] = [A_\mathcal{D}']$ iff $\mathcal{D} \simeq \mathcal{D}'$.
\end{itemize}

5.6. **Some properties of the quotients.** In this subsection we summarize some properties of the quotients $A_\mathcal{D}$ following the study made in [AG1]. Let $\mathcal{D}=(I_+,I_-,N,\Gamma,\sigma,\delta)$ be a subgroup datum of $O_{\alpha,\beta}(GL_n)$. By Thm. 5.23, $A_\mathcal{D}$ fits into the commutative diagram

\begin{equation}
(37) \begin{array}{cccccc}
1 & \longrightarrow & O(GL_n) & \longrightarrow & O_{\alpha,\beta}(GL_n) & \longrightarrow & \hat{u}_{\alpha,\beta}(g_n)^* & \longrightarrow & 1 \\
\downarrow & \searrow & \downarrow \text{res} & \quad & \downarrow & & \downarrow & & \\
1 & \longrightarrow & O(L) & \longrightarrow & O_{\alpha,\beta}(L) & \longrightarrow & \hat{u}_{\alpha,\beta}(l)^* & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow \text{Res} & & \downarrow & \downarrow & \downarrow & & \\
1 & \longrightarrow & O(\Gamma) & \longrightarrow & A_\mathcal{D} & \longrightarrow & H & \longrightarrow & 1.
\end{array}
\end{equation}

and by Lemma 5.21 (b) and the proof of Lemma 5.22 $H \simeq \hat{u}_{\alpha,\beta}(l)^*/(D^m - 1)| m \in \mathbb{N}$). Let $T$ be the diagonal torus of $GL_n(\mathbb{C})$. The following lemma shows that it coincides with the group of characters of $O_{\alpha,\beta}(GL_n)$.

**Lemma 5.30.**
\begin{itemize}
  \item[(a)] $\text{Alg}(O_{\alpha,\beta}(GL_n), \mathbb{C}) \simeq T$.
  \item[(b)] $j$ induces a group map $\iota_j : \text{Alg}(A_\mathcal{D}, \mathbb{C}) \rightarrow \Gamma$ and $\text{Im}(\sigma \circ \iota_j) \subseteq T \cap \sigma(\Gamma)$.
\end{itemize}

**Proof.** (a) Let $\Lambda \in T$ with main diagonal $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^\times)^n$. It defines an element $\Lambda \in \text{Alg}(O_{\alpha,\beta}(GL_n), \mathbb{C})$ by setting $\Lambda(x_{ij}) = \delta_{i,j} \lambda_i$ for all $1 \leq i, j \leq n$. Hence we may define a group homomorphism $\varphi : T \rightarrow \text{Alg}(O_{\alpha,\beta}(GL_n), \mathbb{C})$ by $\varphi(\Lambda) = \hat{\Lambda}$ for all $\Lambda \in T$. This group map is clearly injective, so we need to prove that it is also surjective. Let $\theta \in \text{Alg}(O_{\alpha,\beta}(GL_n), \mathbb{C})$ and let $x_{ij}$ be a generator of $O_{\alpha,\beta}(GL_n)$ with $1 \leq i \neq j \leq n$. Then from the defining relations it follows that $\theta(x_{ij}) = 0$. Since $\theta(g) \neq 0$, we have that $\theta(x_{kk}) \in \mathbb{C}^\times$ and thus $\theta = \varphi(\Lambda)$ with $\lambda_k = \theta(x_{kk})$ for all $1 \leq k \leq n$.

(b) The bottom exact sequence of (37) induces an exact sequence of groups

\begin{equation}
(38) \quad 1 \rightarrow G(H^*) = \text{Alg}(H, \mathbb{C}) \rightarrow \text{Alg}(A_\mathcal{D}, \mathbb{C}) \rightarrow \text{Alg}(O(\Gamma), \mathbb{C}) = \Gamma,
\end{equation}
which fits into the commutative diagram of group maps

\[
\begin{array}{cccccccc}
1 & \longrightarrow & G(\mathfrak{u}_{\alpha,\beta}(g_n)) & \overset{t_{\pi}}{\longrightarrow} & \text{Alg}(O_{\alpha,\beta}(GL_n), \mathbb{C}) & \overset{t_{\epsilon}}{\longrightarrow} & \text{Alg}(O(GL_n), \mathbb{C}) & \longrightarrow & 1 \\
& & \bigg| & & & \bigg| & & & \bigg| \\
1 & \longrightarrow & G(H^*) & \overset{t_{\pi}}{\longrightarrow} & \text{Alg}(A_D, \mathbb{C}) & \overset{t_{\gamma}}{\longrightarrow} & \Gamma. & &
\end{array}
\]

Since \( q_D \) is surjective, \( \tau q_D : \text{Alg}(A_D, \mathbb{C}) \rightarrow \text{Alg}(O_{\alpha,\beta}(GL_n), \mathbb{C}) \) is injective. Thus the subgroup \((\sigma \circ t_j)(\text{Alg}(A_D, \mathbb{C})) \) of \( \sigma(\Gamma) \) must be a subgroup of \( \text{Im} \ t_\epsilon = \mathbb{T} \subseteq GL_n = \text{Alg}(O(GL_n), \mathbb{C}) \).

We resume some properties of the Hopf algebras \( A_D \) in the following proposition. Recall that a twist of a Hopf algebra \( A \) is an invertible element \( J \in A \otimes A \) such that

\[
(1 \otimes J)(\text{id} \otimes \Delta)J = (J \otimes 1)(\Delta \otimes \text{id})J \quad \text{and} \quad (\text{id} \otimes \epsilon)J = 1 = (\epsilon \otimes 1)J.
\]

If \( A^J \) denotes the Hopf algebra with the same algebra structure as \( A \) but with the comultiplication given by \( \Delta_J = J \Delta J^{-1} \), then we say that \( A^J \) is a twist deformation of \( A \). We say that two Hopf algebras \( A \) and \( B \) are twist equivalent if \( B \simeq A^J \) for some twist \( J \). The Hopf center of a Hopf algebra \( A \) is the maximal central Hopf subalgebra \( \mathcal{H}(A) \) of \( A \), see [A] 2.2.2.

**Proposition 5.31.** Let \( \mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta) \) be a subgroup datum.

(a) If \( A_D \) is pointed, then \( I_+ \cap I_- = \emptyset \) and \( \Gamma \) is a subgroup of the group of upper triangular matrices of some size. In particular, if \( \Gamma \) is finite, then it is abelian.

(b) \( A_D \) is semisimple if and only if \( I_+ \cup I_- = \emptyset \) and \( \Gamma \) is finite.

(c) If \( \dim A_D < \infty \) and \( A_D^\sigma \) is pointed, then \( \sigma(\Gamma) \subseteq \mathbb{T} \).

(d) If \( A_D \) is co-Frobenius then \( \Gamma \) is reductive.

(e) If \( \mathcal{H}(A_D) \not\approx \mathcal{H}(A_D') \) then \( A_D \) and \( A_D' \) are not twist equivalent.

**Proof.** Items (a), . . . , (d) are a small variation of [AGH] Prop. 3.8.

(e) If \( A_D \cong A_D^J \) for some twist \( J \in A_D \otimes A_D' \), then \( \mathcal{H}(A_D) \cong \mathcal{H}(A_D^J) \).

Since \( \mathcal{H}(A_D^J) = \mathcal{H}(A_D') \), the claim follows. \( \square \)

We end the paper with the following theorem that gives a new family of Hopf algebras coming from deformations on two parameters which can not be obtained as quotients of \( O_\epsilon(G) \), with \( G \) a connected, simply connected, simple complex Lie group, \( \epsilon \) a primitive \( s \)-th root of unity, \( s \) odd and \( 3 \nmid s \) if \( G \) is of type \( G_2 \), see [AC2]. Recall that if \( (a_{ij})_{1 \leq i, j \leq n} \) is a Cartan matrix, a subset \( I \subseteq I_n = \{1, \ldots, n - 1\} \) is called connected if for all \( i, j \in I \) there exist \( i = k_1, k_2, \ldots, k_{r+1} = j \) such that \( a_{k_l,k_{l+1}} \neq 0 \).

**Theorem 5.32.** Let \( \mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta) \) be a finite subgroup datum such that \( I_+ \cap I_- \neq \emptyset \) and \( \sigma(\Gamma) \not\subseteq \mathbb{T} \). Then \( A_D \) is non-semisimple and non-pointed with non-pointed dual. If moreover \( I = I_+ = I_- \) and \( I \) is connected, then \( A_D \) cannot be obtained as a quotient of \( O_\epsilon(G) \).
Proof. The first assertion follows from Prop. 5.31 Thus we show that \( A_D \) can not be obtained as a quotient of \( O_e(G) \).

Suppose that \( I = I_+ = I_- \) and let \( \mathfrak{u}_{\alpha,\beta}(l_0) \) be the Hopf subalgebra of \( \mathfrak{u}_{\alpha,\beta}(l) \) generated by \( \{e_i, f_i, w_i, w'_i : i \in I\} \). Then \( \mathcal{H}(\mathfrak{u}_{\alpha,\beta}(l_0)^*) = \mathbb{C} \). Indeed, suppose on the contrary that \( C = \mathcal{H}(\mathfrak{u}_{\alpha,\beta}(l_0)^*) \neq \mathbb{C} \), then we have a central exact sequence \( 1 \rightarrow C \rightarrow \mathfrak{u}_{\alpha,\beta}(l_0)^* \rightarrow B \rightarrow 1 \), where \( B = \mathfrak{u}_{\alpha,\beta}(l_0)^*/C \). Taking duals we get an exact sequence \( 1 \rightarrow B^* \rightarrow \mathfrak{u}_{\alpha,\beta}(l_0) \rightarrow C^* \rightarrow 1 \), with \( C^* \) a group algebra. Since \( \mathfrak{u}_{\alpha,\beta}(l_0) \) is pointed, by [AG2, Cor. 1.12], \( B^* \) is generated by a subset of \( \{e_i, f_i : i \in I\} \) and a subgroup \( F \) of \( \langle w_i, w'_i : i \in I \rangle \) such that \( w_j \in F \) and \( w'_j \in F \) if \( e_j \in B^* \) or \( f_j \in B^* \), respectively. But if \( e_j \) or \( f_j \) belongs to \( B^* \), then \( w_j \in B^* \) or \( w'_j \in B^* \) and by [AS, Lemma A.1] and the fact that \( I \) is connected this would imply that \( B^* = \mathfrak{u}_{\alpha,\beta}(l_0) \), a contradiction. Hence \( B^* \) must be generated by group-likes. This is also not possible, since this would imply that \( \mathfrak{u}_{\alpha,\beta}(l_0) \) is semisimple. Thus we must have \( \mathcal{H}(\mathfrak{u}_{\alpha,\beta}(l_0)^*) \neq \mathbb{C} \).

By Lemma 5.21 (i) the set of group-likes \( X = \{\tilde{x} | \chi \in M_I\} \) of \( \mathfrak{u}_{\alpha,\beta}(l_0)^* \) is central and we have that \( \mathfrak{u}_{\alpha,\beta}(l_0)^*/(\tilde{x} - 1 | \chi \in M_I) \simeq \mathfrak{u}_{\alpha,\beta}(l_0)^* \). Since \( N \subseteq M_I \), \( \mathfrak{u}_{\alpha,\beta}(l_0) \) is a quotient of \( H \) and by Lemma 5.21 (ii) we have \( \mathfrak{u}_{\alpha,\beta}(l_0)^* \simeq H/(\tilde{x} - 1 | \chi \in M_I/N) \). On the other hand, by Lemma 5.22 and Thm. 5.23 \( A_D \) contains a group of central group-likes isomorphic to \( W = \{\tilde{x} | \chi \in M_I/N\} \) and the Hopf subalgebra \( O(\tilde{G}) := O(\tilde{G})W \) is central in \( A_D \). Following [AG1, Lemma 3.10], one can prove that \( \mathcal{H}(A_D) = O(\tilde{G}) \), \( A_D \) is given by a pushout and \( A_D \) fits into the central exact sequence

\[ 1 \rightarrow O(\tilde{G}) \rightarrow A_D \rightarrow \mathfrak{u}_{\alpha,\beta}(l_0)^* \rightarrow 1. \]

Suppose \( A_D \cong A \) as Hopf algebras, with \( A \) a quotient of \( O_e(G) \). Then by [AG1, Lemma 3.10], \( O_e(G) \) fits into a central exact sequence

\[ 1 \rightarrow O(\tilde{G}_1) \rightarrow A \rightarrow \mathfrak{u}_{\alpha,\beta}(l_0)^* \rightarrow 1, \]

with \( \mathfrak{u}_{\alpha,\beta}(l_0) \) a Hopf subalgebra of \( \mathfrak{u}_{\alpha,\beta}(l) \), \( \mathcal{H}(A) = O(\tilde{G}_1) \) and \( \mathcal{H}(\mathfrak{u}_{\alpha,\beta}(l_0)^*) = \mathbb{C} \). This implies in particular that \( \mathfrak{u}_{\alpha,\beta}(l_0) \simeq \mathfrak{u}_{\alpha,\beta}(l_0) \). Since both pointed Hopf algebras are generated by group-likes and skew-primitives, by looking at the linear spaces generated by the skew-primitives we should have that \( \alpha = \beta^{-1} \), which contradicts our assumption (12) on the parameters. \( \square \)

It remains an open question to determine when two quotients given by subgroup data are isomorphic as Hopf algebras. The problem was solved for a special case for quotients of \( O_e(G) \) at [AG1] using algebraic geometry and homological tools. In view of this, it seems to be difficult to solve the problem with all generality. This will be left for future research.

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