A Contact Model of Heterogeneous Material by Method of Images

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Abstract. Contact analyses of heterogeneous materials may find crucial applications in many tribological studies. However, the simulations of such analyses may be difficult to be implemented since the nonlinear behaviors caused by the material heterogeneity and contact may be more complicated. Based on Eshelby's equivalent inclusion method (EIM), an arbitrarily shaped inhomogeneity interacting with a contact load may be evaluated by an equivalent inclusion model with a contact load, where the influences of the boundary surface are taken into account through the method of images. In order to improve the computational efficiency, a two-dimensional fast Fourier transform based algorithms (2D-FFT) with zero padding and wrap-around order are employed to solve the contact problem. To benchmark the present solutions, an elastic half-space containing inhomogeneities under Hertzian-type contact load is considered, and parametric studies are further performed to demonstrate the effects of elastic modulus and depth on the stress field.

1. Introduction
In tribological applications, a highly localized influence zone due to the contact pressure may be found in the sub-surface, where the existence of the impurities in such an influence zone may lead to the failure of the machine components. Analytical solutions for the inclusions or inhomogeneities in a half-space are usually difficult to solve due to the existence of boundary surface. Chiu [1] used the method of images combining with the full space solution of two cuboidal inclusions [2] to derive the analytical solutions of a cuboidal inclusion in a half-space. Hu [3] obtained the exterior stresses for a cuboidal inclusion subjected to thermal eigenstrains in a semi-infinite space. Yu and Sanday [4] provided an alternate way for handling the axisymmetric elastic fields in a half-space with a spheroidal inclusion subjected to dilatational eigenstrains. However, the explicit solution of the elastic field for an arbitrarily shaped inclusion is difficult to achieve. As pointed out by Chiu [1], the computational domain involving such an inclusion can be discretized into a series of cuboidal elements, and the resultant solutions may be obtained by superpositions of the elementary solutions in a half-space.

Based on Chiu's theory [1] and semi-analytical method, Jacq et al. [5] developed a 3D elastic-plastic contact model within the framework of Hertz's hypotheses, where the 2D-FFT based algorithms is used to solve the residual stresses layer by layer. Since the technique used by Jacq et al. [5] is two dimensional while the problem to be solved is three dimensional, the increase of computational efficiency is limited and a huge amount of data storage may be required. To address the problem, Zhou et al. [6] presented a fast algorithm based on the method of images in combination with 3D-FFT and 2D-FFT for solving the 3D arbitrarily shaped inclusions. When the method of images is
implemented, the summation of the two solutions in full space may lead to the boundary surface subjected to normal tractions, which are distributed continuously and decay slowly. The truncation of the boundary surface may result in the inevitable numerical errors which were quantitively studied by Zhou et al. [7]

Compared with the method of images [1], Liu et al. [8-9] developed a straightforward approach to derive the analytical solutions for a half-space inclusion, and the corresponding matrix of the influence coefficients can be numerically processed with the discrete convolution FFT (DC-FFT) and discrete correlation FFT (DCR-FFT) [10-11]. In view of the solutions of Liu et al. [9], Wang et al. [12] presents a fast numerical contact model for solving the elastic-plastic partial slip contact problems, and Zhou et al. [13] provides a mesh differential refinement scheme to deal with the interaction between a contact load and a half-space inclusion.

In this work, the EIM and the method of images are employed to study an ellipsoidal inhomogeneity in a half-space under Hertzian contact pressure. Since the analytical solutions of ellipsoidal inclusion are utilized as the full space solutions in the method of images, it can effectively circumvent the numerical errors due to the discretization for solving the full space component. It should be noted that the analysis of an ellipsoidal inhomogeneity in this work is based on the assumption of a uniform equivalent eigenstrain. The effectiveness of such assumption for solving ellipsoidal inhomogeneities has been demonstrated by many scholars [14-16].

2. Model and Method

2.1. Model Description

Consider an ellipsoidal inhomogeneity, Ω, embedded in a half-space, D, under Hertz pressure (figure 1). The inhomogeneity is centered at (0, 0, d) with the semi-axes of (a1, a2, a3), and the half space is defined by \( x_3 \geq 0 \). The Young’s moduli and Poisson’s ratios for the matrix and the inhomogeneity are \( E_m \) , \( \nu_m \) and \( E_i \), \( \nu_i \), respectively. The contact pressure distribution is given by \( p = p_{max}\sqrt{1 - x^2/a^2 - y^2/b^2} \), where \( a \), \( b \) is the semi-axis of the elliptic contact region, and \( p_{max} \) is the maximum contact pressure.

![Figure 1](image.png)

**Figure 1.** Schematic of an ellipsoidal inhomogeneity in a half-space subjected to Hertz contact load.

2.2. Method of Images

With the help of the method of images, when the solution of an ellipsoidal inclusion in the full space is added to the solution of an image ellipsoidal inclusion with the free surface, \( x_3 = 0 \), as the symmetry plane, the summation may eliminate the tangential traction on the boundary surface and only normal traction remains (figure 2). The details of the implement of the method of images are retained in this section. Specifically, let the region of the ellipsoidal inclusion subjected to uniform eigenstrains, \( \varepsilon_{ij}^* = [\varepsilon_{11}^*, \varepsilon_{22}^*, \varepsilon_{33}^*, \varepsilon_{23}^*, \varepsilon_{13}^*, \varepsilon_{12}^*]^T \), in the original space be \( \Omega_1 \), and the center coordinate of the
inclusion be \( x'(x'_1, x'_2, x'_3) \). The region of the image inclusion is denoted as \( \Omega_2 \), and the corresponding center of the inclusion is located at \( x''(x''_1, x''_2, x''_3) \). The eigenstrains distributed inside \( \Omega_2 \) are set to be \( \varepsilon_{ij}' = [\varepsilon_{11}', \varepsilon_{22}', \varepsilon_{33}', -\varepsilon_{23}', -\varepsilon_{13}']^T \). It can be seen that the components in four directions \( \varepsilon_{11}', \varepsilon_{22}', \varepsilon_{33}', \varepsilon_{12}' \) are equal, while the magnitudes of \( \varepsilon_{23}', \varepsilon_{13}' \) are the same, but the signs are opposite.

**Figure 2.** Schematic of the method of images for an ellipsoidal inhomogeneity in a half-space.

In the process of superposition of solutions of two ellipsoidal inclusions, shear stresses, i.e., \( \sigma_{31} \) and \( \sigma_{23} \), generated by the above eigenstrain components vanish on the boundary surface while the normal stress acting on the surface is twice of that due to a single inclusion in the full space. Therefore, it is necessary to apply the opposite normal surface stress, \(-2\sigma_{33}\), to make the boundary surface of the half-space free of normal traction. It should be noted that the surface plane (the right-most graph in Fig. 2) may be discretized into a number of rectangular patches, and the resultant solutions can be obtained by using the 2D layer by layer DC-FFT.

2.3. Equivalent Inclusion Method

An elastic body with inhomogeneity may be affected by an applied stress, such as the far-field loading or the contact load. However, on the other hand, the stress disturbance may also occur when a heterogeneous material contains its own eigenstrains, e.g. the precipitate or the martensite. The self-equilibrated internal stresses in such inhomogeneous inclusions may be obtained due to the incompatibility of the eigenstrains.

**Figure 3.** Schematic of equivalent inclusion method for an ellipsoidal inhomogeneity subjected to remote applied load.

As stated by Eshelby [17], the equivalent inclusion method (EIM) may provide an effective way to solve the inhomogeneities by means of the inclusion solutions. Figure 3 demonstrates that the stress field caused by the inclusion with the proper equivalent eigenstrains may be utilized to simulate the
stress disturbance of an inhomogeneity. In order to illustrate this point, the perturbation problem of the ellipsoid inhomogeneity subjected to uniform stress field, $\sigma_{ij}^0$, is considered. Note that an inhomogeneity may have its own eigenstrain. The stress disturbance caused by the inhomogeneous inclusion can be also solved by the EIM.

Considering an inhomogeneous inclusion region with eigenstrain $\varepsilon_{ij}^0$, the stiffness modulus of matrix $(D - \Omega)$ and inhomogeneity are $C_{ijkl}$ and $C_{ijkl}^{*}$ respectively, the stress field at the far end is $\sigma_{ij}^0$, the corresponding strain is $\varepsilon_{ij}^0$ and $\sigma_{ij}^0 = C_{ijkl}^* \varepsilon_{ijkl}$. The symbols $\sigma_{ij}$ and $\varepsilon_{ij}$ with the superscript “*” denote the total stress and the total strain, respectively. It should be noted that the perturbation strains may be determined by subtracting the homogeneous strains from the actual total strains, that is $\tilde{\varepsilon}_{ij} = \varepsilon_{ij} - \varepsilon_{ij}^0$, and the following conclusions are obtained according to Hook’s law:

$$
\begin{align*}
\sigma_{ij} &= \sigma_{ij}^0 + \tilde{\sigma}_{ij} = C_{ijkl}^* \left( \varepsilon_{ijkl}^0 - \varepsilon_{ijkl}^0 - \varepsilon_{ijkl}^0 \right) \\
\sigma_{ij}^0 &= C_{ijkl}^* \left( \varepsilon_{ijkl}^0 + \tilde{\varepsilon}_{ijkl} - \varepsilon_{ijkl}^0 \right) \quad \text{in } \Omega \\
\sigma_{ij} &= \sigma_{ij}^0 + \tilde{\sigma}_{ij} = C_{ijkl}^* \left( \varepsilon_{ijkl}^0 + \tilde{\varepsilon}_{ijkl} \right) \quad \text{in } D - \Omega
\end{align*}
$$

(1)

With the help of the EIM, the actual interior and exterior total stresses in the right hand side of figure 3 may be expressed as

$$
\begin{align*}
\sigma_{ij} &= \sigma_{ij}^0 + \tilde{\sigma}_{ij} = C_{ijkl}^* \left( \varepsilon_{ijkl}^0 - \varepsilon_{ijkl}^0 - \varepsilon_{ijkl}^0 \right) \\
\sigma_{ij}^0 &= C_{ijkl} \left( \varepsilon_{ijkl}^0 + \tilde{\varepsilon}_{ijkl} - \varepsilon_{ijkl}^0 \right) \\
\sigma_{ij}^0 &= C_{ijkl} \left( \varepsilon_{ijkl}^0 + \tilde{\varepsilon}_{ijkl} \right)
\end{align*}
$$

(2)

Accordingly, the stress equivalent conditions may be established inside the inhomogeneous inclusion, and the corresponding expression is written as:

$$
C_{ijkl}^* \left( \varepsilon_{ijkl}^0 + \tilde{\varepsilon}_{ijkl} - \varepsilon_{ijkl}^0 \right) = C_{ijkl} \left( \varepsilon_{ijkl}^0 + \tilde{\varepsilon}_{ijkl} - \varepsilon_{ijkl}^0 \right) \quad \text{in } \Omega
$$

(3)

For the uniformly distributed eigenstrains of $\varepsilon_{ij}^p$ and $\varepsilon_{ij}^*$ in $\Omega$, the perturbed strain $\tilde{\varepsilon}_{ijkl}$ are expressed as

$$
\tilde{\varepsilon}_{ijkl} = S_{klmn} \varepsilon_{mn}^*
$$

(4)

where

$$
\varepsilon_{mn}^* = \varepsilon_{mn}^p + \varepsilon_{mn}^*
$$

(5)

When equation (4) is substituted into equation (3), the corresponding expression is represented as

$$
C_{ijkl}^* \left( \varepsilon_{ijkl}^0 + S_{klmn} \varepsilon_{mn}^* - \varepsilon_{ijkl}^0 \right) = C_{ijkl} \left( \varepsilon_{ijkl}^0 + S_{klmn} \varepsilon_{mn}^* - \varepsilon_{ijkl}^0 \right)
$$

(6)

The equivalent $\varepsilon_{ij}^*$ in the above expression are only unknowns for the problem of inhomogeneous inclusion. Equation (6) may be further reduced for the case of an inhomogeneity:

$$
C_{ijkl}^* \left( \varepsilon_{ijkl}^0 + S_{klmn} \varepsilon_{mn}^* \right) = C_{ijkl} \left( \varepsilon_{ijkl}^0 + S_{klmn} \varepsilon_{mn}^* \right) \quad \text{in } \Omega
$$

(7)

3. Results and Discussions
In this section, benchmark examples of an ellipsoidal inhomogeneity subjected to elliptical Hertz pressure on the surface of the half-space are studied, where Poisson’s ratios of the matrix and the inhomogeneity are assumed to be $v_m = v_i = 0.3$. The maximum contact pressure $p_{max}$ is chosen as 5GPa and the semi-axis of the elliptic contact region are set to be $a=100\mu m$ and $b=50\mu m$. The center of a prolate spheroidal inhomogeneity with the semi-axis $a_1 = a_2 = 5\mu m$, $a_3=4\mu m$ is located at (0,
0, 12μm). The results of the resultant stresses are normalized by $\sigma_0 = Ee^*/(1 - \nu)$. Figure 4 demonstrates the variation of the dimensionless stress along the $x_3$-axis from 0 to 40μm for different combinations of elastic moduli. It can be found that the stress concentration occurs in a localized area due to the presence of the inhomogeneity. Furthermore, the effects of different locations of inhomogeneity center on the resultant stress are studied in figure 5, where the depth, $d$, is chosen as 6μm, 9μm, 12μm, 15μm and 18μm.

![Figure 4. Variation of normalized stress along the $x_3$-axis for different combinations of the elastic moduli of the inhomogeneity and the matrix under elliptical contact load in a half-space. (a) Stress $\sigma_{33}$; (b) Stress $\sigma_{13}$.](image)

![Figure 5. Variation of normalized stress along the $x_3$-axis for different depths under elliptical contact load in a half-space. (a) Stress $\sigma_{33}$; (b) Stress $\sigma_{13}$.](image)

### 4. Concluding Remarks
The interaction between a contact load and micro defects, such as inhomogeneities, voids and cracks, is of fundamental importance in solid mechanics. However, the analytical solutions are difficult to obtain due to the material mismatch, various inclusion shapes and the existence of boundary surface. The present work provides a contact model to solve an ellipsoidal inhomogeneity in a half-space subjected to Hertz contact pressure. Since the method of images is based on the analytical solutions of an ellipsoidal inclusion in full space, the numerical errors due to the discretization are greatly reduced. A numerical scheme through the equivalent inclusion method is proposed and the 2D layer by layer FFT is employed to expedite the computations. Furthermore, the effects of the near surface inhomogeneities on the stress field are discussed.

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