Quenched Two Dimensional Supersymmetric Yang-Mills Theory

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By studying the pure Yang-Mills theory on a circle, as well as an adjoint scalar coupled to the gauge field on a circle, we propose a quenching prescription in which the combination of the spatial component of the gauge field and $P$ is treated as a dynamic variable. Averaging over momentum is not necessary, therefore the usual ultraviolet cut-off is eliminated. We then apply this prescription to study the large $N$ two dimensional supersymmetric gauge theory. An one dimensional supersymmetric matrix model is obtained. It is not known whether this model can be solved exactly. However, an extended model with one more complex fermion is exactly solvable, with $N = 1$ supersymmetry as Parisi-Sourlas supersymmetry. The exact solvability may have some implications for the $N = 1$ quenched model.

June 1995
1. Introduction

The two dimensional supersymmetric Yang-Mills theory may turn out to be a two dimensional matrix model which can be solved in the large $N$ limit. A set of loop equations are solved in [1] with a certain assumption, its spectrum is studied numerically in [2] in the light-cone formalism. There seems to be at least two major motivations for studying supersymmetric gauge theories in various dimensions. The first is the hope that deeper understanding of these theories may shed light on several longstanding open problems in particle theory and quantum field theories, such as confinement [3], dynamic supersymmetry breaking [4] and various kinds of duality [3,5]. The second motivation, which is equally important if not more important, is the search for higher dimensional solvable matrix models. As we have learned from the study of zero and one dimensional matrix models, string theory can be reformulated in terms of matrix models. In such models, some spacetime dimensions are dynamically generated, and powerful mathematical tools can be developed to calculate various physical quantities. It may be justified to hope that string theory will be ultimately formulated in this fashion. This latter motivation serves as the primary one for our previous study [1] and the work presented in this paper. Recently, based on the explicit solution of the low energy effective action in the supersymmetric $SU(N)$ Yang-Mills theory in four dimensions [6], Douglas and Shenker extract some interesting large $N$ information about the theory [7], and find that in the large $N$ limit, the validity region of the low energy solution becomes very narrow. Because of the importance of the large $N$ solution of super-gauge theories, it is then desirable to explore all possible valuable methods.

As long as the large $N$ problem is concerned, Eguchi and Kawai showed that the Euclidean lattice gauge theory can be reduced to a model at a single site with $D$ matrices [8], here $D$ is the spacetime dimension. Unfortunately, it was shown subsequently that the EK model suffers breakdown of global U(1) symmetries at weak coupling in dimensions higher than two [9], so the EK model is not capable of reproducing the large $N$ result of the Wilson theory in the physical weak coupling regime. A quenched model was then introduced in [9]. This idea was substantially augmented by Parisi [10] and by Gross and Kitazawa [11], these authors were able to show that the quenched matrix model indeed correctly produces the large $N$ results for correlation functions in the weak coupling regime. We shall follow refs. [12] to work with the Hamiltonian formalism.

The quenching prescription proposed in this paper is slightly different from the conventional one. Instead of first calculating quantities with fixed quenching momentum matrix, and then averaging over momenta, we propose to consider the combination of the momentum matrix with the spatial component of the gauge field as a dynamic variable. The
motivation for doing this is from consideration of the pure Yang-Mills theory on a cylinder, in which the theory already reduces to a quantum mechanics problem. By construction this prescription works only in two dimensions. The quenched super gauge theory, being equivalent to the original theory in the large $N$ limit, is an one dimensional $N = 1$ supersymmetric multi-matrix model. There are two Hermitian bosonic matrices, two Hermitian fermionic matrices, or equivalently one complex bosonic matrix and one complex fermionic matrix. This matrix model is interesting in its own right. It is interesting to study its $1/N$ corrections, even these may have nothing to do with $1/N$ corrections in the original super gauge theory. In the similar spirit, one may try to generalize this supersymmetric matrix model in various ways, and to ask the question whether a doubling scaling limit exists, and if it does, what kind of string theory it describes.

We shall not try to solve the quenched model directly in this paper. The main result of this paper perhaps is the construction of a solvable model by extending the supersymmetric quenched model to include one more complex fermionic matrix. There is still one complex bosonic matrix. The bosonic part of the action differs from the $N = 1$ model by a potential term. The $N = 1$ supersymmetry in the extended model can be explained as Parisi-Sourlas supersymmetry [13]. Since such a model can be reduced to a Gaussian model together with a pair of stochastic equations, Green’s functions of bosonic matrices are calculable in principle, once the first order stochastic equations are solved. It is conceivable that further understanding of the extended model will shed light on understanding of the quenched model.

Since the quenching prescription was proposed more than ten years ago, it is appropriate to review the essential ingredients first. We shall do this in the next section. We then proceed in sect.3 to discuss the prescription for two dimensional gauge theories by starting with the pure Yang-Mills theory defined on a cylinder. We shall argue that by promoting the combination of momentum matrix and the spatial component of the gauge field to a dynamic matrix, the average over momenta is automatically done. Also, it is necessary to rescale the coupling constant. The consistency of this prescription is checked in the case of an adjoint matter coupled to the gauge field. The supersymmetric quenched model is introduced in sect.4. We do not know how to solve this model yet. Then in sect.5 we extend this model to obtain an exactly solvable model. This model exhibits Parisi-Sourlas supersymmetry, and associated first order stochastic equations can be integrated. The last section is devoted to a discussion.
2. A Brief Review of the Quenching Prescription

To illustrate the idea of quenching, consider a two dimensional Hermitian matrix $M(x, t)$ with the following action

$$S(M) = \int d^2x \text{tr} \left( \frac{1}{2}(\partial_t M)^2 - \frac{1}{2}(\partial_x M)^2 + \frac{g}{\sqrt{N}} M^3 \right), \quad (2.1)$$

where the coupling constant $g$ is held fixed in the limit $N \to \infty$.

Instead of quenching all spacetime momenta as in [9,10,11], only the spatial momentum is to be quenched in this paper [12]. The prescription is to replace the spatial dependent matrix field $M(x, t)$ by $\exp(iPx)M(t)\exp(-iPx)$, here $P$ is a diagonal matrix with eigenvalues $p_i$. The derivative $\partial_x M$ is replaced by $i[P,M]$ and the quenched action reads

$$S = a \int dt \text{tr} \left( \frac{1}{2}(\partial_t M)^2 + \frac{1}{2}[P, M]^2 + \frac{g}{\sqrt{N}} M^3 \right), \quad (2.2)$$

where $a = 2\pi/\Lambda$ is the ultraviolet cut-off whose utility we will see shortly, $M$ in the above action depends only on $t$. The propagator derived from (2.2) is

$$\langle M_{ij}(t)M_{lk}(t') \rangle = -\frac{i}{a} \delta_{ik} \delta_{jl} (\partial_t^2 + p_{ij}^2)^{-1} \delta(t - t'), \quad (2.3)$$

where $p_{ij} = p_i - p_j$, the momentum carried by $M_{ij}$. We thus see that this propagator is almost the same as $\langle M_{ij}(p,t)M_{lk}(q,t') \rangle$ in the original model if one lets $p = p_{ij}$, except for lacking of the delta function factor $\delta(p + q)$.

We now show that any planar vacuum diagram in the original model is recovered by averaging the corresponding diagram in the quenched model with an integral

$$a^{-1} \prod_{i=1}^{N} \int_{-\Lambda/2}^{\Lambda/2} \frac{adp_i}{2\pi}, \quad (2.4)$$

each integral in the above product is normalized to 1. To see this, consider a planar diagram with $l$ loops, $n$ vertices and $p$ propagators. With the standard double-line representation, on assigns momentum $p_i$ the line with index $i$, then momentum conservation is automatic in the quenched model. An example of vacuum diagrams is drawn in the following figure.
The propagator is essentially the same as in the original model, with an extra factor $a^{-1}$, so there is an additional factor $a^{-p}$ from all propagators. The $n$ vertices contribute a factor $(ag)^n N^{-n/2}$ and the contraction of matrix indices gives rise to a factor $N^{l+1}$. So altogether there is a factor $a^{n-p} g^n N^{l+1-n/2} = a^{n-p} g^n N^2$, where relations $3n = 2p$ and $l - p + n = 1$ are used. The factor $N^2$ is the right one for a planar diagram. Now multiplying the result by the integral factor (2.4), $l$ integrals in this factor are identified with loop integrals in the original diagram, leaving a factor $a^l$ together $a^{-1}$ in (2.4). The rest integral factors normalize to 1. Thus, the $a$ dependent factor is $a^{l-p+n-1} = 1$, with the planar relation $l - p + n = 1$. This explains why a factor $a$ in the quenched action is introduced. In the limit $N \to \infty$ all planar diagrams are included, thus the free energy calculated with the quenching prescription is the same as in the original model.

Calculating Green’s functions requires a little modification. First, according to the quenching prescription, $\text{tr} \, M^n(x,t) = \text{tr} \, M^n(t)$, so Green’s functions of these quantities will be independent of positions. This is true in the large $N$ limit according to the factorization theorem. Next, one would like to calculate the expectation value of quantities such as $\text{tr} \, [M(x,t)M(y,t')]$ or $\text{tr} \, [M(p,t)M(q,t')]$. By the quenching prescription

$$\langle \text{tr} \, [M(p,t)M(q,t')] \rangle = (2\pi)^2 \delta(p - p_{ij}) \delta(q + p_{ij}) \langle M(t)_{ij} M(t')_{ji} \rangle,$$

where sum over indices is assumed. In the leading order, applying (2.3) and (2.4), one finds that the result is the same as in the original model except for an additional factor $a^{-1}$. It is easy to see that this factor exists for all connected planar diagrams contributing to this Green’s function. Similarly, when one considers a connected Green’s function of this type involving $n$ matrices, an additional factor $a^{-1}$ is to be compensated.
3. Quenched Yang-Mills Theory on a Cylinder

3.1. Pure Yang-Mills Theory on a Cylinder

The pure Yang-Mills theory on a cylinder reduces to a quantum mechanics problem \[16,17\], since the only dynamical degrees of freedom are winding modes of the gauge field. Because the system itself already effectively collapses to a “point” before quenching, it is desirable to compare what obtained in the quenched model to the un-quenched model.

We start by showing how the pure Yang-Mills theory effectively reduces to a quantum mechanical system. The standard action

\[ S = \frac{1}{2g^2} \int d^2x \text{tr} (F_{01})^2 \]  

(3.1)

can be cast into a first order form, upon introducing the canonical momentum \( \Pi^D \)

\[ S = \int d^2x \text{tr} \left( -\frac{g^2}{2}(\Pi^D)^2 + \Pi^D F_{01} \right) \]

\[ = \int d^2x \text{tr} \left( -\frac{g^2}{2}(\Pi^D)^2 + \Pi^D \partial_t A + A_0 D_x \Pi^D \right), \]

(3.2)

where \( A = A_1 \). Integrating out \( A_0 \) in the path integral \( \int [dA_0 dA d\Pi^D] \exp(iS) \) results in a delta function \( \delta(D_x \Pi^D) \). This forces \( \Pi^D \) to satisfy \( D_x \Pi^D = 0 \). The solution is given by

\[ \Pi^D(x) = U^{-1}(x)\Pi^D U(x), \quad U(x) = P \exp(i \int_0^x A dx), \]

where \( \Pi^D \) is \( \Pi^D(0) \), a matrix independent of \( x \). Let the circumference of the circle be \( L \).

By the periodic condition, \( \Pi^D(L) = \Pi^D(0) = \Pi^D \), one finds \([U, \Pi^D] = 0\). Here \( U \) is the holonomy around the circle \( U = U(L) \). Thus, upon substituting this solution into the path integral, the delta function reduces to \( \delta([U, \Pi^D]) \) and the path integral itself becomes

\[ \int [dAd\Pi^D] \delta([U, \Pi^D]) \exp \left( i \int dt \text{tr} \left( -\frac{g^2L}{2}(\Pi^D)^2 + \Pi^D \int dx U(x) \partial_t A(x) U^{-1}(x) \right) \right). \]

It is easy to see that \( \int dx U(x) \partial_t A(x) U^{-1}(x) = -i \partial_t U U^{-1} \), from the definition of \( U \).

Thus, all degrees of freedom in \( A(x) \) collapse to that of \( U \), and the above path integral finally reduces to a path integral of a quantum mechanic system

\[ \int [dU d\Pi^D] \delta([U, \Pi^D]) \exp \left( i \int dt \text{tr} \left( -\frac{g^2L}{2}(\Pi^D)^2 - i\Pi^D \partial_t U U^{-1} \right) \right), \]

(3.3)
where the appropriate measure \([dU]\) is the Haar measure. Instead of working with the holonomy, one can work with \(D\) with the definition \(U = \exp(iDL)\). Using the following formula

\[-i\partial_t UU^{-1} = L \int_0^1 d\tau e^{iDL\tau} \partial_t De^{-iDL\tau}\]

in (3.3) and noting that \(\exp(iDL\tau)\) effectively commutes with \(\Pi_D\), thanks to the delta function in the path integral, one finds that these exponentials cancel upon taking trace. The delta function \(\delta([U, \Pi^D])\) can be replaced by another one \(\delta([D, \Pi^D])\) and finally

\[
\int [dDd\Pi^D] \delta([D, \Pi^D]) \exp \left( i \int dt L \text{tr} \left( -\frac{g^2}{2}(\Pi^D)^2 + \Pi^D \partial_t D \right) \right),
\]

(3.4)

where extra care need be exercised in definition of the measure \([dD]\). One can define the measure as the usual flat one, and the nontrivial factor coming from the Haar measure \([dU]\) can be absorbed into the definition of the delta function. Eq. (3.4) is our final formula for comparison with the quenched model.

Following the prescription given in the previous section, we use the substitution

\[
\Pi^D(x) = e^{iPx} \Pi^D e^{-iPx}, \quad A_0(x) = e^{iPx} A_0 e^{-iPx}, \quad A(x) = e^{iPx} A e^{-iPx}
\]

(3.5)

with a diagonal matrix \(P\) whose entries take the form \(p_i = 2\pi n_i/L\), \(n_i\) is an integer. Furthermore, \(D_x \Pi^D = i[D, \Pi^D]\), where \(D = P + A\). We shall see presently that \(D\) is to be identified with the previously introduced \(D\) in (3.4). The action (3.2) is replaced by

\[
S = a \int dt \text{tr} \left( -\frac{g^2}{2}(\Pi^D)^2 + \Pi^D \partial_t A + iA_0[D, \Pi^D] \right).
\]

(3.6)

Again integration of \(A_0\) results in constraint \([D, \Pi^D] = 0\), and the path integral becomes

\[
\int [dDd\Pi^D] \delta([D, \Pi^D]) \exp \left( i \int dt L \text{tr} \left( -\frac{g^2}{2}(\Pi^D)^2 + \Pi^D \partial_t A \right) \right).
\]

(3.7)

To see what quantity corresponds to \(D\), consider the Wilson line \(U_{xy} = P \exp \int_x^y A(x) dx\). With the quenching substitution (3.5), \(U_{xy} = \exp(iPx) \exp(iD(y-x)) \exp(iPy)\), and in particular for the Wilson line which wraps the circle once \(U = \exp(iDL)\). We thus see that \(D\) is the same quantity as introduced in (3.4). With this in mind, the path integral (3.7) is seen to be almost the same as the un-quenched version (3.4). The differences between the two are that it is \(A\) appears in the action of the quenched version, and that the ultraviolet cut-off \(a\) appears in the quenched action while it is the infrared cut-off \(L\) appears in the
un-quenched action. The first difference is easily resolved by replacing $\partial_tA$ by $\partial_tD$, since $P$ is supposed to be held fixed with discrete eigenvalues.

Before resolving the second difference, we remind ourselves that in the original proposal of [11], the constraint that the eigenvalues of $D$ coincide with those of $P$ is imposed. This amounts to inserting the following factor

$$\int [dU]\delta(D - UPU^{-1})\Delta(p_i), \quad \Delta(p_i) = \prod_{i<j}(p_i - p_j)^2 \tag{3.8}$$

into the path integral. We now argue that such constraint should not be imposed in our case of a compact circle. As we have seen, an eigenvalue of $DL$ is the phase of an eigenvalue of the holonomy $U$. With the above constraint, this eigenvalue would be $2\pi n_i$ and the corresponding phase is trivial in the holonomy. So the constraint (3.8) would lead to a trivial theory.

Now the second difference mentioned before between the two actions is formally resolved by rescaling $\Pi^D \to \frac{L}{a}\Pi^D$ and $g^2 \to \frac{a}{L}g^2$. The rescaling of $\Pi^D$ does not change the quenched model, since it is just a matter of convention. The rescaling of $g^2$ is a little disturbing. By viewing $D$, instead of $A$, as the dynamic degrees of freedom, this rescaling appears necessary in order to recover the original theory at the large $N$ limit. Such modification is not absurd as it might appear. Note that except for the Gauss law constraint $[D,\Pi^D] = 0$, the action in (3.7) is quadratic, so a straightforward perturbative argument as presented in the previous section is lacking. In particular, averaging over $p_i$ with (2.4) can not be introduced directly. Finally, we would like to point out that the ratio $L/a$ can be taken equal to $N$. The reason is the following. The number of possible values of $p_i$ with both a ultraviolet cut-off and an infrared cut-off is just $L/a$. When $P$ is absorbed into $D$, the rank of $D$, $N$, counts effectively the number of possible values of $p_i$. This is pointed out in [14] as well as in [15].

To check the consistency of our special prescription, we shall consider the system of Yang-Mills coupled to an adjoint scalar field in the next subsection.

3.2. Yang-Mills Coupled to an Adjoint Scalar

Let $\phi$ be an adjoint scalar, therefore a Hermitian matrix field for a gauge group $U(N)$. In this subsection we are interested in studying the coupled system with an action

$$S = \frac{1}{2g^2} \int d^2x \text{tr} \left( (F_{01})^2 - (D_\mu \phi)^2 \right), \tag{3.9}$$
with definition of covariant derivatives $D_\mu \phi = \partial_\mu \phi + i[A_\mu, \phi]$. In the above we appropriately rescaled $\phi$ so that the action is weighted by a factor $1/g^2$. This action can also be written in a first order form

$$S = \int d^2x \text{tr} \left( -\frac{g^2}{2}[(\Pi^D)^2 + (\phi^\prime)^2] + \Pi^DF_{01} + \Pi^D\partial_1\phi - \frac{1}{2g^2}(D_x\phi)^2 \right)$$

$$= \int d^2x \text{tr} \left( -\frac{g^2}{2}[(\Pi^D)^2 + (\phi^\prime)^2] + \Pi^D\partial_1\phi + \Pi^\phi\partial_1\phi - \frac{1}{2g^2}(D_x\phi)^2 \right)$$

$$+ \int d^2x A_0 \left( D_x\Pi^D + i[\phi, \Pi^\phi] \right).$$

(3.10)

Again integrating out $A_0$ in the path integral imposes the Gauss law $D_x\Pi^D + i[\phi, \Pi^\phi] = 0$, which is solved by the following expression

$$\Pi^D(x) = U^{-1}(x)\Pi^D U(x) - i \int_0^x dy U(x, y) [\phi(y), \Pi^\phi(y)] U(y, x), \quad (3.11)$$

where $\Pi^D = \Pi^D(0)$, and $U(x)$ is defined as in the last subsection, and $U(x, y) = P \exp(i \int_x^y A(x) dx)$. Now if one puts the system on a circle of circumference $L$, the periodic boundary condition for $\Pi^D(x)$ no longer results in $[U, \Pi^D] = 0$, because of the second term in (3.11). Also, after substituting (3.11) into the action (3.10), the action not only depends on $U = U(0, L)$, but also on $U(x, y)$. Therefore, in the quenched model, the eigenvalues of $D$, as defined in the previous subsection, is no longer restricted to a circle, since $U(x, y)$ is not well defined on this circle. Thus, unlike in the pure Yang-Mills theory, the momentum as part of an eigenvalue of $D$ will enter in the story in an essential way.

The solution (3.11) of the constraint can be simplified by using gauge transformation

$$\phi(x) \rightarrow U^{-1}(x)\phi(x) U(x), \quad \Pi^\phi(x) \rightarrow U^{-1}(x)\Pi^\phi(x) U(x),$$

$$\Pi^D(x) \rightarrow U^{-1}(x)\Pi^D U(x), \quad A(x) \rightarrow U^{-1}(x)A(x) U(x) - iU^{-1}(x)\partial_x U(x).$$

The action (3.10) is invariant under this transformation, since the additional term resulting from the transformation is proportional to the constraint, therefore vanishes. The solution (3.11) after subject to this transformation reads

$$\Pi^D(x) = \Pi^D - i \int_0^x dy [\phi(y), \Pi^\phi(y)]. \quad (3.12)$$

Since $U(x)$ is not necessarily periodic, the periodic boundary conditions for new fields are twisted by $U$, for example $\Pi^D(L) = U\Pi^D(0)U^{-1}$. With the help of the above solution, this condition translates into

$$[U, \Pi^D] = -i \int_0^L dy [\phi(y), \Pi^\phi(y)]. \quad (3.13)$$
One can substitute (3.12) into action (3.10). Once again, one would obtain a term

\(-\frac{g^2 L}{2} \int dt \text{tr} (\Pi^D)^2\)

in which the infrared cut-off \(L\) appears explicitly. This suggests that the quenching pre-
scription introduced in the previous subsection also works in this model. To further confirm
this, we suggest an unconventional perturbation scheme in which the whole term

\(-\frac{1}{2g^2} \int d^2x \text{tr} (D_x \phi)^2\)

is treated as a perturbation term. Certainly there are other terms resulting from solving
the constraint. Let us focus on the above term. To normalize the kinetic terms in (3.10),
one has to rescale \(\phi \rightarrow g\phi, \Pi_\phi \rightarrow \Pi_\phi / g, D_x \rightarrow g D_x\). Thus, the above perturbation becomes

\(-\frac{g^2}{2} \int dt \frac{1}{L} \sum_n \text{tr} (D_x \phi)^2,\)

(3.14)

where we replaced the integral over the spatial dimension by a sum over discrete momentum
modes. Accordingly, the quantity \(\text{tr} (D_x \phi)^2\) should be viewed as the Fourier transform of
the original term. Essentially, there is no ultraviolet cut-off involved in this perturbation
theory. \(D_x \phi\) is effectively replaced by \(ip\phi + ia, \phi\) where \(a\) is the constant mode of the
gauge field. For a detailed discussion we refer to [19] where an adjoint fermion coupled to
the gauge field is discussed. Now one can replace the sum over momentum modes in (3.14)
by a trace, thus the perturbation (3.14) becomes a four point vertex and the combination
of \(p + A\) becomes an independent field.

Finally, we are in a position to propose our quenching prescription. First, as usual,

\[A(x) = e^{iP_x} A e^{-iP_x}, \quad \Pi^D(x) = e^{iP_x} \Pi^D e^{-iP_x},\]
\[\phi(x) = e^{iP_x} \phi e^{-iP_x}, \quad \Pi^\phi(x) = e^{iP_x} \Pi^\phi(x) e^{-iP_x}.\]

(3.15)

With these substitutes, the action in (3.10) becomes

\[S = L \int dt \text{tr} \left( -\frac{g^2}{2} \left[ (\Pi^D)^2 + (\Pi^\phi)^2 \right] + \Pi^D \partial_t D + \Pi^\phi \partial_t \phi + \frac{1}{2g^2} [D, \phi]^2 \right),\]

(3.16)

with constraint

\[[D, \Pi^D] + [\phi, \Pi^\phi] = 0.\]

(3.17)

It is easy to see that the above constraint is a consequence of the old one with prescription
(3.15). What is new here is that the usual ultraviolet cut-off \(a\) is replaced by the infrared
cut-off \( L \), and \( D \) is promoted to a dynamic variable so the average over momentum is not necessary. This is also why a ultraviolet cut-off is not necessary, since the propagator of \( \phi \) does not involve \( P \). The necessity of an infrared cut-off is best shown in study of the pure Yang-Mills theory. To see that this prescription is consistent, we note that any gauge invariant quantity, such as

\[
\text{tr} \left[ \phi(x)U(x,y)\phi(y)U(y,x) \right] = \text{tr} \, \phi \exp(iD(y - x))\phi \exp(iD(x - y)),
\]

contains \( P \) through \( D \). The prescription works only for gauge theories, since in a matrix model with only global symmetry, \( P \) appears explicit in all correlation functions.

Before closing this section, we mention once again that the Gross-Kitazawa constraint (3.8) should not be imposed in this model on a compact circle too. And the eigenvalues of \( D \), unlike in the pure gauge theory, should not be restricted to live on a circle.

4. The Quenched Supersymmetric Yang-Mills Theory

4.1. The Quenched Model

In this section, we shall introduce the quenched supersymmetric gauge theory. First, let us introduce the 2D super-gauge theory. The vector super-multiplet in two dimensions consists of a vector field, a scalar field and a Majorana fermion. The scalar field is needed, since the vector field in two dimensions does not have dynamical degrees of freedom. The action

\[
S = \frac{1}{g^2} \int d^2x \text{tr} \left( -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (D_{\mu} \phi)^2 - i\lambda \tilde{\sigma}^\mu D_{\mu} \lambda - \lambda \sigma^3 [\phi, \lambda] \right),
\]

is invariant under the SUSY transformation:

\[
\begin{align*}
\delta A_\mu & = -2i\lambda \tilde{\sigma}_\mu \epsilon, \\
\delta \phi & = 2i\lambda \sigma^3 \epsilon, \\
\delta \lambda & = \sigma^1 F_{01} \epsilon - \sigma^\mu \sigma^3 D_{\mu} \phi \epsilon.
\end{align*}
\]

We use notations of [18], \( \tilde{\sigma}_\mu = (1, -\sigma^1) \). The constant \( g^2 N \) is held fixed in the limit \( N \to \infty \). Here for simplicity we consider the gauge group \( U(N) \).

It is necessary to fix a gauge in the Hamiltonian formalism. An arbitrary gauge will spoil supersymmetry, thus many advantages brought about by SUSY may disappear. As was shown in [1], there is a gauge in which half of SUSY is broken, but half of SUSY
survives. In this gauge, $A_0 + \phi = 0$. Thus we are left with two bosonic fields $A = A_1$ and $\phi$, and two fermionic fields $\lambda_\alpha$. The conjugate momenta are

$$
\Pi^A = \frac{1}{g^2} F_{01} = \frac{1}{g^2} (\partial_0 A + D_x \phi),
\Pi^\phi = \frac{1}{g^2} \partial_0 \phi, \quad \Pi_\lambda = i \frac{1}{g^2} \lambda.
$$

(4.3)

with the standard commutation relations

$$
[\Pi^A_{ij}(x), A_{lk}(y)] = -i \delta_{ik} \delta_{jl} \delta(x - y), \quad [\Pi^\phi_{ij}(x), \phi_{lk}(y)] = -i \delta_{ik} \delta_{jl} \delta(x - y),
$$

$$
\{\lambda^i_\alpha(x), \lambda^j_\beta(y)\} = \frac{g^2}{2} \delta_{ik} \delta_{jl} \delta(x - y).
$$

(4.4)

Now the Hamiltonian and the unbroken super-charge $Q$ are, respectively

$$
H = g^2 \int dx \text{tr} \left( \frac{1}{2} (\Pi^A)^2 + \frac{1}{2} (\Pi^\phi)^2 \right) - \int dx \text{tr} \Pi^A D_x \phi
+ \frac{1}{g^2} \int dx \text{tr} \left( \frac{1}{2} (D_x \phi)^2 - i \lambda \sigma^1 D_x \lambda + \lambda (\sigma^3 - 1)[\phi, \lambda] \right),
$$

(4.5)

and

$$
Q = \int dx \text{tr} \left( \lambda_1 \Pi^\phi + \lambda_2 (\Pi^A - \frac{1}{g^2} D_x \phi) \right).
$$

(4.6)

The relation between these two quantities is $H = \{Q, Q\}$.

Now we are in a position to introduce the quenched model. The model is obtained by replacing every matrix field $F(x)$ with $\exp(iPx) F \exp(-iPx)$, where $F$ is independent of $x$ and $P$ is a real diagonal matrix, as in the previous section. The covariant derivative $D_x F(x)$ is then replaced by $\exp(iPx) [D, F] \exp(-iPx)$, where $D = P + A$. On a circle, the eigen-values of $P$ take discrete values $2\pi n/L$, $n$ is an integer. With these substitutions, the Hamiltonian and the super-charge are truncated to a point

$$
H = \frac{g^2 L}{2} \text{tr} \left( (\Pi^D)^2 + (\Pi^\phi)^2 \right) - i L \text{tr} \Pi^D [D, \phi]
+ \frac{L}{g^2} \text{tr} \left( -\frac{1}{2} [D, \phi]^2 + \lambda \sigma^1 [D, \lambda] + \lambda (\sigma^3 - 1)[\phi, \lambda] \right),
$$

(4.7)

and

$$
Q = L \text{tr} \left( \lambda_1 \Pi^\phi + \lambda_2 (\Pi^D - \frac{1}{g^2} i [D, \phi]) \right),
$$

(4.8)

where we replaced $\Pi^A$ by $\Pi^D$. Still these quantities are not convenient to work with, since the commutation relations are given by (4.4) with $\delta(x - y)$ replaced by $1/L$. Note that here
our prescription is different from the usual one, in that the delta function is not regularized by $1/a$ with a short-distance cut-off $a$. The argument is the same as we presented in the last section. The commutators are simplified by substitutions

$$D \rightarrow \frac{g}{\sqrt{L}}D, \quad \phi \rightarrow \frac{g}{\sqrt{L}}\phi, \quad \lambda \rightarrow \frac{g}{\sqrt{L}}\lambda,$$

$$\Pi^D \rightarrow \frac{1}{g\sqrt{L}}\Pi^D, \quad \Pi^\phi \rightarrow \frac{1}{g\sqrt{L}}\Pi^\phi.$$

Explicitly

$$[\Pi^D_{ij}, D_{lk}] = -i\delta_{ik}\delta_{jl}, \quad [\Pi^\phi_{ij}, \phi_{lk}] = -i\delta_{ik}\delta_{jl},$$

$$\{\lambda^{ij}_{\alpha}, \lambda^{lk}_{\beta}\} = \frac{1}{2}\delta_{ik}\delta_{jl}\delta_{\alpha\beta}. \quad (4.9)$$

The super-charge reads

$$Q = \text{tr} \left( \lambda_1 \Pi^\phi + \lambda_2 (\Pi^D - il[D, \phi]) \right), \quad (4.10)$$

where the parameter $l = g/\sqrt{L}$. The final ingredient is the gauge transformation

$$D \rightarrow UDU^{-1}, \quad \phi \rightarrow U\phi U^{-1}, \quad \lambda \rightarrow U\lambda U^{-1},$$

$$\Pi^D \rightarrow U\Pi^D U^{-1}, \quad \Pi^\phi \rightarrow U\Pi^\phi U^{-1}, \quad (4.11)$$

where the original gauge transformation parameter $U(x)$ is also replaced by

$$\exp(iPx) U \exp(-iPx).$$

The above gauge transformation is generated by the generator

$$G = i[D, \Pi^D] + i[\phi, \Pi^\phi] + 2\lambda^2. \quad (4.12)$$

A physical state is annihilated by $G$.

4.2. The Superspace Formulation

The super-charge \(\Pi D\Pi\) can be written in a more symmetric fashion, if one introduces

$$\lambda = \lambda_1 - i\lambda_2, \quad \bar{\lambda} = \lambda_1 + i\lambda_2,$$

$$M = \phi - iD, \quad \overline{M} = \phi + iD,$$

$$A = \Pi^M - \frac{il}{4}[\overline{M}, M], \quad \bar{A} = \Pi \overline{M} + \frac{il}{4}[\overline{M}, M], \quad (4.13)$$

then

$$Q = \text{tr} \left( \lambda A + \bar{\lambda}\bar{A} \right), \quad (4.14)$$
with corresponding action
\[
S = \int dt \text{tr} \left[ \frac{1}{2} \partial_t M \partial_t \bar{M} - \frac{i l}{4} [M, M] \partial_t (M - M) + i \bar{\lambda} \partial_t \lambda \\
- \frac{l}{2} \lambda [\bar{M}, \lambda] - \frac{l}{2} \bar{\lambda} [M, \bar{\lambda}] + \frac{l}{2} [M + \bar{M}, \lambda] \right].
\] (4.15)

This action is invariant under the following SUSY transformation
\[
\delta M = -2i \epsilon \lambda, \quad \delta \bar{M} = -2i \epsilon \bar{\lambda}, \\
\delta \lambda = \epsilon \partial_t M, \quad \delta \bar{\lambda} = \epsilon \partial_t \bar{M}.
\] (4.16)

Since the above transformation has a simple form, one would guess there might be a simple superspace formulation. Indeed there is. Introduce a real fermionic coordinate \(\theta\) as the superpartner of \(t\). Let \(D = \partial_\theta - i \theta \partial_t\) be the super-covariant derivative. The supercharge commuting with \(D\) is \(Q = \partial_\theta + i \theta \partial_t\). There is \(\{Q, Q\} = 2i \partial_t = 2H\). Introduce the following super-fields
\[
\Phi = M + i \theta \lambda, \quad \bar{\Phi} = \bar{M} + i \bar{\theta} \bar{\lambda}.
\] (4.17)

It is easy to see that the free part of the action (4.13) is
\[
-\frac{1}{2} \int dt d\theta \text{tr} \ D\bar{\Phi} D^2 \Phi = \int dt \text{tr} \ \frac{1}{2} \left( \partial_t M \partial_t \bar{M} + i \bar{\lambda} \partial_t \lambda \right),
\]
where there is an extra factor 1/2 for the fermionic part, which is due to a rescaling \(\lambda \to \lambda / \sqrt{2}\). The cubic terms in (4.13) can also be sorted out easily. We simply write down the whole action in terms of the super-fields
\[
S = \int dt d\theta \text{tr} \ \left( -\frac{1}{2} D\bar{\Phi} D^2 \Phi + \frac{l}{4} \bar{[\Phi, \bar{\Phi}] D(\bar{\Phi} - \Phi)} \right).
\] (4.18)

5. An Extended Model With Parisi-Sourlas Supersymmetry

We have tried to solve the quenched model introduced in the last section, for example, by constructing some conserved quantities. So far we have not succeeded in tackling this model directly. The reason why any simple minded method of constructing conserved quantities does not work is that all \(A\) and \(\bar{A}\) introduced in (4.13) do not commute. For example, a commutator of two matrix elements of \(A\) depends on \(\bar{M}\).

In this section, we suggest to study an exactly solvable model with one more complex fermion. This model is very similar to the \(N = 1\) model introduced in the previous section. We believe that better understanding of this model shall shed light to the \(N = 1\) quenched model.
The supersymmetry to be introduced here is Parisi-Sourlas supersymmetry. Therefore it is convenient to follow the line of the original papers \[13\] to introduce this model. We start with an action obtained by adding a potential term

\[
\frac{l^2}{8} \text{tr} [\overline{M}, M]^2,
\]

to the bosonic part of (4.15). This term is positive definite, since $[\overline{M}, M]$ is Hermitian. This means that a negative definite term is added to the quenched Hamiltonian. The so obtained bosonic action is

\[
S(M, \overline{M}) = \int dt \text{tr} \left[ \frac{1}{2} (\partial_t M - \frac{i l}{2} [\overline{M}, M]) (\partial_t \overline{M} + \frac{i l}{2} [\overline{M}, M]) \right].
\]

(5.1)

Note that this action is a complete square.

Define the complex Gaussian fields

\[
\eta = \partial_t M - \frac{i l}{2} [\overline{M}, M], \quad \bar{\eta} = \partial_t \overline{M} + \frac{i l}{2} [\overline{M}, M],
\]

(5.2)

with the measure

\[
\int [d\eta d\bar{\eta}] e^{\frac{i}{2} \int dt \text{tr} \eta \bar{\eta}}.
\]

(5.3)

The Green’s functions of $M$ and $\overline{M}$ are defined through solution to (5.2) together with the above Gaussian path integral. Denote the solution (with appropriate initial conditions) of (5.2) by $M = M(\eta, \bar{\eta})$, $\overline{M} = \overline{M}(\eta, \bar{\eta})$. A Green’s function is then

\[
\int [dM d\overline{M}] \det \left[ \frac{\partial (\eta, \bar{\eta})}{\partial (M, \overline{M})} \right] F[M, \overline{M}] e^{iS},
\]

where in the second line we changed the integral variables from the $\eta$’s to the $M$’s, and $S$ is the action given by (5.1). The Jacobian, following relations (5.2), is given by

\[
\frac{\partial (\eta, \bar{\eta})}{\partial (M, \overline{M})} = \begin{pmatrix}
\partial_t - \frac{i l}{2} [\overline{M}, \cdot] & -\frac{i l}{2} [\cdot, M] \\
-\frac{i l}{2} [\cdot, \overline{M}] & \partial_t - \frac{i l}{2} [M, \cdot]
\end{pmatrix}.
\]

(5.4)

Its determinant can be expressed as a fermionic path integral. Let $\Psi = [\lambda, \psi]^t$, then

\[
det \left[ \frac{\partial (\eta, \bar{\eta})}{\partial (M, \overline{M})} \right] = \int [d\Psi d\overline{\Psi}] \exp \left( iS(\Psi, \overline{\Psi}) \right)
\]
with

\[ S(\Psi, \overline{\Psi}) = i \int dt \text{tr} \left( \psi, \overline{\psi} \right) \frac{\partial (\eta, \overline{\eta})}{\partial (M, \overline{M})} (\lambda, \overline{\lambda})^t \]

\[ = \int dt \text{tr} \left( i\psi \partial_t \lambda + i\overline{\psi} \partial_t \overline{\lambda} + \frac{l}{2} (\psi[M, \lambda] + \overline{\psi}[\overline{M}, \overline{\lambda}] = \psi[M, \overline{\lambda}] - \psi[M, \lambda]) \right). \]

(5.5)

Note that the structure of this action is almost the same as the fermionic part of action \((4.13)\), except here there are two complex fermionic matrices. (Formally if \(\psi \rightarrow \overline{\lambda}\) and \(\overline{\psi} \rightarrow \lambda\), then the above action is the same as in \((4.13)\).) Now a Green’s function of \(M\) and \(\overline{M}\) can be written as

\[ \langle F[M, \overline{M}] \rangle = \int [dM \overline{M} d\Psi d\overline{\Psi}] F[M, \overline{M}] e^{i[S(M, \overline{M}) + S(\Psi, \overline{\Psi})]} . \]

(5.6)

As always, the combined action \(S(M, \overline{M}) + S(\Psi, \overline{\Psi})\) possesses supersymmetry, Parisi-Sourlas supersymmetry. Unlike in the simplest cases, the supersymmetry parameter is a real Grassmanian:

\[ \delta M = -2i\epsilon \lambda, \quad \delta \overline{M} = -2i\epsilon \overline{\lambda}, \]

\[ \delta \psi = \epsilon \left( \partial_t \overline{M} + \frac{i}{2} [\overline{M}, M] \right), \]

\[ \delta \overline{\psi} = \epsilon \left( \partial_t M - \frac{i}{2} [M, \overline{M}] \right), \]

(5.7)

\(\lambda\) and \(\overline{\lambda}\) are invariant under this supersymmetry. We suspect that there is one more supersymmetry, perhaps a nonlinear one, but have not been able to sort it out. The supersymmetry \((5.7)\) itself does not generate the Hamiltonian, precisely because the \(\lambda\)'s are invariant under it. To be able to generate the Hamiltonian, one perhaps need another supersymmetry.

The first order nonlinear differential equations in \((5.2)\) are integrable. Observe that the “friction” term \(il/2[M, M]\) is anti-Hermitian, so the Hermitian part of \(M\) satisfies a linear equation. Use the same notation as in \((4.13)\), denote this Hermitian part by \(\phi\) and the anti-Hermitian part by \(D\). A general solution for \(\phi\) is

\[ \phi(t) = \int_{-\infty}^{t} dt' \text{Re}\eta(t') + \phi(-\infty), \]

(5.8)

where \(\text{Re}\eta\) denotes the Hermitian part of \(\eta\), \(\phi(-\infty)\) is the value of \(\phi\) at the infinite past. Now \(D\) satisfies the following equation

\[ \partial_t D - il[\phi, D] = -\text{Im}\eta, \]

(5.9)
with Im\(\eta\) being the anti-Hermitian part of \(\eta\). Since \(\phi(t)\) is known as in (5.8), the solution to the above equation can be expressed in terms of \(\phi\),

\[
D(t) = -\int_{-\infty}^{t} dt' U^{-1}(t', t) \operatorname{Im}\eta(t') U(t', t) + D(-\infty),
\]

(5.10)

where

\[
U(t', t) = P \exp(-il \int_{t'}^{t} \phi(\tau)d\tau).
\]

That \(\phi\) appears in the above path ordered integral is not surprising. Recall that in the quenched model, \(\phi\) is just \(-A_0\) in the gauge in which one supersymmetry is preserved. Thus, \(U(t', t)\) is just the Wilson line along the time direction.

Now any Green’s function involving only matrices \(\phi\) and \(D\), or equivalently \(M\) and \(\overline{M}\), can be calculated using (5.8) and (5.10) together with the Gaussian path integral (5.3). We shall not try to develop a systematic technique to do this in the present paper, but shall leave it for future work. Since the \(\eta\)’s are Gaussian fields, it is possible to treat the large \(N\) problem with the help of the master field, for these are the so-called free variables. Some Green’s functions involving fermionic matrices can also be calculated, using the Ward identities associated with SUSY in (5.7). For instance, starting with

\[
\langle \delta_{\epsilon} \operatorname{tr} [\psi M] \rangle = 0
\]

one derives

\[
\langle \operatorname{tr} [\psi \lambda] \rangle = \frac{i}{2} \langle \operatorname{tr} [M(\partial_t \overline{M} + \frac{il}{2} [\overline{M}, M])] \rangle.
\]

(5.11)

This Ward identity holds for finite \(N\). There is no anomalous term, since SUSY is not spontaneously broken. This is guaranteed by the fact that the solution to the Langevin equations (5.2) is unique for given initial data.

Finally, a technical issue to be addressed is the projection to the singlet sector. The solution is not difficult and again we leave it to future work.

6. Discussion

We have shown that the two dimensional supersymmetric Yang-Mills theory truncates to an one dimensional supersymmetric matrix model in the large \(N\) limit. Although the quenched model may have nothing to do with the original SYM theory beyond the large \(N\) limit, it is tempting to treat itself as an interesting matrix model, and to ask various questions as often arise in dealing with a matrix model. Before the large \(N\) problem becomes tractable, asking these questions remain un-practical.
The extended matrix model studied in sect.5, exhibiting Parisi-Sourlas supersymmetry, can be solved exactly. This model is very similar to the quenched model, so further study will undoubtedly shed light on understanding of the quenched model. Some mechanism is needed to truncate the number of fermionic matrices, in order to come down to the quenched model. We have studied the quenched $N = 2$ two dimensional SYM theory, and the resulting theory contains the same number of fermions as in the extended model. There is $N = 2$ supersymmetry and an additional bosonic Hermitian matrix. We intend to publish some results concerning this model elsewhere. For now, we just mention that the $N = 2$ 2D SYM theory is the dimensional reduction of the $N = 1$ 4D SYM theory, and the latter is shown to possess the Nicolai mapping [20]. In a sense, the Nicolai mapping is nothing but a generalized Parisi-Sourlas mapping. Therefore, we suspect that this model might be interconnected with the two models studied in this paper.

It is our hope that some higher dimensional string theory will eventually emerge from these supersymmetric matrix models. With multi-matrices at hand, it is possible to generate more than one dimensions. Also, so far only supersymmetry can help in overcoming the so-called $c = 1$ barrier, and in eliminating problems such as tachyon. We hope to return to models introduced in this paper in near future.

Acknowledgments

We would like to thank M. Douglas for initial encouragement, and S. Das and A. Jevicki for very useful discussions. This work was supported by DOE grant DE-FG02-91ER40688-Task A.
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