ILM gluons in perturbative QCD

M. Musakhanov, N. Rakhimov

Theoretical Physics Department, National University of Uzbekistan, Tashkent 100174, Uzbekistan

(Dated: December 13, 2021)
Abstract

In this paper we extend our previous work on gluon propagator in the Instanton Liquid Model (ILM) of the QCD vacuum. This objects presents a lot of interest for studies of the heavy quarkonium $Q\bar{Q}$ observables in the framework of potential Nonrelativistic QCD (pNRQCD). Our goal is to evaluate the gluon polarization operator in ILM, and understand if it gets contributions from infrared (IR) renormalons. We perform a systematic analysis, taking into account both perturbative and nonperturbative effects, and making a double series expansion in terms of the strong coupling $\alpha_s(\rho) \sim 0.5$ (the scale is given by average instanton size $\rho \approx 1/3$ fm) and the instanton gas packing fraction $\lambda = \rho^4/R^4 \sim 0.01$ ($R \approx 1$ fm is average inter-instanton distance). We demonstrate that there are no IR renormalon related to ILM gluon propagator, since instantons generate a ILM gluon dynamical mass.

I. INTRODUCTION

The quasi-classical approach to QCD establishes importance of the topologically non-trivial classical solutions of chromodynamics in Euclidean space – instantons [1–4]. They have a quantum meaning of the paths in the internal Chern-Simons space, connecting classical vacuum states with different Chern-Simons numbers [5, 6]. Accordingly in quantum mechanics these paths correspond to tunneling processes between different classical vacuum states of chromodynamics. On the base of these ideas, it was formulated the Instanton Liquid Model (ILM) for the QCD vacuum (see the reviews [7–9]). In ILM framework the four-dimensional Euclidean space-time is populated by randomly distributed QCD instantons and anti-instantons. Their sizes and densities are controlled by instanton-instanton and instanton-antiinstanton interactions. The average instanton size $\bar{\rho}$ and average inter-instanton distance $\bar{R}$ have been independently estimated using different variational, phenomenological and numerical methods, yielding $\bar{\rho} \approx 1/3$ fm, and $\bar{R} \approx$ fm. These values were confirmed by lattice measurements [10–13]. The instanton size distribution $n(\rho)$ has been studied by the lattice simulations [14] (see Fig.1).

The main success of ILM framework in the last years was a clear explanation of the Spontaneous Breaking of Chiral Symmetry [15, 16], including Chiral Perturbation Theory

* musakhanov@gmail.com
results of light quarks physics \cite{17–19}. Further extension of ILM – Dyon Liquid Model framework even provided a possible way to understanding of the confinement \cite{20–25}.

FIG. 1. The dependence of the instanton size distribution function $n$ on the instanton size parameter $\rho$. The dots correspond to the calculations in the framework of ILM while the continuous lines correspond to the lattice simulations \cite{14}.

**Details of Instanton Liquid Model for QCD Vacuum.**

In this section we will remind briefly the details of ILM approach (see the reviews \cite{7–9} and references therein for more details). These results will be used below for the calculations.

Due to diluteness of the instanton gas, the background field in ILM approach is given by a simple sum of instanton and antiinstanton fields,

$$A(\xi) = \sum_i A_i(\xi_i)$$

where $\xi_i = (z_i, U_i, \rho_i)$ are collective coordinates of instantons. It is also necessary to note that the instanton field has a specific strong coupling dependence given by $A \sim 1/g$. The quantization of the gluonic field in the instanton background is done extending the total field is $A(\xi) + a$, where the quantum fluctuation $a$ might be treated perturbatively. In what follows we will need to average over all collective degrees of freedom of instantons $\xi = (\xi_1, \ldots, \xi_N)$, and we will use for this a shorthand notation

$$\langle \ldots \rangle_\xi = \int D\xi \ldots, \quad \int D\xi = 1.$$ 

The perturbative evaluation of the loop corrections introduces dependence on the regularization scale $\mu$, which determines the magnitude of the gauge coupling constant given at
that scale $\alpha_s(\mu)$. This dependence might be related to the scale of strong interactions $\Lambda$ given by

$$\Lambda = \mu \exp \left( -\frac{2\pi}{b_1 \alpha_s(\mu)} \right) \left( \frac{4\pi}{b_1 \alpha_s(\mu)} \right)^{b_2/2b_1^2} (1 + O(\alpha_s)), \quad (3)$$

where $N_c = 3$ is the number of quark colors and $N_f$ is the number of acting quark flavors, while $b_1$, $b_2$ are the coefficients of QCD $\beta$-function defined as

$$\mu^2 \frac{d\alpha_s(\mu)}{d\mu^2} = \sum_{k=0}^{\infty} \beta_k \alpha_s^{k+2} = -\frac{1}{4\pi} b_1 \alpha_s^2(\mu) - \frac{1}{(4\pi)^2} b_2 \alpha_s^3(\mu) + ..., \quad \beta_0 = -\frac{1}{4\pi} b_1, \beta_1 = -\frac{1}{4\pi} b_2. \quad (5)$$

For many practical applications the relevant values of normalization scale $\mu$ significantly exceed the soft scale $\Lambda$. For this reason, using smallness of $\alpha_s(\mu)$, it is possible to rewrite $\alpha_s(\mu)$ in a conventional asymptotic form

$$\frac{2\pi}{\alpha_s(\mu)} = b_1 \ln \frac{\mu}{\Lambda} + b_2 \frac{\ln \mu^2}{2b_1} \ln \frac{\mu^2}{\Lambda^2} + O \left( \frac{1}{\ln \frac{\mu}{\Lambda}} \right). \quad (6)$$

It is natural to expect that all dimensional physical observables in QCD are proportional to $\Lambda$ in the appropriate power. In the ILM approach the natural regularization scale is set by the instanton size $\rho$, so the coupling $\alpha_s(\rho)$, which controls the dynamics of strong interactions in the instanton background, is given to the 1-loop accuracy by

$$2\pi/\alpha_s^{(1)}(\rho) = b_1 \ln \frac{1}{\Lambda \rho}, \quad (7)$$

whereas inclusion of 2-loop corrections modifies it as

$$2\pi/\alpha_s^{(2)}(\rho) = b_1 \ln \frac{1}{\Lambda \rho} + b_2 \frac{\ln \ln \frac{1}{\frac{\mu^2}{\Lambda^2}}}{2b_1} \ln \frac{1}{\Lambda^2 \rho^2}. \quad (8)$$

We need to mention that the estimates for the scale $\Lambda$ depend on the accepted regularization scheme: for example, in $\overline{\text{MS}}$ scheme its values are slightly smaller than in Pauli-Villars regularization, $\Lambda_{\overline{\text{MS}}} = e^{-\frac{1}{64}} \Lambda = 0.955\Lambda$. The distribution over the sizes of instantons (“instanton weight function”) in two-loop approximation is given by

$$d_0(\rho) = \frac{C(N_c)}{\rho^5} \left( \frac{2\pi}{\alpha_s^{(1)}(\rho)} \right)^{2N_c} \exp \left[ -\frac{2\pi}{\alpha_s^{(1)}(\rho)} + \left( 2N_c - \frac{b_2}{2b_1} \right) \frac{\alpha_s^{(1)}(\rho) \ln (2\pi/\alpha_s^{(1)}(\rho))}{2\pi} \right] \frac{1}{\rho^5 (\Lambda \rho)^{\frac{1}{4}N_c - \frac{5}{2}N_f}}, \quad (9)$$
and clearly is divergent at large $\rho$. This divergence is a mere consequence of dilute gas approximation and disappears when the inter-instantons interactions are taken into account. We need to mention that for large instantons the strong coupling increases drastically, for this reason it is complicated to evaluate this modification from the first principles. However, the estimates based on variational principle suggest that suppression of large-size dipoles is quite fast and might be described by a Gaussian cutoff,

$$d_0(\rho) \rightarrow d(\rho) = d_0(\rho) \exp \left( -c \frac{\rho^2}{\bar{R}^2} \right), \quad (10)$$

where $c$ is some constant. In fact, the function $d(\rho)$ is a rather narrow distribution peaked around $\bar{\rho}$ (11); therefore for practical estimates we may just neglect the width of this distribution. In what follows we will use for our estimates the average instanton size and the average separation between instantons [9]

$$\bar{\rho} \simeq 0.48/\Lambda_{\text{MS}} \simeq 0.35 \text{ fm}, \quad \bar{R} \simeq 1.35/\Lambda_{\text{MS}} \simeq 0.95 \text{ fm}, \quad (11)$$

where the scale $\Lambda_{\text{MS}} = 280 \text{ MeV}$ is extracted from phenomenological studies of strong coupling. These values agree with estimates from the lattice [10–13], as well as phenomenological applications of instantons [7].

**Application of ILM to light quark physics.**

The ILM framework provides a very natural nonperturbative explanation of the Spontaneous Breaking of the Chiral Symmetry (SBCS) in QCD (see [7, 9] for the review), and as a consequence provides a consistent framework for microscopic description of the pions, giving the possibility to evaluate microscopically all the couplings in chiral lagrangians. Technically, the ILM approach the possibility to explain SBCS and the goldstone nature of the pion are closely related to the fact that the dynamics of light quarks in the instanton background is strongly affected by the presence of zero-modes in light quark propagator. A consistent way for the derivation of light quark determinant and on this base the light quark partition function was proposed in the number of works [17–19, 26–28]. In what follows we will extend this approach for analysis of hadrons involving heavy quarks.

**Radiative corrections to gluon propagator in QCD.**

While formally in the heavy quark mass limit we could expect that the quarkonia might be described perturbatively, the for practical applications, especially in charm sector, the numerical values of $\alpha_s$ still are quite significant. For this reason a successful application
of potential Non-Relativistic QCD (pNRQCD) to heavy quarks physics requires to take into account higher order corrections in $\alpha_s$, as could be evidenced from analysis of the so-called IR renormalons problem (see recent work [29] and the references therein). While at short distances the interaction potential between quarks is still dominated by the one-gluon exchange, we understand that behaviour of the running strong coupling $\alpha_s(q)$ at small momentum becomes more pronounced.

The present study is the extension of our previous work on gluon propagator in ILM [30, 31], which allowed to evaluate the dynamical momentum-dependent gluon mass $M_g(q)$. Furthermore, ILM gluon propagator was applied for the calculations of lowest order on strong coupling $\alpha_s$ one-gluon exchange potential in addition to direct instanton contribution potential for the problem of heavy quarkonium in QCD [32]. Our aim is to calculate gluon polarization operator in ILM and understand if there are IR renormalons in the $Q\bar{Q}$ observables. A systematic analysis including both perturbative and nonperturbative effects requires a double expansion series in terms of $\alpha_s(\rho) \sim 0.5$ and $\lambda \sim 0.01$. In order to perform such an analysis we assume that $\alpha_s \sim \lambda^{1/4}$ which is quite reasonable according to the phenomenological studies.

II. RADIATIVE CORRECTIONS TO GLUON PROPAGATOR IN ILM

In QED the lowest order polarization operator certainly is related only to fermion one-loop Feynman diagrams, while in QCD we have also gluon and ghost contributions. In QCD insertion of quark loops leads to the $N_f$ depended part in $b_1$ Eq. (4). The substitution of the QCD full $b_1$ in quark loops-chain diagrams is referred to as “non-abelianization” [33, 34].

In ILM it is natural to split the light quark determinant into the low- and high-frequency parts according to $\text{Det} = \text{Det}_{\text{high}} \times \text{Det}_{\text{low}}$ (see the reviews [7, 9]) and concentrate on the evaluation of $\text{Det}_{\text{low}}$, which is responsible for the low-energy domain. The high-energy part $\text{Det}_{\text{high}}$ is responsible mainly for the perturbative coupling renormalization discussed above. As was demonstrated before in our previous papers [17, 26], a proper inclusion of current quark mass and external fields needs some care and leads to the fermionic representation of $\text{Det}_{\text{low}}$ in the presence some external vector field $a_\mu$ as

$$\text{Det}_{\text{low}} = \int \prod_f D\psi_f D\psi^\dagger_f \exp \left( \int \sum_f \bar{\psi}^\dagger_f (\hat{p} + \hat{a} + im) \psi_f \right) \prod_{\pm} \tilde{V}_{\pm,f}[\xi, \psi^\dagger, \psi, a] , \quad (12)$$
where
\[ \hat{V}_\pm[\xi, \psi^\dagger, \psi, a] = \int d^4x \left( \psi^\dagger(x) L^{-1}(x, z_\pm) \hat{p} \Phi_{\pm,0}(x; \xi_\pm) \right) \int d^4y \left( \Phi_{\pm,0}^\dagger(y; \xi_\pm)(\hat{p} L^{-1}(y, z_\pm) \psi(y) \right) \]

\( \Phi_{\pm,0} \) are the light quarks zero-modes and the gauge links \( L_i \) are defined as
\[ L_i(x, z_i) = P \exp \left( i \int_{z_i}^x dy_\mu a_\mu(y) \right), \quad L_i(x, z_i) = \gamma_4 L_i^\dagger(x, z_i) \gamma_4. \]

The partition function in ILM \( Z[j] \) (normalized as \( Z[0] = 1 \)) is given by
\[ Z[j] = \frac{1}{\langle \text{Det}_{\text{low}}[\xi, m] \rangle_\xi} \int D\xi Da \text{Det}_{\text{low}}[\xi, a, m] e^{-[S_{eff}\{a, A(\xi)\} + (ja)]} \]
\[ = \frac{1}{\langle \text{Det}_{\text{low}}[\xi, m] \rangle_\xi} \int D\xi Da \text{Det}_{\text{low}}[\xi, \frac{\delta}{\delta j_\mu}, m] e^{-[S_{eff}\{a, A(\xi)\} + (ja)]} \]
\[ \approx \frac{1}{\langle \text{Det}_{\text{low}}[\xi, m] \rangle_\xi} \int D\xi \text{Det}_{\text{low}}[\xi, \frac{\delta}{\delta j_\mu}, m] e^\frac{1}{2}(j_\mu S_{\mu\nu}(\xi) j_\nu), \]

where \( a_\mu \) are perturbative gluons (quantum fluctuations around instanton background) introduced earlier, \( j_\mu \) are their external sources, and light quarks contribute via their determinant \( (12) \). We also used the shorthan notation \( (ja) = \int d^4x j_\mu^a(x) a_\mu^a(x) \). The measure of integration in ILM is given explicitly as \( D\xi = \prod_i d\xi_i = V^{-1} \prod_i dz_i dU_i \), and the integration over the instantons’ sizes \( \rho_i \) is disregarded in view of the above-mentioned smallness of the width of the instanton distribution.

In order to simplify further discussion, temporarily we will replace the real gluon field \( a_\mu \) with a scalar ”gluon” field \( \phi \). This allows us to suppress the gauge links \( L_i \). Furthermore, in Eq. \( (12) \) we will change \( \phi \) to \( \frac{\delta}{\delta j} \), so the partition function might be rewritten as
\[ Z[j] = \frac{1}{\langle \text{Det}_{\text{low}}[\xi, m] \rangle_\xi} \int D\xi D\phi D\text{Det}_{\text{low}}[\xi, \frac{\delta}{\delta j}, m] e^{-[S_{eff}\{\phi, \xi\} + (j\phi)]} \]
\[ = \frac{1}{\langle \text{Det}_{\text{low}}[\xi, m] \rangle_\xi} \int D\xi D\text{Det}_{\text{low}}[\xi, \frac{\delta}{\delta j}, m] e^\frac{1}{2}(jA) \]

The scalar ”gluon” propagator \( \Delta \) in background field \( A \) of the instanton gas is given by
\[ \Delta = (p + A)^{-2} = (p^2 + \sum_i \{p, A_i\} + A_i^2) + \sum_{i \neq j} A_i A_j)^{-1}, \quad \Delta_0 = p^{-2}, \]
\[ \Delta = (p^2 + \sum_i \{p, A_i\} + A_i^2)^{-1}, \quad \Delta_i = P_i^{-2} = (p^2 + \{p, A_i\} + A_i^2)^{-1}. \]

where \( \Delta_0 \) is the free propagator, \( \Delta_i \) is the propagator in the field of a single instanton \( i \), and \( \Delta \) is the propagator in the field of instanton gas in dilute approximation (when overlap of the neighbour instantons is neglected). There are no zero modes in \( \Delta_i^{-1} = P_i^2 \) and \( \Delta^{-1} = P^2 \),
which means the existence of the inverse operators $\Delta_i$ and $\Delta$. Now we would like to discuss evaluation of $\tilde{\Delta}$. Expanding it over $\{\{p, A_i\} + A_i^2\}$ carrying out further re-summation, we obtain the multi-scattering series

$$\tilde{\Delta} = \Delta_0 + \sum_\pm (\Delta_\pm (\xi_\pm) - \Delta_0) + ..., \quad (18)$$

where the expansion is done over the packing fraction $\lambda = \rho^4/R^4 \sim 0.01$, which in essence characterizes the fraction of 4D space occupied by instantons. The difference between exact and dilute gas approximation propagators is suppressed in this limit, $\Delta = \tilde{\Delta} + O(\lambda^2)$, so, the partition function (16) might be rewritten as

$$Z[j] = \frac{1}{Z[0]} \int \prod_f D\psi_f D\psi^\dagger_f \exp \left( \sum_f \psi^\dagger_f (\hat{p} + g \frac{\delta}{\delta j} + i m) \psi_f \right) \exp \left[ \frac{1}{2} (j \Delta_0 j) \right] \quad (19)$$

$$\times \prod_{\pm} \left( \int d\xi_\pm \exp \left[ \frac{1}{2} (j (\Delta_\pm (\xi_\pm) - \Delta_0) j) \right] \prod_f V_{\pm, f}[\xi_\pm, \psi^\dagger, \psi] \right)$$

$$= \frac{1}{Z[0]} \int \prod_f D\psi_f D\psi^\dagger_f \exp \left( \sum_f \psi^\dagger_f (\hat{p} + g \frac{\delta}{\delta j} + i m) \psi_f \right) \exp \left[ \frac{1}{2} (j \Delta_0 j) \right] \quad (19)$$

$$\times \prod_{\pm} \left( \int d\xi_\pm \exp \left[ \frac{1}{2} (j (\Delta_\pm (\xi_\pm) - \Delta_0) j) \right] \prod_f V_{\pm, f}[\xi_\pm, \psi^\dagger, \psi] \right)^{N_{\pm}}$$

We may rewrite the last term in the bracket as

$$\langle \exp \left[ \frac{1}{2} (j (\Delta_\pm (\xi_\pm) - \Delta_0) j) \right] \prod_f V_{\pm, f}[\xi, \psi^\dagger, \psi] \rangle_{\xi} \quad (20)$$

$$= \langle \exp \left[ \frac{1}{2} (j (\Delta_\pm (\xi_\pm) - \Delta_0) j) \right] \prod_f V_{\pm, f}[\xi, \psi^\dagger, \psi] \rangle_{\xi}$$

$$+ \left( \langle \exp \left[ \frac{1}{2} (j (\Delta_\pm (\xi_\pm) - \Delta_0) j) \right] \prod_f V_{\pm, f}[\xi, \psi^\dagger, \psi] \rangle_{\xi} \right.$$

$$- \langle \exp \left[ \frac{1}{2} (j (\Delta_\pm (\xi_\pm) - \Delta_0) j) \right] \prod_f V_{\pm, f}[\xi, \psi^\dagger, \psi] \rangle_{\xi} \right)$$

We see that the integration over $\xi_\pm$ in the second term leads to the interaction terms between "gluons" and light quarks. For a moment we will neglect this contribution. Furthermore, we will neglect the "gluon"-"gluon" interactions generated by instantons, which appear due to integration over $\xi_\pm$.

$$\langle \exp \left[ \frac{1}{2} (j (\Delta_\pm (\xi_\pm) - \Delta_0) j) \right] \rangle_{\xi} \approx \exp \left[ \frac{1}{2} (j \langle (\Delta_\pm (\xi_\pm) - \Delta_0) \rangle_{\xi} j) \right] \quad (21)$$
Now we may exponentiate $V^N$ by using Stirling-like formula
\[ V^N = \int d\eta \exp(N \ln \frac{N}{\eta} - N + \eta V), \] (22)
in order to rewrite the partition function $Z$ as
\[
Z[j] \approx \frac{1}{Z[0]} \int \prod_\pm d\eta_\pm \prod_f D\psi_f D\psi^\dagger_f \exp \left( \sum_f \psi^\dagger_f (\hat{\rho} + g \frac{\delta}{\delta j} + im) \psi_f \right) \times \exp \sum_\pm \left( N_\pm \ln \frac{N_\pm}{\eta_\pm} - N_\pm + \eta_\pm \langle \prod_f V_{+,f} [\xi, \psi^\dagger, \psi] \rangle \right) \times \exp \left[ \frac{1}{2} (j \Delta_0 j) \right] \exp \left[ \frac{1}{2} j \sum_\pm N_\pm (\Delta_\pm (\xi_\pm) - \Delta_0) \xi_j \right],
\] (23)
where in the last string we see ILM "gluon" propagator
\[
\bar{\Delta} = \Delta_0 + \sum_\pm N_\pm (\Delta_\pm (\xi_\pm) - \Delta_0) \xi + O(\lambda^2).
\]
For a moment we'll consider a theory with just a single quark flavour $N_f = 1$ and equal number of instantons and antiinstantons $N_\pm = N/2$. The integration over $\eta_\pm$ at saddle-point approximation yields $\eta_\pm = \eta$, so we may get
\[
\int \prod_\pm d\eta_\pm D\psi D\psi^\dagger \exp \left( \psi^\dagger (\hat{\rho} + g \frac{\delta}{\delta j} + im) \psi \right) \times \exp \sum_\pm \left( N_\pm \ln \frac{N_\pm}{\eta_\pm} - N_\pm + \eta_\pm \langle \prod_f V_{+,f} [\xi, \psi^\dagger, \psi] \rangle \xi \right) = \exp \left[ \text{Tr} \ln \left( \hat{\rho} + g \frac{\delta}{\delta j} + i(m + M(p)) \right) + N \ln \frac{N/2}{\lambda} - N \right],
\]
\[
N = \text{Tr} \frac{i M(p)}{\hat{\rho} + i(m + M(p))}, \quad M(p) = \frac{\eta}{N_c} (2\pi \rho F(p))^2, \quad F(q) = \rho K_1(q \rho).
\] (24)
where $M(p)$ is the dynamical (constituent) quark mass. The partition function in this approximation becomes:
\[
Z[j] \approx \frac{1}{Z[0]} \exp \left[ \text{Tr} \ln \left( \hat{\rho} + g \frac{\delta}{\delta j} + i(m + M(p)) \right) + N \ln \frac{N/2}{\lambda} - N \right] \exp \left[ \frac{1}{2} j \bar{\Delta} j \right].\] (25)
Since
\[
Z[0] = \exp \left[ \text{Tr} \ln (\hat{\rho} + i(m + M(p))) + N \ln \frac{N/2}{\lambda} - N \right]
\] (26)
the Eq. (26) might be rewritten as
\[
Z[j] \approx \exp \left[ \text{Tr} \ln \left( 1 + g \frac{\delta}{\delta j} (\hat{\rho} + i(m + M(p))^{-1}) \right) \right] \exp \left[ \frac{1}{2} j \bar{\Delta} j \right].
\] (27)
Next we will consider the case of two quark flavours ($N_f = 2$). As earlier, we assume the equality of number of instantons and antiinstantons ($N_\pm = N/2$), and integrate over $\eta_\pm$ at saddle-point approximation. In this approximation we may find that $\eta_\pm = \eta$. For $N_f = 2$ case the effective action includes a nonlocal 4-quark interaction vertex. The latter might be rewritten in a simpler form, making a bosonisation, which essentially replaces the 4-quark interaction with a new interaction vertices of quarks with scalar and pseudoscalar fields of different isospin ($\sigma$, $\eta$, $\vec{\sigma}$, $\vec{\phi}$). Due to spontaneous violation of chiral symmetry, the scalar meson field $\sigma$ has non-zero vacuum expectation $\sigma_0$, and in what follows we will use notation $\Phi' = (\sigma', \vec{\phi}', \eta', \vec{\sigma}')$ for the quantum fluctuations of this bosonic field around the vacuum $\sigma_0$.

Straightforward evaluation shows that similar to $N_f = 1$ case, the quarks acquire dynamical (constituent) mass, and its $p$-dependence is given by an expression similar to (25),

$$M(p) = \frac{\eta^{0.5}}{2c} (2\pi p)^2 F^2(p) \sigma_0, \quad c^2 = \frac{(N_c^2 - 1)2N_c}{2N_c - 1}.$$

(29)

The magnitude of the mass is controlled by the non-zero vacuum expectation $\sigma_0$, which might be fixed from the so-called gap equation

$$N = 0.5\text{Tr} \frac{iM(p)}{\hat{p} + im + iM(p)}, \quad V\sigma_0^2 = \text{Tr} \frac{iM(p)}{\hat{p} + im + iM(p)} \quad (30)$$

where $\text{Tr}(...) = \text{tr}_D \text{tr}_c \text{tr}_f \int d^4x < x|(...)|x >$. The partition function in this case might be rewritten as an effective interaction of quarks with mesonic fields $\Phi' = (\sigma', \vec{\phi}', \eta', \vec{\sigma}')$

$$Z[j] \approx \frac{1}{Z[0]} \int D\Phi' \prod_f D\psi_f D\psi_f^\dagger \exp \left[ N/2 \ln \frac{N}{2\eta} - N/2 - \frac{1}{2} V\sigma_0^2 - \frac{1}{2} \int dx \Phi'^2 \right] \quad (31)$$

$$+ \sum_f \psi_f^\dagger \left( \hat{p} + g \frac{\delta}{\delta j} + im + iM_f(p) + \frac{iM}{\sigma_0} F(p)\Phi' F(p) \right) \psi_f \exp \left[ \frac{1}{2} j \Delta j \right]$$

where $\Phi'^2 = \sigma'^2 + \vec{\phi}'^2 + \vec{\sigma}'^2 + \eta'^2$. We may assume that if we neglect by meson fluctuations, at any $N_f$ we may approximate the action of light quarks by Eq. (31).

Up to now we considered the case of scalar "gluons". The extension of these results for the case of real gluons is straightforward and as was shown in our previous paper [31], yields for the partition function

$$Z[j] \approx \exp \left[ \text{Tr} \ln \left( 1 + g \frac{\delta}{\delta j^\mu} (\hat{p} + i(m + M(p))^{-1}) \right) \right] \exp \left[ \frac{1}{2} j_\mu \hat{S}\mu\epsilon j_\epsilon \right].$$

(32)
where we neglected the gauge links $L_i$ contributions (see Eq. (14)), and the ILM gluon propagator is given by

\[ \bar{S}_{\mu\nu}(q) = \left( \delta_{\mu\nu} - (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right) \frac{1}{q^2 + M_g^2(q)}, \quad M_g(q) = F_g(q) M_g, \]

\[ M_g = \left[ \frac{6\rho^2}{(N_c^2 - 1) R^4 4\pi^2} \right]^{1/2}, \quad F_g(q) = q \rho K_1(q \rho). \]  

It is obvious that Eq. (15) generate light quarks loops contributions to the gluon propagator, which can be summed-up to geometrical progression as:

\[ \bar{S}_{\mu\nu}(q) \frac{1}{1 - \frac{q^2}{q^2 + M_g^2(q)} \pi(q)} \]  

where we used Landau gauge $\xi = 0$. The gluon polarization operator $\pi_{ab,\mu\nu}(q) = \delta_{ab} q^2 \delta_{\mu\nu} - q_\mu q_\nu \pi(q)$ in the lowest order in $\alpha_s$ is given by contribution of light quarks loops,

\[ \pi_{0,\mu\nu}(q) = 4\pi \alpha_s \int \frac{d^d p}{(2\pi)^d} \text{tr} t_a \gamma_\mu \left( \hat{p} - i(m + M(p)) \right) \frac{\gamma_\nu}{p^2 + (m + M(p))^2} \frac{\hat{p} - \hat{q} - i(m + M(p - q))}{(p - q)^2 + (m + M(p - q))^2}; \]

where $d = 4 - \epsilon$ is the dimension in $\overline{\text{MS}}$ scheme, $\mu$ is thenormalization point, and we regularized the polarization operator as $\pi(q) \rightarrow \pi(q) - \pi(\mu)$ in order to remove the ultraviolet logarithmic divergence $(1/\epsilon)$ in Eq. (35). Straightforward evaluation leads to the standard answer [29]:

\[ \pi_0(q) = \beta_0 \alpha_s(\mu) \ln \frac{q^2 e^{-C}}{\mu^2} \]

where $C = 5/3$ in $\overline{\text{MS}}$ scheme of regularization, and we use full $\beta_0$ which is meaning “non-abelianization” [33, 34]. Similarly we can consider radiative correction $\Delta m$ to the quark mass $m$. For its evaluations we have to use ILM gluon propagator with radiative corrections Eq. (34). The calculation is similar to [29] and leads to

\[ \Delta m = 4\pi \alpha_s(\mu) C_F \mu^{2\epsilon} \int \frac{d^d q}{(2\pi)^d} \gamma_\mu \hat{p} \gamma_\nu \gamma_\nu \frac{\hat{p} - \hat{q} - i m}{(p - q)^2 + m^2} \left( \delta_{\mu\nu} - q_\mu q_\nu / q^2 \right) \sum_{n=0}^{\infty} \left[ \beta_0 \alpha_s(\mu) \frac{q^2}{q^2 + M_g^2(q)} \ln \left( \frac{q^2 e^{-C}}{\mu^2} \right) \right]^n + \text{counterterms}, \]

where the color factor $C_F = 4/3$.

**Infrared region contribution to $\Delta m$.**

In the infrared region ($q \leq \mu$) the dynamical gluon mass $M_g(q)$ might be approximated as a constant, $M_g(q) \approx M_g(0) \equiv M_g$. 

11
The evaluation of the typical integrals in a series (37) in this region yields

\[ \Delta m_{\text{IR}} = -\frac{4\pi C_F}{\beta_0} (-\beta_0 \alpha_s(\mu)) \sum_{n=0}^{\infty} (-\beta_0 \alpha_s(\mu))^n c_n, \]

(38)

where \( a_g = M_g/\mu < 1 \) and it is taken into account that \( \beta_0 < 0 \). Also, simple estimations show that typical \( q \leq \mu \exp(-n) \).

Asymptotic series (38) sometimes can be summed using the Borel transform. Formally, the Borel transform of a series \( f(\alpha) = \alpha \sum_{n=0}^{\infty} c_n \alpha^n \), with respect to \( \alpha \) is defined as \( B[f](t) = \sum_{n=0}^{\infty} c_n t^n/n! \). If this Borel series converges, then the integral \( I[f] = \int_0^\infty e^{-t/\alpha} B[f](t) \, dt \) gives the Borel sum of the original series.

The corresponding Borel transform of Eq. (38) in respect to \( (-\beta_0 \alpha(\mu)) \) is

\[
B[\Delta m_{\text{IR}}](t) = -\frac{4\pi C_F}{\beta_0} \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \int_0^1 dx \left( \frac{x^2}{x^2 + a_g^2} \right)^{n+1} \ln^n x^2
\]

\[
= -\frac{4\pi C_F}{\beta_0} \int_0^1 dx \left( \frac{x^2}{x^2 + a_g^2} \right) \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( -x^2 \ln x^2 \right)^n
\]

\[
= -\frac{4\pi C_F}{\beta_0} \int_0^1 dx \left( \frac{x^2}{x^2 + a_g^2} \right) \exp \left( -t x^2 \ln x^2 \right). \tag{40}
\]

The Eq. (40) define the function \( B[\Delta m](t) \) without singularities at least at any positive \( t \).

So, we may conclude that in ILM there is no IR renormalons in \( \Delta m \).

For massless gluons \( a_g = 0 \) the Borel transform of \( \Delta m_{\text{IR}} \) is

\[
B[\Delta m_{\text{IR}, a_g=0}](t) = -2\frac{4\pi C_F}{\beta_0} \int_0^1 dx x^{-2t} = -2\frac{4\pi C_F}{\beta_0} \frac{1}{1 - 2t} \tag{41}
\]

and have the pole \( t = 1/2 \). This pole correspond IR renormalon, which inhibits evaluation of \( \Delta m_{\text{IR}, a_g=0} \) using inverse Borel transform (see recent paper [29] and references therein).

In ILM situation is much more comfortable, we may restore \( \Delta m_{\text{IR}} \) by the calculation of the Borel integral

\[
I[\Delta m_{\text{IR}}] = -\frac{4\pi C_F}{\beta_0} \int_0^\infty dt \int_0^1 dx \frac{x^2}{x^2 + a_g^2} \exp \left( -t x^2 \ln x^2 + \frac{t}{\beta_0 \alpha(\mu)} \right). \tag{42}
\]

Since the integrand of \( I[\Delta m_{\text{IR}}] \) has no poles on \( 0 < t < \infty \), we may change the order of the integration and make the integration on \( t \) first, which gives

\[
I[\Delta m_{\text{IR}}] = -\mu \alpha(\mu) 4\pi C_F \int_0^1 dx \frac{x^2}{x^2 + a_g^2 - \beta_0 \alpha(\mu)x^2 \ln x^2}, \tag{43}
\]
and further integration can be done numerically for any given values of $\alpha_s(\mu)$.

The same conclusion can be made about one-gluon exchange potential $V(r)$ for colorless $Q\bar{Q}$ with account of radiative corrections

$$V(r) = -4\pi\alpha_s(\mu)C_F \int \frac{d^3q}{(2\pi)^3} \exp(i\vec{q}\vec{r}) S_{44}(q) \left(1 - \frac{q^2}{q^2 + M^2_g(q)}\pi(q)\right)^{-1},$$  \hspace{1cm} (44)

since again at IR region the typical integrals will be the same as shown at Eq. (40).

III. CONCLUSION

We see from calculations above that in ILM framework it is safe to use the pole heavy quark mass $m_Q$ and the perturbative potential $V(r)$ for $Q\bar{Q}$-oniums, since there are no IR renormalons. We plan to calculate perturbatively in ILM the total energy of $Q\bar{Q}$ color singlet system $E(r) = 2m_Q + V(r)$, since there is essential cancellation of IR region contributions to the $2m_Q$ and $V(r)$, which is improving the convergence of perturbation series in $\alpha_s$.

ACKNOWLEDGEMENTS

M.M. is thankful to Marat Siddikov for the useful and helpful communications.

[1] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, Y. S. Tyupkin, Pseudoparticle Solutions of the Yang-Mills Equations, Phys. Lett. B 59 (1975) 85–87. doi:10.1016/0370-2693(75)90163-X.

[2] A. M. Polyakov, Quark Confinement and Topology of Gauge Groups, Nucl. Phys. B 120 (1977) 429–458. doi:10.1016/0550-3213(77)90086-4.

[3] G. ’t Hooft, Symmetry Breaking Through Bell-Jackiw Anomalies, Phys. Rev. Lett. 37 (1976) 8–11. doi:10.1103/PhysRevLett.37.8.

[4] G. ’t Hooft, Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle, Phys. Rev. D 14 (1976) 3432–3450, [Erratum: Phys.Rev.D 18, 2199 (1978)]. doi:10.1103/PhysRevD.14.3432.

[5] L. D. Faddeev, In Search for Multidimensional Solitons, in: 4th International Conference on Nonlocal Quantum Field Theory, JINR Dubna, 1976, pp. 207–223.
[6] R. Jackiw, C. Rebbi, Vacuum Periodicity in a Yang-Mills Quantum Theory, Phys. Rev. Lett. 37 (1976) 172–175. doi:10.1103/PhysRevLett.37.172.

[7] T. Schäfer, E. V. Shuryak, Instantons in QCD, Rev. Mod. Phys. 70 (1998) 323–426. arXiv: hep-ph/9610451, doi:10.1103/RevModPhys.70.323.

[8] E. Shuryak, Lectures on nonperturbative QCD (Nonperturbative Topological Phenomena in QCD and Related Theories) (12 2018). arXiv:1812.01509.

[9] D. Diakonov, Instantons at work, Prog. Part. Nucl. Phys. 51 (2003) 173–222. arXiv:hep-ph/0212026, doi:10.1016/S0146-6410(03)90014-7.

[10] M. C. Chu, J. M. Grandy, S. Huang, J. W. Negele, Evidence for the role of instantons in hadron structure from lattice QCD, Phys. Rev. D 49 (1994) 6039–6050. arXiv:hep-lat/9312071, doi:10.1103/PhysRevD.49.6039.

[11] J. W. Negele, Instantons, the QCD vacuum, and hadronic physics, Nucl. Phys. B Proc. Suppl. 73 (1999) 92–104. arXiv:hep-lat/9810053, doi:10.1016/S0920-5632(99)85010-5.

[12] T. A. DeGrand, Short distance current correlators: Comparing lattice simulations to the instanton liquid, Phys. Rev. D 64 (2001) 094508. arXiv:hep-lat/0106001, doi:10.1103/PhysRevD.64.094508.

[13] P. Faccioli, T. A. DeGrand, Evidence for instanton induced dynamics, from lattice QCD, Phys. Rev. Lett. 91 (2003) 182001. arXiv:hep-ph/0304219, doi:10.1103/PhysRevLett.91.182001.

[14] R. Millo, P. Faccioli, Computing the Effective Hamiltonian of Low-Energy Vacuum Gauge Fields, Phys. Rev. D 84 (2011) 034504. arXiv:1105.2163, doi:10.1103/PhysRevD.84.034504.

[15] D. Diakonov, V. Y. Petrov, P. V. Pobylitsa, A Chiral Theory of Nucleons, Nucl. Phys. B 306 (1988) 809. doi:10.1016/0550-3213(88)90443-9.

[16] D. Diakonov, V. Y. Petrov, A Theory of Light Quarks in the Instanton Vacuum, Nucl. Phys. B 272 (1986) 457–489. doi:10.1016/0550-3213(86)90011-8.

[17] K. Goeke, M. M. Musakhanov, M. Siddikov, Low energy constants of chi PT from the instanton vacuum model, Phys. Rev. D 76 (2007) 076007. arXiv:0707.1997, doi:10.1103/PhysRevD.76.076007.

[18] H.-C. Kim, M. Musakhanov, M. Siddikov, Magnetic susceptibility of the QCD vacuum, Phys. Lett. B 608 (2005) 95–106. arXiv:hep-ph/0411181, doi:10.1016/j.physletb.2004.12.
[19] H.-C. Kim, M. M. Musakhanov, M. Siddikov, Meson-loop contributions to the quark condensate from the instanton vacuum, Phys. Lett. B 633 (2006) 701–709. arXiv:hep-ph/0508211, doi:10.1016/j.physletb.2005.11.054.

[20] T. C. Kraan, P. van Baal, Exact T duality between calorons and Taub - NUT spaces, Phys. Lett. B 428 (1998) 268–276. arXiv:hep-th/9802049, doi:10.1016/S0370-2693(98)00411-0.

[21] T. C. Kraan, P. van Baal, Periodic instantons with nontrivial holonomy, Nucl. Phys. B 533 (1998) 627–659. arXiv:hep-th/9805168, doi:10.1016/S0550-3213(98)00590-2.

[22] K.-M. Lee, C.-h. Lu, SU(2) calorons and magnetic monopoles, Phys. Rev. D 58 (1998) 025011. arXiv:hep-th/9802108, doi:10.1103/PhysRevD.58.025011.

[23] D. Diakonov, Topology and confinement, Nucl. Phys. B Proc. Suppl. 195 (2009) 5–45. arXiv:0906.2456, doi:10.1016/j.nuclphysbps.2009.10.010.

[24] Y. Liu, E. Shuryak, I. Zahed, Confining dyon-antidyon Coulomb liquid model. I., Phys. Rev. D 92 (8) (2015) 085006. arXiv:1503.03058, doi:10.1103/PhysRevD.92.085006.

[25] Y. Liu, E. Shuryak, I. Zahed, Light quarks in the screened dyon-antidyon Coulomb liquid model. II., Phys. Rev. D 92 (8) (2015) 085007. arXiv:1503.09148, doi:10.1103/PhysRevD.92.085007.

[26] M. Musakhanov, Improved effective action for light quarks beyond chiral limit, Eur. Phys. J. C 9 (1999) 235–243. arXiv:hep-ph/9810295, doi:10.1007/s100529900017.

[27] M. Musakhanov, Current mass dependence of the quark condensate in instanton vacuum, Nucl. Phys. A 699 (2002) 340–343. doi:10.1016/S0375-9474(01)01516-0.

[28] M. M. Musakhanov, H.-C. Kim, A Test of the instanton vacuum with low-energy theorems of the axial anomaly, Phys. Lett. B 572 (2003) 181–188. arXiv:hep-ph/0206233, doi:10.1016/j.physletb.2003.08.022.

[29] M. Beneke, Pole mass renormalon and its ramifications, (8 2021). arXiv:2108.04861, doi:10.1140/epjs/s11734-021-00268-w.

[30] M. Hutter, Gluon mass from instantons, (11 1993). arXiv:hep-ph/9501335.

[31] M. Musakhanov, O. Egamberdiev, Dynamical gluon mass in the instanton vacuum model, Phys. Lett. B 779 (2018) 206–209. arXiv:1706.06270, doi:10.1016/j.physletb.2018.01.080.
[32] M. Musakhanov, N. Rakhimov, U. T. Yakhshiev, Heavy quark correlators in an instanton liquid model with perturbative corrections, Phys. Rev. D 102 (7) (2020) 076022. arXiv:2006.01545, doi:10.1103/PhysRevD.102.076022.

[33] D. J. Broadhurst, A. G. Grozin, Matching QCD and HQET heavy - light currents at two loops and beyond, Phys. Rev. D 52 (1995) 4082–4098. arXiv:hep-ph/9410240, doi:10.1103/PhysRevD.52.4082.

[34] M. Beneke, V. M. Braun, Naive nonAbelianization and resummation of fermion bubble chains, Phys. Lett. B 348 (1995) 513–520. arXiv:hep-ph/9411229, doi:10.1016/0370-2693(95)00184-M.