Phase transitions of hyperbolic black holes in anti-de Sitter space

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We find analytic solutions of hyperbolic black holes with scalar hair in AdS space, and they do not have spherical or planar counterparts. The system is obtained by taking a neutral limit of an Einstein-Maxwell-dilaton system whose special cases are maximal gauged supergravities, while the dilaton is kept nontrivial. There are phase transitions between these black holes and the hyperbolic Schwarzschild-AdS black hole. We discuss two AdS/CFT applications of these hyperbolic black holes. One is phase transitions of Rényi entropies, and the other is phase transitions of QFTs in de Sitter space. In addition, we give a C-metric solution as a generalization of the hyperbolic black holes with scalar hair.

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I. INTRODUCTION

The AdS/CFT correspondence provides a powerful tool to study strongly coupled CFTs in a given space-time background \( B_d \) in terms of a classical gravity whose boundary is conformal to \( B_d \). A basic example is the Schwarzschild-AdS solution

\[
 ds^2 = - \left( k - \frac{2M}{r} + \frac{r^2}{L^2} \right) dt^2 + \left( k - \frac{2M}{r} + \frac{r^2}{L^2} \right)^{-1} dr^2 + r^2 d\Sigma_{2,k}^2, \quad (1)
\]

where \( k = 1, 0, \) and \( -1 \) are for positive, zero, and negative curvatures of the two-dimensional space \( d\Sigma_{2,k}^2 \), respectively. Properties of these three cases are as follows.

- Spherical black hole \((k = 1)\): The AdS boundary is \( \mathbb{R} \times S^{d-1} \), so the dual CFT lives on a sphere. Black holes have a minimum temperature. There is a Hawking-Page phase transition between a black hole and a thermal gas \([2, 3]\).
- Planar black hole \((k = 0)\): The AdS boundary is \( \mathbb{R}^d \), so the dual CFT lives on a Minkowski space. There is no phase transition at finite temperature. If a spatial dimensional is compactified to \( S^1 \), there will be a phase transition between a black hole and the AdS soliton \([3]\).
- Hyperbolic black hole \((k = -1)\): The AdS boundary is \( \mathbb{R} \times \mathbb{H}^{d-1} \), so the CFT lives on a hyperboloid. There is no phase transition \([4, 5]\). The zero mass solution is at finite temperature, and the zero temperature solution is reached by a negative mass \([6]\). The AdS boundary is conformal to a Rindler space \([6]\) or a de Sitter space \([7]\).

If we include a scalar field in the system, the hyperbolic black hole described by \( (1) \) may have a phase transition to a hyperbolic black hole with scalar hair. The IR (near-horizon) geometry of the extremal hyperbolic black hole described by \( (1) \) is \( \text{AdS}_2 \times \mathbb{H}^2 \), and instability will happen when the IR Breitenlohner-Freedman (BF) bound is violated \([8]\). (As a comparison, the extremal spherical or planar black hole is the pure AdS.) Numerical solutions of hyperbolic black holes with scalar hair were obtained \([8-10]\).

A generalization of the Schwarzschild solution is Einstein-Maxwell-dilaton (EMD) systems. There are analytic solutions in maximal gauged supergravities whose...
special cases are EMD systems [11]. The most notable cases in AdS$_4$ are 1-charge, 2-charge, 3-charge, and 4-charge black holes, which are summarized in Appendix C. The thermodynamics of black holes in STU supergravity was studied in [12, 13], and there are phase transitions. Both gauge fields and dilaton fields are in the system; if we set the gauge fields to zero, the dilaton fields will also become zero, apparently. An EMD system whose special cases intersect with STU supergravity was found in [14–19], and more properties of the system was studied by [17–19].

We find a class of analytic solutions that describe phase transitions of hyperbolic black holes. These solutions are related to supergravity and do not have spherical or planar counterparts. We observe that there are two neutral limits for charged hyperbolic black holes that are solutions to the EMD system. One neutral limit is the solution [1], in which the dilaton becomes zero. The other neutral limit is a black hole with scalar hair. We show that there exist both first-order and second-order phase transitions between these two hyperbolic black holes at sufficiently low temperatures.

We discuss two applications of the hyperbolic black holes in terms of the AdS/CFT correspondence. One is phase transitions of the Rényi entropies. Rényi entropies as a generalization of the entanglement entropy play a key role in describing the quantum entanglement. If the entangling surface is a sphere, Rényi entropies can be calculated in terms of hyperbolic black holes [20, 21]. The parameter $n$ of Rényi entropies $S_n$ is related to the temperature of hyperbolic black holes: larger $n$ corresponds to a lower temperature. Hyperbolic black holes developing a scalar hair imply that Rényi entropies have a phase transition in $n$. While previous studies constructed numerical solutions to describe such a phase transition, this work provides analytic examples.

The hyperbolic black holes can also be used to study strongly coupled quantum field theories (QFTs) in de Sitter space by the AdS/CFT correspondence, because the AdS boundary of a hyperbolic black hole is conformal to a de Sitter space in the static patch [1]. A hyperbolic black hole describes a QFT in de Sitter space in the static patch at a temperature that may differ from the de Sitter temperature. Since there are phase transitions between hyperbolic black holes with and without scalar hair, the dual QFTs in de Sitter space will also have phase transitions. This result may shed some light on phase transitions of QFTs in the early universe.

This paper is organized as follows. In Sec. II, we present the analytic solution of neutral hyperbolic black holes with scalar hair. In Sec. III, we study the thermodynamics of these hyperbolic black holes and their phase transitions. In Sec. IV, we give the AdS$_5$ and higher-dimensional solutions of hyperbolic black holes with scalar hair. In Sec. V, we give a C-metric solution as a generalization of the hyperbolic black holes with scalar hair. In Sec. VI, we discuss two applications of hyperbolic black holes in terms of the AdS/CFT correspondence. Finally, we summarize and discuss some open questions.

In Appendix A, we present new insights on Einstein-scalar systems. In Appendix B, we derive an auxiliary solution. In Appendix C, we summarize some special cases of STU supergravity. In Appendix D, we discuss some properties of planar black holes.

II. TWO NEUTRAL LIMITS OF AN EINSTEIN-MAXWELL-DILATON SYSTEM

We study the AdS$_4$ case in detail, and put higher-dimensional cases in Sec. VI. The action is

$$S = \int d^4x \sqrt{-g} \left( R - \frac{1}{4} e^{-\alpha \phi} F^2 - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right),$$

where $F = dA$. The potential of the dilaton field is

$$V(\phi) = -\frac{2}{(1 + \alpha^2)^2 L^2} \left( \alpha^2 (3\alpha^2 - 1) e^{-\phi/\alpha} + 8 \alpha e^{(\alpha - 1)/\alpha} \right),$$

where $\alpha$ is a parameter, and the values of $\alpha = 0, 1/\sqrt{3}, 1$, and $\sqrt{3}$ correspond to special cases of STU supergravity. This potential was found in [13]. A simpler derivation of this potential is in Appendix A, in which we will explain why this potential is “privileged.”

The three exponentials in $V(\phi)$ are ordered by their importance. The $\phi \to 0$ behavior is $V(\phi) = -6L^2 - (1/L^2)\phi^2 + \cdots$, where the first term is the cosmological constant, and the second term shows that the mass of the scalar field satisfies $m^2 L^2 = -2$. The scaling dimension of the dual scalar operator in the CFT is $\Delta_\phi = 1$ and $\Delta_\phi = 2$.

The solution of the metric $g_{\mu \nu}$, gauge field $A_\mu$, and dilaton field $\phi$ is [14, 19]

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + U(r) d\Sigma^2_{k},$$

$$A = 2 \sqrt{\frac{bc}{1 + \alpha^2}} \left( \frac{1}{r_h} - \frac{1}{r} \right) dt,$$

$$e^{\alpha \phi} = \left( 1 - \frac{b}{r} \right)^{\frac{2\alpha^2}{1 + \alpha^2}},$$

with

$$f = \left( k - \frac{c}{r} \right) \left( 1 - \frac{b}{r} \right)^{\frac{1 - \alpha^2}{1 + \alpha^2}} + r^2 \left( 1 - \frac{b}{r} \right)^{\frac{2\alpha^2}{1 + \alpha^2}},$$

$$U = r^2 \left( 1 - \frac{b}{r} \right)^{\frac{2\alpha^2}{1 + \alpha^2}}.$$

The solution has two parameters $b$ and $c$ in addition to $\alpha$.

A key observation in this work is that we can eliminate the gauge field while keeping the dilaton field nontrivial.
If we take \( b = 0 \), this solution will be reduced to a neutral black hole described by (1). If we take \( c = 0 \), only in the case of \( k = -1 \) can we obtain a black hole solution. By taking \( c = 0 \), the gauge field vanishes, but the dilaton field is still nontrivial. This is a neutral hyperbolic black hole with scalar hair.

The solution of hyperbolic black holes are obtained by taking a nontrivial neutral limit of the above EMD system. We find an analytic solution of hyperbolic black holes given by

\[
\begin{align*}
  ds^2 &= -f(r)dt^2 + \frac{1}{f(r)}dr^2 + U(r)d\Sigma^2, \\
  U &= r^2 \left( 1 - \frac{b}{r} \right)^{\frac{2\alpha^2}{1 + \alpha^2}}, \\
  f &= 1 - \frac{c}{r} + \frac{r^2}{L^2}, \\
  \phi &= 0.
\end{align*}
\]

Interestingly, the same type of the neutral limit of the spherical or planar black holes in the same EMD system does not give a black hole.

Another solution to the system (2) without the gauge field is the hyperbolic Schwarzschild-AdS black hole (without scalar hair) given by

\[
\begin{align*}
  f &= 1 - \frac{c}{r} + \frac{r^2}{L^2}, \\
  U &= r^2, \\
  \phi &= 0.
\end{align*}
\]

For a given \( \alpha \), there are two solutions. The one with lower free energy is thermodynamically preferred. In the following, we will calculate thermodynamic quantities of the hyperbolic black holes with and without scalar hair, and demonstrate that there are phase transitions between the two solutions as the temperature is varied. We take \( L = 1 \) in the following.

### III. PHASE TRANSITIONS OF HYPERBOLIC BLACK HOLES

Consider the hyperbolic black hole with scalar hair. First, the curvature singularity is at \( r = b \). The horizon of the black hole is determined by \( f(r_h) = 0 \), from which the parameter \( b \) is expressed in terms of \( r_h \):

\[
b = r_h - r_h^{-\frac{3 - \alpha^2}{1 + \alpha^2}}.
\]

This ensures that \( b < r_h \), and the curvature singularity is always enclosed by a horizon (except for \( \alpha = 1/\sqrt{3} \), in which case \( r_h = 1 \)). The temperature is given by

\[
T = \frac{f'(r_h)}{4\pi} = \frac{1}{4\pi(1 + \alpha^2)} \left[ (3 - \alpha^2)r_h^{\frac{1 + \alpha^2}{1 + \alpha^2}} - (1 - 3\alpha^2)r_h^{-\frac{1 + \alpha^2}{1 + \alpha^2}} \right],
\]

where we have used (11) to replace \( b \) with \( r_h \). The system is invariant under \( \alpha \to -\alpha \) and \( \phi \to -\phi \). We assume \( \alpha > 0 \), and there are two distinctive cases as follows. See Fig. 1

- \( \alpha < 1/\sqrt{3} \) or \( \alpha > \sqrt{3} \). The temperature reaches zero when

\[
r_h = \left( \frac{1 - 3\alpha^2}{3 - \alpha^2} \right)^{\frac{1 + 3\alpha^2}{2(1 + \alpha^2)}}.
\]

The extremal geometry is given by (8) with

\[
b = \frac{2(1 + \alpha^2)}{3 - \alpha^2} \left( \frac{1 - 3\alpha^2}{3 - \alpha^2} \right)^{\frac{1 + 3\alpha^2}{2(1 + \alpha^2)}}.
\]

- \( 1/\sqrt{3} < \alpha < \sqrt{3} \). The temperature can never reach zero. There is a minimum temperature at

\[
r_h = \left( \frac{3\alpha^2 - 1}{3 - \alpha^2} \right)^{\frac{1 + 3\alpha^2}{2(1 + \alpha^2)}}.
\]

For a given temperature above the minimum temperature, there are two values of \( r_h \). (When \( \alpha = 1 \), the two values coincide.) At the minimum temperature, the geometry is given by (8) with

\[
b = \frac{4(1 - \alpha^2)}{3 - \alpha^2} \left( \frac{3\alpha^2 - 1}{3 - \alpha^2} \right)^{\frac{1 + 3\alpha^2}{2(1 + \alpha^2)}}.
\]

Other thermodynamic quantities are calculated in the following. The mass per unit area is

\[
M = -8\pi \frac{1 - \alpha^2}{1 + \alpha^2} r_h.
\]

The entropy per unit area is

\[
S = 16\pi^2 U(r_h) = 16\pi^2 r_h^{\frac{2(1 - \alpha^2)}{3 - \alpha^2}}.
\]

It can be checked that the first law of thermodynamics \( dM = TdS \) is satisfied by (12), (17), and (18). The free energy is calculated by

\[
F = M - TS.
\]

The free energy as a function of temperature is plotted in Fig. 2.

For the hyperbolic Schwarzschild-AdS black hole, the thermodynamic quantities can be calculated by setting \( c = -b \) and \( \alpha = 0 \). They are

\[
\begin{align*}
  T &= \frac{3r_h^2 - 1}{4\pi r_h}, \\
  S &= 16\pi^2 r_h^2, \\
  M &= -8\pi r_h (1 - r_h^2).
\end{align*}
\]

1 The mass and entropy in arbitrary dimensions are (47) and (48) in Sec. [IV]. Note that the mass and entropy depend on Newton’s constant \( G_N \). Here we have set \( 16\pi G_N = 1 \) in the action.
FIG. 1. Temperature as a function of $r_h$. In the left plot ($\alpha < 1/\sqrt{3}$), there is one branch of black hole solutions, and the temperature can reach zero. In the right plot ($1/\sqrt{3} < \alpha < \sqrt{3}$), there are two branches of black hole solutions, and there is a minimum temperature.

FIG. 2. Free energy as a function of temperature for different values of $\alpha$. The solid line is for the hyperbolic black hole with scalar hair. The dashed line is for the hyperbolic Schwarzschild-AdS black hole. A crossing point of the solid and the dashed lines is the pure AdS.

The free energy is $\bar{F} = \bar{M} - \bar{T}\bar{S}$. The free energy as a function of temperature is plotted in Fig. 2. The solid line is for the black hole with scalar hair, and the dashed line is for the black hole without scalar hair. The two solutions (9) and (10) share the same geometry when $b = c = 0$, which is the pure AdS. This is at $r_h = \bar{r}_h = 1$, which gives

$$T_c = \frac{1}{2\pi}, \quad F_c = -8\pi. \quad (21)$$

The two curves cross at this point. There is a second-order phase transition at $T_c$. Besides this, there is a first-order phase transition at the minimum temperature $T_{\text{min}}$ when $1/\sqrt{3} < \alpha < \sqrt{3}$.

The asymptotic behavior of the scalar field near the AdS boundary $r \to \infty$ is

$$\phi = \frac{A}{r} + \frac{B}{r^2} + \cdots. \quad (22)$$

The boundary condition for the solution (9) can be chosen as [22]

- $B/A = b$. This corresponds to a double-trace deformation of the dual CFT.
- $B/A^2 = 1 + \frac{\alpha^2}{2\alpha}$, which is dimensionless. This corresponds to a triple-trace deformation of the dual CFT.

The source of the scalar operator dual to $\phi$ is zero in either case. We can take either $\langle O_1 \rangle = A$ or $\langle O_2 \rangle = B$ as
the order parameter. Near $T_c$ we have $\langle O_1 \rangle \sim 1 - T/T_c$ and $\langle O_2 \rangle \sim (1 - T/T_c)^2$. Fig. 3 shows the order parameter as a function of temperature.

IV. HIGHER-DIMENSIONAL SOLUTIONS

The above result can be generalized to higher-dimensional spacetimes. The $(d + 1)$-dimensional action is

$$S = \int d^{d+1}x \sqrt{-g} \left( R - \frac{1}{4} e^{-\alpha \phi} F^2 - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right),$$

where $F = dA$. The privileged potential of the dilaton field is given as follows. The solutions and their thermodynamic quantities were obtained in [14–19]. We observe that neutral hyperbolic black holes can be obtained by taking a neutral limit of the EMD systems while the dilaton field is kept nontrivial. We put a backlash on the parameter $c$ to indicate that it will be set to zero in this neutral limit.

In the AdS$_5$ ($d = 4$) case, the potential of the dilaton field is

$$V(\phi) = \frac{12}{(4 + 3\alpha^2)^2 L^2} \left[ 3\alpha^2(3\alpha^2 - 2)e^{-\frac{4\phi}{4 + 3\alpha^2}} + 36\alpha^2 e^{-3\alpha^2 - 4\phi} + 2(8 - 3\alpha^2)e^{\alpha \phi} \right].$$

The $\phi \rightarrow 0$ behavior is $V(\phi) = -12/L^2 - (2/L^2)\phi^2 + \cdots$, where the first term is the cosmological constant, and the second term shows that the mass of the scalar field satisfies $m^2L^2 = -4$. The scaling dimension of the dual scalar operator is $\Delta_{\pm} = 2$. We study the limit that the gauge field vanishes while the scalar field is nontrivial. The solution of the metric $g_{\mu \nu}$, gauge field $A_\mu$, and dilaton field $\phi$ is

$$ds^2 = -f(r) dt^2 + \frac{1}{g(r)} dr^2 + U(r) d\Sigma^2_{3,k},$$

$$A = 2\sqrt{\frac{3b^2k^2}{4 + 3\alpha^2}} \left( \frac{1}{r^2_h} - \frac{1}{r^2} \right) dt,$$

$$e^{\alpha \phi} = \left( 1 - \frac{b^2}{r^2} \right) \frac{e^{\frac{2\phi}{4 + 3\alpha^2}}}{r^2},$$

with

$$f = \left( k - \frac{\alpha^2}{r^2} \right) \left( 1 - \frac{b^2}{r^2} \right)^{\frac{4 + 3\alpha^2}{4 + 3\alpha^2}} + \frac{r^2}{L^2} \left( 1 - \frac{b^2}{r^2} \right)^{\frac{4 + 3\alpha^2}{4 + 3\alpha^2}},$$

$$g = f(r) \left( 1 - \frac{b^2}{r^2} \right)^{\frac{4 + 3\alpha^2}{4 + 3\alpha^2}} U = r^2 \left( 1 - \frac{b^2}{r^2} \right)^{\frac{4 + 3\alpha^2}{4 + 3\alpha^2}},$$

where $b$ and $c$ are parameters.

Another solution to the system [23] is the hyperbolic Schwarzschild-AdS black hole (without scalar hair) given by

$$f = -1 - \frac{c^2}{r^2} L^2, \quad g = f^{-1}, \quad U = r^2, \quad \phi = 0.$$  

(29)

For a given $\alpha$, there are two solutions. The one with lower free energy is thermodynamically preferred. We demonstrate that there are phase transitions between the two solutions as the temperature is varied. We take $L = 1$ in the following.

Consider the hyperbolic black hole with scalar hair first. The curvature singularity is at $r = b$. The horizon of the black hole is determined by $f(r_h) = 0$, from which the parameter $b$ is expressed in terms of $r_h$:

$$b^2 = r_h^2 - \frac{8 - 3\alpha^2}{2 + 3\alpha^2}.$$

This ensures that $b < r_h$, and the curvature singularity is always enclosed by a horizon (except for $\alpha = 2/\sqrt{6}$, in
which case \( r_h = 1 \). The temperature is given by

\[
T = \frac{\sqrt{f'}g'}{4\pi r_h}, \quad (31)
\]

\[
t = \frac{1}{2\pi(4 + 3\alpha^2)} \left[ (8 - 3\alpha^2)\sqrt{\frac{4 + 3\alpha^2}{4 - 6\alpha^2}} - (4 - 6\alpha^2)r_h \sqrt{\frac{4 + 3\alpha^2}{4 - 6\alpha^2}} \right],
\]

where we have used (30) to replace \( b \) with \( r_h \). The system is invariant under \( \alpha \rightarrow -\alpha \) and \( \phi \rightarrow -\phi \). We assume \( \alpha > 0 \), and there are two distinctive cases as follows.

- \( \alpha < 2/\sqrt{6} \) or \( \alpha > 4/\sqrt{6} \). The temperature reaches zero when

\[
r_h = \left( \frac{4 - 6\alpha^2}{8 - 3\alpha^2} \right)^{\frac{2 - 3\alpha^2}{4 + 3\alpha^2}}. \quad (32)
\]

- \( 2/\sqrt{6} < \alpha < 4/\sqrt{6} \). The temperature can never reach zero. There is a minimum temperature at

\[
r_h = \left( \frac{6\alpha^2 - 4}{8 - 3\alpha^2} \right)^{\frac{2 - 3\alpha^2}{4 + 3\alpha^2}}. \quad (33)
\]

For a given temperature above the minimum temperature, there are two values of \( r_h \). (When \( \alpha = 2/\sqrt{3} \), the two values coincide.)

Other thermodynamic quantities are calculated in the following. The mass per unit volume is

\[
M = -4\pi \frac{4 - 3\alpha^2}{4 + 3\alpha^2} b^2. \quad (34)
\]

The entropy per unit volume is

\[
S = 16\pi^2 U(r_h) = \frac{16\pi^2}{3} \frac{\alpha^2}{r_h^{2-3\alpha^2}}. \quad (35)
\]

It can be checked that the first law of thermodynamics \( dM = TdS \) is satisfied by (33) and (34). The relation between the free energy \( F = M - TS \) and temperature \( T \) is qualitatively similar to the AdS\(_d\) case shown in Fig. 2.

For the hyperbolic Schwarzschild-AdS black hole, the thermodynamic quantities can be calculated by setting \( c^2 = -b^2 \) and \( \alpha = 0 \). They are

\[
\tilde{T} = \frac{2\tilde{r}_h^2}{2\pi \tilde{r}_h}, \quad \tilde{S} = \frac{16\pi^2}{3} \tilde{r}_h^3, \quad \tilde{M} = -4\pi\tilde{r}_h^2(1 - \tilde{r}_h^2).
\]

The two solutions (28) and (29) share the same geometry when \( b = c = 0 \), which is the pure AdS. This is at \( r_h = \tilde{r}_h = 1 \), which gives

\[
T_c = \frac{1}{2\pi}, \quad F_c = \frac{8\pi}{3}.
\]

There is a second-order phase transition at \( T_c \). Besides this, there is a first-order phase transition at the minimum temperature \( T_{\min} \) when \( 2/\sqrt{6} < \alpha < 4/\sqrt{6} \).

The asymptotic behavior of the scalar field near the AdS boundary \( r \rightarrow \infty \) is

\[
\phi = \frac{A}{r^2} \ln r + \frac{B}{r^2} + \cdots.
\]

The boundary condition for the solution (25) is \( A = 0 \). The source \( A \) is zero, and the expectation value \( B \) is the order parameter.

Generally, in the action (23) for an arbitrary dimensional spacetime AdS\(_{d+1}\), the potential of the dilaton field is

\[
V(\phi) = V_1 e^{-\frac{2(4-d-2)\phi}{d-1}} + V_2 e^{-\frac{2(4-d-2)\phi}{d-1}} + V_3 e^{\alpha \phi}, \quad (39)
\]

where

\[
V_1 = \frac{-(d-1)^2[2(4-d-2)\alpha^2 - 2(4-d-2)\alpha^2]}{L^2[2(d-2) + (d-1)\alpha^2]^2},
\]

\[
V_2 = -\frac{2(4-d-2)(d-1)\alpha^2}{L^2[2(d-2) + (d-1)\alpha^2]^2}, \quad (40)
\]

\[
V_3 = -\frac{2(4-d-2)(d-1)[2(d-2) + (d-1)\alpha^2]}{L^2[2(d-2) + (d-1)\alpha^2]^2}.
\]

The solution is

\[
ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + U(r)d\Sigma_{d-1,k}, \quad (41)
\]

\[
A = 2 \sqrt{(d-1)b^{d-2}x^{d-2}} \frac{1}{y^{d-2}} - \frac{1}{r^{d-2}} \frac{d}{dt}, \quad (42)
\]

\[
e^{\alpha \phi} = \left( 1 - \frac{b^{d-2}}{r^{d-2}} \right)^{\frac{2(4-d-2)}{2(4-d-2) + (d-1)\alpha^2}}, \quad (43)
\]

with

\[
f = \left( k - \frac{\phi^2}{r^{d-2}} \right) \left( 1 - \frac{b^{d-2}}{r^{d-2}} \right)^{\frac{2(4-d-2)}{2(4-d-2) + (d-1)\alpha^2}}.
\]

The potential (39) can be generated by the same procedure as in Appendix A. The metric ansatz is

\[
ds^2 = e^{2A(\phi)}(-h(\phi)dt^2 + d\Sigma_{d,1,k}^2) + \frac{e^{2B(\phi)}}{h(\phi)}dr^2. \quad (45)
\]

The generating function is

\[
e^A = b \left( e^{\frac{d-2}{4-d-2} \phi} - e^{-\frac{d-2}{4-d-2}} \right)^{-\frac{1}{d-2}}. \quad (46)
\]

The temperature as a function of \( r_h \) has two distinctive cases as follows.
• $0 < \alpha < (d - 2)\sqrt{\frac{2}{d(d - 1)}}$ or $\alpha > \sqrt{\frac{2d}{d - 1}}$. The temperature can reach zero.

• $(d - 2)\sqrt{\frac{2}{d(d - 1)}} < \alpha < \sqrt{\frac{2d}{d - 1}}$. The temperature cannot reach zero, and there is a minimum temperature.

The mass and entropy per unit volume are

$$M = \frac{(d - 1)V_{d-1}}{16\pi G_N} \left( k + k \frac{2(d - 2) - (d - 1)\alpha^2}{2(d - 2) + (d - 1)\alpha^2} b^{d-2} \right),$$

$$S = \frac{V_{d-1}U^{d/2} (r_h)}{4G_N},$$

where $V_{d-1}$ is the volume of the $(d - 1)$-dimensional unit sphere.

V. C-METRIC SOLUTION AS A GENERALIZATION

When a black hole solution is available, we can try to find a C-metric solution as a generalization with one more parameter. We will obtain the C-metric solution for the hyperbolic black holes with scalar hair. As a comparison, the C-metric without scalar hair in Ricci-flat spacetime is

$$ds^2 = \frac{1}{a^2(y - x)^2} \left( G(y)dy^2 + G(x)dx^2 + G(x)dx^2 \right),$$

where

$$G(\xi) = 1 - k\xi^2 - 2a\xi.$$

The $a \to 0$ limit of this solution gives the Schwarzschild solution. The C-metric solution with a cosmological constant and different $k$ was given in [23].

The C-metric solution for an EMD system with $V(\phi) = 0$ was given in [24]. The C-metric solution for the EMD system with $V(\phi)$ as [23] in the $k = 1$ case was given in [25]. A charged C-metric is significantly more sophisticated than a neutral one, and thus it is desirable to analyze a neutral C-metric with scalar hair before the charged one.

We keep the gauge field and the general $k = 0, \pm 1$ in the following solutions. Only in the $k = -1$ case can we obtain a neutral black hole with scalar hair in the $a \to 0$ limit. The C-metric solution to the system (2) with (3) is

$$ds^2 = \frac{1}{a^2(x - y)^2} \left[ h_x^{2\alpha \omega} \left( -F(y)dy^2 + \frac{dy^2}{F(y)} \right) + h_y^{2\alpha \omega} \left( \frac{dx^2}{G(x)} + G(x)dx^2 \right) \right],$$

where

$$F = -(1 - ky^2 - acy^2)h_y^{1 - \omega^2} + \frac{1}{a^2L^2} h_y^{2\alpha \omega},$$

$$G = (1 - kx^2 - acx^2)h_x^{1 - \omega^2},$$

$$h_y = 1 + aby, \quad h_x = 1 + abx,$$

$$A = 2\sqrt{\frac{bc}{\alpha + 1}} ydt, \quad e^{\alpha \phi} = \left( \frac{h_y}{h_x} \right)^{\frac{2\alpha \omega}{1 + \omega^2}}.$$

There are two neutral limits: one is $b = 0$, in which the dilaton field vanishes; the other is $c = 0$. When we take $c = 0$ and $k = -1$, we obtain the C-metric generalization of the solution in Sec. II.

After a coordinate transformation [26]

$$y = \frac{1}{ar}, \quad \tau = at,$$

the above solution can be written as

$$ds^2 = \frac{1}{(1 + ar^2)^2} \left[ h_x^{2\alpha \omega} \left( -F(r)dr^2 + \frac{dr^2}{F(r)} \right) + r^2 h_r^{2\alpha \omega} \left( \frac{dx^2}{G(x)} + G(x)dx^2 \right) \right],$$

with

$$F = \left( k - \frac{c}{r} - a^2 r^2 \right)h_x^{1 - \omega^2} + \frac{r^2}{L^2} h_r^{2\alpha \omega},$$

$$G = (1 - kx^2 - acx^2)h_x^{1 - \omega^2},$$

$$h_r = 1 - \frac{b}{r}, \quad h_x = 1 + abx,$$

$$A = 2\sqrt{\frac{bc}{1 + \alpha^2}} \left( \frac{1}{r_h} - \frac{1}{r} \right) dt, \quad e^{\alpha \phi} = \left( \frac{h_r}{h_x} \right)^{\frac{2\alpha \omega}{1 + \omega^2}}.$$

The $a \to 0$ limit of this solution is explicitly the same as [4]-[7].

We find that there is a cosmic brane with nonconstant tension at $x = x_0$, where $x_0$ is a root of the algebraic equation

$$4a^2bcx^3 + (3ac(\alpha^2 + 1) - abk(\alpha^2 - 3)) x^2 + 2k(\alpha^2 + 1)x + ab(3a^2 - 1) = 0.$$

The extrinsic curvature and the induced metric of the hypersurface $x = x_0$ satisfy

$$K_{\mu \nu} = -\lambda(y)h_{\mu \nu},$$

where

$$\lambda(y) = \frac{a\sqrt{1 - kx_0^2 - acx_0^2} (1 + abx_0 + \alpha^2(1 + aby))}{(1 + \alpha^2)(1 + abx_0) \frac{1}{2\alpha \omega \pi r_h^2} (1 + aby)^{\frac{2\alpha \omega}{1 + \omega^2}}},$$

assuming $\text{sgn}(x_0 - y) > 0$. The brane tension is

$$\mathcal{T} = \frac{\lambda(y)}{2\pi G_4},$$

where $G_4$ is the 4-dimensional Newton’s constant. When $\alpha = 0$ [27], the brane is at $x = 0$, and we have $\lambda = a$. 
VI. ADS/CFT APPLICATIONS

A. Phase transitions of Rényi entropies

Consider a QFT in a state described by a density matrix $\rho$, and divide the system into two parts, A and B. The reduced density matrix for the subsystem A is $\rho_A = \text{Tr}_{B}\rho$. The nth Rényi entropy is defined by

$$S_n = \frac{1}{1-n} \log \text{Tr}[\rho_A^n].$$

(60)

The entanglement entropy $S_{EE}$ can be obtained by taking the $n \to 1$ limit of the Rényi entropy: $S_{EE} = \lim_{n \to 1} S_n = -\text{Tr}\rho_A \log \rho_A$. Rényi entropies are usually difficult to calculate in QFTs.

In terms of the AdS/CFT correspondence, Rényi entropies with the entangling surface being a sphere can be calculated by hyperbolic black holes [20, 21]. Suppose we want to calculate the Rényi entropies of a CFT with a hyperbolic black hole: $S_{EE} = \lim_{n \to 1} S_n$. For $1 < n < \alpha < \sqrt{3}$, the entanglement entropy is related to the free energy of a hyperbolic black hole:

$$S_n(\mu) = \frac{n}{1-n} \left( F(T_0) - F(T_0/n) \right) = \frac{n}{1-n} T_0 \int_{T_0/n}^{T_0} S_{\text{therm}}(T)dT,$$

(61)

where $T_0 = \frac{3}{2\pi L}$ is the temperature of a zero-mass hyperbolic black hole, $S_{\text{therm}}$ is the thermal entropy of the hyperbolic black hole, and $S_{\text{therm}} \to -\partial F/\partial T$. In Secs. IV and IV we have set $R = 1$.

A phase transitions of hyperbolic black holes at a sufficiently low temperature imply a phase transition of Rényi entropies in $n$, i.e., $S_n$ is non-analytic at some $n = n_c$. As shown in Fig. 2, there is always a second-order phase transition at $T = 1/2\pi$, i.e., $n = 1$ (except for $\alpha = 1$). However, $\partial_n S_n$ is continuous and the Rényi entropies give a well-defined entanglement entropy in the $n \to 1$ limit [20]. For $1/\sqrt{3} < \alpha < \sqrt{3}$, there is a minimum temperature $T_{\text{min}}$ for the hairy black hole, and there is a first-order phase transition at $T = T_{\text{min}}$, corresponding to the parameter $n_c$ of Rényi entropies.

B. Phase transitions of QFTs in de Sitter space

Strongly coupled QFTs in de Sitter space can be studied in terms of the AdS/CFT correspondence. The goal is to find an AdS solution whose boundary is conformal to a de Sitter space in static coordinates:

$$ds^2 = -(1 - H^2 \rho^2) dt^2 + \frac{d\rho^2}{1 - H^2 \rho^2} + \rho^2 d\Omega_{d-2}^2,$$

(62)

where $d\Omega_{d-2}$ is the metric for a $(d - 2)$-dimensional sphere, and $H$ is the Hubble parameter. The de Sitter space has a temperature given by

$$T_{\text{ds}} = \frac{H}{2\pi}.$$

(63)

The foliation of de Sitter space gives a solution

$$ds^2 = dr^2 + H^2 L^2 \sinh^2 \frac{r}{L} ds^2.$$  

(64)

This solution is limited to a special case, in which the temperature $T$ for the QFT and the temperature $T_{\text{ds}}$ for the de Sitter space are the same. This solution is equivalent to a zero-mass hyperbolic black hole; the general hyperbolic black hole described by $[1]$ is used for $T \neq T_{\text{ds}}$. However, there is no phase transition even in this general case [11, 15].

Here we have a more general class of neutral hyperbolic black holes. The solution of the hyperbolic black hole in AdS4 is [39] and [40]. After a coordinate transformation $H \rho = \tanh \theta$, the hyperbolic space $d\Sigma_2$ can be written as

$$d\Sigma_2 = d\theta^2 + \sin^2 \theta d\varphi^2 = \frac{H^2 d\rho^2}{(1 - H^2 \rho^2)^2} + \frac{H^2 \rho^2}{1 - H^2 \rho^2} d\varphi^2.$$  

(65)

By substituting (65) to (68), the hyperbolic black hole in AdS4 can be written as

$$ds^2 = \frac{H^2 U(r)}{1 - H^2 \rho^2} \left( -\frac{f(r)}{H^2 U(r)} (1 - H^2 \rho^2) dt^2 + \frac{d\rho^2}{1 - H^2 \rho^2} + \rho^2 d\varphi^2 \right) + \frac{dr^2}{f(r)}.$$  

(66)

The conformal boundary is at $r \to \infty$, where $f(r) \to r^2/L^2$ and $U(r) \to r^2$. The QFT lives on the AdS boundary, which is conformal to

$$ds_0^2 = -\frac{1}{H^2 L^2} (1 - H^2 \rho^2) dt^2 + \frac{d\rho^2}{1 - H^2 \rho^2} + \rho^2 d\varphi^2.$$  

(67)

According to Sec. III, there are phase transitions between hyperbolic black holes with and without scalar hair. Therefore, there will be phase transitions of QFTs in de Sitter space [67].

In the case of AdS5, the AdS boundary of the spacetime is conformal to $dS_4$, which can be made explicit by the following coordinates:

$$ds^2 = \frac{H^2 U(r)}{1 - H^2 \rho^2} \left( -\frac{f(r)}{H^2 U(r)} (1 - H^2 \rho^2) dt^2 + \frac{d\rho^2}{1 - H^2 \rho^2} + \rho^2 d\Omega_2^2 \right) + \frac{dr^2}{g(r)}.$$  

(68)

2 The static patch of $dS_3$ is conformal to the Lorentzian hyperbolic cylinder $\mathbb{R} \times H^2$ [13].
VII. DISCUSSION

We have found a class of hyperbolic black holes with scalar hair in AdS space, by taking a particular limit of an EMD system. For the spherical and planar black holes in the EMD solution, the same type of limit does not give a black hole. The main conclusions are summarized as follows.

• For the Einstein-scalar system we consider, there is an analytic solution for hyperbolic black holes with scalar hair. The system is obtained by taking a neutral limit of an EMD system whose special cases include maximal gauged supergravities, while the dilaton field is kept nontrivial.

• There are phase transitions between the hyperbolic black hole with scalar hair and the hyperbolic Schwarzschild-AdS black hole. Phase transitions can be first or second order.

• By holography, the system we study describes (i) phase transitions of Rényi entropies; (ii) phase transitions of QFTs in de Sitter space.

• We give a C-metric solution for the system (A1) as a generalization. This neutral solution is less complicated than the full EMD solution, while the dilaton field is nontrivial.

• We propose two constraints for Einstein-scalar systems. Consequently, the potential of the scalar field is highly restricted, and analytic solutions are available. See Appendix A.

The following topics need further investigation: (i) the special cases of $\alpha = 1/\sqrt{3}$, 1, $\sqrt{3}$; (ii) the dual CFT of the hyperbolic black holes [31]; (iii) the relation to QFTs in cosmological backgrounds [32]; (iv) the black funnels and droplets [33, 34] from the C-metric solutions.

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Appendix A: Constraints on Einstein-scalar systems

We find that the Einstein-scalar system is significantly simplified under some reasonable constraints. We consider the AdS$_4$ spacetime, and the generalization to higher-dimensional cases is straightforward. The action is

$$S = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right), \quad (A1)$$

where $V(\phi)$ is the potential of the scalar field $\phi$. We consider the following metric ansatz:

$$ds^2 = e^{2A(\bar{r})} (-h(\bar{r}) dt^2 + 2 \sqrt{2} \frac{\bar{r}}{h(\bar{r})} d\Sigma_{2,k}^2), \quad (A2)$$

where $\bar{r}$ is the AdS radial coordinate, and the metric for the 2-dimensional sphere, plane, and hyperboloid can be written as

$$d\Sigma_{2,k}^2 = \frac{ds^2}{1-kx^2} + x^2 dy^2, \quad (A3)$$

where $k = 1$, 0, and $-1$, respectively. There are four unknown functions $A(\bar{r})$, $B(\bar{r})$, $h(\bar{r})$, and $\phi(\bar{r})$ and one gauge degree of freedom. We propose the following constraints.

Constraint. (i) The potential $V$ is independent of $k$; (ii) The function $h$ depends on $k$, and other functions are independent of $k$.

In other words, for a given $V(\phi)$, the only difference between the $k = 0$ solution and the $k \neq 0$ solution is some terms $h^{(k)}$ in $h$. Justification of these constraints includes special cases of STU supergravity in Appendix C.

Theorem 1. For a given cosmological constant $V(0) = -6/L^2$, the general potential satisfying the above constraints is a two-parameter family of the potential given by

$$V_{\alpha,\beta}(\phi) = V_{\alpha}(\phi) + \beta (V_{\alpha}(\phi) - V_{-\alpha}(\phi)), \quad (A4)$$

where $\alpha$ and $\beta$ are parameters, and $V_{\alpha}(\phi)$ is a one-parameter family of the potential given by

$$V(\phi) = -\frac{2}{(1+\alpha^2)2L^2} \left[ \alpha^2 (3\alpha^2 - 1)e^{-\phi/\alpha} + 8\alpha^2 e^{(\alpha-1/\alpha)\phi/2} + (3 - \alpha^2)e^{\alpha\phi} \right]. \quad (A5)$$

Theorem 2. For the one-parameter family of the potential $V_{\alpha}(\phi)$, the general solution under the above constraints for the hyperbolic black hole is obtained as $[^8]$ and $[^9]$ in Sec. II. and solution under the above constraints for the spherical or planar black hole does not exist.

A brief proof is as follows. The Einstein-scalar system has a weak form of integrability, which was presented in $[^5]$ and $[^6,^7]$. We will review this procedure and give more insights that enable us to find a “privileged” potential. Consequently, we find a way to derive the potential (A5) by relating the $k = 0$ and $k \neq 0$ solutions.
Equations of motion are obtained by the action (A1) with the metric ansatz (A2). The Einstein’s equation gives
\[ A'B' = \frac{1}{4} \phi'^2 + A'', \quad \text{(A6)} \]
\[ e^{3A-B} h' + 2e^{A+B} k = 0. \quad \text{(A7)} \]
The first equation comes from eliminating the potential \( V \) from \( G_{tt} = \frac{1}{2} T_t t \) and \( G_{\bar{r}\bar{r}} = \frac{1}{2} T_{\bar{r}\bar{r}} \). The second equation comes from eliminating the potential \( V \) from \( G_{\bar{r}\bar{r}} = \frac{1}{2} T_{\bar{r}\bar{r}} \) and \( G_{xx} = \frac{1}{2} T_{xx} \). Solving \( V \) from \( G_{\bar{r}\bar{r}} = \frac{1}{2} T_{\bar{r}\bar{r}} \) gives
\[ V = \frac{1}{2} e^{-2B} (\phi'^2 \bar{h} - 12A'^2 h - 4A' h') + 2ke^{-2A}. \quad \text{(A8)} \]

There is a gauge freedom, which is fixed by \( \phi = \tilde{r} \). Other equations can be derived from (A6), (A7), and (A8). Starting from a given \( A(\tilde{r}) \), we can obtain \( V(\tilde{r}) \) in the following way
\[ A \rightarrow \tilde{A}, B \rightarrow \tilde{B}, h \rightarrow \tilde{h}, V. \quad \text{(A9)} \]
The function \( A(\tilde{r}) \) plays the role of a generating function. Finally, replacing the function \( V(\tilde{r}) \) with \( V(\phi) \) gives the potential. This method can be generalized to EMD systems in a straightforward way. Only careful choices of \( A \) can we obtain a relatively simple \( V(\phi) \). Related methods are used in [38, 39], in which the potential can be generated by choices of either the metric or the scalar field.

Let \( h^{(0)} \) be the solution of \( h \) at \( k = 0 \), and we can decompose \( h \) into two parts:
\[ h(\tilde{r}) = h^{(0)} + h^{(k)}. \quad \text{(A10)} \]
The potential \( V \) solved by (A8) apparently depends on \( k \). Constraint (i) requires that the terms dependent on \( k \) must cancel:
\[ V = V^{(0)} + V^{(k)}, \quad V^{(k)} = 0. \quad \text{(A11)} \]
If \( A, B, \) and \( h^{(0)} \) satisfy the equations of motion with \( V = V^{(0)} \) in the \( k = 0 \) case, then \( A, B, \) and \( h^{(k)} \) satisfy the equations of motion with \( V = 0 \) in the \( k = 1 \) case. The key point is that these two cases share the same generating function \( A \), and the latter one is simpler to solve. A solution with \( V = 0 \) and \( k = 1 \) has been given in [10].

The solution for \( h \) from (A7) is given by
\[ h = \int e^{-3A+B} \left(-2k \int e^{A+B} d\tilde{r} + C_2\right) d\tilde{r} + C_1, \quad \text{(A12)} \]
where the \( C_1 = C_2 = 0 \) solution is the solution for the system with \( V = 0 \). To satisfy constraint (i), \( h \) can only linearly depend on \( k \). If we take \( C_1 = 1 \) and \( C_2 \propto k \), we obtain the potential \( V \) as (A5). Let \( V_\alpha(\phi) \) be the potential (A5) with parameter \( \alpha \), the general solution for the potential is
\[ V(\phi) = \beta_1 V_\alpha(\phi) + \beta_2 V_{-\alpha}(\phi), \quad \text{(A13)} \]
where \( \beta_1 \) and \( \beta_2 \) are constants. This potential can be rewritten as (A4), where the terms proportional to \( \beta \) do not change the cosmological constant and the mass of the scalar field. A nonzero \( \beta \) will give a cumbersome solution of \( h \), in which \( C_1 = 1 \), and \( C_2 \) is related to \( \beta \). This potential is expected to be consistent with [39], in which (B3) was used as an assumption.

Since (A8) and (A7) are independent of the cosmological constant, the two integration constants \( C_1 \) and \( C_2 \) are related to the cosmological constant. A shortcut to obtain the potential (A5) is as follows. Start from a solution with \( V(\phi) = 0 \) and \( k = 1 \), and then use the procedure (A9) with \( h = 1 \) and \( k = 0 \).

Appendix B: Solution to Einstein-scalar systems with \( V(\phi) = 0 \)

To solve the equations of motion with \( V = 0 \) and \( k = 1 \), it is more convenient to choose another gauge, \( B = -A \). The metric ansatz is (1). Consider Einstein’s equation with left-hand side being \( R_{\mu\nu} \), and we take the following procedure:

(a) From the \( tt \) and \( \theta\theta \) components, a simple equation is obtained (\( fU'' \)) = 2. So \( fU \) is a second-order polynomial of \( r \), and we can parameterize it as
\[ fU = (r - r_1)(r - r_2). \quad \text{(B1)} \]

(b) The \( tt \) component gives (\( f'U' \)) = 0. The general solution for \( f \) is
\[ f = f_0 \left( \frac{r - r_1}{r - r_2} \right)^{\nu_1}, \quad \text{(B2)} \]
where \( f_0 \) and \( \nu_1 \) are integration constants. At the AdS boundary \( r \to \infty, f = 1 \) gives \( f_0 = 1 \).

(c) The equation of motion for the scalar field is (\( fU\phi' \)) = 0. The general solution for \( \phi \) is
\[ e^\phi = e^{\phi_0} \left( \frac{r - r_1}{r - r_2} \right)^{\nu_2}, \quad \text{(B3)} \]
where \( \phi_0 \) and \( \nu_2 \) are integration constants. At the AdS boundary \( r \to \infty, \phi = 0 \) gives \( \phi_0 = 0 \).

(d) Equations (B1), (B2), and (B3) are solution to the equations of motion, provided that the condition \( \nu_1^2 + \nu_2^2 = 1 \) is satisfied.

(e) The coordinate \( \tilde{r} \) in the \( \phi = \tilde{r} \) gauge and the coordinate \( r \) in the \( A = -B \) gauge are related by \( \phi(\tilde{r}) = \tilde{r} \). By comparing (4) and (A2), we have
\[ e^A = (r_1 - r_2) \left( e^{\frac{1+\nu_1}{2\nu_2}} - e^{-\frac{1-\nu_1}{2\nu_2}} \right)^{-1}. \quad \text{(B4)} \]
After taking \( r_1 = b, r_2 = 0, \alpha = (1 - \nu_1)/\nu_2 \), we obtain the generating function for the potential (A5) as
\[ e^A = b \left( e^{\frac{1}{\nu_2}} - e^{-\frac{1}{\nu_2}} \right)^{-1}. \quad \text{(B5)} \]
Appendix C: Special cases of STU supergravity

In STU supergravities, there are $U(1)^4$ gauge fields in AdS$_4$, $U(1)^3$ gauge fields in AdS$_5$, and $U(1)^2$ gauge fields in AdS$_7$. Special cases of them can be reduced to EMD systems. They are 1-charge, 2-charge, and 3-charge black holes in AdS$_4$; 1-charge and 2-charge black holes in AdS$_5$; 1-charge black hole in AdS$_7$.

The AdS$_4$ Lagrangian is

$$\mathcal{L} = R - \frac{1}{2}(\partial \vec{\phi})^2 + 8\varphi^2(\cosh \phi_1 + \cosh \phi_2 + \cosh \phi_3) - \frac{1}{4} \sum_{i=1}^{4} e^{\vec{a}_i \cdot \vec{x}}(F^2_{i})^2 , \quad (C1)$$

where $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, $\vec{a}_1 = (1, 1, 1)$, $\vec{a}_2 = (1, -1, -1)$, $\vec{a}_3 = (-1, 1, -1)$, and $\vec{a}_4 = (-1, -1, 1)$. More details can be found in [11]. The solution is given by [11, 42]

$$ds^2 = -(H_1 H_2 H_3 H_4)^{-1/2} f dt^2 + (H_1 H_2 H_3 H_4)^{1/2}(f^{-1} dr^2 + r^2 d\Sigma_{2,k}), \quad (C2)$$

$$X_i = H_i^{-1}(H_1 H_2 H_3 H_4)^{1/4} , \quad (C3)$$

$$A_i^{(1)} = \sqrt{k}(1 - H_i^{-1}) \coth \beta_i dt , \quad (C4)$$

with $X_i = e^{-\frac{1}{2} \vec{a}_i \cdot \vec{r}}$ and

$$f = k - \frac{\mu}{\bar{r}} + \frac{4}{L^2} r^2 (H_1 H_2 H_3 H_4) , \quad H_i = 1 + \frac{\mu \sinh^2 \beta_i}{k r} . \quad (C5)$$

The following special cases are obtained when some of the $U(1)^4$ charges are the same, and others are zero:

- 1-charge black hole ($\alpha = \sqrt{3}$): $H_1 = H$, $H_2 = H_3 = H_4 = 1$. The Lagrangian is

$$\mathcal{L} = R - \frac{1}{2}(\partial \phi)^2 + \frac{6}{L^2} \cosh \phi \sqrt{3} - \frac{1}{4} e^{-\sqrt{3} \phi} F^2 . \quad (C6)$$

- 2-charge black hole ($\alpha = 1$): $H_1 = H_2 = H$, $H_3 = H_4 = 1$. The Lagrangian is

$$\mathcal{L} = R - \frac{1}{2}(\partial \phi)^2 + \frac{2}{L^2} (\cosh \phi + 2) - \frac{1}{4} e^{-\phi} F^2 . \quad (C7)$$

- 3-charge black hole ($\alpha = 1/\sqrt{3}$): $H_1 = H_2 = H_3 = H$, $H_4 = 1$. The Lagrangian is

$$\mathcal{L} = R - \frac{1}{2}(\partial \phi)^2 + \frac{6}{L^2} \cosh \phi \sqrt{3} - \frac{1}{4} e^{-\sqrt{3} \phi} F^2 . \quad (C8)$$

- 4-charge black hole ($\alpha = 0$): $H_1 = H_2 = H_3 = H_4 = H$. This is the RN-AdS$_4$ black hole.

The AdS$_5$ Lagrangian is

$$\mathcal{L} = R - \frac{1}{2}(\partial \vec{\phi})^2 + 4\varphi^2 \sum_i X_i^{-1} - \frac{1}{4} \sum_{i=1}^{4} X_i^{-2}(F^2_{i})^2 . \quad (C9)$$

The solution is [43]

$$ds^2 = -(H_1 H_2 H_3)^{-2/3} f dt^2 + (H_1 H_2 H_3)^{1/3}(f^{-1} dr^2 + r^2 d\Sigma_{2,k}) , \quad (C10)$$

$$X_i = H_i^{-1}(H_1 H_2 H_3)^{1/3} , \quad (C11)$$

$$A_i^{(1)} = \sqrt{k}(1 - H_i^{-1}) \coth \beta_i dt , \quad (C12)$$

with

$$f = k - \frac{\mu}{\bar{r}} + \frac{4}{L^2} \bar{r}^2 (H_1 H_2 H_3) , \quad H_i = 1 + \frac{\mu \sinh^2 \beta_i}{k \bar{r}} . \quad (C13)$$

The following special cases are obtained when some of the $U(1)^3$ charges are the same, and others are zero:

- 1-charge black hole ($\alpha = 4/\sqrt{6}$): $H_1 = H_2 = H_3 = 1$. The Lagrangian is

$$\mathcal{L} = R - \frac{1}{2}(\partial \phi)^2 + \frac{4}{L^2} (2e^{-\frac{1}{\sqrt{6}} \phi} + e^{-\frac{2}{\sqrt{6}} \phi}) - \frac{1}{4} e^{-\frac{1}{\sqrt{6}} \phi} F^2 . \quad (C14)$$

- 2-charge black hole ($\alpha = 2/\sqrt{6}$): $H_1 = H_2 = H$, $H_3 = 1$. The Lagrangian is

$$\mathcal{L} = R - \frac{1}{2}(\partial \phi)^2 + \frac{4}{L^2} (2e^{-\frac{1}{\sqrt{6}} \phi} + e^{-\frac{2}{\sqrt{6}} \phi}) - \frac{1}{4} e^{-\frac{1}{\sqrt{6}} \phi} F^2 . \quad (C15)$$

- 3-charge black hole ($\alpha = 0$): $H_1 = H_2 = H_3 = 1$. This is the RN-AdS$_5$ black hole.

In the AdS$_7$ case, a similar analysis can be done. There will be 1-charge black hole and 2-charge black hole, and the latter is the RN-AdS$_7$ black hole.

If we set $\mu = 0$ in the above solutions, both gauge fields and dilaton fields will vanish. However, in the above EMD systems, it is possible to make the gauge field vanish while keeping the dilaton field nontrivial in the $k = -1$ case, which is a key observation made in this paper. To see this explicitly, we can replace $\beta$ with $i\beta$ and make the following coordinate transformation:

$$r = \bar{r} + \mu \sin^2 \beta . \quad (C16)$$

Appendix D: Properties of the planar solution ($k = 0$)

The hyperbolic black hole solution is related to a planar black hole in the following way. The action with $(d-1)$ axion (massless scalar) fields $\chi_i$ is

$$S = \int d^{d+1}x \sqrt{-g} \left( R - \frac{1}{4} e^{-\alpha \phi} F^2 - \frac{1}{2}(\partial \phi)^2 - V(\phi) - \frac{1}{2} \sum_{i=1}^{d-1}(\partial \chi_i)^2 \right) . \quad (D1)$$
where $\chi_i = \kappa x_i$ satisfies the equation of motion of $\chi_i$. This system was used as a simple way to introduce momentum dissipation, since $\kappa x_i$ breaks the translation symmetry \cite{14, 15}. As pointed out in \cite{15}, for the potential $\left(\text{39}\right)$, the solution to the functions with the metric ansatz \cite{11} in the $k = 0$ case is the same as that of a hyperbolic black hole without the axions $\chi_i$.

For an arbitrary potential $V(\phi)$ and the metric ansatz \cite{15}, we observe that the equations of motion for a planar black hole with axions are the same as the equations of motion for a hyperbolic black hole without axions, provided that we make the identification

$$k = -\frac{1}{2(d-1)}\kappa^2. \quad \text{(D2)}$$

Here $\sqrt{|k|}$ is the inverse curvature radius of the hyperbolic space, and we set it to 1 previously.

The planar solution without axions has an intriguing (and peculiar) property: for $\alpha > (d - 2)\sqrt{\frac{2}{d(d-1)}}$, the gauge field is automatically eliminated from the EMD system in the extremal limit $r_h \rightarrow b$. In this case, the gauge field is proportional to a positive power of $(r_h - b)$. Examples include 1-charge black hole in AdS and 2-charge black holes in AdS$_d$. Some explanations are in section 2.3 of \cite{16}.

Consider the AdS$_d$ solution. When $\alpha > 1/\sqrt{3}$, the gauge field has to vanish when we take the extremal limit $r_h \rightarrow b$, because the gauge field $A_i$ is proportional to a positive power of $(r_h - b)$. The extremal solution is

$$d^2 = f(-dt^2 + dx^2) + f^{-1}dr^2, \quad A = 0,$$

$$f = \frac{r^2}{L^2} \left(1 - \frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \quad e^{-\alpha \phi} = \left(1 - \frac{b}{r}\right)^{-\frac{2\alpha^2}{1+\alpha^2}}. \quad \text{(D3)}$$

This is the neutral limit of \cite{1, 2, 3, 4} in the $k = 0$ case, i.e., the planar counterpart of \cite{8} and \cite{9}. This solution does not have a horizon, and has a spacetime singularity at $r = b$. We expect that this solution can be taken as an extremal limit of a finite temperature solution, when the Gubser criterion \cite{17} is not violated. The finite temperature solution will not satisfy the constraints proposed in Appendix A, according to Theorem 2. By a coordinate transformation $r = \tilde{r} + b$, the above solution becomes

$$d^2 = f(-dt^2 + dx^2) + f^{-1}d\tilde{r}^2, \quad A = 0,$$

$$f = \frac{\tilde{r}^2}{L^2} \left(1 + \frac{b}{\tilde{r}}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \quad e^{-\alpha \phi} = \left(1 + \frac{b}{\tilde{r}}\right)^{-\frac{2\alpha^2}{1+\alpha^2}}. \quad \text{(D4)}$$

The spacetime singularity is now at $\tilde{r} = 0$.

The planar black hole solution to the EMD system \cite{23} has the following distinctive IR geometries in the extremal case.

- $0 < \alpha < \sqrt{\frac{2}{d(d-1)}}$. The IR geometry is AdS$_d \times \mathbb{R}^{d-1}$.
- $\alpha = \sqrt{\frac{2}{d(d-1)}}$. The IR geometry is conformal to AdS$_2 \times \mathbb{R}^{d-1}$.
- $\alpha > \sqrt{\frac{2}{d(d-1)}}$. The IR geometry is a hyperscaling-violating geometry \cite{48, 49}.
- $\alpha > (d - 2)\sqrt{\frac{2}{d(d-1)}}$. The extremal limit of the EMD system \cite{23} is the same as an Einstein-scalar system.

For the Einstein-scalar (neutral) system, we draw the following conclusions by analyzing the IR geometry according to \cite{49}. Assuming $b > 0$, which implies $\alpha \phi < 0$, the leading term in $V(\phi)$ in the IR is the first term, and we have

- $\alpha > \sqrt{\frac{d-1}{2}}$. The spectrum is gapless.
- $\sqrt{\frac{d-1}{2}} < \alpha < \sqrt{\frac{d-1}{2}}$. The extremal geometry is at $T \rightarrow \infty$. The spectrum is potentially gapped.
- $0 < \alpha \leq \sqrt{\frac{d-1}{2}}$. It violates the Gubser criterion, and thus unacceptable holographically. However, the IR geometry can be changed by introducing extra fields.

[1] D. Marolf, M. Rangamani and T. Wiseman, Holographic thermal field theory on curved spacetimes, Class. Quant. Grav. 31 (2014) 063001 [arXiv:1312.0612].
[2] S. W. Hawking and D. N. Page, Thermodynamics of black holes in anti-de Sitter space, Commun. Math. Phys. 87 (1983) 577.
[3] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. 2 (1998) 505 [hep-th/9803131].
[4] R. Emparan, AdS membranes wrapped on surfaces of arbitrary genus, Phys. Lett. B 432 (1998) 74 [hep-th/9804031].
[5] D. Birmingham, Topological black holes in anti-de Sitter space, Class. Quant. Grav. 16 (1999) 1197 [hep-th/9808032].
[6] R. Emparan, AdS/CFT duals of topological black holes and the entropy of zero energy states, JHEP 9906 (1999) 036 [hep-th/9906040].
[7] D. Marolf, M. Rangamani and M. Van Raamsdonk, Holographic models of de Sitter QFTs, Class. Quant. Grav. 28.
R. Emparan, G. T. Horowitz and R. C. Myers, *Holographic phases of Rényi entropies*, JHEP **1312** (2013) 050 [arXiv:1306.2640].

T. Fang, S. He and D. Li, *Note on stability of new hyperbolic AdS black holes and phase transitions in Rényi entropies*, Nucl. Phys. B **923** (2017) 1 [arXiv:1601.05649].

M. Cvetic et al., *Embedding AdS black holes in ten-dimensions and eleven-dimensions*, Nucl. Phys. B **558** (1999) 96 [hep-th/9903214].

M. Cvetic and S. S. Gubser, *Phases of R-charged black holes, spanning branes and strongly coupled gauge theories*, JHEP **9904** (1999) 024 [hep-th/9902195].

R. G. Cai and A. Wang, *Thermodynamics and stability of hyperbolic charged black holes*, Phys. Rev. D **70** (2004) 064013 [hep-th/0406057].

C. J. Gao and S. N. Zhang, *Dilaton black holes in de Sitter or anti-de Sitter universe*, Phys. Rev. D **70** (2004) 124019 [hep-th/0411104].

C. J. Gao and S. N. Zhang, *Higher dimensional dilaton black holes with cosmological constant*, Phys. Lett. B **605** (2005) 185 [hep-th/0411105].

C. J. Gao and S. N. Zhang, *Topological black holes in dilaton gravity theory*, Phys. Lett. B **612** (2005) 127.

A. Sheykh, M. H. Dehghani and S. H. Hendi, *Thermodynamic instability of charged dilaton black holes in AdS spaces*, Phys. Rev. D **81** (2010) 084040 [arXiv:0912.4199].

S. H. Hendi, A. Sheykh and M. H. Dehghani, *Thermodynamics of higher dimensional topological charged AdS black branes in dilaton gravity*, Eur. Phys. J. C **70** (2010) 703 [arXiv:1002.0202].

K. Goto, H. Marrochio, R. C. Myers, L. Queimada and B. Yoshida, *Holographic complexity equals which action?*, JHEP **1902** (2019) 160 [arXiv:1901.00014].

H. Casini, M. Huerta and R. C. Myers, *Towards a derivation of holographic entanglement entropy*, JHEP **1105** (2011) 036 [arXiv:1102.0440].

L. Y. Hung, R. C. Myers, M. Smolkin and A. Yale, *Holographic calculations of Rényi entropy*, JHEP **1112** (2011) 047 [arXiv:1110.1084].

E. Witten, *Multitrace operators, boundary conditions, and AdS/CFT correspondence*, hep-th/0112258.

R. B. Mann, *Pair production of topological anti-de Sitter black holes*, Class. Quant. Grav. **14** (1997) L109 [gr-qc/9607071].

P. Dowker, J. P. Gauntlett, D. A. Kastor and J. H. Traschen, *Pair creation of dilaton black holes*, Phys. Rev. D **49** (1994) 2909 [hep-th/9309075].

H. Lii and J. F. Vazquez-Poritz, *Dynamic C-metrics in gauged supergravities*, Phys. Rev. D **91** (2015) 064004 [arXiv:1408.3124].

J. B. Griffiths, P. Krtous and J. Podolsky, *Interpreting the C-metric*, Class. Quant. Grav. **23** (2006) 6745 [gr-qc/0609056].

R. Emparan, G. T. Horowitz and R. C. Myers, *Exact description of black holes on branes*, JHEP **01** (2000) 007 [hep-th/9911043].

R. Emparan, G. T. Horowitz and R. C. Myers, *Exact description of black holes on branes. 2. Comparison with BTZ black holes and black strings*, JHEP **01** (2000) 021 [hep-th/9912135].

A. Belin, L. Y. Hung, A. Maloney and S. Matsuura, *Charged Rényi entropies and holographic superconductors*, JHEP **1501** (2015) 059 [arXiv:1407.5630].

G. W. Gibbons and S. W. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, Phys. Rev. D **15** (1977) 2738.

X. Huang, S. J. Rey and Y. Zhou, *Three-dimensional SCFT on conic space as hologram of charged topological black hole*, JHEP **1403** (2014) 127 [arXiv:1401.5421].

J. Erdmenger, K. Ghoroku and R. Meyer, *Holographic (de)confinement transitions in cosmological backgrounds*, Phys. Rev. D **84** (2011) 026004 [arXiv:1105.1776].

V. E. Hubeny, D. Marolf and M. Rangamani, *Hawking radiation in large N strongly-coupled field theories*, Class. Quant. Grav. **27** (2010) 095015 [arXiv:0908.2270].

V. E. Hubeny, D. Marolf and M. Rangamani, *Black funnels and droplets from the AdS C-metrics*, Class. Quant. Grav. **27** (2010) 025001 [arXiv:0909.0005].

S. S. Gubser and A. Nellore, *Mimicking the QCD equation of state with a dual black hole*, Phys. Rev. D **78** (2008) 086007 [arXiv:0804.0434].

D. Li, S. He, M. Huang and Q. S. Yan, *Thermodynamics of deformed AdS model with a positive/negative quadratic correction in granton-dilaton system*, JHEP **1109** (2011) 041 [arXiv:1103.5389].

R. G. Cai, S. He and D. Li, *A hQCD model and its phase diagram in Einstein-Maxwell-Dilaton system*, JHEP **1203** (2012) 033 [arXiv:1201.0829].

A. Anabalon, *Exact black holes and universality in the backreaction of non-linear sigma models with a potential in (A)dS*, JHEP **1206** (2012) 127 [arXiv:1204.2720].

X. H. Feng, H. Lü and Q. Wen, *Scalar hairy black holes in general dimensions*, Phys. Rev. D **89** (2014) 044014 [arXiv:1312.5374].

D. Garfinkle, G. T. Horowitz and A. Strominger, *Charged black holes in string theory*, Phys. Rev. D **43** (1991) 3140 [Erratum ibid 45 (1992) 3888].

M. J. Duff and J. T. Liu, *Anti-de Sitter black holes in gauged $\mathcal{N} = 8$ supergravity*, Nucl. Phys. B **554** (1999) 237 [hep-th/9901149].

W. A. Sabra, *Anti-de Sitter BPS black holes in $\mathcal{N} = 2$ gauged supergravity*, Phys. Lett. B **458** (1999) 36 [hep-th/9903143].

K. Behrndt, M. Cvetic and W. A. Sabra, *Nonextreme black holes of five-dimensional $\mathcal{N} = 2$ AdS supergravity*, Nucl. Phys. B **553** (1999) 317 [hep-th/9810227].

A. Andrade and B. Withers, *A simple holographic model of momentum relaxation*, JHEP **1405** (2014) 101 [arXiv:1311.5157].

B. Goutéraux, *Charge transport in holography with momentum dissipation*, JHEP **1404** (2014) 181 [arXiv:1401.5436].

O. DeWolfe, S. S. Gubser and C. Rosen, *Fermi surfaces in $N = 4$ Super-Yang-Mills theory*, Phys. Rev. D **86** (2012) 106002 [arXiv:1207.3352].

S. S. Gubser, *Curvature singularities: The Good, the bad, and the naked*, Adv. Theor. Math. Phys. **4** (2000) 679 [hep-th/0002160].

X. Dong, S. Harrison, S. Kachru, G. Torroba and H. Wang, *Aspects of holography for theories with hyperscaling violation*, JHEP **1206** (2012) 041 [arXiv:1201.1905].

C. Charmousis, B. Goutéraux, B. S. Kim, E. Kir-
itsis and R. Meyer, Effective holographic theories for low-temperature condensed matter systems, \textit{JHEP 1011} (2010) 151 [arXiv:1005.4690].

[50] E. Kiritsis and J. Ren, On holographic insulators and supersolids, \textit{JHEP 1509} (2015) 168 [arXiv:1503.03481].