Quantum systems subject to projective measurements at random times

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What happens when a quantum system undergoing unitary evolution in time is subject to repeated projective measurements to the initial state at random times? A question of general interest is: How does the survival probability $S_m$, namely, the probability that an initial state survives even after $m$ number of measurements, behave as a function of $m$? In the context of two paradigmatic quantum systems, one evolving in discrete and the other in continuous time, we present a wealth of numerical and analytical results that hint at the curious nature of quantum measurement dynamics. In particular, we unveil that when evolution after every projective measurement continues with the projected component of the instantaneous state, the average and the typical survival probability decay as an exponential in $m$ for large $m$. By contrast, if the evolution continues with the leftover component, namely, what remains of the instantaneous state after a measurement has been performed, the survival probability decays asymptotically as a power law in $m$ with exponent 3/2. These results hold independently of the choice of the distribution of times between successive measurements, implying robustness and ubiquity of our derived results.

I. INTRODUCTION

Consider a quantum system described by a time-independent Hamiltonian $H$. The state of the system at any time $t$ is characterized by a state vector $|\psi(t)\rangle$ defined in the Hilbert space $\mathcal{H}$ of the system (S). The state vector undergoes unitary evolution in time, as $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$; $t > t_0$, where $U(t,t_0) \equiv \exp(-iH(t-t_0))$ is the unitary time-evolution operator [1]. Consider next a series of instantaneous measurements performed on the system at random times. Following the measurement postulate of quantum mechanics [2], each measurement involves projecting the instantaneous state of the system onto the Hilbert space $\mathcal{H}_D \subset \mathcal{H}$ of the measuring device (D). Starting at time $t = 0$ with a state vector with unit norm, it is evident that each projective measurement would reduce the magnitude of the norm, and we may ask: what is the fate of the norm after a certain number $m$ of measurements have been performed on the system?

In the aforementioned protocol, the norm of the initial state vector at any time $t > 0$ has the physical interpretation of survival probability of the initial state at that time instant, as may be appreciated from the following illustrative example. Consider a quantum particle undergoing motion in a potential field defined in a given three-dimensional spatial domain $\mathcal{D}$, on which we perform a set of instantaneous projective measurements at random times to learn about the location of the particle. If $|\psi(t)\rangle$ is the state vector of the particle at time $t$, then $\int_\mathcal{D} |\psi(r,t)\rangle^2 d^3r$ gives the probability of finding the particle between locations $r$ and $r + dr$ in domain $\mathcal{D}$ at time $t$, with $\psi(r,t) \equiv \langle r | \psi(t) \rangle$. With $\int_\mathcal{D} d^3r |\psi(r,t = 0)\rangle^2 = 1$ so that the particle is initially somewhere for sure within the domain $\mathcal{D}$, and with $m$ instantaneous projective measurements performed at random times $t_1, t_2, t_3, \ldots, t_m$, with $0 < t_1 < t_2 < t_3 < \ldots < t_m$, the quantity $\int_\mathcal{D} d^3r |\psi(r,t_m)\rangle^2$ would give the probability $S_m$ that the particle is still inside the domain $\mathcal{D}$ at the end of $m$ projective measurements. In other words, $S_m$ is the survival probability that the particle has survived in $\mathcal{D}$ up to time $t_m$. How does $S_m$ vary from one realization of random times $\{t_1, t_2, \ldots, t_m\}$ to another? What is the dependence on $m$ of $S_m$, the survival probability averaged over different realizations of these random times? We may consider the successive time gaps $\tau_\alpha = t_\alpha - t_{\alpha-1}$; $\alpha = 1, 2, 3, \ldots, m$; $t_0 = 0$ to be random variables sampled independently from a common distribution $p(\tau)$, that is to say, the $\tau_\alpha$’s are independent and identically-distributed (i.i.d.) random variables. In this backdrop, we may ask: How does the behavior of $S_m$ versus $m$ depend on the choice of $p(\tau)$? What can one say about the behavior of the typical survival probability $S_m^\tau$ (namely, the survival probability observed in a typical realization $\{\tau_\alpha\}_{1 \leq \alpha \leq m}$)? In this work, we seek answers to these questions within the ambit of two representative quantum systems defined on a lattice, one, the quantum random walk (QRW) model evolving in discrete times [3], and the other, the tight-binding model (TBM), evolving in continuous time [4, 5]. We unveil a plethora of interesting results, numerical as well as analytical, including universal features in the late-time behavior of both the average and the typical survival probability, all of which point to the intriguing nature of quantum measurement process.

The paper is organized as follows. We choose to study first the QRW, which is described in detail in Section II A. The so-called Leggett-Garg inequalities (LGIs) constrain correlations between measurements performed on a system at different times, and are derived on the assumptions of macroscopic realism (namely, that the physical properties of a system are well defined at every time regardless of whether one performs measurements on the system or not) and noninvasive measurability (namely, any measurement does not affect the future evolution of the system) [6]. Violation of the LGIs in a system implies that its evolution cannot be understood classically. Following a derivation of exact results for the site occupation

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probability of the QRW in Section II A, we use these results in Section II B to elucidate the quantum nature of its evolution through demonstration of violation of the LGIs. In Section II C, we present our main results on the survival probability of a generic initial state subject to instantaneous projective measurements to the initial state at random times. We report extensive numerical results demonstrating that the average as well as the typical survival probability decays asymptotically as a function of \( m \), the number of measurements, either as an exponential or as a power-law with universal exponent 3/2, depending respectively on whether the evolution following each projective measurement is carried on with the projected component of the instantaneous state or the leftover component (namely, what remains of the instantaneous state after a measurement has been performed). These results hold independently of the choice of the distribution of the time between successive measurements. For the projected case, we support our numerical findings with explicit analytical calculations using large deviation theory (LDT) well known in probability theory [7, 8], while for the leftover case, a semi-analytical approach reproduces correctly our numerical findings.

Next, we turn to a description of the TBM and a derivation of its site occupation probability in Section III A. This is followed in Section III B by a discussion of our numerical results for the survival probability for the case in which the evolution following each projective measurement is carried on with the projected component as well as for the case in which it is the leftover component of the instantaneous state that undergoes subsequent evolution until next measurement. Here too we support our findings with analytical results derived using the LDT for the projected case and with semi-analytical calculations for the leftover case. Similar to the case of the QRW, we find that for representative \( p(\tau) \), the average as well as the typical survival probability decays asymptotically as a function of \( m \) as an exponential for the projected case and as a power-law with universal exponent 3/2 for the leftover case, thereby hinting at the robustness of our results with respect to both discrete and continuous time evolution. The paper ends by a unitary operator \( C \mathbb{1} \), while for the unitary operator \( U \), with both operations constituting one time step of evolution of the walker state. Here, \( U \) is given by

\[
U \equiv |\uparrow\rangle\langle\uparrow| \otimes \sum_n |n+1\rangle\langle n| + |\downarrow\rangle\langle\downarrow| \otimes |n-1\rangle\langle n|, \quad (1)
\]

while \( C \) has the form

\[
C \equiv \left( \cos \theta |\uparrow\rangle\langle\uparrow| + \sin \theta |\uparrow\rangle\langle\downarrow| \right.
\]

\[
- \sin \theta |\downarrow\rangle\langle\uparrow| + \cos \theta |\downarrow\rangle\langle\downarrow| \left. \right), \quad (2)
\]

where \( \theta \in (0,2\pi) \) is a given parameter. The operator \( U \) generates a translation of the walker on the lattice that is conditioned on the state of the quantum coin, while the operator \( C \) generates a rotation in the coin space.

A large class of initial conditions corresponds to having the walker on a given site \( n_0 \) and in an arbitrary superposition of the coin states:

\[
|\psi(0)\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}} (a |\uparrow\rangle + b |\downarrow\rangle) \otimes |n_0\rangle, \quad (3)
\]

with \( a, b \in \mathbb{C} \). Following the rules of evolution of the random walker detailed above, we may write down explicitly the state of the walker at the first and the second time step:

\[
|\psi(1)\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}} \left( [a \cos \theta |\uparrow\rangle + b \sin \theta |\uparrow\rangle] \otimes |1\rangle \right.
\]

\[
+ \left[ -a \sin \theta |\downarrow\rangle + b \cos \theta |\downarrow\rangle \right] \otimes |1\rangle \left. \right), \quad (4)
\]

\[
|\psi(2)\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}} \left( [a \cos^2 \theta |\uparrow\rangle + b \cos \theta \sin \theta |\uparrow\rangle] \otimes |2\rangle \right.
\]

\[
+ \left[ -b \sin^2 \theta |\downarrow\rangle - a \cos \theta \sin \theta |\downarrow\rangle \right] \otimes |0\rangle \left. \right)
\]

\[
+ \left[ -a \sin^2 \theta |\uparrow\rangle + b \cos^2 \theta |\uparrow\rangle \right] \otimes |0\rangle \left. \right)
\]

\[
+ \left[ -a \cos \theta \sin \theta |\downarrow\rangle + b \cos^2 \theta |\downarrow\rangle \right] \otimes |2\rangle \left. \right).
\]

It is evident then that the state at time \( t \) has the general structure

\[
|\psi(t)\rangle = |\uparrow\rangle \otimes \sum_n \Psi_u(n,t) |n\rangle + |\downarrow\rangle \otimes \sum_n \Psi_d(n,t) |n\rangle, \quad (5)
\]

where the sum extends over all the lattice sites. Here, \( \Psi_u(n,t) \) and \( \Psi_d(n,t) \) are respectively the probability amplitude to find
In terms of its eigenvalues and orthonormal eigenvectors:

\[ \lambda_1^{(1)} = e^{i\omega_k}, \quad \phi_1^{(1)} = \frac{1}{\sqrt{1+h_+^2(k)}} \begin{bmatrix} -ie^{i2\pi k/N} h_+ (k) \\ 1 \end{bmatrix}, \]

\[ \lambda_1^{(2)} = e^{-i\omega_k}, \quad \phi_1^{(2)} = \frac{1}{\sqrt{1+h_+^2(k)}} \begin{bmatrix} ie^{i2\pi k/N} h_- (k) \\ 1 \end{bmatrix}, \]

with \( \cos \omega_k \equiv \cos (2\pi k/N) \cos \theta \), and \( h_{\pm} (k) \equiv \cot \theta \sin (2\pi k/N) \pm \csc \theta \sin \omega_k \). Note that \( h_{\pm} (-k) = -h_{\pm} (k) \), and \( h_+ (k) h_- (k) = -1 \). The initial condition corresponding to Eq. (3) is \( |\Psi(0,0)\rangle \equiv \left(\exp\{i2\pi kn_0/N\}/\sqrt{|a|^2 + |b|^2}\right) [a \ b]^T \forall \ k \). Using these results to obtain \( |\Psi(k,t)\rangle = (\lambda_1^{(1)} )^t \phi_1^{(1)} |\Psi(0,0)\rangle + (\lambda_1^{(2)} )^t \phi_1^{(2)} |\Psi(0,0)\rangle \) and then performing inverse Fourier transformation, we finally obtain for odd \( N \) the result

\[ |\Psi_u(n,t)\rangle = \sum_{k=-(N-1)/2}^{(N-1)/2} \mathcal{A}_u(k,n,t), \quad |\Psi_d(n,t)\rangle = \sum_{k=-(N-1)/2}^{(N-1)/2} \mathcal{A}_d(k,n,t), \]

while for even \( N \), we get

\[ |\Psi_u(n,t)\rangle = (1)^{n-m_0+t} \left\{ a \cos (\theta t) + b \sin (\theta t) \right\} \frac{N\sqrt{|a|^2 + |b|^2}}{N^{N/2} \sum_{k=-N/2+1}^{N/2-1} \mathcal{A}_u(k,n,t)} \\
+ \sum_{k=-N/2+1}^{N/2-1} \mathcal{A}_u(k,n,t), \]

\[ |\Psi_d(n,t)\rangle = (1)^{n-m_0+t} \left\{ -a \sin (\theta t) + b \cos (\theta t) \right\} \frac{N\sqrt{|a|^2 + |b|^2}}{N^{N/2} \sum_{k=-N/2+1}^{N/2-1} \mathcal{A}_d(k,n,t)}. \]

Here, the quantities \( \mathcal{A}_u(k,n,t) \) and \( \mathcal{A}_d(k,n,t) \) are given by

\[ \mathcal{A}_u(k,n,t) = \mathcal{N}(a,b,k) \left[ a \cos \{2\pi k(n-n_0)/N + \omega_k \} \\
+ bh_+ (k) \sin \{2\pi k(n-n_0-1)/N + \omega_k \} \right], \]

\[ \mathcal{A}_d(k,n,t) = \mathcal{N}(a,b,k) \left[ -ah_+ (k) \sin \{-2\pi k(n-n_0+1)/N + \omega_k \} \\
+ b \cos \{-2\pi k(n-n_0)/N + \omega_k \} \right], \]

with \( \mathcal{N}(a,b,k) = 2/N \left[ N(1+h_+^2(k)) \sqrt{|a|^2 + |b|^2} \right]. \)

Let us remark that implementing the transformation \( \theta \to \theta + \pi \) in the expression for \( C \) is tantamount to multiplying \( C \) and consequently the matrix \( M_k \) by the factor \(-1\). As a result, the eigenvalues of \( M_k \) get both multiplied by the factor \(-1\), although the corresponding eigenvectors remain the same. All
of these would however leave \(|\Psi_a(n,t)|^2\) and \(|\Psi_d(n,t)|^2\) and consequently the site occupation probability \(P_n(t)\) unchanged, thereby making us conclude that the QRW is invariant with respect to the transformation \(\theta \to \theta + \pi\). Hence, we will in the rest of the paper restrict the values of \(\theta\) to the range \([0, \pi]\).

\[
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\end{array}
\]

FIG. 1: Site occupation probability \(P_n(t)\) for the QRW on a one-dimensional periodic lattice of \(N\) sites and at time \(t = 20\), while starting from the state (3) with \(n_0 = 0\). The values of the parameters \(a\) and \(b\) defining the initial state are \(a = 1\), \(b = i\) for panels (a) and (b), and \(a = b = 1\) for panels (c) and (d). The value of \(N\) is \(N = 6\) for panels (a) and (c), and \(N = 7\) for panels (b) and (d). The angle \(\theta\) has a value in radian that corresponds to 80 degrees. Numerics correspond to bare evolution of the unitary dynamics of the model, while analytical results are given by Eqs. (12) and (13). The lines are a guide to the eye.

In Fig. 1, we show a comparison between numerical results and theory for the site occupation probability \(P_n(t)\), demonstrating a perfect match. In the figure, numerics correspond to bare evolution of the unitary dynamics of the model, while analytical results are given by Eqs. (12) and (13).

B. Leggett-Garg inequalities and nonclassicality of the QRW

While the celebrated Bell’s inequalities probe entanglement between spatially-separated systems, the issue of correlations of a single system measured at different times is addressed by a class of inequalities derived by Leggett and Garg [6, 11]. In order to introduce the Leggett-Garg Inequalities (LGIs), let us first summarize a physicist’s view of the macroscopic world, which consists of the following two underlying principles:

- \(P1\): Macroscopic realism
  A macroscopic system that may exist in two or more macroscopically distinct states will at all times actually exist in one or the other of these states.

- \(P2\): Noninvasive measurability
  It is possible in principle to determine the state in which the system is existing at a given time instant, with the measurement involving arbitrarily small perturbation that does not affect at all the future evolution of the system.

Let us note that classical mechanics is consistent with both the above principles, while quantum mechanics is not: existence of macroscopic superposition violates the first, while quantum-mechanical collapse under measurement does not conform to the second. The LGIs refer to a class of inequalities that any system behaving in accord with our intuition of the macroscopic world as summarized above should obey. Violation of the inequalities implies that at least one of the aforementioned principles does not apply to the system under consideration, which must then be viewed not from a macroscopic point of view. Consequently, the LGIs offer a way to investigate the existence of quantum coherence in macroscopic systems, and to test the possibility of applying quantum mechanics to the macroscopic world. Violation of LGIs has been demonstrated in a wide range of physical systems, e.g., superconducting qubits [12], nuclear spins [13], phosphorus impurities in silicon [14], massive quantum particle propagating on a discrete lattice [15], and many more.

To introduce the LGIs, consider a system that may exist in two distinct macroscopic states characterized by a macroscopic dichotomic variable \(Q = \pm 1\). Next, consider a collection of experimental runs all starting from the same initial state at \(t = 0\). We may then straightforwardly define the two-time probabilities \(P_{12}(Q_1, Q_2), P_{23}(Q_2, Q_3), P_{13}(Q_1, Q_3)\) corresponding to \(Q\) assuming values \(Q_\alpha\) at times \(t_\alpha\) (with \(0 < t_1 < t_2 < t_3\)). Note that these are classical probabilities and may be obtained, e.g., from the set of runs in which we decide to measure \(Q\) at times \(t_1\) and \(t_2\) (yielding thereby the probability \(P_{12}(Q_1, Q_2)\)), from the set of runs in which we decide to
measure $Q$ at times $t_2$ and $t_1$ (yielding thereby the probability $P_{23}(Q_2, Q_3)$), and so on. On the basis of these joint probabilities, one may then define the correlations

$$C_{\alpha \beta} = \langle Q_\alpha Q_\beta \rangle = \sum_{Q_\alpha, Q_\beta = \pm 1} Q_\alpha Q_\beta P_{\alpha \beta}(Q_\alpha, Q_\beta).$$

(15)

with $\alpha, \beta \in \{1, t_2, t_3\}$ and $\alpha < \beta$. Let us now consider the quantity $K_3$, defined as

$$K_3 = C_{12} + C_{23} - C_{13}.$$  

(16)

Now, principle $PI$ implies that the observable $Q$ has a well-defined value at all times even when no measurement has been performed on the system, so that the two-time probability may be obtained from the three-time probability as

$$P_{\alpha \beta}(Q_\alpha, Q_\beta) = \sum_{Q_\gamma = \pm 1, \gamma \neq \alpha, \beta} P_{\alpha \beta}(Q_\alpha, Q_\beta, Q_\gamma).$$

(17)

Suppose we consider only principle $PI$. In this case, the three probabilities, namely, $P_{23}(Q_2, Q_3)$, $P_{23}(Q_1, Q_2, Q_3)$, $P_{13}(Q_1, Q_2, Q_3)$, required to evaluate $K_3$ in Eq. (16) would be independent of one another, since measurements at different times may lead to different amount of perturbation introduced in the subsequent dynamics of the system. However, if we consider also principle $P2$, the latter possibility is ruled out, and consequently, we would have $P_{23}(Q_1, Q_2, Q_3) = P_{23}(Q_1, Q_2, Q_3) = P_{13}(Q_1, Q_2, Q_3) = P(Q_1, Q_2, Q_3)$. We may then use the probability $P(Q_1, Q_2, Q_3)$ to evaluate the various terms occurring in $K_3$, to get [11]

$$-3 \leq K_3 \leq 1,$$

(18)

the LGI for $K_3$. The above inequality is one of a class of inequalities, the most general form of which concerns the $n$-measurement string

$$K_n = C_{12} + C_{23} + \cdots + C_{(n-1)} - C_{1n}.$$  

(19)

which is bounded as [11]

$$-n \leq K_n \leq n - 2; \quad n \geq 3, \quad \text{if } n \text{ is odd},$$

$$-(n - 2) \leq K_n \leq n - 2; \quad n \geq 4, \quad \text{if } n \text{ is even}. $$

(20)

Let us note that for these bounds to hold, the variable $Q$ does not necessarily have to be dichotomic, but must be bounded: $|Q| \leq 1$ [11].

In the above backdrop, let us now discuss the LGIs in the context of the QWR. We specifically focus on the LGI for $K_3$, although our method detailed below lets us obtain exact analytic expressions for any $K_n$. In the spirit of LGIs, let $Q$ be a dichotomic variable that takes the value $+1$ (respectively, the value $-1$) depending on whether the walker when on site 0 has its spin in the up state (respectively, in the down state). Now, from Eq. (15), one may write for a given initial condition at $t = 0$ that

$$C_{12} = \overline{Q_1 Q_2}$$

$$= P_0(+1, t_1) \left[ P(+1, t_2 | +1, t_1) - P(-1, t_2 | +1, t_1) \right]$$

$$+ P_0(-1, t_1) \left[ P(-1, t_2 | -1, t_1) - P(+1, t_2 | -1, t_1) \right],$$

(21)

where $P_0(+1, t_1)$ is the probability of measuring $Q = +1$ at time $t_1 > 0$, $P(-1, t_2 | +1, t_1)$ is the probability of measuring $Q = -1$ at time $t_2 > t_1$, conditioned on having measured $Q = +1$ at time $t_1$, etc.

For the initial state (3), the one-time probabilities $P_0(+1, t_1)$ and $P_0(-1, t_1)$ occurring on the right-hand side of Eq. (21) are given by

$$P_0(+1, t_1) = |\Psi_u(0, t_1)|^2, \quad P_0(-1, t_1) = |\Psi_d(0, t_1)|^2.$$

(22)

In order to find the conditional probabilities in Eq. (21), we first note that a measurement at time $t = t_1$ that yields $Q = +1$ collapses the state of the system to state

$$|\psi\rangle = |\uparrow\rangle \otimes |0\rangle.$$  

(23)

As a result, subsequent evolution starts from state (23) and the state at time $t > t_1$ has the generic structure (see Eq. (5)):

$$|\psi(t)\rangle = |\uparrow\rangle \otimes \sum_n |\psi_u^{(0)}(n, t)| n \rangle + |\downarrow\rangle \otimes \sum_n |\psi_d^{(0)}(n, t)| n \rangle.$$  

(24)

where the superscript is an indicator of what has been the initial state for the evolution, namely, the state (23) that has the spin in the up (u) state. Equation (24) implies that the probability of finding the walker on site 0 with spin up (respectively, spin down) is $|\psi_u^{(0)}(0, t)|^2$ (respectively, $|\psi_d^{(0)}(0, t)|^2$). Similarly, a measurement that yields $Q = -1$ collapses the state of the system to state

$$|\psi\rangle = |\downarrow\rangle \otimes |0\rangle,$$

(25)

evolving which for time $t$, one may obtain the probabilities $|\psi_u^{(0)}(0, t)|^2$ and $|\psi_d^{(0)}(0, t)|^2$.

Now, invoking time translation invariance of the QWR dynamics, we see that $P(+1, t_2 | +1, t_1)$ is given by the probability $P(+1, t_2 - t_1 | +1, 0)$ of having $Q = +1$ after evolution for time $t_2 - t_1$ while starting from the initial state (23). As for the other conditional probabilities, we have, e.g., $P(-1, t_2 | -1, t_1)$ given by the probability $P(-1, t_2 - t_1 | -1, 0)$ of measuring $Q = -1$ after evolution for time $t_2 - t_1$ while starting from the initial state (25). We then have

$$P(+1, t_2 | +1, t_1) = |\psi_u^{(0)}(0, t_2 - t_1)|^2,$$

$$P(-1, t_2 | +1, t_1) = |\psi_d^{(0)}(0, t_2 - t_1)|^2,$$

$$P(+1, t_2 | -1, t_1) = |\psi_u^{(0)}(0, t_2 - t_1)|^2,$$

$$P(-1, t_2 | -1, t_1) = |\psi_d^{(0)}(0, t_2 - t_1)|^2.$$  

(26)

The quantities $\psi_u^{(0)}(0, t)$ and $\psi_d^{(0)}(0, t)$ are obtained from Eqs. (12) and (13) with the substitution $n = 0, a = 1, b = 1$. 


0, \ n_0 = 0. \text{ On the other hand, we may obtain } \Psi_u^{(d)}(0,t) \text{ and } \Psi_d^{(d)}(0,t) \text{ from Eqs. (12) and (13) with the substitution } n = 0, \ a = 0, \ b = 1, \ n_0 = 0. \text{ Lastly, } \Psi_u(0,t) \text{ and } \Psi_d(0,t) \text{ are given by Eqs. (12) and (13) by substituting } n = 0 \text{ and using the values of } a, \ b, \ n_0 \text{ that one chooses in order to test the LGIs.}

Using the aforementioned procedure, we may obtain expressions for \( C_{\text{I}2}, C_{\text{I}3}, C_{\text{I}4}, C_{\text{I}6} \). These expressions when substituted in the expression for \( K_4 \) gives an exact analytic expression for the latter, which may be evaluated numerically. For numerical purpose, we choose even \( \Delta t \) between consecutive measurements: \( t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \Delta t \). Note that as we have displayed in Table I, we have, in this case, the probability for the walker to be on site 0 being zero at odd times for all values of \( a, b \) defining the initial state (3), so that we need to choose \( \Delta t \) to be even. Figure 2 shows the results for \( K_4 \) for the choice \( a = 1, \ b = i \), the angle \( \theta \) defining the operator \( C \) (see Eq. (2)) having the value in radian corresponding to 10, 45, and 80 degrees, and for system size \( N = 6 \). We observe LGI violation for the value \( \theta = 80 \) degrees (Note that the LGI for \( K_4 \) reads: \( -2 \leq K_4 \leq 2 \)), and choose this particular value of \( \theta \) for our further work reported below on projective measurements on the QRW. Figure 3 illustrates that \( K_4 \) is independent of the parameters \( a \) and \( b \) defining the initial state, as evidenced in the figure by the independence of the data with respect to different values of the parameters. Our results summarized in this subsection are a clear illustration of the non-classical nature of the QRW dynamics, and serve as yet another application of the LGIs in elucidating the quantum nature of a given dynamics.

\[ \begin{align*}
C_{\text{I}2}, C_{\text{I}3}, C_{\text{I}4}, C_{\text{I}6} & \text{. These expressions when substituted in the expression for } K_4 \text{ gives an exact analytic expression for the latter, which may be evaluated numerically. For numerical purpose, we choose even } \Delta t \text{ between consecutive measurements: } t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \Delta t \text{. Note that as we have displayed in Table I, we have, in this case, the probability for the walker to be on site 0 being zero at odd times for all values of } a, b \text{ defining the initial state (3), so that we need to choose } \Delta t \text{ to be even. Figure 2 shows the results for } K_4 \text{ for the choice } a = 1, \ b = i \text{, the angle } \theta \text{ defining the operator } C \text{ (see Eq. (2)) having the value in radian corresponding to 10, 45, and 80 degrees, and for system size } N = 6 \text{. We observe LGI violation for the value } \theta = 80 \text{ degrees (Note that the LGI for } K_4 \text{ reads: } -2 \leq K_4 \leq 2 \text{)), and choose this particular value of } \theta \text{ for our further work reported below on projective measurements on the QRW. Figure 3 illustrates that } K_4 \text{ is independent of the parameters } a \text{ and } b \text{ defining the initial state, as evidenced in the figure by the independence of the data with respect to different values of the parameters. Our results summarized in this subsection are a clear illustration of the non-classical nature of the QRW dynamics, and serve as yet another application of the LGIs in elucidating the quantum nature of a given dynamics.}

\[ \begin{align*}
\text{FIG. 2: Checking the LGI for } K_4 \text{ in the case of the QRW. The system size is } N = 6. \text{ Here } \Delta t \text{ denotes the time gap (taken here to be equal) between successive measurements: } t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \Delta t. \text{ In the figure, the value of } \Delta t \text{ changes in steps of two units while starting with the value } \Delta t = 2. \text{ The initial state is given by Eq. (3) with } n_0 = 0 \text{ and } a \text{ taking the values } a = 0, \ b = 1 \text{ (blue dashed line) and } a = 1, \ b = 1 \text{ (black dotted line). The angle has the value in radian corresponding to 80 degrees. The figure suggests that LGI holds independent of parameters } a \text{ and } b \text{ defining the initial state (3). The data are obtained by using the expression (21) and similar ones for } C_{\text{I}3}, C_{\text{I}4} \text{ and } C_{\text{I}6} \text{ in the definition of } K_4. \text{ The angle has a value in radian corresponding to 80 degrees.}

\[ \begin{align*}
\text{FIG. 3: The LGI for } K_4 \text{ in the case of the QRW and for different initial conditions. The system size is } N = 6. \text{ Here } \Delta t \text{ denotes the time gap between successive measurements: } t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \Delta t. \text{ In the figure, the value of } \Delta t \text{ changes in steps of two, while starting with the value } \Delta t = 2. \text{ The initial state is given by Eq. (3) with } n_0 = 0 \text{ and } a, b \text{ taking the values } a = 1, \ b = 0 \text{ (red continuous line), } a = 0, \ b = 1 \text{ (blue dashed line) and } a = 1, \ b = 1 \text{ (black dotted line). The angle has the value in radian corresponding to 80 degrees. The figure suggests that LGI holds independent of parameters } a \text{ and } b \text{ defining the initial state (3). The data are obtained by using the expression (21) and similar ones for } C_{\text{I}3}, C_{\text{I}4} \text{ and } C_{\text{I}6} \text{ in the definition of } K_4. \text{ The data are obtained by using the expression (21) and similar ones for } C_{\text{I}3}, C_{\text{I}4} \text{ and } C_{\text{I}6} \text{ in the definition of } K_4. \text{}

\[ \begin{align*}
\text{C. Random projective measurements and survival probability}

We consider a generic state of the QRW undergoing unitary evolution in time, which is interspersed with instantaneous projective measurements at random time instants. Specifically, starting with the initial state } |\psi(0)\rangle \text{ given in Eq. (3), the dynamics involves the following: We let the system evolve for a random number of time steps } \tau_1 \in \mathbb{Z}_+, \text{ where a single-step evolution involves acting on } |\psi(0)\rangle \text{ by the operator } C \otimes I \text{ followed by the operator } U, \text{ see Eqs. (1) and (2). Since the QRW evolves in discrete times, the phrases “time” and “timestep” would have the same meaning in the context of the QRW and would be used interchangeably in the following. The state at the end of time } \tau_1 \text{ is thus the result of the action of a single unitary operator } U_1 \equiv |U(C \otimes I)\rangle \langle I| \text{ on } |\psi(0)\rangle. \text{ The evolved state is then subject to an instantaneous projective measurement according to a given projection operator } P. \text{ Subsequent evolution may then proceed with either the projected component } PU_1 |\psi(0)\rangle \text{ of the instantaneous state } U_1 |\psi(0)\rangle \text{ or its leftover component given by } (I - P)U_1 |\psi(0)\rangle \text{, where } I \text{ is the identity operator. We then iterate a given number of times (which we denote by the integer } m > 0 \text{) the aforementioned set of events: unitary evolution for a random time step } \tau_\alpha \in \mathbb{Z}_+; \ \alpha = 1, 2, \ldots, m, \text{ projective measurement, subsequent evolution with either the projected or the leftover component. Thus, we have two different schemes of time evolution of the system that involve one of the following repetitive sequence of event-pair:}

- **Scheme 1:** A unitary evolution for a random time step \( \tau_\alpha \in \mathbb{Z}_+ \) according to the unitary operator \( U_\alpha \equiv |U(C \otimes I)\rangle \langle I| \) follow the action of the operator } P. \text{ This scheme corresponds to subsequent evolution with the projected component after each measurement.}
• **Scheme 2**: A unitary evolution for a random time $\tau_\alpha \in \mathbb{Z}_+ \times$ according to the unitary operator $U_\alpha$, followed by the action of the operator $\tilde{P} = I - P$. This scheme corresponds to subsequent evolution with the leftover component after each measurement.

Measurements: $1 \quad 2 \quad 3 \quad m - 1 \quad m$  
Initial state $|\psi(0)\rangle$

![Diagram of Scheme 1](image)

![Diagram of Scheme 2](image)

FIG. 4: A typical time evolution of a quantum system subject to projective measurements at random times, as detailed in the text. Starting from the state $|\psi(0)\rangle$, the evolution involves the following repetitive sequence of events: unitary evolution for time $\tau_\alpha$ according to operator $U_\alpha$ for time $\tau_\alpha$ followed by a projective measurement (denoted by the down arrows), with $U_\alpha = [U(C \otimes \mathbb{I})]^{\tau_\alpha}$ for the QWR defined in Section II, and $U_\alpha = \exp(-i H \tau_\alpha)$ for the TBM defined in Section III. Here, the operators $U$ and $C$ are given respectively by Eqs. (1) and (2), while the operator $H$ is given by Eq. (40). The different $\tau_\alpha$'s are i.i.d. random variables sampled from a given distribution. In **Scheme 1** (respectively, **Scheme 2**), subsequent evolution at the end of each measurement is carried out with the projected component of the instantaneous state (respectively, the leftover component).

A typical time evolution for a given realization $\{\tau_\alpha\}_{1 \leq \alpha \leq m}$ is shown in Fig. 4. We let the different $\tau_\alpha$'s, denoting the time interval between two consecutive application of the operator $P$ (**Scheme 1**) or the operator $\tilde{P}$ (**Scheme 2**), to be i.i.d. random variables sampled from a common distribution $p_\tau$ normalized as $\sum_\tau p_\tau = 1$ [16].

We take the projection operator to be

$$P = |\psi(0)\rangle \langle \psi(0)|. \quad (27)$$

In this way, starting with the state $|\psi(0)\rangle$, the system evolves according to the following repetitive sequence of events: unitary evolution for a random time interval, then a measurement that projects the evolved state into the initial state, followed by subsequent evolution with either the projected component (**Scheme 1**) or the leftover component (**Scheme 2**).

In the following, we will use the notation $|\psi^{(b)}_\alpha\rangle$ to denote the state of the system at the end of evolution for time $\tau_\alpha$ and just before the $\alpha$-th projective measurement, while $|\psi^{(a)}_\alpha\rangle$ will be taken to denote the state just after the $\alpha$-th measurement. It is evident from the dynamical rules of evolution that $|\psi^{(b)}_\alpha\rangle = U_\alpha |\psi^{(a)}_{\alpha-1}\rangle$, while $|\psi^{(a)}_\alpha\rangle$ will be either $|\psi^{(a)}_\alpha\rangle = P |\psi^{(b)}_\alpha\rangle$ (**Scheme 1**) or $|\psi^{(a)}_\alpha\rangle = \tilde{P} |\psi^{(b)}_\alpha\rangle$ (**Scheme 2**), and $\alpha = 1, 2, \ldots, m$. Moreover, the inner product $\langle \psi^{(a)}_m | \psi^{(a)}_m \rangle$ for any $m \geq 1$ would have a value smaller than or equal to unity owing to the application of the operators $P, \tilde{P}$. The precise value that the inner product assumes would vary from one realization of the dynamics to another. We then define the random variable

$$S_m = S_m(\{\tau_\alpha\}_{1 \leq \alpha \leq m}) = \langle \psi^{(a)}_m | \psi^{(a)}_m \rangle \quad (28)$$

as the survival probability of the initial state after $m$ projective measurements for the realization $\{\tau_\alpha\}_{1 \leq \alpha \leq m}$. Note that different values of $S_m$ correspond to different total duration of evolution given by

$$\mathcal{T} = \sum_{\alpha=1}^m \tau_\alpha. \quad (29)$$

We are interested in the question: What is the average survival probability $\bar{S}_m$, namely, the survival probability $S_m$ averaged with respect to different realizations of either **Scheme 1** or **Scheme 2** of the dynamics? How do the results depend on the particular scheme we adopt, namely, whether we consider evolution to proceed throughout with the projected or with the leftover component of the instantaneous state? What is the essential role played by the underlying distribution $p_\tau$ in dictating the value of the average survival probability obtained in the two schemes?

The so-called quantum Zeno effect, first discussed in a seminal paper by Sudarshan and Misra in 1977 [17], is a curious quantum mechanical phenomenon involving a quantum system subject to projective measurements to a given initial state at uniform time intervals. In the extreme case of a frequent-enough series of measurements over a fixed total duration (implying thereby that the successive measurements are infinitesimally close to one another), the survival probability for the system to remain in the initial state may be shown to approach unity in the limit of an infinite number of measurements, i.e., the dynamical evolution of the system gets completely frozen in time. Subsequently, stochastic quantum Zeno effect was introduced to study the situation in which the measurements are randomly spaced in time [18, 19]. It is important to remark in the context of the present work that the Zeno effect involves evolution following each measurement to be carried out with the projected component of the instantaneous state, while it is evidently of interest to investigate the dynamics in the complimentary case in which it is the leftover component of the instantaneous state that undergoes evolution subsequent to each measurement. These two dynamical scenarios correspond respectively to **Scheme 1** dynamics and **Scheme 2** dynamics defined above. Performing measurements at random times is not quite an issue of only theoretical curiosity, but may be motivated on the ground that after all, any experiment that aims to employ projective measurements to demonstrate the Zeno effect would typically use a timer to time the gap between successive measurements. Because the timer would invariably be of finite precision, it would not be possible to ensure that measurements are performed at exactly uniform time intervals. On the other hand, in the context of this paper, it is relatively easy to control the number of times that the projective measurement is repeated, thus justifying our dynamical setups. The average time between two intervals (averaged over different realizations of the intervals) would of course be finite in experiments.
1. Numerical results

We now present numerical results on the survival probability $S_m$, and in particular, on the average and the typical survival probability, and their dependence on the number of measurements $m$. As a representative case, we take the QRW lattice size $N$ to be even with $n_0 = 0$. It then follows from Table I that the site occupation probability for our choice of the initial state is nonzero on site 0 only at even time steps. Consequently, we must choose the i.i.d. random variables $\tau_\alpha$ as even numbers, as otherwise any projective measurement on the instantaneous state would yield null result. For our purpose, we make for the $\tau_\alpha$ distribution $p_r$ two representative choices, namely, that of a discrete exponential distribution

\[ p_r = r(1-r)^{\tau-1}; \quad \tau = 2,4,6,\ldots; \quad 0 < r < 1, \quad (30) \]

and that of a power-law distribution

\[ p_r = \frac{2s}{\zeta(s)\tau^s}; \quad \tau = 2,4,6,\ldots; \quad s > 1, \quad (31) \]

where $\zeta(s) \equiv \sum_{n=1}^{\infty} 1/n^s$ is the Riemann zeta function. Both the aforementioned distributions satisfy the normalization $\sum_{\tau=2,4,6,\ldots} p_r = 1$. The exponential distribution (30) has all its moments, and in particular, the mean finite for all values of the parameter $r$. By contrast, the power-law distribution (31) has a finite mean only for $s > 2$. Since any reasonable experimental setup would allow measurements to be performed at random time intervals that have a finite average, as argued above, we will in this work consider only values of $s$ larger than 2.

In Fig. 5, we show in case of the evolution under **Scheme 1** our numerical results on the average survival probability $\overline{S_m}$ when averaged over typically hundreds of realizations $\{\tau_\alpha\}_{1\leq\alpha\leq m}$, and the survival probability $S_m$ obtained in a typical realization of the $\tau_\alpha$’s, both plotted as a function the $m$, the number of measurements. In the figure, panel (a) corresponds to $\tau_\alpha$’s distributed according to the exponential distribution (30) with $r = 0.5$, while panels (b) and (c) are for the power-law distribution (31) with $s$ having values 2.5 and 3.5, respectively. The panels suggest for both the quantities $\overline{S_m}$ and $S_m$ an exponential decay with $m$ for large $m$.

In case of **Scheme 2** dynamics, Fig. 6 shows our numerical results on the average survival probability $\overline{S_m}$ and the typical survival probability $S_m^*$ for the exponential distribution (30) with $r = 0.5$ (panels (a) and (d)), and for the power-law distribution (31) with $s$ = 2.5 (panels (b) and (e)) and $s = 3.5$ (panels (c) and (f)). These plots suggest that independent of the form of $p_r$, both the probabilities decay asymptotically as a power-law in $m$ with exponent $3/2$. On the basis of the foregoing, we see a stark contrast between our obtained results on the survival probability under the two choices of the dynamics. It is our objective in the following to offer an analytical treatment of these results.

2. Analytical results

In **Scheme 1**, we obtain $|\psi_m^{(a)}\rangle$ for arbitrary $m$ as

\[ |\psi_m^{(a)}\rangle = PU_m \ldots PU_3 PU_2 PU_1 |\psi(0)\rangle. \quad (32) \]

From Eqs. (28), (27), and (32), we see that $S_m$ may be expressed as

\[ S_m = \prod_{\alpha=1}^{m} q(\tau_\alpha), \quad (33) \]

where the (quantum) probability $q(\tau_\alpha)$ is given by

\[ q(\tau_\alpha) \equiv |\langle \psi(0)| U_\alpha |\psi(0)\rangle|^2 = \frac{1}{|a|^2 + |b|^2} |a^\dagger \Psi_a(n_0, \tau_\alpha) + b^\dagger \Psi_d(n_0, \tau_\alpha)|^2, \quad (34) \]

with $\Psi_a(n_0, \tau_\alpha)$ and $\Psi_d(n_0, \tau_\alpha)$ given by Eqs. (12) and (13). Note that $q(\tau)$ is nothing but the probability to be found in the initial state $|\psi(0)\rangle$ after evolution for time $\tau$. Using Eq. (33), one obtains the average survival probability as

\[ \overline{S_m} = \prod_{\tau} \left( \sum_{\tau} p_\tau q(\tau) \right) = \exp \left( m \log \sum_{\tau} p_\tau q(\tau) \right). \quad (36) \]

As shown in Appendix A, the most probable value of the survival probability $S_m^*$ is obtained as

\[ S_m^* = \exp \left( m \sum_{\tau} p_\tau \log q(\tau) \right). \quad (37) \]

Using the Jensen’s inequality $\exp(x) \geq \exp(\overline{x})$, we obtain that

\[ \overline{S_m} \geq S_m^*, \quad (38) \]

with the equality holding only when there is no randomness in $\tau$, that is, when $\tau$ can take on only a single value: $p_\tau = \delta_{\tau, \tau_0}$, with $\tau_0 > 0$. On performing a large number $m$ of projective measurements, the value of the survival probability to remain in the initial state that is measured in a single experimental run will equal $S_m^*$ in the limit $m \to \infty$. On the other hand, averaging the survival probability over a large (ideally infinite) number of experimental runs would yield the value $\overline{S_m}$.

For the case $p_\tau = \delta_{\tau, \tau_0}$, we get $\overline{S_m} = S_m^* = \exp(m \log q(\tau_0))$, and $\mathcal{F} = m \tau_0$. In the limit $\tau_0 \to 0$, $m \to \infty$ with $\mathcal{F}$ kept constant (frequent measurements at close intervals), using Eq. (34) and the fact that $|\psi(0)\rangle$ is normalized to unity, it then follows that $S_m^* = S_m^* = 1 - m \theta(\tau_0^2)$. This result implies that to leading order, there is no evolution of the initial state, an illustration of the quantum Zeno effect [17].

The theoretical results are compared against those obtained in numerical implementation of the **Scheme 1** dynamics in Fig. 5: In this figure, the continuous lines in red obtained from Eq. (36) show the behavior of $\overline{S_m}$, while the blue-dashed lines obtained from Eq. (37) depict that of $S_m^*$. We see from the figure a very good match between theoretical and numerical results for the average survival probability, whereas the
In this scheme, to find an analytical closed form of the typical survival probability \( S \) given by Eqs. (36), (37), and (35). The implementation of the dynamics; while the average survival probability corresponds to results obtained in a typical realization of the \( \tau_\alpha \)'s. The lines in the plots correspond to analytical results given by Eqs. (36), (37), and (35).

In Scheme 2, we obtain \( |\psi_m^{(a)}\rangle \) for arbitrary \( m \) as

\[
|\psi_m^{(a)}\rangle = PU_m \ldots PU_3PU_2PU_1 |\psi(0)\rangle. \tag{39}
\]

This scheme, to find an analytical closed form of \( S_m \) and \( S_m^\star \) is nontrivial, as we explain below. Consequently, we rely on a semi-analytical approach that involves implementation of the following four steps:

1. For a given choice of \( a, b, \) and \( n_0 \) specifying the initial state \( |\psi(0)\rangle \) in Eq. (3), we first obtain the vector \( |\Psi(n,0)\rangle \) by using Eqs. (5) and (7). Then, we implement the discrete Fourier transform given by

\[
|\tilde{\Psi}(k,0)\rangle = \sum_n |\Psi(n,0)\rangle e^{i2\pi kn/N}. \tag{40}
\]

2. Subsequently, we use Eq. (9) to obtain \( |\tilde{\Psi}(k,\tau_1)\rangle \) as the result of dynamical evolution for a random time \( \tau_1 \) sampled according to either the exponential distribution (30) or the power-law distribution (31), and with \( |\tilde{\Psi}(k,0)\rangle \) as the initial condition. At the end of the evolution, one has the set \( \{ |\tilde{\Psi}(k,\tau_1)\rangle \} \).

3. Next, we perform inverse discrete Fourier transform of the set \( \{ |\tilde{\Psi}(k,\tau_1)\rangle \} \) to obtain the set \( \{ |\Psi(n,\tau_1)\rangle \} \). To implement a projective measurement at the end of evolution for time \( \tau_1 \) and obtaining the corresponding left-over component of the state, we first obtain the state \( |\psi(n,\tau_1)\rangle \) by using the obtained values of the elements of \( |\Psi(n,\tau_1)\rangle \) in Eq. (5), and then compute the difference \( |\psi(\tau_1)\rangle - |\psi(0)\rangle \langle \psi(0)| \psi(\tau_1)\rangle \), which yields the state \( |\psi_{1}^{(a)}\rangle \).

4. Steps 1–3 are applied in turn to the leftover component of the state corresponding to last projection; \( m \geq 1 \) number of repetitions would generate the leftover component of the instantaneous state after \( m \) projections, and this allows us to obtain the survival probability \( S_m \) for a given realization \( \{ \tau_\alpha \} \leq m \) of the dynamics.

The method is semi-analytical in the sense that while the dynamical evolution in the Fourier space follows the exact solution (9), inverse Fourier transform to obtain \( |\psi(n, \tau)\rangle \) is performed numerically. The semi-analytical results for the average and the typical survival probability are compared in Fig. 6 against numerical results, demonstrating a very good match.

We remark that in Scheme 1, each time a projective measurement is made, the dynamical evolution starts afresh from the initial state (indeed, the state for subsequent dynamical evolution after say the first measurement is \( \langle \psi(0)|\psi(\tau_1)\rangle \langle \psi(0)| \psi(0)\rangle \)). Consequently, dynamical evolution over the different time intervals \( \tau_\alpha \) are completely uncorrelated, and this results in the survival probability \( S_m \) being a product of the quantum probabilities \( q(\tau_\alpha) \) over different times \( \tau_\alpha \), see Eq. (33). On the contrary, in Scheme 2, the left-over component of the instantaneous state after say the \( \alpha \)-th measurement (given by \( |\psi(\tau_\alpha)\rangle - |\psi(0)\rangle \langle \psi(0)| \psi(\tau_\alpha)\rangle \)) depends in an essential way on the time \( \tau_\alpha \) of evolution of the leftover component following the \( (\alpha - 1) \)-th measurement. It then follows that the dynamical evolution over the different \( \tau_\alpha \)'s are strongly correlated, and this defines the survival probability \( S_m \) to be written as a product of probabilities for different \( \tau_\alpha \)'s and does not allow a straightforward analytical estimate of the average and the typical survival probability for Scheme 2. It is this presence of correlation that manifests in the average and the typical survival probability to have a power-law decay in Scheme 2, while the absence of it gives rise to an exponential decay of these quantities in Scheme 1.
III. TIGHT-BINDING MODEL (TBM)

A. Model and site-occupation probability

The tight-binding model (TBM) involves quantum evolution of a particle on a lattice that we consider here to be of \( N \) sites with periodic boundary conditions. The dynamics in continuous time is generated by the Hamiltonian \([4, 5]\)

\[
H = -\gamma \sum_{j=0}^{N-1} (|j+1\rangle \langle j| + |j\rangle \langle j+1|); \quad |N\rangle = |0\rangle. \tag{40}
\]

Here, \( \gamma > 0 \) is a real parameter, while the index \( j \) denotes the lattice sites. Let \( \psi_{n,n_0}(t); \quad n = 0,1,2,\ldots,N-1 \) be the probability amplitude to find the particle on site \( n \) at time \( t \) while starting from site \( n_0 \) at time \( t = 0 \), with the normalization \( \sum_{n=0}^{N-1} |\psi_{n,n_0}(t)|^2 = 1 \ \forall \ t \). From the evolution equation \( |\psi(t+\Delta t)|^2 = \exp(-iH\Delta t)|\psi(t)|^2; \quad |\psi(0)| = |n_0\rangle \), one obtains the time evolution of \( \psi_{n,n_0}(t) \) in a small time interval \( \Delta t \) as

\[
\psi_{n,n_0}(t+\Delta t) = \sum_{j=0}^{N-1} \langle j| e^{-iH\Delta t} |j\rangle \psi_{j,n_0}(t), \tag{41}
\]

where we have \( \psi_{j,n_0}(t) = \langle j|\psi(t)\rangle \). Expanding in powers of \( \Delta t \) the right hand side of the above equation and then taking the limit of continuous time, \( \Delta t \to 0 \), one obtains the evolution

\[
\frac{\partial \psi_{n,n_0}(t)}{\partial t} = -i\gamma (\delta_{j,j+1} + \delta_{j,j-1}), \tag{42}
\]

yielding

\[
\frac{\partial \psi_{n,n_0}(t)}{\partial t} = i\gamma (\psi_{n+1,n_0}(t) + \psi_{n-1,n_0}(t)). \tag{42}
\]

In order to solve Eq. (42) for \( \psi_{n,n_0}(t) \), we perform discrete Fourier transform of the set \( \{\psi_{n,n_0}(t)\}_{0 \leq n \leq N-1} \), given by the set \( \{\tilde{\psi}_{q,n_0}(t)\}_{0 \leq q \leq N-1} \), with \( \tilde{\psi}_{q,n_0}(t) = \sum_{n=0}^{N-1} \psi_{n,n_0}(t) \text{exp}(-i2\pi qt/N) \); \( \psi_{j,n_0}(t) = (1/N) \sum_{q=0}^{N-1} \tilde{\psi}_{q,n_0}(t) \text{exp}(i2\pi jqt/N) \). From Eq. (42), one then
obtains
\[
\frac{\partial \hat{\psi}_{j_m}(t)}{\partial t} = 2i\gamma \cos \left( \frac{2\pi q}{N} \hat{\psi}_{j_m}(t) \right). \tag{43}
\]

Subject to the initial condition \( \hat{\psi}_{n_0}(0) = \delta_{n,n_0} \) implying \( \hat{\psi}_{j_m}(0) = \exp(-i2\pi n_0q/N) \), Eq. (43) has the solution \( \hat{\psi}_{j_m}(t) = \exp(i2\gamma \cos(2\pi q/N) - i2\pi n_0q/N) \), inverting which yields
\[
\hat{\psi}_{j,n_0}(t) = \frac{1}{N} \sum_{q=0}^{N-1} e^{2i\pi \cos(2\pi q/N) + i2\pi q(j-m)/N}. \tag{44}
\]

It is easily checked that \( \sum_{j=0}^{N-1} |\hat{\psi}_{j,n_0}(t)|^2 = 1 \), as required. In particular, starting with the particle on site \( n_0 \), we may ask for the probability \( P_n(t) \) to be on site \( n \) at time \( t \), the so-called site occupation probability. It is given by
\[
P_n(t) = |\hat{\psi}_{n,n_0}(t)|^2 = \left| \frac{1}{N} \sum_{q=0}^{N-1} e^{2i\pi \cos(2\pi q/N) + i2\pi q(n-n_0)/N} \right|^2. \tag{45}
\]

In Fig. 7, we show a comparison between numerical results and theory, demonstrating a perfect match. In the figure, numerics correspond to bare evolution of the unitary dynamics of the model, while analytical results are given by Eq. (45). The line is a guide to the eye.

![Site occupation probability](image)

**FIG. 7:** Site occupation probability \( P_n(t) \) in the TBM on a one-dimensional periodic lattice of \( N \) sites and at time \( t = 10 \), while starting from initial site \( n_0 = 0 \). Here, we have \( N = 50, \gamma = 1 \). Numerics correspond to bare evolution of the unitary dynamics of the model, while analytical results are given by Eq. (45). The line is a guide to the eye.

### B. Random projective measurements and survival probability

#### 1. Numerical results

Here, we report numerical results on the average and the typical survival probability for the TBM subject to projective measurements at random times. A typical time evolution of the system is shown in Fig. 4. The initial state corresponds to the particle located on site \( n_0 \) (thus, \( |\psi(0)\rangle = |n_0\rangle \)), and as in the case of the QRW reported above, we consider the projective measurement to be involving projection to the initial state \( n_0 \) implemented by the projection operator \( P = |n_0\rangle\langle n_0| \). From Eq. (45), we get the quantity \( q(\tau) \) as nothing but the probability to be on site \( n_0 \) after time \( \tau \) while starting from the same site:
\[
q(\tau) = |\hat{\psi}_{n_0,n_0}(\tau)|^2, \tag{46}
\]

with the latter obtained from Eq. (45). Here, the random variable \( \tau \) between two successive measurements, now a continuous variable, is taken to be distributed according to either an exponential distribution:
\[
p_{\text{exp}} = r \exp(-r\tau); \quad \tau \in [0,\infty), \quad r > 0, \tag{47}
\]
or a power-law distribution:
\[
p_{\text{pow}}(\tau) = \frac{\alpha}{\tau_{\text{ch}}^{\alpha+1}}; \quad \tau \in [\tau_{\text{ch}},\infty), \quad \alpha > 0. \tag{48}
\]

In Eq. (47), the parameter \( r \) equals the inverse of average \( \tau \), while in Eq. (48), the parameter \( \tau_{\text{ch}} \) sets the lower cut-off scale. The exponential distribution (47) has a finite mean for all values of \( r \). On the other hand, the power-law distribution (48) has a finite mean only for \( \alpha > 2 \), and since we would like in the view of discussions presented in Section II C to have the \( \tau_{\text{ch}} \)'s to have a finite average, we will in the following consider values of \( \alpha \) to be larger than 2.

Figure 8 shows in case of the evolution under **Scheme 1** our numerical results on the average survival probability \( S_m \) and the typical survival probability \( S_m^\star \), both plotted as a function of \( m \), the number of measurements. In the figure, panel (a) corresponds to the exponential distribution (47) for the \( \tau \) with \( r = 2 \), while panels (b) and (c) are for the power-law distribution (48) with \( \alpha \) having values 2.5 and 3.5, respectively. The panels suggest for both the quantities \( S_m \) and \( S_m^\star \) an exponential decay with \( m \) for large \( m \).

As regards **Scheme 2** dynamics, Fig. 9 shows numerical results for the average survival probability \( S_m \) and the typical survival probability \( S_m^\star \) for the exponential distribution (panels (a) and (d)), and for the power-law distribution with \( s = 2.5 \) (panels (b) and (e)) and \( s = 3.5 \) (panels (c) and (f)). From the plots, we see that irrespective of the distribution \( p(\tau) \), both the probabilities decay asymptotically as a power-law in \( m \) with exponent 3/2. Similar to the results for the QRW reported in Section II C, we see a stark contrast in the behavior of the survival probability under **Scheme 1** and **Scheme 2** of the measurement dynamics.

#### 2. Analytical results

For **Scheme 1**, on using the continuous-\( \tau \) equivalent of Eqs. (36) and (37), one may obtain the average and the typical survival probability for the TBM subject to projective measurements at random times intervals \( \tau \) distributed according to the exponential and the power-law distribution, Eqs. (47) and (48), respectively. One has
FIG. 8: Average and typical survival probability for the TBM subject to instantaneous projective measurements at random times (Scheme 1). The plots correspond to the initial state $|0\rangle$ (particle located on site $n_0 = 0$) that is subject to repeated projective measurements at random times to the initial state and subsequent evolution with the projected component of the instantaneous state. Here, the time intervals $\tau_n$ between two consecutive measurements are i.i.d. random variables sampled from the exponential distribution (47) with $r = 2.0$ (panel (a)) and from the power-law distribution (48) with $\tau_{th} = 1$ and $\alpha = 2.5$ (panel (b)) and $\alpha = 3.5$ (panel (c)). The system size is $N = 200$, while we have taken $\gamma = 1$. In the plots, the points are based on results obtained from numerical implementation of the dynamics; while the average survival probability $S_m$ involves averaging $10^4$ realizations of the set $\{\tau_n\}_{1 \leq n \leq m}$, the typical survival probability $S^*_m$ corresponds to results obtained in a typical realization of the $\tau_n$’s. The lines in the plots correspond to analytical results given by Eqs. (49) and (46).

FIG. 9: Average survival probability (panels (a) – (c)) and typical survival probability (panels (d) – (f)) for the TBM subject to instantaneous projective measurements at random times (Scheme 2). The plots correspond to the initial state $|0\rangle$ (particle located on site $n_0 = 0$) that is subject to repeated projective measurements at random times to the initial state and subsequent evolution with the leftover component of the instantaneous state after the measurement. Here, the time intervals $\tau_n$ between two consecutive measurements are i.i.d. random variables sampled from the exponential distribution (47) with $r = 2$ (panels (a) and (d)), and from the power-law distribution (48) with $\tau_{th} = 1$ and $\alpha = 2.5$ (panels (b) and (e)) and $\alpha = 3.5$ (panels (c) and (f)). The system size is $N = 200$, while we have taken $\gamma = 1$. In the main plots, the points are based on results obtained from numerical implementation of the dynamics; while the average survival probability $S_m$ involves averaging over 25 realizations of the set $\{\tau_n\}_{1 \leq n \leq m}$, the typical survival probability $S^*_m$ corresponds to results obtained in a typical realization of the $\tau_n$’s. The lines in the main plots correspond to those obtained from the semi-analytical approach described in the text, see Section III B 2. In the insets in the upper row, the points correspond to numerically-evaluated average survival probability, while the line represents an $m^{-3/2}$ behavior. In the insets in the lower row, the continuous lines correspond to numerically-evaluated survival probability for five typical realizations of $\{\tau_n\}_{1 \leq n \leq m}$, while the dashed line represents an $m^{-3/2}$ behavior. We conclude from the insets in both the upper and the lower row that the average as well as the typical survival probability behaves at large $m$ as $m^{-3/2}$.
We see from the figure a very good match of the average and the typical survival probability results. The Zeno limit discussed for the QRW in Section II C 2 also holds for the TBM with the choice $p(\tau) = \delta(\tau - \tau_0)$.

For Scheme 2, a semi-analytic approach to obtain $|\psi_m^{(a)}\rangle$ along the one employed for the QRW and detailed in Section II C 2, involves the following steps:

1. For the initial state $|\psi(0)\rangle = |n_0\rangle$ so that $\psi_{n,n_0}(0) = \delta_{n,n_0}$, we have the discrete Fourier transform $\hat{\psi}_{q,n_0}(0) = \exp(-i2\pi q n_0/N)$ for $0 \leq q \leq N - 1$.

2. Subsequently, $\hat{\psi}_{q,n_0}(\tau_1)$, as the outcome of evolution according to (43) for a random time $\tau_1$ sampled according to either the exponential distribution (47) or the power-law distribution (48) and with $\hat{\psi}_{q,n_0}(0)$ as the initial condition, is obtained as

$$\hat{\psi}_{q,n_0}(\tau_1) = \hat{\psi}_{q,n_0}(0) e^{i2\gamma \tau_1 \cos(2\pi q/N)}.$$ (50)

3. Inverse discrete Fourier transform of the set $\{\hat{\psi}_{q,n_0}(\tau_1)\}_{0 \leq q \leq N - 1}$ yields the set $\{|\psi_{n,n_0}(\tau_1)\rangle\}_{0 \leq n \leq N - 1}$. The result of a projective measurement at the end of evolution for time $\tau_1$ to obtain the corresponding leftover component of the state is then given by the set $\{|\psi_{n,n_0}(\tau_1)\rangle\}_{0 \leq n \leq N - 1}$ with $|\psi_{n_0,n_0}(\tau_1)\rangle = 0$.

4. We apply steps 1–3 in turn to the leftover component of the state corresponding to last projection, to finally obtain the survival probability $S_m$ for a given realization $\{\tau_\alpha\}_{1 \leq \alpha \leq m}$ of the dynamics.

Figure 9 shows a very good agreement for both the average and the typical survival probability between the numerical results and those obtained based on the aforementioned semi-analytic approach.

**IV. CONCLUSIONS**

In this work, we studied the issue of what happens when a quantum system undergoing unitary evolution in time is subject to repeated projective measurements to the initial state at random times. We considered two distinct dynamical scenarios: Scheme 1, in which the evolution after every projective measurement continues with the projected component of the instantaneous state, and Scheme 2, in which the evolution continues with the leftover component of the instantaneous state after a measurement has been performed. We focused on a physical quantity of relevance, namely, the survival probability of the initial state after a certain number $m$ of measurements have been performed on the system. Based on results derived for two representative quantum systems, one, the quantum random walk, evolving in discrete time and the other, the tight-binding model, evolving in continuous time, we showed that in Scheme 1, both the average (averaged with respect to different realizations of the set of random time intervals $\{\tau_\alpha\}_{1 \leq \alpha \leq m}$ between successive measurements) and the typical survival probability (obtained in a typical realization of the set $\{\tau_\alpha\}_{1 \leq \alpha \leq m}$) decay as an exponential in $m$ for large $m$. One obtains by stark contrast a power-law decay as $m^{-3/2}$ of both the average and the typical survival probability under Scheme 2 of the dynamics. These results hold independently of the choice of the distribution of times $\tau_\alpha$. It would be interesting to extend our studies to the case of a many-body quantum system where additional dynamical timescales may interplay with the average time between successive measurements to dictate rich static and dynamical behavior.

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Appendix A: Derivation of Eq. (37) of the main text

In this appendix, we briefly discuss the large deviation (LD) formalism to obtain Eq. (37) of the main text, following Ref. [19]. To proceed, let us specialize to the case of $p_\tau$ being a $d$-dimensional Bernoulli distribution. In other words, we consider the situation in which $\tau$ takes on $d$ possible discrete values $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(d)}$, with corresponding probabilities $p^{(1)}, p^{(2)}, \ldots, p^{(d)}$ satisfying $\sum_{\alpha=1}^{d} p^{(\alpha)} = 1$. To invoke the LD formalism for the survival probability, consider the quantity

$$\mathcal{L} = \log (S_m(\tau^{(1)} \ldots \tau^{(d)}) = \sum_{\alpha=1}^{d} n^{(\alpha)} \log (p^{(\alpha)}),$$

where we have denoted by $n^{(\alpha)}$ the number of times the value $\tau^{(\alpha)}$ occurs in the sequence $\{\tau^{(\alpha)} \ldots \tau^{(d)}$. The quantity $\mathcal{L}$ is a sum of i.i.d. random variables, and its probability distribution is evidently given by (see Ref. [19])

$$\mathcal{P}(\mathcal{L}) = \frac{m!}{n_1! n_2! \ldots n_d!} \prod_{\alpha=1}^{d} (p^{(\alpha)})^{n^{(\alpha)},}$$

where the quantities $n^{(\alpha)}$ satisfy the two constraints $\sum_{\alpha=1}^{d} n^{(\alpha)} = d$ and $\sum_{\alpha=1}^{d} n^{(\alpha)} \log (p^{(\alpha)} = \mathcal{L}$, implying that one has

$$m \log (p^{(\alpha)} - \mathcal{L} = \sum_{\alpha=1}^{d} n^{(\alpha)} \lambda (\tau^{(\alpha)}), \text{ with } \lambda (\tau^{(\alpha)}) \equiv \log (p^{(\alpha)}) - \log (p^{(\alpha)}).$$

The solution is [19]

$$n^{(\alpha)} = \frac{m \log (\tau^{(\alpha)} - \mathcal{L}}{(d-1) \lambda (\tau^{(\alpha)})}, \quad \alpha = 1, 2, \ldots, d-1,$$

and $n^{(d)} = m - \sum_{\alpha=1}^{d-1} n^{(\alpha)}$. Using this solution in Eq. (A2), it may be shown that in the limit $m \to \infty$, the distribution $\mathcal{P}(\mathcal{L})$ has the following LD form [19]

$$\mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L}/m) \approx \exp (-m \mathcal{I}(\mathcal{L}/m),$$

with

$$I(\mathcal{L}) \equiv \sum_{\alpha=1}^{d} f^{(\alpha)} \log (f^{(\alpha)}) / p^{(\alpha)}),$$

$$f^{(\alpha)} = \log (q^{(\alpha)}) - x \lambda (\tau^{(\alpha)}); \quad \alpha = 1, \ldots, (d-1),$$

$$f^{(d)} = 1 - \sum_{\alpha=1}^{d-1} f^{(\alpha)}.$$}

From Eq. (A4), it follows that the minimum of the function $I(\mathcal{L}/m)$ corresponds to the most probable value $\mathcal{L}^*$ of $\mathcal{L}$ as $m \to \infty$. From Eq. (A5), the condition $\partial I(\mathcal{L}/m) / \partial \log (q^{(\alpha)}) |_{\mathcal{L}=\mathcal{L}^*} = 0$ gives, on performing a series of algebraic manipulations, that [19]

$$\mathcal{L}^* = m \sum_{\alpha=1}^{d} p^{(\alpha)} \log (q^{(\alpha)}).$$

Using Eqs. (A1) and (A4), one may obtain an LD form for the distribution of the survival probability $S_m$ as [19]

$$\mathcal{P}(S_m) \approx \exp (-m J(S_m)),$$

with $J(S_m) \equiv \min_{\mathcal{L}=\mathcal{L}^*} | \mathcal{L}/m |$. The value $S_m^*$ that minimizes the function $J(S_m)$ is the most probable value of the survival probability in the limit $m \to \infty$; one gets [19]

$$S_m^* = \exp \left( m \sum_{\alpha=1}^{d} p^{(\alpha)} \log (q^{(\alpha)}) \right).$$