The Exact Evolution Equation of the Curvature Perturbation for Closed Universe

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Abstract:

As is well known, the exact evolution equation of the curvature perturbation plays a very important role in investigation of the inflation power spectrum of the flat universe. However, its corresponding exact extension for the non-flat universes has not yet been given out clearly. The interest in the non-flat, specially closed, universes is being aroused recently. The need of this extension is pressing. We start with most elementary physical consideration and obtain finally this exact evolution equation of the curvature perturbation for the non-flat universes, as well as the evolutionary controlling parameter and the exact expression of the variable mass in this equation. We approximately do a primitive and immature analysis on the power spectrum of non-flat universes. This analysis shows that this exact evolution equation of the curvature perturbation for the non-flat universes is very complicated, and we need to do a lot of numerical and analytic work for this new equation in future in order to judge whether the universe is flat or closed by comparison between theories and observations.

Key words: exact evolution equation, curvature perturbation, closed universe.
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1 Introduction

In recent years the cosmology acquires a dazzling development. Unexpectedly, an elusive ingredient in the eyes of physicists, i.e., a tiny non-zero vacuum energy or so-called “dark energy”, turns out to be a main component of our present universe. Some evidence for the once recondite inflation has likely emerged now. A nice observed curve about the power spectrum of the Cosmic Microwave Background Radiation (CMBR) is actually fine consistent with the theoretical prediction[1]. The best model may be the slow-roll inflation of the flat universe. All seem faultless. A new era of the precision cosmology has been coming.
However, there is still some little bother. It seems the central value of the total density of the universe is slightly larger, $\Omega_T = 1.02 \pm 0.02$. The large angle power spectrum looks like rather low\cite{2}\cite{3}. Whether or not these clues come into real questions? Maybe these are only observed errors, the future experiments will tell us that the universe is indeed flat, and the low power of the larger angle scales comes only from imprecise data. However, no matter how precise our future data are, we shall probably never be able to rule out the non-flat universe models. A couple of years ago, when one didn’t find out enough matter density in the universe and didn’t believe the cosmological constant will play important role in it, the open universe models were once voguish. Recently, the possibility of the closed universe model is being reconsidered\cite{4}. Indeed, the idea of the closed universe model has a long history \cite{5}\cite{6}\cite{7}, some reasons seem attractive\cite{8}. Anyhow, no matter the universe is flat, open or closed, it is important for us that we must know how to calculate exactly its primary power spectrum, in order for us to do a rigour comparison between observations and theories.

As is well known, the inflation spectrum for the flat universe with single inflaton field has been understood very well, which is expressed by an evolution equation of the perturbation

$$\chi''_k + (k^2 - z_0''/z_0)\chi_k = 0, \quad z_0 = a\dot{\phi}/H,$$

as well as the curvature spectrum

$$P^R_k = \frac{k^3 |\chi_k|^2}{2\pi^2 z_0^2},$$

and forms a standard paradigm\cite{9}, where the variable mass $m_v^2 = -z_0''/z_0$ has an exact expression even for a no slow-roll case

$$z_0''/z_0 = a^2 H^2 (2 + 2\epsilon_1 - 3\epsilon_2 + 2\epsilon_2^2 - 4\epsilon_1\epsilon_2 + \epsilon_3).$$

All notations can be found later. The fundamentality of all these results is never overestimated in the analysis of the primary power spectrum.

However, for the non-flat case, especially for closed universe, the corresponding exact equations have not been written down clearly up to now, although some results are close to this goal nearly. Only when we know how to extend these equations to the non-flat case, and make theoretical predictions based on these new equations, we are able to judge whether the universe was flat, open or closed by observational confirmations.

The goal of this letter is to provide the exact evolution equation of the scalar perturbation for non-flat universes from some elementary physical consideration. We do further some primitive, certainly immature, approximative analysis based on our exact equations. This letter is organized as follows. In section 2, we shall start with the perturbative formalism of the Einstein equation to obtain the curvature perturbative evolution equation for non-flat universes. In the section 3, we try to
do an analogic slow-roll analysis for these equations, look into whether the slow-roll approximation is suitable. In the section 4, we want to do a brief conclusion. Since our result may lay a foundation of further analysis for non-flat universes in future, the correctness of our result is very key. So it is necessary to give out the middle derivation as detailed as possible in order for interested someone to check it conveniently. In this paper, we only concentrate on derivation of the main formal results. A detailed analysis of the perturbation spectrum in phenomenology will be the subject of a forthcoming work.

2 Perturbative evolution equation for non-flat universes

Our starting point is the standard effective action of the simple coupling system of Einstein gravity and single inflaton scalar field with an arbitrary inflation potential \( V(\phi) \),

\[
S = \int \sqrt{g} \left[ -\frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] d^4x. \tag{4}
\]

We shall take \( 8\pi G = M_{\text{pl}}^{-2} = 1 \). Now, supposing that inflaton field owns perturbation, we have

\[
\phi(t, \vec{x}) = \phi_0(t) + \delta\phi(t, \vec{x}). \tag{5}
\]

The perturbative metric in the longitudinal gauge, in terms of the Newtonian gravitational potential \( \Phi \), is

\[
ds^2 = [1 + 2\Phi(t, \vec{x})]dt^2 - [1 - 2\Phi(t, \vec{x})] \frac{a^2(t)}{(1 + K_c r^2/4)^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2). \tag{6}
\]

The background motion equations are

\[
H^2 + \frac{K_c}{a^2} = \frac{1}{3} \left( \frac{\dot{\phi}_0^2}{\phi_0^2} + V \right), \quad \dot{\phi}_0 + 3H \dot{\phi}_0 + V_{,\phi} = 0, \quad H \equiv \frac{\dot{a}}{a}, \tag{7}
\]

hereafter a “dot” denotes derivative with respect to the physical time, \( \dot{f} = df/dt \), and so on. The parameter \( K_c \) takes 1, 0 or \(-1\) for the closed, flat or open universe respectively.

From the standard literatures, we can get the Einstein tensor and the energy-momentum tensor in the perturbative form

\[
\delta G^0_i = (\dot{\Phi} + H\Phi)_i, \quad \delta G^0_0 = \frac{1}{a^2} \Delta \Phi - 3H \dot{\Phi} + 3 \left( \frac{K_c}{a^2} - H^2 \right) \Phi, \tag{8}
\]

\[
\delta T^0_0 = \dot{\phi}_0^2 \frac{\delta \phi}{\phi_0}, \quad \delta T^0_i = \dot{\phi}_0^2 \left( \frac{\delta \phi}{\phi_0} \right)_i - \Phi - 3H \dot{\phi}_0^2 \frac{\delta \phi}{\phi_0}. \tag{9}
\]

Thus the linearized Einstein equation \( \delta G^j_i = 4\pi G \delta T^j_i \) for the component of 0i is,

\[
(a\Phi)^* = \frac{1}{2} a\dot{\phi}_0^2 \left( \frac{\delta \phi}{\phi_0} \right). \tag{10}
\]
Another Einstein equation for the component of 00, having used (10), is

\[ \left( \frac{\delta \phi}{\phi_0} \right) \cdot (1 + \frac{2(\Delta + 3K_c)}{a^2 \phi_0^2}) \Phi. \]  

(11)

Both equations have been given in Ref.[10].

The motion equation of \( \Phi \) is simple. In order to express this equation in a more convenient way for later use we introduce a new variable \( S \equiv a \Phi \). Using (10) and (11) we have an equation with regard to \( \Phi \) in fact

\[ \ddot{S} = \left[ \frac{1}{2} \frac{\dot{\phi}_0^2}{\phi_0} + a^{-2}(\Delta + 3K_c) \right] S + \frac{d}{dt} \left[ \ln(a \dot{\phi}_0) \right] \cdot \dot{S}. \]  

(12)

However, what we need really is the motion equation about the intrinsic curvature perturbation \( R \) of comoving hypersurfaces, which can expressed as, seeing in Ref.[11],

\[ R = \Phi + H \frac{\delta \phi}{\phi_0}. \]  

(13)

In order to gain the exact equation about \( R \) in a clear way, we need to define the rolling parameters, as we emphasize, which are not only valid for slow-roll case, but also for no slow-roll case

\[ \eta \equiv a^{-2}H^{-2}, \quad \epsilon_1 \equiv \frac{\dot{\phi}_0^2}{2H^2}, \quad \epsilon_2 \equiv -\frac{\dot{\phi}_0}{H \phi_0}, \quad \epsilon_3 \equiv \frac{\phi_0^{(3)}}{H^2 \phi_0}, \quad \epsilon_4 \equiv \frac{\phi_0^{(4)}}{H^3 \phi_0}. \]  

(14)

The background motion equations (7) can be rewritten as

\[ H^2 = \frac{V}{3} (1 + K_c \eta - \frac{1}{3} \epsilon_1)^{-1}, \quad \dot{\phi}_0 = -\frac{V \phi}{3H} (1 - \frac{1}{3} \epsilon_2)^{-1}, \]  

(15)

which formalism is particularly suitable for parameter expansion if the rolling parameters are small, in order to be used later. During the progress of getting the equation about \( R \), we shall meet many derivatives of these various rolling parameters with respect to time, which should be listed here, seeing Ref.[12],

\[ \dot{H} = H^2 (K_c \eta - \epsilon_1), \quad \dot{\eta} = -2H \eta (1 + K_c \eta - \epsilon_1), \quad \dot{\epsilon}_1 = 2H \epsilon_1 (\epsilon_1 - \epsilon_2 - K_c \eta), \]  

(16)

\[ \dot{\epsilon}_2 = -H (\epsilon_3 - \epsilon_1 \epsilon_2 - \epsilon_2^2 + K_c \epsilon_2 \eta), \quad \dot{\epsilon}_3 = H [\epsilon_4 + \epsilon_3 (2 \epsilon_1 + \epsilon_2 - 2K_c \eta)]. \]  

(17)

Then the Eq.(12) can be rewritten as

\[ \ddot{S} = H^2 [\epsilon_1 + \eta (-k^2 + 4K_c)] S + H (1 - 2 \epsilon_2) \dot{S}, \]  

(18)

where we have use the eigenfunction expansion for some fluctuation \( F(t, \vec{x}) \)

\[ F(t, \vec{x}) = \sum_{klm} F_{klm}(t) Q_{kl}(r) Y_{lm}(\theta, \phi), \]  

(19)

\[ (\Delta + k^2 - K_c) Q_{kl} = 0. \]  

(20)
All definitions here can be found in Ref.[13]. Strictly speaking, \( \bar{k} \equiv \sqrt{k^2 - K_c} \) is just the dimensionless comoving wavenumber of a fluctuation. Hereafter our notations \( S \) and \( R \) mean their expanded amplitudes \( S_k(t) \) and \( R_k(t) \), since the isotropy implies independent on the indices \( l \) and \( m \).

Using Eq.(10), the equation of curvature perturbation, i.e. Eq.(13), is recast as
\[
R = a^{-1} S + (aH\epsilon_1)^{-1}\dot{S}.
\]
(21)
Doing derivative of it and using (18), we get
\[
\dot{R} = \eta(a\epsilon_1)^{-1}[-(k^2 - 4K_c)HS + K_c\dot{S}] .
\]
(22)
Solve \( R \) and \( \dot{R} \) from above two equations, we have
\[
S = a\epsilon_1(k^2 - 4K_c + K_c\epsilon_1)^{-1}[K_c R - aH\dot{R}],
\]
(23)
\[
\dot{S} = aH\epsilon_1(k^2 - 4K_c + K_c\epsilon_1)^{-1}[(k^2 - 4K_c)R + aH\epsilon_1\dot{R}].
\]
(24)
We can continue to apply derivative to (22), again using (18), yield
\[
\ddot{R} = (a^3 \epsilon_1)^{-1}\{[K_c\epsilon_1 + (k^2 - 4K_c)(3 + \epsilon_1 - 2\epsilon_2 - 2K_c\eta)]S - H^{-1}(k^2 - 2K_c)\dot{S}\}. 
\]
(25)
In the following we want to use the conformal time \( \tau = \int dt/a \), and hereafter a “prime” denotes derivative with respect to the conformal time, \( f' = df/d\tau \), and so on. We have a transformation concerned \( R \)
\[
\dot{R} = a^{-1} R', \quad \ddot{R} = a^{-2} R'' - a^{-1}H R'.
\]
(26)
Substituting Eqs.(23) and (24) into the right hand side of Eq.(25), and changing to the conformal time, we establish
\[
R'' = -k^2 R + C_0 R + C_1 R',
\]
(27)
where the coefficients are obtained finally after some calculation
\[
C_0 = 5K_c + 2K_c(\epsilon_1 - \epsilon_2 - K_c\eta)[1 + K_c\epsilon_1/(k^2 - 4K_c)]^{-1},
\]
(28)
\[
C_1 = -2aH\{K_c\epsilon_1 + (1 + \epsilon_1 - \epsilon_2 - K_c\eta)[1 + K_c\epsilon_1/(k^2 - 4K_c)]^{-1}\}. 
\]
(29)
The equation (27) has the first order term \( R' \), we hope to cancel it by introducing a new variable \( \chi_k \),
\[
\chi_k \equiv zR.
\]
(30)
Thus equation (27) becomes
\[
\chi''_k + k^2\chi_k - C_0\chi_k - z^{-1}z''\chi_k = 0,
\]
(31)
if we take
\[
C_1 = -2z^{-1}z'.
\]
(32)
Eq. (31) is a standard equation without the damping term and with an equivalent varying mass term $m_v^2 = -C_0 - z''/z$. Solving $z$ from (32) and (29), we get

$$z = -a\sqrt{2c_1[1 + K_c\epsilon_1/(k^2 - 4K_c)]}^{-1}.$$  \hfill (33)

This is important parameter which controls evolution of the scalar perturbation. Its first and second derivatives with respect to the conformal time are given by

$$z'/aHz = K_c\epsilon_1 + (1 + \epsilon_1 - \epsilon_2 - K_c\eta)[1 + K_c\epsilon_1/(k^2 - 4K_c)]^{-1}, \hfill (34)$$

$$z''/z = a^2H^2(k^2 - 4K_c + K_c\epsilon_1)^{-2}[4^4(2 + 2\epsilon_1 - 3\epsilon_2 + 2\epsilon_1^2 - 4\epsilon_1\epsilon_2 + \epsilon_3) + K_c\epsilon_1(-16 - 12\epsilon_1 + 24\epsilon_2 - 15\epsilon_1^2 + 29\epsilon_1\epsilon_2 - 8\epsilon_3 - \epsilon_3^3 + 2\epsilon_1^2\epsilon_2 - 3\epsilon_1\epsilon_2^2 + \epsilon_1\epsilon_3) + K_c\epsilon_2(32 + 16\epsilon_1 - 48\epsilon_2 + 30\epsilon_1^2 - 52\epsilon_1\epsilon_2 + 16\epsilon_3 + 3\epsilon_3^3 - 8\epsilon_1^2\epsilon_2 + 12\epsilon_1\epsilon_2^2 - 4\epsilon_1\epsilon_3) + K_c\eta(-68\epsilon_1 - 4k^4\epsilon_1 + 32\epsilon_2 + 2k^4\epsilon_2 - 7\epsilon_1^2 + 16\epsilon_1\epsilon_2) + K_c\epsilon_1^2\eta^2(-16 - \epsilon_1) + K_c\epsilon_2^2\eta^2(33\epsilon_1 - 16\epsilon_2 + 2\epsilon_1^2 - 4\epsilon_1\epsilon_2) + K_c\epsilon_2^2\eta^2(32 + 2k^4 + 4\epsilon_1)]. \hfill (35)$$

The equation (27) or (31) becomes finally

$$\chi_k'' + [k^2 - 3K_c - 2K_cz'/aHz - z''/z]\chi_k = 0,$$  \hfill (36)

with an equivalent varying mass $[-3K_c - 2K_cz'/aHz - z''/z]^{1/2}$. Its spectrum, referring [14] and [15], is

$$P_k^R = \frac{k(k^2 - K_c)[|\chi_k|^2]}{2\pi^2} z^2.$$  \hfill (37)

If the universe is closed, the parameter $k$, almost to be dimensionless comoving wavenumber, takes integer value $3, 4, \ldots$, to infinity. It is notable that the results gotten until now is exact, especially for (34) and (35), i.e., there is no any approximation to be taken. Of course a precondition of such statement is to recognize the validity of the first order approximation for the perturbative Einstein equation. If $K_c = 0$, all return to standard result for flat case, i.e., (1), (2) and (3), specially $z \rightarrow z_0$. Eqs.(33)-(36) are our main results, which are the foundation for further analysis. In fact, these results were able to be obtained almost by Refs.[15] and [16], if they would like to finish their last steps. However, we adopt a rather different method to obtain the exact evolution equation of the comoving curvature perturbation for non-flat case, this increases the reliability of the results. The exact evolution equation of the tensor perturbation for non-flat universe will be found in a forthcoming work.

3 A try to treat new evolution equation

The evolution equation obtained by us for non-flat case is obviously more complicated than flat one. New parameter $\eta$ appears in an unmanageable way. How to treat it brings us a new challenge. What is a boundary condition on this equation? How to define the vacuum for this quantized system? Noted that $\eta = a^{-2}H^{-2} \simeq \tau^2$, the coefficient of $\chi_k$ term, i.e., variable mass square, not only contains a single term
of $\tau^{-2}$ like the flat case, but also contains other terms which is very complicate function of the conformal time $\tau$ and even various $\epsilon$ varied slowly. It is difficult to apply the slow-roll method in this case. We must think out some better methods to treat it. However, before finding out better outlet, we still hope to do a comparison if we father upon so-called slow-roll approximation on it. In such way we can appraise how the slow-roll approximation is partly invalid.

Let us introduce the potential parameters which are very suitable for slow-roll approximation, see Ref.[12],

$$u_1 \equiv (V_{,\phi}/V)^2, \quad v_1 \equiv V_{,\phi\phi}/V, \quad v_2 \equiv V_{,\phi\phi\phi}/V^2,$$

(38)

to express the rolling parameters mentioned in (14),

$$\epsilon_1 \simeq \frac{1}{2} u_1 - \frac{1}{3} u_1^2 + \frac{7}{3} u_1 v_1 + \frac{14}{9} K_c u_1 \eta + \frac{4}{9} u_1^3 - \frac{5}{6} u_1^2 v_1 + \frac{5}{18} u_1 v_1 + \frac{1}{9} u_1 v_2 - \frac{67}{27} K_c u_1^2 \eta + \frac{53}{27} K_c u_1 v_1 \eta + \frac{190}{81} K_c^2 u_1^2 \eta^2,$$

(39)

$$\epsilon_2 \simeq \frac{1}{2} u_1 - v_1 - \frac{5}{3} K_c \eta - \frac{2}{3} u_1^2 + \frac{4}{3} u_1 v_1 - \frac{1}{3} v_1^2 - \frac{1}{3} v_2 + 3 K_c u_1 \eta - \frac{20}{9} K_c v_1 \eta - \frac{22}{27} K_c^2 \eta^2 + \frac{3}{9} u_1^3 - 4 u_1^2 v_1 + \frac{23}{6} u_1 v_1^2 - \frac{2}{9} v_1^3 + 16 v_1 u_2 - \frac{4}{3} v_1 v_2 - \frac{1}{9} v_3 - \frac{167}{27} K_c u_1^2 \eta + \frac{50}{9} K_c u_1 v_1 \eta - \frac{7}{9} K_c v_1 \eta^2 + \frac{366}{27} K_c^2 u_1 \eta^2 + \frac{49}{81} K_c^2 v_1 \eta^2 + \frac{8}{81} K_c^3 \eta^3,,$$

(40)

the terms with $K_c$ are our new results. Then we have, up to the second order,

$$z'(aHz) \simeq 1 + v_1 + \frac{2}{3} K_c \eta - \left(\frac{1}{2} K_c u_1 v_1 + \frac{1}{3} K_c^2 u_1 \eta \right)/(k^2 - 4 K_c) + \frac{1}{9} (3 u_1^2 - 9 u_1 v_1 + 3 v_1^2 + 3 v_2 - 13 K_c u_1 \eta + 20 K_c v_1 \eta + \frac{22}{3} K_c^2 \eta^2),$$

(42)

$$z''/z \simeq a^2 H^2 \left[ (2 + 7 K_c \eta - \frac{1}{2} u_1 + 3 v_1) - \left(\frac{3}{2} K_c u_1 v_1 + 2 K_c^2 u_1 \eta \right)/(k^2 - 4 K_c) + \frac{1}{18} (33 u_1^2 - 69 u_1 v_1 + 36 v_1^2 + 36 v_2 - 142 K_c u_1 \eta + 150 K_c v_1 \eta + 64 K_c^2 \eta^2) \right].$$

(43)

Moreover for the conformal time we have an approximation if the rolling parameters are small

$$\tau \simeq -a^{-1} H^{-1} \left[ 1 + (\epsilon_1 - \frac{1}{3} K_c \eta) + (3 \epsilon_1^2 - 2 \epsilon_1 \epsilon_2 - \frac{4}{3} K_c \epsilon_1 \eta + \frac{1}{3} K_c^2 \eta^2) \right],$$

(44)

this result can be checked by $d\tau/dt = a^{-1}$ up to the appropriate orders by using Eqs.(16) and (17). If we take the slow-roll approximation, we have

$$\tau = -\frac{1 + \mu}{aH} \simeq -\frac{1}{aH} (1 + \frac{1}{2} u_1 - \frac{1}{12} u_1^2 + \frac{4}{3} u_1 v_1 - \frac{1}{3} K_c \eta + \frac{23}{9} K_c u_1 \eta),$$

(45)
then we get
\[
z''/z \simeq \tau^{-2}[2 + \frac{17}{3}K_c\eta + \frac{3}{2}u_1 + 3v_1] - \left(\frac{3}{2}K_cu_1v_1 + 2K_c^2u_1\eta\right)/(k^2 - 4K_c) \\
+ \frac{3}{2}u_1^2 + 9u_1v_1 + 2v_1^2 + 2v_2 + 9K_cu_1\eta + \frac{19}{3}K_cv_1\eta - \frac{4}{45}K_c^2\eta^2].
\] (46)

The equation (36) is approximated as
\[
\chi''_k + \left\{k^2 - \frac{32}{3}K_c + K_c \cdot O(u_1, v_i) - \frac{1}{\tau^2}[2 + \frac{3}{2}u_1 + 3v_1 + \frac{3}{2}u_1^2 + 9\frac{2}{2}u_1v_1 + 2v_1^2 \\
+ 2v_2 - \frac{3K_cu_1v_1}{2(k^2 - 4K_c)}] + K_c \cdot (\tau^2 \text{ or higher terms})\right\} \chi_k = 0.
\] (47)

Notice that we shall omit the \(O(u_1, v_i)\) terms in the term of \(\tau^0\) order, and omit the higher terms such as \(\tau^2\), since they should be small in the limit case \(|\tau| \to 0\). Thus we obtained an equation very similar with flat case. Since we omitted the \(\tau^2\) and higher terms, it is obvious that we are not able to do analysis on the limit case of \(|\tau| \to \infty\). We then see that an Bessel asymptotic analysis loses its validity.

If a Bessel analysis is forcibly adopted for the equation (47), then we have the spectrum
\[
P_k^R = f(k) \frac{H^4}{4\pi^2\delta^2} \cdot 2^{2\beta-3} \frac{\Gamma^2(\beta)}{\Gamma^2(3/2)}(1 + \mu)^{1-2\beta},
\] (48)
where
\[
f(k) = k(k^2 - K_c)(k^2 - \frac{32}{3}K_c)^{-3/2}[1 + K_c\epsilon_1/(k^2 - 4K_c)],
\] (49)
and
\[
\beta = \frac{3}{2} + \frac{1}{2}u_1 + v_1 + \frac{5}{12}u_1^2 + \frac{7}{6}u_1v_1 + \frac{1}{3}v_1^2 + \frac{2}{3}v_2 - \frac{1}{2}K_cu_1v_1/(k^2 - 4K_c).
\] (50)

It is unnecessary for us to go further, not due to mathematical difficulty, but since the evolution is not slow-roll in the small \(k\) region, where is of greatest interest for us. In other words, we don’t reassure our approximation, in spite of it is very efficient for flat case. However, these equations still give us some elicitation. We have seen that there will be some change in the forthcoming formulæ of the spectral index and its running, which will be different from the flat case. Even if the universe is slight curved, the slow-rolling is still a good approximation for the power spectrum of the small angle part.

We must note the relationship between the dimensionless comoving wavenumber \(k\) and the multipole \(l\) in CMBR spectrum if the universe is closed. If the horizon of our universe is about the present Hubble distance \(H_0^{-1}\), the length corresponded to the multipole \(l\) is about \(\pi H_0^{-1}/l\), therefore this length should be equal to the physical half-wavelength \(\pi a_0/k\) corresponded to wavenumber \(k\), thus we have \(k \sim a_0H_0l\), where \(a_0\) is the present cosmic scale factor with length dimensional. If the total density \(\Omega_T\) is about 1.01, then \(a_0H_0 \simeq 10\). For the quadrupole fluctuation of \(l = 2\) in CMBR, which has the real cosmological signification, the wavenumber \(k\) is about
20. Therefore the enhanced effect of the factor $f(k)$ for spectrum is not too large. The leading behavior of the spectrum is controlled by the parameter $z \sim z_0$,

$$k^3 \chi^2 \sim \eta^{-1}, \quad P^R_k \sim z^{-2} \eta^{-1} \sim H^4/\dot{\phi}_0^2,$$  \tag{51}

i.e., the factor $H^4/\dot{\phi}_0^2$ in Eq.(48) governs the spectrum, just the same as the flat case. This key feature has been captured by Ref.[17], specially seeing their figure 15. They think this can explain the low power in small multipole $l$ for CMBR spectrum (figure 20 in [17]).

If the universe is born from nothing to a finite size one through a tunnelling effect, as described by a model of Ref.[18], at this beginning the universe has an equilibrium of $K_c a^{-2} = V/3$ from Eq.(7), we then have a asymptotic behavior near the time starting point $t \to 0$

$$H \sim \dot{\phi} \sim t, \quad H^4/\dot{\phi}^2 \sim t^2.$$  \tag{52}

The quantity $H^4/\dot{\phi}^2$ will increase from zero up to its maximum at some time point, keeping this high value for a period with a very slowly decreasing, i.e., an approximated de Sitter phase, then fall off to end its inflation. Before this time point the spectrum is suddenly low for small $k$, i.e., for a sky large angle or low multipole $l$, looks like a cut-off. It seems possible that the closed universe can explain the large angle lack of the CMBR spectrum and the little up-departure of the cosmic total density from the exact flat case. Anyhow we must remember that the result (47)-(50) obtained by us is only a try. In order to obtain an assured consequence, it is necessary to investigate numerically on the exact evolution equation (33)-(37), which will be our further serious tasks.

4 Conclusion

We obtained an exact evolution equation of the scalar perturbation for non-flat universe by using a direct method. The exact evolution equation of the scalar perturbation for flat universe has been playing an important role in the getting a primordial inflation spectrum. Its non-flat extension will play a key role in future to analysis whether the universe is closed or flat. The idea of the closed universe is attractive. After all, this idea is the simplest one among the many ideas to introduce a new scale. We should start with our investigation from as simple idea as possible.

The exact evolution equation obtained by us is rather complicated. We have to do a try, in spite of which is immature, however, surely beneficial. It is impossible that the slow-roll approximation will lost wholly its efficiency even in non-flat case. Maybe some revelation can be drawn out from this rather simplified treatment.

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