ON VECTOR SOLUTIONS FOR COUPLED NONLINEAR SCHRÖDINGER EQUATIONS WITH CRITICAL EXPOIENTS

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Abstract. In this paper, we study the existence and asymptotic behavior of a solution with positive components (which we call a vector solution) for the coupled system of nonlinear Schrödinger equations with doubly critical exponents

\[
\begin{align*}
\Delta u + \lambda_1 u + \mu_1 u^{\frac{N+2}{N+2}} + \beta u^{\frac{2}{N}} v^{\frac{N}{N-2}} &= 0 \quad \text{in } \Omega, \\
\Delta v + \lambda_2 v + \mu_2 v^{\frac{N+2}{N+2}} + \beta u^{\frac{N}{N-2}} v^{\frac{N}{N-2}} &= 0
\end{align*}
\]

as the coupling coefficient \( \beta \in \mathbb{R} \) tends to 0 or \( +\infty \), where the domain \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is smooth bounded and certain conditions on \( \lambda_1, \lambda_2 > 0 \) and \( \mu_1, \mu_2 > 0 \) are imposed. This system naturally arises as a counterpart of the Brezis-Nirenberg problem (Comm. Pure Appl. Math. 36: 437-477, 1983).

1. Introduction. Recently, there has been considerable interest on the coupled system of time-independent nonlinear Schrödinger equations

\[
\begin{align*}
\Delta u + \lambda_1 u + \mu_1 u^3 + \beta uv^2 &= 0 \quad \text{in } \Omega, \\
\Delta v + \lambda_2 v + \mu_2 v^3 + \beta u^2 v &= 0
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N (N \leq 3) \) is the whole domain \( \mathbb{R}^N \) or a smooth bounded domain, \( \lambda_1, \lambda_2 < \lambda_2(\Omega) \) (\( \lambda_2(\Omega) \) is the first eigenvalue of the Dirichlet Laplacian \(-\Delta\) in \( \Omega \)), \( \mu_1, \mu_2 > 0 \) and the coupling coefficient \( \beta \in \mathbb{R} \) (see [16], [5], [18], [1], [22], [6], [24], [17], [21], [14], [4] and references therein). Its solution is a solitary wave for the coupled time-dependent nonlinear Schrödinger equations which appears in various branches of physics, especially from the Hartree-Fock theory for a binary mixture of Bose-Einstein condensates in the different hyperfine states [15] and an application of nonlinear optics to birefringent optical fibers [19].

One thing that makes it difficult to prove the existence of solutions with nonzero components, or vector solutions for the system (1) is the fact that it admits solutions of the form \((u,0)\) or \((0,v)\) with \(u,v \neq 0\), which we call semi-trivial solutions. Therefore, in order to obtain its vector solution, there should be means to distinguish the solution which we get from semi-trivial solutions.

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As far as the author know, the first attempt to obtain a vector solution for the coupled system (1), (2) was the work of Lin and Wei [16] under the assumption \( \Omega = \mathbb{R}^N \) \((N \leq 3)\) and \( \beta > 0 \) sufficiently small. Right after, subsequent works of [1], [5], [6], [18], [22] and others showed that, in particular, if \( \lambda_1 = \lambda_2 = -1 \) and \( \Omega = \mathbb{R}^N \) \((N \leq 3)\), then the following holds:

1. If \( 0 \leq \beta < \min\{\mu_1, \mu_2\} \), then a vector ground state exists, that is, there is a vector solution whose energy level is smallest among all vector solutions for the system.

2. If \( \min\{\mu_1, \mu_2\} \leq \beta \leq \max\{\mu_1, \mu_2\} \) and \( \mu_1 \neq \mu_2 \), then there is no vector solution of positive components.

3. If \( \beta > \max\{\mu_1, \mu_2\} \), then a vector ground state exists. In fact, it is a ground state (or a least energy solution), which means that it has the least energy among all nonzero solutions (containing semi-trivial solutions).

As we can expect, a similar result holds for any smooth bounded domain \( \Omega \). From this result, we can make a crude but helpful observation on the effect of the coupling coefficient \( \beta \) to overall structure of a coupled nonlinear Schrödinger equation. If \( \beta \) is near 0, we can regard the coupled system as a perturbation of two independent nonlinear Schrödinger equations, and, as a result, it becomes a natural approach to find a vector ground state for the system in a neighborhood of the family of pairs of ground states for each nonlinear Schrödinger equation. Moreover, we can expect that the energy of a vector ground state is approximately the sum of the energy of ground states for each nonlinear Schrödinger equation. On the other hand, if \( \beta \) is large enough, two equations in the coupled system are strongly tied by the coupling term and the system acts as if it is a scalar equation, thereby a vector ground state becoming a ground state. Our approach in dealing with the case \( \beta \) is near 0 or sufficiently large is based on this viewpoint.

The main objective of this paper is to extend the results for the equation (1) to the case when the nonlinearities are replaced by one of critical growth. In other words, we will study how to recover the lack of compactness of the inclusion map \( H^1(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega) \) for \( N \geq 3 \) and, hence, to obtain vector ground solutions of the doubly critical nonlinear Schrödinger system

\[
\begin{align*}
\Delta u + \lambda_1 u + \mu_1 u^{\frac{N+2}{2}} + \beta u^{\frac{N}{2}} v^{\frac{N}{N-2}} &= 0 \quad \text{in } \Omega \\
\Delta v + \lambda_2 v + \mu_2 v^{\frac{N+2}{2}} + \beta u^{\frac{N}{2}} v^{\frac{N}{N-2}} &= 0 \quad \text{in } \Omega \\
u, v > 0 \quad \text{in } \Omega, \quad u, v = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

(3)

where \( \Omega \) is a smooth and bounded domain. Furthermore, we will study not only existence of vector solutions, but their asymptotic behavior as \( \beta \to 0 \) or \(+\infty\), stressing on the relation of a vector solution to a least energy solution of the nonlinear Schrödinger equation

\[
\Delta w + \lambda w + \mu w^{\frac{N+2}{2}} = 0 \quad \text{in } \Omega
\]

(5)

\[
w > 0 \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial \Omega
\]

(6)

where \( \Omega \) is bounded and \( \lambda \in \mathbb{R} \).

For the equation (3), (4) with large \( \beta \) and \( \Omega \subset \mathbb{R}^N \), we will see that the existence of a ground state is relatively easily obtained when the order of nonlinearities are all cubic (corresponding to the dimension \( N = 4 \)), but showing the existence of a ground state turns out to be quite delicate if the order is not cubic \((N \geq 3, \ N \neq 4)\). It is because the algebraic structure of the equation with non-cubic nonlinearities
has less symmetry than cubic one. In this manner, it can be told that the coupled nonlinear Schrödinger equation (1), (2) with cubic nonlinearities, which has several physical meanings, also is relatively simpler than one with non-cubic nonlinearities.

The organization of this paper is as follows:

• In Section 2, we state our main results.
• In Section 3, we list several facts related to the equation (3) and its functional (7) that are analogs of results on the scalar equation (5) and (6).
• In Section 4, we consider the case that the coupling coefficient $\beta$ is large. In particular, we will see that a vector ground solution has the least energy among all possible solutions except the trivial solution $(u, v) = (0, 0)$, provided $\beta$ sufficiently large.
• In Section 5, we regard the problem as a regularly perturbed problem of two independent (scalar) nonlinear Schrödinger equations and prove the existence of vector solutions for sufficiently small $|\beta|$.
• Lastly, in Appendix A, we give a Brezis-Kato type estimate on a coupled nonlinear Schrödinger equation which is necessary in the proof of our main theorem.

Notations.

• $2^* = 2N/(N - 2)$ : the Sobolev critical exponent.
• $S$ : the Sobolev constant in $\mathbb{R}^N$, i.e.,

$$S := \inf_{w \in H^1_0(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{(\int_{\mathbb{R}^N} u^2)^{\frac{N}{2}}}$$

It is well-known that

$$S(\Omega) := \inf_{w \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{(\int_{\Omega} u^2)^{\frac{N}{2}}}$$

is equal to $S$ for any domain $\Omega \subset \mathbb{R}^n$.
• $u_+ = \max(u, 0)$, $u_- = \max(-u, 0)$.
• $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$, $\overline{\mathbb{R}}_+ = \{r \in \mathbb{R} : r \geq 0\}$.
• $\lambda_1(\Omega)$ : the first eigenvalue of Laplacian with Dirichlet boundary condition in $\Omega$.
• We write $p = p(q_1, \cdots, q_k)$ to denote that the parameter $p$ depends (only) on $q_1, \cdots, q_k$ ($k \in \mathbb{N}$).
• The small letter $c$ will be used to denote critical values of a functional, while the capital letter $C$ will appear as a constant which may vary from line to line.

2. Main results. We will seek a solution of (3) in the Sobolev space $(H^1_0(\Omega))^2$ endowed with the norm

$$\|(u, v)\| := \int_\Omega (|\nabla u|^2 + |\nabla v|^2).$$

Let $E_\beta$ denote the energy functional corresponding to (3)

$$E_\beta(u, v) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + |\nabla v|^2 - \lambda_1 u^2 - \lambda_2 v^2)$$

$$- \frac{N}{2N} \int_\Omega (\mu_1 |u|^{\frac{2N}{N-2}} + \mu_2 |v|^{\frac{2N}{N-2}}) - \beta \int_\Omega |uv|^{\frac{N}{N-2}} \quad (7)$$
for \((u, v) \in (H^1_0(\Omega))^2\). It is a \(C^1\)-functional in \((H^1_0(\Omega))^2\) and its critical point is a weak solution of (3) satisfying Dirichlet boundary condition.

**Remark 1.** 1. If the coupling term \(|uv|^{\frac{N}{N-2}}\) in (7) is replaced by \(|uv|^p\) with \(p \in (1, N/(N - 2))\), then the problem becomes easier since now \((u_n, v_n) \to (u_0, v_0)\) in \((H^1_0(\Omega))^2\) implies the strong convergence \(\beta \int_\Omega |u_n v_n|^p \to \beta \int_\Omega |u_0 v_0|^p\), along a subsequence. This paper will cover this case only when \(\beta\) is near 0 (see Theorem 2.3), and we recommend the interested reader to see the paper [23] which covers the case that \(p\) is subcritical and \(\beta\) is sufficiently large, given \(N = 4\).

2. In [3], the authors studied the existence of a ground state for a coupled elliptic system involving Hardy-type singular potential and critical power nonlinearities.

\[
\begin{aligned}
\Delta u + \lambda_1 \frac{u}{|x|^2} + u^{\frac{N+2}{N-2}} + \beta h(x) u^{\frac{2}{N-2}} v^{\frac{N}{N-2}} &= 0 \\
\Delta v + \lambda_2 \frac{v}{|x|^2} + v^{\frac{N+2}{N-2}} + \beta h(x) u^{\frac{2}{N-2}} v^{\frac{N}{N-2}} &= 0
\end{aligned}
\]

where \(N \geq 3, \lambda_1, \lambda_2 \in (-\frac{(N-2)^2}{4}, 0)\) and \(h\) is assumed to be a nonnegative, nonzero, bounded function vanishing at 0 and \(\infty\).

We state our main results dividing into two parts according to magnitude of the coupling coefficient \(\beta\). We will use the term that \((u, v)\) is positive (nonnegative, respectively) to mean that each component \(u\) and \(v\) is positive (nonnegative, respectively).

The first result deals with when the coupling coefficient \(\beta\) is sufficiently large. If \(N = 3\), let \(\lambda_*(\Omega)\) be a positive number such that the equation (5), (6) admits a least energy solution for all \(\lambda_*(\Omega) < \lambda < \lambda_2(\Omega)\). For instance, we can define \(\lambda_*(\Omega) = \lambda(B_{R_0})/4 = \frac{\pi^2}{(4R_0^2)}\) where

\[R_0 := \sup\{R > 0 : a \text{ ball } B_R \text{ of radius } R \text{ can be contained in } \Omega\}\]

(see Theorem 1.2’ in [10]).

**Theorem 2.1.** Suppose that \(0 < \mu_1 \leq \mu_2 \leq \beta_1 < \beta\) for a number \(\beta_1 = \beta_1(N, \mu_1, \mu_2)\) defined in Section 4 and \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain. If

\[
\begin{aligned}
0 < \lambda_1, \lambda_2 < \lambda_1(\Omega) \\
\lambda_*(\Omega) < \frac{\lambda_1 + \lambda_2 \theta^2}{1 + \theta^2} \quad \text{and} \quad \lambda_1, \lambda_2 < \lambda_2(\Omega)
\end{aligned}
\]

for \(N \geq 4\), and \(\lambda_1, \lambda_2 < \lambda_2(\Omega)\) for \(N = 3\), where \(\theta = \theta(\beta) > 0\) is a number converging to 1 as \(\beta \to +\infty\) determined in Proposition 4, then the system of equations (3) has a nonnegative least energy solution in \((C^2(\Omega))^2\) (nontrivial but possibly semi-trivial), which tends to \((0, 0)\) in \((H^1_0(\Omega))^2\) as \(\beta \to +\infty\). Moreover, there exists sufficiently large \(\beta_0 = \beta_0(N, \Omega, \lambda_1, \lambda_2, \mu_1, \mu_2)\) such that if \(\beta > \beta_0\), every least energy solution is in fact a vector solution.

**Remark 2.** 1. As we will see in the proof for Theorem 2.1, we can prove not only the existence of \(\beta_0\) and \(\beta_1\), but also give a concrete description for them. See Section 4.

2. If \(N = 3, \Omega\) is strictly star-shaped with respect to some point in \(\Omega\) and (3), (4) admits a solution, then \(\max(\lambda_1, \lambda_2) > \lambda_0(\Omega) > 0\) for some positive number \(\lambda_0(\Omega)\) depending on the domain. (See Theorem 1.2” in [10].)

If \(\beta\) is located near 0, regardless of its sign, we have the following result.
**Theorem 2.2.** Suppose that $\mu_1, \mu_2 > 0$, $0 < \lambda_1, \lambda_2 < \lambda_2(\Omega)$ if $N \geq 4$ or $\lambda_2(\Omega) < \lambda_1, \lambda_2 < \lambda_2(\Omega)$ if $N = 3$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. If $|\beta|$ is small enough, there is a solution $(u_\beta, v_\beta) \in (C^2(\Omega))^2$ of (3), (4), which converges to $(u_0, v_0) \in (C^2(\Omega))^2$ in $(H_0^1(\Omega))^2$ where $u_0$ is a positive ground state for
\[
\Delta w + \lambda_1 w + \mu_1 w^{\frac{N+2}{N-2}} = 0 \quad \text{in } \Omega
\]
and $v_0$ is a positive ground state for
\[
\Delta w + \lambda_2 w + \mu_2 w^{\frac{N+2}{N-2}} = 0 \quad \text{in } \Omega,
\]
respectively, as $\beta$ tends to 0 in $(H_0^1(\Omega))^2$.

Actually, we can also cover the case that $H$ is of subcritical growth when $\beta$ is near 0, as the following theorem indicates.

**Theorem 2.3.** Impose the assumption on the parameters $\mu_1, \mu_2, \lambda_1, \lambda_2$ and the domain $\Omega$ as in Theorem 2.2. Also, assume that a function $H \in C^{1,\alpha}(\mathbb{R} \times \mathbb{R})$ for some $\alpha \in (0, 1)$ satisfies
\[
H(u, v) = 0 \text{ if } u \leq 0 \text{ or } v \leq 0, \ H \text{ is not identically zero}
\]
and
\[
|H(u, v)|, |H_u(u, v)| \cdot |u|, |H_v(u, v)| \cdot |v| \leq C(1 + |u|^p + |v|^p)
\]
for some $p \in [1, 2^*)$ and $C > 0$. Then for $|\beta|$ small enough, there is a solution $(u_\beta, v_\beta) \in (C^2(\Omega))^2$ of
\[
\begin{align*}
\Delta u + \lambda_1 u + \mu_1 u^{\frac{N+2}{N-2}} + \beta H_u(u, v) &= 0 \quad \text{in } \Omega \\
\Delta v + \lambda_2 v + \mu_2 v^{\frac{N+2}{N-2}} + \beta H_v(u, v) &= 0
\end{align*}
\]
\[u, v > 0 \quad \text{in } \Omega, \quad u, v = 0 \quad \text{on } \partial \Omega
\]
which converges to $(u_0, v_0) \in (C^2(\Omega))^2$ in $(H_0^1(\Omega))^2$ where each $u_0$ and $v_0$ is a ground state for (9) and (10), respectively, as $\beta$ goes to 0 in $(H_0^1(\Omega))^2$. Here $H_u$ and $H_v$ denote partial derivatives of $H$ with respect to $u$ and $v$, respectively.

The proof of Theorem 2.2 and Theorem 2.3 are identical except only a small portion (Proposition 10).

**Remark 3.** While it is known that (1), (2) has no vector solution for a certain range of $\beta$ (see Proposition 2), we suspect that (3), (4) with $N \neq 4$ has a vector solution for all $\beta > 0$. It will be an interesting problem to find conditions for the coupled nonlinear Schrödinger equations or certain weakly coupled systems which guarantee for them to admit a vector solution.

3. **General results.** For the standard coupled nonlinear Schrödinger equations (1) with $N = 3$ or 4, the following results are readily checked. (See the former part of the proof of Theorem 2.1 in [14] or Page 217 in [22]. For more general type of non-existence results including Proposition 2 below, refer to [5].)

**Proposition 1.** (i) If $\beta > 0$ and there exists a positive solution for (1), then $\lambda_1, \lambda_2 < \lambda_2(\Omega)$.

(ii) If the condition $\lambda_1 = \lambda_2$ is also true, then the existence of a positive solution for (1) implies $\lambda_1 = \lambda_2 < \lambda_2(\Omega)$ for any $\beta > -\sqrt{\mu_1 \mu_2}$. 

Lemma 3.1. Suppose that \( \lambda_1 \leq \lambda_2, \mu_1 \leq \beta \leq \mu_2 \) and \( \mu_1 \neq \mu_2 \), then there is no positive solution for (1).

For coupled nonlinear Schrödinger equations with general nonlinearities, we can derive Pohozaev’s identity as in scalar nonlinear Schrödinger equations.

**Proposition 3 (Pohozaev’s identity).** Suppose that \( f, g \in C(\Omega), H \in C^1(\Omega), u, v \in L^\infty(\Omega) \) and \( \nabla u, \nabla v \in L^2(\Omega) \). Also, assume \( F(u), G(v) \in L^1(\Omega) \) for the primitives \( F \) and \( G \) for \( f \) and \( g \), respectively. If \( (u, v) \) is a solution of

\[
\begin{align*}
\Delta u + f(u) + H_u(u, v) = 0 \\
\Delta v + g(v) + H_v(u, v) = 0 \\
u, v = 0 \text{ on } \partial \Omega,
\end{align*}
\]

then it holds that

\[
\frac{N - 2}{2} \int_\Omega (|\nabla u|^2 + |\nabla v|^2) - N \int_\Omega (F(u) + G(v) + H(u, v)) + \frac{1}{2} \int_{\partial \Omega} \left( \left| \frac{\partial u}{\partial n} \right|^2 + \left| \frac{\partial v}{\partial n} \right|^2 \right) \nu \cdot xdS(x) = 0
\]

where \( \nu \) denotes the outward unit normal vector on \( \partial \Omega \) and \( dS \) the standard surface measure on \( \partial \Omega \).

**Proof.** Adapt the proof of Proposition 1 in [7]. \( \square \)

From Pohozaev’s identity, we get

**Corollary 1.** Suppose \( \Omega \subset \mathbb{R}^N, N \geq 3 \) is a smooth star-shaped domain. If \( \lambda_1, \lambda_2 \leq 0 \), then the only solution of (3) is the trivial solution.

Moreover, an analog of the Brezis-Lieb lemma [9] in the product space of \( L^p \)-space can be obtained.

**Lemma 3.1.** Suppose that \( \Omega \) is an open set in \( \mathbb{R}^N \) and \( (u_n, v_n) \) is a bounded sequence in \( (L^p(\Omega))^2 \) for \( 2 < p < \infty \). If \( a, b > 1, a + b = p \) and \( (u_n, v_n) \to (u, v) \) almost everywhere, then

\[
\int_\Omega |u_n - u|^a |v_n - v|^b = \int_\Omega |u_n|^a |v_n|^b - \int_\Omega |u|^a |v|^b + o(1).
\]

**Proof.** Fix \( \epsilon > 0 \). Then, by the mean value theorem and Young’s inequality, the following chain of inequalities holds for all \( s_1, s_2, t_1, t_2 \in \mathbb{R} \):

\[
\begin{align*}
|s_1 + s_2|^a |t_1 + t_2|^b - |s_1|^a |t_1|^b & \leq C_1 \epsilon^{1-p} \max_{0 \leq r \leq 1} (|s_1 + s_2|^{a-1} |t_1 + rt_2|^b + |s_1 + rs_2|^{a} |t_1 + rt_2|^{b-1}(|s_2| + |t_2|)) \\
& = C_1 \epsilon^{1-p} \max_{0 \leq r \leq 1} (|\epsilon(s_1 + rs_2)|^{a-1} |\epsilon(t_1 + rt_2)|^b + \cdots) (|s_2| + \cdots) \\
& \leq C_2 \epsilon^{1-p} \max_{0 \leq r \leq 1} \{ (|\epsilon(s_1 + rs_2)|^p + |\epsilon(t_1 + rt_2)|^p + |s_2|^p + \cdots) \} \\
& \leq C_3 \epsilon(|s_1|^p + |s_2|^p + |t_1|^p + |t_2|^p) + C_4 \epsilon^{1-p} (|s_2|^p + |t_2|^p)
\end{align*}
\]

where \( C_i > 0 (i = 1, 2, 3, 4) \) are constants. Therefore if we let \( w_n = u_n - u \) and \( z_n = v_n - v \),

\[
f^* := \left[ |u_n|^a |v_n|^b - |w_n|^a |z_n|^b - |u|^a |v|^b \right] - C_3 \epsilon (|w_n|^p + |z_n|^p + |u|^p + |v|^p)
\]

\[
\leq |u|^a |v|^b + C_4 \epsilon^{1-p} (|u|^p + |v|^p).
\]
Hence \( \int_{\Omega} f_n^+ \to 0 \) by the dominated convergence theorem and
\[
\int_{\Omega} \left( |u_{n1}|^a |v_{n1}|^b - |w_{n1}|^a |z_{n1}|^b - |u|^a |v|^b \right)
\leq \int_{\Omega} f_n^+ + \int_{\Omega} C_{\varepsilon} (|w_{n1}|^p + |z_{n1}|^p + |u|^p + |v|^p) \leq o(1) + eK
\] as \( n \to \infty \) for some constant \( K \) independent of \( n \) and \( \varepsilon \). Finally, take \( \varepsilon \to 0 \).

4. The case that the coupling coefficient \( \beta \) is near \( \infty \). In this section, we prove Theorem 2.1. In particular, to get a ground state of (3) with \( \beta \) large, we will apply the mountain pass theorem of Ambrosetti and Rabinowitz \cite{2}. The most intricate part is to recuperate the loss of compactness of the Sobolev embedding from \( (H^1_0(\Omega))^2 \) into \( (L^2(\Omega))^2 \) and it is main consideration of Proposition 4.

Before presenting Proposition 4, we derive the following elementary lemma.

**Lemma 4.1.** Fix \( 0 < \mu_1 \leq \mu_2 \) and let \( \beta > \mu_2 \). Also, define a function \( \xi_\beta : \mathbb{R}_+ \to \mathbb{R} \) by
\[
\xi_\beta(t) = \beta t \frac{\mu_1}{\mu_2} - \mu_2 t \frac{\mu_1}{\mu_2} - \beta t \frac{\mu_1}{\mu_2} + \mu_1.
\]
Then there is a constant \( \beta_2 = \beta_2(\mu_1, \mu_2) \geq \mu_2 \) such that
- if \( \beta > \beta_2 \) and \( N = 3 \), \( \xi_\beta \) has three zeros \( \theta_{1,\beta}, \theta_{2,\beta}, \theta_{3,\beta} > 0 \) such that \( \theta_{1,\beta} \to 0, \theta_{2,\beta} \to 1, \theta_{3,\beta} \to +\infty \) as \( \beta \to \infty \);
- if \( \beta > \beta_2 \) and \( N \geq 4 \), \( \xi_\beta \) has a unique zero \( \theta_\beta \) which converges to 1 as \( \beta \to \infty \).
Moreover, if \( N \geq 4 \), the lowest possible value for \( \beta_2 \) is the unique positive solution of
\[
\frac{4N \mu_1^2}{(N-2)^2 \beta} - \frac{2^{\frac{2N}{N-2}} \mu_2^{\frac{2N}{N-2}}}{(N-2)^{\frac{4N}{N-2}}} + (N-4) \beta = 0
\]
and in particular we can choose \( \beta_2 = \mu_2 \) if \( N = 4 \).

**Proof.** The lemma follows from direct computation. The description for \( \beta_2 \) for \( N \geq 4 \) comes from the observation that if \( \beta > \beta_2 \), then \( \xi_\beta'(t) > 0 \) for all \( t > 0 \).

**Proposition 4.** Suppose that \( \Omega \subset \mathbb{R}^N \) is smooth and bounded. Fix parameters \( 0 < \mu_1 \leq \mu_2 \leq \beta_2 < \beta \) (\( \beta_2 \) is defined in Lemma 4.1) and set
\[
c^* = c^*(\theta) := \frac{1}{N} \cdot \frac{(1 + \theta^2)^{\frac{\pi}{2}}}{(\mu_1 + \mu_2 \theta^{\frac{2N}{N-2}} + 2\beta \theta^{\frac{2N}{N-2}})^{\frac{N-2}{2}}} \cdot S^\frac{N}{2}
\]
with \( \theta > 0 \) equal to \( \theta_{2,\beta} \) if \( N = 3 \) and \( \theta_\beta \) if \( N \geq 4 \), a zero of \( \xi_\beta \) defined in Lemma 4.1. Then there is a constant \( \beta_1 = \beta_1(\mu_1, \mu_2, N) \geq \beta_2 \) such that if \( \beta > \beta_1 \), then any sequence \( (u_n, v_n) \in (H^1_0(\Omega))^2 \) such that \( E_\beta(u_n, v_n) \to c < c^* \) and \( E'_\beta(u_n, v_n) \to 0 \) has a convergent subsequence. Besides, \( \beta_1 \) can be chosen as
\[
\overline{\beta}_1 := \inf \left\{ \beta \geq \beta_0 : \left( \frac{\mu_1 + \mu_2 \theta^{\frac{2N}{N-2}}}{\mu_2} \right) \cdot \frac{N-2}{N} \cdot 2^{\frac{2}{N-2}} - (1 + \theta^2)^{\frac{2}{N-2}} \cdot \mu_2 > 0 \right\}
\]
if \( N \geq 4 \). In particular, we can take \( \beta_1 = \mu_2 \) if \( N = 4 \) (see Remark 4).

**Remark 4.1.** If \( N = 4 \), then (15) has a unique positive zero
\[
\theta = \theta_\beta = \sqrt{\frac{\beta - \mu_1}{\beta - \mu_2}}.
\]
which satisfies
\[
\frac{(1 + \theta^2)^2}{\mu_1 + \mu_2 \theta^2 + 2 \beta \theta^2} = \frac{2 \beta - \mu_1 - \mu_2}{\beta^2 - \mu_1 \mu_2}.
\]
By using these explicit formulae, we may check that $\beta_1$ can be taken as $\mu_2$.

2. If $N = 3$ or $N \geq 5$, the situation becomes involved. The main problem is that (15) may have more than one positive zero and even when it has a unique zero, we cannot solve it in an explicit form in general. Fortunately, thanks to Lemma 4.1, we know there is a zero of $\xi$ which tends to 1 as $\beta \to \infty$ and this information is enough to draw out the proposition.

As one might expect, when $N = 3$, numerical simulation suggests that the energy $c^*(\theta_{2,\beta})$ is less than both $c^*(\theta_{1,\beta})$ and $c^*(\theta_{3,\beta})$. On the other hand, although we can compute a possible value for $\beta_1$ if $N = 3$, it is too complicated to describe so we omit it here.

**Proof of Proposition 4.** The proof is a ramification of the argument which appeared in [10]. By the assumption, $(u_n, v_n)$ is bounded in $(H^1_0(\Omega))^2$, so we may assume $(u_n, v_n)$ converges to a $C^2$-solution $(u_0, v_0)$ of (3) weakly in $(H^1_0(\Omega))^2$ (see Appendix A).

Write $w_n = u_n - u_0$ and $z_n = v_n - v_0$. By using the fact $E(u_0, v_0) \geq 0$, the Brezis-Lieb lemma (refer to [9]) and Lemma 3.1, we obtain that
\[
\frac{1}{2} \int_\Omega (|\nabla w_n|^2 - \lambda_1 w_n^2 + |\nabla z_n|^2 - \lambda_2 z_n^2) - \frac{1}{2} \int_\Omega (\mu_1 |w_n|^2 + \mu_2 |z_n|^2) - \frac{2}{\beta} \int_\Omega \beta |w_n z_n|^2/2 \leq c + o(1).
\]

Here, $o(1)$ denotes any sequence which tends to vanish as $n \to \infty$. The Brezis-Lieb type lemmata also imply (Recall that $w_n, z_n \to 0$ in $L^2(\Omega)$.)
\[
\int_\Omega (|\nabla w_n|^2 + |\nabla z_n|^2) - \int_\Omega (\mu_1 |w_n|^2 + \mu_2 |z_n|^2) - 2\beta \int_\Omega |w_n z_n|^2/2 = o(1),
\]

hence we can assume that
\[
\int_\Omega (|\nabla w_n|^2 + |\nabla z_n|^2), \int_\Omega (\mu_1 |w_n|^2 + \mu_2 |z_n|^2) + 2\beta \int_\Omega |w_n z_n|^2/2 \to \tau \text{ as } n \to \infty,
\]

for some constant $\tau \geq 0$.

We now claim that if $\tau \neq 0$, then $\tau \geq Nc^*$. By (17), the Sobolev inequality and the Cauchy-Schwarz inequality, the following inequalities holds:
\[
\tau + o(1) \geq S \left[ \left( \int_\Omega w_n^2 \right)^{1/2} + \left( \int_\Omega z_n^2 \right)^{1/2} \right]
\]

and
\[
\tau + o(1) \leq \mu_1 \int_\Omega w_n^2 + \mu_2 \int_\Omega z_n^2 + 2\beta \left( \int_\Omega w_n^2 \right)^{1/2} \left( \int_\Omega z_n^2 \right)^{1/2}.
\]

If we stand for
\[
A_n = \left( \int_\Omega w_n^2 \right)^{1/2} \text{ and } B_n = \left( \int_\Omega z_n^2 \right)^{1/2},
\]

then
\[
A_n^2 = \frac{1}{2} \int_\Omega (|\nabla w_n|^2 - \lambda_1 w_n^2), B_n^2 = \frac{1}{2} \int_\Omega (|\nabla z_n|^2 - \lambda_2 z_n^2).
\]
(18) and (19) can be rewritten as
\[
\tau + o(1) \geq S \left[ \frac{2(N-2)}{N} A_n \right] + B_n^{2(N-2)} \quad \text{and} \quad \tau + o(1) \leq \mu_1 A_n^2 + \mu_2 B_n^2 + 2\beta A_n B_n.
\]
Hence, denoting limits of \((A_n)\) and \((B_n)\) as \(A\) and \(B\), respectively, it suffices to ascertain that
\[
\left(1 + \frac{\theta^2}{N} \right) \left( \mu_1 + \mu_2 \frac{2}{N} \right) + 2\beta A \frac{2(N-2)}{N} \leq A \frac{2(N-2)}{N} + B \frac{2(N-2)}{N}
\]
or equivalently
\[
(1 + \frac{\theta^2}{N} ) \left( \mu_1 + \mu_2 \frac{2}{N} + 2\beta A \right) \frac{2(N-2)}{N} \leq \left( \mu_1 + \frac{\beta N}{N} \right) \frac{2(N-2)}{N} \left( 1 + \frac{B}{A} \right) \frac{2(N-2)}{N}
\]
(20)

or equivalently
\[
(1 + \theta^2) \frac{2}{N} \left( \mu_1 + \mu_2 \frac{2}{N} + 2\beta A \right) \frac{2(N-2)}{N} \leq \left( \mu_1 + \frac{\beta N}{N} \right) \frac{2(N-2)}{N} \left( 1 + \frac{B}{A} \right) \frac{2(N-2)}{N}
\]
(21)

to derive \(\tau \geq N\epsilon^*\). (Here, the equivalence between (20) and (21) comes from the
\text{fact that } \theta \text{ is a zero of } \xi.) \text{ However, since it seems to be hard to extract any useful
information on } A \text{ and } B, \text{ we will instead show that a somewhat stronger result holds; that is, we will prove}
\[
(1 + \theta^2) \frac{2}{N} \left( \mu_1 + \mu_2 t^2 + 2\beta t \right) \frac{2(N-2)}{N} \leq \left( \mu_1 + \frac{\beta N}{N} \right) \frac{2(N-2)}{N} \left( 1 + \frac{B}{A} \right) \frac{2(N-2)}{N} \]
(22)
is true for every \(t \geq 0\), if \(\beta\) is large enough.

\textbf{Case I } \((N = 4)\): Under the case \(N = 4\), we know \(\theta\) explicitly as given by (16), and
from it, we can deduce that (22) holds for any \(\beta \geq \mu_2(\geq \mu_1)\).

\textbf{Case II } \((N = 3 \text{ or } N \geq 5)\): Suppose that \(N = 3\) in which case showing that a
function \(\chi_3 : \mathbb{R}_+ \to \mathbb{R}\) defined by
\[
\chi_3(t) = (\mu_1 + \beta t^2) \cdot (1 + t^2) - (1 + \theta^2)^2 \cdot (\mu_2 t^6 + 2\beta t^3 + \mu_1)
\]
is nonnegative is identical to checking (22) for every \(t > 0\). By using elementary calculus and that \(\theta = \theta_{2,\beta} = 1\) as \(\beta \to \infty\) (if \(\mu_1 = \mu_2\), then \(\theta = 1\) for any \(\beta\) large),
we obtain that \(\chi_3\) has a global minimum \(t_\beta\) which goes to 1 as \(\beta \to \infty\) (and one
local maximum in \(\mathbb{R}_+\)). In fact, we may get \(\chi_3(t_{2,\beta}) = 0\) and \(\chi_3''(t_{2,\beta}) > 0\) by
inspection, thus \(t_\beta = \theta_{2,\beta}\). From \(\chi_3(t_{2,\beta}) = 0\), we deduce that
\[
\chi_3(t_{2,\beta}) = \chi_3(t_{2,\beta}) = (1 + t_{2,\beta}^2)^2 (\mu_2 t_{2,\beta}^6 + 2\beta t_{2,\beta}^3 + \mu_1) = 0
\]
for sufficiently large \(\beta\) and this implies (22) for every \(t \geq 0\).

Similarly, if \(N \geq 5\) and \(\beta \geq \beta_1\), a function \(\chi_N : \mathbb{R}_+ \to \mathbb{R}\) defined by
\[
\chi_N(t) = \left( \mu_1 + \beta t^2 \right) \frac{2(N-2)}{N} \left( 1 + t^2 \right) \frac{2(N-2)}{N} \left( \frac{2(N-2)}{N} \right) \left( 2\beta t^{\frac{2(N-2)}{N}} + \mu_1 \right)
\]
has one global minimum at \(t_\beta = \beta\) with \(\chi_N(t_{2,\beta}) = 0\) (and no local maximum in
\(\mathbb{R}_+\)). Again (22) is true and this leads to the validity of our claim.

The conclusion now follows immediately. Suppose that \(\tau \neq 0\). Then
\[
c^* \leq \frac{\tau}{N}
\]
\[
eq \frac{1}{2} \int_\Omega (|\nabla w_n|^2 + |\nabla z_n|^2) - \frac{1}{2} \int_\Omega (\mu_1 |w_n|^2 + \mu_2 |z_n|^2 + 2\beta |w_n z_n|^{\frac{N}{2}}) + o(1)
\]
\[
\leq c + o(1) < c^*
\]
for sufficiently large \( n \), which is absurd. Thus \( \tau = 0 \) and we may extract a subsequence \((u_{nk}, v_{nk}) \to (u_0, v_0)\) strongly in \((H_0^1(\Omega))^2\).

The following proposition shows that the mountain pass value is smaller than \( c^* \). This enables to apply the mountain pass theorem.

**Proposition 5.** Suppose that \( \lambda_1, \lambda_2 > 0 \) if \( N \geq 4 \) or \( \frac{\lambda_1 + \lambda_2 \theta^2}{1 + \theta^2} > \lambda_*(\Omega) \) if \( N = 3 \) where \( \theta \) is the solution of the equation (15) which was determined in the proof of the previous proposition. Let

\[
c_\beta = \inf_{(u, v) \in (H_0^1(\Omega))^2 \setminus \{(0, 0)\}} \sup_{s \geq 0} E_\beta(s u, s v).
\]

Then we have \( c_\beta < c^* \).

**Proof.** Let \( W \) be the Talenti instanton, a positive solution of \( \Delta u + u^{2^*-1} = 0 \) in \( \mathbb{R}^N \). Its explicit form is given by

\[
W(x) = \left(\frac{N(N - 2)}{2(N - 1)}\right)^{\frac{N - 2}{4}} \left(1 + |x|^2\right)^{-\frac{N - 2}{4}}
\]

(refer to [20]).

**Case I** \((N \geq 4)\): Choose a nonnegative compactly supported smooth function \( \phi \in C_c^\infty(\Omega) \) such that \( 0 \leq \phi \leq 1 \) in \( \Omega \) and identically 1 in a small ball in \( \Omega \), and then define \( W_\epsilon(x) = \epsilon^{\frac{2-N}{2}} W(x/\epsilon) \) and \( w_\epsilon(x) = \phi(x) W_\epsilon(x) \) for \( \epsilon > 0 \). By Lemma 1.1 in [10], if \( N \geq 4 \), then we have

\[
\int_\Omega |\nabla w_\epsilon|^2 = S^2 + O(\epsilon^{N-2}), \quad \int_\Omega w_\epsilon^{2^*} = S^2 + O(\epsilon^N)
\]

and

\[
\int_\Omega w_\epsilon^2 = \begin{cases} 
C_\epsilon^2 + O(\epsilon^{N-2}) & \text{if } N \geq 5 \\
C_\epsilon^2 \log \epsilon + O(\epsilon^2) & \text{if } N = 4
\end{cases}
\]

where \( C > 0 \) is a constant. With this, we obtain

\[
c_\beta \leq \sup_{s \geq 0} E_\beta(s w_\epsilon, s \theta w_\epsilon)
\]

\[
= \frac{1}{N} \frac{\left[1 + \theta^2\right] \cdot \int_\Omega |\nabla w_\epsilon|^2 - (\lambda_1 + \lambda_2 \theta^2) \cdot \int_\Omega w_\epsilon^{2^*} S^2}{\left(\mu_1 + \mu_2 \theta^{\frac{N}{N-2}} + 2 \beta \theta^{\frac{N}{N-2}}\right) \cdot \left(\int_\Omega w_\epsilon^2\right)^{\frac{N-2}{4}}} < c^*.
\]

(23)

for sufficiently small \( \epsilon > 0 \).

**Case II** \((N = 3)\): We consider first when the domain is the open unit ball \( B_1 \). Set \( \phi(x) = \cos \left(\frac{\pi |x|}{2}\right) \) and \( w_\epsilon(x) = \phi(x) W_\epsilon(x) \) for any fixed \( \epsilon > 0 \). Then the following estimation can be obtained as in Lemma 1.3 in [10]:

\[
\int_{B_1} |\nabla w_\epsilon|^2 = S^2 + \frac{\sqrt{3}}{2} \pi^3 \epsilon + O(\epsilon^2), \quad \int_{B_1} w_\epsilon^6 = S^3 + O(\epsilon^2)
\]

and

\[
\int_{B_1} w_\epsilon^2 = 2\sqrt{3} \pi \epsilon + O(\epsilon^2).
\]
This leads to the desired relation
\[ c_{\beta}^2 \leq \frac{1}{9} \left[ (1 + \theta^2)S^2 + 2\sqrt{3}\pi \epsilon \left( (1 + \theta^2) \frac{x^2}{4} - (\lambda_1 + \lambda_2 \theta^2) \right) + O(\epsilon^2) \right]^3 \] for \( \epsilon > 0 \) small enough, whenever \( (\lambda_1 + \lambda_2 \theta^2) > \frac{\pi^2}{4} \cdot (1 + \theta^2) \) is true. Recall that \( \lambda_\beta(B_1) = \lambda_\beta(B_1)/4 = \pi^2/4 \).

For arbitrary domain \( \Omega \), applying the above argument, we can get \( c_{\beta} < c^* \) if \( \lambda_1 + \lambda_2 \theta^2 > \frac{\pi^2}{4} \) where \( R_0 \) is defined in (8). The proof of Proposition 5 is finished. \( \square \)

**Remark 5.** Let \( W \) and \( W_\epsilon \) as above. Then \((\eta W_\epsilon, \eta_\theta W_\epsilon)\) with \( \theta \) a positive zero of (15) and \( \eta = \left( \mu_1 + \beta \theta \frac{\pi}{\sqrt{N}} \right)^{-\frac{2}{N}} \) satisfies

\[ \begin{cases} \Delta u + \mu_1 u^{\frac{N+2}{N-2}} + \beta u \frac{\pi}{\sqrt{N}} v^{\frac{N}{N-2}} = 0 & \text{in } \mathbb{R}^N, \\ \Delta v + \mu_2 v^{\frac{N+2}{N-2}} + \beta u \frac{\pi}{\sqrt{N}} v^{\frac{N}{N-2}} = 0 & \text{in } \mathbb{R}^N, \end{cases} \]

\(<0 \text{ in } \mathbb{R}^N, \lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0. \)

We conclude the proof of Theorem 2.1.

**Proof of Theorem 2.1.** It is easy to check all hypotheses of the mountain pass theorem are satisfied with the assumption \( \lambda_1, \lambda_2 < \lambda_\beta(\Omega) \) and \( \beta > \beta_1 \). Therefore we obtain a nontrivial solution of equation (3), which is possibly semi-trivial. From the mountain pass geometry of \( E_\beta \), we can further deduce that the solution which we found is a ground state, in other words, a minimizer of the Nehari manifold

\[ \mathcal{N}_\beta = \left\{ (u, v) \in (H^1_0(\Omega))^2 \setminus \{(0, 0)\} : \int_\Omega \left( |\nabla u|^2 + |\nabla v|^2 - \lambda_1 u - \lambda_2 v \right) = \int_\Omega (\mu_1 |u|^2 + \mu_2 |v|^2) - 2\beta \int_\Omega |uv|^2 \right\}. \]

It is a standard procedure to show that the Nehari manifold is a natural constraint of \( E_\beta \) \(^1\) and that if \( (u, v) \) is the minimizer of \( E_\beta \) on \( \mathcal{N}_\beta \), then so is \((|u|, |v|)\). Thus we may assume that our ground state has nonnegative components.

On the other hand, we can find from (23) (or (24) if \( N = 3 \)) a constant \( \beta_0 \geq \beta_1 \) such that

\[ c_{\beta} < \min \left\{ \frac{1}{2} \int_\Omega (|\nabla w_i|^2 - \lambda_i w_i^2) - \frac{1}{2\pi} \int_\Omega w_i^2 : i = 1, 2 \right\} \]

for all \( \beta > \beta_0 \) where each \( w_i \) (\( i = 1, 2 \)) is a least energy solution for

\[ \Delta w_i + \lambda_i w_i + \mu_i w_i = 0 \quad \text{in } \Omega, \quad w_i = 0 \quad \text{on } \partial \Omega, \]

respectively, hence every vector ground state should be a ground state for such \( \beta \) (here we can write down an explicit form for \( \beta_0 \) by combining (23) and (26) if \( N \geq 4 \), and (24) and (26) if \( N = 3 \); see Remark 2). Moreover, the maximum principle implies that our ground state is positive and the condition \( \lambda_1, \lambda_2 < \lambda_\beta(\Omega) \) and the estimation due to (23) or (24)

\[ c_{\beta} = \frac{1}{N} \int_\Omega \left( |\nabla u|^2 + |\nabla v|^2 + \lambda_1 u^2 + \lambda_2 v^2 \right) \to 0 \]

\(^1\)This means that if \((u, v) \in \mathcal{N}_\beta \) satisfies \( E_\beta|_{\mathcal{N}_\beta} \) \((u, v) = 0\), it should be \( E_\beta^\prime(u, v) = 0\). (\( E_\beta|_{\mathcal{N}_\beta} \) denotes the restriction of \( E_\beta \) into \( \mathcal{N}_\beta \).)
Lastly, a ground state is of $X^{(3)}$ or $(13)$ in a neighborhood of Theorem 2.1. This enables us to pick $w$ for $(\tilde{s}, \tilde{v})$ with $\tilde{s} = s$, $\tilde{v} = v$. Therefore, if $|\beta|$ is sufficiently small, it is natural to search a solution of the modified equation near solutions of the original equation. Therefore, if $|\beta|$ is sufficiently small, it is natural to search a solution of (3) or (13) in a neighborhood of $X := S_1 \times S_2$.

To begin with, let

$$S_i = \{ w \in H^1_0(\Omega) : w \text{ is a positive ground state of } (5), (6) \text{ with } \lambda = \lambda_i \text{ and } \mu = \mu_i \}. $$

It is nonempty and compact in $H^1_0(\Omega)$ for each $i = 1, 2$ as shown in [10]. (See Section 1.10 of [25].) In general, if an equation is modified slightly, then one may expect that a solution of the modified equation lies near solutions of the original equation. Therefore, if $|\beta|$ is sufficiently small, it is natural to search a solution of (3) or (13) in a neighborhood of $X := S_1 \times S_2$.

Set $\tilde{\gamma}(s, t) = (\tilde{\gamma}_1(s), \tilde{\gamma}_2(t)) = (s\tilde{u}, \tilde{v})$ with $(\tilde{u}, \tilde{v})$ a fixed element in $X$ and define

$$E_i^S(w) = \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda_i u^2) - \frac{N - 2}{2N} \int_\Omega \mu_i^{\frac{2N}{N-2}} - \beta \int_\Omega H(u, v)$$

for $w \in H^1_0(\Omega)$ and $i = 1, 2$. Moreover, let $H(u, v) = (u_+ v_+)^{\frac{2N}{N-2}}$ when we consider (3) (namely, Theorem 2.2) so that it can be seen as a special case of (13).

If we denote the energy functional corresponding to (13) by $E_\beta$ again, that is, if we set

$$E_\beta(u, v) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + |\nabla v|^2 - \lambda_1 u^2 - \lambda_2 v^2)$$

for $(u, v) \in (H^1_0(\Omega))^2$, then we have

$$E_0(\tilde{\gamma}(s, t)) = E_i^S(\tilde{\gamma}_1(s)) + E_2^S(\tilde{\gamma}_2(t))$$

$$= \frac{s^2}{2} \int_\Omega (|\nabla \tilde{u}|^2 - \lambda_1 \tilde{u}^2) - \frac{s^2}{2} \int_\Omega \mu_i \tilde{u}^{2^*_s} + \frac{t^2}{2} \int_\Omega (|\nabla \tilde{v}|^2 - \lambda_2 \tilde{v}^2) - \frac{t^2}{2} \int_\Omega \mu_i \tilde{v}^{2^*_s}$$

$$= \left( \frac{s^2}{2} - \frac{s^2}{2^{*_s}} \right) \int_\Omega |\nabla \tilde{u}|^2 + \left( \frac{t^2}{2} - \frac{t^2}{2^{*_s}} \right) \int_\Omega |\nabla \tilde{v}|^2.$$  \hspace{1cm} (28)

This enables us to pick $s_0, t_0 > 1$ such that $E_0(\tilde{\gamma}(s, t)) < 0$ for $s \geq s_0$ and $t \geq t_0$.

We now fix a small number $\delta > 0$ which will be defined explicitly shortly, and choose $b > 0$ such that if $s, t \in (1 - b, 1 + b)$, then $\text{dist}(\tilde{\gamma}(s, t), X) \leq 4\delta$ and conversely if $\text{dist}(\tilde{\gamma}(s, t), X) \leq 2\delta$, then $s, t \in (1 - b, 1 + b)$.

Furthermore, for given $\delta, c > 0$, let us define

$$X^\delta = \left\{ (u, v) \in (H^1_0(\Omega))^2 : \text{dist}((u, v), X) \leq \delta \right\},$$

$$E_\beta^c = \left\{ (u, v) \in (H^1_0(\Omega))^2 : E_\beta(u, v) \leq c \right\},$$

5. The case that the coupling coefficient $\beta$ is near 0. We will prove Theorem 2.2 and Theorem 2.3 in this section, motivated by [12].
\[
\Phi = \begin{cases}
\gamma \in C \left([0, 1]^2 \setminus (H^1_0(\Omega))^2\right) : \\
\gamma(s, t) = \tilde{\gamma}(s, t) & \text{for } (s, t) \in [0, s_0] \times [0, t_0] \setminus (1 - b, 1 + b)^2 \\
\gamma(s, t) \in X^4 & \text{otherwise}
\end{cases}
\]

and minimax values

\[c_\beta = \inf \max_{r \in \Phi} E_\beta(\gamma(s, t))\]

and

\[d_\beta = \max_{(s, t) \in [0, s_0] \times [0, t_0]} E_\beta(\tilde{\gamma}(s, t)),\]

noting that \(\tilde{\gamma}(s, t) = (\tilde{\gamma}_1(s), \tilde{\gamma}_2(t)) \in \Phi\).

Finally, we denote \(E_{1_m} := E_1^{\beta}(\hat{u})\) and \(E_{2_m} := E_2^{\beta}(\hat{v})\) where \((\hat{u}, \hat{v}) \in X\), or equivalently, we define \(E_{i_m}\) as the energy level achieved by an element of \(S_i\) \((i = 1, 2)\).

We can now start the proof of Theorem 2.2 and Theorem 2.3.

As a first step, we select small \(\delta > 0\) which we shall use in the remainder of the proof in the next proposition. It follows from the argument presented in Lemma 1.2 of [9] and Lemma 1.44 of [25].

**Proposition 6.** Let \(\delta > 0\) be sufficiently small. There exist constants \(\omega > 0\) and \(\beta_0 > 0\) such that \(\|E'_{\beta}(u, v)\| \geq \omega\) for \((u, v) \in E_{\beta_0}^\delta \cap (X^\delta \setminus X^\delta/2)\) and \(\beta \in (0, \beta_0)\).

**Proof.** To the contrary, let us assume that there exist sequences \(\beta_n \to 0\) and \((u_n, v_n) \in E_{\beta_n}^\delta \cap (X^\delta \setminus X^\delta/2)\) such that \(\|E_{\beta_n}(u_n, v_n)\| \to 0\) as \(n \to \infty\). By norm-boundedness of the sequence \(\{(u_n, v_n)\}_{n=1}^\infty\), \((u_n, v_n)\) converges to \((u_0, v_0)\) weakly up to a subsequence in \((H^1_0(\Omega))^2\) as \(n \to \infty\), where \(u_0\) and \(v_0\) are nonnegative and solve equations (9) and (10) respectively. Moreover, we can make \(u_0, v_0\) nonzero and

\[\sup_{n \in \mathbb{N}} E_{1_m}^S(u_n) < \frac{S_{N, \bar{\mu}_1}}{N\bar{\mu}_1^2} \text{ and } \sup_{n \in \mathbb{N}} E_{2_m}^S(v_n) < \frac{S_{N, \bar{\mu}_2}}{N\bar{\mu}_2^2}\]

by picking \(\delta\) sufficiently small, since \(E_{i_m}^S(w_i) < S_{N, \bar{\mu}_i}/(N\bar{\mu}_i^2)\) for any \(w_i \in S_i\) \((i = 1, 2)\) by [10].

Set \(z_n = u_n - u_0\). Then by modifying the proof of the Brezis-Lieb lemma as in the proof of Lemma 3.1 above, one may show that

\[\int_{\Omega} (z_n)^2_+ = \int_{\Omega} (u_n)^2_+ - \int_{\Omega} (u_0)^2_+ + o(1).\]

This and \(E_{1_m}(u_0) \geq 0\) imply that

\[\frac{1}{2} \int_{\Omega} (|\nabla z_n|^2 - \lambda_1 z_n^2) - \frac{1}{2} \int_{\Omega} \mu_1 (z_n)^2_+ \leq c + o(1)\]

for some \(c < \frac{S_{N, \bar{\mu}_1}}{N\bar{\mu}_1^2}\) and again (29) and \(E'_{\beta_n}(u_n, v_n)(u_n, 0) \to 0\) gives us that

\[\int_{\Omega} (|\nabla z_n|^2 - \lambda_1 z_n^2) - \int_{\Omega} \mu_1 (z_n)^2_+ = o(1).\]

Hence we may assume that

\[\int_{\Omega} (|\nabla z_n|^2 - \lambda_1 z_n^2), \quad \int_{\Omega} \mu_1 (z_n)^2_+ \to \tau\]
for some nonnegative constant $\tau$, and by recalling the definition of the Sobolev constant $S$, we get $\tau = 0$ or $\tau \geq S^{\frac{N}{2}} / \mu_{1}^{\frac{N-2}{2}}$. However, if the latter holds,

$$\frac{S^{\frac{N}{2}}}{N \mu_{1}^{\frac{N-2}{2}}} \leq \frac{\tau}{N} = \frac{1}{2} \int_{\Omega} \left( (\nabla z_{n})^{2} - \lambda_{1} z_{n}^{2} \right) - \frac{1}{2^{*}} \int_{\Omega} \mu_{1}(z_{n})^{2^{*}} \leq c + o(1) < \frac{S^{\frac{N}{2}}}{N \mu_{1}^{\frac{N-2}{2}}}$$

a contradiction arises, so it must be $\tau = 0$, i.e., $u_{n}$ converges to $u_{0}$ strongly in $H_{0}^{1}(\Omega)$. Similarly, it can be shown that $v_{n}$ converges to $v_{0}$ strongly in $H_{0}^{1}(\Omega)$.

On the other hand, by using the facts that $E_{\beta_{n}}(u_{n}, v_{n}) \leq d_{\beta_{n}} \rightarrow E_{m}^{1} + E_{m}^{2}$, $E'_{\beta_{n}}(u_{n}, v_{n}) \rightarrow 0$ in $\left( H_{0}^{1}(\Omega) \right)^{2}$ and $(u_{n}, v_{n}) \rightarrow (u_{0}, v_{0})$ in $(H_{0}^{1}(\Omega))^{2}$ as $n \rightarrow \infty$, we can deduce that $(u_{0}, v_{0}) \in X$.

In summary, $(u_{n}, v_{n})$ converges to $(u_{0}, v_{0}) \in X$ strongly in $(H_{0}^{1}(\Omega))^{2}$ as $n \rightarrow \infty$, but it contradicts to $(u_{n}, v_{n}) \notin X^{3/2}$ so our assumption is false. Therefore the proposition holds.

**Proposition 7.** $\lim_{\beta \rightarrow 0} c_{\beta} = E_{m}^{1} + E_{m}^{2}$.

**Proof.** As a preliminary step, we first prove that

$$\max_{(s,t) \in [0,s_{0}] \times [0,t_{0}]} E_{0}(\gamma(s,t)) \geq E_{m}^{1} + E_{m}^{2}.$$  \hfill (30)

for every element $\gamma = (\gamma_{1}, \gamma_{2}) \in \Phi$. Choose a sufficiently small $\epsilon > 0$, and define a continuous map $\Gamma(\gamma) : [\epsilon, s_{0}] \times [\epsilon, t_{0}] \rightarrow \mathbb{R}^{2}$ as

$$\Gamma(\gamma)(s,t) = \left( \int_{\Omega} \left( |\nabla \gamma_{1}(s,t)|^{2} - \lambda_{1} \gamma_{1}(s,t)^{2} \right) - \int_{\Omega} \mu_{1} \gamma_{1}(s,t)^{2^{*}} \right),$$

$$\int_{\Omega} \left( |\nabla \gamma_{2}(s,t)|^{2} - \lambda_{2} \gamma_{2}(s,t)^{2} \right) - \int_{\Omega} \mu_{2} \gamma_{2}(s,t)^{2^{*}} \right).$$

(We added the argument $\gamma$ in $\Gamma(\gamma)$ to display dependence of $\Gamma$ upon $\gamma$.) Since $(0,0) \notin \Gamma(\gamma)(\partial([\epsilon, s_{0}] \times [\epsilon, t_{0}]))$, the Brouwer degree $\deg(\Gamma(\gamma)), [\epsilon, s_{0}] \times [\epsilon, t_{0}], (0,0))$ is well-defined and equal to $\deg(\Gamma(\gamma)), [\epsilon, s_{0}] \times [\epsilon, t_{0}], (0,0)) = 1$ where $\tilde{\gamma}$ is one defined in the fore part of this section. Therefore, there is an element $(s_{1}, t_{1}) \in [\epsilon, s_{0}] \times [\epsilon, t_{0}]$ such that $\Gamma(\gamma)(s_{1}, t_{1}) = (0,0)$, or the two dimensional surface $\gamma$ and the Nehari-type manifold

$$\tilde{N}_{0} = \left\{ (u,v) \in (H_{0}^{1}(\Omega))^{2}, u,v \neq 0 : \int_{\Omega}(|\nabla u|^{2} - \lambda_{1} u^{2}) - \int_{\Omega} \mu_{1} u^{2} = 0 \right\}$$

link. It is a standard procedure to check that $\tilde{N}_{0}$ is a natural constraint hence if $(u_{0}, v_{0})$ is a minimizer of $E_{0}$ in $\tilde{N}_{0}$, then it is a positive vector ground state of $(9)$ and $(10)$ with $E_{0}(u_{0}, v_{0}) = E_{m}^{1} + E_{m}^{2}$. This proves (30).

The conclusion now follows easily. First of all, since $\tilde{\gamma} \in \Phi$, it holds that $\limsup_{\beta \rightarrow 0} c_{\beta} \leq E_{m}^{1} + E_{m}^{2}$. If $\liminf_{\beta \rightarrow 0} c_{\beta} > E_{m}^{1} + E_{m}^{2}$, then there is $\epsilon > 0, \beta_{n} \rightarrow 0$ and $\gamma_{n} = (\gamma_{1,n}, \gamma_{2,n}) \in \Phi$ satisfying $E_{\beta_{n}}(\gamma_{n}(s,t)) \leq E_{m}^{1} + E_{m}^{2} - \epsilon$ for all $(s,t) \in [0,s_{0}] \times [0,t_{0}]$. However, we observe that (30) implies

$$\max_{(s,t) \in [0,s_{0}] \times [0,t_{0}]} E_{\beta_{n}}(\gamma_{n}(s,t)) \geq \frac{1}{2} \max_{(s,t) \in [0,s_{0}] \times [0,t_{0}]} E_{0}(\gamma_{n}(s,t)) - \frac{\epsilon}{2} \geq E_{m}^{1} + E_{m}^{2} - \frac{\epsilon}{2}$$

for sufficiently large $n$, which is inconsistent with the definition of $\gamma_{n}$. \hfill \Box

Notice that $c_{\beta} \leq d_{\beta}$ and by Proposition 7, $\lim_{\beta \rightarrow 0} c_{\beta} = \lim_{\beta \rightarrow 0} d_{\beta} = E_{m}^{1} + E_{m}^{2}$. 
**Proposition 8.** There is $\alpha > 0$ such that for sufficiently small $\beta > 0$, $E_\beta(\tilde{\gamma}(s,t)) \geq c_\beta - \alpha$ implies that $\tilde{\gamma}(s,t) \in X^{\delta/2}$.

**Proof.** Choose $\beta > 0$ small enough so that $|E_\beta(\tilde{\gamma}(s,t)) - E_0(\tilde{\gamma}(s,t))| < \alpha$ for all $(s,t) \in [0,s_0] \times [0,t_0]$ and $|c_\beta - (E_1^1 + E_2^2)| < \alpha$. By the hypothesis,

$$E_0(\tilde{\gamma}(s,t)) \geq c_\beta - 2\alpha \geq (E_1^1 + E_2^2) - 3\alpha. \quad (31)$$

Due to the shape of the graph of $t \mapsto t^2/2 - t^2/2^\ast$ (see (28)), in order to satisfy (31), $(s,t)$ should be contained in some neighborhood $(s_1, s_2) \times (t_1, t_2) \subset [0, s_0] \times [0, t_0]$ of $(1,1)$ with $|s_i - 1|, |t_i - 1| (i = 1, 2)$ sufficiently small and converge to 0 as $\alpha$ goes to 0. Thus we may choose $\alpha$ > 0 small enough to hold

$$\|\tilde{\gamma}(s,t) - \tilde{\gamma}(1,1)\| \leq \|\tilde{\gamma}_1(s) - \tilde{\gamma}_1(1)\| \mathcal{H}_1^1(\Omega) + \|\tilde{\gamma}_2(t) - \tilde{\gamma}_2(1)\| \mathcal{H}_2^1(\Omega) < \frac{\delta}{2}$$

However, $(\tilde{\gamma}_1(1), \tilde{\gamma}_2(1)) \in X$, so $\tilde{\gamma}(s,t)$ must be contained in $X^{\delta/2}$.

The following proposition assures the existence of a norm-bounded Palais-Smale sequence for $E_\beta$.

**Proposition 9.** For fixed sufficiently small $\beta > 0$, there exists $\{(u_n, v_n)\} \in E^d_\beta \cap X^\delta$ such that $E^d_\beta(u_n, v_n) \to 0$ as $n \to \infty$.

**Proof.** Suppose that there exists a positive constant $a = a(\beta)$ such that $|E^d_\beta(u, v)| \geq a$ for any $(u, v) \in E^d_\beta \cap X^\delta$.

Let $Q_\beta$ be a pseudo-gradient vector field of $E_\beta$ and $\varphi : (H^1_0(\Omega))^2 \to \mathbb{R}$ a smooth function such that $\varphi \equiv 1$ in $X^\delta$, supp$(\varphi) \subset X^{2\delta}$ and $0 \leq \varphi \leq 1$. Also, define $\Psi_\beta : (H^1_0(\Omega))^2 \times \mathbb{R} \to (H^1_0(\Omega))^2$ by the solution of the initial value problem

$$\begin{align*}
\frac{\partial}{\partial \tau} \Psi_\beta((u,v),\tau) &= -\varphi(\Psi_\beta((u,v),\tau))Q_\beta(\Psi_\beta((u,v),\tau)) \\
\Psi_\beta((u,v),0) &= (u,v)
\end{align*}$$

and

$$A_\beta(\tau) = \max_{(s,t) \in [0,s_0] \times [0,t_0]} E_\beta(\Psi_\beta((\tilde{\gamma}(s,t),\tau))).$$

By definition of $\Psi_\beta$, $A_\beta$ is non-increasing. As a matter of fact, we can show that

$$A_\beta(\tau_0) < c_\beta - \frac{\alpha}{4}$$

as follows. Since

$$A_\beta(0) = \max_{(s,t) \in [0,s_0] \times [0,t_0]} E_\beta(\tilde{\gamma}(s,t)) = d_\beta \geq c_\beta - \alpha,$$

by Proposition 8, $\tilde{\gamma}(s_0, t_0)$ attaining the maximum of $E_\beta$ in $[0, s_0] \times [0, t_0]$ is in $X^{\delta/2}$. If $\Psi_\beta(\tilde{\gamma}(s,t), \tau)$ remains in $X^\delta$ for all $\tau \geq 0$, then (32) must hold since $\|E^d_\beta(\tilde{\gamma}(s,t), \tau)\| (H^1_0(\Omega))^2 \geq a(\beta) > 0$. Otherwise, $\Psi_\beta(\tilde{\gamma}(s,t), \tau)$ needs to escape $X^\delta$ within a finite time, and in particular, goes through $X^\delta \setminus X^{\delta/2}$. In that case, if we take $\tau_0$ as the infimum of $\tau$ such that $\Psi_\beta(\tilde{\gamma}(s,t), \tau)$ is outside of $X^\delta$ for all $(s,t) \in [0, s_0] \times [0, t_0]$, then $A_\beta(\tau_0) < c_\beta - \frac{\alpha}{4}$ by Proposition 6 (choose $\alpha$ smaller if necessary).

However, $\Psi_\beta(\tilde{\gamma}(s,t), \tau_0) \in \Phi$, so $A_\beta(\tau_0) \geq c_\beta$, which contradicts (32). This ends the proof. \qed
Proposition 10. For fixed sufficiently small $\beta > 0$, $E_\beta$ has a critical point $(u, v) \in E_\beta^{d_\beta} \cap X^\delta$. Also, $u, v > 0$.

Proof. Let $\{(u_n, v_n)\}_n \subset E_\beta^{d_\beta} \cap X^\delta$ be a Palais-Smale sequence obtained in Proposition 9. Then, along a subsequence, it converges to $(u, v)$ weakly in $(H^1_0(\Omega))^2$ and $E'_\beta(u, v) = 0$. Also, by compactness of $X$, $(u, v) \in X^\delta$.

If we write $(u_n, v_n) = (u, v) + ((r_1)_n, (r_2)_n)$, then

$$\|((r_1)_n, (r_2)_n)\| \leq 3\delta$$

for sufficiently large $n$ and

$$E_\beta(u_n, v_n) = E_\beta(u, v) + E_\beta((r_1)_n, (r_2)_n) + o(1)$$

by Lemma 3.1 (precisely speaking, we need its variant that corresponds to the product-space version of (29)) if $H$ satisfies the hypotheses given in the statement of Theorem 2.3. Furthermore, if $\delta$ is sufficiently small, we have

$$E_\beta((r_1)_n, (r_2)_n) \geq 0$$

for all $|\beta|$ small. Therefore

$$E_\beta(u, v) \leq \limsup_{n \to \infty} E_\beta(u_n, v_n) \leq d_\beta.$$

Finally, because $(u, v)$ solves (13), assumption (11) and the strong maximum principle imply $u, v > 0$. \qed

To emphasize its dependence on $\beta$, we will attach the subscript $\beta$ on the solution $(u, v)$ found in the above proposition. It remains to study asymptotic behavior of the solutions $(u_\beta, v_\beta)$ for (13) as $\beta \to 0$.

Proposition 11. Along a subsequence, $(u_\beta, v_\beta)$ converges to $(u_0, v_0)$ in $(H^1_0(\Omega))^2$ as $\beta \to 0$ where each $u_0$ and $v_0$ is a positive ground state for (9) and (10), respectively.

Proof. By Corollary 2, we know that $(H^1_0(\Omega))^2$-boundedness of $\{(u_\beta, v_\beta)\}_\beta$ guarantees its $(C^{2,\alpha}(\overline{\Omega}))^2$-boundedness for some $\alpha \in (0, 1)$. Thus the Arzela-Ascoli theorem can be applied to show that $(u_\beta, v_\beta)$ converges to some $(u_0, v_0)$ in $(H^1_0(\Omega))^2$ along a subsequence as $\beta \to 0$ and each of the $u_0$ and $v_0$ is a solution for (9) and (10), respectively. Since $u_\beta$ and $v_\beta$ are positive and their $H^1_0(\Omega)$-norms are away from 0 uniformly in $\beta$, the strong maximum principle implies that $u_0$ and $v_0$ are positive. Moreover, since we have

$$E_0(u_0, v_0) \leq \inf_{\beta \to 0} \lim E_\beta^{d_\beta} = E_1^1 + E_2^2$$

by Proposition 10, it holds that $E_1^S(u_0) = E_1^m$ and $E_2^S(v_0) = E_2^m$. \qed

This completes the proof of Theorem 2.2 and Theorem 2.3.
Appendix A. Regularity of solutions of (3) with Dirichlet boundary condition. In this appendix, we will show the following Brezis-Kato type estimation based on the Moser iteration technique. We refer to [8] and Proposition 3.5 in [11].

**Proposition 12.** If $N \geq 3$ and parameters and a function $H \in C^1(\mathbb{R} \times \mathbb{R})$ for some $\alpha \in (0,1)$ satisfies (11) and

$$|H(u,v)| \cdot |u|, |H_v(u,v)| \cdot |v| \leq C(1 + |u|^{2^*} + |v|^{2^*}), \quad (12')$$

a solution of (13) in $(H_0^1(\Omega))^2$ is in $(L^q(\Omega))^2$ for any $q \geq 1$.

**Proof.** Let $t$ be a nonnegative number. To prove the proposition, it suffices to show that a solution $(u,v) \in (H_0^1(\Omega))^2$ of (13) in $(L^{(t+1)}(\Omega))^2$ is actually contained in $(L^{(t+1)}(\Omega))^2$. To this aim, we let $u_t = \min\{u,l\}$ and $v_t = \min\{v,l\}$ for fixed number $t > 0$ and multiply the first equation of (13) by $uu_t^2$, getting

$$\int_\Omega \nabla u \cdot \nabla (uu_t^2) = \int_\Omega \lambda_1 u^2 u_t^2 + \int_\Omega \mu_1 u^{\frac{2N}{N-2}} u_t^2 + \int_\Omega \beta H(u,v)uu_t^2. \quad (33)$$

(Notice that a solution of (13) is necessarily nonnegative.)

It is easily checked that

$$\int_\Omega \nabla u \cdot \nabla (uu_t^2) \geq \frac{2t + 1}{(t + 1)^2} \int_\Omega |\nabla (uu_t^2)|^2,$$

while

$$\int_\Omega \lambda_1 u^2 u_t^2 + \int_\Omega \mu_1 u^{\frac{2N}{N-2}} u_t^2 + \int_\Omega \beta H(u,v)uu_t^2$$

$$\leq \int_\Omega \lambda_1 u^2 u_t^2 + \mu_1 \int_\Omega u^{\frac{2N}{N-2}} u_t^2 + |\beta| C \int_\Omega (1 + u^{\frac{2N}{N-2}} + v^{\frac{2N}{N-2}}) uu_t^2$$

$$\leq \int_\Omega \lambda_1 u^2 u_t^2 + (\mu_1 + |\beta| C) \int_\Omega u^{\frac{2N}{N-2}} u_t^2 + |\beta| C \int_{\{u \geq v\}} u^{\frac{2N}{N-2}} u_t^2$$

$$+ |\beta| C \int_{\{u \leq v\}} v^{\frac{2N}{N-2}} v_t^2 + |\beta| C \int_\Omega u_t^2,$$

for some positive constant $C > 0$. Therefore, by the Sobolev embedding theorem,

$$\left( \int_\Omega (uu_t^2)^{\frac{N}{N-2}} \right)^{\frac{N-2}{N}} \leq C(t+1) \left[ \lambda_1 \int_\Omega u^2 u_t^2 + (\mu_1 + |\beta| C) \int_\Omega u^{\frac{2N}{N-2}} u_t^2$$

$$+ |\beta| C \int_{\{u \geq v\}} u^{\frac{2N}{N-2}} u_t^2 + |\beta| C \int_{\{u \leq v\}} v^{\frac{2N}{N-2}} v_t^2 + |\beta| C \int_\Omega u_t^2 \right]. \quad (34)$$

Similarly, we have

$$\left( \int_\Omega (v^2 v_t^2)^{\frac{N}{N-2}} \right)^{\frac{N-2}{N}} \leq C(t+1) \left[ \lambda_2 \int_\Omega v^2 v_t^2 + (\mu_2 + |\beta| C) \int_\Omega v^{\frac{2N}{N-2}} v_t^2$$

$$+ |\beta| C \int_{\{u \geq v\}} u^{\frac{2N}{N-2}} u_t^2 + |\beta| C \int_{\{u \leq v\}} v^{\frac{2N}{N-2}} v_t^2 + |\beta| C \int_\Omega v_t^2 \right]. \quad (35)$$
Putting (34) and (35) together and then using Hölder’s inequality, we find
\[
\left( \int_{\Omega} (u_{t})^{2N} \right)^{\frac{N-2}{N}} + \left( \int_{\Omega} (v_{t})^{2N} \right)^{\frac{N-2}{N}} \leq C(t+1) \cdot \left[ \lambda_{1} \int_{\Omega} u^{2} + \lambda_{2} \int_{\Omega} v^{2} \right]
\]
\[
(\mu_{1} + 3|\beta|C) \left\{ K^{\frac{4}{N-2}} \int_{\Omega} u_{t}^{2} + \left( \int_{\Omega} u_{t} \right)^{\frac{2}{N}} \left( \int_{\Omega} (u_{t})^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \right\}
\]
\[
(\mu_{2} + 3|\beta|C) \left\{ K^{\frac{4}{N-2}} \int_{\Omega} v_{t}^{2} + \left( \int_{\Omega} v_{t} \right)^{\frac{2}{N}} \left( \int_{\Omega} (v_{t})^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \right\}
\]
\[
+ |\beta|C \int_{\Omega} u_{t}^{2(t+1)} + |\beta|C \int_{\Omega} v_{t}^{2(t+1)} + 2|\beta|C|\Omega|^{\frac{1}{1+\epsilon}} \right]
\]
for any $K > 0$. We now take $K > 0$ which makes
\[
\int_{\{u \geq K\}} u^{\frac{2N}{N-2}} \quad \text{and} \quad \int_{\{v \geq K\}} v^{\frac{2N}{N-2}}
\]
small enough and let $l \to \infty$ to deduce
\[
\left( \int_{\Omega} |u|^{2(t+1)} \right)^{\frac{N-2}{N}} + \left( \int_{\Omega} |v|^{2(t+1)} \right)^{\frac{N-2}{N}} \leq C(t+1) \left\{ \lambda_{1} + (\mu_{1} + 3|\beta|C)K^{\frac{4}{N-2}} + |\beta|C \right\} \int_{\Omega} |u|^{2(t+1)}
\]
\[
+ \left\{ \lambda_{2} + (\mu_{2} + 3|\beta|C)K^{\frac{4}{N-2}} + |\beta|C \right\} \int_{\Omega} |v|^{2(t+1)} + 2|\beta|C|\Omega|^{\frac{1}{1+\epsilon}} \right].
\]
Consequently, the proposition is proven. \qed

**Corollary 2.** If $N \geq 3$ and $H(u, v) = (u_{+}v_{+})^{\frac{N}{N-2}}$ or $H$ satisfies the assumption imposed in Theorem 2.2, a solution in $(H_{0}^{1}(\Omega))^{2}$ of (13) is in $(C^{2, \alpha}(\Omega))^{2}$.

**Proof.** Since $H \in C^{1,\alpha}(\mathbb{R} \times \mathbb{R})$ for some $\alpha \in (0, 1)$, the standard elliptic regularity result can be applied. \qed

**Note.** After this work was completed, the author became aware of [13] in which Chen and Zou studied the existence of a positive least energy solution of (3) for $N = 4$. They also dealt with the case $\beta < 0$ and the phase separation phenomenon as $\beta \to -\infty$.

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