DOUBLING CONSTRUCTIONS FOR COVERING GROUPS AND TENSOR PRODUCT $L$-FUNCTIONS

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Abstract. This is a research announcement concerning a series of constructions obtained by applying the “doubling method” from the theory of automorphic forms to covering groups. Using these constructions, we obtain partial tensor product $L$-functions attached to generalized Shimura lifts, which may be defined in a natural way since at almost all places the representations are unramified principal series.

1. Introduction

In their famous paper $L$-Functions for the Classical Groups (in [G-PS-R]), Piatetskii-Shapiro and Rallis introduced a new type of global integral which represents the standard $L$-function for any split classical group. To describe their construction, let $\mathbb{A}$ denote the ring of adeles of a global field $F$. Let $G$ denote any split classical group, $\pi_1$ and $\pi_2$ denote two irreducible cuspidal automorphic representations of $G(\mathbb{A})$, and $\varphi_{\pi_1}, \varphi_{\pi_2}$ be automorphic forms in the corresponding spaces. Then the global integral they introduced is

$$\int_{G(F) \times G(F)/G(\mathbb{A})} \varphi_{\pi_1}(g_1) \varphi_{\pi_2}(g_2) E((g_1, g_2), s) \, dg_1 \, dg_2.$$ 

Here $E(h, s)$ is an Eisenstein series defined on the adelic points of another classical group $H$, which depends on the choice of $G$ and contains the direct product $G \times G$ as a subgroup. In the case $G = GL_n$, a slight modification is required to handle the issue of convergence. Since the domain of integration is two copies of the group $G$, this type of construction has become known as the doubling method. Moreover, the doubling method also works when $G = \tilde{Sp}_{2n}$, the two-fold metaplectic cover of the symplectic group $Sp_{2n}$. (This was known to Piatetski-Shapiro and Rallis.) In all cases, after unfolding the integral it is easy to see that it is not zero only if $\pi_2$ is the contragredient of $\pi_1$.

The doubling method is general in two aspects. First, as indicated above, it is valid for all split classical groups. Second, and maybe more important, this construction works for any irreducible cuspidal automorphic representation $\pi_1$. This is a rare phenomenon in these types of constructions. Usually, such integrals unfold to some special model which is afforded by some but not all cuspidal automorphic representations, such as the Whittaker model. (Although every cuspidal automorphic representation of $GL_n(\mathbb{A})$ is globally generic,

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this is not the case for other classical groups.) In the above construction, after carrying out the unfolding process, one obtains the inner product

\[ < \pi_1(g) \varphi_{\pi_1}, \varphi_{\pi_2} > = \int_{G(F) \backslash G(A)} \varphi_{\pi_1}(g_1 g) \varphi_{\pi_2}(g_1) \, dg_1. \]

This inner product is identically zero unless \( \pi_2 \) is the contragredient of \( \pi_1 \), and then it is nonzero for some choices of data for any \( \pi_1 \).

In this note we describe a generalization of the doubling method to covering groups. Let \( G \) be a split classical group, \( m \geq 1 \) and let \( G^{(m)}(\mathbb{A}) \) denote a metaplectic \( m \)-fold cover of \( G(\mathbb{A}) \). This is defined provided the underlying field has a full set of \( m \)-th roots of unity. When \( m = 1 \), we understand \( G^{(1)} \) to be the linear group \( G \). Let \( \pi^{(m)} \) denote a genuine irreducible cuspidal automorphic representation of \( G^{(m)}(\mathbb{A}) \), and let \( \tau^{(r)} \) denote a genuine irreducible cuspidal automorphic representation of \( GL_k^+(\mathbb{A}) \). Here \( r = m \) except when \( G = Sp_{2n} \) and \( m \) is even, in which case we set \( r = m/2 \). Via doubling, we shall describe a global construction which represents the partial standard tensor product \( L \)-function attached to \( \pi^{(m)} \times \tau^{(r)} \). This partial \( L \)-function is the product of local \( L \)-functions over all finite places of \( F \) for which the representations \( \pi^{(m)} \) and \( \tau^{(r)} \) are unramified. The local \( L \)-functions are Rankin-Selberg \( L \)-functions attached to the local Shimura lifts. These will be described in [12] below, and may also be obtained from the work of Savin [Sa].

The construction we introduce is a generalization of the construction described by the integral [11], keeping track of covers. It is given by

\[ \int_{G(F) \times G(F) \backslash G(A) \times G(A)} \varphi_1^{(m)}(g_1) \varphi_2^{(m)}(g_2) E_{\tau^{(r)}}^{U,\psi_U}((g_1, g_2), s) \, dg_1 \, dg_2. \]  

Here, for \( i = 1, 2 \), the functions \( \varphi_i^{(m)} \) are vectors in the space of \( \pi^{(m)} \). There is an implicit choice of section \( G(\mathbb{A}) \to G^{(m)}(\mathbb{A}) \) in (2), and the choice of \( r \) in terms of \( m \) implies that the integral is in fact independent of this section. The Eisenstein series \( E \) is defined on \( H^{(r)}(\mathbb{A}) \) where \( H \) is a certain suitable split classical group, and \( E_{\tau^{(r)}}^{U,\psi_U} \) is a certain Fourier coefficient of this Eisenstein series. The induction data of \( E \) depends on a certain representation which we refer to as a Whittaker-Speh-Shalika representation, which is defined on a cover of the general linear group. This representation, which depends on \( \tau^{(r)} \), is defined by means of a residue of an Eisenstein series, and when \( r = 1 \) it reduces to the well-known Speh representation, studied for example by Jacquet [J]. We discuss these representations in Section 2 below.

Unfortunately, and this is a weak point of this construction, while these residue representations are conjectured to exist for our covering groups, this is not proved in general. This is because, in the absence of a general theory of Shimura lifts for covers of the general linear group, the study of the necessary partial \( L \)-functions needed to analyze the Eisenstein series attached to automorphic representations on covers of the general linear group is not available. We discuss this issue in Section 5 and suggest some solutions in various cases. See also Conjecture [11].

In the sections below we sketch the general construction of the global integrals for all classical groups \( G \). We discuss the symplectic case in more detail and in Theorem 3 we state the result of the unramified computations. Work carrying out the global unfolding and the
computation of the local unramified integrals for other classical groups is in progress and we shall report on it in detail in a follow-up article.

2. Whittaker-Speh-Shalika Representations

In this section we define and construct a set of representations which we will then use in the global integrals.

2.1. Definition of Whittaker-Speh-Shalika representations. Let \( m, a \) and \( b \) denote three positive integers. Let \( GL_{ab}^{(m)}(\mathbb{A}) \) denote an \( m \)-fold metaplectic cover of \( GL_{ab}(\mathbb{A}) \), as in Kazhdan and Patterson [K-P]. When \( m = 1 \) we take \( GL_{ab}^{(m)} \) to be \( GL_{ab} \). Recall that the cover is split over unipotent subgroups. For example, we can choose a section \( GL_{ab}^{(m)}(\mathbb{A}) \to GL_{ab}^{(m)}(\mathbb{A}) \) which is a splitting of \( GL_{ab}^{(m)}(\mathbb{F}) \), where \( V(\mathbb{A}) \) is the subgroup of upper triangular unipotent matrices. Therefore notions involving unipotent orbits transfer to covering groups mutatis mutandis.

Let \( V_{a,b} \) be the unipotent radical of the standard parabolic subgroup of \( GL_{ab}^{(m)} \) whose Levi part is \( GL_b \times GL_b \times \cdots \times GL_b \). Here \( GL_b \) appears \( a \) times. In terms of matrices, this group consists of all unipotent matrices \( X \) of \( GL_{ab}^{(m)} \) of the form

\[
X = \begin{pmatrix}
I & X_{1,2} & X_{1,3} & \cdots & X_{1,a} \\
I & X_{2,3} & \cdots & X_{2,a} \\
& & \ddots & \ddots \\
& & & & X_{a-1,a} \\
& & & & I
\end{pmatrix}, \quad X_{i,j} \in \text{Mat}_{b \times b},
\]

where \( I \) is the \( b \times b \) identity matrix. Fix a nontrivial character \( \psi \) of \( F \backslash \mathbb{A} \), and let \( \psi_{a,b} \) be the character of \( V_{a,b}(\mathbb{F}) \backslash V_{a,b}(\mathbb{A}) \) given by

\[
\psi_{a,b}(X) = \psi(\text{tr}(X_{1,2} + X_{2,3} + \cdots + X_{a-1,a}))
\]

with \( X \) as in (3).

Suppose \( \sigma^{(m)} \) is an automorphic representation of the group \( GL_{ab}^{(m)}(\mathbb{A}) \). If \( \varphi_{\sigma^{(m)}} \) is in the space of \( \sigma^{(m)} \), then the integral

\[
W(\varphi_{\sigma^{(m)}})(g) = \int_{V_{a,b}(\mathbb{F}) \backslash V_{a,b}(\mathbb{A})} \varphi_{\sigma^{(m)}}(vg) \psi_{a,b}(v) \, dv
\]

is a Fourier coefficient corresponding to the unipotent orbit \((a^b)\). We refer to this Fourier coefficient as the Whittaker-Speh-Shalika coefficient of the representation \( \sigma^{(m)} \). Notice that when \( b = 1 \), the Fourier coefficient (4) reduces to the well-known Whittaker coefficient, and when it is nonzero we say that the representation \( \sigma^{(m)} \) is globally generic. For each completion \( F_\nu \) of \( F \) we have a similar character of \( V_{a,b}(F_\nu) \) which we also denote \( \psi_{a,b} \).

We make the following definition.

**Definition 1.** An irreducible genuine automorphic representation \( \sigma^{(m)} \) of the group \( GL_{ab}^{(m)}(\mathbb{A}) \) is a Whittaker-Speh-Shalika representation of type \((a, b)\) if:
1) This representation has a nonzero Fourier coefficient corresponding to the unipotent orbit \((a^b)\), and moreover, for all orbits which are greater than or not related to \((a^b)\), all corresponding Fourier coefficients are zero for all choices of data. In the notation of Ginzburg [GI], this statement can be written as \(O_{GL_{ab}^m}(\sigma_{(m)}) = (a^b)\).

2) For a finite place \(\nu\), let \(\sigma_{\nu}^{(m)}\) denote the irreducible constituent of \(\sigma^{(m)}\) at \(\nu\). Suppose that \(\sigma_{\nu}^{(m)}\) is an unramified representation. Then \(O_{GL_{ab}^m}(\sigma_{\nu}^{(m)}) = (a^b)\). (That is, the local analogue of part [1] holds.) Moreover, the dimension of

\[
\text{Hom}_{V_{a,b}(F_{\nu})}(\sigma_{\nu}^{(m)}; \psi_{a,b})
\]

is one.

When \(m = b = 1\), it is well known that condition [2] in Definition [1] is satisfied. However, on covering groups not every generic representation satisfies condition [2].

Before constructing examples of Whittaker-Speh-Shalika representations, we note that the Fourier coefficient \(W(\varphi_{\sigma^{(m)}})(g)\) enjoys an extra invariance property. Indeed, let \(GL_{ab}^m\) denote the image of \(GL_b\) inside \(GL_{ab}\) under the diagonal embedding \(h \mapsto h_0 := \text{diag}(h,h,\ldots,h)\). Then \(GL_{ab}^m\) is the stabilizer of the character \(\psi_{a,b}\) inside the group \(GL_b \times GL_b \times \cdots \times GL_b\). Consider the group \(SL_{ab}^m\) embedded inside \(GL_{ab}^m\). Then the function \(W(\varphi_{\sigma^{(m)}})(g)\) is left invariant by all matrices \(h_0\) as above with \(h \in SL_b(\mathbb{A})\), i.e. \(W(\varphi_{\sigma^{(m)}})(h_0g) = W(\varphi_{\sigma^{(m)}})(g)\) for all \(h_0 \in SL_b(\mathbb{A})\). This follows since the unipotent orbit attached to \(\sigma^{(m)}\) is \((a^b)\). Moreover, if we expand along any unipotent subgroup of \(SL_b(\mathbb{A})\), then the nontrivial contribution to the expansion is trivial. This follows since the nontrivial term of the expansion is associated with a unipotent orbit which is strictly greater than \((a^b)\). See Friedberg and Ginzburg. [F-G], Proposition 3, for details. This means that if \(m > 1\), then the group \(SL_0(\mathbb{A})\), embedded diagonally in \(GL_{ab}^{(m)}(\mathbb{A})\), must split under the \(m\)-fold cover. This implies \(m \mid a\).

2.2. Construction of Whittaker-Speh-Shalika representations. In this subsection we shall construct examples of Whittaker-Speh-Shalika representations, by means of residues of Eisenstein series.

Let \(k\) and \(c\) be two positive integers. Denote \(b = mc\). Let \(\tau^{(m)}\) denote a genuine irreducible cuspidal automorphic representation of the group \(GL_k^{(m)}(\mathbb{A})\), and \(s = (s_1,\ldots,s_b) \in \mathbb{C}^b\). We construct an Eisenstein series \(E_{\tau^{(m)}}(g, \mathbf{s})\) defined on the group \(GL_{kk}^{(m)}(\mathbb{A})\) as follows. Let \(P_{k,b}\) denote the standard parabolic subgroup of \(GL_{kk}\) whose Levi part is \(GL_k \times GL_k \times \cdots \times GL_k\). As in [F-G], Section 2, we construct the induced representation

\[
\text{Ind}_{\mathcal{P}_{k,b}^{(m)}(\mathbb{A})}^{GL_{kk}^{(m)}(\mathbb{A})}(\tau^{(m)}; \mathbf{s}_1 \otimes \tau^{(m)}; \mathbf{s}_2 \otimes \cdots \otimes \tau^{(m)}; \mathbf{s}_b)\delta_{P_{k,b}}^{1/2}.
\]

We remark that the induction process here is more complicated when \(m > 1\) since the \(GL_k\) blocks do not commute in the covering group. Let \(E_{\tau^{(m)}}(g, \mathbf{s})\) denote the Eisenstein series associated with this induced representation. When \(m = 1\), these representations and their residues were studied by various authors. See for example [J]. When \(m > 1\), these Eisenstein series were constructed and studied in Suzuki [Su], Section 8.

We start with the following conjecture.
Conjecture 1. Given $\tau^{(m)}$ as above, the Eisenstein series $E_{\tau^{(m)}}^{(m)}(g, z)$ has a simple multi-residue at the point

$$s_1 + s_2 + \cdots + s_b = 0; \quad m(s_i - s_{i+1}) = 1; \quad 1 \leq i \leq b - 1.$$  

We shall discuss this conjecture in Section 5 below.

Assuming Conjecture 1, denote by $E_{\tau^{(m)}}^{(m)}(g)$ the residue of the Eisenstein series $E_{\tau^{(m)}}^{(m)}(g, z)$ at the above point (this depends on the choice of test vector, but we suppress this from the notation). Let $L_{\tau^{(m)}}^{(m)}$ denote the representation of $GL_{km}^{(m)}(\mathbb{A})$ generated by all the residue functions $E_{\tau^{(m)}}^{(m)}(g)$. Also, let $Z_{km}^{(m)}$ denote the center of the group $GL_{km}^{(m)}(\mathbb{A})$. We have the following two results.

**Proposition 1.** The automorphic representation $L_{\tau^{(m)}}^{(m)}$ lies in the discrete spectrum of the space $L^2(Z_{km}GL_{km}^{(m)}(E)\backslash GL_{km}^{(m)}(\mathbb{A}))$.

**Proposition 2.** We have $\mathcal{O}_{GL_{km}^{(m)}}(L_{\tau^{(m)}}^{(m)}) = ((km)^c)$.  

From these results we deduce that the representation $L_{\tau^{(m)}}^{(m)}$ has at least one irreducible summand which has a nonzero Fourier coefficient corresponding to the unipotent orbit $((km)^c)$. Denote this summand by $S_{\tau^{(m)}}^{(m)}$. Then we have

**Theorem 1.** The representation $S_{\tau^{(m)}}^{(m)}$ is a Whittaker-Speh-Shalika representation of type $(km, c)$.  

The first condition of Definition 1 for representation $S_{\tau^{(m)}}^{(m)}$ follows from the above discussion. The main content of Theorem 1 is the verification of 2) in Definition 1.

3. The Global Construction

In this section we introduce the general global integral. In the next section we treat the case of $Sp_{2n}$ in detail.

Let $m$, $n$, and $k$ denote three positive integers. Let $G$ denote one of the split groups $GL_n$, $Sp_{2n}$, $SO_{2n+1}$ and $SO_{2n}$. Let $c(n) = n$ if $G = GL_n$, $c(n) = 2n$ if $G = Sp_{2n}$ or $G = SO_{2n}$, and $c(n) = 2n + 1$ if $G = SO_{2n+1}$. Let $G^{(m)}(\mathbb{A})$ denote an $m$-fold metaplectic cover of $G(\mathbb{A})$. There is a difference in the symplectic group case depending on the parity of $m$; in this section we shall assume that if $G = Sp_{2n}$ then $m$ is odd.

Depending on $G$, we introduce another classical group $H$ on which we shall construct an Eisenstein series. Let

$$H = \begin{cases} 
GL_{2nkm} & G = GL_n, \\
Sp_{4nkm} & G = Sp_{2n}, \\
SO_{4nkm} & G = SO_{2n}, \\
SO_{2(2n+1)km} & G = SO_{2n+1}.
\end{cases}$$

Let $P$ denote the maximal parabolic subgroup of $H$ whose Levi part is

$$\begin{cases} 
GL_{km(n)} \times GL_{km(n)} & G = GL_n, \\
GL_{km(n)} & \text{otherwise}.
\end{cases}$$
For later use, let \( U(P) \) denote the unipotent radical of \( P \). Let \( \tau^{(m)} \) denote a genuine irreducible cuspidal representation of \( GL_k^{(m)}(\mathbb{A}) \). As in the previous section, construct the residue representation \( \mathcal{E}_{\tau^{(m)}}^{(m)} \) defined on the group \( GL_k^{(m)}(\mathbb{A}) \). This is a Whittaker-Speh-Shalika representation of type \((km,c(n))\). Form the Eisenstein series \( E_{\tau^{(m)}}^{(m)}(h,s) \) defined on the group \( H^{(m)}(\mathbb{A}) \) attached to the induced representation

\[
\begin{align*}
\text{Ind}_{H^{(m)}(\mathbb{A})}^{\mathbb{A}}(\mathcal{E}_{\tau^{(m)}}^{(m)} \otimes \mathcal{E}_{\tau^{(m)}}^{(m)}) \delta^s \quad & G = GL_n, \\
\text{Ind}_{H^{(m)}(\mathbb{A})}^{\mathbb{A}} \mathcal{E}_{\tau^{(m)}}^{(m)} \delta^s & \text{otherwise.}
\end{align*}
\]

To introduce the global integral, let \( \pi_1^{(m)} \) and \( \pi_2^{(m)} \) denote two genuine irreducible cuspidal automorphic representations defined on the group \( G^{(m)}(\mathbb{A}) \). When \( G = GL_n \), one has to be careful with the center of the group, and also with the issue of convergence. To avoid these minor complications, in describing the integral we shall assume that \( G \) is not the group \( GL_n \). Then the global integral we consider is

\[
\int_{G/F \times G/F \times G(\mathbb{A}) \times G(\mathbb{A}) U(F) \times U(\mathbb{A})} \varphi_1^{(m)}(g_1) \varphi_2^{(m)}(g_2) E_{\tau^{(m)}}^{(m)}(u(g_1, g_2), s) \psi_U(u) \, du \, dg_1 \, dg_2.
\]

Here, \( \varphi_i^{(m)} \) are vectors in the spaces of \( \pi_i^{(m)} \).

We still need to check that the integral is well defined. From the description below it follows that the embeddings of the two copies of \( G^{(m)}(\mathbb{A}) \) in \( H^{(m)}(\mathbb{A}) \) commute with each other. It also follows that the function

\[
(g_1, g_2) \mapsto \varphi_1^{(m)}(g_1) \varphi_2^{(m)}(g_2) E_{\tau^{(m)}}^{(m)}((g_1, g_2), s) \quad g_i \in G(\mathbb{A})
\]

does not depend on the choice of section \( G(\mathbb{A}) \to G^{(m)}(\mathbb{A}) \). Hence the integral (7) is well defined and converges absolutely for \( \text{Re}(s) \) large.

Finally, we describe in some detail the unipotent group \( U \), the character \( \psi_U \) and the embedding of \( G \times G \) inside \( H \). Let

\[
H_{c(n)} = \begin{cases} 
GL_{2n} \times GL_n \times \ldots \times GL_n & G = GL_n, \\
Sp_{4n} & G = Sp_{2n}, \\
SO_{4n} & G = SO_{2n}, \\
SO_{4n+2} & G = SO_{2n+1}.
\end{cases}
\]

When \( G = GL_n \), the group \( GL_n \) appears \( mk - 1 \) times in \( H_{c(n)} \), and we mention this case for the sake of completeness. Let \( Q_{n,m,k} \) denote the standard parabolic subgroup of \( H \) whose Levi part is \( GL_{c(n)} \times GL_{c(n)} \times \ldots GL_{c(n)} \times H_{c(n)} \) where \( GL_{c(n)} \) appears \( mk - 1 \) times. Let \( U \) denote the unipotent radical of the parabolic subgroup \( Q_{n,m,k} \).

To define the character \( \psi_U \), consider the unipotent orbit \(((2km - 1)^{c(n)} 1^{c(n)}) \) associated with the group \( H \). It follows from Collingwood and McGovern [C-M] that this is a well defined orbit for every group \( H \), and that the stabilizer of this orbit is the group \( G \times G \). From [GI] we obtain that a Fourier coefficient associated with this orbit can be constructed using the group \( U \), and one can define a character \( \psi_U \) such that the stabilizer inside the Levi part of \( Q_{n,m,k} \) is the split group \( G \times G \). In the next section we shall construct this Fourier coefficient explicitly in the symplectic group case. As mentioned above, it follows from the way the group \( G \times G \) is embedded inside \( H \) that the covers in integral (7) are compatible.
Here is our main theorem. (Notation not defined above is similar to that in other doubling method computations.)

**Theorem 2.** The integral (7) is well defined, converges absolutely for Re(s) large, and admits a meromorphic continuation to the whole complex plane. It is not identically zero only if \( \pi_1^{(m)} = \pi_2^{(m)} = \pi^{(m)} \). In this case, for Re(s) large it is equal to

\[
\int \int_{G(\mathbb{A}) \backslash GL_2(\mathbb{A})} \langle \pi^{(m)}(g) \varphi_{\pi^{(m)}}, \overline{\varphi_{\pi^{(m)}}} \rangle > f_{W(\mathcal{E}_{\pi^{(m)}})}(\delta u_0(1, g), s) \psi_U(u_0) \, du_0 \, dg.
\]

Here

\[
\langle \pi^{(m)}(g) \varphi_{\pi^{(m)}}, \varphi_{\pi^{(m)}} \rangle = \int_{G(F)/G(\mathbb{A})} \varphi_{\pi^{(m)}}(g_1 g) \varphi_{\pi^{(m)}}(g_1) \, dg_1.
\]

In particular the integral (7) represents an Euler product.

### 4. The Symplectic Group Case

In this section we give some details for the symplectic group. Let \( G = Sp_{2n} \). For this group there is a difference depending on the parity of \( m \). To give a uniform construction, let \( r = m \) if \( m \) is odd, and \( r = m/2 \) if \( m \) is even. Thus, the integral will consider is integral (2) which agrees with integral (7) when \( m = r \) is an odd number. Let \( H = Sp_{4nrk} \).

Let \( \mathcal{E}_{\pi^{(r)}} \) be the Whittaker-Speh-Shalika representation of type \( (rk, 2n) \), as constructed in subsection 2.2. This representation is defined on the group \( GL_{2nrk}(\mathbb{A}) \), and in the notations of Definition 1 condition 1, we have \( \mathcal{O}_{GL_{2nrk}}(\mathcal{E}_{\pi^{(r)}}) = ((rk)^2n)^L \).

To describe the group \( U \) in integral (2), let \( Q_{n,r,k} \) denote the parabolic subgroup of \( Sp_{4nrk} \) whose Levi part is \( GL_{2n} \times \ldots \times GL_{2n} \times Sp_{4n} \). Here \( GL_{2n} \) appears \( rk - 1 \) times. Denote by \( U_{n,r,k} \) or simply by \( U \) the unipotent radical of \( Q_{n,r,k} \). We may identify the quotient \( U/[U,U] \) with the group

\[
L = \text{Mat}_{2n} \oplus \ldots \oplus \text{Mat}_{2n} \oplus \text{Mat}_{2n \times 4n}.
\]

Here \( \text{Mat}_{2n} \) appears \( rk - 2 \) times. To define the character \( \psi_U \) it is enough to specify it on \( L \). For \( (X_1, \ldots, X_{rk-2}, Y) \in L \) define \( \psi_L \) as \( \psi(\text{tr}(X_1 + \ldots + X_{rk-1}) + \text{tr}'(Y)) \). Here \( X_i \in \text{Mat}_{2n} \) and \( Y \in \text{Mat}_{2n \times 4n} \). To define \( \text{tr}'(Y) \), write

\[
Y = \begin{pmatrix} Y_1 & Z_1 & Y_2 & Z_3 & Y_4 \\ Y_3 & Z_2 & Y_4 & Z_4 & Y_4 \\
\end{pmatrix}, \quad Y_i \in \text{Mat}_{n \times n}; \quad Z_j \in \text{Mat}_{n \times 2n}.
\]

Then \( \text{tr}'(Y) = \text{tr}(Y_1 + Y_4) \). Let \( \psi_U \) denote the extension of \( \psi_L \) to \( U \) which is trivial on \([U,U]\). It follows from [G1] that the corresponding Fourier coefficient given by \( U \) and \( \psi_U \) is associated with the unipotent orbit \(((2rk-1)^2n)^L)^L\).

Finally, we specify the embedding of \( (g_1, g_2) \in Sp_{2n} \times Sp_{2n} \) in \( Sp_{4nrk} \). It is given by \( \text{diag}(g_1, \ldots, g_1, g_1, g_2, g_1^*, \ldots, g_1^*) \). Here \( g_1 \) appears \( 2rk - 1 \) times, and by \( (g_1, g_2) \) we mean the usual embedding inside \( Sp_{4n} \), i.e.

\[
(g_1, g_2) \mapsto \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{1,2} & g_{2} \\ g_{1,3} & g_{1,4} \\ g_{1,4} & g_{1,3} \end{pmatrix}, \quad g_1 = \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{1,2} & g_{1,1} \\ g_{1,3} & g_{1,4} \\ g_{1,4} & g_{1,3} \end{pmatrix}; \quad g_{1,i} \in \text{Mat}_{n \times n}.
\]
The entries with asterisk are determined so that embedding is symplectic. It is now easy to check that the coverings are compatible, and hence that integral (2) is well defined.

In [G2], Ginzburg introduced the dimension equation, which is strongly correlated with global integrals that unfold to Euler products. Roughly speaking, this equation states that the sum of dimensions of the representations involved in the integral is equal to the sum of the dimensions of the groups involved. We note that the dimension equation is satisfied in our case. To see what is involved, let us verify this when \( m = r \) is odd. The sum of the dimensions of the representations involved is the sum of the dimension of the Eisenstein series \( E^{(m)}_{\tau(m)}(\cdot, s) \) and the dimension of the functional \( \langle \pi^{(m)}(g) \varphi_{\tau(m)}, \varphi_{\tau(m)} \rangle \) obtained after unfolding the integral. It follows from (9) that the dimension of this functional is \( \dim S p_{2n} \).

Thus the equation we need to verify is

\[
(11) \quad \dim S p_{2n} + \dim E^{(m)}_{\tau(m)}(\cdot, s) = 2 \dim S p_{2n} + \dim U.
\]

From [G1] we see that

\[
\dim E^{(m)}_{\tau(m)}(\cdot, s) = \dim \mathcal{E}^{(m)}_{\tau(m)} + \dim U(P) = \frac{1}{2} \dim ((km)^{2n}) + \dim U(P).
\]

Here \( \dim ((km)^{2n}) \) is the dimension of the unipotent orbit \((km)^{2n}\) and \(U(P)\) is the unipotent radical of the maximal parabolic \(P\) defined in Section 3. The number \( \frac{1}{2} \dim ((km)^{2n}) \) is equal to the dimension of the unipotent radical of the parabolic subgroup of \(GL_{2nmk}\) whose Levi part is \(GL_{2n} \times \ldots \times GL_{2n}\) where \(GL_{2n}\) appears \(mk\) times. Thus its dimension is equal to \(2n^2km(km-1)\). Comparing this with the dimension of \(U(P)\), equation (11) follows.

We end this section with a theorem regarding the unramified computation. To simplify notation we shall assume that \( m = r \) is odd. We first define the local \(L\)-functions under consideration. In general, if \( \tau \) is a local unramified representation of an \(m\)-fold cover of \(G\), it is a constituent of an unramified principal series representation, and corresponds to some \(k\)-tuple \( \chi = (\chi_1, \ldots, \chi_k) \), where \(\chi_i\) is an unramified character of \(F^*\). When \(m = 1\), the values of \(\chi_i\) at a local uniformizer are simply the Satake parameters of \(\tau\). For the \(m\)-fold cover, we define local \(L\)-functions at unramified places by using the standard definition in the case of linear groups, but replacing each \(\chi_i\) with its \(m\)-th power.

For example, let \(\tau^{(m)}_\nu = \text{Ind}_{B_{GL}(m)^{1/2}}^{GL_{B_{GL}}(m)} \chi^{1/2} \) denote the local unramified component of \(\tau^{(m)}\) at a finite place \(\nu\). Here \(B_{GL}\) is the Borel subgroup of \(GL\) and \(\chi = (\chi_1, \ldots, \chi_k)\) is an unramified character. Similarly let \(\pi^{(m)}_\nu = \text{Ind}_{B_{Sp}(m)}^{Sp_{B_{Sp}}(m)} \mu^{1/2}\). Let \(p\) be a generator of the maximal ideal in the ring of integers of the field \(F_{\nu}\) and \(q = |p|_{\nu}^{-1}\). Then the local standard tensor product \(L\) function is defined by

\[
L(s, \pi^{(m)}_\nu \times \tau^{(m)}_\nu) = \prod_{i=1}^{n} \prod_{j=1}^{k} \frac{1}{(1 - \mu_i^{m}(p)\chi_j^{m}(p)q^{-s})(1 - \mu_i^{-m}(p)\chi_j^{-m}(p)q^{-s})(1 - \chi_j^{m}(p)q^{-s})}.
\]

In a similar way one can define the local standard \(L\)-function \(L(s, \tau^{(m)}_\nu)\), the local exterior square \(L\)-function \(L(s, \tau^{(m)}_\nu, \lambda^2)\) and the local symmetric square \(L\)-function \(L(s, \tau^{(m)}_\nu, \nu^2)\).
It follows from Theorem 2 that for Re(s) large, integral (8) is a product of local integrals. At a finite unramified place \( \nu \), the local integral is given by

\[
Z(s, \pi^{(m)}, \tau^{(m)}) = \int_G \int_{U_0} \omega_{\pi^{(m)}}(g) f_W(\delta u_0(1, g), s) \psi_U(u_0) du_0 dg.
\]

Here \( \omega_{\pi^{(m)}}(g) \) is the spherical function of \( \pi^{(m)} \), and \( f_W \) is the unramified function obtained at the place \( \nu \) from the factorizable function \( f_W(\varepsilon^{(m)}_\nu) \). (We have dropped the subscripts \( \nu \) to condense the notation.) We have

**Theorem 3.** Suppose that \( m = r \) is odd. Then for Re(s) large, the integral 
\[
Z(s, \pi^{(m)}, \tau^{(m)})
\]

is equal to

\[
L(\alpha s - \frac{\alpha - 1}{2}, \pi^{(m)} \times \tau^{(m)})
\]

\[
\frac{L(\alpha(s - \frac{1}{2}) + nm + \frac{1}{2}, \tau^{(m)}) \prod_{j=1}^{nm} L(\alpha(2s - 1) + 2j, \tau^{(m)}, \lambda^2) \prod_{j=1}^{nm-1} L(\alpha(2s - 1) + 2j + 1, \tau^{(m)}, \nu^2)}
\]

where \( \alpha = m(2nmk + 1) \).

We remark that, working in the context of Brylinski-Deligne extensions, Gao [Ga] also attaches Euler products to pairs of automorphic representations on covering groups. He does so by studying the constant term of Eisenstein series on covering groups, that is, by means of a generalization of the Langlands-Shahidi method. However, it is not clear if his Euler products are the same as those obtained in this note or not.

5. Residues of Eisenstein Series on Covers of the General Linear Group

In this section we discuss Conjecture 1 of subsection 2.2 above. One way to study this Conjecture is by considering the various constant terms of the corresponding Eisenstein series. This reduces the problem of determining the poles to the study of the poles of certain intertwining operators. This in turn reduces to the study of the poles of the partial \( L \)-function \( L^S(s, \tau^{(m)} \times \tilde{\tau}^{(m)}) \), where \( S \) is a finite set of places including all archimedean places such that the representation \( \tau^{(m)}_\nu \) is unramified for \( \nu \notin S \). Here \( \tilde{\tau}^{(m)} \) is the contragredient representation of \( \tau^{(m)} \). The partial \( L \)-function is by definition the product over \( \nu \notin S \) of local \( L \)-functions defined similarly to (12).

When \( m = 1 \), Jacquet, Piatetski-Shapiro and Shalika studied this \( L \)-function by means of the Rankin-Selberg method. See for example Gelbart and Shahidi [G-S], Section 1.7, for an overview of these constructions. Using this method one may establish that the partial \( L \)-function has a simple pole at \( s = 1 \). Then it follows from [1], for example, that Conjecture 1 holds. The key property that makes this work when \( m = 1 \) is the uniqueness of the Whittaker model. Unfortunately, this uniqueness does not hold when \( m > 1 \).

Suzuki studies these Eisenstein series and their residues for higher covers in the last section of his paper [Su]. In order to establish that the residue exists, he assumes the existence of a generalized Shimura lifting, that is, a correspondence between irreducible cuspidal automorphic representations of the group \( GL_n^{(m)}(\mathbb{A}) \) and automorphic representations of the group \( GL_n(\mathbb{A}) \) which satisfies certain properties. At the moment, such a lifting has only been proved in full for covers of \( GL_2 \) (Flicker, [F]). If one had the lift in general and if the lifted automorphic representation of \( GL_n(\mathbb{A}) \) was cuspidal, then it would follow from the...
case $m = 1$ that the partial $L$-function $L^S(s, \tau^{(m)} \times \hat{\tau}^{(m)})$ has a simple pole at $s = 1$. This would then imply Conjecture [1]. We summarize:

**Proposition 3.** Assume that $\pi^{(m)}$ satisfies the two conditions in [Su], Section 8.5, p. 752. Then Conjecture [1] holds.

An alternative method to study the residues of Eisenstein is given in Jacquet and Rallis [J-R]. In this method one constructs a certain global nonzero integral which involves the residue representation. The non-vanishing of the global period then implies that the residue is nonzero. When $m = 2$ we propose the following construction. For simplicity let us consider the case of the maximal parabolic Eisenstein series. Let $E^{(2)}(g, s)$ denote the Eisenstein series defined on the group $GL_{2k}(\mathbb{A})$, which is associated with the induced representation

$$\text{Ind}_{P_{2,k}(\mathbb{A})}^{GL_{2k}(\mathbb{A})} (\tau^{(2)} \otimes \hat{\tau}^{(2)}) \delta_{P_{2,k}}.$$

Let $E^{(2)}_{\tau(2)}(g)$ denote the residue of this series at the point $s = (2k + 1)/4k$. Our goal is to prove that the representation generated by these residues is nonzero. Let $\Theta^{(2)}_{Sp}$ denote the theta representation of the group $Sp_{2k}(\mathbb{A})$. It is the minimal representation of this group. We have the following result.

**Proposition 4.** Let $\theta^{(2)}_{Sp}$ be a vector in the space of $\Theta^{(2)}_{Sp}$. Then the integral

$$\int_{Sp_{2k}(F) \backslash Sp_{2k}(\mathbb{A})} E^{(2)}_{\tau(2)}(g) \theta^{(2)}_{Sp}(g) \, dg$$

converges absolutely. Moreover, for some choice of data the integral is not identically zero. In particular the residue representation is nonzero.

Unfortunately, at this point we do not have a way to extend this result to higher order covers.

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