A QUANTITATIVE SUBSPACE BALIAN-LOW THEOREM

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Abstract. Let $G \subset L^2(\mathbb{R})$ be the subspace spanned by a Gabor Riesz sequence $(g, \Lambda)$ with $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$ of rational density. It was shown recently that if $g$ is well-localized both in time and frequency, then $G$ cannot contain any time-frequency shift $\pi(z)g$ of $g$ with $z \in \mathbb{R}^2 \setminus \Lambda$. In this paper, we improve the result to the quantitative statement that the $L^2$-distance of $\pi(z)g$ to the space $G$ is equivalent to the Euclidean distance of $z$ to the lattice $\Lambda$, in the sense that the ratio between those two distances is uniformly bounded above and below by positive constants. On the way, we prove several results of independent interest, one of them being closely related to the so-called weak Balian-Low theorem for subspaces.

1. Introduction

The Balian-Low theorem is a well known and fundamental result in time-frequency analysis, which asserts that a Gabor system cannot be a Riesz basis for $L^2(\mathbb{R})$ if its generating window is well localized both in time and frequency. More precisely, it states the following:

**Theorem 1.1** (Balian-Low Theorem). Let $g \in L^2(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice such that the Gabor system $\{e^{2\pi ibx}g(x-a) : (a,b) \in \Lambda\}$ is a Riesz basis for $L^2(\mathbb{R})$ (and therefore $\Lambda$ is of density 1). Then

$$\left(\int x^2 |g(x)|^2 \, dx\right) \left(\int \omega^2 |\hat{g}(\omega)|^2 \, d\omega\right) = \infty.$$ (1.1)

Recently, the following generalization of the Balian-Low theorem was proved in [4] (see also [6] for a similar generalization of the amalgam Balian-Low theorem).

**Theorem 1.2** ([4]). Let $g \in L^2(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice of rational density such that the Gabor system $\{e^{2\pi ibx}g(x-a) : (a,b) \in \Lambda\}$ is a Riesz basis for its closed linear span $G(g, \Lambda)$. If there exists a time-frequency shift $e^{2\pi iux}g(x-u)$, $(u,\eta) \in \mathbb{R}^2 \setminus \Lambda$, of $g$ which is contained in $G(g, \Lambda)$, then (1.1) holds.

Note that condition (1.1) is equivalent to having $g \notin H^1(\mathbb{R})$ or $\hat{g} \notin H^1(\mathbb{R})$, where $H^1(\mathbb{R})$ denotes the usual Sobolev space in $L^2(\mathbb{R})$ of regularity order 1. Therefore, Theorem 1.2 can be rephrased as follows: if $g, \hat{g} \in H^1(\mathbb{R})$, then the time-frequency shift $e^{2\pi iux}g(x-u)$ has a positive $L^2$-distance to the space $G(g, \Lambda)$ whenever $(u,\eta) \in \mathbb{R}^2$ has a positive Euclidean distance to the lattice $\Lambda$. As our main result, we are going to prove the following quantitative version of Theorem 1.2 which relates the two mentioned distances. In the sequel, we denote by $\mathbb{H}^1(\mathbb{R})$ the set of all $g \in H^1(\mathbb{R})$ satisfying $\hat{g} \in H^1(\mathbb{R})$.

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Theorem 1.3. Let $g \in H^1(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice of rational density such that \( \{e^{2\pi ibx}g(x-a) : (a,b) \in \Lambda \} \) is a Riesz basis for its closed linear span $\mathcal{G}(g, \Lambda)$. Then there exist constants $C_1, C_2 > 0$ such that for all $(u, \eta) \in \mathbb{R}^2$ we have

$$C_1 \cdot \text{dist} \left( (u, \eta), \Lambda \right) \leq \text{dist} \left( e^{2\pi ibx}g(x-u), \mathcal{G}(g, \Lambda) \right) \leq C_2 \cdot \text{dist} \left( (u, \eta), \Lambda \right). \quad (1.2)$$

The upper bound in (1.2) in fact holds for any $g \in H^1(\mathbb{R})$ and any lattice $\Lambda \subset \mathbb{R}^2$, regardless of \( \{e^{2\pi ibx}g(x-a) : (a,b) \in \Lambda \} \) being a Riesz sequence or the lattice $\Lambda$ having rational density; besides, an explicit constant $C_2$ can be found easily; see Proposition 4.1 below. On the other hand, finding an explicit constant $C_1$ is more elusive. Even in the case where $(g, \Lambda)$ forms an orthonormal system, we were only able to derive a constant $C_1$ such that (1.2) holds for $(u, \eta)$ close to the lattice $\Lambda$; see Theorem 5.3. We expect such a constant to depend on the Riesz bounds of \( \{e^{2\pi ibx}g(x-a) : (a,b) \in \Lambda \} \) and on the norms \( \|g\|_{L^2}, \|g\|_{H^1}, \) and \( \|g\|_{H^1}. \)

Quantitative Balian-Low estimates for general elements in the Gabor space. Writing $\pi(u, \eta)f(x) = e^{2\pi ibx}f(x-u)$, one might wonder whether the estimate

$$\text{dist} \left( (\pi(u, \eta)f, \mathcal{G}(g, \Lambda)) \right) \asymp \|f\|_{L^2}$$

holds for general $f \in \mathcal{G}(g, \Lambda)$ and not just for $f = g$. In general this is not the case. Indeed, if $g$ is a Gaussian, then $(g, 2\mathbb{Z} \times \mathbb{Z}^2)$ is a Riesz basis for its closed linear span $\mathcal{G}(g, 2\mathbb{Z} \times \mathbb{Z}^2)$, but there exists a function \( 0 \neq f \in \mathcal{G}(g, 2\mathbb{Z} \times \mathbb{Z}^2) \) satisfying $f(-1) \in \mathcal{G}(g, 2\mathbb{Z} \times \mathbb{Z}^2)$ (see Example A.17 for details). Therefore, we see that the distance $\text{dist} \left( (1, 0)f, \mathcal{G}(g, 2\mathbb{Z} \times \mathbb{Z}^2) \right)$ vanishes, even though $\text{dist}((1,0), 2\mathbb{Z} \times \mathbb{Z}^2) \cdot \|f\|_{L^2} \neq 0$.

Implications regarding the OFDM communication scheme. One motivation for analyzing the distance of the time-frequency shift $\pi(u, \eta)g$ to the Gabor space $\mathcal{G}(g, \Lambda)$ stems from the communication scheme called orthogonal frequency division multiplexing (OFDM). In OFDM, the sender wants to transmit the sequence $g \in \mathcal{G}(g, \Lambda)$ to the receiver. This is done by selecting a fixed Riesz sequence $(\pi(k\alpha, \ell\beta)g)_{k,\ell \in \mathbb{Z}}$ to form the transmission signal $Fg = \sum_{k,\ell \in \mathbb{Z}} c_{k,\ell} (\pi(k\alpha, \ell\beta)g)$, which is then sent to the receiver through a communication channel. Mathematically, the effect of the channel is modeled as a linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$; that is, the signal that arrives at the receiver is $TFg$ instead of $Fg$.

The first step of the reconstruction procedure in OFDM is to apply the reconstruction operator $R$ given by $RFf = (\langle f, \pi(k\alpha, \ell\beta)g^* \rangle)_{k,\ell \in \mathbb{Z}}$ to the signal $TFg$, thereby obtaining the sequence $\tilde{c} = RTFg \in \ell^2(\mathbb{Z}^2)$. Here, $g^*$ is the dual window for the Riesz sequence $(\pi(k\alpha, \ell\beta)g)_{k,\ell \in \mathbb{Z}}$; i.e., $g^* \in \mathcal{G} : = \mathcal{G}(g, \alpha \mathbb{Z} \times \beta \mathbb{Z})$ satisfies the biorthogonal property $\langle \pi(k\alpha, \ell\beta)g, \pi(k'\alpha, \ell'\beta)g^* \rangle = \delta_{k,k'} \delta_{\ell,\ell'}$ for $k, k', \ell, \ell' \in \mathbb{Z}$. At least for the ideal reconstruction channel $T = \text{Id}_{L^2(\mathbb{R})}$, this guarantees perfect reconstruction, meaning that $\tilde{c} = c$. For more general channels, this is not the case, but one might hope to reconstruct $c$ by applying a suitable (linear) post-processing operator $P$ to $\tilde{c}$.

In fact, there exists such a bounded post-processing operator $P$ satisfying $PRTFg = c$ for all $c \in \ell^2(\mathbb{Z}^2)$ if and only if the operator $RT$ is bounded below on the Gabor space $\mathcal{G}$, meaning that $\|RTf\|_{L^2} \gtrsim \|f\|_{L^2}$ for all $f \in \mathcal{G}$. It is not hard to see that $\|Rfh\|_{L^2} \asymp \|Pfh\|_{L^2}$ for $h \in L^2(\mathbb{R})$, where we denote by $P$ the orthogonal projection onto the Gabor space $\mathcal{G}$. For the important special case that $T$ is a pure time-frequency shift $\pi(u, \eta)$, reconstruction is thus possible if and only if $\|P\pi(u, \eta)f\|_{L^2} \gtrsim \|f\|_{L^2}$ for all $f \in \mathcal{G}$, which is equivalent to the existence of a constant $c < 1$ satisfying

$$\text{dist} \left( (\pi(u, \eta)f, \mathcal{G}) \right) = \|(I - P)\pi(u, \eta)f\|_{L^2} \leq c \|f\|_{L^2}, \quad \forall f \in \mathcal{G}. \quad (1.3)$$
The term dist(\(\pi(u, \eta)f, \mathcal{G}\)) measures the off-band energy loss caused by the time-frequency shift \(\pi(u, \eta)\), that is, the proportion of the signal energy that gets “pushed out of the Gabor space” by applying the time-frequency shift \(\pi(u, \eta)\). Even in the case where the off-band energy loss is small enough so that (1.3) holds, it is interesting to know more precise upper and lower bounds for this quantity, since it influences the stability of the reconstruction. Theorem 1.3 shows that in the case \(f = g\), the off-band energy loss is of the order dist((u, \eta), \alpha \mathbb{Z} \times \beta \mathbb{Z}).

**Structure of the proof.** The outline of the proof of Theorem 1.3 is as follows. First of all, we note that it suffices to establish the inequality (1.2) for \((u, \eta)\) in a neighborhood of the origin \((0, 0)\); the inequality then holds for all \((u, \eta) \in \mathbb{R}^2\) (with possibly different constants \(C_1\) and \(C_2\)) by Theorem 1.2 and a compactness argument. In order to analyze the behavior of the quantity dist(\(\pi(u, \eta)g, \mathcal{G}(g, \Lambda)\)) for \((u, \eta)\) close to the origin, we first show that the time-frequency map \(S_g : (a, b) \mapsto \pi(a, b)g\) is differentiable at \((0, 0)\) with (Fréchet) derivative \((a, b) \mapsto -ag' + 2\pi ibXg\), where \(X\) is the position operator defined formally by \(Xf(x) = xf(x)\); see Lemma 3.2. We then prove in Proposition 4.4 that \(-ag' + 2\pi ibXg\) is not contained in \(\mathcal{G}(g, \Lambda)\) unless \(a = b = 0\). Denoting by \(P\) the orthogonal projection from \(L^2(\mathbb{R})\) onto \(\mathcal{G}(g, \Lambda)\), this implies that there exists a constant \(\gamma > 0\) with \(\|(I - P)(-ag' + 2\pi ibXg)\|_{L^2} \geq \gamma \|\pi(a, b)\|_{L^2}\) for all \((a, b) \in \mathbb{R}^2\). The claim then follows immediately because \((a, b) \mapsto (I - P)(-ag' + 2\pi ibXg)\) linearizes the map \((a, b) \mapsto (I - P)(e^{2\pi ibx}g(x - a))\) in a neighborhood of \((0, 0)\).

The main ingredients of the proof are thus the differentiability of the time-frequency map (see Section 3) and the fact that none of its directional derivatives \(-ag' + 2\pi ibXg\) with \((a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}\) are contained in \(\mathcal{G}(g, \Lambda)\) (see Proposition 4.4). While the former is probably folklore (although we could not find a reference), the latter seems to be a new result and should be interesting in its own right. We also point out a close relationship between Proposition 4.4 and the weak Balian-Low theorem for subspaces from [19]; see Remark 4.5 for a detailed discussion.

As mentioned above, we were unable to derive a closed-form formula for the constant \(C_1\) in Equation 1.2. However, if we assume the Gabor system \(\{e^{2\pi ibx}g(x - a) : (a, b) \in \Lambda\}\) to be orthonormal, then we can find an explicit constant \(C_1 > 0\) such that (1.2) holds for all \((u, \eta)\) in a neighborhood of the lattice \(\Lambda\); see Theorem 5.4. This result then leads to a statement similar to Theorem 1.3 but without assuming the rational density of \(\Lambda\); see Corollary 5.5.

The paper is organized as follows: In Section 2 we show how the main properties that we are interested in (the regularity of \(g\), the property of \((g, \Lambda)\) being a Riesz sequence, and the distance dist(\(\pi(\mu), \mathcal{G}(g, \Lambda)\))) can be described via the Zak transform and certain associated matrix multiplication operators. Section 3 contains the aforementioned differentiability result for the time-frequency map of \(\mathbb{H}^1(\mathbb{R})\) functions. The proof of Theorem 1.3 is given in Section 4. Finally, in Section 5 we provide an explicit local lower bound \(C_1\) in the case where the Gabor system is orthonormal.

Several results that are technical or only tangentially related to the core arguments are deferred to A. Although most of them should be well-known or be considered folklore, we either give detailed references or include their proofs for the sake of completeness.
2. Preparations

Notation. Let us begin with collecting some notation which will be used throughout the paper. We set $\mathbb{N} := \{1, 2, \ldots \}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The closure of a subset $M \subset X$ of a metric space $X$ will be denoted by $\overline{M}$. The Lebesgue measure of a Borel set $E \subset \mathbb{R}^n$ is denoted by $\lambda(E)$. If $g : \mathbb{R} \to \mathbb{C}$ is measurable, we write $Xg$ for the function $x \mapsto xg(x)$, that is, $(Xg)(x) = xg(x)$, $x \in \mathbb{R}$.

Let $\mathcal{H}$ be a Hilbert space, and $\Phi = (\varphi_i)_{i \in I}$ be a family of vectors in $\mathcal{H}$. This family is called a frame for $\mathcal{H}$ if $A \| f \|_{\mathcal{H}}^2 \leq \sum_{i \in I} |\langle f, \varphi_i \rangle|^2 \leq B \| f \|_{\mathcal{H}}^2$ for all $f \in \mathcal{H}$ and certain constants $A, B \in (0, \infty)$. If $\Phi$ is a frame for its closed linear span $\text{span}(\varphi_i : i \in I)$, then we say that $\Phi$ is a frame sequence. We say that $\Phi$ is a Riesz sequence if there are $A, B \in (0, \infty)$ such that $A \| c \|_{\ell^2} \leq \| \sum_{i \in I} c_i \varphi_i \| \leq B \| c \|_{\ell^2}$ for all finitely supported sequences $c = (c_i)_{i \in I} \in \ell^2(I)$. If $\Phi$ is a Riesz sequence and span$\{\varphi_i : i \in I\}$ is dense in $\mathcal{H}$, we say that $\Phi$ is a Riesz basis for $\mathcal{H}$. Each Riesz basis is a frame.

Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a (complex) Hilbert space $\mathcal{H}$. The spectrum of $T$ will be denoted by $\sigma(T)$; that is, $\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not boundedly invertible} \}$.

We denote by $g(T)$ the complement set of $\sigma(T)$ in $\mathbb{C}$ which is called the resolvent set of $T$. For a bounded linear operator $A : \mathcal{H} \to \mathcal{K}$ between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we define

$$\sigma_0(A) := \sqrt{\min \sigma(A^*A)} \in [0, \infty) \quad \text{and} \quad \sigma_1(A) := \sqrt{\inf \{\sigma(A^*A) \setminus \{0\}\}} \in [0, \infty]. \quad (2.1)$$

Note that $\sigma(A^*A) = \{0\}$ if and only if $A = 0$, in which case we have $\sigma_1(A) = \infty$; on the other hand, we have $\sigma_1(A) < \infty$ for $A \neq 0$. If $A$ is a matrix, then $\sigma_0(A)$ is the smallest singular value of $A$, while $\sigma_1(A)$ is the smallest positive singular value of $A$.

We occasionally consider the vector-valued $L^2$ space $L^2(\Omega; \mathbb{C}^k)$, which we equip with the inner product $(f, g) = \int \langle f(\omega), g(\omega) \rangle_{\mathbb{C}^k} d\mu(\omega)$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}^k}$ denotes the standard inner product on $\mathbb{C}^k$.

The Fourier transform $\hat{g}$ of $g \in L^2(\mathbb{R})$ is defined by

$$\hat{g}(\omega) := \lim_{R \to \infty} \int_{-R}^R g(x) e^{-2\pi i x \omega} \, dx,$$

where the limit is taken in $L^2(\mathbb{R})$. For $a, b \in \mathbb{R}$ we also define the time-frequency shift operator

$$[\pi(a, b)g](x) := e^{2\pi ibx} g(x - a), \quad x \in \mathbb{R},$$

which can be expressed as $\pi(a, b) = M_b T_a$ where $T_a$ and $M_b$ denote the operators of translation by $a \in \mathbb{R}$ and modulation by $b \in \mathbb{R}$, respectively. For $k \in \mathbb{N}$, we set $\mathbb{H}^k(\mathbb{R}) := \{ f \in H^k(\mathbb{R}) : f \in H^k(\mathbb{R}) \}$, with the usual (complex-valued) $L^2$-Sobolev space $H^k(\mathbb{R}) = W^{k,2}(\mathbb{R})$.

A lattice in $\mathbb{R}^2$ is a set $\Lambda = A \mathbb{Z}^2$ with $A \in \text{GL}(2, \mathbb{R})$. Its density is defined as $|\det A|^{-1}$. If $\Lambda$ is a lattice in $\mathbb{R}^2$ and $g \in L^2(\mathbb{R})$, we denote by $(g, \Lambda)$ the Gabor system generated by $g$ and $\Lambda$, that is,

$$(g, \Lambda) := \{ \pi(\lambda)g : \lambda \in \Lambda \}.$$

The Gabor space generated by $g$ and $\Lambda$ is defined as $G(g, \Lambda) := \text{span}(g, \Lambda)$, with the closure taken in $L^2(\mathbb{R})$. 


The Zak transform of \( g \in L^2(\mathbb{R}) \) is defined as

\[
Zg(x, \omega) = \lim_{N \to \infty} \sum_{k=-N}^{N} e^{2\pi i k \omega} g(x - k), \quad (x, \omega) \in (0, 1)^2,
\]

where the limit is taken in \( L^2((0, 1)^2) \). The Zak transform \( g \mapsto Zg \) is a unitary operator from \( L^2(\mathbb{R}) \) to \( L^2((0, 1)^2) \). In the following, we will consider the Zak transform \( Zg \) of \( g \in L^2(\mathbb{R}) \) as (a.e. defined) function on \( \mathbb{R}^2 \), by using Equation (2.2) on all of \( \mathbb{R}^2 \), where the limit is taken in \( L^2_{loc}(\mathbb{R}^2) \). This extended Zak transform has the following properties (all of which hold for a.e. \((x, \omega) \in \mathbb{R}^2\)):

(a) \( Zg(x + m, \omega + n) = e^{2\pi i m \omega} Zg(x, \omega) \) for all \( m, n \in \mathbb{Z} \).
(b) \( Z[\pi(u, \eta)]g(x, \omega) = e^{2\pi i n \omega} Zg(x - u, \omega - \eta) \) for all \((u, \eta) \in \mathbb{R}^2\).
(c) \( (Z[\pi(m, n)]g)(x, \omega) = e^{2\pi i (nx - mw)} Zg(x, \omega) \) for all \( m, n \in \mathbb{Z} \).
(d) \( Zg(x, \omega) = e^{2\pi i x \omega} Zg(-\omega, x) \).
(e) \( g(x) = \int_{0}^{1} Zg(x, \omega) d\omega \) and \( \hat{g}(\omega) = \int_{0}^{1} e^{-2\pi i x \omega} Zg(x, \omega) \, dx \).

For all these properties, we refer to [12, Chapter 8]. The property (a) of \( Zg \) is called quasi-periodicity.

### 2.1. Reduction to matrix multiplication operators

In this subsection, we show that the properties and quantities that we are interested in—the distance \( \text{dist}(\pi(\mu)g, G(g, \Lambda)) \) and whether \((g, \Lambda)\) is a Riesz sequence—can be conveniently reformulated using certain matrix multiplication operators

\[
M_{\Lambda} : L^2(\Omega; \mathbb{C}^k) \to L^2(\Omega; \mathbb{C}^\ell), \quad f \mapsto A(\cdot) f(\cdot).
\]

Here, the matrix function \( A : \Omega \to \mathbb{C}^{\ell \times k} \) will be defined using the Zak transform. More details regarding these matrix multiplication operators can be found in [11].

We start by considering the Gabor system \((g, \Lambda)\) associated to the lattice \( \Lambda = \frac{1}{d} \mathbb{Z} \times \mathbb{Z} P \) (where \( P, Q \in \mathbb{N} \)) and connect the spectral properties of the frame operator

\[
S : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g
\]

and the Gram operator \( G : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2) \) defined by

\[
G(c_{n,k})_{n,k \in \mathbb{Z}} = \left( \sum_{n,k \in \mathbb{Z}} c_{n,k} \pi(Q^{-1}n, Pk)g, \pi(Q^{-1}m, P\ell)g \right)_{m,\ell \in \mathbb{Z}}
\]

to matrix multiplication operators on the domain \( R_P := (0, \frac{1}{P}) \times (0, 1) \). This relies on using the unitary operators \( V : L^2((0,1)^2) \to L^2(R_P, \mathbb{C}^P) \) and \( U : \ell^2(\mathbb{Z}^2) \to L^2(R_P, \mathbb{C}^Q) \), defined by

\[
(Vf)(x, \omega) := (f(x + \frac{k}{P}, \omega))^P_{k=0} \quad \text{and} \quad Uc = \left( \sum_{s,n \in \mathbb{Z}} c_{s+Q\ell,n} \right)_{\ell=0}^{Q-1},
\]

where \( f \in L^2((0,1)^2) \) and \( c = (c_{n,m})_{n,m \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2) \), and where we use the function \( e_{s,n}(x, \omega) := P^{1/2} e^{2\pi i (nPx - sw)} \) defined for \((x, \omega) \in R_P \). Furthermore, we denote by \( S_n \in \mathbb{C}^{n \times n} \) the cyclic shift operator satisfying \( S_n e_i = e_{i-1} \) for \( i \in \{1, \ldots, n - 1\} \) and \( S_n e_0 = e_{n-1} \) for the standard basis \( \{e_0, \ldots, e_{n-1}\} \) of \( \mathbb{C}^n \). Finally, for \( \omega \in \mathbb{R} \) we define the matrices

\[
L_\omega := S_P \text{ diag}(e^{2\pi i \omega}, 1, \ldots, 1) \in \mathbb{C}^{P \times P}
\]
and

\[ R_\omega = \text{diag}(e^{-2\pi \iota \omega}, 1, \ldots, 1) S_Q^{-1} \in \mathbb{C}^{Q \times Q}. \]

**Lemma 2.1.** For \( P, Q \in \mathbb{N} \) and \( g \in L^2(\mathbb{R}) \), \( g \neq 0 \), let us define the matrix function \( A_g : \mathbb{R}^2 \to \mathbb{C}^{P \times Q} \) by

\[ A_g(x, \omega) := \frac{1}{P} (Zg(x + \frac{k}{P} - \frac{\ell}{Q}, \omega))_{k,\ell=0}^{P-1,Q-1}. \]

Then for a.e. \((x, \omega) \in \mathbb{R}^2\) we have

\[ A_g(x + \frac{k}{P}, \omega) = L_\omega A_g(x, \omega) \quad \text{and} \quad A_g(x - \frac{\ell}{Q}, \omega) = A_g(x, \omega) R_\omega. \tag{2.4} \]

In particular, \( A_g(A_g) \) is \((\frac{1}{P}, 1)\)-periodic and \( A_g A_g^* \) is \((\frac{1}{Q}, 1)\)-periodic.

If \( \Lambda = \frac{1}{Q} \mathbb{Z} \times P \mathbb{Z} \), then \((g, \Lambda)\) is a Bessel sequence if and only if \( Zg \in L^\infty(\mathbb{R}^2) \). In this case, the synthesis operator

\[ T : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R}), \quad (c_{m,n})_{n,m \in \mathbb{Z}} \mapsto \sum_{n,m \in \mathbb{Z}} c_{m,n} \pi(Q^{-1}n, Pm)g, \]

the frame operator \( S \), and the Gram operator \( G \) of \((g, \Lambda)\) satisfy

\[ T = (VZ)^* M_{A_g} U, \quad S = (VZ)^* M_{A_g A_g^*} (VZ), \quad \text{and} \quad G = U^* M_{A_g A_g^*} U, \tag{2.5} \]

respectively, where \( M_{A_g A_g^*} \) (respectively \( M_{A_g^* A_g} \) or \( M_{A_g} \)) is the matrix multiplication operator (cf. [A.1]) with respect to \( A_g A_g^* \) (resp. \( A_g^* A_g \) or \( A_g \)) acting on \( L^2(R_P; \mathbb{C}^P) \) (resp. \( L^2(R_P; \mathbb{C}^Q) \)).

If \( Zg \in L^\infty(\mathbb{R}^2) \), the following statements hold:

(a) \((g, \Lambda)\) is a Riesz sequence if and only if \( \text{essinf}_{z \in \mathbb{R}^2} \sigma_0(A_g(z)) > 0 \).

(b) \((g, \Lambda)\) is a frame sequence if and only if \( \text{essinf}_{z \in \mathbb{R}^2} \sigma_1(A_g(z)) > 0 \).

(c) \((g, \Lambda)\) is a frame for \( L^2(\mathbb{R}) \) if and only if \( \text{essinf}_{z \in \mathbb{R}^2} \sigma_0(A_g(z)) > 0 \).

**Proof.** Let \( A := A_g \). We have \( A(x + \frac{1}{P}, \omega) = P^{-\frac{1}{2}} \cdot (Zg(x + \frac{k+1}{P} - \frac{\ell}{Q}, \omega))_{k,\ell=0}^{P-1,Q-1} \), where—due to the quasi-periodicity of \( Zg \)—we see that

\[ Zg(x + \frac{k+1}{P} - \frac{\ell}{Q}, \omega) = \begin{cases} \sqrt{P} \cdot (A(x, \omega))_{k+1,\ell} & \text{if } k < P - 1, \\ e^{2\pi i \omega} Zg(x - \frac{k}{P}, \omega) = \sqrt{P} \cdot e^{2\pi i \omega} \cdot (A(x, \omega))_{0,\ell} & \text{if } k = P - 1. \end{cases} \]

In matrix notation, this means precisely that \( A \) satisfies the first relation in (2.4), and the \((\frac{1}{P}, 1)\)-periodicity of \( A^* A \) follows from \( L^*_0 L_\omega = \text{Id}_{C^P} \) and from \( A(x, \omega + 1) = A(x, \omega) \). The second relation in (2.4) can be proved similarly and shows that \( AA^* \) is \((\frac{1}{Q}, 1)\)-periodic.

Let \( T_0 \) denote the pre-synthesis operator of \((g, \Lambda)\), that is,

\[ T_0 : \ell_0(\mathbb{Z}^2) \to L^2(\mathbb{R}), \quad T_0(c_{m,n})_{n,m \in \mathbb{Z}} := \sum_{n,m \in \mathbb{Z}} c_{m,n} \pi(\frac{m}{P}, nP) g, \]

where \( \ell_0(\mathbb{Z}^2) \) is the space of all elements of \( \ell^2(\mathbb{Z}^2) \) with only finitely many non-zero entries. For \( c \in \ell_0(\mathbb{Z}^2) \), the properties of the Zak transform listed after Equation (2.2) show that

\[ (Z T_0 c)(x, \omega) = \sum_{m,n \in \mathbb{Z}} c_{m,n} Z[\pi(\frac{m}{P}, nP)] g(x, \omega). \]
\[ = P^{-1/2} \sum_{\ell=0}^{Q-1} h_\ell(x,\omega) Zg(x - \frac{x}{Q},\omega) = \langle A(x,\omega)h(x,\omega),e_0\rangle_{\mathbb{C}^P}, \]

where \( h_\ell(x,\omega) := P^{1/2} \sum_{k,n\in\mathbb{Z}} e^{2\pi i (nPx - \omega)} \) and \( h := (h_\ell)_{\ell=0}^{Q-1} = \mathcal{U}c \) with \( \mathcal{U} \) defined in Equation (2.3). Since \( h \) is \((\frac{1}{P},1)\)-periodic, we obtain for \( k \in \{0,\ldots,P-1\} \) that

\[
(Z \mathbf{T}_0 c)(x + \frac{k}{P},\omega) = \langle A(x + \frac{k}{P},\omega)h(x,\omega),e_0\rangle_{\mathbb{C}^P} = \langle A(x,\omega)h(x,\omega),e_k\rangle_{\mathbb{C}^P}.
\]

Here, we used the identity \( A(x + \frac{k}{P},\omega) = L_\omega A(x,\omega) \) from the beginning of the proof to get

\[
\langle A(x + \frac{k}{P},\omega)h(x,\omega),e_0\rangle = \langle L_k^k A(x,\omega)h(x,\omega),e_0\rangle = \langle A(x,\omega)h(x,\omega),(L_k^k e_0)\rangle
\]

for \( k = 0,\ldots,P-1 \). With the operator \( V \) defined in Equation (2.3), we have thus shown

\[
(VZ \mathbf{T}_0 c)(x,\omega) = A(x,\omega)h(x,\omega); \text{ that is, } VZ \mathbf{T}_0 = M_A \mathcal{U}|_{\ell_0(\mathbb{Z}^2)}.
\]

Since the operators \( V, Z, \mathcal{U} \) are unitary, this shows that \( \mathbf{T}_0 \) is bounded if and only if \( M_A \) is bounded, that is, if and only if each entry of \( A \) is essentially bounded (on \( \mathbb{R}^2 \)), which—by quasi-periodicity—exactly means that \( Zg \in L^\infty(\mathbb{R}^2) \). In particular, this shows that \( (g,\Lambda) \) is a Bessel sequence if and only if \( \mathbf{T}_0 \) is bounded, if and only if \( Zg \in L^\infty(\mathbb{R}^2) \).

Let us assume for the rest of this proof that \( Zg \in L^\infty(\mathbb{R}^2) \). Then \( VZ \mathbf{T} = M_A \mathcal{U} \), where \( \mathbf{T} = \mathbf{T}_0 = (VZ)^* M_A \mathcal{U} \) is the synthesis operator of \( (g,\Lambda) \). Clearly, \( M_A = M_{A^*} \) is the (bounded) multiplication operator with \( A^* \); thus \( M_A^* M_A = M_{A^*} A_A \) and \( M_A^* M_A = M_{AA^*} \).

By definition, \( (g,\Lambda) \) is a Riesz sequence if and only if the synthesis operator \( \mathbf{T} \) is bounded below. Lemma [A.3] shows that this holds if and only if \( G = \mathbf{T}^* \mathbf{T} \) is boundedly invertible, that is, if and only if \( 0 \in \varrho(G) \). Similarly, \( (g,\Lambda) \) is a frame for \( L^2(\mathbb{R}) \) if and only if \( 0 \in \varrho(S) \). Likewise, \( (g,\Lambda) \) is a frame sequence if and only if \( (0,\varepsilon_0) \subset \varrho(G) \) for some \( \varepsilon_0 > 0 \) (see Lemmas [A.2] and [A.3]). Hence, \( (g,\Lambda) \) is a Riesz sequence if and only if \( 0 \in \varrho(M_{A^*} A) \), a frame sequence if and only if \( (0,\varepsilon_0) \subset \varrho(M_{A^*} A) \) for some \( \varepsilon_0 > 0 \), and a frame for \( L^2(\mathbb{R}) \) if and only if \( 0 \in \varrho(M_{AA^*}) \).

The statements (a)–(c) now follow from Lemma [A.1] (ii) and (iii). Here, it is used for properties (a) and (b) that \( \sigma_i(\Lambda^g) \) only depends on \( \Lambda^g \), which is \((P^{-1},1)\)-periodic, so that \( \text{essinf}_{z \in \mathbb{C}} \sigma(A^g(z)) = \text{essinf}_{z \in R^*_p} \sigma(A^g(z)) \) for \( i \in \{1,2\} \). Finally, for property (c), it is used that if \( (g,\Lambda) \) is a frame for \( L^2(\mathbb{R}) \), then \( P/Q \leq 1 \) (see [12] Corollary 7.5.1), so that \( R_Q \subset R_P \). This implies \( \text{essinf}_{z \in \mathbb{C}} \sigma(\Lambda^g(z)) = \text{essinf}_{z \in R^*_p} \sigma(\Lambda^g(z)) \), since \( z \mapsto \Lambda^g(z) \) is \((Q^{-1},1)\)-periodic. Conversely, if \( \text{essinf}_{z \in \mathbb{C}} \sigma(\Lambda^g(z)) > 0 \), then we also have \( \text{essinf}_{z \in R^*_p} \sigma(\Lambda^g(z)) > 0 \), so that \( 0 \in \varrho(M_{AA^*}) \) by Lemma [A.1].

In the next lemma, we derive a formula for the matrix function \( A^g \) associated to the dual window \( \tilde{g} \) of the Riesz sequence \( (g,\Lambda) \). This means—that considered on the Gabor space \( \mathcal{G}(g,\Lambda) \)—the Gabor system \( (\tilde{g},\Lambda) \) is the canonical dual frame to \( (g,\Lambda) \). In the proof of this lemma, we will use that \( \tilde{g} = S^* g \) satisfies this property, where \( S^* \) is the pseudo-inverse (see [A.2]) of the (pre)frame operator \( S \) of \( (g,\Lambda) \), which is given by \( Sf = \sum_{\lambda \in \Lambda} \langle f,\pi(\lambda)g \rangle \pi(\lambda)g \). For completeness, we sketch a proof of this fact. Let \( S_0 := |S_{g}(g,\Lambda)| : \mathcal{G}(g,\Lambda) \to \mathcal{G}(g,\Lambda) \) denote the restriction of \( S \) to \( \mathcal{G}(g,\Lambda) \). Note that \( S_0 \) is invertible since \( (g,\Lambda) \) is a frame for \( \mathcal{G}(g,\Lambda) \), and that \( \mathcal{G}(g,\Lambda) = \text{ran} \, S = (\text{ker} \, S^* \mathbb{P}) \) since \( S \) is self-adjoint. Therefore, the pseudo-inverse of \( S \) is given by \( S^* = S_0^{-1} \mathbb{P} \), where \( \mathbb{P} \) denotes
the orthogonal projection from $L^2(\mathbb{R})$ onto $G(g, \Lambda)$. Hence, $g_0 := S^1 g = S_0^{-1} g$. Finally, a straightforward but tedious computation shows that $\pi(\lambda) S = S_0^{\lambda}$ for all $\lambda \in \Lambda$, which also implies that $\pi(\lambda) G(g, \Lambda) = G(\lambda g, \Lambda)$. Therefore, $S_0^{\lambda} g_0 = \pi(\lambda) g_0 = \pi(\lambda) g$, showing that $(\pi(\lambda) g_0)_{\lambda \in \Lambda} = (S_0^{\lambda} g)_{\lambda \in \Lambda}$ is indeed the canonical dual frame of $(g, \Lambda)$.

With this preparation, we can now prove the announced lemma.

**Lemma 2.2.** Let $g \in L^2(\mathbb{R})$, $P, Q \in \mathbb{N}$, $A = \frac{1}{2} \mathbb{Z} \times P \mathbb{Z}$, and assume that $(g, \Lambda)$ is a Riesz sequence. Let $\tilde{g}$ be the dual window of $(g, \Lambda)$ and $G := Zg$, $\tilde{G} := Z\tilde{g}$, $A := A_g$, and $A := A_{\tilde{g}}$, with $A_g$ and $A_{\tilde{g}}$ as in Lemma 2.1. Then

\[ \tilde{A} = A(A^*)^{-1} \quad \text{almost everywhere on } \mathbb{R}^2. \]  

Moreover, for arbitrary $\mu = (u, \eta) \in \mathbb{R}^2$ we have

\[ \text{dist}^2(\pi(\mu) g, G(g, \Lambda)) = \|g\|^2_{L^2} - \int_0^1 \int_0^1 |H_\mu(x, \omega) e_0|_{E'}^2 \, dx \, d\omega, \]

where $e_0 = (1, 0, \ldots, 0)^T \in \mathbb{C}^Q$,

\[ H_\mu(z) = P^{1/2} \cdot A(z) \cdot (A(z)^* A(z))^{-1} \cdot A(z)^* \cdot e^{2\pi i n D P} \cdot A(z - \mu) \in \mathbb{C}^P \times Q, \]

and $D P = \text{diag}(k/P)_{k=0}^{P-1}$, with the notation $e^{2\pi i n D P} := \text{diag}((e^{2\pi i n k/P})_{k=0}^{P-1})$.

**Proof.** In this proof we shall make use of the notion of the pseudo-inverse $T^\dagger$ of an operator $T$ with closed range. For the definition of this notion and a review of some of its properties, we refer to A.2.

Let $S$ be the frame operator of $(g, \Lambda)$. Then the range of $S$ is ran $S = G(g, \Lambda)$, which is closed in $L^2(\mathbb{R})$. Hence, by Lemma A.6 we have $S^1 = \varphi(S)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by $\varphi(0) = 0$ and $\varphi(t) = 1/t$ for $t \neq 0$. As seen before the statement of the lemma, $\tilde{g} = S^1 g = \varphi(S) g$. Furthermore, Lemma A.1 shows that $\varphi(A(z) A(z)^*) = (A(z) A(z)^*)^\dagger$ for every $z \in R_P$. Hence, an application of Equation (2.5) and of Lemma A.1 (iv) shows that

\[ \tilde{G} = Z\tilde{g} = Z[\varphi(S)]g = Z(VZ)^* \varphi(M_{A^*} VZ)g = V^* M_{\varphi(A^*)} V G, \]

with $V$ as defined in Equation (2.3). Therefore,

\[ \text{V} \tilde{G} = (A A^*)^\dagger (V G) \quad \text{a.e. on } R_P. \]  

(2.7)

In order to extend this relation to $\mathbb{R}^2$, define $\tilde{(Vf)}(x, \omega) := (f(x + \frac{k}{P}, \omega))_{k=0}^{P-1}$ for $f : \mathbb{R} \to \mathbb{C}$ and $(x, \omega) \in \mathbb{R}^2$. Let $z = (x, \omega) \in R_P$ be arbitrary, and set $z_{n,k} = (x + \frac{n}{P}, \omega)$ for $k \in \{0, \ldots, P-1\}$ and $n \in \mathbb{Z}$. Using Equation (2.4), we see that

\[ \tilde{G}(z_{n,k}) = (Z \tilde{g})(x + \frac{n}{P}, \omega) = \sqrt{P} \cdot \left( A_{\tilde{g}}(x + \frac{n}{P}, \omega) \right)_{k=0}^{k,0} \]

\[ = \sqrt{P} \cdot \left( L^n \omega A_{\tilde{g}}(x, \omega) \right)_{k=0}^{k,0} = \left( L^n \omega \left( \tilde{G}(z_{0,k}) \right)_{k=0}^{P-1} \right)_{k=0}^{k}. \]

Similarly, Equation (2.4) shows that $(G(z_{n,k}))_{k=0}^{P-1} = L^n \omega(G(z_{0,k}))_{k=0}^{P-1}$. Thus, we get for $(x, \omega) \in R_P$ and $n \in \mathbb{Z}$ that

\[ (\text{V} \tilde{G})(x + \frac{P}{P}, \omega) = \left( \tilde{G}(z_{n,k}) \right)_{k=0}^{P-1} = L^n \omega \left( \tilde{G}(z_{0,k}) \right)_{k=0}^{P-1} = L^n \omega (V \tilde{G})(x, \omega) \]

(2.7) = $L^n \omega \left[ A(x, \omega) A(x, \omega)^* \left( G(z_{0,k}) \right)_{k=0}^{P-1} \right.$

(2.7) = $L^n \omega \left[ L^n \omega A(x + \frac{P}{P}, \omega) A(x + \frac{P}{P}, \omega)^* L^n \omega \right] \left. \right]_k$. 

8

A. CARAGEA, D.G. LEE, F. PHILIPP, AND F. VOIGTLAENDER
In particular, projection from $\mathcal{G}(g, \Lambda)$ is the
\[ \langle \mathcal{G}(g, \Lambda) \rangle = \left[ A(x + \frac{\mu}{2}, \omega)A(x + \frac{\mu}{2}, \omega)^* \right]\left( G(z_{n,k}) \right)_{k=0}^{P-1} \]
\[ = \left[ A(x + \frac{\mu}{2}, \omega)A(x + \frac{\mu}{2}, \omega)^* \right]\left( \tilde{V}G \right)(x + \frac{\mu}{2}, \omega). \]

In combination with the 1-periodicity in the second variable of all involved functions, this implies
\[ \tilde{V}G = (AA^*)^\dagger (\tilde{V}G) \quad \text{a.e. on } \mathbb{R}^2. \]

Since $(\tilde{V}G)(x + \frac{\mu}{2}, \omega)$ is the $\ell$-th column of the matrix $\sqrt{P} \cdot A(x, \omega)$, since $(\tilde{V}G)(x + \frac{\mu}{2}, \omega)$ is the $\ell$-th column of $\sqrt{P} \cdot \tilde{A}(x, \omega)$, and because $AA^*$ is $(\frac{1}{2}, 1)$-periodic, we obtain the identity $\tilde{A} = (AA^*)^\dagger A = A(A^*)^\dagger A^{-1} = 1$, see Lemma A.5 (iv). Here, we used that $A^* A$ is invertible almost everywhere by Lemma 2.1 (a). We have thus proved Equation (2.6).

Now, denote the orthogonal projection from $L^2(\mathbb{R})$ onto $\mathcal{G}(g, \Lambda) = \text{ran } S$ by $P$. Then, for any $\mu = (u, \eta) \in \mathbb{R}^2$ we have
\[ \text{dist}^2(\pi(\mu)g, \mathcal{G}(g, \Lambda)) = \| (I - P) \pi(\mu)g \|_{L^2}^2 = \| g \|_{L^2}^2 - \| P \pi(\mu)g \|_{L^2}^2. \]

Next, Lemmas A.6 and A.1 show $M_{AA^*}^{1} = \varphi(M_{AA^*}^{2}) = M_{\varphi(A_{A^*}^{2})} = M_{\varphi(A_{A^*}^{2})}^1$, which implies $S^1 = (\sqrt{Z})^* M_{\varphi(A_{A^*}^{2})}^1 \sqrt{Z} = (\sqrt{Z})^* M_{\varphi(A_{A^*}^{2})}^1 \sqrt{Z}$ thanks to Equation (2.5) and Corollary A.2. Now, Lemma A.3 shows $P = SS^\dagger$. Hence, Equations (2.3) and (2.6) show
\[ P = (\sqrt{Z})^* M^{\dagger}_{AA^*} M_{(AA^*)^\dagger}^1 (\sqrt{Z}) = (\sqrt{Z})^* M_{(AA^*)^\dagger}^1 (\sqrt{Z}) = (\sqrt{Z})^* M^{\dagger}_{ran A} (\sqrt{Z}) \]
and $P_{ran A} = P_{ran (AA^*)} = (AA^*)^\dagger A^* A = A(A^*)^\dagger A^* = (AA^*)^{-1} A^*$. For arbitrary $f \in L^2(\mathbb{R})$, we thus see that
\[ \| \pi(\mu)g \|_{L^2}^2 = \| M_{ran A} \sqrt{Z}f \|_{L^2(\mathbb{R}, dx dw)}^2 = \int_0^1 \int_0^{1/P} \| A(A^*)^{-1} A^* \sqrt{Z}f \|_{L^2(\mathbb{R}, dx dw)}^2. \]

Finally, since $Z(\pi(\mu)g)(x, \omega) = e^{2\pi igx} Zg(x - u, \omega - \eta)$ for $\mu = (u, \eta)$, we see
\[ (\sqrt{Z}(\pi(\mu)g))(x, \omega) = \left( e^{2\pi igx + \frac{k}{2}} Zg(x + \frac{k}{2} - u, \omega - \eta) \right)_{k=0}^{P-1} = e^{2\pi igx} e^{2\pi ig D^P} (Zg(x + \frac{k}{2} - u, \omega - \eta))_{k=0}^{P-1} = e^{2\pi igx} e^{2\pi ig D^P} A(x - u, \omega - \eta) \xi_{0}. \]

Now, the claim follows from $|e^{2\pi igx}| = 1$. \qed

In proving the next result, we crucially use that if $\lambda = (\alpha, \beta) \in \mathbb{R}^2$ and $\mu = (a, b) \in \mathbb{R}^2$, then $\pi(\lambda) \pi(\mu) f = e^{-2\pi i a b} \pi(\lambda + \mu) f$, as can be verified by a direct calculation. In particular, this implies $\| T \pi(\lambda) \pi(\mu) f \|_{L^2} = \| T \pi(\lambda + \mu) f \|_{L^2}$ for any linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

**Lemma 2.3.** Let $g \in L^2(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}$ be a lattice. If $P$ denote the orthogonal projection from $L^2(\mathbb{R})$ onto $\mathcal{G}(g, \Lambda)$. Then $P$ commutes with the operators $\pi(\lambda)$, $\lambda \in \Lambda$. In particular
\[ \text{dist} (\pi(\mu) f, \mathcal{G}(g, \Lambda)) = \text{dist} (\pi(\mu) f, \mathcal{G}(g, \Lambda)) \quad \forall \mu \in \mathbb{R}^2, f \in L^2(\mathbb{R}), \lambda \in \Lambda. \]

**Proof.** Let $\mathcal{G} = \mathcal{G}(g, \Lambda)$. Lemma A.10 shows $\pi(-\lambda) \mathcal{G} \subset \mathcal{G}$ for all $\lambda \in \Lambda$. This implies $\pi(\lambda) \mathcal{G}^\perp \subset \mathcal{G}^\perp$: Indeed, $\langle \pi(\lambda) f, h \rangle = e^{2\pi i \lambda_1 \lambda_2} \langle f, \pi(-\lambda) h \rangle = 0$ for $f \in \mathcal{G}^\perp$ and $h \in \mathcal{G}$. Now, if $f \in L^2(\mathbb{R})$, then we can write $f = f_1 + f_2$ with $f_1 \in \mathcal{G}$ and $f_2 \in \mathcal{G}^\perp$; hence, $\pi(\lambda) f = \pi(\lambda) f_1 + \pi(\lambda) f_2$ with $\pi(\lambda) f_1 \in \mathcal{G}$ and $\pi(\lambda) f_2 \in \mathcal{G}^\perp$, which implies $P[\pi(\lambda) f] = \pi(\lambda) f_1 = \pi(\lambda)[P f]$. \qed
As to the “in particular”-part, we observe for $\mu \in \mathbb{R}^2$ and $\lambda \in \Lambda$ that
\[
\| (I - \mathcal{P}) \pi (\mu + \lambda) f \|_{L^2} = \| (I - \mathcal{P}) \pi (\lambda) \pi (\mu) f \|_{L^2} \\
= \| \pi (\lambda) (I - \mathcal{P}) \pi (\mu) f \|_{L^2} = \| (I - \mathcal{P}) \pi (\mu) f \|_{L^2}.
\]
The claim now follows by noting $\text{dist}(f, \mathcal{G}) = \| (I - \mathcal{P}) f \|_{L^2}$ for $f \in L^2(\mathbb{R})$. □

2.2. Describing the regularity of $g$ via the Zak transform

The following lemma is probably folklore. However, since we could not find any reference for it (one direction is proved in [8, Proof of Thm. 2.3]), we give a full proof here. Recall that $\mathbb{H}^1(\mathbb{R}) = \{ f \in H^1(\mathbb{R}) : \hat{f} \in H^1(\mathbb{R}) \}$.

**Lemma 2.4.** Let $g \in L^2(\mathbb{R})$. Then $g \in \mathbb{H}^1(\mathbb{R})$ if and only if $Zg \in H^1_{\text{loc}}(\mathbb{R}^2)$. In this case, the weak derivatives of $Zg$ are given by
\[
\partial_1 Zg = Z(g') \quad \text{and} \quad \partial_2 Zg(x, \omega) = -2\pi i [Z(Xg)(x, \omega) - x \cdot Zg(x, \omega)]. \quad (2.8)
\]

**Proof.** “$\Rightarrow$” Assume that $g \in \mathbb{H}^1(\mathbb{R})$ and let $V \subset \mathbb{R}$ be nonempty, open, and bounded. Let us first assume that $g \in C_c^\infty(\mathbb{R})$ (such a function of course is in $\mathbb{H}^1(\mathbb{R})$). Recalling the definition (2.2) of the Zak transform, we see that on $V$, $Zg$ is defined by a finite sum (hence $Zg \in C^\infty(V)$), and the first relation in (2.8) is easily verified. For the second relation, we note
\[
\partial_2 [Zg(x, \omega)] = \sum_{k=-\infty}^{\infty} \partial_\omega [e^{2\pi ik\omega} g(x - k)] = \sum_{k=-\infty}^{\infty} 2\pi i e^{2\pi ik\omega} g(x - k)
\]
\[
= \sum_{k=-\infty}^{\infty} e^{2\pi ik\omega} [2\pi i x g(x - k) - 2\pi i Xg(x - k)]
\]
\[
= 2\pi i \cdot [x \cdot Zg(x, \omega) - Z[Xg](x, \omega)],
\]
as claimed in (2.8).

Now, let $g \in \mathbb{H}^1(\mathbb{R})$ be arbitrary. Since $C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$ (see for instance [2, Section E10.8]), we find a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R})$ which converges to $g$ in $H^1(\mathbb{R})$, that is, $\varphi_n \to g$ and $\varphi'_n \to g'$ in $L^2(\mathbb{R})$. For $\phi \in C_c^\infty(V)$ we then have (with $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(V)}$)
\[
|\langle Zg, \partial_1 \phi \rangle + |\langle Zg', \phi \rangle| \\
\leq |\langle Z(g - \varphi_n), \partial_1 \phi \rangle| + |\langle Z\varphi_n, \partial_1 \phi \rangle| + |\langle Z(g' - \varphi'_n), \phi \rangle|.
\]
The middle term vanishes by partial integration and since $\partial_1 (Z\varphi_n) = Z(\varphi'_n)$; the other two terms tend to zero as $n \to \infty$. Hence, $\langle Zg, \partial_1 \phi \rangle = -\langle Zg', \phi \rangle$.

The relation $\langle Zg, \partial_2 \phi \rangle = 2\pi i \langle Z(Xg) - XZg, \phi \rangle$ is proven similarly, by noting that since $(1 + |X|)g \in L^2(\mathbb{R})$, one can find a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R})$ satisfying $\|(1 + |X|)(g - \varphi_n)\|_{L^2} \to 0$, whence $\varphi_n \to g$ in $L^2$ and $X \varphi_n \to Xg$ in $L^2$, and therefore $Z(X\varphi_n) \to Z(Xg)$ and $X Z\varphi_n \to XZg$ with convergence in $L^2_{\text{loc}}(\mathbb{R}^2)$.

Because of $Zg' \in L^2(V)$ and $Z(Xg) - XZg \in L^2(V)$, this proves that $Zg \in H^1(V)$ and that (2.8) holds on $V$. Since $V \subset \mathbb{R}^2$ was an arbitrary non-empty, open, bounded set, we have proved one implication.

$^3$Indeed, given $\varepsilon > 0$, there is a compactly supported $h$ such that $\|(1 + |X|)(g - h)\|_{L^2} \leq \varepsilon$, say $\text{supp } h \subset [-N, N]$. Pick $\varphi \in C_c^\infty(\mathbb{R})$ satisfying $\| \varphi - h \|_{L^2} < \varepsilon/(1 + 2N)$ and supp $\varphi \subset [-2N, 2N]$, whence $\|(1 + |X|)(\varphi - h)\|_{L^2} \leq (1 + 2N) \| \varphi - h \|_{L^2} < \varepsilon$, so that finally $\|(1 + |X|)(g - \varphi)\|_{L^2} < 2\varepsilon$. 
“⇐” Assume that $G := Zg \in H^1_{\text{loc}}(\mathbb{R}^2)$. Lemma 14 shows that, after changing $G$ on a null-set, we can assume that $G(x, \cdot)$ is locally absolutely continuous on $\mathbb{R}$ with derivative $(\partial_2 G)(x, \cdot) \in L^2_{\text{loc}}(\mathbb{R})$ for almost every $x \in \mathbb{R}$ and simultaneously that $G(\cdot, \omega)$ is locally absolutely continuous on $\mathbb{R}$ with derivative $(\partial_1 G)(\cdot, \omega) \in L^2_{\text{loc}}(\mathbb{R})$ for almost every $\omega \in \mathbb{R}$.

According to the properties of the Zak transform, $g(x) = \int_{\mathbb{R}}^1 G(x, \omega) \, d\omega$ for almost all $x \in \mathbb{R}$; see the list of properties below Equation (2.2). Let us fix one $x_0 \in \mathbb{R}$ for which this is true. Hence, for almost all $x \in \mathbb{R}$ we have

$$g(x) = \int_{0}^{1} G(x, \omega) \, d\omega = \int_{0}^{1} \left( G(x_0, \omega) + \int_{x_0}^{x} \partial_1 G(t, \omega) \, dt \right) \, d\omega$$

$$= g(x_0) + \int_{x_0}^{x} \int_{0}^{1} \partial_1 G(t, \omega) \, d\omega \, dt = g(x_0) + \int_{x_0}^{x} \phi(t) \, dt,$$

where $\phi(t) := \int_{0}^{t} \partial_1 G(t, \omega) \, d\omega$. Note that $\phi \in L^1_{\text{loc}}(\mathbb{R})$ since $\partial_1 G \in L^1_{\text{loc}}(\mathbb{R}^2)$. Hence, possibly after redefining $g$ on a set of measure zero, $g$ is locally absolutely continuous on $\mathbb{R}$. To see that actually $\phi \in L^2(\mathbb{R})$ (and hence $g \in H^1(\mathbb{R})$), recall from the properties of the Zak transform that $G(t + n, \omega) = e^{2\pi i n \omega} G(t, \omega)$ for almost all $(t, \omega) \in \mathbb{R}^2$. Hence,

$$\|\phi\|_{L^2}^2 = \sum_{n \in \mathbb{Z}} \int_{0}^{1} \left| \int_{0}^{1} \partial_1 G(t + n, \omega) \, d\omega \right|^2 \, dt = \int_{0}^{1} \sum_{n \in \mathbb{Z}} \left| \int_{0}^{1} e^{2\pi i n \omega} \partial_1 G(t, \omega) \, d\omega \right|^2 \, dt.$$

Now, set $g_t(\omega) := \partial_1 G(t, \omega)$ (which is in $L^2((0,1))$ for a.e. $t \in \mathbb{R}$). Then

$$\int_{\mathbb{R}} |\phi(t)|^2 \, dt = \int_{0}^{1} \sum_{n \in \mathbb{Z}} |g_t(n)|^2 \, dt = \int_{0}^{1} \int_{0}^{1} |g_t(\omega)|^2 \, d\omega \, dt = \|\partial_1 G\|^2_{L^2([0,1]^2)}.$$

Hence, $g \in H^1(\mathbb{R})$ with $g'(x) = \int_{0}^{1} \partial_1 G(x, \omega) \, d\omega$.

To see that also $\widehat{g} \in H^1(\mathbb{R})$, define $F : \mathbb{R}^2 \to \mathbb{C}, (x, \omega) \mapsto e^{-2\pi i x \omega} G(x, \omega)$. Since $G_x := G(x, \cdot)$ is locally absolutely continuous for almost all $x \in \mathbb{R}$, the product rule for Sobolev functions (see for instance [2 Section 4.25]) shows that also $F_x := F(x, \cdot)$ satisfies this property. Moreover, the product rule also shows for almost all $x \in \mathbb{R}$ that we have

$$F_x'(\omega) = e^{-2\pi i x \omega} (-2\pi i x G_x(\omega) + G_x'(\omega))$$

$$= e^{-2\pi i x \omega} \left(-2\pi i x G(x, \omega) + (\partial_2 G)(x, \omega)\right) = H(x, \omega)$$

for almost all $\omega \in \mathbb{R}$. Note that $H \in L^2_{\text{loc}}(\mathbb{R}^2)$, since $G \in H^1_{\text{loc}}(\mathbb{R}^2)$. This easily implies that the function $\psi : \mathbb{R} \to \mathbb{C}, \omega \mapsto \int_{0}^{1} H(x, \omega) \, dx$, is almost everywhere well-defined and satisfies $\psi \in L^1_{\text{loc}}(\mathbb{R})$.

Next, recall the inversion formula of the Zak transform (see the list of properties below Equation (2.2)), stating $\widehat{g}(\omega) = \int_{0}^{1} e^{-2\pi i x \omega} G(x, \omega) \, dx = \int_{0}^{1} F(x, \omega) \, dx$ for almost all $\omega \in \mathbb{R}$. Fix some $\omega_0 \in \mathbb{R}$ for which this holds, and note for almost all $\omega \in \mathbb{R}$ that

$$\widehat{g}(\omega) = \int_{0}^{1} F_x(\omega) \, dx = \int_{0}^{1} \left( F_x(\omega_0) + \int_{\omega_0}^{\omega} F'_x(\gamma) \, d\gamma \right) \, dx = \widehat{g}(\omega_0) + \int_{\omega_0}^{\omega} \psi(\gamma) \, d\gamma.$$

Hence—possibly after changing $\widehat{g}$ on a null-set—we see that $\widehat{g}$ is locally absolutely continuous, with $\widehat{g}'(\omega) = \psi(\omega)$, so that it remains to show $\psi \in L^2(\mathbb{R})$.

To see this, note for arbitrary $n \in \mathbb{Z}$ that $G_x(\omega + n) = Zg(x, \omega + n) = Zg(x, \omega) = G_x(\omega)$, and hence also $G'_x(\omega + n) = G'_x(\omega)$, which finally implies for almost all $x \in \mathbb{R}$ that
\[
H(x, \omega + n) = e^{-2\pi inx}H(x, \omega) \quad \text{for almost all } \omega \in \mathbb{R}.
\]
Therefore, we see for any \(n \in \mathbb{Z}\) that \(\psi(x + n) = \int_{0}^{1} e^{-2\pi inx}H(x, \omega)\, dx = \hat{H}_\omega(n)\), where \(H_\omega\) is defined by \(H_\omega(x) := H(x, \omega)\) for \(x \in [0, 1]\), so that \(H_\omega \in L^2([0, 1])\) for almost all \(\omega \in \mathbb{R}\). Thus, we finally arrive at
\[
\int_{\mathbb{R}} |\psi(\omega)|^2 \, d\omega = \sum_{n \in \mathbb{Z}} \int_{0}^{1} |\psi(\omega + n)|^2 \, d\omega = \int_{0}^{1} \sum_{n \in \mathbb{Z}} |\hat{H}_\omega(n)|^2 \, d\omega
\]
\[
= \int_{0}^{1} \|H_\omega\|_{L^2}^2 \, d\omega = \int_{0}^{1} \int_{0}^{1} |H(x, \omega)|^2 \, dx \, d\omega < \infty. \quad \square
\]

### 2.3. Symplectic operators and the regularity of the dual window

In this subsection, we show that if \((g, \Lambda)\) is a Riesz sequence with \(g \in \mathbb{H}^1(\mathbb{R})\), then the canonical dual window \(\hat{g}\) belongs to \(\mathbb{H}^1(\mathbb{R})\) as well.

For proving this—and also several other results—we shall make use of so-called symplectic operators to generalize statements involving lattices of the form \(Q^{-1}\mathbb{Z} \times P\mathbb{Z}\), \(P, Q \in \mathbb{N}\), to general lattices of rational density. To explain this, let \(\Lambda \subset \mathbb{R}^2\) be such a general lattice of rational density. Then there exists a matrix \(B \in \mathbb{R}^{2 \times 2}\) with \(\det B = 1\) such that \(BA = Q^{-1}\mathbb{Z} \times P\mathbb{Z}\) with \(P, Q \in \mathbb{N}\) co-prime. Indeed, we have \(\Lambda = AZ^2\) for some \(A \in \mathbb{R}^{2 \times 2}\) with \(\det A \in \mathbb{Q}\backslash\{0\}\), that is, \(|\det A| = P/Q\) for some co-prime \(P/Q \in \mathbb{N}\).

Now define \(B_0 := |\det A|^{1/2} \cdot A^{-1}\) if \(\det A > 0\), and if instead \(\det A < 0\), then let \(B_0 := |\det A|^{1/2} \cdot \text{diag}(-1, 1) \cdot A^{-1}\). It is not hard to check that \(\det B_0 = 1\), and that \(B_0\Lambda = |\det A|^{1/2}\mathbb{Z}^2\). Thus, the matrix \(B := \text{diag}((PQ)^{-1/2}, (PQ)^{1/2})B_0\) satisfies \(\det B = 1\) and \(BA = Q^{-1}\mathbb{Z} \times P\mathbb{Z}\).

Next, since \(\det B = 1\), we see from [12] Lemma 9.4.1 and Equation (9.39)] that there is a unitary operator \(U_B : L^2(\mathbb{R}) \to L^2(\mathbb{R})\) satisfying
\[
U_B \rho(z) = \rho(Bz)U_B \quad \text{for all } z \in \mathbb{R}^2,
\]
where
\[
\rho(a, b)f := e^{-\pi iab} \pi(a, b)f, \quad \text{for } f \in L^2(\mathbb{R}) \text{ and } a, b \in \mathbb{R}.
\]
(10.0)

Such an operator \(U_B\) is called symplectic. As a consequence of Schur’s Lemma (see [12] Lemma 9.3.2)], the operator \(U_B\) is unique up to multiplication with unimodular constants; thus, we see for \(B, B_1, B_2 \in \text{SL}(2, \mathbb{R})\) that
\[
U_{B_1B_2} = c_{B_1,B_2}U_{B_1}U_{B_2} \quad \text{and} \quad U_B^* = c_B \cdot U_{B^{-1}}
\]
(11.1)

for certain constants \(c_{B_1,B_2}, c_B \in \mathbb{C}\) with \(|c_{B_1,B_2}| = 1 = |c_B|\).

For us, an important property of symplectic operators is that they leave \(\mathbb{H}^1(\mathbb{R})\) invariant. To see this, recall from [4] discussion around Equation (4.5)] that each matrix \(B \in \text{SL}(2, \mathbb{R})\) can be written as a product of matrices of the form
\[
B_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B_0^{(1)} := \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}, \quad \text{and} \quad B_0^{(2)} := \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}
\]
with \(\alpha, \beta \in \mathbb{R}\backslash\{0\}\). Furthermore, if we define operators \(D_\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R})\) and \(C_\beta : L^2(\mathbb{R}) \to L^2(\mathbb{R})\) by \(D_\alpha f(x) := |\alpha|^{1/2} \cdot f(\alpha x)\) and \(C_\beta f(x) := e^{\pi i \beta x^2} \cdot f(x)\), then a direct computation shows that the choices \(U_{B_0^{(1)}} := D_\alpha\) and \(U_{B_0^{(2)}} := C_\beta\) make (2.9) valid. Likewise, if we let \(U_{B_0} := \mathcal{F}\) be the Fourier transform, then (2.9) is satisfied as well.

Thus, in view of (2.11), it suffices to show that \(\mathbb{H}^1(\mathbb{R})\) is invariant under the operators \(\mathcal{F}, D_\alpha, \) and \(C_\beta\). For \(\mathcal{F}\) and \(D_\alpha\), this is trivial. Finally, for \(C_\beta\) recall that \(f \in L^2(\mathbb{R})\)
is in $\mathbb{H}^1(\mathbb{R})$ if and only if $Xf \in L^2(\mathbb{R})$ and if $f$ is locally absolutely continuous with $f' \in L^2(\mathbb{R})$. As a consequence of the product rule for Sobolev functions (see for instance [2] Section 4.25), it follows that if $g \in \mathbb{H}^1(\mathbb{R})$, then $C_\beta g$ is locally absolutely continuous, with
\[
(C_\beta g)'(x) = 2\pi i \beta x \cdot e^{\pi i / \beta x^2} \cdot g(x) + e^{\pi i / \beta x^2} \cdot g'(x) \in L^2(\mathbb{R}).
\]
Since $XC_\beta g \in L^2(\mathbb{R})$ holds trivially, we have $C_\beta g \in \mathbb{H}^1(\mathbb{R})$, as desired.

To see an application of symplectic operators, note that if $\Lambda$ is a lattice of rational density with $BA = Q^{-1}\mathbb{Z} \times P\mathbb{Z}$ for some $B \in \text{SL}(2, \mathbb{R})$, and if $g \in L^2(\mathbb{R})$ is such that $(g, \Lambda)$ is a Riesz sequence, one may define
\[
g_1 := UBg \quad \text{and} \quad \Lambda_1 := B\Lambda = \frac{1}{c}\mathbb{Z} \times P\mathbb{Z}.
\]
Then (2.4) implies $\pi(B\lambda)g_1 = c_\lambda U_B \pi(\lambda)g$, $\lambda \in \Lambda$, where $c_\lambda = c_\lambda(B)$ is a unimodular constant. Hence, $(g_1, \Lambda_1)$ is a Riesz basis for its closed linear span $G(g_1, \Lambda_1) = U_B G(g, \Lambda)$. This reduction to the separable lattice $\Lambda_1$ will be crucial in the proof of the following proposition.

**Proposition 2.5.** Let $g \in L^2(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice of rational density such that $(g, \Lambda)$ is a Riesz sequence. Let $\tilde{g}$ be the dual window of $(g, \Lambda)$. Then $g \in \mathbb{H}^1(\mathbb{R})$ if and only if $\tilde{g} \in \mathbb{H}^1(\mathbb{R})$.

**Proof.** Let us first prove the claim for $\Lambda = Q^{-1}\mathbb{Z} \times P\mathbb{Z}$, where $P, Q \in \mathbb{N}$. Assume that $g \in \mathbb{H}^1(\mathbb{R})$. By Lemma 2.3, $Zg \in H^1_{\text{loc}}(\mathbb{R}^2)$. Let us denote by $A_g$ and $A_\beta$ the matrix functions introduced in Lemma 2.1. Using that lemma, we conclude that each entry of $A_g$ is contained in $L^\infty(\mathbb{R}^2)$ and that there exists $c > 0$ such that $\sigma_0(A_g(z)) \geq c$ for a.e. $z \in \mathbb{R}^2$. Therefore, a combination of Equation (2.6) and Lemmas A.3 and A.10 shows that each entry of $A_g = A_g A_\beta^{-1} = (A_\beta^*)^{-1}$ is contained in $H^1_{\text{loc}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. In view of the definition of $A_\beta$, this shows that $Zg \in H^1_{\text{loc}}(\mathbb{R}^2)$, whence Lemma 2.4 implies $\tilde{g} \in \mathbb{H}^1(\mathbb{R})$. Since $(\tilde{g}, \Lambda)$ is also a Riesz basis for $G(g, \Lambda) = G(\tilde{g}, \Lambda)$ with $(g, \Lambda)$ being the dual Riesz basis, interchanging the roles of $g$ and $\tilde{g}$ in the above arguments shows that $\tilde{g} \in \mathbb{H}^1(\mathbb{R})$ implies $g \in \mathbb{H}^1(\mathbb{R})$.

Now, let $\Lambda \subset \mathbb{R}^2$ be an arbitrary lattice of rational density. As seen before Equation (2.4), there is a matrix $B \in \text{SL}(2, \mathbb{R})$ such that $\Lambda_1 := B\Lambda = Q^{-1}\mathbb{Z} \times P\mathbb{Z}$ for certain $P, Q \in \mathbb{N}$. Let $g_1 := UBg$. Then $(g_1, \Lambda_1)$ is a Riesz basis for $G(g_1, \Lambda_1) = U_B G(g, \Lambda)$. Furthermore, since $\pi(B\lambda)g_1 = c_\lambda UB \pi(\lambda)g$ for $\lambda \in \Lambda$, where $|c_\lambda| = 1$, it is not hard to see that the frame operator $S$ for $(g_1, \Lambda_1)$ is given by $S = UB SU_B^*$, where $S$ is the frame operator of $(g, \Lambda)$. Hence, as discussed before Lemma 2.2, the dual window of $(g_1, \Lambda_1)$ is given by via the pseudo-inverse as $\tilde{g}_1 = S_1 g_1 = UB S_1^* UBg = UB\tilde{g}$, where used that $S_1 = UB S_1 U_B^*$ due to Corollary A.7.

Now, suppose that $g \in \mathbb{H}^1(\mathbb{R})$. As seen in the discussion before this proposition, symplectic operators leave $\mathbb{H}^1(\mathbb{R})$ invariant; thus, $g_1 = UBg \in \mathbb{H}^1(\mathbb{R})$. Hence, by what we showed above, we see that $\tilde{g}_1 \in \mathbb{H}^1(\mathbb{R})$, which implies $\tilde{g} = U_B^* \tilde{g}_1 = c_B U_{B^{-1}} \tilde{g}_1 \in \mathbb{H}^1(\mathbb{R})$. Finally, by interchanging the roles of $g$ and $\tilde{g}$ we see that $\tilde{g} \in \mathbb{H}^1(\mathbb{R})$ implies $g \in \mathbb{H}^1(\mathbb{R})$. \qed

### 3. Differentiability of the Time-Frequency Map

In this section, we show that for $g \in \mathbb{H}^1(\mathbb{R})$ the map $(a, b) \mapsto e^{2\pi ibx}g(x - a)$ is differentiable at the origin, with the derivative given by $(a, b) \mapsto -a g' + 2\pi ibXg$. In the
proof, we will make use of the following simple estimate. Recall that the sinc function is defined by \( \operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x} \) for \( x \in \mathbb{R}\setminus\{0\} \) and \( \operatorname{sinc}(0) := 1 \).

**Lemma 3.1.** We have \( |\operatorname{sinc}(x) - e^{-ix}| \leq |x| \) for all \( x \in \mathbb{R}\setminus\{0\} \). Consequently, 
\[
|\operatorname{sinc}(x) - e^{-ix}| \leq \min\{2, \pi|x|\} \quad \text{for all } x \in \mathbb{R}.
\]  

**Proof.** The first inequality is equivalent to \( |\sin(x) - xe^{-ix}| \leq x^2 \) and thus to \( f(x) := (\sin(x) - x \cos(x))^2 + x^2 \sin^2(x) - x^4 \leq 0 \).

Since \( f \) is even, it suffices to prove \( f(x) \leq 0 \) for \( x > 0 \). We have
\[
f'(x) = 2x \sin(x) \left( \sin(x) - x \cos(x) \right) + 2x \sin^2(x) + 2x^2 \sin(x) \cos(x) - 4x^3 \]
\[
= 4x \sin^2(x) - 4x^3 = 4x(\sin(x) - x)(\sin(x) + x). 
\]

As \( \sin(x) < x \) and \( \sin(x) + x > 0 \) for \( x > 0 \), we have that \( f'(x) < 0 \) for \( x > 0 \). Since \( f(0) = 0 \), this proves the claim. Equation (3.1) is a direct consequence of the first estimate combined with \( |\operatorname{sinc}(x)| \leq 1 \) and \( |e^{-ix}| \leq 1 \).

For \( g \in L^2(\mathbb{R}) \) define the map \( S_g : \mathbb{R}^2 \to L^2(\mathbb{R}) \) by
\[
S_g(a, b) := \pi(a, b)g = e^{2\pi ib} g(\cdot - a), \quad a, b \in \mathbb{R}.
\]

It is well known (see e.g. [7, Lemma 2.9.2]) that \( S_g \) is continuous for every \( g \in L^2(\mathbb{R}) \). Here, we will show that \( S_g \) is differentiable if \( g \in H^1(\mathbb{R}) \). We will first prove the differentiability of \( S_g \) at the origin and then use it to prove the differentiability of \( S_g \) at arbitrary points \((u, \eta) \in \mathbb{R}^2\). For the convenience of the reader, we recall the notion of (Fréchet)-differentiability: The map \( S_g : \mathbb{R}^2 \to L^2(\mathbb{R}) \) is (Fréchet)-differentiable at \((u, \eta) \in \mathbb{R}^2\) if there is a bounded linear map \( T : \mathbb{R}^2 \to L^2(\mathbb{R}) \) satisfying \( \frac{S_g(u + a, \eta + b) - S_g(u, \eta) - T(a, b)}{\|a, b\|} \to 0 \) as \((a, b) \to 0\). Such a map \( T \) is referred to as the derivative of \( S_g \) at \((u, \eta) \), denoted by \( S'_g(u, \eta) \); see e.g. [17 Chapter XIII, §2] for more details.

**Lemma 3.2.** For any \( g \in H^1(\mathbb{R}) \), the map \( S_g \) is (Fréchet)-differentiable at \((0, 0)\) with
\[
S'_g(0, 0) \left( \begin{smallmatrix} a \\ \eta \end{smallmatrix} \right) = -ag' + 2\pi ibXg,
\]
where \( X \) is the position operator defined formally by \( Xf(x) = xf(x) \).

If \( g \in H^2(\mathbb{R}) \) (that is, \( g \in H^2(\mathbb{R}) \) and \( \hat{g} \in H^2(\mathbb{R}) \)), then
\[
\|S_g(a, b) - g - (-ag' + 2\pi ibXg)\|_{L^2} \leq C_g \cdot \|a, b\|_2^2 \quad \forall (a, b) \in \mathbb{R}^2, \quad (3.2)
\]
where
\[
C_g := 3\pi^2 \max\left\{ \|X^2g\|_{L^2}, \|\omega^2 \hat{g}\|_{L^2}, \|Xg'\|_{L^2} \right\}. \quad (3.3)
\]

**Remark.** As shown in Lemma 3.1, we indeed have \( Xg' \in L^2(\mathbb{R}) \) if \( g \in H^2(\mathbb{R}) \).

**Proof.** Let \( \Psi_g : \mathbb{R} \to L^2(\mathbb{R}) \) be defined by \( \Psi_g \left( \begin{smallmatrix} a \\ \eta \end{smallmatrix} \right) := -ag' + 2\pi ibXg \); in particular, \( \Psi_g \) is linear. We have to prove that
\[
\lim_{(a, b) \to (0, 0)} \frac{\|S_g(a, b) - g - \Psi_g \left( \begin{smallmatrix} a \\ \eta \end{smallmatrix} \right) \|_{L^2}}{\sqrt{a^2 + b^2}} = 0. \quad (3.4)
\]

To see this, we write
\[
[S_g(a, b) - g - \Psi_g \left( \begin{smallmatrix} a \\ \eta \end{smallmatrix} \right)](x)
\]
\[
= e^{2\pi ibx} g(x - a) - g(x) + ag'(x) - 2\pi ibxg(x)
\]
\[
= e^{2\pi ibx} (g(x - a) - g(x)) + (e^{2\pi ibx} - 1 - 2\pi ibx)g(x) + ag'(x)
\]
\[ e^{2\pi ibx}(T_ag-g+ag')(x) + [e^{2\pi ibx}-1-2\pi ibx]g(x) + a[1-e^{2\pi ibx}]g'(x). \] (3.5)

To estimate the middle term in (3.3), recall that \( \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} = \frac{e^{\pi ix} - e^{-\pi ix}}{2\pi ix} \) and hence
\[ e^{2\pi ibx} - 1 - 2\pi ibx = 2\pi ibxe^{\pi ibx} \left[ \text{sinc}(bx) - e^{-\pi ibx} \right]. \] (3.6)

Therefore,
\[ \int |e^{2\pi ibx} - 1 - 2\pi ibx|^2 |g(x)|^2 dx = 4\pi^2 b^2 \int x^2 \left| \text{sinc}(bx) - e^{-\pi ibx} \right|^2 |g(x)|^2 dx. \] (3.7)

Using the estimate (3.11), we find that this expression is not larger than
\[ 4\pi^4 |b|^3 \int_{|x| < 1/\sqrt{|b|}} x^2 |g(x)|^2 dx + 16\pi^2 b^2 \int_{|x| \geq 1/\sqrt{|b|}} x^2 |g(x)|^2 dx. \]

Hence, we obtain
\[ \left( \frac{\|(e^{2\pi ibx} - 1 - 2\pi ibx) \cdot g\|_{L^2}}{\sqrt{a^2 + b^2}} \right)^2 \leq \frac{1}{b^2} \int |e^{2\pi ibx} - 1 - 2\pi ibx|^2 |g(x)|^2 dx \]
\[ \leq 4\pi^4 |b| \| Xg \|_2^2 + 16\pi^2 \int_{|x| \geq |b|^{-1/2}} x^2 |g(x)|^2 dx, \] (3.8)

which tends to zero as \( b \to 0 \) as a consequence of \( Xg \in L^2 \) and the dominated convergence theorem.

For the first term in (3.5), observe that Plancherel’s theorem yields
\[ \|T_ag-g+ag'\|_{L^2}^2 = \int |e^{2\pi i(-a)x} - 1 - 2\pi i(-a)x|^2 |\hat{g}(x)|^2 dx. \] (3.9)

Thus, using that \( (\omega \mapsto \omega \cdot \hat{g}(\omega)) \in L^2 \), we can conclude from our calculations in (3.8) that
\[ 0 \leq \left( \frac{\|T_ag-g+ag'\|_{L^2}}{\sqrt{a^2+b^2}} \right)^2 \leq \frac{\|T_ag-g+ag'\|_{L^2}^2}{a^2} \xrightarrow{a \to 0} 0. \]

Finally, using the estimates \( |e^{2\pi ibx} - 1| \leq 2\pi |bx| \) and \( |e^{2\pi ibx} - 1| \leq 2 \), we can treat the last summand in (3.5) as follows:
\[ \|a \cdot [1 - e^{2\pi ibx}] \cdot g'(x)\|_{L^2}^2 \leq 4\pi^2 a^2 |b| \int_{|x| \leq |b|^{-1/2}} |g'(x)|^2 dx + 4a^2 \int_{|x| \geq |b|^{-1/2}} |g'(x)|^2 dx. \]

Hence,
\[ \left( \frac{\|a \cdot g'(x) \cdot (1 - e^{2\pi ibx})\|_{L^2}}{\sqrt{a^2+b^2}} \right)^2 \leq 4\pi^2 |b| \cdot \|g'\|_{L^2}^2 + 4 \int_{|x| \geq |b|^{-1/2}} |g'(x)|^2 dx, \]

which tends to zero as \( (a, b) \to (0, 0) \), again as a consequence of the dominated convergence theorem and \( g' \in L^2 \). By recalling (3.3), we thus see that (3.1) holds.

Assume now that \( g \in \mathbb{H}^2(\mathbb{R}) \). In order to prove (3.2), we recall Equations (3.7) and (3.1) to see that
\[ \int |e^{2\pi ibx} - 1 - 2\pi ibx|^2 |g(x)|^2 dx = 4\pi^2 b^2 \int x^2 \left| \text{sinc}(bx) - e^{-\pi ibx} \right|^2 |g(x)|^2 dx \]
\[ \leq 4\pi^4 b^4 \int x^4 |g(x)|^2 dx = 4\pi^4 b^4 \| X^2 g \|_{L^2}^2. \]
Likewise, we use Equations (3.9), (3.1), and (3.6) to obtain
\[ \| T_u g - g + ag'\|_{L^2}^2 \leq 4\pi^4 a^4 \int \omega^4 |\tilde{g}(\omega)|^2 d\omega = 4\pi^4 a^4 \|\omega^2 \tilde{g}(\omega)\|_{L^2}^2. \]

Furthermore,
\[ a^2 \int |e^{2\pi i b x} - 1|^2 |g'(x)|^2 dx \leq 4\pi^2 a^2 b^2 \int x^2 |g'(x)|^2 dx = 4\pi^2 a^2 b^2 \| Xg' \|_{L^2}^2. \]

Thus, Equation (3.5), combined with the elementary estimate \(|ab| \leq \frac{1}{4}(a^2 + b^2)\), shows that
\[ \| S_g(a, b) - g - (-ag' + 2\pi i b Xg) \|_{L^2} \leq \frac{3}{4} C_g \cdot (a^2 + b^2 + |a||b|) \leq C_g \cdot \|(a, b)\|_2^2, \]
and the lemma is proved.

**Corollary 3.3.** For any \( g \in H^1(\mathbb{R}) \), the map \( S_g \) is continuously (Fréchet)-differentiable with
\[ S'_g(\mu) (\varepsilon) = -a\pi(\mu)g' + 2\pi i b X\pi(\mu)g, \quad \mu \in \mathbb{R}^2. \]

**Proof.** Let \( \mu, \lambda \in \mathbb{R}^2 \), \( \mu = (u, \eta) \), \( \lambda = (a, b) \). Then
\[ \pi(\mu + \lambda)g - \pi(\mu)g = M_{\eta + b} T_{u + a} g - M_{\eta} T_{u} g = M_{\eta} (\pi(a, b) - I) T_{u} g. \]

Now, since \( T_u g \in H^1(\mathbb{R}) \) with \( (T_u g)' = T_u g' \), Lemma 3.2 shows that
\[ (\pi(a, b) - I) T_u g = -a T_u g' + 2\pi i b X T_u g + \varepsilon(a, b), \]
where \( \varepsilon(a, b) = \varepsilon_u(a, b) \in L^2(\mathbb{R}) \) satisfies \( \lim_{(a, b) \to (0, 0)} \frac{\|\varepsilon(a, b)\|_{L^2}}{\|(a, b)\|_2} = 0 \). Thus,
\[ \pi(\mu + \lambda)g - \pi(\mu)g = -a M_{\eta} T_u g' + 2\pi i b M_{\eta} X T_u g + M_{\eta} \varepsilon(a, b) \]
\[ = -a\pi(\mu)g' + 2\pi i b X\pi(\mu)g + \tilde{\varepsilon}(a, b), \]
where \( \tilde{\varepsilon} := M_{\eta} \varepsilon \). As \( \|\tilde{\varepsilon}(a, b)\|_{L^2} = \|\varepsilon(a, b)\|_{L^2} \), the claim is proved. \( \Box \)

4. Proof of Theorem 1.3

As mentioned in the introduction, an upper bound in (1.2) is not difficult to achieve. It even holds without the additional assumptions of \( \Lambda \) having rational density or \((g, \Lambda)\) forming a Riesz sequence.

**Proposition 4.1.** Let \( g \in H^1(\mathbb{R}) \) and let \( \Lambda \) be a lattice in \( \mathbb{R}^2 \). Then
\[ \text{dist} (\pi(\mu)g, G(\mu, \Lambda)) \leq \sqrt{\|g'\|_{L^2}^2 + \|2\pi i Xg\|_{L^2}^2 \cdot \text{dist}(\mu, \Lambda)} \quad \text{for all } \mu \in \mathbb{R}^2. \]

**Proof.** Let \( \lambda \in \Lambda \) be a closest point (in Euclidean distance) in \( \Lambda \) to \( \mu \). Then \( (0, 0) \) is a closest point in \( \Lambda \) to \( z := \mu - \lambda \), and thus \( \text{dist}(\mu, \Lambda) = \|z\|_2 \). By Lemma 2.3 we have
\[ \text{dist} (\pi(\mu)g, G(\mu, \Lambda)) = \text{dist} (\pi(z)g, G(\mu, \Lambda)) \leq \|\pi(z)g - g\|_{L^2}. \]

Now, if \( z = (u, \eta) \), then Plancherel’s theorem shows that
\[ \|\pi(z)g - g\|_{L^2} \leq \|\pi(u, \eta - (0, \eta))g\|_{L^2} + \|\pi(0, \eta)g - g\|_{L^2} \]
\[ = \|M_{\eta}(T_u - I)g\|_{L^2} + \|(M_{\eta} - I)g\|_{L^2} \]
\[ = \|(M_{-u} - I)\tilde{g}\|_{L^2} + \|(M_{\eta} - I)g\|_{L^2}. \]
Next, recall that $|e^{2ix} - 1| = \left| e^{ix} - e^{-ix} \right| = 2|\sin(x)| \leq 2|x|$ for $x \in \mathbb{R}$. Using this estimate, we observe for $f \in L^2(\mathbb{R})$ with $f \in H^1(\mathbb{R})$ and $\alpha \in \mathbb{R}$ that

$$\|(M_\alpha - I)f\|_{L^2}^2 = \int \left| e^{2\pi i \alpha x} - 1 \right|^2 |f(x)|^2 \, dx \leq 4\pi^2 \alpha^2 \int x^2 |f(x)|^2 \, dx,$$

that is, $\|(M_\alpha - I)f\|_{L^2} \leq 2\pi |\alpha| \cdot \|Xf\|_{L^2} = |\alpha| \cdot \|2\pi i Xf\|_{L^2}$. Hence, if we define $\Omega \widehat{f}(\omega) := \omega \widehat{f}(\omega)$ for $\omega \in \mathbb{R}$ and $f \in H^1(\mathbb{R})$, we find that

$$\text{dist} \left( (\pi(\mu)g, G(g, \Lambda)) \right) \leq |u| \cdot \|2\pi i \Omega \widehat{g}\|_{L^2} + |\eta| \cdot \|2\pi i Xg\|_{L^2}.$$

Since $2\pi \Omega \widehat{g} = \mathcal{F}[g']$, Plancherel’s theorem and the Cauchy-Schwarz inequality yield the claim. \hfill \square

**Remark 4.2.** If $\Lambda = AZ^2$ with $A \in \text{GL}(2, \mathbb{R})$, then the maximal distance of a point $\mu \in \mathbb{R}^2$ to the lattice $\Lambda$ is bounded above by $2^{-1/2} \|A\|_{op}$. Therefore, for each time-frequency shift $\pi(\mu)g$ of $g$ we have that

$$\text{dist} \left( (\pi(\mu)g, G(g, \Lambda)) \right) \leq \sqrt{\frac{\|g'\|_{L^2}^2 + \|2\pi i Xg\|_{L^2}^2}{2}} \|A\|_{op},$$

In other words, the better $g$ is localized in both time and frequency, the closer the time-frequency shifts of $g$ scatter around $G(g, \Lambda)$. However, due to the uncertainty principle (see e.g., [12, Theorem 2.2.1]), the constant in the above inequality is easily seen to satisfy $\sqrt{\frac{\|g'\|_{L^2}^2 + \|2\pi i Xg\|_{L^2}^2}{2}} \geq \sqrt{\pi}$.

In the proof of the next proposition we consider matrix-valued ordinary differential equations (ODEs) of the form

$$X'(t) = X(t)M(t), \quad (4.1)$$

where $X : \mathbb{R} \to \mathbb{C}^{m \times n}$ and where $M : \mathbb{R} \to \mathbb{C}^{m \times n}$ has locally integrable entries. A solution of this ODE is a matrix function $X : \mathbb{R} \to \mathbb{C}^{m \times n}$ with (locally) absolutely continuous entries for which $X'(t) = X(t)M(t)$ holds for a.e. $t \in \mathbb{R}$.

**Lemma 4.3.** If $X_1$ and $X_2$ are two solutions to the ODE (4.1) such that $X_1(0) = X_2(0)$, then $X_1(t) = X_2(t)$ for all $t \in \mathbb{R}$.

**Proof.** Since the classical ODE theory deals with continuously differentiable solutions to equations with coefficient functions fulfilling a Lipschitz condition, we cannot quite apply that theory. As we will see, however, the same proof idea still works.

Indeed, since $X := X_1 - X_2$ is a solution to the ODE $X' = X \cdot M$ with $X(0) = 0$, it suffices to show that any such function satisfies $X \equiv 0$. Since $X$ is continuous, the set $\Gamma := \{ t \in \mathbb{R} : X(t) = 0 \}$ is closed. Since $\mathbb{R}$ is connected and since $0 \in \Gamma \neq \emptyset$, it is therefore enough to show that $\Gamma$ is also open.

Thus, let $x_0 \in \Gamma$ be fixed but arbitrary. Since $M$ is locally integrable, there is some $\varepsilon > 0$ such that $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \|M(t)\|_{op} \, dt \leq \frac{1}{2}$. Now, set $I := [x_0 - \varepsilon, x_0 + \varepsilon]$, and denote by $X := C(I; \mathbb{C}^{m \times n})$ the space of all continuous functions $f : I \to \mathbb{C}^{m \times n}$, equipped with the norm $\|f\|_{X} := \sup_{t \in I} \|f(t)\|_{op}$. It is not hard to see that $X$ is a Banach space. Furthermore, define the linear operator

$$T : X \to X, f \mapsto Tf \quad \text{where} \quad (Tf)(t) := \int_{x_0}^{t} f(s) M(s) \, ds \quad \text{for} \ t \in I.$$
Note that indeed $Tf \in X$ if $f \in X$, since $M$ is locally integrable, so that $f \cdot M$ is integrable on $I$. Next, observe

$$\|Tf(t)\|_{op} \leq \left| \int_{x_0}^t \|f(s)\|_{op} \cdot \|M(s)\|_{op} \, ds \right| \leq \|f\|_X \cdot \int_I \|M(s)\|_{op} \, ds \leq \frac{1}{2} \|f\|_X,$$

and hence $\|T\|_{X \to X} \leq \frac{1}{2} < 1$. From this, it follows using a Neumann series argument that $\text{id} - T : X \to X$ is invertible.

Finally, since $X(x_0) = 0$ and $X'(t) = X(t)M(t)$, we have

$$X(t) = X(t) - X(x_0) = \int_{x_0}^t X'(s) \, ds = \int_{x_0}^t X(s)M(s) \, ds = (T[X|_I])(t)$$

for all $t \in I$, which means that $f := X|_I$ satisfies $(\text{id} - T)f = 0$. Hence $f = 0$, which means that $X \equiv 0$ on $(x_0 - \varepsilon, x_0 + \varepsilon)$. Thus, $(x_0 - \varepsilon, x_0 + \varepsilon) \subset \Gamma$, so that $\Gamma$ is open. □

The following proposition can be seen as a weak Balian-Low-type theorem for subspaces. For a comparison with related results, see Remark 4.5 below.

**Proposition 4.4.** Let $g \in \mathbb{H}^1(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice of rational density such that $(g, \Lambda)$ is a Riesz basis for its closed linear span $\mathcal{G}(g, \Lambda)$. Then

$$-ag'' + 2\pi ib X g \notin \mathcal{G}(g, \Lambda) \quad \text{for all } (a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

**Proof.** Let us assume towards a contradiction that $\gamma := -ag'' + 2\pi ib X g \in \mathcal{G}(g, \Lambda)$ for some $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. We divide the proof into five steps.

**Step 1:** In the first four steps of the proof, we only consider separable lattices of the form $\Lambda = \frac{1}{Q} \mathbb{Z} \times P \mathbb{Z}$ for certain $P, Q \in \mathbb{N}$.

Let $G := Zg \in L^2_{\text{loc}}(\mathbb{R}^2)$ denote the Zak transform of $g$, and recall from Lemma 2.1 the definition of the function $A_g \in L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{C}^{P \times Q})$ given by

$$A_g(x, \omega) = P^{-1/2} \cdot (G(x + \frac{k}{P}, \frac{\ell}{Q}, \omega))_{k, \ell}^{P-1, Q-1}.$$

Since $g \in \mathbb{H}^1(\mathbb{R})$, Lemma 2.4 shows that $G \in H^1_{\text{loc}}(\mathbb{R}^2)$, so that all component functions of $A_g$ are in $H^1_{\text{loc}}(\mathbb{R}^2)$ as well. In this step, we show that $A_g$ satisfies a certain differential equation; see Equation 4.2 below.

Since $\gamma \in \mathcal{G}(g, \Lambda)$ and $\Lambda = \frac{1}{Q} \mathbb{Z} \times P \mathbb{Z}$, Lemma A.16 shows $\pi(\frac{P}{Q}, 0) \gamma \in \mathcal{G}(g, \Lambda)$ for each $L \in \{0, \ldots, Q - 1\}$. This means that for each $L \in \{0, \ldots, Q - 1\}$ there is a sequence $(c_{m,n}^{(L)})_{m,n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$ such that

$$\pi(\frac{P}{Q}, 0) \gamma = \sum_{m,n \in \mathbb{Z}} c_{m,n}^{(L)} \pi(\frac{P}{Q}, Pm) g = \sum_{\ell=0}^{Q-1} \sum_{m,s \in \mathbb{Z}} c_{m,sQ+\ell}^{(L)} \pi(\frac{P}{Q}, Pm) g.$$ 

By using the properties (a)–(c) of the Zak transform listed below Equation 2.2, this implies for each $L \in \{0, \ldots, Q - 1\}$ that

$$(Z\gamma)(x - \frac{\ell}{Q}, \omega) = Z \left[ \pi(\frac{P}{Q}, 0) \gamma \right](x, \omega) = \sum_{\ell=0}^{Q-1} \sum_{m,s \in \mathbb{Z}} c_{m,sQ+\ell}^{(L)} Z[\pi(\frac{P}{Q}, Pm) g](x, \omega) = \sum_{\ell=0}^{Q-1} f_{\ell}^{(L)}(x, \omega) \cdot G(x - \frac{\ell}{Q}, \omega)$$
where $f^L(t)(x,\omega) := -\sum_{m,s \in \mathbb{Z}} e^{2\pi i (Pmx - s\omega)}$. Note that each $f^L(t)$ is locally square-integrable on $\mathbb{R}^2$ and $(\frac{1}{2},1)$-periodic.

Now, recall from Lemma 4.1 that $\partial_2 G) = 2\pi i(xG(x,\omega) - Z(Xg)(x,\omega))$ and $\partial_1 G = Zg'$. Therefore,

\[(Z\gamma)(x,\omega) = Z[-ag' + 2\pi ibXg](x,\omega) = -a \cdot \partial_1 G(x,\omega) + 2\pi ib \cdot xG(x,\omega) - b \cdot \partial_2 G(x,\omega).\]

Thus, we arrive at

\[a \partial_1 G(x - \frac{L}{Q},\omega) + b \partial_2 G(x - \frac{L}{Q},\omega) = 2\pi ib (x - \frac{L}{Q}) G(x - \frac{L}{Q},\omega) + \sum_{i=0}^{Q-1} f^L(t)(x,\omega) G(x - \frac{L}{Q},\omega).\]

Denoting by $e_0, \ldots, e_{Q-1}$ the standard basis vectors of $\mathbb{C}^Q$, plugging $x + \frac{k}{T}$ instead of $x$ into the preceding displayed equation, and recalling that $f^L(t)$ is $(\frac{1}{2},1)$-periodic, we obtain for each $L \in \{0, \ldots, Q-1\}$ that

\[a \partial_1 A_g(x,\omega) e_L + b \partial_2 A_g(x,\omega) e_L = 2\pi ib[(x - \frac{L}{Q}) A_g(x,\omega) e_L + D_p A_g(x,\omega) e_L] + A_g(x,\omega) f^L(t)(x,\omega),\]

where $f^{(L)} := (f^L(\ell))_{\ell=0}^{Q-1}$ and $D_p := \text{diag}(k/p^{p-1})$. This leads to

\[a \partial_1 A_g(x,\omega) + b \partial_2 A_g(x,\omega) = 2\pi ib[(x A_g(x,\omega) + D_p A_g(x,\omega)] + A_g(x,\omega) (F(x,\omega) - 2\pi ib D_Q),\]

where $D_Q := \text{diag}(L/Q)_{\ell=0}^{Q-1}$ and $F := \begin{bmatrix} f(0) & \cdots & f(Q-1) \end{bmatrix} \in L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{C}^{Q \times Q})$. As a consequence of Fubini’s theorem (and since $(a,b) \neq (0,0)$), there is a null-set $N_0 \subset \mathbb{R}^2$ such that $(t \mapsto F(x + ta,\omega + tb)) \in L^2_{\text{loc}}(\mathbb{R}; \mathbb{C}^{Q \times Q})$ for all $(x,\omega) \in \mathbb{R}^2 \setminus N_0$.

Note that the preceding displayed equation holds for almost all $(x,\omega) \in \mathbb{R}^2$. Therefore, if we let $v_i := v + t(a,b)$ for $v \in \mathbb{R}^2$ and $t \in \mathbb{R}$, then Lemma A.15 yields a null-set $N_1 \subset \mathbb{R}^2$ such that if $v = (x,\omega) \in \mathbb{R}^2 \setminus N_1$, then

\[a (\partial_1 A_g)(v) + b (\partial_2 A_g)(v) = 2\pi ib[(x + ta) A_g(v) + D_p A_g(v)] + A_g(v)(F(v) - 2\pi ib D_Q)\]

\[\quad = 2\pi ib D_p A_g(v) + A_g(v)[2\pi ib(x + ta) + F(v) - 2\pi ib D_Q] = 2\pi ib D_p A_g(v) + A_g(v)W_v(t) \tag{4.2}\]

for almost all $t \in \mathbb{R}$. In the last step we introduced the matrix

\[W_v(t) := 2\pi ib(x + ta)I_Q + F(v) - 2\pi ib D_Q \in \mathbb{C}^{Q \times Q}, \quad t \in \mathbb{R},\]

where $I_Q$ denotes the $Q$-dimensional identity matrix. Note $W_v \in L^2_{\text{loc}}(\mathbb{R}; \mathbb{C}^{Q \times Q})$ for all $v \in \mathbb{R}^2 \setminus N_0$.

**Step 2:** In this step, we construct a particularly nice representative of $G = Zg$.

Recall from Step 1 that $G \in H^1_{\text{loc}}(\mathbb{R})$. Next, define $\gamma := (a,b) \in \mathbb{R}^2 \setminus \{0\}$, and choose $\theta \in \mathbb{R}^2$ with $\|\theta\|_2 = 1$ and $\theta \perp \gamma$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$, $(t,s) \mapsto t\theta + s\gamma$, and note that $T$ is linear and bijective, so that the same holds also for $T^{-1}$. In particular, $T$ and $T^{-1}$ are Lipschitz continuous, and thus map null-sets to null-sets. Furthermore, since $T$ and
$T^{-1}$ are Lipschitz continuous, the change-of-variables formula for Sobolev functions (see for instance [22, Theorem 2.2.2]) shows that $\widetilde{G} := G \circ T \in H^1_{\text{loc}}(\mathbb{R}^2)$, and that

$$
D\widetilde{G}(t, s) = DG(T(t, s)) \cdot DT(t, s) = \left( a [\partial_1 G](t \theta + s \theta), [\partial_2 G](t \theta + s \theta) \right) \cdot \left( \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \right)
$$

(4.3)

for almost all $(t, s) \in \mathbb{R}^2$. By Lemma A.12, there is a null-set $N_2 \subset \mathbb{R}$ such that for all $s \in \mathbb{R} \setminus N_2$, Equation (4.3) holds for almost all $t \in \mathbb{R}$.

Lemma A.14 yields a null-set $N_3 \subset \mathbb{R}$, and a (pointwise defined) Borel function $\widetilde{G}_0 : \mathbb{R}^2 \to \mathbb{C}$ such that $\widetilde{G}_0 = \widetilde{G}$ almost everywhere, and such that for all $s \in \mathbb{R} \setminus N_3$, the function $t \mapsto \widetilde{G}_0(t, s)$ is continuous and in $H^1_{\text{loc}}(\mathbb{R})$ with $\frac{d}{dt} \widetilde{G}_0(t, s) = (\partial_1 \tilde{G})(t, s)$ almost everywhere. In view of Equation (4.3), we thus see for all $s \in \mathbb{R} \setminus (N_2 \cup N_3)$ that

$$
\frac{d}{dt} \widetilde{G}_0(t, s) = a [\partial_1 G](t \theta + s \theta) + b [\partial_2 G](t \theta + s \theta) \quad \text{for almost all } t \in \mathbb{R}.
$$

Note that since $\widetilde{G}_0 = \widetilde{G} = G \circ T$ almost everywhere and since $T$ and $T^{-1}$ map null-sets to null-sets, we have $G = \widetilde{G}_0 \circ T^{-1} = : G_0$ almost everywhere. By Lemma A.15, there is thus a null-set $N_4 \subset \mathbb{R}^2$ such that

$$
\forall (x, \omega) \in \mathbb{R}^2 \setminus N_4: \quad G_0(x + ta, \omega + tb) = G(x + ta, \omega + tb) \quad \text{for a.e. } t \in \mathbb{R}. \quad (4.4)
$$

Since $T$ is Lipschitz continuous, the set $N_5 := T(\mathbb{R} \times (N_2 \cup N_3)) \subset \mathbb{R}^2$ is a null-set. For any $(x, \omega) \in \mathbb{R}^2 \setminus N_5$, we have $(x, \omega) = T(t_0, s_0) = t_0 \theta + s_0 \theta$ for certain $(t_0, s_0) \in \mathbb{R} \times (\mathbb{R} \setminus (N_2 \cup N_3))$. By the properties from above, this means that the map

$$
\mathbb{R} \to \mathbb{C}, \quad t \mapsto G_0(x + ta, \omega + tb) = G_0((x, \omega) + t \theta) = G_0((t + t_0) \theta + s_0 \theta) = \widetilde{G}_0(t + t_0, s_0)
$$

is continuous and in $H^1_{\text{loc}}(\mathbb{R})$ with derivative

$$
\frac{d}{dt} G_0(x + ta, \omega + tb) = \frac{d}{dt} \widetilde{G}_0(t + t_0, s_0) = a [\partial_1 G](t + t_0) \theta + s_0 \theta) + b [\partial_2 G](t + t_0) \theta + s_0 \theta)
$$

for almost all $t \in \mathbb{R}$, for each fixed $(x, \omega) \in \mathbb{R}^2 \setminus N_5$.

Finally, let $N_6 := \bigcup_{k, \ell \in \mathbb{Z}} \left( (N_4 \cup N_5) + (\frac{k}{Q} - \frac{\ell}{P}, 0) \right) \subset \mathbb{R}^2$, which is a null-set. If $(x, \omega) \in \mathbb{R}^2 \setminus N_6$, then $(x + \frac{k}{P} - \frac{\ell}{Q}, \omega) \in \mathbb{R}^2 \setminus (N_4 \cup N_5)$ for all $k, \ell \in \mathbb{Z}$.

**Step 3:** In this step, we use the “nice” representative $G_0$ of $G$ to construct for almost all $v = (x, \omega) \in \mathbb{R}^2$ two locally absolutely continuous functions $R_v : \mathbb{R} \to \mathbb{C}^{P \times Q}$ and $L_v : \mathbb{R} \to \mathbb{C}^{P \times Q}$ which satisfy the differential equations $R_v'(t) = R_v(t)W_v(t)$ and $L_v'(t) = L_v(t)W_v(t)$ for almost all $t \in \mathbb{R}$, for the matrix function $W_v \in L^2_{\text{loc}}(\mathbb{R} : \mathbb{C}^{Q \times Q})$ defined in Step 1. We then use this differential equation to deduce $R_v = L_v$. In Step 4 we will finally employ this identity to complete the proof for the case $\Lambda = \frac{1}{2} \mathbb{Z} \times \mathbb{Z}$.

First, define

$$
A : \mathbb{R}^2 \to \mathbb{C}^{P \times Q}, \quad (x, \omega) \mapsto P^{-1/2} \cdot \left( G_0(t + \frac{k}{P} - \frac{\ell}{Q}, \omega) \right)^{P-1} Q^{-1},
$$

noting $A = A_g$ almost everywhere. Next, note for $v = (x, \omega) \in \mathbb{R}^2 \setminus (N_0 \cup N_1 \cup N_6)$ that $(x + \frac{k}{P} - \frac{\ell}{Q}, \omega) \in \mathbb{R}^2 \setminus (N_4 \cup N_5)$ for all $k, \ell \in \mathbb{Z}$, so that Equations (1.5), (1.2), and
show that the function $E_v : \mathbb{R} \to \mathbb{C}^{P \times Q}$, $t \mapsto A(v_t) = A(x + ta, \omega + tb)$ is locally absolutely continuous and satisfies

$$E_v'(t) = P^{-1/2} \left( a (\partial_1 G)(x + \frac{t}{P} - \frac{t}{Q} + ta, \omega + tb) + b (\partial_2 G)(x + \frac{t}{P} - \frac{t}{Q} + ta, \omega + tb) \right)_{k,t=0}^{P-1,Q-1}$$

Hence, for almost all $t \in \mathbb{R}$, and note that the matrix function $L$ for almost all $(x, \omega)$, the last equality follows from Equation (4.6) combined with the elementary identity $e^{-2\pi itbD_P} = D_P e^{-2\pi ib D_P}$.

Step 4: We complete the proof for the case $\Lambda = Q^{-1} \mathbb{Z} \times P \mathbb{Z}$. To this end, let $t \in \mathbb{R}$ be arbitrary, and note that the matrix function $H_{(-ta,-tb)}$ defined in Lemma 2.2 satisfies for almost all $v = (x, \omega) \in \mathbb{R}^2$ that

$$H_{(-ta,-tb)}(x, \omega) = P^{1/2} \cdot A_g(v) \cdot (A_g^*(v)A_g(v))^{-1} A_g^*(v) A(v) = A(v) = R_v(0).$$

Therefore, Lemma 4.3 shows $L_v(t) = R_v(t)$ for all $v \in \mathbb{R}^2 \setminus (N_0 \cup N_1 \cup N_2 \cup N_3)$ and all $t \in \mathbb{R}$.

Step 4: We complete the proof for the case $\Lambda = Q^{-1} \mathbb{Z} \times P \mathbb{Z}$. To this end, let $t \in \mathbb{R}$ be arbitrary, and note that the matrix function $H_{(-ta,-tb)}$ defined in Lemma 2.2 satisfies for almost all $v = (x, \omega) \in \mathbb{R}^2$ that

$$H_{(-ta,-tb)}(x, \omega) = P^{1/2} \cdot A_g(v) \cdot (A_g^*(v)A_g(v))^{-1} A_g^*(v) A(v) = A(v) = R_v(0).$$

Therefore, Lemma 4.3 shows $L_v(t) = R_v(t)$ for all $v \in \mathbb{R}^2 \setminus (N_0 \cup N_1 \cup N_2 \cup N_3)$ and all $t \in \mathbb{R}$.
In particular, we have (see (2.9))

\[ \sum_{k=0}^{P-1} |G(x + ta + \frac{k}{P}, \omega + tb)|^2 \text{ for a.e. } (x, \omega) \in \mathbb{R}^2. \]

By Lemma 2.2 and by the quasi-periodicity of \( G = Zg \) (which implies that |\( G \) is (1, 1)-periodic), this implies that

\[
\text{dist}^2 (\pi(-ta, -tb)g, \mathcal{G}(g, \Lambda)) = \|g\|_{L^2}^2 - \int_0^1 \int_0^1 \sum_{k=0}^{P-1} |G(x + ta + \frac{k}{P}, \omega + tb)|^2 \, dx \, d\omega
\]

\[
= \|g\|_{L^2}^2 - \int_0^1 \int_0^1 |G(x, \omega)|^2 \, dx \, d\omega
\]

\[
= \|g\|_{L^2}^2 - \int_0^1 \int_0^1 |G(x, \omega, \omega + tb)|^2 \, dx \, d\omega = 0.
\]

That is, \( \pi(-ta, -tb)g \in \mathcal{G}(g, \Lambda) \) for each \( t \in \mathbb{R} \). By Theorem 1.2, this means that \( (-ta, -tb) \in \Lambda \) for every \( t \in \mathbb{R} \). Because of \( (a, b) \neq (0, 0) \) and since \( \Lambda \subset \mathbb{R}^2 \) is discrete, this yields the desired contradiction.

**Step 5:** Let \( \Lambda \subset \mathbb{R}^2 \) be an arbitrary lattice of rational density, and assume again that \(-ag' + 2\pi ibXg \in \mathcal{G}(g, \Lambda) \) for some \( a, b \in \mathbb{R} \). Then there exists a matrix \( B \in \text{GL}(2, \mathbb{R}) \) with \( \det B = 1 \) and certain \( P, Q \in \mathbb{N} \) such that \( \Lambda_1 := B\Lambda = Q^{-1}Z \times PZ \). With the symplectic operator \( U_B \) (see (2.9)), set \( g_1 := U_B g \). Then \( (g_1, \Lambda_1) \) is a Riesz basis for \( \mathcal{G}(g_1, \Lambda_1) = U_B \mathcal{G}(g, \Lambda) \) and, as \( \mathbb{H}^1(\mathbb{R}) \) is invariant under symplectic operators (see the discussion after Equation (2.11)), we have \( g_1 \in \mathbb{H}^1(\mathbb{R}) \). For \( f \in \mathbb{H}^1(\mathbb{R}) \), let us set \( T_f(x, \omega) := \rho(x, \omega) f, x, \omega \in \mathbb{R} \), cf. (2.10). Using Corollary 3.3 we find that

\[
T_f'(x, \omega) \left( \frac{\alpha}{\beta} \right) = -a f' + 2\pi bXf.
\]

We have (see (2.10))

\[
U_B T_g(x, \omega) = \rho(B (\frac{x}{\beta})) g_1 = T_{g_1}(B (\frac{x}{\beta})).
\]

Differentiating this with respect to \( x, \omega \) gives \( U_B T_g'(x, \omega) = T_{g_1}'(B (\frac{x}{\beta})) \circ B \). Hence, by Equation (4.7), we see that

\[
U_B (-ag' + 2\pi bXg) = U_B T_{g_1}'(0, 0) \left( \frac{\alpha}{\beta} \right) = T_{g_1}'(0, 0) (B (\frac{\alpha}{\beta}) = -\alpha g_1' + 2\pi \beta Xg_1,
\]

where \( (\frac{\alpha}{\beta}) = B (\frac{\alpha}{\beta}) \). That is, \(-ag_1' + 2\pi \beta Xg_1 \in U_B \mathcal{G}(g, \Lambda) = \mathcal{G}(g_1, \Lambda_1) \), which, by the first part of this proof, implies that \( \alpha = \beta = 0 \) and thus \( a = b = 0 \). \( \square \)

**Remark 4.5.** Proposition 4.4 is closely related to the so-called weak subspace Balian-Low Theorem (cf. [13] Thm. 8) which states that if \( g \in L^2(\mathbb{R}) \) and \( \Lambda \subset \mathbb{R}^2 \) is a lattice such that \( (g, \Lambda) \) is a Riesz basis for its closed linear span \( \mathcal{G} \), then at least one of the distributions \( g', Xg, g', X\bar{g} \) is not contained in \( \mathcal{G} \), where \( \bar{g} \) denotes the dual window of \( (g, \Lambda) \). More precisely, Proposition 4.4 implies that if \( g', Xg \in L^2(\mathbb{R}) \) and \( \Lambda \subset \mathbb{R}^2 \) is a lattice of rational density such that \( (g, \Lambda) \) is a Riesz sequence (and hence also \( g', X\bar{g} \in L^2(\mathbb{R}) \) by Proposition 2.3), then none of \( g', Xg, g', X\bar{g} \) is contained in \( \mathcal{G} \). In fact, it even asserts that none of the real linear combinations of \( ig' \) and \( Xg \) except \( 0 \) can
belong to $\mathcal{G}$. Similarly, none of the real linear combinations of $i\vec{g}'$ and $\vec{Xg}$ except 0 can belong to $\mathcal{G}$.

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3** Let us denote by $\mathcal{P}$ the orthogonal projection from $L^2(\mathbb{R})$ onto $\mathcal{G} := \mathcal{G}(g, \Lambda)$. Proposition 1.4 implies that the $\mathbb{R}$-linear mapping

$$\mathbb{R}^2 \to L^2(\mathbb{R}), \quad (a, b) \mapsto (\text{Id} - \mathcal{P})(-ag' + 2\pi ibXg),$$

with $L^2(\mathbb{R})$ considered as an $\mathbb{R}$-linear space, is injective. Since $\mathbb{R}^2$ is finite-dimensional, this implies $\| (\text{Id} - \mathcal{P})(-ag' + 2\pi ibXg) \|_{L^2} \geq 2\gamma \| (a, b) \|_2$ for some $\gamma > 0$ and all $(a, b) \in \mathbb{R}^2$. On the other hand, Lemma 3.2 gives a family of functions $\{ \varepsilon(a, b) \}_{(a, b) \in \mathbb{R}^2} \subset L^2(\mathbb{R})$ such that

$$\pi(a, b)g - g = -ag' + 2\pi ibXg + \varepsilon(a, b)$$

and

$$\lim_{(a, b) \to (0, 0)} \frac{\| \varepsilon(a, b) \|_{L^2}}{\| (a, b) \|_2} = 0.$$ 

In particular, there exists some $\delta > 0$ such that $\| \varepsilon(a, b) \|_{L^2} \leq \gamma \| (a, b) \|_2$ for $\| (a, b) \|_2 < \delta$. Combining these observations and the fact that $(\text{Id} - \mathcal{P})g = 0$, we see for $\| (a, b) \|_2 < \delta$ that

$$2\gamma \| (a, b) \|_2 \leq \| (\text{Id} - \mathcal{P})(-ag' + 2\pi ibXg) \|_{L^2} = \| (\text{Id} - \mathcal{P})\pi(a, b)g - \varepsilon(a, b) \|_{L^2} \leq \| (\text{Id} - \mathcal{P})\pi(a, b)g \|_{L^2} + \| \varepsilon(a, b) \|_{L^2} \leq \text{dist}(\pi(a, b)g, \mathcal{G}) + \gamma \| (a, b) \|_2,$$

that is, $\text{dist}(\pi(a, b)g, \mathcal{G}) \geq \gamma \| (a, b) \|_2$ for $\| (a, b) \|_2 < \delta$.

Now, consider the compact set $R := \{ \mu \in \mathbb{R}^2 : \| \mu \|_2 = \text{dist}(\mu, \Lambda) \}$ and denote by $B = B_2(0, 0) \subset \mathbb{R}^2$ the open ball of radius $\delta > 0$ centered at $(0, 0)$. By possibly shrinking $\delta$, we may assume that $B \subset R$; in fact, since $\Lambda$ is discrete, there is some $\delta_0 > 0$ such that $\| \lambda \|_2 \geq 2\delta_0$ for all $\lambda \in \Lambda \setminus \{0\}$. We then have $B \subset R$ as soon as $0 < \delta \leq \delta_0$.

We will show that $\| (\text{Id} - \mathcal{P})\pi(a, b)g \|_{L^2} \geq \gamma' \| (a, b) \|_2$ for a suitable $\gamma' > 0$ and all $(a, b) \in R \setminus B$. Towards a contradiction, suppose that there is no such $\gamma' > 0$. Then there exists a sequence $(\mu_n)_{n \in \mathbb{N}} \subset R \setminus B$ such that $(\text{Id} - \mathcal{P})\pi(\mu_n)g \to 0$ as $n \to \infty$. As $R \setminus B$ is compact, we may assume that $\mu_n \to \mu_0$ as $n \to \infty$ for some $\mu_0 \in R \setminus B$. But then, since $\mu \mapsto \pi(\mu)g$ is continuous, it follows that $(\text{Id} - \mathcal{P})\pi(\mu_0)g = 0$, that is, $\pi(\mu_0)g \in \mathcal{G}$, which by Theorem 1.2 is only possible if $\mu_0 \in \Lambda$; but this implies $\| \mu_0 \|_2 = \text{dist}(\mu_0, \Lambda) = 0$, in contradiction to $\mu_0 \in R \setminus B$.

Hence, dist$(\pi(\mu_0)g, \mathcal{G}) = \| (\text{Id} - \mathcal{P})\pi(\mu_0)g \|_{L^2} \geq \gamma' \| (a, b) \|_2$ for some $\gamma' > 0$ and all $(a, b) \in R \setminus B$. As a consequence, we have with $C_1 := \min \{ \gamma, \gamma' \} > 0$,

$$\text{dist}(\pi(\mu_0)g, \mathcal{G}) \geq C_1 \cdot \| \mu \|_2 = C_1 \cdot \text{dist}(\mu, \Lambda) \quad \text{for all } \mu \in R.$$

Finally, we note that for each $\mu \in \mathbb{R}^2$ there exist $\lambda \in \Lambda$ and $\nu \in R$ with $\mu = \lambda + \nu$; indeed, there exists $\lambda \in \Lambda$ with $\| \mu - \lambda \|_2 = \text{dist}(\mu, \Lambda)$, and then $\nu := \mu - \lambda$ satisfies $\| \nu \|_2 = \text{dist}(\nu, \Lambda) = \text{dist}(\nu, \Lambda)$. Thus, we obtain (see Lemma 2.3)

$$\text{dist}(\pi(\mu_0)g, \mathcal{G}) = \text{dist}(\pi(\nu)g, \mathcal{G}) \geq C_1 \cdot \text{dist}(\nu, \Lambda) = C_1 \cdot \text{dist}(\mu, \Lambda).$$

In view of Proposition 1.1 this completes the proof. 

\[\square\]

5. An Explicit Local Bound

As mentioned in the introduction, we were unable to derive an explicit constant $C_1$ for (1.2). Nevertheless, we can find a constant $C_1$ that is valid for $(u, \eta)$ close to the lattice $\Lambda$. For this, however, we have to assume that $(g, \Lambda)$ is an orthonormal sequence.
The following result makes a first step towards finding such a constant $C_1$; it improves Proposition 4.4 under the additional assumption of orthonormality.

**Proposition 5.1.** Let $g \in H^1(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice such that $(g, \Lambda)$ is an orthonormal basis of its closed linear span $\mathcal{G}(g, \Lambda)$. Then for any $(a, b) \in \mathbb{R}^2$, 

$$\text{dist} (-ag' + 2\pi ibXg, \mathcal{G}(g, \Lambda)) \geq \frac{\pi}{\sqrt{\|g'\|_{L^2}^2 + \|2\pi iXg\|_{L^2}^2}} \|(a, b)\|_2.$$ 

**Remark 5.2.** The classical uncertainty principle (see e.g., [12] Theorem 2.2.1), combined with elementary computations, implies because of $\|g\|_{L^2} = 1$ that the lower bound appearing in Proposition 5.1 is bounded by 

$$\frac{\pi}{\sqrt{\|g'\|_{L^2}^2 + \|2\pi iXg\|_{L^2}^2}} \leq \sqrt{\frac{\pi}{2}}.$$ 

The proof of Proposition 5.1 hinges crucially on the following lemma which describes a general property of Hilbert spaces.

**Lemma 5.3.** Let $\mathcal{H}$ be a Hilbert space, and let $f, g \in \mathcal{H}$ with $f \neq 0$ or $g \neq 0$. Then 

$$\|af + bg\|^2 \geq \frac{\|f\|^2 \cdot \|g\|^2 - (\text{Re}(f, g))^2}{\|f\|^2 + \|g\|^2} \cdot \|(a, b)\|_2^2 \geq \frac{(\text{Im}(f, g))^2}{\|f\|^2 + \|g\|^2} \cdot \|(a, b)\|_2^2$$ 

for all $a, b \in \mathbb{R}$.

**Proof.** Let $\alpha := \|f\|^2_H$, $\gamma := \|g\|^2_H$, and $\beta := \text{Re}(f, g)$. Moreover, set $A := \alpha + \gamma$ and $B := \alpha \gamma - \beta^2$. Because of $f \neq 0$ or $g \neq 0$, we have $A > 0$. Besides, the Cauchy-Schwarz inequality shows $\beta \leq |\beta| \leq \sqrt{\alpha \gamma}$, and thus $B \geq 0$. Finally, a direct computation shows $A^2 - 4B = (\alpha + \gamma)^2 - 4(\alpha \gamma - \beta^2) = (\alpha - \gamma)^2 + 4\beta^2 \geq 0$.

Given these notations, another direct computation shows for $a, b \in \mathbb{R}$ that 

$$\|af + bg\|^2 = (af + bg, af + bg)_{\mathcal{H}} = \langle (\xi \alpha \beta), M (\eta \beta \gamma) \rangle_{\mathbb{R}^2} \quad \text{where} \quad M := \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}. \quad (5.1)$$

Note that the matrix $M$ is real-symmetric, with characteristic polynomial

$$\chi_M(\lambda) = \det \begin{pmatrix} \lambda - \alpha & -\beta \\ -\beta & \lambda - \gamma \end{pmatrix} = \lambda^2 - A \lambda + B,$$

which has the roots 

$$\lambda_{1/2} = \frac{A}{2} \pm \frac{\sqrt{A^2 - 4B}}{2} = \frac{A \pm \sqrt{A^2 - 4B}}{2}.$$ 

Therefore, and because of $\sqrt{A^2 - 4B} \leq \sqrt{A^2} = A$, the smallest eigenvalue of $M$ satisfies 

$$\lambda_{\text{min}} = \frac{A - \sqrt{A^2 - 4B}}{2} = \frac{1}{2} \frac{A^2 - (A^2 - 4B)}{A + \sqrt{A^2 - 4B}} = \frac{2B}{A + \sqrt{A^2 - 4B}} \geq \frac{B}{A} \geq 0.$$ 

Since $M$ is real symmetric, this implies $\langle x, Mx \rangle_{\mathbb{R}^2} \geq \frac{B}{4} \|x\|_{\mathbb{R}^2}^2$ for all $x \in \mathbb{R}^2$.

Now, Equation (5.1) shows that $\|af + bg\|^2_{\mathcal{H}} = \langle (\xi \alpha \beta), M (\eta \beta \gamma) \rangle_{\mathbb{R}^2} \geq \frac{B}{A} \cdot \|(a, b)\|_2^2$ for all $a, b \in \mathbb{R}$, which establishes the first part of the claim. For the second part, note that the Cauchy-Schwarz inequality implies 

$$B = \alpha \gamma - \beta^2 = \|f\|^2_H \|g\|^2_H - (\text{Re}(f, g))^2 \geq |\langle f, g \rangle_{\mathcal{H}}|^2 - (\text{Re}(f, g))^2 = (\text{Im}(f, g))^2.$$ \hfill \qed
Proof of Proposition 5.1. Denote by \( P \) the orthogonal projection from \( L^2(\mathbb{R}) \) onto \( G(g, \Lambda) \) in \( L^2(\mathbb{R}) \). Since \((g, \Lambda)\) is an orthonormal sequence,

\[
P f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g,
\]
whence \( \langle Pg', iXg \rangle = \sum_{\lambda \in \Lambda} \langle g', \pi(\lambda) g \rangle \langle \pi(\lambda) g, iXg \rangle \).

Let \( a, b \in \mathbb{R} \). By integration by parts and translation, and by using the elementary identity \( \langle \pi(a, b) \rangle^* = e^{-2\pi iab} \pi(-a, -b) \), we see that

\[
\langle g', \pi(a, b) g \rangle = \left\{ \begin{array}{ll} -\langle g, g' \rangle, & \text{if } (a, b) = 0, \\
e^{-2\pi iab} \langle \pi(-a, -b) g, g' \rangle, & \text{otherwise} \end{array} \right.
\]

as well as

\[
\langle \pi(a, b) g, iXg \rangle = \left\{ \begin{array}{ll} -\langle iXg, g \rangle, & \text{if } (a, b) = 0, \\
e^{2\pi iab} \langle iXg, \pi(-a, -b) g \rangle, & \text{otherwise} \end{array} \right.
\]

From Equations (5.2) and (5.3), we see by orthonormality of \((g, \Lambda)\) for arbitrary \((a, b) \in \Lambda\) that

\[
\langle g', \pi(a, b) g \rangle = -\langle g, g' \rangle, \quad \text{if } (a, b) = 0,
\]

\[
= e^{-2\pi iab} \langle \pi(-a, -b) g, g' \rangle, \quad \text{otherwise}
\]

and

\[
\langle \pi(a, b) g, iXg \rangle = -\langle iXg, g \rangle, \quad \text{if } (a, b) = 0,
\]

\[
= e^{2\pi iab} \langle iXg, \pi(-a, -b) g \rangle, \quad \text{otherwise}
\]

Combining these identities, we arrive at

\[
\langle g', \pi(a, b) g \rangle \langle \pi(a, b) g, iXg \rangle = \langle \pi(-a, -b) g, g' \rangle \langle iXg, \pi(-a, -b) g \rangle,
\]

for all \((a, b) \in \Lambda\). Therefore, with \( \mu = -\lambda \), we see that

\[
\langle Pg', iXg \rangle = \sum_{\lambda \in \Lambda} \langle g', \pi(\lambda) g \rangle \langle \pi(\lambda) g, iXg \rangle
\]

\[
= \sum_{\mu \in \Lambda} \langle \pi(\mu) g, g' \rangle \langle iXg, \pi(\mu) g \rangle = \langle iXg, Pg' \rangle,
\]

which shows that \( \text{Im} \langle Pg', iXg \rangle = 0 \).

We now intend to use partial integration to get \( \langle g', Xg \rangle = -\|g\|_2^2 - \langle Xg, g' \rangle \); however, since \( Xg \notin H^1(\mathbb{R}) \), we cannot directly apply such a partial integration. Instead, pick \( \varphi \in C_c^\infty(\mathbb{R}) \) with \( 0 \leq \varphi \leq 1 \), \( \text{supp} \varphi \subset (-2, 2) \), and \( \varphi \equiv 1 \) on \((-1, 1)\), and set \( \varphi_n : \mathbb{R} \to [0, 1], x \mapsto \varphi(x/n) \). We then have \( \varphi_n \to 1 \) pointwise, so that the dominated convergence theorem implies \( \langle f, \varphi_n \cdot h \rangle \to \langle f, h \rangle \) for all \( f, h \in L^2(\mathbb{R}) \). Likewise, we have
Let \( H^1(\mathbb{R}) \) be a lattice such that \((g,\lambda)\) is an orthonormal basis of its closed linear span \(G(g,\lambda)\). Then there exists \( \varepsilon > 0 \) such that
\[
\text{dist} \left( \pi(\mu)g, G(g,\lambda) \right) \geq \frac{\pi/2}{\sqrt{\|g'\|_{L^2}^2 + 2\|2\pi i Xg\|_{L^2}^2}} \text{dist}(\mu,\lambda) \quad \forall \mu \in \Lambda + B_\varepsilon(0).
\]

If \( g \in H^2(\mathbb{R}) \), then \( \varepsilon \) can be chosen as \( \varepsilon := \frac{\pi}{\sqrt{2C_g \left( \|g'\|_{L^2}^2 + 2\|2\pi i Xg\|_{L^2}^2 \right)}} \) with \( C_g \) as in Equation \( \ref{eq:cg} \).

**Proof.** For \((a,b) \in \mathbb{R}^2\) let \( \gamma(a,b) := \pi(a,b)g - g - (-ag' + 2\pi ib Xg) \). Denote by \( \mathcal{P} \) the orthogonal projection from \( L^2(\mathbb{R}) \) onto \( G(g,\lambda) \). Due to Proposition \( \ref{prop:projection} \) we have
\[
\sqrt{\|g'\|_{L^2}^2 + 2\|2\pi i Xg\|_{L^2}^2} \geq \pi \sqrt{\|g'\|_{L^2}^2 + 2\|2\pi i Xg\|_{L^2}^2} \geq \|g'\|_{L^2}^2 + 2\|2\pi i Xg\|_{L^2}^2.
\]

In the last inequality we used that \((I - \mathcal{P})g = 0 \) and \( \|I - \mathcal{P}\| = 1 \). By Lemma \( \ref{lem:projection} \) there exists \( \varepsilon > 0 \) such that
\[
\|\gamma(a,b)\|_{L^2} \leq \frac{\pi/2}{\sqrt{\|g'\|_{L^2}^2 + 2\|2\pi i Xg\|_{L^2}^2}} \| (a,b) \|_2 \quad \text{for } \| (a,b) \|_2 < \varepsilon.
\]

Moreover, this is satisfied in the case \( g \in H^2(\mathbb{R}) \) if \( \varepsilon \) is as given in the theorem (see Lemma \( \ref{lem:projection} \)). Hence, if \((\alpha,\beta) \in \Lambda + B_\varepsilon(0)\), say \((\alpha,\beta) = \lambda + (a,b)\) with \( \lambda \in \Lambda \) and \((a,b) \in B_\varepsilon(0)\), then (see Lemma \( \ref{lem:projection} \))
\[
\|(I - \mathcal{P})\pi(\alpha,\beta)g\|_{L^2} \geq \frac{\pi/2}{\sqrt{\|g'\|_{L^2}^2 + 2\|2\pi i Xg\|_{L^2}^2}} \| (a,b) \|_2.
\]

This proves the theorem. \( \square \)
Remark. In the case $g \in \mathbb{H}^1(\mathbb{R})$, the value of $\varepsilon$ in Theorem 5.4 depends on the convergence to zero of the following quantities (see the proof of Lemma 3.2):

$$
\int_{|x|>b} x^2|g(x)|^2 \, dx, \quad \int_{|x|>b} |g'(x)|^2 \, dx \quad \text{and} \quad \int_{|\omega|>b} \omega^2 \hat{g}(\omega)^2 \, d\omega \quad \text{as} \quad b \to \infty.
$$

Note that the lattice $\Lambda$ in Theorem 5.4 is not necessarily of rational density. The following corollary suggests that the rational density condition of $\Lambda$ in Theorems 1.2 and 1.3 might be redundant.

**Corollary 5.5.** Let $g \in \mathbb{H}^1(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice such that $(g, \Lambda)$ is an orthonormal basis of its closed linear span $\mathcal{G}(g, \Lambda)$. Then there exists an $N \in \mathbb{N}$ such that $\pi(\mu)g \notin \mathcal{G}(g, \Lambda)$ for all $\mu \in \mathbb{R}^2 \setminus \frac{1}{N}\Lambda$; that is, $\mathcal{G}(g, \Lambda)$ is invariant only under time-frequency shifts with parameters in a subset of $\frac{1}{N}\Lambda$.

**Proof.** This follows by combining Theorem 5.4 with [5, Lemma 3.1].

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**A. Auxiliary results**

**A.1. Matrix multiplication operators**

Let $(\Omega, \Sigma, \mu)$ be a measure space. To avoid trivialities, we assume that there exists $S \in \Sigma$ with $0 < \mu(S) < \infty$. Now, let $B : \Omega \to \mathbb{C}^{n \times m}$ be a measurable matrix-valued function. Then the multiplication operator

$$
M_B : \quad \text{dom}(M_B) \subset L^2(\Omega, \mathbb{C}^m) \to L^2(\Omega, \mathbb{C}^n)
$$

is defined by

$$(M_B f)(\omega) := B(\omega)f(\omega), \quad \omega \in \Omega, \ f \in \text{dom}(M_B),$$

where

$$
\text{dom}(M_B) := \left\{ f \in L^2(\Omega; \mathbb{C}^m) : \int \|B(\omega)f(\omega)\|_{\mathbb{C}^n}^2 \, d\mu(\omega) < \infty \right\}.
$$

It is easy to see that the operator $M_B$ is bounded if and only if each entry of $B(\cdot)$ is essentially bounded as a function on $\Omega$, if and only if $\text{dom}(M_B) = L^2(\Omega; \mathbb{C}^m)$.

Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator in a Hilbert space $\mathcal{H}$. Then for any continuous, real-valued function $\varphi \in C(\sigma(A); \mathbb{R})$, the operator $\varphi(A)$ is defined by $\varphi(A) := \lim_{n \to \infty} p_n(A)$, where $(p_n)_{n \in \mathbb{N}}$ is a sequence of real-valued polynomials converging uniformly to $\varphi$ on $\sigma(A) \subset \mathbb{R}$ and the limit is taken with respect to the operator norm. Since $\|p(A)\| = \|p\|_{C(\sigma(A))}$ for polynomials $p$, this definition is meaningful. One then has $\|\varphi(A)\| = \|\varphi\|_{C(\sigma(A))}$ and $\sigma(\varphi(A)) = \{\varphi(\lambda) : \lambda \in \sigma(A)\}$. Furthermore, $\varphi(A)$ is self-adjoint for all $\varphi \in C(\sigma(A); \mathbb{R})$, since this is easily seen to hold for all polynomials $p_n$. For more details on this continuous functional calculus, see [20, Section VII.1].

For the case $n = m$, the next lemma connects the spectral properties of the multiplication operator $M_B$ to those of the matrices $B(\omega)$, $\omega \in \Omega$.

**Lemma A.1.** Let $B : \Omega \to \mathbb{C}^{n \times n}$ be a measurable, essentially bounded matrix-valued function satisfying $B(\omega) = B(\omega)^*$ for a.e. $\omega \in \Omega$. Then the following statements hold:

(i) The operator $M_B$ is bounded and self-adjoint.

(ii) For a.e. $\omega \in \Omega$ we have

$$
\sigma(B(\omega)) \subset \sigma(M_B).
$$
(iii) For every set $N \subset \Omega$ of zero measure,

$$\sigma(M_B) \subset \bigcup_{\omega \in \Omega \cap N} \sigma(B(\omega)).$$

(iv) For every function $\varphi \in C(\sigma(M_B); \mathbb{R})$ we have

$$\varphi(M_B) = M_{\varphi(B)},$$

where $(\varphi(B))(\omega) := \varphi(B(\omega))$ is well-defined for almost all $\omega \in \Omega$.

Proof. Part (i) follows easily from $(B(\omega)f(\omega), g(\omega))^c = (f(\omega), B(\omega)g(\omega))^c$, which holds for almost all $\omega \in \Omega$.

For the proofs of both (ii) and (iii), we will use \cite{16, Proposition 1}, which shows for any $\lambda \in \mathbb{C}$ that

$$\lambda \in \varrho(M_B) \iff \exists \varepsilon > 0: \mu(\{\omega \in \Omega: \sigma(B(\omega)) \cap B_\varepsilon(\lambda) \neq \emptyset\}) = 0. \quad \text{(A.1)}$$

To prove (ii), let us assume towards a contradiction that the claim is false; that is, the set

$$\Omega_0 := \{\omega \in \Omega: \sigma(B(\omega)) \cap \varrho(M_B) \neq \emptyset\}$$

is not a null-set. Since $\varrho(M_B)$ is an open set, we have $\varrho(M_B) = \bigcup_{k \in \mathbb{N}} I_k$ for certain compact sets $I_k \subset \varrho(M_B) \subset \mathbb{C}$. Setting $\Omega_k := \{\omega \in \Omega: \sigma(B(\omega)) \cap I_k \neq \emptyset\}$, we then have $\Omega_0 = \bigcup_{k \in \mathbb{N}} \Omega_k$, so that there is some $k \in \mathbb{N}$ for which $\Omega_k$ is not a null-set. Let us choose a dense subset $\{\lambda_n: n \in \mathbb{N}\}$ of $I_k$, and define

$$\Omega_{m,n} := \{\omega \in \Omega: \sigma(B(\omega)) \cap B_{1/m}(\lambda_n) \neq \emptyset\} \quad \text{for } m, n \in \mathbb{N}.$$ 

By density, we have $I_k \subset \bigcup_{n \in \mathbb{N}} B_{1/m}(\lambda_n)$ for every $m \in \mathbb{N}$, and hence $\Omega_k \subset \bigcup_{n \in \mathbb{N}} \Omega_{m,n}$. Thus, for each $m \in \mathbb{N}$, there is some $n_m \in \mathbb{N}$ such that $\Omega_{m,n_m}$ is not a null-set.

Since $(\lambda_{n_m})_{m \in \mathbb{N}}$ is a sequence in the compact set $I_k$, there is a subsequence $(\lambda_{n_{m_\ell}})_{\ell \in \mathbb{N}}$ such that $\lambda_{n_{m_\ell}} \to \lambda \in I_k \subset \varrho(M_B)$ as $\ell \to \infty$. By \textbf{(A.1)}, there is some $\varepsilon > 0$ such that $\Theta := \{\omega \in \Omega: \sigma(B(\omega)) \cap B_\varepsilon(\lambda) \neq \emptyset\}$ is a null-set. But for $\ell \in \mathbb{N}$ large enough, we have $\frac{1}{m_\ell} + |\lambda_{n_{m_\ell}} - \lambda| < \varepsilon$, and hence $B_{1/m_\ell}(\lambda_{n_{m_\ell}}) \subset B_\varepsilon(\lambda)$, which shows that $\Omega_{m_\ell,n_{m_\ell}} \subset \Theta$ is a null-set. This is the desired contradiction.

To prove (iii) let $\lambda \in \sigma(M_B)$ and let $N \subset \Omega$ be of zero measure. If $k \in \mathbb{N}$ is arbitrary, then by \textbf{(A.1)}, the set $\{\omega \in \Omega: \sigma(B(\omega)) \cap B_{1/k}(\lambda) \neq \emptyset\}$ does not have measure zero, and thus has non-empty intersection with $\Omega \setminus N$. Hence, we can pick $\omega_k \in \Omega \setminus N$ and $\lambda_k \in \sigma(B(\omega_k))$ such that $|\lambda_k - \lambda| < 1/k$. This proves the inclusion in (iii).

Part (iv) is obvious for polynomials $\varphi$. Given general $\varphi \in C(\sigma(M_B); \mathbb{R})$, we can approximate $\varphi$ uniformly on $\sigma(M_B) \subset \mathbb{R}$ by polynomials $p_n$. Then $\varphi(M_B) - p_n(M_B)$ converges to zero in operator norm, and $p_n(M_B) = M_{p_n(B)}$. Hence, we see for every $f \in L^2(\Omega; \mathbb{C}^n)$ that $\varphi(M_B)f = \lim_{n \to \infty} M_{p_n(B)}f$. But by (ii), we have $\sigma(B(\omega)) \subset \sigma(M_B)$ for almost every $\omega \in \Omega$. For each such $\omega \in \Omega$,

$$\|p_n(B(\omega)) - \varphi(B(\omega))\| = \|p_n - \varphi\|_{C(\sigma(B(\omega)))} \leq \|p_n - \varphi\|_{C(\sigma(M_B))}.$$ 

Thus, we see $[M_{p_n(B)}f](\omega) = p_n(B(\omega))f(\omega) \to \varphi(B(\omega))f(\omega) = [M_{\varphi(B)}f](\omega)$ for almost every $\omega$, for every $f \in L^2(\Omega; \mathbb{C}^n)$. Since also have $M_{p_n(B)}f \to \varphi(M_B)f$ with convergence in $L^2(\Omega, \mathbb{C}^n)$, this implies $\varphi(M_B)f = M_{\varphi(B)}f$, as claimed. \qed
A.2. Operators with closed range and their pseudo-inverse

In this subsection, we review the notion of the pseudo-inverse of an operator with closed range and some of its elementary properties. All of these properties are well-known in general; yet, as some readers might not be familiar with them we decided to include the essentials. Throughout this subsection $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{L}$ denote Hilbert spaces.

Lemma A.2. Let $A : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator. Then

$$\ker A = \overline{\operatorname{ran} A^*}. \quad (A.2)$$

Moreover, the following statements are equivalent:

(a) $\operatorname{ran} A$ is closed in $\mathcal{K}$.
(b) $\operatorname{ran}(AA^*)$ is closed in $\mathcal{K}$.
(c) $\operatorname{ran}(A^*A)$ is closed in $\mathcal{H}$.
(d) $\operatorname{ran} A^*$ is closed in $\mathcal{H}$.
(e) $\sigma_1(A) > 0$.
(f) $\sigma_1(A^*) > 0$.

In one of these properties holds, then the following identities hold:

$$\operatorname{ran}(AA^*) = \operatorname{ran} A, \quad \operatorname{ran}(A^*A) = \operatorname{ran} A^*, \quad \text{and} \quad \sigma_1(A) = \sigma_1(A^*). \quad (A.3)$$

Proof. The identity $(A.2)$ is a simple exercise (see [14, Theorem 58.2]).

For the equivalence of (a)–(f), we refer to [19, Theorem 2].

Next, if (a)–(f) hold, then Equation $(A.2)$ shows $(\ker A)^\perp = \operatorname{ran}(A^*)$. This implies $\operatorname{ran} A = \operatorname{ran}(A\vert_{(\ker A)^\perp}) = \operatorname{ran}(A\vert_{\operatorname{ran} A^*}) = \operatorname{ran}(AA^*)$, which proves the first part of Equation $(A.3)$. The second part follows by applying the first part to $A^*$ instead of $A$.

The last identity in $(A.3)$ follows directly from the definition of $\sigma_1$ and the well-known Jacobson lemma which states that for arbitrary bounded linear operators $S : \mathcal{H} \to \mathcal{K}$ and $T : \mathcal{K} \to \mathcal{H}$ we have $\sigma(ST)\setminus\{0\} = \sigma(TS)\setminus\{0\}$. It can indeed be easily seen that $\lambda \in \sigma(TS)\setminus\{0\}$ implies $\lambda \in \sigma(ST)$, by virtue of the identity

$$(ST - \lambda I_\mathcal{K})^{-1} = \frac{1}{\lambda} \left[ S(TS - \lambda I_\mathcal{H})^{-1}T - I_\mathcal{K} \right].$$

By symmetry, this implies $\sigma(TS)\setminus\{0\} = \sigma(ST)\setminus\{0\}$. \hfill $\square$

Lemma A.3. A bounded operator $A : \mathcal{K} \to \mathcal{H}$ is bounded below (meaning that there is $c > 0$ with $\|Ax\|_\mathcal{H} \geq c \|x\|_\mathcal{K}$ for all $x \in \mathcal{K}$) if and only if $A^*A : \mathcal{K} \to \mathcal{K}$ is bounded below.

Furthermore, a bounded self-adjoint operator $T : \mathcal{H} \to \mathcal{H}$ is bounded below if and only if $T$ is boundedly invertible.

Proof. Using the bounded inverse theorem, it is easy to see that a bounded operator $T$ between two Hilbert spaces is bounded below if and only if $\ker T = \{0\}$ and if $\operatorname{ran} T$ is closed. Lemma A.2 shows that $\operatorname{ran} A$ is closed if and only if $\operatorname{ran}(A^*A)$ is closed. Since furthermore $\ker A = \ker(A^*A)$, we obtain the first claim.

For the second part of the claim, let $T : \mathcal{H} \to \mathcal{H}$ be bounded, self-adjoint, and bounded below. As seen above, this implies that $\operatorname{ran} T$ is closed and that $\ker T = \{0\}$. Therefore, Equation $(A.2)$ shows $\mathcal{H} = (\ker T)^\perp = \overline{\operatorname{ran} T^*} = \operatorname{ran} T$. Hence, $T : \mathcal{H} \to \mathcal{H}$ is bijective, so that the bounded inverse theorem shows that $T$ is boundedly invertible. It is clear that if $T$ is boundedly invertible, then $T$ is bounded below. \hfill $\square$

The next lemma follows directly from [11, Cor. 5.5.2 and Cor. 5.5.3].

Lemma A.4. A Bessel sequence $(\varphi_i)_{i \in I}$ in a Hilbert space $\mathcal{H}$ is a frame sequence if and only if its analysis operator $A : \mathcal{H} \to \ell^2(I)$, $f \mapsto (\langle f, \varphi_i \rangle)_{i \in I}$ has closed range.
Let $A : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator with closed range. Then the operator $A_0 : (\ker A)^\perp \to \text{ran } A, \quad x \mapsto Ax,$ \hfill (A.4) 

is boundedly invertible by the bounded inverse theorem. Hence, the pseudo-inverse $A^\dagger := \iota_{(\ker A)^\perp} \circ A_0^{-1} \circ \text{ran } A$ of $A$ defines a bounded linear operator from $\mathcal{K}$ to $\mathcal{H}$. Here, $\iota_{(\ker A)^\perp}$ is the inclusion map $(\ker A)^\perp \to \mathcal{H}, x \mapsto x$.

In the following lemma we list some of the properties of the pseudo-inverse.

**Lemma A.5.** Let $A : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator with closed range. Then the following hold:

\begin{enumerate}[(i)]  
\item $A^\dagger A = P_{(\ker A)^\perp}$.  
\item $AA^\dagger = P_{\text{ran } A}$.  
\item $(A^\dagger)^* = (A^*)^\dagger$.  
\item $(A^*A)^\dagger A^* = A^\dagger A^* = A^*(AA^*)^\dagger$.  
\end{enumerate}

**Proof.** Properties (i)–(iii) can be found in \[2, Lemma 2.5.2].

For the first identity in (iv), we refer to \[3, Theorem 2.1]. The remaining identity follows from the first one and (iii) by applying the first part of (iv) on the right-hand side of the identity $A^\dagger = ((A^*)^\dagger)^*$. \hfill $\square$

**Lemma A.6.** Let $A : \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator with closed range and set $c := \sigma_1(A)$. Then $\sigma(A) \subset \{0\} \cup (\mathbb{R}\setminus(-c,c))$ and $A^\dagger = \varphi(A)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by $\varphi(t) = \frac{1}{t}$ for $t \neq 0$ and $\varphi(0) = 0$.

**Remark.** Since $0$ is an isolated point of $\sigma(A) \subset \{0\} \cup (\mathbb{R}\setminus(-c,c))$, $\varphi|_{\sigma(A)}$ is continuous.

**Proof.** Lemma A.2 shows $c = \sigma_1(A) > 0$. By definition of $\sigma_1(A)$ (see Equation (2.1)), we thus see that $A^2 = A^*A$ satisfies $\sigma(A^2) \subset \{0\} \cup [c^2, \infty)$. As $\sigma(A^2) = \{\lambda^2 : \lambda \in \sigma(A)\}$ and since $\sigma(A) \subset \mathbb{R}$ because of $A^* = A$, it follows that $\sigma(A) \subset \{0\} \cup (\mathbb{R}\setminus(-c,c))$. In particular, this entails that $\varphi|_{\sigma(A)}$ is continuous.

To prove $A^\dagger = \varphi(A)$, define $\psi := \mathds{1}_{\{0\}}$ and note $\psi \in C(\sigma(A); \mathbb{R})$ since $0$ is an isolated point of $\sigma(A)$ (or even $0 \notin \sigma(A)$). Since $\psi^2 = \psi$, we see that $P := \psi(A)$ satisfies $P^2 = P = P^*$, so that $P = P_V$ is the orthogonal projection onto a closed subspace $V \subset \mathcal{H}$. For $x \in \ker A$ we have $Ax = 0x$, so that \[20, Theorem VII.1(d)] shows $P_x = \psi(A)x = \psi(0)x = x$; hence, $\ker A \subset V$. Conversely, we have $\text{id}_{\sigma(A)} \cdot \psi \equiv 0$ and hence $0 = (\text{id}_{\sigma(A)} \cdot \psi)(A) = AP$, which shows $V = \text{ran } P \subset \ker A$ and hence $V = \ker A$.

Next, observe that $\varphi \cdot \text{id}_{\sigma(A)} = 1 - \psi$, whence $\varphi(A) = \text{id}_H - P = P_{V^\perp} = A^\dagger$, where the last step used Lemma A.3(a). Hence, $\varphi(A) = A^\dagger$ on $\text{ran } A$. Finally, we have $\varphi(A)P_V = (\varphi \cdot \psi)(A) = 0$, meaning $\varphi(A) = 0 = A^\dagger$ on $V = \ker A = (\text{ran } A)^\perp$. Overall, this shows $\varphi(A) = A^\dagger$, as claimed. \hfill $\square$

**Corollary A.7.** Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint operator with closed range, and let $U : \mathcal{K} \to \mathcal{H}$ be unitary. Then $U^*AU : \mathcal{K} \to \mathcal{K}$ is also bounded and self-adjoint with closed range, and we have $(U^*AU)^\dagger = U^*A^\dagger U$.

**Proof.** It is clear that $U^*AU$ is bounded and self-adjoint with closed range. Furthermore, a direct calculation shows $p(U^*AU) = U^*p(A)U$ for every polynomial $p \in \mathbb{R}[x]$. By definition of the continuous spectral calculus, we thus get $\varphi(U^*AU) = U^*\varphi(A)U$ for all $\varphi \in C(\sigma(A); \mathbb{R})$, where we note $\sigma(A) = \sigma(U^*AU)$. Now, the claim follows from Lemma A.6 \hfill $\square$
A.3. Some properties of Sobolev functions

A.3.1. Essentially bounded (matrix-valued) Sobolev functions

Our main objective in this subsection is to prove that the space of matrix-valued functions with all entries in $H^1(\Omega) \cap L^\infty(\Omega)$ is stable under matrix multiplication and inversion. For this, the following lemma will be crucial.

**Lemma A.8** ([3 Cor. 2.7]). Let $\Omega \subset \mathbb{R}^n$ be open and let $\gamma : \mathbb{C} \to \mathbb{C}$ be a Lipschitz continuous map. In case of $\lambda(\Omega) = \infty$, assume additionally that $\gamma(0) = 0$. If $f \in H^1(\Omega)$, then $\gamma \circ f \in H^1(\Omega)$.

**Lemma A.9.** Let $\Omega \subset \mathbb{R}^n$ be open and let $f, g \in H^1(\Omega) \cap L^\infty(\Omega)$. Then:

(a) $f \cdot g \in H^1(\Omega) \cap L^\infty(\Omega)$.

(b) If $\text{essinf } |f| > 0$, then also $1/f \in H^1(\Omega) \cap L^\infty(\Omega)$.

**Proof.** (a) Clearly, $fg \in L^2(\Omega) \cap L^\infty(\Omega)$. Further, [2 Section 4.25] shows that the weak derivatives of $fg$ exist and satisfy

$$\partial_j (fg) = (\partial_j f) \cdot g + f \cdot (\partial_j g) \quad \text{for } j \in \{1, \ldots, n\}.$$

As $\partial_j f, \partial_j g \in L^2(\Omega)$ and $f, g \in L^\infty(\Omega)$ it follows that $\partial_j (fg) \in L^2(\Omega)$.

(b) Let $r := \text{essinf } |f| > 0$. We trivially have $1/f \in L^\infty(\Omega)$. Note that $\lambda(\Omega) < \infty$ since $f \in H^1(\Omega) \subset L^2(\Omega)$ and $|f(x)| \geq r > 0$ almost everywhere. Let $B := \{z \in \mathbb{C} : |z| < r\}$ and define $\gamma_0 : \mathbb{C} \setminus B \to \mathbb{C}, z \mapsto z^{-1}$. Then $\gamma_0$ is well-defined and Lipschitz continuous, since $|z^{-1} - w^{-1}| = \frac{|w-z|}{|zw|} \leq r^{-2}|w-z|$ for $z, w \in \mathbb{C} \setminus B$. Now, [10] Theorem 1 in Section 3.1] shows that there exists a Lipschitz continuous extension $\gamma : \mathbb{C} \to \mathbb{C}$ of $\gamma_0$. Since $|f(x)| \geq r$ almost everywhere, we have $\gamma \circ f = \gamma_0 \circ f = 1/f$ almost everywhere, and Lemma A.8 shows $1/f = \gamma \circ f \in H^1(\Omega)$. 

In the following we denote by $H^1(\Omega; \mathbb{C}^{k \times \ell})$ the space of all matrix-valued functions $A : \Omega \to \mathbb{C}^{k \times \ell}$ for which each component function is in $H^1(\Omega)$. We similarly define $L^p(\Omega; \mathbb{C}^{k \times \ell})$ for $p \in [1, \infty]$.

**Lemma A.10.** Let $\Omega \subset \mathbb{R}^n$ be open and let $A \in H^1(\Omega; \mathbb{C}^{k \times \ell}) \cap L^\infty(\Omega; \mathbb{C}^{k \times \ell})$ and $B \in H^1(\Omega; \mathbb{C}^{\ell \times m}) \cap L^\infty(\Omega; \mathbb{C}^{\ell \times m})$. Then the following statements hold:

(a) $AB \in H^1(\Omega; \mathbb{C}^{k \times m}) \cap L^\infty(\Omega; \mathbb{C}^{k \times m})$.

(b) If $k = \ell$ and $\text{essinf } \sigma_0(A(x)) > 0$, then $A^{-1} \in H^1(\Omega; \mathbb{C}^{k \times k}) \cap L^\infty(\Omega; \mathbb{C}^{k \times k})$.

(c) If $\text{essinf } \sigma_0(B(x)) > 0$, then $B^\dagger \in H^1(\Omega; \mathbb{C}^{m \times k}) \cap L^\infty(\Omega; \mathbb{C}^{m \times k})$.

**Proof.** Statement (a) follows from Lemma [A.9] (a), since $(AB)_{j,n} = \sum_i A_{j,t}B_{t,n}$. For (b) we first observe that Leibniz’s formula

$$\det A(x) = \sum_{\sigma \in S_k} \text{sign}(\sigma) \prod_{j=1}^{k} A_{\sigma(j),j}(x)$$

and Lemma [A.9] (a) yield $\det A \in H^1(\Omega) \cap L^\infty(\Omega)$. Now, the condition on $A$ implies that $A(x)$ is invertible for a.e. $x \in \Omega$ so that $(A(x))^{-1}$ indeed exists for a.e. $x \in \Omega$. Moreover, for a.e. $x \in \Omega$, for the smallest eigenvalue $\lambda(x)$ of $A(x)^*A(x)$ we have that $\lambda(x) \geq c := \text{essinf } \sigma_0(A(x))^2 > 0$. Therefore, we conclude that $|\det A(x)|^2 = \det(A(x)^*A(x)) \geq \lambda(x)^k \geq c^k$. 

A QUANTITATIVE SUBSPACE BALIAN-LOW THEOREM 31
for a.e. \( x \in \Omega \). Hence, Lemma A.9 (b) shows that \((\det A)^{-1} \in H^1(\Omega) \cap L^\infty(\Omega)\). Also, 
\[
\|A(x)^{-1}\|^2 = \|A(x)^* A(x)\| = \frac{1}{c}
\]
for a.e. \( x \in \Omega \) implies that \( A^{-1} \in L^\infty(\Omega; \mathbb{C}^{k \times k})\). Finally, \( A^{-1} \in H^1(\Omega; \mathbb{C}^{k \times k}) \) follows from Lemma A.9 (a), combined with the so-called cofactor formula for the inverse of a matrix (see for instance [15] Equations (5-22) and (5-23)). It states for \( A \in \mathbb{C}^{k \times k} \) with \( k > 1 \) and \( \det A \neq 0 \) that 
\[
A^{-1} = \frac{\text{adj} A}{\det A} \quad \text{with} \quad (\text{adj} A)_{i,j} = (-1)^{i+j} \cdot \det A^{(j,i)},
\]
where \( A^{(j,i)} \) is obtained from \( A \) by deleting its \( j \)-th row and its \( i \)-th column. In the remaining case \( k = 1 \), we have \( A^{-1} = (\det A)^{-1} \in H^1(\Omega) \cap L^\infty(\Omega) \) as well.

The condition on \( B \) in (c) implies that \( B(x)^* B(x) \) is invertible for a.e. \( x \in \Omega \) with \( \text{essinf}_{x \in \Omega} \sigma_0(B(x)^* B(x)) = \text{essinf}_{x \in \Omega} \sigma_0(B(x))^2 > 0 \). The claim now follows from (a), (b), and the identity \( B(x)^1 = (B(x)^* B(x))^{-1} B(x)^* \) (see Part (iv) of Lemma A.5). \(\square\)

### A.3.2 A certain property of the space \( \mathbb{H}^2(\mathbb{R}) \)

**Lemma A.11.** If \( g \in \mathbb{H}^2(\mathbb{R}) \), then \( Xg' \in L^2(\mathbb{R}) \) with the estimate 
\[
\|Xg'\|_{L^2} \leq 45 \cdot (\|g''\|_{L^2}^2 + \|X^2 g\|_{L^2}^2 + \|g'\|_{L^2}^2)^{1/2} 
\leq 45 \cdot (1 + 4\pi^2) \|g''\|_{L^2}^2 + \|X^2 g\|_{L^2}^2 + 4\pi^2 \|g\|_{L^2}^2)^{1/2}.
\]

**Proof.** It follows from [11] Lemma 5.4] that for any \( \eta > 0 \) and \( f \in C^2([0, \eta]) \),
\[
|f'(0)|^2 \leq \frac{C}{\eta} \cdot \left( \eta^2 \cdot \int_0^\eta |f''(t)|^2 dt + \eta^{-2} \cdot \int_0^\eta |f(t)|^2 dt \right),
\]
where \( C := 2 \cdot 9^2 \). One can see that this remains true for \( f \in H^2((0, \eta)) \), by a density argument since \( H^2((0, \eta)) \subset C^1([0, \eta]) \) (see for instance [11] Thm. 4.12, Part II).

Given \( g \in \mathbb{H}^2(\mathbb{R}) \) and \( x \in [1, \infty) \), we can apply the above estimate to the function \( t \mapsto g(x + t) \) to obtain 
\[
|g(x)|^2 \leq C \cdot \int_0^\eta \left( \eta^2 \cdot |g''(x + t)|^2 + \eta^{-2} \cdot |g(x + t)|^2 \right) dt,
\]
where we denote by \( \int_\Omega f(x) \, dx = \frac{1}{\lambda(\Omega)} \int_\Omega f(x) \, dx \) the average of \( f \) over \( \Omega \), with \( \lambda(\Omega) \) denoting the Lebesgue measure of \( \Omega \).

Now, fix \( n \in \mathbb{N} \) for the moment, and let \( x \in [2^n, 2^{n+1}] \). If we set \( \eta = 2^{-n} \), then 
\[
2^n \eta = 2^{2n} \leq x^2 \leq (x + t)^2 \quad \text{for all} \quad t \in [0, \eta].
\]
Therefore,
\[
|x \cdot g'(x)|^2 \leq 4 \cdot 2^{2n} \cdot |g'(x)|^2 \leq 4C \int_0^{2^{-n}} ((\eta^2 \cdot |g''(x + t)|^2 + (x + t)^2 \cdot |g(x + t)|^2) dt. \quad (A.5)
\]

For brevity, set \( F(y) := |g''(y)|^2 + |y^2 \cdot g(y)|^2 \). Then, for any \( t \in [0, 2^{-n}] \subset [0, 1] \), we have \( 2^{n+1} + t \leq 2^{n+2} \) and hence 
\[
\int_{2^n}^{2^{n+1}} F(x + t) \, dx \leq \int_{2^n}^{2^{n+2}} F(y) \, dy.
\]
By combining this observation with the trivial estimate \( \int_\Omega G(t) \, dt \leq \|G\|_{L^\infty(\Omega)} \), and by integrating Equation (A.5) over \( x \in [2^n, 2^{n+1}] \), we arrive at 
\[
\int_{2^n}^{2^{n+1}} |x \cdot g'(x)|^2 \, dx \leq 4C \cdot \int_0^{2^{-n}} \int_{2^n}^{2^{n+1}} F(x + t) \, dx \, dt \leq 4C \cdot \int_{2^n}^{2^{n+2}} F(y) \, dy.
\]
Summing over \( n \in \mathbb{N}_0 \), we conclude that
\[
\int_1^\infty |x \cdot g'(x)|^2 \, dx = \sum_{n=0}^\infty \int_1^{2^{n+1}} |x \cdot g'(x)|^2 \, dx \\
\leq 4C \cdot \int_1^\infty F(y) \cdot \sum_{n=0}^\infty \mathbb{I}_{(2^n, 2^{n+2})}(y) \, dy \\
\leq 12C \cdot (\|g''\|_{L^2((1, \infty))}^2 + \|X^2g\|_{L^2((1, \infty))}^2).
\]
Here the last step used that \( \sum_{n=0}^\infty \mathbb{I}_{(2^n, 2^{n+2})}(y) \leq 3 \); indeed, if \( 2^n < y < 2^{n+2} \), then each \( k \) for which also \( 2^k < y < 2^{k+2} \) satisfies \( 2^n < 2^{k+2} \) and \( 2^k < 2^{n+2} \), so that \( k \in \{n-1, n, n+1\} \).

By applying estimate \( \text{(A.6)} \) to \( h : \mathbb{R} \to \mathbb{C}, x \mapsto g(-x) \) instead of \( g \), we easily get \( \int_{-\infty}^1 |x \cdot g'(x)|^2 \, dx \leq 12C \cdot (\|g''\|_{L^2((1, \infty))}^2 + \|X^2g\|_{L^2((1, \infty))}^2) \). Adding this to \( \text{(A.6)} \) and using the trivial estimate \( \int_1^\infty |x \cdot g'(x)|^2 \, dx \leq \|g''\|_{L^2}^2 + 12C \cdot (\|g''\|_{L^2}^2 + \|X^2g\|_{L^2}^2) \).

This easily implies the first part of the stated estimate.

For the last part, recall that \( \mathcal{F}[g'](\xi) = 2\pi\xi \hat{g}(\xi) \) and \( \mathcal{F}[g''](\xi) = (2\pi\xi)^2 \hat{g}(\xi) \). Thanks to Plancherel’s theorem and the elementary estimate \( |\xi|^2 \leq 1 + |\xi|^4 \), we thus see
\[
\|g''\|_{L^2}^2 = \int_{\mathbb{R}} |2\pi\xi \cdot \hat{g}(\xi)|^2 \, d\xi \leq (2\pi)^2 \cdot \int_{\mathbb{R}} |\hat{g}(\xi)|^2 + (2\pi\xi)^2 \hat{g}(\xi)^2 \, d\xi \\
= (2\pi)^2 \cdot (\|g\|_{L^2}^2 + \|g''\|_{L^2}^2).
\]
Together with the first part of the lemma, this implies the second part. \( \square \)

**A.3.3. Sobolev functions on slices and the AC-property**

Let \( A \subset \mathbb{R}^n \) be Borel measurable, where \( n > 1 \). For \( i \in \{1, \ldots, n\} \) and \( x \in \mathbb{R}^{n-1} \) we define the following Borel measurable subset of \( \mathbb{R}^n \):
\[
A_{i,x} = \{ t \in \mathbb{R} : (x_1, \ldots, x_{i-1}, t, x_i, \ldots, x_{n-1}) \in A \}.
\]

Note that \( A_{i,x} \) is open if \( A \) is so. The following lemma is an easy consequence of Fubini’s theorem.

**Lemma A.12.** A Borel set \( N \subset \mathbb{R}^n \) has measure zero if and only if for some (and then all) \( i \in \{1, \ldots, n\} \) and a.e. \( x \in \mathbb{R}^{n-1} \) the set \( N_{i,x} \) has measure zero in \( \mathbb{R}^n \).

We say that a function \( h : U \to \mathbb{C} \), where \( U \subset \mathbb{R} \) is open, is **locally absolutely continuous (LAC)** on \( U \) if it is LAC on each connected component of \( U \); this is equivalent to \( h \) being LAC on each open subinterval of \( U \). Here, a function \( f : I \to \mathbb{C} \) with an open interval \( I \subset \mathbb{R} \) is called locally absolutely continuous if there is a function \( g \in L^1_{loc}(I) \) such that \( f(x) - f(y) = \int_y^x g(t) \, dt \) for all \( x, y \in I \). In particular, each LAC function is continuous.

**Definition A.13.** Let \( \Omega \subset \mathbb{R}^n \) be an open set. A (pointwise defined) function \( f : \Omega \to \mathbb{C} \) is said to have the **AC-property** (on \( \Omega \)), if for each \( i \in \{1, \ldots, n\} \) and almost all \( x \in \mathbb{R}^{n-1} \) the function
\[
f_{i,x} : \Omega_{i,x} \to \mathbb{C}, \quad t \mapsto f(x_1, \ldots, x_{i-1}, t, x_i, \ldots, x_{n-1})
\]
is LAC on \( \Omega_{i,x} \).
Note that the classical partial derivatives $\partial_k f$ of a function $f : \Omega \to \mathbb{C}$ having the AC-property exist a.e. on $\Omega$ by [11, Theorem 3.35] and Lemma A.12.

**Lemma A.14.** Let $\Omega \subset \mathbb{R}^n$ be open and let $f \in W^{1,1}_{loc}(\Omega)$. Then there is a representative $g : \Omega \to \mathbb{C}$ of $f$ which has the AC-property on $\Omega$. In particular, we have $\partial_i g = D_i f$ a.e. on $\Omega$, $i = 1, \ldots, n$. Here, $D_i f$ denotes the weak derivative of $f$.

**Proof.** Let $\Omega^{(0)} := \emptyset$ and $\Omega^{(k)} := \{ x \in \Omega : \text{dist}(x, \Omega^c) > 1/k \} \cap (-k, k)^n$, $k \in \mathbb{N}$. Then each $\Omega^{(k)}$ is open in $\mathbb{R}^n$, $\Omega^{(k)} \subset \Omega$ is compact, $\Omega^{(k)} \subset \Omega^{(k+1)}$, and $\bigcup_k \Omega^{(k)} = \Omega$. By [18, Thm. 1.41], for each $k \in \mathbb{N}$ there exists a representative $f^{(k)}$ of $f$ which has the AC-property on $\Omega^{(k)}$. It is clear that the function $g : \Omega \to \mathbb{C}$,

$$g := \sum_{k=1}^{\infty} \mathbb{1}_{\Omega^{(k)}} \cdot f^{(k)}$$

is a representative of $f$. Let us show that it has the AC-property on $\Omega$.

First of all, for each $k \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$ there exists a set $L^{(k)}_i \subset \mathbb{R}^{n-1}$ of measure zero such that $f^{(k)}_{i,x}$ is LAC on $\Omega^{(k)}_{i,x}$ for every $x \in \mathbb{R}^{n-1} \setminus L^{(k)}_i$. Let $L := \bigcup_i L^{(k)}_i$.

Fix $k \in \mathbb{N}$. Then $f^{(k+1)} = f^{(k)}$ a.e. on $\Omega^{(k)}$. In particular, by Lemma A.12 for each $i \in \{1, \ldots, n\}$ there exists a set $M^{(k)}_i \subset \mathbb{R}^{n-1}$ of measure zero such that for all $x \in \mathbb{R}^{n-1} \setminus M^{(k)}_i$ we have that $f^{(k+1)}_{i,x} = f^{(k)}_{i,x}$ a.e. on $\Omega^{(k)}_{i,x}$. Let $M := \bigcup_i M^{(k)}_i$.

Let $N := L \cup M \subset \mathbb{R}^{n-1}$. Then $N$ is a null-set, and for each $i \in \{1, \ldots, n\}$, each $k \in \mathbb{N}$, and each $x \in \mathbb{R}^{n-1} \setminus N$ we have that $f^{(k)}_{i,x}$ is LAC on $\Omega^{(k)}_{i,x}$ and $f^{(k+1)}_{i,x} = f^{(k)}_{i,x}$ almost everywhere on $\Omega^{(k)}_{i,x}$, and since both functions are continuous on the open set $\Omega^{(k)}_{i,x}$, they agree everywhere on $\Omega^{(k)}_{i,x}$. Now, if $i \in \{1, \ldots, n\}$, $x \in \mathbb{R}^{n-1} \setminus N$, and if $K \subset \Omega^{(k)}_{i,x}$ is compact, then $\bigcup_k \Omega^{(k)}_{i,x}$ is an open cover of $K$. Thus, there is some $k = k(i, x, K) \in \mathbb{N}$ such that $K \subset \Omega^{(k)}_{i,x}$ and $g_{i,x} = f^{(k)}_{i,x}$ on $K$. Therefore, $g_{i,x}$ is LAC on $\Omega^{(k)}_{i,x}$.

For the “in particular”-part, it suffices to prove that $\partial_i g = D_i f$ almost everywhere on every open rectangular cell $R = \prod_{j=1}^{n} (a_j, b_j)$ satisfying $\overline{R} \subset \Omega$. To see this, set $R_i := \prod_{j \neq i} (a_j, b_j) \subset \mathbb{R}^{n-1}$, and observe that for any $\varphi \in C^\infty_c(R)$ we have

$$\int_R (\partial_i g) \cdot \varphi \, dy + \int_R g \cdot (\partial_i \varphi) \, dx = \int_{R_i} \int_{a_i}^{b_i} g_{i,x}(t) \varphi_{i,x}(t) \, dt \, dx + \int_{R_i} \int_{a_i}^{b_i} g_{i,x}(t) \varphi'_{i,x}(t) \, dt \, dx = 0 .$$

Hence, $\int_R (\partial_i g - D_i f) \varphi \, dx = \int_R (f - g) \partial_i \varphi \, dx = 0$ for every $\varphi \in C^\infty_c(R)$. The claim thus follows from the fundamental lemma of the calculus of variations (see for instance [2, Section 4.22]).

We close with this subsection with a result that generalizes Lemma A.12 in the case $n = 2$ to sections of $\mathbb{R}^2$ that are not necessarily parallel to the coordinate axes.

**Lemma A.15.** Let $N \subset \mathbb{R}^2$ be a null-set, and let $(a, b) \in \mathbb{R}^2 \setminus \{0\}$. Then there is a null-set $N_0 \subset \mathbb{R}^2$ such that for all $(x, \omega) \in \mathbb{R}^2 \setminus N_0$, we have

$$(x + ta, \omega + tb) \in \mathbb{R}^2 \setminus N \quad \text{for almost all } t \in \mathbb{R}.$$
Proof. Set \( \theta := (a,b) \in \mathbb{R}^2 \setminus \{0\} \), and choose \( \varphi \in \mathbb{R}^2 \setminus \{0\} \) with \( \varphi \perp \theta \). Let us define \( T : \mathbb{R}^2 \to \mathbb{R}^2, (t,s) \mapsto t\theta + s\varphi \). Note that \( T \) is linear and bijective, so that the same holds for \( T^{-1} \). In particular, \( T \) and \( T^{-1} \) are Lipschitz continuous, and thus map null-sets to null-sets.

Let \( \tilde{N} := T^{-1}N \subset \mathbb{R}^2 \). By Lemma A.12 there is a null-set \( \tilde{N}_1 \subset \mathbb{R} \) such that for all \( s \in \mathbb{R} \setminus \tilde{N}_1 \), the set \( \tilde{N}_{1,s} = \{ t \in \mathbb{R} : (t,s) \in \tilde{N} \} \) is a null-set. Let \( N_0 := T(\mathbb{R} \times \tilde{N}_1) \), and note that \( N_0 \subset \mathbb{R}^2 \) is indeed a null-set. We claim that if \( (x,\omega) \in \mathbb{R}^2 \setminus N_0 \), then \( (x+ta,\omega+tb) \in \mathbb{R}^2 \setminus N \) for almost all \( t \in \mathbb{R} \). To see this, let \( (x,\omega) \in \mathbb{R}^2 \setminus N_0 \). This implies \( (x,\omega) = T(t_0,s_0) \) for certain \( (t_0,s_0) \in \mathbb{R} \times (\mathbb{R} \setminus \tilde{N}_1) \), so that \( \tilde{N}_{1,s_0} \) is a null-set. Finally, if \( t \in \mathbb{R} \setminus (\tilde{N}_1,s_0 - t_0) \) (which holds for almost all \( t \in \mathbb{R} \)), then \( t+t_0 \notin \tilde{N}_{1,s_0} \), which means that \( (t+t_0,s_0) \notin \tilde{N} = T^{-1}N \), and hence \( (x+ta,\omega+tb) = T(t+t_0,s_0) \in \mathbb{R}^2 \setminus N \), as claimed. \( \square \)

A.4. Invariance properties of Gabor spaces

Lemma A.16. Let \( g \in L^2(\mathbb{R}) \) and let \( \Lambda \subset \mathbb{R}^2 \) be a lattice. Define \( \mathcal{G} := \mathcal{G}(g,\Lambda) \) Then \( \pi(\lambda)\mathcal{G} \subset \mathcal{G} \) for all \( \lambda \in \Lambda \).

Proof. For \( \lambda,\lambda' \in \Lambda \), there exists a unimodular constant \( c = c(\lambda,\lambda') \in \mathbb{C} \) satisfying \( \pi(\lambda)\pi(\lambda') = c\pi(\lambda + \lambda') \). Hence, \( \pi(\lambda)\pi(\lambda')g \in \mathcal{G} \). Since \( \mathcal{G} \) is spanned by the elements \( \pi(\lambda')g, \lambda' \in \Lambda \), this shows \( \pi(\lambda) \subset \mathcal{G} \) for all \( \lambda \in \Lambda \). \( \square \)

A.5. Failure of the main result for general elements of \( \mathcal{G}(g,\Lambda) \)

We close this paper with an example showing that the relation

\[
\text{dist} \left( \pi(u,\eta)f, \mathcal{G}(g,\Lambda) \right) \asymp \text{dist}(\pi(u,\eta)\Lambda) \cdot \|f\|_{L^2},
\]

which holds for \( f = g \), does not extend to general \( f \in \mathcal{G}(g,\Lambda) \). The example is constructed based on a footnote in [13].

Example A.17. Let \( \varphi : \mathbb{R} \to \mathbb{R}, x \mapsto e^{-\pi x^2} \) denote the Gaussian. We will repeatedly make use of the following two facts: First, [12] Theorem 7.5.3 shows that if \( \alpha,\beta > 0 \), then \( (\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z}) \) is a frame for \( L^2(\mathbb{R}) \) if and only if \( \alpha/\beta < 1 \). By Ron-Shen duality (see [12] Theorem 7.4.3), this implies that \( (\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z}) \) is a Riesz sequence (a Riesz basis for its closed linear span) if and only if \( \alpha/\beta > 1 \).

Set \( \Lambda := 2\mathbb{Z} \times \frac{3}{4}\mathbb{Z} \) and \( \Lambda_0 := \Lambda \cup ((1,0) + \Lambda) = \mathbb{Z} \times \frac{3}{4}\mathbb{Z} \). Then \( (\varphi,\Lambda_0) \) is a frame for \( L^2(\mathbb{R}) \) but not a Riesz sequence. Thus, the synthesis operator

\[
T : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R}), \quad (c_k,\ell)_{k,\ell \in \mathbb{Z}} \mapsto \sum_{k,\ell \in \mathbb{Z}} c_k,\ell \pi(k, \frac{3}{4}\ell)\varphi
\]

is surjective, but not injective, since otherwise the bounded inverse theorem would imply that \( T \) is boundedly invertible, meaning that \( (\varphi,\Lambda_0) \) is a Riesz basis for \( L^2(\mathbb{R}) \). In other words, there exist \( \ell^2 \) sequences \( c = (c_{m,n})_{m,n \in \mathbb{Z}} \) and \( d = (d_{m,n})_{m,n \in \mathbb{Z}} \) with \( c, d \neq 0 \) and

\[
\sum_{m,n \in \mathbb{Z}} c_{m,n} \pi(2m, \frac{3}{4}n)\varphi = \sum_{m,n \in \mathbb{Z}} d_{m,n} \pi(2m + 1, \frac{3}{4}n)\varphi
= \pi(1,0) \left[ \sum_{m,n \in \mathbb{Z}} \tilde{d}_{m,n} \pi(2m, \frac{3}{4}n)\varphi \right],
\]
where $\tilde{d}_{m,n} := e^{4\pi i m}d_{m,n}$ for $m, n \in \mathbb{Z}$. Then $f := \sum_{m,n \in \mathbb{Z}} \tilde{d}_{m,n} \pi(2m, \frac{2}{3}n)\varphi$ satisfies $f \in \mathcal{G}(\varphi, \Lambda)$ and $\pi(1, 0)f \in \mathcal{G}(\varphi, \Lambda)$. Now, once we show that $f \neq 0$, we will have disproved (A.7).

To see that $f \neq 0$, we note that $(\varphi, \Lambda)$ is a Riesz sequence. If $f = 0$, we would have $\tilde{d} = 0$ and therefore $d = 0$. In turn, the above identity gives $0 = \sum_{m,n \in \mathbb{Z}} c_{m,n} \pi(2m, \frac{2}{3}n)\varphi$, whence $c = 0$, again since $(\varphi, \Lambda)$ is a Riesz sequence. Therefore, $f = 0$ implies $(c, d) = 0$ which is a contradiction.

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