Primordial black hole formation in $F(R)$ bouncing cosmology

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Abstract. The phenomenology of primordial black holes (PBHs) physics, and the associated PBH abundance constraints, can be used in order to probe the early-universe evolution. In this work, we focus on the bounce realization within $F(R)$ modified gravity and we investigate the corresponding PBH behavior. In particular, we calculate the energy density power spectrum at horizon crossing time as a function of the involved theoretical parameters, and then we extract the PBH abundance in the context of peak theory, considering the non-linear relation between the density contrast and the comoving curvature perturbation, as well as the critical collapse law for the PBH masses. We first calculate the PBH mass function, and then we extract the PBH abundance $\Omega_{\text{PBH},f}$ at formation time as a function of the model parameters, namely the involved $F(R)$ parameter $\alpha$ and the Hubble parameter at the transition time from the bounce to the radiation dominated epoch $H_{\text{RD}}$. Interestingly, we find that in order to have a significant black hole production, namely $10^{-10} < \Omega_{\text{PBH},f} < 1$, $H_{\text{RD}}$ and $\alpha$ should lie roughly within the ranges $10^{-7} M_{\text{Pl}} \leq H_{\text{RD}} \leq 10^{-6} M_{\text{Pl}}$, $10^{-9} M_{\text{Pl}} \leq H_{\text{RD}} \leq 2 \times 10^{-9} M_{\text{Pl}}$ and $10^{-30} M_{\text{Pl}}^2 \leq \alpha \leq 10^{-12} M_{\text{Pl}}^2$ respectively. Finally, we show that the excluded region corresponding to PBH overproduction forms a closed ring.

Keywords: Primordial Black Holes, $F(R)$ Gravity, Bouncing Cosmology
1 Introduction

The theory of cosmological inflation [1–5] constitutes a promising paradigm to describe the physical conditions that prevailed in the very early universe, being able to address a number of cosmological issues like the horizon and the flatness problems. However, inflationary theories face the problem of initial singularity [6]. One attractive alternative to inflation is the non-singular bouncing cosmological paradigm [7, 8], which assumes that the universe existed forever before the Hot Big Bang (HBB) era in a contracting phase and at some point transitioned into the expanding universe that we observe today. Apart from solving the singularity problem the bounce realization can also address the usual flatness and horizon problems of standard Big Bang cosmology (for a review on bouncing cosmologies, see [9]) and give rise to an observationally compatible cosmological power spectrum [10–12].

In order to acquire a non-singular bouncing phase, violation of the null energy condition is necessary. Consequently, modified gravity theories [13–15] provide an ideal framework for obtaining a bouncing universe. Hence, such bouncing solutions have been constructed through various approaches to modified gravity, such as the Pre-Big-Bang [16] and the Ekpyrotic [17, 18] models, gravitational theories whose gravity actions contain higher order corrections [19, 20], $F(R)$ gravity [21, 22], $f(T)$ gravity [23] models,
braneworld scenarios [24, 25], non-relativistic gravity [26, 27], massive gravity [28], etc. The above scenarios can be further extended to the paradigm of cyclic cosmology [29–31].

As a potential candidate, the bounce scenario is expected to be consistent with current cosmological observations and to be distinguishable from the experimental predictions of cosmic inflation as well as other paradigms [32, 33]. One interesting way of distinguishing various approaches from the bounce scenario is the study of primordial black holes (PBH) [34, 35] forming within non-singular bouncing cosmological models.

Primordial black holes, first proposed in early ’70s [36–38], are considered to form in the very early universe out of the gravitational collapse of very high overdensity regions, whose energy density is higher than a critical threshold [39–45]. According to recent arguments, PBHs can naturally act as a viable dark matter candidate [46, 47] and potentially explain the generation of large-scale structures through Poisson fluctuations [48, 49], while they can also seed the supermassive black holes residing in galactic centres [50, 51]. Furthermore, they are associated with numerous gravitational-wave (GW) signals, from black-hole merging events [52–56] up to primordial second-order scalar induced GWs from primordial curvature perturbations [57–62] (for a recent review see [63]) or from Poisson PBH energy density fluctuations [64–66]. Other indications in favor of the PBH scenario can be found in [67]. Their abundance is constrained from a wide variety of probes [47, 68–72] over a range of masses from 10g up to $10^{20} M_\odot$, thus giving us access to a very rich phenomenology.

Up to now, the majority of the literature studied PBH formation within single-field [73–76] or multi-field [77–79] inflationary cosmology. However, the study of PBHs in bouncing scenarios is limited [80–84], most of which has been done with a generalised approach, without any falsification of the bouncing scenarios. Therefore, given the aforementioned rich phenomenology and the associated PBH abundance constraints over a range of masses which span more than 50 orders of magnitude, PBHs can clearly provide a novel promising way to test and constrain various bounce scenarios.

In this work, we investigate the bounce realization within one of the simplest modifications of general relativity which can violate the null energy condition and thus give rise to a bouncing phase, namely the $F(R)$ gravity theory. $F(R)$ gravity forms a particular class of theories in which the Einstein–Hilbert action is upgraded to a general function of the Ricci scalar $R$ [14]. $F(R)$ theories have been studied extensively in the context of inflation [85–87], bounce [21, 22, 88] and late-time acceleration [88–90]. Additionally, this class of theories has been highly successful in explaining both late and early time acceleration along with the intermediate thermal history of the Universe (see [91, 92] for reviews). Therefore, it would be very interesting to examine how such theories can be constrained or ruled out through the study of PBH formation within them.

The manuscript is organised as follows: In Sec. 2 we present the class of $F(R)$ gravity which can induce a bouncing scale factor. In Sec. 3 we extract the associated curvature power spectrum, which can potentially lead to PBH production as a function of the theoretical parameters involved, namely the bouncing parameter $\alpha$ and the energy scale $H_{RD}$ at the onset of the radiation dominated HBB era. Subsequently, in Sec. 4,
after extracting the energy density power spectrum $\mathcal{P}_\delta(k)$ from the curvature power spectrum, we compute the PBH mass function $\beta(M)$ within peak theory. Then, in Sec. 5 we investigate the behavior of $\mathcal{P}_\delta(k)$ and $\beta(M)$ on the different regions of the parameter space $(\alpha, H_{\text{RD}})$, and we extract constraints on $\alpha$ and $H_{\text{RD}}$ by demanding that PBHs are not overproduced at PBH formation time. Finally, Sec. 6 is devoted to conclusions.

2 Bounce cosmology through $F(R)$ gravity

For the present analysis we consider the flat Friedman-Lémaitre-Robertson-Walker (FLRW) background metric
\[ ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \] 
where $a(t)$ is the scale factor while the gravitational action for $F(R)$ gravity in vacuum can be written as:
\[ S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} F(R), \] 
where $\kappa^2 = 8\pi G = \frac{1}{M_{\text{Pl}}^2}$, with $M_{\text{Pl}}$ being the reduced Planck mass. Re-writing $F(R)$ as $F(R) = R + f(R)$, with $f(R)$ representing the deviation from General Relativity, the corresponding Friedmann equations turn out to be
\[ 3H^2 = -\frac{f(R)}{2} + 3(H^2 + \dot{H}) f'(R) - 18(4H^2 \dot{H} + H\ddot{H}) f''(R) \] 
\[ \frac{f(R)}{2} = (3H^2 + \dot{H}) f'(R) - 6(8H^2 \dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \dddot{H}) f''(R) \] 
\[ -36(4H \dot{H} + \dddot{H})^2 f'''(R), \]
where $H(t) \equiv \dot{a}/a$ is the Hubble parameter.

Since we are interested in studying the bounce realization in $F(R)$ gravity, we choose the scale factor accordingly. The general evolution of the universe in bouncing cosmology consists of a period of contraction followed by a cosmological bounce and then by the standard expanding universe. Any form of the scale factor satisfying $a(t_b) > 0$, $\dot{a}(t_b) = 0$, $\ddot{a}(t_b) > 0$, is capable of giving rise to bouncing cosmology, where $t_b$ corresponds to the time when the bounce occurs.

Let us now present the bounce realization at the background level. Without loss of generality we consider a bouncing scale factor of the form
\[ a_b(t) = 1 + \alpha t^2, \] 
with $\alpha$ being a free parameter and the bounce happening at $t = 0$. The above form of scale factor has been obtained by keeping terms up to quadratic order in $t$ in the Taylor expansion of $a(t)$ near the bounce. We neglect higher order terms as we are interested for solutions near the bounce. Finally, note that the bounce realization conditions mentioned above indicate that $\alpha > 0$. 

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Using the above form of the scale factor, we obtain the expressions for the Hubble parameter and the Ricci scalar (keeping terms up to $\mathcal{O}(t^2)$) as:

$$H(t) = \frac{2\alpha t}{1 + \alpha t^2} \simeq 2\alpha t,$$

$$R(t) = 12H^2 + 6\dot{H} = \frac{12\alpha(1 + 3\alpha t^2)}{(1 + \alpha t^2)^2} \simeq 12\alpha + 12\alpha^2 t^2. \quad (2.6)$$

As we can see from the above relations, the Hubble parameter varies linearly with time around the bounce, and becomes zero at the bounce point, as expected. Moreover, the Ricci scalar at the bounce is $R(0) = 12\alpha$. Inserting the above expressions into Eq. (2.3) we acquire

$$24\alpha(R - 12\alpha)f''_b(R) + (R - 24\alpha)f'_b(R) + f_b(R) + 2(R - 12\alpha) = 0, \quad (2.7)$$

where the index $b$ refers to background quantities. Finally, solving the above equation for $f_b(R)$ and keeping terms up to $\mathcal{O}(t^2)$, the solution for $F_b(R)$ near the bounce can be recast as

$$F_b = R + e^{-\frac{R}{24\alpha}}\left(\frac{12\alpha - C}{216\alpha}\right)\left[12e^{\frac{R}{24\alpha}}R + \sqrt{\frac{6\pi}{\alpha}}(R - 12\alpha)^{3/2}\text{Erfi}\left(\sqrt{\frac{R - 12\alpha}{24\alpha}}\right)\right], \quad (2.8)$$

where Erfi$(z)$ is the imaginary error function defined as Erfi$(z) = -i\text{Erf}(iz)$ and $C$ is an integration constant which will be fixed later. Hence, from now on the parameter $\alpha$ can be considered as the $F(R)$ model parameter.

### 3 The curvature power spectrum

Since we have studied the background behavior of a bounce scenario realized within $F(R)$ gravity, and we have extracted the function $F(R)$ around the bounce, we proceed to the calculation of the curvature power spectrum by deriving the corresponding comoving curvature perturbation.

#### 3.1 The curvature perturbation

Before launching our methodology, we should examine which primordial perturbation modes are relevant for present-day observation. As we saw above, the Hubble parameter vanishes at the bounce point, thus giving rise to an infinite comoving Hubble radius ($1/aH$) there. In the following, we match the bouncing phase with the standard Hot Big Bang radiation phase, which in turn, according to the standard cosmological evolution as dictated by the current cosmological probes, is connected to a matter epoch and then at late times with an accelerated expansion phase. Consequently, the Hubble horizon decreases and tends to zero for late times, while for cosmic times near the bouncing point the Hubble horizon has an infinite size. Therefore, all the perturbation modes at
that time are contained within the horizon, and at later epochs they cross the Hubble radius becoming relevant for current observations. Hence, in the following we focus on the perturbation equations near the bounce, namely near $t = 0$.

Choosing to work in the comoving gauge, the spatial part of the perturbed scalar metric tensor reads as

$$\delta g_{ij} = a^2(t) \left[ 1 - 2\zeta(\vec{x}, t) \right] \delta_{ij}, \quad (3.1)$$

where $\zeta(\vec{x}, t)$ denotes the comoving curvature perturbation. The corresponding action for the scalar perturbations reads as [93–95]

$$\delta S_\zeta = \int dt d^3 \vec{x} a(t) z(t)^2 \left( \dot{\zeta}^2 - \frac{1}{a^2} (\partial_i \zeta)^2 \right), \quad (3.2)$$

with $z(t)$ given by the following expression [88]:

$$z(t) = \frac{a(t)}{\kappa \left[ H(t) + \frac{1}{2F'(R)} \frac{dF'(R)}{dt} \right]} \sqrt{\frac{3}{2F'(R)} \left( \frac{dF'(R)}{dt} \right)^2}, \quad (3.3)$$

where $\kappa \equiv 1/M_{\text{Pl}}$. Using the solution for $F(R)$, i.e. Eq. (2.8), the expressions for $F'(R)$ and $F''(R)$ are now given by

$$F'(R(t)) = \frac{(12\alpha - C) \left[ -2t\alpha - t^3\alpha^2 + \sqrt{2}\alpha \sqrt{t} \alpha (-3 + t^2\alpha) F_D \left( \frac{\sqrt{\alpha}}{\sqrt{2}} \right) \right]}{36t\alpha^2}, \quad (3.4)$$

$$F''(R(t)) = \frac{t(12\alpha - C) \left( \frac{t^2\alpha (5 - t^2\alpha) + \sqrt{2}\alpha \sqrt{3t} [3 + t^2\alpha (-6 + t^2\alpha)] F_D \left( \frac{\sqrt{\alpha}}{\sqrt{2}} \right) \right)}{36t^2\alpha}, \quad (3.5)$$

where $F_D(x)$ is the Dawson function.

As mentioned earlier, the perturbation modes are generated close to the bounce, therefore we solve the above equation for cosmic times near the bouncing point. As a result, we keep terms up to $O(t^2)$ for the rest of our analysis. The corresponding expression for $z(t)$, keeping terms up to $O(t^2)$ in $F'(R)$ and $F''(R)$, becomes

$$z(t) = \frac{(1/\alpha)^{3/2} \sqrt{12\alpha - C} \sqrt{2\alpha \sqrt{\alpha}}}{3^{1/2}(t^2 + 1)\kappa} + \frac{2\alpha^2 \sqrt{12\alpha - C} \sqrt{2\alpha}}{3^{1/2}4\alpha^{3/2}\kappa}. \quad (3.6)$$

At the end, the perturbed action leads to the following Lagrange equation for the Fourier mode of the comoving curvature perturbation, $\zeta_k$:

$$\frac{1}{a(t)z^2(t)} \frac{d}{dt} \left[ a(t)z^2(t)\dot{\zeta}_k \right] + \frac{k^2}{a^2} \zeta_k(t) = 0. \quad (3.7)$$

In the above equation, by using (3.6) and keeping terms upto $O(t^2)$, the quantity $a(t)z(t)^2$ becomes:

$$a(t)z(t)^2 = U + Vt^2, \quad (3.8)$$
with \( U = \frac{(12\alpha - C)}{12\alpha} \), \( V = \frac{(12\alpha - C)}{4\alpha} \).

Evaluating then the scalar perturbations near the bounce, at leading order in \( t \) the Lagrange equation for \( \zeta_k \) can be recast as

\[
\ddot{\zeta}_k + \frac{2V}{U} t \dot{\zeta}_k + k^2 \zeta_k(t) = 0,
\]

whose solution is

\[
\zeta_k(t) = C_1(k) e^{-\frac{V}{U} t^2} H \left( -1 + \frac{k^2U}{2V}, \sqrt{\frac{U}{V}} t \right) + C_2(k) e^{-\frac{V}{U} t^2} _1 F_1 \left( \frac{1}{2} - \frac{k^2U}{4V}, \frac{1}{2}, \sqrt{\frac{V}{U}} t^2 \right),
\]

where \( C_1(k), C_2(k) \) are integration constants, \( H(n, x) \) is the \( n \)-th order Hermite polynomial, and \( _1 F_1(a, b, x) \) is the Kummer confluent hypergeometric function.

The expressions for the integration constants \( C_1(k), C_2(k) \) are obtained by setting the initial conditions for the scalar perturbations. Given the fact that close to the bounce the Hubble radius is infinitely large as mentioned above, the primordial modes are well inside the Hubble radius thus satisfying the condition \( k \gg aH \). Therefore, the initial conditions for \( \zeta_k \) will be set through the Mukhanov-Sasaki variable, defined in the present context as \( v_k(t) \equiv z(t) \zeta_k(t) \) \(^{[88]} \), and whose value on sub-Hubble scales is set by the Bunch-Davies vacuum state, i.e.

\[
v_{k, k \ll aH} = \frac{e^{-ik\eta}}{\sqrt{2k}},
\]

where the time variable \( \eta \) is the conformal time defined by \( d\eta \equiv dt/a(t) \). Using the expression (2.5) for the scale factor near the bounce, we obtain from Eq. (3.11) that

\[
\eta = \int_0^t dt'/a(t') = \frac{\arctan(\sqrt{\alpha} t)}{\sqrt{\alpha}}.
\]

Consequently, the initial conditions satisfied by \( v_k \) and its derivative become:

\[
v_k(t \to 0) = \frac{1}{\sqrt{2k}},
\]
\[
v'_k(t \to 0) = -\frac{ik\sqrt{\alpha}}{\sqrt{2k}}.
\]

Using these conditions and the fact that \( z'(t \to 0) = 0 \), we finally acquire straightforwardly the expressions for the integration constants \( C_1, C_2 \) as

\[
C_1(k) = \frac{3i\kappa 2^{\frac{3}{2}} - \frac{\kappa^2}{m^2} \sqrt{\kappa} \alpha^{3/2} \Gamma \left( \frac{3}{2} - \frac{k^2}{12\alpha} \right)}{\sqrt{\pi} (6\alpha - k^2) \sqrt{12\alpha - C}},
\]

\[
C_2(k) = \frac{\sqrt{2} \kappa \left[ -6i\kappa \alpha^{3/2} \Gamma \left( \frac{3}{2} - \frac{k^2}{12\alpha} \right) + \sqrt{3\alpha} (6\alpha - k^2) \Gamma \left( 1 - \frac{k^2}{12\alpha} \right) \right]}{k^{1/2} (6\alpha - k^2) \sqrt{12\alpha - C} \Gamma \left( 1 - \frac{k^2}{12\alpha} \right)},
\]
where $\Gamma(x)$ denotes the Gamma function. At the end, the corresponding curvature power spectrum can be recast as follows:

$$\mathcal{P}_\zeta(k, t) \equiv \frac{k^3}{2\pi^2} \left| \zeta_k(t) \right|^2 = \frac{k^3}{2\pi^2} \left| C_1(k) e^{-\frac{V}{2} t^2} H \left[ -1 + \frac{k^2 U}{2V} \sqrt{\frac{V}{U}} t \right] + C_2(k) e^{-\frac{V}{4} t^2} _1 F_1 \left[ \frac{1}{2} - \frac{k^2 U}{4V} \frac{1}{2} \sqrt{\frac{V}{U}} t^2 \right] \right|^2. \quad (3.16)$$

### 3.2 Matching the bounce with a radiation dominated era

Close to the bounce point the scale factor is given by (2.5), which is equivalent with a fluid with $w = -2/3$. Since PBH formation in negative-$w$ background is something not well studied and corresponds to a cosmic period prevailed by an exotic form of matter, in the following we match the bounce with a radiation-dominated (RD) era during which, once the characteristic size of the collapsing overdensity region crosses the Hubble radius, one is expecting PBH production if $\delta > \delta_c \simeq 0.45$ [40]. In order to achieve this we assume that the intermediate phase between the bounce and RD era is described by the scale factor (2.5). Thus, in summary we can recast the scale factor evolution as:

$$a(t) = \begin{cases} 
1 + \alpha t^2, & t < t_{\text{RD}} \\
\text{a}_{\text{RD}} \left( \frac{t}{t_{\text{RD}}} \right)^{1/2}, & t > t_{\text{RD}},
\end{cases} \quad (3.17)$$

with $t_{\text{RD}} = 1/(2H_{\text{RD}})$ being the transition time between the exotic phase with $w = -2/3$ and the RD phase, and with $a_{\text{RD}}$ the respective scale factor. We mention that in order to keep the scale factor continuous during the transition we choose $a_{\text{RD}}$ to be $a_{\text{RD}} = 1 + \alpha t_{\text{RD}}^2$.

Given the fact that in the following we elaborate the power spectrum at the horizon crossing time during the RD era, i.e. $k = a(t) H(t)$ with $t > t_{\text{RD}}$, one can find the horizon crossing time $t_{\text{HC}}(k, \alpha, H_{\text{RD}})$ by solving $k = a H$ with $a(t) = a_{\text{RD}} \left( \frac{t}{t_{\text{RD}}} \right)^{1/2}$ and $H(t) = \frac{1}{2t}$. At the end, we extract that

$$t_{\text{HC}}(k, \alpha, H_{\text{RD}}) = \frac{H_{\text{RD}}}{2k^2} \left( 1 + \frac{\alpha}{4H_{\text{RD}}^2} \right)^2. \quad (3.18)$$

At this point it is important to stress out that in the expression (3.16) we derived the power spectrum close to the bounce by parametrizing the scale factor as (2.5). Hence, assuming as mentioned above that (2.5) describes efficiently also the intermediate era which connects the bounce with the RD phase, we can compute $\mathcal{P}_\zeta(k, t)$ at horizon exiting time during the intermediate phase before the RD era, namely when $k = a(t) H(t)$ with $t < t_{\text{RD}}$. At the end, accounting for the fact that on superhorizon scales the comoving curvature perturbation is conserved [96, 97], the curvature power spectrum at horizon crossing time during the RD era will be the same as the curvature power spectrum at horizon exiting time during the intermediate phase between the bounce and the RD era, namely

$$\mathcal{P}_\zeta[k, t_{\text{HC}}(k, \alpha, H_{\text{RD}})] = \mathcal{P}_\zeta[k, t_{\text{exit}}(k, \alpha)], \quad (3.19)$$
where $t_{HC}(k, \alpha, H_{RD})$ is given by (3.18) and $t_{exit}(k, \alpha) = \frac{k}{2\pi}$. Finally, we can then use $P_\zeta[k, t_{HC}(k, \alpha, H_{RD})]$ and proceed to the calculation of the PBH abundance at horizon crossing time, which is considered to be the PBH formation time.

### 3.3 The scales involved

Regarding the relevant scales for the problem at hand, here we consider modes crossing the horizon during the RD era after the bounce. Therefore, given the fact that the PBH mass is of the order of the horizon mass at the time of PBH formation, which is here considered as the horizon crossing time, one should require that the PBH masses involved should be higher than the mass associated to the scale crossing the horizon at the onset of the RD era, which is actually the minimum PBH mass, $M_{RD}$. At the end, by requiring that $M_{PBH} > M_{RD}$ we obtain an upper bound on the comoving scales $k$ reading as

$$k < H_{RD} + \frac{\alpha}{4H_{RD}}. \quad (3.20)$$

### 3.4 Investigating different bouncing scale factor parametrisations

Up to now, we have considered that the scale factor close to the bounce is parametrized by (2.5), by keeping terms up to quadratic order in $t$ in the Taylor expansion for $a(t)$ near the bounce. Thus, a legitimate question to ask is how our results will change by changing the scale factor parametrisation near the bounce. In general, the scale factor near a non-singular bounce can be parameterized as [88]

$$a_b(t) \simeq (1 + \alpha t^2)^n, \quad (3.21)$$

where $n$ is a real number. In the following we study the cases where $n = 2$ and $n = 3$ and we examine how the curvature power spectrum changes accordingly.

1) $a(t) = (1 + \alpha t^2)^2$:

Using this parametrisation for the scale factor near the bounce, and solving Eq. (2.3) for $F(R)$ we find that

$$F_b(R(t)) = \frac{1}{420} t^2 \alpha^2 \left\{ 99225 + t^2 \alpha \left\{ -814275 + t^2 \alpha \left\{ 91875 + t^2 \alpha \left\{ 3360 + t^2 \alpha \{ 245 

+ t^2 \alpha \left\{ 75 + t^2 \alpha (-25 + t^2 \alpha) \right\} \right\} \right\} \right\} \right\}

+ \frac{1}{8 \sqrt{t^2 \alpha}} \left\{ 105 + t^2 \alpha \{ 525 + t^2 \alpha \{ -1050 + t^2 \alpha \{ 350 + t^2 \alpha (-35 + t^2 \alpha) \} \} \} \} \right\} C

+ \frac{1}{8 \sqrt{t^2 \alpha}} \left\{ 105 + t^2 \alpha \{ 525 + t^2 \alpha \{ -1050 + t^2 \alpha \{ 350 + t^2 \alpha (-35 + t^2 \alpha) \} \} \} \right\}

\cdot \text{Erfc} \left( \frac{t \sqrt{t \alpha}}{\sqrt{2}} \right) \text{ Erfi} \left( \frac{t \sqrt{t \alpha}}{\sqrt{2}} \right)

+ \frac{9}{8} \alpha \left\{ -e^{t^2 \alpha / 2} \sqrt{2\pi t \sqrt{\alpha}} \left\{ 105 + t^2 \alpha \{ -790 + t^2 \alpha \{ 318 + t^2 \alpha (-34 + t^2 \alpha) \} \} \right\} \right\}.$$
\[
\pm\left\{105 + t^2\alpha \left[525 + t^2\alpha \left[-1050 + t^2\alpha \left[350 + t^2\alpha (-35 + t^2\alpha)\right]\right]\right]\right\}
\cdot \text{Erfi}\left(\frac{t\sqrt{t\alpha}}{\sqrt{2}}\right) \cdot \text{Erf}\left(\frac{t\sqrt{t\alpha}}{\sqrt{2}}\right)
- \frac{1}{840} e^{t^2\alpha/2} t^{12} \alpha^7 \left\{105 + t^2\alpha \left[-790 + t^2\alpha \left[318 + t^2\alpha (-34 + t^2\alpha)\right]\right]\right\}
\cdot \text{ExpIntegralE}\left(-\frac{9}{2}, -\frac{t^2\alpha}{2}\right),
\]
(3.22)

where \(\text{ExpIntegralE}\) is the exponential integral function \(E_n(z)\).

Similar to the previous case, keeping terms up to \(O(t^2)\) the expression for \(z(t)\) becomes
\[
z(t) = U + Vt - Xt^2,
\]
(3.23)
where
\[
U = \sqrt{\frac{105\alpha}{2\alpha K\sqrt{\alpha'/C}}}, \quad V = \frac{\sqrt{\alpha'/C}(2592\pi\alpha^{12} - 1225C^2)}{12\alpha^{12}\sqrt{210\pi}},
\]
\[
X = \frac{(\alpha'/C)^{3/2}(419904\pi^2\alpha^{18} + 45360\pi\alpha^{12}C + 1190700\pi\alpha^6C^2 + 42875C^3)}{1890\alpha^{18}\sqrt{105}}.
\]

Thus, evaluating the curvature perturbation near the bounce, at leading order in \(t\), we obtain
\[
\zeta_k = C_1(k)H\left[\frac{k^2U^2}{2V^2 + 4UX - 4U^2\alpha}, \frac{-UV + tV^2 + 2tUX - 2tU^2\alpha}{U\sqrt{V^2 + 2U(X - U\alpha)}}\right]
\cdot 1F1\left\{-\frac{k^2U^2}{4[V^2 + 2U(X - U\alpha)]}, \frac{1}{2}, \frac{-tV^2 + U(V - 2tX) + 2tU^2\alpha}{U^2[V^2 + 2U(X - U\alpha)]}\right\}.
\]
(3.26)

The forms of \(C_1(k)\) and \(C_2(k)\) are determined using the initial conditions given in (3.13) modified appropriately for the present case where
\[
a(t) = (1 + \alpha t^2)^2.
\]

As one may notice, the general form of the comoving curvature perturbation is qualitatively the same as what we had extracted in (3.10) for the parametrization \(a(t) = (1 + \alpha t^2)\). Hence, the general results for the power spectrum remain qualitatively the same with those obtained above.

2) \(a(t) = (1 + \alpha t^2)^3\):

With the same reasoning as before, the solution for \(F(R)\) around the bounce reads as
\[
F_b(t) = 6\alpha + 324\alpha^2 t^2 + \left\{324t^4\alpha^3(-3 + t^2\alpha) - \frac{54}{288} e^{\frac{3t^2\alpha}{2}} \left[1 + t^2\alpha(-8 + 3t^2\alpha)\right] C
\right. + 9\sqrt{6\pi} \alpha^{9/2} \left\{1 + 9t^2\alpha \left[1 + t^2\alpha(-3 + t^2\alpha)\right]\right\} \text{Erfi}\left(\sqrt{\frac{3\alpha^3}{2}} t\right)\right\}.
\]
(3.27)
Once again, keeping up to $O(t^2)$ terms in the scalar perturbation, we extract the form of $z(t)$ as

$$z(t) = U + Vt - Xt^2,$$

where

$$U = \sqrt{\frac{6}{\kappa}}, \quad V = \frac{(-124416 + \alpha C^2)}{24\sqrt{6}\kappa C},$$

$$X = \frac{\alpha(746496 + \alpha C^2)}{6912(\sqrt{6}\kappa)}.$$ (3.28) (3.29) (3.30)

The corresponding solution for the curvature perturbation, at leading order in $t$, is

$$\zeta_k = C_1(k) H \left[ \frac{k^2 U^2}{2V^2 + 4UX - 4U^2\alpha} \frac{-tV^2 + 2tUX - 2tU^2\alpha}{U\sqrt{V^2 + 2U(X - U\alpha)}} \right]$$

$$+ C_2(k) \frac{k^2 U^2}{4[V^2 + 2U(X - U\alpha)]} \left[ \frac{-tV^2 + U(V - 2tX) + 2tU^2\alpha]}{U^2[V^2 + 2U(X - U\alpha)]} \right].$$ (3.31)

Hence, we again see that the final form of the curvature perturbation is qualitatively the same as in the other parametrisations. Consequently, one can argue that our results are nearly the same for $(1 + \alpha t^2)$, $(1 + \alpha t^2)^2$, $(1 + \alpha t^2)^3$ and other values of $n$ in $(1 + \alpha t^2)^n$. We mention here that other possible bouncing scale factor forms that have been studied in the literature are $\cosh(1 + \alpha t^2)$, and $e^{\alpha t^2}$. However, when expanded around $t$ their forms become similar to $(1 + \alpha t^2)^n$, hence our above results become quite general, being valid for any parametrization of the scale factor giving rise to a bounce.

4 The PBH Formation Formalism

In this section we present a general formalism for the computation of the mass function of PBHs formed due to the collapse of enhanced cosmological perturbations once they reenter the cosmological horizon. Basically, this happens when the energy density contrast of the collapsing overdensity region, or the respective comoving curvature perturbation, becomes greater than a critical threshold $\delta_c$ or $\zeta_c$. In the following, we firstly describe how the comoving curvature perturbation is connected to the energy density contrast, extracting the non-linear relation between them, and then we proceed by presenting the formalism for the computation of the PBH mass function and the PBH abundance within the context of peak theory [98].

4.1 From the comoving curvature perturbation to the energy density contrast

Assuming spherical symmetry on superhorizon scales, the local region of the universe describing the aforementioned collapsing cosmological perturbations is described by the following asymptotic form of the metric

$$ds^2 = -dt^2 + a^2(t)e^{\zeta(r)} \left[ dr^2 + r^2 d\Omega^2 \right],$$ (4.1)
where \( a(t) \) is the scale factor and \( \zeta(r) \) is the comoving curvature perturbation which is conserved on superhorizon scales. In this regime one can perform a gradient expansion approximation, where all the hydrodynamic and metric quantities are nearly homogeneous, and their perturbations are small deviations away from their background values [99–102]. In this approximation, the energy density perturbation profile is related to the comoving curvature perturbation through the following expression [40, 103, 104]:

\[
\frac{\delta \rho}{\rho_b} = \frac{\rho(r, t) - \rho_b(t)}{\rho_b(t)} = -\left( \frac{1}{aH} \right)^2 \frac{4(1 + w)}{5 + 3w} e^{-\frac{5\zeta(r)}{2}} e^{\frac{5\zeta(r)}{2}}
\]

where \( w \) is the total equation-of-state parameter defined as the ratio between the total pressure \( p \) and the total energy density \( \rho \), i.e. \( w = p/\rho \). Note that the last expression is obtained by Fourier transforming the curvature perturbation \( \zeta \).

In the linear regime, where \( \zeta \ll 1 \), the above expression is reduced to

\[
\frac{\delta \rho}{\rho_b} \approx -\left( \frac{k}{aH} \right)^2 \frac{2(1 + w)}{5 + 3w} \left[ \zeta_k + \frac{\zeta_k^2}{2} \right] e^{-2\zeta_k},
\]

(4.2)

From the above form we can see that there is a one-to-one relation between the comoving curvature perturbation and the energy density contrast. Thus, if the curvature perturbation is a Gaussian variable then the same is true for the density contrast within the linear regime described by (4.3). However, the amplitude of the critical threshold \( \delta_c \) or \( \zeta_c \) is in general non-linear, and as a consequence one should consider the full non-linear expression between \( \zeta \) and \( \delta \), namely (4.2).

At this point we should stress the fact that the use of \( \zeta \) for the computation of the PBH abundance vastly overestimates the number of PBHs, since scales larger than the PBH scale, which are unobservable, are not properly removed when the PBH distribution is smoothed [105]. Therefore, one should instead use the energy density contrast, given the fact that with this prescription the superhorizon scales are naturally damped by \( k^2 \), as it can be seen by (4.2).

Consequently, smoothing the energy density contrast with a top-hat window function over scales larger than the horizon scale and using (4.2), we can straightforwardly find that the smoothed energy density contrast is related to the comoving curvature perturbation in radiation era, where \( w = 1/3 \), as

\[
\delta_m = -\frac{2}{3} r_m \zeta'(r_m) \left[ 2 + r_m \zeta'(r_m) \right].
\]

(4.4)

The scale \( r_m \) is the comoving scale of the collapsing overdensity, which can be found by maximizing the compaction function \( C \) defined as [40]

\[
C(r, t) = 2 \frac{M(r, t) - M_b(r, t)}{R(r, t)}.
\]

(4.5)
where \( R(r, t) \) is the areal radius, \( M(r, t) \) is the Misner-Sharp mass \([106, 107]\) within a sphere of a radius \( R \), and \( M_b = 4\pi R^3(r, t)/3 \) is the background mass with respect to a FLRW metric. Finally, by maximizing the compaction function, namely \( C'(r_m) = 0 \), the \( r_m \) scale will be given by the solution of the following equation:

\[
\zeta'(r_m) + r_m \zeta(r_m) = 0. \tag{4.6}
\]

Now, given the fact that \( \zeta \) is assumed to have a Gaussian distribution, its derivative will have a Gaussian distribution, too. Hence, we can identify a linear Gaussian variable \( \delta_l = -\frac{4}{3} r_m \zeta'(r_m) \) with a probability distribution function (PDF) given by

\[
P(\delta_l) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\delta_l^2}{2\sigma^2}}, \tag{4.7}
\]

where \( \sigma \) is the smoothed variance of \( \delta_l \) written as

\[
\sigma^2 = \langle \delta_l^2 \rangle = \int_0^\infty \frac{dk}{k} \mathcal{P}_\delta(k, R) = \frac{16}{81} \int_0^\infty \frac{dk}{k} (kR)^4 \tilde{W}^2(k, R) T^2(k, R) P_\zeta(k). \tag{4.8}
\]

The function \( \tilde{W}(k, R) \) is the Fourier transformation of the top-hat window function mentioned above, and reads as

\[
\tilde{W}(k, R) = 3 \sin\left(\frac{kr}{\sqrt{3}}\right) - \frac{kr}{\sqrt{3}} \cos\left(\frac{kr}{\sqrt{3}}\right) \left(\frac{kr}{\sqrt{3}}\right)^3. \tag{4.9}
\]

while \( T(k, R) \) is the linear transfer function during a RD era, which can be recast as \([108]\)

\[
T(k, R) = 3 \sin\left(\frac{kr}{\sqrt{3}}\right) - \frac{kr}{\sqrt{3}} \cos\left(\frac{kr}{\sqrt{3}}\right) \left(\frac{kr}{\sqrt{3}}\right)^3. \tag{4.10}
\]

Finally, the smoothed energy density contrast is related with the linear Gaussian energy density contrast through the following expression:

\[
\delta_m = \delta_l - \frac{3}{8} \delta_l^2. \tag{4.11}
\]

### 4.2 The PBH mass function within peak theory

In order to extract the mass function of PBHs which form due to the gravitational collapse of non-Gaussian energy density perturbations, we work with the Gaussian component of the smoothed non-Gaussian energy density contrast denoted as \( \delta_l \). Regarding the critical threshold of the linear Gaussian component, this can be found by solving Eq. (4.11) for \( \delta_l \) with \( \delta_m = \delta_c \). Hence, we find that

\[
\delta_{c, l} = \frac{4}{3} \left(1 \pm \sqrt{\frac{2 - 3\delta_c}{2}}\right). \tag{4.12}
\]
From the above expression we acquire a critical threshold for $\delta_l$. As explained in [109], only $\delta_{c,l-}$ corresponds to a physical solution, and since the argument of the square root should be positive we require $\delta_c < 2/3$. In summary, we find that the physical range of $\delta_l$ is $\delta_{c,l-} < \delta_l < 4/3$.

Regarding the PBH mass, it should be of the order of the horizon mass at PBH formation time, which is considered as the horizon crossing time. More precisely, the PBH mass spectrum, as it has been shown in previous works [110, 111], should follow a critical collapse scaling law which can be recast as

$$M_{\text{PBH}} = M_H K (\delta - \delta_c)^\gamma,$$  \hspace{1cm} (4.13)

where $M_H$ is the mass within the cosmological horizon at horizon crossing time, and $\gamma$ is the critical exponent which depends on the equation-of-state parameter at the time of PBH formation and for radiation it is $\gamma \approx 0.36$. The parameter $K$ is a parameter that depends on the equation-of-state parameter and on the particular shape of the collapsing overdensity region. In the following we consider a representative value of $K \approx 4$, which corresponds to an initial Mexican-hat energy density perturbation profile with $\delta_c \approx 0.55$.

Thus, working with the Gaussian linear component of the energy density contrast, we can calculate the PBH abundance in the context of peak theory, where the density of sufficiently rare and large peaks for a random Gaussian density field in spherical symmetry is given by [98]

$$N(\nu) = \frac{\mu^3 \nu^3}{4\pi^2 \sigma^3} e^{-\nu^2/2}.$$  \hspace{1cm} (4.14)

In this expression, $\nu \equiv \delta/\sigma$ and $\sigma$ is given by (4.8), while the parameter $\mu$ is the first moment of the smoothed power spectrum given by

$$\mu^2 = \int_0^\infty \frac{dk}{k} P_\delta(k, R) \left(\frac{k}{aH}\right)^2 = \frac{16}{81} \int_0^\infty \frac{dk}{k} (kR)^4 \tilde{W}^2(k, R) T^2(k, R) P_\zeta(k) \left(\frac{k}{aH}\right)^2.$$  \hspace{1cm} (4.15)

Finally, the fraction $\beta_\nu$ of the energy of the universe at a peak of a given height $\nu$, which collapses to form a PBH, will be given by

$$\beta_\nu = \frac{M_{\text{PBH}}(\nu)}{M_H} N(\nu) \Theta(\nu - \nu_c),$$  \hspace{1cm} (4.16)

and the total energy fraction of the universe contained in PBHs of mass $M$ can be recast as

$$\beta(M) = \int_{\nu_{c-}}^{\nu_c} d\nu \frac{\mu}{4\pi^2} \left(\frac{\nu \sigma - \frac{3}{8} \nu^2 \sigma^2 - \delta_c}{\mu aH \sigma}\right)^\gamma \left(\frac{\mu}{aH \sigma}\right)^3 \nu^3 e^{-\nu^2/2},$$  \hspace{1cm} (4.17)

where $\nu_{c-} = \delta_{c,l}/\sigma$. Lastly, the overall PBH abundance, defined as $\Omega_{\text{PBH}} \equiv \frac{\rho_{\text{PBH}}}{\rho_{\text{tot}}}$, where $\rho_{\text{tot}}$ is the total energy density of the universe, will be the integrated PBH mass function

$$\Omega_{\text{PBH}} = \int_{M_{\text{min}}}^{M_{\text{max}}} d\ln M \beta(M).$$  \hspace{1cm} (4.18)
5 Results

In the previous sections we extracted the curvature power spectrum and we presented the mathematical setup through which one can calculate the PBH mass function and abundance. Thus, in this section we present the main results of our work. Initially, we study the behaviour of the energy density power spectrum by varying the parameters of the problem at hand, namely the bounce and thus the $F(R)$ model parameter $\alpha$ and the energy scale $H_{RD}$. Then, we compute numerically the PBH mass function and we show how it varies by changing the theoretical parameters involved. Finally, by demanding that PBHs should not be overproduced at formation time, namely that $\Omega_{PBH,i} < 1$, we constrain the parameter space $(\alpha, H_{RD})$.

5.1 The energy density power spectrum

Given the fact that the scales collapsing to PBHs are initially super-Hubble before crossing the Hubble radius and collapse to PBHs, we perform a Taylor expansion of the co-moving curvature perturbation (3.10) on super-Hubble scales, i.e. when $k \ll aH$. By keeping terms up to $O\left(\frac{k}{aH}\right)^{3/2}$ we obtain

$$\zeta_{k,k\ll aH} \simeq \kappa e^{3\alpha} \left[\frac{3}{t(12\alpha - C)} \left(\frac{k}{a(t)H(t)}\right)^{-1/2} \right] - i \frac{\kappa \alpha t}{\sqrt{t(12\alpha - C)}} e^{3\alpha} \left( -1, \sqrt{3\alpha} \right) \left( -H(t) \right)$$

$$- \frac{\kappa \alpha t^{3/2}}{\sqrt{3(12\alpha - C)}} \left[ F_1 \left( \frac{1}{2}, \frac{1}{2}, 3\alpha \right) \left( \frac{k}{a(t)H(t)} \right)^{3/2} \right]. \quad (5.1)$$

Therefore, inserting this expression in Eq. (4.2) and following the procedure described in Sec. 4, we can calculate the energy density power spectrum $P_\delta(k)$ at horizon crossing time [see (3.18)] by fixing the bouncing parameter $\alpha$, the integration constant $C$, and the energy scale at the transition to the radiation era $H_{RD}$. As it was checked numerically, $P_\delta(k)$ is independent on the value of $C$ and in the following we will fix its value to $C = 0.1\alpha$.

In Fig. 1, we compare the exact numerical solution of the energy density power spectrum with the approximate one on super-Hubble scales, where $k \ll aH$. The curves were obtained for $H_{RD} = 10^{-7}M_{Pl}$ and for $\alpha = 10^{-10}M_{Pl}^2$. As we can see, for small values of $k$, the numerical and the approximate energy density power spectrum on super-Hubble scales match quite well. However, on large $k$ values, i.e. on small scales, the analytical expression for $P_\delta(k)$ vastly underestimates the numerical $P_\delta(k)$, and subsequently the PBH mass function. Thus, the PBH abundance, being the integrated PBH mass function, will be underestimated and in this sense the constraints that will be derived later for $(\alpha, H_{RD})$ by demanding that PBHs should not be overproduced at their formation time must be considered as rather conservative. It is also worth noticing that for other choices of the parameters $\alpha$ and $H_{RD}$ the numerical and the approximate power spectra
match quite well all along the $k$ range. In the following, we will use the approximate expression for $P_\delta(k)$ for our numerical computations.

$$H_{RD} = 10^{-7} M_{Pl}, \quad \alpha = 10^{-10} M_{Pl}^2$$

Figure 1: Comparison graph between the complete numerical solution for the energy density power spectrum and the curvature power spectrum in the limit $k \ll aH$. The energy density power spectra were obtained for $H_{RD} = 10^{-7} M_{Pl}$ and for $\alpha = 10^{-10} M_{Pl}^2$.

In Figs. 2 and 3, we depict the power spectrum $P_\delta(k)$, having taken into account the non-linear relation between $\delta$ and $\zeta$, for different values of $\alpha$ and $H_{RD}$ and for $C = 0.1\alpha$. As we can see, for a given value of $H_{RD}$, the power spectrum increases by increasing the value of $\alpha$. On the other hand, for constant $\alpha$, the $P_\delta(k)$ decreases with increasing $H_{RD}$. This behaviour can be understood if one sees how the maximum allowed value of $k$, which corresponds to the lowest scale of the problem at hand, varies with $\alpha$ and $H_{RD}$. In particular, using (3.20), we can easily show that

$$k_{\max} = H_{RD} + \frac{\alpha}{4H_{RD}}. \quad (5.2)$$

As it can be seen from Eq. (5.2), for a given value of $H_{RD}$, with an increase in $\alpha$ the value of $k_{\max}$ also increases, therefore the power spectrum shifts to higher values of $k$, i.e. to smaller scales. Consequently, as approaching smaller and smaller scales one starts to probe the granularity of the energy density field, entering in this way the non-linear regime where $P_\delta \gg 1$. Hence, one can clearly understand the tendency of the power spectrum to increase with increasing $\alpha$, given that it probes smaller scales which become non-linear.
Figure 2: The energy density power spectrum versus $k$, for different values of $\alpha$ and for $H_{\text{RD}} = 10^{-7} M_{\text{Pl}}$.

Figure 3: The energy density power spectrum versus $k$, for different values of $H_{\text{RD}}$ and for $\alpha = 10^{-8} M_{\text{Pl}}^2$. 
Similarly, for a given value of \( \alpha \), which for our case is \( 10^{-8} M_{Pl}^2 \), \( k_{\text{max}} \sim \alpha/(4H_{\text{RD}}) \) i.e. inversely proportional to \( H_{\text{RD}} \). Therefore, as we increase the value of \( H_{\text{RD}} \), \( k_{\text{max}} \) decreases and hence the power spectrum shifts to smaller values of \( k \) or larger scales. Following the previous reasoning, \( P_{\delta}(k) \) tends to decrease by increasing \( H_{\text{RD}} \) given that it shifts to larger scales away from the non-linear regime.

5.2 The PBH mass function

Since we have extracted above the energy density power spectra for different values of \( \alpha \) and \( H_{\text{RD}} \), we proceed to the calculation of the PBH mass function within peak theory. In particular, we follow the mathematical formalism presented in Sec. 4.2, and we incorporate the non-linear relation between \( \delta \) and \( \zeta \) as well as the critical collapse law for the PBH masses. Below, we show how the PBH mass function changes by varying the parameters \( \alpha \) and \( H_{\text{RD}} \).

![Figure 4: The PBH mass function \( \beta(M) \) as a function of the PBH mass \( M_{\text{PBH}} \), for different values of \( \alpha \) and for \( H_{\text{RD}} = 10^{-7} M_{Pl} \).](image)

In Fig. 4, by fixing the value of the energy scale at the transition to the RD era at \( H_{\text{RD}} = 10^{-7} M_{Pl} \), we show how the PBH mass function changes by varying the parameter \( \alpha \). In particular, by increasing \( \alpha \) the mass function decreases its overall amplitude and shifts to smaller PBH masses. This behavior is rather expected, since following the

1In the x axis we plot the the mass within the horizon at horizon crossing time which is roughly equal with the actual PBH mass as it can be seen from Eq. (4.13) for values of \( \delta \sim \delta_c \). Masses much larger than the horizon mass with \( \delta \gg \delta_c \) are in general exponential suppressed due to the Gaussian distribution of \( \delta_l \).
discussion of Sec. 5.1 an increase in $\alpha$ leads to an increase in the energy density power spectrum and shifts it to smaller scales, as it can already be seen by (5.2). Thus, due to the strong smoothing on these small scales through the window and the transfer functions, the PBH mass function is expected to have the tendency to decrease, and this behavior can be observed in Fig. 4.

\[ \alpha = 10^{-22} M_{Pl}^2 \]

\[ H_{RD}/M_{Pl} = 10^{-10} \]
\[ H_{RD}/M_{Pl} = 10^{-9} \]
\[ H_{RD}/M_{Pl} = 10^{-8} \]
\[ H_{RD}/M_{Pl} = 10^{-7} \]

\[ \beta(M_{PBH}) \]

**Figure 5:** The PBH mass function $\beta(M)$ as a function of the PBH mass $M_{PBH}$, for different values of $H_{RD}$ and for $\alpha = 10^{-22} M_{Pl}^2$.

On the other hand, in Fig. 5, fixing the value of the parameter $\alpha$ at $\alpha = 10^{-22} M_{Pl}^2$, we examine how the PBH mass function changes by varying the energy scale $H_{RD}$. Interestingly, decreasing the parameter $H_{RD}$ we see that the mass function decreases its overall amplitude, in contrast with the behavior that was shown in Fig. 3, where a decrease in $H_{RD}$ leads to an increase in the energy density power spectrum. This behavior can be explained as in the previous case where $H_{RD}$ was constant, by the fact that decreasing the energy scale $H_{RD}$ the energy density power spectrum shifts to smaller scales, since in the case where $\alpha = 10^{-8} M_{Pl}^2$ and $10^{-9} M_{Pl} \leq H_{RD} \leq 10^{-6} M_{Pl}$, $k_{max} \sim \alpha/H_{RD}$. At these small scales the smoothing performed through the window function and the transfer function is very efficient in reducing the PBH mass function drastically, despite a highly increased power spectrum. That is why by decreasing $H_{RD}$ the mass function has the tendency to decrease too.

### 5.3 Constraining $\alpha$ and $H_{RD}$

We can now proceed to perform a full parameter-space analysis by calculating the PBH abundance at formation time $\Omega_{PBH,f}$, for a wide range of values of the $F(R)$ parameter
\( \alpha \) and \( H_{\text{RD}} \). In particular, in Fig. 6 we show how \( \Omega_{\text{PBH},f} \) varies as a function of the bouncing parameter \( \alpha \) and the energy scale \( H_{\text{RD}} \). Interestingly, we find that in order to have a significant black hole production, i.e. \( 10^{-10} < \Omega_{\text{PBH},f} < 1 \), \( H_{\text{RD}} \) and \( \alpha \) should predominantly lie roughly within the ranges \( 10^{-7} M_{\text{Pl}} \leq H_{\text{RD}} \leq 10^{-6} M_{\text{Pl}} \), \( 10^{-9} M_{\text{Pl}} \leq H_{\text{RD}} \leq 2 \times 10^{-9} M_{\text{Pl}} \) and \( 10^{-30} M_{\text{Pl}}^2 \leq \alpha \leq 10^{-12} M_{\text{Pl}}^2 \) respectively. This is the main result of the present work. Furthermore, we found a finely-tuned lateral region for \( H_{\text{RD}} \leq 10^{-9} M_{\text{Pl}} \), where one faces a significant PBH production too.

We should mention here that one can extract constraints on the parameters \( \alpha \) and \( H_{\text{RD}} \) by excluding regions where \( \Omega_{\text{PBH},f} > 1 \). Indeed, within our cosmological bouncing model in the context of \( F(R) \) gravity we have found such regions. As we can see from Fig. 6, the excluded region shown in grey is forming a closed ring-like surface. In particular, one can say that the region of PBH overproduction, i.e. \( \Omega_{\text{PBH},f} > 1 \), mainly lies within the region roughly defined by \( 2 \times 10^{-9} M_{\text{Pl}} \leq H_{\text{RD}} \leq 10^{-7} M_{\text{Pl}} \) and \( 10^{-30} M_{\text{Pl}}^2 \leq \alpha \leq 10^{-14} M_{\text{Pl}}^2 \), as well as by the lateral grey region where \( H_{\text{RD}} \leq 10^{-9} M_{\text{Pl}} \).

![Figure 6](image.png)

**Figure 6:** The PBH abundance at formation time \( \Omega_{\text{PBH},f} \) as a function of the \( F(R) \) parameter parameter \( \alpha \) and the energy scale \( H_{\text{RD}} \). The grey region is excluded since there we have PBH overproduction.

### 6 Conclusions

The non-singular bouncing cosmological paradigm is one of the most appealing alternatives to inflation. Since the bounce realization requires the violation of the null energy condition, it can be typically implemented in the framework of modified gravity. On the
other hand, the phenomenology of PBH physics, and the associated PBH abundance constraints which span a range of masses over more than 50 orders of magnitude, has recently started to be investigated in detail, since it can be used in order to probe and extract constraints on the early-universe behavior. Hence, studying PBH both at inflationary and bounce scenarios, could be helpful in extracting possible distinguishable features.

In this work we focused on the bounce realization within $F(R)$ modified gravity and we investigated the corresponding PBH phenomenology. We postulated a radiation domination era, which connects the bouncing phase with the Hot Big Bang universe, and we calculated the energy density power spectrum at horizon crossing time, during the radiation dominated era, as a function of the involved theoretical parameters. These are the bounce parameter $\alpha$, which is the involved $F(R)$ parameter, and $H_{RD}$, i.e. the Hubble parameter at the time of the transition from the intermediate phase after the bounce to the radiation dominated epoch.

We started by extracting the behaviour of the energy density power spectrum for different combinations of the model parameters, and then we calculated the PBH abundance in the context of peak theory, considering the non-linear relation between $\delta$ and $\zeta$ as well as the critical collapse law for the PBH masses. In Figs. 4 and 5 we showed how the PBH mass function changes by varying $\alpha$ and $H_{RD}$. In particular, we found that in parameter regions where the energy density power spectrum moves to very small scales, where the smoothing performed is very efficient, the PBH mass function has the tendency to decrease.

Additionally, by making a full parameter-space analysis, in Fig. 6 we presented the PBH abundance at formation time $\Omega_{PBH,f}$ as a function of the bouncing parameter $\alpha$ and the energy scale $H_{RD}$. Interestingly enough, we found that in order to have a significant black hole production, namely $10^{-10} < \Omega_{PBH,f} < 1$, $H_{RD}$ and $\alpha$ should lie roughly within the ranges $10^{-7}M_{Pl} \leq H_{RD} \leq 10^{-6}M_{Pl}$, $10^{-9}M_{Pl} \leq H_{RD} \leq 2 \times 10^{-9}M_{Pl}$ and $10^{-30}M_{Pl}^2 \leq \alpha \leq 10^{-12}M_{Pl}^2$ respectively. Furthermore, within the parameter space that we have analysed, we showed that the excluded region corresponding to PBH overproduction, i.e. with $\Omega_{PBH,f} > 1$, forms a closed ring bounded by the allowed regions on both sides. We mention that the explored parameter space can be further constrained by evolving the PBH abundance $\Omega_{PBH}$ up to later times, and accounting for current observational constraints on $\Omega_{PBH}$. Moreover, one can extract more stringent constraints by studying additionally the scalar induced stochastic gravitational-wave background (SGWB) associated to the primordial curvature perturbations which gave rise to PBHs (see [63] for a review), as well as the SGWB induced from PBH Poisson fluctuations [64, 112, 113].

Since the PBH production during the bounce realization may serve as a novel tool to study alternative theories of gravity, one should perform a similar analysis in other modified gravity scenarios, and examine whether there are qualitative and quantitative differences amongst them. Such a detailed investigation will be performed elsewhere.
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References

[1] A. A. Starobinsky, A New Type of Isotropic Cosmological Models Without Singularity, Phys. Lett. B91 (1980) 99–102.
[2] A. H. Guth, The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems, Phys. Rev. D23 (1981) 347–356.
[3] A. D. Linde, A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems, Phys. Lett. B108 (1982) 389–393.
[4] A. Albrecht and P. J. Steinhardt, Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking, Phys. Rev. Lett. 48 (1982) 1220–1223.
[5] A. D. Linde, Chaotic Inflation, Phys. Lett. B129 (1983) 177–181.
[6] A. Borde and A. Vilenkin, Singularities in inflationary cosmology: A Review, Int. J. Mod. Phys. D 5 (1996) 813–824, [gr-qc/9612036].
[7] V. Mukhanov and R. Brandenberger, A nonsingular universe, Phys. Rev. Lett. 68 (Mar, 1992) 1969–1972.
[8] R. H. Brandenberger, V. F. Mukhanov and A. Sornborger, A Cosmological theory without singularities, Phys. Rev. D 48 (1993) 1629–1642, [gr-qc/9303001].
[9] M. Novello and S. E. P. Bergliaffa, Bouncing Cosmologies, Phys. Rept. 463 (2008) 127–213, [0802.1634].
[10] M. Lilley and P. Peter, Bouncing alternatives to inflation, Comptes Rendus Physique 16 (2015) 1038–1047.
[11] D. Battefeld and P. Peter, A Critical Review of Classical Bouncing Cosmologies, Phys. Rept. 571 (2015) 1–66, [1406.2790].
[12] P. Peter and N. Pinto-Neto, Cosmology without inflation, Phys. Rev. D 78 (Sep, 2008) 063506.
[13] CANTATA collaboration, E. N. Saridakis et al., Modified Gravity and Cosmology: An Update by the CANTATA Network, 2105.12582.
[14] S. Nojiri and S. D. Odintsov, Introduction to modified gravity and gravitational alternative for dark energy, eConf C0602061 (2006) 06, [hep-th/0601213].
[15] S. Capozziello and M. De Laurentis, Extended Theories of Gravity, Phys. Rept. 509 (2011) 167–321, [1108.6266].
[16] G. Veneziano, *Scale factor duality for classical and quantum strings*, Phys. Lett. B 265 (1991) 287–294.

[17] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, *The Ekpyrotic universe: Colliding branes and the origin of the hot big bang*, Phys. Rev. D 64 (2001) 123522, [hep-th/0103239].

[18] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, *From big crunch to big bang*, Phys. Rev. D 65 (2002) 086007, [hep-th/0108187].

[19] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, *The Ekpyrotic universe: Colliding branes and the origin of the hot big bang*, Phys. Rev. D 64 (2001) 123522, [hep-th/0103239].

[20] T. Biswas, A. Mazumdar and W. Siegel, *Bouncing universes in string-inspired gravity*, JCAP 03 (2006) 009, [hep-th/0508194].

[21] J. Khoury and B. A. Ovrut, *From big crunch to big bang*, Phys. Rev. D 65 (2002) 086007, [hep-th/0108187].

[22] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, *From big crunch to big bang*, Phys. Rev. D 65 (2002) 086007, [hep-th/0108187].

[23] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, *The Ekpyrotic universe: Colliding branes and the origin of the hot big bang*, Phys. Rev. D 64 (2001) 123522, [hep-th/0103239].

[24] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, *From big crunch to big bang*, Phys. Rev. D 65 (2002) 086007, [hep-th/0108187].

[25] T. Biswas, A. Mazumdar and W. Siegel, *Bouncing universes in string-inspired gravity*, JCAP 03 (2006) 009, [hep-th/0508194].

[26] J. Khoury and B. A. Ovrut, *From big crunch to big bang*, Phys. Rev. D 65 (2002) 086007, [hep-th/0108187].

[27] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, *From big crunch to big bang*, Phys. Rev. D 65 (2002) 086007, [hep-th/0108187].

[28] Y.-F. Cai, S.-H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, *Matter Bounce Cosmology with the f(T) Gravity*, Class. Quant. Grav. 28 (2011) 215011, [1104.4349].

[29] Y. Shtanov and V. Sahni, *Bouncing brane worlds*, Phys. Lett. B 557 (2003) 1–6, [gr-qc/0208047].

[30] E. N. Saridakis, *Cyclic Universes from General Collisionless Braneworld Models*, Nucl. Phys. B 808 (2009) 224–236, [0710.5269].

[31] E. N. Saridakis, *Horava-Lifshitz Dark Energy*, Eur. Phys. J. C 67 (2010) 229–235, [0906.3532].

[32] E. N. Saridakis, *Cyclic Universes from General Collisionless Braneworld Models*, Nucl. Phys. B 808 (2009) 224–236, [0710.5269].

[33] E. N. Saridakis, *Cyclic Universes from General Collisionless Braneworld Models*, Nucl. Phys. B 808 (2009) 224–236, [0710.5269].

[34] Y.-F. Cai, S.-H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, *Matter Bounce Cosmology with the f(T) Gravity*, Class. Quant. Grav. 28 (2011) 215011, [1104.4349].

[35] Y. Shtanov and V. Sahni, *Bouncing brane worlds*, Phys. Lett. B 557 (2003) 1–6, [gr-qc/0208047].

[36] E. N. Saridakis, *Cyclic Universes from General Collisionless Braneworld Models*, Nucl. Phys. B 808 (2009) 224–236, [0710.5269].

[37] E. N. Saridakis, *Cyclic Universes from General Collisionless Braneworld Models*, Nucl. Phys. B 808 (2009) 224–236, [0710.5269].

[38] E. N. Saridakis, *Cyclic Universes from General Collisionless Braneworld Models*, Nucl. Phys. B 808 (2009) 224–236, [0710.5269].

[39] E. N. Saridakis, *Cyclic Universes from General Collisionless Braneworld Models*, Nucl. Phys. B 808 (2009) 224–236, [0710.5269].

[40] E. N. Saridakis, *Cyclic Universes from General Collisionless Braneworld Models*, Nucl. Phys. B 808 (2009) 224–236, [0710.5269].

[41] E. N. Saridakis, *Cyclic Universes from General Collisionless Braneworld Models*, Nucl. Phys. B 808 (2009) 224–236, [0710.5269].
Y. B. Zel’dovich and I. D. Novikov, *The Hypothesis of Cores Retarded during Expansion and the Hot Cosmological Model*, Soviet Astronomy 10 (Feb., 1967) 602.

B. J. Carr and S. W. Hawking, *Black holes in the early Universe*, Mon. Not. Roy. Astron. Soc. 168 (1974) 399–415.

B. J. Carr, *The primordial black hole mass spectrum*, ApJ 201 (Oct., 1975) 1–19.

T. Harada, C.-M. Yoo and K. Kohri, *Threshold of primordial black hole formation*, Phys. Rev. D88 (2013) 084051, [1309.4201].

I. Musco, *Threshold for primordial black holes: Dependence on the shape of the cosmological perturbations*, Phys. Rev. D 100 (2019) 123524, [1809.02127].

A. Kehagias, I. Musco and A. Riotto, *Non-Gaussian Formation of Primordial Black Holes: Effects on the Threshold*, JCAP 12 (2019) 029, [1906.07135].

I. Musco, V. De Luca, G. Franciolini and A. Riotto, *Threshold for primordial black holes. II. A simple analytic prescription*, Phys. Rev. D 103 (2021) 063538, [2110.05982].

A. Addazi et al., *Quantum gravity phenomenology at the dawn of the multi-messenger era—A review*, Prog. Part. Nucl. Phys. 125 (2022) 103948, [2111.05659].

T. Papanikolaou, *Towards the primordial black hole formation threshold in a time-dependent equation-of-state background*, 2205.07748.

G. F. Chapline, *Cosmological effects of primordial black holes*, Nature 253 (1975) 251–252.

S. Clesse and J. Garc´ıa-Bellido, *Seven Hints for Primordial Black Hole Dark Matter*, Phys. Dark Univ. 22 (2018) 137–146, [1711.10458].

P. Meszaros, *Primeval black holes and galaxy formation*, Astron. Astrophys. 38 (1975) 5–13.

N. Afshordi, P. McDonald and D. Spergel, *Primordial black holes as dark matter: The Power spectrum and evaporation of early structures*, Astrophys. J. Lett. 594 (2003) L71–L74, [astro-ph/0302035].

B. J. Carr and M. J. Rees, *How large were the first pregalactic objects?*, Monthly Notices of Royal Astronomical Society 206 (Jan., 1984) 315–325.

R. Bean and J. Magueijo, *Could supermassive black holes be quintessential primordial black holes?*, Phys. Rev. D 66 (2002) 063505, [astro-ph/0204846].

T. Nakamura, M. Sasaki, T. Tanaka and K. S. Thorne, *Gravitational waves from coalescing black hole MACHO binaries*, Astrophys. J. 487 (1997) L139–L142, [astro-ph/9708060].

K. Ioka, T. Chiba, T. Tanaka and T. Nakamura, *Black hole binary formation in the expanding universe: Three body problem approximation*, Phys. Rev. D58 (1998) 063003, [astro-ph/9807018].

Y. N. Eroshenko, *Gravitational waves from primordial black holes collisions in binary systems*, J. Phys. Conf. Ser. 1051 (2018) 012010, [1604.04932].
[55] J. L. Zagorac, R. Easther and N. Padmanabhan, GUT-Scale Primordial Black Holes: Mergers and Gravitational Waves, *JCAP* **1906** (2019) 052, [1903.05053].

[56] M. Raidal, V. Vaskonen and H. Veermäe, Gravitational Waves from Primordial Black Hole Mergers, *JCAP* **1709** (2017) 037, [1707.01480].

[57] E. Bugaev and P. Klimai, Induced gravitational wave background and primordial black holes, *Phys. Rev. D* **81** (2010) 023517, [0908.0664].

[58] R. Saito and J. Yokoyama, Gravitational-wave background as a probe of the primordial black-hole abundance, *Physical Review Letters* **102** (Apr, 2009).

[59] T. Nakama and T. Suyama, Primordial black holes as a novel probe of primordial gravitational waves, *Physical Review D* **92** (Dec, 2015).

[60] C. Yuan, Z.-C. Chen and Q.-G. Huang, Probing primordial–black-hole dark matter with scalar induced gravitational waves, *Phys. Rev. D* **100** (2019) 081301, [1906.11549].

[61] Z. Zhou, J. Jiang, Y.-F. Cai, M. Sasaki and S. Pi, Primordial black holes and gravitational waves from resonant amplification during inflation, *Phys. Rev. D* **102** (2020) 103527, [2010.03537].

[62] J. Fumagalli, S. Renaux-Petel and L. T. Witkowski, Oscillations in the stochastic gravitational wave background from sharp features and particle production during inflation, *JCAP* **08** (2021) 030, [2012.02761].

[63] G. Domènech, Scalar Induced Gravitational Waves Review, *Universe* **7** (2021) 398, [2109.01398].

[64] T. Papanikolaou, V. Vennin and D. Langlois, Gravitational waves from a universe filled with primordial black holes, *JCAP* **03** (2021) 053, [2010.11573].

[65] G. Domènech, C. Lin and M. Sasaki, Gravitational wave constraints on the primordial black hole dominated early universe, *JCAP* **04** (2021) 062, [2012.08151].

[66] J. Kozaczuk, T. Lin and E. Villarama, Signals of primordial black holes at gravitational wave interferometers, 2108.12475.

[67] S. Clesse and J. García-Bellido, Seven hints for primordial black hole dark matter, *Physics of the Dark Universe* **22** (Dec., 2018) 137–146, [1711.10458].

[68] B. Carr, M. Raidal, T. Tenkanen, V. Vaskonen and H. Veermäe, Primordial black hole constraints for extended mass functions, 1705.05567.

[69] F. Kühnel and K. Freese, Constraints on Primordial Black Holes with Extended Mass Functions, *Phys. Rev. D* **95** (2017) 083508, [1701.07223].

[70] N. Bellomo, J. L. Bernal, A. Raccanelli and L. Verde, Primordial Black Holes as Dark Matter: Converting Constraints from Monochromatic to Extended Mass Distributions, *JCAP* **01** (2018) 004, [1709.07467].

[71] A. M. Green, Primordial Black Holes: sirens of the early Universe, *Fundam. Theor. Phys.* **178** (2015) 129–149, [1403.1198].

[72] M. Sasaki, T. Suyama, T. Tanaka and S. Yokoyama, Primordial black holes—perspectives in gravitational wave astronomy, *Class. Quant. Grav.* **35** (2018) 063001, [1801.06235].

[73] J. García-Bellido and E. Ruiz Morales, Primordial black holes from single field models of inflation, 1702.03901.
[74] H. Motohashi and W. Hu, *Primordial Black Holes and Slow-roll Violation*, 1706.06784.
[75] J. M. Ezquiaga, J. Garcia-Bellido and E. Ruiz Morales, *Primordial Black Hole production in Critical Higgs Inflation*, Phys. Lett. B 776 (2018) 345–349, [1705.04861].
[76] J. Martin, T. Papanikolaou and V. Vennin, *Primordial black holes from the preheating instability in single-field inflation*, JCAP 01 (2020) 024, [1907.04236].
[77] S. Clesse and J. García-Bellido, *Massive Primordial Black Holes from Hybrid Inflation as Dark Matter and the seeds of Galaxies*, Phys. Rev. D92 (2015) 023524, [1501.07565].
[78] J. Martin, T. Papanikolaou and V. Vennin, *Primordial black holes from the preheating instability in single-field inflation*, JCAP 01 (2020) 024, [1907.04236].
[79] S. Clesse and J. Garc´ıa-Bellido, *Massive Primordial Black Holes from Hybrid Inflation as Dark Matter and the seeds of Galaxies*, Phys. Rev. D92 (2015) 023524, [1501.07565].
[80] G. A. Palma, S. Sypsas and C. Zenteno, *Seeding primordial black holes in multifield inflation*, Phys. Rev. Lett. 125 (2020) 121301, [2004.06106].
[81] J. Fumagalli, S. Renaux-Petel, J. W. Ronayne and L. T. Witkowski, *Turning in the landscape: a new mechanism for generating Primordial Black Holes*, 2004.08369.
[82] B. J. Carr and A. A. Coley, *Persistence of black holes through a cosmological bounce*, Int. J. Mod. Phys. D 20 (2011) 2733–2738, [1104.3796].
[83] B. J. Carr, *Primordial Black Holes and Quantum Effects*, Springer Proc. Phys. 170 (2016) 23–31, [1402.1437].
[84] J. Quintin and R. H. Brandenberger, *Black hole formation in a contracting universe*, JCAP 11 (2016) 029, [1609.02556].
[85] J.-W. Chen, J. Liu, H.-L. Xu and Y.-F. Cai, *Tracing Primordial Black Holes in Nonsingular Bouncing Cosmology*, Phys. Lett. B 769 (2017) 561–568, [1609.02571].
[86] S. Nojiri and S. D. Odintsov, *Unifying inflation with LambdaCDM epoch in modified f(R) gravity consistent with Solar System tests*, Phys. Lett. B 657 (2007) 238–245, [0707.1941].
[87] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, *Constant-roll Inflation in F(R) Gravity*, Class. Quant. Grav. 34 (2017) 135005, [1701.05750].
[88] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, *Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models*, Phys. Rept. 505 (2011) 59–144, [1011.0544].
[89] J.-c. Hwang and H. Noh, *Classical evolution and quantum generation in generalized gravity theories including string corrections and tachyon: Unified analyses*, Phys. Rev. D 71 (2005) 063536, [gr-qc/0412126].
[94] H. Noh and J.-c. Hwang, Inflationary spectra in generalized gravity: Unified forms, Phys. Lett. B 515 (2001) 231–237, [astro-ph/0107069].

[95] J.-c. Hwang and H. Noh, Cosmological perturbations in a generalized gravity including tachyonic condensation, Phys. Rev. D 66 (2002) 084009, [hep-th/0206100].

[96] V. F. Mukhanov and G. Chibisov, Quantum Fluctuation and Nonsingular Universe., JETP Lett. 33 (1981) 532–535.

[97] H. Kodama and M. Sasaki, Cosmological Perturbation Theory, Prog. Theor. Phys. Suppl. 78 (1984) 1–166.

[98] J. M. Bardeen, J. R. Bond, N. Kaiser and A. S. Szalay, The Statistics of Peaks of Gaussian Random Fields, Astrophys. J. 304 (1986) 15–61.

[99] M. Shibata and M. Sasaki, Black hole formation in the friedmann universe: Formulation and computation in numerical relativity, Physical Review D 60 (Sep, 1999).

[100] D. S. Salopek and J. R. Bond, Nonlinear evolution of long wavelength metric fluctuations in inflationary models, Phys. Rev. D42 (1990) 3936–3962.

[101] D. Wands, K. A. Malik, D. H. Lyth and A. R. Liddle, A New approach to the evolution of cosmological perturbations on large scales, Phys.Rev. D62 (2000) 043527, [astro-ph/0003278].

[102] D. H. Lyth, K. A. Malik and M. Sasaki, A General proof of the conservation of the curvature perturbation, JCAP 0505 (2005) 004, [astro-ph/0411220].

[103] T. Harada, C.-M. Yoo, T. Nakama and Y. Koga, Cosmological long-wavelength solutions and primordial black hole formation, Phys. Rev. D 91 (2015) 084057, [1503.03934].

[104] C.-M. Yoo, T. Harada, J. Garriga and K. Kohri, Primordial black hole abundance from random Gaussian curvature perturbations and a local density threshold, PTEP 2018 (2018) 123E01, [1805.03946].

[105] S. Young, C. T. Byrnes and M. Sasaki, Calculating the mass fraction of primordial black holes, JCAP 1407 (2014) 045, [1405.7023].

[106] C. W. Misner and D. H. Sharp, Relativistic equations for adiabatic, spherically symmetric gravitational collapse, Phys. Rev. 136 (1964) B571–B576.

[107] S. A. Hayward, Gravitational energy in spherical symmetry, Phys. Rev. D 53 (1996) 1938–1949, [gr-qc/9408002].

[108] A. S. Josan, A. M. Green and K. A. Malik, Generalised constraints on the curvature perturbation from primordial black holes, Phys. Rev. D 79 (2009) 103520, [0903.3184].

[109] S. Young, I. Musco and C. T. Byrnes, Primordial black hole formation and abundance: contribution from the non-linear relation between the density and curvature perturbation, JCAP 11 (2019) 012, [1904.00984].

[110] J. C. Niemeyer and K. Jedamzik, Near-critical gravitational collapse and the initial mass function of primordial black holes, Phys. Rev. Lett. 80 (1998) 5481–5484, [astro-ph/9709072].

[111] J. C. Niemeyer and K. Jedamzik, Dynamics of primordial black hole formation, Phys. Rev. D 59 (1999) 124013, [astro-ph/9901292].
[112] T. Papanikolaou, C. Tzerefos, S. Basilakos and E. N. Saridakis, *Scalar induced gravitational waves from primordial black hole Poisson fluctuations in Starobinsky inflation*, 2112.15059.

[113] T. Papanikolaou, C. Tzerefos, S. Basilakos and E. N. Saridakis, *No constraints for f(T) gravity from gravitational waves induced from primordial black hole fluctuations*, 2205.06094.