Quantum dissipative chaos in the statistics of excitation numbers

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Abstract

A quantum manifestation of chaotic classical dynamics is found in the framework of oscillatory numbers statistics for the model of nonlinear dissipative oscillator. It is shown by numerical simulation of an ensemble of quantum trajectories that the probability distributions and variances of oscillatory number states are strongly transformed in the order-to-chaos transition. A nonclassical, sub-Poissonian statistics of oscillatory excitation numbers is established for chaotic dissipative dynamics in the framework of Fano factor and Wigner functions. It is proposed to use these results in experimental studies and tests of the quantum dissipative chaos.

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The quantum effects inherent in the chaotic behavior of classical systems is an interesting field pertaining to many problems of fundamental interest [1]. Though these problems have been posed long ago, they are still of great interest. The study of quantum dynamics of isolated or so-called Hamiltonian systems, the classical counterparts of which are chaotic, has a long history. In the majority of studies the attention has been focused on static properties such as spectral statistics of energy levels and the transition probabilities between eigenstates of the system. A variety of studies have been also carried out for understanding the features of time-dependent chaotic systems [2]. By contrast to that, few works dealt with investigations of the quantum chaos for open nonlinear systems. The early studies of open chaotic systems date back from the work by Ott et al. [3], and papers of Graham and Dittrich [4], where the authors have analyzed the kicked rotor and similar discrete-time systems. The investigations of quantum chaotic systems are distinctly connected with the quantum-classical correspondence problem in general and with environment induced decoherence and dissipation in particular. Recently this topic has been in the focus of theoretical investigations. As part of these studies it has been recognized by Zurek and Paz [5] and in later works [6] that the decoherence has rather unique properties for systems the classical analogues of which are chaotic ones. Among the criteria suggested for definition of chaos in open quantum systems one is to separate those based on entropy production and Wigner functions [4]-[6]. From experimental viewpoint, the observation of dissipative effects and environment induced decoherence of dynamically localized states in the quantum delta-kicked rotor is made using the gas of ultracold cesium atoms in a magneto-optical trap subjected to a pulsed standing wave [7,8]. Recently, new problems of chaotic motion were studied in an experimental scheme with ultra-cold atoms in magneto-optical double-well potential [9]. In spite of these important developments in the investigation of chaos for open quantum systems, there are still many open questions, and there is a clear need in new models that admit experimental verification, as well as comparatively more simple physical criterions for testing the dissipative quantum chaos.

The first purpose of this Letter is to investigate the order-to-chaos transition at the
level of statistics of elementary excitations for the quantum model of nonlinear oscillator. We show below that the distributions of oscillatory occupation numbers can be used to distinguish between the ordered and chaotic quantum dissipative dynamics. The need in realization of this study is to have a proper quantum model showing both the regular and chaotic dynamics in the classical limit. This aim in view we propose a nonlinear oscillator driven by two forces at different frequencies. This model was proposed to study the quantum stochastic resonance and quantum chaos in our previous papers [10], where it was shown in details that the model is suited to verification in experiments. Our second purpose is to identify the kind of statistics of oscillatory number states taking place in the quantum chaos. Our central result here is that the nonclassical, sub-Poissonian statistics can be realized for chaotic dynamics of the system under consideration.

The open quantum systems are usually studied in the framework of reduced density matrix obtained by tracing it over the degrees of freedom of a reservoir. The evolution of the system of interest is governed by the following master equation for the reduced density matrix in the interaction picture

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H, \rho] + \sum_{i=1,2} \left( L_i \rho L_i^+ - \frac{1}{2} L_i^+ L_i \rho - \frac{1}{2} \rho L_i^+ L_i \right),$$

(1)

where the Hamiltonian

$$H = \hbar \Delta a^+ a + \hbar \left[ (\Omega_1 + \Omega_2 \exp (-i\delta t)) a^+ + (\Omega_1^* + \Omega_2^* \exp (i\delta t)) a \right] + \hbar \chi (a^+ a)^2$$

(2)

describes an anharmonic oscillator with oscillatory frequency $\omega_0$ driven by two periodic forces at frequencies $\omega_1$ and $\omega_2$. The couplings with two driving forces are given by Rabi frequencies $\Omega_1$ and $\Omega_2$, and $\chi$ is the strength of anharmonicity. Here $\Delta = \omega_0 - \omega_1$ is the detuning, $\delta = \omega_2 - \omega_1$ is the difference between driving frequencies, that plays the role of modulation frequency in the interaction picture, and $a, a^+$ are boson annihilation and creation operators. The last terms in Eq. (2) pertain to the influence of the environment induced diffusion. $L_i$ are the Lindblad operators: $L_1 = \sqrt{(N + 1)\gamma} a, L_2 = \sqrt{N\gamma} a^+$, where $\gamma$ is the effective decay rate of the dissipation process and $N$ denotes the mean number
of quanta of the heat bath. We have followed the standard approach \[11\] to dissipative quantum dynamics in the range of weak coupling of system with the reservoir under the condition: \( \gamma \ll k_B T/\hbar \), where \( k_B T \) is the Boltzman’s constant times temperature. Equation (1) is obtained in both the rotating wave and Markov approximations, without regard for the driving-induced noise effects \[12\].

For \( \Omega_2 = 0 \) this equation describes the single driven, dissipative anharmonic oscillator, which is a well-known and archetypal model in nonlinear physics \[13\]. In case of double driven oscillator (\( \Omega_2 \neq 0 \)), the Hamiltonian (2) is explicitly time-dependent and the system exhibits regions of regular and chaotic motion. In the classical limit, the corresponding equation of motion for the dimensionless amplitude \( \alpha(t) = \langle a(t) \rangle \) has the form

\[
\frac{d}{dt} \alpha = -\frac{1}{2} \gamma \alpha - i \left( \Delta + \chi (1 + 2 |\alpha|^2) \right) \alpha - i (\Omega_1 + \Omega_2 \exp (-i\delta t)).
\] (3)

It should be noted that our model corresponds to a modified model of Duffing oscillator. Indeed, it is easy to demonstrate using the result of \[14\] that Eq.(3) is the rotating-wave approximation of the equation of Duffing oscillator driven by two periodic forces. At first, we study the order-to-chaos transition of the system in question using a constant phase map in the phase space. Our numerical analysis of Eq.(3) in \((X, Y)\) plane \((X = \text{Re} \alpha, Y = \text{Im} \alpha)\) shows that the classical dynamics of the system is regular in domains of small and large values of modulation frequency, i.e. \( \delta \ll \gamma \) and \( \delta \gg \gamma \), and also when one of the perturbation forces is much greater than the other: \( \Omega_1 \ll \Omega_2 \) or \( \Omega_2 \ll \Omega_1 \). The dynamics is chaotic in the range of parameters \( \delta \gtrsim \gamma \) and \( \Omega_1 \simeq \Omega_2 \), where the classical strange attractors for the Poincaré section are realized (see Fig.1).

The proposed model does not allow an analytical description. Our numerical analysis is based on quantum state diffusion approach that represents the reduced density operator by the mean over the projectors onto the stochastic states \( |\Psi_\xi\rangle \) of the ensemble: \( \rho(t) = M \langle |\Psi_\xi\rangle \langle \Psi_\xi| \rangle \), where \( M \) denotes the ensemble averaging. The corresponding equation of motion is
\[|d\Psi_\xi\rangle = -\frac{i}{\hbar} H |\Psi_\xi\rangle \, dt - \frac{1}{2} \sum_{i=1,2} \left( L_i^+ L_i - 2 \langle L_i^+ \rangle L_i + \langle L_i \rangle \langle L_i^+ \rangle \right) |\Psi_\xi\rangle \, dt + \sum_{i=1,2} (L_i - \langle L_i \rangle) |\Psi_\xi\rangle \, d\xi_i, \]

where \( \xi \) indicates the dependence on the stochastic process, the complex Wiener variables \( d\xi_i \) satisfy the fundamental properties \( M(d\xi_i) = 0, \ M(d\xi_i d\xi_j) = 0, \ M(d\xi_i d\xi_j^*) = \delta_{ij} dt, \) and the expectation value \( \langle L_i \rangle = \langle \Psi_\xi | L_i | \Psi_\xi \rangle. \)

Numerical studies of the oscillatory mean excitations number \( \langle n \rangle = M(\langle \Psi_\xi | a^+ a | \Psi_\xi \rangle) \) show that in both cases of regular and chaotic dynamics this quantity exhibits a periodic time-dependent behavior that is approximately sinusoidal with a period of \( 2\pi/\delta. \) We see that the quantum manifestation of chaotic dissipative dynamics is not evident on the mean oscillatory number. We will study the macroscopic quantum effects assisting to chaotic behavior by consideration of both the probability distribution of oscillatory excitation numbers \( P_n = \langle n | \rho | n \rangle, \) where \( |n\rangle \) is the number states, and the Fano factor which describes the excitation number uncertainty, normalized to the level of fluctuations for coherent states, i.e. \( F = \langle (\Delta n)^2 \rangle / \langle n \rangle, \langle (\Delta n)^2 \rangle = \langle (a^+ a)^2 \rangle - \langle a^+ a \rangle^2. \) This investigation will be complemented by testing the quantum chaos in phase space with the help of numerical calculations of the Wigner function.

To study the pure quantum effects we focus on the cases of very low reservoir’s temperatures which, however, ought to be still larger than the characteristic temperature \( T \gg T_{ch} = \hbar \gamma / k_B. \) This restriction implies that dissipative effects can be described self-consistently in the frame of the Lindblad equation (1). For clarity, in our numerical calculation we choose the mean number of reservoir photons \( N = (e^{\hbar \omega / k_B T} - 1)^{-1} \) equaling to \( N = 0.002. \) Note here that for \( N \ll 1 \) the above restriction is valid for the majority of problems of quantum optics and, particularly, for the scheme involving a trapped electron. In this scheme, \( N = 0.002 \) for the microwave spectral range corresponds to \( T \approx 0.16 \) K, while \( T_{ch} = 10^{-9} \) K, as \( \gamma \sim 10^2 \) s\(^{-1}. \) The detailed discussion of possible experimental realizations is postponed to the end of the Letter.

Let us first consider the quantities of interest for the region of classically regular behavior
with parameters: $\chi/\gamma = 0.1$, $\Delta/\gamma = -15$, $\Omega_1/\gamma = 27$, $\Omega_2/\gamma = 35$, and $\delta/\gamma = 5$. In Fig.2 the Wigner function is shown at the fixed moments $t_n = [7.13 + (2\pi/\delta)n]_{\gamma^{-1}}$, $(n = 0, 1, 2, \ldots)$ exceeding the transient time. We find that the Wigner function located around the point $X = 0$, $Y = -10$ and its contour-plot has a narrow crescent form with the origin of phase space as its centrum. The radial squeezing that reproduces the known property of the anharmonic oscillator model to generate the excitation number squeezing is also clearly seen in the figure. The important novelty here is that the radial squeezing effect in this model is much stronger, than an analogous one for the model of single driven anharmonic oscillator [13]. Below we will quantitatively demonstrate this by analyzing the Fano factor. Another peculiarity here is that the Wigner function is nonstationary. As the calculations show, during the modulation period $2\pi/\delta$ it is rotated around the origin of the phase space. In Fig.3 (curve 1) one can see the time evolution of the Fano factor, which shows the formation of nonclassical sub-Poissonian statistics ($\langle (\Delta n)^2 \rangle < \langle n \rangle$) for time intervals exceeding the transient time. The Fano factor reaches its minimum $F_{\min} \simeq 0.12$ and maximum $F_{\max} \simeq 0.65$ values in definite time intervals. One can account for surprisingly high sub-Poissonian statistics only by the quantum nature of oscillatory excitations under the influence of two driving forces. Indeed, in case of $\Omega_2 = 0$ we have $F \simeq 0.35$ for the same parameters as those in Fig.3 (curve 1).

We are now in a position to study the rise of quantum chaos, which is expected to manifest itself as crucial changes in above results at the passage into the classically chaotic operational regime, with parameter values: $\chi/\gamma = 0.1$, $\Delta/\gamma = -15$, $\Omega_1/\gamma = \Omega_2/\gamma = 27$, and $\delta/\gamma = 5$. In this range the oscillatory mean excitations number oscillates between $\langle n \rangle = 70 \div 130$. Now consider the behavior of the Fano factor. Its time evolution is shown in Fig.3 (curve 2). Surprisingly, the excitation-number fluctuations are also squeezed below the coherent level under the considered chaotic regime. However, opposite to the previous regular regime, the excitation number exhibits both the sub-Poissonian ($F < 1$) and super-Poissonian ($F > 1$) statistics, that are alternating in definite time intervals. The minimum and maximum values of $F$ in time intervals during one modulation period are equal to
$F_{\text{min}} \simeq 0.30$ and $F_{\text{max}} \simeq 1.98$. Thus, Fig.3 shows the drastic difference between the behavior of Fano factor for regular and chaotic dynamics.

It is tempting to explain the emergence of nonclassical sub-Poissonian statistics in the double driven nonlinear oscillator at the transition from regular to chaotic dynamics using the phase space symmetry properties of the Wigner function. The results of ensemble-averaged numerical calculations of the contour-plot of Wigner function at fixed time intervals $t_n = [6.96 + (2\pi/\delta)n]^{-1}$, $(n = 0, 1, 2, ...)$ is shown in Fig.4. It is seen that the contour-plot for chaotic motion still has the radial squeezed form (see Fig.4). This result takes place for $t_n = [6.96 + (2\pi/\delta)n]^{-1}$, $(n = 0, 1, 2, ...)$, at which the Fano factor reaches its minimum value $F_{\text{min}} \simeq 0.30$. In the next time intervals during the period of modulation, the level of excitation number fluctuations increases, and as a result the radial squeezing in contour-plot decreases. It is easy to see that the contour-plot is generally similar to the correspondent classical Poincaré section (Fig.1).

In the search for a criterion of quantum chaos that is more promising and easily attainable in experiments, we consider the probability distribution of oscillatory excitation numbers $P_n = \langle n | \rho | n \rangle$. We give in Fig.5 the results for both regular (a) and chaotic (b) regimes at two time moments corresponding to $F_{\text{min}}$ (curve 1) and $F_{\text{max}}$ (curve 2). One can conclude from the comparison of these figures that the probability distributions $P_n$ are strongly transformed at the order-to-chaos transition. While $P_n$ for regular dynamics is clearly bell-shaped and localized in narrow intervals of oscillatory numbers, the distribution for chaotic dynamics is flat-topped with oscillatory numbers from $n = 0$ to $n_{\text{max}} \simeq 200$. Moreover, the shape of distributions changes irregularly in time during the period $2\pi/\delta$. Especially typical for chaotic motion is the result shown in Fig. 5(b) (curve 2), where the probability distribution is almost equally probable.

We also studied the reliability of our results depending on noise intensity. We found that the results had not changed visibly in the range of $0 \leq N \lesssim 0.05$. For example, $F_{N=0.05} - F_{N=0} \sim 0.1$. What concerns the Wigner function, it is only slightly smeared in the phase space under the influence of such an amount of noise.
We emphasize, that the number of possible experimental schemes demonstrating the proposed model is rather large. One of those is that of a single relativistic electron in Penning trap, which is a realization of anharmonic oscillator as was predicted theoretically by Kaplan [15] and experimentally realized by Gabrielse and co-authors [16]. In the presence of two microwave electromagnetic fields this system gives an example of double driven anharmonic oscillator and may be used for demonstration of quantum dissipative chaos. This system is governed by Eq. (1), where the operators $a$ and $a^+$ describe the cyclotron quantized motion, Rabi frequencies $\Omega_1$ and $\Omega_2$ characterize the amplitudes of the microwave driving fields, $\chi$ is the strength of the anharmonicity due to relativistic effects, and $\gamma$ is the spontaneous decay rate of the cyclotron motion.

In conclusion, we should like to summarize that the quantum-statistical effects that accompany the chaotic dynamics have been identified. These results were obtained for the model of dissipative anharmonic oscillator, which has been proposed for studies of quantum chaos. It was demonstrated that the oscillatory excitation numbers statistics could be used for the diagnostic of quantum chaos. Indeed, we have shown that such measurable quantities as the Fano factor and probability distributions of number states are drastically changed at the order-to-chaos transition. But perhaps even more intriguing are the results that nonclassical, sub-Poissonian statistics of oscillatory number states is realized for the chaotic dissipative dynamics. The results of our numerical work were obtained under conditions of strong anharmonicity $\chi/\gamma \lesssim 1$, for the value $\chi/\gamma = 0.1$, which is close to those actually achieved in the experiments with trapped relativistic electron. We believe that the results obtained are applicable to more general quantum systems, the classical analogues of which exhibit chaotic dynamics.
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FIGURE CAPTIONS

Fig. 1. The Poincaré section (approximately 20000 points) of a solution to (3) plotted at times of the constant phase $t_n = 6.96 + (2\pi/\delta)n$, $n = 0, 1, 2, ...$ The dimensionless parameters are in the range of chaos: $\chi/\gamma = 0.1$, $\Delta/\gamma = -15$, $\Omega_1/\gamma = \Omega_2/\gamma = 27$, $\delta/\gamma = 5$.

Fig. 2 The Wigner function for the regular regime averaged over 3000 trajectories.

Fig. 3. The Fano factor versus dimensionless time for the regular (curve 1) and chaotic (curve 2) regimes. The parameters are: $\chi/\gamma = 0.1$, $\Delta/\gamma = -15$, $\Omega_1/\gamma = 27$, $\delta/\gamma = 5$, and $\Omega_2/\gamma = 35$ (curve 1), $\Omega_2/\gamma = 27$ (curve 2)

Fig. 4. The contour-plot of the Wigner function in the chaotic regime.

Fig. 5. Probability $P_n$ of finding the system in the state $|n\rangle$ at different time intervals (curves 1 and 2) and for regular (a) and chaotic (b) regimes. The parameters for (a) and (b) coincides with ones for Fig. 2 curve1 and curve 2 respectively.
