On the robustness of hybrid control systems to measurement noise and actuator disturbances

Alfonso Baños*, Miguel A. Davó®, Cristian D. Cánovas*

*Universidad de Murcia, Dept. Informática y Sistemas, 30100 Murcia, Spain
®Université Grenoble Alpes, CNRS, GIPSA-Lab, F-38000 Grenoble, France

Abstract

Robustness of hybrid control systems to measurement noise, actuator disturbances, and more generally perturbations, is analyzed. The relationship between the robustness of a hybrid control system and of its implementations is emphasized. Firstly, a formal definition of implementation of a hybrid control system is provided, based on the uniqueness of the solutions. Then, two examples are analyzed in detail, showing how the previously developed robustness property fails to guarantee that the implementations, necessarily used in control practice, are also robust. A new concept of strong robustness is proposed, which guarantees that at least jumping-first and flowing-first implementations are robust when the hybrid control system is strongly robust. In addition, we provide a sufficient condition for strong robustness based on the previously developed hybrid relaxation results.

Keywords: Hybrid dynamical systems, Hybrid control systems, Reset control systems, Robustness to measurement noise, Robustness to perturbations.

1. INTRODUCTION

This work is focused on the hybrid system framework developed in [1] (see also App. B) for a detailed exposition of the HI framework. The reader is referred to [1] for a detailed exposition of the HI framework. For the sake of completeness, a minimal background is given in the appendices. In particular, the work is centered in hybrid control systems, measurement noise is embedded in a type of perturbation referred to as outer perturbation ([3]). It is known that the recent work [8] gives a related definition of robustness to measurement noise in an implicit way (Theorem 5.5), for a specific type of hybrid systems. In the HI framework, robustness to perturbations is usually approached by imposing the hybrid basic conditions on H, and the convergence property on its perturbation H_δ ([4],[5],[3]). This results in a regularization of H that usually leads to a non-deterministic system, in the sense that there may exist several solutions to H from some initial points.

However, when H is a feedback control system, and specially when the hybrid dynamics is determined by the feedback controller, its practical implementation entails a decision mechanism such that a unique solution is selected within all the possible solutions. A key question in control practice is whether the different implementations of a robust hybrid control system keep the robustness property or not. This is a non-trivial question that apparently has not been previously approached. The

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Notation: R_≥^n is the set of non-negative real numbers, R^n is the n-dimensional Euclidean space, and x = (x_1, · · · , x_n) ∈ R^n is a column vector; ||x|| is the euclidean norm. B is the closed unit ball in R^n centered at the origin. For a set K ⊂ R^n, con(K) denotes the convex hull of K, K is its interior, and int K is the interior of K. S_K(ξ) is the set of maximal solutions φ to the hybrid system H with φ(0,0) = ξ, dom stands for domain, and \ denotes sets difference.

For H, the so-called hybrid basic conditions defined in [1] (see also App. B) are trivially satisfied if C and D are both closed subsets of R^n, and for example, f and g are continuous functions. On the other hand, a perturbation of H is a family of hybrid systems H_δ with data (C_δ, f_δ, D_δ, g_δ, and perturbation parameter δ > 0.

Robustness of H to perturbations is developed in [4], being this related with the closeness of solutions to H and solutions to H_δ for small enough values of the parameter δ. Although an explicit definition of the robustness property has not been given, the property is implicitly defined in [4] (see Corollary 5.5), and in ([1], Prop. 6.34) named as “dependence on initial conditions and perturbation”. More specifically, for hybrid control systems, measurement noise is embedded in a type of perturbation referred to as outer perturbation ([3]). It is known that the recent work [8] gives a related definition of robustness to measurement noise in an implicit way (Theorem 5.5), for a specific type of hybrid systems.

In the HI framework, robustness to perturbations is usually approached by imposing the hybrid basic conditions on H, and the convergence property on its perturbation H_δ ([4],[5],[3]). This results in a regularization of H that usually leads to a non-deterministic system, in the sense that there may exist several solutions to H from some initial points.

However, when H is a feedback control system, and specially when the hybrid dynamics is determined by the feedback controller, its practical implementation entails a decision mechanism such that a unique solution is selected within all the possible solutions. A key question in control practice is whether the different implementations of a robust hybrid control system keep the robustness property or not. This is a non-trivial question that apparently has not been previously approached. The
main goal of this work is to analyze this question, and specifically to investigate the relation between the robustness of a hybrid control system and of its implementations.

In Section 2, the hybrid control system setup, including measurement noise and actuator disturbances as exogenous signals, is given. The model is embedded in a more general hybrid system with perturbations. Then, robustness to perturbations is formally defined by using a definition similar to the implicit concept given in (11), and a basic HI result is recalled. In Section 3, we define the concept of implementation of a hybrid control system; in addition, we use two examples to show that robustness of a hybrid control system does not guarantee the existence of robust implementations. Finally, Section 4 introduces the new concept of robustness to perturbations, that leads to robustness of implementations; moreover, some hybrid relaxation results are used to derive sufficient conditions for strong robustness. Some concluding remarks are also elaborated.

2. Preliminaries and background

In this work, the main focus is on hybrid control systems that can be modeled by (11). This is, for example, the case in which a continuous-time plant is controlled by a hybrid controller (see Fig. 1), a type of control which appears in a broad class of industrial applications (3). The plant is described by the differential equation:

\[ x_p = f_p(x_p, u) \] (2)

where \( x \in \mathbb{R}^n_p, u \in \mathbb{R}^n_u, \) and \( f_p \) is continuous. The hybrid controller, with state \( x \in \mathbb{R}^n \), is defined by a flow set \( C \subseteq \mathbb{R}^n_{x_0} \), a flow map \( f_c : \mathbb{R}^n_{x_0} \rightarrow \mathbb{R}^n \), a jump set \( D \subseteq \mathbb{R}^n_{x_0} \), a jump map \( g_c : \mathbb{R}^n_{x_0} \rightarrow \mathbb{R}^n \), and a feedback control law \( k_c : \mathbb{R}^n_{x_0} \rightarrow \mathbb{R}^n \) that specifies the control signal \( u \). Defining the state of the closed-loop system as \( x = (x_p, x) \) and \( n = n_p + n_c \), it results that the closed-loop system is a hybrid system \( \mathcal{H}^3 = (C, f, D, g) \) as given by (11) with flow map \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined as

\[ f(x_p, x) = \left( \begin{array}{c} f_p(x_p, k_c(x_p, x)) \\ f_c(x_p, x) \end{array} \right) \] (3)

and jump map \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined as

\[ g(x_p, x) = \left( \begin{array}{c} x_p \\ g_c(x_p, x) \end{array} \right) \] (4)

The main goal is to analyze the robustness properties of this hybrid control system with respect to measurement noise, and more generally with respect to external disturbances. Considering the measurement noise \( d_1 \in \mathbb{R}^{n_u} \) and the actuator disturbance \( d_2 \in \mathbb{R}^{n_o} \) as perturbations (Fig. 1), the perturbed hybrid control system \( \mathcal{H}^{cl}_{(d_1, d_2)} \) is given by:

\[
\begin{align*}
\mathcal{H}^{cl}_{(d_1, d_2)} & : \begin{cases} 
\dot{x} = \left( f_p(x_p, k_c(x_p, x_p) + d_1, x_p) + d_2 \right), \quad (x_p + d_1, x_p) \in C, \\
x^* = g_c(x_p + d_1, x_p)
\end{cases}, \\
(x_p + d_1, x_p) & \in \mathcal{D}.
\end{align*}
\] (5)

In order to embed this perturbed hybrid control system in a more general perturbation, let us consider perturbations \( e_1, e_2, e_3 \in \mathbb{R}^n \) and the following perturbed hybrid system:

\[
\mathcal{H}_{(e_1, e_2, e_3)} : \begin{cases} 
\dot{x} = f(x + e_1) + e_2, \\
x^* = g(x + e_1) + e_3, \\
x + e_1 & \in \mathcal{D},
\end{cases}
\] (6)

Then, the perturbed hybrid control system \( \mathcal{H}^{cl}_{(d_1, d_2)} \) is embedded in the general system (6) by simply considering the following perturbations:

\[
e_1 = (d_1, 0), \quad e_2 = (f_p(x_p, k_c(x_p + d_1, x_p) + d_2), 0), \quad e_3 = -(d_1, 0),
\] (7)

where it directly follows that \( \|e_1\| \) and \( \|e_2\| \) are arbitrarily small, and also \( \|e_3\| \) by continuity of \( f_p \), when \( \|d_1\| \) and \( \|d_2\| \) are small enough.

The perturbed hybrid system (6) is considered in this work as a general perturbation mode-\( \mathcal{H}^{cl}_{(d_1, d_2)} \), which corresponds to the hybrid system like (6) with \( e_1, e_2, e_3 = 0 \), when \( e_3 \in \mathbb{R}^n \), and \( \mathcal{H}_0 \) is a set of hybrid systems. In some cases, perturbation signals only depends on time \( t \) and on \( j \); then, the admissible perturbation signal \( n \) can be considered as given by setting \( n(t, j) = n_0(t) \) for all \( t, j \in E \), and for some function \( n_0 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \), and any arbitrary hybrid time domain \( E \).

\[\text{Note that since } f_p \text{ is continuous, then for any } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such as if } \|x_p, k_c(x_p + d_1, x_p) + d_2\| = \|x_p + d_1, x_p\| < \delta \text{ then } \|e_1\| < \epsilon. \text{ This is obtained for example for } \|d_1\| < \min(\delta/2, \epsilon/2) \text{ and } \|d_2\| < \min(\delta/2, \epsilon/2), \text{ which on the other hand directly makes } \|e_1\| < \epsilon \text{ and } \|e_3\| < \epsilon.} \]

A model similar to \( \mathcal{H}_{(e_1, e_2, e_3)} \) has been used to represent different hybrid feedback control systems with external perturbations due to measurement noise, actuator error, and other external disturbances (see [3]) for the continuous-time systems case. Obviously, the model may also represent perturbed hybrid systems that are not necessarily feedback control systems.
In the following, we define robustness in the spirit of the HI framework with the goal of developing a precise analysis in the next section. For the sake of simplicity, $n = (n_1, n_2, n_3)$, and thus $H_{in} = H_{in}(n_1, n_2, n_3)$, is used in the following. In addition, robustness to perturbations is used to denote robustness of the hybrid system \( H \) to vanishing perturbations signals.

**Definition 2.1 (Robustness to perturbations)** For a compact set $K \subset \mathbb{R}^n$ such that $H$ is forward complete from $K$, the hybrid system $H$ is robust to perturbations if for any $\epsilon > 0$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta > 0$ with the following property: for any admissible perturbation signal $n$, any $\delta \in (0, \delta']$, and any $x_{in} \in S_{H_{in}}(K + \delta B)$ there exists a solution $x$ to $H$, with $x(0, 0) \in K$ such that $x_{in}$ and $x$ are $(T, J, \epsilon)$-close.

Note that the definition of robustness includes a notion of uniformity with respect to continuous dependence on a set of initial conditions $K$. In particular, when the hybrid system is a hybrid control system $H$, robustness to perturbations means, in particular, robustness to the perturbation signals given by (7), then robustness to perturbations implies robustness of the hybrid control system to measurement noise and actuator disturbances.

A significant result, that directly follows from Th. 5.4 and Corollary 5.5 in [4], embedding the perturbation \( p \) in a more general outer perturbation, is stated in the following corollary.

**Corollary 2.2** A hybrid system $H$ is robust to perturbations for some compact set $K \subset \mathbb{R}^n$, where $H$ is forward complete from $K$, if it satisfies the basic hybrid conditions.

3. Robustness of hybrid control system implementations

Robustness to perturbations is a sound contribution of the HI framework, since it applies to hybrid systems with some simple properties (the hybrid basic conditions), and thus it may be applied to a generality of cases. In fact, the hybrid basic conditions are used as a mean to regularize hybrid systems and to equip them with the robustness to perturbation property (and some other useful properties like robust stability, see e.g. [11]). The result is that, in general, hybrid systems are non-deterministic in the sense that several solutions may exist for a given initial point.

In control practice, the implementation of a hybrid control system $H$ (see Fig. 1), satisfying the hybrid basic conditions, entails a decision mechanism for the hybrid controller such that a unique solution is selected within all the theoretical possible solutions. Such mechanism basically consists in choosing to jump or to flow at each instant in which both jumping and flowing are possible.

For the HI framework to be useful in control practice, we may expect that there exist implementations of the hybrid control system that inherits the robustness property; otherwise, any possible implementation will be sensitive to arbitrarily small measurement noise signals and/or actuator disturbances. Next, the notion of hybrid control system implementation is formalized; in addition, two examples are developed, showing that, in fact, the robustness property of hybrid control systems is not necessarily inherited by its implementations.

3.1. Implementation of a hybrid control system

In the rest of this work, and with some abuse of notation, $H$ will be indistinguishably used to denote a hybrid control system $H^0$ or more generally a hybrid system like \( H \). An implementation of $H$ will be defined as a hybrid system $H$ that has unique solutions, and those solutions are also solutions to $H$. While $H^0$ is an implementation of $H$, we refer to the hybrid system $H$ as an abstraction of $H^0$.

**Definition 3.1 (Implementation of a hybrid control system)** Consider a hybrid control system $H = (C, f, D, g)$ that satisfies the hybrid basic conditions. A hybrid system $H^i = (C_i, f_i, D_i, g_i)$ is an implementation of $H$ if

1. $C_i \cup D_i = C \cup D$;
2. for every $\xi \in C_i \cup D_i$, each solution $\phi \in S_{H^i}(\xi)$ is unique, and in addition, $\phi \in S_{H}(\xi)$.

Although this work is mainly focused on hybrid control system, the above definition is also valid for hybrid systems $H$ like \( H \) that are not necessarily hybrid control systems. The main motivations of the above definition are: firstly, to obtain implementations $H^i$ which share the same state-space that its abstractions $H$ (in this case $\mathbb{R}^n$), and the same set of initial conditions $C \cup D$; and secondly, that implementations be deterministic hybrid systems in the sense that they have unique solutions for any initial point.

On the other hand, it directly follows from Def. 3.1 that

$$ \text{int } C_i \cap \text{int } D_i = \emptyset, $$

otherwise $H$ would have several solutions $\phi$ with $\phi(0, 0) \in \text{int } C_i \cap \text{int } D_i$. For those systems $H$ such that $\text{int } C \cap \text{int } D = \emptyset$, condition 1 of Def. 3.1 would be directly guaranteed by considering implementations such that $C = C_i$ and $D = D_i$. In addition, the abstraction $H$ would be a Krasovskii regularization of $H^i$ if $f_i = f$ and $g_i = g$.

The implementations of a hybrid control system differ in the sequence of elections of jumping and flowing. Therefore, we can think about two particular implementations obtained by simply always choosing to jump (jumping-first solution) or always choosing to flow (flowing-first solution). Following this idea, for $H = (C, f, D, g)$, let us define the following two hybrid systems:

$$ H^0 = (C \setminus D, f, D, g) $$

$$ H^C = (C, f, D \setminus C^*, g), $$

where

$$ C^* = \{ x \in C : T_C(x) \cap f(x) \neq \emptyset \}, $$

being $T_C(x)$ the tangent cone \( \mathbb{R}^n \) to $C$ at the point $x$.

Finally, note that $f$ and $g$ are not set-valued functions, and thus, considering the hybrid basic conditions and the basic uniqueness conditions (see App. A and B), it directly follows the following result on the existence of implementations.

**Corollary 3.2** Consider a hybrid control system $H$, satisfying the basic hybrid conditions, and that for every $\xi \in C$ there

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3. The tangent cone to the set $\Sigma \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$, $T(x)$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exist $\xi, r, r_i > 0$, for all $i = 1, 2, \ldots$ such that $x_\tau \rightarrow x$, $r_i \rightarrow 0$, and $(x_r - x_i)/r_i \rightarrow w$ as $i \rightarrow \infty$. Informally speaking, $C^*$ is basically the set of points of $C$ from which flowing to $C$ is possible.
exists \( \varepsilon > 0 \) and a unique maximal solution \( z : [0, \varepsilon) \to \mathbb{R}^n \) to \( \dot{z}(t) = f(z(t)) \) satisfying \( z(0) = \xi \) and \( z(t) \in \mathcal{C} \) for all \( t \in [0, \varepsilon) \). The hybrid systems \( \mathcal{H}^C \) and \( \mathcal{H}^D \) are implementations of \( \mathcal{H} \).

Thereafter, we refer to \( \mathcal{H}^C \) and \( \mathcal{H}^D \) as flowing-first implementation and jumping-first implementation, respectively. The definition of \( \mathcal{H}^C \) is a bit more involved, since a simple definition of the jump set, as \( \mathcal{J} \) is not necessary. In order to build the jump set of the implementation it is necessary to add to \( \mathcal{D}_t \) all the points for which \( \dot{z} \) is not possible.

Note that when \( \mathcal{H} \) has unique solutions then there is no possibility of jumping/flowing choice. In this case \( \mathcal{H}^C = \mathcal{H}^D = \mathcal{H} \), in the sense that the three hybrid systems produce the same unique solution for each initial point.

3.2. A simple example

Although this example is not a hybrid control system, it is developed here to make the clear relation between hybrid systems and its implementations, that will make full sense in the hybrid control system example of Section 3.3. Consider the hybrid system \( \mathcal{H} \) on \( \mathbb{R}^2 \) given by

\[
\mathcal{H} : \begin{cases}
    \dot{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & x \in \mathcal{C}, \\
    \dot{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & x \in \mathcal{D},
\end{cases}
\]

where the jump set is the convex polytope \( \mathcal{D} = \{(x_1, x_2) : x_1 - x_2 \leq -1, -x_1 - x_2 \leq -1\} \), and the flow set is \( \mathcal{C} = \mathbb{R}^2 \setminus \overline{\mathcal{D}} \) (see Fig. 2). It is easy to see that all the maximal solutions to \( \mathcal{H} \) are complete, and thus \( \mathcal{H} \) is forward complete from any compact set \( \mathcal{K} \subset \mathbb{R}^2 \). In addition, since \( \mathcal{H} \) satisfies the basic hybrid conditions (note that the jump arc maps are constant, and the flow and jump sets are closed). Thus, by Corollary 2.2.2, \( \mathcal{H} \) is robust to perturbations for any compact set \( \mathcal{K} \subset \mathbb{R}^2 \). Next, we analyze the robustness of its implementations for the set \( \mathcal{K} = \{\xi\} = \{(-1, 1)\} \).

Perturbation-free solutions. It directly follows that there are only two solutions for the initial point \( \xi \), which are \( \phi^C : [0, \varepsilon) \times [0] \to \mathbb{R}^2 \) with \( \phi^C(t, 0) = (-1 + t, 1) \), \( t \in [0, \varepsilon) \) and \( \phi^D : [0, 1] \times [0] \to \mathbb{R}^2 \) with \( \phi^D(t, 0) = (-1 + t, 1) \) for all \( t \in [0, 1] \) and \( \phi^D(t, 1) = (-1 + t, 0) \) for all \( t \in [1, \infty) \). The solutions \( \phi^C \) and \( \phi^D \) are plotted in Fig. 2(a) and 2(b) respectively.

Perturbed solutions. The perturbed hybrid system is \( \mathcal{H}(\xi_n, \mathcal{K}, \nu) \), where by simplicity \( \xi_n = \xi = 0 \), that is only state perturbations are considered. Consider the admissible perturbation signal \( \xi_0 : [0, 1] \times [0] \to \mathbb{R}^2 \), given by \( \xi_0(t, 0) = (0, 1) \) and \( \xi_0(t, 1) = (0, 0) \) otherwise. For any \( \delta > 0 \), the hybrid arc \( \phi_{\xi_0, n} : [0, 1] \times [0] \to \mathbb{R}^2 \) is the unique solution to \( \mathcal{H}(\xi_0, n, \nu) \) with \( \phi_{\xi_0, n}(t, 0) = (\xi_0, \xi_0(0, 0) = (\delta - 1, 1) \) for \( t \in [0, 1] \) and \( \phi_{\xi_0, n}(t, 1) = (\delta - 1, 1) \) for \( t \in [1, \infty) \). Now consider the admissible perturbation signal \( \xi_1 : [0, \infty) \times [0] \to \mathbb{R}^2 \), with \( \xi_1(t, 0) = (0, 1) \) and \( \xi_1(t, 1) = (0, 0) \) if \( t \neq 1 \). For any \( \delta > 0 \), the hybrid arc \( \phi_{\xi_1, \nu} : [0, \infty) \times [0] \to \mathbb{R}^2 \) is the unique solution to \( \mathcal{H}(\xi_1, \nu, \nu) \), with \( \phi_{\xi_1, \nu}(0, 0) = (\xi, 0) \), is \( \phi_{\xi_1, \nu}(t, 0) = (-1 + t, 1) \) for \( t \in [0, \infty) \). The solutions \( \phi_{\xi_0, n} \) and \( \phi_{\xi_1, \nu} \) are plotted in Fig. 2(a) and 2(b) respectively.

Robustness analysis. For the chosen \( \mathcal{K} = \{\xi\} \), that \( \mathcal{H} \) is robust to perturbation means that for any perturbed solution there exists a close perturbation-free solution. For example, considering the solution \( \phi_{\xi_1, \nu} \), it directly follows that \( \phi_{\xi_1, \nu} \) is \( (T, J, \epsilon) \)-close for any \( T, J \) and \( \epsilon \). The same applies to \( \phi_{\xi_0, n} \) and \( \phi_{\xi_1, \nu} \). This is the exact meaning of robustness to perturbations in the HI framework. Now let us analyze the implementations. First, note that for any implementation \( \mathcal{H}^1 \), the hybrid arcs \( \phi_{\xi_0, n} \) and \( \phi_{\xi_1, \nu} \) are solutions to \( \mathcal{H}^1(\xi_0, n, \nu) \) and \( \mathcal{H}^1(\xi, \nu, \nu) \), respectively. In addition, for any implementation, one of the hybrid arcs \( \phi^C \) or \( \phi^D \) is solution to \( \mathcal{H}^1 \). Suppose that \( \phi^D \) is solution to an implementation \( \mathcal{H}^1 \), then for the implementation to be robust to perturbations, both solutions \( \phi_{\xi_0, n} \) and \( \phi_{\xi_1, \nu} \) should be \( (T, J, \epsilon) \)-close to \( \phi^D \) for a small enough \( \delta \), since \( \phi^D \) is the unique solution. However, it is clear that \( \phi^D \) and \( \phi_{\xi_0, n} \) are not \( (T, J, \epsilon) \)-close for \( J = 1 \) independently of \( \delta \) (see Fig. 2(b)). Similarly for \( \phi^C \) and \( \phi_{\xi_1, \nu} \) (see Fig. 2(a)). As a result, there are not implementations of \( \mathcal{H} \) that are robust to perturbations for the set \( \mathcal{K} \).

3.3. A hybrid control system example

The FORE (first order reset element) controller was introduced in [9], and since then it has been used in a number of works (see for example [10] and references therein). Different versions of FORE has been devised in the literature, some of them in the HI framework; here, the FORE proposed in [11] is used:

\[
\begin{cases}
    t = 1, & x_v = -\lambda x_v + v, \quad \alpha^2 + 2x_v \geq 0 \quad \alpha \leq \rho, \\
    t^r = 0, & x_v^r = 0, \quad \varepsilon^2 + 2x_v \leq 0 \quad \varepsilon \geq \rho,
\end{cases}
\]

where \((x_v, \tau) \in \mathbb{R} \times \mathbb{R}_{\geq 0}\) is the state, \(x_v\) is the output, and \(v \in \mathbb{R}\) is the input. In addition, \(\lambda \in \mathbb{R}\) defines the pole of the base system, and \(\rho\) and \(\varepsilon\) are some positive constants. See [11]-Section III for details and motivation.

Now, consider a hybrid control system \( \mathcal{H}^1 \) (Fig. 1) consisting of the feedback interconnection of a plant \( P \) with transfer function \( F(s) = \frac{1}{s^2 + \alpha s + 1} \) and a FORE. This feedback control system has been analyzed in a number of works, including several works in the HI framework [12][13].

If the input and state of \( P \) are \( a \) and \( (x_1, x_2) \), respectively, then the feedback interconnection is given by \( u = x_v \) and \( v = \)
−x2. The closed-loop hybrid system \( \mathcal{H}^C \), with state \((x, \tau) = (x_1, x_2, x_3, \tau) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \), is given by
\[
\begin{align*}
\dot{\tau} &= 1, \quad x = Ax = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -0.2 & 1 \\ 0 & -1 & -1 \end{pmatrix} x, \quad (x, \tau) \in \mathcal{C}, \\
\tau^+ &= 0, \quad x^+ = A_b x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x, \quad (x, \tau) \in \mathcal{D},
\end{align*}
\]
where \( I = 1 \) has been chosen. Here, the flow and jump sets are given by \( \mathcal{C} = \{(x, \tau) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0} : \epsilon x_2^2 - 2x_2x_3 \geq 0 \text{ or } \tau \leq \rho \} \), and \( \mathcal{D} = \{(x, \tau) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0} : \epsilon x_2^2 - 2x_2x_3 \leq 0 \text{ and } \tau \geq \rho \} \), respectively. Note that if \((x, \tau) \in \mathcal{D} \) then \((x^+, \tau^+) = (x_1, x_2, 0, 0) \in \mathcal{C} \setminus \mathcal{D} \), and thus only flowing is possible after a jump.

Although \( \mathcal{H} \) is defined on \( \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \), \( \tau \) is a controller state component that acts simply as a timer to avoid that two consecutive jumps are performed in lesser time than the minimum dwell time \( \rho \). Note that \( \mathcal{H} \) satisfies the hybrid basic conditions and is forward complete from any compact set \( K \subset \mathbb{R}^3 \times \{0\} \); thus, Corollary 2.2 guarantees that \( \mathcal{H} \) is robust to perturbations for any compact set \( K \). In this example, we only consider the measurement noise \( d_1 \) as the unique perturbation affecting the state \( x_3 \) in the feedback path, that is \( d_1 = (e, 0) \) and \( d_2 = (0, 0) \), for some scalar perturbation signal \( e \), and thus the perturbed hybrid control system is \( \mathcal{H}_{\mathcal{H}^C, \mathcal{D}} \) (see Fig. 1).

As a result, the perturbed control system takes the form (7), where by using (6) it results that \( e_1 = (0, 0, e, 0) \), \( e_2 = (0, 0.2e, 0, 0) \), and \( e_3 = (0, -e, 0, 0) \), and thus the admissible perturbation signals are \( n_1 = (0, n, 0, 0) \), \( n_2 = (0, 0.2n, 0, 0) \), and \( n_3 = (0, -n, 0, 0) \), for some scalar admissible perturbation signal \( n \). In the following, different noise-free and noisy solutions to the hybrid control system are analyzed. Two admissible perturbation signals \( n \) will be used: \( \tau(t) = e^{-t} \cos(10\pi t) \) and \( \tau(t) = \cos(10\pi t) \). For simplicity, the notation \( \mathcal{H}_{\mathcal{H}^C, \mathcal{D}} \) will be used for the perturbed hybrid control systems, respectively. On the other hand, \( K \) will be any compact subset of \( \mathbb{R}^3 \times \{0\} \) such that \((\xi, 0) \in K \), where \( \xi = (1, 0, -1) \). In addition, the values \( e = 0.1 \) and \( \rho = 0.1 \) have been chosen.

Noise-free solutions. Any solution \( \phi \) to \( \mathcal{H}^C \), with \( \phi(0, 0) = (\xi, 0) \), has either a domains given by \( \text{dom } \phi = [0, t_1] \times \{0\} \cup [t_1, t_2] \times \{1\} \cup \cdots \) with \( t_1 \geq \rho \), or a domain \( \text{dom } \phi = [0, \infty) \times \{0\} \). By Def. 3.1, for any implementation \( \mathcal{H}^{cl} \), there is a solution \( \phi \) to \( \mathcal{H}^C \) with one of the above domains, that is also solution to \( \mathcal{H}^{cl} \). By convenience, define the implementations set \( \mathcal{H}_{t_1}^{cl} \) with parameter \( t_1 \geq \rho \), as the set of all implementations for which the solution \( \phi \) has the first jump at \( t_1 \) (if the domain of the solution is \( \text{dom } \phi = [0, \infty) \times \{0\} \) then the set is \( \mathcal{H}^{cl}_w \)). On the other hand, solutions \( \phi^{\mathcal{H}} \) and \( \phi^C \) of the jumping-first implementation \( \mathcal{H}^{cl, D} \in \mathcal{H}^{cl}_{\mathcal{D}} \) and the flowing-first implementation \( \mathcal{H}^{cl, C} \in \mathcal{H}^{cl}_{\mathcal{C}} \) are plotted in Fig. 3 and 4, respectively.

Noisy solutions. First, let us focus on the solutions \( \phi_{d_0} \) to \( \mathcal{H}^{cl}_{d_0} \). It is always possible to find a small enough \( e \) such that \( \max(|l_1 - s_1|, |l_1 - l_1|) > \epsilon \) (note that \( s_1 \neq l_1 \)), and thus, \( \phi \) is not \((T, J, \epsilon)\)-close to \( \phi_{d_0} \) or \( \phi_{d_0} \). This means that any implementa-

![Figure 3: Noise-free solution \( \phi^{cl}(0,0) = (1,0,-1,0) \) and noisy solutions \( \phi_{d_0}(0,0) = (1,1,-1,0) \) and \( \phi_{d_0}(0,0) = e^{-t} \cos(10\pi t) \): (top) \( \delta = 0.1 \), (middle) \( \delta = 0.01 \), (down) \( \delta = 0.001 \). The perturbation signal \( \phi_1 = (0, \delta, 0, 0) \) (black) is shown added to the noisy solution.](image-url)
4. A new definition of robustness to perturbations

Although hybrid basic conditions are sufficient for a hybrid control system to be robust to perturbations (according to Def. 2.1), this sense of robustness is not enough in control practice. It has been shown that implementations of a robust hybrid control system are not necessarily robust to perturbations. To overcome this limitation, a narrower notion of robustness to perturbations, that will be useful to characterize robustness of implementations, is proposed. In addition, a relationship with previously developed relaxations results for hybrid systems is developed.

4.1. Strong Robustness to perturbations

**Definition 4.1** (Strong robustness to perturbations) For a compact set $K \subset \mathbb{R}^n$ such that the hybrid system $\mathcal{H}$ is forward complete from $K$, $\mathcal{H}$ is strongly robust to perturbations if it is robust to perturbations and, in addition, for any $\epsilon > 0$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta^* > 0$ with the following property: for any admissible noise signal $n$, any $\delta \in (0, \delta^*)$, any $\xi \in K$, any $\xi_{0} \in \xi + \delta B$, and any solution $x \in S(\xi)$, there exists a solution $x_{in}$ to $\mathcal{H}_{in}$, with $x_{in}(0, 0) = \xi_{0}$, such that $x$ and $x_{in}$ are $(T, J, \epsilon)$-close.

Note that for any implementation, the properties of robustness and strong robustness to perturbations are equivalent, since they have unique solutions for any initial point. Next, we show that a sufficient condition for the jumping-first and flowing-first implementations of a hybrid control system to be robust is that the hybrid control system be strongly robust.

**Proposition 4.2** Suppose that the hybrid control system $\mathcal{H}$, satisfying the assumptions of Corollary 3.2, is strongly robust to perturbations for some compact set $K \subset \mathbb{R}^n$, and in addition, $\mathcal{H}$ is forward complete from $K$. Then the flowing-first implementation, $\mathcal{H}^C$, and the jumping-first implementation, $\mathcal{H}^D$, are robust to perturbations for the set $K$.

**Proof:** Note that assumptions of Corollary 3.2 guarantee the existence of implementations $\mathcal{H}^C$ and $\mathcal{H}^D$. Let us prove that $\mathcal{H}^C$ is robust to perturbations for the set $K$. A similar approach can be applied to $\mathcal{H}^D$. Consider any $\epsilon > 0$, any $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, any admissible perturbations signal $n$, any $\xi \in K$ and the unique solution $x \in S(\xi)$, then we aim at finding $\delta^* > 0$ in Def. 4.1, which may depend on $\epsilon, T$, and $J$. From the definition of implementation (Def. 3.1), we get $x \in S(\xi)$.

Since $\mathcal{H}$ is strongly robust to perturbations for the set $K$ then there exists $\delta^*$ such that for any $\delta \in (0, \delta^*)$, any $\xi_{0} \in \xi + \delta B$, there exists a solution $x_{in}$ to $\mathcal{H}_{in}$, with $x_{in}(0, 0) = \xi_{0}$, such that $x$ and $x_{in}$ are $(T, J, \epsilon)$-close. Consider the truncation $X_{in}$ of $x_{in}$ with $dom X_{in} = \{t, j\} \in dom x_{in}: t \leq T, j \leq J$, if the truncation is also a solution to $\mathcal{H}^C_{in}$ then it directly follows that $\mathcal{H}^C$ is strongly robust perturbations for the set $K$ by taking $\delta^* = \delta^*$.

By way of contradiction, suppose that any $x_{in} \in S(\xi)$, that is $(T, J, \epsilon)$-close to $x$, its truncation $X_{in}$, truncated at $x_{in} = \{t, j\} \in dom x_{in}: t \leq T, j \leq J$ is not a solution to $\mathcal{H}^C_{in}$, then for any $\xi_{0}, \xi_{1} \in S(\xi)$, there exists $(t, j) \in dom x_{in}$ with $(t, j) + 1 \in dom x_{in}$ such that $X_{in}(t, j) + \delta n(t, j) \notin D \cap C^*$. From the strong robustness of $\mathcal{H}$, for any of the previous $(t, j)$ there exist $\tilde{\epsilon} > 0$ depending on $\epsilon$ with $\lim_{\epsilon \to 0} \tilde{\epsilon} = 0$ and $s$ such that $|t-s| < \tilde{\epsilon}$, $(s, j) \in dom x_{in}$ and $(s, j) + 1 \in dom x_{in}$ and $\|x_{in}(t, j) - x(s, j)\| < \tilde{\epsilon}$. Therefore, we get

$$\tilde{\epsilon} + \delta > \|x_{in}(t, j) - x(s, j)\| + \delta \geq \|x_{in}(t, j) + \delta n(t, j) - x(s, j)\| \geq \inf_{y \in D^C} \|y - x(s, j)\|$$

Since $x(s, j) \in D \setminus C^*$ and $s < T + \tilde{\epsilon}$ there exists $y(T, \tilde{\epsilon})$ such
that

\[ \dot{\epsilon} + \delta > \inf_{y \in D \cap C^*} \| y - x(s, j) \| \geq \gamma(T, \bar{\epsilon}) \]

Note that \( \gamma \) is nonincreasing in \( \bar{\epsilon} \). The above inequality must hold for any \( \bar{\epsilon} > 0 \). It is evident that \( \epsilon \) is a subset of \( \gamma(T, \bar{\epsilon}) \). Therefore, the truncation \( x_{in} \) is solution to \( H^C \), and the proof is complete. \( \square \)

A direct application of the strong robustness definition to Example 3.2 results in that \( H \), given by (12), is strongly robust to perturbations for any compact set \( K \subset \mathbb{R}^2 \setminus X \), where \( X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = x_1 + 1 \} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0, x_2 = 1 \} \). Moreover, both implementations \( H^B \) and \( H^C \) are robust for any compact set \( K \subset \mathbb{R}^3 \setminus X \). Example 3.3 shows that besides avoiding grazing, the set \( K \) cannot contain some specific initial points; in general, higher order hybrid control systems requires a deeper analysis. In the following, we use the path for characterization of conditions that implies strong robustness.

### 4.2. Relationship with hybrid relaxation results

In [15], several relaxation results are used to analyze continuous dependence on initial conditions of solutions to hybrid systems. Although the scope is more general than hybrid systems given by (1), it turns out that some of these relaxation results may be helpful to analyze strong robustness to perturbations of hybrid systems like (1). A first result in that direction is the following proposition, that follows by using some relaxation properties (an extension of the strong relaxation property in [15], see Appendix C).

**Proposition 4.3** Consider a hybrid system \( H \) satisfying the hybrid basic conditions and a compact set \( K \subset \mathbb{R}^n \) such that \( H \) is forward complete from \( K \). If for each \( \xi \in K \) total strong relaxation is possible \( \xi \) for solutions from \( \xi \) then \( H \) is strongly robust to perturbations for the set \( K \).

**Proof.** Since \( H \) satisfies the hybrid basic conditions, the robustness to perturbations is directly obtained by Corollary 2.2.2, and thus the proof is centered on the additional property for strong robustness according to Def. 4.1. In first place, using similar arguments to the proof of Th. 3.4 in [15], it can be shown that total strong relaxation implies 4 that given \( \xi \in \mathbb{R}^n \), for any compact solution \( x : \text{dom } x \rightarrow \mathbb{R}^n \) to \( H \) with \( x(0,0) = \xi \) and for any \( \epsilon > 0 \), there exist \( \delta > 0 \) such as for any admissible perturbation signal \( n \), any \( \xi_n \in (\xi + \delta B) \cap (C \cup D) \) there exist a solution \( x_{in} \) to the perturbed hybrid system \( H_{in} \) such as if \( x(T, J) \in D \), where \( (T, J) = \text{max dom } x \), then \( x_{in}(T, J) \in D \), where \( (T, J) = \text{max dom } x_{in} \) and \( x \) and \( x_{in} \) are \( (T, J, \epsilon) \)-close.

4 See Appendix C, the name is inspired in the classical concept of total stability for ordinary differential equations (also referred to as stability under persistent disturbances) [16].

5 This property may be referred to as that total strong relaxation for initially flowing (respectively, initially jumping) solutions from \( \xi \) relative to \( C \) (respectively, relative to \( D \)) is possible (using a direct analogy with Def. 3.1 in [13]).

It remains to show uniformity with respect to the initial condition, that is that \( \delta \) works for all solutions from \( \xi \in \mathcal{K} \). By contradiction, if \( H \) is robust to perturbations but not strongly robust to perturbations then for some \( \epsilon > 0 \), and \( (T, J) \in \mathbb{R}_{>0} \times \mathbb{N} \), there exist a sequence \( x_i : \text{dom } x_i \rightarrow \mathbb{R}^n \) of solutions to \( H \) with \( x_i = \xi \), a sequence of admissible perturbation signal \( n_i \), a sequence \( \delta_i \rightarrow 0 \), and a sequence \( \xi_n \in \xi + \delta B \), such as all solutions \( x_{in,0} \) to the hybrid system \( H_{in,0} \), with \( x_{in,0}(0,0) = \xi_n \), satisfy that \( x_i \) and \( x_{in,0} \) are not \( (T, J, \epsilon) \)-close. Similar arguments to proof of Proposition 6.2 in [15] may be applied, resulting in a contradiction of the total strong relaxation at \( \xi \). Finally, the uniformity of \( \delta \) in \( \mathcal{K} \) comes of an argument similar to the one used in Corollary 6.4 in [15], which ends the proof. \( \square \)

In [15], some hybrid relaxation conditions are developed for strong relaxation for any \( \xi \in (C \setminus D) \cup (D \setminus C) \cup (\text{int } C \cap \text{int } D) \); it can be checked that these conditions are not satisfied for the Examples of Section 3, basically due to initial points that produce grazing in 3.2, and to the existence of a non-empty unobservable subspace in Section 3.3. This fact prevents the use of an extension of hybrid relaxation conditions to include perturbations, which would be of limited use in control practice.

Note that the difference between total strong relaxation and strong robustness is the uniformity in the latter, that is that \( \delta \) works for all solutions from \( \xi \) and for any \( \xi \in K \), rather than for each solution we have a \( \delta_i \); and thus, total strong relaxation is a property easier to check in principle. For example, for the hybrid control system of Section 3.2, it is not difficult to see that total strong relaxation is possible for solutions from any \( \xi \in K \), for any compact \( K \subset (R^3 \times R_{>0}) \setminus ((C \setminus D) \cup (\text{span}(1,0,-1)) \times R_{>0}) \).

### 5. Conclusions

Robustness of hybrid systems to perturbations is a sound contribution of the HI framework, since it develops a property that may be applied to a generality of cases (hybrid systems satisfying the hybrid basic conditions). Although in general, this property of robustness is suitable for hybrid systems, hybrid control systems demand a narrower property, since its implementations are not necessarily robust to perturbations, which is a clear limitation in control practice. This fact has been proved with two counterexamples (one specifically related to hybrid control systems), showing that for two robust hybrid systems none of their implementations are robust. A new concept of robustness referred to as strong robustness to perturbations has been proposed; moreover, it has been shown that this new property is a sufficient condition for jumping-first and flowing-first implementations to be robust. Finally, a relationship between strong robustness and previously developed hybrid relaxation results has been found.

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A solution $\phi$ to a hybrid system is nontrivial if dom $\phi$ contains at least one point different to (0, 0); maximal if it cannot be extended, that is there is no solution $\phi'$ with dom $\phi'$ contains dom $\phi$ as a proper subset, and such that $\phi'(t, j) = \phi(t, j)$ for any $(t, j) \in$ dom $\phi$; and complete if dom $\phi$ is unbounded.

There exists nontrivial solutions from $\xi \in C \cup D$ if there exist a discrete-time nontrivial solution or a continuous-time nontrivial solution, that if either $\xi \in D$ or there exist a solution $z(t) = f(z(t))$ in some interval $[0, \epsilon)$, for some $\epsilon > 0$, and satisfying $z(0) = \xi$ and $z(t) \in C$ for $t \in [0, \epsilon]$. In addition, solutions are unique if and only if the following basic uniqueness conditions (see [3] hold:

- for every $\xi \in C \setminus D$ there exists $\epsilon > 0$ and a unique maximal solution $z : [0, \epsilon) \rightarrow R^n$ such that $z(0) = \xi$, $z(t) = f(z(t))$ satisfying $z(0) = \xi$ and $z(t) \in C$ for all $t \in [0, \epsilon]$;
- for every $\xi \in C \cap D$, there does not exist $\epsilon > 0$ and an absolutely continuous $z : [0, \epsilon) \rightarrow R^n$ such that $z(0) = \xi$, $z(t) = f(z(t))$ for almost all $t \in [0, \epsilon]$, and $z(t) \in C$ for all $t \in [0, \epsilon]$.

Given $T \geq 0$, $J \geq 0$, and $\epsilon > 0$, two hybrid arcs $\phi_1$ and $\phi_2$ are $(T, J, \epsilon)$-close if: (a) for all $(t, j) \in$ dom $\phi_1$ with $t \leq T$, $J \leq j$, there exists $s$ such that $(s, j) \in$ dom $\phi_1$, $|t - s| < \epsilon$, and $\phi_1(t, j) - \phi_1(s, j) < \epsilon$; (b) for all $(t, j) \in$ dom $\phi_2$ with $t \leq T$, $J \leq j$, there exists $s$ such that $(s, j) \in$ dom $\phi_2$, $|t - s| < \epsilon$, and $\phi_2(t, j) - \phi_2(s, j) < \epsilon$.

### Appendix A. Hybrid systems solutions and basic properties (1, 4, 11, 13)

A subset $E$ of $R^n \times [0, \infty)$ is a hybrid time domain if it is the union of infinitely many intervals $[t_j, t_{j+1}] \times j$, where $0 = t_0 \leq t_1 \leq t_2 \leq \cdots$ or of finitely many such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times j$, $[t_j, t_{j+1}] \times j$, or $[t_j, \infty) \times j$. A hybrid arc $\phi$ is a function $\phi : dom \phi \rightarrow R^n$, where dom $\phi$ is a hybrid time domain and, for each $t, t + \phi(t, j)$ is a locally absolutely continuous function on the interval $I_j = [t : (t, j) \in$ dom $\phi$].

The hybrid arc $\phi$ is a solution to the hybrid system $H = (C, F, D, g)$ given by (1) (see [3]) if $\phi(0, 0) \in C \cup D$, and

- (Flow condition) For each $j \in N$ such that int $I_j = \emptyset$, $\phi(t, j) = f(\phi(t, j))$, for almost all $t \in I_j$,
- (Jump condition) For each $(t, j) \in$ dom $\phi$ such that $(t, j + 1) \in$ dom $\phi$,

$$\phi(t, j + 1) = g(\phi(t, j)), \quad \phi(t, j) \in D.$$

### Appendix B. Hybrid basic conditions (1, 4)

A hybrid system with the data $(C, F, D, G)$ in $R^n$, satisfies the hybrid basic conditions if

1. $C$ and $D$ are closed sets.
2. $F : R^n \rightarrow R^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in C$.
3. $G : R^n \rightarrow R^n$ is outer semicontinuous and locally bounded, and $G(x)$ is nonempty for all $x \in D$.

For the hybrid system $H = (C, F, D, g)$ given by (1), hybrid basic conditions are satisfied if $C$ and $D$ are closed sets, and $f$ and $g$ are continuous functions.

### Appendix C. Relaxation properties for hybrid inclusions

For a hybrid system $H$ with the data $(C, F, D, G)$ in $R^n$, $H_{\text{const}}$ is defined by the relaxed hybrid inclusion

$$H_{\text{const}} : \begin{cases} \xi \in \text{conv}(F(x)), & x \in C, \\ x' \in G(x), & x' \in D. \end{cases} \quad \text{(C.1)}$$

Given $x_0 \in C \cup D$, strong relaxation for all solutions from $x_0$ is possible (15) if for any compact $x : dom x \rightarrow R^n$ with $x(0, 0) = x_0$ that is a solution to $H_{\text{const}}$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y_0 \in (x_0 + \delta B) \cap (C \cup D)$ there exist a hybrid arc $y : dom y \rightarrow R^n$ with compact dom $y$ and $y(0, 0) = y_0$ that is a solution to $H$ and $d_{\text{relax}}(x, y) < \epsilon$, and moreover, if $x(T, J) \in D$, then $y(T, J) = \text{max dom } x$, then $y(r, J) \in D$, where $(T, J) = \text{max dom } y$.

In this work, it is used an extension of the strong relaxation property to cope with the problem of measurement noise and external disturbances. Total strong relaxation for all solutions from $x_0$ is possible (15) if for any compact $x : dom x \rightarrow R^n$ with $x(0, 0) = x_0$ that is a solution to $H_{\text{const}}$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y_0 \in (x_0 + \delta B) \cap (C \cup D)$ and any admissible perturbations signals $n_1, n_2,$ and $n_3$ there exist a hybrid arc $y : dom y \rightarrow R^n$ with compact dom $y$
and \( y(0,0) = y_0 \) that is a solution to \( H_{\text{con}}^{\text{in}}(n_1, n_2, n_3) \) and \( d_{\text{ph}}(x, y) \leq \varepsilon \), and moreover, if \( x(T,J) \in \mathcal{D} \), where \((T,J) = \max \text{ dom } x\), then \( y(\tau, J) \in \mathcal{D} \), where \((\tau, J) = \max \text{ dom } y\).

Note that for \( \mathcal{H} = (C, f, D, g) \), in which \( f : \mathbb{R}^n \to \mathbb{R}^n \), simply \( H_{\text{con}} = \mathcal{H} \).