Numerical simulation of a class of space fractional bistable systems based on the Fourier spectral method

Haitao Liu¹, Wang Yulan², Li Cao³ and Wei Zhang⁴

Abstract
Nonlinear vibration arises everywhere in a bistable system. The bistable system has been widely applied in physics, biology, and chemistry. In this article, in order to numerically simulate a class of space fractional-order bistable system, we introduce a numerical approach based on the modified Fourier spectral method and fourth-order Runge-Kutta method. The fourth-order Runge-Kutta method is used in time, and the Fourier spectrum is used in space to approximate the solution of the space fractional-order bistable system. Numerical experiments are given to illustrate the effectiveness of this method.

Keywords
Space fractional bistable system, Fourier spectral method, fractional Laplacian, numerical simulation

Introduction
Fractional partial differential equations are becoming widely used as a suitable modeling approach for many fields in science and engineering.¹⁻⁶ We mainly study models arising in the application areas of theoretical biology, physics, and chemistry, modeled by some space fractional bistable systems

\[ \frac{\partial u}{\partial t} = d_1 \Delta^\alpha x u + a_{11} u + a_{12} v + f(u, v), \]
\[ \frac{\partial v}{\partial t} = d_1 \Delta^\alpha x v + a_{21} u + a_{22} v + g(u, v) \]  

(1)

where \( u = u(x, y, t) \) and \( v = v(x, y, t) \) are unknown functions. \( (x, y) \in \Omega = [a, b] \times [c, d], t > 0 \), the smooth boundary is \( \partial \Omega \). The parameters \( d_i, i = 1, 2 \in \mathbb{R}^+ \) are diffusion coefficients, and \( f(u, v) \) and \( g(u, v) \) are the reaction terms. The system (1) is subjected to some initial condition \( u(x, y, 0) = u_0(x, y), v(x, y, 0) = v_0(x, y) \), and the homogeneous Neumann boundary condition, namely, \( \frac{\partial u}{\partial n} \cdot \partial \Omega = \frac{\partial v}{\partial n} \cdot \partial \Omega = 0 \). The functions \( u(x, y, t) \) and \( v(x, y, t) \) are assumed to be a causal function of time, that is, vanishing for \( t < 0 \). The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses. In this article, we use the fractional Laplacian operator by the Riesz fractional derivatives as follows

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\[ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{2} \cos \frac{\alpha \pi}{2} \frac{d_x u}{dx} + \frac{d_y u}{dy} \]

with \( D^\alpha_L u \) and \( D^\alpha_R u \) being the Riemann-Liouville fractional operators.

Many approaches are used to solve the reaction-diffusion system. These methods include the finite difference method, \(^7\) reproducing the kernel method (RKM), \(^8\)–\(^17\) variational iteration method (VIM), \(^18\)–\(^19\) homotopy perturbation methods (HPM), \(^20\)–\(^22\) etc. \(^23\)–\(^27\) There are few numerical methods for higher order space fractional reaction-diffusion system.

In this article, in order to numerically simulate space fractional-order reaction-diffusion system, we introduce a novel numerical approach based on the modified Fourier spectral method \(^28\)–\(^30\) and fourth-order Runge-Kutta method. Some patterns are shown by using this new approach, and the results have good agreement with theoretical results. Simulation results show the effectiveness of the method.

### Bifurcation analysis of system

In this section, we give the Turing bifurcation conditions of system (1). We assume the equilibrium point of the non-diffusive system (1) is \( E^* = (u_0, v_0) \), so

\[ a_{11} u_0 + a_{12} v_0 + f(u_0, v_0) = 0, \]

\[ a_{11} u_0 + a_{12} v_0 + g(u_0, v_0) = 0 \]

(3)

Now, we do linear stability analysis of the equilibrium point \( E^* \). We evaluate the Jacobian matrix \( A_0 \) of the system at \( E^* \) as

\[ g(t - \bar{u}) = \]

The characteristic roots for \( A_0 \) are given by

\[ \lambda_{1,2}(0) = \frac{1}{2} \left[ tr(A)_0 \pm \sqrt{[tr(A)_0]^2 - 4 \det(A)_0} \right] \]

(5)
For any integer \( N \geq 0 \), consider \( x_j = j \Delta x = \frac{2 \pi j}{N}, y_j = j \Delta y = \frac{2 \pi j}{N} L = b - a, j = 0, 1, \ldots, N - 1 \). \( u(x,y,t) \) is transformed into the discrete Fourier space as

\[
\hat{u}(k_x, k_y, t) = F(u) = \frac{1}{N^2} \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} u(x_j, y_j, t) e^{-i(k_x x_j + k_y y_j)} - \frac{N}{2} \leq k_x, k_y \leq \frac{N}{2} - 1
\]
and the inverse formula is
\[
  u(x_j, y_j, t) = F^{-1} \left( \hat{u} \right) = \sum_{k=-N}^{N-1} \sum_{l=-N}^{N-1} \hat{u}(k, l, t) e^{-i(kx_j - ly_j)} \quad 0 \leq j \leq N - 1
\]  

(9)

It is easy to know \( u(x_j, y_j, t) = F^{-1} \{ F[u(x, y, t)] \} \).

For system (1) with \( f(u, v) = r(a - u + u^2 v) \) and \( g(u, v) = r(b - u^2 v) \), using Fourier transform, we can get
\[
\frac{\partial \hat{u}}{\partial t} = d_1 (ik_x)^a + (ik_y)^a \hat{u} + a_{12} F \left\{ F^{-1} \left[ \hat{u} \right] \right\} \rightleftharpoons a_{12} F \left\{ F^{-1} \left[ \hat{v} \right] \right\} + r \left( a - F \left\{ F^{-1} \left[ \hat{u} \right] + F^{-1} \left[ \hat{u} \right]^2 F^{-1} \left[ \hat{v} \right] \right\} \right)
\]
\[
\frac{\partial \hat{v}}{\partial t} = d_2 (ik_x)^a + (ik_y)^a \hat{u} + a_{22} F \left\{ F^{-1} \left[ \hat{u} \right] \right\} \rightleftharpoons a_{22} F \left\{ F^{-1} \left[ \hat{v} \right] \right\} + r \left( b - F \left\{ F^{-1} \left[ \hat{u} \right]^2 F^{-1} \left[ \hat{v} \right] \right\} \right)
\]  

(10)

We use the fourth-order Runge-Kutta method to solve the ordinary differential equation (10) which is as follows
\[
k_1 = g \left( t_n - \tilde{u} \right),
\]
\[
k_2 = g \left( t_n + \frac{\tau}{2}, \tilde{u}_n + \frac{\tau k_1}{2} \right),
\]
\[
k_3 = g \left( t_n + \frac{\tau}{2}, \tilde{u}_n + \frac{\tau k_2}{2} \right),
\]
\[
k_4 = g \left( t_n + \tau, \tilde{u}_n + \tau k_3 \right),
\]
\[
\tilde{u}_{n+1} = \tilde{u}_n + \frac{\tau}{6} (k_1 + 2k_2 + 2k_3 + k_4)
\]  

(11)
where \( \tau \) is step-size and \( g(t-u) = \frac{\partial u}{\partial t} \). For convenience of expression, we denote

\[
\mathbf{U} = \left( \tilde{u}_0(t), \tilde{u}_1(t), \ldots, \tilde{u}_{N-1}(t) \right)^T, \quad G(t, \mathbf{U}) = \left( g(\tilde{u}(t)), g(\tilde{u}(t)) \right)^T, \quad n = 1, \ldots, T
\]  

Equation (10) is reduced to

\[
\frac{\partial \mathbf{U}}{\partial t} = G(t, \mathbf{U})
\]

Next, we can obtain the standard fourth-order Runge-Kutta formula

\[
K_1 = G(t_n - U_n),
\]
\[
K_2 = G(t_n + \frac{\tau}{2}, U_n + \frac{\tau K_1}{2}),
\]
\[
K_3 = G(t_n + \frac{\tau}{2}, U_n + \frac{\tau K_2}{2}),
\]
\[
K_4 = G(t_n + \tau, U_n + \tau K_3),
\]

\[
U_{n+1} = U_n + \frac{\tau}{6} (K_1 + 2K_2 + 2K_3 + K_4)
\]

Then, we can derive that by solving the following formula

\[
k_{j1} = g\left( t_n, \tilde{u}_{0,\alpha}, \tilde{u}_{1,\alpha}, \ldots, \tilde{u}_{N-1,\alpha} \right),
\]
\[
k_{j2} = g\left( t_n + \frac{\tau}{2}, \tilde{u}_{0,\alpha} + \frac{\tau k_{j1}}{2}, \tilde{u}_{1,\alpha} + \frac{\tau k_{j1}}{2}, \ldots, \tilde{u}_{N-1,\alpha} + \frac{\tau k_{j1}}{2} \right),
\]
\[
k_{j3} = g\left( t_n + \frac{\tau}{2}, \tilde{u}_{0,\alpha} + \frac{\tau k_{j2}}{2}, \tilde{u}_{1,\alpha} + \frac{\tau k_{j2}}{2}, \ldots, \tilde{u}_{N-1,\alpha} + \frac{\tau k_{j2}}{2} \right),
\]
\[
k_{j4} = g\left( t_n + \tau, \tilde{u}_{0,\alpha} + \tau k_{j3}, \tilde{u}_{1,\alpha} + \tau k_{j3}, \ldots, \tilde{u}_{N-1,\alpha} + \tau k_{j3} \right),
\]
\[
\tilde{u}_{j,\alpha+1} = \tilde{u}_{j,\alpha} + \frac{\tau}{6} (k_{j1} + 2k_{j2} + 2k_{j3} + k_{j4})
\]

Finally, we find the numerical solution using the inverse discrete Fourier transform.  

### Numerical simulation

In this section, numerical simulation of the fractional Gray-Scott (GS) model with a perturbation to the spatially homogeneous steady-state equation is obtained. The domain of interest is taken to be \( \Omega = [-1, 1]^2 \), discretized using \( N = 128 \) points in each spatial coordinate. On account of the evolution of the numerical solution, \( u \) is similar with that of the \( v \), but we will not present it in this article.
Numerical experiment Consider the following fractional Gray-Scott (GS)\textsuperscript{31} model:

\[
\frac{\partial u}{\partial t} = d_1 \left( \Delta^{\alpha} \right) u - uv^2 + F(1 - u),
\]

\[
\frac{\partial v}{\partial t} = d_2 \left( \Delta^{\alpha} \right) v - uv^2 + (F + k) v, \quad (x, y, t) \in \Omega \times [0, T]
\]

(16)

We take \(d_1 = 2 \times 10^{-5}, d_2 = 1 \times 10^{-5}, F = 0.045, k = 0.0625, u = \text{ones} (N), v = \text{zeros} (N), v(N/2, N/2 : N) = 0.5, v(N/2 - 1, 1 : N/2) = 1\), the numerical results are shown in Figures 1 and 2. We take \(d_1 = 2 \times 10^{-5}, d_2 = 1 \times 10^{-5}, F = 0.045, k = 0.0625, u = \text{ones} (N), v = \text{zeros} (N), v(N/2, N/2 : N) = 0.5, v(N/2 - 2, 1 : N/2) = 1\), the numerical results are shown in Figure 3. We take \(d_1 = 3 \times 10^{-5}, d_2 = 1 \times 10^{-5}, F = 0.015, k = 0.055, u = \text{ones} (N), v = \text{zeros} (N), v(N/2, N/2 : N) = 0.5, v(N/2 - 2, 1 : N/2) = 1\); the numerical results are shown in Figures 4–6.

Conclusion and remarks

In the article, a numerical method that combines the Fourier spectral method with the Runge-Kutta method is proposed to study a class of space fractional bistable system. This approach has general meanings and thus can be used to solve same types of nonlinear space fractional partial differential equations with periodic boundary condition in science and engineering. Some pattern formations are shown by using this new approach, and the results have good agreement with theoretical results. Simulation results show the effectiveness of the method.

All computations are performed by the MATLAB R2017b software.

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