Wall-crossing from Lagrangian Cobordisms

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Abstract

Biran and Cornea showed that monotone Lagrangian cobordisms give an equivalence of objects in the Fukaya category. We take a first look at extending this theory to the non-monotone case and show that Lagrangian surfaces related by mutation along an antisurgery disk are equivalent after incorporation of an appropriate bounding cochain. An interpretation of these corrections as a wall-crossing formula is given in the spirit of [Aur07]. We additionally provide some constructions for filtered $A_\infty$ algebras, specifically the construction of mapping cocylinders.

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1 Introduction

Wall-Crossing...

The wall-crossing phenomenon for Lagrangian submanifolds is an observation that the count of holomorphic disks with boundary on a family of Lagrangian submanifolds need not be continuous over the family. The count of these disks and the manner in which this count changes over families play an important role in describing the space of Lagrangian submanifolds up to Hamiltonian isotopy.

The moduli space of Lagrangian submanifolds can be understood locally by constructing coordinate charts. Nearby any Lagrangian, the flux of a Lagrangian isotopy builds a local \((\mathbb{C}^\ast)^k\) chart, where \(k = \dim(H^1(L; \mathbb{R}))\). An expectation which has been proven in good examples (\cite{Aur07, PW19}) is that the open Gromov-Witten (OGW) potential, which records an area-weighted count of the holomorphic disks with boundary on a given Lagrangian \(L\), is a holomorphic map in these coordinates. Both the symplectic area of these disks and the flux of an isotopy are complexified by choosing a unitary local system on the Lagrangian submanifold.

Locally, the flux-charts and the OGW potential are holomorphic. However, neither of these can consistently provide coordinates or functions globally. An inconsistency occurs when a holomorphic disk “bubbles” over a Lagrangian isotopy, causing a discontinuity in the OGW potential. Similarly, the coordinates constructed via the flux of an isotopy will accrue discrepancies from the monodromy of \(H^1(L, \mathbb{R})\) over the moduli space.

In order to consistently construct coordinates on the moduli space of Lagrangian submanifolds, one must incorporate “instanton corrections” to the count of holomorphic disks and flux computation. In \cite{KS01, Aur07} these corrections were phrased in terms of a wall-crossing formula. In this paradigm the moduli space of Lagrangian submanifolds is divided into chambers of Lagrangians which do not bound Maslov index 0 disks. Separating these chambers are “walls” consisting of the Lagrangian submanifolds which bound Maslov index 0 disks. To transition from coordinates on one chamber to another, one computes a wall-crossing formula given by the count of Maslov index 0 disks, which appropriately modifies the OGW potential and flux computation.

A particular example of wall-crossing occurs when a Lagrangian torus \(L\) bounds a Lagrangian disk \(D\). For such pairs, there exists another Lagrangian, called the mutation \(\mu_D(L)\), which lies in a different chamber. \cite{PT17, PW19, Ton19} explicitly compute the wall-crossing formula between these two chambers. Notably, this computation gives a wall-crossing formula between two Lagrangians which are not Hamiltonian isotopic.

The geometric justification for these wall-crossing transformations comes from the homological mirror symmetry conjecture of \cite{Kon94}. Associated to a symplectic space \(X\) is the Fukaya category \(\text{Fuk}(X)\) whose objects are Lagrangian submanifolds. The conjecture predicts that to a Calabi-Yau manifold \(X\), there exists a mirror Calabi-Yau manifold \(\hat{X}\) so that the symplectic geometry of \(X\) as recorded by \(\text{Fuk}(X)\) is interchanged with the complex geometry of \(\hat{X}\) as recorded by \(\text{D}^b\text{Coh}(\hat{X})\). One way to recover the space \(\hat{X}\) is to study the moduli of points on \(\hat{X}\). A useful perspective comes from the SYZ conjecture \cite{SYZ96},
which presents mirror spaces as dual Lagrangian torus fibrations. From this viewpoint, a candidate mirror to the skyscraper sheaf of a point in $\mathcal{X}$ is a Lagrangian torus fiber of the SYZ fibration $X \to \mathcal{Q}$.

As Hamiltonian isotopic Lagrangian submanifolds give quasi-isomorphic objects of the Fukaya category, the moduli of objects in the Fukaya category is a good proxy for the Hamiltonian isotopy classes of Lagrangian submanifolds. This Fukaya-categorical interpretation has the advantage that the incorporation of “instanton corrections” naturally arises in the construction of the Fukaya category. The presence of holomorphic disks with boundary on a Lagrangian $L$ complicates the construction of the Fukaya category considerably. To account for the presence of holomorphic disks, one must equip each Lagrangian submanifold with additional data. The objects of the Fukaya category are pairs $(L, b)$, where $b$ is a “bounding cochain” deforming $L$ as an object of the category. This deformation encodes an algebraic cancellation of holomorphic disks with boundary on $L$. Hamiltonian isotopy still produces an equivalence of such pairs, where the bounding cochain is allowed to change over the isotopy. In the examples of crossing a wall, Lagrangians $(L, 0)$ in one chamber are equivalent to $(L', b')$ in the second chamber. The perspective of [FOOO10] is that the non-trivial deformation $b'$ should be considered as the correction which occurs in the wall-crossing formula.

... and Lagrangian Cobordisms

In the previous discussion, we focused on Hamiltonian isotopy as the geometric equivalence relation on objects of the Fukaya category. A weaker equivalence relation was exhibited by [BC14], who proved that Lagrangian submanifolds related by monotone Lagrangian cobordism are equivalent objects in the Fukaya category. As every Hamiltonian isotopy gives an example of a Lagrangian cobordism, this indeed generalizes the previously considered equivalences. However, the monotonicity condition precludes the existence of Maslov-index 0 disks so the examples considered by [BC14] do not realize the wall-crossing phenomenon.

A natural extension is to consider Lagrangian cobordisms which are equipped with bounding cochains. Such a cobordism $(K, b)$ is predicted to yield equivalences in the Fukaya category between the ends $(L^-, b|_{L^-})$ and $(L^+, b|_{L^+})$. One interesting example of a Lagrangian cobordism is the mutation cobordism constructed in [Hau15], which relates mutant Lagrangians. This is an example of a non-monotone Lagrangian cobordism, as mutant Lagrangians $L, \mu_D(L)$ are generally non-isomorphic as objects of the Fukaya category.

1.1 Results

The goal of this paper is to extend our understanding of wall-crossing between Hamiltonian isotopic Lagrangians to wall-crossing between cobordant Lagrangians. We use the pearly Floer complex $CF^\bullet(L)$ as a receptacle for counting holomorphic disks. This is a deformation of the Morse complex $CM^\bullet(L)$ constructed by inserting holomorphic disks into the flow lines of the Morse function. $CF^\bullet(L)$ is a filtered $A_\infty$ algebra and given a pair $(L, b)$, we denote by $CF^\bullet_b(L)$ the algebraic deformation of the pearly Floer complex of $L$ by the bounding
cochain $b$. Our first result is an extension of [BC14] to the non-monotone and cylindrical setting.

**Theorem 1.1.1** (Paraphrasing proposition 4.2.1). Suppose that $K : L^- \rightsquigarrow L^+$ is a Lagrangian cobordism whose topology is $L^- \times \mathbb{R}$. Then there is a filtered map of $A_\infty$ algebras

$$\Theta_K : CF^*(L^-) \to CF^*(L^+).$$

As every Hamiltonian isotopy of Lagrangians produces a cylindrical suspension Lagrangian cobordism realizing that isotopy, we obtain another proof that Hamiltonian isotopic Lagrangians have quasi-isomorphic pearly Floer complexes.

The first non-monotone and non-cylindrical example that we look at is the mutation cobordism. We similarly prove that this cobordism gives an equivalence.

**Theorem 1.1.2** (Paraphrasing theorem 5.2.5). Let $K_{\mu_D} : L \rightsquigarrow \mu_D(L)$ be a mutation cobordism satisfying definition 5.2.10. Then there exists a deforming cochain $b \in CF^*(K_{\mu_D})$ so that the $CF^*_b(K_{\mu_D})$ is a mapping cocylinder, yielding a map of filtered $A_\infty$ algebras:

$$\Theta_{\mu_D} : CF^*_b(L^-) \to CF^*_b(\mu_D(L)).$$

We verify that this bounding cochain gives a correction to the Floer homology of these mutant Lagrangians in a way which matches the wall-crossing transformations constructed in [Aur08]. This gives (to our knowledge) the first example of a non-cylindrical two-ended Lagrangian cobordism yielding an equivalence in the Fukaya category. This also gives a Fukaya category interpretation to the wall-crossing result of [PT17]. Our result can be considered as an application of the [FOOO10] viewpoint on wall-crossing to non-Hamiltonian isotopic Lagrangian submanifolds. The theorem includes the important example of monotone Lagrangian tori in $\mathbb{C}^2 \setminus \{z_1z_2 = 1\}$, which is a toy model for understanding the general wall-crossing phenomenon.

Additionally, we include an appendix developing some language for filtered $A_\infty$ algebras which may be of independent interest. The tools developed are general statements relating morphisms and mapping cocylinders of filtered $A_\infty$ algebras.

**Theorem 1.1.3** (Paraphrasing theorem B.0.1). Let $B$ be a filtered $A_\infty$ algebra and let $(A, d_A)$ be a graded $A$-module with differential (not necessarily squaring to zero). Suppose that we have (an appropriate generalization) of chain maps $\alpha, \beta$ and homotopy $h$:

$$A \xrightarrow{\alpha} B \xleftarrow{\beta} B \xrightarrow{h}$$

so that $\beta \circ \alpha = id_A$ and $h$ is a homotopy from $\alpha \circ \beta$ to the identity. Then the maps $\alpha, \beta, h$ and the differential on $A$ have filtered $A_\infty$ extensions.

This allows us to prove the existence of mapping cocylinders in the category of filtered $A_\infty$ algebras.
Theorem 1.1.4 (Paraphrasing theorem D.1.3): Let $A^-$ and $A^+$ be filtered $A_\infty$ algebras.

1. To every mapping cocylinder $A^- \leftrightarrow B \rightarrow A^+$, we can associate a morphism

$$\Theta_B : A^- \rightarrow A^+.$$ 

2. To every morphism $f : A^- \rightarrow A^+$, we can associate a cocylinder

$$A^- \leftrightarrow B_f \rightarrow A^+.$$ 

3. These constructions are compatible in the sense that $\Theta_{B_f} = f$.

1.2 Structure of the paper

We will frequently refer to the example of wall-crossing between the Chekanov and product tori charts on $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$. Therefore, we introduce this example and the naive coordinates on the moduli space of Lagrangian tori computed via Lagrangian flux in section 2.1. The exposition of this example follows closely that of [Aur07], but we include it here for convenience and to establish notation.

Lagrangian Cobordisms and Wall-crossing

Section 3 provides background describing both the model we use for Floer cohomology and key results for Lagrangian cobordisms. In section 3.1 we give an outline of the construction of the pearly Floer complex $CF^*(L, h)$ including key properties of this Floer complex that we will use. Section 3.2 defines Lagrangian cobordisms and reviews expectations for how the pearly Floer complex of a Lagrangian cobordism should interact with the ends. In particular, we state assumption 3.2.6, a relationship between the moduli space of disks with boundary on a Lagrangian cobordism and the moduli space of disks with boundary on the ends of the cobordism. This assumption holds whenever disks on Lagrangian cobordisms can be regularized with a split almost complex structure near the ends of the cobordism, but is expected to hold more generally when abstract perturbation techniques are required to regularize the moduli space of holomorphic disks. The example of wall-crossing on $\mathbb{C}^2 \setminus \{z_1 z_2\}$ bounds no holomorphic disks at the ends of the cobordism and therefore satisfies this assumption.

In section 4 we recover continuation maps for open Gromov-Witten invariants of Hamiltonian isotopy using non-monotone Lagrangian cobordisms. We start by showing that there are certain correspondences between the holomorphic disks on the suspension cobordism of a Hamiltonian isotopy of Lagrangians and the disks with boundary on the Lagrangians in the isotopy itself. These correspondences (in section 4.1) are not necessary for the construction of continuation maps, but will later be useful for explicitly computing the continuation maps. Section 4.2 constructs a continuation map for the Lagrangian mutation cobordism (see proposition 4.2.1). The cobordism is our first example of a non-monotone cobordism giving an equivalence of objects in the Fukaya category.
Section 5 contains the main ideas of the paper. Section 5.1 introduces the Lagrangian mutation cobordism from [Hau15]. We take some care to work out the parameterization of this cobordism explicitly in the setting of $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$, which allows us to compute the flux swept between the ends of the cobordism. Section 5.2 looks at the mutation cobordism constructed from a mutation pair – a Lagrangian $L$ bounding a Lagrangian disk $D$. The remainder of the section is spent proving theorem 5.2.5 which shows that this mutation cobordism $K_{\mu D}$, when equipped with an appropriate bounding cochain, gives an equivalence in the Fukaya category. We outline the proof here.

- Section 5.2.1 we construct a Morse function for $K_{\mu D}$ and point out how this cobordism fails to be topological cylinder between its ends.
- Section 5.2.2 we find a holomorphic disk with boundary on $K_{\mu D}$.
- Section 5.2.3 we show that this holomorphic disk contributes to $m^0_{K_{\mu D}}$ in such a way that its lowest order contributions can be cancelled out by a deforming cochain $d_c$.
- Section 5.2.4 we show that there exists a homotopy between $\text{CF}^\bullet_{d_c}(K_{\mu D})$ and its left end $\text{CF}^\bullet_{d_c|L^-}(L^-)$ which uses the component of the deformed differential arising from the holomorphic disk.
- Section 5.2.5 we present $K_{\mu D}$ as a filtered $A_\infty$ mapping cocylinder (definition D.1.2) using the homotopy constructed above and conclude the existence of a continuation map.

The last section, section 6 explores examples and non-examples of proposition 4.2.1 and theorem 5.2.5. In section 6.1 we look at the correction from bounding cochain for a Hamiltonian isotopy passing through the wall in $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$. We use this to recover the wall-crossing formula for disk counts in section 6.2. Section 6.3 justifies why Lagrangian mutation is constructed from anti-surgery along a disk whose boundary sits in a non-trivial homology class. In section 6.4 we compute the change in coordinates from the mutation cobordism on $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$ between monotone Chekanov and product tori.

Appendices: some homological algebra

In order to obtain the geometric results outlined above, it was necessary to develop some tools for filtered $A_\infty$ algebras aimed at defining $A_\infty$ mapping cocylinders and exhibiting their expected properties. We hope that this appendix motivates the use of filtered $A_\infty$ algebras with the perspective that it is necessary to develop $A_\infty$ algebras as a generalization of differential graded algebras in order to obtain a better-behaved algebraic deformation theory and a homotopy transfer theorem. As the geometric goals of this paper are to compute wall-crossing formulae, our exposition of filtered $A_\infty$ algebras is rather formulaic and concrete.

Appendix A serves to fix notation for filtered $A_\infty$ algebras. We introduce notation for expressing the $A_\infty$ relations “operadically” via trees in appendix A.3. We additionally introduce deforming cochains, bounding cochains, and pushforwards of deformations in appendix A.4.
The core of our exposition is in appendix B, which proves a homotopy transfer theorem for filtered \( A_\infty \) algebras (theorem B.0.1). This theorem states that given a (appropriately generalized) homotopy equivalence between a filtered \( A_\infty \) algebra \( B \) and a filtered module \( A \), one can construct a filtered \( A_\infty \) structure on \( A \). We construct this filtered \( A_\infty \) structure as a sum of morphisms determined by trees: in appendix B.2 the \( A_\infty \) relations are recovered by comparing the sum of morphisms to the gluing relations for the associahedra. As we work with filtered (rather than uncurved) \( A_\infty \) algebras, our proof differs from the standard proof of the homotopy transfer theorem by counting trees with marked leaves corresponding to the curvature terms. These terms need to be handled delicately and correspond to the main difficulty of the proof.

We then define homotopy fiber products for curved \( A_\infty \) algebras in appendix C. The purpose of this section is to give a construction of a mapping cocylinder associated to an \( A_\infty \) homomorphism, which is the fiber product of \( A \xrightarrow{f} B \xleftarrow{id} B \). We construct this by proving a change-of-base theorem for filtered \( A_\infty \) bimodules.

The previous two sections are tied together in appendix D. From our understanding of fiber products, we are able to construct a mapping cocylinder object from each morphism of filtered \( A_\infty \) algebras; and by applying the curved homotopy transfer theorem we are able to show that every cocylinder object gives a morphism of filtered \( A_\infty \) algebras (see theorem D.1.3).

1.3 Notation

We introduce here several pieces of notation that are used throughout the paper.

- Let \( A_1, \ldots, A_k \) be vector spaces. The component-wise projection is the map
  \[
  \pi^{k_1|\cdots|k_i} : (A_1 \oplus \cdots \oplus A_i)^{\otimes k_1+\cdots+k_i} \to A_1^{\otimes k_1} \otimes \cdots \otimes A_i^{\otimes k_i}.
  \]

- Let \( A = A^- \oplus A^0 \oplus A^+ \) and let \( m^k : A^{\otimes k} \to A \) be a multilinear map. When we want to restrict the domain and codomain, we’ll write
  \[
  m_{\ell_0}^{\ell_1 \cdots \ell_k} : \bigotimes_{i=1}^k A^{\ell_i} \to A^{\ell_0}
  \]
  where \( \ell_i \) are chosen from the labels \{+, −, 0\}. In this notation, \( m_\ell \) denotes an element in \( A^\ell \).

- Let \( f : X^+ \to X^- \) be a map of topological spaces. The mapping cylinder is the topological space \( \text{cyl}(f) = (X^+ \times I) \cup_{\{(x^+) \times \{1\} \sim f(x^+)\}} X^- \). This comes with inclusion maps \( i_{X^\pm} : X^\pm \to \text{cyl}(f) \). Additionally, there is a pushforward map \( \pi : \text{cyl}(f) \to X^- \). We denote the maps on the cochains by
  \[
  \beta^\pm := (i_{X^\pm})^* : C^*(\text{cyl}(f)) \to C^*(X^\pm)
  \]
  \[
  \alpha^- := (\pi)^* : C^*(X^-) \to C^*(\text{cyl}(f)).
  \]
• Unless specifically noted, all Lagrangian submanifolds considered are spin, graded, and embedded.

• In appendices A to D we work up to sign.

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2 Running example: wall-crossing on $\mathbb{C}^2 \setminus \{ z_1 z_2 = 1 \}$

2.1 Wall-crossing

We review an example from [Aur07], computing wall-crossing for Chekanov and product tori in the complement of an anticanonical divisor. Consider the Lefschetz fibration with total space $\mathbb{C}^2 \setminus \{ z_1 z_2 = 1 \}$,

$$W : \mathbb{C}^2 \setminus \{ z_1 z_2 = 1 \} \to (\mathbb{C} \setminus \{ 1 \})$$

$$(z_1, z_2) \mapsto z_1 z_2$$

as drawn in Figure 1. We symplectically inflate this manifold at the removed divisor by taking the completion along the removed hypersurface. This resulting manifold has the same topology as $(\mathbb{C}^*)^2 \setminus \{ z_1 z_2 = 1 \}$. The new symplectic form can be chosen so that the projection $W : (\mathbb{C}^2 \setminus \{ xy = 1 \}) \to (\mathbb{C} \setminus \{ 1 \})$ remains a symplectic fibration. The regular fibers of this map have the topology of $\mathbb{C}^*$ and can be given an SYZ fibration

$$\text{val}_{W^{-1}(z)} : W^{-1}(z) \to \mathbb{R}$$

$$(z_1, z_2) \mapsto |z_1|^2 - |z_2|^2.$$

which is a restriction of the global Hamiltonian to a fiber. The Lefschetz fibration has a single degenerate fiber $z_1 z_2 = 0$. The base of the fibration can also be equipped with an SYZ fibration. We will take the fibration of the base given by loops $\gamma_r(\theta) := 1 + re^{2\pi i \theta}$. The symplectic parallel transport map given by the Lefschetz fibration preserves the isotopy class of SYZ fibers of $W^{-1}(z)$; as a result, one can build an SYZ fibration for the total space.
Figure 1: Symplectic fibration over \( W: (\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}) \to (\mathbb{C} \setminus \{1\}) \). Lagrangian tori are created by parallel transporting cycles in the fibers by loops in the base. Loops on the left of the critical value are of Chekanov type, while those on the right are product type.

\[
\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}
\]
by taking the circles \( \gamma_r^{-1}(s) \) and parallel transporting them along circles \( \gamma_r(\theta) \) of the second fibration to obtain Lagrangian tori

\[
L_{\gamma, |w|} = \{(z_1, z_2) \in \mathbb{C}^2 \setminus \{z_1 z_2 = 1\} \mid z_1 z_2 \in \gamma_r, |z_1|^2 - |z_2|^2 = \log |w|\}.
\]

The resulting SYZ fibration has one degenerate fiber which occurs when \( \log |w| = 0 \) and \( r \) approaches 1. The degenerate fiber \( L_{\gamma_1,1} \) is the Whitney sphere, an immersed Lagrangian sphere with a single double point. We may generalize this construction to Lagrangians \( L_{\gamma, |w|} \) for curves \( \gamma: S^1 \to \mathbb{C} \setminus \{1\} \) which wind around the removed point a single time. Such curves are divided into three types: the Chekanov type curves which additionally wind around the origin, the product type curves which do not, and those curves \( \gamma \) which contain the origin. If the curve \( \gamma \) contains the origin, we say the Lagrangian \( L_{\gamma, |w|} \) is on the wall between Chekanov and product type. By lifting these curves to Lagrangian submanifolds via parallel transport of cycles in the fibers, we obtain the Chekanov and product type Lagrangian tori in \( (\mathbb{C}^*) \setminus z_1 z_2 = 1 \). We will denote the Chekanov-type Lagrangian tori as \( L_{-\gamma, |w|}^+ \), while the product type tori will be decorated as \( L_{\gamma, |w|}^- \).

### 2.2 Flat charts on the moduli space

Wall-crossing for Lagrangian submanifolds is phenomenon which occurs when we try to parameterize the space of these Lagrangian submanifolds with coordinates. These coordinates should be constructed over the Novikov ring, although for the purposes of this exposition we will use complex coordinates and unitary local systems. We consider tuples \( (L_{\gamma, |w|}, b) \), where \( b \in H^1(L, i\mathbb{R}) \) gives us a unitary local system on \( L \) via deformation of the Floer cohomology following [Aur07, Lemma 4.1]. Such a pair is called a Lagrangian brane, although we will frequently refer to this data simply by the Lagrangian \( L \). The space of Hamiltonian isotopy classes of Chekanov (resp. product) Lagrangian branes comes with local coordinates from measuring the flux of an isotopy, which we now describe.
Let $(L_0, b_0)$ and $(L_1, b_1)$ be two Lagrangian submanifolds equipped with local systems and let $L_t$ be a Lagrangian isotopy between these two Lagrangians. Fix $c \in C_1(L_0)$. The Lagrangian isotopy gives a cylinder $c \times I \subset X$ with boundary $(c \times \{0\}) \sqcup (c \times \{1\}) \subset L_0 \sqcup L_1$. The flux of this isotopy along $c_0$ is the quantity

$$\text{Flux}_{L_t}(c) := -\left( \int_{c \times \{0\}} b_0 \right) + \left( \int_{c \times \{1\}} b_1 \right) + \left( \int_{c_0 \times I} \omega \right).$$

This defines a complex valued cohomological class $\text{Flux}_{L_t} \in H^1(L_0, \mathbb{C})$. After picking a base point for our moduli space and a basis for homology, the flux cohomology class gives us local coordinates on the moduli space of Lagrangians up to Hamiltonian isotopy.

In our example of $X = \mathbb{C}^2 \setminus \{z_1 z_2 = 0\}$, one choice of basis comes from compactifying $X$ to $\mathbb{C}^2$ and noting that $L_{\gamma, |w|}^\pm \subset (\mathbb{C}^2)$ now bound holomorphic disks whose boundary class pick out a basis for homology.

- On the Chekanov family, we call these two classes $c_w$ and $c_u$. The class $c_w \in H_1(L_{\gamma, |w|}^-)$ is the class of the circle in a fiber of the moment map which is parallel transported around to obtain the Lagrangian $L_{\gamma, |w|}^-$. When $|w| = 1$, this is the vanishing cycle of the Lefschetz fibration. The second class is obtained by compactifying the total space to $\mathbb{C}^2$. In $\mathbb{C}^2$, there is a family of Maslov-2 disks holomorphic disks with boundary on $L_{\gamma, |w|}^- \subset \mathbb{C}^2$. The homology class of the boundary of such a class is called $c_u$.

- For the product family, we call these two classes $c_r, c_s$. They are both obtained from considering $L_{\gamma, |w|}^+$ inside the compactification $\mathbb{C}^2$, where the Lagrangian torus bounds 2 families of Maslov-2 holomorphic disks. If $L_{\gamma, |w|}^+$ is the standard product torus $(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$, then the classes $c_r$ and $c_s$ correspond to the meridional and longitudinal classes of the product torus. We call the corresponding classes of disks $c_r, c_s \subset H_1(L_{\gamma, |w|}^+)$.

Once we have fixed these homology classes, we can construct coordinates on the space of Chekanov (resp. product) Lagrangian branes by measuring flux against a fixed Lagrangian. Fix the loop $\gamma_0 = 1 + e^{i\theta}$. This loop is neither of Chekanov or product type; however, it still makes sense to measure the flux against the Whitney sphere $L_{\gamma_0, 1}$ for both the Chekanov and product Lagrangians.

**Claim 2.2.1.** Let $L_t$ be a Lagrangian isotopy with $L_0 = L_{\gamma_0, 1}$ and $L_t$ of Chekanov (resp product) type for all $t \in (0, 1]$. Then the flux class $\text{Flux}_{L_t} \in H^1(L_1, \mathbb{C})$ depends only on the Lagrangian $L_1$.

The flux constructs local coordinates on the space of Chekanov (resp product) type tori.

**Definition 2.2.2.** For $(u, w) \in (\mathbb{C}^*)^2$, define the classes

$$[L_{u, w}]_{\text{Flux}} := \{(L_{\gamma, |w|}^-, b) : (*)\exp(\text{Flux}_{L_t}(c_u)) = u, \exp(\text{Flux}_{L_t}(c_w)) = w\}.$$
For \((r,s) \in (\mathbb{C}^*)^2\), define the class of Lagrangians

\[
[L_{r,s}]_{\text{Flux}} := \{(L_{r,s}^+, b) : (**), \exp(\text{Flux}_{L_t}(c_r)) = r, \exp(\text{Flux}_{L_t}(c_s)) = s\}.
\]

The condition (***) means that \(L_t\) is an isotopy of Chekanov type tori starting at \(L_{\gamma,|w|, 0}\) and ending at \((L_{\gamma,|w|, b}^-)\). The condition (***) means that \(L_t\) is an isotopy of product type tori starting at \(L_{\gamma,|w|, 0}\) and ending at \((L_{\gamma,|w|, b}^+ )\).

These classes described are subsets of the equivalence classes of Lagrangians under the relation of Hamiltonian isotopy.

**Claim 2.2.3.** The classes \([L_{u,w}^-]_{\text{Flux}}\) (resp. \([L_{r,s}^+]_{\text{Flux}}\)) are the equivalence classes of Chekanov (resp. product) tori under the equivalence relation of Hamiltonian isotopy through Chekanov (resp. product) tori. Furthermore, with the standard complex structure, no member of such an isotopy will bound a holomorphic disk in \((\mathbb{C}^*)^2 \setminus \{z_1 z_2 = 1\}\).

The classes \([L_{u,w}^-]_{\text{Flux}}\) and \([L_{r,s}^+]_{\text{Flux}}\) allow us to use the coordinates \((u,w)\) and \((r,s)\) to parameterize charts on the moduli space Lagrangian tori. We will frequently refer to a specific representative of each class as \(L_{u,w}^-\) or \(L_{r,s}^+\). Notice that each Lagrangian torus \(L_{\gamma,|w|}\) from fig. 1 is either of Chekanov type, product type, or lies on a “wall” between these two types of Lagrangian tori. If \(L_{\gamma,r}\) is not on the wall, then \(L_{\gamma,|w|}\) belongs to a distinct \([L^\pm]_{\text{Flux}}\) class.

### 2.3 Gluing charts together

Our goal is to loosen the equivalence to Hamiltonian isotopies which pass through the wall. Note that there exists \(\gamma, \gamma'\) so that \(L_{\gamma,|w|} \sim L_{\gamma',|w|}\) are Hamiltonian isotopic, but \(L_{\gamma,|w|}\) is of Chekanov type, while \(L_{\gamma',|w|}\) is of product type. The perspective that we take is that this corresponds to an overlap between the Chekanov and product charts. Our goal is to understand how the \((u,w)\) and \((r,s)\) coordinates glue these two different charts to give complex coordinates on the moduli space of Lagrangians up to Hamiltonian isotopy.

The monodromy of the fibration prevents us from simultaneously trivializing both \(H^1(L_{u,w}^-)\) and \(H^1(L_{r,s}^+)\) and so we cannot construct global coordinates in this way. A single global coordinate may be extracted in this case, as the monodromy fixes the vanishing cycle of the singularity, namely the class \(c_w \in H_1(L^-_{u,w}) = c_r - c_s \in H_1(L^+_{r,s})\). This gives a globally defined coordinate by measuring the flux of vanishing cycle.

Despite the lack of global trivializations, let us go forward and attempt to naively match up the coordinates \((u,w)\) and \((r,s)\) by identifying Hamiltonian isotopic Lagrangians on the regions where we can make the \((u,w)\) and \((r,s)\) charts overlap. Given a Chekanov torus \(L^-_{u,w}\), with \(|w| \geq 0\), one can obtain a product torus via Hamiltonian isotopy by “sliding” the curve \(\gamma\) over the singular value of 1. This allows us to identify the portion of the \((u,w)\) chart with the \((r,s)\) chart where \(|w| \geq 0\). To identify the homology classes, we note that \(\log |w| \geq 0\) means that the area of the \(c_r\) disk is larger than the area of the \(c_s\) disk. As we take a Hamiltonian isotopy of Lagrangian submanifolds, the family of smallest area disks
must persist through the isotopy; therefore the $c_s$ disks are matched to the $c_u$ disks in the compactification. This gives us the identification of homology classes:

\[ c_w = c_r - c_s \quad \text{and} \quad c_u = c_s \]

yielding the naive identification of Hamiltonian isotopy classes of Lagrangians:

\[ L_{u,w}^- \sim_{|w| > 0} L_{u,w,u}^+. \]  \hspace{1cm} (1)

Similarly, when $\log|w| \leq 0$ we can identify $L_{u,w}^-$ with $L_{r,s}^+$ with a Hamiltonian isotopy. However, in this setting the $c_s$ family of disks is largest, giving us the identification:

\[ c_w = c_r - c_s \quad \text{and} \quad c_u = c_r \]

yielding a different identification of the Chekanov and product charts:

\[ L_{u,w}^- \sim_{|w| < 0} L_{u,uw}^+. \]

When $w = 0$, the monotone Chekanov Lagrangians $L_{u,1}^-$ are not Hamiltonian isotopic to a product torus. So we cannot identify these with Lagrangians in the other chart. When one tries to extend the two naive identifications to the monotone Chekanov Lagrangians, we run into a mismatch and as a result we cannot holomorphically identify these two charts. The solution posed in [Aur07] is that the transformation between the coordinates $(u, w)$ and $(r, w)$ needs to be corrected due to the presence of Maslov index 0 disk bubbling. We aim to characterize these wall-crossing transformations in terms of the continuation maps and Lagrangian cobordisms.

### 3 Background

For background and notation related to $A_\infty$ algebras, see appendix A.

#### 3.1 Background: pearly model for open Gromov-Witten

The receptacle for the count of holomorphic disks with boundary on a compact Lagrangian $L$ is the pearly Floer homology of a Lagrangian. From here on, we assume that our Lagrangians are equipped with the choice of $\mathbb{Z}$-grading and spin structure. This algebra $CF^\bullet(L, h)$ is specified by a choice of an admissible Morse function $h : L \to \mathbb{R}$. The algebra structure is determined by taking a count of treed-disks, which are flow trees of the Morse function $h$ accessorized with insertions of holomorphic disks. This algebra has been constructed by several authors in different settings: a non-complete set of references is [FO97, CL06, LW14, CW15].

We take as an assumption that a version of this algebra has been constructed for the examples we consider. In particular, this assumes the existence of a regularization technique.
which produces for critical points $x_0; x_1, \ldots, x_k \in \text{Crit}(h)$ and class $\beta \in H_2(X, L)$ a moduli space of treed disks
\[ \mathcal{M}_\beta(L, x_0; x_1, \ldots, x_k) \]
with appropriate compactification $\overline{\mathcal{M}}_\beta(L, x_0; x_1, \ldots, x_k)$.
The codimension 1 components of the boundary of $\mathcal{M}_\beta(L, x_0; x_1, \ldots, x_k)$ are
\[ \bigsqcup_{\beta_1 + \beta_2 = \beta \atop j_1 + j_2 + j_3 = k} \overline{\mathcal{M}}_{\beta_1}(L, x_0; x_1, \ldots, x_{j_1}, y, x_{j_1+1}, \ldots, x_{j_1+j_2}) \times \overline{\mathcal{M}}_{\beta_2}(L, y; x_{j_1+1}, \ldots, x_{j_1+j_2}). \]

**Definition 3.1.1.** Let $L \subset X$ be an admissible Lagrangian brane equipped with admissible Morse function $h : L \to \mathbb{R}$. Define the chains of the Floer complex to be
\[ CF^\bullet(L, h) = \Lambda\langle (\text{Crit}(h)) \rangle, \]
the $\Lambda$-module generated on critical points of $h$. For each homology class $\beta \in H_2(X, L)$, we define the contribution of the class $\beta$ to the higher product homologically graded in $\beta$ by structure coefficients counting treed disks:
\[ m^k_\beta : CF^\bullet(L, h)^\otimes k \to CF^\bullet(L, h) \]
\[ \langle x_0, m^k_\beta(x_1, \ldots, x_k) \rangle := \left( \int_T dh \right) T^\omega(\beta) \cdot \# \overline{\mathcal{M}}_{\beta T}(L, x_0; x_1, \ldots, x_k). \]
We define the Fukaya-Floer higher products by taking a sum over all classes:
\[ m^k(x_1, \ldots, x_k) := \sum_{\beta \in H_2(X, L)} m^k_\beta(x_1, \ldots, x_k). \]
We call the pair $(CF^\bullet(L, h), m^k)$ the pearly Floer algebra of $L$.

The grading can be recovered by introducing a formal variable which records the Maslov index of the classes $\beta$, or by reducing to a $\mathbb{Z}_2$ grading. For the examples that we consider, the only disks which occur are of Maslov index 0 and so these operations remain graded. This is a deformation of the $A_\infty$ Morse algebra of $L$ (which is recovered by looking at the 0-valuation terms of the algebra).

**Theorem 3.1.2** ([FO97, CL06, LW14, CW15]). The pearly Floer algebra is a filtered $A_\infty$ algebra.

Due to the technical difficulties of working with pearly Floer algebras, we stress that for the examples we consider in sections 6.4 and 6.4 there is only a single holomorphic disk with boundary on $L$ and so one can construct the algebra $CF^\bullet(L, h)$ without having to appeal to abstract perturbation theory or stabilizing divisors. The more general results of section 5.2 should be taken in the settings where theorem 3.1.2 holds.
3.2 Background: Lagrangian cobordisms

We now look at the pearly algebra for Lagrangian cobordisms. This algebra for cobordisms is similar to that for compact Lagrangians, with an admissibility condition on the Morse function to ensure compactness of the moduli space of disks.

Notation 3.2.1. We will interchangeably use $X \times T^*\mathbb{R} = X \times \mathbb{C}$ for the cobordism parameter space. We identify the imaginary direction of $\mathbb{C}$ with the cotangent fiber direction of $T^*\mathbb{R}$.

Definition 3.2.2 ([Arn80]). Let $L^-, L^+$ be Lagrangian submanifolds of $X$. A Lagrangian cobordism between $L^-$ and $L^+$ is a Lagrangian $K \subset X \times T^*\mathbb{R}$ which satisfies the following conditions:

- Fibered over ends: There exist constants $c^+, c^- \in \mathbb{R}$, as well as constants $t^- < t^+ \in \mathbb{R}$ such that
  
  $$K \cap \{(x, z) : \text{Re}(z) \geq t^+\} = L^+ \times \{(t + ic^+) : t \geq t^+\}$$
  
  $$K \cap \{(x, z) : \text{Re}(z) \leq t^-\} = L^- \times \{(t + ic^-) : t \leq t^-\}$$

- Compactness: The projection $\text{Im}_z : L_K \rightarrow i\mathbb{R} \subset \mathbb{C}$ is bounded.

We denote such a cobordism $K : L^- \leadsto L^+$.

There is a generalization of this definition to cobordisms with multiple ends, which requires that the Lagrangian cobordism fibers over rays of fixed argument outside of a compact subset of $\mathbb{C}$. A specialization of the result of [BC14] proves that the equivalence relation of cobordance descends to the Fukaya category.

Theorem 3.2.3. Suppose that $K : L^- \leadsto L^+$ is a monotone Lagrangian cobordism. Then $L^-$ and $L^+$ are equivalent objects of the Fukaya category.

In [BC14], it is stated that this result is expected to hold in a more general setting where the Lagrangians are equipped with the additional data of orientations, local systems, bounding cochains, etc. We will look to understand this result in the context of pearly algebras of cobordant Lagrangians, where the Lagrangian cobordism may be non-monotone but unobstructable by bounding cochain.

Definition 3.2.4. Let $(L^-, h^-)$ and $(L^+, h^+)$ be two geometric Lagrangian branes. An admissible Lagrangian cobordism brane between these is the data of $(K, h)$ where

- $K$ is a Lagrangian cobordism between $L^-$ and $L^+$ and;
- $h : K \rightarrow \mathbb{R}$ is a Morse function and;
- $K$ admits a relative spin structure and grading compatible with those of the ends.

The Morse function $h$ is required to satisfy the following admissibility and compatibility conditions:
Figure 2: The profile of the Morse function for a cobordism. Bottlenecks are inserted on the ends of the cobordism.

- **Admissibility:** Let \( L^- \times (-\infty, t^-] \) be a chart for the left end of the cobordism such that
  \[
  L^- \times (-\infty, t^-] \to X \times T^*\mathbb{R} \\
  (x, t) \mapsto (\iota(x), t + ic^-)
  \]
  where \( \iota : L^- \to X \) is the explicit inclusion of \( L^- \) as a Lagrangian submanifold on \( X \). We require the existence of a bump function \( \rho_\epsilon \) and constants \( A^- \) and \( C^- \) so that we may rewrite the Morse function \( h \) along the ends of the cobordism as:
  \[
  h(x, t)|_{L^- \times (-\infty, t^-]} = \rho_{B_\epsilon(t^- - \epsilon)} h^-(x) + A^- \cdot (t - (t^- - \epsilon))^2 + C^-.
  \]

- **Furthermore,** we require the constant \( A^- \) to be chosen large enough so that on the chart \( L^- \times (-\infty, t^-) \) we have a correspondence
  \[
  \text{Crit}(h) \cap (L^- \times (-\infty, t^-]) = (\text{Crit}(h^-) \times \{(t^- - \epsilon) + ic^-\}).
  \]

We place similar requirements on the positive ends of the cobordism.

One way to construct an admissible Morse function for a Lagrangian cobordism is to first construct a Morse-Bott function \( \tilde{h} \) for \( K \) which is only dependent on the real coordinate. The Morse-Bott function \( \tilde{h} \) has a critical submanifold at each boundary of the cobordism. We call these critical submanifolds the bottlenecks at the boundaries of the cobordism (see fig. 2). One then Morsifies \( \tilde{h} \) by using \( h^\pm \) as a perturbation at the critical submanifolds. With this setup, flow lines with output on a critical point in the bottleneck must also have inputs in the bottleneck. Additionally, every Morse flow line which has an output on a critical point of \( h \) has real part bounded in the region \( (t^- - \epsilon) < \text{Re}(\gamma) < (t^+ + \epsilon) \).

In Morse theory, we use these Morse functions for the following desirable properties:
• Even though $K$ is non-compact, the moduli space of Morse flow lines between two critical points admits a compactification by broken flow lines and,

• The Morse flow lines $\gamma$ of $CM^\bullet(K, h)$ for which $\tilde{h}(\gamma) = t^\pm$ are exactly the Morse flow lines of $CM(L^\pm, h^\pm)$. This allows us to construct projections of the Morse cochain complexes to the ends:

\[
\begin{array}{ccc}
CM^\bullet(K, h) & \xrightarrow{\beta^-} & CM^\bullet(L^-, h^-) \\
& & \xleftarrow{\beta^+} & CM^\bullet(L^+, h^+).
\end{array}
\]

We expect that a similar kind of result should carry over to the setting of treed disks and open Gromov-Witten chain complexes. In the setting where there is a regularizing split almost complex structure, we have the analogous result for holomorphic disks:

**Claim 3.2.5.** Let $X \times \mathbb{C}$ be equipped with a split holomorphic structure $J \times j$.

- Every pseudoholomorphic treed disk $u : T \to K$ with ends limiting to critical points of $H$ has boundary contained inside
  \[\partial u \subset K \cap (X \times [t^-, t^+ + \epsilon]).\]

- Every disk with the property that $\tilde{h}(u) = t^\pm \pm \epsilon$ corresponds to a pseudoholomorphic disk of $X$ with boundary on $L^\pm$.

The proof of the claim uses the open-mapping principle for holomorphic disks. Near the ends of the cobordism this means that the image of holomorphic disks need to be constant in the cobordism factor (as they have boundary on a real line,) and so a flow disk can only be non-compact if the Morse flow components are non-compact. However, the Morse flow components are bounded by the bottlenecks chosen for the Morse function.

Due to the need to use abstract perturbation techniques to construct a Floer theory, we cannot assume that $X \times \mathbb{C}$ is equipped with a split holomorphic structure and split perturbation. We assume that there exists a regularization scheme for the abstract perturbation theory used to define the Floer theory so that the following matching of moduli spaces holds:

**Assumption 3.2.6.** Let $(x_0; x_1, \ldots, x_k)$ be critical points of $h$. Suppose that $x_0$ is a critical point of $h^-$. Fix a class $\beta \in H_2(X, L^-)$ giving us a class $i_*(\beta) \in H_2(X \times \mathbb{C}, K)$. There exists a regularization of the moduli space of treed disks so that there is an identification

\[
\mathcal{M}_{i_*(\beta)}(K, x_0; x_1, \ldots, x_k) = \left\{ \begin{array}{ll}
\mathcal{M}_\beta(L^-, x_0; x_1, \ldots, x_k) & \text{if all } x_j \in \text{Crit}(h^-) \\
\emptyset & \text{otherwise.}
\end{array} \right.
\]
This assumption is needed to obtain the following maps of filtered $A_\infty$ algebra.

**Claim 3.2.7.** Given assumption 3.2.6, the projections of vector spaces

$$
\begin{align*}
CF^\bullet(K, h) \\
\beta^- \\
\beta^+ \\
CF^\bullet(L^-, h^-) \quad CF^\bullet(L^+, h^+)
\end{align*}
$$

are $A_\infty$ homomorphisms with $(\beta^\pm)^k = 0$ for all $k \neq 1$.

From here on out we drop the Morse function and denote the pearly complex by $CF^\bullet(L)$.

## 4 Toy model: continuation maps from cylindrical cobordisms

We now give an extension of the results of [BC14] to the non-monotone setting. While we now allow our Lagrangian cobordisms to possibly bound holomorphic disks, we require that the cobordism’s topology be cylindrical. We discuss Lagrangian suspension cobordisms and their holomorphic disks in section 4.1. In section 4.2 we apply the machinery of theorem D.1.3 to the suspension of a Hamiltonian isotopy to produce a continuation map.

### 4.1 Characterizing disks on suspension cobordisms

**Definition 4.1.1 (Suspension of a Hamiltonian Isotopy).** Let $H_t : X \times \mathbb{R} \rightarrow \mathbb{R}$ be a Hamiltonian which is constant on the intervals $t \leq t^-$ and $t \geq t^+$. Let $\theta_t$ be the flow of $H_t$. The suspension of $H_t$ is the Lagrangian cobordism $K_{H_t}$ given by the embedding

$$
L \times \mathbb{R} \hookrightarrow X \times \mathbb{C} \\
(x, t) \mapsto (\theta_t(x), t + iH_t(x)).
$$

It is useful to equip $X \times T^*\mathbb{R}$ with a modified almost complex structure when studying the suspension of Hamiltonian isotopy which relates holomorphic disks with boundary on $K_{H_t}$ to holomorphic disks with boundary on $\theta_t(L)$.

**Definition 4.1.2.** Let $H_t : X \times \mathbb{R} \rightarrow \mathbb{R}$ be a time-dependent Hamiltonian. Let $J_t$ be a family of almost complex structures for $X$. We call

$$
J_{H_t} := \begin{pmatrix}
J_t \\
\frac{dH_t}{dt} + iH_t J_t \\
0
\end{pmatrix} : T(X \times T^*\mathbb{R}) \rightarrow T(X \times T^*\mathbb{R})
$$

the $H_t$-adjusted almost complex structure. This is an almost complex structure compatible with the symplectic form $\omega_X + \omega_{T^*\mathbb{R}}$. 

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Note that this matches the standard split complex structure over the ends of the suspension cobordism where $H_t$ is constant.

**Lemma 4.1.3 (Disk Lifting).** Whenever 

$$u : (D^2, \partial D^2) \to (X, \theta_{t_0}(L))$$

is a regular $J_{t_0}$ holomorphic disk; the lifting 

$$\tilde{u} : (D^2, \partial D^2) \to (X \times T^*\mathbb{R}, K_{H_t})$$

$$z \mapsto (u(z), t_0 + i \cdot H_t \circ u(z))$$

is a regular $J_{H_t}$ disk.

**Proof.** To simplify notation, let $L_{t_0} = \theta_{t_0}(L)$. To show that $\tilde{u}$ is a $J_{H_t}$ pseudoholomorphic disk is a computation of the Cauchy-Riemann equation in local coordinates.

We summarize an argument here for why $\tilde{u}$ is regular. Regularity of a pseudoholomorphic disk is determined by surjectivity of a linearized Fredholm operator. In this instance, the operator that we need to show is surjective is 

$$D_{\bar{\partial}J_{H_t},\tilde{u}} : (TB_{(D^2, X \times T^*\mathbb{R}, K_{H_t})}) \tilde{u} \to TE_{(D^2, X \times T^*\mathbb{R}, K_{H_t})}.$$

Here $B_{D^2, X, L}$ is the Banach manifold of $W^{1,p}$ disks in $X$ with boundary on $L$. $E_{D^2, X, L}$ is the Banach bundle over $B_{D^2, X, L}$ whose fibers are $L^p(D^2, \Omega^0_{D^2} \otimes u^*TX)$. Because $u : (D^2, \partial D^2) \to (X, L_{t_0})$ is $J_{t_0}$ regular, the linearized $\bar{\partial}J_{t_0}$ equation 

$$D_{\bar{\partial}J_{t_0},u} : (TB_{(D^2, X, L_{t_0})}) u \to TE_{D^2, X, L_{t_0}}$$

is known to be surjective.

The Banach manifold $B_{(D^2, X \times T^*\mathbb{R}, K_{H_t})}$ does not decompose along the factors of $X \times T^*\mathbb{R}$ due to the nonconstant boundary condition $K_{H_t}$. However, since $\tilde{u}$ is transverse to the $\mathbb{R}$ direction of $T^*\mathbb{R}$, at $\tilde{u}$ we can decompose the tangent bundle of $B_{(D^2, X \times T^*\mathbb{R}, K_{H_t})}$ at $\tilde{u}$ with a fiber direction and $T^*\mathbb{R}$ direction.

$$T_{\tilde{u}}B_{(D^2, X \times T^*\mathbb{R}, K_{H_t})} = T_uB_{(D^2, X, L_{t_0})} \oplus T_{t_0}B_{(D^2, T^*\mathbb{R}, \mathbb{R})}$$

We additionally have a splitting of the bundle 

$$T_{\tilde{u}}E_{(D^2, X \times T^*\mathbb{R}, K_{H_t})} = T_uE_{(D^2, X, L_{t_0})} \oplus T_{t_0}E_{(D^2, T^*\mathbb{R}, \mathbb{R})}$$

In this decomposition, we can rewrite the linearized $\bar{\partial}J_t$ equation as 

$$D_{\bar{\partial}J_{H_t},\tilde{u}} = \begin{pmatrix} D_{\bar{\partial}J_{t_0},u} & 0 \\ A & D_{\bar{\partial}J_{H_t}(u)} \end{pmatrix},$$

where $A : T_uB \to T_{t_0}E$ is some map. To show surjectivity, it suffices to show that $D_{\bar{\partial}J_{H_t}(u)}$ surjects. This is the linearized $\bar{\partial}$ equation for a constant almost-holomorphic disk being mapped to $\mathbb{C}$, which is always regular (and therefore surjects.)
Somewhat unexpectedly, every holomorphic disk with boundary on \( K_t \) can be related to holomorphic disks on \( L_t \) with this complex structure.

**Lemma 4.1.4 (Disk Falling).** Let \( \tilde{u} : (D^2, \partial D^2) \to (X \times T^* \mathbb{R}, K_{H_t}) \) be any \( J_{H_t} \)-holomorphic disk. There exists a (possibly non-regular) \( J_{t_0} \)-holomorphic disk \( u : (D^2, \partial D^2) \to (X, L_{t_0}) \) so that \( \tilde{u}(z) = (u(z), t_0 + i \cdot H_t \circ u(z)) \).

**Proof.** Let \( \tilde{u}(z) = (u^X(z), u^p(z) + i u^q(z)) \) be the coordinate functions of this disk. Consider the map

\[
v : D^2 \to \mathbb{C} \\
v(z) = (u^p(z) + i \cdot (u^q(z) - H_{u^p(z)}(u^X(z))))
\]

By explicit computation \( v \) is an almost complex map. Observe that along the boundary of the disk we additionally know that \( H_{u^p}(u^X) |_{\partial u} = u^q |_{\partial u} \). Therefore, \( \text{Im}(v(z)) |_{\partial D^2} \) is constant. By the open mapping principle \( v \) must be the constant map. Therefore \( u^p(z) \) is constant, proving the lemma.

In this setting, there is no reason that the fallen disk \( u \) need be regular. However, this lemma is still useful for proving the non-existence of holomorphic disks with boundary on \( K_{H_t} \).

### 4.2 Continuation maps for Hamiltonian isotopies

In a similar vein to the equivalences constructed in [BC14], the suspension Lagrangian cobordism gives a continuation map for the pearly algebra of \( L_t \).

**Proposition 4.2.1.** Let \( L^-, L^+ \subset X \) be two Lagrangian submanifolds, with Morse functions \( h^\pm : L^\pm \to \mathbb{R} \). Suppose that \( K \subset X \times T^* \mathbb{R} \) is a Lagrangian cobordism with ends on \( L^-, L^+ \). Furthermore, suppose that topologically \( K \cong L^- \times \mathbb{R} \). Then there exists a filtered \( A_\infty \) homomorphism \( \Theta_K : CF^*(L^-; h^-) \to CF(L^+; h^+) \).

**Proof.** We pick a particular Morse function \( h \) for \( K \). Take a preliminary function

\[
\tilde{h}(t) : \mathbb{R} \to \mathbb{R}
\]

which has maximums at \( t^- \) and \( t^+ + \epsilon \) and a minimum at \( t^+ \). See fig. After taking a Hamiltonian perturbation on \( X \times (T^*[-t^-, t^+]) \) the function \( \tilde{h}(t) \) is a Morse-Bott function on \( K \). We Morsify this function by taking perturbations based on the functions \( h^\pm \). Let \( \rho^\pm : \mathbb{R} \to \mathbb{R} \) be a bump function which is one on \( B_2(t^\pm) \). Consider the Morse function

\[
h(x, t) = (\rho^+(t) \cdot h^+(x)) + (\rho^-(t) \cdot h^-(x)) + A \cdot \tilde{h}(t)
\]

for a choice of \( A \) large enough so that the only critical points which occur are in the fibers \( t \in \{t^-, t^+, t^+ + \epsilon\} \). By construction, we have an identification

\[
CF^*(K, h) = CF^*(L^-, h^-) \oplus CF^*(L^+, h^+)[1] \oplus CF^*(K^+, h^+)
\]
as graded vector spaces. From assumption 3.2.6, the pearly differential on \( K_H \) decomposes as

\[
m^1_K = \begin{pmatrix}
m_+ & 0 & 0 \\
m_0 & m_0^0 & m_0^+ \\
0 & 0 & m_+^+
\end{pmatrix}.
\]

where \( m_{\pm}^\pm \) matches \( m_{L_{\pm}}^\pm \). See section 1.3 for notation. We additionally have the projection \( A_\infty \) homomorphisms \( \beta^\pm : CF^*(K, h) \to CF^*(L^\pm, h^\pm) \). The map \( m_0^+ \) is a deformation of the Morse differential on \( K \) by terms of non-zero valuation. Since the zero valuation portion \( (m_0^+)_{-0} : CM^*(L^+, h^+) \to CM^*(L^+, h^+)[1] \) is the identity, the map \( m_0^+ \) is an isomorphism. Therefore \( CF^*(L^-, h^-) \leftarrow CF^*(K, H) \to CF^*(L^+, h^+) \) is a filtered \( A_\infty \) mapping cocylinder (definition D.1.2). We obtain a map \( \alpha : CF^*(L^-, h^-) \to CF^*(K, h) \) and a pushforward-pullback map (theorem D.1.3)

\[
\Theta_K := \beta^+ \circ \alpha : CF^*(L^-, h^-) \to CF(L^+, h^+).
\]

**Corollary 4.2.2.** Let \( H_t \) be a Hamiltonian. Let \( L^-, L^+ \) be the ends of the suspension cobordism \( K_{H_t} \). There exists a pullback-pushforward map associated to this mapping cocylinder

\[
\Theta_{H_t} : CF^*(L^-, h^-) \to CF^*(L^+, h^+)
\]

called the continuation map of \( H_t \).

**Proof.** Follows from the existence of a suspension cobordism associated to a Hamiltonian isotopy.

**Corollary 4.2.3.** If \( CF^*(L^-) \) is unobstructed by bounding cochain \( b^- \), then \( L^+ \) is unobstructed by the bounding cochain \( b^+ = (\Theta_{H_t})_* b^- \). Furthermore, the continuation map can be made a map of uncurved \( A_\infty \) algebras \( (\Theta_{H_t})_* : CF^*_{b^-}(L^-) \to CF^*_{b^+}(L^+) \).

**Proof.** Follows from claim A.4.10.

## 5 Cobordisms and mutations

In this section, we look at cobordisms arising from mutation configurations. We prove that in certain scenarios such a cobordism gives an \( A_\infty \) mapping cocylinder between the Floer homologies of its ends. In section 5.1, we introduce a local model for mutation in \( \mathbb{C}^2 \setminus \{z_1 z_2 = 1\} \). This local description is used to construct a holomorphic disk with boundary on the Lagrangian cobordism. We then use this disk to build a continuation map from a Lagrangian mutation cobordism in section 5.2.
5.1 Review of the Haug cobordism

This is a review of [Hau15]'s mutation cobordism between the Chekanov and product torus. We assemble this cobordism from pieces built in Lefschetz fibrations. The fibration we consider is \( W = z_1 z_2 : \mathbb{C}^2 \to \mathbb{C} \), which has a single singular fiber at the origin with a single node.

We first give a description of Lagrangian surgery in Lefschetz fibrations. Consider the paths

\[ \ell_{\uparrow/\downarrow} : [0, R] \to \mathbb{C} \quad r \mapsto \pm i \cdot r^2 \]

in the base of the Lefschetz fibration with ends on the critical value of the fibration. We take lifts of these paths to give Lagrangian thimbles \( L_{\uparrow}, L_{\downarrow} \subset \mathbb{C}^2 \). These Lagrangians have the topology of \( D^2 \) and we will parameterize them with coordinates as:

\[ L_{\uparrow} = \{(x + iy, y + ix) : x^2 + y^2 \leq R^2\} \quad L_{\downarrow} = \{(x + iy, -y - ix) : x^2 + y^2 \leq R^2\}. \]

We now recall the construction of Lagrangian surgery and the associated suspension cobordism from [BC13]. We pick a surgery profile curve, which is a map \( (a(t) + ib(t)) : [-R, R] \to \mathbb{C} \), as drawn in fig. 3. We will use a parameterization of the curve so that the neck region of the curve is drawn in the interval \([-c, c]\). This neck provides a local model for the Lagrangian surgery. To insert this neck at the transverse intersection point of two Lagrangian submanifolds, we take a complex chart around the intersection point which identifies these Lagrangian submanifolds with the totally real and imaginary subspaces of \( \mathbb{C}^n \) and interpolate between these two subspaces using the surgery profile curve. For the example we are concerned with, the surgery \( L_{\uparrow} \# L_{\downarrow} \) is explicitly described in coordinates by

\[ \{((a(t) + ib(t))(\hat{x} + i\hat{y}), (a(t) + ib(t))(\hat{y} + i\hat{x})) : \hat{x}^2 + \hat{y}^2 = 1\}. \]

With this construction, we take special care to the order of the summands in the connect sum. Our convention is that the end of the neck which corresponds to \( t \in [-R, -c] \) corresponds to the first summand of the Lagrangian connect sum. The neck width is the area...
bounded by the curve \( a(t) + ib(t) \) and the axes. The neck width measures the amount of flux swept out as one takes an isotopy from \( L_\uparrow \# L_\downarrow \) to \( L_\uparrow \cup L_\downarrow \). The projection of this cobordism under \( W \) is

\[
W(L_\uparrow \# L_\downarrow) = i(a(t) + ib(t))^2(x^2 + y^2) = (a(t) + ib(t))^2
\]

whose image under \( W \) is a curve in the complex plane:

\[
\ell_\uparrow \# \ell_\downarrow : [-R, R] \rightarrow \mathbb{C} \quad t \mapsto (2a(t)b(t) + i(a(t)^2 - b(t)^2))
\]

Notice that when \( t \in [-R, -c] \) the coordinate \( b(t) = 0 \) and the curve matches the positive imaginary axis. Similarly, when \( t \in [c, R] \) we have that \( a(t) = 0 \) and the curve matches the negative imaginary axis. The real component of \( \ell_\uparrow \# \ell_\downarrow \) is always non-negative. See fig. 5 for diagrams of the pieces described above. We can similarly construct a profile for \( L_\downarrow \# L_\uparrow \) and corresponding path \( \ell_\downarrow \# \ell_\uparrow \). Let \( \text{width}_{L_\downarrow \# L_\uparrow} \) and \( \text{width}_{L_\uparrow \# L_\downarrow} \) be the widths of these two surgeries.

Associated to a Lagrangian surgery there is a trace cobordism of the surgery (\cite{BC}), giving us the Lagrangian cobordisms \( K_\uparrow : L_\uparrow \# L_\downarrow \rightsquigarrow L_\downarrow \sqcup L_\uparrow \) and \( K_\downarrow : L_\downarrow \sqcup L_\uparrow \rightsquigarrow L_\uparrow \# L_\downarrow \). There is a comparison between the neck width of the cobordism and the area of the projection of the cobordism to \( \mathbb{C} \), which is that the area of the cobordism is at least the neck-width. By concatenating these cobordisms together, we obtain a Lagrangian cobordism \( K_\uparrow \circ K_\downarrow : L_\downarrow \# L_\downarrow \rightsquigarrow L_\downarrow \# L_\uparrow \). The projection of \( K_\uparrow \circ K_\downarrow \) to \( \mathbb{C} \) is the eye-shaped Lagrangian drawn in fig. 4. When constructing the Lagrangian in this fashion, the sum of the widths \( \text{width}_{L_\downarrow \# L_\uparrow} + \text{width}_{L_\uparrow \# L_\downarrow} \) is bounded by the area of the eye-portion of the Lagrangian.

**Remark 5.1.1.** Another way of stating this is that the shadow of the cobordism \( K_\uparrow \circ K_\downarrow \) as defined in \cite{CS} is an upper bound for the widths of the two surgeries.

We now wish to make comparisons between the surgery of the thimbles \( \ell^\pm \), Chekanov and product tori, and the Whitney sphere. In the base of the Lefschetz fibration, one can construct the Whitney sphere by taking a path \( \ell_c : [-S, S] \rightarrow \mathbb{C} \) from the negative end of \( \ell_\downarrow \) to the positive end of \( \ell_\uparrow \) in a clockwise fashion so that the concatenation \( \ell_\downarrow \cdot \ell_c \cdot \ell_\uparrow \) is a matching path from the single critical point of the fibration to itself. Let \( L_c : [-S, S] \times S^1 \rightarrow \mathbb{C} \) be the corresponding parallel transport of the vanishing cycle. The union \( L_c \sqcup L_\downarrow \sqcup L_\uparrow \) is the Whitney sphere \( L_{S^2} \). For the simplicity of exposition, we will assume that \( L_{S^2} = L_{\gamma_0, 1} \), where \( \gamma_0 = 1 + e^{i\theta} \). This means that measuring flux against \( L_{S^2} \) will produce coordinates matching those constructed in section 2.2. By attaching \( L_c \) to \( L_\downarrow \# L_\uparrow \), we will obtain the monotone Chekanov torus. The Chekanov tori we can build we will denote \( L_{u,1} \) for an appropriate
choice of $u$. By attaching $L_c$ to $L_\uparrow \# L_\downarrow$, we obtain the monotone product torus, which will be denoted $L_{\uparrow, z}^+$ for some choice of $z$. See fig. 5 for a picture of these Lagrangian submanifolds.

Ideally at this point we would glue the cobordism $L_c \times \mathbb{R}$ to the concatenation $K_r \circ K_l$ to obtain a cobordism between the Chekanov and product torus. However the Lagrangian $L_c \times \mathbb{R}$ does not glue to the cobordism $K_r \circ K_l$, as the boundary of $K_r \circ K_l$ corresponding to the ends of the thimbles $L_{\uparrow, \downarrow}$ move around the parameter space of the cobordism along curves dictated by the gluing profile. One should modify $L_c \times \mathbb{R}$ by a Hamiltonian isotopy so these pieces overlap and we can glue together.

To achieve this, we need a better description of the construction of the Lagrangian surgery trace cobordism. The surgery trace cobordism is obtained by performing the surgery in one dimension higher and stretching the neck of the surgery (see section 6.1 of [BC13]). We will build our cobordism between Chekanov and product tori by performing surgery on a immersed Lagrangian cobordism. We now describe the pieces drawn in fig. 6 which are assembled to produce this Lagrangian cobordism.

Take paths $\gamma_{\uparrow, \downarrow}(t) : \mathbb{R} \to \mathbb{C}$ as drawn in fig. 6 and consider the Lagrangian cobordisms $L_\downarrow \times \gamma_\downarrow$ and $L_\uparrow \times \gamma_\uparrow$. Choose a parameterization of these paths so that $\gamma_{\uparrow, \downarrow}(t) = t + f_{\uparrow, \downarrow}(t)$. By taking a translation of these paths, we assume that $f_{\uparrow, \downarrow}(0) = 0$ and $f_{\uparrow, \downarrow}(t_0) = 0$. We will now glue to $(L_\downarrow \times \gamma_\downarrow) \sqcup (L_\uparrow \times \gamma_\uparrow)$ a Lagrangian which is topologically $L_c \times \mathbb{R}$. This cobordism will be the suspension of a time dependent Hamiltonian isotopy of $L_c$. We pick a Hamiltonian $H_t : L_c \times \mathbb{R} \to \mathbb{R}$ which satisfies the following properties:

- The value of $H_t$ on $L_c$ factors through the $s$-component parameterizing $L_c = [-S, S] \times S^1$ and is increasing in $s$.

- There exists a parameter $\epsilon$ so that the restriction of $H_t$ on $L_c$ is $H_t(L_c|_{s \in [-S, -S + \epsilon)}) = f_{\downarrow}(t)$ and $H_t(L_c|_{s \in [S, S - \epsilon)}) = f_{\uparrow}(t)$

We define the cobordism $K_c$ to be the suspension of this Hamiltonian isotopy. By design, the Hamiltonian flow $X_{H_t}$ is constant on the region $[\pm S, \pm S \mp \epsilon] \times S^1$ of $L_c$. Therefore the portion of $K_c$ corresponding to the $[\pm S, \pm S \mp \epsilon] \times S^1 \times \mathbb{R}$ is exactly $L_c|_{s \in [\pm S, \pm S \mp \epsilon]} \times \gamma_{\uparrow, \downarrow}$. See figs. 5 and 6.
Claim 5.1.2. The union $K_c \cup (L_\uparrow \times \gamma_\uparrow) \cup (L_\downarrow \times \gamma_\downarrow)$ is a smooth Lagrangian cobordism with 2 transverse self-intersections.

We will now make some computations which will help us determine the Hamiltonian isotopy class of the ends of the cobordism. Consider the Lagrangian cylinder $\theta^f_{H_t}(L_c)$. Since the Hamiltonian $H_t$ factors through the projection of the Lefschetz fibration, this Lagrangian can again be described by a matching path, drawn in fig. 5. $L_c$ and $\theta^f_{H_t}(L_c)$ share the same boundary components, so we can measure the amount of flux traced out between these two Lagrangians relative to their boundary.

Claim 5.1.3. Let $\gamma_c \subset L_c$ be a path with $W(\gamma_c) = \ell_c$. Let $S(x, y)$ be the strip parameterized by $\theta^\theta_{H_t}\gamma(x)$. The symplectic area of this strip is

$$ \int_S \omega = \int_0^\ell_c (f_\uparrow - f_\downarrow) dt. $$

Here is a quick heuristic for working out the flux swept out by this isotopy. Since $H_t$ increases along the path $\ell_c$ and since $\text{grad} \, H_t$ is perpendicular to the fiber of the Lefschetz fibration, the direction of Hamiltonian flow in the base of the fibration can be determined by taking $J_c \text{grad} \, H_t$. Based on our choices, this flow proceeds in the positive radial direction on $\mathbb{C}^2$, which is reflected in fig. 5.

We now resolve the Lagrangian cobordism $K_c \cup (L_\uparrow \times \gamma_\uparrow) \cup (L_\downarrow \times \gamma_\downarrow)$ at its two immersed points to obtain a preliminary Lagrangian cobordism $K^{\text{pre}}$. We have choices of neck-sizes at each of the two immersed points. With the construction given, the sizes of necks we choose will have width bounded above by

$$ \text{width}_{L_\downarrow \# L_\uparrow} + \text{width}_{L_\uparrow \# L_\downarrow} < \int_S \omega. $$

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We then apply the same cutting and stretching as in [BC13, Lemma 6.1.1] at each of the surgery regions to obtain cylindrical surgery necks. The conclusion of this construction is a cobordism $K$ between $L^- := (L_{\uparrow} \# L_{\downarrow}) \cup L_c$ and $L^+ := (L_{\downarrow} \# L_{\uparrow}) \cup \partial H_{\uparrow}(L_c)$. These are product and Chekanov tori respectively. Then the width of the surgery and flux of the Hamiltonian isotopy relative to boundary determine the coordinates of the Lagrangians $L^\pm$.

**Claim 5.1.4.** Let

$$u_- = \exp(-\text{width}_{L_{\downarrow} \# L_{\uparrow}}) \quad u_+ = \exp(\text{width}_{L_{\uparrow} \# L_{\downarrow}}) \quad u_\Delta = \exp \left( \int_S \omega \right).$$

The ends of the cobordism constructed above correspond to the following Chekanov and product tori:

$$L^- = L_{u_-}^- \quad L^+ = L_{u_+ \# u_\Delta}^+.$$ 

We call the cobordism constructed the *mutation cobordism*, $K_{\mu_D}$. Note that the bound on the width of the necks means that $u_- \neq \frac{u_+}{u_\Delta}$.

**Remark 5.1.5.** We will later look at this discrepancy (which is measured by the area of the red circle drawn in fig. 7) which we call $z_{\text{ex}} := u_\Delta u_- / u_+$. In the limit where the necks are made as large as possible, $z_{\text{ex}} \to 1$.

**Remark 5.1.6.** One can also perform these computations where $L^-$ and $L^+$ are equipped with unitary local systems restricted from a local system on $K$. Note that not every local system on $L^\pm$ comes as a restriction of one on $K$, as the holonomy along the loop associated to the vanishing cycle of the Lefschetz fibration must be trivial.

### 5.2 General mutations

In this section we look at a generalization of the above construction.

**Theorem 5.2.1** ([Hau15]). Suppose that $D^k$ is an isotropic disk with boundary contained in $L$ and cleanly intersecting $L$ along the boundary. There exists an immersed Lagrangian $\alpha_D(L) \subset X$ called the Lagrangian antisurgery of $L$ along $D$, which satisfies the following properties:

- $\alpha_D(L)$ is topologically obtained from $L$ by performing surgery along $\partial D^k$
- $\alpha_D(L)$ agrees with $L$ outside of a small neighborhood of $D^k$
- If $L$ was embedded and disjoint from the interior of $D^k$, then $\alpha_D(L)$ has a single self-intersection point.

\footnote{The character $\alpha$ is chosen for antisurgery as it looks like an immersed Lagrangian.}
Antisurgery is inverse to Lagrangian surgery in the sense that if \( L' \) is obtained from \( L \) by resolving a self-intersection point, there exists an antisurgery disk on \( L' \) so that \( \alpha_D(L') = L \). However, the choices in both surgery and antisurgery mean that the process of applying antisurgery followed by surgery need not construct the same Lagrangian. By combining anti-surgery with surgery, we can obtain a new embedded Lagrangian. We now restrict to complex dimension two.

**Definition 5.2.2.** Let \( L \) be an embedded Lagrangian and \( D^2 \) a surgery disk. Let \( \alpha_D(L) \) be obtained from \( D^2 \) by antisurgery. A mutation of \( L \) along \( D^2 \) is the Lagrangian \( \mu_D(L) \) obtained from \( \alpha_D(L) \) by resolving the resulting single self-intersection point in opposite direction. The width \( \epsilon \) of the mutation is the sum of the fluxes swept from \( L \) to \( \alpha_D(L) \) to \( \mu_D(L) \).

An example of Lagrangians obtained by mutation are the monotone Chekanov and product tori in \( \mathbb{C}^2 \setminus \{ z_1z_2 = 1 \} \). It is expected that Lagrangians which are related by mutation give different charts on the moduli space of Lagrangian submanifolds in the Fukaya category and that these charts are related by a wall-crossing formula tied to cluster transformation [PTT17].

**Theorem 5.2.3** ([Hau15]). Let \( L \) be a Lagrangian and let \( D \) be an antisurgery disk for \( L \). There exists a Lagrangian cobordism between \( L \) and \( \mu_D(L) \).

**Definition 5.2.4.** A mutation configuration is a pair \((L, D)\), where \( D \) is an antisurgery disk for \( D \) with boundary in a primitive homology class. The mutation cobordism is the cobordism \( K_{\mu_D} \) with ends on \( L \) and \( \mu_D(L) \). We call \([\partial D^2] \subset H_1(L, \mathbb{Z})\) the mutation class or mutation direction.

A key feature of Lagrangian mutation is that the Lagrangians \( L \) and \( \mu_D(L) \) are Lagrangian isotopic. We will show that the Lagrangian cobordism \( K_{\mu_D} \) can induce a continuation map between the Lagrangians \( L \) and \( \mu_D(L) \). Unlike the suspension cobordism for a Hamiltonian isotopy considered in section 6.1, this cobordism is not topologically a cylinder so we cannot simply copy our proof from section 4. To overcome this difference, will prove the following replacements for cylindricity of the cobordism \( K_{\mu_D} \):

- We first describe the Morse theory of \( K_{\mu_D} \) in section 5.2.1.
- Section 5.2.2 constructs a holomorphic disk with boundary on this cobordism and section 5.2.3 shows that the \( m^0 \) contribution of this disk can be removed.
- We show that the restriction of the differential to the right end of the cobordism can be inverted in section 5.2.4, allowing us to apply a theorem on mapping cocylinders for \( A_\infty \) algebras in section 5.2.5.

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This will give us enough leverage to construct a mapping cocylinder from the Lagrangian cobordism $K_{\mu D}$ and prove the following wall-crossing theorem:

**Theorem 5.2.5.** Suppose that $(L, D)$ is a mutation configuration satisfying definition 5.2.16. Then there exists a deforming cochain $d_\epsilon$ for $CF^\bullet(K_{\mu D})$ so that $CF^\bullet_{d_\epsilon}(K_{\mu D})$ is an $A_\infty$ mapping cocylinder between $CF^\bullet_{\beta^-_\epsilon}(L^-)$ and $CF^\bullet_{\beta^+_\epsilon}(L^+)$. 

### 5.2.1 A Morse Function for $K_{\mu D}$

The Floer cochain complex of $K_{\mu D}$ is dependent on the data of a Morse function. By picking an appropriate Morse function, we can simplify the later discussion.

**Notation 5.2.6.** In this section, let $L^- = L, L^+ = \mu_D(L)$. $K_{\mu D}$ is constructed from $L^-$ by attachment of a 1 cell and a 0 cell. However, there is a slightly more symmetric treatment to $K_{\mu D}$. Consider an intermediate Lagrangian $L^0$, which is obtained by performing antisurgery on $L^-$. Both $L^-$ and $L^+$ may be obtained from $L^0$ via attachment of a 1-cell.

Consider a Morse-Bott function $\tilde{h}: K_{\mu D} \to \mathbb{R}$ which has a minimum along $L^0$, maximums at the ends $L^\pm$ and two index 2 critical points corresponding to the attached 1-cells. See, for instance, fig. 8 for an example when $L = T^2$.

We now give a decomposition of the Morse cochain complex, with the goal of understanding where $CM^\bullet(K_{\mu D})$ fails to be a mapping cocylinder. The Morse-Bott stratification allows us to decompose $CM^\bullet(K_{\mu D})$ as

$$CM^\bullet(K_{\mu D}) = \begin{pmatrix} CM^\bullet(L^-) & CM^\bullet(L^+) \\ x^- & CM^\bullet(L^0)[1] & x^+ \end{pmatrix}$$

---

2We note here that our definition of mutation differs slightly from that in [PT17] as there is some symplectic flux swept between $L, \alpha_D(L)$ and $\mu_D(L)$. Usually, one requires that the amount of symplectic flux between these Lagrangians is zero. This ensures that properties like monotonicity are preserved by the operation of Lagrangian mutation.
Figure 8: Critical submanifolds of the Morse-Bott function for the mutation cobordism. The critical points associated to the handle attachments are marked by $x^\pm$. Notice that as $L^0$ separates the cobordism into two parts, the restriction of the product $m^{-+} : CM^\bullet(L^-) \otimes CM^\bullet(L^+) \to CM^\bullet(K_{\mu_D})$ has image in $CM^\bullet(L^0)$.

where $x^-$ and $x^+$ are the critical points corresponding to the handles. We will call the subspace spanned by $x^-$ and $x^+$

$$E^0 := \Lambda(x^-, x^+).$$

**Claim 5.2.7.** The spaces spanned by

$$CF^\bullet(L^-) \oplus CF^\bullet(L^0) \oplus E^0$$
$$CF^\bullet(L^+) \oplus CF^\bullet(L^0) \oplus E^0$$

are curved $A_\infty$ ideals of $CF^\bullet(K_{\mu_D})$.

**Proof.** These are the kernels of $\beta^\pm : CF^\bullet(K_{\mu_D}) \to CF^\bullet(L^\pm)$, which by assumption 3.2.6 are $A_\infty$ homomorphisms.

**Notation 5.2.8.** For this section, whenever we write an $A_\infty$ product map, it will always mean the $A_\infty$ product structure on $CF^\bullet(K_{\mu_D})$.

Setting $A^\pm = CM^\bullet(L^\pm)$ and $A^0 = CM^\bullet(L^0) \oplus E^0$, we can write the differential on $CM^\bullet(K_{\mu_D}) = A^- \oplus A^0 \oplus A^+$ as

$$\begin{pmatrix}
(m^-)_0 & 0 & 0 \\
(m^-)_0 & (m^0)_0 & (m^0)_0 \\
0 & 0 & (m^0)_0
\end{pmatrix}.$$ 

where $(m^\pm)_0 = m^1_{CM^\bullet(L^\pm)}$ are the Morse differentials. If the ideal $A^+ \to A^0$ was null-homotopic — equivalently if $(m^0)_0$ was an isomorphism — then $CM^\bullet(K_{\mu_D})$ would be a mapping cocylinder. We will further decompose this cochain complex by identifying subspaces of $CM^\bullet(L^\pm)$ which correspond to the mutation direction and tease out exactly how our complex fails to be a mapping cocylinder.
The kernel of \((m_0^+)_0 = 0\) is one dimensional. We pick an element \(c_u^+\) which generates the kernel so that we may write:
\[
\ker((m_0^+)_0) = \Lambda \cdot (c_u^+).
\]
Let us now define \(E^+ := ((m_0^+)_0)^{-1}(E^0)\). Pick an element \(c_w^+\) so that \((m_0^+)_0(c_w^+) = x^+\) and the classes \(c_w^+\) and \(c_u^+\) now span the \(E^+\).

**Remark 5.2.9.** The class \(c_u^+\) is determined completely by the cobordism as the Morse homology class whose downward flow space is the surgery disk. However, the selection of \(c_w^+\) requires a choice similar to the choices required to produce local coordinates on the moduli space of Lagrangian tori.

Furthermore, we pick a splitting
\[
CM^\bullet(L^+) \simeq (CM^\bullet(L^+)/E^+) \oplus E^+.
\]
This can be done so that the restriction of the Morse differential produces an isomorphism \(CM^\bullet(L^+)/E^+ \to CM^\bullet(L^0)\). We repeat the same choices for the \(L^-\) side. Notice that if \(L^+, L^-\) are tori, then \(E^\pm = CM^1(L^\pm)\). With this description, the non-cylindricity of \(CM^\bullet(K_{\mu_D})\) exactly corresponds to the non-surjectivity of the map \((m_0^+)_0\) in the diagram:
\[
\begin{array}{c}
E^+ & \xrightarrow{(m_0^+)_0|_{E^+}} & E^0 \\
E^- & \xleftarrow{(m_0^0)_0|_{E^-}} & E^+. \\
\end{array}
\]
Having fixed basis \(\{c_u^+, c_w^+\}\) for \(E^\pm\) and \(\{x^+, x^-\}\) for \(E^0\), the maps \((m_0^\pm)_0\) are
\[
(m_0^0)_0|_{E^-} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (m_0^0)_0|_{E^+} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

### 5.2.2 A Holomorphic Disk

We now show that for a specific choice of almost complex structure there exists a pseudo-holomorphic disk with boundary on \(K_{\mu_D}\).

**Proposition 5.2.10.** There exists a choice of complex structure so that \(K_{\mu_D}\) bounds a Maslov index 0 disk
\[
u_{ex} : (D^2, \partial D^2) \to (X \times \mathbb{C}, K_{\mu_D}).
\]
The downward flow space of this disk is in the cohomological class \(x^+ + x^-\).

**Proof.** We prove the proposition by exhibiting the disk in the local model constructed in section 5.1. Following [Hau15, Section 2.3] we can construct a standard neighborhood \(U\) of the Lagrangian antisurgery disk \(D\) identifying \(U\) with a subset of \(\mathbb{C}^2\) (which, by abuse of notation, we shall also call \(U\)). We choose a complex structure for \(X\) which matches the standard complex structure on \(U \subset \mathbb{C}^2\). We can pick this identification so that the
Lagrangians $L^-$ and $L^+$ when restricted to $U$ match the local model based on the Lefschetz fibration (see [Hau15] section 2.3 for implanting the local model).

$$(L^- \cap U) = ((L\uparrow \# L\downarrow) \cap U)$$

$$(L^+ \cap U) = ((L\downarrow \# L\uparrow) \cap U).$$

The restriction $K_U := K_{\mu_D} \cap (U \times C)$ is the eye-shaped Lagrangian cobordism drawn in fig. 4. Let $\pi_C : X \times C \to C$ be projection onto the cobordism parameter. Let $\pi_U : U \times C \to U \subset C^2$ be the standard projection. In this local model, the two Lagrangian thimbles have intersection $L_\uparrow \cap L_\downarrow = (0, 0)$. One observes that $\pi_U^{-1}(0, 0) \cap K$ is a closed curve. This closed curve can be explicitly described as the union of the surgery profile from fig. 3 and the curves $\gamma_{\uparrow/\downarrow}$ used to define the preliminary cobordism (see also fig. 6). The fiber $\pi_U^{-1}(0, 0)$ describes a complex line which intersects $K_{\mu_D}$ cleanly along this curve. We call the corresponding disk $u_{ex} : (D^2, \partial D^2) \to (U \times C, K_{\mu_D})$. A calculation shows that the Maslov index of this disk is zero and methods from [CO+06] show that this disk is regular. The homological class of this disk is the loop in the cobordism $K$ which is introduced from the two 1-handle attachments. Since our local model was chosen to have the same $J$-structure as $X$, this gives us an isolated Maslov index 0 disk $u_{ex} : D^2 \to X \times C$ whose boundary lies on the cobordism $K$. The downward flow space of the boundary of this disk meets the upward flows space of $x^+$ and $x^-$ once.

The area of the disk $u_{ex}$ is bounded by the area of the projection of the cobordism to the $C$ factor. In the setting of $X = C^2 \setminus \{z_1 z_2 = 1\}$, the area of this disk is computed in remark 5.1.5

$$\exp(u_{ex}) = z := \frac{u_\Delta u_+}{u_-},$$

where it describes the only disk with boundary on $K_{\mu_D}$ (proposition 6.4.2). More generally, this disk is isolated in the following sense:

**Claim 5.2.11.** For the standard holomorphic structure, the disk $u_{ex}$ is the only disk contained in $U$ with boundary on $K_{\mu_D}$ in this relative homology class.

**Proof.** This follows from the fact that every disk with boundary on $(K_{\mu_D} \cap (U \times C)) \subset U \times C$ with same relative homology class as $u_{ex}$ has boundary which travels around a loop of the projection $\pi_C : (K_{\mu_D} \cap (U \times C)) \to C$ drawn in fig. 4. The area of the red loop drawn is $\omega(u_{ex})$. Therefore, any other holomorphic disk $u : (D^2, \partial D^2) \to (U \times C, K_{\mu_D})$ with boundary in the same class as $u_{ex}$ will have projection $\pi_C \circ u$ with area at least $\omega(u_{ex})$. As this honestly is the area of the disk, we know that the projection $\pi_U \circ u$ must have zero area. However, $\pi_U \circ u$ is a holomorphic map so the zero-area condition means that $\pi_U \circ u$ is constant. The proof is completed upon checking that $\pi_U^{-1}(p) \cap (K \cap U)$ contains a loop (and therefore may bound a disk) if and only if $p = (0, 0)$. 

The presence of this disk is the first step to proving that $CF^\bullet(K_{\mu_D})$ is a mapping co-cylinder.
• The holomorphic disk contributes to $m^0$ as $m^0_{[u_{ex}]} = T^{w([u_{ex}])}(x^+ + x^-)$.

• The boundary $\partial u_{ex}$ intersects the downward flow space of $e^{u^+}$ at a point and the upward flow space of $x^-$ at a point. This will deform the Morse differential on $CM^*(K_{\mu D})$. After applying this deformation, the map $m^+_0 : CF^*(L^+) \to CF^*(L^0) \oplus E^0$ is surjective. This gives us a chance to prove that $CF^*(K_{\mu D})$ is a mapping cocylinder. In particular, if $\langle m^0_{[u_{ex}]}(e^{u^+}), x^- \rangle = T^\epsilon$, the map $m^+_0|_{E^+} : E^+ \to E^0$ will become invertible.

This nearly proves that $CF^*(K_{\mu D})$ is a mapping cocylinder. However, theorem D.1.3 requires that the valuation of $(m^+_0)^{-1}(m^0)$ be positive. As a result, we cannot immediately apply theorem B.0.1 This leaves the following steps to construct a mapping cocylinder:

1. **Deforming to increase** $\text{val}(m^0|_{E^+})$. We first equip $K_{\mu D}$ with a deforming chain $d_\epsilon$ which will cancel out the contribution of the disk $u_{ex}$, so that the valuation of $(m^0)_d\epsilon$ will be greater than $\epsilon$.

2. **Inverting** $(m^+_0)_d\epsilon|_{E^+}$. We show that the map $(m^+_0)_d\epsilon|_{E^+}$ can now be inverted. This is complicated by the introduction of the deforming cochain $d_\epsilon$.

3. **Applying HTT** We conclude that $CF^*_{d\epsilon}(K_{\mu D})$ is a filtered mapping cocylinder. As a corollary, we conclude that $K_{\mu D}$ is unobstructed whenever $L^-$ is.

This outline is complicated by the possibility of other disks with boundary appearing in the cobordism. We will make a simplifying assumption (definition 5.2.16) to complete item 2.

### 5.2.3 Deforming to remove $m^0|_{E^0}$.

When working over the Novikov field (appendix A) we can prove that a map is invertible by showing that it is invertible at low valuation. We introduce some notation for filtered $\Lambda$-modules that will help us use this method of proof. We assume that our $\Lambda$ modules are finitely generated.

**Definition 5.2.12.** Let $A$ and $B$ be filtered $\Lambda$-modules. The valuation of a map $\Theta : A \to B$ is the largest jump in valuation under $\Theta$.

$$\text{val}(\Theta) := \sup_{v : \text{val}(v) = 0} \{\text{val}(\Theta(v))\}.$$  

The **leading order** of a map $\Theta : A \to B$ is the smallest jump in valuation under $\Theta$.

$$\text{ord}(\Theta) := \inf_{v | \text{val}(v) = 0} \{\text{val}(\Theta(v))\}.$$  

Our reason for using valuations of maps will be to construct inverses.

**Claim 5.2.13.** If $\text{val}(\Theta) < \infty$, then $\Theta$ has a left inverse. If this is also a right inverse then $\text{val}(\Theta) = -\text{ord}(\Theta^{-1})$ and $\text{val}(\Theta^{-1}) = -\text{ord}(\Theta)$.
Proof. The condition $\text{val}(\theta) < \infty$ states that $\theta(v) \neq 0$ for any $v$, so our map has a left inverse.

Suppose that $\Theta^{-1}$ is the inverse of $\Theta$. Let $v$ be the valuation $0$ element on which the maximum $\text{val}(\Theta(v)) = \text{val}(\Theta)$ is realized. Since $\text{val}(\Theta^{-1} \circ \Theta(v)) = \text{val}(v) = 0$, we obtain that $\text{ord}(\Theta^{-1}) \leq -\text{val}(\Theta)$. Similarly, let $w$ be the valuation zero vector so that $\text{val}(\Theta^{-1}(w)) = \text{ord}(\Theta^{-1})$. Since $\text{val}(\Theta \circ \Theta^{-1}(w)) = \text{val}(w) = 0$, we obtain that $\text{val}(\Theta) \geq -\text{ord}(\Theta^{-1})$.

If one possesses a bound on the valuation and order of a map, then one can obtain a bound on their sum.

**Claim 5.2.14.** Suppose that $\text{val}(\Theta_1) < \text{ord}(\Theta_2)$. Then $\text{val}(\Theta_1 + \Theta_2) = \text{val}(\Theta_1)$.

**Proof.** Let $v$ be the vector on which the maximum $\text{val}(\Theta(v)) = \text{val}(\Theta)$ is realized.

$$\text{val}(\Theta_1 + \Theta_2) \geq \text{val}(\Theta_1(v) + \Theta_2(v))$$

Since $\text{val}(\Theta_2(v)) \geq \text{ord}(\Theta_2) > \text{val}(\Theta_1)$,

$$= \text{val}(\Theta_1(v)) = \text{val}(\Theta_1).$$

For the other direction, let $w$ be the vector on which the maximum $\text{val}((\Theta_1 + \Theta_2)(w)) = \text{val}(\Theta_1 + \Theta_2)$.

$$\text{val}(\Theta_1 + \Theta_2) = \text{val}(\Theta_1(w) + \Theta_2(w)) \leq \min(\text{val}(\Theta_1), \text{val}(\Theta_2)) = \text{val}(\Theta_1).$$

\[ \square \]

**Definition 5.2.15.** Let $\Theta : A \to B$ be a map. We say that $\Theta$ has a leading term of valuation $\lambda$ and write

$$\Theta = \Theta_{\leq \lambda} + O(\lambda)$$

if $\Theta = \Theta_{\leq \lambda} + R$, with

$$\text{val}(\Theta_{\leq \lambda}) = \lambda < \text{ord}(R).$$

**Definition 5.2.16.** Let $\epsilon = \omega(u_{ex})$. We say that the mutation $\mu_D$ is isolated if every disk $u$ with boundary on $K_{\mu_D}$ which is not $u_{ex}$ with $([\partial u], x^\pm) \neq 0$ has $\omega(u_{ex}) > \epsilon$.

We will now assume that $\mu_D$ is an isolated mutation.

**Claim 5.2.17.** Suppose that $\mu_D(L)$ is an isolated mutation. Let $\pi_{E^0} : CF^*(K_{\mu_D}) \to E_0$ be the standard projection. There exists a deforming cochain $d_\epsilon \in CF^*(K_{\mu_D})$ which increases the valuation of the curvature in the following way:

$$\text{val}_{\Lambda}(\pi_{E^0} \circ m^{0}_{d_\epsilon}) > \epsilon.$$
Proof. The restriction $\pi_{E^0} \circ m^1|_{E^- \oplus E^+} : E^- \oplus E^+ \to E^0$ surjects. Pick $d_\epsilon \in E^- \oplus E^+$ so that $\pi_{E^0} \circ (m^1)(d_\epsilon) = -m^0|_{\Lambda(x^-, x^+)}$. Note that $\text{val}(d_\epsilon) \geq \epsilon$.

\[
\text{val}(\pi_{E^0} \circ m^0_{d_\epsilon}) = \text{val} \left( (\pi_{E^0} \circ m^0 + \pi_{E^0} \circ m^1(d_\epsilon) + \sum_{k} \pi_{E^0} \circ m^k(d_\epsilon^\otimes k)) \right) \\
\geq \min \left( \text{val}(\pi_{E^0} \circ (m^0 + m^1(d_\epsilon))), \text{val} \left( \sum_{k} \pi_{E^0} \circ m^k(d_\epsilon^\otimes k) \right) \right)
\]

By assumption, the lowest valuation term of $m^0$ is $T_{\omega_{x^+}} \cdot (x^- + x^+)$, which exactly cancels $m^1_{\omega_0}(d_\epsilon)$. Therefore, $\text{val}(\pi_{E^0}(m^0 + m^1(d_\epsilon))) > \epsilon$. Since $\text{val}(m^k(d_\epsilon^\otimes k)) \geq k\epsilon$, we conclude:

$$\epsilon > \min(\epsilon, 2\epsilon) > \epsilon.$$  

\[\square\]

5.2.4 Inverting $m^+_0|_{E^+}$.

We now choose the deforming cochain from claim 5.2.17 so that $\text{val}(\langle m^+_0, E^0 \rangle) > \epsilon$. Because of definition 5.2.16, we obtain a lower bound for the valuation of the deforming cochain,

$$\text{val}(d_\epsilon) \geq \epsilon.$$

Claim 5.2.18. The map $(m^+_0)_d$ is invertible, with $\text{ord}((m^+_0)_d)^{-1} \geq -\epsilon$.

Proof. It suffices to show that the restriction $\pi_{E^0} \circ (m^+_0)_d|_{E^+} : E^+ \to E^0$ is invertible. To do so, we expand $(m^+_0)_d$ term wise:

$$(m^+_0)_d|_{E^+} = m^+_0 + \pi_{E^0} \circ \left( \sum_{k \geq 2} m^k_{\lambda^k} \left( d_\epsilon^\otimes k_1 \otimes \text{id} \otimes d_\epsilon^\otimes k_2 \right) \right)|_{E^+}$$

and use this to compute the lowest-order portion of the map. We will compute this map in the coordinates $\{c^{u+}, c^{w+}\}, \{x^-, x^+\}$.

- The first term can be expanded by valuation. By definition 5.2.16, there is only one disk which can contribute to this term at a valuation of $\epsilon$. As a result,

$$\langle (m^+_0)_d(c^{u+}), x^- \rangle = T^\epsilon + O(\epsilon).$$

- The next terms in the above expansion will necessarily have higher valuation. We split into two cases: $k = 2$ and $k > 2$. When $k > 2$, $\langle \sum_{k \geq 2} m^k(d_\epsilon^\otimes k_1 \otimes \text{id} \otimes d_\epsilon^\otimes k_2), x^- \rangle$ necessarily will be at least $2\epsilon$ by eq. (3).
• The remaining term is $m^2 \circ (d_\epsilon \otimes c^{u+}) + m^2 \circ (c^{u+} \otimes d_\epsilon)$. We can further split this into terms given by the homology class of the disks which deform the product.

$$m^2 \circ (d_\epsilon \otimes c^{u+}) = \sum_{\beta \in H^2(X,L)} m^2_\beta (d_\epsilon \otimes c^{u+})$$

Whenever $\beta \neq 0$, the term $m^2_\beta (d_\epsilon \otimes c^{u+})$ has valuation at least $\text{val}(d_\epsilon) + \text{val}(\beta) > \epsilon$. So we need only worry about the classical portion product, $(m^2)_{=0} (d_\epsilon \otimes c^{u+})$. As the space spanned by $CM^\bullet (L^+) \oplus \langle x^+ \rangle \oplus CM^\bullet (L^0)$ is an ideal of $CM^\bullet (K_{\mu_D})$,

$$\langle (m^2)_{=0} (d_\epsilon \otimes c^{u+}), x^- \rangle = 0.$$  

We obtain

$$\langle m^1_\epsilon (c^{u+}), x^- \rangle = T^\epsilon + O(\epsilon)$$

A similar argument for the other components of our map allows us to express:

$$\pi_{E^0} \circ (m^+_0)_{d_c} \mid_{E^+} = \begin{pmatrix} T^\epsilon & 0 \\ T^\epsilon & 1 \end{pmatrix} + O(\epsilon).$$

Since the lowest order portion of this map is invertible, the map $\pi_{E^0} \circ (m^+_0)_{d_c} \mid_{E^+}$ is invertible. The order of the inverse is at least $-\epsilon$.

Note that a similar argument shows that $(m^+_0)_{d_c} : CF^\bullet (d_\epsilon) (L^+) \to CF^\bullet (d_\epsilon) (L^0) \oplus E^0$ is invertible.

5.2.5 Checking conditions of theorem [D.1.3]

To show that $CF^\bullet (K_{\mu_D})$ is an $A_\infty$ mapping cocylinder (definition [D.1.2]) set $A^\pm = CF^\bullet (d_\epsilon) (L^\pm)$ and $A^0 = CM^\bullet (d_\epsilon) (L^0) \oplus E^0$. We’ve proven that $CF^\bullet (K_{\mu_D}) = A^- \oplus A^0 \oplus A^+$ as a vector space. By assumption 3.2.6 differential on this complex is of the form

$$\begin{pmatrix} (m^1_{A^-})_{d_\epsilon} & 0 & 0 \\ (m^0_0)_{d_\epsilon} & (m^+_0)_{d_\epsilon} & (m^+_1)_{d_\epsilon} \\ 0 & 0 & (m^1_{A^+})_{d_\epsilon} \end{pmatrix}.$$  

By claim 5.2.18 the map $(m^+_0)_{d_\epsilon}$ is invertible with order greater than $-\epsilon$. By definition 5.2.16 $\text{val}((m^+_0)_{d_\epsilon} \circ m^0_{d_\epsilon}) \geq 0$. We therefore satisfy the definition [D.1.2] and may apply theorem [D.1.3]. This concludes the proof of theorem 5.2.5.

6 Examples: wall-crossings and mutations

We now explore some applications of proposition 4.2.1 and theorem 5.2.5.
• In section 6.1, we look at how disk bubbling during an isotopy between non-monotone Chekanov and product type tori can be interpreted as a bounding cochain on the suspension of the Hamiltonian isotopy. We show that this bounding cochain recovers the wall-crossing formula for holomorphic disk counts in section 6.2.

• The definition of mutation configuration requires that the homology class of the anti-surgery disk occupy a non-trivial homology class. This is because we use this class to construct the bounding cochain for the mutation cobordism. In section 6.3 we give a non-example of Lagrangian mutation highlighting the necessity of this condition.

• When constructing the mutation cobordism, we found a relation between the width of the surgeries used in the mutation construction and the size of the holomorphic disk on the mutation cobordism. We’ll use this relation to recover the wall-crossing formula for Chekanov and product tori in section 6.4.2.

For purposes of exposition, we will assume for the remainder of this discussion that we can work safely with complex coefficients and avoid convergence issues. Furthermore, we make the following “wishful” identification of bounding cochains as local systems on Lagrangians.

**Assumption 6.0.1.** The following Lagrangian branes are isomorphic as objects of the Fukaya category with complex coefficients.

\[(L^+_{r,s}, a \cdot c^r + b \cdot c^s) \mapsto (L^+_{r-\exp(a), s-\exp(b)}, 0)\]

A short justification for this assumption was provided in [Aur08, Lemma 4.1] when the chain model for the open Gromov-Witten invariants is the de Rham complex. More generally, [Fuk10] constructs a version of Lagrangian Floer theory based on the de Rham complex for which this holds. The comparison of the \(A_\infty\) structure between the Morse complex and de Rham complex is non-trivial and explicitly described in [KS01]. A proof of assumption 6.0.1 would combine these two techniques, however we are content to take the relation between bounding cochains and local systems in the pearly-model as a given.

### 6.1 Continuation example: \(\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}\)

#### 6.1.1 Computing the Transition Map

We now compute the uncurved \(A_\infty\) continuation map \(\Theta_\gamma : CF^\bullet_0(L^-_{u,w}) \to CF^\bullet_0(L^+_r)\), as well as the bounding cochain \(b^+\) that \(L^+_r\) inherits. Let \(L^-_{u,w}\) be a Lagrangian torus on the left side of the chart with \(\log |w| > 0\).

Consider a Hamiltonian isotopy \(\theta\) which deforms the path \(\gamma\) so that it passes around the critical value on the right side. We will let \(K\) be the associated suspension cobordism between \(L^-_{u,w}\) and \(\theta(L^-_{u,w}) = L^+_r\). We choose this Hamiltonian isotopy in such a way that it
only contains one member on the wall. As we use the pearly model for homology, we now fix an explicit Morse function for $K$ in order to ease this computation. On the left hand side of the cobordism we have the homology classes $c_u$ and $c_w$. We choose the Morse function $h^-$ in such a way that we have a dual basis $c^u, c^w$ for the Morse cohomology in the sense that assumption 6.0.1 relates holonomy along $c_u, c_w$ to bounding cochains on $c^u, c^w$. We use $h^-$ and $h^+$ to construct to a Morse function for $K$ in the usual way, giving a Morse function with 12 critical points labelled as follows:

$$CM^\bullet(L^-) \xrightarrow{(m_0^-) = 0} CM^\bullet(L^0)[1] \xrightarrow{(m_0^+) = 0} CM^\bullet(L^+)$$

Note that in the basis $\{c^u, c^w\}$, $\{c^r_0, c^s_0\},$ the Morse portion of the differential from the left side of the cobordism to the minimum can be written as

$$(m_0^-)_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This will be deformed by the inclusion of a count of holomorphic disks.

Claim 6.1.1. For the adjusted complex structure $J_{H_t}$, there is a single simple disk $u_{wc}$ with boundary on $K_{H_t}$.

Proof. By the falling lemma, any disk with boundary on the cobordism $K$ is a disk with boundary on one of the Lagrangians $L_t$. A holomorphic disk with boundary on $L_t$ must project to a holomorphic disk in $\mathbb{C}^* \setminus \{1\}$ with boundary on $\gamma_t$. Because every $\gamma_t$ winds around $z = 1$, the only such disk is the constant disk. This would require that the disk map to the fiber. For topological reasons, this can only occur when the fiber is $z_1 z_2 = 0$. Therefore, there is only one disk $u_{wc}$ parameterized by $z_1 = 0$ (resp. $z_2 = 0$ should log $|w| < 0$) with boundary on the Lagrangians $L_t$, which occurs exactly when we cross the wall.

Claim 6.1.2. For log $|w| > 0$, the curvature on $K_{H_t}$ is

$$m_{CF^\bullet}(K) = \sum_{k=1}^{\infty} \left( \frac{(-1)^k}{k} \right) T^{k \omega(u_{wc})} (c^r_0 + c^s_0)$$

Proof. We first consider the types of flow trees which could possibly contribute to the count of $m^0$. Because the direction of the Morse flow displaces the disk $u_{wc}$ from itself there is at
most one copy of \( u_{wc} \) (or its multiple covers) in any flow tree. The only configurations which can contribute to \( m^0_K \) are those which are one of the disk \( u_{wc} \) (or its multiple covers) flow to a critical point of the cobordism.

The boundary of the class \( u_{wc} \) is \( c_w \). Our Morse function was chosen so that the downward flow space \( c_w \) was equal to the downward flow space of the cohomology class \( c^w \). Therefore, the cohomological class of \( m^0_K \) is \( (c^r_0 + c^s_0) \).

It remains to account for the factor \( \left( \frac{(-1)^k}{k} \right) T^{\kappa w(u_{wc})} \). The factor \( k \) counts the multiplicity of the covering. The symplectic area of the \( k \)-fold cover of \( u_{wc} \) is \( k \cdot \omega(u_{wc}) \) and the \( \frac{1}{k} \) factor accounts for the need to quotient out by automorphisms of the configuration. The choice of sign is justified in section 6.2.

By abuse of notation, we can make the replacement \( T^{\kappa w(u_{wc})} = w^k \) so that

\[
m^0_{CF^\bullet(K)} = - \log(1 + w)(c^r_0 + c^s_0).
\]

This Maslov index 0 disk will also deform the higher products as well.

Since \( CF^\bullet(L_{u,w}^-) \) is tautologically unobstructed, we are in the scenario of corollary 4.2.3 giving us an uncurved map

\[
\Theta_\theta : CF_0(L_{u,w}^-) \to CF_{b_{wc}u,w}(L_{r,s}^+)
\]

where

\[
b_{wc}u,w := (\Theta_K)_\ast(0) = (\beta^+ \circ \alpha)^0
\]

\[
= (\beta^+)^1 \circ \alpha^0 = (\beta^+)^1 \circ (m^+_0)^{-1} \circ m^0_K
\]

\[
= - \log(1 + w)(c^r_+ + c^s_+)
\]

We can repeat the same discussion with the chart where \( \log |w| < 0 \). Summarizing both cases into one claim:

**Claim 6.1.3.** Let \( \log |w| \neq 0 \) and let \( \theta \) be the Hamiltonian isotopy given above. The Hamiltonian isotopy \( \theta \) identifies pairs of Lagrangians and bounding cochains,

\[
(L_{u,w}^-, 0) \simeq (\theta(L_{u,w}^-), b_{wc}u,w).
\]

The bounding cochain depends on whether the Hamiltonian isotopy passes “above” or “below” the wall:

\[
b_{wc}u,w = \begin{cases} 
- \log(1 + w)(c^r_+ + c^s_+) & \log |w| > 0 \\
- \log(1 + w^{-1})(c^r_+ + c^s_+) & \log |w| < 0
\end{cases}
\]
6.1.2 Recovering the Wall-Crossing formula

Our construction now accounts for the discrepancy in eq. (1), the naive identification of coordinate charts. On the portion of the chart where \( \log |w| > 0 \), the composition of identification of Lagrangian branes with the wishful map sends

\[
L^-_{u,w} \mapsto (\theta(L^-_{u,w}), b^\text{wc}_{u,w}) \mapsto (L^+_{1+u, 1+u} - w, u).
\]

Compare this to the portion where \( \log |w| < 0 \)

\[
L^-_{u,w} \mapsto (\theta(L^-_{u,w}), b^\text{wc}_{u,w}) \mapsto L^+_{1+u, 1+u} - \frac{w-1}{1+w}.
\]

Notice now that this composition consistently glues together the objects regardless of whether we pass above or below the monotone locus. This correction (to our previous inconsistency) is the discrepancy from monodromy of the fibration exactly cancelling out the contribution of the bounding cochain. In summary, we get a gluing formula for the coordinates on these two charts

\[
(u, w) \mapsto \left( \frac{u}{1+w-1}, \frac{u}{1+w} \right).
\]

This coordinate change can be better understood by making the change of variables \( v = r^{-1} \) and the identification \( s = rw \). The charts then parameterize the variety

\[
wv = 1 + w.
\]

This matches previous constructions for charts on the mirror space.

6.2 Maslov index 2 disks in \( \mathbb{C}^2 \)

We now interpret the wall-crossing formula for open Gromov-Witten superpotential using these bounding cochains. This occurs in the setting of \( W : \mathbb{C}^2 \to \mathbb{C}, (z_1, z_2) \mapsto z_1 z_2 \). Following [Aur08], we consider Lagrangian tori \( \tilde{L}^-_{u,w} \) and \( \tilde{L}^+_{r,s} \) in \( \mathbb{C}^2 \). These tori now bound Maslov index 2 disks and have non-trivial curvature term. We define the open Gromov-Witten potential,

\[
W_{\text{OGW}}(\tilde{L}^\pm) := \langle m^0, e \rangle
\]

to be the area and local system weighted counts of Maslov index 2 holomorphic disks with boundary on \( L^\pm \). The wall-crossing phenomenon is the observation that Chekanov type tori only bound 1 family of Maslov index 2 holomorphic disks while the product tori bound two families of holomorphic disks. The counts of these disks give the tori the following OGW potentials:

\[
W_{\text{OGW}}(\tilde{L}^-_{u,w}) = u \quad \quad W_{\text{OGW}}(\tilde{L}^+_{r,s}) = r + s
\]

We now look at \( K \), the suspension of a Hamiltonian isotopy between the Chekanov and product tori, equipped with the same Morse function as considered before. Let \( \tilde{L}^-_{u,w} \) be the
left end of this cobordism and let \( \bar{L}_{rs}^+ \) be the right end of this cobordism, with \( \log |w| > 0 \).

As before, we construct \( b\omega_c \), which turns \( CF^\bullet(K, b\omega_c) \) into a mapping cocylinder between \( CF^\bullet(\bar{L}_{u,w}) \) and \( CF^\bullet(b^+)(\bar{L}_{rs}^+) \). This induces a morphism of weakly unobstructed \( A^\infty \) algebras; in particular, we have that \( W_{OGW}(\bar{L}_{u,w}) = W_{OGW}(\bar{L}_{rs}^+, b^+) \) (the weight of the \( b^+ \) deformed curvature term). We will use this to justify the sign in eq. (4) as there is only one choice of signs for eq. (4) for which this equality will hold. To check the choice of sign, it suffices to check that the \( b^+ \) deformed OGW potential for product torus matches that of the Chekanov torus. By assumption 6.0.1, we can reinterpret this bounding cochain instead as a local system on the \( L \) whose holonomy is given by \( \exp(b^+ \cdot \gamma) \). The weights of the disks are then modified by the holonomy induced by the bounding cochain.

\[
W_{OGW}(\bar{L}_{rs}^+, b^+) = r \cdot \exp(b^+ \cdot c^r) + s \cdot \exp(b^+ \cdot c^s) = r \cdot \exp(-\log(1 + w)) + s \cdot \exp(-\log(1 + w))
\]

Using the relation the disk areas, \( r = uw, s = u \)

\[
= \frac{uw}{1 + w} + \frac{u}{1 + w} = u.
\]

As this matches \( W_{OGW}(\bar{L}_{u,w}) \), we can conclude that the choice of signs in eq. (4) is correct.

### 6.3 A non-example: not a mutation Configuration

We now give an example of an antisurgery which does not give a mutation. Let \( X = T^*S^2 \cup_p T^*S^2 \) be the plumbing of the cotangent bundle of two spheres. Let \( L_1, L_2 \) be the two Lagrangian spheres inside the plumbing. Then there is a Lagrangian cobordism between \( L_{S^2}^- := L_1 \# L_2 \) and \( L_{S^2}^+ := L_2 \# L_1 \). So these Lagrangian spheres can be related by anti-surgery resolved by surgery similar to Lagrangian mutation. However, the Lagrangian antisurgery disk \( D \) which relates the \( L_{S^2}^\pm \) does not have boundary in a non-trivial class of \( H_1(L_{S^2}^\pm, \mathbb{Z}) \). As \( H_1(L_{S^2}^\pm) \) fails to generate \( H^1(K_{\mu_D}) \) the proof of claim 5.2.17 fails.

This is expected as \( L_{S^2}^\pm \) are different objects of the Fukaya category and do not have a rich enough space of bounding cochains to make them isomorphic objects. Furthermore, unlike in the setting where we have a mutation configuration, the Lagrangians \( L_{S^2}^\pm \) are not Lagrangian isotopic.

### 6.4 An example: monotone Chekanov and product tori

We’ll now look at the specific example of \((\mathbb{C}^2 \setminus z_1 z_2 = 1)\), where we will show that the inclusion of the deforming cochain \( d_e \) recovers the wall-crossing formula. The main computation for this example is the following sharpening of the minimal energy disk requirement of definition 5.2.16, which was necessary to apply our wall-crossing computation.

**Remark 6.4.1.** As \( L_{u,v}^- \) and \( L_{rs}^+ \) bound no holomorphic disks, assumption 3.2.6 trivially holds.
Figure 9: Disks in the Lagrangian cobordism $K_{w,s}$ must either have boundary contained inside of the red region, or outside of the red region.

**Proposition 6.4.2.** Let $L_{u,1}^- := L_{u,-1}^-$ and $L_{s,s}^+ := L_{u,s}^+ \frac{u}{\sqrt{\Delta}}$ be the Chekanov and product tori from claim 5.1.3. The only disk with boundary on $K_{\mu D}$ is the one described in proposition 5.2.10 and its multiple covers. The area of this disk is given by remark 5.1.5,

$$z_{ex} := \exp \left( \int_{u_{ex}} \omega \right) = u s^{-1}.$$

**Proof.** We look at the Lagrangian cobordism $K_{\mu D} \subset X \times \mathbb{C}$ under the projections $\pi_X : X \times \mathbb{C} \to X$, $\pi_{\mathbb{C}} : X \times \mathbb{C} \to \mathbb{C}$ and $W : X \to \mathbb{C}$. The first projection that we look at is $W \circ \pi_X : K_{\mu D} \to \mathbb{C}$. The regions where we have performed Hamiltonian isotopy and surgery sweep out flux corresponding to shaded regions in the projection fig. 9. We let $U$ be the neighborhood drawn in fig. 9. Every disk with boundary on $K_{\mu D}$ must either have boundary contained within the red region $U$, or will have boundary completely disjoint from the red region $U$ by the open mapping principle.

We now look at $K_{\mu D} \cap (W \circ \pi_X)^{-1}(U)$. This is the eye-shaped cobordism from fig. 9. By claim 5.2.11 the only disks which appear here are $u_{ex}$ and its multiple covers. The complement of the region given by $U$ cannot bound holomorphic disks for topological reasons. This completely classifies the disks which may appear on $K_{\mu D}$.

**6.4.1 Orientations**

For the computation of the continuation map arising from the mutation cobordism we need to make a short remark about spin structures on the Chekanov and product tori.

**Definition 6.4.3.** If $L$ is a $n$-manifold with $n \geq 3$, a spin structure on $L$ is a trivialization of $TL$ over the 1-skeleton of $L$ which extends to a trivialization over the 2-skeleton. If $n = 2$, then it is a trivialization of $\mathbb{R} \oplus TL$ over the 1 skeleton which extends over the 2 skeleton.

Let $L_{u,w}^-$ be a Chekanov type torus. The standard spin structure on $L_{u,w}^-$ is the one arising by first trivializing the tangent bundle, then trivializing $\mathbb{R}$. The induced spin structure on
the torus is the spin structure arising from the pullback along \( T^2 \hookrightarrow \mathbb{R}^3 \) of the standard trivialization of the tangent bundle on \( \mathbb{R}^3 \).

Note that if \( K : L^- \to L^+ \) is a Lagrangian cobordism, a choice of spin structure for \( K \) induces a spin structure on the ends. In the case where \( K_{H_t} \) is the suspension of a Hamiltonian isotopy, this matches the standard spin structures.

As the Lagrangian cobordism \( K_{\mu_D} \) admits an embedding into \( \mathbb{R}^3 \), it has a trivial tangent bundle is therefore spin. We say that this is the induced spin structure on \( K_{\mu_D} \), which restricts to the induced spin structures on \( L^{\pm} \).

6.4.2 Wall-Crossing computation

We now compute \( \pi^\pm_*(d_\epsilon) \) from theorem 5.2.5. From our discussion on orientations, we have that the Lagrangian cobordism \( K_{\mu_D} \) does not identify \( L_{u,1}^- \) with \( L_{s,s}^+ \); to signify that these Lagrangians have a different spin structure than the Lagrangians previously considered. This is important as Lagrangians equipped with different spin structures can represent distinct objects in the Fukaya category. Let \( z = T^w(u_{ex}) \). We choose the Morse functions for the \( \hat{L}_{u,1}^- \) and \( \hat{L}_{s,s}^+ \) matching the one chosen in section 6.1.2 so that the coordinates for the moduli spaces can be identified with our previous computation.

The vanishing cycle on the Chekanov side is \( c^w \), while the vanishing cycle on the product side is \( c^s + c^e \). To construct of the continuation map from a mutation cobordism, we must pick a splitting of the vector spaces \( E^\pm \) from section 5.2.1. On the Chekanov side we take the class \( c_w \) and on the product side we choose the class \( c^s \).

With this choice of Morse function, the curvature term is:

\[
m^0_{K_{\mu_D}} = \log(1 + z)(x^+ + x^-).
\]

The deformation from claim 5.2.17 can be extended to a bounding cochain by including the contributions from multiple covers:

\[
b = \log(1 + z) \cdot c^w + \log(1 + z) \cdot c^s
\]

By restricting \( b_\epsilon \) to the ends of the cobordism, we obtain a continuation map between \( CF^*_{b^-}(\hat{L}_{u,1}^-) \to CF^*_{b^+}(\hat{L}_{s,s}^+) \). To mix-and-match coordinates on the geometric moduli space of Lagrangians and the coordinates on the bounding cochains, we need to again apply the “wishful map” of assumption 6.0.1. If we assume convergence over complex coefficients, we obtain a correspondence between Lagrangians

\[
(\hat{L}_{u,1+z}) \sim (L_{i,1}^-, \log(1 + z) \cdot c^w) \sim (\hat{L}_{s,s}^+, \log(1 + z) \cdot c^s) \sim (\hat{L}_{s,s}(1+z))
\]

Setting \( w = (1 + z) \) and noting that \( s = u/z \) from 5.1.5

\[
(\hat{L}_{u,w}^-) \sim \hat{L}_{u/(w-1),uw/(w-1)}^+
\]

This closely matches the identification from section 6.4, with the only difference being the signs. This discrepancy is a result of using the induced rather than standard spin structure.
A  A refresher on curved $A_\infty$ algebras

In this section we review some statements on filtered $A_\infty$ algebras. In appendix A.1, we fix notation for filtered $A_\infty$ algebras. Appendices B to D prove that $A_\infty$ algebras satisfy a homotopy transfer theorem giving rise to mapping cocylinders. Our exposition of $A_\infty$ algebras is tailored to understanding the following properties of $A_\infty$ algebras:

- **Deformations:** Given an anticommutative differential graded algebra $A$ and $a \in A$ a closed class, we can deform the differential to $d + a \wedge$. In order to remove the requirement that $a$ be closed and obtain a reasonable theory, we must work with filtered $A_\infty$ algebras. We show that deformations of this type give reasonable $A_\infty$ algebras in appendix A.4.

- **Homotopy Transfer Principle** Given $B$ a chain complex and maps $\alpha : A \to B$ and $\beta : B \to A$ so that $\alpha \circ \beta$ is homotopic to the identity and $\beta \circ \alpha = \text{id}_A$, we can transfer the chain structure on $B$ to a chain structure on $A$. There is no such analogous statement for differential graded algebras – instead, one must work with $A_\infty$ algebras to obtain an analogous homotopy transfer lemma. To that end, we spend appendix A.3 defining $A_\infty$ homomorphisms and developing a notation for expressing compositions of these morphisms using diagrams of planar trees.

A.1  An $A_\infty$ refresher

These notes are partly based on an exposition on uncurved $A_\infty$ algebras [Kel99], as well as [FOOO10]. We will review curved $A_\infty$ algebras, their morphisms and deformations. For reasons related to the convergence (we will frequently use infinite sums) we will work with the theory of filtered $A_\infty$ algebras.

Definition A.1.1 ([FOOO10]). Let $R$ be a commutative ring with unit. The universal Novikov ring over $R$ is the set of formal sums

$$\Lambda_{\geq 0} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \Bigg| a_i \in R, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i \to \infty} \lambda_i = \infty \right\}$$

Let $k$ be a field. The Novikov Field is the set of formal sums

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \Bigg| a_i \in k, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = \infty \right\}.$$ 

An energy filtration on a graded $\Lambda$-module $A^*$ is a filtration $F^{\lambda_i} A^k$ so that

- Each $A^k$ is complete with respect to the filtration and has a basis with valuation zero over $\Lambda$.
- Multiplication by $T^\lambda$ increases the filtration by $\lambda$. 

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There is a valuation map \( \text{val} : A^\bullet \to \mathbb{R} \) which is defined by the smallest exponent \( \lambda \) which appears in the basis expansion of an element.

**Definition A.1.2.** A filtered \( A_\infty \) structure \( (A^\bullet, m^k) \) is a graded \( \Lambda \)-module \( A^\bullet \) with \( \Lambda \)-linear cohomologically graded higher products for each \( k \geq 0 \)

\[
m^k : (A^\bullet)^{\otimes k} \to (A^\bullet)^{\otimes 2-k}
\]
satisfying the following properties:

- **Energy Filtration:** The product respects the energy filtration in the sense that:

\[
m^k(F^{\lambda_1}A^\bullet, \ldots, F^{\lambda_k}A^\bullet) \subset F^{\sum_{i=1}^k \lambda_i}A^\bullet
\]

- **Non-Zero Energy Curvature:** The obstructing curvature term has positive energy,

\[
m^0 \in F^{\lambda_0 > 0}(A^\bullet)
\]

- **Quadratic \( A_\infty \) relations:** For each \( k \geq 0 \),

\[
\sum_{j_1+i+j_2=k} (-1)^{\clubsuit}m^{j_1+1+i+j_2}(\text{id}^{\otimes j_1} \otimes m^i \otimes \text{id}^{\otimes j_2}) = 0.
\]

The value of \( \clubsuit \) is determined on an input element \( a_1 \otimes \cdots \otimes a_k \) by

\[
\clubsuit = |a_{k-j_1}| + \cdots + |a_k| - i.
\]

We say that \( A^\bullet \) is unital if there exists an element \( e \) so that

\[
m^{k_1+1+k_2}(\text{id}^{\otimes k_1} \otimes e \otimes \text{id}^{\otimes k_2}) = \begin{cases} 
\text{id} & k_1+k_2 = 1 \\
0 & k_1+k_2 \neq 1
\end{cases}
\]

For the purposes of exposition, we will work up to signs from here on out and ignore the parity from \( (-1)^{\clubsuit} \). If \( m^0 = 0 \), then \( (A^\bullet, m^1) \) is a chain complex and we say that \( A^\bullet \) is uncurved or tautologically unobstructed. We from now on suppress the cohomological index and, when the product structure is clear, we will simply notate an \( A_\infty \) algebra by \( A \).

**Definition A.1.3.** Let \( A \) be a filtered \( A_\infty \) algebra. An ideal of \( A \) is a subspace \( I \subset A \) so that for every \( b \in I \) and \( a_1, \ldots, a_{k-1} \in A \),

\[
m^k(a_1 \otimes \cdots \otimes a_j \otimes b \otimes a_{j+1} \otimes \cdots \otimes a_{k-1}) \in I.
\]

Note that we do not require \( m^0 \in I \).

The quotient of an \( A_\infty \) algebra by an ideal is again a filtered \( A_\infty \) algebra. Given \( (A, m^k) \) a filtered \( A_\infty \) algebra, define the positive filtration ideal

\[
A_{>0} := \{ a \in A : \text{val}(a) > 0 \}.
\]

We may recover uncurved \( A_\infty \) algebra by taking the quotient,

\[
A_{=0} := A/A_{>0}.
\]

This is always an uncurved as the \( m^0 \) term is required to have positive valuation. An example of this relation comes from Lagrangian Floer theory and Morse cohomology, where \( CM^\bullet(L) = CF^\bullet_{=0}(L) \).
A.2 Trees to the $A_\infty$ relations

The $A_\infty$ relations are described by large compositions of multilinear maps and it is convenient to notate these large compositions of multilinear maps using diagrams of trees.

Definition A.2.1. A planer rooted tree with some semi-infinite leaves is a tree $T$ with the following additional data:

- An ordering of the leaves of $T$ arising from a planar embedding of $T$.
- A choice of leaf $v_0$ called the root of $T$.
- A choice $E^c$ of non-root leaves called the semi-infinite leaves or external leaves.

When we refer to the vertices $V(T)$ of a planar rooted tree with semi-infinite leaves, we will always mean the vertices which do not belong solely to a semi-infinite leaf or root. If $T$ is a planar rooted tree with some semi-infinite leaves and has at least 1 vertex, we denote by $v_0$ the interior vertex which is connected to the root edge. We say that a planar rooted tree is stable if it has at least one interior vertex and no vertex has degree 2.

See fig. 10 for an example of planar rooted tree. One should imagine that a planar rooted tree is a rooted tree with a planar embedding into the disk with some subset of the leaf vertices on the boundary of the disk. From now on we will always use the word “tree” to describe a planar rooted tree with some semi-infinite leaves. We define the valence of a tree $T$ to be the number of external leaves and write

$$\nu(T) := |E^c|.$$ 

The external leaf set $E^c$ inherits an ordering $\{1, 2, \ldots, \nu(T)\}$ from the ordering of the leaves. Since $T$ is a rooted tree, to each vertex we have an ordered upward edge set, $E^+_v$ and a downward edge $e^+_v$. Similarly, to each edge we have an upward vertex $v^+_e$ and downward vertex $v^-_e$.

Definition A.2.2. A labelling $L$ of a tree $T$ is an assignment to

- Each edge $e \in E$ a vector space $A_e$. 

Figure 10: An example of a planar rooted tree with some semi-infinite leaves. Critical leaves are drawn in green and the root is marked in red. The vertex set $V(T)$ is marked in black.
• Each vertex \( v \in V \) a morphism from the edges immediately above \( v \) to the edge below \( v \),

\[
f^v : \bigotimes_{e \in E^v} A_e \to A_{e^\downarrow}.
\]

In the case where \( \text{deg}(v) = 1 \), we prescribe an element of \( A_{e^\downarrow} \), which should be thought of as a morphism \( \Lambda \to A_{e^\downarrow} \).

To each labelled tree \((T, L)\), we obtain a morphism

\[
f^{(T,L)} : \left( \bigotimes_{e \in E^c} A_e \right) \to A_{e_0}
\]

by composing the morphisms associated to the vertices according to their edge adjacency.

**Notation A.2.3.** If there is a fixed algebra \( A \) so that for every \( e \in E^c \cup e^0 \) the edge algebras are \( A_e = A \), we will use the letter \( m \) instead of \( f \) to signify that this should be interpreted as a product relation on \( A \),

\[
m^{(T,L)} : A^{\otimes \nu(T)} \to A.
\]

If the trees are supposed to describe some kind of relation between the operations labelling the vertices, we will label the morphism \( q \).

**Remark A.2.4.** To specify the data of a labelled tree \((T, L)\), it suffices to specify labels of compatible morphisms on the internal vertices, as one can recover the edge data from the domain and codomains of these morphisms.

The quadratic \( A_\infty \) relations may be restated as the following sum over all trees with two internal vertices, with each vertex labelled as \( m^{\text{deg}(v)-1} \). The set of such trees for the uncurved \( k = 2 \) relation is displayed in fig. 11:

\[
\sum_{(T,L) : \nu(T)=k \atop |V(T)|=2, f^e=m^{\text{deg}(v)-1}} q^{(T,L)} = 0.
\]
A.3 Morphisms of filtered $A_\infty$ algebras

The definition of morphisms between filtered $A_\infty$ algebras is similar to the definition of morphisms of differential graded algebras, except that the homomorphism relation is relaxed by homotopies.

**Definition A.3.1.** Let $(A, m^k_A)$ and $(B, m^k_B)$ be $A_\infty$ algebras. A weakly-filtered $A_\infty$ homomorphism from $A$ to $B$ is a sequence of graded maps 

$$f^k : A^\otimes k \to B$$

satisfying the following conditions:

- **Weakly Filtered:** The maps nearly preserve energy
  $$f^k(F^{\lambda_1} A, \ldots, F^{\lambda_k} A) \subset F^{-c \cdot k + \sum_{i=1}^k \lambda_i} B$$
  for some fixed constant $c$ called the energy loss of $f$ with $c < \text{val}(m^0_A)$.

- **Quadratic $A_\infty$ relations:** The $f^k, m^k_A$ and $m^k_B$ mutually satisfy the quadratic filtered $A_\infty$ homomorphism relations
  $$\sum_{(j_1+j_2=k)} \pm f^{j_1+1+j_2}(\text{id} \otimes m^j_A \otimes \text{id} \otimes m^{j_2}) = \sum_{i_1+\cdots+i_l=k} \pm m^l_B(f^{i_1} \otimes \cdots \otimes f^{i_l})$$

Suppose that $A$ and $B$ are $A_\infty$ algebras with unit. We say that $f^k$ is a unital $A_\infty$ homomorphism if

$$f^1(e_A) = e_B \quad \quad f^k(\text{id} \otimes m^j_A \otimes \text{id} \otimes m^{j_2}) = 0.$$  

The quadratic $A_\infty$ homomorphism relation may also be written as

$$\sum_{(T,L) : \nu(T)=k, |V^c|=2} q^{(T,L)} = 0,$$

where $v_0$ is labelled $f^j$, every other vertex is labelled $m^j_A$ or $m^j_B$.

This can be re-expressed as:

$$\sum_{(T,L) : \nu(T)=k, v_0 \text{ labelled } m^j_A \text{ or } m^j_B} q^{(T,L)} = 0.$$  

Our tree notation becomes more useful for constructing new morphisms out of old.
Claim A.3.2. Let $f^k : A^\otimes k \to B$ and $g^k : B^\otimes k \to C$ be two filtered $A_\infty$ homomorphisms. Then
\[
(g \circ f)^k := \sum_{j_1 + \cdots + j_l = k} g^l(f^{j_1} \otimes \cdots \otimes f^{j_l})
\]
is an $A_\infty$ homomorphism.

See fig. 12 for a typical term which appears in the composition.

A.4 Deformations of $A_\infty$ algebras

The presence of higher product structures gives us additional wiggle room to deform the product structures on a filtered $A_\infty$ algebra. We will be mainly interested in the case when we can deform a given filtered $A_\infty$ algebra into an uncurved one. This is useful as the theory of uncurved $A_\infty$ algebras is easier to work with than the theory of curved $A_\infty$ algebras. In particular, the uncurved setting reduces to algebra on the level of cohomology.

Notation A.4.1. As a shorthand, we write
\[
(id \oplus a)^{\binom{n+k}{n}}_a = \sum_{j_0 + \cdots + j_k = n} (a^{\otimes j_0} \otimes \text{id} \otimes a^{\otimes j_1} \otimes \text{id} \otimes \cdots \otimes a^{\otimes j_k-1} \otimes \text{id} \otimes a^{\otimes j_k})
\]
for the sum over all monomials containing $n + k$ terms, $n$ of which are $a$ and $k$ for which are $\text{id}$.

Definition A.4.2. Let $a \in A$ be an element of positive valuation. Define the $a$-deformed product $m^k_a : A^\otimes k \to A$ by the sum
\[
m^k_a := \sum_n m^{k+n} \left( (id \oplus a)^{\binom{n+k}{n}}_a \right).
\]
We call this a graded deformation if the element $a$ has homological degree 1.
In the notation of trees the deformed product can be expressed as the sum

\[ m^k_a = \sum_{(T,L) \mid \nu(T)=k, \nu^0 \text{ is labelled } m^k} m^{(T,L)}. \]

Every internal leaf is labelled \( a \).

See fig. 13 for one of the terms of this sum. Note that this will be an infinite sum, as the number of trees with a bounded number of external leaves need not be bounded in the number of internal leaves. However, each internal leaf contributes some valuation to the composition, so at a bounded valuation the number of trees contributing to \( m^k_a \) is finite. This ensures convergences.

**Claim A.4.3.** \((A, m^k_a)\) is again a filtered curved \( A_\infty \) algebra.

**Example A.4.4.** The simplest example of a deformation is in a DGA where \( m^i = 0 \) for \( i \neq \{1, 2\} \). Assuming that the product \( \wedge \) is anticommutative, the deformed product

\[ d^1_a(x) = d(x) + (a \wedge x) \]

gives a chain complex when \( a \) is an element of cohomology. To preserve the cohomological grading, \( a \in H^1(A) \). This is the standard twisting of the differential on a differential graded algebra.

We are interested in the cases where \((A, m_a)\) gives us a well defined homology theory even though \( A \) itself may be curved.

**Definition A.4.5.** We say that \( a \in A \) is a bounding cochain or Maurer-Cartan solution if

\[ m^0_a = \sum_k m^k(a \otimes k) = 0. \]

If \( A \) has a bounding cochain, we say that \( A \) is unobstructed\(^3\) Suppose that \( A \) has a unit. We say that \( a \in A \) is a weak bounding cochain or weak Maurer-Cartan solution if

\[ m^0_a = W \cdot e_A, \]

where \( e_A \) is a unit and \( W \) is some constant called the obstruction superpotential. If \( A \) has a weak bounding cochain, we say that \( A \) is weakly unobstructed.

\(^3\)Compare this to when \( m^0 = 0 \), when we say that \( A \) is tautologically unobstructed.
The presence of either a bounding cochain or weak bounding cochain ensures that \( m_a^1 \circ m_a^1 = 0 \). In the weak bounding case, this follows from the short computation:

\[
m_a^1 \circ m_a^1 = m^2 \circ (\text{id} \otimes (W \cdot e)) - m^2 \circ ((W \cdot e) \otimes \text{id}) = 0.
\]

In both the unobstructed or weakly unobstructed case, we have a well defined cohomology theory of \((A, m^k)\).

**Definition A.4.6.** Let \( A \) be an \( A_\infty \) algebra. The space of Maurer-Cartan elements is defined as

\[
\mathcal{MC}(A) := \{ a \in A : m^0_a = 0 \}.
\]

Similarly, the space of weak Maurer-Cartan elements is the space

\[
\mathcal{MC}_W(A) := \{ a \in A : m^0_a = W \cdot e \}.
\]

The Maurer-Cartan equation is non-linear. In the event that the Maurer-Cartan space contains a linear subspace, then 0 is a Maurer-Cartan element and the algebra \( A \) is uncurved.

**Lemma A.4.7.** Let \( f : A \to B \) be a weakly filtered \( A_\infty \) morphism (preserving units) of energy loss \( c \). Then there exists a pushforward map between the (weak) bounding cochains with valuation at least \( c \) on \( A \) the (weak) bounding cochains of \( B \) given by

\[
f_* : \mathcal{MC}_W(A) \to \mathcal{MC}_W(B)
\]

\[
b_A \mapsto \sum_k f^k(b^\otimes A^k)
\]

**Proof.** In order for \( \sum_k f^k(b^\otimes A^k) \) to converge, it suffices for the valuation of \( b_A \) to be greater than the energy loss of \( f \), which was assumed. We want to show that \( b_B = \sum_k f^k(b^\otimes A^k) \) satisfies the (weak) Maurer-Cartan equation

\[
\sum_k m_B^k(b^\otimes B^k) = \sum_k m_B^k \left( \left( \sum_{j_1} f^{j_1}(b^\otimes A^{j_1}) \right) \otimes \cdots \otimes \left( \sum_{j_k} f^{j_k}(b^\otimes A^{j_k}) \right) \right)
\]

\[
= \sum_l \sum_k \left( \sum_{j_1 + \cdots + j_k = l} m_B^k(f^{j_1} \otimes \cdots \otimes f^{j_k}) \right) \circ (b^\otimes l)
\]

\[
= \sum_l \sum_{i_1 + j_2 = l} f^{i_1 + i_2 + 1}(\text{id} \otimes m^i_A \otimes \text{id} \otimes j_2) \circ (b^\otimes l)
\]

\[
= \sum_{i_1, i_2} f^{i_1 + i_2 + 1}(b^\otimes A \otimes (W \cdot e_A) \otimes b^\otimes A)
\]

\[
=W \cdot e_B + W \cdot \sum_{i_1 + i_2 > 0} f^{i_1 + i_2 + 1}(b^\otimes A \otimes e_A \otimes b^\otimes A)
\]

In the case where \( b_A \) is a bounding cochain, \( W = 0 \) and we are finished. In the case where \( W \neq 0 \), the requirement that \( f \) be unital means that the right terms vanish.

\[\square\]
Surprisingly, deformations commute with each other in the following sense:

Claim A.4.8. Let \( a_1, a_2 \) be elements of \( A \). Then \((A, (m_{a_1})_{a_2}) = (A, m_{a_1 + a_2})\).

Proof. A calculation shows that

\[
(m_{a_1}^k)_{a_2} = \sum_n m_{a_1}^{k+n} (\text{id} \oplus a_2)^{\binom{n+k}{n}}_{a_2} = \sum m \sum_n m^{k+m+n} (\text{id} \oplus a_1)^{\binom{n+k+m}{m}}_{a_1} \circ (\text{id} \oplus a_2)^{\binom{n+k}{n}}_{a_2} = \sum m^{k+m+n} (\text{id} \oplus a_1 \oplus a_2)^{\binom{n+m+k}{m+n}}_{a_1 + a_2} = m_{a_1 + a_2}^k
\]

Remark A.4.9. Because the space of Maurer-Cartan elements is cut out by a non-linear equation it is unlikely that if \( a_0 \) and \( a_1 \) are bounding cochains that \( a_0 + a_1 \) is similarly a bounding cochain. However, this remark states that at a Maurer-Cartan element \( b \), there may well be a linear tangent space of deformations corresponding to the first homology of \((A, m^1_b)\)

Claim A.4.10. Let \( f : A \to B \) be a filtered \( A_\infty \) algebra morphism. Then there exists an \( A_\infty \) homomorphism

\[
f_{\flat} : (A, (m_A)) \to (B, (m_B)_{f,(0)})
\]

where \( f_{\flat} \) is defined\(^4\) by

\[
f_{\flat}^k = \begin{cases} f^k & \text{for } k > 0 \\ 0 & \text{if } k = 0 \end{cases}
\]

Claim A.4.11. Let \( f : A \to B \) be a filtered \( A_\infty \) algebra morphism. Let \( a \in A \) be a deforming element. Then there is a map

\[
f_a : (A, (m_A^k)_a) \to (B, m_B^k).
\]

Proof. Define \( f_a^k \) to be the map

\[
f_a^k := \sum_n f^{k+n} \circ (\text{id} + a)^{\binom{n+k}{n}}_a.
\]

\(^4\)The the notation is read “f-flat”, as this is the flat version of the curved \( A_\infty \) homomorphism \( f \).
We show that this satisfies the quadratic $A_\infty$ relations by explicit computation.
\[
\sum_{j_1+i+j_2=k} f_{j_1+1+j_2} (\text{id} \otimes (m^i_A)_a \otimes \text{id} \otimes j_2)
\]
\[
= \sum_{j_1+i+j_2=k} f_{j_1+n_1+1+j_2+n_2} \left( \left( (\text{id} \oplus a)^{(j_1+n_1)}_a \right) \otimes m^{i+m} \left( (\text{id} \oplus a)^{(i+m)}_a \otimes (\text{id} \oplus a)^{(j_2+n_2)}_a \right) \right)
\]
\[
= \sum_{n} \sum_{n_1+m+n_2=n} \sum_{j_1+i+j_2=k} \left( f^{k+n} \left( (\text{id} \otimes (j_1+n_1) \otimes m^{i+m} \otimes \text{id} \otimes (j_2+n_2)) \right) \right) \circ (\text{id} \oplus a)^{(k+n)}_a
\]
\[
= \sum_{n} \sum_{i_1+\cdots+i_l=n+k} m^h_B (f^{i_1} \otimes \cdots \otimes f^{i_l}) \circ (\text{id} \oplus a)^{(k+n)}_a
\]
\[
= \sum_{h_1+\cdots+h_l=k} m_{j_2}^h (f^h_{j_1} \otimes \cdots \otimes f^h_{j_l})
\]

One may use the previous two claims to construct the pushforward map on bounding cochains, as
\[
f_\ast (b) = (f_\ast)(0)
\]
This, along with the statement on the pushforward of a bounding cochain, proves the following characterization of unobstructed $A_\infty$ algebras.

**Corollary A.4.12.** Let $A, B$ be filtered $A_\infty$ algebras. There exists a zero-morphism $0 : A \to B$ with $f^i = 0$ for $i \geq 1$ if and only if $B$ is unobstructed.

## B Curved homotopy transfer theorem

**Theorem B.0.1 (Curved homotopy transfer theorem).** Let $B$ be a filtered $A_\infty$ algebra and let $(A, d_A)$ be a graded $\Lambda$-module with differential (not necessarily squaring to zero). Suppose that we have maps:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\text{\textcircled{h}} & \text{\textcircled{h}}
\end{array}
\]

which satisfy the following relations:

1. $\beta \circ m^0_B - d_A \circ \beta = \beta \circ (m^2_B \circ (h \otimes m^0_B) + m^2_B \circ (m^0 \otimes h))$
2. $\alpha \circ d_A - m^0_B \circ \alpha = h \circ (m^2_B \circ (\alpha \otimes m^0_B) + m^2_B \circ (m^0_B \otimes \alpha))$
3. $h \circ m^1_B + m^1_B \circ h = \text{id} - \alpha \circ \beta + h \circ (m^2_B \circ (m^0_B \otimes h) + m^2_B \circ (h \otimes m^0_B))$
4. $\beta \circ h = 0$
5. $\beta \circ \alpha = \text{id}_A$
6. $h \circ \alpha = 0$
7. $\text{val}(h \circ m^0_B) > 0$

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Then we can extend the maps $\beta$ and $d_A$ to $A_\infty$ homomorphisms and differentials so that

- $A$ comes with a filtered $A_\infty$ structure $(A, m^k_A)$ which matches $d_A$ at zero valuation and;
- for this choice of filtered $A_\infty$ structure, one can extend the map $\beta$ to an $A_\infty$ homomorphism $\beta^k : B^\otimes k \to A$ and;
- we can extend $\alpha$ to a $A_\infty$ morphism $\alpha^k : A^\otimes k \to B$.

Note that this relates to the homotopy transfer theorem of [Kad80] when $m^0_B = 0$, as eqs. (6) to (8) become the chain map relations and homotopy relation. The additional conditions on the compositions of $\beta \circ h, h \circ \alpha$ are called the strong deformation retract conditions. The strong deformation retract conditions can always be arranged for chain complexes after applying a homotopy. While $\beta$ and $\alpha$ are not chain maps, they satisfy a weaker relation:

**Claim B.0.2.** $\beta \circ m^1_B \circ m^1_B \circ \alpha = d_A \circ d_A$

**Proof.**

\[
\beta \circ m^1_B \circ m^1_B \circ \alpha = \beta \circ m^1_B \circ \alpha \circ d_A + \beta \circ m^1_B \circ h \circ (m^2_B \circ (\alpha \otimes \text{id}) + m^2_B \circ (\text{id} \otimes \alpha)) \\
= \beta \circ \alpha \circ d_A \circ d_A + \beta \circ h \circ (m^2_B \circ ((\alpha \circ d_A) \otimes \text{id}) + m^2_B \circ ((\text{id} \otimes \alpha) \circ d_A) + \\
+ \beta \circ \left( h \circ m^1_B + \alpha \circ \beta - \text{id} \right) \circ (m^2_B(\alpha \otimes \text{id}) + m^2_B(\text{id} \otimes \alpha)) \\
= d_A \circ d_A
\]

For future applications, it will be useful to have the following special case of the curved homotopy transfer theorem, where we only construct the $A_\infty$ homomorphism $\alpha^k$, but use a given $A_\infty$ product structure on $A$.

**Claim B.0.3.** Let $A$ and $B$ be filtered $A_\infty$ algebras. Let $\beta : B \to A$ be an $A_\infty$ homomorphism with no higher terms. Suppose that there is a map $\alpha : A \to B$ so that $\beta \circ \alpha = \text{id}_A$. Furthermore, suppose we have a map $h : B \to B$ so that $\alpha, \beta,$ and $h$ satisfy the conditions of theorem B.0.1. Then the $A_\infty$ extension of $d_A = m^1_A$ constructed by the homotopy transfer theorem matches $m^k_A$.

### B.1 Definition of $m^k_A$ and $\phi^k_A$.

The remainder of this appendix is devoted to the proof of theorem B.0.1.

We now describe $\alpha^k : A^\otimes k \to B$ and $m_A^k : A^\otimes k \to A$ which mutually satisfy the $A_\infty$ relations. The maps can be constructed inductively (see [Sei08] for the tautologically unobstructed setting) but we will describe them using trees. We start by defining preliminary maps $\phi^{T,L_{hm}} : A^\otimes k \to B$ associated to a tree $T$. We define the labelling $L_{hm}$ of $T$ by the following rules:
Figure 14: A typical example of a tree with the homotopy transfer theorem labelling contributing to $f$.

- We label each external leaf with the vector space $A$. We label every other edge $B$.
- Every vertex $v$ not equal to the root vertex $v_0$ obtains the label $h \circ m_B^{\deg(v)-1}$. The root is labeled as $m_B^1$.
- If $v$ is a vertex of $T$ incident to an external leaf, we pre-compose with the appropriate tensor product of inclusions $\alpha : A \to B$ and $\text{id} : B \to B$ so that the domain of the label of $v$ matches its upward edges.

See fig. 14 for a typical tree. Recall that a stable tree must have at least one interior vertex and no vertices of degree 2. From this preliminary map, we define the following deformation terms:

$$\alpha^k := \sum_{\text{Stable trees } T} h \circ \phi^{T,L,hm}_{\nu(T)=k}$$

$$m_A^k := \sum_{\text{Stable trees } T} \beta \circ \phi^{T,L,hm}_{\nu(T)=k}$$

We then define the maps $m_A^k : A^{\otimes k} \to A$ and $i^k : A^{\otimes k} \to B$ as:

$$\alpha^k := \begin{cases} 
\alpha + \alpha^1 & \text{if } k = 1 \\
\alpha^k & \text{if } k \neq 1
\end{cases}$$

$$m_A^k := \begin{cases} 
d_A + m_A^1 & \text{if } k = 1 \\
m_A^k & \text{if } k \neq 1
\end{cases}$$

Both $\alpha^k$ and $m_A^k$ are defined by infinite sums, but these sums converge over $\Lambda$, as the valuation of a morphism associated by a tree can be bounded below by a function of the number of internal leaves when $h$ is weakly filtered. This is sufficient to ensure convergence in the Novikov field.

We can now check that claim B.0.3 holds.
\textbf{Proof of claim B.0.3}. As the map $\beta : B \to A$ is a filtered $A_\infty$ homomorphism, the kernel of this map is an $A_\infty$ ideal. If the morphism $\phi(T,L^{hm}_B)$ contains a labelling of $h$ — in particular a non-root vertex — then $\beta \circ \phi(T,L^{hm}_B) = 0$. The only trees which contain no non-root vertices are those with just one vertex. If $T^k_1$ is the tree with a single vertex, 
\[ \beta \circ \phi(T^k_1,L^{hm}) = \beta \circ m^k_B \circ (\alpha^\otimes k) = m^k_A \circ ((\beta \circ \alpha))^\otimes k = m^k_A. \]

We now proceed to the main portion of the proof of the homotopy transfer theorem, which is to verify that the morphisms $\alpha^k, m^k_A, m^k_B$ and $\beta^k$ mutually satisfy the quadratic $A_\infty$ homomorphism relations.

\textbf{B.1.1 Observation and Motivation.}

We now will look at some modifications of the labelling of our trees which motivate why these trees are relevant to the $A_\infty$ homomorphism relations. We consider the following “broken” labelling of trees.

- \textit{\alpha \circ \beta broken Trees}: At a specified vertex $v$, we substitute the label $\alpha \circ \beta \circ m^{\deg(v)-1}_B$ instead of the standard label $h \circ m^k_B$. We call the corresponding labelling an “$\alpha \circ \beta$ tree broken at $v$” and write $L^{\alpha \circ \beta}_v$. The $\alpha \circ \beta$ broken trees can be also expressed in the following way. Let $T^\uparrow_v$ be the tree which is obtained by taking all edges upwards of $v$ attached to root edge $e^\downarrow_v$ so that $v$ is the root vertex of the tree $T^\uparrow_v$. Let $T^\downarrow_v$ be the tree which consists of all edge not upward of $v$, so that $v$ is an external leaf of this new tree. We note that an $\alpha \circ \beta$ broken tree is equivalent to the composition 
\[ \phi(T,L^{\alpha \circ \beta}_v) = \phi(T^\uparrow_v,L^{hm}) \circ (\text{id}^\otimes k^1 \otimes (\beta \circ \phi(T^\downarrow_v,L^{hm})) \otimes \text{id}^\otimes k^2), \]
where $k^1$ is the number of leaves left of the vertex $v$ and $k^2$ is the number of leaves right of $v$, so that $\text{val}(T) = k^1 + \text{val}(T^\downarrow_v) + k^2$. See fig. 15.

- \textit{id broken trees}: At a specified vertex $v$, we substitute in the label $\text{id} \circ m^{\deg(v)-1}_B$. We call the corresponding label $L^{\text{id}}_v$.

The following observations become the framework for proving the homotopy transfer theorem:

1. The sum over all id-broken stable trees at $v_0$ of fixed valence nearly gives the right hand side of the $A_\infty$ relations,
\[ \sum_{T : \text{val}(T) = k} h \circ \phi(T,L^{\text{id}}_{v_0}) = \sum_{i_1 + \cdots + i_j = k, k > 1} m^j_B (\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_j}) \quad (9) \]

2. The sum over all $\alpha \circ \beta$ broken trees of fixed valence nearly gives the left hand side of the $A_\infty$ homomorphism relations,
\[ \sum_{v \in T | \text{val}(T) = k} h \circ \phi(T,L^{\alpha \circ \beta}_v) = \sum_{(j_1 + 1 + j_2 = k, i > 1)} \alpha^{j_1 + 1 + j_2} (\text{id}^\otimes j_1 \otimes m^i_A \otimes \text{id}^\otimes j_2) \quad (10) \]
Figure 15: The $\alpha \circ \beta$ broken tree is equivalent to a composition of two morphisms from trees.

**B.2 A relation between the $\phi^{(T,L)}$**

In this section we expand the boundary $m_B^1 \circ \phi^{(T,L^m_h)}$ as a sum of broken trees and show that these boundary components have many terms which mutually cancel out. The correspondences between the boundary components arise from matching expansions and contractions of stable trees.

**Definition B.2.1.** We say that $T$ is a 1-unstable tree if it has a single vertex $v$ of degree two. If a tree $T$ is 1-unstable, we say that the vertex $v$ the unstable vertex of $T$. Let $T$ be a 1-unstable tree. The instability distance of $T$ is the distance from the root to the unstable vertex.

We show that the quadratic $A_\infty$ relations allow us to reexpress a term $\phi^{(T,L^m_h)}$ where $T$ is an 1-unstable trees as a sum of unstable trees with greater instability distance.

Let $v_e$ be the unstable vertex of an unstable tree $T \div e$. Let $T_{\downarrow e}$ and $T_{\uparrow e}$ be the two stable trees obtained from $T \div e$ by attaching a leaf to the vertex $v_e$. On such a tree, let $w_{-e}$ and $w_{+e}$ be the attached vertex. The subdivision of a tree is never stable, as the new vertex $v_e$ has degree 2.

**Claim B.2.2 (Homotopy Relation).** Let $T$ be a tree. Let $e$ be an interior edge with upward vertex $v_{\uparrow e}$.

$$
\phi^{(T \div e, L_{v_{\downarrow e}}} + \phi^{(T \div e, L_{v_{\uparrow e}}} + \phi^{(T, L_{v_{\downarrow e}^{\uparrow e}})} + \phi^{(T, L_{v_{\uparrow e}^{\downarrow e}})} + \phi^{((T)_{-e}, L_{w_{-e}}} + \phi^{((T)_{+e}, L_{w_{+e}})} = 0
$$

This is a restatement of eq. (8) in terms of trees. See fig. 16.

Given a stable tree $T$ and a vertex $v \in T$, an expansion of $T$ at $v$ is a planar tree $T'$ with two vertices $v_1, v_\uparrow \subset V(T')$ so that the contraction $T'/\{v_1, v_\uparrow\} = T$ and under this contraction

---

5These two trees are isomorphic as graphs, but not necessarily as planar graphs.
both $v_+^i$ and $v_+^j$ are identified with $v$. Note that the number of vertices increases by one after taking an expansion. If $T'$ is an expansion of $T$ at a vertex $v$, we label it $(T', L_{\text{id} v_+^i})$.

Claim B.2.3 (Associativity Identity). Let $T$ be a tree. Let $v \in T$ be any vertex. Then the sum over all expansions of $T$ at $v$,

$$\sum_{(T', L_{\text{id} v_+^i}) \in \frac{T'/\{v_+^i v_+^j\}}{T'}} \phi^{(T', L_{\text{id} v_+^i})} = 0.$$ 

This is a restatement of the $A_\infty$ product relations at the fixed vertex $v$.

Lemma B.2.4 (Boundary of a cell). Let $T$ be a stable tree.

$$\phi^{(T \uplus_{e_0} L_{\text{id} e_0})} = \sum_{(T', L_{\text{id} v_+^i}) \in \frac{T'/\{v_+^i v_+^j\}}{T'}} \phi^{(T', L_{\text{id} v_+^i})} + \sum_{e \in E(T), \ e \neq e_0} \phi^{(T \uplus_{e} L_{\text{id} w_{e}})} + \phi^{(T \uplus_{e} L_{\text{id} w_{e}})}$$

\[\begin{align*}
\text{Bubbling at edges} \\
\sum_{v \in V(T), v \neq v_0} \phi^{(T, L_{v_0})} + \sum_{v \in V(T)} \phi^{(T, L_{v_0}^\alpha \beta)} + \phi^{(T, L_{\text{id} v_0} \otimes d_A \otimes \text{id}^\otimes k_2)}.
\end{align*}\]

\[\begin{align*}
\text{id broken trees} \\
\alpha \circ \beta \text{ broken trees}
\end{align*}\]

\[\begin{align*}
\text{Again, the planarity condition is important here— there are multiple isomorphic trees which are different as expansions of a fixed tree.}
\end{align*}\]
Proof. The proof of this relation comes from repeatedly applying the homotopy and associativity relation to the tree labeled $\phi^{(T \div e_0, L_{\text{id}v_0})} = m_1 \circ \phi^{(T, L)}$.

- If we have an unstable tree id-broken above the unstable vertex, an application of the associativity relation produces expansions of $T$ which are id-broken at the expanded vertex. We sort these expansions into two types:
  - If the expansion is stable, we keep this as a boundary term.
  - If the expansion is unstable, we pass this tree to the next step in the algorithm. Note that the unstable expansions have greater instability distance.

- If we have an unstable tree id-broken at the unstable vertex, we can apply the homotopy relation. This produces contractions of $T$ which are stable, and an unstable tree id-broken above the unstable vertex.
  - We keep the contractions which are stable as boundary terms.
  - We pass the unstable tree which is id-broken above the unstable vertex to the first step of the algorithm.

Each time we apply these two steps, we replace the 1-unstable tree $T \div e$ with stable broken trees and 1-unstable trees of greater instability distance. If at any point during this algorithm the unstable vertex proceeds to the boundary of the tree, we apply eq. (8). See fig. 17 for a diagram of this algorithm.

**Proposition B.2.5** (Cell Incidence relations). For all $k$, we have the following relation:

$$\sum_{\nu(T) = k} \left( \sum_{v \in V(T), v \neq v_0} \phi^{(T, L_{\text{id}v})} \right) = \sum_{\nu(T) = k} \left( \sum_{(T', L') \text{ a stable expansion of } T} \phi^{(T', L_{\text{id}v})} + \sum_{e \in E(T)} \phi^{(T \div e, L_{\text{id}w})} \phi^{(T \div e, L_{\text{id}w})} \right)$$

(11)

Proof. This comes from the bijections for trees:

$$\bigcup_{T' \text{ stable}, \nu(T') = k} \{(T', e) : e \in T', T' / e \text{ is stable}\} \rightarrow \bigcup_{T \text{ stable}, \nu(T) = k} \{\text{Stable expansions } T' \text{ of } T\}$$

and

$$\bigcup_{T' \text{ stable}, \nu(T') = k} \{(T', e) : e \in T', T' / e \text{ is not stable}\} \rightarrow \left\{ \begin{array}{ll} \bigcup_{T \text{ stable or } |V(T)| = 0} & \{T_{-e} : e \in T\} \\ & \bigcup_{\nu(T) = k} \{T_{-e} : e \in T\} \\ \bigcup_{T \text{ stable or } |V(T)| = 0} & \{T_{e} : e \in T\} \\ \bigcup_{\nu(T) = k} & \{T_{e} : e \in T\} \end{array} \right\}$$
Figure 17: Repeated applications of homotopy and associative relations.
These are bijections of infinite sets, and so we need to check that using this bijection to equate two sums will converge over the Novikov field. While the bijection does not respect the grading on each set given by number of internal leaves, the difference in the number of internal leaves in these bijections is at most one, which is sufficient to prove Novikov convergence.

Applying the border relations eq. (III) to the boundaries in lemma B.2.4 gives us the following relation between the maps \( f \).

**Lemma B.2.6** (Fundamental relation on the maps \( f \)). For each \( k \), the following sum of maps is zero:

\[
\sum_{\nu(T)=k} \phi^{(T\div e_0,L_{e=0}^{id})} + \sum_{\nu\in V(T)} \phi^{(T,L_{e=0}^{id}\circ\beta)} + \sum_{k_1+1+k_2=\nu(T)} \phi^{(T,L)} \circ (\text{id} \otimes k_1 \otimes d_A \otimes \text{id} \otimes k_2)
\]

\[
- \sum_{\nu(T)=k} \phi^{(T\div e_0,L_{e=0}^{id})} + \phi^{(T,e_0^{id}\div)} + \phi^{(T,e_0^{id}\div e_0^{id})}
\]

We now apply this lemma to prove the \( A_\infty \) relations needed for theorem B.0.1.

**B.3 \( A_\infty \) relations from fundamental relation**

We only check the relations for the \( m_A^k \) and \( \alpha^k \), as the other relations are proven in a similar fashion.

**B.3.1 Proving that \( m_A^k \) satisfy the \( A_\infty \) relations.**

Before we show that the \( m_A^k \) satisfy the \( k \)-quadratic \( A_\infty \) relations, we restate the terms from lemma B.2.6

\[
\phi^{(T\div e_0,L_{e=0}^{id})} = m_B^1 \circ \phi^{(T,L_{e=0}^{id})}
\]

\[
\sum_{e \in E} \phi^{(T\div e,L_{e=0}^{id})} = \sum_{j_1+1+j_2=k} \phi^{(T\div e,L_{e=0}^{id})} \circ (\text{id} \otimes j_1 \otimes d_A \otimes \text{id} \otimes j_2)
\]

\[
\phi^{(T,L_{e=0}^{id}\circ\beta)} = \phi^{(T,L)} \circ (\text{id} \otimes k_1 \otimes (m(T_{e=0}^{id},L_{e=0}^{id}) \otimes \text{id} \otimes k_2)
\]

\[
\phi^{(T\div e_0,L_{e=0}^{id})} + \phi^{(T,e_0^{id}\div)} + \phi^{(T,e_0^{id}\div e_0^{id})}
\]

\[
\beta \circ m_B^2(m_B^2(\alpha \otimes m_B^0) + m_B^2(m_B^0 \otimes \alpha)) = d_A \circ d_A
\]
Recall that for all $k, m_A^{(T,L)} := \beta \circ \phi^{(T,L)}$. After applying $\beta$ to the relation in lemma B.2.6
we obtain:

$$0 = \beta \circ m_B^1 \circ \phi^k + \left( \sum_{j_1+i+j_2=k} \beta \circ \phi^{j_1+1+j_2} \circ ( id^\otimes j_1 \otimes ( \beta \circ \phi^i ) \otimes id^\otimes j_2 ) \right)$$

$$+ \left( \sum_{j_1+i+j_2=k} \beta \circ \phi^k \circ ( id^\otimes j_1 \otimes d_A \otimes id^\otimes j_2 ) \right)$$

$$- \beta \circ ( m_B^2 \circ ( m_B^0 \otimes ( h \circ \phi^k ) ) + m_B^2 \circ (( h \circ \phi^k ) \otimes m_B^0 ) )$$

$$+ \frac{ \beta \circ m_B^2 ( \alpha \otimes m_B^0 ) + m_B^2 ( m_B^0 \otimes \alpha ) }{ \text{if } k = 1 }$$

$$= d_A \circ m_A^k \circ m_B^k + \beta \circ ( ( h \circ \phi^k ) \otimes m_B^0 ) + m_B^2 \circ ( m_B^0 \otimes ( h \circ \phi^k ) )$$

$$+ \left( \sum_{j_1+i+j_2=k} m_B^{j_1+1+j_2} \circ ( id^\otimes j_1 \otimes m_A^i \otimes id^\otimes j_2 ) \right)$$

$$+ \left( \sum_{j_1+i+j_2=k} m_A^k \circ ( id^\otimes j_1 \otimes d_A \otimes id^\otimes j_2 ) \right)$$

$$- \beta \circ ( m_B^2 ( m_B^0 \otimes ( h \circ \phi^k ) ) + m_B^2 (( h \circ \phi^k ) \otimes m_B^0 ) ) + d_A \circ d_A$$

$$= \sum_{j_1+i+j_2=k} m_A^{j_1+1+j_2} \circ ( id^\otimes j_1 \otimes m_A^i \otimes id^\otimes j_2 ).$$

**B.3.2 Proving that $m_A^k$ and $\alpha^k$ satisfy the $A_\infty$ relations**

Recall that for all $k \neq 1, \alpha_A^{(T,L)} := h \circ \phi^{(T,L)}$. After composing $h$ with lemma B.2.6 we obtain:
\[ 0 = h \circ m_B^1 \circ \phi^k + \left( \sum_{j_1 + i + j_2 = k} h \circ \phi^{j_1 + 1 + j_2} \circ (\text{id} \otimes j_1 \otimes (\beta \circ \phi^i) \otimes \text{id} \otimes j_2) \right) \]
\[ + \left( \sum_{j_1 + i + j_2 = k} h \circ \phi^k \circ (\text{id} \otimes \phi \otimes \text{id} \otimes j_2) \right) \]
\[ - h \circ (m_B^2 \circ (h \circ \phi^k)) + m_B^2 \circ (h \circ \phi^k \otimes m_B^0) \]
\[ + h \circ m_B^2 \circ (\alpha \otimes m_B^0) + m_B^2 \circ (m_B^0 \otimes \alpha) \]
\[ \text{if } k = 1 \]
\[ = (\text{id} + (m_B^1 \circ h) + (\alpha \circ \beta) + h \circ m_B^2 \circ (h \otimes m_B^0) + h \circ m_B^2 \circ (m_B^0 \otimes h)) \circ \phi^k \]
\[ + \left( \sum_{j_1 + i + j_2 = k} \alpha^{j_1 + 1 + j_2} \circ (\text{id} \otimes \phi \otimes j_1 \otimes \text{id} \otimes j_2) \right) + \sum_{j_1 + i + j_2 = k} \alpha^k \circ (\text{id} \otimes \phi \otimes \text{id} \otimes j_2) \]
\[ - h \circ (m_B^2 \circ (m_B^0 \otimes (h \circ \phi^k)) + m_B^2 \circ (h \circ \phi^k \otimes m_B^0)) \]
\[ + \alpha \circ d_A + m_B^1 \circ \alpha \]
\[ \text{if } k = 1 \]
\[ = \phi^k + m_B^1 \circ \alpha^k + \alpha \circ m_A^1 + \left( \sum_{j_1 + i + j_2 = k} \alpha^{j_1 + 1 + j_2} \circ (\text{id} \otimes \phi \otimes m_A^1 \otimes \text{id} \otimes j_2) \right) \]
\[ + \sum_{j_1 + i + j_2 = k} \alpha^k \circ (\text{id} \otimes \phi \otimes d_A \otimes \text{id} \otimes j_2) + \alpha \circ d_A + m_B^1 \circ \alpha \]
\[ \text{if } k = 1 \]

Note that \( \phi^k = \sum_{i_1 + \cdots + i_j > 1} m_B^j(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_j}) \)
\[ = \left( \sum_{j_1 + \cdots + i_j} m_B^j(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_j}) \right) + \left( \sum_{j_1 + i + j_2 = k} \alpha^{j_1 + 1 + j_2} \circ (\text{id} \otimes j_1 \otimes (m_A^i) \otimes \text{id} \otimes j_2) \right) \]

This concludes the proof of theorem B.0.1

C Fiber products

Definition C.0.1. Let \( A, B \) be \( A_\infty \) algebras. An \((A, B)\)-bimodule is a filtered graded \( \Lambda \)-module \( M \), along with a set of maps
\[ m_{A|M|B}^{k_1 |l_1 |k_2} : A^{\otimes k_1} \otimes M \otimes B^{\otimes k_2} \rightarrow M \]
Figure 18: A typical term contributing to $m^5_{\mathbb{C}|B}$.

satisfying filtered quadratic $A_\infty$ module relations for each triple $(k_1|1|k_2)$

$$0 = \sum_{j_1+j_2+j_2 = k_1+1+k_2 \atop j_1+j_2 \leq k_1} m_{A|M|B}^{k_1-j+1} \circ (\text{id}_A^{\otimes j_1} \otimes m_A^{j_2} \otimes \text{id}_M^{k_1-j} \otimes \text{id}_B^{k_2}) + \sum_{j_1+j_2+j_2 = k_1+1+k_2 \atop j_1 \leq k_1+j_1-j_1} m_{A|M|B}^{j_1|1|j_2} \circ (\text{id}_A^{\otimes j_1} \otimes m_A^{k_1-j_1} \otimes \text{id}_M^{k_2-j_2} \otimes \text{id}_B^{\otimes j_2}) + \sum_{j_1+j_2+j_2 = k_1+1+k_2 \atop k_2 + 1 < j_1} m_{A|M|B}^{k_1|1|k_2-j+1} \circ (\text{id}_A^{\otimes k_1} \otimes \text{id}_M^{k_2-j_2-j} \otimes m_B^{j_2} \otimes \text{id}_B^{\otimes j_2}).$$

This appears to be three separate sums, but can be restated as one sum in the language over all trees with 2 vertices, one of which is labelled $m_{A|M|B}^{j_1|1|j_2}$.

Remark C.0.2. If $A$ and $B$ are uncurved, an $(A, B)$ bimodule can be made into a left $A$ module by restricting

$$m_{A|M}^{k|1} := m_{A|M|B}^{k|1|0}.$$

The $A_\infty$ relations follow from the quadratic $A_\infty$ relations for the bimodule where the $B$-inputs have been evaluated at 0. It is important to note that this does not hold if the modules $A$ and $B$ are curved, as $m_0^B$ terms may contribute to the quadratic $A_\infty$ on the “right” side of the module relations, causing $M$ to fail as a left $A$ module!

Claim C.0.3 (Change of Basis). Let $C$ be an $A_\infty$ algebra and let $f : A \to C$ and $g : B \to C$ be filtered $A_\infty$ morphisms. Then $C$ has the structure of a $(A, B)$ bimodule.

Proof. The bimodule structure is given by the higher product maps

$$m_{A|C|B}^{k_1|1|k_2} = \sum_{h_1+\ldots+h_{\alpha_1} = k_1 \atop i_1+\ldots+i_{\alpha_2} = k_2} m_{C}^{\alpha_1+1+\alpha_2} \circ (f^{h_1} \otimes \ldots \otimes f^{h_{\alpha_1}} \otimes \text{id}_C \otimes g^{i_1} \otimes \ldots \otimes g^{i_{\alpha_2}}).$$

These correspond to trees labelled in fig. [18]. We show that these satisfy the $A_\infty$ bimodule relations by explicit computation. Figure [19] operadically explains the origin of this computation. We now apologize for an additional piece of notation. Given a fixed sequence of
non-negative integers \( \mathbf{h} = (h_1, \ldots, h_\alpha) \) we will write \( f^\mathbf{h} = f^{\otimes h_1} \otimes \cdots \otimes f^{\otimes h_\alpha} \). Examining the terms of the \( A_\infty \) relations gives us the following preliminary relations for each \( k \),

\[
\sum_{j_1+j_2+j_3=k \atop k_1+1+k_2=1+1+j_2 \atop k_1+j<k_1} m_{A|C|B}^{k_1|k_2} \circ (\text{id}_{A}^{\otimes j_1} \otimes m_{A}^{j} \otimes \text{id}^{\otimes k_1-j_1-j} \otimes \text{id}_{M} \otimes \text{id}_{B}^{k_2})
\]

\[
= \sum_{\mathbf{h}=(h_1, \ldots, h_\alpha) \atop i=(i_1, \ldots, i_\alpha) \atop j'_1+j'_2 \leq \alpha_1} m_{C}^{\alpha_1+1+\alpha_2} (\text{id}^{\otimes j'_1} \otimes m_{C}^{j'} \otimes \text{id}^{\otimes j'_2}) \circ (f_{\mathbf{h}} \otimes \text{id}_{C} \otimes g^4).
\]

\[
\sum_{j_1+j_2+j_3=k \atop k_1+1+k_2=1+1+j_2 \atop j_1 \leq k_1 \leq j_1+j} m_{A|C|B}^{k_1|k_2} \circ (\text{id}_{A}^{\otimes j_1} \otimes m_{A|C|B}^{k_1-j_1|1|k_2-j_2} \otimes \text{id}_{B}^{j_2})
\]

\[
= \sum_{\mathbf{h}=(h_1, \ldots, h_\alpha) \atop i=(i_1, \ldots, i_\alpha) \atop j'_1+a_1<j'_1+j'_2} m_{C}^{\alpha_1+1+\alpha_2} (\text{id}^{\otimes j'_1} \otimes m_{C}^{j'} \otimes \text{id}^{\otimes j'_2}) \circ (f_{\mathbf{h}} \otimes \text{id}_{C} \otimes g^4).
\]
\[
\sum_{j_1+j_2=k, \ k_1+1+k_2=j_1+1+j_2} m_{A|C|B}^{k_1|k_2} \circ (\text{id}_A^{\otimes k_1} \otimes \text{id}_M \otimes \text{id}_B^{k_2-j_2-j} \otimes m_B^{j_2} \otimes \text{id}_{B}^{\otimes j_2})
\]
\[
= \sum_{h=(h_1, \ldots, h_{\alpha_1}) \atop i=(i_1, \ldots, i_{\alpha_2})} m_{C}^{\alpha_1+1+\alpha_2} \circ (\text{id}_C^{\otimes j_1} \otimes m_C^{j_1} \otimes \text{id}_C^{\otimes j_2}) \circ (f_h^{i_1} \otimes \text{id}_C \otimes g_i^{i_2}).
\]

Making these substitutions into the quadratic \(A_\infty\) bimodule relation for fixed \((k_1|k_2)\),
\[
\sum_{j_1+j_2=k_1+1+k_2} \left( \sum_{j_1< j_2} m_{A|C|B}^{k_1-j+1|j|k_2} \circ (\text{id}_A^{\otimes j_1} \otimes m_A^{j_1} \otimes \text{id}_A^{\otimes k_1-j_1-j} \otimes \text{id}_M \otimes \text{id}_B^{k_2}) \right) \\
+ \sum_{j_1< j_2} m_{A|C|B}^{j_1|k_1|k_2-j_2-j} \circ (\text{id}_A^{\otimes j_1} \otimes m_{A|C|B}^{j_1|k_1|k_2-j_2-j} \otimes \text{id}_B^{j_2}) \\
= \sum_{\alpha_1, \alpha_2 \geq 0, \ j_1+j_2=\alpha_1+1+\alpha_2} m_C^{j_1+1+j_2} \circ (\text{id}_C^{\otimes j_1} \otimes m_C^{j_1} \otimes \text{id}_C^{\otimes j_2}) \circ \left( \sum_{h_1, \ldots, h_{\alpha_1}=k_1 \atop i_1, \ldots, i_{\alpha_2}=k_2} (f_h^{i_1} \otimes \text{id}_C \otimes g_i^{i_2}) \right)
\]
\[
=0.
\]

This bimodule construction allows us to construct fiber products in the category of \(A_\infty\) algebras.

**Claim C.0.4.** Suppose we have a diagram of \(A_\infty\) algebras,
\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
B & \xrightarrow{g} & C
\end{array}
\]
Then \(A \times_C B := A \oplus C[1] \oplus B\) can be given the structure of an \(A_\infty\) algebra, called the homotopy fiber product which fits into the diagram

\[
\begin{array}{ccc}
A \times_C B & \xrightarrow{\pi_A} & A \\
\downarrow \pi_B & & \downarrow f \\
B & \xrightarrow{g} & C
\end{array}
\]

**Proof.** We denote by
\[
\pi_{A|C|B}^{k_1|k_2} : (A \oplus C[1] \oplus B) \otimes (k_1+k_2) \to A^{\otimes k_1} \otimes (C[1])^{\otimes k} \otimes B^{\otimes k_2}
\]

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the standard projection. We define an $A\infty$ product on $A \oplus C[1] \oplus B$ by:

$$m_{A \times C B}^k = (m_A^k \circ \pi_{A\mid C \mid B}^{k\mid 0\mid 0})$$

$$\oplus \left( f^k \circ \pi_{A\mid C \mid B}^{k\mid 0\mid 0} + \left( \sum_{k_1+1+k_2=k} m_{A\times C B}^{k_1\mid 1\mid k_2} \circ \pi_{A\mid C \mid B}^{k_1\mid 1\mid k_2} \right) + g^k \circ \pi_{0\mid 0\mid k} \right)$$

$$\oplus \left( m_B^k \circ \pi_{0\mid 0\mid k} \right).$$

The proof is a verification of the $A\infty$ structure. It is immediate from the quadratic relations for $A$ and $C$ that

$$\pi_{A\mid C \mid B}^{1\mid 0\mid 0} \circ \left( \sum_{j_1+j_2=k} m_{A \times C B}^{j_1+1+j_2} (\text{id}^\otimes j_1 \otimes m_{A \times C B}^j \otimes \text{id}^{j_2}) \right) = 0$$

$$\pi_{A\mid C \mid B}^{0\mid 0\mid 1} \circ \left( \sum_{j_1+j_2=k} m_{A \times C B}^{j_1+1+j_2} (\text{id}^\otimes j_1 \otimes m_{A \times C B}^j \otimes \text{id}^{j_2}) \right) = 0$$

It therefore suffices to look at the $B$-component of this relation. We need a useful equality:

$$\sum_{k_1+j+k_2=k} m_{A \times C B}^{k_1\mid 1\mid k_2} \circ (\text{id}_{A}^{\otimes k_1} \otimes f_j \otimes \text{id}_{B}^{\otimes k_2}) \circ \pi_{k_1+j\mid 0\mid k_2}$$

$$+ \sum_{k_1+j+k_2=k} m_{A \times C B}^{k_1\mid 1\mid k_2} (\text{id}_{A}^{\otimes k_1} \otimes g^j \otimes \text{id}_{B}^{\otimes k_2}) \circ \pi_{k_1\mid 0\mid k_2+j}$$

$$= \sum_{h_1+\cdots+h_{\alpha_1}=k_1+j} \sum_{i_1+\cdots+i_{\alpha_2}=k_2} m_{A}^{\alpha_1+\alpha_2} (f^h \otimes g^i)$$

$$+ \sum_{h_1+\cdots+h_{\alpha_1}=k_1} \sum_{i_1+\cdots+i_{\alpha_2}=k_2} m_{B}^{\alpha_1} (f^h)$$

(12)

We start with the preliminary relations:

$$\pi_{A\mid C \mid B}^{j_1+1+j_2\mid 0\mid 0} \circ (\text{id}_{A\times C B}^{\otimes j_1} \otimes m_{A \times C B}^j \otimes \text{id}_{A\times C B}^{\otimes j_2}) = (\text{id}_{A}^{\otimes j_1} \otimes m_{A}^j \otimes \text{id}_{B}^{\otimes j_2}) \circ \pi_{A\mid C \mid B}^{k\mid 0\mid 0}$$

$$\pi_{A\mid C \mid B}^{0\mid 0\mid j_1+1+j_2} \circ (\text{id}_{A\times C B}^{\otimes j_1} \otimes m_{A \times C B}^j \otimes \text{id}_{A\times C B}^{\otimes j_2}) = (\text{id}_{B}^{\otimes j_1} \otimes m_{B}^j \otimes \text{id}_{B}^{\otimes j_2}) \circ \pi_{A\mid C \mid B}^{0\mid 0\mid k}$$

For fixed $(k_1\mid 1\mid k_2)$ and $j_1+j_2 = k_1+k_2$, we have

$$\pi_{A\times C B}^{k_1\mid 1\mid k_2} \circ (\text{id}_{A\times C B}^{\otimes j_1} \otimes m_{A \times C B}^j \otimes \text{id}_{A\times C B}^{\otimes j_2}) = (\text{id}_{A}^{\otimes k_1} \otimes f_j \otimes \text{id}_{B}^{\otimes k_2}) \circ \pi_{k_1+j\mid 0\mid k_2}$$

$$+ (\text{id}_{A}^{\otimes k_1} \otimes g^j \otimes \text{id}_{B}^{\otimes k_2}) \circ \pi_{k_1\mid 0\mid k_2+j}$$

$$+ \left\{(\text{id}_{A}^{\otimes j_1} \otimes m_{A}^{k_1-j_1-j} \otimes \text{id}_{C} \otimes \text{id}_{B}^{\otimes k_2}) \circ \pi_{k_1\mid 1\mid k_2} \right\}$$

$$+ \left\{(\text{id}_{A}^{\otimes j_1} \otimes m_{A\mid C \mid B}^{k_1-j_1-j_2} \otimes \text{id}_{B}^{\otimes k_2}) \circ \pi_{k_1\mid 1\mid k_2} \right\}$$

$$+ \left\{(\text{id}_{A}^{\otimes k_1} \otimes \text{id}_{C} \otimes m_{B}^{k_2-j_2-j} \otimes \text{id}_{B}^{\otimes j_2}) \circ \pi_{k_1\mid 1\mid k_2} \right\}$$

(12)
where (*) means to take the term which is a well-defined composition under the constraints of indices $j_1, j_2, k_1, k_2$. The $B$-component of the quadratic $A_\infty$ relations is

$$
\pi_{A[C|B]}^{0|1} \circ \sum_{j_1 + j + j_2 = k} m_{A \times C}^{j_1 + j + j_2} (\id_{A \times C} \otimes m_{A \times C}^j \otimes \id_{A \times C}^j) \\
= \sum_{j_1 + j + j_2 = k} \left( f_{j_1 + j + j_2} \circ \pi_{k|0}^{k|0} + m_{k|1|k_2} \circ \pi_{k|1|k_2} + g_{j_1 + j + j_2} \circ \pi_{0|0}^{0|0} \right) \circ (\id_{A \times C} \otimes m_{A \times C}^j \otimes \id_{A \times C}^j) \\
+ \sum_{j_1 + j + j_2 = k} g_{j_1 + j + j_2} \circ (\id_{A} \otimes m_{A}^j \otimes \id_{A}^j) \circ \pi_{A[C|B]}^{0|k} \circ m_{A[C|B]}^{k_1|k_2} \circ \left( \begin{array}{c}
(id_{A} \otimes m_{A}^j \otimes \id_{B}^k) \circ \pi_{k_1+j|0}^{k_1+j|0} \\
+ (id_{A} \otimes g^j \otimes \id_{B}^k) \circ \pi_{k_1+j|0}^{k_1+j|0} \\
(id_{A} \otimes m_{A}^j \otimes \id_{C} \otimes \id_{B}^k) \\
(id_{A} \otimes \id_{C} \otimes m_{A}^j \otimes \id_{B}^k) \\
(id_{A} \otimes \id_{C} \otimes \id_{B}^j \otimes m_{B}^j \otimes \id_{B}^j) \end{array} \right) \right) \right) \right)
$$

The three terms highlighted in blue give the $A_\infty$ bimodule relation and therefore cancel. The two terms highlighted in orange fit into the left hand side of eq. (12). The right hand side of eq. (12), along with the red terms, form the quadratic $A_\infty$ relations for $f$ and $g$ and therefore cancel as well. \hfill \Box

**Remark C.0.5.** In the category of differential graded algebras, there is a well defined fiber product given by

$$A \times_C B := \{ (a, b) : f(a) = g(b) \}.$$

This definition does not carry over to $A_\infty$ algebras as this construction implicitly uses the fact that morphisms of DGAs have well defined images. However, a morphism of $A_\infty$ algebras do not have a well defined image, as the homotopies described by the $f^k$ need not lie in the image of $f^1$.

### D Mapping cocylinders

In the category of chain complexes, there is a dictionary between morphisms and mapping cocylinders. We now extend this dictionary to filtered $A_\infty$ algebras.
D.1 Morphisms are mapping cocylinders

Definition D.1.1. Let \( f : A^- \to A^+ \) be a morphism of \( A_\infty \) algebras. Let \( \text{id} : A^+ \to A^+ \) be the identity. The mapping cocylinder of \( f \) is the \( A_\infty \) fiber product

\[
B_f := A^- \times_{A^+} A^+.
\]

When we want to state that \( B_f \) is a mapping cocylinder, we will write:

\[
A^- \leftrightarrow B_f \to A^+.
\]

Definition D.1.2. Let \( A^+ \) and \( A^- \) be two filtered \( A_\infty \) algebras. A cocylinder from \( A^+ \) to \( A^- \) is a filtered \( A_\infty \) algebra \( B \) which:

- as a vector space is isomorphic to \( A^- \oplus A^0 \oplus A^+ \);
- has differential is of the form:

\[
\begin{pmatrix}
  m_{-}^- & 0 & 0 \\
  m_{0}^- & m_{0}^0 & m_{0}^+ \\
  0 & 0 & m_{-}^+
\end{pmatrix}.
\]

where \( m_{0}^+ \) is an isomorphism with inverse satisfying \( \text{val}((m_{0}^+)^{-1} \circ m^0) > 0 \);

- has projections of chain complexes

\[
\begin{array}{ccc}
A^- & \xrightarrow{\beta^-} & B \\
& \beta^+ \leftarrow & \\
& \beta^+ \rightarrow & A^+
\end{array}
\]

which can be extended to \( A_\infty \) homomorphisms \( \beta^k_\pm \), with \( \beta^k_\pm = 0 \) for all \( k \neq 1 \).

We denote such a mapping cocylinder

\[
A^- \leftrightarrow B \to A^+.
\]

The cylinders from \( A^- \) to \( A^+ \) are in correspondence with morphisms \( f : A^- \to A^+ \).

Theorem D.1.3 (Cylinders are mapping cocylinders). Let \( A^- \) and \( A^+ \) be two filtered \( A_\infty \) algebras.

1. To every cocylinder \( A^- \leftrightarrow B \to A^+ \), we can associate a morphism \( \Theta_B : A^- \to A^+ \).
2. To every morphism \( f : A^- \to A^+ \), we can associate a cocylinder

\[
A^- \leftrightarrow B_f \to A^+.
\]
3. These constructions are compatible in the sense that $\Theta_B f = f$.

Proof. Each statement is proven using statements from appendix [B] and appendix [C].

Proof of item 7: We show that the cocylinder satisfies the conditions of the homotopy transfer theorem. Namely, we construct a map $\alpha$ and homotopy $h$ which satisfy eqs. (6) to (8). From our hypothesis, we are in the scenario where claim [B.0.3] applies.

The construction of the first-order inclusion and homotopy is similar to the construction in the chain complex case. The component of the differential $m^+_0$ is invertible by assumption. We will for convenience denote the inverse by $h^+_0 := (m^+_0)^{-1}$. We define the map

$$\alpha : A^- \to A^- \oplus A^0 \oplus A^+$$

$$x \mapsto (x, 0, h^+_0 \circ m_0 (x))$$

Homotopy Identity: We show that the map $h = (0 \oplus 0 \oplus h^+_0)$ is a curved homotopy between $\alpha \circ \beta$ satisfying eq. (8).

$$m^+_B \circ h + h \circ m^+_B = (0 \oplus (m^+_0 \circ h^+_0) \oplus (m^+_+ \circ h^+_0 + h^+_0 \circ (m^-_0 + m^+_0 + m^+_0)))$$

$$= (0 \oplus \text{id} \oplus (id + h^+_0 \circ m^-_0)) + (0 \oplus 0 \oplus (m^+_+ \circ h^+_0 + h^+_0 \circ m^-_0))$$

$$= \text{id} - \alpha \circ \beta + (0 \oplus 0 \oplus (m^+_+ \circ h^+_0 + h^+_0 \circ m^-_0))$$

The $+$ component of the third term in this sum can be rearranged as

$$m^+_+ \circ h^+_0 + h^+_0 \circ m^-_0 = h^+_0 \circ m^+_+ \circ h^+_0 + h^+_0 \circ m^-_0$$

$$= h^+_0 \circ \left(\alpha \circ \beta + h \circ m^+_B \circ (m^+_0 \otimes h) + m^+_B \circ (h \otimes m^+_0)\right)$$

$$= h \circ (m^+_B \circ (m^+_0 \otimes h) + m^+_B \circ (h \otimes m^+_0))$$

So that all together we satisfy eq. (8):

$$m^+_B \circ h + h \circ m^+_B = \text{id} - \alpha \circ \beta + h \circ (m^+_B \circ (m^+_0 \otimes h) + m^+_B \circ (h \otimes m^+_0)).$$

$\alpha$ is a morphism  The more involved relation to check is eq. (7):

$$\alpha \circ m^+_A - m^+_B \circ \alpha = h \circ (m^+_B \circ (\alpha \otimes m^+_B) + m^+_B \circ (m^+_0 \otimes \alpha)).$$

When expanded out the terms on the left hand side in more detail:

$$\alpha \circ m^+_A = m^-_+ \oplus 0 \oplus (h^+_0 \circ m^-_0 \circ m^-_+)$$

$$m^+_B \circ \alpha = m^-_+ \oplus (m^-_0 \circ m^+_0 \circ h^+_0 \circ m^-_0) \oplus (m^+_+ \circ h^+_0 \circ m^-_0)$$

$$= m^-_+ \oplus 0 \oplus (m^+_+ \circ h^+_0 \circ m^-_0)$$
Note that the − and 0 already cancel out when we take the difference of these two terms. The only difficulty will be in verifying the + coordinate of this relation. We therefore expand out these terms a bit further. The first term is:

\[
h_+^0 \circ m_0^- \circ m_-^0 = h_+^0 \circ \left( m_0^- \circ (m_- \circ \text{id}) + m_0^- \circ (\text{id} \circ m_-) + m_0^- \circ (m_0 \circ \text{id}) + m_0^- \circ (\text{id} \circ m_0 \circ \text{id}) \right)
\]

while the second term is:

\[
m_+^+ \circ h_+^0 \circ m_0^- = h_+^0 \circ m_0^+ \circ m_+^+ \circ h_+^0 \circ m_-
\]

\[
= h_+^0 \circ \left( m_0^+ \circ (m_+ \circ h_+^0 \circ m_0^-) + m_0^+ \circ (h_+^0 \circ m_0^- \circ m_+) \right)
\]

We match these with the terms from the right hand side of eq. (7).

\[
h \circ (m_B^2(\alpha \otimes m_B^0)) = h \circ (m_-^0 \circ (m_- \circ \text{id}))
\]

\[
\oplus \left( m_0^- \circ (m_- \circ \text{id}) + m_0^- \circ (m_0 \circ \text{id}) + m_0^+ \circ (m_0 \circ (h_+^0 \circ m_0^-)) + m_0^+ \circ (h_+^0 \circ m_0^- \circ m_+) \right)
\]

\[
= 0 \oplus 0 \oplus h_+^0 \circ \left( m_0^- \circ (m_- \circ \text{id}) + m_0^- \circ (m_0 \circ \text{id}) + m_0^+ \circ (m_0 \circ (h_+^0 \circ m_0^-)) + m_0^+ \circ (m_+ \circ (h_+^0 \circ m_0^-)) \right)
\]

and similarly we obtain

\[
h \circ (m_B^2(m_B^0 \otimes \alpha)) = 0 \oplus 0 \oplus h_+^0 \circ \left( (m_0^- \otimes m_-) + m_0^0 \circ ((h_+^0 \circ m_0^-) \otimes m_0) + m_0^0 \circ ((h_+^0 \circ m_0^-) \otimes m_+) \right)
\]

Cancelling the matching highlighted terms shows that the relation eq. (7) is satisfied.

**Applying HTT** The homotopy transfer theorem allows us to construct the following associated \(A_\infty\) homomorphisms
By taking the composition $\beta^+ \circ \alpha$, we get a new map from $A^- \to A^+$ called the pullback-pushforward map, which we will denote

$$\Theta_B := \beta^+ \circ \alpha.$$ 

Proof of item $\blacksquare$: From construction, the chain structure on $B_f$ fits the definition of a mapping cocylinder.

Proof of item $\blacksquare$ It remains to show that $\Theta_{B_f} = f$. An explicit computation suffices. One checks that

$$\beta^+ \circ \alpha^{(T,L)} = \begin{cases} 
0 & \text{T has more than 1 vertex} \\
\h \circ m^k_B|_{(A^-)^{\otimes k}} & \text{T has exactly 1 vertex}
\end{cases}$$

and

$$\h \circ m^k_B|_{(A^-)^{\otimes k}} = f^k.$$ 

which shows that the pullback-pushforward map agrees with $f$. $\square$

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