M-REGULARITY OF Q-TWISTED SHEAVES AND ITS APPLICATION TO LINEAR SYSTEMS ON ABELIAN VARIETIES

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Abstract. G. Pareschi and M. Popa give criterions for global generations and surjectivity of multiplication maps of global sections of coherent sheaves on abelian varieties in the theory of M-regularity. In this paper, we generalize some of their criterions via the M-regularity of Q-twisted sheaves. As an application, we show that the M-regularity of a suitable Q-twisted sheaf implies property (N_p) and jet-ampleness for ample line bundles on abelian varieties.

1. Introduction

Throughout this paper we work over an algebraically closed field \(K\). In [PP03], G. Pareschi and M. Popa introduce the notion of M-regularity as follows:

For a coherent sheaf \(F\) on an abelian variety \(X\) defined over \(K\), set

\[ V^i(F) = \{ \alpha \in \hat{X} | h^i(X, F \otimes P_\alpha) > 0 \}, \]

where \(\hat{X}\) is the dual abelian variety of \(X\) and \(P_\alpha\) is the algebraically trivial line bundle on \(X\) corresponding to \(\alpha \in \hat{X}\). Then \(F\) is said to be GV if \(\text{codim} \hat{X} V^i(F) \geq i\) for any \(i > 0\). It is said to be M-regular if \(\text{codim} \hat{X} V^i(F) > i\) for any \(i > 0\). It is said to be IT(0) if \(V^i(F) = \emptyset\) for any \(i > 0\).

In [PP03], [PP04], [PP11], etc., Pareschi and Popa develop the theory of M-regularity and give many applications. M-regularity is useful since it implies suitable globally generation and surjectivity of multiplication maps of global sections, as in the case of Castelnuovo-Mumford regularity on projective spaces. Among others, they prove the following results:

**Theorem 1.1** ([PP03, Theorem 6.3], [PP11, Theorem 7.34]). Let \(A\) be an ample line bundle on an abelian variety \(X\) and \(E, F\) be coherent sheaves on \(X\).

1. If \(F \otimes A^{-1}\) is M-regular, then \(F\) is globally generated.
2. If \(F \otimes A^{-2}\) is M-regular, then the natural map \(H^0(A^n) \otimes H^0(F) \to H^0(A^n \otimes F)\) is surjective for any \(n \geq 2\).
3. If \(E, F\) are locally free and \(E \otimes A^{-2}, F \otimes A^{-2}\) are M-regular, then the natural map \(H^0(E) \otimes H^0(F) \to H^0(E \otimes F)\) is surjective.

Recently Z. Jiang and G. Pareschi [JP20] extend the notions such as GV, M-regular, IT(0) to a Q-twisted sheaf \(F(xl)\), where \(x \in \mathbb{Q}\) and \(l \in \text{Pic} X/\text{Pic}^0 X\) is the class of an ample line bundle \(L\) (see [2.2] for the definition of Q-twisted sheaves). In [JP20], the authors also define an invariant \(0 < \beta(l) \leq 1\), which is characterized as

\[ \beta(l) < x \iff I_0(xl) \text{ is IT(0)} \]

for \(x \in \mathbb{Q}\), where \(I_0 \subset \mathcal{O}_X\) is the maximal ideal corresponding to the origin \(o \in X\).

The first purpose of this paper is to generalize **Theorem 1.1** to the \(\mathbb{Q}\)-twisted setting as follows:

**Theorem 1.2** (Propositions 3.1, 4.4). Let \(L\) be an ample line bundle, \(E\) be a locally free sheaf, and \(F\) be a coherent sheaf on an abelian variety \(X\). Take \(x \in \mathbb{Q}\) such that \(x \geq \beta(l)\).

1. If \(F(-xl)\) is M-regular, then \(F\) is globally generated.

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(2) If \( x < 1 \) and \( \mathcal{F}(\frac{1}{x} l) \) is M-regular, then the natural map \( H^0(L) \otimes H^0(\mathcal{F}) \rightarrow H^0(L \otimes \mathcal{F}) \) is surjective.

(3) If there exist rational numbers \( s, t > 0 \) such that \( \mathcal{E}(-s l), \mathcal{F}(-t l) \) are M-regular and \( st/(s + t) \geq \beta(l) \), then the natural map \( H^0(\mathcal{E}) \otimes H^0(\mathcal{F}) \rightarrow H^0(\mathcal{E} \otimes \mathcal{F}) \) is surjective.

We note that \( \mathcal{F}(ml) \) is M-regular if and only if so is \( \mathcal{F} \otimes L^m \) for \( m \in \mathbb{Z} \). Hence Theorem 1.1 (1) is nothing but the case \( A = L, x = 1 \) of Theorem 1.2 (1). Similarly, Theorem 1.1 (2) follows from the case \( L = A^n, x = 1/n \) of Theorem 1.3 (2), and Theorem 1.1 (3) follows from the case \( A = L, s = t = 2 \) of Theorem 1.2 (3).

The second purpose of this paper is to study linear systems on abelian varieties by using Theorem 1.2 as in [PP04]. For an ample line bundle \( A \) on an abelian variety \( X \),

- \( A^n \) is basepoint free if \( n \geq 2 \) (Lefschetz Theorem),
- \( A^n \) is projectively normal if \( n \geq 3 \) ([Mum70], [Koi76]),
- the homogeneous ideal of \( X \) embedded by \( |A^n| \) is generated by quadrics if \( n \geq 4 \) ([Mum66], [Kum89]).

As a generalization of these results, R. Lazarsfeld conjectures that \( A^n \) satisfies property \((N_p)\) if \( n \geq p + 3 \) for \( p \geq 0 \). See §2.1 for the definition of \((N_p)\). We just note here that \((N_0)\) holds for an ample line bundle \( L \) if and only if \( L \) defines a projectively normal embedding, and \((N_1)\) holds if and only if \((N_0)\) holds and the homogeneous ideal of the embedding is generated by quadrics.

Lazarsfeld’s conjecture is proved by Pareschi [Par00] in \( \text{char}(\mathbb{K}) = 0 \). In [PP04], Pareschi and Popa strengthen Lazarsfeld’s conjecture when \( A \) has no base divisor, that is, when \( \text{codim}_X \text{Bs}|A| \geq 2 \), by using Theorem 1.1.

**Theorem 1.3** ([PP04] Theorem 6.2). Let \( p \geq 1 \) be an integer such that \( \text{char}(\mathbb{K}) \) does not divide \( p + 1 \) and \( p + 2 \). Let \( A \) be an ample line bundle with no base divisor on an abelian variety \( X \).

1. If \( n \geq p + 2 \), then \( A^n \) satisfies \((N_p)\).
2. More generally, if \( n \geq (p + r + 2)/(r + 1) \), then \( A^n \) satisfies \((N_p^r)\) for \( r \geq 0 \).

Property \((N_p^r)\) for \( p, r \geq 0 \) is introduced in [Par00], where it is proved that \( A^n \) satisfies \((N_p^r)\) if \( n \geq (p + r + 3)/(r + 1) \) without the assumption on the base divisors of \( A \). See §2.1 for the definition of \((N_p^r)\). We just note here that \((N_0^0)\) is equivalent to \((N_p)\) and \((N_p^r)\) is a property of “being off” by \( r \) from \((N_p)\).

On the other hand, the following theorem by Jiang-Pareschi and F. Caucci generalizes Lazarsfeld’s conjecture to ample line bundles which is not necessarily multiples of another line bundles:

**Theorem 1.4** ([JP20], Section 8, [Cau20a], Theorem 1.1). Let \( L \) be an ample line bundle on an abelian variety \( X \) and \( p \geq 0 \). Let \( \mathcal{I}_o \subset \mathcal{O}_X \) be the maximal ideal sheaf corresponding to the origin \( o \) in \( X \). Then

1. If \( \mathcal{I}_o(xl) \) is IT(0) for \( 1 < x \in \mathbb{Q} \), and \( \mathcal{I}_o(l) \) is IT(0) if and only if \( L \) is basepoint free.
2. If \( \mathcal{I}_o(\frac{1}{p+2} l) \) is IT(0), then \( L \) satisfies \((N_p)\).

Theorem 1.4 gives a quick and characteristic-free proof of Lazarsfeld’s conjecture as follows: If \( L = A^n \) for some \( A \) and \( n \geq 1 \), then \( \mathcal{I}_o(\frac{1}{p+2} l) = \mathcal{I}_o(\frac{1}{p+2} a) \) is IT(0) if \( n/(p + 2) > 1 \) by Theorem 1.4 (1). Hence \( A^n \) satisfies \((N_p)\) if \( n \geq p + 3 \) by Theorem 1.4 (2). Furthermore, the proof of Theorem 1.4 shows that \( L \) satisfies \((N_p^r)\) for \( r \geq 0 \) if \( \mathcal{I}_o(\frac{r+1}{p+r+2} l) \) is IT(0).

Applying Theorem 1.2 we obtain the following theorem, which contains Theorem 1.3 and the case \( p \geq 1 \) of Theorem 1.4 (2):

**Theorem 1.5.** Let \( L \) be an ample line bundle on an abelian variety \( X \) and \( p \geq 1 \).

1. If \( \mathcal{I}_o(\frac{1}{p} l) \) is M-regular, then \( L \) satisfies \((N_p)\).
2. More generally, if \( \mathcal{I}_o(\frac{r+1}{p+r+2} l) \) is M-regular, then \( L \) satisfies \((N_p^r)\) for \( r \geq 0 \).
By [PP04] Remark 3.6, \(I_o(l)\) is M-regular if and only if \(L\) has no base divisor. Hence Theorem 1.5 gives a characteristic-free proof of Theorem 1.3. If \(A\) has no base divisor and \(L = A^n\), then \(I_o(xa)\) is M-regular for \(x \geq 1\) and hence \(I_o(\frac{1}{x+2}l) = I_o(\frac{n}{n+2}a)\) is M-regular if \(n/(p+2) \geq 1\). Thus Theorem 1.3 (1) holds in any characteristic. Similarly, Theorem 1.5 (2) recovers Theorem 1.3 (2) in any characteristic.

Other than \((N_p)\), we can study jet-ampleness via M-regularity as in [PP04]. Recall that a line bundle \(L\) is called \(k\)-jet ample for \(k \geq 0\) if the restriction map
\[
H^0(L) \rightarrow H^0(L \otimes O_X/I_l^1 \cdots I_{l+1})
\]
is surjective for any (not necessarily distinct) \(k + 1\) points \(p_1, \ldots, p_{k+1} \in X\). In particular, 0-jet ampleness is equivalent to basepoint freeness and 1-jet ampleness is equivalent to very ampleness.

For an ample line bundle \(A\) on an abelian variety, [BS97a] proves that \(A^n\) is \(k\)-jet ample if \(n \geq k + 2\), and the same holds if \(A\) has no base divisor and \(n \geq k + 1 \geq 2\). In [PP04 Theorem 3.8], the authors generalize this result using the theory of M-regularity. On the other hand, Caucci shows that \(L\) is \(k\)-jet ample if \(I_o(\frac{1}{k+1}l)\) is \(IT(0)\) [Cau20b Theorem D], which generalizes the first statement of the above result in [BS97a]. The following theorem generalizes the result in [BS97a] and improves [Cau20b Theorem D]. See Proposition 5.1 for a generalization of [PP04 Theorem 3.8].

**Theorem 1.6.** Let \(L\) be an ample line bundle on an abelian variety \(X\) and \(k \geq 1\) be an integer. If \(I_o(\frac{1}{k+1}l)\) is M-regular, then \(L\) is \(k\)-jet ample.

**Remark 1.7.** We note that the statement of Theorem 1.3 (1) does not hold for \(p = 0\) as we will see Example 7.1. Hence Theorem 1.3 (1) also fails for \(p = 0\), that is, the M-regularity of \(I_o(\frac{1}{l+1}l)\) does not imply the projective normality of \(L\) in general. However, the M-regularity of \(I_o(\frac{1}{l+1}l)\) implies the very ampleness of \(L\) by Theorem 1.6. See [S7] for the relations between these notions.

This paper is organized as follows. In [S2] we recall some notation. In [S3] we show Theorem 1.2 (1). In [S4] we show Theorem 1.2 (2), (3). In [S9] we prove Theorem 1.6. We also give an alternative proof of a characterization of polarized abelian varieties whose Seshadri constants are one by M. Nakamaye [Nak96]. In [S6] we prove Theorem 1.5. In [S7] we study projective normality.

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2. Preliminaries

Throughout this paper, \(X\) is an abelian variety of dimension \(g\). We denote the origin of \(X\) by \(O_X\) or \(o \in X\). Let \(L\) be an ample line bundle on \(X\) and \(l \in NS(X) = \text{Pic}(X)/\text{Pic}^0(X)\) be the class of \(L\) in the Neron-Severi group. Then we have an isogeny
\[
\varphi_l : X \rightarrow \hat{X} := \text{Pic}^0(X), \quad p \mapsto t_p^*L \otimes L^{-1}
\]
which depends only on the class \(l\), where \(t_p : X \rightarrow X\) is the translation by \(p\).

For \(b \in \mathbb{Z}\), we denote the multiplication-by-\(b\) isogeny by
\[
\mu_b = \mu_b^X : X \rightarrow X, \quad p \mapsto bp.
\]
It holds that \(\mu_b^X = b^2 \mu_1\). Since \(\varphi_l\) is a group homomorphism, we have \(\varphi_{bl} = \hat{\mu}_b \circ \varphi_l = \varphi_l \circ \mu_b\), where \(\hat{\mu}_b\) is the multiplication-by-\(b\) isogeny on \(\hat{X}\).

For two line bundles \(L, L'\), \(L \equiv L'\) means that \(L, L'\) are algebraically equivalent.
2.1. Properties \((N_p)\) and \((N^*_p)\). Let \(L\) be a basepoint free ample line bundle \(L\) on a projective variety \(Y\) and set \(S_L := \text{Sym} H^0(Y, L)\). Take a minimal free resolution of \(R_L := \bigoplus_{n \geq 0} H^0(Y, L^n)\) as an \(S_L\)-module.

\[\cdots \to E_p \to \cdots \to E_1 \to E_0 \to R_L \to 0 \quad \text{with} \quad E_i \cong \bigoplus_j S_L(−a_{ij}).\]

Then \(L\) is said to satisfy property \((N_p)\) if \(E_0 = S_L\) and \(a_{ij} = i + 1\) for any \(1 \leq i \leq p\) and any \(j\). For example, \((N_0)\) holds for \(L\) if and only if \(L\) defines a projectively normal embedding, and \((N_1)\) holds if and only if \((N_0)\) holds and the homogeneous ideal of the embedding is generated by quadrics.

More generally, \(L\) is said to satisfy property \((N^*_p)\) if \(a_{0j} \leq 1 + r\) for any \(j\) in \(\{1, \ldots, p\}\). Inductively, \(L\) is said to satisfy property \((N^*_{p+1})\) if \(L\) satisfies \((N^*_p)\) and \(a_{pq} \leq p + 1 + r\) for any \(p, q\). For example, \((N^*_0)\) is equivalent to \((N_0)\) and \(L\) satisfies \((N^*_0)\) if and only if \(H^0(L) \otimes H^0(L^n) \to H^0(L^{n+1})\) is surjective for \(n \geq r + 1\) and \(L\) is projectively normal, \(L\) satisfies \((N^*_1)\) if and only if the homogeneous ideal of \(X\) embedded by \(|L|\) is generated by homogeneous polynomials of degree at most \(r\).

We note that this equivalence about \((N^*_1)\) and the degrees of generators of the homogeneous ideal does not hold in general without assuming the projectivity normally. For example, there exists a polarized abelian surface \((X, A)\) of type \((1, 2)\) over the complex number field \(\mathbb{C}\) such that \(A\) has no base divisor, \(A^2\) is very ample, and the homogeneous ideal of \(X\) embedded by \(|A^2|\) is generated by quadrics and quartics [Ago17, Remark 3] (see also [Bar87, §2]). Since \(A\) has no base divisor, \(A^2\) satisfies \((N^*_1)\) by [Theorem 1.3]. However, the homogeneous ideal of \(X\) is not generated by polynomials of degree at most \(3\).

In particular, we need to assume that \(A^2\) is projectively normal in [PP04, Theorem 6.1 (b)].

2.2. Generic vanishing, M-regularity and IT(0) of \(\mathbb{Q}\)-twisted sheaves. Let \(l \in \text{NS}(X)\) be an ample class. For a coherent sheaf \(\mathcal{F}\) on \(X\) and \(x \in \mathbb{Q}\), a \(\mathbb{Q}\)-twisted coherent sheaf \(\mathcal{F}(xl)\) is the equivalence class of the pair \((\mathcal{F}, xI)\), where the equivalence is defined by

\[(\mathcal{F} \otimes L^m, xl) \sim (\mathcal{F}, (x + m)l)\]

for any line bundle \(L\) representing \(l\) and any \(m \in \mathbb{Z}\).

As explained in Introduction, a coherent sheaf \(\mathcal{F}\) on \(X\) is said to be GV (resp. M-regular, resp. IT(0)) if \(\text{codim}_X V^i(\mathcal{F}) \geq i\) (resp. \(\text{codim}_X V^i(\mathcal{F}) > i\), resp. \(V^i(\mathcal{F}) = \emptyset\)) for any \(i > 0\), where \(V^i(\mathcal{F}) = \{a \in X \mid h^i(X, \mathcal{F} \otimes P_a) > 0\}\).

In [JP20], these notions are extended to the \(\mathbb{Q}\)-twisted setting. A \(\mathbb{Q}\)-twisted coherent sheaf \(\mathcal{F}(xl)\) for \(x = \frac{a}{b}\) with \(b > 0\) is said to be GV, M-regular, or IT(0) if so is \(\mu^n_\mathbb{Q}(\mathcal{F} \otimes L^{ab})\). This definition does not depend on the representation \(x = \frac{a}{b}\) nor the choice of \(L\) representing \(l\).

We note that this is true even in \(\text{char}(\mathbb{K}) \geq 0\) by [Can20a, Remark 3.2]. Furthermore, for an isogeny \(f : Y \to X\),

\[(2.2) \quad \mathcal{F}(xl) \text{ is GV, M-regular, or IT(0)} \iff \text{so is } f^*(\mathcal{F}(xl)) := f^*\mathcal{F}(xf*l)\]

by [Can20b, Proposition 1.3.3] or arguments in [Can20a, Remark 3.2].

By [JP20, Theorem 5.2], we also have an equivalence

\[(2.3) \quad \mathcal{F}(xl) \text{ is GV } \iff \mathcal{F}((x + x')l) \text{ is IT(0)} \text{ for any rational number } x' > 0.\]

**Example 2.1.** For a line bundle \(B\) on \(X\), \(B(xl)\) is M-regular if and only if \(B(xl)\) is IT(0) if and only if \(B + xL\) is ample by [PP08, Example 3.10 (1)]. Hence \(B(xl)\) is GV if and only if \(B + xL\) is nef.

**Example 2.2.** For an ample line bundle \(L\) on \(X\), \(I_o \otimes L = I_o(l)\) is GV, and \(I_o \otimes L = \text{IT(0)}\) if and only if \(L\) is basepoint free by [Theorem 1.4 (1)]. By [PP03, Remark 3.6], \(I_o \otimes L\) is M-regular if and only if \(L\) has no base divisor.
For a polarized abelian variety \((X, l)\), Jiang and Pareschi introduce an invariant \(\beta(l) \in \mathbb{R}\). It is defined by using cohomological rank functions, which is also introduced by them, but \(\beta(l)\) is characterized by the notion IT(0) as follows:

**Lemma 2.3** ([JP20] Section 8, [Cau20a] Lemma 3.3). Let \((X, l)\) be a polarized abelian variety and \(x \in \mathbb{Q}\). Then \(\beta(l) < x\) if and only if \(I_p(xl)\) is IT(0) for some (and hence for any) \(p \in X\).

By Lemma 2.3 and (2.3), it also holds that \(\beta(l) \leq x\) if and only if \(I_p(xl)\) is GV for \(x \in \mathbb{Q}\).

We note that Theorem 1.4 is stated by using \(\beta(l)\) in [JP20], [Cau20a], but the original statement is equivalent to that in Theorem 1.4 by Lemma 2.3.

**Example 2.4.** If \(\beta(l) < 1/2\), \(L\) satisfies \((N_0)\) by Theorem 1.4 (2). Contrary to Theorem 1.4 (1), the converse does not hold in general. For example, let \((X, A)\) be a general polarized abelian variety of type \((1, \ldots, 1, d)\) with \(3 \leq d \leq 6\) and set \(L := A^2\). Since \(h^0(A) = d \leq g\), \(A\) is not basepoint free, which is equivalent to \(\beta(a) = 1\) by Theorem 1.4 (1). Hence we have \(\beta(l) = \beta(a)/2 = 1/2\). On the other hand, \(L\) is projectively normal by [Rub98].

**Example 2.5.** If \(\beta(l) < 1/2\), \(L\) satisfies not only \((N_0)\) but also \((N_1^1)\) as in the following paragraph of Theorem 1.4 (see also the proof of Theorem 1.5 in [S]). Hence the homogeneous ideal of \(X\) embedded by \(|L|\) is generated by quadrics and cubics if \(\beta(l) < 1/2\).

It might be interesting to ask if the homogeneous ideal of \(X\) is generated by quadrics and cubics under the condition that \(L\) is projectively normal, which is weaker than the condition \(\beta(l) < 1/2\). At least, the answer is yes if \(\dim X = 2\) by [Ago17] Lemma 2.2]. The answer is yes as well when \(L = A^2\) for some ample line bundle \(A\). In fact, \(A^2\) is very ample if and only if \(A\) has no base divisor by [Ohb87]. Thus if \(A^2\) is projectively normal, then \(A\) has no base divisor and hence \(A^2\) satisfies \((N_1^1)\) by Theorem 1.3 (2). Therefore the homogeneous ideal of \(X\) embedded by \(|A^2|\) is generated by quadrics and cubics. We note that Theorem 1.3 is true in any characteristic by Theorem 1.5 which we will prove in [S].

We will use the following proposition.

**Proposition 2.6** ([PP11] Proposition 3.1, [PP11] Theorem 3.2, [Cau20a] Proposition 3.4). Assume that \(\mathcal{F}\) and \(\mathcal{G}\) are coherent sheaves on \(X\), and that one of them is locally free.

(i) If \(\mathcal{F}(xl)\) is IT(0) and \(\mathcal{G}(yl)\) is GV, then \(\mathcal{F}(xl) \otimes \mathcal{G}(yl) := (\mathcal{F} \otimes \mathcal{G})(xl)\) is IT(0).

(ii) If \(\mathcal{F}(xl)\) and \(\mathcal{G}(yl)\) are M-regular, then \(\mathcal{F}(xl) \otimes \mathcal{G}(yl)\) is M-regular.

We also note that for an isogeny \(f : Y \to X\) and a coherent sheaf \(\mathcal{F}\) on \(Y\),

\[(2.4)\]

\[f_*\mathcal{F} \text{ is GV, M-regular, or IT(0) } \iff \text{ so is } \mathcal{F}\]

since \(h^i(f_*\mathcal{F} \otimes P_\alpha) = h^i(\mathcal{F} \otimes f^*P_\alpha) = h^i(\mathcal{F} \otimes P_{f(\alpha)})\) holds for any \(\alpha \in \hat{X}\), where \(\hat{f} : \hat{X} \to \hat{Y}\) is the dual isogeny. We see a \(\mathbb{Q}\)-twisted version of this fact in Lemma 4.6.

2.3. Fourier-Mukai transforms. Let \(\mathcal{P}\) be the Poincaré line bundle on \(X \times \hat{X}\). Let \(D^h(X)\) be the bounded derived category of coherent sheaves on \(X\) and

\[\Phi_p = \Phi_p^X : D^b(X) \to D^b(\hat{X})\]

be the Fourier-Mukai functor associated to \(\mathcal{P}\). We note that \(\Phi_p(\mathcal{F})\) is a locally free sheaf (concentrated in degree 0) for an IT(0) sheaf \(\mathcal{F}\). For an isogeny \(f : Y \to X\),

\[(2.5)\]

\[f^* \circ \Phi_p^Y \simeq \Phi_p^X \circ f_*\]

holds by [Muk81] (3.4)].
24. (Skew) Pontrjagin products. For coherent sheaves $\mathcal{E}, \mathcal{F}$ on $X$, their Pontrjagin product $\mathcal{E} \ast \mathcal{F}$ is defined as

$$\mathcal{E} \ast \mathcal{F} = (p_1 + p_2)_* (p_1^* \mathcal{E} \otimes p_2^* \mathcal{F}),$$

where $p_i$ is the natural projection from $X \times X$ to the $i$-th factor for $i = 1, 2$. By definition, $\mathcal{E} \ast \mathcal{F} = \mathcal{F} \ast \mathcal{E}$ holds. Similarly, their skew Pontrjagin product $\mathcal{E} \ast \mathcal{F}$ is defined as

$$\mathcal{E} \ast \mathcal{F} = p_1_* ((p_1 + p_2)^* \mathcal{E} \otimes p_2^* \mathcal{F}).$$

As in [Par00, Remark 1.2], $\mathcal{E} \ast \mathcal{F} \simeq \mathcal{E} \ast (-1_X)^* \mathcal{F}$ and $\mathcal{F} \ast \mathcal{E} \simeq (-1_X)^* (\mathcal{E} \ast \mathcal{F})$, where $-1_X = \mu_X^{-1}$.

We will use the following properties of (skew) Pontrjagin products. For simplicity, we assume locally freeness or IT(0) for some sheaves. In particular, all the objects are sheaves and $\otimes$ is the usual (non-derived) tensor product in the following proposition.

**Proposition 2.7** ([Muk81, (3.7),(3.10)], [Par00, Proposition 1.1], [PP04, Proposition 5.2]). Let $L$ be an ample line bundle, $\mathcal{E}$ be a vector bundle and $\mathcal{F}$ be a coherent sheaf on $X$.

(i) If $\mathcal{E}, \mathcal{F}$ are IT(0), then $\Phi_p (\mathcal{E} \ast \mathcal{F}) = \Phi_p (\mathcal{E}) \otimes \Phi_p (\mathcal{F})$.

(ii) Assume that $h^i ((t_0^* \mathcal{E}) \otimes \mathcal{F}) = 0$ for any $q \in X$ and $i > 0$. For $p \in X$, the natural map $H^0 (t_0^* \mathcal{E}) \otimes H^0 (\mathcal{F}) \to H^0 ((t_0^* \mathcal{E}) \otimes \mathcal{F})$ is surjective if and only if $\mathcal{E} \ast \mathcal{F}$ is generated by global sections at $p$.

(iii) If $L \otimes \mathcal{F}$ is IT(0), then $L \ast \mathcal{F} = L \otimes \varphi^*_p (\Phi_p ((-1_X)^* \mathcal{F}) \otimes L)$ and $L \ast \mathcal{F} = L \otimes \varphi^*_p (\Phi_p (\mathcal{F} \otimes L)$.

3. **Criterion for global generations**

In this section, we prove [Theorem 1.2] (1), which is nothing but (iii) in the following proposition.

**Proposition 3.1.** Let $L$ be an ample line bundle, $\mathcal{E}$ be a vector bundle and $\mathcal{F}$ be a coherent sheaf on $X$.

(i) If $\mathcal{F} \otimes L^{-ab}$ is M-regular, $\mathcal{I}_{\mu^{-1}_b (p)} \mathcal{F}$ is IT(0) for any $p \in X$, where $\mu_b^{-1} (p) \subset X$ is the scheme-theoretic fiber over $p$ and $\mathcal{I}_{\mu^{-1}_b (p)} \mathcal{F}$ is the image of the natural homomorphism $\mathcal{I}_{\mu^{-1}_b (p)} \mathcal{F} \to \mathcal{F}$.

(ii) If $\mathcal{F} \langle y \rangle$ is M-regular, $(\mathcal{I}_{\mu^{-1}_b (p)} \mathcal{F} \langle (x + y) \rangle)$ is IT(0) for any $p \in X$.

(iii) If $\mathcal{F} \langle -x \rangle$ is M-regular, $\mathcal{F}$ is globally generated.

**Proof.** (i) Fix $p \in X$ and consider the exact sequence

$$0 \to \mathcal{I}_{\mu^{-1}_b (p)} \mathcal{F} \otimes P_{\alpha} \to \mathcal{F} \otimes P_{\alpha} \to \mathcal{F} \otimes P_{\alpha} \mid_{\mu^{-1}_b (p)} \to 0$$

for $\alpha \in \hat{X}$. Since $\mathcal{F} \otimes L^{-ab}$ is M-regular and $ab > 0$, $\mathcal{F}$ is IT(0) by [Proposition 2.6]. Since the support of $\mathcal{F} \otimes P_{\alpha} \mid_{\mu^{-1}_b (p)}$ is zero dimensional, $h^i (\mathcal{I}_{\mu^{-1}_b (p)} \mathcal{F} \otimes P_{\alpha}) = 0$ for any $i \geq 2$ and $\mathcal{I}_{\mu^{-1}_b (p)} \mathcal{F}$ is IT(0) if and only if the restriction map

$$H^0 (\mathcal{F} \otimes P_{\alpha}) \to \mathcal{F} \otimes P_{\alpha} \mid_{\mu^{-1}_b (p)}$$

is surjective for any $\alpha \in \hat{X}$. Since $\mathcal{F} \otimes P_{\alpha} \otimes L^{-ab}$ is also M-regular, it suffices to show the surjectivity of [3.1] for $\alpha = o \hat{X}$.

By $x = a/b \geq \beta (l)$, $L^{ab} \otimes \mathcal{I}_{\mu^{-1}_b (p)} = L^{ab} \otimes \mu_b^0 \mathcal{F}$ is GV. Hence

$$V^1 := \{ \alpha \in \hat{X} \mid h^1 (L^{ab} \otimes \mathcal{I}_{\mu^{-1}_b (p)} \otimes P_{\alpha}) > 0 \}$$

is a proper closed subset of $\hat{X}$.

Since $\mathcal{F} \otimes L^{-ab}$ is M-regular, there exists an integer $N > 0$ such that for any general $\alpha_1, \ldots, \alpha_N \in \hat{X}$ the natural map

$$\bigoplus_{i=j}^N H^0 (\mathcal{F} \otimes L^{-ab} \otimes P_{\alpha_j} \otimes P_{\alpha_j}) \to \mathcal{F} \otimes L^{-ab}$$

is surjective by [PP03, Proposition 2.13]. We take general $\alpha_j$ so that $-\alpha_j \notin V^1$. 

Consider the following diagram:
\[
\begin{array}{ccc}
\bigoplus_{j=1}^{N} H^0(F \otimes L^{-ab} \otimes P_{\alpha_j}) \otimes H^0(L^{ab} \otimes P_{\alpha_j}^\vee) \otimes O_X & \longrightarrow & H^0(F) \otimes O_X \\
\bigoplus_{j=1}^{N} H^0(F \otimes L^{-ab} \otimes P_{\alpha_j}) \otimes L^{ab} \otimes P_{\alpha_j}^\vee & \longrightarrow & F
\end{array}
\]
Since the bottom map is surjective, \(H^0(F) \to F|_{\mu^{-1}_n(p)}\) is surjective if so is
\[
\bigoplus_{j=1}^{N} H^0(F \otimes L^{-ab} \otimes P_{\alpha_j}) \otimes H^0(L^{ab} \otimes P_{\alpha_j}^\vee) \to \bigoplus_{j=1}^{N} H^0(F \otimes L^{-ab} \otimes P_{\alpha_j}) \otimes L^{ab} \otimes P_{\alpha_j}^\vee|_{\nu^{-1}_n(p)}.
\]
This map is surjective since so is \(H^0(L^{ab} \otimes P_{\alpha_j}^\vee) \to L^{ab} \otimes P_{\alpha_j}^\vee|_{\mu^{-1}_n(p)}\) for any \(j\) by \(-\alpha_j \notin \mathbb{V}^1\). Hence (i) holds.

(ii) If \(F(yl)\) is M-regular, so is \(\mu^*_b F \otimes L^{ab}\) by definition. By (i), \(\mathcal{I}_{\mu^{-1}_n(p)} \cdot \mu^*_b F \otimes L^{ab}\) is IT(0) for any \(p \in X\). Since \(\mu_b\) is flat, \(\mathcal{I}_{\mu^{-1}_n(p)} \cdot \mu_b F = \mu^*_b (\mathcal{I}_{p^*} F)\) holds. Hence \(\mu^*_b (\mathcal{I}_{p^*} F) \otimes L^{ab} = \mu^*_b (\mathcal{I}_{p^*} F) \otimes L^{(a+\alpha)_b}\) is IT(0), which means that \((\mathcal{I}_{p^*} F)(x + y l)\) is IT(0).

(iii) The global generation of \(F\) follows from the vanishing \(h^1(\mathcal{I}_{p^*} F) = 0\) for any \(p \in X\). Hence it suffices to show that \(\mathcal{I}_{p^*} F\) is IT(0) for any \(p \in X\). This follows from (ii) for \(y = -x\). \(\Box\)

4. Surjectivity of multiplication maps on global sections

Throughout this section, \(L\) is an ample line bundle on an abelian variety \(X\). If \(L\) is basepoint free, we can define a vector bundle \(M_L\) on \(X\) by the exact sequence
\[
(4.1) \quad 0 \to M_L \to H^0(L) \otimes O_X \to L \to 0.
\]
We note that we do not assume the basepoint freeness of \(L\) otherwise stated. The following proposition is essentially proved in [JP20 Proposition 8.1]:

**Proposition 4.1** ([JP20 Proposition 8.1]). Let \(F\) be an IT(0) sheaf on \(X\) and \(y \in \mathbb{Q}_{>0}\). Then \(F(-yl)\) is GV, M-regular, or IT(0) if and only if so is \(\varphi_1 F(\mathcal{I}_y)\).

In particular, if \(L\) is basepoint free, \(\varphi_{\alpha} x_{\alpha}\) is GV, M-regular, or IT(0) if and only if so is \(M_L(\varphi_{\alpha} x_{\alpha})\) for a rational number \(0 < x < 1\).

To show Proposition 4.1 we recall some results in [JP20]. For a coherent sheaf \(F\) on \(X\), \(y \in \mathbb{Q}\) and \(i \geq 1\), Jiang and Pareschi define a rational number \(h^i_F(yl) \geq 0\) and study the function \(\mathbb{Q} \to \mathbb{Q} : y \mapsto h^i_F(yl)\), which is called the cohomological rank function of \(F\). See [JP20] for the definition. We note that \(h^i_F(yl)\) can be defined in \(\text{char}(k) \geq 0\) by [Cau20a Section 2]. By [JP20 Theorem 5.2 (c), Proposition 5.3, \(F(xyl)\) is GV (resp. M-regular, resp. IT(0)) if and only if for all \(i \geq 1\) it holds that
\[
h^i_F((x_0 - t)) = O(t^i) \quad (\text{resp. } O(t^{i+1}), \text{resp. } 0) \quad \text{for sufficiently small } t \in \mathbb{Q}_{>0}.
\]

**Proof of Proposition 4.1** By [JP20 Proposition 2.3], it holds that
\[
h^i_F(-yl) = \frac{y^g}{\chi(l)} h^{i^*}_F(\varphi_1^* F) \left( \frac{1}{y} l \right)
\]
for any \(y \in \mathbb{Q}_{>0}\) and hence
\[
h^i_F((-y - t)l) = \frac{(y + t)^g}{\chi(l)} h^{i^*}_F(\varphi_1^* F) \left( \frac{1}{y + t} l \right)
\]
for any \(y, t \in \mathbb{Q}_{>0}\). Since
\[
\frac{1}{y + t} = \frac{1}{y} - \frac{1}{y(y + t)},
\]
for a fixed $y > 0$, $h_t^\frac{1}{y}((y-t)l) = O(t^i)$, or $O(t^{i+1})$, or $0$ for sufficiently small $t > 0$ if and only if so is $h_t^i\varphi_\ast \Phi_P(F)(\frac{1}{y+t})$ if and only if so is $h_t^i\varphi_\ast \Phi_P(F)(\frac{1}{y-t}l)$. By (4.2) the first statement of this proposition holds.

When $L$ is basepoint free, $L_o(L)$ is IT(0) and $\varphi_\ast \Phi_P(L_o(L)) = M_L \otimes L^{-1}$ as shown in the proof of [JP20, Proposition 8.1]. Hence the second statement follows from the first one by considering $F = L_o(L)$ and $y = 1 - x$.

**Remark 4.2.** Assume that $L$ is projectively normal. Then $H^0(L) \otimes H^0(L) \to H^0(L^2)$ is surjective and hence $h^1(M_L \otimes L) = 0$ by (4.1). Since $h^1(M_L \otimes L \otimes P_\alpha) = 0$ for any $\alpha \in X$ and $i \geq 2$ by (4.1), $M_L \otimes L$ is GV and hence so is $L_o(\frac{1}{2}l)$ by Proposition 4.1, which is equivalent to $\beta(l) \leq 1/2$.

More generally, we have $\beta(l) \leq n/(n+1)$ if $H^0(L) \otimes H^0(L^n) \to H^0(L^{n+1})$ is surjective for $n \geq 1$ by the same argument. For example, if $X$ is simple and $h^0(L) > ((n+1)/n)^g \cdot g!$ in char($k$) = 0, this multiplication map is surjective by [Blo19 Theorem 1.2], and hence $\beta(l) \leq n/(n+1)$ holds.

**Lemma 4.3.** Let $E$ be a locally free sheaf and $F$ be a coherent sheaf on $X$. If $E$ and $F$ are IT(0), then $E \ast F$ is also IT(0).

**Proof.** Since $E \ast F \simeq (-1_X)^*(F \ast E)$, it suffices to show that $F \ast E$ is IT(0). For $\alpha \in X$ and $i > 0$, we have

$$h^i((F \ast E) \otimes P_\alpha) = h^i((F \ast E) \otimes E) = h^i(H^0(F \otimes P_\alpha) \otimes E) = 0,$$

where the first equality follows from [PP04, Proposition 5.5 (b)(i)], the second one from [Par00, Remark 3.5 (c)], and the third one holds since $E$ is IT(0). We note that we need

$$h^i(t_p^*F \otimes P_\alpha) = h^i(t_p^*F \otimes E) = 0$$

for any $p \in X$ and $i > 0$ to apply [PP04, Proposition 5.5 (b)(i)]. This condition is satisfied since $t_p^*F$ and $t_p^*F \otimes E$ are IT(0). \[\Box\]

Theorem 1.2 (2), (3) follow from the following proposition:

**Proposition 4.4.** Let $E$ be a locally free sheaf and $F$ be a coherent sheaf on $X$. Assume

(a) $E, F$ are IT(0), and

(b) $\varphi_\ast \Phi_P(E) \otimes \varphi_\ast \Phi_P((-1_X)^*F)(\frac{1}{x}l)$ is M-regular for some rational number $x \geq \beta(l)$.

Then the natural map $H^0(t_p^*E) \otimes H^0(F) \to H^0(t_p^*E \otimes F)$ is surjective for any $p \in X$.

Furthermore, the assumptions (a), (b) are satisfied if

1. $E = L$, and $F(-\frac{1}{x-1}l)$ is M-regular for some $\beta(l) \leq x < 1$, or
2. there exist rational numbers $s, t > 0$ such that $E(-sl)$ and $F(-tl)$ are M-regular and $st/(s+t) \geq \beta(l)$.

**Proof.** Assume that (a), (b) are satisfied. To show the surjectivity of $H^0(t_p^*E) \otimes H^0(F) \to H^0(t_p^*E \otimes F)$ for any $p \in X$, it suffices to show that $E \ast F$ is globally generated by the Proposition 2.7 (ii). By Theorem 1.2 (1), it is enough to show the M-regularity of $E \ast F(-xl)$. Since $E, F$ are IT(0), so is $E \ast F$ by Lemma 4.3. Hence $E \ast F(-xl)$ is M-regular if and only if so is $\varphi_\ast \Phi_P(E \ast F)(\frac{1}{x}l)$ by Proposition 4.1.

Since

$$\Phi_P(E \ast F) = \Phi_P(E \ast (-1_X)^*F) = \Phi_P(E) \otimes \Phi_P((-1_X)^*F),$$

by Proposition 2.7 (i), we have $\varphi_\ast \Phi_P(E \ast F)(\frac{1}{x}l) = \varphi_\ast \Phi_P(E \otimes \Phi_P((-1_X)^*F)(\frac{1}{x}l)$, which is M-regular by (b). Hence $E \ast F(-xl)$ is M-regular and $H^0(t_p^*E) \otimes H^0(F) \to H^0(t_p^*E \otimes F)$ is surjective for any $p \in X$.

The rest is to show that (a), (b) are satisfied in the cases (1), (2).

1. Since $x/(1-x) > 0$ and $F(-\frac{1}{x}l)$ is M-regular, $F$ is IT(0). Since $L$ is an ample line bundle and hence IT(0), (a) is satisfied.
Since \( \varphi_1^* \Phi_P(E) = \varphi_1^* \Phi_P(L) = H^0(L) \otimes L^{-1} \) by \cite[Proposition 3.11 (1)]{Muk}, we have
\[
\left( \varphi_1^* \Phi_P(E) \otimes \varphi_1^* \Phi_P((-1)^* F) \left( \frac{1}{x} \right) \right) = H^0(L) \otimes \varphi_1^* \Phi_P((-1)^* F) \left( \frac{1-x}{x} \right).
\]

By \cite[Proposition 4.1]{Proposition 4.1}, this is M-regular if and only if so is \(((-1)^* F)(-\frac{x}{1-x})\), which is equivalent to the M-regularity of \(F(-\frac{x}{1-x})\) by \((-1)^* l = l\). Hence (b) is satisfied.

(2) Since \( s, t > 0 \) and \( E(-sl) \) and \( F(-tl) \) are M-regular, (a) is satisfied.

For (b), set \( x = st/(s + t) \geq \beta(l) \). Then \( 1/x = 1/s + 1/t \) holds and
\[
\varphi_1^* \Phi_P(E) \otimes \varphi_1^* \Phi_P((-1)^* F) \left( \frac{1}{x} \right) = \left( \varphi_1^* \Phi_P(E) \left( \frac{1}{s} \right) \right) \otimes \left( \varphi_1^* \Phi_P((-1)^* F) \left( \frac{1}{t} \right) \right).
\]

By \cite[Proposition 4.1]{Proposition 4.1}, \( \varphi_1^* \Phi_P(E) \left( \frac{1}{s} \right) \) and \( \varphi_1^* \Phi_P((-1)^* F) \left( \frac{1}{t} \right) \) are M-regular if and only if so are \( E(-sl) \) and \((-1)^* F(-tl)\). Hence (b) is satisfied by \cite[Proposition 2.6]{Proposition 2.6}.

\textbf{Remark 4.5.} The M-regularity of \( L \cdot F(-xl) \) in the case (1) of \cite[Proposition 4.4]{Proposition 4.4} can be proved slightly easier as follows:

By \cite[Proposition 2.7 (iii)]{Proposition 2.7}, we have \( L \cdot F(-xl) = L \otimes \varphi_1^* \Phi_P(F \otimes L)(-xl) = \varphi_1^* \Phi_P(F \otimes L)((1-x)l) \) by \cite[Proposition 4.1]{Proposition 4.1} this is M-regular if and only if so is \( F \otimes L(-\frac{x}{1-x}) = F(-\frac{x}{1-x}) \).

\cite[Proposition 4.1]{Proposition 4.1} also implies the following lemma, which is a \( Q \)-twisted version of \cite[(2.4)]{Lemma 4.6}.

\textbf{Lemma 4.6.} Let \( f : Y \to X \) be an isogeny and \( F \) be a coherent sheaf on \( Y \). For \( x \in \mathbb{Q}, f_* F(xl) \) is GV, M-regular or IT(0) if and only if so is \( F(x^* f^! l) \).

In particular, for a coherent sheaf \( F \) on \( X \), an integer \( m > 0 \) and \( x \in \mathbb{Q}, \mu_{m*} F(xl) \) is GV, M-regular or IT(0) if and only if so is \( F(m^2 xl) \).

\textbf{Proof.} For a sufficiently large integer \( n, F \otimes f^* L^n \) is IT(0). Since
\[
f_* F(xl) = f_* (F \otimes f^* L^n)(n - x - n)l, \quad F(x^* f^! l) = F \otimes f^* L^n((x - n) f^* l),
\]
we may assume that \( F \) is IT(0) and \( x < 0 \) by replacing \( F \) and \( x \) with \( F \otimes f^* L^n \) and \( x - n \) respectively.

Set \( y := -x > 0 \). By \cite[Proposition 4.1]{Proposition 4.1} and \cite[(2.2)]{Lemma 4.6}, \( f_* F(xl) = f_* F(-yl) \) is GV, M-regular or IT(0) if and only if so is \( f^* \varphi_1^* \Phi_P^Y(f_* F)(\frac{y}{f} \frac{f}{y} f^! t) \).

We have
\[
f^* \varphi_1^* \Phi_P^Y(f_* F) = f^* \varphi_1^* \Phi_P^Y(f_* F) = \varphi_1^* \Phi_P^Y(F)
\]
by \cite[(2.5)]{Lemma 4.6} and \( f \circ \varphi_1 \circ f = \varphi_1 f^* l \). Hence \( f_* F(xl) \) is GV, M-regular or IT(0) if and only if so is \( \varphi_1^* \Phi_P^Y(F)\left(\frac{f}{y} f^! l\right) \) if and only if so is \( F(-y f^* l) = F(x^* f^! l) \) by \cite[Proposition 4.1]{Proposition 4.1}.

The last statement is nothing but the spacial case \( f = \mu_m \) since \( \mu_m l = m^2 l \). \hfill \Box

\section{On jet ampleness}

In this section, we study \( k \)-jet ampleness using \cite[Proposition 3.1]{Proposition 3.1}. Throughout this section, \( L \) is an ample line bundle on an abelian variety \( X \). First, we prove \cite[Theorem 1.6]{Theorem 1.6}.

\textbf{Proof of \cite[Theorem 1.6]{Theorem 1.6}.} If \( \mathcal{I}_p \langle \frac{1}{p+1} l \rangle \) is M-regular, so is \( \mathcal{I}_{p_1} \langle \frac{1}{p_1+1} l \rangle \) for any \( p_1 \in X \). Furthermore, we have \( \beta(l) \leq \frac{1}{p+1} \). Applying \cite[Proposition 3.1 (ii)]{Proposition 3.1} to \( x = \frac{1}{p+1} \) and \( F(yl) = \mathcal{I}_{p_1} \langle \frac{1}{p_1+1} l \rangle \), we see that \( \mathcal{I}_{p_2} \mathcal{I}_{p_1} \langle \frac{2}{p_2+p_1+2} l \rangle \) is IT(0) for any \( p_2 \in X \). We note that \( p_2 \) can be \( p_1 \). Since IT(0) implies M-regularity, we can apply \cite[Proposition 3.1 (ii)]{Proposition 3.1} to \( \mathcal{I}_{p_2} \mathcal{I}_{p_1} \langle \frac{2}{p_2+p_1+2} l \rangle \) and hence \( \mathcal{I}_{p_1} \mathcal{I}_{p_2} \mathcal{I}_{p_1} \langle \frac{3}{p_3+p_2+p_1+3} l \rangle \) is IT(0) for any \( p_3 \in X \). Repeating this, we obtain that \( \mathcal{I}_{p_{k+1}} \cdots \mathcal{I}_{p_2} \mathcal{I}_{p_1} \langle \frac{k+1}{k+2} l \rangle \) is IT(0), i.e., \( \mathcal{I}_{p_{k+1}} \cdots \mathcal{I}_{p_2} \mathcal{I}_{p_1} \otimes L \) is IT(0) for any (not necessarily distinct) \( p_1, \ldots, p_{k+1} \in X \). By \cite[Lemma 3.3]{Lemma 3.3}, this is equivalent to the \( k \)-jet ampleness of \( L \). \hfill \Box

In \cite{Lemma 3.3}, the authors introduce an invariant \( m(L) \), called the M-regularity index, as
\[
m(L) := \max \{ m \geq 0 \mid L \otimes \mathcal{I}_{p_1} \mathcal{I}_{p_2} \cdots \mathcal{I}_{p_m} \text{ is M-regular} \}
\]
for any (not necessarily distinct) \( m \) points \( p_1, \ldots, p_m \in X \).
Theorem 3.8] states that if \( A, L_1, \ldots, L_{k+1-m(A)} \) are ample line bundles on \( X \) and \( k \geq m(A) \geq 1 \), then \( A \otimes L_1 \otimes \cdots \otimes L_{k+1-m(A)} \) is \( k \)-jet ample. In particular, \( A^{k+2-m(A)} \) is \( k \)-jet ample for \( k \geq m(L) \geq 1 \). The following is a generalization of this result:

**Proposition 5.1.** Let \( n, k_1, \ldots, k_n \) be positive integers and \( A, L_1, \ldots, L_n \) be ample line bundles on \( X \). If \( \beta(l) \leq \frac{1}{k_i} \) for any \( 1 \leq i \leq n \), then \( A \otimes L_1 \otimes \cdots \otimes L_n \) is \( k \)-jet ample, where \( k = m(A) + \sum_{i=1}^{n} k_i - 1 \).

**Proof.** For simplicity, set \( m = m(A) \). By definition, \( A \otimes \mathcal{I}_{p_1} \mathcal{I}_{p_2} \cdots \mathcal{I}_{p_m} \) is M-regular for any \( p_i \in X \). By Proposition 3.1(ii), \( A \otimes \mathcal{I}_{p_1} \mathcal{I}_{p_2} \cdots \mathcal{I}_{p_m} \mathcal{I}_{p_{m+1}} \mathcal{I}_{k_1} \) is IT(0) for any \( p_{m+1} \in X \). Since IT(0) implies M-regularity, \( A \otimes \mathcal{I}_{p_1} \mathcal{I}_{p_2} \cdots \mathcal{I}_{p_m} \mathcal{I}_{p_{m+1}} \mathcal{I}_{p_{m+2}} \mathcal{I}_{k_1} \) is IT(0) for any \( p_{m+2} \in X \) by Proposition 3.1(ii). Repeating this, \( A \otimes \mathcal{I}_{p_1} \mathcal{I}_{p_2} \cdots \mathcal{I}_{p_{m+k_1}} \mathcal{I}_{k_1+k_2} \) is IT(0) for any \( p_1 \). Repeating this argument, \( A \otimes L_1 \otimes \cdots \otimes L_n \otimes \mathcal{I}_{p_1} \mathcal{I}_{p_2} \cdots \mathcal{I}_{p_{m+k_1}} \) is IT(0) for any \( p_i \in X \). By [PP04] Lemma 3.3, this is equivalent to the \( k \)-jet amenable of \( A \otimes L_1 \otimes \cdots \otimes L_n \).

We note that \( A \otimes L_1 \otimes \cdots \otimes L_n \) is a tensor product of two or more ample line bundles since we assume \( n \geq 1 \). Hence Proposition 5.1 do not contain Theorem 1.6.

In the rest of this section, we see some relation between \( m(L) \), the M-regularity of \( \mathcal{I}_o(xl) \), and Seshadri constants.

**Corollary 5.2.** If \( \mathcal{I}_o(\frac{1}{m}l) \) is M-regular for an integer \( m \geq 1 \), then \( m(L) \geq m \) holds.

**Proof.** If \( m = 1 \), this holds from definition. Hence we may assume that \( m \geq 2 \). Then we already see that \( \mathcal{I}_{p_1} \mathcal{I}_{p_2} \cdots \mathcal{I}_{p_m} \mathcal{I}_o \mathcal{I}_o(L) \) is IT(0) for any \( p_1, \ldots, p_m \in X \) in the proof of Theorem 1.6. Hence \( m(L) \geq m \) holds.

In general, the converse of Corollary 5.2 does not hold, that is, \( \mathcal{I}_o(\frac{1}{m}l) \) is not M-regular in general. For example, let \( (X, L) \) be a general polarized abelian surface of type \((1,4)\) by [PP04] Example 3.7, we have \( m(L) = 2 \). On the other hand, \( \mathcal{I}_o(\frac{1}{2}l) \) is not M-regular by Theorem 1.6 since \( X \) is not very ample.

In fact, \( \mathcal{I}_o(\frac{1}{m}l) \) is not even GV, equivalently \( \beta(l) \leq 1/m(L) \) does not hold in general. For example, let \( (X, L) \) be a polarized abelian surface of type \((1,d)\) with Picard number one. Then \( L \) is \( k \)-jet ample if and only if \( d > \frac{1}{2}(k+2)^2 \) by [BS97]. Hence we have \( m(L) \geq k+1 \) if \( d > \frac{1}{2}(k+2)^2 \) by [PP04] Proposition 3.5. Thus \( m(L) \geq \lceil \sqrt{2d-1} \rceil - 1 \). On the other hand, \( \beta(l) \geq 1/\sqrt{d} \) holds by [IT20a] Lemma 3.4, hence \( 1/m(L) \), which is bounded from above by roughly \( 1/\sqrt{2d} \), is strictly smaller than \( \beta(l) \) for \( d \gg 1 \).

However, we can show that \( \mathcal{I}_o(\frac{\dim X}{m}l) \) is IT(0) at least in \( \text{char}(K) = 0 \). To see this, recall that the Seshadri constant \( \varepsilon(L) \) of \( L \) is defined as
\[
\varepsilon(X, L) = \varepsilon(L) := \max \{ t \geq 0 \mid \exists^* L - tE \text{ is nef} \},
\]
where \( \pi : \tilde{X} \rightarrow X \) is the blow-up at \( o \in X \) and \( E \subset \tilde{X} \) is the exceptional divisor. By [PPTT] Theorem 7.4], \( \varepsilon(L) \geq m(L) \) holds. On the other hand, \( \beta(L) \cdot \varepsilon(L) \geq 1 \) holds by [Can20a] Proposition 1.6.7, that is, \( \varepsilon(L) > x^{-1} \) holds if \( \mathcal{I}_o(xl) \) is IT(0). The following is a refinement of these results:

**Proposition 5.3.** Assume \( \text{char}(K) = 0 \).

(i) If \( \mathcal{I}_o^n(xl) \) is M-regular for an integer \( m \geq 0 \) and \( x \in \mathbb{Q}_{>0} \), then \( \varepsilon(L) > x^{-1} \cdot m \) holds.

(ii) If \( \mathcal{I}_o^n(xl) \) is GV for an integer \( m \geq 0 \) and \( x \in \mathbb{Q}_{>0} \), then \( \varepsilon(L) \geq x^{-1} \cdot m \) holds.

(iii) \( \varepsilon(L) > m(L) \) holds.

(iv) If \( m(L) \geq 1 \), \( \mathcal{I}_o(\frac{\dim X}{m}l) \) is IT(0), where \( g = \dim X \).

**Proof.** (i) Since \( \varepsilon(L) > 0 \), this is clear if \( m = 0 \). Hence we may assume \( m \geq 1 \). Let \( \pi : \tilde{X} \rightarrow X \) be the blow-up at \( o \in X \) and \( E \subset \tilde{X} \) be the exceptional divisor. What we need to show is the amplementness of \( \pi^* L - x^{-1} \cdot mE \), i.e. the amplementness of \( \pi^* xL - mE \).
Let $x = a/b$. Then $\mu_b^* T_o^m \otimes L^{ab}$ is M-regular. Hence $\mu_b^* T_o^m \otimes L^{ab}$ is ample, that is, the
tautological line bundle $\mathcal{O}(1)$ of

$$\mathbb{P}_X(\mu_b^* T_o^m \otimes L^{ab}) := \text{Proj}_X \left( \bigoplus_{n \geq 0} \text{Sym}^n(\mu_b^* T_o^m \otimes L^{ab}) \right)$$
is ample by [PP03 Proposition 2.13] and [Deb06 Corollary 3.2]. Since

$$\text{Proj}_X \left( \bigoplus_{n \geq 0} \text{Sym}^n(\mu_b^* T_o^m \otimes L^{ab}) \right) \simeq \text{Proj}_X \left( \bigoplus_{n \geq 0} \text{Sym}^n(\mu_b^* T_o^m) \right) \simeq \text{Proj}_X \left( \bigoplus_{n \geq 0} \text{Sym}^n(\mu_b^* L_o) \right),$$

$\mathbb{P}_X(\mu_b^* T_o^m \otimes L^{ab}) \to X$ is isomorphic to the blow-up $\pi' : \tilde{X}' \to X$ along the ideal $\mu_b^* T_o$. Under this isomorphism, the tautological line bundle $\mathcal{O}(1)$ of $\mathbb{P}_X(\mu_b^* T_o^m \otimes L^{ab})$ corresponds to $\mathcal{O}(-mE') \otimes \pi'^* L^{ab}$, where $E' \subset \tilde{X}'$ is the exceptional divisor of $\pi'$. Hence $\pi'^* \text{ab}L - mE'$ is ample.

Since $\pi'$ is the blow-up along $\mu_b^* T_o$, there exists a morphism $\tilde{\mu}_b : \tilde{X}' \to \tilde{X}$ such that $\pi \circ \tilde{\mu}_b = \mu_b \circ \pi'$ and $\tilde{\mu}_b^* \mathcal{O}(-E) = \mathcal{O}(-E')$. Hence we have $\pi'^* \text{ab}L - mE' \equiv \tilde{\mu}_b^* (\pi'xL - mE)$ and the ampleness of $\pi'xL - mE$ follows from that of $\pi'^* \text{ab}L - mE'$.

(ii) If $T_o^m(xl)$ is GV, then $T_o^m((x + x')l)$ is IT(0) for any rational number $x' > 0$. Hence $\varepsilon(L) > (x + x')^{-1} \cdot m$ holds by (1). By $x' \to 0$, we have $\varepsilon(L) \geq x^{-1} \cdot m$.

(iii) By definition, $T_o^m(l)$ is M-regular and hence $\varepsilon(L) > m(L)$ holds by (1).

(iv) By [He21a Proposition 3.1] and (3), we have $\beta(l) \leq g \cdot \varepsilon(L)^{-1} < g \cdot m(L)^{-1}$. Hence (4) follows from [Lemma 2.3].

Example 5.4. (1) By definition, $T_o^2(l)$ is GV (resp. M-regular) if and only if the codimension of $\{ p \in X | h^1(X, T_o^2 \otimes L) > 0 \}$ in $X$ is at least one (resp. at least two). Since $h^1(X, T_o^2 \otimes L) = 0$ holds if and only if the rational map $f_{[L]}$ defined by $[L]$ is an immersion at $p$, we have $\varepsilon(L) \geq 2$ if $f_{[L]}$ is generically finite, and $\varepsilon(L) > 2$ if $f_{[L]}$ is an immersion outside a codimension two subset. If (Proposition 5.3) (i), (ii) in $\text{char}(K) = 0$. (2) In [Nak96 Theorem 1.1, Lemma 2.6], Nakamaye proves that for a polarized abelian variety $(X, L)$ in $\text{char}(K) = 0$, $\varepsilon(L) \geq 1$ and the equality holds if and only if

$$(X, L) \simeq (E, L_E) \times (X', L') := (E \times X', p_E^* L_E \otimes p_X^* L'),$$

for a principally polarized elliptic curve $(E, L_E)$ and a polarized abelian variety $(X', L')$. We can recover this result from [Proposition 5.3] as follows:

We show this by the induction on $g = \dim X$. If $g = 1$, this is clear since $\varepsilon(X, L) = \deg(L)$ by definition. Assume $g \geq 2$.

If $(X, L) \simeq (E, L_E) \times (X', L')$ as (5.1) then we have $\varepsilon(X, L) = \min\{\varepsilon(E, L_E), \varepsilon(X', L')\}$ (see [MR15 Proposition 3.4] for example). Since $\varepsilon(E, L_E) = \deg L_E = 1$ and $\varepsilon(X', L') \geq 1$ by induction hypothesis, we have $\varepsilon(X, L) = 1$.

Conversely, assume $\varepsilon(L) \leq 1$. Then $m(L) = 0$ by [Proposition 5.3] (ii) and hence $L$ has a base divisor.

Assume that $(X, L)$ is indecomposable, that is, $(X, L)$ is not isomorphic to any product of polarized abelian varieties of positive dimensions. Then $L$ is an indecomposable principal polarization by [BL88 Theorem 4.3] since $L$ has a base divisor. Replacing $L$ with an algebraically equivalent line bundle, we may assume that $L$ is symmetric. Then the morphism $f_{[L]}$ defined by $[L]$ is the natural morphism $\pi : X \to X/(1X) \subset \mathbb{P}^{2g-1}$ to the Kummer variety $X/(1X)$ by [BL88 Theorem 4.8.1]. Since $\pi$ is an immersion outside two-torsion points in $X$ and $g \geq 2$, we have $2 \varepsilon(L) = \varepsilon(L^2) > 2$ by (1) of this example. This contradicts the assumption $\varepsilon(L) \leq 1$.

Thus $(X, L)$ is decomposable, that is, $(X, L)$ is isomorphic to a product $(X_1, L_1) \times (X_2, L_2)$ with $\dim X_i \geq 1$. Then $\varepsilon(X, L) = \min\{\varepsilon(X_1, L_1), \varepsilon(X_2, L_2)\}$ and hence we may assume $\varepsilon(X_1, L_1) = \varepsilon(X, L) \leq 1$. Since $\varepsilon(X_1, L_1) \geq 1$ by induction hypothesis, $\varepsilon(X_1, L_1) = 1$ holds.
and \((X_1, L_1)\) has a decomposition as \((5.1)\) Hence \(\varepsilon(X, L) = \varepsilon(X_1, L_1) = 1\) and \((X, L)\) has a decomposition as \((5.1)\)

6. On property \((N_p)\)

Throughout this section, \(L\) is an ample line bundle on an abelian variety \(X\). If \(\mathcal{I}_p(x)\) is M-regular for some rational number \(0 < x < 1\), then \(L\) is basepoint free by Theorem 1.4 (1). Furthermore, \(M_L(x_{1-x})\) is M-regular by Proposition 4.1. Hence \(M_L^{\otimes m}(mx_{1-x})\) is also M-regular for any \(m \geq 1\) by Proposition 2.6. In fact, we can show that \(M_L^{\otimes m}(mx_{1-x})\) is IT(0) if \(m \geq 2\) as follows:

**Proposition 6.1.** Assume that \(\mathcal{I}_p(x)\) is M-regular for a rational number \(0 < x < 1\). Then \(M_L^{\otimes m}(mx_{1-x})\) is IT(0) for any integer \(m \geq 2\).

**Proof.** Let \(1 - x = \frac{b}{a}\) for integers \(b > a > 0\). Then

\[
(6.1) \quad M_L\left\langle \frac{x}{1-x} \right\rangle = M_L\left\langle \frac{b-a}{a} \right\rangle \quad \text{and} \quad \mu^*_a M_L \otimes L^{a(b-a)} \quad \text{are M-regular}
\]

by Proposition 4.1. If \(M_L^{\otimes 2}(2x_{1-x})\) is IT(0), so is

\[
M_L^{\otimes m}\left\langle \frac{mx}{1-x} \right\rangle = M_L^{\otimes 2}\left\langle \frac{2x}{1-x} \right\rangle \otimes M_L\left\langle \frac{x}{1-x} \right\rangle^{\otimes m-2}
\]

for \(m \geq 2\) by Proposition 2.6 since \(M_L^{\otimes 2}(2x_{1-x})\) is IT(0) and \(M_L(x_{1-x})\) is GV. Hence it suffices to show the case \(m = 2\).

By definition and (2.4), \(M_L^{\otimes 2}(2x_{1-x}) = M_L^{\otimes 2}(2(b-a)/a)\) is IT(0) if and only if so is \(\mu^*_a M_L^{\otimes 2} \otimes L^{2a(b-a)}\) if and only if so is \(\mu^*_a (\mu^*_a M_L^{\otimes 2} \otimes L^{2a(b-a)}) = M_L^{\otimes 2} \otimes \mu^*_a L^{2a(b-a)}\). Consider the exact sequence

\[
0 \to M_L^{\otimes 2} \otimes \mu^*_a L^{2a(b-a)} \to H^0(L) \otimes M_L \otimes \mu^*_a L^{2a(b-a)} \to L \otimes M_L \otimes \mu^*_a L^{2a(b-a)} \to 0
\]

obtained by tensoring \(M_L \otimes \mu^*_a L^{2a(b-a)}\) with (4.1). Since \(\mu^*_a M_L \otimes L^{a(b-a)}\) is M-regular by (6.1), \(\mu^*_a M_L \otimes L^{2a(b-a)}\) is IT(0). Hence \(M_L \otimes \mu^*_a L^{2a(b-a)} = \mu^*_a (\mu^*_a M_L \otimes L^{2a(b-a)})\) and \(L \otimes M_L \otimes \mu^*_a L^{2a(b-a)}\) are IT(0) by (2.4) and Proposition 2.6. Thus \(h^1(M_L^{\otimes 2} \otimes \mu^*_a L^{2a(b-a)} \otimes P_\alpha) = 0\) for any \(i \geq 2\) and \(\alpha \in \hat{X}\), and \(M_L^{\otimes 2} \otimes \mu^*_a L^{2a(b-a)}\) is IT(0) if and only if the natural map

\[
H^0(L) \otimes H^0(M_L \otimes \mu^*_a L^{2a(b-a)} \otimes P_\alpha) \to H^0(L \otimes M_L \otimes \mu^*_a L^{2a(b-a)} \otimes P_\alpha)
\]

is surjective for any \(\alpha \in \hat{X}\). By Theorem 1.2 (2), this map is surjective if

\[
M_L \otimes \mu^*_a L^{2a(b-a)} \otimes P_\alpha \left\langle \frac{-x}{1-x} \right\rangle = M_L \otimes \mu^*_a L^{2a(b-a)} \otimes P_\alpha \left\langle \frac{b-a}{a} \right\rangle
\]

\[
= \mu^*_a (\mu^*_a M_L \otimes L^{2a(b-a)} \otimes \mu^*_a P_\alpha) \left\langle \frac{-b-a}{a} \right\rangle
\]

is M-regular. By Lemma 4.6, this is equivalent to the M-regularity of

\[
\mu^*_a M_L \otimes L^{2a(b-a)} \otimes \mu^*_a P_\alpha \left\langle \frac{b-a}{a} \right\rangle = \mu^*_a M_L \langle a(b-a) \rangle,
\]

which is nothing but (6.1).

**Proposition 6.2.** Assume that \(\mathcal{I}_p(x)\) is M-regular for a rational number \(0 < x < 1\). Then \(h^1(M_L^{\otimes m} \otimes B) = 0\) for an integer \(m \geq 1\) and a line bundle \(B\) on \(X\) if \(B - \frac{mx}{a} L\) is ample or \(m \geq 2\) and \(B - \frac{mx}{a} L\) is nef.

In particular, if \(\mathcal{I}_p(x)\) is M-regular for an integer \(p \geq 0\), then \(h^1(M_L^{\otimes m} \otimes L^h) = 0\) for positive integers \(m, h\) if \(h > m/(p+1)\) or \(m \geq 2\) and \(h \geq m/(p+1)\).
Proof. The proof is essentially the same as that of [Cau20a, Proposition 3.5] other than we use Proposition 6.1 when $m\geq 2$.

As a $\mathbb{Q}$-twisted sheaf, $M_L^L \otimes B$ is written as

$$
\left(M_L\left(\frac{x}{1-x}\right)^{\otimes m}\right) \otimes B\left(\frac{-mx}{1-x}\right).
$$

By Proposition 4.1, $M_L\left(\frac{x}{1-x}\right)$ is $M$-regular.

If $B - \frac{mx}{1-x}L$ is ample, $B\left(-\frac{mx}{1-x}\right)$ is IT(0) by Example 2.1. Thus $M_L^L \otimes B$ is IT(0) by Proposition 2.6 and hence the vanishing $h^i(M_L^L \otimes B) = 0$ holds.

When $m \geq 2$ and $B - \frac{mx}{1-x}L$ is nef, $M_L\left(\frac{x}{1-x}\right)^{\otimes m}$ is IT(0) by Proposition 6.1 and $B\left(-\frac{mx}{1-x}\right)$ is GV by Example 2.1. Hence $M_L^L \otimes B$ is IT(0) by Proposition 2.6 and the vanishing $h^i(M_L^L \otimes B) = 0$ holds.

The last statement is just a special case when $x = 1/(p + 2)$ and $B = L^h$. \hfill $\Box$

Proof of Theorem 1.5 (1) Since $\beta(l) \leq 1/(p + 2) < 1$, $L$ is basepoint free. By [Cau20a, Proposition 4.1], $(N_p)$ holds for $L$ if $h^1(M_L^L \otimes L^h) = 0$ for any $h \geq 1$ in char($\mathbb{K}$) $\geq 0$. Since $p + 1 \geq 2$, here we use the assumption $p \geq 1$, and $h \geq 1 = (p + 1)/(p + 1)$, this vanishing follows from Proposition 6.2.

For (2), $(N_p)$ holds for $L$ if $h^1(M_L^L \otimes L^h) = 0$ for any $h \geq r + 1$ by [PP04, Proposition 6.3] when char($\mathbb{K}$) does not divide $p + 1$ and by [Cau20a, Section 4] in any characteristic. We can prove (2) similarly. \hfill $\Box$

We give an example which does not follow from Theorem 1.3, Theorem 1.4.

Example 6.3. Let $(X, L)$ be a general polarized abelian variety of dimension $g \geq 2$ and of type $(1, 3, \ldots, 3)$ in char($\mathbb{K}$) = 0. Then $L^n$ satisfies $(N_p)$ if $n \geq \frac{2g+2}{3}$. To see this, it suffices to show that $\mathcal{I}_{o}(\frac{1}{3}l)$ is $M$-regular by Theorem 1.5. In fact, we can show that $\mathcal{I}_{o}(\frac{1}{3}l)$ is $M$-regular but not IT(0) as follows:

Take an isogeny $\pi: Y \rightarrow X$ with kernel $\pi^{-1}(o) \simeq \mathbb{Z}/3\mathbb{Z}$ such that there exists a principally polarization $\Theta$ such that $\pi^{*}L \equiv 3\Theta$. Then $\mathcal{I}_{o}(\frac{1}{3}l)$ is M-regular or IT(0) if and only if so is $\pi^{*}\mathcal{I}_{o}(\frac{2}{3}l) = \mathcal{I}_{o}\left(\frac{2}{3}l\right)$ by [2.2]. Hence it suffices to show that $\mathcal{I}_{o}\left(\frac{2}{3}l\right) \otimes 2\Theta$ is $M$-regular but not IT(0). We may assume that $\Theta$ is symmetric. Since $h^i(\mathcal{I}_{o}(\frac{2}{3}l) \otimes 2\Theta) = 0$ for any $p \in X$ and $i \geq 1$, it suffices to show that

$$
V^1 := \{p \in Y \mid h^1(\mathcal{I}_{o}(\frac{2}{3}l) \otimes 2\Theta) > 0\} \subset Y
$$

is not empty and the codimension is greater than 1.

Since $(X, L)$ is general, $(Y, \Theta)$ is indecomposable, i.e. $(Y, \Theta)$ is not a product of smaller dimensional principally polarized abelian varieties. Hence $2\Theta$ gives a double cover $f: Y \rightarrow Y/(-1_Y) \subset \mathbb{P}^{2g-1}$. Thus $p \in Y$ is contained in $V^1$ if and only if the restriction map

$$
H^0(Y, 2\Theta) = H^0(\mathbb{P}^{2g-1}, \mathcal{O}(1)) \rightarrow 2\Theta|_{\pi^{-1}(o)+p}
$$

is not surjective if and only if $f(\pi^{-1}(o)+p)$ is contained in a line in $\mathbb{P}^{2g-1}$. Hence we have

$$
V^1 = \{p \in Y \mid f(\pi^{-1}(o)+p) \text{ is contained in a line}\}.
$$

Let $\varepsilon \in \pi^{-1}(o) \simeq \mathbb{Z}/3\mathbb{Z}$ be a generator. If $2p = \varepsilon$ for $p \in Y$, then $f(p) = f(-p) = f(-\varepsilon + p)$ holds and hence $f(\pi^{-1}(o)+p) = \{f(p), f(\varepsilon+p), f(-\varepsilon+p)\}$ is contained in a line. Thus $\mu_2^{-1}(\varepsilon)$ is contained in $V^1$. By a similar argument, we have

$$
(6.2) \quad \mu_2^{-1}(\pi^{-1}(o)) = \mu_2^{-1}(\{o, \varepsilon, -\varepsilon\}) \subset V^1.
$$

In particular, $V^1$ is not empty and hence $\mathcal{I}_{o}(\frac{1}{3}l)$ is not IT(0).
When \( g = 2 \), we need to see \( \dim V^1 = 0 \) for the M-regularity of \( \mathcal{I}_o(\frac{2}{3}l) \). Assume \( \dim V^1 \geq 1 \). Then \( Y \) is the Jacobian \( J(C) \) of a smooth curve \( C \) of genus two and \( \mu_2(V^1) \subset Y = J(C) \) is a theta divisor, i.e., \( \mu_2(V^1) \) is the image of \( C \) by \( a \in C \mapsto \mathcal{O}_C(a) \otimes Q \in J(C) \) for a line bundle \( Q \) of degree \(-1\) on \( C \) by [Weh84 Theorem (0.5)]. On the other hand, \( \pi^{-1}(a) = \{ o_Y, \varepsilon, -\varepsilon \} \) is contained in \( \mu_2(V^1) \) by (6.2). Hence there exist \( a, b, c \in C \) such that
\[
\mathcal{O}_C(a) \otimes Q = o_Y, \quad \mathcal{O}_C(b) \otimes Q = \varepsilon, \quad \mathcal{O}_C(c) \otimes Q = -\varepsilon.
\]
Thus \( \varepsilon = \mathcal{O}_C(b-a), \varepsilon = \mathcal{O}_C(c-a) \) and hence \( \mathcal{O}_C(b+c) \simeq \mathcal{O}_C(2a) \). Since \( b \neq a, c \neq a \) by \( \varepsilon \neq o_Y \), \( \mathcal{O}_C(2a) \) is basepoint free of degree two on a curve \( C \) of genus two. Thus \( \mathcal{O}_C(2a) \) is linearly equivalent to the canonical line bundle \( \omega_C \).

Furthermore, we also have \( \mathcal{O}_C(c+a) \simeq \mathcal{O}_C(2b) \) by \( \mathcal{O}_C(b+c) \simeq \mathcal{O}_C(2a) \) and \( o_Y = 3\varepsilon = \mathcal{O}_C(3b-3a) \). Thus \( \mathcal{O}_C(2b) \) is also basepoint free of degree two, and hence \( \mathcal{O}_C(2b) \simeq \omega_C \). Then
\[
\mathcal{O}_C(2b) \simeq \omega_C \simeq \mathcal{O}_C(2a) \simeq \mathcal{O}_C(b+c),
\]
which implies \( b = c \). This contradicts \( \varepsilon = \mathcal{O}_C(b-a) \neq -\varepsilon = \mathcal{O}_C(c-a) \). Hence \( V^1 \) cannot be one dimensional and we have the M-regularity of \( \mathcal{I}_o(\frac{2}{3}l) \).

Remark 6.4. (1) In [PP04 Conjecture 6.4], it is conjectured that if \( A \) is ample, \( m(A) \geq m \), and \( p \geq m \), then \( A^n \) satisfies \((N_p)\) for any \( n \geq p + 3 - m \). The case \( m = 0 \) is nothing but Lazarsfeld’s conjecture and the case \( m = 1 \) is nothing but Theorem 1.3. If we want to apply Theorem 1.5, we need to show the M-regularity of \( \mathcal{I}_o(\frac{p+3-m}{p+2}a) \). Since \( m(A) \geq m \) and \( p \geq m \), it is enough to see the M-regularity of \( \mathcal{I}_o(\frac{3}{m(A)+2}a) \). However, the author does not know this holds or not in general.

(2) Let \((X, L)\) be a general polarized abelian variety of dimension \( g \) and of type \((1, \ldots, 1, d)\). In [Ito03], the author shows that for an integer \( m \geq 1 \), \( \mathcal{I}_o(\frac{1}{m}l) \) is GV if and only if \( d \geq m^g \) and \( \IT(0) \) if \( d \geq m^g + m^{g-1} + \cdots + m + 1 \).

On the other hand, \( L \) has no base divisor if \( d > 1 = 1^g \), and \( L \) is very ample if \( d > 2^g \) by [DHS94 Remark 3, Corollary 25], and \( L \) satisfies \((N_1)\) if \( d > 9 = 3^2 \) when \( g = 2 \) by [GP98]. Hence it might be natural to guess that \( \mathcal{I}_o(\frac{1}{m}l) \) is M-regular if \( d > m^g \). If this is true, we can recover the above results in [DHS94], [GP98] from Theorems 1.3, 1.6.

We also note that \( \mathcal{I}_o(\frac{1}{m}l) \) is not M-regular if \( d \leq m^g \). More generally, \( \mathcal{I}_o(\lambda l) \) is not M-regular if \( x \leq 1/\sqrt[3]{x(l)} \) for any polarized abelian variety \((X, L)\). In fact, we have an inequality
\[
h^1_{\mathcal{I}_o}(yl) \geq h^0_{\mathcal{O}_X/\mathcal{I}_o}(yl) - h^0_{\mathcal{O}_X}(yl) = 1 - \chi(l)y^g
\]
for \( y \in \mathbb{Q}_{>0} \) by the exact sequence \( 0 \to \mathcal{I}_o \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_o \to 0 \) (see [Jia20 Section 2.3]). By this inequality, it is easy to see that \( h^1_{\mathcal{I}_o}((x-t)l) \neq O(l^2) \) for sufficiently small \( t \in \mathbb{Q}_{>0} \) if \( x \leq 1/\sqrt[3]{x(l)} \) and hence \( \mathcal{I}_o(\lambda l) \) is not M-regular by (4.2).

7. On Projective normality

Contrary to Lazarsfeld’s conjecture, the statement of Theorem 1.3 does not hold for \( p = 0 \), that is, \( A^2 \) might not be projectively normal even if \( A \) has no base divisor as in the following example. Hence the statement of Theorem 1.5 does not hold for \( p = 0 \) as well. Equivalently, the M-regularity of \( \mathcal{I}_o(\frac{1}{l}l) \) does not imply the projective normality of \( L \) in general.

Example 7.1. If \( A \) is a symmetric ample line bundle on an abelian variety \( X \), \( A^2 \) is projectively normal if and only if the origin \( o \in X \) is not contained in the base locus of \([A \otimes P]\) for any line bundle \( P \) on \( X \) with \( P^2 \simeq \mathcal{O}_X \) by [Ohb88] in char(\( K \)) \( \neq 2 \) and [Ohb96] in char(\( K \)) \( \geq 0 \). Thus \( A^2 \) might not satisfy \((N_0)\) even if \( A \) has no base divisor. More explicitly, if \((X, A)\) is a general polarized abelian surface of type \((1, 2)\), then \( A \) has no base divisor but \( A^2 \) is not projectively normal by [Ohb93 Lemma 6].

In [PP04 §5], the authors give an alternative proof of the above result in [Ohb88] in char(\( K \)) \( \neq 2 \) using the theory of M-regularity. For general \( L \), which is not necessarily written as \( A^2 \), we can show the following lemma:
Lemma 7.2. Assume $\mathrm{char}(\mathbb{K}) \neq 2$ and let $L$ be an ample line bundle on an abelian variety $X$. Then the following are equivalent:

(i) $L$ is projectively normal,

(ii) $L \otimes \mu_2(L^{-2})$ is globally generated at the origin $o \in X$,

(iii) Let $X_2 := \mu_2^{-1}(a)$ be the set of two torsion points in $X$ and $f : X \to \mathbb{P}^N$ be the morphism defined by $[\mu_2 L \otimes L^{-2}]$. Then $f(X_2) \subset \mathbb{P}^N$ spans a linear subspace of dimension $\#X_2 - 1 = 4^d - 1$.

Proof. (ii) $\iff$ (iii): We note that the line bundle $\mu_2^* L \otimes L^{-2}$ is basepoint free since $\mu_2^* L \otimes L^{-2} \cong L^2$. Since $L \otimes \mu_2(L^{-2}) = \mu_2^* (\mu_2^* L \otimes L^{-2})$ by projection formula, the natural map

(7.1) $H^0(L \otimes \mu_2(L^{-2})) \to H^0(L \otimes \mu_2(L^{-2}) \otimes O_X / \mathcal{I}_o$

can be identified with the natural map

(7.2) $H^0(\mu_2^* L \otimes L^{-2}) \to \mu_2^* L \otimes L^{-2} \otimes O_X / \mu_2^* \mathcal{I}_o = \mu_2^* L \otimes L^{-2} \otimes O_X / \mathcal{I}_X$.

Since (ii) and (iii) are equivalent to the surjectivity of (7.1) and (7.2) respectively, we have the equivalence (ii) $\iff$ (iii).

(i) $\iff$ (ii): By [Iye03] Proposition 2.1, $L$ is projectively normal if and only if the natural map $H^0(L) \otimes H^0(L) \to H^0(L^2)$ is surjective. We note that the proof of [Iye03] Proposition 2.1 works in any characteristic. Hence $L$ is projectively normal if and only if $L^* L = L \otimes \varphi_i^* \Phi_p(L^2)$ is globally generated at $o$ by Proposition 2.7.

For simplicity, set $\mathcal{E} = \varphi_i^* \Phi_p(L^2)$ and $\mathcal{F} = \mu_2(L^{-2})$. Since $\varphi_1 \circ \varphi_2 = \varphi_2$, we have

$$\mu_2 \circ \mu_2^* \mathcal{E} = \mu_2(L) \otimes \varphi_2^* \mathcal{F} \Phi_p(L^2) = \mu_2^* \varphi_2^* \Phi_p(L^2) = \mu_2^* (L^{-2}) \otimes h^0(L^2) = \mathcal{F} \otimes h^0(L^2),$$

where the third equality follows from [Muk81] Proposition 3.11 (1). Since we assume $\mathrm{char}(\mathbb{K}) \neq 2$, $\mathcal{E}$ is a direct summand of $\mu_2^* \mathcal{E} = \mathcal{F} \otimes h^0(L^2)$. Hence $L^* L = L \otimes \mathcal{E}$ is globally generated at $o$ if so is $L \otimes \mathcal{F} = L \otimes \mu_2(L^{-2})$, which shows (ii) $\Rightarrow$ (i).

On the other hand, we have

$$\varphi_i^* \varphi_i \mathcal{F} = \varphi_i^* \varphi_i \mu_2^* L^{-2} = \varphi_i^* \varphi_2^* \mu_2(L^{-2}) = \varphi_i^* (\Phi_p(L^2)) \otimes h^0(L^2) = \mathcal{E} \otimes h^0(L^2),$$

where the third equality follows from [Muk81] Proposition 3.11 (2). Since the natural homomorphism $\varphi_i^* \varphi_i \mathcal{F} \to \mathcal{F}$ is surjective, $L \otimes \mathcal{F} = L \otimes \mu_2(L^{-2})$ is globally generated at $o$ if so is $L^* L = L \otimes \mathcal{E}$, which shows (i) $\Rightarrow$ (ii).

Remark 7.3. (1) If $L = A^2$ for a symmetric $A$, we have $\mu_2^* A = A^4$ and hence

$L \otimes \mu_2(A^{-2}) = \mu_2(L) \otimes \mu_2^* (A^{-2}) = \mu_2^* (A^2 \otimes A^{-4}) = \mu_2^* (A \otimes P)$

in $\mathrm{char}(\mathbb{K}) \neq 2$, where we take the direct sum of all $P$ with $P^2 \cong O_X$. Thus Lemma 7.2 recovers Ohbuchi’s result when $\mathrm{char}(\mathbb{K}) \neq 2$.

(2) By Theorem 1.4 Remark 4.2, $L$ is projectively normal if $\beta(l) < 1/2$ and is not projectively normal if $\beta(l) > 1/2$. Hence the projective normality of $L$ is determined by $\beta(l)$ when $\beta(l) \neq 1/2$. It might be interesting to find examples with $\beta(l) = 1/2$ to which we can apply Lemma 7.2 to show the projective normality, other than the case $L = A^2$. We note that we cannot use Theorem 1.2 (1) to show the globally generation of $L \otimes \mu_2(L^{-2})$ at $o$ when $\beta(l) = 1/2$. In fact,

$L \otimes \mu_2(L^{-2}) \langle - \frac{1}{2} \rangle = \mu_2(L) \otimes L^{-2} \langle - \frac{1}{2} \rangle$

is not M-regular by Lemma 4.6 since $\mu_2^* L \otimes L^{-2}(-2l) = L^2(-2l)$ is not M-regular.

(3) Finally, we summarize relations between projective normality and related notions:

(a) $\mathcal{I}_o(\frac{1}{2})$ is IT(0), i.e. $\beta(l) < \frac{1}{2}$,

(b) $\mathcal{I}_o(\frac{1}{2})$ is M-regular,

(c) $\mathcal{I}_o(\frac{1}{2})$ is GV, i.e. $\beta(l) \leq \frac{1}{2}$,
(d) $L$ is projectively normal,
(e) $L$ is very ample.

For these five notions, we have the following relations:

\[ IT(0) \rightarrow M\text{-reg} \rightarrow GV \]

\[ p.n. \rightarrow v.a. \]

In fact, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) hold by definition. It is easy to see that the converses of these implications do not hold in general by considering suitable $L = A^2$.

(a) $\Rightarrow$ (d), (b) $\Rightarrow$ (e), and (d) $\Rightarrow$ (c) follow from Theorem 1.4, Theorem 1.6, and Remark 4.2, respectively.

On the other hand, (b) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a) do not hold in general by Example 7.1 and Example 2.4, respectively. (c) $\Rightarrow$ (e) does not hold in general since a principal polarization $A$ satisfies (c) but is not very ample. (e) $\Rightarrow$ (c) also does not hold in general since a general polarized abelian 4-fold $(X, L)$ of type $(1, 1, 1, 15)$ is very ample by [DHS94, Remark 26] but $\beta(l) \geq 1/\sqrt{1/15} > 1/2$ by [Ito20b, Lemma 3.4].

The author does not know whether (d) $\Rightarrow$ (b) holds or not in general.

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