ON THE RIGIDITY OF SOUSLIN TREES AND THEIR GENERIC BRANCHES

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Abstract. We show it is consistent that there is a Souslin tree $S$ such that after forcing with $S$, $S$ is Kurepa and for all clubs $C \subset \omega_1$, $S \upharpoonright C$ is rigid. This answers the questions in $[2]$. Moreover, we show it is consistent with ♣ that for every Souslin tree $T$ there is a dense $X \subseteq T$ which does not contain a copy of $T$. This is related to a question due to Baumgartner in $[1]$.

1. Introduction

Recall that an $\omega_1$-tree is said to be Souslin if it has no uncountable chain or antichain. In $[2]$ and $[3]$, Fuchs and Hamkins considered various notions of rigidity of Souslin trees and studied the following question: How many generic branches can Souslin trees introduce, when they satisfy certain rigidity requirements? In $[2]$, Fuchs asks a few questions which motivate the following theorem.

Theorem 1.1. It is consistent with GCH that there is a Souslin tree $S$ such that $\Vdash_S “S$ is Kurepa and $S \upharpoonright C$ is rigid for every club $C \subset \omega_1”$.

Theorem 1.1 answers all questions in $[2]$. We refer the reader to $[2]$ and $[3]$ for motivation and history.

In $[1]$, Baumgartner proves that under ♣+ there is a lexicographically ordered Souslin tree which is minimal as a tree and as an uncountable linear order. At the end of his construction he asks the following question: Does there exist a minimal Aronszajn line if ♣ holds? This question is not settled here but motivates the following proposition.

Proposition 1.2. It is consistent with ♣ that if $S$ is a Souslin tree then there is a dense $X \subset S$ which does not contain a copy of $S$.

Proposition 1.2 shows it is impossible to follow the same strategy as Baumgartner’s in $[1]$, in order to show ♣ implies that there is a minimal Aronszajn line. More precisely, it is impossible to find a lexicographically ordered Souslin tree which is minimal as a tree and as an uncountable linear order.

Key words and phrases. Souslin trees, Kurepa trees.
This paper is organized as follows. In the next section we prove Proposition [1.2]. In the third section we introduce a Souslin tree which makes itself a Kurepa tree. This tree is used in the last section, where we prove Theorem [1.1].

Let’s fix some definitions, notations and conventions. Assume $T, S$ are trees and $f : T \to S$ is injective. Then $f$ is said to be an embedding when $t <_T s \iff f(t) <_S f(s)$. $T$ is called an $\omega_1$-tree if its levels are countable and $\text{ht}(T) = \omega_1$. $T$ is said to be pruned if for all $t \in T$ and $\alpha \in \omega_1 \setminus \text{ht}(t)$ there is $s \geq t$ such that $\text{ht}(s) = \alpha$. If $t \in T$ and $\alpha \leq \text{ht}(t)$, $t \upharpoonright \alpha$ refers to the $\leq_T$ predecessor of $t$ in level $\alpha$. $C \subset T$ is called a chain if it consists of pairwise comparable elements. A chain $b \subset T$ is called a branch if it intersects all levels of $T$. An $\omega_1$-tree $U$ is called minimal if for every uncountable $X \subset U$, $U$ embeds into $X$. If $T$ is a tree and $\alpha$ is an ordinal, $T(\alpha) = \{ t \in T : \text{ht}(t) = \alpha \}$ and $T(< \alpha) = \{ t \in T : \text{ht}(t) < \alpha \}$. If $A$ is a set of ordinals, $T \upharpoonright A = \{ t \in T : \text{ht}(t) \in A \}$. If $t \in T$ and $U \subset T$ then $U_t = \{ u \in U : t \leq_T u \}$. Assume $Q$ is a poset and $\theta$ is a regular cardinal. We say $M < H_\theta$ is suitable for $Q$ if $Q$ and the power set of the transitive closure of $Q$ are in $M$.

2. Minimality of Souslin trees and ♦

This section is devoted to the proof of Proposition [1.2]. We will use the following terminology and notation in this section. By $N$ we mean the set of all countable infinite successor ordinals, and $\mathbb{P}$ refers to the countable support iteration $\langle P_i, Q_j : i < \omega_2, j < \omega_2 \rangle$, where $Q_j = 2^{< \omega_1}$ for each $j \in \omega_2$.

**Lemma 2.1.** Assume $U = (\omega_1, <)$ is a Souslin tree, $p \in \mathbb{P}$, $X$ is the canonical $\mathbb{P}_1$-name for the generic subset of $\omega_1$, $p \Vdash \text{“$f$ is an embedding from } U \text{ to } X \text{”}$ and for every $t \in U$ define $\varphi(p, t) = \{ s \in U : \exists \bar{p} \leq p \bar{p} \Vdash f(t) = s \}$. Then there is an $\alpha \in \omega_1$ such that for all $t \in U \setminus U(< \alpha)$, $\varphi(p, t)$ is not a chain.

**Proof.** Let $Y_p = \{ y \in U : \varphi(p, y) \text{ is a chain} \}$. $Y_p$ is downward closed and if it is countable we are done. Fix $p \in \mathbb{P}$ and assume for a contradiction that $Y_p$ is uncountable. Let $A_p = \{ t \in U : p \Vdash t \in X \text{ or } p \Vdash t \notin X \}$. $A_p$ is countable. Fix $\alpha > \sup \{ \text{ht}(a) : a \in A_p \}$ and $y \in Y_p \setminus U(\leq \alpha)$. Since $U$ is an Aronszajn tree and $\varphi(p, y)$ is a chain, we can choose $\beta \in \omega_1 \setminus \sup \{ \text{ht}(s) : s \in \varphi(p, y) \}$. For all $s \in \varphi(p, y)$, $\alpha < \text{ht}(s) < \beta$ since $\emptyset \Vdash \text{ht}(y) \leq \text{ht}(\hat{f}(y))$. Then we can extend $p$ to $q$ such that $q \Vdash X \cap (U(\leq \beta) \setminus U(< \alpha)) = \emptyset$, which contradicts $p \Vdash \hat{f}(y) \in \varphi(p, y)$.

\[ \Box \]
Lemma 2.2. Assume $U \in V$ is a pruned Souslin tree and $G \subset \mathbb{P}$ is $V$-generic. Then in $V[G]$, there is a dense $X \subset U$ which does not have a copy of $U$.

Proof. Let $\dot{X}$ be as in Lemma 2.1. Since $U$ is pruned, $1_{\mathbb{P}} \models \dot{X} \subset U$ is dense. We will show $1_{\mathbb{P}} \not\models X$ has no copy of $U$. Assume for a contradiction that $p \Vdash \dot{f}$ is an embedding from $U$ to $\dot{X}$. Fix a regular cardinal $\theta$ and a countable $M \prec H_\theta$ which contains $U, p, \dot{f}, 2^{\mathbb{P}}$. Also let $\langle D_n : n \in \omega \rangle$ be an enumeration of all dense open subsets of $\mathbb{P}$ in $M$, $\delta = M \cap \omega_1$ and $t \in U(\delta)$. For each $\sigma \in 2^{\omega}$, find $p_\sigma \in D_{|\sigma|} \cap M$, $s_\sigma$ and $t_{|\sigma|} < t$, such that:

1. $\sigma \sqcap \tau$ then $p_\tau \leq p_\sigma$ and $s_\tau \leq s_\sigma$,
2. $\sigma \perp \tau$ then $s_\sigma \perp s_\tau$,
3. $p_\sigma \Vdash \dot{f}(t_{|\sigma|}) = s_\sigma$.

In order to see how these sequences are constructed, let $t_0 < t$ be arbitrary and $p_0, s_0$ be such that $p_0 \Vdash \dot{f}(t_0) = s_0$ and $p_0 \in D_0 \cap M$. Assuming these sequences are given for all $\sigma \in 2^n$, use Lemma 2.1 to find $t_{n+1} < t$ such that $\varphi(p_\sigma, t_{n+1})$ is not a chain, for all $\sigma \in 2^n$. Let $s_{\sigma \rightarrow 0}, s_{\sigma \rightarrow 1}$ be in $\varphi(p_\sigma, t_{n+1}) \cap M$ such that $s_{\sigma \rightarrow 0} \perp s_{\sigma \rightarrow 1}$. Now find $p_{\sigma \rightarrow i}, p_{\sigma \rightarrow \neg i}$ in $M \cap D_{n+1}$ which are extensions of $p_\sigma$ such that $p_{\sigma \rightarrow i} \Vdash \dot{f}(t_{n+1}) = s_{\sigma \rightarrow i}$, for $i = 0, 1$.

For each $r \in 2^\omega$, let $p_r$ be a lower bound for $\{p_\sigma : \sigma \sqcap r\}$ and let $b_r \subset U \cap M$ be a downward closed chain such that $p_r \Vdash \dot{f}([s \in U : s < t]) \subset b_r$. Note that $b_r$ intersects all the levels of $U$ below $\delta$. It is obvious that $p_r$ is an $(M, \mathbb{P})$-generic condition below $p$. Moreover, if $r, r'$ are two distinct real numbers then $b_r \neq b_{r'}$. Let $r \in 2^\omega$ such that $U$ has no element on top of $b_r$. Then $p_r$ forces that $\dot{f}(t)$ is not defined, which is a contradiction.

Now we are ready for the proof of Proposition 1.2. Let $V$ be a model of ZFC + GCH and $G \subset \mathbb{P}$ be $V$-generic. Since $\mathbb{P}$ is a countable support iteration of $\sigma$-closed posets of size $\aleph_1$, it preserves all cardinals. The same argument as in Theorem 8.3 in [4] shows that $\diamondsuit$ holds in $V[G]$.

Let $U$ be a Souslin tree in $V[G]$. For some $\alpha \in \omega_2$, $U \in V[G \cap P_\alpha]$ since $|U| = \aleph_1$. Let $\dot{R}$ be the canonical $P_\alpha$-name such that $\mathbb{P} = P_\alpha \ast \dot{R}$. Then $1_{P_\alpha} \Vdash \dot{R}$ is isomorphic to $\mathbb{P}$. By Lemma 2.2 there is a dense $X \subset U$ in $V[G]$ which has no copy of $U$, as desired.

3. A Souslin Tree with Many Generic Branches

Definition 3.1. The poset $Q$ is the set of all $p = (T^p, \Pi_p)$ such that:
Proof. The first part of the lemma is obvious. Assume $Q$ is countable, so consider a descending $(M, Q)$-generic sequence with $ht(M, Q)$ be a descending $(M, Q)$-generic sequence with $ht(M, Q)$. Let $\mathcal{F}$ be the set of all finite compositions of functions of the form $\bigcup_{n<\omega} \pi^n_\xi$ with $\xi \in M \cap \omega_2$. For each $\xi \in M \cap \omega_2$, let $\langle \pi^n_\xi : \xi \in R \rangle$ be an enumeration of $\mathcal{F}$ with infinite repetition and $A = \{t \in R : \exists n \in \omega (p_n \Vdash t \in \tau)\}$. Observe that for all $t \in R$ there is $a \in A$ such that $a, t$ are comparable.

Let $\langle \alpha_m : m \in \omega \rangle$ be an increasing cofinal sequence in $\delta$. For each $t \in R$ we build an increasing sequence $\bar{t} = \langle t_m : m \in \omega \rangle$ as follows. Let $t_0 = t$. Assume $t_m$ is given. If $R_{t_m} \cap dom(f_m) = \emptyset$, choose $t_{m+1} > t_m$ with $ht(t_{m+1}) > \alpha_m$. If $R_{t_m} \cap dom(f_m) \neq \emptyset$, let $s \in dom(f_m) \cap R_{t_m}$. Let $a \in A$ such that $a, f_m(s)$ are comparable. Let $x = \max\{f_m(s), a\}$ and $t_{m+1} > f_m^{-1}(x)$ with $ht(t_{m+1}) > \alpha_m$. Let $b_t$ be the downward closure of $\bar{t}$.

Let $B = \{ f_n[b_t] : t \in R \text{ and } n \in \omega \}$. Let $q$ be the lower bound for $\langle p_n : n \in \omega \rangle$ described as follows. $T^q = R \cup T^q(\delta)$ and for each cofinal branch $c \subset R$ there is a unique $y \in T^q(\delta)$ above $c$ if and only if $c \in B$. For each $\xi \in M \cap \omega_2$, let $\pi^q_\xi \restriction R = \bigcup_{n<\omega} \pi^n_\xi$. Note that this determines $\pi^q_\xi \restriction T^q(\delta)$ as well and $\pi^q_\xi(y)$ is defined for all $y \in T^q(\delta)$.

The condition $q$ forces that for each $y \in T(\delta) = T^q(\delta)$ there is $a \in A$ with $a < y$. In other words $q$ forces that $\tau = A$. Since $p$ was arbitrary, $1_Q$ forces that every maximal antichain has to be countable. \qed
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4. HIGHLY RIGID DENSE SUBSETS OF T

In this section we show the tree $T$, in the forcing extensions by $P = (2^{<\omega_1}, \supset)$, has dense subsets which are witnesses for Theorem 1.1.

**Lemma 4.1.** Let $U = (\omega_1, <)$ be a pruned Souslin tree and $S \subset \omega_1$ be generic for $P$. Then in $V[S]$ the following hold.

1. $S$ is a Souslin tree when it is considered with the inherited order from $U$.
2. $S \subset U$ is dense.
3. For all clubs $C \subset \omega_1$, $S \upharpoonright C$ is rigid.

**Proof.** In order to see that $S$ is Souslin, note that $\sigma$-closed posets do not add uncountable antichains to Souslin trees. Moreover by standard density arguments $S \subset U$ is dense.

Assume for a contradiction $p \Vdash_P \langle f : \dot{S} \upharpoonright \dot{C} \longrightarrow \dot{S} \upharpoonright \dot{C} \rangle$ is a nontrivial tree embedding.” Let $\langle M_\xi : \xi \in \omega + 1 \rangle$ be a continuous $\in$-chain of countable elementary submodels of $H_\theta$ where $\theta$ is regular and $p, f, 2^U$ are in $M_0$. For each $\xi \leq \omega$, let $\bar{\delta}_\xi = M_\xi \cap \omega_1$ and $t \in U(\bar{\delta}_\omega)$. Let $t_n = t \upharpoonright \bar{\delta}_n$. For each $\sigma \in 2^{<\omega}$ we find $q_\sigma \in M_{|\sigma| + 1} \cap P$, $s_\sigma$ such that:

1. $q_0 \leq p$, and if $\sigma \subset \tau$ then $q_\tau \leq q_\sigma$,
2. $q_\sigma$ is $(M_{|\sigma|}, P)$-generic and $q_\sigma \subset M_{|\sigma|}$,
3. $q_\sigma$ forces that $\dot{f}(t_{|\sigma| - 1}) = s_\sigma$,
4. if $\sigma \upharpoonright \tau$ then $s_\sigma \downarrow s_\tau$,
5. if $\sigma \subset \tau$ then $q_\tau$ forces that $t_{|\sigma|} \in \dot{S} \upharpoonright \dot{C}$.

Assuming $q_\sigma$ and $s_\sigma$ are given for all $\sigma \in 2^n$, we find $q_{\sigma \upharpoonright 0}, q_{\sigma \upharpoonright 1}, s_{\sigma \upharpoonright 0}, s_{\sigma \upharpoonright 1}$, and $s_{\sigma \upharpoonright -1}$. Let $\tilde{q}_\sigma = q_\sigma \cup \{(t_n, 1)\}$. Obviously, $\tilde{q}_\sigma \Vdash t_n \in \dot{S} \upharpoonright \dot{C}$ and for all $\sigma \in 2^n$, $\{s \in U : \exists r \leq \tilde{q}_\sigma \ r \Vdash \dot{f}(t_n) = s\}$ is uncountable.

In $M_{n+1}$, find $r_0, r_1$ below $\tilde{q}_\sigma$ and $s_{\sigma \upharpoonright 0}, s_{\sigma \upharpoonright 1}$ such that $s_{\sigma \upharpoonright 0} \upharpoonright s_{\sigma \upharpoonright 1}$ and $r_i \Vdash \dot{f}(t_n) = s_{\sigma \upharpoonright i}$. Let $q_{\sigma \upharpoonright i} < r_i$ be $(M_{n+1}, P)$-generic with $q_{\sigma \upharpoonright -1} \subset M_{n+1}$, and $q_{\sigma \upharpoonright i} \in M_{n+2}$.

Let $r \in 2^\omega$ such that $\{s_\sigma : \sigma \subset r\}$ does not have an upper bound in $U$. Let $p_r$ be a lower bound for $\{p_\sigma : \sigma \subset r\}$. Then $p_r$ forces that $\dot{f}(t)$ is not defined which is a contradiction. \qed

**Lemma 4.2.** Suppose $M$ is suitable for $Q$ and $\delta = M \cap \omega_1$. Let $\langle q_n : n \in \omega \rangle$ be a decreasing $(M, Q)$-generic sequence. Define a condition $q \in Q$ by setting $T^q = \bigcup_{n \in \omega} T^q_n$, $D_q = \bigcup_{n \in \omega} D_q_n$ and for each $\xi \in D_q$
let \( \pi_q^q = \bigcup_{n<\omega} \pi_{x,n}^q \). Also let \( \Pi_q = \langle \pi_q^q : \xi \in D_q \rangle \). Let \( \mathcal{F} \) be the set of all finite compositions of functions of the form \( \pi_q^q \) with \( \xi \in D_q \). Assume \( m \in \omega \) and \( \langle b_i : i \in m \rangle \) are branches through \( T^q \). Then there is an extension \( q' \leq q \) such that \( \alpha_{q'} \geq \delta + 1 \) and for all branches \( c \subset T^{q'} \), \( c \) has an upper bound \( f \) for some \( f \in \mathcal{F} \) and \( i \in m \), \( f(b_i) \) is cofinal in \( c \).

Proof. Note that \( D_q = M \cap \omega_2 \) and \( \alpha_q = \delta \). Let \( T^{q'} \upharpoonright \delta = T^q \). Let \( B = \{ f(b_i) : i \in m \text{ and } f \in \mathcal{F} \} \). Obviously \( B \) is countable and we can fix an enumeration of \( B \) with \( n \in \omega \). Let \( T^{q'}(\delta + 1) = [\delta, \delta + \omega) \) and put \( \delta + n \) on top of the \( n \)’th element in \( B \). It is obvious how we should extend \( \Pi_q \) to \( \Pi_{q'} \) with \( D_q = D_{q'} \). \( \square \)

**Lemma 4.3.** Let \( G \subset Q \) be \( V \)-generic, \( p \in P \) and \( \dot{S} \) be the canonical \( P \)-name for the generic subset of \( \omega_1 \). Let \( \dot{f}, \dot{C} \) be \( P \star T \)-names in \( V[G] \) and \( t, x, y \) be pairwise incompatible in \( T \). Suppose \( (p,t) \) forces \( f \) is an embedding from \( \dot{S}_x \upharpoonright \dot{C} \) to \( \dot{S}_y \upharpoonright \dot{C} \). For every \( u \in T_x \) define \( \psi(p,t,u) = \{ s \in T : \exists t' > t \exists \bar{p} \leq p \ (\bar{p},t') \Vdash [u \in \dot{S}_x \upharpoonright \dot{C} \land \dot{f}(u) = s] \}. \) Then for any \( u \in T_x \) there is \( u' > u \) such that \( \psi(p,t,u') \) is not a chain.

Proof. Fix \( p, t, u, u' \) as above and assume for a contradiction that for all \( u' > u \) in \( T \), \( \psi(p,t,u') \) is a chain. Since \( T \) is ccc, without loss of generality we can assume that for all \( q \in P \) and \( \alpha \in \omega_1 \), there is \( \bar{q} \leq q \) such that \( (\bar{q},1_T) \) decides the statement \( \alpha \in \dot{C} \). For each \( q \in P, r \in T, v \in T \) let \( \alpha_{q,r,v} = \sup \{ \text{ht}_T(s) : s \in \psi(q,r,v) \} \). Note that if \( \bar{q} \leq q \) and \( \bar{r} \geq r \) then \( \psi(\bar{q},\bar{r},v) \subseteq \psi(q,r,v) \) and \( \alpha_{\bar{q},\bar{r},v} \leq \alpha_{q,r,v} \).

Let \( M_0, M_1 \) be countable elementary submodels of \( H_\theta, \theta \) be a regular cardinal and \( \{ p, t, u, x, y, \dot{f}, \dot{C} \} \in M_0 \Subset M_1 \). Suppose \( \langle p_n : n \in \omega \rangle \) is an \((M_0,P)\)-generic sequence which is in \( M_1 \) and \( p_0 \leq p \). Let \( p' = \bigcup_{n \in \omega} p_n \) and \( \delta_i = M_i \cap \omega_1, i \in 2 \). Note that \( p' \Vdash \delta_0 \in \dot{C} \).

Let \( \bar{p} < p' \) such that:

1. \( \bar{p} \Vdash \forall v \in T_x \cap (M_1 \setminus M_0) [v \in \dot{S}] \)
2. \( \bar{p} \Vdash \forall v \in T_y \cap (M_1 \setminus M_0) [v \notin \dot{S}] \).

Let \( u_0 > u \) be in \( T(\delta_0) \). Since \( \bar{p} \) is \((M_0,Q)\)-generic, it forces that \( \delta_0 \in \dot{C} \land u_0 \in \dot{S} \land \text{ht}_\dot{S}(u_0) = \delta_0 \). In particular, by elementarity of \( M_0 \) and basic facts on ordinal arithmetic, \( \bar{p} \Vdash u_0 \in \dot{S}_x \upharpoonright \dot{C} \).

Suppose \( q < \bar{p}, r > t \) such that \( (q,r) \) decides \( \dot{f}(u_0) \). Then the condition \( (q,r) \) forces that \( \text{ht}(\dot{f}(u_0)) \geq \delta_1 \). So, \( \delta_1 \leq \alpha_{\bar{p},t,u_0} \leq \alpha_{p',t,u_0} \in M_1 \). But this is a contradiction. \( \square \)

In the next lemma we use the following standard fact: If \( U \) is a Souslin tree and \( X \subset U \) is uncountable and downward closed, then there is \( x \in U \) such that \( U_x \subset X \). In order to see this assume for all \( v \in
$U$, $U_v$ is not contained in $X$. Let $A$ be the set of all minimal $a$ outside of $X$. Observe that $A$ is an uncountable antichain, contradicting the fact that $U$ was Souslin. Lemma 4.4 finishes the proof of Theorem 1.1.

**Lemma 4.4.** Assume $G * S * b$ is $V$-generic for $Q * P * \dot{T}$. Let $x, y$ be incomparable in $T$. Then in $V[G * S * b]$ for all clubs $C \subset \omega_1$, $S_x \upharpoonright C$ does not embed into $S_y \upharpoonright C$.

**Proof.** Assume for a contradiction that $(q_0, p_0, t_0)$ is a condition in $Q * P * T$ which forcing $\dot{f}: \dot{S}_x \upharpoonright C \rightarrow \dot{S}_y \upharpoonright C$ is a tree embedding and $x, y$ are incompatible in $T$. Note that $\dot{f}(\dot{S}_x)$ is an uncountable subset of $\dot{T}_y$ and $\dot{T}$ is a Souslin tree in $V[G][S]$. So the downward closure of $\dot{f}(\dot{S}_x)$ contains $\dot{T}_z$ for some $z > y$. Therefore, by extending $x, y, (q_0, p_0, t_0)$ if necessary, we can assume that $\dot{f}(\dot{S}_x)$ is dense in $\dot{S}_y$.

Again by extending $x, y, (q_0, p_0, t_0)$ we may assume $(q_0, p_0, t_0) \VDash [x, y$ are in $\dot{S} \upharpoonright \dot{C}$ and $\dot{f}(x) = y]$. Furthermore, by extending $t_0$ if necessary we can assume that $\text{ht}(t_0) > \text{ht}(y)$ and $x, y, t_0$ are pairwise incompatible. Since $T$ is a ccc poset we can assume that for all $\alpha \in \omega_1$, for all $u, v \in T$ and for all $(a, b) \in P * Q$ we have $(a, b, u) \Vdash \alpha \in \dot{C} \longleftrightarrow (a, b, v) \Vdash \alpha \in \dot{C}$.

Let $M$ be a countable elementary submodel of $H_\theta$ such that $\theta$ is regular and $(q_0, p_0, t_0), \dot{f}$ are in $M$. Let $\langle q_n : n \in \omega \rangle$ be a decreasing $(M, Q)$-generic sequence. Define $q \in Q$ as in Lemma 3.2 Let $\mathcal{F}$ be the set of all finite compositions of functions of the form $\pi^i_\xi$ with $\xi \in M \cap \omega_2$. Let $\Pi_q = \langle \pi^i_\xi : \xi \in M \cap \omega_2 \rangle$. Obviously, $q$ is an $(M, Q)$-generic condition. Let $\langle q_n : n \in \omega \rangle$ be an enumeration of $\mathcal{F}$ with infinite repetition. Let $\langle \gamma_n : n \in \omega \rangle$ be an increasing cofinal sequence in $\delta = M \cap \omega_1$ with $\gamma_0 = 0$.

We find a decreasing sequence $\langle p_n \in P \cap M : n \in \omega \rangle$ and increasing sequences $\langle \delta_n \in \delta : n \in \omega \rangle$, $\langle t_n \in T^q : n \in \omega \rangle$, $\langle u_n \in T^q : n \in \omega \rangle$ $\langle s_n \in T^q : n \in \omega \rangle$ such that:

1. $\delta_n \geq \gamma_n$ for all $n \in \omega$,
2. $(q, p_n, t_n) \Vdash \min\{\text{ht}_{\dot{S}}(s_n), \text{ht}_{\dot{S}}(u_n), \text{dom}(p_n)\} \geq \delta_n$,
3. $\text{ht}_{\dot{T}^q}(t_n) \geq \text{ht}_{\dot{T}^q}(s_n) + 1$,
4. $(q, p_n, 1_{\mathcal{F}}) \Vdash \delta_n \in \dot{C}$,
5. $(q, p_n, t_n) \Vdash \dot{f}(u_n) = s_n$,
6. if $n \in \omega \setminus 1$ and $t_{n-1} \in \text{dom}(g_n)$ then $g_n(t_n) \perp s_n$,
7. if $n \in \omega \setminus 1$ and $u_{n-1} \in \text{dom}(g_n)$ then $g_n(u_n) \perp s_n$.

We let $u_0 = x, s_0 = y, \delta_0 \in \omega_1$ such that $(q_0, p_0, t_0)$ forces that $\min\{\text{ht}_{\dot{S}}(x), \text{ht}_{\dot{S}}(y), \alpha_{p_0}\} = \delta_0$. It is easy to see that this choice together with $p_0, t_0$ will satisfy the corresponding conditions. For given $p_n, t_n, s_n, u_n, \delta_n$ we introduce $p_{n+1}, t_{n+1}, s_{n+1}, u_{n+1}, \delta_{n+1}$.
If \( t_n \notin \text{dom}(g_{n+1}) \) let \( v = s_n \). If \( t_n \in \text{dom}(g_{n+1}) \), let \( v \geq s_n \) such that \( v \perp g_{n+1}(t_n) \). Such a \( v \) exists because \( \text{ht}(t_n) > \text{ht}(s_n) \), \( g_{n+1} \) is level preserving and the tree \( T^q \) is binary.

**Claim 4.5.** There are \( t'_n > t_n \), \( p'_n < p_n \), \( u'_n > u_n \) such that if \( u_n \in \text{dom}(g_{n+1}) \) then \( (q,p'_n,t'_n) \) forces \[ u'_n \in \text{dom}(\hat{f}) \land v < \hat{f}(u'_n) \land \hat{f}(u'_n) \perp g_{n+1}(u'_n) \].

**Proof of Claim.** Assume \( u_n \in \text{dom}(g_{n+1}) \). Recall that \( \hat{f}(\hat{S}_y) \) is forced to be dense in \( \hat{S}_y \). Let \( \bar{p}_n \leq p_n, \bar{t}_n \geq t_n, a_0 > u_n \), \( v' > v \) such that \( (q,\bar{p}_n,\bar{t}_n) \vdash \hat{f}(a_0) = v' \). This is possible because \( q \) is \((M,Q)\)-generic. Let \( a > a_0, t^0_n, t^1_n \) be extensions of \( \bar{t}_n \), and \( p^0_n, p^1_n \) be extensions of \( \bar{p}_n \) such that \( (q, p^i_n, t^i_n) \vdash \hat{f}(a) = s^i \) where \( i \in 2 \) and \( s^0_n \perp s^1_n \). Again, this is possible because of Lemma 4.3 and the fact that \( q \) is \((M,Q)\)-generic. Let \( a' > a \) such that \( \text{ht}(a') > \max\{\text{ht}(s^0_n), \text{ht}(s^1_n)\} \). Fix \( i \in 2 \) such that \( g_{n+1}(a') = s^i_n \). Then for all \( e > a' \), \( (q, p^i_n, t^i_n) \) forces that if \( e \in \text{dom}(\hat{f}) \) then \( \hat{f}(e) > s^i_n \). Moreover it forces that \( g_{n+1}(e) \perp s^i_n \). Therefore, \( (q, p^i_n, t^i_n) \vdash [\forall e > a' \exists e \in \text{dom}(\hat{f}) \rightarrow g_{n+1}(e) \perp \hat{f}(e)] \). Let \( u'_n > a' \), \( p'_n < p^i_n \) and \( t'_n > t^i_n \) such that \( (q, p'_n, t'_n) \vdash [u'_n \in \text{dom}(\hat{f})] \). Then this condition will also force \( \hat{f}(u'_n) \perp g_{n+1}(u'_n) \) and \( v < \hat{f}(u'_n) \).

Fix \( p'_n, t'_n, u'_n \) as in the claim above. By extending \( p'_n \) if necessary, we can assume that \( (q,p'_n,1_{T^q}) \) decides the \( \gamma_{n+1}' \)st element of \( \bar{C} \setminus \delta_n \) and we let \( \delta_{n+1} \) be this ordinal. Let \( u_{n+1} > u'_n \) such that for some \( p_{n+1} < p'_n \) with \( \text{dom}(p_{n+1}) \geq \delta_{n+1} \), the condition \( (q,p_{n+1},1_{T^q}) \) forces that \( u_{n+1} \in \hat{S} \upharpoonright \bar{C} \) and \( \text{ht}(\hat{S}(u_{n+1})) \geq \delta_{n+1} \). Let \( r > t'_n \). By extending \( (q,p_{n+1},r) \) if necessary, we can assume this condition decides \( \hat{f}(u_{n+1}) \). Let \( s_{n+1} \in T^q \) such that \( (q,p_{n+1},r) \vdash \hat{f}(u_{n+1}) = s_{n+1} \). Let \( t_{n+1} \geq r \) such that \( \text{ht}(t_{n+1}) > \text{ht}(s_{n+1}) \). We leave it to the reader to verify that all of the conditions above hold.

Let \( b_0, b_1 \) be the downward closure of \( \{u_n : n \in \omega\} \) and \( \{t_n : n \in \omega\} \) respectively. By Lemma 1.2 there is \( q' < q \) such that \( \alpha_{q'} \geq \delta + 1 \) and for all branches \( c \subset T^q \), \( c \) has an upper bound in \( T^{q'} \) if and only if \( g_n(b_i) \) is cofinal in \( c \) for some \( n \in \omega \) and \( i \in 2 \). Fix such a \( q' \) for the rest of the argument.

We claim that \( \langle s_n : n \in \omega \rangle \) does not have an upper bound in \( T^{q'} \). Suppose for a contradiction that it has an upper bound. Then for some \( m \in \omega \), either

\[ (1) \{g_m(t_n) : n \in \omega \land t_n \in \text{dom}(g_m)\} \text{ is cofinal in the downward closure of } \{s_n : n \in \omega\} \text{ or} \]

\[ (2) \{g_m(u_n) : n \in \omega \land u_n \in \text{dom}(g_m)\} \text{ is cofinal in the downward closure of } \{s_n : n \in \omega\}. \]
Due to similarity of the arguments, let’s assume that the first alternative happens. Since we enumerated the elements of $F$ with infinite repetition, by increasing $m$ if necessary, we can assume that $t_m \in \text{dom}(g_m)$. But then $g_m(t_m) \perp s_m$, meaning that the first alternative cannot happen, which is a contradiction. Hence $\{s_n : n \in \omega\}$ does not have an upper bound in $T^{q'}$.

Let $t$ be the upper bound of $\langle t_n : n \in \omega \rangle$ in $T^{q'}$, and $u$ be the upper bound for $\langle u_n : n \in \omega \rangle$ which has the lowest height $\delta$. Let $p$ be a lower bound for $\langle p_n : n \in \omega \rangle$ which forces that $u \in \dot{S}$. It is easy to see that $(q',p,t) \models [\delta \in \dot{C} \land u \in \dot{S} \land \text{ht}_\delta(u) = \delta]$. Also by $\boxdot$ $(q',p,t)$ forces $f(u_n) = s_n$ for all $n \in \omega$. Hence $(q',p,t)$ forces that $f(u)$ is an upper bound for $\langle s_n : n \in \omega \rangle$ which is a contradiction. \hfill $\square$

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