ITÔ ISOMORPHISMS FOR $L^p$-VALUED POISSON STOCHASTIC INTEGRALS

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Abstract. Motivated by the study of existence, uniqueness and regularity of solutions to stochastic partial differential equations driven by jump noise, we prove Itô isomorphisms for $L^p$-valued stochastic integrals with respect to a compensated Poisson random measure. The principle ingredients for the proof are novel Rosenthal type inequalities for independent random variables taking values in a (noncommutative) $L^p$-space, which may be of independent interest. As a by-product of our proof, we observe some moment estimates for the operator norm of a sum of independent random matrices.

1. Introduction

In the functional analytic approach to stochastic partial differential equations (SPDEs), one studies an SPDE by reformulating it as a stochastic ordinary differential equation in a suitable infinite-dimensional state space $X$. This approach was developed by the schools of da Prato and Zabczyk for SPDEs driven by Gaussian noise in Hilbert spaces [5] and has since proven effective in obtaining existence, uniqueness and regularity results for large classes of SPDEs with Gaussian noise. In the last decade there has been increased interest in SPDEs driven by Poisson-type noise, see for instance [2, 9, 21, 22] and the recent book [26]. To obtain existence, uniqueness and regularity results for such equations, one requires as a basic tool $L^p$-estimates for vector-valued Poisson stochastic integrals. Concretely, one needs to answer the following fundamental question. Suppose we are given a compensated Poisson random measure $\tilde{\mathcal{N}}$ on $\mathbb{R}_+ \times J$, where $J$ is a $\sigma$-finite measure space, and a simple, adapted $X$-valued process $F$. Can one find a suitable Banach space $I_{p,X}$ such that

\[ c_{p,X} \| F \|_{I_{p,X}} \leq \left( \mathbb{E} \left[ \int_{\mathbb{R}_+ \times J} F \ d\tilde{\mathcal{N}} \right]^p \right)^{\frac{1}{p}} \leq C_{p,X} \| F \|_{I_{p,X}}, \tag{1} \]

for constants $c_{p,X}, C_{p,X}$ depending only on $p$ and $X$? In the SPDE literature the right hand side inequality is often referred to as a Bichteler-Jacod inequality. This estimate allows one to define an Itô-type stochastic integral, sometimes called a strong or $L^p$-stochastic integral in the literature [1, 32], for all elements in the closure of the simple adapted processes in $I_{p,X}$. The left hand side inequality in (1) shows that the Bichteler-Jacod inequality is the best possible one that one can use and hence, that $I_{p,X}$ provides the proper framework to study well-posedness and regularity questions. In case both inequalities in (1) hold we shall speak of an Itô isomorphism. In the case of Gaussian noise Itô isomorphisms in UMD Banach spaces were obtained in [25]. The optimality of these estimates proved crucial in

Key words and phrases. Poisson stochastic integration in Banach spaces, decoupling inequalities, vector-valued Rosenthal inequalities, noncommutative $L^p$-spaces, norm estimates for random matrices.

This research was supported by VICI subsidy 639.033.604 of the Netherlands Organisation for Scientific Research (NWO) and the Hausdorff Center for Mathematics.
obtaining maximal regularity results for stochastic parabolic evolution equations driven by Gaussian noise [24]. Although Bichteler-Jacod inequalities are fundamental to the study of SPDEs and have not yet been investigated, not even in the scalar-valued case. The main aim of this paper is to provide optimal estimates of the form (1) in the important case where \( X \) is an \( L^q \)-space, or even a noncommutative \( L^q \)-space associated with a semi-finite von Neumann algebra \( \mathcal{M} \), for any \( 1 < q < \infty \). To keep our exposition accessible to readers who have little familiarity with noncommutative analysis, we choose to focus on classical \( L^q \)-spaces and only later indicate the modifications needed to prove our results in full generality.

Our main result for classical \( L^q \)-spaces reads as follows. Let \( (S, \Sigma, \sigma) \) be any measure space. We consider the completions \( S^p_q \), \( D^p_q \), and \( D^p_{p,q} \) of the space of all simple functions in the respective norms

\[
\frac{\|F\|_{S^p_q}}{\|F\|_{D^p_q}} = \left( \mathbb{E}\left( \int_{\mathbb{R}^+ \times S} |F|^p \, dt \times d\sigma \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]

\[
\frac{\|F\|_{D^p_q}}{\|F\|_{D^p_{p,q}}} = \left( \mathbb{E}\left( \int_{\mathbb{R}^+ \times S} \|F\|^p_{L^q(S)} \, dt \times d\sigma \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]

\[
\frac{\|F\|_{D^p_{p,q}}}{\|F\|_{S^p_q}} = \left( \mathbb{E}\left( \int_{\mathbb{R}^+ \times S} \|F\|^p_{L^q(S)} \, dt \times d\sigma \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]

We use the following notation. If \( A, B \) are quantities depending on a parameter \( \alpha \), then we write \( A \lessapprox B \) if there is a constant \( c_\alpha > 0 \) depending only on \( \alpha \) such that \( A \leq c_\alpha B \). We write \( A \approx B \) if both \( A \lessapprox B \) and \( B \lessapprox A \) hold. Also, we use \( \chi_A \) to denote the indicator of a set \( A \).

**Theorem 1.1.** (*Itô isomorphism*) Let \( 1 < p, q < \infty \). For any \( B \in \mathcal{J} \), any \( t \geq 0 \) and for any simple, adapted \( L^q(S) \)-valued process \( F \),

\[
\left( \mathbb{E}\sup_{0 \leq s \leq t} \left\| \int_{[0,s] \times B} F \, d\tilde{N} \right\|_{L^q(S)}^p \right)^{\frac{1}{p}} \lessapprox_{p,q} \| F \chi_{[0,t] \times B} \|_{D^p_{p,q}},
\]

where \( I_{p,q} \) is given by

\[
S^p_q \cap D^p_{q,q} \cap D^p_{p,q} \quad \text{if} \quad 2 \leq q \leq p < \infty;
\]

\[
S^p_q \cap (D^p_{q,q} + D^p_{p,q}) \quad \text{if} \quad 2 \leq p \leq q < \infty;
\]

\[
(S^p_q \cap D^p_{q,q}) + D^p_{p,q} \quad \text{if} \quad 1 < p < 2 \leq q < \infty;
\]

\[
(S^p_q + D^p_{q,q}) \cap D^p_{p,q} \quad \text{if} \quad 1 < q < 2 \leq p < \infty;
\]

\[
S^p_q + (D^p_{q,q} + D^p_{p,q}) \quad \text{if} \quad 1 < q \leq p \leq 2;
\]

\[
S^p_q + D^p_{q,q} + D^p_{p,q} \quad \text{if} \quad 1 < p \leq q \leq 2.
\]

Moreover, the estimate \( \lessapprox_{p,q} \) in (3) remains valid if \( q = 1 \).

To understand the estimates in (3), recall that if \( X \) and \( Y \) are two Banach spaces which are continuously embedded in some Hausdorff topological vector space, then their intersection \( X \cap Y \) and sum \( X + Y \) are Banach spaces under the norms

\[
\|z\|_{X \cap Y} = \max\{\|z\|_X, \|z\|_Y\}
\]

and

\[
\|z\|_{X + Y} = \inf\{\|x\|_X + \|y\|_Y : z = x + y, \ x \in X, \ y \in Y\}.
\]
In Section 2.4.1. Observe that for a simple process \( F \) the right hand side can be written as a sum of conditionally independent, mean zero random variables. These inequalities are a special case of the decoupling inequalities for martingale difference sequences in UMD Banach spaces due to McConnell [23] and Hitczenko [11]. A relatively simple direct proof of (4) can be found in e.g. [34], Theorem 2.4.1. Observe that for a simple process \( F \), the decoupled stochastic integral on the right hand side can be written as a sum of conditionally independent, mean zero random variables. Thus, the key to obtaining an Itô isomorphism as in [11] lies in answering the following question: given \( 1 \leq p < \infty \) and a Banach space \( X \), can we find constants \( c_{p,X} \), \( C_{p,X} \) depending only on \( p \) and \( X \) such that for any sequence of independent, mean zero \( X \)-valued random variables \( (\xi_i) \)

\[
(\mathbb{E} \left[ \sum_{i=1}^{n} |\xi_i|^p \right] )^{\frac{1}{p}} \leq C_{p,X} \left[ \frac{1}{n} \sum_{i=1}^{n} (\max \{ |\xi_i|, |\xi_i|^2 \} ) \right]^{\frac{1}{2}}.
\]

for a suitable norm \( \| \cdot \|_{p,X} \) which can be computed explicitly in terms of the (moments of the) individual summands \( \xi_i \)? These kind of inequalities can be termed Rosenthal inequalities, since in the case \( X = \mathbb{C} \) the well-known answer to this question is due to Rosenthal [31]: For \( 2 \leq p < \infty \), there exists an absolute constant \( c \) such that

\[
\left( \mathbb{E} \left[ \sum_{i=1}^{n} |\xi_i|^p \right] \right)^{\frac{1}{p}} \leq \frac{C_{p,X}}{\log p} \max \left\{ \left( \sum_{i=1}^{n} \mathbb{E} |\xi_i|^p \right)^{\frac{1}{p}}, \left( \sum_{i=1}^{n} \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{2}} \right\}.
\]

These inequalities are a special case of the decoupling inequalities for martingale difference sequences in UMD Banach spaces due to McConnell [23] and Hitczenko [11]. A relatively simple direct proof of (4) can be found in e.g. [34], Theorem 2.4.1. Observe that for a simple process \( F \), the decoupled stochastic integral on the right hand side can be written as a sum of conditionally independent, mean zero random variables. Thus, the key to obtaining an Itô isomorphism as in [11] lies in answering the following question: given \( 1 \leq p < \infty \) and a Banach space \( X \), can we find constants \( c_{p,X} \), \( C_{p,X} \) depending only on \( p \) and \( X \) such that for any sequence of independent, mean zero \( X \)-valued random variables \( (\xi_i) \)

\[
(\mathbb{E} \left[ \sum_{i=1}^{n} |\xi_i|^p \right] )^{\frac{1}{p}} \leq \frac{C_{p,X}}{\log p} \max \left\{ \left( \sum_{i=1}^{n} \mathbb{E} |\xi_i|^p \right)^{\frac{1}{p}}, \left( \sum_{i=1}^{n} \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{2}} \right\}.
\]
A version of (1) for noncommutative random variables, as well as a version for $1 < p \leq 2$, was recently observed by Junge and Xu [12]. Their main results yield two-sided bounds of the form (5) if $X$ is a noncommutative $L^q$-space and $p = q$. Various upper bounds for the moments of a martingale with values in a uniformly 2-smooth Banach space were obtained by Pinelis [27]. However, these results lead to a two-sided estimate of the form (5) only if $X$ is a Hilbert space (see [27], Theorem 5.2). Beyond the stated results, it is not known whether (5) holds.

Our main result in this direction provides Rosenthal-type inequalities for independent random variables taking values in a noncommutative $L^q$-space. We state the version for classical $L^q$-spaces. We consider the following norms on the linear space of all finite sequences $(f_i)$ of random variables in $L^\infty(\Omega; L^q(S))$. For $1 \leq p, q < \infty$ we set

$$
\| (f_i) \|_{S_q} = \left\| \left( \sum_i |\mathbb{E}[f_i]|^2 \right)^{\frac{1}{2}} \right\|_{L^q(S)},
$$

(7)

$$
\| (f_i) \|_{D_{p,q}} = \left( \sum_i \| \mathbb{E}[f_i] \|_{L^q(S)}^p \right)^{\frac{1}{p}}.
$$

Theorem 1.2. Let $1 < p, q < \infty$ and let $(S, \Sigma, \sigma)$ be a measure space. If $(\xi_i)$ is a sequence of independent, mean zero random variables taking values in $L^\infty(S)$, then

$$
\left( \mathbb{E} \left[ \left( \sum_i |\xi_i|^p \right)^{\frac{1}{p}} \right] \right) \lesssim_{p,q} \| (\xi_i) \|_{S_{p,q}},
$$

(8)

where $s_{p,q}$ is given by

- $S_q \cap D_{q,q} \cap D_{p,q}$ if $2 \leq q \leq p < \infty$;
- $S_q \cap (D_{q,q} + D_{p,q})$ if $2 \leq p \leq q < \infty$;
- $(S_q \cap D_{q,q}) + D_{p,q}$ if $1 < p < 2 \leq q < \infty$;
- $(S_q + D_{q,q}) \cap D_{p,q}$ if $1 < q < 2 \leq p < \infty$;
- $S_q + (D_{q,q} \cap D_{p,q})$ if $1 < q \leq p \leq 2$;
- $S_q + D_{q,q} + D_{p,q}$ if $1 < p \leq q \leq 2$.

Moreover, the estimate $\lesssim_{p,q}$ in (5) remains valid if $p = 1, q = 1$ or both.

The notational similarity between the spaces introduced in (2) and (7) is intentional. Indeed, when applying Theorem 1.2 to the decoupled Poisson stochastic integral on the right hand side of (1), the spaces $S_q$, $D_{q,q}$, and $D_{p,q}$ give rise to $S^p_q$, $D^p_{q,q}$ and $D^p_{p,q}$, respectively.

If $p = q$, then the result in Theorem 1.2 (as well as its generalization in Theorem 5.4) is a special case of the noncommutative Rosenthal inequalities in [12] and the only novelty here is a new proof. However, in the application we are interested in one typically needs $p \neq q$.

As said before, we can even prove an extension of the Itô isomorphism in Theorem 1.1 in which $L^q(S)$ is replaced by a general noncommutative $L^q$-space associated with a semi-finite von Neumann algebra $\mathcal{M}$. This result is stated and proved in Theorem 7.1 below. The proof proceeds along the same lines as the result for classical $L^q$-spaces and in particular requires a version of the Rosenthal-type inequalities stated above for random variables taking values in a noncommutative $L^q$-space, which we prove in Theorem 5.4. As a by-product of the proof of Theorem 5.4 we take the opportunity to observe the following estimates for the moments of the operator norm of a sum of independent, mean zero $d_1 \times d_2$ random matrices $(x_i)$,
which may be of independent interest. If \( 2 \leq p < \infty \) and \( d = \min\{d_1, d_2\} \), then

\[
\left( \mathbb{E} \left| \sum_i x_i \right|^p \right)^{\frac{1}{p}} \leq C_{p,d} \max \left\{ \left\| \left( \sum_i \mathbb{E}|x_i|^2 \right)^{\frac{1}{2}} \right\|, \left\| \left( \sum_i \mathbb{E}|x_i|^2 \right)^{\frac{1}{2}} \right\| \right\},
\]

where \( C_{p,d} \) is of order \( \max\{\sqrt{p}, \sqrt{\log d}\} \). This result is discussed in Section 6.

Applications of Theorem 1.1 will be discussed separately in forthcoming work.

\[\text{Lemma 2.1.} \text{ Suppose } 1 \leq p, q < \infty \text{ and } 0 < r < \infty. \text{ Then, for any } 0 < p, q < \infty \text{ and any finite sequence } (x_i) \text{ in } L^q(S) \text{ we have}
\]

\[
\left( \mathbb{E} \left| \sum_i r_i x_i \right|^p \right)^{\frac{1}{p}} \simeq_{p,q} \left( \sum_i |x_i|^2 \right)^{\frac{1}{2}} \left\| L^q(S) \right\|^p.
\]

We will frequently use this result in combination with the following well-known symmetrization inequalities (see e.g. [19], Lemma 6.3). Let \( 1 \leq p < \infty \), let \( X \) be a Banach space and \( (\xi_i) \) a sequence of independent, mean zero random variables. If \( (r_i) \) is a Rademacher sequence defined on a probability space \( (\Omega_r, \mathcal{F}_r, \mathbb{P}_r) \), then

\[
\frac{1}{2} \left( \mathbb{E} \left| \sum \xi_i \right|^p_X \right)^{\frac{1}{p}} \leq \left( \mathbb{E}, \mathbb{E} \left| \sum_i r_i \xi_i \right|^p_X \right)^{\frac{1}{p}} \leq 2 \left( \mathbb{E} \left| \sum_i \xi_i \right|^p_X \right)^{\frac{1}{p}}.
\]

As a first consequence, we find the following useful estimates.

\[\text{Lemma 2.1. Suppose } 1 \leq p, q \leq 2. \text{ Let } (\xi_i) \text{ be a finite sequence of independent, mean zero } L^q(S)-\text{valued random variables. Then,}
\]

\[
\left( \mathbb{E} \left| \sum_i \xi_i \right|^p_{L^q(S)} \right)^{\frac{1}{p}} \lesssim_{p,q} \left( \sum_i \mathbb{E}|\xi_i|^2 \right)^{\frac{1}{2}} \left\| L^q(S) \right\|^p.
\]

On the other hand, if \( 2 \leq p, q < \infty \) then

\[
\left( \sum_i \mathbb{E}|\xi_i|^2 \right)^{\frac{1}{2}} \left\| L^q(S) \right\|^p \lesssim_{p,q} \left( \mathbb{E} \left| \sum_i \xi_i \right|^p_{L^q(S)} \right)^{\frac{1}{p}}.
\]

\[\text{Proof. Suppose } 1 \leq p, q \leq 2. \text{ Combining (10) and (11) yields,}
\]

\[
\left( \mathbb{E} \left| \sum_i \xi_i \right|^p_{L^q(S)} \right)^{\frac{1}{p}} \simeq_{p,q} \left( \mathbb{E} \left( \sum_i |\xi_i|^2 \right)^{\frac{1}{2}} \left\| L^q(S) \right\|^p \right)^{\frac{1}{p}}
\]

\[
= \left( \mathbb{E} \left| \sum_i \xi_i \right|^2 \left\| L^q(S) \right\|^p \right)^{\frac{1}{2}}
\]

\[
\leq \left( \sum_i \mathbb{E}|\xi_i|^2 \right)^{\frac{1}{2}} \left\| L^q(S) \right\|^p = \left( \sum_i \mathbb{E}|\xi_i|^2 \right)^{\frac{1}{2}} \left\| L^q(S) \right\|^p.
\]

Note that in the final inequality we apply Jensen’s inequality, using that \( \frac{2}{p}, \frac{2}{q} < 1 \). If we assume \( 2 \leq p, q < \infty \) then this inequality is reversed. \(\square\)
We recall the notions of type and cotype. A Banach space $X$ is said to have type $s$ for some $1 \leq s \leq 2$ if for any finite sequence $(x_i)$ in $X$

$$\left( \mathbb{E} \left\| \sum_i r_i x_i \right\|_X^2 \right)^{\frac{1}{2}} \lesssim_{s,X} \left( \sum_i \|x_i\|_X^s \right)^{\frac{1}{s}}.$$ 

A Banach space $X$ is said to have cotype $s$ for some $2 \leq s < \infty$ if for any finite sequence $(x_i)$ in $X$

$$\left( \sum_i \|x_i\|_X^s \right)^{\frac{1}{s}} \lesssim_{s,X} \left( \mathbb{E} \left\| \sum_i r_i x_i \right\|_X^2 \right)^{\frac{1}{2}}.$$ 

It is well-known that any $L^q$-space with $1 \leq q < \infty$ has type $\min\{q, 2\}$ and cotype $\max\{q, 2\}$. The following observation is well known, we include a proof for the convenience of the reader. The main ingredient are Kahane’s inequalities (see e.g. [10], Theorem 4.7): for any $0 < p, q < \infty$ there exists a constant $\kappa_{p,q}$ such that for any Banach space $X$ and $x_1, \ldots, x_n \in X$,

$$\left( \mathbb{E} \left\| \sum_{i=1}^n r_i x_i \right\|_X^p \right)^{\frac{1}{p}} \leq \kappa_{p,q} \left( \mathbb{E} \left\| \sum_{i=1}^n r_i x_i \right\|_X^q \right)^{\frac{1}{q}}.$$  

(11)

**Lemma 2.2.** Fix $1 \leq p < \infty$. Let $X$ be a Banach space and $(\xi_i)$ be a finite sequence of independent, mean zero $X$-valued random variables. If $X$ has type $1 \leq s \leq 2$, then

$$\left( \mathbb{E} \left\| \sum_i \xi_i \right\|_X^p \right)^{\frac{1}{p}} \lesssim_{p,s,X} \left( \mathbb{E} \left( \sum_i \|\xi_i\|_X^s \right)^{\frac{r}{s}} \right)^{\frac{1}{r}}.$$ 

On the other hand, if $X$ has cotype $2 \leq s < \infty$, then

$$\left( \mathbb{E} \left( \sum_i \|\xi_i\|_X^s \right)^{\frac{1}{s}} \right)^{\frac{1}{r}} \lesssim_{p,s,X} \left( \mathbb{E} \left\| \sum_i \xi_i \right\|_X^p \right)^{\frac{1}{r}}.$$ 

**Proof.** Suppose $X$ has type $s$. By symmetrization, Kahane’s inequalities and the type $s$ inequality we obtain

$$\left( \mathbb{E} \left\| \sum_i \xi_i \right\|_X^p \right)^{\frac{1}{p}} \simeq \left( \mathbb{E} \mathbb{E}_r \left\| \sum_i r_i \xi_i \right\|_X^p \right)^{\frac{1}{p}} \simeq_p \left( \mathbb{E} \left( \mathbb{E}_r \left\| \sum_i r_i \xi_i \right\|_X^2 \right)^{\frac{r}{2}} \right)^{\frac{1}{r}} \lesssim_{p,s,X} \left( \mathbb{E} \left( \sum_i \|\xi_i\|_X^s \right)^{\frac{r}{s}} \right)^{\frac{1}{r}}.$$ 

The second assertion is proved similarly. \hfill \Box

The following result is the key to the Rosenthal-type inequalities in the cases where $2 \leq p, q < \infty$.

**Theorem 2.3.** Suppose that $2 \leq p, q < \infty$. If $(\xi_i)$ is a finite sequence of independent, mean zero $L^q(S)$-valued random variables, then

$$\left( \mathbb{E} \left\| \sum_i \xi_i \right\|_{L^q(S)}^p \right)^{\frac{1}{p}} \simeq_{p,q} \max \left\{ \left( \sum_i \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \left( \sum_i \|\xi_i\|_{L^q(S)}^q \right)^{\frac{r}{2}} \right)^{\frac{1}{r}} \right\},$$  

(12)

**Proof.** We first prove the estimate $\lesssim_{p,q}$. By Lemma 2.1

$$\left( \sum_i \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \left( \sum_i \|\xi_i\|_{L^q(S)}^q \right)^{\frac{r}{2}} \right)^{\frac{1}{r}}.$$ 


Moreover, since $L^q(S)$ has cotype $q$ Lemma 2.2 implies
\[
\left( \mathbb{E} \left( \sum_i \|\xi_i\|_{L^q(S)}^q \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \lesssim_{p,q} \left( \mathbb{E} \left( \sum_i \|\xi_i\|_{L^p(S)}^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}.
\]
We now prove the reverse inequality in (12). By symmetrization and the Khintchine inequalities (10),
\[
\left( \mathbb{E} \left( \sum_i \|\xi_i\|_{L^q(S)}^q \right) \right)^{\frac{1}{q}} \lesssim_{p,q} \left( \mathbb{E} \left( \left( \sum_i |\xi_i|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right)^{\frac{1}{2}}.
\]
By the triangle inequality we obtain
\[
\left( \mathbb{E} \left( \sum_i |\xi_i|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \left( \sum_i |\xi_i|^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}.
\]
(13)
\[
\left( \mathbb{E} \left( \sum_i |\xi_i|^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} \lesssim_{p,q} \left( \mathbb{E} \left( \sum_i \|\xi_i\|^p_{L^q(S)} \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]
Suppose first that $q \leq 4$. Then $L^q(S)$ has type $\frac{q}{2}$, so by Lemma 2.2
\[
\left( \mathbb{E} \left( \sum_i |\xi_i|^2 - \mathbb{E}|\xi_i|^2 \right) \right)^{\frac{q}{2}} \lesssim_{p,q} \mathbb{E} \left( \sum_i \|\xi_i\|^q_{L^q(S)} \right)^{\frac{1}{q}}
\]
(14)
\[
\left( \mathbb{E} \left( \sum_i \|\xi_i\|^q_{L^q(S)} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \lesssim_{p,q} \left( \mathbb{E} \left( \left( \sum_i |\xi_i|^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} + \left( \mathbb{E} \left( \sum_i \|\xi_i\|^p_{L^q(S)} \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]
where in the final step we apply Jensen’s inequality.

Suppose now that $q > 4$. By applying symmetrization and the Khintchine inequalities (10) we find
\[
\left( \mathbb{E} \left( \sum_i |\xi_i|^2 - \mathbb{E}|\xi_i|^2 \right) \right)^{\frac{q}{2}} \lesssim_{p,q} \mathbb{E} \left( \left( \sum_i |\xi_i|^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}
\]
(15)
\[
\left( \mathbb{E} \left( \left( \sum_i |\xi_i|^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \lesssim_{p,q} \left( \mathbb{E} \left( \sum_i \|\xi_i\|^p_{L^q(S)} \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} + \left( \sum_i \mathbb{E} \|\xi_i\|^p_{L^q(S)} \right)^{\frac{1}{p}}
\]
Since $q > 4$ there is some $0 < \theta < \frac{1}{2}$ such that $\frac{1}{4} = \frac{\theta}{2} + \frac{1-\theta}{q}$. By applying Hölder’s inequality three times we obtain
\[
\left( \mathbb{E} \left( \sum_i |\xi_i|^4 \right)^{\frac{p}{4}} \right)^{\frac{1}{p}} \lesssim \left( \mathbb{E} \left( \sum_i |\xi_i|^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} \left( \sum_i \mathbb{E} \|\xi_i\|^p_{L^q(S)} \right)^{\frac{1}{p}}
\]
Combining (14), (15) and (16) we arrive at the inequality
\[ a \geq b \]
Hence, we may assume
\[ \text{this result we shall make use of the fact that for any } 1 \]
holds isometrically. This follows from the following general principle. Suppose
\[ \sum \]
Recall the space \( s_{p,q} \) defined in the statement of Theorem (1.2). In the proof of
this result we shall make use of the fact that for any \( 1 < p, q < \infty \)
holds isometrically. This follows from the following general principle. Suppose that
\( X \) and \( Y \) are two Banach spaces which are continuously embedded in some
Hausdorff topological vector space and assume moreover that \( X \cap Y \) is dense in
both \( X \) and \( Y \). Then we have
\[ (X \cap Y)^* = X^* + Y^*, \quad (X + Y)^* = X^* \cap Y^* \]
isometrically. The duality brackets under these identifications are given by
\[ \langle x, x^* \rangle = \langle x, x^* \rangle_{X \cap Y} \quad (x^* \in X^* + Y^*) \]
and
\[ \langle x^*, x \rangle = \langle y, x^* \rangle + \langle z, x^* \rangle \quad (x^* \in X^* \cap Y^*, \ x = y + z \in X + Y), \]
respectively, see e.g. (16), Theorem I.3.1. In our case of interest, the spaces \( S_q, D_{p,q} \) and \( D_{q,q} \) have dense intersection, and therefore the duality of these individual
spaces imply together with (18) that (17) holds, with associated duality bracket
\[ \langle (f_i), (g_i) \rangle = \sum_i E \int f_i g_i d\sigma. \]
We need two more ingredients for the proof of Theorem 1.2. The first are the hypercontractive-type inequalities due to Hoffmann-Jørgensen [12] (see also [17, 13] for a proof yielding a constant of optimal order)

$$\left( E\|\sum_{i} \xi_{i}\|_{L^{q}(S)}^{p} \right) \lesssim \frac{p}{p-2q} E\left( \sum_{i} \xi_{i}\right)_{X} + \left( E\max_{i} \|\xi_{i}\|_{X}^{p} \right)^{\frac{1}{p}},$$

valid for any $1 \leq p < \infty$ and any sequence $(\xi_{i})$ of independent, mean zero random variables taking values in a Banach space $X$. Finally, let us recall the Rosenthal inequalities for a sequence $(f_{i})$ of positive scalar-valued random variables: if $1 \leq p < \infty$, then

$$\left( E\|\sum_{i} f_{i}\|_{p} \right)^{\frac{1}{p}} \lesssim \max \left\{ \left( \sum_{i} E|f_{i}|^{p} \right)^{\frac{1}{p}}, \sum_{i} E|f_{i}| \right\}.$$  

We are now ready to prove our first main result.

**Proof.** (Of Theorem 1.2) Let us note that the inequalities ‘$\lesssim_{p,q}$’ in (8) follow by duality once the reverse inequalities have been established. Indeed, if $(\eta_{i})$ is a finite sequence in $s_{\varphi,\varphi'}$ of norm 1, then

$$\langle (\xi_{i}), (\eta_{i}) \rangle = \sum_{i} E\int (\xi_{i}\eta_{i}) d\mu$$

$$= \sum_{i} E\int (\xi_{i}(E(\eta_{i}|\xi_{i}) - E(\eta_{i}))) d\mu$$

$$= \sum_{i,j} E\int (\xi_{i}(E(\eta_{j}|\xi_{j}) - E(\eta_{j}))) d\mu$$

$$= E\int \left( \sum_{i} \xi_{i} \right) \left( \sum_{j} E(\eta_{j}|\xi_{j}) - E(\eta_{j}) \right) d\mu$$

$$\leq \left( E\left\| \sum_{i} \xi_{i} \right\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}} \left( E\left\| \sum_{j} E(\eta_{j}|\xi_{j}) - E(\eta_{j}) \right\|_{s_{\varphi,\varphi'}}^{p'} \right)^{\frac{1}{p'}}$$

Since the elements $E(\eta_{j}|\xi_{j}) - E(\eta_{j})$ are independent and mean zero,

$$\langle (\xi_{i}), (\eta_{i}) \rangle \lesssim_{\varphi,\varphi'} \left( E\left\| \sum_{i} \xi_{i} \right\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}} \left\| \sum_{j} E(\eta_{j}|\xi_{j}) - E(\eta_{j}) \right\|_{s_{\varphi,\varphi'}}$$

$$\leq 2 \left( E\left\| \sum_{i} \xi_{i} \right\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}}.$$  

By (17), the claim now follows by taking the supremum over all $(\eta_{i})$ as above. We now prove the estimates $\lesssim_{p,q}$ case by case.

**Case 2 $\leq q < p$:** Recall that Theorem 2.1 says that

$$\left( E\left\| \sum_{i} \xi_{i} \right\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}} \lesssim_{p,q} \max \left\{ \left( \sum_{i} E|\xi_{i}|^{p} \right)^{\frac{1}{p}}, \left( E\left( \sum_{i} \|\xi_{i}\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}} \right) \right\}.$$  

Since $q \leq p$, applying (20) with $f_{i} = \|\xi_{i}\|_{L^{q}(S)}$ yields

$$\left( E\left( \sum_{i} \|\xi_{i}\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \lesssim_{p,q} \max \left\{ \left( \sum_{i} E|\xi_{i}| \right)^{\frac{1}{p}}, \left( E\left( \sum_{i} \|\xi_{i}\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}} \right) \right\}.$$  

**Case 2 $\leq q < p$:** If $p \leq q$ the contractive embeddings $L^{q}(\Omega) \subset L^{p}(\Omega)$ and $L^{p} \subset L^{q}$ imply

$$\left( E\left( \sum_{i} \|\xi_{i}\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \leq \left( E\left( \sum_{i} \|\xi_{i}\|_{L^{q}(S)}^{q} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}.$$
and
\[(24) \quad \left( E \left( \sum_i \| \xi_i \|^q_{L^q(S)} \right)^\frac{1}{q} \right)^\frac{1}{p} \leq \left( \sum_i E \| \xi_i \|^p_{L^p(S)} \right)^\frac{1}{p}. \]

By the triangle inequality,
\[\left( E \left( \sum_i \| \xi_i \|^q_{L^q(S)} \right)^\frac{1}{q} \right)^\frac{1}{p} \leq \| \xi_i \|_{D_{p,q} + D_{q,q}}.\]

The asserted estimate now follows from Theorem 2.3.

Similarly, Lemma 2.2, (23) and Jensen’s inequality yield
\[(24) \quad \left( E \left( \sum_i \| \xi_i \|^q_{L^q(S)} \right)^\frac{1}{q} \right)^\frac{1}{p} \leq \| \xi_i \|_{D_{p,q} + D_{q,q}}.\]

By the triangle inequality,
\[\left( E \left( \sum_i \| \xi_i \|^q_{L^q(S)} \right)^\frac{1}{q} \right)^\frac{1}{p} \leq \| \xi_i \|_{D_{p,q} + D_{q,q}}.\]

Case 1 \(1 \leq p \leq q \leq 2\): Let \((\eta_i) \in S_q, (\theta_i) \in D_{p,q}\) and \((\kappa_i) \in D_{q,q}\) be such that
\[\xi_i = \eta_i + \theta_i + \kappa_i.\]

Then,
\[\xi_i = E(\eta_i|\xi_i) - E(\eta_i) + E(\theta_i|\xi_i) - E(\theta_i) + E(\kappa_i|\xi_i) - E(\kappa_i).\]

By Lemma 2.1,
\[\left( E \left( \sum_i E(\eta_i|\xi_i) - E(\eta_i) \right)^p_{L^p(S)} \right)^\frac{1}{p} \lesssim_{p,q} \left( \sum_i \| E(\eta_i|\xi_i) - E(\eta_i) \|^q_{L^q(S)} \right)^\frac{1}{q} \]
\[\leq 2 \left( \sum_i \| \eta_i \|^2 \right)^\frac{1}{q} \left( \sum_i \| \xi_i \|^q_{L^q(S)} \right)^\frac{1}{q}.\]

where the final step follows from the triangle inequality and Jensen’s inequality.

Now apply Lemma 2.2 (using that \(L^q(S)\) has type \(q\)), (24) and the vector-valued Jensen inequality to find
\[\left( E \left( \sum_i \| \theta_i|\xi_i \right) - E(\theta_i) \right)^p_{L^p(S)} \right)^\frac{1}{p} \lesssim_{p,q} \left( \sum_i \| E(\theta_i|\xi_i) - E(\theta_i) \|^q_{L^q(S)} \right)^\frac{1}{q} \]
\[\leq 2 \left( \sum_i \| \theta_i \|^p_{L^p(S)} \right)^\frac{1}{p} \]

Similarly, Lemma 2.2 and Jensen’s inequality yield
\[\left( E \left( \sum_i \| \kappa_i|\xi_i \right) - E(\kappa_i) \right)^p_{L^p(S)} \right)^\frac{1}{p} \lesssim_{p,q} \left( \sum_i \| \kappa_i \|^q_{L^q(S)} \right)^\frac{1}{q}.\]

The asserted estimate now follows by the triangle inequality.

Case 1 \(1 \leq q \leq p \leq 2\): The proof is very similar to the previous case. Let \((\eta_i) \in S_q\) and \((\theta_i) \in D_{p,q} \cap D_{q,q}\) be such that \(\xi_i = \eta_i + \theta_i\), then
\[\xi_i = E(\eta_i|\xi_i) - E(\eta_i) + E(\theta_i|\xi_i) - E(\theta_i).\]

By the same argument as in (25),
\[\left( E \left( \sum_i \| \eta_i|\xi_i \right) - E(\eta_i) \right)^p_{L^p(S)} \right)^\frac{1}{p} \lesssim_{p,q} \left( \sum_i \| \eta_i \|^2 \right)^\frac{1}{q} \left( \sum_i \| \xi_i \|^q_{L^q(S)} \right)^\frac{1}{q}.\]

Moreover, successively applying Lemma 2.2 the Rosenthal inequality (20) (using that \(q \leq p\)) and the vector-valued Jensen inequality yields
\[\left( E \left( \sum_i \| \theta_i|\xi_i \right) - E(\theta_i) \right)^p_{L^p(S)} \right)^\frac{1}{p} \lesssim_{p,q} \left( \sum_i \| \theta_i \|^p_{L^p(S)} \right)^\frac{1}{p} \left( \sum_i \| E(\theta_i|\xi_i) - E(\theta_i) \|^q_{L^q(S)} \right)^\frac{1}{q} \]
\[\lesssim_{p,q} \max \left\{ \left( \sum_i \| E(\theta_i|\xi_i) - E(\theta_i) \|^p_{L^p(S)} \right)^\frac{1}{p}, \left( \sum_i \| E(\theta_i|\xi_i) - E(\theta_i) \|^q_{L^q(S)} \right)^\frac{1}{q} \right\}.\]
implies Definition 3.1.

Let us first define the Poisson stochastic integral. On the other hand, as proof.

By the previous Case (with $p = q$) we have

$$\left( E\left\| \sum_{i} \xi_{i}\right\| _{L^{q}(S)}^{p}\right)^{\frac{1}{p}} \leq p \max \left\{ \left( E\left\| \sum_{i} \xi_{i}\right\| _{L^{q}(S)}^{q}\right)^{\frac{1}{q}}, \left( E\max_{i} \left\| \xi_{i}\right\| _{L^{q}(S)}^{p}\right)^{\frac{1}{p}} \right\}.$$ 

By the previous Case (with $p = q$) we have

$$\left( E\left\| \sum_{i} \xi_{i}\right\| _{L^{q}(S)}^{q}\right)^{\frac{1}{q}} \sim_{p,q} \left\| (\xi_{i})\right\| _{S_{q}+D_{q,\varepsilon}}$$

and obviously

$$\left( E\max_{i} \left\| \xi_{i}\right\| _{L^{q}(S)}^{p}\right)^{\frac{1}{p}} \leq \left( \sum_{i} E\left\| \xi_{i}\right\| _{L^{q}(S)}^{p}\right)^{\frac{1}{p}}.$$ 

Case 1 $\leq p \leq 2 \leq q < \infty$: Let $\xi_{i} = \eta_{i} + \theta_{i}$. Then, $\xi_{i} = E(\eta_{i}|\xi_{i}) - E(\eta_{i}) + E(\theta_{i}|\xi_{i}) - E(\theta_{i})$. By our estimate in the case $p = q \geq 2$ we have

$$\left( E\left\| \sum_{i} E(\eta_{i}|\xi_{i}) - E(\eta_{i})\right\| _{L^{p}(S)}^{p}\right)^{\frac{1}{p}} \leq \left( E\left\| \sum_{i} E(\eta_{i}|\xi_{i}) - E(\eta_{i})\right\| _{L^{q}(S)}^{q}\right)^{\frac{1}{q}} \leq p \max \left\{ \left( \sum_{i} E\left\| E(\eta_{i}|\xi_{i}) - E(\eta_{i})\right\| _{L^{q}(S)}^{p}\right)^{\frac{1}{p}}, \left( \sum_{i} E\left\| E(\eta_{i}|\xi_{i}) - E(\eta_{i})\right\| _{L^{q}(S)}^{q}\right)^{\frac{1}{q}} \right\} \leq 2 \max \left\{ \left( \sum_{i} E\left\| \eta_{i}\right\| _{L^{q}(S)}^{q}\right)^{\frac{1}{q}}, \left( \sum_{i} E\left\| \eta_{i}\right\| _{L^{q}(S)}^{q}\right)^{\frac{1}{q}} \right\}.$$ 

On the other hand, as $L^{q}(S)$ has type 2, it has type $p$ and therefore Lemma 2.2 implies

$$\left( E\left\| \sum_{i} E(\theta_{i}|\xi_{i}) - E(\theta_{i})\right\| _{L^{p}(S)}^{p}\right)^{\frac{1}{p}} \leq p \max \left\{ \left( \sum_{i} E\left\| E(\theta_{i}|\xi_{i}) - E(\theta_{i})\right\| _{L^{q}(S)}^{p}\right)^{\frac{1}{p}}, \left( \sum_{i} E\left\| E(\theta_{i}|\xi_{i}) - E(\theta_{i})\right\| _{L^{q}(S)}^{q}\right)^{\frac{1}{q}} \right\} \leq 2 \left( \sum_{i} E\left\| \theta_{i}\right\| _{L^{q}(S)}^{p}\right)^{\frac{1}{p}}.$$ 

The claimed inequality now follows by the triangle inequality. This completes the proof.

3. Itô-Isomorphisms: Classical $L^{q}$-Spaces

In this section we present a proof of the Itô isomorphism stated in Theorem 1.1. Let us first define the Poisson stochastic integral.

**Definition 3.1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(E, \mathcal{E}, \mu)$ be a measure space. We say that a random measure $N$ on $E$ is a Poisson random measure if the following conditions hold:

(i) For disjoint sets $A_{1}, \ldots, A_{n}$ in $\mathcal{E}$ the random variables $N(A_{1}), \ldots, N(A_{n})$ are independent and

$$N\left( \bigcup_{i=1}^{n} A_{i}\right) = \sum_{i=1}^{n} N(A_{i}),$$

(ii) For any $A \in \mathcal{E}$ with $\mu(A) < \infty$ the random variable $N(A)$ is Poisson distributed with parameter $\mu(A)$. 

Let us define the Poisson stochastic integral.
Let $E_\mu = \{ A \in E : \mu(A) < \infty \}$. Then the random measure $\bar{N}$ on $(E, E_\mu, \mu)$ defined by

$$\bar{N}(A) := N(A) - \mu(A) \quad (A \in E_\mu),$$

is called the compensated Poisson random measure associated with $N$.

Throughout, we let $(J, \mathcal{F}, \nu)$ be a $\sigma$-finite measure space and we fix a Poisson random measure $N$ on $\mathbb{R}_+ \times J$. To arrive at a satisfactory stochastic integration theory with respect to the associated compensated Poisson random measure, we need to impose the following standard compatibility assumption.

**Assumption 3.2.** Throughout we fix a filtration $(\mathcal{F}_t)_{t \geq 0}$ such that for any $0 \leq s < t < \infty$ and any $A \in \mathcal{F}$ the random variable $\bar{N}((s, t] \times A)$ is independent of $\mathcal{F}_s$.

**Definition 3.3.** Fix a Banach space $X$ and let $F : \Omega \times \mathbb{R}_+ \times J \to X$. We say that $F$ is a simple, adapted $X$-valued process if there is a finite partition $\pi = \{0 \leq t_1 < \ldots < t_{l+1} < \infty\}$ of $\mathbb{R}_+$, $F_{i,k} \in L^\infty(\mathcal{F}_{t_i})$, $x_{i,j,k} \in X$ and disjoint sets $A_1, \ldots, A_m$ in $\mathcal{F}$ satisfying $\nu(A_i) < \infty$ for $i = 1, \ldots, l$, $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$F = \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n F_{i,k} \chi_{(t_i, t_{i+1})} x_{i,j,k}. \tag{26}$$

Given $t \geq 0$ and $B \in \mathcal{F}$, we define the (compensated) Poisson stochastic integral of $F$ on $[0, t] \times B$ with respect to $\bar{N}$ by

$$\int_{[0, t] \times B} F \, d\bar{N} = \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n F_{i,k} \bar{N}((t_i \wedge t, t_{i+1} \wedge t] \times (A_j \cap B)) x_{i,j,k},$$

where $s \wedge t := \min\{s, t\}$.

The following elementary observation will be important for our proof.

**Lemma 3.4.** Let $N$ be a Poisson distributed random variable with parameter $0 \leq \lambda \leq 1$. Then for every $1 \leq p < \infty$ there exist constants $b_p, c_p > 0$ such that

$$b_p \lambda \leq \mathbb{E}|N - \lambda|^p \leq c_p \lambda. \tag{27}$$

**Proof.** The inequalities are trivial if $\lambda = 0$, so we may assume $\lambda > 0$. We first prove the inequality on the left hand side of (27). Suppose first that $2 \leq p < \infty$. Then we have

$$\mathbb{E}|N - \lambda|^p = \sum_{k=0}^\infty |k - \lambda|^p \frac{\lambda^k e^{-\lambda}}{k!} \geq \sum_{k=2}^\infty |k - \lambda|^2 \frac{\lambda^k e^{-\lambda}}{k!} + |\lambda|^p e^{-\lambda} + |1 - \lambda|^p \lambda e^{-\lambda}. \tag{28}$$

Hence,

$$\mathbb{E}|N - \lambda|^p \geq \mathbb{E}|N - \lambda|^2 - |\lambda|^2 e^{-\lambda} - (1 - \lambda)^2 \lambda e^{-\lambda} + |\lambda|^p e^{-\lambda} + |1 - \lambda|^p \lambda e^{-\lambda} = \lambda + \lambda e^{-\lambda}(-\lambda - (1 - \lambda)^2 + \lambda^{p-1} + (1 - \lambda)^p) = \lambda(1 + e^{-\lambda} f_p(\lambda)), \tag{29}$$

where

$$f_p(\lambda) = \lambda^{p-1} - \lambda^2 + \lambda - 1 + (1 - \lambda)^p. \tag{30}$$

One easily sees that $\min_{0 \leq \lambda \leq 1} (1 + e^{-\lambda} f_p(\lambda)) = b_p > 0$. Indeed,

$$1 + e^{-\lambda} f_p(\lambda) > 1 + e^{-\lambda}(-\lambda^2 + \lambda - 1) + e^{-\lambda}(1 - \lambda)^p. \tag{31}$$

Now,

$$1 + e^{-\lambda}(-\lambda^2 + \lambda - 1) + e^{-\lambda}(1 - \lambda)^p > 0$$

for $0 < \lambda < 1$.
if and only if
\[(1 - \lambda)^p > -e^\lambda + \lambda^2 - \lambda + 1 = -2\lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{6} - \frac{\lambda^4}{24} - \cdots.
\]
Clearly this holds if 0 \leq \lambda \leq 1. This proves the left hand side inequality if 2 \leq p < \infty. We now consider the right hand side inequality. It suffices to prove this in the case where p is an even integer. Since the moment generating function of \(N - \lambda\) is given by
\[
E(e^{t(N-\lambda)}) = e^{\lambda(e^{t-1}-1)} = \exp(\lambda \sum_{n=2}^{\infty} \frac{t^n}{n!}),
\]
it is easy to see that the \(n\)-th moment of \(N - \lambda\) can be written as \(\lambda p_n(\lambda)\) for some polynomial \(p_n\) with positive coefficients. Since \(\max_{0 \leq \lambda \leq 1} |p_n(\lambda)| \leq c_n\) for some constant \(c_n > 0\), our proof for the case 2 \leq p < \infty is complete.

Suppose now that 1 \leq p < 2. Then, by the Cauchy-Schwartz inequality,
\[
\lambda^2 \leq \mathbb{E}[|N - \lambda|^2] = \mathbb{E}[|N - \lambda|^p] |N - \lambda|^{2-p} \leq (\mathbb{E}[|N - \lambda|^p] \mathbb{E}[|N - \lambda|^{4-p}])^{\frac{1}{2}}.
\]
Since 4 - p \geq 2 we find by the above that
\[
\lambda^2 \leq \mathbb{E}[|N - \lambda|^p] \mathbb{E}[|N - \lambda|^{4-p}] \leq \mathbb{E}[|N - \lambda|^p c_{4-p} \lambda].
\]
To prove the right hand side inequality in (27), note that if 1 \leq p < 2 the inequalities in (28) and (29) reverse and therefore
\[
E[N - \lambda]^p \leq \lambda \max_{0 \leq \lambda \leq 1} (1 + e^{-\lambda} f_p(\lambda)),
\]
where \(f_p\) is the continuous function defined in (30).

**Remark 3.5.** By refining the partition \(\pi\) in Definition 3.5 if necessary, we can and will always assume that \((t_{i+1} - t_i)\nu(A_j) \leq 1\) for all \(i = 1, \ldots, l, j = 1, \ldots, m\). This will allow us to apply Lemma 2.4 to the compensated Poisson random variables \(\bar{N}((t_i \land t, t_{i+1} \land t) \times (A_j \cap B))\).

We are now ready to prove Theorem 1.1.

**Proof.** (Of Theorem 1.1) Observe that the map
\[
s \mapsto \left\| \int_{[0,s] \times B} F \, d\bar{N} \right\|_{L^q(S)}
\]
defines a positive submartingale in \(L^p(\Omega)\) and hence by Doob’s maximal inequality (see e.g. [30], Theorem 1.7) we have for any \(p > 1\),
\[
\left( \mathbb{E} \sup_{0 \leq s \leq t} \left\| \int_{[0,s] \times B} F \, d\bar{N} \right\|_{L^q(S)}^p \right)^{\frac{1}{p}} \leq p \left( \mathbb{E} \int_{[0,t] \times B} F \, d\bar{N} \right)^{\frac{1}{p}}.
\]
where \(\frac{1}{p} + \frac{1}{q} = 1\). Moreover, \(L^q(S)\) has the UMD property if \(1 < q < \infty\), so in view of the decoupling inequalities (1) it suffices to prove
\[
(31) \quad \left( \mathbb{E} \left\| \int_{[0,t] \times B} F \, d\bar{N} \right\|_{L^q(S)}^p \right)^{\frac{1}{p}} \approx_{p,q} \| F \chi_{[0,t] \times B} \|_{L^p(S)},
\]
where \(\bar{N}\) is a separable copy of \(\bar{N}\) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We show this in the cases \(2 \leq q \leq p < \infty\) and \(1 < p \leq q \leq 2\) in detail. All the main technical difficulties occur in these two cases, and the similar proof in the other cases is left to the reader. Let \(F\) be the simple adapted process given in (20), taking Remark 3.5 into account. We may assume that \(t = t_{i+1}\) and \(B = \cup_{j=1}^{p} A_j\). We write \(\bar{N}^c_{i,j} := \bar{N}^c((t_i, t_{i+1}) \times A_j)\) for brevity.
Case 2 ≤ q ≤ p < ∞: Let \( \mathcal{G} \) be the sub-\( \sigma \)-algebra \( F \times \{ \Omega_c, \emptyset \} \) of \( F \times F_c \). If we set \( y_{i,j} = \sum_{k=1}^{n} F_{i,k} x_{i,j,k} \), then the doubly indexed sequence \( d_{i,j} = y_{i,j} \mathcal{N}_{i,j}^{c} \) is conditionally independent with respect to \( \mathcal{G} \) and mean zero, and moreover,

\[
\int_{[0,1] \times B} F \, d\tilde{N} = \sum_{i,j} d_{i,j}.
\]

Applying Theorem 1.2 conditionally on \( \mathcal{G} \) and taking \( L^p \)-norms yields

\[
\left( \mathbb{E}_{E} \left\| \sum_{i,j} d_{i,j} \right\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}} \geq \left( \mathbb{E}_{E} \left( \sum_{i,j} \mathbb{E}_{\mathcal{G}}[d_{i,j}]^{2} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}},
\]

\[
\left( \mathbb{E}_{E} \left( \sum_{i,j} \mathbb{E}_{\mathcal{G}}[d_{i,j}]^{2} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} = \left( \mathbb{E} \left( \sum_{i,j} |y_{i,j}|^{2} \mathbb{E}_{c}[\mathcal{N}_{i,j}^{c}]^{2} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}},
\]

\[
(32) \quad \left( \mathbb{E}_{E} \left( \sum_{i,j} \mathbb{E}_{\mathcal{G}}[d_{i,j}] \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} = \left( \mathbb{E} \left( \sum_{i,j} |y_{i,j}|^{2}(t_{i+1} - t_{i}) \nu(A_{j}) \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} = \|F\|_{S_{p}^{q}}.
\]

Using Lemma 3.1 and Remark 3.3 we compute

\[
\left( \mathbb{E} \left( \sum_{i,j} \mathbb{E}_{\mathcal{G}}[d_{i,j}] \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \geq \left( \mathbb{E} \left( \sum_{i,j} |y_{i,j}|^{q} \mathbb{E}_{c}[\mathcal{N}_{i,j}^{c}]^{q} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.
\]

(33) \quad \left( \mathbb{E} \left( \sum_{i,j} |y_{i,j}|^{q} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} = \|F\|_{D_{p,q}^{q}}.

Finally,

\[
(34) \quad \left( \sum_{i,j} \mathbb{E}_{E} \left\| d_{i,j} \right\|_{L^{q}(S)}^{p} \right)^{\frac{1}{p}} = \left( \sum_{i,j} \mathbb{E} \left| y_{i,j} \right|^{p} \mathbb{E}_{c}[\mathcal{N}_{i,j}^{c}]^{p} \right)^{\frac{1}{p}} \geq \left( \sum_{i,j} \mathbb{E} \left| y_{i,j} \right|^{p} \right)^{\frac{1}{p}} \geq \left( \sum_{i,j} \mathbb{E} \left| y_{i,j} \right|^{p} \right)^{\frac{1}{p}} = \|F\|_{D_{p,q}^{q}}.
\]

We conclude that (31) holds.

Case 1 < p ≤ q ≤ 2: Let \( \mathcal{I}_{\text{elem}} \) denote the algebraic tensor product

\[ \mathcal{I}_{\text{elem}} = L^{\infty}(\Omega) \otimes L^{\infty}(\mathbb{R}_{+}) \otimes (L^{1} \cap L^{\infty})(J) \otimes (L^{1} \cap L^{\infty})(S).\]

This linear space is dense in \( S_{p}^{q}, D_{p,q}^{q} \) and \( D_{p,q}^{q} \). Therefore, if we fix \( \varepsilon > 0 \), we can find a decomposition \( F = F_{1} + F_{2} + F_{3} \) with \( F_{\alpha} \in \mathcal{I}_{\text{elem}} \) for \( \alpha = 1, 2, 3 \) such that

\[
\|F\|_{\mathcal{I}_{p,q}} = \|F_{1}\|_{S_{p}^{q}} + \|F_{2}\|_{D_{p,q}^{q}} + \|F_{3}\|_{D_{p,q}^{q}} - \varepsilon.
\]

Let \( \mathcal{A} \) be the sub-\( \sigma \)-algebra of \( B(\mathbb{R}_{+}) \times \mathcal{J} \) generated by the sets \( (t_{i}, t_{i+1}] \times A_{j} \). The associated conditional expectation \( \mathbb{E}(\cdot|\mathcal{A}) \) is well-defined, as \( \mathcal{J} \) is \( \sigma \)-finite. Note that
By the computations in (32), (33) and (34), we conclude that
\[ y_{i,j,\alpha} = \sum_{k=1}^{n} F_{i,k,\alpha} x_{i,j,k,\alpha} \] and set \( d_{i,j,\alpha} = y_{i,j,\alpha} N_{i,j}^c \), so that
\[
\int_{[0,t] \times B} F \, d\tilde{N}^c = \sum_{i,j} d_{i,j,1} + d_{i,j,2} + d_{i,j,3}.
\]
If we apply Theorem 1.2 conditionally on \( G \) and \( q \) where \( \langle \langle \rangle \rangle_{G} \) denote the associated duality bracket. If \( G \) from (18) that
\[
\langle \langle \rangle \rangle_{G} = \sum_{i,j,k} c_{i,j,k} \int_{[0,t] \times B} F \, d\tilde{N}^c \]
we find
\[
\left( \mathbb{E}_c \left\| \sum_{i,j} d_{i,j} \right\|_{L^p(S)}^p \right)^{\frac{1}{p}} \\lesssim_{p,q} \left( \mathbb{E}_c \left\| \sum_{i,j} E_G |d_{i,j,1}|^2 \right\|_{L^q(S)}^p \right)^{\frac{1}{p}} + \left( \mathbb{E}_c \left\| \sum_{i,j} E_G |d_{i,j,2}|^2 \right\|_{L^q(S)}^p \right)^{\frac{1}{p}} + \left( \mathbb{E}_c \left\| \sum_{i,j} E_G |d_{i,j,3}|^2 \right\|_{L^q(S)}^p \right)^{\frac{1}{p}}.
\]
By the computations in (32), (33) and (34),
\[
\left( \mathbb{E}_c \left\| \sum_{i,j} E_G |d_{i,j,1}|^2 \right\|_{L^q(S)}^p \right)^{\frac{1}{p}} \approx \| E(F_1 |A) \|_{S^p_{q}} \leq \| F_1 \|_{S^p_{q}},
\]
\[
\left( \mathbb{E}_c \left\| \sum_{i,j} E_G |d_{i,j,2}|^2 \right\|_{L^q(S)}^p \right)^{\frac{1}{p}} \approx p \| E(F_2 |A) \|_{D^p_{q}} \leq \| F_2 \|_{D^p_{q}},
\]
\[
\left( \mathbb{E}_c \left\| \sum_{i,j} E_G |d_{i,j,3}|^2 \right\|_{L^q(S)}^p \right)^{\frac{1}{p}} \approx q \| E(F_3 |A) \|_{D^p_{q}} \leq \| F_3 \|_{D^p_{q}}.
\]
We conclude that
\[
\left( \mathbb{E} \left\| \int_{[0,t] \times B} F \, d\tilde{N}^c \right\|_{L^p(S)}^p \right)^{\frac{1}{p}} \lesssim_{p,q} \| F_1 \|_{S^p_{q}} + \| F_2 \|_{D^p_{q}} + \| F_3 \|_{D^p_{q}} = \| F \|_{T^p_{q}} + \varepsilon.
\]
We deduce the reverse inequality by duality. If \( p', q' \) are the Hölder conjugates of \( p \) and \( q \), then \( (S^p_{q})^* = S^{p'}_{q'} \), \((D^p_{q})^* = D^{p'}_{q'} \) and \((D^p_{q})^* = D^{p'}_{q'} \). Therefore, it follows from (18) that \( T^{p,q} = T^{p',q'} \). We let
\[
\langle F, G \rangle = \int \int \int_{R \times R \times J \times S} FG \, dP \, dt \, d\sigma
\]
denote the associated duality bracket. If \( G \in \mathcal{I}_{elem} \), then \( E(G|A) \) is of the form
\[
E(G|A) = \sum_{i,j,k} G_{i,k} \chi_{(t_i, t_{i+1})} \chi_{A_j} x_{i,j,k}^*,
\]
where \( G_{i,k} \in L^\infty(\Omega) \). Now,
\[
\langle F, G \rangle = \langle F, E(G|A) \rangle
\]
\[
= \sum_{i,j,k} \mathbb{E}(F_{i,k} G_{i,k}) dt \times \nu((t_i, t_{i+1]) \times A_j)(x_{i,j,k}, x_{i,j,k}^*)
\]
\[
= \sum_{i,j,k} \mathbb{E}(F_{i,k} G_{i,k}) dt \times \nu((t_i, t_{i+1}) \times A_j)(x_{i,j,k}, x_{i,j,k}^*)
\]
\[
= \sum_{i,j,k,l,m,n} \mathbb{E}(F_{i,k} G_{l,n}) \mathbb{E}_c(\tilde{N}^c_{i,j} \tilde{N}^c_{l,m})(x_{i,j,k}, x_{i,m,n}^*)
\]
\[
= \sum_{i,j,k,l,m,n} \mathbb{E}_c(F_{i,k} \tilde{N}^c_{i,j} G_{l,n} \tilde{N}^c_{l,m})(x_{i,j,k}, x_{i,m,n}^*)
\]
\[
\langle \sum_{i,j,k} F_{i,k} \tilde{N}^c_{i,j,k}, \sum_{l,m,n} G_{l,n} \tilde{N}^c_{l,m,n} \rangle \\
\leq \left\| \int_{[0,t] \times B} F \ d\tilde{N} \right\|_{L^p(\Omega \times H^s(S))} \left\| \sum_{l,m,n} G_{l,n} \tilde{N}^c_{l,m,n} \right\|_{L^{p'}(\Omega \times H^s(S))},
\]

Since \(2 \leq q' \leq p' < \infty\), our previously established case implies that

\[
\left\| \sum_{l,m,n} G_{l,n} \tilde{N}^c_{l,m,n} \right\|_{L^{p'}(\Omega \times H^s(S))} \lesssim_{p,q} \left\| \mathbb{E}(G|\mathcal{A}) \right\|_{L^{p',q'}},
\]

Summarizing, we find

\[
\langle F, G \rangle \lesssim_{p,q} \left\| \int_{[0,t] \times B} F \ d\tilde{N} \right\|_{L^p(\Omega \times H^s(S))} \left\| G \right\|_{L^{p',q'}}.
\]

Taking the supremum over all \(G \in \mathcal{I}_{\text{elem}}\) yields the result.

For the proof of the final assertion, note that \(L^1(S)\) is not a UMD space. However, for any \(1 \leq p < \infty\), the one-sided decoupling inequality

\[
\left( \mathbb{E} \left\| \int_{[0,t] \times B} F \ d\tilde{N} \right\|_{L^1(S)}^p \right)^{\frac{1}{p}} \lesssim_{p,q} \left( \mathbb{E} \left\| \int_{[0,t] \times B} F \ d\tilde{N} \right\|_{L^1(S)}^q \right)^{\frac{1}{q}}
\]

still holds, see [5]. The remainder of the proof is the same as in the case \(q > 1\).

\[\square\]

**Remark 3.6.** It is clear from the proof that the inequality

\[
\left( \mathbb{E} \left\| \int_{[0,t] \times B} F \ d\tilde{N} \right\|_{L^s(S)}^p \right)^{\frac{1}{p}} \lesssim_{p,q} \left\| F \right\|_{L^{p,q}}
\]

is valid if \(p = 1, 1 \leq q < \infty\).

4. Preliminaries on noncommutative \(L^q\)-spaces

We begin by reviewing some facts on noncommutative \(L^q\)-spaces. References for proofs of the results presented below can be found in the survey [29]. Let \(\mathcal{M}\) be a von Neumann algebra acting on a complex Hilbert space \(H\), which is equipped with a normal, semi-finite faithful trace \(\tau\). We say that a closed, densely defined linear operator \(x\) on \(H\) is affiliated with the von Neumann algebra \(\mathcal{M}\) if \(ux = xu\) for any unitary element \(u\) in the commutant \(\mathcal{M}'\) of \(\mathcal{M}\). For such an operator we define its distribution function by

\[
d(v; x) = \tau(\chi_v)(v, \infty) \quad (v \geq 0),
\]

where \(\chi_v\) is the spectral measure of \(v\). The decreasing rearrangement of \(x\) is defined by

\[
\mu_t(x) = \inf\{v > 0 : d(v; x) \leq t\} \quad (t \geq 0).
\]

We call \(x\) \(\tau\)-measurable if \(d(v; x) < \infty\) for some \(v > 0\). We let \(S(\tau)\) denote the linear space of all \(\tau\)-measurable operators. One can show that \(S(\tau)\) is a metrizable, complete topological \(*\)-algebra with respect to the measure topology. Moreover, the trace \(\tau\) extends to a trace (again denoted by \(\tau\)) on the set \(S(\tau)_+\) of positive \(\tau\)-measurable operators by setting

\[
\tau(x) = \int_0^\infty \mu_t(x) \, dt \quad (x \in S(\tau)_+).
\]

For \(0 < q < \infty\) we define

\[
\|x\|_{L^q(\mathcal{M})} = (\tau(\|x\|^q)_+)^{\frac{1}{q}} \quad (x \in S(\tau)).
\]

The linear space \(L^q(\mathcal{M}, \tau)\) of all \(x \in S(\tau)\) satisfying \(\|x\|_{L^q(\mathcal{M})} < \infty\) is called the noncommutative \(L^q\)-space associated with the pair \((\mathcal{M}, \tau)\). We usually denote \(L^q(\mathcal{M}, \tau)\) by \(L^q(\mathcal{M})\) for brevity. The map \(\|\cdot\|_{L^q(\mathcal{M})}\) in (37) defines a norm (or
q-norm if 0 < q < 1) on the space $L^q(\mathcal{M})$ under which it becomes a Banach space (respectively, quasi-Banach space). It can alternatively be viewed as the completion of $\mathcal{M}$ in the (quasi-)norm $\| \cdot \|_{L^q(\mathcal{M})}$. We use the expression $L^\infty(\mathcal{M})$ to denote $\mathcal{M}$ equipped with its operator norm. By (36) and using that $\mu(|x|^q) = \mu(x)^q$, the noncommutative $L^q$-(quasi-)norm can alternatively be computed as

$$
\|x\|_{L^q(\mathcal{M})} = \left( \int_0^\infty \mu_e(x)^q \, dt \right)^{\frac{1}{q}} \quad (x \in L^q(\mathcal{M})).
$$

Below we shall use the following facts. First recall Hölder’s inequality: if 0 < $q, r, s \leq \infty$ are such that $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$ and $x \in L^q(\mathcal{M})$, $y \in L^r(\mathcal{M})$, then $xy \in L^s(\mathcal{M})$ and

$$
\|xy\|_{L^s(\mathcal{M})} \leq \|x\|_{L^r(\mathcal{M})} \|y\|_{L^q(\mathcal{M})}.
$$

For 1 ≤ $q < \infty$ and $\frac{1}{q} + \frac{1}{r} = 1$, the familiar duality $L^q(\mathcal{M})^* = L^q(\mathcal{M})$ holds isometrically, with the duality bracket given by $\langle x, y \rangle = \tau(xy)$. In particular, $L^q(\mathcal{M})$ is reflexive if and only if 1 < $q < \infty$ and $L^q(\mathcal{M}) = \mathcal{M}_e$ isometrically, where $\mathcal{M}_e$ is the predual of $\mathcal{M}$. We recall that $L^q(\mathcal{M})$ is a UMD Banach space if and only if 1 < $q < \infty$. If 1 ≤ $q < \infty$, then $L^q(\mathcal{M})$ has type $\min\{q, 2\}$ and cotype $\max\{q, 2\}$.

We conclude this section by describing the column and row spaces and their conditional versions. Let 1 ≤ $q < \infty$. For a finite sequence $(x_i)$ in $L^q(\mathcal{M})$ we define

$$
\|(x_i)\|_{L^q(\mathcal{M}; \mathcal{E})} = \left\| \left( \sum_{i=1}^n x_i^* x_i \right)^{\frac{1}{2}} \right\|_{L^q(\mathcal{M})};
$$

$$
\|(x_i)\|_{L^q(\mathcal{M}; \mathcal{E})} = \left\| \left( \sum_{i=1}^n x_i x_i^* \right)^{\frac{1}{2}} \right\|_{L^q(\mathcal{M})}.
$$

Given $x_1, \ldots, x_n$, we let $\text{diag}(x_i)$, $\text{row}(x_i)$ and $\text{col}(x_i)$ denote the matrix with the $x_i$ on its diagonal, first row and first column, respectively, and zeroes elsewhere. Let $\mathcal{M} \otimes B(l^2)$ be the von Neumann tensor product equipped with its product trace $\tau \otimes \text{Tr}$. By noting that

$$
\left\| \left( \sum_{i=1}^n x_i^* x_i \right)^{\frac{1}{2}} \right\|_{L^q(\mathcal{M})} = \|\text{col}(x_i)\|_{L^q(\mathcal{M}; \mathcal{E})};
$$

$$
\left\| \left( \sum_{i=1}^n x_i x_i^* \right)^{\frac{1}{2}} \right\|_{L^q(\mathcal{M})} = \|\text{row}(x_i)\|_{L^q(\mathcal{M}; \mathcal{E})},
$$

one sees that the expressions in (39) define two norms on the linear space of all finitely nonzero sequences in $L^q(\mathcal{M})$. The completion of this space in these norms are called the column and row space, respectively.

We shall need a conditional version of these two spaces. Let 1 ≤ $q < \infty$. Suppose that $\mathcal{N}$ is a von Neumann algebra equipped with a normal, semi-finite faithful trace $\sigma$ and let $\mathcal{K}$ be a von Neumann subalgebra such that $\sigma|_{\mathcal{K}}$ is again semi-finite. Let $\mathcal{E} : \mathcal{N} \to \mathcal{K}$ be the conditional expectation with respect to $\mathcal{K}$. For a finite sequence $(x_i)$ in $\mathcal{N}$ we define

$$
\|(x_i)\|_{L^q(\mathcal{N}; \mathcal{E}; \mathcal{E})} = \left\| \left( \sum_{i=1}^n \mathcal{E}|x_i|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\mathcal{N})};
$$

$$
\|(x_i)\|_{L^q(\mathcal{N}; \mathcal{E}; \mathcal{E})} = \left\| \left( \sum_{i=1}^n \mathcal{E}|x_i|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\mathcal{N})}.
$$

Using techniques from Hilbert $C^*$-modules it was shown by M. Junge [13] that

$$
\{(x_i)_{i=1}^n : x_i \in \mathcal{N}, n \geq 1, \|(x_i)\|_{L^q(\mathcal{N}; \mathcal{E}; \mathcal{E})} < \infty\}
$$

and

$$
\{(x_i)_{i=1}^n : x_i \in \mathcal{N}, n \geq 1, \|(x_i)\|_{L^q(\mathcal{N}; \mathcal{E}; \mathcal{E})} < \infty\}$$
We denote by $D$ sequences (42) and (43). By (41) we have the duality $\langle f, \cdot \rangle$ for any finite sequence $(E)$. The associated conditional expectation. Under the identification (44), the element $L$ show that for any $1 < q < \infty$ $L \leq L^{\infty}$ that, for any $1 < q < \infty$ extends to an isometric isomorphism $L$. We refer to Section 2 of [13] for more information.

Let $K$ be the von Neumann subalgebra of $N$, equipped with the tensor product trace $E \otimes \tau$. Let us recall that, for any $1 \leq q < \infty$, the map defined on simple functions in the Bochner space $L^q(\Omega; L^q(M))$ by

$$I_q\left(\sum_i \chi_{A_i} x_i\right) = \sum_i \chi_{A_i} \otimes x_i$$

extends to an isometric isomorphism

$$L^q(\Omega; L^q(M)) = L^q(L^\infty(\Omega) \otimes M).$$

Let $\mathcal{K}$ be the von Neumann subalgebra of $\mathcal{N}$ given by $\mathcal{K} = C1 \otimes M$ and let $\mathcal{E}$ be the associated conditional expectation. Under the identification (44), the element $\mathcal{E}(f)$ coincides with the Bochner integral $E(f)$, whenever $f \in L^q(\mathcal{N})$. In particular, for any finite sequence $(f_i)$ in $\mathcal{N}$,

$$\|f_i\|_{L^q(\mathcal{N}; L^q(\Omega))} = \|f_i\|_{S_{q,c}}, \quad \|f_i\|_{L^q(\mathcal{N}; L^q(\Omega))} = \|f_i\|_{S_{q,r}}.$$  

We denote by $D_{p,q}$, $S_{q,c}$ and $S_{q,r}$ the completion of the linear space of all finite sequences $(f_i)$ of random variables in $L^\infty(\Omega; L^q(M))$ with respect to the norms in (42) and (43). By (41) we have the duality

$$(S_{q,c})^* = S_{q',r}, \quad (S_{q,r})^* = S_{q',c} \quad (1 < q < \infty, \frac{1}{q} + \frac{1}{q'} = 1).$$

We are now ready to state the extension of Theorem 1.2...
Theorem 5.1. Let $1 < p, q < \infty$. If $(\xi_i)$ is a finite sequence of independent, mean zero $L^q(M)$-valued random variables, then

$$\left( \mathbb{E} \left| \sum_{i} \xi_i \right|_L^p \right)^{\frac{1}{p}} \simeq_{p,q} \| (\xi_i) \|_{s_{p,q}},$$

where $s_{p,q}$ is given by

- $S_{q,c} \cap S_{q,r} \cap D_{p,q} \cap D_{p,q}$ if $2 \leq q < p < \infty$;
- $S_{q,c} \cap S_{q,r} \cap (D_{q,q} + D_{p,q})$ if $2 \leq p \leq q < \infty$;
- $(S_{q,c} \cap S_{q,r} \cap D_{p,q})$ if $1 < p < 2 \leq q < \infty$;
- $(S_{q,c} + S_{q,r} + D_{q,q} \cap D_{p,q})$ if $1 < q < 2 \leq p < \infty$;
- $S_{q,c} + S_{q,r} + D_{q,q} + D_{p,q}$ if $1 < q \leq p \leq 2$;
- $S_{q,c} + S_{q,r} + D_{q,q} + D_{p,q}$ if $1 < p < 2 \leq 2$.

Remark 5.3. Theorem 5.1 was proved by Buchholz that

To prove Theorem 5.1 we shall need to generalize Lemma 2.1 and Theorem 2.3.

Let us first recall the noncommutative version of Khintchine’s inequalities (9).

To prove Theorem 5.2 we shall need to generalize Lemma 2.1 and Theorem 2.3.

Let $(r_i)$ be a Rademacher sequence and fix $1 \leq p < \infty$. If $2 \leq q < \infty$, then, for any finite sequence $(x_i)$ in $L^q(M)$,

$$\left( \mathbb{E} \left| \sum_{i} r_i x_i \right|_L^p \right)^{\frac{1}{p}} \leq K_{p,q} \max \left\{ \left\| \left( \sum_{i} |x_i|^2 \right)^{\frac{1}{2}} \right\|_{L^q(M)} \right\}$$

and

$$\left( \mathbb{E} \left| \sum_{i} r_i x_i \right|_L^{q/2} \right)^{\frac{1}{q}} \geq \min \left\{ \left\| \left( \sum_{i} |x_i|^2 \right)^{\frac{1}{2}} \right\|_{L^q(M)} \right\}.$$
Lemma 5.4. Let \((\xi_i)\) be a finite sequence of independent, mean zero \(L^2(\mathcal{M})\)-valued random variables. If \(1 \leq p, q < 2\),
\[
\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_{L^p(\mathcal{M})}^p\right)^{\frac{1}{p}} \leq 4 \inf \left\{ \left(\sum_i \mathbb{E}\left|\xi_i\right|^2\right)^{\frac{1}{2}} \left\|\sum_i \mathbb{E}\left|\xi_i\right|^2\right\|_{L^q(\mathcal{M})} \right\},
\]
where the infimum is taken over all sequences \((\eta_i) \in S_{q,c}\) and \((\theta_i) \in S_{q,r}\) such that \(\xi_i = \eta_i + \theta_i\). On the other hand, if \(2 \leq p, q < \infty\), then
\[
2 \left(\mathbb{E}\left\|\sum_i \xi_i\right\|_{L^p(\mathcal{M})}^p\right)^{\frac{1}{p}} \geq \max \left\{ \left(\sum_i \mathbb{E}\left|\xi_i\right|^2\right)^{\frac{1}{2}} \left\|\sum_i \mathbb{E}\left|\xi_i\right|^2\right\|_{L^q(\mathcal{M})} \right\}.
\]

Proof. Suppose \(1 \leq p, q < 2\). Let \((\alpha_i)\) be a finite sequence in \(S_{q,c}\) of independent, mean zero \(L^2(\mathcal{M})\)-valued random variables. By symmetrization \([10]\) and Theorem 5.2,
\[
\left(\mathbb{E}\left\|\sum_i \alpha_i\right\|_{L^p(\mathcal{M})}^p\right)^{\frac{1}{p}} \leq 2 \left(\mathbb{E}\left\|\sum_i r_i \alpha_i\right\|_{L^p(\mathcal{M})}^p\right)^{\frac{1}{p}}
\]
\[
\leq 2 \left(\mathbb{E}\left\|\left(\sum_i |\alpha_i|^2\right)^{\frac{p}{2}}\right\|_{L^{q,c}(\mathcal{M})}\right)^{\frac{1}{p}}
\]
\[
= 2 \left(\mathbb{E}\left\|\sum_i |\alpha_i|^2\right\|_{L^{q,c}(\mathcal{M})}\right)^{\frac{1}{2}}
\]
\[
\leq 2 \left(\mathbb{E}\left\|\sum_i |\alpha_i|^2\right\|_{L^{q,c}(\mathcal{M})}\right)^{\frac{1}{2}} = 2 \left(\sum_i \mathbb{E}|\alpha_i|^2\right)^{\frac{1}{2}} \left\|\sum_i \mathbb{E}|\alpha_i|^2\right\|_{L^q(\mathcal{M})}.
\]

Note that in the final two inequalities we apply Jensen’s inequality and \([17]\), respectively, using that \(\frac{p}{2}, \frac{q}{2} < 1\). Applying this for \((\alpha_i)\) yields
\[
\left(\mathbb{E}\left\|\sum_i \alpha_i\right\|_{L^p(\mathcal{M})}^p\right)^{\frac{1}{p}} \leq \left(\sum_i \mathbb{E}|\alpha_i|^2\right)^{\frac{1}{2}} \left\|\sum_i \mathbb{E}|\alpha_i|^2\right\|_{L^q(\mathcal{M})}.
\]

Let \((\eta_i)\) and \((\theta_i)\) be finite sequences in \(S_{q,c}\) and \(S_{q,r}\), respectively, such that \(\xi_i = \eta_i + \theta_i\), then \(
\xi_i = \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i).
\)
Since \((\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i))\) and \((\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i))\) are sequences of independent, mean zero random variables, we obtain by the triangle inequality and the above,
\[
\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_{L^p(\mathcal{M})}^p\right)^{\frac{1}{p}} \leq 2 \left(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\right)^{\frac{1}{2}} \left\|\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\right\|_{L^q(\mathcal{M})}.
\]

Therefore, by the triangle inequality in \(S_{q,c}\) and \(S_{q,r}\) we find
\[
\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_{L^p(\mathcal{M})}^p\right)^{\frac{1}{p}} \leq 2 \left(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i)|^2\right)^{\frac{1}{2}} \left\|\sum_i \mathbb{E}|\mathbb{E}(\eta_i)|^2\right\|_{L^q(\mathcal{M})}
\]
\[
+ \left(\sum_i \mathbb{E}|\mathbb{E}(\theta_i^*|\xi_i)|^2\right)^{\frac{1}{2}} \left\|\sum_i \mathbb{E}|\mathbb{E}(\theta_i^*)|^2\right\|_{L^q(\mathcal{M})}
\]
\[
\leq 4 \left(\sum_i \mathbb{E}|\eta_i|^2\right)^{\frac{1}{2}} \left\|\sum_i \mathbb{E}|\eta_i|^2\right\|_{L^q(\mathcal{M})} + \left(\sum_i \mathbb{E}|\theta_i|^2\right)^{\frac{1}{2}} \left\|\sum_i \mathbb{E}|\theta_i|^2\right\|_{L^q(\mathcal{M})}.
\]

Note that the final step follows directly from Kadison’s inequality for (noncommutative) conditional expectations if \(\eta_i, \theta_i\) are, in addition, in \(L^{\infty} \otimes \mathcal{M}\). For general
\[ \eta_t \text{ and } \theta_t \text{ as above the asserted inequality then follows by a density argument. This proves the first statement.} \]

Suppose now that \( 2 \leq p, q < \infty \). By symmetrization (10) and Theorem 6.2

\[
2 \left( \mathbb{E} \left\| \sum_i \xi_i \right\|_{L^q(M)}^p \right)^{\frac{1}{p}} \\
\geq \left( \mathbb{E} \mathbb{E} \left\| \sum_i r_i \xi_i \right\|_{L^q(M)}^p \right)^{\frac{1}{p}} \\
\geq \max \left\{ \left( \mathbb{E} \left\| \left( \sum_i |\xi_i|^2 \right)^{\frac{1}{2}} \right\|_{L^q(M)}^p \right)^{\frac{1}{p}}, \left( \mathbb{E} \left\| \left( \sum_i |\xi_i|^2 \right)^{\frac{1}{2}} \right\|_{L^q(M)}^p \right)^{\frac{1}{p}} \right\} \\
= \max \left\{ \left( \mathbb{E} \left\| \sum_i |\xi_i|^2 \right\|_{L^q(M)}^p \right)^{\frac{1}{p}}, \left( \mathbb{E} \left\| \sum_i |\xi_i|^2 \right\|_{L^q(M)}^p \right)^{\frac{1}{p}} \right\} \\
\geq \max \left\{ \left( \sum_i \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{p}} \left\| \sum_i \mathbb{E} |\xi_i|^2 \right\|_{L^q(M)}^p \right\} \\
= \max \left\{ \left( \sum_i \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{p}} \left\| \sum_i \mathbb{E} |\xi_i|^2 \right\|_{L^q(M)}^p \right\}. \]

This completes the proof. \( \square \)

For our discussion in Section 8 we will keep track of the dependence of the constants on \( p \) and \( q \) in the inequalities (48) and (49) below.

**Theorem 5.5.** Suppose that \( 2 \leq p, q < \infty \). If \( (\xi_i) \) is a finite sequence of independent, mean zero \( L^q(M) \)-valued random variables, then

\[
\left( \mathbb{E} \left\| \sum_i \xi_i \right\|_{L^q(M)}^p \right)^{\frac{1}{p}} \leq C_{p,q} (1 + \sqrt{2} \max \left\{ \left( \sum_i \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{2}} \left\| \sum_i \mathbb{E} |\xi_i|^2 \right\|_{L^q(M)}^p \right\}^{\frac{1}{p}} \}
\]

\[
\left( \mathbb{E} \left\| \sum_i \xi_i \right\|_{L^q(M)}^p \right)^{\frac{1}{p}} \geq \frac{1}{2} \max \left\{ (\kappa_{p,q})^{-1} \left( \mathbb{E} \left( \sum_i \xi_i \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \right\}.
\]

Moreover, if \( \kappa_{p,q} \) is the constant in (47) then

\[
\left( \mathbb{E} \left\| \sum_i \xi_i \right\|_{L^q(M)}^p \right)^{\frac{1}{p}} \geq \frac{1}{2} \max \left\{ (\kappa_{p,q})^{-1} \left( \mathbb{E} \left( \sum_i \xi_i \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \right\}.
\]

**Proof.** We first prove (49). By Lemma 5.4.

\[
\max \left\{ \left( \mathbb{E} \left\| \sum_i \xi_i \right\|_{L^q(M)}^p \right)^{\frac{1}{p}} \left\| \sum_i \mathbb{E} |\xi_i|^2 \right\|_{L^q(M)}^p \right\} \leq 2 \left( \mathbb{E} \left\| \sum_i \xi_i \right\|_{L^q(M)}^p \right)^{\frac{1}{p}}.
\]

By successively applying the cotype \( q \) inequality for \( L^q(M) \), Kahane’s inequalities (11) and (10) we see that

\[
\left( \mathbb{E} \left( \sum_i \xi_i \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \left( \sum_i r_i \xi_i \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq \kappa_{q,p} \left( \mathbb{E} \left( \sum_i r_i \xi_i \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq 2\kappa_{q,p} \left( \mathbb{E} \left( \sum_i \xi_i \right)^{\frac{1}{2}} \right)^{\frac{1}{p}}.
\]
We refer to [3] for a proof that (50) holds with constant 1.

We now prove (55). By (10) and Theorem 5.2 we have
\[
\left( \mathbb{E} \left( \sum_i |\xi_i|^4 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq 2K_{p,q} \max \left\{ \left( \mathbb{E} \left( \sum_i |\xi_i|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} , \left( \mathbb{E} \left\| \left( \sum_i \xi_i^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)}^{\frac{1}{p}} \right)^{\frac{1}{p}} \right\},
\]
(51)
By the triangle inequality in \(L^\frac{q}{2}(\Omega; L^\frac{p}{2}(M))\) it follows that
\[
\left( \mathbb{E} \left( \sum_i |\xi_i|^2 - \mathbb{E}|\xi_i|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \left( \sum_i |\xi_i|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} + \left( \mathbb{E}|\xi|^2 \right)^{\frac{1}{p}},
\]
(52)
We now estimate the first term on the right hand side. By applying (10) and Theorem 5.2 once again we obtain
\[
\left( \mathbb{E} \left( \sum_i |\xi_i|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq 2K_{p,q} \left( \mathbb{E} \left( \sum_i \left| \xi_i^2 \right| - \mathbb{E}|\xi_i|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}}
\]
(53)
where the final inequality is a consequence of the triangle inequality in \(L^\frac{q}{2}(\Omega; L^\frac{p}{2}(M; l^2))\).

Note that the second term on the right hand side is smaller than the first one. Indeed,
\[
\left( \sum_i |\mathbb{E}|\xi_i|^2 | \right)^{\frac{1}{2}} \leq \|\text{col}(\mathbb{E}|\xi_i|^2)\|_{L^\frac{p}{2}(M \otimes B(l^2))}.
\]
(54)
Write \(x = \text{col}(\xi_i)\) and \(y = \text{diag}(\xi_i)\) for the matrices with the \(|\xi_i|\) in their first column and diagonal, respectively, and zeroes elsewhere. By the noncommutative Hölder inequality [53],
\[
\left( \mathbb{E} \left( \sum_i |\xi_i|^4 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq (\mathbb{E} \left| x^* y^* y x \right|^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq (\mathbb{E} \left| y^* y \right|^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \left| y \right|_{L^p(M \otimes B(l^2))} \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \left| y \right|_{L^p(M \otimes B(l^2))} \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \left| x \right|_{L^p(M \otimes B(l^2))} \right)^{\frac{1}{p}}.
Collecting our estimates (52), (53), (54) and (55), we obtain the quadratic equation

\[ a^2 \leq (2C_{p,q})ab + c^2, \]

where we set \( a = \left( \mathbb{E}\left( \sum_i |\xi_i|^2 \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \), \( b = \left( \mathbb{E}\left( \sum_i |\xi_i|^2 \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \) and \( c = \left( \sum_i \mathbb{E}|\xi_i|^2 \right)^{\frac{1}{q}} \) in (55). Solving this quadratic equation we obtain

\[ a \leq \frac{1}{2}(2C_{p,q}b + ((2C_{p,q}b)^2 + 4c^2)^{\frac{1}{2}}) \leq \frac{1 + \sqrt{2}}{2} \max\{2C_{p,q}b, 2c\}, \]

that is,

\[
\left( \mathbb{E}\left( \sum_i |\xi_i|^2 \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq (1 + \sqrt{2}) \max \left\{ \left( \mathbb{E}\left( \sum_i |\xi_i|^2 \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}, C_{p,q} \left( \mathbb{E}\left( \sum_i |\xi_i|^2 \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \right\}.
\]

Applying this to the sequence \((\xi_i^*)\) we obtain

\[
\left( \mathbb{E}\left( \sum_i |\xi_i^*|^2 \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \leq (1 + \sqrt{2}) \max \left\{ \left( \mathbb{E}\left( \sum_i |\xi_i^*|^2 \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}, C_{p,q} \left( \mathbb{E}\left( \sum_i |\xi_i^*|^2 \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \right\}.
\]

The inequality (48) now follows from (51). \(\square\)

Note that even if \(\mathcal{M}\) is commutative, the proof of Theorem 5.5 is different from the one presented for Theorem 5.3. We are now ready to prove Theorem 5.4.

**Proof.** (Of Theorem 5.4) Observe that the spaces \(S_{q,c}, S_{q,r}, D_{p,q}\) and \(D_{q,q}\) have dense intersection, and therefore the duality of these individual spaces imply together with (18) that

\[
(s_{p,q})^* = s_{p',q'}, \frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1,
\]

with associated duality bracket

\[
\langle (f_i), (g_i) \rangle = \sum_i \mathbb{E}\tau(f_i, g_i).
\]

Thus, the lower estimates \(\gtrsim_{p,q}\) in (45) can be deduced from the upper ones using the duality argument presented in (21) and (22).

The upper estimates \(\lesssim_{p,q}\) follow essentially as in the proof of Theorem 1.2 once we replace the use of Lemma 2.4 and Theorem 5.3 by their noncommutative versions Lemma 5.4 and Theorem 5.5 respectively. The straightforward modifications are left to the reader. \(\square\)

Before deducing Itô isomorphisms for Poisson stochastic integrals taking values in a noncommutative \(L^p\)-space from Theorem 5.4, we take the opportunity to observe some moment estimates for the norm of a sum of random matrices.
Let us recall the following noncommutative Khintchine inequality for the operator norm of a Rademacher sum of matrices. Let \( d_1, d_2 \in \mathbb{N} \) and set \( d = \min\{d_1, d_2\} \). If \( x_1, \ldots, x_n \) are \( d_1 \times d_2 \) random matrices, then there is a constant \( C_{p,d} \) depending only on \( p \) and \( d \) such that

\[
\left( \mathbb{E} \left\| \sum_{i=1}^{n} r_i x_i \right\|^p \right)^{\frac{1}{p}} \leq C_{p,d} \max \left\{ \left\| \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \right\|, \left\| \left( \sum_{i=1}^{n} |x_i^*|^2 \right)^{\frac{1}{2}} \right\| \right\}.
\]

Indeed, this inequality can readily be deduced from the noncommutative Khintchine inequalities for Schatten spaces. Since \( \|x\|_{\log d} \leq e\|x\|_{\log d} \) for any \( d_1 \times d_2 \) matrix \( x \),

\[
\left( \mathbb{E} \left\| \sum_{i=1}^{n} r_i x_i \right\|^p \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \left\| \sum_{i=1}^{n} r_i x_i \right\|_{\log d}^p \right)^{\frac{1}{p}} \leq K_{p,\log d} \max \left\{ \left\| \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \right\|_{\log d}, \left\| \left( \sum_{i=1}^{n} |x_i^*|^2 \right)^{\frac{1}{2}} \right\|_{\log d} \right\} \leq eK_{p,\log d} \max \left\{ \left\| \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \right\|, \left\| \left( \sum_{i=1}^{n} |x_i^*|^2 \right)^{\frac{1}{2}} \right\| \right\}.
\]

By the remark following Theorem 5.2, if \( 2 \leq \log d \leq p \) then

\[
C_{p,d} \leq eK_{p,\log d} \leq e\sqrt{2} \sqrt{p - 1}
\]

and \( C_{p,d} \leq e\sqrt{\log d} \) if \( 2 \leq p \leq \log d \).

**Remark 6.1.** The Khintchine inequality (57) cannot hold with a constant independent of the dimensions \( d_1, d_2 \). Indeed, it was shown by Seginer ([33], Theorem 3.1) that there is an absolute constant \( C \) such that for any \( a_{ij}, i = 1, \ldots, d_1 \), \( j = 1, \ldots, d_2 \) in \( \mathbb{C} \) and any \( 1 \leq p \leq 2 \log \max\{d_1, d_2\} \) the rank one matrices \( x_{ij} = a_{ij} \otimes e_{ij} \) satisfy

\[
\left( \mathbb{E} \left\| \sum_{i,j} r_{ij} x_{ij} \right\|^p \right)^{\frac{1}{p}} \leq C(\log d)^{\frac{1}{2}} \max \left\{ \left\| \left( \sum_{i,j} |x_{ij}|^2 \right)^{\frac{1}{2}} \right\|, \left\| \left( \sum_{i,j} |x_{ij}^*|^2 \right)^{\frac{1}{2}} \right\| \right\}.
\]

Moreover, the order of growth \( (\log d)^{\frac{1}{2}} \) in (57) is optimal ([33], Theorem 3.2).

**Theorem 6.2.** Let \( 2 \leq p < \infty \). If \( (\xi_i) \) is a finite sequence of independent, mean zero \( d_1 \times d_2 \) random matrices, then

\[
\left( \mathbb{E} \left\| \sum_{i} \xi_i \right\|^p \right)^{\frac{1}{p}} \leq (1 + \sqrt{2})C_{p,d} \max \left\{ \left\| \left( \sum_{i} \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{2}} \right\|, \left\| \left( \sum_{i} \mathbb{E} |\xi_i^*|^2 \right)^{\frac{1}{2}} \right\| \right\},
\]

where \( d = \min\{d_1, d_2\} \). The reverse inequality holds with constant \( 2^{1+1/p} \).

**Proof.** By repeating the proof of Theorem 5.5 using (56) instead of the noncommutative Khintchine inequality (46), we find

\[
\left( \mathbb{E} \left\| \sum_{i} \xi_i \right\|^p \right)^{\frac{1}{p}} \leq (1 + \sqrt{2})C_{p,d} \max \left\{ \left\| \left( \sum_{i} \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{2}} \right\|, \left\| \left( \sum_{i} \mathbb{E} |\xi_i^*|^2 \right)^{\frac{1}{2}} \right\| \right\},
\]

Clearly, \( \|\text{diag}(\xi_i)\| = \max_{i} \|\xi_i\| \), so the first assertion holds.
For the second assertion, let \((r_i)\) be a Rademacher sequence on a probability space \((\Omega_r, F_r, P_r)\). Then,
\[
\left( \mathbb{E} \max_i \| \xi_i \|_X^p \right)^{\frac{1}{p}} = \left( \mathbb{E} \mathbb{E}_r \max_i \| r_i \xi_i \|_X^p \right)^{\frac{1}{p}}
\leq 2^{\frac{1}{p}} \left( \mathbb{E} \mathbb{E}_r \left\| \sum_i r_i \xi_i \right\|_X^p \right)^{\frac{1}{p}} \leq 2^{1+\frac{1}{p}} \left( \mathbb{E} \| \sum_i \xi_i \|_X^p \right)^{\frac{1}{p}},
\]
where the first inequality follows by the Lévy-Octaviani inequality in [17], Proposition 1.1.1. Moreover,
\[
\left\| \left( \sum_i \mathbb{E} |\xi_i|^2 \right)^{\frac{1}{2}} \right\| = \left\| \mathbb{E}_r \sum_{i,j} r_i r_j \xi_i^* \xi_j \right\| \leq \left( \mathbb{E}_r \left\| \sum_{i,j} r_i r_j \xi_i^* \xi_j \right\| \right)^{\frac{1}{p}}
\leq \left( \mathbb{E}_r \left( \sum_i r_i \xi_i \right)^2 \right)^{\frac{1}{2}} \leq 2 \left( \mathbb{E} \| \sum_i \xi_i \|^p \right)^{\frac{1}{p}},
\]
where the final inequality follows from \((10)\). □

As a consequence, we find the following moment inequalities for the norm of a random matrix with independent, mean zero entries.

**Corollary 6.3.** Let \(2 \leq p < \infty\). Suppose that \(x_{ij}, i = 1, \ldots, d_1, j = 1, \ldots, d_2\) are independent, mean zero random variables in \(L^p(\Omega)\). If \(x\) is the \(d_1 \times d_2\) random matrix \((x_{ij})\), then
\[
\left( \mathbb{E} \|x\|^p \right)^{\frac{1}{p}} \leq (1 + \sqrt{2}) C_{p,d} \max \left\{ \max_{i,j} \left( \sum_{i=1}^{d_1} \mathbb{E} x_{ij}^2 \right)^{\frac{1}{p}}, \max_{i,j} \left( \sum_{j=1}^{d_2} \mathbb{E} x_{ij}^2 \right)^{\frac{1}{p}} \right\},
\]
with \(C_{p,d} < 2e \max\{ \sqrt{\log d}, \sqrt{2p-1} \}\) as in Theorem 6.2.

**Proof.** Let \(e_{ij}\) be the \(d_1 \times d_2\) matrix having 1 in entry \((i, j)\) and zeroes elsewhere. Set \(y_{ij} = x_{ij} \otimes e_{ij}\), then \((y_{ij})\) is a doubly indexed sequence of independent, mean zero random matrices and \(x = \sum_{i,j} y_{ij}\). Notice that
\[
y_{ij}^* y_{ij} = x_{ij}^2 \otimes e_{ij} e_{ij} = x_{ij}^2 \otimes e_{ij},
\]
so
\[
\left\| \left( \sum_{i,j} \mathbb{E} |y_{ij}|^2 \right)^{\frac{1}{2}} \right\| = \left\| \sum_j \left( \sum_i \mathbb{E} x_{ij}^2 \right)^{\frac{1}{2}} \otimes e_{jj} \right\| = \max_j \left( \sum_i \mathbb{E} x_{ij}^2 \right)^{\frac{1}{2}}.
\]
Moreover,
\[
y_{ij}^* y_{ij} = x_{ij}^2 \otimes e_{ij} e_{ij} = x_{ij}^2 \otimes e_{ii}
\]
and therefore
\[
\left\| \left( \sum_{i,j} \mathbb{E} |y_{ij}|^2 \right)^{\frac{1}{2}} \right\| = \left\| \sum_i \left( \sum_j \mathbb{E} x_{ij}^2 \right)^{\frac{1}{2}} \otimes e_{ii} \right\| = \max_i \left( \sum_j \mathbb{E} x_{ij}^2 \right)^{\frac{1}{2}}.
\]
Finally, it is clear that
\[
\left( \mathbb{E} \max_{i,j} \|y_{ij}\|_X^p \right)^{\frac{1}{p}} = \left( \mathbb{E} \max_{i,j} |x_{ij}|^p \right)^{\frac{1}{p}}.
\]
The result now follows from Theorem 6.2. □
In [18] Latała showed that there is a universal constant $C > 0$ such that

$$\mathbb{E}\|x\| \leq C \left( \max_{i=1,\ldots,d_1} \left( \sum_{j=1}^{d_2} \mathbb{E} x_{ij}^2 \right)^{\frac{1}{2}} + \max_{j=1,\ldots,d_2} \left( \sum_{i=1}^{d_1} \mathbb{E} x_{ij}^2 \right)^{\frac{1}{2}} + \left( \sum_{i,j} \mathbb{E} x_{ij}^4 \right)^{\frac{1}{4}} \right),$$

for any random matrix $x = (x_{ij})$ with independent, mean zero entries in $L^4(\Omega)$. To compare this result to Corollary 6.3 observe that (59) implies together with (19) that there is a universal constant $C > 0$ such that for all $1 \leq p < \infty$,

$$\mathbb{E}\|x\|^p \leq C \frac{p}{\log p} \left( \max_{i=1,\ldots,d_1} \left( \sum_{j=1}^{d_2} \mathbb{E} x_{ij}^2 \right)^{\frac{1}{2}} + \max_{j=1,\ldots,d_2} \left( \sum_{i=1}^{d_1} \mathbb{E} x_{ij}^2 \right)^{\frac{1}{2}} + \left( \sum_{i,j} \mathbb{E} x_{ij}^4 \right)^{\frac{1}{4}} \right).$$

The upper bound in Corollary 6.3 exhibits different growth behaviour in $p$ and does not contain the factor $\left( \sum_{i,j} \mathbb{E} x_{ij}^4 \right)^{\frac{1}{4}}$. In particular, the bound (59) is applicable to random matrices having entries with infinite fourth moment. On the other hand, note that the bound in (60) is of order $\sqrt{d}$ for matrices with uniformly bounded entries, which is optimal for $d \to \infty$ (see the discussion in [18]). Through the use of the noncommutative Khintchine inequality in our proof, we incur an extra factor of order $\sqrt{\log d}$. As the order $(\log d)^{\frac{1}{4}}$ of the constant in (59) is optimal, this additional factor is an inevitable product of our method.

7. Itô-isomorphisms: Noncommutative $L^q$-spaces

We now present an extension of Theorem 1.1 for integrands taking values in a noncommutative $L^q$-space. In the statement of our main result we will use the following noncommutative $L^2$-valued $L^q$-spaces, which were introduced by Pisier in [28] and treated in more detail in [14]. For any simple function on a measure space $(E, \mathcal{E}, \mu)$ with values in $L^q(M)$, $F = \sum x_i \chi_{E_i}$ say, we set

$$\|F\|_{L^q(M; L^2(\mathbb{R}_+ \times J))} = \left\| \left( \sum_i |x_i|^2 \mu(E_i) \right)^{\frac{1}{2}} \right\|_{L^q(M)},$$

$$\|F\|_{L^q(M; L^q(\mathbb{R}_+ \times J))} = \left\| \left( \sum_i |x_i|^2 \mu(E_i) \right)^{\frac{1}{q}} \right\|_{L^q(M)}.$$

It can be shown that these expressions define two norms on the simple functions, and we let $L^q(M; L^2(E))$ and $L^q(M; L^2(E)_c)$ denote the respective completions in these norms. Alternatively, one can describe these spaces as complemented subspaces of $L^q(\mathcal{M} \otimes B(L^2(E)))$ and in this way one can show that for $1 < q, q' < \infty$ with $\frac{1}{q} + \frac{1}{q'} = 1$,

$$L^q(M; L^2(E)_c)^* = L^{q'}(M; L^2(E)_c), \quad (L^q(M; L^2(E))_c)^* = L^{q'}(M; L^2(E)).$$

We refer to Chapter 2 of [14] for details. Now, for any $1 \leq p, q < \infty$ we set

$$S_{q,c}^p = L^p(\Omega; L^q(M; L^2(\mathbb{R}_+ \times J)_c)), \quad S_{q,c}^p = L^p(\Omega; L^q(M; L^2(\mathbb{R}_+ \times J))).$$

Since $L^q(M; L^2(\mathbb{R}_+ \times J)_c)$ and $L^q(M; L^2(\mathbb{R}_+ \times J))$ can be identified with closed subspaces of $L^q(\mathcal{M} \otimes B(L^2(\mathbb{R}_+ \times J)))$ they are reflexive if $1 < q < \infty$. Therefore, it follows from (17) and (15) that for any $1 < p, q < \infty$, $1 + \frac{1}{q} = 1, \frac{1}{q} + \frac{1}{q'} = 1$.

If $\mathcal{M}$ is commutative, then $S_{q,c}^p$ and $S_{q,c}^{p'}$ coincide and are equal to the Bochner space $S_{q}^p = L^p(\Omega; L^q(S; L^2(\mathbb{R}_+ \times J)))$ considered earlier.
We are now ready to prove our main theorem.

**Theorem 7.1.** Let $1 < p, q < \infty$. For any $B \in \mathcal{F}$, any $t \geq 0$ and for any simple, adapted $L^q(\mathcal{M})$-valued process $F$, 

$$
\left( \mathbb{E} \sup_{0 \leq s \leq t} \left\| \int_{[0,s] \times B} F \, d\tilde{N} \right\|_{L^p(\mathcal{M})}^p \right)^{1/p} \approx_{p,q} \left\| F \chi_{[0,t] \times B} \right\|_{\mathcal{I}_{p,q}},
$$

where $\mathcal{I}_{p,q}$ is given by

$$
\mathcal{I}_{p,q} = \left\{ \mathcal{S}_{q,c} \cap \mathcal{S}_{q,r} \cap \mathcal{D}_{p,q} \cap \mathcal{D}_{p,q}^c, \quad \text{if} \quad 2 \leq p \leq q < \infty; \right.
$$

$$
\mathcal{S}_{q,c} \cap \mathcal{S}_{q,r} \cap (\mathcal{D}_{q,q} \cap \mathcal{D}_{p,q}^c), \quad \text{if} \quad 2 \leq p \leq q < \infty; \right.
$$

$$
(\mathcal{S}_{q,c} \cap \mathcal{S}_{q,r} \cap \mathcal{D}_{q,q}) \cap \mathcal{D}_{p,q}, \quad \text{if} \quad 1 < p < 2 \leq q < \infty; \right.
$$

$$
(\mathcal{S}_{q,c} \cap \mathcal{S}_{q,r} \cap \mathcal{D}_{q,q}^c) \cap \mathcal{D}_{p,q}, \quad \text{if} \quad 1 < q \leq 2; \right.
$$

$$
\mathcal{S}_{q,c} \cap \mathcal{S}_{q,r} + \mathcal{D}_{q,q} \cap \mathcal{D}_{p,q}, \quad \text{if} \quad 1 < p < 2 \leq q < \infty; \right.
$$

$$
\mathcal{S}_{q,c} \cap \mathcal{S}_{q,r} + \mathcal{D}_{q,q} + \mathcal{D}_{p,q}, \quad \text{if} \quad 1 < p \leq q \leq 2.
$$

**Proof.** The proof is similar to the one for Theorem 1.1, we sketch the main differences in the cases $2 \leq q \leq p < \infty$ and $1 < p \leq q \leq 2$. Since $L^q(\mathcal{M})$ is a UMD space if $1 < q < \infty$, by the decoupling inequality \[\text{[12]}\] and Doob’s maximal inequality it suffices to show that

$$
\left( \mathbb{E} \mathbb{E}_c \left\| \int_{[0,t] \times B} F \, d\tilde{N} \right\|_{L^q(\mathcal{M})}^p \right)^{\frac{1}{p}} \approx_{p,q} \left\| F \chi_{[0,t] \times B} \right\|_{\mathcal{I}_{p,q}}.
$$

Let $F$ be the simple adapted process given in \[\text{[20]},\] taking Remark \[\text{[3.5]}\] into account. We may assume that $t = t_{i+1}$ and $B = \cup_{j=1}^n A_j$. We write $\tilde{N}_{i,j} := N((t_i, t_{i+1}) \times A_j)$ for brevity.

**Case 2 \leq q \leq p < \infty:** Let $y_{i,j} = \sum_{k=1}^n F_{i,k} x_{i,j,k}$ and let $\mathcal{G} = \mathcal{F} \times \{\Omega_c, \emptyset\}$. The doubly indexed sequence $d_{i,j} = y_{i,j} \tilde{N}_{i,j}$ is conditionally independent with respect to $\mathcal{G}$ and mean zero, and moreover

$$
\int_{[0,t] \times B} F \, d\tilde{N} = \sum_{i,j} d_{i,j}.
$$

Applying Theorem \[\text{[5.1]}\] conditionally on $\mathcal{G}$ and taking $L^p$-norms yields

$$
\left( \mathbb{E} \mathbb{E}_c \left\| \sum_{i,j} d_{i,j} \right\|_{L^q(\mathcal{M})}^p \right)^{\frac{1}{p}} \leq_{p,q} \left\| \sum_{i,j} d_{i,j} \right\|_{L^q(\mathcal{M})}^p,
$$

$$
\leq_{p,q} \max \left\{ \left( \mathbb{E} \mathbb{E}_c \left\| \sum_{i,j} \mathbb{E}_G \left| d_{i,j} \right|^2 \right\|_{L^q(\mathcal{M})} \right)^{\frac{1}{2}}, \left( \mathbb{E} \mathbb{E}_c \left\| \left( \sum_{i,j} \mathbb{E}_G \left| d_{i,j} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\mathcal{M})} \right)^{\frac{1}{2}}, \left( \mathbb{E} \mathbb{E}_c \left\| \sum_{i,j} \mathbb{E}_G \left| d_{i,j} \right|^{\frac{p}{q}} \right\|_{L^q(\mathcal{M})} \right)^{\frac{1}{p}}, \left( \sum_{i,j} \mathbb{E} \mathbb{E}_c \left\| d_{i,j} \right\|_{L^q(\mathcal{M})} \right)^{\frac{1}{p}} \right\}
$$

$$
\approx_{p,q} \max \{\|F\|_{s_{p,q}^c \cap \mathcal{D}_{p,q}}, \|F\|_{s_{p,q}^c \cap \mathcal{D}_{p,q}}, \|F\|_{s_{p,q}^c \cap \mathcal{D}_{p,q}}, \|F\|_{s_{p,q}^c \cap \mathcal{D}_{p,q}}, \|F\|_{p_{p,q}^c} \}.
$$

The final step follows by the calculations in \[\text{[52], [33]}\] and \[\text{[54]}\] and \[\text{[54]}\].

**Case 1 < p \leq q \leq 2:** Let $\mathcal{T}_{\text{elem}}$ denote the algebraic tensor product

$$
\mathcal{T}_{\text{elem}} = L^\infty(\Omega) \otimes L^\infty(\mathbb{R}_+) \otimes (L^1 \cap L^\infty)(\mathcal{F}) \otimes (L^1 \cap L^\infty)(\mathcal{M}).
$$

Since this linear space is dense in $\mathcal{S}_{q,c} \cap \mathcal{S}_{q,r} \cap \mathcal{D}_{p,q}$ and $\mathcal{D}_{p,q}$, we can find, for any fixed $\varepsilon > 0$, a decomposition $F = F_1 + F_2 + F_3 + F_4$ with $F_i \in \mathcal{T}_{\text{elem}}$ such that

$$
\|F\|_{p_{p,q}} = \|F_1\|_{s_{p,q}^c \cap \mathcal{D}_{p,q}} + \|F_2\|_{s_{p,q}^c \cap \mathcal{D}_{p,q}} + \|F_3\|_{p_{p,q}^c} + \|F_4\|_{p_{p,q}^c} - \varepsilon.
$$
Let $\mathcal{A}$ be the sub-$\sigma$-algebra of $\mathcal{B}(\mathbb{R}) \times \mathcal{F}$ generated by the sets $(t_i, t_{i+1}) \times A_j$. Then $E(F_n|\mathcal{A})$ is of the form

$$E(F_n|\mathcal{A}) = \sum_{i,j,k} F_{i,k,\alpha} \chi_{(t_i, t_{i+1})} \chi_{A_j} x_{i,j,k,\alpha} \quad (\alpha = 1, 2, 3, 4).$$

Let $y_{i,j,\alpha} = \sum_{k=1}^{n} F_{i,k,\alpha} x_{i,j,k,\alpha}$ and set $d_{i,j,\alpha} = y_{i,j,\alpha} \tilde{N}_{i,j}$, so that

$$\int_{[0,t] \times B} F \, d\tilde{N} = \sum_{i,j} d_{i,j,1} + d_{i,j,2} + d_{i,j,3} + d_{i,j,4}.$$

By the computations in (32), (33) and (34), $H$ the estimate (63) for $E$ the bound Theorem 7.1 implies (63) but not vice versa. In [22] Marinelli and Röckner proved (64)

$$\|d_{i,j,\alpha}\|_{S^\alpha_{p,r}} = |A| \leq \|E(F_1|\mathcal{A})\|_{S^\alpha_{p,r}} \leq \|F_1\|_{S^\alpha_{p,r}},$$

$$\|d_{i,j,\alpha}\|_{S^\alpha_{p,r}} = \|E(F_2|\mathcal{A})\|_{S^\alpha_{p,r}} \leq \|F_2\|_{S^\alpha_{p,r}},$$

$$\|d_{i,j,\alpha}\|_{D^\alpha_{p,q}} \leq \|E(F_3|\mathcal{A})\|_{D^\alpha_{p,q}} \leq \|F_3\|_{D^\alpha_{p,q}},$$

$$\|d_{i,j,\alpha}\|_{D^\alpha_{p,q}} \leq \|E(F_4|\mathcal{A})\|_{D^\alpha_{p,q}} \leq \|F_4\|_{D^\alpha_{p,q}}.$$

By applying Theorem 5.1 conditionally on $G$ we conclude that

$$\left( \mathbb{E} \left( \int_{[0,t] \times B} F \, d\tilde{N} \right)^p \right)^{\frac{1}{p}} \leq \|F_1\|_{S^\alpha_{p,r}} + \|F_2\|_{S^\alpha_{p,r}} + \|F_3\|_{D^\alpha_{p,q}} + \|F_4\|_{D^\alpha_{p,q}}$$

For the reverse inequality, observe that if $p', q'$ are the Hölder conjugates of $p$ and $q$, then in view of (63) and (13) we have $I_{p,q} = I_{p', q'}$, with associated duality bracket

$$\langle F, G \rangle = \int_{\Omega \times \mathbb{R} \times \mathcal{A}} \tau(FG) \, d\mathbb{P} dtd\nu.$$

The reverse inequality can therefore be deduced using the duality argument in [35] given in the proof of Theorem 1.1.

Let us make a detailed comparison of our main result with the existing results in the literature. We restrict our attention to [2, 10, 21, 22] and refer to the papers for earlier achievements. In [21], Marinelli, Prevôt and Röckner showed using Itô’s formula that if $H$ is a Hilbert space and $2 \leq p < \infty$, then

$$\left( \mathbb{E} \sup_{0 \leq s \leq t} \| \int_{[0,s] \times B} F \, d\tilde{N} \right)^{\frac{p}{2}} \leq \|F_1\|_{S^\alpha_{p,r}} + \|F_2\|_{S^\alpha_{p,r}} + \|F_3\|_{D^\alpha_{p,q}} + \|F_4\|_{D^\alpha_{p,q}}$$

Due to the first term on the right hand side, this estimate is only near optimal. Indeed, since

$$\left( \mathbb{E} \left( \int_{[0,t] \times B} \|F\|_{L^p(H)}^2 \, d\nu \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \int_{[0,t]} \left( \int_{B} \|F\|_{L^p(H)}^2 \, d\nu \right)^{\frac{p}{2}} \, dt \right)^{\frac{1}{p}},$$

Theorem 7.1 implies (63) but not vice versa. In [22] Marinelli and Röckner proved the bound

$$\left( \mathbb{E} \sup_{0 \leq s \leq t} \| \int_{[0,s] \times B} F \, d\tilde{N} \right)^{\frac{p}{2}} \leq \|F_1\|_{S^\alpha_{p,r}} + \|F_2\|_{S^\alpha_{p,r}} + \|F_3\|_{D^\alpha_{p,q}} + \|F_4\|_{D^\alpha_{p,q}}$$

valid for any $2 \leq p < \infty$. This result is deduced by a Fubini-type argument from the estimate (63) for $H = \mathbb{R}$. Of course, such an argument can only work if $p = q$ (in our notation). Observe that the optimal bound in Theorem 7.1 improves upon
Also note that the constants in (62) do not depend on $t$, in contrast to (63) and (64). Finally, let us recall the following bounds valid for a Banach space $X$ with martingale type $1 < q \leq 2$. Brzeźniak and Hausenblas showed ([2], Corollary B.6) that if $1 < p \leq q$ then

$$
(65) \quad \left( \mathbb{E} \sup_{0 \leq s \leq t} \left\| \int_{[0,s] \times B} F \, d\tilde{N} \right\|_X^p \right)^{\frac{1}{p}} \lesssim_{p,q,X} \left( \mathbb{E} \left( \int_{[0,t] \times B} \|F\|_X^q \, dt \, d\nu \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}.
$$

Moreover, Hausenblas proved ([10], Proposition 2.14) that if $p = q^n$ for some $n \in \mathbb{N}$, then

$$
(66) \quad \left( \mathbb{E} \sup_{0 \leq s \leq t} \left\| \int_{[0,s] \times B} F \, d\tilde{N} \right\|_X^p \right)^{\frac{1}{p}} \lesssim_{p,q,X} \left( \mathbb{E} \left( \int_{[0,t] \times B} \|F\|_X^n \, dt \, d\nu \right)^{\frac{p}{n}} \right)^{\frac{1}{p}} + \left( \mathbb{E} \int_{[0,t] \times B} \|F\|_X^n \, dt \, d\nu \right)^{\frac{1}{p}}.
$$

If $X = L^2(\mathcal{M})$, so that $q = 2$, and $p = 2^n$ then (66) reproduces the optimal upper bound in Theorem 7.1. In all other cases, however, both (65) and (66) yield sub-optimal bounds for $L^q$-spaces.

**Acknowledgement**

I would like to thank Sonja Cox, Jan Maas, Jan van Neerven and Mark Veraar for helpful discussions on this topic.

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