Improved sensing with a single qubit

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We consider quantum metrology with arbitrary prior knowledge of the parameter. We demonstrate that a single sensing two-level system can act as a virtual multi-level system that offers increased sensitivity in a Bayesian, single-shot, metrology scenario and that allows one to estimate (arbitrary) large parameter values by avoiding phase wraps. This is achieved by making use of additional degrees of freedom or auxiliary systems not participating in the sensing process. The joint system is manipulated by intermediate control operations in such a way that an effective Hamiltonian, with an arbitrary spectrum, is generated that mimics the spectrum of a multi-level system interacting with the field. We show how to use additional internal degrees of freedom of a single trapped ion to achieve a high-sensitivity magnetic field sensor for fields with arbitrary prior.

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Introduction. — Quantum metrology deals with one of the most fundamental problems in science—the measurement or estimation of an unknown physical quantity. In the widely studied case of phase estimation it was found that quantum mechanics allows for a significant improvement in the achievable precision over classical schemes. This applies both to the local estimation scenario \cite{1-3}, where the experiment is repeated sufficiently many times without any modification or the phase is already known to lie within a localized set of values, and to reference frame alignment scenario \cite{4, 5}, where the phase is totally unknown prior to the estimation and a single measurement is performed. In this letter we consider the problem of frequency estimation, a variant of the phase estimation scenario, which, however, reduces to neither of the two scenarios above. This is due to the additional freedom of choosing the interaction time at each run, and to the occurrence of phase ambiguities (known as phase-wraps) due to the periodicity of unitary evolution.

Here we propose a simple scheme that uses a single sensing particle and additional auxiliary systems (or internal degrees of freedom) that do not participate in the sensing process together with (fast) quantum control. We note that one can also assume without loss of generality that the prior probability density has zero mean. The precision of the estimation is quantified by the mean posterior variance, or mean squared error (MSE) of the states. In the case where multiple sensing systems interact with the field. We show how to use additional internal degrees of freedom of a single trapped ion to achieve a high-sensitivity magnetic field sensor for fields with arbitrary prior.

We remark that in the absence of noise quantum control does not allow to increase the quantum Fisher information (QFI), since it only depends on the extremal eigenvalues of the Hamiltonian. However, as we demonstrate, the QFI is never the directly relevant quantity in the noiseless Bayesian frequency estimation as it corresponds to the second order expansion of the precision in the inquiry time, and it is always better to increase the latter beyond this regime.

Background and setting of the problem. — Consider a single sensing system whose evolution is governed by $H_S = \omega \hbar$, where the frequency $\omega$ is known with prior probability $p_0(\omega)$ with variance $V_0$. The goal is to estimate the value of $\omega$ by preparing the system in an initial state, $\rho = |\psi\rangle \langle \psi|$, allowing it to freely evolve under the unitary $U = \exp(-i t H_S)$ for a time $t$ before measuring the state of the system, gaining some information about $\omega$, and updating the corresponding probability density. We note that one can also assume without loss of generality that the prior probability density has zero mean. The precision of the estimation is quantified by the mean posterior variance, or mean squared error (MSE) $\langle V(\omega) \rangle_t = \sum_\omega p_m \text{Var}(p(\omega|m))$, where $p(\omega|m)$ is the posterior probability density upon obtaining outcome $m$ and we also define the estimators $\omega_m = \int \omega p(\omega|m) d\omega$.

Consider the decrease of the MSE with time. For well-behaved priors it can be lower-bounded using the Bayesian Cramér-Rao inequality \cite{6}

\begin{equation}
\langle V(\omega) \rangle_t \geq \frac{1}{\mathcal{I}(p_0(\theta)) + t^2 \mathcal{F}_h(\rho)},
\end{equation}

where $\mathcal{F}_h(\rho)$ is the QFI of the state $\rho$ with respect to the generator $h$, and $\mathcal{I}(p_0(\theta)) \equiv \int \frac{d\omega}{p_0(\omega)} \frac{\partial p_0(\omega)}{\partial \omega}^2 d\omega$ is the (time independent) Fisher information of the prior. Moreover, for short times the average variance decreases quadratically with $t$ with a pre-factor given by

\begin{equation}
\langle V(\omega) \rangle_t = V_0 - t^2 V_0^2 \mathcal{F}_h(\rho) + O(t^3),
\end{equation}

as we show in App. 1.

However, $\langle V(\omega) \rangle_t$ cannot decrease indefinitely with $t$. Indeed, Holevo’s theorem \cite{7} places an upper bound on the mutual information $I(m : \omega)$ between the measurement outcome and the parameter, based on the maximal
amount of information that the output state can encode. As we show in App. 2 Holevo’s bound implies the following bound on the average variance

$$\langle V(\omega) \rangle_t \geq \frac{1}{d^2} \frac{e^{2H(p_0(\omega))}}{2\pi e} = \frac{1}{d^2} \frac{e^{2H(p_0(\omega))}}{2\pi e}, \quad (3)$$

where $H(p_0(\omega))$ is the Shannon entropy of the prior. For a Gaussian prior of width $\sigma$ Eq. (3) simplifies to

$$\langle V(\omega) \rangle_t \geq \left( \frac{\sigma}{2} \right)^2.$$  

Hence, the number of non-degenerate levels of the probe state (with respect to the eigenbasis of $H_S$) directly bounds the attainable precision, regardless of the inquiry time $t$. To determine the optimal time to perform the measurement we must first determine the optimal measurement.

For von Neumann measurements, i.e., POVMs comprised of orthogonal rank-one projectors $\hat{E}_m$, it was shown in [8] that the measurement minimizing the MSE has to satisfy

$$\Gamma_t S_t + S_t \Gamma_t = 2\eta_t, \quad (4)$$

where $\Gamma_t = \int e^{-itH_S} \rho e^{itH_S} \rho(\omega) d\omega$, $\eta_t = \int e^{-itH_S} \rho e^{itH_S} \rho(\omega) d\omega$, and $S_t = \sum_m \omega_m \hat{E}_m$. In addition, the average variance attained by such a measurement is given by

$$\langle V(\omega) \rangle_t = V_0 - \text{tr} \eta_t S_t. \quad (5)$$

Moreover, it was later proven that von Neumann measurements are generally optimal for the MSE figure of merit [9], so that the solution $S_t = \sum_m \omega_m \hat{E}_m$ of (4) contains the optimal measurement and estimators, and provides the best attainable MSE via Eq. (5).

Notice for any $H_S$ with a finite gap the decrease in the average variance vanishes in the limit $t \to \infty$ whenever the prior has a bounded spectrum. This follows from the fact that $\lim_{t \to \infty} \eta_t = 0$. Intuitively this happens because beyond a certain time phase wrap ambiguities render the information obtained via the measurement useless for the MSE. Hence, there exists an optimal time, $t_{\text{max}} = \arg\max_t \langle V(\omega) \rangle_t$ for which the error is minimized, and after which the phase begins to wrap due to the periodicity of the unitary evolution $e^{itH_S}$.

Now consider the case where the probe is a single qubit and $h = \frac{1}{2} \sigma_z$. The optimal initial state of the probe always lies on the equator of the Bloch sphere (see App. 3) and, due to the symmetry of the problem with respect to simultaneous rotations about the $z$-axis, one can choose it to be $|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$. This results in

$$\Gamma_t = \frac{1}{2} \begin{pmatrix} \rho_0(t) & 1 \\ 1 & \rho_0(t) \end{pmatrix}, \quad \eta_t = \frac{1}{2} \begin{pmatrix} 0 & i \rho_0(t) \\ -i \rho_0(t) & 0 \end{pmatrix}, \quad (6)$$

where $\rho_0(t) = \int e^{it\omega} \rho_0(\omega) d\omega$, and $\rho_0(t) = \frac{d}{dt} \rho_0(t)$. For this simple case an exact analytical solution for both the optimal measurement strategy, $S_t$, as well as the optimal time can be found in App. 4. For a symmetric prior $p_0(\omega) = p_0(-\omega)$, the solution of Eq. (4) is particularly simple: $S_t = 2\eta_t$, and the average variance reads

$$\Delta \langle V(\omega) \rangle_t \equiv \langle V(\omega) \rangle_t - V_0(\omega) = \text{tr} \eta_t S_t = (\rho_0(t))^2, \quad (7)$$

making it easy to determine $t_{\text{max}}$.

A good strategy, then, is to prepare the system in the state $|+\rangle$, subject it to the noiseless unitary evolution and measure the subsequent state shortly before $t_{\text{max}}$. Depending on the measurement outcome, $m$, we update our knowledge of the parameter to $p_1(\omega) = p(\omega|m)$, and repeat the protocol again for the updated distribution [10]. Though this strategy is easy to formulate, finding the optimal performance and sequence of optimal measurement settings is infeasible beyond a certain number of steps, as one has to keep track of the exact form of the prior at each step and for each measurement outcome (just keeping track of the variance is insufficient). This is why to assess the performance one is constrained to numerical search [11]. In App. 5 we find the optimal performance under the wrong assumption that the prior knowledge stays Gaussian at every step. Our result suggests that after a large number of steps such a recursive strategy achieves the optimal scaling precision $\langle V(\omega) \rangle_t \propto 1/T^2$, where $T$ is the total running time.

However, the practical applicability of such sequential schemes may be strongly limited, as they require multiple measurements and preparation procedures. In most set-ups state preparation and measurements are rather time consuming, and hence cannot be neglected. For instance, for ion traps cooling of ions to the motional ground state as well as measurements in an ion trap setup take place on a much longer time scale than unitary control pulses. In practice, this limits the minimal evolution time, and the number of intermediate measurements.

We now present an alternative scheme that requires only one state preparation step at the beginning and one measurement step at the end, whereas at intermediate times only some limited unitary control is required.

Degeneracy lifting.— We consider a single sensing qubit whose evolution is governed by $H_S$. Moreover, we assume that there are additional auxiliary systems or internal degrees of freedom available that are not affected by $H_S$ and that we have fast control over the sensing plus ancilla systems or additional degrees of freedom. We describe the latter as a $n$-level system, so that the total Hamiltonian is given by $H_{SA} = H_S \otimes \mathbb{1}_A$, and a basis of the joint system is given by $|\psi_{j,k}\rangle = |j\rangle \otimes |k\rangle$ with $j \in \{0, 1\}$, $k \in \{1, 2, \ldots, n\}$. The total Hamiltonian still has only two distinct eigenvalues, however the eigenvectors are $n$-fold degenerate, $|\psi_{0,0}\rangle$ and $|\psi_{1,1}\rangle$.

We first show that for such a set-up, with one sensing qubit plus an $n$-level auxiliary system, and any fixed evolution time $t$ one can obtain an effective $2n$-level system that evolves under an effective Hamiltonian with an arbitrary spectrum $|\lambda\rangle$ with $2n$ eigenvalues, where $|\lambda| \leq 1$, $H_{\text{eff}} = \frac{1}{\omega} \sum_{\ell=1}^{2n} \lambda_\ell |\ell\rangle \langle \ell|$. That is, an $n$-level state

$$|\psi\rangle = \sum_{\ell=1}^{2n} c_\ell |\ell\rangle \quad (8)$$

evolves to $|\psi_t\rangle = \sum_{\ell=1}^{2n} c_\ell e^{-i\frac{\omega}{2} \lambda_\ell t} |\ell\rangle$. In addition, any state $|\psi\rangle$ with real positive coefficients $c_\ell$ can be mimicked within the same procedure [12] as we now explain.

This is achieved by preparing the probe in initial state $\rho_0(\omega) = p_0(\omega) |\psi_0\rangle \langle \psi_0|$, and at times $t_{\text{jk}} \leq t_{\text{jk+1}}$ control pulses $U_{j,k}$ rotating between the levels $|j,1\rangle$ and $|j+1,0\rangle$ are applied. In the basis $\{|j,1\}, |j+1,0\rangle$, the control unitary reads $U_{j,k} = \begin{pmatrix} \cos(\theta_{j,k}) - \sin(\theta_{j,k}) & -\sin(\theta_{j,k}) \\ \sin(\theta_{j,k}) & \cos(\theta_{j,k}) \end{pmatrix}$. At time $t$ each level $|j,k\rangle$ picks the phase exp $\left(-i\frac{\omega}{2} \lambda_{j,k}(1-2j)\right)$.
with \( \lambda_{j,k} = 1 - 2 \frac{\omega}{\tau} \), resulting from the sum of phases picked before and after the time \( t_{j,k} \) when the control was applied (to shorten the notation we formally introduced \( t_{j,0} = 0 \)). Consequently the final state reads

\[
|\psi\rangle_t = \sum_{j,k} c_{j,k} e^{-i\frac{\pi}{2} \lambda_{j,k} t} |j,k\rangle,
\]

with \( c_{0,k} = s_0 \sin(\theta_{0,k}) \prod_{j<k} \cos(\theta_{0,j}) \) and \( c_{1,k} = c_0 \sin(\theta_{1,k}) \prod_{j<k} \cos(\theta_{1,j}) \) for all \( k > 1 \), and \( c_{0,1} = c_0 \prod \cos(\theta_{1,j}) \) and \( c_{1,1} = s_0 \prod \cos(\theta_{0,k}) \). It is easy to see that with this parametrization the coefficients \( \{c_{j,k}\} \) span all normalized vectors with positive coefficients. Remark that any adaptive sequential scheme with \( N \) measurement steps and predefined running times for each step corresponds to a particular case of the general strategy above, with a \( H_{\text{eff}} \) that has \( 2N \) levels (with additional constraints on the spectrum), a flat input state \(|\psi\rangle = \frac{1}{\sqrt{2N}} \sum_i |\ell\rangle \) and the final measurement that is also constrained.

We numerically evaluate the performance of the above strategy for the case of a symmetric prior density and a state \(|\psi\rangle \) with real coefficients. For this case one finds that \( \Gamma_t \) is a real symmetric matrix, whereas \( \eta_k = i \eta_A \) with \( \eta_A \) a real antisymmetric matrix. Solving Eq. (5) to find the optimal measurement yields \( S = iA \) with real antisymmetric \( A \). To find the optimal pair \( \{|\psi\rangle, A\} \) we numerically minimize the variance decrease for a fixed time \( t \) and number of non-degenerate levels \( n \). Fig. 1 shows the performance of our protocol for up to \( n = 9 \) equally spaced levels. To simplify the analysis from now on we shall only consider the case of engineering Hamiltonians with an equally gapped spectrum. This simplification is based on numerical evidence suggesting that equally gapped spectra are nearly optimal in time regimes where the average variance grows, i.e., before one encounters the phase wraps. However, we stress that the degeneracy lifting protocol described above allows one to engineer any spectrum.

We would like to note that fast control techniques to boost estimation precision using a single spin only were also considered in [13]. However, contrary to our approach, the surrounding auxiliary particles were also assumed to be subjected to the evolution.

On-the-fly spectral engineering.— As mentioned earlier in order to achieve the maximum gain in the average variance one must measure the system at the right time \( t_{\text{max}} \), before one encounters the phase wrap. Reciprocally for any fixed measurement time \( t \) there exists an optimal spectrum and state minimizing the MSE that can be attained with the strategy above. However this scheme requires the measurement time to be known before the sensing process even starts. We now show how the same can be achieved with an on-the-flight scheme, where in order to avoid phase wraps and repel the bound of Eq. (3) additional virtual levels are introduced during the sensing process, keeping the extractable MSE nearly optimal for all times.

As suggested by Eq. (2) and can be seen explicitly form Fig. 1, no additional levels are needed in order to minimize the average variance for \( t \leq \tau \). In the next time-slot, between \( \tau \) and \( 2\tau \), the three-level strategy performs optimally. But from \( 2\tau \) to \( 3\tau \) the performance of the three-level state starts to decay, however it is enough to go for the four-level one in order to recover the optimal MSE. This is due to the fact that as time increases, one needs to populate more and more intermediate levels in order to avoid phase wraps. Indeed, the results of Fig. 1 show that the number of intermediate levels required to maintain an optimally decreasing average variance grows linearly with time (such that the spectral gap of the effective Hamiltonian times the inquiry time \( t \) stays constant).

Consider a stroboscopic scenario where the controls are performed at times which are multiples of \( \tau \) (for a Gaussian prior we heuristically determined a good time step duration to be around \( \tau = 0.775/\sigma \)). One initially prepares the state \(|+\rangle, \sigma \rangle \), i.e., an equal superposition of the \( \pm 1/2 \) eigenstates of the sensing Hamiltonian and ground state in the auxiliary degree of freedom. The state evolves for a time \( \tau \) and the two levels pick a phase \( e^{\pm i\omega/2} \). At this point in time the state is optimal for sensing—it attains the minimal variance among all states given the evolution time \( \tau \) letting this state evolve for a longer time results in a reduction in precision (see Fig. 1). Thus, at time \( \tau \) one intrudes an additional auxiliary level, by implementing the degeneracy lifting procedure described above and maps the state \(|\Psi_1\rangle = \sqrt{0.5} e^{i\omega/2} |\uparrow, 0\rangle + \sqrt{0.5} e^{-i\omega/2} |\downarrow, 0\rangle \) to \(|\Phi_1\rangle = e^{+i\omega/2} (\sqrt{0.3} |\uparrow, 0\rangle + \sqrt{0.2} |\downarrow, 1\rangle) + e^{-i\omega/2} (\sqrt{0.3} |\downarrow, 0\rangle + \sqrt{0.2} |\uparrow, 1\rangle) \) at time \( 2\tau \) the state evolves to \(|\Psi_2\rangle = e^{+2i\omega/2} \sqrt{0.3} |\uparrow, 0\rangle + \sqrt{0.4} |+1, 1\rangle + e^{-2i\omega/2} \sqrt{0.3} |\downarrow, 0\rangle \), which is optimal three-level-state for this time. This procedure can be carried on by introducing an additional auxiliary level at each time step and mapping \(|\Psi_2\rangle \) onto \(|\Psi_3\rangle \) chosen such that it evolves to the optimal state \(|\Phi_{k+1}\rangle \) for the next time step. In order to be able to do this it is necessary that the weights of all levels of the state \(|\Psi_k\rangle \) correspond to a reshuffling between adjacent levels of the state \(|\Psi_k\rangle \). Fig. 2 depicts the phase evolution of different levels and shows how the above procedure works for constructing the optimal states at each time step for up to nine additional intermediate levels. The attained precision is given in Fig. 1.

Whilst we did not perform the numerical optimization of the states for more intermediate levels, we note that this can be easily done using more powerful numerical
techniques such as those in [9, 14]. Alternatively, we note that the single states [4, 15] known to attain the optimal scaling of precision with time [16, 17] satisfy the weight reshuffling property and are thus achievable via this strategy.

Multi-qubit systems. We now turn to multi-qubit systems and show that intermediate local control also allows one to increase the effective number of levels, even without using additional degrees of freedom or auxiliary systems. The spectrum of an N-qubit system with respect to the Hamiltonian \( H = \omega \sum_{k=1}^{N} \sigma_z^{(k)} \) has \( N + 1 \) distinct eigenvalues \( \lambda_k = k \). The corresponding eigenstates are given by permutations of the state \( |0\rangle^\otimes N-k \otimes |k\rangle^\otimes k \). For each \( \lambda_k \), there are \( \binom{N}{k} \) eigenstates \( |\chi_{k,l}\rangle \). This degeneracy can be lifted by picking two states, \( |\chi_{k_1,l_1}\rangle \) and \( |\chi_{k_2,l_2}\rangle \) and swapping them at time \( xT \) of the total evolution time \( T \). This results in two new effective eigenvalues \( x\lambda_{k_1} + (1-x)\lambda_{k_2} \) and \( (1-x)\lambda_{k_1} + x\lambda_{k_2} \). In principle, any eigenvalue can be reached in this way. However, in this case there are limitations on the effective spectrum one can achieve, as the number of degenerate eigenstates is different for different eigenvalues. For \( N \) qubits one can achieve an arbitrary spectrum of exponentially many levels, where the spectrum is small, and only using the middle part of the spectrum where many degenerate eigenstates are available. When using only \( \{\lambda_k\} \) for \( N/4 \leq k \leq 3N/4 \), the spectral width is reduced by a factor of \( 1/2 \), and the number of available levels is given \( \binom{N}{N/4} \approx e^{0.56N} \).

By way of example, consider the simplest non-trivial case of two spin-1/2 particles where the overall spectrum of the Hamiltonian is \( \{1, 0, -1\} \) with a doubly degenerate 0 eigenvalue. The use of dynamical decoupling allows to lift the degeneracy of the 0-level at the price of reducing the spectral range, this allows to decrease the minimal attainable MSE by a factor of 1.36 (see App. 6).

Noisy estimation. Thus far our treatment assumes a noiseless Bayesian estimation scenario. However, the techniques outlined so far can be used in conjunction with fast control and error-correction [17], which serve to combat the effects of noise. For example, by engineering a spectrum such that each distinct level is doubly degenerate [18] one can implement the error-correcting schemes of [17] to eliminate all rank-one Pauli noise processes that act on the single sensing system. For general noise acting on the sensing system, part of the noise can still be eliminated with fast control and/or error-correction. Un-correctable noise will effectively act as correlated noise acting on the effective n-level system.

Realization with trapped ions A possible experimental implementation of our techniques can be realized in ion-trap set-ups to sense magnetic fields. There the internal electronic degrees of freedom of a single ion, \( |g\rangle, |e\rangle \), form the bare qubit, and the motional degrees of freedom of the ion in the trap, labeled by \( |k\rangle, 1 \leq k \leq n \), provide the auxiliary degrees of freedom. Alternatively one can also use more of the electronic levels of the ion that are in principle accessible. The Hamiltonian describing the evolution of both electronic and motional degrees of freedom is given by \( H = \frac{B}{2} \sigma_z \otimes \mathbb{I} \), where \( \sigma_z = |g\rangle \langle g| - |e\rangle \langle e| \) acts on the electronic (motional) degrees of freedom respectively. Thus, the total Hamiltonian contains just two distinct eigenvalues and, in principle, an infinite number of degenerate states for each eigenvalue, which correspond to the motional degrees of freedom of the ion in the trap.

In order to engineer an effective Hamiltonian with an arbitrary spectrum one needs to be able to couple the electronic and motional degrees of freedom of the ion. Moreover, in order to implement the protocols described above, one needs to be able to prepare arbitrary states of both the electronic and motional degrees of freedom as well as be able to perform arbitrary measurements on such states. In App. 7 we provide a detailed description of the corresponding eigenstates, as well as be able to perform arbitrary measurements on such states. We stress that all operations required belong to the standard repertoire of ion trap quantum information processing [19–21].

Conclusion We have demonstrated a single qubit system (a qubit), along with additional degrees of freedom, can be transformed into an effective multi-level system that offers increased sensitivity in a single parameter, noiseless Bayesian estimation scenario. The engineering of such an effective multi-level system can be accomplished on the fly, by suitably coupling the qubit and auxiliary degrees of freedom at the appropriate times. Furthermore, we have demonstrated how such on-the-fly engineering can be accomplished in ion trap experimental set-ups in order to perform high precision sensing of magnetic fields.

We believe that our techniques can be readily applied to multi-parameter estimation problems, such as the precise estimation of an orthogonal triplet of spatial directions. More importantly, we believe that our techniques are highly relevant in the construction of atomic clocks, where the ambiguity of phase wraps is a hindrance to the stability of atomic clocks.

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1. Recovery of the QFI for short times

Consider the Taylor expansion of Eq. (4) in terms of $t$. The two operators are given by

$$\Gamma_t = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \langle \omega^n \rangle [h, \rho]^{(n)} = \rho + \mathcal{O}(t^2) \quad (11)$$

$$\eta_t = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (\omega^{n+1}) [h, \rho]^{(n)} = -itV_0(\omega)[h, \rho] + \mathcal{O}(t^2), \quad (12)$$

where $[h, \rho]^{(n)} = [h, ..., [h, \rho],...]$ is the nested commutator with $h$ appearing $n$ times. This implies for the optimal measurement strategy

$$S_t = tV_0(\omega)\mathcal{L}_h(\rho) + \mathcal{O}(t^3), \quad (13)$$

where $\mathcal{L}_h(\rho)$ is the symmetric logarithmic derivative [22] given by

$$\rho\mathcal{L}_h(\rho) + \mathcal{L}_h(\rho)\rho = -2i[h, \rho], \quad (14)$$

yielding

$$\Delta V_t = \text{tr} S_t \eta_t = t^2V_0(\omega)^2F_h(\rho) + \mathcal{O}(t^3) \quad (15)$$

where $F_h(\rho)$ is the quantum Fisher information.

2. Entropic bound on the MSE

We now use Holevo’s theorem [7] to derive a lower bound on the mean average variance. Holevo’s theorem establishes a relation between the mutual information of the measurement outcomes and the parameter encoded in a quantum state $I(m : \omega)$, and the Von Neumann entropy of the mean state $S(\Gamma_t)$

$$I(m : \omega) \equiv H(p_0(\omega)) - \langle H(m : \omega) \rangle \leq S(\Gamma_t), \quad (16)$$

where $H(p_0(\omega)) = H_0$ is the Shannon entropy of the prior distribution and $\langle H(m : \omega) \rangle = \sum_m p_m H(p(\omega|m))$ is the mean Shannon entropy of the posterior. In addition, a trivial upper bound for the right hand side of Eq. (16) is given by the dimension, $d$, of the system, $S(\Gamma_t) \leq \ln(d)$. Rewriting Eq. (16)

$$\langle H(m : \omega) \rangle \geq H_0 - \ln(d), \quad (17)$$

allows us to obtain a bound for the MSE of the posterior distribution.

First notice that among all probability distributions with the same entropy the Gaussian has the least variance. Hence, for any distribution $p(\omega)$ the following inequality holds

$$\frac{1}{2} \ln \left( 2\pi e \text{Var}(p(\omega)) \right) \geq H(p(\omega)) \quad (18)$$

since $\ln(\sqrt{2\pi e \sigma})$ is the entropy of a Gaussian with variance $\sigma^2$. As the logarithm is a concave function, the same bound in Eq. (18) holds for the mean variance $\langle V(\omega) \rangle$ and mean entropy $\langle H(m : \omega) \rangle$ over an ensemble of posterior distributions $\{p_m, p(\omega|m)\}$. Consequently, one obtains

$$\langle V(\omega) \rangle \geq \frac{e^{2H_0}}{2\pi e} \geq \frac{1}{4 \pi} e^{2H_0}, \quad (19)$$

which for a Gaussian prior of width $\sigma$ takes a simpler form

$$\langle V(\omega) \rangle \geq \frac{\sigma^2}{d^2}. \quad (20)$$

3. Optimality of equatorial states

Consider any qubit state

$$|\psi_\varphi\rangle = \cos(\varphi) |0\rangle + \sin(\varphi) |1\rangle \quad (21)$$

and a POVM element $|n\rangle\langle n| = \frac{1}{2} (\sigma_z + \sigma \cdot n)$, we take $n_y = 0$ without loss of generality for the argument. The probability to observe the outcome $n$ given the evolution $U_\theta = e^{-\frac{i}{2}\sigma_\theta}$ is

$$p_{\varphi}(n|\theta) = \frac{1}{2} (1 + n_z \cos(2\varphi) + n_x \sin(2\varphi) \sin \theta)$$

$$= \sin(2\varphi) p_{+}(n|\theta) + \frac{1}{2} (1 - \sin(2\varphi) + n_z \cos(2\varphi)), \quad (22)$$

where the subscript $+$ stands for $\varphi = \pi/4$ (and thus $|\psi_{\pi/4}\rangle = |+\rangle$). Accordingly, for any von Neumann measurement the statistics $p_{\varphi}(\theta) = (p_{+}(n|\theta), p_{-}(-n|\theta))$ are given by

$$p_{\varphi}(\theta) = \sin(2\varphi)p_{+}(\theta) + (1 - \sin(2\varphi))q(\varphi, n_z), \quad (23)$$

where $q(\varphi, n_z)$ is $\theta$-independent noise. The measurement statistics in Eq. (23) can alternatively be reproduced as follows: one tosses a biased coin and in the case of heads (occurring with probability $\sin(2\varphi)$) runs the estimation with the state $|+\rangle$ yielding the statistics $p_{+}(\theta)$, whereas in the case of tails one generates a random outcome from $q(\varphi, n_z)$ and subsequently forgets about the result of the coin toss. From this construction it is obvious that it is always optimal to set $\sin(2\varphi) = 1$, which shows the optimality of the states on the equator for the estimation of the rotation generated by $\sigma_\varphi$. Note also that this argument is not restricted to von Neumann measurements nor to any particular cost-function.

4. Single qubit solution for general prior

Here we give the solution of Eq. (4) in the main text for a general prior $p_0(\omega)$. For ease of exposition we use the notation $t_r = \text{Re} (\tilde{p}_0(t))$ and $i_t = \text{Im} (\tilde{p}_0(t))$. The optimal measurement operator is given by
\[
S_t = \begin{pmatrix}
\frac{i\omega r_1 - i_r r_1}{2(i_t + r_t r_{-1})} & -\frac{i_1 + i_r (r_1 + 1)}{2(i_t + r_t + r_{-1})}\\
\frac{i_1 + i_r (r_1 + 1)}{2(i_t + r_t r_{-1})} & \frac{i \omega r_1 - i_r r_1}{2(i_t + r_t + r_{-1})}
\end{pmatrix},
\]
which leads to
\[
\text{tr} \eta_t S_t = \frac{2 i_1 r_1 r_1 + (i_1^2 - 1)(i_r^2) + (r_1^2 - 1)(r_r^2)}{2(i_t + r_t + r_{-1})}.
\]

5. Approximative sequential strategy

Here we determine the optimal sequential decrease of a\ sequential strategy under the assumption that the phase\ distribution stays Gaussian after every measurement
\[
p_k(\omega) = \frac{1}{\sqrt{2\pi V_k}} e^{-\omega^2/2V_k}.
\]
It is easy to link the variance at a step \(k\), \(V_k\), to the\ variance at \(V_{k+1}\). Given a measurement time \(t_k\) for the\ prior \(p_k(\omega)\) we get for \(V_{k+1}\), using Eq. (7),
\[
V_{k+1} = V_k(1 - V_k^2 t_k^2 e^{-i_1^2 V_k}).
\]
Hence at each step of the protocol the variance is reduced\ by the factor \(R_k = (1 - A_k e^{-A_k})\) that only depends on\ the product \(A_k = t_k^2 V_k\). A natural choice is to require\ that this product is constant, and the same for all steps, \(A_k = A\). Hence, \(R_k = R = (1 - A e^{-A})\), which requires that\ the time duration grows exponentially with the number of\ steps \(t_k^2 = A / V_k = e^{A^2 / V_k} \). This choice then yields for\ the variance and total time at step \(k\) the expressions
\[
V_k = R^k V_0 \quad \text{(28)}
\]
\[
T_k = \sum_{\ell=0}^{k-1} t_\ell = \sqrt{\frac{A}{V_0}} R^{-k/2} - 1 \quad \text{(29)}
\]
In particular one notes that asymptotically the precision\ scales quadratically with the total time
\[
(1/V) \approx \frac{(R^{-1/2} - 1)^2}{A} T^2, \quad \text{(30)}
\]
which is maximized by \((1/V) \approx 0.08 T^2\) for \(A \approx 0.63\).

6. Degeneracy lifting with two spins-1/2

Given a Gaussian prior \(p_0(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2}\) consider\ the simplest non-trivial degenerate case of two spin-1/2\ particles where the overall spectrum of the Hamiltonian is\ \{+1, 0, -1\} with a doubly degenerate 0 eigenvalue. The\ corresponding eigenstates are \{+1 : |↑↑\rangle, 0 : |↑↓\rangle and\ |↓↑\rangle, -1 : |↓↓\rangle\}. By performing intermediate Rabi flips\ between the states |↑↑\rangle with |↑↓\rangle, and |↓↑\rangle with |↓↓\rangle it is\ possible to lift the degeneracy between the two 0-levels\ (remark that this can be achieved with a simple π-pulse\ on the first qubit), but paying the price of slowing down\ the phase evolution of the outer levels +1 and -1. This\ becomes advantageous after a certain time as one can see\ from Fig. 3 and Fig. 4. In Fig. 3 we compare the\ attainable precision for the case of a degenerate and non-\ degenerate spectrum. In Fig. 4 we plot the optimal spec-\ trum for each running time as compared to the free phase\ evolution. When time increases further the availability of\ four levels is not sufficient to avoid a phase wrap, the\ minimal MSE can however be preserved by freezing the phase\ of each level at the optimal value. This can be done by\ repeatedly applying Rabi flips between states |↑↑\rangle with\ |↓↓\rangle, and |↑↓\rangle with |↓↑\rangle. The optimal final state is then given by
\[
|Ψ\rangle = e^{i1.41\omega 0.42} |↑↑\rangle + e^{i0.46\omega 0.57} |↓↑\rangle + e^{-0.46\omega 0.57} |↑↓\rangle + e^{-i1.41\omega 0.42} |↓↓\rangle,
\]
and the optimal measurement reads
\[
S = \omega_1 |\varphi_1^+ \rangle \langle \varphi_1^+| - \omega_1 |\varphi_1^- \rangle \langle \varphi_1^-| + \\
\omega_2 |\varphi_2^+ \rangle \langle \varphi_2^+| - \omega_2 |\varphi_2^- \rangle \langle \varphi_2^-| \quad (32)
\]
\[
|\varphi_1^\pm \rangle \approx \epsilon^{\pm i \omega_{10} 0.38} |\uparrow \uparrow \rangle + \epsilon^{\pm i \omega_{12} 0.59} |\downarrow \downarrow \rangle + \\
+ \epsilon^{\pm i \omega_{14} 0.59} |\uparrow \downarrow \rangle + \epsilon^{\pm i \omega_{10} 0.38} |\downarrow \uparrow \rangle
\]
\[
|\varphi_2^\pm \rangle \approx \epsilon^{\pm i \omega_{10} 0.38} |\uparrow \downarrow \rangle + \epsilon^{\pm i \omega_{12} 0.59} |\downarrow \uparrow \rangle + \\
+ \epsilon^{\pm i \omega_{14} 0.59} |\downarrow \downarrow \rangle + \epsilon^{\pm i \omega_{10} 0.38} |\uparrow \uparrow \rangle
\]

Observe that the optimal precision, state and measurement achieved by lifting the degeneracy of the Hamiltonian are very close to those for the non-degenerate, equally gapped Hamiltonian with four distinct eigenvalues, but this optimal precision is attained at a later time. This is due to the fact that the spectral radius of the modified Hamiltonian is smaller than that of the 4-level equally gapped one.

7. Ion trap implementation

We now describe a realization of our scheme for trapped ions. We make use of four internal electronic levels, |g⟩, |e⟩, which form the qubit, and two additional levels |g'⟩, |e'⟩, where the energy difference \(E_{g'} - E_{e'} \neq E_g - E_e\). In addition, we consider the motion degrees of freedom that we label by \(|k\rangle\), \(1 \leq k \leq n\), where \(k = 1\) describes the motional ground state. The usage of such levels and their manipulation belongs to the standard repertoire of ion trap quantum information processing (see [19–21]).

The protocol we describe here is a variant of the general protocol outlined in the main text, as the \(n\) level system we will construct consists of only motional degrees of freedom. The electronic degrees of freedom, while useful for the preparation of the state, will in the end act factor out. We define the red de-tuned hiding pulses \(G_i\) and \(E_i\) that couple levels \(|g'⟩|j⟩\ \leftrightarrow |g⟩|j + l⟩\) and \(|e'⟩|j⟩\ \leftrightarrow |e⟩|j + l⟩\). That is,
\[
G_i = \sum_j |g'⟩|j⟩⟨g⟩|j + l⟩ + h.c.
\]
\[
E_i = \sum_j |e'⟩|j⟩⟨e⟩|j + l⟩ + h.c. \quad (33)
\]

(see Fig. 5). In addition, we consider the operators
\[
X' = |g'⟩⟨e'| + |e'⟩⟨g'|
\]
\[
\sigma_z = |g⟩⟨e| + |e⟩⟨g|
\]

which act as identity on the remaining levels. The evolution is governed by \(H = B_x \sigma_z = B(|g⟩⟨e| - |e⟩⟨g|)\). We remark that whilst an evolution of the states \(|g'⟩, |e'⟩\) also takes place, this evolution can be safely ignored as we only populate these states at an intermediate stage for a short time.

Our protocol consists of the following steps

- Perform an arbitrary projective measurement on the motional degrees of freedom.

Notice the difference of this implementation to the general protocol described in the main text. The electronic degrees of freedom are effectively coupled out, and the \(n\)-level system is comprised entirely of motional degrees of freedom.

We now show how to construct the controlled-flip operations \(U_k\) of Eq. (35) by making use of a sequence of operations \(G_i E_i X' G_i E_i\). Notice that the states \(|g⟩|e⟩|k⟩\) for \(k < l \leq n\) are not affected by this operation, while for \(k \geq l + 1\) we obtain an operation \(\sigma_z = |g⟩⟨e| + |e⟩⟨g|\). It follows that \(G_k E_k X' G_k E_k\) together with \(G_{k+1} E_{k+1} X' G_{k+1} E_{k+1}\) implements \(U_k\) for \(k < n\), while for \(k = n\) the operation \(G_n E_n X' G_n E_n\) suffices. Note that \(G_n E_n X' G_n E_n\) can be replaced by \(\sigma_z\).

What remains to be shown is the possibility to prepare arbitrary initial states \(|ψ⟩\) = \(|g⟩ \otimes \sum_k c_k |k⟩\) and the performance of measurements on the resulting states (that are of the same form). A general operation on states \(|ψ⟩ = |g⟩ \otimes \sum_k c_k |k⟩\) can be obtained as follows. Observe that an arbitrary unitary operation \(V^{(k_1 k_2)}\) acting on any pair of levels \(|k_1⟩, |k_2⟩\) can be realized by the sequence \(M = E_{k_1} U_{k_1} X' E_{k_1} U_{k_1}\) that transfers the states \(|g⟩|k_1⟩, |g⟩|k_2⟩\) to auxiliary levels \(|g⟩|1⟩, |e⟩|1⟩\), followed by an arbitrary operation \(V'\) on the auxiliary levels \(|g⟩, |e⟩\) (which can be realized in standard way). The auxiliary levels are mapped back via \(M\) to motional states \(|k_1⟩, |k_2⟩\). Using sequences of operations \(V^{(k_1 k_2)}\) between pairs of levels, one can realize any unitary operation on the \(n\)-level system. This allows one to prepare an arbitrary initial state \(|ψ⟩\) from \(|g⟩|1⟩\).

Finally, a projective measurement of the form \(|P_k = |k⟩⟨k|, P^{\perp}\) can be achieved with \(U_k\), followed by the standard projective measurement that couples the excited state \(|e⟩\) to some meta-stable auxiliary level via a laser pulse and detects the emitted photons. By a sequence of such (commuting) two-outcome measurements, a projective measurement in the basis \(|k⟩\) can be achieved. Together with general unitary operations that can be implemented as shown above, one can realize an arbitrary projective measurement.

We remark that for small \(n\), more efficient schemes can be constructed where more of the electronic levels can be used. For instance, we can consider only states
and $|2\rangle$. In this case, we can make use of all four levels $|g/e\rangle_1/2$. We use $E_1$ to hide the state $|e\rangle|2\rangle$, then perform the red sideband pulse $|e\rangle|1\rangle \leftrightarrow |g\rangle|2\rangle$ and again $E_1$. By performing this pulse sequence at an appropriate time, one can generate an effective symmetric spectrum $\{-1, -\lambda_1, \lambda_1, 1\}$. Also in this case, one can perform any operation on the four level using auxiliary levels $|g'\rangle, |e'\rangle$.

Using $\sigma_x$ and $|g\rangle|1\rangle \leftrightarrow |e\rangle|2\rangle$, $|e\rangle|1\rangle \leftrightarrow |g\rangle|2\rangle$, together with $E_1$ and $X'$, one can transfer any two levels to the auxiliary system and perform an arbitrary two-level operation there. This allows one to prepare arbitrary initial states and perform arbitrary measurements on the four-level system.

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