Factorization theory for Wiener-Hopf plus Hankel operators with almost periodic symbols

A. P. Nolasco and L. P. Castro

Abstract. A factorization theory is proposed for Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols. We introduce a factorization concept for the almost periodic Fourier symbols such that the properties of the factors will allow corresponding operator factorizations. Conditions for left, right, or both-sided invertibility of the Wiener-Hopf plus Hankel operators are therefore obtained upon certain indices of the factorizations. Under such conditions, the one-sided and two-sided inverses of the operators in study are also obtained.

1. Introduction

1.1. Motivation and Historical Background. Wiener-Hopf plus Hankel operators (as well as their discrete analogues based on Toeplitz and Hankel operators) are well known to play an important role in several applied areas. E.g., this is the case in certain wave diffraction problems, digital signal processing, discrete inverse scattering, and linear prediction. For concrete examples of a detailed enrollment of those operators in these (and others) applications we refer to [13, 14, 19, 28, 30, 31, 40, 43].

In view of the needs of the applications, it is natural to expect a corresponding higher interest in mathematical fundamental research for those kind of operators. In fact, in recent years several authors have contributed to the mathematical understanding of Wiener-Hopf plus Hankel (and their discrete analogues) under different types of assumptions (cf. [1, 2, 3, 12, 15, 18, 22, 23, 31, 33, 38, 39, 41]).

We recall that the name “Wiener-Hopf operators” is due to the initial work of Norbert Wiener and Eberhard Hopf [44] where a reasoning to solve integral equations whose kernels depend only on the difference of the arguments was provided:

\[ cf(x) + \int_0^{+\infty} k(x-y)f(y)\,dy = g(x), \quad x \in \mathbb{R}_+, \]
i.e. the so-called integral Wiener-Hopf equations. Here \( c \in \mathbb{C}, k \in L^1(\mathbb{R}) \) and \( f, g \in L^2(\mathbb{R}_+) \), where \( c \) and \( k \) are fixed, \( g \) is given and \( f \) is the unknown element.

From those Wiener-Hopf equations arise the (classical) Wiener-Hopf operators defined by

\[
W_\phi f(x) = c f(x) + \int_0^{+\infty} k(x-y)f(y)dy , \quad x \in \mathbb{R}_+,
\]

where \( \phi \) belongs to the Wiener algebra:

\[
\mathbb{W} = \{ \phi : \phi = c + \mathcal{F}k, c \in \mathbb{C}, k \in L^1(\mathbb{R}) \}
\]

(which is a Banach algebra when endowed with the norm \( \|c+\mathcal{F}k\|_\mathbb{W} = |c|+\|k\|_{L^1(\mathbb{R})} \) and the usual multiplication operation). Having in mind the convolution operation, the definition of \( W_\phi \) in (1.2) gives rise to an understanding of the Wiener-Hopf operators as convolution type operators. Therefore, they can also be represented as

(1.3) \[
W_\phi = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F} : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+). \]

Here \( L^2_+(\mathbb{R}) \) denotes the subspace of \( L^2(\mathbb{R}) \) formed by all the functions supported in the closure of \( \mathbb{R}_+ = (0, +\infty) \), \( r_+ \) is the restriction operator from \( L^2(\mathbb{R}) \) into \( L^2(\mathbb{R}_+) \), and \( \mathcal{F} \) denotes the Fourier transformation. The Wiener-Hopf operators on \( L^2_+(\mathbb{R}) \) may also be written in the form

(1.4) \[
P_+A = \ell_0r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F} : L^2_+(\mathbb{R}) \to L^2_+(\mathbb{R}),
\]

where \( \ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}) \) is the zero extension operator, \( P_+ = \ell_0r_+ \) is the canonical projection of \( L^2(\mathbb{R}) \) onto \( L^2_+(\mathbb{R}) \), and \( A \) is the translation invariant operator \( \mathcal{F}^{-1}\phi \cdot \mathcal{F} \). Looking now to the structure of the operators in (1.3) and (1.4), we recognize that possibilities other than only the Wiener algebra can be considered for the so-called Fourier symbols \( \phi \) of the Wiener-Hopf operators. Namely, we may consider to choose \( \phi \) among the \( L^\infty(\mathbb{R}) \) elements (i.e., as a measurable essentially bounded function in the real line).

Within the context of (1.1) and (1.2), the Hankel integral operators have the form

(1.5) \[
Hf(x) = \int_0^{+\infty} k(x+y)f(y)dy , \quad x \in \mathbb{R}_+.
\]

(for some \( k \in L^1(\mathbb{R}) \)). It is well known that \( H \), as an operator defined between \( L^2 \) spaces, is a compact operator. However, as seen above, it is also possible to provide a rigorous meaning to the expression (1.5) when the kernel \( k \) is a temperate distribution whose Fourier transform belongs to \( L^\infty(\mathbb{R}) \).

We would like to mention that the discrete analogue of \( H \) has its roots in the year of 1861 with the Ph.D. thesis of Hermann Hankel [21]. There the study of finite matrices with entries depending only on the sum of the coordinates was proposed. Determinants of infinite complex matrices with entries defined by \( a_{jk} = a_{j+k} \) (for \( j, k \geq 0 \), and where \( a = \{a_j\}_{j \geq 0} \) is a sequence of complex numbers) were also studied. For these (infinite) Hankel matrices, one of the first main results was obtained by Kronecker in 1881 [27] when characterizing the Hankel matrices of finite rank as the ones that have corresponding power series, \( a(z) = \sum_{j=0}^{\infty} a_j z^j \), which are rational functions. In 1906, Hilbert proved that the operator (induced by the famous Hilbert matrix), \( \mathcal{H} : \ell^2 \to \ell^2 , \{b_j\}_{j \geq 0} \mapsto \{\sum_{k=0}^{\infty} b_k/(j+k+1)\}_{j \geq 0} \),
is bounded on $\ell^2$. This result may be viewed as the origin of (discrete) Hankel operators, as natural objects arising from Hankel matrices. Later on, in 1957, Nehari presented a characterization of bounded Hankel operators on $\ell^2$ [32]. Due to the importance of such characterization, we may say that it marks the beginning of the contemporary period of the study of Hankel operators.

As for combinations between Wiener-Hopf (or Toeplitz) and Hankel operators, an important initial step occurred in 1979 when Power [35] used the $C^*$–algebra generated by Toeplitz and Hankel operators. In particular, Power devoted a particular attention to those kinds of operators having piecewise continuous Fourier symbols. Later on, several other authors considered also interactions between Wiener-Hopf and Hankel operators, as well as the algebra generated by them (see e.g. [1, 2, 3, 8, 36, 37, 39, 41]). As a consequence, the theory of Wiener-Hopf plus Hankel operators is nowadays well developed for some classes of Fourier symbols (like in the case of continuous or piecewise continuous symbols). In particular, the invertibility and Fredholm properties of such kind of operators with piecewise continuous Fourier symbols are now well known (and of great importance for the applications [14, 28, 31, 43]). However, this is not the case for almost periodic Fourier symbols which are also important in the applications in view of their appearance due to, e.g., (i) particular finite boundaries in the geometry of physical problems [13], or (ii) the needs of compositions with shift operators which introduce almost periodic elements in the Fourier symbols of those operators [11, 26].

The present paper introduces characterizations of the invertibility and Fredholm properties of Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols.

1.2. The Wiener-Hopf plus Hankel Operators in Study, Basic Definitions, and Main Results Outline. The main objects of the present work are the Wiener-Hopf plus Hankel operators with Fourier symbols in the algebra of almost periodic functions, and acting between $L^2$ Lebesgue spaces. In a detailed way, we will consider operators with the form

\[ WH_\phi = W_\phi + H_\phi : L^2_+ (\mathbb{R}) \to L^2 (\mathbb{R}_+), \]

with $W_\phi$ and $H_\phi$ being **Wiener-Hopf and Hankel operators** defined by

\[ W_\phi = r_+ F^{-1} \phi \cdot F : L^2_+ (\mathbb{R}) \to L^2 (\mathbb{R}_+) \]
\[ H_\phi = r_+ F^{-1} \phi \cdot F J : L^2_+ (\mathbb{R}) \to L^2 (\mathbb{R}_+) \]

respectively. Here and in what follows, $J$ is the **reflection operator** given by the rule $J \varphi(x) = \bar{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$. According to [11, 26] and [13], we have

\[ WH_\phi = r_+ (F^{-1} \phi \cdot F + F^{-1} \phi \cdot F J) = r_+ F^{-1} \phi \cdot F (I_{L^2_+ (\mathbb{R})} + J), \]

where $I_{L^2_+ (\mathbb{R})}$ denotes the **identity operator** in $L^2_+ (\mathbb{R})$. Furthermore, since

\[ I_{L^2_+ (\mathbb{R})} + J = \ell^c r_+, \]

where $\ell^c : L^2 (\mathbb{R}_+) \to L^2 (\mathbb{R})$ denotes the **even extension operator**, we may write the Wiener-Hopf plus Hankel operator as

\[ WH_\phi = r_+ F^{-1} \phi \cdot F \ell^c r_+. \]
Finally, the Fourier symbol \( \phi \) belongs to the algebra \( AP \) of almost periodic functions, i.e., the smallest closed subalgebra of \( L^\infty(\mathbb{R}) \) that contains all the functions \( e_\lambda \) (\( \lambda \in \mathbb{R} \)) with \( e_\lambda(x) = e^{i\lambda x}, \ x \in \mathbb{R} \).

We will proceed now with some basic definitions which will be needed in what follows. Let \( T : X \to Y \) be a bounded linear operator acting between Banach spaces. The operator \( T \) is said to be \emph{normally solvable} if \( \text{Im} \ T \) is closed. In this case, the \emph{cokernel} of \( T \) is defined as \( \text{Coker} \ T = Y/\text{Im} \ T \). For a normally solvable operator \( T \), the \emph{deficiency numbers} of \( T \) are given by

\[
(1.10) \quad n(T) := \dim \text{Ker} \ T, \quad d(T) := \dim \text{Coker} \ T.
\]

If at least one of the deficiency numbers is finite, the operator \( T \) is said to be a \emph{semi-Fredholm} operator. A normally solvable operator \( T \) is said to be:

(i) a \emph{Fredholm} operator if both \( n(T) \) and \( d(T) \) are finite;
(ii) \emph{left-Fredholm} if \( n(T) \) is finite;
(iii) \emph{right Fredholm} if \( d(T) \) is finite. In the case when only one of the deficiency numbers is finite, the operator \( T \) is said to be a \emph{properly semi-Fredholm} operator. In a more detailed way, a normally solvable operator \( T \) is said to be \emph{properly right-Fredholm} if \( n(T) \) is finite and \( d(T) \) is infinite, and \emph{properly left-Fredholm} if \( d(T) \) is finite and \( n(T) \) is infinite. We point out that in German and Russian literature, (semi-)Fredholm operators are often called (semi-)\emph{Noether operators}. This is due to the pioneering work \[34\] of Fritz Noether who was the first to discover that singular integral operators with nonvanishing continuous symbols are normally solvable, and have finite kernel and cokernel dimensions. Once again, in German and Russian literature (and related with the notation used in \[11\]), right-Fredholm operators and left Fredholm operators are frequently called \emph{n-normal operators} and \emph{d-normal operators}, respectively (see, e.g., \[7, 24, 25, 26\]).

Let us choose the notation \( GB \) for the group of all invertible elements of a Banach algebra \( B \). By Bohr’s theorem, for each \( \phi \in GAP \) there exists a real number \( \kappa(\phi) \) and a function \( \psi \in AP \) such that

\[
(1.11) \quad \phi(x) = e^{i\kappa(\phi)x}e^{\psi(x)}, \ x \in \mathbb{R}.
\]

Since \( \kappa(\phi) \) is uniquely determined, \( \kappa(\phi) \) is usually called the \emph{mean motion} of \( \phi \).

For Wiener-Hopf operators with Fourier symbols in \( GAP \), there is a famous semi-Fredholm and invertibility criterion due to Golberg-Feldman/Coburn-Douglas based on the sign of the mean motion of the Fourier symbol of the operator (cf. \[16, 20\] or \[7, \text{Theorem 2.28}\]):

(a) if the mean motion of the Fourier symbol is negative, then the Wiener-Hopf operator is properly right-Fredholm and right-invertible;
(b) if the mean motion of the symbol is positive, then the Wiener-Hopf operator is properly left-Fredholm and left-invertible;
(c) if the mean motion of the symbol is zero, then the Wiener-Hopf operator is invertible.

This criterion was one of the initial motivations for the present work. Accordingly, the main purpose of this paper is to establish an invertibility and Fredholm criterion for Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols. To reach such criterion, we need to introduce a new factorization concept for \( AP \) functions – the so-called \( AP \) asymmetric factorization. As we will see in Section \[4\] (Definition \[14\]), a function \( \phi \in GAP \) is said to admit an \( AP \) asymmetric factorization if it can be represented in the form \( \phi = \phi_+ e_\lambda \phi_- \), where \( \lambda \in \mathbb{R}, \phi_- \in GAP^- \).
(cf. \((2.2)\)), and \(\varphi_e \in G_{L^\infty}(\mathbb{R})\) with \(\tilde{\varphi}_e = \varphi_e\). Therefore, assuming that \(\phi \in G_{AP}\) admits an AP asymmetric factorization, the obtained invertibility and Fredholm criterion for Wiener-Hopf plus Hankel operators \(WH_\phi\) (cf. Theorem 5.1) has a similar structure as the theorem of Gohberg-Feldman/Coburn-Douglas mentioned before, and states the following:

(a) if \(\lambda < 0\), then the Wiener-Hopf plus Hankel operator is properly right-Fredholm and right-invertible;

(b) if \(\lambda > 0\), then the Wiener-Hopf plus Hankel operator is properly left-Fredholm and left-invertible;

(c) if \(\lambda = 0\), then the Wiener-Hopf plus Hankel operator is invertible.

Under such conditions, we can do even better: we can provide a formula for the one-sided and two-sided inverses of the Wiener-Hopf plus Hankel operators by using the factors of the AP asymmetric factorization. This result is stated in the last section, in Theorem 5.2 and exhibits the importance of having convenient factorizations for the Fourier symbols of the corresponding operators.

We will also present a result on the invertibility dependencies between Wiener-Hopf and Wiener-Hopf plus Hankel operators with the same AP Fourier symbol (cf. Corollary 5.3). At a first glance, this may appear to be a very surprising result. Noticing however that (in this case) we will be dealing with a particular kind of factorization, it is more natural to hope to achieve the invertibility of Wiener-Hopf plus Hankel operators from the invertibility of Wiener-Hopf operators (by using certain relations between different AP factorizations).

Before arriving at the main Section 5 (where the last briefly described results will appear in detail), we will recall in Section 2 several useful details about almost periodic functions, operator identities for Wiener-Hopf plus Hankel operators are exhibited in Section 3 (allowing therefore the understanding of certain compositions of those kind of operators), and factorization concepts are proposed and analyzed in Section 4.

### 2. Almost Periodic Functions

The theory of almost periodic functions (AP functions) was created by Harald Bohr between 1923 and 1925. Since then, many contributions to the development of the theory of AP functions were made (namely by V. V. Stepanov, H. Wyel, A. S. Besicovitch, S. Bochner, J. Von Neumann, C. Corduneanu, and others). The importance of AP functions in problems of differential equations, stability theory and dynamical systems potentiated the development of the theory of these functions.

For defining AP functions, Bohr used the concepts of relative density and translation number (cf. \([4]\)). A set \(E \subset \mathbb{R}\) is said to be relatively dense if there exists a number \(l > 0\) such that any interval of length \(l\) contains at least one number of \(E\). In addition (considering \(\phi\) being a real or complex function defined on the real line), a number \(\tau\) is called a translation number of \(\phi\), corresponding to \(\varepsilon > 0\), if

\[|\phi(x + \tau) - \phi(x)| \leq \varepsilon\]

for all \(x \in \mathbb{R}\). These two last concepts provide the conditions to present the alternative definition of AP function: a continuous function \(\phi\) defined on the real line is called almost periodic if for every \(\varepsilon > 0\) there exists a relatively dense set of translation numbers of \(\phi\) corresponding to \(\varepsilon\). That is to say, \(\phi\) is called an AP
function if for every $\varepsilon > 0$ there exists a number $l > 0$ such that any interval of length $l$ contains at least one number $\tau$ for which
\[ |\phi(x + \tau) - \phi(x)| \leq \varepsilon, \]
for all $x \in \mathbb{R}$.

Like the periodic functions, $AP$ functions can also be represented by Fourier series. To obtain such representation, we have to consider the mean value of an $AP$ function. By the Mean Value Theorem, it follows that for every $\phi \in AP$ there exists the Bohr mean value (or the mean value) of $\phi$:
\[ M(\phi) := \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \phi(x) \, dx. \]

For $\phi \in AP$ and $\lambda \in \mathbb{R}$, the function $\phi e^{-\lambda}$, being the product of two $AP$ functions, is also an $AP$ function (recall that $e^{i\lambda x} = e^{i\lambda x}$, $x \in \mathbb{R}$). Therefore, there exists the mean value of $\phi e^{-\lambda}$. The set
\[ \Omega(\phi) := \{ \lambda \in \mathbb{R} : M(\phi e^{-\lambda}) \neq 0 \} \]
is at most countable, and is called the Bohr-Fourier spectrum of $\phi$. The Fourier series associated with the function $\phi$ is given by
\[ \sum_{j} \phi_j e^{\lambda_j x}, \]
and we may write
\[ \phi(x) \sim \sum_{j} \phi_j e^{i\lambda_j x}, \]
where $\phi_j = M(\phi e^{-\lambda_j})$. The elements $\lambda_j$ of the Bohr-Fourier spectrum are called the Fourier exponents, and the corresponding mean values $\phi_j$ are called the Fourier coefficients.

In [6], we find the Fundamental Theorem of the theory of $AP$ functions. The Fundamental Theorem states that the class of almost periodic functions is identical with the closure of the class of all trigonometric polynomials:
\[ p(x) = \sum_{n=1}^{N} \phi_n e^{i\lambda_n x}, \]
where $\phi_n \in \mathbb{C}$ and $\lambda_n \in \mathbb{R}$. To reach to the Fundamental Theorem, Bohr proved that the limit of a uniformly convergent sequence of $AP$ functions is also an almost periodic function. Thus, being trigonometric polynomials almost periodic functions, it follows that the limit of a uniformly convergent sequence of trigonometric polynomials is an $AP$ function. This means that every function belonging to the closure of the class of all finite sums is in the class of almost periodic functions. The converse also holds. That is, every $AP$ function is the limit of a uniformly convergent sequence of trigonometric polynomials. This result is called the Approximation Theorem. More precisely, the approximation theorem asserts that given $\phi \in AP$, for each $\varepsilon > 0$, there exists a trigonometric polynomial $p$, whose exponents are the Fourier exponents of $\phi$, and such that
\[ |\phi(x) - p(x)| \leq \varepsilon \]
for all $x \in \mathbb{R}$.
From the Approximation Theorem, we obtain a characterization of almost periodic functions which is sometimes presented as the definition of almost periodic functions. See, e.g., [17] where AP functions are those complex-valued functions defined on the real line which can be uniformly approximated by trigonometric polynomials, and [7] where the algebra of the AP functions is defined as the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions $e^{\lambda x}$ ($\lambda \in \mathbb{R}$).

To illustrate the notion of AP function, Figures 1 and 2 exhibit the images of two particular AP functions $\phi_1$ and $\phi_2$.

From Bohr’s Theorem (cf. also [1,14]), it is possible to observe that the argument of an invertible AP function is given by the sum of a linear function and an
AP function. For $\phi \in \mathcal{GAP}$, the mean motion of $\phi$ can be obtained by

$$\kappa(\phi) = \lim_{T \to \infty} \frac{(\arg \phi)(T) - (\arg \phi)(-T)}{2T},$$

where $\arg \phi$ is any continuous argument of $\phi$.

Considering that every function in $\mathcal{AP}$ may be represented by a series, but not every function in $\mathcal{AP}$ may be represented by an absolutely convergent series, it is also useful to consider the subclass $\mathcal{APW}$ of all functions $\varphi \in \mathcal{AP}$ which can be written in the form of an absolutely convergent series:

$$\varphi(x) = \sum_j \varphi_j e^{i\lambda_j x} \quad (x \in \mathbb{R}), \quad \lambda_j \in \mathbb{R}, \quad \sum_j |\varphi_j| < \infty.$$

To end this section about $\mathcal{AP}$ functions, we present the definitions of other special subsets of $\mathcal{AP}$ that will have a preponderant use in the proposed factorizations of $\mathcal{AP}$ functions in below. Let $\mathcal{AP}^-$ ($\mathcal{AP}^+$) denote the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions $e^\lambda$ with $\lambda \leq 0$ ($\lambda \geq 0$). Equivalently, we have

(2.2) $$\mathcal{AP}^- = \{ \phi \in \mathcal{AP} : \Omega(\phi) \subset (\infty, 0] \}$$

and

$$\mathcal{AP}^+ = \{ \phi \in \mathcal{AP} : \Omega(\phi) \subset [0, \infty) \}.$$

For more information about $\mathcal{AP}$ functions we refer to [5, 6, 17, 29].

3. Operator Identities for Wiener-Hopf plus Hankel Operators

The study of relations between different classes of operators is an important subject in Operator Theory since (in particular) it allows the transfer of properties from one class of operators to other ones. An example of these kind of relations is the equivalence between operators: considering two bounded linear operators acting between Banach spaces, $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$, the operators $T$ and $S$ are said to be equivalent [4] if there are two boundedly invertible linear operators, $E : Y_2 \to X_2$ and $F : X_1 \to Y_1$, such that

(3.1) $$T = E S F.$$

Another important example of relation between operators is the equivalence after extension relation (or matricial coupling), which is a generalization of the equivalence relation between operators. In [4], Bart and Tsekanovskii proved that two operators acting between Banach spaces are equivalent after extension if and only if they are matricially coupled. It follows from [31] that if two operators are equivalent, then they belong to the same regularity class [9, 10, 42]. In particular, this implies that one of these operators is invertible, one-sided invertible, Fredholm, properly left-Fredholm, properly right Fredholm or normally solvable, if and only if the other operator enjoys the same property. In our case we are interested in obtaining relations between Wiener-Hopf plus Hankel operators and Wiener-Hopf operators, and this idea will be used ahead in the process of obtaining an invertibility criterion for Wiener-Hopf plus Hankel operators.

Therefore, in order to reach such relation, we will start by identifying certain operator identities for Wiener-Hopf plus Hankel operators and Wiener-Hopf operators.
From the Wiener-Hopf and Hankel operator theory, the following relations are well known:

\[ W_{\phi \varphi} = W_{\phi} \ell_0 W_{\varphi} + H_{\phi} \ell_0 H_{\varphi}, \]
\[ H_{\phi \varphi} = W_{\phi} \ell_0 H_{\varphi} + H_{\phi} \ell_0 W_{\varphi}, \]

where \( \ell_0 : L^2(\mathbb{R}+) \to L^2(\mathbb{R}) \) is the zero extension operator. Additionally, from the last two identities, it follows that

\[ WH_{\phi \varphi} = W_{\phi} \ell_0 WH_{\varphi} + H_{\phi} \ell_0 WH_{\varphi}. \]

and

\[ (3.2) \quad WH_{\phi \varphi} = WH_{\phi} \ell_0 WH_{\varphi} + H_{\phi} \ell_0 WH_{\varphi}. \]

Let \( H^\infty(\mathbb{C}_-) \) denote the set of all bounded and analytic functions in \( \mathbb{C}_- = \{ z \in \mathbb{C} : \text{Im} \, z < 0 \} \), and let \( H^\infty(\mathbb{R}) \) be the set of all functions in \( L^\infty(\mathbb{R}) \) that are non-tangential limits of elements in \( H^\infty(\mathbb{C}_-) \). We will also use \( H^\infty_+(\mathbb{R}) \), which is defined in the obvious corresponding way. Due to (3.2), if we consider \( \phi \in H^\infty(\mathbb{R}) \) or \( \varphi \) being an even function, then we obtain a multiplicative relation

\[ (3.3) \quad WH_{\phi \varphi} = WH_{\phi} \ell_0 WH_{\varphi}. \]

Note that if the Fourier symbol of a Wiener-Hopf operator admits a factorization of the form \( \varphi - \psi \varphi_+ \), where \( \varphi_\pm \in H^\infty_\pm(\mathbb{R}) \) and \( \psi \in L^\infty(\mathbb{R}) \), it is possible to apply to the Wiener-Hopf operator the multiplicative property

\[ WH_{\phi \varphi} = WH_{\phi} \ell_0 WH_{\varphi}. \]

(see e.g. [7, Proposition 2.17]). With a convenient change, it is possible to construct for Wiener-Hopf plus Hankel operators a corresponding result as the one known for Wiener-Hopf operators. In the present case we may apply the multiplicative property on the left if the left factor belongs to \( H^\infty(\mathbb{R}) \) and on the right if the right factor is an even function, like the following proposition asserts.

**Proposition 3.1.** Let \( \varphi, \psi, \phi \in L^\infty(\mathbb{R}) \). If \( \varphi \in H^\infty(\mathbb{R}) \) and \( \phi = \tilde{\phi} \), then the following operator factorization takes place:

\[ WH_{\varphi \psi \phi} = WH_{\varphi} \ell_0 WH_{\psi} \ell_0 WH_{\phi} = W_{\varphi} \ell_0 WH_{\psi} \ell_0 WH_{\phi}. \]

**Proof.** From the hypothesis \( \varphi \in H^\infty(\mathbb{R}) \), we may apply the already presented multiplicative relation for Wiener-Hopf plus Hankel operators, see (3.3). Thus

\[ (3.4) \quad WH_{\varphi \psi \phi} = WH_{\varphi} \ell_0 WH_{\psi}. \]

In addition, since \( \phi = \tilde{\phi} \), it also follows from (3.3) that

\[ (3.5) \quad WH_{\psi \phi} = WH_{\psi} \ell_0 WH_{\phi}. \]

From (3.4) and (3.5), we have that

\[ (3.6) \quad WH_{\varphi \psi \phi} = WH_{\varphi} \ell_0 WH_{\psi} \ell_0 WH_{\phi}. \]

Since \( \varphi \in H^\infty(\mathbb{R}) \), we have \( H_\varphi = 0 \) due to the structure of the Hankel operators. Therefore \( WH_{\varphi} = W_{\varphi} \) and it follows from (3.6) that \( WH_{\varphi \psi \phi} = W_{\varphi} \ell_0 WH_{\psi} \ell_0 WH_{\phi}. \)

\[ \Box \]
From (3.3) we have that if the symbol of the Wiener-Hopf plus Hankel operator is factorized in such a way that the right factor is an even function, this leads to a factorization of the Wiener-Hopf plus Hankel operator, where a Wiener Hopf plus Hankel operator with an even symbol appears. Due to the multiplicative relation for Wiener-Hopf plus Hankel operators (see (3.3)), we conclude that the Wiener Hopf plus Hankel operator with an even symbol is an invertible operator. So, we end this section with this result.

**Proposition 3.2.** If $\phi_e \in GL_\infty(\mathbb{R})$ and $\tilde{\phi}_e = \phi_e$, then $WH_{\phi_e}$ is invertible and its inverse is the operator $\ell_0 WH_{\phi_e^{-1}} : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R})$.

**Proof.** On the one hand, we have

$$WH_{\phi_e^{-1}} \ell_0 = WH_1 \ell_0 = W_1 \ell_0 = I_{L^2(\mathbb{R}_+)};$$

where $I_{L^2(\mathbb{R}_+)}$ represents the identity operator in $L^2(\mathbb{R}_+)$. On the other hand, since $\phi_e \in GL_\infty(\mathbb{R})$ and $\tilde{\phi}_e = \phi_e$, then $\tilde{\phi}_e^{-1} = \phi_e^{-1}$ and therefore we may apply the multiplicative relation for Wiener-Hopf plus Hankel operators. So we have

$$WH_{\phi_e^{-1}} = WH_{\phi_e} \ell_0 WH_{\phi_e}^{-1}.$$  

Thus, combining (3.7) and (3.8), we get that

$$WH_{\phi_e} \ell_0 WH_{\phi_e}^{-1} \ell_0 = I_{L^2(\mathbb{R}_+)}. $$

In the same way, we obtain that

$$\ell_0 WH_{\phi_e}^{-1} \ell_0 WH_{\phi_e} = I_{L^2_+(\mathbb{R})}. $$

Therefore, (3.9)–(3.10) show that $WH_{\phi_e}$ is invertible and its inverse is $\ell_0 WH_{\phi_e}^{-1} \ell_0$. □

### 4. AP Factorizations

We begin this section with the definition of a new kind of AP factorization, the AP asymmetric factorization. This definition was initially motivated by the role of the so-called APW factorization in the theory of Wiener-Hopf operators with APW Fourier symbols [7], and by the recent works about Toeplitz plus Hankel operators [11] [18] and convolution type operators with symmetry [12] [15]. Additionally, it extends the corresponding concept introduced in [33] for the subclass of almost periodic Fourier symbols APW.

**Definition 4.1.** We will say that a function $\phi \in GAP$ admits an AP asymmetric factorization if it can be represented in the form

$$\phi = \phi_- e_\lambda \phi_e$$

where $\lambda \in \mathbb{R}$, $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$, $\phi_- \in GAP$, $\phi_e \in GL_\infty(\mathbb{R})$ with $\tilde{\phi}_e = \phi_e$.

The particular case of an AP asymmetric factorization with $\lambda = 0$ will be referred to as a canonical AP asymmetric factorization.

**Example 4.2.** Consider the function $\phi$ defined by

$$\phi(x) = e^{-i2x} + e^{i\pi x} \ln \left( \left( \arctan(100x^2) + \frac{\pi}{2} \right) e^{-2\sin(\pi x)} \right),$$

From (6.3) we have that if the symbol of the Wiener-Hopf plus Hankel operator is factorized in such a way that the right factor is an even function, this leads to a factorization of the Wiener-Hopf plus Hankel operator, where a Wiener Hopf plus Hankel operator with an even symbol appears. Due to the multiplicative relation for Wiener-Hopf plus Hankel operators (see (3.3)), we conclude that the Wiener Hopf plus Hankel operator with an even symbol is an invertible operator. So, we end this section with this result.
for all $x \in \mathbb{R}$. Figure 3 shows the image of $\phi(x)$ (when $x$ belongs to three different intervals). We may rewrite $\phi$ as

$$
\phi(x) = e^{e^{-i2x} + e^{i\pi x}} \ln \left( \arctan(100x^2) + \frac{\pi}{2} \right)^{-2 \sin(\pi x)}
$$
for all $x \in \mathbb{R}$. Considering $\phi_-$ and $\phi_+$ given by
\[
\phi_-(x) = e^{e^{-i\pi x}}, \\
\phi_+(x) = \ln \left( \arctan(100x^2) + \frac{\pi}{2} \right),
\]
for all $x \in \mathbb{R}$, we have $\phi_- \in \mathcal{GAP}^-$ and $\phi_+ \in \mathcal{GL}^\infty(\mathbb{R})$ such that $\tilde{\phi}_e = \phi_+$. Therefore, it follows that $\phi$ admits an $AP$ asymmetric factorization.

As for more examples, in $APW$ we find an endless number of functions which have an $AP$ asymmetric factorization. Indeed, in [33] it is proved that every function in $APW$ admits an $APW$ asymmetric factorization. Since $AP$ asymmetric factorization is a generalization of $APW$ asymmetric factorization, it results that every function in $APW$ admits an $AP$ asymmetric factorization.

It is interesting to clarify that, when existing, the $AP$ asymmetric factorization of a function is unique up to a constant, like it is stated in the following proposition.

**Proposition 4.3.** Let $\phi \in \mathcal{GAP}$. Suppose that $\phi$ admits two $AP$ asymmetric factorizations:
\[
\phi = \phi_-^{(1)} e_{\lambda_1} \phi_+^{(1)}, \\
\phi = \phi_-^{(2)} e_{\lambda_2} \phi_+^{(2)}.
\]
Then $\lambda_1 = \lambda_2$, $\phi_-^{(1)} = \gamma \phi_-^{(2)}$ and $\phi_+^{(1)} = \gamma^{-1} \phi_+^{(2)}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

**Proof.** The equality $\phi_-^{(1)} e_{\lambda_1} \phi_+^{(1)} = \phi_-^{(2)} e_{\lambda_2} \phi_+^{(2)}$, implies that
\[
(\phi_-^{(2)})^{-1} \phi_-^{(1)} e_{\lambda_1} = e_{\lambda_2} \phi_+^{(2)} (\phi_+^{(1)})^{-1}.
\]
Assume, without loss of generality, that $\lambda_1 \leq \lambda_2$. Then $\lambda = \lambda_1 - \lambda_2 \leq 0$. From (4.1) it follows that
\[
(\phi_-^{(2)})^{-1} \phi_-^{(1)} e_{\lambda} = \phi_+^{(2)} (\phi_+^{(1)})^{-1}.
\]
Since the right-hand side of (4.1) is an even function, $(\phi_-^{(2)})^{-1} \phi_-^{(1)} e_{\lambda}$ is also an even function. Put
\[
\varphi = (\phi_-^{(2)})^{-1} \phi_-^{(1)}.
\]
Thus $\varphi(x) e_{\lambda}(x) = \tilde{\varphi}(x) \tilde{\epsilon}(x)$, i.e. $\varphi(x) e_{\lambda}(x) = \tilde{\varphi}(x) e_{-\lambda}(x)$, or equivalently
\[
\varphi(x) e_{2\lambda}(x) = \tilde{\varphi}(x).
\]
On the one hand, since $\varphi \in \mathcal{GAP}^-$, we may apply the well-known characterization of $\mathcal{GAP}^-$ which assures the existence of a $\psi \in AP^-$ such that $\varphi = e^\psi$ (cf. e.g. [2] Lemma 3.4). On the other hand, because $\tilde{\varphi} \in \mathcal{GAP}^+$, by a corresponding characterization of $\mathcal{GAP}^+$, there exists a $\eta \in AP^+$ such that $\tilde{\varphi} = e^\eta$. From (4.4), it follows that
\[
e^{\psi(x)+i2\lambda x} = e^{\eta(x)},
\]
which implies that $\lambda = 0$ and $\psi \in AP^- \cap AP^+$, i.e., $\lambda_1 = \lambda_2$ and $\psi$ is a complex constant function. From (4.3), we get $\phi_-^{(1)} = \gamma \phi_-^{(2)}$ with $\gamma \in \mathbb{C} \setminus \{0\}$. By (4.4), we obtain $\phi_+^{(1)} = \gamma^{-1} \phi_+^{(2)}$. \qed
Let us recall that \( \phi \in \mathcal{GAP} \) is said to admit a right AP factorization \[2\] if
\[ \phi = \varphi_- e_\lambda \varphi_+ , \]
where \( \varphi_- \in \mathcal{GAP}^- , \varphi_+ \in \mathcal{GAP}^+ , \) and \( \lambda \in \mathbb{R} \). In addition, if \( \lambda = 0 \) this factorization is called a canonical right AP factorization. The AP asymmetric factorization is related to a special case of right AP factorization, which we will call AP antisymmetric factorization. In this new kind of factorization a strong dependence between the left and the right factor occurs (as we may realize in the next definition).

**Definition 4.4.** A function \( \phi \in \mathcal{GAP} \) admits an AP antisymmetric factorization if it is possible to write
\[ \phi = \phi_+ e_\lambda \phi_- \]
where \( \lambda \in \mathbb{R} , e_{2\lambda}(x) = e^{2i\lambda x} , x \in \mathbb{R} , \) and \( \phi_- \in \mathcal{GAP}^- \).

The following proposition shows how the AP asymmetric factorization and the AP antisymmetric factorization are related. That is, \( \phi \) has an AP asymmetric factorization if and only if \( \Phi = \varphi \varphi^{-1} \) has an AP antisymmetric factorization.

**Proposition 4.5.** Let \( \phi \in \mathcal{GAP} \) and put \( \Phi = \varphi \varphi^{-1} \).

(a) If \( \phi \) admits an AP asymmetric factorization, \( \phi = \phi_+ e_\lambda \phi_- \), then \( \Phi \) admits an AP antisymmetric factorization with the same factor \( \phi_- \) and the same index \( \lambda \).

(b) If \( \Phi \) admits an AP antisymmetric factorization, \( \Phi = \psi_+ e_{2\lambda} \psi_-^{-1} \), then \( \phi \) admits an AP asymmetric factorization with the same minus factor \( \psi_- \), the same index \( \lambda \) and the even factor \( \phi_e = e^{-\lambda} \psi^{-1} \).

**Proof.** (a) From the AP asymmetric factorization of \( \phi , \phi = \phi_+ e_\lambda \phi_- \), we have
\[ \varphi^{-1} = \varphi_+ e_\lambda \varphi_-^{-1} , \]
with \( \phi_- \in \mathcal{GAP}^- \). Hence
\[ \Phi = \varphi \varphi^{-1} = \phi_- e_{2\lambda} \phi_-^{-1} . \]

(b) It follows from the definition of the factor \( \phi_e \) that \( \phi = \psi_- e_\lambda \phi_e \). Thus it remains to prove that \( \phi_e \) is an even function. Once again by the definition of \( \phi_e \), we obtain
\[ \varphi_e = e_\lambda \psi^{-1}_- \phi = e_\lambda e_{-2\lambda} \psi^{-1}_- \phi = e_{-\lambda} \psi^{-1}_- \phi = \phi_e , \]
since \( \psi^{-1}_- \phi = e^{-2\lambda} \psi^{-1}_- \phi \) (due to the AP antisymmetric factorization of \( \Phi \)). Therefore \( \phi_e \) is an even function. \( \square \)

Since AP antisymmetric factorization is a special case of right AP factorization and, by Proposition 4.3, we already know how to relate AP antisymmetric factorization with AP asymmetric factorization, it is natural to wonder how AP asymmetric factorization and right AP factorization are related. The next theorem gives the answer to this question.

**Theorem 4.6.** Let \( \phi \in \mathcal{GAP} \). If \( \phi \) admits a right AP factorization,
\[ \phi = \varphi_- e_\lambda \varphi_+ , \]
then \( \phi \) admits an AP asymmetric factorization,
\[ \phi = \phi_+ e_\lambda \phi_- , \]
with \( \phi_- = \varphi_- \varphi^{-1}_+ , \) and \( \phi_e = \varphi^+ \varphi^+ . \)
PROOF. Suppose that \( \phi \) admits a right \( AP \) factorization, i.e., \( \phi = \varphi_- e_\lambda \varphi_+ \), where \( \varphi_- \in \mathcal{G}AP^- \), \( \varphi_+ \in \mathcal{G}AP^+ \). Considering \( \Phi = \phi \phi^{-1} \), we have
\[
(4.5) \quad \Phi = \varphi_- e_\lambda \varphi_+ e_\lambda \varphi_-^{-1} = \varphi_- \varphi_-^{-1} e_\lambda \varphi_+ \varphi_-^{-1}.
\]
Since \( \varphi_- \in \mathcal{G}AP^- \) and \( \varphi_+ \in \mathcal{G}AP^+ \), then \( \varphi_-^{-1} \in \mathcal{G}AP^- \), \( \varphi_+^{-1} \in \mathcal{G}AP^+ \) and therefore \( \varphi_- \varphi_-^{-1} \in \mathcal{G}AP^- \) and \( \varphi_+ \varphi_-^{-1} \in \mathcal{G}AP^+ \). Putting \( \phi_- = \varphi_- \varphi_-^{-1} \), it follows from (4.5) that
\[
\Phi = \phi_- e_\lambda \phi_-^{-1}.
\]
Since \( \phi_- \in \mathcal{G}AP^- \), it results that \( \Phi \) admits a \( AP \) antisymmetric factorization. By Proposition 4.5, that implies that \( \phi \) admits a \( AP \) asymmetric factorization, \( \phi = \phi_- e_\lambda \phi_+ \), with \( \phi_+ = e_\lambda \phi_- \phi_+ \). Rewriting \( \phi_- \) and \( \phi_+ \) by using the factors of the right \( AP \) factorization, \( \varphi_- \) and \( \varphi_+ \), we have
\[
\phi_- = \varphi_- \varphi_-^{-1}, \quad \phi_+ = \varphi_+ \varphi_+.
\]
\[ \square \]

**Corollary 4.7.** Let \( \phi \in \mathcal{G}AP \). If \( \phi \) admits a canonical right \( AP \) factorization,
\[
\phi = \varphi_- \varphi_+,
\]
then \( \phi \) admits a canonical \( AP \) asymmetric factorization,
\[
\phi = \phi_- \phi_+,
\]
with \( \phi_- = \varphi_- \varphi_-^{-1} \), and \( \phi_+ = \varphi_+ \varphi_+ \).

**Proof.** The result is a direct consequence of Theorem 4.6 if we take \( \lambda = 0 \). \[ \square \]

5. Main Results: Invertibility of Wiener-Hopf plus Hankel Operators

In this section, we present an invertibility and Fredholm criterion, as well as an explicit formula for the (one-sided and two-sided) inverses of Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols. Both results are obtained in terms of an \( AP \) asymmetric factorization of the Fourier symbol of the operators in study.

**Theorem 5.1.** Let \( \phi \in \mathcal{G}AP \) admit an \( AP \) asymmetric factorization \( \phi = \phi_- e_\lambda \phi_+ \).

(a) If \( \lambda < 0 \), then \( WH_\phi \) is properly right-Fredholm and right-invertible.

(b) If \( \lambda > 0 \), then \( WH_\phi \) is properly left-Fredholm and left-invertible.

(c) If \( \lambda = 0 \), then \( WH_\phi \) is invertible.

**Proof.** In the case where \( \lambda < 0 \), we have that \( e_\lambda \in AP^- \). Since \( AP^- = AP \cap H^\infty(R) \), it holds that \( e_\lambda \in H^\infty(R) \) and hence
\[
(5.1) \quad WH_\phi = W_{\phi_-} \ell_0 W_{e_\lambda} \ell_0 WH_{\phi_+},
\]
due to Proposition 3.1 and also taking into account that, because \( e_\lambda \in H^\infty(R) \), \( WH_{e_\lambda} = W_{e_\lambda} \). Since \( \phi_- \in \mathcal{G}AP^- \), by the characterization of \( \mathcal{G}AP^- \), there exists a \( \psi \in AP^- \) such that \( \phi_- = e^\psi \). Thus, the mean motion of \( \phi_- \) is zero and due to the Gohberg-Feldman/Coburn-Douglas Theorem (stated in the first section), \( W_{\phi_-} \) is invertible. From Proposition 4.2 we know that \( WH_{\phi_+} \) is invertible. Therefore, since
\( \ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}) \) is also an invertible operator, \( \ref{eq:inv} \) shows that \( WH_\phi \) is equivalent to \( W_{e_\lambda} \). Once again, by the Theorem of Gohberg-Feldman/Coburn-Douglas, since the mean motion of \( e_\lambda \) is \( \lambda < 0 \), we have that the operator \( W_{e_\lambda} \) is properly right Fredholm and right-invertible. Consequently, due to the equivalence relation \( \ref{eq:inv} \), the operator \( WH_\phi \) is also properly right-Fredholm and right-invertible. This completes the proof of part (a).

Part (b) can be derived from part (a) by passage to adjoint operators.

Finally, let us now suppose that \( \lambda = 0 \). Then \( \phi = \phi_e \) and \( WH_\phi = W_{\phi_e} \ell_0 WH_{\phi_e} \). Since \( W_{\phi_e} \) and \( WH_{\phi_e} \) are invertible, then \( WH_\phi \) is also invertible. \( \square \)

In order to proceed to the next result, we recall here the concept of reflexive generalized invertibility. Let \( T : X \to Y \) be a bounded linear operator acting between Banach spaces. \( T \) is said to be reflexive generalized invertible if there exists a bounded linear operator \( T^- : Y \to X \) such that \( TT^-T = T \) and \( T^-TT^- = T^- \). In this case, the operator \( T^- \) is referred to as the reflexive generalized inverse of \( T \). A linear bounded one-sided or two-sided invertible operator is also a reflexive generalized invertible operator.

After having reached an invertibility criterion for the Wiener-Hopf plus Hankel operators, we start looking for an explicit formula for the reflexive generalized inverses of these Wiener-Hopf plus Hankel operators. The obtained formula is expressed in terms of the factors of the \( AP \) asymmetric factorization. From the value of \( \lambda \) of the middle factor of the \( AP \) asymmetric factorization, \( e_\lambda \), it is possible to distinguish that the reflexive generalized inverse is in fact a right-inverse, left-inverse or inverse. I.e., depending on \( \lambda < 0, \lambda > 0 \) and \( \lambda = 0 \), we obtain the right-inverse, the left-inverse and the inverse of \( WH_\phi \), respectively. As we will see in the theorem below, the explicit formula for the reflexive generalized inverse of \( WH_\phi \) is given in terms of an arbitrary extension operator \( \ell : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}) \). This means that the reflexive generalized inverse of \( WH_\phi \) is independent of the choice of \( \ell \) (and therefore several choices are allowed, like for instance \( \ell = \ell_0 \) or \( \ell = \ell_e \)).

**Theorem 5.2.** If \( \phi \in \mathcal{GAP} \) admits an \( AP \) asymmetric factorization

\[
\phi = \phi_e - e_\lambda \phi_e ,
\]
then we obtain a reflexive generalized inverse of \( WH_\phi \) defined by

\[
WH_\phi^- = \ell_0 r_+ \mathcal{F}^{-1} \phi_e^{-1} \cdot \mathcal{F} \ell e r_+ \mathcal{F}^{-1} e_{-\lambda} \cdot \mathcal{F} \ell e r_+ \mathcal{F}^{-1} \phi_e^{-1} \cdot \mathcal{F} \ell : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}) ,
\]
where \( \ell : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}) \) denotes an arbitrary extension operator.

Additionally, in a more detailed way:

(a) if \( \lambda < 0 \), then \( WH_\phi^- \) is the right-inverse of \( WH_\phi \);
(b) if \( \lambda > 0 \), then \( WH_\phi^- \) is the left-inverse of \( WH_\phi \);
(c) if \( \lambda = 0 \), then \( WH_\phi^- \) is the inverse of \( WH_\phi \).

**Proof.** From the \( AP \) asymmetric factorization \( \phi = \phi_e - e_\lambda \phi_e \), it directly follows that

\[
WH_\phi = r_+ A_- E A_e \ell e r_+ ,
\]
where \( A_- = \mathcal{F}^{-1} \phi_e^{-1} \cdot \mathcal{F} \), \( E = \mathcal{F}^{-1} e_{-\lambda} \cdot \mathcal{F} \) and \( A_e = \mathcal{F}^{-1} \phi_e^{-1} \cdot \mathcal{F} \).
(i) If $\lambda \leq 0$, consider
\begin{equation}
WH_{\phi}WH_{\phi} = r_+ A_+ E A_+ e^{\xi r_+} \ell_0 r_+ A_+^{-1} e^{\xi r_+} E^{-1} e^{\xi r_+} A_+^{-1} \ell
\end{equation}

(5.2)
where the term $\ell_0 r_+$ was omitted due to the fact that $r_+ \ell_0 r_+ = r_+$. Since $A_+^{-1}$ preserves the even property of its symbol, we may also drop the first $\ell e^{\xi r_+}$ term in $[5.2]$, and obtain
\begin{equation}
WH_{\phi}WH_{\phi} = r_+ A_+ E e^{\xi r_+} E^{-1} e^{\xi r_+} A_+^{-1} \ell.
\end{equation}

(5.3)
Additionally, since in the present case (due to $\lambda \leq 0$) $E^{-1}$ is a plus type factor $[15, 42]$, we have $e^{\xi r_+} E^{-1} e^{\xi r_+} = e^{-1} e^{\xi r_+}$; also because $A_+$ is a minus type factor it follows
\begin{equation}
WH_{\phi}WH_{\phi} = r_+ A_+ e^{\xi r_+} A_+^{-1} \ell = r_+ \ell = I_{L^2(\mathbb{R})},
\end{equation}
and we can directly realize that such identities do not depend on the particular choice of the extension operator $\ell$.

(ii) If $\lambda \geq 0$, we will now analyze the composition
\begin{equation}
WH_{\phi}WH_{\phi} = \ell_0 r_+ A_+^{-1} e^{\xi r_+} E^{-1} e^{\xi r_+} A_+^{-1} \ell \ r_+ A_+ E A_+ e^{\xi r_+}.
\end{equation}

(5.5)
In the present case $E^{-1}$ is a minus type factor and for this reason $e^{\xi r_+} E^{-1} e^{\xi r_+} = e^{-1} e^{\xi r_+}$. The same reasoning applies to the factor $A_+^{-1}$, and therefore the equality $([5.6])$ takes the form
\begin{equation}
WH_{\phi}WH_{\phi} = \ell_0 r_+ A_+^{-1} e^{\xi r_+} A_+ e^{\xi r_+} \ell_0 r_+ = \ell_0 r_+ = I_{L^2(\mathbb{R})},
\end{equation}
where we have used the identity $e^{\xi r_+} A_+ e^{\xi r_+} = A_+ e^{\xi r_+}$.

(iii) Intersecting the last two cases, (i) and (ii), it follows that for $\lambda = 0$, the operator $WH_{\phi}$ is the (both-sided) inverse of $WH_{\phi}$ (cf. [5.3] and [5.6]).

As a consequence of Theorem 5.1 and of the relation between right $AP$ factorizations and $AP$ asymmetric factorizations presented in Theorem 4.7 and Corollary 4.8, we end up with a curious result on the dependence between the invertibility of Wiener-Hopf and Wiener-Hopf plus Hankel operators with the same $AP$ Fourier symbol. That is, from the invertibility of the Wiener-Hopf operator, we obtain the invertibility of the Wiener-Hopf plus Hankel operator.

**Corollary 5.3.** Let $\phi \in \mathcal{GAP}$. If $W_{\phi}$ is invertible with $\phi$ having a canonical right $AP$ factorization, then $WH_{\phi}$ is invertible.

**Proof.** Suppose that $\phi = \varphi_- \varphi_+$ is a canonical right $AP$ factorization of $\phi$. By Corollary 4.8, $\phi$ admits a canonical $AP$ asymmetric factorization, $\phi = \phi_- \phi_+$, where

$$\phi_- = \varphi_- \varphi_+^{-1}, \quad \phi_+ = \varphi_- \varphi_+.$$

From Theorem 5.1 it follows that $WH_{\phi}$ is an invertible operator. □

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Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

E-mail address: anolasco@mat.ua.pt

Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

E-mail address: lcastro@mat.ua.pt

URL: http://www.mat.ua.pt/lcastro/