Existence of Saddle Points in Discrete Markov Games and Its Application in Numerical Methods for Stochastic Differential Games

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Abstract—This work establishes sufficient conditions for existence of saddle points in discrete Markov games. The result reveals the relation between dynamic games and static games using dynamic programming equations. This result enables us to prove existence of saddle points of non-separable stochastic differential games of regime-switching diffusions under appropriate conditions.

I. INTRODUCTION

The merge of differential games and regime-switching models stems from a wide range of applications in communication networks, complex systems, and financial engineering. Many problems arising in, for example, pursuit-evasion games, queueing systems in heavy traffic, risk-sensitive control, and constrained optimization problems, can be formulated as two-player stochastic differential games [1], [2], [3]. In another direction, recent applications for better describing the random environment leads to the use of the so-called regime-switching models; see [8], [11], [14], [19], [20] and many references therein. Since for many problems arising in applications, closed-form solutions are difficult to obtain. As a viable alternative, one is contended with numerical approximations [10], [12], [15]. A systematic approach of numerical approximation for stochastic differential games was provided in [6] using Markov chain approximation methods. The major difficulty in dealing with such game problems is to prove the existence of the value of the game. To ensure the existence of saddle points, separability with respect to controls for objective function and the drift of the diffusion is required in [6]. It would be nice to be able to relax the separability condition.

Markov chain approximations of stochastic differential games are indeed discrete Markov games. In this paper, we aim to develop sufficient conditions for the existence of saddle point of discrete Markov games. In the proof, we start with dynamic programming equation together with static game results obtained by Sion [13] and von Neumann [9], discover the relations between static games and dynamic games by a series of inequalities. This approach enables us to treat non-separable discrete Markov games with respect to controls. By virtue of results in discrete Markov games, we can easily prove the existence of saddle points of discrete Markov games arising in numerical approximations of stochastic differential games when a discretization parameter \( h \) is used. As \( h \to 0 \), we are able to obtain the existence of saddle points of non-separable stochastic differential games using weak convergence techniques in [7] and [6].

The rest of the paper is arranged as follows. Section II begins with the formulation of the discrete Markov games. Section III presents sufficient conditions for the existence of saddle points of discrete Markov games for both ordinary control and relaxed control spaces, respectively. Section IV applies the results in the discrete Markov games to stochastic differential games. Section V concludes the paper with further remarks.

II. FORMULATION

Consider a two-player discrete Markov zero-sum game. Let \( S \) be a finite state space of a Markov chain, and \( \partial S \subset S \) be a collection of absorbing states. Control space \( U_1 \) and \( U_2 \) for player 1 and player 2 are compact subsets of \( \mathbb{R} \).

For notational simplicity, we have chosen to treat real-valued controls in this paper. Let \( \{\xi_n, n < \infty\} \) be a controlled discrete-time Markov chain, whose time-independent transition probabilities controlled by a pair of sequences \( \{(u_{1,n}, u_{2,n}), n < \infty\} \) is

\[
p(x, y|r_1, r_2) = P\{\xi_{n+1} = y|\xi_n = x, u_{1,n} = r_1, u_{2,n} = r_2\},
\]

where \( u_{i,n} \in U_i \) denote the decision at time \( n \) by player \( i \).

Definition 2.1: A control policy \( \{(u_{1,n}, u_{2,n}), n < \infty\} \) for the chain \( \{\xi_n, n < \infty\} \) is admissible if

\[
P\{\xi_{n+1} = y|\xi_n, u_{1,k}, u_{2,k}, k \leq n\} = p(\xi_n, y|u_{1,n}, u_{2,n}).
\]

If there is a function \( u_i(\cdot) \) such that \( u_{i,n} = u_i(\xi_n) \), then we refer to \( u_i(\cdot) \) as a feedback control of player \( i \).

Given the running cost function \( c(\cdot, \cdot, \cdot) : S \times U_1 \times U_2 \mapsto \mathbb{R}^{+}\cup\{0\} \), and the terminal cost function \( g(\cdot) : S \mapsto \mathbb{R}^{+}\cup\{0\} \), the cost for an initial \( \xi_0 = x \in S \) and an admissible control policy \( (u_1, u_2) = \{(u_{1,n}, u_{2,n}), n < \infty\} \) is defined by

\[
W(x, u_1, u_2) = E^x_1,u_2[\sum_{n=0}^{N-1} c(\xi_n, u_{1,n}, u_{2,n}) + g(\xi_N)],
\]

where \( N = \min\{n : \xi_n \in \partial S\} \) and \( E^x_1,u_2 \) is the expectation given that initial \( \xi_0 = x \) and control \( (u_1, u_2) \).

In the discrete Markov game, player 1 wants to minimize the cost, while player 2 wants to maximize. The two players have different information available depending on who makes the decision first (or who “goes first”). Using \( U_i(1) \) to denote the space of the admissible ordinary controls that player \( i \) goes first. That is, for \( u_i \in U_i(1) \), there exists a
sequence of measurable functions \( F_n(\cdot) \) taking values in \( U_i \) such that \( u_{i,n} = F_n(\xi_k, k \leq n; u_{i,k}, u_{2,k}, k < n) \). Similarly, using \( U_i(2) \) to denote the collection of the admissible ordinary controls that player \( i \) goes last, that is, \( u_i \in U_i(2) \) is determined by a sequence of measurable functions \( F_n(\cdot) \) taking values in \( U_i \) such that \( u_{i,n} = F_n(\xi_k, k \leq n; u_{i,k}, k < n; u_{j,k}, k \leq n, j \neq i) \).

To proceed, we define upper and lower values by

\[
V^+(x) = \min_{u_1 \in U_1(1)} \max_{u_2 \in U_2(2)} W(x, u_1, u_2) \tag{4}
\]

\[
V^-(x) = \max_{u_2 \in U_2(1)} \min_{u_1 \in U_1(2)} W(x, u_1, u_2) \tag{5}
\]

respectively. It is obvious \( V^-(x) \leq V^+(x) \) for all \( x \in S \). If the lower value and upper value are equal, then we say there exists a saddle point for the game, and its value is

\[
V(x) = V^+(x) = V^-(x), \quad \forall x \in S. \tag{6}
\]

The corresponding dynamic programming equation is

\[
V^+(x) = \min_{r_1 \in U_1} \max_{r_2 \in U_2} \{ E_x[V^+(\xi_1)] + c(x, r_1, r_2) \}, \tag{7}
\]

\[
V^-(x) = \max_{r_2 \in U_2} \min_{r_1 \in U_1} \{ E_x[V^-(\xi_1)] + c(x, r_1, r_2) \}. \tag{8}
\]

Practically, we can find \( V^+ \) and \( V^- \) in (4) and (5) by solving (7) and (8) using iterations. This is possible owing to the following lemma. The proof of this lemma can be found in [4, Lemma 2], and a weaker form in [18].

**Lemma 2.2:** \( \{ \xi_n, n < \infty \} \) is Markov chain with state space \( S \), absorbing states \( \partial S \), and transition probability \( p(x, y|r_1, r_2) \). Let there be a real number \( \gamma > 0 \) with

\[
P(\xi_n \in \partial S | \xi_0 = x, u_{1,k}, u_{2,k}, k \leq n) \geq \gamma, \quad \forall x \in S. \tag{9}
\]

\( c(x, r_1, r_2) \) is continuous in \( r_1 \) and \( r_2 \). To each admissible control, \( (u_{1}, u_{2}) \), the cost \( W(x, u_{1}, u_{2}) \) is defined by (4).

Then \( W(x, u_{1}, u_{2}) \) is finite and solutions of (7) and (8) are unique. For any initial value \( \{ V^+_0(x) : x \in S \} \), the sequence

\[
V^+_{n+1}(x) = \min_{r_1 \in U_1} \max_{r_2 \in U_2} \{ E_x[V^+_n(\xi_1)] + c(x, r_1, r_2) \} \tag{10}
\]

converges to \( V^+(x) \), the unique solution of (7) as \( n \to \infty \). Analogously, for any initial \( \{ V^-_0(x) : x \in S \} \), the sequence

\[
V^-_{n+1}(x) = \max_{r_2 \in U_2} \min_{r_1 \in U_1} \{ E_x[V^-_n(\xi_1)] + c(x, r_1, r_2) \} \tag{11}
\]

converges to \( V^-(x) \), the unique solution of (8) as \( n \to \infty \).

### III. Existence of Saddle Points

In this section, we provide sufficient conditions for the existence of saddle points in discrete Markov games. An existence proof is established through a series of inequalities. In addition, the definition of relaxed controls is given as a generalization of ordinary controls. It is shown that saddle points always exist in relaxed control space.

**Definition 3.1:** \( f(r_1, r_2) \) is said to be convex-concave with respect to \( (r_1, r_2) \), if \( f(\cdot, r_2) \) is convex and \( f(r_1, \cdot) \) is concave.

Next, we present a well-known minimax principle in static games, which was obtained by Sion in [13].

**Lemma 3.2:** Let \( M_1 \) and \( M_2 \) be compact spaces, \( \phi(\cdot, \cdot) \) be a convex-concave function on \( M_1 \times M_2 \), then

\[
\min_{r_1 \in M_1} \max_{r_2 \in M_2} \phi(r_1, r_2) = \max_{r_2 \in M_2} \min_{r_1 \in M_1} \phi(r_1, r_2). \tag{15}
\]

One of following two assumptions are needed for the existence theorem.

(H1) \( p(x, y|r_1, r_2) \) and \( c(x, r_1, r_2) \) are continuous and separable in \( r_1 \) and \( r_2 \).

(H2) \( p(x, y|r_1, r_2) \) and \( c(x, r_1, r_2) \) are convex-concave with respect to \( (r_1, r_2) \).

**Theorem 3.3:** Assume either (H1) or (H2), \( \{ \xi_n, n < \infty \} \) is a Markov chain as in Lemma 2.2. Let \( V^+(x) \) and \( V^-(x) \) be associated upper and lower values defined in (4) and (5). Then there exists a saddle points, that is,

\[
V^+(x) = V^-(x), \quad \forall x \in S. \tag{16}
\]

**Proof.** Define two functions \( \phi^+(-) \) and \( \phi^-(\cdot) \) by

\[
\phi^+(x, r_1, r_2) = \sum_{y \in S} p(x, y|r_1, r_2) V^+(y) + c(x, r_1, r_2), \tag{17}
\]

\[
\phi^-(x, r_1, r_2) = \sum_{y \in S} p(x, y|r_1, r_2) V^-(y) + c(x, r_1, r_2). \tag{18}
\]

The dynamic programming equation of (17) and (18) can be rewritten as

\[
V^+(x) = \min_{r_1 \in U_1} \max_{r_2 \in U_2} \{ \phi^+(x, r_1, r_2) \}, \tag{19}
\]

\[
V^-(x) = \max_{r_2 \in U_2} \min_{r_1 \in U_1} \{ \phi^-(-, r_1, r_2) \}. \tag{20}
\]

Under either assumption (H1) or (H2), by Lemma 3.2

\[
\min_{r_1 \in U_1} \max_{r_2 \in U_2} \phi^+(x, r_1, r_2) = \max_{r_2 \in U_2} \min_{r_1 \in U_1} \phi^+(x, r_1, r_2). \tag{21}
\]

Let \( \rho = \max_{x \in S} \{ V^+(x) - V^-(x) \} \geq 0 \), then

\[
V^+(x) \leq V^-(x) + \rho, \quad \forall x \in S. \tag{22}
\]

In particular, there exists \( \hat{x} \in S \), so that equal holds in (22).

\[
V^+(\hat{x}) = V^-(\hat{x}) + \rho. \tag{23}
\]

For \( \hat{x} \) given in (22), a series of inequalities follows,

\[
V^+(\hat{x}) = \min_{r_1 \in U_1} \max_{r_2 \in U_2} \{ \phi^+(x, r_1, r_2) \}
= \max_{r_2 \in U_2} \min_{r_1 \in U_1} \{ \phi^+(x, r_1, r_2) \}
\leq \max_{r_2 \in U_2} \min_{r_1 \in U_1} \{ \sum_{y \in S} p(x, y|r_1, r_2) V^+(y) + c(x, r_1, r_2) \}
\leq \max_{r_2 \in U_2} \min_{r_1 \in U_1} \{ \sum_{y \in S} p(x, y|r_1, r_2) V^-(y) + \rho + c(x, r_1, r_2) \}
= \max_{r_2 \in U_2} \min_{r_1 \in U_1} \{ \phi^-(x, r_1, r_2) \} + \rho
= V^-(\hat{x}) + \rho. \tag{24}
\]
By virtue of (14), we conclude all inequalities are indeed equal in (15), and this implies
\[ V^+(y) = V^-(y) + \rho, \quad \forall y \in S. \]

Note that \( V^+(x) = V^-(x) \) for all \( x \in \partial S \). Hence \( \rho = 0 \).

The existence of the saddle point is established. \( \square \)

The above theorem gives sufficient conditions for the existence of saddle points. We note that there always exist saddle points in relaxed control space with merely continuity assumed.

**Definition 3.4:** A control policy \( \{(m_{1,n}, m_{2,n}), n < \infty \} \) for the chain \( \{\xi_n, n < \infty\} \) is said to be a relaxed control policy, if \( m_{i,n} \) is a probability measure on \( B(U_i) \), a \( \sigma \)-algebra of Borel subsets of \( U_i \).

More general definition of relaxed control is given by Definition 4.4 in the context of stochastic differential games. Let \( P(U_1) \) and \( P(U_2) \) be collection of probability measure on \( B(U_1) \) and \( B(U_2) \). Slightly abusing notations, we generalize real function \( f(\cdot, \cdot) \) on \( U_1 \times U_2\) into a function \( f \) on \( P(U_1) \times P(U_2) \) as following
\[
f(\mu_1, \mu_2) = \int_{U_1} \int_{U_2} f(r_1, r_2)\mu_1(dr_1)\mu_2(dr_2).\]

Using the notation of relaxed control representation, the transition probability function is
\[
p(x, y|\mu_1, \mu_2) = \int_{U_1} \int_{U_2} p(x, y|r_1, r_2)\mu_1(dr_1)\mu_2(dr_2),\]

and the cost under the relaxed control policy \( (m_1, m_2) = \{(m_{1,n}, m_{2,n}), n < \infty \} \) is
\[
W(x, m_1, m_2) = E^{m_1,m_2}\left[ \sum_{n=0}^{N-1} c(\xi_n, m_{1,n}, m_{2,n}) + g(\xi_N) \right].
\]

Using \( \Gamma_i(1) \) to denote the space of admissible relaxed controls that player \( i \) goes first. That is, for \( m_i \in \Gamma_i(1) \), there exists a sequence of measurable function \( H_n(\cdot) \) taking values in \( P(U_i) \) such that
\[
m_{i,n} = H_n(\xi_k, k \leq n, m_{1,k}, m_{2,k}, k < n).
\]

Analogously, using \( \Gamma_i(2) \) to denote the space of admissible relaxed controls that player \( i \) goes last. That is, \( m_i \in \Gamma_i(2) \), there exists a sequence of measurable function \( \tilde{H}_n(\cdot) \) taking values in \( P(U_i) \) such that
\[
m_{i,n} = \tilde{H}_n(\xi_k, k \leq n; m_{1,k}, k < n; m_{j,k}, k \leq n, j \neq i).
\]

The upper and lower values associated with relaxed control space are defined by
\[
V^+_n(x) = \min_{m_1 \in \Gamma_1(1)} \max_{m_2 \in \Gamma_2(2)} W(x, m_1, m_2), \tag{16}
\]
\[
V^-_n(x) = \max_{m_2 \in \Gamma_2(1)} \min_{m_1 \in \Gamma_1(2)} W(x, m_1, m_2), \tag{17}
\]

respectively. To proceed, we present another static game result obtained by von Neumann [9].

**Lemma 3.5:** Let \( M_1 \) and \( M_2 \) be finite sets. Let \( \phi(\cdot, \cdot) \) be a function on \( M_1 \times M_2 \), \( \mu_1 \in P(M_1) \) and \( \mu_2 \in P(M_2) \) be probability measure on \( M_1 \) and \( M_2 \), then
\[
\min_{\mu_1 \in P(M_1)} \max_{\mu_2 \in P(M_2)} \phi(\mu_1, \mu_2) = \max_{\mu_2 \in P(M_2)} \min_{\mu_1 \in P(M_1)} \phi(\mu_1, \mu_2). \tag{18}
\]

**Theorem 3.6:** \( \{\xi_n, n < \infty\} \) is a Markov chain as in Lemma 2.3 with relaxed control used. Assume \( p(x, y|\cdot, \cdot) \) and \( c(x, \cdot, \cdot) \) are continuous on \( U_1 \times U_2 \). Let \( V^+_m(x) \) and \( V^-_m(x) \) be associated upper and lower values of (16) and (17). Then there always exists a saddle point, that is \( V^+_m(x) = V^-_m(x), \quad \forall x \in S. \)

**Proof.** Define two functions \( \phi^+_m(\cdot) \) and \( \phi^-_m(\cdot) \) by
\[
\phi^+_m(x, \mu_1, \mu_2) = \sum_{y \in S} p(x, y|\mu_1, \mu_2) V^+_m(y) + c(x, \mu_1, \mu_2),
\]
\[
\phi^-_m(x, \mu_1, \mu_2) = \sum_{y \in S} p(x, y|\mu_1, \mu_2) V^-_m(y) + c(x, \mu_1, \mu_2).
\]

Then dynamic programming equation in relaxed control space can be written by
\[
V^+_m(x) = \min_{\mu_1 \in P(U_1)} \max_{\mu_2 \in P(U_2)} \{\phi^+_m(x, \mu_1, \mu_2)\},
\]
\[
V^-_m(x) = \max_{\mu_2 \in P(U_2)} \min_{\mu_1 \in P(U_1)} \{\phi^-_m(x, \mu_1, \mu_2)\},
\]

Note that \( c(x, \cdot, \cdot) \) is continuous in compact set \( U_1 \times U_2 \). Hence for \( \forall \epsilon > 0 \), there exists a finite subset \( U_1^\epsilon \times U_2^\epsilon \subset U_1 \times U_2 \), such that
\[
\left| \min_{\mu_1 \in P(U_1)} \max_{\mu_2 \in P(U_2)} c(x, \mu_1, \mu_2) \right| - \min_{\mu'_1 \in P(U_1^\epsilon)} \max_{\mu'_2 \in P(U_2)} c(x, \mu'_1, \mu'_2) < \epsilon. \tag{19}
\]
\[
\left| \min_{\mu_2 \in P(U_2)} \max_{\mu_1 \in P(U_1)} c(x, \mu_1, \mu_2) \right| - \min_{\mu'_2 \in P(U_2^\epsilon)} \max_{\mu'_1 \in P(U_1)} c(x, \mu'_1, \mu'_2) < \epsilon. \tag{20}
\]

Forcing to the limit as \( \epsilon \to 0 \) in (19) and (20), as well as using Lemma 3.5, we have
\[
\min_{\mu_1 \in P(U_1)} \max_{\mu_2 \in P(U_2)} c(x, \mu_1, \mu_2) = \max_{\mu_2 \in P(U_2)} \min_{\mu_1 \in P(U_1)} c(x, \mu_1, \mu_2). \tag{21}
\]

Similarly, we obtain equality for function \( p(x, y|\cdot, \cdot), \)
\[
\min_{\mu_1 \in P(U_1)} \max_{\mu_2 \in P(U_2)} p(x, y|\mu_1, \mu_2) = \max_{\mu_2 \in P(U_2)} \min_{\mu_1 \in P(U_1)} p(x, y|\mu_1, \mu_2). \tag{22}
\]

Equalities in (21) and (22) implies
\[
\min_{\mu_1 \in P(U_1)} \max_{\mu_2 \in P(U_2)} \phi^+_m(x, \mu_1, \mu_2) = \max_{\mu_2 \in P(U_2)} \min_{\mu_1 \in P(U_1)} \phi^+_m(x, \mu_1, \mu_2), \tag{23}
\]
\[
\min_{\mu_1 \in P(U_1)} \max_{\mu_2 \in P(U_2)} \phi^-_m(x, \mu_1, \mu_2) = \max_{\mu_2 \in P(U_2)} \min_{\mu_1 \in P(U_1)} \phi^-_m(x, \mu_1, \mu_2).
\]

The rest of this proof is similar to the lines of inequalities (15). The details are omitted. \( \square \)
IV. NUMERICAL METHODS REGIME-SWITCHING STOCHASTIC DIFFERENTIAL GAMES

In this section, we formulate stochastic differential games with regime switching. Numerical methods using Markov chain approximation lead to a sequence of discrete Markov games discussed in the previous section. The use of Theorem gives sufficient conditions for the existence of saddle points, and facilitates the proof.

A. Formulation

Consider a two-player stochastic game of regime-switching diffusions. For a finite set $\mathcal{M} = \{1, \ldots, m_0\}$, $x \in \mathbb{R}^{m_0}$, $b(\cdot, \cdot, \cdot) : \mathbb{R}^{m_0} \times \mathcal{M} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^{m_0}$, $\sigma(\cdot, \cdot, \cdot) : \mathbb{R}^{m_0} \times \mathcal{M} \mapsto \mathbb{R}^{m_0}$, the dynamic system is given by

$$x(t) = x(0) + \int_0^t b(x(s), \alpha(s), u_1(s), u_2(s))ds + \int_0^t \sigma(x(s), \alpha(s))dw(s),$$

where for each $i = 1, 2$, $u_i(\cdot)$ is a control for player $i$, $w(\cdot)$ is a standard $\mathbb{R}^{m_0}$-valued Brownian motion, and $\alpha(\cdot)$ is a continuous-time Markov chain having state space $\mathcal{M}$ with generator $Q = (q_{i,j}) \in \mathbb{R}^{m_0 \times m_0}$. Let $\mathcal{F}_t : 0 \leq t$ be a filtration, which might depend on controls, and which measures at least $\{\{w(s), \alpha(s)\} : s \leq t\}$. We suppose that for each $i = 1, 2$, $u_i(\cdot)$ is $\mathcal{F}_t$-adapted taking values in a compact subset $U_i$ which are called admissible controls. Denote $A(x, \alpha) = (\sigma(x, \alpha)\sigma(x, \alpha) = (a_{j_0k_0}(x, \alpha), k_0) \in \mathbb{R}^{m_0 \times \mathbb{R}_0}$, which is symmetric and positive definite.

Let $G \subset \mathbb{R}^{m_0}$ be a compact set that is the closure of its interior $G^0$ and $\tau$ be the first exit time of $x(t)$ from $G^0$ with

$$\tau = \min\{t : x(t) \notin G^0\}.$$

Using a real number $\beta > 0$ to denote the discount factor, let the cost function be

$$W(x, \tau, u) = E_{x, \tau}^u \left[ \int_0^\tau e^{-\beta s} \tilde{k}(x(s), \alpha(s), u(s))ds + \tilde{g}(x(\tau), \alpha(\tau)) \right],$$

where $\tilde{k}(\cdot)$ and $\tilde{g}(\cdot)$ are functions representing the running cost and terminal cost, respectively, and $E_{x, \tau}^u$ denotes the expectation taken with the initial data $x(0) = x$ and $\alpha(0) = \alpha$ and given control process $u(\cdot) = (u_1(\cdot), u_2(\cdot))$. Next, we introduce the relaxed control representation; see [6], [7].

Definition 4.1: Let $B(U \times [0, \infty))$ be the $\sigma$-algebra of Borel subsets of $U \times [0, \infty)$. An admissible relaxed control $m(\cdot)$ is a measure on $B(U \times [0, \infty))$ such that $m(U \times [0, t]) = t$ for each $t \geq 0$. Given a relaxed control $m(\cdot)$, there is an $m_\tau(\cdot)$ such that $m(\tau dr dt) = m_\tau(\tau dr dt)$. In fact, we can define $m_\tau(B) = \lim_{\delta \to 0} m(B \times [t - \delta, t])$ for $B \in B(U)$.

To proceed, we need the following assumptions.

(A1) For each $\tau \in \mathcal{M}$, $\tilde{k}(\cdot, \tau, \cdot, \cdot)$ and $b(\cdot, \tau, \cdot, \cdot)$ are continuous functions on the compact set $G \times \mathbb{R}_0 \times \mathbb{R}_0$.

(A2) For each $\tau \in \mathcal{M}$, the functions $\sigma(\cdot, \tau)$ and $\tilde{g}(\cdot, \tau)$ are continuous on $G$.

(A3) Equation (4), where the controls are replaced by relaxed controls, has a unique weak sense solution (i.e., unique in the sense of in distribution) for each admissible triple $(w(\cdot), \alpha(\cdot), m(\cdot))$, where $m(\cdot) = (m_1(\cdot), m_2(\cdot))$.

(A4) For any $\tau \in \mathcal{M}$, $j_0, k_0 \in \{1, 2, \ldots, l_0\}, j_0 \neq k_0$, $a_{j_0k_0}(x, \tau) \geq \max_{k_0 \neq j_0} |a_{j_0k_0}(x, \tau)|$.

(A5) Let $\tilde{r}(\cdot)$ be the first exit time of $x(\cdot) = (x, \tau) \in \mathcal{F}_t : 0 \leq t$ into $G^0$. Then $\tilde{r}(\cdot)$ is continuous in probability, and $\tilde{r}(\cdot)$ is finite a.s. for all $\tau \in \mathcal{F}_t : 0 \leq t \leq T$.

The function $\tilde{r}(\cdot)$ is continuous as a mapping from $D[0, \infty)$ to $[0, \infty]$ with probability one relative to the measure induced by any solution with initial condition $(x, \tau)$, where $D[0, \infty)$ denotes the space of functions that are right continuous and have left limits endowed with the Skorohod topology, and $[0, \infty]$ is the interval $[0, \infty]$ compactified (see [7, p. 259]).

(A6) The functions $b(\cdot)$ and $\tilde{k}(\cdot)$ are separable in $r_1$ and $r_2$ for every $(x, \tau) \in G \times \mathcal{M}$. That is, $b(x, \tau, r_1, r_2) = \sum_{i=1}^2 b_i(x, \tau, r_1, r_2) = \sum_{i=1}^2 \tilde{k}_i(x, \tau, r_1, r_2)$.

(A7) The cost $\tilde{k}(\cdot)$ is convex-concave with respect to $(r_1, r_2)$, and there exist $\mathbb{R}^{m_0}$-valued continuous functions $b_i(x, \tau) (i = 0, 1, 2, 3)$ such that $b_i(x, \tau, r_1, r_2) = r_1 r_2 b_i(x, \tau) + r_1 b_i(x, 0) + r_2 b_i(0, x) + b_i(0, 0)$.

Assumption (A4) is used for construction of transition probabilities of the approximating Markov chain. It requires that the diffusion matrix be diagonally dominated. If the given dynamic system does not satisfy (A4), then we can adjust the coordinate system to satisfy assumption (A4); see [7, p. 110]. (A5) is a broad condition that is satisfied in most applications. The main purpose is to avoid the tangency problem discussed in [7, p. 278]. Later, we will establish the existence of saddle points using either (A6) or (A7) in addition to (A1)–(A5). Condition (A7) allows non-separable differential games with respect to controls.

Now we are ready to define upper values, lower values, and saddle points of differential games; see [6] for the corresponding definitions of systems without regime switching. Let $\mathcal{U}_i$ be collection of all admissible ordinary control with respect to $(w(\cdot), \alpha(\cdot))$ for $\Delta > 0$. Let $\mathcal{U}_i(\Delta) \subset \mathcal{U}_i$ such that $u_i(\cdot)$ are piecewise constant on the intervals $[k\Delta, k\Delta + \Delta), k = 0, 1, 2, \ldots$, and $u_i(k\Delta)$ is $\mathcal{F}_{k\Delta}$-measurable.

Let $L_1(\Delta) \subset \mathcal{U}_i(\Delta)$ denote the set of such piecewise constant controls for player 1 that are determined by measurable real-valued functions $Q_{1,n}(\cdot)$

$$u_1(n\Delta) = Q_{1,n}(w(s), \alpha(s), u(s), s < n\Delta),$$

We can define $L_2(\Delta)$ and the associated rule $u_2$ for player 2 analogous to (27).

Thus we can always suppose that if the control of (for example) player 1 is determined by a form such as (27). Then (in relaxed control terminology) the law of $(w(t), \alpha(t), m_2(t))$ for $n\Delta \leq t < (n + 1)\Delta$ is determined recursively by past information

$$\{w(s), \alpha(s), m_2(s), s < t, m_1(s), s \leq n\Delta\}.$$
Definition 4.2: For initial condition \( x(0) = x, \alpha(0) = \iota \), define the upper and lower values for the game as
\[
V^+(x, \iota) = \lim_{\Delta \to 0} \inf_{u_1 \in \mathcal{L}(1, \Delta)} \sup_{u_2 \in \Omega} W(x, \iota, u_1, u_2),
\]
\[
V^-(x, \iota) = \lim_{\Delta \to 0} \sup_{u_2 \in \mathcal{L}(2, \Delta)} \inf_{u_1 \in \mathcal{L}(1)} W(x, \iota, u_1, u_2).
\]
If the lower and upper value are equal, then we say there exists a saddle point for the game, and its value is
\[
V^+(x, \iota) = V^-(x, \iota) = V(x, \iota), \quad \forall x \in G, \iota \in \mathcal{M}. \tag{31}
\]

B. Markov Chain Approximations

Here, we will construct a two-component Markov chain. The discretization of differential game leads to a sequence of discrete Markov games. The approximation is of finite difference type. The basis of the approximation is a discrete-time, finite-state, controlled Markov chain \( \{ (\xi^h_n, \alpha^h_n) : n < \infty \} \) whose properties are \textit{locally consistent} with that of \( \Xi^G \).

For each \( h > 0 \), let \( G_h \) be a finite subset of \( G \) such that \( d(G_h, G) \to 0 \) as \( h \to 0 \), where \( d(\cdot) \) is a metric defined by
\[
d(G, G_h) = \max_{p \in G, q \in G_h} d(p, q). \tag{32}
\]
Let \( \{ (\xi^h_n, \alpha^h_n) : n < \infty \} \) be a controlled discrete-time Markov chain on a discrete state space \( G \times \mathcal{M} \) with transition probabilities denoted by \( p^h((x, \iota), (y, \ell)) \), where \( r = (r_1, r_2) \in U_1 \times U_2 \). We use \( (u^h_{1,n}, u^h_{2,n}) \) to denote the actual control action for the chain at discrete time \( n \).

We have a positive function \( \Delta^h(\cdot) \) on \( G_h \times \mathcal{M} \times U_1 \times U_2 \) such that \( \Delta^h(x, \iota, r) \to 0 \) as \( h \to 0 \), but \( \inf_{x, \iota, r} \Delta^h(x, \iota, r) > 0 \) for each \( h > 0 \). We take an interpolation of the discrete Markov chain \( \{ (\xi^h_n, \alpha^h_n) \} \) by using interpolation interval \( \Delta^h_n = \Delta^h(\xi^h_n, \alpha^h_n, u^h_{1,n}, u^h_{2,n}) \). Now we give the definition of local consistency.

Definition 4.3: Let \( \{ p^h((x, \iota), (y, \ell)) \} \) for \( (x, \iota) \) and \( (y, \ell) \) in \( G_h \times \mathcal{M} \) and \( r \in U_1 \times U_2 \) be a collection of well-defined transition probabilities for the two-component Markov chain \( \{ (\xi^h_n, \alpha^h_n) \} \), approximation to \( (x(\cdot), (y(\cdot), \eta(\cdot), \iota(\cdot)) \). Define the difference \( \Delta^h_n = \xi^h_{n+1} - \xi^h_n \). Assume \( \lim_{h \to 0} \sup_{x, \iota, r} \Delta^h(x, \iota, r) = 0 \). Denote by \( E^r_{x,n}, \rho^r_{x,n} \) and \( p^r_{x,n} \) the conditional expectation, covariance, and probability given \( \{ \xi^h_k, \alpha^h_k, u^h_{1,k}, u^h_{2,k} \} \leq n, \xi^h_n = x, \alpha^h_n = \iota, (u^h_{1,n}, u^h_{2,n}) = r \}. \) The sequence \( \{ (\xi^h_n, \alpha^h_n) \} \) is said to be \textit{locally consistent} with \( \Xi^G \), for \( \Delta^h_n = \Delta^h(x, \iota, r) \), if
\[
E^r_{x,n} \Delta^h_n = b(x, \iota, r) \Delta^h + o(\Delta^h), \tag{33}
\]
\[
\rho^r_{x,n} \Delta^h_n = A(x, \iota) \Delta^h + o(\Delta^h), \tag{33}
\]
\[
p^r_{x,n} \{ \alpha^h_{n+1} = \ell \} = q_{\ell} \Delta^h + o(\Delta^h), \text{ for } \ell \neq \iota, \tag{33}
\]
\[
p^r_{x,n} \{ \alpha^h_{n+1} = \iota \} = (1 + q_{\iota}) \Delta^h + o(\Delta^h), \tag{33}
\]
\[
\sup_{n, r, \iota} |\Delta^h_n| \to 0 \text{ as } h \to 0. \tag{33}
\]
To approximate the cost defined in \( \Xi^G \), we define a cost function using the Markov chain above. Let
\[
t^h_n = \sum_{j=0}^{n-1} \Delta^h_j \text{ and } N_h = \inf \{ n : \xi^h_n \notin G_h \}.
\]
The cost for \( u^h = \{ (u^h_{1,n}, u^h_{2,n}) \} \) and initial \( (x, \iota) \) is
\[
W^h(x, \iota, u^h) = E_{x,\iota} \left[ \sum_{n=0}^{N_h-1} e^{-\beta t^n_h} \Delta^h_n + \mathcal{K}(\xi^h_n, \alpha^h_n, u^h_{1,n}, u^h_{2,n}) \right]. \tag{34}
\]
Using \( \Lambda_t^h(1) \) to denote the space of the ordinary controls that player \( i \) goes first, and its strategy is defined by measurable functions of the type similar to \( \Xi^n \). That is, for \( u^h \in \Lambda_t^h(1), \)
\[
u^h = \inf_{u^h \in \Lambda_t^h(1)} W^h(x, \iota, u^h_1, u^h_2). \tag{35}
\]
The associated upper and lower values is defined as
\[
V^h+(x, \iota) = \inf_{u^h_1 \in \Lambda_t^h(1)} \sup_{u^h_2 \in \Lambda_t^h(2)} W^h(x, \iota, u^h_1, u^h_2), \tag{36}
\]
C. Saddle Points for the Markov Chain Approximation

In this section, we present a local consistent discrete Markov game of \( \{ (\xi^h_n, \alpha^h_n) \} \) generated by central finite difference scheme for analysis purpose. Under assumptions (A1)–(A5) together with either (A6) or (A7), we can apply Theorem \ref{thm:} to show the existence of saddle points for each \( h \). By forcing the limit \( h \to 0 \), the (upper) (lower) values converge to that of stochastic differential game by Lemma \ref{lem:} and it results in the existence of saddle points.

First, the transition probabilities for \( \{ (\xi^h_n, \alpha^h_n) \} \) are
\[
p^h((x, \iota), (x + e_{j_0} h, \iota)) = \pm \pm b_{j_0} h_{j_0}(x, \iota) + a_{j_0 k_0}(x, \iota) - \sum_{k_0 \neq j_0} |a_{j_0 k_0}(x, \iota)| - \frac{1}{2 h^2} \Delta^h(x, \iota) - \beta h^2 \tag{37}
\]
for \( j_0 = 1, 2, \ldots, l_0 \),
\[
p^h((x, \iota), (x + e_{j_0} h - e_{k_0} h, \iota)) = \pm \pm \frac{1}{2} a_{j_0 k_0}(x, \iota), \Delta^h(x, \iota) - \beta h^2 \tag{37}
\]
for \( j_0 = k_0 \),
\[
p^h((x, \iota), (x + e_{j_0} h - e_{k_0} h, \iota)) = \pm \pm \frac{1}{2} a_{j_0 k_0}(x, \iota), \Delta^h(x, \iota) - \beta h^2 \tag{37}
\]
for \( j_0 \neq k_0 \),
\[
p^h((x, \iota), (x, \iota)) = \frac{q_{\iota} h^2}{D(x, \iota) - \beta h^2}, \quad \ell \neq \iota, \tag{37}
\]
\[
p^h((x, \iota), (y, \ell)) = 0, \quad \text{otherwise.} \tag{37}
\]
where
\[
D(x, \iota) = \sum_{j_0 = 1}^{l_0} a_{j_0 k_0}(x, \iota) - \sum_{j_0 < k_0} |a_{j_0 k_0}(x, \iota)| - q_{\iota} h^2 + \beta h^2.
\]
Set the interpolation interval as $\Delta t^h(x, t) = h^2 / D^h(x, t)$. By (A4), $D^h(x, t) - \beta h^2 > 0$. Also, we have

$$\sum_{(y, \ell)} p^h((x, t), (y, \ell)) r_{\ell} = 1.$$  To ensure that $p^h(\cdot)$ is always nonnegative, we require

$$h \leq \min_{j_0} \left( \frac{a_{j_0 j_0}(x, t) - \sum_{k_0 \neq j_0} |a_{j_0 k_0}(x, t)|}{\max_k |b_{j_0}(x, t, r_1, r_2)|} \right). \tag{38}$$

Lemma 4.4: Assume (A1), (A2), (A4), and $h$ satisfies (38). The Markov chain $(\xi^h, \alpha^h)$ with transition probabilities $(p^h(\cdot))$ and interpolation $\Delta t^h(\cdot)$ defined above is locally consistent with (24).

Proof. The criterion in (33) can be verified through a series of calculations, thus details are omitted. □

Theorem 4.5: Assume (A1)–(A5), either (A6) or (A7), and $G_h$ is a finite set defined above (43). For $x \in G_h$ and $i \in \mathcal{M}$, a Markov chain is defined by (37). Let $V^{h,+}(x, t)$ and $V^{h,-}(x, t)$ be the associated upper and lower values defined in (35) and (36) in the control spaces $\mathcal{U}^h(i)$ and $\bar{U}^h(2)$. Then there exists a saddle point

$$V^{h,+}(x, t) = V^{h,-}(x, t), \tag{39}$$

provided $h$ satisfies (38).

Proof. The contraction condition (9) satisfies for the discount factor $\beta > 0$. Let

$$p((x, t), (y, \ell)) r_{1, r_2} = p^h((x, t), (y, \ell)) r_{1, r_2},$$

$$c(x, t, r_1, r_2) = e^{-\beta \Delta t^h(x, t)} \Delta t^h(x, t) k((x, t), r_1, r_2).$$

Assumptions (A6) and (A7) lead to (H1) and (H2), respectively. The result holds applying Theorem (3,3) □

Although the proof of next lemma is rather complicated and not trivial, the proof is referred to weak convergence techniques in [7], [5], and [6] due to the limit of space.

Lemma 4.6: Assume that the conditions of Theorem (3,5) are satisfied. Then for the approximating Markov chain, we have

$$\lim_{h \to 0} V^{h,+}(x, t) = V^+(x, t), \tag{40}$$

$$\lim_{h \to 0} V^{h,-}(x, t) = V^-(x, t). \tag{41}$$

Theorem 4.7: Assume the conditions of Theorem (3,5) are satisfied. Then the differential game has saddle point in the sense

$$V^+(x, t) = V^-(x, t). \tag{42}$$

V. Further Remarks

The key part of zero-sum game problems is existence of saddle point. This paper is devoted to sufficient condition for the existence of saddle point in discrete Markov game. Using dynamic programming equation method, we are able to use static game results of Sion [13] and von Neumann [9] to discover the sufficient conditions. A direct application is numerical methods for stochastic differential game problems.

The transition probabilities used in (37) requires restriction (38) on $h$. Practically, we develop the transition probabilities by upward finite difference scheme, so that the generated one is well defined without restriction on $h$. It can be routinely calculated to verify the local consistency. This kind of discrete Markov game might have different upper and lower values for some $h$. However, both the upper and lower values in this situation converge to the original saddle point of differential game $V(x)$ by Lemma 4.6 and Theorem 4.7.

Numerical examples in pursuit-evasion games are omitted due to the space limit, although the numerical results clearly verify our works.

For a regime-switching system in which the Markov chain has a large state space, we may use the ideas of two-time-scale approach presented in [16] (see also [17] and references therein) to first reduce the complexity of the underlying system and then construct numerical solutions for the limit systems. Optimal strategies of the limit systems can be used for constructing strategies of the original systems leading to near optimality.

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