EXISTENCE AND STABILITY OF INTERFACIAL CAPILLARY- GRAVITY SOLITARY WAVES WITH CONSTANT VORTICITY

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Abstract. In this paper, we consider capillary-gravity waves propagating on the interface separating two fluids of finite depth and constant density. The flow in each layer is assumed to be incompressible and of constant vorticity. We prove the existence of small-amplitude solitary wave solutions to this system in the strong surface tension regime via a spatial dynamics approach. We then use a variant of the classical Grillakis–Shatah–Strauss (GSS) method to study the orbital stability/instability of these waves. We find an explicit function of the parameters (Froude number, Bond number, and the depth and density ratios) that characterizes the stability properties. In particular, conditionally orbitally stable and unstable waves are shown to be possible.

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In recent years, internal waves have gained a great deal of attention among mathematicians. A plethora of works has been devoted to proving existence of various types of internal traveling waves, see for instance [AT86, AT89, SS93, Nil17] or the survey in [HHS+22]. Largely for reasons of mathematical convenience, many of these results assume the flow in each layer is irrotational; that is, the vorticity is identically zero. The literature pertaining to rotational flows is unsurprisingly far more limited due to the significant increase in mathematical complexity created by the presence of vorticity. Nonetheless, rotational effects play a significant part in many physical situations, such as waves propagating over a background current; see, for example, [Mar22].

The results of the present paper come in two parts. First, we prove the existence of a family of small-amplitude internal solitary waves with constant vorticity. This is done under the assumption of strong surface tension regime, in a sense to be explained shortly. Second, as our primary contribution, we investigate the stability properties of these waves as solutions to the dynamical problem. In particular, we exhibit an explicit function of the physical parameters whose sign determines whether a sufficiently small-amplitude wave is (conditionally) orbitally stable or orbitally unstable. We also consider several specific parameter regimes, and find that both stable and unstable waves exist. These are among a very small number of analytical results concerning the nonlinear stability or instability of internal waves. In particular, Chen and Walsh [CW22] proved the orbital stability of a class of internal waves in the irrotational setting. We will adopt their basic strategy, but incorporating constant vorticity introduces numerous complications.

Mathematically, the problem is formulated as follows. Let \((x, y) \in \mathbb{R}^2\) be a point in the standard Cartesian coordinates system, with \(x\) being the direction of wave propagation and gravity acting in the negative \(y\) direction. For time \(t \geq 0\), we assume that the fluid is confined to a channel and organized into two superposed layers:

\[
\Omega(t) := \Omega_+(t) \cup \Omega_-(t),
\]

bounded above and below by infinitely-long rigid walls, \(\{y = d_+\}\) and \(\{y = -d_-\}\). Here and throughout the paper, we will use subscripts of \(\pm\) to indicate the restriction of a quantity to \(\Omega_\pm\). Both layers share a common boundary \(\mathcal{S} = \mathcal{S}(t)\) that is free. We call this the internal interface and assume it has can be parameterized as the graph of an unknown function \(y = \eta(t, x)\). Our focus will be on solitary waves, meaning \(\eta \to 0\) as \(|x| \to \infty\). We take the density to be constant in each layer with \(0 < \rho_+ \leq \rho_-\). More precisely, the upper and lower regions of the fluid can be written as

\[
\Omega_+(t) = \{(x, y) \in \mathbb{R}^2 : \eta(t, x) < y < d_+\} \tag{1.1}
\]

and

\[
\Omega_-(t) = \{(x, y) \in \mathbb{R}^2 : -d_- < y < \eta(t, x)\}. \tag{1.2}
\]

We will assume that the flow in each layer is incompressible and governed by the incompressible Euler equations. Due to incompressibility, the velocity field \((u_\pm, v_\pm)\) in \(\Omega_\pm\) can be expressed in terms of a stream function \(\psi_\pm = \psi_\pm(t, x, y)\) by

\[
u_\pm = - (\psi_x)_\pm, \quad u_\pm = (\psi_y)_\pm.
\]
Figure 1. Configuration of the fluid domain. Two fluids of different densities are confined in an infinitely long channel.

Suppose that the vorticity in $\Omega_{\pm}$ is the constant $\omega_{\pm} := v_{\pm x} - u_{\pm y} \in \mathbb{R}$. Note that in two-dimensions the vorticity is transported, so this assumption is valid even for the dynamical problem. Taking the curl of the velocity field, we then find that the stream function satisfies the Poisson equations

$$\Delta \psi_{\pm} = -\omega_{\pm} \quad \text{in } \Omega_{\pm}(t).$$

As is common with constant vorticity waves, we wish to subtract out the background shear from $\psi$ to obtain a harmonic function. That is, define $\tilde{\psi}_{\pm} := \psi_{\pm} + \frac{\omega_{\pm} y^2}{2}$, which will satisfy $\Delta \tilde{\psi}_{\pm} = 0$ in $\Omega_{\pm}(t)$. Let $\phi_{\pm}$ be a harmonic conjugate:

$$\phi_x = (\psi_y)_{\pm} + \omega_{\pm} y, \quad \text{and} \quad \phi_y = -(\psi_x)_{\pm}.$$

Then, the rotational incompressible Euler equations can be recast as follows. In the interior we have

$$\Delta \phi_{\pm} = 0 \quad \text{in } \Omega_{\pm}(t).$$

On the internal interface and both rigid walls, the kinematic conditions read

$$\begin{cases}
\eta_t = (\phi_{\pm})_y - ((\phi_{\pm})_x - \omega_{\pm} \eta) \eta_x \quad \text{on } y = \eta(t, x), \\
(\phi_{\pm})_y = 0 \quad \text{on } y = \pm d_{\pm}.
\end{cases}$$

Physically, (1.3b) states that the velocity field is tangential along the boundaries. Finally, via the dynamic condition and Young–Laplace law, we can infer that the pressure jump across the internal interface is proportional to the signed curvature. Using Bernoulli’s principle in each layer, it can therefore be stated as

$$[\rho \phi] = -\left[\frac{1}{2} \rho |\nabla \psi|^2 + g \rho \eta + \rho \omega \psi \right] - \sigma \left(\frac{\eta_x}{\sqrt{1 + (n_x)^2}}\right) x \quad \text{on } y = \eta(t, x),$$

The notation $[\cdot] := (\cdot)_+ - (\cdot)_-$ denotes the difference in trace between two quantities on the internal interface in the upper and lower layer, $g > 0$ is the gravitational constant, and $\sigma > 0$ is the coefficient of surface tension.

Observe that the functions $\phi_{\pm}$ are defined on a moving spatial domain, which complicates the task of finding an appropriate functional analytic setting for the problem. We therefore prefer to work with the unknown

$$\xi_{\pm}(t, x) = \phi_{\pm}(t, x, \eta(t, x))$$

(1.4)
which corresponds to trace on the surface. Using this new variable allows us to push the entire problem to the free boundary, rendering it nonlocal but more tractable for analysis.

1.1. Statement of results. Now, we are ready to state our main results. We record them in Theorem 1.1 and Theorem 1.2. We begin by introducing some important terminology and physical parameters that describe the system.

A steady or traveling wave is a solution to the Euler equations (1.3) that translate in the \( x \)-direction at a fixed wave speed \( c \in \mathbb{R} \) without altering its shape. Thus, in a moving frame of reference, it appears stationary. Concretely, this means the unknowns can be written as

\[
\xi_\pm = \eta_c(t, x) = \eta_c(x - ct), \quad \xi_\pm(x - ct) = \xi_c(x - ct),
\]

for some steady profiles \( \eta_c \) and \( \xi_c \). Recall that we will focus on solitary waves, for which \( \eta_c \) is localized.

There is an extensive body of work devoted to establishing the existence of traveling internal waves in various parameter regimes. Most well studied is the pure gravity case \((\sigma = 0, g > 0)\), where both solitary waves [AT86, BBT83, Mie95] and periodic waves [AT86, AT89] have been constructed. It is, however, important to note that in the absence of surface tension, the water wave problem (1.3) becomes ill-posed dynamically; see for example [Lan13]. Hence, when looking at questions pertaining to stability/instability, one has to assume \( \sigma > 0 \), which we will do throughout this work.

The existence of small-amplitude internal waves in the presence of surface tension was obtained previously by many authors, for instance [Kir22, SS93, Nil17]. However, none of those results allows for vorticity. Since understanding the effects of rotation on the stability is our objective, we spend the first part of our analysis developing an existence theory for small-amplitude internal waves with layer-wise constant vorticity. This is accomplished using a spatial dynamics method: we view the \( x \)-coordinate as a time-like variable, and then use a center manifold reduction approach. The process is closely inspired by the work of Nilsson [Nil17].

Internal solitary waves can be described using four dimensionless parameters. The first two of them are the Bond number \( \beta \) and the inverse square of the Froude number \( \alpha \) defined by

\[
\beta := \frac{\sigma}{d_+ \rho_- c^2}, \quad \alpha := \frac{-g \left\| \rho_+ \right\| d_+}{\rho_- c^2}.
\]

From its definition, we see that the Bond number \( \beta \) measures the strength of the surface tension. In view of (1.5), the Froude number \( 1/\sqrt{\alpha} \) can be thought of as the non-dimensionalized wave speed.

Upon linearizing (1.3) at the trivial solution and inserting the plane-wave ansatz \( \eta = \exp(ik(x - ct)) \), we arrive at the following dispersion relation

\[
\alpha + \beta k^2 = \sum_{\pm} \frac{\rho_\pm}{\rho_-} k \coth \left( \frac{d_\pm}{d_+} k \right) + \left( \frac{\omega_+ d_+ \rho_+}{cp_-} - \frac{\omega_- d_+}{c} \right).
\]
It can be checked easily that \( k = 0 \) is a root of (1.6) exactly when
\[
\beta = \beta_0 := \frac{1}{3} \left( \frac{\rho_+ - d_-}{\rho_-} \right), \quad \alpha = \alpha_0 := \frac{\rho_+ - d_+}{\rho_-} + \frac{\omega_+ d_+ \rho_+}{c \rho_-} - \frac{\omega_- d_+}{c}.
\]
(1.7)

We will regard \( \beta_0 \) as the critical Bond number: the range \( \beta > \beta_0 \) corresponds to the strong surface regime, and \( \beta < \beta_0 \) is the weak surface tension regime. Heuristically, one expects that solitary waves will bifurcate from the trivial solutions at \( \alpha_0 \). We will specifically be concerned with the strong surface tension case \( \beta > \beta_0 \).

Apart from \( \beta \) and \( \alpha \), there are two other physical parameters that have to be considered when studying interfacial waves: the density ratio \( \varrho \) and the asymptotic height ratio \( d \) given as follows
\[
\varrho := \frac{\rho_+}{\rho_-}, \quad d := \frac{d_-}{d_+}.
\]
(1.8)

Unlike \( \alpha \) and \( \beta \), these are specific to the two-layer case. Nilsson [Nil17] proved that for irrotational flow \( (\omega_\pm = 0) \), when \( \varrho - 1/d^2 < 0 \) and \( O(1) \) as \( \alpha \searrow \alpha_0 \), there exist waves of depression \( (\eta < 0) \). On the other hand, if \( \varrho - 1/d^2 > 0 \) and \( O(1) \) as \( \alpha \searrow \alpha_0 \), then waves of elevation exist \( (\eta > 0) \).

That said, our main result on existence of solitary waves is as follows.

**Theorem 1.1 (Existence).** Let \( \Pi = \{ (\rho_{\pm}, d_{\pm}, \omega_{\pm}, \sigma, c) : 0 < \epsilon \ll 1 \} \) be a smooth curve in the physical parameter space such that along \( \Pi \), the corresponding Bond number is supercritical \( \beta > \beta_0 \), and the inverse-square Froude number is \( \alpha = \alpha_0 + \epsilon^2 \). Suppose that
\[
\varrho - \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_+}{cd} + \frac{\omega_+^2 d_+^2 \varrho}{3c^2} - \frac{\omega_-^2 d_+^2}{3c^2} = O(1) \text{ as } \epsilon \searrow 0,
\]
where we have suppressed the \( \epsilon \) dependence of the quantities on the left-hand side. Then for any \( k > 1/2 \) there exists a smooth curve of internal wave solutions
\[
C = \{ (\eta_{\epsilon, \beta}, \xi_{+\epsilon, \beta}, \xi_{-\epsilon, \beta}) : 0 < \epsilon \ll 1 \} \subset H^{k+\frac{1}{2}}(\mathbb{R}) \times \left( \dot{H}^k(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \right)^2.
\]
(1.10)

For every solution on the curve \( C \), the surface profile exhibits the following asymptotics:
\[
\eta_{\epsilon, \beta}(x) = \frac{d_+ \epsilon^2 \text{sech}^2 \left( \frac{\epsilon x}{2d_+ \sqrt{\beta - \beta_0}} \right)}{\varrho - \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_+}{cd} + \frac{\omega_+^2 d_+^2 \varrho}{3c^2} - \frac{\omega_-^2 d_+^2}{3c^2}} + O(\epsilon^3)
\]
(1.11)
in \( H^{k+\frac{1}{2}}(\mathbb{R}) \) as \( \epsilon \searrow 0 \).

We note that, the denominator in (1.11) determines whether the solution is a wave of elevation or depression for \( \epsilon \) sufficiently small. In contrast to the irrotational case, this will depend not only on the relative sizes of \( \varrho \) and \( d \), but also the strength of the vorticity in each layer. It is also important to observe that, while \( \epsilon \) is the appropriate parameter for proving existence, stability is best studied by fixing the physical parameters \( \rho_{\pm}, d_{\pm}, \omega_{\pm}, \sigma \), and varying \( c \). Because \( \epsilon = \sqrt{\alpha - \alpha_0} \), we can solve (1.5) in terms of the wave speed and write \( \alpha = \alpha_\epsilon \) and \( \beta = \beta_\epsilon \).
The main result of the paper characterizes the conditional stability of these solutions in the orbital sense. More precisely, we say a solitary wave \((\eta_c, \xi_{c+}, \xi_{c-})\) is conditionally orbital stable provided that, for all \(R > 0\) and \(r > 0\), there exists \(r_0 > 0\) such that if \((\eta, \xi_+, \xi_-)\) is a solution to the internal wave problem on the time interval \([0, t_0]\) that obeys the a priori bound
\[
\sup_{t \in [0, t_0]} \left( \|\eta(t)\|_{H^{s+}} + \|\xi_+(t)\|_{\dot{H}^{s+}} + \|\xi_-(t)\|_{\dot{H}^{s+}} \right) < R, \tag{1.12}
\]
and whose initial data satisfies
\[
\|\eta(0) - \eta_c\|_{H^1} + \|\xi_+(0) - \xi_{c+}\|_{\dot{H}^{\frac{1}{2}+}} + \|\xi_-(0) - \xi_{c-}\|_{\dot{H}^{\frac{1}{2}+}} < r_0, \tag{1.13}
\]
then
\[
\sup_{t \in [0, t_0]} \inf_{s \in \mathbb{R}} \left( \|\eta(t, \cdot - s) - \eta_c\|_{H^1} + \|\xi_+(t, \cdot - s) - \xi_{c+}\|_{\dot{H}^{\frac{1}{2}+}} + \|\xi_-(t, \cdot - s) - \xi_{c-}\|_{\dot{H}^{\frac{1}{2}+}} \right) < r. \tag{1.14}
\]

The inequality (1.14) measures the distance between the translated solutions \((\eta, \xi_+, \xi_-)\) and the family of traveling waves. The norms in (1.12) represents the lowest regularity required for local well-posedness of the Cauchy problem that is currently available. The meaning of the superscript \((+)\) on the regularity will be made clear later in Section 4. Furthermore, as we will see shortly, the regularity in (1.13) and (1.14) matches the regularity of the energy space.

Notice that this result is conditional in that we must assume a priori that the solution exists on a give time interval, since global well-posedness for the system is not known. However, as \(r_0\) is independent of the life span \(t_0\), the bound in (1.14) is substantially stronger result than merely continuity of the data-to-solution map. In particular, if global existence is known, then we obtain orbital stability in the classical sense.

Conversely, we say a steady solution \((\eta_c, \xi_{c+}, \xi_{c-})\) is orbitally unstable provided there exists \(r > 0\) such that, for all \(r_0 > 0\), there exists initial data
\[(\eta(0), \xi_+(0), \xi_-(0)) \in H^1(\mathbb{R}) \times \left( \dot{H}^{\frac{1}{2}}(\mathbb{R}) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R}) \right)^2 \]
satisfying (1.13) and for which the corresponding solution \((\eta(t), \xi_+(t), \xi_-(t))\) to the Cauchy problem exists the tubular neighborhood of the solitary wave in finite time:
\[
\inf_{s \in \mathbb{R}} \left( \|\eta(t, \cdot - s) - \eta_c\|_{H^1} + \|\xi_+(t, \cdot - s) - \xi_{c+}\|_{\dot{H}^{\frac{1}{2}+}} + \|\xi_-(t, \cdot - s) - \xi_{c-}\|_{\dot{H}^{\frac{1}{2}+}} \right) > r
\]
for some time \(t < \infty\).

We can now state our result on (conditional) orbital stability/instability.

**Theorem 1.2 (Stability/instability).** Fix the physical parameters: \(\rho_{\pm}, d_{\pm, \omega_{\pm}}, \sigma\). There exists an explicit smooth function \(m = m(c)\) such that, any sufficiently small-amplitude solitary internal wave \((\eta_c, \xi_{c+}, \xi_{c-})\) with \(\beta_c > \beta_0\) and \(\alpha_c = \alpha_0 + \epsilon^2\) is conditionally orbitally stable if \(m'(c) > 0\) and orbitally unstable if \(m'(c) < 0\).
The formula for $m$ is given in (6.3). Because it is rather complicated, it is instructive to look at a few special cases. If the Bond number is sufficiently close to critical $0 < \beta_c - \beta_0 \ll 1$, then the waves given by Theorem 1.1 are always orbitally stable. We can also obtain more definitive statements by assuming the vorticity in one layer is 0. Parameter regime for which the waves are stable are given in Figure 2, and unstable in Figure 3. Observe that both involve the relative size of the density ratio and a non-dimensionalized measure of the vorticity strength in the rotational layer. The criterion also changes depending on whether we are considering a wave of elevation or depression, which is determined by the sign of the denominator in (1.11). Note that when $\omega_+ = \omega_- = 0$, we recover the result in [CW22] that all sufficiently small-amplitude waves are orbitally stable.

1.2. Idea of the proof. In Section 2, we prove Theorem 1.1 on the existence of the small-amplitude internal wave solutions. Following the strategy of Nilsson [Nil17], we write the corresponding steady water wave problem as a spatial dynamical Hamiltonian system. This is obtained by identifying a Lagrangian (the flow force), then applying a Legendre transform to arrive at the desired Hamiltonian. In the strong surface tension regime, we find that the linearized operator at the trivial solution has a 0 eigenvalue of multiplicity 2, corresponding to a Hamiltonian $0^2$ resonance. Performing a center manifold reduction, we show that solitary waves of elevation or depression exist when

$$1 - \frac{d}{d^2} + \frac{\omega_+ d_+}{c} + \frac{\omega_- d_-}{cd} + \frac{\omega_+^2 d_+^2}{3c^2} - \frac{\omega_-^2 d_-^2}{3c^2},$$

is positive or negative, respectively. Note that when $\omega_+ = 0$, this analysis coincides with [Nil17, Section 3.3].

Next, we consider the stability or instability of these waves. Similar to approach of the existence theory, we again exploit the Hamiltonian structure of problem (1.3). This time, however, it is the Hamiltonian for the time-dependent problem rather than spatial dynamical. In Section 4, we show that (1.3) can be written as

$$\partial_t u = JDE(u),$$
where \( u = u(t, x) \) is an unknown represented by \((\eta, \xi_+, \xi_-)\), \( J \) is a skew-adjoint operator called the Poisson map, and \( E \) is an energy functional. It is well known that the translation-invariant nature of the problem gives rise to another conserved quantity known as the momentum \( P \). By construction, it is clear that a traveling (steady) wave solution is a critical point of the augmented Hamiltonian given by \( E_c := E - cP \).

The main technique used to prove Theorem 1.2 is based on the seminal works of Grillakis, Shatah, and Strauss \([GSS90a, GSS90b]\). Their approach, known as the GSS method, provides a systematic way to prove nonlinear stability/instability for Hamiltonian systems that are invariant under a continuous symmetry group. Although GSS has been successfully used to treat many model equations for water waves, the full free boundary Euler system exhibits a number of features that have made it resistant to this machinery. For instance, GSS requires the Poisson map \( J \) to be an isomorphism, which is not satisfied in the present setup as we will see shortly. Further, they require the Cauchy problem to be globally well-posed in the energy space. Currently, only local well-posedness of \((1.3)\) is known, and this assumes considerably higher regularity.

In recent work, Varholm, Wahlén, and Walsh \([VWW20]\) obtained a variant of the GSS method with hypotheses sufficiently relaxed so that it can be applied directly to the water wave problem. Their framework allows for the Poisson map \( J \) to only have a dense range. It also permits the mismatch between the local well-posedness space and the energy space that the water wave problem possess. This theory was also the basis for the paper of Chen and Walsh \([CW22]\) on irrotational internal waves. For the benefit of the reader, an abbreviated statement of the abstract result is given in Section 3.

To apply this machinery to our problem, the main difficulty is to characterize the spectrum of the linearized augmented Hamiltonian at a traveling wave. Introducing vorticity increases the complexity of the calculations substantially. However, after a series of non-trivial computations, we find that the linearized operator at a sufficiently small-amplitude wave has Morse index 1, as required by the abstract theory. This analysis is carried out in Section 5. Then, by the general theory, stability or instability of the wave is determined by the convexity or concavity of the moment of instability, a scalar-valued function of the wave speed. In Section 6, we then prove the statement regarding conditional stability/instability in Theorem 1.2. We emphasize that these are considerably more involved than the irrotational regime, and in particular both stable and unstable waves exist, which points toward the importance of including vortical effects in the model.

2. Existence theory

We start this section by proving the existence of small amplitude solitary water waves. In doing that, we pattern the approach presented in \([Nil17]\). In comparison to \([Nil17]\), due to the rotational assumption, the equations that we have to deal with are several order more complex. Seeking traveling wave solutions, we impose a change of variables \((t, x, y) \mapsto (x - ct, y)\). The main governing equation \((1.3)\) can then be recast in terms of the relative streamfunctions \(\psi_{\pm}\) and posed in a frame of reference moving with the waves as follows
\[
\Delta \psi_\pm = -\omega_\pm \quad \text{in } \Omega_\pm, \\
\psi_\pm = \mp m_\pm \quad \text{on } y = \pm d_\pm, \\
\psi_\pm = 0 \quad \text{on } y = \eta(x), \\
\left[ \frac{1}{2} \rho |\nabla \psi|^2 + g \rho \eta \right] = -\sigma \left( \frac{\eta_x}{\langle \eta_x \rangle} \right)_x + Q \quad \text{on } y = \eta(x),
\]

for some constants \( m_\pm \) together with the following asymptotic condition

\[ \eta \to 0 \quad \text{as } x \to \infty. \]

The variable \( Q \) in (2.1) is the hydraulic head constant.

Next, we introduce the following non-dimensionalized variables

\[
(x', y') = \frac{1}{d_+} (x, y), \quad \eta'(x') = \frac{\eta(x)}{d_+}, \quad \psi'_\pm (x', y') = \frac{\psi_\pm (x, y)}{d_+ c}.
\]

Under these variables, the problem now reads

\[
\begin{cases}
\Delta \psi_+ = -\frac{\omega_+ d_+}{c} & \text{for } \eta(x) < y < 1, \\
\Delta \psi_- = -\frac{\omega_- d_+}{c} & \text{for } d < y < \eta(x), \\
\psi_+ = \frac{-m_+}{cd_+} & \text{on } y = 1, \\
\psi_- = \frac{m_-}{cd_+} & \text{on } y = -d, \\
\psi_\pm = 0 & \text{on } y = \eta(x),
\end{cases}
\]

\[
\left[ \frac{1}{2} \rho c^2 |\nabla \psi|^2 + g \rho d_+ \eta \right] + \frac{\sigma}{d_+} \left( \frac{\eta_x}{\langle \eta_x \rangle} \right)_x = Q \quad \text{on } y = \eta(x),
\]

where we have dropped the ' for notational convenience.

Next, to obtain the harmonic function \( \tilde{\psi} \), we subtract the shear flow from \( \psi \):

\[
\tilde{\psi}_\pm := \psi_\pm + \frac{\omega_\pm d_+ y^2}{2c}.
\]

Clearly, \( \tilde{\psi}_\pm \) is harmonic in both layers, that is

\[
\Delta \tilde{\psi} = 0 \quad \text{in } \Omega_\pm.
\]

Moreover, the boundary conditions on the rigid walls and internal interface, respectively, become

\[
\tilde{\psi}_+ = 0 \quad \text{on } y = 1, \\
\tilde{\psi}_- = 0 \quad \text{on } y = -d,
\]
and
\[
\tilde{\psi}_\pm = \frac{\omega_\pm d_+ \eta^2}{2c} \quad \text{on } y = \eta(x),
\]

\[
\left[ \frac{1}{2} \rho c^2 |\nabla \tilde{\psi}|^2 - c \rho \omega d_+ \eta \partial_y \tilde{\psi} + \frac{1}{2} \rho \omega^2 d_+^2 \eta^2 + g \rho d_+ \eta \right] = -\frac{\sigma}{d_+} \left( \frac{\eta_x}{\langle \eta_x \rangle} \right)_x + Q \quad \text{on } y = \eta(x).
\]

(2.6)

Rather than work with \( \tilde{\psi} \), we will use its harmonic conjugate \( \phi \) to reformulate equations (2.4), (2.5), and (2.6). One can view \( \phi \) as the velocity potential. Using the fact that \( \phi_\pm = \tilde{\psi}_\pm \) and \( \phi_\pm = -\tilde{\psi}_\pm \), the equations now become

\[
\begin{cases}
\Delta \phi_\pm = 0 & \text{in } \Omega_\pm, \\
\phi_+ = 0 & \text{on } y = 1, \\
\phi_- = 0 & \text{on } y = -d, \\
\phi_\pm = \phi_\pm \eta_x - \frac{\omega_\pm d_+ \eta_x}{c} - \eta_x & \text{on } y = \eta(x), \\
\partial_y \phi \pm \eta_x^2 + g \rho d_+ \eta \pm \eta_x \phi_\pm + 1 \eta_x \phi_\pm = 0 & \text{on } y = \eta(x).
\end{cases}
\]

(2.7)

Consider the following rescaling of the domain via the mapping \((x, y) \mapsto (x, z)\), where

\[
z(x, y) := \begin{cases}
\frac{y - 1}{\eta(x) - 1} & \text{for } \eta(x) < y < 1, \\
\frac{y + d}{\eta(x) + d} & \text{for } -d < y < \eta(x).
\end{cases}
\]

(2.8)

As a result, we have the following change of variables formulas

\[
\begin{align*}
\partial_y &= \frac{1}{\eta + d} \partial_z, & \partial_y &= \frac{1}{\eta - 1} \partial_z, \\
\partial_x &= \partial_X - \frac{z \eta_x}{\eta - 1} \partial_z, & \partial_x &= \partial_X - \frac{z \eta_x}{\eta + d} \partial_z.
\end{align*}
\]

(2.9)

Recycling notations, let us define \( \phi_\pm (x, z) := \phi_\pm (x, y) \). Under the derivative formulas (2.9), equation (2.4) now read

\[
\begin{align*}
\phi_{++} - \frac{2z \eta_x}{\eta - 1} \phi_{++} - \frac{z \eta_x^2}{\eta - 1} & \phi_{++} + \frac{2z \eta_x^2}{(\eta - 1)^2} \phi_{+z} + \frac{z^2 \eta_x^2}{(\eta - 1)^2} \phi_{zz} + \frac{1}{(\eta - 1)^2} \phi_{zz} = 0 \quad 0 < z < 1, \\
\phi_{--} - \frac{2z \eta_x}{\eta - 1} \phi_{--} - \frac{z \eta_x^2}{\eta - 1} & \phi_{--} - \frac{2z \eta_x^2}{(\eta - 1)^2} \phi_{-z} + \frac{z^2 \eta_x^2}{(\eta - 1)^2} \phi_{zz} + \frac{1}{(\eta - 1)^2} \phi_{zz} = 0 \quad 0 < z < 1.
\end{align*}
\]

(2.10)
The boundary conditions on the rigid walls (2.5) and the internal interface (2.6) translate to
\[ \phi_{\pm z} = 0 \quad \text{on} \quad z = 0, \] (2.12)
and
\[
\begin{align*}
\phi_{+z} &= (\eta - 1) \left( \frac{-\omega_+ d_+ \eta_x}{c} + \phi_{+z} \eta_x - \frac{\eta_x^2 \phi_{+z}}{\eta - 1} - \eta_x \right), \quad \text{on} \quad z = \eta, \\
\phi_{-z} &= (\eta + d) \left( \frac{-\omega_- d_- \eta_x}{c} + \phi_{-z} \eta_x - \frac{\eta_x^2 \phi_{-z}}{\eta + d} - \eta_x \right), \quad \text{on} \quad z = \eta,
\end{align*}
\]
\[
\theta \left[ \frac{1}{2} \left( \phi_{+z} - \frac{\eta_x \phi_{+z}}{\eta - 1} \right)^2 + \frac{1}{2} \left( \phi_{+z} \frac{\eta_x \phi_{+z}}{\eta - 1} \right)^2 - \frac{\omega_+ d_+ \eta}{c} \left( \phi_{+z} - \frac{z \eta_x}{\eta - 1} \phi_{+z} \right) + \frac{1}{2} \frac{\omega_+^2 d_+^2 \eta^2}{c^2} \right]
\]
\[
- \left[ \frac{1}{2} \left( \phi_{-z} - \frac{\eta_x \phi_{-z}}{\eta + d} \right)^2 + \frac{1}{2} \left( \frac{\phi_{-z}}{\eta + d} \right)^2 - \frac{\omega_- d_- \eta}{c} \left( \frac{\phi_{-z}}{\eta + d} - \frac{z \eta_x}{\eta + h} \phi_{-z} \right) + \frac{1}{2} \frac{\omega_-^2 d_-^2 \eta^2}{c^2} \right]
\]
\[
= \alpha \eta - \beta \left( \frac{\eta_x}{\eta_z} \right)_x \quad \text{on} \quad z = \eta, \] (2.13)
where \( \alpha, \beta, \) and \( \theta \) are variables defined earlier in (1.5) and (1.8). The energy can be formulated as
\[
E = K + V
\]
\[
= \frac{c^2 d_+^2 \rho_+}{2} \int_R \int_0^1 \left( \phi_{+z} - \frac{z \eta_x \phi_{+z}}{\eta - 1} - \frac{\omega_+ d_+ (z(\eta - 1) + 1)}{c} \right)^2 \left( \frac{\phi_{+z}}{\eta - 1} \right) (1 - \eta) \, dz \, dx
\]
\[
\frac{c^2 d_-^2 \rho_-}{2} \int_R \int_0^1 \left( \phi_{-z} - \frac{z \eta_x \phi_{-z}}{\eta + d} - \frac{\omega_- d_- (z(\eta + d) - d)}{c} \right)^2 \left( \frac{\phi_{-z}}{\eta + d} \right) (d + \eta) \, dz \, dx
\]
\[
- \frac{1}{2} g \alpha^3 [\rho] \int_R \eta^2 \, dx + \sigma d_+ \int_R \left( \sqrt{1 + \eta_x^2} - 1 \right) \, dx.
\]
Further, the momentum \( P \) is given by
\[
P = d_+^2 c \int_R \int_0^1 \rho_+ \left( \phi_{+z} - \frac{z \eta_x \phi_{+z}}{\eta - 1} - \frac{\omega_+ d_+ (z(\eta - 1) + 1)}{c} \right) (1 - \eta) \, dz \, dx
\]
\[
+ d_-^2 c \int_R \int_0^1 \rho_- \left( \phi_{-z} - \frac{z \eta_x \phi_{-z}}{\eta + d} - \frac{\omega_- d_- (z(\eta + d) - d)}{c} \right) (d + \eta) \, dz \, dx.
\]
From many literature, it is known that solitary waves can be detected by looking at the critical points of the functional \( E - cP \). For this reason, we will study the Hamiltonian that arises from taking the Lagrangian of \( E - cP \). Using the expressions for \( E \) and \( P \), we
state the functional

\[ E - cP = \]

\[
d_x^2 c^2 \rho \left[ \frac{\theta}{2} \int_{\mathbb{R}} \int_0^1 \left( \left( \phi_{+x} - \frac{z \eta_x \phi_{+z}}{\eta - 1} - \frac{\omega_{+z} (z(\eta - 1) + 1)}{c} - 1 \right)^2 + \left( \frac{\phi_{+z}}{(\eta - 1)} \right)^2 \right) (1 - \eta) \ dz \ dx \right. \\
+ \left. \frac{1}{2} \int_{\mathbb{R}} \int_0^1 \left( \left( \phi_{-x} - \frac{z \eta_x \phi_{-z}}{\eta + d} - \frac{\omega_{-z} (z(\eta + d) - d)}{c} - 1 \right)^2 + \left( \frac{\phi_{-z}}{(\eta + d)} \right)^2 \right) (d + \eta) \ dz \ dx \right. \\
- \left. \int_{\mathbb{R}} \left( \frac{1}{2} \alpha \eta^2 - \beta \left( \sqrt{1 + \eta_x^2} - 1 \right) \right) + \frac{\theta}{2} (1 - \eta) + \frac{1}{2} (\eta + d) \ dx \right].
\]

From that, we derive the corresponding Lagrangian

\[ L = \frac{\theta}{2} \int_0^1 \left( \left( \phi_{+x} - \frac{z \eta_x \phi_{+z}}{\eta - 1} - \frac{\omega_{+z} (z(\eta - 1) + 1)}{c} - 1 \right)^2 + \left( \frac{\phi_{+z}}{(\eta - 1)} \right)^2 \right) (1 - \eta) \ dz \\
+ \frac{1}{2} \int_0^1 \left( \left( \phi_{-x} - \frac{z \eta_x \phi_{-z}}{\eta + d} - \frac{\omega_{-z} (z(\eta + d) - d)}{c} - 1 \right)^2 + \left( \frac{\phi_{-z}}{(\eta + d)} \right)^2 \right) (d + \eta) \ dz \\
- \frac{1}{2} \alpha \eta^2 + \beta \left( \sqrt{1 + \eta_x^2} - 1 \right) \right) + \frac{\theta}{2} (1 - \eta) - \frac{1}{2} (\eta + d).
\]

(2.14)

In order to formulate the correct Hamiltonian, we need to know variational derivatives of \( L \) with respect to the individual variable \( \phi_{+x}, \phi_{-x}, \) and \( \eta_x \),

\[
\Phi_+ := \frac{\delta L}{\delta \phi_{+x}} = \theta (1 - \eta) \left( \phi_{+x} - \frac{z \eta_x \phi_{+z}}{\eta - 1} - \frac{\omega_{+z} (z(\eta - 1) + 1)}{c} - 1 \right),
\]

\[
\Phi_- := \frac{\delta L}{\delta \phi_{-x}} = (\eta + d) \left( \phi_{-x} - \frac{z \eta_x \phi_{-z}}{\eta + d} - \frac{\omega_{-z} (z(\eta + d) - d)}{c} - 1 \right),
\]

\[
\gamma := \frac{\delta L}{\delta \eta_x} = \int_0^1 \theta \left( \phi_{+x} - \frac{z \eta_x \phi_{+z}}{\eta - 1} - \frac{\omega_{+z} (z(\eta - 1) + 1)}{c} - 1 \right) z \phi_{+z} \ dz \\
- \int_0^1 \left( \phi_{-x} - \frac{z \eta_x \phi_{-z}}{\eta + d} - \frac{\omega_{-z} (z(\eta + d) - d)}{c} - 1 \right) z \phi_{-z} \ dz + \beta \eta_x \eta_x.
\]

(2.15)
Having the information above, the Hamiltonian can, therefore, be expressed in terms of $u = (\eta, \gamma, \phi_+, \Phi_+, \phi_-, \Phi_-)$ as follows

$$H(u) = \int_0^1 \Phi_+ \phi_+^z \, dz + \int_0^1 \Phi_- \phi_-^z \, dz + \gamma \eta_z - L$$

$$= \int_0^1 \frac{1}{2\rho(1-\eta)} \left( (\Phi_+ + \rho(1-\eta))^2 - \rho^2 \phi_+^2 \right) \, dz$$

$$+ \int_0^1 \frac{1}{2(d+\eta)} \left( (\Phi_- + (d+\eta))^2 - \phi_-^2 \right) \, dz$$

$$+ \int_0^1 \frac{\Phi_+ \omega_d + (z(\eta - 1) + 1)}{c} \, dz + \int_0^1 \frac{\Phi_- \omega_d + (z(\eta + d) - d)}{c} \, dz$$

$$- \sqrt{\beta^2 - \gamma^2} + \beta - \frac{\alpha \eta^2}{2},$$

where,

$$\tilde{\gamma} = \gamma + \int_0^1 \frac{\Phi_+ \phi_+^z}{\eta - 1} \, dz + \int_0^1 \frac{\Phi_- \phi_-^z}{\eta + d} \, dz.$$

To formalize this, for $s \geq 0$, we define the following product spaces

$$\mathcal{X}_s = \mathbb{R} \times \mathbb{R} \times H^{s+1}(0, 1) \times H^{s+1}(0, 1) \times H^{s+1}(0, 1) \times H^{s+1}(0, 1).$$

We would like to point out that the symbol $H^{s+1}$ refers to a Sobolev space of order $s + 1$, not the Hamiltonian $H$. Further, we let $\hat{M} = \mathcal{X}_0$ be a manifold with $m \in \hat{M}$ and let $v = (\eta, \gamma, \phi_+, \Phi_+, \phi_-, \Phi_-) \in T_m \hat{M}$. On $T_m \hat{M} \times T_m \hat{M}$, consider the position independent symplectic form

$$\hat{\Omega}(v, v^*) = \gamma^* \eta - \eta^* \gamma + \int_0^1 (\Phi_+^* \phi_+ - \phi_+^* \Phi_+) \, dz + \int_0^1 (\Phi_-^* \phi_- - \phi_-^* \Phi_-) \, dz. \quad (2.17)$$

One may observe that $(\hat{M}, \hat{\Omega})$ is a symplectic manifold. The corresponding set

$$\hat{N} = \{ m \in \hat{M} : |\hat{\gamma}| < \beta, -d < \eta < 1 \}$$

is a manifold domain of $\hat{M}$ where the Hamiltonian $H$ is a smooth functional on it (i.e., $H \in C^\infty(\hat{N}, \mathbb{R})$). Hence, the tuple $(\hat{M}, H, \hat{\Omega})$ forms a Hamiltonian system. Via the symplectic
form (2.17) and standard computations, the associated Hamilton’s equations read

\[
\dot{\eta} = \frac{\bar{\gamma}}{\sqrt{\beta^2 - \bar{\gamma}^2}},
\]

\[
\dot{\gamma} = \int_0^1 \left[ -\frac{\varrho}{2(1-\eta)^2} \left( \frac{\Phi_+^2 - \phi_{+z}^2}{\xi^2} \right) + \frac{\theta}{2} \right] \, dz
+ \int_0^1 \left[ \frac{1}{2(d+\eta)^2} \left( \Phi_-^2 - \phi_{-z}^2 \right) - \frac{1}{2} \right] \, dz - \int_0^1 \frac{\Phi_+ \omega_+ d_+}{} \, dz - \int_0^1 \frac{\Phi_- \omega_- d_-}{} \, dz
+ \frac{\bar{\gamma}}{\sqrt{\beta^2 - \bar{\gamma}^2}} \left( \int_0^1 z \phi_+ \Phi_+ \, dz + \int_0^1 z \phi_- \Phi_- \, dz \right) + \alpha \eta,
\]

\[
\dot{\phi}_+ = \frac{1}{\eta - 1} \left( -\frac{\Phi_+}{\varrho} + (\eta - 1) + \frac{\bar{\gamma} z \phi_+}{\sqrt{\beta^2 - \bar{\gamma}^2}} \right) + \frac{\omega_+ d_+(z(\eta - 1) + 1)}{c},
\]

\[
\dot{\Phi}_+ = \frac{1}{\eta - 1} \left( \frac{\bar{\gamma}(z \Phi_+)_z}{\sqrt{\beta^2 - \bar{\gamma}^2}} + \varrho \phi_{+zz} \right),
\]

\[
\dot{\phi}_- = \frac{1}{\eta + d} \left( \Phi_- + (\eta + d) + \frac{\bar{\gamma} z \phi_-}{\sqrt{\beta^2 - \bar{\gamma}^2}} \right) + \frac{\omega_- d_+(z(\eta + d) - d)}{c},
\]

\[
\dot{\Phi}_- = \frac{1}{\eta + d} \left( \frac{\bar{\gamma}(z \Phi_-)_z}{\sqrt{\beta^2 - \bar{\gamma}^2}} - \phi_{-zz} \right),
\]

(2.18)

where the Hamiltonian vector field also satisfies the corresponding boundary conditions

\[
\varrho \Phi_+(1) = -\frac{\bar{\gamma} \Phi_+(1)}{\sqrt{\beta^2 - \bar{\gamma}^2}},
\]

\[
\Phi_-(1) = \frac{\bar{\gamma} \Phi_-(1)}{\sqrt{\beta^2 - \bar{\gamma}^2}},
\]

\[
\phi_{\pm}(0) = 0.
\]

(2.19)

In order to set a firmer ground for the latter analysis, we define the product space

\[
Y_s = \mathbb{R} \times \mathbb{R} \times H^{s+1}(0, 1) \times H_0^{s+1}(0, 1) \times H^{s+1}(0, 1) \times H_0^{s+1}(0, 1),
\]

(2.20)

where \(H^{s+1}(0, 1) = \{ f \in H^{s+1}(0, 1) : f(0) = f(1) = 0 \} \). Additionally, let us also define these spaces

\[
M = \{ m \in \mathbb{R}^2 \times H^{1}(0, 1)^4 : \Gamma_+(0) = \Gamma_-(0) = \int_0^1 \phi_+ \, dz = \int_0^1 \phi_- \, dz = 0 \},
\]

\[
\tilde{M} = \{ m \in Y_0 : \int_0^1 \phi_+ \, dz = \int_0^1 \phi_- \, dz = 0 \},
\]

\[
\tilde{N} = \{ m \in \tilde{M} : |\bar{\gamma}| < \beta, -d < \eta < 1 \},
\]

(2.21)
where $m = (\eta, \gamma, \bar{\phi}_+, \Gamma_+, \bar{\phi}_-, \Gamma_-)$. Going back to the Hamilton’s equations (2.18), we can see that it has an equilibrium point

$$
\begin{pmatrix}
\eta \\
\gamma \\
\phi_+ \\
\Phi_+ \\
\phi_- \\
\Phi_-
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\omega_+ d_+ \rho(z - 1) \\
\frac{c}{c} \\
0 \\
\omega_- d_+ d^2 (1 - z)
\end{pmatrix}
- g.
$$

However, in preparation for the Hamiltonian reduction process on a center manifold, we need to shift the equilibrium point obtained before to the origin $(0, 0, 0, 0, 0, 0)$. To achieve that, we impose another change of variables for the unknowns $\Phi_\pm, \bar{\phi}_\pm$, and $\chi_\pm$

$$
\Gamma_+ := \int_0^z (\Phi_+ + \rho - \frac{\omega_+ d_+ \rho(s - 1)}{c}) ds,
$$

$$
\Gamma_- := \int_0^z (\Phi_- + d - \frac{w_- d_+ d^2 (1 - s)}{c}) ds,
$$

$$
\bar{\phi}_+ := \phi_+ - \chi_+,
$$

$$
\bar{\phi}_- := \phi_- - \chi_-,
$$

$$
\chi_+ := \int_0^1 \phi_+ dz,
$$

$$
\chi_- := \int_0^1 \phi_- dz.
$$

(2.22)
This gives rise to a new formulation of the Hamiltonian equation. Precisely, one may think of this change of variables as a mapping that sends \((\eta, \gamma, \phi, \Phi, \chi, \chi) \in \hat{M}\) to \((\eta, \gamma, \bar{\phi}, \Gamma, \bar{\chi}, \chi) \in M \times \mathbb{R}^2\). Thus, in the new variables the new symplectic form \(\tilde{\Omega}\) becomes

\[
\tilde{\Omega}(v, v^*) = \omega^* \eta - \eta^* \omega + \int_0^1 (\Gamma_{+z} \bar{\phi} - \bar{\phi} \Gamma_{+z}) \, dz + \int_0^1 (\Gamma_{-z} \bar{\phi} - \bar{\phi} \Gamma_{-z}) \, dz
+ \Gamma_{+}(1) \chi^* \Gamma_{+}(1) + \Gamma_{-}(1) \chi^* \Gamma_{-}(1).
\]

Further, using variables in (2.22), the Hamiltonian now reads

\[
H = \int_0^1 \frac{1}{2(1 - \eta)} \left( \left( \Gamma_{+} + \frac{\omega \, d_+ \varrho(z - 1)}{c} - \eta \varrho \right)^2 - \varrho^2 \bar{\phi}^2 \right) \, dz
+ \int_0^1 \frac{1}{2(1 + \eta)} \left( \left( \Gamma_{-} + \frac{w \, d_- d^2(1 - z)}{c} + \eta \right)^2 - \bar{\phi}^2 \right) \, dz
+ \frac{1}{c} \int_0^1 \left( \Gamma_{+} - \eta + \frac{\omega_+ d_+ \varrho(z - 1)}{c} \right) \omega_+ d_+(z(\eta - 1) + 1) \, dz
+ \frac{1}{c} \int_0^1 \left( \Gamma_{-} - d + \frac{w_+ d_+ d^2(1 - z)}{c} \right) \omega_- d_+(z(\eta + d) - d) \, dz
- \sqrt{\beta^2 - \bar{\gamma}^2} + \beta - \frac{\alpha}{2} \eta^2 + \frac{\omega_+ d_+ \varrho^2}{6c^2} + \frac{\omega_- d_+ \varrho^2}{6c^2} + \frac{\omega_+ d_+ \varrho}{2c} - \frac{\omega_- d_+ d^2}{2c},
\]

where

\[
\bar{\gamma} = \gamma + \int_0^1 \frac{1}{\eta - 1} \left( \Gamma_{+} + \frac{\omega_+ d_+ \varrho(z - 1)}{c} - \varrho \right) \, dz + \int_0^1 \frac{z \bar{\phi}_{+z}}{\eta + d} \left( \Gamma_{-} + \frac{w_- d_- d^2(1 - z)}{c} \right) - d \right) \, dz.
\]

Notice that we have added the constants in the definition of the Hamiltonian so that \(H(0) = 0\).
The new Hamiltonian structure (2.24) gives rise to the new Hamilton’s equations which are given by

\[ \dot{\eta} = \bar{\gamma} (\beta^2 - \bar{\gamma}^2)^{-1/2}, \]

\[ \dot{\gamma} = \int_0^1 \left[ -\frac{\eta}{2(1-\eta)^2} \left( \frac{(\Gamma_{++} + \frac{\omega_{++} \phi(z-1)}{c} - \eta}{c^2} - \bar{\phi}_+^2 \right) + \frac{\eta}{2} \right] dz \]

\[ + \int_0^1 \left[ \frac{1}{2(d + \eta)^2} \left( \frac{(\Gamma_{--} + \frac{\omega_{--} d^2(1-z)}{c} - d)^2 - \bar{\phi}_{--}^2}{c^2} \right) - \frac{1}{2} \right] dz \]

\[ - \int_0^1 \left( \frac{\omega_{-+} d_{-+} (z-1)}{c} - \eta \right) \omega_{++} dz \]

\[ - \int_0^1 \left( \frac{\omega_{-+} d^2(1-z)}{c} - d \right) \omega_{++} dz \]

\[ + \frac{\bar{\gamma}}{\sqrt{\beta^2 - \gamma^2}} \left( \int_0^1 \frac{\omega_{++} d_{++} (z-1)}{c} - \eta \right) \right) \right] + \alpha \eta, \] (2.25)
along with

\[
\begin{aligned}
\dot{\chi}_+ &= \frac{1}{\eta - 1} \left( -\frac{\Gamma_+}{\eta} + \eta + \frac{\bar{\gamma} \bar{\phi}_+(1)}{\sqrt{\beta^2 - \bar{\gamma}^2}} \right), \\
\dot{\chi}_- &= \frac{1}{\eta + d} \left( \Gamma_- (1) + \eta + \frac{\bar{\gamma} \bar{\phi}_-(1)}{\sqrt{\beta^2 - \bar{\gamma}^2}} \right),
\end{aligned}
\]  

(2.26)

and the boundary conditions

\[
\begin{aligned}
\bar{\phi}_+(1) &= -\frac{\bar{\gamma}}{\eta} \left( \frac{\Gamma_+}{\sqrt{\beta^2 - \bar{\gamma}^2}} \right), \\
\bar{\phi}_-(1) &= \frac{\bar{\gamma}}{\eta} \left( \frac{\Gamma_- (1) - d}{\sqrt{\beta^2 - \bar{\gamma}^2}} \right), \\
\bar{\phi}_\pm(0) &= 0.
\end{aligned}
\]  

(2.27)

Note that, the two equations in (2.26) can be neglected since they can be recovered from the rest of the equations in (2.25).

We now proceed with linearizing (2.25) around the equilibrium point \((0, 0, 0, 0, 0, 0)\). This leads us to the linearized problem stated in terms of the operator \(L\)

\[
L \left[ \begin{array}{c}
\eta \\
\gamma \\
\bar{\phi}_+ \\
\bar{\phi}_+ \\
\bar{\phi}_- \\
\Gamma_-
\end{array} \right] = \left[ \begin{array}{c}
\frac{1}{\beta} \left( \gamma + \theta \bar{\phi}_+(1) + \int_0^1 \frac{2z \bar{\phi}_+ \omega d_+ \theta}{c} dz + \int_0^1 \frac{2z \bar{\phi}_- \omega d_- \theta}{c} dz - \bar{\phi}_-(1) \right) \\
\frac{2}{\beta} \int_0^1 \frac{\omega_+ d_+}{c} dz + 2 \int_0^1 \frac{\omega_- d_-}{c} dz \\
- \beta \left( \frac{\omega_+ d_+ \theta^2}{3c^2} + \frac{\omega_- d_- \theta^2}{c^2} \right) - \frac{1}{\beta} \int_0^1 \frac{\omega_+ d_+ \theta ^2}{c} dz + \frac{\omega_- d_- \theta ^2}{c} dz + \alpha \right] \eta \\
\frac{z(\gamma + \theta \bar{\phi}_+(1) + \int_0^1 \frac{2z \bar{\phi}_+ \omega d_+ \theta}{c} dz}{\beta} \\
\frac{z(\Gamma_+ + \omega d_+ \eta(2z - 1))}{\beta} \\
z(\frac{\gamma + \theta \bar{\phi}_+(1) + \int_0^1 \frac{2z \bar{\phi}_+ \omega d_+ \theta}{c} dz}{\beta} \\
z(\frac{2z \bar{\phi}_- \omega d_- \theta}{c} dz - \bar{\phi}_-(1)) - \theta \bar{\phi}_+ \\
z(\frac{\Gamma_- + \omega d_- \eta(2z - 1)}{d} \\
z(\frac{z(-d + \omega d_+ \theta (1 - z))}{d \beta} \left( \gamma + \theta \bar{\phi}_+(1) + \int_0^1 \frac{2z \bar{\phi}_+ \omega d_+ \theta}{c} dz \right) \\
+ \int_0^1 \frac{2z \bar{\phi}_- \omega d_- \theta}{c} dz - \bar{\phi}_-(1)) - \frac{\bar{\phi}_-}{\bar{d}} \\
\end{array} \right]
\]  

(2.28)
coupled with the linearized boundary conditions

\[
\begin{align*}
\bar{\phi}_+ (1) &= \frac{1}{\beta} \left( \gamma + \phi \bar{\phi}_+ (1) + \int_0^1 \left( \frac{2 \bar{\phi} \omega d + \varrho}{c} + \frac{2 \varphi d - \omega}{c} \right) \, dz - \bar{\phi}_-(1) \right), \\
\bar{\phi}_- (1) &= -\frac{d}{\beta} \left( \gamma + \phi \bar{\phi}_+ (1) + \int_0^1 \left( \frac{2 \bar{\phi} \omega d + \varrho}{c} + \frac{2 \varphi d - \omega}{c} \right) \, dz - \bar{\phi}_-(1) \right), \\
\bar{\phi}_\pm (0) &= 0.
\end{align*}
\]

Consider the eigenvalue problem \( L u = \lambda u \), together with the boundary conditions (2.29). Upon setting \( \lambda = ik \), we obtain the dispersion relation

\[
\alpha + \beta k^2 = \frac{k \varrho}{\tanh (k)} + \frac{k}{\tanh (kd)} + \left( \frac{\omega d + \varrho}{c} - \frac{\omega - d}{c} \right).
\]

We would like to mention that this dispersion relation is equivalent and consistent to the dispersion relation obtained in (1.6).

Our next objective in the construction of small-amplitude solutions is to apply the center manifold approach due to Mielke [Mie88] to the system (2.28) and (2.29). For convenience, the main approach used for this is outlined in the theorem stated in Appendix A, which is a version used, for instance, in [Nil17, Section 3]. Due to nonlinear boundary conditions (2.29), however, we are not able to crudely implement the theorem right away. As an intermediate step, we do change of variables via the operator \( \Phi \). This linearizes the boundary conditions at the cost of complication of the problem in the bulk. Explicitly, the operator \( \Phi \) takes the form

\[
\Phi(\eta, \gamma, \bar{\phi}_+, \Gamma_+, \bar{\phi}_-, \Gamma_-) = (\eta, \nu, \varphi_+, \Gamma_+, \varphi_-, \Gamma_-),
\]

where,

\[
\begin{align*}
\nu &= \varrho \bar{\phi}_+ (1) - \bar{\phi}_- (1), \\
\varphi_+ &= \varrho \bar{\phi}_+ + W \left( A[\Gamma_+] (z) - \frac{\varrho}{2} (z^2 - \frac{1}{3}) \right), \\
\varphi_- &= \bar{\phi}_- - W \left( A[\Gamma_-] (z) - \frac{d}{2} (z^2 - \frac{1}{3}) \right),
\end{align*}
\]

and

\[
\begin{align*}
W &= \frac{\tilde{\gamma}}{\sqrt{\beta^2 - \tilde{\gamma}^2}}, \\
A[f](s) &= \int_0^s s f(s) \, ds = \int_0^1 \int_0^s s f(s) \, ds \, dz.
\end{align*}
\]

One may check easily that \( \varphi_\pm (1) = \varphi_\pm (0) = 0 \). Further, via the definition of \( \varphi_\pm \) in (2.32), one can check that

\[
\int_0^1 \varphi_\pm \, dz = 0 \quad \text{provided} \quad \int_0^1 \bar{\phi}_\pm \, dz = 0.
\]
It is also worth noting that the operator $\mathfrak{G}$ is invertible in some neighborhood of the origin and its inverse is explicitly given by

$$
\mathfrak{G}^{-1} \begin{bmatrix}
\eta \\
\nu \\
\varphi_+ \\
\Gamma_+ \\
\varphi_- \\
\Gamma_-
\end{bmatrix} = \begin{bmatrix}
\beta R & \eta & -I - \Pi \\
\frac{\beta R}{\sqrt{1 + R^2}} & \eta & -I - \Pi \\
\varphi_+ - R & \frac{A[\Gamma_+](z)}{c} & \Gamma_+ \\
\varphi_- + R & \frac{A[\Gamma_-](z)}{c} & \Gamma_-
\end{bmatrix},
$$

where

$$
I = \int_0^1 \frac{z}{\eta - 1} \left( \varphi_+ - Rz(\Gamma_+ - \varphi_+ + \frac{\omega_+ d_+ g(z - 1)}{c}) \right) \left( \Gamma_+ - \varphi_+ + \frac{w_+ d_+ g}{c}(z - 1) \right) dz,
$$

$$
II = \int_0^1 \frac{z}{\eta + d} \left( \varphi_- - Rz(\Gamma_- - d + \frac{\omega_- d_+ d^2(1 - z)}{c}) \right) \left( \Gamma_- - d + \frac{w_- d_+ d^2}{c}(1 - z) \right) dz,
$$

$$
R = \frac{\varphi_+(1) - \varphi_-(1) - \nu}{A[\Gamma_+](1) + A[\Gamma_-](1) - \frac{\eta + \varphi}{3}}.
$$

In this new coordinate system, the Hamiltonian exhibits a new expression:

$$
H = \int_0^1 \frac{1}{2 \varphi(1 - \eta)} \left( \left( \Gamma_+ + \frac{\omega_+ d_+ g(z - 1)}{c} - \eta \varphi_+ \right)^2 - (\varphi_+ - Rz(\Gamma_+ - \varphi))^2 \right) dz \\
+ \int_0^1 \frac{1}{2(d + \eta)} \left( \left( \Gamma_- + \frac{\omega_- d_+ d^2(1 - z)}{c} + \eta \right)^2 - (\varphi_- + Rz(\Gamma_- - d))^2 \right) dz \\
+ \frac{1}{c} \int_0^1 \left( \Gamma_+ - \varphi_+ + \frac{\omega_+ d_+ g(z - 1)}{c} \right) \omega_+ d_+(z(\eta - 1) + 1) dz \\
+ \frac{1}{c} \int_0^1 \left( \Gamma_- - d_+ + \frac{\omega_- d_+ d^2(1 - z)}{c} \right) \omega_- d_+(z(\eta + d) - d) dz \\
- \frac{\beta}{\sqrt{1 + R^2}} + \beta + \frac{\alpha}{2} \eta^2 + \frac{\gamma^2 \varphi^2}{6c^2} + \frac{\omega^2 d^2 d^3}{6c^2} + \frac{\omega_+ d_+ \varphi}{2c} - \frac{\omega_- d_+ d^2}{2c}.
$$

(2.33)
Consequently, the Hamilton’s equations now become

\[
\dot{\eta} = R,
\]

\[
\dot{v} = \frac{1}{\eta - 1} \left( -\Gamma_+ (1) + \frac{\omega_+ d_+}{2c} + R \left[ -R(\Gamma_+ - d) - \varphi_+ (1) + R \left( A[\Gamma_+] (1) - \frac{\varrho}{3} \right) \right] \right) - \frac{1}{\eta + d} \left( -\Gamma_- (1) + \frac{\omega_- d_- d^2}{2c} + R \left[ R(\Gamma_- - d) - \varphi_- (1) - R \left( A[\Gamma_-] (1) - \frac{d}{3} \right) \right] \right),
\]

\[
\dot{\varphi}_+ = \frac{1}{\eta - 1} \left( -\Gamma_+ + \frac{\omega_+ d_+ (2z - 1)}{2c} + R \left[ z \varphi_+ - Rz^2 (\Gamma_+ d - \varrho) \right] - \varphi_+ (1) + R \left( A[\Gamma_+] (1) - \frac{\varrho}{3} \right) \right) + \frac{\dot{\gamma} (1 + R^2)^{3/2}}{\eta} \left( A[\Gamma_+] (z) - \frac{\varrho}{3} (z^2 - 1/3) \right) + \frac{RA[\varphi_+] (z)}{\eta - 1},
\]

\[
\dot{\Gamma}_+ = \frac{1}{\eta - 1} \varphi_+,
\]

\[
\dot{\varphi}_- = \frac{1}{\eta + d} \left( -\Gamma_- + \frac{\omega_- d_- d^2 (1 - 2z)}{2c} + R \left[ z \varphi_- + Rz^2 (\Gamma_- - d) \right] - \varphi_- (1) - R \left( A[\Gamma_-] (1) - \frac{d}{3} \right) \right) + \frac{\dot{\gamma} (1 + R^2)^{3/2}}{\eta} \left( A[\Gamma_-] (z) - \frac{d}{3} (z^2 - 1/3) \right) + \frac{RA[\varphi_-] (z)}{\eta + d},
\]

\[
\dot{\Gamma}_- = \frac{-1}{\eta + d} \varphi_-,
\]

where

\[
\dot{\gamma} = (1 + R^2) \left( \frac{(\Gamma_+ + \frac{\omega_+ d_+ \varrho (z - 1)}{c} - \varrho)^2}{2\varrho (\eta - 1)^2} + \frac{(\Gamma_- + \frac{\omega_- d_- d^2 (1 - z)}{c} - d)^2}{2(\eta + d)^2} \right) + \frac{\varrho - 1}{2} + \alpha \eta.
\]

Recall that we are interested in the solutions that satisfy the condition \((\beta, \alpha) = (\beta, \alpha_0) + (0, \epsilon^2)\) with \(\beta > \beta_0\) where these parameters are defined in (1.5). It can be shown that the imaginary part of the spectrum of the linearized operator \(\mathcal{L}\) consists of zero, which is an eigenvalue of (algebraic) multiplicity 2 when \(\alpha = \alpha_0\) and \(\beta = \beta_0\), as given by (1.7). The associated eigenvector and the generalized eigenvector, namely \(e_1\) and \(e_2\), of the zero
eigenvalue are then computed. Explicitly, they take the form

\[
e_1 = \begin{pmatrix}
1 \\
0 \\
0 \\
\frac{1}{c} \omega_+ d_+ \theta (z - z^2) \\
1 \\
0 \\
0 \\
\frac{1}{c} \omega_- d_- (z - z^2)
\end{pmatrix},
\]

\[
e_2 = \begin{pmatrix}
1 \\
0 \\
\frac{0}{\beta - \frac{\varrho + d}{3} - \frac{1}{12c} (z^2 - 1/3)} \\
\frac{2}{\beta - \frac{\varrho + d}{3} - \frac{1}{12c} (z^2 - 1/3)} \\
\frac{0}{\beta - \frac{\varrho + d}{3} - \frac{1}{12c} (z^2 - 1/3)} \\
\frac{0}{\beta - \frac{\varrho + d}{3} - \frac{1}{12c} (z^2 - 1/3)}
\end{pmatrix}.
\]

It is straightforward to check that \( L e_1 = 0 \) and \( L e_2 = e_1 \) with \( \tilde{\Omega}(e_1, e_2) = \beta - \frac{\varrho + d}{3} =: \beta_* \), provided \( \alpha = \varrho + \frac{1}{d} + \frac{\omega_+ d_+ \varrho}{c} - \frac{\omega_- d_+}{c} \).

Let

\[ v_1 = \frac{e_1}{\sqrt{\beta_*}} \quad \text{and} \quad v_2 = \frac{e_2}{\sqrt{\beta_*}}. \]

They, indeed, form a symplectic basis of the vector space spanned by the eigenvectors \( e_1 \) and \( e_2 \). Let us define \( f_i := d\Theta(0)(v_i) \).

Upon applying the center manifold theorem along with Darboux’s theorem, we obtain a Hamiltonian system \((X^\mu_C, \Psi, \tilde{H}^\mu)\),

\[ X^\mu_C = \{ u_1 + r(u_1, \mu) : u_1 \in \tilde{U}_1 \} \]

and \( \tilde{U}_1 \) is a neighborhood of 0 as stated in Appendix A. We would like to note here that all hypothesis \( H1 - H4 \) in the center manifold theorem are satisfied. As a conclusion, we obtain small bounded solutions on the two-dimensional center manifold. Precisely, every solution \( u_1 \) can be represented as

\[ u_1 = (q, p) = q f_1 + p f_2, \]

where

\[
f_1 = \frac{1}{\sqrt{\beta_*}} \begin{pmatrix}
1 \\
0 \\
0 \\
\frac{1}{c} \omega_+ d_+ \theta (z - z^2) \\
1 \\
0 \\
0 \\
\frac{1}{c} \omega_- d_- (z - z^2)
\end{pmatrix},
f_2 = \frac{1}{\sqrt{\beta_*}} \begin{pmatrix}
0 \\
\frac{\varrho + d}{3} \\
0 \\
\frac{0}{3} \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Upon completing a number of nontrivial computations, the Taylor expansion of the reduced Hamiltonian is derived and explicitly given by

\[
\tilde{H}^\mu(q, p) = \frac{1}{2} p^2 - \frac{1}{2 \beta_*} e^2 q^2 + \frac{\varrho - \frac{1}{d^2}}{c} + \frac{\omega_+ d_+ \varrho}{c d} + \frac{\omega_- d_+ \varrho}{c d^2} + \omega_+^2 d_+^2 \varrho - \frac{\omega_- d_+^2}{3} \varrho^2 - \frac{\omega_+ d_+^2}{3} \varrho^2
\]

\[ + O(|p||q|(|e^2, p, q|) + O(|(p, q)|^2(|e^2, p, q|^2)). \]
From there, we obtain the corresponding Hamilton’s equations
\[ q_x = p + O(\langle p, q \rangle \epsilon^2, p, q), \]
\[ p_x = \frac{\epsilon^2 q}{\beta_*} + \frac{-\varrho + \frac{1}{\Lambda_d^2} (\omega_d \varrho - \frac{\omega_d d^2 \varrho}{c} - \frac{\omega_d d^2 \varrho}{cd}) + O(\langle p, q \rangle \epsilon^2, p, q)}{2\beta_*^{3/2}} + \frac{O(\langle p \rangle \epsilon^2, p, q)}{\Lambda_d^2} + O(\langle p, q \rangle^2 \epsilon^2). \] (2.38)

Consider the following rescaling
\[ X = \frac{\epsilon}{\sqrt{\beta_*}} x, \quad q(x) = \beta_*^2 \epsilon^2 Q(X), \quad p(x) = \epsilon^3 \beta_*^{3/2} P(X). \] (2.39)

Under these rescaling, the Hamilton’s equations in (2.38) read
\[ Q_X = P + O(\epsilon), \]
\[ P_X = Q + 3K(q, d, \omega_+, \omega_-, c) \frac{Q^2}{2} + O(\epsilon), \] (2.40)
where
\[ K(q, d, \omega_+, \omega_-, c) = \beta_*^{3/2} \left( -\varrho + \frac{1}{\Lambda_d^2} (\omega_d \varrho - \frac{\omega_d d^2 \varrho}{c} - \frac{\omega_d d^2 \varrho}{cd}) + O(\langle p, q \rangle \epsilon^2, p, q) \right). \] (2.41)

Upon truncating the rescaled Hamilton’s equations in (2.40), we obtain
\[ Q_X = P, \quad P_X = Q + 3K(q, d, \omega_+, \omega_-, c) \frac{Q^2}{2}, \] (2.42)
which has solutions
\[ Q(X) = -\text{sech}^2(X/2) \frac{1}{K(q, d, \omega_+, \omega_-, c)}, \]
\[ P(X) = \text{sech}^2(X/2) \frac{\tanh(X/2)}{K(q, d, \omega_+, \omega_-, c)}. \] (2.43)

Thanks to the structure of the symplectic basis in (2.35), we then obtain the profile of \( \eta \) in the original variables
\[ \eta(x) = \frac{d_+ \epsilon^2 \text{sech}^2\left( \frac{\epsilon x}{2d_+ \sqrt{\beta_*}} \right)}{\varrho - \frac{1}{\Lambda_d^2} (\omega_d \varrho - \frac{\omega_d d^2 \varrho}{c} - \frac{\omega_d d^2 \varrho}{cd}) + O(\epsilon^3)}. \]

Observe that, depending on the sign of the denominator in the expression above, we obtain a wave of depression or elevation. Hence, the proof of Theorem 1.1 is now complete.
3. **Stability**

3.1. **General theory.** Having proved the existence of small-amplitude waves, in the remaining part of the work, we will investigate the aspect concerning their orbital stability/instability. For that, we are using the general theory stated in [VWW20], which is a variant of the well known GSS machinery introduced in [GSS90a, GSS90b]. As it is to any mathematical approach, there are a number of preliminary assumptions that first have to hold before applying the theory. For the water wave problem, however, there are some conditions that obstruct a direct use of the classical GSS approach. The variant in [VWW20] essentially solves these issues by weakening some of the requirements in GSS which then permits its application to the water wave problem. Although, the assumptions are weakened, the final conclusions of the both approaches in [VWW20] and [GSS90a, GSS90b] remain the same. We have outlined all the required hypothesis below and we will refer to them again later in the paper. For more in-depth and detailed explanations on the general theory, see [VWW20, Section 2] and the references therein.

**Assumption 1** (Spaces). Let $X, V, W$ be spaces defined by (4.13), (4.14), and (4.16). Assume there exists a constant $C > 0$ and $\theta \in (0, 1]$, so that the following inequality holds

$$\|u\|_V^3 \leq C \|u\|_X^{2+\theta} \|u\|_W^{1-\theta},$$

for any $u \in W$. (3.1)

Let $O \subset X$ be an open set where solutions live. Assume that $J : D(J) \subset X^* \rightarrow X$ is a closed linear operator and for any $u \in O \cap V$.

**Assumption 2** (Poisson map).

1. The domain $D(J)$ is dense in $X^*$.
2. $J$ is injective.
3. For each $u \in O \cap V$, $J(u)$ is skew-adjoint, that is

$$\langle J(u)v, w \rangle = -\langle v, J(u)w \rangle,$$

for all $v, w \in D(J)$.

**Assumption 3** (Derivative extension). Assume that there exist (extension) mappings $\nabla E, \nabla P \in C^0(O \cap V,V^*)$ of $DE(u)$ and $DP(u)$ respectively for all $u \in O \cap V$.

Suppose also that there exists a family of affine maps $T(s) : X \rightarrow X$ parameterized by $s$ where the linear part $dT(s) := T(s)u - T(s)0$ enjoys a number of properties.

**Assumption 4** (Symmetry group). The symmetry group $T(\cdot)$ satisfies

1. (Invariance) The neighborhood $O$, and the subspaces $V$ and $W$, are all invariant under the symmetry group. Moreover $T^{-1}D(J)$ is invariant under the linear symmetry group i.e. $D(J)$ is invariant under the adjoint $dT^*(s) : X^* \rightarrow X^*$.
2. (Flow property) Assume $T(0) = dT(0) = Id_X$, for any $r, s \in \mathbb{R}$, we have

$$T(s + r) = T(s)T(r), \quad dT(s + r) = dT(s)dT(r).$$

(3.3)

3. (Unitary) The linear part $dT(s)$ is a unitary operator on $X$ and an isometry on $V$ and $W$ for each $s \in \mathbb{R}$.
STABILITY OF SOLITARY WAVES IN TWO-LAYER WATER

(4) (Strong continuity) The symmetry group is strongly continuous on $X, V$, and $W$. 

(5) (Affine part) The function $T(\cdot)0$ belongs to $C^3(\mathbb{R}; W)$ and there exists an increasing function $\iota : [0, \infty) \to [0, \infty)$ such that 

$$\|T(s)0\|_W \neq \iota(\|T(s)0\|_X), \text{ for all } s \in \mathbb{R}. \quad (3.4)$$

(6) (Commutativity with $J$) For all $s \in \mathbb{R}$, 

$$JIT(s) = T(s)JI. \quad (3.5)$$

(7) (Infinitesimal generator) The infinitesimal generator of $T$ is the affine mapping 

$$T'(0)u = \lim_{s \to 0} \frac{T(s)u - u}{s} = dT'(0) + T'(0)0, \quad (3.6)$$

with dense domain $D(T'(0)) \subset X$ consisting of all $u \in X$ such that the limit above exists in $X$ (similarly for the spaces $V$ and $W$). Assume also that $\nabla P(u) \in D(J)$ and that 

$$T'(0)u = J(u)\nabla P(u), \quad (3.7)$$

for all $u \in D(T'(0)|_V) \cap O$. 

(8) (Density) The subspace 

$$D(T'(0))|_W \cap \text{Rng.} J \quad (3.8)$$

is dense in $X$. 

(9) (Conservation) For all $u \in O \cap V$, the energy is conserved by the flow of the symmetry group, meaning 

$$E(u) = E(T(s)u), \text{ for all } s \in \mathbb{R}. \quad (3.9)$$

We call $u \in C^1(\mathbb{R}; O \cap W)$ to be a bound state of the abstract Hamiltonian (1.16), if $u$ is a solution of the form 

$$u(t) = T(ct)U, \quad (3.10)$$

for some $c \in \mathbb{R}$ and $U \in O \cap W$. 

We would like to point out that the water wave problem in the present setting satisfies Assumption 4. In order to see this, one has to go through a series of lengthy, yet elementary, computations. Therefore, we will avoid checking the majority of the requirements in Assumption 4 here. Instead, we will focus more on showing that Assumption 4(8) holds: this is precisely one of requirements from the general theory in [GSS90a] that has been weakened and modified in [VWW20].

Assumption 5 (Bound states). There exists a one-parameter family of bound state solutions $\{U_c : c \in \mathcal{I}\}$ to the Hamiltonian system (1.16),

(1) The mapping $c \in \mathcal{I} \mapsto U_c \in O \cap W$ is of class $C^1$.

(2) The non-degeneracy condition $T'(0)U_c \neq 0$ holds for every $c \in \mathcal{I}$. Equivalently, $U_c$ is never a critical point of the momentum.
(3) For all $c \in \mathcal{I}$,
\[ U_c \in \mathcal{D}(T''''(0)) \cap \mathcal{D}(JIT'(0)), \tag{3.11} \]
and
\[ JIT''(0)U_c \in \mathcal{D}(T'(0)|_{\mathcal{W}}). \tag{3.12} \]

(4) It holds that $\lim \inf_{|s| \to \infty} \|T(s)U_c - U_c\|_{X} > 0$.

**Assumption 6** (Spectrum). The operator $D^2 E_c(U_c) \in \text{Lin}(\mathcal{V}, \mathcal{V}^*)$ extends uniquely to a bounded linear operator $H_c : X \to X^*$ such that:

1. $I^{-1}H_c$ is a self-adjoint operator on $X$.
2. The spectrum of $I^{-1}H_c$ satisfies
   \[ \text{spec}(I^{-1}H_c) = \{-\mu_c^2\} \cup \{0\} \cup \Sigma_c, \tag{3.13} \]
   where $-\mu_c^2 < 0$ is a simple eigenvalue that correspond to a unit eigenvector $\chi_c$, $0$ is a simple eigenvalue generated by $T$, and $\Sigma_C \subset (0, \infty)$ is bounded away from 0.

### 3.2. Notion on stability/instability.

After outlining required assumptions for the general theory, we now proceed to define the notion on stability/instability concerned here. At this point, the functions spaces that we are working with are still abstract and will be specified soon in the next subsection. Fix a bound state $U_c$ and radius $r > 0$, we define the following sets
\[ U^X_r := \{ u \in \mathcal{O} : \inf_{s \in \mathbb{R}} \|u - T(s)U_c\|_X < r \}, \]
\[ U^\mathcal{W}_r := \{ u \in \mathcal{O} \cap \mathcal{W} : \inf_{s \in \mathbb{R}} \|u - T(s)U_c\|_{\mathcal{W}} < r \}. \tag{3.14} \]

Fix $R > 0$, let $B^\mathcal{W}_R$ denote the intersection between the ball of radius $R$ centered at the origin in $\mathcal{W}$ and the set $\mathcal{O}$.

**Definition 3.1.** The bound state $U_C$ is conditionally orbitally stable provided that for any $r > 0$ and $R > 0$, there exists $r_0 > 0$ such that if $u : [0, t_0] \to B^\mathcal{W}_R$ is a solution to (1.3) where $u(0) \in U^X_{r_0}$ then $u(t) \in U^X_r$ for all $t \in [0, t_0)$.

Note that the definition of orbital stability above is the same as the definition of stability introduced earlier in Section 1. Here, we are presenting it in a more general and abstract manner.

From the general theory [GSS90a, GSS90b, VW20], the conclusion on stability/instability can be determined by looking at the sign of the second derivative of a scalar-valued function known as **moment instability**
\[ d(c) := E_c(U_c) = E(U_c) - cP(U_c). \tag{3.15} \]

This leads us to state the following theorem.

**Theorem 3.1** (Stability/Instability). Let all Assumptions 1-6 be satisfied. The bound state $U_c$ under consideration is conditionally orbitally stable (unstable), provided $d''(c) > 0$ ($< 0$).
4. Hamiltonian Formulation

4.1. Nonlocal operators. We begin this section by reformulating the governing equations (1.3) in terms of variables restricted to the interface \( y = \eta(x, t) \) in the spirit of Zakharov–Craig–Sulem. Although this way of formulating the problem forces us to work with some complicated non-local operators (pseudo-differential operators), it simplifies the problem by pushing all the unknowns to the boundary, in this case, the internal interface. The idea was then adopted by a number of authors studying internal waves, for instance, [BB97], [CG00].

Let \( \xi \) be defined as the trace of \( \phi \) on the interface \( y = \eta(x, t) \) for the upper and lower regions of the fluid. It is clear that

\[
\xi_\pm = (\partial_x \phi_\pm)|_{y=\eta} + \eta'(\partial_y \phi_\pm)|_{y=\eta}.
\]  

(4.1)

Additionally, we denote \( \mathcal{H}_\pm \) as the Hilbert transform acting on \( \xi \):

\[
\mathcal{H}_\pm(\eta)\xi_\pm = \tilde{\psi}_\pm(t, x, \eta) = \psi_\pm(t, x, \eta) + \frac{\omega_\pm}{2} \eta^2.
\]  

(4.2)

For later use, let us also introduce the Dirichlet–Neumann operator in \( \Omega_+ \) and \( \Omega_- \) (for a fixed \( \eta \)):

\[
G_\pm(\eta)\xi_\pm := \langle \eta' \rangle (N_\pm \cdot \nabla \mathcal{H}_\pm(\eta)\xi_\pm),
\]

where \( N_\pm \) is the outward unit normal relative to the domain \( \Omega_\pm \) along the internal interface \( \mathcal{S} \) and the Japanese bracket \( \langle \cdot \rangle := \sqrt{1 + |\cdot|^2} \). Further, the notation \( \mathcal{H}_\pm(\eta)\xi_\pm \) denotes the harmonic extension of \( \xi_\pm \) to \( \Omega_\pm \) and uniquely solves

\[
\begin{cases}
\Delta \mathcal{H}_\pm(\eta)\xi_\pm = 0 & \text{in } \Omega_\pm, \\
\mathcal{H}_\pm(\eta)\xi_\pm = \xi_\pm & \text{on } y = \eta(x, t), \\
\partial_y \mathcal{H}_\pm(\eta)\xi_\pm = 0 & \text{on } y = \pm d_\pm.
\end{cases}
\]

(4.3)

It is known that for \( \eta \in H^{k_0 + 1/2}(\mathbb{R}) \), the Dirichlet–Neumann operator \( G_\pm(\eta) \) is an isomorphism \( \dot{H}^k(\mathbb{R}) \to \dot{H}^{k-1}(\mathbb{R}) \), where \( \dot{H}^k \) denotes the usual homogeneous Sobolev space of order \( k \) and \( k \in [1/2 - k_0, 1/2 + k_0] \) for any real number \( k_0 > 1/2 \). The operator \( \mathcal{H}_\pm(\eta) \) is a bounded mapping from \( H^k(\mathbb{R}) \) to \( H^{k+1/2}(\Omega_\pm) \) and from \( \dot{H}^k(\mathbb{R}) \) to \( \dot{H}^{k+1/2}(\Omega_\pm) \).

Remark 4.1. The spaces \( H^k \) and \( \dot{H}^k \) are the Sobolev and homogeneous Sobolev spaces, respectively.

Using the operators \( G_\pm(\eta), \mathcal{H}_\pm(\eta) \), and the new unknown \( \xi \), the water wave problem can be pushed to the boundary \( y = \eta(x, t) \). Now, it reads

\[
\begin{cases}
\eta_t = \mp G_\pm(\eta)\xi_\pm + \omega_\pm \eta \eta_x & \text{on } y = \eta(x, t), \\
[\rho \xi_\pm] = -g \eta [\rho] - \sigma \left( \frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right)_x & \text{on } \pm \mathcal{S}, \\
-\sum_{\pm} \pm \frac{\rho_\pm}{2} \Gamma_\pm(\eta, \xi_\pm) \mp \rho_\pm \omega_\pm \eta \xi_\pm \mp \rho_\pm \omega_\pm \mathcal{H}_\pm(\eta)\xi_\pm & \text{on } y = \eta(x, t),
\end{cases}
\]

(4.4)
where
\[ \Gamma_\pm(\eta, \xi_\pm) = \left( \xi^2_{x_\pm} - (G_\pm(\eta)\xi_\pm)^2 \pm 2\eta_x \xi_x \pm G_\pm(\eta)\xi_\pm \right) / (1 + \eta_x^2). \] (4.5)

Next, in order to reformulate the problem in a Hamiltonian language, we introduce the variable \( \tilde{\xi} := -[\rho \xi] \). Recall, from the kinematic boundary condition in (4.4) we have
\[ G_-(\eta)\xi_- + G_+(\eta)\xi_+ = [\omega] \eta_{tx}. \] (4.6)

This, together with the definition of \( \tilde{\xi} \), yields
\[ \pm G_\pm \tilde{\xi} = B(\eta)\xi_\pm - \rho_\pm [\omega] \eta_{tx}, \] (4.7)

where \( B(\eta) := \rho_+ G_-(\eta) + \rho_- G_+(\eta) \). Following the property of \( G_\pm(\eta) \), the operator \( B(\eta) \) is also bounded and linear from \( H^k(\mathbb{R}) \) to \( H^{k-1}(\mathbb{R}) \) and from \( \dot{H}^k(\mathbb{R}) \) to \( \dot{H}^{k-1}(\mathbb{R}) \) for any \( \eta \in H^{k_0+1/2}(\mathbb{R}) \) with \( k_0, k \) given as before. Moreover, \( B(\eta) \) is an isomorphism from \( \dot{H}^k(\mathbb{R}) \) to \( \dot{H}^{k-1}(\mathbb{R}) \). Therefore, using (4.7) and solving for \( \xi_\pm \) we obtain
\[ \xi_\pm = \mp B(\eta)^{-1} G_\mp(\eta)\tilde{\xi} + \rho_\mp B(\eta)^{-1} [\omega] \eta_{tx}. \] (4.8)

The kinematic and dynamic conditions now read
\[
\begin{align*}
\eta_t &= A(\eta)\tilde{\xi} + \rho_\pm G_\pm(\eta)B(\eta)^{-1} [\omega] \eta_{tx} + \omega_\pm \eta_{tx} & \text{on } y = \eta(x, t), \\
\tilde{\xi}_t &= g\eta [\rho] + \sigma \left( \frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right)_x + \sum_{\pm} \left[ \pm \frac{\rho_\pm}{2} \Gamma_\pm(\eta, \xi_\pm) \mp \rho_\pm \omega_\pm \eta_x \pm \rho_\pm \omega_\pm \mathcal{H}_\pm(\eta)\xi_\pm \right] & \text{on } y = \eta(x, t), 
\end{align*}
\] (4.9)

where \( A(\eta) := G_\pm(\eta)B(\eta)^{-1} G_\mp(\eta) \).

The above equations can alternatively be written as
\[
\begin{align*}
\eta_t &= A(\eta)\tilde{\xi} + G_-(\eta)B(\eta)^{-1} \rho_+ [\omega] \eta_{tx} + \omega_- \eta_{tx} & \text{on } y = \eta(x, t), \\
\tilde{\xi}_t &= g\eta [\rho] + \sigma \left( \frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right)_x + \frac{[\rho|\nabla \psi|^2]}{2} - \eta_t [\rho \phi_y] + [\rho \omega \psi] & \text{on } y = \eta(x, t).
\end{align*}
\] (4.10)

We would like to mention that similar formulation involving constant (non-vanishing) vorticity for a one-fluid and two-fluid case can be found, for instance, in \([\text{Wah07}] \) and \([\text{Com16, CI15}] \) respectively.

**Remark 4.2.** The spaces \( H^k \) and \( \dot{H}^k \) are the Sobolev and homogeneous Sobolev spaces, respectively.

4.2. Function spaces. The formulation stated in (4.10) is crucial in helping us exploit the Hamiltonian structure of the problem. In light of that, let us informally introduce the required function spaces where the internal water wave problem will be posed. Fix \( k \geq 1/2 \), we define a product space for \( u \):
\[ \mathcal{X}_k := \mathcal{X}_1 \times \mathcal{X}_2 := H^{k+1/2}(\mathbb{R}) \times \left( \dot{H}^k(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \right). \] (4.11)
For future reference, we will denote $X^{k+}$ to mean $X^{k+\epsilon}$ for any $0 < \epsilon \ll 1$, similarly for $H^{k+}$.

**Remark 4.3.** Note that the space $H^p(\mathbb{R}) \cap \dot{H}^q(\mathbb{R})$ is dense in $H^p(\mathbb{R})$ and $\dot{H}^q(\mathbb{R})$ for all $p,q \in \mathbb{R}$. We will use this fact to verify Assumption 1 in the general theory.

Consider the following sequence of (continuously) embedded spaces

$$
\mathbb{W} \hookrightarrow \mathbb{V} \hookrightarrow X,
$$

(4.12)

where $X$ is a Hilbert space, while $\mathbb{W}$ and $\mathbb{V}$ are reflexive Banach spaces. In practice, $\mathbb{W}$ will be the local well-posedness space of the internal water wave problem. Its regularity follows from the available results on local well-posedness. The space $X$ is the natural energy space of the problem with $X^*$ being its continuous dual:

$$
X := H^1(\mathbb{R}) \times \dot{H}^{1/2}(\mathbb{R}), \quad X^* := H^{-1}(\mathbb{R}) \times \dot{H}^{-1/2}(\mathbb{R}).
$$

(4.13)

Indeed, if $u \in X$, then $\nabla \phi \pm \in L^2(\Omega \pm)$. Furthermore, it also informs us that $\eta \in H^1(\mathbb{R})$ which ensures the finiteness of the potential energy. Both combined confirms that the energy is finite on $X$. The space $\mathbb{V}$ is an intermediate space that lies between the spaces $\mathbb{W}$ and $X$ where all the conserved quantities are smooth there.

The regularity of $X$ turns out to be slightly insufficient for our analysis. This is because the map $u \mapsto G_\pm(\eta)$ is not smooth with $X$ being its domain. To fix this, the profile $\eta$ has to be, at least, Lipschitz continuous and bounded away from the rigid walls $\{y = \pm d_\pm\}$. In order to satisfy this level of smoothness condition, we define an intermediate space

$$
\mathbb{V} := X^{1+} = H^{3/2+}(\mathbb{R}) \times \left(\dot{H}^{1+}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\right)
$$

(4.14)

along with a neighborhood

$$
\mathcal{O} := \{(\eta, \tilde{\xi}) \in X : -d_- < \eta < d_+\},
$$

(4.15)

which makes sure that $\eta$ is bounded away from the rigid walls. Moreover, observe that $H^{3/2+}(\mathbb{R}) \hookrightarrow W^{1,\infty}(\mathbb{R})$, meaning if $(\eta, \tilde{\xi}) \in \mathbb{V}$, then $\eta$ is Lipschitz continuous.

Finally, since the current result on the Cauchy problem for water wave is not yet available with the level of regularity in $\mathbb{V}$, thus we define a smoother space

$$
\mathbb{W} := X^{5/2+} = H^{3+}(\mathbb{R}) \times \left(\dot{H}^{5/2+}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\right).
$$

(4.16)

It is worth mentioning that the work of Shatah and Zeng [SZ11] proves the local-wellposedness of the water wave problem at the same regularity as in $\mathbb{W}$. Additionally, we would like to point out that the regularity of these spaces is the same and largely follows from the recent work of Chen and Walsh [CW22].

Having specified the function spaces, we express the relationship between the trio function spaces $X, \mathbb{V},$ and $\mathbb{W}$ via an inequality recorded in the next lemma. Moreover, the content of the lemma shows that Assumption 1 in the general theory is satisfied.
Lemma 4.4 (Interpolation). Consider the following spaces: \( \mathbb{X}, \mathbb{V}, \) and \( \mathbb{W} \) defined by (4.13), (4.14), and (4.16), respectively. Then there exists a constant \( C > 0 \) and \( \theta \in (0, 1) \) such that

\[
\|u\|_\mathbb{V}^3 \leq C \|u\|_\mathbb{X}^{2+\theta} \|u\|_{\mathbb{W}}^{1-\theta},
\]

for all \( u \in \mathbb{W} \).

Proof. This follows from the Gagliardo–Nirenberg interpolation theorem.

The above lemma shows that a small cubic term in \( \mathbb{V} \) norm can be bounded using a quadratic term in \( \mathbb{X} \). This fact is needed in the general theory when bounding some of the terms resulted from Taylor expanding functionals whose domain is \( \mathbb{V} \cap \mathcal{O} \).

4.3. Hamiltonian Structure. In [BB97] Benjamin and Bridges formulated the internal water wave problem as a Hamiltonian system in the style of Zakharov–Craig–Sulem. Inspired by the aforementioned paper, we show that the water wave problem also exhibits a Hamiltonian structure (with a non-canonical Poisson map) that can be exploited for the stability analysis. Following the same idea as in [CW22, Section 3.3], we derive the energy functional of (4.10)

\[
E(\eta, \tilde{\xi}) = \frac{1}{2} \int_\mathbb{R} \tilde{\xi} A(\eta) \tilde{\xi} + 2\rho_+ \|\omega\| \eta \xi G^-(\eta) B^{-1}(\eta) - \rho_+ \rho_+ \|\omega\| \eta \xi B^{-1}(\eta) \|\omega\| \xi \eta_x \\
- g(\rho) \eta^2 + 2\tilde{\xi} \omega \eta_x - \frac{\eta^3}{3} \rho \omega^2 + 2\sigma(\sqrt{1 + \eta^2} - 1) \, dx.
\]

(4.17)

Observe that \( E \in C^\infty(\mathcal{O} \cap \mathbb{V}; \mathbb{R}) \). Moreover, we will show that there is an extension mapping \( \nabla E(u) \) of \( DE(u) \) defined on the dual space \( \mathbb{X}^* \) which is the content of the lemma below.

Lemma 4.5 (Energy Extension). There exists a mapping \( \nabla E \in C^\infty(\mathcal{O} \cap \mathbb{V}; \mathbb{X}^*) \) such that

\[
\langle \nabla E(u), v \rangle_{\mathbb{X}^* \times \mathbb{X}} = DE(u)v, \quad \text{for all } u \in \mathcal{O} \cap \mathbb{V}, \quad v \in \mathbb{V}.
\]

(4.18)

Proof. Fix \( u = (\eta, \tilde{\xi}) \in \mathcal{O} \cap \mathbb{V} \) and let \( \tilde{u} = (\tilde{\eta}, \tilde{\xi}) \in \mathbb{V} \) be given. Using the definition of the energy (4.17) together with the self-adjointness properties of \( A(\eta), G_\pm(\eta), \) and \( B^{-1}(\eta) \), one can show that

\[
\begin{align*}
\mathcal{D}_\eta E(u)\tilde{u} &= \\
&= \frac{1}{2} \int_\mathbb{R} \tilde{\xi} (\mathcal{D} A(\eta) \tilde{\eta} + \tilde{\xi}) \, dx - \int_\mathbb{R} \left( g(\|\omega\|) \eta + \sigma \left( \frac{\eta'}{\langle\eta'\rangle} \right) \right) \tilde{\eta} \, dx \\
&\quad + \int_\mathbb{R} \rho_+ \tilde{\xi} (\mathcal{D} G^-(\eta) \tilde{\eta} + \tilde{\xi}) \mathcal{G}_-^{-1}(\eta) \|\omega\| \eta \xi_x + \rho_+ \mathcal{G}_-^{-1}(\eta) \tilde{\xi} \mathcal{D} B^{-1}(\eta) \|\omega\| \eta \xi_x \, dx \\
&\quad + \int_\mathbb{R} \rho_+ \tilde{\xi} \mathcal{G}_-^{-1}(\eta) \mathcal{B}^{-1}(\eta) \|\omega\| (\rho \eta_x + \eta \xi_x) - \rho_+ \rho_+ \|\omega\| (\rho \eta_x + \rho \xi_x) \mathcal{G}_-^{-1}(\eta) \|\omega\| \eta \xi_x \, dx \\
&\quad - \frac{1}{2} \int_\mathbb{R} \rho_+ \rho_+ \|\omega\| \eta \xi_x \mathcal{G}_-^{-1}(\eta) \xi_x + \|\omega\| \eta \xi_x + \eta \xi_x \mathcal{D} B^{-1}(\eta) \|\omega\| (\eta \xi_x) \, dx \\
&\quad - \frac{1}{2} \left( \tilde{\xi} \omega_-(\rho \eta_x + \eta \xi_x) - \frac{\eta^2 \eta^2}{2} \rho \omega^2 \right) \, dx.
\end{align*}
\]

(4.19)
and
\[
D_\xi E(u) \dot{u} = \int_R \left( A(\eta) \dot{\xi} + \omega_- \eta x_\xi \right) \dot{\xi} \, dx + \int_R \rho_+ \dot{\xi} G_-(\eta) B^{-1}(\eta) \xi \eta \eta_x \, dx. \tag{4.20}
\]
First of all, let us look at the expression in (4.20), it is easy to see that
\[
A(\eta) \xi, \omega_\eta \eta_x, G_-(\eta) B^{-1}(\eta) \xi \eta \eta_x \in L^2(\mathbb{R}).
\]
Moreover, it is clear that the last integral in (4.20) is an element of the dual space \( X^* \) acting on \( \dot{u} \).

In (4.19), the first integral can be written as
\[
\int_R \dot{\xi} (D A(\eta) \eta, \dot{\xi}) \, dx = \sum_{\pm} \rho_\pm \int_R a_\pm^1(\eta, \theta \pm) \eta_x' \dot{\eta} \, dx + \sum_{\pm} \rho_\pm \int_R (a_\pm^2(\eta \pm, \theta \pm) A(\eta) \dot{\xi}) \dot{\eta} \, dx.
\]
Since \( u \in \mathcal{O} \cap \mathcal{V} \), we obtain
\[
a_\pm^1(\eta, \theta \pm), a_\pm^2(\eta \pm, \theta \pm) \in L^2(\mathbb{R}), \quad \theta \pm \in H^1(\mathbb{R}),
\]
where \( \theta \pm = A(\eta) G_{\pm}(\eta)^{-1} \dot{\xi} \). Further, one can directly see that the second and last integral in (4.19) is an element of the dual space \( X^* \) acting on \( \dot{u} \).

Hence the extension \( \nabla E(u) \) can be thought to have an \( L^2 \) gradient structure, that is,
\[
\langle \nabla E(u), v \rangle_{X^* \times X} = (E'(u), v)_{L^2}, \tag{4.21}
\]
where the \( L^2 \) gradient \( E'(u) = (E'_\eta(u), E'_\xi(u)) \) takes the form
\[
E'\eta(u) := \\
\frac{1}{2} \sum_{\pm} \rho_\pm a_\pm^1(\eta, \theta \pm) \theta \pm' + \sum_{\pm} \rho_\pm (a_\pm^2(\eta \pm, \theta \pm) A(\eta) \dot{\xi}) - \left( g \rho \eta + \sigma \left( \eta \rho' \eta' \right) \right) \\
+ \rho_+ \left( a_1^1(\eta, B^{-1}(\eta) \xi \eta_x) \xi' + a_2^1(\eta, B^{-1}(\eta) \eta \eta_x) G_-(\eta) \dot{\xi} \right) \\
- \rho_+ \sum_{\pm} \rho_\pm \left( a_1^1(\eta, B^{-1}(\eta) \xi \eta_x) (B^{-1}(\eta) G_-(\eta) \dot{\xi})_x \\
+ B^{-1}(\eta) G_{\pm}(\eta)(a_2^1(\eta, B^{-1}(\eta) \xi \eta_x) G_-(\eta) \dot{\xi}) \right) \\
- \rho_+ \left( \eta \xi_x \left(B^{-1}(\eta) G_-(\eta) \dot{\xi} \right)_x + \rho_+ \rho_- \left( \eta (B^{-1}(\eta) \xi \eta_x) \right)_x \\
+ \frac{1}{2} \rho_+ \rho_- \sum_{\pm} \rho_\pm \left( a_1^1(\eta, B^{-1}(\eta) \xi \eta_x) (B^{-1}(\eta) \xi \eta_x)_x \\
+ a_2^\pm(\eta, B^{-1}(\eta) \xi \eta_x) B^{-1}(\eta) G_{\pm}(\eta) \xi \eta_x \right) \\
- \omega_- \dot{\xi} \eta - \left( \rho \omega^2 \right) \eta^2, \right)
\]
\[
E'_\xi(u) := A(\eta) \dot{\xi} + \omega_- \eta x_\xi + \rho_+ G_-(\eta) B^{-1}(\eta) \xi \eta \eta_x.
\]
Hence, the proof is now complete.

The function space $\mathbb{X}$ (energy space) will be equipped with a symplectic form via a Poisson map

$$J = \begin{pmatrix} 0 & 1 \\ -1 & -\sum_{\pm} \pm \rho_{\pm} \omega_{\pm} \partial_x^{-1} \end{pmatrix} : \text{Dom } J \subset \mathbb{X}^* \to \mathbb{X},$$

(4.23)

where

$$\text{Dom } J := (H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})) \times (H^1(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R})).$$

(4.24)

Observe that from the structure of $J$ in (4.23), the difference in regularity, and homogeneity of the the spaces in (4.24), one can conclude that $J$ is not a bijection. This fact clearly violates one of the assumption in the classical GSS approach [GSS87]. This is one of the reasons why we instead use the relaxed GSS [VWW20] as it only requires that $J$ to be an injection, as stated in Assumption 2. Due to the presence of vorticity, the Poisson map is not canonical anymore. In other words, it is different with the Poisson map obtained in [CW22]. When $\omega_+ = 0$ (one-layer fluid), we recover the map presented in [Wah07].

**Lemma 4.6** (Poisson map). The Poisson map $J$ (4.23) satisfies Assumption 2 in the general theory.

**Proof.** The density of Domain $J$ (4.24) in $\mathbb{X}^*$ is a direct consequence of Remark 4.3. Further, the injectiveness and skew-adjointness of $J$ follows easily from the definition of $J$ in (4.23).

**Theorem 4.7** (Hamiltonian formulation). Let $u = (\eta, \tilde{\xi}) \in O \cap W$. Consider the following Hamiltonian system given by

$$\partial_t u = JDE(u), \quad u|_{t=0} = u_0,$$

(4.25)

with $u_0 \in O \cap W$, where $E$ is given by (4.17) and $J$ is the Poisson map in (4.23). Then $u \in C^0([0, t_0); O \cap W)$ is a (weak) solution to (4.25) if and only if the profiles $(\eta, \phi_{\pm})$ solve the governing equations (1.3).

**Proof.** Suppose that $u(t) = (\eta(t), \tilde{\xi}(t)) \in C^0([0, t_0); O \cap W)$ is a weak solution to the abstract Hamiltonian (1.16). Using (4.8), we define

$$\phi_{\pm} := \mathcal{H}(\eta) \left( \pm \mathcal{B}(\eta)^{-1} \mathcal{G}_{\pm}(\eta) \tilde{\xi} + \rho_{\pm} \mathcal{B}(\eta)^{-1} [\omega] \eta_{\pm x} \right).$$

Clearly, this satisfies the Laplace equation in (1.3a) and the boundary conditions on the walls $\{ y = \pm d_{\pm} \}$ by definition of $\mathcal{H}_{\pm}$ in (4.3). Further, recalling the expression for $E'(u)$ in Lemma 4.5, we have

$$\eta_t = E_{\tilde{\xi}}'(u) = \mathcal{A}(\eta) \tilde{\xi} + \rho_{\pm} \mathcal{G}_{\pm}(\eta) \mathcal{B}^{-1}(\eta) \omega \eta_{\pm x} + \omega_{\pm} \eta_{\pm x},$$

which holds in the distributional sense and it is equivalent to the kinematic boundary conditions in (4.10), therefore leads to the kinematic condition in (1.3b). Next, we claim that the Bernoulli condition is equivalent to

$$\tilde{\xi}_t = -E_{\eta}'(u) - \sum_{\pm} \pm \rho_{\pm} \omega_{\pm} \partial_x^{-1} E_{\tilde{\xi}}'(u).$$
We begin by writing the integrand in (4.17) in terms of $\xi_{\pm}$ instead of $\tilde{\xi}$:

$$
\frac{1}{2} \left[ \tilde{\xi} A(\eta) \tilde{\xi} + 2 \rho_+ [\omega] \eta \eta_x \mathcal{B}^{-1} \tilde{\xi} - \rho_+ \rho_- [\omega] \eta \eta_x \right]
$$

$$
- g [\rho] \eta^2 + 2 \tilde{\xi} \omega \eta \eta_x - \frac{\eta^3}{3} \left[ \frac{\rho \omega^2}{3} + 2\sigma(\sqrt{1 + \eta^2} - 1) \right]
$$

$$\begin{align*}
&= \frac{1}{2} \left[ \rho_+ \xi - \rho_+ \xi_+ G_+(\eta) \xi_+ - \rho_+ \xi_+ G_+(\eta) \xi_+ - 2 \rho_+ \rho_+ [\omega] \eta \eta_x \\
&- g [\rho] \eta^2 + 2 \tilde{\xi} \omega \eta \eta_x - \frac{\eta^3}{3} \left[ \frac{\rho \omega^2}{3} + 2\sigma(\sqrt{1 + \eta^2} - 1) \right] \right] .
\end{align*}
$$

Therefore, equivalently, the Hamiltonian (4.17) can be written as

$$
E(\xi_{\pm}, \eta) = \frac{1}{2} \int_{\mathbb{R}} \rho_+ \xi - \rho_+ \xi_+ G_+(\eta) \xi_+ - \rho_+ \xi_+ G_+(\eta) \xi_+ - 2 \rho \rho_+ [\omega] \eta \eta_x \\
$$

$$- g [\rho] \eta^2 - \frac{\eta^3}{3} \left[ \frac{\rho \omega^2}{3} + 2\sigma(\sqrt{1 + \eta^2} - 1) \right] dx .
$$

(4.26)

Thanks to the derivative formula in the Appendix B for the operator $G_\pm(\eta)$ with respect to $\eta$:

$$
\int_{\mathbb{R}} \xi_\pm (D G_\pm(\eta) \eta, \xi_\pm) dx = \int_{\mathbb{R}} \tilde{\eta} (\mp \Gamma_\pm(\eta, \xi_\pm)) dx ,
$$

(4.27)

where $\Gamma_\pm$ is defined in (4.5). Due to formula (4.27), it follows that the Bernoulli equation is satisfied.

4.4. **The symmetry group and momentum.** It is well known that the internal water wave problem is invariant under the horizontal translations. For this reason, we define a one-parameter symmetry group:

$$
T(s)u := u(\cdot - s) \quad \text{for} \; u \in X .
$$

In addition to that, this invariance also gives rise to another conserved quantity known as the momentum, $P_\pm$ in each layer:

$$
P_\pm = \pm \int_{\mathbb{R}} \left( \rho_\pm \eta \xi_\pm + \frac{1}{2} \rho_\pm \omega^2 \eta^2 \right) dx .
$$

Summing both momentum in each layers leads to the total momentum equation given by

$$
P(\eta, \tilde{\xi}) = - \int_{\mathbb{R}} \left( \eta \tilde{\xi} - \frac{1}{2} \rho [\omega] \eta^2 \right) dx .
$$

(4.28)
Observe that $P$ is a smooth functional in $O \cap V$.

The following lemma shows that $T$ and $P$ satisfy a number of properties as required by Assumption 4.

Lemma 4.8 (Conserved quantities and symmetry). The energy $E$, momentum $P$, and the translation symmetry group $T$ given above satisfy Assumptions 3 and 4. Specifically, the infinitesimal generator of $T_{X^k}$ is the unbounded linear operator $T'(0)|_{X^k} : \text{Dom} T'(0) \subset X^k \to X^k$ with dense domain $\text{Dom} T'(0)|_{X^k} := X^{k+1}$, and

$$T'(0)u = J\nabla P(u) \quad \text{for all } u \in O \cap \text{Dom} T'(0).$$

Proof. In light of Assumption 3, we have shown that the energy has an extension as stated by Lemma 4.5. Here, we will show that the momentum can also be extended. Let $u = (\eta, \tilde{\xi}) \in O \cap V$ and $\dot{u} = (\dot{\eta}, \dot{\tilde{\xi}}) \in V$. Recalling the definition of $P$ in (4.28) and computing its first variation yield

$$\mathcal{D}P(u) \dot{u} = \int_{\mathbb{R}} \left( \tilde{\xi}' + [\rho \omega] \eta \right) \dot{\eta} \, dx - \int_{\mathbb{R}} \eta_x \dot{\xi} \, dx =: \langle \nabla P(u), \dot{u} \rangle_{X^k \times X^k}.$$  

The above expression has an $L^2$ gradient

$$P'(u) = (P'_\eta(u), P'_\tilde{\xi}(u)) = (\tilde{\xi} + [\rho \omega] \eta, -\dot{\eta}').$$

Further, it is obvious that $\nabla P(u) \in \text{Dom} J$ for $u \in O \cap V$. Observe that $\text{Dom} T'(0) = X^{3/2} \subset X$. The expression (4.30) follows easily from the definition of $J$ in (4.23) and $T'(0)$ in (4.29). \hfill \Box

As it is mentioned previously, most of the requirements in Assumption 4 can be shown to hold in a straightforward manner. Hence, we will omit the details here. However, we want to focus more on Assumption 4(8). First, recall that

$$\text{Rng} J = \left( H^1(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R}) \right) \times \left( H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \right).$$

Therefore, by Lemma 4.8, we obtain

$$\text{Dom} T'(0)|_{W \cap \text{Rng} J} = \left( H^4(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R}) \right) \times \left( H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \cap \dot{H}^{7/2+}(\mathbb{R}) \right).$$

Hence, by Remark 4.3, Dom $T'(0)|_{W \cap \text{Rng} J}$ is dense in $X$.

4.5. Bound states. The existence result in Theorem 1.1 was obtained by fixing the value of $\beta$ and letting the rest of the parameters vary. Unfortunately, when using the general theory, this choice is not ideal: essentially, one might study the stability of two waves that solve two different internal water waves problem. To avoid such degenerate and nonphysical scenario, instead, we require a family of solutions parameterized only by the variable $c$, known as bound states, while fixing the rest of the physical parameters.
Let the parameters \((\rho_{\pm}, d_{\pm}, \omega_{\pm}, \sigma_{s}, c_{s})\) be fixed. We define

\[
(\beta_c, \alpha_c) := \left(\frac{\sigma_s}{d_{s\pm} \rho_{-\pm} c_{s}^2}, -\frac{g \left[\rho_{\pm}\right] d_{s\pm}}{\rho_{-\pm} c_{s}^2}\right), \quad \epsilon_c := \sqrt{\alpha_c - \alpha_0} \quad \text{for } |c - c_s| \ll 1. \tag{4.31}
\]

The pair \((\beta_c, \alpha_c)\) parameterizes a line segment joining the fixed point \((\beta_s, \alpha_s)\) to the origin in the \((\beta, \alpha)\) plane. Meanwhile, \(\epsilon_c\) plays a role as a bifurcation parameter in terms of \(c\).

Throughout this week, \(\epsilon_c\) will be kept sufficiently small.

**Corollary 4.9** (Bound states). Fix \((\rho_{\pm}, d_{\pm}, \omega_{\pm}, \sigma_{s}, c_{s})\) such that

\[g_s - \frac{1}{d_s^2} + \frac{\omega_{+} d_{+} d_s g_s}{c_s} + \frac{\omega_{-} d_{+} d_s}{c_s d_s} + \frac{\omega^2_{+} d^2_{+} d_s g_s}{3 c^2_s} - \frac{\omega^2_{-} d^2_{+} d_s}{3 c^2_s} \neq 0\]

and the fixed non-dimensional parameters \((\beta_s, \alpha_s)\) satisfies the condition \(\beta_s > \beta_0\) and \(\alpha_s = \alpha_0 + \epsilon^2\). Then there exists an open interval \(I \ni c_s\) and a family of bound states \(\{U_c\}_{c \in I} \subset \mathcal{O} \cap \mathcal{W}\) having the non-dimensional parameter values \((\beta_c, \alpha_c)\). The free surface of the bound states is

\[\eta_c := \eta_{c, \beta_c} \quad \text{for } c \in I.\]

Moreover, the family of bound states \(\{U_c\}\) satisfies Assumption 5.

**Proof.** Let \((\rho_{\pm}, d_{\pm}, \omega_{\pm}, \sigma_{s}, c_{s})\) be given. Suppose that \(\beta_s > \beta_0\) and \(0 < \alpha_s - \alpha_0 \ll 1\). For any \(c\) such that \(0 < c - c_s \ll 1\), then by Theorem 1.1, the bound states can be taken to be \(U_c := u_{c, \beta_c}\) where the surface profile depends on the parameter \(\beta_c, \alpha_c\). Observe that from the explicit expression of the profile in Theorem 1.1, it is exponentially localized. Further, due to the translation invariance of the problem, the profile \(\eta_c\) is of class \(C^\infty\). Thus, \(\eta_c \in \mathbb{X}_1^k\) for all \(k \geq 1/2\). Similarly, via the kinematic condition, \(\xi_c\) is also smooth and belongs to \(\mathbb{X}_2^k\) for all \(k \geq 1/2\). The first part of Assumption 5 now follows. Assumption 5(3) also follows from the regularity considered here. From the knowledge on the expression of \(P\) and the profile \(\eta_c\), Assumption 5(2),(4) clearly hold. \(\square\)

5. **Spectral Analysis**

If \(u(t) = T(\sigma t)U\) is a traveling wave solutions for any bound state solution \(U \in \mathcal{O} \cap \mathcal{W}\) with a wave speed \(c \in \mathbb{R}\), then by the Hamiltonian structure, Lemma 4.5, and Assumption 5(6), we have

\[
\frac{du}{dt} = c T'(0) U = JDE(U). \tag{5.1}
\]

Furthermore, recall that via (4.30), the infinitesimal generator of \(T\) satisfies the following relation

\[T'(0)(u) = J \nabla P(u),\]

where the operator \(T'(0)\) maps \(u \mapsto -\partial_x u\). In concert with (5.1), they yield

\[
\frac{du}{dt} = c T'(0) U = DE(U) = cDP(U) = DE(U).
\]
The above equation leads us to define the following functional known as the augmented Hamiltonian for a fixed speed $c$:

$$E_c(u) = E(u) - cP(u).$$

Let $u_* = (\eta_*, \xi_*)$ be the critical point of the functional $E_c$, then $D_u E(u_*) = D_\xi P(u_*)$. From the Kinematic conditions in (4.4) and (4.9), for traveling waves, we have useful expressions for $\xi_{\pm*}$ and $\bar{\xi}_*$

$$\xi_{\pm*} = \pm cG^\pm_1(\eta)\eta_x \pm \omega_\pm G^\pm_1(\eta)\eta_t = \pm cG^\pm_1(\eta)(c\eta_x + \omega_\pm \eta_t),$$

$$\bar{\xi}_* = -cA^{-1}(\eta)\eta_x - \rho_\pm [\omega] A^{-1}(\eta)G^-_-(\eta)B^{-1}(\eta)\eta_t - \omega_\pm A^{-1}(\eta)\eta_t.$$

Inspired by the notation used in [CW22], we define the $a^\pm_1(\eta, \xi)$ and $a^\pm_2(\eta, \xi)$ as follows:

$$a^\pm_1(\eta, \phi) := \mp (\partial_\xi H_\pm(\eta)\phi)|_{y=\eta} \quad a^\pm_2(\eta, \phi) := -(\partial_\phi H_\pm(\eta)\phi)|_{y=\eta}.$$

Note that these two quantities represent the horizontal and vertical velocities respectively when $\phi$ is being replaced with $\xi_{\pm*}$. Further, in relations to $a^\pm_1$ and $a^\pm_2$, we define the following functions which represent the relative velocities in horizontal and vertical directions:

$$b^\pm_1 := \mp a^\pm_1(\eta, \xi_{\pm*}) - c - \omega_\pm \eta, \quad b^\pm_2 := -a^\pm_2(\eta, \xi_{\pm*}). \tag{5.2}$$

For traveling waves solutions, the Kinematic condition can now be recast as

$$b^\pm_2 = (b^\pm_1) \eta_x.$$

Via some standard computation (in the notes), we obtain

$$D\xi_{\pm*}(\eta)\hat{\eta} = \mp G^\pm(\eta)^{-1}(b^\pm_1)\eta_x + b^\pm_2 \hat{\eta}$$

Using the definition of $\bar{\xi}_*$ and the formula for $D\xi_{\pm*}(\eta)$, we can infer

$$D\bar{\xi}_*(\eta)\hat{\eta} = \sum_\pm \rho_\pm G^\pm_1(\eta) (b^\pm_1) \eta_x - \sum_\pm \rho_\pm b^\pm_2 \hat{\eta} = : S\hat{\eta} - T\hat{\eta}. \tag{5.3}$$

Let us now define a smooth functional known as the augmented potential $V^\text{aug}_c$ as follows:

$$V^\text{aug}_c := E_c(\eta, \bar{\xi}_*) = \min_{\bar{\xi}_*} E_c(\eta, \bar{\xi}_*).$$

In the rest of this article, we shall compute the spectrum of $D^2 V^\text{aug}_c$, which will determine the spectrum of $D^2 E_c$.

**Lemma 5.1** (Second derivative of $V^\text{aug}_c$). For all $(\eta, \bar{\xi}_*(\eta)) \in \Omega \cap \forall$ and $\hat{\eta} \in H^{3/2+}$, we have the following formula

$$D^2 V^\text{aug}_c(\eta)[\hat{\eta}, \hat{\eta}] = D^2 E_c(\eta, \bar{\xi}_*)(\eta)\hat{\eta} - \int_\mathbb{R} (S - T)\hat{\eta}\cdot A(\eta)(S - T)\hat{\eta} dx. \tag{5.4}$$

**Proof.** We begin by differentiating in the direction of $\hat{\eta}$,

$$D V^\text{aug}_c(\eta)\hat{\eta} = D_\eta E_c(u_*)\hat{\eta} + D_\xi E_c(u_*) D\xi_*(\eta)\hat{\eta} = D_\eta E_c(u_*)\hat{\eta}. \tag{5.5}$$
where \( u_* = (\eta, \xi_*) \). Observe that the second term in the summation above has vanished due to its evaluation at \( u_* = (\eta, \xi_*(\eta)) \) which is a critical point of \( E_c \). Differentiating again in the direction of \( \hat{\eta} \) gives us

\[
D^2\mathcal{V}_{\text{aug}}^\eta(\eta, \hat{\eta}) = D^2\mathcal{V}_c(\eta, \hat{\eta}) + D_x^2\mathcal{V}_c(\eta, \hat{\eta})[D\xi_*(\eta)\hat{\eta}, \hat{\eta}]
\]

\[
= D^2\mathcal{V}_c(\eta, \hat{\eta}) - D^2\mathcal{V}_c(\eta, \hat{\eta})[D\xi_*(\eta)\hat{\eta}, D\xi_*(\eta)\hat{\eta}].
\]

From the definition of \( E_c \) and the fact that the momentum is linear in \( \xi \), we obtain

\[
D^2\mathcal{V}_c(\eta, \hat{\eta}) = \int_\mathbb{R} D\xi_*(\eta)\hat{\eta}A(\eta)D\xi_*(\eta)\hat{\eta} \, dx. \tag{5.6}
\]

Combining (5.6) and (5.3) leads us to the formula (5.4).

\[ \Box \]

**Lemma 5.2 (Quadratic form).** For all \((\eta, \xi_*(\eta)) \in \mathcal{O} \cap V \) and \( c \in \mathbb{R} \), there exists a self-adjoint operator \( Q_c(\eta) \in \text{Lin}(X_1; X_1^*) \) such that

\[
D^2\mathcal{V}_{\text{aug}}^\eta(\eta, \hat{\eta}) = \langle Q_c(\eta)\hat{\eta}, \hat{\xi} \rangle_{X_1^* \times X_1},
\]

for all \( \hat{\eta}, \hat{\xi} \in \mathbb{V}_1 \) and

\[
Q_c(\eta)\hat{\eta} = -
\left(
\left(a_1^+ b_1^+ + \rho_1 b_1^\pm \right)^2 + \omega_1 \eta_1 \right) \hat{\eta} + \left(a_1^+ (\eta, \xi_*(\eta)) \hat{\eta} \right)
- \left(\rho_1 \xi_*(\eta) \right) \hat{\xi} + \left(a_1^+ (\eta, \xi_*(\eta)) \hat{\xi} \right)
+ \left(\rho_1 \xi_*(\eta) \right) \hat{\eta} + \left(a_1^+ (\eta, \xi_*(\eta)) \hat{\eta} \right)
- \left(\rho_1 \xi_*(\eta) \right) \hat{\xi} + \left(a_1^+ (\eta, \xi_*(\eta)) \hat{\xi} \right)
+ \left(\rho_1 \xi_*(\eta) \right) \hat{\eta} + \left(a_1^+ (\eta, \xi_*(\eta)) \hat{\eta} \right)
- \left(\rho_1 \xi_*(\eta) \right) \hat{\xi} + \left(a_1^+ (\eta, \xi_*(\eta)) \hat{\xi} \right)
+ \left(\rho_1 \xi_*(\eta) \right) \hat{\eta} + \left(a_1^+ (\eta, \xi_*(\eta)) \hat{\eta} \right) + \left(\rho_1 \xi_*(\eta) \right) \hat{\xi} + \left(a_1^+ (\eta, \xi_*(\eta)) \hat{\xi} \right)
\]

\[ \tag{5.8} \]

**Proof.** First, it is clear that the Hamiltonian can alternatively be written in the following way

\[
E(\eta, \xi) = \frac{1}{2} \int_\mathbb{R} \xi A(\eta)\xi + \rho_1 \xi \left[\omega \eta_1 B^{-1}(\eta) \right] \eta_1 x - \rho_1 \xi \left[\omega \eta_1 B^{-1}(\eta) \right] \eta_1 x + \xi \omega_1 \eta_1 x
- g \left[\rho \right] \eta^2 - \frac{\eta^3 \rho \omega^2}{3} + 2\sigma(\sqrt{1 + \eta^2} - 1) \, dx. \tag{5.9}
\]
From 5.4, it is therefore useful to start doing an expansion on $D^2_\eta E_c(\eta, \tilde{\xi}_s)[\dot{\eta}, \ddot{\eta}]$. Using the expression of the Hamiltonian (5.9), we can see that

$$D^2_\eta E_c(\eta, \tilde{\xi}_s)[\dot{\eta}, \ddot{\eta}] = \frac{1}{2} \int_{\mathbb{R}} \tilde{\xi}_s \langle D^2 A(\eta)[\dot{\eta}, \ddot{\eta}] \tilde{\xi}_s \rangle \, dx - \int_{\mathbb{R}} g[\rho] \, \dot{\eta}^2 \, dx + \int_{\mathbb{R}} \left( \frac{\dot{\eta}_x}{\langle \dot{\eta}_x \rangle^3} \right) \, dx$$

$$- \int_{\mathbb{R}} \left[ \dot{c} [\rho \omega] \, \dot{\eta}^2 \, dx - \int_{\mathbb{R}} [\omega \xi_x \rho] \, \dot{\eta}^2 \, dx - \int_{\mathbb{R}} [\rho \omega^2] \, \dot{\eta} \dot{\eta} \, dx \right] - 2 \rho - \rho_+ \left[ \omega \partial_x \left( B^{-1}(\eta) \, \omega \right) \right] \dot{\eta} \, dx$$

$$- 2 \rho - \rho_+ \left[ \omega \partial_x \left( \eta \partial_x \left( B^{-1}(\eta) \, \omega \right) \right) \right] \dot{\eta} \, dx$$

$$- 4 \rho - \rho_+ \left[ \omega \partial_x \left( \eta \partial_x \left( B^{-1}(\eta) \, \omega \right) \right) \right] \dot{\eta} \, dx$$

$$+ \int_{\mathbb{R}} \rho - \rho_+ \left[ \omega \right] \eta \partial_x \left[ D^2 B^{-1}(\eta)[\dot{\eta}, \ddot{\eta}] \right] \eta \, dx.$$

(5.10)

It is important to note that when we let $\omega_\pm = 0$, we recover back the expression found in [CW22]. To arrive at the expression stated in (5.8), we first begin by looking at the term involving $\tilde{\xi}_s \langle D^2 A(\eta)[\dot{\eta}, \ddot{\eta}] \tilde{\xi}_s \rangle$ in (5.10). Since we are going to use some formula in [CW22], let us define the following variable

$$\theta_\pm(u_s) = G_\pm(\eta)^{-1} A(\eta) \tilde{\xi} = \mp \xi_\pm \pm \rho_+ B(\eta)^{-1} \left[ \omega \right] \eta \, dx =: \mp \xi_\pm \pm \mathbf{Y}_\mp.$$

This implies

$$a_1^\pm(\eta, \theta_\pm) = b_1^\pm + c + \omega_\pm \eta + a_1^\pm(\eta, \pm \mathbf{Y}_\mp), \quad a_2^\pm(\eta, \theta_\pm) = \pm b_2^\pm.$$

We further define the following expressions for later use:

$$S_\pm(\eta) \zeta := G_\pm(\eta)^{-1} (b_1^\pm \zeta), \quad T_\pm(\eta) \zeta := \pm b_2^\pm \zeta.$$

Upon using the formula in [CW22], we can infer that

$$\frac{1}{2} \int_{\mathbb{R}} \tilde{\xi}_s \langle D^2 A(\eta)[\dot{\eta}, \ddot{\eta}] \tilde{\xi}_s \rangle \, dx = \sum_{\pm} \mp \rho_+ \left[ \left( (b_1^\pm)_x b_2^\pm + \omega_\pm \eta b_2^\pm + (a_1^\pm(\eta, \pm \mathbf{Y}_\mp))_x b_2^\pm \right) \eta^2 + T \dot{\eta} G_\pm(\eta) \dot{\eta} \right] \, dx$$

$$+ \int_{\mathbb{R}} (-\dot{\eta} M(u_s) \dot{\eta} + \dot{\eta} N(u_s) \dot{\eta}) \, dx.$$

(5.11)

To compute the terms on the second row in (5.11), we would need to define the following expressions:

$$L_\pm(u_s) \dot{\eta} = T \dot{\eta} - S \dot{\eta} - G_\pm(\eta)^{-1} \partial_x \left( c \dot{\eta} + \omega_\pm \eta \dot{\eta} + a_1^\pm(\eta, \pm \mathbf{Y}_\mp) \dot{\eta} \right)$$

$$L(u_s) \dot{\eta} = T \dot{\eta} - S \dot{\eta} - A(\eta)^{-1} \partial_x \left( c \dot{\eta} + \omega_\pm \eta \dot{\eta} + a_1^\pm(\eta, \pm \mathbf{Y}_\mp) \dot{\eta} \right).$$

(5.12)
Furthermore,

\[
\int_{\mathbb{R}} \dot{\eta} \mathcal{M} \dot{\eta} \, dx = \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \left( (b_{1}^{\pm} + c + \omega_{\pm} \eta + a_{1}^{\pm}(\eta, \pm Y_{\mp}))(\mathcal{L}_{\pm} \dot{\eta} \right) \right)^{\prime} b_{2}^{\pm} \mathcal{G}_{\pm}(\eta) \mathcal{L}_{\pm} \dot{\eta} \, dx \\
= \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \mathcal{L}_{\pm} \dot{\eta} \mathcal{G}_{\pm}(\eta) \mathcal{L}_{\pm} \dot{\eta} \, dx.
\]

Via the definition of \( \mathcal{L}_{\pm} \) in (5.12), we can infer

\[
\int_{\mathbb{R}} \dot{\eta} \mathcal{M} \dot{\eta} \, dx = \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} (S_{\pm} \dot{\eta} \mathcal{G}_{\pm}(\eta)S_{\pm} \dot{\eta} - 2S_{\pm} \dot{\eta} \mathcal{G}_{\pm}(\eta)T_{\pm} \dot{\eta} + T_{\pm} \dot{\eta} \mathcal{G}_{\pm}(\eta)T_{\pm} \dot{\eta}) \, dx \\
+ \int_{\mathbb{R}} \left( \partial_{x} (c \dot{\eta} + \omega_{\pm} \eta \dot{\eta} + a_{1}^{\pm}(\eta, \pm Y_{\mp})), \mathcal{A}(\eta) \right) A^{-1} \partial_{x} (c \dot{\eta} + \omega_{\pm} \eta \dot{\eta} + a_{1}^{\pm}(\eta, \pm Y_{\mp})), \mathcal{A}(\eta) \right) \, dx \\
+ \int_{\mathbb{R}} 2 \partial_{x} (c \dot{\eta} + \omega_{\pm} \eta \dot{\eta} + a_{1}^{\pm}(\eta, \pm Y_{\mp})), \mathcal{A}(\eta) \right) (S - T) \dot{\eta} \, dx.
\]

For the importance of simplification later, we display the following formula

\[
\int_{\mathbb{R}} S_{\pm} \dot{\eta} \mathcal{G}_{\pm}(\eta)T_{\pm} \dot{\eta} \, dx = \pm \int_{\mathbb{R}} \mathcal{G}_{\pm}(\eta) \mathcal{G}_{\pm}(\eta)(b_{1}^{\pm} \dot{\eta}) \, dx \\
= \pm \int_{\mathbb{R}} (b_{1}^{\pm} \dot{\eta}) (b_{2}^{\pm} \dot{\eta}) \, dx \\
= \pm \frac{1}{2} \int_{\mathbb{R}} (b_{1}^{\pm})(b_{2}^{\pm})(b_{1}^{\pm})(b_{1}^{\pm}) \dot{\eta}^{2} \, dx.
\]

At last, following the formula in Appendix B, we obtain

\[
\int_{\mathbb{R}} \dot{\eta} \mathcal{N} \dot{\eta} \, dx = \int_{\mathbb{R}} \left( \mathcal{T} \dot{\eta} - 2S \dot{\eta} - \mathcal{A}(\eta) \right) \mathcal{A}(\eta) \, dx \\
= \int_{\mathbb{R}} \left( \mathcal{T} \dot{\eta} - 2S \dot{\eta} - \mathcal{A}(\eta) \right) \mathcal{A}(\eta) \, dx \\
= \int_{\mathbb{R}} \left( \mathcal{D} \dot{\xi} \mathcal{A}(\eta) \mathcal{D} \dot{\xi} \mathcal{A}(\eta) \dot{\eta} + 2 \partial_{x} (c \dot{\eta} + \omega_{\pm} \eta \dot{\eta} + a_{1}^{\pm}(\eta, \pm Y_{\mp})), \mathcal{A}(\eta) \right) (S - T) \dot{\eta} \, dx \\
+ \partial_{x} (c \dot{\eta} + \omega_{\pm} \eta \dot{\eta} + a_{1}^{\pm}(\eta, \pm Y_{\mp})), \mathcal{A}(\eta) \right) \, dx.
\]
Combining together all the computations above and the formula in (5.4), we get

\[ D^2 Y_{\text{aug}}(\eta)[\dot{\eta}, \ddot{\eta}] = D^2 E_c(\eta, \xi^*)_x[i\eta, \dot{\eta}] - \int_{\mathbb{R}} (S - T) \dot{\eta} A(\eta)(S - T) \dot{\eta} \, dx \]

\[ = \int_{\mathbb{R}} \left( \sigma \frac{(\eta_x)^2}{\langle \eta_x \rangle^3} - \left( g \frac{[\rho]}{[\omega]} + \sum_{\pm} \rho_{\pm} \left( \pm b_{\pm} \eta (b_{\pm} \eta)' + \omega_{\pm} \eta (b_{\pm} \eta)' + a_{\pm} \eta (\eta, \pm \gamma (b_{\pm} \eta)') \right) \right) \dot{\eta}^2 \right. \]

\[ - \left( \left[ \rho \xi_x \omega \right] - \eta \left[ \rho \omega^2 \right] - c \left[ \rho \omega \right] \right) \dot{\eta}^2 - \sum_{\pm} \rho_{\pm} S_{\pm} \dot{\eta} G_{\pm}(\eta) \dot{\eta} \right) \, dx \]

\[ - 2 \int_{\mathbb{R}} \rho_{-\rho} \left[ \omega \right] \partial_x \left( B^{-1}(\eta) \left[ \omega \right] \eta \eta_x \right) \dot{\eta} \, dx \]

\[ - 2 \int_{\mathbb{R}} \rho_{-\rho} \left[ \omega \right] \eta \partial_x \left( B^{-1}(\eta) \left[ \omega \right] \eta \eta_x \right) \dot{\eta} \, dx \]

\[ - 4 \int_{\mathbb{R}} \rho_{-\rho} \left[ \omega \right] \eta \partial_x \left( DB^{-1}(\eta) \dot{\eta} \left[ \omega \right] \eta \eta_x \right) \dot{\eta} \, dx \]

\[ + \int_{\mathbb{R}} \rho_{-\rho} \left[ \omega \right] \eta \partial_x \left( D^2 B^{-1}(\eta)[\dot{\eta}, \ddot{\eta}] \right) \dot{\eta} \, dx. \]  

(5.13)

To compute the last two integrals in (5.13), we need to derive the formula for

\[ \langle DB^{-1}(\eta) \dot{\eta} \left[ \omega \right] \eta \eta_x \rangle \text{ and } \langle D^2 B^{-1}(\eta)[\dot{\eta}, \ddot{\eta}] \left[ \omega \right] \eta \eta_x \rangle. \]

We begin with the expansion of the formula below

\[ \langle DB(\eta) \dot{\eta}, \dot{\xi} \rangle = \rho_+ \langle DG_+(\eta) \eta, \dot{\xi} \rangle + \rho_- \langle DG_-(\eta) \eta, \dot{\xi} \rangle \]

\[ = \rho_+ \left( - \partial_x (a_1^+ (\eta, \dot{\xi}) \dot{\eta}) + G_-(\eta) a_2^- (\eta, \dot{\xi}) \dot{\eta} \right) \]

\[ + \rho_- \left( - \partial_x (a_1^+ (\eta, \dot{\xi}) \dot{\eta}) + G_+(\eta) a_2^+ (\eta, \dot{\xi}) \dot{\eta} \right). \]  

(5.14)

Using the equation

\[ \langle DB^{-1}(\eta) \dot{\eta}, \dot{\xi} \rangle = -B^{-1}(DB(\eta) \dot{\eta}, \xi); \]

in combination with (5.14), we derive the representation formula for the Fréchet derivative of \( B^{-1}(\eta) \) (for any given \( \zeta \):}
\[
\int_{\mathbb{R}} \zeta \langle D \mathcal{B}^{-1}(\eta) \dot{\eta}, \dot{\xi} \rangle \, dx = - \int_{\mathbb{R}} \mathcal{B}^{-1} \left( \rho_+ \left( - \partial_x (a_1^-(\eta, \mathcal{B}^{-1} \dot{\xi}) \dot{\eta}) \zeta + a_2^-(\eta, \mathcal{B}^{-1} \dot{\xi}) \eta \mathcal{G}_- \zeta \right) \right) \, dx \\
- \int_{\mathbb{R}} \mathcal{B}^{-1} \left( \rho_- \left( - \partial_x (a_1^+(\eta, \mathcal{B}^{-1} \dot{\xi}) \dot{\eta}) \zeta + a_2^+(\eta, \mathcal{B}^{-1} \dot{\xi}) \mathcal{G}_+ \eta \zeta \right) \right) \, dx \\
= - \sum_{\pm} \rho_+ \int_{\mathbb{R}} \left( a_1^+(\eta, \mathcal{B}^{-1} \dot{\xi}) (\mathcal{B}^{-1} \zeta) \right) \, \dot{\eta} \dot{x} \\
+ \left( a_2^+(\eta, \mathcal{B}^{-1} \dot{\xi}) \mathcal{B}^{-1} \mathcal{G}_\pm(\eta) \zeta \right) \, \dot{\eta} \, dx.
\]
(5.15)

From (5.15), we arrive the expression in the second row from the bottom of equation (5.8).

Finally, it remains to show that the last integral in (5.13) yields the last expression in (5.8). First, recall that \( \mathcal{B}(\eta) := \sum_{\pm} \rho_\pm \mathcal{G}_\mp(\eta) \). Exploiting the second derivative formula for \( \mathcal{G}_\pm(\eta) \), we obtain

\[
\int_{\mathbb{R}} \tilde{\xi} \langle D^2 \mathcal{B}(\eta)[\dot{\eta}, \dot{\eta}], \tilde{\xi} \rangle \, dx = \sum_{\pm} \rho_\pm \int_{\mathbb{R}} \tilde{\xi} \langle D^2 \mathcal{G}_\mp(\eta)[\dot{\eta}, \dot{\eta}], \tilde{\xi} \rangle \, dx \\
= \sum_{\pm} \rho_\pm \int_{\mathbb{R}} a_1^\mp(\eta, \tilde{\xi}) \eta^2 + 2a_2^\mp(\eta, \tilde{\xi}) \eta \mathcal{G}_\pm(\eta) (a_1^\mp(\eta, \tilde{\xi}) \dot{\eta}) \, dx.
\]
(5.16)

Additionally, we have the following identity

\[
D^2 \mathcal{B}(\eta)[\dot{\eta}, \dot{\eta}] = -\mathcal{B}(\eta) D^2 \mathcal{B}^{-1}(\eta)[\dot{\eta}, \dot{\eta}] \mathcal{B}(\eta) + 2D \mathcal{B}(\eta)[\dot{\eta}] \mathcal{B}^{-1}(\eta) D \mathcal{B}(\eta)[\dot{\eta}].
\]

After rearranging terms and applying (5.16), we obtain

\[
\int_{\mathbb{R}} \rho_- \rho_+ \left[ \tilde{\omega} \right] \eta \eta_x \langle D^2 \mathcal{B}^{-1}(\eta)[\dot{\eta}, \dot{\eta}], \left[ \tilde{\omega} \right] \eta \eta_x \rangle \, dx \\
= \int_{\mathbb{R}} 2\rho_- \rho_+ \left( \sum_{\pm} a_1^\mp(\eta, \tilde{\eta}_\pm) \dot{\eta} \right) \mathcal{B}^{-1}(\eta) \left( \sum_{\pm} a_1^\mp(\eta, \tilde{\eta}_\pm) \dot{\eta} \right) \, dx
\]

Combining this with the statements above, yields the claimed formula for \( \mathcal{Q}_c(\eta) \) and completes the proof. \( \square \)

Based on the formula derived above, we can now determine the continuous spectrum of the linearized operator.

**Lemma 5.3** (Continuous spectrum). Let \( u = (\eta, \tilde{\xi}) \in \mathcal{O} \cap \mathcal{V} \) be given. Then the operator \( \mathcal{Q}_c(\eta) \) is a self-adjoint operator on \( L^2(\mathbb{R}) \) with domain \( H^2(\mathbb{R}) \). Further, the spectrum of
this operator is equal to the one of $Q_c(0)$, that is $[\tau_*, +\infty]$, where

$$\tau_* = \begin{cases} 
-g \|\rho\| \left(1 - \frac{\alpha_0}{\alpha}\right) & \text{for } \beta \geq \beta_0, \\
g \|\rho\| \left[1 - \frac{1}{\alpha} \max_{\xi \in \mathbb{R}} \left( \sum_{\pm} \frac{\rho_{\pm}}{\rho_{-}} d_{\pm} \coth(d_{\pm} \xi) - \beta d_{\pm}^2 \xi^2 + \left(\frac{\omega_{\pm} d_{\pm}}{c} - \frac{\omega_{\pm} d_{\pm}}{c}\right)\right) \right] & \text{for } \beta < \beta_0.
\end{cases}$$

Proof. The fact that $Q_c(\eta)$ is self-adjoint on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$ follows directly from the regularity of $\eta$. Also, since $\eta(x) \to 0$ as $x \to \infty$, then the continuous spectrum of $Q_c(\eta)$ coincides with that of $Q_c(0)$. Further, due to the fact that $Q_c(0)$ is translation invariant, the whole spectrum is therefore continuous. The symbol of $Q_c(0)$ is

$$q_c(\xi) = -g \|\rho\| \left[1 - \frac{1}{\alpha} \left( \sum_{\pm} \frac{\rho_{\pm}}{\rho_{-}} d_{\pm} \coth(d_{\pm} \xi) - \beta d_{\pm}^2 \xi^2 + \left(\frac{\omega_{\pm} d_{\pm}}{c} - \frac{\omega_{\pm} d_{\pm}}{c}\right)\right) \right].$$

The continuous spectrum results from looking at the range of the mapping $\xi \mapsto q_c(\xi)$, and $\tau_* = \min \{q_c(\xi) : \xi \in \mathbb{R}\}$. For $\beta \geq \beta_0$, the minimum is attained at $\xi = 0$. But for $\beta < \beta_0$, the minimum is attained at some value $\xi \neq 0$. □

5.1. Rescaled operator. Now, we will make a use of a long-wave rescaling to obtain the information of the leading-order form of the operator $Q_c(\eta)$. Assume $\beta > \beta_0$ and $\alpha = \alpha_0 + \epsilon^2$, consider the following rescaling operator:

$$S_\epsilon := f\left(\frac{\epsilon}{d_+}\right).$$

It is clear to see that $S_\epsilon$ is an isomorphism on $H^k(\mathbb{R})$. Note also that

$$\partial_x S_\epsilon = \frac{\epsilon}{d_+} S_\epsilon \partial_x, \quad \partial_x S_\epsilon^{-1} = \frac{d_+}{\epsilon} S_\epsilon^{-1} \partial_x.$$ This shows that $\partial_x S_\epsilon$ and $\partial_x S_\epsilon^{-1}$ are uniformly bounded in $\text{Lin}(H^{k+1}, H^k)$. 

Lemma 5.4 (Expansion of $\tilde{Q}_\epsilon(\eta_\epsilon)$). The operator $\tilde{Q}_\epsilon(\eta_\epsilon)$ admits the following expansion

$$\tilde{Q}_\epsilon(\eta_\epsilon) = \tilde{Q}_\epsilon(0) + \tilde{R}_\epsilon,$$

where

$$\tilde{R}_\epsilon = -3 \left( g - \frac{1}{d^2} \frac{\omega_{+} d_{+} g}{c} + \frac{\omega_{-} d_{+} g}{c d} + \frac{\omega_{+}^2 d_{+}^2 g}{3 c^2} - \frac{\omega_{-}^2 d_{+}^2 g}{3 c^2} \right) \tilde{\eta} + O(\epsilon), \quad (5.17)$$

in $\text{Lin}(H^{k+2}, H^k)$. 

Proof. Let $P_\epsilon$ be the operator obtained by evaluating operator $Q_c$ at $\eta_\epsilon$:
\[ Q_\varepsilon(\eta_k) := -\partial_x \left( \frac{a \partial_x}{\langle \eta_k \rangle^3} \right) - \left( g \| \rho \| + \sum_{\pm} \left( \pm \rho \pm b_{1e}^\pm (b_{2e}^\pm) \right) + \| \rho_\varepsilon \omega \| - \eta_k \| \rho \omega^2 \| + c \| \rho \omega \| \right) + \sum_{\pm} \rho \pm b_{1e}^\pm \partial_x G_{\pm}(\eta_k)^{-1} \partial_x b_{1e}^\pm - \sum_{\pm} \rho \pm (\mp \omega \pm \eta (b_{2e}^\pm) - a_1^\pm (\eta, \Upsilon_{\mp})(b_{2e}^\pm)') + 2 \rho_+ \rho_- \| \omega \| \partial_x B(\eta_k)^{-1} \{ \omega \} \eta_k \bar{\eta}_k + 2 \rho_+ \rho_- \| \omega \| \eta_k \partial_x (B(\eta_k)^{-1} \{ \omega \} \eta_k) \right) - 4 \rho_+ \rho_- \sum_{\pm} \left( a_1^\pm (\eta_k, \Upsilon_{\pm}) B^{-1} (\eta_k) \left( \sum_{\pm} a_1^\pm (\eta_k, \Upsilon_{\pm}) \hat{\eta}_k \right) \right).
\]

Following the same rescaling argument in [CW22], the rescaled surface tension term becomes

\[ \frac{-1}{\varepsilon^2} \frac{d_+}{\varepsilon^2 \rho_-} S_\varepsilon^{-1} \partial_x \left( \alpha \frac{\partial_x}{\langle \eta_k \rangle^3} \right) S_\varepsilon = -\partial_x \left( \frac{\beta}{\varepsilon^3 (\bar{\eta} + \bar{\eta})^3} \right). \]

Further, let us define the non-dimensionalized and rescaled relative velocity field

\[ b_{1e}^\pm := c S_\varepsilon \bar{b}_{1e}^\pm, \quad b_{2e}^\pm := c S_\varepsilon \bar{b}_{2e}^\pm. \]

Since \( b_{2e}^\pm = \eta_k (b_{1e}^\pm) \), we have \( \bar{b}_{1e}^\pm = \varepsilon^3 \bar{\eta} b_{1e}^\pm \). Using this, we obtain the following rescaled expression of the second and the third terms:

\[ \frac{-1}{\varepsilon^2} \frac{d_+}{\varepsilon^2 \rho_-} S_\varepsilon^{-1} \left( g \| \rho \| + \sum_{\pm} \left( \pm \rho \pm b_{1e}^\pm (b_{2e}^\pm) \right) \right) S_\varepsilon = \frac{\alpha}{\varepsilon^2} - \frac{1}{\varepsilon^2} \left( \frac{\omega + \rho \bar{\eta}}{c} - \frac{\omega - \rho \bar{\eta}}{c} \right) - \varepsilon^2 \sum_{\pm} \pm \rho \pm \bar{b}_{1e}^\pm (\bar{\eta} \bar{b}_{1e}^\pm)' \]

For later use in dealing with the non-local terms in \( Q_\varepsilon(\eta_k) \), we define the following two operators:

\[ \tilde{M}_\varepsilon(\eta_k) := \frac{d_+}{\varepsilon^2} S_\varepsilon^{-1} \partial_x G_{\pm}(\eta_k)^{-1} \partial_x S_\varepsilon \]

and

\[ \tilde{Z}_\varepsilon(\eta_k) := \frac{d_+}{\varepsilon^2} S_\varepsilon^{-1} \partial_x B(\eta_k)^{-1} \partial_x S_\varepsilon \]

For any \( f \in H^{k+2} \), we have

\[ F(\tilde{M}_\varepsilon(0) f)(\xi) = \frac{d_+}{\varepsilon^2} \frac{1}{d_+} F(\partial_x G_{\pm}(\eta_k)^{-1} \partial_x S_\varepsilon f)(\frac{\rho \bar{\eta}}{d_+} \xi) = \frac{d_+}{\varepsilon^2} m_{\pm}(\frac{\rho \bar{\eta}}{d_+} \xi) \tilde{f}(\xi), \]
where \( m_\pm := -\xi \coth(d_\pm \xi) \) is the symbol for \( \partial_x \mathcal{G}_\pm(0)^{-1} \partial_x \). Thus, \( \tilde{M}_\epsilon^\pm(0) \) is a Fourier multiplier with a symbol given by
\[
\tilde{m}_\epsilon^\pm(\xi) := \frac{-1}{\epsilon^2} \frac{\epsilon \xi}{\tanh(d_\pm \epsilon \xi / d_\pm)}.
\]

Additionally, for any \( f \in H^{k+2} \), we have
\[
\mathcal{F}(\tilde{Z}_\epsilon(0)f)(\xi) = \frac{d_+}{\epsilon^2} \mathcal{F}(\partial_x \mathcal{B}(\eta_\epsilon)^{-1} \partial_x S_x f)(\frac{\epsilon}{d_+} \xi) = \frac{d_+}{\epsilon^2} \tilde{b}_\epsilon^\pm(\frac{\epsilon}{d_+} \xi) \hat{f}(\xi),
\]
where \( b_\pm := -\xi / (\rho_+ \tanh(d_- \xi) + \rho_- \tanh(d_+ \xi)) \) is the symbol for \( \partial_x \mathcal{B}(0)^{-1} \partial_x \). Thus, \( \tilde{Z}_\epsilon(0) \) is a Fourier multiplier with a symbol given by
\[
\tilde{b}_\epsilon^\pm(\xi) := \frac{-1}{\epsilon^2} \frac{\epsilon \xi}{(\rho_+ \tanh(d\epsilon \xi) + \rho_- \tanh(\epsilon\xi))}.
\]
As a result we get
\[
\| \epsilon^2 \tilde{M}_\epsilon^\pm(0) \|_{\text{Lin}(H^{k+2}, H^k)} \leq \| \frac{1}{\epsilon^2} \left( \epsilon^2 \tilde{m}_\epsilon^\pm + \frac{d_+}{d_\pm} \right) \| \lesssim \epsilon^2, \quad (5.22)
\]
and
\[
\| \epsilon^2 \tilde{Z}_\epsilon(0) \|_{\text{Lin}(H^{k+2}, H^k)} \leq \| \frac{1}{\epsilon^2} \left( \epsilon^2 \tilde{b}_\epsilon^\pm + \frac{d_+}{d_- + d_+} \right) \| \lesssim \epsilon^2. \quad (5.23)
\]
In particular, we obtain
\[
\hat{Q}_\epsilon(0) = \frac{1}{\epsilon^2} \left( -\epsilon^2 \beta \partial_x^2 + \alpha - \left( \frac{\omega_+ d_+ + \theta}{c} - \frac{\omega_- d_+}{c} \right) + \sum_{\pm} \frac{\rho_+}{\rho_-} \epsilon^2 \tilde{M}_\epsilon^\pm(0) \right).
\]
We are now ready to carefully analyze the remainder operator:
\[
\hat{R}_\epsilon := \hat{Q}_\epsilon(\eta_\epsilon) - \hat{Q}_\epsilon(0)
\]
\[
= -\beta \partial_x \left( \frac{\beta}{\epsilon^3 (\eta_\epsilon' + R_\epsilon')} - 1 \right) \partial_x - \epsilon^2 \sum_{\pm} \frac{\rho_+}{\rho_-} \tilde{b}_1^\pm(\eta_\epsilon'; \eta_\epsilon) - \frac{d_-}{\epsilon^2 \rho_-} \left[ \rho \xi, \omega \right] + \left[ \frac{\rho \omega_\epsilon^2}{c^2 \rho_-} \right]. \quad (5.24)
\]
It is straightforward to see that
\[
-\beta \partial_x \left( \frac{\beta}{\epsilon^3 (\eta_\epsilon' + R_\epsilon')} - 1 \right) \partial_x = O(\epsilon^9) \quad \text{in Lin} \left( H^{k+2}, H^k \right).
\]
In view of (5.2) and (5.19), we know that
\[
\tilde{b}_1^\pm = \frac{1}{c} S_{\epsilon^{-1}} \left( \partial_x \phi_{\epsilon \pm} | y - c - \omega_{\pm} \eta_\epsilon \right).
\]
Expanding the Dirichlet–Neumann operator in $H^k$, we obtain
\[
\xi'_{\pm} = \pm \partial_x \mathcal{G}_\pm(\eta_\epsilon)^{-1}(c\eta'_\epsilon + \omega_\pm \eta_\epsilon') \\
= \pm \partial_x \left[ \mathcal{G}_\pm(0)^{-1}(c\eta'_\epsilon + \omega_\pm \eta_\epsilon' + (DG_\pm(0)^{-1}\eta_\epsilon, (c\eta'_\epsilon + \omega_\pm \eta_\epsilon'))] + O(\epsilon^6) \right]
\]
\[
(\partial_x \phi_\pm)|_\eta = \frac{1}{1 + (\eta'_\epsilon)^2} (\xi'_{\pm} \pm \eta'_\epsilon \mathcal{G}_\pm(\eta_\epsilon) \xi_{\pm}) \\
= \pm \partial_x \left[ \mathcal{G}_\pm(0)^{-1}(c\eta'_\epsilon + \omega_\pm \eta_\epsilon' + (DG_\pm(0)^{-1}\eta_\epsilon, (c\eta'_\epsilon + \omega_\pm \eta_\epsilon'))] + O(\epsilon^6) \right]
\]
(5.25)

Combining the formula
\[
\langle DG_\pm(0)^{-1}\eta_\epsilon, f, f\rangle = -\mathcal{G}_\pm(0)^{-1}\langle DG_\pm(0)\eta_\epsilon, \mathcal{G}_\pm(0)^{-1}f \rangle
\]
with the first derivative formula for $\mathcal{G}_\pm(\eta)$, we can infer that
\[
\langle DG_\pm(0)\eta_\epsilon, \mathcal{G}_\pm(0)^{-1}\partial_x S_\epsilon f \rangle = \pm \partial_x S_\epsilon \epsilon^4 \left( \tilde{\mathcal{M}}_\epsilon(0) f \right) \eta \pm \epsilon^3 \mathcal{G}_\pm(0)S_\epsilon (\tilde{\eta}\partial_x f).
\]
Paterning the computation done in [CW22], therefore we can say that
\[
\tilde{b}_1^\pm = -1 + \epsilon^2 \frac{d_+}{d_\pm} \tilde{\eta} - \epsilon^2 \frac{\omega_\pm d_\pm}{e} \tilde{\eta} - \epsilon^4 \frac{d_+^2}{d_\pm^2} \tilde{\eta}^2 + O(\epsilon^6).
\]
(5.26)

Therefore, the second term in the remainder operator (5.24)
\[
\epsilon^2 \sum_\pm \frac{\rho_\pm}{\rho_-} \tilde{b}_1^\pm (\tilde{\eta} \tilde{b}_1^\pm) = \epsilon^2 (1 - \varrho) \tilde{\eta}'' + \epsilon^4 \left( \frac{1}{d} - \varrho + \frac{\omega_- d_+}{e} - \frac{\omega_+ d_-}{e} \right) [2\tilde{\eta}'' + (\tilde{\eta}')^2] + O(\epsilon^6).
\]
(5.27)

Utilizing the expansion in (5.26) yields
\[
\tilde{b}_1^\pm \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) \tilde{b}_1^\pm = \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) \pm \epsilon^2 \frac{d_+}{d_\pm} \left( \tilde{\eta} \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) + \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) \tilde{\eta} \right) \\
+ \epsilon^2 \frac{\omega_\pm d_\pm}{e} \left( \tilde{\eta} \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) + \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) \tilde{\eta} \right) \\
+ \epsilon^4 \frac{d_+^2}{d_\pm^2} \left( \tilde{\eta} \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) \tilde{\eta} + \tilde{\eta}^2 \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) + \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) \tilde{\eta}^2 \right) \\
+ \epsilon^4 \frac{\omega_\pm d_\pm}{e} \left( \pm \frac{d_+}{d_\pm} + \frac{\omega_\pm d_\pm}{c} \right) \tilde{\eta} \tilde{\mathcal{M}}_\epsilon(\eta_\epsilon) \tilde{\eta} + O(\epsilon^6) \text{ in Lin}(H^{k+2}, H^k).
\]
(5.28)
Furthermore, for \( f \in H^{k+2} \) with \( \|f\|_{H^{k+2}} = 1 \), we have
\[
\left( \tilde{M}^\pm_\epsilon(\eta_\epsilon) - \tilde{M}^\pm_\epsilon(0) \right) f = \frac{d_+}{\epsilon^2} S_\epsilon^{-1} \partial_x (G_\pm(\eta_\epsilon) - G_\pm(0)^{-1}) \partial_x \tilde{S}_\epsilon f
\]
\[
= \frac{d_+}{\epsilon^2} S_\epsilon^{-1} \partial_x (D G_\pm(0)^{-1} \eta_\epsilon) \partial_x \tilde{S}_\epsilon f
\]
\[
+ \frac{d_+}{2\epsilon^2} S_\epsilon^{-1} \partial_x (D^2 G_\pm(0)^{-1} [\eta_\epsilon, \partial_x \tilde{S}_\epsilon f]) + O(\epsilon^4)
\]
in \( H^k \). In order to understand the third term in (5.17) better, further expansion needs to be done to the equation above. In [CW22] such computation has been done carefully. From that we can infer
\[
\left( \tilde{M}^\pm_\epsilon(\eta_\epsilon) - \tilde{M}^\pm_\epsilon(0) \right) f = \frac{d_+^2}{ \rho_+ - d_+^2} \tilde{f} + \epsilon^2 \partial_x (\tilde{f} \partial_x f) - \epsilon^2 \frac{d_+^3}{\rho_+ - d_+^2} \tilde{f} + O(\epsilon^3),
\]
in \( H^k \). The expression in (5.28) becomes
\[
\tilde{b}_1^\pm \tilde{M}^\pm_\epsilon(\eta_\epsilon) \tilde{b}_1^\pm f = \tilde{M}^\pm_\epsilon(0) f \pm \epsilon^2 \frac{d_+^2}{ \rho_+ - d_+^2} \left( \tilde{\eta} \tilde{M}^\pm_\epsilon(0) f + \tilde{M}^\pm_\epsilon(0) \tilde{\eta} f \right)
\]
\[
+ \epsilon^2 \frac{\omega_+ d_+^2}{c} \left( \tilde{\eta} \tilde{M}^\pm_\epsilon(0) f + \tilde{M}^\pm_\epsilon(0) \tilde{\eta} f \right) + \frac{d_+^2}{\rho_+ - d_+^2} \tilde{f} + O(\epsilon^3).
\]
Finally, replacing \( \tilde{M}^\pm_\epsilon(0) \) with \(-d_+/\rho_+ \epsilon^2 \) as in (5.22), we obtain
\[
\tilde{b}_1^\pm \tilde{M}^\pm_\epsilon(\eta_\epsilon) \tilde{b}_1^\pm f = \tilde{M}^\pm_\epsilon(0) f \pm \frac{d_+^2}{ \rho_+ - d_+^2} \tilde{f} - 2 \frac{\omega_+ d_+^2}{\rho_+ - d_+^2} \tilde{f} + O(\epsilon^2).
\]

Hence, the third term in (5.24) reads
\[
\sum_\rho \frac{\rho_\pm}{\rho_-} \left( \tilde{b}_1^\pm \tilde{M}^\pm_\epsilon(\eta_\epsilon) \tilde{b}_1^\pm - \tilde{M}^\pm_\epsilon(0) \right) f
\]
\[
= 3 \sum_\pm \frac{\rho_\pm}{\rho_-} \frac{d_+^2}{d_+^2} \tilde{f} - 2 \sum_\pm \frac{\rho_\pm}{\rho_-} \frac{\omega_+ d_+^2}{d_+ c} \tilde{f} + O(\epsilon^2)
\]
\[
= -3 \left( \rho - \frac{1}{d^2} \right) \tilde{f} - 2 \left( \frac{\omega_+ d_+}{c} + \frac{\omega_- d_+}{c d} \right) \tilde{f} + O(\epsilon^2).
\]
The next term to analyze is \([\rho \xi_\epsilon^\pm \omega] \). For that, we are going to exploit the expression in (5.25). Via the same type expansion procedure, one can infer
\[
\xi_\epsilon^\prime = \pm \frac{d_+^2}{c \rho_- d_+} + O(\epsilon^2)
\]
in \( H^k \).

Hence,
\[
- [\rho \xi_\epsilon^\prime \omega] = - \left( \frac{\omega_+ d_+}{c} + \frac{\omega_- d_+}{c d} \right).
\]

Another term in \( Q_\epsilon(\eta_\epsilon) \) that we still have yet to handle is \(-\eta_\epsilon [\rho \omega^2] \). It is straightforward to see that
\[-\frac{1}{\epsilon^2} \frac{d_+}{c^2 \rho_-} S^{-1}_\epsilon \left[ \rho \omega^2 \right] S_\epsilon = -\frac{1}{\epsilon^2} \frac{d_+}{c^2 \rho_-} \epsilon^2 d_+ \left[ \rho \omega^2 \right] + O(\epsilon) = -\frac{\omega_+^2 \rho d_+^2}{c^2} + \frac{\omega_-^2 d_+^2}{c^2} + O(\epsilon). \tag{5.35} \]

Finally, it remains to deal with all the expressions of the last four rows in (5.18). Again, going through the same analysis as before and using the fact that $\hat{z}_+(0) = -d_+/(\epsilon^2 d_\pm)$, one can show that those terms are lower order (sufficiently small).

Combining (5.33), (5.34), and (5.35) we arrive at (5.17). The proof is then complete.

\[\square\]

**Lemma 5.5** (Limiting rescaled operator). Consider the rescaled operator $\hat{Q}_\epsilon(\eta_\epsilon)$. Assume that $\beta > \beta_0$ and $\alpha = \alpha_0 + \epsilon^2$. Then for any $k > 1/2$ and $\zeta \in H^{k+2}$, we have

$$
\|\hat{Q}_\epsilon(\eta_\epsilon) \zeta - \hat{Q}_0 \zeta\|_{H^k} \to 0 \quad \text{as } \epsilon \searrow 0,
$$

where the operator

$$
\hat{Q}_0 = - (\beta - \beta_0) \partial_x^2 + 1 - 3 \left( \frac{\rho}{d^2} + \frac{\omega_+ d_+ \theta}{c} - \frac{\omega_- d_+}{c d} + \frac{\omega_+^2 d_+^2 \theta}{3c^2} - \frac{\omega_-^2 d_+^2}{3c^2} \right) \hat{\eta}. \tag{5.36}
$$

**Proof.** Let $k > 1/2$. Recall from the expansion of $Q_\epsilon$ in Lemma 5.4, we have $\hat{Q}_\epsilon(\eta_\epsilon) = \hat{Q}_0(0) + \hat{R}_\epsilon$. It is also important to note that $\hat{R}_\epsilon$ has a uniform limit at linear map from $H^{k+1}$ to $H^k$ as $\epsilon \searrow 0$. Moreover, the operator $\hat{Q}_\epsilon(0)$ is a Fourier multiplier in $H^{k+2}$ (as $\hat{M}_\epsilon(0)$ is). Precisely,

$$
\mathcal{F}(\hat{Q}_\epsilon(0) f)(\xi) = \frac{1}{\epsilon^2} \left( \epsilon^2 \beta \xi^2 + \alpha - \frac{[\rho \omega d_+]}{\rho_- c} - \sum_\pm \frac{\rho_\pm}{\rho_-} \epsilon \xi \coth \left( \frac{d_\pm}{d_+} \epsilon \xi \right) \right) \hat{f}(\xi).
$$

Further the symbol $\hat{q}_\epsilon$ can be re-written in the following way

$$
\hat{q}_\epsilon(\xi) = \epsilon^{2-n} (\beta - \beta_0) \xi^2 + \frac{\alpha - \alpha_0}{\epsilon^2} + \frac{1}{\epsilon^2} \left( \beta_0 (\epsilon \xi)^2 + \alpha_0 - \frac{[\rho \omega d_+]}{\rho_- c} - \sum_\pm \frac{\rho_\pm}{\rho_-} \epsilon \xi \coth \left( \frac{d_\pm}{d_+} \epsilon \xi \right) \right)
$$

$$
= \epsilon^{2-n} (\beta - \beta_0) \xi^2 + \frac{\alpha - \alpha_0}{\epsilon^2} + O((\epsilon \xi)^4),
$$

as $\epsilon \xi \to 0$. Since $\alpha = \alpha_0 + \epsilon^2$, hence for each fixed $\xi \in \mathbb{R}$, we obtain

$$
\hat{q}_\epsilon(\xi) \to (\beta - \beta_0) \xi^2 + 1 \text{ as } \epsilon \searrow 0.
$$

Combining this result with the expression for $\hat{R}_\epsilon$ in (5.17), we obtain the formula of $\hat{Q}_0$. \[\square\]

**5.2. Spectrum of the linearized augmented potential.** Having established the limiting behavior above, we will now analyze the spectrum of the operator $Q_\epsilon(\eta_\epsilon)$. Recall that, the operator $Q_\epsilon(\eta_\epsilon)$ only converges point-wise to the operator $Q_0(0)$ whose essential spectrum is $[0, \infty)$. This is certainly creates a challenge in deducing the spectrum of $Q_\epsilon(\eta_\epsilon)$. Thanks to the rescaled operator $\hat{Q}_\epsilon(\eta_\epsilon)$ introduced earlier. It is known that such rescaled
operator converges point-wise to $\tilde{Q}_0$, which has a gap between 0 and the positive essential spectrum.

**Lemma 5.6** (Spectrum of the Limiting Operator $\tilde{Q}_0$). *Let the assumptions in Lemma 5.5 hold. The limiting rescaled operator $\tilde{Q}_0$ satisfies*

$$\text{ess spec } \tilde{Q}_0 = [1, \infty), \quad \text{spec } \tilde{Q}_0 = \{-\tilde{\tau}^2, 0\} \cup \tilde{\Lambda},$$

(5.37)

*where the first two eigenvalues $-\tilde{\tau}^2 < 0$ and 0 are both simple eigenvalues with the corresponding eigenfunctions $\tilde{g}_1$ and $\tilde{g}_2 = \tilde{\eta}'$, respectively; and there exists $\tau_* > 0$ such that $\tilde{\Lambda} \subset [\tau_*, \infty)$.*

**Proof.** The spectra condition above is a classic result and can be found, for instance, in the surveys of Pava [Pav09]. It is important to mention that the proof of that is not trivial by any means. The fact that $-\tilde{\tau}^2$ and 0 are simple follows from the result which says that: the Wronskian of any two solutions in $L^2$ of the eigenvalue problem $\tilde{Q}_0 f = \tilde{\tau}$ must be 0. □

**Theorem 5.7.** *Suppose that the assumptions of Lemma 5.5 hold. For each $a \in (0, \tau_*)$, there exists some $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the operator $Q_{\varepsilon}(\eta_\varepsilon)$ satisfies*

$$\text{ess spec } Q_{\varepsilon}(\eta_\varepsilon) \subset [\varepsilon^2c^2 \rho_-/d_+, \infty), \quad \text{spec } Q_{\varepsilon}(\eta_\varepsilon) = \{-\tau^2, 0\} \cup \Lambda,$$

where $\Lambda \subset [a \varepsilon^2 c^2 \rho_-/d_+, \infty)$, and

$$\tau^2 = \frac{\varepsilon^2 c^2 \rho_-}{d_+} - \tilde{\tau}^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \searrow 0.$$  

*The first two eigenvalues $\tau_1 := -\tilde{\tau}^2 < 0$ and $\tau_2 := 0$ are both simple with the corresponding eigenfunctions given by $g_i := S_\varepsilon \tilde{g}_i + o(1)$ in $H^k$ as $\varepsilon \searrow 0$.  

**Proof.** The proof of this theorem can be done in a similar manner as the one in [CW22, Theorem 3.2] which is mainly inspired by the proof of [Mie02, Theorem 4.3]. Therefore, we opt to avoid rewriting it here. □

**Lemma 5.8** (Extension of $D^2E_c$). *Let $\{U_c\}$ be a family of bound states, then the operator $D^2E_c(U_c)$ can be extended uniquely to a bounded linear operator $H_c : X \to X^*$ such that*

$$D^2E_c(U_c)[\dot{u}, \dot{v}] = \langle H_c \dot{u}, \dot{v} \rangle, \quad \text{for all } \dot{u}, \dot{v} \in \mathbb{V},$$

(5.38)

*and $I^{-1}H_c$ is self-adjoint on $X$.***
Proof. Let \( U_c = (\eta_c, \xi_c) \) be a bound state and \( \dot{u} = (\dot{\eta}, \dot{\xi}) \in V \) be given. From Lemma 5.1 and Lemma 5.2, we know that

\[
D^2 E_c(U_c)[\dot{u}, \dot{u}] = D^2 V^\text{aug}(\eta_c)[\dot{\eta}, \dot{\eta}] + \int_{\mathbb{R}} (S_c - T_c) \dot{\eta} A(\eta_c)(S_c - T_c) \dot{\eta} \, dx \\
+ 2D_\xi D_\eta E_c(U_c)[\dot{\eta}, \dot{\xi}] + D^2_\xi E_c(U_c)[\dot{\xi}, \dot{\xi}] \\
= \langle Q_C(\eta_c) \dot{\eta}, \dot{\eta} \rangle + \int_{\mathbb{R}} (S_c - T_c) \dot{\eta} A(\eta_c)(S_c - T_c) \dot{\eta} \, dx + 2 \int_{\mathbb{R}} \dot{\xi} \langle D A(\eta_c) \dot{\eta}, \dot{\xi} \rangle \, dx \\
+ 2 \int_{\mathbb{R}} \hat{\xi} \mathcal{G}_-(\eta_c) B^{-1}(\eta_c) [\omega](\eta_c \dot{\eta})_x \, dx \\
+ 2 \int_{\mathbb{R}} \hat{\xi} \langle D \mathcal{G}_-(\eta_c) \dot{\eta}, B^{-1}(\eta_c) [\omega] \eta_c \eta'_c \rangle \, dx \\
+ 2 \int_{\mathbb{R}} \dot{\xi} \omega_-(\eta_c) \eta'_c \, dx + 2c \int_{\mathbb{R}} \dot{\eta} \hat{\xi} \, dx + \int_{\mathbb{R}} \hat{\xi} A(\eta_c) \hat{\xi} \, dx.
\]

Expanding some of the terms using the first derivative formula for \( \mathcal{G}_-(\eta) \) and \( B^{-1}(\eta) \) yields

\[
D^2 E_c(U_c)[\dot{u}, \dot{u}] = \langle Q_C(\eta_c) \dot{\eta}, \dot{\eta} \rangle + \int_{\mathbb{R}} (S_c - T_c) \dot{\eta} A(\eta_c)(S_c - T_c) \dot{\eta} \, dx + 2c \int_{\mathbb{R}} \dot{\eta} \hat{\xi} \, dx + \int_{\mathbb{R}} \hat{\xi} A(\eta_c) \hat{\xi} \, dx \\
+ 2 \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} (b_{1c}^\pm + c + \omega_{\pm} \eta_c \pm a_{1c}^\pm(\eta_c, \Upsilon_\pm)) \left( \mathcal{G}_{\pm}(\eta_c)^{-1} A(\eta_c) \hat{\xi} \right)' \dot{\eta} \, dx \\
+ 2 \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \pm b_{2c}^\pm \dot{\eta} A(\eta_c) \hat{\xi} \, dx \\
+ 2 \int_{\mathbb{R}} \hat{\xi} \mathcal{G}_-(\eta_c) B^{-1}(\eta_c) [\omega](\eta_c \dot{\eta})_x \, dx + 2 \int_{\mathbb{R}} \hat{\xi} \langle D \mathcal{G}_-(\eta_c) \dot{\eta}, B^{-1}(\eta_c) [\omega] \eta_c \eta'_c \rangle \, dx \\
+ 2 \int_{\mathbb{R}} \dot{\xi} \omega_-(\eta_c) \eta'_c \, dx.
\]

Upon cancellation and regrouping, we get

\[
D^2 E_c(U_c)[\dot{u}, \dot{u}] \\
= \langle Q_C(\eta_c) \dot{\eta}, \dot{\eta} \rangle + \int_{\mathbb{R}} \left( (S_c - T_c) \dot{\eta} + \dot{\xi} \right) A(\eta_c) \left( (S_c - T_c) \dot{\eta} + \dot{\xi} \right) \, dx \\
+ 2 \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \mathcal{G}_{\pm}(\eta_c)^{-1} \left( a_{1c}^{-1}(\eta_c, \Upsilon_\pm)' \dot{\eta} \right) A(\eta_c) \hat{\xi} \, dx - \int_{\mathbb{R}} 2(a_{1c}^{-1}(\eta_c, \Upsilon_\pm)' \dot{\eta})' \hat{\xi} \, dx. \tag{5.39}
\]

Using the fact that

\[
A(\eta)^{-1} = \rho_+ \mathcal{G}_+^{-1}(\eta)^{-1} + \rho_- \mathcal{G}_-^{-1}(\eta)^{-1},
\]
the expression on the second row (5.39) vanishes. Therefore, we obtain

$$D^2E_c(U_c)[\tilde{u}, \tilde{u}] = \langle \mathcal{Q}_c(\eta_c)\dot{\eta}, \dot{\eta} \rangle + \int_{\mathbb{R}} \left((\mathcal{S}_c - \mathcal{T}_c)\dot{\eta} + \dot{\xi} \right) A(\eta_c) \left((\mathcal{S}_c - \mathcal{T}_c)\dot{\eta} + \dot{\xi} \right) \, dx. \quad (5.40)$$

From the expression above, $D^2E_c(U_c)$ extends to an element in $\mathbb{X}^*$. \hfill \Box

**Theorem 5.9** (Spectrum). Let $\{U_c\}$ be one family of bound states given in the Corollary (4.9). Then

$$\text{spec } I^{-1}H_c = \{-\mu_c^2, 0\} \cup \Sigma_c,$$

where $-\mu_c^2 < 0$ is a simple eigenvalue that corresponds to a unique eigenvector $\chi_c$; 0 is simple eigenvalue generated by $T$; and $\Sigma_c \subset (0, \infty)$ is uniformly bounded away from 0.

**Proof.** The following proof relies on the nice structure of $H_c$ in Lemma 5.8, therefore $I^{-1}H_c$ and the idea presented in [Mie02, Proposition 5.3] and [CW22, Theorem 4.1.3]. Consider the following operator

$$\mathcal{Q}_c(\eta_c) + (\lambda - \tau_c^2)(\cdot, g_{1c})g_{1c} + \lambda(\cdot, \eta_c'\eta_c').$$

One can easily check that for $\lambda > 0$, $-\tau_c^2$ is the negative eigenvalue of $\mathcal{Q}_c(\eta_c)$and $g_{1c}$ is the corresponding eigenfunction. Recall that $A(\eta_c)$ is a positive definite operator. Therefore, using the expression for $H_c$ from Lemma 5.8, we can conclude that

$$\langle H_c u, u \rangle_{\mathbb{X}^* \times \mathbb{X}} + (\alpha - \tau_c^2)(I^{-1}(g_{1c}, 0), u)_{\mathbb{X}^* \times \mathbb{X}} + \lambda(I^{-1}(\eta_c', 0), u)_{\mathbb{X}^* \times \mathbb{X}} \geq_c \|u\|_{\mathbb{X}}^2,$$

for any $u \in \mathbb{X}$. Hence $I^{-1}H_c$ is positive definite on a codimension 2 space. Further, from (5.40), for $u = (g_{1c}, (\mathcal{S}_c - \mathcal{T}_c)g_{1c})$, we have

$$\langle H_c u, u \rangle_{\mathbb{X}^* \times \mathbb{X}} = \langle \mathcal{Q}_c(\eta_c)g_{1c}, g_{1c} \rangle_{\mathbb{X}^* \times \mathbb{X}} = -\tau_c^2 < 0.$$

Hence, we can conclude that $I^{-1}H_c$ has a one-dimensional kernel generated by $T'(0)U_c$, a one-dimensional negative definite subspace. Moreover, It is positive definite on the orthogonal complement. This then proves Assumption 6. \hfill \Box

## 6. Proof of theorem

In this section, we will prove the (conditional) orbital stability of the bound states presented in Corollary (4.9). As mentioned previously, it requires us to check the sign of the second derivative of the scalar valued function $d$.

**Theorem 6.1** (Stability/instability for strong surface tension). Fix $c_*$ such that $0 < \alpha_{c_*} - \alpha_0 \ll 1$, the conditional orbital stability/instability of the bound states $U_{c_*}$ in Corollary 4.9 can be determined by looking at the sign of the derivative of a scalar-valued function $m := m(c)$ stated in (6.3).

**Proof.** Having confirmed all the assumptions, The conclusion on (conditional) orbital stability is drawn by showing $d''(c_*) > 0$. Recall that since $U_{c_*}$ is a critical point of $E_c$, we therefore have

$$d'(c_*) = -P(U_{c_*}). \quad (6.1)$$
We need to show that $d'(c)$ is strictly increasing at $c = c_*$. We start by defining a rescaling operator:

$$S_c f := f \left( \frac{c \epsilon}{d_+ \sqrt{\beta_c - \beta_0}} \right).$$

Using the asymptotics for the free surface, we define

$$\eta_c =: c^2 d_+ S_c (\tilde{\eta}_c + \tilde{r}_c), \quad \text{for } \tilde{r}_c = O(\epsilon_c).$$

Via the definition of $P$ in (4.28), we can express $d'(c)$ as follows

$$d'(c) = c \int \eta_c \partial_x \left( A(\eta_c)^{-1} \eta_c' \right) \, dx + \int \eta_c \partial_x \left( A(\eta_c)^{-1} G_-(\eta_c) B(\eta_c)^{-1} \rho_+ [\omega] \, \eta_c \eta_c' \right) \, dx$$

$$+ \int \eta_c \omega \partial_x \left( A(\eta_c)^{-1} \eta_c \eta_c' \right) \, dx - \frac{1}{2} \int \rho_0 \eta_c^2 \, dx$$

$$= c \epsilon_c^4 d_+^2 \int S_c (\tilde{\eta}_c + \tilde{r}_c) \partial_x \left( A(\eta_c)^{-1} \partial_x S_c (\tilde{\eta}_c + \tilde{r}_c) \right) \, dx$$

$$+ \epsilon_6^4 d_+^3 \int S_c (\tilde{\eta}_c + \tilde{r}_c) \partial_x \left( G_+ (\eta_c)^{-1} \rho_+ [\omega] S_c (\tilde{\eta}_c + \tilde{r}_c) \partial_x S_c (\tilde{\eta}_c + \tilde{r}_c) \right) \, dx$$

$$+ \epsilon_6^4 d_+^3 \int S_c (\tilde{\eta}_c + \tilde{r}_c) \omega \partial_x \left( A(\eta_c)^{-1} S_c (\tilde{\eta}_c + \tilde{r}_c) \partial_x S_c (\tilde{\eta}_c + \tilde{r}_c) \right) \, dx$$

$$- \epsilon_4^4 d_+^2 \int \frac{1}{2} \rho_0 \eta_c^2 \, dx.$$

Undoing the scaling,

$$= c \epsilon_c^3 d_+^2 \sqrt{\beta_c - \beta_0} \int (\tilde{\eta}_c + \tilde{r}_c) S_c^{-1} \partial_x \left( A(\eta_c)^{-1} \partial_x S_c (\tilde{\eta}_c + \tilde{r}_c) \right) \, dx$$

$$+ \frac{c_6^3 d_+^4 \sqrt{\beta_c - \beta_0}}{2} \int (\tilde{\eta}_c + \tilde{r}_c) S_c^{-1} \partial_x \left( G_+ (\eta_c)^{-1} \rho_+ [\omega] \partial_x (S_c (\tilde{\eta}_c + \tilde{r}_c))^2 \right) \, dx$$

$$+ \frac{c_6^3 d_+^4 \sqrt{\beta_c - \beta_0}}{2} \omega \int (\tilde{\eta}_c + \tilde{r}_c) S_c^{-1} \partial_x \left( A(\eta_c)^{-1} \partial_x (S_c (\tilde{\eta}_c + \tilde{r}_c))^2 \right) \, dx$$

$$- \epsilon_4^3 d_+^3 \sqrt{\beta_c - \beta_0} \int \frac{1}{2} (\tilde{\eta}_c + \tilde{r}_c) \rho_0 \eta_c \partial_x (\tilde{\eta}_c + \tilde{r}_c) \, dx.$$

Using the idea in [CW22], we define $\tilde{M}_c^\pm(\eta_c) := d_+ S_c^{-1} \partial_x G_\pm (\eta_c)^{-1} \partial_x S_c$. Following similar line of argument as in Lemma 5.4, we find that

$$\|\tilde{M}_c^\pm(0) + \frac{d_+}{d_\pm} \|_{L^{\infty}(H^{k+2}, L^2)} \lesssim \epsilon_c^2, \quad \|\tilde{M}_c^\pm(\eta_c) - \tilde{M}_c^\pm(0)\|_{L^{\infty}(H^2, L^2)} \lesssim \epsilon_c^2,$$

which yields

$$= -c \epsilon_4^3 d_+^3 \sqrt{\beta_c - \beta_0} \sum_{d_\pm} \rho_\pm \frac{d_+}{d_\pm} \int (\tilde{\eta}_c)^2 \, dx - \frac{c_6^3 d_+^4 \sqrt{\beta_c - \beta_0}}{2} [\rho_\omega] \int (\tilde{\eta}_c)^2 \, dx + O(\epsilon_c^4)$$

$$= -c \left( \frac{1}{d_+} + \frac{\omega_+ \rho d_+}{2c} - \frac{\omega_- d_+}{2c} \right) \left[ \epsilon_4^3 d_+^3 \sqrt{\beta_c - \beta_0} \int (\tilde{\eta}_c)^2 \, dx \right] + O(\epsilon_c^4).$$

(6.2)
Recall that

\[
\tilde{\eta}_c = \frac{\text{sech}^2 (x/2)}{\varrho - \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_+}{cd} + \frac{\omega_+^2 d_+^2 \varrho}{3c^2} - \frac{\omega_-^2 d_+^2}{3c^2}}.
\]

Thus,

\[
d'(c) = -4 \left( \varrho - \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{2c} - \frac{\omega_- d_+}{2c} \right) + O(\epsilon_c^4)
\]

\[
= m(c) + O(\epsilon_c^4).
\]

For notation simplicity, let us define the following new variables:

\[
\mathfrak{A} := \varrho + \frac{\omega_+ d_+ \varrho}{2c} - \frac{\omega_- d_+}{2c},
\]

\[
\mathfrak{B} := \varrho - \frac{1}{d^2} + \frac{\omega_+ d_+ \varrho}{c} + \frac{\omega_- d_+}{cd} + \frac{\omega_+^2 d_+^2 \varrho}{3c^2} - \frac{\omega_-^2 d_+^2}{3c^2}.
\]

The derivative of \( m \) with respect to \( c \) in terms of \( \mathfrak{A} \) and \( \mathfrak{B} \) reads

\[
m'(c) =
\]

\[
-4 \frac{\mathfrak{A}^2 \epsilon_c^3 c (\beta_c - \beta_0) + 3 \mathfrak{A} \epsilon_c \epsilon_c' \epsilon_c (\beta_c - \beta_0) + 2 \mathfrak{A} c \epsilon_c^3 \beta_c'}{\mathfrak{B}^2}
\]

\[
- 2 \mathfrak{A} \mathfrak{B} \mathfrak{B}' \epsilon_c^3 (\beta_c - \beta_0).
\]

From Theorem 3.1, we see that for \( 0 < \epsilon_c \ll 1 \), the wave is stable provided \( m'(c) > 0 \) and unstable if \( m'(c) < 0 \). Notice that the sign is determined by the quantity in square brackets on the right-hand side of (6.4).

Next, let us consider the special cases described in Figures 2 and 3. From the expression of \( m' \) (essentially \( d'' \) without the lower order term) together with definitions of \( \mathfrak{A}, \mathfrak{B}, \epsilon_c \), and their derivatives, we can conclude that for any \( c \in \mathcal{I} \), orbital stability (\( d''(c) > 0 \)) holds in each of the three cases below.

- Sufficiently small-amplitude and near critical Bond number waves: \( \beta_c - \beta_0 \ll 1 \).
- Waves of elevation with \( \{ c < 0, \omega_+ \geq 0, \frac{2a(\varrho - 2)}{c} > \omega_- \text{ and } \omega_+ = 0 \} \) OR \( \{ c > 0, \omega_- \leq 0, \frac{2a(\varrho - 2)}{c} < \omega_- \text{ and } \omega_+ = 0 \} \).
- Waves of depression with \( \{ c < 0, \omega_+ \leq 0, \frac{2a(\varrho - 2)}{c} > -\omega_+ \rho_+ \text{ and } \omega_- = 0 \} \) OR \( \{ c > 0, \omega_+ \geq 0, \frac{2a(\varrho - 2)}{c} < -\omega_+ \rho_+ \text{ and } \omega_- = 0 \} \).

Moreover, we find some cases where orbital instability (\( d''(c) < 0 \)) occur:

- Waves of depression with \( \{ c < 0, \omega_+ \geq 0, \frac{2a(\varrho - 2)}{c} < \omega_- \text{ and } \omega_+ = 0 \} \) OR \( \{ c > 0, \omega_- \leq 0, \frac{2a(\varrho - 2)}{c} > \omega_- \text{ and } \omega_+ = 0 \} \).
Waves of elevation with \( \{ c < 0, \omega_+ \leq \frac{2\rho_+}{c} < -\omega_+\rho_+ \text{ and } \omega_- = 0 \} \) OR \( \{ c > 0, \omega_+ \geq 0, \frac{2\rho_+}{c} > -\omega_+\rho_+ \text{ and } \omega_- = 0 \} \).

This concludes the proof. \( \square \)

Appendix A. Center Manifold

The first part of appendix records the center manifold reduction theorem introduced in [Mie95] that was implemented in, for example, [Nil17]. It is used in proving the existence of small-amplitude internal water waves.

**Theorem A.1** (Center manifold). Consider the differential equation of the form

\[
\dot{u} = Lu + F(u, \mu),
\]

where the unknown \( u \in E \) for some Hilbert space \( E \), \( \mu \in \mathbb{R}^n \) is a parameter and \( L : \mathcal{D}(L) \subset E \rightarrow E \) is a closed linear operator. Assume that the differential equation (A.1) is Hamilton’s equations that correspond to the Hamiltonian system \((E, \Omega, H)\) with 0 being its fix point. Moreover, assume also the following:

**H1** The space \( E \) has two closed and \( L \)-invariant subspaces, namely \( E_1 \) and \( E_2 \) such that

\[
E = E_1 \oplus E_2,
\]

\[
\dot{u}_1 = L_1 u_1 + F_1(u_1 + u_2, \mu),
\]

\[
\dot{u}_2 = L_2 u_2 + F_2(u_1 + u_2, \mu),
\]

where \( L_i = L|_{\mathcal{D}L_i \cap E_i} : \mathcal{D}L_i \cap E_i \rightarrow E_i \), for \( i = 1, 2 \) and \( F_1 = PF \), \( F_2 = (I - P)F \), where the operator \( P \) is a projection of \( E \) onto \( E_1 \).

**H2** \( E_1 \) is a finite dimensional Hilbert space and the spectrum of \( L_1 \) is purely imaginary.

**H3** The imaginary axis lies in the resolvent of \( L_2 \) and

\[
\| (L_2 - iaI)^{-1} \| \leq \frac{C}{1 + |a|}, \quad \text{for } a \in \mathbb{R}. \tag{A.3}
\]

**H4** There exists \( k \in \mathbb{N} \) and neighborhoods \( \Lambda \subset \mathbb{R}^n \) and \( U \subset \mathcal{D}(L) \) of 0 such that \( F \) is \( k + 1 \) continuously differentiable on \( U \times \Lambda \) and the derivatives of \( F \) are all bounded and uniformly continuous on \( U \times \Lambda \) with

\[
F(0, \mu_0) = 0, \quad \text{d}F[0, \mu_0] = 0.
\]

Under the hypothesis H1-H4 there exist neighborhoods \( \tilde{\Lambda} \subset \Lambda \) and \( \tilde{U}_1 \subset U \cap E_1 \), \( \tilde{U}_2 \subset U \cap E_2 \) of zero and a reduction function \( r : \tilde{U}_1 \times \tilde{\Lambda} \rightarrow \tilde{U}_2 \) with the following properties. The reduction function \( r \) is \( k \) times continuously differentiable on \( \tilde{U}_1 \times \tilde{\Lambda} \) and the derivatives of \( r \) are bounded and uniformly continuous on \( \tilde{U}_1 \times \tilde{\Lambda} \) with

\[
r(0, \mu_0) = 0, \quad \text{dr}[0, \mu_0] = 0.
\]

The graph

\[
X_C^\mu = \{ u_1 + r(u_1, \mu) \in \tilde{U}_1 \times \tilde{U}_2 : u_1 \in \tilde{U}_1 \},
\]

is a Hamiltonian center manifold for (A.1) with the following properties:
• Through every point in $X_C^\mu$, there passes a unique solution of (A.1) that stays on $X_C^\mu$ as long as it remains in $\tilde{U}_1 \times \tilde{U}_2$. We say that $X_C^\mu$ is a locally invariant manifold of (A.1).
• Every small bounded solution $u(x)$, $x \in \mathbb{R}$ of (A.1) that satisfies $u_1(x) \in \tilde{U}_1$ and $u_2(x) \in \tilde{U}_2$ lies completely in $X_C^\mu$.
• Every solution $u_1$ of the reduced equation
  \[ \dot{u}_1 = L_1 u_1 + F_1(u_1 + r(u_1, \mu), \mu), \tag{A.4} \]
generates a solution
  \[ u(x) = u_1(x) + r(u_1(x), \mu) \]
of (A.1).
• $X_C^\mu$ is a symplectic submanifold $E$, and the flow determined by the Hamiltonian system $X_C^\mu, \tilde{\Omega}, \tilde{H}$, where the tilde denotes the restriction to $X_C^\mu$, coincides with the flow on $X_C^\mu$ determined by $(E, \Omega, H)$. The reduced equation (A.4) represents Hamilton’s equations for $(X_C^\mu, \tilde{\Omega}, \tilde{H})$.
• If (A.1) is reversible, that is if there exists a linear symmetry $S$ that anti-commutes with the right hand side of (A.1), then the reduction function $r$ can be chosen so that it commutes with $S$.

**Appendix B. Formulae**

Many of the computations in the present work make use of the first and second derivative formulas of the non-local operators $G_\pm(\eta)$ and $A(\eta)$. We record them in a series of lemmas below. A derivation can be found in [CW22].

**Lemma B.1 (First Derivatives).** Let $\eta, \xi \in \mathcal{O} \cap \mathcal{V}, \dot{\eta} \in \mathcal{V}_1$ and be given.

(a) The Fréchet derivative of $G_\pm(\eta)$ admits the representation formula
  \[ \int_\mathbb{R} \zeta \langle D G_\pm(\eta) \dot{\eta}, \dot{\xi} \rangle \, dx = \int_\mathbb{R} \left( a_1^\pm(\eta, \dot{\xi}) \zeta' + a_2^\pm(\eta, \dot{\xi}) G_\pm(\eta) \zeta \right) \, dx, \tag{B.1} \]
  with
  \[ a_1^\pm(\eta, \dot{\xi}) := \frac{1}{1 + (\eta')^2} \left( \mp \dot{\xi'} - \eta' G_\pm(\eta) \dot{\xi} \right), \]
  \[ a_2^\pm(\eta, \dot{\xi}) := \frac{1}{1 + (\eta')^2} \left( \pm G_\pm(\eta) \dot{\xi} - \eta' \dot{\xi}' \right). \tag{B.2} \]
(b) The Fréchet derivative of $A(\eta)$ admits the representation formula

$$
\int_R \xi \langle D_2 A(\eta)| \hat{\eta}, \hat{\xi} \rangle \, dx \\
= \sum_{\pm} \rho_{\pm} \left( a_{4}^{\pm}(\eta, A(\eta)G_{\pm}(\eta)^{-1}\hat{\xi}) \left( A(\eta)G_{\pm}(\eta)^{-1}\hat{\zeta} \right) \right) \, dx \\
+ \sum_{\pm} \rho_{\pm} \int_R \left( \frac{a_{2}^{\pm}(\eta, A(\eta)G_{\pm}(\eta)^{-1}\hat{\zeta})}{A(\eta)\hat{\zeta}} \right) \, dx.
$$

\textbf{Lemma B.2} (Second derivative of $G_{\pm}$). For all $u = (\eta, \tilde{\xi}) \in O \cap \mathbb{V}$ and $\hat{\eta} \in \mathbb{V}_1$, it holds that

$$
\int_R \xi \langle D_2^2 G_{\pm}(\eta)[\hat{\eta}, \hat{\eta}], \hat{\xi} \rangle \, dx \\
= \int_R \left( a_{4}^{\pm}(u)\hat{\eta}^2 + 2a_{2}^{\pm}(u)\hat{\eta}G_{\pm}(\eta) \left( a_{2}^{\pm}(u)\hat{\eta} \right) \right) \, dx,
$$

where

$$a_{4}^{\pm}(u) := -2a_{1}\pm(u) a_{2}\pm(u), \quad (B.5)$$

and $a_{1}^{\pm}, a_{2}^{\pm}$ are given by (B.2).

\textbf{Lemma B.3} (Second derivative formula of $A$). For all $u = (\eta, \tilde{\xi}) \in O \cap \mathbb{V}$ and $\hat{\eta} \in \mathbb{V}_1$, it holds that

$$
\int_R \xi \langle D_2^2 A(\eta)| \hat{\eta}, \hat{\eta} \rangle \, dx \\
= \int_R \left( a_{4}(u)\hat{\eta} + 2 \sum_{\pm} \rho_{\pm} a_{2}^{\pm}(u, \theta_{\pm}) G_{\pm}(\eta) \left( a_{2}^{\pm}(\eta, \theta_{\pm}) \hat{\eta} \right) \right) \, dx,
$$

where

$$\begin{align*}
\theta_{\pm}(u) &= G_{\pm}(\eta)^{-1} A(\eta) \xi, \\
a_{4}(u) &= \sum_{\pm} \rho_{\pm} a_{4}^{\pm}(\eta, \theta_{\pm}), \quad (B.7) \\
L_{\pm}(u)\hat{\eta} &= -G_{\pm}(\eta)^{-1} a_{4}^{\pm}(\eta, \theta_{\pm}) \hat{\eta} \quad a_{4}^{\pm}(u) \hat{\eta} \hat{\eta}^2 + a_{2}^{\pm}(\eta, \theta_{\pm}) \hat{\eta} G_{\pm}(\eta) \left( a_{2}^{\pm}(u)\hat{\eta} \right) \\
M(u)\hat{\eta} &= \sum_{\pm} \rho_{\pm} \left( a_{4}^{\pm}(\eta, \theta_{\pm}) \left( a_{4}^{\pm}(\eta, \theta_{\pm}) \right) \hat{\eta} \right) \quad L(u) := \sum_{\pm} \rho_{\pm} L_{\pm}(u) \\
M(u)\hat{\eta} &= \sum_{\pm} \rho_{\pm} \left( a_{4}^{\pm}(\eta, \theta_{\pm}) \left( a_{4}^{\pm}(\eta, \theta_{\pm}) \right) \hat{\eta} \right) \quad a_{4}^{\pm}(u) \hat{\eta} \hat{\eta}^2 + a_{2}^{\pm}(\eta, \theta_{\pm}) \hat{\eta} G_{\pm}(\eta) \left( a_{2}^{\pm}(u)\hat{\eta} \right) \\
N(u)\hat{\eta} &= \sum_{\pm} \rho_{\pm} \left( a_{4}^{\pm}(\eta, \theta_{\pm}) \left( a_{4}^{\pm}(\eta, \theta_{\pm}) \right) \hat{\eta} \right) \quad L(u) := \sum_{\pm} \rho_{\pm} L_{\pm}(u) \\
N(u)\hat{\eta} &= \sum_{\pm} \rho_{\pm} \left( a_{4}^{\pm}(\eta, \theta_{\pm}) \left( a_{4}^{\pm}(\eta, \theta_{\pm}) \right) \hat{\eta} \right) \quad a_{4}^{\pm}(u) \hat{\eta} \hat{\eta}^2 + a_{2}^{\pm}(\eta, \theta_{\pm}) \hat{\eta} G_{\pm}(\eta) \left( a_{2}^{\pm}(u)\hat{\eta} \right) .
\end{align*}$$

**Acknowledgments**

The work of DS was supported in part by the NSF through the award NSF DMS-1812436.
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