Nonperturbative effects of a Topological $\Theta$-term on Principal Chiral Nonlinear Sigma Models in (2+1) dimensions

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We study the effects of a topological $\Theta$-term on 2+1 dimensional principal chiral models, which are nonlinear sigma models defined on Lie group manifolds. We find that when $\Theta = \pi$, the nature of the disordered phase of the principal chiral model is strongly affected by the topological term: it is either a gapless conformal field theory, or it is gapped and two-fold degenerate.

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Introduction: It is well known that topological terms in field theories are responsible for many profound phenomena in condensed matter physics. For instance, the one-dimensional SU(2) spin-1/2 Heisenberg quantum spin chain is known to be described by the 1+1d O(3) Nonlinear Sigma Model (NLSM) with a $\Theta$-term, for a (real) three-dimensional unit vector $\vec{n}$ on the two-dimensional sphere $S^2$ with action

$$S = \int d^2x \frac{1}{g} \sum_{i,j} \left( \nabla_i \vec{n} \cdot \nabla_j \vec{n} \right)^2 + \frac{i\Theta}{8\pi} \epsilon_{abc} \nabla_i n^a \nabla_j n^b \nabla_k n^c. \quad (1)$$

Throughout this article we will always work in imaginary time. The $\Theta$-term contributes a factor $\exp(i\Theta q)$ to the partition function of the NLSM for every field configuration $\vec{n}(x)$ in 1+1d space-time which has $q$ topological instantons. For a spin-$s$ chain $\Theta = 2\pi s$. This describes the Haldane conjecture which states that the integer spin chain is gapped while half-integer spin chain is gapless. The qualitative difference between integer and half-integer spin chains was attributed to the constructive and destructive interference between even and odd number of instantons at the two different values of $\Theta$.

A similar NLSM with a $\Theta$-term can be used to describe the integer quantum Hall state in two dimensions, which possesses a fixed point at finite coupling $g$ for $\Theta = \pi$ fixed. For arbitrary values of $\Theta$ the NLSM is invariant under a $G_L \times G_R$ symmetry, denoting left and right multiplication of $U$ by group elements. When $\Theta = \pi k$ with integer $k$, the system also has time-reversal symmetry that transforms $i \rightarrow -i$ (throughout the paper we assume $U$ carries a trivial representation of time-reversal). Thus the time-reversal symmetry guarantees that when $\Theta = \pi k$, $\Theta$ does not flow under RG; while $\Theta$ in principle can flow for any other value due to nonperturbative effects.

Below, we will present our results first for the special case of $G = SU(2)$. Subsequently, we will explain that our arguments and conclusions are in fact generally applicable to a (simple) compact Lie group $G$. In the special case of $G = SU(2)$ one can parametrize any group element in terms of a four-dimensional (real) unit vector $\vec{\phi} = (\phi^0, \phi^1, \phi^2, \phi^3)$ on the 3-dimensional sphere $S^3$ as $U = e^{i\phi^a \sigma^a}$, where $\sigma^a = (\sigma^x, \sigma^y, \sigma^z)$ is the Pauli matrix. The $G_L \times G_R$ symmetry is isomorphic to $SO(4)$. This implies that for $G = SU(2)$, the NLSM in Eq. (2) can also be thought of as the (2+1)-d O(4) NLSM with action

$$S = \int d^3x \frac{1}{g} \sum_{ij} \left( \partial_i \phi^j \right)^2 + \frac{i\Theta}{12\pi^2} \epsilon_{abcd} \phi^a \partial_i \phi^b \partial_j \phi^c \partial_k \phi^d. \quad (3)$$

The goal of this paper is to provide a non-perturbative argument for the phase diagram of the (2+1)-d PCM with theta term, Eq. (2) in terms of the two coupling constants $g$ and $\Theta$. Our result for this phase diagram is depicted in Fig. 2 below. Note that in the absence of space-time boundaries (i.e. what the bulk physics is concerned) we are allowed to compactify (2+1)-d space-time into a three dimensional sphere $S^3$. First consider the case where $\Theta$ is an integer multiple of $2\pi$. $\Theta = 2\pi k$, so that the $\Theta$-term contributes a factor of unity to the partition function for any nontrivial instanton configuration in the space-time. Thus, in the absence of boundaries
in space-time the phase diagram of PCM in Eq. 3 at \( \Theta = 2\pi k \) is identical to the model at \( \Theta = 0 \). For small values of the coupling \( g \), the system is in an O(4) ordered phase with non-vanishing order parameter \( \langle \hat{n} \rangle \neq 0 \) and three gapless Goldstone modes. For large \( g \), on the other hand, the system is in a quantum disordered phase with a non-degenerate ground state and a fully gapped spectrum. The quantum phase transition between the O(4) ordered and disordered phases is an ordinary 2nd order transition in the 3D O(4) Wilson-Fisher (WF) universality class.

The presence of a \( \Theta \)-term is not expected to affect the ordered phase, because in it instantons are suppressed. Therefore, it can only play a role for the transition into the disordered phase and in the disordered phase itself. In order to understand the disordered phase and the phase transition in the presence of a theta term, standard perturbative methods fail [11]. Thus in order to understand the disordered phase of the PCM in Eq. 3, a non-perturbative argument must be developed, which we do in this article. Our conclusion is that there are two possibilities for the disordered phase of the PCM with a theta term (Eq. 3 and Eq. 2): It is either a gapless phase with power-law correlations for the fields \( U \) (or \( \hat{n} \)), or it is a gapped phase but with a two-fold ground state degeneracy.

(1+1)-d O(3) NLSM at \( \Theta = \pi \): Before we start our argument in (2+1)-d, let us first consider the (1+1)-d O(3) NLSM with the \( \Theta \)-term in Eq. 1 and focus on \( \Theta = \pi \). Since this model describes the SU(2) spin-1/2 chain, we know from the Lieb-Schultz-Mattis (LSM) theorem [12] that this NLSM is either a gapless CFT, or gapped but two-fold degenerate. Since the goal of this paper is to understand the (2+1)-d PCM field theories of Eq. 2 without recourse to any lattice spin model representations, we will first use a new argument to understand the behavior of the (1+1)-d O(3) NLSM without using any microscopic lattice representations such as spin chains. Subsequently, we will generalize this argument to the (2+1)-d models.

Our argument proceeds in four steps:

Step (1). In order to understand the O(3) NLSM Eq. 1 at \( \Theta = \pi \), let us first look at \( \Theta = 0 \) and \( \Theta = 2\pi \). At these values of \( \Theta \), the system also has the discrete symmetry \( n \rightarrow -n \), in addition to the SO(3) rotation symmetry. The bulk spectra for \( \Theta = 0 \) and \( \Theta = 2\pi \) are identical, possessing a non-degenerate ground state and a gap to all excitations.

Step (2). Now let us consider the system on a spatial interval with open boundaries at \( x = 0 \) and \( x = L \). Although the models with \( \Theta = 0 \) and \( \Theta = 2\pi \) have identical bulk spectra, they behave very differently at the boundaries. Since the bulk is gapped, we can safely ignore the bulk, and focus on the boundary because the gap in the bulk will protect the effective boundary theory from any singular contributions. Since the boundary is a point in space, it is effectively described by a (0+1)-d O(3) NLSM model. When \( \Theta = 0 \) this (0+1)-d NLSM model is completely trivial. However, when \( \Theta = 2\pi \), the \( \Theta \)-term in Eq. 1 can be viewed as the O(3) Wess-Zumino-Witten (WZW) term for the (0+1)-d O(3) NLSM model at each of the two boundaries, at \( x = 0 \) and at \( L \):

\[
\int_0^L dx \int dt \frac{i2\pi}{8\pi} \epsilon_{\mu\nu} n^\alpha \partial_{\mu} n^b \partial_{\nu} n^c = \text{WZW}_0 - \text{WZW}_L (4)
\]

The WZW term for a 0+1 dimensional O(3) NLSM, appearing on the right hand side, is defined as follows: In the (0+1)-d ) NLSM the O(3) vector \( \vec{n} \) is a function only of imaginary time \( \tau \). Consider a periodic evolution of \( \vec{n}(\tau) \), namely \( \vec{n}_{\tau=0} = \vec{n}_{\tau=\beta} \). Then \( \vec{n} \) is a mapping from a closed loop \( S^1 \) parametrized by \( \tau \in [0, \beta] \) to the target space \( S^2 \). The WZW term is defined as the solid angle on the target space \( S^2 \) enclosed by the closed loop \( \tau \in [0, \beta] \). The WZW term at level \( k \) can be explicitly written as

\[
\text{WZW} = 2\pi \int du d\tau \frac{i k}{8\pi} \epsilon_{abc} n^a \partial_{\mu} n^b \partial_{\nu} n^c, \quad \vec{n}(\tau, 0) = \hat{z}. \quad (5)
\]

Here, the function \( \vec{n}(\tau) \) has been extended to a mapping \( \vec{n}(\tau, u) \) from a disc \( (\tau, u) \), where \( 0 \leq u \leq 1 \) and \( \tau \in S^1 \), to the target space \( S^2 \). The second line of Eq. 5 is the only constraint on this extended mapping. Unlike the \( \Theta \)-term, the coefficient \( k \) in Eq. 5 has to be an integer [11], regardless of whether any discrete symmetry is present or not. By simply identifying \( u \) with \( x \), one arrives at Eq. 1.

It is well known that if a (0+1)-d O(3) NLSM, describing the quantum mechanics of a point particle on a sphere, has a WZW term at level \( k \), the ground state of this quantum mechanics is \((k + 1)\)-fold degenerate. In fact, the ground state of the (0+1)-d O(3) NLSM with a WZW term at level \( k \) precisely describes a single SU(2) spin with \( S = k/2 \). In Eq. 5 the WZW term at each boundary is at level \( k = 1 \). This implies that the model in Eq. 1 has two fold degeneracy at each boundary when \( \Theta = 2\pi \). This conclusion again agrees with Haldane’s conjecture, which states that the model with \( \Theta = 2\pi \) describes the spin-1 chain. Moreover it recovers the well-known fact that the Haldane phase of the spin-1 chain has an unpaired spin-1/2 degree of freedom at each of its boundaries [13, 12].

Step (3). Now let us tune \( \Theta \) in Eq. 1 continuously from \( 2\pi \) to 0. Then the spin-1/2 boundary state has to disappear at a certain value of \( \Theta \). When \( \Theta \) is tuned away from \( 2\pi \), the discrete symmetry \( \vec{n} \rightarrow -\vec{n} \) of the system is broken (in a spin-1/2 chain, this discrete symmetry is translation by one lattice constant). One important fact is that the spin-1/2 boundary state cannot be destroyed without going through a bulk transition, even when the...
discrete symmetry is broken. This is because, given a single spin-1/2, as long as the SO(3) symmetry is preserved, the spin-1/2 doublet degeneracy cannot be lifted. This conclusion can also be drawn by noticing that the coefficient of the WZW term has to be quantized, no matter whether the discrete symmetry is broken or not.

Step (4). We have concluded that in order to destroy the boundary spin-1/2 state, a bulk transition has to occur. Here we assume the simplest case, i.e. that there is one single transition between $\Theta = 0$ and $2\pi$. Then in this case there are exactly two possibilities for this bulk transition:

4A. The transition is of second order, meaning that the bulk gap closes continuously for some values of $\Theta$ between $\Theta = 0$ and $2\pi$. Because the bulk spectrum is identical for $\Theta$ and $(2\pi - \Theta)$, this transition has to occur at $\Theta = \pi$ if the bulk gap closes only at one value of $\Theta$. This implies that the bulk is gapless when $\Theta = \pi$. When $\Theta$ is approaching $\pi$ from $2\pi$, the boundary spin-1/2 state will become more and more delocalized, and eventually gets absorbed by the gapless bulk states at $\Theta = \pi$.

4B. The transition is of first order, meaning there is always a bulk gap. However, at this first order transition, the two phases with $\Theta = 0$ and $\Theta = 2\pi$ will have a crossing of their ground state energies, and this level crossing also has to occur at $\Theta = \pi$. This implies that the ground state at $\Theta = \pi$ is two-fold degenerate. In this case, when $\Theta$ is tuned from $2\pi$ to $\pi$, the boundary spin-1/2 states will never delocalize, they will simply disappear abruptly at $\Theta = \pi$. An example of this phenomenon is the (1+1)-d CP$^N$ model which is known to have a first order transition at $\Theta = \pi$ when $N \geq 3$.

These two possibilities that we have arrived at above are completely consistent with the conclusion drawn from the LSM theorem for the SU(2) spin-1/2 chain.

(2+1)-d O(4) NLSM at $\Theta = \pi$. We will now generalize the arguments given in the previous paragraph to the (2+1)-d O(4) NLSM with a $\Theta$-term, Eq. 3. Since we are only interested in the nature of the disordered phase of Eq. 3 we will consider the case with large values of the coupling $g$.

Step (1). In order to investigate the disordered phase at $\Theta = \pi$, we first look, as before, at $\Theta = 0$ and $2\pi$. Again, in the bulk these two disordered phases are both gapped with a non-degenerate ground state, while they have different boundary states. In order to look at the boundary states, we let the $x$ direction be a finite interval $0 \leq x \leq L$ while the $y$ direction is periodic, so that the system is defined on a finite 2d cylinder (Fig. 1).

Step (2). At each boundary located at $x = 0$ and $x = L$ there is a (1+1)-d theory defined on $(y, \tau)$-space-time. Since the bulk is gapped when $\Theta = 0$ and $2\pi$, the kinetic term of the effective boundary theory is still that of a local O(4) NLSM, Eq. 3, but now in (1+1)-d. When $\Theta = 0$, there is no nontrivial topological term at the boundary. However, when $\Theta = 2\pi$, the $\Theta$-term of the (2+1)-d bulk O(4) NLSM, Eq. 3, can as before be viewed as a WZW term of the (1+1)-d O(4) NLSM appearing on the boundary. Thus, the boundary theory is described by the following 1+1d O(4) NLSM with a WZW term:

\[
S = \int dyd\tau \frac{1}{g} (\partial_{\mu} \phi^a)^2 + \frac{i2\pi k}{12\pi^2} \int dudyd\tau \epsilon_{\mu\nu\rho} \epsilon_{abcd} \partial^a \phi^b \partial^c \phi^d \partial^d \phi^d. \tag{6}
\]

When $\Theta = 2\pi$, the boundary WZW term has level $k = 1$. It is well known that the long-distance behavior of this (1+1)-d O(4) NLSM with level $k = 1$ WZW term is controlled by a stable fixed point at finite $g^*$, and that this fixed point is precisely the SU(2)$_1$ CFT which describes the nearest neighbor spin-1/2 Heisenberg chain. When $\Theta = 2\pi k$ and $k = \text{integer}$, the boundary is described at long scales by the SU(2)$_k$ CFT\cite{16, 17}. Thus once again, when $\Theta$ is a non-vanishing integer multiple of $2\pi$ the system possesses nontrivial gapless boundary states.

Step (3). The same strategy that we used before can now be applied: When we tune $\Theta$ continuously from $2\pi$ to 0, then the boundary state has to disappear through a bulk phase transition. This is because SO(4) symmetry of the (1+1)-d theory in Eq. 3 is, as mentioned in the introduction, isomorphic to the SU(2)$_L \times SU(2)_R$ symmetry of the PCM. It is this symmetry that protects the finite-coupling fixed point at $g = g^*$ of the boundary NLSM, Eq 3, from being gapped out. In order to gap out this fixed point CFT, we need to break the SU(2)$_L \times SU(2)_R$
symmetry down to the diagonal SU(2) symmetry, i.e., we need to induce a relevant back-scattering between left and right moving boundary modes. However, since our model Eq. 3 has $O(4) \sim SU(2)_L \times SU(2)_R$ symmetry for any value of the coupling $g$, such backscattering processes are absent. Thus, although tuning $\Theta$ away from $2\pi$ breaks a discrete symmetry of the system, the boundary CFT cannot be gapped out without going through a bulk transition.

Step (4). Since the boundary states can only be destroyed through a bulk transition, there are the following two possibilities for this transition:

4A. This bulk transition is of second order, and it has to occur at $\Theta = \pi$. This implies that the disordered phase of the PCM in Eq. 3 is gapless at $\Theta = \pi$. Since the transition is of second order, the ground state energy $E(\Theta)$ has a singularity in its second derivative $\partial^2 E(\Theta) / \partial \Theta^2$ at $\Theta = \pi$ (Fig. 1).

4B. This bulk transition is first order and occurs at $\Theta = \pi$. At this transition the two gapped phases with $\Theta = 0$ and $\Theta = 2\pi$ will have a crossing of their ground state energies, and this level crossing has to appear at $\Theta = \pi$. This implies, as before, a gapped spectrum and a two-fold degenerate ground state at $\Theta = \pi$. In this case, the ground state energy $E(\Theta)$ has a kink at $\Theta = \pi$, i.e., the first order derivative $\partial E(\Theta) / \partial \Theta$ is discontinuous at $\Theta = \pi$ (Fig. 1).

It is straightforward to generalize these arguments to other $(2+1)$-d PCMs with a Theta term, as in Eq. 2 defined on more general compact Lie group manifolds such as e.g. $G = SU(N), SO(N)$ and $Sp(N)$. The key argument rests on the gaplessness of the CFT (often denoted by $G_k$) that describes the long-distance behavior at the finite-coupling fixed point of the $(1+1)$-d PCM with WZW term at level $k$ and which appears that the boundary of the $(2+1)$-d bulk PCM at $\Theta = 2\pi k$. The gaplessness of this CFT is protected by the $G_L \times G_R$ symmetry of the PCM with WZW term, which forbids all operators which are relevant in the renormalization group (RG) sense.

Based on these arguments we obtain the two possibilities for the RG flow diagram of the two coupling constants $g$ and $\Theta$ of model Eq. 2 sketched in Fig. 2.

So far, in our argument for $G = SU(2)$, we have assumed full SO(3) symmetry in (1+1)-d, and full O(4) symmetry in (2+1)-d. Our arguments only relied on the stability of the boundary states at $\Theta = 2\pi$ under discrete symmetry breaking (e.g. time-reversal symmetry in 2+1d, $\vec{n} \rightarrow -\vec{n}$ in 1+1d). In fact, the stability of the boundary states is still guaranteed even if the continuous symmetry of the system is relaxed. For instance, in the (1+1)-d NLSM, the SO(3) symmetry can be relaxed to O(2)$_{xy} \times Z_2$, or even $Z_2 \times Z_2 \times Z_2$. Here O(2)$_{xy}$ denotes spin rotations in the XY plane, while $Z_2$ is the $Z_2$ symmetry that transforms $z \rightarrow -z$. To break SO(3) to O(2)$_{xy} \times Z_2$, one can simply add a term of the form $(n_x^2) + (n_y^2) - 2(n_z^2)$ in Eq. 1 to the Lagrangian. One can show that these symmetries guarantee that the two-fold degeneracy of a single spin-1/2 cannot be lifted, thus our conclusions about $\Theta = \pi$ are still valid.

In the 2+1d case, the O(4) symmetry can be relaxed to O(2) x O(2), or O(3) x $Z_2$. Here O(2) x O(2) stands for the two separate O(2) rotations between ($\phi^1$, $\phi^2$), and ($\phi^0$, $\phi^3$) respectively. This O(4) to O(2) x O(2) symmetry breaking can be induced by the following term in the Lagrangian: $(\phi^1)^2 + (\phi^2)^2 - (\phi^0)^2 - (\phi^3)^2$. Under this symmetry breaking, the bulk state with $\Theta = 2\pi$ is analogous to the Quantum Spin Hall state [21, 22] with both charge conservation and spin $S^z$ conservation. Also, these two O(2) symmetries are the U(1) global symmetry and U(1) chiral (‘axial’) symmetry of the boundary states. It is known that with these two separate U(1) symmetries the boundary state of the quantum spin Hall state is stable even if time-reversal symmetry is broken. This is because if we want to gap out the boundary state without a bulk transition, then backscattering at the boundary has to be induced, but it is forbidden by the U(1) conservation of $S^z$, or the unbroken U(1) chiral symmetry.

In the case of O(4) to O(3) x $Z_2$ symmetry breaking, the O(3) is the symmetry that rotates between ($\phi^1$, $\phi^2$, $\phi^3$), while the $Z_2$ corresponds to $\phi^0 \rightarrow -\phi^0$. This symmetry breaking can be induced by adding following term to the bulk Lagrangian: $\sum_{a=1}^3 (\phi^a)^2 - 3(\phi^0)^2$. This induces a term of the form $\lambda \sum_{a=1}^3 J_{L_a}^2 J_{R_a}^2$, in the boundary CFT Lagrangian, where $J_{L_a}^2$ and $J_{R_a}^2$ denote the Noether currents for left- and right-multiplication by $SU(2)$. This term is a marginal perturbation and, depending on the sign of $\lambda$, is either marginally irrelevant or marginally relevant. But in both cases, the boundary with $\Theta = 2\pi$ is non-trivial: it is either a gapless CFT or two-fold degenerate with $\langle \phi^0 \rangle \neq 0$ at the boundary. In either case, a bulk

![FIG. 2: The two possible RG flows for the coupling constants $g$ and $\Theta$ of the (2+1)-d PCM on a compact Lie group G with Theta term, Eq. 2 (and Eq. 3 for $G = SU(2)$). There is always an ordered phase with small $g$. For $G = SU(2)$, the phase transition is in the conventional three-dimensional O(4) Wilson-Fisher universality class when $\Theta = 0$ and 2$\pi$, while the fixed points at $\Theta = \pi$ are presumably in the different universality classes.](image-url)
transition has to occur in order to gap out the boundary CFT, or to destroy the two-fold degeneracy at boundary.

If the bulk spectrum of the $G = SU(2)$ PCM of Eq. 3 is gapped at $\Theta = \pi$, after rescaling the system to the infrared limit this gap becomes infinity. Thus the coupling $g$ can be taken to infinity effectively. Thus we can ignore the first term in Eq. 3 and focus on the second, topological term only. Since we are interested in the bulk properties, we may compactify two dimensional space to a two-sphere $S^2$; i.e., any spatial configuration of $\phi(x,y,\tau)$ at fixed time is a mapping from the two-sphere $S^2$ to the target space $S^3$. Now, we can write the $\Theta$-term in the following form:

$$ S = \int d\tau \left( \partial_\tau \Phi \right)^2 + i\Theta \partial_\tau \Phi + V(\Phi), \quad U \to \infty, \quad \Phi(\tau) = \int d^2x \int du \frac{1}{12\pi^2} \epsilon_{abcd} \epsilon_{\mu\nu\rho} \partial^a \phi^b \partial_\mu \phi^c \partial_\nu \phi^d \partial_\rho (7) $$

where $\epsilon_{abcd} \epsilon_{\mu\nu\rho}$ is the WZW term in Eq. 3. In Eq. 4 we have extended the spatial dependence of $\phi(x,y,\tau)$ at fixed time to a mapping from the three dimensional ball $D^3$ parameterized by $(x,y,u)$ to the target space $S^3$, so that the spatial two-sphere $S^2$ is the boundary of $D^3$. Just like the WZW term, the only constraint on this extended mapping is $\phi(x,y,1,\tau) = \phi(x,y,\tau)$, and $\phi(x,y,0,\tau) = (0,0,0,1)$, for all times $\tau$.

Now, as $g$ is taken to infinity, the action Eq. 3 simplifies to that of a quantum mechanics problem,

$$ S = \int d\tau \left( \partial_\tau \Phi \right)^2 + i\Theta \partial_\tau \Phi + V(\Phi), \quad U \to \infty, \quad \Phi(\tau) = \int d^2x \int du \frac{1}{12\pi^2} \epsilon_{abcd} \epsilon_{\mu\nu\rho} \partial^a \phi^b \partial_\mu \phi^c \partial_\nu \phi^d \partial_\rho (7) $$

where $\Phi(\tau)$ is related to the Skyrmion number of $\bar{n}$ evaluated at fixed time $\tau$. Once the $O(4)$ symmetry is broken down to $O(3) \times Z_2$, $\Phi(\tau)$ is no longer a continuous variable, instead it is a discrete number, because the $O(3)$ Skyrmion number is quantized on the space $S^2$. More precisely, when the instantaneous field $\bar{n}(x,y,\tau)$ at time $\tau$ has $q$ Skyrmions in 2d space, then $\Phi(\tau) = q/2$. Thus this $O(4)$ to $O(3) \times Z_2$ symmetry breaking is equivalent to turning on a strong periodic potential $V(\Phi)$ in Eq. 8 with periodicity 1/2, and the Lagrangian Eq. 8 reduces to a one dimensional tight-binding model with lattice constant 1/2 (Fig. 3) i.e. the size of the Brillouin zone of this tight binding model is 4$\pi$. Nearest neighbor hopping on this lattice is physically equivalent to changing the Skyrmion number on space $S^2$ by 1.

To change the Skyrmion number, one needs a space-time hedgehog-like monopole configuration of $\bar{n}$. The core of the monopole has $\phi^b \neq 0$, and there are two different types of monopole, depending on the sign of $\phi^0$ at the monopole core. The $\Theta$-term will contribute a factor $\exp(i\Theta/2)$ and $\exp(-i\Theta/2)$ to the monopole with $\phi^0 > 0$ and $\phi^0 < 0$ respectively [22]. Thus when $\Theta = \pi$ these two types of monopoles have complete destructive interference with each other, and it is forbidden to change the Skyrmion number by 1, i.e. hopping by one lattice constant in Fig. 3 is forbidden. However, hopping by two lattice constants is still allowed, but the band structure will be doubly degenerate, namely on this one dimensional lattice the states with momentum $p = 0$ and $p = 2\pi$ are degenerate. The wavefunctions of these two states are

$$ |0\rangle \sim \sum_q |q\rangle, \quad |1\rangle \sim \sum_q (-1)^q |q\rangle, \quad \Phi(\tau) = \int d^2x \int du \frac{1}{12\pi^2} \epsilon_{abcd} \epsilon_{\mu\nu\rho} \partial^a \phi^b \partial_\mu \phi^c \partial_\nu \phi^d \partial_\rho (7) $$

where $q$ is the Skyrmion number on $S^2$.

Ref. [23] has considered the $O(4)$ to $O(3) \times Z_2$ symmetry breaking in Eq. 3 and using a different formalism the authors argued that the disordered phase has two fold degeneracy at $\Theta = \pi$. Although the argument was different, the two degenerate ground states derived in Ref. [23] are precisely the two states in Eq. 10.

**Summary.** We have found in this article, using non-perturbative arguments, that the quantum disordered phase of the (2+1)-d principal chiral models defined on (simple) Lie group manifolds with a topological $\Theta$-term is either gapless or two-fold degenerate at $\Theta = \pi$. Our conclusion is analogous to the LSM theorem for the one-dimensional SU(2) spin-1/2 quantum chain [12]. The LSM theorem has been generalized to higher dimensional lattice spin systems in Ref. [24]. The reasoning presented in this article, on the other hand, is entirely based on continuum field theories, without recourse to lattice spin representations.

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