REGULARITY OF CERTAIN VERTEX OPERATOR ALGEBRAS
WITH TWO GENERATORS

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Abstract. For every \( m \in \mathbb{C} \setminus \{0, -2\} \) and every nonnegative integer \( k \) we define the vertex operator (super)algebra \( D_{m,k} \) having two generators and rank \( \frac{3m}{m+2} \). If \( m \) is a positive integer then \( D_{m,k} \) can be realized as a subalgebra of a lattice vertex algebra. In this case, we prove that \( D_{m,k} \) is a regular vertex operator (super)algebra and find the number of inequivalent irreducible modules.

1. Introduction

In the theory of vertex operator (super)algebras, the classification and the construction of rational vertex operator (super)algebras are main problems. The rationality of certain familiar vertex operator (super)algebras was proved in papers [W], [FZ], [D], [DL], [Li1], [A1], [A2], [A3].

It is natural to consider rational vertex operator (super)algebras of certain rank. In particular, in the rank one case for every positive integer \( k \) we have the well-known rational vertex operator (super)algebra \( F_k \) associated to the lattice \( \sqrt{k} \mathbb{Z} \). These vertex operator (super)algebras are generated by two generators.

In the present paper we will be concentrated on vertex operator (super)algebras of rank \( c_m = \frac{3m}{m+2} \), \( m \in \mathbb{C} \setminus \{0, -2\} \). This rank has the vertex operator algebra \( L(\ell, 0) \) associated to the irreducible vacuum \( \hat{sl}_2 \)–module of level \( m \) and the vertex operator superalgebra \( L_{cm} \) associated to the vacuum module for the \( N = 2 \) superconformal algebra with central charge \( c_m \) ( cf. [A2], [A3], [FST1], [FST2], [EG], [HM]). In the case \( m = 1 \) these vertex operator (super)algebras are included into the family \( F_k \), \( k \in \mathbb{N} \), since \( L(1, 0) \cong F_2 \) and \( L_{c_1} \cong F_3 \). The main purpose of this article is to include \( L(\ell, 0) \) and \( L_{cm} \) into the family \( D_{m,k} \), \( k \in \mathbb{N} \), of rational vertex operator (super)algebras of rank \( c_m \) for arbitrary positive integer \( m \).

In fact, for every \( m \in \mathbb{C} \setminus \{0, -2\} \) we define the vertex operator (super)algebra \( D_{m,k} \) as a subalgebra of the vertex operator (super)algebra \( L(\ell, 0) \otimes F_k \) (cf. Section
In the special case $k = 1$, $D_{m,1}$ is in fact the $N = 2$ vertex operator superalgebra $L_{c_n}$ constructed using the Kazama-Suzuki mapping (cf. [KS], [FST1]). We also have that $D_{m,0} \cong L(m,0)$ and $D_{1,k} \cong F_{k+2}$.

Moreover, we shall demonstrate that $D_{m,k}$ has many properties similar to affine and $N = 2$ superconformal vertex algebras.

When $m$ is not a nonnegative integer, then $D_{m,k}$ has infinitely many irreducible representations. Thus, it is not rational (cf. Section 3). In order to construct new examples of rational vertex operator (super)algebras we shall consider the case when $m$ is a positive integer. Then $D_{m,k}$ can be embedded into a lattice vertex algebra (cf. Section 4). In fact, we shall prove that

\begin{align}
(1.1) & \quad D_{m,k} \otimes F_{-k} \cong L(m,0) \otimes F_{-\frac{k}{2}(mk+2)}, \quad (k \text{ even}), \\
(1.2) & \quad D_{m,k} \otimes F_{-k} \cong L(m,0) \otimes F_{-k(mk+2)} \oplus L(m,m) \otimes MF_{-k(mk+2)}, \quad (k \text{ odd}).
\end{align}

These relations completely determine the structure of $D_{m,k} \otimes F_{-k}$ as a weak $L(m,0)$–module.

In [DLM] was introduced the notion of regular vertex operator algebra, i.e. rational vertex operator algebra with the property that every weak module is completely reducible. The relations (1.1) and (1.2), together with the regularity results from [DLM] and [Li2] imply that $D_{m,k}$ is a regular vertex operator algebra if $k$ is even, and a regular vertex operator superalgebra if $k$ is odd.

Let us here discuss the case $k = 2n$, where $n$ is a positive integer. The relation (1.1) suggests that one can study the dual pair $(D_{m,2n}, F_{-2n})$ directly inside $L(m,0) \otimes F_{-2n(nm+1)}$. This approach requires many deep results on the structure of the vertex operator algebra $L(m,0)$ and deserves to be investigated independently. Instead of this approach, we realize the vertex algebra $D_{m,2n} \otimes F_{-2n}$ inside a larger lattice vertex algebra. Then the formulas for the generators are much simpler (cf. Section 5). The similar analysis can be done when $k$ is odd (cf. Section 6). This approach was also used in [A3] for studying the fusion rules for the $N = 2$ vertex operator superalgebra $D_{m,1}$.
2. LATTICE AND AFFINE VERTEX ALGEBRAS

In this section, we shall recall the lattice construction of vertex superalgebras from [DT], [K].

Let $L$ be a lattice. Set $h = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-form $\langle \cdot, \cdot \rangle$ on $L$ to $h$. Let $\hat{h} = \mathbb{C}[t, t^{-1}] \otimes h \oplus \mathbb{C}c$ be the affinization of $h$. We also use the notation $h(n) = t^n \otimes h$ for $h \in h, n \in \mathbb{Z}$.

Set $h^+ = t\mathbb{C}[t] \otimes h; \quad h^- = t^{-1}\mathbb{C}[t^{-1}] \otimes h$. Then $h^+$ and $h^-$ are abelian subalgebras of $\hat{h}$. Let $U(h^-) = \mathcal{S}(h^-)$ be the universal enveloping algebra of $h^-$. Let $\lambda \in h$. Consider the induced $\hat{h}$-module

$$M(1, \lambda) = U(\hat{h}) \otimes_{U(\mathbb{C}[t] \otimes h \oplus \mathbb{C}c)} C_\lambda \simeq S(h^-) \quad {\text{(linearly)}},$$

where $t\mathbb{C}[t] \otimes h$ acts trivially on $\mathbb{C}$, $h$ acting as $\langle h, \lambda \rangle$ for $h \in h$ and $c$ acts on $\mathbb{C}$ as multiplication by 1. We shall write $M(1)$ for $M(1, 0)$. For $h \in h$ and $n \in \mathbb{Z}$ write $h(n) = t^n \otimes h$. Set $h(z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n-1}$.

Then $M(1)$ is a vertex operator algebra which is generated by the fields $h(z), h \in h,$ and $M(1, \lambda)$, for $\lambda \in h$, are irreducible modules for $M(1)$.

Let $\hat{L}$ be the canonical central extension of $L$ by the cyclic group $\langle \pm 1 \rangle$:

$$(2.1) \quad 1 \to \langle \pm 1 \rangle \to \hat{L} \to L \to 1$$

with the commutator map $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in L$. Let $e : L \to \hat{L}$ be a section such that $e_0 = 1$ and $e : L \times L \to \langle \pm 1 \rangle$ be the corresponding 2-cocycle. Then $e(\alpha, \beta)e(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle}$,

$$(2.2) \quad e(\alpha, \beta)e(\alpha + \beta, \gamma) = e(\beta, \gamma)e(\alpha, \beta + \gamma)$$

and $e_\alpha e_\beta = e(\alpha, \beta)e_{\alpha + \beta}$ for $\alpha, \beta, \gamma \in L$. Form the induced $\hat{L}$-module

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\langle \pm 1 \rangle} \mathbb{C} \simeq \mathbb{C}[L] \quad {\text{(linearly)}},$$

where $\mathbb{C}[\cdot]$ denotes the group algebra and $-1$ acts on $\mathbb{C}$ as multiplication by $-1$. For $a \in \hat{L}$, write $\iota(a)$ for $a \otimes 1$ in $\mathbb{C}\{L\}$. Then the action of $\hat{L}$ on $\mathbb{C}\{L\}$ is given by:

$$a \cdot \iota(b) = \iota(ab) \quad \text{and} \quad (-1) \cdot \iota(b) = -\iota(b) \quad \text{for} \ a, b \in \hat{L}.$$

Furthermore we define an action of $h$ on $\mathbb{C}\{L\}$ by: $h \cdot \iota(a) = \langle h, a \rangle \iota(a)$ for $h \in h, a \in \hat{L}$. Define $z^h \cdot \iota(a) = z^{\langle h, a \rangle} \iota(a)$. 

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The untwisted space associated with $L$ is defined to be

$$V_L = \mathbb{C}\{L\} \otimes_{\mathbb{C}} M(1) \simeq \mathbb{C}[L] \otimes S(\hat{h}^-) \; \text{(linearly)}.$$ 

Then $\hat{L}, \hat{h}, z^h \; (h \in \mathfrak{h})$ act naturally on $V_L$ by acting on either $\mathbb{C}\{L\}$ or $M(1)$ as indicated above. Define $1 = \iota(e_0) \in V_L$. We use a normal ordering procedure, indicated by open colons, which signify that in the enclosed expression, all creation operators $h(n) \; (n < 0), a \in \hat{L}$ are to be placed to the left of all annihilation operators $h(n), z^h \; (h \in \mathfrak{h}, n \geq 0)$. For $a \in \hat{L}$, set

$$Y(\iota(a), z) = e^{f(\bar{a}(z) - a(0)z^{-1})}az^a :.$$ 

Define vertex operator $Y(v, z)$ with

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-2} (v_n \in \text{End} V_L).$$ 

This gives us a well-defined linear map

$$Y(\cdot, z) : \quad V_L \to (\text{End} V_L)[[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, \; (v_n \in \text{End} V_L).$$

Let $\{ h_i \mid i = 1, \ldots, d \}$ be an orthonormal basis of $\mathfrak{h}$ and set

$$\omega = \frac{1}{2} \sum_{i=1}^d h_i(-1)h_i(-1) \in V_L.$$ 

Then $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$ gives rise to a representation of the Virasoro algebra on $V_L$ with the central charged $d$ and

$$L(0) (\iota(a) \otimes h_1(-n_1) \cdots h_n(-n_k))$$

$$= \left( \frac{1}{2}(\bar{a}, \bar{a}) + n_1 + \cdots + n_k \right) (\iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k)).$$ 

The following theorem was proved in [DL] and [K].

**Theorem 2.1.** The structure $(V_L, Y, 1, L(-1))$ is a vertex superalgebra.
Define the Schur polynomials \( p_r(x_1, x_2, \cdots) \) in variables \( x_1, x_2, \cdots \) by the following equation:

\[
\exp \left( \sum_{n=1}^{\infty} \frac{x_n}{n} y^n \right) = \sum_{r=0}^{\infty} p_r(x_1, x_2, \cdots) y^r. 
\]  

(2.5)

For any monomial \( x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r} \) we have an element \( h(-1)^{n_1} h(-2)^{n_2} \cdots h(-r)^{n_r} \cdot 1 \) in both \( M(1) \) and \( V_L \) for \( h \in \mathfrak{h} \). Then for any polynomial \( f(x_1, x_2, \cdots) \), \( f(h(-1), h(-2), \cdots) \cdot 1 \) is a well-defined element in \( M(1) \) and \( V_L \). In particular, \( p_r(h(-1), h(-2), \cdots) \cdot 1 \) for \( r \in \mathbb{N} \) are elements of \( M(1) \) and \( V_L \).

Suppose \( a, b \in \hat{L} \) such that \( \bar{a} = \alpha, \bar{b} = \beta \). Then

\[
\exp \left( \sum_{n=1}^{\infty} \frac{\alpha(-n)}{n} z^n \right) \iota(ab) = \sum_{r=0}^{\infty} p_r(\alpha(-1), \alpha(-2), \cdots) \iota(ab) z^{r+\langle \alpha, \beta \rangle}.
\]  

(2.6)

Thus

\[
\iota(a)_i \iota(b) = 0 \quad \text{for} \quad i \geq -\langle \alpha, \beta \rangle.
\]  

(2.7)

Especially, if \( \langle \alpha, \beta \rangle \geq 0 \), we have \( \iota(a)_i \iota(b) = 0 \) for \( i \geq 0 \), and if \( \langle \alpha, \beta \rangle = -n < 0 \), we get

\[
\iota(a)_{-i} \iota(b) = p_{n-i}(\alpha(-1), \alpha(-2), \cdots) \iota(ab) \quad \text{for} \quad i \in \{0, \ldots, n\}.
\]  

(2.8)

Let \( n \in \mathbb{Z} \), and \( \langle \beta, \beta \rangle = n \). Define

\[
L_n = \mathbb{Z} \beta, \quad F_n = V_{L_n}.
\]

Then \( F_n \) is a vertex algebra if \( n \) is even, and a vertex superalgebra if \( n \) is odd. For \( i \in \mathbb{Z} \), let \( \bar{i} = i + n \mathbb{Z} \in \mathbb{Z}/n \mathbb{Z} \). We define \( \overline{F_n} = V_{\mathbb{Z} \beta + \frac{i}{n} \beta} \). Clearly \( F_n = \overline{F_n} \). It is well-known (cf. \[DL\], \[DL\], \[Xu\]) that the set \( \{\overline{F_n}\}_{i=0, \ldots, |n|-1} \) provides all irreducible \( F_n \)-modules. In particular, \( F_n \) has \(|n|\) inequivalent irreducible modules.

The fusion algebra is (cf. \[DL\])

\[
\overline{F_n} \times \overline{F_n^J} = \overline{F_n^{n+J}}.
\]  

(2.9)

If \( n = 2k \) is even, we define \( \bar{L}_{2k} = \frac{3}{2} + \mathbb{Z} \beta \), and \( MF_{2k} = V_{\bar{L}_{2k}} = \overline{F_{2k}} \). Then \( F_{2k} \) is a vertex algebra, and \( MF_{2k} \) is a \( F_{2k} \)-module.
We shall also need the following result from \[\text{DLM}\].

**Proposition 2.1.** \[\text{DLM}\] The vertex (super)algebra \(F_n\) is regular, i.e. any (weak) \(F_n\)-module is completely reducible.

Let \(g\) be the Lie algebra \(sl_2\) with generators \(e, f, h\) and relations \([e, f] = h, [h, e] = 2e, [h, f] = -2f\). Let \(\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K\) be the corresponding affine Lie algebra of type \(A^{(1)}_1\). As usual we write \(x(n)\) for \(x \otimes t^n\) where \(x \in g\) and \(n \in \mathbb{Z}\). Let \(L(m, 0)\) denote the irreducible highest weight \(A^{(1)}_1\)-modules with the highest weight \((m - j)\Lambda_0 + j\Lambda_1\). Then \(L(m, 0)\) has a natural structure of a simple vertex operator algebra. Let \(1_m\) denote the vacuum vector in \(L(m, 0)\).

If \(m\) is a positive integer then \(L(m, 0)\) is a regular vertex operator algebra, and the set \(\{L(m, j)\}_{j=0,\ldots,m}\) provides all inequivalent irreducible \(L(m, 0)\)-modules. The fusion algebra (cf. \[\text{FZ}\]) is given by

\[
L(m, j) \times L(m, k) = \sum_{i=\max\{0, j+k-m\}}^{\min\{j,k\}} L(m, j+k-2i).
\]

In particular, \(L(m, m) \times L(m, j) = L(m, m-j)\).

We shall now recall the lattice construction of the vertex operator algebra \(L(m, 0)\).

Define the following lattice

\[
A_{1,m} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_m
\]

\[
\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j},
\]

for every \(i, j \in \{1, \ldots, m\}\). Define also \(\tilde{A}_{1,m} = \frac{\alpha_1 + \cdots + \alpha_m}{2} + A_{1,m}\). We have:

**Lemma 2.1.** \[\text{DL}\] The vectors \(E = \iota(e_{\alpha_1}) + \cdots + \iota(e_{\alpha_m})\), \(F = \iota(e_{-\alpha_1}) + \cdots + \iota(e_{-\alpha_m})\), span a subalgebra of \(V_{A_{1,m}}\) isomorphic to \(L(m,0)\). Moreover, \(L(m, m)\) is a \(L(m, 0)\) submodule of \(V_{\tilde{A}_{1,m}}\).

**3. The definition of \(D_{m,k}\)**

In this section we give the definition of the vertex operator (super)algebra \(D_{m,k}\).

Let the vertex (super)algebras \(L(m, 0)\) and \(F_k\) be defined as in Section 2.
Definition 3.1. Let $m \in \mathbb{C}\setminus \{0, -2\}$, and let $k$ be a nonnegative integer. Then $D_{m,k}$ is a vertex subalgebra of the vertex operator (super)algebra $L(m, 0) \otimes F_k$ generated by the vectors:

\[ X = e(-1)1_m \otimes \iota(e_\beta) \quad \text{and} \quad Y = f(-1)1_m \otimes \iota(e_{-\beta}). \]

Let $1_{m,k} = 1_m \otimes 1 \in D_{m,k} \subset L(m, 0) \otimes F_k$. Define also

\[ H = h(-1)1_m \otimes 1 + 1 \otimes \beta(-1)1, \]

\[ \omega_{m,k} = \frac{1}{2(m+2)} \left( X_{k-1}Y + Y_{k-1}X + \frac{1-k}{mk+2}H^2_{-1}1_{m,k} \right). \]

It is easy to see that the components of the field

\[ Y(\omega_{m,k}, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \]

give rise a representation of the Virasoro algebra of central charge $c_m = \frac{3m}{m+2}$. For $n \geq 0$ one has

\[ L(n)x = \delta_{n,0}(1+\frac{k}{2})x \quad \text{and} \quad L(n)y = \delta_{n,0}(1+\frac{k}{2})y. \]

Thus, $L(0)$ defines on $D_{m,k}$ a $\mathbb{Z}_+$-gradation if $k$ is even, and a $\frac{1}{2} \mathbb{Z}_+$-gradation if $k$ is odd. In this way we get the following theorem.

Theorem 3.1. Let $m \in \mathbb{C}\setminus \{0, -2\}$, and let $k$ be a nonnegative integer. Then $D_{m,k}$ is a vertex operator algebra if $k$ is even and a vertex operator superalgebra if $k$ is odd. The Virasoro vector is $\omega_{m,k}$, the vacuum vector is $1_{m,k}$ and the rank is $c_m$.

Remark 3.1. Let $k = 0$. Then $D_{m,0}$ is isomorphic to the $\hat{sl}_2$ vertex operator algebra $L(m,0)$. Note also that the vector

\[ \omega_{m,0} = \frac{1}{2(m+2)} \left( X_{-1}Y + Y_{-1}X + \frac{1}{2}H^2_{-1}1_{m,0} \right) \]

coincides with the Virasoro vector in $L(m,0)$ constructed using the Sugawara construction.
Remark 3.2. For $k = 1$, $D_{m,1}$ is in fact the vertex operator superalgebra associated to the vacuum representation of the $N = 2$ superconformal algebra constructed using the Kazama-Suzuki mapping (cf. [FST], [KS]). The Virasoro vector in $D_{m,1}$ is

$$\omega_{m,1} = \frac{1}{2(m+2)}(X_0Y + Y_0X).$$

Its representation theory was studied in [EG] and [A2]. In particular, when $m$ is not a nonnegative integer then $D_{m,1}$ is not rational. In Theorem 3.2 we will generalize this fact for every positive integer $k$.

It was proved in [A3] that if $m$ is a positive integer, then $D_{m,1}$ is a regular vertex operator superalgebra and that the vertex superalgebra $D_{m,1} \otimes F_{-1}$ is a simple current extension of the vertex algebra $L(m,0) \otimes F_{-2(m+2)}$.

The definition of $D_{m,k}$ implies that for every weak $L(m,0)$–module $M$, $M \otimes F_k$ is a weak module for $D_{m,k}$. Thus, the representation theory of $D_{m,k}$ is closely related to the representation theory of the vertex operator algebra $L(m,0)$. The case when $m$ is an nonnegative integer will be studied in following sections. When $m \neq -2$ and $m$ is not an admissible rational number, then every highest weight $\hat{sl}_2$–module of level $m$ is a module for the vertex operator algebra $L(m,0)$. This easily gives that $D_{m,k}$ is not rational. In the case when $m$ is an admissible rational number, using the similar arguments as in [A2], and using the representation theory of the vertex operator algebra $L(m,0)$ in this case (cf. [AM]) one can construct infinitely many inequivalent irreducible $D_{m,k}$–modules. In order to be more precise, we shall state the following lemma.

Lemma 3.1. Assume that $m$ is not a nonnegative integer and $m \neq -2$. Let $k \geq 1$. Then for every $t \in \mathbb{C}$ there is a ordinary $D_{m,k}$–module $N_t$ such that $N = \oplus_{n \in \frac{1}{2} \mathbb{Z}_+} N_t(n)$, and the top level $N_t(0)$ satisfies

$$N_t(0) = \mathbb{C}w, \quad L(n)w = t\delta_{n,0}w \quad \text{for } n \geq 0.$$ 

Proof. The proof will use a similar consideration to those in [A2], Section 6.

Assume that $m$ is not a positive integer and $t \in \mathbb{C}$. The results from [AM] give that for every $q \in \mathbb{C}$ there is a $\mathbb{Z}_+$–graded $L(m,0)$–module $M_q = \oplus_{n \in \mathbb{Z}_+} M_q(n)$ and a
weight vector \( v_q \in M_q(0) \) such that

\[
\Omega(0)|M_q(0) \equiv \frac{(m+2)m}{2} \text{Id}, \quad h(0)v_q = qv_q,
\]

where \( \Omega(0) = e(0)f(0) + f(0)e(0) + \frac{1}{2}h(0)^2 \) is the Casimir element acting on the \( sl_2 \)-module \( M_q(0) \). Then \( M_q \otimes F_k \) is a weak \( D_{m,k} \)-module. Choose \( q \in \mathbb{C} \) such that

\[
\frac{m}{4} - \frac{1}{4(m+2)}q^2 + \frac{1}{2(m+2)(mk+2)}q^2 = t.
\]

Let \( N_t \) be the \( D_{m,k} \)-submodule of \( M_q \otimes F_k \) generated by the vector \( w = v_q \otimes 1 \). Then for \( n \geq 0 \) we have that

\[
L(n)w = \delta_{n,0}(\frac{m}{4} - \frac{1}{4(m+2)}q^2 + \frac{1}{2(m+2)(mk+2)}q^2)w = \delta_{n,0}tw.
\]

Now it is easy to see that \( N_t \) is an ordinary \( \frac{1}{2}\mathbb{Z}_+ \)-graded \( D_{m,k} \)-module with the top level \( N_t(0) = \mathbb{C}w \) and that \( L(0)|N_t(0) \equiv t \text{Id} \). Thus, the lemma holds.

In fact, Lemma 3.1 gives that there is uncountably many inequivalent irreducible \( D_{m,k} \)-modules. Thus, we conclude that the following theorem holds.

**Theorem 3.2.** Assume that \( m \) is not a nonnegative integer and \( m \neq -2 \). Then for every positive integer \( k \), the vertex operator (super)algebra \( D_{m,k} \) is not rational.

**Remark 3.3.** In what follows we will prove that if \( m \) is a positive integer, then \( D_{m,k} \) is rational. In fact, we will establish more general complete reducibility theorem, which will imply that \( D_{m,k} \) is regular in the sense of [DLM].

4. **The lattice construction of \( D_{m,k} \) for \( m \in \mathbb{N} \)**

In this section we give the lattice construction of the vertex operator (super)algebra \( D_{m,k} \). This construction is a generalization of the lattice constructions of the vertex operator algebra \( L(m,0) \) (cf. [DL] and our Lemma 2.1) and of the N=2 vertex operator superalgebra \( L_{c_m} \) (cf. [A3]).

Let \( m, k \) be positive integers. Define the lattice

\[
\Gamma_{m,k} = \mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_m
\]

\[
\langle \gamma_i, \gamma_j \rangle = 2\delta_{i,j} + k,
\]

for every \( i, j \in \{1, \ldots, m\} \).
Then $V_{Γ_{m,k}}$ is a vertex operator algebra if $k$ is even and a vertex operator superalgebra if $k$ is odd.

**Proposition 4.1.** Let $m, k$ be positive integers. The vertex operator (super)algebra $D_{m,k}$ is isomorphic to the subalgebra of the vertex operator (super)algebra $V_{Γ_{m,k}}$ generated by the vectors

$$X = ι(e_{γ_1}) + \cdots + ι(e_{γ_m});$$
$$Y = ι(e_{-γ_1}) + \cdots + ι(e_{-γ_m}).$$

Set $H = X_k Y$. Then the Virasoro vector in $D_{m,k}$ is given by

$$\bar{\omega}_{m,k} = \frac{1}{2(m+2)}(\bar{X}_{k-1} Y + \bar{Y}_{k-1} X + \frac{1-k}{mk+2} \bar{H}_{k-1}^2 1)$$
$$= \frac{1}{2(m+2)} \sum_{i=1}^{m} γ_i (-1)^2 1 + \frac{1}{m+2} \sum_{i \neq j} ι(e_{γ_i-γ}) +$$
$$+ \frac{1-k}{2(m+2)(mk+2)} \left( \sum_{i=1}^{m} γ_i (-1) \right)^2 1.$$

**Proof.** Define the lattice $Γ_1$ by

$$Γ_1 = \mathbb{Z}α_1 + \cdots + \mathbb{Z}α_m + \mathbb{Z}β,$$
$$\langle α_i, α_j \rangle = 2δ_{i,j}, \quad \langle α_i, β \rangle = 0, \quad \langle β, β \rangle = k.$$  

For $i = 1, \ldots, m$ set $γ_i = α_i + β$. It is clear that the lattice $Γ_{m,k}$ can be identified with the sublattice $\mathbb{Z}γ_1 + \cdots + \mathbb{Z}γ_m$ of the lattice $Γ_1$. In the same way $V_{Γ_{m,k}}$ can treated as a subalgebra of the vertex operator (super)algebra $V_{Γ_1}$. Lemma 2.1 implies that $\bar{E} = ι(e_{α_1}) + \cdots + ι(e_{α_m}), \bar{F} = ι(e_{-α_1}) + \cdots + ι(e_{-α_m})$, span a subalgebra of $V_{Γ_1}$ isomorphic to $L(m,0)$, and the elements $ι(β), ι(β)$ span a subalgebra isomorphic to $F_k$. Since

$$\bar{X} = \bar{E}_{-1} ι(β) \quad \text{and} \quad \bar{Y} = \bar{F}_{-1} ι(β),$$

we conclude that the vertex subalgebra generated by the elements $\bar{X}, \bar{Y} ∈ V_{Γ_{m,k}} ⊂ V_{Γ_1}$ is isomorphic to the vertex operator (super)algebra $D_{m,k}$. This concludes the proof of the theorem. □
The previous result implies that we can identify the generators of $D_{m,k}$ in $L(m,0) \otimes F_k$ with the generators of $D_{m,k}$ in $V_{\Gamma_{m,k}}$. Thus we can assume that $X \approx \bar{X}$, $Y \approx \bar{Y}$. This identification implies that $H \approx \bar{H}$ and $\omega_{m,k} \approx \bar{\omega}_{m,k}$.

We shall also prove an interesting proposition which identifies some regular subalgebras of $D_{m,k}$.

**Proposition 4.2.** For every positive integer $n$ we have that

$$\iota(e_n(\gamma_1+\ldots+\gamma_m)), \ i\iota(e_{-n}(\gamma_1+\ldots+\gamma_m)) \in D_{m,k}.$$  

In particular, $D_{m,k}$ has a vertex subalgebra isomorphic to $F_{n^2(mk+2)}$.

**Proof.** Using relations (2.7) and (2.8), it is easy to prove that:

$$\left( X_{-(nm-1)k-2n+1} \ldots X_{-(n-1)m-2n+1} \right) \cdot \left( X_{-(nm-1)k-3} \ldots X_{-mk-3} \right) \cdot \left( X_{-(m-1)k-1} \ldots X_{-1}X_{-1} \right) = C\iota(e_n(\gamma_1+\ldots+\gamma_m))$$

for some constant $C$. Thus $\iota(e_n(\gamma_1+\ldots+\gamma_m)) \in D_{m,k}$. Similarly we prove that $\iota(e_{-n}(\gamma_1+\ldots+\gamma_m)) \in D_{m,k}$. The second assertion of the proposition follows from the fact that the vectors $\iota(e_{\pm n}(\gamma_1+\ldots+\gamma_m))$ generate a subalgebra of $V_{\Gamma_{m,k}}$ isomorphic to $F_{n^2(mk+2)}$. \hfill \Box

5. **Regularity of the vertex operator algebra $D_{m,2n}$**

In this section we study the vertex algebra $L(m,0) \otimes F_{-2n(mn+1)}$ where $m,n$ are positive integers. We now that $L(m,0) \otimes F_{-2n(mn+1)}$ is a regular vertex algebra; i.e. every module for this vertex algebra is completely reducible. Its irreducible modules are:

$$L(m,r) \otimes F_{-2n(mn+1)}, \quad r \in \{1, \ldots, m\}, \ s \in \frac{Z}{-2n(mn+1)Z}.$$ 

The fusion rules can be calculated easily from the fusion rules for $L(m,0)$ and $F_{-2n(mn+1)}$.

Our main goal is to show that the vertex operator algebra $D_{m,2n}$ is isomorphic to a subalgebra of $L(m,0) \otimes F_{-2n(mn+1)}$. In order to do this, we shall first give the lattice construction of the vertex algebra $L(m,0) \otimes F_{-2n(mn+1)}$. 

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Define the following lattice:

\[ L = \mathbb{Z} \alpha_1 + \cdots + \mathbb{Z} \alpha_m + \mathbb{Z} \beta \]

\[ \langle \alpha_i, \alpha_j \rangle = 2 \delta_{i,j}, \quad \langle \alpha_i, \beta \rangle = 0, \quad \langle \beta, \beta \rangle = -2n(mn + 1) \]

for every \( i, j \in \{1, \ldots, m\} \).

We shall now give another description of the lattice \( L \).

For \( i = 1, \ldots, m \), we define

\[ \delta = n\alpha_1 + \cdots + n\alpha_m + \beta, \]

\[ \gamma_i = \alpha_i + \delta. \]

Since

\[ \alpha_i = \gamma_i - \delta, \quad \beta = (nm + 1)\delta - n(\gamma_1 + \cdots + \gamma_m), \]

we have that

\[ L = \mathbb{Z} \gamma_1 + \cdots + \mathbb{Z} \gamma_m + \mathbb{Z} \delta \]

\[ \langle \gamma_i, \gamma_j \rangle = 2\delta_{i,j} + 2n, \quad \langle \gamma_i, \delta \rangle = 0, \quad \langle \delta, \delta \rangle = -2n \]

for every \( i, j \in \{1, \ldots, m\} \).

In fact, we have proved that

\[ (5.1) \quad L \cong \Gamma_{m,2n} + L_{-2n} \cong A_{1,m} + L_{-2n(mn+1)}, \]

which implies that

\[ (5.2) \quad V_L \cong V_{\Gamma_{m,2n}} \otimes F_{-2n} \cong V_{A_{1,m}} \otimes F_{-2n(mn+1)}. \]

Define the following vectors in the vertex algebra \( V_L \):

\[ E = \iota(e_{\alpha_1}) + \cdots + \iota(e_{\alpha_m}); \]

\[ F = \iota(e_{-\alpha_1}) + \cdots + \iota(e_{-\alpha_m}). \]

These vectors span a subalgebra of \( V_L \) isomorphic to \( L(m,0) \).

As in Section 4 we define:

\[ X = \iota(e_{\gamma_1}) + \cdots + \iota(e_{\gamma_m}); \]

\[ Y = \iota(e_{-\gamma_1}) + \cdots + \iota(e_{-\gamma_m}). \]
Clearly $X, Y$ span a subalgebra isomorphic to $D_{m,2n}$. In fact, the definition of elements $E, F, X, Y$ together with relations (5.1) and (5.2) imply the following lemma.

**Lemma 5.1.**

1. Let $V$ be the subalgebra of $V_L$ generated by the vectors 
   $$E, F, \iota(e_{\beta}), \iota(e_{-\beta}).$$
   Then $V \cong L(m,0) \otimes F_{-2n(mn+1)}$.

2. Let $W$ be the subalgebra of $V_L$ generated by the vectors 
   $$X, Y, \iota(e_{\delta}), \iota(e_{-\delta}).$$
   Then $W \cong D_{m,2n} \otimes F_{-2n}$.

Now using standard calculations in lattice vertex algebras one easily gets the following important lemma.

**Lemma 5.2.** In the vertex algebra $V_L$ the following relations hold:

1. $X = (E_{-2n-1}(-1)(e_{n(1+\cdots+\alpha_m)}))^1$;
2. $Y = (F_{-2n-1}(1)(e_{-n(1+\cdots+\alpha_m)}))^1$;
3. $\iota(e_{\delta}) = \iota(e_{n(1+\cdots+\alpha_m)})^1 \iota(e_{-\beta})$;
4. $\iota(e_{-\delta}) = \iota(e_{-n(1+\cdots+\alpha_m)})^1 \iota(e_{\beta})$;
5. $E = X_{-1} \iota(e_{-\delta})$;
6. $F = Y_{-1} \iota(e_{\delta})$;
7. $\iota(e_{\beta}) = \iota(e_{nm+1})^1 \iota(e_{-n(n(1+\cdots+\gamma_m))})$;
8. $\iota(e_{-\beta}) = \iota(e_{-(nm+1)})^1 \iota(e_{n(n(1+\cdots+\gamma_m))})$.

**Theorem 5.1.** The vertex subalgebras $V$ and $W$ coincide. In particular, we have the following isomorphism of vertex algebras:

$$L(m,0) \otimes F_{-2n(mn+1)} \cong D_{m,2n} \otimes F_{-2n}.$$ 

**Proof.** Using the same arguments as in the proof of Proposition 4.2 we get 

$$\iota(e_{\pm n(1+\cdots+\alpha_m)}) \in V, \quad \iota(e_{\pm n(n(1+\cdots+\gamma_m))}) \in W.$$
Then the relations (1) - (4) in Lemma 5.2 implies that \( X, Y, \iota(e_{\pm\delta}) \in V \). Thus \( W \subset V \). Similarly, the relations (5) - (8) in Lemma 5.2 gives that \( V \subset W \). Hence, \( V = W \). Then Lemma 5.1 implies that \( L(m, 0) \otimes F_{-2n(mn+1)} \cong D_{m,2n} \otimes F_{-2n} \). \( \square \)

The following proposition was essentially proved in [DLM] and [DMZ].

**Proposition 5.1.** Let \( V \) be a vertex operator (super) algebra and \( s \in \mathbb{Z} \). Then \( V \otimes F_s \) is a regular vertex superalgebra if and only if \( V \) is a regular vertex operator (super)algebra.

**Theorem 5.2.** Let \( m, m_1, \ldots, m_r \) be positive integers and let \( k, k_1, \ldots, k_r \) be positive even integers.

1. The vertex operator algebra \( D_{m,k} \) is regular; i.e. every weak \( D_{m,k} \)-module is completely reducible.
2. The vertex operator algebra \( D_{m_1,k_1} \otimes \cdots \otimes D_{m_k,k_r} \) is regular.

**Proof.** Since \( L(m, 0) \) and \( F_{-2n(mn+1)} \) are regular vertex algebras, Proposition 5.1 implies that \( L(m, 0) \otimes F_{-2n(mn+1)} \) is regular. Since

\[
L(m, 0) \otimes F_{-2n(mn+1)} \cong D_{m,2n} \otimes F_{-2n},
\]

using again Proposition 5.1 we get that \( D_{m,2n} \) is a regular vertex operator algebra. This gives (1). The proof of (2) is now standard (cf. [DLM]). \( \square \)

Since \( L(m, 0) \) has \((m+1)\) inequivalent irreducible modules, and for every \( k \in \mathbb{Z} \) \( F_k \) has \(|k|\) inequivalent irreducible modules, we get:

**Corollary 5.1.** The vertex operator algebra \( D_{m,2n} \) has exactly \((m+1)(nm+1)\) inequivalent irreducible representations.

6. **Regularity of the vertex operator superalgebra \( D_{m,k} \) for \( k \) odd**

In this section, we shall consider the case when \( k \) is an odd natural number. When \( k = 1 \), then \( D_{m,1} \) is the vertex operator superalgebra associated to the unitary vacuum representation for the \( N = 2 \) superconformal algebra. This case was studied in [A3].

First we see that the following relation between lattices holds:
\[(6.1) \quad \Gamma_{m,k} + L_{-k} \cong (A_{1,m} + L_{-2k(mk+2)}) \cup (\tilde{A}_{1,m} + \tilde{L}_{-2k(mk+2)}), \]

which implies the following isomorphism of vertex algebras:

\[(6.2) \quad V_{\Gamma_{m,k}} \otimes F_{-k} \cong (V_{A_{1,m}} \otimes F_{-2k(mk+2)}) \oplus (V_{\tilde{A}_{1,m}} \otimes MF_{-2k(mk+2)}). \]

Using (6.1), (6.2) and a completely analogous proof to that of Theorem 7.1 in [A3], we get the following result.

**Theorem 6.1.** We have the following isomorphism of vertex superalgebras:

\[ D_{m,k} \otimes F_{-k} \cong L(m,0) \otimes F_{-2k(mk+2)} \oplus L(m,m) \otimes MF_{-2k(mk+2)}. \]

In other words, the vertex superalgebra \( D_{m,k} \otimes F_{-k} \) is a simple current extension of the vertex algebra \( L(m,0) \otimes F_{-2k(mk+2)} \).

Using Proposition 5.1, Theorem 6.1 and the fact that a simple current extension of a regular vertex algebra is a regular vertex (super)algebra (cf. [Li2]) we get the following theorem.

**Theorem 6.2.** Let \( m, m_1, \ldots, m_r \) be positive integers and let \( k, k_1, \ldots, k_r \) be positive odd integers.

(1) The vertex operator superalgebra \( D_{m,k} \) is regular.

(2) The vertex operator superalgebra \( D_{m_1,k_1} \otimes \cdots \otimes D_{m_k,k_r} \) is regular.

We also have:

**Corrolary 6.1.** The vertex operator superalgebra \( D_{m,k} \) has exactly \( \frac{(m+1)(km+2)}{2} \) inequivalent irreducible representations.

**Proof.** The results from [Li2] imply that the extended vertex superalgebra

\[ L(m,0) \otimes F_{-2k(mk+2)} \oplus L(m,m) \otimes MF_{-2k(mk+2)} \]

has exactly \( \frac{1}{2}(m+1)k(km+2) \) inequivalent irreducible representations (see also [A3], [Li3]). Since the vertex superalgebra \( F_{-n} \) has \( n \) inequivalent irreducible representations, we conclude that \( D_{m,k} \) has to have \( \frac{(m+1)(km+2)}{2} \) inequivalent irreducible representations. \( \square \)
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