Abstract

We explore the diversity of warped metric function in five-dimensional gravity including a scalar field and a 3-brane. We point out that the form of the function is determined by a parameter introduced here. For a particular value of the parameter, the warped metric function is smooth without having a singularity, and we show that the bulk cosmological constant have a upper bound and must be positive and that the lower bound of five-dimensional fundamental scale is controlled by both the brane tension and four-dimensional effective Planck scale. The general warp factor obtained here may relate to models inspired by SUGRA or M-theory.
1 Introduction

Randall and Sundrum proposed the setup of 3-branes embedded in the five-dimensional theory with warped metric and discussed an alternative explanation of the hierarchy problem in the form of a fine-tuning between the negative bulk cosmological constant and brane tension \[^1\,^2\]. The warped metric function in the model is an exponential scaling of the metric along the fifth dimension compactified on \(S^1/Z_2\) orbifold. Originally, the introduction of warped metric have been made by Raba kov and Shaposhnikov who discussed the cosmological constant problem in six-dimensional theory with warped metric containing a singularity \[^3\]. Recently, the extensions of the Randall-Sundrum scenario are widely made \[^4\,^5\,^6\], in particular, several brane world scenarios with bulk scalar field are investigated \[^4\,^7\,^8\,^9\,^10\,^11\]. Moreover, it is expected that this setup may connect with the AdS/CFT correspondence or string theory \[^7\,^8\,^12\,^13\,^14\,^15\].

In this paper, we consider an alternative extension of warped metric function in the framework of five-dimensional gravity with a scalar field \(\phi\). In ref. \[^15\], it was shown that Randall-Sundrum brane-worlds arise form extremal D-brane configuration. Moreover, from a field theory point of view these brane-worlds consist of a warped geometry with a bulk scalar field, and the dual theories turn out to be non-conformal. To study the connection between RS model and SUGRA theories, it is important to investigate various types of warp factor in the setup with a bulk scalar fields. The metric taken here corresponds to the most general metric appearing to SUGRA theories. We show that the form of the warped metric is controlled by a parameter introduced here and investigate the behavior of the metric for the general value as well as particular value of the parameter.

This paper is organized as follows. In section 2, we describe the setup and derive the warped metric function and the jump conditions due to the existence of a brane. Moreover, we compute the four dimensional effective Planck scale by integrating out a fifth dimension. In section 3, a conclusion is described.

2 The Setup

The physics of this model is governed by the following action

\[
S = \int d^5x \sqrt{-G} \left\{ \frac{1}{2\kappa_5^2} \mathcal{R} - \frac{1}{2} (\nabla \phi)^2 - \Lambda \right\} + \int d^4x \sqrt{-g} \left\{ -f(\phi) \right\}, \quad \text{(1)}
\]

where \(\mathcal{R}\) and \(\Lambda\) is the curvature and the cosmological constant in the bulk, respectively. Here \(1/\kappa_5^2 = M_*^3\) where \(M_*\) is the fundamental scale of five-dimensional theory, and \(G\) is the five-dimensional metric and \(g\) is the induced four-dimensional metric on the brane which is located at \(y = 0\), where \(y\) is the coordinate of fifth
dimension. In Eq. (1), the second term denotes the brane tension with dependence of a scalar field. We take the following metric

\[ ds^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{2B(y)} dy^2 \]

\[ \equiv G_{MN} dx^M dx^N , \]  

where \( M, N = 0, \ldots, 3, 5 \) and \( g_{\mu\nu} = \text{diag}(-, +, +, +) \). We shall use the notation \( \{ x^\mu \} \) with \( \mu = 0, \ldots, 3 \) for the coordinates on the four-dimensional spacetime, and \( x^5 = y \) for the fifth coordinate on an extra dimension. Note that the appropriate change of \( y \)-coordinate can lead to the metric of the original Randall-Sundrum model, however, the original metric is not inspired by SUGRA theories. Therefore we can take most general metric Eq. (2) appearing in SUGRA theories.

Using the metric, Einstein equations are given by

\[ R_{MN} - \frac{1}{2} G_{MN} R = \kappa^2 \left[ \partial_M \phi \partial_N \phi - G_{MN} \left\{ \frac{1}{2} (\nabla \phi)^2 + \Lambda \right\} - \frac{\sqrt{-g}}{\sqrt{-G}} g_{\mu\nu} \delta^\mu_M \delta^\nu_N f(\phi) \delta(y) \right]. \]  

While, the equation of motion with respect to a scalar field \( \phi \) with \( y \)-dependence is given by

\[ \partial_M \left( \sqrt{-G} G^{MN} \partial_N \phi \right) = \sqrt{-g} \frac{\partial f}{\partial \phi} \delta(y). \]  

With the metric ansatz, these equations can be written as

\[ A'' + 2 (A')^2 - A'B' = -\frac{\kappa_5^2}{3} \left\{ \frac{1}{2} (\phi')^2 + e^{2B} \Lambda \right\} - \frac{\kappa_5^2}{3} e^B f(\phi) \delta(y), \]  

\[ (A')^2 = \frac{1}{12} \kappa_5^2 (\phi')^2 - \frac{\kappa_5^2}{6} e^{2B} \Lambda, \]  

\[ 4A'\phi' - B'\phi' + \phi'' = e^B \frac{\partial f}{\partial \phi} \delta(y), \]

where the prime represents the derivative with respect to the \( y \).

We take the simple ansatz:

\[ B = \alpha A, \]  

where \( \alpha \) is a parameter at this stage. Below, we solve the equation of motion in the bulk and study the warped metric function for arbitrary \( \alpha \). Integrating out the equation in the bulk of Eq. (7), we have

\[ \phi' = c e^{(\alpha-4)A}. \]
Hence $c$ is the integration constant and it has mass dimension $[5/2]$. Substituting the above equation into Eq.(6), the equation is expressed as

$$A' = \varepsilon \frac{\sqrt{3}}{6} \kappa_5 |c| e^{(\alpha-4)A} \sqrt{1 - \frac{2\Lambda}{c^2} e^{8A}},$$

(10)

where $\varepsilon = \pm$, and the selection of the sign $\varepsilon$ determines the branch of the square root. Note that this solution make sense when the argument of the square root in Eq.(10) is positive.

For $\alpha \neq 4, 12$, this equation can be obtained by using hypergeometric function† as follows

$$e^{(4-\alpha)A} \ _2F_1 \left( \frac{1}{2} - \frac{4-\alpha}{8}; \frac{12 - \alpha}{8}; \frac{2\Lambda}{c^2} e^{8A} \right) = \varepsilon \frac{\sqrt{3}}{6} \kappa_5 |c| (4 - \alpha) \ (y + d),$$

(11)

where $d$ is the integration constant and $e^{A(0)}$ can be normalized to be unity. Note that the above equation is solution to be consistent with Eq.(5), consequently. In the case of $\alpha = 4, 12$, we cannot use the integral representation of hypergeometric function due to vanishing of second or third argument. Later, we describe the case of $\alpha = 4$. For $4 < \alpha < 12$, since the second argument in the hypergeometric function is negative, it becomes positive by using the transformation formulas of the function. In the case of $\alpha > 12$, both the second and the third argument become negative. The simplest warped metric of $\alpha = 0$ have been already investigated in Ref. [4, 5]. Note that the value of $\alpha$ determines the form of the metric function $A(y)$. Furthermore, the solution of Eq.(11) is well-defined on one side of $y = -d$ due to $\ _2F_1 \geq 0$ ( depending on the sign of $\varepsilon(4 - \alpha)$ ) [7, 8].

The existence of a brane at $y = 0$ leads to the jump conditions with respect to the first derivative of $A$ and $\phi$. From Eqs.(3) and (7), the jump conditions are

$$A'(0+) - A'(0-) = -\frac{1}{3} \kappa_5^2 e^{\alpha A(0)} f(\phi(0)),
\phi'(0+) - \phi'(0-) = e^{\alpha A(0)} \frac{\partial f}{\partial \phi}(\phi(0)).$$

(12)

We denote the sign and the integration constants for the positive region ($y > 0$) by $\epsilon_+, c_+, d_+$ and those for the negative region ($y < 0$) by $\epsilon_-, c_-, d_-$. Imposing the fact with $A(0) = 0$, the jump conditions yield

$$\epsilon_+ \frac{1}{d_+} - \epsilon_- \frac{1}{d_-} = \frac{\partial f}{\partial \phi}(\phi(0)),$$

† The integral representation of hypergeometric function is given by

$$\ _2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 dt \ t^{b-1} (1 - t)^{c-b-1} (1 - z t)^{-a}, \text{Re } c > \text{Re } b > 0.$$
\begin{equation}
\epsilon_+ \sqrt{\frac{1}{d_+^2} - 2\Lambda} - \epsilon_- \sqrt{\frac{1}{d_-^2} - 2\Lambda} = -\frac{2}{\sqrt{3}}\kappa_5 f(\phi(0)) ,
\end{equation}

where

\begin{equation}
\frac{1}{d_{\pm}} = \frac{6}{\sqrt{3\kappa_5(4 - \alpha)}} \, _2F_1\left(\frac{1}{2}, \frac{4 - \alpha}{8}; \frac{12 - \alpha}{8}; \frac{2\Lambda}{c_{\pm}^2}\right) \frac{1}{d_{\pm}} .
\end{equation}

Furthermore, if the fifth dimension is assumed to have certain symmetry, the sign \(\epsilon_{\pm}\) and the integration constants \(c_{\pm}, d_{\pm}\) are related each other.

If this setup arises from string theory, it is natural that a scalar field \(\phi\) in the bulk is regarded as dilaton field. Namely, in the framework of perturbative string theory, \(f(\phi)\) corresponds to the tree-level dilaton coupling with dependence of exponential type \(f(\phi) \propto e^{a\phi}\). Thus, the jump conditions of Eq. (13) gives the information on the general form of dilating coupling [7]. As for this point, we discuss elsewhere.

For \(\alpha = 4\), we can directly solve the differential equation in Eq. (10), the warped function with \(Z_2\) symmetry is given by

\begin{equation}
\begin{split}
e^{8A_y(y)} = & \frac{c^2}{2\Lambda} \left\{ 1 - \tanh^2 \frac{2}{\sqrt{3}}\kappa_5 |c|(y \pm d) \right\} ,
\end{split}
\end{equation}

where plus ( minus ) corresponds to the function for \(y > 0\) ( \(y < 0\) ) region. Note that this warped metric function is smooth without having a singularity. The normalization condition \(A(0) = 0\) gives the constraint on the integration constants

\begin{equation}
\tanh^2 \frac{2}{\sqrt{3}}\kappa_5 |c|d = 1 - \frac{2\Lambda}{c^2} ,
\end{equation}

therefore, the allowed range of the bulk cosmological constant is

\begin{equation}
0 < \Lambda \leq \frac{c^2}{2} ,
\end{equation}

where we assume \(\Lambda \neq 0\). From Eq. (12), the jump condition of \(A'\) yields the relation between the bulk cosmological constant and the brane tension

\begin{equation}
V = \frac{\sqrt{3}(c^2 - 2\Lambda)}{\kappa_5} ,
\end{equation}

where \(f(\phi(0)) = V\).

It assumes that the fifth dimension is noncompact. Integrating out \(y\) in the action Eq. (1), the resulting four dimensional effective Planck scale \(M_{Pl}\) is finite

\begin{equation}
M_{Pl}^2 = \frac{1}{\kappa_5^2} \int_{-\infty}^{\infty} dy \ e^{(2 + \alpha)A} \bigg|_{\alpha = 4} = \frac{\sqrt{3}}{2\kappa_5^2 |c|} \, _2F_1\left(\frac{3}{4}, \frac{1}{2}; \frac{7}{4}; \frac{2\Lambda}{c^2}\right) .
\end{equation}
Taking account for the range of $\Lambda$ in Eq.(17), since $1 < 2F_1\left(\frac{3}{4}, \frac{1}{2}; \frac{7}{4}; z\right) \leq \frac{r(\frac{1}{2})r(\frac{7}{4})}{r(\frac{3}{4})} \sim 1.79721$ for $0 < z \leq 1$, we obtain

$$M_{\text{Pl}}^2 \lesssim \frac{M_*^{9/2}}{|c|},$$

(20)

where we used $1/\kappa_5^2 = M_*^3$. From Eqs.(18) and (21), the integration constant can be eliminated, and the bulk cosmological constant is expressed as

$$2\Lambda \lesssim \frac{M_*^9}{M_{\text{Pl}}^4} - \frac{V^2}{3M_*^3},$$

(21)

imposing the fact that $\Lambda$ is positive, we have

$$M_* > V^\frac{1}{6} M_{\text{Pl}}^\frac{1}{3}. $$

(22)

Thus, the lower bound of the five dimensional fundamental scale is controlled by both the brane tension and four dimensional effective Planck scale. For instance, if the value of brane tension $V$ is TeV scale, we then obtain

$$M_* > 10^8 \text{ GeV}. $$

(23)

Consequently, the lower bound of $M_*$ decreases as the value of the brane tension decreases. Especially, in the case of vanishing brane tension ($V = 0$), if we take $M_* \sim 1 \text{ TeV}$, the inequality Eq.(21) becomes

$$\Lambda \lesssim (1 \text{ eV })^5, $$

(24)

therefore, Eq.(18) leads that $c^2 \lesssim 2(eV)^5$.

As for the value of $\alpha = 12$, solving Eq.(10), since the exact form of the warped metric function is complicated, we are going to investigate it in next paper [16]. Furthermore, whether the effective four-dimensional Planck scale is finite depends on the value of $\alpha$ [16].

3 Conclusion

Under the conjecture that Randall-Sundrum model arises from the underlying theories (string or M-theory), we showed that warp factors has various types in the setup with a bulk scalar field. Namely, we explore the diversity of the five-dimensional warped geometry, where the warped metric function with $y$-dependence of four-dimensional spacetime metric part and the one of fifth dimension metric part are tied through a parameter $\alpha$. The metric taken here corresponds to the metric
appearing in SUGRA theories to be low energy effective theories of D-brane configurations. For arbitrary $\alpha$, the form of the warped metric function is implicitly described in terms of $y$ by using a hypergeometric function. Taking a specific value of $\alpha = 4$, the warped metric function without having a singularity can be explicitly obtained, and we pointed out that the resulting four-dimensional effective Planck scale is finite and that the lower bound of five dimensional fundamental scale is determined by the brane tension. Furthermore, there exist the allowed bound on the bulk cosmological constant. Although we concretely investigate a simple case of $\alpha = 4$ as mentioned above, it is necessary to explore the behavior of the warped metric function for various $\alpha$.\cite{16}. However, it is unknown whether the value of $\alpha$ corresponds to concrete types of D-brane configuration. As for this point, we are going to describe elsewhere in future. As preliminary study of connection between RS model and SUGRA theories, the general results obtained here are important.

References

[1] L. Randall and R. Sundrum, “A large mass hierarchy from a small extra dimension,” Phys. Rev. Lett. 83, 3370 (1999), hep-ph/9905221.

[2] L. Randall and R. Sundrum, “An alternative to compactification,” Phys. Rev. Lett. 83, 4690 (1999), hep-th/9906064.

[3] V. A. Rubakov and M. E. Shaposhnikov, “Extra Space-Time Dimensions: Towards A Solution To The Cosmological Constant Problem,” Phys. Lett. B 125, 139 (1983).

[4] H. Collins and B. Holdom, “The Randall-Sundrum scenario with an extra warped dimension,” Phys. Rev. D 64, 064003 (2001), hep-ph/0103103.

[5] H. Collins and B. Holdom, “The cosmological constant and warped extra dimensions,” Phys. Rev. D 63, 084020 (2001), hep-th/0009127.

[6] M. Ito, “Warped geometry in higher dimensions with an orbifold extra dimension,” hep-th/0105180.

[7] S. Kachru, M. Schulz and E. Silverstein, “Bounds on curved domain walls in 5d gravity,” Phys. Rev. D 62, 085003 (2000), hep-th/0002121.

[8] S. Kachru, M. Schulz and E. Silverstein, “Self-tuning flat domain walls in 5d gravity and string theory,” Phys. Rev. D 62, 045021 (2000), hep-th/0001206.

[9] C. Csaki, J. Erlich, C. Grojean and T. Hollowood, “General properties of the self-tuning domain wall approach to the cosmological constant problem,” Nucl. Phys. B 584, 359 (2000), hep-th/0004133.
[10] P. Binetruy, J. M. Cline and C. Grojean, “Dynamical instability of brane solutions with a self-tuning cosmological constant,” Phys. Lett. B 489, 403 (2000), hep-th/0007029.

[11] S. M. Carroll and L. Mersini, “Can we live in a self-tuning universe?,” hep-th/0105007.

[12] S. Nojiri, S. D. Odintsov and S. Zerbini, “Quantum (in)stability of dilatonic AdS backgrounds and holographic renormalization group with gravity,” Phys. Rev. D 62, 064006 (2000), hep-th/0001192.

[13] L. Anchordoqui, J. Edelstein, C. Nunez, S. Perez Bergliaffa, M. Schvellinger, M. Trobo and F. Zyserman, “Brane worlds, string cosmology, and AdS/CFT,” hep-th/0106127.

[14] L. Anchordoqui, C. Nunez and K. Olsen, “Quantum cosmology and AdS/CFT,” JHEP 0010, 050 (2000), hep-th/0007064.

[15] T. Gherghetta and Y. Oz, “Supergravity, non-conformal field theories and brane-worlds,” hep-th/0106255.

[16] M. Ito, “Various types of five dimensional warp factor and effective Planck scale,” hep-th/0109140, to be published in PLB.