Poset Ramsey number $R(P, Q_n)$. I.
Complete multipartite posets

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Abstract

A poset $(P', \leq_{P'})$ contains a copy of some other poset $(P, \leq_P)$ if there is an injection $f: P' \to P$ where for every $X, Y \in P$, $X \leq_P Y$ if and only if $f(X) \leq_{P'} f(Y)$. For any posets $P$ and $Q$, the poset Ramsey number $R(P, Q)$ is the smallest integer $N$ such that any blue/red coloring of a Boolean lattice of dimension $N$ contains either a copy of $P$ with all elements blue or a copy of $Q$ with all elements red. We denote by $K_{t_1, \ldots, t_\ell}$ a complete $\ell$-partite poset, i.e. a poset consisting of $\ell$ pairwise disjoint sets $A_i$ of size $t_i$, $1 \leq i \leq \ell$, such that for any $i, j \in \{1, \ldots, \ell\}$ and any two $X \in A_i$ and $Y \in A_j$, $X < Y$ if and only if $i < j$. In this paper we show that $R(K_{t_1, \ldots, t_\ell}, Q_n) \leq n + \frac{(2+o(1))\log n}{\log \log n}$.

1 Introduction

Ramsey theory is a field of combinatorics that asks whether in any coloring of the elements in a discrete host structure we find a particular monochromatic substructure. This question offers a lot of variations depending on the chosen sub- and host structure. While originating from a result of Ramsey [7] on uniform hypergraphs from 1930, the most well-known setting considers monochromatic subgraphs in edge-colorings of complete graphs. In contrast, this paper considers a Ramsey-type problem using partially ordered sets, or posets for short, as the host structure. A poset is a set $P$ which is equipped with a relation $\leq_P$ on the elements of $P$ that is transitive, reflexive, and antisymmetric. Whenever it is clear from the context we refer to such a poset $(P, \leq_P)$ just as $P$. Given a non-empty set $X$, the poset consisting of all subsets of $X$ equipped with the inclusion relation $\subseteq$ is the Boolean lattice $Q(X)$ of dimension $|X|$. We use $Q_n$ to denote a Boolean lattice with an arbitrary $n$-element ground set.

We say that a poset $P_1$ is an induced subposet of another poset $P_2$ if $P_1 \subseteq P_2$ and for every two $X, Y \in P_1$, $X \leq_{P_1} Y$ if and only if $X \leq_{P_2} Y$.

A copy of $P_1$ in $P_2$ is an induced subposet $P'$ of $P_2$ which is isomorphic to $P_1$.

Here we consider color assignments of the elements of a poset $P$ using the colors blue and red, i.e. mappings $c: P \to \{\text{blue}, \text{red}\}$, which we refer to as a blue/red coloring of $P$. A poset is colored monochromatically if all its elements have the same color. If a poset is colored monochromatically in blue [red], we say that it is a blue [red] poset. The elements of a poset $P$ are usually referred to as vertices.

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Axenovich and Walzer \[1\] were the first to consider the following Ramsey variant on posets. For posets $P$ and $Q$, the poset Ramsey number of $P$ versus $Q$ is given by

$$R(P,Q) = \min\{N \in \mathbb{N} : \text{ every blue/red coloring of } Q_N \text{ contains either}
\text{ a blue copy of } P \text{ or a red copy of } Q\}.$$ 

As a central focus of research in this area, bounds on the poset Ramsey number $R(Q_n,Q_n)$ were considered and gradually improved with the best currently known bounds being $2n + 1 \leq R(Q_n,Q_n) \leq n^2 - n + 2$, see listed chronologically Walzer \[8\], Axenovich and Walzer \[1\], Cox and Stolee \[4\], Lu and Thompson \[6\], Bohman and Peng \[3\]. The related off-diagonal setting $R(Q_m,Q_n)$, $m < n$, also received considerable attention over the last years.

When both $m$ and $n$ are large, the best known upper bound is due to Lu and Thompson \[6\], yielding together with a trivial lower bound that $m + n \leq R(Q_m,Q_n) \leq (m - 2 + o(1))n + m$. When $m$ is fixed and $n$ is large, an exact result is only known in the trivial case $m = 1$ where $R(Q_1,Q_n) = n + 1$. For $m = 2$, after earlier estimates by Axenovich and Walzer \[1\] as well as Lu and Thompson \[6\], the best known upper bound is due to Grósz, Methuku, and Tompkins \[5\], which is complemented by a lower bound shown recently by Axenovich and the present author \[2\]:

$$n \left(1 + \frac{1}{15 \log n}\right) \leq R(Q_2,Q_n) \leq n \left(1 + \frac{2 + o(1)}{\log n}\right).$$

In this paper we generalize the upper bound of Grósz, Methuku and Tompkins \[5\] on $R(Q_2,Q_n)$ to a broader class of posets, namely we discuss the poset Ramsey number of a complete multipartite poset versus the Boolean lattice $Q_n$. A complete $\ell$-partite poset $K_{t_1,\ldots,t_\ell}$ is a poset on $\sum_{i=1}^\ell t_i$ vertices obtained as follows. Consider $\ell$ pairwise disjoint layers $A^1,\ldots,A^\ell$ of vertices, where layer $A^j$ consists of $t_j$ distinct vertices. Now for any two indexes $i,j \in \{1,\ldots,\ell\}$ and any vertices $X \in A^i, Y \in A^j$, let $X < Y$ if and only if $i < j$. Such a poset can be seen as a complete blow-up of a chain. Note that $Q_2 = K_{1,2,1}$.

![Hasse diagram of the complete 3-partite poset $K_{3,4,2}$](image)

**Theorem 1.** For $n \in \mathbb{N}$, let $\ell \in \mathbb{N}$ be an integer such that $\ell = o(\log n)$ and for $i \in \{1,\ldots,\ell\}$, let $t_i \in \mathbb{N}$ be integers with $\sup_i t_i = n^{o(1)}$. Then

$$R(K_{t_1,\ldots,t_\ell},Q_n) \leq n \left(1 + \frac{2 + o(1)}{\log n}\right)^\ell \leq n + \frac{(2 + o(1))\ell n}{\log n}.$$

Here and throughout this paper, the $O$-notation is used exclusively depending on $n$, i.e. $f(n) = o(g(n))$ if and only if $\frac{f(n)}{g(n)} \to 0$ for $n \to \infty$. For parameters as above, this theorem implies that $R(K_{t_1,\ldots,t_\ell},Q_n) = n + o(n)$. Under the precondition that $\ell$ is fixed, we even obtain a bound that is asymptotically tight in the first and second summand: We say that a complete
ℓ-partite poset \( K = K_{t_1,\ldots,t_\ell} \) is non-trivial, if it is neither a chain nor an antichain, i.e. if \( \ell \geq 2 \) and \( t_i \geq 2 \) for some \( i \in \{1,\ldots,\ell\} \). Observe that such a non-trivial \( K \) contains either a copy of \( K_{1,2} \) or \( K_{2,1} \), so Theorem 2 of \cite{2} yields \( R(K, Q_n) \geq n + \frac{n}{15 \log n} \). Thus for non-trivial \( K \),

\[
R(K, Q_n) = n + \Theta \left( \frac{n}{\log n} \right).
\]

For trivial \( K \), it is known that \( R(K, Q_n) = n + \Theta(1) \). In detail, if \( K \) is a chain on \( \ell \) vertices, then \( R(K, Q_n) = n + \ell - 1 \), where the upper bound is a consequence of Lemma 4 stated later on and the lower bound is easy to see using a layered coloring of the host lattice. If \( K \) is an antichain on \( t \) vertices, then a trivial lower bound, Lemma 3 in Axenovich and Walzer’s \cite{1}, and Sperner’s Theorem imply \( n \leq R(K, Q_n) \leq n + \alpha(t) \) where \( \alpha(t) \) is the smallest integer such that

\[
\left\lfloor \frac{\alpha(t)}{2} \right\rfloor \geq t.
\]

We shall first consider a special complete multipartite poset that we call a spindle. Given \( r \geq 0, s \geq 1 \) and \( t \geq 0 \), an \((r,s,t)\)-spindle \( S_{r,s,t} \) is defined as the complete multipartite poset

\[
K_{t'_1,\ldots,t'_r+1+1,\ldots,t'_{r+1+t}} = 1 \quad \text{and} \quad t'_{r+1+t} = s.
\]

In other words this poset on \( r+s+t \) vertices is constructed using an antichain \( A \) of size \( s \) and two chains \( C_r, C_t \) on \( r \) and \( t \) vertices, respectively, combined such that every vertex of \( A \) is larger than every vertex from \( C_r \) but smaller than every vertex from \( C_t \).

![Figure 2: Hasse diagram of the spindle \( S_{2,5,3} \)](image)

**Theorem 2.** Let \( r, s, t \) be non-negative integers with \( r + t = o(\sqrt{\log n}) \) and \( 1 \leq s = n^{o(1)} \) for \( n \in \mathbb{N} \). Then

\[
R(S_{r,s,t}, Q_n) \leq n + \frac{(1 + o(1))(r + t)n}{\log n}.
\]

The spindle \( S_{1,s,1} \) is known in the literature as an \( s \)-diamond \( D_s \), while the poset \( S_{1,s,0} \) is usually referred to as an \( s \)-fork \( V_s \).

**Corollary 3.** Let \( s \in \mathbb{N} \) with \( s = n^{o(1)} \) for \( n \in \mathbb{N} \). Then

\[
R(D_s, Q_n) \leq n + \frac{(2 + o(1))n}{\log n} \quad \text{and} \quad R(V_s, Q_n) \leq n + \frac{(1 + o(1))n}{\log n}.
\]

For a positive integer \( n \in \mathbb{N} \), we use \([n]\) to denote the set \( \{1,\ldots,n\} \), additionally let \([0] = \emptyset \). Here ‘log’ always refers to the logarithm with base 2. We omit floors and ceilings where appropriate.

The structure of the paper is as follows. First, in Section 2 we introduce some notation and two preliminary lemmas. In Section 3 we show the bound for spindles and afterwards the generalization for general complete multipartite posets.
2 Preliminaries

2.1 Red $Q_n$ versus blue chain

Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets. Then the vertices of the Boolean lattice $Q(\mathcal{X} \cup \mathcal{Y})$, i.e. the subsets of $\mathcal{X} \cup \mathcal{Y}$, can be partitioned with respect to $\mathcal{X}$ and $\mathcal{Y}$ in the following manner. Every $Z \subseteq \mathcal{X} \cup \mathcal{Y}$ has an $\mathcal{X}$-part $X_Z = Z \cap \mathcal{X}$ and a $\mathcal{Y}$-part $Y_Z = Z \cap \mathcal{Y}$. In this setting, we refer to $Z$ alternatively as the pair $(X_Z, Y_Z)$. Conversely, for all $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$, the pair $(X,Y)$ corresponds uniquely to the vertex $X \cup Y \in Q(\mathcal{X} \cup \mathcal{Y})$. One can think of such pairs as elements of the Cartesian product $2^X \times 2^Y$ which has a canonical bijection to $2^{\mathcal{X} \cup \mathcal{Y}} = Q(\mathcal{X} \cup \mathcal{Y})$. Observe that for $X_i \subseteq \mathcal{X}$, $Y_i \subseteq \mathcal{Y}$, $i \in [2]$, we have $(X_1, Y_1) \subseteq (X_2, Y_2)$ if and only if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$.

We shall need the following lemma.

**Lemma 4.** Let $\mathcal{X}$, $\mathcal{Y}$ be disjoint sets with $|\mathcal{X}| = n$ and $|\mathcal{Y}| = k$, for some $n, k \in \mathbb{N}$. Let $Q = Q(\mathcal{X} \cup \mathcal{Y})$ be a blue/red colored Boolean lattice. Fix some linear ordering $\pi = (y_1, \ldots, y_k)$ of $\mathcal{Y}$ and define $Y(0), \ldots, Y(k)$ by $Y(0) = \emptyset$ and $Y(i) = \{y_1, \ldots, y_i\}$ for $i \in [k]$. Then there exists at least one of the following in $Q$:

(a) a red copy of $Q_n$, or

(b) a blue chain of length $k + 1$ of the form $(X_0, Y(0)), \ldots, (X_k, Y(k))$ where $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k \subseteq \mathcal{X}$.

Note that a version of this lemma was used implicitly in a paper of Grósz, Methuku and Tompkins [5]. It was stated explicitly and reproved by Axenovich and the author, see Lemma 8 in [2].

2.2 Gluing two posets

By identifying vertices of two posets, they can be “glued together” creating a new poset. We will later construct complete multipartite posets by gluing spindles on top of each other using the following definition. Given a poset $P_1$ with a unique maximal vertex $Z_1$ and a poset $P_2$ disjoint from $P_1$ with a unique minimal vertex $Z_2$, let $P_1 \dot\cup P_2$ be the poset obtained by identifying $Z_1$ and $Z_2$. Formally speaking, $P_1 \dot\cup P_2$ is the poset $(P_1 \setminus \{Z_1\}) \cup (P_2 \setminus \{Z_2\}) \cup \{Z\}$ for a $Z \notin P_1 \cup P_2$ where for any two $X, Y \in P_1 \dot\cup P_2$, $X \prec_{P_1 \dot\cup P_2} Y$ if and only if one of the following five cases hold: $X, Y \in P_1$ and $X \prec_{P_1} Y$; $X, Y \in P_2$ and $X \prec_{P_2} Y$; $X \in P_1$ and $Y \in P_2$; $X \in P_1$ and $Y = Z$; or $X = Z$ and $Y \in P_2$.

![Figure 3: Creating $P_1 \dot\cup P_2$ from $P_1$ and $P_2$](image-url)
Lemma 5. Let $P_1$ be a poset with a unique maximal vertex and let $P_2$ be a poset with a unique minimal vertex. Then $R(P_1 \mapsto P_2, Q_n) \leq R(P_1, Q_{R(P_2, Q_n)}).

Proof. Let $N = R(P_1, Q_{R(P_2, Q_n)})$. Consider a blue/red colored Boolean lattice $Q$ of dimension $N$ which contains no blue copy of $P_1 \mapsto P_2$. We shall prove that there exists a red copy of $Q_n$ in this coloring. We say that a blue vertex $X$ in $Q$ is $P_1$-clear if there is no red copy of $P_1$ in $Q$ containing $X$ as its maximal vertex. Similarly, a blue vertex $X$ in $Q$ is $P_2$-clear if there is no blue copy of $P_2$ in $Q$ with minimal vertex $X$. Observe that every blue vertex is $P_1$-clear or $P_2$-clear (or both), since there is no blue copy of $P_1 \mapsto P_2$.

We introduce an auxiliary coloring of $Q$ using colors green and yellow. Color all blue vertices which are $P_1$-clear in green and all other vertices in yellow. Then this coloring does not contain a monochromatic green copy of $P_1$, since otherwise the maximal vertex of such a copy is not $P_1$-clear. Recall that $N = R(P_1, Q_{R(P_2, Q_n)})$, thus $Q$ contains a monochromatic yellow copy of $Q_{R(P_2, Q_n)}$, which we refer to as $Q'$.

Consider the original blue/red coloring of $Q'$. Every blue vertex of $Q'$ is yellow in the auxiliary coloring, i.e. not $P_1$-clear. Thus every blue vertex of $Q'$ is $P_2$-clear. This coloring of $Q'$ does not contain a blue copy of $P_2$, since otherwise the minimal vertex of such a copy is not $P_2$-clear. Note that the Boolean lattice $Q'$ has dimension $R(P_2, Q_n)$, thus there exists a monochromatic red copy of $Q_n$ in $Q'$, hence also in $Q$.

Corollary 6. Let $P_1$ be a poset with a unique maximal vertex and let $P_2$ be a poset with a unique minimal vertex. Suppose that there are functions $f_1, f_2 : \mathbb{N} \to \mathbb{R}$ with $R(P_1, Q_n) \leq f_1(n)n$ and $R(P_2, Q_n) \leq f_2(n)n$ for any $n \in \mathbb{N}$ and such that $f_1$ is monotonically non-increasing. Then for every $n \in \mathbb{N},

\[ R(P_1 \mapsto P_2, Q_n) \leq f_1(n)f_2(n). \]

Proof. For an arbitrary $n \in \mathbb{N}$, let $n' = f_2(n)n$. Note that for any poset $P$, $R(P, Q_n) \geq n$, so $n' \geq n$. Hence $f_1(n') \leq f_1(n)$, and Lemma 5 provides

\[ R(P_1 \mapsto P_2, Q_n) \leq R(P_1, Q_{n'}) \leq f_1(n')n' \leq f_1(n)f_2(n)n \]

3 Proofs of Theorem 2 and Theorem 1

Proof of Theorem 2

Let $\epsilon = \frac{\log s}{\log n}$, so $s = n^\epsilon$ and $\epsilon = o(1)$. We can suppose that $n$ is large and hence $\epsilon < 1$. Then let $c = \frac{r+2s}{2(\log n)}$ where $\delta = \frac{2(r+1)}{\log n}$. Since $r + t = o(\sqrt{\log n})$, $\delta = o(1)$. Let $k = \frac{cn}{\log n}$. We show for sufficiently large $n$ that $R(S_{r,s,t}, Q_n) \leq n + k$.

If $s = 1$, $S_{r,s,t}$ is a chain and $R(S_{r,s,t}, Q_n) \leq n + r + s \leq n + k$ by Lemma 4, so suppose $s \geq 2$.

Claim: For sufficiently large $n$, $k! > 2^{(r+t)(n+k)} \cdot (s-1)^{k+1}$.

Note that $k! > \left(\frac{k}{e}\right)^k = 2^{k(k - \log 2)}$ and $(s - 1)^{k+1} = 2^{k+1}(s-1)^{k+1}$. Thus we shall prove that $k(k - \log 2) > (r + t + \log(s - 1))k + \log(s - 1) + (r + t)n$. Using that $k = \frac{cn}{\log n}$ and
\( s - 1 \leq n^\epsilon \), we obtain
\[
\begin{align*}
k \cdot \left( \log k - \log (s - 1) \right) - k \cdot \left( r + t + \log \epsilon \right) - \log (s - 1) - (r + t)n \\
\geq \frac{cn}{\log n} \left( \log c + \log n - \log \log n - \epsilon \log n \right) - \frac{cn}{\log n} \left( r + t + \log \epsilon \right) - \epsilon \log n - (r + t)n \\
\geq cn(1 - \epsilon) - (r + t)n - \frac{cn}{\log n} \left( \log \log n + r + t + \log \epsilon \right) - \epsilon \log n \\
> \delta n - \frac{2(r + 1)n}{\log n} \left( \log \log n + r + t \right) = 0,
\end{align*}
\]
where the last inequality holds for sufficiently large \( n \).

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be disjoint sets with \( |\mathcal{X}| = n \) and \( |\mathcal{Y}| = k \). We consider a blue/red coloring of \( Q = Q(\mathcal{X} \cup \mathcal{Y}) \) with no red copy of \( Q_n \). We shall show that there is a monochromatic blue copy of \( S_{r,s,t} \) in \( Q \). For every linear ordering \( \pi = (y_1^r, \ldots, y_k^r) \) of \( \mathcal{Y} \), Lemma 4 provides a blue chain \( C^\pi \) of the form \( Z_0^\pi = (X_0^\pi, \emptyset), Z_1^\pi = (X_1^\pi, \{y_1^r\}), \ldots, Z_k^\pi = (X_k^\pi, \mathcal{Y}) \), where \( X_k^\pi \subseteq \mathcal{X} \).

For every ordering \( \pi \) of \( \mathcal{Y} \), we consider the \( r \) smallest vertices \( Z_0^\pi, \ldots, Z_{r-1}^\pi \) and the \( t \) largest vertices \( Z_{k-t+1}^\pi, \ldots, Z_k^\pi \) of its corresponding chain \( C^\pi \), so let \( I = \{0, \ldots, r-1\} \cup \{k-t+1, \ldots, k\} \). Each \( Z_i^\pi \) is a vertex of \( Q \), so one of the \( 2^{r+k} \) distinct combinations of the \( Z_i^\pi \), \( i \in I \). Recall that \( k! > 2^{(r+t)(n+k)} \cdot (s-1)^{k+1} \). By pigeonhole principle, we find a collection \( \pi_1, \ldots, \pi_m \) of \( m = (s-1)^{k+1} + 1 \) distinct linear orderings of \( \mathcal{Y} \) such that for all \( j \in [m] \) and \( i \in I \), \( Z_i^{\pi_j} = Z_i \) for some \( Z_i \subseteq \mathcal{X} \cup \mathcal{Y} \) independent of \( j \). In other words, we find many chains with same \( r \) smallest vertices \( Z_i, i \in \{0, \ldots, r-1\} \), and same \( t \) largest vertices \( Z_i, i \in \{k-t+1, \ldots, k\} \). Let \( \mathcal{P} \) be the poset induced in \( Q \) by the chains \( C^{\pi_j}, j \in [m] \).

If there is an antichain \( A \) of size \( s \) in \( \mathcal{P} \), then none of the vertices \( Z_i, i \in A \), because they are contained in every chain \( C^\pi \) and therefore comparable to all other vertices in \( \mathcal{P} \). Now \( A \) together with the vertices \( Z_i, i \in A \), form a copy of \( S_{r,s,t} \) in \( \mathcal{P} \). Recall that all vertices in every \( C^\pi \) are blue, i.e. \( \mathcal{P} \) is monochromatic blue. Thus we obtain a blue copy of \( S_{r,s,t} \), which we are done. From now on, suppose that there is no antichain of size \( s \) in \( \mathcal{P} \).

By Dilworth’s Theorem we obtain \( s - 1 \) chains \( C_1, \ldots, C_{s-1} \) which cover all vertices of \( \mathcal{P} \), i.e. all vertices of the \( C^{\pi_j} \)’s. Note that the chains \( C_i \) might consist of significantly more vertices than the \( (k + 1) \)-element chains \( C^{\pi_j} \).

Now we consider the restriction to \( \mathcal{Y} \) of each vertex in \( \mathcal{P} \), i.e. the sets \( Z_i^\pi \cap \mathcal{Y} \), in order to apply the pigeonhole principle once again. Assume for a contradiction that for some \( i \in [s-1] \) there are \( Z, Z' \in C_i \) with \( Z \cap \mathcal{Y} = |Z \cap \mathcal{Y}| = |Z' \cap \mathcal{Y}| \) but \( Z \cap \mathcal{Y} \neq Z' \cap \mathcal{Y} \). This implies that \( Z \cap \mathcal{Y} \subseteq Z' \cap \mathcal{Y} \) and \( Z \cap \mathcal{Y} \supseteq Z' \cap \mathcal{Y} \), so \( Z \) and \( Z' \) are incomparable, a contradiction as they are both contained in the chain \( C_i \). Consequently, there is only at most one \( \ell \)-element set \( Y^\ell \subseteq Y^s \), \( \ell \in \{0, \ldots, k\} \), for which there exists a \( Z \in C_i \) with \( Z \cap \mathcal{Y} = Y^\ell \).

Note that for all \( j \in [m] \) and for all \( \ell \in \{0, \ldots, k\} \), \( |Z_i^{\pi_j} \cap \mathcal{Y}| = \ell \), i.e. \( Z_i^{\pi_j} \cap \mathcal{Y} = Y^\ell \) for some \( i \in [s-1] \). In other words, for fixed \( j \), each of the \( k + 1 \) sets \( Z_i^{\pi_j} \cap \mathcal{Y}, i \in \{0, \ldots, k\} \), is equal to one of at most \( s - 1 \) \( Y^\ell \)’s. Recall that we have chosen \( m = (s-1)^{k+1} + 1 \) distinct linear orderings \( \pi_j \) of \( \mathcal{Y} \). Using pigeonhole principle we find two indexes \( j_1, j_2 \) such that \( Z_i^{\pi_j_1} \cap \mathcal{Y} = Z_i^{\pi_j_2} \cap \mathcal{Y} \) for all \( \ell \in \{0, \ldots, k\} \). This implies that \( y_i^{\pi_j_1} = y_i^{\pi_j_2} \), i.e. \( \pi_{j_1} \) and \( \pi_{j_2} \) are equal. But this is a contradiction to the fact that all orderings \( \pi_j \) are distinct.

\( \square \)
Now we extend Theorem 2 to general complete multipartite posets using Corollary 6.

Proof of Theorem 2. Let \( t = \sup_i t_i \). Then Theorem 2 shows the existence of a function \( \epsilon(n) = o(1) \) with \( R(K_{1,t,1}, Q_n) \leq n \left( 1 + \frac{2 + \epsilon(n)}{\log n} \right) \). We can suppose that \( \epsilon \) is monotonically non-increasing by replacing \( \epsilon(n) \) with \( \max_{N > n} \{ \epsilon(N), 0 \} \) where necessary. In order to prove the theorem, we show a stronger statement using the auxiliary \((2\ell + 1)\)-partite poset \( P = K_{1,t,1,t,...,1,t,1} \). Note that \( K_{t,1,t,...,t} \) is an induced subposet of \( P \), thus \( R(K_{t,1,t,...,t}, Q_n) \leq R(P, Q_n) \). In the following we verify that

\[
R(P, Q_n) \leq n \left( 1 + \frac{2 + \epsilon(n)}{\log n} \right) ^\ell.
\]

We use induction on \( \ell \). If \( \ell = 1 \), then \( P = K_{1,t,1} \), so \( R(P, Q_n) \leq n \left( 1 + \frac{2 + \epsilon(n)}{\log n} \right) \). If \( \ell \geq 2 \), we “deconstruct” the poset into two parts. Consider \( P_1 = K_{1,t,1} \) and the complete \((2\ell - 1)\)-partite poset \( P_2 = K_{1,t,1,t,...,1,t,1} \). Then \( P_1 \) has a unique maximal vertex and \( P_2 \) has a unique minimal vertex. Observe that \( P_1 \uplus P_2 = P \). Using the induction hypothesis

\[
R(P_1, Q_n) \leq n \left( 1 + \frac{2 + \epsilon(n)}{\log n} \right) \text{ and } R(P_2, Q_n) \leq n \left( 1 + \frac{2 + \epsilon(n)}{\log n} \right) ^{\ell-1}.
\]

Now Corollary 6 provides the required bound. \( \square \)

4 Conclusive remarks

In this paper we considered \( R(K, Q_n) \) where \( K \) is a complete multipartite poset. Although the presented bounds hold if the parameters of \( K \) depend on \( n \), the original motivation for these results concerned the case where \( K \) is fixed, i.e. independent from \( n \):

After \( R(Q_2, Q_n) \) was bounded asymptotically sharply by Grósz, Methuku and Tompkins [5] and Axenovich and the present author [2], the examination of \( R(Q_3, Q_n) \) is an obvious follow-up question. The best known upper bound is due to Lu and Thompson [6], while the best known lower bound can be deduced from a bound on \( R(K_{1,2}, Q_n) \) in [2],

\[
n + \frac{n}{15 \log n} \leq R(K_{1,2}, Q_n) \leq R(Q_3, Q_n) \leq 37 \frac{n}{16} + 39 \frac{n}{16}.
\]

In order to find better bounds and answer the question whether or not \( R(Q_3, Q_n) = n + o(n) \), the consideration of \( R(P, Q_n) \) for small posets \( P \) might prove helpful as building blocks for Boolean lattices. For example, \( Q_3 \) can be partitioned into a copy of \( K_{1,3} \) and a copy of \( K_{3,1} \) which interact in a proper way. Both of these posets are complete 2-partite posets with, as shown here, Ramsey numbers bounded by

\[
R(K_{1,3}, Q_n) = R(K_{3,1}, Q_n) = n + \Theta \left( \frac{n}{\log n} \right).
\]

However, it remains open how to use our estimate to tighten the bounds on \( R(Q_3, Q_n) \).

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