Constraints on proton structure from precision atomic physics measurements

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Ground-state hyperfine splittings in hydrogen and muonium are very well measured. Their difference, after correcting for magnetic moment and reduced mass effects, is due solely to proton structure—the large QED contributions for a pointlike nucleus essentially cancel. The rescaled hyperfine difference depends on the Zemach radius, a fundamental measure of the proton, computed as an integral over a product of electric and magnetic proton form factors. The determination of the Zemach radius, \((1.043 \pm 0.016)\) fm, from atomic physics tightly constrains fits to accelerator measurements of proton form factors. Conversely, we can use muonium data to extract an “experimental” value for QED corrections to hydrogenic hyperfine data; we find that measurement and theory are consistent.

Introduction. Quantum Electrodynamics, QED, stands out as the most precisely tested component of the Standard Model. QED predictions for the classic Lamb shift, and hyperfine splittings (hfs) in hydrogen, positronium, and muonium have been confirmed to better than 10 parts per million (ppm) \([1, 2]\), 2 ppm \([3, 4]\), and 1 part in 10 million \([1]\), respectively. The measurements of the electron and positron g-2, magnetic ratios agree with order-\(\alpha^4\) perturbative QED predictions to 1 part in 10\(^{11}\) \([4]\). QED and gauge theory have thus been validated to extraordinary precision.

The polarization transfer results are at variance with the published Rosenbluth measurements of \(G_E\). The difference may well be due to corrections from hard two-photon exchange \([16, 17]\). One wants to examine with the maximum possible precision whether the new determinations of \(G_E(Q^2)\), falling with respect to \(G_M(Q^2)\), is compatible with other information on the form factor. The extraction of the Zemach radius to be described here provides such a constraint.

A sum rule for proton structure. We now show how one can use the hfs of the muonium atom \((e^-\mu^+)\) to expose the hadronic structure contributions to the hydrogen hfs. For an electron bound to a positively charged particle of mass \(m_N\), magnetic moment \(\mu_N = (g_N/2)\langle e/2m_N\rangle\), and Landé g-factor \(g_N\), the leading term in the hfs is the Fermi energy,

\[
E_F^N = \frac{8}{3\pi} \alpha^3 \mu_B \mu_N \frac{m_e^3 m_N^3}{(m_N + m_e)^3},
\]

where \(\mu_N\) represents the proton magnetic moment, \(\mu_N\) is used for the proton or muon, but only the lowest order term, the Bohr magneton \(\mu_B\), is inserted for the \(e^-\).

The ground-state hydrogen hfs can be written as

\[
E_{\text{hfs}}(e^- p) = (1 + \Delta_{\text{QED}} + \Delta_R + \Delta_S + \Delta_{\text{hvp}} + \Delta_{\mu\mu p} + \Delta_{\text{weak}})E_F^p,
\]

where \(\Delta_{\text{QED}}\) represents QED corrections, \(\Delta_R\) represents recoil effects, including finite-size recoil corrections, \(\Delta_S\) represents the proton structure contributions, and \(\Delta_{\text{hvp}}, \Delta_{\mu\mu p}\), and \(\Delta_{\text{weak}}\) represent the effects of hadronic vacuum polarization, muonic vacuum polarization, and weak interactions, respectively. The corresponding quantity for muonium is simply

\[
E_{\text{hfs}}(e^- \mu^+) = (1 + \Delta_{\text{QED}} + \Delta_R + \Delta_{\text{hvp}} + \Delta_{\text{weak}})E_F^\mu.
\]

We define the fractional difference between the hydrogen and rescaled muonium hfs as

\[
\Delta_{\text{hfs}} = \frac{E_{\text{hfs}}(e^- p)}{E_{\text{hfs}}(e^- \mu^+)} \frac{\mu_p}{\mu_e} \frac{1 + m_e/m_p}{1 + m_e/m_\mu} - 1
\]

The large contributions from QED corrections cancel in \(\Delta_{\text{hfs}}\). Since the hfs of hydrogen and muonium, as well as the ratio of
muon and proton magnetic moments, have been measured to better than 30 ppb, $\Delta_{\text{hfs}}$ can be determined to high precision from experiment.

From Eqs. (2) and (3), we have

$$\frac{E_{\text{hfs}}(e^{-p})}{E_{\text{hfs}}(e^{-\mu^+})} = \left(\frac{1 + \Delta_{\text{QED}} + \Delta_R + \Delta_S + \Delta_{\text{hfs}}^{\text{pol}} + \Delta_{\text{hfs}}^{\text{weak}}}{1 + \Delta_{\text{QED}} + \Delta_R + \Delta_S + \Delta_{\text{hfs}}^{\text{pol}} + \Delta_{\text{hfs}}^{\text{weak}}}\right).$$

Thus we can obtain a result for the proton structure contribution in terms of quantities measurable to high precision in atomic physics:

$$\Delta_S = \Delta_{\text{hfs}} + \Delta_R^{\text{pol}} + \Delta_{\text{hfs}}^{\text{weak}} - \left(\Delta_R^{\text{pol}} + \Delta_{\text{hfs}}^{\text{pol}} - \Delta_{\text{hfs}}^{\text{weak}}\right) + \Delta_{\text{hfs}} \left(\frac{\Delta_{\text{QED}} + \Delta_R + \Delta_{\text{hfs}}^{\text{pol}} + \Delta_{\text{hfs}}^{\text{weak}}}{1 + \Delta_{\text{QED}} + \Delta_R + \Delta_S + \Delta_{\text{hfs}}^{\text{pol}} + \Delta_{\text{hfs}}^{\text{weak}}}\right).$$

The cross terms are smaller than the uncertainties in the leading terms, and here $\Delta_{\text{QED}}$ can be approximated as $\alpha/2\pi$.

The proton structure contributions consist of the classic Zemach term computed from a convolution of elastic form factors and the polarization contribution from the inelastic hadronic states contributing to the spin-dependent virtual Compton scattering: $\Delta_S = \Delta_R + \Delta_{\text{pol}}$. In addition, as we discuss below, the relativistic recoil corrections of order $\alpha m_{\text{e}}/m_p$ are modified by the finite size of the proton. The Zemach term takes into account the finite-size correction to the proton magnetic interactions as well as the finite-size distortions of the electron’s orbit in the hydrogen atom [1, 2]: $\Delta_R = -2\alpha m_{\text{e}}(e^+Z)(1 + \delta_{Z,\text{rad}})$, where $\langle e^+Z\rangle$ is the radius of the proton as calculated from the Zemach integral

$$\langle e^+Z\rangle = \frac{4}{\pi} \int_0^\infty \frac{dQ}{Q^2} \left[G_E(Q^2)G_{\text{M}}(Q^2) - 1\right].$$

with $G_E$ and $G_M$ the electric and magnetic form factors of the proton, normalized with $G_E(0) = G_M(0)/(1 + \kappa_p) = 1$, and $\kappa_p = g_p/2 - 1$. Additionally, $\delta_{Z,\text{rad}}$ is a radiative correction to the Zemach term estimated in [11].

It has been calculated analytically in [13] for the case where the form factors are represented by dipole forms: $\delta_{Z,\text{rad}} = (\alpha/3\pi) [2\ln(\Lambda^2/m_p^2) - 4111/420]$. With $\Lambda^2 = 0.71$ GeV$^2$, this yields $\delta_{Z,\text{rad}} = 0.0153$.

The main part of the inelastic contribution can be constructed from the work of Iddings [2] and Drell and Sullivan [8]. Compact expressions are given by De Rafael [9], Gnädig and Kuti [10], and Faustov and Martynenko [11] in terms of the Pauli form factor $F_2$ and spin-dependent structure functions $g_1$ and $g_2$ of the proton.

**Evaluation of the constraint.** We will consider each term on the right hand side of Eq. (6). To compute $\Delta_{\text{hfs}}$ from [4], we use the measured hydrogen hfs of [19]; $E_{\text{hfs}}(e^-p) = 1.420.405.751.766.7(9)$ MHz and muonium hfs of [20]; $E_{\text{hfs}}(e^-\mu^+) = 4.463.302.765(53)$ MHz. The measured masses are $m_p = 938.272.029(80)$ MeV, $m_\mu = 105.658.369(9)$ MeV, and $m_e = 0.510.998.918(44)$ MeV. The ratio of magnetic moments has been measured to high precision, $\pm 0.028$ ppm; the value obtained without input from the muonium hfs is $\mu_p/\mu_p = 3.183.345.20(20)$. From these values we find $\Delta_{\text{hfs}} = 145.51(4)$ ppm.

The recoil corrections $\Delta_R$ are separated into relativistic corrections $\Delta_{\text{rel}}$ and additional radiative corrections $\Delta_{\text{rad}}$.

The order-$\alpha$ relativistic recoil correction $\Delta_{\text{rel}}$ has been computed by Baur and Yennie [11]; Bodwin and Yennie [11] have computed the corrections to second order in $\alpha$ in their Eq. (1.10), which is analogous to Eq. (8) below. Expressions for the radiative correction $\Delta_R^{\text{rad}}$ are given in [24] and [25]. With use of [21] $\alpha^{-1} = 137.035.999.11(46)$ and [26] $\kappa_\mu = 0.001.165.920.8(6)$, the total correction is evaluated to be $\Delta_R^{\text{rel}} = -178.34$ ppm.

Bodwin and Yennie [11] have also computed the corrections to their formula in the hydrogen case due to the finite size of the proton from elastic intermediate states. Note that these are finite-size corrections to the recoil correction and are distinct from the Zemach correction. A mark of the distinction is that after scaling out the lowest order Fermi hfs, the recoil corrections go to zero as $(m_p/m_e) \to \infty$, whereas the Zemach correction does not. The Bodwin–Yennie pointlike result to order $\alpha^2$ is [11]

$$\Delta_{\text{rel}}^{\text{rel}} = \frac{\alpha}{\pi} \frac{m_e}{m_p} \frac{m_p}{m_e} \left[-3 + 3\kappa_p - \frac{9}{4} \kappa_p^2\right] \ln \frac{m_p}{m_e}$$

$$+ \alpha^2 \frac{m_e}{m_p} \left[2 \ln \frac{1}{2\alpha} - 6 \ln 2 + \frac{65}{18}\right]$$

$$+ \kappa_p \left[\frac{7}{4} \ln \frac{1}{2\alpha} - \ln 2 + \frac{31}{36}\right]$$

$$+ \frac{\kappa_p}{1 + \kappa_p} \left[\frac{7}{4} \ln \frac{1}{2\alpha} + 4 \ln 2 - \frac{31}{8}\right],$$

with [21] $\kappa_p = 1.792.847.351(28)$. This gives $\Delta_{\text{rel}}^{\text{rel}} = (-2.01 + 0.46)$ ppm, where the two terms are from $O(\alpha)$ and $O(\alpha^2)$. Quoting [11], finite-size corrections change this to $\Delta_{\text{rel}}^{\text{rel}} = (+5.22(1) + 0.46)$ ppm $= 5.68(1)$ ppm, where the quoted error is an estimate using the dipole form factor for the proton (both $G_E$ and $G_M$) with mass parameter $\Lambda^2 = 0.71 \pm 0.02$ GeV$^2$. An additional radiative correction [13] of 0.09 ppm brings $\Delta_{\text{rel}}^{\text{rel}}$ to 5.77 ppm.

Volutka et al. [27] have reevaluated the finite-size corrections to the proton recoil corrections with the same magnetic radius, but with a charge radius taken from Ref. [28], and find $\Delta_{\text{rel}}^{\text{rel}} = 5.86$ ppm, or 0.18 ppm larger than Bodwin and Yennie. By forcing the magnetic form factor to reproduce their result for the Zemach integral, Volutka et al. obtain a second value of 6.01 ppm. We shall use the first Volutka result and include an uncertainty of 0.15 ppm to cover the difference between the modified Bodwin–Yennie and the second Volutka determinations. Note that structure-dependence uncertainty within the recoil corrections is still well under the uncertainty of the polarization terms, and that this uncertainty in the recoil term
can be reduced as knowledge of the form factors improves.

Estimates of the weak and vacuum polarization corrections are also given by Volotka et al. [22]. From these and from the individual values for $\Delta_{\text{hfs}}, \Delta^V_{\pi, r}$, and $\Delta^p_{\pi}$, we obtain $\Delta_S = -38.62(16)$ ppm. Thus the contribution of proton structure is constrained by atomic physics with an uncertainty well under one percent.

The Zemach term. We shall subtract the polarization contributions to isolate the Zemach term and then explore its relevance to new form factor parameterizations. Although the polarization contributions have been long known to be small [3, 10], the error in $\Delta_Z$ is essentially all due to the uncertainty in $\Delta_{\text{pol}}$. From Faustov and Martynenko [3], we take $\Delta_{\text{pol}} = 1.4$ ppm $\pm$ 0.6 ppm, which implies $\Delta_Z = -(40.0 \pm 0.6)$ ppm and thus $(\langle r \rangle)_Z = (1.043 \pm 0.016)$ fm. The unit conversion used $\hbar c = 197.326$ 968(17) MeV-fm.

Predictions for $\Delta_Z$ and $(\langle r \rangle)_Z$ as computed from a selection of parameterizations of the form factors are given in Table I. The first row is the textbook standard, wherein both $G_M$ and $G_E$ are given by the dipole form. The result, $\Delta_Z = -39.32$ ppm $\pm$ 38.72(1 $+ \sigma^M_{2nd}$) ppm, can already be found in [11]. New analytic fits to the form factors [24, 30] make a significant change in the Zemach integral, of up to 6%. The form factor parameterization given in [28] yields $\langle r \rangle_Z = 1.086(12)$ fm. It is not clear why the large difference exists. The scattering data is subject to radiative and other corrections; any difference highlights the usefulness of having the precise value that we have derived. Not all of the $\Delta_Z$ or $(\langle r \rangle)_Z$ for the different models in the table are compatible with the results extracted from the analysis of the atomic data. However, the $G_M-G_E$ combination suggested in the third row from the end of the Table shows that fully compatible models exist.

The table also shows results for the charge radius $\sqrt{\langle r^2 \rangle} = \sqrt{-6 \frac{\alpha Z}{4\pi} G_E(Q^2)|Q^2=0}$.

**TABLE I: Proton electric charge radius $\sqrt{\langle r^2 \rangle}$, Zemach contribution $\Delta_Z$ to the hfs, and Zemach radius $\langle r \rangle_z$ for various parameterizations of $G_E$ and $G_M$. The results should be compared to $\Delta_Z = -(40.0 \pm 0.6)$ ppm or $\langle r \rangle_Z = (1.043 \pm 0.016)$ fm, as obtained from analysis of atomic hfs data. The dipole form is $G_M(Q^2) = (1 + \kappa_p)/(1 + Q^2/0.71 \text{GeV}^2)^2$. The $G_E$ labeled JLab is [13] $\left(1 - 0.13 \frac{G_E}{G_M} \right)\frac{1}{1 + \kappa_p}$. Parameterizations A-I and A-II are from [29]. Those labeled Brodsky-Carlson-Hiller-Hwang (BCHH), I and II, use $F_2/F_1 = [1/\kappa^2_p + Q^2/(1.25 \text{GeV}^2)]^{-1/2}$ and $F_2/F_1 = \kappa_p [1 + (Q^2/0.79 \text{GeV}^2)^2 \ln^2(1 + Q^2/4m^2)]/[1 + (Q^2/0.38 \text{GeV}^2)^2 \ln^2(1 + Q^2/4m^2)]$, respectively [30]. The last column gives the contribution to $(\langle r \rangle)_Z$ from $Q > 0.8$ GeV.**
functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$, we obtain a precise value for the Zemach radius $\langle r \rangle_Z = (1.043 \pm 0.016) \text{ fm}$, which is defined from a convolution of the $G_E$ and $G_M$ form factors. This new determination gives an important constraint on the analytic form and fits to the proton form factors at small $Q^2$. The precision of the Zemach radius will be further improved when new, more precise data for $g_1$ and $g_2$, especially at small $\nu$ and $Q^2$, becomes available.

The proton structure terms can also be extracted using the hydrogen hfs alone [24,35]. The Zemach radius obtained this way is slightly smaller but consistent with our result.

Conversely, by combining the muonium and hydrogen hfs data, one can obtain an “experimental” value for the purely QED bound-state radiative corrections: $\Delta_{\text{QED}} = 1136.09(14) \text{ ppm}$. To minimize the uncertainty, we take advantage of the measured ratio [33] $m_p/m_e = 1836.152 \pm 0.672 \pm 0.618(85)$. This value of $\Delta_{\text{QED}}$ is consistent with the calculated QED correction used in [22,35].

Our method of combining experimental atomic physics has other applications; for example, measurements of the difference of the Lamb shifts (or Rydberg spectra) of muonium and hydrogen could potentially give a very precise value for the proton’s electric charge radius, since again the QED radiative corrections essentially cancel. Similarly, the difference of lepton anomalous moments $a_\mu - a_e$ directly exposes the hadronic and weak corrections to the muon moment.

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