On dynamics of geometrically thin accretion disks

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Abstract

Axisymmetric accretion disks in vicinity of a central compact body are studied. For the simple models such as vertically isothermal disks as well as adiabatic ones the exact solutions to the steady-state MHD (magneto-hydrodynamic) system were found under the assumption that the radial components of velocity and magnetic field are negligible. On the basis of the exact solution one may conclude that vertically isothermal disks will be totally isothermal. The exact solution for the case of adiabatic disk corroborates the view that thin disk accretion must be highly nonadiabatic. An intermediate approach, that is between the above-listed two, for the modeling of thin accretion disks is developed. In the case of non-magnetic disk, this approach enables to prove, with ease, that all solutions for the midplane circular velocity are unstable provided the disk is non-viscous. Hence, this approach enables to demonstrate that the pure hydrodynamic turbulence in accretion disks is possible. It is interesting that a turbulent magnetic disk tends to be Keplerian. This can easily be shown by assuming that the turbulent gas tends to flow with minimal losses, i.e. to have the Euler number as small as possible.

1 Introduction

We will consider the dynamics of axisymmetric accretion disk around a compact object. A successful theory of the process in question is mainly developed (see, e.g. [6], [8], [29], [27], [28]). Extensive use is made of simple models such as vertically isothermal disks as well as adiabatic ones. To estimate possible errors and limits associated to these models, the exact solutions to the steady-state MHD system were found (see Sec. 2 and Sec. 3) under the assumption that the radial components of velocity and magnetic field are negligible. We will

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also consider an intermediate case (see Sec. [4]) that is between the above-listed two axisymmetric flows. Such an approach for the modeling of thin accretion disks turns out to be more flexible and efficient. In particular, the question of pure hydrodynamic turbulence was an open question [4], [8], [21], [24] until recent years. The possibility for finite disturbances to develop turbulence in the nonlinear regime was demonstrated by O. A. Kuznetsov in [9] and in doing so he has disproved the well-known arguments that pure hydrodynamic turbulence cannot be a self-sustaining source of viscosity in accretion disks (see, e.g., [8] and references therein). The other possible origins of pure hydrodynamic turbulence have been investigated in [21], [24]. Using the approach of Sec. [4] we will also demonstrate that the pure hydrodynamic turbulence in accretion disks is possible.

The input system of MHD equations is the following (see, e.g. [10], [16], [17], [18], [20], [25], [28]):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$  

(1)

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[ \rho \mathbf{v} \mathbf{v} + \left( P + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{1}{4\pi} \mathbf{BB} \right] = \nabla \cdot \mathbf{\tau} - \rho \mathbf{\Phi},$$  

(2)

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[ \mathbf{v} \left( E + P + \frac{B^2}{8\pi} \right) - \frac{1}{4\pi} \mathbf{B} (\mathbf{v} \cdot \mathbf{B}) \right] = -\rho \mathbf{v} \cdot \nabla \Phi - \dot{Q} - \nabla \cdot \mathbf{q},$$  

(3)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),$$  

(4)

$$\nabla \cdot \mathbf{B} = 0,$$  

(5)

$$\mathbf{E} = -\frac{1}{c} (\mathbf{v} \times \mathbf{B}),$$  

(6)

where $\rho$, $\mathbf{v}$, $P$, $\mathbf{B}$, $E$, $\mathbf{\tau}$, $\mathbf{q}$, and $\Phi$ denote the density, velocity, pressure, magnetic induction field, electric field, stress tensor, heat current, and gravitational potential, respectively, $B = |\mathbf{B}|$, $v = |\mathbf{v}|$, $E = \rho e_p + 0.5 \rho v^2 + B^2/(8\pi)$ denotes the total energy per unit volume with $e_p$ being the internal energy per unit mass for the plasma, $\dot{Q}$ denotes the local cooling rate [6, p. 142]. It is, mainly, assumed that

$$\Phi = -G \frac{M}{\sqrt{r^2 + z^2}}, \quad G, M = \text{const.}$$  

(7)

The heat conductive flux, $\mathbf{q}$, may be expressed as

$$\mathbf{q} = -\lambda_T \nabla \cdot T,$$  

(8)

where $\lambda_T$ denotes the thermal conductivity, $T$ denotes the temperature. The stress tensor, $\mathbf{\tau}$, is the sum of two tensors, $\mathbf{\tau} = \mathbf{\tau}_v + \mathbf{\tau}_t$, namely, the viscous, $\mathbf{\tau}_v$, and the turbulent, $\mathbf{\tau}_t$, stress tensors:

$$\mathbf{\tau}_v \approx \mu_v \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^* \right] - \frac{2}{3} \mu_v \nabla \cdot \mathbf{v} \mathbf{I},$$  

(9)
\[ \tau_t \approx \mu_t \left[ \nabla v + (\nabla v)^\ast \right] - \frac{2}{3} (\mu_t \nabla \cdot v + \rho \kappa) \mathbf{I}, \]  

(10)

where \((\ )^\ast\) denotes a conjugate tensor, \(\mu_t\) denotes the dynamic viscosity, \(\mu_t\) and \(\kappa\) denote the turbulent viscosity and the kinetic energy of turbulence, respectively (see, e.g., [1], [9] and references therein). We will also use the viscosity \(\mu = \mu_v + \mu_t\). Obviously, if the flow is laminar, then \(\kappa = 0\) and \(\mu\) is the dynamic viscosity.

In this paper, the following three axisymmetric flows in cylindrical coordinates, \((r, \varphi, z)\), will be considered.

1) A vertically isothermal disk, where the temperature is a pre-assigned value,

\[ P = \rho RT, \quad \frac{\partial T}{\partial z} = 0, \quad R = \text{const}. \]  

(11)

2) An adiabatic disk, i.e.

\[ P = K \rho^\gamma, \quad \gamma > 1, \quad \gamma, K = \text{const}. \]  

(12)

3) An intermediate case that is between the above-listed two, in some measure opposite, axisymmetric flows. Such an approach permits us to avoid the solution of energy equation.

Let us note that the number densities of ions and electrons at any point are approximately equal, and, hence, a plasma must always be close to charge neutrality. Even a small charge imbalance would create huge electric fields which would move the plasma particles so as to restore neutrality very quickly [10], [8]. The plasma maintains charge neutrality to a high degree of accuracy. However, local charge imbalances may be produced by thermal fluctuations [10]. To estimate their size, it can be used the Debye length, \(\lambda_D\), which is the typical size of a region over which the charge imbalance may occur [9], [10], [8]:

\[ \lambda_D \approx 70 \sqrt{T / n_e} \text{ m}, \]  

where \(n_e\) denotes the number density of electrons, \(T\) denotes the temperature. The length scale, \(\lambda_s\), of plasma dynamics should be much larger than \(\lambda_D\). For example, inserting the numbers for coronal plasma, we find [10] \(\lambda_D = 0.07 \text{ m}\). Considering typical length scales of coronal loops [10], \(\lambda_s = 10000 \text{ km}\), we can see that the condition \(\lambda_s \gg \lambda_D\) is easily satisfied. We will, mainly, consider the flow at the periphery of accretion disk and, hence, the length scale of plasma dynamics will be much larger than the Debye length. Actually, the mean density of gas in the Milky Way is a million per cubic metre [6]. Then, assuming that the gas temperature is close to 100°K, we find that the Debye length will be less than 0.7 m. If, however, inside the disk there is a charge density, then it gives rise to an electric field outside the disc which is available to pull charges out of the disc [8]. Hence, in the case of at least steady-state flow, we may write (see, e.g., [6] p. 188, [28] p. 275) that

\[ \nabla \cdot \mathbf{E} = 0 \Rightarrow \nabla \cdot (\mathbf{v} \times \mathbf{B}) = 0. \]  

(13)

Let us introduce the following characteristic quantities: \(t_s, l_s, \rho_s, v_s, p_s, T_s, \mu_s, \kappa_s,\) and \(B_s\) for, respectively, time, length, density, velocity, pressure,
temperature, viscosity, kinetic energy of turbulence, and magnetic field. The following notation will also be used:

\[ S_h = \frac{l_s}{v_s l_s}, \quad E_u = \frac{p_s}{\rho_s v_s^2}, \quad \beta = \frac{4\pi p_s}{B_s^2}, \quad F_r = \frac{v_s^2 l_s}{GM}, \quad R_e = \frac{\rho_s v_s l_s}{\mu_s}, \quad \vartheta_{ke} = \frac{2\pi_s}{3v_s^2}, \quad (14) \]

where \( S_h, \) \( E_u, \) \( F_r, \) and \( R_e \) denote, respectively, Strouhal, Euler, Froude, and Reynolds numbers. For axisymmetrical flow, we have, in view of \((14),\) the following non-dimensional system of PDEs (Partial Differential Equations).

\[
S_h \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \rho v_r \right) + \frac{\partial}{\partial z} \left( \rho v_z \right) = 0, \quad (15)
\]

\[
S_h \frac{\partial v_r}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( \rho v_r^2 - \frac{E_u}{\beta} B_r^2 \right) + \frac{\partial}{\partial z} \left( \rho v_r v_z - \frac{E_u}{\beta} B_r B_z \right) + \frac{E_u B_z^2}{\beta} \frac{v_r^2}{r} = - \frac{\partial}{\partial r} \left( E_u P + \frac{E_u B_z^2}{2} \right) - \frac{\rho}{F_r} \frac{\partial \Phi}{\partial r} + \frac{1}{R_e} \left\{ \frac{\partial}{\partial r} \left[ 2\mu \frac{\partial v_r}{\partial r} - 2 \frac{\mu}{r} \left( \frac{\partial v_r}{\partial r} + \frac{v_r}{r} \right) \right] + \frac{\partial}{\partial z} \left( \mu \frac{\partial v_z}{\partial z} + 2\mu \frac{\partial v_z}{\partial r} \right) \right\}, \quad (16)
\]

\[
S_h \frac{\partial v_z}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( \rho v_r v_z - \frac{E_u}{\beta} B_r B_z \right) + \frac{\partial}{\partial z} \left( \rho v_z^2 - \frac{E_u}{\beta} B_z^2 \right) = - \frac{\partial}{\partial z} \left( E_u P + \frac{E_u B_z^2}{2} \right) - \frac{\rho}{F_r} \frac{\partial \Phi}{\partial z} + \frac{1}{R_e} \left\{ \frac{\partial}{\partial r} \left[ \mu \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \right] + \frac{\partial}{\partial z} \left[ \mu \frac{\partial v_z}{\partial z} - 2 \frac{\mu}{r} \left( \frac{\partial v_z}{\partial r} + \frac{v_z}{r} \right) \right] \right\}, \quad (17)
\]

\[
S_h \frac{\partial B_r}{\partial t} + \frac{\partial}{\partial r} \left( v_r B_r - v_z B_z \right) = 0, \quad (19)
\]

\[
S_h \frac{\partial B_z}{\partial t} + \frac{\partial}{\partial r} \left( v_r B_z - v_z B_r \right) + \frac{\partial}{\partial z} \left( v_z B_z - v_z B_z \right) = 0, \quad (20)
\]
\[ S_B \frac{\partial B_z}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (v_r B_z - v_z B_r) = 0, \]
\[ \frac{1}{r} \frac{\partial r B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0. \]

It is, mainly, assumed that
\[ \Phi = -\frac{1}{\sqrt{r^2 + z^2}}. \]

Notice, if \( \kappa = 0 \) and \( \mu = \mu_\ast \) in (16)-(18), then the flow is laminar.

We will consider, in general, accretion disks. Hence, it is assumed that the values \( \rho, v_r, v_\varphi, P, \Phi \) are even functions of \( z \), whereas \( v_z \) is an odd one. In such a case, in view of (15)-(23), there exist two possibilities: 1) \( B_z \) is an even function of \( z \), whereas \( B_\varphi \) and \( B_r \) are odd ones; 2) the values \( B_z \) and \( B_r \) are even functions of \( z \), whereas \( B_\varphi \) is an odd one. We will consider the first possibility.

If \( v_r = v_z = 0 \), then there exists the third possibility, namely, the values \( B_\varphi \) and \( B_z \) are even functions of \( z \), whereas \( B_r \) is an odd one. In such a case the magnetic field will be unstable provided \( B_\varphi \neq 0 \). Actually, if \( v_r \neq 0 \), then \( B_\varphi \) will be an odd function of \( z \), since we consider the case when \( B_z \) is an even function of \( z \). It is very important to note that the solution such that \( B_\varphi (\neq 0) \) is an even function of \( z \) can not be obtained as a limiting case (namely, as \( v_r \to 0 \)) of the motion under \( v_z \neq 0 \). Hence, any solution for \( B_\varphi (\neq 0) \) such that it is not an odd function of \( z \) may be seen as unstable, as any infinitesimal variation, \( \delta v_r (\neq 0) \), gives rise to a finite response in the magnetic field.

In the case of steady-state flow, we sometimes assume that the flow is charge-neutral, (13), i.e.
\[ \frac{\partial r (v_\varphi B_z - v_z B_\varphi)}{r \partial r} + \frac{\partial (v_r B_\varphi - v_\varphi B_r)}{\partial z} = 0. \]

2 Vertically isothermal disk

In this section we construct steady-state solutions for the system (15)-(22) provided \( R_e \to \infty \), \( \partial \kappa \varphi = 0 \), and
\[ B_r = 0, \quad v_r = 0, \quad \frac{\partial T}{\partial z} = 0. \]

Let us note that the equality \( v_r = 0 \) implies \( v_z = 0 \). It could be easily seen from the steady-state version of (15). We take \( p_* = \rho_* R T_* \) and, hence, we obtain from (11) that
\[ P = \rho T. \]

The temperature in (26) is assumed to be a preassigned function of \( r \).

In such a case, the MHD system is reduced to the following.
\[ \frac{E_u B_r^2}{B^2} \frac{\rho v_z^2}{r} = -\frac{\partial}{\partial r} \left( E_u P + \frac{E_u B_\varphi^2}{\beta - 2} \right) - \rho \frac{\partial \Phi}{F_r \partial r}. \]
\[
\frac{\partial (B_\varphi B_z)}{\partial z} = 0, \quad (28)
\]
\[
- \frac{\partial}{\partial z} \left( \frac{E_\mu}{\beta} B_z^2 \right) = - \frac{\partial}{\partial z} \left( E_\mu P + \frac{E_\mu B^2}{\beta} \right) - \frac{\rho}{F_r} \frac{\partial \Phi}{\partial z}, \quad (29)
\]
\[
\frac{\partial (v_\varphi B_z)}{\partial z} = 0, \quad (30)
\]
\[
\frac{\partial B_z}{\partial z} = 0. \quad (31)
\]

In view of (31), (30), and (28), we obtain:

\[
B_z = B_z (r), \quad v_\varphi = v_\varphi (r), \quad B_\varphi = B_\varphi (r). \quad (32)
\]

Then we obtain, instead of (27)–(31):

\[
\frac{E_\mu B_\varphi^2}{\beta} = - \frac{\partial}{\partial r} \left( E_\mu P + \frac{E_\mu B^2 + B_\varphi^2}{2} \right) - \frac{\rho}{F_r} \frac{\partial \Phi}{\partial r}, \quad (33)
\]

\[
0 = - \frac{\partial}{\partial z} (E_\mu P) - \frac{\rho}{F_r} \frac{\partial \Phi}{\partial z}. \quad (34)
\]

Let

\[
C_\rho = \rho |_{z=0}, \quad \phi = \Phi |_{z=0}, \quad (35)
\]

By virtue of (26), we find from (34) that

\[
\rho = C_\rho \exp \left( \frac{\phi - \Phi}{TF_r E_\mu} \right), \quad C_\rho = C_\rho (r), \quad \phi = \phi (r) \equiv \Phi |_{z=0}. \quad (36)
\]

If (23) is valid, then

\[
\rho = C_\rho \exp \left[ \frac{1}{TF_r E_\mu} \left( \frac{1}{\sqrt{r^2+z^2}} - \frac{1}{r} \right) \right], \quad C_\rho = C_\rho (r). \quad (37)
\]

Let, in general,

\[
\frac{\partial \Phi}{\partial z} \neq 0. \quad (38)
\]

Eq. (33) must be valid under all values of \( z \geq 0 \). After differentiation (33) over \( z \), in view of (34) we obtain

\[
\frac{v_\varphi^2}{r} \frac{\rho}{TE_\mu} = - \frac{\partial \rho}{\partial r} + \frac{\rho}{TF_r E_\mu} \frac{\partial \Phi}{\partial r}, \quad (39)
\]

Since, in view of (33),

\[
\frac{\partial \rho}{\partial r} = \left[ \frac{\partial C_\rho}{\partial r} + C_\rho \frac{\partial}{\partial r} \left( \frac{\phi - \Phi}{TF_r E_\mu} \right) \right] \exp \left( \frac{\phi - \Phi}{TF_r E_\mu} \right), \quad (40)
\]
we find from (39) that

\[ \frac{v_\varphi^2}{r} \frac{C_\rho}{T E_u} = \frac{\partial C_\rho}{\partial r} + \frac{C_\rho}{T T_r E_u} \frac{\partial \phi}{\partial r} - \frac{C_\rho (\phi - \Phi)}{T^2 T_r E_u} \frac{\partial T}{\partial r}. \]  

(41)

After differentiation (41) with respect to \( z \), we obtain:

\[ \frac{C_\rho}{T^2 T_r E_u} \frac{\partial \Phi}{\partial z} \frac{\partial T}{\partial r} = 0. \]  

(42)

Since \( C_\rho \neq 0 \), we obtain from (42), in view of (38), that

\[ T = \text{const}, \]  

(43)

and, hence,

\[ C_\rho = \text{const} \exp \left( \int \frac{v_\varphi^2}{r T E_u} dr - \frac{\phi}{T T_r E_u} \right). \]  

(44)

If (23) is valid, then \( \phi = -1/r \). Let the disk be Keplerian, i.e.

\[ v_\varphi = \frac{1}{\sqrt{T r}}, \]  

(45)

then

\[ C_\rho = \text{const} \exp \left( \int \frac{dr}{r^2 T T_r E_u} + \frac{1}{r T T_r E_u} \right) = \text{const}. \]  

(46)

Thus, the assumption that the motion is Keplerian leads to a constant density at the midplane. Let us consider a vortex motion, i.e.

\[ v_\varphi = \frac{C_\varphi}{r}, \quad C_\varphi = \text{const}, \]  

(47)

and let, for the sake of simplicity, \( C_\varphi = 1/\sqrt{T r} \), then, in general, we have

\[ \frac{\partial C_\rho}{\partial r} < 0. \]  

(48)

The density at the midplane, in view of (44), will be the following.

\[ C_\rho = \text{const} \exp \left( \frac{1}{r T T_r E_u} - \frac{1}{2 r^2 T T_r E_u} \right). \]  

(49)

Substituting (36), (43), and (44) into (33), we obtain the following equation in \( B_\varphi \) and \( B_z \).

\[ \frac{B_\varphi^2}{r} + \frac{\partial}{\partial r} \left( \frac{B_\varphi^2 + B_z^2}{2} \right) = 0. \]  

(50)

We can see from (32) that \( B_\varphi (\neq 0) \) is an even function of \( z \), and, hence, \( B_\varphi \) is unstable (see Sec. 1). Thus, \( B_\varphi = 0 \) is a possibly stable solution, and, by virtue of (50), we find that

\[ B_z = \text{const}. \]  

(51)
The semi-thickness, $H$, of disk is often (e.g. \cite{7}) defined as

$$H = \frac{1}{C_\rho} \int_0^\infty \rho dz, \quad C_\rho = \rho \big|_{z=0}. \quad (52)$$

Notice, using the exact solution \cite{37} in (52) we find that $H \to \infty$ provided that $TF, E_u \neq 0$. Thus, even if the value of $E_u$ be small but finite, the disk cannot be thin in terms of (52). Instead of the exact solution, (37), it can be used the following approximation (Cf. \cite{22}, \cite{28}) for small values of $z$.

$$\rho \approx C_\rho \exp \left( -\frac{z^2}{2TF, E_u r^3} \right) \equiv C_\rho \exp \left[ -\frac{1}{2} (z/H)^2 \right], \quad H = r \sqrt{T F, E_u}. \quad (53)$$

Let us note that the semi-thickness $H \propto r^{1.5}$ in (53). Analogous formulae can be found in many monographs (see, e.g., \cite{7}, \cite{22}, \cite{28} and references therein). Thus, in the case of isothermal flow we have

$$H \propto r^{1.5}. \quad (54)$$

Till now we did not use the assumption that the flow is electrically neutral. Let us now assume that (24) is valid. Then, in view of (25), we find

$$\frac{\partial r}{\partial r} (v_\varphi B_z) = 0. \quad (55)$$

Hence, in view of (51), we obtain from (55) that (47) is valid, i.e. we have the vortex flow.

### 3 Adiabatic flow

In this section we intend to find an exact steady-state solution to the MHD system (15)-(22) provided $Re \to \infty$, $\vartheta_{ke} = 0$, and

$$v_r = 0, \quad B_r = 0. \quad (56)$$

Let us remind that the equality $v_r = 0$ implies $v_z = 0$. It could be easily seen from the steady-state version of (15). We take $p_\ast = K \rho^\gamma$ and, hence, we obtain from (12) that

$$P = \rho^\gamma, \quad \gamma > 1. \quad (57)$$

Then, by analogy with Sec. \cite{2} we obtain:

$$B_z = B_z (r), \quad v_\varphi = v_\varphi (r), \quad B_\varphi = B_\varphi (r). \quad (58)$$

$$\frac{E_u B_z^2}{\beta} - \frac{\rho v_\varphi^2}{r} = -\frac{\partial}{\partial r} \left( E_u P + \frac{E_u B_z^2 + B_\varphi^2}{2} \right) - \frac{\rho}{F_r} \frac{\partial \Phi}{\partial r}, \quad (59)$$
\[ 0 = -\frac{\partial}{\partial z} (E_u P) - \frac{\rho \partial \Phi}{F_r \partial z}. \]  

(60)

It is assumed that \( P(r, z)_{z=H} = 0 \) and, hence, \( \rho(r, z)_{z=H} = 0 \), where \( 2H(r) \) denotes the height of disk. Let

\[ \Psi = \frac{\gamma}{\gamma - 1} \rho^{\gamma - 1} \Rightarrow \frac{1}{\rho} \nabla P = \nabla \Psi. \]  

(61)

By virtue of (61), we rewrite (60) to read

\[ \frac{\partial \Psi}{\partial z} + \frac{1}{F_r E_u} \frac{\partial \Phi}{\partial z} = 0. \]  

(62)

We find from (62)

\[ \Psi + \frac{\Phi}{F_r E_u} = C(r). \]  

(63)

Since \( \rho(r, z)_{z=H} = 0 \) and, hence, \( \Psi(r, z)_{z=H} = 0 \), we obtain from (63) that

\[ \Psi = \frac{1}{F_r E_u} \left( \frac{1}{\sqrt{r^2 + z^2}} - \frac{1}{\sqrt{r^2 + H^2}} \right). \]  

(64)

Hence

\[ \rho = \left[ \frac{\gamma - 1}{\gamma F_r E_u} \left( \frac{1}{\sqrt{r^2 + z^2}} - \frac{1}{\sqrt{r^2 + H^2}} \right) \right]^{1/(\gamma - 1)}, \quad |z| \leq H. \]  

(65)

It is clear from (65) that \( \rho \to 0 \) as \( H \to 0 \).

By virtue of (65) and (61), we find

\[ \frac{E_u B_z^2}{\beta r} - \frac{\rho v_z^2}{r} = -\rho E_u \frac{\partial \Psi}{\partial r} - \frac{\partial}{\partial r} \left( \frac{E_u B_z^2 + B_x^2}{\beta} \right) - \frac{\rho \partial \Phi}{F_r \partial r}. \]  

(66)

Eq. (66) must be valid under all values of \( z \geq 0 \). Since \( \rho(r, z)_{z=H} = 0 \), we obtain from (66) that

\[ \frac{B_z^2}{r} = -\frac{\partial}{\partial r} \left( \frac{B_z^2 + B_x^2}{2} \right). \]  

(67)

By virtue of (61) and (67) we obtain from (66):

\[ \frac{\nu^2}{r E_u} = \frac{\partial \Psi}{\partial r} + \frac{1}{F_r E_u} \frac{\partial \Phi}{\partial r} \equiv \frac{\partial}{\partial r} \left( \frac{\Phi}{F_r E_u} \right) \equiv -\frac{1}{F_r E_u} \frac{1}{\partial r \sqrt{r^2 + H^2}} \]  

(68)

In view of (68), we have

\[ \frac{\nu^2}{r} = -\frac{1}{F_r \partial r \sqrt{r^2 + H^2}}. \]  

(69)
Let, for instance, the motion is Keplerian, i.e.

\[ v_\phi = \frac{1}{\sqrt{rF_r}}, \quad (70) \]

then, by virtue of (69), we obtain that

\[ H^2 = 0. \quad (71) \]

Hence, in view of (65), \( \rho = 0 \).

We can see from (58) that \( B_\phi (\neq 0) \) is an even function of \( z \), and, hence, \( B_\phi \) is unstable (see Sec. 1). Thus, the possibly stable solution is: \( B_\phi = 0 \), and, by virtue of (67), we find that \( B_z = \text{const} \).

Assuming that the flow is electrically neutral [28], we find from (24) that the flow is the vortex, i.e.

\[ v_\phi = \frac{C_\phi}{r}. \quad (72) \]

Let, for the sake of simplicity,

\[ C_\phi^2 = \frac{1}{F_r}, \quad (73) \]

then, by virtue of (69), we obtain that

\[ H = r \sqrt{4r^2 - 1} \Rightarrow H \approx 2r^2. \quad (74) \]

Thus, in the case of electrically neutral adiabatic flow, we find that

\[ H \propto r^2. \quad (75) \]

4 Perfect gas. Pre-assigned midplane temperature

As it can be seen from Sec. 2, the vertically isothermal disk will, in fact, be totally isothermal under the assumption that the radial components, \( v_r \) and \( B_r \), of velocity and magnetic field, respectively, are negligible. Furthermore, the disk cannot be thin in terms of, e.g., [7]. Adiabatic disks (see Sec. 3), in contrast to vertically isothermal ones, are more trustworthy. However, thin disk accretion must be highly nonadiabatic, as emphasized in [27]. Because of this, we will consider an intermediate case that is between the above-listed two axisymmetric flows.

In this section we will, mainly, deal with geometrically thin disks. In the case of, e.g., adiabatic flow (see Sec 3) it will be valid if \( E_u \ll 1 \). We take \( p_* = \rho_*RT_* \) and, hence, we obtain from (11) that

\[ P = \rho T. \quad (76) \]
We will consider symmetric disks and, hence, it is, in general, assumed that 
0 \leq z \leq H. Here H denotes the disk semi-thickness. The temperature at the
midplane, i.e.

\[ T_0 \equiv T|_{z=0} = T_0(r,t), \]  

is assumed to be a preassigned function of \( r \) and \( t \). It is also assumed that \( \rho \to 0 \) implies \( T \to 0 \) as well as \( \rho = 0 \) and \( T = 0 \) if \( |z| > H \), i.e., there is a vacuum
outside the disk.

4.1 Non-magnetic disk

It is assumed that 

\[ B_r = B_z = B_\phi = 0. \]  

(78)

Let \( v_z = v_r = 0 \), and let \( R_c \to \infty \), \( \psi_{bc} = 0 \). In such a case, the steady-state
version of (15)-(23) will be the following system of PDEs.

\[ \rho v_\phi^2 = E_u \frac{\partial P}{\partial r} + \frac{\rho \Phi}{F_r} \partial \Phi \partial r, \quad P = \rho T, \quad E_u, F_r = \text{const}, \]  

(79)

\[ E_u \frac{\partial P}{\partial z} + \frac{\rho \Phi}{F_r} \partial \Phi \partial z = 0, \quad |z| < H(r), \quad \Phi \equiv -\frac{1}{\sqrt{r^2 + z^2}} = -\frac{1}{r} + \frac{1}{2r} z^2 + \ldots, \]  

(80)

where \( H(r) \) denotes the free surface of disk. The boundary conditions at the
free surface are the following:

\[ \rho|_{z=H} = 0, \quad T|_{z=H} = 0. \]  

(81)

Let us note that outside the disk the system (79)-(80) is fulfilled identically,
since \( \rho = 0 \) and \( T = 0 \) at any point with \( |z| > H \).

The radial pressure force is usually assumed to be negligible in comparison
with gravity as well as with inertia force and, hence, the circular velocity,
\( v_\phi \), will be Keplerian with a great precision (see, e.g., [6], [9], [11], [29]). Let us
consider this assertion in more detail. Since Eqs. (79)-(80) are written in a
non-dimensional form, the assumption that gravitational and centrifugal forces
in (79) dominate is equivalent to the assumption that \( E_u \ll 1/F_r \) and \( E_u \ll 1. \)
The degenerate equation corresponding to (80), i.e., the equation obtained from
(80) by putting \( E_u = 0 \), has the following solution: \( \rho = 0 \) for all \( z \neq 0 \) and
\( \rho|_{z=0} \geq 0 \), i.e., \( \rho|_{z=0} \) can be equal to an arbitrary non-negative real number.
We have from Eq. (80) that

\[ E_u \frac{\partial P}{\partial z}|_{z=0} = 0, \quad \forall E_u \geq 0. \]  

(82)

From the above it follows that the function \( P = P(z) \) can have a weak disconti-
nuity at \( z = 0 \) provided \( E_u = 0 \). Hence, \( P(z) \) is continuous for all \( z \) and, hence,
\( P(z) \equiv 0 \) since \( \rho = 0 \) for all \( z \neq 0 \). We emphasize that with \( \rho|_{z=0} > 0 \) and
\( P|_{z=0} = 0 \) the temperature \( T|_{z=0} = 0 \). Let \( E_u > 0 \). In such a case, the exact solution to Eq. (80) is the following.

\[
\rho = \frac{\rho_0 T_0}{T} \exp \left( - \int_0^z \frac{1}{E_u F_r T} \frac{\partial \Phi}{\partial z} \, dz \right), \quad 0 \leq z \leq H,
\]

(83)

where \( \rho_0 = \rho|_{z=0}, \) \( T_0 = T|_{z=0} \). Because of the symmetry, it is assumed in (83) that \( 0 \leq z \leq H(r) \) instead of \( |z| \leq H(r) \). We obtain from (83) that \( \rho(z) \to 0 \) as \( E_u \to 0 \) provided that \( z > 0 \). Hence, \( H \to 0 \) as \( E_u \to 0 \). Thus, if \( E_u \to 0 \), then we obtain an infinitely thin disk with \( \rho|_{z=0} = 0 \) or \( \rho|_{z=0} \) can be equal to an arbitrary positive real number. If the latter is the case, the disk would consist of non-interacting particles. As an illustration we refer to a cold dust disk consisting of small non-interacting grains. Since the disk is gaseous, \( H \to 0 \) as \( E_u \to 0 \), and in view of (81), we may assume that

\[
\rho_0 \equiv \rho|_{z=0} \xrightarrow{E_u \to 0} 0.
\]

(84)

An additional justification of (84) will be done in what follows.

Since the values \( \rho, v_\phi, \) and \( T \) are even functions of \( z \), we will use the following asymptotic expansion in the limit \( z/r \to 0 \):

\[
\rho = \rho_0 + \rho_2 z^2 + \ldots, \quad T = T_0 + T_2 z^2 + \ldots, \quad v_\phi = v_\phi^0 + v_\phi^2 z^2 + \ldots,
\]

(85)

where the coefficients \( \rho_0, \rho_2, T_0, T_2, v_\phi^0, \) and \( v_\phi^2 \) depend on \( r \), only. Let us note that namely \( z/r \ll 1 \) and, hence, the power series in (85) must be represented in \( z/r \). However, for simplicity sake, the denominators \( (r^i, i = 0, 2, \ldots) \) are, mainly, included into the coefficients.

By virtue of (80), (81), we obtain the following equalities:

\[
T_0 \rho_2 + T_2 \rho_0 + \frac{\rho_0}{2E_u F_r r^3} = 0,
\]

(86)

\[
T_0 + T_2 H^2 \approx 0,
\]

(87)

\[
\rho_0 + \rho_2 H^2 \approx 0.
\]

(88)

Solving the system (86)-(88), we find that

\[
H^2 \approx 4T_0 E_u F_r r^3,
\]

(89)

\[
T = T_0 - \frac{1}{4E_u F_r r^3} z^2 + O(z^4) \equiv T_0 - \frac{1}{4E_u F_r r} \left( \frac{z}{r} \right)^2 + O \left( \left( z/r \right)^4 \right),
\]

(90)

\[
\rho = \rho_0 \left( 1 - \frac{z^2}{H^2} \right) + O(z^4) \equiv \rho_0 - \frac{\rho_0}{4T_0 E_u F_r r} \left( \frac{z}{r} \right)^2 + O \left( \left( z/r \right)^4 \right).
\]

(91)
By virtue of (85) and (79), we find the following differential equation in the function $\rho_0 = \rho_0(r)$.

$$\frac{\rho_0 v^2}{r} = E_u \frac{\partial T_0}{\partial r} + \frac{\rho_0}{r^2 F_r}, \quad r > r_0,$$

(92)

where $T_0 = T_0(r)$ is a preassigned function. We will use the following notation $T^0_0 = T_0(r_0)$. The boundary condition is the following

$$\rho_0^0 = \rho_0(r_0).$$

(93)

The degenerate equation corresponding to (92) has the following solution.

1) $\rho_0 = 0$ and the midplane circular velocity $v_{\varphi 0} = v_{\varphi 0}(r)$ is an arbitrary function.

2) $\rho_0 = \rho_0(r)$ is an arbitrary function such that $\rho_0 > 0$ and the motion at the midplane is Keplerian:

$$v_{\varphi 0} = \pm \frac{1}{\sqrt{r F_r}}.$$  

(94)

If the latter is the case, i.e. $\rho_0 > 0$, then the disk is infinitely thin, consists of non-interacting particles and, hence, it is natural that all particles are in Keplerian motion.

Let $E_u > 0$, then the solution to (92), (93) is the following.

$$\rho_0 = \rho_0^0 \frac{T^0_0}{T_0(r)} \exp \int_{r_0}^{r} \frac{1}{E_u T_0} \left( v_{\varphi 0}^2 \right) dr.$$  

(95)

Notice, be the disk Keplerian, we would obtain, by virtue of (95), that

$$\rho_0 = \rho_0^0 \frac{T^0_0}{T_0(r)}, \quad \rho_0^0, T^0_0 = \text{const}.$$  

(96)

Thus, if $\rho_0(r) \to 0$ as $r \to \infty$, then, in view of (96), $T_0(r) \to \infty$ as $r \to \infty$, and vice versa, i.e. if $T_0(r) \to 0$, then $\rho_0(r) \to \infty$ as $r \to \infty$. Assuming that $T_0 = T^0_0 = \text{const}$, we obtain that $\rho_0 = \rho_0^0 = \text{const}$ too. Hence, the assumption that the motion is Keplerian leads to improbable density and temperature distributions at the midplane under $r \geq r_0$. To avoid such unlikely solutions, we have to accept that the motion is not Keplerian and, hence

$$v_{\varphi 0}^2 < \frac{1}{r F_r}, \quad r \to \infty.$$  

(97)

It is well known (see, e.g., [6], [27], [29]) that a gaseous disk formed around a central star is in an almost Keplerian rotation with a small inward drift velocity. Let us investigate this “almost Keplerian” rotation. Let $v_r \neq 0$, then, instead of (79), (80), we should consider the following system.

$$\frac{1}{r} \frac{\partial (r \rho v_r)}{\partial r} + \frac{\partial (\rho v_z)}{\partial z} = 0.$$  

(98)
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \rho v_r^2 \right) + \frac{\partial}{\partial z} \left( \rho v_r v_z \right) - \frac{\rho v_z^2}{r} = -\frac{\partial}{\partial r} \left( E_u P \right) - \frac{\rho}{F_r} \frac{\partial \Phi}{\partial r},
\]
(99)

\[
\frac{1}{r} \frac{\partial}{\partial r} r \left( \rho v_r v_z \right) + \frac{\partial}{\partial z} \left( \rho v_r v_z \right) + \frac{\rho v_z v_r}{r} = 0,
\]
(100)

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \rho v_r^2 \right) + \frac{\partial}{\partial z} \left( \rho v_z^2 \right) = -\frac{\partial}{\partial z} \left( E_u P \right) - \frac{\rho}{F_r} \frac{\partial \Phi}{\partial z}.
\]
(101)

For the sake of simplicity, we take the characteristic quantity, \( l_* \), for length such that \( r_0 = 1 \).

Since \( v_r \) is an even function of \( z \), but \( v_z \) is an odd one, we can write for small values of \( z \):

\[
v_r = v_{r0} + v_{r2} z^2 + \ldots, \quad v_z = v_{z1} z + v_{z3} z^3 + \ldots,
\]
(103)

where the coefficients depend on \( r \), only. By virtue of (85), (103), we obtain from (98), (100) that

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \rho v_{r0} \right) + \rho_0 v_{z1} = 0,
\]
(104)

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \rho_0 v_{r0} v_{r0} \right) + \rho_0 v_{r0} v_{z1} + \frac{\rho_0 v_{r0} v_{r0}}{r} = 0.
\]
(105)

By virtue of (104)-(105), we obtain:

\[
\rho_0 v_{r0} \frac{\partial v_{r0}}{\partial r} = 0.
\]
(106)

Thus, we find the following exact solution for the midplane value, \( v_{r0} \), of circular velocity:

\[
v_{r0} = \frac{C_{\psi0}}{r}, \quad C_{\psi0} = \text{const}.
\]
(107)

For the sake of convenience, let us represent \( C_{\psi0} \) as a function of the Froude number, \( F_r \). Let the point \( r = r_m \) be the only point where the motion is Keplerian as well as the vortex, (107). Then

\[
\frac{C_{\psi0}}{r_m} = \frac{1}{\sqrt{r_m F_r}} \Rightarrow C_{\psi0} = \frac{r_m}{\sqrt{F_r}}.
\]
(108)

Hence,

\[
v_{\psi0} = v_{\psi0} |_{z=0} = \frac{\sqrt{r_m}}{r \sqrt{F_r}}, \quad r_m = \text{const}.
\]
(109)

Notice, in the case of a non-viscous flow with \( v_{r0} \neq 0 \) we have the only solution, (107), for the midplane circular velocity, \( v_{r0} \). If, however, \( v_{r0} = 0 \), then we have infinitely many solutions for \( v_{\psi0} \). The only limitations are the boundary conditions. In particular, the function \( v_{\psi0} = v_{\psi0} (r) \) must be such that \( \rho_0 (r) \to 0 \) as \( r \to \infty \) in (85). Thus, if \( v_{r0} = 0 \), then we have the ill-posed problem, i.e.
the problem is not well-posed in the sense of Hadamard [1]. It is important to note that any solution (excluding the vortex) for \( v_{\varphi 0} \) cannot be obtained as a limiting case (namely, as \( v_{\varphi 0} \to 0 \)) of the motion under \( v_{\varphi 0} \neq 0 \). Hence, any solution that does not coincide with the vortex will be unstable, as any infinitesimal variation, \( \delta v_{\varphi 0} \neq 0 \), gives rise to a finite response in the gas flow. The stability of the solution (109) should be investigated. Some stability aspects of the vortex flow will be discussed in Sec. 4.1.1. It is necessary to stress that, by virtue of the solution (107), we obtain from (95) the validity of (84).

Thus, in contrast to the widely known assertion that the circular velocity will be Keplerian with a great precision (see, e.g., [6], [9], [11], [29]), the foregoing proves that in the case of steady-state non-viscous disk we obtain the vortex velocity distribution (107) even if the radial pressure force is negligible in comparison with gravity and inertia forces. The solutions similar to (107) are used to describe a large variety of flow patterns, in particular, to model cosmic swirling jets that develop near accretion disks [26].

Let us note that the power-law model [29, p. 374]

\[
v_{\varphi 0} = \frac{C_{\varphi}}{r^\kappa}, \quad \kappa, C_{\varphi} = \text{const},
\]

(110)
is assumed to be stable (Rayleigh stable [8]) to pure hydrodynamic perturbations under \( 0.5 \leq \kappa < 1 \), since it satisfies the following necessary and sufficient condition for the so-called Rayleigh stability [2, p. 78].

\[
f^2_R \equiv \frac{1}{r^3} \frac{\partial}{\partial r} \left( r^2 \Omega^2 \right) > 0,
\]

(111)
where \( \Omega \) denotes the angular velocity, \( f^2_R \) denotes the Rayleigh frequency. Such an assertion contradicts to the foregoing prove that any solution that does not coincide with the vortex velocity distribution (107) will be unstable. It should be stressed that the Rayleigh criterion, (111), is not strictly applicable to a nebular accretion disk [4]. This remark, [4], is correct because the Rayleigh criterion, (111), is justified for subsonic flows [2], whereas we consider highly supersonic flows. Actually, since \( E_u \ll 1 \) the Mach number \( M_s = \frac{1}{\sqrt{E_u}} \gg 1 \).

Let us estimate \( v_{\varphi 2} \) in (85). We will use the following asymptotic expansion

\[
T_0(r) \approx \sum_{n=0}^{\infty} a_n r^{-n}
\]

(112)
in the limit \( r \to \infty \). As usually, the series in (112) may converge or diverge, but its partial sums are good approximations to \( T_0(r) \) for large enough \( r \). Assuming that \( T_0(r) \to 0 \) as \( r \to \infty \), we find that \( a_0 = 0 \). Then, assuming the characteristic quantity, \( T_*, \) for temperature such that \( a_1 = 1 \), we represent \( T_0(r) \) in the following form:

\[
T_0(r) = \frac{1}{r} + O \left( r^{-2} \right), \quad r \to \infty.
\]

(113)
Thus, we can use the following approximation:

\[
T_0(r) \approx \frac{1}{r}, \quad r \to \infty.
\]

(114)
One can easily see, for instance, that the midplane temperature of the adiabatic disks considered in Sec. 3 can be approximated by (114).

By virtue of (114), we find from (89) that
\[ H \propto r. \]  \hspace{1cm} (115)

Let \( v_z \) is a linear function of \( z \), i.e. \( v_z = v_z z \) with \( v_z = v_z (r) \). From the kinematic boundary condition at the free surface, i.e.
\[ v_z\big|_{z=H} = \frac{\partial H}{\partial r} v_r\big|_{z=H}, \]  \hspace{1cm} (116)
we find, by virtue of (114) and (89), that
\[ v_z H \approx \frac{\partial H}{\partial r} v_r \Rightarrow v_z \approx \frac{v_r}{r}. \]  \hspace{1cm} (117)

By virtue of (103) and (85), we obtain from (98), (100) that
\[ v_r \frac{\partial v_{\varphi}^2}{\partial r} + 2v_z v_{\varphi^2} + \frac{v_{\varphi^2} v_r}{r} = 0. \]  \hspace{1cm} (118)
Thus, by virtue of (117), we find
\[ v_{\varphi^2} \approx \text{const} \frac{r}{r^3}. \]  \hspace{1cm} (119)

By virtue of (85), (89), (114), and (119), we find that
\[ v_{\varphi}\big|_{z=H} \approx \frac{C_{\varphi^0}}{r} \left( 1 - C_{\varphi^2} E_u F_r \right), \quad C_{\varphi^2} = \text{const} > 0. \]  \hspace{1cm} (120)
Since \( E_u \ll 1 \), it can be assumed that
\[ v_{\varphi} \approx \frac{C_{\varphi^0}}{r}, \quad 0 \leq z \leq H. \]  \hspace{1cm} (121)

It is easy to see from (121) that the Rayleigh frequency \( f_R^2 = 0 \) and, hence, the vortex is Rayleigh unstable.

Let us now consider the stability problem of the power-law model from another point of view.

4.1.1 Instability of non-magnetic disk. Power-law model

Let \( \zeta \) denote the value of a dependent variable, \( \zeta \), for the case of a steady-state solution. In view of (110), (95), (97) and (102), the solution to the system (79)-(81) can be written in the form:
\[ v_z = v_r = 0, \quad v_{\varphi} = C_{\varphi\zeta} \frac{C_{\varphi}}{r^\kappa}, \quad \kappa, C_{\varphi} = \text{const}, \quad 0.5 < \kappa \leq 1, \]  \hspace{1cm} (122)
where \( \rho_0^0 = \rho_0|_{r=1}, T_0^0 = T_0|_{r=1} \). To represent \( C_\varphi \) in (122) as a function of the Froude number, \( F_r \), we assume that the point \( r = r_m \) will be the only point where the motion is Keplerian as well as the power-law model, (122). Hence

\[
C_\varphi = \frac{1}{\sqrt{r_m F_r}} \Rightarrow C_\varphi = \frac{r_m^{-0.5}}{\sqrt{F_r}}.
\]

The non-linear system in perturbations \( \tilde{\varphi} (\zeta = \tilde{\varphi} + \tilde{\zeta}) \) for (15)-(18) will be, under \( R_e \rightarrow \infty \) and \( \vartheta_{ke} = 0 \), as follows.

\[
S_h \frac{\partial (\tilde{\varphi}_0 + \tilde{\varphi}_0)}{\partial t} + \frac{1}{r} \frac{\partial r (\tilde{\varphi}_0 + \tilde{\varphi}_0)}{\partial r} \tilde{\varphi}_r + (\tilde{\varphi}_0 + \tilde{\varphi}_0) \tilde{\varphi}_{z1} = 0,
\]

\[
S_h \frac{\partial (\tilde{\varphi}_0 + \tilde{\varphi}_0) \tilde{\varphi}_r}{\partial t} + \frac{1}{r} \frac{\partial r (\tilde{\varphi}_0 + \tilde{\varphi}_0)}{\partial r} \tilde{\varphi}_{r0} + (\tilde{\varphi}_0 + \tilde{\varphi}_0)(\tilde{\varphi}_{r0} + \tilde{\varphi}_{r0}) = \frac{(\tilde{\varphi}_0 + \tilde{\varphi}_0)(\tilde{\varphi}_{r0} + \tilde{\varphi}_{r0})^2}{r}
\]

\[
- E_u \frac{\partial}{\partial r}(\tilde{\varphi}_0 + \tilde{\varphi}_0) (\tilde{T}_0 + \tilde{T}_0) - \frac{\tilde{\varphi}_0 + \tilde{\varphi}_0}{r^2 F_r},
\]

\[
S_h \frac{\partial (\tilde{\varphi}_0 + \tilde{\varphi}_0) \tilde{\varphi}_{z0} + \tilde{\varphi}_{z0}}{\partial t} + \frac{1}{r} \frac{\partial r (\tilde{\varphi}_0 + \tilde{\varphi}_0) (\tilde{\varphi}_{z0} + \tilde{\varphi}_{z0})}{\partial r} \tilde{\varphi}_{z0} + (\tilde{\varphi}_0 + \tilde{\varphi}_0)(\tilde{\varphi}_{z0} + \tilde{\varphi}_{z0}) = \frac{(\tilde{\varphi}_0 + \tilde{\varphi}_0)(\tilde{\varphi}_{z0} + \tilde{\varphi}_{z0})}{r}
\]

\[
- 2E_u \frac{\partial}{\partial z}[\tilde{\varphi}_0 + \tilde{\varphi}_0] (\tilde{T}_2 + \tilde{T}_2) + \rho_2 + \tilde{\varphi}_2 (\tilde{T}_0 + \tilde{T}_0) - \frac{\tilde{\varphi}_0 + \tilde{\varphi}_0}{F_r r^3}.
\]

With the basic Parker’s assumption \[23\], as applied to perturbations \( \tilde{\varphi} \), we may assume that, within the disk, radial derivatives, \( \partial \tilde{\varphi}/\partial r \), are negligible in comparison to vertical ones. Using the Parker’s assumption \[23\] we can reduce (125)-(128) to an ODE (ordinary differential equation) system for a subsequent stability investigation. In this connection we note that the following question remains to be answered. What a class of admissible functions, \( \tilde{\varphi}_i = \tilde{\varphi}_i (r, t) \), in the expansions

\[
\tilde{\varphi} = \sum_{i=0}^{\infty} \tilde{\varphi}_i z^i,
\]

should be taken into account? Obviously, it must be such that, at least, the main terms of expansions may be dropped under the derivative over the coordinate \( r \). Assuming \( \partial \tilde{\varphi}_0 / \partial r = 0 \) in (125)-(128) we find that \( \tilde{\varphi}_0 \equiv 0 \), since \( \tilde{\varphi}_0 |_{r=\infty} = 0 \). In order to circumvent this problem, we will consider a function as admissible if it
will be a singular function \[15\]. That is, this function is continuous on \(1 \leq r < \infty\) and the derivative over the coordinate \(r\) exists and is zero almost everywhere. It can be also assumed that the function is strictly monotone decreasing, e.g. \(\tilde{v}_{r0}|_{r \to \infty} \to 0\). Cantor staircase \[15\] and Lebesgue singular function \[13\] are well known examples of such functions. Singular functions occur in physics, dynamical systems, etc. (see e.g. the references in \[13\]).

We restrict ourself to the case of linear system. Assuming the non-linear terms, as well as \(\tilde{T}_0\) and \(\tilde{T}_2\), in (125)-(128) as negligible, we get:

\[
S_h \frac{\partial \tilde{p}_0}{\partial t} + \frac{1}{r} \frac{\partial r \tilde{p}_0 \tilde{v}_{r0}}{\partial r} + \tilde{p}_0 \tilde{v}_{z1} = 0, \tag{130}
\]

\[
S_h \frac{\partial \tilde{v}_{r0}}{\partial t} - \frac{2 \tilde{p}_0 \tilde{v}_{z0} \tilde{v}_{r0} + \tilde{p}_0 \tilde{v}_{z0}^2}{r} = -E_u \frac{\partial}{\partial r} (\tilde{p}_0 \tilde{T}_0) - \frac{\tilde{p}_0}{r^2 F_r}, \tag{131}
\]

\[
S_h \frac{\partial \tilde{v}_{z0}}{\partial t} = (x - 1) \frac{C_{r} \tilde{v}_{r0}}{r^x + 1}, \tag{132}
\]

\[
S_h \frac{\partial \tilde{v}_{z1}}{\partial t} = -2 E_u (\tilde{p}_0 \tilde{T}_2 + \tilde{p}_2 \tilde{T}_0) - \frac{\tilde{p}_0}{r^3 F_r}. \tag{133}
\]

It easy to see from (85), (89)-(91) that

\[
\tilde{T}_2 = \frac{1}{4 E_u F_r r^3}, \quad \tilde{p}_2 = -\frac{\tilde{p}_0}{4 T_0 E_u F_r r^3}, \quad \tilde{p}_2 = -\frac{\tilde{p}_0}{4 T_0 E_u F_r r^3}. \tag{134}
\]

By virtue of (134), we rewrite (130)-(133) to read:

\[
S_h \frac{\partial \tilde{p}_0}{\partial t} = A_{12} \tilde{v}_{r0} + A_{14} \tilde{v}_{z1} = 0, \tag{135}
\]

\[
S_h \frac{\partial \tilde{v}_{r0}}{\partial t} = A_{21} \tilde{p}_0 + A_{23} \tilde{v}_{z0}, \tag{136}
\]

\[
S_h \frac{\partial \tilde{v}_{z0}}{\partial t} = A_{32} \tilde{v}_{r0}, \tag{137}
\]

\[
S_h \frac{\partial \tilde{v}_{z1}}{\partial t} = 0, \tag{138}
\]

where

\[
A_{12} = -\frac{\partial \tilde{p}_0}{r \partial r}, \quad A_{14} = -\tilde{p}_0, \quad A_{21} = \frac{C_{r}^2}{\tilde{p}_0 r^{2x+1}} - E_u \frac{\partial \tilde{T}_0}{\tilde{p}_0 \partial r} - \frac{1}{\tilde{p}_0 r^2 F_r}, \quad A_{23} = 2 \frac{C_{r}}{r^{x+1}}, \quad A_{32} = (x - 1) \frac{C_{r}}{r^{x+1}}. \tag{139}
\]

We will use the following notation: \(x = \{x_1, x_2, x_3, x_4\}^* = \{\tilde{p}_0, \tilde{v}_{r0}, \tilde{v}_{z0}, \tilde{v}_{z1}\}^*\). Let \(t \to S_h t\), then the system (135)-(138) can be rewritten to read:

\[
\frac{dx}{dt} = A \cdot x, \tag{140}
\]
where
\[
A = \begin{bmatrix}
0 & A_{12} & 0 & A_{14} \\
A_{21} & 0 & A_{23} & 0 \\
0 & A_{32} & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\] (141)

The eigenvalues of $A$ are the following:
\[
\lambda_1 = \lambda_2 = 0, \quad \lambda_{3,4} = \pm \sqrt{A_{23}A_{32} + A_{12}A_{21}}.
\] (142)

Inasmuch as the eigenvalue $\lambda = 0$ has algebraic multiplicity $m = 2$ and rank $A = 3$ provided $0.5 < \varkappa < 1$, the solution to (140) contains secular terms, namely the terms proportional to the time. Hence, the solution, (122)-(123), under $0.5 < \varkappa < 1$, to the system (79)-(80) is unstable since the secular terms grow without bound as $t \to \infty$. Let us note that we have already proven that any solution that does not coincide with the vortex will be unstable (see the text between Eqs. (109) and (110)). On top of that, we have just now proven the linear instability of the power-law model. However, this model satisfies (111) under $0.5 < \varkappa < 1$ and, hence, a Rayleigh stable flow may be unstable in the sense of Lyapunov stability [19]. Let us now consider the solution, (122)-(123), to the system (79)-(80) under $\varkappa = 1$. In such a case rank $A = 2$ since $A_{32} = 0$ and, hence, the solution to (140) does not contain secular terms. Thus, this solution will be unstable [19] if
\[
A_{12}A_{21} > 0.
\] (143)

By virtue of (114) and (124), we find that (143) will be valid if
\[
r > \max \left( \frac{r_m}{1 - E_u F_r}, \frac{r_m}{2E_u F_r - 1} \right).
\] (144)

Since the disk is thin, i.e. $E_u \ll 1$, we obtain the following condition of linear instability:
\[
r > \frac{r_m}{1 - E_u F_r}.
\] (145)

Thus, the vortex will be linearly unstable, at least, on the disk’s periphery, provided $R_e \to \infty$. Let us remind that the vortex is unconditionally Rayleigh unstable because the Rayleigh frequency $f_R^2 = 0$.

### 4.2 Magnetic accretion disk

We will use the following asymptotic expansion in the limit $z/r \to 0$. Then, in view of the symmetry, we may write:
\[
B_\varphi = B_{\varphi 1} z + \ldots, \quad B_z = B_{z 0} + B_{z 2} z^2 + \ldots, \quad B_r = B_{r 1} z + B_{r 3} z^3 + \ldots,
\]
\[
\rho = \rho_0 + \rho_2 z^2 + \ldots, \quad T = T_0 + T_2 z^2 + \ldots, \quad v_\varphi = v_{\varphi 0} + v_{\varphi 2} z^2 + \ldots,
\]
\[
v_r = v_{r 0} + v_{r 2} z^2 + \ldots, \quad v_z = v_{z 1} z + v_{z 3} z^3 + \ldots,
\]
\[
\Phi = -\frac{1}{\sqrt{r^2 + z^2}} = \Phi_0 + \Phi_2 z^2 + \ldots = -\frac{1}{r} + \frac{1}{2r^3} z^2 + \ldots , \tag{146}
\]

where all coefficients depend on \( r \), only. It is also assumed in this section that \( B_{z0} \neq 0 \).

A steady-state magnetic accretion disk with \( v_{r0} \neq 0, \rho_0 \neq 0 \), and with a negligible dynamic viscosity, i.e. \( \mu = 0 \), will be our initial concern. The magnetic field will be called as almost poloidal if

\[
B_{\varphi 1} = 0 \Rightarrow B_\varphi = O (z^3) , \tag{147}
\]

and the magnetic field will be called as almost axial if

\[
B_{\varphi 1} = 0, \ B_{r 1} = 0 \Rightarrow B_\varphi = O (z^3) , \ B_r = O (z^3) . \tag{148}
\]

Let us prove that the circular velocity at the midplane will be the vortex velocity, i.e.

\[
v_{\varphi 0} = v_{\varphi |z=0} = \frac{C_{\varphi 0}}{r}, \quad C_{\varphi 0} = \text{const}, \tag{149}
\]

if and only if the magnetic field will be almost poloidal. Actually, since \( \mu = 0 \), the steady-state version of Eqs. (15), (17) is the following.

\[
1 \frac{1}{r} \frac{\partial}{\partial r} \left( \rho v_r \right) + \frac{\partial}{\partial z} (\rho v_z) = 0, \tag{150}
\]

\[
1 \frac{1}{r} \frac{\partial}{\partial r} r \left( \rho v_\varphi v_r - \frac{E_u}{\beta} B_\varphi B_r \right) + \frac{\partial}{\partial z} \left( \rho v_\varphi v_z - \frac{E_u}{\beta} B_\varphi B_z \right) + \frac{\rho v_\varphi v_r}{r} - \frac{E_u}{\beta} \frac{B_\varphi B_r}{r} = 0. \tag{151}
\]

In view of (150), (151), and (146), we obtain

\[
1 \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) + \rho_0 v_{z1} = 0 , \tag{152}
\]

\[
\frac{v_{z0}}{r} \frac{\partial}{\partial r} (r \rho v_r) + \frac{E_u}{\beta} B_{\varphi 1} B_{z0} + \frac{\rho_0 v_{z0} v_{r0}}{r} = 0. \tag{153}
\]

By virtue of (152), we obtain from (153):

\[
\rho_0 v_{r0} \frac{\partial v_{z0}}{\partial r} - \frac{E_u}{\beta} B_{\varphi 1} B_{z0} + \frac{\rho_0 v_{z0} v_{r0}}{r} = 0. \tag{154}
\]

Let the magnetic field will be almost poloidal, i.e. \( B_{\varphi 1} = 0 \). Then, in view of (154), we have

\[
\rho_0 v_{r0} \frac{\partial v_{z0}}{r \partial r} = 0. \tag{155}
\]

Equality (155) proves (149).
Let \((149)\) be valid. In view of \((154)\), we have
\[
\rho_0 v_{r0} \frac{\partial v_{z0}}{\partial r} - \frac{E_u}{\beta} B_{\phi 1} B_{z0} = 0.
\] (156)

By virtue of \((149)\), we find from \((156)\):
\[
B_{\phi 1} B_{z0} = 0.
\] (157)

Equality \((157)\) proves \((147)\).

Let us now consider a steady-state magnetic disk with \(v_r = 0\) (and, hence, \(v_z = 0\)) and with a negligible dynamic viscosity, i.e. \(\mu = 0\). Let us remind that such assumptions for the case of non-magnetic disk lead to the ill-posed problem in the sense of Hadamard (see Sec. 4.1). In particular, the solution to the mathematical model (Sec. 4.1) is not unique. We intend to find a steady-state solution to the MHD system \((15)-(22)\) provided \(R \to \infty, \vartheta_{ke} = 0\). In such a case, the MHD system is reduced to the following.

\[
-\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{E_u B^2}{\beta} \right) - \frac{\partial}{\partial z} \left( \frac{E_u}{\beta} B_r B_z \right) + \frac{E_u B^2}{\beta} B_{\phi} - \rho v_{z0}^2 = -\frac{\partial}{\partial r} \left( E_u P + \frac{E_u B^2}{\beta} \right) - \rho \frac{\partial \Phi}{F_r} \frac{\partial}{\partial r},
\] (158)

\[
-\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{E_u}{\beta} B_{\phi} B_r \right) - \frac{\partial}{\partial z} \left( \frac{E_u}{\beta} B_{\phi} B_z \right) - \frac{E_u}{\beta} B_{\phi} B_r = 0,
\] (159)

\[
\frac{\partial}{\partial r} \left( \frac{E_u}{\beta} B_{\phi} B_r \right) + \frac{\partial}{\partial z} \left( \frac{E_u}{\beta} B_{\phi} B_z \right) = \frac{\partial}{\partial z} \left( E_u P + \frac{E_u B^2}{\beta} \right) + \rho \frac{\partial \Phi}{F_r} \frac{\partial}{\partial z}
\] (160)

\[
\frac{\partial}{\partial r} \left( v_{\phi} B_r \right) + \frac{\partial}{\partial z} \left( v_{\phi} B_z \right) = 0,
\] (161)

\[
\frac{1}{r} \frac{\partial r B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0.
\] (162)

Eq. (24) is reduced to the following
\[
\frac{\partial}{\partial r} \left( v_{\phi} B_z \right) - \frac{\partial}{\partial z} \left( v_{\phi} B_r \right) = 0.
\] (163)

Let \(B_{\phi}\) be an odd function of \(z\) and let \(B_r = O(z^5)\). Then we find from \((162)\) that
\[
B_{z0} = B_{z0} (r), \quad B_{z2} = B_{z4} = 0.
\] (164)

By virtue of \((164)\), we find from \((161)\) and \((159)\) that
\[
v_{\phi 0} = v_{\phi 0} (r), \quad v_{\phi 2} = v_{\phi 4} = 0, \quad B_{\phi 1} = B_{\phi 3} = 0.
\] (165)
Taking into account that \( B_r = O(z^5) \) and using (164), we obtain from Eq. (160) that (86) is valid for the case of the magnetic disk as well. Hence, using (90), (91), and (164)-(165), we obtain from Eq. (158) that
\[
\rho v^2 \frac{\partial}{\partial r} \phi = E_u T \frac{\partial}{\partial r} \rho + E_u \frac{\partial}{\partial r} T + \frac{E_u}{\gamma} \frac{\partial}{\partial r} \rho_T + \frac{\rho}{F_r} \frac{\partial}{\partial r} \phi + O(z^4). \quad (166)
\]
Eq. (166) must be valid under all values of \( z \). Assuming \( z = H \), we obtain, in view of (81), the following equation in \( B_{z0} \):
\[
\frac{E_u}{\gamma} \frac{\partial}{\partial r} B_{z0}^2 + O \left( \frac{(z/r)^4}{\gamma} \right) = 0. \quad (167)
\]
Then we find:
\[
B_{z0} \approx \text{const.} \quad (168)
\]
Since \( B_r = O(z^5) \), we obtain from (163), by virtue of (164)-(165), that
\[
\frac{\partial}{\partial r} \left( v_{\varphi0} B_{z0} \right) = 0. \quad (169)
\]
We find, in view of (168), (169), that
\[
v_{\varphi0} \approx \text{const} \frac{r}{v}. \quad (170)
\]
Analogously, if \( B_r = 0 \) (and \( v_r = 0 \)), then we find from (158)-(162) that \( B_{\varphi} = 0 \) and \( B_z = \gamma \text{const} \). Thus, in view of (163), the vortex velocity distribution
\[
v_{\varphi} = \text{const} \frac{r}{v}. \quad (171)
\]
will be the only solution for the circular velocity, and, hence, the motion will be Rayleigh unstable. Let us note, the density and temperature distributions can be easily found from Eqs. (158), (160), which can be written, in view of the foregoing, as the following:
\[
\frac{\rho v^2}{r} = \frac{\partial}{\partial r} \left( E_u \rho T_0 \right) + \frac{\rho}{F_r} \frac{\partial}{\partial r} \rho, \quad (172)
\]
\[
\frac{\partial}{\partial z} \left( E_u \rho T \right) + \frac{\rho}{F_r} \frac{z}{(r^2 + z^2)^{3/2}} = 0, \quad (173)
\]
where the circular velocity, \( v_{\varphi} \), is calculated from Eq. (171). Notice, Eqs. (172), (173) coincide with Eqs. (79), (80), and, hence, we obtain (90), (91), and (95).

Notice, we have assumed \( v_{r0} \neq 0 \) in the above-proven assertion that the midplane circular velocity will be \( \text{const} \) if and only if the magnetic field will be almost poloidal. The following counter-example demonstrates that the condition \( v_{r0} \neq 0 \) is essential. We consider the case when \( v_r = 0 \) and \( B_z = B_{z0} + O(z^4) \) with \( B_{z0} = \text{const} \). In view of (162), we have
\[
B_{r1} = \frac{C_{r1}}{r}, \quad C_{r1} = \text{const}. \quad (174)
\]
Then, by virtue of (174), we obtain from (163) the power-law model:

\[ v_{\phi 0} = \frac{\text{const}}{r^\kappa}, \quad \kappa \equiv 1 - \frac{C_{r1}}{B_{z0}} = \text{const} \tag{175} \]

which is the only solution for the midplane circular velocity in the frame of our assumptions. However, in view of (159), \( B_{\phi 1} = 0 \), i.e. the magnetic field is almost poloidal.

### 4.2.1 Viscous disk

Let us, first, demonstrate that the midplane circular velocity will be the vortex one, (149), if the dynamic viscosity \( \mu = \text{const} \neq 0 \), provided that the magnetic field will be almost axial. Actually, if \( B_{\phi} = O \left( z^3 \right) \) and \( B_r = O \left( z^3 \right) \), then, in view of (146), we obtain from the steady-state version of Eq. (17) that

\[
\rho_0 v_{r0} \frac{\partial r v_{\phi 0}}{\partial r} = \frac{1}{R_e} \frac{\partial}{\partial r} \left[ \mu r \frac{\partial}{\partial r} \left( \frac{v_{\phi 0}}{r} \right) \right] + 2 \mu \frac{v_{z2}}{R_e} + 2 \mu \frac{\partial}{\partial r} \left( \frac{v_{\phi 0}}{r} \right). \tag{176} \]

In view of (22) and (146), we have

\[
\frac{1}{r} \frac{\partial r B_{z1}}{\partial r} + 2 B_{z2} = 0. \tag{177} \]

Hence

\[ B_{z2} = 0. \tag{178} \]

By virtue of (146), we find from (20) that

\[
\frac{\partial}{\partial r} (v_{r0} B_{z1} - v_{\phi 0} B_{r1}) + 2 (v_{z1} B_{z1} - v_{\phi 2} B_{z0} - v_{\phi 0} B_{z2}) = 0. \tag{179} \]

Hence, in view of (178), we obtain:

\[ v_{\phi 2} B_{z0} = 0 \Rightarrow v_{\phi 2} = 0. \tag{180} \]

By virtue of (176) and (180), we find that

\[
\rho_0 v_{r0} \frac{\partial r v_{\phi 0}}{\partial r} = \frac{\mu}{R_e} \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} \left( \frac{v_{\phi 0}}{r} \right) \right] + 2 \frac{v_{z0}}{r}. \tag{181} \]

Obviously, the vortex velocity, (149), fulfills Eq. (181), no matter whether \( v_{r0} \neq 0 \) or \( v_{r0} = 0 \). Hence, in contrast to the non-viscous disk, we obtain the only solution, (149), to Eq. (181) provided that \( v_{r0} = 0 \).

As indicated above, the midplane circular velocity can be the vortex velocity if the dynamic viscosity \( \mu = \text{const} \). Let us now consider the case when the midplane circular velocity is not the vortex velocity, but the power-law model:

\[ v_{\phi 0} = \frac{C_{\phi}}{r^\kappa}, \quad \kappa = \text{const} \neq 1, \ C_{\phi} = \text{const}. \tag{182} \]
The power-law model, (182), can fulfill Eq. (181) if \( \mu = \mu(r) \). The following inequality must be valid to fulfil (97).

\[ \kappa > 0.5. \]  

(183)

It is also assumed that the magnetic field will be almost axial, i.e. \( B_x = O\left(z^3\right) \) and \( B_r = O\left(z^3\right) \). To estimate \( \mu = \mu(r) \) at \( z = 0 \) we assume \( v_r = 0 \). We obtain from the steady-state version of Eq. (17) that

\[ \frac{1}{R_e} \frac{\partial}{\partial r} \left[ \mu r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \right] + 2\mu \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) = 0. \]  

(184)

Then, by virtue of (182), we find from (184) that

\[ \mu = C_{\mu r}^{1-\kappa}, \quad C_{\mu} = const. \]  

(185)

Assuming the characteristic quantity, \( \mu_* \), for viscosity such that \( C_{\mu} = 1 \) and using (114), we represent the viscosity, \( \mu \), in the following form:

\[ \mu = T^{1-\kappa}. \]  

(186)

The power law (e.g., [5], [18], [30]) for the laminar viscosity, \( \mu_l \), of dilute gases can be written as the following

\[ \mu_l = T^{\theta}, \]  

(187)

where typically \( \theta = 0.76 \) [5]. It is, also, assumed that \( \theta = 8/9 \) if \( 90 < T < 300 \) °K, and \( \theta = 0.75 \) if \( 250 < T < 600 \) °K. If we assume that the flow in question is laminar, then \( \mu = \mu_l \), and, hence, \( \kappa = 1 - \theta \). Since we consider the flow at the periphery of the disk, i.e. under a low temperature, we find that

\[ \kappa < 0.25. \]  

(188)

The inequality (188) contradicts to (183), and, hence, in contrast to the non-viscous disk under \( v_r = 0 \), the power-law model, (182), can not be assumed as a correct midplane circular velocity in the case of laminar viscous flow, provided that \( T \to 0 \) as \( r \to \infty \) (namely, \( T \propto r^{-1} \)).

Let us now consider a possibility for the power-law model, (182), to be a correct midplane circular velocity provided the flow is turbulent.

The Euler number characterizes “losses” in a flow [12], and it is higher in a turbulent flow than that in the laminar regime [8]. Let us now estimate the value of \( \kappa \) in (182) by assuming that the turbulent gas tends to flow with minimal losses, i.e. to have the Euler number as small as possible. Using Prandtl and Kolmogorov suggestion [1] p. 230 that the turbulent viscosity, \( \mu \), is proportional to the square root of the kinetic energy of turbulence, \( \kappa \), we evaluate \( \mu_0 = \mu|_{z=0} \) as

\[ \mu_0 = C_{\kappa} L_\kappa \rho_0 \kappa^{0.5}, \quad C_{\kappa}, L_\kappa = const. \]  

(189)
Let the disk be non-magnetic, \( v_{r0} = 0 \), the kinetic energy of turbulence \( \kappa = \text{const} \), and let \( \kappa^* \ll RT^* \) (i.e., \( \theta_{ke} \ll E_u \)). Then, by virtue of (114), (102), and (182), we rewrite (95) to read:

\[
\rho_0 = \frac{\rho_0^0}{r^{\alpha-1}} \exp \left[ \frac{1}{r^{2\kappa-1}} - 1 \right], \quad \zeta = \frac{C_0^2 \phi}{E_u (2\kappa - 1)}, \quad \alpha = \frac{1}{E_u F_r}.
\] (190)

We obtain from (189) that

\[
\mu_0 = \frac{C_\rho}{r^{\alpha-1}} \exp \left[ \frac{1}{r^{2\kappa-1}} - 1 \right], \quad C_\rho \equiv \rho_0^0 C_\kappa L_{\alpha \kappa_0}^{0.5} = \text{const}.
\] (191)

Equating (185) and (191) at \( r = 1 \), we find that \( C_\rho = C_\mu \). Let us now assume that \( \mu \) of (185) and \( \mu_0 \) of (191) coincide each other in the vicinity of \( r = 1 \), i.e., at \( r = 1 + \varepsilon \) (\( \varepsilon \ll 1 \)), with accuracy \( O(\varepsilon^2) \). In such a case we obtain that

\[
E_u \propto \frac{1}{2 - \kappa}. \tag{192}
\]

As we can see from (192) and (183), \( E_u \to \min \) if \( \kappa \to 0.5 \). Thus, in the frame of our assumptions, we find that the turbulent flow tends to be Keplerian.

5 Concluding remarks

On the basis of the exact solution to the MHD system in Sec. 2 we may conclude that vertically isothermal disks will, in fact, be totally isothermal under the assumption that the radial components, \( v_r \) and \( B_r \), of velocity and magnetic field, respectively, are negligible. Furthermore, the disks cannot be considered as thin in terms of, e.g., [7] even if the Euler number \( 0 < E_u \ll 1 \).

The exact solution to the MHD system in Sec. 3 corroborates the view [27] that thin disk accretion must be highly nonadiabatic. Despite of the fact that adiabatic disks (see Sec. 3) are more trustworthy than isothermal ones, we find that the non-dimensional semi-thickness \( H \propto r^2 \) instead of the ratio \( H \propto r \) [27].

The exact solutions to the MHD systems in Sec. 2 and Sec. 3 prove that the vortex velocity will be the only solution for the circular velocity provided that the flow is charge-neutral. Let us note that the exact solutions are found under the assumption that \( v_r = 0 \) and \( B_r = 0 \).

The approach developed in Sec. 4 for the modeling of thin accretion disks turns out to be efficient. In the case of non-magnetic disk, this approach enables to obtain, with ease: the solution for the steady-state non-viscous disk with a good accuracy, to find the non-dimensional semi-thickness \( H \propto r \), to prove that all solutions for the midplane circular velocity are unstable provided the disk is non-viscous. Using this approach one can prove (under \( v_r \neq 0 \)) that the midplane circular velocity will be the vortex velocity if and only if the magnetic field will be almost poloidal. The approach of Sec. 4 enables one to demonstrate that the pure hydrodynamic turbulence in accretion disks is possible, and to demonstrate that the turbulent flow tends to be Keplerian.
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