\textbf{CP–Violation in }B_q\text{ Decays and Final State Strong Phases}

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\textbf{Abstract}

Using the unitarity, $SU(2)$ and $C$-invariance of hadronic interactions, the bounds on final state phases are derived. It is shown that values obtained for the final state phases relevant for the direct CP-asymmetries $A_{CP}(B^0 \rightarrow K^+\pi^-, K^0\pi^0)$ are compatible with experimental values for these asymmetries. For the decays $B^0 \rightarrow D^{(*)-}\pi^+$ ($D^{(*)+}\pi^-$) described by two independent single amplitudes $A_f$ and $A'_f$ with different weak phases ($0$ and $\gamma$) it is argued that the $C$-invariance of hadronic interactions implies the equality of the final state phase $\delta_f$ and $\delta'_f$. This in turn implies, the CP-asymmetry \(\frac{S_f+S_{\bar{f}}}{2}\) is determined by weak phase \(2(\beta+\gamma)\) only whereas $\frac{S_f-S_{\bar{f}}}{2} = 0$. Assuming factorization for tree graphs, it is shown that the $B \rightarrow D^{(*)}$ form factors are in excellent agreement with heavy quark effective theory. From the experimental value for \(\left(\frac{S_f+S_{\bar{f}}}{2}\right)_{D^{(*)}\pi}\), the bound $\sin(2(\beta+\gamma)) \geq 0.69$ is obtained and \(\left(\frac{S_f+S_{\bar{f}}}{2}\right)_{D^{(*)}\pi} \approx -(0.41 \pm 0.08)\sin \gamma\) is predicted. For the decays described by the amplitudes $A_f \neq A'_{\bar{f}}$ such as $B^0 \rightarrow \rho^+\pi^- : A_f$ and $B^0 \rightarrow \rho^-\pi^+ : A'_{\bar{f}}$ where these amplitudes are given by tree and penguin diagrams with different weak phases, it is shown that in the limit $\delta_f, \delta'_{\bar{f}} \rightarrow 0, r_{f,\bar{f}} \cos \delta_f, \delta'_{\bar{f}} = \cos \alpha$ and $\frac{S_f}{S_{\bar{f}}} = \frac{S_f+\Delta S}{S-\Delta S} = -\frac{\sqrt{1-C_f^2}}{\sqrt{1-C_{\bar{f}}^2}}$.

1 Introduction

The CP asymmetries in the hadronic decays of B and K mesons involve strong final state phases. Thus strong interactions in these decays play a crucial role. The short distance strong interactions effects at quark level are taken care of by perturbative QCD in terms of Wilson coefficients. The CKM matrix, which connects the weak eigenstates with mass eigenstates, is another aspect of strong interactions at quark level. In the case of semi leptonic decays, the long distance strong interaction effects manifest themselves in the form factors of final states after hadronization. Likewise the strong interactions final state phases are long distance effects. These phase shifts essentially arise in terms of S-matrix which changes an 'in' state into an 'out' state viz.

\[ |f\rangle_{in} = S|f\rangle_{out} = e^{2i\delta_f}|f\rangle_{out} \]  \hspace{1cm} (1)
In fact, the CPT invariance of weak interaction Lagrangian gives for the weak decay \( B(\bar{B}) \to f(\bar{f}) \)

\[
\bar{A}_f \equiv \text{out} \langle \bar{f}|\mathcal{L}_W|\bar{B} \rangle = \eta_f e^{2i\delta_f} A_f^* \tag{2}
\]

Taking out the weak phase \( \phi \), the amplitude \( A_f \) can be written as

\[
A_f = e^{i\phi} F_f = e^{i\phi} e^{i\delta_f} |F_f|
\]

Then Eq. (2) implies

\[
\bar{A}_f = e^{-i\phi} e^{2i\delta_f} F_f^* = e^{-i\phi} F_f
\]

It is difficult to reliably estimate the final state strong phase shifts. It involves the hadronic dynamics. However, using isospin, C-invariance of S-matrix and unitarity, we can relate these phases. In this regard, following cases are of interest:

Case (i): The decays \( B^0 \to f, \bar{f} \) described by two independent single amplitudes \( A_f \) and \( A'_{\bar{f}} \) with different weak phases:

\[
\begin{align*}
A_f &= \langle f |\mathcal{L}_W| B^0 \rangle = e^{i\phi} F_f = e^{i\phi} e^{i\delta_f} |F_f| \\
A'_{\bar{f}} &= \langle \bar{f} |\mathcal{L}_W| B^0 \rangle = e^{i\phi'} F'_{\bar{f}} = e^{i\phi'} e^{i\delta_{\bar{f}}} |F'_{\bar{f}}|
\end{align*}
\]

where the states \( |\bar{f} \rangle \) and \( |f \rangle \) are C conjugate of each other such as states \( D^{(s)^-} \pi^+ (D^{(s)^+} \pi^-), D_s^{(s)^-} K^+(D_s^{(s)^+} K^-), D^- \rho^+ (D^+ \rho^-) \).

For case (i), there is an added advantage that the decay amplitudes \( A_f \) and \( A_{\bar{f}} \) are given by tree graphs. Assuming factorization for tree amplitudes, it is shown that the form factors \( f_0^{B-D}(m_0^2), A_0^{B-D}(m_0^2), f_{\perp}^{B-D}(m_\rho^2) \) obtained from the experimental branching ratios are in excellent agreement with Heavy Quark Effective Theory (HQET). Hence factorization assumption is experimentally on sound footing for these decays.

Case (ii): The weak amplitudes \( A_f \neq A_{\bar{f}} \),

\[
\begin{align*}
A_f &= \langle f |\mathcal{L}_W| B^0 \rangle = [e^{i\phi_1} F_1 f + e^{i\phi_2} F_2 f] \\
A_{\bar{f}} &= \langle \bar{f} |\mathcal{L}_W| B^0 \rangle = [e^{i\phi_1} F_1 \bar{f} + e^{i\phi_2} F_2 \bar{f}]
\end{align*}
\]

as is the case for the following decays,

\[
\begin{align*}
B^0 \to \rho^- \pi^+ (f) & : A_f, \quad B^0 \to \rho^+ \pi^- (\bar{f}) : A_{\bar{f}} \\
B_s^0 \to K^- \pi^+ & , \quad B_s^0 \to K^+ \pi^- \\
B^0_s \to D^{*-} D^+ & , \quad B^0_s \to D^{+*} D^- \\
B^0_s \to D^{*-} D^+_s & , \quad B^0_s \to D^{+*} D^-_s
\end{align*}
\]

The \( C^- \) invariance of S-matrix gives \( S_{\bar{f}} = S_f \) which implies

\[
\delta_f = \delta'_{\bar{f}}, \quad \delta_{1f} = \delta_{1\bar{f}}, \quad \delta_{2f} = \delta_{2\bar{f}}
\]

2 Unitarity and Final State Strong Phases

The time reversal invariance gives

\[
F_f = \text{out} \langle f |\mathcal{L}_W| B \rangle = \text{in} \langle f |\mathcal{L}_W| B \rangle^*	ag{4}
\]
where $\mathcal{L}_W$ is the weak interaction Lagrangian without the CKM factor such as $V_{ud}V_{ub}$. From Eq. (4), we have

$$F^*_f = \text{out} \langle f | S^\dagger \mathcal{L}_W | B \rangle = \sum_n S^*_n f_n$$  \tag{5}$$

It is understood that the unitarity equation which follows from time reversal invariance holds for each amplitude with the same weak phase. Above equation can be written in two equivalent forms:

1. Exclusive version of Unitarity \[1, 2\]

Writing

$$S_{nf} = \delta_{nf} + i M_{nf}$$ \tag{6}$$

we get from Eq (5),

$$\text{Im} F_f = \frac{1}{2} \sum_n M_{nf}^* F_n$$ \tag{7}$$

where $M_{nf}$ is the scattering amplitude for $f \rightarrow n$ and $F_n$ is the decay amplitude for $B \rightarrow n$. In this version, the sum is over all allowed exclusive channels. This version is more suitable in a situation where a single exclusive channel is dominant one. To get the final result, one uses the dispersion relation. In dispersion relation two particle unitarity gives dominant contribution. From Eq.(7), using two particle unitarity, we get \[1\],

$$\text{Disc } F(B \rightarrow f') \approx \frac{1}{16\pi s} \int_{-\infty}^{0} M_{f'f}^* F(B \rightarrow f) dt$$ \tag{8}$$

where $t = -2\bar{p}^2(1 - \cos \theta)$, $|\bar{p}| \approx \frac{1}{2}\sqrt{s}$. Eq.(8) is especially suitable to calculate rescattering corrections to color suppressed $T$-amplitude in terms of color favored $T$-amplitude as for example rescattering correction to color suppressed decay $B^0 \rightarrow \pi^0 D^0(f)$ in terms of dominant decay mode $B^0 \rightarrow \pi^+ D^- (f)$. Before using two particle unitarity in this form, it is essential to consider two particle scattering processes.

$SU(3)$ or $SU(2)$ and $C$-invariance of $S$-matrix can be used to express scattering amplitudes in terms of two amplitudes $M^+$ and $M^-$ which in terms of Regge trajectories are given by \[3, 4, 5\]

$$M^{(+)} = P + f + A_2 = -C_p \frac{e^{-i\pi \alpha_p(t)/2}}{\sin \pi \alpha_p(t)/2} \left( s/s_0 \right)^{\alpha(t)}$$
$$-2C_{\rho} \frac{1 + e^{-i\pi \alpha(t)}}{\sin \pi \alpha(t)} \left( s/s_0 \right)^{\alpha(t)}$$ \tag{9}$$

$$M^{(-)} = \rho + \omega = 2C_{\rho} \frac{1 - e^{-i\pi \alpha(t)}}{\sin \pi \alpha(t)} \left( s/s_0 \right)^{\alpha(t)}$$ \tag{10}$$

For linear Regge trajectories, using exchange degeneracy, we have

$$\alpha_p(t) = \alpha_{A_2}(t) = \alpha_{\omega}(t) = \alpha_f(t) = \alpha(0) + \alpha't,$$
$$\alpha_p(t) = \alpha_p(0) + \alpha'_p(t),$$
$$C_f = C_\omega; \ C_{A_2} = \frac{2}{3} C_\rho; \ C_\omega = C_\rho$$ \tag{11}$$
We take $\alpha_0 \approx 1/2$, $\alpha' \approx 1$ GeV$^{-2}$, $\alpha_p(0) \approx 1, \alpha_p' \approx 0.25$ GeV$^{-2}$. Using SU(3) and taking $\gamma_{D+D^{-}} = \gamma_{D+D^{-}}$, we get $C_D = \gamma_{D+D^{-}}\gamma_{D+D^{-}} = 2\gamma_0$; $\gamma_0 = \gamma_{D+D^{-}}; \gamma_0^2 \approx 72^3$. Hence for $\pi^+D^-$ or $\pi^-K^+$ scattering we get

$$M = M(+) + M(-) = iC_p e^{bt}(s/s_0) + 2\gamma_0^2 iC_p e^{i(\ln(s/s_0) - i\pi)}(s/s_0)^{1/2}$$

where $b = \alpha_p \ln(s/s_0)$

For $\pi^0D^0 \rightarrow \pi^+D^-, \pi^0K^0 \rightarrow \pi^-K^+$

$$M = \pm \sqrt{2}M(-) = \pm 2\sqrt{2}C_p e^{-\frac{i\alpha(t)}{2}}(s/s_0)\alpha(t)$$

From Eq.(8) and (13) with the use of dispersion relation, we obtain

$$A(B^0 \rightarrow \pi^0D^0)_{FSI} = \frac{\sqrt{2}\gamma_0^2(1-i)}{16\pi} A(B^0 \rightarrow \pi^+D^-) \frac{1}{\ln(m_B^2/s_0^2)} + i\pi/2 \int_{(m_B^2+m_D^2)^2}^{s} \frac{ds}{s-m_B^2} (s/s_0)^{\alpha(t)}$$

$$= -\sqrt{2}\epsilon A(B^0 \rightarrow \pi^+D^-) e^{i\theta}$$

We get $\epsilon \approx 0.06, \theta \approx 33^\circ$ by putting $s \approx m_B^2$ in ln$(s/s_0)$. Now $A(B^0 \rightarrow \pi^+D^-) = T$. Hence with rescattering correction

$$A(B^0 \rightarrow \pi^0D^0) = -\frac{1}{\sqrt{2}} C - \sqrt{2}\epsilon T e^{i\theta}$$

$$= -\frac{C}{\sqrt{2}} \left[ 1 + \frac{\epsilon}{b} e^{i\theta} \right]$$

where $2b = C/T$. Hence the final state phase shift $\delta_C$ for the color suppressed amplitude induced by the final state interaction is given by

$$\tan \delta_C = \frac{\epsilon/b \sin \theta}{1 + \epsilon/b \cos \theta} \rightarrow \delta_C \approx 8^\circ$$

with $b \approx 0.174$, which we get from

$$\frac{\Gamma(B^0 \rightarrow \pi^+D^-)}{\Gamma(B^+ \rightarrow \pi^+D^0)} = \frac{1}{(1 + 2b)^2} \approx 0.55 \pm 0.03$$

For $B^0 \rightarrow \pi^0K^0$, the color suppressed $T$-amplitude with rescattering correction is given by

$$-\frac{1}{\sqrt{2}} C + \sqrt{2}\epsilon T e^{i\theta} = -\frac{1}{\sqrt{2}} C \left[ 1 - \frac{\epsilon}{b} e^{i\theta} \right]$$

where $2b = C/T \approx 0.37$. Hence $\delta_C$ generated by the final state interaction is given by

$$\tan \delta_C = \frac{-\epsilon/b \sin \theta}{1 - \epsilon/b \cos \theta} \rightarrow \delta_C \approx -8^\circ$$

To conclude: The scattering amplitude $M(s,t)$ for the two particle final state obtained in eq.(13) is used in the unitarity equation to generate the final state strong phase by rescattering for the color suppressed tree amplitude.
2. Inclusive version of Unitarity \[2\]

This version is more suitable for our analysis. For this case, we write Eq. (5) in the form

\[ F^*_f S_{ff} F_f = \sum_{n\neq f} S^*_{nf} F_n \quad (20) \]

Parametrizing \( S_{ff} \equiv S = \eta e^{i\Delta} \), \( 0 \leq \eta \leq 1 \), we get after taking the absolute square of both sides of Eq. (20)

\[ |F|^2 \left[ (1 + \eta^2) - 2 \eta \cos 2(\delta_f - \Delta) \right] = \sum_{n,n'} F_n S^*_{nf} F_{n'} S_{n'f} \quad (21) \]

The above equation is an exact equation. In the random phase approximation \[2\], we can put

\[ \sum_{n',n\neq f} F_{n'} S^*_{n'f} F_{n} S_{nf} = \sum_{n\neq f} |F_n|^2 |S_{nf}|^2 \]

\[ = |F_n|^2 (1 - \eta^2) \quad (22) \]

We note that in a single channel description \[5, 8\]:

\[(F_{\text{flux}})_{\text{in}} - (F_{\text{flux}})_{\text{out}} = 1 - |\eta e^{i\Delta}|^2 = 1 - \eta^2 = \text{Absorption} \]

The absorption takes care of all the inelastic channels.

Similarly for the amplitude \( F_f \), we have

\[ F^*_f S_{ff} F_f = \sum_{\bar{n}\neq f} S^*_{nf} F_{\bar{n}} \quad (23) \]

The C-invariance of \( S \)-matrix gives:

\[ S_{fn} = \langle f | S | n \rangle = \langle f | C^{-1} CSC^{-1} C | n \rangle \]

\[ = \langle \bar{f} | S | \bar{n} \rangle = S_{f\bar{n}} \quad (24) \]

Thus in particular C-invariance of \( S \)-matrix gives

\[ S_{ff} = S_{\bar{f}\bar{f}} = \eta e^{2i\Delta} \quad (25) \]

Hence from Eq. (21), using Eqs. (22)–(25), we get

\[ \frac{1}{1 - \eta^2} \left[ (1 + \eta^2) - 2 \eta \cos 2(\delta_{f,\bar{f}} - \Delta) \right] = \rho^2, \bar{\rho}^2 \quad (26) \]

where

\[ \rho^2 = \frac{|F_n|^2}{|F_f|^2}, \quad \bar{\rho}^2 = \frac{|F_{\bar{n}}|^2}{|F_f|^2} \quad (27) \]

From Eq. (26), we get

\[ \sin(\delta_{f,\bar{f}} - \Delta) = \pm \sqrt{\frac{1 - \eta^2}{3\eta}} \left[ \rho^2, \bar{\rho}^2 - \frac{1 - \eta}{1 + \eta} \right]^{1/2} \quad (28) \]
The maximum value for $\rho^2, \bar{\rho}^2$ is 1 and the minimum value for them is $\frac{1-\eta}{1+\eta}$. Hence we get the following bounds:

$$\frac{1-\eta}{1+\eta} \leq \rho^2, \bar{\rho}^2 \leq 1$$

$$0 \leq \delta_{f,f} - \Delta \leq \theta$$

$$-\theta \leq \delta_{f} - \Delta \leq 0$$

(29)

$$\theta = \sin^{-1} \frac{1-\eta}{2}$$

(30)

From now on, we will confine ourself to positive square root in Eq.(28).

The strong interaction parameter $\Delta$ and $\eta$ in the above bounds can be obtained from the scattering amplitude $M(s, t)$ given in Eq.(12) obtain from Regge pole analysis. The $s$–wave scattering amplitude $f$ is given by

$$f \approx \frac{1}{16\pi s} \int_{-s}^{0} M(s, t)$$

(31)

For the scattering amplitude $M = M^+ + M^-$ relevant for $\pi^+D^-, \pi^-K^+$ and $\pi^+\pi^-$, we obtain from Eq.(31) using Eq.(12)

$$f = f_P + f_{\rho} = \frac{1}{16\pi s} \frac{i C_P}{b} \left( \frac{s}{s_0} \right) + \frac{1}{16\pi} \frac{\gamma_0}{\ln(s/s_0) - i\pi}(s/s_0)^{-1/2}$$

$$= \begin{bmatrix} 0.12i + (-0.08 + 0.08i) \\ 0.17i + (-0.08 + 0.08i) \\ 0.16i + (-0.16 + 0.16i) \end{bmatrix}$$

(32)

(33)

where we have used $s \approx m_B^2 \approx (5.27)^2$ GeV$^2$. For $C_P$ we have used the values of reference [2] whereas for $C_\rho = \gamma_{\rho\pi^+\pi^-} - \gamma_{\rho K^+K^-} = \gamma_{\rho\pi^+\pi^-} - \gamma_{\rho D^+D^-} = \frac{1}{2} \gamma_0^2$ and $C_\rho = \gamma_{\rho\pi^+\pi^-} - \gamma_{\rho\pi^+\pi^-} = \gamma_0^2 \approx 72$ for $\pi D, \pi K$ and $\pi\pi$ respectively.

Using the relation $S = \eta e^{2i\Delta} = 1 + 2if$, where $f$ is given by Eq.(33), the phase shift $\Delta$, the parameter $\eta$ and the phase angle $\theta$ can be determined. One gets

$$\pi^+D^- (\pi^-D^+) : \Delta \approx -7^\circ, \eta \approx 0.62, \theta \approx 26^\circ$$

$$\pi^-K^+ \text{ or } \pi^0 K^0 : \Delta \approx -9^\circ, \eta \approx 0.52, \theta \approx 29^\circ$$

$$\pi^+\pi^- : \Delta \approx -21^\circ, \eta \approx 0.48, \theta \approx 31^\circ$$

(34)

Hence we get the following bounds

$$\pi^+D^- (\pi^-D^+) : 0 \leq \delta_{f,f} - \Delta \leq 26^\circ$$

$$\pi^-K^+ \text{ or } \pi^0 K^0 : 0 \leq \delta_{f} - \Delta \leq 29^\circ$$

$$\pi^+\pi^- : 0 \leq \delta_{f} - \Delta \leq 31^\circ$$

(35)

Further we note that for these decays, $b$-quark is converted into $c$ or $u$ quark : $b \rightarrow c(u) + \bar{u} + d(s)$. In particular for the tree graph, the configuration is such that $\bar{u}$ and $d(s)$ essentially go together into a color singlet state with the third quark $c(u)$ recoiling; there is a significant probability that
the system will hadronize as a two body final state \[9\]. This physical picture has been put on the strong theoretical basis \[10, 11\], where in these references the QCD factorization have been proved. For the tree amplitude, factorization implies \(\delta_f^T = 0\). We, therefore take the point of view that effective final state phase shift is given by \(\delta_f - \Delta\). We take the lower bound for the tree amplitude so that final state effective phase shift \(\delta_f^T = 0\). Thus for \(\pi^+D^- (\pi^- D^+)\), \(\delta_f^T = 0\).

The decay \(B^0 \rightarrow \pi^- K^+\) is described by two amplitudes \[7\]

\[
A(B^0 \rightarrow \pi^- K^+) = - [P + e^{i\gamma} T] = |P| \left[ 1 - r e^{i(\gamma + \delta_{++})} \right]
\]

where

\[
P = -|P| e^{-i\delta_P}, \ T = |T| e^{i\delta_T}, \ \delta_{++} = \delta_P, \ r = \frac{|T|}{|P|}
\]

The decay \(B^0 \rightarrow \pi^0 K^0\) is described by the two amplitudes \[7\]

\[
A(B^0 \rightarrow \pi^0 K^0) = - \frac{1}{\sqrt{2}} |P| \left[ 1 + r_0 e^{i(\gamma + \delta_{00})} \right]
\]

where

\[
C = |C| e^{i\delta_C}, \ \delta_{00} = \delta_C + \delta_P, \ r_0 = \frac{|C|}{|P|}
\]

For these decays, we use the lower bounds in Eq.(35) for the tree amplitude so that the effective final state phase \(\delta_f = 0\). The phase \(\delta_C\) is generated by rescattering correction and its value is -8°. For the direct \(CP\) asymmetries, the relevant phases are \(\delta_{++}\) and \(\delta_{00}\). For the penguin amplitude, we assume that the effective final state phase \(\delta_P\) has the value near the upper bound. Thus we have \(\delta_{++} \approx 29°\), \(\delta_{00} \approx 21°\).

Now \[7\]

\[
A_{CP}(B^0 \rightarrow \pi^- K^+) = - \frac{2r \sin \gamma \sin \delta_{++}}{R}
\]

\[
R = 1 - 2r \cos \gamma \cos \delta_{++} + r^2_{++}
\]

Neglecting the terms of order \(r^2\), we have

\[
\tan \gamma \tan \delta_{++} = \frac{-A_{CP}(B^0 \rightarrow \pi^- K^+)}{1 - R}
\]

For \(B^0 \rightarrow \pi^0 K^0\)

\[
A_{CP}(B^0 \rightarrow \pi^0 K^0) = (R_0 - 1) \tan \gamma \tan \delta_{00}
\]

\[
R_0 = 1 + 2r_0 \cos \gamma \cos \delta_{00} + r^2_{00}
\]

Now the experimental values of \(A_{CP}, R\) and \(R_0\) are \[12\]

\[
A_{CP}(B^0 \rightarrow \pi^- K^+) = -0.101 \pm 0.015 \ ( -0.097 \pm 0.012)
\]

\[
A_{CP}(B^0 \rightarrow \pi^0 K^0) = -0.14 \pm 0.11 \ ( -0.00 \pm 0.10)
\]

\[
R = 0.899 \pm 0.048
\]

\[
R_0 = 0.908 \pm 0.068
\]
where the numerical values in the bracket are the latest experimental values as given in ref [7]. With \( \delta_{+-} \approx 29^\circ \), we get from Eq.(39), \( \gamma = (60 \pm 3)^\circ \). However for \( \delta_{+-} \approx 20^\circ \) which one gets from Eq.(38) for \( \rho^2 = 0.65 \), \( \gamma = (69 \pm 3)^\circ \). We obtain the following values for \( A_{CP}(B^0 \rightarrow \pi^0 K^0) \) from Eqs. (39) and (40)

\[
A_{CP}(B^0 \rightarrow \pi^0 K^0) = \frac{(1 - R_0) \tan \delta_{00}}{(1 - R) \tan \delta_{+-}} A_{CP}(B^0 \rightarrow \pi^- K^+)
\]

\[
= \begin{cases} 
-0.06 \pm 0.01, & \delta_{+-} = 29^\circ \\
-0.05 \pm 0.01, & \delta_{+-} = 20^\circ \\
\delta_{00} = 21^\circ 
\end{cases}
\]

We conclude: The phase shift \( \delta_{+-} \approx (20 - 29)^\circ \) for \( \pi^- K^+ \) is compatible with experimental value of the direct \( CP \)-asymmetry for \( \pi^- K^+ \) decay mode. For \( \pi^+\pi^- \), \( \delta_{+-} \approx 31^\circ \) is compatible with the value \( (33 \pm 7^{+8}_{-10})^\circ \) obtained by the authors of ref.[7]. Finally we note that the actual value of the effective phase shift \( (\delta_f - \Delta) \) depends on one free parameter \( \rho \), factorization implies \( \delta_f^T = 0 \) i.e. \( \delta_f - \Delta = 0 \) for the tree amplitude; for the penguin amplitude, \( \delta_f^P \) depends on \( \rho \). However, from the experimental values of the direct \( CP \)-violation for \( \pi^- K^+ \), \( \pi^+\pi^- \), it is near the upper bound.

Finally we note that \( \pi^+D^- (\pi^- D^+) \), \( \pi^- K^+ \), \( \pi^-\pi^+ \) decays are \( s \)-wave decay whereas \( B^0 \rightarrow \rho^+\pi^- (\rho^-\pi^+) \) decays are \( p \)-wave decays. For \( p \)-wave, the decay amplitude

\[
f = \frac{1}{16\pi s} \int_{-s}^{s} M(s,t)(1 + \frac{2t}{s}) dt
\]

\[
= \frac{1}{16\pi s} iC_P \left[ \frac{1}{b} + \frac{2}{b^2} \right] (s/s_0)
\]

\[
+ \frac{2\gamma_0^2}{16\pi} i \left[ \frac{1}{\ln(s/s_0) - i\pi} - \frac{2}{s} \frac{1}{\ln(s/s_0) - i\pi}^2 (s/s_0)^{-1/2} \right]
\]

\[
\approx \frac{1}{16\pi s} iC_P \frac{1}{b} (s/s_0) + \frac{2\gamma_0^2}{16\pi} i \frac{1}{\ln(s/s_0) - i\pi} (s/s_0)^{-1/2} + O \left( \frac{1}{s} \right)
\]

to be compared with Eq.(32). Now for the \( B \rightarrow \rho \pi \) decay, only longitudinal polarization of \( \rho \) is effectively involved. Since the longitudinal \( \rho \)-meson emulates a pseudoscalar meson and if we assume same couplings as for pions, we conclude that the final state phase for \( \rho \pi \) should be of the order \( 30^\circ \); in any case it should not be greater than \( 30^\circ \). The upper bound \( \delta_f \leq 30^\circ \) can be used to select the several possible solutions in Table-2 [Section-4] obtained from the analysis of weak decays \( B \rightarrow \rho^+\pi^- (\rho^-\pi^+) \).

### 3 CP Asymmetries and Strong Phases

In this section, we discuss the experimental tests to verify the equality (implied by C-invariance of \( S \)-matrix) of phase shifts \( \delta_f \) and \( \delta_f \) for the weak decays of B mesons mentioned in section 1.
It is convenient to write the time-dependent decay rates in the form [13, 6]

\[ [\Gamma_f(t) - \bar{\Gamma}_f(t)] + [\Gamma_f - \bar{\Gamma}_f(t)] \]

\[ = e^{-rt} \left\{ \cos \Delta mt \left[ (|A_f|^2 - |\bar{A}_f|^2) + (|A_f|^2 - |\bar{A}_f|^2) \right] + 2 \sin \Delta mt \left[ \text{Im} \left( e^{2i\phi_M A_f^* \bar{A}_f} \right) + \text{Im} \left( e^{2i\phi_M A_f^* \bar{A}_f} \right) \right] \right\} \]

\[ \text{(41)} \]

\[ [\Gamma_f(t) - \bar{\Gamma}_f(t)] + [\Gamma_f(t) - \bar{\Gamma}_f(t)] \]

\[ = e^{-rt} \left\{ \cos \Delta mt \left[ (|A_f|^2 - |\bar{A}_f|^2) - \left( |A_f|^2 + |\bar{A}_f|^2 \right) \right] + 2 \sin \Delta mt \left[ \text{Im} \left( e^{2i\phi_M A_f^* \bar{A}_f} \right) - \text{Im} \left( e^{2i\phi_M A_f^* \bar{A}_f} \right) \right] \right\} \]

\[ \text{(42)} \]

**Case (i):** Eqs. (41) and (42) give

\[ \mathcal{A}(t) \equiv \frac{[\Gamma_f(t) - \bar{\Gamma}_f(t)] + [\Gamma_f(t) - \bar{\Gamma}_f(t)]}{[\Gamma_f(t) + \bar{\Gamma}_f(t)] + [\Gamma_f(t) + \bar{\Gamma}_f(t)]} \]

\[ = \frac{2|F_f||F'_f|}{|F_f|^2 + |F'_f|^2} \sin \Delta mt \sin \left( 2\phi_M - \phi - \phi' \right) \cos (\delta_f - \delta'_f) \]

\[ \mathcal{F}(t) \equiv \frac{[\Gamma_f(t) - \bar{\Gamma}_f(t)] - [\Gamma_f(t) + \bar{\Gamma}_f(t)]}{[\Gamma_f(t) + \bar{\Gamma}_f(t)] + [\Gamma_f(t) + \bar{\Gamma}_f(t)]} \]

\[ = \frac{|F_f|^2 - |F'_f|^2}{|F_f|^2 + |F'_f|^2} \cos \Delta mt - \frac{2|F_f||F'_f|}{|F_f|^2 + |F'_f|^2} \sin \Delta mt \cos \left( 2\phi_M - \phi - \phi' \right) \sin (\delta_f - \delta'_f) \]

\[ \text{(43)} \]

\[ \text{(44)} \]

The effective Lagrangians \( \mathcal{L}_W \) and \( \mathcal{L}_W' \) are given by \((q = d, s)\)

\[ \mathcal{L}_W = V_{cb} V_{ub}^* \bar{q} \gamma^\mu (1 - \gamma^5) u [\bar{c} \gamma^\mu (1 - \gamma^5) b] \]

\[ \mathcal{L}_W' = V_{ub} V_{cd}^* \bar{q} \gamma^\mu (1 - \gamma^5) c [\bar{u} \gamma^\mu (1 - \gamma^5) b] \]

\[ \text{(45)} \]

\[ \text{(46)} \]

Hence for these decays

\[ \phi = 0, \quad \phi' = \gamma \]

and

\[ \phi_M \begin{cases} -\beta, & \text{for } B^0 \\ -\beta_s, & \text{for } B^0_s \end{cases} \]

\[ A_f = \langle D^- \pi^+ | \mathcal{L}_W | B^0 \rangle = F_f \]

\[ A_f = \langle D^+ \pi^- | \mathcal{L}_W' | B^0 \rangle = e^{i\gamma} F_f \]

\[ A_{fs} = \langle K^+ D^- | \mathcal{L}_W | B^0_s \rangle = F_{fs} \]

\[ A_{fs} = \langle K^- D^+ | \mathcal{L}_W' | B^0_s \rangle = e^{i\gamma} F_{fs} \]

\[ \text{(47)} \]

\[ \text{(48)} \]
Thus, we get from Eqs. (43) – (48) for $B^0$ decays,

$$A(t) = -\frac{2r_D}{1 + r_D^2} \sin \Delta m_B t \sin (2\beta + \gamma) \cos \left(\delta_f - \delta'_f\right)$$

$$F(t) = \frac{1 - r_D^2}{1 + r_D^2} \cos \Delta m_B t - \frac{2r_D}{1 + r_D^2} \sin \Delta m_B t \cos (2\beta + \gamma) \sin \left(\delta_f - \delta'_f\right)$$

(49)

$$A = -\frac{2r_D}{1 + r_D^2} \sin(2\beta + \gamma) \frac{\Delta m_B / \Gamma}{1 + (\Delta m_B / \Gamma)^2} \cos(\delta_f - \delta'_f)$$

(50)

where

$$r_D = \lambda^2 R_b \left|\frac{F'_f}{F_f}\right|$$

(51)

For the decays,

$$B^0_s (B^0_s) \rightarrow D^+_s K^- (D^-_s K^+)$$

$$B^0_s (B^0_s) \rightarrow D^-_s K^+ (D^+_s K^-)$$

we get,

$$A_s(t) = -\frac{2r_D}{1 + r_D^2} \sin \Delta m_{B_s} t \sin (2\beta_s + \gamma) \cos \left(\delta_{f_s} - \delta'_{f_s}\right)$$

$$F_s(t) = \frac{1 - r_D^2}{1 + r_D^2} \cos \Delta m_{B_s} t - \frac{2r_D}{1 + r_D^2} \sin \Delta m_{B_s} t \cos (2\beta_s + \gamma) \sin \left(\delta_{f_s} - \delta'_{f_s}\right)$$

(52)

where

$$r_{D_s} = R_b \left|\frac{F'_{f_s}}{F_{f_s}}\right|$$

(53)

We note that for time integrated $CP$-asymmetry,

$$A_s = \int_0^\infty \frac{\Gamma_{f_s}(t) - \bar{\Gamma}_{f_s}(t)}{\Gamma_{f_s}(t) + \bar{\Gamma}_{f_s}(t)} dt$$

$$= -\frac{2r_{D_s} r}{1 + r_{D_s}^2} \sin (2\beta_s + \gamma) \frac{\Delta m_{B_s}/\Gamma_s}{1 + (\Delta m_{B_s}/\Gamma_s)^2} \cos(\delta_{f_s} - \delta'_{f_s})$$

(54)

The experimental results for the B decays are as follows [12]

$$\begin{align*}
S_{- + S_+} : & -0.046 \pm 0.023 & -0.037 \pm 0.012 & -0.024 \pm 0.031 \pm 0.009 \\
S_{- - S_+} : & -0.022 \pm 0.021 & -0.006 \pm 0.016 & -0.098 \pm 0.055 \pm 0.018 
\end{align*}$$

(55)

where

$$\frac{S_{- + S_+}}{2} = -\frac{2r_D}{1 + r_D^2} \sin(2\beta + \gamma) \cos(\delta_f - \delta'_f)$$

$$\frac{S_{- - S_+}}{2} = -\frac{2r_D}{1 + r_D^2} \cos(2\beta + \gamma) \sin(\delta_f - \delta'_f)$$

(56)
For $B^0 \rightarrow D_s^- K^+, D_s^- K^+, D_s^- K^{*+}$, replace $r_D \rightarrow r_s$, $\beta \rightarrow \beta_s$, $\delta_f \rightarrow \delta_{fs}$, $\delta'_f \rightarrow \delta'_{fs}$ in Eq. (56).

Since for $B_s^0$, in the standard model, with three generations, gives $\beta_s = 0$, so we have for the CP-asymmetries $\sin \gamma$ or $\cos \gamma$ instead of $\sin(2\beta + \gamma)$, $\cos(2\beta + \gamma)$. Hence $B_s^0$-decays are more suitable for testing the equality of phase shifts $\delta_{fs}$ and $\delta'_{fs}$ as for this case neither $r_s$ nor $\cos \gamma$ is suppressed as compared to the corresponding quantities for $B^0$. To conclude, for $B^0_s$ decays, the equality of phases $\delta_f$ and $\delta'_f$ for $B^0_d$ gives

$$-\frac{S_- + S_+}{2} = 2r_D \sin(2\beta + \gamma)$$
$$-\frac{S_- - S_+}{2} = 0$$

whereas for $B^0_s$ decays, we get

$$-\frac{S_- + S_+}{2} = \frac{2r_{DK}}{1 + r_{DK}^2} \sin(2\beta + \gamma)$$
$$-\frac{S_- - S_+}{2} = 0$$

Corresponding to the decays $B^0_s \rightarrow D_s^- K^+, D^+_s K^-$ described by the tree diagrams, we have the color suppressed decays $B^0 \rightarrow \bar{D}^0 K^0, D^0 K^0$. For these decays,

$$-\frac{S_- + S_+}{2} = 2r_{DK} \sin(2\beta + \gamma) \cos(\delta_{D^0 K^0} - \delta'_{D^0 K^0})$$
$$-\frac{S_- - S_+}{2} = 2r_{DK} \cos(2\beta + \gamma) \sin(\delta_{D^0 K^0} - \delta'_{D^0 K^0})$$
$$r_{DK} = R_b \left| \frac{C_{D^0 K^0}}{C_{D^0 K^0}} \right|$$

and the corresponding expression for $B^0_s \rightarrow \bar{D}^0 \phi, D^0 \phi$. For the color suppressed decays $B^0 \rightarrow D^0 \pi^0, D^0 \pi^0$, we get similar expression as for $B^0 \rightarrow D^- \pi^+, D^+ \pi^-$, with

$$r_D \equiv r_{D^- \pi^-}, \delta_{D^- \pi^-}, \delta'_{D^- \pi^-} \quad \text{replaced by} \quad r_{D^0 \pi^0}, \delta_{D^0 \pi^0}, \delta'_{D^0 \pi^0}$$

To determine the parameter $r_D$ or $r_{D_s}$, we assume factorization for the tree amplitude [7]. Factorization gives for the decays $B^0 \rightarrow D^+ \pi^-, D^{*-} \pi^-, D^+ \rho^-, D^+ a_1^-$:

$$|\tilde{f}_f| = |\tilde{T}_f| = |G(f_\pi(m_B^2 - m_B^2)f_0^{B-D}(m_\pi^2), 2f_\pi m_B|\bar{p}|A_0^{B-D^*}(m_\pi^2)|$$
$$2f_\rho m_B|\bar{p}|f_0^{B-D}(m_\rho^2), 2f_\pi m_B|\bar{p}|f_0^{1/2}(a_1^2)|$$

$$|\tilde{f}_f'| = |\tilde{T}_f'| = |G(f_\pi(m_B^2 - m_B^2)f_0^{B-D}(m_\pi^2), 2f_\pi m_B|\bar{p}|A_0^{B-D^*}(m_\pi^2)|$$
$$2f_\rho m_B|\bar{p}|A_0^{B-D^*}(m_B^2), 2f_\pi m_B|\bar{p}|A_0^{B-D^*}(m_B^2)|$$

$$G = \frac{G_F}{\sqrt{2}} |V_{ud}| |V_{cs}| a_1, \quad G' = \frac{G_F}{\sqrt{2}} |V_{ud}| |V_{ub}|$$

The decay widths for the above channels are given in the table 1 where we have used

$$a_1^2 |V_{ud}|^2 \approx 1, \quad f_\pi = 131 \text{MeV}, \quad 11 f_\rho = 209 \text{MeV}, \quad f_{a_1} = 229 \text{MeV}$$
| Decay     | Decay Width ($10^{-9}$ MeV × | Form Factor | Form Factors $h(w^{(*)})$ |
|-----------|--------------------------------|-------------|---------------------------|
| $B^0 \rightarrow D^ππ^-$ | $(2.281)|f_{B^0→D^π}(m^2_π)|^2$ | $0.58 \pm 0.05$ | $0.51 \pm 0.03$ |
| $B^0 \rightarrow D^{π+}π^-$ | $(2.129)|A_{B^0→D^{π+}}(m^2_{π})|^2$ | $0.61 \pm 0.04$ | $0.54 \pm 0.03$ |
| $B^0 \rightarrow D^\astρ^-$ | $(5.276)|f_{B^0→D^\ast}(m^2_\rho)|^2$ | $0.65 \pm 0.11$ | $0.57 \pm 0.10$ |
| $B^0 \rightarrow D^\ast a_1^-$ | $(5.414)|f_{B^0→D^\ast}(m^2_{a_1})|^2$ | $0.57 \pm 0.31$ | $0.50 \pm 0.27$ |

Table 1: Form Factors

Using the experimental branching ratios and $^{12}$

$$|V_{cb}| = (38.3 \pm 1.3) \times 10^{-3}$$ (62)

we obtain the corresponding form factors given in Table 1.

In terms of variables $^{14, 15}$:

$$ω = v \cdot v', \quad v^2 = v'^2 = 1, \quad t = q^2 = m^2_B + m^2_D - 2m_Bm_Dω$$ (63)

the form factors can be put in the following form

$$f_{B^0→D^π}(t) = \frac{m_B + m_D}{2\sqrt{m_Bm_D}}h_+(ω), \quad f_{B^0→D^{π+}}(t) = \frac{\sqrt{m_Bm_D}}{m_B + m_D}(1 + ω)h_0(ω)$$

$$A_{2→D^\ast}(t) = \frac{m_B + m^{D^\ast}}{2\sqrt{m_Bm^{D^\ast}}}(1 + ω)h_A(ω), \quad A_{B^0→D^\ast}(t) = \frac{m_B + m^{D^\ast}}{2\sqrt{m_Bm^{D^\ast}}}h_{A0}(ω)$$

$$A_{1→D^\ast}(t) = \frac{\sqrt{m_Bm^{D^\ast}}}{m_B + m^{D^\ast}}(1 + ω)h_{A1}(ω)$$ (64)

Heavy Quark Effective Theory (HQET) gives $^{14, 15}$:

$$h_+(ω) = h_0(ω) = h_{A0}(ω) = h_{A1}(ω) = h_{A2}(ω) = ζ(ω)$$

where $ζ(ω)$ is the form factor, with normalization $ζ(1) = 1$. For

$$t = m^2_π, m^2_\rho, m^2_{a_1}$$

$$ω^{(*)} = 1.589(1.504), 1.559, 1.508$$ (65)

In reference $^{16}$, the value quoted for $h_{A1}(ω_{max}^{*})$ is

$$|h_{A1}(ω_{max}^{*})| = 0.52 \pm 0.03$$ (66)

Since $ω_{max}^{*} = 1.504$, the value for $|h_{A1}(ω_{max}^{*})|$ obtained in Table 1 is in remarkable agreement with the value given in Eq. (66) showing that factorization assumption for $B^0 → πD^{(*)}$ decays is experimentally on solid footing and is in agreement with HQET.

From Eqs. (59) and (60), we obtain

$$r_D = \frac{λ^2R_b|T^f_T|}{|T^f_T|} = \frac{λ^2R_b}{|T^f_T|}$$

$$\left[ \frac{f_D(m^2_B - m^2_π)f_{B^0→π}(m^2_D)}{f_D(m^2_B - m^2_π)f_{B^0→π}(m^2_D)}, \frac{f_Df_{B^0→π}(m^2_D)}{f_Df_{B^0→π}(m^2_D)}, \frac{f_Df_{B^0→π}(m^2_D)}{f_Df_{B^0→π}(m^2_D)}, \frac{f_Df_{B^0→π}(m^2_D)}{f_Df_{B^0→π}(m^2_D)} \right]$$ (67)
where
\[
\frac{|V_{ub}| |V_{cd}|}{|V_{cb}| |V_{ud}|} = \lambda^2 R_b \approx (0.227)^2(0.40) \approx 0.021
\] (68)

To determine \( r_D \), we need information for the form factors \( f_B^{-\pi}(m^2_D) \), \( f_B^{+\pi}(m^2_D) \), \( A_{b^0}(m^2_D) \). For these form factors, we use the following values [17, 18]:

\[
A_{b^0}(0) = 0.30 \pm 0.03, A_{b^0}(m^2_D) = 0.38 \pm 0.04
\]

\[
f_B^{+\pi}(0) = f_B^{-\pi}(0) = 0.26 \pm 0.04, \quad f_B^{-\pi}(m^2_D) = 0.32 \pm 0.05, \quad f_B^{+\pi}(m^2_D) = 0.28 \pm 0.04
\]

Along with the values of remaining form factors given in Table 1, we obtain

\[
r_{D^{(*)}} = [0.018 \pm 0.002, \quad 0.017 \pm 0.003, \quad 0.012 \pm 0.002]
\] (69)

The above value for \( r_{D^{(*)}} \) gives

\[
- \left( \frac{S_+ + S_-}{2} \right)_{D^{*\pi}} = 2(0.017 \pm 0.003) \sin(2\beta + \gamma)
\] (70)

The experimental value of the CP asymmetry for \( B^0 \rightarrow D^{*\pi} \) decay has the least error. Hence we obtain the following bounds

\[
\sin(2\beta + \gamma) > 0.69 \quad (71)
\]

\[
44^\circ \leq (2\beta + \gamma) \leq 90^\circ \quad (72)
\]

or

\[
90^\circ \leq (2\beta + \gamma) \leq 136^\circ \quad (73)
\]

Selecting the second solution, and using \( 2\beta \approx 43^\circ \), we get

\[
\gamma = (70 \pm 23)^\circ
\] (74)

Further, we note that the factorization for the decay \( \bar{B}^0 \rightarrow D_{s^0}^{-\pi^+} \) gives

\[
\bar{T} = |V_{ub}| |V_{cs}| f_{D_{s^0}}^2 m_B |\bar{p}| f_B^{-\pi}(m^2_{D_{s^0}})
\] (75)

Using the experimental branching ratio for this decay, we get

\[
\left( \frac{f_{D_{s^0}}}{f_{\pi}} \right)^2 | \frac{f_B^{-\pi}(m^2_{D_{s^0}})}{f_B^{-\pi}(0)} |^2 = 7.7 \pm 1.9
\] (76)

On using

\[
\frac{f_B^{+\pi}(0)}{f_B^{+\pi}(m^2_{D_{s^0}})} = 0.77 \pm 0.09
\] (77)

we get

\[
f_{D_{s^0}} = 279 \pm 79MeV
\] (78)

Similar analysis for \( \bar{B}^0 \rightarrow D_{s}^{-\pi^+} \) gives

\[
\left( \frac{f_{D_{s}}}{f_{\pi}} \right)^2 | \frac{f_0^{B^{-\pi}}(m^2_{D_{s}})}{f_0^{B^{-\pi}}(0)} |^2 = 2.72 \pm 0.64
\] (79)
On using
\[
\frac{f_0^{B_s^0}(0)}{f_0^{B_s^0}(m_{D_s^2})} = 0.93 \pm 0.05
\]
we get
\[
f_{D_s} = 201 \pm 47 \text{MeV}
\] (80)

Finally from the experimental branching ratio for the decay \( B_s^0 \to D_s^+ \pi^- \), we obtain
\[
f_0^{B_s^0}(0) = 0.62 \pm 0.18
\] (81)

To end this section, we discuss the decays \( B_s^0 \to D_s^{(*)}K^- \), \( D_s^{(*)}K^- \) for which no experimental data are available. However, using factorization, we get
\[
\Gamma(B_s^0 \to D_s^+ K^-) = (1.75 \times 10^{-10}) |V_{cb} f_0^{B_s^0-D_s^0}(m_{D_s}^2)|^2 \text{MeV}
\] (82)

\[
\Gamma(B_s^0 \to D_s^{(*)} K^-) = (1.57 \times 10^{-10}) |V_{cb} A_0^{B_s^0-D_s^{(*)}}(m_{K}^2)|^2 \text{MeV}
\] (83)

SU(3) gives
\[
|V_{cb} f_0^{B_s^0-D_s^0}(m_{K}^2)|^2 \approx |V_{cb}| |f_0^{B-D}(m_{\pi}^2)|^2 = (0.50 \pm 0.04) \times 10^{-3}
\]

\[
|V_{cb} A_0^{B_s^0-D_s^{(*)}}(m_{K}^2)|^2 \approx |V_{cb}| |A_0^{B-D^{(*)}}(m_{\pi}^2)|^2 = (0.56 \pm 0.04) \times 10^{-3}
\] (84)

From the above equations, we get the following branching ratios
\[
\frac{\Gamma(B_s^0 \to D_s^{(*)} K^-)}{\Gamma(B_s^0)} = (1.94 \pm 0.07) \times 10^{-4}[1.96 \pm 0.07] 
\] (85)

For \( B_s^0 \to D_s^{(*)} K^- \)
\[
r_{D_s} = \frac{f_{D_s^*} f_0^{B_s^0-D_s^0}(m_{D_s}^2)}{f_K A_0^{B_s^0-D_s^{(*)}}(m_{K}^2)}
\] (86)

Hence we get
\[
-(S_+ + S_-)_{D_s K} = (0.41 \pm 0.08) \sin(2\beta_s + \gamma)
\]

\[
= (0.41 \pm 0.08) \sin \gamma
\] (87)

where we have used
\[
R_b = 0.40, \quad \frac{f_{D_s^*}}{f_{K^*}} = 1.75 \pm 0.06, \quad \frac{f_0^{B_s^0-D_s^0}(m_{D_s}^2)}{f_K} = 0.34 \pm 0.06
\]

\[
A_0^{B_s^0-D_s^0}(m_{K}^2) = A_0^{B_s^0-D_s^{(*)}}(0) = \frac{m_{B_s} + m_{D_s^*}}{2 \sqrt{m_{B_s} m_{D_s^*}}} [h_0(\omega_s^* = 1.453) = 0.52 \pm .03]
\]

\[
= 0.58 \pm 0.03
\] (88)
4 CP Asymmetries for $A_f \neq A_{\bar{f}}$

We now discuss the decays listed in case (ii) where $A_f \neq A_{\bar{f}}$. Subtracting and adding Eqs. (42) and (11), we get,

$$\frac{\Gamma_f(t) - \bar{\Gamma}_f(t)}{\Gamma_f(t) + \bar{\Gamma}_f(t)} = C_f \cos \Delta mt + S_f \sin \Delta mt$$

$$= (C - \Delta C) \cos \Delta mt + (S - \Delta S) \sin \Delta mt$$

(91)

$$\frac{\bar{\Gamma}_f(t) - \bar{\Gamma}_{\bar{f}}(t)}{\bar{\Gamma}_f(t) + \bar{\Gamma}_{\bar{f}}(t)} = C_{\bar{f}} \cos \Delta mt + S_{\bar{f}} \sin \Delta mt$$

$$= (C + \Delta C) \cos \Delta mt + (S + \Delta S) \sin \Delta mt$$

(92)

where

$$C_{f,\bar{f}} = (C \pm \Delta C)$$

$$= \frac{|A_{f,\bar{f}}|^2 - |\bar{A}_{f,\bar{f}}|^2}{|A_{f,\bar{f}}|^2 + |\bar{A}_{f,\bar{f}}|^2}$$

$$= \frac{\Gamma_{f,\bar{f}} - \bar{\Gamma}_{f,\bar{f}}}{\Gamma_{f,\bar{f}} + \bar{\Gamma}_{f,\bar{f}}}$$

$$= \frac{R_{f,\bar{f}}(1 - A_{\bar{C}}^{f,\bar{f}}) - R_{f,\bar{f}}(1 + A_{\bar{C}}^{f,\bar{f}})}{\Gamma(1 \pm A_{\bar{C}}^{f,\bar{f}})}$$

(93)

$$S_{f,\bar{f}} = (S \pm \Delta S)$$

$$= \frac{2 \text{Im}[e^{2i\phi M} A_{f,\bar{f}}^* \bar{A}_{f,\bar{f}}]}{\Gamma_{f,\bar{f}} + \bar{\Gamma}_{f,\bar{f}}}$$

(94)

$$A_{\bar{C}}^{f,\bar{f}} = \frac{\bar{\Gamma}_f - \Gamma_f}{\Gamma_f + \bar{\Gamma}_f}$$

$$A_{\bar{C}}^f = \frac{\bar{\Gamma}_f - \Gamma_f}{\Gamma_f + \bar{\Gamma}_f}$$

(96)

$$A_{\bar{C}} = \frac{(\Gamma_f + \bar{\Gamma}_f) - (\bar{\Gamma}_f + \Gamma_f)}{(\Gamma_f + \bar{\Gamma}_f) - (\bar{\Gamma}_f + \Gamma_f)}$$

$$= \frac{R_f A_{\bar{C}}^{f,\bar{f}} - R_{\bar{f}} A_{\bar{C}}^{f,\bar{f}}}{\Gamma}$$

(97)

where

$$R_f = \frac{1}{2}(\Gamma_f + \bar{\Gamma}_f), \quad R_{\bar{f}} = \frac{1}{2}(\Gamma_f + \bar{\Gamma}_f)$$

$$\Gamma = R_f + R_{\bar{f}}$$

(98)
The following relations are also useful which can be easily derived from above equations

\[ \frac{R_{f,j}}{R_f + R_{ar{f}}} = \frac{1}{2}[(1 \pm \Delta C) \pm A_{CP}C] \]  
(100)

\[ \frac{R_{f,j} - R_f}{R_f + R_{ar{f}}} = [\Delta C + A_{CP}C] \]  
(101)

\[ \frac{R_f A_{CP}^f + R_f A_{CP}^{\bar{f}}}{R_f + R_{ar{f}}} = [C + A_{CP}\Delta C] \]  
(102)

For these decays, the decay amplitudes can be written in terms of tree amplitude \( e^{i\phi_T}T_f \) and the penguin amplitude \( e^{i\phi_P}P_f \):

\[ A_f = e^{i\phi_T}e^{i\delta_f^T}T_f [1 + r_f e^{i(\phi_P - \phi_T)}e^{i\delta_f}] \]

\[ A_{\bar{f}} = e^{i\phi_T}e^{i\delta_{\bar{f}}^T} \bar{T}_f [1 + r_{\bar{f}} e^{i(\phi_P - \phi_T)}e^{i\delta_{\bar{f}}}] \]  
(103)

where \( r_{f,\bar{f}} = \frac{|P_{f,\bar{f}}|}{T_{f,\bar{f}}} \), \( \delta_{f,\bar{f}} = \delta_{f,\bar{f}}^T - \delta_{f,\bar{f}}^T \).

\[ A_f = e^{-i\phi_T}e^{i\delta_f^T}T_f [1 + r_f e^{-i(\phi_P - \phi_T)}e^{i\delta_f}] \]

\[ A_{\bar{f}} = e^{-i\phi_T}e^{i\delta_{\bar{f}}^T} \bar{T}_f [1 + r_{\bar{f}} e^{-i(\phi_P - \phi_T)}e^{i\delta_{\bar{f}}}] \]  
(104)

For \( B^0 \to \rho^-\pi^+ : A_f \);

\[ B^0 \to \rho^+\pi^- : A_{\bar{f}} \]

\[ \phi_T = \gamma, \phi_P = -\beta \]  
(105)

Hence for \( B^0 \to \rho^-\pi^+ , B^0 \to \rho^+\pi^- \), we have

\[ A_f = |T_f| e^{+i\gamma}e^{i\delta_f^T} [1 - r_f e^{i(\alpha + \delta_f)}] \]

\[ A_{\bar{f}} = |T_{\bar{f}}| e^{-i\gamma}e^{i\delta_{\bar{f}}^T} [1 - r_{\bar{f}} e^{i(\alpha + \delta_{\bar{f}})}] \]  
(108)

For \( B^0 \to D^*^-D^+ : A_f^D \);

\[ B^0 \to D^*+D^- : A_{\bar{f}}^D \]

\[ \phi_T = 0, \phi_P = -\beta \]  
(106)

Hence for \( B^0 \to D^*^-D^+ , B^0 \to D^*+D^- \), we have

\[ A_f^D = |T_f^D| e^{i\delta_f^{T_D}} [1 - r_f^D e^{i(\beta + \delta_f^D)}] \]

\[ A_{\bar{f}}^D = |T_{\bar{f}}^D| e^{i\delta_{\bar{f}}^{T_D}} [1 - r_{\bar{f}}^D e^{i(\beta + \delta_{\bar{f}}^D)}] \]  
(109)

We now confine ourselves to \( B^0(\bar{B}^0) \to \rho^-\pi^+, \rho^+\pi^- (\rho^+\pi^-, \rho^-\pi^+, \pi^+) \) decays only \[ 19 \] \[ 20 \]. The experimental results for these decays are \[ 12 \] as

\[ \Gamma = R_f + R_{\bar{f}} = (22.8 \pm 2.5) \times 10^{-6} \]  
(110)

\[ A_{CP}^f = -0.16 \pm 0.23, \quad A_{CP}^{\bar{f}} = 0.08 \pm 0.12 \]  
(111)

\[ C = 0.01 \pm 0.14, \quad \Delta C = 0.37 \pm 0.08 \]  
(112)

\[ S = 0.01 \pm 0.09, \quad \Delta S = -0.05 \pm 0.10 \]  
(113)
With the above values, it is hard to draw any reliable conclusion. Neglecting the term $A_{CP}$ in Eqs. (100) and (101), we get

$$R_{f,f} = \frac{1}{2} \Gamma (1 \pm \Delta C)$$ (114)

$$R_{f} - R_{f} = \Delta C$$

Using the above value for $\Delta C$, we obtain

$$R_{f} = (15.6 \pm 1.7) \times 10^{-6}$$
$$R_{f} = (7.2 \pm 0.8) \times 10^{-6}$$ (115)

We analyze these decays by assuming factorization for the tree graphs [10, 11]. This assumption gives

$$T_{f} = \bar{T}_{f} \sim 2m_{B}f_{\rho}p_{\frak{p}}f_{\frak{p}}(m_{\rho}^{2})$$ (116)

$$T_{f} = \bar{T}_{f} \sim 2m_{B}f_{\pi}p_{\frak{p}}A_{0}(m_{\pi}^{2})$$ (117)

Using $f_{\pi}(m_{\rho}^{2}) \approx 0.26 \pm 0.04$ and $A_{0}(m_{\pi}^{2}) \approx A_{0}(0) = 0.29 \pm 0.03$ and $|V_{ub}| = (3.5 \pm 0.6) \times 10^{-3}$, we get the following values for the tree amplitude contribution to the branching ratios

$$\Gamma_{f}^{\text{tree}} = (15.6 \pm 1.1) \times 10^{-6} \equiv |T_{f}|^{2}$$ (118)

$$\Gamma_{f}^{\text{tree}} = (7.6 \pm 1.4) \times 10^{-6} \equiv |T_{f}|^{2}$$ (119)

$$t = \frac{R_{f}}{T_{f}} = \frac{f_{\pi}A_{0}(m_{\pi}^{2})}{f_{\rho}f_{\frak{p}}(m_{\rho}^{2})} = 0.70 \pm 0.12$$ (120)

Now

$$B_{f} = \frac{R_{f}}{|T_{f}|^{2}} = 1 - 2r_{f} \cos \alpha \cos \delta_{f} + r_{f}^{2}$$ (121)

$$B_{f} = \frac{R_{f}}{|T_{f}|^{2}} = 1 - 2r_{f} \cos \alpha \cos \delta_{f} + r_{f}^{2}$$ (122)

Hence from Eqs. (115) and (119), we get

$$B_{f} = 1.00 \pm 0.12$$
$$B_{f} = 0.95 \pm 0.11$$ (123)

In order to take into account the contribution of penguin diagram, we introduce the angles $\alpha_{ef}^{f}$ [21], defined as follows

$$e^{i\beta}A_{f,f} = |A_{f,f}|e^{-i\alpha_{ef}^{f}}$$

$$e^{-i\beta}A_{f,f} = |\bar{A}_{f,f}|e^{i\alpha_{ef}^{f}}$$ (124)

With this definition, we separate out tree and penguin contributions:

$$e^{i\beta}A_{f,f} - e^{-i\beta}A_{f,f} = |A_{f,f}|e^{-i\alpha_{ef}^{f}} - |\bar{A}_{f,f}|e^{i\alpha_{ef}^{f}} = 2iT_{f,f} \sin \alpha$$ (125)

$$e^{i(\alpha + \beta)}A_{f,f} - e^{-i(\alpha + \beta)}A_{f,f} = |A_{f,f}|e^{-i(\alpha_{ef}^{f} - \alpha)} - |\bar{A}_{f,f}|e^{i(\alpha_{ef}^{f} - \alpha)} = (2iT_{f,f} \sin \alpha)\rho_{f,f}e^{i\delta_{f,f}} = 2iP_{f,f} \sin \alpha$$ (126)
From Eq. (125), we get
\[
2\frac{T_{f,f}}{R_{f,f}} \sin^2 \alpha \equiv \frac{2 \sin^2 \alpha}{B_{f,f}} = 1 - \sqrt{1 - A_{CP}^2 \cos 2\alpha_{eff}} \tag{127}
\]
\[
\sin 2\delta_{f,f} = -A_{CP}^2 \sin 2\alpha_{eff} \tag{128}
\]
\[
\cos 2\delta_{f,f} = \frac{\sqrt{1 - A_{CP}^2 \cos 2\alpha_{eff}}}{1 - \sqrt{1 - A_{CP}^2 \cos 2\alpha_{eff}}} \tag{129}
\]

From Eqs. (125) and (126), we get
\[
r_{f,f}^2 = \frac{1 - \sqrt{1 - A_{CP}^2 \cos(2\alpha_{eff} - 2\alpha)}}{1 - \sqrt{1 - A_{CP}^2 \cos 2\alpha_{eff}}} \tag{130}
\]
\[
r_{f,f} \cos \delta_{f,f} = \frac{\cos \alpha - \sqrt{1 - A_{CP}^2 \cos(2\alpha_{eff} - \alpha)}}{1 - \sqrt{1 - A_{CP}^2 \cos 2\alpha_{eff}}} \tag{131}
\]
\[
r_{f,f} \sin \delta_{f,f} = \frac{-A_{CP}^2 \sin \alpha}{1 - \sqrt{1 - A_{CP}^2 \cos 2\alpha_{eff}}} \tag{132}
\]

Now factorization implies [22]
\[
\delta_{f} = 0 = \delta_{f}^T \tag{133}
\]

Thus in the limit \(\delta_{f}^T \to 0\), we get for Eq. (129)
\[
\cos 2\alpha_{eff} = -1, \quad \alpha_{eff} = 90^\circ \tag{134}
\]
\[
r_{f,f} \cos \delta_{f,f} = \cos \alpha \tag{135}
\]
\[
r_{f,f} \sin \delta_{f,f} = \frac{-A_{CP}^2 \sin \alpha}{1 + \sqrt{1 - A_{CP}^2 \cos 2\alpha}} \tag{136}
\]
\[
r_{f,f}^2 = \frac{1 + \sqrt{1 - A_{CP}^2 \cos 2\alpha}}{1 + \sqrt{1 - A_{CP}^2}} \approx \cos^2 \alpha + \frac{1}{4} A_{CP}^2 \sin^2 \alpha \tag{137}
\]

The solution of Eq. (135) is graphically shown in Fig. 1 for \(\alpha\) in the range \(80^\circ \leq \alpha < 103^\circ\) for \(r_{f,f} = 0.10, 0.15, 0.20, 0.25, 0.30\). From the figure, the final state phases \(\delta_{f,f}\) for various values of \(r_{f,f}\) can be read for each value of \(\alpha\) in the above range. Few examples are given in Table 2

For \(\alpha > 90^\circ\), change \(\alpha \to \pi - \alpha\), \(\delta_{f} \to \pi - \delta_{f}\). For example, for \(\alpha = 103^\circ\)
\[
r_f = 0.25, \quad \delta_f = 154^\circ, \quad A_{CP}^f \approx -0.22
\]
\[
r_f = 0.30, \quad \delta_f = 138^\circ, \quad A_{CP}^f \approx -0.40
\]
These examples have been selected keeping in view that final state phases $\delta_f, \bar{\delta}_f$ are not too large. For $A_{CP}^f$, we have used Eq. (136) neglecting the second order term. An attractive option is $A_{CP}^f = A_{CP}^\bar{f}$ for each value of $\alpha$; although $A_{CP}^f \neq A_{CP}^\bar{f}$ is also a possibility. $A_{CP}^f = A_{CP}^\bar{f}$ implies $r_f = r_{\bar{f}}, \delta_f = \delta_{\bar{f}}$.

Neglecting terms of order $r_{\bar{f}}, r_f, \bar{r}_f, \bar{r}_\bar{f}$, we have

$$A_{CP} \approx \frac{2 \sin \alpha (r_f \sin \delta_f - t^2 r_f \sin \delta_{\bar{f}}) - A_{CP}^\bar{f} - t^2 A_{CP}^f}{1 + t^2}$$

$$C \approx -\frac{2t^2}{(1 + t^2)^2} (A_{CP}^f + A_{CP}^\bar{f})$$

$$\Delta C \approx \frac{1 - t^2}{1 + t^2} - \frac{4t^2 \cos \alpha}{(1 + t^2)^2} (r_f \cos \delta_{\bar{f}} - r_f \cos \delta_f)$$

Now the second term in Eq. (141) vanishes and using the value of $t$ given in Eq. (120), we get

$$\Delta C \approx 0.34 \pm 0.06$$

Assuming $A_{CP}^f = A_{CP}^\bar{f}$, we obtain

$$A_{CP} = -\frac{1 - t^2}{1 + t^2} A_{CP}^\bar{f}$$

$$= (0.34 \pm 0.06)(-A_{CP}^f)$$

$$C \approx -\frac{4t^2}{(1 + t^2)^2} A_{CP}^f \approx -(0.88 \pm 0.14)A_{CP}^f$$

Finally the CP asymmetries in the limit $\delta_{f,\bar{f}} \to 0$

$$S_f = S + \Delta S = \frac{2 \text{Im}[\epsilon^{2i\phi_M} A_f^* A_{\bar{f}}]}{\Gamma(1 + A_{CP})}$$

$$= \sqrt{1 - C_f^2 \sin(2\alpha_{\text{eff}}^\bar{f} + \delta)}$$

$$= -\sqrt{1 - C_f^2 \cos \delta}$$

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\[ S_f = S - \Delta S = \frac{2\text{Im}[e^{2i\phi_M} A_f^* \bar{A}_f]}{\Gamma(1 - A_{CP})} \]
\[ = \sqrt{1 - C_f^2} \sin(2\alpha_{\text{eff}} - \delta) \]
\[ = \sqrt{1 - C_f^2} \cos \delta \quad (146) \]

The phase \( \delta \) is defined as
\[ \bar{A}_f = \frac{|\bar{A}_f|}{|A_f|} A_f e^{i\delta} \quad (147) \]

To conclude:

The final state strong phases essentially arise in terms of \( S \)-matrix, which converts an “in” state into an “out” state. The isospin, \( C \)-invariance of hadronic dynamics and the unitarity together with two particle scattering amplitudes in terms of Regge trajectories are used to get information about these phases. In particular two body unitarity is used to calculate the final state phase \( \delta_C \) generated by rescattering for the color suppressed decays in terms of the color favored decays.

In the inclusive version of unitarity, the information obtained for \( s \)-wave scattering from Regge trajectories is used to derive the bounds on the final state phases. In particular, the value obtained for the final state phases \( \delta_{+} = \delta^{P} \approx 29^\circ - 20^\circ \) and \( \delta_{00} = \delta^{C} + \delta^{F} \approx 20^\circ, 12^\circ \) is found to be compatible with the experimental values for direct \( CP \) asymmetries \( A_{CP}(B^0 \to \pi^- K^+, \pi^0 K^0) \).

For \( B^0 \to D^\pm \pi^\mp(D^\pm \pi^\mp) \), \( B^0_s \to D_s^{\pm} \to K^+ (D_s^{\pm} K^-) \) decays described by two independent single amplitudes \( A_f, A'_{f} \) and \( A_{fs}, A'_{fs} \) with different weak phases viz. 0 and \( \gamma \), equality of phases \( \delta_f = \delta'_{f} \) implies, the time dependent \( CP \) asymmetries
\[ - \left( \frac{S_+ + S_-}{2} \right) = \frac{2r_D^{(s)}}{1 + r_D^{(s)2}} \sin(2\beta_{(s)} + \gamma) \quad (148) \]
\[ \frac{S_+ - S_-}{2} = 0 \quad (149) \]

An added advantage is that these decays are described by tree graphs. Assuming factorization, the decay amplitude \( A_f \) can be determined in term of the form factors \( f_0^{B-D}(m^2_{D}) \) and \( A_0^{B-D'}(m^2_{D}) \). The parameter \( r_{D^{(s)}} \) can be expressed in terms of the ratios of the form factors \( f_D, f_0^{B-D}(m^2_{\pi D})/f_{D_s} f_0^{B-D}(m^2_{\pi D}) \) and \( f_{D_s}, f_0^{B-D}(m^2_{\pi D})/f_{D_s} A_0^{B-D'}(m^2_{D}) \). From the experimental branching ratios, we have obtained the form factors \( f_0^{B-D}(m^2_{D}) \) and \( A_0^{B-D'}(m^2_{D}) \) which are in excellent agreement with the prediction of HQET. We have also determined \( r_{D^{*}} \). For \( r_{D^*} \) we get the value \( r_{D^*} = 0.017 \pm 0.003 \). Using this value we get the following bound from the experimental value of \( \frac{S_+ + S_-}{2} \) for \( B^0 \to D^{*+} \pi^+ \) decay:
\[ \sin(2\beta + \gamma) > 0.69 \]

Using SU(3), for the form factors for \( B^0_s \to D_{s}^{*+} K^+ (D_{s}^{*+} K^-) \) decays, we predict
\[ - \left( \frac{S_+ + S_-}{2} \right) = (0.41 \pm 0.08) \sin(2\beta + \gamma) = (0.41 \pm 0.08) \sin \gamma \]
in the standard model.

In section-4, the decays \( B \to \rho^+ \pi^- (\rho^- \pi^+) \) for which decay amplitudes \( A_f \) and \( A_f \) are given in terms of tree and penguin diagrams are discussed. We have analyzed these decays assuming
factorization for the tree graph. Factorization implies $\delta^T_f = \delta^T_{\bar{f}}$. In the limit $\delta^T_{f,\bar{f}} \to 0$, we have shown that

$$r_{f,\bar{f}} \cos \delta_{f,\bar{f}} = \cos \alpha$$
$$r_{f,\bar{f}}^2 \approx \cos^2 \alpha + A_{CP}^2 \sin^2 \alpha$$

The first equation has been solved graphically, from which the final state phases $\delta_{f,\bar{f}}$ corresponding to various values of $r_{f,\bar{f}}$ can be found for a particular value of $\alpha$. The upper bound $\delta_{f,\bar{f}} \leq 30^0$ obtained in Section-2, using unitarity and strong interaction dynamics based on Regge pole phenomenology can be used to select the solutions given in Table-2. Neglecting the terms of order $r_{f,\bar{f}}^2$, we get using factorization

$$\Delta C = 0.34 \pm 0.06$$

Finally, in the limit $\delta_{f,\bar{f}}^T \to 0$, we get

$$\frac{S_{\bar{f}}}{S_f} = \frac{S + \Delta S}{S - \Delta S} = -\frac{\sqrt{1 - C_{\bar{f}}^2}}{\sqrt{1 - C_f^2}}$$

With the present experimental data, it is hard to draw any definite conclusion.

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**Figure Caption:**

Plot of equation $r_f \cos \delta(f) = \cos \alpha$ for different values of $r$. For $80^\circ \leq \alpha \leq 103^\circ$. Where solid curve, dashed curve, dashed dotted curve, dashed boouble doted and double dashed doted curve are corresponding to $r = 0.1$, $r = 0.15$, $r = 0.2$, $r = 0.25$ and $r = 0.3$ respectively.
This figure "KMEMEJ0B.jpg" is available in "jpg" format from:

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