Application of generalized equations of the method of finite differences to the calculation of continuous orthotropic plates

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Abstract. The paper considers the calculation of continuous orthotropic plates under various boundary conditions. The numerical solution is constructed using generalized equations of the finite difference method (FDM), which allow solving the problem within the integrable domain, taking into account the discontinuities of the required function, its first derivative and the right-hand side of the original differential equation without special reduction of the grid pitch. The differential equation for the equilibrium of orthotropic plates in fourth-order partial derivatives is written as a second-order differential equation with respect to the second partial derivatives of the deflection function, the difference approximation of which follows from the general difference approximation of the partial differential equation. The second difference equation is obtained by considering the compatibility of deformations. The results of the calculation of continuous slabs for hinged support and for rigid jamming are given. The reliability of the calculation was confirmed by comparing the solutions on several grids, performing static and kinematic checks.

1. Introduction
The contemporary design and construction are unthinkable without a clear and precise understanding of the work specificity, a number of constructive elements of the designed facility. The most stringent requirements are applied to the design schemes in respect of their strength, reliability and durability. The uniqueness of the contemporary construction requires the improvement of the calculation methods, taking into account the multiple factors of operational, technological and constructive nature. A significant part of the calculations of the structures of such facilities is connected with the calculation of continuous orthotropic plates, and any advance in the precision of the calculations may significantly affect the cost.

Orthotropic plates have different elastic characteristics in three mutually perpendicular directions. Calculation of plates from corrugated panels, reinforced concrete slabs, plates reinforced with ribbed stiffeners located in one or two directions come down to calculation of orthotropic plates. Calculation of orthotropic plates is based on the same assumptions as calculation of isotropic plates.

The development of numerical methods for calculation of orthotropic plates is a crucial task, as the design experience shows the need for calculating structures with a wider variety of methods. The purpose of this work is to develop a methodology for calculating continuous bending orthotropic plates on hinge supports and with rigid fixing. The numerical solution is based on the application of generalized equations of the finite difference method (FDM) in respect of the second partial derivatives of the deflection.
2. Calculation Methodology

The resolving differential equation of the bending of orthotropic plates in the Cartesian coordinates looks as follows [1]:

\[
D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = q, \tag{1}
\]

where \( W \) – is the deflection; \( q \) – is the intensity of the distributed load; \( D_x, D_y \) – is the flexural rigidity relative to the axes \( x, y \); \( H \) – is the torsional rigidity.

Let us introduce nondimensional values and present a differential equation of fourth order (1) as a differential equation of second order in respect of the second partial derivatives of the deflection

\[
a \frac{\partial^2 \tilde{w}}{\partial \xi^2} + \gamma \frac{\partial^2 \tilde{w}}{\partial \eta^2} = -p. \tag{3}
\]

Here \( \tilde{w} = \frac{WD_y}{q_0 a^4} ; \quad \alpha = \frac{D_x}{D_y} ; \quad \gamma = \frac{H}{D_y} ; \quad p = \frac{q}{q_0} ; \quad \xi = \frac{x}{a} ; \quad \eta = \frac{y}{a} ; \quad a \) – is the length of the smaller side, \( q_0 \) – is the intensity of the distributed load at a fixed point.

Equation (3) may be considered as a special case of a differential equation of second order (2.1.1) [2]

\[
a \frac{\partial^2 \omega}{\partial \xi^2} + \beta \frac{\partial^2 \omega}{\partial \eta^2} + \sigma \frac{\partial \omega}{\partial \xi \eta} + \tau \frac{\partial \omega}{\partial \eta} = -p. \tag{4}
\]

The difference approximation (4) is presented by the generalized equation of the finite difference method in (2.1.17) [2].

For approximation (3) by the generalized FDM equation, it is enough to write in the left part (2.1.17) its linear combination with the substitution of \( \omega \) with \( \tilde{w} \) when \( \sigma = \beta = \gamma = 0 \) and of \( \omega \) with \( \tilde{w} \) when \( \alpha = \gamma, \beta = \sigma = 0, \gamma = 1 \), and in the right part to substitute \( p \) with \( -p \).

In the case of a square grid, we have:

\[
\begin{align*}
\gamma w_{i,j-1} + 2(\alpha + \gamma) w_{i,j} + \gamma w_{i,j+1} + \alpha w_{i-1,j} + \alpha w_{i+1,j} + \\
+ w_{i-1,j} + w_{i+1,j} - 2(1 + \gamma) w_{i,j} - \gamma w_{i-1,j} + \gamma w_{i+1,j} + \\
+ \frac{1}{2} \left( h^{I-III} \Delta q_{i,j}^{(\xi)} + h^{III-IV} \Delta q_{i,j}^{(\xi)} \right) + \frac{1}{2} \gamma \left( h^{I-III} \Delta q_{i,j}^{(\eta)} + h^{III-IV} \Delta q_{i,j}^{(\eta)} \right) = \\
= h^2 \left( p_I + p_{II} + p_{III} + p_{IV} \right) \frac{1}{4}
\end{align*}
\]

Here \( I-III \Delta q_{i,j}^{(\eta)}, \quad II-IV \Delta q_{i,j}^{(\eta)} \) are the discontinuities in the value of the transversal load at point \( i, j \), it may be a linear load or reactive forces on intermediate supports in a continuous plate.

The fragment of the grid on which the numerical solution is built is shown in “Figure 1”; digits I, II, III, IV denote the numbers of the elements having a common point \( i, j \).
The second resolving equation obtained from the condition of strain compatibility \[2\] looks as follows:

\[-w_{i+1,j}^{\eta} + w_{i,j-1}^{\xi} - 2w_{i,j}^{\xi} + 2w_{i,j}^{\eta} + w_{i,j+1}^{\xi} - w_{i+1,j}^{\eta} = 0\] \hspace{1cm} (6)

Thus, for each inner point of the grid, we have two algebraic equations (5) and (6) with respect to the unknowns \(w^{\xi}, w^{\eta}\).

With the hinged support of the deck, where the moments on the contour are zero, and therefore \(w^{\xi} = w^{\eta} = 0\), the above equations are enough to solve the task, that is, the system of equations has completeness. In other boundary conditions, to equations for fields (5) and (6) equations for the boundary points should be added.

Non-dimensional moments of flection at the calculated points of the orthotropic plates are determined by the formulas:

\[m^{(\xi)} = -(\alpha w^{\xi} + \mu w^{\eta}), \quad m^{(\eta)} = -(w^{\xi} + \mu w^{\xi})\] \hspace{1cm} (7)

Where \(m^{(\xi)} = \frac{M_x}{q_0 a^2}\), \(m^{(\eta)} = \frac{M_y}{q_0 a^2}\).

For calculation of the deflections, the known FDM equations are used:

\[w_{i,j}^{\xi} = \frac{w_{i-1,j} - 2w_{i,j} + w_{i+1,j}}{h^2}, \quad w_{i,j}^{\eta} = \frac{w_{i,j-1} - 2w_{i,j} + w_{i,j+1}}{h^2}\] \hspace{1cm} (8)

The difference approximation of the boundary conditions for rigid fixing will be obtained. For the left edge, with \(\eta = 0\) given that \(\frac{\partial w}{\partial \eta} = w_{ij}^\eta = 0; w_\eta = w_\xi = 0\) and denoting \(m_{ij} = -w_{ij}^{\xi \eta} - w_{ij}^{\eta \eta}\) (7) from (3.1.6) \[2\] we have:

\[w_{i,j+1} = \frac{h^2}{3} w_{i,j}^{\eta} - \frac{h^2}{6} w_{i,j+1}^{\xi} - \frac{h^2}{6} w_{i,j+1}^{\eta} = -\frac{h^4}{12} (5 p_{i,j} + p_{i,j+1})\] \hspace{1cm} (9)

Here \(i, j\) is the edge point.

For the upper fixed edge, with \(\xi = 0\), equation (9) should be written interchanging the indices \((i, j + 1)\) to \((i + 1, j)\) and, respectively, the indices \(\xi \eta\) to \(\eta \xi\) and \(\eta \eta\) to \(\xi \xi\).

It is seen from (9) that when calculating plates with fixed edges, the required values become \(w^{\xi \xi}, w^{\eta \eta}, w^{\eta \xi}\).

3. Results

As the results illustrating the practical application of the developed calculation algorithm, let us consider the examples of calculations for plates.
Example 1. Calculation of a two-span continuous hinge-supported square orthotropic deck loaded with a uniformly distributed load, the nondimensional value of which is \( p = 1 \). “Figure 2” presents a quarter of a deck where intermediate supports are arranged along line 13-33; the grid step \( h = 1/4 \). Let us assume \( \alpha = 0.4823; \gamma = 0.6944; \mu = 0.2083 \) [3].

![Figure 2](image)

**Figure 2. A quarter of a two-span orthotropic plate**

Boundary conditions: on the contour \( w_{\xi}^{\xi} = w_{\eta}^{\eta} = 0 \), while along the intermediate support line \( w_{23}^{\xi} = w_{33}^{\xi} = 0 \).

Let us consider a quarter of the deck. For points of the field equations (5) and (6) will be made with consideration of symmetry.

Generalized FDM equations make it possible to calculate the linear loads, in this case these are the reactions of the intermediate supports \( R_{23} \) and \( R_{33} \), that is, in (5) for points 23 and 33, the following items must be taken into account:

\[
\frac{1}{2} \gamma (h^{I-III} \Delta q_{i,j}^{(\eta)} + h^{II-IV} \Delta q_{i,j}^{(\eta)}) = \frac{\gamma}{2} h 2R_{ij}.
\]

where \( \Delta q_{i,j}^{(\eta)} = \Delta q_{13}^{(\eta)} = \Delta q_{24}^{(\eta)} = R_{23} \)

From the solution of the equations let us determine the values of the second derivatives and reactions, and by (7) the values of the moments.

\[
\begin{align*}
  w_{22}^{\xi} &= -0.0055; w_{32}^{\xi} = -0.0036; w_{33}^{\eta} = 0.0181; w_{23}^{\eta} = 0.0146; \\
  w_{22}^{\eta} &= -0.0146; w_{32}^{\eta} = -0.0181; R_{23} = -0.7833; R_{33} = -0.8352; \\
  m_{33}^{(\eta)} &= -0.0181; m_{33}^{(\xi)} = 0.0037; m_{23}^{(\eta)} = 0.0188; m_{23}^{(\xi)} = 0.0055.
\end{align*}
\]

Let us make a kinematic check. For known values of the second derivatives \( W_{23}^{\xi}; W_{33}^{\eta}; W_{32}^{\eta} \) We define the deviation at point (3.2) along two directions using (8), as a result we'll obtain the same value \( w_{32} = 0.0022 \).

To perform an integral check of the balance, the projections of all forces onto the axis perpendicular to the plane of the plate are to be found.

The non-dimensional transverse forces on the contour \( m_{i,j}^{\eta} \) with \( \eta = 0 \) are calculated by (3.1.4) [2].
\[-h \eta_{i,j} - m_{i,j} + m_{i,j+1} = -\frac{h^2}{12} (5p_{i,j} + 5p_{i,j+1}). \]  

(10)

To determine \( m_{i,j} \) with \( \zeta = 0 \) the indices \( i, j + 1 \) in (10) should be replaced with \( i + 1, j \). Then, using the Simpson formula, the resultant of the reactions along the contour and the reactive forces on the intermediate supports are determined, the sum of which should be equal to the resultant of the external load, that is 1. The static check is carried out with an accuracy of 5.4%.

Let us note that the solution of this example was preceded by the calculation of an orthotropic hinged plate on a uniformly distributed load with step \( h = 0.5 \) and \( h = 0.25 \), which showed the convergence of the numerical solution and the discrepancy with the solution of the MSA [4] at the center on \( m^{\xi} \) by 4.35%, and on \( m^\eta \) and in the deflection by 0.71%.

Example 2. Calculation of a two-span continuous rectangular orthotropic plate with a rigid fixing along the contour with an aspect ratio of 1:2, with a minimum number of partitions with a step of \( h = 0.5 \), loaded with a uniformly distributed load \( p = 1 \). “Figure 3” shows the calculation scheme of the plate. The values of \( \alpha, \gamma, \mu \) are the same as above. On line 13-33 there are intermediate supports.

![Figure 3. Two-span orthotropic plate fixed along the contour](image)

Let us write down three groups of difference equations; considering that \( w_{23}^{\xi} = 0; w_{23}^\eta = 0 \). For points of the field we make equations (5) and (6), for points of the contour - equations (9), and equation (8) for point (22). In equation (5) for point (23) we take into account the jump from the desired reaction by analogy with the first example by the value \( -\eta h2R_{23} \). From the solution of the equations we find \( w^{\xi}; w^{\eta}; w \) and \( R_{23} \). The calculated values of the quantities are \( w_{22} = 0.0020 \), \( R_{23} = -1.0361 \) and then calculate the moments at the next points:

\[
\begin{align*}
  m_{21}^{\eta} &= -0.0717; m_{22}^{\eta} = 0.0195; m_{23}^{\eta} = -0.0162; m_{12}^{\xi} = -0.0345; m_{22}^{\xi} = 0.0112.
\end{align*}
\]

Since there is no data to compare the results, in subsequent calculations it is necessary to reduce the grid interval.

The above algorithm for calculating orthotropic plates can be used to calculate isotropic plates, treating it as a special case, assuming that \( \alpha = \gamma = 1 \).

The application of generalized FDM equations makes it possible to build a relatively simpler solution than a solution using the equations of the method of successive approximations (MSA), which
results in smaller errors in solving tasks on the same grids, but to a larger volume of computational work [2], [4, 5, 6].

Along with the method of calculating orthotropic plates, we should mention papers [1], [3], [7, 8, 9, 10], should be mentioned, in which variational, analytical methods, pseudo-loads method and Fourier transforms are applied; in [11] the finite element method is applied for modelling the operation of orthotropic plates. A number of sources [12, 13, 14] present the developed engineering methods of calculation and the results of the experiments.

4. Conclusions
There was developed an algorithm for calculating continuous orthotropic plates with hinged supports and rigid fixing on the base of generalized FDM equations. The calculation examples illustrate the algorithm and demonstrate that it may be used to obtain a solution on a rare grid. The accuracy of the solution will be increased by decreasing the grid step. The method may be recommended to be applied in engineering calculations and verification of calculations for plates performed by other method.

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