THE THRESHOLD FOR INTEGER HOMOLOGY IN RANDOM $d$-COMPLEXES

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ABSTRACT. Let $Y \sim Y_d(n, p)$ denote the Bernoulli random $d$-dimensional simplicial complex. We answer a question of Linial and Meshulam from 2003, showing that the threshold for vanishing of homology $H_{d-1}(Y; \mathbb{Z})$ is less than $80d \log n / n$. This bound is tight, up to a constant factor.

1. Introduction

Define $Y_d(n, p)$ to be the probability distribution on all $d$-dimensional simplicial complexes with $n$ vertices, with complete $(d-1)$-skeleton and with each $d$-dimensional face included independently with probability $p$. We use the notation $Y \sim Y_d(n, p)$ to mean that $Y$ is chosen according to the distribution $Y_d(n, p)$; note the 1-dimensional case $Y_1(n, p)$ is equivalent to the Erdős–Rényi random graph $G \sim G(n, p)$.

Results in this area are usually as $n \to \infty$ and $p = p(n)$. We say that an event occurs with high probability (abbreviated w.h.p.) if the probability approaches one as the number of vertices $n \to \infty$. Whenever we use big-$O$ or little-$o$ notation, it is also understood as $n \to \infty$.

A function $f = f(n)$ is said to be a threshold for a property $P$ if whenever $p/f \to \infty$, w.h.p. $G \in P$, and whenever $p/f \to 0$, w.h.p. $G \notin P$. In this case, one often writes that $f$ is the threshold, even though technically $f$ is only defined up to a scalar factor.

It is a fundamental fact of random graph theory (see for example Section 1.5 of [6]) that every monotone property has a threshold. However, not every monotone property has a sharp threshold. For example, $1/n$ is the threshold for the appearance of triangles in $G(n, p)$, but this threshold is not sharp. In contrast, the Erdős–Rényi theorem asserts that $\log n / n$ is a sharp threshold for connectivity. Classifying which graph properties have sharp thresholds is a problem which has been extensively studied; see for example the paper of Friedgut with appendix by Bourgain [3].

The first theorem concerning the topology of $Y_d(n, p)$ was in the influential paper of Linial and Meshulam [9]. Their results were extended by Meshulam and Wallach.

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to prove the following far reaching extension of the Erdős–Rényi theorem \[10\], where they described sharp vanishing thresholds for homology with field coefficients.

**Linial–Meshulam–Wallach theorem.** Suppose that \(d \geq 2\) is fixed and that \(Y \sim Y_d(n, p)\). Let \(\omega\) be any function such that \(\omega \to \infty\) as \(n \to \infty\).

1. If \(p \leq \frac{d \log n - \omega}{n}\)
    then w.h.p. \(H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) \neq 0\), and
2. if \(p \geq \frac{d \log n + \omega}{n}\)
    then w.h.p. \(H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0\).

The \(d = 1\) case is equivalent to the Erdős–Rényi theorem. The Linial–Meshulam theorem is the case \(d = 2, q = 2\), and the Meshulam–Wallach theorem is the general case \(d \geq 2\) arbitrary and \(q\) any fixed prime. In closing remarks of \[9\], Linial and Meshulam asked "Where is the threshold for the vanishing of \(H_1(Y, \mathbb{Z})\)?" By the universal coefficient theorem, \(H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0\) for every prime \(q\) implies that \(H_{d-1}(Y; \mathbb{Z}) = 0\), so one may be tempted to conclude that the Meshulam–Wallach theorem already answers the question of the threshold for \(\mathbb{Z}\)-coefficients. This is not the case, however, since we are concerned with not just a single simplicial complex, but with a sequence of complexes as \(n \to \infty\), and there might very well be torsion growing with \(n\). The Meshulam–Wallach Theorem holds for \(q\) fixed, and can be made to work for \(q\) growing slowly enough compared with \(n\). But it does not seem possible to extend the cocycle-counting arguments from \[9\] and \[10\] to cover the case when \(q\) is growing much faster than polynomial in \(n\).

On the surface of things, this might actually be a big problem. A complex \(X\) is called \(\mathbb{Q}\)-acyclic if \(H_0(X, \mathbb{Q}) = \mathbb{Q}\) and \(H_i(X, \mathbb{Q}) = 0\) for \(i \geq 1\). Kalai showed that for a uniform random \(\mathbb{Q}\)-acyclic 2-dimensional complex \(T\) with \(n\) vertices and \(\binom{n-1}{2}\) edges, the expected size of the torsion group \(|H_1(T; \mathbb{Z})|\) is of order at least \(\exp(cn^2)\) for some constant \(c > 0\) \[8\]. On the other hand, the largest possible torsion for a 2-complex on \(n\) vertices is of order at most \(\exp(Cn^2)\) for some other constant \(C > 0\), so Kalai’s random \(\mathbb{Q}\)-acyclic complex provides a model of random simplicial complex which is essentially the worst case scenario for torsion.

We mention in passing that another approach to homology-vanishing theorems for random simplicial complexes is “Garland’s method” \[4\], with various refinements due to Żuk \[13, 12\], Ballman–Świątkowski \[2\], and others. These methods have been applied in the context of random simplicial complexes, see for example \[5, 7\]. However, it must be emphasized that these methods only work over a field of
characteristic zero; they do not detect torsion in homology. A different kind of argument is needed to handle homology with $\mathbb{Z}$ coefficients.

The fundamental group $\pi_1(Y)$ of the random 2-complex $Y \sim Y_2(n, p)$ was studied earlier by Babson, Hoffman, and Kahle [1], and the threshold face probability for simple connectivity was shown to be of order $1/\sqrt{n}$. Until now, there seems to have been no upper bound on the vanishing threshold for integer homology for random 2-complexes, other than this.

Our main result is that the threshold for vanishing of integral homology agrees with the threshold for field coefficients, up to a constant factor. In particular we have the following.

**Theorem 1.** Let $d \geq 2$ be fixed and $Y \sim Y_d(n, p)$. If

$$p \geq \frac{80d \log n}{n},$$

then $H_{d-1}(Y; \mathbb{Z}) = 0$ w.h.p.

**Remark.** For the sake of simplicity, we make no attempt here to optimize the constant $80d$. We conjecture that the best possible constant is $d$; in other words we would guess that the Linial–Meshulam–Wallach theorem is still true with $\mathbb{Z}/q\mathbb{Z}$-coefficients replaced by $\mathbb{Z}$-coefficients. But to prove this, it seems that another idea will be required.

Our main tool in proving Theorem 1 is the following.

**Theorem 2.** Let $d \geq 2$ be fixed and let $q = q(n)$ be a sequence of primes. If $Y \sim Y_d(n, p)$ where

$$p \geq \frac{40d \log n}{n},$$

then

$$\mathbb{P}(H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) \neq 0) \leq \frac{1}{n^{d+1}}.$$

**Remark.** Theorem 2 is similar to the main result in Meshulam–Wallach, but the statement and proof differ in fundamental ways. The main point is that the bound on the probability that $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) \neq 0$ holds uniformly over all primes $q$, even if $q$ is growing very quickly compared to the number of vertices $n$.

### 2. Proof

We first prove Theorem 1. The proof relies on Theorem 2 plus one additional fact — a bound on the size of the torsion subgroup in the degree $(d-1)$ homology of a simplicial complex, which only depends on the number of vertices $n$. Let $A_T$ denote the torsion subgroup of an abelian group $A$. 
Lemma 3. Let \( d \geq 2 \) and suppose that \( X \) is a \( d \)-dimensional simplicial complex on \( n \) vertices. Then \( |(H_{d-1}(X; \mathbb{Z}))_\mathcal{T}| = \exp\left(O(n^d)\right)\).

Proof of Lemma 3. We include a proof here for the sake of completeness, but such bounds on the order of torsion groups are known. See, for example, Proposition 3 in Soulé [11], which he attributes in turn to Gabber.

We assume without loss of generality that \( H_d(X) = 0 \). Indeed, if there is a nontrivial cycle \( Z \) in \( H_d(X) \), then delete one face \( \sigma \) from the support of \( Z \). Then in the subcomplex \( X - \sigma \), the rank of \( H_d(X - \sigma) \) is one less than the rank of \( H_d(X) \). So we have

\[
\dim[H_{d-1}(X-\sigma,k)] = \dim[H_{d-1}(X,k)]
\]

over every field \( k \), and then the isomorphism \( H_{d-1}(X-\sigma,\mathbb{Z}) = H_{d-1}(X,\mathbb{Z}) \) follows by the universal coefficient theorem.

We may further assume that the number of \( d \)-dimensional faces \( f_d \) is bounded by \( f_d \leq \binom{n}{d} \), since if there were more faces than this, then we would have \( f_d > f_{d-1} \) and there would have to be nontrivial homology in degree \( d \), by dimensional considerations.

Let \( C_i \) denote the space of chains in degree \( i \), i.e. all formal \( \mathbb{Z} \)-linear combinations of \( i \)-dimensional faces, and let \( \delta_i : C_i \to C_{i-1} \) be the boundary map in simplicial homology. If \( Z_i \) is the kernel of \( \delta_i \) and \( B_i \) is the image of \( \delta_{i+1} \), then by definition \( H_i(X; \mathbb{Z}) = Z_i/B_i \).

Let \( M_i \) be a matrix for the boundary map \( \delta_i \), with respect to the preferred bases of faces in the simplicial complex. Then the order of the torsion subgroup \( |(C_i/B_i)_\mathcal{T}| \) is bounded by the product of the lengths of the columns of \( M_i \), as follows.

We begin by writing \( M_i \) in its Smith normal form, i.e. \( M_i = PDQ \) with \( P \) and \( Q \) invertible matrices over \( \mathbb{Z} \) and \( D \) a rectangular matrix with entries only on its diagonal. Let \( r \) be the rank of \( D \) over \( \mathbb{Q} \); note this is also the \( \mathbb{Q} \)-rank of \( M_i \). By removing the all 0 rows and columns from \( D \) (and some columns of \( P \) and some rows of \( Q \)), we may write \( M_i = P'D'Q' \) where \( D' \) is an \( r \times r \) diagonal matrix, and all of \( P', D', \) and \( Q' \) have \( \mathbb{Q} \)-rank \( r \). By the definition of \( D \), we have \( \det D' = |(C_i/B_i)_\mathcal{T}| \).

As \( P' \) and \( Q' \) both have \( \mathbb{Q} \)-rank \( r \), we can find a collection of \( r \) rows from \( P' \) that are linearly independent over \( \mathbb{Q} \) and \( r \) columns of \( Q' \) that are linearly independent over \( \mathbb{Q} \). Write \( \tilde{P} \) and \( \tilde{Q} \) for the \( r \times r \) submatrices of \( P' \) and \( Q' \) given by these rows and columns. As \( \tilde{P} \) and \( \tilde{Q} \) are full \( \mathbb{Q} \)-rank, they are invertible over \( \mathbb{Q} \) and have nonzero determinant. As they are additionally integer matrices, they each have determinants at least 1. Thus,

\[
\det(D') \leq |\det(\tilde{P})\det(D')\det(\tilde{Q})| = |\det(\tilde{P}D'\tilde{Q})|.
\]

On the other hand \( \tilde{M} = \tilde{P} D' \tilde{Q} \) is an \( r \times r \) submatrix of \( M_i \). Thus, applying the Hadamard bound to \( \tilde{M} \), we may bound \( \det(\tilde{M}) \) by the product of the lengths of the
columns of $\tilde{M}$. As the columns of $M_i$ all have lengths at least 1, the product of the lengths of the columns of $\tilde{M}$ are at most the product of the lengths of the columns of $M_i$, completing the proof.

Since $Z_i/B_i$ is isomorphic to a subgroup of $C_i/B_i$, this also gives a bound on the torsion in homology. In particular, for any simplicial complex $X$ on $n$ vertices, we have that

$$|H_{d-1}(X; T)| \leq \sqrt{d+1} \binom{n}{d} = \exp(O(n^d)).$$

Now define

$$Q(X) = \{q \text{ prime} : H_{d-1}(X; \mathbb{Z}/q\mathbb{Z}) \neq 0\}.$$

An immediate consequence of Lemma 3 is that

$$|Q(X)| = O(n^d),$$

and this is the fact which we will use.

**Proof of Theorem 1**

Our strategy is as follows. Let $Y_1, Y_2 \sim Y_d(n, 40d \log n/n)$ be two independent random $d$-complexes and let $Y \sim Y_d(n, 80d \log n/n)$

**Step 1** First we note that we can couple $Y, Y_1$ and $Y_2$ such that

$$F_d(Y_1) \cup F_d(Y_2) \subset F_d(Y).$$

By (1) if $H_{d-1}(Y_1; \mathbb{Z}/q\mathbb{Z}) = 0$ or $H_{d-1}(Y_2; \mathbb{Z}/q\mathbb{Z}) = 0$ then $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0$.

**Step 2** By Lemma 3, $Q(Y_1)$ has cardinality $O(n^d)$.

**Step 3** Applying a union bound, the probability that either $H_{d-1}(Y_1; \mathbb{Q}) \neq 0$ or there exists $q \in Q(Y_1)$ such that

$$H_{d-1}(Y_1; \mathbb{Z}/q\mathbb{Z}) \neq 0$$

is at most $O(n^d \cdot n^{-(d+1)}) = O(1/n) = o(1)$.

**Step 4** Thus if

(a) $H_{d-1}(Y_1; \mathbb{Q}) = 0$, and

(b) $H_{d-1}(Y_2; \mathbb{Z}/q\mathbb{Z}) = 0$ for all $q \in Q(Y_1)$,

then by the coupling in Step 1, we have that $H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0$ for all primes $q$. By the universal coefficient theorem we have that $H_{d-1}(Y; \mathbb{Z}) = 0$. Each of these two conditions happens with probability $1 - o(1)$ which completes the proof.

**Proof of Theorem 2**

Throughout this paper we are always working with $d$-dimensional simplicial complexes on vertex set $[n]$, with complete $(d-1)$-skeleton. Such a complex $Y$ is defined by $F_d(Y)$, its set of $d$-dimensional
faces. We often associate the two in the following way. If \( f \in \binom{[n]}{d+1} \) (i.e. \( f \) is a \( d \)-dimensional simplex) and \( Y \) is as above then we write \( Y \cup f \) for the simplicial complex with \( F_d(Y \cup f) = F_d(Y) \cup f \).

Let \( q \) be a prime and \( Y \) be as above. Define
\[
q\text{-reducing set } (Y) = \{ f : H_{d-1}(Y \cup f; \mathbb{Z}/q\mathbb{Z}) \neq H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) \}.
\]

In other words, \( q\)-reducing set \( (f) \) is precisely the set of \( d \)-dimensional faces which, when added to \( Y \), drop the dimension of \( H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) \) by one.

**Lemma 4.** A \( d \)-dimensional simplex \( f \in q\text{-reducing set } (Y) \) if and only if the boundary of \( f \) is not a \( (\mathbb{Z}/q\mathbb{Z}) \) boundary in \( Y \). Thus if \( Y \subset Y' \), where \( Y \) and \( Y' \) are \( d \)-dimensional complexes sharing the same \( d-1 \)-skeleton, then
\[
q\text{-reducing set } (Y') \subset q\text{-reducing set } (Y).
\]

**Proof.** If \( \partial f \) is not a boundary in \( Y \) then \( H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) \neq H_{d-1}(Y \cup f; \mathbb{Z}/q\mathbb{Z}) \). If \( \partial f \) is a boundary in \( Y \) then \( H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = H_{d-1}(Y \cup f; \mathbb{Z}/q\mathbb{Z}) \). \( \square \)

**Lemma 5.** \( H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0 \) if and only if \( q\text{-reducing set } (Y) = \emptyset \).

**Proof.** Clearly, \( H_{d-1}(\ast, \mathbb{Z}/q\mathbb{Z}) = 0 \) is monotone with respect to inclusion of \( d \)-faces, so \( H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0 \) implies that \( q\text{-reducing set } (Y) = \emptyset \).

But we also have that the \( d-1 \)-skeleton of \( Y \) is complete, so once all possible \( d \)-faces have been added, homology is vanishing. Once again applying the monotonicity of Lemma 4, \( q\text{-reducing set } (Y) = \emptyset \) also implies that \( H_{d-1}(Y; \mathbb{Z}/q\mathbb{Z}) = 0 \). \( \square \)

Instead of working directly with the Linial–Meshulam distribution \( Y_d(n, p) \) where each face is included independently with probability \( p \), it is convenient to work with the closely related distribution \( Y_d(n, m) \), where the complex is chosen uniformly over all \( \binom{n}{d+1} \binom{m}{d+1} \) simplicial complexes on \( n \) vertices with complete \( d-1 \)-skeleton, and with exactly \( m \) faces of dimension \( d \). As with the random graphs we have that if \( m \approx p \binom{n}{d+1} \) then for many properties the two models are very similar. After doing our analysis with \( Y_d(n, m) \), we convert our results back to the case of \( Y_d(n, p) \).

Let
\[
\tilde{m} = \tilde{m}(n, q) = \min \left\{ m' : \mathbb{E}[q\text{-reducing set } (Y(n, m'))] \leq \frac{1}{2} \binom{n}{d+1} \right\}
\]
This next lemma points out an easy consequence of our definition of \( \tilde{m} \).
Lemma 6. For every $d$-face $f$

$$\mathbb{P}(f \in q\text{-reducing set } (Y(n, \tilde{m}))) \leq 1/2.$$  

Proof. This follows easily by symmetry. \qed

If $Z$ and $Z'$ are random $d$-complexes with vertex set $[n]$ and the complete $(d-1)$-skeleton then we say $Z$ stochastically dominates $Z'$ if there exists a coupling of the two random variables with $\mathbb{P}(F_d(Z') \subset F_d(Z)) = 1$.

Lemma 7. Let $m = \sum_{i=1}^{k} m_i$ with $m_i \in \mathbb{N}$. Also let $Y \sim Y_d(n, m)$ and $Y^i \sim Y_d(n, m_i)$ for all $i$. Then $Y$ stochastically dominates $\bigcup_{i=1}^{k} Y^i$ and

$$q\text{-reducing set } (Y) \subset q\text{-reducing set } \left( \bigcup_{i=1}^{k} Y^i \right).$$

Proof. The first claim is a standard argument; see for example Section 1.1 of [6]. The second follows from the first and the monotonicity of the $q$-reducing set (Lemma 4). \qed

Lemma 8. For any $q$, sufficiently large $n$, $d$-face $f$ and $k \geq 2(d+1) \log_2(n)$, then for $Y \sim Y_d(n, k\tilde{m})$

$$\mathbb{P}(f \in q\text{-reducing set } (Y)) \leq \frac{1}{n^{2(d+1)}}.$$  

Proof. Let $Y^1, \ldots, Y^k$ be i.i.d. complexes with distribution $Y_d(n, \tilde{m})$. Then by Lemma 7 we can find a coupling so that a.s.

$$q\text{-reducing set } (Y) \subset q\text{-reducing set } \left( \bigcup_{i=1}^{k} Y^i \right).$$
Then by Lemmas 4, 5 and 6
\[ P\left( f \in q\text{-reducing set } (Y) \right) \leq P\left( f \in q\text{-reducing set } \left( \bigcup_{i=1}^{k} Y_i \right) \right) \]
\[ \leq P\left( \bigcap_{i=1}^{k} \{ f \in q\text{-reducing set } (Y^i) \} \right) \]
\[ \leq \prod_{i=1}^{k} P\left( f \in q\text{-reducing set } (Y^i) \right) \]
\[ \leq \left( \frac{1}{2} \right)^k \]
\[ \leq \frac{1}{n^{2(d+1)}}. \]

Now the main task that remains is to estimate $\tilde{m}$. Before we do so, we give a heuristic that indicates that $\tilde{m} \leq 2\binom{n}{d}$. We consider the process where we start with $Y_0$ the complex with the complete $(d-1)$-skeleton and no $d$-dimensional faces. Then we inductively generate $Y_{i+1}$ by taking $Y_i$ and independently adding one new $d$-dimensional face. Note that when we are adding faces one at a time, the dimension $\dim H_{d-1}(Y_i, \mathbb{Z}/q\mathbb{Z})$ is monotone decreasing.

As $H_{d-1}(Y_0; \mathbb{Z}/q\mathbb{Z})$ is generated by the $(d-1)$-cycles its dimension is at most $\binom{n}{d}$. Heuristically this indicates that $\tilde{m}$ should be no larger than $2\binom{n}{d}$, because if we were to add $2\binom{n}{d}$ faces and half of them reduce the dimension of the homology, then the dimension has dropped $\binom{n}{d}$ times. This would make the homology trivial, and would leave no faces remaining in the $q$-reducing set. We now make this heuristic rigorous, albeit with a slightly worse constant.

**Lemma 9.** Let $Y$ be a $d$-complex and let $f_1, f_2, \ldots$ be an ordering of $F_d(Y)$. Let $Y_i$ be the $d$-complex with
\[ F_d(Y_i) = \bigcup_{j=1}^{i} \{ f_j \}. \]
Then there are at most $\binom{n}{d}$ $i$ such that
\[ f_i \in q\text{-reducing set } (Y_{i-1}). \]

**Proof.** By induction. If there exist a subsequence $0 < i_1 < i_2 < \cdots < i_s$ with
\[ f_{i_s} \in q\text{-reducing set } (Y_{i_s-1}) \]
then

\[ |H_{d-1}(Y_i, \mathbb{Z}/q\mathbb{Z})| \leq q^{(n \choose d) - s}. \]

Thus the longest possible subsequence has length \( (n \choose d) \).

Lemma 10. For any \( q \) and any \( n > d \) we have \( \tilde{m} \leq 4(n \choose d) \).

Proof. Let \( f_1, f_2, \ldots, f_{n \choose (d+1)} \) be a uniformly random ordering of all the possible \( d \)-faces. Again we define the complexes \( Y_i \) by

\[ F_d(Y_i) = \bigcup_{j=1}^{i} \{ f_j \}, \]

and we remark that each \( F_d(Y_i) \) is distributed as \( Y_d(n, m) \). Define the random variables

\[ Z_i = 1_{\{ f_i \in q\text{-reducing set } (Y_{i-1}) \}}. \]

and \( \{ X_i \} \) be an i.i.d. sequence of Bernoulli(1/3) random variables. We can couple the events so that \( Z_i \) stochastically dominates \( X_i \) up until the random time \( m^* \), where

\[ m^* = \min \left( m' : |q\text{-reducing set } (Y_{m'})| \leq \frac{1}{3} \left( n \choose (d+1) \right) \right). \]

Thus by Lemma 9 we have a.s. that

\[ (n \choose d) \geq \sum_{i=1}^{m^*} Z_i \geq \sum_{i=1}^{m^*} X_i. \]

So either

1. \( m^* \leq 4(n \choose d) \) or
2. \( \sum_{i=1}^{n \choose (d+1)} X_i < (n \choose d) \).

The sum on the left hand side of 2 has expected value \( \frac{4}{3} (n \choose d) \) which is a constant factor larger than \( (n \choose d) \). Thus the probability of the last event is exponentially decreasing in \( (n \choose d) \), and so it is certainly less than 1/10. Thus \( P(m^* > 4(n \choose d)) < 1/10 \) as well.
$$\mathbb{E}|q\text{-reducing set } (Y_d) | \leq \frac{1}{3}\binom{n}{d+1} \cdot \mathbb{P}(m^* \leq 4\binom{n}{d})$$
$$+ \binom{n}{d+1} \mathbb{P}(m^* > 4\binom{n}{d})$$
$$\leq \frac{1}{3}\binom{n}{d+1} + \frac{1}{10}\binom{n}{d+1}$$
$$\leq \frac{1}{2}\binom{n}{d+1}.$$
a random $d$-complex $Y \sim Y_d(n, p)$ has at least $(12d + 12)(\log n) \binom{n}{d}$ faces of dimension $d$. Then the theorem follows from Lemma 11.

\[\square\]

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