ON THE MEASURE OF SIMPLICIAL QUANTUM GRAVITY
IN FOUR DIMENSIONS

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ABSTRACT

We study quantum gravity in the path-integral formulation using the Regge calculus. In spite of the unbounded gravitational action, the existence of an entropy-dominated phase is confirmed. The influence of various types of measures on this phase structure is investigated and our results are compared with those obtained by dynamical triangulation.

Introduction and Theory

The Regge calculus is a useful tool to study non-perturbative aspects of the quantum-gravity path-integral

\[
 Z = \int D\mu e^{-I_E}
\]

in a systematic way \(^1\). \(^2\) \(^3\). \(^4\). Thereby the continuum Einstein-Hilbert action

\[
 -I_E = L_P^{-2} \int d^3x \sqrt{g} R - \lambda \int d^3x \sqrt{g},
\]

with \(L_P\) the Planck length, \(R\) the curvature scalar, \(g\) the determinant of the metric, and \(\lambda\) a cosmological constant, is replaced by the discrete Euclidean action

\[
 -I_E = \beta \sum_t A_t \delta_t - \lambda \sum_s V_s .
\]

Triangle areas \(A_t\), deficit angles \(\delta_t\), and 4-simplex volumes \(V_s\) are calculated from the squared link lengths \(q_t\) given in units of \(L_P\) and being the dynamical quantities \(^5\). The constant \(\lambda\) fixes the expectation value of the lattice volume and the parameter \(\beta\) determines the scale. We define the expectation value of the lattice spacing in units of the Planck length as

\[
 \ell = (\beta^2 \langle q_t \rangle)^{1/2} ,
\]

which is an observable rather than a parameter since the simplicial lattice itself is a quantum object. Another important observable is the average curvature measured in units of the average link length. It is defined as

\[
 \bar{R} = \left( \frac{1}{N_t} \sum_t q_t \right) \frac{\sum_t A_t \delta_t}{\sum_s V_s} \]

with \(N_t\) the total number of links.

Obviously, the unpleasant feature of an unbounded gravitational action is also present in simplicial quantum gravity, but this does not rule out a priori a well-defined path integral. As a matter of fact it is possible that the entropy of the system compensates the unbounded action leading to the occurrence of a well-defined phase as mentioned first by Berg \(^6\). This can be seen if \(^7\) is rewritten as

\[
 Z = \int_{-\infty}^{+\infty} dI_E n(I_E) e^{-I_E} ,
\]

where \(n(I_E)\) denotes the state density for a given value \(I_E\) of the action, i.e. the number of configurations with the same Euclidean action. If \(n(I_E)\) vanishes fast enough for \(I_E \to -\infty\) the integral \(^7\) stays finite in a certain range of \(\beta\). This means that there are many configurations with small action and only few giving a large average curvature; the larger the curvature the smaller its probability. Configurations with large curvature contain distorted 4-simplices and are near to leave the Euclidean sector. On the other hand configurations with almost equilateral simplices correspond to small average curvatures.

To investigate this mechanism we have set a lower limit

\[
 \phi_s \geq f \geq 0
\]

for the fatness of each 4-simplex

\[
 \phi_s \sim \frac{V_s^2}{\max_{l \in s} (q_l^2)}
\]

restricting the configuration space \(^8\). The gravitational action \(^3\) is therefore bounded for a finite number of 4-simplices as long as \(f > 0\). The convergence of expectation values in the limit \(f \to 0\) indeed supports the above entropy hypothesis \(^8\). Since the configuration space grows with smaller values of \(f\) an increasing number of iterations are necessary to reach equilibrium. In order to make numerical simulations easier a lower limit \(f = 10^{-5}\) has been applied to the fatness \(\phi_s\) in the following.

We address now the fundamental question about the influence of the measure on the entropy-dominated phase described above. A unique definition of the gravitational measure does not exist since it is not clear

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\(^1\) Supported in part by "Fonds zur Förderung der wissenschaftlichen Forschung" under Contract P9522-PHY.
which quantities have to be identified with the ‘true’
physical degrees of freedom [7-11]. Therefore, we ex-
mamine different measures of the form
\[
D\mu = (\prod_\ell q_\ell^{-1}dq)\mathcal{F}(q)
\]
(9)
by varying the parameter \(\sigma\). This corresponds for \(\sigma = 0\)
to the scale-invariant measure of Faddeev and Popov \[1\]
and for \(\sigma = 1\) to the uniform measure of deWitt \[1\]. The function \(\mathcal{F}\) is equal one for Euclidean configu-
rations and zero otherwise.

**Results and Discussion**

Computations have been performed on a hypercu-
bic triangulated 4-torus with \(4^4, 6^4\) and \(8^4\) vertices. It
turned out that finite-volume effects on one-point func-
tions are small. The behavior of the average curvature
versus the average lattice spacing is given in Figure 1
for the 4\(^4\)-vertex system. The parameter \(\sigma\) in \(\mathcal{F}\) was
increased step by step from 0 to 1.5 observing two dif-
ferent regimes. For \(\sigma \leq 1\) and small lattice spacing,
\(0 \leq \ell \leq 0.3\), the expectation value \(\langle \tilde{R} \rangle\) is negative and
seems to be independent of \(\sigma\). Near the transition point
to positive curvature, \(\ell \approx 0.4\), the influence of \(\sigma\) be-
comes more pronounced. The result for \(\sigma = 1.5\) differs
over the entire range of \(\ell\) from those obtained for \(\sigma \leq 1\).
Even at \(\ell = 0\) the curvature \(\langle \tilde{R} \rangle\) is significantly larger
for \(\sigma > 1\) as illustrated in Figure 2.

![Figure 1: Average curvature \(\langle \tilde{R} \rangle\) as a function of the lattice spacing \(\ell\) within the Regge approach for different types of the measure parametrized by \(\sigma \geq 0\). The behavior of \(\langle \tilde{R} \rangle\) in the region of small \(\ell\) is almost independent of \(\sigma\) as long as \(\sigma \leq 1\).](image1)

![Figure 2: Average curvature \(\langle \tilde{R} \rangle\) versus measure parameter \(\sigma\) in the case of pure entropy, \(\ell = 0 \leftrightarrow \beta = 0\). For \(0 \leq \sigma \leq 1\) \(\langle \tilde{R} \rangle\) stays rather constant whereas for \(\sigma > 1\) significant deviations are observed.](image2)

To yield more information about the lattice geom-
etry the behavior of the triangle areas and deficit an-
gles is examined separately. Figure 3 displays the scale-
invariant quantity \(\langle A_t \rangle / \langle q_t \rangle\) as a function of \(\ell\) for differ-
ent values of \(\sigma\). For small lattice spacing, \(0 \leq \ell \leq 0.4\),
this ratio stays almost constant and somewhat below
the value \(\sqrt{\frac{3}{4}} \approx 0.433\) corresponding to equilateral trian-
gles. The curve for \(\sigma = 1.5\) lies significantly below the others. Across the transition at \(\ell \approx 0.45\) the ratio
\(\langle A_t \rangle / \langle q_t \rangle\) decreases indicating a distortion of the triangles.

![Figure 3: Ratio \(\langle A_t \rangle / \langle q_t \rangle\) as a function of the lattice spacing \(\ell\) for different measures parametrized by \(\sigma \geq 0\). In the well-
defined phase this scale-invariant quantity stays almost con-
stant and somewhat below the maximum value \(\sqrt{\frac{3}{4}} \approx 0.433\)
for equilateral triangles. The expectation values decrease
slightly with increasing \(\sigma\).](image3)
The expectation value of the average deficit angle $\langle \delta_t \rangle$ as a function of $\ell$ is depicted in Figure 4. Surprisingly, $\langle \delta_t \rangle$ stays negative even above the transition to large positive curvature. In the well-defined phase the absolute value of $\langle \delta_t \rangle$ is rather small compared to $\pi$. This can be understood by considering the hypercubic triangulation of the 4-torus that contains two different types of triangles. The number of 4-simplices sharing a triangle of the first type is 6 while it is 4 for the second type. The contributions of these two types to the average deficit angle almost cancel each other leaving a small negative value $\sigma$.

The dependence of the simplicial path integral on the measure within the framework of dynamical triangulation has been studied by Brügmann [12]. By means of an additional term $n \sum_v \ln |o(v)|$ in the action the type of the measure is controlled via the parameter $n$. Although the correspondence with the continuum is not entirely clear, it is plausible that the cases with $n = -5$ and $n = 0$ reproduce the scale-invariant and the uniform measure, respectively. Besides a shift in $\ell$ the picture has a striking similarity to Figure 1 if one identifies $\sigma = 0$ with $n = -5$ (scale-invariant measure) and $\sigma = 1$ with $n = 0$ (uniform measure). Notice the small influence of $n$ in the range $-5 \leq n \leq +1$ and the exceptional behavior of $n = +5$.

The expectation value of the average deficit angle $\langle \delta_t \rangle$ as a function of $\ell$ is depicted in Figure 4. Remarkably, $\langle \delta_t \rangle$ stays always negative even across the transition to positive curvature. The curves lie close together for $0 \leq \sigma \leq 1$ and differ significantly for $\sigma = 1.5$.

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All these results show that the considered family of measures falls into two qualitatively different classes $0 \leq \sigma < 1$ and $\sigma > 1$. The behavior of the considered expectation values suggests that uniform and scale-invariant measure seem to belong to the same class.\footnote{We use $\ell = \text{sgn}(\beta) \sqrt{\frac{1}{2} \langle \beta \rangle}$ as a definition of the lattice spacing whenever $\ell < 0$.}