A REMARK ON THE CHOW RING OF SOME HYPERKÄHLER FOURFOLDS

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ABSTRACT. Let $X$ be a hyperkähler variety. Voisin has conjectured that the classes of Lagrangian constant cycle subvarieties in the Chow ring of $X$ should lie in a subring injecting into cohomology. We study this conjecture for the Fano variety of lines on a very general cubic fourfold.

1. INTRODUCTION

For a smooth projective variety $X$ over $\mathbb{C}$, let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Q}$–coefficients, modulo rational equivalence). Let $A^i_{hom}(X)$ denote the subgroup of homologically trivial cycles.

As is well–known, the world of Chow groups is still largely shrouded in mystery, its map containing vast unexplored regions only vaguely sketched in by conjectures [6], [9], [10], [11], [14], [23], [15]. One region on this map that holds particular interest is that of hyperkähler varieties (i.e. projective irreducible holomorphic symplectic manifolds [3], [2]). Here, motivated by results for $K3$ surfaces and for abelian varieties, in recent years significant progress has been made in the understanding of Chow groups [4], [22], [24], [21], [18], [19], [16], [17], [7], [12], [13], [8].

It is expected that for a hyperkähler variety $X$, the Chow groups split in a finite number of pieces

$$A^i(X) = \bigoplus_j A^i_{(j)}(X),$$

such that $A^i_{(0)}(X)$ is a bigraded ring and $A^i_{(0)}(X)$ injects into cohomology. This was first conjectured by Beauville [5], who conjectured more precisely that the piece $A^i_{(j)}(X)$ should be isomorphic to the graded $Gr_{F^j}A^i(X)$ for the conjectural Bloch–Beilinson filtration.

What kind of cycles are contained in the subring $A^i_{(0)}(X)$? Certainly divisors and the Chern classes of $X$ should be in this subring. In addition to this, Voisin has stated the following conjecture:

**Conjecture 1.1** (Voisin [24]). Let $X$ be a hyperkähler variety of dimension $2m$. 
(i) Let $Y \subset X$ be a Lagrangian constant cycle subvariety (i.e., $dim Y = m$ and the pushforward map $A_0(Y) \to A_0(X)$ has image of dimension 1). Then

$$Y \in A^m_{00}(X).$$
(ii) The subring of $A^*(X)$ containing divisors, Chern classes and Lagrangian constant cycle subvarieties injects into cohomology.

(NB: part (ii) follows from part (i), provided the bigrading $A^*_{(s)}(X)$ has the desirable property that $A^*_{(0)}(X) \cap A^*_{\text{hom}}(X) = 0$, which is expected from the Bloch–Beilinson conjectures.)

Evidence for conjecture 1.1 is presented in [24]. The modest aim of this note is to determine how far conjecture 1.1 can be solved unconditionally in the special case where $X$ is the Fano variety of lines on a cubic fourfold. Here, the Fourier decomposition of Shen–Vial [18] provides an unconditional splitting $A^*_{(s)}(X)$ of the Chow ring. The main result is as follows:

**Proposition** (=proposition 3.1). Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let $X$ be the Fano variety of lines in $Z$. Assume $Y \subset X$ is a Lagrangian constant cycle subvariety.

Then

$$Y \in A^2_{(0)}(X)$$

(where $A^*_{(s)}(X)$ denotes the Fourier decomposition of [18]).

This doesn’t settle conjecture 1.1(ii) (because it is not known whether $A^2_{(0)}(X) \cap A^2_{\text{hom}}(X) = 0$). However, this at least implies some statements along the lines of conjecture 1.1(ii):

**Corollary** (=corollaries 4.2 and 4.1). Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let $X$ be the Fano variety of lines in $Z$.

(i) Let $a \in A^3(X)$ be a 1–cycle of the form

$$a = \sum_{i=1}^{r} Y_i \cdot D_i \in A^3(X),$$

where $Y_i$ is a Lagrangian constant cycle subvariety and $D_i \in A^1(X)$. Then $a$ is rationally trivial if and only if $a$ is homologically trivial.

(ii) Let $a \in A^4(X)$ be a 0–cycle of the form

$$a = \sum_{i=1}^{r} Y_i \cdot b_i \in A^4(X),$$

where $Y_i$ is a Lagrangian constant cycle subvariety and $b_i \in A^2(X)$. Then $a$ is rationally trivial if and only if $a$ is homologically trivial.

**Conventions.** In this article, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A_j(X)$ the Chow group of $j$–dimensional cycles on $X$ with $\mathbb{Q}$–coefficients; for $X$ smooth of dimension $n$ the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably.

The notations $A^j_{\text{hom}}(X)$, $A^j_{AJ}(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles.

We use $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$. 
2. Preliminaries

2.1. The Fourier decomposition.

**Theorem 2.1** (Shen–Vial [18]). Let \( Z \subset \mathbb{P}^5(\mathbb{C}) \) be a smooth cubic fourfold, and let \( X \) be the Fano variety of lines in \( Z \). There is a decomposition

\[
A^i(X) = \bigoplus_{0 \leq j \leq i} A^j_{(i)}(X),
\]

with the following properties:

(i) \( A^j_{(i)}(X) = (\Pi_{X}^X)^{2i-j} A^j(X) \), where \( \{\Pi_{X}^X\} \) is a certain self–dual Chow–Künneth decomposition;

(ii) \( A^j_{(i)}(X) \subset A^j_{\text{hom}}(X) \) for \( j > 0 \);

(iii) if \( Z \) is very general, \( A^*_{(i)}(X) \) is a bigraded ring.

**Proof.** The decomposition is defined in terms of a Fourier transform, involving the cycle \( L \in A^2(X \times X) \) representing the Beauville–Bogomolov class (cf. [18, Theorem 2]). Points (i) and (ii) follow from [18, Theorem 3.3]. Point (iii) is [18, Theorem 3]. \( \Box \)

2.2. Multiplicative structure.

**Theorem 2.2** (Shen–Vial [18]). Let \( Z \subset \mathbb{P}^5(\mathbb{C}) \) be a smooth cubic fourfold, and let \( X \) be the Fano variety of lines in \( Z \). There is a distinguished class \( l \in A^2_{(0)}(X) \) such that intersection induces an isomorphism

\[
\cdot l : A^2_{(2)}(X) \xrightarrow{\cong} A^4_{(2)}(X).
\]

The inverse isomorphism is given by

\[
\frac{1}{25} L_* : A^4_{(2)}(X) \xrightarrow{\cong} A^2_{(2)}(X),
\]

where \( L \in A^2(X \times X) \) is the class defined in [18, Equation (107)].

**Proof.** This follows from [18, Theorems 2.2 and 2.4]. \( \Box \)

2.3. The class \( c \).

**Lemma 2.3** (Voisin [21], Shen–Vial [18]). Let \( Z \subset \mathbb{P}^5(\mathbb{C}) \) be a smooth cubic fourfold, and let \( X \) be the Fano variety of lines in \( Z \). Let \( c := c_2(\mathcal{E}_2) \in A^4(X) \), where \( \mathcal{E}_2 \) is the restriction to \( X \) of the tautological rank 2 vector bundle on the Grassmannian of lines in \( \mathbb{P}^5(\mathbb{C}) \). There exists a constant cycle surface \( Y_0 \subset X \) such that 

\[
Y_0 = c \text{ in } A^2(X).
\]

(In particular, \( \cdot c : A^2_{\text{hom}}(X) \to A^4(X) \) is the zero–map.)

Moreover, if \( Z \) is very general then the class \( c \) is in \( A^2_{(0)}(X) \) (where \( A^*_{(0)}(X) \) is the Fourier decomposition of [18]).
Proof. This is well–known. As explained in [21, Lemma 3.2], the idea is to consider $Y \subset X$ defined as the Fano surface of lines contained in $Z \cap H$, where $H$ is a hyperplane in $\mathbb{P}^5$. For general $H$, the surface $Y$ is a smooth surface of general type which is a Lagrangian subvariety of class $c$ in $A^2(X)$. However, if one takes $H$ such that $Z \cap H$ acquires 5 nodes, then one obtains a singular surface $Y_0$ which is rational, hence $A_0(Y_0) = \mathbb{Q}$. It follows that $Y_0 \subset X$ is a constant cycle subvariety of class $c$ in $A^2(X)$.

The last statement is [18, Theorem 21.9(iii)]. □

2.4. A result in cohomology.

Definition 2.4 (Voisin [24]). Let $X$ be a hyperkähler variety of dimension $2m$. A Hodge class $a \in H^{2m}(X) \cap F^m$ is coisotropic if

$$\cup a : H^{2,0}(X) \to H^{m+2,m}(X)$$

is the zero–map.

(This is [24, Definition 1.5], where coisotropic cohomology classes are defined in any degree 2i.)

Proposition 2.5. Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let $X$ be the Fano variety of lines in $Z$. Assume $a \in H^4(X)$ is coisotropic. Then

$$a = \lambda \cdot c \quad \text{in } H^4(X),$$

where $\lambda \in \mathbb{Q}$ and $c \in A^2(X)$ is as in lemma 2.3.

Proof. For very general $Z$, it is known that $N^2H^4(X)$ (which is the subspace of Hodge classes, as the Hodge conjecture is known for $X$) has dimension 2. This is all that we need for the proof.

For any ample class $g \in A^1(X)$, the $\mathbb{Q}$–vector space $N^2H^4(X)$ is generated by $g^2$ and $c$. (These two elements cannot be proportional, as cupping with $g^2$ induces an isomorphism $H^{2,0}(X) \cong H^{4,2}(X)$ by hard Lefschetz, whereas cupping with $c$ is the zero–map $H^{2,0}(X) \to H^{4,2}(X)$.) Let us write

$$a = \lambda_1 c + \lambda_2 g^2 \quad \text{in } N^2H^4(X).$$

The coisotropic condition forces $\lambda_2$ to be 0, and we are done. □

Remark 2.6. In particular, proposition 2.5 implies that any Lagrangian subvariety $Y \subset X$ is proportional to $c$ in cohomology:

$$Y = \lambda \cdot c \quad \text{in } H^4(X).$$

This was first observed by Amerik [1, Remark 9].

3. MAIN RESULT

Proposition 3.1. Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let $X$ be the Fano variety of lines in $Z$. Assume $Y \subset X$ is a constant cycle subvariety of codimension 2. Then

$$Y \in A^2_{(0)}(X).$$
Proof. We assume there is a decomposition

\[ Y = b_0 + b_2 \text{ in } A^2_{(0)}(X) \oplus A^2_{(2)}(X), \]

with \( b_i \in A^2_{(i)}(X) \). We will show that \( b_2 \) must be 0.

First, we claim that

\[ Y \cdot a \in A^4_{(0)}(X) \quad \forall a \in A^2(X). \tag{1} \]

Indeed, the subvectorspace \( Y \cdot A^2(X) \subset A^4(X) \) has dimension 1, as \( Y \subset X \) is a constant cycle subvariety. To prove (1), it remains to exclude the possibility that \( (Y \cdot A^2(X)) \cap A^4_{(0)}(X) = 0 \).

But we know (proposition 2.5) that

\[ Y = \lambda c \text{ in } H^4(X), \]

for some \( \lambda \in \mathbb{Q}^* \). Since \( c \in A^2_{(0)}(X) \), this implies there is a further decomposition

\[ Y = \lambda c + b'_0 + b_2 \text{ in } A^2(X), \]

with \( b'_0 \in A^2_{(0)}(X) \cap A^2_{\text{hom}}(X) \) (which is conjecturally, but not provably, zero). Consider the intersection

\[ Y \cdot c = \lambda c^2 + b'_0 \cdot c + b_2 \cdot c = \lambda c^2 \text{ in } A^4(X). \]

(Here we have used that \( c \cdot A^2_{\text{hom}}(X) = 0 \) in \( A^4(X) \), which is lemma 2.3 or [18, Lemma A.3(iii)].) Since \( c^2 = 27 \sigma_X \) where \( \sigma_X \) is a certain distinguished generator of \( A^2_{(0)}(X) \) [18, Lemma A.3(i)], the intersection \( Y \cdot c \) defines a non–zero element in \( A^4_{(0)}(X) \). This proves the claim.

To prove the proposition, consider the intersection

\[ Y \cdot \ell = b_0 \cdot \ell + b_2 \cdot \ell \text{ in } A^4(X), \]

where \( \ell \) is the class of theorem 2.2. Since \( \ell \in A^2_{(0)}(X) \) and \( A^4_{(*)}(X) \) is a bigraded ring, we have that \( b_i \cdot \ell \in A^4_{(i)}(X) \). It follows from (1) that \( Y \cdot \ell \in A^4_{(0)}(X) \) and so

\[ b_2 \cdot \ell = 0 \text{ in } A^4_{(2)}(X). \]

But then, applying theorem 2.2, we find that \( b_2 = 0 \) and we are done. \( \square \)

Remark 3.2. Let \( X \) be the Fano variety of a very general cubic fourfold. We have seen (proposition 2.5) that any Lagrangian constant cycle subvariety \( Y \) is proportional to the class \( c \) in cohomology. Proposition 3.1 suggests that the same should be true modulo rational equivalence: indeed, \( Y \) is proportional to \( c \) in \( A^2(X) \) modulo the “troublesome part” \( A^2_{(0)}(X) \cap A^2_{\text{hom}}(X) \) (which is conjecturally zero).
We present three corollaries that provide weak versions of conjecture 1.1(ii).

**Corollary 4.1.** Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let $X$ be the Fano variety of lines in $Z$. Let $a \in A^4(X)$ be a 0–cycle of the form

$$a = \sum_{i=1}^r Y_i \cdot b_i \in A^4(X),$$

where $Y_i$ is a Lagrangian constant cycle subvariety and $b_i \in A^2(X)$. Then $a$ is rationally trivial if and only if $a$ is homologically trivial.

**Proof.** We know from claim 1 that $a$ is in $A^4_{(0)}(X)$. But $A^4_{(0)}(X) \cong \mathbb{Q}$ injects into cohomology. □

**Corollary 4.2.** Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let $X$ be the Fano variety of lines in $Z$. Let $a \in A^3(X)$ be a 1–cycle of the form

$$a = \sum_{i=1}^r Y_i \cdot D_i \in A^3(X),$$

where $Y_i$ is a Lagrangian constant cycle subvariety and $D_i \in A^1(X)$. Then $a$ is rationally trivial if and only if $a$ is homologically trivial.

**Proof.** We know from proposition 3.1 that each $Y_i$ is in $A^2_{(0)}(X)$. Since $D_i \in A^1(X) = A^1_{(0)}(X)$, it follows that $a$ is in $A^3_{(0)}(X)$. But we know [18] that

$$A^3_{(0)}(X) \cap A^3_{hom}(X) = 0.$$

□

**Corollary 4.3.** Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let $X$ be the Fano variety of lines in $Z$. Let $\phi : X \dashrightarrow X$ be the degree 16 rational map defined in [20]. Let $a \in A^2(X)$ be a 2–cycle of the form

$$a = \phi^*(b) - 4b \in A^2(X),$$

where $b$ is a linear combination of Lagrangian constant cycle subvarieties and intersections of 2 divisors. Then $a$ is rationally trivial if and only if $a$ is homologically trivial.

**Proof.** We know from proposition 3.1 that $b$ is in $A^2_{(0)}(X)$. Let $V^2_\lambda$ denote the eigenspace

$$V^2_\lambda := \{ \alpha \in A^2(X) \mid \phi^*(\alpha) = \lambda \cdot \alpha \}.$$

Shen–Vial have proven that there is a decomposition

$$A^2_{(0)}(X) = V^2_{31} \oplus V^2_{14} \oplus V^2_4$$

[18] Theorem 21.9. The “troublesome part” $A^2_{(0)}(X) \cap A^2_{hom}(X)$ is contained in $V^2_4$ [18] Lemma 21.12. This implies that

$$(\phi^* - 4(\Delta_X)^*)A^2_{(0)}(X) = V^2_{31} \oplus V^2_{-14}$$
injects into cohomology.

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