Discussion about the characteristics of flow in atmospheric airflow field using the complex function

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1. Introduction

As a classical meteorological problem, the airflow over mountains has aroused interest among many scholars at home and abroad (Scorer and Klieforth 1959; Chao, Chang, and Yan 1964; Гутман 1976; Zhu, Yu, and Lu 1999; Zang and Zhang 2008; Torres, Lanz, and Larios 2016). It is very important to understand and study the impact of topography on atmospheric motion when establishing local weather forecasting methods, studying anthropogenic influences on weather and climate, and advising on aviation and glider safety (Yu, Sun, and Xiang 2013). In developing our understanding of mesoscale dynamics, many scholars have carried out a lot of relevant work in this respect.

Theoretical and numerical research on lee waves began in the 1940s and 1950s. Based on the equations of atmospheric motion, Scorer (1949) discussed lee wave theories and was the first to combine the two most important factors influencing lee waves (atmospheric stratification and horizontal wind speed) into the so-called Scorer parameter, which formed the research basis for many subsequent scholars. Long (1953, 1954) carried out a series of studies (both theoretical and experimental) that discussed the situation of stratified flow across mountains. Following that, in terms of the problem of a nonlinear two-dimensional constant lee wave, he assumed there was a fixed horizontal plane in the upper bound of the air, and a particular solution was obtained under that assumption (Long 1955). Later that same decade, Yeh (1956) systematically summarized the knowledge at that time on lee wave theories. Ignoring the rotation of the Earth, he discussed the main generative mechanism of a lee wave. Chao, Chang, and Yan (1964) discussed the influence of mountains on airflow in a two-layer model and investigated the formation of pressure jumps. Гутман (1976) summarized nonlinear theories in mesoscale meteorology, and studied the effect of topography on airflow, including Long’s work (Long 1955) and the case of the air having an infinite upper bound. A decade later, Taylor, Mason, and Bradley (1987) comprehensively summarized research on airflow...
over mountains. Since then, scholars have studied the lee waves of mesoscale terrain through theoretical analyses, practical individual cases, and numerical simulations (Zhu, Yu, and Lu 1999; Galewsky and Sobel 2005; Moore, Wilson, and Bell 2005; Zang and Zhang 2008; Xiang, Chen, and Spedding 2017), obtaining broadly consistent conclusions.

It is well-known that complex variable functions can also be used to describe the basic flow in a passive and irrotational planar vector field (Zhong 2009). With this powerful mathematical tool, weather and climate diagnosis, as well as well as data assimilation, can be carried out (Cao, Xu, and Gao 2015). Also, the movement of airflow through an obstacle can be simulated, including that through a cylindrical obstacle (Johnson and McDonald 2004a; Crowdy 2006) or a solid wall with gaps (Johnson and McDonald 2004b; Crowdy and Marshall 2006). Several studies have also been carried out based on conformal mapping (Burton, Gratus, and Tucker 2004; Ryzhov, Izrailskii, and Koshel 2014), in which a target area is transformed into a regular plane to facilitate discussion about the specific flow situation.

In this paper, the movement of a parallel airflow over an arc-shaped mountain is studied firstly; the upper half plane with a circular region removed is mapped to the regular upper half plane, and the influence of the airflow velocity and of the angle between the mountain and the negative direction of horizontal ground on the airflow movement is discussed. Then, a parallel flow is superimposed onto a point vortex and the combined movement is investigated when passing a linear boundary. To ensure that the line \( y = 0 \) is a zero-streamline, a virtual vortex is required for the point vortex. Now, the vortex core is an extreme point, and its velocity is infinite, which induces the motion of the vortex core to remain unchanged, no matter how fast the velocity of the superimposed parallel flow; that is, the vortex core remains stationary. We are clear about the motion of basic flow on a horizontal plane. On the one hand, for an irregular region, it can be mapped to the regular upper half plane by conformal mapping, and its movement in the target region can be understood. On the other hand, for some flows that can be represented as a combination of certain basic flows, according to the known flow conditions, the unknown movement can be determined, and the streamline chart can be drawn correspondingly.

2. Theoretical basis of the motion equation determined by the complex potential

Based on complex function theory (Zhong 2009), we assume that \( w(z) = \phi(x, y) + i\psi(x, y) \) is the complex potential of a certain flow. Its real part \( \phi(x, y) \) and imaginary part \( \psi(x, y) \) are defined as the potential function and the stream function respectively; and correspondingly, \( \psi(x, y) = c \) \((c \text{ is a constant})\) and \( \psi(x, y) = c \) \((c \text{ is a constant})\) are the potential line and streamline respectively. As we know, the stream function and potential function are conjugate harmonic functions. Therefore, once the complex potential is determined, the potential lines and streamlines can be obtained on the basis of its real part and imaginary part, and the flow situation can be understood.

On the other hand, in a passive and irrotational planar vector field, complex potential and complex velocity have the following relationship:

\[
w(z) = \int_C \frac{\nu(z)}{z} \, dz
\]

where \( \nu(z) \) is complex velocity, \( \frac{\nu(z)}{z} \) is its complex conjugate. The complex velocity can be determined by the complex potential according to the following equation:

\[
\frac{d\nu}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = u - iv = \tilde{V}
\]

Therefore,

\[
\frac{dx}{dt} = u, \quad (3a)
\]

\[
\frac{dy}{dt} = v. \quad (3b)
\]

The motion trajectory of a particle can be obtained by integrating Equation (3).

3. Analysis of a parallel flow and a point vortex motion

3.1. Movement of a parallel flow over a mountain

The atmosphere is assumed as an ideal fluid, ignoring the turbulence. In reality, the width of a mountain is usually (often many times) larger than its height, the wind speed and vertical temperature gradient change considerably, and the basic flow is also often restricted by the inversion layer. However, given that the flow at infinity is a horizontal structure, for simplicity, we assume that the parallel flow velocity is constant.

In order to map the arc-shaped mountain to the upper half plane, we use the following composite mapping:

\[
z_1 = \frac{z - a}{z + a}, \quad (4a)
\]

\[
z_2 = z_1^{i\frac{a}{R}}, \quad (4b)
\]
ζ = b \frac{1}{1 - z_2}, \quad (4c)

where \( z, z_1, z_2, \) and \( ζ \) are complex variables; \( a = R \sin \pi α \) (approximately equal to the half-value of the mountain range (suppose \( R = 1 \)), in which \( π α \) is the angle between the mountain and the negative direction of horizontal ground); and \( b \) is a parameter to be determined. Based on the conformal properties of the analytical transform, we can obtain ‘when \( z = ∞, \ ζ = 1 \)’. Thus, we can also obtain \( b = \frac{2a}{α} \).

Figure 1(a–d) represent the \( z \)-plane, \( z_1 \)-plane, \( z_2 \)-plane, and \( ζ \)-plane respectively, where the \( x \)-axis and \( y \)-axis represent the real axis and imaginary axis respectively. Based on the conformal transformation, Equation (4a), the upper half plane with a circular region removed in the \( z \)-plane is mapped to an angle-shaped region in the \( z_1 \)-plane. Based on Equation (4b), the above region is mapped to the regular upper half plane in the \( z_2 \)-plane. Based on Equation (4c), the points \( -∞, -a, a, \) and \( +∞ \) in the \( z \)-plane are mapped to the points \( -∞, 0, b, \) and \( +∞ \) in the \( ζ \)-plane respectively.

The complex potential of a parallel flow is
\[ \omega = V_∞ ζ, \quad (5) \]
where \( V_∞ \) is constant, \( ζ \) is the function of the complex variable \( z \); that is, \( ζ = ζ(z) \).

Then, the movement of the parallel flow over a semicircular mountain can be discussed.

Firstly, the effect of velocity on streamlines is studied. We assume that \( a = 1/2 \), and draw the streamlines at 0.5 intervals, where the zero-streamline is approximately the terrain.

Then, the effect of the angle between the mountain and the negative direction of horizontal ground on the flow is studied. We assume that \( V_∞ = 1 \), change \( a \), and then draw the streamlines at 0.5 intervals.

The results indicate that the airflow moves forward at a constant speed on the windward side of the mountain, fluctuations begin to appear when the airflow is close enough to the mountain, and the intensity decreases with the increase of height. Fluctuations disappear on the leeward side of the mountain, and the airflow continues to move forward. Figure 2 shows that when the shape of the mountain is certain, streamlines become dense with the increase of parallel flow velocity. Figure 3 shows that when the parallel flow velocity is certain, the fluctuation degree of streamlines increases with the decrease of the angle between the mountain and the negative direction of horizontal ground.

3.2. Movement of a parallel flow superimposed onto a point vortex when passing a linear boundary

Here, we consider a barotropic non-viscous flow. The movement of a point vortex passing a linear boundary is discussed. In order to guarantee the symmetry of the upper and lower half plane, we need to ensure that the line \( y = 0 \) is 0-streamline (while the horizontal ground is the terrain function), and a virtual vortex with the

![Figure 1](image-url)
same intensity and local symmetry along the boundary is taken into consideration. We can thus obtain the following complex potential:

\[ w = \frac{\Gamma}{2\pi i} \ln \left( \frac{z}{z_0} \right) / C_0 \left( \frac{1}{z} \right) / C_0 \left( \frac{1}{z_0} \right) \]  

(6)

where \( \Gamma \) represents the motion intensity of the point vortex, \( z \) is a complex variable, \( z_0 \) is the coordinates of the vortex core, and \( \bar{z}_0 \) is its complex conjugate. The complex velocity of every point can be determined by the expression as follows:

\[ \vec{V} = \frac{\Gamma}{2\pi i} \left( \frac{1}{z} - \frac{1}{z_0} \right) \]  

(7)

Assume that the coordinate of the point vortex core is \((-5, 0.5)\), and the velocity is \( \infty \). The streamlines chart can be seen in Figure 4(a).

When a parallel flow with velocity \( V_\infty \) is composited, the complex potential can be expressed by

\[ \omega = V_\infty z + \frac{\Gamma}{2\pi i} \left( \ln(z - z_0) - \ln(z - \bar{z}_0) \right) \]  

(8)

Now, the coordinates of the vortex core are \((-5, 0.5)\). We change \( V_\infty \), and draw streamlines at 0.5 intervals as follows:

Without any boundary conditions, each point moves along a certain circle, the movement speed increases with the decrease of the distance to the vortex core. Near the horizontal boundary, the bottom of the circle will be deformed, and the movement speed will become larger (Figure 4(a)). When the velocity of the composited parallel flow is relatively small, it is not enough to change the motion of the vortex core and the particles near it; that is, these particles still move according to the original streamlines: the closer they are to the ground or to the vertex core, the greater the velocity. With the increase of the parallel flow velocity, the quantity of the particles that can maintain the original state of motion gradually decreases, but the vortex core remains stationary (Figure 4(b–d)).

Figure 2. Streamline charts of the parallel flow with different velocities over a semicircular mountain: (a) \( V_\infty = 1 \); (b) \( V_\infty = 2 \); (c) \( V_\infty = 3 \).

Figure 3. Streamline charts of a parallel flow with constant velocity over a mountain with different angles between the mountain and the negative direction of horizontal ground: (a) \( a = 3/4 \); (b) \( a = 2/3 \); (c) \( a = 1/2 \).
4. Conclusions

Based on complex function theory, the motion of the parallel flow over mountains and the motion of a parallel flow superimposed onto a point vortex through a linear boundary are discussed. The main conclusions are as follows:

(1) For the case where the parallel flow moves forward at a constant speed on the windward side of the mountain, when it is close enough to the mountain, fluctuations occur and their intensity increases with the increase of the velocity and the decrease of the angle between the mountain and the negative direction of horizontal ground. After passing the mountain, the fluctuations disappear and the airflow continues to move forward (Figures 2 and 3).

(2) When there is no boundary condition, the particle moves along a circle. The movement speed increases with the decrease of the distance to the vortex core. When the horizontal ground is the boundary, the bottom of the circle will be flattened, and the speed will increase (Figure 4(a)). Because of the singularity of the vortex core, the combined parallel flow is not enough to change the motion state of the vortex core; that is, the vortex core remains stationary. With the increase of the horizontal velocity, the particles surrounding the vortex core will not move in a circle due to the effect of the horizontal velocity, but will move forward along the parallel flow (Figure 4(b–d)).

(3) This research is a theoretical result in an idealized state. The real atmosphere is viscous, so it is unsuitable to simulate airflow and vortices over mountains using complex potential functions; however, it is nevertheless an approximation of the real atmosphere and can serve as a reference in numerical simulations of airflow over mountains.

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