The Solutions of Affine and Conformal Affine Toda Field Theories

G. Papadopoulos

II. Institute for Theoretical Physics
Luruper Chaussee 149
22761 Hamburg Germany

and

B. Spence

School of Physics
University of Melbourne
Parkville 3052 Australia

ABSTRACT

We give new formulations of the solutions of the field equations of the affine Toda and conformal affine Toda theories on a cylinder and two-dimensional Minkowski space-time. These solutions are parameterised in terms of initial data and the resulting covariant phase spaces are diffeomorphic to the Hamiltonian ones. We derive the fundamental Poisson brackets of the parameters of the solutions and give the general static solutions for the affine theory.

⋆ E-mail: gpapas@vxdesy.desy.de
† E-mail: spence@tauon.ph.unimelb.edu.au
1. Introduction

It has been known for many years that the field equations of certain two-dimensional field theories can be solved exactly. Some of these theories are the Wess-Zumino-Witten model, Liouville field theory and and the various versions of Toda field theory. The field equations of these theories are non-linear, partial, hyperbolic differential equations and various discussions of their solutions have been presented in the literature. One method for solving such integrable systems is based on the theory of Lax pairs and this has been extensively studied for Toda systems [1-3]. These authors solve the field equations on two-dimensional Minkowski space-time and describe their solutions in terms of functions that depend on the light-cone co-ordinates $x^\pm$ of the Minkowski space-time. Some difficulties with these solutions are that the ranges of the solution parameters are not clearly specified, and the relation of the parameters to the initial data of these theories is not given.

To deal with the above deficiencies, new parameterisations of solutions of the field equations of the Wess-Zumino-Witten, Liouville, Toda and non-Abelian Toda conformal field theories were given in refs. [4-5]. These parameterisations have the advantage that the solutions of these theories are explicitly well-defined for all the values of their parameters, the Poisson brackets of their parameters can be directly calculated, and the spaces of these parameters are isomorphic to the spaces of the associated initial data, i.e. the solutions of these theories are parameterised directly in terms of their initial data. This made it possible to give covariant phase space descriptions of these theories which are explicitly diffeomorphic to the Hamiltonian phase space formulations.

A lot of attention has been focused recently on integrable, but not necessarily conformal, two-dimensional field theories. An important class is the affine Toda field theories, which are integrable perturbations of conformal field theories [6]. These models have a rich algebraic structure, involving exact S-matrices and soliton solutions (see refs. [7-9] and references therein). The solutions of the classical field equations of these theories were discussed recently in ref. [9], following the Leznov-
In this paper, we will give a new formulation of the solutions of affine Toda field theory. This formulation will be along the lines of similar solution space parameterisations given by us for the WZW and Toda-type field theories. In this new parameterisation, the affine Toda solutions are well-defined over all space, which is taken to be a cylinder or two-dimensional Minkowski space-time. The Poisson brackets of the parameters of the solutions are calculated. An explicit isomorphism is also constructed between the space of initial data and the space of parameters of the solutions of affine Toda field theory. A similar formulation of the solutions of the field equations of the conformal affine Toda field theory is also given. Finally, using our new approach, the general static solution of the affine Toda theory is presented.

2. The Solutions of Affine Toda Field Theory

Let $g$ be a (semi-) simple Lie algebra and $\mathcal{H}$ be a Cartan subalgebra of $g$. We introduce a Chevalley basis $(H_i, E_{\alpha^i}, E_{-\alpha^i})$ in $g$, where $\Delta \equiv \{\alpha^i, i = 1, \ldots, l = \text{rank } g\}$ is the set of simple roots, $H_i \equiv \frac{2\alpha^i \cdot H}{|\alpha^i|^2}$, $H \in \mathcal{H}$, $E_{\pm \alpha^i}$ are the step operators for the simple roots and $[H_i, H_j] = 0$, $[E_{\alpha^i}, E_{-\alpha^i}] = H_i$ and $[H_i, E_{\pm \alpha^i}] = \pm K_{ij} E_{\pm \alpha^j}$ (with no summation over $j$). The matrix $K \equiv \{K_{ij}\}$ is the Cartan matrix of $g$, i.e. $K_{ij} = \frac{2\alpha^i \cdot \alpha^j}{|\alpha^i|^2}$. The symbols $\Phi^+$ ($\Phi^-$) will denote the sets of positive (negative) roots, respectively, and $\Phi \equiv \Phi^+ \cup \Phi^-$ is the space of all roots of $g$. We will also use the symbols $\mathcal{L}^+(g)$ and $\mathcal{L}^-(g)$ to denote the sets of step operators for the positive and negative roots respectively. Finally, we normalise the Killing form $(\cdot, \cdot)$ as follows: $(H_i, H_j) \equiv \text{Tr}(H_i \cdot H_j) = C_{ij}$, $(E_{\alpha^i}, E_{-\alpha^i}) \equiv \text{Tr}(E_{\alpha^i} \cdot E_{-\alpha^i}) = \frac{2}{|\alpha^i|^2} \delta_{ij}$ and $(E_{\alpha^i}, H_j) \equiv \text{Tr}(E_{\alpha^i} \cdot H_j) = 0$, where $C_{ij} = \frac{2}{|\alpha^i|^2} K_{ij}$.

The Lagrangian of affine Toda field theory is

$$L = -\frac{\kappa^2}{8\pi} \left( (\partial_+ \phi, \partial_- \phi) + \frac{2M^2}{\kappa^2} \left( \sum_i \frac{m_i}{|\alpha^i|^2} \exp \left[ \frac{\kappa}{2} (\alpha^i)^2 (H_i, \phi) \right] + \frac{1}{\psi^2} \exp \left[ -\frac{\kappa}{2} \psi^2 (m, \phi) \right] \right) \right)$$

(1)
or equivalently in components

\[ L = -\frac{\kappa^2}{8\pi} \left( C_{ij} \partial_+ \phi^i \partial_- \phi^j + \frac{2M^2}{\kappa^2} \left( \sum_i \frac{m_i}{|\alpha_i|^2} \exp[\kappa K_{ij} \phi^j] + \frac{1}{\psi^2} \exp[-\kappa \psi^2 C_{ij} \frac{m_i}{2} \phi^j] \right) \right), \]

(2)

where \( \phi \) is a map from a cylinder \( S^1 \times \mathcal{R} \) to \( \mathcal{R}^l \) (\( \phi \equiv \phi^i H_i \)) and \( \kappa, M \) are non-zero coupling constants. The symbol \( \psi \) denotes the highest root of \( g \), the integers \( m_i \) being defined by the relation \( \frac{\psi}{\bar{\psi}} = \sum_i m_i \frac{\alpha'_i}{|\alpha'_i|^2} \), with \( m \equiv m^i H_i \). The pairs \((x, t) : 0 \leq x < 1, -\infty < t < \infty\) are the co-ordinates of \( S^1 \times \mathcal{R} \) and \( x^\pm = x \pm t, \partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t) \).

The equations of motion following from the Lagrangian (2) are

\[ \partial_+ \partial_- \phi^i - \frac{M^2 m_i}{2\kappa} \left( \exp(\kappa K_{ij} \phi^j) - \exp(-\frac{\kappa}{2} \psi^2 C_{ij} m^k \phi^k) \right) = 0. \]

(3)

Let \( \hat{g} \) be the affine Lie algebra associated to the Lie algebra \( g \). We denote by \( \alpha^r, r = 0, \ldots, l \), and \( \Lambda_r, r = 0, \ldots, l \) the simple roots and the lowest fundamental weights of \( \hat{g} \) respectively. The step operators for the simple roots of \( \hat{g} \) are denoted by \( \hat{E}_{\alpha^r} \). The new formulation of the solutions of the affine Toda field equations (3) takes the form

\[ \exp(-\kappa \phi^i(x, t)) = \exp(-\kappa \phi^i_R(x^-)) \frac{\langle \Lambda_i \mid W(A; x^+, x^-) \mid \Lambda_i \rangle}{\langle \Lambda_0 \mid W(A; x^+, x^-) \mid \Lambda_0 \rangle^{m_r}}, \]

(4)

where \( W \) is the holonomy of a connection \( A \) and \( \phi^i_R \) is a periodic function on the real line. The components of the connection \( A \equiv A_0 + A_< + A_> \) are

\[ A_0 = \kappa u, \]
\[ A_< = \mu E_{-1}, \]
\[ A_> = \nu \exp(\kappa \phi_R) \hat{E}_1 \exp(-\kappa \phi_R), \]

(5)

where \( u \) is a periodic one-form on the real line with values in the Cartan subalgebra \( \mathcal{H} \) of \( g \), \( \hat{E}_{\pm 1} \equiv \sum_{r=0}^l \sqrt{m_r} \hat{E}_{\pm \alpha^r} \) and \( \mu, \nu \) are non-zero real constants satisfying the
relation $M^2 = 2\mu\nu$. That the expression for $\phi$ in equation (4) solves the field equations (3) of affine Toda theory follows from exactly the same argument as that presented in ref. [9], where the Leznov-Saveliev formulation of the solutions was considered. Our notation follows this reference. We will not repeat this argument here. The periodicity of the solution $\phi$ in (4) follows from the periodicity of its independent parameters $u$ and $\phi_R$. The space of independent parameters of the solutions is diffeomorphic to $T^*L\mathcal{R}_l$, i.e. the cotangent bundle of the loop space of $\mathcal{R}_l$. Note that the solutions of the affine Toda field theory on two-dimensional Minkowski space-time are also given by equation (4). The only difference in this case is that the independent parameters $u$ and $\phi_R$ of the solutions are functions on the real line which are not necessarily periodic.

The Lagrangian symplectic form of the affine Toda theory is

$$
\Omega = \frac{\kappa^2}{16\pi} \int_0^1 dx \, C_{ij} \, \delta \phi^i \partial_t \delta \phi^j,
$$

(6)
evaluated at $t = 0$, where the map $\phi$ in eqn. (6) satisfies the affine Toda field equations of motion (4). To express $\Omega$ in terms of the independent parameters of the solutions, we insert the Toda solution (4) into the form $\Omega$, giving

$$
\Omega = -\frac{\kappa^2}{16\pi} \int_0^1 dx \, C_{ij} \left( \delta \phi^i_R \partial_x \delta \phi^j_R - 2\delta \phi^i_R \delta u^j \right),
$$

(7)

where $u^i = -<\lambda_i|u|\lambda_i>$ and $\{\lambda^i; i = 1,\ldots,l\}$ is the set of fundamental lowest weights of $g$. The Poisson brackets are easily obtained from eqn. (7), and are

$$
\{\phi^i_R(x), \phi^j_R(y)\} = 0,
$$

$$
\{\phi^i_R(x), u^j(y)\} = \frac{8\pi}{\kappa^2} (C^{-1})^{ij} \delta(x,y),
$$

(8)

$$
\{u^i(x), u^j(y)\} = \frac{8\pi}{\kappa^2} (C^{-1})^{ij} \partial_x \delta(x,y).
$$

Note that these brackets are the Poisson brackets on the cotangent bundle of the
loop space of $\mathcal{R}^t$.

The space of initial data of affine Toda field theory is the space of affine Toda fields $\phi$, at $t = 0$, and their time derivatives $\frac{\partial \phi^i}{\partial t}$ at $t = 0$, and it can be equipped with the symplectic form induced from eqn. (6). It is straightforward to see that the Hamiltonian phase space of affine Toda field theory is isomorphic as symplectic manifold to the space of initial data of this theory. There is also an isomorphism between the space of initial data of affine Toda field theory and the space of parameters of the solutions (4) of the field equations (3). From the solutions (4) of affine Toda theory, we find immediately that the map between the initial data $f^i(x) \equiv \phi^i(x,0)$, $w^i(x) \equiv \frac{\partial \phi^i}{\partial t}(x,0)$ and our covariant phase space parameters $\phi^i_R, u^i$ is

$$f^i(x) = \phi^i_R(x),$$

$$w^i(x) = -\partial_x \phi^i_R(x) + 2u^i(x).$$

This map is clearly a diffeomorphism. Moreover it is a symplectic diffeomorphism from the space of initial data to the space of parameters of the solutions of the affine Toda theory because, as it is easy to show, it maps the symplectic form (4) to the form $\Omega = \frac{\kappa^2}{16\pi} \int_0^1 dx C_{ij} \delta f^i(x) \delta w^j(x)$. Finally, we can use the same argument as in the case of Toda theory of ref. [5] to prove that the solutions (4) of the affine Toda field theory are real and well-defined for all the values of the parameters $\phi_R$ and $u$.

3. The Conformal Affine Toda Field Theory

Let $d \equiv -L_0$ be the derivation associated with the affine Lie Algebra $\tilde{\mathfrak{g}}$. We denote by $k$ the central charge generator and extend the inner product $(\cdot, \cdot)$ from $\mathfrak{g}$ to the vector space $\mathfrak{g}_0 \equiv \mathfrak{g} \oplus \mathcal{R}[k, d]$ by taking $(d, k) = 1$, $(d, d) = 0$, $(k, k) = 0$, $\mathfrak{g}$ orthogonal to $\mathcal{R}[k, d]$, and $\mathcal{H}_0 \equiv \mathcal{H} \oplus \mathcal{R}[k, d]$. If $V_1, V_2 \in \mathcal{H}_0$ and $V_1 = v_1 + v'_1 k + \bar{v}_1 d$, $v_1 \in \mathcal{H}$, similarly for $V_2$, we define $V_1(V_2) \equiv (v_1, v_2) + v'_1 v'_2 + \bar{v}_1 \bar{v}_2$. 
The Lagrangian of conformal affine Toda field theory is [10]

\[
L = -\frac{\kappa^2}{8\pi} \left( (\partial_+ \Phi, \partial_- \Phi) + \frac{2M^2}{\kappa^2} \sum_{r=0}^{l} \frac{m^r}{|a^r|^2} \exp(\kappa a^r(\Phi)) \right),
\]

(10)

where \( \Phi \) is a map from a cylinder \( S^1 \times \mathcal{R} \) to \( \mathcal{H}_0 \), \( \hat{\Delta} \equiv \{ a^r; r = 0, \ldots, l \} \) is the set of simple roots of the affine Lie algebra \( \hat{\mathfrak{g}} \), the positive integers \( \{ m^r; r > 0 \} \) are defined as in the previous section, \( m^0 = 1 \), and \( \kappa \) and \( M \) are real, non-zero coupling constants. The pairs \( (x, t): 0 \leq x < 1, -\infty < t < \infty \) are the co-ordinates of \( S^1 \times \mathcal{R} \) and \( x^\pm = x \pm t, \partial^\pm = \frac{1}{2} (\partial_x \pm \partial_t) \).

The field equations of conformal affine Toda theory following from the Lagrangian (10) are

\[
\partial_+ \partial_- \Phi - \frac{M^2}{\kappa} \sum_{r=0}^{l} \frac{m^r H^a r}{|a^r|^2} \exp(\kappa a^r(\Phi)) = 0.
\]

(11)

It is known that the above field equation can be truncated to the field equation of affine Toda field theory [10]. To do this, we rewrite the field \( \Phi \) in the basis \( \{ H, k, d' = hd + \theta H \} \) as

\[
\Phi = \phi H + \xi k + \eta d',
\]

(12)

where \( h \) is the Coxeter number of \( \mathfrak{g} \) and \( \theta = \sum_{i=1}^{l} \frac{\lambda^i}{|\alpha^i|^2} \), we identify the affine Toda field with the first component \( \phi \) of \( \Phi \), and set the fields \( \eta \) and \( \xi \) to zero.

The solutions of the field equations (11) of conformal affine Toda field theory are

\[
\exp(\kappa \Lambda^r(\Phi(x, t))) = \exp(\kappa \Lambda^r(\Phi_R(x^-))) \langle \Lambda_r | \mathcal{W}(A;x^+, x^-)| \Lambda_r \rangle
\]

\[
\eta = \int_{x^-}^{x^+} b(s) \, ds + \eta_R(x^-),
\]

(13)

where \( \mathcal{W} \) is the holonomy of a connection \( A \), \( \Phi_R \) is a map from the real line into \( \mathcal{H}_0 \), \( \{ \eta_R, b \} \) are maps from the real line into the real line \( \mathcal{R} \) and \( \{ \Lambda_r; r = 0, \ldots, \text{rank} \mathfrak{g} \} \).
are the lowest fundamental weights of the affine Lie algebra \( \hat{\mathfrak{g}} \). Note that \( \eta \) in eqn. (13) is a free field, i.e. satisfies \( \partial_+ \partial_- \eta = 0 \). The addition of the equation for \( \eta \) in (13) is necessary because the lowest fundamental weights \( \Lambda_r \) span a subspace of \( \mathcal{H}_0 \) of co-dimension one. The components of the connection \( A \equiv A_0 + A_{>0} + A_{<0} \) in equation (13) are

\[
\begin{align*}
A_0 &= \kappa u, \\
A_{>0} &= \mu \hat{E}_1, \\
A_{<0} &= \nu e^{\kappa \Phi_R} \hat{E}_{-1} e^{-\kappa \Phi_R},
\end{align*}
\]

where \( u \) is a periodic map from the real line into \( \mathcal{H}_0 \). The independent parameters of the solutions of the conformal affine Toda field theory are \( \{ u^r \equiv -\langle \Lambda_r \mid u \mid \Lambda_r \rangle ; r = 0, \ldots, l \} \), \( \{ \Phi^r_R \equiv -\Lambda^r(\Phi_R) ; r = 0, \ldots, l \} \), \( b \), and \( \eta_R \), and the space of independent parameters is diffeomorphic to \( T^* L \mathcal{H}_0 \). Note the difference in the definitions of \( u^r \) and \( \Phi^r_R \). The periodicity of the solutions \( \Phi \) of the conformal affine Toda theory in the co-ordinate \( x \) follows from the periodicity of the independent parameters \( u^r, b, \Phi^r_R \) and \( \eta_R \). If we consider the conformal affine Toda field theory on two-dimensional Minkowski space-time, rather than on the cylinder, then the solutions of this theory are still given by equation (13), but in this case the independent parameters of the solutions are functions on the real line which are not necessarily periodic.

The Lagrangian symplectic form of the conformal affine Toda theory is

\[
\Omega = \frac{\kappa^2}{16\pi} \int_0^1 dx \left( \delta \Phi, \partial_t \delta \Phi \right),
\]

evaluated at \( t = 0 \), where the map \( \Phi \) in eqn. (6) satisfies the field equations (11) of conformal affine Toda theory. To express the symplectic form \( \Omega \) in terms of the free parameters \( u^r, \Phi^r_R, b \) and \( \eta_R \) of the solutions (13), we insert these solutions into \( \Omega \), giving

\[
\Omega = -\frac{\kappa^2}{16\pi} \int_0^1 dx \left( C_{ij} (\delta \varphi^i_R \partial_x \delta \varphi^j_R - 2 \delta \varphi^i_R \delta \tilde{u}^j) + D_{pq} (\delta \pi^p_R \partial_x \delta \pi^q_R - 2 \delta \pi^p_R \delta \tilde{b}^q) \right),
\]
where $\tilde{u}^i = u^i - m^i u^0$, $\pi_R = (\xi_R, \eta_R)$, $\tilde{b} = (\frac{2}{\psi^2} u^0, b)$, the non-zero components of the matrix $D$ are $D_{01} = D_{10} = h$, and $p, q = 0, 1$. Here we have used the decomposition $\Phi_R = \varphi_R^i H_i + \xi_R k + \Phi_R^0 d$ of the vector $\Phi_R \in \mathcal{H}_0$ and $\{u^r; r = 0, \ldots, l\} = \{u^0, u^i; i = 1, \ldots, l\}$. Note that $\Phi_R^0$ does not appear in the symplectic form (16).

The symplectic form (16) can be inverted to calculate the Poisson brackets of the theory. We find them to be

$$\{\varphi_R(x), \varphi_R(y)\} = 0, \quad \{\pi_R(x), \pi_R(y)\} = 0,$$

$$\{\varphi_R^i(x), \tilde{u}^j(y)\} = \frac{8\pi}{\kappa^2} (C^{-1})^{ij} \delta(x, y), \quad \{\pi_R^p(x), \tilde{b}^q(y)\} = \frac{8\pi}{\kappa^2} (D^{-1})^{pq} \delta(x, y),$$

$$\{\tilde{u}^i(x), \tilde{u}^j(y)\} = \frac{8\pi}{\kappa^2} (C^{-1})^{ij} \partial_x \delta(x, y), \quad \{\tilde{b}^p(x), \tilde{b}^q(y)\} = \frac{8\pi}{\kappa^2} (D^{-1})^{pq} \partial_x \delta(x, y).$$

(17)

These Poisson brackets can be written in a form similar to that obtaining for the affine Toda theory of the previous section, by defining $\chi_R \equiv (\varphi_R, \pi_R)$ and $z \equiv (\tilde{u}, \tilde{b})$. Expressing the above Poisson brackets in terms of $\chi_R$ and $z$, we get

$$\{\chi_R^I(x), \chi_R^J(y)\} = 0,$$

$$\{\chi_R^I(x), z^J(y)\} = \frac{8\pi}{\kappa^2} (\hat{C}^{-1})^{IJ} \delta(x, y),$$

$$\{z^I(x), z^J(y)\} = \frac{8\pi}{\kappa^2} (\hat{C}^{-1})^{IJ} \partial_x \delta(x, y).$$

(18)

where the matrix $\hat{C} \equiv C \oplus D$ is the inner product $(\cdot, \cdot)$ of $G_o$ restricted on $\mathcal{H}_o$ and evaluated in the basis $\{H_i, k, d\}$ of $\mathcal{H}_o$, and $I, J = 1, \ldots, \dim \mathcal{H}_o$. It can be shown as in the Toda case that the space of independent parameters of the solutions (13) of the field equations of the conformal affine Toda field theory is isomorphic as a symplectic manifold to the space of initial data of this theory, and that the solutions of this theory are real and well-defined for all the values of these parameters.
4. Affine Toda Particles and Static Solutions

The static solutions of the field equations of affine Toda field theory are those that are independent of the time co-ordinate $t$ of the space-time. The static solutions of affine Toda field theory obey the ordinary differential equation

$$\partial_x^2 \phi^i - \frac{2M^2m^i}{\kappa} \left( \exp(\kappa K_{ij}\phi^j) - \exp\left(-\kappa\psi^2 C_{kl}\frac{m^k}{2}\phi^l\right) \right) = 0,$$

(19)

which follows from setting $\partial_t \phi = 0$ in the field equation of affine Toda field theory (3). One way to find the solutions of this equation is to use the formula (4) for all solutions of affine Toda field theory, and find which solutions are independent of $t$. However, this turns out to be a rather complicated algebraic problem. Instead, it is easier to solve equation (19) directly by comparing it with the equation of the affine Toda particle, which is

$$\partial_t^2 \phi^i + \frac{2M^2m^i}{\kappa} \left( \exp(\kappa K_{ij}\phi^j) - \exp\left(-\kappa\psi^2 C_{kl}\frac{m^k}{2}\phi^l\right) \right) = 0.$$  

(20)

The solutions of equation (20) are those of equation (4) with the parameters $\phi_R$ and $u$ constant. These are the most general solutions of this equation and the space of parameters of the solutions is isomorphic as a symplectic manifold to the cotangent bundle of $\mathcal{R}^l$.

Next observe that if we transform $t \rightarrow x$ and $M^2 \rightarrow -M^2$ in equation (20), it becomes equation (19). To solve equation (19), we must find a way to change the overall sign of the potential term in equation (20). But we know that $M^2 = 2\mu\nu$, so this can be achieved by setting either $\mu \rightarrow -\mu$ or $\nu \rightarrow -\nu$ in the definition of the components of the connection $A$ in equation (5). Choosing the former, we deduce that the solutions of equation (19) are

$$\exp\left(-\kappa\dot{\phi}^i(x)\right) = \exp\left(-\kappa\dot{\phi}_R^i\right) \frac{\langle \Lambda_i | \exp(2x\dot{A}) | \Lambda_i \rangle}{\langle \Lambda_0 | \exp(2x\dot{A}) | \Lambda_0 \rangle^{m_i}}$$  

(21)

where $\phi_R$ is a constant parameter and the components of $\dot{A}$ ($\dot{A} \equiv \dot{A}_0 + \dot{A}_{<0} + \dot{A}_{>0}$)
are
\[ \hat{A}_o = \kappa u, \quad u \in \mathcal{H} \]
\[ \hat{A}_{<0} = -\mu \hat{E}_{-1}, \]
\[ \hat{A}_{>0} = \nu \exp(\kappa \phi_R) \hat{E}_1 \exp(-\kappa \phi_R). \]  

(22)

Note the sign difference in the definitions of \( \hat{A}_{<0} \) (eqn. (22)) and \( A_{<0} \) (eqn. (5)).

The independent parameters of the solutions (21) are \( \phi_R \) and \( u \) and the space of parameters is diffeomorphic to the cotangent bundle of \( \mathcal{R}^l \). Although there is a natural symplectic structure on this space, it is not the one induced by the symplectic structure on the space of all solutions of the affine Toda field theory studied in section two.

Acknowledgements

G.P. was funded by a grant from the European Union and would like to thank H. Nicolai for support. B.S. was supported by a QEII Fellowship from the Australian Research Council, and acknowledges the hospitality of the Theory Group at Queen Mary & Westfield College London, where part of this work was carried out.

REFERENCES

1. A.N. Leznov and M.V. Saveliev, Lett. Math. Phys. 3 (1979) 489; Commun. Math. Phys. 74 (1980) 111; Lett. Math. Phys. 6 (1982) 505; Commun. Math. Phys. 83 (1983) 59; J. Sov. Math. 36 (1987) 699; Acta Appl. Math. (1989) 1.
2. D.I. Olive and N. Turok, Nucl. Phys. B220 (1983) 491, B257 (1985) 277, B265 (1986) 469.
3. P. Mansfield, Nucl. Phys. B208 (1982) 277, B222 (1983) 419.
4. G. Papadopoulos and B. Spence, Phys. Lett. B295 (1992) 44; Phys. Lett. B308 (1993) 253.
5. G. Papadopoulos and B. Spence, Class. Quantum Gravity, in press.
6. A.B. Zamolodchikov, Int. J. Mod. Phys. Lett. A1 (1989) 4235;  
   T. Eguchi and S.-K. Yang, Phys. Lett. B224 (1989) 373;  
   T.J. Hollowood and P. Mansfield, Phys. Lett. B226 (1989) 73.

7. H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Nucl. Phys. B338 (1990) 689, B356 (1991) 469; P.E. Dorey, Nucl. Phys. B358 (1991) 654, B374 (1992) 74;  
   T.R. Klassen and E. Melzer, Nucl. Phys. B338 (1990) 485;  
   M.D. Freeman, Phys. Lett. B261 (1991) 57;  
   A. Fring, H.C. Liao and D.I. Olive, Phys. Lett. B266 (1991) 82; A. Fring, G. Mussardo and P. Simonetti, Imperial preprint TP/91-92/31.

8. T.J. Hollowood, Nucl. Phys. B384 (1992) 523;  
   D.I. Olive, N. Turok and J. Underwood, Affine Toda Solitons and Vertex Operators, [hep-th/9305160];  
   J. Underwood, Aspects of Non-Abelian Toda Theory, [hep-th/9304156];  
   M. Kneipp and D.I. Olive, Crossing and Anti-Solitons, [hep-th/9305154].

9. D.I. Olive, N. Turok and J. Underwood, Nucl. Phys. B362 (1993) 294.

10. O. Babelon and L. Bonora, Phys. Lett. B244 (1990) 220;  
   L. Bonora, M. Martellini and Y.-Z. Zhang, Phys. Lett. B253 (1991) 373.