Time Delay in Gravitational Lensing by a Charged Black Hole of String Theory

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Abstract

We calculate the time delay between different relativistic images formed by the gravitational lensing produced by the Gibbons-Maeda-Garfinkle-Horowitz-Stromiger (GMGHS) charged black hole of heterotic string theory. Modeling the supermassive central objects of some galaxies as GMGHS black holes, numerical values of the time delays are estimated and compared with the correspondient Reissner-Nordström black holes. The time difference amounts to hours, thus being measurable and permitting to distinguish between General Relativity and String Theory charged black holes.

1 Introduction

Gravitational lensing is one of the first and more important applications of General Relativity. The first recognized effect was the light deflection by the sun, and after that, there was the lensing of quasars by galaxies. Now it is an ordinary phenomenon in astronomical observations.

Because of the non-linearity of General Relativity, gravitational lensing has been developed in the weak field approximation. However, in the last years the literature is starting to look at the lensing in the strong field limit, because of the necessity of looking for the behavior of light near massive objects, for example near the event horizon of black holes.

The development of strong-field lensing theory was started by Virbhadra and Ellis\textsuperscript{[3]}, and more recently by Bozza\textsuperscript{[6]}, that shows an analytical technique for obtaining the deflection angle in the strong field situation and showed that the deflection angle diverge logarithmically as light approach the photon sphere of a general class of static spherically symmetric metrics.

Following the work of Bozza; A. Bhadra\textsuperscript{[1]} studies the gravitational lensing due to the Gibbons-Maeda-Garfinkle-Horowitz-Stromiger (GMGHS) charged
black hole of heterotic string theory and compares its estimated observables lensing quantities with those due to the Reissner-Nordström solution of General relativity. As a conclusion, Bhadra finds that there is no significant string effect present in the angular position and magnification of the relativistic images in the strong-gravity scenario.

Very recently, Bozza and Mancini\cite{7} extended the analytical theory of strong lensing by calculating the time delay between different relativistic images, and shows that different types of black holes are characterized by different time delays. Thus, this quantity can, eventually, become available for the classification of the black holes.

In the present work, we follow the method of \cite{7} to study the time delay between the relativistic images produced by the GMGHS black hole gravitational lensing. In order to get a clear idea of the string contribution, we estimate the expected time delay for several interesting supermassive extragalactic black holes and compare the with the expected for the Reissner-Nordström solution.

The paper is structured as follows. In Sec. 2, we review the GMGHS black hole and its strong-field lensing solution found by Bhadra. In Sec. 3, we present the time delay computation and the estimated results for several supermassive extragalactic black holes. These are compared with the time delays obtained for Schwarzschild and Reissner-Nordström metrics. Finally, a discussion of the results is given in Sec. 4.

2 Lensing due to the GMGHS Charged Black Hole of String Theory

The effective action of heterotic string theory in four dimensions (in Planck units $c = \hbar = G = 1$), in the low energy sector, is given by\cite{2}:

$$\mathcal{A} = \int d^4x \sqrt{-g}e^{-2\phi} \left(-R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - 4 (\nabla \phi)^2 + F_{\mu\nu} F^{\mu\nu}\right)$$ (1)

where $R$ is the Ricci scalar, $\phi$ is the dilaton field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell Field and $H_{\mu\nu\rho}$ is a three form related to an antisymmetric tensor gauge field $B_{\mu\nu}$ and the gauge field $A_\mu$ by $H = dB - A \wedge F$.

From the low energy action is clear that $e^{\phi}$ plays the role of a coupling constant, so it governs the strength of the quantum corrections. The complete string action includes higher order corrections $R^2, F^4$, etc. but these can be neglected when discussing black hole solutions if the size of the hole is much larger than the Planck length.
Assuming that the field $H_{\mu
u}$ is zero, and making the conformal transformation $g_{\mu\nu}^E = e^{-2\phi}g_{\mu\nu}$ to the Einstein frame, the action can be written as:

$$A = \int d^4x \sqrt{-g^E} \left(-R^E - 2(\nabla\phi)^2 + e^{-2\phi}F_{\mu\nu}F^{\mu\nu}\right)$$

One class of solution of the above theory is the static charged black hole configuration, which is often called the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) black hole:

$$ds^2 = -(1 - \frac{2M}{r}) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(1 - \frac{Q^2 e^{-2\phi_o}/Mr}{1 - \frac{Q^2 e^{-2\phi_o}/Mr}{1 - \frac{Q^2 e^{-2\phi_o}}{Mr}}}ight) d\Omega^2$$

where $\phi_o$ is the asymptotic constant value of the dilaton field; and $M$ and $Q$ are the mass and electric charge of the black hole. The causal structure of this space-time is identical to Schwarzschild; there is an event horizon at $r = 2M$ and a curvature singularity at $r = 0$. (The vector potential $A_t$ is actually finite at $r = 0$, although the invariant $F^\mu\nu F_{\mu\nu}$ diverges there.)

If we compare this solution with the Reissner-Nordstr"om black hole:

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2}) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

is easy to see that charged black holes of string theory are not Reissner-Nordstr"om when $\phi = 0$.

For the lensing study we will consider, as usual, a light ray from a source ($S$) that is deflected by a lens ($L$) and reaches the observer ($O$). We will take the background space-time asymptotically flat. The line joining the lens and the observer ($OL$) is the optical axis of the system. The distances between observer and lens, lens and source and observer and source are $D_{OL}$, $D_{LS}$ and $D_{OS}$ respectively. $\beta$ and $\theta$ are the angular position of the source and the image respect to the optical axis; and $\alpha$ is the deflection angle.

The position of the source and the image are related through the lens equation obtained by Virbhadra and Ellis:

$$\tan \theta - \tan \beta = \frac{D_{LS}}{D_{OS}} \left[\tan \theta + \tan (\alpha - \theta)\right]$$

For a spherically symmetric metric,

$$ds^2 = -A(r) \, dt^2 + B(r) \, dr^2 + C(r) \, d\Omega^2$$

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the deflection angle \( \alpha \), is given by

\[
\alpha (r_o) = I (r_o) - \pi
\]

where \( r_o \) is the closest approach of the light ray, and

\[
I (r_o) = 2 \int_{r_o}^{\infty} \left[ \frac{B (r)}{C (r)} \right]^{\frac{1}{2}} \left[ \frac{C (r) A (r_o)}{C (r_o) A (r)} - 1 \right]^{-\frac{1}{2}} \, dr
\]

For the GMGHS black hole, we have:

\[
I (x_o) = 2 \int_{x_o}^{\infty} \frac{dx}{\sqrt{\left( \frac{x}{x_o} \right)^2 - 1}} \left( \frac{x}{x_o} \right)^2 \left( \frac{1}{2} \right)^{\frac{1}{2}} \left( \frac{1}{2} \right)^{-\frac{1}{2}} - \left( \frac{1}{2} \right)^{-\frac{1}{2}} \left( \frac{1}{2} \right)^{-\frac{1}{2}}
\]

where \( x_o = \frac{r_o}{2M} \); \( x = \frac{r}{2M} \); \( \xi = 2q^2 e^{-2\phi_o} \) and \( q = \frac{Q}{2M} \).

The impact parameter is given by:

\[
u (x_o) = \frac{1}{2M} \sqrt{C (x_o) A (x_o)} = x_o \sqrt{\left( \frac{1 - \frac{\xi}{x_o}}{1 - \frac{1}{x_o}} \right)}
\]

The deflection angle increase when the closest approach \( x_o \) decreases. For a certain value of \( x_o \) the deflection angle will be \( 2\pi \) and then, a light ray will make a complete loop around the black hole. If \( x_o \) decreases further, the light ray will give more loops, and when \( x_o \) is equal to the radius of the photon sphere \( (x_m) \) the deflection angle becomes infinite, and the light ray is captured by the black hole. For the GMGHS black hole, the photon sphere is at the radius:

\[
x_m = \frac{\xi + 3 + \eta}{4}
\]

with

\[
\eta = \sqrt{\xi^2 - 10\xi + 9}\]

Following Bozza, the first step to solve the integral \( 11 \) in the strong field limit is to make the change of variable \( z = 1 - \frac{x}{x_o} \), obtaining:

\[
I (x_o) = \int_{0}^{1} R (z, x_o) f (z, x_o) \, dz
\]

where

\[
R (z, x_o) = \frac{2 \sqrt{1 - \frac{\xi}{x_o}}}{1 - \frac{\xi}{x_o} + z \frac{\xi}{x_o}}
\]
\[ f(z, x_o) = \left[ 1 - \frac{1}{x_o} - \left( 1 - \frac{1}{x_o} + \frac{z}{x_o} \right) \right] (1 - z)^2 \left( 1 + \frac{z\xi}{x_o - \xi} \right)^{-1} \]  

(17)

The function \( R(z, x_o) \) is regular for all values of \( z \), while \( f(z, x_o) \) diverges for \( z \to 0 \). Hence, the second step is to find out the order of divergence, expanding the argument of the square root in \( f(z, x_o) \) up to the second order in \( z \). This gives:

\[ f(z, x_o) \sim f_o(z, x_o) = \frac{1}{\sqrt{p(x_o) z + q(x_o) z^2}} \]  

(18)

where:

\[ p(x_o) = \frac{\eta (x_o - 2) + x_o (3 - 2x_o)}{x_o (\eta - x_o)} \]  

(19)

\[ q(x_o) = \frac{\eta^2 - 3\eta x_o - (x_o - 3)x_o^2}{x_o (\eta - x_o)^2} \]  

(20)

Bozza\cite{6} has proved that for a static spherically symmetric space-time with the form of equation (8), the deflection angle always diverge logarithmically for \( x_o \to x_m \); and can be written as:

\[ \alpha(u) = -\pi \log \left( \frac{u}{u_m} - 1 \right) + \tilde{b} + O(u - u_m) \]  

(21)

All quantities with the subscript \( m \) are evaluated at \( x_o = x_m \). The coefficients of this expansion are calculated by Bhadra\cite{11} for the GMGHS Black hole as:

\[ \pi = \frac{\sqrt{3 - 3\xi + \eta (3 - \xi + \eta)}}{2a_o} \]  

(22)

\[ \tilde{b} = -\pi + \pi \left\{ 2 \left[ \log (4\sqrt{a_o}) - \log \left( \frac{a_o + a_1}{a_2} + 2\sqrt{a_o + a_1 + a_2} \right) \right] + \log \frac{a_o + a_1}{a_2} \right\} \]  

(23)

\[ u_m = \frac{1}{2\sqrt{2}} \sqrt{(9 - \xi) \eta + 27 - 18\xi - \xi^2} \]  

(24)

where:

\[ a_o = 2 \left( 1 - \xi \right) \left( \xi^2 + \xi (\eta - 12) + 9 (\eta + 3) \right) \]  

(25)

\[ a_1 = 4 \left( \xi^3 + \xi^2 (\eta - 15) + \eta (23 + 6\eta) - 3 (3 + \eta) \right) \]  

(26)

\[ a_2 = 24\xi^2 - 8\xi (3 + \eta) \]  

(27)
3 Time Delay Computation

Bozza and Mancini\[2\] derived the time delay between different relativistic images, following an approach very similar to the one used for the deflection angle. For an observer at infinity, the time taken for the photon to travel from the source to the observer is:

\[
T = \int_{t_0}^{t_f} dt
\]  

(28)

This integral can be solved following the same scheme, to obtain finally, that the time delay between the \(i\)-th and the \(j\)-th relativistic images (in Schwarzschild units) is:

\[
\Delta T^s_{i,j} = 2\pi (i - j) \frac{\tilde{a}}{\tilde{c}} + 2 \sqrt{\frac{B_m}{A_m}} \sqrt{\frac{u_m}{c}} e^{\frac{\bar{x}}{m}} \left( e^{-2\pi i j\gamma - 2\pi i j\gamma} - e^{2\pi i j\gamma - 2\pi i j\gamma} \right)
\]  

(29)

when the two images are on the same side of the lens, and:

\[
\Delta T^o_{i,j} = [2\pi (i - j) - 2\gamma] \frac{\tilde{a}}{\tilde{c}} + 2 \sqrt{\frac{B_m}{A_m}} \sqrt{\frac{u_m}{c}} e^{\frac{\bar{x}}{m}} \left( e^{-2\pi i j\gamma - 2\pi i j\gamma} - e^{2\pi i j\gamma - 2\pi i j\gamma} \right)
\]  

(30)

when the two images are on opposite sides of the lens. Here \(\gamma\) is the angular separation between the source and the optical axis, as seen from the lens. In the first case, the upper sign before \(\gamma\) applies if both images are on the same side of the source, while the lower sign if both images are on the other side.

For spherically symmetric metrics, we have:

\[
\frac{\tilde{a}}{\tilde{c}} = \sqrt{\frac{C_m}{A_m}}
\]  

(31)

\[
c = \frac{C_m A_m'' - C_m A_m' A_m''}{4 \sqrt{A_m' C_m}}
\]  

(32)

When the source is almost aligned with the lens, we have \(\gamma \sim D_{OL}^{-1} \ll 2\pi\); and therefore, in this case, we have for \(i \neq j\):

\[
\Delta T^s_{i,j} \simeq \Delta T^o_{i,j}
\]  

(33)

3.1 Time Delay in Supermassive GMGHS Black Hole Lensing

By applying this scheme for the GMGHS black hole and assuming that the source is almost aligned with the lens, we obtain the time delay between the second and the first relativistic images as:
\[ \Delta T_{2,1} = 2\pi \frac{\tilde{a}}{\alpha} + 2 \sqrt{\frac{B_m}{A_m}} \sqrt{\frac{u_m}{c}} e^{\frac{\xi}{\alpha}} \left( e^{-\frac{\xi}{\alpha}} - e^{-\frac{\tilde{a}}{\alpha}} \right) \]  

(34)

where the coefficients are:

\[ A_m = \left( 1 - \frac{1}{x_m} \right) = B_m^{-1} \]  

(35)

\[ \frac{\tilde{a}}{\alpha} = \sqrt{\frac{x_m^2 - \xi x_m}{\left( 1 - \frac{1}{x_m} \right)}} \]  

(36)

\[ c = \frac{2 \left( 1 - \frac{1}{x_m} \right) + \frac{1}{x_m} \left( 1 - \frac{\xi}{x_m} \right)}{4 \sqrt{\left( 1 - \frac{1}{x_m} \right)^3(\xi x_m)}} \]  

(37)

And \( \alpha, \beta \) and \( u_m \) given by (22), (23), and (24) respectively. Thus, the time delay for the GMGHS black hole depends on its mass and electric charge.

Now we will give some realistic values by modelling some supermassive black holes as GMGHS. Taking the case \( q = 0 \) (i.e. no electric charge) we obtain the same time delays reported by Bozza and Macini\cite{7} for the Schwarzschild metric:

| Galaxy           | Mass \((M_\odot)\) | GMGHS \((q = 0)\) |
|------------------|-------------------|--------------------|
| NGC4486 (M87)    | \(3.3 \times 10^9\) | 149.3 h.           |
| NGC3115          | \(2.0 \times 10^9\) | 90.5 h.            |
| NGC4374 (M84)    | \(1.4 \times 10^9\) | 63.3 h.            |
| NGC4594 (M104)   | \(1.0 \times 10^9\) | 45.2 h.            |
| NGC4486B (M104)  | \(5.7 \times 10^8\) | 25.8 h.            |
| NGC4261          | \(4.5 \times 10^8\) | 20.4 h.            |
| NGC4342 (IC3256) | \(3.9 \times 10^8\) | 17.6 h.            |
| NGC7052          | \(3.3 \times 10^8\) | 14.9 h.            |
| NGC3377          | \(1.8 \times 10^8\) | 8.1 h.             |

Table I. Time delay for GMGHS Black Hole without electric charge

This result just shows that the GMGHS black hole without electric charge reduces to the Schwarzschild metric.

Now, if we put some electric charge \((q \neq 0)\) the estimated time delays for the same black holes differ from the obtained by Bozza and Mancini\cite{7} for the Reissner-Nordström metric:
Table II. Time delays for GMGHS and Reissner-Nordström Black Holes.

| Galaxy    | Reissner-Nordström $\Delta T_{2,1}$ (hours) | GMGHS $\Delta T_{2,1}$ (hours) |
|-----------|---------------------------------------------|---------------------------------|
|           | $q = 0.1$ | $q = 0.2$ | $q = 0.3$ | $q = 0.4$ | $q = 0.1$ | $q = 0.2$ | $q = 0.3$ | $q = 0.4$ |
| NGC4486   | 148.3     | 145.4     | 140.1     | 131.7     | 148.4     | 145.5     | 140.5     | 133.1     |
| NGC3315   | 89.9      | 88.1      | 84.9      | 79.8      | 90.0      | 88.2      | 85.2      | 80.7      |
| NGC4374   | 62.9      | 61.7      | 59.4      | 55.9      | 63.0      | 61.7      | 59.6      | 56.5      |
| NGC4594   | 44.9      | 44.0      | 42.4      | 39.9      | 45.0      | 44.1      | 42.6      | 40.3      |
| NGC4486B  | 26.6      | 25.1      | 24.2      | 22.7      | 25.6      | 25.1      | 24.3      | 23.0      |
| NGC4261   | 20.2      | 19.8      | 19.1      | 18.0      | 20.2      | 19.8      | 19.2      | 18.1      |
| NGC4342   | 17.5      | 17.2      | 16.6      | 15.6      | 17.5      | 17.2      | 16.6      | 15.7      |
| NGC7052   | 14.8      | 14.5      | 14.0      | 13.2      | 14.8      | 14.6      | 14.1      | 13.3      |
| NGC3377   | 8.1       | 7.9       | 7.6       | 7.2       | 8.1       | 7.9       | 7.7       | 7.3       |

4 Conclusion

Since the charged black holes of General Relativity and String Theory are qualitatively different, it is expected that there will be some differences in the observational features of each of them, particularly in the strong field limit.

Here, we have used the analytical model of Bozza to obtain the time delay in the strong field scenario for the GMGHS black hole. Modeling some supermassive objects at the center of galaxies as charged string theory black holes, we have seen that there are some differences in the numerical estimates of the time delay between the second and the first relativistic images, when compared with the obtained in the Reissner-Nordström case by Bozza and Mancini.

As can be seen from Table II, the string effects in the time delay are more evident for great masses and great electric charges. The differences in the estimates are of the order of hours, so they can be easily measured, in principle; providing a possible method for distinguishing between General Relativity and String Theory charged black holes.

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