On Conserved Quantities at Spatial Infinity*

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Abstract

There is a well-known short list of asymptotic conserved quantities for a physical system at spatial infinity. We search for new ones. This is carried out within the asymptotic framework of Ashtekar and Romano, in which spatial infinity is represented as a smooth boundary of space-time. We first introduce, for physical fields on space-time, a characterization of their asymptotic behavior as certain fields on this boundary. Conserved quantities at spatial infinity, in turn, are constructed from these fields. We find, in Minkowski space-time, that each of a Klein-Gordon field, a Maxwell field, and a linearized gravitational field yields an entire hierarchy of conserved quantities. Only certain quantities in this hierarchy survive into curved space-time.

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I. INTRODUCTION

In the description of isolated systems in flat space-time, conserved quantities have often been found to be useful. Examples of such conserved quantities include electric charge, energy-momentum, angular momentum, and, in certain circumstances, various multipole moments. These conserved quantities are usually expressed as surface integrals in the limit as the surface approaches infinity. In general relativity, by contrast, the construction of such conserved quantities is more complicated. Not least of these complications is that “infinity” is so much more difficult to pin down in the presence of curvature.

The study of isolated systems in general relativity was pioneered by Arnowitt, Deser and Misner. They defined asymptotic flatness of a space-time in terms of the existence of an initial-data set which, expressed in suitable coordinates, has the initial data approach the flat values at suitable rates. Conserved quantities, such as energy-momentum and angular momentum, were then be expressed as limits of certain surface integrals.

One unfortunate aspect of the approach of Arnowitt, Deser and Misner is that their asymptotic conditions are tied so closely to coordinates. Their approach was subsequently geometrized and extended by Geroch via a conformal completion by a single point “at spatial infinity”. Multipole moments for certain fields in flat space-time were generalized to static asymptotically flat space-times within this framework. An alternative geometrical framework, which unifies spatial and null infinity and is thus adapted to the relation between these two asymptotic regimes, was introduced by Ashtekar and Hansen. This framework involves a conformal completion of the entire space-time, null infinity becoming a null cone with spatial infinity its vertex. This framework is used, for example, both to formulate and to prove the assertion that the ADM mass is the past limit of the future Bondi mass.

In both of the geometrical frameworks outlined above spatial infinity is squeezed into a point, and there smoothness of the completed manifold fails. So, inevitably, one is forced to deal with complicated differentiable structures there. This circumstance is less satisfactory than that of null infinity, which is formulated as a smooth boundary of space-time. Early attempts
to restore smoothness to spatial infinity include those of Sommers\textsuperscript{6} and Persides\textsuperscript{7}. Beig and Schmidt\textsuperscript{8,9}, using a coordinate-dependent treatment similar to that of Bondi et. al\textsuperscript{9}, obtained fields on the surface at spatial infinity order by order, and noticed that these fields there satisfy hyperbolic equations. This work culminates in that of Ashtekar and Romano\textsuperscript{11}, who introduced a new geometrical framework for asymptotic flatness in which spatial infinity was indeed expressed as a smooth boundary of space-time. Their definition also provided a natural geometrical setting for the results of Beig and Schmidt. Ashtekar and Romano’s framework is somewhat of a hybrid, in that it involves both the conformal and projective structure. By their definition, a space-time is asymptotically flat at spatial infinity provided one can attach to it a smooth boundary $\mathcal{H}$ and introduce a smooth function $\Omega$ vanishing at $\mathcal{H}$ such that the induced metric on and the normal to the constant-$\Omega$ surfaces are, after rescaling by suitable powers of $\Omega$, smoothly extendible to $\mathcal{H}$. This new definition has proven to be useful in the study of asymptotic properties of space-time at spatial infinity since various physical fields turn out to be smooth there.

We return now to conserved quantities. It is natural to ask: Do, in some sense, the well-known conserved quantities—energy-momentum, angular momentum, electric charge—at spatial infinity exhaust all conserved quantities that could possibly be defined there? To settle this question would clearly provide insights into the asymptotic properties of the physical fields and of the space-time. The framework introduced by Ashtekar and Romano is perfectly suited to addressing this question. One has a simple, universal smooth structure at spatial infinity enabling one to investigate fields at spatial infinity order by order. The notion of a conserved quantity had already been formulated by Ashtekar and Romano: Each conserved quantity is to be expressed as an integral over a 2-sphere section of spatial infinity where the value of the integral is independent of section. In particular, the well-known conserved quantities are so expressed. We seek others.

This paper is organized as follows. Section II contains the basic framework, which underlies the rest of the paper. We first review briefly (a slight modification of) the Ashtekar-Romano
definition of asymptotic flatness. We then formulate within this framework the asymptotic structure of the physical fields. In particular, we introduce the notion of a conserved quantity, and give some familiar examples. In section III, we consider the special but important case of fields in Minkowski space-time. We construct all linear conserved quantities associated with a Klein-Gordon field, with a Maxwell field, and with a linearized gravitational field and having a certain “polynomial dependence” on asymptotic translations. We then study the symmetry properties and the “gauge behavior” (dependence on a certain freedom in the formulation of asymptotic structure) of these quantities. In section IV, we consider fields in a curved, asymptotically flat space-time. We first derive the equations, at spatial infinity, satisfied by the asymptotic fields. We then show that—at least in the Klein-Gordon and Maxwell cases—certain of the conserved quantities found in section III for Minkowski space-time can be generalized to these curved space-times. In section V, we discuss various related issues. In particular, we formulate two conjectures. One asserts that a certain conserved quantity for linearized gravity in Minkowski space-time can be generalized to curved space-time. The other asserts that we have here found all conserved quantities in curved space-time for Klein-Gordon, Maxwell and gravitational fields.

II. PRELIMINARIES

A. Asymptotic Flatness

Fix a space-time \((\tilde{M}, \tilde{g}_{ab})\).

**Definition 1** By a completion of \((\tilde{M}, \tilde{g}_{ab})\), we mean (c.f. Ashtekar and Romano in Ref. 10): A manifold \(M\) with boundary \(\mathcal{H}\), a smooth function \(\Omega\) defined on \(M\) vanishing on \(\mathcal{H}\), and a diffeomorphism from \(\tilde{M}\) to \(M - \mathcal{H}\) (by means of which we identify \(\tilde{M}\) with its image in \(M\)) satisfying the following three classes of conditions:

1. The combinations (i) \(\nabla_a \Omega\), (ii) \(\Omega^{-4} \tilde{g}^{ab} \tilde{\nabla}_b \Omega (\equiv n^a)\), and (iii) \(\Omega^2 [\tilde{g}_{ab} - (\tilde{g}^{cd} \nabla_c \Omega \nabla_d \Omega)^{-1} \nabla_a \Omega \nabla_b \Omega]\) (\(\equiv q_{ab}\)) admit smooth, nowhere-vanishing extensions to \(\mathcal{H}\) such that (iv) \(n^a \nabla_a \Omega (\equiv \lambda^{-2})|_{\mathcal{H}} = 1\) and (v) \(L_{\lambda}[(n^m \nabla_m \Omega)^{-1} q_{ab}]|_{\mathcal{H}} = 0\).
(2)\( (\mathcal{H}, q_{ab}\lvert_\mathcal{H}) \) is a standard time-like hyperboloid, i.e., \( \mathcal{H} \) has topology \( S^2 \times \mathbb{R}, q_{ab}\lvert_\mathcal{H} \equiv 0 \) is of constant positive curvature and is geodesically complete.

(3) The combinations (i)\( n^k n^l \tilde{G}_{kl} \), (ii)\( \Omega^{-1} q_a^k n^l \tilde{G}_{kl} \), and (iii)\( \Omega^{-2} q_a^k q_b^l \tilde{G}_{kl} \), are smoothly extendible to \( \mathcal{H} \), where \( \tilde{G}_{ab} \) is the Einstein tensor of \( \tilde{g}_{ab} \).

The boundary \( \mathcal{H} \) represents spatial infinity. Conditions (1) describe the fall-off behavior of the metric \( \tilde{g}_{ab} \) and conditions (3) that of its second derivative. Conditions (2) ensure, among other things, that we are dealing with (all of) spatial infinity. There is some redundancy in the above conditions. Specifically, the constancy both of the left side of (1)(iv) and of the curvature of \( q_{ab} \) already follow from the other conditions. In light of this, the choice of the constant “1” in condition (1)(iv) (which is equivalent to the demand that \( 0 \) be the metric of a unit hyperboloid) serves only to restrict the freedom of multiplying \( \Omega \) by a constant factor. Condition (1)(v) is essentially the condition, \( \tilde{B}_{ab} = 0 \) (c.f. eqn.(3)), introduced by Ashtekar and Hansen\(^4\) in order to define angular momentum. More precisely, when \( \tilde{B}_{ab} = 0 \), condition (1)(v) can always be achieved without affecting the other conditions by choosing a suitable \( \Omega \).

Definition \( \square \) is essentially the same as the definition, given by Ashtekar and Romano\(^1\), of what they call an asymptotically Minkowskian space-time. However, there are three differences. First, our conditions on the Einstein tensor are weaker than the corresponding condition, namely \( \lim_{\Omega \to 0} \Omega^{-1} \tilde{G}_{ab} = 0 \), in their definition. Their condition, expressed in the present language, is equivalent to the smooth extendibility to \( \mathcal{H} \) of \( \Omega^{-2} n^k n^l \tilde{G}_{kl}, \Omega^{-2} q_a^k n^l \tilde{G}_{kl}, \) and \( \Omega^{-2} q_a^k q_b^l \tilde{G}_{kl} \). Indeed, our condition holds while theirs fails (for \( n^k n^l \tilde{G}_{kl}\lvert_\mathcal{H} \neq 0 \)) in the Reissner-Nordstrom solution. Second, we impose condition (1)(iv) which, as mentioned above, is effectively a gauge restriction on the conformal factor \( \Omega \), a restriction that is absent in the definition of Ashtekar and Romano. Finally, we impose condition (1)(v), which Ashtekar and Romano omit from the general definition of asymptotic Minkowskian space-times, but subsequently impose for their discussion of angular momentum.

We give a few simple examples of completion. As a first example, let \( (\tilde{M}, \tilde{g}_{ab}) \) be Minkowski space-time, and let \( (t, r, \theta, \phi) \) be ordinary spherical polar coordinates. Set \( \Omega = (r^2 - t^2)^{-1/2} \) and
\[ \tanh \chi = t/r. \] Let \( M \) be \( \tilde{M} \) together with the boundary \( \mathcal{H} \) consisting of the points labeled by \( \Omega = 0 \) in the (hyperbolic coordinate) chart \((\Omega, \chi, \theta, \phi)\), with differentiable structure given by that chart. Then this \((M, \Omega)\) is a completion of Minkowski space-time. As a second example, let \((\tilde{M}, \tilde{g}_{ab})\) be Reissner-Nordstrom solution, and let \((t, r, \theta, \phi)\) be the usual Schwarzschild-like coordinates therein. Repeat the same construction as in Minkowski space-time to obtain a manifold with boundary \((M, \Omega)\). Then this choice of \( \Omega \) satisfies all conditions in definition 1 except condition (1)(v). Condition (1)(v) in turn can be achieved, without violating other conditions, by choosing a new conformal factor \( \Omega' \) of the form \( \Omega' = \Omega(1 + \omega \Omega) \) with a suitable smooth function \( \omega \). In general, all stationary vacuum space-times asymptotically flat by the usual definition admit completions in the present sense.

Two completions \((M, \Omega), (M', \Omega')\) of \((\tilde{M}, \tilde{g}_{ab})\) are said to be equivalent if the identity map of \( \tilde{M} \) extends to a diffeomorphism from \( M \) to \( M' \). It turns out that a space-time may admit inequivalent completions. Minkowski space-time, for instance, has a four-parameter family of inequivalent completions related to each other by what are called “logarithmic translations.” Indeed, let \( x^\mu \) be a usual Minkowskian coordinate system in Minkowski space-time \( \tilde{M} \), and \( c^\mu \) any constant vector. Then the hyperbolic coordinates associated with \( x'^\mu \) given by \( x'^\mu = x^\mu - c^\mu \log \Omega' \) yield a new completion of \( \tilde{M} \) inequivalent to that arising from \( x^\mu \). In this case we can single out the usual completion to be the preferred one among this four-parameter family since it is the only one in which all Killing fields are smoothly extendible to the boundary at spatial infinity. Similarly, any stationary asymptotically flat space-time admits at least a one-parameter family of inequivalent completions, arising from logarithmic time-translations. There is also a sort of converse to this: the existence of two inequivalent completions related by such a logarithmic translation implies that the space-time admits an asymptotic translational Killing field—a vector field \( \tilde{\xi}^a \) with the properties that \( \Omega^{-1} \tilde{\xi}^a \) is smoothly extendible to, and vanishes nowhere on, \( \mathcal{H} \); and that \( \tilde{\nabla}_{(a} \tilde{\xi}_{b)} \) and all its derivatives vanishes on \( \mathcal{H} \). In the spatial-infinity framework of Geroch and Ashtekar-Hansen, it has been shown by Chrusciel that these logarithmic translations are the only kind of inequivalent completions that may arise. We
conjecture in the present framework: Any two inequivalent completions are related by such a logarithmic translation. If this conjecture is true, then our work will not be affected by the possible existence of inequivalent completions. In what follows we will always fix a specific completion and only consider completions smoothly related (i.e., equivalent) to the fixed one.

B. Physical Fields and Their Remnants

We now set up the framework for dealing with the asymptotic structure of physical fields. Let \((\tilde{M}, \tilde{g}_{ab})\) be a space-time, with \((M, \Omega)\) a completion. Let \(\tilde{\nu}_{a_1 \cdots a_m}\) be a smooth, covariant, \(m\)-th rank tensor field on \(\tilde{M}\), and consider the \(2^m\) tensor fields that result from contracting each index of \(\tilde{\nu}\) with either \(\Omega^2 n^a\) or \(\Omega q^a_b\). We say \(\tilde{\nu}\) is *asymptotically regular of order* \(s\) provided each of these \(2^m\) tensor fields, multiplied by \(\Omega^{-s}\), is smoothly extendible to \(\mathcal{H}\). Asymptotic regularity of a general tensor field is defined by lowering any contra-variant indices with \(\tilde{g}_{ab}\) and applying the definition above to the resulting covariant field. Note that conditions (1)(ii), (iii) above are precisely the statement that \(\tilde{g}_{ab}\) is asymptotically regular of order 0; and conditions (3)(i)–(iii) are precisely the statement that \(\tilde{G}_{ab}\) is asymptotically regular of order 4. The outer product of two asymptotically regular fields, of respective orders \(s\) and \(s'\), is asymptotically regular, of order \(s + s'\). Contractions using \(\tilde{g}^{ab}\) preserve asymptotic regularity, and order.

Thus, an asymptotically regular physical field gives rise, on \(M\), to \(2^m\) smooth fields, with ranks ranging from \(m\) down to zero, whose behavior near \(\mathcal{H}\) reflects the asymptotic behavior of the physical field. Let \(u_{a_1 \cdots a_m}\) denote any one of these fields. Then set, for \(k\) any non-negative integer,

\[
^k u_{a_1 \cdots a_m} \equiv \frac{\psi^*}{\xi^m} \left[ (\mathcal{L}_{(\eta \cdot \nabla \Omega)^{-1} n})^k u_{a_1 \cdots a_m} \right],
\]

where \(\frac{\psi}{\xi^m}\) stands for the pull-back to \(\mathcal{H}\) via the natural embedding map \(\mathcal{H} \xrightarrow{\psi} M\). Note the right side of eqn.\((1)\) exists since \(u_{a_1 \cdots a_m}\) (and therefore each of its derivatives) is smoothly extendible to \(\mathcal{H}\). The \(^k u_{a_1 \cdots a_m}\) so defined will be called the \(k\)-th order remnant of \(u_{a_1 \cdots a_m}\). These remnants, \((k = 0, 1, \ldots)\), clearly carry, order by order, the asymptotic information contained in \(u_{a_1 \cdots a_m}\), and, therefore, the asymptotic information in the original physical field \(\tilde{\nu}\). Suppose, next that
the physical field $\tilde{v}$ satisfies various field equations. Then these field equations yield partial differential equations on $M$ on the $u$’s that arise via asymptotic regularity from $\tilde{v}$, and so partial differential equations on $\mathcal{H}$ on the remnants $\tilde{u}$ that arise via eqn.(1) from the $u$’s. We will refer to these as the remnant field equations.

We give some examples of asymptotically regular fields and their associated remnants. Fix a space-time $(\tilde{M}, \tilde{g}_{ab})$ with a completion $(M, \Omega)$. For the first example, consider the space-time metric $\tilde{g}_{ab}$. Then, as we mentioned above, this field is asymptotically regular of order 0. The corresponding $u$’s are $q_{ab}(\equiv \Omega^2 q^k_a q^l_b \tilde{g}_{kl})$, $0(\equiv \Omega^3 q^k_a n^l \tilde{g}_{kl})$, and $\lambda^{-2}(\equiv [\Omega^4 n^a n^b \tilde{g}_{ab}])$. Their corresponding remnants, $\tilde{q}_{ab}$ and $\tilde{\lambda}$, carry the asymptotic information contained in the space-time geometry. We note that conditions (1)(iv) and (1)(v) in the definition of a completion are actually conditions on these remnants: namely $\tilde{0}\lambda = 1$, and $\tilde{1}q_{ab} = -\tilde{2}1\lambda q_{ab}$ respectively. For the second example, consider the Einstein tensor $\tilde{G}_{ab}$. Then, as we mentioned above, this field is asymptotically regular of order 4. The corresponding $u$’s, written in terms of the stress-energy tensor $\tilde{T}_{ab}$ ($= \tilde{G}_{ab}/\kappa$, with $\kappa = 8\pi G/c^4$) are

$$T \equiv \lambda^2 n^a n^b (\kappa \tilde{T}_{ab}), \quad T_a \equiv \lambda \Omega^{-1} q^k_a n^l (\kappa \tilde{T}_{kl}), \quad T_{ab} \equiv \Omega^{-2} q^k_a q^l_b \kappa (\tilde{T}_{kl} - \frac{1}{2} \tilde{T} \tilde{g}_{kl}),$$

(2)

where we have introduced certain powers of $\lambda$ and have used the trace-reversed version of $\tilde{T}_{ab}$ in defining $T_{ab}$ for later convenience. We denote by $\tilde{T}$, $\tilde{T}_a$, $\tilde{T}_{ab}$, the remnants of $T$, $T_a$, and $T_{ab}$ respectively. For the third example, consider the Weyl tensor, $\tilde{C}_{abcd}$ of this space-time. It is shown in Appendix C (c.f. the discussion around eqn.(C14)) that this field is asymptotically regular of order 3. The corresponding $u$’s, written in terms of the stress-energy tensor $\tilde{T}_{ab}$ ($= \tilde{G}_{ab}/\kappa$, with $\kappa = 8\pi G/c^4$) are

$$E_{ab} \equiv \Omega^3 \lambda^2 q^j_a q^l_b n^k n^m \tilde{C}_{jklm}, \quad B_{ab} \equiv \Omega^3 \lambda^2 q^j_a q^l_b n^k n^m \tilde{C}_{jklm}.$$  

(3)

Denote their remnants $\tilde{E}_{ab}$ and $\tilde{B}_{ab}$. Note that condition (1)(v) in the definition of a completion is actually a condition on one of these remnants, namely $\tilde{0}B_{ab} = 0$ (c.f. eqn.(C17)). For the final example, consider a Maxwell field $\tilde{F}_{ab}$ in this space-time. We demand that it be asymptotically regular of order 2, i.e., that each of

$$E_a \equiv \Omega \lambda n^b \tilde{F}_{ab}, \quad B_a \equiv \Omega \lambda n^b \tilde{F}_{ab}$$

(4)
be smoothly extendible to $\mathcal{H}$. These are effectively the $u$’s. This demand reflects the idea that a physically reasonable Maxwell field must fall off like $1/r^2$ near spatial infinity. We denote by $E_a$, $B_a$ the remnants of $E_a$, $B_a$ respectively. Note that it follows that the stress-energy tensor of this Maxwell field has the fall-off rate consistent with that of eqn.(2). Indeed, from

$$\tilde{T}_{ab} = \frac{1}{2}(\tilde{F}_{am}\tilde{F}_{b}^{\,m} - \frac{1}{4}\tilde{F}^{2}{g}_{ab})$$

we have

$$T = \frac{1}{2}\kappa(E^2 + B^2), \quad T_a = -\kappa\epsilon_{amn}E^mB^n, \quad T_{ab} = \kappa[E_aE_b + B_aB_b - \frac{1}{2}(E^2 + B^2)q_{ab}]. \quad (5)$$

There remains, as it turns out, some gauge freedom in the present framework. Fix a space-time $(\tilde{M}, \tilde{g}_{ab})$, and let $(M, \Omega)$ and $(M, \Omega')$, be two completions of $(\tilde{M}, \tilde{g}_{ab})$ It then follows that $\Omega' = \Omega(1 + \omega\Omega)$, for some smooth function $\omega$ on $M$ such that $\omega|_H$ satisfies eqn.(7) below (i.e., $\omega$ is an “asymptotic translation”); and, conversely, for $(M, \Omega)$ any completion and $\omega$ and $\Omega'$ as above, then $(M, \Omega')$ is also a completion. Thus the gauge freedom consists precisely of such $\omega$-fields. The asymptotic gauge freedom, then, is described by the remnants, $\omega$, of $\omega$. It turns out\(^8\), that one can, utilizing this gauge freedom, always achieve

$$k\lambda = 0, \quad k \geq 2. \quad (6)$$

and that this exhausts the gauge freedom associated with the remnants $\omega$, for $k \geq 1$. Thus, making this gauge choice, the remaining gauge freedom is represented by a single $\omega$ satisfying eqn.(7).

C. Asymptotic Translations

In order to construct conserved quantities, it will be convenient to have at hand some facts about asymptotic translations. Denote by $\mathcal{T}$ the set of functions $v$ on $\mathcal{H}$ satisfying the differential equation

$$D_a D_b v + v\hat{q}_{ab} = 0, \quad (7)$$

where $D_a$ denotes the derivative operator of $\hat{q}_{ab}$. This $\mathcal{T}$ is a 4-dimensional vector space (since, by virtue of the fact that the curl of eqn.(7) is an identity, $v$ is completely determined by its
value and derivative at any one point) equipped with a Lorentz metric $\langle v, w \rangle \equiv q^{ab}D_a v D_b w + vw$ (since, by virtue of eqn.(7), the right side is a constant ). Elements of $\mathcal{T}$ can be interpreted as asymptotic translations on $M$ in the following sense: For $\tilde{\xi}^a$ a vector field on $\tilde{M}$ asymptotically regular of order 0 such that $\mathcal{L}_{\tilde{\xi}} \tilde{g}_{ab}$ is asymptotically regular of order 2, then $\Omega^{-2}\mathcal{L}_{\tilde{\xi}}\Omega|_{\mathcal{H}} \in \mathcal{T}$.

It is convenient to introduce an index notation for tensors over $\mathcal{T}$: Greek superscripts and subscripts denote, respectively, elements of $\mathcal{T}$ and its dual $\mathcal{T}^*$. Thus a solution $v$ of eqn.(7) might be denoted $v^\mu$, while a linear map $\mathcal{T} \to \mathbb{R}$ might be denoted $w_\mu$. The action of $w$ on $v$ would be expressed by contraction: $w(v) = w_\mu v^\mu$. We denote by $\eta_{\mu \nu}$ the above Lorentz metric on $\mathcal{T}$, i.e., we set $\eta_{\mu \nu}v^\mu w^\nu = \langle v, w \rangle$. We shall use $\eta_{\mu \nu}$ (and its inverse) to lower and raise indices of tensors over $\mathcal{T}$. The objects with which we shall be concerned are fields on $\mathcal{H}$ that may have Latin indices (indicating tensor character over the manifold $\mathcal{H}$) and Greek indices (indicating tensor character over the vector space $\mathcal{T}$). Thus, for example, $\zeta_\alpha$ would denote a $\mathcal{T}^*$-valued function on $\mathcal{H}$, $\zeta^a$ would denote an ordinary tangent vector field on $\mathcal{H}$, and $\zeta^a_\alpha$ would denote a $\mathcal{T}^*$-valued tangent vector field on $\mathcal{H}$. In particular, an element $v^\mu$ in $\mathcal{T}$ is now viewed as a $\mathcal{T}$-valued constant function on $\mathcal{H}$. We lower and raise Greek indices of such fields with $\eta^{\alpha \beta}$ and its inverse, and lower and raise Latin indices with $q^{0}_{ab}$ and its inverse. There is a natural field, $\alpha_\mu$, defined by the property that, for any $v^\mu \in \mathcal{T}$, $\alpha_\mu v^\mu$ is the corresponding solution of eqn.(7). Then, e.g., $\alpha_\mu \alpha^\mu = 1$. The derivative operator $D_a$ on $\mathcal{H}$ associated with $q^{0}_{ab}$ extends to a derivative operator on our indexed fields by demanding that $D_a v^\alpha = 0$, for $v^\alpha$ any constant field. There now follows $D_a q^{0}_{bc} = 0$, $D_a \eta^{0}_{\alpha \beta} = 0$,

\begin{equation}
D_a D_b \alpha_\mu + \alpha_\mu q^{0}_{ab} = 0
\end{equation}

(from eqn.(7)),

\begin{equation}
D_a \alpha_\mu D^a \alpha_\nu + \alpha_\mu \alpha_\nu = \eta_{\mu \nu}
\end{equation}

(from the definition of $\eta_{\mu \nu}$), and $\eta_{\mu \nu} D_a \alpha_\mu D_b \alpha_\nu = q^{0}_{ab}$.\textsuperscript{17}
D. Conserved Quantities

Now imagine that we were somehow able to find a divergence-free vector field, constructed from (the remnants of) some physical fields and the background geometry of $\mathcal{H}$. Integrating (the dual of) this vector field over a 2-sphere cut (i.e., a non-contractible 2-sphere sub-manifold) of $\mathcal{H}$, we obtain a number—one clearly independent of choice of cut. Think of such an integral as being the limit of an integral over a space-like 2-sphere in space-time, as the 2-sphere approaches the cut at spatial infinity. These integrals we call conserved quantities. In each of the examples we shall consider, the divergence-free vector field is multi-linear in $\alpha_{\mu}$, and so the conserved quantities may be viewed as a tensor over $T$.

We now give three well-known\textsuperscript{1,2,4,11} examples of conserved quantities. Some of the computations are relegated to section IV and Appendix C. Fix a space-time $(\tilde{M}, \tilde{g}_{ab})$, a completion $(M, \Omega)$ thereof and a cut $C$ of $\mathcal{H}$.

For the first example, let $\tilde{F}_{ab}$ be a Maxwell field on $\tilde{M}$, regular of order 2. Consider the right side of
\begin{equation}
Q = \frac{1}{4\pi} \int_{C} \tilde{E}^{a} dS_{a},
\end{equation}
where $\tilde{E}^{a}$ is the (zero-th order) remnant of $E_{a}$ given by eqn.(\textsuperscript{11}). Maxwell’s equations imply the integrand above is divergence-free (c.f. eqn.(\textsuperscript{51})). Thus eqn.(\textsuperscript{11}) defines a conserved quantity. This $Q$ is precisely the electric charge, for the right side of eqn.(\textsuperscript{11}) is the limit of the integral of $\ast F_{ab}$ over a large space-like 2-sphere in the space-time as that 2-sphere approaches the cut $C$. For the second example, consider the right side of
\begin{equation}
\mathcal{P}^{\mu} = \frac{1}{8\pi} \int_{C} \tilde{E}^{ab} D_{b} \alpha^{\mu} dS_{a},
\end{equation}
where $\tilde{E}^{ab}$ is the remnant of $E_{ab}$, a portion of the Weyl tensor, given in eqn.(\textsuperscript{8}). The remnant field equation (eqn.(\textsuperscript{20})) together with eqn.(\textsuperscript{8}) on $\alpha_{\mu}$, imply that the integrand above is divergence-free Thus eqn.(\textsuperscript{11}) defines a conserved quantity, which is a vector over $\mathcal{T}$. This $\mathcal{P}^{\mu}$ is precisely\textsuperscript{1,2} the total mass-momentum of the space-time. For the third example, consider the
right side of
\[
M_{\mu\nu} = -\frac{1}{16\pi} \epsilon_{\mu\nu}^{\tau\sigma} \int_{C} B^{ab}_{a\tau} D_{b\sigma} dS_{a},
\] (12)

where \( \epsilon_{\mu\nu\tau\sigma} \) denotes the \( \eta \)-alternating tensor on \( \mathcal{T} \). In order for the integrand above to be divergence-free, we must impose on the space-time the additional condition\(^{19} \) that
\[
D^{a}_{[aT_{b}]} = 0.
\] (13)

Under this additional condition, eqn.(12) defines a conserved quantity, which is a two-form over \( \mathcal{T} \). This \( M_{\mu\nu} \) is precisely\(^{4} \) the total angular momentum of the space-time.

Finally, we revisit the issue of gauge. Fix a space-time \((\tilde{M}, \tilde{g}_{ab})\), and a completion \((M, \Omega)\) thereof. Demand further that the completion satisfy the gauge condition (6), so the remaining gauge freedom is represented by the choice of some \( \omega \in \mathcal{T} \). Applying such a gauge transformation, the remnants of any physical field, and thus also of any conserved quantities associated with that field, will in general change. Specifically, let \( Q_{A} \) be any conserved quantity or any remnant field, where the subscript \( A \) is an abbreviation for all the indices of \( Q \). Then, for each translation \( \omega \in \mathcal{T} \), there corresponds a “gauge-transformed” quantity—\( Q_{A}[\omega] \). Thus, we may regard our quantity \( Q_{A} \) as a tensor field on the 4-manifold \( \mathcal{T} \) so defined that its value at \( \omega \in \mathcal{T} \) is \( Q_{A}[\omega] \). In short, the gauge behavior of our original quantity \( Q_{A} \) is coded in the position dependence of this tensor field on \( \mathcal{T} \). The derivative of this tensor field reflects the behavior of the quantity under “infinitesimal gauge transformation”. Indeed, from
\[
Q_{A}[\omega + \delta \omega] = Q_{A}[\omega] + (\delta \omega)^\mu Q^{(1)}_{\mu A}[\omega] + O((\delta \omega)^2),
\] (14)
we have
\[
\nabla_{\mu} Q_{A} = Q^{(1)}_{\mu A},
\] (15)

where \( \nabla_{\mu} \) denote the natural derivative operator on the 4-manifold \( \mathcal{T} \). As examples, consider the conserved quantities (10)–(12). Under a gauge transformation, \( \Omega' = \Omega(1 + \omega \Omega) \) with \( \omega \in \mathcal{T} \) the remnants \( \tilde{E}_{a}, \tilde{E}_{ab} \) remain unchanged, while \( \tilde{B}_{ab} \) changes to \( \tilde{B}'_{ab} = \tilde{B}_{ab} - 2\epsilon_{(a}^{\ k} \tilde{E}_{b)}_{k} \tilde{D}_{l} \omega \).
In terms of the corresponding tensor fields on the 4-manifold $\mathcal{T}$, these become, $\nabla_\mu E_a = 0$, $\nabla_\mu E_{ab} = 0$, and $\nabla_\mu B_{ab} = -2\epsilon_{(kl} E_{b)k} D_l \alpha_\mu$. It follows that the total electric charge $Q$ (10) and the 4-momentum $P_\mu$ (11) are gauge invariant, and that the angular momentum $M_{\mu\nu}$ (12) changes via

$$M'_{\mu\nu} = M_{\mu\nu} - \omega_{[\mu} P_{\nu]}.$$  

(16)

In terms of the corresponding tensor fields on the 4-manifold $\mathcal{T}$, these become, respectively, $\nabla_\lambda Q = 0$, $\nabla_\lambda P_\mu = 0$, and

$$\nabla_\lambda M_{\mu\nu} = -\eta_{[\mu} P_{\nu]}.$$  

(17)

III. MINKOWSKI SPACE-TIME

We now apply the framework developed in the previous section to the study of conserved quantities associated with physical fields in Minkowski space-time. Minkowski space-time is a good starting point: It is simple, and suggestive of what might happen in the presence of curvature. We shall take as the physical field successively a Klein-Gordon field, a Maxwell field and a linearized gravitational field. We will write down, for each of these cases, all conserved quantities linear in the physical fields and multi-linear in asymptotic translations.

Let $(\tilde{M}, \tilde{\eta}_{ab})$ be Minkowski space-time. Fix a point $p \in \tilde{M}$, let $\Omega$ be the inverse geodesic distance from $p$. Then this $\Omega$ yields a completion $(M, \Omega)$ of Minkowski space-time which we call the standard completion. In this completion, we have $n_\lambda = 0$ and $n_{ab} = 0$, for $n \geq 1$.

A. Remnant field equations

Here we derive the remnant field equations for Klein-Gordon, Maxwell, and linearized gravitational fields for later use in constructing conserved quantities. For what follows we fix a standard completion of Minkowski space-time $\tilde{M}$ and denote by $D_a$ the derivative operator associated with $q_{ab}$ on constant-$\Omega$ surfaces.
Let $\tilde{\phi}$ be a Klein-Gordon field on $\tilde{M}$ asymptotically regular of order 1. Setting $\phi = \Omega^{-1}\tilde{\phi}$, we have

$$0 = \tilde{\nabla}^2 \tilde{\phi} = \Omega^2 \left[ (D^2 - 1)\phi + \Omega L_n \phi + \Omega^2 (L_n)^2 \phi \right].$$

(18)

Taking the remnants of the above equation, we obtain,

$$D^n_a \phi = (-n^2 + 1)\phi,$$

(19)

for $n = 0, 1, 2, \ldots$.

Let $\tilde{F}_{ab}$ be a Maxwell field on $\tilde{M}$ asymptotically regular of order 2, with remnants $\tilde{E}_a$ and $\tilde{B}_a$. Using eqn.(4), Maxwell’s equation can be written as

$$0 = \tilde{\nabla}^m \tilde{F}_{ma} = \Omega D^m E_m \nabla_a \Omega - \Omega^2 \left( \Omega L_n E_a - \epsilon_{akl} D^k B^l \right),$$

(20)

and

$$0 = \tilde{\nabla}^m \ast \tilde{F}_{ma} = \Omega D^m B_m \nabla_a \Omega - \Omega^2 \left( \Omega L_n B_a + \epsilon_{akl} D^k E^l \right).$$

(21)

Taking the remnants of the above equations, we obtain

$$D^0_a \tilde{E}_a = 0, \quad D^0_a \tilde{B}_a = 0,$$

(22)

and

$$\epsilon^{abc} D_b \tilde{E}_c = n \tilde{E}_a, \quad -\epsilon^{abc} D_b \tilde{E}_c = n \tilde{B}_a,$$

(23)

for $n = 0, 1, 2, \ldots$. Note that eqn.(23) imply

$$D^n_a \tilde{E}_a = (-n^2 + 2)\tilde{E}_a, \quad D^n_a \tilde{B}_a = (-n^2 + 2)\tilde{B}_a,$$

(24)

for $n = 0, 1, 2, \ldots$.

Let $\tilde{K}_{abcd}$ be a linearized gravitational field on $\tilde{M}$, i.e., a tensor field on $\tilde{M}$ having the same symmetry and contractions as the Weyl tensor and satisfying the linearized Bianchi identity:

$$\tilde{\nabla}_{[a} \tilde{K}_{bc]de} = 0.$$  

(25)

Let $\tilde{K}_{abcd}$ be asymptotically regular of order 3, so $E_{ab} \equiv \Omega^3 \tilde{K}_{abkl} n^k n^l$ and $B_{ab} \equiv \Omega^3 \ast \tilde{K}_{abkl} n^k n^l$ are smoothly extendible to $\mathcal{H}$. Their remnants, denoted $\tilde{E}_{ab}$ and $\tilde{B}_{ab}$, are symmetric and
trace-free. The linearized Bianchi identity can be written as
\[
0 = \tilde{\nabla}^m \tilde{K}^{*}_{mabc}
\]
\[
= [\Omega^{-1} \nabla_a \Omega D^m B_{m[b} - (\Omega \mathcal{L}_n B_{a|b} + \epsilon_{akl} D^k E^l |_b)] \nabla_c \Omega
\]
\[
+ \frac{1}{2} [\nabla_a \Omega D^k E_{km} - \Omega (\Omega \mathcal{L}_n E_{ma} - \epsilon_{mkl} D^k B^l_a)] \epsilon^m_{bc}.
\]
(26)

Taking the remnants of the above equation, we obtain
\[
\epsilon_{lma} D^l n B^{ab} = n n E^{ab},
\]
\[
-\epsilon_{lma} D^l n E^{ab} = n n B^{ab},
\]
(27)

for \( n = 0, 1, 2, \ldots \). Note that, eqn.(27) imply
\[
D^n E^{ab} = (-n^2 + 3) E^{ab},
\]
\[
D^n B^{ab} = (-n^2 + 3) B^{ab},
\]
(28)

for \( n = 0, 1, 2, \ldots \).

Recall that the present framework is subject to a class of restricted gauge transformations (namely, replacements of \( \Omega \) by \( \Omega' = \Omega (1 + \omega \Omega) \)), which preserve the gauge conditions \( n^2 = 0, n \geq 2 \), and that each such gauge transformation is completely characterized by an \( \omega \in T \). For completeness, we summarize the behavior of the remnants above under such a gauge transformation:

\[
\nabla^n \nabla^0 \phi = n (\mathcal{L}_{D\alpha\mu} \phi - n \alpha^0 \phi),
\]
(29)

\[
\nabla^n E^a = n (\mathcal{L}_{D\alpha\mu} E^a - n \alpha^0 E^a + \epsilon_{akl} B^l_k D^a(\alpha_l)),
\]
(30)

\[
\nabla^n B^a = n (\mathcal{L}_{D\alpha\mu} B^a - n \alpha^0 B^a - \epsilon_{akl} E^l_k D^a(\alpha_l)),
\]
(31)

\[
\nabla^n E^{ab} = n (\mathcal{L}_{D\alpha\mu} E^{ab} - n \alpha^0 E^{ab} + 2 \epsilon^{kl} (a \times B^l_k) D^a(\alpha_l)),
\]
(32)

\[
\nabla^n B^{ab} = n (\mathcal{L}_{D\alpha\mu} B^{ab} - n \alpha^0 B^{ab} - 2 \epsilon^{kl} (a \times E^l_k) D^a(\alpha_l)),
\]
(33)

for \( n = 0, 1, 2, \ldots \). Note that \( \phi^0, \tilde{E}_a, \tilde{B}_a, \tilde{E}_{ab}, \tilde{B}_{ab} \) are gauge invariant. As a consistency check, we note also that the \( \nabla_{\mu} \)-curl of the right side of each of the above equation vanishes, by virtue of \( \nabla_{\mu} \alpha_{\nu} = 0 \), as it must. Of course, these gauge-transformed fields satisfy the same equations as the original fields.
B. Remnant Radiation Multipoles

It is perhaps most natural to seek conserved quantities that are linear in the remnants, since, e.g., this category includes all well-known conserved quantities. In this section we shall find all such conserved quantities for a Klein-Gordon field, a Maxwell field and a linearized gravitational field in Minkowski space-time $\tilde{M}$. Again, we fix the standard completion of $\tilde{M}$.

We begin with the Klein-Gordon field. Let $\tilde{\phi}$ be a Klein-Gordon field asymptotically regular of order 1, with remnants $n_{\tilde{\phi}}$.

**Theorem 1**

(i) The conserved quantities linear in this Klein-Gordon field consist precisely of the family

$$\mathcal{K}_{\mu_1 \cdots \mu_{n-1}}[\tilde{\phi}] \equiv \int_C \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}})D^a \bar{n}^{\mu_1} \cdots \bar{n}^{\mu_{n-1}} \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) dS_a, \ n \geq 1, \quad (34)$$

where $\mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}})$ denotes the symmetric, trace-free part of $\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}$.\(^{22}\)

(ii) The $\mathcal{K}_{\mu_1 \cdots \mu_{n-1}}$ are totally symmetric and trace-free.

(iii) The behavior of $\mathcal{K}_{\mu_1 \cdots \mu_{n-1}}$ under restricted gauge transformations is given by

$$\nabla_\mu \mathcal{K}_{\mu_1 \cdots \mu_{n-1}} = \frac{1}{2} n(n-2)\eta(\mu_1 \mu_2)\mathcal{K}_{\mu_3 \cdots \mu_{n-1}} - n(n-1)\eta(\mu_1)\mathcal{K}_{\mu_2 \cdots \mu_{n-1}}. \quad (35)$$

We will refer to these $\mathcal{K}$’s as the remnant radiation multipoles of a Klein-Gordon field.

To see that eqn.(34) indeed defines a conserved quantity, take the divergence of the integrand, and use that both $\tilde{\phi}$ and $\mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}})$ satisfy eqn.(19). To prove (iii), use eqn.(29), the definition of $\mathcal{K}$, and a certain identity on $\mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}})$. See Appendix B for details. Note the right side of eqn.(35) is, up to an overall factor, the only $(n-1)$-th rank, symmetric, trace-free tensor linear in $n^{-2}$. Eqn.(35) states that the dependence of the $\mathcal{K}$’s on position in $\mathcal{\mathcal{T}}$ is exactly that of ordinary multipole moments. The proof that the family given by eqn.(34) exhausts the linear conserved quantities in the Klein-Gordon case is outlined in Appendix B.

As an example of these remnant radiation multipoles, let $\tilde{\phi} = (f(t + r) - f(t - r))/r$. Then, provided $k_{\pm}(x) \equiv f(\pm x), x > 0,$ are both smoothly extendible to zero, this $\tilde{\phi}$ will be asymptotically regular of order 1. Then the remnants of $\tilde{\phi}$ are given by $\tilde{\phi} = (1 + \zeta^2)^{-1/2}k_{\pm}^{(n)}(0)\{(1 + $
\(\zeta^2/2 - \zeta^n - k^{(n)}(0)(1 + \zeta^2)^{1/2} + \zeta^n\), where we have set \(\zeta = -\Omega^{-2}\partial_\Omega|_{\mathcal{H}} \in \mathcal{T}\). The \(\mathcal{K}\)'s in this example involve various derivatives of \(k_\pm\) at zero. Explicitly, the first two are given by 

\[\mathcal{K} = 4\pi k'_0 + k''(0)\]\n
\[\mathcal{K}_\mu = 4\pi [k''(0) + k''(0)]\langle \alpha_\mu, \zeta \rangle.\]

Thus the \(\mathcal{K}\)'s in this example describe radiation emanating from future and past time-like infinity.

We turn next to the Maxwell case. Let \(\tilde{\mathcal{F}}_{ab}\) be a Maxwell field asymptotically regular of order 2, with remnants \(\tilde{n}_a E_a\) and \(\tilde{n}_a B_a\).

**Theorem 2**

(i) The conserved quantities linear in this Maxwell field consist precisely of the electric charge (given by eqn. (10)), the magnetic charge (obtained by replacing \(0_E_a\) by \(0_B_a\) in eqn. (10)), and the family

\[\mathcal{E}_{\mu_1\cdots\mu_{n-1}} \equiv \mathcal{K}_{\mu_1\cdots\mu_{n-1}} [\tilde{n}_m D_m \alpha_\mu]\]

\[= \int \left[ D^\nu (\tilde{n}_m D_m \alpha_\mu) C(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) - \tilde{n}_m D_m \alpha_\mu D^\nu C(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \right] dS_a,\]

for \(n = 1, 2, 3, \ldots\).

(ii) The \(\mathcal{E}_{\mu_1\cdots\mu_{n-1}}\) are trace-free in all indices, totally symmetric in the indices \(\mu_1 \cdots \mu_{n-1}\), and satisfy

\[\mathcal{E}_{(\mu_1\cdots\mu_{n-1})} = 0.\]

(iii) The gauge behavior of \(\mathcal{E}_{\mu_1\cdots\mu_{n-1}}\) is given by

\[\nabla^\nu \mathcal{E}^\nu_{\mu_1\cdots\mu_{n-1}} = \frac{1}{2} n(n-2) \eta(\mu_1, \mu_2) \mathcal{E}^{\nu_{\mu_3 \cdots \mu_{n-1}}} - n(n-1) \delta^\nu_{\mu_1} \mathcal{E}^\nu_{\mu_2 \cdots \mu_{n-1}}
\]

\[+ \frac{n(n-2)}{n-1} \eta(\mu_1, \mu_2) \mathcal{E}^{[\mu \nu]_{\mu_3 \cdots \mu_{n-1}}} - 2n \delta^\mu_{\mu_1} \mathcal{E}^{\nu}_{\mu_2 \cdots \mu_{n-1}}.\]

We will refer to the \(\mathcal{E}\)'s as the remnant radiation multipoles of a Maxwell field.

To see that eqn. (36) indeed defines a conserved quantity, take the divergence of the integrand and use that \(\tilde{n}_m D_m \alpha_\mu\) and \(\tilde{n}_m D_m \alpha_\mu\) satisfy eqn. (13). To prove eqn. (37), we note that its integrand is the divergence of an anti-symmetric tensor. Eqn. (37) implies, in particular, that \(\mathcal{E}_\mu\) is zero, and that \(\mathcal{E}_{\mu\nu}\) is anti-symmetric. While a second family of conserved quantities, \(*\mathcal{E}\),
associated similarly with $B_a$ could be defined, they yield nothing new, for we have

$$^* \mathcal{E}_{\mu_1 \ldots \mu_{n-1}} = \frac{n-1}{n} \epsilon_{\mu(\mu_1} \mathcal{E}_{|\nu_1|\mu_2| \ldots |\mu_{n-1})}. \quad (39)$$

Note that $^* \mathcal{E}_{\mu_1 \ldots \mu_{n-1}}$ has the symmetries $(i)$ in theorem 2 above and that $^{**} \mathcal{E}_{\mu_1 \ldots \mu_{n-1}} = - \mathcal{E}_{\mu_1 \ldots \mu_{n-1}}$. The gauge behavior, eqn.(38), is proved in Appendix B. Note eqn.(38) yields, in particular, that $\mathcal{E}_{\mu\nu}$ is gauge invariant. The proof that the quantities given by eqn.(36) exhaust the linear, Maxwell conserved quantities is outlined in Appendix B. Here is an example of these electro-magnetic conserved quantities. Let $\tilde{\phi}$ be a Klein-Gordon field asymptotically regular of order 1, $\tilde{w}^{ab}$ a constant anti-symmetric tensor field on $\tilde{M}$, and set $\tilde{F}_{ab} = \tilde{\nabla}_{[a}(\tilde{w}_{b]}\tilde{\nabla}^{m}\tilde{\phi})$. Then this $\tilde{F}_{ab}$ is a solution of Maxwell’s equations, asymptotically regular of order 2. Its remnants are given in terms of those of $\tilde{\phi}$ by

$$E_a = D_a D_b \tilde{\phi} \tilde{\xi}^b - (n + 1)D_m \xi_a D^n \tilde{\phi} + n^2 \tilde{\phi} \xi_a, \quad (40)$$

where we have set $\xi^a = \tilde{w}^{ab} x_b$. Then, the remnant radiation multipoles of $\tilde{F}_{ab}$ can be expressed in terms of those of $\tilde{\phi}$. For instance, we have $\mathcal{E}_{\mu\nu} = Kw_{\mu\nu}$, where we have set $w_{\mu\nu} \equiv 2\xi_a\alpha_{[\mu}D^a\alpha_{\nu]} + D_a \xi_b D^a\alpha_{\mu} D^b\alpha_{\nu}$.

We turn finally to linearized gravity. Let $K_{abcd}$ be a linearized gravitational field asymptotically regular of order 3, with remnants $E_{ab}$ and $B_{ab}$.

**Theorem 3** (i) The conserved quantities linear in this linearized gravitational field consist precisely of the mass-momentum (given by eqn.(11)), the angular momentum, (given by eqn.(12)), and

$$\mathcal{G}_{\mu\nu\mu_1 \ldots \mu_{n-1}} \equiv K_{\mu_1 \ldots \mu_{n-1}} [E_{k}^{|k|D_{k}\alpha_{\mu}D_{1}\alpha_{\nu}}] = E_{|\nu_1|\mu_2| \ldots |\mu_{n-1})}[E_{ab}D^b\alpha_{\mu}]$$

$$= \int \left[ D^a (E_{k}^{|k|D_{k}\alpha_{\mu}D_{1}\alpha_{\nu}})C(\alpha_{\mu_1} \cdot \cdot \cdot \alpha_{\mu_{n-1}}) \right. \left. - \tilde{E}_{k}^{|k|D_{k}\alpha_{\mu}D_{1}\alpha_{\mu}}D^aC(\alpha_{\mu_1} \cdot \cdot \cdot \alpha_{\mu_{n-1}}) \right] dS_a, \quad (41)$$

for $n = 1, 2, 3, \ldots$.
(ii) The $G_{\mu_1\mu_2\cdots\mu_{n-1}}$ are trace-free in all indices, totally symmetric in the indices $\mu_1 \cdots \mu_{n-1}$, symmetric in indices $\mu, \nu$, and satisfy

$$G_{\mu_1\cdots\mu_{n-1}} = 0.$$  \hfill (42)

(iii) The gauge behavior of $G_{\mu_1\cdots\mu_{n-1}}$ is given by

$$\nabla^\sigma G^{\mu_1\cdots\mu_{n-1}} = \frac{1}{2}n(n-2)\eta_{\mu_1\mu_2}G^{\mu_\sigma_\mu_3\cdots\mu_{n-1}} - n(n-1)\delta^\sigma_{\mu_1}G^{\mu_2\cdots\mu_{n-1}} + \frac{n(n-2)}{n-1}\eta_{\mu_1\mu_2}(G^{\mu_\sigma}[\mu_3\cdots\mu_{n-1}] + G^{\nu_\sigma][\mu_3\cdots\mu_{n-1}})

- 2n\delta^\sigma_{\mu_1}(G^{\mu_\sigma}[\mu_2\cdots\mu_{n-1}] + G^{\nu_\sigma][\mu_2\cdots\mu_{n-1}}). \hfill (43)$$

We will refer to the $G$’s as the remnant radiation multipoles of a linearized gravitational field.

To see that eqn.\(41\) indeed defines a conserved quantity, take the divergence of the integrand and use that $E_k^{\alpha_\mu_1}D_k^{\alpha_\mu_2}$ and $B_k^{\alpha_\mu_1}D_k^{\alpha_\mu_2}$ satisfy eqn.\((19)\). Eqn.\((42)\), which is actually equivalent to $E^{\nu_\mu_1\cdots\mu_{n-1}} = 0$, implies, in particular, that $G_{\mu_\nu} = G^{\nu_\mu_1\cdots\mu_{n-1}} = 0$. While a second family of conserved quantities, $G_1^{\mu_\nu_1\cdots\mu_{n-1}}$, associated similarly with $B_1^{\alpha_\beta}$ could be defined, they yield nothing new, for we have\(25\)

$$^*G^{\mu_1\cdots\mu_{n-1}} = \frac{n}{n-1}\epsilon^{\sigma_\mu_1\nu_2 \cdots \nu_{n-1}}G^{\mu_\sigma_\nu_2 \cdots \nu_{n-1}}.$$  \hfill (44)

Note that $^*G^{\mu_\nu_1\cdots\mu_{n-1}}$ also satisfies (ii) in theorem 3 and that $^**G^{\mu_\nu_1\cdots\mu_{n-1}} = -G^{\mu_\nu_1\cdots\mu_{n-1}}$. The proof that the quantities given by eqn.\((41)\) exhaust the linear, gravitational conserved quantities is outlined in Appendix B. We omit the proof of the gauge behavior (eqn.\((43)\)), which is similar to the Maxwell case. Examples of linearized gravitational fields, their remnants, and their remnant radiation multipoles can be constructed in a manner similar to that of the Maxwell case.\(26\)

One might expect, on physical grounds, that a static field would be characterized completely by its static multipole moments and that its remnant radiation multipoles would all vanish identically. This indeed turns out to be the case. See Appendix A.
IV. CURVED SPACE-TIME

It is natural to ask whether the remnant radiation multipoles constructed above for various fields in Minkowski space-time can be generalized to curved space-time. To address this issue, we first obtain the remnant equations. Let \( \tilde{\phi} \) be a Klein-Gordon field asymptotically regular of order 1, so, \( \phi(\equiv \Omega^{-1}\phi) \) is smoothly extendible to \( \mathcal{H} \). Then the Klein-Gordon equation on \( \tilde{\phi} \) yields

\[
0 = \tilde{\nabla}^2 \tilde{\phi} = \Omega^3[D^2\phi + \lambda^{-1}D^a\lambda D_a\phi + \Omega\lambda^{-2}(2\phi + \Omega\phi) + (\phi + \Omega\phi)(-\lambda^{-2} - \Omega\lambda^{-3}\lambda + \frac{1}{2}\Omega\lambda^{-2}q^{ab}q_{ab})],
\]

(45)

where \( D_a \) is, as before, the derivative operator on constant-\( \Omega \) surfaces induced from \( \tilde{\nabla}_a \). Evaluating (45) and its first two normal derivatives on \( \mathcal{H} \), we obtain, respectively,

\[
0 = (D^2 - 1)^0\phi, \quad (46)
\]

\[
0 = (D^2)^1\phi, \quad (47)
\]

\[
0 = (D^2)^2\phi + 3\phi - \tilde{q}^{ab}D_a D_b\phi - 16\lambda^1\phi
\]

\[
-14\lambda D^a\lambda D_a\phi - 2(D_a D_b\lambda)D^a\lambda D^b\phi + 32\lambda^2\phi + 2(D\lambda)^2\phi.
\]

(48)

For the \( n \)-th derivative, the equation that results has the form

\[
0 = (D^2 + n^2 - 1)^n\phi - 4n^2(n - 1)^{n-1}\lambda^1\phi + ..., \quad (49)
\]

where ... involves only remnants of \( \phi \) of order \( \leq n - 2 \).

Next, let \( \tilde{F}_{ab} \) be a Maxwell field asymptotically regular of order 2. Then Maxwell’s equations yield

\[
0 = \tilde{\nabla}^m \tilde{F}_{ma} = -\lambda\Omega D^m E_m \nabla_a \Omega - \lambda^{-1}\Omega^2\epsilon_a^{bc}[D_b(\lambda B_c) + \frac{1}{2}\Omega\mathcal{L}_{\chi^2}(E^m\epsilon_{mbc})].
\]

(50)

Evaluating (50) and its first two normal derivatives on \( \mathcal{H} \), we obtain, respectively,

\[
D_a E^a = 0, \quad D_a B^a = 0, \quad (51)
\]

\[
D_{[a} E_{b]} = 0, \quad D_{[a} B_{b]} = 0, \quad (52)
\]
\[ D_{[a}^{\ 1} B_{b]} = \ -\frac{1}{2} \epsilon_{ab} c (E_c - 2 \lambda E_c), \]  
(53)  
\[ D_{[a}^{\ 1} E_{b]} = \ \frac{1}{2} \epsilon_{ab} c (E_c - 2 \lambda E_c), \]  
(54)  
\[ D_{[a}^{\ 2} B_{b]} = \ -\epsilon_{ab} c [\ \frac{2}{2} E_c - 4 \lambda E_c + w_{cd} E^d], \]  
(55)  
\[ D_{[a}^{\ 2} E_{b]} = \ \epsilon_{ab} c [B_c - 4 \lambda B_c + w_{cd} B^d], \]  
(56)

where we have set \[ E_a = E_a + \lambda E_a, \ E_a = E_a + 2 \lambda E_a + \lambda E_a, \ B_a = B_a + \lambda B_a, \ B_a = B_a + 2 \lambda B_a + \lambda B_a, \]
and \[ w_{ab} = -\frac{2}{2} q_{ab} + (\frac{1}{2} q + 3 \lambda^2 - 2) q_{ab}. \] For the \( n \)-th derivative, the equations that result have the form

\[ D_{[a}^{\ n} (B_{b]} + n \lambda B_{b]} = -\frac{n}{2} \epsilon_{ab} c (E_c - n \lambda E_c + ...), \]  
(57)  
\[ D_{[a}^{\ n} (E_{b]} + n \lambda E_{b]} = \frac{n}{2} \epsilon_{ab} c (B_c - n \lambda B_c + ...), \]  
(58)

where \( \ldots \) involves only remnants of \( E_a, B_a \) of order \( \leq n - 2 \).

We turn finally to the gravitational field. The remnant field equations of order \( \leq 2 \) were obtained by Beig and Schmidt\(^8,9\) in the vacuum case under the gauge conditions (1). We here drop the assumption of vacuum and the gauge condition. See Appendix C for outline of our derivation. The zeroth-order equations are satisfied identically. The first-order equations are

\[ q_{ab} = -2 \lambda q_{ab} \]  
(59)  
\[ (D^2 + 3) \lambda = 0. \]  
(60)

The second-order equations are

\[ q^{2} = 2(D^{\lambda})^2 + 24 \lambda^2 - D^2 \lambda - 6 \lambda - D^m T_m - 2 T \]  
(61)  
\[ D_b q^a = 32 \lambda D_a \lambda + 4 D^b \lambda D_a D_b \lambda - 4 D_a \lambda - 6 D_a (D^2 \lambda) \]  
(62)  
\[ + D_a (-D^m T_m - 2 T) + 2 T, \]  
\[ (D^2 - 2) q^{2} = 8(D^{\lambda})^2 q_{ab} + 20 D_a \lambda D_b \lambda + 28 \lambda D_a D_b \lambda - 36 \lambda^2 q_{ab} + 4 D_a D^c \lambda D_b D_c \lambda \]  
(63)  
\[ + 4 D^c \lambda D_a D_b D_c \lambda - 4 D_a D_b \lambda + 4 \lambda q_{ab} - D_a D_b (D^2 \lambda) \]  
\[ - 4 T_{ab} + D_a D_b (-D^m T_m - 2 T) + 4 D_{(a T_b)}. \]
The third- and fourth-order equations are not used in what follows and are collected in Appendix C, where we also rewrite the second-order equations in terms of the remnants of the Weyl tensor.

We turn now to the issue of whether or not the various conserved quantities that we defined in flat space-time can be generalized to curved space-time. Recall that a conserved quantity is given by the integral over a cut of $\mathcal{H}$ of a vector field $v^a\Gamma$ on $\mathcal{H}$, where that field is expressed as an algebraic function of the preferred field $\alpha_\mu$ of the universal background geometry of $\mathcal{H}$, the remnants of the physical field, the remnants of the geometry, and their derivatives. The divergence of this $v^a\Gamma$ must, for independence of cut, vanish by virtue of the equations satisfied by $\alpha_\mu$ and the various remnants. In the special case of flat space-time, we have (or, at least, achieved via gauge) $k^k = 0$, $\tilde{q}_{ab} = 0$, for $k \geq 1$, i.e., we have effectively no “remnants of the geometry”. Clearly, every conserved quantity in general remains a conserved quantity in the special case of flat space-time. But the converse need not hold. Given a conserved quantity in flat space-time—i.e., given a divergence-free field $v^a\Gamma$ constructed from the preferred field $\alpha_\mu$, the remnants of the physical field, and their derivatives—then it may or may not be the case that it is the specialization to flat space-time of some conserved quantity in curved space-time. When it is, we say we have produced a generalization of our given flat-space conserved quantity.

Consider first the Klein-Gordon case. We have immediately from eqn. (47):

**Theorem 4** The conserved quantity $K$ ($n = 1$ in eqn. (34)) for the Klein-Gordon field in flat space-time admits a generalization, in the sense described above, to a conserved quantity in curved space-time, namely that given by

$$K = \int D^a\phi dS_a.$$  \hspace{1cm} (64)

For the higher-order Klein-Gordon remnant radiation multipoles, we have

**Theorem 5** Fix $n \geq 2$. Then the conserved quantity $K_{\mu_1 \cdots \mu_{n-1}}$ for the Klein-Gordon field in flat space-time, given by eqn. (34), does not admit generalization to curved space-time.

**Proof**: Let, for contradiction, $v^{a}_{\mu_1 \cdots \mu_{n-1}}$ be a generalization to curved space-time. By a simple scaling argument (using, respectively, linearity of the Klein-Gordon remnant field equations in
the $\phi$ and homogeneity of all remnant field equations in order), we may assume that $v^{a}_{\mu_1 \cdots \mu_{n-1}}$ is linear in the $\phi$, and of total order $n$ in all remnants. Thus, $v^{a}_{\mu_1 \cdots \mu_{n-1}}$ contains no $\phi$, for $k > n$, and the term involving $\phi$ is, because $v^{a}_{\mu_1 \cdots \mu_{n-1}}$ must reduce to the integrand of $\mathcal{K}_{\mu_1 \cdots \mu_{n-1}}$ in flat space-time, precisely $\psi_{\mu_1 \cdots \mu_{n-1}} D^a \phi - \phi D^a \psi_{\mu_1 \cdots \mu_{n-1}}$, where we have set $\psi_{\mu_1 \cdots \mu_{n-1}} = \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}})$. Denote by $u^{a}_{\mu_1 \cdots \mu_{n-1}}$ the term of $v^{a}_{\mu_1 \cdots \mu_{n-1}}$ involving $\phi$. Then, the vanishing of divergence of $v^{a}_{\mu_1 \cdots \mu_{n-1}}$ implies

$$D_a u^{a}_{\mu_1 \cdots \mu_{n-1}} = - 4n^2 (n-1) \lambda \psi_{\mu_1 \cdots \mu_{n-1}} \phi,$$

where $\hat{=} = \text{equality modulo Klein-Gordon remnants of order} \leq n - 2$. But there exists no such $u^{a}_{\mu_1 \cdots \mu_{n-1}}$, as one sees by the following steps. First, add to $v^{a}_{\mu_1 \cdots \mu_{n-1}}$ a divergence of an antisymmetric tensor field to achieve the form

$$u^{a}_{\mu_1 \cdots \mu_{n-1}} = w_{\mu_1 \cdots \mu_{n-1}} D^a \phi - \phi D^a w_{\mu_1 \cdots \mu_{n-1}},$$

with $w_{\mu_1 \cdots \mu_{n-1}}$ linear in $\lambda$ and $\psi_{\mu_1 \cdots \mu_{n-1}}$, and from eqn.(65) and (49) satisfying

$$(D^2 + n^2 - 2n) w_{\mu_1 \cdots \mu_{n-1}} = 4n^2 (n-1) \lambda \psi^{1}_{\mu_1 \cdots \mu_{n-1}}. \quad (67)$$

Second, replace every occurrence of $\lambda$ in eqn.(67) by $\alpha_{\mu}$. Then, under this substitution, $w_{\mu_1 \cdots \mu_{n-1}}$ reduces to the form $w_{\mu_1 \cdots \mu_{n-1}} = c D_a \alpha_{\mu} D^a \psi_{\mu_1 \cdots \mu_{n-1}} + c' \alpha_{\mu} \psi_{\mu_1 \cdots \mu_{n-1}}$, for some constants $c$, $c'$. Since $\alpha_{\mu}$ satisfies eqn.(60), which is the only property of $\lambda$ that may be used in establishing (67), it follows that eqn.(67) must continue to hold after replacing $\lambda$ therein by $\alpha_{\mu}$. However, under this substitution, eqn.(67) becomes

$$2[(n+1)c + c'] [D_a \alpha_{\mu} D^a \psi_{\mu_1 \cdots \mu_{n-1}} + (n-1) \alpha_{\mu} \psi_{\mu_1 \cdots \mu_{n-1}}] = 4n^2 (n-1) \alpha_{\mu} \psi_{\mu_1 \cdots \mu_{n-1}}, \quad (68)$$

which can never hold. \Box

We turn next to Maxwell fields. We have

**Theorem 6** Let $B_0 = 0$, and let the the stress-energy tensor $T_{ab}$ satisfy

$$T_a^0 = 0, \quad T_{ab} E^b - T E_a = 0. \quad (69)$$

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Then the conserved quantity $\mathcal{E}_{\mu\nu}$ (of eqn.(36)) for the Maxwell field in flat space-time admits a generalization to a conserved quantity in curved space-time, namely that given by

$$\mathcal{E}_{\mu\nu} = \int s^{ab}_{\alpha[\mu} D_{b]\alpha_{\nu]} dS_a,$$

(70)

where we have set

$$s_{ab} = 2D_{(a} \hat{\mathcal{E}}_{b)} - 2D^c \hat{\mathcal{E}}_{c[ab} + 16\hat{\mathcal{E}}_{(a} D_{b)} \lambda - 8\hat{\mathcal{E}}_{c[} D_{c]} \lambda_{ab} + 16\psi \lambda D_a D_b \lambda + 8\lambda E_{(a} D_{b)} \lambda + 8\lambda^2 D_a E_b$$

$$+ [12\psi \lambda^2 - 20\lambda^2 E^c D_c \lambda - 4\psi (D^1 \lambda)^0 q_{ab} + 4\psi w_{ab} + 4\psi T_{ab} - 4\psi T q_{ab}$$

$$- 11(\psi D_a D_b \lambda + \lambda D_a D_b \psi - 2D_{(a} \psi D_{b)} \lambda) - \psi (11D^2 \lambda + 22\lambda) q_{ab},$$

(71)

where $\psi$ is so chosen to satisfy $D_a \psi = E_a.$\textsuperscript{27}

The integrand reduces, in flat space-time, to the integrand of $\mathcal{E}_{\mu\nu}$ therein plus a divergence, namely $D_b (\hat{\mathcal{E}}^{[a}_{\mu} \hat{\mathcal{E}}^{\nu]}_{\alpha}),$ of an antisymmetric tensor field. The demonstration of $D_b s_{ab} = 0$ is given in Appendix B. The above conditions, (69), on $T_{ab}$ are satisfied when the space-time is vacuum, and also when the source is the Maxwell field itself. But the condition (69) need not be satisfied in the presence of other matter sources. It is readily verified that this generalized $\mathcal{E}_{\mu\nu}$ is again gauge invariant.

For higher-order Maxwell remnant radiation multipoles, we have

**Theorem 7** Fix $n \geq 3$. Then the conserved quantity $\mathcal{E}_{\mu_1 \cdots \mu_{n-1}}$ for the Maxwell field in flat space-time, given by eqn.(36), does not admit generalization to curved space-time.

The proof of theorem (7) is similar to that of theorem (5) and is therefore omitted.

V. CONCLUSION

We have constructed, for each of a Klein-Gordon field, a Maxwell field, and a linearized gravitational field in Minkowski space-time, a hierarchy of conserved quantities which we call the remnant radiation multipoles. In the cases of Klein-Gordon and Maxwell, we have generalized the remnant radiation monopoles to curved space-time. There follows a discussion of some outstanding issues.
Does the remnant gravitational monopole admit generalization to curved space-time? We conjecture that the answer is yes. In Appendix C we give the remnant field equations necessary for addressing this question. We also there display a candidate for a curved-space gravitational remnant monopole. This candidate has the attractive feature that its divergence, which could in principle have contained remnants or order as high as 3, contains only remnants of order $\leq 2$. Although the existence of this candidate lends some support to the conjecture, it is of course far from a proof of it. Work is in progress to settle this conjecture. We further conjecture that none of the higher-order remnant radiation multipoles for linearized gravitational fields admit generalization to curved space-time.

The way we introduced the Klein-Gordon and Maxwell remnant radiation monopoles in a curved space-time involves a quite strong fall-off condition, namely $B_{ab} = 0$, on the gravitational remnants. In the Klein-Gordon case, this restriction is in fact unnecessary. Indeed, in the absence of this condition, the first-order remnant field equation on $\phi$ becomes $D_a v^a = 0$, with

$$v^a = D^a \phi - q^{ab} D_b \phi + \frac{1}{2} q D^a \phi + \lambda D^a \phi.$$  

Thus, $K \equiv \int_C v^a dS_a$ remain conserved in asymptotic conditions weaker than the ones presently imposed. Can other conserved quantities be defined with such weaker asymptotic conditions?

Do there exist conserved quantities analogous to our remnant radiation multipoles, but defined at null rather than spatial infinity? And if so, are there any simple relations between the values of corresponding quantities at spatial and null infinity? In Minkowski space-time, it should not be too difficult to answer these two questions. A relevant observation is that, in the case of Minkowski space-time corresponding conserved quantities in general take different values at spatial and null infinity. This result suggests that “remnant radiation” is capable of escaping between spatial infinity and null infinity. Recall that Newman, Penrose and Exton have introduced certain conserved quantities at null infinity in curved space-time. Are these quantities analogs, in any sense, of the remnant radiation monopoles?

It is unfortunate that the present treatment of asymptotic quantities involves such complicated algebra. It is not entirely clear whether these complications are inherent in the subject
itself, or merely a reflection of the present techniques. One case in which we know that these techniques are the culprit is that of stationary space-times. It is not hard to convince oneself that the present framework, in the case of stationary space-times, is essentially equivalent to the usual formalism involving a 3-dimensional manifold of trajectories. Since the stationary gravitational multipole moments of all order can be defined within this 3-dimensional formalism, it should also be possible, in principle, to define these very same moments within the present framework. However, it already seems difficult to define even the first few stationary multipoles in the present framework. Unlike the 3-dimensional formalism, the present framework is not well adapted to the presence of Killing fields. For example, to treat Killing’s equation order by order yields a complicated set of remnant equations. Finding a more natural way of dealing with stationary space-times within the present framework may give some clue as to how to tame its algebraic complexity. Indeed, it may further lead to a generalization of the stationary multipoles to more general asymptotically flat space-times.

We have here restricted our consideration to conserved quantities that are linear in (the highest-order part of the remnants of) the physical fields. More generally, one might allow polynomial dependence on the remnants. A candidate for a conserved quantity quadratic in the remnants has been given by R. Beig$^9$: \[ \int \epsilon^{kl(a}D_k \lambda E_{l)b)}\alpha_{[\mu}D_{\nu]}\alpha_{\nu]}dS_a, \] where \( \xi^a \) is any Killing field in \( \mathcal{H} \). However, as shown in Ref. 8, this quantity vanishes identically by virtue of the second-order gravitational remnant equations. It would be of interest to carry out a systematic search for polynomial conserved quantities. One might even search for conserved quantities with non-polynomial dependence on the remnant fields, but the fact that these remnant fields have complicated gauge behavior rather suggests that no such quantities will exist.

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Appendix A: Stationary Fields in Minkowski Space-time

Consider in Minkowski space-time, a physical field that is static, i.e., that it is invariant under a time-translation in the space-time. In this appendix, we do two things: Express within the present framework the ordinary static multipole moments of such a field; and show that, in this static case, all the remnant radiation multipoles vanish. We will only discuss linearized gravity here since the treatment of Klein-Gordon and Maxwell fields is similar and simpler.

Let \( \tilde{K}_{abcd} \) be a linearized gravitational field in Minkowski space-time \( \tilde{M} \), asymptotically regular of order 3. Further, let \( \tilde{K}_{abcd} \) be static, i.e., let

\[
\mathcal{L}_t \tilde{K}_{abcd} = 0,
\]

(A1)

where \( \tilde{t}^a \) is a (unit) time-like Killing field in \( \tilde{M} \). Denote by \( \zeta = \Omega - 2 L_{\tilde{t}} \Omega \) the corresponding unit time-translation on \( H \). Taking the remnants of eqn.(A1), we obtain the following equations on the remnant fields

\[
\mathcal{L}_{D\zeta} \tilde{E}_{ab} - (n + 1) \zeta \tilde{E}_{ab} + 2 \epsilon_{(a} \tilde{B}_{b)l} D_m \zeta = 0,
\]

(A2)

for \( n = 0, 1, 2, \ldots \). Set \( \tilde{n} \phi_E = \tilde{E}_{ab} D^a \zeta D^b \zeta \). Then this \( \tilde{n} \phi_E \) satisfies eqn.(19), and, from eqn.(A2), also

\[
\mathcal{L}_{D\zeta} \tilde{n} \phi_E - (n + 1) \zeta \tilde{n} \phi_E = 0.
\]

(A3)

Under a gauge transformation \( \tilde{n} \phi_E \) changes according to eqn.(29). The same equations hold, similarly, on \( \tilde{n} \phi_B \equiv \tilde{B}_{ab} D^a \zeta D^b \zeta, n = 0, 1, \ldots \). We note that the \( \tilde{E}_{ab} \) and \( \tilde{B}_{ab} \) for this static linearized gravitational field can be expressed in terms of \( \tilde{n} \phi_E \) and \( \tilde{n} \phi_B \). Indeed, we have, from eqn.(A2) and eqn.(27), that

\[
\tilde{E}_{ab} = \frac{2 \zeta^2 + 1}{(n + 1)(n + 2)} D_a D_b \phi_E + \frac{3 D_{(a} \zeta D_{b)} D_m \phi_E D^m \zeta}{2(n + 1)(n + 2)} - \frac{D^k D^l \phi_E D_k \zeta D_l \zeta \phi_E}{2(n + 1)(n + 2)}
\]

\[
+ \frac{5 \zeta D_{(a} \zeta D_{b)} \phi_E}{2(n + 1)} + \frac{4n + 5}{2(n + 2)} \phi_E D_a \zeta D_b \zeta + \frac{(n - 3) \zeta^2 + 2n + 1}{2(n + 2)} \phi_E D_l \zeta
\]

\[
+ \frac{2 \zeta \epsilon_{(a} D_{b)} D_k \phi_B D_l \zeta}{(n + 1)(n + 2)} + \frac{2 \epsilon_{(a} D_{b)} \zeta D_k \phi_B D_l \zeta}{n + 1},
\]

(A4)
and similarly for $B_{ab}$.

Now consider, for $n = 0, 1, 2, \ldots$

$$M_{\mu_1 \cdots \mu_n}[\phi_E] \equiv \frac{(2n + 1)!}{2^{n+1}(n!)^2} \int_C \left[ \phi_E (\alpha_{\mu_1} + \langle \alpha_{\mu_1}, \zeta \rangle \zeta) \cdots (\alpha_{\mu_n} + \langle \alpha_{\mu_n}, \zeta \rangle \zeta)(1 + \zeta^2)^{-1} D^a \zeta \right] dS_a. \tag{A5}$$

The integrand on the right above is divergence-free, by eqn.(A3), and so eqn.(A5) defines, for each $n$, a conserved quantity, $M_{\mu_1 \cdots \mu_n}[\phi_E]$. These are precisely the ordinary static electric multipole moments of the linearized gravitational field. They are totally symmetric, and satisfy

$$0 = \zeta^{\mu_1} M_{\mu_1 \mu_2 \mu_3 \cdots \mu_n}, \tag{A6}$$

$$0 = \eta^{\mu_1 \mu_2} M_{\mu_1 \mu_2 \mu_3 \cdots \mu_n}, \tag{A7}$$

$$\nabla_\mu M_{\mu_1 \cdots \mu_n} = -(2n - 1) h_{\mu(\mu_1} M_{\mu_2 \cdots \mu_n)} + (n - 1) h_{(\mu_1 \mu_2} M_{\mu_3 \cdots \mu_n)\mu}, \tag{A8}$$

where we have set $h_{\mu\nu} = \eta_{\mu\nu} + \zeta_\mu \zeta_\nu$. To prove eqn.(A7), use that $\phi_E$ satisfies eqn.(19); To prove eqn.(A8), use the gauge behavior, (29), of $\phi_E$. Similarly, we obtain the magnetic multipole moments, $M_{\mu_1 \cdots \mu_n}[\phi_B]$. These two sets of moments are the linearized versions of Hansen’s mass and angular momentum multipole moments, respectively.

Finally, we show that all of the gravitational remnant radiation multipoles (the $G$’s introduced in eqn.(41), Section III), vanish for a static linearized gravitational field in Minkowski space-time. To see this, substitute eqn.(A4) into the integrand of $G_{\mu_\nu \mu_1 \cdots \mu_{n-1}}$, to obtain

$$G_{\mu_\nu \mu_1 \cdots \mu_{n-1}} = K [c_1 \eta_{\mu_\nu} \eta_{\mu_1} \phi_E + c_2 \eta_{\mu_\nu} \phi_E + c_3 \eta_{(\mu} \mathcal{L}_{\xi_{\nu}^a)} \phi_E + c_4 \mathcal{L}_{\xi_{(\mu}^a} \mathcal{L}_{\xi_{\nu)}^a} \phi_E + c_5 \eta_{(\mu} \mathcal{L}_{\xi_{\nu}^a)} \phi_E + c_6 \mathcal{L}_{\xi_{(\mu}^a} \mathcal{L}_{\xi_{\nu)}^a} \phi_E + c_7 \mathcal{L}_{\xi_{(\mu}^a} \mathcal{L}_{\xi_{\nu)}^a} \phi_E]_{\mu_1 \cdots \mu_{n-1}}, \tag{A9}$$

where $c_1, \ldots, c_7$ are certain constants, and $\xi_{\mu}^a$ and $\xi_{\mu}^{*a}$ are the Killing fields given by $\xi_{\mu}^a = \zeta D^a \alpha_\mu - \alpha_\mu D^a \zeta$ and $\xi_{\mu}^{*a} = \epsilon^{abc} D_b \zeta D_c \alpha_\mu$. Let $C$ denote the $\zeta = 0$ 2-sphere section of $H$. We show that each term on the right in eqn.(A9) contributes zero by evaluating the integral over $C$. The first four terms contribute zero by virtue of the fact that each of the term satisfies the same equations as a static $n$-th order Klein-Gordon remnant field $\phi$, and that, for any such $\phi$,

$$\int_C \phi \alpha_{\mu_1} \cdots \alpha_{\mu_k} D^a \zeta dS_a = 0,$$

for $0 \leq k < n$. The fifth and sixth terms contribute zero because
for any \( \phi \) as above, \( \mathcal{L}_{\xi^\mu} \phi \) vanishes on \( C \), and \( \mathcal{L}_{D\xi}(\mathcal{L}_{\xi^\mu} \phi) \) is a sum of two terms, one of which (namely \((n+1)\xi^\mu \phi\)) vanishes on \( C \) and the other (namely \(-\phi^\mu = -D_a \alpha^\mu D_a \phi + (n+1) \alpha^\mu \phi\)) satisfies the \((n+1)\)th remnant field equation and is static. Finally the last term contributes zero because \( \mathcal{L}_{\xi^\mu} \mathcal{L}_{\xi^\nu} \phi_E \) is a translation times a term which satisfies the \((n+1)\)th remnant field equation and is static, and because \( \mathcal{L}_{D\xi}(\mathcal{L}_{\xi^\mu} \mathcal{L}_{\xi^\nu} \phi_E) \) is a sum of two terms, one of which (namely \((n+1)\xi^\mu \mathcal{L}_{\xi^\nu} \phi_E - \mathcal{L}_{\xi^\mu} (\phi_E)^\nu\)) vanishes on \( C \) and the other (namely \(-[\mathcal{L}_{\xi^\nu} \phi_E]^\nu\)) satisfies the \((n+1)\)th remnant field equation and is equal to a static field \( \phi^{n+1} \) on \( C \). \( \square \)

Appendix B: Miscellaneous Results

Appendix B.1 contains the proofs of item (i) of each of theorems 1–3. Appendix B.2 contains the proofs of item (iii) of each of theorems 1–2. Appendix B.3 completes the proof that the \( \mathcal{E}_{\mu \nu} \) we introduced in Theorem 3 is indeed conserved.

B.1 The Remnant Radiation Multipoles Exhaust the Conserved Quantities in Minkowski Space-time

We first show that, in the Klein-Gordon case, the \( \mathcal{K}'s \) of eqn. exhaust all conserved quantities in Minkowski space-time linear in \( \phi \) and multi-linear in \( \mathcal{T} \). Sketch of proof: Let \( v^a_{\Gamma} \) be a divergence-free vector field on \( \mathcal{H} \), constructed linearly in the \( \phi \), and multi-linearly in \( \mathcal{T} \). (We introduce the subscript \( \Gamma \) to stand for any Greek indices that may be attached to \( v^a \).) Since the various \( \phi \) are uncoupled in eqn. we may take \( v^a_{\Gamma} \) to depend on just a single remnant field, say \( \phi \). Then \( v^a_{\Gamma} \) takes the form

\[
v^a_{\Gamma} = \sum_{k=0}^{s} w^{\alpha a_1 \cdots a_s}_{\Gamma k} D_{a_1} \cdots D_{a_k} \phi, \tag{B1}\]

where \( s \) is the order of the highest derivative in \( v^a_{\Gamma} \). We may assume \( w^{\alpha a_1 \cdots a_s}_{\Gamma} = w^{(a,a_1 \cdots a_s)}_{\Gamma} \), since any parts of \( w^{\alpha a_1 \cdots a_s}_{\Gamma} \) antisymmetric between “\( a \)” and another index can be eliminated by adding to \( v^a_{\Gamma} \) the divergence of an antisymmetric second-rank tensor field, and any parts antisymmetric between two indices neither “\( a \)” can be eliminated using the definition of the Riemann tensor. It now follows, from \( D_a v^a_{\Gamma} = 0 \), that \( w^{\alpha a_1 \cdots a_s}_{\Gamma} = q^{(a_1 a_2 \cdots a_s)}_{( \Gamma a)}, \) for some tensor
field $u^{a_2 \cdots a_s}$). Were $s \geq 2$, then this term could now be eliminated in its entirety by adding to $v^a_{\Gamma}$ a divergence, namely $D_a [2(D^{[a}D_{a_3} \cdots D_{a_s}]_{a}]$, of an antisymmetric tensor. So we may assume $s = 1$ in eqn. (B1), i.e., we may set $v^a = w_\Gamma D^a \phi - \phi w^a_{\Gamma}$. It now follows, again from $D_a v^a_{\Gamma} = 0$, that $w^a_{\Gamma} = D^a w_\Gamma$, where $w_\Gamma$ is some solution of eqn. (19). But this $w_\Gamma$ must be multi-linear in $T_a$, and the only such solution of eqn. (19) is $C(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}})$.

We next show that, in the Maxwell case, the $E$'s of eqn. (37) together with the electric and magnetic charges exhaust all conserved quantities in Minkowski space-time linear in the remnants of the Maxwell field and multi-linear in $T$. Sketch of proof: Let $v^a_{\Gamma}$ be a divergence-free vector field on $\mathcal{H}$, constructed linearly in the $E_a$ and multi-linearly in $T$. As before, we may take $v^a_{\Gamma}$ to depend on a single remnant field, $E_a$. If $n = 0$, the result, that $v^a = 0$, follows by setting $E_a = D_a \psi$ with $D^2 \psi = 0$, and using the Klein-Gordon result. So, let $n \geq 1$. Then $v^a_{\Gamma}$ takes the form

$$v^a_{\Gamma} = \sum_{k=0}^{s} w^{a_1 \cdots a_k+1}_{\Gamma} D_{a_1} \cdots D_{a_k} E_{a_k+1}, \quad n \geq 1,$$

where $s$ is the order of the highest derivative in $v^a_{\Gamma}$. An argument similar to the Klein-Gordon case shows that $v^a_{\Gamma}$ can be brought to the form

$$v^a_{\Gamma} = w_\Gamma D^a E^b - \frac{n}{n^2} D^a w^b_{\Gamma} + \mu T^a_{\Gamma},$$

where $(D^2 + n^2 - 2)w_{a\Gamma} = D_a \mu_\Gamma$. We may achieve $\mu = 0$ in (B2) by adding to $v^a_{\Gamma}$ a divergence of an antisymmetric tensor field, namely $D_b (2E^{[a}D^{b]}w_{\Gamma} + 2w_\Gamma D^{[a} E^{b]} + \frac{2}{n^2} c_\Gamma D^{[a} E^{b]} D^{c]} E^{d])$ where $c_\Gamma$ is a certain constant and where we have set $w_\Gamma = \frac{1}{n^2} (-\mu + D_a w^a_{\Gamma})$. Now $w^a_{\Gamma}$ satisfies precisely the same equations as $E^a_{\Gamma}$. The conserved quantity thus arises from the “symplectic product” between $E^a_{\Gamma}$ and $w^a_{\Gamma}$. But this $w^a_{\Gamma}$ must be multi-linear in $T$, and the only such solution of eq. (23) is $w^a_{\mu \mu_1 \cdots \mu_{n-1}} = C(\alpha_{\mu} \cdots \alpha_{\mu_{n-1}}) D^a \alpha_{\mu} + \frac{1}{n^2} D^a (D^\beta \alpha_{\mu} D_b C(\alpha_{\mu} \cdots \alpha_{\mu_{n-1}}) - \alpha_{\mu} C(\alpha_{\mu} \cdots \alpha_{\mu_{n-1}})).$

Finally, we show that, in the case of linearized gravity, the $G$'s (of eqn. (41)) together with the mass-momentum and angular momentum exhaust all conserved quantities in Minkowski space-time linear in remnants of the linearized gravitational field and multi-linear in $T$. Sketch of proof: Let $v^a_{\Gamma}$ be a divergence-free vector field on $\mathcal{H}$, constructed linearly in the $E_{ab}$ and
multi-linearly in $\mathcal{T}$. As before, we may take $v^a_{\Gamma}$ to depend on a single remnant field, $\vec{E}_{ab}$. If $n = 0$, the result, that $v^a_{\mu} = E^a_{\mu b} \alpha_{\mu}$, follows by setting $E_{ab} = D_a D_b \psi + \psi q_{ab}$ with $D^2 \psi = -3 \psi$, and using the Klein-Gordon result. If $n = 1$, the result, that $v^a_{\mu \nu} = E^a_{\mu \nu} D_b \alpha_{\nu}$, follows by setting $E_{ab} = (\alpha u_b)$ with $D^2 u_a = -2 u_a$, $D_a u^a = 0$, and using the Maxwell result. So, let $n \geq 2$. Then $v^a_{\Gamma}$ takes the form

$$v^a_{\Gamma} = \sum_{k=0}^{s} w^{aa_1 \cdots a_{k+2}}_{a_1 \cdots a_{k+2}} D_{a_1} \cdots D_{a_k} \vec{E}_{a_{k+1} a_{k+2}}, \quad n \geq 2,$$

where $s$ is the order of the highest derivative in $v^a_{\Gamma}$. An argument similar to the Klein-Gordon case shows that $v^a_{\Gamma}$ can be brought to the form

$$v^a_{\Gamma} = w_{bc} D^a E^b_{\Gamma} - E^b_{\Gamma} w^{bc} D^a_{\Gamma} - E^a_{\Gamma} u_b,$$

where $w_{abc}$ is symmetric and trace-free, and satisfies $(D^2 + n^2 - 3)w_{ab} = D_{(a} u_{b)} - \frac{1}{2} q_{ab} D_m u_m{\Gamma}$. We may achieve $u_{a\Gamma} = 0$ in (B5) by adding to $v^a_{\Gamma}$ a divergence of an antisymmetric tensor field.

The result now follows from an argument similar to the Maxwell case.

### B.2 Gauge Behavior of Remnant Radiation Multipoles

We here prove that the gauge behavior of the Klein-Gordon and Maxwell remnant radiation multipoles is that given respectively by eqn.s (B5) and (B8).

For the Klein-Gordon case, denote by $k^a_{\mu_1 \cdots \mu_{n-1}}$ the integrand of eqn.(34). Then we have

$$\nabla_\mu k^a_{\mu_1 \cdots \mu_{n-1}} = n D^n (L_{D \alpha_\mu} \phi - \alpha_\mu \phi) C(\alpha_{\mu_1}, \cdots, \alpha_{\mu_{n-1}}) - n (L_{D \alpha_\mu} \phi - \alpha_\mu \phi) D^n C(\alpha_{\mu_1}, \cdots, \alpha_{\mu_{n-1}}),$$

$$= \frac{1}{2} n (n - 2) \eta_{\mu_1 \mu_2} k^a_{\mu_3 \cdots \mu_{n-1} \mu} - n - 1 \eta_{\mu_1 \mu_2} k^a_{\mu_3 \cdots \mu_{n-1}}$$

$$+ D_m \left[ 2 n D^n \phi D^m \alpha_\mu C(\alpha_{\mu_1}, \cdots, \alpha_{\mu_{n-1}}) + 2 n \phi D^m \alpha_\mu C(\alpha_{\mu_1}, \cdots, \alpha_{\mu_{n-1}}) \right].$$

where we used, in the first step, eqn.s (34) and (29), and, in the second, the identity

$$D^m \alpha_\mu D_m C(\alpha_{\mu_1}, \cdots, \alpha_{\mu_{n-1}}) = -(n - 1) \alpha_\mu C(\alpha_{\mu_1}, \cdots, \alpha_{\mu_{n-1}}) + (n - 1) \eta_{\mu_1 \mu_2} C(\alpha_{\mu_2}, \cdots, \alpha_{\mu_{n-1}})$$

$$- \eta_{\mu_1 \mu_2} C(\alpha_{\mu_3}, \cdots, \alpha_{\mu_{n-1}} \alpha_\mu),$$

(B7)
Integrate over a cut of $\mathcal{H}$.

For the Maxwell case, denote by $(e^E)^a_{\mu_1 \cdots \mu_n}^a$ the integrand of eqn.\((34)\), and set $\xi_{\mu \nu} = 2\alpha_{\mu}D^a\alpha_{\nu}$. Then we have

$$
\nabla^a[(e^E)^a_{\mu_1 \cdots \mu_n}^a] = nD^a \left[ (\mathcal{L}_{\alpha} n_{\mu_1} n_{\mu_2} \cdots n_{\mu_n} - n_{\alpha_1} n_{\mu_2} \cdots n_{\mu_n} + \epsilon_{mkl} B^k D^\lambda (\alpha_{\mu_1} D^m \alpha_{\mu_}\nu) \right] C(\alpha_{\mu_1} \cdots \alpha_{\mu_n})
$$

$$
- n \left[ (\mathcal{L}_{\alpha} n_{\mu_1} n_{\mu_2} \cdots n_{\mu_n} - n_{\alpha_1} n_{\mu_2} \cdots n_{\mu_n} + \epsilon_{mkl} B^k D^\lambda (\alpha_{\mu_1} D^m \alpha_{\mu_}\nu) \right] D^a C(\alpha_{\mu_1} \cdots \alpha_{\mu_n})
$$

$$
= \nabla^a [(\mathcal{L}_{\alpha} n_{\mu_1} n_{\mu_2} \cdots n_{\mu_n} - n_{\alpha_1} n_{\mu_2} \cdots n_{\mu_n} + \epsilon_{mkl} B^k D^\lambda (\alpha_{\mu_1} D^m \alpha_{\mu_}\nu) \right] C(\alpha_{\mu_1} \cdots \alpha_{\mu_n})
$$

where we used, in the first step, eqn.s \((30)\) and \((36)\), and, in the second, the identity eqn.\((B7)\) again. Integrate over a cut of $\mathcal{H}$.

\subsection*{B.3 Completion of Proof of Theorem \ref{thm6}}

In our proof of Theorem \ref{thm6}, we omitted one step: The demonstration the the $s_{ab}$ of eqn.\((71)\) is indeed divergence-free. We here supply that step. We have

$$
D^b s_{ab} = -2\epsilon_{abc} D^b (w^{cd} B_d) + 16\epsilon_{abc} D^b \lambda B^c - 8\psi^0 T_a + 4(T_{ab} E^b - T E_a)
$$

$$
= 0,
$$

where, in the first step, we used the following six equations

$$
D^b (2D_{(a} \varepsilon_{b)} - 2D^c \varepsilon_{c d} g_{a b}) = 16(D_{(a} \varepsilon_{b)} D^b \lambda + 24\lambda \varepsilon_a - 16\lambda E_a - 4w_{ab} E^b - 2\epsilon_{abc} D^b (w^{cd} B_d)
$$

$$
+ 16\epsilon_{abc} D^b \lambda B^c, \quad \text{(B10)}
$$

$$
D^b (16\varepsilon_{a (D_b \lambda - 8\varepsilon^c D_c \lambda g_{a b})} = -16(D_{(a} \varepsilon_{b)} D^b \lambda - 24\lambda \varepsilon_a + 16(\varepsilon^c D_c \lambda) D_{a \lambda}, \quad \text{(B11)}
$$

$$
D^b (4\psi w_{a b}) = 4w_{ab} E^b - 8\psi^1 \lambda D_a \lambda - 8\psi D_a D_b \lambda + 22\psi D_a (D^2 \lambda) + 8D_a (2D_b T^b + 4T) - 8\psi T_a, \quad \text{(B12)}
$$

\[32\]
\[
D^b[16\psi\lambda D_a D_b \lambda + 8\lambda E_{(a} D_{b)} \lambda + 8\lambda^2 D_a E_b + (12\psi \lambda^2 - 20\lambda E_c D_c \lambda - 4\psi (D \lambda)^2)q_{ab} \\
= 16\lambda E_a - 16(E^b D_b \lambda) D_a \lambda + 8\psi \xi D_a \lambda + 8\psi D_a D_b \lambda D^b \lambda.
\] (B13)

\[
D^b[-11(\psi D_a D_b \lambda + \lambda D_a D_b \psi - 2D_{(a} \psi D_{b)} \lambda) + \psi(-11D^2 \lambda + 22\lambda)q_{ab},] = -22\psi D_a (D^2 \lambda),
\] (B14)

\[
D^b(4\psi T_{ab} - 4\psi T q_{ab}) = 4(T_{ab} E_b - T E_a) + 4\psi (D^b T_{ab} - D_a T),
\] (B15)

(themselves consequences of the remnant field equations (50)–(56), (60)–(63), (C12), (C13)),
and, in the second step, \(\rho_\text{a} = 0\) and eqn. (69) of the theorem.

Appendix C: Gravitational Remnant Equations

In Appendix C, we discuss the issue of generalizing the remnant radiation multipoles from linearized to full gravitation. In C.2, we outline the derivation of gravitational remnant field equations. In C.3 we present an alternative version of the second-order gravitational remnant field equations, involving the remnants of the Weyl tensor.

C.1 Generalization of Gravitational Remnant Radiation Monopole

In Section IV, we generalized the flat-space Klein-Gordon and Maxwell remnant radiation multipoles to curved space-time. However, we have been unable to determine whether there exists a similar generalization for linearized gravity. Here is how far we have gotten.

The remnant equations for gravitation were given, up to second order, in (60)–(63). We shall need the next two orders. The third-order remnant equations are

\[
D^3 q = \frac{2}{3}q_{ab} D_a D_b \lambda + 2D^a D^b \lambda D_a \lambda D_b \lambda + 12\lambda(D \lambda)^2 - 24\lambda^3,
\] (C1)

\[
D^b D^3 q_{ab} = D_a q - 3q_{ab} D^1 + 4(D \lambda)^2 D_a \lambda + 64\lambda^2 D_a \lambda,
\] (C2)

\[
D^2 D^3 q_{ab} = -3q_{ab} + D_a D_b \lambda - 12D \lambda \cdot D^2 q_{ab} + 24\lambda q_{ab} + 16D_{(a} q_{b)m} D^m \lambda \\
+ 16D_{(a} \lambda D_{b)} \lambda D^c \lambda - 4(D \lambda)^2 D_a D_b \lambda + 160\lambda D_a \lambda D_b \lambda - 52\lambda^2 D_a D_b \lambda \\
- [5q_{ab} - 12\lambda(D \lambda)^2 + 372\lambda^3] q_{ab}.
\] (C3)
The fourth-order remnant equations are

\[
\frac{4}{3} \dot{q} = \frac{2}{3} q^{ab} D_{(a} D_{b)} \frac{1}{\lambda} + \frac{1}{3} D^{\lambda} \cdot D \ddot{q} + \frac{10}{3} q^{ab} D_a \frac{1}{\lambda} D_b \frac{1}{\lambda} + 5 \lambda q^{ab} D_a D_b \frac{1}{\lambda} + 2 q^{ab} q^{cd} \frac{1}{\lambda},
\]

\[D^{(b} q_{ab} = D_a \frac{1}{4} q + 4 q_{ab} D^b \frac{1}{\lambda} + 2 q D_a \frac{1}{\lambda} + 3 D^b (q_{ac} q^{cb}) - \frac{9}{4} D_a (q_{bc} q^{ac})
- \frac{3}{2} q_{ab} D^b \ddot{q} - 10 \lambda q_{ab} D^b \frac{1}{\lambda} + 76 \lambda (D^a \frac{1}{\lambda}) D_a \frac{2}{\lambda} + 832 \lambda^3 D_a \frac{1}{\lambda},
\]

\[D^{a} q_{ab} = -6 q_{ab} + 2 D_{(a} D^{m} q_{b)m} + 2 q D_{(a} D^{m} q_{b)m} + 16 D_{(a} D^{m} q_{b)m} + 16 D_{(a} q_{b)m} D^{m} \frac{1}{\lambda} + 72 \lambda q_{ab}
- 20 \lambda q_{ab} - 8 \lambda D^2 q_{ab} - 16 D \frac{1}{\lambda} \cdot D_{(a} D_{b)} \frac{1}{\lambda} + 96 \lambda D_{(a} D^{m} q_{b)m} + 96 \lambda q_{ab} - 96 \lambda^2 q_{ab}
+ 120 \lambda D_{(a} q_{b)m} D^{m} \frac{1}{\lambda} - 48 \lambda^2 D^2 q_{ab} - 156 \lambda D_{(a} \frac{1}{\lambda} \cdot D_{b)} \frac{1}{\lambda} + 16 D^c (a \lambda q_{bc})
+ 16 D_c q_{(c} D_{d)} q_{ab} - 8 q_{cd} D_c D_d \frac{1}{\lambda} q_{ab} - 8 D^c q_{cd} D^d \frac{1}{\lambda} q_{ab} - 12 D^c q_{cd} D_{(a} q_{b)d}
- 12 q^c_{ad} D_{(a} q_{b)d} + 6 D^c q_{cd} D_{d} q_{ab} + 2 q^c_{cd} D_{d} q_{ab} + 192 \lambda D_{(a} \lambda D^2 q\b_{c})
+ 192 \lambda D_{(a} \lambda q_{b)c} + 192 \lambda D_{(a} q_{b)c} D C \frac{1}{\lambda} - 96 \lambda D_{(a} \lambda q_{cd} D_{d} q_{ab} - 96 \lambda^2 q_{cd} D_{d} q_{ab}
- 96 q^c_{ad} D_{(a} \lambda D_{b\lambda} + 432 \lambda^3 D_{a} \lambda D_{b\lambda} + 1440 \lambda^2 D_{a} \lambda D_{b\lambda} + 1488 \lambda^2 (D \frac{1}{\lambda})^2 q_{ab}
+ 1296 \lambda^4 q_{ab} + 48 \lambda D_{a} \lambda D_{b} \frac{20}{\lambda} q_{ab} - 3 D \frac{2}{\lambda} \cdot D_{ab} + 6 D_{(a} q_{b)c} D^c q_{ab}
+ 4 D \lambda \cdot D q q_{ab} + 48 (D \frac{1}{\lambda})^2 (D \frac{1}{\lambda})^2 q_{ab} + 24 q_{ac} \frac{2}{\lambda} q_{b} - 12 q^c_{cd} q^c_{cd} q_{ab}
- 48 (D \frac{1}{\lambda})^2 q_{ab} - 6 D^c q_{(a} D^{c} q_{b)c} - 3 D_{ac} q_{cd} D_{b} q_{d} - 6 D_{ac} q_{da} D_{b} q_{d} - D_{a} q_{d} q_{b}
- 8 D_{a} D_{b} \frac{1}{\lambda} + 3 D_{a} D_{b} \frac{2}{\lambda} q_{cd} - 48 D_{a} D_{b} (\frac{1}{\lambda} \frac{2}{\lambda}).
\]

We begin by noting that, introducing a potential \( h_{ab} \) for the linearized gravitational field, the integrand of \( G_{\mu\nu\lambda\sigma} \) in flat space-time in eqn. (11) is a multiple of \( D(a h_{bc}) \xi^b \), where we have set \( \xi^{a}_{\mu\nu} = \alpha_{\mu} D^{a} \alpha_{\nu} \). Note also that \( D(a h_{bc}) \) is divergence-free, by virtue of the remnant field equations on \( h_{ab} \). Thus, the problem of generalizing to curved space-time the \( G_{\mu\nu\lambda\sigma} \) of flat space-time is equivalent to that of finding a third-rank, totally symmetric, divergence-free tensor field \( s_{abc} \) on \( \mathcal{H} \), constructed from the gravitational remnants, such that \( s_{abc} \) reduces, in the case of linearized gravity, to \( D(a h_{bc}) \). Consider, in this connection, the candidate \( \hat{s}_{abc} \) given...
by

\[
\hat{s}_{abc} = D_{(a} q_{bc)}
+ \left( \frac{82}{3} + 4c \right) \lambda_{(a} q_{bc)}^{\lambda_{(d} q_{c)} d} \lambda_{d}^d + \left( \frac{20}{3} + 2c \right) \lambda D_{(a} q_{bc)}^{\lambda_{(d} q_{c)} d}
- \left( \frac{10}{3} + c \right) \lambda^{d} q_{(a} \lambda_{c)} d + \left( \frac{2}{3} + \frac{c_0}{2} \right) q_{(a} \lambda_{c)} d e q_{d e} + \frac{4}{3} \lambda D_{a} q_{(c)} d
- \left( \frac{10}{3} + \frac{c_0}{2} \right) q_{(a} q_{c)}^d \lambda^{d} q_{d e} + \frac{8}{3} q_{(a} q_{c)}^d \lambda_{d e} + (2 + c) \lambda D_{a} q_{(c)} d
- \left( \frac{8}{3} + c \right) \lambda^{d} q_{d a} q_{(c)} d + c \lambda_{d} q_{(c)} d e q_{d e},
\]

(C7)

where \( c \) is any constant, and where we have set \( \lambda_a \equiv D_a \lambda \), \( \lambda_{ab} \equiv D_a D_b \lambda + \lambda q_{ab} \) and \( \lambda_{abc} \equiv D_a \lambda_{bc} \).

This \( \hat{s}_{abc} \) has all the required properties, except that its divergence, instead of vanishing, includes remnants of order not exceeding 2. The issue, then, is whether one can add to this \( \hat{s}_{abc} \) terms of order not exceeding two to achieve vanishing divergence. In any case, the mere existence of this field \( \hat{s}_{abc} \) lends support to the conjecture that \( G_{\mu \nu \lambda \sigma} \) admits a generalization to curved space-time. Work is in progress to settle this conjecture.

### C.2 Derivation of Gravitational Remnant Field Equations

The Einstein equation gives rise to certain differential equations on the gravitational remnants, \( k_{\lambda_{ab}}, k_{q_{ab}} \). These equations for a vacuum space-time were first systematically studied by Beig and Schmidt.\(^8\)\(^9\). We have here utilized the non-vacuum equations, of order one (C5)–(C6), two (C1)–(C3), and vacuum equations of order three (C4)–(C6). We summarize how these were derived. First write Einstein’s equation in 3 + 1-form, adapted to the surfaces \( \Omega = \text{constant} \):

\[
\Omega^2 T = -\frac{1}{2} [\mathcal{R} + p^{mn} p_{mn} - p^2],
\]

(C8)

\[
\Omega^2 T_a = D^m (p_{am} - p q_{am}),
\]

(C9)

\[
\Omega^2 T_{ab} = \mathcal{R}_{ab} + 2 p_a^m p_{mb} - p p_{ab} - \lambda^{-1} D_a D_b \lambda + \lambda^{-1} p_{ab} - \Omega L_{\lambda_{nn} p_{ab}},
\]

(C10)
where $D_a$ denotes the derivative operator of the metric $q_{ab}$ of these surfaces, $\mathcal{R}_{ab}$ its Ricci curvature, and $p_{ab}$ the rescaled extrinsic curvature of these surfaces, defined by

$$p_{ab} \equiv \Omega q^{k} q^{l} \tilde{\nabla}_{k} (\Omega^{-2} \lambda \tilde{\nabla}_{l} \Omega) = -\lambda^{-1} q_{ab} + \frac{1}{2} \Omega \mathcal{L}_{\lambda} q_{ab}. \quad (C11)$$

Taking the remnants of eqn.s (C8)–(C10) through fourth order, we obtain eqs. (59)–(60), (60)–(63), (C1)–(C3), and (C4)–(C6).

We remark, finally, that the conservation equation of the stress-energy tensor, \( \tilde{\nabla}^a \tilde{T}_{ab} = 0 \), yields, for the zeroth order remnants of \( \tilde{T}_{ab} \), the following equations

$$0 = D^{a} \tilde{T}_{a} + 2 \tilde{T} + 2 \tilde{T}_{m}^{m}, \quad (C12)$$

$$0 = D^{b} \tilde{T}_{ab} - D_{a}(T + \tilde{T}_{m}^{m}). \quad (C13)$$

### C.3 Second Order Equations in Weyl Remnants

We first remark that, for any space-time with completion, the Weyl tensor is asymptotically regular of order 3. To see this, rewrite \( 2 \tilde{\nabla}_{[a} \tilde{\nabla}_{b]} n_{c} = \tilde{R}_{abc}^{d} n_{d} \) as

$$E_{ab} = \Omega^{-1} ( -\tilde{R}_{ab} - p_{a}^{m} p_{bm} + pp_{ab}) + \Omega [ \frac{1}{2} (T_{ab} - \frac{1}{3} q_{ab} T_{m}^{m}) - \frac{2}{3} q_{ab} T], \quad (C14)$$

$$B_{ab} = \Omega^{-1} \epsilon_{mn(a} L_{b}^{m} p_{n)}, \quad (C15)$$

with $E_{ab}$ and $B_{ab}$ given by eqn.(3), $p_{ab}$ given by eqn.(C11), and $T_{ab}$ given by eqn.(2). But, by the conditions in definition 4, the right sides are smooth on $M$.

We next remark that the gravitational remnant equations, (B0)–(B3), can be written in terms of the Weyl remnants, $E_{ab}$, $B_{ab}$. To see this, first take the zeroth-order remnants of eqn.s (C14), (C15) above, to obtain

$$0 \quad E_{ab} = -(D_{a} D_{b} \lambda + \lambda q_{ab}), \quad (C16)$$

$$0 \quad B_{ab} = \epsilon_{kl(a} D_{b}^{k} (q_{l}^{b}) + 2 \lambda q_{b}^{l}) = 0, \quad (C17)$$
and the first-order remnants, to obtain

$$E_{ab} = -\frac{1}{2}q_{ab} + \left[(D\lambda)^2 + 5\lambda^2\right]q_{ab} + \lambda D_a D_b \lambda - 2 D_a \lambda D_b \lambda - \frac{1}{2} T_{ab} - \left(\frac{2}{3} T + \frac{1}{6} T^m_n\right)q_{ab},$$

(C18)

$$B_{ab} = \frac{1}{2} \varepsilon_{mn(a} D^m q^n_{b)}.$$  

(C19)

These Weyl remnants satisfy, by virtue of eqn.s (C16)–(C19), the equations

$$D_{[a} E_{b]c} = 0.$$  

(C20)

$$D_{[a} E_{b]c} = \frac{1}{2} \varepsilon_{ab}^m \left[B_{mc} + 4 \epsilon_{kl(m} (D^k \lambda) E^l_c + \frac{1}{2} \epsilon_{mc}^n T_n\right],$$

(C21)

$$D_{[a} B_{b]c} = -\frac{1}{2} \varepsilon_{ab}^m \left[B_{mc} - 2 T_{mc} - T q_{mc} + D_c T_m\right],$$

(C22)

where we have set

$$E_{ab} = E_{ab} - \lambda E_{ab} + \frac{1}{2} T_{ab} + \left(\frac{1}{6} T^m_n - \frac{2}{3} T\right)q_{ab}.$$  

(C23)

Now fix a space-time with completion, and define $E_{ab}, E_{ab},$ and $B_{ab}$ by eqn.s (C16)–(C19). Then eqn.s (C16)–(C19) are equivalent to the statements that the $E_{ab}, E_{ab},$ $B_{ab},$ so defined are trace-free and satisfy (C20)–(C22).
1. R. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. 117, 1695, (1960); Phys. Rev. 121, 1566, (1961); Phys. Rev. 122, 997, (1961); Gravitation, an Introduction to Current Research, ed. L. Witten (New York, Wiley, 1962).

2. R. Geroch, J. Math. Phys. 13, 956, (1972).

3. R. Geroch, J. Math. Phys. 11, 2580, (1970).

4. A. Ashtekar and R. O. Hansen J. Math. Phys. 19, 1542 (1978).

5. A. Ashtekar and A. Magnon-Ashtekar, J. Math. Phys. 20(5), 793 (1979).

6. P. Sommers, J. Math. Phys. 19, 549, (1978).

7. S. Persides, J. Math. Phys. 20, 1731, (1979), J. Math. Phys. 21, 135, (1980), J. Math. Phys. 21, 142, (1980).

8. R. Beig and B. G. Schmidt, Comm. Math. Phys. 87, 65, (1982).

9. R. Beig, Proc. R. Soc. London, A 391, 295, (1984).

10. H. Bondi, A. W. K. Metzner and M. J. G. Van Der Berg, Proc. Roy. Soc. London, A 269, 21 (1962).

11. A. Ashtekar and J. D. Romano, Class. Quantum Grav. 9, 1069–1100, (1992).

12. Sketch of proof: Let \((\tilde{\mathcal{M}}, \tilde{g}_{\alpha\beta})\) be a stationary asymptotically flat vacuum space-time with Killing field \(\xi^a\). Denote by \(\mu\) and \(\omega\) the norm and twist of the Killing field respectively. Let \((\tilde{V}, \tilde{h}_{ab})\) denote the (Riemannian) manifold of orbits of the Killing field, \((V, \Lambda)\) its completion, \(\Omega_G\) a conformal factor and \(h_{ab} = \Omega_G^2 \tilde{h}_{ab}\) (See Ref. 3). It follows\(^{36}\) that each of \(\Omega_G^{-1/2} \mu^{1/4}(\mu - \mu^{-1} + \mu^{-1}\omega^2), \Omega_G^{-1/2} \mu^{-1/4}(\mu^{-1}\omega), \) and \(\mu + \mu^{-1} + \mu^{-1}\omega^2\) is smooth on \(V\). Fix any smooth coordinates \(x^i\) on \(V\) near \(\Lambda\) such that \(h_{ij}\big|_\Lambda = \delta_{ij}\), and Lie drag them into \(\tilde{\mathcal{M}}\) by \(\xi^a\). Perform an inversion on these coordinates to obtain \(\tilde{x}^i\) on \(\tilde{M}\). Pick a \(\tau'_a\) on \(V\) satisfying \(D_a \tau'_b = -1/2 \mu^{-3/2} \epsilon_{abc} D^c \omega\) and such that \(\tau'_a\) is smooth in \(y'\) and vanishes at \(\Lambda\) (See, e.g., the appendix of Ref. 36 for motivation). Let \(\tau_a\) be the pull-back of \(\tau'_a\) to \(\tilde{M}\). Define \(\tilde{x}^0\) on \(\tilde{M}\) such that \(\nabla_a \tilde{x}^0 = \mu^{-1} \xi_a - \tau_a\) (note the right side is curl-free and yields 1 when contracted with \(\tilde{\xi}^a\)). Then the hyperbolic coordinates associated with the \(\tilde{x}^\mu\) coordinates yield a completion of \(\tilde{M}\) in the sense above.

13. Such freedom in choices of (inequivalent) completion are known to exist also for other frameworks.
such as that of Geroch and that of Ashtekar-Hansen.

14 P. G. Bergmann, Phys. Rev. 124, 274, (1961).

15 P. T. Chrusciel, J. Math. Phys. 30(9), 2094, (1989).

16 More generally, for a spin-$s$ field, we would demand asymptotic regularity of order $s + 1$.

17 To see this, evaluate $D_{[a}D_{b]}(\eta^{\mu\nu}D_c\alpha_\mu D_d\alpha_\nu)$ using eqn.(8) and equate the result to $2R_{ab(c}^m(\eta^{\mu\nu}D_d\alpha_\mu D_m\alpha_\nu)$, to obtain $\eta^{\mu\nu}D_a\alpha_\mu D_b\alpha_\nu = \eta^{\mu\nu}\alpha_\mu\alpha_\nu 0^{a}$. Now contract with $\hat{q}^{ab}$ using eqn.(9).

18 One might be tempted to consider, in addition, those divergence-free vector fields that are multi-linear in the Killing fields on $\mathcal{H}$. However, this adds nothing new since every anti-symmetric second rank tensor $F^{\mu\nu}$ in $\mathcal{T}$ yields a Killing field in $\mathcal{H}$ when contracted with $\alpha_{[\mu}D^a\alpha_{\nu]}$ and, conversely, for every Killing field $\xi^a$ in $\mathcal{H}$, there exists an anti-symmetric second rank tensor over $\mathcal{T}$ (namely $F_{\mu\nu} = 2\xi^a\alpha_{[\mu}D^a\alpha_{\nu]} + D_a\xi_bD^a\alpha_\mu D^b\alpha_\nu$) that gives rise to it. Similarly, multilinearity in conformal Killing fields yields nothing new, for every vector $\upsilon^\mu$ in $\mathcal{T}$ yields a curl-free conformal Killing field in $\mathcal{H}$ when contracted with $D^a\alpha_\mu$ and conversely, for every curl-free conformal Killing field $\zeta^a$ in $\mathcal{H}$, there exists a vector over $\mathcal{T}$ (namely $\upsilon^\mu = \zeta^aD_a\alpha_\mu - \frac{1}{3}(D_a\zeta^a)\alpha^\mu$) that gives rise to it.

19 In Ref. 10, Ashtekar and Romano used instead the condition $\lim_{\Omega \to 0} \Omega^{-1}\tilde{G}_{ab} = 0$ to show that the angular momentum is conserved. As we have noted earlier, their condition is too strong. The condition we are imposing is the necessary and sufficient condition for $B_{ab}$ to be divergence-free on $\mathcal{H}$ (c.f. eqn.(C22)). An example of a space-time satisfying our additional condition is the Kerr-Newman solution. In fact, the Kerr-Newman solution satisfies a stronger condition: $\hat{T}_a = 0$. In general, it is not clear how restrictive is the condition given by eqn.(13). However, the condition is presumably satisfied for all stationary asymptotically flat space-times since in that case one expects the angular momentum is well-defined and equal to Hansen’s angular-momentum dipole moment.
To see this, note that
\[ \mathcal{M}^\prime_{\mu\nu} - \mathcal{M}_{\mu\nu} = -\frac{1}{16\pi} \epsilon_{\mu\nu} \tau^\sigma \int_C \{ (-2e^{mn(a)}E^0_{m\omega})\alpha_{z} D_{\nu} \alpha_{\sigma} \} dS_{\alpha}. \]
\[ = -\frac{1}{8\pi} \omega_{\mu} \int_C \epsilon^{\alpha\beta}_{\mu} D_{[\beta} \alpha_{\nu]} dS_{\alpha}. \]
\[ = -\omega_{\mu} \mathcal{P}_{\nu}. \]

The linearity is clear for electric charge. For total energy-momentum and angular momentum, we have in mind the linearized gravity in Minkowski space-time in which these quantities are linear in the gravitational field and are expressible as surface integrals.

That is,
\[ \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \equiv \sum_{m=0}^{\lceil n/2 \rceil} \left( -\frac{1}{4} \right)^m \left( \begin{array}{c} n - m - 1 \\ m \end{array} \right) \eta_{(\mu_1 \mu_2 \cdots \mu_{m-1})} \eta_{\mu_1 \mu_2 \cdots \mu_{m-1} \mu_{m+1} \cdots \mu_{n-1}}, \]
with \( \lceil n/2 \rceil \) denoting the largest integer not exceeding \( n/2 \).

Indeed, we have
\[ D^n(\hat{E}^m D_m \alpha_{(\mu)} \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}})) - \hat{E}^m D_m \alpha_{(\mu)} D^n \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \]
\[ = D_k \left\{ 2\hat{E}^n [D^a \alpha_{(\mu)} \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}})] + \epsilon^{abc} D_c \left[ \alpha_{(\mu)} \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) - \frac{1}{2} \eta_{(\mu_1} \mathcal{C}(\alpha_{\mu_2} \cdots \alpha_{\mu_{n-1}})) \right] \right\}, \]
which can be seen by using eqn.\((37)\) and the identity
\[ \alpha_{(\mu} D^n \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}})) = (n - 1) \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) D^n \alpha_{\mu} + \frac{1}{2} \eta_{\mu_{m+1}} D^n \mathcal{C}(\alpha_{\mu_2} \cdots \alpha_{\mu_{n-1}}). \]

To see this, note that the difference between the integrands of \( ^*\mathcal{E}_{\mu_1 \cdots \mu_{n-1}} \) and that of the right of eqn.\((33)\) is given by
\[ D_k \left\{ \frac{2}{n} D_m \alpha_{(\mu} \epsilon^{lm[\alpha} \left[ (D^k)_{\nu]} \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) - E_{\nu} D^k \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \right] \right\}. \]
Integrate over a cut of \( \mathcal{H} \).

To see this, note that the difference between the integrands of \( ^*\mathcal{G}_{\mu\nu\mu_1 \cdots \mu_{n-1}} \) and the right side of eqn.\((44)\) is given by
\[ D_k \left\{ \frac{1}{n} \epsilon^{akl} \epsilon^{n_{\mu_1} \cdots \mu_{n-1}} \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \right\} + \frac{2}{n} D_m \alpha_{(\mu} D^\nu \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \epsilon^{lm[\alpha} \left[ (D^k)_{\nu]} \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) - E_{\nu} D^k \mathcal{C}(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \right] \right\}. \]
Integrate over a cut of $\mathcal{H}$.

Indeed, we have the following. Let $n^{-2} \phi$ satisfy the $n - 2$-th remnant equation for a Klein-Gordon field, $\xi^a$ any Killing field. Denote by $\psi_{a_1 \cdots a_s}$ the symmetric and trace-free part of $D_{a_1} \cdots D_{a_s} n^{-2} \phi$. Then

$$E_{ab} \equiv n^{-2} \psi_{abcd} \xi^c \xi^d + \frac{12}{7} (n+1)(n+2) [\xi^m n^{-2} \psi_{m(a} \xi_{b)} + \xi^m n^{-2} \psi_{m(a} \xi^*_{b)} - \frac{2}{3} \epsilon_{ab} n^{-2} \psi_c \xi^c \xi^d]$$

$$- \frac{4}{5} n(n+1)(n+2) \psi^m D_m (\xi(a) \xi^*_b) + \frac{4}{5} n(n-1)(n+1)(n+2) \psi \xi(a) \xi^*_b$$

$$+ (n+2) \psi_{cd(a} D_b) (\xi^c \xi^*_d)$$

satisfies the $n$-th remnant equations for a linearized gravitational field.

Adding to $\psi$ a constant changes the integrand of $\mathcal{E}_{\mu\nu}$ in eqn.(70) by a divergence of an antisymmetric tensor.

Let $\tilde{\phi}$ be a Klein-Gordon field in Minkowski space-time asymptotically regular of order 1. Consider

$$I = \int_{S_\infty} -\tilde{\epsilon}_{abcd} x^e \nabla^d [(x^e \nabla_e + 1) \tilde{\phi}],$$

where $S_\infty$ denotes a two-sphere at infinity, and $x^a$ a position vector field. When $S_\infty$ is any two-sphere cut at spatial infinity, the above integral reproduces the remnant radiation monopole $K$ associated with $\tilde{\phi}$. However, in general the integral evaluates to a different value when the two-sphere $S_\infty$ is at null infinity. For example, let $\tilde{\phi} = (f(t + r) - f(t - r))/r$, with $k_\pm(x) \equiv f(\pm \frac{1}{2}x), x > 0$, both smoothly extendible to zero. Then, when $S_\infty$ is at spatial infinity, we have $I = K = 4\pi [k'_+(0) + k'_-(0)]$, while for $S_\infty$ any cut at future null infinity, $I = 4\pi k'_+(0)$, and for $S_\infty$ any cut at past null infinity, $I = 4\pi k'_-(0)$.

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The analogous equation for $\bar{B}_{ab}$,

$$\mathcal{L}_{D\zeta} \bar{B}_{ab} - (n + 1) \bar{\zeta} \bar{B}_{ab} - 2 \epsilon_{lm} \bar{B}_{ab} D_m \zeta = 0,$$
follows from eqn. (A2) and the remnant field equations. For a Maxwell field the corresponding equation is

\[ L^n D_a \zeta^n E_a - (n + 1) \zeta^n E_a + \epsilon_a^{kl} n B_k D_l \zeta = 0. \]

and for a Klein-Gordon field, eqn. (A3).

32 We remark that the multipole moment \( M \) defined here is related to the \( Q \) defined by Geroch in Ref. 38 by a normalization factor:

\[ Q_{i_1 \cdots i_n} = (-\frac{1}{3})^n n! M_{i_1 \cdots i_n}. \]

33 We are concerned only with “irreducible” solutions. Thus, for example, \( C(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \) and \( \eta_{\mu \nu} C(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \) are viewed as equivalent solutions to eqn. (19).

34 To see this, embed \( H \) as the unit hyperboloid in Minkowski space-time \( M' \). Let \( x^a \) denote the position vector field from some origin. Then \( \nabla_a x^b = \delta_a^b \) and \( H \) is specified by \( x^a x_a = 1 \). Let \( k^a \) be a constant vector field in \( M' \). Then \( k^a x_a \) is a translation on \( H \). Thus the most general function multi-linear in translations is a sum of terms of the form \( w(s) \equiv w_{a_1 \cdots a_s} x_{a_1} \cdots x_{a_s} \), with \( w_{a_1 \cdots a_s} \) some symmetric, trace-free constant tensor. This \( w(s) \) satisfies the Klein-Gordon equation in \( M' \). Using \( \nabla^2 w = [D^2 + (x \cdot x)^{-1}((x \cdot \nabla)^2 + 2x \cdot \nabla)] w, \) we see that \( w(n-1) \) satisfies eqn. (19) on \( H \). Such \( w(n-1)'s \) clearly exhaust all solutions of eqn. (19) which are multi-linear in translations. But each such \( w(n-1)'s \) on \( H \) is the contraction of \( C(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \) with some tensor over \( T \).

35 One way to prove the identity is to note: \( D^a \alpha_\mu D_a C(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) + (n-1) \alpha_\mu C(\alpha_{\mu_1} \cdots \alpha_{\mu_{n-1}}) \) satisfies eqn. (19) with \( n \) replaced by \( n-1 \) and is trace-free in \( \mu_1 \cdots \mu_{n-1} \). The overall normalization factor can be fixed by comparing, say, the coefficients of the term \( \eta_{\mu_1 \mu_2} C(\alpha_{\mu_2} \cdots \alpha_{\mu_{n-1}}) \) on both sides.

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