Thermodynamics of the universe filled with perfect fluid having variable equation of state

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1 Introduction

In the semi-classical description of black hole physics it is found that a black hole behaves as a black body and emits thermal radiation. The temperature (known as the Hawking temperature) and the entropy are proportional to the surface gravity at the horizon and area of the horizon (Hawking 1975; Bekenstein 1973), respectively. The Hawking temperature, entropy and mass of the black hole satisfy the first law of thermodynamics (Bardeen et al. 1973). As the temperature and entropy are determined by purely geometric quantities (namely surface gravity and horizon area respectively), i.e. the case is characterized by the space-time geometry and hence by Einstein field equations, it is natural to speculate on some relationships between black hole thermodynamics and the Einstein equations. Jacobson (1995) was able to derive the Einstein equations from the proportionality of the entropy to the horizon area together with the fundamental relation $\delta Q = T \, ds$ (Clausius relation), assuming that the relation holds for all local Rindler causal horizons through each space-time point. Here $\delta Q$ and $T$ are the energy flux and the Unruh temperature seen by an accelerated observer just inside the horizon. For a general static spherically symmetric space-time, Padmanabhan (2002, 2005) was able to derive the first law of thermodynamics on the horizon, starting from the Einstein equations.

Subsequently, this equivalence between Einstein equation and thermodynamical laws has been generalized in the context of cosmology. If we assume the universe to be a thermodynamical system and consider at the apparent horizon $R_A$ the Hawking temperature $T_A = \frac{1}{2\pi R_A}$ and the entropy $S_A = \frac{\pi R_A^2}{G}$, then it was shown that the first law of thermodynamics on the apparent horizon and the Friedmann equations are equivalent (Cai and Kim 2005)—one can be derived from other. Then, in a higher dimensional space–time, the relation was established for gravity with a Gauss–Bonnet term and for the Lovelock theory of gravity (Cai and Kim 2005; Akbar and Cai 2006; Paranjape et al. 2006). As a result, it is speculated that such a deep relationship between the thermodynamics at the apparent horizon and the Einstein equations may give some clue on the properties of dark energy.

There is a large difference between astrophysical thermodynamics and cosmological thermodynamics. The first one concerns the stars as astrophysical objects and the second one relates to the universe in its totality. The study of the cosmological thermodynamics concerns the universe for the reason that the natural starting point for the thermodynamical study is the characterization of possible equilibrium states of the system under consideration. It is well known that equilibrium states in the expanding universe are only possible under certain conditions.

In the present work, we are trying to study the thermodynamics of the universe with matter in the form of a perfect fluid having variable equation of state.
fluid having an equation of state in the form: \( p = \omega \rho \). We consider \( \omega \) as a function of the red shift \( z \) in the following forms of two index parametrization (Johri and Rath 2007):

(i) \( \omega = \omega_0 + \omega_1 z \) (linear red shift parametrization)

(ii) \( \omega = \omega_0 + \frac{\omega_1 z}{(1+z)} \) (Chevallier–Polarski linear parametrization)

(iii) \( \omega = \omega_0 + \frac{\omega_1 z^2}{(1+z)^2} \) (Jassal–Bagla–Padmanavan parametrization)

(iv) \( \omega = \omega_0 + \omega_1 z \) if \( z < 1 \), \( \omega_0 + \omega_1 \) if \( z \geq 1 \) (Upadhya–Ishak–Steinhardt parametrization)

The thermal quantities are expressed either as a function of volume or temperature and due to the adiabatic nature of the thermodynamical system the entropy turns out to be a constant. Subsequently, we examine the validity of the thermodynamical laws for universe bounded by the horizon (event or apparent).

2 General thermodynamical description

Let us consider a thermodynamical system bounded in a volume \( V \) and suppose that \( \rho \), \( p \) and \( T \) are the energy density, thermodynamical pressure and temperature of the fluid bounded by the volume. In an expanding universe one may assume for simplicity that the energy density \( \rho \) and the pressure \( p \) are functions of temperature only, i.e. \( \rho = \rho(T) \) and \( p = p(T) \). Also the entropy is assumed to be a function of temperature and volume, i.e. \( S = S(T, V) \). Here the volume and temperature are considered as independent variables. Then from the first law of thermodynamics (Myung 2008)

\[
T \, dS = d(\rho V) + p \, dV = d[(\rho + p) V] - V \, d\rho,
\]

where \( S \) is the entropy of the thermodynamical system. Now the integrability condition (Myung 2008)

\[
\frac{\partial^2 S}{\partial T \, \partial V} = \frac{\partial^2 S}{\partial V \, \partial T}
\]

demands the following differential relation (Gong et al. 2007):

\[
\frac{dp}{\rho + p} = \frac{dT}{T}.
\]

Combining (1) and (3) and integrating once we obtain (except for an additive constant)

\[
S = \frac{(\rho + p) V}{T}.
\]

However, for an adiabatic process the entropy is constant and consequently, the first law of thermodynamics (1) becomes the conservation law

\[
d[(\rho + p) V] = V \, dp.
\]

Note that one can obtain the relation (4) easily using the conservation relation (5) in the integrability condition (3). Hence for an adiabatic process, (4) may be considered as the temperature-defining equation. If we suppose our universe to be homogeneous and isotropic being a FRW space–time with line element

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right]
\]

then the Einstein’ equations (known as Friedmann equations) and the energy conservation law are

\[
H^2 = \frac{8\pi G \rho}{3} - \frac{k}{a^2},
\]

\[
\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2}
\]

and

\[
\dot{\rho} + 3H(\rho + p) = 0,
\]

where \( H = \frac{\dot{a}}{a} \) is the Hubble parameter, \( k \) is the curvature scalar, having values 0, ±1, for three dimensional space flat or having positive or negative spatial curvature, and \( d\Omega^2 \) is the metric on the unit 2-sphere. Now we consider our universe filled with a perfect fluid having the equation of state \( p = \omega \rho \) where \( \omega \) is not constant but a function of the red shift \( z \), i.e. \( \omega = \omega(z) \). We choose the following forms for \( \omega(z) \):

(I) \( \omega = \omega_0 + \omega_1 z \)

(II) \( \omega = \omega_0 + \frac{\omega_1 z}{(1+z)} \)

(III) \( \omega = \omega_0 + \frac{\omega_1 z^2}{(1+z)^2} \)

(IV) \( \omega = \omega_0 + \omega_1 z \) if \( z < 1 \), \( \omega_0 + \omega_1 \) if \( z \geq 1 \) where \( \omega_0 \) and \( \omega_1 \) are constants

Case I: \( \omega(z) = \omega_0 + \omega_1 z \)

From the energy conservation equation (8), integrating once, we have

\[
\rho(z) = \rho_0 e^{3\omega_1 z} (1+z)^{3(1+\omega_0-\omega_1)},
\]

where \( \rho_0 \) is an integration constant. As the system is an adiabatic process, the entropy is constant \( (S_0, \text{say}) \) and hence from (4)

\[
T(z) = T_0 e^{3\omega_1 z} (1+z)^{3(1+\omega_0-\omega_1)} (1 + \omega_0 + \omega_1 z)
\]

with \( T_0 = \frac{S_0}{\rho_0} \) a constant. The squared speed of sound \( v_s^2 \) is given by

\[
v_s^2(z) = \frac{d\rho}{d\rho} = \rho + \omega(1+z) = \frac{3\omega(1+z)}{3(1+\omega)}.
\]

Finally, the heat capacity is calculated to be

\[
C_v(z) = V \frac{\partial p}{\partial T} = \frac{3S_0}{1+z} \left[ \frac{3\omega}{1+z} + \frac{\omega_1}{1+\omega} \right]^{-1}.
\]