ORDER POLARITIES

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Abstract. We define an order polarity to be a polarity \((X, Y, R)\) where \(X\) and \(Y\) are partially ordered, and we define an extension polarity to be a triple \((e_X, e_Y, R)\) such that \(e_X : P \rightarrow X\) and \(e_Y : P \rightarrow Y\) are poset extensions and \((X, Y, R)\) is an order polarity. We define a hierarchy of increasingly strong coherence conditions for extension polarities, each equivalent to the existence of a pre-order structure on \(X \cup Y\) such that the natural embeddings, \(\iota_X\) and \(\iota_Y\), of \(X\) and \(Y\), respectively, into \(X \cup Y\) preserve the order structures of \(X\) and \(Y\) in increasingly strict ways. We define a Galois polarity to be an extension polarity where \(e_X\) and \(e_Y\) are meet- and join-extensions respectively, and we show that for such polarities there is a unique pre-order on \(X \cup Y\) such that \(\iota_X\) and \(\iota_Y\) satisfy particularly strong preservation properties. We define morphisms for polarities, providing the class of Galois polarities with the structure of a category, and we define an adjunction between this category and the category of \(\Delta_1\)-completions and appropriate homomorphisms. We formalize the theory of extension polarities and prove a duality principle to the effect that if a statement is true for all extension polarities then so too must be its dual statement.

1. Introduction

1.1. Background. The concept of a polarity, i.e. a pair of sets \(X\) and \(Y\) and a relation \(R\) between them, was known to Birkhoff at least as far back as 1940 [3]. While, according to [4, p122], originally defined as a generalization of the dual isomorphism between polars in analytic geometry, the generality of the definition has lent itself to diverse applications in mathematics and computer science. For example, polarities under the name of formal concepts are fundamental in formal concept analysis [10]. As another example, polarities appear bearing the name classification in the theory of information classification [2, Lecture 4], where they are again a foundational concept.

For a more purely mathematical application, a particular kind of polarity, referred to as a polarization, was used in [35] to produce poset completions. The same paper also proves various results connecting properties of polarizations with properties of the resulting completion. More recently, this technique has been exploited to construct canonical extensions for bounded lattice expansions [12], and also for posets [7], where they provide a tool for ‘completeness via canonicity’ results for substructural logics. Something similar also appears implicitly in [18], though neither polarizations nor polarities in general are mentioned explicitly.

The general idea behind these completeness results is, given a poset \(P\) equipped with additional operations that are either order preserving or reversing in each
coordinate, to show that there exists a completion of \( P \) to which the additional operations can be extended. This roots of this technique appear in [20], though not in the context of ‘completeness via canonicity’ results, as a generalization of Stone’s representation theorem to Boolean algebras with operators (BAOs). The approach there was to first (non-constructively) dualize to relational structures, then construct the canonical extension from these.

Early generalizations to distributive lattices used Priestly duality [26, 27] in a similar way (see for example [14, 29, 30]). More recent approaches using polarities bypass the dual construction, which is significantly more complicated outside of the distributive setting, and have the additional advantage of being constructive [12, 7]. Indeed, an innovation of [7] is to use the canonical extension of a poset to construct a dual, which can then play the same role in providing completeness results for substructural logics as the canonical frame does in the modal setting (see e.g. [5, Chapters 4 and 5]). For more on the development of the theory of canonical extensions see, for example, [17] or the introduction to [19].

We note that for operations that are not operators in the sense of [20], the canonical extension construction is ambiguous, as there are often several non-equivalent choices for the lifts of each operation, each of which may be ‘correct’ depending on the situation (see for example the epilogue of [15] for a brief discussion of this). Moreover, for posets, what is meant by the canonical extension is even less clear than it is in the lattice case. This is a consequence of ambiguities surrounding the notions of ‘filters’ and ‘ideals’ in the more general setting. See [21] for a thorough investigation of this issue.

For canonical extensions in their various guises to play a role in ‘completeness via canonicity’ arguments, general results concerning the preservation of equations and inequalities are extremely useful. Some results of this sort can be found in [31, 32], where arguments from [18] are extended to more general settings. One component of these arguments is the exploitation of the so called intermediate structure, an extension of the original poset intermediate between it and the canonical extension. The idea is that operations are, in a sense, lifted first to the intermediate structure, and then to the canonical extension.

More generally, the class of \( \Delta_1 \)-completions [13] includes canonical extensions (however we define them), and also others such as the MacNeille (aka Normal) completion. Given a poset \( P \), the \( \Delta_1 \)-completions of \( P \) are, modulo suitable concepts of isomorphism, in one-to-one correspondence with certain kinds of polarities constructed from the poset [13, Theorem 3.4]. Here also the intermediate structure appears. Indeed, a \( \Delta_1 \)-completion is the MacNeille completion of its intermediate structure [13, Section 3].

1.2. **What is done here.** In the existing literature, the intermediate structure emerges almost coincidentally from the construction of a completion. Given a polarity \((X, Y, R)\), first a complete lattice \( G(X, Y, R) \) is constructed using the antitone Galois connection between \( \mathcal{P}(X) \) and \( \mathcal{P}(Y) \) induced by \( R \), as we explain in more detail in Section 2.3. The intermediate structure is then found sitting inside it as a subposet. There are natural maps from \( X \) and \( Y \) into the intermediate structure, and, if these are injective, partial orderings are thus induced on \( X \) and \( Y \). When \( G(X, Y, R) \) is an extension of a poset \( P \), it will also follow that \( X \) and \( Y \) are extensions of \( P \). It turns out that the pre-order on \( X \cup Y \) induced by the intermediate structure agrees with \( R \) on \( X \times Y \).
The broad goal of this paper is to take the idea of a polarity involving order extensions \( e_X : P \to X \) and \( e_Y : P \to Y \) as primitive, and develop a theory from this. More explicitly, we are interested in the interaction between the relation \( R \) and the orders on \( X \) and \( Y \), and, in particular, under what circumstances something corresponding to the ‘intermediate structure’ can be defined on \( X \cup Y \). This issue raises several questions, depending on exactly what properties we think an ‘intermediate structure’ should have.

Based on our answers to these questions, we define a sequence of so-called coherence conditions for polarities. The bulk of this work is done in Section \( \text{S} \) where the main definitions are made, and in Section \( \text{S} \) where, among other things, we prove our defined conditions are strictly increasing in strength.

In Section \( \text{S} \) we define a Galois polarity to be a triple \( (e_X, e_Y, R) \) satisfying the strongest of our coherence conditions, and with the additional property that \( e_X \) is a meet-extension, and \( e_Y \) is a join-extension. The ‘aptness’ of this definition is partly demonstrated by the fact that, if \( (e_X, e_Y, R) \) is a Galois polarity, there is one and only one possible pre-order structure on \( X \cup Y \) that agrees with the orders on \( X \) and \( Y \), agrees with \( R \) on \( X \times Y \), and also preserves meets and joins from the base poset \( P \) (see Theorem \( \text{S} \) for a more precise statement).

Galois polarities are studied further in Section \( \text{S} \). First we justify the choice of terminology by demonstrating that, for Galois polarities, the unique pre-order structure described above can be defined in terms of a Galois connection between any join-preserving join-completion of \( Y \) and any meet-preserving meet-completion of \( X \). This requires some technical results on extending and restricting polarity relations, which we provide in Section \( \text{S} \). Here we investigate the ‘simplest’ way we might hope to extend a relation between posets to a relation between meet- and join-extensions of these posets, and conversely the simplest way we might restrict a relation between extensions to a relation between the original posets. In particular we prove that it is rather common for coherence properties of a polarity to be preserved by extension and restriction as we define them.

By defining suitable morphisms, we can equip the class of Galois polarities with the structure of a category. This can be seen as a generalization of the concept of a \( \delta \)-homomorphism from \( \text{S} \) Section \( \text{S} \). We define an adjunction between this category and the category of \( \Delta_1 \)-completions (see Theorem \( \text{S} \)). This produces the correspondence between \( \Delta_1 \)-completions of a poset and certain kinds of polarity from \( \text{S} \) Theorem 3.4] via the categorical equivalence between fixed subcategories.

Finally, in Section \( \text{S} \) we characterize order polarities with various coherence levels as models of certain first- and second-order theories, and using this formulate a ‘duality principle’ for order polarities. This generalizes the familiar order duality for posets, and formalizes a labour-saving intuition to which we frequently appeal in proofs throughout the document.

In the long term we imagine handling lifting of operations, and the preservation of inequalities and so on, to ‘intermediate structures’ induced by polarities, and Galois polarities in particular. This is, of course, not an entirely new idea. Indeed, we have mentioned previously that lifting operations to canonical extensions is often done by first lifting to the intermediate structure. The hope is that, by shifting the focus a little from intermediate structures as they emerge in the construction of completions, to intermediate structures as algebraic objects of interest in their
own right, some new insight might be gained. However, to control the length of this document, we leave the pursuit of this rather vague goal to future work.

2. Orders and completions

2.1. A note on notation. We use the following not entirely standard notations:

- Give a poset $P$ and $p \in P$, we define $p^\uparrow = \{q \in P : q \geq p\}$ and $p^\downarrow = \{q \in P : q \leq p\}$.
- Given a function $f : X \to Y$, and given $S \subseteq X$, we define $f[S] = \{f(x) : x \in S\}$.
- With $f$ as above and with $y \in Y$ and $T \subseteq Y$ we define $f^{-1}(y) = \{x \in X : f(x) = y\}$ and $f^{-1}(T) = \{x \in X : f(x) \in T\}$.
- If $P$ is a poset then $P^\partial$ is the order dual of $P$.
- If $X$ and $Y$ are sets, then we may refer to a relation $R \subseteq X \times Y$ as being a relation on $X \times Y$.

2.2. Extensions and completions. We assume familiarity with the basics of order theory. Textbook exposition can be found in [6]. In this subsection we provide a brisk introduction to some more advanced order theory concepts. This serves primarily to establish the notation we will be using.

Definition 2.1. Let $P$ and $Q$ be posets. We say an order embedding $e : P \to Q$ is a poset extension, or just an extension. If $Q$ is also a complete lattice we say $e$ is a completion. If for all $q \in Q$ we have $q = \bigwedge e[e^{-1}(q)^\uparrow]$ then we say $e$ is a meet-extension, or a meet-completion when $Q$ is complete.

Note that it is common in the literature to refer to completions using the codomain of the function. For example, we might say “$Q$ is a completion of $P$” when talking about the completion $e : P \to Q$. This has the disadvantage of obfuscating the issue of what it means for two extensions to be isomorphic, as an isomorphism between codomains is not sufficient for extensions to be isomorphic in the sense used here. This rarely causes significant problems in practice, as it is usually clear from context what kind of isomorphism is required. However, we find the identification of extensions with maps to be more elegant, and will generally use this approach.

Definition 2.2 (Morphisms between monotone maps and extensions). Given posets $P_1, P_2, Q_1, Q_2$, and monotone maps $f_1 : P_1 \to Q_1$ and $f_2 : P_2 \to Q_2$, a map, or morphism, from $f_1$ to $f_2$ is a pair of monotone maps $g_P : P_1 \to P_2$ and $g_Q : Q_1 \to Q_2$ such that the diagram in Figure 1 commutes. If $g_P$ and $g_Q$ are both order isomorphisms then we say $f_1$ and $f_2$ are isomorphic.

If $f_1 : P \to Q_1$ and $f_2 : P \to Q_2$ are extensions of a poset $P$, then $f_1$ and $f_2$ are isomorphic as extensions of $P$ if they are isomorphic in the sense described above and the map $g_P$ is the identity on $P$. 
Definition 2.2 equips the class of monotone maps between posets, and in particular the subclass of poset extensions, with the structure of a category. We will make frequent use of the idea of extensions being isomorphic, and we will return to the idea of a category of extensions in Section 7.4.

![Diagram]

**Figure 1.**

**Definition 2.3.** Given a poset $P$, the **MacNeille completion** of $P$ is a map $e : P \rightarrow \mathcal{N}(P)$ that is both a meet- and a join-completion.

The MacNeille completion was introduced in [23] as a generalization of Dedekind’s construction of $\mathbb{R}$ from $\mathbb{Q}$, it is unique up to isomorphism. The characterization used here is due to [1]. See e.g. [6, Section 7.38] for more information.

**Definition 2.4.** The **canonical extension** of a lattice $L$ is a completion $e : L \rightarrow L^\delta$ such that:

1. $e[L]$ is *dense* in $L^\delta$. I.e. Every element of $L^\delta$ is expressible both as a join of meets, and as a meet of joins, of elements of $e[L]$.
2. $e$ is *compact*. I.e. for all $S, T \subseteq L$, if $\bigwedge e[S] \leq \bigvee e[T]$ then there are finite $S' \subseteq S$ and $T' \subseteq T$ with $\bigwedge S' \leq \bigvee T'$.

Canonical extensions are also unique up to isomorphism. This characterization, and the proof that such a completion exists for all $L$, is due to [12]. It generalizes the definition of the canonical extension for Boolean algebras [20], and distributive lattices [14]. The construction used in [12] can, as noted in Remark 2.8 of that paper, also be used for posets, and will again result in a dense completion. However, the kind of compactness obtained is weaker. This idea is expanded upon in [7]. The differences between the lattice and poset cases arise from the fact that definitions for filters and ideals which are equivalent for lattices are not so for posets. This issue is discussed in detail in [24]. One way to address this systematically is to talk about the canonical extension of $P$ with respect to $\mathcal{F}$ and $\mathcal{I}$, where $\mathcal{F}$ and $\mathcal{I}$ are sets of ‘filters’ and ‘ideals’ of $P$ respectively. By making the definitions of ‘filter’ and ‘ideal’ weak enough, this allows all notions of the canonical extension of a poset to be treated in a uniform fashion. This is the approach taken in [25], for example.

**Definition 2.5.** Given a poset $P$, a **$\Delta_1$-completion** of $P$ is a completion $e : P \rightarrow D$ such that $e[P]$ is dense in $D$.

$\Delta_1$-completions, introduced in [13], include both MacNeille completions and canonical extensions. As such they are not usually unique up to isomorphism, so it doesn’t make sense to talk about the $\Delta_1$-completion.

**Definition 2.6.** Let $P$ and $Q$ be posets. Then a **monotone Galois connection**, or just a **Galois connection**, between $P$ and $Q$ is a pair of monotone maps $\alpha : P \rightarrow Q$ and...
and \( \beta : Q \to P \) such that, for all \( p \in P \) and \( q \in Q \), we have
\[
\alpha(p) \leq q \iff p \leq \beta(q).
\]
The map \( \alpha \) is the left adjoint, and \( \beta \) is the right adjoint.

An antitone Galois connection between \( P \) and \( Q \) is a Galois connection between \( P \) and the order dual, \( Q^\circ \), of \( Q \).

**Definition 2.7.** A pre-order on a set is a binary relation that is reflexive and transitive. Every pre-order induces a canonical partial order by identifying pairs elements that break anti-symmetry.

### 2.3. Polarities for completions.

Following [2], we define a polarity to be a triple \( (X, Y, R) \), where \( X \) and \( Y \) are sets, and \( R \subseteq X \times Y \) is a binary relation. For convenience we will assume also that \( X \) and \( Y \) are disjoint. See the section on polarities in [9] for several examples. Polarities have also been called polarity frames in [34]. Given any polarity \( (X, Y, R) \), there is an antitone Galois connection between \( \varphi(X) \) and \( \varphi(Y) \). This is given by the order reversing maps \((-)^R : \varphi(X) \to \varphi(Y) \) and \( R(-) : \varphi(Y) \to \varphi(X) \) defined as follows:
\[
(S)^R = \{ y \in Y : x R y \text{ for all } x \in S \}.
\]
\[
R(T) = \{ x \in X : x R y \text{ for all } y \in T \}.
\]

The set \( G(X, Y, R) \) of subsets of \( X \) that are fixed by the composite map \( R(-) \circ (-)^R \) is a complete lattice. Indeed, this is a closure operator on \( \varphi(X) \).

Polarities in the special case where \( X \) and \( Y \) are sets of subsets of some common set \( Z \), where the relation \( R \) is that of non-empty intersection, and which also satisfy some additional conditions, have been referred to as polarizations in the literature [35, 25]. Polarizations play an important role in the construction of canonical extensions.

There are maps \( \Xi : X \to G(X, Y, R) \) and \( \Upsilon : Y \to G(X, Y, R) \) defined by:
\[
\Xi(x) = R(\{ x \}^R) \text{ for } x \in X, \text{ and}
\]
\[
\Upsilon(y) = R(\{ y \}) \text{ for } y \in Y.
\]
\( \Xi[X] \) and \( \Upsilon[Y] \) join- and meet-generate \( G(X, Y, R) \) respectively [11] Proposition 2.10. Moreover, the (not usually disjoint) union \( \Xi[X] \cup \Upsilon[Y] \) inherits an ordering from \( G(X, Y, R) \). Thus the inclusion of the poset \( \Xi[X] \cup \Upsilon[Y] \) into \( G(X, Y, R) \) can be characterized as the MacNeille completion of \( \Xi[X] \cup \Upsilon[Y] \). The order on \( \Xi[X] \cup \Upsilon[Y] \) can be defined without first constructing \( G(X, Y, R) \). We expand on this in Proposition 2.8 below.

**Proposition 2.8.** A pre-order on \( \Xi[X] \cup \Upsilon[Y] \) is defined below. The partial ordering of \( \Xi[X] \cup \Upsilon[Y] \) inherited from \( G(X, Y, R) \) is the canonical partial order induced by this pre-ordering.

1. For \( x_1, x_2 \in X \) we have \( \Xi(x_1) \leq \Xi(x_2) \iff (x_2 R y \implies x_1 R y \text{ for all } y \in Y) \).
2. For \( y_1, y_2 \in Y \) we have \( \Upsilon(y_1) \leq \Upsilon(y_2) \iff (x R y_1 \implies x R y_2 \text{ for all } x \in X) \).
3. For \( x \in X \) and \( y \in Y \) we have \( \Xi(x) \leq \Upsilon(y) \iff x R y \).
4. For \( x \in X \) and \( y \in Y \) we have
\[
\Upsilon(y) \leq \Xi(x) \iff (x' R y \text{ and } x R y' \implies x' R y', \text{ for all } x' \in X \text{ and } y' \in Y).
\]
Definition 2.11. We have the following definitions.

Proof. This is essentially [11 Proposition 2.7]. □

Proposition 2.9 below provides another perspective on the conditions from Proposition 2.8.

Proposition 2.9. Let $(X,Y,R)$ be a polarity. Then the following are equivalent:

1. $\preceq$ is the least pre-order definable on $\Xi[X] \cup \Upsilon[Y]$ such that:
   (a) $\Xi(x) \preceq \Upsilon(y) \iff x\, R\, y$ for all $x \in X$ and $y \in Y$.
   (b) The restrictions of $\preceq$ to $\Xi[X]$ and $\Upsilon[Y]$ agree with the orders on these sets inherited from $G(X,Y,R)$.

2. $\preceq$ satisfies the conditions from Proposition 2.8.

Proof. Suppose $\preceq$ is any pre-order on $X \cup Y$ satisfying conditions 1(a) and 1(b). Then, by Proposition 2.8 we have

$$\Xi(x_1) \preceq \Xi(x_2) \iff \Xi(x_1) \subseteq \Xi(x_2) \iff (x_2 \, R \, y \implies x_1 \, R \, y \text{ for all } y \in Y),$$

and thus Proposition 2.8(1) is satisfied. A similar argument works for Proposition 2.8(2), and Proposition 2.8(3) holds automatically. Finally, as $\preceq$ is transitive, we must have

$$\Upsilon(y) \preceq \Xi(x) \implies \left(\Xi(x') \preceq \Upsilon(y) \text{ and } \Xi(x) \preceq \Upsilon(y') \implies \Xi(x') \preceq \Upsilon(y') \right)$$

for all $x' \in X$ and $y' \in Y$.

Thus, by 1(a), any such pre-order $\preceq$ satisfies Proposition 2.8(1)-(3), and the ‘forward implication only’ version of Proposition 2.8 tells us.

To complete the proof it is sufficient to show that the ‘minimal’ $\preceq$ defined from $R$ using conditions Proposition 2.8(1)-(4) defines a pre-order on $\Xi[X] \cup \Upsilon[Y]$ satisfying conditions 1(a) and 1(b). But this is what Proposition 2.8 tells us. □

Lemma 2.10. The following are equivalent:

1. The map $\Xi : X \to G(X,Y,R)$ is injective.
2. The map $\Upsilon : Y \to G(X,Y,R)$ is injective.
3. Whenever $x_1 \neq x_2 \in X$ there is $y \in Y$ such that either $(x_2,y) \in R$ and $(x_1,y) \notin R$, or vice versa.
4. Whenever $x_1 \neq x_2 \in X$ we have either $x_1 \notin \Xi(x_2)$ and/or $x_2 \notin \Xi(x_1)$.

The following are also equivalent:

5. The map $\Xi : X \to G(X,Y,R)$ is injective.
6. The map $\Upsilon : Y \to G(X,Y,R)$ is injective.
7. Whenever $y_1 \neq y_2 \in Y$ there is $x \in X$ such that either $(x,y_2) \in R$ and $(x,y_1) \notin R$, or vice versa.
8. Whenever $y_1 \neq y_2 \in Y$ we have either $y_1 \notin \Upsilon(y_2)$ and/or $y_2 \notin \Upsilon(y_1)$.

Proof. Observe that $\Xi(x) = \{z \in X : x \, R \, y \implies z \, R \, y \text{ for all } y \in Y\}$ for all $x \in X$. Let $x_1 \neq x_2$ and suppose without loss of generality that there is $z \in \Xi(x_1) \setminus \Xi(x_2)$. Then $(z,y) \in R$ for all $y \in Y$ with $(x_1,y) \in R$, but there is $y' \in Y$ with $(x_2,y') \in R$ and $(z,y') \notin R$. For this $y'$ must have $(x_2,y') \in R$ and $(x_1,y') \notin R$. Thus (1.a) $\implies$ (1.b). That (1.b) $\implies$ (1.c) and (1.c) $\implies$ (1.a) is automatic. The proof for $\Upsilon$ is similar, but even more straightforward. □

What if $X$ and $Y$ are not merely sets but also have a poset structure? We make the following definition.

Definition 2.11. A polarity $(X,Y,R)$ is an order polarity if $X$ and $Y$ are posets.

In this situation we might, for example, want the maps $\Xi$ and $\Upsilon$ to be order embeddings, which places constraints on $R$. Building on lemma 2.10 we have the following result.
Theorem 3.3. Let \((X, Y, R)\) be an order polarity. Then the map \(\Xi : X \to G(X, Y, R)\) is an order embedding if and only if, for all \(x_1, x_2 \in X\), we have
\[x_1 \leq x_2 \iff \text{for all } y \in Y \text{ we have } x_2 R y \Rightarrow x_1 R y.\]

The map \(\Upsilon : Y \to G(X, Y, R)\) is an order embedding if and only if, for all \(y_1, y_2 \in Y\), we have
\[y_1 \leq y_2 \iff \text{for all } x \in X \text{ we have } x R y_1 \Rightarrow x R y_2.\]

Proof. We could appeal to proposition 2.8, but the direct argument is also extremely simple. Explicitly, \(\Xi\) is an order embedding if and only if \(\Xi(x_1) \subseteq \Xi(x_2)\), and a little consideration reveals that \(\Xi(x_1) \subseteq \Xi(x_2)\) if and only if \(x_2 R y \Rightarrow x_1 R y\) for all \(y \in Y\). Again, the argument for \(\Upsilon\) is even more straightforward. \(\square\)

Propositions 2.9 and 2.12, while essentially trivial in themselves, contain, in a sense, the seed of inspiration for the rest of the paper. In broad terms, we want to investigate the conditions for the existence of pre-orders on \(X \cup Y\) such that similar results can be proved. This we do in the next section and onwards. First a little more notation.

Definition 2.13 \((X \cup Y)\). Given disjoint sets \(X\) and \(Y\), we sometimes write \(X \cup Y\) to specify that we are talking about \(X \cup Y\) ordered by a given pre-order \(\preceq\).

3. Coherence conditions for order polarities

3.1. The basic case. In the previous section we discussed polarities and order polarities from the perspective of \(G(X, Y, R)\), and the inherited order structure on \(\Xi[X] \cup \Upsilon[Y]\). In this situation the maps \(\Xi\) and \(\Upsilon\) may fail to be monotone, order reflecting, or even injective. In this section we forget about \(G(X, Y, R)\), and ask instead, given an order polarity \((X, Y, R)\), under what circumstances can we define pre-orders on \(X \cup Y\) that agree with \(R\) on \(X \times Y\), and also extend the order structures of \(X\) and \(Y\)? In other words, when are there pre-orders on \(X \cup Y\) agreeing with \(R\) on \(X \times Y\) such that the natural inclusions of \(X\) and \(Y\) into \(X \cup Y\) are monotone? What about if we require the inclusions to be order embeddings, or to have stronger preservation properties? We will address these questions, but first some definitions.

Definition 3.1 \((R^d)\). Given a relation \(R \subseteq X \times Y\) we define the relation \(R^d \subseteq Y \times X\) by
\[y R^d x \iff (x' R y \text{ and } x R y' \Rightarrow x' R y', \text{ for all } x' \in X \text{ and } y' \in Y).\]

Definition 3.2 \((P_R)\). Let \((X, Y, R)\) be an order polarity. Define \(P_R\) to be the set of pre-orders on \(X \cup Y\) agreeing with \(R\) on \(X \times Y\), and extending the orders on \(X\) and \(Y\). I.e. \(\preceq \in P_R\) if and only if:
(i) \(\preceq |_{X \times Y} = R\), and
(ii) the orders on \(X\) and \(Y\) are contained in \(\preceq |_{X \times X}\) and \(\preceq |_{Y \times Y}\) respectively.

Theorem 3.3. Let \((X, Y, R)\) be an order polarity. Then \(P_R\) is non-empty if and only if:
(A0) For all \(x_1, x_2 \in X\) we have \(x_1 \leq x_2 \Rightarrow (x_2 R y \Rightarrow x_1 R y \text{ for all } y \in Y)\).
(A1) For all \(y_1, y_2 \in Y\) we have \(y_1 \leq y_2 \Rightarrow (x R y_1 \Rightarrow x R y_2 \text{ for all } x \in X)\).
In addition, for all relations $\preceq$ on $(X \cup Y)^2$ we have $\preceq \in \mathcal{P}_R$ only if the following conditions are satisfied:

(A2) For $x_1, x_2 \in X$ we have $x_1 \preceq x_2 \implies x_1 \preceq x_2$.

(A3) For $y_1, y_2 \in Y$ we have $y_1 \preceq y_2 \implies y_1 \preceq y_2$.

(A4) For all $x \in X$ and $y \in Y$ we have $x \preceq y \iff x \preceq y$.

(A5) $\mathcal{P}_R^d$ extends $\preceq$ on $Y \times X$.

Moreover, $\mathcal{P}_R$ is closed under non-empty intersections, and, if it is non-empty, has a minimal element $\leq_0$, defined by:

(A6) For all $x_1, x_2 \in X$ we have $x_1 \leq_0 x_2 \iff x_1 \preceq x_2$.

(A7) For all $y_1, y_2 \in Y$ we have $y_1 \leq_0 y_2 \iff y_1 \preceq y_2$.

(A8) For all $x \in X$ and $y \in Y$ we have $x \leq_0 y \iff x \preceq y$.

(A9) There is no $x \in X$ and $y \in Y$ with $x \leq_0 x$. I.e. $\leq_0 |Y \times X = \emptyset$.

Proof. Suppose first that (A0) does not hold. Then there are $x_1 \preceq x_2 \in X$, and $y \in Y$ with $x_2 \preceq y$ but not $x_1 \preceq y$. But this is impossible if there is a pre-order $\preceq$ on $X \cup Y$ agreeing with $R$ and extending the order on $X$, as it would have to be transitive, and we would have $x_1 \preceq x_2$, and $x_2 \preceq y$, but not $x_1 \preceq y$. By a duality argument, which we discuss in Remark 3.4 below, it follows that if either (A0) or (A1) fails then $\mathcal{P}_R$ is empty.

Now suppose $\preceq$ is a relation on $(X \cup Y)^2$. Conditions (A2) and (A3) are just the statements that $\preceq$ extends the orders on $X$ and $Y$ respectively, and (A4) is just the statement that $\preceq$ agrees with $R$ on $X \times Y$. Condition (A5) amounts to demanding a kind of transitivity:

$$y \preceq x \implies y \preceq x \implies (x' \preceq y \implies x \preceq y')\).$$

I.e. if $x' \preceq y$, $y \preceq x$, and $x \preceq y'$, then $x' \preceq y'$. Thus all these condition must certainly hold for $\preceq \in \mathcal{P}_R$. It follows directly from this that any relation $\preceq \in \mathcal{P}_R$ must contain $\leq_0$, so to complete the proof it remains only to show that, assuming (A0) and (A1), the relation $\preceq$ is in $\mathcal{P}_R$.

It follows from (A6) and (A7) that $\leq_0$ is reflexive, so it remains only to check transitivity. To do this we consider triples $(z_1, z_2, z_3) \in (X \cup Y)^3$, with $z_1 \preceq_0 z_2$, and $z_2 \preceq_0 z_3$. A simple counting argument reveals there are eight cases, depending on the containment of each $z_i$ in $X$ or $Y$. The cases where the $z$ values are either all in $X$ or all in $Y$ follow from the fact that $\leq_0$ agrees with the orders on $X$ and $Y$.

The cases that require $y \preceq_0 x$ are ruled out by (A9), so the only remaining cases are $(x_1, x_2, y)$, where $x_1, x_2 \in X$ and $y \in Y$, and $(x, y_1, y_2)$ where $x \in X$ and $y_1, y_2 \in Y$. These cases are covered by the assumption of (A0) and (A1), so we are done.

Finally, that $\mathcal{P}_R$ is closed under non-empty intersections follows almost immediately from the definition of $\mathcal{P}_R$ and the fact that intersections of pre-orders are also pre-orders.  

\[\square\]

Remark 3.4. In the proof of Theorem 3.3 we appealed to a duality principle. This arises from the fact that (A0) and (A1) are, in a sense, dual to each other. Informally, it means something like “by switching some conditions to their (intuitively obvious) duals we could prove this using essentially the same argument”, and this ad hoc approach usually suffices to reconstruct proofs as necessary. We formulate the concept precisely in Section 8.
Note that the pre-order $\preceq_0$ defined above is such that the inclusions of $X$ and $Y$ into $X \cup \preceq_0 Y$ are not only monotone but order embeddings. Note also that conditions (A2)-(A5) are necessary but not sufficient for a relation on $(X \cup Y)^2$ to be in $\mathcal{P}_R$. For example, a relation could satisfy these conditions but fail to be transitive when restricted to $X$.

**Definition 3.5 ($\mathcal{P}_R^e$).** Let $(X, Y, R)$ be an order polarity. Define $\mathcal{P}_R^e$ to be the set of pre-orders on $X \cup Y$ agreeing with $R$ on $X \times Y$, and agreeing with the orders on $X$ and $Y$.

**Corollary 3.6.** If $(X, Y, R)$ is an order polarity then $\mathcal{P}_R^e$ is non-empty if and only if $\mathcal{P}_R$ is non-empty. Moreover, $\mathcal{P}_R^e$ is also closed under arbitrary non-empty intersections.

**Proof.** The first part follows directly from the definition of $\preceq_0$ in Theorem 3.3. That $\mathcal{P}_R^e$ is closed under arbitrary non-empty intersections is obvious. □

In light of the discussion above we make the following definition.

**Definition 3.7.** A polarity $(X, Y, R)$ is 0-coherent if $\mathcal{P}_R$ (or, equivalently, $\mathcal{P}_R^e$) is non-empty. We may sometimes abuse notation slightly by referring to the relation $R$ as being 0-coherent.

**Definition 3.8 ($X \uplus \preceq Y$).** Given an order polarity $(X, Y, R)$ and $\preceq \in \mathcal{P}_R$, we use $X \uplus \preceq Y$ to denote the canonical partial order arising from $X \cup \preceq Y$.

### 3.2. Extension polarities.

Suppose in addition that $X$ and $Y$ are both extensions of some poset $P$. In other words, that there are order embeddings $e_1 : P \to X$ and $e_2 : P \to Y$. What conditions must $R$ satisfy in order for there to be $\preceq \in \mathcal{P}_R$ such that the diagram in Figure 2 commutes? Note that in this figure $i_X$ and $i_Y$ stand for the compositions of the natural inclusion functions into $X \cup \preceq Y$ with the canonical map from $X \cup \preceq Y$ to $X \uplus \preceq Y$.

![Figure 2](image)

As this situation will be the focus of most of the rest of the document, we make the following definition.

**Definition 3.9.** An extension polarity is a triple $(e_X, e_Y, R)$, where $e_X : P \to X$ and $e_Y : P \to Y$ are order extensions of the same poset $P$, and $(X, Y, R)$ is an order polarity. When both $e_X$ and $e_Y$ are completions, we say $(e_X, e_Y, R)$ is a completion polarity. We sometimes say an extension polarity of form $(e_X, e_Y, R)$ extends $P$. The concept of 0-coherence from Definition 3.7 also applies, *mutatis mutandis*, to extension polarities.

Note that an order polarity is an extension polarity in the special case where $P$ is empty.
Definition 3.10 (\(\hat{\mathcal{P}}_R\)). Let \((e_X,e_Y,R)\) be an extension polarity. Define \(\hat{\mathcal{P}}_R\) to be the subset of \(\mathcal{P}_R\) containing all \(\preceq\) such that the diagram in Figure 2 commutes. We define \(\hat{\mathcal{P}}_R^c\) similarly (recalling Definition 3.5).

We will use the following lemma.

Lemma 3.11. Let \((e_X,e_Y,R)\) be an extension polarity. Suppose \((e_X,e_Y,R)\) satisfies
\[
(\hat{\tau}_0) \quad e_X(p) \triangleleft R e_Y(p) \text{ for all } p \in P.
\]
Then, if \((e_X,e_Y,R)\) satisfies (A0) from Theorem 3.3 it also satisfies \((\hat{\tau}_1)\) below. Similarly, if \((e_X,e_Y,R)\) satisfies (A1) then it also satisfies \((\hat{\tau}_2)\).

\[
(\hat{\tau}_1) \quad x \leq e_X(p) \implies x \triangleleft R e_Y(p) \text{ for all } p \in P \text{ and for all } x \in X.
\]

\[
(\hat{\tau}_2) \quad e_Y(p) \preceq y \implies e_X(p) \triangleleft R y \text{ for all } p \in P \text{ and for all } y \in Y.
\]

Moreover, if a polarity \((e_X,e_Y,R)\) satisfies either \((\hat{\tau}_1)\) or \((\hat{\tau}_2)\) then it also satisfies \((\hat{\tau}_0)\).

Proof. Suppose \((e_X,e_Y,R)\) satisfies (A0) and \((\hat{\tau}_0)\), and let \(x \leq e_X(p)\) for some \(x \in X\) and \(p \in P\). Then \(e_X(p) \triangleleft R e_Y(p)\) by \((\hat{\tau}_0)\), and so \(x \triangleleft R e_Y(p)\) by (A0). Thus \((e_X,e_Y,R)\) satisfies \((\hat{\tau}_1)\). The case where we assume (A1) and \((\hat{\tau}_0)\) to prove \((\hat{\tau}_2)\) is dual. Suppose now that \((e_X,e_Y,R)\) satisfies \((\hat{\tau}_1)\), and let \(p \in P\). Then, as \(e_X(p) \preceq e_Y(p)\), we have \(e_X(p) \triangleleft R e_Y(p)\) by \((\hat{\tau}_1)\), and thus \((e_X,e_Y,R)\) satisfies \((\hat{\tau}_0)\). The case where we assume \((\hat{\tau}_2)\) and prove \((\hat{\tau}_0)\) is again dual. \(\square\)

Theorem 3.12. Let \((e_X,e_Y,R)\) be a 0-coherent extension polarity. Then \(\hat{\mathcal{P}}_R\) is non-empty if and only if:

\[
(\hat{B}_0) \quad e_X(p) \triangleleft R e_Y(p) \text{ for all } p \in P.
\]

\[
(\hat{B}_1) \quad x \triangleleft R e_Y(p) \text{ and } e_X(p) \preceq R y \implies x \preceq R y \text{ for all } p \in P \text{ and for all } x \in X \text{ and for all } y \in Y.
\]

In addition, if \((e_X,e_Y,R)\) satisfies \((\hat{B}_0)\) and \((\hat{B}_1)\), then, given \(\preceq \in \mathcal{P}_R\) we have \(\preceq \in \hat{\mathcal{P}}_R\) if and only if it satisfies either \((\hat{B}_2)\) or \((\hat{B}_5)\), which are equivalent modulo these assumptions. In this case it also satisfies \((\hat{B}_3)\) and \((\hat{B}_4)\).

\[
(\hat{B}_2) \quad e_Y(p) \preceq e_X(p) \text{ for all } p \in P.
\]

\[
(\hat{B}_3) \quad \text{For all } x_1,x_2 \in X \text{ and for all } p \in P, \text{ if either}
\]

\[
(i) \quad x_1 \leq x_2, \text{ or}
\]

\[
(ii) \quad x_1 \triangleleft R e_Y(p) \text{ and } e_X(p) \preceq x_2,
\]

\[
\text{then } x_1 \preceq x_2.
\]

\[
(\hat{B}_4) \quad \text{For all } y_1,y_2 \in Y \text{ and for all } p \in P, \text{ if either}
\]

\[
(i) \quad y_1 \leq y_2, \text{ or}
\]

\[
(ii) \quad y_1 \preceq e_Y(p) \text{ and } e_X(p) \preceq y_2,
\]

\[
\text{then } y_1 \preceq y_2.
\]

\[
(\hat{B}_5) \quad \text{For all } x \in X \text{ and for all } y \in Y, \text{ if there are } p,q \in P \text{ with } y \preceq e_Y(p), \text{ with}
\]

\[
e_X(p) \triangleleft R e_Y(q), \text{ and with } e_X(q) \leq x, \text{ then } y \preceq x.
\]

Moreover, if \(\hat{\mathcal{P}}_R\) is non-empty then it is closed under arbitrary non-empty intersections, and has a least element \(\preceq_1\) defined by the following conditions:

\[
(\hat{B}_6) \quad \text{For all } x_1,x_2 \in X \text{ we have } x_1 \preceq_1 x_2 \iff \text{ either (i) } x_1 \leq x_2, \text{ or (ii) there is } p \in P \text{ with } x_1 \triangleleft R e_Y(p) \text{ and } e_X(p) \preceq x_2.
\]

\[
(\hat{B}_7) \quad \text{For all } y_1,y_2 \in Y \text{ we have } y_1 \preceq_1 y_2 \iff \text{ either (i) } y_1 \leq y_2, \text{ or (ii) there is } p \in P \text{ with } y_1 \preceq e_Y(p) \text{ and } e_X(p) \preceq y_2.
\]
we must proceed case by case, and each case may have several subcases.

Consider the eight relevant cases of the triples \((z, \text{automatic})\), so we show now that it is transitive. As in the proof of Theorem 3.3, we necessary one.

\(\forall \in \) (B5) then setting \(y\) with \(e\) and so \(y\) Similarly, assuming (B2) holds and that \(\in \) by transitivity. It follows that \(\leq \) satisfies (B3), and the argument for (B4) is dual.

Similarly, assuming (B2) holds and that \(\in \) if there are \(p, q \in P\) with \(y \leq e_Y(p)\), \(e_X(p)\) and \(e_Y(q)\), and with \(e_X(q) \leq x\), then

\(y \leq e_Y(p) \leq e_X(p) \leq e_Y(q) \leq x,\)

and so \(y \leq x\) by transitivity, and thus (B2) \(\implies \) (B5). Conversely, if we assume (B5) then setting \(y = e_Y(p)\) and \(x = e_X(p)\) produces (B2), and thus (B2) and (B5) are equivalent as claimed.

If \(\leq \in \mathcal{P}_R\), and \(\leq \) satisfies (B2), then, assuming (B0), the diagram in Figure 2 clearly commutes, and so \(\leq \in \mathcal{P}_R\). Thus (B2) is a sufficient condition, as well as a necessary one.

We now show that, assuming (B0) and (B1) hold, \(\leq\) as defined above is a pre-order such that the corresponding diagram commutes. That it is reflexive is automatic, so we show now that it is transitive. As in the proof of Theorem 3.3 we consider the eight relevant cases of the triples \((z_1, z_2, z_3)(X \cup Y)^3\). Unfortunately we must proceed case by case, and each case may have several subcases.

- \((x_1, x_2, x_3): \) Here \(x_1 \leq x_2,\) and \(x_2 \leq x_3\). This case breaks down into subcases, depending on the reason \(\leq \) holds for each pair.
  - If \(x_1 \leq x_2\) and \(x_2 \leq x_3\) in \(X\), then we have \(x_1 \leq x_3\), and thus \(x_1 \leq x_3\), by transitivity of \(\leq\).
  - Suppose instead that \(x_1 \leq x_2\), and that there is \(p \in P\) with \(e_X(p) \leq x_3\) and \(x_2 \R e_Y(p)\). Then \(x_1 \R e_Y(p)\) by (A0) of Theorem 3.3, and so \(x_1 \leq x_3\) by (B6).
  - Alternatively, if \(x_1 \R e_Y(p)\), \(e_X(p) \leq x_2\) and \(x_2 \leq x_3\), then \(e_X(p) \leq x_3\), and so \(x_1 \leq x_3\) by (B6).
  - Finally, suppose there are \(p, q \in P\) with \(x_1 \R e_Y(p)\), with \(e_X(p) \leq x_2\), with \(x_2 \R e_Y(q)\) and with \(e_X(q) \leq x_3\). Then \(e_X(p) \R e_Y(q)\) by (A0), and so \(x_1 \R e_Y(q)\) by (B1), and thus \(x_1 \leq x_3\) by (B6).

- \((y_1, y_2, y_3): \) This case is dual to the previous one.

- \((x_1, x_2, y): \) Here we have \(x_2 \leq y\), and thus \(x_2 \R y\). We also have \(x_1 \leq x_2\), which breaks down into two cases.
  - First suppose \(x_1 \leq x_2\). Then \(x_1 \R y\) by (A0), and so \(x_1 \leq y\) as required.
  - Suppose instead that there is \(p \in P\) with \(x_1 \R e_Y(p)\) and \(e_X(p) \leq x_2\). Then \(e_X(p) \R y\) by (A0), and so \(x_1 \R y\) by (B1), and thus \(x_1 \leq y\) as required.
• \((y, x_1, x_2)\): Here we have \(y \preceq_1 x\), and so there are \(p, q \in P\) with \(y \leq e_Y(p)\), with \(e_X(p) \preceq_Y(q)\), and with \(e_X(q) \leq x_1\), and \(x_1 \preceq_1 x_2\). There are two subcases.
  
  - Suppose first that \(x_1 \leq x_2\). Then \(e_X(q) \leq x_2\) and the result is an immediate application of (B9).
  
  - Suppose instead that there is \(r \in P\) with \(x_1 \preceq_Y(r)\) and \(e_X(r) \leq x_2\).
    Then an application of (A0) produces \(e_X(q) \preceq_Y(r)\), and using this with (B1) provides \(e_X(p) \preceq_Y(r)\). Thus we get \(y \preceq_1 x_2\) from (B9).

• \((x_1, y, x_2)\): We have \(x_1 \preceq_Y y\), and, by (B9), there are \(p, q \in P\) with \(y \leq e_Y(p)\), with \(e_X(p) \preceq_Y(q)\), and with \(e_X(q) \leq x_2\). Then (A1) gives us \(x_1 \preceq_Y e_Y(p)\), and consequently (B1) produces \(x_1 \preceq_Y e_Y(q)\). Thus \(x_1 \preceq_1 x_2\) by (B6).

• \((y_1, x, y_2)\): Dual to the previous case.

• \((x, y_1, y_2)\): Dual to the \((x_1, x_2, y)\) case.

• \((y_1, y_2, x)\): Dual to the \((y, x_1, x_2)\) case.

From the above argument we conclude that \(\preceq_1\) is transitive, and thus defines a pre-order. To complete the argument that \(\preceq_1 \in \hat{P}_R\), note first that \(\preceq_1\) obviously extends the orders on \(X\) and \(Y\), and so \(\preceq_1 \in P_R\). Finally, that \(\preceq_1\) satisfies (B2) follows easily from (B9) and the fact that \(e_Y(p) \leq e_Y(q)\), \(e_X(p) \preceq_Y e_Y(p)\) and \(e_X(q) \leq e_X(p)\). Thus \(\preceq_1 \in P_R\) by a part of this theorem proved previously.

It follows from the fact that \(\preceq_1 \in \hat{P}_R\) must satisfy conditions (B3)-(B5) that \(\preceq_1\) is the smallest element of \(\hat{P}_R\) when \(\hat{P}_R\) is non-empty. That \(\hat{P}_R\) is closed under non-empty meets is again essentially obvious.

\[\Box\]

**Definition 3.13.** An extension polarity \((e_X, e_Y, R)\) is **1-coherent** if it is 0-coherent and also satisfies conditions (B0)-(B1) of Theorem 3.12. I.e. if \(\hat{P}_R\) is non-empty.

**Corollary 3.14.** Let \((e_X, e_Y, R)\) be a 1-coherent extension polarity. Then \(\hat{P}_R\) is non-empty if and only if the following conditions are both satisfied:

- (C0) For all \(x_1, x_2 \in X\) and for all \(p \in P\), if \(x_1 \preceq_Y e_Y(p)\) and \(e_X(p) \leq x_2\), then \(x_1 \leq x_2\).
- (C1) For all \(y_1, y_2 \in Y\) and for all \(p \in P\), if \(y_1 \leq e_Y(p)\) and \(e_X(p) \preceq_Y y_2\), then \(y_1 \leq y_2\).

In this case conditions (B3) and (B4) of Theorem 3.12 are equivalent, respectively, to:

- (B3′) For all \(x_1, x_2 \in X\), if \(x_1 \leq x_2\) then \(x_1 \leq x_2\).
- (B4′) For all \(y_1, y_2 \in Y\), if \(y_1 \leq y_2\) then \(y_1 \leq y_2\).

Moreover, if \(\hat{P}_R^e\) is non-empty then it is closed under non-empty intersections, and its least element is \(\preceq_1\) as in Theorem 3.12.

**Proof.** Let \(x_1, x_2 \in X\), and let \(p \in P\) with \(x_1 \preceq_Y e_Y(p)\) and \(e_X(p) \leq x_2\). Suppose \(\preceq_1 \in \hat{P}_R^e\). Then from condition (B3) of Theorem 3.12 we see that \(x_1 \leq x_2\). Thus, since we are assuming the map induced by the inclusion of \(X\) into \(X \cup \preceq_1 Y\) is an order embedding we must have \(x_1 \leq x_2\). So (C0) is indeed a necessary condition. A similar argument shows the necessity of (C1).

Moreover, if (C0) holds then (B3) is clearly equivalent to (B3′), as the disjunction of (i) and (ii) from (B3) is then equivalent to \(x_1 \leq x_2\), and similarly if (C1) holds then the same is true for (B4) and (B4′). It is easily seen that \(\preceq_1\) is indeed an element of \(\hat{P}_R^e\) whenever \((e_X, e_Y, R)\) satisfies (C0) and (C1), and so must be
minimal, as it is minimal in $\hat{P}_R$. That $\hat{P}_R^g$ is closed under non-empty intersections is obvious.

\begin{definition}
\textbf{Definition 3.15.} An extension polarity $(e_X, e_Y, R)$ is \textbf{2-coherent} if it is 1-coherent and also satisfies conditions (C0) and (C1) of Corollary 3.14. I.e. if $\hat{P}_R^g$ is non-empty.
\end{definition}

As mentioned in the proof of Corollary 3.14 in the case of 2-coherent extension polarities, the conditions (B3), (B4), (B6) and (B7) of Theorem 3.12 simplify, as (i) ∨ (ii) and (i) are equivalent in all cases.

We will provide examples showing that the strengths of the coherence conditions defined so far are strictly increasing, but we defer this till section 5.2.

\begin{definition}[\hat{P}_R^g]
\textbf{Definition 3.16 (\hat{P}_R^g).} Let $(e_X, e_Y, R)$ be an extension polarity. Define $\hat{P}_R^g$ to be the subset of $\hat{P}_R^g$ such that for all $\preceq \in \hat{P}_R^g$ the following both hold:

1. The induced map $\iota_X : X \to X \uplus Y$ has the property that, for all $S \subseteq P$, if $\bigwedge e_X[S]$ is defined in $X$ then $\iota_X(\bigwedge e_X[S]) = \bigwedge (\iota_X \circ e_X)[S]$.
2. The induced map $\iota_Y : Y \to Y \uplus X$ has the property that, for all $T \subseteq P$, if $\bigvee e_Y[T]$ is defined in $Y$ then $\iota_Y(\bigvee e_Y[T]) = \bigvee (\iota_Y \circ e_Y)[T]$.

\end{definition}

\begin{theorem}
\textbf{Theorem 3.17.} Let $(e_X, e_Y, R)$ be a 2-coherent extension polarity. Then $\hat{P}_R^g$ is non-empty if and only if $(e_X, e_Y, R)$ satisfies the following conditions:

(D0) For all $x \in X$ and $y_1, y_2 \in Y$, and for all $S \subseteq P$ with $\bigwedge e_X[S] = x$, if $x R y_2$ and $y_1 \leq e_Y(p)$ for all $p \in S$, then $y_1 \leq y_2$.

(D1) For all $x_1, x_2 \in X$ and $y \in Y$, and for all $T \subseteq P$ with $y = \bigvee e_Y[T]$, if $x_1 R y$ and $e_X(q) \leq x_2$ for all $q \in T$, then $x_1 \leq x_2$.

Moreover, if $(e_X, e_Y, R)$ satisfies (D0) and (D1), then, given $\preceq \in \hat{P}_R^g$ we have $\preceq \in \hat{P}_R^g$ if and only if it satisfies conditions (D2) and (D3) below.

(D2) For all $y \in Y$, and for all $S \subseteq P$ with $\bigwedge e_X[S]$ defined in $X$, we have $\left(y \leq e_Y(p) \text{ for all } p \in S\right) \implies y \leq \bigwedge e_X[S]$.

(D3) For all $x \in X$, and for all $T \subseteq P$ with $\bigvee e_Y[T]$ defined in $Y$, we have $\left(e_X(q) \leq x \text{ for all } q \in T\right) \implies \bigvee e_Y[T] \geq x$.

$\hat{P}_R^g$ is closed under non-empty intersections, and, when non-empty, has a least element $\preceq_3$ defined by:

(D4) For all $x_1, x_2 \in X$ we have $x_1 \preceq_3 x_2 \iff x_1 \leq x_2$.

(D5) For all $y_1, y_2 \in Y$ we have $y_1 \preceq_3 y_2 \iff y_1 \leq y_2$.

(D6) For all $x \in X$ and $y \in Y$ we have $x \preceq_3 y \iff x R y$.

(D7) For all $x \in X$ and for all $y \in Y$ we have $y \preceq_3 x \iff$ either

1. there is $S \subseteq P$ with $\bigwedge e_X[S]$ defined in $X$, $\bigwedge e_X[S] \leq x$ and $y \leq e_Y(p)$ for all $p \in S$, or
2. there is $T \subseteq P$ with $\bigvee e_Y[T]$ defined in $Y$, $\bigvee e_Y[T] \geq y$ and $e_X(q) \leq x$ for all $q \in T$.

\end{theorem}

\begin{proof}
Proof. If (D1) does not hold for some $x_1, x_2 \in X$, then for any $\preceq \in \hat{P}_R^g$ we would have $x_1 \leq x_2$, but not $x_1 \leq x_2$, which would contradict the definition of $\hat{P}_R^g$. Thus (D1) is indeed a necessary condition for $\hat{P}_R^g$ to be non-empty. (D0) is necessary by duality.

Suppose now that $\hat{P}_R^g$ is not empty, and let $\preceq \in \hat{P}_R^g$. To prove the necessity of (D2), let $S \subseteq P$ and suppose $\bigwedge e_X[S]$ exists. Suppose that $y \leq e_Y(p)$ for all
Then, by the assumption that \( \preceq \in \hat{\mathcal{P}}_R \) it follows that \( y \preceq e_X(p) \) for all \( p \in S \), and thus that \( y \) is a lower bound for \( e_X[S] \). By Definition 3.16, the map \( \iota_X : X \to X \uplus Y \) preserves meets, so we must have \( y \preceq \bigwedge e_X[S] \) as claimed. Duality proves the necessity of (D3).

Conversely, suppose \( (e_X, e_Y, R) \) satisfies (D0) and (D1), let \( \preceq \in \hat{\mathcal{P}}_R \) and suppose \( \preceq \) satisfies (D2) and (D3). Let \( S \subseteq P \) and suppose \( \bigwedge e_X[S] \) exists in \( X \). If \( x \in X \) and \( \iota_X(x) \preceq \iota_X \circ e_X(p) \) for all \( p \in S \), then \( x \preceq \bigwedge e_X[S] \), as \( \iota_X \) is an order embedding (because \( \preceq \in \hat{\mathcal{P}}_R \)), and so \( \iota_X(x) \preceq \iota_X \big( \bigwedge e_X[S] \big) \). Moreover, if \( y \in Y \) and \( \iota_Y(y) \preceq \iota_Y \circ e_Y(p) \) for all \( p \in S \), then \( y \preceq e_Y(p) \) for all \( p \in S \), and so \( \iota_Y(y) \preceq \iota_X \big( \bigwedge e_X[S] \big) \), by (D2). From this and the dual result we see that \( \preceq \in \hat{\mathcal{P}}_R \) as claimed.

We must now show that \( \preceq_3 \) as defined here induces a pre-order \( X \cup_\preceq Y \) such that the diagram in Figure 2 commutes, and the maps \( \iota_X \) and \( \iota_Y \) are order embeddings with the required preservation properties. First of all, it’s obvious that \( \preceq_3 \) is reflexive and that the \( \iota \) maps are order embeddings. That the diagram commutes follows by using (D6) to get \( e_X(p) \preceq_3 e_Y(p) \), and using (D7) with \( S = \{ p \} \) and \( x = e_X(p) \) to give \( e_Y(p) \preceq_3 e_X(p) \). Moreover, that \( \preceq_3 \) satisfies (D2) and (D3) is automatic from the definition.

The main work now is showing that \( \preceq_3 \) is transitive. Again this breaks down into eight cases of form \((z_1, z_2, z_3)\). The cases where a \( y \) value does not appear before an \( x \) value are covered by the proof of Theorem 3.12 (noting the result of Corollary 3.13), so the proofs need not be repeated. There are four remaining cases.

- \((y, x_1, x_2)\): We have \( x_1 \preceq x_2 \), and two subcases.
  - Suppose there is \( S \subseteq P \) with \( \bigwedge e_X[S] \preceq x_1 \) and \( y \preceq e_Y(p) \) for all \( p \in S \).
    Then, since \( x_1 \preceq x_2 \) we have \( y \preceq_3 x_2 \) by (D7)(1).
  - Suppose instead that there is \( T \subseteq P \) with \( \bigvee e_Y[T] \preceq y \) and \( e_X(q) \preceq x_1 \) for all \( q \in T \). Then, as \( x_1 \preceq x_2 \) we have \( e_Y(q) \preceq x_2 \) for all \( t \in T \), and so we have \( y \preceq_3 x_2 \) by (D7)(2).
- \((y_1, y_2, x)\): Dual to the previous case.
- \((x_1, y, x_2)\): We have \( x_1 R y \) and two subcases.
  - Suppose there is \( S \subseteq P \) with \( \bigwedge e_X[S] \preceq x_2 \) and \( y \preceq e_Y(p) \) for all \( p \in S \). Then, given \( p \in S \) we have \( x_1 R e_Y(p) \) by (A1) of Theorem 3.3.
    It then follows from (C0) of Corollary 3.14 that \( x_1 \preceq e_X(p) \), and so \( x_1 \leq \bigwedge e_X[S] \preceq x_2 \) as required.
  - Suppose instead that there is \( T \subseteq P \) with \( y \preceq \bigvee e_Y[T] \) and \( e_X(q) \leq x_2 \) for all \( q \in T \). Then we have \( x_1 R \bigvee e_Y[T] \) by (A1), and so \( x_1 \preceq x_2 \) by (D1).
- Dual to the previous case.

Thus \( \preceq_3 \) is indeed a pre-order, and so is in \( \hat{\mathcal{P}}_R^q \) as claimed. Finally, that \( \hat{\mathcal{P}}_R^q \) is closed under non-empty intersections is again obvious.

\[ \square \]

**Definition 3.18.** An extension polarity \((X, Y, R)\) is 3-coherent if it is 2-coherent and also satisfies conditions (D0) and (D1) of Theorem 3.17. I.e. \( \hat{\mathcal{P}}_R^q \) is non-empty.

Note that, when \((e_X, e_Y, R)\) is 2-coherent, given \( x \in X \) and \( y \in Y \), and given \( p, q \in P \) such that

1. \( y \preceq e_Y(p) \),
2. \( e_X(p) R e_Y(q) \), and
3. \( e_X(q) \leq x \),
Galois polarities have several strong properties, as we shall see.

and \((A1')\) both hold. Let \(x_1 \not\leq x_2 \in X\). Then, by entanglement, there is \(y \in Y\) with \((x_2, y) \in R\) and \((x_1, y) \not\in R\). So we cannot have \(x_1 \not\leq x_2\), as otherwise transitivity would produce \(x_1 \leq y\), and consequently \(x_1 R y\). So \(x_1 \not\leq x_2\), and thus \(\hat{P}_R^e = \hat{P}_R\) as required. That \(\hat{P}_R^e = \hat{P}_R\) also follows from this argument.

4.2. Defining Galois polarities.

Definition 4.4. A Galois polarity is a 3-coherent extension polarity \((e_X, e_Y, R)\) such that \(e_X : P \rightarrow X\) is a meet-extension, and \(e_Y : P \rightarrow Y\) is a join-extension.

The motivation for the name Galois polarity will become clear in section 7.1. Galois polarities have several strong properties, as we shall see.
Lemma 4.5. Galois polarities are entangled.

Proof. Let \((e_X, e_Y, R)\) be a Galois polarity, and let \(x_1 \not\leq x_2 \in X\). Then, as \(e_X\) is a meet-extension there is \(p \in P\) with \(x_1 \not\leq e_X(p)\), and \(x_2 \leq e_X(p)\). Thus \(x_2 \leq e_Y(p)\) by \((\circ_1)\) of Lemma 4.11. Moreover, if \(x_1 \leq e_Y(p)\) then \(x_1 \leq e_X(p)\) by \((C0)\) from Corollary 4.13 which contradicts the choice of \(p\). We conclude that \((E0)\) holds. A dual argument works for \((E1)\).

\[\square\]

Corollary 4.6. If \((e_X, e_Y, R)\) is a Galois polarity then \(\hat{\mathcal{P}}_R^{g} = \hat{\mathcal{P}}_R\).

Proof. This follows immediately from Lemma 4.5 and Corollary 4.3.

For Galois polarities, the structure of \(\hat{\mathcal{P}}_R^{g}\) is trivial, as we show in Theorem 4.8.

First, the following technical lemma will be useful.

Lemma 4.7. If \((e_X, e_Y, R)\) is 3-coherent then \(a)\) and \(b)\) below both imply \(c)\) for all \(x \in X\) and for all \(y \in Y\). Moreover, if \((e_X, e_Y, R)\) is Galois then \(a)\), \(b)\) and \(c)\) are all equivalent for \(x\) and \(y\).

\(a)\) There is \(S \subseteq P\) with \(\bigwedge e_X[S] \leq x \) and \(y \leq e_Y(p)\) for all \(p \in S\).
\(b)\) There is \(T \subseteq P\) with \(\bigvee e_Y[T] \geq y \) and \(e_X(q) \leq x\) for all \(q \in T\).
\(c)\) For all \(p, q \in P\), if \(e_Y(p) \leq y\) and \(x \leq e_X(q)\), then \(p \leq q\).

Proof. As \((e_X, e_Y, R)\) is 3-coherent, \(\hat{\mathcal{P}}_R^{g}\) is non-empty, so let \(\leq \in \hat{\mathcal{P}}_R^{g}\). Suppose first that \(a)\) holds for \(x\) and \(y\), and let \(p, q \in P\) with \(e_Y(p) \leq y\) and \(x \leq e_X(q)\). Then we have

\[e_Y(p) \leq y \leq \bigwedge e_X[S] \leq x \leq e_X(q)\]

for some \(S \subseteq P\), by appealing to \((D2)\) from Theorem 3.17. By commutativity of the diagram in Figure 2, we must therefore have \(e_X(p) \leq e_X(q)\), and so \(p \leq q\). This shows \(a) \implies c)\), and a dual argument shows \(b) \implies c)\).

Suppose now that \((e_X, e_Y, R)\) is Galois and that \(c)\) holds for \(x\) and \(y\). As \((e_X, e_Y, R)\) is Galois we have \(x = \bigwedge e_X[S]\) for \(S = e_X^{-1}(x')\), and \(y = \bigvee e_Y[T]\) for \(T = e_Y^{-1}(y')\). By \(c)\) we have \(q \leq p\) for all \(q \in T\) and \(p \in S\). Given \(\leq \in \hat{\mathcal{P}}_R^{g}\) we thus have \(e_X(q) \leq e_Y(p)\), and as \(\iota_X\) and \(\iota_Y\) are meet- and join-preserving respectively, we must have \(y = \bigvee e_Y[T] \leq \bigwedge e_X[S] = x\), and thus \(y \leq e_Y(p)\) for all \(p \in S\), and \(e_X(q) \leq x\) for all \(q \in T\). It follows that \(c)\) implies both \(a)\) and \(b)\), and so we have the claimed equivalence.

\[\square\]

Theorem 4.8. If \((e_X, e_Y, R)\) is a Galois polarity then \(\hat{\mathcal{P}}_R^{g}\) contains only the element \(\leq_3\) defined in Theorem 3.17 and the maps \(\iota_X : X \to X \uparrow \leq_3 Y\) and \(\iota_Y : Y \to X \downarrow \leq_3 Y\) are completely meet- and join-preserving respectively. Moreover, in this case \(\leq_3\) can be defined as follows:

\(G0)\) For all \(x_1, x_2 \in X\) we have \(x_1 \leq_3 x_2 \iff x_1 \leq x_2 \) in \(X\).
\(G1)\) For all \(y_1, y_2 \in Y\) we have \(y_1 \leq_3 y_2 \iff y_1 \leq y_2 \) in \(Y\).
\(G2)\) For all \(x \in X\) and \(y \in Y\) we have \(x \leq_3 y \iff x \leq y \) in \(X\).
\(G3)\) For all \(x \in X\) and \(y \in Y\) we have \(y \leq_3 x \iff p \leq q \) for all \(p \in e_Y^{-1}(y')\) and \(q \in e_X^{-1}(x')\).

Proof. We will start by proving that the alternative definition of \(\leq_3\) is correct. Since \((G0)\), \((G1)\) and \((G2)\) are identical to \((D4)\), \((D5)\) and \((D6)\) respectively, and that \((G3)\) is equivalent to \((D7)\) follows immediately from Lemma 4.7.
That $\iota_X$ and $\iota_Y$ are completely meet- and join-preserving is a simple consequence of the definition of $\hat{P}_R^g$ (Definition 3.16) and the fact that $x = \bigwedge e_X(e_X^{-1}(x^+))$ and $y = \bigvee e_Y(e_Y^{-1}(y^+))$ for all $x \in X$ and $y \in Y$.

To see that $\preceq_3$ is the only element of $\hat{P}_R^g$, note first that it must be the smallest element, by definition. Moreover, if $\preceq \in \hat{P}_R^g$, then $\preceq$ is determined either by the orders on $X$ and $Y$, or by $R$, everywhere except on $Y \times X$. So $\preceq \neq \preceq_3$ if and only if there is $x \in X$ and $y \in Y$ with $y \preceq x$ and $y \npreceq x$. But this is impossible, as for any $p, q \in P$ with $e_Y(p) \leq y$ and $x \leq e_X(q)$ we are forced to have $p \leq q$ by the transitivity of $\preceq$ and the commutativity of the diagram in Figure 2.

$\square$

Given a 0-coherent extension polarity $E = (e_X, e_Y, R)$ where $e_X$ and $e_Y$ are meet- and join-extensions respectively, there is a simple necessary and sufficient condition for $E$ to be Galois.

**Proposition 4.9.** Let $(e_X, e_Y, R)$ be 0-coherent and let $e_X$ and $e_Y$ be, respectively, meet- and join-extensions of $P$. Then $(e_X, e_Y, R)$ is Galois if and only if the following both hold:

1. $(S0)$ For all $p \in P$ and for all $x \in X$ we have $x \leq e_X(p) \iff x R e_Y(p)$.
2. $(S1)$ For all $p \in P$ and for all $y \in Y$ we have $e_Y(p) \leq y \iff e_X(p) R y$.

**Proof.** Suppose first that $(e_X, e_Y, R)$ is Galois, and let $p \in P$ and $x \in X$. Suppose $x \leq e_X(p)$. Then $x R e_Y(p)$ by Lemma 3.11. Conversely, if $x R e_Y(p)$ then $x \leq e_X(p)$ by (C0) of Corollary 3.14. Thus (S0) holds, and (S1) holds by a dual argument.

Suppose now that $(e_X, e_Y, R)$ is 0-coherent and satisfies (S0) and (S1), and also that $e_X$ and $e_Y$ are meet- and join-extensions respectively. We will show that the necessary conditions from Theorems 3.12 and 3.14 and Corollary 3.14 are satisfied.

1. $(B0)$: This is trivial.
2. $(B1)$: This follows from (S0) and (A0) from Theorem 3.13.
3. $(C0)$: Let $x_1 R e_Y(p)$. Then $x_1 \leq e_X(p)$ by (S0), and so if $e_X(p) \leq x_2$ then $x_1 \leq x_2$ by transitivity of $\preceq$.
4. $(C1)$: This is dual to (C0).
5. $(D0)$: Let $\bigwedge e_X(S) = x$, let $x R y_2$, and suppose $y_1 \leq e_Y(p)$ for all $p \in S$. Let $q \in P$ and suppose $e_Y(q) \leq y_1$. Then $q \leq p$ for all $p \in S$, and so $e_X(q) \leq x$. Thus $e_X(q) R y_2$ by (A0), and so $e_Y(q) \leq y_2$ by (S1). As $e_Y$ is a join-extension it follows that $y_1 \leq y_2$ as required.
6. $(D1)$: This is dual to (D0).

$\square$

It follows from Proposition 4.9 that what we call a Galois polarity corresponds to what [13 Section 4] calls a $\Delta_1$-polarity. See also [13 Proposition 4.1], which tells us that the pre-order $\preceq_3$ described in Theorem 4.8 is the one arising naturally from $G(X, Y, R)$. Theorem 4.8 says that this is in fact the only way we can pre-order $X \cup Y$ if we want the properties defining $\hat{P}_R^g$ to hold. Note of course that if $(e_X, e_Y, R)$ is not Galois then $\preceq_3$ may not be a pre-order.

Since for Galois polarities $\hat{P}_R^g$ has only the one member, to lighten the notation we will from now on write e.g. $X \bowtie Y$ in place of $X \bowtie_{\preceq_3} Y$ when working with Galois polarities.

5. The satisfaction and separation of the coherence conditions
5.1. **Sets of coherent relations.** If $X$ and $Y$ are posets, it’s easy to see that the set of relations on $X \times Y$ such that the induced order polarity is 0-coherent is closed under arbitrary unions and intersections, and has $\emptyset$ and $X \times Y$ as least and greatest elements respectively. The situation for extension polarities and more restrictive forms of coherence is a little more delicate, as illustrated by Proposition 5.2 below. First we introduce another definition.

**Definition 5.1 ($R^{(e_X,e_Y)}_*$).** Let $e_X : P \to X$ and $e_Y : P \to Y$ be poset extensions. For $* \in \{0,1,2,3\}$, define $R^{(e_X,e_Y)}_*$ to be the set of relations on $X \times Y$ such that $R \in R^{(e_X,e_Y)}_*$ is $*$-coherent.

**Proposition 5.2.** Let $e_X : P \to X$ and $e_Y : P \to Y$ be poset extensions. Then $R^{(e_X,e_Y)}_*$ is closed under arbitrary non-empty intersections for all $* \in \{0,1,2,3\}$.

Moreover, define the relation $R_I$ by

$$x R_I y \iff e_X^{-1}(x^I) \cap e_Y^{-1}(y^I) \neq \emptyset.$$  

Then $R_I$ is the minimal element of $R^{(e_X,e_Y)}_*$ for $* \in \{0,1,2\}$. If $e_X$ and $e_Y$ are meet- and join-extensions respectively, then the same is true for $* = 3$.

**Proof.** We dealt with the case where $* = 0$ in the preamble to this section. So, let $* \in \{1,2,3\}$, let $I$ be an indexing set, and, for all $i \in I$, let $R_i$ be a relation on $X \times Y$ such that $(e_X,e_Y,R_i)$ is $*$-coherent. To show that $(e_X,e_Y,\bigcap_i R_i)$ is also $*$-coherent involves only a routine check of the relevant conditions from Theorems 3.12 and 3.17 and Corollary 3.14.

If $R$ is a relation such that $(e_X,e_Y,R)$ is 1-coherent, then $R$ must satisfy (A0), (A1) and (B0), and it follows that $R_I \subseteq R$. We should show that $R_I$ indeed produces a 2-coherent polarity $(e_X,e_Y,R_I)$ for every choice of $e_X$ and $e_Y$, but this again is a routine check of the relevant conditions, so we omit the details.

Finally, suppose that $e_X$ is a meet-extension and $e_Y$ is a join-extension. We will check that $R_I$ also satisfies (D0). Let $S \subseteq P$, and let $x = \bigwedge e_X[S]$ in $X$. Let $y_1,y_2 \in Y$ and suppose that $y_1 \leq e_Y(p)$ for all $p \in S$, and that $x R_I y_2$. Let $q \in P$ and suppose $e_Y(q) \leq y_1$. Then $e_Y(q) \leq e_Y(p)$, and thus $q \leq p$, for all $p \in S$. It follows that $e_X(q) \leq x$. Also, by definition of $R_I$, there is $q' \in P$ with $x \leq e_X(q')$ and $e_Y(q') \leq y_2$. But then $q \leq q'$, and consequently $e_Y(q) \leq y_2$. This is true for all $q \in e_X^{-1}(y_1^I)$, and so $y_1 \leq y_2^I$ as $e_Y$ is a join-extension. $R_I$ also satisfies (D1) by duality, and so the proof is complete. □

Note that when $e_Y$ is not a join-extension $R_I$ may not satisfy (D0), as Example 5.3 demonstrates. In this case $R^{(e_X,e_Y)}_3$ is empty, as every $R$ such that $(e_X,e_Y,R)$ is 3-coherent must contain $R_I$, by the proof of Proposition 5.2. By duality, when $e_X$ is not a meet-extension $R_I$ may not satisfy (D1), and in this case too $R^{(e_X,e_Y)}_3$ will be empty, for the same reason.

**Example 5.3.** Let $P$ be the poset in Figure 3 and let $e_X$ and $e_Y$ be the extensions defined in Figures 2 and 3 respectively. Note that $e_X$ is a meet-extension, but $e_Y$ is not a join-extension. Let $S = \{p,q\}$. Then $x = \bigwedge e_X[S]$, and $x R_I e_Y(r)$. But we also have $y \leq e_Y(p)$ and $y \leq e_Y(q)$, but $y \not\leq e_Y(r)$. So (D0) does not hold for $R_I$. 


5.2. A strict hierarchy for coherence. Example 5.3 taken with Proposition 5.2 also demonstrates that it is possible for an order polarity to be 2-coherent but not 3-coherent (take \((e_X, e_Y, R_l)\) from this example). Thus 3-coherence is a strictly stronger condition than 2-coherence. However, this example only applies when either \(e_Y\) fails to be a join-extension, or, by duality, when \(e_X\) fails to be a meet-extension. Example 5.4 below demonstrates that, even when \(e_X\) and \(e_Y\) are meet- and join-extensions respectively, there may be choices of \(R\) for which \((e_X, e_Y, R)\) is 2-coherent but not 3-coherent.

Example 5.4. Let \(e_X\) and \(e_Y\) be as in Figures 6 and 7 respectively, where the embedded images of elements of \(P\) are represented using \(\bullet\), and the extra elements of \(X\) and \(Y\) using \(\circ\). Then it’s easy to see that \(e_X\) and \(e_Y\) are meet- and join-extensions respectively. Moreover, if we define \(R = R_l \cup \{(x, y_2)\}\) then \((e_X, e_Y, R)\) is 2-coherent, as can be observed by noting the pre-order on \(X \cup Y\) defined in Figure 8. However, \((e_X, e_Y, R)\) is not 3-coherent, as any \(\preceq \in \hat{P}^e_R\) preserving the meet that defines \(x\) would necessarily have \(y_1 \preceq y_2\), which would contradict the definition of \(\hat{P}^e_R\).

2-coherence is also a strictly stronger condition than 1-coherence, as witnessed by Example 5.5 below.
Example 5.5. Let $P$ be the two element antichain $\{p, q\}$. Define $X \cong Y \cong P$, and let $e_X$ and $e_Y$ be isomorphisms. Define $R = R_l \cup \{(e_X(p), e_Y(q))\}$. Then $(e_X, e_Y, R)$ is 1-coherent, but not 2-coherent.

5.3. Separating the classes of pre-orders. We have seen that the classes of extension polarities defined by the coherence conditions are strictly separated. It is also true that, even for a Galois polarity $(e_X, e_Y, R)$ we may have $\hat{\mathcal{P}}^g_{\text{R}} \subset \hat{\mathcal{P}}^e_{\text{R}}$, and for a 3-coherent $(e_X, e_Y, R)$ we may have $\hat{\mathcal{P}}^g_{\text{R}} \subset \hat{\mathcal{P}}^e_{\text{R}} \subset \hat{\mathcal{P}}_{\text{R}}$ (from Corollary 4.3 we know this is not true for Galois polarities). This is demonstrated in Examples 5.6 and 5.7 respectively.

Example 5.6. Let $e_X$ and $e_Y$ be as in Figures 9 and 10 respectively. Let $R = R_l$. Then $(e_X, e_Y, R)$ is Galois, by Proposition 5.2 but the pre-order represented in Figure 11 is in $\hat{\mathcal{P}}^e_{\text{R}}$, but is not in $\hat{\mathcal{P}}^g_{\text{R}}$.

Example 5.7. Let $P$ and $e_X$ be as in Example 5.6 and let $e_Y$ be defined by the diagram in Figure 12. Again let $R = R_l$. Then $(e_X, e_Y, R)$ is 3-coherent by Proposition 5.2. However, there is a pre-order in $\hat{\mathcal{P}}^e_{\text{R}} \setminus \hat{\mathcal{P}}^g_{\text{R}}$ based on that in Figure 11, and a pre-order in $\hat{\mathcal{P}}_{\text{R}} \setminus \hat{\mathcal{P}}^e_{\text{R}}$ obtained by additionally setting $z_2 \preceq z_1$.

6. Extending and restricting polarity relations

6.1. Extension. If $e : P \to Q$ is an order extension, then given another order extension $e' : Q \to Q'$, the composition $e' \circ e$ is also an order extension. It is natural to ask whether an extension polarity $(e_X, e_Y, R)$ can be extended to something like $(e'_X \circ e_X, e'_Y \circ e_Y, R')$, and under what circumstances the level of coherence of $R$ transfers to $R'$. This is of particular interest, for example, if we wish to extend $e_X$ and $e_Y$ to completions, as we shall do in Section 7.1. The next theorem provides some answers.
Theorem 6.1. Let \((e_X, e_Y, R)\) be an extension polarity, let \(i_X : X \to \overline{X}\) and \(i_Y : Y \to \overline{Y}\) be order extensions. Let \(\overline{R}\) be the relation on \(\overline{X} \times \overline{Y}\) defined by
\[x' \overline{R} y' \iff \text{there is } x \in X \text{ and } y \in Y \text{ with } x' \leq i_X(x), i_Y(y) \leq y', \text{ and } x \mathbin{R} y.\]
Then:

1. \((\overline{X}, \overline{Y}, \overline{R})\) is 0-coherent.
2. For all \(x \in X\) and for all \(y \in Y\) we have \(x \mathbin{R} y \implies i_X(x) \mathbin{\overline{R}} i_Y(y)\), and the converse is true if and only if \((X, Y, R)\) is 0-coherent.
3. If \((e_X, e_Y, R)\) is *-coherent then \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) is *-coherent, for \(* \in \{1, 2\}\).
4. If \((e_X, e_Y, R)\) is Galois, and if \(i_X : X \to \overline{X}\) and \(i_Y : Y \to \overline{Y}\) are meet- and join-extensions respectively, then \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) is also Galois.
5. Let \(S \subseteq \overline{X} \times \overline{Y}\) satisfy (A0) and (A1), and suppose \(x \mathbin{R} y \implies i_X(x) S i_Y(y)\). Then \(\overline{R} \subseteq S\).
6. If \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) is not *-coherent, then there is no \(S \subseteq \overline{X} \times \overline{Y}\) satisfying the conditions from part (5) such that \((i_X \circ e_X, i_Y \circ e_Y, S)\) is *-coherent, for \(* \in \{2, 3\}\).

Proof:
(1) We check that \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) is 0-coherent using Theorem 3.3. We need only check (A0) as (A1) is dual. Let \(x'_1 \leq x'_2 \in \overline{X}\), let \(y' \in \overline{Y}\), and suppose \(x'_2 \overline{R} y'\). Then there is \(x \in X\) and \(y \in Y\) with \(x'_2 \leq i_X(x)\), with \(i_Y(y) \leq y'\), and with \(x \mathbin{R} y\). But then \(x'_1 \overline{R} y'\), by definition of \(\overline{R}\), so (A0) holds.
(2) If \(x \mathbin{R} y\) then that \(i_X(x) \overline{R} i_Y(y)\) follows directly from the definition. Conversely, suppose \((e_X, e_Y, R)\) is 0-coherent and \(i_X(x_1) \overline{R} i_Y(y_1)\). Then there is \(x_2 \in X\) and \(y_2 \in Y\) with \(x_1 \leq x_2\), with \(x_2 \mathbin{R} y_2\), and with \(y_2 \leq y_1\). It follows from 0-coherence of \((e_X, e_Y, R)\) that \(x_1 \mathbin{R} y_1\) as required. Moreover, \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) is always 0-coherent by (1), so, if the converse holds \((e_X, e_Y, R)\) inherits 0-coherence from \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\).
(3) Now suppose \((e_X, e_Y, R)\) is 1-coherent. Appealing to Theorem 3.12 we check that (B0) and (B1) hold for \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\).
(B0): Let \(p \in P\). Then \(e_X(p) R e_Y(p)\) as \((e_X, e_Y, R)\) is 1-coherent, and it follows easily that \(i_X \circ e_X(p) \overline{R} i_Y \circ e_Y(p)\). Thus (B0) holds for \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) as required.
(B1): Let \(x' \in \overline{X}\), let \(y' \in \overline{Y}\), and let \(p \in P\). Suppose \(x' \overline{R} (i_Y \circ e_Y(p))\) and \((i_X \circ e_X(p)) \overline{R} y'\). Then there are \(x_1 \in X\) and \(y_1 \in Y\), with \(x' \leq i_X(x_1)\), with \(x_1 \mathbin{R} y_1\), and with \(i_Y(y_1) \leq i_Y \circ e_Y(p)\), and also \(x_2 \in X\) and \(y_2 \in Y\) with \(i_X \circ e_X(p) \leq i_X(x_2)\), with \(x_2 \mathbin{R} y_2\), and with \(i_Y(y_2) \leq y'\). As \(i_X\) and \(i_Y\) are order embeddings we have \(y_1 \leq e_Y(p)\) and \(e_X(p) \leq x_2\). As \((e_X, e_Y, R)\) is 1-coherent we have \(\leq_1 \in P_{\overline{R}}\), and thus
\[x_1 \leq_1 y_1 \leq_1 e_Y(p) \leq_1 e_X(p) \leq_1 x_2 \leq_1 y_2.\]
So \(x_1 \mathbin{R} y_2\) by transitivity of \(\leq_1\) and the fact that it agrees with \(R\) on \(X \times Y\). It follows that \(x' \overline{R} y'\), by the definition of \(\overline{R}\), and so (B1) holds.
Thus \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) is 1-coherent. Suppose now that \((e_X, e_Y, R)\) is 2-coherent. Appealing to Corollary 3.13, we check that (C0) holds for \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\). Let \(x'_1, x'_2 \in \overline{X}\), and let \(p \in P\). Suppose \(x'_1 \overline{R} (i_Y \circ e_Y(p))\),
and \( i_X \circ e_X(p) \leq x'_2 \). Then there are \( x \in X \) and \( y \in Y \) with \( x'_1 \leq i_X(x) \), with \( i_Y(y) \leq i_Y \circ e_Y(p) \), and with \( x \ R \ y \). As \( (e_X,e_Y,R) \) is 2-coherent we know \( \preceq_1 \in \mathcal{P}_R^E \), by Corollary 3.14 and we have
\[
x \preceq_1 y \preceq_1 e_Y(p) \preceq_1 e_X(p).
\]
So \( x \leq e_X(p) \), by definition of \( \mathcal{P}_R^E \), and consequently
\[
x'_1 \leq i_X(x) \leq i_X \circ e_X(p).
\]
Thus \( x'_1 \leq x'_2 \), as \( i_X \circ e_X(p) \leq x'_2 \), and so \((C0)\) holds. By duality \((C1)\) also holds, and so \((i_X \circ e_X, i_Y \circ e_Y, R)\) is 2-coherent as claimed.

(4) Suppose now that \( (e_X,e_Y,R) \) is 3-coherent, and that the \( i_X \) and \( i_Y \) are meet- and join-extensions respectively. First, that \( i_X \circ e_X \) and \( i_Y \circ e_Y \) are meet- and join-extensions respectively follows from the corresponding properties of \( i_X, e_X, i_Y \) and \( e_Y \). It remains only to check that \((D0)\) and \((D1)\) hold for \((i_X \circ e_X, i_Y \circ e_Y, R)\) and appeal to Theorem 3.17.

Let \( x' \in X \), let \( y'_1, y'_2 \in Y \), and let \( S \subseteq P \). Suppose \( \bigwedge (i_X \circ e_X[S]) = x' \).
Suppose also that \( x' \ R \ y'_2 \), and that \( y'_1 \leq i_Y \circ e_Y(p) \) for all \( p \in S \). Then there are \( x \in X \) and \( y \in Y \) with \( x' \leq i_X(x) \) and \( i_Y(y) \leq y'_2 \), and with \( x \ R \ y \).
We aim to prove that \( y'_1 \leq y'_2 \).

Let \( y_0 \in Y \) be such that \( i_Y(y_0) \leq y'_1 \), and let \( p \in e_Y^{-1}(y'_0) \). Then
\[
e_Y(q) \leq y_0 \leq e_Y(p) \text{ for all } p \in S,
\]
and so \( i_X \circ e_X(q) \leq x' \leq i_X(x) \), and consequently \( e_X(q) \leq x \). Since \((e_X,e_Y,R)\) is Galois, \( \mathcal{P}_R^E \) contains \( \preceq_3 \), and we have \( y_0 \preceq_3 x \) as the map \( i_Y : Y \rightarrow X \ \preceq_3 \ Y \) preserves joins of sets in \( e_Y[P] \) and \( y_0 = \bigvee e_Y[y_0^{-1}(y'_0)] \).
So we have
\[
y_0 \preceq_3 x \preceq_3 y,
\]
and thus \( y_0 \leq y \) for all \( y_0 \) with \( i_Y(y_0) \leq y'_1 \). But, as \( i_Y \) is a join-extension, we have
\[
y'_1 = \bigvee i_Y[y^{-1}(y'_0)],
\]
and so \( y'_1 \leq i_Y(y) \leq y'_2 \), which is what we are trying to prove. It follows that \((D0)\) holds for \((i_X \circ e_X, i_Y \circ e_Y, R)\), and thus by duality \((D1)\) also holds.

(5) Suppose \( x' \ R \ y' \). Then there is \( x \in X \) and \( y \in Y \) with \( x' \leq i_X(x) \), \( x \ R \ y \), and \( i_Y(y) \ R \ y' \). Let \( S \subseteq X \times Y \) satisfy the conditions from (5). Then \( i_X(x) S i_Y(y) \), and the result follows from \((A0)\) and \((A1)\).

(6) From (5) we know that any relation on \( X \times Y \) that ‘extends \( R \)’ must contain \( R \). Examination of the conditions \((C0), (C1), (D0) \) and \((D1)\) reveals that if they fail for \( R \) they will also fail for any relation containing \( R \).

\( \square \)

Theorem 6.1 specifically parts (5) and (6), tells us that if we want to find a 2- or 3-coherent polarity extending \((e_X,e_Y,R)\), then it suffices to look at \( R \), as if this does not produce the desired result then nothing will. Note that this does not apply for 1-coherence. To see this let \( P = \{p\} \cong X \cong Y \cong X \cong Y \), and let \( R = \emptyset \). Then \((B0)\) fails for \( R \), but if \( S = \{(i_X \circ e_X(p)), i_Y \circ e_Y(p)\} \) then \((i_X \circ e_X, i_Y \circ e_Y, S)\) is obviously 1-coherent.
For 0-coherent polarities we can add converses to some of the statements in Theorem 6.1, but we will leave this till Corollary 6.8. Note that for \((i_X \circ e_X, i_Y \circ e_Y, R)\) to be 3-coherent it is not sufficient for \((e_X, e_Y, R)\) to be 3-coherent, or even Galois. The additional restrictions on the extensions \(i_X\) and \(i_Y\) from Theorem 6.4 are necessary, as Example 6.2 demonstrates below.

Example 6.2. Let \(P\) be the three element antichain from Figure 4 and let \(X \cong Y \cong P\). Let \(\overline{X}\) and \(\overline{Y}\) be the poset extensions illustrated in Figures 4 and 3 respectively. Define \(R\) on \(X \times Y\) by \(x R y \iff \text{there is } p \in P \text{ with } x = e_X(p) \text{ and } y = e_Y(p)\).

We can put a poset structure on \(X \cup Y\) just by identifying copies elements of \(P\) appropriately, in which case we end up with something isomorphic to \(P\). Clearly the natural maps \(i_X\) and \(i_Y\) are meet- and join-preserving order embeddings, and so \((e_X, e_Y, R)\) is Galois. However, \((i_X \circ e_X, i_Y \circ e_Y, R)\) is not 3-coherent. Indeed, it follows from Example 6.3 that there is no relation \(S\) such that \((i_X \circ e_X, i_Y \circ e_Y, S)\) is 3-coherent.

The following lemma says, roughly, that the extension of the ‘minimal’ polarity relation \(R_i\) is again the minimal polarity relation.

Lemma 6.3. Let \((e_X, e_Y, R_i)\) be an extension polarity, where \(R_i\) is as in Proposition 6.2, and let \(i_X : X \to \overline{X}\) and \(i_Y : Y \to \overline{Y}\) be order extensions. Then \(\overline{R_i} = S_i\), where \(S_i \subseteq \overline{X} \times \overline{Y}\) is defined analogously to \(R_i\).

Proof. Let \(x' \in \overline{X}\) and let \(y' \in \overline{X}\). Then
\[
x' \overline{R_i} y' \iff x' \leq i_X(x), x R_i y \text{ and } i_Y(y) \leq y' \text{ for some } x \in X \text{ and } y \in Y
\]
\[
\iff x' \leq i_X(x), i_Y(y) \leq y' \text{ and } e_X^{-1}(x') \cap e_Y^{-1}(y') \neq \emptyset \text{ for } x \in X, y \in Y
\]
\[
\iff (i_X \circ e_X)^{-1}(x') \cap (i_Y \circ e_Y)^{-1}(y') \neq \emptyset
\]
\[
\iff x' S_i y'.
\]

6.2. Restriction. If \(i_X : X \to \overline{X}\) and \(i_Y : Y \to \overline{Y}\) are order extensions, then a polarity \((\overline{X}, \overline{Y}, S)\) can be restricted in a natural way to a polarity \((X, Y, S)\). The following theorem makes this precise.

Theorem 6.4. Let \(X\) and \(Y\) be posets, and let \(i_X : X \to \overline{X}\) and \(i_Y : Y \to \overline{Y}\) be order extensions. Let \(S\) be a relation on \(\overline{X} \times \overline{Y}\). Then there is a relation \(S\) on \(X \times Y\) defined by
\[
x S y \iff i_X(x) S i_Y(y)
\]
such that the following hold:

1. \((X, Y, S)\) is an order polarity.
2. If \((\overline{X}, \overline{Y}, S)\) is 0-coherent then so is \((X, Y, S)\).

Moreover, if \(P\) is a poset, and if \(e_X : P \to X\) and \(e_Y : P \to Y\) are order extensions, then both \((e_X, e_Y, S)\) and \((i_X \circ e_X, i_Y \circ e_Y, S)\) are extension polarities, and:

3. If \((i_X \circ e_X, i_Y \circ e_Y, S)\) is \(\ast\)-coherent then so is \((e_X, e_Y, S)\) for \(\ast \in \{1, 2\}\).
4. Suppose \(i_X\) preserve meets in \(X\) of subsets of \(e_X[P]\) whenever they exist, and let \(i_Y\) likewise preserve joins in \(Y\) of subsets of \(e_Y[P]\). Then, if \((i_X \circ e_X, i_Y \circ e_Y, S)\) is 3-coherent, so is \((e_X, e_Y, S)\), and the same is true if we replace ‘3-coherent’ with ‘Galois’.
Proof. First of all, \((X, Y, S)\) is obviously an order polarity as the definition requires only that \(X\) and \(Y\) are posets and \(S\) is a relation between \(X\) and \(Y\).

Now, let \(\preceq\) be a pre-order on \(\overline{X} \cup \overline{Y}\), and consider the diagram in Figure 13. Here \(\overline{X} \preceq \overline{Y}\) is the poset structure induced on \(\overline{X} \cup \overline{Y}\) by the maps \(i_X \circ i_X\) and \(i_Y \circ i_Y\), and \(\phi\) is the associated order embedding (which we can think of as an inclusion). It is easy to see that if \(\preceq\) is in \(P_S\), \(P_S\) or \(P_S^e\) then the restriction of \(\preceq\) to \(X \cup Y\) will be in \(P_S\), \(P_S\) or \(P_S^e\) appropriately. This proves (1), (2), and (3).

For (4) we can take the same approach to prove 3-coherence. Because of the restriction on \(i_X\), meets in \(X\) of subsets of \(e_X[P]\) correspond to meets in \(\overline{X}\) of subsets of \(i_X \circ e_X[P]\), and similarly joins in \(Y\) of subsets of \(e_Y[P]\) correspond to joins in \(\overline{Y}\), so the meet- and join-preservation properties of \(i_X\) and \(i_Y\), respectively, also apply to \(i_X\) and \(i_Y\).

Finally, suppose \((i_X \circ e_X, i_Y \circ e_Y, S)\) is Galois, and let \(x_1, x_2 \in \overline{X}\). Then, as \(i_X \circ e_X\) is a meet-extension, there is \(p \in P\) with \(x_2 \leq i_X \circ e_X(p)\) and \(x_1 \leq i_X \circ e_X(p)\). By writing \(i_X \circ e_X(p)\) as \(i_X(e_X(p))\) we see immediately that \(i_X\) is a meet-extension, and \(i_Y\) is a join-extension by duality. Similarly, let \(x_1, x_2 \in X\). Then \(i_X(x_1) \not\leq i_X(x_2)\), so there is \(q \in P\) with \(i_X(x_2) \leq i_X \circ e_X(q)\) and \(i_X(x_1) \not\leq i_X \circ e_X(q)\), and thus \(x_2 \leq e_X(q)\) and \(x_1 \not\leq e_X(q)\). So \(e_X\) is also a meet-extension, and \(e_Y\) is a join-extension by duality. The result then follows, as we have already proved that \((e_X, e_Y, S)\) will be 3-coherent. \(\square\)

\[ \begin{array}{c}
\begin{array}{c}
P \longrightarrow \overline{X} \quad i_Y \quad \overline{Y} \\
X \quad i_X \quad \overline{Y} \\
X \quad \phi \quad \overline{Y}
\end{array}
\end{array} \]

Figure 13.

Unlike the situation in Theorem 6.1, partial converses for the implications in Theorem 6.4 do not hold, as Example 6.5 demonstrates.

Example 6.5. Let \(P\) be the poset represented by the \(\bullet\) elements in Figure 14, let \(P \cong X \cong Y \cong \overline{X}\), and let \(\overline{Y}\) be represented by Figure 14. Then the implicit maps \(i_X\) and \(i_Y\) are obviously meet- and join-extensions respectively, and are also, respectively, trivially completely meet- and join-preserving. Let \(S = R_1 \cup \{(p, y)\}\), where \(R_1 \subseteq \overline{X} \times \overline{Y}\) is as in Proposition 5.3. Then \(S = R_1\), where \(R_1 \subseteq X \times Y\) is defined analogously, and \((e_X, e_Y, S)\) is Galois by Proposition 5.3. However, \((i_X \circ e_X, i_Y \circ e_Y, S)\) is not even 0-coherent, as we have \((p, y) \in S\) but \((q, y) \notin S\), so (A0) from Theorem 5.3 fails.

Using the notation of Theorems 6.1 and 6.4 we can define a map \(\overline{\pound}\) from the complete lattice of relations on \(X \times Y\) to the complete lattice of relations on \(\overline{X} \times \overline{Y}\), by taking \(R\) to \(\overline{R}\). Similarly, we can define a map \(\pound\) going back the other way.
by taking $S$ to $\mathbb{S}$. These maps are obviously monotone. We also have the following result.

**Lemma 6.6.** Let $X$ and $Y$ be posets, let $i_X : X \to \overline{X}$ be an extension of $X$, and let $i_Y : Y \to \overline{Y}$ be an extension of $Y$. Then:

1. Let $R$ be a relation on $X \times Y$. Then $R \subseteq (\overline{R})$. Moreover, if $(X,Y,R)$ is 0-coherent then $R = (\overline{R})$.

2. Let $S$ be a relation on $\overline{X} \times Y$. If $(i_X,i_Y,S)$ is 0-coherent then $(\overline{S}) \subseteq S$. Moreover, the opposite inclusion may fail, even when $(i_X \circ e_X,i_Y \circ e_Y,S)$ is Galois.

**Proof.** We start with (1). Let $x \in X$, let $y \in Y$ and suppose $x R y$. Then $i_X(x) \overline{R} i_Y(y)$ by definition of $\overline{R}$, and so $x \overline{R} y$ by definition of $\overline{R}$. Suppose now that $(X,Y,R)$ is 0-coherent and let $x \overline{R} y$. Then $i_X(x) \overline{R} i_Y(y)$ by definition of $\overline{R}$, and thus $x R y$ by Theorem 6.1(2).

For (2), suppose first that $(i_X,i_Y,S)$ is 0-coherent, and let $x' \in \overline{X}$ and $y' \in \overline{Y}$ with $x' \overline{S} y'$. Then, by definition of $\overline{S}$ there are $x \in X$ and $y \in Y$ with $x' \leq i_X(x)$, with $i_Y(y) \leq y'$, and with $x \overline{S} y$. But then $i_X(x) \overline{S} i_Y(y)$ by definition of $\overline{S}$, and so $x' \leq i_X(x) \overline{S} i_Y(y) \leq y'$, and thus $x' \overline{S} y'$ by 0-coherence of $(i_X,i_Y,S)$. To see that the opposite inclusion may fail, see Example 6.9 below. \hfill \Box

Note that the polarity $(i_X,i_Y,S)$ from Example 6.5 is not 0-coherent, but, appealing to Lemma 6.3 we have $\overline{\mathbb{S}} \subseteq S$. Thus $(i_X,i_Y,S)$ being 0-coherent is strictly stronger than having $\overline{\mathbb{S}} \subseteq S$.

**Corollary 6.7.** Using the notation of Lemma 6.6, let $L$ be the complete lattice of relations on $X \times Y$, and let $M$ be the complete lattice of 0-coherent relations on $\overline{X} \times \overline{Y}$. Then the maps $(-) : L \to M$ and $(\overline{-}) : M \to L$ are, respectively, the left and right adjoints of a Galois connection.

**Proof.** First, recall the discussion at the start of Section 5.1 for the lattice structure of $M$. Moreover, $(-) : L \to M$ is well defined by Theorem 6.1(1). By Lemma 6.6 we have $R \subseteq (\overline{R})$ for all $R \in L$, and $(\overline{S}) \subseteq S$ for all $S \in M$, which is one of the equivalent conditions for two monotone maps to form a Galois connection (see e.g. [6, Lemma 7.26]). \hfill \Box

Using Theorem 6.4 and Lemma 6.6 we can get partial converses for Theorem 6.1.
Corollary 6.8. With notation as in Theorem 6.7, suppose \((X,Y,R)\) is 0-coherent. Then:

1. If \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) is *-coherent then so is \((e_X, e_Y, R)\) for \(* \in \{1, 2\}\).
2. Suppose \(i_X\) preserves meets in \(X\) of subsets of \(e_X[P]\) whenever they exist, and let \(i_Y\) likewise preserve joins in \(Y\) of subsets of \(e_Y[P]\). Then, whenever \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) is 3-coherent, so is \((e_X, e_Y, R)\), and this is also true if we replace ’3-coherent’ with ’Galois’.

Proof. For (1), if \((i_X \circ e_X, i_Y \circ e_Y, \overline{R})\) is *-coherent, then so is \((X,Y, \overline{(R)})\), by Theorem 6.4, and as \((X,Y,R)\) is 0-coherent we have \(R = \overline{(R)}\), by Lemma 6.6. The proof of (2) is essentially the same. □

Example 6.9. Let \(P\) be the poset in Figure 15, and let \(X \cong Y \cong P\). Let \(X\) and \(Y\) be the posets in Figures 16 and 17 respectively, denoting embedded images of elements of \(X\) and \(Y\) with \(\bullet\), and extension elements with \(\circ\). Define \(S\) on \(X \times Y\) so that \(i_X \circ e_X(p) S e_Y \circ i_Y(p)\) for all \(p \in P\), and also \(x' S y'\). Then \((i_X \circ e_X, i_Y \circ e_Y, S)\) is Galois, as can be seen by considering the poset in Figure 18, and defining \(\iota_X\) and \(\iota_Y\) in the obvious way. However, there is no \(x \in X\) and \(y \in Y\) with \(x' \leq \iota_X(x)\), with \(\iota_Y(y) \leq y'\), and with \(x \triangleright y\). Thus \((x', y') \notin (S)\).

7. Galois polarities revisited

7.1. Galois polarities via Galois connections. Galois polarities are so named because the associated (unique) pre-order on \(X \cup Y\) can be described in terms of a Galois connection. This idea is precisely articulated in Corollary 7.4 below.

When \((e_X, e_Y, R)\) is Galois, we know from Theorem 4.8 that \(P^R_0\) contains only a single element, \(\leq_3\). As mentioned previously, to lighten the notation we write e.g. \(X \uplus Y\) in place of \(X \uplus_{\leq_3} Y\) when working with Galois polarities. The following theorem collects together some useful facts.

Theorem 7.1. Let \((e_X, e_Y, R)\) be a Galois polarity, let \(i_X : X \to \overline{X}\) be a completely meet-preserving meet-extension, and let \(i_Y : Y \to \overline{Y}\) be a completely join-preserving join-extension. Then:
(1) \((i_X \circ e_X, i_Y \circ e_Y, \mathbb{R})\) is Galois.
(2) The map \(\gamma : P \to X \uplus Y\) defined by \(\gamma = i_X \circ e_X = i_Y \circ e_Y\) is an order embedding. Moreover, if \(S, T \subseteq P\) and \(\land S\) and \(\lor T\) exist in \(P\), then
   \((a)\) \(\gamma(\land S) = \land \gamma[S] \iff e_X(\land S) = \land e_X[S]\), and
   \((b)\) \(\gamma(\lor T) = \lor \gamma[T] \iff e_Y(\lor T) = \lor e_Y[T]\).
(3) \(\gamma[P] = \iota_X[X \cap \iota_Y[Y]\).
(4) Define the map \(\phi : X \uplus Y \to \overline{X} \uplus \overline{Y}\) by

\[
\phi(z) = \begin{cases} 
\overline{\overline{X}} \circ i_X(z) & \text{if } z \text{ is (the equivalence class of) an element of } X. \\
\overline{\overline{Y}} \circ i_Y(z) & \text{if } z \text{ is (the equivalence class of) an element of } Y.
\end{cases}
\]

Then \(\phi\) is a well defined order embedding, and the diagram in Figure 17 commutes.

Proof.

(1) That \((i_X \circ e_X, i_Y \circ e_Y, \mathbb{R})\) is Galois is Theorem 6.1(4).
(2) That \(\gamma\) is well defined follows from 1-coherence of \((e_X, e_Y, \mathbb{R})\), and that \(\gamma\) is an order embedding follows from 2-coherence of \((e_X, e_Y, \mathbb{R})\), as \(\gamma\) is the composition of two order embeddings, \(i_X \circ e_X\). That (a) and (b) hold follows from 3-coherence of \((e_X, e_Y, \mathbb{R})\), as, for example, \(\gamma = i_X \circ e_X\) and \(\iota_X\) preserves meets in \(X\) of subsets of \(e_X[P]\).
(3) We obviously have \(\gamma[P] \subseteq \iota_X[X \cap \iota_Y[Y]\), so let \(z \in \iota_X[X \cap \iota_Y[Y]\). Then there are \(x \in X\) and \(y \in Y\) with \(z = \iota_X(x) = \iota_Y(y)\). Thus, as \(\iota_X(x) \leq \iota_Y(y)\) we have \(e_X^{-1}(x) \cap e_Y^{-1}(y) \neq \emptyset\). Suppose \(p \in e_X^{-1}(x) \cap e_Y^{-1}(y)\), and that \(e_X(p) \not\leq x\). Then there is \(q \in P\) with \(x \leq e_X(q)\) and \(e_X(p) \not\leq e_X(q)\). But this is a contradiction, as, since \(\iota_Y(y) \leq \iota_X(x)\), (G3) of Theorem 4.8 tells us that \(p \leq q\). Thus \(x = e_X(p)\), and so \(z = \gamma(p)\). It follows that \(\iota_X[X \cap \iota_Y[Y] \subseteq \gamma[P]\) as claimed.
(4) First note that since \((i_X \circ e_X, i_Y \circ e_Y, \mathbb{R})\) is Galois, it makes sense to write \(\overline{X} \uplus \overline{Y}\). Now, \(R = (\mathbb{R})\) by Lemma 6.6, so the unique element of \(\mathbb{P}_R^\beta\) must be the same as the unique element of \(\mathbb{P}_R^\alpha\), which is induced by the embeddings of \(X\) and \(Y\) into \(\overline{X} \uplus \overline{Y}\) via \(\overline{\iota_X} \circ i_X\) and \(\overline{\iota_Y} \circ i_Y\) respectively, as discussed in the proof of Theorem 6.4. The result then follows from the commutativity of the diagram in Figure 13.

The following fact will be useful.

**Proposition 7.2.** Let \(P\) and \(Q\) be posets, let \(e_1 : P \to J\) be a join-completion, and let \(e_2 : Q \to M\) be a meet-completion. Then any Galois connection \(\alpha : P \leftrightarrow Q : \beta\) extends uniquely to a Galois connection \(\alpha' : J \leftrightarrow M : \beta'\).

**Proof.** This is [28, Corollary 2].

**Lemma 7.3.** Let \(P\) be a poset, and let \(e_X : P \to X\) and \(e_Y : P \to Y\) be meet- and join-completions respectively. Then there is a unique Galois connection \(\Gamma : Y \leftrightarrow
Figure 19.

$X: \Delta$ such that $e_X = \Gamma \circ e_Y$ and $e_Y = \Delta \circ e_X$. The left and right adjoints of this Galois connection are defined, respectively, by

$$
\Gamma(y) = \bigvee e_X[e_Y^{-1}(y^\uparrow)]
$$

$$
\Delta(x) = \bigwedge e_Y[e_X^{-1}(x^\uparrow)].
$$

Proof. $\Gamma$ and $\Delta$ are well defined as $X$ and $Y$ are complete. Using the fact that $e_X$ and $e_Y$ are, respectively, meet- and join-completions, we have

$$
\Gamma(y) \leq x \iff \bigvee e_X[e_Y^{-1}(y^\uparrow)] \leq x
$$

$$
\iff x \leq e_X(q) \implies e_X(p) \leq e_X(q) \text{ for all } p \in e_Y^{-1}(y^\uparrow) \text{ and for all } q \in P
$$

$$
\iff q \in e_X^{-1}(x^\uparrow) \text{ and } p \in e_Y^{-1}(y^\uparrow) \implies p \leq q
$$

$$
\iff e_Y(p) \leq y \implies e_Y(p) \leq e_Y(q) \text{ for all } p \in e_X^{-1}(x^\uparrow) \text{ and for all } q \in P
$$

$$
\iff y \leq \bigwedge e_Y[e_X^{-1}(x^\uparrow)]
$$

$$
\iff y \leq \Delta(x).
$$

To see that this is the only such Galois connection between $X$ and $Y$ we apply Proposition [7.2] with $P = Q$ and the Galois connection produced by the identity function on $P$. \hfill \Box

Corollary 7.4. Let $(e_X, e_Y, R)$ be a Galois polarity, let $i_X : X \to \overline{X}$ be a completely meet-preserving meet-completion of $X$, and let $i_Y : Y \to \overline{Y}$ be a completely join-preserving join-completion of $Y$. Let $\leq R$ be the unique element of $\mathcal{P}_R^g$. Let $\Gamma$ and $\Delta$ be as defined in Lemma [7.3], with respect to the maps $i_X \circ e_X$, and $i_Y \circ e_Y$. Then to define $\leq_3$ we can replace (G3) from Theorem [4.8] by:

\begin{itemize}
  \item [(G3')] For all $x \in X$ and $y \in Y$ we have
  \end{itemize}

$$
y \leq x \iff \Gamma(i_Y(y)) \leq i_X(x) \iff i_Y(y) \leq \Delta(i_X(x)).
$$

Proof. First note that the maps $\Gamma$ and $\Delta$ exist as $i_X \circ e_X : P \to \overline{X}$ and $i_Y \circ e_Y : P \to \overline{Y}$ are meet- and join-completions respectively. That (G3') and (G3) are equivalent is an immediate consequence of the following equivalence:
\[\Gamma(i_Y(y)) \leq i_X(x)\]
\[\iff \bigvee i_X \circ e_X ((i_Y \circ e_Y)^{-1}(i_Y(y)^\dagger)) \leq \bigwedge i_X \circ e_X ((i_Y \circ e_Y)^{-1}(i_X(x)^\dagger))\]
\[\iff \left(i_Y \circ e_Y(p) \leq i_Y(y) \text{ and } i_X(x) \leq i_X(q) \implies i_X \circ e_X(p) \leq i_X \circ e_X(q)\right)\]
\[\iff \left(e_Y(p) \leq y \text{ and } x \leq e_X(q) \implies p \leq q\right).\]

\[\square\]

Corollary 7.4 justifies the terminology ‘Galois polarity’, as the upshot of this result is that, for any Galois polarity \((e_X,e_Y,R)\), the unique element of \(\hat{P}_R^Y\) is directly defined by \(R\), the orders on \(X\) and \(Y\), and the Galois connection from Lemma 7.3. Note that the choice of meet- and join-completions of \(X\) in \(\hat{\Delta}\) is constrained only by the requirement that they preserve meets and joins respectively. Indeed, we can weaken these conditions to just the preservation of meets and joins respectively. So long as these requirements are met, the ordering induced by \((G3')\) will be the same as the one induced by \((G3)\).

7.2. Polarity morphisms.

**Definition 7.5.** A Galois polarity \((e_X,e_Y,R)\) is complete if \(e_X\) and \(e_Y\) are completions.

Noting Proposition 4.9, we see that Theorem 3.4 establishes a one-to-one correspondence between what we call complete Galois polarities and \(\Delta_1\)-completions of a poset. Theorem 7.18 below expands on the proof of this result, and in Section 7.4 we reformulate it in terms of an adjunction between categories. First we need to define a concept of morphism between Galois polarities.

**Definition 7.6.** Let \(P\) and \(P'\) be posets, let \((e_X,e_Y,R)\) be a Galois polarity extending \(P\), and let \((e_X',e_Y',R')\) be a Galois polarity extending \(P'\). Then a polarity morphism between \((e_X,e_Y,R)\) and \((e_X',e_Y',R')\) is a triple of monotone maps \((h_X : X \to X', h_P : P \to P', h_Y : Y \to Y')\) such that:

1. The diagram in Figure 20 commutes.
2. For all \(x \in X\) and \(y \in Y\) we have
   \[\iota_Y(y) \leq \iota_X(x) \implies \iota_Y \circ h_Y(y) \leq \iota_X \circ h_X(x).\]
3. For all \(x' \in X'\) and for all \(y' \in Y'\), if \((x',y') \notin R'\) then there is \(x \in X\) and \(y \in Y\) such that:
   - (i) \(h_X^{-1}(x'^\dagger) \subseteq x^\dagger.\)
   - (ii) \(h_Y^{-1}(y'^\dagger) \subseteq y^\dagger.\)
   - (iii) \(h_X(a) R' y' \implies a R y\) for all \(a \in X.\)
   - (iv) \(x' R' h_Y(b) \implies x R b\) for all \(b \in Y.\)
   - (v) \((x,y) \notin R.\)

If \(h_X, h_P\) and \(h_Y\) are all order embeddings, and also \(h_X(x) R' h_Y(y) \implies x R y\) for all \(x \in X\) and \(y \in Y\), then \((h_X,h_P,h_Y)\) is a polarity embedding. If, in addition, all maps are actually order isomorphisms then \((h_X,h_P,h_Y)\) is a polarity isomorphism, and we say \((e_X,e_Y,R)\) and \((e_X',e_Y',R')\) are isomorphic.

Sometimes we want to fix a poset \(P\) and deal exclusively with isomorphism classes of Galois polarities extending \(P\). In this case we say Galois polarities \(E_1\)
and $E_2$ are **isomorphic as Galois polarities extending** $P$ if there is a polarity isomorphism $(h_X, h_P, h_Y) : E_1 \to E_2$ where $h_P$ is the identity on $P$.

![Figure 20.](image)

Note that if $h_X$ and $h_Y$ are order embeddings then $h_P$ will be too, but this is not necessarily the case for order isomorphisms. Note also that Definition 7.6 while being similar in some respects, is largely distinct from the notion of a **bounded morphism between polarity frames** from [33]. It is also completely different to the frame morphisms of [7, 11], which are duals to complete lattice homomorphisms, rather than `decomposed' versions of certain maps $X \sqcup Y \to X' \sqcup Y'$. We will make this clear in Theorem 7.9 later.

**Lemma 7.7.** If $h = (h_X : X \to X', h_P : P \to P', h_Y : Y \to Y')$ is a polarity morphism, then for all $x \in X$ and for all $y \in Y$ we have $x R y \implies h_X(x) R' h_Y(y)$.

**Proof.** Suppose $(h_X(x), h_Y(y)) \notin R'$. Then, by 7.6(3) there are $x_0 \in X$ and $y_0 \in Y$ with $h_X^{-1}(h_X(x))^\uparrow \subseteq x_0^\uparrow$, with $h_Y^{-1}(h_Y(y))^\downarrow \subseteq y_0^\downarrow$ and with $(x_0, y_0) \notin R$. From $h_X^{-1}(h_X(x))^\uparrow \subseteq x_0^\uparrow$ it follows that $x_0 \leq x$, and similarly we have $y \leq y_0$. Thus $(x, y) \notin R$, as otherwise (A0) and (A1) of Theorem 3.3 would force $x_0 \leq y$.

The following definition is due to Erné [8]. This will be of interest to us as it precisely characterizes those maps between posets that lift (uniquely) to complete homomorphisms between their MacNeille completions [8, Theorem 3.1].

**Definition 7.8.** A monotone map $f : P \to Q$ is **cut-stable** if whenever $q_1 \not\leq q_2 \in Q$, there are $p_1 \not\leq p_2 \in P$ such that $f^{-1}(q_1) \subseteq p_1^\uparrow$ and $f^{-1}(q_2) \subseteq p_2^\uparrow$.

Condition (3) of Definition 7.6 is related to cut-stability, as we shall see in the proof of Theorem 7.9. We can think of this as an adaptation of ideas from [10, Section 4]. We extend from what, according to our terminology, is the special case of $(e_X, e_Y, R)$ where $e_X$ and $e_Y$ are the free directed meet- and join-completions respectively and $R = R_\uparrow$, to Galois polarities in general.

**Theorem 7.9.** Let $P$ and $P'$ be posets, let $(e_X, e_Y, R)$ be a Galois polarity extending $P$, and let $(e_X', e_Y', R')$ be a Galois polarity extending $P'$. Let $\gamma : P \to X \sqcup Y$ and $\gamma' : P' \to X' \sqcup Y'$ be the canonical maps as in Theorem 7.1. Then, given a polarity morphism $(h_X, h_P, h_Y) : (e_X, e_Y, R) \to (e_X', e_Y', R')$, there is a unique, cut-stable monotone map $\psi : X \sqcup Y \to X' \sqcup Y'$ such that the diagram in Figure 21 commutes. Moreover, $\psi$ satisfies conditions (1)-(3) below.

1. $\psi \circ \gamma[P] \subseteq \gamma'[P]$,
2. $\psi \circ \iota_X[X] \subseteq \iota_X'[X']$, and
3. $\psi \circ \iota_Y[Y] \subseteq \iota_Y'[Y']$,
Conversely, given a cut-stable monotone map \( \psi : X \sqcup Y \to X' \sqcup Y' \) satisfying (1)-(3), there is a unique polarity morphism \((h_X, h_P, h_Y)\) such that the diagram in Figure 21 commutes.

Finally, if \( \psi \) and \((h_X, h_P, h_Y)\) are, respectively, a cut stable monotone map and a polarity morphism uniquely specifying each other according to the correspondence described above, then:

(a) \( \psi \) is an order embedding if and only if:

(i) \( h_X \) and \( h_Y \) are order embeddings, and

(ii) for all \( x \in X \) and for all \( y \in Y \) we have \( h_X(x) R' h_Y(y) \implies x R y \).

I.e. if and only if \((h_X, h_P, h_Y)\) is a polarity embedding.

(b) If \( h_X \) and \( h_Y \) are both surjective then \( \psi \) is surjective, but the converse does not hold in general.

Proof. Given \((h_X, h_P, h_Y)\), the commutativity of the diagram in Figure 21 demands that \( \psi \) can only be defined by

\[
\psi(z) = \begin{cases} 
\iota_X \circ h_X(z) & \text{when } z \in \iota_X[X] \\
\iota_Y \circ h_Y(z) & \text{when } z \in \iota_Y[Y] 
\end{cases}
\]

Abusing notation slightly, let \( \leq \) stand for the unique element of both \( \hat{\mathcal{P}}_R^g \) and \( \hat{\mathcal{P}}_{R'} \). If \( x \in X \) and \( y \in Y \), then, using Lemma 7.7 we have

\[
x \leq y \iff x R y \iff h_X(x) R' h_Y(y) \iff h_X(x) \leq h_Y(y).
\]

If \( y \leq x \), then \( \iota_Y(y) \leq \iota_X(x) \) by definition, and so \( h_Y(y) \leq h_X(x) \) by Definition 7.6(2). This shows \( \psi \) is well defined, and along with the fact that \( h_X \) and \( h_Y \) are monotone proves \( \psi \) is monotone.

To see that \( \psi \) is cut-stable, let \( z_1 \nleq z_2 \in X' \sqcup Y' \). Since \( \iota_X[X'] \) and \( \iota_Y[Y'] \) are, respectively, join- and meet-dense in \( X' \sqcup Y' \), there are \( x' \in X' \) and \( y' \in Y' \) with \( \iota_X(x') \leq z_1 \), with \( z_2 \leq \iota_Y(y') \), and with \( \iota_X(x') \nleq \iota_Y(y') \) (i.e. \( (x', y') \notin \mathcal{R}' \)). Thus by Definition 7.6(3) there are \( x \in X \) and \( y \in Y \) with the five properties described in that definition. We will satisfy the condition of Definition 7.8 using the pair \( \iota_X(x) \nleq \iota_Y(y) \).

Let \( z \in \psi^{-1}(z_1^+) \). We must show that \( z \in \iota_X(x)^+ \). We have \( \psi(z) \geq z_1 \geq \iota_X(x') \).

There are two cases. If \( z = \iota_X(a) \) for some \( a \in X \), then \( \psi(z) = \iota_X \circ h_X(a) \), and so \( h_X(a) \geq x' \). Thus \( a \in h_X^{-1}(x') \), and so \( a \in \iota_X(x)^+ \), by Definition 7.6(3)i. It follows that \( z = \iota_X(a) \in \iota_X(x)^+ \) as claimed. Alternatively, suppose \( z = \iota_Y(b) \) for some \( b \in Y \). Then \( \psi(z) = \iota_Y \circ h_Y(b) \), and so \( \iota_Y \circ h_Y(b) \geq \iota_X(x') \), and consequently \( x' R' h_Y(b) \).

It follows from Definition 7.6(3) that \( x R b \), and thus that \( \iota_X(x) \leq \iota_Y(b) = z \) as required. That \( \psi^{-1}(z_1^+) \subseteq \iota_Y(y)^+ \) follows by a dual argument, and so \( \psi \) is cut-stable.

To see that condition (1) holds for \( \psi \) note that

\[
\gamma(p) = \psi \circ \iota_Y \circ e_Y(p)
\]

\[
= \iota_Y \circ h_Y \circ e_Y(p)
\]

\[
= \iota_Y \circ e_Y \circ h_P(p)
\]

\[
= \gamma'(h_P(p)).
\]

That (2) and (3) hold is automatic.

Conversely, given monotone \( \psi \) satisfying (2) and (3), if the diagram in Figure 21 is to commute, then \( h_X \) and \( h_Y \) must be \( \iota_X^{-1} \circ \psi \circ \iota_X \) and \( \iota_Y^{-1} \circ \psi \circ \iota_Y \) respectively. Here \( \iota_X^{-1} \) and \( \iota_Y^{-1} \) are the partial inverse maps, which are total on \( \psi \circ \iota_X[X] \) and
ψ \circ \iota_Y[Y]$ by (2) and (3) respectively. The commutativity of this diagram also demands that, if $h_P$ exists, we have
\[ e_X \circ h_P = h_X \circ e_X = \iota_X^{-1} \circ \psi \circ \iota_X \circ e_X = \iota_X^{-1} \circ \psi \circ \gamma, \]
and thus $h_P = \iota_X^{-1} \circ \iota_X^{-1} \circ \psi \circ \gamma$, if this is well defined. Consequently, assuming $\psi$ also satisfies (1), we can, and must, define
\[ h_P = \gamma^{-1} \circ \psi \circ \gamma. \]
That $(h_X, h_P, h_Y)$ satisfies Definition 7.6(2) follows immediately from the definitions of $h_X$ and $h_Y$ and the fact that $\psi$ is monotone. If $\psi$ is also cut-stable, then to prove that $(h_X, h_P, h_Y)$ is a polarity morphism it remains only to check Definition 7.6(3).

So let $x' \in X'$, let $y' \in Y'$, and suppose $(x', y') \notin R'$. Then $\iota_X'(x') \leq \iota_Y'(y')$, and thus by cut-stability there are $z_1 \leq z_2 \in X \cup Y$ with $\psi^{-1}(\iota_X'(x')) \subseteq z_1$, and $\psi^{-1}(\iota_Y'(y')) \subseteq z_2$. As $\iota_X[X]$ and $\iota_Y[Y]$ are, respectively, join- and meet-dense in $X \cup Y$, there are $x \in X$ and $y \in Y$ with $\iota_X(x) \leq z_1$, with $z_2 \leq \iota_Y(y)$, and with $(x, y) \notin R$. It follows that $\psi^{-1}(\iota_X'(x')) \subseteq \iota_X(x)^\uparrow$ and $\psi^{-1}(\iota_Y'(y')) \subseteq \iota_Y(y)^\downarrow$.

We will check the conditions required by Definition 7.6(3) are satisfied by the pair $(x, y)$:

(i) Let $a \in X$ and suppose $a \in h_X^{-1}(x')$. Then $\iota_X(a) \in \psi^{-1}(\iota_X'(x')) \subseteq \iota_X(x)^\uparrow$, and thus $a \in x^\uparrow$ as required.

(ii) Dual to (i).

(iii) Let $a \in X$ and suppose $h_X(a) \in h_Y'(y')$. Then $\iota_X \circ h_X(a) \leq \iota_Y'(y')$, and thus $\psi \circ \iota_X(a) \leq \iota_Y'(y')$. It follows that $\iota_X(a) \in \psi^{-1}(\iota_Y'(y')^\downarrow) \subseteq \iota_Y(y)^\downarrow$, and so $a \in y^\downarrow$ as required.

(iv) Dual to (iii).

(v) By choice of $(x, y)$.

Finally, we check the claims (a) and (b). For (a), if $\psi$ is an order embedding then that $h_X$ and $h_Y$, and thus also $h_P$, are order embeddings follows directly from the commutativity of the diagram in Figure 21. Moreover, (‡) holds for the same reason. Conversely, suppose (‡) and (‡) hold and consider the map $\psi$. Since we already know $\psi$ is monotone, suppose $z, z' \in X \cup Y$ and that $\psi(z) \leq \psi(z')$. There are four cases.

If either $z, z' \in \iota_X[X]$, or $z, z' \in \iota_Y[Y]$, then that $z \leq z'$ follows again from the commutativity of the diagram in Figure 21. In the case where $z = \iota_X(x)$ and $z' = \iota_Y(y)$ for some $x \in X$ and $y \in Y$, then
\[ \psi(z) \leq \psi(z') \iff h_X(x) \mid h_Y(y) \iff x \mid y \iff \iota_X(x) \leq \iota_Y(y) \iff z \leq z'. \]

In the final case we have $z = \iota_Y(y)$ and $z' = \iota_X(x)$ for some $x \in X$ and $y \in Y$. Then
\[ \psi(z) \leq \psi(z') \iff \iota_Y \circ h_Y(y) \leq \iota_X \circ h_X(x). \]

If $p, q \in P$, and $e_Y(p) \leq y$ and $x \leq e_X(q)$, then
\[ \iota_Y \circ h_Y \circ e_Y(p) \leq \iota_Y \circ h_Y(y) \leq \iota_X \circ h_X(x) \leq \iota_X \circ h_X \circ e_X(q), \]
and thus $\iota_Y \circ e_Y(p) \leq \iota_X \circ e_X(q)$, by the commutativity of the diagram in Figure 20 and it follows that $p \leq q$. Thus by the definition of $\preceq$ we have $y \preceq x$ as required.

For (b), if $h_X$ and $h_Y$ are onto then given $z' \in X' \cup Y'$ we must have either $z = \iota_X'(h_X(x))$ for some $x \in X$, or $z = \iota_Y'(h_Y(y))$ for some $y \in Y$. In either case it
follows there is $z \in X \uplus Y$ with $\psi(z) = z$, and thus that $\psi$ is onto. To see that the converse may not hold see Example 7.10.

\[ \text{Figure 21.} \]

**Example 7.10.** Let $P = X$ be a two element antichain, and let $Y$ be this two element antichain extended by adding a join for the two base elements. Let $P' = X' = Y' = Y$. Then the inclusion maps and the relation $R_1$ define Galois polarities, and $X \uplus Y \simeq Y \simeq X' \uplus Y'$. Let $\psi : X \uplus Y \to X' \uplus Y'$ be map induced by the identity function on $X \uplus Y$. Then $\psi$ is clearly monotone, surjective, cut-stable and satisfies (1)-(3) from Theorem 7.9. However, the induced map $h_X$ cannot be surjective, as $2 = |X| < |X'| = 3$.

**Lemma 7.11.** The class of Galois polarities and polarity morphisms forms a category.

**Proof.** Identity morphisms obviously exist, so we need only check composition. We will use Theorem 7.9. It’s straightforward to show that the composition of maps satisfying conditions (1)-(3) of that theorem also satisfies these conditions, and compositions of monotone maps are obviously monotone. Moreover, cut-stability is preserved by composition [8, Corollary 2.10]. Thus it follows from Theorem 7.9 that polarity morphisms compose appropriately.

We will expand on this categorical viewpoint in Section 7.4.

7.3. Galois polarities and $\Delta_1$-completions. Recall that given a poset $P$ we write, for example, $e : P \to \mathcal{N}(P)$ for the MacNeille completion of $P$ (see Definition 2.3).

**Lemma 7.12.** If $(e_X, e_Y, R)$ is a Galois polarity, and if $e : X \uplus Y \to \mathcal{N}(X \uplus Y)$ is the MacNeille completion of $X \uplus Y$, then $e \circ \gamma : P \to \mathcal{N}(X \uplus Y)$ is a $\Delta_1$-completion (where $\gamma$ is as in Theorem 7.7).

**Proof.** $X \uplus Y$ is join-generated by $\iota_X[X]$, and meet-generated by $\iota_Y[Y]$, and $X$ and $Y$ are meet- and join-generated by $e_X[P]$ and $e_Y[P]$ respectively. $\mathcal{N}(X \uplus Y)$ is both join- and meet-generated by $e[X \uplus Y]$. Thus every element of $\mathcal{N}(X \uplus Y)$ is both a join of meets, and a meet of joins, of elements of $e \circ \gamma[P]$ as required.

**Definition 7.13.** If $(e_X, e_Y, R)$ is a Galois polarity, define the $\Delta_1$-completion $e \circ \gamma$ constructed from $(e_X, e_Y, R)$ in Lemma 7.12 to be the $\Delta_1$-completion generated by $(e_X, e_Y, R)$. 
Lemma 7.14. Let $d : P \to D$ be a $\Delta_1$-completion. Define $X_D$ and $Y_D$ to be (disjoint isomorphic copies of) the subsets of $D$ meet- and join-generated by $e[P]$ respectively. Define $e_{X_D} : P \to X_D$ and $e_{Y_D} : P \to Y_D$ by composing $d$ with the isomorphisms into $X_D$ and $Y_D$ respectively. Abusing notation by identifying $X_D$ and $Y_D$ with their images in $D$, define $R_D$ on $X_D \times Y_D$ by $x R_D y \iff x \leq y$ in $D$. Then $(e_{X_D}, e_{Y_D}, R_D)$ is a complete Galois polarity.

Proof. The inherited order from $D$ defines a pre-order on $X_D \cup Y_D$, and it is a Galois polarity. Moreover, $X_D$ and $Y_D$ are complete because $D$ is.

Definition 7.15. If $d : P \to D$ is a $\Delta_1$-completion, define the complete Galois polarity $(e_{X_D}, e_{Y_D}, R_D)$ to be the Galois polarity generated by $d$.

Lemma 7.16. Let $(e_X, e_Y, R)$ be a Galois polarity. Then there is a polarity embedding from $(e_X, e_Y, R)$ to $(e_{X'}, e_{Y'}, R_{X'})$, where the latter object is the Galois polarity generated by the $\Delta_1$-completion generated by $(e_X, e_Y, R)$. Moreover, if $(e_X, e_Y, R)$ is complete then this embedding is an isomorphism of polarities extending $P$.

Proof. Using Lemma 7.12 $e \circ \gamma : P \to (X \sqcup Y)$ is the $\Delta_1$-completion generated by $e_X, e_Y, R$, where $e : X \sqcup Y \to (X \sqcup Y)$ is the MacNeille completion. Recall that $\gamma = \iota_X \circ e_X = e_Y \circ e_Y$ by definition. To lighten the notation we write $e_X$, $e_Y$, $X_N(Y_W)$ for $X_N(Y_W)$, $X_N(Y_W)$. Define the map $h_X : X \to X_N$ by $h_X = \mu_X \circ e \circ \iota_X$, where $\mu_X$ is the isomorphism used to define $X_N$, as in Lemma 7.14. This is clearly an order embedding. Similarly define an order embedding $h_Y : Y \to Y_N$ by $h_Y = \mu_Y \circ e \circ \iota_Y$. Define $h_P$ to be the identity map. Note that $e_{X_N}$ is just $\mu_X \circ e \circ \gamma = \mu_X \circ e \circ \iota_X \circ e_X$, and similarly $e_{Y_N} = \mu_Y \circ e \circ \iota_Y \circ e_Y$. Thus we trivially have the commutativity required by Definition 7.1.(1).

To show that (2) is also satisfied, let $x \in X$, let $y \in Y$, and suppose $\iota_Y(y) \leq \iota_X(x)$. Then $e \circ \iota_Y(y) \leq e \circ \iota_X(x)$. The unique pre-order $\preceq\preceq_3$ on $X_N \sqcup Y_N$, can only be the order inherited from $N(X \sqcup Y)$, so $\mu_X \circ e \circ \iota_X(x) \preceq \mu_Y \circ e \circ \iota_Y(y)$, and thus $\iota_X(x) \circ h_Y(y) \preceq \iota_X(x) \circ h_X(x)$ as required.

For (3), let $x' \in X_N$, let $y' \in Y_N$, and suppose $(x', y') \not\in R_X$. Then $\mu_X^{-1}(x') \not\leq \mu_Y^{-1}(y')$. As $e \circ \iota_X[X]$ and $e \circ \iota_Y[Y]$ are, respectively, join- and meet-dense in $N(X \sqcup Y)$, there is $x \in X$ and $y \in Y$ with $e \circ \iota_X(x) \not\leq e \circ \iota_Y(y)$, with $e \circ \iota_X(x) \leq \mu_X^{-1}(x')$, and with $\mu_Y^{-1}(y') \leq e \circ \iota_Y(y)$. We check the necessary conditions are satisfied for this choice of $x$ and $y$:

(i) Let $a \in X$. Then

$$h_X(a) \geq x' \iff \mu_X^{-1} \circ \mu_X \circ e \circ \iota_X(a) \geq \mu_X^{-1}(x')$$

$$\Rightarrow e \circ \iota_X(a) \geq e \circ \iota_X(x)$$

$$\iff a \geq x,$$

and so $h_X^{-1}(x') \subseteq x^+$ as required.

(ii) Dual to (i).

(iii) Let $a \in X$, let $y' \in Y_N$, and suppose $h_X(a) R_{X'} y'$. Then

$$\mu_X^{-1} \circ \mu_X \circ e \circ \iota_X(a) \leq \mu_Y^{-1}(y') \leq e \circ \iota_Y(y),$$

and so $\iota_X(a) \leq \iota_Y(y)$, and thus $a R y$ as required.
(iv) Dual to (iii).

(v) Since \( e \circ \iota_X(x) \not\leq e \circ \iota_Y(y) \) we must have \((x, y) \not\in R\).

Now let \( x \in X \), let \( y \in Y \), and suppose \( h_X(x) R x h_Y(y) \). Then

\[
\mu_X^{-1} \circ \mu_X \circ e \circ \iota_X(x) \leq \mu_Y^{-1} \circ \mu_Y \circ e \circ \iota_Y(y),
\]

and so \( \iota_X(x) \leq \iota_Y(y) \), and thus \( x R y \), and we conclude that \((h_X, \text{id}_P, h_Y)\) is a polarity embedding as claimed. Finally, when \( X \) and \( Y \) are complete, as taking MacNeille completions preserves all meets and joins, the maps \( h_X \) and \( h_Y \) will be surjective, and thus isomorphisms. As \( h_P \) is the identity on \( P \) the result follows. \( \qed \)

**Lemma 7.17.** Let \( d : P \to D \) be a \( \Delta_1 \)-completion. Then \( d \) is isomorphic, as an extension of \( P \), to the \( \Delta_1 \)-completion generated by \((e_{XD}, \gamma_{XD}, R_D)\), where the latter object is the complete Galois polarity generated by \( d \).

**Proof.** \( X_D \) and \( Y_D \) are (disjoint isomorphic copies of) the subsets of \( D \) meet- and join-generated by \( d[P] \) respectively. By definition, and abusing notation slightly, the inclusion of \( X_D \cup Y_D \) into \( D \) is a MacNeille completion. The unique pre-order \( \preceq \in \mathcal{P}_R \) on \( X_D \cup Y_D \) is just the one inherited from \( D \). So composing the embedding \( \gamma_D : P \to X_D \cup Y_D \) with the MacNeille completion of \( X_D \cup Y_D \) we get something isomorphic to \( d \) as an extension of \( P \). \( \qed \)

**Theorem 7.18.** Let \( P \) be a poset. There is a 1-1 correspondence between (isomorphism classes of) \( \Delta_1 \)-completions and (isomorphism classes of) complete Galois polarities. Moreover, for a fixed poset \( P \) this correspondence restricts to a 1-1 correspondence between (isomorphism classes of) \( \Delta_1 \)-completions of \( P \) and (isomorphism classes of) complete Galois polarities extending \( P \).

**Proof.** Let, for example, \([d]\) stand for an isomorphism class of \( \Delta_1 \)-completions, let \( E \) stand for a complete Galois polarity, and let \( \Theta \) be the map defined by \( E \in \Theta([d]) \) if and only if \( E \) is isomorphic to a complete Galois polarity that generates a \( \Delta_1 \)-completion isomorphic to \( d \).

Now, if \( E_1 \in \Theta([d]) \) and \( E_1 \cong E_2 \), then \( E_2 \in \Theta([d]) \) by definition of \( \Theta \). Moreover, if \( E_1, E_2 \in \Theta([d]) \) then there are complete Galois polarities \( E'_1 \cong E_1 \) and \( E'_2 \cong E_2 \) which generate \( \Delta_1 \)-completions \( d_1 \) and \( d_2 \) respectively, and such that \( d_1 \cong d \cong d_2 \). If \( d_1 \) and \( d_2 \) are isomorphic \( \Delta_1 \)-completions, then it’s easy to construct a polarity isomorphism between the complete Galois polarities they generate, so we have \( E_1 \cong E'_1 \cong E'_2 \cong E_2 \). Thus \( \Theta([d]) \) is an isomorphism class of complete Galois polarities, and this class does not depend on the choice of representative of \( [d] \).

By Lemma 7.17, every \( \Delta_1 \)-completion is isomorphic to the \( \Delta_1 \)-completion generated by the Galois polarity it generates, so \( \Theta \) is a well defined map from the class of isomorphism classes of \( \Delta_1 \)-completions to the class of isomorphism classes of complete Galois polarities.

By Lemma 7.16 if \((e_X, e_Y, R)\) is a complete Galois polarity then it is isomorphic to the complete Galois polarity generated by the \( \Delta_1 \)-completion it generates, so \( \Theta \) is surjective. It’s also easy to see that isomorphic Galois polarities generate isomorphic \( \Delta_1 \)-completions, so \( \Theta \) is injective.

Appealing to Proposition 4.9 the second claim is essentially [13, Theorem 3.4], and we can also obtain this result with the proof above by working modulo isomorphisms of extensions of \( P \) and polarities extending \( P \). \( \qed \)
Before moving on we pause to consider a technical question regarding the polarity extensions discussed in Section 6.1. Given a Galois polarity \((e_X, e_Y, R)\) which is not complete, by Lemma 7.10 there is a polarity embedding from \((e_X, e_Y, R)\) to \((e_{X_D}, e_{Y_D}, R_D)\), where \(e_{X_D}\) and \(e_{Y_D}\) are meet- and join-completions respectively.

By the definition of polarity embeddings, there are order embeddings \(h_X : X \to X_D\) and \(h_Y : Y \to Y_D\), and it’s easy to see these will be meet- and join-completions respectively.

Thus Theorem 6.1 applies and produces a Galois polarity \((e_{X_D}, e_{Y_D}, \overline{R})\). We certainly have \(\overline{R} \subseteq R_D\), by Theorem 6.1 (5), but does the other inclusion also hold? The answer, in general, is no. To see this we borrow [13, Example 2.2], and lean heavily on the discussion at the start of Section 4 in that paper. The MacNeille completion of a poset \(P\) can be constructed from the Galois polarity \((e_{X_0}, e_{Y_0}, R_0)\), where \(e_{X_0} : P \to \mathcal{F}_p\) and \(e_{Y_0} : P \to \mathcal{I}_p\) are the natural embeddings into the sets of principal upsets and downsets of \(P\) respectively.

Consider the poset \(P = \omega \cup \omega^\partial\). I.e. \(P\) is made up of a copy of \(\omega\) below a disjoint copy of the dual \(\omega^\partial\). Then \(\mathcal{N}(P) = \omega \cup \{z\} \cup \omega^\partial\). I.e. \(P\) with an additional element above \(\omega\) and below \(\omega^\partial\). Let \(X = \mathcal{F}_p\) and let \(Y = \mathcal{I}_p\), so \(\mathcal{N}(P)\) is generated by \((e_X, e_Y, R)\). Let the complete Galois polarity arising from Theorem 7.18 be \((e_{X_D}, e_{Y_D}, R_D)\), where \(X_D \cong X \cup \{z_X\}\), and \(Y_D \cong Y \cup \{z_Y\}\). Now, to produce \(\mathcal{N}(P)\) it is necessary that \(z_X R_D z_Y\), but \((z_X, z_Y) \notin \overline{R}\), and thus \(R_D \neq \overline{R}\) in this case.

It also follows from this that the \(\Delta_1\)-completion generated by \((e_X, e_Y, R)\) may not be isomorphic to the one generated by \((e_X \circ e_X, e_Y \circ e_Y, \overline{R})\) from Theorem 6.1 even when \(e_X\) and \(e_Y\) are meet- and join-completions respectively.

7.4. A categorical perspective. Here we assume some familiarity with the basic concepts of category theory. The standard reference is [22], and an accessible introduction can be found in [21].

**Definition 7.19.** We define a pair of categories, \(\text{Pol}\) and \(\text{Del}\) as follows:

**Pol:** Let \(\text{Pol}\) be the category of Galois polarities and polarity morphisms (from Definition 7.6).

**Del:** Let \(\text{Del}\) be the category whose objects are \(\Delta_1\)-completions, and whose maps are commuting squares as described in Definition 2.2, with the additional property that \(g_Q\) in that diagram is a complete lattice homomorphism.

**Theorem 7.20.** Let \(F : \text{Pol} \to \text{Del} \) and \(G : \text{Del} \to \text{Pol}\) be defined as follows:

- **F:** Let \(F : \text{Pol} \to \text{Del}\) be the map that takes a Galois polarity \((e_X, e_Y, R)\) to the \(\Delta_1\)-completion it generates (described in Lemma 7.12), and takes a polarity morphism \((h_X : X_1 \to X_2, h_Y : Y_1 \to Y_2)\) to the map between the \(\Delta_1\)-completions described in Figure 22, where \(\mathcal{N}(\psi)\) is the unique complete lattice homomorphism lift of the map \(\psi : X_1 \uplus Y_1 \to X_2 \uplus Y_2\) from Theorem 7.9 to the respective MacNeille completions, as described in [8, Theorem 3.1].

- **G:** Let \(G : \text{Del} \to \text{Pol}\) be the map that takes a \(\Delta_1\)-completion \(d : P \to D\) to the Galois polarity it generates (described in Lemma 7.14), and takes a \(\Delta_1\)-completion morphism as in Figure 23 to the triple \((g)_{X_D}, f, g)_{Y_D}\), where, for example, \(g)_{X_D}\) is (modulo isomorphism) the restriction of \(g\) to \(X_D\), where this is as defined in Lemma 7.14.
Then \( F \) and \( G \) are functors and form an adjunction \( F \dashv G \).

**Proof.** For ease of reading we will break the proof down into discrete statements which obviously add up to a proof of the claimed result.

- **“\( F \) is well defined”**. \( F \) is certainly well defined on objects. For maps, as Theorem 7.9 says \( \psi \) will be cut-stable, [8] Theorem 3.1 says that \( \mathcal{N} (\psi) \) will be a complete lattice homomorphism. Moreover, the conjunction of these theorems also implies that the diagram in Figure 22 commutes.

- **“\( F \) is a functor”**. To see that \( F \) lifts identity maps to identity maps note that the identity morphism on \((e_X, e_Y, R)\) clearly lifts via Theorem 7.9 to the identity on \( X \uplus Y \), and since taking MacNeille completions is functorial for cut-stable maps (see [8] Corollary 3.3), that \( F \) maps identity morphisms appropriately follows immediately.

Similarly, it follows from the uniqueness of the map \( \psi \) in Theorem 7.9 that if \( h_1 = (h_{X_1}, h_{P_1}, h_{Y_1}) \) induces the map \( \psi_1 \), and if \( h_2 = (h_{X_2}, h_{P_2}, h_{Y_2}) \) induces the map \( \psi_2 \), then the composition \( h_2 \circ h_1 \), if it exists, induces the map \( \psi_2 \circ \psi_1 \). So \( F \) respects composition as the MacNeille completion functor does.

- **“\( G \) is well defined”**. \( G \) is also clearly well defined on objects. Consider now a map as in Figure 23. We must show that \( g|_{X_D_1} : X_{D_1} \to X_{D_2} \), that \( g|_{Y_D_1} : Y_{D_1} \to Y_{D_2} \), and that \((g|_{X_D_1}, f, g|_{Y_D_1})\) satisfies the conditions of Definition 7.6. First, to lighten the notation define \( g X = g|_{X_D_1} \), and \( g Y = g|_{Y_D_1} \).

Now, by commutativity of Figure 23 we have \( g \circ d_1[P_1] \subseteq d_2[P_2] \). Since \( X_{D_1} \) and \( X_{D_2} \) are (modulo isomorphism) the meet-closures of \( d_1[P_1] \) and \( d_2[P_2] \) respectively, and since \( g \) is a complete lattice homomorphism, it follows that \( g X \) does indeed have codomain \( X_{D_2} \), and \( Y_{D_2} \) is the codomain of \( g Y \) by duality. We now check the conditions of Definition 7.6:

1. This follows immediately from the definitions of \( g X \) and \( g Y \) and the commutativity of Figure 23.

2. Abusing notation slightly, we can think of the \( i \) maps as inclusion functions, and so the claim is just the statement that \( y \leq x \implies g(y) \leq g(x) \), and thus is true as \( g \) is monotone.

3. Abusing notation in the same way as before, let \( x' \in X_{D_2} \), let \( y' \in Y_{D_2} \), and suppose \( x' \nleq y' \in D_2 \). Using the completeness of \( D_1 \), let \( z_1 = \bigwedge g^{-1}(x') \), and let \( z_2 = \bigvee g^{-1}(y') \). It follows easily from the fact that \( g \) is a complete lattice homomorphism that \( x' \leq g(z_1) \) and \( g(z_2) \leq y' \), so if \( z_1 \leq z_2 \) then \( x' \leq g(z_1) \leq g(z_2) \leq y' \), as \( g \) is monotone. Thus to avoid contradiction we must have \( z_1 \nleq z_2 \).

As \( X_{D_1} \) and \( Y_{D_1} \) are, respectively, join- and meet-dense in \( D_1 \), there is \( x \in X_{D_1} \cap z_1 \) and \( y \in Y_{D_1} \cap z_2 \) with \( x \nleq y \). Now, as \( x \leq z_1 = \bigwedge g^{-1}(x') \), we have \( g X(x') \subseteq z_1 \subseteq x' \). Thus (i) holds for this choice of \( x \), and (ii) holds for \( y \) by a dual argument.

Moreover, suppose \( a \in X_{D_1} \), and that \( g(a) \leq y' \). Then \( a \in g^{-1}(y') \), and so \( a \leq z_2 \), by definition of \( z_2 \), and consequently \( a \leq y \). Thus (iii) holds, and (iv) is dual. That (v) holds is automatic from the choice of \( x \) and \( y \).

- **“\( G \) is a functor”**. \( G \) obviously sends identity maps to identity maps, and almost as obviously respects composition.
The components of counit of the adjunction between \( F \) and \( G \) are provided by the isomorphisms produced in Lemma \( 7.17 \). Thus the subcategory, \( \text{Fix}(FG) \), of

- \( \text{"} F \dashv G \text{"} \). The unit \( \eta \) is defined so that its components are the embeddings of \((e_X, e_Y, R)\) into \( GF(e_X, e_Y, R)\) described in Lemma \( 7.16 \). We first show that \( \eta \) is indeed a natural transformation. Let \( A = (e_X, e_Y, R_1) \) and \( B = (e_{X_2}, e_{Y_2}, R_2) \) be Galois polarities, and let \( g = (g_X, g_P, g_Y) \) be a polarity morphism from \( A \) to \( B \). We aim to show that the diagram in Figure 23 commutes.

Consider the diagram in Figure 25. Here, for example, \( h_{X_1} : X_1 \to X_{N_1} \) takes the role of \( h_X \) from Lemma \( 7.16 \) embedding \( X_1 \) into \( X_{N_1} \). The map \( g_{X}^2 : X_{N_1} \to X_{N_2} \) is the \( X \) component of \( GFg \), which is, modulo isomorphism, the restriction of \( Fg \) to \( X_{N_1} \), and so on. The inner squares commute by definition of \( g \), and that the outer squares commute can be deduced from the commutativity of the diagram in Figure 26, the commutativity of whose right square follows from the commutativity of the diagram in Figure 27.

Now, \( GFg \circ \eta_A \) is the polarity morphism \( (g_X^2 \circ h_{X_1}, g_P, g_Y^2 \circ h_{Y_1}) \), and \( \eta_B \circ g \) is the polarity morphism \( (h_{X_2} \circ g_X, g_P, h_{Y_2} \circ g_Y) \), and these are equal by the commutativity of the diagram in Figure 26.

Now, let \( E = (e_X, e_Y, R) \) be a Galois polarity extending \( P \). We will show that \( \eta_E \) has the appropriate universal property (see e.g. \[ 21 \] Theorem 2.3.6). Let \( d : Q \to D \) be a \( \Delta_0 \)-completion, and let \( h = (h_X, h_P, h_Y) : E \to G(d) \) be a polarity morphism. We must find a map \( g : F(E) \to d \) such that \( Gg \circ \eta_E = h \), and show that \( g \) is unique with this property.

Consider the diagram in Figure 28. The upper triangle commutes as \( \eta \) is a natural transformation (see Figure 21). The isomorphism between \( G(d) \) and \( GFg(d) \) is just \( \eta_{G(d)} \), by Lemma \( 7.16 \) and the fact that \( G(d) \) is a complete Galois polarity. Note that this is an isomorphism of Galois polarities extending \( Q \), so is the identity map on \( Q \). By Lemma \( 7.17 \) there is an isomorphism, \( \phi : FG(d) \to d \), of extensions of \( Q \), and it follows that \( \eta^{-1}_{G(d)} = G\phi \). Thus \( \phi \circ Fh : F(E) \to d \) has the property that

\[
G(\phi \circ Fh) \circ \eta_E = G\phi \circ GFh \circ \eta_E = \eta_{G(d)}^{-1} \circ GFh \circ \eta_E = h.
\]

We must show that \( \phi \circ Fh \) is unique with this property, so let \( f : F(E) \to d \) be another Del morphism with \( Gf \circ \eta_E = h \). Recall that \( F(E) = e \circ \gamma : P \to \mathcal{N}(X \uplus Y) \). Then \( f \) must agree with \( \phi \circ Fh \) on \( e[X \uplus Y] \), by definition of \( G \). But \( (\phi \circ Fh)|_{e[X \uplus Y]} \) is cut-stable, so extends uniquely to a complete lattice homomorphism (by \[ 8 \] Theorem 3.1). Thus \( f = \phi \circ Fh \) as required, and so \( F \dashv G \) as claimed.

\[
\begin{array}{ccc}
P_1 & \xrightarrow{\gamma_1} & X_1 \uplus Y_1 \xrightarrow{e_1} \mathcal{N}(X_1 \uplus Y_1) \\
& \downarrow{h_P} & \downarrow{\psi} \\
P_2 & \xrightarrow{\gamma_2} & X_2 \uplus Y_2 \xrightarrow{e_2} \mathcal{N}(\psi) \\
\end{array}
\]

\[
\begin{array}{ccc}
P_1 & \xrightarrow{d_1} & D_1 \\
\downarrow{f} & & \downarrow{g} \\
P_2 & \xrightarrow{d_2} & D_2 \\
\end{array}
\]

Figure 22. Figure 23.

The components of counit of the adjunction between \( F \) and \( G \) are provided by the isomorphisms produced in Lemma \( 7.17 \). Thus the subcategory, \( \text{Fix}(FG) \), of
Del is just Del itself. The canonical categorical equivalence between \( \text{Fix}(GF) \) and \( \text{Fix}(FG) \) produces a categorical version of the correspondence in Theorem 7.18.

We end the section with a universal property for Galois polarities whose relation is the minimal \( R_l \).

**Proposition 7.21.** Let \((e_X, e_Y, R_l)\) be a Galois polarity extending \( P \), let \( Q \) be a poset, and let \( f : X \to Q \) and \( g : Y \to Q \) be monotone maps such that \( f \circ e_X = g \circ e_Y \). Let \( \preceq \) be the unique element of \( \overset{\sim}{\mathcal{P}}_{R_l} \). Then the following are equivalent:

1. \( y \preceq x \implies g(y) \leq f(x) \).
2. There is a unique monotone map \( u : X \sqcup Y \to Q \) such that the diagram in Figure 29 commutes.

**Proof.** Suppose (1) holds. We define \( u' : X \cup Y \to Q \) by

\[
\begin{align*}
u'(z) = \begin{cases} f(z) & \text{if } z \in X \\ g(z) & \text{if } z \in Y \end{cases}
\end{align*}
\]

We show that \( u' \) is monotone with respect to the pre-order \( \preceq \) and the order on \( Q \). Let \( z_1 \preceq z_2 \in X \cup Y \). If \( z_1 \) and \( z_2 \) are both in \( X \), or both in \( Y \), then that \( u'(z_1) \preceq u'(z_2) \) follows immediately from the definition of \( u \) and the fact that both \( f \) and \( g \) are monotone. If \( z_1 = x \in X \), and \( z_2 = y \in Y \) then there is
If \( z \) these properties is automatic from the required commutativity of the diagram. Let \( \epsilon_X \) and \( \epsilon_Y \) be fixed meet- and join-extensions respectively, the pair of maps \((\epsilon_X, \epsilon_Y)\) arising from \((e_X, e_Y, R_i)\) is initial in the category whose objects are pairs of monotone maps \((f : X \to Q, g : Y \to Q)\) such that \( f \circ e_X = g \circ e_Y \) and \( y \leq x \implies g(y) \leq f(x) \), and whose maps are commuting triangles as in Figure 30 (here \( h \) is monotone, and commutativity means \( f_2 = h \circ f_1 \) and \( g_2 = h \circ g_1 \)). In particular this category contains all \((\iota_X, \iota_Y)\) arising from Galois polarities \((e_X, e_Y, R)\) based on \( e_X \) and \( e_Y \).

8. A Duality Principle for Order Polarities

8.1. The theory of order polarities. We want to think of order polarities in their various forms as the classes of models for certain theories. A similar approach is taken for ordinary (i.e., not ‘order’) polarities in [19, Section 5], but we must extend this system to deal with the additional features of extension polarities. For convenience we will use \( \approx \) as a logical symbol representing equality.

**Definition 8.1.** Let \( \mathcal{L} = \{P, \mathcal{X}, \mathcal{Y}, \mathcal{R}, \triangleleft, e_X, e_Y\} \), where \( P, \mathcal{X}, \mathcal{Y} \) are unary predicates, and \( \mathcal{R}, \triangleleft, e_X, e_Y \) are binary predicates. Then \( \mathcal{L} \) is the signature of extension polarities.

**Definition 8.2** (Tpol). Let \( E \) be an \( \mathcal{L} \)-structure. We can write down a first-order \( \mathcal{L} \)-sentence guaranteeing that:

1. For all \( z \in E \) exactly one of \( P(z) \), \( \mathcal{X}(z) \) and \( \mathcal{Y}(z) \) holds.
2. \( \triangleleft \) defines a partial ordering on \( E \).
3. For all \( z_1, z_2 \in E \), if \( z_1 \triangleleft z_2 \) then either \( P(z_1) \) and \( P(z_2) \), \( \mathcal{X}(z_1) \) and \( \mathcal{X}(z_2) \), or \( \mathcal{Y}(z_1) \) and \( \mathcal{Y}(z_2) \).
4. For all \( z_1, z_2 \in E \), if \( z_1 \mathcal{R} z_2 \) then \( \mathcal{X}(z_1) \) and \( \mathcal{Y}(z_2) \).
5. \( e_X \) corresponds to an order embedding from \( \{z \in E : P(z)\} \) to \( \{z \in E : \mathcal{X}(z)\} \).
It will help us later to be explicit here, so define $\mathcal{L}$-sentences as follows:

1. $\forall z_1 z_2 z_3 \left( (z_1 \triangleleft z_2) \land (z_2 \triangleleft z_3) \to z_1 \approx z_2 \right) \land \left( (z_1 \triangleleft z_2) \land (z_2 \triangleleft z_3) \to (z_1 \triangleleft z_3) \right)$.

2. $\forall z_1 z_2 \left( (z_1 \triangleleft z_2) \to \left( (\mathcal{P}(z_1) \land \mathcal{X}(z_1)) \lor (\mathcal{X}(z_1) \land \mathcal{Y}(z_1)) \right) \right)$.

3. $\forall z_1 z_2 \left( (z_1 \triangleleft z_2) \to \left( (\mathcal{P}(z_1) \land \mathcal{P}(z_2)) \lor (\mathcal{X}(z_1) \land \mathcal{X}(z_2)) \lor (\mathcal{Y}(z_1) \land \mathcal{Y}(z_2)) \right) \right)$.

4. $\forall z_1 z_2 \left( R(z_1, z_2) \to (\mathcal{X}(z_1) \land \mathcal{Y}(z_2)) \right)$.

5. $\forall z_1 \left( \mathcal{P}(z_1) \to \exists z_2 (\mathcal{X}(z_2) \land e_\mathcal{X}(z_1, z_2)) \right) \land \left( \exists z_2 (e_\mathcal{X}(z_1, z_2)) \to \mathcal{P}(z_1) \right)$
   \[ \land \forall z_2 z_3 z_4 \left( (z_1 \triangleleft z_2) \land e_\mathcal{X}(z_1, z_3) \land e_\mathcal{X}(z_2, z_4) \to (z_3 \triangleleft z_4) \right) \land \forall z_2 z_3 z_4 \left( (z_3 \triangleleft z_4) \land e_\mathcal{X}(z_1, z_3) \land e_\mathcal{X}(z_2, z_4) \to (z_1 \triangleleft z_2) \right) \right)$.

6. Like (5) but substituting $\mathcal{Y}$ for $\mathcal{X}$, and $e_\mathcal{Y}$ for $e_\mathcal{X}$.

We get a single sentence by taking the conjunction of all the sentences we have defined. Note that (5) and (6) requires (2) to guarantee that $e_\mathcal{X}$ and $e_\mathcal{Y}$ are well defined and injective. Denote the set of these axioms $\text{Tpol}$.

**Proposition 8.3.** If $E$ is an $\mathcal{L}$-structure and $E \models \text{Tpol}$, then

$\{e_\mathcal{X} : \{ z \in E : P(z) \} \to \{ z \in E : X(z) \}, e_\mathcal{Y} : \{ z \in E : P(z) \} \to \{ z \in E : Y(z) \}, R \}$

defines an extension polarity when $P$, $X$, $Y$, $R$ are the interpretations of $\mathcal{P}$, $\mathcal{X}$, $\mathcal{Y}$, and where $e_\mathcal{X}$, $e_\mathcal{Y}$, $\leq$ are defined using $e_\mathcal{X}$, $e_\mathcal{Y}$, $\triangleleft$ in the obvious way. Moreover, if $(e_\mathcal{X}, e_\mathcal{Y}, R)$ is an extension polarity then $(e_\mathcal{X}, e_\mathcal{Y}, R)$ can be naturally understood as an $\mathcal{L}$-structure, and $(e_\mathcal{X}, e_\mathcal{Y}, R) \models \text{Tpol}$.

**Proof.** This is straightforward. \qed

**Definition 8.4.** Define $\text{EP}$ to be the class of $\mathcal{L}$-structures where $\text{Tpol}$ holds.

8.2. Dual formulas and dual polarities.

**Definition 8.5** ($E^\partial$). Let $E$ be an $\mathcal{L}$-structure, and suppose the symbols of $\mathcal{L}$ are interpreted in $E$ as $P$, $X$, $Y$, $\leq$, $e_\mathcal{X}$, $e_\mathcal{Y}$, $R$. Define $E^\partial$ to be the $\mathcal{L}$-structure whose underlying set is that of $E$, and whose interpretations of the symbols of $\mathcal{L}$ are as follows:

- $\mathcal{P}$ is $P$.
- $\mathcal{X}$ is $Y$.
- $\mathcal{Y}$ is $X$.
- $\triangleleft$ is $\leq^\partial$, which is defined using $z_1 \leq^\partial z_2 \iff z_2 \leq z_1$.
- $R$ is $R^\partial$, which is defined by $R^\partial(z_1, z_2) \iff R(z_2, z_1)$.
- $e_\mathcal{X}$ is $e_\mathcal{Y}$.
- $e_\mathcal{Y}$ is $e_\mathcal{X}$. 

(6) $e_\mathcal{Y}$ corresponds to an order embedding from $\{ z \in E : \mathcal{P}(z) \}$ to $\{ z \in E : \mathcal{Y}(z) \}$. 


Definition 8.6 ($\theta^\varnothing$, $\Gamma^\varnothing$). Let $\theta$ be a second-order $\mathcal{L}$-formula. We define the dual, $\theta^\varnothing$, recursively. As $\mathcal{L}$ is relational the only terms are variables. We define the dual for atomic $\mathcal{L}$-formulas by:

- $(\mathcal{P}(z))^\varnothing = \mathcal{P}(z)$.
- $(\mathcal{X}(z))^\varnothing = \mathcal{X}(z)$.
- $(\mathcal{Y}(z))^\varnothing = \mathcal{Y}(z)$.
- $(eX(z_1, z_2))^\varnothing = eX(z_1, z_2)$.
- $(eY(z_1, z_2))^\varnothing = eX(z_1, z_2)$.
- $(z_1 < z_2)^\varnothing = z_2 < z_1$.
- $(R(z_1, z_2))^\varnothing = R(z_2, z_1)$.
- $(z_1 \approx z_2)^\varnothing = z_1 \approx z_2$.

We extend this to first-order $\mathcal{L}$-formulas by defining:

- $(-\theta)^\varnothing = \neg(\theta^\varnothing)$.
- $(\theta_1 \land \theta_2)^\varnothing = \theta_1^\varnothing \land \theta_2^\varnothing$.
- $(\forall \theta)^\varnothing = \forall \theta^\varnothing$.

Finally, to extend to second-order formulas, suppose $Z$ is a predicate variable and define:

- $(\forall Z \theta)^\varnothing = \forall Z \theta^\varnothing$.

If $\Gamma = \{ \theta_i : i \in I \}$ is a set of $\mathcal{L}$-formulas, then we define $\Gamma^\varnothing = \{ \theta_i^\varnothing : i \in I \}$.

Lemma 8.7 (Duality lemma). Let $E$ be an $\mathcal{L}$-structure, and let $\theta$ be a second-order $\mathcal{L}$-formula. Let $v$ be an assignment of variables for $E$, and note that $v$ also defines an assignment of variables for $E^\varnothing$. Then $E, v \models \theta \iff E^\varnothing, v \models \theta^\varnothing$.

Proof. We induct on formula construction. Again, as Tpol is a relational signature the only terms are variables. If $\theta$ is atomic there are nine cases:

- $\theta = z_1 \approx z_2$: We have $v(z_1) = v(z_2)$ in both $E$ and $E^\varnothing$.
- $\theta = \mathcal{P}(z)$: We $P(v(z))$, and there is nothing to prove.
- $\theta = \mathcal{X}(z)$: We have $\theta^\varnothing = \mathcal{Y}(z)$, and $\mathcal{Y}$ is interpreted in $E^\varnothing$ as $X$. Since $X(v(z))$ there is nothing to do.
- $\theta = \mathcal{Y}(z)$: Similar to the preceding case.
- $\theta = eX(z_1, z_2)$: Here $\theta^\varnothing = eY(z_1, z_2)$, and $eY$ is interpreted in $E^\varnothing$ as $eX$. Since $eX(v(z_1), v(z_2))$ there is nothing to do.
- $\theta = eY(z_1, z_2)$: Similar to the preceding case.
- $\theta = z_1 < z_2$: We have
  
  $$E, v \models z_1 < z_2 \iff v(z_1) \leq v(z_2) \iff v(z_2) \leq^\varnothing v(z_1) \iff E^\varnothing, v \models z_2 < z_1.$$  

- $\theta = R(z_1, z_2)$: We have $R(v(z_1), v(z_2)) \iff R^\varnothing(v(z_2), v(z_1))$.
- $\theta = Z(z_1, \ldots, z_n)$ for some $n$-ary predicate variable $Z$: This is automatic.

Now for the inductive step we have four cases:

- $\theta = \neg \psi$: In this case
  
  $$E, v \models \neg \psi \iff E, v \not\models \psi \iff E^\varnothing, v \not\models \psi^\varnothing \iff E^\varnothing, v \models \neg \psi^\varnothing.$$  

- $\theta = \psi_1 \lor \psi_2$: We have either $E, v \models \psi_1$ or $E, v \models \psi_2$, and the result follows.
- $\theta = \forall z \psi$: If $u$ is an assignment of variables for $E$ agreeing with $v$ everywhere except, possibly, at $z$, we have $E, u \models \psi$, and so $E^\partial, u \models \psi^\partial$, and thus $E^\partial, v \models \forall z \psi^\partial$. So $E, v \models \forall z \psi \Rightarrow E^\partial, v \models \forall z \psi^\partial$, and the argument for the converse is similar.

- $\theta = \forall S \psi$: Essentially the same argument as in the preceding case.

\[ \square \]

Lemma 8.8. Consider the axioms Tpol from Definition 8.2. (1)-(4) are self-dual, and (5) and (6) are dual to each other.

Proof. This is a routine check. \[ \square \]

Corollary 8.9. $E \in EP \iff E^\partial \in EP$.

Proof. This follows immediately from Lemmas 8.7 and 8.8. \[ \square \]

The conditions from Theorems 3.3, 3.12, 3.17, and Corollary 3.14 all correspond to $L$-sentences, as do the conditions that $e_X$ and $e_Y$ are meet- and join-extensions respectively. These conditions are all first-order, except (D0) and (D1) which require quantification over sets. Note that an order polarity $(X, Y, R)$ is an extension polarity $(e_X, e_Y, R)$ where $P$ is empty. I.e. which is a model of the self-dual $L$-sentence

$$\forall z (\neg R(z)).$$

Bearing this in mind, and recalling Proposition 8.3, we make the following definition.

Definition 8.10. For $* \in \{0, 1, 2, 3\}$ define Tpol$_*$ to be the finite set of $L$-sentences defining $*$-coherence. Similarly, define Tpol$_g$ to be the finite set of $L$-sentences defining Galois polarities. Moreover, for $* \in \{0, 1, 2, 3, g\}$ define EP$_*$ to be the class of $L$-structures satisfying Tpol$_*$. 

Note that Tpol$_3$ uses some second-order axioms, as mentioned previously.

Lemma 8.11. Let $* \in \{0, 1, 2, 3\}$ and let $E$ be an $L$-structure. Then

$$E \in EP_* \iff E^\partial \in EP_*.$$

Proof. This follows from Lemma 8.7 and the fact that the $L$-sentences involved are all either self-dual or come in dual pairs. Lemma 8.8 proves this for EP, and for the EP$_*$ cases we just need to look at the conditions from the corresponding theorems. For example, both (B0) and (B1) are self-dual. \[ \square \]

Theorem 8.12 (Duality principle for order polarities). Let $* \in \{0, 1, 2, 3, g\}$, and let $\theta$ be a second-order $L$-sentence. Then Tpol$_* \models \theta \iff$ Tpol$_* \models \theta^\partial$.

Proof. Suppose Tpol$_* \models \theta$, and let $E \in EP_*$. Then $E^\partial \in EP_*$. by Lemma 8.11. So $E^\partial \models \theta$, and it follows from Lemma 8.7 that $E^{\partial \partial} \models \theta^\partial$. But $E^{\partial \partial} = E$, so we have $E \models \theta^\partial$ as required. Thus Tpol$_* \models \theta \Rightarrow$ Tpol$_* \models \theta^\partial$. The argument for the converse is similar. \[ \square \]
8.3. Applying the duality principle. We have appealed to this duality principle several times during the course of the paper. We go through the details of a pair of representative examples here.

**Example 8.13.** Consider the proof of Lemma 8.11. First it is shown that

\[(A0) \land (\bar{1}_0) \models (\bar{1}_1).
\]

So, by Lemma 8.7 we have \((A0)^0 \land (\bar{1}_0)^0 \models (\bar{1}_1)^0\), but \((A0)\) and \((\bar{1}_1)\) are dual to \((A1)\) and \((\bar{1}_2)\) respectively, and \((\bar{1}_0)\) is self-dual. So we have

\[(A1) \land (\bar{1}_0) \models (\bar{1}_2)
\]
as claimed.

In the next example we abuse notation slightly by writing, for example, \(e_X(p)\) as a shorthand way to specify “the element \(z\) such that \(e_X(p, z)\)”.

**Example 8.14.** Consider the proof of Theorem 8.12 specifically the proof of the transitivity of \(\leq_1\), and, even more specifically, the \((x_1, y, x_2)\) case. Let \(\theta_1\) be the \(L\)-formula defined by

\[\theta_1 = \mathcal{X}(x_1) \land \mathcal{X}(x_2) \land \mathcal{Y}(y) \land \mathcal{R}(x_1, y) \land \exists p q (\mathcal{P}(p) \land \mathcal{P}(q) \land \mathcal{R}(e_X(p), e_Y(q)) \land (y \leq e_Y(p)) \land (e_X(q) \leq x_2)),\]

and define \(\theta_2\) by

\[\theta_2 = \mathcal{X}(x_1) \land \mathcal{X}(x_2) \land \exists p \mathcal{R}(x_1, e_Y(p)) \land (e_X(p) \leq x_2)).\]

Define \(\theta\) by

\[\theta = \forall x_1 x_2 y (\theta_1 \rightarrow \theta_2).\]

Then the \((x_1, y, x_2)\) case of the transitivity proof is showing that \(T_{\text{pol}} \models \theta\), and so by the duality principle we have \(T_{\text{pol}}^0 \models \theta^0 = \forall x_1 x_2 y (\theta_1^0 \rightarrow \theta_2^0)\).

But, if we substitute the variable symbols \(y_2, y_1, x\) for \(x_1, x_2, y\) respectively we get

\[\theta_1^0 = \mathcal{Y}(y_2) \land \mathcal{Y}(y_1) \land \mathcal{X}(x) \land \mathcal{R}(x_1, y_2) \land \exists p q (\mathcal{P}(p) \land \mathcal{P}(q) \land \mathcal{R}(e_X(q), e_Y(p)) \land (e_X(q) \leq x) \land (y_1 \leq e_Y(q))),\]

and

\[\theta_2^0 = \mathcal{Y}(y_2) \land \mathcal{Y}(y_1) \land \exists p (\mathcal{P}(p) \land \mathcal{R}(e_X(p), y_2) \land (y_1 \leq e_Y(p))).\]

So

\[T_{\text{pol}} \models \forall x y_1 y_2 \theta^0,\]

which proves the \((y_1, x, y_2)\) case of transitivity.

As the reader has no doubt observed from these examples, formal application of the duality principle can involve some rather tedious bookkeeping. Fortunately, it is usually fairly easy to see where it applies, and the details can be safely suppressed.

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