INCREASING PROPERTY AND LOGARITHMIC CONVEXITY
OF FUNCTIONS INVOLVING RIEMANN ZETA FUNCTION

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Abstract. Let $\alpha > 0$ be a constant, let $\ell \geq 0$ be an integer, and let $\Gamma(z)$
denote the classical Euler gamma function. With the help of the integral representation for the Riemann
czeta function $\zeta(z)$, by virtue of a monotonicity rule for the ratio of two integrals with a parameter,
and by means of complete monotonicity and another property of the function $\frac{1}{e^t - 1}$ and its derivatives,
the authors present that,

1) for $\ell \geq 0$, the function $x \mapsto \frac{(x + \alpha + \ell)}{\alpha} \zeta(x + \alpha) \zeta(x)$
is increasing from $(1, \infty)$ onto $(0, \infty)$, where $\binom{z}{w}$ denotes the extended
binomial coefficient;

2) for $\ell \geq 1$, the function $x \mapsto \Gamma(x + \ell) \zeta(x)$ is logarithmically convex on
$(1, \infty)$.

1. Motivations and main results

In this paper, we use the notation

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$,

$\mathbb{N} = \{1, 2, \ldots\}$,

$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$,

$\mathbb{N}_- = \{-1, -2, \ldots\}$.

It is well known that the classical Euler gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{\prod_{k=0}^{n} (z + k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}.$$ 

For more information and recent developments of the gamma function $\Gamma(z)$ and its logarithmic derivatives $\psi^{(n)}(z)$ for $n \geq 0$, please refer to [1, Chapter 6], [25, Chapter 3], or recently published papers [14, 18, 20, 21, 31] and closely related references therein.

According to [4, Fact 13.3], for $z \in \mathbb{C}$ such that $\Re(z) > 1$, the Riemann zeta function $\zeta(z)$ can be defined by

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} = \frac{1}{1 - 2^{-z}} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^z} = \frac{1}{1 - 2^{1-z}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^z} \quad (1.1)$$

and has the integral representation

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_{0}^{\infty} \frac{t^{z-1}}{e^t - 1} \, dt, \quad \Re(z) > 1. \quad (1.2)$$

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The last two definitions in (1.1) tell us some reasons why many mathematicians investigated the Dirichlet eta and lambda functions

\[ \eta(z) = \left(1 - \frac{1}{2^z}\right)\zeta(z) \quad \text{and} \quad \lambda(z) = \left(1 - \frac{1}{2^z}\right)\zeta(z). \]

According to discussions in [25, Section 3.5, pp. 57–58], the Riemann zeta function \( \zeta(z) \) has an analytic continuation which has the only singularity \( z = 1 \), a simple pole with residue 1, on the complex plane \( \mathbb{C} \).

We collect several known properties and applications of the Riemann zeta function \( \zeta(x) \), the Dirichlet eta function \( \eta(x) \), and the Dirichlet lambda function \( \lambda(x) \) as follows.

1. In 1998, Wang [27] proved that the Dirichlet eta function \( \eta(x) \) is logarithmically concave on \((0, \infty)\). In 2018, Qi [12, 17] used this result to establish a double inequality for bounding the ratio \( \frac{|B_{2n+1}|}{B_{2n}} \) for \( n \in \mathbb{N} \), where the Bernoulli numbers \( B_{2n} \) for \( n \geq 0 \) are generated by

\[ \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi. \]

2. In 2009, Cerone and Dragomir [5] proved that the reciprocal \( \frac{1}{\zeta(x)} \) is concave on \((1, \infty)\).

3. In 2010, Zhu and Hua [35] proved that the sequence \( \lambda(n) \) for \( n \in \mathbb{N} \) is decreasing. In 2018, Qi [12, 17] used also this result while he established a double inequality for bounding the ratio \( \frac{|B_{2n+1}|}{B_{2n}} \) for \( n \in \mathbb{N} \). In 2020, Zhu [34] used this result once to discuss those conclusions in [12, 17].

4. In 2015, Adell–Lekuona [2] and Alzer–Kwong [3] proved that the Dirichlet eta function \( \eta(x) \) is concave on \((0, \infty)\).

5. In 2019, Hu and Kim [9] obtained a number of infinite families of linear recurrence relations and convolution identities for the Dirichlet lambda function \( \lambda(2n) \) for \( n \in \mathbb{N} \).

6. In 2020, Yang and Tian [33] proved that the function

\[ \frac{1}{2^x} \frac{\zeta(x) - 2^{-p} \zeta(x + p)}{\zeta(x) - \zeta(x + p)} \]

is increasing from \((1, \infty)\) onto \((\frac{1}{2}, 1)\). By this result, Yang and Tian [33] extended and sharpened the double inequality established in [12, 17] for bounding the ratio \( \frac{|B_{2n+1}|}{B_{2n}} \) for \( n \in \mathbb{N} \).

In this paper, we consider

1. the function

\[ x \mapsto \left(\frac{x + \alpha + \ell}{\alpha}\right) \frac{\zeta(x + \alpha)}{\zeta(x)} \]  

and its monotonicity on \((1, \infty)\), where \( \alpha > 0 \) is a constant, \( \ell \in \mathbb{N}_0 \),

\[ \begin{align*}
\Gamma(z + 1) \quad & \text{if } z \not\in \mathbb{N}_-, \quad w, z - w \not\in \mathbb{N}_- \\
\Gamma(w + 1)\Gamma(z - w + 1) & \text{if } z \not\in \mathbb{N}_-, \quad w \in \mathbb{N}_- \text{ or } z - w \in \mathbb{N}_- \\
0 & \text{if } z \in \mathbb{N}_-, \quad w \in \mathbb{N}_0 \\
\frac{\zeta(z)}{w!} & \text{if } z, w \in \mathbb{N}_-, \quad z - w \in \mathbb{N}_0 \\
\frac{\zeta(z - w)}{(z - w)!} & \text{if } z, w \in \mathbb{N}_-, \quad z - w \in \mathbb{N}_- \\
0 & \text{if } z \in \mathbb{N}_-, \quad w \not\in \mathbb{Z}
\end{align*} \]  

(1.3)
for $z, w \in \mathbb{C}$ denotes the extended binomial coefficient [28], and

\[
q_n = \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} \binom{k}{n-k} (z-w)^k
\]

for $\beta \in \mathbb{C}$ is called the falling factorial.

(2) the function $\Gamma(x + \ell)\zeta(x)$ on $(1, \infty)$ for $\ell \in \mathbb{N}$ and its logarithmic convexity.

2. Lemmas

For proving our main results in this paper, we need the following lemmas.

Lemma 2.1 (Monotonicity rule for the ratio of two integrals with a parameter [15, Lemma 2.8 and Remark 6.3] and [19, Remark 7.2]). Let $U(t), V(t) > 0$, and $W(t, x) > 0$ be integrable in $t \in (a, b)$,

1. if the ratios $\frac{\partial W(t, x)/\partial t}{W(t, x)}$ and $\frac{U(t)}{V(t)}$ are both increasing or both decreasing in $t \in (a, b)$, then the ratio

\[
R(x) = \int_a^b W(t, x)U(t) \, dt \int_a^b W(t, x)V(t) \, dt
\]

is increasing in $x$;

2. if one of the ratios $\frac{\partial W(t, x)/\partial t}{W(t, x)}$ and $\frac{U(t)}{V(t)}$ is increasing and another one of them is decreasing in $t \in (a, b)$, then the ratio $R(x)$ is decreasing in $x$.

Lemma 2.2 ([7, Theorem 2.1], [8, Theorem 2.1], and [32, Theorem 3.1]). Let $\vartheta \neq 0$ and $\vartheta \neq 0$ be real constants and $k \in \mathbb{N}$. When $\vartheta > 0$ and $t \neq -\frac{\ln q}{\vartheta}$ or when $\vartheta < 0$ and $t \in \mathbb{R}$, we have

\[
\frac{d^k}{dt^k} \left( \frac{1}{\vartheta e^{\vartheta t} - 1} \right) = (-1)^k \vartheta^k \sum_{p=1}^{k+1} (p-1)! S(k+1, p) \left( \frac{1}{\vartheta e^{\vartheta t} - 1} \right)^p,
\]

where

\[
S(k, p) = \frac{1}{p!} \sum_{q=1}^{p} (-1)^{p-q} \left( \begin{array}{c} p \end{array} \right) q^k, \quad 1 \leq p \leq k
\]

are the Stirling numbers of the second kind.

For detailed information on the Stirling numbers of the second kind $S(k, m)$ for $1 \leq m \leq k$, please refer to [1, pp. 824–825, 24.1.4], [25, pp. 18–21, Section 1.3], the papers [13, 16], or the monograph [22] and closely related references therein.

Recall from [11, Chapter XIII], [23, Chapter I], [30, Chapter IV], and recently published papers [14, 18, 20, 21] that

1. a function $q(x)$ is said to be completely monotonic on an interval $I$ if it is infinitely differentiable and $(-1)^n q^{(n)}(x) \geq 0$ for $n \geq 0$ on $I$.

2. a positive function $q(x)$ is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it is infinitely differentiable and its logarithm $\ln f(x)$ satisfies $(-1)^n \ln f^{(n)}(x) \geq 0$ for $n \in \mathbb{N}$ on $I$.

Lemma 2.3 ([6, p. 98] and [26, p. 395]). If a function $q(x)$ is non-identically zero and completely monotonic on $(0, \infty)$, then $q(x)$ and its derivatives $q^{(k)}(x)$ for $k \in \mathbb{N}$ are impossibly equal to 0 on $(0, \infty)$.

Lemma 2.4 ([29, Theorem 1]). For $k \in \{0\} \cup \mathbb{N}$, the functions

\[
\mathcal{F}_k(t) = (-1)^k \left( \frac{1}{e^t - 1} \right)^{(k)}
\]
are both increasing on \((0, \infty)\). More strongly, the function \(F_0(t)\) is logarithmically completely monotonic on \((0, \infty)\).

3. Increasing Property and Logarithmic Convexity of Two Functions Involving the Riemann Zeta Function

We are now in a position to state and prove our main results in this paper.

**Theorem 3.1.** Let \(\alpha > 0\) be a constant and let \(\ell \in \mathbb{N}_0\) be an integer. Then the function defined in (1.3) is increasing from \((1, \infty)\) onto \((0, \infty)\). Consequently, for fixed \(\ell \in \mathbb{N}\), the function \(\Gamma(x + \ell)\) is logarithmically convex in \(x \in (1, \infty)\).

**Proof.** By virtue of the recurrence relation \(\Gamma(z + 1) = z\Gamma(z)\) and the integral representation (1.2), integrating by parts yields

\[
\frac{\Gamma(x + \alpha + 1) \zeta(x + \alpha)}{\Gamma(x + 1)} \cdot \zeta(x) = \frac{\Gamma(x + \alpha + 1)}{\Gamma(x + 1)} \cdot \frac{\Gamma(\frac{x + \alpha - 1}{e^t - 1})}{\Gamma(\frac{x}{e^t - 1})} \int_0^\infty \frac{t^x + \alpha - 1}{e^t - 1} \, dt
\]

\[
= \frac{(x + \alpha)}{x} \cdot \frac{\Gamma(x + \alpha + 1)}{\Gamma(x + 1)} \int_0^\infty \frac{t^{x + \alpha - 1}}{e^t - 1} \, dt
\]

\[
= \int_0^\infty \frac{1}{e^t - 1} \left( \frac{t^x + \alpha}{t^x} \right) \, dt
\]

\[
= \int_0^\infty \frac{1}{e^t - 1} \left( t^x + \alpha \right) \, dt
\]

\[
= \int_0^\infty \frac{1}{e^t - 1} \left( t^x \right) \, dt
\]

\[
= \int_0^\infty \frac{e^t}{(e^t - 1)^2} \, dt
\]

Applying Lemma 2.1 to

\[
U(t) = \frac{e^t t^\alpha}{(e^t - 1)^2}, \quad V(t) = \frac{e^t}{(e^t - 1)^2} > 0, \quad W(t, x) = t^x > 0,
\]

and \((a, b) = (0, \infty)\), since \(\frac{U(t)}{V(t)} = t^\alpha\) and

\[
\frac{\partial W(t, x)}{\partial x} = \ln t
\]

are both increasing on \((0, \infty)\), we conclude that the ratio

\[
\frac{\int_0^\infty \frac{e^t}{(e^t - 1)^2} \, dt}{\int_0^\infty \frac{e^t}{(e^t - 1)^2} \, dt} = \frac{\Gamma(x + \alpha + 1) \zeta(x + \alpha)}{\Gamma(x + 1) \zeta(x)}
\]

\[
= \Gamma(x + 1) \left( \frac{x + \alpha}{\alpha} \right) \frac{\zeta(x + \alpha)}{\zeta(x)}
\]

is increasing in \(x \in (1, \infty)\), where we used the definition (1.4). Consequently, the function in (1.3) for \(\ell = 0\) is increasing in \(x \in (1, \infty)\).

Inductively, for \(\ell, m > 1\), we obtain

\[
\frac{\Gamma(x + \alpha + \ell)}{\Gamma(x + m)} \cdot \zeta(x + \alpha) = \frac{\Gamma(x + \alpha + \ell)}{\Gamma(x + m)} \cdot \frac{\Gamma(x + \alpha + 1)}{\Gamma(x + 1)} \cdot \zeta(x + \alpha)
\]

\[
= \frac{\Gamma(x + \alpha + \ell)}{\Gamma(x + m)} \cdot \frac{\Gamma(x + \alpha + 1)}{\Gamma(x + 1)} \cdot \frac{\Gamma(\frac{x + \alpha - 1}{e^t - 1})}{\Gamma(\frac{x}{e^t - 1})} \int_0^\infty \frac{t^{x + \alpha}}{e^t - 1} \, dt
\]

\[
= \frac{\Gamma(x + \alpha + \ell)}{\Gamma(x + m)} \cdot \frac{\Gamma(x + \alpha + 1)}{\Gamma(x + 1)} \cdot \int_0^\infty \frac{t^{x + \alpha}}{e^t - 1} \, dt
\]

\[
= \frac{\Gamma(x + \alpha + \ell)}{\Gamma(x + m)} \cdot \frac{\Gamma(x + \alpha + 1)}{\Gamma(x + 1)} \cdot \int_0^\infty \frac{e^t}{(e^t - 1)^2} \, dt
\]
where $2 \leq i \leq \ell - 1$, $2 \leq j \leq m - 1$, and we used (2.1) in Lemma 2.2 for $\theta = \vartheta = 1$ for reaching the limits

$$
\left[ (-1)^k F_k(t) t^{x+k} \right]_{t=0^+}^{t=\infty} = \left[ \left( \frac{1}{e^t - 1} \right)^k t^{x+k} \right]_{t=0^+}^{t=\infty} = (-1)^k \left( \sum_{p=1}^{k+1} (p-1)! S(k+1,p) \left( \frac{1}{e^t - 1} \right)^p \right)_{t=0^+}^{t=\infty} = (-1)^k \left( \sum_{p=1}^{k+1} (p-1)! S(k+1,p) \left( \frac{1}{e^t - 1} \right) \right)_{t=0^+}^{t=\infty} = 0
$$

for $k \in \mathbb{N}$ and $F_k(t)$ is defined by (2.2) in Lemma 2.4.

By Lemmas 2.3 and 2.4, we see that the functions $F_k(t)$ for $k \geq 0$ are all positive on $(0, \infty)$. Once applying Lemma 2.1 to

$$
U(t) = F_x(t) t^{\alpha+\ell}, \quad V(t) = F_m(t) t^m > 0, \quad W(t, x) = t^{x} > 0,
$$

and $(a, b) = (0, \infty)$, since $U(t) V(t) = F_x(t) t^{\alpha+\ell}$ for $m = \ell$ and the partial derivative in (3.1) are both increasing on $(0, \infty)$, we acquire that the ratio

$$
\frac{\Gamma(x + \alpha + \ell)}{\Gamma(x + \ell)} \frac{\zeta(x + \alpha + \ell)}{\zeta(x)} = \frac{\Gamma(\alpha+1)}{\alpha} \left( \frac{x + \alpha + \ell - 1}{\alpha} \right) \frac{\zeta(x + \alpha + \ell)}{\zeta(x)} = \frac{\int_0^\infty F_x(t) t^{\alpha+\ell} \, dt}{\int_0^\infty F_x(t) t^\ell \, dt}
$$
for $\ell > 1$ and $\alpha > 0$ is increasing in $x \in (1, \infty)$, where we used the definition (1.4).

Consequently, the function in (1.3) for $\ell > 0$ is increasing in $x \in (1, \infty)$.

Because the function

$$\frac{\Gamma(x + \alpha + \ell) \zeta(x + \alpha)}{\Gamma(x + \ell) \zeta(x)} = \frac{\Gamma(x + \alpha + \ell) \zeta(x + \alpha)}{\Gamma(x + \ell) \zeta(x)}$$

for fixed $\ell \in \mathbb{N}$ is increasing in $x \in (1, \infty)$, its derivative

$$\left[ \frac{\Gamma(x + \alpha + \ell) \zeta(x + \alpha)}{\Gamma(x + \ell) \zeta(x)} \right]' = \left[ \frac{\Gamma(x + \alpha + \ell) \zeta(x + \alpha)}{\Gamma(x + \ell) \zeta(x)} \right]'$$

is positive for $x \in (1, \infty)$. This means that

$$\frac{\Gamma(x + \alpha + \ell) \zeta(x + \alpha)'}{\Gamma(x + \alpha + \ell) \zeta(x + \alpha)} > \frac{\Gamma(x + \ell) \zeta(x)'}{\Gamma(x + \ell) \zeta(x)},$$

that is, the logarithmic derivative

$$\langle \ln[\Gamma(x + \ell) \zeta(x)] \rangle = \frac{\Gamma(x + \ell) \zeta(x)'}{\Gamma(x + \ell) \zeta(x)}$$

is increasing in $x \in (1, \infty)$. Consequently, for fixed $\ell \in \mathbb{N}$, the function $\Gamma(x + \ell) \zeta(x)$ is logarithmically convex in $(1, \infty)$. The proof of Theorem 3.1 is complete. \hfill \Box

4. A SHORT APPENDIX

In this section, we slightly strengthen [29, Theorem 3] as follows.

**Proposition 4.1.** For $k \in \{0\} \cup \mathbb{N}$, the ratio

$$\mathcal{F}_k(t) = \frac{\mathcal{F}_{k+1}(t)}{\mathcal{F}_k(t)}$$

is decreasing from $(0, \infty)$ onto $(1, \infty)$, where the function $\mathcal{F}_k(t)$ is defined by (2.2) in Lemma 2.4.

**Proof.** In [29, Theorem 3], the decreasing property of the ratio $\mathcal{F}_k(t)$ in (4.1) and the limit $\lim_{t \to 1} \mathcal{F}_k(t) = 1$ has been proved.

Making use of the equation (2.1) in Lemma 2.2 for $\vartheta = \theta = 1$ yields

$$\mathcal{F}_k(t) = \frac{(-1)^{k+1}((1-x)^{k+1})}{(-1)^{k}((1-x)^{k})}$$

$$= \frac{\sum_{p=1}^{k+2} (p-1)!S(k+2, p)(\frac{1}{x^{p-1}})^p}{\sum_{p=1}^{k+1} (p-1)!S(k+1, p)(\frac{1}{x^{p-1}})^p}$$

$$= \frac{\sum_{p=1}^{k+2} (p-1)!S(k+2, p)(\frac{1}{x^{p-1}})^p}{\sum_{p=1}^{k+1} (p-1)!S(k+1, p)(\frac{1}{x^{p-1}})^p}$$

$$\to \frac{k!S(k+2, k+1) + (k+1)!S(k+2, k+2) \lim_{t \to 0^+} t^{-1}}{k!S(k+1, k+1)}$$

as $t \to 0^+$. The proof of Proposition 4.1 is thus complete. \hfill \Box

**Remark 4.1.** This paper is a companion of the papers [10, 24].
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