Topological robustness of quantization of the anomalous Hall conductance of a two-dimensional disordered Chern insulator

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(Dated: April 9, 2021)

The robustness of the anomalous Hall conductance, $\sigma_H$, quantization in the model of a two-dimensional disordered gas of massive Dirac electrons subjected to an external orthogonal magnetic field is investigated in the framework of Kubo–Středa formalism. Using the momentum representation for the averaged one-electron Green functions in a magnetic field, an explicit analytical expression for the Středa term of $\sigma_H$ is obtained. It is shown that this term is proportional to the topological Chern number, $\text{Ch} = \pm 1/2$, if the Fermi level is in the energy gap. In this case, the total $\sigma_H$ takes the half-integer quantized value, $\sigma_H = \pm e^2/4\pi\hbar$, that does not depend on either the magnitude of the disorder or the strength of the external magnetic field. As an example, we calculated the densities of states and intrinsic anomalous Hall conductance, $\sigma_H^{\text{int}}$, of a two-dimensional disordered gas of massive Dirac electrons subjected to an external magnetic field in the self-consistent Born approximation. A numerical analysis of the field and energy dependencies of these expressions is carried out for various values of the parameters of the model under consideration. In particular, the results of this analysis show that the Středa term of $\sigma_H$ is susceptible to disorder in a sufficiently wide vicinity of the point of transition to the quantization regime.

PACS numbers: 72.15.Gd, 72.25.-b, 72.25.Dc, 72.80.Ng

Keywords: quantum anomalous Hall effect, Chern number, Berry curvature, Chern insulator, Dirac electrons, Středa term, Moyal product

I. INTRODUCTION

In the last years, a widespread interest was attracted to investigation of the quantum anomalous Hall effect (quantum AHE or QAHE) in the topological non-trivial materials\textsuperscript{1,2}. The possibility of quantizing Hall conductance without Landau levels was first demonstrated by Cui-Zu Chang et al\textsuperscript{3} based on a tight-binding model with a zero net magnetic flux through the unit cell of the honeycomb lattice. Subsequently, several authors predicted the quantization of anomalous Hall conductance in two-dimensional (2D) magnets with spin-orbit interaction\textsuperscript{4-8}. This effect was first discovered experimentally by Cui-Zu Chang with co-workers\textsuperscript{9} in thin films of the (Bi\textsubscript{1-x}Sb\textsubscript{x})\textsubscript{2}Te\textsubscript{3} topological insulator doped by Cr. An overview of recent experimental and theoretical studies of QAHE can be found in review by Culcer et al.\textsuperscript{10}

Assuming $T = 0$, the expression for the anomalous Hall conductance $\sigma_H = \sigma_{yx}$ of a clean electron system with broken time reversal symmetry can be written as\textsuperscript{5,11}

$$\sigma_H = \sigma_0 \sum_i \int_{\varepsilon_n(p) < \varepsilon_F} \frac{d\varepsilon}{2\pi} \frac{\Omega_{n,z}(p)}{\varepsilon} \tag{1}$$

Here, $\sigma_0 = e^2/2\pi\hbar$ is the conductance quantum, $\Omega_{n,z}(p) = i\langle np|\hat{\nabla}_p \times \hat{\nabla}_p^\dagger|np\rangle$ is the $z$-component of Berry’s curvature vector\textsuperscript{12} of the electron states $|np\rangle$ manifold in $n$th energy band. The arrows above the nabla operators indicate the directions of their actions.

If the Fermi level $\varepsilon_F$ is in an energy gap, then the integration in the each non-zero summand in (1) is performed over the closed manifold without boundary. As a result, these integrals become equal to integers Chern numbers $\text{Ch}$ (or $\pm 1/2$ for the model of 2D massive Dirac electrons), and the Hall conductance takes on a quantized value that is a multiple of $\sigma_0$. However, there is an alternative expression for the quantized anomalous Hall conductance $\sigma_H$ in the thermodynamic limit if the Fermi level is in the mobility gap.

The fundamental problem of the influence of an external magnetic field on the QAHE was investigated by Böttcher et al.\textsuperscript{14,15}. Using the effective field theory, the authors of these works showed that quantization of the anomalous Hall conductance $\sigma_H^\text{eff}$ survives in an external orthogonal magnetic field and can be distinguished from the quantization of normal Hall conductance due to the parity $\sigma_H^\text{eff}(B) = -\sigma_H^\text{eff}(-B)$.

In both these cases, the quantized anomalous Hall conductance is proportional to the Chern number. But the equation (1) holds for a free system in the clean limit when the momentum is a well-defined quantum number. However, there is an alternative expression for the Chern number in terms of one-electron Green’s functions...
where $E_p = p^2/2m$ is the kinetic energy of a free non-relativistic electron. The quantum number $s = \pm 1$ determines the helicity of the electron eigenstate with energy $E_{s,n}$. $\omega_c = |e|B/mc$ is the cyclotron frequency of a free electron, $n$ is the integer number of the Landau level $E_n$. The typical one-particle dispersion \cite{1} and two fans of the Landau levels \cite{1} are shown in Fig. 1.

According to the Kubo–Středa formalism \cite{1}, the expression for the Hall component of the conductivity tensor of the system under consideration can be written as

$$
\sigma_{HI} = \sigma_{HI}^1 + \sigma_{HI}^R = -\frac{\hbar e^2}{2\pi} \int \frac{dE}{dE} \text{Tr} \left( V_y G^R V_z G^A \right) dE - \frac{\hbar e^2}{2\pi} \text{Re} \left[ \int f(E) \text{Tr} \left( V_y G^A V_z \frac{\partial G^A}{\partial E} - (x \leftrightarrow y) \right) dE \right].
$$

Here, $f(E)$ is the Fermi–Dirac function, $G^{R(A)}(E) = 1/(E - \mathcal{H} - U(\pm i0))$ is the exact (i.e., non-averaged) retarded ($R$) or advanced ($A$) GF operator of the Schrödinger equation with total Hamiltonian $\mathcal{H} + U(\pm i0)$, $V_y(x) = \nu v_0 \sigma_{xy}$ are Cartesian components of the velocity operator, the symbol $\text{Tr} = \text{tr} \text{Sp}$ denotes the trace over both the spatial (Sp) and (pseudo-) spin (tr) degrees of freedom, the symbol $(x \leftrightarrow y)$ denotes term that is obtained from the previous one due to permutation $V_z \leftrightarrow V_y$, and, finally, angular brackets $\langle \ldots \rangle$ denote averaging over the random field $U$ configurations. The normalization volume (area) is assumed to be unity.
The first term of conductivity \( [5] \), \( \sigma_H^1 \), results from the electrons at the Fermi surface, whereas \( \sigma_H^{11} \) is determined by the contribution of all occupied states of the Fermi sea. Streda et al.\( \text{[22]} \) were first to show that, for spinless electrons, this part of the conductivity is equal to

\[
\sigma_H^{11} = |e|c \left( \frac{\partial n}{\partial B} \right) \zeta ,
\]

where \( n \) is the electron concentration, \( B \) is a magnetic field orthogonal to the considered system (||OZ), and \( \zeta \) is the chemical potential of the electron gas. It should be pointed out that Eq. (6) is exact, and with thermodynamic Maxwell relation \( \sigma_H^{11} \) can be expressed through \( \left( \partial M/\partial \zeta \right)_B \), where \( M \) is the magnetization of the electron gas. Detailed discussion of \( \sigma_H^{11} \) and its physical interpretation can be found in Pruisken’s survey\( \text{[28]} \). As shown in Ref. \( \text{[28]} \), the expression (6) is converted to

\[
\sigma_H^{11} = |e|c \left[ \left( \frac{\partial n}{\partial B} \right) \zeta - \left( \frac{\partial M}{\partial \zeta} \right)_B \right] ,
\]

if the electron spin degree of freedom taken into account. Here, \( M_p \) is the spin magnetization of the electron gas. It follows that in the general case, \( \sigma_H^1 \) is determined only by the orbital (diamagnetic) part of the electron gas magnetization.

III. ONE-ELECTRON GREEN’S FUNCTION AND DENSITIES OF STATES

The terms of the perturbation theory series for the electrical conductivity \( [14] \) are expressed in terms of the averaged retarded (advanced) GFs, which can be represented in the Dyson form

\[
G^{(R,A)}(E) = \langle G^{(R,A)}(E) \rangle = \frac{1}{E - \mathcal{H} - \Sigma^{(R,A)}(E)} ,
\]

where \( \Sigma^{(R,A)}(E) \) is the electron self-energy operator. Below, in specific calculations, we restrict ourselves to the SCBA in which \( \Sigma^{(R,A)}(E) = W\text{Sp}G^{(R,A)}(E) \), where \( W \) is the amplitude of the pair correlator of the Gaussian random field \( U \). It is easy to verify that, in this approximation, \( \Sigma(E) \) is diagonal in spin space and has the following matrix structure\( \text{[28]} \):

\[
\Sigma = \Sigma_e + \Sigma_m \sigma_z , \quad \Sigma_e = \frac{W}{2} \text{Tr} G , \quad \Sigma_m = \frac{W}{2} \text{Tr} \sigma_z G .
\]

Hence, it follows that the averaged GF \( [8] \) can be obtained from the GF of the clean model \( G^0 = 1/(E - \mathcal{H}) \) using the substitutions\( \text{[28]} \):

\[
\mathcal{E} \rightarrow \tilde{\mathcal{E}} = \mathcal{E} - \Sigma_e , \quad M \rightarrow \tilde{M} = M + \Sigma_m .
\]

Here, \( \Sigma_e = \Delta_e \equiv i\hbar/2\tau_e \) describes the perturbation (shift \( \Delta_e \) and broadening \( \hbar/\tau_e \)) of the one-electron energy levels by a random field. The real part of the

\[
\Sigma_m = \Delta_m \equiv i\hbar/2\tau_m \text{ determines the renormalization of the Dirac electron mass } M \text{, while its imaginary part } \propto \hbar/\tau_m \text{ makes a contribution to the overall lifetime of the one-electron states in the spin-split subbands. Thus, the one-particle GF operator averaged over random field configurations takes the following form}\( \text{[31-33]} \):

\[
G = \frac{\tilde{\mathcal{E}} + \tilde{M} \sigma_z + v(\pi \cdot \sigma)}{\varepsilon^2 - M^2 - 2mv^2\mathcal{H}_0} = \left[ \tilde{\mathcal{E}} + \tilde{M} \sigma_z + v(\pi \cdot \sigma) \right] \mathcal{H}_0 ,
\]

where

\[
\mathcal{H}_0 = \frac{1}{2m}(\pi \cdot \sigma)^2 = \frac{\pi^2}{2m} + \frac{\hbar\omega}{2} \sigma_z \]

is the Hamiltonian of a free electron with ideal value of Zeeman coupling (\( g = 2 \)) in an orthogonal magnetic field. In the limit \( B \rightarrow 0 \), \( \pi \) and \( \mathcal{H}_0 \) in (11) are replaced by \( p \) and \( \mathcal{E}_p \), respectively. It should be emphasized that the second term in Hamiltonian \( \mathcal{H}_0 \) (12) appears due to commutation properties of the operators \( \pi \) and \( \sigma \) and has no relation to the true Zeeman energy of an electron in a magnetic field.

Eqs. (9) and (11) form a system of the self-consistent transcendental equations for the \( \Sigma_e \) and \( \Sigma_m \). In the absence of an external magnetic field, the traces \( \text{Tr} \sigma_z G \) are proportional to each other; therefore, both of these self-energies in SCBA are expressed through the same function\( \text{[28]} \):

\[
\Sigma_e = \gamma_0 \mathcal{E} \Phi , \quad \Sigma_m = \gamma_0 M \Phi , \quad |\gamma_0 \Phi| << 1
\]

that satisfies the equation

\[
\Phi = \ln \left[ (1 + \gamma_0 \Phi)^2 - (1 - \gamma_0 \Phi)^2 \frac{\mathcal{E}^2}{M^2} \right] .
\]

Here \( \gamma_0 = W\mathcal{N}_F/2mv^2 \) is the dimensionless parameter of disorder, \( \mathcal{N}_F = m/2\pi\hbar^2 \) is the density of states (DOS) of two-dimensional free non-relativistic spinless electrons in the absence of a magnetic field. The numerical solution of equation (14) allows us to calculate the self-energies \( \Sigma_e \) and \( \Sigma_m \), as well as the total DOS \( \mathcal{N}(\mathcal{E}) \) and the difference of partial DOSs \( \mathcal{N}_m(\mathcal{E}) \) with opposite values of the pseudospin projections onto the OZ-axis (SDOS).

\[
\mathcal{N}(\mathcal{E}) = \frac{1}{\pi} \text{Im Tr} G^A(\mathcal{E}) , \quad \mathcal{N}_m(\mathcal{E}) = \frac{1}{\pi} \text{Im Tr} \sigma_z G^A(\mathcal{E}) .
\]

Some results of this calculation at the different values of the parameter \( \gamma_0 \) are shown in Fig. 2. As can be seen from this figure, the energy gap between valence and conductivity bands narrows as \( \gamma_0 \) increases.

Now, we consider the calculation of the electron self-energies \( [9] \) and DOSs \( [15] \) in the case of \( B \neq 0 \). The spectrum of Hamiltonian \( \mathcal{H}_0 \) (12) consists of two unbounded from above sets of the equidistant Landau levels \( \mathcal{E}_n = \hbar\omega_c(n + 1/2 \pm 1/2) \), where \( n = 0, 1, 2, \ldots \). As a consequence, the traces that determine the electronic self-energies \( [9] \) diverge logarithmically. These divergences
The energy dependence of the DOS calculated in SCBA for various values of the disorder parameter $\gamma_0 = WNp/2mv^2$. Three curves are calculated for $B = 0$ and $\gamma_0 = 0.0$ (1), 0.02 (2), 0.04 (3). Vertical dashed lines represent unperturbed edges of the electronic spectrum $\varepsilon = \pm |M|$. Left bottom panel: The same for the SDOS $N_m(\varepsilon) = N_\uparrow(\varepsilon) - N_\downarrow(\varepsilon)$. Central panel: The energy dependence of the DOS calculated in SCBA for $\gamma_0 = 0.02$ and $\hbar \omega_c / M = 0.50, 0.33, 0.25, 0.17$ (from top to bottom). The dashed curves depict the energy dependence of DOS for $\gamma_0 = 0.02$ and $B = 0$. For clarity, the curves are shifted relative to each other along the abscissa axis. Right panel: The same for the SDOS $N_m(\varepsilon)$.

IV. TOPOLOGICAL NATURE OF STŘEDA-LIKE TERM OF CONDUCTANCE

Let us take Středa-like formula (7) as the initial expression for calculating of the $\sigma_{12}^H$ contribution to the intrinsic anomalous Hall conductance. With this purpose, we have to find the explicit expressions for the thermodynamic derivatives $(\partial n / \partial B)$ and/or $(\partial M_B / \partial \varepsilon)_{B=0}$ in the presence of an orthogonal magnetic field $B || OZ$ [see Eq. (7)]. For example, we consider the derivative $(\partial n / \partial B)_{\Sigma}$. By definition, the electron concentration is equal to

$$n = \int f(\varepsilon) N(\varepsilon) \, d\varepsilon = \frac{1}{2 \pi i} \int_C f(z) \text{Tr} G(z) \, dz. \quad (18)$$

Here, $G(z)$ is the one-electron GF defined on the complex $z$-plane in such a way that $\lim_{z \to \varepsilon \pm 0} G(z) = G^{R}(\varepsilon)$. The integration contour $C$ encircles counterclockwise the energy intervals belonging to the spectrum of the eigenvalues of the Hamiltonian (2). Using the analytical properties of the integrand in Eq. (18), we can deform $C$ as shown in Fig. 4. This deformed integration contour encircles clockwise the poles of the Fermi–Dirac function $\zeta_n = \zeta + i \pi k_B T (2n + 1)$ (see Figs. 31 or 32). The integrals along the parts of $C$ located to the right of the poles $\zeta_n$ vanish as $T \to 0$. In this case, the integration contour in Eq. (18) looks as shown in Figs. 33 or 34.

Thus, the problem is reduced to finding of an explicit expression for derivative $\partial \text{Tr} G / \partial B$. This can be done in the momentum representation in which Dyson’s equation
In the momentum representation, the operations $\text{Tr}$ and $\partial/\partial B$ commute with each other. So, we just need to find an explicit expression for $\partial G/\partial B$ to calculate $(\partial n/\partial B)_c$. Direct differentiation of Dyson equation (19) with respect to the magnetic field induction gives the following result

$$\frac{\partial G}{\partial B} = -G \star \frac{\partial Q}{\partial B} \star G + i \frac{|e|}{2c} \epsilon_{\alpha\beta} G \star \frac{\partial Q}{\partial p_\alpha} \star \frac{\partial G}{\partial p_\beta}$$

$$= -G \star \frac{\partial Q}{\partial B} \star G - i \frac{|e|}{2c} \epsilon_{\alpha\beta} G \star \frac{\partial Q}{\partial p_\alpha} \star G \star \frac{\partial Q}{\partial p_\beta} \star G. \quad (21)$$

Here, the last equality is obtained using the first of the identities

$$\frac{\partial G}{\partial B} = -G \star \frac{\partial Q}{\partial p} \star G, \quad \frac{\partial Q}{\partial z} = -G \star \frac{\partial Q}{\partial z} \star G, \quad (22)$$

which are also derived from Dyson equation (19).

Let us take a trace of Eq. (21). Then, using the second identity from (22) and the trace invariance under cyclic permutations, we obtain

$$\text{Tr} \frac{\partial G}{\partial B} = \text{Tr} \frac{\partial Q}{\partial B} \star \frac{\partial G}{\partial B} \star G \star \frac{\partial G}{\partial B} \star G \star \frac{\partial Q}{\partial B} \star G \star \frac{\partial Q}{\partial B} \star G$$

$$\quad - i \frac{|e|}{2c} \epsilon_{\alpha\beta} \text{Tr} G \star \frac{\partial Q}{\partial p_\alpha} \star G \star \frac{\partial Q}{\partial p_\beta} \star G. \quad (23)$$

Now we substitute the explicit expression

$$\frac{\partial Q}{\partial B} = -\frac{|e|}{4mc} s_z \otimes g - \frac{\partial \Sigma}{\partial B} \quad (24)$$

in the first term of the right-hand side of Eq. (23) and transfer the Zeeman term to the left-hand side of this equality. The electron self energy satisfies relations like Ward identity

$$\frac{\partial \Sigma}{\partial B} = \sum_{n,n'} U_{n,n'} \frac{\partial G_{n'}}{\partial z}, \quad \frac{\partial \Sigma}{\partial B} = \sum_{n,n'} U_{n,n'} \frac{\partial G_{n'}}{\partial B}, \quad (25)$$

where $U_{n,n'}$ is the two-particle irreducible interaction vertex. Hence, it follows that

$$\text{Tr} \frac{\partial \Sigma}{\partial B} \star \frac{\partial G}{\partial z} = \text{Tr} \frac{\partial G}{\partial B} \star \frac{\partial \Sigma}{\partial z}. \quad (26)$$

This is more strong version of the relation previously proposed by Pruisken. With this in mind Eq. (23) can be rewritten as

$$\text{Tr} \frac{\partial G}{\partial B} + \frac{|e|}{4mc} \text{Tr} s_z \otimes g \frac{\partial G}{\partial z} = -i \frac{|e|}{2c} \epsilon_{\alpha\beta} \times$$

$$\times \text{Tr} \frac{\partial Q}{\partial p_\alpha} \star G \star \frac{\partial Q}{\partial p_\beta} \star G \star \frac{\partial Q}{\partial z} \star G. \quad (27)$$

After integration with respect to $z$ of this equation with Fermi–Dirac weight function $f(z)$, the terms of its
left-hand side determine derivatives of the electron concentration and spin magnetization included in Eq. (7). Therefore, after some cosmetic transformations, the expression for the Středa-like term of the Hall conductance can be written in the following form

\[
\sigma_{H}^{II} = - \frac{\hbar e^2}{24\pi} \int_C f(z) \times \epsilon_{\alpha\beta\gamma} \text{Tr} \left( \frac{\partial Q}{\partial p_\alpha} \right) G \left( \frac{\partial Q}{\partial p_\beta} \right) G \left( \frac{\partial Q}{\partial p_\gamma} \right) G \left( \frac{\partial Q}{\partial x} \right) G \left( \frac{\partial Q}{\partial y} \right) \left( \frac{\partial Q}{\partial z} \right). \tag{28}
\]

Here, integration is carried out along the contour C shown in Fig. 31 or Fig. 32; \(\epsilon_{\alpha\beta\gamma}\) is the unit antisymmetric pseudotensor of the third rank, indices \(\alpha, \beta, \gamma\) run values 0, 1, 2 with \(p_0 = z, p_1 = p_x,\) and \(p_2 = p_y.\)

Thus, we obtain the expression for the Středa-like part of the Hall conductance in which the averaging over random field configurations is performed exactly. Eq. (28) is true regardless of the disorder type and the approximation for the averaged one-particle GF. This is possible owing to special analytical structure of starting expression for \(\sigma_{H}^{II}\) that does not contain the products \(G^R G^A.\)

It should be emphasized that both expressions for \(\sigma_{H}^{II}\) and (28) have the identical structure. The only difference is that Eqs. (28) and (28) include exact and averaged GFs, respectively. In other words, the averaging procedure of the \(\times\)-products in this equation by the usual multiplication and treat \(G\) and \(Q\) as functions depending on the operators of the corresponding dynamic variables.

Eq. (28) indicates a close connection between the intrinsic anomalous Hall conductance and topological properties of the one-electron states. Indeed, in the case \(T = 0,\) the Fermi–Dirac function \(f(z)\) coincides with the Heaviside unit step \(\Theta(\mathcal{E}_F - \text{Re} z),\) and the expression for \(\sigma_{H}^{II}\) takes the form

\[
\sigma_{H}^{II} = \sigma_0 \int_C \left\{ -\epsilon_{\alpha\beta\gamma} \frac{1}{24\pi^2} \right\} \times \left( \frac{\partial Q}{\partial p_\alpha} \right) G \left( \frac{\partial Q}{\partial p_\beta} \right) G \left( \frac{\partial Q}{\partial p_\gamma} \right) dp_1 dp_2 \right\} dp_0, \tag{29}
\]

where the integration contour C is shown in Figs. 33 or 34 and trace over the spatial degrees of freedom is presented as the integral with respect to momenta \(p_{1,2}\) (38).

Now imagine that the Fermi level lies inside the energy gap of the system under consideration. In this case, the integration contour C in Eq. (29) is the straight line parallel to the imaginary axis \(\text{Re} z = \mathcal{E}_F - 0, \) \(-\infty < \text{Im} z < +\infty\) (See Fig. 34) and the Středa term of the Hall conductance \(\sigma_{H}^{I}\) takes the disorder-independent quantized value proportional to the Chern number, i.e., \(\sigma_{H}^{II} = \sigma_0 \mathcal{C}_H,\)

\[
\mathcal{C}_H = -\frac{\epsilon_{\alpha\beta\gamma}}{2\pi} \int \left[ \frac{\partial Q}{\partial p_\alpha} \right] G \left( \frac{\partial Q}{\partial p_\beta} \right) G \left( \frac{\partial Q}{\partial p_\gamma} \right) G \left( \frac{\partial Q}{\partial x} \right) G \left( \frac{\partial Q}{\partial y} \right) G \left( \frac{\partial Q}{\partial z} \right) G \left( \frac{\partial Q}{\partial x} \right)^2 \right] dp. \tag{30}
\]

Integration is performed here over the entire \((2 + 1)\)-dimensional space of points \((\text{Im} z, p_x, p_y).\) Eq. (30) is an analogue of the expression for the Chern number obtained by Zubkov et al. in the Wigner representation, which is necessary for describing spatially nonuniform systems. Writing the Chern number in the momentum representation is more appropriate in the practically important spatially uniform case.

We represent the result of integration along the straight line \(\text{Re} z = \mathcal{E}_F - 0\) in Eq. (30) in the form of increment of the corresponding antiderivative, i.e.

\[
\mathcal{C}_H = \Delta F(\mathcal{E}_F + i\infty) - F(\mathcal{E}_F - i\infty), \tag{31}
\]

where [See Eq. (29)]

\[
\frac{\partial F}{\partial z} = -\epsilon_{\alpha\beta\gamma} \mathcal{C}_H = \int \left( \frac{\partial Q}{\partial p_\alpha} \right) G \left( \frac{\partial Q}{\partial p_\beta} \right) G \left( \frac{\partial Q}{\partial p_\gamma} \right) G \left( \frac{\partial Q}{\partial x} \right) G \left( \frac{\partial Q}{\partial y} \right) G \left( \frac{\partial Q}{\partial z} \right) \left( \frac{\partial Q}{\partial x} \right)^2 \right] dp. \tag{32}
\]

Obviously, the asymptotic values \(F(\mathcal{E}_F \pm i\infty)\) do not depend on the disorder and coincide with their values in the clean limit.

If the Fermi level lies outside the gap, then the contour of integration in Eq. (30) consists of two semi-infinite straight lines parallel to the imaginary axis \(\text{Re} z = \mathcal{E}_F - 0, \) \(-\infty < \text{Im} z < -\infty\) and \(+\infty < \text{Im} z < +\infty\) (See Fig. 33). In this case, a term proportional to the discontinuity of \(F(z)\) across the cut line corresponding to the continuous spectrum of the electron is added to the expression for the Středa part of the Hall conductance. So, we have

\[
\sigma_{H}^{II} = \sigma_0 \begin{cases} 
\mathcal{C}_H, & \mathcal{E}_F \in \text{Gap}, \\
\mathcal{C}_H + \Delta F(\mathcal{E}_F), & \mathcal{E}_F \notin \text{Gap}, \end{cases} \tag{33}
\]

where \(\Delta F(\mathcal{E}_F) = F(\mathcal{E}_F - i\infty) - F(\mathcal{E}_F + i\infty).\) The first term in the bottom line of (33) remains unchanged since \(\Delta F(\mathcal{E}_F \pm i\infty)\) does not depend on placing the Fermi level. In the next section, we will illustrate the validity of (33) using the massive Dirac electrons model as an example.

V. INTRINSIC ANOMALOUS HALL CONDUCTANCE IN SCBA

The intrinsic Hall conductance consists of two terms \(\sigma_{H}^{II} = \sigma_{H}^{II} + \sigma_{H}^{II},\) where \(\sigma_{H}^{II}\) is the bare bubble part of \(\sigma_{H}^{II}\) and \(\sigma_{H}^{II}\) is the Středa term [see. Eq. (33)].

In the presence of an external magnetic field, the Hall conductance consists of the normal \(\sigma_{H}^{II}\) and anomalous \(\sigma_{H}^{II}\) terms, at that \(\sigma_{H}^{II}(-B) = -\sigma_{H}^{II}(B)\) and \(\sigma_{H}^{II}(-B) = -\sigma_{H}^{II}(B).\) Below, we restrict ourselves to calculating the even in B parts of \(\sigma_{H}^{II}\) and \(\sigma_{H}^{II}.\) Of course, the parity

\[
\mathcal{C}_H = -\frac{\epsilon_{\alpha\beta\gamma}}{2\pi} \int \left[ \frac{\partial Q}{\partial p_\alpha} \right] G \left( \frac{\partial Q}{\partial p_\beta} \right) G \left( \frac{\partial Q}{\partial p_\gamma} \right) G \left( \frac{\partial Q}{\partial x} \right) G \left( \frac{\partial Q}{\partial y} \right) G \left( \frac{\partial Q}{\partial z} \right) G \left( \frac{\partial Q}{\partial x} \right)^2 \right] dp. \tag{30}
\]
of the anomalous Hall conductance in the magnetic field does not mean violation of the Onsager relation. The point is that this relation is valid for the both normal and anomalous Hall conductances with simultaneous reversal of the signs of both the magnetic field $B$ and the Dirac mass $M$, since the introduction of each of them breaks the time reversal symmetry. Thus, the Onsager relation for both parts of $\sigma_H$ must have the form

$$\sigma_H(-B, -M) = -\sigma_H(B, M). \quad (34)$$

The intrinsic anomalous Hall conductance calculated below satisfies the relation (34) not only in the QAHE regime\textsuperscript{14,15}, but also outside it.

### A. Středa like part of the anomalous Hall conductance

It is usually assumed that $\sigma_H^{11}$ is weakly dependent on disorder, and it is calculated without taking into account the scattering of charge carriers\textsuperscript{30–32,40–42}. It will be shown below that this approximation is violated in a fairly wide vicinity of the Hall plateau. Let us rewrite the expression (28) for the Středa term in a more convenient for calculating operator notation

$$\sigma_H^{11} = \frac{\hbar^2}{2} \int_C \text{Tr} \left[ \sigma_y G \frac{\partial G}{\partial z} - (x \leftrightarrow y) \right] dz. \quad (35)$$

Here, it is taken into account that $\partial Q/\partial p_{\sigma(y)} = -v_{\sigma(y)}$ in SCBA considered below. For simplicity, we assume the temperature to be zero, the generalization to the case $T > 0$ is obvious.

In the article of Dugaev et al.\textsuperscript{30} the Středa term of $\sigma_H$ was calculated for the case of a clean two-dimensional Rashba model in absence of a magnetic field. We will apply their approach to calculate the Středa part of the anomalous Hall conductance of the disordered Chern insulator (2) in SCBA in the presence of an orthogonal magnetic field. Direct differentiation of the averaged GF\textsuperscript{11} with respect to $z$ gives the following result

$$\frac{\partial G}{\partial z} = \left( 1 - \frac{\partial \Sigma_x}{\partial z} + \frac{\partial \Sigma_m}{\partial z} \sigma_z \right) \tilde{G} + G \frac{\partial \tilde{G}}{\partial z} (\tilde{z}^2 - M^2), \quad (36)$$

where $\tilde{z} = z - \Sigma_x(z)$. In absence of a magnetic field, the second term in this expression does not contribution to the Hall conductance\textsuperscript{30}. In the presence of a magnetic field, this is no longer the case, but only the first term from Eq. (36) makes an even in $B$ contribution to the Středa part of the Hall conductance. The substitution of this term into the integrand of (35) gives, after some simple algebra, the following result

$$\text{Tr}[\ldots] = \frac{i}{mv^2 \hbar \omega_x} \times$$

$$\times \text{Tr}(\tilde{G} - \tilde{G})(\tilde{z} - M \sigma_z) \left( 1 - \frac{\partial \Sigma_x}{\partial z} + \frac{\partial \Sigma_m}{\partial z} \sigma_z \right),$$

where $\tilde{G} = \sigma_{\epsilon(y)} \tilde{G} \sigma_{\epsilon(y)}$, i.e., this is the result of transformation $\sigma_z \mapsto -\sigma_z$.

$$= 2i \frac{N_F}{mv^2} \frac{M}{\tilde{z}^2 - M^2} \left( 1 - \frac{\partial \Sigma_x}{\partial z} \right) - \sigma_z \frac{\partial \Sigma_m}{\partial z}$$

$$= i \frac{N_F}{mv^2} \frac{\partial \ln M - \tilde{z}}{M + \tilde{z}}. \quad (37)$$

Thus, we obtain the explicit expression for the antiderivative [see Eq. (32)]

$$F(z) = \frac{1}{4\pi i} \ln \frac{M + \tilde{z}}{M - \tilde{z}} \quad (38)$$

in the case of the model of disordered massive Dirac electrons. Obviously, its increment along the straight line $\text{Re} z = \varepsilon_F$ does not depend on the placing of the Fermi level $\varepsilon_F$, nor on the external magnetic field $B$, nor on the electronic self-energies $\Sigma_{\epsilon(m)}$, and is equal to the Chern number $\Delta F(\varepsilon_F \pm i\infty) = 1/2 = \text{Ch}$\textsuperscript{31}.

If $\varepsilon_F \in \text{Gap}$, i.e., when $\text{Im} \Sigma_{\epsilon(m)} = 0$, then $F(z)$ is continuous across the real axis. As a result, the anomalous part of the Středa term takes on quantized value $\sigma_H^{11} = \sigma_0 \text{Ch}$ in this energy range [see Eq. (33)]. Otherwise ($\varepsilon_F \notin \text{Gap}$), $\sigma_H^{11}$ acquires additional term $\Delta F(\varepsilon_F) = F(\varepsilon_F - i0) - F(\varepsilon_F + i0)$ [see Eq. (33)].

Gives the explicit form of the function $F(z)$ (38), we obtain the following expression for the Středa term of the anomalous Hall conductance

$$\sigma_H^{11} = \frac{\sigma_0}{2} [1 + 2 \Delta F(\varepsilon_F)] = \frac{\sigma_0}{2} \text{Im} \ln \frac{\tilde{\varepsilon} + A}{\tilde{\varepsilon} - A}, \quad (39)$$

which turns out to be valid both in the quantization regime and beyond it. This expression for $\sigma_H^{11}$ has exactly the same form as in absence of an external magnetic field\textsuperscript{31}. The entire dependence on the magnetic field of (39) is contained in the electronic self-energies $\Sigma_{\epsilon}$ and $\Sigma_{\sigma}$\textsuperscript{30}. The results of numerical analysis of Eq. (39) are represented in Fig. 4 (upper left and central panels).

### B. Bare bubble part of the anomalous Hall conductance

Now we turn to the calculation of the bare bubble contribution $\sigma_H^{1b}$ to the intrinsic anomalous Hall conductance. In the case of the clean $[U(r) = 0]$ massive Dirac electrons, this term was calculated by Sinitsyn and co-workers\textsuperscript{32} in the basis of the Hamiltonian $\mathbb{H}$ (2) in absence of a magnetic field. Here, we apply the simple algebraic approach to calculating $\sigma_H^{1b}$ of the considered model in the same approximation as the Středa term, that is, in SCBA.

Thus, we start with the expression

$$\sigma_H^{1b} = \frac{\hbar e^2 v^2}{2\pi} \text{Tr} \sigma_y G^R \sigma_z G^A. \quad (40)$$
The same for the total intrinsic part of the anomalous Hall conductance \( \sigma_{II}^{int} \) calculated in SCBA for various values of the disorder parameter \( \gamma_0 = WN_{EF}/2mv^2 \). Three curves are calculated for \( B = 0 \) and \( \gamma_0 = 0.0 \) (1), \( 0.02 \) (2), \( 0.04 \) (3). Vertical dashed lines represent unperturbed edges of the electronic spectrum \( E = \pm |M| \).

Left bottom panel: The same for the total intrinsic part of the anomalous Hall conductance \( \sigma_{II}^{int} = \sigma_{II}^{ba} + \sigma_{II}^{IIa} \). The dashed line shows the energy dependence of \( \sigma_{II}^{int} \) of the clean \((\gamma_0 = 0)\) model of massive Dirac fermions for \( B = 0 \). The solid curves calculated for \( B = 0 \) and \( \gamma_0 \to 0 \) (1), \( 0.02 \) (2), \( 0.04 \) (3) are practically indistinguishable at this scale. The inset shows, on an enlarged scale, the behavior of curves (1-3) in vicinity of the point \( E = |M| \) (circled in the main graph).

Central panel: The energy dependence of the Streda term of the anomalous Hall conductance \( \sigma_{II}^{ba} \) calculated in SCBA for \( \gamma_0 = 0.02 \) and \( \hbar \omega_c/M = 0.50, 0.33, 0.25, 0.17 \) (from top to bottom). The dashed curves depict the energy dependence of \( \sigma_{II}^{ba} \) for \( \gamma_0 = 0.02 \) and \( B = 0 \). For clarity, the curves are shifted relative to each other along the abscissa axis.

Right panel: The same for the total intrinsic part of the anomalous Hall conductance \( \sigma_{II}^{int} = \sigma_{II}^{ba} + \sigma_{II}^{IIa} \).

Given the properties of the Pauli matrices, it can be shown that the terms \( \propto (\mathbf{\pi} \cdot \mathbf{\sigma}) \) in the numerator of GF \( \sigma^{Ib}_{II} \) do not contribute to \( \text{Tr} \sigma_y G^R \sigma_x G^A \) due to isotropy of the considered system. So, substitution of the one-particle GFs \( \sigma^{Ib}_{II} \) in Eq. (40) gives the result

\[
\sigma_{II}^{Ib} = \frac{\hbar e^2 v^2}{2\pi} \text{Tr} \sigma_y G^R (\tilde{\mathbf{e}}^R + \tilde{\mathbf{M}}^R \sigma_z) \sigma_x (\tilde{\mathbf{e}}^A + \tilde{\mathbf{M}}^A \sigma_z) G^A
\]

We are interested in the even in \( B \) part of this expression

\[
\sigma_{II}^{Ib} = \frac{\hbar e^2 v^2}{2\pi} \text{Im} (\tilde{\mathbf{e}}^R \tilde{\mathbf{M}}^A) \text{Tr} \tilde{G}^R \tilde{G}^A.
\]  

Using the resolvent identity we write the product \( \tilde{G}^R \tilde{G}^A \) in the following form

\[
\tilde{G}^R \tilde{G}^A = \frac{1}{i\hbar} \frac{\tau}{2mv^2} \left( 1 + i\omega \tau \sigma_z \right)^{-1} (\tilde{G}^A - \tilde{G}^R).
\]  

The quantity \( \tau \) that is defined by relation

\[
\frac{1}{\tau} = \frac{1}{mv^2} \left( \frac{E}{\tau_c} + \frac{M}{\tau_m} \right)
\]  

plays the role of the transport time in the Drude-like denominator \( 1 + \omega_c^2 \tau^2 \). Let us introduce one more characteristic time

\[
\frac{1}{\tau'} = \frac{2}{\hbar} \text{Im} (\tilde{\mathbf{e}}^R \tilde{\mathbf{M}}^A) = \frac{1}{mv^2} \left( \frac{E}{\tau_m} + \frac{M}{\tau_c} \right).
\]  

Substitution of expressions (42), (43) into Eq. (41) gives the following result

\[
\sigma_{II}^{Ib} = \frac{\sigma_0}{2\pi} \frac{\tau}{\tau'} \text{Im} \Phi^A + \omega_c \tau \text{Re} \frac{mv^2 \hbar \omega_c}{M^2 - E^2}.
\]  

where the function \( \Phi^A \) is defined in (17).

Obviously, expression (45) as a function of \( \mathcal{E}_F \) vanishes inside the gap of the one-electron energy spectrum. Thus, the quantization of the \( \sigma_{II}^{int} = \sigma_{II}^{Ib} + \sigma_{II}^{IIa} \) (39), (45) survives when an external magnetic field is turned on that is consistent with recent results by Böttcher et al.14,15.

The results of numerical analysis of \( \sigma_{II}^{int} \) are represented in Fig. 4 (bottom left and right panels).
VI. RESULTS AND DISCUSSION

Let us summarize briefly the main results obtained in this work. The explicit expressions in SCBA \( \text{(16)} \) are obtained for the total DOS and SDOS of the minimal model of a two-dimensional disordered Chern insulator. The results of a numerical analysis of these expressions are shown in Fig. 2. In the case \( B = 0 \), the curves in the left panels demonstrate a narrowing of the gap in the spectrum of one-electron states as the disorder parameter increases. The graphs presented in central and right panels in Fig. 2 show two effects caused by an external magnetic field. First, the gap in the electronic spectrum is shifted towards higher energies as a magnetic field increases. This is due to the presence of the anomalous Landau level \( E_{-1,0} = -M \) at the top of the valence band. Second, in the region of sufficiently strong magnetic fields, the de Haas–van Alphen oscillations manifest themselves in the energy dependencies of the DOSs.

Section IV provides the simple derivation of expressions [see Eq. (25) for the case \( T > 0 \) and Eq. (29) for the case \( T = 0 \)] for the Středa-like term \( \sigma^c_H \) of the anomalous Hall conductance in terms of the averaged one-particle GFs in the momentum representation. These expressions are valid both in absence of an external magnetic field and in its presence and describe the dependence of \( \sigma^c_H \) on the location of the Fermi level \( E_F \) both inside the gap of the one-electron spectrum and outside it. In the first case \( (E_F \in \text{Gap}) \), \( \sigma_H \) is proportional to the Chern number (in the case \( T = 0 \)), which takes on quantized values regardless of the presence or absence of disorder and/or an external magnetic field. The half-integer \( (\text{Ch} = \pm 1/2) \) quantization of the anomalous Hall conductance of massive Dirac electrons \( \sigma^c_H \) is a consequence of the fermionic numbers fractionalization \( \text{(14)} - \text{(15)} \). It should be emphasized that the bare bubble part of \( \sigma_H \), as well as the contributions to \( \sigma_H \) due to extrinsic mechanisms, vanish under these conditions. Thus, the survival of the gap in the electronic spectrum of the Chern insulator is the only condition for quantizing its anomalous Hall conductance in the presence of disorder, external magnetic field, and other perturbations.

As an illustration, we calculate in Section V the even in a magnetic field intrinsic anomalous Hall conductance \( \sigma^{\text{int}}_H = \sigma^{\text{lb}}_H + \sigma^{\text{ll}}_H \) of a two-dimensional disordered gas of massive Dirac electrons in SCBA [see Eqs. (39) and (45)]. As noted above, the parity of \( \sigma^c_H \) in a magnetic field does not mean the violation of the Onsager relation, since it requires both normal and anomalous conductances be odd in all parameters that break the time reversal symmetry of the considered system [see Eq. (51)].

The results of a numerical analysis of the dependencies of \( \sigma^{\text{ll}}_H \) \( \text{(39)} \), and \( \sigma^{\text{int}}_H = \sigma^{\text{lb}}_H + \sigma^{\text{ll}}_H \) \( \text{(39)} \), \( \text{(45)} \) on the location of the Fermi level at various values of the disorder parameter and magnetic field induction are shown in Fig. 3. As expected, the quantization of the anomalous Hall conductance survives in the presence of sufficiently weak perturbations that leave open the gap in the electronic energy spectrum.

Contrary to conventional opinion, the presence of disorder has a significant effect on the \( \sigma^{\text{int}}_H \) behavior beyond the QAHE-plateau (see the upper left panel in the Fig. 3). But, as can be seen from the graphs on the lower left panel of Fig. 3, the corrections to \( \sigma^c_H \) and \( \sigma^b_H \) due to scattering of electrons in a random field of impurities cancel out with a very high precision. In particular, the relative deviation of the sum of these terms from its quantized value \( \sigma^c_0/2 \) for \( |E| < M \), and from its behavior in the clean limit

\[
\sigma^{\text{int}}_H = \sigma_0 \frac{|E_M|}{E^2 + M^2}
\]

for \( |E| > M \) is at least an order of magnitude smaller than the dimensionless disorder parameter \( \gamma_0 \).

As already mentioned, the quantization of \( \sigma^{\text{int}}_H \) survives in an external magnetic field \( B \). But, as can be seen from Fig. 4, the lower boundary of the Hall plateau shifts towards higher energies due to the broadening of the zero Landau level \( E_{0,-1} = -M \) as magnetic field \( B \) increases. At the same time, the upper boundary remains almost fixed that leads to a narrowing of the Hall plateau.

In the region of sufficiently strong magnetic fields, the upper boundary of the Hall plateau turns out to be inside the gap of the electron spectrum. In this case, it represents the threshold at which the \( \sigma^{\text{int}}_H \) as a function of the Fermi energy suddenly drops from its quantized value to zero as in the clean limit. A similar dependence of the anomalous Hall conductance on the magnetic field was predicted in Ref. [13].

The main features of the behavior of the intrinsic anomalous Hall conductance \( \sigma^{\text{int}}_H \) beyond the plateau are determined by the energy dependence of the relaxation time \( \tau \). Indeed, on the one hand, this parameter plays the role of transport time in the Drude denominator \( 1 + \omega^2 e^2 \tau^2 \) \( \text{(35)} \), and, on the other hand, it is related to the Dingle temperature \( 1/\omega e \tau \propto k_B T_D/\hbar \omega_e \). Consequently, \( \sigma^{\text{int}}_H \) exhibits Shubnikov-de Haas (ShdH) oscillations against the background of a pronounced dip in its energy dependence in vicinity of the plateau if the condition of a strong magnetic field, \( \omega_e \tau > 1 \), is satisfied in this region. The relaxation time \( \tau \) \( \text{(13)} \) is proportional to \( E_F^{-2} \). As a result, the amplitudes of ShdH oscillations rapidly tend to zero, and the conductance asymptotically approaches its value in absence of a magnetic field as the Fermi level increases (See right panel in Fig. 4).

Of course, we are talking here about the behavior outside the plateau of only intrinsic conductance \( \sigma^{\text{int}}_H \). Calculations of the contributions of known external mechanisms to the anomalous Hall conductance of a two-dimensional gas of massive Dirac fermions in the absence of a magnetic field can be found in Refs. [31] and [32] (side jump and skew scattering mechanisms) and in Ref. [41] (coherent skew scattering of electrons by impurity pairs).

In conclusion, we note that the SCBA used in this article is correct provided that \( k_F l \gg 1 \), where \( k_F \) is the Fermi momentum and \( l \) is the mean free path. This
inequality is violated near the edges of the valence and conduction bands $|E| \simeq M$. In this energy region, SCBA gives only a qualitative description of the behavior of the DOSs and Hall conductance. Besides that, SCBA does not describe the so-called "tails" of the DOS within the energy gap $|E| < M$. Therefore, the question of how the presence of these "tails" affects the quantization of anomalous Hall conductance remains open.

ACKNOWLEDGMENTS

Author thanks V. V. Ustinov, I. I. Lyapilin, and N. G. Bebenin for helpful discussions. This research was carried out within the state assignment of Ministry of Science and Higher Education of the Russian Federation (theme "Spin" No. AAAA-A18-118020290104-2).

Appendix A: Momentum representation in a magnetic field

An averaged one-electron GF in the spatially uniform disordered system subject to an external magnetic field has in coordinate representation $G(r, r′) = \langle r | G | r′ \rangle$ the following general structure

$$G(r, r′) = e^{i\Phi(r, r′)}G(\mathbf{q}) = \exp \left[ i \frac{e}{\hbar c} \int_{r′}^{r} A(x) \cdot dx \right] G(\mathbf{q}),$$

where $G(\mathbf{q}) = G(r - r′)$ is a translation invariant and gauge-independent multiplier, $\Phi(r, r′)$ is the gauge-dependent phase, and integral in the exponent is calculated along the straight line connecting points $r′$ and $r$. For example, in symmetric gauge, this phase is equal to $\Phi(r, r′) = eB \cdot (r′ - r)/2\hbar c$. Obviously, any two-point function invariant in magnetic translations must have such factorization. For example, the electron self-energy operator in coordinate representation can be written in the form $\Sigma(r, r′) = \exp[\Phi(r, r′)]\Sigma(\mathbf{q})$. Therefore, given the gauge-independent identity

$$e^{-i\Phi(r, r′)} \left[ p - \frac{e}{c} A(r) \right] e^{i\Phi(r, r′)} = p - \frac{e}{2c} B \times \mathbf{q},$$

we obtain equation for the translation invariant multiplier of the one-particle GF:

$$[\mathcal{E} - \mathcal{H}(\pi)]G(\mathbf{q}) - \int e^{i\Theta(\mathbf{q}, \mathbf{q′})}\Sigma(\mathbf{q} - \mathbf{q′})G(\mathbf{q′})d\mathbf{q′} = \delta(\mathbf{q}),$$

where $\mathcal{H}(\pi)$ is the Hamiltonian of an electron moving in an external orthogonal ($\perp$ OXY) magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, $\pi = -i\hbar \nabla_\mathbf{q} - e(B \times \mathbf{q})/2c$ is the operator of gauge-invariant mechanical momentum, $\Theta(\mathbf{q}, \mathbf{q′}) = eB \cdot (\mathbf{q′} - \mathbf{q})/2c$, $\mathbf{q} = (x, y)$.

The equation [A3] is translation invariant that allows us to passage to the momentum representation with help of Fourier transform

$$G_p = \int e^{-ip \cdot \mathbf{q}/\hbar}G(\mathbf{q})d\mathbf{q}, \quad G(\mathbf{q}) = \int e^{ip \cdot \mathbf{q}/\hbar}G_p \frac{dp}{(2\pi \hbar)^2}.$$  

As a result, we obtain the equation

$$[\mathcal{E} - \mathcal{H}(\mathbf{p} - \frac{ie \hbar}{2c} B \times \nabla_p)]G_p = \Sigma_p \ast G_p = 1,$$  

where symbol $F \ast G$ denotes the Moyal $\ast$-product of the functions dependent on the momentum $p$

$$F_p \ast G_p = \frac{l_B^2}{\pi^2} \int \int \int \int F_{p+hk}e^{-i2\pi \frac{b}{l_B^2} \mathbf{k} \cdot \mathbf{G}}G_{hk+p}dkdk′,$$

where $l_B = \sqrt{\hbar/(eB)}$ is the magnetic length and $b = B/B$ is the unit vector parallel to the magnetic field $B$. In the general case, the $\ast$-product [A6] is noncommutative, but it satisfies the requirements of associativity and distributivity. An useful representation of the $\ast$-product can be obtained by rewriting [A6] with help of the argument shift operator $F_{p+hk} = \exp(\hbar k \cdot \nabla_p)F_p$. After calculating the resulting Gaussian integral, we obtain the definition of the $\ast$-product [20], which is equivalent to [A6]. Since $\ast$-product [20] includes an exponential function dependent on the gradient operator, it can be represented through shift of the argument in one of its multipliers, for example,

$$F_p \ast G_p = F \left( p - \frac{ie \hbar}{2c} B \times \nabla_p \right) G_p.$$  

Given the last definition of the $\ast$-product, we can rewrite the equation [A3] in the form used in main text [See Eq. (19)]. It looks almost like an equation for the averaged one-electron GF in absence of a magnetic field. The only difference is that [19] contains the $\ast$-product instead of usual multiplication. The momentum determined by the Fourier transform [A3] has the meaning of gauge-invariant mechanical momentum, which is proportional to the electron velocity, i.e., $p = mv$. Indeed the turning on a magnetic field results in this representation to replacing conventional products by the $\ast$-products of momentum dependent functions $F_p G_p \mapsto F_p \ast G_p$. In particular, $p_\alpha p_\beta \mapsto p_\alpha \ast p_\beta = p_\alpha p_\beta - i\epsilon_{\alpha\beta}h^2/2\pi l_B^2 (\alpha, \beta = x, y)$. It follows that the $\ast$-commutator of the Cartesian components of the momentum is equal to $[p_\alpha \ast p_\beta] = p_\alpha p_\beta - p_\beta p_\alpha = -ih^2/l_B^2$. Thus, the algebra of the momentum dependent $\ast$-numerical functions with respect to $\ast$-multiplication is equivalent to the algebra of the mechanical momentum operators of an electron in a magnetic field.

The transition to the Fourier transform in a magnetic field is useful, since $G_p$ possesses some properties of the one-electron GF in the momentum representation. In particularly, the trace over spatial degrees of freedom is expressed through integral with respect to momentum

$$\text{Sp} G = \int G_p \frac{dp}{(2\pi \hbar)^2}. $$
A similar representation also holds for the trace of the two-particle correlation function $S_p \langle V^R_\alpha G^A \rangle$ through which the components of the electrical conductivity tensor are expressed.

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