ASYMPTOTIC BEHAVIOR OF GRAFTING RAYS

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Abstract. In this paper we study the convergence behavior of grafting rays to the Thurston boundary of Teichmüller space. When the grafting is done along a weighted system of simple closed curves or along a maximal uniquely ergodic lamination this behavior is the same as for Teichmüller geodesics and lines of minima. We also show that the ray grafted along a weighted system of simple closed curves is at bounded distance from Teichmüller geodesic.

1. Introduction

A complex projective structure on a surface is a maximal atlas with charts modelled on $\mathbb{CP}^1$ whose transition maps are restrictions of automorphisms of $\mathbb{CP}^1$. The space $P(S)$ of projective structures on an oriented surface $S$ can be parametrized by the bundle of quadratic differentials over the Teichmüller space $T(S)$, through the Schwarzian derivative of the developing map of the projective structure. Another more geometric parametrization of $P(S)$ was given by Thurston using grafting. Indeed $P(S)$ is homeomorphic to $T(S) \times \mathcal{ML}(S)$ where $\mathcal{ML}(S)$ is a measured lamination space [9]. His grafting technique bridges together Teichmüller theory, projective structure and Kleinian group theory in geometric point of view. This connection appears between the boundary of the convex core of a hyperbolic 3-manifold which records a hyperbolic metric $\sigma$ together with a bending measured lamination $\lambda$, and its corresponding ideal boundary admitting a projective structure which is the result of the grafting of $\sigma$ along $\lambda$. After this technique was introduced, there has been a tremendous study over this subject. See [4, 10, 16, 21] for example for excellent expositions. The

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difficult part is to know the hyperbolic metric underlying the projective structure obtained by grafting. In this paper, we estimate the hyperbolic lengths of simple closed curves in a grafted structure and draw some useful information.

On the other hand, Teichmüller space has been studied more than several decades in relation to Teichmüller geodesics, dynamics and related combinatorial structures. The complication lies often in the thin part of it. Analytic tools like quadratic differentials, Beltrami differentials have been studied by Gardiner, Strebel and many others [6, 19]. Also the relation between quadratic differentials and measured laminations have been studied by Gardiner, Masur and Kerckhoff [7, 11]. More recently Teichmüller space has been studied by using harmonic maps by Minsky, Scannell and Wolf [17, 20].

Teichmüller space $T(S)$ has some common aspects with negatively curved spaces. Nevertheless, Masur [14] proved that Teichmüller space does not have negative curvature. His method consists on showing the existence of two geodesic rays starting at the same point and remaining at bounded distance from each other. Also by studying the limit points of some Teichmüller geodesics, Masur proved in [15] that the Teichmüller and Thurston boundaries of Teichmüller space are almost the same. Indeed Kerckhoff had proved that they are different [11]. There are also examples of Teichmüller geodesics which do not converge in Thurston compactification [13].

There is another object in Teichmüller space, lines of minima, which were introduced by Kerckhoff [12], with a more hyperbolic-geometric meaning (along these lines some function in terms of hyperbolic lengths is minimized, while along Teichmüller geodesics, some function in terms of extremal lengths is minimized). The convergence properties of these lines towards Thurston boundary has been studied by Díaz-Series [3], and their similar asymptotic behavior with Teichmüller geodesics by Choi-Rafi-Series [2].

In this paper, we study the third object, grafting rays and their convergence properties, and prove analogous results as in [15] for Teichmüller geodesics and in [3] for lines of minima. More precisely we prove that

**Theorem A** (Theorems 3.3 and 3.6) For a fixed hyperbolic surface $X$, if either $\lambda$ is a maximal uniquely ergodic measured lamination or a weighted system $\sum c_i \gamma_i$ of simple closed curves, then $gr_{\lambda}(X)$ converges to $[\lambda]$ or to $[\sum \gamma_i]$ respectively in Thurston boundary.

A natural question is then whether grafting rays are quasi-geodesics with respect to the Teichmüller metric.
Theorem B (Theorem 4.3) If \( \lambda = \sum c_i \gamma_i \) is a weighted system of simple curves, then the grafting ray \( gr \lambda(X) \) is within a bounded distance from a geodesic ray in \( T(S) \).

The proofs follow the similar lines as in [2, 15, 3]. To prove the convergence properties, we need to estimate the length of simple closed curves along the ray. We use the general estimates of the lengths of curves given by Proposition 2.1. Then we analyze in Propositions 3.4 and 3.5 how the different terms in this estimate behave along the grafting rays.

To prove that grafting rays are quasi-geodesics, we estimate the distance of these rays with Teichmüller geodesics by using Minsky’s product region theorem [18].

2. Preliminaries

2.1. Teichmüller space, metric and geodesics. Let \( S \) be a smooth orientable surface with finitely many punctures, of hyperbolic type. The Teichmüller space of \( S \) is the set \( T(S) \) of conformal structures on \( S \) up to conformal homeomorphisms homotopic to the identity. Each conformal structure determines a class of metrics compatible with the conformal structure. By the uniformization theorem, among these metrics there is a unique Riemannian hyperbolic metric, and conversely, each hyperbolic structure \( \sigma \) on \( S \) has an underlying conformal structure. Thus, we can also consider \( T(S) \) as the set of hyperbolic metrics on \( S \), where two metrics \( \sigma, \sigma' \) are equivalent if there is an isometry \((S, \sigma) \to (S, \sigma')\) isotopic to the identity. As notation, when we have a curve \( \gamma \) on a surface \( S \) and we consider a hyperbolic structure \( \sigma \) on \( S \), then we will denote by \( \ell_\sigma(\gamma) \) the hyperbolic length of the geodesic representative of \( \gamma \) on \( (S, \sigma) \). By abuse of notation, most times we will also denote by \( \gamma \) this geodesic representative.

Given two different points \( X, X' \in T(S) \), Teichmüller showed that there is a unique quasiconformal homeomorphism \( f_0 \) between them, homotopic to the identity, with the smallest possible maximal dilatation (or quasiconformal constant) \( K[f_0] \). The Teichmüller distance between \( X, X' \) is then defined to be \( d_{T(S)}(X, X') = \frac{1}{2} \log K[f_0] \).

To describe the extremal map \( f_0 \) Teichmüller made use of quadratic differentials. A meromorphic quadratic differential \( \varphi \) on a Riemann surface \( X \in T(S) \) has in local coordinates the expression \( \varphi(z) dz^2 \), where \( \varphi(z) \) is meromorphic with simple poles at most in the punctures of \( S \). At points \( z_0 \in S \) that are not zeros or poles of \( \varphi \), we can consider another coordinate chart \( w(z) = \int_{z_0}^z \sqrt{\varphi(z)} dz \), that determines two well defined foliations on \( S \): the horizontal foliation along whose leaves
Im\(w\) is constant, and the \textit{vertical foliation} along whose leaves \(Re\(w\) is constant. There is also a transverse measure defined on these foliations, making them two measured foliations \(\nu^+, \nu^-\): the \textit{horizontal measure} of an arc \(\alpha = z(t)\) transverse to the horizontal foliation is defined as the integral \(\int_0^1 |Im\sqrt{(\varphi(z(t))})| dt\), and similarly for the vertical measure. The space \(Q(X)\) of quadratic differentials on \(X\) is a complex vector space of complex dimension \(3g - 3 + p\), with \(g\) the genus of \(S\) and \(p\) the number of punctures.

We consider the norm given by \(\int_S |\varphi| dx dy\), and let \(B^1(X)\) be the open unit ball and \(\Sigma^1(X)\) the unit sphere. Given a quadratic differential \(\varphi \in \Sigma^1\) and a number \(0 \leq k < 1\), the \textit{Teichmüller deformation} of \(X\) determined by \(\varphi\) and \(k\) is the new complex structure \((X, \varphi, k)\) on \(S\) defined by the charts \(w'(w) = K^{1/2}Re\(w\) + \(iK^{-1/2}Im\(w\), where \(K = \frac{1 + k}{1 - k}\); that is, we expand the horizontal lamination of \(\varphi\) by a factor of \(K^{1/2}\) and contract the vertical lamination by a factor of \(K^{-1/2}\). Notice that the construction gives also a quadratic differential \(\varphi'\) on \((X, \varphi, k)\), with norm 1, whose horizontal and vertical measured foliations are \(K^{1/2}\nu^+\) and \(K^{-1/2}\nu^-\), respectively. The identity on the topological surface \(S\) is the extremal map between \(X\) and \((X, \varphi, k)\), with maximal dilatation \(K\).

Teichmüller existence and uniqueness theorems imply that the map \(\varphi \mapsto (X, \frac{\varphi}{\|\varphi\|}, \|\varphi\|)\) is a homeomorphism between \(B^1(X)\) and \(T(S)\).

**Teichmüller geodesics.** Given a point \(X \in T(S)\), and a measured foliation, it is proven in \([8]\) that there is a unique quadratic differential \(\varphi \in \Sigma^1(X)\) whose vertical foliation \(\nu^-\) is projectively equivalent to the given one. Then, the map \(t \mapsto (X, \varphi, \frac{1}{t+2})\) with \(t \geq 0\) parameterizes a Teichmüller geodesic ray starting at \(X\). We use this parametrization for later purpose. We denote this geodesic ray by \(G(\nu^-, X)\), and by \(G_t(\nu^-, X)\) the point on this geodesic that is the image of \(t\).

### 2.2. Thurston’s boundary of Teichmüller space.

Let \(\mathcal{ML}(S)\) denote the space of measured laminations of \(S\). The hyperbolic length of closed geodesics extends by linearity and continuity to the hyperbolic length of measured laminations. Equally, the intersection number between closed geodesics extends to the intersection number of measured laminations. Thurston proved that Teichmüller space can be compactified by the space \(PML\) of projective measured laminations, so that \(\overline{T(S)} = T(S) \cup PML\) is homeomorphic to a closed ball. Precisely, a sequence of marked hyperbolic surfaces \((S, \sigma_n)\) converges to a projective measured lamination \([\nu]\) if there is a sequence \(c_n \to 0\) such that
for any other measured lamination $\alpha$,
\[
\lim_{n \to \infty} c_n \ell_{\sigma_n}(\alpha) = i(\alpha, \nu).
\]
As a consequence of the definition, if $(S, \sigma_n) \to [\nu]$, then the length of any lamination $\beta$ satisfying $i(\beta, \nu) \neq 0$ tends to infinity along the sequence.

In this paper, we will freely use the natural identification between measured laminations and measured foliations.

### 2.3. Twisting numbers and Fenchel-Nielsen coordinates

Following Minsky [18] (see also [2]), we define the twisting number of a closed geodesic around another curve in a hyperbolic surface. Let $\sigma$ be a hyperbolic metric on a surface $S$ and $\gamma$ a homotopically nontrivial oriented simple closed curve. Suppose a simple closed curve $\alpha$ intersects $\gamma$ and geodesic representatives $\gamma^\sigma$ and $\alpha^\sigma$ intersects at $x$. Take lifts of them to $\mathbb{H}^2$ and denote them by $L_\gamma, L_\alpha$ respectively, so that they intersect at a point which projects to $x$. $L_\alpha$ has an orientation because $\gamma$ is oriented. Let $a_r, a_l$ be endpoints of $L_\alpha$ to the right and left of $L_\gamma$. Let $p: \mathbb{H}^2 \to L_\gamma$ be the orthogonal projection, and define the signed twisting number of $\alpha$ around $\gamma$ as
\[
tw_\sigma(\alpha, \gamma) = \min_{\gamma^\sigma \cap \alpha^\sigma} \frac{p(a_r) - p(a_l)}{\ell_\sigma(\gamma)}.
\]
We will denote by $T \sigma(\alpha, \gamma)$ the absolute value of $tw_\sigma(\alpha, \gamma)$.

We remark that one can also define the twisting number of a measured lamination around a $\gamma$ (in fact this number only depends on the support of the measured lamination).

Fenchel-Nielsen coordinates are global coordinates for Teichmüller space associated to a fixed marking of the surface. We recall its definition here. Let $\{\gamma_1, \ldots, \gamma_N\}$ denote a system of simple, oriented, closed curves that decompose $S$ into a union of pairs of pants. A pair of pants $P$ with boundary curves $\gamma_1, \gamma_2, \gamma_3$ contains three unique homotopy classes of simple arcs $\alpha_{12}, \alpha_{23}, \alpha_{13}$, called seams, such that $\alpha_{ij}$ joins $\gamma_i$ to $\gamma_j$. Fix a set of representatives of the seams which match on opposite sides of $\gamma_i$; this will determine a system of curves $\mu$, and we call the oriented curves $\gamma_i$ together with $\mu$ a marking of $S$. Note that these are topological data.

Now, given a hyperbolic metric $\sigma$ on $S$, each $\gamma_i$ and each seam have a unique geodesic representative $\gamma_i^\sigma, \sigma_{ij}^\sigma$, and $\mu$ has a unique representative going along the $\sigma_{ij}^\sigma$ and some arcs along $\gamma_i^\sigma$. For each $j = 1, \ldots, N$, let $m_j(\sigma)$ be the signed length of the arc on $\gamma_j^\sigma$ mentioned above (the
sign is given according to the orientation of $\gamma_j$. The twist parameter $t_j(\sigma)$ is defined to be $\frac{m_j(\sigma)}{\ell_\sigma(\gamma_j)}$.

The Fenchel-Nielsen coordinates for $\sigma$ with respect to the marking $\{\gamma_1, \cdots, \gamma_N; \mu\}$ are defined to be $(\ell_\sigma(\gamma_1), \cdots, \ell_\sigma(\gamma_N), t_1(\sigma), \cdots, t_N(\sigma))$.

Remark. Minsky proved in [18] that the twist parameter $t_j(\sigma)$ is almost the same as the twisting number of $\mu$ around $\gamma_j$; precisely, 

$$|t_j(\sigma) - tw_\sigma(\mu, \gamma_j)| \leq 1.$$ 

He also showed that for any two curves $\alpha, \beta$ intersecting $\gamma_j$, the difference $|tw_\sigma(\alpha, \gamma_j) - tw_\sigma(\beta, \gamma_j)|$ is independent of $\sigma$ up to a bounded error (of 1). As a consequence, along a family of hyperbolic surfaces, bounding the quantity $t_j(\sigma)\ell_\sigma(\gamma_j)$ is equivalent to bounding $tw_\sigma(\alpha, \gamma_j)\ell_\sigma(\gamma_j)$ for any curve $\alpha$ intersecting $\gamma_j$ if $\ell_\sigma(\gamma_j)$ is bounded.

2.4. Estimate of the length of a curve. In this paper we will always be dealing with hyperbolic surfaces with the property that one specific pants decomposition system of curves $\{\gamma_i\}$ have length bounded above. Under this condition, the length of any other curve $\alpha$ can be approximated up to a bounded additive error by the length of a polygonal curve. We describe here this estimate, which is mainly obtained in [3].

For notations we will say that two functions $f, g$ have the same order, denoted by $f \sim g$, if there exists a constant $C > 0$ so that $\frac{1}{C}f \leq g \leq Cf$. We will also use $f = O(1)$ to indicate that $f$ is bounded.

Let $(S, \sigma)$ be a hyperbolic surface and $\{\gamma_1, \cdots, \gamma_N\}$ a pants decomposition system. When $\gamma_i, \gamma_j$ bound a pant, denote by $H_{ij}$ the simple common perpendicular arc between them. For any pant $P$ and any boundary curve $\gamma_j$, we also consider the arc $H_{jj}$ perpendicular to $\gamma_j$ and separating the other two boundary components of $P$.

Now, any simple closed curve $\alpha$ can be homotoped to a polygonal curve $BA_\alpha$ made up of arcs $V_j$ running along the geodesics $\gamma_j$, and arcs $H_{ij}$ ($i$ may be equal to $j$). This polygonal curve is uniquely determined if we don’t allow backtrackings; i.e., if, for instance, $BA_\alpha$ goes along $H_{ij}$, then once around $\gamma_j$, and then comes back along $H_{ij}$, then we change this part by $H_{ii}$. (The notation $BA$ stands for the term “broken arc” used in [3]).

**Proposition 2.1.** Let $\{\gamma_1, \cdots, \gamma_N\}$ be a pants decomposition of $S$, let $M > 0$, and let $\alpha$ be a simple closed curve of $S$. Then there exists a constant $C$ (depending on $\alpha$ and $M$) such that for any hyperbolic
surface \((S, \sigma)\) with \(\ell_\sigma(\gamma_j) < M\) for all \(j = 1, \ldots, N\), we have

\[
|\ell_\sigma(\alpha) - \sum_{j=1}^{N} i(\alpha, \gamma_j) [2 \log \frac{1}{\ell_\sigma(\gamma_j)} + T_{w_\sigma}(\alpha, \gamma_j)\ell_\sigma(\gamma_j)]| \leq C.
\]

**Proof.** Because the lengths of all the curves \(\gamma_i\) are bounded above, the orthogonal arcs between each two of them are greater than some constant \(D\), and we can apply Lemma 5.1 in [3]. Then we get that

(1) \[
|\ell_\sigma(\alpha) - \ell_\sigma(BA_\alpha)| \leq C',
\]

for some constant \(C'\).

Next, we analyze the lengths of the arcs \(H_{ij}\) and \(V_j\) in \(BA_\alpha\). Since \(\ell_\sigma(\gamma_j)\) are bounded above, it can be shown by using the trigonometric formulae for pairs of pants that

\[
\ell_\sigma(H_{ij}) = \log \frac{1}{\ell_\sigma(\gamma_i)} + \log \frac{1}{\ell_\sigma(\gamma_j)} + O(1)
\]

(see Lemma 5.4 in [3]).

On the other hand, the length of the each arc \(V_j\) is close to \(T_{w_\sigma}(\alpha, \gamma_j)\ell_\sigma(\gamma_j)\). To see this, observe that we can choose a marking \(\mu\) associated to \(\{\gamma_i\}\) so that, for any surface,

\[
\left| \frac{\ell_\sigma(V_j)}{\ell_\sigma(\gamma_j)} - t_j(\sigma) \right| < 1.
\]

Since \(|t_j(\sigma) - T_{w_\sigma}(\mu, \gamma_j)| < 1\), we obtain that the difference between \(\frac{\ell_\sigma(V_j)}{\ell_\sigma(\gamma_j)}\) and \(T_{w_\sigma}(\mu, \gamma_j)\) is independent of the surface, up to bounded error. Since this is also true for the difference between \(T_{w_\sigma}(\mu, \gamma_j)\) and \(T_{w_\sigma}(\alpha, \gamma_j)\), we finally get that

\[
\left| T_{w_\sigma}(\alpha, \gamma_j) - \frac{\ell_\sigma(V_j)}{\ell_\sigma(\gamma_j)} \right| < O(1).
\]

Plugging the previous estimates in (1), we finally obtain the result. \(\square\)

An immediate consequence of the previous estimate that we will use later is the following.

**Corollary 2.2.** Let \(\{\gamma_1, \ldots, \gamma_N; \mu\}\) be a marking for a surface \(S\), and \((S, \sigma_n) \in \mathcal{T}(S)\) a sequence such that \(\ell_{\sigma_n}(\gamma_j)\) is bounded above for all \(j\). Suppose further that, for some \(k \in \{1, \ldots, N\}\), \(\ell_{\sigma_n}(\gamma_k)\) is bounded below away from zero and that there is a curve \(\beta\) intersecting \(\gamma_k\) with \(\ell_{\sigma_n}(\beta)\) bounded above. Then, the twisting number \(T_{w_{\sigma_n}(\beta, \gamma_k)}\) and the twist parameter \(t_k(\sigma_n)\) are bounded.
Remark. For arbitrary surfaces (not satisfying the condition explained above) the length of a simple closed curve can also be estimated in terms of a “thin-thick” decomposition of the surface. For the length of the curve in the thin parts, the estimates are the same as the one given above, but now it appear new terms corresponding to the length of the arcs crossing the thick parts. See [1] or [2] for detailed explanations.

2.5. Product region theorem. Let \( \mathcal{A} = \{ \gamma_1, \ldots, \gamma_k \} \) be a collection of disjoint, homotopically distinct, simple closed curves on \( S \). Given \( \epsilon_0 > 0 \) let \( T_{\text{thin}}(\mathcal{A}, \epsilon_0) \subset T(S) \) be the subset on which all curves in \( \mathcal{A} \) have length at most \( \epsilon_0 \). We extend \( \mathcal{A} \) to a pants decomposition system \( \{ \gamma_1, \ldots, \gamma_k, \gamma_{k+1}, \ldots, \gamma_N \} \), choose a marking \( \mu \) and consider Fenchel-Nielsen coordinates with respect to this marking. Let \( S_A \) denote the surface (possibly disconnected) obtained from \( S \) by pinching all the curves in \( \mathcal{A} \). The marking chosen on \( S \) induces a marking \( \mu_A \) on \( S_A \), and we take Fenchel-Nielsen coordinates on \( T(S_A) \) with respect to this marking. Then we can define the map \( \Pi_0 : T(S) \rightarrow T(S_A) \) by associating to a surface \( (S, \sigma) \) the surface in \( T(S_A) \) with Fenchel-Nielsen coordinates \( (\ell_\sigma(\gamma_{k+1}), \ldots, \ell_\sigma(\gamma_N), t_{\gamma_{k+1}}(\sigma), \ldots, t_{\gamma_N}(\sigma)) \) with respect to the \( \mu_A \) (that is, we just forget the coordinates corresponding to the curves in \( \mathcal{A} \)). In [18], the following map is considered

\[
\Pi : T(S) \rightarrow T(S_A) \times H_{\gamma_1} \times \cdots \times H_{\gamma_k},
\]

where \( H_{\gamma_j} \) is a copy of the upper half plane, the first component of \( \Pi \) is \( \Pi_0 \), and the remaining components \( \Pi_{\gamma_j} : T(S) \rightarrow H_{\gamma_j} \) are defined by

\[
\Pi_{\gamma_j}(\sigma) = t_{\gamma_j}(\sigma) + i \frac{1}{\ell_\sigma(\gamma_j)}.
\]

Let \( d_{H_{\gamma_j}} \) be half the usual hyperbolic metric on \( H_{\gamma_j} \). Minsky proved [18] that the Teichmüller metric on the thin parts can be approximated by the sup metric on the previous product space. More precisely:

**Theorem 2.3.** Given \( \epsilon_0 \) sufficiently small, then for any metrics \( \sigma, \tau \in T_{\text{thin}}(\mathcal{A}, \epsilon_0) \), we have

\[
d_{T(S)}(\sigma, \tau) = \max_{\gamma \in \mathcal{A}} \{ d_{T(S_A)}(\Pi_0(\sigma), \Pi_0(\tau)), d_{H_\gamma}(\Pi(\gamma)(\sigma), \Pi(\gamma)(\tau)) \} + O(1).
\]

A consequence of the theorem (see [2]) that we will use later is: if \( \sigma_1, \sigma_2 \in T_{\text{thin}}(\gamma, \epsilon_0) \) and for some \( \nu \in \mathcal{ML}(S) \) with \( Tw_\sigma(\nu, \gamma)\ell_\sigma(\gamma) = O(1) \), then

\[
d_{H_\gamma}(\Pi(\gamma)(\sigma_1), \Pi(\gamma)(\sigma_2)) = | \log \frac{\ell_{\sigma_1}(\gamma)}{\ell_{\sigma_2}(\gamma)} | + O(1).
\]
2.6. Grafting. We will describe a map $Gr: \mathcal{ML}(S) \times \mathcal{T}(S) \to P(S)$ and its composition with the projection to $\mathcal{T}(S)$, $gr: \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$. For a simple closed geodesic $\gamma$ on a hyperbolic surface $X$, $gr_{\gamma}(X)$ is constructed by cutting $X$ along $\gamma$ and inserting a Euclidean right cylinder $A(t)$ of height $t$ and circumference $\ell_X(\gamma)$, with no twist. The Euclidean and hyperbolic metric piece together continuously to give a well-defined conformal structure. To define the projective surface $Gr_{\gamma}(X)$, we first define a projective model $A(t)$ of the inserted cylinder as the quotient of the sector $\tilde{A}(t) = \{z \in \mathbb{C}^*: \operatorname{Arg}(z) \in \left[\frac{\pi}{2}, \frac{\pi}{2} + t\right]\}$ by the group generated by $\tau(z) = e^{i\ell_X(\gamma)}z$ (when $t \geq 2\pi$, $\tilde{A}(t)$ must be interpreted as multi-sheeted). Then we glue the projective cylinder $A(t)$ with the fuchsian structure on $X - \{\gamma\}$. Identifying the universal cover of $X$ with $\mathbb{H}^2$ so that $i\mathbb{R}$ is a component of the lift of $\gamma$, a local model for grafting is obtained by cutting along $i\mathbb{R}$, applying the map

$$z \mapsto \begin{cases} e^{it}z, & \text{if } \operatorname{Arg}(z) > \pi/2 \\ z, & \text{if } \operatorname{Arg}(z) \leq \pi/2 \end{cases}$$

and inserting the sector $\tilde{A}(t)$. Notice that the metric $|dz|/|z|$ on $\tilde{A}(t)$ makes $\tilde{A}(t)/\langle \tau \rangle$ into a Euclidean cylinder of height $t$ and circumference $\ell_X(\gamma)$, so that $Gr_{\gamma}(X)$ is conformally identical to $gr_{\gamma}(X)$.

Grafting can be extended to general measured laminations by continuity, giving

$$gr: \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S).$$

Given a projective structure $M$ on a surface, we can associate two metrics to it: the projective (or Thurston) metric, and the Kobayashi metric which coincides with the hyperbolic metric compatible with the underlying complex structure. We recall their definitions (see [21]).

For a tangent vector $v$ at a point $x \in M$, the projective length is defined as the infimum of the hyperbolic length of vectors $v'$ in $T\mathbb{H}^2$ such that there exists a projective map $f: \mathbb{H}^2 \to M$ sending $v'$ to $v$. The hyperbolic length is defined in the same way, but allowing $f$ to be any holomorphic map. It is clear from the definition that the hyperbolic length is less than or equal to the projective length. On the other hand, the projective metric on $Gr_{\gamma}(X)$, obtained from $X$ with a Euclidean cylinder inserted along $\gamma$, is just the combination of the hyperbolic metric of $X$ and the flat metric of the cylinder. Also, as a consequence of the definition of the Kobayashi metric, any holomorphic map between hyperbolic surfaces is distance decreasing.

Recall that the modulus of the annulus $A_r = \{z \in \mathbb{C}: r < |z| < 1\}$ is $\operatorname{mod}(A) = \frac{1}{2\pi} \log \frac{1}{r}$, and the length of its core geodesic with respect to its hyperbolic structure is $\ell_{A_r}(\gamma) = \frac{\pi}{\operatorname{mod}(A_r)}$. Also, the modulus of a
Euclidean right cylinder $E$ of height $t$ and circumference $\ell$ is $\text{mod}(E) = t/\ell$.

For a grafted surface $\text{gr}_{t\gamma}(X)$ along a closed curve $\gamma$, we can always consider a conformal homeomorphism $g$ from $A_r$ to the inserted Euclidean cylinder $E$ (where $r$ is chosen so that $\text{mod}(A_r) = \text{mod}(E)$). Since $g$ is conformal, it is distance decreasing with respect to the hyperbolic metric on $A_r$ and the Kobayashi metric on $\text{Gr}_{t\gamma}(X)$. Therefore, we have the following estimate for the length of $\gamma$ on $\text{gr}_{t\gamma}(X)$

$$\ell_{\text{gr}_{t\gamma}(X)}(\gamma) \leq \frac{\pi}{t} \ell_X(\gamma).$$

We will obtain a slightly better estimate in Proposition 3.4.

Given a measured lamination $\lambda$ and a point $X \in T(S)$, we define the grafting ray with base $X$ determined by $\lambda$ as the set $R_{t\lambda}(X) = \{\text{gr}_{t\lambda}(X) | t \geq 0\}$, and denote by $R_t$ the surface $\text{gr}_{t\lambda}(X)$.

3. grafting and Thurston boundary of Teichmüller space

We begin with some estimates of the length of measured laminations along grafting rays.

**Proposition 3.1** (McMullen [16]). For any $\alpha, \gamma \in \mathcal{ML}(S)$ and $X \in T(S)$, we have

$$\ell_{\text{gr}_{t\gamma}(X)}(\alpha) \leq \ell_X(\alpha) + i(\alpha, \gamma)$$

The inequality is strict if both $\alpha$ and $\gamma$ are nonzero.

**Proof.** When $\gamma$ is a closed curve, then the projective length of $\alpha$ at $\text{Gr}_{t\gamma}(X)$ is $\ell_X(\alpha) + t i(\alpha, \gamma)$, and the result follows because the hyperbolic length on $\text{Gr}_{t\gamma}(X)$ is smaller than the projective length. For general laminations, the result follows by continuity. \qed

**Corollary 3.2.** Consider the grafting ray $R_{\gamma}(X)$. We have:

(a) if $\alpha, \gamma \in \mathcal{ML}$ and $i(\alpha, \gamma) = 0$, then $\ell_{\text{gr}_{t\gamma}(X)}(\alpha) < \ell_X(\alpha)$ (so, the length of $\alpha$ is bounded above along the grafting ray);

(b) if $\alpha, \gamma$ are disjoint simple closed curves, then there exists a constant $C > 0$ such that $\ell_{\text{gr}_{t\gamma}(X)}(\alpha) > C$, for all $t > 0$.

**Proof.** Part (a) is direct application of the previous proposition. For (b), first observe that, by (a), $\ell_{\text{gr}_{t\gamma}(X)}(\alpha)$ is bounded above. Since $\alpha, \gamma$ are disjoint, then there exists a measured lamination $\beta$ intersecting $\alpha$ but disjoint from $\gamma$; by (a), $\ell_{\text{gr}_{t\gamma}(X)}(\beta)$ is bounded above. Suppose now that $\ell_{\text{gr}_{t\gamma}(X)}(\alpha) \to 0$; then, since $\beta$ intersects $\alpha$, it should be $\ell_{\text{gr}_{t\gamma}(X)}(\beta) \to \infty$, arriving to a contradiction. \qed
From this corollary we immediately find that, for \( \lambda \) maximal uniquely ergodic lamination, the grafting ray \( R_\lambda(X) \) converges to \([\lambda]\). The same thing holds for Teichmüller geodesics and lines of minima. A measured lamination is maximal if its support is not properly contained in the support of any other measured lamination. A measured lamination is uniquely ergodic if every measured lamination with the same support is in the same projective class.

**Theorem 3.3.** Let \( \lambda \) be a maximal uniquely ergodic measured lamination, and \( X \in \mathcal{T}(S) \). Then the grafting ray \( R_\lambda(X) \) converges to \([\lambda]\) in Thurston’s compactification of \( \mathcal{T}(S) \).

**Proof.** Since \( \mathcal{T}(S) \) is compact, the grafting ray has a convergent subsequence. We will show that the limits of all the convergent subsequences of the grafting ray are equal to \([\lambda]\).

So, suppose that \( \text{gr}_{t,\lambda}(X) \to [\nu] \). If \(|\nu| \neq |\lambda|\), since \( \lambda \) is maximal, then \( i(\nu, \lambda) \neq 0 \) and hence \( \ell_{\text{gr}_{t,\lambda}(X)}(\lambda) \) tends to infinity. Since this contradicts Corollary 3.2(a) for \( \alpha = \gamma = \lambda \), we must have that \(|\nu| = |\lambda|\). Since \( \lambda \) is uniquely ergodic, this implies that \([\nu] = [\lambda]\). \(\square\)

Next, we are going to obtain a better estimate of the lengths of the grafting curves, when we do grafting along a simple closed curve or a system of disjoint closed curves \( \gamma_i \). In particular, we will see that along the grafting ray determined by \( \sum c_j \gamma_j \), the length of each \( \gamma_j \) tends to zero in the order of \( 1/t \).

**Proposition 3.4.** Let \( \{\gamma_1, \ldots, \gamma_k\} \) be a system of disjoint simple closed curves and let \( \lambda = \sum_j c_j \gamma_j \), with \( c_j > 0 \). Consider \( X \in \mathcal{T}(S) \). Then, for each \( j = 1, \ldots, k \) we have

\[
\frac{2\theta_0}{2\theta_0 + ct} \ell_X(\gamma_j) \leq \ell_{\text{gr}_{t,\lambda}(X)}(\gamma_j) \leq \frac{\pi}{\pi + c_j t} \ell_X(\gamma_j),
\]

where \( c = \max(c_1, \ldots, c_k) \) and \( \theta_0 \) is some positive fixed number depending only on \( X \) (smaller than \( \pi/2 \)).

**Proof.** To prove the right side inequality, assume first that \( \lambda \) is equal to a single simple closed curve \( \gamma \). To compare the length of \( \gamma \) in the surfaces \( X \) and \( X_t = \text{gr}_{t,\gamma}(X) \), we look at the universal covers \( \tilde{X}, \tilde{X}_t \) of these surfaces. The surface \( \tilde{X}_t \) is the underlying conformal structure of the universal cover of \( \text{Gr}_{t,\gamma}(X) \), and this projective surface is built from \( \mathbb{H}^2 \) by inserting “sectors” \( \tilde{A}_j \) at any component \( \tilde{\gamma}_j \) of the lift of \( \gamma \) (when \( \tilde{\gamma}_j \) is not a vertical line, \( \tilde{A}_j \) is a region bounded by \( \tilde{\gamma}_j \) and an equidistant line from \( \tilde{\gamma}_j \), giving the effect that we get a bubble, see Figure 1). If
Figure 1. Universal cover of $Gr_{t\gamma}(X)$ when $t$ is small

$t$ is large, the universal cover $\tilde{G}r_{t\gamma}(X)$ should be understood as multi-sheeted. For both universal covers, the map $z \mapsto e^{t}z$ is a covering transformation, where we are using the notation $\ell = \ell_{X}(\gamma)$.

Consider the holomorphic map $\phi(z) = z^{\frac{\pi + t}{\pi}}$ from $\mathbb{H}^2$ to $\tilde{G}r_{t\gamma}(X)$. Let $L$ be the vertical segment of $\mathbb{H}^2$ with endpoints $i, e^{\frac{\pi}{\pi + t}}i$. Its image $\phi(L)$ projects onto a closed curve homotopic to $\gamma$ in $X_t$. Therefore,

$$\ell_{X_t}(\gamma) \leq \ell_{X_t}(\phi(L)) \leq \ell_{\mathbb{H}^2}(L) = \frac{\ell_{\pi}}{\pi + t} = \frac{\pi}{\pi + t}\ell_{X}(\gamma),$$

where the second inequality comes from the definition of the Kobayashi metric on $Gr_{t\gamma}(X)$. Then, we have the desired inequality. For the general case, $\lambda = \sum_{j} c_{j}\gamma_{j}$, note that doing more graftings along the other curves means that one has to insert more sectors to build $\tilde{G}r_{t\gamma}(X)$, and then the hyperbolic length on this domain is smaller than the one obtained in the previous computation by the definition of Kobayashi metric.

For the left side inequality, we use the following general formula that compares the length of a closed curve in two different hyperbolic surfaces in terms of the Teichmüller distance of these surfaces (see [22]).
Let \( X, X' \in T(S) \) with Teichmüller distance less than \( C > 0 \). Then, for any closed curve \( \gamma \), we have that
\[
(3) \quad e^{-2C\ell_{X'}(\gamma)} \leq \ell_X(\gamma) \leq e^{2C\ell_{X'}(\gamma)}.
\]

To apply this formula in our case, we construct the following quasi-conformal map \( \phi : X \to X_t = \text{gr}_{t\lambda}(X) \). Take \( \epsilon_0 > 0 \) sufficiently small so that the \( \epsilon_0 \)-neighborhood of \( \gamma_i \) is embedded on \( X \). Denote by \( N_i \) this \( \epsilon_0 \)-neighborhood, which is made up of two annuli \( A_{i,1}, A_{i,2} \) of height \( \epsilon_0 \) glued at \( \gamma_i \). Take similar annuli \( N_{i,t} \) in \( X_t \), namely, \( N_{i,t} \) is obtained from a Euclidean cylinder of height \( tc_i \) with circumference \( \ell_X(\gamma_i) \) by attaching \( A_{i,1} \) and \( A_{i,2} \) at each side. Fundamental domains of \( N_i, N_{i,t} \) are conformally equivalent to the sets
\[
\tilde{N}_i = \{ re^{i\theta} \in \mathbb{C} : r \in [1, e^{\ell_X(\gamma)}], \theta \in [-\theta_0, \theta_0] \}
\]
\[
\tilde{N}_{i,t} = \{ re^{i\theta} \in \mathbb{C} : r \in [1, e^{\ell_X(\gamma)}], \theta \in [-\theta_0 - c_i t, \theta_0] \},
\]
where \( \theta_0 \) is a concrete value in terms of \( \epsilon_0 \) and, for large \( t \), \( \tilde{N}_{i,t} \) should be understood as multi-sheeted. Then the map \( r \exp(i\theta) \mapsto r \exp(i(\frac{2\theta_0 + ct}{2\theta_0} \theta - \frac{c_i t}{2})) \) is a quasiconformal homeomorphism from \( \tilde{N}_i \) to \( \tilde{N}_{i,t} \) with maximal dilatation \( \frac{2\theta_0 + ct}{2\theta_0} \), and that glues well to give a quasiconformal homeomorphism \( \phi \) between the quotient annuli. Extending the homeomorphisms \( \phi \) to \( X - (\cup N_i) \) by the identity map, we obtain a quasiconformal homeomorphism \( \phi \) with maximal dilatation \( \frac{2\theta_0 + ct}{2\theta_0} \), where \( c = \max\{c_1, \ldots, c_k\} \). Therefore the Teichmüller distance between \( X, X_t \) is smaller than \( \frac{1}{2} \log \frac{2\theta_0 + ct}{2\theta_0} \), and by using formula (3), we get the desired inequality. \( \square \)

**Remark 1.** In [16], McMullen sketched the proof of the right side inequality of the above proposition by comparing the moduli of the respective covers of \( X \) and \( \text{gr}_{t\gamma}(X) \) corresponding to the subgroup of the fundamental group generated by \( \gamma \).

2. For the left side inequality, one can use some inequalities derived in [21], but not as sharp as in this proposition. When \( \lambda \) is a weighted simple closed curve, an asymptotic length estimate is given in [5].

Next, we bound the twisting numbers of some curves around the curves \( \gamma_j \) along which we do grafting. We will do it for some special curves associated to a pants decomposition, called dual curves. If \( \gamma_1, \ldots, \gamma_N \) is a pants decomposition, a curve \( \delta_j \) is dual to \( \gamma_j \) if \( i(\delta_j, \gamma_h) = 0 \) for all \( h \neq j \) and \( i(\delta_j, \gamma_j) \) is 1 or 2 depending on whether the two pants of the decomposition glued along \( \gamma_j \) are equal or different, respectively.
Proposition 3.5. Let \( \{\gamma_1, \ldots, \gamma_k\} \) be a set of disjoint simple closed curves and \( \lambda = c_1 \gamma_1 + \cdots + c_k \gamma_k \). We add curves \( \gamma_{k+1}, \ldots, \gamma_N \) to obtain a pants decomposition system, and let \( \delta_j \) be dual curves to \( \gamma_j \). Then, for each \( j = 1, \ldots, N \) we have that \( T_{\text{gr}_1}(\delta_j, \gamma_j) \ell_{\text{gr}_1}(\gamma_j) \) is bounded above as \( t \to \infty \).

Proof. If \( j = k + 1, \ldots, N \), the result follows from Corollary 2.2, since curves not intersecting \( \gamma_1, \ldots, \gamma_k \) have length bounded above and below away from zero along the grafting ray by Corollary 3.2.

Consider then \( j = 1, \ldots, k \), and assume that \( i(\delta_j, \gamma_j) = 2 \), that is, the two pairs of pants \( P_1, P_2 \) glued along \( \gamma_j \) are different (the other case is similar). Let \( X_t = \text{gr}_t(X) \). Since \( \ell_{X_t}(\gamma_j) \) is bounded above for all \( j \), we can apply Proposition 2.1; then, there exists a constant \( C \) independent of \( t \) such that

\[
(4) \quad \ell_{X_t}(\delta_j) > 4 \log \frac{1}{\ell_{X_t}(\gamma_j)} + 2 T w_{X_t}(\delta_j, \gamma_j) \ell_{X_t}(\gamma_j) - C
\]

To estimate the length of \( \delta_j \) from above, recall that \( X_t \) is a uniformization of the projective surface constructed by cutting \( X \) along \( \gamma_j \) and inserting a Euclidean cylinder \( E_j \); then the geodesic \( \delta_j \) of \( X \) is split into two arcs \( a_1, a_2 \); we join these arcs by two horizontal arcs \( L_t, L_t' \) of the inserted cylinder, to get the curve \( a_1 \cup L_t \cup a_2 \cup L_t' \), that represents \( \delta_j \) in \( X_t \) (see Figure 2). Then, the hyperbolic length of the geodesic representing \( \delta_j \) satisfies

\[
\ell_{X_t}(\delta_j) \leq \ell_{X_t}(a_1) + \ell_{X_t}(a_2) + \ell_{X_t}(L_t) + \ell_{X_t}(L_t').
\]

Now, \( \ell_{X_t}(a_i) < \ell_X(a_i) \) because the hyperbolic metric in \( X_t \) is smaller than the projective metric, and the latter coincides with the hyperbolic metric of \( X \) on the arcs \( a_1, a_2 \).

It is left to estimate \( \ell_{X_t}(L_t') \) when \( t \to \infty \). In order to do this, take \( t_0 \) some fixed positive number (independent of \( t \)) and divide the arc \( L_t \) as union of three segments \( L_1^t \cup L_2^t \cup L_3^t \), where both \( L_1^t \) and \( L_3^t \) have Euclidean length \( t_0 \). Since the Kobayashi metric is smaller than the projective metric, and the latter coincides with the Euclidean metric on the inserted annuli, we have that \( \ell_{X_t}(L_i^t) < t_0, \) for \( i = 1, 3 \).

We finally estimate \( \ell_{X_t}(L_2^t) \). As in section 2.6 consider a holomorphic embedding \( g: A_r \to E_j \), where \( r \) is such that \( \text{mod}(A_r) = \text{mod}(E_j) \); thus \( r = e^{-2 \pi \ell_{X}(\gamma_j)}, \) where \( \ell = \ell_X(\gamma_j) \). Taking \( E_j \) as the rectangle \([0, c_j \ell] \times [0, \ell]\) with the horizontal sides being identified, then the map \( g \) has inverse \( g^{-1}(z) = \exp(-\frac{2\pi}{\ell_{X}(\gamma_j)}z) \). On the other hand, the uniformizing map for \( A_r \) is \( \Phi: \mathbb{H}^2 \to A_r \) defined as \( \Phi(z) = z^{-\frac{\text{mod}(A_r)}{\text{mod}(E_j)}} \). Then, we
can see that $g^{-1}(L^2_i)$ is a geodesic arc on the hyperbolic metric of $A_r$, whose length is equal to the hyperbolic length of one of the connected components of $\Phi^{-1}(g^{-1}(L^2_i))$ (see Figure 3).
We compute this length by elementary hyperbolic geometry and, using that $g$ is distance decreasing, we get

$$\ell_{X_t}(L_t^2) \leq \ell_{A_t}(g^{-1}(L_t^2)) = 2 \log \frac{\cos \frac{\ell_{X_t}(L_t^2)}{2c_jt}}{\sin \frac{\ell_{X_t}(L_t^2)}{2c_jt}}$$

Finally, notice that

$$\frac{\cos \frac{\ell_{X_t}(L_t^2)}{2c_jt}}{\sin \frac{\ell_{X_t}(L_t^2)}{2c_jt}} \sim t \text{ as } t \to \infty,$$

so that we have that $\ell_{X_t}(L_t^2) \leq 2 \log(t) + O(1)$. The same estimate holds for $L_t'$. Thus, putting all these together, we have

$$\ell_{X_t}(\delta_j) \leq 4 \log(t) + O(1). \tag{5}$$

Combining (4) and (5), taking into account that $\frac{1}{\ell_{X_t}(\gamma_j)} \sim t$ (by Proposition 3.4), and cancelling out the terms $\log(t)$, we get that the term $Tw_{X_t}(\delta_j, \gamma_j)\ell_{X_t}(\gamma_j)$ is bounded as $t \to \infty$. \qed

Using the previous results we can now study the convergence to the Thurston’s boundary of the grafting rays determined by rational laminations.

**Theorem 3.6.** Let $\{\gamma_1, \ldots, \gamma_k\}$ be a system of disjoint simple closed curves, let $\lambda = \sum c_i \gamma_i$ be a weighted system, and consider any $X \in \mathcal{T}(S)$. Then

$$grt_{\lambda}(X) \to [\sum \gamma_i]$$

in Thurston boundary of Teichmüller space.

**Proof.** We need to show that for any two simple closed curves $\beta, \beta'$,

$$\frac{\ell_{X_t}(\beta)}{\ell_{X_t}(\beta')} \to \frac{i(\beta, \sum \gamma_i)}{i(\beta', \sum \gamma_i)};$$

where $X_t = grt_{\lambda}(X)$.

Extend $\{\gamma_1, \ldots, \gamma_k\}$ to a pants decomposition $\{\gamma_1, \ldots, \gamma_N\}$. By Corollary 3.2 all the $\gamma_j$ have length bounded above along the grafting ray. Therefore, applying Proposition 2.1 we have

$$\ell_{X_t}(\beta) = \sum_{j=1}^N i(\beta, \gamma_j)[2 \log \frac{1}{\ell_{X_t}(\gamma_j)} + Tw_{X_t}(\beta, \gamma_j)\ell_{X_t}(\gamma_j)] + O(1).$$

By Proposition 3.5 all the terms $Tw_{X_t}(\beta, \gamma_j)\ell_{X_t}(\gamma_j)$ are bounded above. Since $\ell_{X_t}(\gamma_j)$ is bounded below away from zero for $j = k +$
1, . . . , N (Corollary 3.2), the terms of the previous summation from
$j = k + 1$ to $j = N$ are bounded above. Therefore we get
\[
\ell_{X_t}(\beta) = 2 \sum_{j=1}^{k} i(\beta, \gamma_j) \log \frac{1}{\ell_{X_t}(\gamma_j)} + O(1),
\]
and similarly for $\beta'$. By Proposition 3.4 we have that $\ell_{X_t}(\gamma_i) \sim 1/t$, for all $i = 1, . . . , k$, so that
\[
\log \frac{1}{\ell_{X_t}(\gamma_i) \log(t)} \rightarrow 1.
\]
Therefore, to compute the limit of the quotient $\frac{\ell_{X_t}(\beta_i)}{\ell_{X_t}(\beta_j)}$, we first
divide numerator and denominator by $\log(t)$, then all the bounded terms
vanish and we get the desired result.
\[\square\]

4. Grafting ray is a Quasi-geodesic in Teichmüller space

In this section we will use Minsky’s product region theorem to estimate the distance between a grafting ray $R_{\lambda}(X)$ based on a surface $X \in T(S)$ and the Teichmüller geodesic ray $G(\lambda, X)$ based on the same surface and with vertical foliation $\lambda$, when $\lambda = c_1 \gamma_1 + \cdots + c_k \gamma_k$.

We first give some results needed to analyze the different components of Minsky’s formula.

Proposition 4.1. Consider the situation described in section 2.5 to define the map $\Pi_0: T(S) \rightarrow T(S_A)$; let $X_t \in T(S)$ be a family such that the length of any closed curve $\beta$ disjoint from the curves in $A$ is bounded above. Then, the family $\Pi_0(X_t)$ is contained in a compact subset of $T(S_A)$.

Proof. Let $(\ell_{X_t}(\gamma_j), t_j(X_t))_{j=1,\ldots,N}$ be the Fenchel-Nielsen coordinates of $X_t$ with respect to the marking $\{\gamma_j; \mu\}$. Because the lengths of all simple closed curves disjoint from $\gamma_1, \ldots, \gamma_k$ are bounded above, then these lengths are also bounded below away from zero, and the twist parameters $t_{k+1}(X_t), \ldots, t_N(X_t)$ are all bounded by Corollary 2.2. Hence, we have that $(\ell_{X_t}(\gamma_j), t_j(X_t))_{j=k+1,\ldots,N}$ is contained in a compact subset of $(\mathbb{R}_+ \times \mathbb{R})^{2N-2k}$, and this last set parameterizes $T(S_A)$. \[\square\]

Next, we give some properties of Teichmüller geodesics, mainly extracted from [13] (see also [2]).

Proposition 4.2. Let $\{\gamma_1, \ldots, \gamma_k\}$ be a system of disjoint simple closed curves and $\lambda = c_1 \gamma_1 + \cdots + c_k \gamma_k$, $c_i > 0$. Let $X \in T(S)$ and consider $\varphi \in \Sigma^1(X)$ with vertical foliation projectively equivalent to $\lambda$. Let $G(\lambda, X)$
be the Teichmüller geodesic ray determined by \( \varphi \), and \( G_t = G_t(\lambda, X) = (X, \lambda, \frac{t}{t+2}) \). Then we have:

i) the length of any simple closed geodesic disjoint from \( \gamma_i \) \( (i = 1, \ldots, k) \) is bounded above.

ii) for all \( i = 1, \ldots, k \), we have \( \ell_{G_t}(\gamma_i) \sim \frac{1}{t+1} \) for \( t \to \infty \);

iii) for any \( \alpha \) intersecting \( \gamma_i \), \( Tw_{G_t}(\alpha, \gamma_i) \ell_{G_t}(\gamma_i) \) is bounded.

Proof. i) By [15] Lemma 3 (v), the hyperbolic metrics on the points of the Teichmüller geodesic ray converge to a specific hyperbolic metric on the punctured surface \( S_A \). Therefore, any curve disjoint from \( \gamma_i \) converges to a curve in the limiting surface, and hence its length is bounded above.

ii) A quadratic differential like the one in the statement, with all the leaves of the vertical foliation being closed curves, is called a Jenkins-Strebel differential. The leaves of this foliation are grouped into conformal annuli \( A_i \) \( (i = 1, \ldots, k) \) with core curve homotopic to \( \gamma_i \). Let \( M_i \) be the moduli of these annuli (which depends on the coefficient \( c_i \) and the hyperbolic length of \( \gamma_i \) on \( X \). The surface \( G_t = (X, \lambda, \frac{t}{t+2}) \) is also decomposed into conformal annuli \( A_{i,t} \), whose moduli are now \( (t+1)M_i \). The hyperbolic length of the core geodesic of \( A_{i,t} \) is \( \frac{\pi}{(t+1)M_i} \).

Now, the result follows from Lemma 4 in [15], where the author proves that the hyperbolic metric on \( G_t \) and the hyperbolic metric on \( A_{i,t} \) have the same order (as \( t \to \infty \)) at all the points of a certain subannulus \( A_\delta \) of \( A_{i,t} \), which contains the geodesic core of \( A_{i,t} \).

iii) The proof of Proposition 3.5 works also in the situation explained in [15] for Teichmüller geodesic rays.

We finally prove the theorem.

**Theorem 4.3.** Let \( \{\gamma_1, \ldots, \gamma_k\} \) be a system of disjoint simple closed curves on \( S \), let \( \lambda = \sum c_i\gamma_i \) be a weighted system, and consider any \( X \in \mathcal{T}(S) \). Then, the grafting ray \( \mathcal{R}_\lambda(X) \) is at bounded distance from the Teichmüller geodesic ray \( G(\lambda, X) \).

**Proof.** We will show that \( d_{\mathcal{T}(S)}(\mathcal{R}_t, G_t) \) is bounded above. As in section 2.5, extend the \( \gamma_i \) to a pants decomposition and take a marking. By Corollary 3.2 and Proposition 4.2(i), the length of curves disjoint from \( \gamma_1, \ldots, \gamma_k \) have bounded length both along the grafting and the Teichmüller rays. Then, by Proposition 4.1 \( \{\Pi_0(\mathcal{R}_t), \Pi_0(G_t) : t \geq 0\} \) is contained in a compact set of \( \mathcal{T}(S_A) \), and therefore \( d_{\mathcal{T}(S_A)}(\Pi_0(\mathcal{R}_t), \Pi_0(G_t)) \) is bounded as \( t \to \infty \).
On the other hand, by Propositions 3.4 and 3.5 we have
\[ \ell_{R_t}(\gamma_i) \sim 1/t, \]
\[ Tw_{R_t}(\delta_{\gamma_i}, \gamma_i)\ell_{R_t}(\gamma_i) = O(1); \]
and by Proposition 4.2,
\[ \ell_{G_t}(\gamma_i) \sim \frac{1}{t+1}, \]
\[ Tw_{G_t}(\delta_{\gamma_i}, \gamma_i)\ell_{G_t}(\gamma_i) = O(1). \]
Then, by equation (2), we have
\[ d_{H_{\gamma_i}}(\Pi_{\gamma_i}(R_t), \Pi_{\gamma_i}(G_t)) = |\log \frac{\ell_{G_t}(\gamma_i)}{\ell_{R_t}(\gamma_i)}| \pm O(1). \]

Putting all these estimates into Minsky’s product region theorem, we finally obtain that the Teichmüller distance between \( G_t \) and \( R_t \) is bounded. \( \square \)

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