CONGRUENT NUMBERS ON THE RIGHT TRAPEZOID

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Abstract. We introduce and study a new kind of congruent number problem on the right trapezoid.

1. Introduction

A congruent number is a positive integer that is the area of a right triangle with three rational number sides. In other words, $n$ is a congruent number if and only if there is a right triangle with rational sides $a, b, c \in \mathbb{Q}^+$ such that
$$a^2 + b^2 = c^2, \ ab = 2n.$$ Let $x = n(a + c)/b, y = 2n^2(a + c)/b^2$, we get a family of elliptic curves
$$E_n : y^2 = x^3 - n^2x.$$ Conversely, if we have rational solutions $(x, y)$ on the elliptic curve $E_n$ with $y \neq 0$, then the rational numbers
$$a = \left| \frac{2nx}{y} \right|, b = \left| \frac{x^2 - n^2}{y} \right|, c = \left| \frac{x^2 + n^2}{y} \right|$$
are the sides of a right triangle with area $n$.

Due to the homogeneity of the condition $a^2 + b^2 = c^2$, we only need consider the square-free positive integers. Determining whether a given square-free positive integer is a congruent is the congruent number problem. It has not yet been solved in general. Many mathematicians studied this problem, such as Fibonacci, Fermat and Euler. For more information about this problem, we can refer to [1, 2, 4, 5, 6].

In particular, there is a generalized congruent number problem. A positive integer $n$ is called $t$-congruent number (see [6]) if there are positive rational numbers $a, b, c$ such that
$$a^2 = b^2 + c^2 - 2bc\frac{t^2 - 1}{t^2 + 1}, bc \frac{2t}{t^2 + 1} = 2n.$$ The case $t = 1$ corresponds to the classical congruent number problem. Its corresponding elliptic curve is
$$E_{n,t} : y^2 = x(x - \frac{n}{t})(x + nt).$$

Now we consider a new kind of congruent number problem which related with the area of right trapezoid.

Definition 1.1. A positive integer $n$ is called $i$-congruent number, if it is the area of the right trapezoid (Figure 1) with $a, b, c \in \mathbb{Z}^+, d \in \mathbb{N}, (b, c) = 1$. 

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When \( d = 0 \), \( n \) is the classic congruent number with \( a, b, c \in \mathbb{Z}^+ \). By this definition, we have
\[
(1.1) \quad n = (a + d)b/2, (a - d)^2 + b^2 = c^2, (b, c) = 1.
\]

Figure 1.

From the property of Pythagoras triples, we have \((a - d, b) = 1\), and \( a - d, b \) have opposite parity. Then
\[
(a - d, b) = (2xy, x^2 - y^2), \text{ or } (x^2 - y^2, 2xy),
\]
where \( x > y, (x, y) = 1 \), and \( x, y \) have opposite parity. Observing this, we have

**Proposition 1.1.** A positive integer \( n \) is \( i \)-congruent number if and only if \( n = pk \), where \( p \) is any odd prime and \( k \geq \frac{p^2 - 1}{4} \), or \( n = 2^i k, k \geq 2^i - 1, i \geq 1 \), where \( k \) is any odd integer.

The \( i \)-congruent numbers in Proposition 1.1 may coincide in some cases, the intersection set of them less than 100 is
\[
\{6, 18, 30, 42, 50, 54, 60, 70, 78, 84, 90, 98, 100\}.
\]

On the other hand, \( n > 1 \) is non-\( i \)-congruent number if and only if \( n \) has the following form
\[
p, p^2(p \neq 3), pq(5 < p < q < \frac{p^2 - 1}{4}), (\ast)
\]
\[
2^i(i \geq 0), 2^ip(i \geq 2, 2^{1+i/2} < p < 2^{2i} - 1),
\]
where \( p, q \) are primes.

For examples, the number of non-\( i \)-congruent numbers less than 100 is 46, where the non-primes have 21 which are
\[
1, 4, 8, 16, 20, 25, 28, 32, 49, 52, 56, 58, 62, 64, 74, 77, 82, 86, 88, 91, 94.
\]

But we have

**Proposition 1.2.** Almost every positive integer is \( i \)-congruent number.

**Theorem 1.3.** Let \( f(x) \) be the number of non-\( i \)-congruent numbers less than \( x \), then
\[
f(x) \sim \frac{cx}{\log x},
\]
where \( c = 1 + \ln 2 \).
For some $i$-congruent numbers $m$, there might be many right trapezoids having the area $n$. By Proposition 1.1 and its proof, for any positive integer $m$, there are infinitely many congruent integers $n$ such that we have $m$ or more right trapezoids which have the area $n$, it is only to need take $n = p_1 \cdots p_m n'$, where $p_i$ is the $i$-th prime and $n' \geq \left( \frac{p_m - 1}{4} \right)^2$ is an arbitrary integer.

We find 16 $i$-congruent numbers with $d = 0$ less than 1000, i.e.,

$$6, 30, 60, 84, 96, 180, 210, 330, 480, 486, 504, 546, 630, 840, 924, 960,$$

where there are two right triangles with $(a, b, c) = (21, 20, 29), (35, 12, 37)$ have the same area $n = 210$. Then we have

**Proposition 1.4.** Let $g(x)$ be the number of $i$-congruent numbers less than $x$ for $d = 0$ with $a > b$ and $a, b \in \mathbb{Z}^+$, including the repeated cases, then

$$\frac{\sqrt{x}}{2} + O(1) < g(x) \leq \frac{1}{2 \sqrt{4} x^{3/4}} + O(x^{3/9}).$$

Next, we give the definitions of two new kinds of congruent numbers.

**Definition 1.2.** A positive integer $n$ is called $k$-congruent number, if it is the area of the right trapezoid (Figure 1) with $a, b, c, d \in \mathbb{Q}^+, k \in \mathbb{Z}^+, k \geq 1, a = kd$.

By this definition, we have

\begin{equation}
(1.2) \quad n = (a + d)b/2, (a - d)^2 + b^2 = c^2.
\end{equation}

**Definition 1.3.** A positive integer $n$ is called $d$-congruent number, if it is the area of the right trapezoid (Figure 2) with $a, b, c \in \mathbb{Q}^+, d \in \mathbb{N}$.

When $d = 0$, $n$ is the classic congruent number. By this definition, we have

\begin{equation}
(1.3) \quad n = (a + 2d)b/2, a^2 + b^2 = c^2.
\end{equation}

![Figure 2](image)

**Figure 2.**

For $k$-congruent number and $d$-congruent number, we have

**Proposition 1.5.** For every positive integer $n$, there exists a $k$ such that $n$ is $k$-congruent number.

**Theorem 1.6.** For every positive integer $n$, there exists a $d$ such that $n$ is $d$-congruent number.

In section 2, we will give the proofs of propositions and theorems. We consider the family of elliptic curves about $k$-congruent numbers in section 3, and prove two propositions about $d$-congruent numbers in section 4.
2. Proofs of Propositions and Theorems

Proof of Proposition 1.1. By Definition 1.1, \( n \) is an \( i \)-congruent number if and only if the Diophantine system

\[
a + d = \frac{2n}{x^2 - y^2}, a - d = 2xy, \quad \text{or} \quad a + d = x^2 - y^2, a - d = \frac{n}{xy}
\]

has non-negative integer solutions.

Solving them, we get

\[
a = \frac{n}{x^2 - y^2} + x, d = \frac{n}{x^2 - y^2} - x, \quad \text{or} \quad 2a = \frac{n}{xy} + x^2 - y^2, 2d = \frac{n}{xy} - x^2 - y^2.
\]

Hence, when \( a - d \) be even, \( a, d \) are non-negative integers if and only if \( n = (x^2 - y^2)k, k \geq xy \), and when \( a - d \) be odd \( a, d \) are non-negative integers if and only if \( n = (x^2 - y^2)k, k \geq xy \).

Because every odd prime can be presented by the difference of two coprime integers, so the positive integer \( n \) is an \( i \)-congruent number, where \( p \) is odd prime and \( k \geq \frac{p^2 - 1}{4} \). Next, we will prove the opposite direction is also right. Let \( n = (x^2 - y^2)k, k \geq xy \), \( x > y \), \( (x, y) = 1 \), and \( x, y \) have opposite parity. If \( p | x^2 - y^2 \), put \( x^2 - y^2 = ps \), then \( n = p(sk) \). If \( p > s \), let

\[
x = \frac{p + s}{2}, y = \frac{p - s}{2}, xy = \frac{p^2 - s^2}{4},
\]

hence \( sk \geq s^2 - \frac{p^2 - s^2}{4} \geq \frac{p^2 - 1}{4} \). If \( p < s \), let

\[
x = \frac{p + s}{2}, y = \frac{s - p}{2}, xy = \frac{s^2 - p^2}{4},
\]

hence \( sk \geq s^2 - \frac{p^2 - s^2}{4} \geq \frac{p^2 - 1}{4} \). If \( p = s \), let

\[
x = \frac{p^2 + 1}{2}, y = \frac{p^2 - 1}{2}, xy = \frac{p^4 - 1}{4},
\]

hence \( sk \geq \frac{p^4 - 1}{4} \geq \frac{p^2 - 1}{4} \). So in any case, we have \( sk \geq \frac{p^2 - 1}{4} \). Let \( k' = ks \), we get \( n = pk' \), \( k' \geq \frac{p^2 - 1}{4} \).

By the same method, the positive integer \( n = 2^i k, k \geq 2^{2i} - 1, i \geq 1 \) is \( i \)-congruent number, where \( k \) is odd integer, only if we let \( x = 2^i, y = 1 \) in \( n = xyk \). Next, we want to prove the opposite direction is also right. Let \( n = xyk, k \geq x^2 - y^2, x > y \), \( (x, y) = 1 \), and \( x, y \) have opposite parity. If \( 2^i \| xy \), we can put \( xy = 2^t t, 2 \not| t \), then \( n = 2^t tk \). If \( 2^t > t \), let

\[
x = 2^t, y = t, x^2 - y^2 = 2^{2^i} - t^2,
\]

hence \( tk \geq t(2^{2^i} - t^2) \geq 2^{2^i} - 1 \). If \( 2^i < t \), let

\[
x = t, y = 2^i, x^2 - y^2 = t^2 - 2^{2i},
\]

hence \( tk \geq t(t^2 - 2^{2i}) \geq 2^{2i} - 1 \). So in any case, we have \( tk \geq 2^{2i} - 1 \). Let \( k' = kt \), we get \( n = 2^{i}k', k' \geq 2^{2i} - 1 \), where \( k' \) is an odd integer. \( \square \)

Proof of Proposition 1.2. We only need to prove the case for

\[
a = \frac{n}{x^2 - y^2} + xy, d = \frac{n}{x^2 - y^2} - xy.
\]
It’s easy to see that for a fix odd prime, when \( x \) is large enough, the number of \( i \)-congruent numbers \( n \) in the interval \([1, x]\) satisfying
\[
1 \leq n = pk \leq x, \quad k \geq \frac{p^2 - 1}{4}
\]
has the proportion \( 1/p \). By the Pigeonhole principle, taking \( p \) be the first \( s \) primes, under the above condition, the number of \( i \)-congruent numbers in the interval \([1, x]\) has the proportion
\[
1 - \prod_{j=1}^{s} \left( 1 - \frac{1}{p_j} \right),
\]
where \( p_j \) is the \( j \)-th prime. Since
\[
\lim_{s \to \infty} \left( 1 - \prod_{j=1}^{s} \left( 1 - \frac{1}{p_j} \right) \right) = 1,
\]
hence almost every positive integer is \( i \)-congruent number.

**Proof of Theorem 1.3.** It’s easy to see that the number of the 2-th, 4-th and 5-th term in (*) are \( O(\sqrt{x}) \), \( O(\log x) \) and \( O(x^{3/4}) \). Let \( \pi(x) \) be the prime function, the number of the first term in (*) is
\[
\pi(x) = \frac{x}{\log x} + O\left( \frac{x}{\log^2 x} \right).
\]
In the following, we consider the third term, i.e.,
\[
\sum \frac{1}{pq \leq x, 5 < p < q < p^2/4} = \sum \sum \frac{1}{5 < p \leq \sqrt{\frac{x}{4}}} \frac{1}{p < q < p^2/4} \sum \frac{1}{\sqrt{\frac{x}{4}} < p \leq \sqrt{x}} \frac{1}{p < q < x/p} = \sum_1 + \sum_2.
\]
In view of \( \frac{x^2}{\log x} \) is an increasing function, then
\[
\sum_1 = O\left( \sum \frac{\pi(p^2/4) - \pi(p)}{5 < p \leq \sqrt{4x}} \right) = O\left( \sum \frac{p^2}{\log p} \right) = O\left( \frac{x}{\log^2 x} \right).
\]
And
\[
\sum_2 = \sum \frac{\pi(x/p) - \pi(p)}{\sqrt{\frac{x}{4}} < p \leq \sqrt{x}} = \sum \frac{\pi(x/p)}{\sqrt{\frac{x}{4}} < p \leq \sqrt{x}} - \sum \frac{\pi(x/p)}{\sqrt{\frac{x}{4}} < p \leq \sqrt{x}} + \sum \frac{\pi(p)}{\sqrt{\frac{x}{4}} < p \leq \sqrt{x}} = \sum_3 - \sum_4 - \sum_5.
\]
Similarly, by the prime number theorem and the identity
\[
\sum \frac{1}{p} = \log \log x + c_1 + O\left( \frac{1}{\log x} \right),
\]
where \( c_1 \) is a constant, we get
\[
\sum_4 = \sum_5 = O\left( \frac{x}{\log^2 x} \right).
\]
At last, we estimate $\sum 3$. Let
\[
c_n = \begin{cases} 
1, & n \text{ is a prime}, \\
0, & n \text{ is not a prime}, 
\end{cases}
\]
and
\[
f(n) = \frac{x/n}{\log(x/n)}.
\]
By the prime number theorem and the sum formula of Abel, we get
\[
\sum_{\sqrt{x} < p \leq \sqrt{x}} \frac{x/p}{\log(x/p)} = \sum_{\sqrt{x} < n \leq \sqrt{x}} \frac{x/n}{\log(x/n)} c_n
\]
\[
= \pi(\sqrt{x})f(\sqrt{x}) - \pi(\sqrt{x})f(\sqrt{x}) - \int_{\sqrt{x}}^{\sqrt{x}} \pi(x) f'(t) dt
\]
\[
= \int_{\sqrt{x}}^{\sqrt{x}} \frac{x}{t \log t \log \frac{x}{t}} dt + O\left(\frac{x}{\log^2 x}\right).
\]
Taking $t = x^u$ in the above integral, we have
\[
\sum_3 = \frac{x}{\log x} \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{du}{u(1-u)} + O\left(\frac{x}{\log^2 x}\right)
\]
\[
= \frac{x \log 2}{2} + O\left(\frac{x}{\log^2 x}\right).
\]
Combining the above sum formulas, the number of non-$i$-congruent numbers less than $x$ is
\[
\frac{cx \log x}{\log x},
\]
where $c = 1 + \ln 2$. □

**Proof of Proposition 1.4.** It’s easy to see that for $d = 0$ the $i$-congruent numbers have the form $st(s^2 - t^2)$, where $s > t > 0$, $\frac{x}{(s,t)}$ and $\frac{t}{(s,t)}$ has the opposite parity. Let us consider
\[
st(s^2 - t^2) \leq x,
\]
because of the repeated cases and the parity, the number of the pair $(s, t)$ satisfying the above inequity great than $g(x)$. Noting that
\[
st(s^2 - t^2) \geq (t + 1)t(2t + 1) > 2t^3,
\]
then $t < \sqrt[3]{x/2}$. From
\[
st(s^2 - t^2) > t(s - t)^3,
\]
we have $s \leq \sqrt[3]{x/t + t}$. Then
\[
g(x) \leq \sum_{t=1}^{\sqrt[3]{x/t}} (\sqrt[3]{x/t} + t) \leq \frac{1}{2\sqrt[3]{4}} x^{\frac{2}{3}} + O(x^{\frac{5}{6}}).
\]
On the other hand, for any integer $t$, taking $s = \lceil \sqrt[3]{x/t} \rceil$, we have
\[
st(s^2 - t^2) < s^3t \leq x.
Then
\[ g(x) \leq \frac{1}{2} \sum_{t \leq \sqrt{x}} \left( \left\lfloor \frac{\sqrt{x}}{t} \right\rfloor - t \right) = \frac{\sqrt{x}}{2} + O(1). \]

Combining these, we complete the proof of Proposition 1.4. \( \square \)

Next, we use some facts about congruent numbers to prove Proposition 1.5 and the theory of elliptic curve to prove Theorem 1.6.

**Proof of Proposition 1.5.** From (1.2), we know that for \( k = 1 \), the right trapezoid degenerates into a rectangle. It’s easy to see that there are \( a = d = \frac{2}{t}, c = b = \frac{t}{2} \) such that the area is 1, then 1 is a \( k \)-congruent number. This is a trivial case. In the following, we consider the case \( k \geq 2 \).

Let
\[ b = \left\lfloor \frac{x^2 - (k^2 - 1)^2n^2}{(k + 1)y} \right\rfloor, \quad d = \frac{2nx}{y}, \]
we get a family of elliptic curves
\[ E_{n,k} : y^2 = x^3 - (k^2 - 1)^2n^2x, \]
and
\[ a = \left\lfloor \frac{2knx}{y} \right\rfloor, \quad c = \left\lfloor \frac{x^2 + (k^2 - 1)^2n^2}{(k + 1)y} \right\rfloor. \]
For a given \( k \geq 2, E_{n,k} \) is the special case of congruent number curve, we call it \( k \)-congruent number curve. Noting that \( n^3 - n \) is a congruent number, let \( k = n \), then \( E_{n,n} \) has positive rank, which leads to \( n \) is a \( n \)-congruent number.

Therefore, for every positive integer \( n \), there exists a \( k \) such that \( n \) is \( k \)-congruent number. \( \square \)

For examples, when \( k = n = 2 \), from \( E_{2,2} \), we have
\[ (a, b, c, d) = \left( \frac{8}{3}, \frac{5}{3}, \frac{1}{3}, \frac{4}{3} \right), \left( \frac{80}{7}, \frac{7}{30}, \frac{1201}{210}, \frac{40}{7} \right), \left( \frac{6808}{4653}, \frac{1551}{851}, \frac{7776485}{3959703}, \frac{3404}{851} \right) \]
such that 2 is a 2-congruent number.

When \( k = n = 3 \), from \( E_{3,3} \), we have
\[ (a, b, c, d) = \left( \frac{9}{4}, \frac{5}{2}, \frac{3}{4} \right), \left( \frac{21}{40}, \frac{60}{7}, \frac{1201}{140}, \frac{7}{40} \right), \left( \frac{851}{517}, \frac{4653}{1702}, \frac{7776485}{2639802}, \frac{851}{1551} \right) \]
such that 3 is a 3-congruent number.

**Proof of Theorem 1.6.** Let
\[
\begin{align*}
    a &= \frac{(3x - d^2 - 3n)(3x - d^2 + 3n)}{3(-3y + 3dx - d^3)}, \\
    b &= \frac{2n(3x - d^2)}{(-3y + 3dx - d^3)}, \\
    c &= \frac{(9 - 6d^2)x^2 + 9n^2 + d^4}{3(-3y + 3dx - d^3)}.
\end{align*}
\]
in (1.3), we get a family of elliptic curves

\[ E_{n,d} : y^2 = x^3 - \frac{3n^2 + d^4}{3} x + \frac{(9n^2 + 2d^4)d^2}{27}. \]

We call it \( d \)-congruent number curve. To prove for every positive integer \( n \) there are \( a, b, c, d \) satisfying (1.3), let \( d = 3n \), we get

\[ E_{n,3n} : y^2 = x^3 - (1 + 27n^2)n^2 x + 3n^4(1 + 18n^2). \]

The discriminant of \( E_{n,3n} \) is \( \Delta = (4 + 81n^2)n^6 \). When \( n \geq 1 \), we have \( \Delta > 0 \), this means that \( E_{n,3n} \) is nonsingular.

To find a solution \( a, b, c, d \) satisfying (1.3), we need to find a suitable point on \( E_{n,3n} \). Noting that the point \( P = (-6n^2, 3n^2) \) lies on \( E_{n,3n} \), using the group law on the elliptic curve, we obtain the point

\[ [2]P = \left( \frac{(27n^2 + 1)(243n^2 + 1)}{36}, -\frac{(81n^2 + 1)(6561n^4 + 324n^2 - 1)}{216} \right). \]

It’s easy to see that the \( x \)-coordinate of the point \([2]P\) is not in \( \mathbb{Z} \) for every \( n \geq 1 \). By the Nagell-Lutz Theorem (see p. 56 of [3]), for all \( n \geq 4 \) the point \([2]P\) is of infinite order. Then there are infinitely many rational points on \( E_{n,3n} \). Moreover, the point \([2]P\) is the exact point such that \( a, b, c, d = 3n \) satisfying (1.3), which leads to

\[
\begin{align*}
  a &= \frac{(729n^3 - 81n^2 + 27n + 1)(9n - 1)}{6(1 + 81n^2)}, \\
  b &= \frac{12n(1 + 81n^2)}{(1 + 9n)(729n^3 + 81n^2 + 27n - 1)}, \\
  c &= \frac{43046721n^8 + 2125764n^6 + 39366n^4 + 1620n^2 + 1}{6(1 + 81n^2)(1 + 9n)(729n^3 + 81n^2 + 27n - 1)}. 
\end{align*}
\]

For every \( n \geq 1 \), we have \( a, b, c \in \mathbb{Q}^+ \).

Therefore, for every positive integer \( n \), there exists a \( d \) such that \( n \) is \( d \)-congruent number. \( \square \)

For examples, when \( n = 1, 2, 3 \), from the elliptic curve \( E_{n,3n} \), we have

\[ (a, b, c, d) = \left( \frac{1352}{123}, \frac{123}{1045}, \frac{1412921}{128535}, 3 \right) \]

such that 1 is a 3-congruent number,

\[ (a, b, c, d) = \left( \frac{94571}{1950}, \frac{7800}{117971}, \frac{11156645809}{230043450}, 6 \right) \]

such that 2 is a 6-congruent number and

\[ (a, b, c, d) = \left( \frac{123734}{1095}, \frac{3285}{71722}, \frac{8874450677}{78535590}, 9 \right) \]

such that 3 is a 9-congruent number.
3. Further Consideration on $E_{n,k}$

In the proof of Proposition 1.5, we get a family of elliptic curves

$$(3.1) \quad E_{n,k} : y^2 = x^3 - (k^2 - 1)^2 n^2 x.$$ 

$E_{n,k}$ has four integer points

$$(x, \pm y) = (0, 0), (((k^2 - 1)n, 0), (-((k^2 - 1)n, 0))$$

and the point at infinity. In [1], the author listed some classes of congruent numbers. For examples,

$$(3.2) \quad n = \alpha^4 + 4\beta^4, 2\alpha^4 + 2\beta^4, \alpha^4 - \beta^4$$

are congruent numbers for $\alpha, \beta \in \mathbb{Z}^+$. Then we have

**Proposition 3.1.** For a fixed $k \geq 2$, there are infinitely many positive integers $n$ which are $k$-congruent numbers.

**Proof.** Let $(k^2 - 1)n = \alpha^4 - \beta^4$. Put

$$\alpha = k^2, \beta = 1,$$

then

$$n = k^2 + 1.$$ 

Therefore, for a fixed $k \geq 2$, there are infinitely many positive integers $n = k^2 + 1$ which are $k$-congruent numbers. \qed

Next, we consider the problem for a fixed $n$ whether there are infinitely many $k$ such that $n$ is a $k$-congruent number. Noting that $m^3 - m$ is a congruent number, we consider the Diophantine systems

$$\lambda(k + 1) = \alpha^2 - \beta^2, n(k - 1) = \lambda(\alpha^2 + \beta^2),$$

which lead to the Pell’s equation

$$(n - \lambda^2)\alpha^2 - (n + \lambda^2)\beta^2 = 2n\lambda.$$ 

When $\lambda = 1$, it’s easy to prove they have infinitely many integer solutions for $n = 2, 5$. When $\lambda = 2$, the same result holds for $n = 10, 13, 52$. When $\lambda = 3$, the same result holds for $n = 13, 17, 27, 30, 45$. When $\lambda = 4$, the same result holds for $n = 17, 18, 26, 32, 50, 68, 80$. Then there are infinitely many $k$ such that $2, 5, 10, 13, 17, 18, 26, 27, 30, 32, 45, 50, 52, 68, 80$ are $k$-congruent numbers.

In the following table, we give some integer solutions of $(k^2 - 1)n = \alpha^4 - \beta^4$ for $1 < n \leq 10$. 

Problem 3.2. For each \( n \geq 2 \), whether there are infinitely many \( k \) such that \( n \) is a \( k \)-congruent number.

By Tunnell’s theorem [7] about congruent numbers, we have

Corollary 3.3. Assuming the validity of the BSD Conjecture for \( E_n : y^2 = x^3 - m^2x \), the following statements are equivalent:

1. For a given \( k \geq 2 \), \( n \) is a \( k \)-congruent number.

2. If \( (k^2 - 1)n \) is odd, then the number of triples of integers \((x, y, z)\) satisfying
   \[ 2x^2 + y^2 + 8z^2 = (k^2 - 1)n \]
   is equal to twice the number of triples satisfying
   \[ 2x^2 + y^2 + 32z^2 = (k^2 - 1)n. \]
   If \( (k^2 - 1)n \) is even, then the number of triples of integers \((x, y, z)\) satisfying
   \[ 8x^2 + 2y^2 + 16z^2 = (k^2 - 1)n \]
   is equal to twice the number of triples satisfying
   \[ 8x^2 + 2y^2 + 64z^2 = (k^2 - 1)n. \]

4. Further Consideration on \( E_{n,d} \)

In the proof of Theorem 1.6, we get a family of elliptic curves

\[
E_{n,d} : y^2 = x^3 - \frac{3n^2 + d^4}{3}x + \frac{(9n^2 + 2d^4)d^2}{27},
\]
which seems complicated, but there are some interesting things. Multiply 729 on both sides of (4.1), we have

\[
E'_{n,d} : y^2 = x^3 - (81n^2 + 27d^4)x + 27d^2(9n^2 + 2d^4).
\]
For \( n \neq d^2 \), we have the following proposition.

Proposition 4.1. If \( n \neq d^2 \), then \( n \) is a \( d \)-congruent number.

Proof. By some calculations, we find that the points \( Q = (-6d^2, 27dn) \) and \( R = (3d^2 - 9n, 27dn) \) lie on \( E'_{n,d} \). From the group law, we get

\[
[2]Q = \left( \frac{3(n^2 + 3d^4)(3n^2 + d^4)}{4d^2n^2}, \frac{-27(n^2 + d^4)(d^8 + 4d^4n^2 - n^4)}{8d^2n^3} \right).
\]
Let the point $S$ be the intersection of the line, which goes through $Q, R$, and the elliptic curve $E_{n,d}$, we have
\[
S = \left( \frac{3(d^6 - nd^4 + 7d^2n^2 - 3n^3)}{(n + d^2)^2}, -\frac{27dn(-n + d^2)(d^4 + 3n^2)}{(n + d^2)^3} \right),
\]
which leads to
\[
\begin{cases}
a = \frac{2(d^4 + n^2)d}{(n - d^2)(n + d^2)}, \\
b = \frac{(n - d^2)(n + d^2)}{2nd}, \\
c = \frac{n^4 + 6d^4n^2 + d^8}{2(n - d^2)(n + d^2)d}.
\end{cases}
\]
it’s easy to see that $a, b, c \in \mathbb{Q}^+$ when $n > d^2$.

From the point
\[
-S = [-1]S = \left( \frac{3(d^6 - nd^4 + 7d^2n^2 - 3n^3)}{(n + d^2)^2}, -\frac{27dn(-n + d^2)(d^4 + 3n^2)}{(n + d^2)^3} \right),
\]
we get
\[
\begin{cases}
a = \frac{4dn^2}{(-n + d^2)(n + d^2)}, \\
b = \frac{n(-n + d^2)(n + d^2)}{(d^4 + n^2)d}, \\
c = \frac{n(n^4 + 6d^4n^2 + d^8)}{2(-n + d^2)(n + d^2)(d^4 + n^2)d},
\end{cases}
\]
it’s easy to see that $a, b, c \in \mathbb{Q}^+$ when $n < d^2$.

Therefore, for $n \neq d^2$, all other positive integers are $d$-congruent numbers. □

For example, when $d = 1$, for $n \geq 2$, we have
\[
\begin{cases}
a = \frac{2(n^2 + 1)}{(n - 1)(n + 1)}, \\
b = \frac{(n - 1)(n + 1)}{2n}, \\
c = \frac{n^4 + 6n^2 + 1}{2(n - 1)(n + 1)n},
\end{cases}
\]
i.e., all $n \geq 2$ are 1-congruent numbers.

For a fixed $n$, from the proof of Proposition 4.1, we have

**Proposition 4.2.** For any $n \in \mathbb{Z}^+$, except $d^2 = n$, all other $d$ such that $n$ is a $d$-congruent number.

**Remark 4.3.** The $k$-congruent number curve $E_{n,k}$ and the $d$-congruent number curve $E_{n,d}$ have a big difference. The $j$-invariant of $E_{n,k}$ is 1728, but the $j$-invariant of $E_{n,d}$ depends on $n$. 
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