The Wick theorem for non-Gaussian distributions and its application for noise filtering of correlated $q$-Exponentially distributed random variables

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We derive the Wick theorem for the $q$-Exponential distribution. We use the theorem to derive a numerical algorithm for finding parameters of the correlation matrix of $q$-Exponentially distributed random variables given empirical spectral moments of the time series.

Keywords: Many-point correlation functions; Wick theorem; Portfolio management;

1. Introduction

$q$-Exponential distributions ($q$-Exponentials defined in the next section) are used[1] in modeling distributions of stocks. The $q$-Exponentials possess two desirable features namely exhibit power law tails in the high end of the distribution and tend toward a Gaussian when $q = 1 + 1/K \to 1$. These facts provide enough motivation to derive a variant of a Wick theorem for linear combinations of independent identically distributed (iid) $q$-Exponentials (correlated $q$-Exponentials).

Recent growth of interest in applications of physics to economics and to the theory of finance (econophysics) yields the Wick theorem useful for the following purposes:

(1) Deriving exact relations between spectra of eigenvalues of financial covariance matrices related to $q$-exponentially distributed time series and of estimators of covariance matrices. This will be a generalization of existing exact relations [8] for Gaussian distributed time series.

(2) Deriving algorithms [6] for optimizing a portfolio (minimizing the variance and/or the higher moments of a portfolio subject to a presumed return from a portfolio) of stocks whose time series are $q$-exponentially distributed.

(3) Verifying empirical findings [5] regarding a very fast time decay of the autocorrelation function in time series and regarding a power-law time dependence of the autocorrelation function of the volatility; investigating many point correlations in financial time series.
2. The Wick theorem

The objective of this section is to derive the Fourier transform of \( N \) correlated \( q \)-Exponentially distributed variables.

A random variable \( X \) is \( q \)-Exponentially distributed, \( (X \sim q - \text{Exp}) \) if \( q = 1 + 1/K \) and the probability density function (pdf) \( D_K(x) \) of \( X \) reads:

\[
D_K(x) = \frac{N_K}{\sqrt{2\pi\sigma_K^2}} e^{-x^2/(2\sigma_K^2)}
\]  

where \( e_K^2 := (1 - z/K)^{-K} \), \( N_K = \Gamma(K)/(\sqrt{K} \cdot \Gamma(K - 1/2)) \) and \( \sigma_K = \sigma\sqrt{(K - 3/2)/K} \).

The pdf (2.1) has (equation (20.52) in [1]) is a continuous superposition of Gaussians and has a following integral representation:

\[
D_K(x) = \frac{1}{2^D} \int_{\mathcal{D}} d^D\xi e^{-\xi^2(D+1)/2} \cdot \frac{e^{-\xi^2x^2/(2\sigma_D^2)}}{\sqrt{2\pi\sigma_D^2/\xi^2}}
\]  

where \( \mathcal{D} := \sqrt{2\pi/(D+1)} \) and \( \sigma_D = \sigma\sqrt{(D-2)/(D+1)} = \sigma_K \) and the integral runs over the \( D \)-dimensional space \((\xi_1, \ldots, \xi_D)\) with:

\[
D = 2K - 1
\]

We define correlated \( q \)-Exponentials as linear combinations of iid \( q \)-Exponentials. Therefore the joint pdf \( \rho_\vec{X}(\vec{x}) \) of \( NT \) correlated \( q \)-Exponentials

\[
X_{i,t} = \sum_{j=1}^{N} \sum_{t=1}^{T} O_{i,t}^{j,\theta} Y_{j,\theta}
\]

where \( i = 1, \ldots, N, \ t = 1, \ldots, T \) and \( Y_{j,\theta} \sim q - \text{Exp} \) are iid, reads:

\[
\rho_\vec{X}(\vec{x}) = \int_{\mathcal{O}^{NT}} \delta(\vec{x} - \sum_{i=1}^{N} \sum_{t=1}^{T} O_{i,t}^{j,\theta} y_{i,t}) d\mathcal{O}^{NT}
\]

\[
= (\det \mathcal{O})^{-1} \prod_{i=1}^{N} \prod_{t=1}^{T} D_K \left( \sum_{j=1}^{N} (O^{-1})_{i,t}^{j,\theta} x_{j,\theta} \right)
\]

Here we call \( \mathcal{O} := \{O_{i,t}^{j,\theta}\} \) a rotation tensor. We take \( \vec{k} := \{k_{i,t}\} \) and

\[
\int_{\mathcal{D}^{NT}} \det \mathcal{O} e^{\vec{k} \cdot \vec{x}}
\]

we calculate the Fourier transform \( \kappa(\vec{k}) = \mathcal{F}_\vec{X} [\rho_\vec{X}](\vec{k}) := \int_{\mathcal{O}^{NT}} d^{NT} \vec{\rho}_\vec{X}(\vec{x}) e^{i\vec{k} \cdot \vec{x}} \). It reads:

\[
(\mathcal{D}^{NT} \det \mathcal{O}) \kappa(\vec{k}) = (\mathcal{D}^{NT} \det \mathcal{O}) \int_{\mathcal{R}^{NT}} d^{NT} \vec{x} \rho_\vec{X}(\vec{x}) e^{i\vec{k} \cdot \vec{x}} =
\]

\[
\int_{\mathcal{D}^{NT}} \prod_{i=1}^{N} \prod_{t=1}^{T} d\xi_{i,t} \left( \prod_{i=1}^{N} \prod_{t=1}^{T} \frac{e^{-\xi_{i,t}^2(D+1)/2}}{\sqrt{2\pi\sigma_D^2/\xi_{i,t}^2}} \right) \int_{\mathcal{R}^{NT}} d^{NT} \vec{x} \exp \left( -\frac{\xi_{i,t}^2}{2\sigma_D^2} \right) \left( \sum_{j=1}^{N} \sum_{\theta=1}^{T} (O^{-1})_{i,t}^{j,\theta} x_{j,\theta} \right)^2 \right) e^{i\vec{k} \cdot \vec{x}}
\]
\[ \int_{\mathbb{R}^D} \cdots \int_{\mathbb{R}^D} \prod_{i=1,\ldots,N} d\xi_{i,\theta} \left( \prod_{i=1,\ldots,N} e^{-\xi_{i,\theta}^2(D+1)/2} \right) \int_{\mathbb{R}^{NT}} d^{NT}x \exp \left\{ -\frac{1}{2} \overset{T}{\bar{x}}^T \cdot \mathbb{C}^{-1}(\bar{\xi}) \cdot \bar{x} + i\bar{k}\bar{x} \right\} \]  

(2.8)

where we introduced \( \mathbb{C}(\bar{\xi}) := \mathbb{Q} \cdot \mathbb{D} \cdot \mathbb{Q}^T \) (a correlation tensor) that is related to the rotation tensor \( \mathbb{Q} \) and to a diagonal tensor \( \mathbb{D}_{\lambda,\mu} = \delta_{\lambda,\mu} \sigma_{\lambda,\mu}^2 / \xi_{\mu,1}^2 \). The transposition \( T \) operation is defined as \( (O^T)_{\lambda,\mu} := O_{\mu,\lambda} \). This means that:

\[ C_{i,t}^{\mu,\nu}(\bar{\xi}) := \sum_{p=1,\ldots,N} \sum_{\lambda=1,\ldots,T} O_{i,t}^{\mu,\lambda} \frac{\sigma_{\nu,\lambda}^2}{\xi_{p,\lambda}} O_{p,\lambda}^{\nu,\mu} \]  

(2.9)

The last integral on the right hand side in (2.8) is evaluated by “completing to a square” and it reads:

\[ (2\pi)^{NT/2} \left( \det(\mathbb{C}^{-1}(\bar{\xi}))^{-1/2} \right) \cdot \exp \left\{ -\frac{1}{2} \bar{k}^T \mathbb{C}(\bar{\xi}) \bar{k} \right\} \]  

(2.10)

\[ = (2\pi)^{NT/2} \left( \det(\mathbb{Q}) \sigma_D^{NT} \right) \left( \prod_{j=1}^{N} \prod_{\theta=1}^{T} \xi_{j,\theta}^{-1} \right) \cdot \exp \left\{ -\frac{1}{2} \bar{k}^T \mathbb{C}(\bar{\xi}) \bar{k} \right\} \]  

(2.11)

Therefore the Fourier transform (2.7) reads:

\[ \kappa(\bar{k}) := \left( \frac{2\pi}{\mathbb{D}^{\overset{T}{\bar{x}}} \sigma_D^{NT}} \right)^{1/2} \int_{\mathbb{R}^D} \cdots \int_{\mathbb{R}^D} d\xi_{j,\theta} \left( \frac{e^{-\xi_{j,\theta}^2(D+1)/2}}{\sqrt{2\pi} \sigma_{2,j,\theta}^2 / \xi_{j,\theta}^2} \right) \cdot \exp \left\{ -\frac{1}{2} \bar{k}^T \mathbb{C}(\bar{\xi}) \bar{k} \right\} \]  

(2.12)

\[ = \frac{1}{\mathbb{D}^{\overset{T}{\bar{x}}} \sigma_D^{NT}} \int_{\mathbb{R}^D} \cdots \int_{\mathbb{R}^D} d\xi_{j,\theta} \left( e^{-\xi_{j,\theta}^2(D+1)/2} \right) \exp \left\{ -\frac{1}{2} \bar{k}^T \mathbb{C}(\bar{\xi}) \bar{k} \right\} \]  

(2.13)

\[ = \left\langle \exp \left\{ -\frac{1}{2} \bar{k}^T \mathbb{C}(\bar{\xi}) \bar{k} \right\} \right\rangle_{\bar{\xi}} \]  

(2.14)

The weight \( \omega(\bar{\xi}) \) used in the average \( \left\langle \cdot \right\rangle_{\bar{\xi}} \) in (2.14) reads:

\[ \omega(\bar{\xi}) = (\mathcal{N})^{NT} \prod_{j=1,\ldots,N} \prod_{\theta=1,\ldots,T} \xi_{j,\theta}^{D-1} \exp \left\{ -\frac{\xi_{j,\theta}^2(D+1)}{2} \right\} \]  

(2.15)

where \( \mathcal{N} := (D+1)^{D/2} / (2^{D/2-1} \Gamma(D/2)) \).

In (2.15) we expressed the integral over \( \xi_{j,\theta} \) in radial coordinates \( \int_{\mathbb{R}^D} d^D \xi = 2(\pi)^{D/2} / (\Gamma(D/2)) \int_0^\infty \xi^{D-1} d\xi \).

Now we differentiate (2.14) \( 2k \) times with respect to variables \( k_{(j),\theta(1)}, \ldots, k_{(j),\theta(2k)} \) and evaluate the result at \( \bar{k} = 0 \). We get:
Theorem

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\[ \langle X_j(1), \theta(1) \ldots X_j(2k), \theta(2k) \rangle = \frac{1}{i^{2k}} \frac{\partial}{\partial k_j(1), \theta(1)} \ldots \frac{\partial}{\partial k_j(2k), \theta(2k)} \langle \kappa(\vec{k}) \rangle \mid_{\vec{k}=\vec{0}} \] (2.16)

\[ = \frac{1}{(-1)^k} \frac{\partial}{\partial k_j(1), \theta(1)} \ldots \frac{\partial}{\partial k_j(2k), \theta(2k)} \left( \frac{1}{2k!} \left( - \sum_{i,j=1}^N \sum_{\lambda, \theta} C_{i,t}^{j,\theta} (\vec{k}) \right)^k \right) \mid_{\vec{k}=\vec{0}} \] (2.17)

\[ = \sum_{\sigma = \epsilon^{(1)} \ldots \epsilon^{(k)}} \left\langle \prod_{q=1}^k C_j^{\epsilon^{(q)}}, \theta(\epsilon^{(q)}) (\vec{k}) \right\rangle \right|_{\vec{k} = \vec{0}} \] (2.18)

\[ = \sum_{\sigma = \epsilon^{(1)} \ldots \epsilon^{(k)}} \left( \prod_{q=1}^k O^{\sigma(q), \lambda(q)} (\epsilon^{(q)}, \theta^{(q)}) O^j(\epsilon^{(q)}, \lambda^{(q)}) \prod_{l=1}^N \prod_{m=1}^T (\mathcal{N} \int_0^\infty \xi^{D-2\nu(l,m)-1} e^{-\xi^2 D/2} d\xi) \right) \] (2.19)

\[ = \sum_{\sigma = \epsilon^{(1)} \ldots \epsilon^{(k)}} \left( \prod_{q=1}^k O^{\sigma(q), \lambda(q)} (\epsilon^{(q)}, \theta^{(q)}) O^j(\epsilon^{(q)}, \lambda^{(q)}) \prod_{l=1}^N \prod_{m=1}^T \left( \frac{(D+1)}{2} \frac{\nu(l,m)}{\Gamma(D/2)} \right) \right) \] (2.20)

where the sum in (2.18) runs over all 2k-permutations σ that are composed exclusively of cycles \( \epsilon^{(i)} := (\epsilon_1^{(i)}, \epsilon_2^{(i)}) \) for \( i = 1, \ldots, k \) of lengths two. Since the functions \( C_{i,t}^{j,\theta} \) are symmetric with respect to exchanging the above and lower pairs and multiplication is commutative the whole of \( (2k)! \) terms in the sum in (2.17) decomposes into \( (2k)!/(2^k k!) = (2k-1)!! \) distinct terms who occur \( (2k)! \) times each. In this way the factor \( 2^{kk} \)! in the denominator in (2.17) cancels out. In (2.19) we used the definition (2.9) of the correlation tensor and we introduced new indices \( p(q) = 1, \ldots, N \) and \( \lambda(q) = 1, \ldots, T \) Since the average over \( \xi \) in (2.18) consists in performing integrals of the kind \( \mathcal{N} \int_{\mathbb{R}^d} \prod_{q=1}^k \xi^{2-l} \rho(p(q), \lambda(q)) \omega(\xi) d\xi \) it is readily seen that the result (2.19) is expressed via a number \( \nu(l, m) \) that depends on the sequence \( \{ p(q), \lambda(q) \} \) and that is equal to the multiplicity of the pair \( (l, m) \) in that sequence:

\[ \nu(l, m) = \# \{ \text{pairs } (p_q, \lambda_q) \text{ such that } p_q = l \text{ and } \lambda_q = m \} \] (2.21)

3. Covariances their estimators and noise filtering

In this section we discuss definitions of averages over ensembles of stochastic variables (resolvents) whose properties may be compared to measured properties of financial time series.

Definition 3.1. The resolvent function is a complex function \( G(z) \) such that:

\[ \text{Im}[G(z + i\epsilon)] = \delta_\epsilon(z), \quad \epsilon > 0 \] (3.1)

where \( \delta_\epsilon(z) \) is a representation of the delta function.
The function $G(z)$ is used for finding the density of eigenvalues $\rho_{\Lambda}(\lambda)$ of the covariance $\langle \epsilon(X)_{i,j} \rangle$ where

$$c(X)_{i,j} := \frac{1}{T} \sum_{t=1}^{T} X_{i,t} X_{j,t}$$

and the average is over the random ensemble $X_{i,t}$. We have:

$$\rho_{\Lambda}(\lambda) = \lim_{\epsilon \to 0} N \sum_{i=1}^{N} \delta_\epsilon(\lambda - \lambda_i) = \lim_{\epsilon \to 0} \sum_{i=1}^{N} \text{Im}[G(\lambda - \lambda_i + i\epsilon)]$$

$$= \lim_{\epsilon \to 0} \text{Tr} \left[ \text{Im} \left[ G(\lambda - \epsilon + i\epsilon) \right] \right]$$

**Lemma 3.1.** The resolvent function $G(z)$ has $\text{res}_{z=1} G(z) = 1/2\pi$. Here res denotes a residue.

**Proof.** The function $G(z)$ can be expanded in a Laurent series around $z = 0$. This means that

$$\exists k \geq 0 \quad G(z) = \sum_{j=1}^{k} \frac{a_j}{z^j} + G(z)$$

where the function $G(z)$ is analytic.

We need to prove that $\text{Im} \left[ \int_{-\infty}^{\infty} G(x + i\epsilon) \right] = 1$ for $\epsilon > 0$. We consider at first the non-analytic term in (3.5).

$$\text{Im} \left[ \int_{-\infty}^{\infty} \frac{1}{(x+i\epsilon)^j} dx \right] = \begin{cases} 1 & j = 1 \\ 0 & j > 1 \end{cases}$$

where the above equality in (3.6) is straightforward and the lower equality is derived by means of the Cauchy integral theorem. Now we analyse the analytic term. For $R > 0$ we compute

$$\int_{-R}^{R} G(z + i\epsilon) dz = \int_{-R+\epsilon}^{R+\epsilon} G(z) dz = \int_{-R}^{R} G(z) dz + \int_{0}^{\epsilon} [G(R+i\epsilon) - G(-R+i\epsilon)] d\xi$$

where the last equality in (3.7) follows from the application of the Cauchy integral theorem to a contour consisting of four intervals $[-R, R], [R, R+i\epsilon], [R+i\epsilon, -R+\epsilon]$ and $[-R+\epsilon, -R]$. From the last expression on the right hand side in (3.7) we see that the imaginary part of that integral is $O(\epsilon)$ and hence disappears when $\epsilon \to 0$. This fact and (3.6) suffices to finish the proof.

The class of functions $G(z)$ becomes narrowed down subject to the following condition:
Lemma 3.2. The resolvent function \( G(z) = 1/z \), if and only if

\[
\forall n \in \mathbb{N} \frac{G^{(n)}(z)}{n!}(-1)^n = (G(z))^{n+1}
\]

(3.8)

The necessity follows in a straightforward manner from substituting \( G(z) = 1/z \) into (3.8). To prove the sufficiency we take \( x \in \mathbb{R} \), multiply both sides of (3.8) by \( x^n \) and sum over \( n = 0, 1, \ldots, \infty \). Since the left hand side is the Taylor expansion of \( G(z - x) \) around \( z \) and the right hand side form a geometric series we obtain a functional equation:

\[
G(z - x) = \frac{G(z)}{1 - xG(z)} = \frac{1}{\alpha} \frac{1}{G(\alpha z)} - x
\]

(3.9)

We substitute \( x = (1 - \alpha)z \) into (3.9) for some \( \alpha \in \mathbb{R}_+ \) and get:

\[
\frac{1}{G(\alpha z)} = \frac{1}{G(z)} + (\alpha - 1)z \quad \text{(3.10)}
\]

\[
= (\alpha - 1) \left( z + \frac{z}{\alpha} + \frac{z}{\alpha^2} + \ldots \right) + \frac{1}{G(0)} \quad \text{(3.11)}
\]

\[
= (\alpha - 1)z \left( \frac{1}{1 - \frac{1}{\alpha}} \right) = \alpha z \quad \text{(3.12)}
\]

where in (3.11) we have iterated the equation (3.10) and we used the fact that \( 1/G(0) = 0 \). Finally in (3.12) we summed a geometric series and completed the proof.

4. Computation of the expansion of the resolvent

We calculate a function

\[
g(z) = \left\langle G \left( z \cdot 1 - \frac{1}{T} \bar{X} \cdot \bar{X}^T \right) \right\rangle
\]

(4.1)

where the average is performed over random variables \( \langle X \rangle_{i,\theta} \sim q - \text{Exp} \) that are correlated in \( i \) and in time \( \theta \) and \( G(z) \) is a resolvent function. The function \( g(z) \) is termed the resolvent. For the purpose of the calculation we fix \( \bar{\xi}_{i,t} \in \mathbb{R}^D \), we project the joint pdf \( \rho_{\bar{X}}(\bar{x}) \) onto Gaussians Normal(0, \( \sigma_{D,i}^2 / \bar{\xi}_{i,t}^2 \)) using the integral representation (2.2), average over the Gaussians and obtain

\[
\left\langle X_{i,\theta} X_{i',\theta'}^T \right\rangle = C_{i,\theta}^{i',\theta'}(\bar{\xi}) = \sum_{p=1,\ldots,N}^{\lambda=1,\ldots,T} O_{i,\theta}^{p,\lambda} \frac{\sigma_{p}^2}{\bar{\xi}_{p,\lambda}} O_{i',\theta'}^{p,\lambda}
\]

(4.2)

Note that the two-point correlation function (4.2) does not factorize into functions depending on times \( \theta, \theta' \) and on \( i, j \) only. It factorises for \( |\bar{\xi}_{i,t}| = 1 \) if the rotation tensor \( O_{i,j}^{p,\lambda} \) factorises, ie \( O_{i,j}^{p,\lambda} = I_i^p T_{j}^\lambda \). For generic value of \( \bar{\xi} \), however, correlations in \( i \) and in time are coupled with each other. In the following we sum over repeated indices (use Einstein’s summation convention). The N-type indices and the T-type
indices (running from one to \( N \) and to \( T \)) are denoted by Latin and by Greek letters respectively. We expand the resolvent function \( G() \) in a series around \( z \). We get:

\[
g(z)_{i,j} = \sum_{n=0}^{\infty} \frac{G^{(n)}(z)}{n!} (-1)^n \left\langle \left( \mathbf{X} \cdot \mathbf{X}^T \right)^n \right\rangle_{i,j} \tag{4.3}
\]

\[
= \sum_{n=0}^{\infty} \frac{G^{(n)}(z)}{T^n n!} (-1)^n \delta_{i,i(1)} \left\langle \left( \mathbf{X} \cdot \mathbf{X}^T \right)^n \right\rangle_{i(1),i(2n+1)} \delta_{i(2n+1),j} \tag{4.4}
\]

\[
= \sum_{n=0}^{\infty} \frac{G^{(n)}(z)}{T^n n!} (-1)^n \delta_{i,i(1)} \left\langle \prod_{j=1}^{n} \left( X_{i(2j-1),\xi(2j-1)} X_{\xi(2j-1),i(2j+1)}^T \right) \right\rangle_{i(2n+1),j} \delta_{i(2n+1),j} \tag{4.5}
\]

\[
= \sum_{n=0}^{\infty} \sum_{\sigma} \frac{1}{z} \delta_{i,i(1)} \left\langle \prod_{j=1}^{n} \left( C^{i(i_j)^{\phi_j}(\xi(\epsilon_j^{(j)})),i(\epsilon_j^{(j)})}_{i(\epsilon_j^{(j)}),\xi(\epsilon_j^{(j)})} \right) \prod_{j=1}^{n-1} \left( \frac{1}{z} \delta_{i(2j-1),i(2j+1)} \right) \prod_{j=1}^{n} \left( \frac{1}{z} \delta_{i(2j),i(2j+1)} \right) \right\rangle \frac{1}{z} \delta_{i(i_n),j} \tag{4.7}
\]

In (4.4) we introduced two new \( N \)-type indices \( i(1) \) and \( i(2n+1) \); in (4.5) we expanded the \( n \)-th power \( \left( \mathbf{X} \cdot \mathbf{X}^T \right)^n \) and we introduced indices \( i(1), i(3), \ldots, i(2n-1) \) together with \( \xi(1), \ldots, \xi(2n-1) \). In (4.6) we inserted \( n \) \( T \)-type Kronecker delta functions between the ordered \( X, X^T \) pairs and \( n-1 \) \( N \)-type delta functions between the pairs \( X^T, X \). This resulted in introducing indices \( \xi(2), \ldots, \xi(2n) \) and indices \( i(2), \ldots, i(2n) \).

Finally in (4.7) we made use of the Wick theorem (2.18) (the sum over \( \sigma \) runs over all \( 2n \)-permutations composed entirely of cycles \( \epsilon^{(j)} \) of length two), of Lemmas 3.1 and 3.2 and we eliminated the index \( i(2n+1) \). Note that the sum over \( \sigma \) contains \((2n-1)!!\) terms each containing \( 4n \) variable indices \( i(1), \ldots, i(2n) \) and \( \xi(1), \ldots, \xi(2n) \) and two fixed indices \( i \) and \( j \).

Now we construct a pictorial representation (Feynmann diagrams) [2,3,4,7] of terms in the sum (4.7) according to a following recipe. The \( i(...) \) indices are denoted by bullets \( \bullet \), and the \( \xi(...) \) indices are denoted by open circles \( \circ \). The factors in the product on the right hand side in (4.7) are assigned to graphs as follows:
In this way the sum (4.7) can be represented as a sum over graphs that are constructed from building blocks (4) in such a way that for every node, except for the nodes \(i\) and \(j\) (external nodes), there are exactly two edges abutting at it (this follows from the Einstein’s summation convention). All graphs contributing to the second and to the third order of the expansion are listed, together with their weights, in Figs. 3 and 4.

Each graph consists of a number of closed solid, a number of closed dashed loops and of a solid line that starts at index \(i\) and ends at \(j\). Since a closed loop corresponds to a contraction (setting two indices of the tensor equal and summing over them) \(A_{i(1),\theta(1)}^{i(2),\theta(2)}\) of a tensor \(A_{i(1),\theta(1)}^{i(2),\theta(2)}\) that is constructed by multiplying the weights of the edges of the graph, the weight of the closed loop is proportional to \(N\) and to \(T\) for solid and a dashed closed loops respectively. In other words the weight of a closed solid (dashed) loop is equal to a trace of a \(N\times N\) or \((T\times T)\) matrix and thus proportional to \(N\) or \((T)\). In the following we assume that \(N/T = r\) is fixed and investigate the expansion (4.7) in the limit \(N \to \infty\). Only such graphs contribute to the expansion whose number of loops (either dashed or solid) is equal to the order of the expansion \(n\) (planar graphs). Graphs that consist of intersecting lines, like the second graph from the top in Fig.3 are negligible in the limit \(N \to \infty\) since their weight is inverse proportional to a certain power of \(N\).

We define one-line irreducible graphs as graphs that cannot be split into two distinct graphs by cutting a certain edge. The usefulness of this definition follows from the fact that the weight of a graph (the term in the sum (4.7) for fixed \(n\) and fixed \(\sigma\)) factorizes into a product of weights corresponding to one-line irreducible components. Note that this would not be the case if the weight depended explicitly on \(n\) (e.g. the function \(G(z)\) satisfied Lemma 3.1 but not Lemma 3.2) or if it depended on some function of indices \(i(\ldots)\) or \(\xi(\ldots)\). Denoting by \(\Sigma = \{\Sigma_{i,j}\}_{i,j}\) the sum of weights of all one-line irreducible graphs (self-energy) with external nodes \(i\) and \(j\) we realize from (4.7) and from Fig.1 that the resolvent is a sum of a geometric series in self-energy:
\[ g(z) = \frac{1}{z} + \frac{1}{z^2} \Sigma + \frac{1}{z^3} \Sigma^2 + \ldots = (z1 - \Sigma)^{-1} \] (4.8)

This result is quite general, i.e., it holds also if non-planar graphs are taken into account. However, the self-energy can only be calculated in the case when non-planar graphs are neglected as we see in Fig. 2 and in equation (4.9).

\[ \Sigma_i^j = \frac{1}{T} C^{j,\theta}_i,\theta \cdot \frac{1}{T^2} C^{j,\theta_2}_i,\theta_1 C^{q,\theta_2}_p,\theta_1 g(z)_{p,q} + \frac{1}{T^3} C^{j,\theta_2}_i,\theta_1 C^{p_2,\theta_2}_i,\theta_1 C^{p_4,\theta_2}_i,\theta_1 g(z)_{p_3,p_4} + \ldots \]

\[ = C^{j,\theta_2}_i,\theta_1 ((T1 - \mathcal{B})^{-1})^{\theta_2}_{\theta_1} \] (4.9)

where \( \mathcal{B}_{\theta_1}^{\theta_2} := C^{q,\theta_2}_p,\theta_1 g(z)_{p,q} \). Denoting \( i = i(0) \) and \( j = i(n) \) we write the result in a compact way:

\[ g(z)_{i,j} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \prod_{q=0}^{n-1} \Sigma_{i(q+1)}^{i(q)} \] (4.10)

\[ \Sigma_i^j = \sum_{m=0}^{\infty} \frac{C^{j,\theta(m+1)}_{i,\theta(1)}}{T^{m+1}} \prod_{q=1}^{m} C^{j(2q),\theta(q+1)}_{j(2q-1),\theta(q)} (\xi) g(z)_{j(2q-1),j(2q)} \] (4.11)

and we make following comments:

1. The results (4.10) and (4.11) are to be understood as follows. We fix \( \xi = (\xi_{1,1}, \ldots, \xi_{N,T}) \in \mathbb{R}^{NT-D} \) and for a given rotation tensor \( O \) and \( \sigma_D \) we compute the correlation tensor \( C(\xi) \) from (2.9), we insert the self-energy from equation (4.11) into (4.10) and we iterate the result until convergence, in the expansion in \( 1/z \) to a given order, is obtained. Then we average the result over \( \xi \in \mathbb{R}^{NT-D} \) according to the rule (2.15). Since the weight in (2.15) factorises and the resolvent as a function of \( \xi \) is a polynomial of inverse powers of \( \xi_{i,t} \) the weighting can be done analytically.

2. The correlation tensor \( C(\xi) \) is not a tensor product of \( i \)- and \( t \)-dependent matrices (does not factorize) (as noted in the paragraph under equation (4.2)) even if the rotation tensor \( O \) factorises.

3. The result (4.11) is only valid in the limit \( N \to \infty \). For finite values of \( N \) there will be corrections proportional to inverse powers of \( N \), corrections resulting from non-planar graphs (see Figs. 3 and 4).
5. The direct and inverse problems and the moment expansion of the resolvent

The purpose of this section is to make connection between quantities that are measured from financial time series, namely estimators of correlations $c := c(X)_{i,j}$ (definition (3.2)), and between the underlying rotation tensor $O_{\lambda,\theta}^{p,\lambda}$ and the variance $\sigma_D$ of the q-Exponential distribution. We define spectral moments $m_n$ of the estimator of correlations $c$ as traces of powers of the estimator:

$$m_n := \frac{1}{N} \text{Tr}[c^n]$$  \hspace{1cm} (5.1)

for $n = 0, 1, 2, \ldots$. Having done that we readily see from the definition of the resolvent (4.1) that the trace of the resolvent $t(z) := \frac{1}{N} \text{Tr}[g(z)]$ is a generating function of the spectral moments $t(z) = \sum_{n=0}^{\infty} m_n / z^{n+1}$.

Following the terminology from [8] we define two problems the direct and the indirect one. The direct problem consists in computing the spectral moments from the rotation tensor $O_{\lambda,\theta}^{p,\lambda}$ and the variance $\sigma_D$. The indirect problem, which is more interesting from the point of view of applications in quantitative finance, is defined as determining the correlation tensor and the variance from the sole knowledge of the spectral moments. To what extent it is possible, what additional assumptions about the structure of the tensor have to be made before deriving a numerical algorithm and what is the error estimate in the algorithm will be discussed in future work.

5.1. Finding the resolvent

We solve equations (4.10) and (4.11) for $g(z)_{i,j}$ according to the recipe in point (1) at the end of section 4 and obtain following results:
Theorem

The resolvent $g^{(n)}(z)_{i,j}$

| Order $n$ | The resolvent $g^{(n)}(z)_{i,j}$ |
|-----------|----------------------------------|
| 0         | $\delta_{i,j}/z$                | (5.2) |
| 1         | $\frac{1}{z} \delta_{i,j} + \frac{1}{z^2} \left( \frac{1}{T} C_{i,j}^{i,j} \right)$ | (5.3) |
| 2         | $\frac{1}{z} \delta_{i,j} + \frac{1}{z^2} \left( C_{i,j}^{i,j} + \frac{1}{z} \frac{1}{T} \left( C_{i,j}^{j,i} + C_{k,l}^{k,l} C_{i,j}^{k,l} C_{i,j}^{i,j} \right) \right)$ | (5.5) |
| 3         | $\frac{1}{z} \delta_{i,j} + \frac{1}{z^2} \left( C_{i,j}^{i,j} + \frac{1}{z} \frac{1}{T} \left( C_{i,j}^{j,i} + C_{k,l}^{k,l} C_{i,j}^{k,l} C_{i,j}^{i,j} \right) \right)$ | (5.6) |

Comparing the coefficients of the expansion (5.6) in powers of $1/z$ with the spectral moments $m_n$ of the estimator of the correlations we obtain a set of non-linear equations that relate certain contractions of the correlation tensor to the spectral moments. If we assumed that the correlation tensor factorized, which is not the case as we discussed in point (1) in section 4, then we would have obtained equations (34) from [8], equations that relate the spectral moments to moments of the underlying correlation matrices both in $i$ and in time. In our case the relations are averaged over $\vec{\xi}$ and will be related to some contractions of the rotation tensor $\Omega$.

Before proceeding further we note that the relations solve the direct problem but they do not provide enough information to solve the indirect problem.

6. Conclusions

We have derived a variant of the Wick theorem that expresses the many-point correlation function of $q$-Exponential distributed random variables through two-point correlation functions. This theorem will be used for solving the indirect problem in quantitative finance, ie for determining the correlations of time series from the knowledge of the spectral moments of the estimator of covariance.

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Fig. 3. All diagrams of the second order that contribute to the expansion of the resolvent and their weights.

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Fig. 4. The same as in Fig. 3 but for diagrams of third order.