Supplementary Material

Consistent testing for pairwise dependence in time series

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1 Some further remarks on applications

Choice of Kernel: The kernel functions chosen for the simulations are the following:

- The Daniell kernel (DAN), given by

\[ k(z) = \frac{\sin(\pi z)}{\pi z}, \quad z \in \mathbb{R}, \]

- The Parzen kernel (PAR), given by

\[
k(z) = \begin{cases} 
1 - 6(\pi z/6)^2 + 6 |\pi z/6|^3, & |z| \leq 3/\pi, \\
2(1 - |\pi z/6|)^3, & 3/\pi \leq |z| \leq 6/\pi, \\
0, & \text{otherwise}; 
\end{cases}
\]

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Table 1: Computational time (in seconds) for computing the test statistics $T_n$ and $M_n^{(2)}$.

| n:  | 100 | 200 | 500 |
|-----|-----|-----|-----|
| p:  | 3   | 7   | 16  |
|     | 3   | 9   | 25  |
|     | 4   | 13  | 42  |

$T_n$
- **BAR**: 0.13 0.14 0.18 1.20 1.12 1.12 18.39 18.56 17.94
- **PAR**: 0.13 0.12 0.11 0.83 0.79 0.78 18.19 17.97 18.69
- **DAN**: 0.11 0.12 0.11 0.80 0.80 0.78 18.93 18.48 18.47

$M_n^{(2)}$
- **BAR**: 1.20 1.21 1.20 4.79 4.74 4.72 29.61 29.58 29.58
- **PAR**: 1.21 1.22 1.22 4.82 4.79 4.77 29.62 29.58 29.69
- **DAN**: 1.21 1.24 1.24 4.82 4.77 4.81 29.61 29.60 29.61

- The Bartlett kernel (BAR), given by
  \[
  k(z) = \begin{cases} 
  1 - |z|, & |z| \leq 1, \\
  0, & \text{otherwise}.
  \end{cases}
  \]

**Computational Speed:** Table 1 shows the computational time taken to obtain $T_n$ and $M_n^{(2)}$, for various choices of sample size $n$ and bandwidth $p$ on a standard laptop with Intel Core i5 system and CPU 2.30 GHz. Clearly, $M_n^{(2)}$ is computationally more expensive than $T_n$, especially when $n$ and $p$ are large.

**Power:** Recall the NMA(2) model

\[ X_t = \epsilon_t \epsilon_{t-1} \epsilon_{t-2}, \quad (1) \]
where \( \{\epsilon_t\} \) is a sequence of i.i.d standard normal random variables. Figure 1 shows bootstrap p-values for various values of the bandwidth parameter and sample sizes. We note again that the performance of both test statistics, \( T_n \) and \( M_n^{(2)} \), is superior to that of the other test statistics.

Figure 1: Bootstrap p-values of all test statistics as a function of the bandwidth \( p \). The data are generated by the NMA(2) model given by (1). The results are based on \( B = 499 \) bootstrap replications and 100 simulations. The test statistics \( T_n, M_n^{(1)}, M_n^{(2)} \) are calculated by employing the Daniell kernel.

**Plot of ACF and ADCF of S&P 500 index:** Figure 2 shows the ACF (upper plot) and the ADCF (lower plot) of the original series and the squared original series.
Figure 2: Upper plots: The ACF of the original series and the squared original series. Lower plots: The ADCF of the original and the squared original series.
2 Proofs

In this section, we prove the main theorems of the paper. For following the presentation it is more convenient to recall the main assumptions.

**Assumption 1** \( \{X_t\} \) is a strictly stationary \( \alpha \)-mixing process with mixing coefficients \( \alpha(j), j \geq 1 \).

**Assumption 2** \( E|X_t| < \infty \).

**Assumption 3** The mixing coefficients of \( \{X_t\}, \alpha(j) \), satisfy (i) \( \sum_{j=\infty}^{-\infty} \alpha(j) < \infty \), (ii) \( \alpha(j) = O(1/j^2) \).

**Assumption 4** Suppose that \( k(.) \) is a kernel function such that \( k: \mathbb{R} \to [-1, 1] \), is symmetric and is continuous at 0 and at all but a finite number of points, with \( k(0) = 1 \), \( \int_{-\infty}^{\infty} k^2(z)dz < \infty \) and \( |k(z)| \leq C \cdot |z|^{-b} \) for large \( z \) and \( b > 1/2 \).

We first prove two Lemmas which will be employed in the proof of the main results of the paper. In what follows \( C \), possibly with a subscript, denotes a generic constant. We define the pseudoestimator:

\[
\bar{f}_n(\omega, u, v) = \frac{1}{2\pi} \sum_{j=-(n-1)}^{(n-1)} k(j/p)(1 - |j|/n)^{1/2} \tilde{\sigma}_j(u, v)e^{-ij\omega},
\]

where

\[
\tilde{\sigma}_j(u, v) = \frac{1}{n - |j|} \sum_{t=|j|+1}^{n} \psi_t(u)\psi_{t-|j|}(v)
\]

and

\[
\psi_t(u) \equiv e^{iuX_t} - \phi(u).
\]
Lemma 1 Suppose that \( \{X_t, t \geq 1\} \) satisfies Assumptions 1 and 3(ii). Then,
\[
(n - j)^2 E |\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)|^2 \leq C,
\]
and
\[
(n - |j|) E |\tilde{\sigma}_j(u, v)|^2 \leq C,
\]
uniformly in \((u, v) \in \mathbb{R}^2\). The result of the Lemma is also true under independence.

Proof of Lemma 1 Note that 
\[
\tilde{\sigma}_j(u, v) = \frac{1}{n - |j|} \sum_{t=|j|+1}^{n} e^{iuX_t + vX_{t-|j|}} - \frac{\phi(v)}{n - |j|} \sum_{t=|j|+1}^{n} e^{iuX_t} - \frac{\phi(u)}{n - |j|} \sum_{t=|j|+1}^{n} e^{ivX_{t-|j|}} + \phi(u)\phi(v).
\]
Therefore, by subtracting \(\tilde{\sigma}_j(u, v)\) from \(\hat{\sigma}_j(u, v)\) we get:
\[
\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v) = -\frac{1}{(n - |j|)^2} \sum_{t=|j|+1}^{n} \psi_t(u) \sum_{t=|j|+1}^{n} \psi_{t-|j|}(v).
\]
The Cauchy - Schwarz inequality implies that
\[
E |\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)|^2 \leq \frac{1}{(n - j)^4} \left\{ E \left| \sum_{t=|j|+1}^{n} \psi_t(u) \right|^4 E \left| \sum_{t=|j|+1}^{n} \psi_{t-|j|}(v) \right|^4 \right\}^{1/2} \leq \frac{C}{(n - j)^2}
\]
uniformly because \(E \left| \sum_{t=|j|+1}^{n} \psi_t(u) \right|^4 \leq C(n - j)^2\) (Doukhan and Louhichi, 1999, Lemma 6).

From the definition of \(\tilde{\sigma}_j(u, v)\) in (2), it follows that:
\[
E |\tilde{\sigma}_j(u, v)|^2 \leq E \left| \frac{1}{n - |j|} \sum_{t=|j|+1}^{n} \psi_t(u)\psi_{t-|j|}(v) \right|^2 \leq \frac{C}{n - |j|}
\]
uniformly. \(\square\)
Lemma 2 Suppose that \( \{X_t, t \geq 1\} \) satisfies Assumptions 1 and 3(ii). For each \( \gamma > 0 \), denote by \( D(\gamma) \) the region \( D(\gamma) = \{(u, v) : \gamma \leq |u| \leq 1/\gamma, \gamma \leq |v| \leq 1/\gamma\} \). Then, under Assumption 4, for any fixed \( \gamma > 0 \),

\[
\int_{D(\gamma)} \sum_{j=1}^{n-1} k^2(j/p)(n-j) \left\{ |\hat{\sigma}_j(u, v)|^2 - |\tilde{\sigma}_j(u, v)|^2 \right\} dW = O_P(p/\sqrt{n}) = o_P(\sqrt{p})
\]
as \( p/n \to 0 \). The result of the Lemma is also true under independence.

Proof of Lemma 2 From the properties of kernels we obtain that

\[
\sum_{|j|<n} \frac{1}{n-|j|} k^2(j/p) = O(p/n)
\]
for bandwidth \( p = cn^\lambda, \lambda \in (0, 1) \). In addition,

\[
\sum_{j=1}^{n-1} \frac{1}{n-j} k^2(j/p) = O(p/n) \quad (4)
\]

and

\[
\sum_{j=1}^{n-1} \frac{1}{\sqrt{n-j}} k^2(j/p) = O(p/\sqrt{n}) \quad (5)
\]

Now, define

\[
Z_{n;p} = \int_{D(\gamma)} \sum_{j=1}^{n-1} k^2(j/p)(n-j) \left\{ |\hat{\sigma}_j(u, v)|^2 - |\tilde{\sigma}_j(u, v)|^2 \right\} dW,
\]
and observer that, for any fixed \( \gamma > 0 \), the chosen weight function \( W(u, v) \) is bounded on \( D(\gamma) \). Thus, \( W(u, v) \) is an integrable function in this region.

Using

\[
|\hat{\sigma}_j(u, v)|^2 - |\tilde{\sigma}_j(u, v)|^2 = |\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)|^2 + 2Re\{|\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)| \bar{\sigma}_j(u, v)\},
\]

where * denotes complex conjugate, we get the following:

\[
E |Z_{n,p}^\gamma| \leq \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j)E\left( |\hat{\sigma}_j(u,v)|^2 - |\tilde{\sigma}_j(u,v)|^2 \right) \right\} dW
\]

\[
= \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j)^2 E\left( |\hat{\sigma}_j(u,v) - \tilde{\sigma}_j(u,v)|^2 \right) \right\} dW
\]

\[
+ 2 \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j)\text{Re}\{E|\hat{\sigma}_j(u,v) - \tilde{\sigma}_j(u,v)||\tilde{\sigma}_j(u,v)^*|\} \right\} dW.
\]

Now Lemma 1, the Cauchy - Schwarz inequality and the fact that \( \int_{D(\gamma)} dW < \infty \) show that,

\[
E |Z_{n,p}^\gamma| \leq \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j)^2 E\left( |\hat{\sigma}_j(u,v) - \tilde{\sigma}_j(u,v)|^2 \right) \right\} dW
\]

\[
+ 2 \int_{D(\gamma)} \sum_{j=1}^{n-1} \frac{k^2(j/p)}{\sqrt{n-j}} \sqrt{(n-j)^2 E|\hat{\sigma}_j(u,v) - \tilde{\sigma}_j(u,v)|^2 \sqrt{(n-j)E|\tilde{\sigma}_j(u,v)|^2}} dW
\]

\[
\leq C_1 \int_{D(\gamma)} dW \sum_{j=1}^{n-1} \frac{k^2(j/p)}{n-j} + C_2 \int_{D(\gamma)} dW \sum_{j=1}^{n-1} \frac{k^2(j/p)}{\sqrt{n-j}}
\]

\[
= O\left( \frac{p}{\sqrt{n}} \right)
\]

as \( p/n \to 0 \). By Markov’s inequality this implies the first result. The second result follows immediately.

**Proof of Proposition 1** Recall the region \( D(\gamma) \) defined in Lemma 2, and for each \( \gamma > 0 \), define the random variables

\[
\hat{V}_{X,\gamma}^2(j) = \int_{D(\gamma)} |\hat{\sigma}_j(u,v)|^2 dW.
\]
Because of the SLLN for $\alpha$-mixing random variables and the fact that $\int_{D(\gamma)} dW < \infty$, we obtain

$$
\lim_{n \to \infty} \hat{V}_{X,\gamma}^2(j) = V_{X,\gamma}^2(j) = \int_{D(\gamma)} |\sigma_j(u, v)|^2 dW,
$$

almost surely. Clearly, $V_{X,\gamma}^2(j) \to V_X^2(j)$ as $\gamma$ tends to zero. So, it remains to prove that almost surely

$$
\limsup_{\gamma \to 0} \limsup_{n \to \infty} \left| \hat{V}_{X,\gamma}^2(j) - \hat{V}_X^2(j) \right| = 0. \quad (7)
$$

For each $\gamma > 0$, we obtain that

$$
\left| \hat{V}_{X,\gamma}^2(j) - \hat{V}_X^2(j) \right| \leq \int_{|u| < \gamma} |\hat{\sigma}_j(u, v)|^2 dW + \int_{|u| > 1/\gamma} |\hat{\sigma}_j(u, v)|^2 dW
+ \int_{|v| < \gamma} |\hat{\sigma}_j(u, v)|^2 dW + \int_{|v| > 1/\gamma} |\hat{\sigma}_j(u, v)|^2 dW. \quad (8)
$$

Now, for $z \in \mathbb{R}$, define $H(y) = \int_{|z| < y} (1 - \cos z)/|z|^2 dz$. This is bounded by $1/\pi^2$ and $\lim_{y \to 0} H(y) = 0$. Because of $|x + y|^2 \leq 2|x|^2 + 2|y|^2$ and the Cauchy-Schwarz inequality

$$
|\hat{\sigma}_j(u, v)|^2 \leq 4 \left\{ \frac{1}{(n - |j|)} \sum_{t = |j| + 1}^n |\psi_t(u)|^2 \right\} \left\{ \frac{1}{(n - |j|)} \sum_{t = |j| + 1}^n |\psi_{t - |j|}(v)|^2 \right\}. \quad (9)
$$

But, the first summand in (8) satisfies:

$$
\int_{|u| < \gamma} |\hat{\sigma}_j(u, v)|^2 dW \leq \left\{ \frac{4}{(n - |j|)} \sum_{t = |j| + 1}^n \int_{|u| < \gamma} |\psi_t(u)|^2 du \right\}
\times \left\{ \frac{1}{(n - |j|)} \sum_{t = |j| + 1}^n \int_{\mathbb{R}} \frac{|\psi_{t - |j|}(v)|^2}{\pi |v|^2} dv \right\}. \quad (10)
$$

However, (3) yields

$$
|\psi_{t - |j|}(v)|^2 = 1 + |\phi(v)|^2 - \phi(v)^* e^{i\psi_{X_t - |j|}} - \phi(v) (e^{i\psi_{X_t - |j|}})^*.
$$
and similarly for $|\psi_t(u)|^2$. Now, letting $X_{t-[j]} \equiv Y_t$ and using Székely et al. (2007, Lemma 1) we get

$$
\int \frac{|\psi_{t-[j]}(v)|^2}{\pi |v|^2} dv = 2E_Y |Y_t - Y| - E |Y - Y'| \leq 2(|Y_t| + E|Y|) = 2(|X_{t-[j]}| + E|X_1|),
$$

where the expectation $E_Y$ is taken with respect to $Y$ and we denote by $Y'$ the random variable which is a copy of $Y$ and independent of $Y_t$. Similarly

$$
\int \frac{|\psi_t(u)|^2}{\pi |u|^2} \leq 2E_X |X_t - X| H(|X_t - X| \gamma)
$$

where the expectation $E_X$ is taken with respect to $X$. Therefore, from (10)

$$
\int |\hat{\sigma}_j(u,v)|^2 dW \leq 16 \frac{1}{(n - |j|)} \sum_{t=|j|+1}^n (|X_{t-[j]}| + E|X_1|) \frac{1}{(n - |j|)} \sum_{t=|j|+1}^n E_X [|X_t - X| H(|X_t - X| \gamma)]
$$

Because of Assumptions 1 and 2 and the ergodic theorem for $\alpha$-mixing processes we obtain

$$
\frac{1}{n - |j|} \sum_{t=|j|+1}^n (|X_{t-[j]}| + E|X_1|) \to E|X_1| + E|X_1| = 2E|X_1|,
$$

$$
\frac{1}{n - |j|} \sum_{t=|j|+1}^n E_X |X_t - X| H(|X_t - X| \gamma) \to E|X_0 - X_1| H(|X_0 - X_1| \gamma),
$$

as $n \to \infty$, almost surely. Therefore,

$$
\lim sup_{n \to \infty} \int_{|u|<\gamma} |\hat{\sigma}_j(u,v)|^2 dW \leq 32E|X_1| E|X_0 - X_1| H(|X_0 - X_1| \gamma)
$$

and by Lebesgue’s dominated convergence theorem,

$$
\lim sup_{\gamma \to 0} \lim sup_{n \to \infty} \int_{|u|<\gamma} |\hat{\sigma}_j(u,v)|^2 dW = 0.
$$

For the second term of (8), (9) implies that $|\psi_t(u)|^2 \leq 4$ and $1/(n - |j|) \sum_{t=|j|+1}^n |\psi_t(u)|^2 \leq 4$. 

Therefore
\[
\int_{|u|>1/\gamma} |\hat{\sigma}_j(u,v)|^2 \, dW \leq 16 \int_{|u|>1/\gamma} \frac{du}{\pi |u|^2} \int_{\mathbb{R}} \frac{1}{n-|j|} \sum_{t=|j|+1}^{n} \frac{|\psi_{t-|j|}(v)|^2}{\pi |v|^2} \, dv
\]

\[
\leq 16\gamma \frac{2}{n-|j|} \sum_{t=|j|+1}^{n} (|X_{t-|j|}| + E|X_1|).
\]

Then, almost surely
\[
\lim_{\gamma \to 0} \lim_{n \to \infty} \int_{|u|>1/\gamma} |\hat{\sigma}_j(u,v)|^2 \, dW = 0.
\]

The other two summands of (8) can be dealt in a similar way to obtain (7). \qed

**Proof of Theorem 1** Arguing as in Hong (1999), we get the following result:

\[
\sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u,v)|^2 = \mathcal{C}(u,v) + \mathcal{V}(u,v),
\]

(11)

where

\[
\mathcal{C}(u,v) = \sum_{j=1}^{n-1} \frac{k^2(j/p)}{n-j} \left[ \sum_{t=|j|+1}^{n} C_{ttj}(u,v) \right],
\]

\[
\mathcal{V}(u,v) = \sum_{j=1}^{n-1} \frac{k^2(j/p)}{n-j} \left[ \sum_{t=|j|+2}^{n} \sum_{s=|j|+1}^{t-1} V_{tsj}(u,v) \right],
\]

with

\[
V_{tsj}(u,v) = C_{tsj}(u,v) + C_{stj}(u,v)^a
\]

and

\[
C_{tsj}(u,v) = \psi_t(u)\psi_s(u)^a\psi_{t-j}(v)\psi_{s-j}(v)^a.
\]
Consider the first term of (11). We observe that \( \int_{D(\gamma)} C_{ttj}(u,v) dW \) and \( \int_{D(\gamma)} C_{ssj}(u,v) dW \) are two independent integrals except from the cases where \( t = s \) and \( s = \pm j \). In addition

\[
E \int_{D(\gamma)} C_{ttj}(u,v) dW = C_0^2 \equiv \int_{D(\gamma)} \sigma_0(u,-u)\sigma_0(v,-v) dW < \infty.
\]

Therefore, \( E \left\{ \sum_{t=j+1}^{n} \left[ \int_{D(\gamma)} C_{ttj}(u,v) dW - C_0^2 \right]\right\}^2 \leq C(n-j), \) under independence. Hence, by Markov’s inequality, the Cauchy-Schwarz inequality and (4) we obtain that

\[
\int_{D(\gamma)} \hat{C}(u,v) dW - C_0^2 \sum_{j=1}^{n-1} k^2(j/p) = O_P(p/\sqrt{n}). \quad (12)
\]

Lemma 2 and Equations (11) and (12) yield that

\[
\int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u,v)|^2 \right\} dW = \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u,v)|^2 \right\} dW + O_P(p/\sqrt{n})
\]

\[
= \int_{D(\gamma)} \hat{\mathcal{C}}(u,v) dW + \int_{D(\gamma)} \hat{V}(u,v) dW + O_P(p/\sqrt{n})
\]

\[
= C_0^2 \sum_{j=1}^{n-1} k^2(j/p) + \hat{V}_n^\gamma + O_P(p/\sqrt{n}), \quad (13)
\]

where \( \hat{V}_n^\gamma \equiv \int_{D(\gamma)} \hat{V}(u,v) dW \).

Because of Assumption 4, we obtain by applying Hong (1999, Thm. A3) on \( D(\gamma) \) that

\[
\hat{V}_n^\gamma = \hat{V}_{ng}^\gamma + o_P(\sqrt{p}) \quad (14)
\]

where

\[
\hat{V}_{ng}^\gamma = \sum_{t=g+2}^{n} \sum_{s=1}^{t-g-1} \sum_{j=1}^{g} \frac{k^2(j/p)}{n-j} \int_{D(\gamma)} V_{tsj}(u,v) dW
\]

and \( g \equiv g(n) \) such that \( g/p \to 0, g/n \to 0 \). Now, by applying Hong (1999, Thm. A4) on \( D(\gamma) \) we get the following:

\[
\left[ pD_0^{\gamma} \int_0^\infty k^4(z) dz \right]^{-1/2} \hat{V}_{ng}^\gamma \to N(0,1) \quad (15)
\]
as \( n \to \infty \) in distribution, where

\[
D_0^\gamma = 2 \left[ \int_{D(\gamma)} |\sigma_0(u,u')|^2 dW \right]^2.
\]

Then, using (13), (14) and (15) we have the following:

\[
\int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n - j) |\hat{\sigma}_j(u,v)|^2 \right\} dW - C_0^\gamma \sum_{j=1}^{n-1} k^2(j/p)
\frac{\left[ pD_0^\gamma \int_0^{\infty} k^4(z)dz \right]^{1/2}}
\rightarrow N(0,1).
\]

as \( n \to \infty \) in distribution.

Observe that \( \hat{C}_0^\gamma - C_0^\gamma = O_P(1/\sqrt{n}) \) and that \( \sum_{j=1}^{n-1} k^2(j/p) = O(p) \). Hence \( C_0^\gamma \) can be replaced by \( \hat{C}_0^\gamma \) asymptotically given \( p/n \rightarrow 0 \). Furthermore, \( p^{-1} \sum_{j=1}^{n-2} k^4(j/p) \rightarrow \int_0^{\infty} k^4(z)dz \) and \( \hat{D}_0^\gamma \to D_0^\gamma \), in probability. We conclude that the factor \( pD_0^\gamma \int_0^{\infty} k^4(z)dz \) can be replaced by \( \hat{D}_0^\gamma \sum_{j=1}^{n-2} k^4(j/p) \), by Slutsky’s theorem. Thus (16) becomes

\[
\int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n - j) |\hat{\sigma}_j(u,v)|^2 \right\} dW - \hat{C}_0^\gamma \sum_{j=1}^{n-1} k^2(j/p)
\frac{\left[ \hat{D}_0^\gamma \sum_{j=1}^{n-2} k^4(j/p) \right]^{1/2}}
\rightarrow N(0,1).
\]

Write \( T_{n;\gamma} \) as the test-statistic \( T_n \) defined on \( D(\gamma) \) rather than on \( \mathbb{R}^2 \), i.e.

\[
T_{n;\gamma} = \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n - j) |\hat{\sigma}_j(u,v)|^2 \right\} dW.
\]

We note that

\[
T_n - T = T_n - T_{n;\gamma} + T_{n;\gamma} - T_{\gamma} + T_{\gamma} - T
\]
where $T_{\gamma}$ is defined as an asymptotically distributed normal random variable such that

$$T_{\gamma} - \frac{\hat{C}_0^\gamma}{\hat{D}_0^\gamma} \sum_{j=1}^{n-1} k^2(j/p) \left[ \frac{\hat{D}_0^\gamma}{\hat{D}_0^\gamma} \sum_{j=1}^{n-2} k^4(j/p) \right]^{1/2} \to N(0, 1),$$

in distribution, as $n \to \infty$ such that $p/n \to 0$. In addition, $T$ is an asymptotically normally distributed random variable with

$$T - \frac{\hat{C}_0}{\hat{D}_0} \sum_{j=1}^{n-1} k^2(j/p) \left[ \frac{\hat{D}_0}{\hat{D}_0} \sum_{j=1}^{n-2} k^4(j/p) \right]^{1/2} \to N(0, 1)$$

as $\gamma \to 0$. Obviously, (17) shows that

$$T_{n;\gamma} - T_{\gamma} = o_P(1),$$

as $n \to \infty$. But, it is also true that

$$T_{\gamma} - T = o_P(1),$$

as $\gamma \to 0$, because $\hat{C}_0^\gamma \to \hat{C}_0$ and $\hat{D}_0^\gamma \to \hat{D}_0$. If we show that

$$\limsup_{\gamma \to 0} \limsup_{n \to \infty} |T_n - T_{n;\gamma}| = 0,$$

almost surely, then the proof of Theorem 1 will be complete.
For each $\gamma > 0$:

$$\left| T_n - T_{n,\gamma} \right| = \int_{\mathbb{R}^2} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u,v)|^2 \right\} dW - \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u,v)|^2 \right\} dW$$

$$= \sum_{j=1}^{n-1} k^2(j/p)(n-j) \left| \int_{\mathbb{R}^2} |\hat{\sigma}_j(u,v)|^2 dW - \int_{D(\gamma)} |\hat{\sigma}_j(u,v)|^2 dW \right|$$

$$= \sum_{j=1}^{n-1} k^2(j/p)(n-j) \left| \hat{V}_{X,\gamma}^2(j) - \hat{V}_X^2(j) \right|$$

(19)

where $\hat{V}_{X,\gamma}^2(j)$ is defined as in (6). Now recall result (7) and combine (19) and (7) to finally get the required relation (18).

Proof of Theorem 2 We need to show the following: (i) $1/p \sum_{j=1}^n k^4(j/p) \to \int_0^\infty k^4(j/p)$. This follows from Assumption 4, $p \to \infty$ and $p/n \to 0$. In addition, we need that (ii) $E \int_{D(\gamma)} \int_{-\pi}^{\pi} \left| \hat{f}_n(\omega, u, v) - f(\omega, u, v) \right|^2 d\omega dW(u,v) \to 0$ where

$$\hat{f}_n(\omega, u, v) = \frac{1}{2\pi} \sum_{j=-(n-1)}^{(n-1)} (1 - |j|/n)^{1/2} k(j/p) \hat{\sigma}_j(u,v) e^{-ij\omega}.$$ 

This condition is established along the lines of Hong (1999, Thm. 2) on $D(\gamma)$ given Assumptions 1, 3(i) and 4. Furthermore, by applying Markov’s inequality we get (iii) $\hat{C}_0^\gamma = O_P(1)$ and (iv) $\hat{D}_0^\gamma \to D_0^\gamma$ in probability.

Combining (i) and (iv) and by using Slutsky’s theorem we get

$$\frac{1}{\sqrt{p}} \left[ \hat{D}_0 \sum_{j=1}^{n-2} k^4(j/p) \right]^{1/2} \to \left[ D_0^\gamma \int_0^\infty k^4(z) dz \right]^{1/2},$$

(20)

in probability. Next (ii) and (iii) and after some calculations we get

$$\frac{1}{n} \left[ T_{n,\gamma} - \hat{C}_0^\gamma \sum_{j=1}^{n-1} k^2(j/p) \right] \to \frac{\pi}{2} \int_{D(\gamma)} \int_{-\pi}^{\pi} \left| f(\omega, u, v) - f_0(\omega, u, v) \right|^2 d\omega dW(u,v)$$

(21)
in probability. Combining (20) and (21) we get the required result on $D(\gamma)$. However, considering $\hat{C}_0^\gamma \to \hat{C}_0$ and $\hat{D}_0^\gamma \to \hat{D}_0$ as $\gamma \to 0$ and (18), the required result is now proved on $\mathbb{R}^2$. □

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