ANALYTIC PROPERTIES OF MARKOV SEMIGROUP GENERATED BY STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY PROCESSES

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ABSTRACT. We consider the stochastic differential equations of the form
\[
\begin{cases}
    dX^x(t) = \sigma(X(t^-))dL(t) \\
    X^x(0) = x, \quad x \in \mathbb{R}^d,
\end{cases}
\]
where \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \) is Lipschitz continuous and \( L = \{ L(t) : t \geq 0 \} \) is a Lévy process. Under this condition on \( \sigma \) it is well known that the above problem has a unique solution \( X \). Let \((P_t)_{t \geq 0}\) be the Markovian semigroup associated to \( X \) defined by \((P_t f)(x) := \mathbb{E}[f(X^x(t))], t \geq 0, x \in \mathbb{R}^d, f \in B_0(\mathbb{R}^d)\). Let \( B \) be a pseudo-differential operator characterized by its symbol \( q \). Fix \( \rho \in \mathbb{R} \). In this article we investigate under which conditions on \( \sigma \), \( L \) and \( q \) there exist two constants \( \gamma > 0 \) and \( C > 0 \) such that
\[
|B P_t u|_{H^\rho_2} \leq C t^{-\gamma} |u|_{H^\rho_2}, \quad \forall u \in H^\rho_2(\mathbb{R}^d), t > 0.
\]

1. Introduction

The Blumenthal–Getoor index was first introduced in [5] in order to analyze the Hölder continuity of the sample paths, the \( r \)–variation, \( r \in (0,2] \) and the Hausdorff-dimension of the paths of Lévy processes. Straightforward calculations gives that the Blumenthal–Getoor index of an \( r \)–stable process is \( r \). Lévy processes with Blumenthal–Getoor index less than 1 (resp. greater that 1) have paths of finite variation (resp. infinite variation). The Brownian motion has finite 2-variation. By using Höh’s symbol, Schilling introduced in [18] a generalized Blumenthal–Getoor index which enabled him to characterize the Hölder continuity of the samples paths of a stochastic process. In [19], Schilling and Schnurr described the long term behaviour in terms of this generalized index, for more details see also [3]. In [7] Glau gives a classification of Lévy processes via their symbols. To be more precise, Glau defines the Sobolev index of a Lévy process by a certain growth condition on its symbol.

In the present paper we investigate analytic properties of the Markovian semigroup generated by a stochastic differential equation driven by a Lévy process. The main result in this article is Theorem 2.1 which is important,
for instance, in nonlinear filtering with Lévy noise where one has to analyze the Zakai equation with jumps. The leading operator of the Zakai equation is a pseudo–differential operator defined by the Hoh symbol of the driving noise in the state process. Thus, the uniqueness of the mild solution of the Zakai equation and its regularity depends very much on the estimate we obtain in Theorem 2.1.

Let \( L = \{ L(t) : t \geq 0 \} \) be a \( d \)-dimensional Lévy process. We consider the stochastic differential equations of the form

\[
\begin{align*}
\text{d}X^x(t) &= b(X^x(t)) \text{d}t + \sigma(X^x(t)) \text{d}L(t), \\
X^x(0) &= x, \quad x \in \mathbb{R}^d,
\end{align*}
\]

where \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \) are Lipschitz continuous. In this case, the existence and uniqueness of a solution to equation (1.1) is well established. Let \( (P_t)_{t \geq 0} \) be the \( X \) associated Markovian semigroup defined by

\[
(P_t f)(x) := \mathbb{E}[f(X^x(t))], \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad f \in \mathcal{B}_b(\mathbb{R}^d).
\]

Then, it is known that \( (P_t)_{t \geq 0} \) is a Feller semigroup and its infinitesimal generator is given by

\[
Au(x) = \int_{\mathbb{R}} e^{ix \cdot \xi} \psi(x, \xi) \hat{u}(\xi) \text{d}\xi, \quad u \in C_c^\infty(\mathbb{R}^d),
\]

where the symbol \( \psi \) is defined by

\[
\psi(x, \xi) := -\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E} \left[ e^{i(X^x(t)-x) \cdot \xi} - 1 \right], \quad x \in \mathbb{R}^d.
\]

In case \( L \) is a \( d \)-dimensional Brownian and \( \sigma \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d) \) is bounded from below and above, \( A \) is a second order partial differential operator on \( L^2(\mathbb{R}^d) \) with domain \( D(A) = H^2_0(\mathbb{R}^d) \). Moreover, \( (P_t)_{t \geq 0} \) is an analytic semigroup on \( L^2(\mathbb{R}^d) \) and the following inequality holds for \( B = \nabla \)

\[
|B P_t x|_{L^2} \leq \frac{1}{\sqrt{t}} |x|_{L^2}, \quad x \in L^2(\mathbb{R}^d), \quad t > 0.
\]

In this article we investigate under which conditions such an estimate holds, if \( L \) is a pure jump Lévy process and \( B \) a pseudo–differential operator induced by a symbol.

**Notation 1.1.** For any nonnegative integers \( \alpha_1, \alpha_2, \ldots, \alpha_d \) we set \( \alpha = (\alpha_1, \ldots, \alpha_d) \) and \( |\alpha| = \sum_{j=1}^d \alpha_j \). Moreover for a function \( f : \mathbb{R}^d \to \mathbb{C} \) we write \( \partial_\alpha^f f(x) \) for

\[
\frac{\partial^{\alpha}}{\partial x_1 \partial x_2 \cdots \partial x_d} f(x).
\]

For any \( \rho \geq 0 \) we define the function \( (\cdot)^\rho : \mathbb{R} \ni \xi \mapsto (\xi)^\rho := (1+|\xi|^2)^{\frac{\rho}{2}} \in \mathbb{R} \). The following inequality, also called the Peetre inequality is used in several places

\[
(x+y)^s \leq c_s (x)^s (y)^{|s|}, \quad x, y \in \mathbb{R}^d, \quad s \in \mathbb{R}.
\]
Let $X$ be a non empty set and $f, g : X \to [0, \infty)$. We set $f(x) \lessgtr g(x)$, $x \in X$, iff there exists a $C > 0$ such that $f(x) \leq C g(x)$ for all $x \in X$. Moreover, if $f$ and $g$ depend on a further variable $z \in Z$, the statement for all $z \in Z$, $f(x, z) \lessgtr g(x, z)$, $x \in X$ means that for every $z \in Z$ there exists a real number $C_z > 0$ such that $f(x, z) \leq C_z g(x, z)$ for every $x \in X$. Also we set $f(x) \asymp g(x)$, $x \in X$, iff $f(x) \lessgtr g(x)$ and $g(x) \lessgtr f(x)$ for all $x \in X$. Finally, we say $f(x) \gtrless g(x)$, $x \in X$, iff $g(x) \lessgtr f(x)$, $x \in X$. Similarly as above, we handle the case if the functions depend on a further variable.

Let $S(\mathbb{R}^d)$ be the Schwartz space of functions $C^\infty(\mathbb{R}^d)$ where all derivatives decreases faster than any power of $|x|$ as $|x|$ approaches to infinity. Let $S'(\mathbb{R}^d)$ be the dual of $S(\mathbb{R}^d)$. For any pair of functions $f, g \in S(\mathbb{R}^d)$, we define $(f, g)$ by

$$(f, g) := \int_{\mathbb{R}^d} f(x)g(x) \, dx.$$ 

The Fourier transform operator is denoted by $\mathcal{F}$ and its inverse is $\mathcal{F}^{-1}$. Let $1 \leq p < \infty$ and $s \in \mathbb{R}$, then $H^s_p(\mathbb{R}^d)$ denotes the Bessel Potential spaces or Sobolev spaces of fractional order defined by

$$H^s_p(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) : |f|_{H^s_p} = |\mathcal{F}^{-1}((1 + \xi^2)^{\frac{s}{2}}(\mathcal{F}f))|_{L^p} < \infty \right\}.$$ 

Let $(X, d)$ be a metric space. By $C(X)$ we denote the set of all complex valued continuous function $f : X \to \mathbb{C}$. Further, if $m = 1, 2, \ldots$ we define

$$C^{m}(X) = \{ f \in C(X) : \partial^m f \in C(X) \text{ for all } |\alpha| \leq m \}.$$ 

2. Hoh's Symbols associated to Lévy processes

Throughout the remaining article, let $L = \{ L^x(t) : t \geq 0, x \in \mathbb{R}^d \}$ be a family of $d$-dimensional Lévy processes $L^x$, where $L^x$ is a Lévy process starting at $x \in \mathbb{R}^d$. Then $L$ generates a Markovian semigroup $(T_t)_{t \geq 0}$ on $C_b(\mathbb{R}^d)$ by

$$T_t f(x) := \mathbb{E} f(L^x(t)), \quad f \in C_b(\mathbb{R}^d).$$ 

Let $A$ be the infinitesimal generator of $(T_t)_{t \geq 0}$ acting on $C^{(2)}_b(\mathbb{R}^d)$ defined by

$$(2.1) \quad A f := \lim_{h \to 0} \frac{1}{h} (T_h - T_0) f, \quad f \in C^{(2)}_b(\mathbb{R}^d).$$ 

An alternative way of defining $A$ makes use of the Lévy symbols. In particular, let $\psi : \mathbb{R}^d \to \mathbb{C}$ be defined by

$$\psi(\xi) = \frac{1}{t} \ln(\mathbb{E} e^{i\xi^T L(t)}), \quad \xi \in \mathbb{R}^d.$$ 

Then,

$$\psi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{i\xi^T z} - 1 \right) \nu(dz), \quad \xi \in \mathbb{R}^d,$$
and \( \psi \) is called the Lévy symbol of the Lévy process \( L \). It can be shown, that if \( L \) is Lévy process with symbol \( \psi \), then the infinitesimal generator defined by (2.1) can also be written as (see e.g. [2, 11]):

\[
Af(x) = -\int_{\mathbb{R}} e^{ix\xi} \psi(\xi) \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^d, \ f \in C^2_b(\mathbb{R}^d).
\]

(2.2)

The operator \( A \) is well defined in \( C^2_b(\mathbb{R}^d) \), has values in \( B^b(\mathbb{R}^d) \) and satisfies the positive maximum principle (see e.g. [11, Theorem 4.5.1 3]). Therefore, \( A \) generates a Feller semigroup on \( C^\infty_b(\mathbb{R}^d) \) and a sub–Markovian semigroup on \( L^2(\mathbb{R}^d) \) (see e.g. [12, Theorem 2.6.9 and Theorem 2.6.10]).

A further important property of a Lévy symbol is, that it is negative definite.

**Definition 2.1.** A function \( \psi : \mathbb{R}^d \rightarrow \mathbb{C} \) is called negative definite function iff for all \( m \in \mathbb{N} \) and all \( m \)-tuples \( (\xi_1, \ldots, \xi_m) \), \( \xi_j \in \mathbb{R}^d \), \( 1 \leq j \leq m \), the matrix

\[
(\psi(\xi_j) + \psi(\xi_j) - \psi(\xi_k - \xi_j))_{k,j=1,\ldots,m}
\]

is positive Hermitian, i.e. for all \( c_1, \ldots, c_m \in \mathbb{C} \)

\[
\sum_{j,k=1}^{m} (\psi(\xi_j) + \psi(\xi_j) - \psi(\xi_k - \xi_j))c_k \bar{c}_j \geq 0.
\]

If a negative definite function \( \psi \) is continuous, then

\[
\psi(\xi) \lesssim \langle |\xi|^2 \rangle, \quad \xi \in \mathbb{R}^d.
\]

Moreover, every continuous negative definite function \( \psi \) has a representation

\[
\psi(\xi) = c + q(\xi) + \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(y, \xi)) \nu(dy),
\]

where \( c \geq 0 \) is a constant, \( q \geq 0 \) is a quadratic form and \( \nu \) is a symmetric Borel measure on \( \mathbb{R}^d \setminus \{0\} \) called Lévy measure having the property that

\[
\int_{\mathbb{R}^d \setminus \{0\}} \frac{|y|^2}{1 + |y|^4} \nu(dy) < \infty.
\]

The Blumenthal–Getoor index of a Lévy process with Lévy measure \( \nu \) is defined by the

\[
\inf\{p : \int_{|x| \leq 1} |x|^p \nu(dx) < \infty\}.
\]

It has been generalized in several direction. Here, in this article we use also a generalization of the Blumenthal–Getoor index similarly to the characterization of pseudo–differential operators.

**Definition 2.2.** Let \( L \) be a Lévy process with symbol \( \psi \) and \( \psi \in C^k(\mathbb{R}^d \setminus \{0\}) \) for some \( k \in \mathbb{N}_0 \). Then the upper generalized Blumenthal–Getoor index \( s^+ \)
and lower index $s^-$ of $L$ up to order $k$ are defined by

$$s^+ := \inf_{|\alpha| \leq k} \left\{ \lambda : \limsup_{|\xi| \to \infty} \frac{\partial^\alpha \psi(\xi)}{|\xi|^{|\lambda|}} = 0 \right\}$$

and

$$s^- := \inf_{|\alpha| \leq k} \left\{ \lambda : \liminf_{|\xi| \to \infty} \frac{\partial^\alpha \psi(\xi)}{|\xi|^{|\lambda|}} = 0 \right\}.$$

Here $\alpha$ denotes a multi-index.

In order to define the resolvent of the symbol $\psi$ associated to an operator $A$, we need to know the range of the symbol of $\psi$.

**Definition 2.3.** Let $L$ be a Lévy process with symbol $\psi$. Let $R_g(\psi)$ be the essential range of $\psi$, i.e.

$$R_g(\psi) := \{ y \in \mathbb{C} \mid \text{Leb}(\{ s \in \mathbb{C} : |\psi(s) - y| < \varepsilon \}) > 0 \text{ for each } \varepsilon > 0 \}.$$

Finally, to characterize the spectrum of the associated operator, one can introduce the type of a symbol. Therefore, we introduce the following definition. For $\delta \in [0, \pi]$ we define $\Sigma_\delta := \{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \delta \}$.

**Definition 2.4.** Let $L$ be a Lévy process with symbol $\psi$. We say the symbol $\psi$ is of type $(\omega, \theta)$, $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$, iff

$$-R_g(\psi) \subset \mathbb{C} \setminus \omega + \Sigma_{\theta + \frac{\pi}{2}}.$$

**Remark 2.1.** If a symbol $\psi$ is of type $(0, \theta)$, then there exists a constant $c > 0$ such that

$$|\Im(\psi(\xi))| \leq c \Re(\psi(\xi)), \quad \xi \in \mathbb{R}^d.$$

**Remark 2.2.** Let $R(\lambda : A) = [\lambda + A]^{-1}$ be the resolvent of the operator $A$. For $\lambda \in \mathbb{C} \setminus R_g(\psi)$ we have for any $\rho \in \mathbb{R}$

$$\|R(\lambda, A)\|_{L(H_\rho^*)} \leq \frac{1}{\text{dist}(R_g(\psi), \lambda)}.$$

(see Theorem 1.4.2 of [R]). Moreover, the set $R_g(\psi)$ equals the spectrum of $A$.

The generalized Blumenthal–Getoor index of order 0 and type of a symbol can be calculated in many cases. Here, we give some examples.

**Example 2.1.** Let $\alpha \in (0, 2)$ and $L$ be a symmetric $\alpha$–stable process without drift, is given by

$$\psi(\xi) = |\xi|^\alpha,$$

the upper and lower index is $\alpha$, and $\psi$ is of type $(0, \delta)$ for any $\delta > 0$.\footnote{Here, Leb denotes the Lebesgue measure.}
Example 2.2. Let $L$ be the Meixner process as described in [20] (see also [13, p. 136]). In particular, let $L$ be a real–valued Lévy process with symbol

$$
\psi_{m, \delta, a, b}(\xi) = -im\xi + 2\delta \left( \log \cosh \left( \frac{a\xi - ib}{2} \right) - \log \cos \left( \frac{b}{2} \right) \right), \quad \xi \in \mathbb{R},
$$

where $m \in \mathbb{R}$, $\delta, a > 0$, $b \in (-\pi, \pi)$. Then the upper and lower index is 1 (see [13, p. 137-(3.226)]. Moreover, the symbol $\psi$ is of type $(\omega, \theta)$ with $\omega = 0$ and $\theta = \arctan(m/\delta a)$.

Example 2.3. Let $L$ be the normal inverse Gaussian process as described in [4] (see also [13, p. 138]). In particular, let $L$ be a real–valued Lévy process with symbol

$$
\psi_{\text{NIG}}(\xi) = -im\xi + \delta \left( \sqrt{a^2 - (b + i\xi) - \sqrt{a^2 - b^2}} \right), \quad \xi \in \mathbb{R},
$$

where $m \in \mathbb{R}$, $\delta > 0$, $0 < |b| < a$. This process is comparable with the Cauchy process, but has finite expectation. Next, the upper and lower index is 1 (see [13, p. 137-(3.228)]. Moreover, for $m = 0$ the symbol $\psi$ is of type $(\omega, \theta)$ with $\omega = 0$ and $\theta < \pi$.

Let $L = \{L(t) : t \geq 0\}$ be a $d$–dimensional Lévy process without any Gaussian component. We consider the stochastic differential equations of the form

$$
\begin{cases}
    dX^x(t) = \sigma(X^x(t^-))dL(t) \\
    X^x(0) = x, \quad x \in \mathbb{R}^d,
\end{cases}
$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is Lipschitz continuous. Let $(\mathcal{P}_t)_{t \geq 0}$ be the associated Markovian semigroup of $X$ defined by

$$
(\mathcal{P}_t f)(x) := \mathbb{E} [f(X^x(t))], \quad t \geq 0.
$$

Then, $(\mathcal{P}_t)_{t \geq 0}$ is a Feller semigroup and one can compute its infinitesimal generator. Again, one way of computing $A$ is done by Hoh’s symbols (see [10]). In particular, one has

$$
Au(x) = \int_{\mathbb{R}^d} e^{ixT\xi}p(x,\xi)\hat{u}(\xi)\,d\xi \quad u \in C^\infty_c(\mathbb{R}^d),
$$

where the symbol $p$ is defined by

$$
p(x,\xi) := -\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E} \left[ e^{i(X^x(t)-x)T\xi} - 1 \right], \quad x \in \mathbb{R}^d.
$$

Alternatively, we can give an explicit form of $p$ in term of the Lévy symbol of the driving noise $L$. In fact if $\psi$ is Lévy symbol of the Lévy process $L = \{L(t) : t \geq 0\}$, then it is shown in [19, Theorem 3.1], that the symbol $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is given by

$$
p(x,\xi) = \psi(\sigma^T(x)\xi), \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d.
$$

Symbols also arises in the context of pseudo–differential operators, whereas the term symbol is defined in the following way (in Appendix [B] we give a short summary of some Definition and Theorems we need for our proof).
Definition 2.5. (compare [23, p.28, Def. 4.1], [15, Def. 1.1.1, p. 19], [21, p.3 Def.1.1]) Let $X \subset \mathbb{R}^d$, $m \in \mathbb{R}$, and $\rho, \delta$ two real numbers such that $0 \leq \rho \leq 1$ and $0 \leq \delta \leq 1$. Let $S^m_{\rho,\delta}(X, \mathbb{R}^d)$ be the set of all functions $a : X \times \mathbb{R}^d \to \mathbb{C}$, where

- $a$ is infinitely often differentiable;
- for any two multi-indices $\alpha$ and $\beta$ there exists a positive constant $C_{\alpha,\beta} > 0$ depending only on $\alpha$ and $\beta$ such that
  $$ |\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq C_{\alpha,\beta} (|\xi| + |x|)^{m-\rho|\beta|+\delta|\alpha|}, \quad x \in X, \xi \in \mathbb{R}^d. $$

The symbolic calculus for pseudo-differential operators is well established, see [15, 21, 23]. However, in case one considers symbols arising by solution to stochastic differential equations driven by Lévy processes, the derivatives of the symbol will not be necessarily continuous in the origin, i.e. in $\{0\}$. The behavior of $\xi$ in the origin corresponds to the perturbation of the solution $X$ of equation (2.3) by the large jumps of the Lévy process $L$. To illustrate this fact, let us assume that the symbol $p(x, \xi) = a(\xi)$ is independent from $x$ and positive definite. Then the symbol corresponds to a Lévy process. Now, let us assume that the Lévy process has a symmetric Lévy measure $\nu$ such that for all $\ell \geq 2$ the moments $\int_{\mathbb{R}^d \setminus \{0\}} |y|^{\ell} \nu(dy)$ are bounded. Then by [10, Proposition 2; p. 793] the Lévy symbol $a$ is infinitely often differentiable and we have

$$ |\partial^\alpha_\xi a(\xi)| \lesssim \begin{cases} 
 a(\xi) & \text{if } \alpha = 0, \\
 |a(\xi)|^{\frac{1}{2}} & \text{if } |\alpha| = 1, \alpha \in \{1, \ldots, d\} \\
 1 & \text{if } |\alpha| \geq 2.
\end{cases} $$

That means, if all moments of the Lévy measure are bounded, i.e. the moments of the large jumps are bounded, then the Lévy symbol will be infinite often differentiable in the origin. Now, let us assume that the Lévy process is symmetric and $r$-stable with $r < 2$. It follows that the Lévy symbol is $|\xi|^r$ and the large jumps have only bounded moments up to order $\ell$ with $\ell < r$. In case $r < 1$ the Lévy symbol is only once continuously differentiable in the origin and in case $r > 1$, twice continuous differentiable in the origin.

Now, one may ask the question: does the non differentiability at the origin have any effect on the smoothing property of the corresponding Markovian semigroup $(P_t)_{t\geq0}$. Again, let us assume that the Lévy process is symmetric and $r$-stable with $r < 2$. Then, the infinitesimal generator of the corresponding Markovian semigroup $(P_t)_{t\geq0}$ is $-(\Delta)^{\frac{r}{2}}$. However, it is well known that

$$ |P_t u|_{H^2} \leq \frac{c}{t} |u|_{L^2}. $$

That means, the discontinuity of the derivatives in the origin will not have an effect on the smoothing property of the corresponding semigroup $(P_t)_{t\geq0}$. However, the discontinuity at the origin have to be taken into account and we will relax the definition of symbols slightly and define the wider class of Hoh symbols.
Definition 2.6. Let \( m \in \mathbb{R} \), and \( \rho, \delta \) two real numbers such that \( 0 \leq \rho \leq 1 \) and \( 0 \leq \delta \leq 1 \) and \( k \in \mathbb{N}_0 \). Let \( S_{h,k;\rho,\delta}^m(\mathbb{R}^d,\mathbb{R}^d) \) be the set of all functions \( a : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{C} \), where

- for all multi-indices \( \alpha \) and \( \beta \) with \(|\alpha|, |\beta| \leq k\) we have \( \partial^\alpha_x \partial^\beta_\xi a \in C(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}) \) and
  \[ \left| \partial^\alpha_x \partial^\beta_\xi a(x,\xi) \right| \lesssim |\xi|^{-|\alpha|}, \quad |\xi| \leq 1; \]
- for any two multi-indices \( \alpha \) and \( \beta \) with \(|\alpha|, |\beta| \leq k\), there exists a positive constant \( C_{\alpha,\beta} > 0 \) depending only on \( \alpha \) and \( \beta \) such that
  \[ \left| \partial^\alpha_x \partial^\beta_\xi a(x,\xi) \right| \leq C_{\alpha,\beta}(|\xi| + |x|)^{m-\rho|\beta|+\delta|\alpha|}, \quad x,\xi \in \mathbb{R}^d, |\xi| \geq 1. \]

Remark 2.3. Let us assume that a Lévy symbol \( \psi \) has generalized Blumenthal–Getoor index \( s \) of order \( k \geq 1 \) and \( \sigma \asymp 1 \). Direct computation shows that the symbol \( a(x,\xi) := \psi(\sigma^T(x)\xi) \) belongs to \( S_{h,k;1,0}^s(\mathbb{R}^d,\mathbb{R}^d) \).

Now, we can formulate the following Theorem.

Theorem 2.1. Let \( k \geq \frac{d}{2} \). Let \( q \) and \( \psi \) be two Hoh symbols, \( k \) times differentiable on \( \mathbb{R}^d \setminus \{0\} \). Suppose \( b \in C^k_b(\mathbb{R}^d) \) such that for all multi-indices \( |\alpha| \leq k \) we have \( |\partial^\alpha_x b(x)| \lesssim 1, \quad x \in \mathbb{R}^d \). Let the operator \( B : S \to S' \) be defined by

\[ (Bu)(x) := \int_{\mathbb{R}^d} e^{ix^T \xi} q(b^T(x)\xi) \hat{u}(\xi) \, d\xi, \quad u \in S. \]

Suppose \( \sigma \in C^k(\mathbb{R}^d) \) such that for all multiindices \( |\alpha| \leq k \) we have \( |\partial^\alpha_x \sigma(x)| \asymp 1, \quad x \in \mathbb{R}^d \). Moreover, let us assume that there exists a unique solution \( X \) to (1.1), where \( \sigma \) is given above and the Lévy process has symbol \( \psi \) given also above, generating a Markovian semigroup \( \mathcal{P} = (\mathcal{P}_t)_{t \geq 0} \).

If \( q \) and \( \psi \) are \( k \)-times differentiable on \( \mathbb{R}^d \), then no extra condition has to be satisfied. If \( \psi \) or \( q \) is only \( k \)-times differential on \( \mathbb{R}^d \setminus \{0\} \), we assume that \( \psi(0) = 0 \) and \( q(0) = 0 \), and there exists some \( \gamma_\psi > 0 \) and \( \gamma_q > 0 \) such that and there exists some \( \gamma > 0 \) for which

\[ \sup_{|\xi| \leq 1, \xi \neq 0} \left| \xi|^{\alpha}\partial^\alpha_x \psi(\xi) \right| < \infty, \]

and

\[ \sup_{|\xi| \leq 1, \xi \neq 0} \left| \xi|^{\alpha}\partial^\alpha_x q(\xi) \right| < \infty, \]

for all multi-indices \( \alpha \) with \(|\alpha| \leq k \).

If \( \psi \) is of type \((\omega, \theta)\) and has generalized Blumenthal–Getoor index \( s_1 \) and \( q \) has upper Blumenthal–Getoor index less or equal to \( s_2 \) with \( s_2 < s_1 \), then we have for any \( \rho \in \mathbb{R} \)

\[ |B \mathcal{P}_t u|_{H^\rho_2} \leq \frac{C}{\sin \theta} t^{-\frac{\alpha}{s_1}} |u|_{H^\rho_2}, \quad u \in H^\rho_2(\mathbb{R}^d), \quad t > 0. \]
Remark 2.4. The same holds if one of the operators is replaced by its adjoint. In fact, for a symbol \( a \in S_{\delta,\rho}^m(\mathbb{R}^d, \mathbb{R}^d) \) we know by Theorem B.3 that there exists a symbol \( a^*(x, \xi) \in S_{\delta,\rho}^m(\mathbb{R}^d, \mathbb{R}^d) \) such that the to \( a(x, D) \) adjoint operator \( a^*(x, D) \) is described by the symbol \( a^*(x, \xi) \). In addition, by Remark B.3 we know that if \( a \in \text{Hyp}_{\delta,\rho}^m(\mathbb{R}^d, \mathbb{R}^d) \), then \( a^*(x, \xi) \in \text{Hyp}_{\delta,\rho}^{m,\text{mo}}(\mathbb{R}^d, \mathbb{R}^d) \). Thus, if the symbol \( \psi(\sigma^T(x)\xi) \) and \( q(b^T(x)\xi) \) satisfy the assumption of Theorem 2.7, then \( \psi^*(\sigma^T(x)\xi) \) and \( q^*(b^T(x)\xi) \), respectively, satisfy also the assumption of Theorem 2.7.

Proof of Theorem 2.1. First, note that we use within the proof the notation introduced in appendix B.

Let \( a(x, \xi) := \psi(\sigma(x)T\xi), A = a(x, D), \text{Dom}(A) = \{ f \in L^2(\mathbb{R}^d) : Af \in L^2(\mathbb{R}^d) \} \) and \( R(\lambda : A) \) the resolvent. The proof of Theorem 2.1 relies on Proposition A.1 and the fact that there exists a constant \( C > 0 \) such that for \( \varepsilon = s_2/s_1 \)

\[
|BR(\lambda : A)u|_{H^s} \leq C |\lambda|^{-1} |u|_{H^s}, \quad \forall u \in H^s(\mathbb{R}^d).
\]

In particular, if \( \Sigma = \Sigma_{\alpha+\frac{\beta}{2}} \). Since \( \psi, q, \sigma \) and \( b \) are Schwartz functions defined on \( \mathbb{R}^d \), i.e. \( \psi, q, \sigma \in \mathcal{S}(\mathbb{R}^d) \). In the second step we replace \( \psi \) and \( q \) by sequences \( \{ \psi_n : n \in \mathbb{N} \} \subset \mathcal{S}(\mathbb{R}^d) \) and \( \{ q_n : n \in \mathbb{N} \} \subset \mathcal{S}(\mathbb{R}^d) \) converging to \( \psi \) and \( q \), respectively, and \( b \) and \( \sigma \) by sequences \( \{ \sigma_n : n \in \mathbb{N} \} \subset \mathcal{S}(\mathbb{R}^d) \) and \( \{ b_n : n \in \mathbb{N} \} \subset \mathcal{S}(\mathbb{R}^d) \) converging to \( \sigma \) and \( b \), in appropriate sense that will be made precise later.

**Step 1:** Let us assume that \( \omega = 0 \) and \( \sigma, b, q, \psi \in \mathcal{S}(\mathbb{R}^d) \). Let us put \( \Sigma = \Sigma_{\alpha+\frac{\beta}{2}} \). Since \( \sigma \in C^k(\mathbb{R}^d) \) and for all multi-indices \( |\alpha| \leq k, \partial^\alpha_x \sigma(x) \geq 1, x \in \mathbb{R}^d \), and since \( \psi \) has generalized Blumenthal–Getoor index \( s_1 \), one see by the product and chain rule, that \( a(x, \xi) = \psi(\sigma^T(x)\xi) \in S_{1,0}^{s_1}(\mathbb{R}^d, \mathbb{R}^d) \). In particular,

1. the mapping \( (x, \xi) \mapsto a(x, \xi) := \psi(\sigma^T(x)\xi) \) belongs to \( \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d) \).
2. for all multi-indices \( \alpha, \beta \) with \( |\alpha|, |\beta| \leq k \) we have
   \[
   \left| \partial^\alpha_x \partial^\beta_\xi \psi(\sigma^T(x)\xi) \right| \leq |\xi|^{s_1-|\alpha|}, \quad |\xi| + |x| \geq 1;
   \]
3. since \( \sigma \geq 1 \), for all multi-indices \( \alpha, \beta \) with \( |\alpha|, |\beta| \leq k \) we have
   \[
   \left| \partial^\alpha_x \partial^\beta_\xi \psi(\sigma^T(x)\xi) \right| \leq |\xi|^{-|\alpha|}, \quad |\xi| \leq 1.
   \]

Moreover, \( \psi \) has generalized lower Blumenthal–Getoor index \( s_1 \), it follows that

\[
|\xi|^{s_1} \leq |\psi(\sigma^T(x)\xi)|, \quad \xi, x \in \mathbb{R}^d, |\xi| \geq R,
\]

and, therefore, \( \psi(\sigma^T(x)\xi) \in \text{Hyp}_{1,0}^{s_1}(\mathbb{R}^d, \mathbb{R}^d) \).

Let us denote by \( a(x, D) \) the pseudo–differential operator induced by the symbol \( a(x, \xi) = \psi(\sigma^T(x)\xi) \) and let us define the parameterized family of
symbol \( \{a(x, \xi, \lambda) := \psi(\sigma(x)\xi) + \lambda : \lambda \in \Sigma\} \). Observe, that \( a(x, D, \lambda) := (\lambda + A) \) for all \( \lambda \in \Sigma \). A short computation gives, that we have for all \( \xi, x \in \mathbb{R}^d \) and \( \lambda \in \Sigma \)
\[
(\lambda^{1 \over s} + |\xi|^{s_1}) \lesssim |a(x, \xi, \lambda)|,
\]
\[
|a(x, \xi, \lambda)| \lesssim (\lambda^{1 \over s} + |\xi|)^{s_1},
\]
and
\[
\left|\left(\partial_\xi^\alpha \partial_x^\beta a(x, \xi, \lambda)\right) \right| \lesssim (\lambda^{1 \over s} + |\xi|)^{-|\alpha|},
\]
for \( \xi \) and \( \lambda \) large enough.

It follows that \( a(x, \xi, \lambda) \in \text{Hyp}_{1,0,1/s_1}^s(\mathbb{R}^d, \mathbb{R}^d, \Sigma) \). By Theorem \ref{thm:13.5} we know that the symbol \( r(x, \xi, \lambda) \) of the resolvent \((A + \lambda I)^{-1} = R(\lambda, A)\) belongs to \( \text{Hyp}_{1,0,1/s_1}^{s_1-s_1}(\mathbb{R}^d, \mathbb{R}^d, \Sigma) \). Moreover, since the upper generalized Blumenthal–Getoor index of \( q(\xi) \) is smaller or equal to \( s_2 \) and \( |\partial_\xi^\alpha b(x)| \lesssim 1 \) with \( |\alpha| \leq k \), it follows that \( \tilde{q}(x, \xi) := q(b(x)\xi) \in S_{1,0}^{s_2}(\mathbb{R}^d, \mathbb{R}^d) \). Therefore, by Theorem \ref{thm:13.1} we know that the product
\[
\tilde{q}(x, \xi)r(x, \xi, \lambda)
\]
belongs to \( S_{1,0}^{s_2-s_1}(\mathbb{R}^d, \mathbb{R}^d, \Sigma) \). Put \( \Phi = (|\xi| + |x|) \) and \( \Psi(\xi, x) = 1 \). Using the refined symbol class given in Definition \ref{def:13.9} we have \( \tilde{q}(x, \xi) \in S(M_q, \phi, \psi) \) with \( M\tilde{q}(x, \xi) = (|\xi| + |x|)^{s_2} \) and \( r(x, \xi, \lambda) \in S(M_r, \lambda, \phi, \psi) \) with \( M_r, \lambda(x, \xi) = (|\xi| + |x|)^{s_1} \). In particular, by Theorem \ref{thm:13.6} we get
\[
|\tilde{q}(x, \xi)r(x, \xi, \lambda)| \lesssim \frac{(|\xi| + |x|)^{s_2}}{(1 + |\lambda|^{s_1} + |\xi|)^{s_1}}, \quad x, \xi \in \mathbb{R}^d, \lambda \in \Sigma.
\]

We first show that for all multi-indices \( \alpha \) and \( \beta \) with \( |\alpha| \leq k \) and \( |\beta| \leq k \) we have
\[
\sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}} \left|\partial_\xi^\alpha \partial_x^\beta \tilde{q}(x, \xi)r(x, \xi, \lambda)\right| |\xi|^{|\alpha|} < \infty.
\]
First, to get an estimate independent of \( \xi \) and \( x \), we prove that there exists a \( R > 0 \) such that
\[
(1 + |\lambda|^{s_1} + |\xi|)^{s_1} \lesssim \lambda^{s_2} x \in \mathbb{R}^d, \lambda \in \Sigma_R := \Sigma \cap \{\lambda \in \mathbb{C} : |\lambda| \geq R\}.
\]
In fact, differentiation gives that \( f_\lambda \) attains it maximum for
\[
|\xi_\lambda| = \left( \frac{s_2 \lambda}{s_1 - s_2} \right)^{1 \over s_1}.
\]
where
\[
f_\lambda(\eta) := \frac{\eta^{s_2}}{|\lambda + \eta|^{s_1}}, \quad \xi \in \mathbb{R}.
\]
Hence, there exists a constant \( C = C(s_1, s_2) > 0 \) such that
\[
|\tilde{q}(x, \xi)r(x, \xi, \lambda)| \lesssim \lambda^{s_2 - s_1}, \quad x, \xi \in \mathbb{R}^d, \lambda \in \Sigma.
\]
Hence \( \tilde{q}(x, \xi) r(x, \xi, \lambda) \in S^0_{0,1}(\mathbb{R}^d \setminus \{0\}) \). Short calculations shows that
\[
\sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}} \left| \partial_x^\alpha \partial_\xi^\beta \tilde{q}(x, \xi) r(x, \xi, \lambda) \right| |\xi|^{1-|\alpha|} \lesssim \lambda^{\frac{2}{n+1}} - 1,
\]
for all \( \lambda \in \Sigma \). Applying Theorem B.4 gives (2.7).

In order to generalize the inequality to the Bessel Potential spaces \( H^\rho_2(\mathbb{R}^d) \), where \( \rho \in \mathbb{R} \), we notice first that for \( \lambda \in \Sigma \) the symbol
\[
|\langle \xi \rangle|^\rho p(x, \xi) r(x, \xi, \lambda) |\langle \xi \rangle|^{-\rho}
\]
belongs also to \( S^{s_2-s_1}_{0,0,1/s_1}(\mathbb{R}^d, \mathbb{R}^d) \) with norm
\[
|||\langle \xi \rangle|^\rho p(x, \xi) r(x, \xi, \lambda) |\langle \xi \rangle|^{-\rho}||_{S^{s_2-s_1}_{0,0,1/s_1}} \leq \lambda^{\frac{2}{n+1}} - 1.
\]

Next, note that \((I + \Delta)^{\frac{s}{2}}\) is an isomorphism from \( H^\rho_2(\mathbb{R}^d) \) to \( H^{\tau+\rho}_2(\mathbb{R}^d) \). Thus for any \( v \in H^\rho_2(\mathbb{R}^d) \) there exists a \( u \in L^2(\mathbb{R}^d) \) such that \((I + \Delta)^{-\frac{s}{2}} u = v\).

Now,
\[
|\langle I + \Delta \rangle^{\frac{s}{2}} B R(\lambda : A)(I + \Delta)^{-\frac{s}{2}} u|_{L^2} \leq C |\lambda|^{-1} |u|_{L^2}, \quad \forall u \in L^2(\mathbb{R}^d).
\]

In addition, we have
\[
|B R(\lambda : A)(I + \Delta)^{-\frac{s}{2}} u|_{H^2_\rho} = |B R(\lambda : A) v|_{H^\rho_2},
\]
and
\[
|\langle I + \Delta \rangle^{\frac{s}{2}} B R(\lambda : A)(I + \Delta)^{-\frac{s}{2}} u|_{L^2} = |B R(\lambda : A)(1 + \Delta)^{-\frac{s}{2}} u|_{H^2_\rho}.
\]

By \( |u|_{L^2} = |v|_{H^2_\rho} \) and Theorem B.4 we get
\[
|B R(\lambda : A) u|_{H^2_\rho} \leq C |\lambda|^{-1} |u|_{H^\rho_2}, \quad \forall u \in H^\rho_2(\mathbb{R}^d).
\]

Finally, Theorem A.1 gives the assertion for general \( \rho \in \mathbb{R} \).

**Step 2:** If \( \psi \) and \( q \) belong to \( C^k(\mathbb{R}^d) \), since \( \mathcal{S}(\mathbb{R}^d) \) is dense in \( C^k(\mathbb{R}^d) \), there exist sequences \( \{\psi_n : n \in \mathbb{N}\} \), \( \{\sigma_n : n \in \mathbb{N}\} \), \( \{q_n : n \in \mathbb{N}\} \) and \( \{b_n : n \in \mathbb{N}\} \subset \mathcal{S}(\mathbb{R}^d) \) converging to \( \psi \), \( \sigma \), \( q \), and \( b \). If \( \psi, q \in C^k(\mathbb{R}^d \setminus \{0\}) \), then we show that there exist \( \{\psi_n : n \in \mathbb{N}\} \) and \( \{q_n : n \in \mathbb{N}\} \subset \mathcal{S}(\mathbb{R}^d) \) such that for all multi-indices \( \alpha \) and \( \beta \) with \( |\alpha| \leq k \) and \( |\beta| \leq k \) following limit holds
\[
\sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}} \lambda^{1-|\alpha|} |\xi|^{|\alpha|} |\partial_x^\beta \partial_\xi^\alpha \left[ \frac{q_n(b_n^{\#}(x)\xi)}{\lambda + \psi_n(\sigma_n(x)^T \xi)} - \frac{q(b^{\#}(x)\xi)}{\lambda + \psi(\sigma(x)^T \xi)} \right] | \to 0,
\]
as \( n \to \infty \). Let \( h \in C^k(\mathbb{R}^d) \) be a function such that
\begin{itemize}
  \item \( h(x) = 0, \quad x = 0 \);
  \item \( h(x) = 1, \quad |x| \geq R \) for some \( R > 0 \);
  \item \( |h|_{C^k_b(\mathbb{R}^d)} \leq 1 \);
  \item \( \lim_{|x| \to 0} |\partial_x^\alpha h(x)| = 0 \) for all multi-index \( \alpha \) with \( |\alpha| \leq k \);
  \item \( |\partial_x^\alpha h(x)| \leq |x|^{-|\alpha|+1} \) for all multi-index \( \alpha \) with \( |\alpha| \leq k \).
\end{itemize}
Put 
\[ \tilde{\psi}_n(x) := h(xn)\psi(x), \quad x \in \mathbb{R}^d, \]
and 
\[ \tilde{q}_n(x) := h(xn)q(x), \quad x \in \mathbb{R}^d. \]
For each \( n \) there exists sequences \( \{\hat{\psi}_n^k: k \in \mathbb{N}\} \subseteq S(\mathbb{R}^d) \), such that \( \hat{\psi}_n^k \to \tilde{\psi}_n \) in \( C^k(\mathbb{R}^d) \) and \( \hat{q}_n^k \to \tilde{q}_n \) in \( C^k(\mathbb{R}^d) \). Put \( \psi_n = \hat{\psi}_n^k \) and \( q_n = \hat{q}_n^k \).

Now, a straightforward calculations shows that for any multi-index \( \alpha \) with \( |\alpha| \leq k \) we have
\[ \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} |\xi|^{\alpha} \left| \partial_\xi^\alpha \left[ \psi_n(\xi) - \psi(\xi) \right] \right| \to 0, \]
and
\[ \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} |\xi|^{\alpha} \left| \partial_\xi^\alpha \left[ q_n(\xi) - q(\xi) \right] \right| \to 0, \]
for \( n \to \infty \). Furthermore, we have
\[ \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} |\xi|^{\alpha} \left| \partial_\xi^\alpha \left[ \psi_n(\sigma_n^T(x)\xi) - \psi(\sigma^T(x)\xi) \right] \right| \to 0, \]
and
\[ \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} |\xi|^{\alpha} \left| \partial_\xi^\alpha \left[ q_n(b_n^T(x)\xi) - q(b^T(x)\xi) \right] \right| \to 0, \]
for \( n \to \infty \). Let \( \pi_n \) be the symbol of the composition \( q_n \circ r_n \) where \( q_n \) and \( r_n \) are as above. Put
\[ \pi_n(x, \xi, \lambda) = \frac{q_n(b_n^T(x)\xi)}{\lambda + \psi_n(\sigma_n^T(x)\xi)}. \]
Because of (B.1) and (B.2) we have
\[ \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} |\xi|^{\alpha} \left| \partial_\xi^\alpha \left[ \pi_n(x, \xi) - \pi(x, \xi) \right] \right| \to 0 \]
as \( n \to \infty \), with
\[ \pi(x, \xi, \lambda) = \frac{q(b^T(x)\xi)}{\lambda + \psi(\sigma^T(x)\xi)}. \]
Let us now replace in Step I \( \psi \) and \( q \) by the sequences \( \{\psi_n : n \in \mathbb{N}\} \) and \( \{q_n : n \in \mathbb{N}\} \) constructed above, and \( \sigma \) and \( b \) by sequences \( \{\sigma_n : n \in \mathbb{N}\} \) and \( \{b_n : n \in \mathbb{N}\} \) converging to \( \sigma \) and \( b \) in \( C^k(\mathbb{R}^d) \). The Lebesgue domination Theorem gives the assertion.

To tackle the case where \( \omega \neq 0 \) we have to shift \( \lambda \) in order to get the desired result. \[ \square \]
Appendix A. $C_0$–semigroups

For $\delta \in [0, \pi]$ we define $\Sigma_\delta := \{ z \in \mathbb{C} \setminus \{0\} : |\text{arg}(z)| < \delta \}$. A (degenerate) analytic $C_0$-semigroup is defined as follows (see also [16, Chapter 2.5]):

**Definition A.1.** A degenerate $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$ is called analytic in $\Sigma_\theta$ for some $\theta \in (0, \frac{\pi}{2})$ if

1. $T$ extends to an analytic function $T : \Sigma_{\theta + \frac{\pi}{2}} \to \mathcal{L}(X)$;
2. $\lim_{z \to 0, z \in \Sigma_\theta} T(z)x = T(0)x$ for all $x \in X$.

Let $D(A) = \{ x \in X : \lim_{h \to 0} \frac{T(h)x - x}{h} \text{ exists} \}$ and $A$ be an infinitesimal generator of a semigroup $T$, i.e.

$$Ax := \lim_{h \to 0} \frac{T(h)x - x}{h}, \quad x \in D(A).$$

Let $\rho(A)$ be the set of all complex numbers $\lambda$ for which $\lambda - A$ is invertible, i.e.

$$\rho(A) := \{ \lambda \in \mathbb{C} : (\lambda - A)^{-1} \text{ is a bounded operator} \}.$$

In order to characterize a $C_0$–semigroup, we introduce the following definition.

**Definition A.2.** Let $X$ be a Banach space and let $A$ be the generator of a degenerate analytic $C_0$-semigroup on $X$. We say that $A$ is of type $(\omega, \theta, K)$, where $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$ and $K > 0$, if $\omega + \Sigma_{\theta + \frac{\pi}{2}} \subseteq \rho(A)$ and

$$|\lambda - \omega|\|R(\lambda : A)\|_{L(X,X)} \leq K \quad \text{for all } \lambda \in \omega + \Sigma_{\theta + \frac{\pi}{2}}.$$

The theorem below gives some characterizations of analytic $C_0$-semigroups that we will use later on.

**Theorem A.1.** Let $(T(t))_{t \geq 0}$ be a degenerate $C_0$-semigroup of type $(M, \omega)$ for some $M > 0$ and $\omega \in \mathbb{R}$. Let $A$ be the generator of $T$. Let $\omega' > \omega$. The following statements are equivalent:

1. $T$ is an analytic $C_0$-semigroup on $\Sigma_\theta$ for some $\theta \in (0, \frac{\pi}{2})$ and for every $\theta' < \theta$ there exists a constant $C_{1,\theta'}$ such that $\|e^{-\omega'z}T(z)\| \leq C_{1,\theta'}$ for all $z \in \Sigma_{\theta'}$.
2. There exists a $\delta \in (0, \frac{\pi}{2})$ such that:

\[
\omega' + \Sigma_{\frac{\pi}{2} + \delta} \subseteq \rho(A),
\]

and for every $\delta' \in (0, \delta)$ there exists a constant $C_{2,\delta'} > 0$ such that:

\[
|\lambda - \omega'|\|R(\lambda : A)\| \leq C_{2,\delta'}, \quad \text{for all } \lambda \in \omega' + \Sigma_{\frac{\pi}{2} + \delta'}.
\]

3. $T$ is differentiable for $t > 0$ and there exists a constant $C_3$ such that:

$$t\|AT(t)\| \leq C_3 e^{\omega't}, \quad \text{for all } t > 0.$$
The proof can be found in [16, Theorem 2.5.2] for exponentially stable analytic $C_0$-semigroups with boundedly invertible generator and can be transferred to arbitrary analytic $C_0$-semigroup by the following observation. If $T$ is a $C_0$-semigroup of type $(M, \omega, K)$ and $A$ is the generator of $T$, then for any $\omega' > \omega$ the $C_0$-semigroup $(e^{-\omega' t}T(t))_{t \geq 0}$ is exponentially stable and the generator of this semigroup, $A - \omega'I_X$, is invertible.

For our purpose we need an estimate which is very similar to estimate (3) of Theorem A.1.

**Proposition A.1.** Let $X$ be a Banach space. Let $A_0$ be the generator of a degenerate analytic $C_0$-semigroup $T$ on $X$ and let $B$ be a possible unbounded operator acting on $X$. Suppose $A_0$ is of type $(\omega, \theta, K)$ for some $\omega \in \mathbb{R}, \theta \in (0, \frac{\pi}{2})$ and $K > 0$. Suppose there exist an $\varepsilon \in [0, 1)$ and a constant $C(A_0, B)$ such that for all $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\theta}$ one has:

$$
\| R(\lambda : A_0)B \|_{L(X,X)} \leq C(A_0, B) |\lambda - \omega|^{-\varepsilon}. 
$$

Then for all $t > 0$ we have:

$$
\| T(s)B \|_{L(X,X)} \leq \begin{cases} 
\frac{\Gamma(\varepsilon)}{\pi} [\sin \theta]^{-\varepsilon} C(A_0, B) e^{\omega s} - \varepsilon & \text{if } \varepsilon \neq 0, \\
\frac{1}{\pi} C(A_0, B) e^{\omega s} \int_0^\infty e^{-r \sin \theta r^{-1}} dr & \text{if } \varepsilon = 0.
\end{cases}
$$

**Proof of Proposition A.1.** First assume that $\omega = 0$. Let $\theta' \in (0, \theta)$, $\rho \in (0, \infty)$, and $\Gamma_{\theta', \rho} = \Gamma_{\theta', \rho}^{(1)} + \Gamma_{\theta', \rho}^{(2)} + \Gamma_{\theta', \rho}^{(3)}$, where $\Gamma_{\theta', \rho}^{(1)}$ and $\Gamma_{\theta', \rho}^{(2)}$ are the rays $re^{i(\frac{\pi}{2} + \theta')}$ and $re^{-i(\frac{\pi}{2} + \theta')}$, $\rho \leq r < \infty$, and $\Gamma_{\theta', \rho}^{(3)} = \rho^{-1}e^{i\phi}, \phi \in [-\frac{\pi}{2} - \theta', \frac{\pi}{2} + \theta']$. It follows from [16, Theorem 1.7.7] that for $s > 0$

$$
T(s) = \frac{1}{2\pi i} \int_{\Gamma_{\theta', \rho}} e^{\lambda s} R(\lambda : A_0) d\lambda.
$$

For any $x \in X$ we have

$$
T(s)Bx = \frac{1}{2\pi i} \int_{\Gamma_{\theta', \rho}} e^{\lambda s} R(\lambda : A_0) Bx d\lambda.
$$

Therefore, for any $\varepsilon \in [0, 1)$ and $s > 0$ we have the following chain of equalities/inequalities

$$
\| T(s)B \|_{L(X,X)} = \left\| \frac{1}{2\pi i} \int_{\Gamma_{\theta', \rho}} e^{\lambda s} R(\lambda : A_0) Bd\lambda \right\|_{L(X,X)} \leq \frac{1}{2s \pi} \left\| \int_{\rho}^\infty e^{-r e^{-i(\frac{\pi}{2} + \theta')}} R\left( \frac{r}{s} e^{-i(\frac{\pi}{2} + \theta')} : A_0 \right) B e^{i(\frac{\pi}{2} + \theta')} dr \right\|_{L(X,X)} + \frac{1}{2s \pi} \left\| \int_{\rho}^\infty e^{r e^{i(\frac{\pi}{2} + \theta')}} R\left( \frac{r}{s} e^{i(\frac{\pi}{2} + \theta')} : A_0 \right) B e^{-i(\frac{\pi}{2} + \theta')} dr \right\|_{L(X,X)}
$$
Appendix B. Symbol Classes and pseudo–differential operators

In this section we shortly introduce pseudo–differential operators and their symbols. In addition we introduce the definitions and Theorems which are necessary to for our purpose. For a detailed introduction on pseudo–differential operators and their symbols we recommend the books [15, 21, 23].

In order to treat pseudo–differential operators different classes of symbols are necessary to for our purpose. For many estimates, one does not need that the function is infinitely often differentiable. Due to this reason, one introduce also the following classes.

**Definition B.1.** Let $X \subset \mathbb{R}^d$, $m \in \mathbb{R}$, and $\rho, \delta$ two real numbers such that $0 \leq \rho \leq 1$ and $0 \leq \delta \leq 1$. Let $S^m_{\rho,\delta}(X, \mathbb{R}^d)$ be the set of all functions $a : X \times \mathbb{R}^d \to \mathbb{C}$, where

- $a(x, \xi)$ is infinitely often differentiable, i.e. $a \in C^\infty(X \times \mathbb{R}^d)$;
- for any two multi-indices $\alpha$ and $\beta$ and any compact set $K \subset X$ there exists $C_{K,\alpha,\beta}$ such that

$$\left| \partial_\xi^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{K,\alpha,\beta}(|\xi| + |x|)^{m-|\beta|+\delta|\alpha|}, \quad x \in X, \xi \in \mathbb{R}^d.$$  

We call any function $a(x, \xi)$ belonging to $\bigcup_{m\in\mathbb{R}} S^m_{0,0}(\mathbb{R}^d, \mathbb{R}^d)$ a symbol.

Therefore, we can define the following classes.

$$||T(s)B||_{L(X,X)} \leq \frac{1}{s^{\varepsilon}} C(A_0, B) [\sin \theta']^{-\varepsilon} \int_0^\infty [r \sin \theta']^{\varepsilon-1} e^{-r\sin \theta'} \sin \theta' dr.$$
Definition B.2. (compare [23] p. 28) Let $X \subset \mathbb{R}^d$. Let $S^m_{k;\rho,\delta}(X,\mathbb{R}^d)$ be the set of all functions $a : X \times \mathbb{R}^d \to \mathbb{C}$, where

- $a(x, \xi)$ is $k$-times differentiable;
- and for any two multi-indices $\alpha$ and $\beta$ with $|\alpha| + |\beta| \leq k$, there exists a positive constant $C_{\alpha,\beta} > 0$ depending only on $\alpha$ and $\beta$ such that
  $$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta}(|\xi| + |x|)^{m-|\beta|+\delta|\alpha|}, \quad x, \xi \in \mathbb{R}^d.$$ 

Moreover, one can introduce a semi-norm in $S^m_{k;\rho,\delta}(X,\mathbb{R}^d)$ by

$$\|a\|_{k,S^m_{k;\rho,\delta}} = \sup_{|\alpha|,|\beta| \leq k} \sup_{(x,\xi) \in X \times \mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \left(|\xi| + |x|\right)^{\rho|\beta|+\delta|\alpha|-m}.$$

Remark B.1. For $m_1 \geq m_2$ it follows that $S^m_{k;\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d) \supseteq S^{m_2}_{k;\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d)$ and $S^{m_1}_{k;\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d) \supseteq S^{m_2}_{k;\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d)$, $k \in \mathbb{N}$.

Definition B.3. (compare [23] p.28, Def. 4.2) Let $a(x, \xi)$ be a symbol. Then, to $a(x, \xi)$ corresponds an operator $a(x, D)$ is defined by

$$a(x, D) u(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) \, d\xi \quad u \in \mathcal{S}(\mathbb{R}^d)$$

is called pseudo-differential operator.

The product of two pseudo-differential operator is again a pseudo-differential operator and can be characterized as follows.

Theorem B.1. (compare [23] Theorem 6.1, p. 54, [15] Theorem 1.2.16, p. 31) Let $a_1(x, \xi) \in S^{m_1}_{1,0}(\mathbb{R}^d,\mathbb{R}^d)$ and $a_2(x, \xi) \in S^{m_2}_{1,0}(\mathbb{R}^d,\mathbb{R}^d)$. Then the product $b(x, D) := a_1(x, D) a_2(x, D)$ is again a pseudo-differential operator such that

$$b(x, \xi) \in S^{m_1+m_2}_{1,0}(\mathbb{R}^d,\mathbb{R}^d).$$

Moreover,

$$b(x, \xi) \sim \sum_{|\alpha| \leq m_1+m_2} \frac{(-i)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha a_1(x, \xi)\right) \left(\partial_\xi^\alpha a_2(x, \xi)\right).$$

The equation (B.1) means that

$$b(x, \xi) - \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha a_1(x, \xi)\right) \left(\partial_\xi^\alpha a_2(x, \xi)\right)$$

belongs to $S^{m_1+m_2-N}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ for every positive integer $N$.

Definition B.4. Let $a(x, \xi)$ be a symbol in $S^m_{0,0}(\mathbb{R}^d,\mathbb{R}^d)$ and $a(x, D)$ the associated pseudo-differential operator. Suppose there exists a linear operator $a^*(x, D) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ such that

$$(a(x, D) f, g) = (f, a^*(x, D) g), \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$
Then we call $a^*(x,D)$ the formal adjoint operator of the operator $a(x,D)$.

The existence of the formal adjoint is given by the following Theorem.

**Theorem B.2.** (compare [10, Corollary 3.6, p. 803], [23, p. 62, Theorem 7.1]) For a symbol $a(x,\xi) \in S^m_{0,0}(\mathbb{R}^d,\mathbb{R}^d)$ there exists a symbol $a^*(x,\xi) \in S^m_{0,0}(\mathbb{R}^d,\mathbb{R}^d)$ such that the operator defined by

$$a^*(x,D)u(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ixT\xi} a^*(x,\xi) \hat{u}(\xi) d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

is the adjoint operator of $a(x,D)$. In addition, $a^*(x,\xi)$ has the following expansion

$$a^*(x,\xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha \partial_x^\beta a)(x,\xi).$$

**Theorem B.3.** (compare [23, Theorem 9.7]) Let $a(x,\xi) \in S^1_{0,0}(\mathbb{R}^d,\mathbb{R}^d)$. Then, for any $1 < p < \infty$ the operator $a(x,D) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a linear and bounded operator.

In fact, analyzing the proof of Theorem 9.7 [23, p. 79] one can see that the condition of the differentiability at the origin can be relaxed. Here, it is important to mention that the proof relies on the Theorem 2.5 [9, p. 120] (see also Theorem 4.23 [1]), from which one can clearly see the extension of the Theorem 9.7 of [23] to symbols, whose derivatives have a singularity at \{0\}. Moreover, analyzing line by line of the proof of Theorem 9.7, one can give an estimate of the norm of the operator.

**Theorem B.4.** Let $k > \frac{d}{2}$ and $a(x,\xi) \in C^k(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\})$. Moreover, let us assume that for any multi-indices $\alpha$ and $\beta$ with $|\alpha|, |\beta| \leq k$

$$\sup_{|\alpha|,|\beta| \leq k} \sup_{(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}} \left| \partial_\xi^\alpha \partial_x^\beta a(x,\xi) \right| < \infty,$$

and

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \lesssim |\xi|^{-|\alpha|}, \quad \xi \neq 0.$$

Then the corresponding operator $a(x,D)$ is bounded on $L^2(\mathbb{R}^d)$ with the uniform estimate

$$\|a(x,D)\|_{L^2(\mathbb{R}^d)} \lesssim \sup_{|\alpha|,|\beta| \leq k} \sup_{(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}} \left| \xi^{||\alpha|\|} \partial_\xi^\alpha \partial_x^\beta a(x,\xi) \right|.$$

To investigate the inverse of a pseudo-differential operator one can introduce set of elliptic and hypoelliptic symbols.
Definition B.5. (compare [15] p. 35) A symbol $a \in S^m_{\rho, \delta}$ is called globally elliptic in the class $S^m_{\rho, \delta}(\mathbb{R}^d, \mathbb{R}^d)$, if for some $R > 0$,
\[ \langle |\xi| \rangle^m \leq |a(x, \xi)|, \quad |x| + |\xi| \geq R. \]

Definition B.6. (compare [15] p. 35) Let $m, m_0, \rho, \delta$ be real numbers with $0 \leq \delta < \rho \leq 1$. The class $\text{Hyp}_{\rho, \delta}^{m, m_0}(X \times \mathbb{R}^d)$ consists of all functions $a(x, \xi)$ such that
\begin{itemize}
  \item $a(x, \xi) \in C^\infty(X \times \mathbb{R}^d)$;
  \item there exists some $R > 0$ such that
    \[ \langle |\xi| \rangle^{m_0} \leq |a(x, \xi)|, \quad |x| + |\xi| \geq R. \]
\end{itemize}
and for an arbitrary multi-indices $\alpha$ and $\beta$ and for any compact set $K \subset X$ there exists a constants $C_{\alpha, \beta, K}$ with
\[ \left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{K, \alpha, \beta} (\langle |\xi| \rangle + |x|)^{m_0 - \rho|\alpha| + \delta|\beta|}. \]
for $x \in K, \xi \in \mathbb{R}^d$.

Remark B.2. Let $a(x, \xi) \in \text{Hyp}_{\rho, \delta}^{m, m_0}(X \times \mathbb{R}^d)$ be a symbol and $a^*(x, \xi)$ the symbol of the formal adjoint operator $a^*(x, D)$. Then, one can see from the expansion in [B.3], that if $a(x, \xi) \in \text{Hyp}_{\rho, \delta}^{m, m_0}(X \times \mathbb{R}^d)$ then $a^*(x, \xi) \in \text{Hyp}_{\rho, \delta}^{m, m_0}(X \times \mathbb{R}^d)$.

Lemma 1.3.5 [15] p. 36] gives under which conditions a symbol belonging to $\text{Hyp}_{\rho, \delta}^{m, m_0}(X \times \mathbb{R}^d)$ is invertible. However, our aim is to characterize the symbol of a resolvent of an operator $a(x, D)$. Then we define a subclass of hypoelliptic symbols and state a Theorem giving sufficient condition for the existence of the symbol of the resolvent.

Definition B.7. (compare [21] Definition 9.1, p. 77] Let $m, \rho, \delta, \gamma$ be real numbers with $0 \leq \delta < \rho \leq 1, 0 < \gamma < \infty$. The class $S^m_{\rho, \delta}(X \times \mathbb{R}^d, \Lambda)$ consists of all functions $a(x, \xi, \lambda) : X \times \mathbb{R}^d \times \Lambda \to \mathbb{C}$ such that
\begin{itemize}
  \item $a(x, \xi, \lambda_0) \in C^\infty(X \times \mathbb{R}^d)$ for every fixed $\lambda_0 \in \Lambda$;
  \item For arbitrary multi-indices $\alpha$ and $\beta$ and for any compact set $K \subset X$ there exists a constants $C_{K, \alpha, \beta}$ such that
    \[ \left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi, \lambda) \right| \leq C_{K, \alpha, \beta} (\langle |\xi| \rangle + |\lambda|)^{m_0 - \rho|\alpha| + \delta|\beta|}. \]
\end{itemize}
for $x \in K, \xi \in \mathbb{R}^d, \lambda \in \Lambda$.

Since the resolvent can be viewed as a parameterized family of symbols, we introduce the following definition.

Definition B.8. (compare [21] p. 78] Let $m, m_0, \rho, \delta, \gamma$ be real numbers with $0 \leq \delta < \rho \leq 1, 0 < \gamma < \infty$. The class $\text{Hyp}_{\rho, \delta, \gamma}^{m, m_0}(X \times \mathbb{R}^d, \Lambda)$ consists of all functions $a(x, \xi, \lambda) : X \times \mathbb{R}^d \times \Lambda \to \mathbb{C}$ such that
\begin{itemize}
  \item $a(x, \xi, \lambda_0) \in C^\infty(X \times \mathbb{R}^d)$ for every fixed $\lambda_0 \in \Lambda$;
For any compact set $K \subset X$ there exists two constants $C_K$ and $\tilde{C}_K$ such that

$$C_K(|\xi| + |\lambda|^{1/2})^{m_0} \leq |a(x, \xi, \lambda)| \leq \tilde{C}_K(|\xi| + |\lambda|^{1/2})^m.$$  

for $x \in K, \xi \in \mathbb{R}^d, \lambda \in \Lambda, |\xi| + |\lambda| \geq R$.

- For arbitrary multi-indices $\alpha$ and $\beta$ and for any compact set $K \subset X$ there exists a constants $C_{K,\alpha,\beta}$ such that

$$\left| \left( \partial_\xi^\alpha \partial_\lambda^\beta a(x, \xi, \lambda) \right) a^{-1}(x, \xi, \lambda) \right| \leq C_{K,\alpha,\beta}(|\xi| + |\lambda|^{1/2})^{m_0 - \rho(|\alpha| + \delta|\beta|)},$$

for $x \in K, \xi \in \mathbb{R}^d, \lambda \in \Lambda$.

**Remark B.3.** Let $a(x, \xi) \in H_{\rho, \delta, \gamma}(\mathbb{R}^d \times \mathbb{R}^d)$ of type $(\theta, \omega)$. Then, it is easy to see that $a(x, \xi) + \lambda \in H_{\rho, \delta, \gamma}(\mathbb{R}^d \times \mathbb{R}^d \times \Lambda)$ with $m_1 = \max(m_0, m)$ and $\Lambda = \omega + \sum \frac{\rho}{2}$.

One can classify the inverse of the each member of $\{a(x, \xi, \lambda) : \lambda \in \Lambda\}$, but we have to introduce the set of properly supported operator. Let $a(x, \xi)$ be a symbol with kernel $K_a$ and let $\text{supp}(K_a)$ denote the support of $K_a$ (the smallest closed subset $Z \subset X \times X$ such that $K_a|_{(X \times X) \setminus Z} = 0$). Consider the canonical projections $\Pi_1, \Pi_2 : \text{supp}(K_a) \to X$, obtained by restricting the corresponding projections of the direct product $X \times X$. Recall that a continuous map $f : M \to N$ between topological spaces $M$ and $N$ is called proper if for any compact $K \subset N$ the inverse image $f^{-1}(K)$ is a compact in $M$. A symbol $a(x, \xi)$ is called properly supported if both projections $\Pi_1, \Pi_2 : \text{supp}(K_a) \to X$ are proper maps. For more details see [21, p. 18]. We will denote by $H_{\rho, \delta, \gamma}(\mathbb{R}^d \times \mathbb{R}^d, \Lambda)$ the class of symbols $a \in H_{\rho, \delta, \gamma}(\mathbb{R}^d \times \mathbb{R}^d, \Lambda)$ being properly supported.

**Theorem B.5.** (compare [21, Theorem 9.2, p. 85]) Let $a(x, \xi, \lambda) \in H_{\rho, \delta, \gamma}(\mathbb{R}^d \times \mathbb{R}^d, \Lambda)$. Then, there exists a $R > 0$ such that for all $|\lambda| \geq R$ the operator $a(x, D, \lambda)$ is invertible. In particular, there exists a symbol $a_{\text{inv}}(x, \xi, \lambda)$ such that

$$a(x, D, \lambda)^{-1}u = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix^T\xi} a_{\text{inv}}(x, \xi, \lambda) \hat{u} \, d\xi \quad u \in \mathcal{S}(\mathbb{R}^d),$$

and $a_{\text{inv}}(x, D, \lambda)$ belongs to $H_{\rho, \delta, \gamma}(\mathbb{R}^d \times \mathbb{R}^d, \Lambda)$, where $\Lambda_R := \Lambda \cap \{|\lambda| \geq R\}$.

In order to deal with operators depending on parameter, one can treat a family of symbols by refining the definition of symbol classes see for instance, [15, p. 19]. For this aim, we introduce some class of functions. A positive continuous function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is called sub-linear weight function if,

$$1 \leq \Phi(x, \xi) \lesssim 1 + |x| + |\xi|, \quad \text{for } x, \xi \in \mathbb{R}^d.$$
It is called a temperate weight, if for some $s > 0$

$$\Phi(x + y, \xi + \eta) \lesssim \Phi(x, \xi) (1 + |y| + |\eta|)^s, \quad \text{for } x, y, \xi, \eta \in \mathbb{R}^d.$$ 

**Definition B.9.** Let $\Phi, \Psi, M : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be temperate weights such that $\Phi, \Psi$ are sub-linear. We denote by $S(M; \Phi, \Psi)$ the space of all smooth functions $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d,$ such that for every $\alpha, \beta \in \mathbb{N}^d$ we have

$$|\partial^\alpha_x \partial^\beta_x a(x, \xi)| \lesssim M(x, \xi) \Psi(x, \xi)^{-|\alpha|} \Phi(x, \xi)^{-|\beta|}, \quad \text{for } x, y, \xi, \eta \in \mathbb{R}^d.$$

Now, one can choose the target weight $M$ in such a way that it depends on a parameter, say $\lambda$. Now the multiplication theorem for the composition operator can be restated as follows.

**Theorem B.6.** (\cite{15}, Theorem 1.2.16, p. 31) Let $\Phi, \Psi, M : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be temperate weights such that $\Phi, \Psi$ are sub-linear. Let $a_1(x, \xi) \in S(M_1 ; \Phi, \Psi)$ and $a_2(x, \xi) \in S(M_2 ; \Phi, \Psi)$. Then the composition $b(x, D) := a_1(x, D) a_2(x, D)$ is again a pseudo-differential operator such that

$$b(x, \xi) \in S(M_1 M_2 ; \Phi, \Psi).$$

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