Skyrmion–skyrmion interaction induced by itinerant electrons in a ferromagnetic strip

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Abstract

Magnetic skyrmions are promising spin textures for building next-generation magnetic memories and spintronic devices. Nevertheless, one of the major challenges in realizing skyrmion-based devices is the stabilization of ordered arrays of these spin textures in different geometries. Here we numerically study the skyrmion–skyrmion interaction potential that arises due to the dynamics of itinerant electrons coupled to the magnetic texture in a ferromagnetic background with racetrack geometry. We consider different topological textures (ferromagnetic (FM) and antiferromagnetic (AFM)), namely: skyrmions, antiskyrmions and biskyrmions. We show that at low electron filling, for sufficiently short separation, the skyrmions strongly couple each other yielding a bound-state bound by electronic dynamics. However, when the filling is increased, the interaction potential energy presents local minima at specific values of the skyrmion–skyrmion distance. Each of these local minima corresponds to energetically stable positions of skyrmions which are ‘protected’ by well-defined energy barriers. By inspecting the local charge density, we find that in the case of AFM skyrmions, the local antiferromagnetic nature prevents electronic penetration into the core, allowing the AFM skyrmions to be seen as infinite potential barriers for electrons.

Keywords: skyrmions, topology, magnetism, Monte Carlo

(Some figures may appear in colour only in the online journal)

1. Introduction

From the theoretical proposal in the last century and the subsequent experimental evidence in chiral magnets [1–7], magnetic skyrmions have garnered enormous interest due to their potential applications as promising information carriers in spintronics, including the design of racetrack memories [8–12]. The advantages presented by skyrmions compared to other standard and extensively studied magnetic textures (magnetic bubbles [13]) are the small currents needed for their propagation, their smaller size (~1–100 nm diameter) and their topological nature which provides a significant energy barrier to avoid skyrmion annihilation [10].

Currently, there is a large family of topological skyrmion-like textures that includes the standard ferromagnetic skyrmions (FM) figure 1(a), antiferromagnetic skyrmions (AFM) [14–23] figure 1(b), antiskyrmions [24] figure 1(c) and biskyrmions [25] figure 1(d), among others (for a recent review, see for example [26–28]).

In particular, AFM skyrmions have become the subject of intense focus in the context of antiferromagnetic spintronics...
The Ruderman–Kittel–Kasuya–Yosida exchange interaction has been previously studied in magnetic monolayers via the itinerant electrons on the skyrmion–skyrmion pair interaction appearing close to each other. The analysis of the contribution when skyrmions are nucleated in experiments they exhibit exponential behavior at large distances. Frustration may induce the attractive interaction at short distances. Thiele’s approach predicts a short range repulsive force. Furthermore, the Thiele’s approach presents a close relation with the double-well potential problem, showing quite similar energy vs distance curves. At low fillings both cases show a very close behavior developing a very similar pattern of local minima; while at higher fillings, the AFM case presents a much more regular distribution of minima that indicates greater positional stability. In addition, we analyzed the effect of considering skyrmions of charge $Q \neq -1$.

**2. Model and ansätze**

To set the stage, we consider a Kondo lattice model on a square lattice of $L_x \times L_y$ sites with mixed boundary conditions, that is, periodic boundary condition along $x$ and open boundary condition along the $y$-axis. The hopping term and the interaction of itinerant electrons with a magnetic texture are described by the following Hamiltonian,

$$
\mathcal{H} = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \hat{c}_{\mathbf{r}, \sigma}^\dagger \hat{c}_{\mathbf{r}' \sigma} - J_B \sum_{\mathbf{r}} \mathbf{s}_\mathbf{r} \cdot \mathbf{s}_\mathbf{r},
$$

where the operator $\hat{c}_{\mathbf{r}, \sigma}^\dagger$ ($\hat{c}_{\mathbf{r}, \sigma}$) creates (annihilates) an electron with spin $\sigma = \pm 1/2$ at site $\mathbf{r}$. The first term $t$ is the transfer integral between nearest-neighbor sites of itinerant electrons. The second term describes the Hund coupling between the spin of itinerant electrons $\mathbf{s}_\mathbf{r} = (1/2) \sum_{\sigma} \hat{c}_{\mathbf{r}, \sigma}^\dagger \sigma_{\mathbf{r}, \sigma} \cdot \hat{c}_{\mathbf{r}, \sigma}$, and localized classical spins with coupling constant $J_B$, where $\sigma$ is the vector of Pauli matrices. In the computations, we take the lattice spacing $a = 1$ and the hopping constant $t = 1$ as an energy unit.
Hereafter, let us focus in the $J_H/t >> 1$ regime, where the spin of the hopping electron is forced to align parallel to the local moment and the low-energy physics can be described by an effective Hamiltonian of spinless fermions as

\[ \mathcal{H}_{\text{eff}} = - \sum_{(r,x')} \epsilon_{\text{eff}}^{r,x'} \hat{d}_{r}^\dagger \hat{d}_{r'}, \]

where $\hat{d}_r^\dagger$ ($\hat{d}_r$) is the creation (annihilation) fermion operator and $\epsilon_{\text{eff}}^{r,x'}$ is the effective transfer integral (see appendix for details).

For the purpose of studying the effective interaction between consecutive skyrmions on the racetrack geometry, we employ artificial skyrmion textures constructed using the finite size skyrmion ansatz (centered at the origin $r = (x,y) \equiv (0,0)$) [48]:

Ferromagnetic: $\text{S}^{\text{FM}} = \begin{bmatrix} \sin(f) \cos(\phi) \\ \sin(f) \sin(\phi) \\ \cos(f) \end{bmatrix}$

Antiferromagnetic: $\text{S}^{\text{AFM}} = (-1)^{x+y} \text{S}^{\text{FM}}$,

where $f \equiv f(r) = \pi(1-r/R)\Theta(R-r)$, $R$ is the skyrmion radius, $\Theta(R-r)$ is the Heaviside step function, $\phi \equiv \phi(r) = Q \times (\arctan(y/x) + \chi)$ with $\chi$ being the helicity and $Q$ the topological charge [48, 49]. With these ansätze it is possible to build different types of skyrmion configurations with a spacing $\Delta x$ between centers of two adjacent skyrmions. In the case of AFM skyrmions (equation (4)), we have used the most simple picture where the spin texture can be visualized as a superposition of two FM skyrmions coupled antiferromagnetically [50] figure 1(b). However, it should be mentioned that it is also possible to combine skyrmions on multiple sublattices [14, 18–20, 23].

After diagonalizing the electronic Hamiltonian in equation (2) by an unitary transformation $U$, we compute the ground state $\langle GS| = \prod_{n=1}^{\nu} |\nu\rangle$, where $\nu$ is the energy of the $\nu$th electronic eigenstate, i.e. $\mathcal{H}_{\text{eff}} \langle \nu | = \epsilon_\nu \langle \nu |$.

The ground state energy $E(\Delta x)$ and the on-site electron density $\rho_r$ at zero temperature for fixed separation $\Delta x$ (see top of figure 2) at a fixed filling $n$, are given by $E(\Delta x) \equiv \langle GS| \mathcal{H}_{\text{eff}} | GS \rangle = \sum_{\nu=1}^{\nu_{\text{min}}} \epsilon_\nu \rho_\nu = \langle GS| \hat{d}_{n}^\dagger \hat{d}_{n} | GS \rangle = \sum_{\nu=1}^{\nu_{\text{min}}} \epsilon_\nu \rho_\nu$.

3. Calculation of the interaction potential

In this section, we present results for systems consisting of a pair of skyrmions with radius $R$ in a ferromagnetic background with size $L_x \times L_y$. Thorough the rest of this work, the term ‘skyrmion’ is used to identify the $Q = -1$ topological texture, which can be either ferromagnetic (FM) or antiferromagnetic (AFM). We have focused on the case of a skyrmion-pair with the same topological charge $\{Q,Q\} = \{-1,+1\}$ and $\{Q,Q\} = \{-2,-2\}$; and larger systems sizes to rule out significant finite size effects. These results allow us to predict the interaction potential between skyrmions assuming that they are displaced along the nanotrack as shown in figure 1.

3.1. FM–FM and AFM–AFM skyrmion pair interactions with $Q = -1$

Firstly, we have considered the situation of a skyrmion-pair (FM–FM and AFM–AFM) with topological charge $Q = -1$ separated by a distance $\Delta x$. We have determined the interaction potential $E(\Delta x)$ as a function of the horizontal distance between the skyrmions at fixed filling $n$. Figures 2(a)–(h) depicts the behavior of $E(\Delta x)$ as a function of $\Delta x$ and for several values of $n$. One can notice that at low fillings, there is a minimum at $\Delta x = 2 \times R$ indicating that the more favorable arrangement corresponds to locating both SKs next to each other (black arrow in figure 2(a)). Therefore, for sufficiently short separation skyrmions strongly couple each other yielding a bound-state bound by electronic dynamics. More remarkable is what happens when $n$ is increased: a sequence of well-defined local minima appear at specific values of $\Delta x$ (black arrows in figure 2(b)). Each of these local minima corresponds to energetically stable distances between skyrmions which are ‘protected’ by well defined energy barriers (dashed lines (i) and (ii)). This general scheme of local minima is observed in both cases (FM–FM and AFM–AFM pairs). Nevertheless, while in the FM case, for some fillings the interaction is repulsive at short distances (figures 2(d) and (e)), in the AFM case the global minimum always occurs in the bound-state configuration (with skyrmions next to each other). This would indicate that the bound state of AFM skyrmions would be much more stable against changes in electronic filling than in the FM case.

To confirm this picture, we construct a density plot of the interaction potential with control parameters $\Delta x$ and $n$, for both cases (FM–FM and AFM–AFM, skyrmions), shown in figure 3. We plot the interaction potential $E(\Delta x) - E_{\text{min}}^{n}$ (in $E_{\text{Fermion}}$ and $E_{\text{n}}$ are the minimum and maximum values of $E(\Delta x)$ that takes a fixed $n$) where blue color indicates the lowest value $E(\Delta x) - E_{\text{min}}^{n} = 0$. As it can be seen in figure 3 (panels left and right), both the FM and AFM skyrmions develop well defined minima at low fillings. However, when filling $n$ is increased, in the case of FM skyrmions the local minima become irregular and diffuse (left), while in the AFM case the minima remain well defined forming a ribbed pattern as a function of $n$ for all cases (right). This result strongly suggests that for AFM skyrmions the interaction with electrons generates an array of energetically stable separations of skyrmions along the track which are “protected” by well defined energy barriers (‘pinning sites’). As a consequence, the positional stability of AFM skyrmions is expected to be greater than for their FM counterpart.

In order to analyze the spatial localization of the electronic states, we calculate the electronic occupation $\rho_r$ with
the model described in section 2, presenting the results in figure 4. At the top of the figure we show the spin texture consisting of a skyrmion pair (FM–FM or AFM–AFM) separated by a distance $\Delta x$. Below, we present the local electron density distributions at different $n$ fillings for both cases. Quite remarkably, our initial observation was that for low
fillings (as an example we show \( n = 1, 5 \)) both cases present very similar distributions, showing zero electronic penetration within the skyrmion region.

However, for larger fillings \( (n = 20, 50 \text{ in figure 4}) \) in the FM–FM case a non-zero local electron density can be seen inside the skyrmion region \( (r < R) \). This is in sharp contrast to the AFM–AFM case, where the charge density inside the skyrmion vanishes.

From a theoretical point of view, let us remember that the Berry phase, i.e. the quantum-mechanical phase picked up by electrons when their spin follows the orientation of the local magnetization \( S_r \), can be rewritten as an effective Aharonov–Bohm phase \( \alpha_{r, r'} \) associated with an ‘emergent’ local magnetic field \( B^e_r = \frac{1}{2} S_r \cdot (\partial_r S_r \times \partial_r S_r) \). In the case of FM skyrmions we expect this effective field to have a smooth dependence with the position, while this would not necessarily be the case for AFM skyrmions. This can be verified using the ansätze of our study to calculate the effective fields for both cases, obtaining \( B^e_r^{\text{AFM}} = (-1)^{x+y} B^e_r^{\text{FM}} \). For the AFM case, there is a rapid oscillation of the effective field, which translates into a strong barrier potential for the electrons that prevents transmission through them. This picture is maintained for different separations between skyrmions as can be seen in figure 5 (panel (a)) showing the electronic distribution for the case of AFM skyrmions for different values of the distance \( \Delta x \).

It should be noted that this behavior is intrinsic to AFM skyrmions. This can be appreciated much better if we consider the electronic occupation for FM–FM, AFM–AFM skyrmions and circular homogeneous ferromagnetic domains (FMD). In figure 5(b) we make the comparison of the three magnetic textures for large fillings: FM–FM skyrmions have a non-zero density inside; zero penetration into its core is observed in AFM–AFM skyrmions; while in the case of FMD–FMD localized circular states are observed inside the FMDs.

### 3.1.1 Analogy with the double-well potential problem.

The exactly vanishing electronic density inside the AFM skyrmions leads us to suppose that they can be treated as impenetrable (circular) barriers for the electrons. Therefore, we may draw...
Figure 5. (a) Electronic occupation in real space for a filling $n = 10$ and for different skyrmion (AF) distances $\Delta x$. (b) Comparison of the electronic occupation between FM–FM, AFM–AFM skyrmion pairs and pairs of ferromagnetic domains FMD–FMD (with net magnetization opposite to the external field) of equal size. It can be seen that for large fillings, electronic states appear located inside the FMDs.

Figure 6. (a) Double potential well problem. In panels (b)–(e), system energy as a function of the distance $\Delta x$. We can see that the interaction potential is almost the same as in the case of a couple of AFM skyrmions with topological charge $\{Q, Q\} = \{+1, +1\}$ in figure 2.

A parallel between the situation of itinerant electrons coupled to two skyrmions in a racetrack geometry (with skyrmion diameter comparable to the width of the racetrack) with the quantum-mechanical textbook problem of a non-relativistic particle confined in an infinite double well (DW), where the AFM skyrmions play the role of the barriers. Let us recall that the electronic energies of non-relativistic free electrons of mass $m$ in a DW configuration (see figure 6) are given
Figure 7. (a) Interaction potential $E(\Delta x)$ as a function of the horizontal distance between the FM skyrmions with topological charge $\{Q, Q\} = \{-1, -1\}$ (red), $\{Q, Q\} = \{-1, +1\}$ (cyan) and $\{Q, Q\} = \{-2, -2\}$ (blue); (b) magnetic texture and (c) the calculated topological charge density.

by $E_{DW}^{(k,l)}(\Delta x) = \left\{ \frac{C^2}{\Delta x^2}, \frac{C^2}{(L-\Delta x)^2} \right\}$, $k, l = 1, \ldots, C = 2h^2\pi^2/2m$. Then the energy of the system at fixed filling $n$ is

$$E_{DW}(\Delta x)s = \sum_{k, l \mid k+l=n} E_{DW}^{(k,l)}(\Delta x).$$  \hspace{1cm} (5)

In figures 6(b)–(e) we present several curves of the interaction potential $E_{DW}(\Delta x) - E_{\text{min}}$. One can quickly appreciate the great similarity between the energy curves in the figures 6(b)–(e) and the interaction potential $E(\Delta x)$ in the case of AFM skyrmions (see figure 2). The presence of a global minimum at $\Delta x = 2 \times R$ and the array of energetically stable positions (separations $\Delta x$) reinforces the idea that AFM skyrmions can be seen as impenetrable barriers for the itinerant electrons.

3.2. FM–FM and AFM–AFM pairs: skyrmions–antiskyrmion and biskyrmion–biskyrmion interaction potential

Up to now, we have focused on FM–FM and AFM–AFM skyrmions with topological charge $Q = -1$. As a final analysis, we study the effect of introducing skyrmions with different topological charge, i.e. antiskyrmions ($Q = +1$) and biskyrmions ($Q = -2$). In order to show this, we have explicitly computed the interaction potential $E(\Delta x)$ as a function of the horizontal distance between the FM–FM and AFM–AFM skyrmions for two specific cases: $\{Q, Q\} = \{-1, +1\}$ (skyrmion–antiskyrmion) and $\{Q, Q\} = \{-2, -2\}$ (biskyrmion–biskyrmion). First, we analyze the interaction potential $E(\Delta x)$ for FM skyrmions is displayed in figure 7(a). We see that the cases $\{Q, Q\} = \{-1, -1\}$ and $\{Q, Q\} = \{-2, -2\}$ exhibit almost the same behavior. The case $\{Q, Q\} = \{-1, +1\}$ presents slight differences that arise
Figure 8. (a) Potential interaction $E(\Delta x)$ as a function of the horizontal distance between the AFM skyrmions with topological charge $\{Q, Q\} = \{-1, -1\}$ (red), $\{Q, Q\} = \{-1, +1\}$ (cyan) and $\{Q, Q\} = \{-2, -2\}$ (blue); (b) magnetic texture and (c) the calculated topological charge density.

from the opposite sign of the topological charges, although it conserves the structure of local minima observed in the other cases. In the case of AFM skyrmions figure 8(a) we obtain a perfect agreement for the three cases studied, suggesting that in the AFM case the topological charge has a much smaller effect than in the FM case.

This great similarity between the three cases of topological charges can be understood from inspection of the local field felt by electrons: $B_z$. In figures 7(c) and 8(c) we present the calculated effective field associated with the configurations 7(b) and 8(b) respectively. We can see that, except for the central site, all the distributions of $B_z$ present the same structure (and opposite sign in the case $Q = +1$). This indicates that the results obtained in the case of $\{Q, Q\} = \{-1, -1\}$ are actually valid for different topological charges.

4. Conclusions

In recent years, magnetic skyrmions have emerged as promising candidates for devices for memory and logic applications. From the experimental point of view, the skyrmions often come in close proximity with each other, opening the natural question about how this can affect their stability, motion and spacial confinement. In this regard, we have studied the itinerant electrons-induced skyrmion–skyrmion interaction in a ferromagnetic racetrack.
film. We have considered several situations: FM–FM and AFM–AFM skyrmions with three possible sets of topological charges: \{Q, Q\} = \{-1, -1\} (skyrmion–skyrmion), \{Q, Q\} = \{-1, +1\} (skyrmion–antiskyrmion) and \{Q, Q\} = \{-2, -2\} (biskyrmion–biskyrmion). We found that, at low fillings, both the FM and AFM skyrmions develop a well-defined array of energetically stable separations of skyrmions along the track which are ‘protected’ by well-defined energy barriers. However, when filling \( n \) is increased, in the case of FM skyrmions the local minima become irregular and diffuse, while in the AFM case the minima remain well defined forming a ribbed pattern as a function of \( n \) for all cases. These minima in energy act as effective ‘pinning sites’ along the racetrack, i.e. preferred positions for the skyrmions along the track, as long as the size of the skyrmions does not exceed the mean separation between minima. For example, typical skyrmions with a diameter of about 6\( a \)–10\( a \) which corresponds to about 5 nm–100 nm. In our study, we find that the average distance between skyrmions minima goes from \( \sim 30a \)–40\( a \) figure 2(b) to \( \sim 20a \) figure 2(t), making it possible to pin the pinning effect at low fillings. In addition, we have investigated the electronic distribution occupation. We found that at low fillings both cases (FM- and AFM-skyrmions), the electronic local density states reside outside the skyrmion region, showing zero electronic penetration within the skyrmion region. However, for larger fillings, in the FM case, a non-zero local electron density emerges inside skyrmion core. This is in sharp contrast to the AFM case even for large fillings showing that the AFM character of the skyrmions has a very large energy cost on the itinerant electrons inside the skyrmion region.

We have confirmed that these results are valid even for other configurations of the topological charge \( Q \neq -1 \). In order to show this, we have explicitly computed the interaction potential \( E(\Delta x) \) as a function of the horizontal distance between the FM and AFM skyrmions for two specific cases: \{Q, Q\} = \{-1, +1\} and \{Q, Q\} = \{-1, -1\}. As a general result, we found that for FM skyrmions the cases \{Q, Q\} = \{-1, +1\} and \{Q, Q\} = \{-1, -1\} exhibit almost the same behavior, while the case \{Q, Q\} = \{-1, +1\} presents slight differences that arise from the opposite sign of the topological charge, although it conserves the structure of local minima observed in the other cases. In the case of AFM skyrmions we observed a perfect agreement for the three cases studied, suggesting that in the AFM case the topological charge has a much smaller effect than in the FM case. Therefore, our results support the idea that AFM skyrmions are good candidates for electronic devices. Future perspectives of this work include studying different geometries, and different skyrmions combinations such as skyrmion–antiskyrmion, FM skyrmion–AFM skyrmion, as well as different topological textures such as merons, which may also be relevant in potential technological applications.

**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

**Appendix. Effective Hamiltonian derivation**

We consider a Kondo lattice model on the square lattice where the itinerant electrons are coupled with the classical magnetic moments by a Hund’s coupling as

\[
\mathcal{H} = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle, \sigma} (\hat{c}_{\mathbf{r} \sigma}^\dagger \hat{c}_{\mathbf{r}' \sigma} + \text{h.c.}) - J_H \sum_{\mathbf{r}} \mathbf{S}_r \cdot \mathbf{s}_r, \tag{A.1}
\]

where \( \hat{c}_{\mathbf{r} \sigma} (\hat{c}_{\mathbf{r} \sigma}^\dagger) \) is the creation (annihilation) operator at the site \( \mathbf{r} \) with spin \( \sigma = \pm 1/2 \), \( t \) is the hopping amplitude between nearest-neighbor sites, \( J_H \) is the Hund’s coupling strength between the electron spin \( \mathbf{s}_r = \frac{1}{2}(\hat{c}^\dagger_{\mathbf{r} \uparrow} \hat{c}_{\mathbf{r} \downarrow} - \hat{c}^\dagger_{\mathbf{r} \downarrow} \hat{c}_{\mathbf{r} \uparrow}) \) and the local magnetic moment \( \mathbf{S}_r \). In the strong Hund coupling limit \( J_H \gg t \), the spin of the itinerant electron is fully aligned with magnetic moment \( \mathbf{S}_r \). Therefore, it is trivial to observe that the electronic spectrum splits into a low- and high-energy band set [32, 53].

In order to obtain an effective Hamiltonian describing the low energy sector we choose the quantization axis in the site \( \mathbf{r} \) pointing along the direction of the local magnetization, so we introduce the unitary transformation \( \mathcal{U}_\mathbf{r} \) between the fermionic operators, \( \hat{c}^\dagger_{\mathbf{r} \sigma} = \mathcal{U}_\mathbf{r} \hat{f}^\dagger_{\mathbf{r} \sigma} \) and \( \hat{f}^\dagger_{\mathbf{r} \sigma} = \mathcal{U}_\mathbf{r}^\dagger \hat{c}^\dagger_{\mathbf{r} \sigma} \), such that \( \hat{c}_{\mathbf{r} \sigma} = \mathcal{U}_\mathbf{r} \hat{f}_{\mathbf{r} \sigma} \) and \( \hat{U}_\mathbf{r} \cdot (\hat{S}_\mathbf{r} \cdot \hat{\sigma}) \cdot \hat{U}_\mathbf{r} = \sigma \cdot \hat{\sigma} \). The general expression of the matrix transformation is given by

\[
\mathcal{U}_\mathbf{r} = \mathbf{m}_\mathbf{r} \cdot \hat{\sigma} = \left( \begin{array}{cc} \cos \frac{\theta_\mathbf{r}}{2} & \sin \frac{\theta_\mathbf{r}}{2} e^{-i\phi_\mathbf{r}} \\ \sin \frac{\theta_\mathbf{r}}{2} e^{i\phi_\mathbf{r}} & -\cos \frac{\theta_\mathbf{r}}{2} \end{array} \right),
\]

where vector \( \mathbf{m}_\mathbf{r} = \{\sin \frac{\theta_\mathbf{r}}{2} \cos \phi_\mathbf{r}, \sin \frac{\theta_\mathbf{r}}{2} \sin \phi_\mathbf{r}, \cos \frac{\theta_\mathbf{r}}{2}\} \) is defined from the local magnetic moment \( \mathbf{S}_\mathbf{r} = (\cos \phi_\mathbf{r} \sin \theta_\mathbf{r}, \sin \phi_\mathbf{r} \sin \theta_\mathbf{r}, \cos \theta_\mathbf{r}) \). With all this, the transformed Hamiltonian reads

\[
\mathcal{H} = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \mathcal{U}^\dagger_\mathbf{r} \mathcal{U}_\mathbf{r}' \hat{f}^\dagger_{\mathbf{r}' \sigma} \hat{f}_{\mathbf{r} \sigma} - J_H \sum_{\mathbf{r}} \hat{f}^\dagger_{\mathbf{r} \sigma} \mathcal{U}^\dagger_\mathbf{r} \hat{S}_\mathbf{r} \cdot \mathcal{U}_\mathbf{r} \hat{f}_{\mathbf{r} \sigma}, \tag{A.2}
\]

where

\[
\mathcal{U}^\dagger_\mathbf{r} \mathcal{U}_\mathbf{r} = \begin{pmatrix} C_{12} & C_{12} \\ C_{12} & C_{11} \end{pmatrix},
\]

and \( C_{11} = \cos \frac{\theta_\mathbf{r}}{2} \cos \frac{\theta_\mathbf{r}}{2} + \sin \frac{\theta_\mathbf{r}}{2} \sin \frac{\theta_\mathbf{r}}{2} \cos (-\phi_\mathbf{r} - \phi_\mathbf{r}') \) and \( C_{12} = \cos \frac{\theta_\mathbf{r}}{2} \sin \frac{\theta_\mathbf{r}}{2} e^{-i(\phi_\mathbf{r} + \phi_\mathbf{r}')} - \cos \frac{\theta_\mathbf{r}}{2} \sin \frac{\theta_\mathbf{r}}{2} e^{-i\phi_\mathbf{r}'} \).

In the strong coupling regime \( J_H \gg t \), the low-energy sector can be described by effective spinless fermions: \{\hat{d}_\mathbf{r}, \hat{d}^\dagger_\mathbf{r}\}. As a result, the effective Hamiltonian of the system is

\[
\mathcal{H}_{\text{eff}} = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \cos \left( \frac{\theta_\mathbf{r}}{2} \right) e^{i\phi_\mathbf{r}'} \hat{d}^\dagger_\mathbf{r} \hat{d}_\mathbf{r}',
\]

where \( \hat{d}_\mathbf{r}' \) (\( \hat{d}_\mathbf{r} \)) corresponds to the up component of \( \hat{f}^\dagger_{\mathbf{r} \sigma} \) (\( \hat{f}_{\mathbf{r} \sigma} \)), where \( \cos \theta_\mathbf{r} = \hat{S}_\mathbf{r} \cdot \mathbf{s}_\mathbf{r} \) and \( \hat{a}_{\mathbf{r} \mathbf{r}'} \) the phase accumulated by the hopping electron,

\[
\hat{a}_{\mathbf{r} \mathbf{r}'} = \arctan \left( \frac{-\sin (\phi_\mathbf{r} - \phi_\mathbf{r}')}{\cos (\phi_\mathbf{r} - \phi_\mathbf{r}') + \cot \left( \frac{\theta_\mathbf{r}}{2} \right) \cot \left( \frac{\theta_\mathbf{r}'}{2} \right)} \right).
\]
References

[1] Mihlbauer S, Binz B, Jonietz F, Pfleiderer C, Rosch A, Neubauer A, Georgii R and Böni P 2009 Science 323 915–9
[2] Münzer W et al 2010 Phys. Rev. B 81 041203
[3] Yu X, Onose Y, Kanazawa N, Park J, Han J, Matsui Y, Nagaosa N and Tokura Y 2010 Nature 465 901–4
[4] Yu X Kanazawa N, Onose Y, Kimoto K, Zhang W Z, Ishiwata S, Matsui Y and Tokura Y 2011 Nat. Mater. 10 106–9
[5] Heinze S., von Bergmann K, Menzel M, Brede J, Kubetzka A, Wiesendanger R, Bihlmayer G and Blügel S 2011 Nat. Phys. 7 713–8
[6] Seki S, Yu X, Ishiwata S and Tokura Y 2012 Science 336 198–201
[7] Butenko A B, Leonov A A, Rößler U K and Bogdanov A N 2010 Phys. Rev. B 82 052403
[8] Fert A, Cros V and Sampaio J 2013 Skyrmions on the track Nat. Nanotechnol. 8 152–6
[9] Zhang X, Zhou Y, Ezawa M, Zhao G and Zhao W 2015 Sci. Rep. 5 1
[10] Zhang X, Zhou Y, Song K M, Park T-E, Xia J, Ezawa M, Liu X, Zhao W, Zhao G and Woo S 2020 J. Phys.: Condens. Matter 32 143001
[11] Guan S H et al 2022 J. Magn. Magn. Mater. 546 168852
[12] Carvalho-Santos V L, Castro M A, Salazar-Arvaneda D, Laroze D, Corona R M, Allende S and Altbir D 2021 Appl. Phys. Lett. 118 172407
[13] Eschenfelder A H 1980 Magnetic Bubble Technology (Berlin: Springer)
[14] Rosales H D, Cabra D C and Pujol P 2015 Phys. Rev. B 92 214439
[15] Barker J and Tretiakov O A 2016 Phys. Rev. Lett. 116 147203
[16] Osorio S A, Rosales H D, Sturla M B and Cabra D C 2017 Phys. Rev. B 96 024404
[17] Osorio S A, Sturla M B, Rosales H D and Cabra D C 2019 Phys. Rev. B 100 220404(R)
[18] Villalba M E, Gómez Albarracín F A, Rosales H D and Cabra D C 2019 Phys. Rev. B 100 245106
[19] Gao S et al 2020 Nature 586 37–41
[20] Rosales H D, Gómez Albarracín F A, Guratinder K, Tsurkan V, Prodan L, Ressouche E and Zaharko O 2022 Phys. Rev. B 105 224402
[21] Legrand W, Maccariello D, Ajejas F, Collin S, Vecchiola A, Bouzehouane K, Reyren N, Cros V and Fert A 2020 Nat. Mater. 19 34
[22] Liu Z and Yang H 2022 Physica E 135 114978
[23] Mukherjee A, Kathiyat D S and Kumar S 2021 Sci. Rep. 11 1
[24] Nayak A K, Kumar V, Ma T, Werner P, Pippel E, Sahoo R, Damay F, Rößler U K, Felser C and Parkin S S 2017 Nature 548 561
[25] Yu X, Tokunaga Y, Kaneko Y Zhang W Z, Kimoto K, Matsui Y, Taguchi Y and Tokura Y 2014 Nat. Commun. 5 3198
[26] Zhou Y 2019 Natl Sci. Rev. 6 210
[27] Back C et al 2020 J. Phys. D: Appl. Phys. 53 363001
[28] Göbel B, Mertig I and Tretiakov O A 2020 Phys. Rep. 895 1–28
[29] Jungwirth T, Marti X, Wadley P and Wunderlich J 2016 Nat. Nanotechnol. 11 231
[30] Rosales H D, Gómez Albarracín F A and Pujol P 2019 Phys. Rev. B 99 035163
[31] Djavid N and Lake R K 2020 Phys. Rev. B 102 024419
[32] Tomé M and Rosales H D 2021 Phys. Rev. B 103 L020403
[33] Göbel Borge, Mook A, Henk Jurgen and Mertig I 2017 Phys. Rev. B 96 060406(R)
[34] Akosa C A, Tretiakov O A, Tataro G and Manchon A 2018 Phys. Rev. Lett. 121 097204
[35] Chen G 2017 Nat. Phys. 13 112
[36] Zhang X, Zhou Y and Ezawa M 2016 Sci. Rep. 6 24795
[37] Jin C, Song C, Wang J and Liu Q 2016 Appl. Phys. Lett. 109 182404
[38] Akosa C A, Hang Li, Tataro G and Tretiakov O A 2019 Phys. Rev. Appl. 12 054032
[39] Göbel B, Mook A, Henk J and Mertig I 2019 Phys. Rev. B 99 020405
[40] Lin S, Reichhardt C, Batista C D and Saxena A 2013 Phys. Rev. B 87 214419
[41] Lin S and Hayami S 2016 Phys. Rev. B 93 064430
[42] Rózsa L, Deák A, Simon E, Yanes R, Udvardi L, Szunyogh L and Nowak U 2016 Phys. Rev. Lett. 117 157205
[43] Capić D, Garanin D A and Chudnovsky E M 2020 J. Phys.: Condens. Matter 32 415803
[44] Brearton R, van der Laan G and Hesjedal T 2020 Phys. Rev. B 101 134422
[45] Bezvershenko A V, Kolezhuk A K and Ivanov B A 2018 Phys. Rev. B 97 054408
[46] Zhang X, Zhou Y and Ezawa M 2016 Nat. Commun. 7 10293
[47] Cacilhas R, Carvalho-Santos V L, Vojkovic S, Carvalho E B, Pereira A R, Altbir D and Núñez A S 2018 Appl. Phys. Lett. 113 212406
[48] Osorio S A, Sturla M B, Rosales H D and Cabra D C 2019 Phys. Rev. B 99 064439
[49] Bogdanov A N and Yablonovski D 1989 Zh. Eksp. Teor. Fiz. 95 178
[50] Díaz S A, Klinovaja J, Loss D and Hoffman S 2021 Phys. Rev. B 104 214501.
[51] Yu X Z, Onose Y, Kanazawa N, Park J H, Han J H, Matsui Y, Nagaosa N and Tokura Y 2010 Nature 465 901–4
[52] Nagaosa N and Tokura Y 2013 Nat. Nanotechnol. 8 899–911
[53] Ohgushi K, Murakami S and Nagaosa N 2000 Phys. Rev. B 62 R6065

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