1. FOCUSING COHERENT LIGHT THROUGH MULTIPLE SCATTERING AND NONLINEAR LAYERS: SIMULATION DETAILS

In the main text we study focusing of coherent light through a scattering layer followed by a nonlinear layer using wavefront shaping. We propose that this simplified model exhibits the same qualitative behavior as a scattering nonlinear medium, i.e. a medium composed of many successive scattering and nonlinear layers. In order to test this assumption, we performed simulations of focusing coherent light through a composite medium with multiple alternating scattering and nonlinear layers with wavefront shaping. The configuration used for the composite medium simulation is shown in Fig. S1. The configuration is identical to that used in the single-layer simulation, except for the modification of the medium and that there is no free-space propagation between any of the scattering and nonlinear layers. The composite medium consists of five scattering layers with four nonlinear layers in between them, and a fifth nonlinear layer at the end. The scattering layers were modeled just like in the single-layer simulation, as a sheet of randomly distributed phase features in the range of $-\pi < \phi < \pi$, but the phase features were slightly smoothed to avoid sharp phase jumps. The incident beam illuminated $\sim 30 \times 30$ features on the first scattering layer (after smoothing). Despite being multiple scattering, the composite medium is still entirely forward-scattering. The nonlinear layers were modeled with positive nonlinearity ($n_2 > 0$).
and were each a fifth of the length of the layer used in the single-layer simulation. The spatial phase of the field incident upon the first scattering layer (or, equivalently, upon the SLM) was optimized adaptively in order to enhance a single speckle at the output of the composite medium, i.e. at the output of last nonlinear layer. This optimization was repeated for different $\eta_{\text{lin}}$ values, corresponding to different nonlinearity strengths. The linear ($\eta = 0$) enhancement value and focused power fraction were 79 and 0.0283, respectively.

The results presented in Fig. 2c follow the same trend as Fig. 2b, showing a focused power fraction that increases roughly by a factor of 2 as nonlinearity increases. We note that the $\eta$ values in the multiple-layer simulation are larger than those in the single-layer simulation. This is due to the difference between the configurations that the two simulations are based on. Since in the single-layer simulation the speckle field entering the nonlinear medium is fully developed, it propagates under the influence of local nonlinearity throughout the entire medium. In the multi-layer simulation, however, the scattering layers directly follow the nonlinear layers and therefore the scattered field must propagate through a significant length of the composite medium before the speckles develop and local nonlinearity becomes significant. Despite the increase, the $\eta$ values, or nonlinear phases accumulated, in the multi-layer simulation are still of the order of the values typically used in nonlinear microscopy experiments. Additionally, the trend in 2c shows less saturation than that of 2b. We did not explore larger $\eta$ values, which would likely show this saturation trend, due to limited computing power. Since the nonlinearity values are higher for these points, more steps must be performed during propagation using the split-step method. The computational cost of running a genetic algorithm, which must evaluate the fitness of >30,000 phase patterns using such propagation, resulted in impractical run times.

### 2. SIMULATION RESULTS: CROSS-SECTIONS THROUGH 2D SPECKLE PROPAGATION IN NONLINEAR MEDIA WITHOUT WAVEFRONT SHAPING

In Fig. 3 we presented one-dimensional cross-sections through the propagation of simulated two-dimensional speckle fields in linear (Fig. 3a), defocusing nonlinear (Fig. 3b) and focusing nonlinear media (Fig. 3c), after their wavefront was shaped in order to focus at the output of the medium. For comparison purposes, we present cross-sections through the propagation of the same two-dimensional fields, created by the same diffuser, in the same linear (Fig. S2a), defocusing nonlinear (Fig. S2b) and focusing nonlinear media (Fig. S2c) yet without wavefront shaping (hence, the speckle fields entering the nonlinear media are different in Fig. 3 and Fig. S2). We can see that the linear case looks quite similar with and without wavefront shaping, with the speckle field not changing much during propagation. In both nonlinear cases the speckle field evolves quickly during propagation, bright speckles turning into dark and vice versa, as they do without wavefront shaping. The defocusing nonlinear case is similar with and without wavefront shaping in other aspects as well. However, the focusing nonlinear case without wavefront shaping lacks the effect of multiple bright speckles following trajectories leading into a particular bright speckle, as seen with wavefront shaping.
Fig. S2. Simulation results: 1D cuts through the propagation of 2D speckle fields, created by the same diffuser, inside the same (a) linear (b) defocusing nonlinear and (c) focusing nonlinear media as those in Fig. 3, but without wavefront shaping.