A stretched exponential bound on the rate of growth of the number of periodic points for prevalent diffeomorphisms

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1 Introduction

Let \( \text{Diff}^r(M) \) be the space of \( C^r \) diffeomorphisms of a finite-dimensional smooth compact manifold \( M \) with the uniform \( C^r \)-topology, where \( \dim M \geq 2 \), and let \( f \in \text{Diff}^r(M) \). Consider the number of periodic points of period \( n \)

\[
P_n(f) = \# \{ x \in M : \ x = f^n(x) \}.
\]

(1)

The main question of this paper is:

**Question 1.** How quickly can \( P_n(f) \) grow with \( n \) for a “generic” \( C^r \) diffeomorphism \( f \)?

We put the word “generic” in brackets because as the reader will see the answer depends on notion of genericity.

For technical reasons one sometimes counts only isolated points of period \( n \); let

\[
P^n_i(f) = \# \{ x \in M : \ x = f^n(x) \text{ and } y \neq f^n(y) \text{ for } y \neq x \text{ in some neighborhood of } x \}.
\]

(2)

We call a diffeomorphism \( f \in \text{Diff}^r(M) \) an Artin-Mazur diffeomorphism (or simply A-M diffeomorphism) if the number of isolated periodic orbits of \( f \) grows at most exponentially fast, i.e. for some number \( C > 0 \),

\[
P^n_i(f) \leq \exp(Cn) \text{ for all } n \in \mathbb{Z}_+.
\]

(3)

Artin & Mazur \cite{AM} proved the following result.

**Theorem 1.1.** For \( 0 \leq r \leq \infty \), A-M diffeomorphisms are dense in \( \text{Diff}^r(M) \) with the uniform \( C^r \)-topology.

We say that a point \( x \in M \) of period \( n \) for \( f \) is hyperbolic if \( df^n(x) \), the derivative of \( f^n \) at \( x \), has no eigenvalues with modulus 1. (Notice that a hyperbolic solution to \( f^n(x) = x \) must also be isolated.) We call \( f \in \text{Diff}^r(M) \) a strongly Artin-Mazur diffeomorphism if for some number \( C > 0 \),

\[
P_n(f) \leq \exp(Cn) \text{ for all } n \in \mathbb{Z}_+,
\]

(4)

and all periodic points of \( f \) are hyperbolic (whence \( P_n(f) = P^n_i(f) \)). In \cite{K1} an elementary proof of the following extension of the Artin-Mazur result is given.
Theorem 1.2. For $0 \leq r < \infty$, strongly A-M diffeomorphisms are dense in $\text{Diff}^r(M)$ with the uniform $C^r$-topology.

According to the standard terminology, a set in $\text{Diff}^r(M)$ is called residual if it contains a countable intersection of open dense sets and a property is called (Baire) generic if diffeomorphisms with that property form a residual set. It turns out the A-M property is not generic, as is shown in [K2]. Moreover:

Theorem 1.3. [K2] For any $2 \leq r < \infty$ there is an open set $N \subset \text{Diff}^r(M)$ such that for any given sequence $a = \{a_n\}_{n \in \mathbb{Z}^+}$ there is a Baire generic set $R_a$ in $N$ depending on the sequence $a_n$ with the property if $f \in R_a$, then for infinitely many $n_k \in \mathbb{Z}^+$ we have $P_{n_k}^1(f) > a_{n_k}$.

Of course since $P_n(f) \geq P_{n_k}^1(f)$, the same statement can be made about $P_n(f)$. But in fact it is shown in [K2] that $P_n(f)$ is infinite for $n$ sufficiently large, due to a continuum of periodic points, for at least a dense set of $f \in N$.

The proof of this Theorem is based on a result of Gonchenko-Shilnikov-Turaev [GST1]. Two slightly different detailed proofs of their result are given in [K2] and [GST2]. The proof in [K2] relies on a strategy outlined in [GST1].

However, it seems unnatural that if you pick a diffeomorphism at random then it may have an arbitrarily fast growth of number of periodic points. Moreover, Baire generic sets in Euclidean spaces can have zero Lebesgue measure. Phenomena that are Baire generic, but have a small probability are well-known in dynamical systems, KAM theory, number theory, etc. (see [O], [HSY], [K3] for various examples). This partially motivates the problem posed by Arnold [A]:

Problem 1. Prove that “with probability one” $f \in \text{Diff}^r(M)$ is an A-M diffeomorphism.

Arnold suggested the following interpretation of “with probability one”: for a (Baire) generic finite parameter family of diffeomorphisms $\{f_\varepsilon\}$, for Lebesgue almost every $\varepsilon$ we have that $f_\varepsilon$ is A-M (cf. [K3]). As Theorem 1.3 shows, a result on the genericity of the set of A-M diffeomorphisms based on (Baire) topology is likely to be extremely subtle, if possible at all. We use instead a notion of “probability one” based on prevalence [HSY], [K3], which is independent of Baire genericity. We also are able to state the result in the form Arnold suggested for generic families using this measure-theoretic notion of genericity.

For a rough understanding of prevalence, consider a Borel measure $\mu$ on a Banach space $V$. We say that a property holds “$\mu$-almost surely for perturbations” if it holds on a Borel set $P \subset V$ such that for all $v \in V$ we have $v + w \in P$ for almost every $w$ with respect to $\mu$. Notice that if $V = \mathbb{R}^k$ and $\mu$ is Lebesgue measure, then “$\mu$-almost surely for perturbations” is equivalent to “Lebesgue almost everywhere”. Moreover, the Fubini/Tonelli Theorem implies that if $\mu$ is any Borel probability measure on $\mathbb{R}^k$, then a property that holds $\mu$-almost surely for perturbations must also be hold Lebesgue almost everywhere. Based on this observation, we call a property on a Banach space “prevalent” if it holds $\mu$-almost surely for perturbations for some Borel probability measure $\mu$ on $V$, which for technical reasons (cf. [HSY]) we require to have compact support. In order to apply this notion to the Banach manifold $\text{Diff}^r(M)$, we must describe how we make perturbations in this space, which we will do in the next Section.

1 For example, using techniques from [GST2] and [K2] one can prove that for a Baire generic finite-parameter family $\{f_\varepsilon\}$ and a Baire generic parameter value $\varepsilon$ the corresponding diffeomorphism $f_\varepsilon$ is not A-M. Unfortunately, how to estimate from below the measure of non-A-M diffeomorphisms in a Baire generic finite-parameter family is so far an unreachable question.

2 A similar notion of prevalence is used in [K3].
Our first main result is a partial solution to Arnold’s problem. It says that for a prevalent\footnote{In Y, hyperbolicity is introduced as the minimal distance of eigenvalues to the unit circle. This way of defining hyperbolicity does not guarantee the existence of a $M_2^{-2n}\gamma$-neighborhood free from periodic points of the same period [KH].} diffeomorphism $f \in \text{Diff}^r(M)$, with $1 < r \leq \infty$, and all $\delta > 0$ there exists $C = C(\delta) > 0$ such that for all $n \in \mathbb{Z}_+$,
\[
P_n(f) \leq \exp(Cn^{1+\delta}). \quad (5)
\]

The Kupka-Smale theorem (see e.g. [PM]) states that for a generic diffeomorphism all periodic points are hyperbolic and all associated stable and unstable manifolds intersect one another transversally. Ref. [K3] shows that the Kupka-Smale theorem also holds on a prevalent set. So, the Kupka-Smale theorem, in particular, says that a Baire generic (resp. prevalent) diffeomorphism has only hyperbolic periodic points, but how hyperbolic are the periodic points, as function of their period, for a Baire generic (resp. prevalent) diffeomorphism $f$? This is the second main problem we deal with in this paper.

Recall that a linear operator $L : \mathbb{R}^N \to \mathbb{R}^N$ is hyperbolic if it has no eigenvalues on the unit circle $\{|z| = 1\} \subset \mathbb{C}$. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{C}^N$. Then we define the hyperbolicity of a linear operator $L$ by
\[
\gamma(L) = \inf_{\phi \in [0,1]} \inf_{|v|=1} |Lv - \exp(2\pi i \phi)v|.
\]

We also say that $L$ is $\gamma$-hyperbolic if $\gamma(L) \geq \gamma$. In particular, if $L$ is $\gamma$-hyperbolic, then its eigenvalues $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ are at least $\gamma$-distant from the unit circle, i.e. $\min_j |\lambda_j| - 1| \geq \gamma$.

The hyperbolicity of a periodic point $x = f^n(x)$ of period $n$, denoted by $\gamma_n(x, f)$, equals the hyperbolicity of the derivative $df^n(x)$ of $f^n$ at points $x$, i.e. $\gamma_n(x, f) = \gamma(df^n(x))$. Similarly to the number of periodic points $P_n(f)$ of period $n$, define
\[
\gamma_n(f) = \min_{\{x: x = f^n(x)\}} \gamma_n(x, f). \quad (7)
\]

The idea of Gromov [G] and Yomdin [Y] of measuring hyperbolicity is that a $\gamma$-hyperbolic point of period $n$ of a $C^2$ diffeomorphism $f$ has an $M_2^{-2n}\gamma$-neighborhood (where $M_2 = ||f||_{C^2}$) free from periodic points of the same period. One can prove the following slightly more general result.

**Proposition 1.1.** Let $M$ be a compact manifold of dimension $N$, let $f : M \to M$ be a $C^{1+\rho}$ diffeomorphism (where $0 < \rho \leq 1$) that has only hyperbolic periodic points, and let $M_{1+\rho} = \max(||f||_{C^{1+\rho}}, 2^{1/\rho})$. Then there is a constant $C = C(M) > 0$ such that for each $n \in \mathbb{Z}_+$ we have
\[
P_n(f) \leq C (M_{1+\rho})^{nN(1+\rho)/\rho} \gamma_n(f)^{-N/\rho}. \quad (8)
\]

Proposition 1.1 implies that a lower estimate on the decay of hyperbolicity $\gamma_n(f)$ gives an upper estimate on the number of periodic points $P_n(f)$. Therefore, a natural question is:

**Question 2.** How quickly can $\gamma_n(f)$ decay with $n$ for a “generic” $C^r$ diffeomorphism $f$?
The existence of lower bound on a rate of decay of $\gamma_n(f)$ for Baire generic $f \in \text{Diff}^r(M)$ would imply the existence of an upper bound on a rate of growth of the number of periodic points $P_n(f)$, whereas no such bound exists by Theorem [3]. Thus we consider genericity in the measure-theoretic sense of prevalence. Our second main result, which in view of Proposition [1] implies the first main result, is that for a prevalent diffeomorphism $f \in \text{Diff}^r(M)$, with $1 < r \leq \infty$, and all $\delta > 0$ there exists $C = C(\delta) > 0$ such that

$$\gamma_n(f) \geq \exp(-Cn^{1+\delta}).$$

Now we shall discuss in more detail our definition of prevalence (“probability one”) in the space of diffeomorphisms $\text{Diff}^r(M)$.

2 Prevalence in the Space of Diffeomorphisms $\text{Diff}^r(M)$

The space of $C^r$ diffeomorphisms $\text{Diff}^r(M)$ of a compact manifold $M$ is a Banach manifold. Locally we can identify it with a Banach space, which gives it a local linear structure in the sense that we can perturb a diffeomorphism by “adding” small elements of the Banach space. As we described in the previous section, the notion of prevalence requires us to make additive perturbations with respect to a probability measure that is independent of the place that we make the perturbation. Thus although there is not a unique way to put a linear structure on $\text{Diff}^r(M)$, it is important to make a choice that is consistent throughout the Banach manifold.

The way we make perturbations on $\text{Diff}^r(M)$ by small elements of a Banach space is as follows. First we embed $M$ into the interior of the closed unit ball $B^N \subset \mathbb{R}^N$, which we can do for $N$ sufficiently large by the Whitney Embedding Theorem [W]. We emphasize that our results hold for every possible choice of an embedding of $M$ into $\mathbb{R}^N$. We then consider a closed tube neighborhood $U \subset B^N$ of $M$ and the Banach space $C^r(U, \mathbb{R}^N)$ of $C^r$ functions from $U$ to $\mathbb{R}^N$. Next, we extend every element $f \in \text{Diff}^r(M)$ to an element $F \in C^r(U, \mathbb{R}^N)$ that is strongly contracting in the directions transverse to $M$. Again the particular choice of how we make this extension is not important to our results; in the Appendix we describe the conditions we need to ensure that the results of Sacker [Sac] and Fenichel [F] apply as follows. Since $F$ has $M$ as an invariant manifold, if we add to $F$ a small perturbation in $g \in C^r(U, \mathbb{R}^N)$, the perturbed map $F + g$ has an invariant manifold in $U$ that is close to $M$. Then $F + g$ restricted to its invariant manifold corresponds in a natural way to an element of $\text{Diff}^r(M)$, which we consider to be the perturbation of $f \in \text{Diff}^r(M)$ by $g \in C^r(U, \mathbb{R}^N)$. The details of this construction are described in the Appendix.

In this way we reduce the problem to the study of maps in $\text{Diff}^r(U)$, the open subset of $C^r(U, \mathbb{R}^N)$ consisting of those elements that are diffeomorphisms from $U$ to some subset of its interior. The construction we described in the previous paragraph ensures that the number of periodic points $P_n(f)$ and their hyperbolicity $\gamma_n(f)$ for elements of $\text{Diff}^r(M)$ are the same for the corresponding elements of $\text{Diff}^r(U)$, so the bounds that we prove on these quantities for almost every perturbation of any element of $\text{Diff}^r(U)$ hold as well for almost every perturbation of any element of $\text{Diff}^r(M)$. Another justification for considering diffeomorphisms in Euclidean space is that the problem of exponential/superexponential growth of the number of periodic points $P_n(f)$ for a prevalent $f \in \text{Diff}^r(M)$ is a local problem on $M$ and is not affected by a global shape of $M$.

The results stated in the next section apply to any compact domain $U \subset \mathbb{R}^N$, but for simplicity we state them for the closed unit ball $B^N$. In the previous section, we said that a property is prevalent on a Banach space such as $C^r(B^N, \mathbb{R}^N)$ if it holds on a Borel subset $S$ for which there
exists a Borel probability measure $\mu$ on $C^r(B^N, \mathbb{R}^N)$ with compact support such that for all $F, g \in C^r(B^N, \mathbb{R}^N)$ we have $F + g \in S$ for almost every $g$ with respect to $\mu$. The complement of a prevalent set is said to be shy. We then say that a property is prevalent on an open subset of $C^r(B^N, \mathbb{R}^N)$ such as $\text{Diff}^r(B^N)$ if the exceptions to the property in $\text{Diff}^r(B^N)$ form a shy subset of $C^r(B^N, \mathbb{R}^N)$.

In this paper the perturbation measure $\mu$ that we use is supported within the analytic functions in $C^r(B^N, \mathbb{R}^N)$. In this sense we foliate $\text{Diff}^r(B^N)$ by analytic leaves that are compact and overlapping. The main result then says that for every analytic leaf $L \subset \text{Diff}^r(B^N)$ and every $\delta > 0$, for almost every diffeomorphism $f \in L$ in the leaf $L$ both (2) and (3) are satisfied. Now we define an analytic leaf as a “Hilbert brick” in the space of analytic functions, and a natural Lebesgue product probability measure $\mu$ on it.

3 Formulation of Main Results

Fix a coordinate system $x = (x_1, \ldots, x_N) \in \mathbb{R}^N \supset B^N$ and the scalar product $\langle x, y \rangle = \sum_i x_i y_i$. Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a multiindex from $\mathbb{Z}_+^N$, and let $|\alpha| = \sum_i \alpha_i$. For a point $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ we write $x^\alpha = \prod_{i=1}^N x_i^{\alpha_i}$. Associate to a real analytic function $\phi : B^N \to \mathbb{R}^N$ the set of coefficients of its expansion:

$$\phi_{\alpha}(x) = \sum_{\alpha \in \mathbb{Z}_+^N} \hat{\phi}_\alpha x^\alpha. \quad (10)$$

Denote by $W_{k,N}$ the space of $N$-component homogeneous vector-polynomials of degree $k$ in $N$ variables and by $\nu(k, N) = \dim W_{k,N}$ the dimension of $W_{k,N}$. According to the notation of the expansion (11), denote coordinates in $W_{k,N}$ by

$$\hat{\phi}_k = \{\{\hat{\phi}_\alpha\}_{|\alpha|=k}\} \in W_{k,N}. \quad (11)$$

In $W_{k,N}$ we use a scalar product that is invariant with respect to orthogonal transformation of $\mathbb{R}^N \supset B^N$, defined as follows:

$$\langle \hat{\phi}_k, \hat{\psi}_k \rangle_k = \sum_{|\alpha|=k} \binom{k}{\alpha}^{-1} \langle \hat{\phi}_\alpha, \hat{\psi}_\alpha \rangle, \quad ||\hat{\phi}_k||_k = (\langle \hat{\phi}_k, \hat{\phi}_k \rangle_k)^{1/2}. \quad (12)$$

Denote by

$$B^N_k(r) = \{\hat{\phi}_k \in W_{k,N} : ||\hat{\phi}_k||_k \leq r\} \quad (13)$$

the closed $r$-ball in $W_{k,N}$ centered at the origin. Let $\text{Leb}_{k,N}$ be Lebesgue measure on $W_{k,N}$ induced by the scalar product (13) and normalized by a constant so that the volume of the unit ball is one: $\text{Leb}_{k,N}(B^N_k(1)) = 1$.

Fix a nonincreasing sequence of positive numbers $r = (\{r_k\}_{k=0}^\infty)$ such that $r_k \to 0$ as $k \to \infty$ and define a Hilbert brick of size $r$

$$\text{HB}^N(r) = \{\hat{\phi} = \{\hat{\phi}_\alpha\}_{\alpha \in \mathbb{Z}_+^N} : \text{ for all } k \in \mathbb{Z}_+, ||\hat{\phi}_k||_k \leq r_k\}$$

$$= B^N_0(r_0) \times B^N_1(r_1) \times \cdots \times B^N_k(r_k) \times \cdots \subset W_{0,N} \times W_{1,N} \times \cdots \times W_{k,N} \times \cdots \quad (14)$$

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Define a product Lebesgue probability measure $\mu^N_r$ associated to the Hilbert brick $HBN(r)$ of size $r$ by normalizing for each $k \in \mathbb{Z}_+$ the corresponding Lebesgue measure $Leb_{k,N}$ on $W_{k,N}$ to the Lebesgue probability measure on the $r_k$-ball $B_k^N(r_k)$:

$$\mu^N_{k,r} = r^{-\nu(k,N)} \ Leb_{k,N} \quad \text{and} \quad \mu^N_r = \times_{k=0}^\infty \mu^N_{k,r_k}.$$  \hspace{1cm} (15)

**Definition 3.1.** Let $f \in \text{Diff}^r(B^N)$ be a $C^r$ diffeomorphism of $B^N$ into its interior. We call $HBN(r)$ a Hilbert brick of an admissible size $r = (\{r_k\}_{k=0}^\infty)$ with respect to $f$ if:

A) for each $\varepsilon \in HBN(r)$, the corresponding function $\phi_\varepsilon(x) = \sum_{\alpha \in \mathbb{Z}_+^N} \varepsilon_\alpha x_\alpha$ is analytic on $B^N$;

B) for each $\varepsilon \in HBN(r)$, the corresponding map $f_\varepsilon(x) = f(x) + \phi_\varepsilon(x)$ is a diffeomorphism from $B^N$ into its interior, i.e. $\{f_\varepsilon\}_{\varepsilon \in HBN(r)} \subset \text{Diff}^r(B^N)$;

C) for all $\delta > 0$ and all $C > 0$, the sequence $r_k \exp(Ck^{1+\delta}) \to \infty$ as $k \to \infty$.

**Remark 3.1.** The first and second conditions ensure that the family $\{f_\varepsilon\}_{\varepsilon \in HBN(r)}$ lie in an analytic leaf within the class of diffeomorphisms $\text{Diff}^r(B^N)$. The third condition provides us enough freedom to perturb. It is important for our method to have infinitely many parameters to perturb. If $r_k$’s decayed too fast to zero it would make our family of perturbations essentially finite-dimensional.

An example of an admissible sequence $r = (\{r_k\}_{k=0}^\infty)$ is $r_k = \tau/k!$, where $\tau$ is the corresponding Lebesgue product probability measure $\mu^N_r$ on $W_{k,N}$ with respect to $f$ and is chosen sufficiently small to ensure that condition (B) holds. Notice that the diameter of $HBN(r)$ is then proportional to $\tau$, so that $\tau$ can be chosen as some multiple of the distance from $f$ to the boundary of $\text{Diff}^r(B^N)$.

**Main Theorem.** For any $0 < \rho \leq \infty$ and any $C^{1+\rho}$ diffeomorphism $f \in \text{Diff}^{1+\rho}(B^N)$, consider a Hilbert brick $HBN(r)$ of an admissible size $r$ with respect to $f$ and the family of analytic perturbations of $f$

$$\{f_\varepsilon(x) = f(x) + \phi_\varepsilon(x)\}_{\varepsilon \in HBN(r)}.$$  \hspace{1cm} (16)

with the Lebesgue product probability measure $\mu^N_r$ associated to $HBN(r)$. Then for every $\delta > 0$ and for $\mu^N_r$-a.e. $\varepsilon$ there is $C = C(\varepsilon, \delta) > 0$ such that for all $n \in \mathbb{Z}_+$

$$\gamma_n(f_\varepsilon) > \exp(-Cn^{1+\delta}) \quad P_n(f_\varepsilon) < \exp(Cn^{1+\delta}).$$  \hspace{1cm} (17)

**Remark 3.2.** The fact that the measure $\mu^N_r$ depends on $f$ does not conform to our definition of prevalence. However, we can decompose $\text{Diff}^r(B^N)$ into a nested countable union of sets $\mathcal{S}_j$ that are each a positive distance from the boundary of $\text{Diff}^r(B^N)$ and for each $j \in \mathbb{Z}_+$ choose an admissible sequence $r_j$ that is valid for all $f \in \mathcal{S}_j$. Since a countable intersection of prevalent subsets of a Banach space is prevalent $[HS]$, the Main Theorem implies the results stated in terms of prevalence in the introduction.

In the Appendix we deduce from the Main Theorem the following result.

**Theorem 3.1.** Let $\{f_\varepsilon\}_{\varepsilon \in B^m} \subset \text{Diff}^{1+\rho}(M)$ be a generic $m$-parameter family of $C^{1+\rho}$ diffeomorphisms of a compact manifold $M$ for some $\rho > 0$. Then for every $\delta > 0$ and a.e. $\varepsilon \in B^m$ there is a constant $C = C(\varepsilon, \delta)$ such that $[12]$ is satisfied for every $n \in \mathbb{Z}_+$. 6
In the Appendix we also give a precise meaning to the term *generic*.

Let us formulate the most general result we shall prove.

**Definition 3.2.** Let $\gamma \geq 0$ and $f \in \text{Diff}^{1+\rho}(B^N)$ be a $C^{1+\rho}$ diffeomorphism for some $\rho > 0$. A point $x \in B^N$ is called $(n, \gamma)$-periodic if $\|f^n(x) - x\| \leq \gamma$ and $(n, \gamma)$-hyperbolic if $\gamma_n(x, f) = \gamma(d^f_n(x)) \geq \gamma$.

(Notice that a point can be $(n, \gamma)$-hyperbolic regardless of its periodicity, but this property is of interest primarily for $(n, \gamma)$-periodic points.)

For positive $C$ and $\delta$ let $\gamma_n(C, \delta) = \exp(-Cn^{1+\delta})$.

**Theorem 3.2.** Given the hypotheses of the Main Theorem, for every $\delta > 0$ and for $\mu^N_{\varepsilon}$-a.e. $\varepsilon$ there is $C = C(\varepsilon, \delta) > 0$ such that for all $n \in \mathbb{Z}^+$, every $(n, \gamma_n^{1/\rho}(C, \delta))$-periodic point $x \in B^N$ is $(n, \gamma_n(C, \delta))$-hyperbolic. (Here we assume $0 < \rho \leq 1$; in a space $\text{Diff}^{1+\rho}(B^N)$ with $\rho > 1$, the statement holds with $\rho$ replaced by 1.)

This result together with Proposition 1.1 implies the Main Theorem, because every periodic point of period $n$ is $(n, \gamma)$-periodic for all $\gamma > 0$.

**Remark 3.3.** In the statement of the Main Theorem and Theorem 3.2 the unit ball $B^N$ can be replaced by a bounded open set $U \subset \mathbb{R}^N$. After scaling, $U$ can be considered as a subset of the unit ball $B^N$.

One can define a distance on a compact manifold $M$ and almost periodic points of diffeomorphisms of $M$. Then one can cover $M = \bigcup_i U_i$ by coordinate charts and define hyperbolicity for almost periodic points using these charts $\{U_i\}$ (see [1] for details). This gives a precise meaning to the following result.

**Theorem 3.3.** Let $\{f_{\varepsilon}\}_{\varepsilon \in B^m} \subset \text{Diff}^{1+\rho}(M)$ be a generic $m$-parameter family of diffeomorphisms of a compact manifold $M$ for some $\rho > 0$. Then for every $\delta > 0$ and almost every $\varepsilon \in B^m$ there is a constant $C = C(\varepsilon, \delta)$ such that every $(n, \gamma_n^{1/\rho}(C, \delta))$-periodic point $x \in B^N$ is $(n, \gamma_n(C, \delta))$-hyperbolic. (Here again we assume $0 < \rho \leq 1$, replacing $\rho$ with 1 in the conclusion if $\rho > 1$.)

The meaning of the term generic is the same as in Theorem 3.1 and is discussed in the Appendix.

## 4 Formulation of the main result in the 1-dimensional case

The proof of the main result about estimating the rate of growth of the number of periodic points for diffeomorphisms in $N$ dimensions has a lot of complications related to multidimensionality. To describe a model which is from one side nontrivial and from another side is useful for understanding the general technique we apply our method for the 1-dimensional maps. The statement of the main result for the 1-dimensional maps has another important feature: it explains the statement of the main multidimensional result.

Fix the interval $I = [-1, 1]$. Associate to a real analytic function $\phi : I \to \mathbb{R}$ the set of coefficients of its expansion

$$\phi_{\varepsilon}(x) = \sum_{k=0}^{\infty} \varepsilon_k x^k.$$  \hspace{1cm} (18)
For a nonincreasing sequence of positive numbers $r = \{r_k\}_{k=0}^{\infty}$ such that $r_k \to 0$ as $k \to \infty$ following the multidimensional notations we define a Hilbert brick of size $r$

$$HB^1(r) = \{\varepsilon = \{\varepsilon_k\}_{k=0}^{\infty} : \text{ for all } k \in \mathbb{Z}_+, \ |\varepsilon_k| \leq r_k\}$$

and the product probability measure $\mu^1_r$ associated to the Hilbert brick $HB^1(r)$ of size $r$ which considers each $\varepsilon_k$ to be an independent random variable uniformly distributed on $[-r_k, r_k]$.

**Main 1-dimensional Theorem.** For any $0 < \rho \leq \infty$ and any $C^{1+\rho}$ map $f : I \to I$ of the interval $I = [-1, 1]$ consider a Hilbert brick $HB^1(r)$ of an admissible size $r$ with respect to $f$ and the family of analytic perturbations of $f$

$$\{f_\varepsilon(x) = f(x) + \phi_\varepsilon(x)\}_{\varepsilon \in HB^1(r)}$$

with the Lebesgue product probability measure $\mu^1_r$ associated to $HB^1(r)$. Then for every $\delta > 0$ and $\mu^1_r$-a.e. $\varepsilon$ there is $C = C(\varepsilon, \delta) > 0$ such that for all $n \in \mathbb{Z}_+$

$$\gamma_n(f_\varepsilon) > \exp(-Cn^{1+\delta}), \quad P_n(f_\varepsilon) < \exp(Cn^{1+\delta}).$$

In [MMS] Martens-de Melo-Van Strien prove in a sense a stronger statement for $C^2$ maps. They show that for any $C^2$ map $f$ of an interval without “flat” critical points there are some $\gamma > 0$ and $n_0 \in \mathbb{Z}_+$ such that for any $n > n_0$ we have $\gamma_n(f) > 1 + \gamma$. This also implies that the number of periodic points is bounded by an exponential function of the period. The notion of a flat critical point used in [MMS] is a nonstandard one from a point of view of singularity theory. For $C^2$ maps, they call $x_0$ a flat critical point of $f$ if $f'(x_0) = f''(x_0) = 0$; the distance from $f(x)$ to $f(x_0)$ does not have to decay to 0 as $x \to x_0$ faster than any power of $x - x_0$.

In [KK] an example of a $C^2$-unimodal map with a critical point having tangency of order 4 and an arbitrarily fast rate of growth of the number of periodic points is given. Another advantage of the Main 1-dimensional Theorem is that it works for $C^{1+\rho}$ maps with $0 < \rho < 1$, whereas the result in [MMS] works only for $C^2$ maps.

## 5 Strategy of the proof

Here we describe the strategy of the proof of Theorem 3.2. The basic technique is developed and many of the technical difficulties are resolved by the first author in [K4]. The general idea is to fix $C > 0$ and prove an upper bound on the measure of the set of “bad” parameter values $\varepsilon \in HB^N(r)$ for which the conclusion of the theorem does not hold. The upper bound we obtain will approach zero as $C \to \infty$, from which it follows immediately that the set of $\varepsilon \in HB^N(r)$ that are “bad” for all $C > 0$ has measure zero. For a given $C > 0$, we bound the measure of “bad” parameter values inductively as follows.

**Stage 1.** We delete all parameter values $\varepsilon \in HB^N(r)$ for which the corresponding diffeomorphism $f_\varepsilon$ has an almost fixed point that is not sufficiently hyperbolic, and bound the measure of the deleted set.

**Stage 2.** After Stage 1, we consider only parameter values for which all almost fixed points are sufficiently hyperbolic. Then we delete all parameter values $\varepsilon$ for which $f_\varepsilon$ has an almost periodic point of period 2 which is not sufficiently hyperbolic and bound the measure of that set.

**Stage n.** We consider only parameter values for which all almost periodic point of period at most $n - 1$ are sufficiently hyperbolic (we shall call this the Inductive Hypothesis). Then we
Definition 5.1. A diffeomorphism $f \in \text{Diff}^{1+\rho}(B^N)$ satisfies the Inductive Hypothesis of order $n$ with constants $(C, \delta, \rho)$, denoted $f \in IH(n, C, \delta, \rho)$, if for all $k \leq n$, every $(k, \gamma_k^{1/\rho}(C, \delta))$-periodic point is $(k, \gamma_k(C, \delta))$-hyberbolic.

For $f \in \text{Diff}^{1+\rho}(M)$, consider the sequence of sets in the parameter space $HB^N(r)$

$$B_n(C, \delta, \rho, r, f) = \{\varepsilon \in HB^N(r) : f_{\varepsilon} \in IH(n-1, C, \delta, \rho) \text{ but } f_{\varepsilon} \notin IH(n, C, \delta, \rho)\} \quad (22)$$

In other words, $B_n(C, \delta, \rho, r, f)$ is the set of “bad” parameter values $\varepsilon \in HB^N(r)$ for which all almost periodic points of $f_{\varepsilon}$ with period strictly less than $n$ are sufficiently hyperbolic, but there is an almost periodic point of period $n$ that is not sufficiently hyperbolic. Let

$$M_1 = \sup_{\varepsilon \in HB^N(r)} \max\{\|f_{\varepsilon}\|C, \|f_{\varepsilon}^{-1}\|C\}; \quad M_{1+\rho} = \sup_{\varepsilon \in HB^N(r)} \max\{\|f_{\varepsilon}\|C^{1+\rho}, M_1, 2^{1/\rho}\}. \quad (23)$$

Our goal is to find an upper bound $\mu_n(C, \delta, \rho, r, M_{1+\rho})$ for the measure $\mu^N_r(B_n(C, \delta, \rho, r, f))$ of the set of “bad” parameter values. Then $\sum_{n=1}^{\infty} \mu_n(C, \delta, \rho, r, M_{1+\rho})$ is an upper bound on the measure of $\bigcup_{n=1}^{\infty} B_n(C, \delta, \rho, r, f)$, which is the set of all parameter values $\varepsilon$ for which $f_{\varepsilon}$ has for some $n$ an $(n, \gamma_n^{1/\rho}(C, \delta))$-periodic point that is not $(n, \gamma_n(C, \delta))$-hyperbolic. If this sum converges and

$$\sum_{n=1}^{\infty} \mu_n(C, \delta, \rho, r, M_{1+\rho}) = \mu(C, \delta, \rho, r, M_{1+\rho}) \to 0 \text{ as } C \to \infty \quad (24)$$

for every positive $\rho$, $\delta$, and $M_{1+\rho}$, then Theorem 3.2 follows. In the remainder of this announcement we describe the key construction we use to obtain a bound $\mu_n(C, \delta, \rho, r, M_{1+\rho})$ that meets condition (24).

6 Perturbation of recurrent trajectories by Lagrange interpolation polynomials

The approach we take to estimate the measure of “bad” parameter values in the space of perturbations $HB^N(r)$ is to choose a coordinate system for this space and for a finite subset of the coordinates to estimate the amount that we must change a particular coordinate to make a “bad” parameter value “good”. Actually we will choose a coordinate system that depends on a particular point $x_0 \in B^N$, the idea being to use this coordinate system to estimate the measure of “bad” parameter values corresponding to initial conditions in some neighborhood of $x_0$, then cover $B^N$ with a finite number of such neighborhoods and sum the corresponding estimates. For a particular set of initial conditions, a diffeomorphism will be “good” if every point in the set is either sufficiently nonperiodic or sufficiently hyperbolic.
In order to keep the notations and formulas simple as we formalize this approach, we consider the case of 1-dimensional maps, but the reader should always have in mind that our approach is designed for multidimensional diffeomorphisms. Let \( f : I \to I \) be a \( C^1 \) map on the interval \( I = [-1, 1] \). Recall that a trajectory \( \{ x_k \}_{k \in \mathbb{Z}} \) of \( f \) is called recurrent if it returns arbitrarily close to its initial position — that is, for all \( \gamma > 0 \) we have \( |x_0 - x_n| < \gamma \) for some \( n > 0 \). A very basic question is how much one should perturb \( f \) to make \( x_0 \) periodic. Here is an elementary Closing Lemma that gives a simple partial answer to this question.

**Closing Lemma.** Let \( \{ x_k = f^k(x_0) \}_{k=0}^{n-1} \) be a trajectory of length \( n + 1 \) of a map \( f : I \to I \). Let \( u = (x_0 - x_n)/\prod_{k=0}^{n-2}(x_{n-1} - x_k) \). Then \( x_0 \) is a periodic point of period \( n \) of the map

\[
    f_u(x) = f(x) + u \prod_{k=0}^{n-2} (x - x_k).
\]

Of course \( f_u \) is close to \( f \) if and only if \( u \) is sufficiently small, meaning that \( |x_0 - x_n| \) should be small compared to \( \prod_{k=0}^{n-2} |x_{n-1} - x_k| \). However, this product is likely to contain small factors for recurrent trajectories. In general, it is difficult to control the effect of perturbations for recurrent trajectories. The simple reason why is because one cannot perturb \( f \) at two nearby points independently.

The Closing Lemma above also gives an idea of how much we must change the parameter \( u \) to make a point \( x_0 \) that is \( (n, \gamma) \)-periodic not be \( (n, \gamma) \)-periodic for a given \( \gamma > 0 \), which as we described above is one way to make a map that is “bad” for the initial condition \( x_0 \) become “good”. To make use of our other alternative we must determine how much we need to perturb a map \( f \) to make a given \( x_0 \) be \( (n, \gamma) \)-hyperbolic for some \( \gamma > 0 \).

**Perturbation of hyperbolicity.** Let \( \{ x_k = f^k(x_0) \}_{k=0}^{n-1} \) be a trajectory of length \( n \) of a \( C^1 \) map \( f : I \to I \). Then for the map

\[
    f_v(x) = f(x) + v(x - x_{n-1}) \prod_{k=0}^{n-2} (x - x_k)^2
\]

such that \( v \in \mathbb{R} \) and

\[
    \left| \left| (f_v^n)'(x_0) \right| - 1 \right| = \left| \prod_{k=0}^{n-1} f'(x_k) + v \prod_{k=0}^{n-2} (x_{n-1} - x_k)^2 \prod_{k=0}^{n-2} f'(x_k) - 1 \right| > \gamma
\]

we have that \( x_0 \) is an \( (n, \gamma) \)-hyperbolic point of \( f_v \).

One more time we can see the product of distances \( \prod_{k=0}^{n-2} |x_{n-1} - x_k| \) along the trajectory is important quantitative characteristic of how much freedom we have to perturb.

The perturbations (25) and (26) are reminiscent of Lagrange interpolation polynomials. Let us put these formulas into a general setting using singularity theory.

Given \( n > 0 \) and a \( C^1 \) function \( f : I \to \mathbb{R} \) we define an associated function \( j^{1,n} f : I^n \to I^n \times \mathbb{R}^{2n} \) by

\[
    j^{1,n} f(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{n-1}, f(x_0), \ldots, f(x_{n-1}), f'(x_0), \ldots, f'(x_{n-1})).
\]

In singularity theory this function is called the \( n \)-tuple 1-jet of \( f \). The ordinary 1-jet of \( f \), usually denoted by \( j^1 f(x) = (x, f(x), f'(x)) \), maps \( I \) to the 1-jet space \( J^1(I, \mathbb{R}) \simeq I \times \mathbb{R}^2 \). The product
of $n$ copies of $J^1(I, \mathbb{R})$, called the multijet space, is denoted by

$$J^{1,n}(I, \mathbb{R}) = J^1(I, \mathbb{R}) \times \cdots \times J^1(I, \mathbb{R}), \quad (29)$$

and is equivalent to $I^n \times \mathbb{R}^{2n}$ after rearranging coordinates. The $n$-tuple 1-jet of $f$ associates with each $n$-tuple of points in $I^n$ all the information necessary to determine how close the $n$-tuple is to being a periodic orbit, and if so, how close it is to being nonhyperbolic.

The set

$$\Delta_n(I) = \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^{2n} \subset J^{1,n}(I, \mathbb{R}) : \exists i \neq j \text{ such that } x_i = x_j\} \quad (30)$$

is called the diagonal (or sometimes the generalized diagonal) in the space of multijets. In singularity theory the space of multijets is defined outside of the diagonal $\Delta_n(I)$ and is usually denoted by $\mathcal{J}^1(I, \mathbb{R}) = J^{1,n}(I, \mathbb{R}) \setminus \Delta_n(I)$ (see [GG]). It is easy to see that a recurrent trajectory $\{x_k\}_{k \in \mathbb{Z}^+}$ is located in a neighborhood of the diagonal $\Delta_n(I)$ in the space of multijets for a sufficiently large $n$. If $\{x_k\}_{k=0}^{n-1}$ is a part of a recurrent trajectory of length $n$, then the product of distances along the trajectory

$$\prod_{k=0}^{n-2} |x_{n-1} - x_k| \quad (31)$$

measures how close $\{x_k\}_{k=0}^{n-1}$ is to the diagonal $\Delta_n(I)$, or how independently one can perturb points of a trajectory. One can also say that $\text{(31)}$ is a quantitative characteristic of how recurrent a trajectory of length $n$ is. Introduction of this product of distances along a trajectory into the analysis of recurrent trajectories is a new point of our paper.

7 Lagrange interpolation and blow-up along the diagonal in multijet space

Now we present a construction due to Grigoriev and Yakovenko [GY] which puts the “Closing Lemma” and “Perturbation of Hyperbolicity” statements above into a general framework. It is an interpretation of Lagrange interpolation polynomials as an algebraic blow-up along the diagonal in the multijet space. In order to keep the notations and formulas simple we continue in this section to consider only the 1-dimensional case.

Consider the $2n$-parameter family of perturbations of a $C^1$ map $f : I \to I$ by polynomials of degree $2n - 1$

$$f_\varepsilon(x) = f(x) + \phi_\varepsilon(x), \quad \phi_\varepsilon(x) = \sum_{k=0}^{2n-1} \varepsilon_k x^k, \quad (32)$$

where $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{2n-1}) \in \mathbb{R}^{2n}$. The perturbation vector $\varepsilon$ consists of coordinates from the Hilbert brick $HB^1(\mathbb{R})$ of analytic perturbations defined in Section 3. Our goal now is to describe how such perturbations affect the $n$-tuple 1-jet of $f$, and since the operator $j^{1,n}$ is linear in $f$, for the time being we consider only the perturbations $\phi_\varepsilon$ and their $n$-tuple 1-jets. For each $n$-tuple
\( \{x_k\}_{k=0}^{n-1} \) there is a natural transformation \( J^{1,n} : I^n \times \mathbb{R}^{2n} \to J^{1,n}(I, \mathbb{R}) \) from \( \varepsilon \)-coordinates to jet-coordinates, given by

\[
J^{1,n}(x_0, \ldots, x_{n-1}, \varepsilon) = j^{1,n}_\varepsilon(x_0, \ldots, x_{n-1}).
\]

(33)

Instead of working directly with the transformation \( J^{1,n} \), we introduce intermediate \( u \)-coordinates based on Lagrange interpolation polynomials. The relation between \( \varepsilon \)-coordinates and \( u \)-coordinates is given implicitly by

\[
\phi_\varepsilon(x) = \sum_{k=0}^{2n-1} \varepsilon_k x^k = \sum_{k=0}^{2n-1} u_k \prod_{j=0}^{k-1} (x - x_{j \text{mod } n}).
\]

(34)

Based on this identity, we will define functions \( D^{1,n} : I^n \times \mathbb{R}^{2n} \to I^n \times \mathbb{R}^{2n} \) and \( \pi^{1,n} : I^n \times \mathbb{R}^{2n} \to J^{1,n}(I, \mathbb{R}) \) so that \( J^{1,n} = \pi^{1,n} \circ D^{1,n} \), or in other words the diagram in Figure 1 commutes. We will show later that \( D^{1,n} \) is invertible, while \( \pi^{1,n} \) is invertible away from the diagonal \( \Delta_n(I) \) and defines a blow-up along it in the space of multijets \( J^{1,n}(I, \mathbb{R}) \).

\[ DD^{1,n}(I, \mathbb{R}) = I \times \cdots \times I \times \mathbb{R}^{2n} \]

\[ J^{1,n}(I, \mathbb{R}) = I \times \cdots \times I \times \mathbb{R}^{2n} \]

The intermediate space, which we denote by \( DD^{1,n}(I, \mathbb{R}) \), is called the space of divided differences and consists of \( n \)-tuples of points \( \{x_k\}_{k=0}^{n-1} \) and \( 2n \) real coefficients \( \{u_k\}_{k=0}^{2n-1} \). Here are

Figure 1: an Algebraic Blow-up along the Diagonal \( \Delta_n(I) \)
explicit coordinate-by-coordinate formulas defining $\pi^{1,n} : DD^{1,n}(I, \mathbb{R}) \to J^{1,n}(I, \mathbb{R})$.

$$
\begin{align*}
\phi_\varepsilon(x_0) &= u_0, \\
\phi_\varepsilon(x_1) &= u_0 + u_1(x_1 - x_0), \\
\phi_\varepsilon(x_2) &= u_0 + u_1(x_2 - x_0) + u_2(x_2 - x_0)(x_2 - x_1), \\
& \vdots \\
\phi_\varepsilon(x_{n-1}) &= u_0 + u_1(x_{n-1} - x_0) + \cdots + u_{n-1}(x_{n-1} - x_0)(x_{n-1} - x_{n-2}),
\end{align*}
$$

$$
\phi'_\varepsilon(x_0) = \frac{\partial}{\partial x} \left( \sum_{k=0}^{2n-1} u_k \prod_{j=0}^{k-1} (x - x_{j\text{ (mod } n)}) \right) \bigg|_{x=x_0},
$$

$$
\vdots
$$

$$
\phi'_\varepsilon(x_{n-1}) = \frac{\partial}{\partial x} \left( \sum_{k=0}^{2n-1} u_k \prod_{j=0}^{k-1} (x - x_{j\text{ (mod } n)}) \right) \bigg|_{x=x_{n-1}},
$$

These formulas are very useful for dynamics. For a given base map $f$ and initial point $x_0$, the image $f_\varepsilon(x_0) = f(x_0) + \phi_\varepsilon(x_0)$ of $x_0$ depends only on $u_0$. Furthermore the image can be set to any desired point by choosing $u_0$ appropriately — we say then that it depends nontrivially on $u_0$. If $x_0$, $x_1$, and $u_0$ are fixed, the image $f_\varepsilon(x_1)$ of $x_1$ depends only on $u_1$, and as long as $x_0 \neq x_1$ it depends nontrivially on $u_1$. More generally for $0 \leq k \leq n - 1$, if pairwise distinct points $\{x_j\}_{j=0}^k$ and coefficients $\{u_j\}_{j=0}^{k-1}$ are fixed, then the image $f_\varepsilon(x_k)$ of $x_k$ depends only and nontrivially on $u_k$.

Suppose now that an $n$-tuple of points $\{x_j\}_{j=0}^n$ not on the diagonal $\Delta_n(I)$ and Lagrange coefficients $\{u_j\}_{j=0}^{n-1}$ are fixed. Then derivative $f'_\varepsilon(x_0)$ at $x_0$ depends only and nontrivially on $u_n$. Likewise for $0 \leq k \leq n - 1$, if distinct points $\{x_j\}_{j=0}^n$ and Lagrange coefficients $\{u_j\}_{j=0}^{n+k-1}$ are fixed, then the derivative $f'_\varepsilon(x_k)$ at $x_k$ depends only and nontrivially on $u_{n+k}$.

As Figure 2 illustrates, these considerations show that for any map $f$ and any desired trajectory of distinct points with any given derivatives along it, one can choose Lagrange coefficients $\{u_k\}_{k=0}^{2n-1}$ and explicitly construct a map $f_\varepsilon = f + \phi_\varepsilon$ with such a trajectory. Thus we have shown that $\pi^{1,n}$ is invertible away from the diagonal $\Delta_n(I)$ and defines a blow-up along it in the space of multijets $J^{1,n}(I, \mathbb{R})$.

Next we define the function $D^{1,n} : I^n \times \mathbb{R}^{2n} \to DD^{1,n}(I, \mathbb{R})$ explicitly using so-called divided differences. Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^r$ function of one real variable.

**Definition 7.1.** The first order divided difference of $g$ is defined as

$$
\Delta g(x_0, x_1) = \frac{g(x_1) - g(x_0)}{x_1 - x_0}
$$

for $x_1 \neq x_0$ and extended by its limit value as $g'(x_0)$ for $x_1 = x_0$. Iterating this construction we define divided differences of the $m$-th order for $2 \leq m \leq r$,

$$
\Delta^m g(x_0, \ldots, x_m) = \frac{\Delta^{m-1} g(x_0, \ldots, x_{m-2}, x_m) - \Delta^{m-1} g(x_0, \ldots, x_{m-2}, x_{m-1})}{x_m - x_{m-1}}
$$

for $x_{m-1} \neq x_m$ and extended by its limit value for $x_{m-1} = x_m$. 

13
A function loses at most one derivative of smoothness with each application of $\Delta$, so $\Delta^m g$ is at least $C^{r-m}$ if $g$ is $C^r$. Notice that $\Delta^m$ is linear as a function of $g$, and one can show that it is a symmetric function of $x_0, \ldots, x_m$; in fact, by induction it follows that

$$\Delta^m g(x_0, \ldots, x_m) = \sum_{i=0}^{m} g(x_i) \prod_{j \neq i} (x_i - x_j)$$

(38)

Another identity that is proved by induction will be more important for us, namely

$$\Delta^m x^k(x_0, \ldots, x_m) = p_{k,m}(x_0, \ldots, x_m),$$

(39)

where $p_{k,m}(x_0, \ldots, x_m)$ is 0 for $m > k$ and for $m \leq k$ is the sum of all degree $k - m$ monomials in $x_0, \ldots, x_m$ with unit coefficients,

$$p_{k,m}(x_0, \ldots, x_m) = \sum_{r_0 + \cdots + r_m = k-m} \prod_{j=0}^{m} x_j^{r_j}.$$  

(40)

The divided differences form coefficients for the Lagrange interpolation formula. For all $C^\infty$ functions $g : \mathbb{R} \to \mathbb{R}$ we have

$$g(x) = \Delta^0 g(x_0) + \Delta^1 g(x_0, x_1)(x - x_0) + \ldots + \Delta^{n-1} g(x_0, \ldots, x_{n-1})(x - x_0) \ldots (x - x_{n-2}) + \Delta^n g(x_0, \ldots, x_{n-1}, x)(x - x_0) \ldots (x - x_{n-1})$$

(41)

identically for all values of $x, x_0, \ldots, x_{n-1}$. All terms of this representation are polynomial in $x$ except for the last one which we view as a remainder term. The sum of the polynomial terms is
the degree \((n - 1)\) Lagrange interpolation polynomial for \(g\) at \(\{x_k\}_{k=0}^{n-1}\). To obtain a degree \(2n - 1\) interpolation polynomial for \(g\) and its derivative at \(\{x_k\}_{k=0}^{n-1}\), we simply use (41) with \(n\) replaced by \(2n\) and the \(2n\)-tuple of points \(\{x_k(\text{mod } n)\}_{k=0}^{2n-1}\).

Recall that \(D^{1,n}\) was defined implicitly by (34). We have described how to use divided differences to construct a degree \(2n - 1\) interpolating polynomial of the form on the right-hand side of (34) for an arbitrary \(C^\infty\) function \(g\). Our interest then is in the case \(g = \phi_\varepsilon\), which as a degree \(2n - 1\) polynomial itself will have no remainder term and coincide exactly with the interpolating polynomial. Thus \(D^{1,n}\) is given coordinate-by-coordinate by

\[
\begin{align*}
u_m &= \Delta^m \left( \sum_{k=0}^{2n-1} \varepsilon_k x^k \right) (x_0, \ldots, x_m \text{ (mod } n)) \\
&= \varepsilon_m + \sum_{k=m+1}^{2n-1} \varepsilon_k p_{k,m}(x_0, \ldots, x_m \text{ (mod } n))
\end{align*}
\]

for \(m = 0, \ldots, 2n - 1\). We call the transformation given by (42) the Lagrange map. Notice that for fixed \(\{x_k\}_{k=0}^{2n-1}\), the Lagrange map is linear and given by an upper triangular matrix with units on the diagonal. Hence it is Lebesgue volume-preserving and invertible, whether or not \(\{x_k\}_{k=0}^{2n-1}\) lies on the diagonal \(\Delta_n(I)\).

We call the basis of monomials

\[
\prod_{j=0}^{k} (x - x_j(\text{mod } n)) \quad \text{for} \quad k = 0, \ldots, 2n - 1
\]

in the space of polynomials of degree \(2n - 1\) the Lagrange basis defined by the \(n\)-tuple \(\{x_k\}_{k=0}^{n-1}\). The Lagrange map and the Lagrange basis, and their analogues in dimension \(N\), are useful tools for perturbing trajectories and estimating the measure \(\mu_n(C, \delta, \rho, M_1 + \rho)\) of “bad” parameter values \(\varepsilon \in HB^N(r)\).

8 Discretization method

The fundamental problem with using the Lagrange basis to estimate the measure of “bad” parameter values, those for which there is an almost periodic point of period \(n\) that is not sufficiently hyperbolic, is that the Lagrange basis depends on the almost periodic \(n\)-tuple \(\{x_k\}_{k=0}^{n-1}\). For a particular “bad” parameter value we can fix this \(n\)-tuple and the corresponding Lagrange basis, then estimate the measure of the set of parameters for which a nearby \(n\)-tuple is both almost periodic and not sufficiently hyperbolic. But there are a continuum of possible \(n\)-tuples, so how can we account for all of the possible cells of “bad” parameter values \(\varepsilon\) within our parameter brick \(HB^N(r)\)? At the beginning of Section \(\text{3}\), we indicated that for a particular initial condition \(x_0\) we would obtain an estimate on the measure of “bad” parameter values corresponding to an almost periodic point in a neighborhood of \(x_0\), and thus need only to consider a discrete set of initial conditions. But as the parameter vector \(\varepsilon\) varies over \(HB^N(r)\), there is (for large \(n\) at least) a wide range of possible length-\(n\) trajectories starting from a particular \(x_0\), so there is no hope of using a single Lagrange basis to estimate even the measure of “bad” parameter values corresponding to a single \(x_0\).

The solution to this problem is to discretize the entire space of \(n\)-tuples \(\{x_k\}_{k=0}^{n-1}\), considering only those that lie on a particular grid. If we choose the grid spacing small enough, then every
almost periodic orbit of period \( n \) that is not sufficiently hyperbolic will have a corresponding pseudotrajectory of length \( n \) on the grid that also has small hyperbolicity. In this way we reduce the problem to bounding the measure of a set of “bad” parameter values corresponding to a particular length \( n \) pseudotrajectory, and then summing the bounds over all possible length \( n \) pseudotrajectories on the chosen grid.

Returning to the general case of \( C^{1+\varepsilon} \) diffeomorphisms on \( B^n \), where we assume \( 0 < \rho \leq 1 \), the grid spacing we use at stage \( n \) is \( \hat{\gamma}_n(C, \delta, \rho) = N^{-1}(M_{1+\rho}^{-2n}\gamma_n(C, \delta))^{1/\rho} \), where \( M_{1+\rho} > 1 \) is a bound on the \( C^{1+\varepsilon} \) norm of the diffeomorphisms \( f_\varepsilon \) corresponding to parameters \( \varepsilon \in HB^N(r) \). This ensures that when rounded off to the nearest grid points \( \{x_k\}_{k=0}^{n-1} \), an almost periodic orbit of length \( n \) becomes an \( N(M_{1+\rho}+1)\gamma_n(C, \delta, \rho) \)-pseudotrajectory, meaning that \( |f_\varepsilon(x_j) - x_{j+1}| \leq N(M_{1+\rho}+1)\gamma_n(C, \delta, \rho) \) for \( j = 0, 1, \ldots, n-2 \). It also ensures that when rounding, the derivative \( df_\varepsilon \) changes by at most \( M_{1+\rho}^{-2n}\gamma_n(C, \delta) \), which in turn implies that the change in hyperbolicity over all \( n \) points is small compared with \( \gamma_n(C, \delta) \). (Recall that \( \gamma_n(C, \delta) \) is our tolerance for hyperbolicity at stage \( n \).

Roughly speaking, in the case \( N = 1 \) our estimate on the measure of “bad” parameter values for a particular \( n \)-tuple \( \{x_k\}_{k=0}^{n-1} \) is then proportional to \( \gamma_n(C, \delta, \rho) \gamma_n^2(C, \delta) \), whereas the number of possible \( n \)-tuples is proportional to \( \gamma_n(C, \delta, \rho) \gamma_n^2(C, \delta) \), making our bound \( \mu_n(C, \delta, \rho, r, M_{1+\rho}) \) on the total measure of “bad” parameter values at stage \( N \) proportional to \( \gamma_n^2(C, \delta) \). The remaining problem then is to show that for maps satisfying the Inductive Hypothesis of order \( n - 1 \), we can bound the proportionality factor in such a way that \( \mu_n(C, \delta, \rho, r, M_{1+\rho}) \) meets the conditions prescribed in Section 4, namely that it be summable over \( n \) and that the sum approaches 0 as \( C \to \infty \). (Notice that the sequence \( \gamma_n(C, \delta) \) meets these conditions.) The proportionality factor depends on the product of distances described in Section 3, and in Section 4 we proceed as follows.

At the \( n \)th stage we split length \( n \) trajectories of diffeomorphisms satisfying the Inductive Hypothesis into three groups. One group consists of what we call “simple” trajectories for which the product of distances is not too small. For nonsimple trajectories we show that either the \( \gamma_\varepsilon \) changes by at most \( \rho \gamma_n(C, \delta, \rho) \geq 1 \) is a parameter measure is normalized to be 1 on a brick \( HB^1(r) \) whose sides decay rapidly; the normalization increases the measure by a factor of \( r^0 \cdot r_1 \cdots r_{n-1} \cdot r_2^{n-1} \). However, we are able to show that when considering only diffeomorphisms \( f_\varepsilon \) with \( \varepsilon \in HB^1(r) \), the number of \( n \)-tuples we must consider as possible pseudotrajectories of \( f_\varepsilon \) is reduced by the factor \( r_0 \cdot r_1 \cdots r_{n-2} \). Due to our definition of an admissible sequence \( r \), the remaining factor \( r_{n-1} \cdot r_2^{n-1} \) does not affect the necessary summability properties for the bounds \( \mu_n(C, \delta, \rho, r, M_{1+\rho}) \). There is an additional distortion of our estimates that is exponential in \( n \), due to the fact that an image of a finite-dimensional brick of \( \varepsilon \)-parameters under the Lagrange map is a parallelepiped of \( u \)-parameters, but no longer a brick. This exponential factor is also not problematic, because our bound \( \mu_n(C, \delta, \rho, r, M_{1+\rho}) \) decays superexponentially in \( n \).
9 Conclusion

In this announcement we have only been able to outline some of the fundamental tools that are needed for the proof of the main result, which will appear in \textsuperscript{KS} and \textsuperscript{KH}. Here we list some of major difficulties appearing in the proof.

- We must handle almost periodic trajectories of length \( n \) that have a close return after \( k < n \) iterates, so that as discussed above the product of distances along the trajectory is small. The precise definition of a close return is a major problem here. It must not be too restrictive, because we must also show that a trajectory without close returns is simple (the product of distances is not too small).\footnote{This is exactly the place in the proof where we need to impose superexponential decay of our bounds on hyperbolicity and periodicity.}

- In dimension \( N > 1 \), the Lagrange interpolation polynomials involve products of differences of coordinates of points, which may be small even though the points themselves are not close. Thus we must be careful about how we construct the Lagrange basis for a given \( n \)-tuple of points \( x_0, \ldots, x_{n-1} \subset B^N \) and how to incorporate this into the general framework of the space of Lagrange interpolation polynomials.

- At \( n \)-th stage of the induction we need to deal with the \((2n)^N\)-dimensional space \( W_{<2n-1,N} \) of polynomials of degree \( 2n-1 \) in \( N \) variables and handle the distortion properties of the Lagrange map. In such a large dimensional space, even the ratio of volumes of the unit ball and the unit cube is of order \((2n)^N\).\footnote{San}

In \textsuperscript{KS}, \textsuperscript{KH}, based on \textsuperscript{K4}, we first prove the main 1-dimensional result for the case \( N = 1 \), discussed in Sections 5, 6, and 7 of this announcement, and then using additional tools and ideas complete the proof in the general case.

Appendix: Diff\(^r(B^N)\) and Diff\(^r(M)\)

Given a smooth \((C^\infty)\) compact manifold \( M \) of dimension \( D \), for \( N > 2D \) the Whitney Embedding Theorem says that a generic smooth function from \( M \) to \( \mathbb{R}^N \) is a a diffeomorphism between \( M \) and its image. To simplify notation, we identify \( M \) with its image, so that \( M \) becomes a submanifold of \( \mathbb{R}^N \).

Let \( U \subset \mathbb{R}^N \) be a closed neighborhood of \( M \), chosen sufficiently small that there is a well-defined projection \( \pi : U \to M \) for which \( \pi(x) \) is the closest point in \( M \) to \( x \). Then for each \( y \in M \), \( \pi^{-1}(y) \) is an \((N-D)\)-dimensional disk. For \( 0 < \rho < 1 \) and \( y, z \in M \) choose a linear function \( g_{\rho,y,z} : \pi^{-1}(y) \to \pi^{-1}(z) \) that maps \( y \) to \( z \) and contracts distances by a factor of \( \rho \), and such that the dependence of \( g_{\rho,y,z} \) on \( y \) and \( z \) is \( C^r \). Then we can extend each \( f \in \text{Diff}^r(M) \) to a function \( F \in C^r(U) \) that is a diffeomorphism from \( U \) to a subset of its interior by letting

\[
F(x) = g_{\rho,\pi(x),f(\pi(x))}(x)
\]

where \( \rho = ||f^{-1}||_{C^r}/2 \). Then by Fenichel’s Theorem \cite{Fenichel}, every sufficiently small perturbation \( F_\varepsilon \in C^r(U) \) of such an \( F \) has an invariant manifold \( M_\varepsilon \subset U \) for which \( \pi|_{M_\varepsilon} \) is a \( C^r \) diffeomorphism from \( M_\varepsilon \) to \( M \). Then to such an \( F_\varepsilon \) we can associate a diffeomorphism \( f_\varepsilon \in \text{Diff}^r(M) \) by letting

\[
f_\varepsilon(y) = \pi(F_\varepsilon(\pi^{-1}\varepsilon(y))).
\]

Notice that the periodic points of \( F_\varepsilon \) all lie on \( M_\varepsilon \) and are in one-to-one correspondence with the periodic points of \( f_\varepsilon \). Furthermore, because \( f_\varepsilon \) and \( F_\varepsilon|_{M_\varepsilon} \) are conjugate, the hyperbolicity of
each periodic orbit is the same for either map. Thus any estimate on $P_n(F_\epsilon)$ or $\gamma_n(F_\epsilon)$ applies also to $f_\epsilon$.

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