Quantum Dynamical Semigroups and Non-decomposable Positive Maps

Fabio Benatti\textsuperscript{a,b}, Roberto Floreanini\textsuperscript{b}, Marco Piani\textsuperscript{a,b}

\textsuperscript{a}Dipartimento di Fisica Teorica, Università di Trieste, Strada Costiera 11, 34014 Trieste, Italy
\textsuperscript{b}Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, 34100 Trieste, Italy

Abstract

We study dynamical semigroups of positive, but not completely positive maps on finite-dimensional bipartite systems and analyze properties of their generators in relation to non-decomposability and bound-entanglement. An example of non-decomposable semigroup leading to a $4 \times 4$-dimensional bound-entangled density matrix is explicitly obtained.

1 Introduction

Semigroups of dynamical maps are central in the description of open quantum systems in weak interaction with suitable external environments, acting as sources of dissipation and noise. They have been successfully used in many phenomenological applications in quantum chemistry, quantum optics and statistical physics \cite{1, 2, 3, 4}; they have also been applied to dissipative phenomena induced by fundamental dynamics in various elementary particle systems \cite{5, 6, 7, 8, 9}.

In many instances, open systems relevant to physical applications can be modeled as finite $d$-dimensional systems, whose states are described by density matrices $\rho$. Their dynamics takes the form of semigroups of linear maps $\gamma_t$, $t \geq 0$, satisfying the forward in time composition law $\gamma_t \circ \gamma_s = \gamma_{t+s}$, $t, s \geq 0$, and sending any initial state $\rho$ into another state $\gamma_t[\rho]$ in the course of time; in particular, $\gamma_t$ must be a positive map.

In line of principle, positivity is not sufficient to guarantee full physical consistency of the maps $\gamma_t$: a more restrictive property, namely complete positivity, needs to be imposed.
This guarantees that not only the dynamics $\gamma_t$ of any system $S$ be positive, but that such is also the map $\gamma_t \otimes \text{id}$, with “id” the identity operation, describing the time-evolution of $S$ statistically coupled to a generic inert $n$-level system $S_n$.

Complete positivity fully characterizes the form of the map $\gamma_t$ and, consequently, also its infinitesimal generator. On the contrary, if the map $\gamma_t$ is only positive, then very little control is available either on its form or on that of its generator: this partly justifies the reason why only few examples of positive semigroups have been considered so far in dissipative quantum dynamics.

On the other hand, understanding the general structure of the set of positive maps is becoming more and more important, since these maps serve as entanglement witnesses in quantum information. In this respect, particularly interesting from a physical point of view is to study the subset of decomposable maps (see and references therein) which are sums of a completely positive map and another completely positive map composed with the transposition; in fact, the positive maps that are not decomposable are related to the phenomenon of bound-entanglement and its non-distillability.

Although both the mathematical and physical literature on non-decomposable positive maps is rapidly growing, the question of decomposability of positive, continuous semigroups, and not of generic maps, seems not to have so far been raised. The motivation for analysing this problem is twofold: on one hand, the control of the mathematical structure of semigroups can be used to witness the presence of bound-entanglement, on the other, one may hope that they could shed some light on the process of bound-entanglement generation.

In the present investigation we will concentrate on how to reveal bound-entanglement by means of quantum dynamical semigroups and provide some general results relating positivity and decomposability of semigroups to relevant properties of their generators. In the next section, we shall first briefly recall the notions of positivity, complete positivity and decomposability when applied to one-parameter semigroups. Section 3 will be devoted to the study of positivity of product semigroups on bipartite systems; some general results relating this condition to the structure of the corresponding generators will be presented. These considerations will then be used in Section 4 to analyze the structure of positive semigroups in relation to the notion of decomposability. In particular, we shall examine in detail a bipartite system consisting of two 4-dimensional systems whose dynamics is non-decomposable for an initial finite interval of time, but becomes and stays decomposable afterwards. We shall prove this by explicitly constructing a $4 \times 4$-dimensional bound-entangled state that is witnessed by the initial non-decomposability of the dynamics.

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1 We refer to the review and to the references therein for more information concerning separability, entanglement and distillability.
2 Complete positivity vs positivity

According to the statistical interpretation of quantum mechanics, the physical states of a \( d \)-level quantum system \( S \) are represented by density matrices, that is by positive matrices \( \rho \geq 0 \), with \( \text{Tr}(\rho) = 1 \). If we consider open systems and describe their dissipative dynamics by semigroups of maps \( \gamma_t \) sending any state \( \rho \) at \( t = 0 \) into \( \gamma_t[\rho] \) at time \( t > 0 \), then, for physical consistency, the map \( \gamma_t \) must preserve positivity, that is \( \gamma_t[\rho] \geq 0 \) at all times.

We shall focus on probability and positivity preserving semigroups \( \gamma_t \), continuous with respect to the trace-norm \( \|X\|_1 := \text{Tr} \sqrt{X^\dagger X} \) on the matrix algebra \( M_d(\mathbb{C}) \). From \( \text{Tr}(\rho_t) = 1 \) and \( \lim_{t \to 0} \gamma_t = \text{id} \), it follows \([13]\) that \( \gamma_t \) can be represented in exponential form, \( \gamma_t = e^{tL} \), with the generator \( L \) given by

\[
L[\rho] = -i \left[H, \rho\right] + \sum_{a,b=1}^{d^2-1} C_{ab} \left(F_a \rho F_b^\dagger - \frac{1}{2} \left\{ F_b^\dagger F_a, \rho \right\} \right),
\]

where \( H = H^\dagger \in M_d(\mathbb{C}) \), while the \( F_a, a = 1, 2, \ldots, d^2-1, \) are traceless \( d \times d \) matrices forming together with \( F_0 := \frac{1}{\sqrt{d}} \) an orthonormal set in \( M_d(\mathbb{C}) \): \( \text{Tr}(F_{\mu}^\dagger F_{\nu}) = \delta_{\mu\nu}, \mu, \nu = 0, 1, \ldots, d^2-1 \).

In the following, it will prove convenient to isolate the so-called “noise” term

\[
N[\rho] = \sum_{a,b} C_{ab} F_a \rho F_b^\dagger ,
\]

from the pseudo-Hamiltonian contribution

\[
L_h[\rho] = -i \left(H - \frac{i}{2} K\right) \rho + i \rho \left(H + \frac{i}{2} K\right), \quad K = \sum_{a,b=1}^{d^2-1} C_{ab} F_b^\dagger F_a .
\]

Remarks

1.1 A necessary and sufficient condition for the positivity of \( \gamma_t \) is expressed by the following constraint \([24]\):

\[
\text{Tr} \left(P_i L[P_j]\right) \geq 0 , \quad i \neq j ,
\]

for all orthogonal resolutions \( \{P_i\} \) of the identity: \( \sum_i P_i = 1, P_i P_j = \delta_{ij} P_i \).

1.2 The \( (d^2-1) \times (d^2-1) \) matrix \( C \) of coefficients \( C_{ab} \) is usually called the Kossakowski matrix. The condition \([4]\) is too weak to fully characterize the matrix \( C \); the only general algebraic constraint following from \([4]\) is hermiticity: \( C = C^\dagger \).

As already observed, in analyzing open system dynamics, one usually asks for a more stringent condition than positivity, namely that the dynamical map \( \gamma_t \) be completely positive for all \( t \geq 0 \). Essentially, complete positivity guarantees that the map \( \gamma_t \otimes \text{id} \) preserve the positivity of all states of the compound system \( S + S_n \), where \( S_n \) is any \( n \)-level system. As \( S \) is assumed to be a \( d \)-level system, a theorem by Choi \([10]\) ensures that the map \( \gamma_t \) is
completely positive iff the map $\gamma_t \otimes \text{id}$ is positive for $n = d$. The physical argument in support to the necessity of complete positivity is that one cannot exclude that the system of interest $S$ might have interacted with another $d$-level system in the past and become statistically coupled to it. In this case one should consider the two systems together, even though only one of them has a non-trivial time-evolution $\gamma_t$, while the other one is dynamically inert $[12]$. The only states of $S + S_n$ that may develop negative eigenvalues under $\gamma_t \otimes \text{id}$, for $\gamma_t$ not completely positive, are the entangled ones, namely those which cannot be written in the separable form

$$\rho_{\text{sep}} = \sum_{ij} \lambda_{ij} \rho_1^i \otimes \rho_2^j , \quad (5)$$

where, $\lambda_{ij} > 0$, $\sum_{ij} \lambda_{ij} = 1$ and $\rho_1^i$ and $\rho_2^j$ are any states of the two partner systems.

One of the characteristic features of generic, trace-preserving completely positive maps $\Lambda$ is that their structure is uniquely fixed: they can always be cast in the Kraus-Stinespring form $[11, 12]$

$$\Lambda[\rho] = \sum_{\ell} V_{\ell} \rho V_{\ell}^\dagger , \quad (6)$$

where $V_\ell \in M_d(\mathbb{C})$ and $\sum_{\ell} V_{\ell}^\dagger V_\ell = 1$. Equivalently, they can be characterized by the following necessary and sufficient condition $[18]$

$$\Lambda \otimes \text{id}[P^d_+] \geq 0 , \quad (7)$$

where

$$P^d_+ := \frac{1}{d} \sum_{i,j=1}^d |i\rangle\langle j| \otimes |i\rangle\langle j| , \quad (8)$$

is the symmetric projector with respect to a suitable orthonormal basis $\{|j\rangle\}$ in $\mathbb{C}^d$.

Imposing (7) on semigroups of maps $\gamma_t$, with generator as in (1), puts strong constraints on the corresponding Kossakowski matrix $C$ (compare with Remark 1.2):

**Theorem 1** $[15, 14]$ 

The map $\gamma_t$ generated by (1) is completely positive iff $C$ is positive.

To decide whether a given mixed state $\rho$ of $S + S_n$ is entangled or not is a rather subtle task; a very important tool is provided by the partial transposition $[25] \ T \otimes \text{id}$: indeed, $T \otimes \text{id}[\rho]$ can have negative eigenvalues only if $\rho$ is entangled. Having a non-positive partial transposition is sufficient for a state to be entangled and is also necessary when $S$ is a 2-level system and $n = 2, 3$ $[26]$, but not in higher dimension where there can be entangled states with positive partial transposition: they are known as bound-entangled states $[27, 18]$. From a physical point of view, the entanglement contained in bound-entangled states cannot be amplified by any local operation on the two parties $S$ and $S_n$ $[28, 18]$. An operational analytical approach to the description of bound-entangled states is currently being elaborated which is based on the notion of *non-extendible orthonormal basis* $[29]$. Also, an attempt is being developed
at formulating a thermodynamics of entanglement (see the related references in [18]) where bound-entanglement is the counterpart of heat.

As already noticed, from a mathematical point of view, the fact that the algebraic structure of the Kossakowski matrix is fixed by complete positivity and not by positivity is a consequence of the fact that, unlike completely positive maps, positive maps do not in general possess a structural characterization. The exception is provided by positive maps \( \Lambda \) from \( M_2(\mathbb{C}) \) to \( M_{2,3}(\mathbb{C}) \), where the notion of decomposability \([19, 20, 21]\) fully characterizes them. Indeed, any such \( \Lambda \) can be written as

\[
\Lambda = \Lambda_1 + \Lambda_2 \circ T,
\]
where \( \Lambda_{1,2} \) are completely positive maps and \( T \) is the transposition. On the contrary, in higher dimension, there are positive maps that are not decomposable \([19, 20]\), so that the decomposable ones form a cone \( D \) strictly contained in the cone of all positive maps. A useful tool in this context is offered by the duality between \( D \) and the cone \( T \) of positive operators with positive partial transposition \([22]\). The duality is expressed by the fact that \( X \in T \) if and only if

\[
\langle \Lambda, X \rangle := \text{Tr} \left( \Lambda \otimes \text{id} \left[ P_+^d \right] X^T \right) \geq 0
\]

for all \( \Lambda \in D \), where \( X^T \) denotes transposition of \( X \), and viceversa that \( \Lambda \in D \) if and only if \( (10) \) holds for all \( X \in T \).

**Remark 2** Separable states belong to \( T \); further, if \( \rho \) is separable, then every positive map \( \Lambda \) is such that \( \langle \Lambda, \rho \rangle \geq 0 \). Therefore, given a positive map \( \Lambda \), if, for some \( \rho \in T \), \( \langle \Lambda, \rho \rangle < 0 \), it follows that \( \Lambda \) is non-decomposable and \( \rho \) bound-entangled.

### 3 Positivity of product semigroups

In the following, we shall focus on one-parameter semigroups \( \{ \Gamma_t \}_{t \geq 0} \) acting on states of the compound system \( S + S \), consisting of two independent, non-interacting copies of a \( d \)-dimensional system \( S \); the map \( \Gamma_t \) can then be represented in product form, \( \Gamma_t := \gamma_t^1 \otimes \gamma_t^2 \), with \( \{ \gamma_t^{1,2} \}_{t \geq 0} \) dynamical semigroups on \( S \). From the physical point of view, the semigroups \( \Gamma_t = \gamma_t^1 \otimes \gamma_t^2 \) have important applications since they provide suitable dynamics describing the time-evolution of two systems immersed in a same or in two different environments.

Let us first concentrate on the case \( \gamma_t^1 = \gamma_t^2 \); the result of Theorem 1 concerning the positivity of maps of the form \( \Gamma_t = \gamma_t \otimes \text{id} \) can then be extended to semigroups of the form \( \gamma_t \otimes \gamma_t \).

**Theorem 2** \([30]\) The map \( \Gamma_t = \gamma_t \otimes \gamma_t \) is positive iff the map \( \gamma_t \) is completely positive, that is iff the corresponding Kossakowski matrix \( C \) is positive.

**Remark 3** The fact that the map \( \gamma_t \) must be completely positive for \( \gamma_t \otimes \gamma_t \) to be positive provides more physical ground to the necessity of complete positivity in open quantum
Then, the map $\Gamma$ and sufficient condition for the maps $\Gamma_t$.

**Proof:**

The proof is a generalization of that of Theorem 2. From Remark 1.1, a necessary and sufficient condition for the maps $\Gamma_t$ to be positive is that, for any invertible $V \in M_d(\mathbb{C})$, let $V$ be the invertible $(d^2-1) \times (d^2-1)$ matrix implementing the transformation $V F_a^\dag V^{-1} = \sum_{b=1}^{d^2-1} \mathcal{V}_{ab} F_b^\dag$. Then, the map $\Gamma_t = \gamma^1_t \otimes \gamma^2_t$ is positive only if

$$C := C_1 + \mathcal{V}^\dag C_2 \mathcal{V} \geq 0.$$  

**Theorem 3**

Let $\Gamma_t = \gamma^1_t \otimes \gamma^2_t$ and $C_{1,2}$ be the Kossakowski matrices corresponding to the semigroups $\gamma^1_{t,2}$ with the generators as in [7]; for any invertible $V \in M_d(\mathbb{C})$, let $V$ be the invertible $(d^2-1) \times (d^2-1)$ matrix implementing the transformation $V F_a^\dag V^{-1} = \sum_{b=1}^{d^2-1} \mathcal{V}_{ab} F_b^\dag$. Then, the map $\Gamma_t = \gamma^1_t \otimes \gamma^2_t$ is positive only if

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$$0 \leq \sum_{a,b=1}^{d^2-1} \left( C_{1,ab} w_a^* w_b + C_{2,ab} v_a^* v_b \right), \quad w_a = \text{Tr} \left( F_a^\dag \mathcal{V} \mathcal{V}^\dag \right), \quad v_a = \text{Tr} \left( F_a^\dag (\mathcal{V}^\dag \mathcal{V}) \right).$$

Let $W = \sum_{a=1}^{d^2-1} w_a F_a$ be a generic traceless $(d^2 - 1) \times (d^2 - 1)$ matrix and $V$ a generic invertible matrix in $M_d(\mathbb{C})$, then choose $\Phi = VY^{-1} = \mathcal{V}^\dag V^{-1}W$, where $Y$ is the similarity matrix such that $Y V^{-1} W V Y^{-1} = \left(V^{-1} W V\right)^T$. The matrix $Y$ always exists since a given square matrix and its transpose have the same common divisors [35]. It follows that $\Psi = \Phi \Psi T = W$ and $\left(\Psi \Psi^T\right) = V^{-1} W V$, whence $0 \leq \langle w | C_1 + \mathcal{V}^\dag C_2 \mathcal{V} | w \rangle$ for all vectors $|w\rangle \in \mathbb{C}^{d^2-1}$. \[\square\]
Remarks

4.1 When \( C_1 = C_2 \), Theorem 3 reduces to Theorem 2, whereas, when \( \gamma_1^1 \neq \gamma_2^2 \), it indicates that the map \( \Gamma_t \) can be positive without the maps \( \gamma_t^{1,2} \) being both completely positive, since \( C_1 \) and \( C_2 \) need not be both positive. Theorem 3 gives a necessary, but hardly a sufficient condition: even positivity of the maps \( \gamma_t^{1,2} \) may not be enforced.

4.2 From a mathematical point of view, Theorem 3 can be used to construct a rather rich class of positive maps that may be used to study the related notions of non-decomposability and bound-entanglement: notice that this can be achieved by controlling the generators \( L \) which need not be positive.

4.3 From a physical point of view, Theorem 3 might appear to weaken the argument in favour of the necessity of complete positivity in open quantum dynamics: a closer analysis reveals that it is not so. In fact, when two copies of the same system are immersed in the same environment, their time-evolutions \( \gamma_t^{1,2} \) can differ only slightly; then, a perturbative argument leads again to conclude that \( \gamma_t^{1,2} \) must both be completely positive [36].

As already noted positivity is harder to achieve than complete positivity; the only general results are in \( d = 2 \) [24]. They become particularly simple to express with the additional request that the Kossakowski matrix, \( C \), be real and symmetric when \( F_a = \frac{\sigma_a}{\sqrt{2}} \), \( a = 1, 2, 3 \), where the \( \sigma_a \) are the Pauli matrices. In turn, this is equivalent to the fact that the von Neumann entropy \( S(\rho) = -\text{Tr}(\rho \log \rho) \) never decreases [17]. In this case, \( C \) can always be chosen diagonal, \( C = \text{diag}(c_1, c_2, c_3) \); if not, it can be diagonalized by orthogonal matrices that can be used to define new Pauli matrices. Then the map \( \gamma_t \) is positive iff [32, 33]

\[
c_1 + c_2 \geq 0 , \quad c_2 + c_3 \geq 0 , \quad c_1 + c_3 \geq 0 ,
\]

(13) whereas, according to Theorem 1, it is completely positive iff \( c_1 \geq 0, c_2 \geq 0 \) and \( c_3 \geq 0 \).

In higher dimension \( d \geq 3 \), there are no general necessary and sufficient conditions on the eigenvalues of \( C \) that guarantee the positivity of the corresponding semigroup. For example, we know that either \( C_1 \) or \( C_2 \) may have negative eigenvalues and nevertheless lead to a positive, but not completely positive, map \( \Gamma_t = \gamma_1^1 \otimes \gamma_2^2 \).

In order to explicitly construct examples of positive product semigroups, one needs to supplement the necessary conditions of Theorem 3 with appropriate sufficient ones; to this end we have the following result.

Theorem 4 Suppose that the non-Hamiltonian terms in the generators of \( \gamma_t^{1,2} \) are as follows,

\[
D_{1,2}[\rho] = \sum_{\ell=1}^{d^2-1} c_{1,2}^\ell \left( G_{1,2}^{1,2} \rho G_{1,2}^{1,2} - \frac{1}{2} \left\{ (G_{1,2}^{1,2})^2, \rho \right\} \right) , \quad c_{1,2}^\ell \in \mathbb{R} ,
\]

(14) where \( G_{1,2}^{1,2} \in M_d(\mathbb{C}) \), together with \( G_0^{1,2} = \frac{1}{\sqrt{d}} \), constitute two orthonormal sets of hermitian traceless matrices. Suppose that \( c_1^\ell > 0 \) for all \( \ell = 1, 2, \ldots, d^2 - 1 \), and that \( c_2^\ell = -|c_2^k| < 0 \), for one index \( k \), while \( c_2^\ell > 0 \) when \( \ell \neq k \); then, the map \( \Gamma_t = \gamma_1^1 \otimes \gamma_2^2 \) is positive if \( c_1^\ell \geq |c_2^k|, \ell = 1, 2, \ldots, d^2 - 1 \) and \( c_2^\ell \geq |c_2^k|, \ell \neq k \).
Proof: In this case, the right hand side of (11) can be recast as

\[ I(\psi, \phi) := \sum_{\ell=1}^{d^2-1} (c_1^\ell - |c_2^\ell|^2) |\text{Tr} G_\ell^1 \Phi \Psi^\dagger|^2 + \sum_{\ell \neq k=1}^{d^2-1} c_2^\ell |\text{Tr} G_\ell^2(\Psi^\dagger \Phi)^T|^2 \]

\[ + |c_2^\ell| \left( \sum_{\ell=1}^{d^2-1} |\text{Tr} G_\ell^1 \Phi \Psi^\dagger|^2 - |\text{Tr} G_\ell^2(\Psi^\dagger \Phi)^T|^2 \right) . \]

Since \( G_\ell^{1,2}, \ell = 1, 2, \ldots, d^2 - 1, \) as well as \( \Phi \Psi^\dagger \) and \( (\Psi^\dagger \Phi)^T \) are traceless and the \( G_\ell^{1,2} \) form a basis, we can expand \( \Phi \Psi^\dagger = \sum_{\ell=1}^{d^2-1} \text{Tr}(G_\ell^1 \Phi \Psi^\dagger) G_\ell^1 \) and similarly for \( (\Psi^\dagger \Phi)^T \) in terms of \( G_\ell^2 \).

It thus follows that

\[ \sum_{\ell=1}^{d^2-1} \left( \text{Tr} G_\ell^1 \Phi \Psi^\dagger \right)^2 = \text{Tr} \left( \Phi \Psi^\dagger \right)^2 = \text{Tr} \left( \left( \Psi^\dagger \Phi \right)^T \right)^2 = \sum_{\ell=1}^{d^2-1} \left( \text{Tr} G_\ell^2(\Psi^\dagger \Phi)^T \right)^2 . \]

Further, extracting the \( k \)-th contribution and using the triangle inequality one gets

\[ |\text{Tr} G_k^2(\Psi^\dagger \Phi)^T|^2 \leq \sum_{\ell=1}^{d^2-1} |\text{Tr} G_\ell^1 \Phi \Psi^\dagger|^2 + \sum_{\ell \neq k=1}^{d^2-1} |\text{Tr} G_\ell^2(\Psi^\dagger \Phi)^T|^2 , \]

which in turn implies

\[ I(\psi, \phi) \geq \sum_{\ell=1}^{d^2-1} (c_1^\ell - |c_2^\ell|^2) |\text{Tr} G_\ell^1 \Phi \Psi^\dagger|^2 + \sum_{\ell=1}^{d^2-1} (c_2^\ell - |c_2^\ell|^2) |\text{Tr} G_\ell^2(\Psi^\dagger \Phi)^T|^2 \geq 0 . \]

\[ \Box \]

Remark 5. The class of generators of the form (14) is not too narrow; indeed, any generator (1) with \( F_a^\dagger = F_a \) and real symmetric Kossakowski matrices can be cast as in (14): one diagonalizes \( C_{1,2} \) by means of orthogonal transformations that are then used to turn the \( F_a \)'s into \( G_\ell^{1,2} \).

Theorem 5 In dimension \( d = 2 \), if the maps \( \gamma_{1,2} \) are both positive with corresponding real symmetric Kossakowski matrices \( C_{1,2} \), then the conditions of Theorem 4 are also necessary.

Proof: As already observed, we can assume the real symmetric matrix \( C_1 \) to be diagonal, while the matrix \( C_2 \) being also real symmetric can be diagonalized by an orthogonal rotation \( \tilde{V} \). The latter always corresponds to a unitary transformation \( \sigma_a \mapsto V \sigma_a V^\dagger = \sum_{b=1}^3 \tilde{V}_{ab} \sigma_b \) of the orthonormal basis of Pauli matrices. We can now use Theorem 3, with a unitary \( V = U \tilde{V} \), where \( \sum_{b=1}^3 U_{ab} \sigma_b = U \sigma_a U^\dagger \), with \( U = \frac{\sigma_1 + \sigma_2}{2} \) exchanging \( \sigma_1 \) with \( \sigma_2 \) and multiplying \( \sigma_3 \) by \(-1\). In this way \( C_2 \) is first diagonalized by \( \tilde{V} \), \( C_2 = \text{diag}(c_1^{(2)}, c_2^{(2)}, c_3^{(2)}) \), then its eigenvalues \( c_{1,2}^{(2)} \)
are exchanged while \(c_3^{(2)}\) is left unchanged. Varying the Pauli matrices in \(U\), from Theorem 3, one derives \(c_i^{(1)} + c_j^{(2)} \geq 0\), for \(i, j = 1, 2, 3\). Therefore, only one of the Kossakowski matrices, say \(C_2\), may have negative eigenvalues. Moreover, condition (13) enforces the presence of just one negative eigenvalue, whose absolute value must be smaller than the other two.

4 Decomposability of positive semigroups

As already remarked in the Introduction, positive maps that are not completely positive are of great relevance in the physics of quantum information. Much attention has lately been devoted to the study of positive maps that are not decomposable, i.e. that cannot be cast in the form (1), since they signal the phenomenon of bound entanglement. In the light of the results of the previous sections, positive semigroups of the form \(\Gamma_t = \gamma_1^t \otimes \gamma_2^t\) can provide new insights in the study of bound-entanglement since its appearance may be put in relation to some characteristic features of the corresponding generators. We shall first start by discussing an explicit example and then prove some general results.

As an instance of a map \(\Gamma_t = \gamma_1^t \otimes \gamma_2^t\) which is positive, but not completely positive, let us set \(d = 2\) and in (1) choose \(H = 0\) and

\[
C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\] (15)

The generated semigroups have the explicit expressions

\[
\gamma_1^t[\rho] = \alpha \rho + \frac{1 - \alpha}{2} \sigma_0, \quad \gamma_2^t[\rho] = \rho - (1 - \alpha) \rho_2 \sigma_2, \quad \alpha = \exp(-2t),
\] (16)

where the expansion \(\rho = \frac{1}{2} \sigma_0 + \rho_1 \sigma_1 + \rho_2 \sigma_2 + \rho_3 \sigma_3\) has been used, with \(\sigma_0\) the 2 \(\times\) 2 unit matrix; \(\gamma_1^t\) is clearly completely positive, while \(\gamma_2^t\) turns out to be only positive.

It is convenient to define the linear map \(\text{Tr}_2 : M_2(C) \mapsto M_2(C)\), where \(\text{Tr}_2[X] := \text{Tr}(X)\sigma_0\); it is completely positive since it can be written in Kraus-Stinespring form as

\[
\text{Tr}_2[X] = \frac{1}{2} \sum_{\mu=0}^{3} \sigma_\mu X \sigma_\mu.
\] (17)

Also, with respect to the standard representation of the Pauli matrices, the transposition \(T\) does not affect \(\sigma_{1,3}\), while changes the sign of \(\sigma_2\), so that it can be explicitly written as

\[
T[X] = \frac{1}{2} \left( \sum_{\mu=0,\mu\neq2}^{3} \sigma_\mu X \sigma_\mu - \sigma_2 X \sigma_2 \right).
\] (18)
With the help of (16), (17) and (18), $\Gamma_t = \gamma_1^t \otimes \gamma_2^t$ can be written as

$$\Gamma_t = \left( \alpha \text{id}_2 + \frac{1 - \alpha}{2} \text{Tr}_2 \right) \otimes \left( \frac{1 + \alpha}{2} \text{id}_2 + \frac{1 - \alpha}{2} T_2 \right),$$

(19)

where, for the sake of clarity, the identity operation $\text{id}_2$ on $M_2(\mathbb{C})$ has been explicitly inserted.

**Remark 6** In (19), the Kraus-Stinespring form (6) is apparent in the first factor, while the second factor is decomposed as in (9). Since the map $\Gamma_t$ from $M_4(\mathbb{C})$ into itself is positive, but not completely positive, the question whether it is decomposable or not makes sense.

We proceed by rewriting $\Gamma_t$ as $\Gamma^1_t + \Gamma^2_t \circ T_4$ where $T_4 = T_2 \otimes T_2$ is the transposition on $M_4(\mathbb{C})$ and $\Gamma^1_t, \Gamma^2_t$ are the following two linear maps on the same algebra,

$$\Gamma^1_t = \frac{1 + \alpha}{2} \left( \alpha \text{id}_2 + \frac{1 - \alpha}{2} \text{Tr}_2 \right) \otimes \text{id}_2$$

(20)

$$\Gamma^2_t = \frac{1 - \alpha}{2} \left( \alpha T_2 + \frac{1 - \alpha}{2} \text{Tr}_2 \right) \otimes \text{id}_2,$$

(21)

where use has been made of the two identities $T_2 \circ T_2 = \text{id}_2$ and $\text{Tr}_2 \circ T_2 = \text{Tr}_2$. It turns out that the map $\Gamma^1_t$ is completely positive on $M_4(\mathbb{C})$ for any $t \geq 0$ for it is a tensor product of a sum of completely positive maps on $M_2(\mathbb{C})$ with the identity.

In order to check whether the map $\Gamma^2_t$ is also completely positive, we use the criterion (7) with

$$P^+_4 = \frac{1}{4} \sum_{a,b,c,d=1}^{2} \left( |a\rangle \langle b| \otimes |c\rangle \langle d| \right) \otimes \left( |a\rangle \langle c| \otimes |b\rangle \langle d| \right),$$

(22)

where $|a\rangle, a = 1, 2$ is a fixed orthonormal basis in $\mathbb{C}^2$. Then, one explicitly finds

$$\Gamma^2_t \otimes \text{id}_4[P^+_4] = \frac{1 - \alpha^2}{8} P^+_4 \otimes \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 - \alpha & 2\alpha & 0 \\ 0 & 2\alpha & 1 - \alpha & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) + \frac{1 - \alpha}{8} P^+_4 \otimes \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 - \alpha & 2\alpha & 0 \\ 0 & 2\alpha & 1 - \alpha & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

(23)

where $P^+_2 = \frac{1}{2} \sum_{a,b=1}^{2} |a\rangle \langle b| \otimes |a\rangle \langle b|$. It thus follows that $\Gamma^2_t \otimes \text{id}_4[P^+_4]$ is positive for $0 \leq \alpha \leq 1/3$, that is for $t \geq t^*$, $t^* := (\log 3)/2$, while it has a negative eigenvalue for $0 < t < t^*$. Therefore, $\Gamma^2_t$ is completely positive and $\Gamma_t$ decomposable for $t \geq t^*$.

**Remark 7** Because of the non-uniqueness of the decomposition (9), the above result does not necessarily mean that $\Gamma_t$ is not decomposable for $0 < t < t^*$: it may only indicate that the maps $\Gamma^1_t, \Gamma^2_t$ in (20), (21) do not provide the right decomposition in that range of time and that different ones have to be looked for.

\(^2\)Notice that, for sake of compactness, we have chosen to represent the right hand side of (23) as a tensor product which does not respect the splitting in (22).
To prove that the positive map $\Gamma_t$ in (19) is indeed non-decomposable for $0 < t < t^*$, it is convenient to introduce the pure states $Z_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, of the form (13)

$$Z_{\mu\nu} = \left[ 1_4 \otimes \left( \sigma_\mu \otimes \sigma_\nu \right) \right] P_+ \left[ 1_4 \otimes \left( \sigma_\mu \otimes \sigma_\nu \right) \right],$$

constructed using the tensor products of Pauli matrices plus the identity $\sigma_0$.

Recalling the definition of the duality in (10), it can be checked that

$$W_{\mu\nu} := \langle \Gamma_t, Z_{\mu\nu} \rangle = \frac{1}{4} \left( \alpha \delta_{\mu0} + \frac{1-\alpha}{4} \right) \left[ 2(1+\alpha)\delta_{\nu0} + (1-\alpha)(1-2\delta_{\nu2}) \right],$$

and, in particular,

$$W_{02} = \frac{(\alpha - 1)(1 + 3\alpha)}{16}, \quad W_{11} = W_{23} = W_{31} = -W_{32} = W_{33} = \frac{(1 - \alpha)^2}{16}. \quad (26)$$

Let us then consider the following combination

$$\rho_{be} = \frac{1}{6} \left( Z_{02} + Z_{11} + Z_{23} + Z_{31} + Z_{32} + Z_{33} \right)$$

$$= \frac{1}{24} \left( \begin{array}{cccccccc} 1 & . & . & . & . & -1 & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\ . & . & 3 & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & -1 & . & . & . & . & . & . & . & . & . & . & 1 \\ . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & -1 & . & . & . & . & . & . & . & . & 3 \\ . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & -1 \\ . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 3 \\ . & . & -1 & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & -1 & . & . & . & . & . & . & . & . & -1 \\ . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 3 \\ . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & -1 \\ . & -1 & . & . & -1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 3 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\ 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\ \end{array} \right) \quad (27)$$

where only the non-zero entries have been explicitly written down. The matrix $\rho_{be}$ is positive and normalized, and therefore represents a density matrix; one further checks that it has positive partial transposition: $T_4 \otimes 1_4 [\rho_{be}] \geq 0$. In addition, using (20), one finds

$$\langle \Gamma_t, \rho_{be} \rangle = \frac{(1 - \alpha)(1 - 3\alpha)}{48}. \quad (28)$$
As the latter term becomes negative for \( \alpha > 1/3 \), that is for \( 0 < t < t^* \), then, as explained in Remark 2, the state \( \rho_{be} \) cannot be separable. As a consequence, \( \rho_{be} \) is an explicit example of a bound-entangled state in \( 4 \times 4 \) dimension; moreover, in the same interval of time, \( \Gamma_t \) provides a one-parameter family of non-decomposable maps.

**Remarks**

8.1 The idea behind the construction of the bound-entangled state above is as follows. By expanding the pairing in (10) for small \( t \) with \( X \) such that \( \text{Tr}(P_4^1 X^T) = 0 \), one finds that only the “noise” term (2) actually contributes. Thus, in order to make \( \langle \Gamma_t, X \rangle < 0 \), one may restrict to those convex combinations \( X \) of the projectors \( Z_{\mu\nu} \) in (24) that have positive partial transposition (\( X \in T \)), are orthogonal to \( P_4^4 \) and such that \( \langle N, X \rangle < 0 \). Indeed, in the case of (27) and (19), one can check that \( \rho_{be} P_4^4 = 0 \), whereas the noise term reads

\[
N = \left( \text{Tr}_2 - \frac{\text{id}_2}{2} \right) \otimes \text{id}_2 + \text{id}_2 \otimes \left( T_2 - \frac{\text{id}_2}{2} \right),
\]

whence \( \langle N, Z_{22} \rangle = -1/8 \), while all the other pairings vanish, so that \( \langle N, \rho_{be} \rangle = -1/48 \).

8.2 The positivity of \( \Gamma_t \) forces the noise term to fulfill \( \langle \phi | N[|\psi\rangle\langle\psi|] | \phi \rangle \geq 0 \) whenever \( \langle \phi | \psi \rangle = 0 \); this condition puts a strong constraint on the map \( N \), but does not forces it to be positive.

8.3 The state (27) is just one example of a class of bound entangled states witnessed by the map in (19).

Remark 8.1 suggests that if the noise term (2) is positive and decomposable, then the generated map \( \Gamma_t \) is also decomposable.

**Theorem 6** If the noise term (2) in (1) is positive and decomposable as in (4), then the generated semigroup consists of decomposable maps.

**Proof:** First, the composition \( \Lambda \circ \Omega \) of two decomposable positive maps \( \Lambda = \Lambda_1 + \Lambda_2 \circ T \) and \( \Omega = \Omega_1 + \Omega_2 \circ T \), is decomposable. Indeed, since \( T \circ T = \text{id} \), then

\[
\Lambda \circ \Omega = \Lambda_1 \circ \Omega_1 + \Lambda_2 \circ \left( T \circ \Omega_2 \circ T \right) + \left( \Lambda_2 \circ \left( T \circ \Omega_1 \circ T \right) + \left( \Lambda_1 \circ \Omega_2 \right) \right) \circ T.
\]

Compositions and sums of completely positive maps are completely positive; further, when \( \Omega \) is a completely positive map, by using (6), one checks that also the map \( T \circ \Omega \circ T \) can be cast in the Kraus-Stinespring form, and is thus completely positive. Therefore, the first line gives a completely positive map and the second one a completely positive map composed with the transposition \( T \). It thus follows that \( k \)-times composition \( N^k := N \circ N \circ \cdots \circ N \) of the noise in (2) are decomposable and thus also the exponential map \( e^{Nt} = \sum_{k=0}^{\infty} \frac{N^k}{k!}, t \geq 0 \).

The same is true of the strictly contractive completely positive map \( e^{L_h} \), \( t \geq 0 \), generated by
the pseudo-Hamiltonian contribution \( [3] \). Consequently, the linear maps \( (e^{L_{t/n}} \circ e^{N_{t/n}})^n \), \( n \geq 0 \), are decomposable too and, for all \( X \in T \), Trotter formula yields
\[
\langle \Gamma_t, X \rangle = \lim_{n \to +\infty} \text{Tr}\left( (e^{L_{t/n}} \circ e^{N_{t/n}})^n \otimes \text{id}[P^+_T]X^T \right) \geq 0.
\]

**Remarks**

9.1 From Theorem 6, it follows that the noise term in the generator of \( \Gamma_t \) in (19) is not decomposable since the semigroup is not decomposable for \( 0 < t < t^* \).

9.2 The condition of Theorem 6 is sufficient, but not necessary: a simple counterexample to necessity is the semigroup of decomposable positive maps \( \gamma^2_t \) in (16). It is easy to check that the noise term is not a positive map and therefore not decomposable; for instance, \( \langle \phi | N | \psi \rangle \langle \psi | \phi \rangle < 0 \) with \( \psi = \phi = (1, i) \).

9.3 Interestingly, the non-decomposability of the positive map \( \Gamma_t \) in (19) is a property that disappears after a finite time-interval.\(^3\) Similarly, a semigroup \( \gamma_t \) may start positive and become completely positive later. Let \( L \) be a generator of a semigroup on a 2-level system without Hamiltonian term and with Kossakowski matrix
\[
C = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a - b \end{pmatrix},
\]
where \( a, b > 0 \). The positivity conditions \([13]\) are satisfied, whereas \( \gamma_t \otimes \text{id}[P^+_T] \) has eigenvalues \( \mu(t) = 1 - e^{-4bt} \geq 0 \) and \( \lambda_\pm(t) = 1 + e^{-4bt} \pm 2e^{-2at} \). While \( \lambda_+(t) \) is never negative, \( \lambda_-(t) \geq 0 \) when \( a \geq b \), whereas, for \( a < b \), \( \lambda_-(t) \) is non-negative only for \( t \geq \hat{t} \), where \( \hat{t} \) is such that \( \cosh 2b\hat{t} = e^{2(b-a)\hat{t}} \); as a consequence, \( \gamma_t \) is positive, but not completely positive for \( 0 < t < \hat{t} \) and completely positive for \( t \geq \hat{t} \).

5 Discussion

The existence of bound-entanglement (and the related phenomenon of non-distillability) asks for procedures able to recognize its presence. In view of its relation to positive non-decomposable maps, in the present investigation, we have focused on continuous one-parameter semigroups of positive maps, which, unlike the ones consisting of completely positive maps, have so far been little considered in the literature. The advantage of considering semigroups, instead of maps, is that their infinitesimal generators completely characterize their form, which therefore can be controlled, at least to some extent. In particular, we have analyzed bipartite finite dimensional systems and provided sufficient conditions on the

\(^3\)The same phenomenon happens to entanglement: there are semigroups that transform initially entangled states into separable ones in a finite time \([37]\).
structure of the generators giving raise to positive, but not completely positive, product semigroups.

For such one-parameter families of maps, the question of whether they are decomposable or not is of relevance both from the mathematical and physical point of view; indeed, the property of decomposability is related, by duality, to the class of states which possess a positive partial transpose, the non-separable ones providing examples of the phenomenon of bound-entanglement.

In this context, we have constructed a product semigroup of positive, non-decomposable maps witnessing a bound-entangled state in $4 \times 4$ dimensions. The construction is based on the analysis of the corresponding generator when acting on a particular class of states naturally associated with it: this strategy is different from the usual approach relying on positive maps, since it is based on the study of an operator, the infinitesimal generator, that need not be (and indeed most of the times is not) positive.

Finally, besides providing explicit non-decomposable maps and bound-entangled states in $4 \times 4$ dimensions, our construction also indicates a more general procedure for generating similar examples in higher dimensions.

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