HIGHER-ORDER CONVERGENCE OF FINITE ELEMENT METHODS FOR THE STOCHASTIC STOKES EQUATIONS

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Abstract. Numerical analysis for the stochastic Stokes/Navier-Stokes equations is still challenging even though it has been well done for the corresponding deterministic equations. In particular, the existing error estimates of finite element methods for the stochastic equations all suffer from the order reduction with respect to the spatial discretizations. The best convergence result obtained for these fully discrete schemes is only half-order in time and first-order in space, which is not optimal in space in the traditional sense. The purpose of this article is to establish the strong convergence of $O(\tau^{1/2} + h^2)$ and $O(\tau^{1/2} + h^2)$ in the $L^2$ norm for the inf-sup stable velocity-pressure finite element approximations, where $\tau$ and $h$ denote the temporal stepsize and spatial mesh size, respectively. The error estimates are of optimal order for the spatial discretization considered in this article (with MINI element), and consistent with the numerical experiments. The analysis is based on the fully discrete Stokes semigroup techniques and the corresponding new estimates.

1. Introduction

We consider the time-dependent stochastic Stokes equations in a smooth domain $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$, under the stress boundary condition, i.e.,

\begin{equation}
\begin{cases}
    du = [\nabla \cdot \mathcal{T}(u, p) + f] \, dt + B(u) \, dW(t) & \text{in } D \times (0, T], \\
    \nabla \cdot u = 0 & \text{in } D \times (0, T], \\
    \mathcal{T}(u, p)n = 0 & \text{on } \partial D \times (0, T], \\
    u = u^0 & \text{at } D \times \{0\},
\end{cases}
\end{equation}

where $u$ and $p$ denote the velocity and the pressure of the fluid, respectively, $f$ is a given source field and $n$ denotes the outward unit normal vector on the boundary $\partial D$. Moreover, the stress tensor $\mathcal{T}(u, p)$ is defined by

\begin{equation}
\mathcal{T}(u, p) = 2\mathcal{D}(u) - pI \quad \text{and} \quad \mathcal{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),
\end{equation}

where $I$ denotes the identity tensor. The stochastic noise is determined by an $L^2(D)^d$-valued $Q$-Wiener process $\{W(t); t \geq 0\}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and a linear operator $B(u) : L^2(D)^d \to L^2(D)^d$ which depends on the solution nonlinearly.

The numerical approximation of the deterministic Stokes and Navier–Stokes equations has been in large part well understood nowadays; see [18, 19, 24, 26–29]. For the stochastic Stokes and NS equations driven by a multiplicative noise, several time and space discretizations have been studied.

- For sinusoidal noises, i.e., $B(u)$ maps $L^2(D)^d$ into its divergence-free subspace, Carelli & Prohl [2] studied implicit and semi-implicit time discretizations with a divergence-free finite element approximation in two dimensions under the periodic boundary condition, and proved the strong convergence of velocity approximations with an error bound of $O(\tau^{1/2} + h)$.

- For sinusoidal noises, under the Dirichlet boundary condition, Carelli, Hausenblas & Prohl [8] proposed a Chorin-type splitting method, with certain finite element approximations in space, and proved an error bound of $O(\tau^{1/2} + h + h^2\tau^{-1/2})$ for the velocity.

Key words and phrases. stochastic Stokes equation, multiplicative noise, Wiener process, semi-implicit Euler scheme, mixed FEM, analytic semigroup, error estimate.
• For general non-sinusoidal noises, under the Dirichlet boundary condition, Brzeźniak, Carelli & Prohl [7] proposed two semidiscrete time-stepping schemes based on the mixed FEMs for the stochastic NS equations and proved the convergence of velocity approximations based on compactness arguments.

• For general non-sinusoidal noises, under the periodic boundary condition, the half-order convergence in probability and logarithmic-order strong convergence in $L^2$ of a time-splitting method were proved by Bessaih, Brzeźniak & Millet [3] and Bessaih & Millet [4], respectively.

• For general non-sinusoidal noises and the Dirichlet boundary condition, Breit & Prohl [6] proved strong convergence of velocity approximations with an error bound of $O(\tau^{\frac{1}{2}} + h)$ in the energy norm for a fully discrete semi-implicit FEM for the 2D stochastic NS equations.

• The convergence of pressure approximations was studied only recently. In particular, for general non-sinusoidal noises and the periodic boundary condition, Feng & Qiu [13] established strong convergence of order $O(\tau^{\frac{1}{2}} + h^{\frac{1}{2}})$ for both velocity and pressure approximations (the latter is measured in a time-averaged norm) for a fully discrete semi-implicit mixed FEM. The convergence orders for velocity and pressure approximations were improved to $O(\tau^{\frac{1}{2}} + h^2\tau^{-\frac{1}{2}})$ and $O(\tau^{\frac{1}{2}} + h)$, respectively; see [12][14].

The error analysis in the above-mentioned articles is based on different conditions on the noise, which can be viewed as the following Lipschitz continuity and growth conditions in (1.4) and the regularity of the mild solution (see Proposition 3.1). In particular, the strong convergence under the weaker noise condition in (1.4) also has not been investigated yet.

In this article, we establish optimal convergence of fully discrete mixed methods with standard inf-sup stable FE pairs for the stochastic Stokes equations under the weaker noise condition (1.4). The error analysis under the weaker noise condition is more challenging. To the best of our knowledge, the error analysis of fully discrete semi-implicit FEMs based on condition (1.4) was proved only recently by Carelli & Prohl [5] under the periodic boundary condition, where the strong convergence of velocity approximations was proved with an error bound of $O(\tau^{\frac{1}{2}} + h)$. Their proof is based on a stochastic pressure decomposition technique.

In the presence of stochastic noises, the half-order temporal convergence shown in the analyses mentioned above is consistent with the numerical experiments, while the first-order spatial convergence is not optimal and inconsistent with the numerical experiments, mainly due to the limitation of the traditional energy approach used. The convergence order of pressure approximations under the weaker noise condition in (1.4) also has not been investigated yet.

In this article, we establish optimal convergence of fully discrete mixed methods with standard inf-sup stable FE pairs for the stochastic Stokes equations under the weaker noise condition in (1.4) and the regularity of the mild solution (see Proposition 3.1). In particular, the strong convergence in $L^2$ of order $O(\tau^{\frac{1}{2}} + h^2)$ and $O(\tau^{\frac{1}{2}} + h)$ is proved for the velocity and pressure approximations, respectively, where $\tau$ and $h$ denote the temporal stepsize and spatial mesh size, respectively. The error estimates are of optimal order for MINI element in the traditional sense and consistent with the numerical experiments. The analysis presented in this article is based on the fully discrete Stokes semigroup techniques and the corresponding new estimates (Lemma 4.1), and the $H^1$-stability of the orthogonal projection onto the discrete divergence-free finite element subspace (see Section 4.1). The stress boundary condition considered in this article is physically natural.

The rest of this article is organized as follows. In Section 2, we collect the assumptions and describe a fully discrete method with a standard inf-sup stable FE pair for the stochastic
Stokes and then, present our main theorem. In Section 3 we present the abstract formulation of the stochastic Stokes equations under the stress boundary condition and define the mild solution based on the abstract formulation. In Section 4 we present some technical estimates for the discrete semigroup associated to the Stokes operator. The results are used in the error analysis of the fully discrete FEMs for the stochastic problem in Section 5. In Section 6, we present numerical experiments to support our theoretical analysis by illustrating the convergence orders of the velocity and pressure approximations.

2. Main results

In this section, we present the notations and assumptions to be used in this article, as well as the numerical scheme for the stochastic Stokes equations to be analyzed. Then we present the main theoretical result on the convergence of the numerical approximations.

2.1. Basic notations

Let $D$ be a bounded smooth domain in $\mathbb{R}^d$, with $d \in \{2, 3\}$, and let $H^s(D), s \geq 0$, be the conventional Sobolev space of functions defined on $D$, with $L^2(D) = H^0(D)$. The dual space of $H^s(D)$ is denoted by $\dot{H}^{-s}(D)$.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a filtered probability space with probability measure $\mathbb{P}$, $\sigma$-algebra $\mathcal{F}$ and continuous filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The expectation of a random variable $v$ defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ is denoted by $\mathbb{E} v$.

Let $W(t)$ be a standard $L^2(D)^d$-valued $Q$-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$, with expression

$$W(t) = \sum_{\ell} \sqrt{\mu_{\ell}} \phi_{\ell} W_{\ell}(t) \quad \forall t \in [0, T],$$

(2.1)

where $\{W_{\ell}(t)\}_{\ell \geq 1}$ is a family of independent real-valued Wiener processes and the trace operator $Q : L^2(D)^d \rightarrow L^2(D)^d$ is bounded, self-adjoint, positive semi-definite, with eigenvalues $\mu_{\ell}$ and eigenfunctions $\phi_{\ell}, \ell = 1, 2, \ldots$.

Let $\mathcal{L}_2$ and $\mathcal{L}^0_2$ be the spaces of Hilbert-Schmidt operators from $L^2(D)^d$ to $L^2(D)^d$ and from $Q^{1/2}(L^2(D)^d)$ to $L^2(D)^d$, respectively, satisfying

$$\|\Phi\|_{\mathcal{L}^0_2} : = \left( \sum_{\ell} \mu_{\ell} \|\Phi \phi_{\ell}\|^2_{L^2} \right)^{\frac{1}{2}} = \|\Phi Q^\frac{1}{2}\|_{\mathcal{L}_2}. $$

For a progressively measurable process $\Phi : [0, T] \rightarrow \mathcal{L}_2(Q^{1/2}(H), H)$ with $\int_0^T \|\Phi(s)\|^2_{\mathcal{L}^0_2} ds < \infty \mathbb{P}$-a.s., the stochastic integral $\int_0^t \Phi(s) dW(s)$ can be defined and Itô’s isometry holds, i.e.,

$$\mathbb{E} \left\| \int_0^t \Phi(s) dW(s) \right\|^2_{L^2} = \mathbb{E} \int_0^t \|\Phi(s)\|^2_{\mathcal{L}^0_2} ds.$$

(2.2)

For the simplicity of notation, we denote by $x \lesssim y$ or $y \gtrsim x$ the statement “$x \leq Cy$ for some positive constant $C$ (which is independent of the stepsize $\tau$ and the mesh size $h$ in the numerical approximation) ”.

2.2. Assumptions on the noise and nonlinearity

For the existence and uniqueness of mild solutions of problem (3.12), as well as the numerical approximation to the mild solutions, we work with the following assumptions on the noise and nonlinearity.

Assumption 2.1. (The stochastic noise) We assume that the $Q$-Wiener process $W(t)$ has the following property:

$$\|(-\Delta)^{\frac{1}{2}}\|_{\mathcal{L}_2^0} \lesssim 1,$$

(2.3)
where $\Delta : H^2_N(D) \to L^2(D)$ denotes the Neumann Laplacian operator with domain

$$H^2_N(D) = \{ v \in H^2(D) : \partial_n v = 0 \text{ on } \partial D \},$$

and $(-\Delta)^{\frac{1}{2}} : H^1(D) \to L^2(D)$ denotes the fractional power of $-\Delta$.

**Remark 2.1.** In the case that $Q$ and $-\Delta$ having the same eigenfunctions, condition (2.20) is equivalent to

$$\sum_\ell \mu_\ell \lambda_\ell \lesssim 1,$$

where $\lambda_\ell$ are the eigenvalues of $-\Delta$.

**Assumption 2.2.** (The nonlinearity and source term) We assume that $B(v) : L^2(D)^d \to L^2(D)^d$ is a bounded linear operator for any $v \in L^2(D)^d$, with the following properties: There exists $\beta \in (\frac{1}{2}, 2)$ such that

$$\|B(v)\phi_\ell\|_{H^1} \lesssim \|v\|_{H^\beta} \|\phi_\ell\|_{H^1} \quad \forall v \in H^\beta(D)^d, \quad \text{(2.4)}$$

$$\|B(v_1) - B(v_2)\|_{H^{\frac{1}{2}}} \lesssim \|v_1 - v_2\|_{L^2} \|\phi_\ell\|_{H^1} \quad \forall v_1, v_2 \in L^2(D)^d, \quad \text{(2.5)}$$

$$\|B(v_1) - B(v_2)\|_{H^{\frac{1}{2}}} \lesssim \|v_1 - v_2\|_{H^0} \|\phi_\ell\|_{H^1} \quad \forall v_1, v_2 \in H^0(D)^d. \quad \text{(2.6)}$$

Moreover, we assume that the function $f : [0, T] \times L^2(D)^d \to L^2(D)^d$ satisfies

$$\|f(t)\|_{L^2} \lesssim 1 \quad \forall 0 \leq t \leq T, \quad \text{(2.7)}$$

$$\|f(t_1) - f(t_2)\|_{L^2} \lesssim (t_1 - t_2)^{\frac{1}{2}} \quad \forall 0 \leq t_1 \leq t_2 \leq T. \quad \text{(2.8)}$$

**Remark 2.2.** In the case that $(B(v)\phi_\ell)(x) = b_\ell(v(x))\phi_\ell(x)$ for some functions $b_\ell : \mathbb{R} \to \mathbb{R}$, $\ell = 1, 2, \ldots$, conditions (2.4)–(2.6) are satisfied if the functions $b_\ell$ are uniformly Lipschitz continuous with respect to $\ell$, i.e.,

$$|b_\ell(\sigma)| \lesssim 1 + |\sigma| \quad \forall \sigma \in \mathbb{R},$$

$$|b_\ell(\sigma_1) - b_\ell(\sigma_2)| \lesssim |\sigma_1 - \sigma_2| \quad \forall \sigma_1, \sigma_2 \in \mathbb{R}. \quad \text{(2.9)}$$

**Assumption 2.3.** (Initial value) The initial value $u_0 : \Omega \to L^2(D)^d$ is an $\mathcal{F}_0/B(L^2(D)^d)$-measurable function with $u_0 \in L^2(\Omega, \mathcal{D}(A))$.

### 2.3. The numerical method and its convergence

Let $V_h \times Q_h \subset H^1(D)^d \times L^2(D)^d$ be a pair of finite element spaces subject to a quasi-uniform triangulation of $D$ with mesh size $h > 0$, with the following properties (see [16, Chapter II]):

1. There exists a projection operator $\Pi_h : H^1(D)^d \to V_h$, called the Fortin projection, satisfying

$$\nabla \cdot (v - \Pi_h v), q_h) = 0 \quad \forall v \in H^1(D)^d \text{ and } q_h \in Q_h, \quad \text{(2.9)}$$

$$\|v - \Pi_h v\|_{H^m} \lesssim h^{1-m} \|v\|_{H^1} \quad \forall v \in H^1(D)^d \text{ and } m = 0, 1, \quad \text{(2.10)}$$

$$\|\Pi_h v\|_{H^1} \lesssim \|v\|_{H^1} \quad \forall v \in H^1(D)^d. \quad \text{(2.11)}$$

2. The following approximation properties hold:

$$\inf_{q \in Q_h} \|q - Q_h q\|_{L^2} \lesssim h^m \|q\|_{H^m} \quad \forall q \in H^m(D) \text{ and } m = 1, 2, \quad \text{(2.12)}$$

$$\inf_{q \in Q_h \cap H^1_0(D)} \|q - Q_h q\|_{L^2} \lesssim h^m \|q\|_{H^m} \quad \forall q \in H^m(D) \cap H^1_0(D) \text{ and } m = 1, 2. \quad \text{(2.13)}$$

Therefore, the $L^2$ projection $P_{Q_h} : L^2(D) \to Q_h$ satisfies the following estimates:

$$\|q - P_{Q_h} q\|_{L^2} \lesssim h^m \|q\|_{H^m} \quad \forall q \in H^m(D) \text{ and } m = 1, 2, \quad \text{(2.14)}$$

$$\|q - P_{Q_h} q\|_{H^{-1}} = \sup_{\eta \in H^1(D)} \frac{(q - P_{Q_h} q, \eta)}{\|\eta\|_{H^1}} = \sup_{\eta \in H^1(D)} \frac{(q - P_{Q_h} q, \eta - P_{Q_h} \eta)}{\|\eta\|_{H^1}} \lesssim h^2 \|q\|_{H^1}. \quad \text{(2.15)}$$
(3) The following inverse inequality holds
\[ \|v_h\|_{H^1} \lesssim h^{-1}\|v_h\|_{L^2} \quad \forall v_h \in V_h. \] (2.17)

(4) The inf-sup condition holds:
\[ \|q_h\|_{L^2} \lesssim \sup_{v_h \neq 0} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_{H^1}} \quad \forall q_h \in Q_h. \] (2.18)

Several mixed finite element spaces \( V_h \times Q_h \) are known to satisfy the properties above, for example, the mini element space in [2] and the Taylor–Hood finite element space in [30]. We assume that the triangulation may contain curved triangles/tetrahedra which fit the boundary exactly in order to avoid making the problem more complicated with additional errors in approximating the boundary.

Let \( X_h := \{ v_h \in X_h : (\nabla \cdot v_h, q_h) = 0, \forall q_h \in Q_h \} \) be the discrete divergence-free subspace of \( X_h \), and denote by \( P_{X_h} : L^2(D)^d \to X_h \) the \( L^2 \)-orthogonal projection onto \( X_h \). On a uniform partition \( t_n = n\tau, n = 0, 1, \ldots, N \), of the time interval \([0, T] \) with stepsize \( \tau = T/N \), we consider the following fully discrete semi-implicit Euler method for problem (3.12): For the given initial value \( u_0^h = P_{X_h} u_0 \), find a pair of processes \((u_n^h, p_n^h) \in V_h \times Q_h, n = 1, \ldots, N, \) such that the weak formulation
\[
\begin{aligned}
(u_n^h, v_h) + 2\tau \left( \mathbb{D}(u_n^h), \mathbb{D}(v_h) \right) &= (u_{n-1}^h, v_h) + \tau (p_{n-1}^h, \nabla \cdot v_h) + \tau (f(t_n), v_h) \quad \forall v_h \in V_h \\
(\nabla \cdot u_n^h, q_h) &= 0 \quad \forall q_h \in Q_h
\end{aligned}
\] (2.19)
holds \( \mathbb{P} \)-a.s. for all test functions \((v_h, q_h) \in V_h \times Q_h\), where \( \Delta W_n := W(t_n) - W(t_{n-1}) \) is a random variable with \( N(0, \tau Q) \) distribution.

By choosing \( v_h \in X_h \) in (2.19), the fully discrete method in (2.19) can be equivalently written as finding a process \( u_n^h \in X_h, n = 1, \ldots, N, \) such that \( \mathbb{P} \)-a.s.
\[
\begin{aligned}
(u_n^h, v_h) + 2\tau \left( \mathbb{D}(u_n^h), \mathbb{D}(v_h) \right) &= (u_{n-1}^h, v_h) + \tau (f(t_n), v_h) + (B(u_{n-1}^h) \Delta W_n, v_h) \quad \forall v_h \in X_h, \\
u_n^0 &= P_{X_h} u_0
\end{aligned}
\] (2.20)
If we denote by \( A_h : X_h \to X_h \) the discrete Stokes operator defined by
\[ (A_h v_h, w_h) = 2 \left( \mathbb{D}(v_h), \mathbb{D}(w_h) \right) \quad \forall v_h, w_h \in X_h. \] (2.21)
Then the fully discrete method in (2.20) is equivalent to finding a process \( u_n^h \in X_h, n = 1, \ldots, N \) such that \( \mathbb{P} \)-a.s.
\[
\begin{aligned}
u_n^h &= \tilde{E}_{h,\tau} u_{n-1}^h + \tau \tilde{E}_{h,\tau} P_{X_h} f(t_n) + \tilde{E}_{h,\tau} P_{X_h} [B(u_{n-1}^h) \Delta W_n], \\
u_0^0 &= P_{X_h} u_0
\end{aligned}
\] (2.22)
where \( \tilde{E}_{h,\tau} \) denotes the discrete semigroup in the full discretization defined by
\[ \tilde{E}_{h,\tau} v_h = (I + \tau A_h)^{-1} v_h. \] (2.23)

The main result of this article is the following theorem, which provides the convergence of the numerical solution to the mild solution of the stochastic Stokes equations.

**Theorem 2.4.** Let Assumptions 2.1, 2.3 be fulfilled and assume that the finite element space \( V_h \times Q_h \) has Properties (1)–(4). Then the numerical solution \((u_n^h, p_n^h), n = 1, \ldots, N, \) determined by (2.19) has the following error bounds:
\[
\max_{1 \leq n \leq N} \mathbb{E} \left[ \left\| u(t_n) - u_n^h \right\|_{L^2}^2 \right]^{\frac{1}{2}} \lesssim \tau^\frac{3}{2} + h^2,
\] (2.24)
\[
\max_{1 \leq n \leq N} \mathbb{E} \left[ \left\| \int_0^{t_m} p(s) \, ds - \tau \sum_{n=1}^m p_n^h \right\|_{L^2}^2 \right]^{\frac{1}{2}} \lesssim \tau^\frac{3}{2} + h.
\] (2.25)
Remark 2.3. The half-order convergence in time is optimal for the proposed semi-implicit Euler scheme. If we consider a prototypical mini element for spatial discretization, then the error estimate in (2.24) is of optimal order.

The proof of Theorem 2.4 is presented in the next three sections based on the techniques of continuous and discrete analytic semigroups.

3. The abstract formulation under the stress boundary condition

In this section, we present the abstract formulation and functional setting of the stochastic Stokes equations under the stress boundary condition, and define the mild solution of the stochastic Stokes equations to be approximated by the numerical solutions.

The natural function spaces associated to incompressible flow is the divergence-free subspaces of $L^2(D)^d$ and $H^1(D)^d$, defined by

$$X = \{ v \in L^2(D)^d : \nabla \cdot v = 0 \} \quad \text{and} \quad V = X \cap H^1(D)^d. \tag{3.1}$$

It is known that the following orthogonal decomposition holds:

$$L^2(D)^d = X \oplus U \quad \text{with} \quad U = \{ \nabla q : q \in H^1_0(D) \}.$$ 

In particular, any $v \in L^2(D)^d$ can be decomposed as $v = P_X v + \nabla \eta$, where $P_X v = v - \nabla \eta$ denotes the $L^2$-orthogonal projection from $L^2(D)^d$ onto $X$, with $\eta$ being the solution of the equation

$$\begin{cases}
\Delta \eta = \nabla \cdot v \quad \text{in} \quad D, \\
\eta = 0 \quad \text{on} \quad \partial D.
\end{cases} \tag{3.2}$$

Since the solution $\eta$ of the Poisson equation satisfies

$$\| \eta \|_{H^{s+1}} \leq C \| v \|_{H^s} \quad \text{for} \quad s \in [0, 2],$$

it follows that the $L^2$ projection $P_X v = v - \nabla \eta$ has the following properties:

$$\| P_X v \|_{H^s} \lesssim \| v \|_{H^s} \quad \text{for} \quad s \in [0, 2]. \tag{3.3}$$

Since the $L^2$ projection operator $P_X : L^2(D)^d \to X$ is self-adjoint, it follows that (by a duality argument)

$$\| P_X v \|_{H^{-s}} \lesssim \| v \|_{H^{-s}} \quad \text{for} \quad s \in [0, 2]. \tag{3.4}$$

Let $E(t) : X \to X$ be the semigroup of bounded linear operators defined by $E(t) v^0 := v(t)$ as the solution of the linear Stokes equations

$$\begin{cases}
\frac{\partial v}{\partial t} - \nabla \cdot T(v, q) = 0 \quad \text{in} \quad D \times \mathbb{R}_+ \\
\nabla \cdot v = 0 \quad \text{in} \quad D \times \mathbb{R}_+ \\
T(v, q)n = 0 \quad \text{on} \quad \partial D \times \mathbb{R}_+ \\
v(0) = v^0 \quad \text{in} \quad D.
\end{cases}$$

Let $-A$ be the generator of the semigroup $E(t)$ with domain

$$D(A) = \left\{ v^0 \in X : \lim_{t \to 0^+} \frac{E(t) v^0 - v^0}{t} \text{ exists in } X \right\},$$

which is a dense subspace of $X$. Then

$v^0 \in D(A) \iff \partial_t v \in C([0, T]; X)$

$\iff \nabla \cdot T(v, q) \in C([0, T]; X)$ and $\nabla \cdot v = 0$

$\iff -\Delta v + \nabla q \in C([0, T]; X)$ and $\nabla \cdot v = 0$

$$2\mathbb{D}(v)n - qn = 0 \text{ on } \partial D$$

$q$ is a harmonic function with boundary condition $q = 2\mathbb{D}(v)n \cdot n \quad \text{(3.5)}.$
where the last equivalence relation follows from the $H^2$ regularity of the stationary Stokes equations. Since $v^0 \in D(A) \iff Av \in C([0,T];X)$, it follows from (3.6) that
\[ D(A) = \dot{H}^2(D)^d := \{ v \in H^2(D)^d : \nabla \cdot v = 0 \text{ and } D(v)n \times n = 0 \text{ on } \partial D \}. \]
Moreover, from (3.5) we see that the operator $A : D(A) \to X$ can be written as
\[ Av = -\nabla \cdot T(v, q_v) = -\Delta v + \nabla q_v, \quad (3.7) \]
where $q_v$ is determined by $v$ through the following equation:
\[ \begin{cases} \Delta q_v = 0 & \text{in } D, \\ q_v = 2D(v)n \cdot n & \text{on } \partial D. \end{cases} \quad (3.8) \]

For $v \in D(A)$, testing (3.7) with $w$ and using integration by parts, we obtain by utilizing the boundary condition $q_v = 2D(v)n \cdot n$ and $D(v)n \times n = 0$ (which imply $2D(v)n - q_vn = 0$ on $\partial D$)
\[ (Av, w) = (2D(v), D(w)) \quad \forall \ w \in V. \quad (3.9) \]

Therefore, the Stokes operator $A : D(A) \to X$ has an extension $A : V \to V'$ defined by (3.9).

The boundary condition $\mathbb{T}(u, p)n = 0$ in (1.1) implies that $\mathbb{D}(u)n \times n = 0$ and $p = 2\mathbb{D}(u)n \cdot n$, and therefore
\[ -P_X\mathbb{T}(u, p) = -\Delta u + P_X\nabla p = -\Delta u + \nabla(p - \eta), \]
where $\eta$ is the solution of
\[ \begin{cases} \Delta \eta = \Delta p & \text{in } D, \\ \eta = 0 & \text{on } \partial D. \end{cases} \quad (3.10) \]

This implies that
\[ -P_X\mathbb{T}(u, p) = -\Delta u + \nabla q_a = Au, \quad (3.11) \]
where $q_a = p - \eta$ is the harmonic function satisfying $q_a = 2\mathbb{D}(u)n \cdot n$ on $\partial D$.

As a result, applying $P_X$ to (1.1) yields the following abstract formulation of the stochastic Stokes problem in (1.1):
\[ \begin{cases} du = [-Au + P_X f]dt + P_X[B(u)]dW & \text{in } D \times (0,T), \\ u = u^0 & \text{at } D \times \{0\}. \end{cases} \quad (3.12) \]

A predictable process $u \in C([0,T];L^2(\Omega;X))$ is called a mild solution of problem (3.12) if
\[ u(t) = E(t)u^0 + \int_0^t E(t-s)P_X f(s)ds + \int_0^t E(t-s)P_X B(u(s))dW(s) \quad \mathbb{P}\text{-a.s.} \quad (3.13) \]

The proof of the following proposition is presented in Appendix A.

**Proposition 3.1.** (Well-posedness and regularity) Under Assumptions 2.1–2.3 problem (3.12) has a unique mild solution in the sense of (3.13). Moreover, the mild solution satisfies the following regularity estimates:
\[ \sup_{t \in [0,T]} \mathbb{E}\|u(t)\|^2_{H^2} \lesssim (1 + \mathbb{E}\|u^0\|^2_{H^2}), \quad (3.14) \]
\[ \mathbb{E}\|u(t) - u(s)\|^2_{H^2} \lesssim (1 + \mathbb{E}\|u^0\|^2_{H^2})(t-s) \quad \forall 0 \leq s \leq t \leq T \quad (3.15) \]
\[ \mathbb{E}\|u(t) - u(s)\|^2_{H^2} \lesssim (1 + \mathbb{E}\|u^0\|^2_{H^2})(t-s)^{2-\beta} \quad \forall 0 \leq s \leq t \leq T, \ \forall \beta \in \left(\frac{d}{2}, 2\right). \quad (3.16) \]

The error estimates for the numerical approximations will be proved based on the regularity results in Proposition 3.1.

Since the mild solution has the regularity $u \in C([0,T];L^2(\Omega, H^1(D)^d))$, the inf-sup condition [15] Theorem 4.1] implies that there exits $\int_0^t p(s)ds \in C([0,T];L^2(\Omega, L^2(D)))$ satisfying following relation:
\[ \left( \int_0^t p(s)ds, \nabla \cdot v \right) = (u(t) - u^0, v) + 2\int_0^t (\mathbb{D}(u(s)), \mathbb{D}(v)) \ ds \]
\[-\int_0^t (f(s), v) \, ds - \left( \int_0^t B(u(s)) \, dW(s), v \right) \quad \forall v \in H^1(D)^d. \quad (3.17)\]

4. Estimates for the discrete semigroups

In this section, we establish some technical estimates of the discrete analytic semigroup associated to the Stokes operator. The results will be used to prove the optimal-order convergence of the numerical solution in Section 5.

The main results of this section are the following three types of error estimates about approximating the semigroup \( E(t)P_X \) by the discrete semigroup \( E^n_{h,T}P_{X_h} \), \( n = 1, \ldots, N \), defined in \( (2.23) \). These results are the key to the error analysis for the stochastic Stokes problem.

Lemma 4.1. For any \( v \in H^1(D)^d \), there holds
\[
\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| [E(s)P_X - E^n_{h,T}P_{X_h}] v \|_{L^2}^2 \, ds \leq C(\tau + h^4)\| v \|_{H^1}^2. \quad (4.1)
\]

In addition, for all \( v \in L^2(D)^d \), there holds
\[
\left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [E(s)P_X - E^n_{h,T}P_{X_h}] v \, ds \right\|_{L^2}^2 \leq C(\tau + h^4)\| v \|_{L^2}^2, \quad (4.2)
\]
and
\[
\left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \nabla [E(s)P_X - E^n_{h,T}P_{X_h}] v \, ds \right\|_{L^2}^2 \leq C(\tau + h^2)\| v \|_{L^2}^2. \quad (4.3)
\]

Remark 4.1. Note that the error estimates of \( E(t)P_X - E^n_{h,T}P_{X_h} \) in [31] holds for an abstract parabolic equation, including the Stokes equations, but requires the condition \( v \in X \). Since we only require \( B(u) : L^2(D)^d \to L^2(D)^d \) rather than \( B(u) : L^2(D)^d \to X \), we cannot apply the results in Lemma 4.1 to the stochastic Stokes equations under the condition \( v \in X \). By dropping this condition in Lemma 4.1, we manage to prove second-order convergence in space for the numerical solution of the stochastic Stokes equations.

In order to prove Lemma 4.1, we need to extend the \( H^1 \)-stability of the \( L^2 \) projection \( P_{X_h} \) from \( v \in X \cap H^1(D)^d \) to \( v \in H^1(D)^d \). This is obtained by characterizing the orthogonal complement of \( X_h \) in \( V_h \), as discussed in subsection 4.1. The proof of Lemma 4.1 is presented at the end of this section after introducing the orthogonal decomposition and some properties of the discrete semigroup.

4.1. Orthogonal complement of \( X_h \) in \( V_h \)

Let \( X_h^\perp \) be the orthogonal complement of \( X_h \) in \( V_h \), i.e., \( V_h = X_h \oplus X_h^\perp \), namely, any \( v_h \in V_h \) has an orthogonal decomposition
\[
v_h = w_h + z_h \quad \text{with} \quad w_h \in X_h \quad \text{and} \quad z_h \in X_h^\perp \quad \text{satisfying} \quad (w_h, z_h) = 0. \quad (4.4)
\]

This decomposition is stable in the \( H^1 \) norm, as shown in the following lemma.

Lemma 4.2. The orthogonal decomposition in \( (4.4) \) is stable in the \( H^1 \) norm, i.e.,
\[
\| w_h \|_{H^1} + \| z_h \|_{H^1} \lesssim \| v_h \|_{H^1} \quad \forall v_h \in V_h. \quad (4.5)
\]

Proof. Let \( P_{Q_h} : L^2(D) \to Q_h \) be the \( L^2 \)-orthogonal projection onto \( Q_h \). For any given \( v_h \in V_h \), the inf-sup condition \( (2.18) \) implies that there exists a unique solution \( (w_h, q_h) \in V_h \times Q_h \) of the following equations:
\[
(w_h, a_h) - (q_h, \nabla \cdot a_h) = (v_h, a_h) \quad \forall a_h \in V_h, \\
(\nabla \cdot w_h, \eta_h) = 0 \quad \forall \eta_h \in Q_h.
\]
Then $w_h \in X_h$ and

\[(w_h, a_h) = (v_h, a_h) \quad \forall a_h \in X_h.\]

Let $z_h = v_h - w_h$, then $w_h$ and $z_h$ are the functions in the orthogonal decomposition (13). Before estimating $(w_h, q_h)$ directly, we first introduce $(w, q) \in H^1(D)^d \times H^1(D)$ to be the solution of the continuous problem

\[(w, a) - (q, \nabla \cdot a) = (v_h, a) \quad \forall a \in H^1(D)^d,\]

\[\langle \nabla \cdot w, \eta \rangle = 0 \quad \forall \eta \in L^2(D).\]

Via integration by parts in the equation of $w$, one can obtain that $w = v_h - \nabla q$, with $q \in H^1(D)$ being the weak solution of

\[
\begin{cases}
\Delta q = \nabla \cdot v_h & \text{in } D, \\
q = 0 & \text{on } \partial D.
\end{cases}
\]

The equation above has the standard $H^2$ elliptic regularity, i.e.,

\[\|q\|_{H^2} \lesssim \|\nabla \cdot v_h\|_{L^2} \lesssim \|v_h\|_{H^1},\]  

which implies that

\[\|w\|_{H^1} = \|v_h - \nabla q\|_{H^1} \lesssim \|v_h\|_{H^1} \tag{4.6}\]

Denote $\theta_h = w_h - \Pi_h w$ and $\phi_h = q_h - P_{Q_h} q$, and consider the difference between the equations of $w_h$ and $w$, i.e.,

\[(\theta_h, a_h) - (\phi_h, \nabla \cdot a_h) = (w - \Pi_h w, a_h) + (P_{Q_h} q - q, \nabla \cdot a_h) \quad \forall a_h \in V_h,\]

\[\langle \nabla \cdot \theta_h, \eta_h \rangle = 0 \quad \forall \eta_h \in Q_h.\]

Substituting $a_h = \theta_h$ and $\eta_h = \phi_h$ into the equations above and using the inverse inequality (2.17) we obtain

\[\|\theta_h\|_{L^2}^2 = (w - \Pi_h w, \theta_h) + (P_{Q_h} q - q, \nabla \cdot \theta_h) \leq \|w - \Pi_h w\|_{L^2} \|\theta_h\|_{L^2} + \|P_{Q_h} q - q\|_{L^2} \|\nabla \cdot \theta_h\|_{L^2}
\]

\[\lesssim h\|w\|_{H^1} \|\theta_h\|_{L^2} + h^2\|q\|_{H^2} \|\theta_h\|_{H^1}
\]

\[\lesssim h(\|w\|_{H^1} + \|q\|_{H^2}) \|\theta_h\|_{L^2},\]

which together with (4.6) and (4.7) implies

\[\|\theta_h\|_{H^1} \lesssim h^{-1} \|\theta_h\|_{L^2} \lesssim \|w\|_{H^1} + \|q\|_{H^2} \lesssim \|v_h\|_{H^1}.\]

Therefore, the $H^1$-stability (2.11) of the Fortin projection operator gives

\[\|w_h\|_{H^1} = \|\theta_h + \Pi_h w\|_{H^1} \lesssim \|v_h\|_{H^1},\]

which yields

\[\|z_h\|_{H^1} = \|v_h - w_h\|_{H^1} \lesssim \|v_h\|_{H^1}.\]

This proves the desired $H^1$-stability result in (4.3). \qed

**Remark 4.2.** The $H^1$ stability in Lemma (2.2) implies the following properties:

\[\|P_{X_h} v\|_{H^1} \leq C\|v\|_{H^1} \quad \text{for } v \in H^1(D)^d,\]  

\[\|v - P_{X_h} v\|_{H^1} \lesssim \inf_{v_h \in X_h} \|v - v_h\|_{H^1} \leq C h \|v\|_{H^2} \quad \text{for } v \in D(A),\]  

\[\|v - P_{X_h} v\|_{L^2} \lesssim \inf_{v_h \in X_h} \|v - v_h\|_{L^2} \leq C h^2 \|v\|_{H^2} \quad \text{for } v \in D(A).\]  

For any $v \in D(A)$, we denote by $q_v$ the solution of (3.8), and denote by $R_{X_h}: V \to X_h$ the Stokes–Ritz projection defined by

\[(v - R_{X_h} v, w_h) + (A_h(v - R_{X_h} v), w_h) - (q_v, \nabla \cdot w_h) = 0 \quad \forall w_h \in X_h,\]  

which is equivalent to finding $(R_{X_h} v, q_v, h) \in V_h \times Q_h$ satisfying

\[(v - R_{X_h} v, w_h) + 2\langle \mathcal{D}(v - R_{X_h} v), \mathcal{D}(w_h) \rangle - (q_v - q_{v, h}, \nabla \cdot w_h) = 0 \quad \forall w_h \in V_h,\]

\[\langle \nabla \cdot (v - R_{X_h} v), q_v \rangle = 0 \quad \forall q_v \in Q_h.\]  

(4.12)
The Stokes–Ritz projection has the following error bound (cf. [11 Lemma 2.44 and Lemma 2.45] and [11 Proposition 4.18]):

\[ \|v - R_X(v)\|_{L^2} + h\|v - R_Xh\|_{H^1} \lesssim h^2\|v\|_{H^2} + \|q_v\|_{H^1} \leq C h^2\|v\|_{H^2} \quad \forall v \in D(A). \]  

(4.13)

### 4.2. Fractional powers of \( I + A \)

Since the Stokes operator \( A \) defined in (3.7) is self-adjoint and positive semi-definite, it follows that the operator \( z + A \) is invertible on \( X \) for \( z \in \mathbb{C}\setminus(-\infty,0) \), and \( -A \) generates a bounded analytic semigroup \( E(t) = e^{-tA} \) on \( X \); see [11 Example 3.7.5]. The fractional powers of the positive definite operator \( I + A \) (with compact inverse) can be defined by means of the spectral decomposition (see [22 Appendix B.2]), and the following norm equivalence holds.

**Lemma 4.3.** The following equivalence relations hold for \( s \in [0,2] \):

\[ \|v\|_{H^s} \lesssim \|(I + A)^s v\|_{L^2} \lesssim \|v\|_{H^s} \quad \forall v \in D(A^s), \]  

(4.14)

\[ \|v\|_{H^{-s}} \lesssim \|(I + A)^{-s} v\|_{L^2} \lesssim \|v\|_{H^{-s}} \quad \forall v \in D(A^s)' \]  

(4.15)

where \( D(A^s) = (X,D(A))[s/2] \) for \( s \in [0,2] \).

**Remark 4.3.** \( (X,D(A))[\theta] \), with \( \theta \in [0,1] \), denotes the complex interpolation spaces between \( X \) and \( D(A) \). Since \( D(A^s) = X \cap H^1(D)^d \), via complex interpolation it can be shown that \( D(A^s) = X \cap H^s(D)^d \) for \( s \in [0,1] \).

The proof of Lemma 4.3 is presented in Appendix B.

The analyticity of the semigroup \( E(t) = e^{-tA} \) implies the following results.

**Lemma 4.4.** For any \( v \in X \), the following results hold for \( t \in (0,T) \):

\[ \|(I + A)\gamma E(t)v\|_{L^2} \lesssim t^{-\gamma}\|v\|_{L^2} \quad \forall \gamma \geq 0, \]  

(4.16)

\[ \|(I + A)^{-\mu} (I - E(t)) v\|_{L^2} \lesssim t^\mu\|v\|_{L^2} \quad \forall \mu \in [0,1]. \]  

(4.17)

The following results hold for \( 0 \leq t_1 \leq t_2 \leq T \):

\[ \int_{t_1}^{t_2} \|(I + A)^{s/r} E(t_2 - s) v\|_{L^2}^2 \, ds \lesssim (t_2 - t_1)^{1-s/r} \|v\|_{L^2}^2 \quad \forall r \in [0,1], \]  

(4.18)

\[ \|(I + A)^{s/r} \int_{t_1}^{t_2} E(t_2 - s) v \, ds\|_{L^2} \lesssim (t_2 - t_1)^{1-s/r} \|v\|_{L^2} \quad \forall r \in [0,1]. \]  

(4.19)

The proof of Lemma 4.4 is also presented in Appendix B.

### 4.3. The discrete semigroup from spatial discretization

Let \( 0 = \lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,M_h} \) be the eigenvalues of the discrete Stokes operator \( A_h : X_h \to X_h \) with corresponding orthonormal eigenvectors \( \{\varphi_{h,j}\}_{j=1}^{M_h} \subset X_h \) and \( \dim(X_h) = M_h \).

The operator \( -A_h \) generates a bounded analytic semigroup on \( X_h \) defined by

\[ E_h(t) = e^{-tA_h}, \]  

(4.20)

which can be expressed as \( E_h(t)v_h = \sum_{j=1}^{M_h} e^{-t\lambda_{h,j}} (v_h, \varphi_{h,j}) \varphi_{h,j} \) for all \( v_h \in X_h \). Hence, the following two properties can be derived based on the proof of [31 Lemma 3.9] and [23 Lemma 3.2 (iii)], respectively:

\[ \|(I + A_h)^s E_h(t)v_h\|_{L^2} \lesssim t^{-s}\|v_h\|_{L^2} \quad \text{for} \quad v_h \in X_h, \gamma \in [0,1], \]  

(4.21)

\[ \int_0^t \|(I + A_h)^s E_h(s)v_h\|_{L^2}^2 \, ds \lesssim \|v_h\|_{L^2}^2 \quad \text{for} \quad v_h \in X_h. \]  

(4.22)

Let

\[ \Phi_h(t) := E(t)P_X - E_h(t)P_{X_h} \quad \text{for} \quad t \in [0,T], \]  

(4.23)

which is the error in approximating the continuous semigroup. The main results of this subsection are the estimates of \( \Phi_h(t) \) presented in the following lemma.
Lemma 4.5. For $v \in H^1(D)^d$, the following estimates hold:
\[
\|\Phi_h(t)v\|_{L^2} \leq Ct^{-\frac{1}{2}}h^2\|v\|_{H^1},
\]
and
\[
\left(\int_0^t \|\Phi_h(s)v\|_{H^1}^2 \, ds\right)^{\frac{1}{2}} \leq Ch^2\|v\|_{H^1},
\]
For $v \in L^2(D)^d$, the following estimates hold:
\[
\|\Phi_h(t)v\|_{L^2} \leq Ct^{-1}h^2\|v\|_{L^2},
\]
and
\[
\left\|\int_0^t \Phi_h(s)v \, ds\right\|_{L^2} + h\left\|\int_0^t \nabla \Phi_h(s)v \, ds\right\|_{L^2} \leq Ch^2\|v\|_{L^2}.
\]
If $v \in D(A^{\frac{1}{2}})$, then
\[
\|\Phi_h(t)v\|_{L^2} + h\|\nabla \Phi_h(t)v\|_{L^2} \leq Ct^{-\frac{1}{2}}h^2\|v\|_{H^{1+\rho}} \quad \text{for } \rho \in [0,1].
\]

Remark 4.4. Lemma 4.5 is proved later based on an orthogonal decomposition of $v = P_Xv + \nabla q$ for $v \in L^2(D)^d$, and it does not require $\nabla \cdot v = 0$ in the cases $v \in L^2(D)^d$ and $v \in H^1(D)^d$. This is important for the error analysis of the stochastic Stokes equations.

The proof of Lemma 4.5 is based on error estimates for the semi-discrete FEM for the deterministic linear Stokes problem
\[
\begin{cases}
\partial_t u - \nabla \cdot T(u,p) = 0 & \text{in } D \times [0,T], \\
\nabla \cdot u = 0 & \text{in } D \times [0,T], \\
T(u,p)n = 0 & \text{on } \partial D \times [0,T], \\
u(0) = u^0 & \text{in } D.
\end{cases}
\]
Applying $P_X$ to the first equation of (4.29) yields the following abstract formulation:
\[
\partial_t u + Au = 0 \quad \text{with} \quad u(0) = u^0 \in X,
\]
of which the solution can be represented $u(t) = E(t)P_Xu^0$ in terms of the semigroup generated by the Stokes operator. The FEM for (4.29) reads: for given $u_h(0) = P_{Xh}u^0$, find $(u_h(t),p_h(t)) \in V_h \times Q_h$ such that
\[
\begin{cases}
(\partial_t u_h(t), v_h) + 2(\mathbb{D}(u_h(t)), \mathbb{D}(v_h)) - (p_h(t), \nabla \cdot v_h) = 0 & \forall v_h \in V_h, \\
(\nabla \cdot u_h(t), q_h) = 0 & \forall q_h \in Q_h.
\end{cases}
\]
This can be written into the following abstract form by choosing $v_h \in X_h$:
\[
\frac{d}{dt} u_h(t) + A_h u_h(t) = 0 \quad \text{with} \quad u_h(0) = P_{Xh}u^0,
\]
of which the solution can be represented as $u_h(t) = E_h(t)P_{Xh}u^0$.

We shall estimate the error between the solutions of (4.29) and (4.31) by estimating the difference between the continuous and discrete resolvent operators, i.e., $(z + A)^{-1}$ and $(z + A_h)^{-1}$. This is presented in the following lemma. Its proof is presented in Appendix B.

Lemma 4.6. For any $f \in L^2(D)^d$, let $w$ and $w_h$ be the solution and finite element solution of the Stokes equations
\[
\begin{cases}
zw - \nabla \cdot T(w,p) = f & \text{in } D, \\
\nabla \cdot w = 0 & \text{in } D, \\
T(w,p)n = 0 & \text{on } \partial D,
\end{cases}
\]
and
\[
\begin{cases}
(zw_h, v_h) + 2(\mathbb{D}(w_h), \mathbb{D}(v_h)) - (p_h, \nabla \cdot v_h) = (f, v_h) & \forall v_h \in V_h, \\
(\nabla \cdot w_h, q_h) = 0 & \forall q_h \in Q_h.
\end{cases}
\]
where $z \in \Sigma_\phi := \{1 + z' \in \mathbb{C} : |\arg(z')| < \phi\}$ for some $\phi \in (0, \pi)$. Then the following results hold:

\[
\|w - w_h\|_{L^2} + h\|\nabla (w - w_h)\|_{L^2} \leq C h^2 (\|w\|_{H^2} + \|p\|_{H^1}),
\]

\[
\|w\|_{H^2} + \|p\|_{H^1} \leq C \|f\|_{L^2},
\]

where the constant $C$ is independent of $z \in \Sigma_\phi$ (but may depend on $\phi$).

**Remark 4.5.** The first equation in (4.33) can be written as
\[\nabla p = f - (z - \Delta) w.\]
By applying $P_X$ to the first equation in (4.33) and using relation $-P_X T(w, p) = -\Delta w + \nabla q_w$, as shown in (3.11), we obtain
\[\nabla q_w = P_X f - (z - \Delta) w.\]
Combining the two equations above yields
\[\nabla p = f - P_X f + \nabla q_w.\]
In the case $f \in X$, we obtain $p = q_w$ and therefore (4.35) reduces to
\[\|w - w_h\|_{L^2} + h\|\nabla w - \nabla w_h\|_{L^2} \leq C h^2 \|w\|_{H^2},\]
where we have used the inequality $\|q_w\|_{H^1} \lesssim \|w\|_{H^2}$.

Before proving Lemma 4.5 for the general case $v \in H^1(D)^d$, we consider the simpler case $g \in X \cap H^1(D)^d$ in the following lemma.

**Lemma 4.7.** It holds that
\[
\int_0^t \|\Phi_h(s)g\|_{L^2}^2 \, ds \leq C h^4 \|g\|_{H^1}^2 \quad \forall g \in X \cap H^1(D)^d.
\]

**Proof.** Let $(u, p)$ and $(u_h, p_h)$ be the solution of (4.29) and (4.31) with $u^0 = g \in X \cap H^1(D)^d$, respectively. Then the definition of $\Phi_h(s) = E(s)P_X - E_h(s)P_{Q_h}$ implies that
\[\Phi_h(s)g = u(s) - u_h(s).\]
Subtracting (4.31) from (4.29) yields
\[
\begin{cases}
(\partial_t (u - u_h), v_h) + 2(\nabla (u - u_h), \nabla (v_h)) - (p - p_h, \nabla \cdot v_h) = 0 & \forall v_h \in V_h, \\
(\nabla \cdot (u - u_h), \eta_h) = 0 & \forall \eta_h \in Q_h,
\end{cases}
\]
and therefore, choosing $v_h \in X_h$ and using the Ritz projection $R_{Q_h}$ defined in (4.11), we have
\[
\begin{aligned}
(\partial_t (u - u_h), v_h) + ((I + A_h)(R_{Q_h}u - u_h), v_h) \\
+ \left( (u - u_h, v_h) + (q - P_{Q_h}q, \nabla \cdot v_h) - (q - P_{Q_h}q, \nabla \cdot v_h). \right)
\end{aligned}
\]
By denoting $\tilde{e}_h = P_{X_h}u - u_h \in X_h$ and choosing $v_h = (I + A_h)^{-1}\tilde{e}_h \in X_h$ in (4.40), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|(I + A_h)^{-1}\tilde{e}_h\|_{L^2}^2 + \|\tilde{e}_h\|_{L^2}^2
\]
\[
\leq \|(I + A_h)^{-1}\tilde{e}_h\|_{L^2}^2 + C \|R_{Q_h}u - P_{Q_h}u\|_{L^2}^2 + 1/4 \|\tilde{e}_h\|_{L^2}^2 + |R_q(\tilde{e}_h)| + |R_{Q_h}(\tilde{e}_h)|
\]
\[
\leq \|(I + A_h)^{-1}\tilde{e}_h\|_{L^2}^2 + C h^4 \|u\|_{H^2}^2 + 1/4 \|\tilde{e}_h\|_{L^2}^2 + |R_q(\tilde{e}_h)| + |R_{Q_h}(\tilde{e}_h)|,
\]
where
\[
R_q(\tilde{e}_h) = (q - P_{Q_h}q, \nabla \cdot [(I + A_h)^{-1}\tilde{e}_h]) \quad \text{and} \quad R_{Q_h}(\tilde{e}_h) = -(q - P_{Q_h}q, \nabla \cdot [(I + A_h)^{-1}\tilde{e}_h]).
\]
The remainders $R_q(\tilde{e}_h)$ and $R_{Q_h}(\tilde{e}_h)$ can be estimated by considering $w = (I + A)^{-1}P_X \tilde{e}_h$ and $w_h = (I + A_h)^{-1}P_{X_h} \tilde{e}_h$, which are the solutions of (4.33) and (4.34), respectively, with $z = 1$ and $f = \tilde{e}_h$. Then applying Lemma 4.6 yields
\[
\|(I + A)^{-1}P_X - (I + A_h)^{-1}P_{X_h}\|_{H^1} \lesssim h \|\tilde{e}_h\|_{L^2},
\]
which implies that
\[
|R_q(\tilde{e}_h)| = |(q - P_{Q_h}q, \nabla \cdot [(I + A)^{-1} P_X \tilde{e}_h])| \\
+ |(q - P_{Q_h}q, \nabla \cdot [(I + A)^{-1} P_X - (I + A_h)^{-1} P_{X_h}] \tilde{e}_h)| \\
\lesssim \|q - P_{Q_h}q\|_{H^{-1}} \|\nabla \cdot [(I + A)^{-1} P_X \tilde{e}_h]\|_{H^1} \\
+ \|q - P_{Q_h}q\|_{L^2} \|[(I + A)^{-1} P_X - (I + A_h)^{-1} P_{X_h}] \tilde{e}_h\|_{H^1} \\
\lesssim h^2 \|q\|_{H^1} \|(I + A)^{-1} P_X \tilde{e}_h\|_{H^2} + h \|q\|_{H^1} \|\tilde{e}_h\|_{L^2} \\
\lesssim h^2 \|q\|_{H^1} \|\tilde{e}_h\|_{L^2} \\
\leq Ch^4 \|q\|_{H^1} + \frac{1}{8} \|\tilde{e}_h\|_{L^2}^2,
\]
(4.42)
where we have used the inequality \(\|q - P_{Q_h}q\|_{H^{-1}} \lesssim h^2 \|q\|_{H^1}\); see Property (2) of Section 2.5.
Similarly, since \(\|q_u\|_{H^1} \lesssim \|u\|_{H^2}\), it follows that
\[
|R_{Q_u}(\tilde{e}_h)| \leq Ch^4 \|q_u\|_{H^1} + \frac{1}{8} \|\tilde{e}_h\|_{L^2}^2 \leq Ch^4 \|u\|_{H^2} + \frac{1}{8} \|\tilde{e}_h\|_{L^2}^2.
\]
(4.43)
Substituting (4.42)–(4.43) into (4.41), and using Gronwall’s inequality with \(\tilde{e}_h(0) = 0\), we obtain
\[
\|(I + A_h)^{-\frac{1}{2}} \tilde{e}_h(t)\|_{L^2}^2 + \int_0^t \|\tilde{e}_h(s)\|_{L^2}^2 \, ds \lesssim h^4 \int_0^t (\|u\|_{H^2}^2 + \|q\|_{H^1}^2) \, ds \lesssim h^4 \|u^0\|_{H^1}^2
\]
\[
= h^4 \|g\|_{H^1}^2,
\]
(4.44)
where the last inequality is the basic energy inequality for the Stokes equations, which can be obtained by testing (1.30) with \(Au\), respectively. Since \(u(s) - u_h(s) = u(s) - P_{X_h}u(s) + \tilde{e}_h\), by using the inequality above and the triangle inequality we obtain
\[
\int_0^t \|u(s) - u_h(s)\|_{L^2}^2 \, ds \lesssim h^4 \|g\|_{H^1}^2.
\]
This proves the result of Lemma 4.7 in view of (1.39).

Now we prove Lemma 4.8 by utilizing the result of Lemma 4.7.

**Proof of Lemma 4.8.** For \(v \in L^2(D)^d\) we denote
\[
w = (z + A)^{-1} P_X v \quad \text{and} \quad w_h = (z + A_h)^{-1} P_{X_h} v,
\]
which are the solutions of (4.33) and (4.31) with \(f = v\).

Since \(-A\) generates a bounded analytic semigroup on \(X\), there exists an angle \(\phi \in (\frac{\pi}{2}, \pi)\) such that the operator \((z + A)^{-1}\) is analytic with respect to \(z\) in the sector \(\Sigma_\phi = \{z \in \mathbb{C} \setminus \{0\}: |\arg(z)| < \phi\}\). Moreover, the semigroup \(E(t) = e^{-tA}\) can be expressed in terms of the resolvent operator \((z + A)^{-1}\) through a contour integral on the complex plane, i.e.,
\[
E(t)P_X v = \frac{1}{2\pi i} \int_\Gamma (z + A)^{-1} P_X v e^{zt} \, dz, \quad \text{with} \quad \Gamma = \{1 + z : |\arg(z)| = \phi\} \subset \Sigma_\phi.
\]
(4.45)
Similarly,
\[
E_h(t)P_{X_h} v = \frac{1}{2\pi i} \int_\Gamma (z + A_h)^{-1} P_{X_h} v e^{zt} \, dz.
\]
(4.46)
Therefore,
\[
\|\Phi_h(t)v\|_{L^2} = \|E(t)P_X v - E_h(t)P_{X_h} v\|_{L^2} \\
= \left\| \frac{1}{2\pi i} \int_\Gamma [(z + A)^{-1} P_X v - (z + A_h)^{-1} P_{X_h} v] e^{zt} \, dz \right\|_{L^2}
\leq \int_\Gamma |w - w_h| e^{\text{Re}(z)t} |dz|
\lesssim \int_\Gamma h^2 \|v\|_{L^2} e^{\text{Re}(z)t} |dz| \quad \text{(This follows from Lemma 4.6 with } f = v)\]
\[ \lesssim t^{-1}h^2\|v\|_{L^2} \quad \text{for } t \in (0, T). \quad (4.47) \]

This proves (4.26).

If \( v \in \mathcal{D}(A^{1/2}) \) then we use inequality (4.37), which implies that
\[
\|w - w_h\|_{L^2} + h\|\nabla(w - w_h)\|_{L^2} \lesssim h^2\|\nabla w\|_{H^2} = h^2\|(z + A)^{-1}v\|_{H^2} \lesssim h^2\|(I + A)(z + A)^{-1}v\|_{L^2} = h^2\|(I + A)^{1/2}(z + A)^{-1}(I + A)^{1/2}v\|_{L^2} \lesssim h^2\|z\|^{-1/2}\|(I + A)^{1/2}v\|_{L^2} \quad \forall \rho \in [0, 1], \quad (4.48)\]

where we have used the interpolation inequality to get
\[ \|\rho(z + A)^{-1}v\|_{L^2} \leq C_\rho|z|^{-(1-\theta)}\|v\|_{L^2} \quad \forall \theta \in [0, 1], \quad v \in X. \quad (4.49) \]

Substituting (4.48) into (4.37) and using (4.14) gives
\[
\|\Phi_h(t)v\|_{L^2} + h\|\nabla\Phi_h(t)v\|_{L^2} \lesssim h^2\|(I + A)^{1/2}v\|_{L^2} \lesssim t^{-\frac{1}{2}}h^2\|v\|_{H^{1+\rho}} \quad \forall \rho \in [0, 1], \quad v \in \mathcal{D}(A^{1/2}). \quad (4.50) \]

This proves (4.28).

In order to prove (4.24)–(4.25), we consider the orthogonal decomposition \( v = P_X v + \nabla q \) for a function \( v \in H^1(D) \) is the weak solution of
\[
\begin{align*}
\Delta q &= \nabla \cdot v & \text{in } D \\
q &= 0 & \text{on } \partial D,
\end{align*}
\]

which has the classical \( H^1_0(D) \cap H^2(D) \) elliptic regularity, i.e.,
\[ \|q\|_{H^2} \lesssim \|\nabla \cdot v\|_{L^2} \lesssim \|v\|_{H^1}. \quad (4.51) \]

Since \( \Phi_h(t)\nabla q = E(t)P_X \nabla q - E_h(t)P_{X_h} \nabla q = -E_h(t)P_{X_h} \nabla q \in X_h \), by using the self-adjointness of \( E_h(t) \) we have
\[ (\Phi_h(t)\nabla q, a_h) = -(P_{X_h} \nabla q, E_h(t)a_h) = (q, \nabla \cdot [E_h(t)a_h]) \quad (\text{since } E_h(t)a_h \in X_h \text{ and } q = 0 \text{ on } \partial D) = (q - q_h, \nabla \cdot [E_h(t)a_h]) \quad (\text{for any } q_h \in H^1_0(D) \cap Q_h) = -(E_h(t)P_{X_h} \nabla (q - q_h), a_h) = -((I + A_h)^{1/2}E_h(t)(I + A_h)^{-1/2}P_{X_h} \nabla (q - q_h), a_h), \]

which implies that
\[ \Phi_h(t)\nabla q = -(I + A_h)^{1/2}E_h(t)(I + A_h)^{-1/2}P_{X_h} \nabla (q - q_h) \quad \forall q_h \in Q_h \cap H^1_0(D). \quad (4.52) \]

On the one hand, (4.52) can be combined with (4.21) to yield
\[
\|\Phi_h(t)\nabla q\|_{L^2} \lesssim \inf_{q_h \in Q_h \cap H^1_0(D)} t^{-\frac{1}{2}}\|(I + A_h)^{-1/2}P_{X_h} \nabla (q - q_h)\|_{L^2} \lesssim \inf_{q_h \in Q_h \cap H^1_0(D)} t^{-\frac{1}{2}}\|q - q_h\|_{L^2} \lesssim t^{-\frac{1}{2}}h^2\|q\|_{H^2},
\]

where (2.13) is used. On the other hand, inequality (4.52) can be combined with (4.22) to yield
\[
\left( \int_0^T \|\Phi_h(t)\nabla q\|_{L^2}^2 dt \right)^{1/2} \lesssim \inf_{q_h \in Q_h \cap H^1_0(D)} \|(I + A_h)^{-1/2}P_{X_h} \nabla (q - q_h)\|_{L^2} \lesssim \inf_{q_h \in Q_h \cap H^1_0(D)} \|q - q_h\|_{L^2} \lesssim h^2\|q\|_{H^2}.
\]
Substituting (4.51) into the two inequalities above, we obtain
\[ \| \Phi_h(t) \nabla q \|_{L^2} \lesssim t^{-\frac{1}{2}} h^2 \| v \|_{H^1} \quad \text{and} \quad \left( \int_0^t \| \Phi_h(s) \nabla q \|_{L^2}^2 \, ds \right)^{\frac{1}{2}} \lesssim h^2 \| v \|_{H^1}. \]  
(4.53)

By using inequality (4.50) with \( \rho = 0 \) (with \( v \) replaced by \( P_X v \) therein) and Lemma 4.7 with \( g = P_X v \), we have
\[ \| \Phi_h(t) P_X v \|_{L^2} \lesssim t^{-\frac{1}{2}} h^2 \| v \|_{H^1} \quad \text{and} \quad \left( \int_0^t \| \Phi_h(s) P_X v \|_{L^2}^2 \, ds \right)^{\frac{1}{2}} \lesssim h^2 \| v \|_{H^1}. \]  
(4.54)

Then, combining (4.53) and (4.54) in the decomposition \( v = P_X v + \nabla q \), we obtain
\[ \| \Phi_h(t) v \|_{L^2} \lesssim t^{-\frac{1}{2}} h^2 \| v \|_{H^1} \quad \text{and} \quad \left( \int_0^t \| \Phi_h(s) v \|_{L^2}^2 \, ds \right)^{\frac{1}{2}} \lesssim h^2 \| v \|_{H^1}. \]  
(4.55)

This proves (4.21) and (4.26).

It remains to prove (4.27). To this end, we use the expressions in (4.45) and (4.46), which imply that
\[
\int_0^t \Phi_h(s) v \, ds = \int_0^t \frac{1}{2\pi i} \int_{\Gamma} [(z + A)^{-1} P_X v - (z + A_h)^{-1} P_X h v] e^{z s} \, dz \, ds \\
= \frac{1}{2\pi i} \int_{\Gamma} z^{-1} [(z + A)^{-1} P_X v - (z + A_h)^{-1} P_X h v] e^{z s} \, dz \\
= \frac{1}{2\pi i} \int_{\Gamma} z^{-1} [w - w_h] e^{z t} \, dz.
\]

By applying Lemma 4.6 with \( f = v \), we have
\[
\left\| \int_0^t \Phi_h(s) v \, ds \right\|_{L^2} + h \left\| \nabla \int_0^t \Phi_h(s) v \, ds \right\|_{L^2} \\
\lesssim \int_{\Gamma} |z|^{-1} (\| w - w_h \|_{L^2} + h \| \nabla w - \nabla w_h \|_{L^2}) e^{Re(z)t} |dz| \\
\lesssim h^2 \| v \|_{L^2} \int_{\Gamma} |z|^{-1} e^{Re(z)t} |dz| \lesssim h^2 \| v \|_{L^2}.
\]  
(4.56)

This proves (4.27). \( \Box \)

### 4.4. The discrete semigroup in the full discretization

Let \( \lambda^*_{h,j} \geq 0 \) and \( \phi^*_{h,j} \), \( j = 1, \ldots, M_h \), be the eigenvalues and eigenfunctions of the operator \( A : D(A) \to X \). Similarly, let \( \lambda^*_{h,j} \geq 0 \) and \( \phi^*_{h,j} \), \( j = 1, \ldots, M_h \), be the eigenvalues and eigenfunctions of the operator \( A_h : X_h \to X_h \).

We denote by \( R(\tau A_h) : X_h \to X_h \) the linear operator defined by
\[ R(\tau A_h) v_h = \sum_{j=1}^{M_h} R(\tau \lambda^*_{h,j})(v_h, \phi^*_{h,j}) \phi^*_{h,j} \quad \text{with} \quad R(z) = \frac{1}{1 + z} \quad \text{for} \quad z \in \mathbb{R}, \ z \neq -1. \]  
(4.57)

The discrete semigroup \( \hat{E}_{h,\tau} \) defined in (2.23) can be written as
\[ \hat{E}_{h,\tau} v_h = R(\tau A_h) v_h \quad \text{for} \quad v_h \in X_h. \]

As a temporal version of (4.21) and (4.22), the following estimates hold for any \( v_h \in X_h \),
\[ \| (I + A_h)^{\frac{\tau}{2}} \hat{E}_{h,\tau} v_h \|_{L^2} \lesssim t_n \| v_h \|_{L^2} \quad \text{for} \quad 1 \leq n \leq N \quad \text{and} \quad \gamma \in [0, 1], \]  
(4.58)
\[ \tau \sum_{j=1}^{n} \| (I + A_h)^{\frac{\tau}{2}} \hat{E}_{h,\tau} v_h \|_{L^2}^2 \lesssim C \| v_h \|_{L^2}^2 \quad \text{for} \quad 1 \leq n \leq N. \]  
(4.59)

**Remark 4.6.** Inequality (4.58) is equivalent to
\[ |(1 + \lambda^*_{h,j})^\frac{\tau}{2} R(\tau \lambda^*_{h,j}) | \lesssim t_n \]  
for \( \lambda^*_{h,j} \geq 0 \) and \( t_n = n \tau \in [0, T] \).  
(4.60)
The proof of this inequality can be found in [31 Lemma 7.3]). Since the function $w^j_h = E_{h,\tau}^j v_h$, $n = 1, 2, \ldots$, are solutions of the equation

$$\frac{w^j_h - w^{j-1}_h}{\tau} + A_h w^j_h = 0.$$  

Testing the equation with $w^j_h$ and summing up the results for $j = 1, \ldots, n$, yield the basic energy inequality

$$\max_{1 \leq j \leq n} \frac{1}{2} \|w^j_h\|_{L^2}^2 + \tau \sum_{j=1}^n \|A_h^{\frac{1}{2}} w^j_h\|_{L^2}^2 \leq \frac{1}{2} \|w^0_h\|_{L^2}^2 = \frac{1}{2} \|v_h\|_{L^2}^2,$$

which implies (4.59). □

The proof of Lemma 4.1. 

**Lemma 4.8.** Let $\Phi_{h,\tau}^n v := E_h(t_n)P_{X_h} v - E_{h,\tau}^n X_h v$ for $v \in L^2(D)^d$. Then the following estimates hold:

\[
\|\Phi_{h,\tau}^n v\|_{L^2} \lesssim \tau^\frac{1}{2} \|v\|_{H^1} \quad \forall v \in H^1(D)^d, \\
\|\Phi_{h,\tau}^n v\|_{L^2} \lesssim t_n^{-\frac{1}{2}} \|v\|_{L^2} \quad \forall v \in L^2(D)^d, \\
(\tau \sum_{j=1}^n \|\Phi_{h,\tau}^n v\|_{L^2}^2)^{\frac{1}{2}} + \left(\tau \sum_{j=1}^n \|\nabla \Phi_{h,\tau}^n v\|_{L^2}^2\right)^{\frac{1}{2}} \lesssim \tau^\frac{1}{2} \|v\|_{L^2} \quad \forall v \in L^2(D)^d.
\]

**Proof.** It is known that the function $F_n(z) = e^{-nz} - R(z)^n$ has the following upper bound:

$$|F_n(\tau \lambda_{h,j})| \leq (\tau \lambda_{h,j})^{\frac{1}{2}}$$

for the eigenvalues $\lambda_{h,j} \geq 0$. (4.64)

A proof of this result can be found in [31 Theorem 7.1] (with $q = \frac{1}{2}$ therein). Inequality immediately implies the following error estimate:

$$\|E_h(t_n) - E_{h,\tau}^n X_h v\|_{L^2} = \|F_n(\tau A_h) v_h\|_{L^2} \lesssim \tau^\frac{1}{2} \|v_h\|_{H^1} \quad \forall v_h \in X_h.$$  

(4.65)

In view of the $H^1$-stability of $P_{X_h}$ in (4.68), we have

$$\|E_h(t_n) - E_{h,\tau}^n X_h v\|_{L^2} \lesssim \tau^\frac{1}{2} \|P_{X_h} v\|_{H^1} \lesssim \tau^\frac{1}{2} \|v\|_{H^1} \quad \forall v \in H^1(D)^d.$$  

(4.66)

This proves (4.61).

Similarly, the following inequality can be shown (see [31 inequality (7.22)] with $q = \frac{1}{2}$)

$$|F_n(\tau \lambda_{h,j})| \leq \tau^\frac{1}{2} t_n^{-\frac{1}{2}}$$

for the eigenvalues $\lambda_{h,j} \geq 0$, which immediately implies that

$$\|E_h(t_n) - E_{h,\tau}^n X_h v\|_{L^2} = \|F_n(\tau A_h) P_{X_h} v\|_{L^2} \lesssim \tau^\frac{1}{2} t_n^{-\frac{1}{2}} \|P_{X_h} v\|_{L^2} \quad \forall v \in L^2(D)^d.$$  

(4.67)

This proves (4.62).

The first and second terms in (4.63) were estimated in [21 inequality (4.21) with $\rho = 0$] and [21 inequality (4.19) with $\rho = 1$, replacing $A_h^{\frac{1}{2}} P_h x$ by $P_h x$ therein], respectively. □

**Proof of Lemma 4.1.** By using the expressions

$$\Phi_h(t) := E(t) P_X - E_h(t) P_{X_h}$$

and

$$\Phi_{h,\tau}^n v := E_h(t_n) P_{X_h} v - E_{h,\tau}^n P_{X_h} v,$$

and the triangle inequality, we have

$$\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|E(s) P_X - E_{h,\tau}^n P_{X_h} v\|_{L^2}^2 \, ds \lesssim \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left(\|E_h(s) - E_h(t_j)\|_{L^2} \right) \|P_{X_h} v\|_{L^2}^2 \, ds$$

$$+ \int_0^{t_n} \|\Phi_h(s) v\|_{L^2}^2 \, ds + \tau \sum_{j=1}^n \|\Phi_{h,\tau}^n v\|_{L^2}^2.$$  

(4.68)
The first term on the right-hand side (4.68) can be estimated as follows:

\[
\sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} \| [E_{h}(s) - E_{h}(t_{j})] P_{X_{h}} v \|_{L^{2}}^{2} ds = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} \| (1 - e^{-(t_{j} - s)A_{h}}) e^{-sA_{h}} P_{X_{h}} v \|_{L^{2}}^{2} ds
\]

\[
\lesssim \tau \int_{0}^{t_{n}} \| A_{h}^{1/2} e^{-sA_{h}} P_{X_{h}} v \|_{L^{2}}^{2} ds
\]

\[
\lesssim \tau \| v \|_{L^{2}}^{2} \quad \forall \, v \in L^{2}(D)^{d}.
\] (4.69)

The second and third terms on the right-hand side of (4.68) have already been estimated in (1.25) and (1.63), respectively, which imply (4.1).

Similarly, (4.2) can be proved by using (1.27), (1.63) and (4.69).

Similarly, we have

\[
\begin{align*}
\left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} \nabla [E(s) P_{X} - E_{h}(s) P_{X_{h}}] v \right\|_{L^{2}}^{2} & \lesssim \left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} \nabla [E_{h}(s) - E_{h}(t_{j})] P_{X_{h}} v \right\|_{L^{2}}^{2} \\
& + \left\| \int_{0}^{t_{n}} \nabla \Phi_{h}(s) v \right\|_{L^{2}}^{2} + \left\| \sum_{j=1}^{n} \nabla \Phi_{h}(s) v \right\|_{L^{2}}^{2} \\
& \lesssim (\tau + h^{2}) \| v \|_{L^{2}}^{2}.
\end{align*}
\] (4.70)

The first term on the right-hand side of (4.70) was estimated in [21] p. 236, with \( \rho = 1 \) and \( P_{h} x \) replaced by \( A_{h}^{1/2} P_{h} x \), i.e.,

\[
\left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} [E_{h}(s) - E_{h}(t_{j})](I + A_{h})^{1/2} P_{X_{h}} v ds \right\|_{L^{2}}^{2} \lesssim \tau \| P_{X_{h}} v \|_{L^{2}}^{2} \lesssim \tau \| v \|_{L^{2}}^{2}.
\]

The second and third terms on the right-hand side of (4.70) have been estimated in (1.27) and (4.63), respectively. This proves (4.1).

5. Error analysis for the stochastic problem

In this section, we present error estimates for the fully discrete method (2.19) for the stochastic Stokes equations. Some estimates of the noise term in the Hilbert–Schmidt norm are presented in Subsection 5.1 and the error estimates are presented in Section 5.2.

5.1. Estimates in the Hilbert–Schmidt norm

**Lemma 5.1.** Under Assumptions 2.1–2.3, the linear operator \( B(v) : L^{2}(D)^{d} \to L^{2}(D)^{d} \) satisfies

\[
\| E(t) P_{X} [B(u) - B(v)] \|_{L^{2}}^{2} \lesssim t^{-1/2} E \| u - v \|_{L^{2}}^{2} \quad \forall \, u, v \in L^{2}(D)^{d},
\] (5.1)

and

\[
\| B(u) \|_{L^{2}}^{2} \lesssim 1 + \| u \|_{H^{\beta}}^{2} \quad \forall \, u \in H^{\beta}(D)^{d}, \quad \beta \in \left( \frac{3}{2}, 2 \right),
\] (5.2)

\[
\| (I + A)^{1/2} P_{X} B(u) \|_{L^{2}}^{2} \lesssim 1 + \| u \|_{H^{\beta}}^{2} \quad \forall \, u \in H^{\beta}(D)^{d}, \quad \beta \in \left( \frac{3}{2}, 2 \right).
\] (5.3)

**Proof.** By using (1.10) with \( s = \frac{1}{2} \), (2.4), (4.16) with \( \gamma = \frac{1}{4} \), one can show that

\[
\| E(t) P_{X} (B(u) - B(v)) \|_{L^{2}}^{2}
\]

\[
= \sum_{\ell} \mu_{\ell} \| E(t) P_{X} [(B(u) - B(v))\phi_{\ell}] \|_{L^{2}}^{2}
\]

\[
\lesssim \sum_{\ell} \mu_{\ell} \| (I + A)^{1/2} E(t)(I + A)^{-1/2} P_{X} [(B(u) - B(v))\phi_{\ell}] \|_{L^{2}}^{2}
\]
Then, after subtracting (C.1) from (3.13), we obtain the following error equation:

\[
\leq t^{-\frac{1}{2}} \sum_{\ell} \mu_\ell \|(B(u) - B(v))\phi_\ell\|_{H^{-\frac{1}{2}}} \quad \text{(Here (4.16) with } \gamma = \frac{1}{2} \text{ is used)}
\]

\[
\leq t^{-\frac{1}{2}} \sum_{\ell} \mu_\ell \|B(u) - B(v)\|_{L^2} \|\phi_\ell\|_{H^1} \quad \text{(In addition, (3.4) and (4.14) are used)}
\]

and

\[
\|B(u)\|_{L^2}^2 = \sum_{\ell} \mu_\ell \|B(u)\phi_\ell\|_{L^2}^2 \leq \sum_{\ell} \mu_\ell \|B(u)\phi_\ell\|_{H^1}^2
\]

\[
\leq \sum_{\ell} \mu_\ell (1 + \|u\|_{H^{\beta}}^2) \|\phi_\ell\|_{H^1}^2 \quad \text{(Assumption 2.2 is used)}
\]

\[
\leq (1 + \|u\|_{H^{\beta}}^2)(1 + \|(-\Delta)^{\frac{1}{2}}\|_{L^2}^2).
\]

The last two inequalities and Assumption 2.1 imply (5.1)–(5.2). Similarly,

\[
\|(I + A)^{\frac{\tau}{2}} P_X B(u)\|_{L^2}^2 = \sum_{\ell} \mu_\ell \|(I + A)^{\frac{\tau}{2}} P_X B(u)\phi_\ell\|_{L^2}^2 \leq \sum_{\ell} \mu_\ell \|B(u)\phi_\ell\|_{H^1}^2
\]

\[
\leq (1 + \|u\|_{H^{\beta}}^2)(1 + \|(-\Delta)^{\frac{1}{2}}\|_{L^2}^2).
\]

This proves (5.3).

Remark 5.1. As a result of the estimates in Lemma 5.1, the regularity results in Theorem 3.1 imply that

\[
\sup_{t \in [0,T]} E\|B(u(t))\|_{L^2}^2 + \sup_{t \in [0,T]} E\|(I + A)^{\frac{\tau}{2}} P_X B(u(t))\|_{L^2}^2 \lesssim 1.
\]

The following stability estimates for the numerical solution can be proved by using Lemma 5.1 and will be used in the error analysis. The proof is omitted here, and the details can be found in Appendix C.

Lemma 5.2. Under Assumptions 2.1–2.3, the numerical solution determined by the fully discrete method (2.22) satisfies the following energy inequality:

\[
\max_{1 \leq n \leq N} E\|u_h^n\|_{H^{\frac{1}{2}}}^2 + \sum_{n=1}^{N} E\|u_h^n - u_h^{n-1}\|_{L^2}^2 + \tau \sum_{n=1}^{N} E\|u_h^n\|_{H^1}^2 \lesssim 1.
\]

5.2. Error estimates for the velocity

By iterating (2.22) with respect to \(n\), the full discrete method can be rewritten as

\[
u_h^n = E^n_{h,\tau} P_X u_0 + \tau \sum_{i=0}^{n-1} E^{n-i}_{h,\tau} P_X f(t_{i+1}) + \sum_{i=0}^{n-1} E^{n-i}_{h,\tau} P_X [B(u_h^i)\Delta W_{i+1}].
\]

Then, after subtracting (C.1) from (5.13), we obtain the following error equation:

\[
u(t_n) - u_h^n = (E(t_n) P_X - E^n_{h,\tau} P_X) u_0
\]

\[
+ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [E(t_n - s) P_X f(s) - E^{n-i}_{h,\tau} P_X f(t_{i+1})] ds
\]

\[
+ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [E(t_n - s) P_X B(u(s)) - E^{n-i}_{h,\tau} P_X B(u_h^i)] dW(s)
\]

\[
=: T_1 + T_2 + T_3,
\]
which implies that
\[ \mathbb{E}\|u(t_n) - u_0\|_{L^2}^2 \lesssim \sum_{j=1}^{3} \mathbb{E}\|T_j\|_{L^2}^2. \]  

The three terms are estimated below separately below.

Since \( T_1 = \Phi_h(t_n)u^0 + \Phi^0_{h,\tau}u^0 \), by applying (4.28) and (4.61) with \( \rho = 1 \) and \( v = u_0 \in \mathcal{D}(A) \), we obtain
\[ \mathbb{E}\|T_1\|_{L^2}^2 \lesssim \mathbb{E}\|\Phi_h(t_n)u^0\|_{L^2}^2 + \mathbb{E}\|\Phi^0_{h,\tau}u^0\|_{L^2}^2 \lesssim (\tau + h^4)\mathbb{E}\|u^0\|_{L^2}^2. \]

By using the triangle inequality, \( T_2 \) can be further decomposed into the following two parts:
\[ \mathbb{E}\|T_2\|_{L^2}^2 \lesssim \mathbb{E}\left\| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} E(t_n - s)P_X [f(s) - f(t_{i+1})] ds \right\|_{L^2}^2 \]
\[ + \mathbb{E}\left\| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [E(t_n - s)P_X - \bar{E}_{h,\tau}n_iP_{X_h}] f(t_{i+1}) ds \right\|_{L^2}^2 \]
\[ =: T_{21} + T_{22}. \]

By using the Hölder continuity of \( f(t) \) in (2.8), we have
\[ T_{21} \lesssim \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\|f(s) - f(t_{i+1})\|_{L^2}^2 ds \lesssim \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{i+1} - s) ds \lesssim \tau. \]
Through a change of variables \( \sigma = t_n - s \) and \( j = n - i \), we obtain by using the triangle inequality
\[ T_{22} = \mathbb{E}\left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [E(\sigma)P_X - \bar{E}_{h,\tau}n_iP_{X_h}] f(t_{n-j+1}) d\sigma \right\|_{L^2}^2 \]
\[ \lesssim \mathbb{E}\left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [E(\sigma) - E(t_j)] P_X [f(t_{n-j+1}) - f(t_n)] d\sigma \right\|_{L^2}^2 \]
\[ + \mathbb{E}\left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [E(t_j)P_X - \bar{E}_{h,\tau}n_iP_{X_h}] [f(t_{n-j+1}) - f(t_n)] d\sigma \right\|_{L^2}^2 \]
\[ + \mathbb{E}\left\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [E(\sigma)P_X - \bar{E}_{h,\tau}n_iP_{X_h}] f(t_n) d\sigma \right\|_{L^2}^2 \]
\[ =: T_{22}^a + T_{22}^b + T_{22}^c. \]

The term \( T_{22}^a \) can be estimated by using (4.16) with \( \gamma = \frac{1}{2} \), (4.17) with \( \mu = \frac{1}{2} \), and the Hölder continuity of \( f \) in (2.8), i.e.,
\[ T_{22}^a \lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| (I + A)^{\frac{1}{2}} E(\sigma) (I + A)^{-\frac{1}{2}} [I - E(t_j - \sigma)] P_X [f(t_{n-j+1}) - f(t_n)] \|_{L^2}^2 d\sigma \]
\[ \lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-1}(t_j - \sigma) \| f(t_{n-j+1}) - f(t_n) \|_{L^2}^2 d\sigma \]
\[ \lesssim \tau \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-1}(t_n - t_{n-j+1}) d\sigma \lesssim \tau \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-1} d\sigma \lesssim \tau. \]

The term \( T_{22}^b \) can be estimated by using (4.20), (4.22) and the Assumption 2.8 shows
\[ T_{22}^b \lesssim \mathbb{E}\left( \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| \Phi_h(t_j) + \bar{\Phi}_{h,\tau}^j f(t_{n-j+1}) - f(t_n) \|_{L^2}^2 d\sigma \right)^2 \]
The three terms \( \sigma \) Substituting (5.14)–(5.18) into (5.13) yields
\[
20 \tau + h^4 \left( \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_n - t_{n-j+1}) \frac{1}{2} \, \text{d} \sigma \right)^2 
\]
\[
\lesssim (\tau + h^4) \left( \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_n - t_{n-j+1}) \frac{1}{2} \, \text{d} \sigma \right)^2 
\]
\[
\lesssim \tau + h^4.
\]
The term \( T_{22} \) can be estimated by applying (5.12) directly, i.e.,
\[
T_{22} \lesssim (\tau + h^4) \| f(t_n) \|_{L^2}^2 \lesssim \tau + h^4. \tag{5.18}
\]
Substituting (5.14)–(5.18) into (5.13) yields
\[
\mathbb{E} \| T_2 \|_{L^2}^2 \lesssim \tau + h^4. \tag{5.19}
\]
It remains to estimate the term \( T_3 \) in (5.10). By using (2.2) and a change of variables \( \sigma = t_n - s \) and \( j = n - i \), we can decompose \( T_3 \) into three parts as follows:
\[
\mathbb{E} \| T_3 \|_{L^2}^2 = \mathbb{E} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \| E(t_n - s) P_X B(u(s)) - E_{h, \tau}^{n-i} P_X B(u)^j \|_{L^2}^2 \, \text{d}s 
\]
\[
= \mathbb{E} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| E(\sigma) P_X B(u(t_n - \sigma)) - E_{h, \tau}^{j} P_X B(u^j) \|_{L^2}^2 \, \text{d} \sigma 
\]
\[
\lesssim \mathbb{E} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| E(\sigma) P_X [B(u(t_n - \sigma)) - B(u(t_n - j))] \|_{L^2}^2 \, \text{d} \sigma 
\]
\[
+ \mathbb{E} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| [E(\sigma) P_X - E_{h, \tau}^{j} P_X] B(u(t_n - j)) \|_{L^2}^2 \, \text{d} \sigma 
\]
\[
+ \tau \mathbb{E} \sum_{j=1}^{n} \| E_{h, \tau}^{j} P_X [B(u(t_n - j)) - B(u^j)] \|_{L^2}^2 
\]
\[
= : T_{31} + T_{32} + T_{33}. \tag{5.20}
\]
The three terms \( T_{31}, T_{32} \) and \( T_{33} \) are estimated separately below.

The term \( T_{31} \) can be estimated by using (5.11) and Hölder continuity (3.15), which imply that
\[
T_{31} \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{1}{2}} \mathbb{E} \| u(t_n - \sigma) - u(t_n - j) \|_{L^2}^2 \, \text{d} \sigma 
\]
\[
\leq C (1 + \mathbb{E} \| u \|_{H^2}^2) \mathbb{E} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{1}{2}} (t_j - \sigma) \, \text{d} \sigma 
\]
\[
\leq C \tau (1 + \mathbb{E} \| u \|_{H^2}^2). \tag{5.21}
\]

The term \( T_{32} \) can be estimated by decomposing it into three parts, i.e.,
\[
T_{32} \lesssim \mathbb{E} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| [E(\sigma) - E(t_j)] P_X [B(u(t_n - j)) - B(u(t_n - 1))] \|_{L^2}^2 \, \text{d} \sigma 
\]
\[
+ \mathbb{E} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| [E(t_j) P_X - \tilde{E}_{h, \tau}^{j} P_X] B(u(t_n - j)) - B(u(t_n - 1)) \|_{L^2}^2 \, \text{d} \sigma 
\]
\[
+ \mathbb{E} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| [E(\sigma) P_X - \tilde{E}_{h, \tau}^{j} P_X] B(u(t_n - 1)) \|_{L^2}^2 \, \text{d} \sigma 
\]
\[
= : T_{32}^a + T_{32}^b + T_{32}^c. \tag{5.22}
\]
The term $T_{32}^a$ can be estimated by using (4.16) with $\gamma = \frac{1}{2}$, (4.17) with $\mu = \frac{1}{2}$, (5.2) and H"older continuity (3.15), i.e.,

$$T_{32}^a \lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \mathbb{E} \left[ \left\| E(\sigma) [I - E(t_j - \sigma)] P_X [B(u(t_{n-j})) - B(u(t_{n-1}))] \right\|^2_{L^2_2} d\sigma \right]$$

$$\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \mathbb{E} \sum_{\ell} \mu_\ell \left\| (I + A)^{\frac{1}{2}} E(\sigma) [I - E(t_j - \sigma)] (I + A)^{-\frac{1}{2}} P_X [B(u(t_{n-j})) - B(u(t_{n-1}))] \phi_{\ell} \right\|^2_{L^2_2} d\sigma$$

$$\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{1}{2}} (t_j - \sigma) \mathbb{E} \sum_{\ell} \mu_\ell \left\| B(u(t_{n-j})) - B(u(t_{n-1})) \phi_{\ell} \right\|^2_{H^{-\frac{1}{2}}} d\sigma$$

$$\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{1}{2}} (t_j - \sigma) \mathbb{E} \| u(t_{n-j}) - u(t_{n-1}) \|^2_{L^2_2} d\sigma$$

Assumption 2.2 is used

$$\lesssim \tau (1 + \mathbb{E} \| u(0) \|^2_{H^2}) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \sigma^{-\frac{1}{2}} (t_j - t_{j-1}) d\sigma$$

$$\lesssim \tau (1 + \mathbb{E} \| u(0) \|^2_{H^2}).$$

The term $T_{32}^b$ can be estimated by using the expression $E(t_j) P_X - \tilde{E}_{h,\tau}^j P_X = \Phi_h(t_j) + \tilde{\Phi}_{h,\tau}^j$, i.e.,

$$T_{32}^b = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \mathbb{E} \left[ \left\| \Phi_h(t_j) + \tilde{\Phi}_{h,\tau}^j \right\| B(u(t_{n-j})) - B(u(t_{n-1})) \right\|^2_{L^2_2} d\sigma$$

$$\lesssim \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} t_j^{-1} h^4 \mathbb{E} \left[ \left\| (I + A)^{\frac{1}{2}} P_X [B(u(t_{n-j})) - B(u(t_{n-1}))] \right\|^2_{L^2_2} d\sigma$$

Assumption 2.2 is used

$$\lesssim (\tau + h^4) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} t_j^{-1} \mathbb{E} \| u(t_{n-j}) - u(t_{n-1}) \|^2_{H^2} d\sigma$$

$$(5.23)$$

The $T_{32}^c$ term can be estimated by directly applying (4.11) and (6.1), we have

$$T_{32}^c \lesssim (\tau + h^4) \sup_{t \in [0,T]} \mathbb{E} \left\| (I + A)^{\frac{1}{2}} P_X B(u(t)) \right\|^2_{L^2_2} \lesssim \tau + h^4.$$

(5.25)

The term $T_{33}$ in (5.20) can be estimated by using (4.58) with $\gamma = \frac{1}{2}$, i.e.,

$$\mathbb{E} \left\| T_{33} \right\|^2_{L^2_2} \lesssim \tau \sum_{j=1}^{n} \mathbb{E} \left\| (I + A_h)^{\frac{1}{2}} \tilde{E}_{h,\tau}^j (I + A_h)^{-\frac{1}{2}} P_X [B(u(t_{n-j})) - B(u^n_{n-j})] \right\|^2_{L^2_2}$$

$$\lesssim \tau \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \mathbb{E} \sum_{\ell} \mu_\ell \left\| P_X [(B(u(t_{n-j})) - B(u^n_{n-j}))] \phi_{\ell} \right\|^2_{H^{-\frac{1}{2}}}$$

$$\lesssim \tau \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \mathbb{E} \left\| u(t_{n-j}) - u^n_{n-j} \right\|^2_{L^2_2} (-\Delta)^{\frac{1}{2}} \left\|^2_{L^2_2}$$

$$\lesssim \tau \sum_{i=0}^{n-1} (t_n - t_i)^{-\frac{1}{2}} \mathbb{E} \left\| u(t_i) - u^n_{h_i} \right\|^2_{L^2_2}.$$  

(5.26)
In view of (5.20)–(5.26), we have
\[ E\|T_3\|_{L^2} \lesssim \tau + h^4 + \tau \sum_{i=0}^{n-1} (t_n - t_i)^{-\frac{1}{2}} E\|u(t_i) - u_h^i\|_{L^2}^2. \] (5.27)

Substituting (5.12), (5.19) and (5.27) into (5.11) yields
\[ E\|u(t_n) - u_h^n\|_{L^2}^2 \lesssim \tau + h^4 + \tau \sum_{i=0}^{n-1} (t_n - t_i)^{-\frac{1}{2}} E\|u(t_i) - u_h^i\|_{L^2}^2. \] (5.28)

By using the discrete Gronwall inequality, we obtain the desired error estimate in (2.24) for the velocity.

5.3. Error estimates for the pressure.

From the numerical scheme in (2.19) we can derive that
\[ \tau \sum_{n=1}^{m} (p_h^n, \nabla \cdot v_h) = (u_h^m, v_h) - (u_h^0, v_h) + 2\tau \sum_{n=1}^{m} (D(u_h^n), D(v_h)) \\
- \tau \sum_{n=1}^{m} (f(t_n), v_h) - \tau \sum_{n=1}^{m} (B(u_h^{n-1}) \Delta W_n, v_h), \quad v_h \in V_h. \] (5.29)

Subtracting (5.29) from (3.17) yields
\[
\left( \int_0^{t_m} p(t) \, dt - \tau \sum_{n=1}^{m} p_h^n, \nabla \cdot v_h \right) = \left( (u(t_m) - u_h^m) - [u^0 - u_h^0], v_h \right) \\
+ 2 \left( \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} D(u(t) - u_h^n) \, dt, D(v_h) \right) \\
- \left( \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} [f(t) - f(t_n)] \, dt, v_h \right) \\
- \left( \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} [B(u(t)) - B(u_h^{n-1})] \, dW(t), v_h \right) \\
= : J_1(v_h) + J_2(v_h) + J_3(v_h) + J_4(v_h), \quad \forall v_h \in V_h,
\]

where
\[ |J_1(v_h)| \lesssim \left( \|u(t_m) - u_h^m\|_{L^2} + \|u^0 - P_{X_h} u^0\|_{L^2} \right) \|v_h\|_{L^2} =: J^*_1 \|v_h\|_{L^2}, \]
\[ |J_2(v_h)| \lesssim \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} D(u(t) - u_h^n) \, dt \right\| \|\nabla v_h\|_{L^2} =: J^*_2 \|\nabla v_h\|_{L^2} \]
\[ |J_3(v_h)| \lesssim \left\| \int_{0}^{t_m} [f(t) - f(t_n)] \, dt \right\| \|v_h\|_{L^2} =: J^*_3 \|v_h\|_{L^2} \]
\[ |J_4(v_h)| \lesssim \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} P_{X_h} [B(u(t)) - B(u_h^{n-1})] \, dW(t) \right\|_{H^{-1}} \|v_h\|_{H^1} =: J^*_4 \|v_h\|_{H^1}, \]

which imply that (by using the inf-sup condition (2.18) of the finite element space)
\[ E\left| \int_0^{t_m} p(t) \, dt - \tau \sum_{n=1}^{m} p_h^n \right|_{L^2}^2 = E\|J_1\|^2 + E\|J_2\|^2 + E\|J_3\|^2 + E\|J_4\|^2. \] (5.30)

By using (2.24) and (4.10), we have
\[ E\|J_1\|^2 \lesssim (\tau + h^4). \] (5.31)
and

\[ J_2^* \leq \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \mathbb{D}(u(t) - u(t_n)) \, dt \right\|_{L^2} + \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \mathbb{D}(u(t) - u_h^n) \, dt \right\|_{L^2} \tag{5.32} \]

\[ \leq \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \mathbb{D}(u(t) - u(t_n)) \, dt \right\|_{L^2} \quad \text{(using (3.13) with } t = t_n \text{ and (C.1))} \]

\[ + \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \mathbb{D}[E(t_n P_X - E_{h,r} P_{X_h}) u^0] \, dt \right\|_{L^2} \]

\[ + \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{D}(E(t_n - s) P_X f(s) - E_{h,r} P_{X_h} f(t_{i+1})) \, ds \, dt \right\|_{L^2} \]

\[ + \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \sum_{i=0}^{n-1} \mathbb{D} \left( \int_{t_i}^{t_{i+1}} E(t_n - s) P_X B(u(s)) - E_{h,r} P_{X_h} B(u_h) \, dW(s) \right) \, dt \right\|_{L^2} \]

\[ =: J_{21}^* + J_{22}^* + J_{23}^* + J_{24}^* , \]

where

\[ E|J_{21}^*|^2 \lesssim \int_{0}^{t_n} E\|u(t) - u(t_n)\|^2_{H^1} \, dt \]

\[ \lesssim \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} (1 + E\|u^0\|^2_{H^1}) (t - t_n) \, dt \quad \text{(here (3.15) is used)} \]

\[ \lesssim \tau , \]

\[ E|J_{22}^*|^2 \lesssim \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \left\| \nabla[E(t) P_X - \tilde{E}_{h,r} P_{X_h}] u^0 \right\|_{L^2}^2 + E \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \nabla[E(t_n) - E(t)] P_X u^0 \right\|_{L^2}^2 \]

\[ \lesssim \tau + h^2 \quad \text{(here (4.3) is used)} \]

\[ + E \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \left\| (I + A)^{-\frac{1}{2}} [E(t_n - t) - I] (I + A)^{\frac{1}{2}} E(t) (I + A)^{\frac{1}{4}} u^0 \right\|_{L^2}^2 \, dt \]

\[ \lesssim \tau + h^2 + E \int_{0}^{t_n} \tau \left\| (I + A)^{\frac{1}{2}} E(t)(I + A)^{\frac{1}{4}} u^0 \right\|_{L^2}^2 \, dt \quad \text{(here (4.17) is used)} \]

\[ \lesssim \tau + h^2 + E \int_{0}^{t_n} \tau t^{-\frac{1}{2}} \left\| (I + A)^{\frac{1}{4}} u^0 \right\|_{L^2}^2 \, dt \]

\[ \lesssim \tau + h^2 , \]

\[ E|J_{23}^*|^2 \lesssim \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \nabla[E(t_n - s) P_X [f(s) - f(t_{i+1})]] \, ds \, dt \right\|_{L^2}^2 \tag{5.33} \]

\[ + E \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \nabla[E(t_n - s) P_X - \tilde{E}_{h,r} P_{X_h}] f(t_{i+1}) \, ds \right\|_{L^2}^2 \]

\[ =: E|J_{231}^*|^2 + E|J_{232}^*|^2 . \]

The two terms \( E|J_{231}^*|^2 \) and \( E|J_{232}^*|^2 \) can be estimated as follows:

\[ E|J_{231}^*|^2 \lesssim \left( \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left\| (I + A)^{\frac{1}{2}} E(t_n - s) P_X [f(s) - f(t_{i+1})] \right\|_{L^2} \, ds \, dt \right)^2 \tag{5.34} \]

\[ \lesssim \left( \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left\| (t_n - s)^{-\frac{1}{2}} E f(t_{i+1}) - f(s) \right\|_{L^2} \, ds \, dt \right)^2 \]
\[
\lesssim \left( \int_0^{t_m} \int_0^{t_n} (t_n - s)^{-\frac{3}{2} + \frac{1}{2}} \sigma ds \, dt \right)^2
\lesssim \tau
\]
and, by using a change of variables \( \sigma = t_n - s \) and \( j = n - i \),
\[
E[J_{24}^2] = E\left\| \sum_{n=1}^{m-1} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \nabla [E(t_n - s)P_X - \bar{E}_{h,\tau}^{n-i}P_{X_n}]f(t_{i+1}) \, ds \right\|_{L^2}^2
\]
(5.35)
\[
= E\left\| \sum_{n=1}^{m-1} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \nabla [E(\sigma)P_X - \bar{E}_{h,\tau}^{j}P_{X_n}]f(t_{i+1}) \, d\sigma \right\|_{L^2}^2
\]
\[
\lesssim \tau \sum_{n=1}^{m-1} \left\| \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \nabla [E(\sigma)P_X - \bar{E}_{h,\tau}^{j}P_{X_n}]f(t_{i+1}) \, d\sigma \right\|_{L^2}^2
\]
\[
\lesssim \tau \sum_{n=1}^{m-1} (\sigma + h^2)E\left\| f(t_{i+1}) \right\|_{L^2}^2 \quad \text{(here (4.3) is used)}
\]
\[
\lesssim \tau + h^2.
\]

It remains to estimate the term \( J_{24}^* \) in (5.32). Since the integrand in the expression of \( J_{24}^* \) is independent of \( t \) for \( t \in (t_{n-1}, t_n] \), it follows that the integration with respect to \( t \) is actually a summation times \( \tau \). By interchanging the order of integration in the expression of \( J_{24}^* \) and using Itô’s isometry in (2.2), we obtain
\[
E[J_{24}^2] = E\left\| \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \int_{t_i}^{t_{i+1}} E(t_n - s)P_X B(u(s)) - \bar{E}_{h,\tau}^{n-i}P_{X_n} B(u_n) \, dW(s) \, dt \right\|_{L^2}^2
\]
\[
= E\left\| \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \int_{t_i}^{t_{i+1}} [E(t_n - s)P_X B(u(s)) - \bar{E}_{h,\tau}^{n-i}P_{X_n} B(u_n)] \, dW(s) \right\|_{L^2}^2
\]
\[
\lesssim E\left\| \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \int_{t_i}^{t_{i+1}} \nabla [E(t_n - s)P_X B(u(s)) - \bar{E}_{h,\tau}^{n-i}P_{X_n} B(u_n)] \, dW(s) \right\|_{L^2}^2 \, ds.
\]

By using a change of variables \( \sigma = t - t_i \) and \( j = n - i \), we split \( E[J_{24}^*]^2 \) into three parts as follows, by using the triangle inequality,
\[
E[J_{24}^*]^2 = E\left\| \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \int_{t_i}^{t_{i+1}} \nabla [E(t_{i+j} - s)P_X B(u(s)) - \bar{E}_{h,\tau}^{j}P_{X_n} B(u_n)] \, d\sigma \right\|_{L^2}^2 \, ds
\]
\[
\lesssim E\left\| \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \int_{t_i}^{t_{i+1}} \nabla [E(t_{i+j} - s) - E(\sigma)]P_X B(u(s)) \, d\sigma \right\|_{L^2}^2 \, ds
\]
\[
+ E\left\| \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \int_{t_i}^{t_{i+1}} \nabla E(\sigma)P_X [B(u(s)) - B(u_n)] \, d\sigma \right\|_{L^2}^2 \, ds
\]
\[
+ \tau E\left\| \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \int_{t_i}^{t_{i+1}} \nabla [E(\sigma)P_X - \bar{E}_{h,\tau}^{j}P_{X_n}]B(u_n) \, d\sigma \right\|_{L^2}^2 \, ds
\]
\[
=: E[J_{241}^2] + E[J_{242}^2] + E[J_{243}^2].
\]

In order to estimate the term \( E[J_{241}^2] \), we first note that for \( \xi_j, \sigma \in (t_{j-1}, t_j) \) we have \( |\xi_j - \sigma| \leq \tau \) and therefore, by using the triangle inequality,
\[
\sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \| [E(\xi_j) - E(\sigma)]v(\sigma) \|^2_{L^2} \, d\sigma
\]
\[\begin{align*}
v(\tau) \lesssim & \|v(s)\|_{L^2_0}^2 + \sum_{j=3}^{m} \int_{t_j}^{t_{j-1}} \|[E(\xi_j) - E(\sigma - \tau)] - [E(\sigma) - E(\sigma - \tau)]\|_{L^2_0}^2 \, d\sigma \\
\lesssim & \tau \|v(s)\|_{L^2_0}^2 + \sum_{j=3}^{m} \int_{t_j}^{t_{j-1}} \| (I + A)^{-1} [E(\xi_j + \tau - \sigma) - I] (I + A) E(\sigma - \tau) v(s) \|_{L^2_0}^2 \, d\sigma \\
& + \sum_{j=3}^{m} \int_{t_{j-1}}^{t_j} \| (I + A)^{-1} [E(\tau) - I] (I + A) E(\sigma - \tau) v(s) \|_{L^2_0}^2 \, d\sigma \\
\lesssim & \tau \|v(s)\|_{L^2_0}^2 + \tau^2 \int_{2\tau}^{t_m} \| (I + A) E(\sigma - \tau) v(s) \|_{L^2_0}^2 \, d\sigma \quad \text{(here \ref{eq:4.17} with } \mu = 1 \text{ is used)} \\
\lesssim & \tau \|v(s)\|_{L^2_0}^2 + \tau^2 \|v(s)\|_{L^2_0}^2 \int_{2\tau}^{t_m} (\sigma - \tau)^{-2} \, d\sigma \quad \text{(here \ref{eq:4.16} with } \gamma = 1 \text{ is used)} \\
\lesssim & \tau \|v(s)\|_{L^2_0}^2.
\end{align*}\]

Then, the term \(\mathbb{E}|J_{241}\) can be estimated by applying the above estimate with \(\xi = t_{i+j} - s\) and \(v(s) = (I + A)^{\frac{3}{2}} P_X B(u(s)), \) i.e.,

\[\mathbb{E}|J_{241}| \lesssim \tau \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left\| (I + A)^{\frac{3}{2}} P_X B(u(s)) \right\|_{L^2_0}^2 \, ds \lesssim \tau. \quad \text{(here \ref{eq:5.7} is used)}\]

The term \(\mathbb{E}|J_{242}|\) can be estimated by using a variable transform \(t_{m-i} - \sigma = t, \) i.e.,

\[\mathbb{E}|J_{242}| \lesssim \mathbb{E} \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left\| (I + A)^{\frac{3}{2}} \int_{0}^{t_{m-i} - t} E(t_{m-i} - t) (I + A)^{-\frac{3}{2}} P_X [B(u(s)) - B(u_h)] \, dt \right\|_{L^2_0}^2 \, ds \\
\lesssim \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( \int_{0}^{t_{m-i} - t} (t_{m-i} - t)^{-\frac{3}{2}} \, dt \right) \mathbb{E}\left\| (I + A)^{-\frac{1}{2}} P_X [B(u(s)) - B(u_h)] \right\|_{L^2_0}^2 \, ds \\
\lesssim \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left\| B(u(s)) - B(u_h) \right\|_{H^{-\frac{1}{2}}}^2 \, ds \quad \text{(here \ref{eq:4.19} with } \rho = \frac{3}{4} \text{ is used)} \\
\lesssim & \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left\| B(u(s)) - B(u_h) \right\|_{L^2}^2 \, ds \quad \text{(here \ref{eq:3.3} with } s = \frac{1}{2} \text{ is used)} \\
\lesssim & \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left\| u(s) - u_h \right\|_{L^2}^2 \, ds \quad \text{(here \ref{eq:2.3} and \ref{eq:2.5} are used)} \\
\lesssim & \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( \mathbb{E}\left\| u(t) - u_h \right\|_{L^2}^2 + \mathbb{E}\left\| u(t) - u_h \right\|_{L^2}^2 \right) \, ds \\
\lesssim & \tau + \max_{1 \leq n \leq N} \mathbb{E}\left\| u(t_n) - u_h \right\|_{L^2}^2 \quad \text{(here \ref{eq:3.10} is used)} \\
\lesssim & \tau. \quad \text{(here \ref{eq:2.24} is used, which is already proved)} \quad \text{(5.36)}
\]

The term \(\mathbb{E}|J_{243}|\) can be estimated by directly using \ref{eq:4.3}, we obtain

\[\mathbb{E}|J_{243}| \lesssim \tau \sum_{i=0}^{m-1} (\tau + h^2) \mathbb{E}\|B(u_h)\|_{L^2_0}^2 \\
\lesssim (\tau + h^2) \sum_{i=0}^{m-1} \tau (1 + \mathbb{E}\|u_h\|_{H^1}^2) \quad \text{(here \ref{eq:6.8} is used)} \\
\lesssim \tau + h^2. \quad \text{(here \ref{eq:6.8} is used)} \quad \text{(5.37)}\]
The estimates for $J_{21}$, $J_{22}$, $J_{23}$, $J_{24}$ above imply that
\[ \mathbb{E}|J_3|^2 \lesssim \tau + h^2. \] (5.38)

The term $J_3$ can be estimated directly by using the Hölder continuity of $f$ in time, i.e.,
\[ \mathbb{E}|J_3|^2 \lesssim \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \mathbb{E} \|f(t) - f(t_n)\|_{L^2}^2 \, dt \lesssim \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \tau \, dt \lesssim \tau. \] (5.39)

The term $J_4$ can be estimated directly by using Assumptions 2.1 and 2.2, i.e.,
\[ \mathbb{E}|J_4|^2 \lesssim \mathbb{E} \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| \left( I + A_h \right)^{-\frac{1}{2}} P_{X_h}[B(u(t)) - B(u_h^{n-1})] \|_{L^2}^2 \, dt \lesssim \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \tau \, dt \lesssim \tau. \] (5.40)

By substituting (5.31) and (5.38)–(5.40) into (5.30), we obtain
\[ \mathbb{E} \left\| \int_0^{t_{n-1}} p(t) \, dt - \tau \sum_{n=1}^{m} p_h^n \right\|_{L^2}^2 \lesssim \tau + h^2. \]

This proves desired error estimate in (2.26) for the pressure.

6. Numerical experiments

In this section, we present a 2-D numerical test to validate the theoretical convergence rates proved in Theorems 2.4 and to gauge the performance of the proposed fully discrete mixed method. We solve the stochastic Stokes equation (1.1) in the two-dimensional square $[0,1] \times [0,1]$ under the stress boundary condition by the proposed scheme (2.19), up to time $T = 2$. For a stable discretization in space, we use the MINI element in 2.17. All the computations are performed using the software package NGSolve [https://ngsolve.org].

Let $u_0 = (0,0)$, $f = (1,1)^T$ and $B(u_1, u_2) = \left( (u_1^2 + 1)^{\frac{1}{2}}, (u_2^2 + 1)^{\frac{1}{2}} \right)$, which is nonsolenoidal. For simplicity below we take $W$ in (2.1) as
\[ W(t, x_1, x_2) = \sum_{\ell_1=1}^{\infty} \sum_{\ell_2=1}^{\infty} \sqrt{\mu_{\ell_1 \ell_2}} \phi_{\ell_1 \ell_2}(x_1, x_2) w_{\ell_1 \ell_2}(t) \quad \forall \ t \in [0, T], \] (6.1)

where $w_{\ell_1 \ell_2}(t)$ are independent $\mathbb{R}$-valued Wiener processes, $\mu_{\ell_1 \ell_2} = (\ell_1^2 + \ell_2^2)^{-\left(2+\varepsilon\right)}$ with $\varepsilon = 0.1$ and
\[ \phi_{\ell_1 \ell_2} = 2 \sin(\ell_1 \pi x_1) \sin(\ell_2 \pi x_2) \]
are an orthonormal basis of $L^2(D)^d$. As a result, the linear operator $-\Delta$ and the covariance operator $Q$ share the same eigenfunctions, which implies the desired assumption (2.3).

Since the exact solution of the considered problem is not known, we compute the convergence orders of time discretizations by the formulae
\[ \text{order of convergence} = \log \left( \frac{\mathbb{E}\|u_h^{N}\|_D^2 - \mathbb{E}\|u_h^{N/2}\|_D^2}{\mathbb{E}\|u_h^{N/2}\|_D^2 - \mathbb{E}\|u_h^{N/4}\|_D^2} \right)^{\frac{1}{2}} / \log(2) \]
order of convergence \( \log \left( \frac{\left( E \left( \tau \sum_{n=1}^{N} p_{n, \tau}^{n} - \tau/2 \sum_{n=1}^{2N} p_{n, \tau/2}^{n} \right)_{L^2}^{2}} {\left( E \left( \tau \sum_{n=1}^{2N} p_{n, \tau/2}^{n} - \tau/4 \sum_{n=1}^{4N} p_{n, \tau/4}^{n} \right)_{L^2}^{2}} \right)^{1/2}}{\log(2)} \right) / \log(2) \)

respectively, based on the finest three meshes, where \( u_{n, \tau}^{N} \) denotes the one-path simulation at the end point \( t_{N} = T \) computed by using mesh sizes \( \tau \) and \( h \). Note that the expectations are approximated by computing averages over \( M \) samples and the series (6.1) is truncated with finite summation, up to \( L^{d} = h^{-d} \) terms, i.e., the increment satisfies

\[
W^{L}(t_{n+1}, x_1, x_2) - W^{L}(t_{n}, x_1, x_2) = \tau \sum_{\ell_1=1}^{L} \sum_{\ell_2=1}^{L} \sqrt{P_{\ell_1 \ell_2} \phi_{\ell_1 \ell_2}(x_1, x_2)} \xi_{n, \ell_1 \ell_2},
\]

where \( \xi_{n, \ell_1 \ell_2} \sim N(0, 1) \).

To verify the convergence orders of time discretizations, we approximate the expectations by computing averages over three different number of realizations \( M \), and display the numerical results in Figure 1. From Figure 1, we see that the proposed method has \( 1/2 \) order convergence rate in time for both the velocity and pressure approximations, which is optimal and in accordance with the theoretical results proved in Theorem 2.4.

**Figure 1.** Time errors of the numerical solutions at \( T = 2 \) with \( h = 2^{-6} \).

**Figure 2.** Spatial errors of the numerical solutions at \( T = 2 \) with \( \tau = 2^{-8} \).

The spatial discretization errors are presented in Figure 2 based on the finest three spatial meshes. The left graph shows the convergence rate \( O(h^2) \) for the velocity approximation,
while the right graph shows the convergence rate \(O(h)\) for the pressure approximation, both are optimal and consistent with the theoretical results proved in Theorem 2.4.

7. Conclusion

We have proved higher-order strong convergence of fully discrete mixed FEMs for the stochastic Stokes equations under the stress boundary condition driven by a stochastic noise satisfying condition (1.4). The error estimates of \(O(\tau^{\frac{1}{2}} + h^2)\) and \(O(\tau^{\frac{1}{2}} + h)\) are proved for the velocity and pressure approximations, respectively, for a semi-implicit mixed finite element method. The analysis is based on new estimates of the semi-discrete and fully discrete semi-groups associated to the Stokes operator, and the \(H^1\)-stability of the orthogonal projection onto the discrete divergence-free finite element subspace, as shown in Section 4. The improved convergence orders are consistent with the numerical experiments.

For the simplicity of illustration, we have focused on the stochastic Stokes equations in this article. However, the methodology introduced in this article may also be extended to the stochastic NS equations to obtain higher-order convergence in space.

Appendix A: Well-posedness and regularity of the mild solution

In this Appendix we prove Proposition 3.1 including existence and uniqueness of a mild solution to the stochastic Stokes equations Assumptions 2.1–2.3, and the regularity of the mild solution.

Existence and uniqueness: Let \(Y = C([0, T]; L^2(\Omega; H)) \cap L^2(\Omega; \{F_t\}_{t \geq 0}; L^2(0, T; H))\), i.e., the predictable subspace of \(C([0, T]; L^2(\Omega; H))\). For any \(v \in Y\) we denote by \(Mv\) the function defined by

\[
Mv(t) = E(t)u^0 + \int_0^t E(t-s)P_X f(s)ds + \int_0^t E(t-s)P_X B(v(s))dW(s) \quad \text{for } t \in [0, T],
\]

Under Assumptions 2.1–2.3, the following results hold, which are simple modifications of (5.1):

\[
\|E(t)P_X B(v)\|_{L^2}^2 \lesssim t^{-\frac{1}{2}}(1 + \|v\|_{L^2}^2) \quad t > 0, \tag{A.2}
\]

\[
\|E(t)P_X (B(u) - B(v))\|_{L^2}^2 \lesssim t^{-\frac{1}{2}}\|u - v\|_{L^2}^2 \quad t > 0. \tag{A.3}
\]

Then \(Mv\) is predictable and

\[
\mathbb{E}\|Mv(t)\|_{L^2}^2 \lesssim \mathbb{E}\|u^0\|_{L^2}^2 + \mathbb{E}\int_0^t \|f(s)\|_{L^2}^2 ds + \mathbb{E}\int_0^t \|E(t-s)P_X B(v(s))\|_{L^2}^2 ds \lesssim (1 + \|u^0\|_{L^2}^2 + \int_0^t (t-s)^{-\frac{1}{2}}\|v(s)\|_{L^2}^2 ds \lesssim 1 + \|v\|_Y,
\]

which implies that \(Mv \in L^\infty(0, T; L^2(\Omega; L^2(D)^d))\). By considering \(\mathbb{E}\|Mv(t_2) - Mv(t_1)\|_{L^2}^2\) one can also prove the continuity of \(Mv\) in time. As a result, \(Mv \in C([0, T]; L^2(\Omega; L^2(D)^d))\) and therefore the map \(M : Y \to Y\) is well defined.

Clearly, a mild solution of (3.12) is equivalent to a fixed point of \(M\).

To prove the existence and uniqueness of a fixed point of \(M\), we define \(Y_\lambda\) to be \(Y\) equipped with an equivalent norm

\[
\|w\|_{Y_\lambda} := \max_{t \in [0, T]} e^{-\lambda t}(\mathbb{E}\|w\|_{L^2}^2)^{\frac{1}{2}} \quad \text{for } w \in Y_\lambda, \tag{A.4}
\]
If \( v_1, v_2 \in Y \) then
\[
Mv_1(t) - Mv_2(t) = \int_0^t E(t-s)P_X[B(v_1(s)) - B(v_2(s))] \, dW(s) \quad \text{for } t \in [0, T].
\]

By applying (A.3) we obtain
\[
e^{-2\lambda t} \mathbb{E} \|Mv_1(t) - Mv_2(t)\|_{L^2}^2 \lesssim \int_0^t e^{-2\lambda(t-s)} - \frac{1}{2} \mathbb{E} \|v_1(s) - v_2(s)\|_{L^2}^2 \, ds
\]
\[
= \int_0^t e^{-2\lambda(t-s)} \frac{1}{2} e^{-2\lambda s} \mathbb{E} \|v_1(s) - v_2(s)\|_{L^2}^2 \, ds
\]
\[
\lesssim \int_0^t e^{-2\lambda s} \|v_1 - v_2\|_{Y_\lambda} \, ds,
\]
which implies that
\[
\|Mv_1 - Mv_2\|_{Y_\lambda} \lesssim \lambda^{-\frac{1}{2}} \|v_1 - v_2\|_{Y_\lambda}. \tag{A.5}
\]
As a result, by choosing a sufficiently large \( \lambda \), the map \( M : Y_\lambda \to Y_\lambda \) is a contraction. By the Banach fixed point theorem, \( M \) has a unique fixed point in \( Y_\lambda \) (which consists of the same elements as \( Y \)). This proves the existence of a unique mild solution satisfying (3.13).

**Stability:** We first consider the case \( H_\beta \) with parameter \( \beta \in (\frac{3}{2}, 2) \) appeared in (5.3). Itô’s isometry (2.2) together with (4.14) and (4.16) yields
\[
\mathbb{E} \|u(t)\|_{H_\beta}^2 \lesssim \mathbb{E} \|E(t)(I + A)^{\frac{\beta}{2}} u_0\|_{L^2}^2 + \mathbb{E} \left( \int_0^t \|(I + A)^{\frac{\beta}{2}} E(t-s) P_X f(s)\|_{L^2} ds \right)^2
\]
\[
+ \mathbb{E} \left( \int_0^t \|(I + A)^{\frac{\beta}{2}} E(t-s) P_X B(u(s))\|_{L^2} ds \right)^2
\]
\[
\lesssim \mathbb{E} \|u_0\|_{H_\beta}^2 + \mathbb{E} \left( \int_0^t (t-s)^{-\frac{\beta}{2}} \|f(s)\|_{L^2} ds \right)^2
\]
\[
+ \mathbb{E} \left( \int_0^t (I + A)^{\frac{\beta}{2}} E(t-s) (I + A)^{\frac{\beta}{2}} P_X B(u(s))\|_{L^2} ds \right)^2
\]
\[
\lesssim (1 + \mathbb{E} \|u_0\|_{H_\beta}^2) + \int_0^t (t-s)^{-(\beta-1)} \mathbb{E} \|(I + A)^{\frac{\beta}{2}} P_X B(u(s))\|_{L^2}^2 ds
\]
\[
\lesssim (1 + \mathbb{E} \|u_0\|_{H_\beta}^2) + \int_0^t (t-s)^{-(\beta-1)} \mathbb{E} \|u(s)\|_{H_\beta}^2 ds \quad \text{(here (5.3) is used).}
\]
By using the gronwall’s inequality, we arrive at
\[
\sup_{t \in [0, T]} \mathbb{E} \|u(t)\|_{H_\beta}^2 \lesssim 1 + \mathbb{E} \|u_0\|_{H_\beta}^2 \quad \text{for all } \frac{3}{2} \leq \beta < 2. \tag{A.6}
\]
For the regularity in time, it follows from (5.2), (5.3) and (A.6) that
\[
\sup_{t \in [0, T]} \mathbb{E} \|B(u(t))\|_{L^2}^2 + \sup_{t \in [0, T]} \mathbb{E} \|(I + A)^{\frac{\beta}{2}} P_X B(u(t))\|_{L^2}^2 \lesssim 1 + \mathbb{E} \|u_0\|_{H_\beta}^2,
\]
which together with (4.16)–(4.17) shows
\[
\mathbb{E} \|u(t) - u(s)\|_{H_\beta}^2 \lesssim \mathbb{E} \|(I + A)^{-\frac{1}{2}} (E(t-s) - I)(I + A)u_0\|_{L^2}^2
\]
\[
+ \mathbb{E} \left( \int_s^t \|(I + A)^{\frac{\beta}{2}} E(t-\sigma) P_X f(\sigma)\|_{L^2} d\sigma \right)^2
\]
\[
+ \mathbb{E} \left( \int_s^t \|(I + A)^{\frac{\beta}{2}} E(t-\sigma) P_X B(u(\sigma))\|_{L^2}^2 d\sigma \right)
\]
\[
\lesssim (t-s)(1 + \mathbb{E} \|u_0\|_{H_\beta}^2) + \int_s^t \mathbb{E} \|(I + A)^{\frac{\beta}{2}} P_X B(u(\sigma))\|_{L^2}^2 d\sigma
\]
\[
\tag{A.7}
\]

This proves (3.15) and (3.16).

where we use the triangle inequality to get

This proves (3.14). The proof of Proposition 3.1 is complete.

Proof of Lemma 4.3. By Korn’s inequality [20, Theorem 2.4], the following equivalence relation holds:

where \( \beta \) is in \((\frac{3}{2}, 2)\).

Appendix B: Proof of several technical lemmas
This implies the coercivity of the operator $I + A$ and the existence of an inverse operator $(I + A)^{-1} : X \to X \cap H^2(D)^d$ (see the Lax–Milgram Lemma in [10, Theorem 6.2-1]). Since $X \cap H^2(D)^d$ is compactly embedded into $X$, it follows that $(I + A)^{-1}$ is compact. For such a symmetric positive operator $I + A$ with compact inverse, its fractional powers can be defined by means of the spectral decomposition [22, Appendix B.2].

The $H^2$ elliptic regularity of the Stokes equations implies that $(I + A)^{-1} : X \to X \cap H^2(D)^d$, which implies $\|u\|_{H^2} \lesssim \|(I + A)u\|_{L^2}$ for $u \in D(A)$. Since $\|(I + A)u\|_{L^2} \lesssim \|u\|_{H^2}$ for $u \in D(A)$, it follows that the statement in (4.14) holds for $s = 2$. The intermediate case for $s \in (0, 2)$ follows by the real interpolation between the two endpoint cases $s = 0$ and $s = 2$.

(B.1) implies $\|(I + A)^{1/2}v\|_{L^2} \sim \|v\|_{H^1}$ for $v \in D(A)$, i.e., the two norms $\| \cdot \|_{D(A)^{1/2}}$ and $\| \cdot \|_{H^1}$ are equivalent on $D(A)$. Since $D(A)$ is dense in $H^1(D)^d \cap X$, it follows that

$$D(A^{1/2}) = \text{closure of } D(A) \text{ under the norm } \| \cdot \|_{D(A)^{1/2}} \sim \| \cdot \|_{H^1} = H^1(D)^d \cap X.$$ 

Therefore, for $s \in (0, 1)$ we have

$$D(A^{s}) = \text{closure of } D(A^{1/2}) \text{ under the norm } \| \cdot \|_{D(A)^{s}}$$

$$= \text{closure of } H^1(D)^d \cap X \text{ under the norm } \| \cdot \|_{H^s} = H^s(D)^d \cap X.$$ 

The statement in (14.10) follows from the duality argument. □

**Proof of Lemma 4.4.** Let $\tilde{E}(t) := e^{-t(I + A)}$ be the semigroup generated by $I + A$, which has a compact inverse operator. Then [22, Lemma 2.5] is applicable and gives the estimates in (4.16)–(4.19) with $E(t)$ replaced by $\tilde{E}(t)$ therein. Since $E(t) = e^t \tilde{E}(t)$ and $t \in (0, T]$, it follows that (4.16)–(4.19) also hold for $E(t)$. □

**Proof of Lemma 4.6.** Subtracting (4.34) from (4.33) yields

\[
\begin{cases}
(z(w - w_h), v_h) + 2(\mathbb{D}(w - w_h), \mathbb{D}(v_h)) - (p - p_h, \nabla \cdot v_h) = 0 & \forall v_h \in V_h,
\end{cases}
\]

\begin{equation}
(B.2)
\end{equation}

Let $e_h = P_{X_h}w - w_h \in X_h$. Choosing $(v_h, q_h) = (e_h, P_{Q_h}p - p_h) \in X_h \times Q_h$ in (B.2), where $P_{Q_h}$ denotes the $L^2$-orthogonal projection onto $Q_h$, we have

\[
\begin{cases}
\|e_h\|_{L^2}^2 + 2\|D(e_h)\|_{L^2}^2 = -2\|D(w - P_{X_h}w, D(e_h))\|_{L^2} + (p - P_{Q_h}p, \nabla \cdot e_h). 
\end{cases}
\]

\begin{equation}
(B.3)
\end{equation}

By considering the imaginary and the real parts of the above equality, separately, we obtain

\[
\begin{cases}
\Im(z)\|e_h\|_{L^2}^2 \lesssim \|w - P_{X_h}w\|_{H^1}^2 + \|\mathbb{D}(e_h)\|_{L^2}^2 + \|p - P_{Q_h}p\|_{L^2},
\Re(z)\|e_h\|_{L^2}^2 + 2\|\mathbb{D}(e_h)\|_{L^2}^2 \lesssim \|w - P_{X_h}w\|_{H^1}^2 + \|\mathbb{D}(e_h)\|_{L^2}^2 + \|p - P_{Q_h}p\|_{L^2},
\end{cases}
\]

where $\varepsilon > 0$ can be arbitrarily small. For $z \in \Sigma_f := \{1 + z' \in \mathbb{C} : |\arg(z')| < \phi\}$, with some $\phi \in (0, \pi)$, we have $|z| \geq 1$ and $|\Re(z)| \lesssim \|\Im(z)\|$. By using (2.10), (2.11), (4.9) and Korn’s inequality $\|u\|_{H^1} \lesssim \|u\|_{L^2}^2 + \|\mathbb{D}(u)\|_{L^2}^2$, choosing a sufficiently small $\varepsilon$ in the above estimates, we can derive the following result:

\[
\begin{cases}
\|P_{X_h}w - w_h\|_{L^2} + \|\nabla(P_{X_h}w - w_h)\|_{L^2} \lesssim \|w - P_{X_h}w\|_{H^1} + \|p - P_{Q_h}p\|_{L^2} \lesssim h(\|w\|_{H^2} + \|p\|_{H^1}). 
\end{cases}
\]

\begin{equation}
(B.4)
\end{equation}

which, together with triangle inequality and (4.9), implies that

\[
\|\nabla(w - w_h)\|_{L^2} \lesssim h(\|w\|_{H^2} + \|p\|_{H^1}).
\]

\begin{equation}
(B.5)
\end{equation}

In order to obtain an estimate for $\|w - w_h\|_{L^2}$, we consider the following duality argument. Let $\phi \in D(A)$ and $\phi_h \in X_h$ be the solution of the linear Stokes equations

\[
\begin{cases}
z\phi - \nabla \cdot T(\phi, \eta) = w - w_h & \text{in } D,
\nabla \cdot \phi = 0 & \text{in } D,
\phi|_{\partial D} = 0 & \text{on } \partial D.
\end{cases}
\]

\begin{equation}
(B.6)
\end{equation}
and its finite element weak formulation
\[
\begin{cases}
(z\phi_h, v_h) + 2(D(\phi_h), D(v_h)) - (\eta_h, \nabla \cdot v_h) = (w - w_h, v_h) & \forall v_h \in V_h, \\
\n(\nabla \cdot \phi_h, q_h) = 0 & \forall q_h \in Q_h.
\end{cases}
\tag{B.7}
\]

Testing (B.6) by \( \phi \in X \) and considering the imaginary and real parts of the results, we can obtain the following result similarly as (B.4):
\[
|z|\|\phi\|^2_{L^2} + 2\|\nabla \phi\|^2_{L^2} \leq C|z|^{-1}\|w - w_h\|^2_{L^2} + \frac{1}{2}|z|\|\phi\|^2_{L^2},
\]
which implies
\[
|z|\|\phi\|_{L^2} \lesssim \|w - w_h\|_{L^2}.
\]

Through the above estimate and the \( H^2 \) estimate of the linear Stokes equations (B.6) (cf. [25, Corollary 2.1]), we obtain
\[
\|\phi\|_{H^2} + \|\eta\|_{H^1} \lesssim \|w - w_h\|_{L^2} + |z|\|\phi\|_{L^2} \lesssim \|w - w_h\|_{L^2}.
\tag{B.8}
\]

Similar to (B.4)–(B.5), we have
\[
|z|^{\frac{1}{2}}\|P_{X_h}\phi - \phi_h\|_{L^2} + \|\nabla(\phi - \phi_h)\|_{L^2} \lesssim h(\|\phi\|_{H^2} + \|\eta\|_{H^1}) \lesssim h\|w - w_h\|_{L^2},
\tag{B.9}
\]
where the last inequality follows from (B.8).

Since \( (\nabla \cdot v_h, P_{Q_h}\eta) = 0 \) for \( v_h \in X_h \), it follows that
\[(-\eta, \nabla \cdot v_h) = -(\eta - P_{Q_h}\eta, \nabla \cdot v_h) \quad \forall v_h \in X_h.\]

Since \( \phi_h \in X_h \), we can derive the following equation from (B.2):
\[(z\phi_h, w - w_h) + 2(D(\phi_h), D(w - w_h)) - (\nabla \cdot \phi_h, p - P_{Q_h}p) = 0.\]

Then, testing the first relation of (B.6) by \( w - w_h \) and using the two equations above, by utilizing (B.4) and (B.9)–(B.10) we obtain
\[
\|w - w_h\|^2_{L^2} = (z\phi_h, w - w_h) + 2(D(\phi_h), D(w - w_h)) - (\nabla \cdot (\phi - \phi_h), p - P_{Q_h}p) - (\eta - P_{Q_h}\eta, \nabla \cdot (w - w_h))
\lesssim |z|^{\frac{1}{2}}\|\phi - \phi_h\|_{L^2} |z|^{\frac{1}{2}}\|w - w_h\|_{L^2} + \|\nabla (\phi - \phi_h)\|_{L^2}\|\nabla (w - w_h)\|_{L^2}
+ h^2\|\phi\|_{H^2}\|p\|_{H^1} + h\|\eta\|_{H^1}\|\nabla (w - w_h)\|_{L^2}
\lesssim (h^{2} + h|z|^{\frac{1}{2}})\|w - w_h\|_{L^2} + h\|w - w_h\|_{L^2}\|\nabla (w - w_h)\|_{L^2}
+ h^2(\|w\|_{H^2} + \|p\|_{H^1})\|w - w_h\|_{L^2} \quad \text{(here (B.8) and (B.9) are used)}
\lesssim (h^{2} + h^2|z|^{\frac{1}{2}})\|w - w_h\|_{L^2}(\|w\|_{H^2} + \|p\|_{H^1}) \quad \text{(here (B.4) are used)}
\]

When \(|z| \leq h^{-2}\), the inequality above reduces to
\[
\|w - w_h\|_{L^2} \lesssim h^2(\|w\|_{H^2} + \|p\|_{H^1}).
\tag{B.10}
\]

When \(|z| \geq h^{-2}\), (B.4) immediately implies (B.10).

Combining the estimates (B.4) and (B.10), we obtain (4.34). The proof of (4.36) is similar as that for (B.8).

\[
\]
The stability of $\bar{E}^n_{h,\tau}$, $1 \leq n \leq N$ together with (2.2) and (4.58) yields
\[
\mathbb{E}\|u^n_h\|^2_{H^2} \lesssim \mathbb{E}\|\bar{E}^n_{h,\tau}(I + A_h)^{-1}P_{X_h}u^0\|^2_{L^2} + \mathbb{E}\left(\tau \sum_{i=0}^{n-1} \|\bar{E}^n_{h,\tau}P_{X_h}f(t_{i+1})\|^2_{L^2}\right) + \mathbb{E}\left(\tau \sum_{i=0}^{n-1} \|\bar{E}^n_{h,\tau}P_{X_h}B(u^i_h)\|_{L^2}^2\right) + \tau \sum_{i=0}^{n-1} \mathbb{E}\|\bar{E}^n_{h,\tau}P_{X_h}B(u^i_h)\|^2_{L^2}.
\]
By using the discrete Gronwall inequality, we obtain
\[
\max_{1 \leq n \leq N} \mathbb{E}\|u^n_h\|^2_{H^2} \lesssim 1 + \mathbb{E}\|u^0\|^2_{H^2}. \tag{C.3}
\]
This proves the desired estimate for the first term in (5.8).

Setting $v_h = u^n_h \in X_h$ in (2.19) and applying the identity $2(a-b)a = |a|^2 - |b|^2 + |a-b|^2$ for $a, b \in \mathbb{R}$ yield
\[
\frac{1}{2}\|u^n_h\|^2_{H^2} + \frac{1}{2}\|u^n_h - u^{n-1}_h\|^2_{H^2} + 2\tau\|\nabla (u^n_h)\|^2_{L^2} \leq \frac{1}{2}\|u^n_h - u^{n-1}_h\|^2_{H^2} + \frac{1}{2}\tau\|f(t_n)\|^2_{L^2} + \frac{1}{2}\tau\|u^n_h\|^2_{L^2} + (B(u_n^{n-1})\Delta W_n, u^{n-1}_h), \tag{C.4}
\]
where the last term can be estimated by applying the property of a martingale
\[
\mathbb{E}(B(u_n^{n-1})\Delta W_n, u^{n-1}_h) = 0
\]
and using (5.2) and (C.3) as follows:
\[
\mathbb{E}(B(u_n^{n-1})\Delta W_n, u^{n-1}_h) = C\mathbb{E}\|B(u^{n-1}_h)\Delta W_n\|^2_{L^2} + C\mathbb{E}\|u^n_h - u^{n-1}_h\|^2_{L^2} \leq C\mathbb{E}\|B(u^{n-1}_h)\|^2_{L^2} + \frac{1}{4}\mathbb{E}\|u^n_h - u^{n-1}_h\|^2_{L^2} \leq C\tau(1 + \mathbb{E}\|u^{n-1}_h\|^2_{H^2}) + \frac{1}{4}\mathbb{E}\|u^n_h - u^{n-1}_h\|^2_{L^2} \leq C\tau(1 + \mathbb{E}\|u^0_h\|^2_{H^2}) + \frac{1}{4}\mathbb{E}\|u^n_h - u^{n-1}_h\|^2_{L^2}.
\]
By using Korn’s inequality (cf. [20, Theorem 2.4]) $\|u\|^2_{L^2} + \|\nabla(u)\|^2_{L^2} \geq \theta(\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2})$, where $\theta > 0$ is some constant, we obtain
\[
\mathbb{E}\|u^n_h\|^2_{L^2} + \sum_{n=1}^{N} \mathbb{E}\|u^n_h - u^{n-1}_h\|^2_{L^2} + \tau \sum_{n=1}^{N} \mathbb{E}\|\nabla u^n_h\|^2_{L^2} \lesssim 1 + \mathbb{E}\|u^0_h\|^2_{H^2} + \tau \sum_{n=1}^{N} \mathbb{E}\|u^n_h\|^2_{L^2}. \tag{C.6}
\]
Then applying the discrete Gronwall inequality yields the desired estimate for the last two terms in (5.8). This proves Lemma 5.2. \qed
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