Abstract Fractional Calculus for Accretive Operators

Maksim V. Kukushkin

*International Committee Continental,*
Russia, Zheleznovodsk, kukushkinmv@rambler.ru

"You should prefer choosing exact conditions!"
Blessed memory of Adam M. Nakhushev is devoted

Abstract

In this paper we have made an attempt to generalize some results obtained for some class of non-selfadjoint operators by means of using properties of real component. The central point is a main theorem establishing validity of a number of spectral theorems for some type of positive operator functions. The relevance of such consideration is lot of applications to semigroup theory. More precisely, we can treat considered operator as an operator second order with fractional derivative in the lower terms.

**Keywords:** Positive operator; fractional power of operator; semigroup generator; strictly accretive property.

**MSC** 47A10; 47A07; 47B10; 47B25.

1 Introduction

Let $C, C_i$, $i \in N_0$ be positive real constants. We mean that value $C$ can be different in various formulas but values $C_i$, $i \in N_0$ are certain. Let $\mathcal{B}(\mathfrak{H})$ be a set of linear bounded operators acting in a Hilbert space $\mathfrak{H}$ and defined on the whole space $\mathfrak{H}$. Everywhere further we consider linear densely defined operators acting in a separable complex Hilbert space $\mathfrak{H}$. Symbols $D(L), R(L)$ denote respectively a domain of definition and a range of the operator $L$. Symbols $R_L(\zeta), R_L := R_L(0), \zeta \in P(L)$ denote a resolvent of the operator $L$. Define $s$-numbers of a compact operator $L$ as: $s_i(L) = \lambda_i(N), i = 1, 2, \ldots, r(N)$, where $\lambda_i(N)$ are eigenvalues of an operator $N := (L^*L)^{1/2}$, $r(N) = \dim \mathfrak{N}(N)$. If $r(N) < \infty$, we suppose $s_i = 0, i = r(N) + 1, 2, \ldots$. Following terminology [15], we use a term ”algebraic multiplicity” meaning the dimension of a root vectors subspace corresponding to the certain eigenvalue. A sum of all algebraic multiplicities of an operator $L$ we denote by $\nu(L)$. We use the following denotation of a Schatten-von Neumann class $\mathfrak{S}_p(\mathfrak{H}), 1 \leq p \leq \infty$. Following the definition given in the paper [53], denote by $\mu(L)$ an order of the operator $L$ with a compact resolvent, if we have an estimate $\lambda_n(R_L) \leq C n^{-\mu}, n \in N, 0 \leq \mu \leq \infty$. The case $\mu = \infty$ corresponds to...
the Volterra case. Let $L_\Re := (L + L^*)/2$, $L_\Im := (L - L^*)/2i$ be a so-called real and imaginary component of the operator $L$ respectively. We use a notation $\tilde{L}$ for the closure of the operator $L$.

Using terminology [19], we say that an operator $L$ is strictly accretive if the following relation holds $\Re(Lf, f)_{\mathcal{H}} \geq C\|f\|_{\mathcal{H}}^2$, $f \in \text{D}(L)$, also we will use this definition for operators which are defined on subspace of $\mathcal{H}$. Using the definition [19] p.279 we call an operator $L$ $m$-accretive if the next relations hold $(L + \zeta)^{-1} \in \mathcal{B}(\mathcal{H})$, $\|(L + \zeta)^{-1}\| \leq (\Re\zeta)^{-1}$, $\Re\zeta > 0$. Everywhere further if it is not stated otherwise, we use the notations accepted in the literature [15, 19, 52].

1. Spectral theorems

Consider the pair of complex separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ with the assumptions

$$\mathcal{H}_1 \hookrightarrow \mathcal{H}_2.$$  \hspace{1cm} (1)

This denotation means that $\mathcal{H}_1$ is dense in $\mathcal{H}_2$ as a set of the elements and we have a bounded embedding provided by the inequality

$$\|f\|_{\mathcal{H}_2} \leq \|f\|_{\mathcal{H}_1}, \text{ } f \in \mathcal{H}_1,$$  \hspace{1cm} (2)

moreover any set bounded in the sense of the norm $\mathcal{H}_1$ is compact in the sense of the norm $\mathcal{H}_2$. Denote by $\mathcal{H}_{1, L}$, $\Sigma - L$ respectively the energetic space generated by the operator $L$ and the norm in this space (more detailed [55, 47]). We consider the non-selfadjoint operators which can be represented by the sum $W = T + S$ with the certain assumptions relative to a main part - operator $T$ and a lower term - operator $S$, both of these operators act in $\mathcal{H}$. We assume that there exists a linear manifold $\mathcal{M} \subset \mathcal{H}_2$ dense in $\mathcal{H}$ on which the operators $T, S$ are well defined with their adjoint operators. In further we suppose that the following conditions are fulfilled

$$i) \Re(Tf, f)_{\mathcal{H}} \geq C_0\|f\|_{\mathcal{H}_1}^2, \langle Tf, g\rangle_{\mathcal{H}} \leq C_1\|f\|_{\mathcal{H}_1}\|g\|_{\mathcal{H}_2},$$

$$ii) \Re(Sf, f)_{\mathcal{H}} \geq C_2\|f\|_{\mathcal{H}_1}^2, \langle Sf, g\rangle_{\mathcal{H}} \leq C_3\|f\|_{\mathcal{H}_1}\|g\|_{\mathcal{H}_2}, f, g \in \mathcal{M}.$$  \hspace{1cm} (3)

Thus in further we suppose $\text{D}(W) = \mathcal{M}$, what gives us an opportunity to approve that $\text{D}(W) \subset \text{D}(W^*)$. The closure of the operator $W_\Re$ we denote by $W$. Due to conditions [11] and by virtue of Theorem 3.4 [19] p.268 the operator $W$ is closable. Let $\tilde{W}$ be a closure of the operator $W$. It is proved in [35] that under assumptions [11], the operator $\tilde{W}$ is sectorial with the top situated at the point zero and a semi-angle $\theta$. Also the following series of theorems are true

**Theorem 1.** We have the implications

$$\mu > 1 \Rightarrow R_{\tilde{W}} \in \mathcal{G}_1; \mu p > 2 \Rightarrow R_{\tilde{W}} \in \mathcal{G}_p, 1 < p < \infty,$$

where $\mu := \mu(W)$. Moreover under assumptions $\lambda_n(R_H) \geq c n^{-\mu}$, $n \in \mathbb{N}$, we have the implication

$$R_{\tilde{W}} \in \mathcal{G}_p \Rightarrow \mu p > 1, 1 \leq p < \infty.$$ 

**Theorem 2.** The following relation holds

$$\sum_{i=1}^{\nu} |\lambda_i(R_{\tilde{W}})|^p \leq \sec^p \theta \|S^{-1}\| \sum_{i=1}^{n} \lambda_i^p(R_H), 1 \leq p < \infty, \text{ } (n = 1, 2, ..., \nu(R_{\tilde{W}})).$$  \hspace{1cm} (4)

Moreover if $\nu(R_{\tilde{W}}) = \infty$ and the order $\mu(W) \neq 0$, then the following asymptotic formula holds

$$\lambda_i(R_{\tilde{W}}) = o(i^{-\mu+\varepsilon}), i \to \infty, \forall \varepsilon > 0.$$
Theorem 3. Let $\theta < \pi \mu /2$, then a system of root vectors of $R_{\mathfrak{W}}$ is complete in $\mathfrak{H}$.

2. Strictly accretive and m-accretive operators

For a reader convenience, we would like to establish known facts of the operator theory under the point of view required for our next reasonings.

Consider a closed densely defined operator $A$ with the following imposed condition

$$\|(A + t)^{-1}\|_{R \to \mathfrak{H}} \leq \frac{1}{t}, \quad t > 0. \quad (5)$$

If $R := R(A + t) = \mathfrak{H}$, $\text{Re} t > 0$ and in the right part we have $1/\text{Re} t$, $\text{Re} t > 0$,

then in accordance with the definition given in [19] we call the operator $A$ m-accretive. Let us show that having the assumptions (5) is sufficient to prove that the operator $A$ is m-accretive. Using (5), consider

$$\|f\|_{\mathfrak{H}}^2 \leq \frac{1}{t^2} \| (A + t)f \|_{\mathfrak{H}}^2, \quad f \in D(A),$$

we have

$$\|f\|_{\mathfrak{H}}^2 \leq \frac{1}{t^2} \left\{ \| Af \|_{\mathfrak{H}}^2 + 2\text{Re} (Af, f)_{\mathfrak{H}} + t^2 \|f\|_{\mathfrak{H}}^2 \right\}.$$ 

Hence

$$t^{-1} \| Af \|_{\mathfrak{H}}^2 + 2\text{Re} (Af, f)_{\mathfrak{H}} \geq 0.$$ 

Let $t$ be tended to infinity, then we obtain

$$\text{Re} (Af, f)_{\mathfrak{H}} \geq 0, \quad f \in D(A). \quad (6)$$

It means the operator $A$ has an accretive property. Due to (6), we have $\{ \lambda \in \mathbb{C} : \text{Re} \lambda < 0 \} \subset \Delta(A)$, where $\Delta(A) = \mathbb{C} \setminus \Theta(A)$ and $\Theta(A)$ is a closure of the operator $A$ quadratic form on elements with unit norm. Applying Theorem 3.2 [19, p.268], we obtain that $A - \lambda$ has a closed range and $\text{null}(A - \lambda) = 0$, $\text{def}(A - \lambda) = \text{const}, \forall \lambda \in \Delta(A)$. Let $\lambda_0 \in \Delta(A)$, $\text{Re} \lambda_0 < 0$. In consequence of inequality (6), we have

$$\text{Re}(f, (A - \lambda_0)f)_{\mathfrak{H}} \geq -\text{Re} \lambda \|f\|_{\mathfrak{H}}^2, \quad f \in D(A). \quad (7)$$

Since the operator $A - \lambda_0$ has a closed range, then

$$\mathfrak{H} = R(A - \lambda_0) \oplus R(A - \lambda_0)^{\perp}.$$ 

Note that intersection of the sets $D(A)$ and $R(A - \lambda_0)^{\perp}$ is zero, because if we assume otherwise, then applying inequality (7) for any element $f \in D(A) \cap R(A - \lambda_0)^{\perp}$, we get

$$-\text{Re} \lambda_0 \|f\|_{\mathfrak{H}}^2 \leq \text{Re}(f, [A - \lambda_0]f)_{\mathfrak{H}} = 0,$$

hence $f = 0$. It implies that

$$(f, g)_{\mathfrak{H}} = 0, \quad \forall f \in R(A - \lambda_0)^{\perp}, \quad \forall g \in D(A).$$
Since $D(A)$ is a dense set in $\mathcal{H}$, then $R(A - \lambda_0) = 0$. It implies that $\text{def}(A - \zeta_0) = 0$ and if we note Theorem 3.2 [19, p.281] we came to conclusion that $\text{def}(A - \lambda) = 0$, $\forall \lambda \in \Delta(A)$ and

$$
\|(A + \lambda)^{-1}\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq \frac{1}{\Re \lambda}, \Re \lambda > 0.
$$

(8)

It means that the operator $A$ is m-accretive. In accordance with definition made in [36] we can define a positive and negative fractional powers of the positive operator $A$ in the following

$$
A^\alpha := \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{-\alpha - 1}(\lambda + A)^{-1}A d\lambda; \quad A^{-\alpha} := \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1} d\lambda, \quad \alpha \in (0, 1).
$$

(9)

This definition can be correctly extended on m-accretive operators, the corresponding reasoning can be found in [19]. Thus further we define positive and negative fractional powers of m-accretive operators due to formula (9). Let us show that a fractional power $A^\alpha$, $\alpha \in (0, 1)$ of m-accretive operator $A$ is accretive operator. We have the following reasoning

$$
\text{Re} \left( [\lambda + A]^{-1}Af, f \right)_\mathcal{H} = \text{Re} \left( [\lambda + A]^{-1}[\lambda + A]f, f \right)_\mathcal{H} - \text{Re} \left( \lambda[\lambda + A]^{-1}f, f \right)_\mathcal{H}
$$

$$
\geq \|f\|_\mathcal{H}^2 (1 - \lambda \cdot \|(\lambda + A)^{-1}\|), \quad f \in D(A), \lambda > 0.
$$

Using condition (8), we obtain the following inequality

$$
\text{Re} \left( [\lambda + A]^{-1}Af, f \right)_\mathcal{H} \geq 0, \quad f \in D(A).
$$

Hence, we obtain

$$
\text{Re} \left( A^\alpha f, f \right)_\mathcal{H} = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{-\alpha - 1}\text{Re} \left( [\lambda + A]^{-1}Af, f \right)_\mathcal{H} d\lambda \geq 0, \quad f \in D(A).
$$

Since in accordance with Theorem 3.35 [19, p.281] the set $D(A)$ is a core of $A^\alpha$, then we can extend the previous inequality as the following

$$
\text{Re} \left( A^\alpha f, f \right)_\mathcal{H} \geq 0, \quad f \in D(A^\alpha).
$$

Let us show that if an operator $A$ is m-accretive, then the operator $A^*$ is m-accretive. Since it is proved that $\text{def}(A + \lambda) = 0$, $\lambda > 0$, then $\text{nul}(A + \lambda)^* = 0$, $\lambda > 0$. In accordance with well-known theorem, we have $([\lambda + A]^{-1})^* = [\lambda + A]^*$. Since $\lambda + A^* \subset (\lambda + A)^*$, then $(\lambda + A)^{-1} \subset [(\lambda + A)^*]^{-1}$. Also it is clear that $\|(\lambda + A)^{-1}\|_{\mathcal{B} \rightarrow \mathcal{B}} = \|[(\lambda + A)^{-1}]^*\|_{\mathcal{B} \rightarrow \mathcal{B}}$. Hence

$$
\|(\lambda + A^*)^{-1}f\|_\mathcal{B} = \|[(\lambda + A)^*]^{-1}f\|_\mathcal{B} = \|[(\lambda + A)^{-1}]^*f\|_\mathcal{B} \leq \frac{1}{\lambda}\|f\|_\mathcal{B},
$$

$$
f \in R(\lambda + A^*), \lambda > 0.
$$

We can rewrite this relation, as

$$
\|(\lambda + A^*)^{-1}\|_{\mathcal{R} \rightarrow \mathcal{B}} \leq \frac{1}{\lambda}, \lambda > 0,
$$

$$
4
$$
In accordance with proved above, we can extend the previous inequality, as

\[ \| (\lambda + A^*)^{-1} \|_{\mathcal{B}} \leq \frac{1}{\Re \lambda}, \quad \Re \lambda > 0. \] (10)

Let us show that \( A^{\alpha*} \supset A^{\alpha*}. \) Note that formula (9) means that there exists a limit in the sense of norm \( \mathcal{B} \)

\[ \sum_{i=0}^{n} \lambda_i^{\alpha-1} (\lambda_i + A)^{-1} A f \Delta \lambda_i \xrightarrow{\delta} A^{\alpha} f, \quad n \to \infty, \quad \lambda_i \to \infty, \quad \Delta \lambda_i \to 0, \]

\( f \in D(A), \quad \alpha \in (0, 1). \) (11)

Hence, using (1), we can write

\[ (A^\alpha f, g)_{\mathcal{B}} = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha-1} ([\lambda + A]^{-1} A f, g)_{\mathcal{B}} d\lambda = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha-1} (f, ([\lambda + A]^{-1} A^*)^* g)_{\mathcal{B}} d\lambda, \]

\( f \in D(A), \quad g \in D \{([\lambda + A]^{-1} A)^*) \}. \) (12)

It is clear that \([\lambda + A]^{-1} A^* \supset A^* ([\lambda + A]^{-1})^*. \) Since \( \text{def}(A + \lambda), \text{def}(A^* + \lambda) = 0, \lambda > 0, \) then it can be proved easily that \( \lambda + A^* = (\lambda + A)^*, \lambda > 0. \) Hence \([\lambda + A]^{-1} = ([\lambda + A]^*)^{-1} = (\lambda + A^*)^{-1}, \lambda > 0 \) and

\[ ([\lambda + A]^{-1} A^*)^* \supset A^* (\lambda + A^*)^{-1}. \] (13)

It can be checked in an easy way that

\[ A^* (\lambda + A^*)^{-1} f = (\lambda + A^*)^{-1} A^* f, \quad f \in D(A^*), \]

we just should consider the identity \( A^* (\lambda + A^*)^{-1} = I - \lambda (\lambda + A^*)^{-1} \) so the rest part of the proof of relation (14) looks like understandable. Thus we can consider a formal relation

\[ \int_0^\infty \lambda^{\alpha-1} A^* (\lambda + A^*)^{-1} f d\lambda = \int_0^\infty \lambda^{\alpha-1} (\lambda + A^*)^{-1} A^* f d\lambda, \quad f \in D(A^*), \]

but by virtue of (10), the right part of this relation is an operator \( A^{\alpha*}. \) It implies that one is true. In consequence of (1), (13), (14), we obtain \( A^{\alpha*} \subset A^{\alpha}. \)

## 2 Main results

We have the following theorem

**Theorem 4.** Assume that \( J \) is a closed densely defined, \( m \)-accretive operator acting in a separable Hilbert space \( \mathcal{H}, \) \( J^{-1} \) is compact. The operator \( G \) is strictly accretive with bounded sesquilinear form, such that \( T = J^* G J \) is densely defined, \( D(T) \subset D(T^*). \) The operator \( F \) is bounded, a composition \( F J^\alpha \) is strictly accretive and \( F^* : D(J^\alpha) \to D(J^\alpha). \) Then for the operator

\[ \mathcal{L} := J^* G J + F J^\alpha, \quad D(\mathcal{L}) = D(J^* G J), \quad \alpha \in (0, 1/2), \]

theorems \( 1, 2, 3 \) are true.
Proof. Let us define a Hilbert space $\mathcal{H}_J := \{f, g \in D(J), (f, g)_{\mathcal{H}_J} = (Jf, Jg)_{\mathcal{H}_J}\}$. First we should check condition (1) in assumptions that $\mathcal{H}_+ := \mathcal{H}_J$. Since $J^{-1}$ is compact, then it is clear that $\|f\|_{\mathcal{H}_J} \leq C\|Jf\|_{\mathcal{H}_J}$, $f \in D(J)$. Without lose of generality we can consider that $C = 1$. Thus, we obtain fulfillment of inequality (2). In consequence of compactness property of the operator $J^{-1}$ the embedding provided by inequality (2) is compact. Hence we have fulfillment of (1). Let us check condition (1) in assumptions that $H$ results of T.Kato [18], we have $D(J^*) = D(L^*)$. We know that $J^{*a} \subset J^*$, $(FJ^a)^* \supset J^{*a}F^*$. Note that due to the results of T.Kato [18], we have $D(J^a) = D(J^{*a})$. Hence, taking into considerations the conditions of this theorem and said above, we can conclude in an obvious way that $D(L) \subset D(L^*)$.

We should check fulfillment of conditions (3). Consider

$$
|\langle J^*GJf, g \rangle_{\mathcal{H}_J} | = |\langle GJf, Jg \rangle_{\mathcal{H}_J} | \leq C\|Jf\|_{\mathcal{H}_J}\|Jg\|_{\mathcal{H}_J}, f, g \in D(L).
$$

Therefore condition (i) is fulfilled. Due to strictly accretive property of the operator $FJ^a$, we have not doubt in fulfillment of first condition (ii). For the proof the second one, consider

$$
|\langle FJ^a f, g \rangle_{\mathcal{H}_J} | \leq \|FJ^a f\|_{\mathcal{H}_J}\|g\|_{\mathcal{H}_J} \leq C\|J^a f\|_{\mathcal{H}_J}\|g\|_{\mathcal{H}_J}.
$$

(15)

In accordance with [19], we have $J^a = J^{a-1}J$, where

$$
J^{a-1} = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{a-1}(J + \lambda)^{-1}d\lambda.
$$

Let us prove that $J^{a-1}$ is bounded in $\mathcal{H}$, for this purpose consider a decomposition

$$
J^{a-1} = \int_0^1 \lambda^{a-1}(\lambda + J)^{-1}d\lambda + \int_1^\infty \lambda^{a-1}(\lambda + J)^{-1}d\lambda = I_1 + I_2.
$$

Using the fact $J(\lambda + J)^{-1}f = (\lambda + J)^{-1}Jf$, $f \in D(J)$, we can evaluate respectively

$$
\|I_1 f\|_{\mathcal{H}_J} = \left\| \int_0^1 \lambda^{a-1}J^{-1}f(\lambda + J)^{-1}d\lambda \right\|_{\mathcal{H}_J} \leq \|J^{-1}\|_{R \to \mathcal{H}_J} \left\| \int_0^1 \lambda^{a-1}J(\lambda + J)^{-1}f(\lambda + J)^{-1}d\lambda \right\|_{\mathcal{H}_J}
$$

$$
\leq 2\|J^{-1}\|_{R \to \mathcal{H}_J} \|f\|_{\mathcal{H}_J} \int_0^1 \lambda^{a-1}d\lambda, \; f \in D(J);
$$

$$
\|I_2 f\|_{\mathcal{H}_J} = \left\| \int_1^\infty \lambda^{a-1}(\lambda + J)^{-1}f(\lambda + J)^{-1}d\lambda \right\|_{\mathcal{H}_J} \leq \|f\|_{\mathcal{H}_J} \int_1^\infty \lambda^{a-1}d\lambda \leq \|f\|_{\mathcal{H}_J} \int_1^\infty \lambda^{a-2}d\lambda.
$$

Hence $J^{a-1}$ is bounded on $D(J)$, since $D(J)$ is dense in $\mathcal{H}$, then $J^{a-1}$ is bounded in $\mathcal{H}$. Therefore

$$
\|J^a f\|_{\mathcal{H}_J} = \|J^{a-1}Jf\|_{\mathcal{H}_J} \leq C\|Jf\|_{\mathcal{H}_J}, \; f \in D(J).
$$

Hence we have fulfillment of the second condition (ii). The proof is complete. 

\[\square\]
1. Abstract Lebesgue spaces and coefficient-operator \( F \)

Consider a space with a measure \((\Omega, F, \mu)\) and corresponding Hilbert space \( L_2(\Omega, \mu) \) of functions defined on the set \( \Omega \). Let us study more carefully the particular case when \( H \) is \( L_2(\Omega, \mu) \).

Our aim to point out various particular features of Theorem 4 to reader.

Suppose that \( T_t \) is a \( C_0 \) semigroup of contractions acts in \( L_2(\Omega, \mu) \), \( A \) is the semigroup generator. Note that due to Hille-Yosida theorem (Theorem 3.1 [51, p.8]), we have fulfillment of inequality (5), hence positive fractional powers of \( A \) are defined. Also under this assumption we know that so-called Balakrishnian formula takes a place. The following theorem establishes sufficient conditions for fulfillment of the strictly accretive property of the operator \( FA^\alpha \).

**Theorem 5.** Assume that \( T_t \) is \( C_0 \) semigroup acting in the space \( L_2(\Omega, \mu) \), and suppose that \( F \) is bounded, \( Ff \geq 0, (f \geq 0), F^*A^\alpha C > N > 0 \), \( \max_t \Psi_f(t) = \Psi_f(0), \ \Upsilon_f(t) \geq 0 \), where

\[
\Psi_f(t) = \int_{\Omega} (f F T_t f - F f T_t f) \, d\mu, \ \Upsilon_f(t) = \int_{\Omega} [F(T_t f)^2 - T_t F f^2] \, d\mu, \ f \in D(A).
\]

then operator \( FA^\alpha \) is strictly accretive.

**Proof.** First, consider a real case. Note that using the Balakrishnan formula, we get

\[
I = \int_{\Omega} f F A^\alpha f \, d\mu = \frac{\alpha}{\Gamma(1 - \alpha)} \int_{\Omega} f \, d\mu \int_{0}^{\infty} \frac{F(I - T_t)}{t^{\alpha+1}} \, dt, \ f \in D(A).
\]

Since generator \( A \) exists, then representing the inner integral by a sum

\[
\int_{0}^{\infty} \frac{F(I - T_t)}{t^{\alpha+1}} \, dt = \int_{0}^{\varepsilon} \frac{F(I - T_t)}{t^{\alpha+1}} \, dt + \int_{\varepsilon}^{\infty} \frac{F(I - T_t)}{t^{\alpha+1}} \, dt,
\]

we can easily prove that

\[
\left\| \int_{0}^{\infty} \frac{F(I - T_t)}{t^{\alpha+1}} \, dt \right\|_{L_2} \leq C_f \{ \| A f \|_{L_2} + \| f \|_{L_2} \}, \ f \in D(A).
\]

where \( C_f \) is a constant depended on \( f \). Thus applying the Cauchy-Schwartz inequality we obtain

\[
\int_{\Omega} \int_{0}^{\infty} \left| \frac{f F(I - T_t)f}{t^{\alpha+1}} \right| \, dt \, d\mu < \infty, \ f \in D(A).
\]

Also it is clear that the function \( \tilde{f}(Q, t) := f(Q) F(I - T_t) f(Q) t^{-\alpha-1} \) is (L) measurable function on \( \Omega \times (0, \infty) \). It gives us opportunity having applied the Fubini theorem to deduce

\[
I = \frac{\alpha}{\Gamma(1 - \alpha)} \int_{0}^{\infty} t^{-\alpha-1} \int_{\Omega} f F(I - T_t)f \, d\mu, \ f \in D(A).
\]
Let us investigate conditions imposed on the operator $F$ when operator $FA^\alpha$ is strictly accretive. For this purpose consider

$$fF(I - T)f = \{FFf - Ff^2 - (FTf - FfT)\} + \frac{1}{2} [TFf^2 - F(Tf)^2] + \frac{1}{2} (Ff^2 - TfF)^2 + \frac{1}{2} F(f - Tf)^2.$$

By virtue of the theorem conditions, we get

$$\int_\Omega fFA^\alpha f d\mu \geq \frac{\alpha}{2\Gamma(1 - \alpha)} \int_\Omega f d\mu \int_0^\infty \frac{(I - T)f^2}{t^{\alpha+1}} dt = \frac{1}{2} \int_\Omega A^\alpha Ff^2 d\mu \geq C \int_\Omega f^2 d\mu, \ f \in D(A).$$

Since $D(A)$ is a core of the operator $A^\alpha$, then we can extend the previous inequality so that one is true on the set $D(A^\alpha)$. Note that

$$\text{Re} (f, FA^\alpha f)_{L^2(\Omega,\mu)} = (u, FA^\alpha u)_{L^2(\Omega,\mu)} + (v, FA^\alpha v)_{L^2(\Omega,\mu)}, \ f = u + iv, \ f \in D(A^\alpha).$$

Hence

$$\text{Re} (f, FA^\alpha f)_{L^2(\Omega,\mu)} \geq C\|f\|^2_{L^2(\Omega,\mu)}, \ f \in D(A^\alpha).$$

**Remark 1.** Under imposed assumptions of Theorem 5, let us consider the additional condition $Tf = T \cdot Tg, \ f, g \in D(A)$, then

$$\Upsilon_f(t) = \int_{\Omega} [Tf, F] f^2 d\mu, \ f \in D(A),$$

where $[T, F] = T - FT$ is a commutator. If $Ff = \rho f$, $f \in D(F)$, where $\rho$ is some function, when it is obviously that $\Psi_f(t) = 0$, and $\max_t \Psi_f(t) = \Psi_f(0)$. Also we have the following

$$|\Upsilon_f(t)| = \int_{\Omega} T f^2 \cdot (T \rho - \rho) d\mu \leq \|Tf\|^2_{L^2(\Omega,\mu)} \|\rho - Ti\rho\|_{L^\infty(\Omega,\mu)}.$$ 

If we have fulfillment of the conditions $\rho \in L^\infty(\Omega,\mu)$, $\|\rho - Ti\rho\|_{L^\infty(\Omega,\mu)} \leq Mt^\lambda, \lambda > \alpha$, then in the real case, we obtain $\int_0^\infty \Upsilon_f(t) dt \geq -C\|f\|^2_{L^2(\Omega,\mu)}$. Finally, by virtue of such arranging of the conditions, we obtain

$$\text{Re} (f, \rho A^\alpha f)_{L^2(\Omega,\mu)} \geq (N - K)\|f\|^2_{L^2(\Omega,\mu)}, \ f \in D(A^\alpha),$$

where $N$ is positive, $K$ is nonnegative constants depended on the function $\rho$. 

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2. Particular case corresponding to the certain semigroup

Let us demonstrate the certain case within the framework of operator theory that suitable to our abstract main theorem. Let $\Omega \subset \mathbb{R}^n$ be a convex domain with a sufficient smooth boundary of $n$-dimensional Euclidian space. For the sake of the simplicity we consider that $\Omega$ is bounded, but as it will be clear further we can extend obtained results for some type of unbounded domains.

Consider a sum of an uniformly elliptic operator in the divergent form of writing and adjoint of Kipriyanov fractional differentiation operator of order $0 < \alpha < 1/2$ (see [33])

\[
Lf := -Tf + \rho \mathcal{D}_a^\alpha f,
\]

\[
D(L) = H^2(\Omega) \cap H^1_0(\Omega),
\]

where $T := D_j(a^{ij}D_i \cdot)$, with the following assumptions relative to the real-valued coefficients

\[
a^{ij}(Q) \in C^1(\bar{\Omega}), \quad a^{ij}\xi_i \xi_j \geq a|\xi|^2, \quad a > 0,
\]

\[
|\rho(Q) - \rho(N)| \leq M|Q - N|^\mu, \quad N, Q \in \bar{\Omega}, \quad \inf_{Q \in \Omega} \rho(Q) \gg M, \quad \alpha < \mu \leq 1.
\]

Consider the shift semigroup in the direction defined as $T_t f(Q) = f(Q + et)$, where $Q \in \Omega, \ Q = P + er, \ r$ is an Euclidian distance between a fixed point $P$ and the point $Q$. Without lose of generality let us choose a system of coordinate and the point $P$ so that

\[
C_\omega := \sup_{e \in \omega} \max_{i=1}^n \frac{1}{\|e_i\|_{L^\infty}} < \infty,
\]

where $e_i, \ i = 1, 2, \ldots, n$ are basis vectors in $\mathbb{E}^n$. Also we suppose that all functions have a zero extension outside of $\Omega$. By virtue of continuous in average property, we can conclude that $T_t$ is strongly continuous semigroup, also it is clear that $\|T_t\|_{L_2 \to L_2} \leq 1$ and we conclude that $T_t$ is $C_0$ semigroup of contractions (see [51]). Hence due to Corollary 3.6 [51, p.11], we have

\[
\|(\lambda + A)^{-1}\|_{L_2 \to L_2} \leq \frac{1}{\Re \lambda}, \quad \Re \lambda > 0,
\]

(17)

where $A$ is a generator or derivative in a direction $e, \ Af(Q) = -(\nabla f(Q), e)_{\mathbb{E}^n} = -\text{div} f, \ e = (Q - P)/r$. Inequality (17) implies that $A$ is m-accretive. Hence we can define positive fractional powers $\alpha \in (0, 1)$ of operator $A$ using formula (12). Applying the Balakrishnan formula, we obtain

\[
A^\alpha := \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha - 1}(\lambda + A)^{-1} A d\lambda = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{T_t - I}{t^{\alpha + 1}} dt, \quad \alpha \in (0, 1).
\]

Hence, in a concrete form of writing, we have

\[
A^\alpha f(Q) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(Q + et) - f(Q)}{t^{\alpha + 1}} dt = \frac{\alpha}{\Gamma(1 - \alpha)} \int_r^{d(e)} \frac{f(Q) - f(P + et)}{(t - r)^{\alpha + 1}} dt + \frac{f(Q)}{\Gamma(1 - \alpha)} \{d(e) - r\}^{-\alpha} = \mathcal{D}_a^\alpha f(Q), \ f \in D(\mathcal{D}_a^\alpha),
\]

where $r$ is an Euclidian distance between a fixed point $P$ and the point $Q$. Also we suppose that all functions have a zero extension outside of $\Omega$. By virtue of continuous in average property, we can conclude that $T_t$ is strongly continuous semigroup, also it is clear that $\|T_t\|_{L_2 \to L_2} \leq 1$ and we conclude that $T_t$ is $C_0$ semigroup of contractions (see [51]). Hence due to Corollary 3.6 [51, p.11], we have

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\]
where \( d(e) \) is a distance from the point \( P \) to the edge of \( \Omega \) along the direction \( e \). Applying relation (10), we get
\[
\| (\lambda + A^*)^{-1} \|_{L_2 \rightarrow L_2} \leq \frac{1}{\text{Re}\lambda}, \quad \lambda > 0.
\]
Note that we have proved in general \( A^\alpha \subset A^{\alpha^*} \). It was proved in [33] \( \mathcal{D}_d^{\alpha^*} = \mathcal{D}_d^\alpha \), where the last operator is an extension of the Kipriyanov operator of fractional differentiation. If we note the fact \( \mathcal{D}(A^\alpha) = \mathcal{D}(A^{\alpha^*}) \), \( \alpha \in (0, 1/2) \), has been proved by T. Kato [18], then we achieve \( \mathcal{D}_d^\alpha(L_2) = \mathcal{D}_d^\alpha(L_2) \), \( \alpha \in (0, 1/2) \). It should be noticed that the last fact is known in the one dimensional case [52].

Consider a restriction \( A_0 \) of the operator \( A \) defined as \( A_0 \subset A \), \( \mathcal{D}(A_0) = C_0^\infty(\Omega) \). It is well-known fact that \( \mathcal{D}(A) \) is closed, hence \( A_0 \) is closeable.

Let us prove equivalence between norms of the spaces \( H_0^1(\Omega) \) and \( \mathcal{H}_{A_0} \). Suppose \( f \in C_0^\infty(\Omega) \) and consider the following relation
\[
f(Q + e_i \Delta x_i) \cos \beta = f(Q + e \Delta r), \quad \Delta r = \Delta x_i (e, e_i)_{\mathbb{R}^n}, \quad \beta r \sim \Delta x_i,
\]

hence
\[
f(Q + e_i \Delta x_i) - f(Q) = (e, e_i)_{\mathbb{R}^n} f(Q + e \Delta r) - f(Q) \frac{(1 - \cos \beta)}{\Delta x_i} f(Q + e_i \Delta x_i).
\]

Thus, we get
\[
I = \left\| \frac{f(Q + e_i \Delta x_i) - f(Q)}{\Delta x_i} \right\|_{L_2} \leq C_{\omega} \left\| \frac{f(Q + e \Delta r) - f(Q)}{\Delta r} \right\|_{L_2} + C_{P} \frac{(1 - \cos \beta)}{\beta} \| r^{-1} f(Q + e_i \Delta x_i) \|_{L_2} = I_1 + I_2,
\]

where \( C_{P} \) is a some positive constant depended on choosing the point \( P \). Hence
\[
I \rightarrow \left\| \frac{\partial f}{\partial x_i} \right\|_{L_2}, \quad I_1 \rightarrow C_{\omega} \left\| A f \right\|_{L_2}, \quad I_2 \rightarrow 0, \quad \Delta x_i \rightarrow 0; \quad \Rightarrow \left\| \frac{\partial f}{\partial x_i} \right\|_{L_2} \leq C \left\| A_0 f \right\|_{L_2}, \quad i = 1, 2, ..., n
\]

and as a consequence \( \| f \|_{H_0^1(\Omega)} \leq C_0 \left\| A_0 f \right\|_{L_2(\Omega)} \), \( f \in C_0^\infty(\Omega) \). Using a simple relation \( |(\nabla f, e)_{\mathbb{R}^n}| \leq |\nabla f|, f \in H_0^1(\Omega) \) it is easy to see that \( \| A_0 f \|_{L_2(\Omega)} \leq C_1 \| f \|_{H_0^1(\Omega)}, f \in C_0^\infty(\Omega) \). From these facts follows that \( \mathcal{D}(A_0) = H_0^1(\Omega) \), and equivalence of the norms \( H_0^1(\Omega) \) and \( \mathcal{H}_{A_0} \).

Define the operator \( B f(Q) = \int_0^Q f(Q - et)dt, \mathcal{D}(B) = L_2(\Omega) \). It is clear that \( B \) is bounded and there exists \( A_0^{-1} \subset B \). Note that the last fact follows from properties of one dimensional integral defined on smooth functions. Let us prove that \( \bar{A}^{-1} \subset B \), in other word a so-called Newton-Leibniz formula takes place for functions from \( H_0^1(\Omega) \). Let \( f \in \mathcal{D}(\bar{A}_0) \), then there exists \( \{ f_n \} \subset C_0^\infty(\Omega), f_n \rightarrow f \). Due to continuity property, we have \( B A_0 f_n \xrightarrow{L^2} B \bar{A}_0 f, \text{ but } B A_0 f_n = f_n \).

Hence \( B \bar{A}_0 f = f \) almost everywhere. Thus the operator \( \bar{A}_0^{-1} \subset B \) is defined. By virtue of proved above equivalence of the norms \( H_0^1(\Omega), \mathcal{H}_{A_0} \) and well-known fact \( H_0^1(\Omega) \hookrightarrow \hookrightarrow L_2(\Omega) \), we can conclude that \( \bar{A}_0^{-1} \) is compact. Let us find a certain operator \( G \) defined in the abstract form in Theorem 4. It is obvious that the following expression is true
\[
\int_{\bar{\Omega}} A(B T f \cdot g) d\Omega = \int_{\bar{\Omega}} A B T f \cdot g d\Omega + \int_{\bar{\Omega}} B T f \cdot A g d\Omega, \quad f \in C^2(\bar{\Omega}), g \in C_0^\infty(\Omega).
\]
Using the well-known theorem on divergent, we get

\[
\int_{\Omega} A(BTf \cdot g)(Q)d\Omega = \int_{\mathcal{S}} (BTf \cdot g)(\sigma)(e,n)_{\mathbb{E}^n}d\sigma,
\]

but \(g(\partial\Omega) = 0\). Hence

\[
-\int_{\Omega} ABTf \cdot g d\Omega = \int_{\Omega} BTFdg d\Omega,\ f \in C^2(\bar{\Omega}), g \in C_0^\infty(\Omega).
\]  

(20)

Suppose that \(f \in H^2(\Omega)\), then the well-known fact that there exists a sequence \(\{f_n\} \subset C^2(\bar{\Omega})\) such that \(Tf_n \overset{L^2}{\to} Tf\) and as a consequence \(ABTf_n \overset{L^2}{\to} Tf\). On the other hand, since \(B\) is continuous, then \(BTf_n \overset{L^2}{\to} BTf\). Hence \(BTf_n \overset{L^2}{\to} BTf\). Since \(A\) is closed, then \(BTf \in D(A)\) and \(ABTf = Tf\). According to definition \(\forall g \in D(\tilde{A}_0), \exists \{g_n\} \subset C_0^\infty(\Omega), g_n \to g\). Using these facts, we can extend relation (20) to the set \(D(L)\) and rewrite one in the following form

\[
\int_{\Omega} A_0^* BTf \cdot g d\Omega = \int_{\Omega} BTFg d\Omega,\ f, g \in D(L),
\]  

(21)

By virtue of the facts \(A_0^* BTf = -Tf, B\tilde{A}_0f = f, f \in D(L)\), we have \(A_0^* G\tilde{A}_0f = -Tf, f \in D(L)\). Hence, we can rewrite (21) in the following form

\[
\int_{\Omega} G\tilde{A}_0f \tilde{A}_0g d\Omega = -\int_{\Omega} Tf \cdot g d\Omega,\ f, g \in D(L),
\]  

(22)

where \(G = BTB, D(G) := \{\psi : \psi = \tilde{A}_0\varphi, \varphi \in D(L)\}\). Estimating right part of (22), we come to the following inequality

\[
|(Gf, g)_{L^2}| \leq C \cdot \|\nabla Bf\|_{L^2} \|\nabla Bg\|_{L^2},\ f, g \in D(G).
\]

More detailed proof of the last estimate can be found in [33]. Due to equivalence of the norm proved above, finally we obtain boundedness of the form

\[
|(Gf, g)_{L^2}| \leq C\|f\|_{L^2} \|g\|_{L^2},\ f, g \in D(G).
\]

Due to relation (21) it is clear that \(G\) is strictly accretive (it follows from the uniformly elliptic property, see [33]). Also it is obvious that \(D(A_0^* G\tilde{A}_0) \supset C_0^\infty(\Omega)\) is dense in \(L_2(\Omega)\). Hence the operator \(G\) satisfied to the conditions of Theorem 4. Let the operator \(F\) be the operator of multiplication on the real function \(\rho\), then one is selfadjoint. It was proved for the one dimensional case that there exists an equality between the classes \(\rho \cdot \mathcal{J}_{0+}^\alpha(L_2) = \mathcal{J}_{0+}^\alpha(L_2), 0 < \alpha < 1/2\). The known proof of this fact can be successfully applied to \(n\)-dimensional case. Hence \(F^* : D(A^\alpha) \to D(A^\alpha)\). Using Remark 4 we can verify that the composition \(F A^\alpha\) is strictly accretive, also it is clear that \(F\) is bounded. Thus we have fulfillment of Theorem 4 conditions for the operator \(F\). We should note in the remainder an obvious inclusion \((A_0)^\alpha \subset \mathcal{D}^\alpha_{d-}\). Now, if we use a notation \(J := \tilde{A}_0\), then we obtain the following representation

\[ L = J^* GJ + FJ^\alpha. \]
Taking into considerations said above, we come to conclusion that the operator $L$ satisfied to all conditions of Theorem 4. It is very essentially that in the made reasonings we can consider unbounded convex domain $\Omega$ with some restriction related to a solid angle contained $\Omega$. Due to this way we come to generalization of the Kipriyanov operator for a case corresponding to unbounded domain.

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