Ratio vectors of fourth degree polynomials

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Abstract

Let \( p(x) \) be a polynomial of degree 4 with four distinct real roots \( r_1 < r_2 < r_3 < r_4 \). Let \( x_1 < x_2 < x_3 \) be the critical points of \( p \), and define the ratios \( \sigma_k = \frac{x_k - r_k}{r_{k+1} - r_k}, k = 1, 2, 3 \). For notational convenience, let \( \sigma_1 = u, \sigma_2 = v, \) and \( \sigma_3 = w \). \( (u, v, w) \) is called the ratio vector of \( p \). We prove necessary and sufficient conditions for \( (u, v, w) \) to be a ratio vector of a polynomial of degree 4 with all real roots. Most of the necessary conditions were proven in \( [3] \). The main results of this paper involve using the theory of Groebner bases to prove that those conditions are also sufficient.

Key Words: polynomial, real roots, Groebner basis
Introduction

Let \( p(x) \) be a polynomial of degree \( n \geq 2 \) with \( n \) distinct real roots \( r_1 < r_2 < \cdots < r_n \). Let \( x_1 < x_2 < \cdots < x_{n-1} \) be the critical points of \( p \), and define the ratios

\[
\sigma_k = \frac{x_k - r_k}{r_{k+1} - r_k}, \quad k = 1, 2, \ldots, n-1
\]

(\( \sigma_1, \ldots, \sigma_{n-1} \)) is called the ratio vector of \( p \), and \( \sigma_k \) is called the \( k \)th ratio. The following inequality was derived in \((\text{[1]})\) and cited in the author’s paper \((\text{[3]})\). I have since discovered that it was actually derived earlier by Peyser in \((\text{[4]})\).

\[
\frac{1}{n-k+1} < \sigma_k < \frac{k}{k+1}
\]

Solving \( \sigma_k = \frac{x_k - r_k}{r_{k+1} - r_k} \) for \( x_k \) yields

\[
x_k = (r_{k+1} - r_k)\sigma_k + r_k, \quad k = 1, 2, \ldots, n-1
\]

Andrews also defined the sets \( X_n = \prod_{k=1}^{n-1} \left( \frac{1}{n-k+1}, \frac{k}{k+1} \right) \) and \( Y_n \) as the set of elements in \( R^{n-1} \) which are the ratio vectors of polynomials with \( n \) distinct real zeroes. Our main results in this paper concern the case \( n = 4 \). For notational convenience, let \( \sigma_1 = u \), \( \sigma_2 = v \), and \( \sigma_3 = w \). Then \( (4) \) becomes

\[
X_4 = \left\{ (u, v, w) : \frac{1}{4} < u < \frac{1}{2}, \quad \frac{1}{3} < v < \frac{2}{3}, \quad \frac{1}{2} < w < \frac{3}{4} \right\}
\]
In (3) we showed that $Y_4$ lies on the zero set of a polynomial of degree 9, $Q(u, v, w) = (1 - 4v + 4uv)R(u, v, w)$, where $R(u, v, w)$ is defined below (17), which is a necessary condition for $(u, v, w)$ to be a ratio vector. The polynomial $Q$ in (3) was found by directly solving the system of equations (8)–(10) below. This is not difficult to do by solving for $r$ in terms of $s$ in the first two equations and then substituting into the third equation (see [3]). We show below that $1 - 4v + 4uv < 0$ for $(u, v, w) \in X_4$, so that $Y_4$ actually lies on the zero set, $Z$, of $R$. Thus a necessary condition for $(u, v, w)$ to be a ratio vector is $(u, v, w) \in Z \cap X_4$. One can also show that a ratio vector must satisfy certain more restrictive inequalities than those which define $X_4$ (see (11)–(13)). This turns out to be equivalent to the condition that $(u, v, w) \in Z \cap \{Z_1 \cup Z_2 \cup Z_3\}$. Our main result (see Theorem 4) is that $(u, v, w) \in Z \cap \{Z_1 \cup Z_2 \cup Z_3\}$ is also sufficient for $(u, v, w)$ to be a ratio vector. To obtain sufficient conditions, however, one needs more sophisticated methods, like the theory of Gröbner bases, which we use throughout the paper.

Preliminary Material

We shall need a system of algebraic equations which relate the real roots, $r_k$, and the ratios, $\sigma_k$. One can rewrite $p(x) = (x - r_1) \cdots (x - r_n)$ using the elementary symmetric functions $E_j = \frac{e_j}{(n)}$, $e_j \equiv e_j(r_1, ..., r_n) = j$th elementary symmetric function of the $r_j$, $j = 1, 2, ..., n$, starting with $e_1(r_1, ..., r_n) = r_1 + \cdots r_n$, etc.

$$p(x) = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} E_j x^{n-j} + x^n \quad (5)$$

Let $x_1 < x_2 < \cdots < x_{n-1}$ denote the critical points of $p$. By equating coeffi-
It is easy to show (see [5]) that

\[ E_j(r_1, \ldots, r_n) = E_j(x_1, \ldots, x_{n-1}), \quad j = 1, 2, \ldots, n - 1 \tag{6} \]

We shall also make use of

**Lemma 1** Suppose that (6) holds for distinct numbers \( r_1 < r_2 < \cdots < r_n \) and \( x_1 < x_2 < \cdots < x_{n-1} \). Let \( p(x) = (x - r_1)(x - r_2) \cdots (x - r_n) \). Then the \( x_j \) must be the critical points of \( p \).

**Proof.** Since \( \binom{n}{j}(n-j) = \frac{n!}{j!(n-j-1)!} \) = \( n\binom{n-1}{j-1} \), by [5] \( p'(x) = \sum_{j=1}^{n-1} (-1)^j \binom{n}{j} (n-j) E_j(r_1, \ldots, r_n)x^{n-j} + nx^{n-1} = n \sum_{j=1}^{n-1} (-1)^j \binom{n}{j} E_j(x_1, \ldots, x_{n-1})x^{n-j} + nx^{n-1} = n(x-x_1)(x-x_2) \cdots (x-x_{n-1}) \), which implies that the \( x_j \) must be the critical points of \( p \).

We shall now consider the case \( n = 4 \) for the rest of the paper. If \( p \) is a fourth degree polynomial with four distinct real zeroes, then \( p(x + c) \) and \( p(cx)(c \neq 0) \) have the same ratio vectors as \( p \). Thus we may assume that the zeroes of \( p \) are \( r_1 = -1 < r_2 = 0 < r_3 < r_4 \). Also, for notational convenience, let \( r_3 = r, r_4 = s \). Then (6) becomes

\[
\begin{align*}
4x_1x_2x_3 + rs &= 0 \\
4(x_1 + x_2 + x_3) - 3(-1 + r + s) &= 0 \\
2(x_1x_2 + x_1x_3 + x_2x_3) - (-r - s + rs) &= 0
\end{align*}
\]

(7)
Our system of algebraic equations will come from substituting for the $x_k$ in (7) using (3). Using $x_1 = u - 1, x_2 = rv, x_3 = (s - r)w + r$, we have, after some cancellation, the equivalent system of equations

\[(4(1 - u)vw - 1) s + 4(1 - u)v(1 - w)r = 0 \quad (8)\]

\[(-1 + 4u) + (1 + 4v - 4w) r + (4w - 3) s = 0 \quad (9)\]

\[(2u(v - w + 1) - 2v + 2w - 1) r + (2uw - 2w + 1)s - 2v(w - 1)r^2 - (1 - 2vw)rs = 0 \quad (10)\]

**Main Results**

First we state some inequalities for ratio vectors of fourth degree polynomials which are of interest in their own right, and which will also be used to prove our main result below.

**Lemma 2** Suppose that $(u, v, w)$ is a ratio vector of a fourth degree polynomial with four distinct real zeroes. Then

\[\frac{1}{4(1 - u)} < v < \frac{1}{4(1 - u)w} \quad (11)\]

\[\frac{1}{4(1 - v)} < w < \frac{1}{4(1 - u)(1 - v)} \quad (12)\]
and

\[ w < \frac{1}{2(1-u)} \]  

(13)

Proof. As noted above, we may assume that the zeros of \( p \) are 
\(-1 < 0 < r < s\).

Since \( r < s \) and \((u, v, w) \in X_4\), by (8), 
\[ 0 = (4(1-u)vw - 1)s + 4(1-u)v(1-w) \]
\[ r < ((4(1-u)vw - 1)s + 4(1-u)v(1-w))s = (-1 + 4v - 4vu)s. \]
Thus

\[ -1 + 4v - 4vu > 0 \]  

(14), which is equivalent to \( v > \frac{1}{4(1-u)} \) since \((u, v, w) \in X_4 \Rightarrow 1 - u > 0\).

Again, by (8),
\[ \frac{s}{r} = \frac{4(1-u)v(1-w)}{-4(1-u)vw + 1} = \frac{4(1-u)v(1-w)}{-4vw + 4vwu + 1}. \]
Since \( \frac{s}{r} > 0 \) and \( 4(1-u)v(1-w) > 0 \)
\[ -4vw + 4vwu + 1 > 0 \]  

(15), which is equivalent to \( v < \frac{1}{4(1-u)w} \). That proves (11). (12) follows in a
similar fashion by assuming that \( r_1 < r_2 < r_3 = 0 < r_4 \), which yields a system
analogous to (8)–(10). To prove (13), note that the upper bounds in (11) and
(12) imply that \( 4(1-u)vw - 1 < 0 \) and \( 4(1-u)(1-v)w - 1 < 0 \). Thus
\[ \frac{1}{2} (4(1-u)vw - 1 + 4(1-u)(1-v)w - 1) < 0 \]
\[ -1 + 2w - 2wu < 0 \]  

(16), which is equivalent to \( w < \frac{1}{2(1-u)} \). 

Remark 3 We note here that the lower bounds in (11) and in (12) can be used
to prove the monotonicity of the ratios. That is, to prove that \( u < v \) (see (8)).
and that $v < w$.

Define the polynomial in three variables

$$R(u, v, w) = 32u^2v^3w^2 - 48u^2v^2w^2 - 64uv^3w^2 + 24u^2v^2w + 16u^2vw^2 + 120uv^2w^2$$

$$+ 32v^3w^2 - 16u^2vw - 48uw^2w - 64uw^2w^2 - 72v^2w^2 + 52vw + 8uw^2$$

$$+ 24v^2w + 48vw^2 - 6uw - 10uw - 30vw - 8w^2 + 2u + 4v + 6w - 1$$

(17)

along with its zero set

$$Z = \{(u, v, w) : R(u, v, w) = 0\}$$

Also define the following subsets of $X_4$:

$$Z_1 = \{(u, v, w) : \frac{1}{4} < u \leq \frac{1}{3}, \frac{1}{4(1-u)} < v < \frac{1}{2}, \frac{1}{2} < w < \frac{1}{4(1-u)(1-v)}\}$$

$$Z_2 = \{(u, v, w) : \frac{1}{4} < u \leq \frac{1}{3}, \frac{1}{2} < w < \frac{1}{2(1-u)}, \frac{1}{2} \leq v < \frac{1}{4(1-u)w}\} \cap \{(u, v, w) : v < \frac{2}{3}\}$$

$$Z_3 = \{(u, v, w) : \frac{1}{3} < u < \frac{1}{2}, \frac{1}{2} < w < \frac{3}{4}, \frac{1}{4(1-u)} < v < \frac{1}{4(1-u)w}\} \cap \{(u, v, w) : v < \frac{2}{3}\}$$

We now give necessary and sufficient conditions for $(u, v, w)$ to be a ratio vector.
Theorem 4 \( (u, v, w) \) is a ratio vector of a fourth degree polynomial with four distinct real zeroes if and only if \( (u, v, w) \in \mathbb{Z} \cap \{ Z_1 \cup Z_2 \cup Z_3 \} \).

Thus the set of ratio vectors of fourth degree polynomials is precisely equal to \( \mathbb{Z} \cap \{ Z_1 \cup Z_2 \cup Z_3 \} \).

Theorem 4 will follow directly from the following two propositions. Most of Proposition 5 was proved in [3].

**Proposition 5** Suppose that \( (u, v, w) \) is a ratio vector of a fourth degree polynomial with four real distinct zeros. Then \( (u, v, w) \in \mathbb{Z} \cap \{ Z_1 \cup Z_2 \cup Z_3 \} \).

**Proof.** As noted earlier, we showed in [3, Theorem 2] that \( Y_4 \) is contained in the zero set of the polynomial \( Q(u, v, w) = (1 - 4v + 4uv)R(u, v, w) \). By (14), \( R(u, v, w) = 0 \), and thus \( (u, v, w) \in \mathbb{Z} \). \( R(u, v, w) = 0 \) will also follow from the Gröbner basis we use below in the proof of Proposition 9. \( (u, v, w) \in \mathbb{Z} \cap \{ Z_1 \cup Z_2 \cup Z_3 \} \) follows immediately from (4) and Lemma 2.

**Example:** Let \( p(x) = 128x^4 - 752x^3 + 1636x^2 - 1558x + 546 \), so that the roots are \( r_1 = 1, \ r_2 = \frac{3}{2}, \ r_3 = \frac{13}{8}, \ r_4 = \frac{7}{4} \), and the critical points are \( x_1 \approx 1.1506, \ x_2 \approx 1.5560, \ x_3 \approx 1.6996 \). The ratios are \( u = \frac{x_1 - r_1}{r_2 - r_1} \approx 0.3013, \ v = \frac{x_2 - r_2}{r_3 - r_2} \approx 0.4481, \ w = \frac{x_3 - r_3}{r_4 - r_3} \approx 0.5968 \). Then \( u < \frac{1}{3}, \ \frac{1}{4(1 - u)} \approx 0.3578 < v < \frac{1}{2} \) and \( \frac{1}{2} < w < \frac{1}{4(1 - u)(1 - v)} \approx 0.6483 \), so that \( (u, v, w) \in Z_1 \).

We prove a simple lemma about \( Z_1 \cup Z_2 \cup Z_3 \).

**Lemma 6** If \( (u, v, w) \in \{ Z_1 \cup Z_2 \cup Z_3 \} \), then \( (u, v, w) \in X_4 \) and (14) holds, or equivalently, (14) and (15).
Proof. Case 1: \((u, v, w) \in Z_1\). Then \(v > \frac{1}{4(1 - u)} > \frac{1}{3}\) and \(w < \frac{1}{4(1 - u)(1 - v)} < \frac{3}{4}\), which implies that \((u, v, w) \in X_4\). Also, \(\frac{1}{4(1 - u)w} > 1 - v > v\), so that (11) holds.

Case 2: \((u, v, w) \in Z_2\). Then \(w < \frac{1}{2(1 - u)} < \frac{3}{4}\), which implies that \((u, v, w) \in X_4\). Also, \(\frac{1}{4(1 - u)} < \frac{3}{8} < v\), so that (11) holds.

Case 3: \((u, v, w) \in Z_3\). Then \(v > \frac{1}{4(1 - u)} > \frac{3}{8} > \frac{1}{3}\) which implies that \((u, v, w) \in X_4\). It is obvious that (11) holds. ■

Before proving the sufficiency part of Theorem 4 in the form of Proposition 9 below, we define the following polynomial, which will be important in our proof.

\[
k(u, v, w) = 8u^2v^2w - 16uv^2w + 8uvw + 8v^2w - 2uv - 2uw - 8vw + 2u + 2v + 2w - 1
\]

(18)

We now prove the following key lemma about \(k\).

Lemma 7 If \((u, v, w) \in Z \cap \{Z_1 \cup Z_2 \cup Z_3\}\), then \(k(u, v, w) > 0\).

Proof. Write \(k(u, v, w) = -2(1 - u)(4v^2u - 4v^2 + 4v - 1)w - 2uv + 2u + 2v - 1\), and let \(f(v) = \frac{4v^2 - 4v + 1}{4v^2}\). Then, for \(\frac{1}{3} < v < \frac{2}{3}\), \(f''(v) > 0\). Since \(f\left(\frac{1}{3}\right) = \frac{1}{4}\) and \(f\left(\frac{2}{3}\right) = \frac{1}{16}\), \(f(v) \leq \frac{1}{4}\) for \(\frac{1}{3} < v < \frac{2}{3}\). Thus \(u > \frac{1}{4} \geq \frac{1 + 4v^2 - 4v}{4v^2}\), which implies that \(4v^2u - 4v^2 + 4v - 1 > 0\). That proves that \(k(u, v, w)\) is a decreasing function of \(w\) for any \(\frac{1}{4} < u < \frac{1}{2} \), \(\frac{1}{3} < v < \frac{2}{3}\). For \((u, v, w) \in Z_1\), the largest value of \(w\) is \(\frac{1}{4(1 - u)(1 - v)}\). Now \(k\left(u, v, \frac{1}{4(1 - u)(1 - v)}\right) = \frac{1}{2}(1 - 2v)\frac{4u - 1}{1 - v} > 0\) since \(v < \frac{1}{2}\). Thus \(k(u, v, w) > 0\) on all of \(Z_1\). For
(u, v, w) ∈ Z_2, the largest value of w is \(\frac{1}{2(1-u)}\), so we consider \(k \left( u, v, \frac{1}{2(1-u)} \right) = 2(2v-1)(v-u-w)\). If \(u \leq \frac{1}{3} \Rightarrow \frac{u}{1-u} \leq \frac{1}{2} \leq v \Rightarrow v-u-w \geq 0 \Rightarrow k \left( u, v, \frac{1}{2(1-u)} \right) \geq 0\). Again, it follows that \(k(u, v, w) \geq 0\) on all of \(Z_2\). Finally, suppose that \((u, v, w) \in Z_3\). Then \(2k \left( u, v, \frac{3}{4} \right) = H(u, v)\), where \(H(u, v) = 12u^2v^2 - 24uv^2 + 8uv + 12v^2 + u - 8v + 1\). Now \(H(1/3, v) = \frac{4}{3} (2v-1)^2 \geq 0\), \(H(1/2, v) = 3v^2 + \frac{3}{2} - 4v > 0\), \(H(u, 1/3) = \frac{1}{3} (u + 1) (4u - 1) > 0\), and \(H(u, 2/3) = \frac{16}{3} u^2 - \frac{13}{3} u + 1 > 0\). In addition, setting \(\frac{\partial H(u, v)}{\partial u} = \frac{\partial H(u, v)}{\partial v} = 0\) yields only one solution, \(u = 1, v = -\frac{1}{8}\). Thus \(H(u, v) \geq 0\) for \(\frac{1}{3} < u < \frac{1}{2} \frac{1}{3} < v < \frac{2}{3}\), which implies that \(k(u, v, w) \geq 0\) on all of \(Z_3\). Thus \(k(u, v, w) \geq 0\) on \(Z_1 \cup Z_2 \cup Z_3\). Now let \(d(u, v, w) = -16 uv^2 w + 24 uv w + 16 v^2 w - 12 uv - 24 vw + 8v + 4w - 1\), and note the following identity relating \(d, k,\) and \(R:\) \(d(u, v, w)k(u, v, w) = 2v(4u - 1)(1 - w)(1 - 2u) (-4vw + 4vwu + 1) + (-1 + 4v - 4vw) R(u, v, w)\)

Then if \(R(u, v, w) = 0\), \(k(u, v, w) \neq 0\) since \((u, v, w) \in Z_1 \cup Z_2 \cup Z_3\). It follows immediately that \(k(u, v, w) > 0\) if \((u, v, w) \in Z \cap \{Z_1 \cup Z_2 \cup Z_3\}\).

**Remark 8** With a little more effort, one could actually prove that \(k(u, v, w) > 0\) if \((u, v, w) \in Z_1 \cup Z_2 \cup Z_3\), but we did not require that result. Note that \(k(u, v, w)\) is **not** nonnegative on all of \(X_4\). For example, \(k(9/32, 4/9, 7/10) = -\frac{127}{12960}\).

We will now prove the sufficiency part of Theorem 4 in the form of the following proposition.

**Proposition 9** Suppose that \((u, v, w) \in Z \cap \{Z_1 \cup Z_2 \cup Z_3\}\). Then there are unique real numbers \(0 < r < s\) such that the polynomial \(p(x) = (x + 1)x(x - r)(x - s)\) has \((u, v, w)\) as a ratio vector. Furthermore, \(r = \frac{k(u, v, w)}{2v(1 - 2u)(1 - w)}\)
and $s = \frac{2(1-u)vk(u,v,w)}{(-4v^2 + 4vw + 1) v (1-2u)}$.

**Proof.** Let $C[t_1, ..., t_n]$ denote the polynomials in $t_1, ..., t_n$ with complex coefficients, and for any ideal $I \subseteq C[t_1, ..., t_n]$, let $V(I) = \{(t_1, ..., t_n) : f(t_1, ..., t_n) = 0 \forall f \in I\}$.

Our approach is to obtain as much information as one can by viewing $s, r, u, v, w$ as independent variables in (8)–(10), even though in reality they are not by (3).

Let $f, g, h$ denote the LHS of equations (8)–(10), $I = \langle f, g, h \rangle$ = the ideal generated by $f, g, h$ in $C[s, r, u, v, w]$. Let $I_1$ be the first elimination ideal, $I \cap C[r, u, v, w]$, and let $I_2$ equal the second elimination ideal, $I \cap C[u, v, w]$.

We found a Gröbner basis for $I$, denoted by $LEX$, using Maple 7 with the **lexographic ordering** $s > r > w > v > u$. $LEX$ contains 10 elements, which we denote by $LEX_1, ..., LEX_{10}$. Since $I = \langle LEX_1, ..., LEX_{10} \rangle$, it follows that equations (8)–(10) and the system of equations $LEX_1 = 0, ..., LEX_{10} = 0$ have exactly the same set of solutions. $LEX_1$ is the only element of $LEX$ which only depends on $u, v, w$. Hence, by the Elimination Theorem (see (2, Theorem 2, page 114), $LEX_1$ is a Gröbner basis for the second elimination ideal, $I_2$. Since $R(u, v, w) = 0$ by assumption and $LEX_1$ is a multiple of $R$, $(u, v, w)$ is a partial solution of (8)–(10) in $V(I_2)$. Let $S = \{LEX_1, ..., LEX_6\}$, none of which involve $s$. Then by the Elimination Theorem again, $S$ is a Gröbner basis for the first elimination ideal, $I_1$. In particular, $LEX_2 = v(4v-3)(2u-1)(4u-1)(1-4v+4uv)r + c(u, v, w)$ for some polynomial $c(u, v, w)$.

Now $v(4v-3)(2u-1)(4u-1)(1-4v+4uv) \neq 0$ if $(u, v, w) \in Z_1 \cup Z_2 \cup Z_3$.

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1Different computer algebra systems may list the elements of a minimal reduced Grobner basis in a different order, but they must contain the same polynomials.
by Lemma 6. Thus all of the coefficients of the highest powers of $r$ in each element of $S$ cannot vanish. By the Extension Theorem (see (2), Theorem 3, page 117), for each $(u, v, w) \in Z \cap \{Z_1 \cup Z_2 \cup Z_3\}$, there is a complex number $r$ such that $(r, u, v, w) \in V(I_1^2)$. Now $LEX_8 = (4w - 3)s + r + 4rv - 4rw + 4u - 1$.

Since $4w - 3 \neq 0$, by the Extension Theorem again, there is a complex number $s$ such that $(s, r, u, v, w) \in V(I)$. That is, $(s, r, u, v, w)$ is a solution of (8)–(10). The next thing we need to show is that $0 < r < s$, and that $r$ and $s$ are unique for each given $(u, v, w)$. We now find it more convenient to use a Gröbner basis for $I$ using the total degree ordering, which we denote by $TDEG$. $TDEG$ has 7 elements and again, equations (8)–(10) and the system of equations $TDEG_1 = 0, ..., TDEG_7 = 0$ have exactly the same set of solutions. In particular, $TDEG_7 = (4u - 1)A(r, v, u, w)$, where $A(r, v, u, w) = -2v(2u - 1)(w - 1)r + 8u^2v^2w - 16uvw^2w + 8uvw + 8v^2w - 2uw - 2uw - 8vw + 2u + 2v + 2w - 1 = -2v(2u - 1)(w - 1)r + k(u, v, w)$. $TDEG_7 = 0 \Rightarrow -2v(2u - 1)(w - 1)r + k(u, v, w) = 0 \Rightarrow r = \frac{k(u, v, w)}{2v(1 - 2u)(1 - w)}$ (19)

This shows that $r$ is unique and positive, by Lemma 7 and the fact that $2v(2u - 1)(w - 1) > 0$ if $(u, v, w) \in Z_1 \cup Z_2 \cup Z_3 \subset X_4$. As noted in the proof of Lemma 2, $s \frac{4(1 - u)v(1 - w)}{-4vw + 4uvw + 1}$. Now $4(1 - u)v(1 - w) - (-4vw + 4uvw + 1) = 4v - 4uw - 1 > 0$ by (15), which implies that $\frac{4(1 - u)v(1 - w)}{-4vw + 4uvw + 1} > 1$ by (15).
and thus $s > r$. Using (19) yields $s = \frac{2(1 - u)vk(u, v, w)}{(-4vw + 4vwu + 1) v (1 - 2u)}$, which also shows that $s$ is unique. Thus we have a solution $(r, s, u, v, w)$ of (8)–(10) with $0 < r < s$. Let $x_1 = u - 1, x_2 = rv$, and $x_3 = (s - r)w + r$. Then must hold since (8)–(10) and (7) are an equivalent system of equations. If $p(x) = (x + 1)x(x - r)(x - s)$, then $x_1, x_2$, and $x_3$ must be the critical points of $p$ by Lemma 1. Since $u = \frac{x_1 - (-1)}{0 - (-1)}, v = \frac{x_2 - 0}{r - 0}$, and $w = \frac{x_3 - r}{s - r}$, $(u, v, w)$ is a ratio vector of $p$.

**Remark 10** Equivalent necessary and sufficient conditions for $(u, v, w)$ to be a ratio vector is $(u, v, w) \in Z \cap X_4 \cap \left\{(u, v, w) : \frac{1}{4(1 - u)} < v < \frac{1}{4(1 - u)w}\right\} \cap \{(u, v, w) : k(u, v, w) > 0\}$. We preferred to use the sets $Z_1, Z_2, Z_3$ instead since the set where $k(u, v, w) > 0$ is not easy to determine. However, given $(u, v, w) \in Z$, one can easily check if $k(u, v, w) > 0$.

**Remark 11** One can also write the solution for $s$ as $s = \frac{4v(4u - 1)(1 - u)(1 - w)}{d(u, v, w)}$, where $d$ is the polynomial defined in the proof of Lemma 7.

**Example:** Let $u = \frac{15}{32} \approx .4688, v = \frac{5}{9} \approx 0.5556$, and $w = \frac{156303 - 9\sqrt{10054801}}{211888} \approx .6030$. Then $\frac{1}{4(1 - u)} \approx .4706 < v < \frac{1}{4(1 - u)w} \approx .7804$, and by Proposition (u, v, w) is a ratio vector of $p(x) = (x + 1)x(x - r)(x - s)$, where $r \approx 5.9821$ and $s \approx 9.7305$.

**Further Discussion**

Let $S$ be the surface $R(u, v, w) = 0$. Then $S$ contains the family of lines $u = C, v = \frac{1}{2}, w = 1 - C, \frac{1}{4(1 - u)} < v = \frac{1}{2} \iff C < \frac{1}{2}$, \[
\frac{1}{4(1 - u)w} = \frac{1}{4(1 - C)^2} > v = \frac{1}{2} \iff 1 - \frac{1}{2}\sqrt{2} < C < 1 \text{ or } 1 < C < 1 + \frac{1}{2}\sqrt{2}.
\]
and \( k(u, v, w) = C (-1 + 4C - 2C^2) > 0 \iff 0 < C < 1 + \frac{1}{2} \sqrt{2} \) or \( C < 0 \).

By the remark above, \( \left( C, \frac{1}{2}, 1 - C \right) \) is a ratio vector if and only if \( 1 - \frac{1}{2} \sqrt{2} < C < \frac{1}{2} \). However, we can prove more. Since \( R(u, 1/2, w) = (1 - w - u) (2wu - 2w + 1) \)
and \( 2uw - 2w + 1 > 0 \) by (16), \( \left( u, \frac{1}{2}, w \right) \) is a ratio vector only if \( u + w = 1 \).

Thus we have proven

**Theorem 12**  \( \left( u, \frac{1}{2}, w \right) \) is a ratio vector if and only if \( u = C \) and \( w = 1 - C \),
where \( 1 - \frac{1}{2} \sqrt{2} < C < \frac{1}{2} \).

Note that though \( S \) contains a family of lines, \( S \) is not a ruled surface in general. That is easy to see by looking at the second partials \( \frac{\partial^2 R}{\partial u^2}, \frac{\partial^2 R}{\partial v^2}, \frac{\partial^2 R}{\partial w^2} \).

**Correction:** We make a minor correction to equation (11) in (3). It should read

\[
\left( \sigma_1 (\sigma_2 - \sigma_3 + 1) - \sigma_2 + \sigma_3 - \frac{1}{2} \right) r_1 r_3 + \left( \sigma_1 \sigma_3 - \sigma_3 + \frac{1}{2} \right) r_1 r_4 + \sigma_2 (\sigma_3 - 1) r_3^2 + \\
\left( \frac{1}{2} - \sigma_2 \sigma_3 \right) r_3 r_4 = 0
\]

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