The Aharonov-Bohm Effect and Tonomura et al. Experiments. Rigorous Results

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Abstract

The Aharonov-Bohm effect is a fundamental issue in physics. It describes the physically important electromagnetic quantities in quantum mechanics. Its experimental verification constitutes a test of the theory of quantum mechanics itself. The remarkable experiments of Tonomura et al. [“Observation of Aharonov-Bohm effect by electron holography,” Phys. Rev. Lett. 48, 1443 (1982), “Evidence for Aharonov-Bohm effect with magnetic field completely shielded from electron wave”, Phys. Rev. Lett. 56, 792 (1986)] are widely considered as the only experimental evidence of the physical existence of the Aharonov-Bohm effect. Here we give the first rigorous proof that the classical Ansatz of Aharonov and Bohm of 1959 [“Significance of electromagnetic potentials in the quantum theory,” Phys. Rev. 115, 485 (1959)], that was tested by Tonomura et al., is a good approximation to the exact solution to the Schrödinger equation. This also proves that the electron, that is represented by the exact solution, is not accelerated, in agreement with the recent experiment of Caprez et al. in 2007 [“Macroscopic test of the Aharonov-Bohm effect,” Phys. Rev. Lett. 99, 210401 (2007)], that shows that the results of the Tonomura et al. experiments can not be explained by the action of a force. Under the assumption that the incoming free electron is a gaussian wave packet, we estimate the exact solution to the Schrödinger equation for all times. We provide a rigorous, quantitative error bound for the difference in norm between the exact solution and the Aharonov-Bohm Ansatz. Our bound is uniform in time. We also prove that on the gaussian asymptotic state the scattering operator is given by a constant phase shift, up to a quantitative error bound that we provide. Our results show that for intermediate size electron wave packets, smaller than the ones used in the Tonomura et al. experiments, quantum mechanics predicts the results observed by Tonomura et al. with an error bound smaller than $10^{-99}$. It would be quite interesting to perform experiments with electron wave packets of intermediate size. Furthermore, we provide a physical interpretation of our error bound.

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*PACS Classification (2008): 03.65Nk, 03.65.Ca, 03.65.Db, 03.65.Ta. Mathematics Subject Classification(2000): 81U40, 35P25, 35Q40, 35R30.
†Research partially supported by CONACYT under Project P42553F.
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1 Introduction

In classical electrodynamics the force produced by a magnetic field on a charged particle is given by the Lorentz force, \( F = qv \times B \), where \( q \) and \( v \) are, respectively, the charge and the velocity of the particle, and \( B \) is the magnetic field. In regions where the magnetic field is zero the Lorentz force is zero and the particle travels in a straight line. In particular, the dynamics of a classical particle is unaffected by magnetic fields enclosed in regions that are not accessible to the particle. This also means that in classical electrodynamics the relevant physical quantity is the magnetic field and that the magnetic potentials are only a convenient mathematical tool.

The situation is different in quantum mechanics, where the dynamics is described by the Schrödinger equation that cannot be formulated directly in terms of the magnetic field. It is required to introduce the magnetic potential. It was pointed out by Aharonov and Bohm [2] that this implies that in quantum mechanics the magnetic potentials have a real physical significance. Aharonov and Bohm [2] proposed an experiment to confirm the theoretical prediction. They suggested to use a thin, straight solenoid, centered at the origin and with axis in the vertical direction. They supposed that the magnetic field was essentially confined to the solenoid. They advised to employ a coherent electron wave packet that splits in two parts, each one going through one side of the solenoid. Both wave packets should be brought together behind the solenoid, to create an interference pattern due to the difference in phase in the wave function of each part of the wave packet, produced by the magnetic field enclosed inside the solenoid. Actually, the existence of this interference pattern was first predicted by Franz [9]. The Aharonov-Bohm effect plays a prominent role in fundamental physics, among other reasons, because it describes the physically important electromagnetic quantities in quantum mechanics, and since it is a quantum mechanical effect, the verification of its existence constitutes a test of the validity of the theory of quantum mechanics itself.

The case of a solenoid has been extensively studied from the theoretical and experimental points of view. The theoretical analysis is reduced to a two dimensional problem after making the assumption that the solenoid is infinite. Nevertheless, experimentally it is impossible to have an infinite solenoid and, therefore, the magnetic field can not be completely confined into the solenoid. The leakage of the magnetic field was a highly controversial point. To avoid this problem it was suggested to use a toroidal magnet, that can contain a magnetic field inside without a leak. The experiments with toroidal magnets where carried over by Tonomura et al. [17, 25, 26]. In remarkable experiments they were able to superimpose behind the magnet an electron wave packet that traveled inside the hole of the magnet with another electron wave packet that traveled outside the magnet, and they measured the phase shift produced by the magnetic flux enclosed in the magnet, giving a strong evidence of the existence of the Aharonov-Bohm effect. In fact, the Tonomura et al. experiments [17, 25, 26] are widely considered as the only experimental evidence of the existence of the Aharonov-Bohm effect.

In the case of toroidal magnets, several Ansätze have been provided for the solution to the Schrödinger equation and for the scattering matrix without giving error bound estimates for the difference, respectively, between the exact
solution and the exact scattering matrix, and the Ansätze. Most of these works are qualitative, although some of them give numerical values for their Ansätze. Methods like, Fraunhofer diffraction, first-order Born and high-energy approximations, Feynman path integrals and the Kirchhoff method in optics were used to propose the Ansätze. The amount of work related to the Aharonov-Bohm effect is very large. For a review of the literature up to 1989 see [15] and [18]. In particular, in [18] there is a detailed discussion of the large controversy -involving over three hundred papers- concerning the existence of the Aharonov-Bohm effect. For a recent update of this controversy see [23, 27].

The paper [4] presents a discussion of a version of the Aharonov-Bohm Ansatz for an infinite solenoid. For recent rigorous work in the case of an infinite solenoid see [14, 28] where, among other results, it is proven that in the high-velocity limit the scattering operator is given by a constant phase shift, as predicted by Franz [9] and Aharonov and Bohm [2]. In [16] rigorous mathematical ground is given for the presence of the magnetic potential in the Schrödinger operator describing the Aharonov-Bohm effect in the case of a solenoid. In [11], a semi-classical analysis of the Aharonov-Bohm effect in bound-states in two dimensions is given. For a rigorous mathematical analysis of the Aharonov-Bohm effect in three dimensions for toroidal magnets -actually in the general case of handle bodies- see [3], where the high-velocity limit of the scattering operator was evaluated in the case where the direction of the velocity is kept fixed as its absolute value goes to infinity. A rigorous error bound was given for the difference between the scattering operator and its high-velocity limit for incoming asymptotic states that have small interaction with the magnet in the high-velocity limit. The error bound goes to zero as the inverse of the velocity. A detailed analysis of the Aharonov-Bohm effect in the case of the Tonomura et al. experiments [17, 25, 26] was given in [3], as well as other results. The results of [3] give a rigorous qualitative proof that quantum mechanics predicts the interference patterns observed in the Tonomura et al. experiments [25, 26, 17] with toroidal magnets. The papers [3, 14, 28], as well as this paper, use the method introduced in [8] to estimate the high-velocity limit of solutions to Schrödinger equations and of the scattering operator. The papers [21, 22, 29], and [30] study the scattering matrix for potentials of Aharonov-Bohm type in the whole space.

In this paper we give the first rigorous proof that the classical Ansatz of Aharonov and Bohm is a good approximation to the exact solution of the Schrödinger equation. We provide, for the first time, a rigorous quantitative mathematical analysis of the Aharonov-Bohm effect with toroidal magnets under the conditions of the experiments of Tonomura et al. [17, 25, 26]. We assume that the incoming free electron is a gaussian wave packet, what from the physical point of view is a reasonable assumption. The technical advantage of using a gaussian wave packet for the incoming free electrons is that in this case we know very well the dynamics of the free asymptotic gaussian state, and we can carry over the estimates of [3] in a precise manner. We provide a rigorous, simple, quantitative, error bound for the difference in norm between the exact solution and the approximate solution given by the Aharonov-Bohm Ansatz. Our error bound is uniform in time. We also prove that on the gaussian asymptotic state, the scattering operator is given by multiplication by $e^{i\frac{q}{\hbar}\Phi}$ -where $q$ is the charge of the electron, $c$ is the speed of light, $\hbar$ is Planck’s constant, and $\Phi$ is the magnetic flux in a transversal section of the magnet- up to a quantitative error bound, that we provide.
Actually, the error bound is the same in the cases of the exact solution and the scattering operator.

Aharonov and Bohm [2] and Tonomura et al. [17, 25, 26] suggested to split the electron wave packet into the part that goes through the hole of the magnet and the part that goes outside. Tonomura et al. observed that an image was produced behind the magnet that clearly showed that shadow of the magnet and also the hole and the exterior of the magnet. They concluded [25] that this indicates that there was not interference between the part of the electron wave packet that went through the hole and the one that either hit the magnet or traveled outside. The part of the wave packet that goes outside the magnet can be taken as the reference wave packet. Therefore, we only model the part of the electron wave packet that goes through the hole of the magnet. Using the experimental data of Tonomura et al. [17, 25, 26] we provide lower and upper bounds on the variance of the gaussian state in order that the electron wave packet actually goes through the hole. We also rigorously prove that the results of the Tonomura et al. experiments [17, 25, 26], that were predicted by Aharonov and Bohm, actually follow from quantum mechanics. Furthermore, our results show that it would be quite interesting to perform experiments for intermediate size electron wave packets (smaller than the ones used in the Tonomura et al. experiments, that where much larger than the magnet) that satisfies appropriate lower and upper bounds that we provide. One could as well take a larger magnet. In this case, the interaction of the electron wave packet with the magnet is negligible -the probability that the electron wave packet interacts with the magnet is smaller than $10^{-199}$ (See Remark [8.12 and Section [9.2]- and, moreover, quantum mechanics predicts the results observed by Tonomura et al. with an error bound smaller than $10^{-99}$, in norm.

Our error bound has a physical interpretation. For small variances, it is due to Heisenberg’s uncertainty principle. If the variance in configuration space is small, the variance in momentum space is big, and then, the component of the momentum transversal to the axis of the magnet is large. In consequence, the opening angle of the electron wave packet is large, and there is a large interaction with the magnet. If the variance is large, the opening angle is small, but as the electron wave packet is big we have again a large interaction with the magnet.

It has been claimed that the outcome of the Tonomura et al. experiments [17, 25, 26] can be explained by the action of a force acting on the electron that travels through the hole of the magnet. See, for example, [5, 10] and the references quoted there. Such a force would accelerate the electron and it would produce a time delay. In a recent crucial experiment Caprez et al. [6] found that the time delay is zero, thus experimentally excluding the explanation of the results of the Tonomura et al. experiments by the action of a force. In the Aharonov-Bohm Ansatz the electron is not accelerated, it propagates following the free evolution, with the wave function multiplied by a phase. Since, as mentioned above, we prove that the Aharonov-Bohm Ansatz approximates the exact solution with an error bound uniform in time that can be smaller that $10^{-99}$ in norm, we rigorously prove that quantum mechanics predicts that no force acts on the electron, in agreement with the experimental results of Caprez et al. [6].
1.1 Tonomura et al. Experiments

The remarkable experiments of Tonomura et al. [17, 25, 26] are widely considered as the only experimental evidence of the physical existence of the Aharonov-Bohm effect. Tonomura et al. constructed small toroidal magnets such that the magnetic field is practically zero outside them. In [26], the magnets are impenetrable and, furthermore, they are covered by super conductive layers that forbid the leakage of magnetic field outside the magnets. We denote by $\tilde{K} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < \tilde{r}_1 \leq (x_1^2 + x_2^2)^{1/2} \leq \tilde{r}_2, |x_3| \leq 2\tilde{h}\}$ the magnet ($\tilde{r}_1$ is the inner radius, $\tilde{r}_2$ the outer radius and $2\tilde{h}$ is the height), and by $\tilde{B}(x)$ the magnetic field. We suppose that $\tilde{B}(x)$ is zero for $x$ outside the magnet.

An electron wave packet was sent towards the magnet. It was superimposed behind it with a reference electron wave packet to produce the interference pattern. The experiments were set up in such a way that the reference electron wave packet was not influenced by the magnet, and that the electron wave packet and the reference electron wave packet only interfered behind the magnet, were the interference patterns were formed. The observed interference patterns provided a strong evidence of the physical existence of the Aharonov-Bohm effect.

The electron wave packet was much larger than the magnet. It was 3 micrometers in size in the direction of the electron propagation and 20 micrometers in size in a plane perpendicular to the propagation direction [24]. It covered the magnet completely. Recall that it was observed that an image was produced behind the magnet that clearly showed the shadow of the magnet and also the hole and the exterior of the magnet (see [25, 26]) and that it was pointed out by Tonomura et al. [25, 26], that this indicates that there was no interference between the part of the electron wave packet that went through the hole, and the one that either hit the magnet or traveled outside, because of the clear image of the shadow of the magnet [25, 26]. As mentioned before, we will concentrate our analysis on the part of the wave packet that goes through the hole, and we will take it as the electron wave packet itself. It is either the part of the electron wave packet that goes through the hole, or a smaller electron wave packet that really goes through the hole.

1.2 Aharonov-Bohm Ansatz for the Exact Solution

At the time of emission, i.e., as $t \to -\infty$, the electron wave packet is far away from the magnet and it does not interact with it, therefore, it can be assumed that it follows the free evolution,

$$i\hbar \frac{\partial}{\partial t} \phi(x, t) = H_0 \phi(x, t), x \in \mathbb{R}^3, t \in \mathbb{R}.$$ (1.1)

where $H_0$ is the free Hamiltonian.

$$H_0 := \frac{1}{2M} P^2. \quad \text{(1.2)}$$

$M$ is the mass of the electron and $P := -i\hbar \nabla$ is the momentum operator. We represent the emitted electron wave packet by the free evolution of a gaussian wave function, $\varphi_v$, with velocity $v$,
\[ \varphi_v := e^{\frac{i}{\hbar} \mathbf{v} \cdot \mathbf{x}} \varphi, \text{ where } \varphi := \frac{1}{(\sigma^2 \pi)^{3/4}} e^{-\frac{x^2}{2\sigma^2}}, \]  

(1.3)

with variance \( \sigma \) smaller than the inner radius of the magnet. We have chosen the variance transverse to the velocity of propagation, \( v \), equal to the longitudinal variance in the direction of propagation. In fact, the size of the longitudinal variance is not essential for our arguments and we have chosen it equal to the transversal variance only for simplicity.

Notice that in the momentum representation, \( e^{\frac{i}{\hbar} \mathbf{v} \cdot \mathbf{x}} \) is a translation operator by the vector \( M \mathbf{v} \), what implies that the wave function (1.3) is centered at the classical momentum \( M \mathbf{v} \) in the momentum representation,

\[ \hat{\varphi}_v(p) = \hat{\varphi}(p - M \mathbf{v}), \]

where for any state represented by the wave function \( \phi(x) \) in the configuration representation, the momentum representation is given by the Fourier transform,

\[ \hat{\phi}(p) := \frac{1}{(2\pi\hbar)^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{i}{\hbar} p \cdot x} \phi(x) \, dx. \]

By the previous analysis, the electron wave packet is represented at the time of emission by the following gaussian wave packet that is a solution to the free Schrödinger equation (1.1)

\[ \psi_{v,0}(x,t) := e^{-i \frac{\mathcal{H}_0}{\hbar} t} \varphi_v(x). \]  

(1.4)

The (exact) electron wave packet, \( \psi_v(x,t) \), satisfies the interacting Schrödinger equation for all times,

\[ i\hbar \frac{\partial}{\partial t} \psi_v(x,t) = H \psi_v(x,t), \quad x \in \Lambda := \mathbb{R}^3 \setminus \bar{K}, \ t \in \mathbb{R}, \]  

(1.5)

where

\[ H := H(A) := \frac{1}{2M} (\mathbf{P} - \hbar A)^2 \]  

(1.6)

is the Hamiltonian and \( A = \frac{q}{\hbar c} \tilde{A} \), where \( c \) is the speed of light, \( q \) is the charge of the electron, \( \hbar \) is Plank’s constant, and \( \tilde{A} \) is a magnetic potential with curl \( \tilde{A} = \tilde{B} \) where \( \tilde{B} \) is the magnetic field. We define the Hamiltonian (1.6) in \( L^2(\Lambda) \) with Dirichlet boundary condition at \( \partial \Lambda \), i.e. \( \psi = 0 \) for \( x \in \partial \Lambda \). This is the standard boundary condition that corresponds to an impenetrable magnet. It implies that the probability that the electron is at the boundary of the magnet is zero. Note that the Dirichlet boundary condition is invariant under gauge transformations. In the case of the impenetrable magnet the existence of the Aharonov-Bohm effect is more striking, because in this situation there is zero interaction of the electron with the magnetic field inside the magnet. Note, however, that once a magnetic potential is chosen the particular self-adjoint boundary condition taken at \( \partial \Lambda \) does not play an essential role in our
calculations. Furthermore, our results hold also for a penetrable magnet where the interacting Schrödinger equation (1.5) is defined in all space. Actually, this later case is slightly simpler because we do not need to work with two Hilbert spaces, \( L^2(\mathbb{R}^3) \) for the free evolution, and \( L^2(\Lambda) \) for the interacting evolution, what simplifies the proofs. In consequence, the electron wave packet is the unique solution, \( \psi_v \), to the interacting Schrödinger equation (1.5) that is asymptotic to the free gaussian wave packet, \( \psi_{v,0} \), as \( t \to -\infty \),

\[
\psi_v(x,t) \approx \psi_{v,0}(x,t), \quad t \to -\infty.
\] (1.7)

Aharonov and Bohm [2] proposed an approximate solution to the Schrödinger equation over simply connected regions (regions with no holes) where the magnetic field is zero, by a change of gauge formula from the zero vector potential. Of course, it is not possible to have a gauge transformation from the zero potential everywhere because that would imply that the magnetic flux on a transversal section of the magnet would be zero. Hence, the gauge transformation has to be discontinuous somewhere. As mentioned in Section 1.1 in the case of Tonomura et al. [17, 25, 26] experiments the magnet is a cylindrical torus, \( \tilde{K} \).

We take as the surface of discontinuity of the gauge transformation

\[
\mathcal{S} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1^2 + x_2^2)^{1/2} > \tilde{r}_2, x_3 = 0\}
\]

and we define the gauge transformation in the domain, \( \mathcal{D} \), given by

\[
\mathcal{D} := \Lambda \setminus \mathcal{S}.
\]

Without loss of generality we can suppose that the support of \( A \) is contained on the convex hull of \( \tilde{K} \) (see Section 7.1). For every \( x \in \mathcal{D} \), and a fixed point \( x_0 \) in \( \mathcal{D} \) with vertical component less than \(-\tilde{h}\), we define the gauge transformation as follows,

\[
\lambda_{A,0}(x) := \int_{x_0}^{x} A,
\]

where the integral is over a path in \( \mathcal{D} \). Note that for any \( x \in \mathcal{D} \) with \( x_3 > 0 \) the integration contour has to go necessarily through the hole of the magnet.

For any solution to the Schrödinger equation (1.5), \( \phi(x,t) \), that stays in \( \mathcal{D} \), Aharonov and Bohm [2] propose that the solution is given by the following Ansatz, motivated by the change of gauge formula from the zero vector potential,

\[
\phi_{AB}(x,t) := e^{i\lambda_{A,0}(x)} e^{-i\frac{\pi}{2} H_0} e^{-i\lambda_{A,0}(x)} \phi(x,0).
\] (1.8)

Note that if the initial state at \( t = 0 \) is taken as \( e^{-i\lambda_{A,0}(x)} \phi(x,0) \) the Aharonov-Bohm Ansatz is the multiplication of the free solution by the Dirac magnetic factor \( e^{i\lambda_{A,0}(x)} \) [7].
The Aharonov-Bohm Ansatz is expected to be a good approximation to the exact solution if the electron wave packet stays in a connected domain, away from the surface \( S \) where the gauge transformation is discontinuous. This Aharonov-Bohm Ansatz is valid for solutions whose initial data is given at time equal to zero.

For the incoming electron wave packet that satisfies (1.7) the initial data is given as time tends to \(-\infty\) and then, the Aharonov-Bohm Ansatz has to be modified. To formulate the appropriate Ansatz we define the wave operators,

\[
W_\pm(A) := W_\pm := s- \lim_{t \to \pm \infty} e^{i \frac{\tau}{\hbar} H(A)} J e^{-i \frac{\tau}{\hbar} H_0},
\]

where \( J \) is the identification operator from \( L^2(\mathbb{R}^3) \) into \( L^2(\Lambda) \) given by multiplication by the characteristic function of \( \Lambda \), i.e., \( J\phi(x) := \chi_\Lambda(x) \phi(x) \) where, \( \chi_\Lambda(x) = 1, x \in \Lambda, \chi_\Lambda(x) = 0, x \in \mathbb{R}^3 \setminus \Lambda \). It is proved in [3] that the strong limits exist and that we can replace the operator \( J \) by the operator of multiplication by any smooth characteristic cutoff function \( \chi(x) \in C^\infty \) such that \( \chi(x) = 0, x \in \tilde{K} \) and \( \chi(x) = 1 \) for \( x \) in the complement of a bounded set that contains \( \tilde{K} \) on its interior.

The solution to the Schrödinger equation that is asymptotic to the free solution \( e^{-i \frac{\tau}{\hbar} H_0} \varphi_v \) as \( t \to -\infty \) is given by

\[
\psi_v := e^{-i \frac{\tau}{\hbar} H(A)} W_- \varphi_v. \tag{1.9}
\]

It satisfies,

\[
\lim_{t \to -\infty} \|\psi_v - J\psi_{v,0}\| = 0. \tag{1.10}
\]

Using this fact we prove in Section [7] that the Aharonov-Bohm Ansatz for the exact solution to the Schrödinger equation (1.5) with initial data as time tends to \(-\infty\) is given by,

\[
\psi_{AB,v}(x,t) = e^{i \lambda A,0(x)} e^{-i \frac{\tau}{\hbar} H_0} \varphi_v, \tag{1.11}
\]

what, again, is the multiplication of the free incoming solution by the Dirac magnetic factor \( e^{i \lambda A,0(x)} \) [7].

It is expected that if the electron wave packet stays in a connected region of space, away from the surface of discontinuity \( S \), the Aharonov-Bohm Ansatz should be a good approximation to the exact solution, i.e., that,

\[
\psi_v \approx \psi_{AB,v}. \tag{1.12}
\]

The Aharonov-Bohm Ansatz, \( \psi_{AB,v} \), is what is observed in the Tonomura et al. experiments [17, 25, 26]: as the support of the vector potential \( A \) is contained in the convex hull of \( \tilde{K} \), for every \( x \) whose vertical component is bigger than \( \tilde{h} \), \( \lambda A,0(x) \) is equal to the constant \( \frac{e}{\hbar} \Phi \), where \( \Phi \) is the flux of the magnetic field over a transverse section of the magnet. Then, for \( x_3 > \tilde{h} \), the Aharonov-Bohm Ansatz is given by
\[ \psi_{AB,v}(x) = e^{i\frac{q}{\hbar} \Phi} e^{-i\frac{t}{\hbar} H_0} \varphi_v, \quad x_3 > \hat{r}. \] (1.13)

This is exactly what it was observed in the Tonomura et al. experiments \[17, 25, 26\].

The scattering operator is defined as

\[ S := W^*_+ W_. \]

For large positive times, when the exact electron wave packet is far away from the magnet, and it is localized in the region with large positive \( x_3 \), it can be again approximated with an outgoing solution to the free Schrödinger equation,

\[ \psi_{+,v,0} := e^{-i\frac{t}{\hbar} H_0} \varphi_{+,v}, \] (1.14)

such that,

\[ \lim_{t \to \infty} \| \psi_v - J \psi_{+,v,0} \| = 0. \] (1.15)

The initial data of the incoming and the outgoing solutions to the free Schrödinger equation are related by the scattering operator (see Section 3.1),

\[ \varphi_{+,v} = S \varphi_v. \] (1.16)

By equations (1.10) and (1.12-1.16) the Aharonov-Bohm Ansatz suggests that

\[ \varphi_{+,v} = S \varphi_v \approx e^{i\frac{q}{\hbar} \Phi} \varphi_v, \] (1.17)

i.e., that on the gaussian asymptotic state, \( \varphi_v \), the scattering operator is given by multiplication by \( e^{i\frac{q}{\hbar} \Phi} \), to a good approximation. This also is precisely what was observed in the Tonomura et al. experiments \[17, 25, 26\]. Furthermore, in the Aharonov-Bohm Ansatz (1.11) the electron is not accelerated, it propagates along the free evolution, with the wave function multiplied by a phase. This implies that in the Aharonov-Bohm Ansatz no force acts on the electron, and hence, it is not accelerated. This is precisely what was observed in the Caprez et al. \[6\] experiments.

### 1.3 The Main Results

As under the free evolution the electron wave packet is concentrated along the classical trajectory, we can expect that if the velocity \( v \) -that is directed along the positive vertical axis- is large enough, the exact electron wave packet will keep away, for all times, from the surface, \( S \), where the gauge transformation is discontinuous. In consequence, the Aharonov-Bohm Ansatz should be a good approximation, and equations (1.12) and (1.17) should hold. In the following theorem (see also Theorem 8.10) we prove that this is true under the conditions of the Tonomura et al. experiments \[17, 25, 26\], provided that appropriate, quantitative, lower and upper bounds on the variance, \( \sigma \), of the
gaussian wave function are satisfied. The requirement for the variance $\sigma$ to lie within the interval below assures that interaction of the electron with the magnet and the surface $S$ is small.

**THEOREM 1.1.** Aharonov-Bohm Ansatz, Scattering Operator and Tonomura et al. Experiments

Suppose that the magnets and energies are the ones of the experiments of Tonomura et al.. Then, for every gaussian wave function, $\varphi$, with variance $\sigma \in [4.5 \text{ mv}, \frac{r_1}{2}]$ and every $t \in \mathbb{R}$, the solution to the Schrödinger equation, $e^{-i\frac{t}{\hbar}H(A)}W_-\varphi_v$, that behaves as $e^{-i\frac{t}{\hbar}H_0}\varphi_v$ as $t \to -\infty$ is given at the time $t$ by

$$\psi_{AB,v} := e^{i\lambda A,0(x)} e^{-i\frac{t}{\hbar}H_0}\varphi_v, \tag{1.18}$$

up to the following error,

$$\|e^{-i\frac{t}{\hbar}H(A)}\varphi_v - e^{i\lambda A,0}\varphi_v\| \leq$$

$$7e^{-\frac{r_1^2}{2\sigma^2}} + 177 \times 10^3 e^{-\frac{33}{34} \frac{(\sigma mv)^2}{2}} + 10^{-100}, \tag{1.19}$$

where, $m := M/\hbar$. Furthermore, the scattering operator satisfies

$$\|S\varphi_v - e^{i\frac{\alpha}{\pi}}\varphi_v\| \leq$$

$$7e^{-\frac{r_1^2}{2\sigma^2}} + 177 \times 10^3 e^{-\frac{33}{34} \frac{(\sigma mv)^2}{2}} + 10^{-100}. \tag{1.20}$$

The main factors that produce the error bound in equation (1.19, 1.20) are the terms,

- Size of the electron wave packet factor,

$$e^{-\frac{r_1^2}{2\sigma^2}}. \tag{1.21}$$

- Opening angle of the electron wave packet factor,

$$e^{-\frac{33}{34} \frac{(\sigma mv)^2}{2}}. \tag{1.22}$$

When the variance $\sigma$ is close to the inner radius of the magnet (the electron wave packet is big), (1.21) is close to 1 and (1.22) is extremely small (because in this case $\sigma mv$ is big ). Then, when the electron wave packet is big compared to the inner radius, (1.21) is the important term, what justifies our name. When the variance is small (such that $\sigma mv$ is close to 1) the factor (1.22) is close to one and (1.21) is extremely small ( $\frac{r_1^2}{\sigma^2}$ is big) and so, the important factor is (1.22). Note that when the variance in position, $\sigma$, is small, by Heisenberg uncertainly principle the variance in momentum is big. In particular, the transversal component of momentum is large and the electron wave packet spreads a lot as it propagates, what makes the opening angle of the electron wave packet large . This justifies the name that we give to (1.22). Note that in both cases the part of the electron wave packet that hits the obstacle is big.

When $\sigma$ is big, because the wave packet is big, and when $\sigma$ is small, because the opening angle is big, and even if the wave packet was initially small, it spreads rapidly as it propagates inside the magnet and, in consequence, a large part of the wave packet hits the obstacle. For variances, $\sigma$, that are neither to small nor too big the part of the electron
wave packet that hits the obstacle is small and the error is very small. In Section 9 we discuss in detail the physical interpretation of our error bound and we present a detailed quantitative analysis for a large range of $\sigma$.

In particular, we give a rigorous proof that if $1.1592 \times 10^{-9} \leq \sigma \leq 7.7955 \times 10^{-6}$ the error bound is smaller than $10^{-99}$. As mentioned above, it would be quite interesting to perform an experiment with electron wave packets that satisfy our bounds. One could as well take a larger magnet. In this case the probability that the electron wave packet interacts with the magnet is smaller than $10^{-199}$ (See Remark 8.12 and Section 9.2), and quantum mechanics predicts with a very small error bound the interference fringes observed in the experiments of Tonomura et al. [17, 25, 26], and the absence of a force on the electron, as observed in the Caprez et al. experiment [6].

The paper is organized as follows. In Section 2 we introduce notations and definitions that we use along the paper. In Section 3 we study the time evolution of the electron wave packet. We define the wave and the scattering operators, and we introduce the solutions to the Schrödinger equation with initial condition as time goes to $-\infty$. We estimate the solution to the Schrödinger equation when it is incoming, interacting, and outgoing. In Section 4 we use the freedom that we have in the selection of the magnetic field, the magnetic potential and the smooth characteristic cutoff function to make a choice that is convenient for the computation of the error bounds. In Section 5 we make a choice of the free parameters under the experimental conditions of Tonomura et al. [17]. In Section 6 we continue our study of the time evolution of the electron wave packet when it is incoming, interacting, and outgoing. In Section 7 we consider the Aharonov-Bohm Ansatz for initial data at time zero and for initial data at time $-\infty$. In Section 8 we estimate the difference between the exact solution to the Schrödinger equation and the Aharonov-Bohm Ansatz as the electron is incoming, interacting, and outgoing. In particular, in Theorem 8.11 we prove our main result that is quoted as Theorem 1.1 in the Introduction. In Section 9 we give a detailed analysis of the physical interpretation of our error bound with quantitative results. In Section 10 we give the conclusions of our paper. In appendix A we prove estimates for the free evolution of gaussian states that we use in our work. In Appendix B we prove upper bounds for integrals that we need to compute our error bound.

## 2 Notations and Definitions

In this section we collect notations and definitions that are used along the paper.

The magnet $\tilde{K}$ - see Section 1.1 - is defined by the following formula,

$$\tilde{K} := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < \tilde{r}_1 \leq (x_1^2 + x_2^2)^{1/2} \leq \tilde{r}_2, |x_3| \leq \tilde{h} \}.$$  \hspace{1cm} (2.1)

We call $\tilde{D}$ the convex hull of $\tilde{K}$. We use the notation,

$$\Lambda := \mathbb{R}^3 \setminus \tilde{K}.$$  \hspace{1cm} (2.2)
We employ the symbol \( \chi = \chi(x) = \chi(x, \sigma) \) for a twice continuously differentiable cut-off function that depends on the variance of the wave packet, \( \sigma \), see \([1,3]\). The support of \( 1 - \chi \) is contained in the set

\[
K := K(\sigma) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < r_1 \leq (x_1^2 + x_2^2)^{1/2} \leq r_2, |x_3| \leq h(\sigma) \right\},
\]

where \( r_1 \) and \( r_2 \) are some positive numbers such that \( r_1 < \tilde{r}_1, r_2 > \tilde{r}_2, \tilde{r}_1 - r_1 = r_2 - \tilde{r}_2 \) and \( h = h(\sigma) : \mathbb{R}_+ \to \mathbb{R}_+ \) is an increasing function such that \( h(\sigma) > \hat{h} \) for all \( \sigma \in \mathbb{R}_+ \). We will write either \( h \) or \( h(\sigma) \) for the same object.

We designate by \( \epsilon := \tilde{r}_1 - r_1 = r_2 - \tilde{r}_2 \), \( \delta(\sigma) := \delta := h(\sigma) - \hat{h} \), and by \( D := D(\sigma) \) the convex hull of \( K \).

For every \( \zeta, \tilde{\omega}, \sigma \in \mathbb{R}_+ \) such that \( 0 < \tilde{\omega}^{-1} < \sigma mv \), we denote by \( z_{\tilde{\omega}, \sigma}(\zeta) \) the unique solution of the equation,

\[
(z_{\tilde{\omega}, \sigma}(\zeta) - \zeta) \frac{\sigma mv}{(\sigma^4 m^2 v^2 + z_{\tilde{\omega}, \sigma}(\zeta)^2)^{1/2}} = \tilde{\omega}^{-1},
\]

and for every \( \sigma_1, \sigma_2 \in (0, r_1) \) (see \([23]\)) we define

\[
z_{\tilde{\omega}, \sigma_1, \sigma_2}(\zeta) := \max(z_{\tilde{\omega}, \sigma_1}(\zeta), z_{\tilde{\omega}, \sigma_2}(\zeta)), \quad r_{\sigma_1, \sigma_2} := \min_{i \in \{1, 2\}} \{ \lambda > 0 : \frac{(r_i \sigma_i)mv}{\sigma_i^4 (mv)^2 + \lambda^2} = 1 \}. \tag{2.6}
\]

For every \( \sigma \in \mathbb{R}_+ \), we define

\[
\tilde{\omega}(\sigma) := \frac{1}{\min \left( \sqrt{\frac{34}{33}} \sigma mv, \sqrt{2000} \right)}, \quad z(\sigma) := z_{\tilde{\omega}(\sigma), \sigma}(h(\sigma)), \quad \sigma_0 := \sqrt[33]{\frac{2000}{34}} m v. \tag{2.7}
\]

Note that (see equation \([11.23]\) in Appendix A)

\[
z(\sigma) > h(\sigma). \tag{2.8}
\]

For every \( \sigma \in \mathbb{R}_+ \) and every \( z, \zeta, s \in \mathbb{R} \) we use the following notation,

\[
\rho = \rho(z) = \rho(\sigma, z) := \frac{\sigma mv}{(\sigma^4 m^2 v^2 + z^2)^{1/2}},
\]

\[
\theta_{\text{inv}}(\sigma, z, \zeta) := (\zeta - s) \frac{\sigma mv}{(\sigma^4 m^2 v^2 + z^2)^{1/2}}, \quad \theta_{\text{inv}}(\sigma, z) := \theta_{\text{inv}}(\sigma, z, z, h(\sigma)), \tag{2.10}
\]

and

\[
\Upsilon(\sigma, z, s, \zeta) := \int_{\theta_{\text{inv}}(\sigma, z, s, -\zeta)}^{\theta_{\text{inv}}(\sigma, z, s, \zeta)} e^{-\tau^2} d\tau, \quad \Upsilon(\sigma, z) := \Upsilon(\sigma, z, z, h(\sigma)), \tag{2.11}
\]

\[
\Theta(\sigma, z, s, \zeta) := \int_{\theta_{\text{inv}}(\sigma, z, s, -\zeta)}^{\theta_{\text{inv}}(\sigma, z, s, \zeta)} \tau^2 e^{-\tau^2} d\tau, \quad \Theta(\sigma, z) := \Theta(\sigma, z, z, h(\sigma)). \tag{2.12}
\]

We utilize the symbols \( h, c, M \) and \( q \) for the Planck constant, the speed of light and the mass and charge of the electron, respectively. We define,
\[ m := \frac{M}{\hbar}. \]

We denote by \( v \in \mathbb{R}^3 \) the velocity - see \([1,3]\) - and we designate by \( v := |v| \), and \( \hat{v} := v/v \), respectively, the modulus and the direction of the velocity. We suppose that \( \hat{v} = (0, 0, 1) \). We designate by \( p := -i\nabla_x \). The momentum operator is \( P := \hbar p \).

We use the letters \( \tilde{B} \) and \( \tilde{A} \) for the magnetic field and the magnetic potential, respectively. The details of the distribution of the magnetic field inside \( \tilde{K} \) are not relevant for the dynamics of the electron that propagates outside \( \tilde{K} \), as long as \( \tilde{B} \) is contained inside \( \tilde{K} \). Actually, what is relevant is the flux of \( \tilde{B} \) along a transversal section of \( \tilde{K} \) modulo \( 2\pi \). See \([3]\) for this issue. We use this freedom to choose \( \tilde{B} \) and \( \tilde{A} \) in a technically convenient way. Then, unless we specify something else, we assume that \( \tilde{A} \) is contained in \( \tilde{K} \), that the support of \( \tilde{A} \) is contained in the convex hull of \( \tilde{K} \) (what is always possible), and that both are continuously differentiable. In Section 4, for any given flux in the transversal section of the magnet we explicitly construct a magnetic field and a magnetic potential that satisfy our assumptions. We define

\[
\eta(x, \tau) := \int_{\tau}^{\infty} (\hat{v} \times B)(x + \rho \hat{v}) d\rho. \tag{2.13}
\]

We denote by \( \tilde{\Phi} \) the flux of the magnetic field \( \tilde{B} \) over a transversal section (TS) of the magnet,

\[
\tilde{\Phi} := \int_{\text{TS}} \tilde{B}. \tag{2.14}
\]

Then, the flux of \( B \) over a transversal section of the magnet is given by,

\[
\Phi := \int_{\text{TS}} B = \frac{q}{\hbar c} \tilde{\Phi}. \tag{2.15}
\]

By Stokes theorem, for every \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) such that \( \sqrt{x_1^2 + x_2^2} \leq \hat{r}_1 \) we have that,

\[
\tilde{\Phi} = \int_{-\infty}^{\infty} \hat{v} \cdot \hat{A} d\tau, \quad \Phi = \int_{-\infty}^{\infty} \hat{v} \cdot A(x + \tau \hat{v}) d\tau. \tag{2.16}
\]

Given a function \( F \) with domain \( \mathcal{D} \subset \mathbb{R}^n, n = 1, 2, \ldots \) that takes values on a normed space \( \tilde{C} \) with norm \( \| \cdot \| \), we denote by \( \| F \|_\infty := \text{ess sup}\{\| f(x) \| : x \in \mathcal{D}\} \).

The vector \( \vec{M} = \vec{M}(\chi, A, v) = (M_1(\chi, A, v), \ldots, M_5(\chi, A, v)) := (M_1, \ldots, M_5) \in \mathbb{R}^5 \) is given by (\( v \) and \( A \) and \( \chi \) are defined above in this section ).
\[ M_1 := \| p^2 \chi \|_\infty + \| \chi p \cdot A \|_\infty + \| 2(\chi x) \cdot A \|_\infty + \| \chi A^2 \|_\infty, \]
\[ M_2 := \| 2(\chi x) \|_\infty + \| 2\chi A \|_\infty, \]
\[ M_3 = \| (p\chi) \cdot \hat{v} \|_\infty + \| \chi A \cdot \hat{v} \|_\infty, \]
\[ M_4 := \| (\chi p \cdot A)(x + t\hat{v}) \|_\infty + \| (\chi x)A^2(x + t\hat{v}) \|_\infty + 2\| A(x + t\hat{v}) \cdot (p\chi)(x) \|_\infty + 2\| (\chi x)A(x + t\hat{v}) \cdot \eta(x, t) \|_\infty, \]
\[ M_5 := \| A(x + t\hat{v}) \|_\infty. \]

The norms in \( M_4 \) and \( M_5 \) are taken with respect to \( x \in \mathbb{R}^3 \) and \( t \in \mathbb{R} \).

We define the linear function \( A : \mathbb{R}^5 \to \mathbb{R}^5 \) by the following: given a vector \( w := (w_1, \cdots w_5) \in \mathbb{R}^5 \), we take \( A(w) := (A(w)_1, \cdots, A(w)_5) \) as,
\[ A(w)_1 := \frac{1}{\sqrt{2m}}w_1 + \sqrt{2}w_3, \quad A(w)_2 := \frac{4}{\pi^{1/4}}\left[ \frac{\sqrt{2}}{2m}w_1 + \frac{2+\sqrt{2}}{2}w_2 + \sqrt{2}w_3 \right], \]
\[ A(w)_3 := \frac{4+\sqrt{2}^{3/4}}{\sqrt{m^{4/4}}}w_2, \quad A(w)_4 := \frac{1}{\sqrt{2m}}w_4, \quad A(w)_5 := \frac{1}{\sqrt{2m}}w_5. \]

The symbol used on the formulae below where defined in this section. Given \( S_1, v \in \mathbb{R}_+, w \in \mathbb{R}^5 \) and \( j \in \{-\infty, 0, \infty\} \), we define the function \( \tilde{A}_{w,v}^j = \tilde{A}_{w,v} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) by
\[ \tilde{A}_{w,v}^{-\infty}(z, \sigma) = \tilde{A}_{w,v}^{-\infty}(z, \sigma) := \max(z, S_1)^{\frac{A(w)_1}{2}} + \max(z, S_1)^{-1/2}(4\sigma^{1/4})^{1/2}A(w)_2 + \frac{\max(z, S_1)^{1/2}A(w)_3}{\sigma^{1/4}} - zA(w)_1 - \frac{zA(w)_3}{\sigma^{1/2}}, \]
\[ \tilde{A}_{w,v}^0(z, \sigma) = \tilde{A}_{w,v}^0(z, \sigma) := \tilde{A}_{w,v}^{-\infty}(z, \sigma) + zA(w)_4 + \frac{zA(w)_3}{\sigma^{1/2}}, \]
\[ \tilde{A}_{w,v}^{\infty}(z, \sigma) = \tilde{A}_{w,v}^{\infty}(z, \sigma) := 3\tilde{A}_{w,v}^{-\infty}(z, \sigma) + zA(w)_4 + \frac{zA(w)_3}{\sigma^{1/2}}. \]

We will not make explicit the dependence on \( S_1 \) because it will be fixed in our estimates. Actually, \( S_1 \) is a free parameter that we introduce to optimize the error bound for the incoming electron wave packet in Theorem 3.1. We fix \( S_1 \) in Section 3.2. Note, furthermore, that \( \tilde{A}_{w,v}^{-\infty}(z, \sigma) \) is independent of \( w_3 \) and of \( w_4 \). We define it as a function of \( w \in \mathbb{R}^5 \) to simplify the statement of our results.

We define the following quantities,
\[ C_{pp}(\sigma) = C_{pp}(\sigma, B, \chi) := \frac{1}{\pi^{1/4}\sigma m} \left( \| \Delta \chi \|_\infty + 2\| \eta(x, t) \cdot (p\chi)(x) \| + \frac{2}{\pi^{1/4}} \| p\chi(x) \cdot \hat{v} \|_\infty, \right) \]
\[ C_{ps}(\sigma) = C_{ps}(\sigma, B, \chi) := \frac{1}{\pi^{1/4}\sigma m} \left( \| \chi p \cdot \eta(x, t) \|_\infty + \| \chi(x)\eta(x, t) \|_\infty^2, \right) \]
\[ C_{sp}(\sigma) = C_{sp}(\sigma, B, \chi) := \frac{2}{\pi^{1/4}\sigma m} \left( \| p\chi(x) \|_\infty, \right) \]
\[ C_{ss}(\sigma) = C_{ss}(\sigma, B, \chi) := \frac{2}{\pi^{1/4}\sigma m} \left( \| \chi \eta(x, t) \|_\infty, \right) \]
\[ \mathcal{R}(\zeta) = \mathcal{R}(\zeta, Z) = \mathcal{R}(\zeta, Z, A) := \| A \|_\infty e^{\frac{\pi}{2}(\sigma^4m^2\zeta^2)^{1/2}} e^{-\frac{1}{2}(h-Z)^2} e^{-\frac{(\zeta_{max})^2}{\sigma^{1/2}m^2\zeta^2}}. \]
3 Time Evolution of the Electron Wave Packet

3.1 Wave and Scattering Operators

The Hamiltonian operator (1.6) is self-adjoint when it is defined on the domain \( D(H) := \mathcal{H}_2(\Lambda) \cap \mathcal{H}_{1,0}(\Lambda) \), where by \( \mathcal{H}_s(\Lambda), s = 1, 2, \cdots \) we denote the Sobolev spaces and by \( \mathcal{H}_{1,0}(\Lambda) \) we denote the closure in the norm of \( \mathcal{H}_1(\Lambda) \) of the set \( C^\infty_0(\Lambda) \) of all infinitely differentiable functions with compact support in \( \Lambda \) [1]. Note that as the functions in \( \mathcal{H}_{1,0}(\Lambda) \) vanish in trace sense at \( \partial \Lambda \), \( H \) is the positive self-adjoint realization in \( L^2(\Lambda) \) of the formal differential operator \( \frac{1}{2m}(\mathbf{p} - \hbar A)^2 \) with Dirichlet boundary condition at the boundary of \( \Lambda \) [12, 19]. The free Hamiltonian (1.2) is self-adjoint when it is defined on the domain \( D(H_0) := \mathcal{H}_2(\mathbb{R}^3) \). Let \( J \) be the identification operator from \( L^2(\mathbb{R}^3) \) into \( L^2(\Lambda) \) given by multiplication by the characteristic function of \( \Lambda \), i.e.,

\[
J\phi(x) = \chi_\Lambda(x) \phi(x),
\]

(3.1)

where \( \chi_\Lambda(x) = 1, x \in \Lambda, \chi_\Lambda(x) = 0, x \in \mathbb{R}^3 \setminus \Lambda \). As mentioned in the introduction, the wave operators are defined as follows [20],

\[
W_\pm(A) = W_\pm := \lim_{t \to \pm \infty} e^{i\frac{t}{\hbar}H} J e^{-i\frac{t}{\hbar}H_0}.
\]

(3.2)

It is proved in [3] that the strong limits (3.2) exist, that they are partially isometric, and that we can replace \( J \) by the operator of multiplication by any smooth characteristic function, \( \chi(x) \in C^2 \) such that \( \chi(x) = 0, x \in K \) and \( \chi(x) = 1 \) for \( x \) in the complement of a bounded set that contains \( K \) on its interior.

\[
W_\pm(A) = W_\pm = \lim_{t \to \pm \infty} e^{i\frac{t}{\hbar}H} \chi e^{-i\frac{t}{\hbar}H_0}.
\]

(3.3)

It is also known [13] that the wave operators are asymptotically complete, i.e., that the ranges of \( W_\pm \) are the same, and that they coincide with the subspace of absolute continuity of \( H \). Moreover, the \( W_\pm \) are unitary from \( L^2(\mathbb{R}^3) \) onto the subspace of absolute continuity of \( H \), and they satisfy the intertwining relations,

\[
e^{-i\frac{t}{\hbar}H} W_\pm = W_\pm e^{-i\frac{t}{\hbar}H_0}.
\]

(3.4)

Recall that the scattering operator is defined as [20],

\[
S := W_+^* W_-.
\]

(3.5)

3.2 Initial Conditions at Minus Infinity

In scattering experiments we know the wave packet of the electron at the emission time. Thus, if we want to know the evolution of the emitted electron for all times, we have to solve the interacting Schrödinger equation (1.5) with initial conditions at minus infinity. As mentioned in the introduction this is accomplished with wave operator \( W_- \). The
incoming electron wave packet is described at the time of emission \((t \to -\infty)\) by a solution to the free Schrödinger equation, (1.1),
\[
e^{-i \frac{\hat{H}_0}{\hbar} t} \phi_-.
\]
As \(e^{-i \frac{\hat{H}}{\hbar}}\) is unitary, for all \(\phi_- \in L^2(\mathbb{R}^3)\)
\[
\lim_{t \to \pm \infty} \left\| e^{-i \frac{\hat{H}_2}{\hbar} W_\pm \phi_- - J e^{-i \frac{\hat{H}_0}{\hbar} \phi_-}} \right\| = 0.
\]
Then, the solution to (1.5) that behaves as (3.6) as \(t \to -\infty\) is given by,
\[
e^{-i \frac{\hat{H}_2}{\hbar} W_- \phi_-}.
\]
And, moreover,
\[
\lim_{t \to \infty} \left\| e^{-i \frac{\hat{H}_2}{\hbar} W_- \phi_- - J e^{-i \frac{\hat{H}_0}{\hbar} \phi_+}} \right\| = 0, \quad \text{where} \ \phi_+ := W_+ W_- \phi_-.
\]
This means that -as to be expected- for large positive times, when the exact electron wave packet is far away from the magnet, it behaves as the outgoing solution to the free Schrödinger equation (1.1)
\[
e^{-i \frac{\hat{H}_0}{\hbar} t} \phi_+.
\]
where the data at \(t = 0\) of the incoming and the outgoing free wave packets (3.6, 3.10) are related by the scattering operator,
\[
\phi_+ = S \phi_-.
\]

### 3.3 The Incoming Electron Wave Packet

We first introduce concepts that will be used latter in our estimates.

We define the re-scaled boosted Hamiltonians as follows (see (1.2), (1.6)),
\[
H_1 = H_1(v) := \frac{1}{\hbar v} e^{-imv \cdot x} H_0 e^{imv \cdot x}, \quad H_2 = H_2(A, v) := \frac{1}{\hbar v} e^{-imv \cdot x} H(A) e^{imv \cdot x}.
\]
Recall that \(m = \frac{M}{\hbar}\) and \(v\) is the velocity (see (1.3)). Let us denote by
\[
W_{\pm, v} := e^{-imv \cdot x} W_{\pm} e^{imv \cdot x}
\]
the boosted wave operators. We have that,
\[
W_{\pm, v} = s \lim_{\zeta \to \pm \infty} e^{i \zeta H_2} \chi(x) e^{-i \zeta H_1},
\]

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where \( \zeta \) represents the classical \( x_3 \)-coordinate of the electron at the time \( t = \zeta/v \).

We notice that,

\[
e^{-i\zeta H_2} = e^{-imv \cdot z} e^{-i \frac{\zeta t}{m} H(A)} e^{imv \cdot z}, \quad e^{-i\zeta H_1} = e^{-imv \cdot z} e^{-i \frac{\zeta t}{m} H_0} e^{imv \cdot z} = e^{-i \frac{\zeta t}{m}(p+mv)^2}.
\] (3.14)

The following theorem gives us an estimate of the exact electron wave packet \( e^{-i \frac{\zeta t}{m} H} W_\pm(A) \varphi \) for distances \( Z \leq -z(\sigma) < -h(\sigma) \), i.e., where it is incoming.

**Theorem 3.1.** Let \( w = (w_1, \cdots, w_5) \in \mathbb{R}^5 \) be such that \( w_i \geq M_i(\chi, A, \nu) \) for \( i \in \{1, 2, 3\} \). Assume that \( \sigma mv \geq \sqrt{34/33} \). Then, for any \( Z \in \mathbb{R}^+ \) such that \( Z \geq z(\sigma) > h(\sigma) \),

\[
\|e^{\frac{i}{m} H} W_{\pm} \varphi - e^{\frac{i}{m} H_0} \varphi\| \leq e^{-\frac{1}{2m^2v^2} \overline{A}_w(z, \sigma)},
\] (3.15)

**Proof:** First we prove (3.15) for \( W_+(A) \). By Duhamel’s formula and (3.14) we have that,

\[
\| (W_+ - e^{iZH_2} \chi e^{-iZH_1}) \varphi \| \leq \frac{1}{2mv} \int_Z^\infty \left[ |m_1 e^{-izH_1} \varphi| + 2 \| m_2 \cdot p e^{-izH_0} \varphi \| + 2mv \| m_2 \cdot \psi e^{-izH_1} \varphi \| \right] dz,
\] (3.16)

where,

\[
m_1 := (p^2 \chi) - \chi(p \cdot A) - 2(p \chi) \cdot A + A^2 \chi.
\] (3.17)

\[
m_2 := (p \chi) - \chi A.
\] (3.18)

Equation (3.15) for \( W_+(A) \) follows from (3.16), Lemmata 11.3 and 11.5 in Appendix A, the facts that the function \( \theta_{inv}(\sigma, Z) \) is decreasing as a function of \( Z \), for \( Z \geq 0 \), that \( 1/\overline{w}(\sigma) = -\theta_{inv}(\sigma, z(\sigma)) \) and the following estimates:

\[
\int_{max(Z, S_1)}^\infty \left( \frac{1}{\sigma^4 m^2 v^2 \zeta^2} \right)^{3/4} \leq 2 \max(Z, S_1)^{-1/2},
\]

\[
|\theta_{inv}(\sigma, Z)| \leq \sigma mv.
\]

The last inequality follows from the definition of \( \theta_{inv}(\sigma, Z) \), since \( Z \geq z(\sigma) > h(\sigma) \) (see equation (2.8)).

We now consider the case of \( W_-(A) \). Note that by the uniqueness of the solutions to the Schrödinger equation we have that,

\[
e^{-iZH_2(A, \nu)} \overline{\psi} = e^{iZH_2(-A, -\nu)} \overline{\psi}.
\] (3.19)

This is the invariance under time reversal and charge conjugation. Hence,

\[
W_{-, \nu}(-A) \psi = \overline{W_{+, \nu}(A) \overline{\psi}},
\] (3.20)

and then,

\[
(W_{-, \nu}(-A) - e^{-iZH_2(-A, -\nu}) \chi e^{iZH_1(-\nu)}) \varphi = (W_{+, \nu}(A) - e^{iZH_2(A, \nu}) \chi e^{-iZH_1(\nu)}) \varphi.
\] (3.21)
It follows that (3.15) for $W_-(A)$ and $\varphi_-$ follows from (3.15) for $W_+(A)$ and $\varphi_+$, and the fact that $\bar{M}(\chi, A, \nu) = \bar{M}(\chi, -A, -\nu)$.

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $L(x) = -x$, for $x \in \mathbb{R}^3$. Note that,

$$
(e^{-i\zeta H_2(A, \nu)} \psi) \circ L = e^{-i\zeta H_2(-A \circ L, -\nu)} (\psi \circ L).
$$

Equation (3.22) implies that,

$$
(W_-\nu(A) \varphi) \circ L = W_-\nu(-A \circ L) (\varphi \circ L),
$$

where we used that as $\chi(x) = 1$ for $x$ in the complement of a bounded set,

$$
s - \lim_{\zeta \to \pm \infty} (\chi(-x) - \chi(x)) e^{-i\zeta H_1} = 0.
$$

We obtain (3.15) for $W_-(A)$ and $\varphi_-$ from (3.15) for $W_-(A \circ L)$, $\varphi_-$, $\chi \circ L$ instead of $\chi$, and $B \circ L$ instead of $B$ using equations (3.22), (3.23), and observing that $\bar{M}(\chi, A, \nu) = \bar{M}(\chi \circ L, -A \circ L, -\nu)$. For this purpose we use $B \circ L$ instead of $B$ in the definition of $\eta$ in (2.13).

\[\square\]

### 3.4 The Interacting Electron Wave Packet

We first introduce an assumption that we use often.

**Assumption 3.2.** Let $\mu_i$, $i \in \{1, 2, 3\}$ belong to $\mathbb{R}_+$. Suppose that the following conditions hold.

1. Either $\mu_i \leq \sigma_0$, $i \in \{1, 2, 3\}$, or $\mu_i \geq \sigma_0$, $i \in \{1, 2, 3\}$.
2. Either, $\mu_i \leq \mu_3$, $i \in \{1, 2\}$, or $\mu_i \geq \mu_3$, $i \in \{1, 2\}$.

We define $\mu_{\text{max}} := \max(\mu_1, \mu_2)$, $\mu_{\text{min}} := \min(\mu_1, \mu_2)$, and take $\nu = \mu_{\text{min}}$, if $\mu_i \leq \mu_3$, $i \in \{1, 2\}$ and $\nu = \mu_{\text{max}}$, if $\mu_i \geq \mu_3$, $i \in \{1, 2\}$. We denote by $Z := z(\mu_{\text{max}})$, if $\mu_i \leq \sigma_0$, $i \in \{1, 2, 3\}$; and $Z := \max_{i \in \{1, 2\}} \{z \bar{\zeta}(\mu_{\text{max}}), z^{\frac{3}{2}}(h(\mu_{\text{max}}))\}$, if $\mu_i \geq \sigma_0$, $i \in \{1, 2, 3\}$. We suppose that $Z \geq z \sqrt{\bar{\zeta}_i^{\nu \nu_{\mu_3}}(h(\mu_{\text{max}}))}$ and $r_1 \rho(\mu_i, z \sqrt{\bar{\zeta}_i^{\nu \nu_{\mu_3}}(h(\mu_{\text{max}}))}) \geq 1$ for $i \in \{1, 2\}$.

\[\square\]

The quantities $I_{ps}$, $I_{pp}$, $I_{ss}$, and $I_{sp}$ that we use below are defined, respectively, in equations (11.26), (11.32), (11.35), and (11.55) in Appendix A.

**Lemma 3.3.** Suppose that Assumption 3.2 is satisfied and that $\sigma_{\mu v} \geq 1$. Then, for every gaussian wave function $\varphi$ with variance $\sigma \in [\mu_{\text{min}}, \mu_{\text{max}}]$ and every $\zeta \in \mathbb{R}$ with $|\zeta| \leq z(\sigma)$,
\[
\left\| \left( e^{i(z(\sigma) - \zeta)}H_z \chi(x) e^{-i(z(\sigma) - \zeta)H_1} - \chi(x) e^{-i \int_0^{(z(\sigma) - \zeta)} \psi \cdot A(x + \tau \vec{v}) d\tau} \right) e^{-i\zeta H_1} \varphi \right\| \leq
\]
\[
e^{-\frac{1}{2} \frac{1}{2m^2} (Z - \zeta)} \left( \frac{1}{2\sqrt{2\pi}} M_4 + M_5 \left( \frac{(\frac{3\pi}{4})^{1/2} + 2\pi^{1/4}}{\pi^{1/2}} \right) \right) + C_{pp}(\sigma) I_{pp}(\mu_1, \mu_2, \mu_3) +
\]
\[
\frac{C_{ps}(\sigma)}{2} I_{ps}(\mu_1, \mu_2, \mu_3, \zeta) + C_{sp}(\sigma) I_{sp}(\mu_1, \mu_2, \mu_3) + \frac{C_{ps}(\sigma)}{2} I_{ps}(\mu_1, \mu_2, \mu_3, \zeta),
\]
\[
\left\| \left( e^{i(z(\sigma) - \zeta)}H_z \chi(x) e^{-i(z(\sigma) - \zeta)H_1} - \chi(x) e^{-i \int_0^{(z(\sigma) - \zeta)} \psi \cdot A(x + \tau \vec{v}) d\tau} \right) \varphi \right\| \leq
\]
\[
e^{-\frac{1}{2} \frac{1}{2m^2} (Z)} \left( \frac{1}{\sqrt{2\pi}} M_4 + M_5 \left( \frac{(\frac{3\pi}{4})^{1/2} + 2\pi^{1/4}}{\pi^{1/2}} \right) \right) + \frac{C_{pp}(\sigma)}{2} I_{pp}(\mu_1, \mu_2, \mu_3) +
\]
\[
\frac{C_{ps}(\sigma)}{2} I_{ps}(\mu_1, \mu_2, \mu_3) + C_{sp}(\sigma) I_{sp}(\mu_1, \mu_2, \mu_3) + \frac{C_{ps}(\sigma)}{2} I_{ps}(\mu_1, \mu_2, \mu_3).
\]

\[\text{Proof: As in the proof of Lemma 5.6 of } [3] \text{ (see also }[28]\text{) we prove that,}
\]
\[
\left( e^{i(z(\sigma) - \zeta)}H_z \chi(x) e^{-i(z(\sigma) - \zeta)H_1} - \chi(x) e^{-i \int_0^{(z(\sigma) - \zeta)} \psi \cdot A(x + \tau \vec{v}) d\tau} \right) e^{-i\zeta H_1} \varphi = \int_0^{z(\sigma) - \zeta} dz e^{i z H_2} e^{-i \int_0^{z(\sigma) - \zeta} \psi \cdot A(x + \tau \vec{v}) d\tau}
\]
\[
\sum_{i=1}^2 \left( f_i(x, z(\sigma) - \zeta - z) + g_i(x, z(\sigma) - \zeta - z) \cdot \mathbf{p} \right) e^{-i z H_1} + f_3(x) e^{-i z H_1} \right) e^{-i\zeta H_1} \varphi,
\]
\[
\text{where,}
\]
\[
f_1(x, \tau) := \frac{1}{2mv} \left[ -\chi(x)(\mathbf{p} \cdot A)(x + \tau \vec{v}) + \chi(x)(A(x + \tau \vec{v}))^2 - 2A(x + \tau \vec{v}) \cdot (\mathbf{p} \chi)(x) + 2\chi(x)(A(x + \tau \vec{v}) \cdot \eta(x, \tau)),
\]
\[
f_2(x, \tau) := \frac{1}{2mv} \left[ -\chi(x)(\mathbf{p} \cdot \eta)(x, \tau) + \chi(x)(\eta(x, \tau))^2 - (\Delta \chi)(x) - 2\eta(x, \tau) \cdot (\mathbf{p} \chi)(x) \right],
\]
\[
f_3(x) := (\mathbf{p} \chi)(x) \cdot \vec{v},
\]
\[
g_1(x, \tau) := -\frac{1}{mv} \chi(x) A(x + \tau \vec{v}),
\]
\[
g_2(x, \tau) := \frac{1}{mv} \left[ -\chi(x) \eta(x, \tau) + (\mathbf{p} \chi)(x) \right].
\]

\[\text{It follows that,}
\]
\[
\left\| \left( e^{i(z(\sigma) - \zeta)}H_z \chi(x) e^{-i(z(\sigma) - \zeta)H_1} - \chi(x) e^{-i \int_0^{(z(\sigma) - \zeta)} \psi \cdot A(x + \tau \vec{v}) d\tau} \right) e^{-i\zeta H_1} \varphi \right\| \leq
\]
\[
f_0^{z(\sigma) - \zeta} dz \| f_1(x, z(\sigma) - \zeta - z) e^{-i z H_1} e^{-i\zeta H_1} \varphi \| + f_0^{z(\sigma) - \zeta} dz \| f_2(x, z(\sigma) - \zeta - z) e^{-i z H_1} e^{-i\zeta H_1} \varphi \| +
\]
\[
f_0^{z(\sigma) - \zeta} dz \| f_3(x) e^{-i z H_1} e^{-i\zeta H_1} \varphi \| + f_0^{z(\sigma) - \zeta} dz \| g_1(x, z(\sigma) - \zeta - z) \cdot \mathbf{p} e^{-i z H_1} e^{-i\zeta H_1} \varphi \| +
\]
\[
f_0^{z(\sigma) - \zeta} dz \| g_2(x, z(\sigma) - \zeta - z) \cdot \mathbf{p} e^{-i z H_1} e^{-i\zeta H_1} \varphi \|.
\]

We estimate the first integral in the right-hand side of \([3.32]\) using equation \([11.57]\), the second using \([11.25]\) and \([11.31]\), the third using \([11.31]\), the fourth using \([11.59]\), and the fifth using \([11.34]\) and \([11.54]\). To use \([11.59]\) note
that $\theta_{inv}(\sigma, z(\sigma)) = -1/\theta(\sigma)$. Then, as $\sigma_{mv} \geq 1$, $\theta_{inv}(\sigma, z(\sigma))^2 \geq 1/2$. After reordering terms we obtain equation (3.24). Equation (3.25) is obtained in the same way but using (11.33) instead of (11.31) and (11.56) instead of (11.54).

\[ \Box \]

**LEMMA 3.4.** For $Z \geq h$,

\[
\left\| \left( \chi(x)e^{-i \int_0^t \hat{\varphi} \cdot A(x+\tau \hat{\varphi}) d\tau} - \chi(x)e^{-i \int_0^t \hat{\varphi} \cdot A(x+\tau \hat{\varphi}) d\tau} \right) e^{-i H_1 \varphi} \right\| \leq \frac{1}{2} R(\zeta, Z). \quad (3.33)
\]

**Proof:** By Duhamel’s formula and (11.7),

\[
\left\| \left( \chi(x)e^{-i \int_0^t \hat{\varphi} \cdot A(x+\tau \hat{\varphi}) d\tau} - \chi(x)e^{-i \int_0^t \hat{\varphi} \cdot A(x+\tau \hat{\varphi}) d\tau} \right) e^{-i H_1 \varphi} \right\| \leq \int_{Z-\zeta}^{\infty} \left\| \chi(x) \hat{\varphi} \cdot A(x+\tau \hat{\varphi}) e^{-i H_1 \varphi} \right\| d\tau \quad (3.34)
\]

where we used that $(h-\tau)^2 - (h-Z)^2 = (\tau-Z)(\tau+Z-2h)$. Finally since, $(\tau-Z)(\tau+Z-2h) \geq (\tau-Z)^2$,

\[
\left\| \left( \chi(x)e^{-i \int_0^t \hat{\varphi} \cdot A(x+\tau \hat{\varphi}) d\tau} - \chi(x)e^{-i \int_0^t \hat{\varphi} \cdot A(x+\tau \hat{\varphi}) d\tau} \right) e^{-i H_1 \varphi} \right\| \leq \int_{Z}^{\infty} e^{-\frac{1}{2} (h-Z)^2 \frac{\sigma_{mv}^2}{\sigma_{mv}^2 + \zeta^2}} d\tau e^{-\frac{1}{2} (\tau-Z)^2 \frac{\sigma_{mv}^2}{\sigma_{mv}^2 + \zeta^2}}, \quad (3.35)
\]

what proves the lemma.

\[ \Box \]

In the Theorem below we estimate the exact electron wave packet $e^{-i H W \pm (A) \varphi_W}$ for distances $\zeta$ such that $|\zeta| \leq z(\sigma)$. As $z(\sigma) > h(\sigma)$-see equation (2.8)-this is the interaction region.

**THEOREM 3.5.** Suppose that Assumption 1.2 is satisfied, and, furthermore that $\sigma_{mv} \geq 1$. Let $w = (w_1, \cdots, w_5) \in \mathbb{R}^5$ be such that $w_i \geq M_i(\chi, A, \varphi)$ for $i \in \{1, \cdots, 5\}$. Then, for every gaussian wave function $\varphi$ with variance $\sigma \in [\mu_{\min}, \mu_{\max}]$ and every $\zeta \in \mathbb{R}$ with $|\zeta| \leq z(\sigma)$,

\[
\left\| e^{-i \frac{H}{\hbar} W \pm (A) \varphi_W} - \chi e^{-i \int_0^t \hat{\varphi} \cdot A(x+\tau \hat{\varphi}) d\tau} e^{-i \frac{H}{\hbar} H_0 \varphi_W} \right\| \leq e^{-\frac{1}{2|\zeta|^2} \tilde{A}_0^2(z(\sigma), \varphi)} + C_{pp}(\sigma) I_{pp}(\mu, \mu, \mu) + C_{ps}(\sigma) I_{ps}(\mu, \mu, \mu), \quad (3.36)
\]

\[
\left\| W \pm (A) \varphi_W - \chi e^{-i \int_0^t \hat{\varphi} \cdot A(x+\tau \hat{\varphi}) d\tau} \varphi_W \right\| \leq e^{-\frac{1}{2|\zeta|^2} \tilde{A}_w (z(\sigma), \varphi)} + \frac{z(\sigma)}{2 \sigma} A(w) + C_{ps}(\sigma) I_{pp}(\mu, \mu, \mu) + C_{ps}(\sigma) I_{pp}(\mu, \mu, \mu), \quad (3.37)
\]

**Proof:**

We prove (3.36) for $W_+(A)$, the proof for $W_- (A)$ follows as in (3.19-3.23). Note that by the intertwining relations of the wave operators (3.4) and by (3.14) we have that,
Suppose that the conditions of Theorem 3.5 are satisfied. Then,

We first prove the following lemma.

3.5 Estimates for the Scattering Operator

similar, but instead of (3.24) we use (3.25).

Proof:

In the theorem below we approximate the scattering operator by its high-velocity limit (see [3]).

Since the following estimates,

Equation (3.36) is obtained by (3.38), (3.39), Theorem 3.1, equation (3.24) and Lemma 3.4. The proof of (3.37) is similar, but instead of (3.24) we use (3.25).

3.5 Estimates for the Scattering Operator

We first prove the following lemma.

**Lemma 3.6.** Suppose that the conditions of Theorem 3.5 are satisfied. Then,

\[
\left\| (W^*_{+} \mathbf{v}(A) - e^{i(z(\sigma) - \zeta)H_2} \chi e^{-i(z(\sigma) - \zeta)H_1}) \right\| \leq \left\| e^{-i \frac{i(z(\sigma) - \zeta)}{\gamma} H} W^*_+ (A) e^{-i \frac{i(z(\sigma) - \zeta)}{\gamma} H_0} \right\| \leq \left\| e^{-i \frac{i(z(\sigma) - \zeta)}{\gamma} H_0} \right\|
\]

We use the intertwining relations and (3.14) again to obtain,

\[
\left\| e^{-i \frac{i(z(\sigma) - \zeta)}{\gamma} H} W^*_+ (A) \phi \right\| = e^{-i \frac{i(z(\sigma) - \zeta)}{\gamma} H} W^*_+ e^{-i \frac{i(z(\sigma) - \zeta)}{\gamma} H_0} \phi.
\]

Equation (3.36) is obtained by (3.38), (3.39), Theorem 3.1, equation (3.24) and Lemma 3.4. The proof of (3.37) is similar, but instead of (3.24) we use (3.25).

3.5 Estimates for the Scattering Operator

We first prove the following lemma.

**Lemma 3.6.** Suppose that the conditions of Theorem 3.5 are satisfied. Then,

\[
\left\| \left( W^*_{+} e^{-i \int_0^1 \nabla^\cdot A(x + \tau \psi) d\tau - e^{i \phi} \right) \chi(x) \phi \right\| \leq \left\{ e^{-r^2/2} + e^{-r^2/\gamma} \right\} \left\| \left( \nabla^\cdot A - \frac{\phi}{2} \right) \right\| 
\]

Proof: As \( W^*_+ W_+ = I \),

\[
\left\| \left( W^*_{+} e^{-i \int_0^1 \nabla^\cdot A(x + \tau \psi) d\tau - e^{i \phi} \right) \chi(x) \phi \right\| \leq \left\| \left( \nabla^\cdot A - \frac{\phi}{2} \right) \right\| \left\| \left( \nabla^\cdot A - \frac{\phi}{2} \right) \right\| 
\]

Since \( \int_0^1 \nabla^\cdot A(x + \tau \psi) d\tau = \Phi \) for \( x \) in the cylinder \( \{ x \in \mathbb{R}^3 : x_1^2 + x_3^2 \leq r_1^2 \} \), (3.40) follows from Theorem 3.5 and the following estimates,

\[
\left\| (1 - \chi(x)) \phi \right\| \leq e^{-r^2/2\sigma^2}, \quad \left\| \left( e^{-i \int_0^1 \nabla^\cdot A(x + \tau \psi) d\tau - e^{-i \int_0^1 \nabla^\cdot A(x + \tau \psi) d\tau} \right) \chi \phi \right\| \leq 2e^{-r^2/2\sigma^2}.
\]

In the theorem below we approximate the scattering operator by its high-velocity limit (see [3]).
THEOREM 3.7. Suppose that Assumption 3.2 is satisfied. Let \( w = (w_1, \ldots, w_5) \in \mathbb{R}^5 \) be such that \( w_i \geq M_i(\chi, A, \mathbf{v}) \) for \( i \in \{1, \ldots, 5\} \). Then, for every gaussian wave function \( \varphi \) with variance \( \sigma \in [\mu_{\min}, \mu_{\max}] \),

\[
\left\| (S - e^{i\Phi}) \varphi \right\| \leq 3\epsilon^{-\frac{1}{2}} + e^{-\frac{1}{2(\sigma + 1)^2}} (2A_{\infty}^{-\infty}(z(\sigma), \sigma) + z(\sigma)A(w) + \frac{2(\mu)}{\sigma + 2} A(w)) + C_{\text{ps}}(\sigma)I_{pp}(\mu_1, \mu_2, \mu_3) + C_{\text{sp}}(\sigma)I_{sp}(\mu_1, \mu_2, \mu_3) + C_{\text{spp}}(\sigma)I_{sp}(\mu_1, \mu_2, \mu_3) + C_{\text{app}}(\sigma)I_{pp}(\mu_1, \mu_2, \mu_3) + R(0, z(\sigma)).
\]

(3.43)

Proof: We denote,

\[
S_{\varphi} := e^{-imv\cdot x} S e^{imv\cdot x}.
\]

We have that,

\[
\left\| (S - e^{i\Phi}) \varphi \right\| = \left\| (S_{\varphi} - e^{i\Phi}) \varphi \right\| = \left\| W_{++}(W_{-}, x)e^{-i\int_{-\infty}^{0} \varphi H_A(x + \tau v)d\tau} \varphi + \right\|
\]

(3.45)

Equation (3.43) follows from Theorem 3.5, Lemma 3.6 and (3.45).

3.6 The Outgoing Electron Wave Packet

In the following theorem we estimate the exact electron wave packet \( e^{-i\hat{H}_0^H}W_{-}(A)\varphi_{\mathbf{v}} \) for distances \( \zeta \) in the outgoing region, \( \zeta \geq z(\sigma) > h(\sigma) \).

THEOREM 3.8. Suppose that Assumption 3.2 is satisfied. Let \( w = (w_1, \ldots, w_5) \in \mathbb{R}^5 \) be such that \( w_i \geq M_i(\chi, A, \mathbf{v}) \) for \( i \in \{1, \ldots, 5\} \). Then, for every gaussian wave function \( \varphi \) with variance \( \sigma \in [\mu_{\min}, \mu_{\max}] \) and every \( \zeta \in \mathbb{R} \) with \( \zeta \geq z(\sigma) \),

\[
\left\| e^{-i\hat{H}_0^H}W_{-}(A)\varphi_{\mathbf{v}} - \chi e^{i\Phi} e^{-i\hat{H}_0^H} \varphi_{\mathbf{v}} \right\| \leq 3\epsilon^{-\frac{1}{2}} + e^{-\frac{1}{2(\sigma + 1)^2}} A_{\infty}^{-\infty}(z(\sigma), \sigma) + C_{\text{ps}}(\sigma)I_{pp}(\mu_1, \mu_2, \mu_3) + C_{\text{sp}}(\sigma)I_{sp}(\mu_1, \mu_2, \mu_3) + C_{\text{spp}}(\sigma)I_{sp}(\mu_1, \mu_2, \mu_3) + R(0, z(\sigma)).
\]

(3.46)

Proof:

Using the definition of \( S \) (see (3.5)) and the fact that \( e^{-i\hat{H}_0^H} \) is unitary we get,

\[
\left\| e^{-i\hat{H}_0^H}W_{-}(A)\varphi_{\mathbf{v}} - \chi e^{i\Phi} e^{-i\hat{H}_0^H} \varphi_{\mathbf{v}} \right\| = \left\| W_{-}(A)\varphi_{\mathbf{v}} - e^{i\hat{H}_0^H} \chi e^{i\Phi} e^{-i\hat{H}_0^H} \varphi_{\mathbf{v}} \right\| \leq
\]

(3.47)

Furthermore,

\[
\left\| W_{-}(A)\varphi_{\mathbf{v}} - W_{+}(A)e^{i\Phi} \varphi_{\mathbf{v}} \right\| \leq \left\| W_{-}(A)\varphi_{\mathbf{v}} - W_{+}(A)S\varphi_{\mathbf{v}} \right\| + \left\| W_{+}(A)(S - e^{i\Phi})\varphi_{\mathbf{v}} \right\|. \quad (3.48)
\]
Since the wave operators are asymptotically complete [13], the operators $W_{\pm} W_{\pm}^*$ are the orthogonal projector onto the common range of $W_{\pm}$. Then, $W_{+} W_{+}^* W_{-} = W_{-}$, and we have that,

$$W_{-} (A) \varphi \nu - W_{+} (A) S \varphi \nu = W_{-} (A) \varphi \nu - W_{+} (A) W_{+}^* (A) W_{-} (A) \varphi \nu = 0,$$

and by (3.47, 3.48)

$$\| e^{-i \frac{\Phi}{\pi} H} W_{-} (A) \varphi \nu - \chi e^{i \Phi \varepsilon} e^{-i \frac{\Phi}{\pi} H_0} \varphi \nu \| \leq \| W_{+} (A) \varphi \nu - e^{i \frac{\Phi}{\pi} H} \chi e^{-i \frac{\Phi}{\pi} H_0} \varphi \nu \| + \| S \varphi \nu - e^{i \Phi} \varphi \nu \|. \quad (3.49)$$

The inequality (3.46) follows from Theorems 3.1 and 3.7, and from equation (3.49).

4 The Magnetic Field, the Magnetic Potential and the Cutoff Function

We have proven in Theorem 4.1 of [3] that the Hamiltonians (1.6) with Dirichlet boundary condition on $\partial \Lambda$ that correspond to two different magnetic fields contained inside the magnet, and that have the same flux $\Phi$ modulo $2\pi$ are unitarily equivalent. We have also proven in [3] that the scattering operator only depends on the total flux $\Phi$ enclosed inside the magnet, modulo $2\pi$. This implies that without losing generality we can assume that

$$|\Phi| < 2\pi, \quad (4.1)$$

what we do from now on. This also means that we have a large freedom to choose the magnetic field, as long as it is contained inside the magnet. As mentioned in the introduction, we also have a large freedom to choose the smooth cutoff function $\chi$. We use this freedom to choose the magnetic field, the magnetic potential and the smooth cutoff function that is convenient for the computation of the error bounds. Below we construct a magnetic field inspired in the experimental results of Tonomura et. al. [25]. We also choose a magnetic potential and a cutoff function, and we provide bounds for them.

4.1 Mollifiers

We denote for $z \in \mathbb{R}$,

$$\psi(z) := \frac{1}{\ell} \begin{cases} e^{-1/(1-z^2)}, & |z| \leq 1, \\ 0, & |z| \geq 1, \end{cases} \quad (4.2)$$

where,

$$\ell := \int_{-1}^{1} e^{-1/(1-z^2)} dz. \quad (4.3)$$

For $\varepsilon > 0$ we define,

$$\psi_{\varepsilon}(z) := \frac{1}{\varepsilon} \psi(z/\varepsilon), \quad (4.4)$$
and for every \( a, b \in \mathbb{R} \), with \( a < b \) and every \( \varepsilon \in \mathbb{R}_+ \) with \( \varepsilon < \frac{1}{2} (b - a) \), we take,

\[
\psi_{a,b,\varepsilon}(z) := \int_a^b dy \psi_{\varepsilon}(z - y) = \\
\begin{cases} 
1, & z \in [a + \varepsilon, b - \varepsilon], \\
0, & z \notin [a - \varepsilon, b + \varepsilon].
\end{cases}
\] (4.5)

Then,

\[
\|\psi_{a,b,\varepsilon}\|_\infty = 1,
\] (4.6)

\[
\|\psi'_{a,b,\varepsilon}\|_\infty \leq \frac{1}{\iota \varepsilon},
\] (4.7)

\[
\|\psi''_{a,b,\varepsilon}\|_\infty \leq \frac{2N}{\varepsilon^2}, \text{ where } N := 2e^{-(3/2 + \sqrt{3}/4)}(3/2 + \sqrt{3}/4)^2(1 - (3/2 + \sqrt{3}/4)^{-1})^{1/2}.
\] (4.8)

### 4.2 The Magnetic Field

Recall that the magnet is the set,

\[
\tilde{K} := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < \tilde{r}_1 \leq (x_1^2 + x_2^2)^{1/2} \leq \tilde{r}_2, |x_3| \leq \tilde{h}\right\}. \tag{4.9}
\]

We use cylindrical coordinates: for \((x_1, x_2, x_3) \in \mathbb{R}^3\), we take \( r := (x_1^2 + x_2^2)^{1/2}, 0 \leq \theta < 2\pi, x_3 \). For \( \tilde{\varepsilon} < \frac{\tilde{r}_2 - \tilde{r}_1}{4} \), \( \tilde{\delta} < \frac{\tilde{h}}{2} \), we define,

\[
B = B(x, \tilde{\varepsilon}, \tilde{\delta}) := \frac{\Phi}{C_{\tilde{\varepsilon}, \tilde{\delta}}} \psi_{\tilde{r}_1 + \tilde{\varepsilon}, \tilde{r}_2 - \tilde{\varepsilon}, \tilde{\varepsilon}}(r) \psi_{-\tilde{r}_1 + \tilde{\delta}, \tilde{h} - \tilde{\delta}, \tilde{\delta}}(x_3)(- \sin \theta, \cos \theta, 0), \tag{4.10}
\]

where for a transverse section of \( \tilde{K}, TS \),

\[
C_{\tilde{\varepsilon}, \tilde{\delta}} := \int_{TS} \psi_{\tilde{r}_1 + \tilde{\varepsilon}, \tilde{r}_2 - \tilde{\varepsilon}, \tilde{\varepsilon}}(r) \psi_{-\tilde{r}_1 + \tilde{\delta}, \tilde{h} - \tilde{\delta}, \tilde{\delta}}(x_3) \geq 2(\tilde{h} - 2\tilde{\delta}) (\tilde{r}_2 - \tilde{r}_1 - 4\tilde{\varepsilon}). \tag{4.11}
\]

Then, \( \nabla \cdot B = 0 \) and the flux of \( B \) over any transverse section of \( \tilde{K} \) is \( \Phi \).

This choice of \( B \), that is approximately constant along any transverse section of \( \tilde{K} \) and is directed along the unit vector \((- \sin(\theta), \cos(\theta), 0)\) is inspired by the experimental results of Tonomura et al. \[25\]: in Figure 4 (a) of \[25\], the fringes on the shadow of the magnet suggest that the component of the magnetic field that is orthogonal to a transverse section of the magnet is constant over this transverse section.

By \((4.6, 4.7, 4.11)\),

\[
\|B\|_\infty \leq \frac{\pi}{(h - 2\delta)(\tilde{r}_2 - \tilde{r}_1 - 4\tilde{\varepsilon})}, \tag{4.12}
\]

\[
\left\| \frac{\partial}{\partial x_j} B \right\|_\infty \leq \frac{\pi}{(h - 2\delta)(\tilde{r}_2 - \tilde{r}_1 - 4\tilde{\varepsilon})} \left( \frac{1}{\iota \varepsilon} + \frac{1}{\tilde{r}_1} \right), j = 1, 2. \tag{4.13}
\]
\[ \left\| \frac{\partial}{\partial x_j} B \right\|_\infty \leq \frac{\pi}{(h - 2\delta)(\hat{r}_2 - \hat{r}_1 - 4\varepsilon)} \cdot \frac{1}{\nu \delta}, \quad (4.14) \]

With this choice of \( B \) we have that (see (2.13)).

\[ \| \eta(x, \tau) \|_\infty \leq 2\hat{h} \frac{\pi}{(h - 2\delta)(\hat{r}_2 - \hat{r}_1 - 4\varepsilon)}, \quad (4.15) \]

\[ \| \mathbf{p} \cdot \eta(x, \tau) \|_\infty \leq 2\hat{h} \frac{\pi}{(h - 2\delta)(\hat{r}_2 - \hat{r}_1 - 4\varepsilon)} \left( \frac{1}{\nu \delta} + \frac{1}{\hat{r}_1} \right). \quad (4.16) \]

### 4.3 The Magnetic Potential

The potential \( A = A(x, \varepsilon, \delta) \) associated to the field \( B = B(x, \varepsilon, \delta) \) satisfies the differential equation \( \nabla \times A = B \). As \( B \) has no vertical component, we can take \( A \) parallel to the vertical axis.

\[ A = A(x, \varepsilon, \delta) := \frac{-\Phi}{C_{\varepsilon, \delta}} \psi_{-\hat{h}+\delta, \hat{h}-\delta, \delta}(x_3) \left( 0, 0, \int_{(y_1, y_2)}^{(x_1, x_2)} \psi_{\hat{r}_1 + \varepsilon, \hat{r}_2 - \varepsilon, \varepsilon}(r) (\cos \theta, \sin \theta) \right), \quad (4.17) \]

where \((y_1, y_2)\) is any point with \(|(y_1, y_2)| \geq \hat{r}_2\) and the line integral is over any curve in \( \mathbb{R}^2 \) that connects the point \((y_1, y_2)\) with \((x_1, x_2)\). The value of \( A \) is independent of the curve chosen. The potential \( A \) has support in the convex hull of \( \hat{K} \), that we denoted by \( \hat{D} \). Moreover, by (4.6 - 4.7),

\[ \| A \|_\infty \leq \frac{\pi}{(h - 2\delta)(\hat{r}_2 - \hat{r}_1 - 4\varepsilon)}(\hat{r}_2 - \hat{r}_1), \quad (4.18) \]

\[ \left\| \frac{\partial}{\partial x_j} A \right\|_\infty \leq \frac{\pi}{(h - 2\delta)(\hat{r}_2 - \hat{r}_1 - 4\varepsilon)}, j = 1, 2, \quad (4.19) \]

\[ \left\| \frac{\partial}{\partial x_3} A \right\|_\infty \leq \frac{\pi}{(h - 2\delta)(\hat{r}_2 - \hat{r}_1 - 4\varepsilon)} \cdot \frac{1}{\nu \delta}(\hat{r}_2 - \hat{r}_1). \quad (4.20) \]

### 4.4 The Cutoff Function

We use the freedom that we have in the choice of the cutoff function \( \chi(x) \) to select it in a convenient way. Take \( 0 < \varepsilon < \hat{r}_1, \delta > 0 \). We define (see (2.4)),

\[ r_1 := \hat{r}_1 - \varepsilon > 0, \ \ r_2 := \hat{r}_2 + \varepsilon, \ \ h := \hat{h} + \delta. \quad (4.21) \]

We define

\[ \chi(x) := 1 - \psi_{r_1 + \varepsilon/2, r_2 - \varepsilon/2, \varepsilon/2}(r) \psi_{-h + \delta/2, h - \delta/2, \delta/2}(x_3). \quad (4.22) \]
Then (see (2.3)),
\[
\chi(x) = \begin{cases} 
0, & x \in \tilde{K}, \\
1, & x \in \mathbb{R}^3 \setminus K. 
\end{cases}
\] (4.23)
Moreover, by (4.6, 4.7, 4.8),
\[
\|\chi\|_\infty = 1, \\
\left\| \frac{\partial}{\partial x_j} \chi \right\|_\infty \leq \frac{2 \epsilon}{i \delta}, \quad j = 1, 2, \\
\left\| \frac{\partial}{\partial x_3} \chi \right\|_\infty \leq \frac{2 \epsilon}{i \delta},
\] (4.24)
\[
\|p^2 \chi\|_\infty \leq \frac{8 N}{i \delta^2} + \frac{2 \epsilon}{i \epsilon r_1} + \frac{8 N}{i \delta^2}.
\] (4.25)
We denote by
\[
I := \frac{1}{\pi} (\tilde{h} - 2 \tilde{\delta})(\tilde{r}_2 - \tilde{r}_1 - 4 \tilde{\epsilon}), \\
J := \frac{\tilde{r}_2 - \tilde{r}_1}{T}.
\] (4.26)
We designate by \( \bar{m}(\chi) = \bar{m} := (m_1(\chi), \ldots, m_5(\chi)) \in \mathbb{R}^5 \) the vector with the following components,
\[
m_1(\chi) = m_1 := \frac{8 N}{i \epsilon} + \frac{2 \epsilon}{i r_1} + \frac{8 N}{i \delta} + \left( 2 + (\tilde{r}_2 - \tilde{r}_1) \frac{1}{i \delta} \right) I^{-1} + \frac{4 \epsilon}{i \delta} J + J^2, \\
m_2(\chi) = m_2 := 2 \left( \frac{4 \epsilon}{i \epsilon} + \frac{2 \epsilon}{i \delta} \right) + 2J, \\
m_3(\chi) = m_3 := \frac{2 \epsilon}{i \delta} + J, \\
m_4(\chi) = m_4 := \left( 2 + (\tilde{r}_2 - \tilde{r}_1) \frac{1}{i \delta} \right) I^{-1} + J^2 + \frac{4 \epsilon}{i \delta} J, \\
m_5(\chi) = m_5 := 2J.
\] (4.27)
Now we define the following quantities,
\[
c_{pp}(\sigma) := \frac{1}{\pi^{1/4} \sigma v} \left( \frac{8 N}{i \epsilon^2} + \frac{2 \epsilon}{i r_1} + \frac{8 N}{i \delta^2} + \frac{4 \tilde{h}}{T} \frac{4}{i \epsilon} \right) + \frac{4}{\pi^{1/4} i \delta}, \\
c_{ps}(\sigma) := \frac{1}{\pi^{1/4} \sigma v} \left( \frac{2 \epsilon}{T} \left( \frac{1}{i \epsilon} + \frac{1}{\tilde{r}_1} \right) + \left( \frac{2 \tilde{h}}{T} \right)^2 \right), \\
c_{sp}(\sigma) := \frac{1}{\pi^{1/4} \sigma v} \left( \frac{8 \epsilon}{i \epsilon} + \frac{4 \epsilon}{i \delta} \right), \\
c_{ss}(\sigma) := \frac{1}{\pi^{1/4} \sigma v} \frac{4 \tilde{h}}{T}, \\
R(\zeta, Z) = R(\zeta) := \frac{m_5}{2} \frac{(\sigma^4 m^2 v^2 + \zeta^2)^{1/2}}{\sigma v} - \frac{1}{2} \frac{(k - Z)^2 (\sigma v)^2}{\pi^{1/2} e} - \frac{1}{2} \frac{(k - Z)^2 (\sigma v)^2}{\pi^{1/2} e}. 
\] (4.28)
REMARK 4.1. For the field, the potential and cutoff function constructed in this section we have that,

\[ M_i \leq m_i, \quad i \in \{1, \cdots, 5\}, \]
\[ C_{pp}(\sigma) \leq c_{pp}(\sigma), \quad C_{ps}(\sigma) \leq c_{ps}(\sigma), \quad C_{ss}(\sigma) \leq c_{ss}(\sigma), \quad C_{sp}(\sigma) \leq c_{sp}(\sigma), \]
\[ R(\zeta, Z) \leq R(\zeta, Z). \]  

(4.31)

Proof: the Remark follows from explicit computation.

\[ \Box \]

We introduce some notation that we use below. We define the vectors \( \vec{A}_j(v, \vec{m}) = (A_j^1, A_j^{1/2}, A_j^0, A_j^{-1/2}, A_j^{-1}), \) for \( j \in \{-\infty, 0, \infty\} \):

\[ \vec{A}^{-\infty}(v, \vec{m}) = \vec{A}^{-\infty} := (mvr_1 \frac{A(m)}{2} + mvr_2 (\frac{\hat{h}}{m})^{1/2} \frac{1}{(1 - 5 \times 10^{-4})^{1/2}} \frac{A(m)}{2}, mvr_1 \frac{A(m)}{2}, 134.99 \hat{h} \frac{A(m)}{2}, 0), \]
\[ \vec{A}^{-0}(v, \vec{m}) = \vec{A}^{-0} := (A^{-\infty}_1, A^{-\infty}_{1/2}, A^{-\infty}_0 + 135.91 \hat{h} A(\vec{m})_4, A^{-\infty}_{-1/2} + 135.91 \hat{h} A(\vec{m})_5, 0), \]
\[ \vec{A}^{-\infty}(v, \vec{m}) = \vec{A}^{-\infty} := (3A^{-\infty}_1, 3A^{-\infty}_{1/2}, 3A^{-\infty}_0 + 138 \hat{h} A(\vec{m})_4, 3A^{-\infty}_{-1/2} + 138 \hat{h} A(\vec{m})_5, 0). \]

(4.32)

Finally, for \( j \in \{-\infty, 0, \infty\} \) we denote,

\[ A_j^j(\sigma, v, \vec{m}) = A_j^j(\sigma) := \sum_{i \in \{1, 1/2, 0, -1/2, -1\}} A_j^i \sigma^i. \]

(4.33)

5 Tonomura et al. Experiments. Continued

5.1 Experimental Data

We consider the 2 different magnets with their dimensions given in table I of [17]. We denote them by \( \{\hat{K}_j\}_{j \in \{1, 2}\}, \)

\[ \hat{K}_j := \{x = (x_1, x_2, x_3) : \hat{r}_{1,j} \leq \sqrt{x_1^2 + x_2^2} \leq \hat{r}_{2,j}, |x_3| \leq \hat{h}\}. \]

(5.1)

We use the notation

\[ \chi_j, \quad j \in \{1, 2\} \]

(5.2)

for the corresponding cutoff function constructed in Section 4.4.

The height \( \hat{h} \) is \( 10^{-6} \text{cm} \) for both magnets and

\[ \hat{r}_{1,1} = 1.5 \times 10^{-4} \text{cm}, \]
\[ \tilde{r}_{2,1} = 2.5 \times 10^{-4} \text{cm}, \]
\[ \tilde{r}_{1,2} = 1.75 \times 10^{-4} \text{cm}, \]
\[ \tilde{r}_{2,2} = 2.75 \times 10^{-4} \text{cm}. \]

In the Tonomura et al. experiments [26] the electron has an energy of 150 keV. In this experiments they consider impenetrable magnets as we do in this paper. In the experiments [25] they consider penetrable magnets and energies of 80 keV, 100 keV and 125 keV. Since our method applies also in the case of penetrable magnets, we will consider in our estimates below the two extreme energies and an intermediate energy, although the most important one is the one of 150 keV that is the one used for the case of impenetrable magnets. Thus we consider the following energies.

\[ E_1 = 150 \text{ keV}, \]
\[ E_2 = 100 \text{ keV}, \]
\[ E_3 = 80 \text{ keV}. \]

They used an electron wave packet that might be represented at the time of emission \(( t \to -\infty)\) by the gaussian wave function,

\[ \left( \frac{1}{\alpha_r^2 \pi} \right)^{1/4} \left( \frac{1}{\alpha_r^2 \pi} \right)^{2/4} e^{-\frac{i}{\hbar} H_0} e^{\frac{i}{\hbar} \mathbf{v} \cdot \mathbf{x}} e^{-\frac{x_1^2 + x_2^2}{2 \alpha_r^2}} e^{-\frac{x_3^2}{2 \alpha_z^2}}. \] \( (5.3) \)

The transverse variance of the wave function \( \alpha_r \) is several times the radius of the torus \((r_{2,j}, j = 1, 2)\), so the electron wave packet covers the magnet.

The part of the wave packet that goes through the hole of the torus has a different behavior than the one that goes outside the hole. There appears to be no interference between those two parts of the wave packet, because a clear figure of the shadow of magnet is formed behind the torus. This was pointed out by Tonomura et al. [25], [26]. We can, therefore, model only the part of the electron wave packet that goes through the hole of the magnet. Hence, we take the transverse variance \( \alpha_r \) smaller than the inner radius of the magnet. The anisotropy of the variance \((\alpha_z \neq \alpha_r)\) does not introduce new ideas to the analysis and all the proofs that we do assuming that \( \alpha_z = \alpha_r \) can be done in the same way if \( \alpha_z \neq \alpha_r \). We obtain similar results in both situations. Taking \( \alpha_z \neq \alpha_r \) complicates the notations and, therefore, for simplicity, we will assume that \( \alpha_z = \alpha_r = \sigma \). So, when emitted, the electron that goes through the hole is represented by,

\[ \psi_{\nu,0}(x,t) := \frac{1}{(\sigma^2 \pi)^{3/4}} e^{-\frac{i}{\hbar} H_0} e^{\frac{i}{\hbar} \mathbf{v} \cdot \mathbf{x}} e^{-\frac{x_1^2 + x_2^2}{\sigma^2}}, \] \( (5.4) \)
with the variance $\sigma$ smaller than the inner radius of the magnet.

The real electron wave packet, under the experiment conditions, that behaves as \[5.4\] when the time goes to $-\infty$ is given by the wave function (see \[1.9\]),

$$\psi_v(x, t) := e^{-i\frac{\hbar}{2} H} W \varphi_v = e^{-i\frac{\hbar}{2} H} W \varphi_v. \quad (5.5)$$

Remember that we take $v = (0, 0, v)$ and that $\zeta := vt$ is the classical position of the electron, in the vertical direction, at time $t$.

The energy for the free wave packet (or of the perturbed wave packet at $-\infty$) is given by

$$\langle \frac{1}{2} M P^2 \varphi_v, \varphi_v \rangle = \frac{1}{2} M v^2 + \frac{3}{4} \frac{\hbar^2}{M \sigma^2} \approx \frac{1}{2} M v^2. \quad (5.6)$$

When $\sigma$ is big ( $\sigma mv >> 1$ ) the second factor is much smaller than the first. If we take for example $\sigma mv \geq \sqrt{15}$ the second factor is less than $1/10$ times the first. Therefore, when $\sigma mv >> 1$, we can suppose that the energy is given by the classical energy, $\frac{1}{2} M v^2$. With this assumption we can calculate the velocities, and the velocities times $m$ corresponding to the energies $E_1, E_2, E_3$:

- $v_1 = 2.2971 \times 10^{10} \text{cm/s}, \quad mv_1 = 1.9842 \times 10^{10} \text{cm}^{-1}$,
- $v_2 = 1.8755 \times 10^{10} \text{cm/s}, \quad mv_2 = 1.6201 \times 10^{10} \text{cm}^{-1}$,
- $v_3 = 1.6775 \times 10^{10} \text{cm/s}, \quad mv_3 = 1.4491 \times 10^{10} \text{cm}^{-1}$.

For now on we suppose that the obstacle $\tilde{K}$ is either $\tilde{K}_1$ or $\tilde{K}_2$ and that the velocity $v$ is either $v_1, v_2$ or $v_3$.

### 5.2 Selection of the Parameters

We have obtained rigorous upper bounds for the difference between the exact solution to the Schrödinger equation and the Aharonov-Bohm Ansatz, and for the difference between the scattering operator and its high-velocity limit. These bounds hold for any choice of the parameters $S_1, \delta, \varepsilon, \delta$ and $\varepsilon$. We use this freedom to choose these parameters in a convenient way. From now on, we choose the parameter $S_1 > 0$ such that

$$r_1 \rho(S_1) = 1. \quad (5.7)$$

This choice is made to optimize the error bound in Theorem 3.1. This theorem was proven using Lemmata 11.3, 11.5. For example, for the convergence of the integral on the left-hand side of equation (11.10) we need the decay of $\rho(\sigma, z)$ for large $z$, but for $z$ small this factor is very large. For this reason we split this integral in two regions (where we use different estimates) introducing the parameter $S_1$. 

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Furthermore,

\[ \bar{\varepsilon} := \frac{\tilde{r}_2 - \tilde{r}_1}{200}, \]

\[ \bar{\delta} := \frac{\hbar}{100}, \]

\[ \delta := \max(10 \sigma, \tilde{h}), \]

\[ \varepsilon := \frac{\tilde{r}_1}{50}. \]

This selection was obtained using numerical estimates to optimize the error bound for the time evolution of the electron wave packet.

6 The Time Evolution of the Electron Wave Packet. Continued

**Lemma 6.1.** For the data used in the Tonomura et al. experiments, \( v \in \{v_1, v_2, v_3\} \) and \( \tilde{K} \in \{\tilde{K}_1, \tilde{K}_2\} \), suppose that \( \sigma \in [\frac{\tilde{r}_2}{mv}, \tilde{r}_1/2] \) and \( \zeta \in \mathbb{R} \). Then,

\[ e^{-\frac{1}{2(\sigma^2)}} A_m^{-\infty}(z(\sigma), \sigma) \leq e^{-\frac{34}{34} \frac{(\sigma mv)^2}{2}} A^{-\infty}(\sigma) + 10^{-420}, \]

\[ e^{-\frac{1}{2(\sigma^2)}} A_m^{0}(z(\sigma), \sigma) + \frac{1}{2} R(\zeta, z(\sigma)) \leq e^{-\frac{34}{34} \frac{(\sigma mv)^2}{2}} A^{0}(\sigma) + 10^{-420}, \]

\[ e^{-\frac{1}{2(\sigma^2)}} A_m^{-\infty}(z(\sigma), \sigma) + R(0, z(\sigma)) \leq e^{-\frac{34}{34} \frac{(\sigma mv)^2}{2}} A^{-\infty}(\sigma) + 10^{-420}. \]

**Proof:**

- First case, \( \sigma \in [\sigma_0, \frac{\tilde{r}_1}{2}] \).

As \( \tilde{\omega}(\sigma)^{-1} \leq \sqrt{\frac{34}{34} \sigma mv} \), we have that

\[ 1 \leq \frac{(\sigma mv)^2}{(\sigma mv)^2 - \tilde{\omega}(\sigma)^{-2}} \leq 34. \] (6.2)

For these values of \( \sigma \), \( \tilde{\omega}(\sigma)^{-1} = \sqrt{2000} \). Then, using (11.23) and the experimental values we get,

\[ 2.1023 \times 10^{-6} \leq z(\sigma) \leq 0.0673. \] (6.3)

We also have,

\[ .0042 \leq S_1 \leq 303.8306. \] (6.4)

Using (6.3) and (6.4) we get,

\[ (\hbar \nu \tilde{r}_1^2 \sigma^3 m^3 v^3)^{1/2} (\max(z(\sigma), S_1))^{-1/2} \leq 2.9127 \times 10^5, \] (6.5)
and
\[
\frac{(\sigma^4 m^2 v^2 + \zeta^2)^{1/2}}{\sigma m v} \leq (\sigma^2 + \frac{33 z(\sigma)^2}{34 \times 2000})^{1/2} \leq 0.0015. \tag{6.6}
\]

Now we note that (see the definition of $R(\zeta, z(\sigma))$ in (4.30)).

\[
R(\zeta, z(\sigma)) \leq m_5 \frac{(\sigma^4 m^2 v^2 + z(\sigma)^2)^{1/2}}{\sigma m v} \pi^{1/2} e^{-\frac{1}{2\pi^2\sigma^2}}, \quad R(0, z(\sigma)) \leq m_5 \pi^{1/2} \sigma e^{-\frac{(h-z(\sigma))^2}{2\pi^2}}. \tag{6.7}
\]

We bound the quantities $\tilde{A}^{-j}, j \in \{-\infty, 0, \infty\}$ uniformly for $\sigma \in [\sigma_0, \frac{\pi}{4}]$ and for the experimental energies and magnets, using (6.3, 6.4, 6.5, 6.7) and the smaller experimental values of $\tilde{r}_1, (\tilde{r}_2 - \tilde{r}_1), \tilde{h}$ and $m v$ to determine the components of $\tilde{m}$. We use the fact that, for the values of sigma that we consider, $e^{-\frac{1}{2\pi^2\sigma^2}} \leq e^{-1000}$ to obtain,

\[
e^{-\frac{1}{2\pi^2\sigma^2}} \tilde{A}^{-\infty}(z(\sigma), \sigma) \leq 10^{-420},
\]

\[
e^{-\frac{1}{2\pi^2\sigma^2}} \tilde{A}^0(z(\sigma), \sigma) + \frac{1}{2} R(\zeta, z(\sigma)) \leq 10^{-420},
\]

\[
e^{-\frac{1}{2\pi^2\sigma^2}} \tilde{A}^\infty(z(\sigma), \sigma) + R(0, z(\sigma)) \leq 10^{-420}. \tag{6.8}
\]

• Second case, $\sigma \in [\frac{4.5}{m v}, \sigma_0]$. For these values of $\sigma$, $\frac{(\sigma m v)^2}{(\sigma m v)^2 + \sigma^2} = 34$, then by (11.23), $34 \tilde{h} + \sqrt{34} \sqrt{33} \tilde{h} \leq z(\sigma) \leq 34 \tilde{h} + \sqrt{34} \sqrt{33} \tilde{h} - 33 \tilde{h}^2$ and by triangle inequality $z(\sigma) \leq 34h + \sqrt{33} \sigma^2 mv + 34h$ and then, we have that,

\[
134.99 \tilde{h} \leq z(\sigma) \leq 136.82 \tilde{h}. \tag{6.9}
\]

It can be verified that,

\[
\max(z(\sigma), S_1) = S_1 \leq \sigma m v r_1,
\]

\[
\max(z(\sigma), S_1)^{-1/2}(hr_2^2 \sigma^2 m^2 v^3)^{1/2} \leq \sigma m v r_2 \frac{\tilde{h}}{r_1} \frac{1}{(1-5\times10^{-10})^{1/2}}, \tag{6.10}
\]

\[
\frac{(\sigma^4 m^2 v^2 + z(\sigma)^2)^{1/2}}{\sigma m v} \leq (1.11 \times 10^{-6} + 1)^{1/2} \frac{1}{136.82 \tilde{h}},
\]

where in the last inequality we used (6.9). Using (6.9) again we get

\[
R(0, z(\sigma)) \leq m_5 \pi^{1/2} e^{-\frac{1}{2\pi^2\sigma^2}} \leq 10^{-10^8}. \tag{6.11}
\]

Finally we obtain (6.1) using (2.19), (4.32), (6.7), (6.9), (6.10), (6.11) and the fact that $A_i(\tilde{m}) \leq A_i(\tilde{m})$ and $A_5(\tilde{m}) \leq A_3(\tilde{m})$ (note that in this case $e^{-\frac{1}{2\pi^2\sigma^2}} = e^{-\frac{33 \sigma m v}{\pi^2}}$).

**Remark 6.2.** For $j \in \{-\infty, 0, \infty\}$, $e^{-\frac{93 \sigma m v}{\pi^2}} A^j(\sigma)$ is decreasing on the interval $[\frac{4.5}{m v}, \infty]$. 

**Proof:** Calculating the numbers $A_i^j$ we find that $A_i^j \geq 0$ for $i \in \{1, 1/2, -1\}$, and also $A_0^{-1/2} \geq 0$. The other components of the vectors $A^j$ are negative. We suppose that $j \in \{-\infty, \infty\}$, the case $j = 0$ can be done in the
same way (the term $A_{0-1/2}$ is manipulated as the term $A_{0-1}$). Since $A^j(\sigma) \geq 0$ and $\sigma \log(10) \geq 4.5$, we have that

$$
\frac{d}{d\sigma} e^{-\frac{1}{2}(\sigma + 1)^2} A^j \leq e^{-\frac{1}{2}(\sigma + 1)^2} (b_1(\sigma) + b_2(\sigma)),
$$

where $b_1(\sigma) = \frac{4}{3} 4.5 \sigma \log(10) + A_{0-1/2}^j + A_{-1/2}^j(\sigma)^{-1/2} + \sum_{i \in \{1,2,0,0,0-1/2\}} i A_i^j(\sigma)^{-1} \geq 0$. As $b_1$ is increasing and $b_2$ decreasing, $-b_1 + b_2$ is decreasing, as $-b_1(\frac{4}{3} \sigma) + b_2(\frac{4}{3} \sigma) \leq 0$, we have that

$$
\frac{d}{d\sigma} e^{-\frac{1}{2}(\sigma + 1)^2} A^j \leq 0 \text{ for } \sigma \in \left[\frac{4}{3} \sigma, \sigma_0\right].
$$

Below we introduce a partition of an interval that is adapted to the order of magnitude.

**DEFINITION 6.3.** For any number $a > 0$ we designate by $O_a \in \mathbb{Z}$ the order of $a$, (i.e. $O_a$ is such that $10^{O_a} \leq a < 10^{O_a+1}$). For an interval $[a, b], a > 0$ and a positive number $N_0$ we define the partition $P(a, b, N_0) := \{p_i\}_{i=1}^k (p_i < p_{i+1} \forall i \in \{1, \ldots, k - 1\})$ as follows:

- case 1: $b \leq 10^{O_a+1}$. If $b - a \leq N_0 10^{O_a}$ we take $k = 2, p_1 = a, p_2 = b$. If $b - a > N_0 10^{O_a}$ we take $k \geq 3, p_1 = a, p_k = b$ and $p_i, i \in \{2, \ldots, k - 1\}$ such that $p_i < p_{i+1}, p_{i+1} - p_i = N_0 10^{O_a}$ for $i \in \{1, \ldots, k - 2\}$ and $p_k - p_{k-1} \leq N_0 10^{O_a}$.

- case 2: $b > 10^{O_a+1}$. For every $j \in \{0, \ldots, O_b - O_a\}$ we define a set $P^j$ as follows. We take $P^0$ as in the case 1 but taking $10^{O_b+1}$ instead of $b$. $P^{O_b-O_a}$ is taken as in the case 1 taking $10^{O_b}$ instead of $a$. If $O_b - O_a \geq 2$, for $j \in \{1, \ldots, O_b - O_a - 1\}$ we define $P^j$ as in the case 1 taking $10^{O_a+j}$ instead of $a$ and $10^{O_a+j+1}$ instead of $b$.

Now we define $P(a, b, N_0) = \cup_{j \in \{0, \ldots, O_b - O_a\}} P^j$.

**DEFINITION 6.4.** We denote by $\{\Sigma_j\}_{j=1}^{11}$ the following sets:

$$
\Sigma_1 := P(\frac{1}{\log(10)250}, \frac{1}{\log(10)1250}, 0.0003), \Sigma_2 := P(\frac{1}{\log(10)1250}, \frac{1}{\log(10)150}, 0.0005), \Sigma_3 := P(\frac{1}{\log(10)150}, 10^{-5}, 0.0008), \Sigma_4 := P(10^{-5}, 1.1 \times 10^{-5}, 0.0001), \Sigma_5 := P(1.1 \times 10^{-5}, 1.3 \times 10^{-5}, 0.0002), \Sigma_6 := P(1.3 \times 10^{-5}, 1.7 \times 10^{-5}, 0.0004), \Sigma_7 := P(1.7 \times 10^{-5}, 2 \times 10^{-5}, 0.0008), \Sigma_8 := P(2 \times 10^{-5}, \frac{3}{2}, 0.015), \Sigma_9 := P(10^{-6}, \frac{1}{\log(10)250}, 1000), \Sigma_{10} := P(\sigma_0, 10^{-6}, 1000), \Sigma_{11} := P(\frac{4}{3} \sigma, \sigma_0, 1).
$$

**LEMMA 6.5.** Suppose that the energies and magnets are the ones used on Tonomura et al. experiments. Let $\mu_i \in \mathbb{R}_+, i \in \{1, 2, 3\}$. Suppose that $\{\mu_i\}_{i=1}^3$ is contained in one of the sets $\Sigma_j$ for $j \in \{1, \ldots, 11\}$. We take $\mu_3 = 10^{-6}$ if $\{\mu_i\}_{i=1}^3$ is contained in $\Sigma_j$ for $j \in \{1, \ldots, 10\}$ and we take $\mu_3 = \sigma_0$ if $\{\mu_i\}_{i=1}^3$ is contained in the last set. We suppose furthermore, that $\mu_1$ and $\mu_2$ are consecutive numbers in the set in which they belong and $\mu_1 < \mu_2$. Then, for every $\sigma \in [\mu_1, \mu_2]$ and every $\zeta \in \mathbb{R}$ with $|\zeta| \leq \zeta(\sigma)$ we have that,

$$
c_{pp}(\sigma) I_{pp}(\mu_1, \mu_2, \mu_3) + c_{ps}(\sigma) I_{ps}(\mu_1, \mu_2, \mu_3, \zeta) + c_{sp}(\sigma) I_{sp}(\mu_1, \mu_2, \mu_3) + c_{ss}(\sigma) I_{ss}(\mu_1, \mu_2, \mu_3, \zeta) \leq
4e^{-\frac{\sigma}{2}} + 10^{-3} e^{-\frac{1}{2}(\sigma + 2)^2} A^0(\sigma) + 10^{-101},
$$

$$
c_{pp}(\sigma) I_{pp}(\mu_1, \mu_2, \mu_3) + c_{ps}(\sigma) I_{pp}(\mu_1, \mu_2, \mu_3) + c_{sp}(\sigma) I_{sp}(\mu_1, \mu_2, \mu_3) + c_{ss}(\sigma) I_{sp}(\mu_1, \mu_2, \mu_3) \leq
4e^{-\frac{\sigma}{2}} + 10^{-7} e^{-\frac{1}{2}(\sigma + 2)^2} A^\infty(\sigma) + 10^{-101},
$$

(6.12)
where the functions $I_{pp}, I_{ps}, I_{sp}$, and $I_{ss}$ are evaluated at $Z := z(\mu_j)$ if $\mu_j \leq \sigma_0$ and at $Z := \max_{j \in \{1,2\}} \{z(\mu_2), \mu_j (h(\mu_2))\}$, if $\mu_j \geq \sigma_0$.

Proof: We use a computer to calculate $r_1 \rho(\mu_i, Z)$ for $i \in \{1,2,3\}$ and we prove that these quantities are bigger than 1. As $z(\sigma) \leq Z$ (see (11.27), $r_1 \rho(\mu_i, Z) \geq 1$ for $i \in \{1,2,3\}$ implies that $|\zeta| \leq r_{\mu_1,\mu_2}$ and that $r_{\nu,\mu_3} \geq Z$, what simplifies $I_{ss}$ (see equation (11.35)). We estimate the integrals as it is shown in the appendix using a computer, taking $\delta_0 = 1$ if $\mu_1 m v > 10$ and $\delta_0 = \frac{1}{10}$ if $\mu_1 m v \leq 10$. We use the computer again to show that (6.12) is valid with $(4e^{-\frac{r_1^2}{2\pi} + 10^{-3} e^{-\frac{33 (\mu v m)^2}{2}} A_0^0(\mu_2))}$ instead of $(4e^{-\frac{r_1^2}{2\pi} + 10^{-3} e^{-\frac{33 (\mu v m)^2}{2}} A_0^0(\mu_2))}$. We use the computer to calculate $c_T(\mu_1)$ instead of $c_T(\mu_1)$ for $T \in \{pp, ps, sp, ss\}$. Finally by Remark 6.2 and the fact that $c_T(\mu_1), I_T(\mu_1, \mu_2, \mu_3, \zeta) \leq I_T(\mu_1, \mu_2, \mu_3, -Z), T \in \{pp, ps, sp, ss\}$ (see (11.27)), we obtain (6.12).

\[\square\]

6.1 The Incoming Electron Wave Packet. Continued

**THEOREM 6.6.** Suppose that the magnets and energies are the ones of the experiments of Tonomura et al.. Then for every gaussian wave function with variance $\sigma \in \left[\frac{1.5}{m v}, \frac{1.2}{2}\right]$ and every $\zeta \in \mathbb{R}$ with $|\zeta| \leq -z(\sigma)$ we have,

\[
\|e^{-\frac{i}{\hbar} H_{W_0}(A)} \varphi_N - \chi e^{-i \frac{\pi}{2} H_0} \varphi_N\| \leq e^{-\frac{33 (\mu v m)^2}{2}} \sum_{i \in \{1,1/2,0,-1/2,-1\}} A_i^{-\infty} \sigma^i + 10^{-420},
\]

where the quantities $A_i^{-\infty}$ are explicit numbers that depend only on the magnet and the energy that we take (see (4.32)).

Proof: Equation (6.13) is a consequence of Theorem 3.1, Remark 4.1 and Lemma 6.1.

6.2 The Interacting Electron Wave Packet. Continued

**THEOREM 6.7.** Suppose that the magnets and energies are the ones of the experiments of Tonomura et al.. Then for every gaussian wave function with variance $\sigma \in \left[\frac{1.5}{m v}, \frac{1.2}{2}\right]$ and every $\zeta \in \mathbb{R}$ with $|\zeta| \leq z(\sigma)$ we have,

\[
\|e^{-\frac{i}{\hbar} H_{W_0}(A)} \varphi_N - \chi e^{-i \frac{\pi}{2} H_0} \varphi_N\| \leq 4e^{-\frac{r_1^2}{2\pi} + 10^{-3} e^{-\frac{33 (\mu v m)^2}{2}}} \int_{\mathbb{R}} e^{-i \frac{\pi}{2} H_0} \varphi_N \|\leq 4e^{-\frac{r_1^2}{2\pi} + 10^{-3} e^{-\frac{33 (\mu v m)^2}{2}}} \sum_{i \in \{1,1/2,0,-1/2,-1\}} (1 + 10^{-3}) A_i^0 \sigma^i + 10^{-101} + 10^{-420},
\]

where the quantities $A_i^0$ are explicit numbers that depend only on the magnet and the energy that we take (see (4.32)).

Proof: Let $\sigma \in \left[\frac{1.5}{m v}, \frac{1.2}{2}\right]$, then there are $\mu_1, \mu_2$ and $\mu_3$ such that $\mu_1, \mu_2, \mu_3$ and $\sigma$ satisfies the hypothesis of Lemma 6.5. We prove using a computer that they satisfy also the hypothesis of the Theorem 3.5. We obtain (6.14) from Theorem 3.5, Remark 4.1 and Lemmata 6.1 6.5.

\[\square\]
6.3 Outgoing Electron Wave Packet and Scattering Operator. Continued

**THEOREM 6.8.** Suppose that the magnets and energies are the ones of the experiments of Tonomura et al.. Then, for every gaussian wave function with variance \( \sigma \in [\frac{4.5}{m\nu}, \frac{r}{2}] \) and every \( \zeta \in \mathbb{R} \) with \( \zeta \geq z(\sigma) \) we have,

\[
\|e^{-i\hat{\Phi}H_{\nu}}(A)\varphi_{\nu} - e^{i\Phi}\chi e^{-i\hat{\Phi}H_0}\varphi_{\nu}\| \leq 7e^{-r_1^2/4\sigma^2} + e^{-\frac{3\sigma m\nu^2}{2}} \sum_{i \in \{1,1/2,-1/2,-1\}} (1 + 10^{-7})A_1^\infty \sigma^i + 10^{-101} + 10^{-420},
\]

\[
\|S\varphi_{\nu} - e^{i\Phi}\chi\varphi_{\nu}\| \leq 7e^{-r_1^2/4\sigma^2} + e^{-\frac{3\sigma m\nu^2}{2}} \sum_{i \in \{1,1/2,-1/2,-1\}} (1 + 10^{-7})A_1^\infty \sigma^i + 10^{-101} + 10^{-420},
\]

where the quantities \( A_1^\infty \) are explicit numbers that depend only on the magnet and the energy that we take (see (4.32)).

**Proof:** Let \( \sigma \in [\frac{4.5}{m\nu}, \frac{r}{2}] \). Then, there are \( \mu_1, \mu_2 \) and \( \mu_3 \) such that \( \mu_1, \mu_2, \mu_3 \) and \( \sigma \) satisfies the hypothesis of Lemma 6.5. We prove using a computer that they satisfy also the hypothesis of the Theorem 3.8. We obtain (6.14) from Theorem 3.8, Remark 4.1 and Lemmata 6.1, 6.5. To get equation (6.16) we remember that to obtain the error bound in Theorem 3.8 we used the error bound for the scattering operator of Theorem 3.7. Then, the error bound that we get for the outgoing wave function in Theorem 3.8 bounds the error bound for the scattering operator.

7 Aharonov-Bohm Ansatz. Discontinuous Change of Gauge Formula from the Zero Vector Potential

In this section we denote by \( A \) the vector potential constructed in Section 4. We take also the parameters, magnets and energies introduced in Section 5.

7.1 Statement of the Aharonov-Bohm Ansatz

Let \( A_1 \) and \( A_2 \) be two differentiable magnetic potentials defined in \( \mathbb{R}^3 \setminus \tilde{K} \) with curl zero and that have the same flux \( \Phi \). Suppose, furthermore, that

\[
|A_i(x)| \leq C \frac{1}{1 + |x|}, \quad a_i(r) := \max_{x \in \mathbb{R} \setminus K, |x| \geq r} \{|A_i(x) \cdot \hat{x}|\} \in L^1(0, \infty).
\]

Choose any point \( x_0 \in \mathbb{R} \setminus \tilde{K} \). We define

\[
\lambda_{A_2, A_1}(x) := \int_{x_0}^x (A_2 - A_1),
\]

where the integral is over any curve in \( \mathbb{R}^3 \setminus \tilde{K} \) that connects \( x_0 \) with \( x \). This integral does not depends on the curve because both potentials have curl zero, and both have the same flux \( \Phi \). If this last condition is not true we can not define \( \lambda_{A_2, A_1} \). Then,
\[ A_2 = A_1 + \nabla \lambda_{A_2,A_1}. \]  

(7.3)

The solution to the Schrödinger equation with magnetic potential \( A_2 \) and initial condition given when the time is zero by the estate \( \psi \), is obtained in terms of the corresponding one for the magnetic potential \( A_1 \), by the change of gauge formula,

\[ e^{-i \frac{\tau}{\hbar} H(A_2)} \psi = e^{i \lambda_{A_2,A_1}} e^{-i \frac{\tau}{\hbar} H(A_1)} e^{-i \lambda_{A_2,A_1} \psi}. \]  

(7.4)

The solution to the Schrödinger equation for the vector potential \( A_1 \) that behaves as

\[ e^{-i \frac{\tau}{\hbar} H_0} \psi \]  

(7.5)

when the time goes to minus infinity is given by the formula (see equation 1.9),

\[ e^{-i \frac{\tau}{\hbar} H_0} W_-(A_1) \psi. \]  

(7.6)

In other words, (7.6) is the solution to the Schrödinger equation when the initial conditions are taken at time minus infinity by (7.5). Now we give the change of gauge formula for the Schrödinger equation with initial conditions taken at time minus infinity:

\[ e^{-i \frac{\tau}{\hbar} H(A_2)} W_-(A_2) \psi = e^{i \lambda_{A_2,A_1}(x)} e^{-i \frac{\tau}{\hbar} H(A_1)} W_-(A_1) e^{-i \lambda_{A_2,A_1,\infty}(\mathbf{p}) \psi}, \]  

(7.7)

where \( \lambda_{A_2,A_1,\infty}(x) := \lim_{r \to \infty} \lambda_{A_2,A_1}(rx) \). (see equation (5.8) in [3]).

Although the magnetic potential, \( A \), constructed in Section 4 has curl equal zero, it has non zero flux. Therefore, there is no change of gauge between the vector potential zero and \( A \). Suppose now that for every time the electron is practically localized in a region, \( \mathcal{D} \), that has no holes (that is simply connected) or, in other words, in a region where \( \lambda_{A,0} \) can be defined by equation (7.2) if we take curves that connects \( x_0 \) with \( x \) lying on this region. On this region \( A \) is gauge equivalent to the vector potential zero and the change of gauge formulae (7.4) should follow approximately (although not exactly, because there is not a real change of gauge between \( A \) and the zero potential). The error will depend on how much of the electron lies in the complement of \( \mathcal{D} \). This is the Ansatz of Aharonov and Bohm [2]. Let us be more specific. In our case we take,

\[ \mathcal{D} := (\mathbb{R}^3 \setminus \tilde{K}) \setminus \mathcal{S}, \]  

(7.8)

where

\[ \mathcal{S} := \{(x_1, x_2, 0) \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} > \tilde{r}_2\}. \]  

(7.9)

For two vector potentials \( A_1 \) and \( A_2 \) whose curl is zero (and that do not necessarily have the same flux) we define the function given in (7.2) in the simply connected region \( \mathcal{D} \): given \( x_0 = (x_{0,1}, x_{0,2}, x_{0,3}) \in \mathcal{D} \) with \( x_{0,3} < -\tilde{h} \) and \( x \)
in $D$ we define,

$$\lambda_{A_2, A_1}(x) := \int_{x_0}^{x} (A_2 - A_1),$$  \hspace{1cm} (7.10)

where the integral is over any curve in $D$ connecting $x_0$ with $x$. Note that for an electron to cross from the negative vertical axis to the positive one over $D$, it has to go through the hole of the magnet.

Then, we have that,

$$A_2(x) = A_1(x) + \nabla \lambda_{A_2, A_1}(x), \quad x \in D.$$  \hspace{1cm} (7.11)

We extend $\lambda_{A_2, A_1}$ to $\mathbb{R}^3 \setminus \tilde{K}$ by zero without changing notation, i.e., $\lambda_{A_2, A_1}(x) = 0$, for $x \in \mathcal{S}$. Note that $\lambda_{A_2, A_1}$ is discontinuous on $\mathcal{S}$.

The Ansatz of Aharonov and Bohm can be stated in the following way.

**DEFINITION 7.1. Aharonov-Bohm Ansatz with Initial Condition at Zero**

Let $A_1$ be a magnetic potential defined in $\mathbb{R}^3 \setminus \tilde{K}$ such curl $A_1 = 0$, and with flux not necessarily zero. Let $\psi$ the initial data at time zero of a solution to the Schrödinger equation that stays in $D$ for all times. Then, the change of gauge formula ([2], page 487),

$$e^{-i \frac{\hbar}{\epsilon} H(A_1)} \psi \approx \psi_{AB}(x,t) := e^{i \lambda_{A_1,0}(x)} e^{-i \frac{\hbar}{\epsilon} H_0} e^{-i \lambda_{A_1,0}(x)} \psi$$  \hspace{1cm} (7.12)

holds.

Note that if the initial state at $t = 0$ is taken as $e^{-i \lambda_{A_1,0}(x)} \psi$ the Aharonov-Bohm Ansatz is the multiplication of the free solution by the Dirac magnetic factor $e^{i \lambda_{A_1,0}(x)}$ [7].

Equation (7.12) is formulated when the initial conditions are taken at time zero. Now we reformulate it taking initial conditions when the time is minus infinity and for the high velocity state $\varphi_v$. For the high-velocity state $\varphi_v$ and for big $v$, we have that,

$$e^{-i \lambda_{A_2, A_1, \infty}(-p)} \varphi_v \approx e^{-i \lambda_{A_2, A_1, \infty}(-\psi)} \varphi_v.$$  \hspace{1cm} (7.13)

For this statement see the proof of Theorem 5.7 of [3]. Formula (7.11) with $W_-(0) = I$, and equation (7.13) suggest the following formulation of the Aharonov-Bohm Ansatz, with initial condition at time minus infinity and for high-velocity states.

**DEFINITION 7.2. Aharonov-Bohm Ansatz with Initial condition at $-\infty$. General Potentials**

Let $A_1$ be a magnetic potential defined in $\mathbb{R}^3 \setminus \tilde{K}$ such curl$A_1 = 0$, and with flux not necessarily zero. Let $\psi_v(A_1)(x,t)$,

$$\psi_v(A_1)(x, t) := e^{-i \frac{\hbar}{\epsilon} H(A_1)} W_-(A_1) \varphi_v$$

be the solution to the Schrödinger equation that behaves as

$$e^{-i \frac{\hbar}{\epsilon} H_0} \varphi_v$$  \hspace{1cm} (7.14)
when the time goes to minus infinity. We suppose that \( \psi_v(A_1)(x, t) \) is approximately localized in \( D \) for every time. Then, the following change of gauge formula follows,

\[
\psi_v(A_1)(x, t) \approx e^{i\lambda_{A,0}(x)}e^{-i\frac{H_0}{\hbar}t} e^{-i\lambda_{A,0,\infty}(\mathbf{v})} \varphi_v,
\]

where \( \lambda_{A,0,\infty}(x) = \lim_{r \to \infty} \lambda_{A,0}(rx) \).

\[\square\]

Let us show that formula (7.15) can formally be derived from (7.12). We take \( \psi = e^{i\lambda_{A,0}(x)}e^{-i\lambda_{A,0,\infty}(\mathbf{v})} \varphi_v \) in (7.12). Then, we have that

\[
e^{-i\frac{\mathbf{v}}{\hbar}H(A_1)} \psi \approx e^{i\lambda_{A,0}(x)}e^{-i\frac{H_0}{\hbar}t} e^{-i\lambda_{A,0,\infty}(\mathbf{v})} \varphi_v.
\]

For big velocities, the time evolution \( e^{-i\frac{\mathbf{v}}{\hbar}H_0} \varphi_v \) is localized near the classical position \( vt \). Therefore,

\[
e^{i\lambda_{A,0}(x)}e^{-i\frac{\mathbf{v}}{\hbar}H_0} e^{-i\lambda_{A,0,\infty}(\mathbf{v})} \varphi_v \approx e^{i\lambda_{A,0}(vt)} e^{-i\lambda_{A,0,\infty}(\mathbf{v})} \varphi_v,
\]

and thus, \( e^{-i\frac{\mathbf{v}}{\hbar}H(A_1)} \psi \) behaves as (7.14) when the time goes to minus infinity. Then,

\[
\psi_v(A_1)(x, t) \approx e^{-i\frac{\mathbf{v}}{\hbar}H(A_1)} \psi \approx e^{i\lambda_{A,0}(x)}e^{-i\frac{H_0}{\hbar}t} e^{-i\lambda_{A,0,\infty}(\mathbf{v})} \varphi_v
\]

and (7.15) follows.

For a general \( C^1 \) vector potential \( A_1 \) with curl equal zero and flux \( \Phi \), there is a real change of gauge (given by formula (7.2)) between this potential and the vector potential \( A \) with support in the convex hull of \( \tilde{K} \) constructed in Section 4. As the vector potentials \( A \) and \( A_1 \) are gauge equivalent, they define the same physics and, therefore, we can always choose the vector potential \( A \). For this potential, \( \lambda_{A,0,\infty}(\mathbf{v}) = 0 \), and then, the Aharonov-Bohm Ansatz for initial conditions at minus infinity and the potential \( A \) is as follows.

**DEFINITION 7.3. Aharonov-Bohm Ansatz**

Let \( A \) be the magnetic potential constructed in Section 4. Let \( \psi_v(x, t) := e^{-i\frac{\mathbf{v}}{\hbar}H(A)}W_-(A) \varphi_v \) be the solution to the Schrödinger equation that behaves like

\[
\psi_{v,0} := e^{-i\frac{\mathbf{v}}{\hbar}H_0} \varphi_v
\]

when time goes to minus infinity. We suppose that \( \psi_v \) is approximately localized in \( D \) for all times. Then, the following change of gauge formula holds,

\[
\psi_v \approx \psi_{AB,v}(x, t) := e^{i\lambda_{A,0}(x)}e^{-i\frac{\mathbf{v}}{\hbar}H_0} \varphi_v.
\]

\[\square\]

Observe that the Aharonov-Bohm Ansatz is the multiplication of the free solution by the Dirac magnetic factor \( e^{i\lambda_{A,0}(x)} \).
Note that as we noticed before, the electron - when emitted, would follow the free evolution $e^{-i\frac{\tau}{\hbar}H_0}\varphi_v$ under the assumption that we take a representation where the magnetic potential ($A$) vanishes at this time. If we take a representation given by a general vector potential ($A_1$) with flux $\Phi$, we should change the initial conditions at minus infinity by $e^{i\lambda A_{1,0}\infty(-\nabla)}e^{-i\frac{\tau}{\hbar}H_0}\varphi_v$ (notice that $\lambda_{A,0\infty}(-\nabla) = \lambda_{A_1\infty}(-\nabla)$).

In the following sections we give a rigorous proof that (7.17) holds and we obtain error bounds for the difference between the exact solution and the Aharonov-Bohm Ansatz. We also provide a physical interpretation of the error bound and we relate it to the probability for the electron to be outside the region $D$.

8 The Time Evolution of the electron Wave Packet. Final Estimates

In this Section we use the same symbol, $e^{-i\frac{\tau}{\hbar}H_0}$, for the restriction of the free evolution to $\Lambda$ and, moreover, we designate by $\| \cdot \|$ the norm in $L^2(\Lambda)$.

8.1 Incoming Electron Wave Packet. Final Estimates

**Lemma 8.1.** For every gaussian wave function, $\varphi$, with variance $\sigma$ and for every $\zeta \in \mathbb{R}$ with $\zeta \leq -z(\sigma)$, the following estimate holds.

$$\| \chi e^{-i\frac{\tau}{\hbar}H_0}\varphi_v - e^{i\lambda A_{0\infty}}e^{-i\frac{\tau}{\hbar}H_0}\varphi_v \| \leq \sqrt{2}e^{-\frac{33}{34}(m\sigma\zeta)^2} + 10^{-420}. \quad (8.1)$$

**Proof:** Let $\mathcal{D}_{-h}$ be the set $\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 \leq -h\}$. We have that, $\lambda_{A,0}(x) = 0$ and $\chi(x) = 1$ for $x \in \mathcal{D}_{-h}$. Using polar coordinates we obtain (see (3.14), (11.3) and Remark 11.1).

$$\| \chi e^{-i\frac{\tau}{\hbar}H_0}\varphi_v - e^{i\lambda A_{0\infty}}e^{-i\frac{\tau}{\hbar}H_0}\varphi_v \|^2 \leq \frac{4}{\pi^{3/2}} \int_{(\mathbb{R}^3 \setminus \mathcal{D}_{-h} - \varphi_{\zeta})\rho(\sigma,\zeta)} e^{-x^2} dx \leq 2e^{-\theta_{inv}(\sigma,z(\sigma))^2}. \quad (8.2)$$

Finally we notice that $\sqrt{2}e^{-\theta_{inv}(\sigma,z(\sigma))^2} \leq 10^{-434}$ for $\sigma \geq \sigma_0$.

□

Using Theorem 6.6 and Lemma 8.1 we prove that,

**Theorem 8.2.** Aharonov-Bohm Ansatz. Incoming Wave Packet

Suppose that the magnets and energies are the ones of the experiments of Tonomura et al.. Then for every gaussian wave function with variance $\sigma \in \left[\frac{4 \pi^2}{m^2 e^2}, \frac{\epsilon^2}{4}\right]$ and every $\zeta \in \mathbb{R}$ with $\zeta \leq -z(\sigma)$, the solution to the Schrödinger equation that behaves as (7.16) when the time goes to minus infinity, $e^{-i\frac{\tau}{\hbar}H_w(A)}\varphi_v$, is given at the time $t = \frac{\zeta}{v}$ ($\zeta$ being the vertical coordinate) by the Aharonov-Bohm Ansatz,

$$e^{i\lambda A_{0\infty}}e^{-i\frac{\tau}{\hbar}H_0}\varphi_v, \quad (8.3)$$

up to an error bound of the form:
\[
\|e^{-i\frac{\sigma}{2} H} W_\omega (A) \varphi_\omega - e^{i\lambda A} e^{-i\frac{\sigma}{2} H_0} \varphi_\omega \| \leq e^{-\frac{33}{34} (\text{sn}v)^2} (\sum |\epsilon| \in \{1,1/2,0,-1/2,-1\} \tilde{A}_\epsilon^{-\infty} \sigma^1 + \sqrt{2}) + 10^{-419},
\]
(8.4)
where the quantities \( \tilde{A}_\epsilon^{-\infty} \) are explicit numbers that depend only on the magnet and the energy that we take (see \( \frac{33}{34} \)).

### 8.2 Interacting Electron Wave Packet. Final Estimates

**LEMMA 8.3.** For every gaussian wave function, \( \varphi \), with variance \( \sigma \in [4.5/mv, \tilde{r}_1/2] \) and for every \( \zeta \in \mathbb{R} \) with \( |\zeta| \leq z(\sigma) \), the following estimate holds.

\[
\|e^{-i\int_0^{-\infty} \varphi A(x+\tau \tilde{v}) \, d\tau - i\frac{\sigma}{2} H_0} \varphi - e^{i\lambda A} e^{-i\frac{\sigma}{2} H_0} \varphi \| \leq 2e^{-\frac{1}{4} r_1^2 \rho(\sigma, \zeta)^2} \leq 2.0031 e^{-\frac{1}{4} r_1^2} + 2e^{-\frac{33}{34} (\text{sn}v)^2} + 10^{-456}.
\]
(8.5)

**Proof:** We denote by \( \mathcal{H} M := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} \leq r_1 \} \). For \( x \in \mathcal{H} M, -\int_0^{-\infty} \tilde{v} \cdot A(x+\tau \tilde{v}) \, d\tau = \lambda_A(x) \) and \( \chi(x) = 1 \). Using polar coordinates we obtain (see \( 3.14, 11.3 \)),

\[
\|e^{-i\int_0^{-\infty} \varphi A(x+\tau \tilde{v}) \, d\tau - i\frac{\sigma}{2} H_0} \varphi - e^{i\lambda A} e^{-i\frac{\sigma}{2} H_0} \varphi \| \leq \frac{4}{\pi^{3/2}} \int_{(\mathbb{R}^2 \backslash \mathcal{H} M - \tilde{v} \zeta) \rho(\sigma, \zeta)} e^{-x^2} \, dx \leq 4e^{-r_1^2 \rho(\sigma, \zeta)^2}.
\]
(8.6)

The second inequality in (8.5) is proved in three cases:

- \( \sigma \in [\frac{4.5}{mv}, \sigma_0] \).

By \( 6.9 \), see also Sections \( 2 \) and \( 5.2 \),

\[
e^{-r_1^2 \rho(\sigma, \zeta)^2} \leq e^{-\frac{r_1^2}{1 (\text{sn}v)^2} \rho(\sigma, \zeta)^2} \leq e^{-\frac{r_1^2}{1 (\text{sn}v)^2} \frac{33}{34} (\text{sn}v)^2} \leq e^{-\frac{33}{34} (\text{sn}v)^2}.
\]
(8.7)

- \( \sigma \in [\sigma_0, 3.2 \times 10^{-6}] \). For these values of \( \sigma \) we have that \( \tilde{w}(\sigma) = 2000^{-1/2} \). We use \( 6.2, 11.23 \) and the triangle inequality for the square-root term to obtain,

\[
\frac{z(\sigma)}{\sigma^2 mv} \leq \frac{1}{mv} \left( \frac{68h}{\sigma^2} + \frac{\sqrt{2000 \times 34}}{\sigma} \right).
\]
(8.8)

Then,

\[
e^{-\frac{1}{4} \rho(\sigma, \zeta)^2 r_1^2} \leq \exp \left[ -\frac{r_1^2}{2 \sigma^2} \frac{1}{1 \left( \frac{68h}{\sigma^2} + \frac{\sqrt{2000 \times 34}}{\sigma} \right)^2} \right] = \exp \left[ -\frac{r_1^2}{2 \sigma^2} + \frac{1}{(mv)^2} \left( \frac{68h}{\sigma} + \sqrt{2000 \times 34} \right)^2 \right].
\]
(8.9)

The function \( f(\sigma) = \frac{1}{\sigma^2 + \frac{1}{(mv)^2} \left( \frac{68h}{\sigma} + \sqrt{2000 \times 34} \right)^2} \) restricted to the interval \([\sigma_0, 10^{-7}]\) has derivative equal to zero on the positive axis only at the unique point of intersection of the function \( \sigma^4 \) and the line \( \frac{68h}{(mv)^2} (68h + \sqrt{2000 \times 34}) \), see Sections \( 2 \) and \( 5.2 \). For the interval \([10^{-7}, 3.2 \times 10^{-6}]\) the derivative of \( f \) is zero.
over the positive axis in the unique solution of the equation \( \sigma^4 = \frac{68\hbar}{(mv)^2} (68\hat{\nu} + (\sqrt{2000 \times 34} + 680)\sigma) \), see Sections 2 and 5.2. Then, it follows that,

\[
\exp \left[ -\frac{r_1^2}{2} \frac{2}{\sigma^2}, \frac{1}{(mv)^2}, \frac{1}{2\pi} \frac{\hbar}{\sigma^2}, \sqrt{2000 \times 34} \right] \leq \max_{\nu \in \{\sigma_0, 10^{-7}, 3.2 \times 10^{-6}\}} \exp \left[ -\frac{r_1^2}{2} \nu^2 + \frac{1}{(mv)^2} \frac{1}{2\pi} \frac{\hbar}{\sigma^2} + \sqrt{2000 \times 34} \right].
\]  

(8.10)

Evaluating (8.10) using the experimental energies and magnets, we find that,

\[
e^{-\frac{1}{2} \rho(\sigma, \zeta)^2} \leq 10^{-458}.
\]  

(8.11)

- \( \sigma \in [3.2 \times 10^{-6}, \frac{r_1}{2}] \).

Now we use that

\[
e^{-\frac{1}{2} \rho(\sigma, \zeta)^2} \leq e^{-\frac{1}{2} \left( \frac{\rho(\sigma, \zeta)^2}{\sigma^2} \right)} e^{-\frac{1}{2} \left( \frac{\rho(\sigma, \zeta)^2}{\sigma^2} \right)} = e^{-\frac{1}{2} \left( \frac{\rho(\sigma, \zeta)^2}{\sigma^2} \right) + \left( \frac{\rho(\sigma, \zeta)^2}{\sigma^2} \right)}.
\]  

(8.12)

By (11.23), \( \frac{z(\sigma)}{\sigma^2} \) is decreasing as a function of \( \sigma \) (see Sections 2 and 5.2) and notice that \( \frac{\sigma(v)^2}{(\sigma v)^2} \) is decreasing on \( \sigma \) and then, we have that,

\[
\sqrt{2} \exp \left[ \frac{r_1^2}{2\sigma^2} \frac{(\frac{z(\sigma)}{\sigma^2})^2}{1 + (\frac{z(\sigma)}{\sigma^2})^2} \right] \leq \sqrt{2} \exp \left[ \frac{r_1^2}{2(3.2 \times 10^{-6})^2} \frac{(\frac{z(3.2 \times 10^{-6})}{(3.2 \times 10^{-6})^2})^2}{1 + (\frac{z(3.2 \times 10^{-6})}{(3.2 \times 10^{-6})^2})^2} \right] \leq 1.4171.
\]  

(8.13)

**REMARK 8.4.** The term appearing in the middle inequality of equation (8.5) is two times the square root of the probability for the free particle to be outside the hole of the magnet (\( \mathcal{H} \)) when the electron is classically at the position \((0, 0, \zeta)\):

\[
\int_{\mathbb{R}^3 \setminus \mathcal{H}} |(e^{-iH_\sigma^\nu} \chi(x))|^2 dx = e^{-\frac{1}{2} \rho(\sigma, \zeta)^2}.
\]  

(8.14)

Recall that \( \mathcal{H} \) is defined in the proof of Lemma 8.3. Equation (8.14) is a measure of the part of the electron that hits the magnet when the classical electron (the electron under classical mechanics rules) lies within a distance less than \( z(\sigma) \) from the center of the magnet. By the second inequality in (8.5) we can see that the probability of the electron to be outside the hole of the magnet at time \( \zeta/v \) splits in two terms: one, \( e^{-\frac{1}{2} \rho(\sigma, \zeta)^2} \), is due to the probability of the free electron to be outside the hole when \( \zeta = 0 \) (see formula (8.14)). This factor provides us an idea of the influence of the magnet over the electron given by the size of the wave packet (i.e., how much does the electron hits the magnet - see Section 8.4), and the other, \( e^{-\frac{z(\sigma)}{\sigma^2} \frac{m^2}{2} \zeta} \), is related with the spreading of the electron as time increases - see Section 9.5.

This factor is important when \( \sigma \) is small, because by Heisenberg uncertainly principle when the electron is localized in a small region, its momentum is not localized and therefore the electron spreads. Those two factors are essentially the causes of all the error bounds that we have in this paper. The error bounds are mainly produced by the probability of
the electron to hit the magnet when it is classically at the position \((0, 0, \zeta)\), with \(|\zeta| \leq z(\sigma)\). In Section 9 we provide an analysis of these terms and we give precise definitions of the size of the electron wave packet and of the opening angle, that is due to the spreading.

Using Theorem 6.7 and Lemma 8.3 we prove,

**THEOREM 8.5.** Aharonov-Bohm Ansatz. Interacting Electron Wave Packet

Suppose that the magnets and energies are the ones of the experiments of Tonomura et al.. Then, for every gaussian wave function with variance \(\sigma \in [\frac{4.5}{mv}, \frac{7}{2}]\) and every \(\zeta \in \mathbb{R}\) with \(|\zeta| \leq z(\sigma)\) the solution to the Schrödinger equation, \(e^{-i\hat{H}_{-}(A)}\varphi_{v}\), that behaves as \((7.16)\) when time goes to minus infinity is given at the time \(t = \frac{\zeta}{v}\) (\(\zeta\) being the vertical coordinate) by the Aharonov-Bohm Ansatz,

\[
e^{i\lambda_{A,0}}e^{-i\frac{\zeta}{v}H_{0}\varphi_{v}},
\]

up to an error bound of the form:

\[
\|e^{-i\hat{H}_{-}(A)}\varphi_{v} - e^{i\lambda_{A,0}}e^{-i\frac{\zeta}{v}H_{0}\varphi_{v}}\| \leq 6.0031e^{-\frac{r_{2}^{2}}{2\sigma^{2}}} + e^{-\frac{33}{34}(\frac{mv}{\sigma})^{2}}\left(\sum_{i \in \{1,1/2,0,-1/2,-1\}}(1 + 10^{-3})A_{0}^{i}\sigma^{2} + 2\right) + 10^{-101} + 10^{-420} + 10^{-456},
\]

where the quantities \(A_{0}^{i}\) are explicit numbers that depend only on the magnet and the energy that we take (see \((4.32)\)).

**8.3 Outgoing Electron Wave Packet and Scattering Operator. Final Estimates**

**LEMMA 8.6.** For every gaussian wave function, \(\varphi\), with variance \(\sigma\) and for every \(\zeta \in \mathbb{R}\) with \(\zeta \geq z(\sigma)\), the following estimate holds.

\[
\|\chi e^{i\Phi}e^{-i\frac{\zeta}{v}H_{0}\varphi_{v}} - e^{i\lambda_{A,0}}e^{-i\frac{\zeta}{v}H_{0}\varphi_{v}}\| \leq \sqrt{2}e^{-\frac{33}{34}(\frac{mv}{\sigma})^{2}} + 10^{-420}.
\]

**Proof:** Let \(D_{h}\) be the set \(\{(x_{1}, x_{2}, x_{3}) \in \mathbb{R}^{3} : x_{3} \geq h\}\), note that \(\lambda_{A,0}(x) = \Phi\) and \(\chi(x) = 1\) for \(x \in D_{h}\). The proof follows in the same way as the proof of Lemma 8.1.

Theorem 6.8 and Lemma 8.6 imply the following theorem.

**THEOREM 8.7.** Aharonov-Bohm Ansatz. Outgoing Electron Wave Packet

Suppose that the magnets and energies are the ones of the experiments of Tonomura et al.. Then for every gaussian wave function with variance \(\sigma \in [\frac{4.5}{mv}, \frac{7}{2}]\) and every \(\zeta \in \mathbb{R}\) with \(\zeta \geq z(\sigma)\) the solution to the Schrödinger equation, \(e^{-i\hat{H}_{-}(A)}\varphi_{v}\), that behaves as \((7.16)\) when time goes to minus infinity is given at the time \(t = \frac{\zeta}{v}\) (\(\zeta\) being the vertical coordinate) by the Aharonov-Bohm Ansatz,

\[
e^{i\lambda_{A,0}}e^{-i\frac{\zeta}{v}H_{0}\varphi_{v}},
\]
up to an error bound of the form:

$$
\|e^{-i \frac{\zeta}{\hbar} H W_-(A) \varphi} - e^{i \lambda A_0} e^{-i \frac{\zeta}{\hbar} H_0 \varphi}\| \leq 7 e^{-\frac{\zeta^2}{2 \sigma^2}} + e^{-\frac{33 (\sigma \sigma_{mv})^2}{2}} \left( \sum_{i \in \{1,1/2,0,-1/2,-1\}} (1 + 10^{-7}) \mathcal{A}^\infty_i \sigma^i + \sqrt{2} \right) + 10^{-101} + 2 \times 10^{-420},
$$

and, furthermore, the scattering operator satisfies,

$$
\|S(A) \varphi - e^{i \Phi} \varphi\| \leq 7 e^{-\frac{\zeta^2}{2 \sigma^2}} + e^{-\frac{33 (\sigma \sigma_{mv})^2}{2}} \left( \sum_{i \in \{1,1/2,0,-1/2,-1\}} (1 + 10^{-7}) \mathcal{A}^\infty_i \sigma^i + \sqrt{2} \right) + 10^{-101} + 2 \times 10^{-420},
$$

where the quantities $\mathcal{A}^\infty_i$ are explicit numbers that depend only on the magnet and the energy that we take (see (4.32)).

8.4 Uniform in Time Estimates for the Electron Wave Packet

**REMARK 8.8.** The error bound of Theorem 8.2 is smaller that the one of Theorem 8.5 and this last one is bounded by the error bound of Theorem 8.7. This is physically reasonable, because for an electron to be an interacting electron, it has to be first incoming electron and for an electron to be outgoing electron it has to be before an interacting electron, so the error should be accumulative. Let us prove this. That the error of Theorem 8.2 is smaller than the one of Theorem 8.5 follows directly from the definitions (4.32). To prove that the error in Theorem 8.7 bounds the one of Theorem 8.5 we use again (4.32) and that (remember that $\sigma_{mv} \geq 4.5$),

$$
(1 + 10^{-3}) \mathcal{A}_0^0 + (2 - \sqrt{2}) = (1 + 10^{-3}) \left[ \frac{\sqrt{\pi} m_5}{2} (1 + 1.11 \times 10^{-6})^{1/2} \frac{\hbar}{\sigma_{mv}} \right] + (2 - \sqrt{2}) \leq 10^{-3} (150 (1 - \frac{1}{50}) \sigma_{mv} - 134.99) \frac{\hbar}{2} \frac{\sqrt{\pi}}{\sqrt{\sigma_{mv}} m_5} \leq (2 - 10^{-3}) (\sigma^{1/2} \nu r_1 - \sigma^{-1/2} 134.99 \hbar) \mathcal{A}_3^0 (m) \leq (2 - 10^{-3}) (\sigma^{1/2} A_{-1/2}^\infty + \sigma^{-1/2} A_{-1/2}^\infty) \tag{8.21}
$$

This gives us the following theorem.

**THEOREM 8.9.** Aharonov-Bohm Ansatz. Time-Uniform Estimates

Suppose that the magnets and energies are the ones of the experiments of Tonomura et al. Then, for every gaussian wave function with variance $\sigma \in \left[ \frac{4.5}{\sigma_{mv}}, \frac{3\pi}{2} \right]$ and every $\zeta \in \mathbb{R}$ the solution to the Schrödinger equation, $e^{-i \frac{\zeta}{\hbar} H W_-(A) \varphi}$, that behaves as (7.16) when the time goes to minus infinity is given at the time $t = \frac{\zeta}{\pi} \ (\zeta$ being the vertical coordinate) by the Aharonov-Bohm Ansatz,

$$
e^{i \lambda A_0(x)} e^{-i \frac{\zeta}{\hbar} H_0 \varphi}, \tag{8.22}
$$

up to an error bound of the form:

$$
\|e^{-i \frac{\zeta}{\hbar} H W_-(A) \varphi} - e^{i \lambda A_0} e^{-i \frac{\zeta}{\hbar} H_0 \varphi}\| \leq 7 e^{-\frac{\zeta^2}{2 \sigma^2}} + e^{-\frac{33 (\sigma \sigma_{mv})^2}{2}} \left( \sum_{i \in \{1,1/2,0,-1/2,-1\}} (1 + 10^{-7}) \mathcal{A}^\infty_i \sigma^i + \sqrt{2} \right) + 10^{-101} + 2 \times 10^{-420}. \tag{8.23}
$$
Moreover, the scattering operator satisfies,
\[ \|S\varphi - e^{i\Phi}\varphi\| \leq 7e^{-\frac{r^2}{2\sigma^2}} + e^{-\frac{33}{2}(\sigma mv)^2} \left( \sum_{i \in \{1, 1/2, -1/2, -1\}} (1 + 10^{-7})A_i^\infty\sigma^i + \sqrt{2} \right) + 10^{-101} + 2 \times 10^{-420}. \quad (8.24) \]

The quantities \( A_i^\infty \) are explicit numbers that depend only on the magnet and the energy that we take (see \( \text{4.32} \)).

By \( \text{4.29} \) and \( \text{5.2} \), \( m_i(\chi_1) \geq m_i(\chi_2), i \in \{1, \cdots, 5\} \), and as \( \sigma^{1/2}mv_1 \geq \frac{134.99\hbar}{\sigma^{1/2}} \) (\( \sigma mv_1 \geq 134.99\hbar \), remember that \( \sigma mv \geq 4.5 \)), we have that \( A^j(\sigma, v, \bar{m}(\chi_1)) \geq A^j(\sigma, v, \bar{m}(\chi_2)) \) (see \( \text{4.33} \)). We have also (see \( \text{4.32} \) and \( \text{4.33} \)) that \( A^\infty(\sigma, v_i, \bar{m}(\chi_1)) \leq A^\infty(\sigma, v_i, \bar{m}(\chi_2)) \) for \( i \in \{1, 2, 3\} \) and \( j \in \{ 1, 2 \} \) (notice that \( A(\bar{m})_1 \geq A(\bar{m})_4 \) and \( A(\bar{m})_3 = A(\bar{m})_5 \)). So if we write \( A^\infty_i(v_i, \bar{m}(\chi_1)) \) in \( \text{8.23} \) instead of \( A^\infty_i \) we obtain also error bounds, but now the coefficients \( A^\infty_i \) are fixed for all the magnets and velocities. Taking this into consideration we calculate the values of \( A^\infty_i(v_i, \bar{m}(\chi_1)) \) and we obtain the following theorem.

**THEOREM 8.10. Aharonov-Bohm Ansatz and Tonomura et al. Experiments**

Suppose that the magnets are the ones of the experiments of Tonomura et al. Then, for every gaussian wave function with variance \( \sigma \in \left[ \frac{4.5}{mv}, \frac{5}{2} \right] \) and every \( \zeta \in \mathbb{R} \), the solution to the Schrödinger equation, \( e^{-i\hat{\Phi}^H}W_-(A)\varphi, \) that behaves as \( \text{7.16} \) when the time goes to minus infinity is given at the time \( t = \frac{\zeta}{v} \) (\( \zeta \) being the vertical coordinate) by the Aharonov-Bohm Ansatz,
\[ e^{i\lambda_{A,0}(x)}e^{-i\frac{\Phi}{\hbar}}\varphi, \quad (8.25) \]
up to an error bound of the form:
\[ \|e^{-i\frac{\Phi}{\hbar}}W_-(A)\varphi - e^{i\lambda_{A,0}}e^{-i\frac{\Phi}{\hbar}}\varphi\| \leq 7e^{-\frac{r^2}{2\sigma^2}} + e^{-\frac{33}{2}(\sigma mv)^2} \left( 1.04 \times 10^{14}\sigma + 3.91 \times 10^8\sigma^{1/2} - 1.41 \times 10^3 - 1.14 \times 10^{-2}\frac{1}{\sigma^{1/2}} \right) + 10^{-101} + 2 \times 10^{-420}. \quad (8.26) \]

Furthermore, the scattering operator satisfies,
\[ \|S\varphi - e^{i\Phi}\varphi\| \leq 7e^{-\frac{r^2}{2\sigma^2}} + e^{-\frac{33}{2}(\sigma mv)^2} \left( 1.04 \times 10^{14}\sigma + 3.91 \times 10^8\sigma^{1/2} - 1.41 \times 10^3 - 1.14 \times 10^{-2}\frac{1}{\sigma^{1/2}} \right) + 10^{-101} + 2 \times 10^{-420}. \quad (8.27) \]

We now bound the right hand side of \( 8.26 \) by \( 7e^{-\frac{r^2}{2\sigma^2}} + \mathcal{F}(\sigma, mv) \), where \( \mathcal{F}(\sigma, mv) := e^{-\frac{33}{2}(\sigma mv)^2} \left( 1.04 \times 10^{14}\sigma + 3.91 \times 10^8\sigma^{1/2} + 10^{-101} + 2 \times 10^{-420} \right) \). We notice that \( \mathcal{F} \) is decreasing for \( mv \) fixed and \( \sigma mv \geq 4.5 \). We compute \( \mathcal{F}(15.5 \times 10^{-10}, mv_3) \) and show that this quantity is less than \( 10^{-100} \), it follows that \( \mathcal{F}(\sigma, mv) \leq 10^{-100} \) for \( \sigma \geq 15.5 \times 10^{-10} \) and the experimental velocities. Then \( \mathcal{F}(\sigma, mv) \leq e^{-\frac{33}{2}(\sigma mv)^2} \left( 1.04 \times 10^{14}(15.5 \times 10^{-10}) + 3.91 \times 10^8(15.5 \times 10^{-10})^{1/2} + 10^{-100} \right) = 177 \times 10^3 e^{-\frac{33}{2}(\sigma mv)^2} + 10^{-100} \). We obtain the following theorem, that is our main result, and that is quoted as **Theorem 1.1** in the introduction.
THEOREM 8.11. Aharonov-Bohm Ansatz and Tonomura et al. Experiments. Final Estimates

Suppose that the magnets and energies are the ones of the experiments of Tonomura et al. Then, for every gaussian wave function with variance $\sigma \in [\frac{4.5}{\hbar}, \frac{5}{\hbar}]$ and every $\zeta \in \mathbb{R}$, the solution to the Schrödinger equation, $e^{-i\frac{H}{\hbar}}W_-(A)\varphi_v$, that behaves as (7.16) when the time goes to minus infinity is given at the time $t = \frac{\zeta}{v}$ (\zeta being the vertical coordinate) by the Aharonov-Bohm Ansatz,

$$e^{i\lambda_{A,0}(x)}e^{-i\frac{H}{\hbar}t_0}\varphi_v,$$  \hspace{1cm} (8.28)

up to an error bound of the form:

$$\|e^{-i\frac{H}{\hbar}}W_-(A)\varphi_v - e^{i\lambda_{A,0}}e^{-i\frac{H}{\hbar}t_0}\varphi_v\| \leq 7e^{-\frac{\zeta^2}{2\sigma^2}} + 177 \times 10^3 e^{-\frac{33}{34}(\sigma\omega)^2} + 10^{-100}.$$  \hspace{1cm} (8.29)

Furthermore, the scattering operator satisfies,

$$\|S\varphi_v - e^{i\Phi}\varphi_v\| \leq 7e^{-\frac{\zeta^2}{2\sigma^2}} + 177 \times 10^3 e^{-\frac{33}{34}(\sigma\omega)^2} + 10^{-100}.$$  \hspace{1cm} (8.30)

REMARK 8.12. In the experiments of Tonomura et al. \cite{28}, they send an electron wave packet that partially hits the magnet. The part of the electron wave packet that hits the magnet does not go behind the magnet because we can see the black shadow of the magnet behind it. In other words, this part of the electron wave packet will be in the region \{(x_1, x_2, x_3) \in \Lambda : x_3 \leq h\}. We can bound, therefore, the probability of interaction of the electron with the magnet by the probability for the electron to not be behind the magnet for large time. We denote, as in the proof of Lemma \ref{8.6} by $D_h$ the set \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq h\}. Actually $D_h$ is the region behind the magnet. Then the probability of interaction of the electron with the magnet is bounded by,

$$\|\chi_{\Lambda \setminus D_h}e^{-i\frac{H}{\hbar}W_-(A)}\varphi_v\|^2$$  \hspace{1cm} (8.31)

when the time goes to $\infty$, where $\chi_{\Lambda \setminus D_h}$ is the characteristic function of the set $\Lambda \setminus D_h$. We take as before $\zeta = vt$, then we have,

$$\|\chi_{\Lambda \setminus D_h}e^{-i\frac{H}{\hbar}W_-(A)}\varphi_v\|^2 \leq (\|e^{-i\frac{H}{\hbar}W_-(A)}\varphi_v - e^{i\lambda_{A,0}}e^{-i\frac{H}{\hbar}t_0}\varphi_v\| + \|\chi_{\Lambda \setminus D_h}e^{-i\frac{H}{\hbar}t_0}\varphi_v\|)^2.$$  \hspace{1cm} (8.32)

We take $\hat{\omega}(\sigma) := \frac{1}{\sqrt{\mathbb{H}\sigma\omega}}$ and $\hat{z}(\sigma) := z_0(\sigma)\sigma(\hat{h}(\sigma))$, see Section 2. Using polar coordinates we obtain for $\zeta \geq \hat{z}(\sigma)$ (see Section 2, (8.14), (11.3) and Remark 11.1),

$$\|\chi_{\Lambda \setminus D_h}e^{-i\frac{H}{\hbar}t_0}\varphi_v\| \leq \frac{1}{\pi^{3/2}} \int_{(3\mathbb{R}^3, -\times, \rho(\sigma, \zeta))} e^{-x^2} dx \leq \frac{1}{2} e^{-\frac{33}{34}(\sigma\omega)^2}.$$  \hspace{1cm} (8.33)

Letting the time go to $\infty$ in (8.31) and using Theorem 8.11, (8.32) and (8.33) we obtain that the probability of interaction of the electron with the magnet is bounded by,

$$(7e^{-\frac{\zeta^2}{2\sigma^2}} + 177001e^{-\frac{33}{34}(\sigma\omega)^2} + 10^{-100})^2.$$  \hspace{1cm} (8.34)
9 Physical Interpretation of the Error Bounds

We analyze the error bounds given in equations (8.29, 8.30). The error bounds appearing in the whole paper are produced by the same factors. Equations (8.29, 8.30) provide uniform in time error bounds that apply to all experimental magnets and energies. The behaviour of the error bound is the same for the three energies and the two magnets, so there is no loss of generality if we select a magnet and an energy in our analysis. We will use the biggest energy \((E_1)\) and the second magnet \((K_2)\) to provide numbers and graphics. So, for now on we take the magnet \(K_2\) and the energy \(E_1\).

The main factors that produce the error bound in equations (8.29, 8.30) are the terms,

1. Size of electron wave packet factor.

\[
e^{-\frac{r_1^2}{2\sigma^2}}.
\] (9.1)

2. Opening angle of the electron wave packet factor.

\[
e^{-\frac{1}{32} (\sigma mv)^2}.
\] (9.2)

When the variance \(\sigma\) is close to the radius of the magnet, (9.1) is close to 1 and (9.2) is extremely small, because in this case \(\sigma mv\) is big. Then, when the electron is big compared to the inner radius, (9.1) is the important term, which justifies our name. When the variance is very small -such that \(\sigma mv\) is close to 1- the factor (9.2) is close to one and (9.1) is extremely small (\(\frac{r_1}{\sigma}\) is big), and then, the important factor is (9.2). But when the variance in position \((\sigma)\) is small, by Heisenberg uncertainly principle the variance in momentum is big, and then, the component of the momentum transversal to the axis of the magnet is large. In consequence, the opening angle of the electron wave packet is large, and the electron spreads fast as it propagates. This justifies the name of (9.2).

By the previous discussion, we divide the analysis of the error bounds in (8.29, 8.30) in three sections: big sigma \((\sigma\) close to the inner radius of the magnet), small sigma \((\sigma mv\) close to 1) and intermediate sigma (sigma neither big, nor small).

9.1 Big Sigma, \(\sigma \in \left[\frac{r_1}{22}, \frac{\tilde{r}_1}{2}\right]\)

Remember that \(r_1 = \tilde{r}_1 - \varepsilon\) and that \(\varepsilon\) is defined in Section 5.2. Here \(\tilde{r}_1 = 1.75 \times 10^{-4} cm\) (see Section 5.1). Then, in terms of absolute values, big sigma ranges over the interval \([7.7955 \times 10^{-6}, 8.7500 \times 10^{-5}]\). In Figure II we show the graphic of the error bound in (8.29) as a function of \(\frac{\sigma}{r_1}\), for big sigma, and in the table below we give some representative values.
Error Bound as a Function of Sigma Over \( r_1 \) for Big Sigma.

| Sigma Over \( r_1 \) | Error Bound |
|----------------------|-------------|
| .34305               | \( 10^{-3} \) |
| .27626               | \( 10^{-4} \) |
| .23764               | \( 10^{-5} \) |
| .21170               | \( 10^{-6} \) |
| .19274               | \( 10^{-7} \) |
| .17811               | \( 10^{-8} \) |
| .16637               | \( 10^{-9} \) |
| .15668               | \( 10^{-10} \) |
| .14851               | \( 10^{-11} \) |
| .14150               | \( 10^{-12} \) |

9.2 Intermediate Sigma, \( \sigma \in [6.7591 \times 10^{-6} r_1, \frac{r_1}{22}] \), or \( \sigma \in \left[ \frac{23}{mv}, \frac{154678}{mv} \right] \)

Remember that \( mv = 1.9842 \times 10^{10} \) (see Section 5.1). Therefore, in terms of absolute values, intermediate sigma ranges over the interval \([1.1592 \times 10^{-9}, 7.7955 \times 10^{-6}]\). For these values of sigma, \( \frac{r_1}{\sigma} \geq 22 \) and \( \sigma mv \geq 23 \), and therefore, the error bound in (8.29) is less than \( 10^{-99} \).

For intermediate sigma the probability of interaction of the electron with the magnet is less than \( 10^{-199} \) (see Remark 8.12).

9.3 Small Sigma, \( \sigma \in [1.3224 \times 10^{-6} r_1, 6.7591 \times 10^{-6} r_1] \), or \( \sigma \in \left[ \frac{45}{mv}, \frac{23}{mv} \right] \)

In terms of absolute values we have that \( \sigma \in [2.2679 \times 10^{-10}, 1.1592 \times 10^{-9}] \). In Figure 2 we show the graphic of the error bound in (8.29) as a function of \( \frac{r_1}{\sigma} \), for small sigma, and in the table below we give some representative values.

Error Bound as a Function of Sigma Over \( r_1 \) for Small Sigma.

| Sigma Over \( r_1 \) | Error Bound |
|----------------------|-------------|
| \( 1.6001 \times 10^{-6} \) | \( 10^{-4} \) |
| \( 1.7234 \times 10^{-6} \) | \( 10^{-4} \) |
| \( 1.8384 \times 10^{-6} \) | \( 10^{-5} \) |
| \( 1.9467 \times 10^{-6} \) | \( 10^{-5} \) |
| \( 2.0492 \times 10^{-6} \) | \( 10^{-6} \) |
| \( 2.1469 \times 10^{-6} \) | \( 10^{-7} \) |
| \( 2.2403 \times 10^{-6} \) | \( 10^{-8} \) |
| \( 2.3299 \times 10^{-6} \) | \( 10^{-9} \) |
| \( 2.4162 \times 10^{-6} \) | \( 10^{-10} \) |
| \( 2.4996 \times 10^{-6} \) | \( 10^{-10} \) |

9.4 The Radius of the Electron Wave Packet

As before, we denote by \( \mathcal{HM} \) the cylinder \( \{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} \leq r_1 \} \). \( \mathcal{HM} \) is basically the hole of the magnet. The factor \( e^{-\frac{r_1^2}{2\sigma^2}} \) is practically the square root of the probability for the free particle at time zero to be outside the hole of the magnet.
\[ e^{-\frac{x^2}{2\sigma^2}} = \left\| \chi_{\mathcal{R} \setminus \mathcal{M}} \left( \frac{1}{\sigma \sqrt{2\pi}} \right)^{3/4} e^{-\frac{r^2}{2\sigma^2}} \right\|. \]  

(9.3)

This factor represents the part of the electron wave packet that hits the magnet or goes outside (the square root appears because our estimations are in norm and not in probability). It is natural to have this factor in the error bound because we are only modeling the particles that go through the hole. This factor is significant only when the variance is close to the inner radius of the magnet. As the proximity of the electron to the magnet increases the error in equations (8.29, 8.30), it is important to define intuitively what is the meaning of this closeness or, in other words, what is the size of the electron wave packet. We agree that the free electron is actually localized in configuration space in a ball centered in the classical position \( vt \) and with radius chosen in such a way that the probability of finding the electron on this ball is 99%. We measure the radius of the wave packet at the time \( t = 0 \) - when the free particle is in the center of the magnet - and denote it by \( R(\sigma) \). Then, we have:

\[ R := R(\sigma) = 2.382 \sigma. \]

The error due to the part of the electron that hits the magnet \([9.3]\) is practically zero (smaller than \( 10^{-99} \)) when \( R \leq 0.1082 r_1 (R \leq 1.8556 \times 10^{-5}) \). In Figure 3 we show the error bound of equation \([8.29]\) as a function of the radius of the wave packet over \( r_1 \) for big sigma, \( \sigma \in \left[ \frac{r_1}{2}, \frac{r_1}{2} \right] \) \( (.1082 r_1 \leq R \leq .5102 r_1) \).

Even when the size of the wave packet is comparable to the inner radius of the magnet we have error bounds extremely small. We give some data to show this behavior:

| Radius of the Wave Packet over \( r_1 \) | Error Bound |
|----------------------------------------|-------------|
| \( .81716 \)                           | \( 10^{-1} \) |
| \( .65806 \)                           | \( 10^{-2} \) |
| \( .56606 \)                           | \( 10^{-3} \) |
| \( .50427 \)                           | \( 10^{-4} \) |
| \( .45911 \)                           | \( 10^{-5} \) |
| \( .42425 \)                           | \( 10^{-6} \) |
| \( .39629 \)                           | \( 10^{-7} \) |
| \( .37322 \)                           | \( 10^{-8} \) |
| \( .35376 \)                           | \( 10^{-9} \) |
| \( .33703 \)                           | \( 10^{-10} \) |

### 9.5 The Opening Angle of the Electron Wave Packet

Although it is impossible to define an opening angle of the electron, because it is everywhere, we agree to say that the free electron (in momentum representation) is actually in a ball, \( B_P(Mv) \) with center the classical momentum \( (Mv) \) and radius \( P \) such that there is a 99% probability for the electron to have its momentum within this ball. We define the opening angle, \( \omega(\sigma) \), in the obvious way (see Figure 4).
\[ \sin \left( \frac{\omega(\sigma)}{2} \right) := \frac{P}{Mv} = \frac{2.382}{\sigma mv}. \]

When sigma is big, the opening angle is very small and when sigma is small, the opening angle is big, this is nothing more than Heisenberg uncertainty principle.

The factor \( e^{-\frac{33}{34} \left( \frac{\sigma mv}{2} \right)} \) of the error bound (8.29, 8.30) has the following interpretation in terms of the opening angle:

\[ e^{-\frac{33}{34} \left( \frac{\sigma mv}{2} \right)^2} = e^{-2.7535 \left( \frac{1}{\sin(\omega(\sigma)/2)} \right)^2}. \]

This factor is practically zero (smaller than \( 10^{-100} \)) when \( \omega \leq 11.8 \) degrees (\( \sigma \geq 1.1592 \times 10^{-9} \) or \( \sigma mv \geq 23 \)), and then, it begins to increase as \( \omega \) increases (\( \sigma \) decreases). In Figure 5 we show the error bound in equation (8.29) as a function of the opening angle for small sigma, \( \sigma \in [1.3224 \times 10^{-6} r_1, 6.7591 \times 10^{-6} r_1] \), and in the table below we give some representative values.

| Opening Angle (degrees) | Error Bound |
|-------------------------|-------------|
| 51.8407                 | \( 10^{-4} \) |
| 47.8885                 | \( 10^{-4} \) |
| 44.7231                 | \( 10^{-5} \) |
| 42.1135                 | \( 10^{-5} \) |
| 39.9137                 | \( 10^{-6} \) |
| 38.0265                 | \( 10^{-7} \) |
| 36.3842                 | \( 10^{-8} \) |
| 34.9380                 | \( 10^{-9} \) |
| 33.6517                 | \( 10^{-9} \) |
| 32.4979                 | \( 10^{-10} \) |

### 10 Conclusions

In Theorems 8.2, 8.5, 8.7, 8.9, 8.10 and 8.11 we found the time evolution of the electron up to an error bound that we provide explicitly. The approximate wave function of the electron that we give is the one given by the Aharonov-Bohm Ansatz. It coincides also with the part of the electron wave packet that goes through the hole of the magnet in Tonomura et al. experiments [26]. As we noticed before (see Section 7.1) the Aharonov-Bohm Ansatz is valid if the evolution of the exact wave packet is localized at every time in a simply connected region, with no holes, (for example in (7.8)). The main factors that produce the error bounds are the size of the wave packet (see (9.1)) and the opening angle (see (9.2)). These factors can be understood also in terms of the part of the wave packet that hits the magnet when the electron crosses the hole of the magnet (see Remark 8.4) and, therefore, they are related with the part of the electron not localized in a simple connected region (see (7.8)) at every time. In Section 9 we analyzed the error bounds and we have shown that our estimates for the time evolution are valid for a rather big interval that starts when the opening angle is close to 55 degrees (\( \sigma \approx 1.3224 \times 10^{-6} r_1 \)) and ends when the size of
the wave packet is close to the inner radius of the magnet (close to $r_1$). We have shown also that the error bounds decrease very fast -exponentially- as the variance gets away from the extremes of the interval. For intermediate sigma ($\sigma \in [6.7591 \times 10^{-6} r_1, 22]$), the time evolution given by the Aharonov-Bohm Ansatz (8.28) differs from the exact one only by a number less than $10^{-99}$ in norm. As it is shown in Remark 8.12 and Section 9.2, for intermediate sigma, the probability that the electron wave packet interacts with the magnet is smaller than $10^{-199}$ and so, there are no fields in the trajectory of the electron. Nevertheless, the solution is the one given by the Aharonov-Bohm Ansatz (8.28) and it is affected by the vector potential $A$ by a phase factor $e^{i\lambda A}$. This phase factor is the one that appears in Tonomura et al. experiments [26]. Although in the experiments of Tonomura et al. [26] there is no interaction with the magnetic field, there is an interaction with the impenetrable magnet. Tonomura et al. [26] argued that it is not necessary to consider the part of the electron wave packet that hits the magnet -they used a rather big one- because the shadow of the magnet was clearly seen in the hologram. Our results show that it would be quite interesting to perform an experiment with a medium size electron wave packet with an intermediate sigma. One could use, as well, a bigger magnet. Our results show that quantum mechanics predicts in this case the interference patterns observed by Tonomura et al. [26] with extraordinary precision.

In the Aharonov-Bohm Ansatz the electron is not accelerated, it propagates following the free evolution, with the wave function multiplied by a phase. As we prove that the Aharonov-Bohm Ansatz approximates the exact solution with an error bound uniform in time that can be smaller that $10^{-99}$ in norm, we rigorously prove that quantum mechanics predicts that no force acts on the electron, in agreement with the experimental results of Caprez et al. [6].

Summing up, the experiments of Tonomura et al. [17, 25, 26] give a strong evidence of the existence of the interference fringes predicted by Franz [9] and by Aharonov and Bohm [2]. The experiment of Caprez et al. [6] verifies that the interference fringes are not due to a force acting on the electron, and the results of this paper rigorously prove that quantum mechanics theoretically predicts the observations of these experiments in a extremely precise way. This gives a firm experimental and theoretical basis to the existence of the Aharonov-Bohm effect [2], namely, that magnetic fields act at a distance on charged particles, even if they are identically zero in the space accessible to the particles, and that this action at a distance is carried by the circulation of the magnetic potential, what gives magnetic potentials a real physical significance.

11 Appendix A. Estimates for the Free Evolution of gaussian States

In this appendix we prove estimates for the solutions to the boosted free Schrödinger equation,

$$i \frac{\partial}{\partial z} \varphi (x,z) = H_1 \varphi (x,z), \quad \varphi (x,0) = \varphi (x),$$

where the boosted free Hamiltonian $H_1$ is defined in (3.11).

Recall that under the change of variable $t := z/v$, the solutions of (11.1) are solutions of the boosted free Schrödinger
equation with Hamiltonian $e^{-imv \cdot x} H_0 e^{imv \cdot x}$. Classically, a particle that starts at the origin with velocity $v = (0, 0, v)$, will be located at time $t$ at the position $(0, 0, z)$. At the high-velocity limit, the quantum evolution follows the classical one and the parameter $z$ can be taken as the position in the $z-$direction of the particle. We consider the case where the initial state is gaussian,

$$\varphi(x) := \frac{1}{(\sigma^2 \pi)^{3/4}} e^{-x^2/2\sigma^2},$$  \hspace{1cm} (11.2)

with variance $\sigma$. The solution to (11.1) is given by,

$$e^{-izH_1} \varphi = e^{-izmv/2} \frac{\sigma^{3/2}}{\pi^{3/4} (\sigma^2 + i\varepsilon/mv)^{3/2}} e^{-(x-z\hat{v})^2/2(\sigma^2 + i\varepsilon/mv)}.$$  \hspace{1cm} (11.3)

we will often use the following simple result.

**REMARK 11.1.** Suppose that $C_3 \leq C_2 \leq C_1 \leq 0$. Then,

1. $$\int_{C_3}^{C_2} e^{-z^2} dz \leq e^{-C_1^2} \int_{C_3-C_1}^{C_2-C_1} e^{-z^2} dz \leq e^{-C_1^2} \int_{0}^{\infty} e^{-z^2} dz.$$ \hspace{1cm} (11.4)

2. $$\int_{C_3}^{C_2} z^2 e^{-z^2} dz \leq \frac{C_2}{2} e^{-C_1^2} + \frac{C_3}{2} e^{-C_2^2} + \frac{1}{2} e^{-C_1^2} \int_{C_3-C_1}^{C_2-C_1} e^{-z^2} dz \leq e^{-C_1^2} (-\frac{C_2}{2} + \frac{1}{2} \int_{0}^{\infty} e^{-z^2} dz).$$ \hspace{1cm} (11.5)

Proof:

$$\int_{C_3}^{C_2} e^{-z^2} dz \leq e^{-C_1^2} \int_{C_3}^{C_2} e^{-(z^2 - C_1^2)} \leq e^{-C_1^2} \int_{C_3}^{C_2} e^{(z-C_1)^2} dz,$$

where we used that, $z^2 - C_1^2 \geq (z-C_1)^2$. This proves 1. Furthermore, 2 follows from 1 and the following equation.

$$\int_{C_3}^{C_2} \frac{z}{2} 2z e^{-z^2} dz = -\frac{z}{2} e^{-z^2} \bigg|_{C_3}^{C_2} + \int_{C_3}^{C_2} \frac{1}{2} e^{-z^2} dz.$$

**LEMMA 11.2.** Let $f$ be a bounded complex valued function with support contained in $D$. Then, for $z \geq h$ and $d \geq h - z$,

1. $$\| f(x) e^{-izH_1} \varphi \| \leq \frac{\| f \|_{\infty}}{\sqrt{2}} e^{-\theta_{inv}(\sigma,z)^2/2},$$  \hspace{1cm} (11.6)

2. $$\| f(x + d\hat{v}) e^{-izH_1} \varphi \| \leq \frac{\| f \|_{\infty}}{\sqrt{2}} e^{-\theta_{inv}(\sigma,z + d,h(\sigma))^2/2},$$  \hspace{1cm} (11.7)

3. $$\| f(x) e^{-izH_1} \varphi \| \leq \frac{\| f \|_{\infty}}{\pi^{1/4}} e^{-\theta_{inv}(\sigma,z)^2/2} \sqrt{2\pi r_2 \rho(z)}^{3/2}.$$  \hspace{1cm} (11.8)
Proof: We use the function $\rho(z)$ defined in (2.9),

$$\left\| f(x)e^{-izH_1}\varphi \right\|^2 \leq \frac{\|f\|_\infty^2}{\pi^{3/2}} \int_{\overline{D-vz})\rho(z)} e^{-x^2} dx \leq \frac{\|f\|_\infty^2}{\pi^{1/2}} \int_{(-h-z)\rho(z)}^{(h-z)\rho(z)} d\mu e^{-\mu^2} \left(1 - e^{-r_2^2\rho(z)^2}\right) \leq \frac{\|f\|_\infty^2}{\pi^{1/2}} e^{-\theta_{inv}(\sigma, z)^2} \int_{-2h\rho(z)}^0 dz e^{-z^2} \left(1 - e^{-r_2^2\rho(z)^2}\right),$$

(11.9)

where in the last inequality we used (11.4). Equation (11.6) follows from (11.9). Equation (11.7) is obtained similarly. Equation (11.8) follows from (11.9) and the estimate,

$$\int_{-2h\rho(z)}^0 dz e^{-z^2} \left(1 - e^{-r_2^2\rho(z)^2}\right) \leq 2hr_2^2\rho(z)^3.$$

**LEMMA 11.3.** Let $f$ be a bounded complex valued function with support contained in $D$. Then, for $Z \geq h, s \geq 0$,

$$\int_Z \left\| f(x)e^{-izH_1}\varphi \right\|^2 \leq \frac{\|f\|_\infty^2}{\pi^{3/2}} e^{-\theta_{inv}(\sigma, z)^2} / 2 (\max(Z, s) - Z) +$$

$$\frac{\|f\|_\infty^2}{\pi^{1/2}} e^{-\theta_{inv}(\sigma, \max(Z, s))^2} / 2 \sqrt{2hr_2(\sigma m)^3} / 2 \int_{\max(Z, s)}^\infty \frac{1}{(\sigma m)^{2+1/(2\lambda)}} d\zeta.$$  

(11.10)

Proof: We prove the lemma writing the integral in the left hand side of (11.10) as follows

$$\int_Z \left\| f(x)e^{-izH_1}\varphi \right\|^2 = \int_Z \left\| f(x)e^{-izH_1}\varphi \right\| + \int_{\max(Z, s)}^\infty \left\| f(x)e^{-izH_1}\varphi \right\|$$

and using (11.6) in the first integral, (11.8) in the second, and the fact that $\theta_{inv}(\sigma, z)^2$ is increasing in $z$ for $z \geq h$.

**LEMMA 11.4.** Let $g : \mathbb{R}^3 \rightarrow \mathbb{C}^3$ be bounded and with support contained in $D$ and let $z \geq h$. Then,

1. $$\left\| g(x) \cdot \rho e^{-izH_1}\varphi \right\| \leq \frac{\|g\|_\infty}{\pi^{1/4}} e^{-\theta_{inv}(\sigma, z)^2/2} \left[ -\frac{\theta_{inv}(\sigma, z)^2}{2} + \frac{3\sqrt{\pi}}{4} \right]^{1/2}.$$  

(11.11)

2. $$\left\| g(x) \cdot \rho e^{-izH_1}\varphi \right\| \leq \frac{\|g\|_\infty}{\pi^{1/4}} e^{-\theta_{inv}(\sigma, z)^2/2} \left[ 4(\sigma m)^2 + 2 \right]^{1/2} \sqrt{2h r_2 \rho(z)^3/2}.$$  

(11.12)

Proof:

$$\left\| g(x) \cdot \rho e^{-izH_1}\varphi \right\|^2 \leq \frac{\|g\|_\infty^2}{\pi^{3/2}} \int_{\overline{D-vz})\rho(z)} x^2 e^{-x^2} dx \leq \frac{\|g\|_\infty^2}{\pi^{1/2}} (\Theta(\sigma, z) + \Upsilon(\sigma, z)) \left(1 - e^{-r_2^2\rho(z)^2}\right),$$

(11.13)

where $\Theta$ and $\Upsilon$ are defined in Section 2. Equation (11.11) follows from (11.13) applying the last inequality in (11.4) to $\Upsilon(\sigma, z)$ and the last inequality in (11.5) to $\Theta(\sigma, z)$. Furthermore, using the middle inequality in (11.5) we obtain that,

$$\Theta(\sigma, z) \leq \frac{\rho(z)}{2} \left[ (z-h)e^{-(z-h)^2\rho^2(z)} + (z+h)e^{-(z+h)^2\rho^2(z)} \right] + \frac{1}{2} e^{-\theta_{inv}(\sigma, z)^2} \int_{-2h\rho(z)}^0 e^{-z^2} dz.$$  

(11.14)
Note that,
\[ e^{-(z-h)^2}\rho(z)^2 - e^{-(z+h)^2}\rho(z)^2 \leq e^{-\theta_{inv}(\sigma,z)^2}4zh\rho(z)^2, \]  
(11.15)
\[ \int_{-2h\rho(z)}^{0} e^{-z^2} dz \leq 2h\rho(z). \]  
(11.16)
Writing \( z-h = z+h-2h \) in (11.14) we obtain that,
\[ \Theta(\sigma,z) \leq \frac{\rho(z)}{2} (z+h)e^{-\theta_{inv}(\sigma,z)^2}4zh\rho(z)^2 \leq e^{-\theta_{inv}(\sigma,z)^2}4h\rho(z)(\sigma mv)^2. \]  
(11.17)
Moreover, applying the middle inequality in (11.4) to \( \Upsilon(\sigma,z) \) we prove that,
\[ \Upsilon(\sigma,z) \leq e^{-\theta_{inv}(\sigma,z)^2}2h\rho(z). \]  
(11.18)
Equation (11.12) follows from (11.13), (11.17), (11.18).

**Lemma 11.5.** Let \( g : \mathbb{R}^3 \to \mathbb{C}^3 \) be bounded and with support contained in \( D \). Then, for any \( Z \geq h \) with \( \theta_{inv}(\sigma,Z) \geq 1, s \geq 0, \)
\[ \int_{Z}^{\infty} \| g(x) \cdot \mathbf{P} e^{-izH_1} \varphi \| \leq \frac{\| g \|_{\infty}}{\pi^{3/2}} e^{-\theta_{inv}(\sigma,Z)^2/2} \left( \frac{\theta_{inv}(\sigma,Z)^{1/2}}{\sqrt{2}} + \frac{\sqrt{3} \sqrt[4]{2}}{2} \right) (\max(Z,s) - Z) + \frac{\| g \|_{\infty}}{\pi^{3/2}} e^{-\theta_{inv}(\sigma,\max(Z,s))^2/2} (2\sigma mv + \sqrt{2}) \sqrt{h} r_{2}(\sigma mv)^{3/2} \int_{\max(Z,s)}^{\infty} (\sigma^4 m^2 v^2 + \zeta^2)^{-3/4}. \]  
(11.19)

**Proof:** We split the integral in the left hand side of (11.19) as follows
\[ \int_{Z}^{\infty} \| g(x) \cdot \mathbf{P} e^{-izH_1} \varphi \| = \int_{Z}^{\max(Z,s)} \| g(x) \cdot \mathbf{P} e^{-izH_1} \varphi \| + \int_{\max(Z,s)}^{\infty} \| g(x) \cdot \mathbf{P} e^{-izH_1} \varphi \| \]
and using (11.11) in the first integral, (11.12) in the second, and the fact that the functions \( e^{-x^2/\sqrt{2}}, e^{-x^2/\sqrt{2}}x^{1/4} \) are decreasing for \( x \geq 1 \) (notice also that \( \theta_{inv}(\sigma,z)^2 \) is increasing in \( z \) for \( z \geq h \)).

**Remark 11.6.** Suppose that \( z, \zeta \in \mathbb{R}^+, s \) and \( b \) are real numbers such that \( z \geq \zeta, s \geq z - 2\zeta, b > 0. \) Then,

1. In any interval \( I := [\sigma_1, \sigma_2] \) such that \( \forall \sigma \in I, -\theta_{inv}(\sigma,z,s,\zeta) \geq \sqrt{1/2}, \)
\[ \Upsilon(\sigma,z,s,\zeta) e^{-bp(\sigma,z)} \leq \max[\Upsilon(\sigma_1,z,s,\zeta) e^{-bp(\sigma_1,z)}, \Upsilon(\sigma_2,z,s,\zeta) e^{-bp(\sigma_2,z)}]. \]  
(11.20)
2. In any interval \( I := [\sigma_1, \sigma_2] \) such that \( \forall \sigma \in I, -\theta_{inv}(\sigma,z,s,\zeta) \geq \sqrt{3/2}, \)
\[ \Theta(\sigma,z,s,\zeta) e^{-bp(\sigma,z)} \leq \max[\Theta(\sigma_1,z,s,\zeta) e^{-bp(\sigma_1,z)}, \Theta(\sigma_2,z,s,\zeta) e^{-bp(\sigma_2,z)}]. \]  
(11.21)

**Proof:** We give the proof of 1. The proof of 2 is similar. We have that
\[ \frac{\partial}{\partial \sigma} \mathcal{Y}(\sigma, z, s, \zeta) e^{-b\rho(\sigma, z)^2} = \frac{1}{\sigma} \frac{m v^2}{m v^2 + z^2} \left( \frac{z}{mv} \right)^2 - \sigma^4 e^{-b\rho(\sigma, z)^2} \left[ e^{-\left(\zeta + s\right)^2 \rho(\sigma, z)^2} (\zeta + s) \rho(\sigma, z) - e^{-\left(\zeta - s\right)^2 \rho(\sigma, z)^2} (\zeta - s) \rho(\sigma, z) - 2b \rho(\sigma, z)^2 \mathcal{Y}(\sigma, z, s, \zeta) \right]. \] \tag{11.22}

As the function \( e^{-x^2} x \) is decreasing for \( x \geq 1/\sqrt{2} \), the term in the square brackets is \( (11.22) \) is negative. Then, the left-hand side of \((11.22)\) is different from zero for \( \sigma \in I \) if \( \sqrt{z/mv} \notin I \) and otherwise, it is negative for \( \sigma < \sqrt{z/mv} \) and it is positive for \( \sigma > \sqrt{z/mv} \). This proves 1.

\[ \square \]

Remember that \( z_{\omega, \sigma}(h) \) is defined in Section 2. It is given by,

\[ z_{\omega, \sigma}(h) = \frac{h(\sigma mv)^2}{(\sigma mv)^2 - \omega^{-2}} + \frac{\sigma mv}{((\sigma mv)^2 - \omega^{-2})^{1/2}} \left( \omega^{-2} \sigma^2 + h^2 \left( \frac{(\sigma mv)^2}{(\sigma mv)^2 - \omega^{-2}} - 1 \right) \right)^{1/2}. \] \tag{11.23}

**REMARK 11.7.** Suppose that \( \sigma_2 \leq \sigma \leq \sigma_1 \). Then,

\[ z_{\omega, \sigma}(\zeta) \leq \max(z_{\omega, \sigma_1}(\zeta), z_{\omega, \sigma_2}(\zeta)). \] \tag{11.24}

**Proof:** Note that as a function of \( \sigma \), \( \rho(\sigma, z) \) is increasing for \( \sigma \leq \sqrt{z/mv} \) and that it is decreasing for \( \sigma > \sqrt{z/mv} \).

Suppose that \( z_{\omega, \sigma_1}(\zeta) \leq z_{\omega, \sigma}(\zeta) \). Then, \( \sigma \leq \sqrt{z/mv} \), because if \( \sigma > \sqrt{z/mv} \),

\[ \omega^{-1} = -\theta_{inv}(\sigma_1, z_{\omega, \sigma_1}(\zeta), z_{\omega, \sigma_1}(\zeta); \zeta) < -\theta_{inv}(\sigma, z_{\omega, \sigma_1}(\zeta), z_{\omega, \sigma_1}(\zeta), \zeta), \]

since, \( -\theta_{inv}(\sigma, z, \zeta) = (z - \zeta) \rho(\sigma, z) \) and as \( -\theta_{inv} \) is increasing in \( z \geq 0 \), this implies that \( z_{\omega, \sigma}(\zeta) < z_{\omega, \sigma_1}(\zeta) \). Then, \( \sigma_2 < \sigma \leq \sqrt{z/mv} \), and it follows that,

\[ \omega^{-1} = -\theta_{inv}(\sigma, z_{\omega, \sigma}(\zeta), z_{\omega, \sigma}(\zeta); \zeta) \geq -\theta_{inv}(\sigma_2, z_{\omega, \sigma_2}(\zeta), z_{\omega, \sigma_2}(\zeta), \zeta). \]

But as also,

\[ \omega^{-1} = -\theta_{inv}(\sigma_2, z_{\omega, \sigma_2}(\zeta), z_{\omega, \sigma_2}(\zeta), \zeta), \]

and \( -\theta_{inv} \) is increasing in \( z \geq 0 \), we have that \( z_{\omega, \sigma}(\zeta) \leq z_{\omega, \sigma_2}(\zeta) \).

**LEMMA 11.8.** Let \( \mu_i, i \in \{1, 2, 3\} \) belong to \( R_+ \). Suppose that the following conditions are satisfied,

1. Either \( \mu_i \leq \sigma_0, i \in \{1, 2, 3\} \), or \( \mu_i \geq \sigma_0, i \in \{1, 2, 3\} \).

2. \( \mu_i \leq \mu_3, i \in \{1, 2\} \), or \( \mu_i \geq \mu_3, i \in \{1, 2\} \).
We define \( \mu_{\text{max}} := \max(\mu_1, \mu_2) \), \( \mu_{\text{min}} := \min(\mu_1, \mu_2) \) and take \( \nu = \mu_{\text{max}} \), if \( \mu_i \geq \mu_3, i \in \{1, 2\} \) and \( \nu = \mu_{\text{min}} \), if \( \mu_i \leq \mu_3, i \in \{1, 2\} \). We denote by \( Z := z(\mu_{\text{max}}) \), if \( \mu_i \leq \sigma_0 \); and \( Z := \max_{i \in \{1, 2\}} \{z(\mu_{\text{max}}), \mu_i(\mu_{\text{max}})\} \), if \( \mu_i \geq \sigma_0 \).

We suppose that \( Z \geq z_{\sqrt{\sigma}, \nu, \mu_3}(\mu_{\text{max}}) \). Let \( f : \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C} \) be a complex valued function and we take \( f_{\sigma, z}(x) := f(x, \sigma, z) \). Suppose the support of \( f_{\sigma, z} \) is contained in \( K - [0, (z(\sigma) - \zeta - z)] \mathbb{V} \) for some \( \zeta \in \mathbb{R} \), every \( \sigma \in \mathbb{R}_+ \) and every \( z \in \mathbb{R} \) with \( z + \zeta \leq z(\sigma) \). Then, for every gaussian wave function \( \varphi \) with variance \( \sigma \in [\mu_{\text{min}}, \mu_{\text{max}}], \)

\[
\int_0^z (\sigma)^{-\zeta} \left\| f_{\sigma, z}(x) e^{-izH_1 e^{-i\zeta H_1} \varphi} \right\| \leq \frac{\|f\|_{1/4}}{\pi^{1/4}} I_{\text{ps}}(\mu_1, \mu_2, \mu_3, \zeta),
\]

where,

\[
I_{\text{ps}}(\mu_1, \mu_2, \mu_3, \zeta) := \frac{1}{\pi^{1/4}} z_{\sqrt{\sigma}, \nu, \mu_3}(\mu_{\text{max}}) \max_{\mu_i \in \{\mu_1, \mu_2\}} e^{-\frac{\sigma^2}{12} \rho(\mu_i, \zeta) \rho(\mu_i, \rho)} + \pi^{1/4} \max\{\zeta, 0\}
\]

\[
\max_{\mu_i \in \{\mu_1, \mu_2\}} e^{-\frac{\sigma^2}{12} \rho(\mu_i, \zeta) \rho(\mu_i, \rho)} + \sum_{\mu_i \in \{\nu, \mu_3\}} \int_{\sqrt{\sigma}, \nu, \mu_3}(\mu_{\text{max}}) \mathbb{V}(\mu_i, \tau, Z, \rho(\mu_{\text{max}}))^{1/2} e^{-\frac{\sigma^2}{12} \rho(\mu_i, \tau) \rho(\mu_i, \rho, \tau) \rho(\mu_i, \rho, \tau) d\tau.
\]

**Proof:** It follows from equation (11.23) that \( z(\sigma) \leq z_{\sqrt{\sigma}, \sigma}(\mu_{\text{max}}) \). If \( \mu_i \geq \sigma_0 \) then \( \rho(\sigma, \mu_{\text{max}}) = \rho(\mu_{\text{max}}) \) for \( \sigma \in [\mu_{\text{min}}, \mu_{\text{max}}] \). It follows from Remark 11.7 that \( z(\sigma) \leq \max_{\mu_i \in \{\nu, \mu_3\}} z(\mu_{\text{max}}) \). If \( \mu_i \leq \sigma_0 \) then from formula (11.23) and the definition of \( \rho(\sigma) \) we have that \( z_{\sqrt{\sigma}, \sigma}(\mu_{\text{max}}) \) \( \leq z_{\sqrt{\sigma}, \nu, \mu_3}(\mu_{\text{max}}) \) = \( z(\mu_{\text{max}}) \). We conclude that

\[
z(\sigma) \leq Z,
\]

and then,

\[
\int_0^z (\sigma)^{-\zeta} \left\| f_{\sigma, z}(x) e^{-izH_1 e^{-i\zeta H_1} \varphi} \right\| \leq \int_0^z (\sigma)^{-\zeta} \left\| f_{\sigma, z}(x) e^{-izH_1 e^{-i\zeta H_1} \varphi} \right\|.
\]

As in (11.9) we prove that

\[
\left\| f_{\sigma, z}(x) e^{-izH_1 e^{-i\zeta H_1} \varphi} \right\|^2 \leq \frac{\|f\|^2}{\pi^{1/2}} \mathbb{V}(\sigma, z + \zeta, Z, \rho(\mu_{\text{max}})) e^{-\frac{\sigma^2}{12} \rho(\sigma, z + \zeta)^2}.
\]

Then,

\[
\int_0^Z (\sigma)^{-\zeta} \left\| f_{\sigma, z}(x) e^{-izH_1 e^{-i\zeta H_1} \varphi} \right\| d\sigma \leq \frac{\|f\|^2}{\pi^{1/4}} \left[ \max_{\sigma \geq 0} \mathbb{V}(\sigma, z, Z, \rho(\mu_{\text{max}}))^{1/2} e^{-\frac{\sigma^2}{12} \rho(\sigma, z)^2} \right] +
\]

\[
\int_{\sigma \geq 0} \mathbb{V}(\sigma, Z, \rho(\mu_{\text{max}}))^{1/2} e^{-\frac{\sigma^2}{12} \rho(\sigma, z)^2} + \int_{\sigma \geq 0} \mathbb{V}(\sigma, Z, \rho(\mu_{\text{max}}))^{1/2} e^{-\frac{\sigma^2}{12} \rho(\sigma, z)^2} \left[ \mathbb{V}(\sigma, z, Z, \rho(\mu_{\text{max}}))^{1/2} e^{-\frac{\sigma^2}{12} \rho(\sigma, z)^2} \right]
\]

\[
\int_{\sigma \geq 0} \mathbb{V}(\sigma, Z, \rho(\mu_{\text{max}}))^{1/2} e^{-\frac{\sigma^2}{12} \rho(\sigma, z)^2} \left[ \mathbb{V}(\sigma, z, Z, \rho(\mu_{\text{max}}))^{1/2} e^{-\frac{\sigma^2}{12} \rho(\sigma, z)^2} \right],
\]

where we used that \( \mathbb{V} \leq \sqrt{\pi} \). If \( z \geq z_{\sqrt{\sigma}, \nu, \mu_3}(\mu_{\text{max}}) \), it follows from Remark 11.7 that \( z \geq z_{\sqrt{\sigma}, \nu, \mu_3}(\mu_{\text{max}}) \) for every \( \sigma \) belonging to the interval limited by \( \nu \) and \( \mu_3 \). We complete the proof of the lemma using (11.20) in the integral in the right-hand side of (11.30), and for the other two terms we argue as in the proof of (11.20).
Using the proof of the preceding lemma, we prove the following,

**Lemma 11.9.** Suppose that the hypothesis of the Lemma 11.8 are fulfilled and furthermore, assume that the support of \( f_{\sigma,z} \) is contained in \( K \) for every \( \sigma \in \mathbb{R}_+ \) and every \( z \in \mathbb{R} \). Then, for every \( \zeta \in \mathbb{R} \) with \( |\zeta| \leq z(\sigma) \),

\[
\int_0^{z(\sigma)-\zeta} \| f_{\sigma,z}(x)e^{-izH_1}e^{-i\zeta H_1}\varphi \| \leq \frac{\| f \|_{\infty}}{\pi^{1/4}} 2I_{pp}(\mu_1, \mu_2, \mu_3),
\]

where,

\[
I_{pp}(\mu_1, \mu_2, \mu_3) := I_{ps}(\mu_1, \mu_2, \mu_3, 0).
\]

(11.32)

and

\[
\int_0^{z(\sigma)} \| f_{\sigma,z}(x)e^{-izH_1}\varphi \| \leq \frac{\| f \|_{\infty}}{\pi^{1/4}} I_{pp}(\mu_1, \mu_2, \mu_3).
\]

(11.33)

**Lemma 11.10.** Let \( \mu_i, \mu_{\max}, \mu_{\min}, \nu \) and \( Z \) be as in Lemma 11.8. We suppose furthermore that \( Z \geq z \sqrt{\gamma_{x,\mu_3}(h(\mu_{\max}))} \) and \( r_1 \rho(\mu_i, z \sqrt{\gamma_{x,\mu_3}(h(\mu_{\max}))}) \geq 1 \) for \( i \in \{1,2\} \). Let \( g : \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}^3 \) be a complex vector valued function and we take \( g_{\sigma,z}(x) := g(x, \sigma, z) \). Suppose that the support of \( g_{\sigma,z} \) is contained in \( K - [0, z(\sigma) - \zeta - z] \hat{v} \) for some \( \zeta \in \mathbb{R} \), all \( \sigma \in \mathbb{R}_+ \) and for all \( z \) with \( z + \zeta \leq z(\sigma) \). Then, for every gaussian wave function \( \varphi \) with variance \( \sigma \in [\mu_{\min}, \mu_{\max}] \),

\[
\int_0^{z(\sigma)-\zeta} \| g_{\sigma,z}(x) \cdot e^{-izH_1}e^{-i\zeta H_1}\varphi \| dx \leq \frac{\| g \|_{\infty}}{\pi^{1/4}} I_{ps}(\mu_1, \mu_2, \mu_3, \zeta)
\]

(11.34)
where,

\[
I_{ax}(\mu_1, \mu_2, \mu_3, \zeta) := \frac{\pi^{1/4}}{\sqrt{2}} z^{\frac{3}{4}} \rho_{\zeta, \nu, \mu_3}(h(\mu_{\text{max}})) \max_{\mu_i \in \{\mu_1, \mu_2\}} e^{-\frac{\pi}{2} \rho_{\mu_i, z} \sqrt{\frac{1}{\nu, \mu_3}} (h(\mu_{\text{max}}))^2} +
\]

\[
\frac{\pi^{1/4}}{\sqrt{2}} \max\{-\zeta, 0\} \max_{\mu_i \in \{\mu_1, \mu_2\}} e^{-\frac{\pi}{2} \rho_{\mu_i, \zeta}^2} +
\]

\[
\pi^{1/4} z^{\frac{3}{4}} \rho_{\zeta, \nu, \mu_3}(h(\mu_{\text{max}})) \max_{\mu_i \in \{\mu_1, \mu_2\}} (r_1 \rho_{\mu_i, z} \sqrt{\frac{1}{\nu, \mu_3}} (h(\mu_{\text{max}}))) e^{-\frac{\pi}{2} \rho_{\mu_i, z}^2} +
\]

\[
\pi^{1/4} \max\{-\zeta, 0\} \left\{ \max_{\mu_i \in \{\mu_1, \mu_2\}} (r_1 \rho_{\mu_i, \zeta} \rho_{\mu_i, \zeta}^2), \text{ if } |\zeta| \leq r_{\mu_1, \mu_2} \right\} + e^{-1/2}, \text{ if } |\zeta| > r_{\mu_1, \mu_2}
\]

\[
\pi^{1/4} z^{\frac{3}{4}} \rho_{\zeta, \nu, \mu_3}(h(\mu_{\text{max}})) \max_{\mu_i \in \{\mu_1, \mu_2\}} e^{-\frac{\pi}{2} \rho_{\mu_i, z}^2} +
\]

\[
\pi^{1/4} \max\{-\zeta, 0\} \max_{\mu_i \in \{\mu_1, \mu_2\}} e^{-\frac{\pi}{2} \rho_{\mu_i, \zeta}^2} +
\]

\[
\sum_{\mu_i \in \{\nu, \mu_3\}} \int_{\sqrt{\frac{1}{\nu, \mu_3}} (h(\mu_{\text{max}}))}^{Z} \Theta(\mu_i, \tau, Z, h(\mu_{\text{max}}))^{1/2} e^{-\frac{\pi}{2} \rho_{\mu_i, \tau}^2} d\tau +
\]

\[
\sum_{\mu_i \in \{\nu, \mu_3\}} \max\{\int_{\sqrt{\frac{1}{\nu, \mu_3}} (h(\mu_{\text{max}}))}^{\min(r_{\nu, \mu_3}, Z)} \Upsilon(\mu_i, \tau, Z, h(\mu_{\text{max}}))^{1/2} r_1 \rho_{\mu_i, \tau} \rho_{\mu_i, \tau}^2 d\tau, 0\} +
\]

\[
\sum_{\mu_i \in \{\nu, \mu_3\}} \int_{\sqrt{\frac{1}{\nu, \mu_3}} (h(\mu_{\text{max}}))}^{Z} \Upsilon(\mu_i, \tau, Z, h(\mu_{\text{max}}))^{1/2} e^{-\frac{\pi}{2} \rho_{\mu_i, \tau}^2} d\tau +
\]

\[
\sum_{\mu_i \in \{\nu, \mu_3\}} \int_{\sqrt{\frac{1}{\nu, \mu_3}} (h(\mu_{\text{max}}))}^{Z} \Upsilon(\mu_i, \tau, Z, h(\mu_{\text{max}}))^{1/2} e^{-\frac{\pi}{2} \rho_{\mu_i, \tau}^2} d\tau.
\]

Proof: By (11.27),

\[
\int_{0}^{Z-\zeta} \|g_{\sigma, z}(x) \cdot p e^{-i z H_1} e^{-i \zeta H_1} \| \leq \int_{0}^{Z-\zeta} \|g_{\sigma, z}(x) \cdot p e^{-i z H_1} e^{-i \zeta H_1} \|. \tag{11.36}
\]

Estimating as in the proof of (11.13) we prove that,

\[
\|g_{\sigma, z}(x) \cdot p e^{-i z H_1} e^{-i \zeta H_1} \|^2 \leq \|g\|^2 \pi^{1/2} \sum_{\sigma} \left[ \Theta(\sigma, z + \zeta, Z, h(\mu_{\text{max}})) + \Upsilon(\sigma, z + \zeta, Z, h(\mu_{\text{max}}))^{1 + r_2^2 \rho(\sigma, z + \zeta)^2} \right] e^{-r_2^2 \rho(\sigma, z + \zeta)^2}. \tag{11.37}
\]

We have that,

\[
\int_{0}^{Z-\zeta} \|g_{\sigma, z}(x) \cdot p e^{-i z H_1} e^{-i \zeta H_1} \| d\tau \leq \frac{\|g\|}{\pi^{1/4} \sigma} \sum_{j=1}^{7} I_j, \tag{11.38}
\]

where,

\[
I_1 := \max\{\int_{0}^{Z-\zeta} \left( \Theta(\sigma, \tau, Z, h(\mu_{\text{max}})) e^{-r_1^2 \rho(\sigma, \tau)^2} \right)^{1/2} d\tau, 0\} +
\]

\[
\int_{0}^{\sqrt{\frac{1}{\nu, \mu_3}} (h(\mu_{\text{max}}))} \left( \Theta(\sigma, \tau, Z, h(\mu_{\text{max}})) e^{-r_1^2 \rho(\sigma, \tau)^2} \right)^{1/2} d\tau,
\]

\[
I_2 := \int_{\sqrt{\frac{1}{\nu, \mu_3}} (h(\mu_{\text{max}}))}^{Z} \left( \Theta(\sigma, \tau, Z, h(\mu_{\text{max}})) e^{-r_1^2 \rho(\sigma, \tau)^2} \right)^{1/2} d\tau,
\]
Moreover, since \( \Upsilon \leq \sqrt{\pi} \) and \( \Theta \leq \sqrt{\pi}/2 \) we have that,

\[
I_1 + I_6 \leq \frac{e^{-\frac{7}{4}}} \left( \frac{1}{\sqrt{2}} \right) \int_0^\infty e^{-\frac{7}{2}r_1^2\rho(\sigma,\tau)^2} d\sigma d\tau + \frac{1}{\sqrt{2}} \int_0^\infty e^{-\frac{7}{2}r_1^2\rho(\sigma,\tau)^2} \sqrt{2} \sqrt{\pi} \rho(\sigma,\tau) \mu \max(h(\mu)) d\sigma d\tau + \frac{\pi}{4} \int_0^\infty e^{-\frac{7}{2}r_1^2\rho(\sigma,\tau)^2} \sqrt{2} \sqrt{\pi} \rho(\sigma,\tau) \mu \max(h(\mu)) d\sigma d\tau.
\]

By Remark 11.6

\[
I_2 \leq \sum_{\mu \in \{\nu,\mu_3\}} \int_0^\infty e^{-\frac{7}{2}r_1^2\rho(\sigma,\tau)^2} d\sigma d\tau.
\]

\[
I_7 \leq \sum_{\mu \in \{\nu,\mu_3\}} \int_0^\infty e^{-\frac{7}{2}r_1^2\rho(\sigma,\tau)^2} d\sigma d\tau.
\]

Moreover, since \( xe^{-x^2/2} \) is increasing for \( 0 \leq x < 1 \) and decreasing for \( x \geq 1 \),

\[
I_3 \leq \frac{\pi}{2} \max(-\zeta,0) \left\{ \begin{array}{ll} r_1 \rho(\sigma,\zeta) e^{-\frac{7}{2}r_1^2\rho(\sigma,\zeta)^2}, & \text{if } |\zeta| \leq r_{\mu_1,\mu_2} \\ e^{-1/2}, & \text{if } |\zeta| > r_{\mu_1,\mu_2} \end{array} \right\} + \frac{\pi}{2} \int_0^\infty e^{-\frac{7}{2}r_1^2\rho(\sigma,\tau)^2} d\sigma d\tau.
\]

By Remark 11.6 if \( \tau \geq z \sqrt{2} \sqrt{\pi} \rho(\sigma,\tau) \mu \max(h(\mu)) \),

\[
\Upsilon(\sigma,\tau,\mu) \leq \max_{\mu \in \{\nu,\mu_3\}} \Upsilon(\mu,\tau,\mu) \leq \max_{\mu \in \{\nu,\mu_3\}} \Upsilon(\mu,\tau,\mu).
\]
By \( (11.50) \) and as \( x e^{-x} \) takes its maximum at \( x = 1 \),

\[
I_5 \leq \sum_{\mu_i \in \{\nu, \mu_3\}} \int_{\max(z_{\nu, \mu_3} (h(\mu_{\text{max}})), \min(r_{\nu, \mu_3}, Z))}^{Z} \left( Y(\mu_i, \tau, Z, h(\mu_{\text{max}})) e^{-1/2} \right) d\tau. \tag{11.51}
\]

Note that if \( r^2 \rho(\mu_i, \tau)^2 \geq 1, \mu_i \in \{\nu, \mu_3\} \) then, \( r^2 \rho(\sigma, \tau)^2 \geq 1, \forall \sigma \) between \( \nu \) and \( \mu_3 \). Hence, as in the proof of Remark \( 11.6 \) we prove that,

\[
I_4 \leq \sum_{\mu_i \in \{\nu, \mu_3\}} \int_{\max(z_{\nu, \mu_3} (h(\mu_{\text{max}})), \min(r_{\nu, \mu_3}, Z))}^{Z} \left( r_1^2 \rho(\mu_i, \tau)^2 Y(\mu_i, \tau, Z, h(\mu_{\text{max}})) e^{-r_1^2 \rho(\mu_i, \tau)^2} \right)^{1/2} d\tau. \tag{11.53}
\]

To obtain equation \( (11.34) \), we use \( (11.38, 11.46, 11.49, 11.51, 11.53) \) and we argue as in the proofs of Remark \( 11.6 \) to estimate equations \( (11.46) \) and \( (11.49) \).

\( \square \)

**LEMMA 11.11.** Suppose that the hypothesis of the Lemma \( 11.10 \) are fulfilled, assume furthermore, that the support of \( g_{\sigma, z} \) is contained in \( K \), for all \( \sigma \in \mathbb{R}^+ \) and for all \( z \). Then, for every \( \zeta \in \mathbb{R} \) with \( |\zeta| \leq z(\sigma) \) and every gaussian wave function \( \varphi \) with variance \( \sigma \in [\mu_{\text{min}}, \mu_{\text{max}}], \)

\[
\int_0^{z(\sigma)-\zeta} \|g_{\sigma, z}(x) \cdot p e^{-izH_1} e^{-i\zeta^2 H_1} \varphi\| \, dz \leq \frac{\|g\|_{L^\infty}}{\pi^{1/4} \sigma} 2I_{sp}(\mu_1, \mu_2, \mu_3), \tag{11.54}
\]

where,

\[
I_{sp}(\mu_1, \mu_2, \mu_3) := I_{ss}(\mu_1, \mu_2, \mu_3, 0). \tag{11.55}
\]

And

\[
\int_0^{z(\sigma)} \|g_{\sigma, z}(x) \cdot p e^{-izH_1} \varphi\| \, dz \leq \frac{\|g\|_{L^\infty}}{\pi^{1/4}} I_{sp}(\mu_1, \mu_2, \mu_3). \tag{11.56}
\]

**LEMMA 11.12.** Let \( f : \mathbb{R}^3 \to \mathbb{C} \) be bounded and with support contained in \( D \). Then, for \( Z \geq h \), and \( \zeta \) such that \( \zeta \leq Z \),

\[
\int_0^{Z-\zeta} \|f(x + (Z - (z + \zeta)) \hat{\nu}) e^{-izH_1} e^{-i\zeta^2 H_1} \varphi\| \, dz \leq (Z - \zeta) \frac{\|f\|_{L^\infty}}{\sqrt{2}} e^{-\frac{1}{2} \theta_{\text{max}}(\sigma, z)^2}. \tag{11.57}
\]
In this appendix we prove upper bounds for the integrals appearing in the terms \( Z \). Where we used (11.4).

**Proof:** Estimating as in the proof of (11.9) we prove that,

\[
\|f(x + (Z - (z + \zeta))\vec{v})\|_{\mathcal{L}^p} e^{-izH_1\varphi} \leq \frac{\|f\|_p}{\pi^{3/2}} \int_{-Z-h}^{Z+h} e^{-s^2} ds \pi \int_0^{r_2\rho(s,z+\zeta)} e^{-r^2} 2r \, dr
\]

\[
\leq \frac{\|f\|_p}{2} e^{-\theta_{\text{inv}}(\sigma,Z)^2} ,
\]

where we used (11.4).

**Lemma 11.13.** Let \( g : \mathbb{R}^3 \to C^3 \) be bounded and with support contained in \( D \), suppose that \( \theta_{\text{inv}}(\sigma,Z)^2 \geq \frac{1}{2} \). Then, for \( Z \geq h \), and \( \zeta \) such that \( \zeta \leq Z \), we have that,

\[
\int_0^{Z-\zeta} \|g(x + (Z - (z + \zeta))\vec{v})\cdot \mathbf{p} e^{-izH_1 e^{-i\zeta H_1} \varphi} \| \, dz \leq (Z - \zeta) \left\{ \frac{g\|\infty}{\pi^{1/4}2} e^{-\frac{1}{2}\theta_{\text{inv}}(\sigma,Z)^2} \left[ -\frac{\theta_{\text{inv}}(\sigma,Z)}{2} + \frac{3\sqrt{2}}{4} \right] \right\} .
\]

**Proof:** The lemma is proven estimating as in the proof of (11.11) using Remark 11.1.

**12 Appendix B. Upper Bounds for the Integrals**

In this appendix we prove upper bounds for the integrals appearing in the terms \( I_{ps}, I_{pp}, I_{ss} \) and \( I_{sp} \) (see (11.26), (11.32), (11.35), (11.55)).

Suppose that \( Z \geq s \geq \zeta, \delta_0 > 0 \). Designate \( \sqrt{N} := \{0, 1, \sqrt{2}, \sqrt{3}, \ldots\} \). We denote,

\[
\{Z_1, Z_2, \ldots, Z_K\} := \sqrt{0}\sqrt{N} \cap [-\theta_{\text{inv}}(\sigma, s, s, \zeta), -\theta_{\text{inv}}(\sigma, Z, Z, \zeta)],
\]

where \( Z_1 < Z_2 < \cdots < Z_K \). As \( -\theta_{\text{inv}}(\sigma, \tau, \tau, \zeta) \) is increasing as a function of \( \tau \) we have that,

\[
s \leq z_{Z_1^{-1}, \sigma}(\zeta) < z_{Z_2^{-1}, \sigma}(\zeta) < z_{Z_3^{-1}, \sigma}(\zeta) < \cdots < z_{Z_K^{-1}, \sigma}(\zeta) \leq Z.
\]

**Lemma 12.1.** Suppose that \( Z \geq s \geq \zeta, r > 0 \), and let \( f : \mathbb{R} \to \mathbb{R} \) satisfy \( f(\tau) \geq r - 2\zeta \). Then,

\[
\int_s^Z d\tau \Theta(\sigma, \tau, f(\tau), \zeta)^{1/2} \leq \frac{n^{1/4}}{\sqrt{2}} \left[ e^{-\frac{1}{2}\theta_{\text{inv}}(\sigma, s, s, \zeta)^2} (z_{Z_1^{-1}, \sigma}(\zeta) - s) + \sum_{j=1}^{K-1} e^{-\frac{1}{2}\rho(z_{Z_j^{-1}, \sigma}(\zeta) - z_{Z_{j+1}^{-1}, \sigma}(\zeta))} + e^{-\frac{1}{2}\rho(z_{Z_K^{-1}, \sigma}(\zeta) - Z)} \right] .
\]

(12.3)

\[
\int_s^Z d\tau Y(\sigma, \tau, f(\tau), \zeta)^{1/2} e^{-\frac{1}{2}\rho(\tau)^2} \leq \frac{n^{1/4}}{\sqrt{2}} \left[ e^{-\frac{1}{2}\rho(z_{Z_1^{-1}, \sigma}(\zeta) - s)} + \sum_{j=1}^{K-1} e^{-\frac{1}{2}\rho(z_{Z_j^{-1}, \sigma}(\zeta) - z_{Z_{j+1}^{-1}, \sigma}(\zeta))} + e^{-\frac{1}{2}\rho(z_{Z_K^{-1}, \sigma}(\zeta) - Z)} \right] .
\]

(12.4)

\[
\int_s^Z d\tau \Theta(\sigma, \tau, f(\tau), \zeta)^{1/2} e^{-\frac{1}{2}\rho(\tau)^2} \leq \frac{1}{\sqrt{2}} \left[ e^{-\frac{1}{2}\rho(z_{Z_1^{-1}, \sigma}(\zeta) - s)} + \sum_{j=1}^{K-1} e^{-\frac{1}{2}\rho(z_{Z_j^{-1}, \sigma}(\zeta) - z_{Z_{j+1}^{-1}, \sigma}(\zeta))} + e^{-\frac{1}{2}\rho(z_{Z_K^{-1}, \sigma}(\zeta) - Z)} \right] .
\]

(12.5)
If moreover, \( r_1 \rho(Z) \geq 1 \),
\[
\int_s Z \, d\tau \, \Upsilon(\sigma, \tau, f(\tau), \zeta) \frac{1}{2} \leq \frac{1}{\sqrt{2}} |r_1 \rho(z_{Z^{-1}}, \sigma(\zeta)) |
\]
\[
e^{-\frac{r_1^2}{4} \rho(z_{Z^{-1}}, \sigma(\zeta))^2} e^{-\frac{1}{2} \rho(z_{Z^{-1}}, \sigma(\zeta))} (z_{Z^{-1}} - s) + \sum_{j=1}^{K-1} r_1 \rho(z_{Z^{-1}}, \sigma(\zeta)) \frac{1}{2} e^{-\frac{r_1^2}{4} \rho(z_{Z^{-1}}, \sigma(\zeta))^2} e^{-\frac{1}{2} \rho(z_{Z^{-1}}, \sigma(\zeta))} (z_{Z^{-1}} - s) + \]
\[
r_1 \rho(Z) e^{-\frac{r_1^2}{4} \rho(Z)^2} e^{-\frac{1}{2} \rho(z_{Z^{-1}}, \sigma(\zeta) + \}
\]
\[(12.6)\]

**Proof:** We split the integral in the left-hand side of (12.3) as follows,
\[
\int_s Z \, d\tau \, \Upsilon(\sigma, \tau, f(\tau), \zeta) \frac{1}{2} = \int_s z_{Z^{-1}}, \sigma(\zeta) \, d\tau \Upsilon(\sigma, \tau, f(\tau), \zeta) \frac{1}{2} + \sum_{j=1}^{K-1} \int_s z_{Z^{-1}}, \sigma(\zeta) \, d\tau \Upsilon(\sigma, \tau, f(\tau), \zeta) \frac{1}{2} +
\]
\[
\int_s z_{Z^{-1}}, \sigma(\zeta) \, d\tau \Upsilon(\sigma, \tau, f(\tau), \zeta) \frac{1}{2},
\]
and we apply (11.4). This proves (12.3). (12.4) is proved in a similar way. Equation (12.5) is proven in the same way, but using (11.5). Finally, we prove (12.6) as above, using (11.4) and observing that the function \( x e^{-x^2/2} \) is decreasing for \( x \geq 1 \).

**Acknowledgement**

This work was partially done while we were visiting the project POems at Institut National de Recherche en Informatique et en Automatique (INRIA) Paris-Rocquencourt. We thank Patrick Joly for his kind hospitality. We thank Luis Carlos Velázquez for his help in writing the Matlab code.

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Figure 1: Error bound as a function of $\sigma$ over $r_1$ for big sigma.
Error Bound as a Function of Sigma over $r_1$ for Small Sigma.

Figure 2: Error bound as a function of $\frac{\sigma}{r_1} \times 10^{-6}$ for small sigma.
Error Bound as a Function of the Radius of the Wave Packet over $r_1$ for Big Sigma.

Figure 3: Error bound as a function of the radius of the wave packet over $r_1$ for big sigma
Figure 4: Opening angle
Error Bound as a Function of the Opening Angle for Small Sigma.

Figure 5: Error bound as a function of the opening angle for small sigma.