Dilogarithm identities for solutions to Pell’s equation in terms of continued fraction convergents

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Abstract

We describe a new connection between the dilogarithm function and the solutions of Pell’s equation $x^2 - ny^2 = \pm 1$. For each solution $x, y$ to Pell’s equation, we obtain a dilogarithm identity whose terms are given by the continued fraction expansion of the associated unit $x + y \sqrt{n} \in \mathbb{Z}[\sqrt{n}]$. We further show that Ramanujan’s dilogarithm value-identities correspond to an identity for the regular ideal hyperbolic hexagon.

Keywords  Pell’s equation  ·  Dilogarithm  ·  Hyperbolic surfaces  ·  Identities

Mathematics Subject Classification  11D09  ·  11G55  ·  32Q45

1 Dilogarithm and Pell’s equation

Dilogarithm The dilogarithm function $\text{Li}_2(z)$ is the integral function

$$\text{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} \, dt.$$  

It follows that it has power series

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad \text{for} \quad |z| \leq 1.$$
In [16], Rogers introduced the following normalization for the dilogarithm for \( x \) real,

\[
\mathcal{L}(x) = \text{Li}_2(x) + \frac{1}{2} \log |x| \log(1 - x).
\]

The dilogarithm function arises naturally in many areas of mathematics, including hyperbolic geometry and number theory (see [17]). In particular, volumes in the Lie group \( \text{PSL}(2, \mathbb{R}) \) and the symmetric space \( \mathbb{H}^3 \) can be described in terms of the dilogarithm (see Sect. 5.1 for discussion).

Pell’s equation Pell’s equation for \( n \in \mathbb{N} \) is the Diophantine equation \( x^2 - ny^2 = \pm 1 \) over \( \mathbb{Z} \). Pell’s equation has a long and interesting history going back to Archimedes’ cattle problem (see [10]). The equation only has solutions for \( n \) square-free, so we assume \( n \) is square-free. Also, by symmetry, we need only consider solutions with \( x, y > 0 \). A solution is positive/negative depending on whether \( x^2 - ny^2 = 1 \), or, \( x^2 - ny^2 = -1 \). For all square-free \( n \) there is always a positive solution but not necessarily a negative solution. Solutions to Pell’s equation correspond to units in \( \mathbb{Z}[\sqrt{n}] \) by identifying \( x, y \) with \( x + y\sqrt{n} \) and it is natural to identify the two. The smallest positive unit \( u = x + y\sqrt{n} \) is called the fundamental unit and a well-known result is that the set of positive units is exactly \( \{u^k\}, k \in \mathbb{N} \) (see [15, Theorem 7.26]).

In this paper, we prove a new and surprising connection between the dilogarithm and solutions to Pell’s equation. Using earlier work of the author, which gave a dilogarithm identity associated to a hyperbolic surface, we obtain a dilogarithm identity for each solution \( x, y \) to Pell’s equation whose terms are given by the continued fraction expansion of \( x + y\sqrt{n} \).

### 1.1 Dilogarithm identities

The dilogarithm function satisfies a number of classical identities, see [11] for details. In particular, by adding power series termwise, we have the squaring identity

\[
\text{Li}_2(z) + \text{Li}_2(-z) = \frac{1}{2} \text{Li}_2(z^2).
\]

It follows by direct computation that this identity holds for the Rogers dilogarithm with

\[
\mathcal{L}(x) + \mathcal{L}(-x) = \frac{1}{2} \mathcal{L}(x^2) \quad \text{(Squaring Identity)}.
\]

The other classic identities are Euler’s reflection identities

\[
\mathcal{L}(x) + \mathcal{L}(1 - x) = \frac{\pi^2}{6}, \quad \mathcal{L}(x) + \mathcal{L}(x^{-1}) = \frac{\pi^2}{6} \quad \text{(Reflection Identity)}.
\]
Landen’s identity (see [9])
\[ \mathcal{L} \left( -\frac{1}{x} \right) = -\mathcal{L} \left( \frac{1}{x + 1} \right) \text{ for } x > 0 \] (Landen’s identity),
and Abel’s well-known 5-term identity
\[ \mathcal{L}(x) + \mathcal{L}(y) = \mathcal{L}(xy) + \mathcal{L} \left( \frac{x(1 - y)}{1 - xy} \right) + \mathcal{L} \left( \frac{y(1 - x)}{1 - xy} \right) \] (Abel’s Identity).

It can be easily shown that the reflection identities and Landen’s identity follow from Abel’s identity.

A closed form for values of \( \mathcal{L} \) is only known for a small set of values. These are
\[ \mathcal{L}(0) = 0, \quad \mathcal{L}(1) = \frac{\pi^2}{6}, \quad \mathcal{L} \left( \frac{1}{2} \right) = \frac{\pi^2}{12}, \]
\[ \mathcal{L}(\phi^{-1}) = \frac{\pi^2}{10}, \quad \mathcal{L}(\phi^{-2}) = \frac{\pi^2}{15}, \] (1.1)
where \( \phi \) is the golden ratio. In [11], Lewin gave the following remarkable infinite identity
\[ \sum_{k=2}^{\infty} \mathcal{L} \left( \frac{1}{k^2} \right) = \frac{\pi^2}{6}. \] (1.2)

2 Results

Using earlier work of the author, we first prove the below new infinite identity for \( \mathcal{L} \). We prove:

Theorem 2.1 If \( L > 0 \) then
\[ \mathcal{L}(e^{-L}) = \sum_{k=2}^{\infty} \mathcal{L} \left( \frac{\sinh^2 \left( \frac{L}{k^2} \right)}{\sinh^2 \left( \frac{L}{2} \right)} \right). \]

One immediate observation is if we let \( L \to 0 \), we recover the formula of Lewin in Eq. (1.2) above.

We now apply the above identity to solutions of Pell’s equation and units in the ring \( \mathbb{Z}[\sqrt{n}] \).
Dilogarithm identity for solution to Pell’s equation

In order to obtain our identity associated to a given solution \( a^2 - nb^2 = \pm 1 \) of Pell’s equation, we let \( L \) satisfy \( e^{L/2} = a + b\sqrt{n} \). We then show that the summation terms in Theorem 2.1 above are given in terms of the continued fraction expansion of \( a + b\sqrt{n} \). We obtain:

**Theorem 2.2** Let \( u = a + b\sqrt{n} \in \mathbb{Z}[\sqrt{n}] \) be a solution to Pell’s equation.

- If \( u \) is a positive solution with continued fraction convergents \( r_j = h_j/k_j \), then
  \[
  \mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=1}^{\infty} \mathcal{L}\left(\frac{1}{(h_{2k-1})^2}\right).
  \]

- If \( u \) is a negative solution and \( u^2 \) has convergents \( R_j = H_j/K_j \), then
  \[
  \mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=0}^{\infty} \mathcal{L}\left(\frac{1}{b^2n(2H_{2k-1})^2}\right) + \mathcal{L}\left(\frac{1}{(2H_{2k+1} - H_{2k})^2}\right).
  \]

**Examples**

We now consider some examples:

- **Case of \( \mathbb{Z}[\sqrt{2}] \)** For \( \mathbb{Z}[\sqrt{2}] \) the fundamental unit is \( 3 + 2\sqrt{2} \) giving

  \[
  \mathcal{L}\left(\frac{1}{(3 + 2\sqrt{2})^2}\right) = \mathcal{L}\left(\frac{1}{6^2}\right) + \mathcal{L}\left(\frac{1}{35^2}\right) + \mathcal{L}\left(\frac{1}{204^2}\right) + \mathcal{L}\left(\frac{1}{1189^2}\right) + \ldots.
  \]

  We note that \( 3 + 2\sqrt{2} \) has convergents \( r_k \) given by

  \[
  5, 6, 29, 35, 169, 204, 985, 1189, 204.985, 35, 29, 6, 5, 1.
  \]

  It can be further shown that the units of \( \mathbb{Z}[\sqrt{2}] \) are given by \((1 + \sqrt{2})^k\). As \( u = 1 + \sqrt{2} \) is a negative solution to Pell’s equation with \( u^2 = 3 + 2\sqrt{2} \), we get

  \[
  \mathcal{L}\left(\frac{1}{3 + 2\sqrt{2}}\right) = \mathcal{L}\left(\frac{1}{2 \times (2)^2}\right) + \mathcal{L}\left(\frac{1}{7^2}\right) + \mathcal{L}\left(\frac{1}{2 \times (12)^2}\right) + \mathcal{L}\left(\frac{1}{41^2}\right)
  \]

  \[
  + \mathcal{L}\left(\frac{1}{2 \times (70)^2}\right) + \mathcal{L}\left(\frac{1}{2 \times (408)^2}\right) + \ldots.
  \]
Case of $\mathbb{Z}[\sqrt{13}]$ An interesting case of a large fundamental solution occurs for $\mathbb{Z}[\sqrt{13}]$. Here $u = 649 + 180\sqrt{13}$ is the fundamental unit, giving

$$L\left(\frac{1}{842401 + 233640\sqrt{13}}\right) = L\left(\frac{1}{1298^2}\right) + L\left(\frac{1}{1684803^2}\right) + L\left(\frac{1}{2186872996^2}\right) \ldots .$$

The continued fraction convergents of $u$ are

$$\frac{1297}{1}, \frac{1298}{1}, \frac{1683505}{1297}, \frac{1684803}{1298}, \frac{2185188193}{1683505}, \frac{2186872996}{1684803} \ldots .$$

Pell’s equation over $\mathbb{Q}$

Similarly, we consider Pell’s equation over $\mathbb{Q}$. If $a, b \in \mathbb{Q}$ satisfy Pell’s equation $a^2 - nb^2 = \pm 1$, we will identify this with the element $a + b\sqrt{n} \in \mathbb{Q}[\sqrt{n}]$. Applying the identity in Theorem 2.1, we get the following:

**Theorem 2.3** Let $u = a + b\sqrt{n} \in \mathbb{Q}[\sqrt{n}], a, b > 0$ satisfy Pell’s equation and let $u^k = a_k + b_k\sqrt{n}$.

If $u$ is a positive solution, then

$$L\left(\frac{1}{u^2}\right) = \sum_{k=2}^{\infty} L\left(\frac{1}{(b_k/b)^2}\right).$$

Further if $u \in \mathbb{Z}[\sqrt{n}]$, then $b_k/b \in \mathbb{Z}$ for all $k$.

If $u$ is a negative solution, then

$$L\left(\frac{1}{u^2}\right) = \sum_{k=1}^{\infty} L\left(\frac{1}{n(b_{2k}/a)^2}\right) + L\left(\frac{1}{(a_{2k+1}/a)^2}\right).$$

Further, if $u \in \mathbb{Z}[\sqrt{n}]$, then $b_{2k}/a, a_{2k+1}/a \in \mathbb{Z}$ for all $k$.

**Fibonacci numbers** The golden mean $\phi \in \mathbb{Q}[\sqrt{5}]$ corresponds to a negative solution to Pell’s equation over $\mathbb{Q}$. Also, we have

$$\phi^k = \frac{g_k + f_k \sqrt{5}}{2}$$

where $f_k$ is the classic Fibonacci sequence 1, 1, 2, 3, 5, … and $g_k$ is the Fibonacci sequence 1, 3, 4, 7, 11, … .

By Eq. (1.1) we have $L(\phi^{-2}) = \pi^2/15$. Therefore we get the identity,

$$\sum_{k=1}^{\infty} \left( L\left(\frac{1}{5f_{2n}^2}\right) + L\left(\frac{1}{g_{2n+1}^2}\right) \right) = \frac{\pi^2}{15}.$$
Chebyshev polynomials, Pell’s equation and dilogarithms

Chebyshev polynomials arise in numerous areas of mathematics and have a natural interpretation in terms of Pell’s equation. The Chebyshev polynomial of the first kind $T_n$ is the unique polynomials satisfying $T_n(\cos(\theta)) = \cos(n\theta)$. The Chebyshev polynomials of the second kind $U_n$ is given by

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$ 

We obtain the following corollary:

**Corollary 2.4** Let $x > 1$, then

$$\mathcal{L} \left( \frac{1}{(x + \sqrt{x^2 - 1})^2} \right) = \sum_{n=1}^{\infty} \mathcal{L} \left( \frac{1}{U_n(x^2)} \right).$$

The reader interested in knowing more about the dilogarithm function and its generalizations, is referred to the book [11] and the aforementioned article [17].

3 Units in $\mathbb{Z}[\sqrt{n}]$, Pell’s equation

We assume $n$ is not a perfect square. If $a + b\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$ is a unit, then $\pm a \pm b\sqrt{n}$ are also units. Therefore, we need only consider solutions $(a, b) \in \mathbb{N}^2$. It follows easily that $a \pm b\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$ is a unit if and only if $(a, b)$ satisfy Pell’s equation over $\mathbb{Z}$

$$a^2 - nb^2 = \pm 1.$$ 

We call a solution $(a, b)$ (or the unit $a + b\sqrt{n}$) positive/negative, depending on whether the right-hand side of the Pell equation is positive/negative. Whereas there is always a solution to the positive Pell equation $x^2 - ny^2 = 1$, it can be shown that there are no solutions to $x^2 - ny^2 = -1$ for certain $n$ (see [15, Chapter 7]).

Continued fraction convergents

If $u \in \mathbb{R}_+$, we say $u$ has continued fraction expansion $u = [c_0, c_1, c_2, c_3, \ldots]$ if $c_i \in \mathbb{Z}$ and

$$u = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \ldots}}}. $$
By this we mean that if we define \( r_n = [c_0, c_1, c_2, \ldots, c_n] \in \mathbb{Q} \) to be the \( n \)th convergent, then \( r_n \to u \) as \( n \to \infty \). If the continued fraction coefficients satisfy \( c_{n+r} = c_n \) for \( n > k \), we say \( u \) is periodic with period \( r \) and write \( u = [c_0, c_1, \ldots, c_k, c_{k+1}, \ldots, c_{k+r}] \). We have the following standard description of \( r_n \):

**Theorem 3.1** [15, Theorems 7.4, 7.5] Let \( u \in \mathbb{R}_+ \) with \( u = [c_0, c_1, c_2, \ldots] \). Define \( h_n, k_n \) by

\[
  h_i = c_i h_{i-1} + h_{i-2} \quad k_i = c_i k_{i-1} + k_{i-2} \quad i \geq 0
\]

with \((h_{-2}, k_{-2}) = (0, 1)\) and \((h_{-1}, k_{-1}) = (1, 0)\). Then \( \gcd(h_i, k_i) = 1 \) and

\[
  r_n = [c_0, c_1, c_2, \ldots, c_n] = \frac{h_n}{k_n}.
\]

The positive units in \( \mathbb{Z}[\sqrt{n}] \) have the following elegant description.

**Theorem 3.2** [15, Theorem 7.26] Let \( n \in \mathbb{N} \) not be a perfect square. Then there is a unique solution \((a, b) \in \mathbb{N}^2\) of Pell’s equation \( x^2 - ny^2 = 1 \) such that the set of solutions to \( x^2 - ny^2 = 1 \) in \( \mathbb{N}^2 \) is \( \{(a_k, b_k)\}_{k=1}^\infty \) where

\[
  a_k + b_k \sqrt{n} = (a + b \sqrt{n})^k.
\]

The pair \((a, b)\) is called the fundamental solution of \( x^2 - ny^2 = 1 \). One consequence of the above is, if \( u \) is the fundamental unit, then \( \{u^k\} \) gives the set of all positive solutions to Pell’s equation. Thus the dilogarithm identity in Theorem 2.2 can be interpreted as a sum over all solutions to Pell’s equation.

### 4 The orthospectrum identity

In a prior paper, the author proved a dilogarithm identity for a hyperbolic surface with geodesic boundary. In [6] the identity was generalized to hyperbolic manifolds by the author and Kahn. The relation to other identities on hyperbolic manifolds, such as the Basmajian identity (see [3]), the McShane–Mirzakhani identity (see [13,14]), and the Luo–Tan identity (see [12]), is discussed in [7].

#### 4.1 Hyperbolic geometry

We will use two models for the hyperbolic plane \( \mathbb{H}^2 \), the upper half-plane model \( \mathbb{H} = \{z \mid \text{Im}(z) > 0\} \), with hyperbolic metric \( ds = |dz|/\text{Im}(z) \), and the Poincaré model \( \mathbb{D} = \{z \mid |z| < 1\} \) with the hyperbolic metric \( ds = 2|dz|/(1 - |z|^2) \). In each model, the group of orientation preserving isometries correspond to the group of conformal automorphisms and is therefore isomorphic to \( \text{PSL}(2, \mathbb{R}) \).

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In $\mathbb{H}$ the geodesics are semi-circles which are orthogonal to the boundary $\partial \mathbb{H} = \mathbb{R}$ (including vertical lines). Thus a geodesic can be identified with its endpoints in $\mathbb{R} \times \mathbb{R}$. Two disjoint geodesics are \textit{ultra parallel} if they do not have a common endpoint in $\mathbb{R}$, and \textit{asymptotically parallel} if they have a common endpoint. If $g, h$ are ultra parallel, then by a Möbius transformation $m \in \text{PSL}(2, \mathbb{R})$, $g, h$ can be mapped to geodesics $m(g), m(h)$ where $m(g)$ has endpoints $1, -1$ and $m(h)$ has endpoints $el, -el$ for some $l > 0$. Then the $y$-axis is a common perpendicular geodesic to $m(g), m(h)$ in $\mathbb{H}$, showing that $g, h$ have a common perpendicular. Also by simple integration, we have that $l$ is the length of the common perpendicular. If $g, h$ are asymptotically parallel, then there is no common perpendicular and the region between $g, h$ is said to form a \textit{cusp} at the common endpoint in $\mathbb{R}$.

4.2 Orthogeodesics and orthospectrum

In order to state the orthospectrum identity, we recall some basic terms.

Let $S$ be a finite area hyperbolic surface with totally geodesic boundary. Then an \textit{orthogeodesic} for $S$ is a proper geodesic arc $\alpha$ which is perpendicular to the boundary $\partial S$ at its endpoints. The set of orthogeodesics of $S$ is denoted $O(S)$. Each boundary component is either a closed geodesic or an infinite geodesic whose endpoints are \textit{boundary cusps} of $S$. We let $N(S)$ be the number of boundary cusps of $S$. Further, let $\chi(S)$ be given by $\text{Area}(S) = -2\pi \chi(S)$. We note that if there are no boundary cusps, then $\chi(S)$ is the Euler Characteristic of $S$.

We note that for $S$ a finite area hyperbolic surface with totally geodesic boundary, the universal cover $\tilde{S} \subseteq \mathbb{H}^2$ is a simply connected convex region bounded by a countable collection of geodesics (see Fig. 1). A lift of an orthogeodesic is then a common perpendicular to two boundary components of $\tilde{S}$ that are ultra parallel.

One elementary example of a surface is an ideal hyperbolic $n$-gon. In this case, $N(S) = n$ and $O(S)$ is a finite set. Also as $\text{Area}(S) = (n-2)\pi$, then $\chi(S) = 1-n/2$. In fact, ideal hyperbolic $n$-gons are the only surfaces with $O(S)$ finite.

The dilogarithm orthospectrum identity is as follows:
Theorem 4.1 (Dilogarithm Orthospectrum Identity, [5]) Let $S$ be a finite area hyperbolic surface with totally geodesic boundary $\partial S \neq 0$. Then

$$\sum_{\alpha \in O(S)} L \left( \frac{1}{\cosh^2 \left( \frac{l(\alpha)}{2} \right)} \right) = -\frac{\pi^2}{12} (6\chi(S) + N(S)),$$

and equivalently,

$$\sum_{\alpha \in O(S)} L \left( \frac{1}{\sinh^2 \left( \frac{l(\alpha)}{2} \right)} \right) = \frac{\pi^2}{12} (6\chi(S) + N(S)).$$

4.3 A geometric decomposition using orthogeodesics

For completeness, we now give a sketch of the proof of the orthospectrum identity. We will see that it follows from an elementary decomposition of the unit tangent bundle of $S$.

Let $\mathcal{T}_1(S)$ be the unit tangent bundle of $S$. Given $v \in \mathcal{T}_1(S)$, we let $\alpha_v$ be the maximal geodesic with tangent vector $v$. Generically (except for a set of measure zero), $\alpha_v$ will be a geodesic arc with endpoints on the boundary of $S$. We define an equivalence relation on $\mathcal{T}_1(S)$, by defining $v \sim w$ if the geodesics $\alpha_v, \alpha_w$ are homotopic rel. $\partial S$.

This gives a partition of (a full measure subset of) $\mathcal{T}_1(S)$ into equivalence classes of two types, one type corresponding to the orthogeodesics and the other type corresponding to boundary cusps. For each orthogeodesic $\gamma \in O(S)$ we have an equivalence class $E_\gamma$ corresponding to all $w \in \mathcal{T}_1(S)$ such that $\alpha_w$ is homotopic rel. boundary to $\gamma$. For each boundary cusp $c$, we have an equivalence class $E_c$ corresponding to all $w \in \mathcal{T}_1(S)$ such that $\alpha_w$ is homotopic rel boundary out the cusp $c$. Then the equivalence relation gives a volume relation

$$\text{Vol}(\mathcal{T}_1(S)) = \sum_{\gamma \in O(S)} \text{Vol}(E_\gamma) + \sum_{c \text{ boundary cusp}} \text{Vol}(E_c).$$

For the left-hand side, we have $\text{Vol}(\mathcal{T}_1(S)) = 2\pi \text{Area}(S) = -4\pi^2 \chi(S)$.

Lifting an orthogeodesic $\gamma$ to $\tilde{\gamma}$ in the universal cover $\tilde{S}$, we have $\tilde{\gamma}$ is the common perpendicular to two geodesic components $g, h$ of $\partial \tilde{S}$. Then $E_\gamma$ lifts to the set $\tilde{E}_\gamma$ of vectors which are between $g$ and $h$ in the following sense. The vector $v \in \mathcal{T}_1(\mathbb{H}^2)$ is between $g$ and $h$, if the unique geodesic $\alpha_v$ tangent to $v$, intersects both $g$ and $h$. Thus it follows that $\text{Vol}(E_\gamma)$ only depends on $l(\gamma)$ and, by direct calculation (see [5]), we have

$$\text{Vol}(E_\gamma) = 8L \left( \frac{1}{\cosh^2 \left( \frac{l(\gamma)}{2} \right)} \right). \quad (4.3)$$

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Similarly, the equivalence class corresponding to a cusp $E_c$ lifts to the set of tangent vectors between two geodesics $g, h$ that have a common ideal endpoint. Therefore, as $\text{PSL}(2, \mathbb{R})$ acts transitively on triples on $\mathbb{R}$, we can assume the endpoints of $g$ are 0, 1 and $h$ are 1, 2. Therefore each $E_c$ are isometric and have the same volume. Then applying the identity to an ideal triangle, which has no orthogeodesics, 3 boundary cusps and area $\pi$, we get

$$\text{Vol}(E_c) = \frac{2\pi^2}{3}. $$

Substituting these gives the orthospectrum identity,

$$\text{Vol}(T_1(S)) = -4\pi^2 \chi(S) = \sum_{\gamma \in O(S)} 8L \left( \frac{1}{\cosh^2 \left( \frac{l(\gamma)}{2} \right)} \right) + N(S) \frac{2\pi^2}{3}.$$

In the original paper [5], we showed that the orthospectrum identity above recovers the reflection identities, Landen’s identity and Abel’s identity, by considering the elementary cases of the ideal quadrilateral and ideal pentagon, respectively.

5 An infinite dilogarithm identity

Given $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ distinct points we define the cross-ratio by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}. $$

Let $\mathbb{H}$ be the upper half-plane model for the hyperbolic plane and $x_1, x_2, x_3, x_4 \in \partial \mathbb{H} = \mathbb{R}$ be distinct points, ordered counterclockwise on $\mathbb{R}$. If $g$ is the geodesic with endpoints $x_1, x_2$, and $h$ is the geodesic with endpoints $x_3, x_4$, then $g, h$ are disjoint. We let $l$ be the perpendicular distance between $g$ and $h$. Then we can choose a Möbius transformation $m \in \text{PSL}(2, \mathbb{R})$ such that $m(g)$ has endpoints $-1, 1$ and $m(h)$ has endpoints $-e^l, e^l$. Then by invariance of the cross-ratio under Möbius transformations, we have

$$[x_1, x_2, x_3, x_4] = [-1, 1, e^l, -e^l] = \frac{1}{\cosh^2(l/2)} \quad (5.4)$$

We now prove Theorem 2.1.

Proof of Theorem 2.1 Let $S$ be an annulus with two geodesic boundary components $g, h$. Let $g$ be a closed geodesic of length $L$, and $h$ an infinite geodesic with a single boundary cusp (see Fig. 2).

We lift $S$ to the upper half-plane with $g$ lifted to the y-axis. Further let $\lambda = e^L$. Then $\tilde{S}$ is an infinite-sided ideal polygon invariant under multiplication by $\lambda$ (see Fig. 3).
We normalize so that one of the ideal vertices is at \( z = 1 \). Then the vertices of \( \tilde{S} \) are 0, \( \infty \) and \( \lambda^k \) for \( k \in \mathbb{Z} \). The edges of \( \tilde{S} \) are the lift of \( g \), denoted \( \tilde{g} \), which has vertices 0, \( \infty \), and the lifts of \( h \), labelled \( \tilde{h}_k \), which has vertices \( \lambda^k, \lambda^{k+1} \).

We now compute the orthospectrum of \( S \). Every orthogeodesic lifts to a geodesic that is the common perpendicular between two boundary components of \( \tilde{S} \). We consider two types.

If \( \alpha \) is an orthospectrum with an endpoint on \( g \), then it lifts to \( \tilde{\alpha} \) which is a perpendicular between two edges of \( \tilde{S} \), with one edge being \( \tilde{g} \). By the \( \mathbb{Z} \) action, which preserves \( \tilde{g} \), we can choose \( \tilde{\alpha} \) to have the other endpoint on \( \tilde{h}_0 \). Therefore \( \alpha \) has length \( l \) satisfying

\[
\frac{1}{\cosh^2(l/2)} = [\infty, 0, 1, \lambda] = \frac{\lambda - 1}{\lambda} = 1 - e^{-L}.
\]

Any other orthogeodesic \( \alpha \) has both endpoints in \( h \). Therefore \( \alpha \) lifts to \( \tilde{\alpha} \) which is the perpendicular between \( \tilde{h}_j, \tilde{h}_k \) for some \( j < k \). By the action of \( \mathbb{Z} \), we can assume \( j = 0 \). Also, as adjacent sides do not have a common perpendicular, we have that
Denoting the length $l_k$ of the perpendicular between $\tilde{h}_0$ and $\tilde{h}_k$, we have
\[
\frac{1}{\cosh^2(l_k/2)} = [1, \lambda, \lambda^k, \lambda^{k+1}] = \frac{(1-\lambda)(\lambda^{k+1} - \lambda^k)}{(1-\lambda^k)(\lambda^{k+1} - \lambda)} = \lambda^{k-1} \frac{(\lambda - 1)^2}{(\lambda^k - 1)^2}
\]
\[
= \frac{(\lambda^{1/2} - \lambda^{-1/2})^2}{(\lambda^{k/2} - \lambda^{-k/2})^2} = \frac{\sinh^2(L/2)}{\sinh^2(kL/2)}.
\]

As $Area(S) = \pi$, then $\chi(S) = -1/2$. Furthermore $N(S) = 1$. Thus by Theorem 4.1, we have the dilogarithm identity for $S$ is
\[
\mathcal{L}(1 - e^{-L}) + \sum_{k=2}^{\infty} \mathcal{L} \left( \frac{\sinh^2(L/2)}{\sinh^2(kL/2)} \right) = -\frac{\pi^2}{12} (-6(1/2) + 1) = \frac{\pi^2}{6}.
\]

Using the reflection identity $\mathcal{L}(1 - x) + \mathcal{L}(x) = \pi^2/6$, we get
\[
\mathcal{L}(e^{-L}) = \sum_{k=2}^{\infty} \mathcal{L} \left( \frac{\sinh^2(L/2)}{\sinh^2(kL/2)} \right).
\]

\[\square\]

### 5.1 Hyperbolic volume and $\text{PSL}(2, \mathbb{R})$ volume

Another important normalization of the dilogarithm is the Bloch–Wigner dilogarithm $D : \mathbb{C} - \{0, 1\} \rightarrow \mathbb{R}$ by
\[
D(z) = \text{Im}(\text{Li}_2(z)) + \arg(1 - z) \log |z|.
\]

This was introduced by Bloch on his work in K-theory and regulators and by Wigner in his work on Lie groups (see [4]).

The Bloch–Wigner dilogarithm function also arises naturally in the formula for the volume of an ideal hyperbolic tetrahedron. If $T$ is an ideal hyperbolic tetrahedron $T$ in $\mathbb{H}^3$, with ideal vertices $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$, then a classical result (see [8, Equation 4.13]) states that the volume of $T$ is given by
\[
\text{Vol}(T) = D([z_1, z_2, z_3, z_4]).
\]

Similarly, in the orthospectrum identity, we see that $\mathcal{L}(x)$ is also a volume. If $x_1, x_2, x_3, x_4$ are distinct points ordered counterclockwise on $\partial \mathbb{H}^2$, we let $g$ be the geodesic with endpoints $x_1, x_2$ and $h$ the geodesic with endpoints $x_3, x_4$. We let $T$ be the set of tangent vectors in $T_1(\mathbb{H}^2)$ between $g, h$ as described in Sect. 4.3. Then, considering the volume measure on $T_1(\mathbb{H}^2)$, by Eq. (4.3), we have
\[
\text{Vol}(T) = 8 \mathcal{L}([x_1, x_2, x_3, x_4]).
\]
Interpreting $T_1(\mathbb{H}^2)$ as $\text{PSL}(2, \mathbb{R})$, we see that the volume of an ideal tetrahedron in $\text{PSL}(2, \mathbb{R})$ is given by the Rogers dilogarithm of the cross-ratio of its vertices.

### 6 Proof of identity for solutions to Pell’s equation over $\mathbb{Q}$

We now prove the dilogarithm identity for solutions to Pell’s equation over $\mathbb{Q}$ given in Theorem 2.3.

**Proof of Theorem 2.3** Let $e^{L/2} = u = a + b\sqrt{n}$, then $e^{-L/2} = u^{-1} = \pm(a - b\sqrt{n})$ with the sign depending on if $u$ is a positive or negative unit. If $u$ is a positive unit, then

$$\cosh(L/2) = a \quad \text{and} \quad \sinh(L/2) = b\sqrt{n}.$$  

If $u$ is a negative unit, then

$$\sinh(L/2) = a \quad \text{and} \quad \cosh(L/2) = b\sqrt{n}.$$  

In both cases we have

$$u^k = e^{kL/2} = \cosh(kL/2) + \sinh(kL/2).$$

We let $m_k = \sinh(kL/2)$ and $n_k = \cosh(kL/2)$. The dilogarithm identity gives

$$\mathcal{L} \left( \frac{1}{u^2} \right) = \sum_{k=2}^{\infty} \mathcal{L} \left( \frac{\sinh^2(L/2)}{\sinh^2(kL/2)} \right) = \sum_{k=2}^{\infty} \mathcal{L} \left( \frac{m_1^2}{m_k^2} \right).$$

If $u$ is a positive root, then $m_1 = \sinh(L/2) = b\sqrt{n}$ and $n_1 = \cosh(L/2) = a$. Then by the addition formulae, we have

$$m_{k+1} = am_k + bn_k\sqrt{n} \quad \text{and} \quad n_{k+1} = n_k a + bm_k\sqrt{n}.$$  

By induction, we have $n_k = a_k$ and $m_k = b_k\sqrt{n}$, and

$$b_{k+1} = ab_k + ba_k \quad \text{and} \quad a_{k+1} = aa_k + nbb_k.$$  

Substituting, we get

$$\mathcal{L} \left( \frac{1}{u^2} \right) = \sum_{k=1}^{\infty} \mathcal{L} \left( \frac{b_k^2}{b_k^2} \right) = \sum_{k=1}^{\infty} \mathcal{L} \left( \frac{1}{(b_k/b)^2} \right).$$

If $u$ is a negative solution, then $m_1 = \sinh(L/2) = a$ and $n_1 = \cosh(L/2) = b\sqrt{n}$. Then, by the addition formulae we have,

$$m_{k+1} = bm_k\sqrt{n} + an_k \quad \text{and} \quad n_{k+1} = bn_k\sqrt{n} + am_k.$$
Therefore
\[ n_{2k} = a_{2k}, \quad n_{2k+1} = b_{2k+1}\sqrt{n}, \quad m_{2k} = b_{2k}\sqrt{n}, \quad \text{and} \quad m_{2k+1} = a_{2k+1}. \]

It follows that
\[ b_{2k} = ba_{2k-1} + ab_{2k-1} \quad \text{and} \quad a_{2k+1} = bb_{2k} + aa_{2k}. \]

Therefore,
\[ \mathcal{L}\left(\frac{1}{u^{2}}\right) = \sum_{k=1}^{\infty} \mathcal{L}\left(\frac{m_{k}^{2}}{m_{k}^{2}}\right) = \sum_{k=1}^{\infty} \mathcal{L}\left(\frac{1}{n(b_{2k}/a)^{2}}\right) + \mathcal{L}\left(\frac{1}{(a_{2k+1}/a)^{2}}\right). \]

We now prove Corollary 2.4 relating the identity to the Chebyshev polynomials \( U_n \) of the second kind.

**Proof of Corollary 2.4** We have the Chebyshev polynomials \( T_n(x), U_n(x) \in \mathbb{R}[x] \). We let \( x = \cos(\theta) \), then \( \sin(\theta) = \sqrt{1 - x^2} \). Therefore
\[ e^{i\theta} = \cos(\theta) + i \sin(\theta) = x + i \sqrt{1 - x^2} = x + \sqrt{x^2 - 1}. \]

Thus
\[ e^{in\theta} = (x + \sqrt{x^2 - 1})^n \quad \text{and} \quad e^{-in\theta} = (x - \sqrt{x^2 - 1})^n. \]

Substituting, we get
\[ T_n(x) = \cos(n\theta) = \frac{1}{2} \left( (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right) \]
and
\[ U_{n-1}(x) = \frac{\sin(n\theta)}{\sin(\theta)} = \frac{1}{2\sqrt{x^2 - 1}} \left( (x + \sqrt{x^2 - 1})^n - (x - \sqrt{x^2 - 1})^n \right). \]

As this holds for \( |x| < 1 \), it also holds for all \( x \in \mathbb{R} \). If \( x > 1 \), we define \( L > 0 \) to be given by \( x = \cosh(L/2) \). Then \( \sqrt{x^2 - 1} = \sinh(L/2) \), giving
\[ x + \sqrt{x^2 - 1} = e^{L/2} \quad \text{and} \quad x - \sqrt{x^2 - 1} = e^{-L/2}. \]

Therefore, by the above formulae
\[ T_k(x) = \frac{e^{kL/2} + e^{-kL/2}}{2} = \cosh(kL/2) \quad \text{and} \]
\[ U_{k-1}(x) = \frac{e^{kL/2} - e^{-kL/2}}{2 \sinh(L/2)} = \frac{\sinh(kL/2)}{\sinh(L/2)}. \]
Thus
\[ L \left( \frac{1}{(x + \sqrt{x^2 - 1})^2} \right) = \sum_{k=2}^{\infty} L \left( \frac{\sinh^2(L/2)}{\sinh^2(kL/2)} \right) = \sum_{k=1}^{\infty} L \left( \frac{1}{U_k(x)^2} \right). \]

\[ \square \]

7 Identity for continued fraction convergents

We now consider the case where \( u \in \mathbb{Z} [\sqrt{n}] \). We prove Theorem 2.2 expressing the above in terms of the convergents \( r_j = h_j/k_j \) of their continued fractions expansion.

First, we have the following lemma.

**Lemma 7.1** Let \( u = a + b \sqrt{n} \in \mathbb{Z} [\sqrt{n}] \) be a solution to Pell’s equation with \( a, b \in \mathbb{N} \). If \( u \) is a positive solution, then \( u = [2a - 1, 1, 2a - 2] \). If \( u \) is a negative solution, then \( u = [0, 2a] \).

**Proof** If \( u \) is a negative solution, then \( u = a + \sqrt{a^2 + 1} \). Therefore \( u^2 - 2au - 1 = 0 \). Therefore
\[ u = 2a + \frac{1}{u}. \]

Thus \( u = [\overline{2a}] \).

If \( u \) is a positive solution, then \( u = a + \sqrt{a^2 - 1} \). Therefore \( u \) satisfies the quadratic \( u^2 - 2au + 1 = 0 \). Rewriting, we have
\[ u = 2a - \frac{1}{u} = 2a - 1 + 1 - \frac{1}{u} = 2a - 1 + \frac{u - 1}{u}. \]

Now we have
\[ \frac{u - 1}{u} = \frac{1}{u - 1} = \frac{1}{1 + \frac{1}{u - 1}} = \frac{1}{1 + \frac{1}{2a - 2 + \frac{1}{u}}} . \]

Therefore \( u = [2a - 1, 1, 2a - 2] \). \[ \square \]

Using the above description of the continued fraction, we will show the relation between the approximates \( r_j = h_j/k_j \) for \( u \) and the coefficients \( a_j, b_j \) given by \( u^j = a_j + b_j \sqrt{n} \). This will allow us to prove Theorem 2.2.

**Lemma 7.2** Let \( u = a + b \sqrt{n} \in \mathbb{Z} [\sqrt{n}] \) be a solution to Pell’s equation.

If \( u \) is a positive solution and \( u \) has continued fraction convergents \( r_j = h_j/k_j \), then \( k_j = h_{j-2} \) and
\[ L \left( \frac{1}{u^2} \right) = \sum_{j=1}^{\infty} L \left( \frac{1}{(h_{2j-1})^2} \right). \]
If $u$ is a negative solution and $u^2$ has continued fraction convergents $R_j = H_j/K_j$, then
\[ \mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{j=0}^{\infty} \left( \mathcal{L}\left(\frac{1}{nb^2(2H_{2j-1})^2}\right) + \mathcal{L}\left(\frac{1}{(2H_{2j+1} - H_{2k})^2}\right) \right). \]

**Proof** Let $u = a + b\sqrt{n} = e^{L/2}$, then $u^2 = a_k + b_k\sqrt{n} = \cosh(kL/2) + \sinh(kL/2)$.

If $u$ is a positive solution, then $u = [2a - 1, 1, 2a - 2]$. Therefore we have $(h_0, h_{-1}) = (2a - 1, 1)$. By Theorem 3.1 describing the continued fraction convergents, for $k > 0$
\[ \begin{bmatrix} h_{2k} \\ h_{2k-1} \end{bmatrix} = A^k \begin{bmatrix} 2a - 1 \\ 1 \end{bmatrix} \]
where
\[ A = \begin{bmatrix} 2a - 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2a - 1 & 2a - 2 \\ 1 & 1 \end{bmatrix}. \]

The matrix $A$ has characteristic polynomial $x^2 - 2ax + 1$. Therefore $A$ has eigenvalues $u, u^{-1}$ and eigenvectors $(u - 1, 1), (1 - u, u)$. Diagonalizing, we get
\[ \begin{bmatrix} h_{2k} \\ h_{2k-1} \end{bmatrix} = \frac{1}{u^2 - 1} \begin{bmatrix} u - 1 & 1 - u \\ 1 & u \end{bmatrix} \begin{bmatrix} u^k & 0 \\ 0 & u^{-k} \end{bmatrix} \begin{bmatrix} u - 1 & 1 - u \\ 1 & u \end{bmatrix} = \begin{bmatrix} 2a - 1 \\ 1 \end{bmatrix}. \]

As $u = e^{L/2}$, we have
\[ h_{2k} = \frac{(u - 1)\left(u^{k+2} + u^{-(k+1)}\right)}{u^2 - 1} = \frac{\cosh((k + \frac{3}{2})L/2)}{\cosh(L/4)}, \]
\[ h_{2k-1} = \frac{u^{k+2} - u^{-k}}{u^2 - 1} = \frac{\sinh((k + 1)L/2)}{\sinh(L/2)}. \]

It follows that for $k \geq 1$
\[ h_{2k-3} = \frac{\sinh(kL/2)}{\sinh(L/2)} = \frac{b_k}{b}. \]

Therefore
\[ \mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=2}^{\infty} \mathcal{L}\left(\frac{1}{(b_k/b)^2}\right) = \sum_{j=1}^{\infty} \mathcal{L}\left(\frac{1}{(h_{2j-1})^2}\right). \]

Similarly, we note that as $(k_0, k_1) = (1, 0)$, then applying the above analysis we get
\[ k_{2j} = \frac{\cosh((j + \frac{1}{2})L/2)}{\cosh(L/4)} = h_{2j-2}. \]
and
\[ k_{2j-1} = \frac{\sinh(jL/2)}{\sinh(L/2)} = h_{2j-3}. \]

Therefore \( k_j = h_{j-2}. \)

Let \( u \) be a negative solution. Then for \( k \) odd, \( a_k = \sinh(kL/2), b_k\sqrt{n} = \cosh(kL/2) \) and for \( k \) even, \( b_k\sqrt{n} = \sinh(kL/2), a_k = \cosh(kL/2). \)

As \( u = [2a], \) by Theorem 3.1 we have the formula
\[ h_{j+1} = 2ah_j + h_{j-1} \quad \text{and} \quad k_{j+1} = 2ak_j + k_{j+1}, \]
with \( (h_{-2}, k_{-2}) = (0, 1) \) and \( (h_{-1}, k_{-1}) = (1, 0). \) Iterating, we get \( h_j = 0, 1, 2a, \ldots \) and \( k_j = 1, 0, 1, 2a, \ldots \) Therefore \( k_j = h_{j-1} \) for \( j \geq -1. \) We focus on calculating \( h_k. \) As \( (h_{-1}, h_{-2}) = (1, 0), \) we have the recursion
\[
\begin{bmatrix}
    h_k \\
    h_{k-1}
\end{bmatrix} = A^{k+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 2a & 1 \\ 1 & 0 \end{bmatrix}.
\]

The matrix \( A \) has characteristic polynomial \( x^2 - 2ax - 1 = 0. \) Therefore \( A \) has eigenvalues \( u, -u^{-1} \) and eigenvectors \((u, 1), (1, -u). \) Thus,
\[
\begin{bmatrix}
    h_k \\
    h_{k-1}
\end{bmatrix} = \frac{1}{u^2 + 1} \begin{bmatrix} u & 1 \\ 1 & -u \end{bmatrix} \begin{bmatrix} u^{k+1} & 0 \\ 0 & (u)^{-k-1} \end{bmatrix} \begin{bmatrix} u & 1 \\ 1 & -u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Multiplying, we get
\[
h_k = \frac{1}{u^2 + 1} \left( u^{k+3} + (-1)^{k+1} u^{-(k+1)} \right) = \frac{1}{u + u^{-1}} \left( u^{k+2} + (-1)^{k+1} u^{-(k+2)} \right) .
\]

For \( k \) odd, we have
\[
h_k = \frac{\cosh((k + 2)L/2)}{\cosh(L/2)} = \frac{b_{k+2}}{b}.
\]

Similarly, for \( k \) even, we have
\[
h_k = \frac{\sinh((k + 2)L/2)}{\cosh(L/2)} = \frac{b_{k+2}}{b}.
\]

Thus for all \( k \geq 0 \)
\[
\frac{b_k}{b} = h_{k-2}.
\]

We let \( H_j, K_j \) be the convergents for the continued fraction expansion of \( u^2. \) Then \( u^2 = e^L \) is a positive solution to Pell’s equation. Applying Eqs. 7.5 and 7.6 above we
have,
\[
H_{2k} = \frac{\cosh((2k + 3)L/2)}{\cosh(L/2)} = \frac{b_{2k+3}}{b} = h_{2k+1}.
\]
\[
H_{2k-1} = \frac{\sinh((k + 1)L)}{\sinh(L)} = \frac{1}{2} \left( \frac{\sinh(2k + 1)L/2}{\sinh(L/2)} + \frac{\cosh(2k + 1)L/2}{\cosh(L/2)} \right)
\]
\[= \frac{1}{2} \left( \frac{a_{2k+1}}{a} + h_{2k-1} \right). \]

Also, if \((u^2)^k = A_k + B_k \sqrt{n}\) then \(A_k = a_{2k}, B_k = b_{2k}\). Then by Eq. (7.7),
\[
H_{2k-3} = \frac{B_k}{B_1} = \frac{b_{2k}}{2ab} = \frac{h_{2k-2}}{2a}.
\]

Therefore
\[
h_{2k} = 2aH_{2k-1} \quad \text{and} \quad h_{2k+1} = H_{2k}.
\]

Also
\[
\frac{b_{2k}}{a} = 2bH_{2k-3} \quad \text{and} \quad \frac{a_{2k+1}}{a} = 2H_{2k-1} - h_{2k-1} = 2H_{2k-1} - H_{2k-2}.
\]

Thus, if \(u\) is a negative solution to Pell’s equation
\[
\mathcal{L} \left( \frac{1}{u^2} \right) = \sum_{k=0}^{\infty} \mathcal{L} \left( \frac{1}{n(2bH_{2k-1})^2} \right) + \mathcal{L} \left( \frac{1}{(2H_{2k-1} - H_{2k-2})^2} \right).
\]

\[\square\]

8 Ideal n-gon identities

We now describe the orthospectrum identity for a general ideal hyperbolic \(n\)-gon. We show that the case of the regular ideal hyperbolic \((2n+1)\)-gon recovers an identity of Richmond and Szekeres (see [11, Equation 2.51]). We also show that the regular ideal hyperbolic hexagon case recovers the following value-identities.

Ramanujan gave the following value-identities for linear combinations of specific values of \(\mathcal{L}\) (see [1, Entry 39]):

1. \(\text{Li}_2 \left( \frac{1}{4} \right) - \frac{1}{6} \text{Li}_2 \left( \frac{1}{5} \right) = \frac{\pi^2}{18} - \frac{\log^2 3}{6}\)
2. \(\text{Li}_2 \left( -\frac{1}{4} \right) + \frac{1}{6} \text{Li}_2 \left( \frac{1}{5} \right) = -\frac{\pi^2}{18} + \log 2 \log 3 - \frac{\log^2 2}{2} - \frac{\log^2 3}{3}\)
3. \(\text{Li}_2 \left( \frac{1}{4} \right) + \frac{1}{3} \text{Li}_2 \left( \frac{1}{6} \right) = \frac{\pi^2}{18} + 2 \log 2 \log 3 - 2 \log^2 2 - 2 \log^2 3\)
4. \(\text{Li}_2 \left( -\frac{1}{3} \right) - \frac{1}{3} \text{Li}_2 \left( \frac{1}{9} \right) = -\frac{\pi^2}{18} - \frac{\log^2 3}{6}\)
5. \(\text{Li}_2 \left( -\frac{1}{8} \right) + \text{Li}_2 \left( \frac{1}{9} \right) = -\frac{\log^2 (9/8)}{2}\)
More recently, in the article [2], Bailey, Borwein and Plouffe gave the identity

\[ 36 \text{Li}_2 \left( \frac{1}{2} \right) - 36 \text{Li}_2 \left( \frac{1}{4} \right) - 12 \text{Li}_2 \left( \frac{1}{8} \right) + 6 \text{Li}_2 \left( \frac{1}{64} \right) = \pi^2. \]  

Applying Landen’s identity, we have \( \mathcal{L}(-1/3) = -\mathcal{L}(1/4) \) and \( \mathcal{L}(-1/8) = -\mathcal{L}(1/9) \). This reduces the value-identities of Ramanujan to the two equations

\[ \mathcal{L} \left( \frac{1}{4} \right) + \frac{1}{3} \mathcal{L} \left( \frac{1}{9} \right) = \frac{\pi^2}{18} \quad \text{and} \quad \mathcal{L} \left( \frac{1}{3} \right) - \frac{1}{6} \mathcal{L} \left( \frac{1}{9} \right) = \frac{\pi^2}{18}. \]

We recall the dilogarithm identity in [5] for ideal hyperbolic polygons. Let \( P \) be an ideal polygon in \( \mathbb{H}^2 \) with vertices in counterclockwise order \( x_1, \ldots, x_n \) about \( \partial \mathbb{H}^2 \). If \( l_{ij} \) is the length of the orthogeodesic joining side \([x_i, x_{i+1}]\) to \([x_j, x_{j+1}]\), then by Eq. (5.4), we have

\[ [x_i, x_{i+1}, x_j, x_{j+1}] = \frac{1}{\cosh^2 \left( l_{ij}/2 \right)}. \]

As \( \text{Area}(P) = (n - 2)\pi \), then \( \chi(P) = -n/2 \). Furthermore, \( N(P) = n \).

Applying the orthospectrum identity in Theorem 4.1 to \( P \), we obtain the equation

\[ \sum_{|i-j| \geq 2} \mathcal{L}([x_i, x_{i+1}, x_j, x_{j+1}]) = \frac{\pi^2}{12} \left( -6 \left( \frac{n - 2}{2} \right) + n \right) = \frac{(n - 3)\pi^2}{6}. \]

If \( P \) is the regular ideal \( n \)-gon, then in the Poincaré disk model for \( \mathbb{H}^2 \), we can choose \( P \) to have vertices \( e^{\pi ik/n} \) for \( k = 0, \ldots, n - 1 \). Therefore, taking cross-ratios and grouping terms, we obtain the equation

\[ \frac{e_n}{2} \mathcal{L} \left( \sin^2 \left( \frac{\pi}{n} \right) \right) + \sum_{k=2}^{\left\lceil \frac{n}{2} \right\rceil} \mathcal{L} \left( \frac{\sin^2(\pi/n)}{\sin^2(k\pi/n)} \right) = \frac{(n - 3)\pi^2}{6n} \]  

where \( e_n = 0 \) if \( n \) is odd and \( e_n = 1 \) if \( n \) is even. In the case of \( n \) odd, Eq. (8.9) recovers the identity of Richmond and Szekeres (see [11, Equation 2.51]) which they derived using Rogers–Ramanujan partition identities.

### 8.1 Ideal hexagons and Ramanujan’s value-identities

We now show that Ramanujan’s value-identities 1–5, and identity 8.8 of Bailey, Borwein, Plouffe, correspond to identities for the regular ideal hexagon.

For the regular 6-gon \( H_{\text{reg}} \) the orthospectrum identity gives

\[ 6\mathcal{L} \left( \frac{1}{3} \right) + 3\mathcal{L} \left( \frac{1}{4} \right) = \frac{\pi^2}{2}. \]
By Landen’s identity, $\mathcal{L}(-1/3) = -\mathcal{L}(1/4)$. Therefore applying the squaring identity we get

$$
\frac{1}{2} \mathcal{L} \left( \frac{1}{9} \right) = \mathcal{L} \left( \frac{1}{3} \right) + \mathcal{L} \left( -\frac{1}{3} \right) = \mathcal{L} \left( \frac{1}{3} \right) - \mathcal{L} \left( \frac{1}{4} \right).
$$

Thus, we obtain

$$
\mathcal{L} \left( \frac{1}{3} \right) - \mathcal{L} \left( \frac{1}{4} \right) = \frac{1}{2} \mathcal{L} \left( \frac{1}{9} \right).
$$

Combining this and the identity above for the regular hexagon, we obtain Ramanujan’s value-identities

$$
\mathcal{L} \left( \frac{1}{4} \right) + \frac{1}{3} \mathcal{L} \left( \frac{1}{9} \right) = \frac{\pi^2}{18}, \quad \mathcal{L} \left( \frac{1}{3} \right) - \frac{1}{6} \mathcal{L} \left( \frac{1}{9} \right) = \frac{\pi^2}{18}.
$$

To recover the identity 8.8, we note that by Landen’s identity $\mathcal{L}(-1/8) = -\mathcal{L}(1/9)$. Then by the squaring identity, we have

$$
\frac{1}{2} \mathcal{L} \left( \frac{1}{64} \right) = \mathcal{L} \left( \frac{1}{8} \right) + \mathcal{L} \left( -\frac{1}{8} \right) = \mathcal{L} \left( \frac{1}{8} \right) - \mathcal{L} \left( \frac{1}{9} \right).
$$

Therefore, substituting for $\mathcal{L}(1/8)$, we get

$$
36 \mathcal{L} \left( \frac{1}{2} \right) - 36 \mathcal{L} \left( \frac{1}{4} \right) - 12 \mathcal{L} \left( \frac{1}{8} \right) + 6 \mathcal{L} \left( \frac{1}{64} \right) = 36 \mathcal{L} \left( \frac{1}{2} \right) - 36 \mathcal{L} \left( \frac{1}{4} \right) - 12 \mathcal{L} \left( \frac{1}{9} \right).
$$

As $\mathcal{L}(1/2) = \pi^2/12$, and applying the hexagon identity $3\mathcal{L}(1/4) + \mathcal{L}(1/9) = \pi^2/6$, we recover identity 8.8.

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