IRREDUCIBLE JET MODULES FOR THE VECTOR FIELD LIE ALGEBRA ON $S^1 \times \mathbb{C}$

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Abstract. For a commutative algebra $A$ over $\mathbb{C}$, denote $\mathfrak{g} = \text{Der}(A)$. A module over the smash product $A \# U(\mathfrak{g})$ is called a jet $\mathfrak{g}$-module, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. In the present paper, we study jet modules in the case of $A = \mathbb{C}[t_1 \pm 1, t_2]$. We show that $A \# U(\mathfrak{g}) \cong D \otimes U(L)$, where $D$ is the Weyl algebra $\mathbb{C}[t_1 \pm 1, t_2, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}]$, and $L$ is a Lie subalgebra of $A \# U(\mathfrak{g})$ called the jet Lie algebra corresponding to $\mathfrak{g}$. Using a Lie algebra isomorphism $\theta : L \to m_{1,0} \Delta$, where $m_{1,0} \Delta$ is the subalgebra of vector fields vanishing at the point $(1,0)$, we show that any irreducible finite dimensional $L$-module is isomorphic to an irreducible $\mathfrak{gl}_2$-module. As an application, we give tensor product realizations of irreducible jet modules over $\mathfrak{g}$ with uniformly bounded weight spaces.

Keywords: Smash product, jet module, weight module, Weyl algebra, jet algebra.

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1. Introduction

Among the theory of infinite dimensional Lie algebras, Lie algebras $\mathcal{V}_X$ of polynomial vector fields (i.e., the derivation Lie algebras of the affine coordinate algebras $A_X$) on irreducible affine algebraic varieties $X$ is an important class of Lie algebras. This kind of Lie algebras have been studied in the conformal field theory, see [7]. Unlike finite dimensional simple Lie algebras, the representation theory of vector fields Lie algebras at large is still not well developed. The centerless Virasoro algebra $W_1$ is the Lie algebra of polynomial vector fields on the circle $S^1$ whose irreducible modules with finite-dimensional weight spaces were classified in [22]. Higher rank Witt algebras $W_n$ are simple Lie algebras of polynomial vector fields on $n$-dimensional torus. There are quite a lot of studies on representations for $W_n$, see [3, 10, 13, 15, 16, 17, 18, 19, 20, 24, 25, 30]. Billig and Futorny classified all irreducible Harish-Chandra $W_n$-module, see [2]. Weight modules for the Lie algebra of vector fields on $\mathbb{C}^n$ were studied in [11, 13, 12, 26, 27, 28, 31]. Recently, there is a systematic study on representations of the Lie algebra $\mathcal{V}_X$ for arbitrary smooth affine varieties $X$, see [3, 4, 5, 6, 8].

Let $A$ be a commutative associative algebra, $\mathfrak{g}$ the derivation algebra $\text{Der}(A)$ of $A$. We say that a module $M$ is a jet module (also called $AV$-module in [3, 8]) if it is a module both for the Lie algebra $\mathfrak{g}$ and for the commutative associative algebra $A$, and the action between $A$ and $\mathfrak{g}$ are compatible in the following rule: $X(fv) = f(Xv) + X(f)v$, $X \in \mathfrak{g}, f \in A, v \in M$. This definition of jet modules is more general, without the condition that $M$ is free over $A$. The name of jet modules (which are also free over the affine coordinate algebras) for vector field
Lie algebras was firstly introduced in [1]. Jet modules for $\text{Der}\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ were classified in [15]. In [1], this classification was also given by the technique of polynomial modules. Since $A$ is a left module algebra over the Hopf algebra $U(g)$, we have the smash product algebra $A\#U(g)$. Then jet modules are exactly modules over the associative algebra $A\#U(g)$. When $A = \mathbb{C}[x_1, \ldots, x_n]$, the structure of the algebra $A\#U(g)$ was independently described by the classical Weyl algebra and the Lie algebra of vector fields vanishing at the origin in [31] and [8]. We found that the ways of approaching jet modules for $\text{Der}\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ and $\text{Der}\mathbb{C}[t_1, \ldots, t_n]$ are different, see [1, 15, 8, 31]. Our motivation in the present paper is to find a general method to handle jet modules for any vector field Lie algebra $V_X$. As a tentative research, we study the structure of $A\#U(g)$ for the mixed type algebra $\mathbb{C}[t_1^{\pm 1}, t_2]$. We show that $A\#U(g) \cong D \otimes U(L)$, where $D$ is the differential operator algebra $\mathbb{C}[t_1, t_1^{-1}, t_2, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}]$, and $L$ is the jet Lie algebra of $g$. Using a Lie algebra isomorphism $\theta : L \to m_{1,0}\Delta$, rather than the technique of polynomial modules, we show that any irreducible finite dimensional $L$-module is actually an irreducible $g_{t_2}$-module. As a corollary, we give a classification of irreducible jet modules over $g$ with uniformly bounded weight spaces. These known results reveal that the key point in researching jet modules for any vector field Lie algebra $V_X$ lies in clarifying the structure of the jet algebra associated with $V_X$.

The paper is organized as follows. In Section 2, we give some basic facts about $g$ and our main result, see Theorem 2.3. In Section 3, we study the structure of the jet Lie algebra $L$ for $g$. We also classify finite dimensional irreducible modules over $L$, see Theorem 3.6. The isomorphism $L \cong m_{1,0}\Delta$, in Lemma 3.3 makes the structure of $L$ become very clear. In Section 4, we give a proof of Theorem 2.3. It should be noted that to separate the operator $\frac{\partial}{\partial t_1}$ in $D = \mathbb{C}[t_1^{\pm 1}, t_2, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}]$ from $A\#U(g)$, we use $t_1^{-1} \cdot t_1 \partial_1$ instead of the element $\partial_1$ in $g$. Finally, we give tensor product realizations of irreducible jet modules over $g$ with uniformly bounded weight spaces, see Theorem 4.5.

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{C}$ the sets of all integers, nonnegative integers, positive integers and complex numbers, respectively. For a Lie algebra $L$ over $\mathbb{C}$, we use $U(L)$ to denote the universal enveloping algebra of $L$.

2. Preliminaries and main results

We fix the vector space $\mathbb{C}^2$ of $2 \times 1$ complex matrices. Denote the standard basis by $\{e_1, e_2\}$. For $m = (m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}_+$, denote $t^m = t_1^{m_1}t_2^{m_2}$. Let $A = \mathbb{C}[t_1^{\pm 1}, t_2], g = \text{Der}(A)$. Denote $\partial_i = \frac{\partial}{\partial t_i}, \ i = 1, 2$. Then the algebra $g$ can be defined as follows:

$$g = \text{Span}\{t^\alpha \partial_i | \alpha \in \mathbb{Z} \times \mathbb{Z}_+, i = 1, 2\}.$$ 

We can write the Lie bracket in $g$ as follows:

$$[t^\alpha \partial_i, t^\beta \partial_j] = \beta_i t^{\alpha+\beta-e_i} \partial_j - \alpha_j t^{\alpha+\beta-e_j} \partial_i, \forall \alpha, \beta \in \mathbb{Z} \times \mathbb{Z}_+, 1 \leq i, j \leq 2.$$ 

Note that the subspace $h = \text{Span}\{t_1 \partial_1, t_2 \partial_2\}$ is a Cartan subalgebra of $g$, i.e., a self-normalizing nilpotent Lie subalgebra.

We denote the semidirect Lie algebra $g \ltimes A$ by $\tilde{g}$.
**Definition 2.1.** A $\mathfrak{g}$-module $M$ is called a weight module if the action of $\mathfrak{h}$ on $M$ is diagonalizable, i.e., $M = \bigoplus_{\lambda \in \mathfrak{h}}^* M_\lambda$, where  
$$M_\lambda = \{ v \in M | (h - \lambda(Id)v = 0, \forall h \in \mathfrak{h} \}.$$  

**Definition 2.2.** A module $M$ over $\tilde{\mathfrak{g}}$ is a jet $\mathfrak{g}$-module if the action of $A$ is associative, i.e.,  
$$g(fv) = (gf)v, \text{ for any } f, g \in A, v \in M.$$  

Since $A$ is a left module algebra over the Hopf algebra $U(\mathfrak{g})$, we have the smash product algebra $A \# U(\mathfrak{g})$. A jet module is actually a module over $A \# U(\mathfrak{g})$. We use $\cdot$ to denote the multiplication between $A$ and $U(\mathfrak{g})$ in $A \# U(\mathfrak{g})$. To find homogenous elements in $A \# U(\mathfrak{g})$ that commute with $t_1 \cdot 1, t_2 \cdot 1, 1 \cdot t_1 \partial_1, 1 \cdot \partial_2$, we define the following element:  

\[(2.1) \quad X_k(m) = \sum_{i=0}^{m_2} (-1)^i \left(\begin{array}{c} m_2 \\ i \end{array} \right) t_1^{-m_1} t_2^{i} t_1^{m_1 + \delta_{k1}} t_2^{m_2 - i} \partial_k - \delta_{m_2, 0} 1 \cdot t_1^{\delta_{k1}} \partial_k,\]

where $m = (m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}_+$, $k = 1, 2$. Note that $X_k(0, 0) = 0$ for $k = 1, 2$.

Let $L$ be the Lie subalgebra of $A \# U(\mathfrak{g})$ spanned by $X_k(m)$, for $k = 1, 2, m \in \mathbb{Z} \times \mathbb{Z}_+$, which is called the jet Lie algebra of $\mathfrak{g}$. In the present paper, we will show the following algebra isomorphism.

**Theorem 2.3.** The linear map $\phi : A \# U(\mathfrak{g}) \to D \otimes U(L)$ defined by  

\[
\phi(1 \cdot t_1^{m_1 + \delta_{k1}} \partial_k) = t_1^{m_1 + \delta_{k1}} t_2^{m_2} \partial_k \otimes 1 + \sum_{i=0}^{m_2} \left(\begin{array}{c} m_2 \\ i \end{array} \right) t_1^{m_1} t_2^{i} \partial_k \otimes X_k(m - i e_2),
\]

\[
\phi(t_1^m \cdot 1) = t_1^m \otimes 1,
\]

is an associative algebra isomorphism, where $m \in \mathbb{Z} \times \mathbb{Z}_+$, $k = 1, 2$.

The isomorphism in Theorem 2.3 tells us that there is a close relationship between vector field Lie algebras and Weyl algebras.

### 3. The structure of $L$

In this section, we study the structure of the Lie algebra $L$. We also classify finite dimensional irreducible modules over $L$.

In order to break up $A \# U(\mathfrak{g})$ into subalgebras: $D$ and $U(L)$, which commute with each other, we need give the following lemma.

**Lemma 3.1.** For $m \in \mathbb{Z} \times \mathbb{Z}_+, k = 1, 2$ we have

(a) $[X_k(m), t_1] = [X_k(m), t_2] = 0$.

(b) $[X_k(m), t_1 \partial_1] = [X_k(m), \partial_2] = 0$.

**Proof.** (a). It is easy to see that $[X_k(m), t_i] = 0$ if $k \neq i$. 

For $k = 1, 2$, we have that
\[
[X_k(m), t_k] = \sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} t_1^{-m_1} t_2^i \cdot t_1^{m_1 + \delta \delta_k} t_2^{m_2 - i} \partial_k - \delta_{m_2, 0} t_1^{\delta \delta_k} \partial_k, t_k
\]
\[
= \sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} t_1^{-m_1} t_2^i \cdot t_1^{m_1} t_2^{-i} - \delta_{m_2, 0} t_1^{\delta \delta_k}
\]
\[
= \sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} t_1^{\delta \delta_k} t_2^{m_2 - i} = 0.
\]

(b) The equality $[X_k(m), t_1 \partial_1] = 0$ follows from the fact that $X_k(m)$ is a homogenous element of degree zero with respect to the degree derivation $t_1 \partial_1$.

From $(i + 1)\binom{m_2}{i+1} = (m_2 - i)\binom{m_2}{i}$, we can check that
\[
[X_k(m), \partial_2] = \sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} t_1^{-m_1} t_2^i \partial_2 \cdot t_1^{m_1} t_2^{-i} \partial_k
\]
\[
+ \sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} t_1^{-m_1} t_2^i \cdot [t_1^{m_1} t_2^{-i} \partial_k, \partial_2]
\]
\[
= \sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} t_1^{-m_1} t_2^{i-1} \cdot t_1^{m_1 + \delta \delta_k} t_2^{m_2 - i} \partial_k
\]
\[
- \sum_{i=0}^{m_2} (m_2 - i)(-1)^i \binom{m_2}{i} t_1^{-m_1} t_2^i \cdot t_1^{m_1} t_2^{m_2 - i-1} \partial_k
\]
\[
= 0.
\]

We complete the proof of Lemma 3.1.

By straightforward calculations, we give the Lie bracket of $L$ in the following lemma.

Lemma 3.2. For $m, s \in \mathbb{Z} \times \mathbb{Z}_+$, $k = 1, 2$, we have that

(a) $[X_1(m), X_1(s)] = m_1 \delta \delta_{s_2, 0} X_1(m) - s_1 \delta \delta_{m_2, 0} X_1(s) + (s_1 - m_1) X_1(m + s)$.

(b) $[X_2(m), X_2(s)] = -s_2 \delta \delta_{m_2, 0} X_2(s - e_2) + m_2 \delta \delta_{s_2, 0} X_2(m - e_2) + (s_2 - m_2) X_2(m + s - e_2)$.

(c) $[X_1(m), X_2(s)] = -s_1 \delta \delta_{m_2, 0} X_2(s) + m_2 \delta \delta_{s_2, 0} X_1(m - e_2) + s_1 X_2(m + s) - m_2 X_1(m + s - e_2)$.

Let $\mathfrak{m}_{1,0}$ be the maximal ideal of $A$ generated by $t_1 - 1, t_2$ and $\Delta = \text{span}\{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\}$. Then $\mathfrak{m}_{1,0} \Delta$ is a Lie subalgebra of $\mathfrak{g}$.

Inspired by the isomorphism $\psi$ in [32] for Lie superalgebras, we have the following isomorphism for the Lie algebra $L$.

Lemma 3.3. The linear map $\theta : L \to \mathfrak{m}_{1,0} \Delta$ defined by
\[
\theta(X_1(m)) = (t^m - \delta \delta_{m_2, 0}) t_1 \frac{\partial}{\partial t_1}, \quad \theta(X_2(m)) = (t^m - \delta \delta_{m_2, 0}) \frac{\partial}{\partial t_2},
\]
is a Lie algebra isomorphism, where $m = (m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}_+, k = 1, 2.$
Proof. For any \( m, s \in \mathbb{Z} \times \mathbb{Z}_+ \), we can check that
\[
\theta([X_1(m), X_2(s)])
\]
\[
= -s_1 \delta_{m_2,0}(t^s - \delta_{s_2,0}) \frac{\partial}{\partial t_2} + m_2 \delta_{s_2,0}(t^{m-s} - \delta_{m_2-1,0})t_1 \frac{\partial}{\partial t_1}
\]
\[
+ s_1(t^{m+s} - \delta_{m_2+s_2,0}) \frac{\partial}{\partial t_2} - m_2(t^{s-m} - \delta_{m_2+s_2-1,0})t_1 \frac{\partial}{\partial t_1}
\]
\[
= s_1(t^{m+s} - \delta_{m_2,0}t^s) \frac{\partial}{\partial t_2} - m_2(t^{s-m} - \delta_{m_2,0}t^{m-s}) \frac{\partial}{\partial t_1}
\]
\[
= [(t^m - \delta_{m_2,0})t_1 \frac{\partial}{\partial t_1}, (t^s - \delta_{s_2,0}) \frac{\partial}{\partial t_2}]
\]
\[
= [\theta(X_1(m)), \theta(X_2(s))].
\]

Similarly, we can verify that
\[
\theta([X_1(m), X_1(s)]) = [\theta(X_1(m)), \theta(X_1(s))],
\]
and
\[
\theta([X_2(m), X_2(s)]) = [\theta(X_2(m)), \theta(X_2(s))].
\]

Therefore the map \( \theta \) is a homomorphism. Since \((t^m - \delta_{m_2,0})t_1 \frac{\partial}{\partial t_1}, (t^s - \delta_{s_2,0}) \frac{\partial}{\partial t_2}\) are linearly independent for different \( m, s \in \mathbb{Z} \times \mathbb{Z}_+ \), \( \theta \) is injective. Moreover we can see that \( \mathfrak{m}_{1,0}\Delta \) is spanned by \((t^m - \delta_{m_2,0})t_1 \frac{\partial}{\partial t_1}, (t^s - \delta_{s_2,0}) \frac{\partial}{\partial t_2}\) with \( m, s \in \mathbb{Z} \times \mathbb{Z}_+ \). So \( \theta \) is surjective, and hence it is an isomorphism.

\[
\square
\]

**Lemma 3.4.** The linear map \( \pi : \mathfrak{m}_{1,0}\Delta / \mathfrak{m}_{1,0}^2\Delta \to \mathfrak{gl}_2 \) such that
\[
\pi((t_1 - 1) \frac{\partial}{\partial t_1}) = E_{11}, \pi((t_1 - 1) \frac{\partial}{\partial t_2}) = E_{12},
\]
\[
\pi(t_2 \frac{\partial}{\partial t_1}) = E_{22}, \pi(t_2 \frac{\partial}{\partial t_2}) = E_{21},
\]
is a Lie algebra isomorphism.

**Lemma 3.5.** If \( M \) is a finite dimensional irreducible \( \mathfrak{m}_{1,0}\Delta \)-module. Then \( \mathfrak{m}_{1,0}^2\Delta M = 0 \). So \( M \) is an irreducible \( \mathfrak{gl}_2 \)-module via the isomorphism in Lemma 3.4.

**Proof.** Let \( A^+ = \mathbb{C}[t_1, t_2], \mathfrak{m}_{1,0}^+ = A^+ \cap \mathfrak{m}_{1,0} \), and \( d = (t_1 - 1) \frac{\partial}{\partial t_1} + t_2 \frac{\partial}{\partial t_2} \). Then \( \mathfrak{m}_{1,0}^+ = \bigoplus_{i \in \mathbb{Z}_+} (\mathfrak{m}_{1,0}^+)_i \) is a \( \mathbb{Z}_+ \)-graded Lie algebra with respect to the adjoint action of \( d \), where \( (\mathfrak{m}_{1,0}^+)_i = \{ X \in \mathfrak{m}_{1,0}^+ \mid [d, X] = iX \} \). More precisely,
\[
(\mathfrak{m}_{1,0}^+)_i = \text{span}\{ (t_1 - 1)^{m_1} t_2^{m_2} \mid m_1, m_2 \in \mathbb{Z}_+, m_1 + m_2 = i \}.
\]

Since \( M \) is finite dimensional, the action of \( d \) on \( M \) has finite eigenvalues, hence there is some integer \( l \geq 2 \) such that \((\mathfrak{m}_{1,0}^+)_i \Delta M = 0 \) for any \( i \in \mathbb{Z}_+ \). So \((\mathfrak{m}_{1,0}^+)^l \Delta M = 0 \). Let \( I \) be the ideal of \( \mathfrak{m}_{1,0}\Delta \) generated by \((\mathfrak{m}_{1,0}^+)^l \Delta \). Then \( IM = 0 \).

For any \( k \in \mathbb{Z}, u_1 \in (\mathfrak{m}_{1,0}^+)_1, v_i \in (\mathfrak{m}_{1,0}^+)^i \), we have
\[ [t^k_1u_1d, v_l d] - [t^k_1d, u_1v_l d] = t^k_1u_1d(v_l)d - v_l(d(t^k_1)u_1 + t^k_1u_1d)
- t^k_1(u_1v_l + u_1d(v_l))d + u_1v_l d(t^k_1)d
= -2t^k_1u_1v_l d \in I. \]

So \( m^{t+1}_1dM = 0 \). Then \([v_l d, u_1 \frac{\partial}{\partial v_l}] + u_1 \frac{\partial v_l}{\partial v_l} d = v_l u_1 \frac{\partial}{\partial v_l} \in I \), for any \( v_l \in (m^{+}_1, i = 1, 2 \).

Hence \( m^{t+1}_{1,0} \Delta M = 0 \). Consequently \( M \) is an irreducible module over the quotient algebra \( m^{+}_{1,0} \Delta / m^{t+1}_{1,0} \Delta \). Since \( m^{+}_{1,0} \Delta / (m^{+}_{1,0})^{t+1} \Delta \cong m^{+}_{1,0} \Delta / m^{t+1}_{1,0} \Delta \), the module \( M \) is also an irreducible \( m^{+}_{1,0} \Delta / (m^{+}_{1,0})^{t+1} \Delta \)-module. Since the adjoint action of \( d \) on \( m^{2}_{1,0} \Delta \) is diagonalizable, so is the action of \( d \) on \( M \). From that the eigenvalues of \( ad \) on \( m^{2}_{1,0} \Delta \) are all positive, \( m^{2}_{1,0} \Delta M \) is a proper submodule of \( M \). The irreducibility of \( M \) forces that \( m^{2}_{1,0} \Delta M = 0 \). Then we can complete the proof.

□

By Lemma 3.3 and Lemma 3.4, any irreducible \( gl_2 \)-module \( V \) can be lifted to an \( L \)-module denoted by \( V^L_{gl_2} \) in the following way:

\[ X_k(i, 0)v = i E_{1k}v, \quad X_k(i, 1) = E_{2k}v, \quad X_k(m_1, m_2)v = 0, \]

where \( v \in V, i, m_1 \in \mathbb{Z}, k = 1, 2, m_2 \in \mathbb{Z}_{\geq 2} \).

By Lemma 3.5 we obtain the classification of finite dimensional irreducible \( L \)-modules.

**Theorem 3.6.** Let \( M \) be a finite dimensional irreducible \( L \)-module. Then \( M \cong V^L_{gl_2} \) for some irreducible \( gl_2 \)-module.

### 4. The map \( \varphi \) is an isomorphism

In this section, we will show that \( \varphi \) is an isomorphism. Then using the classification of irreducible finite dimensional modules over \( L \), we give the classification of irreducible jet \( g \)-modules with finite dimensional weight spaces.

**Lemma 4.1.** The linear map \( \phi : A \# U(g) \to D \otimes U(L) \) defined in Theorem 2.3 is an associative algebra homomorphism.

**Proof.** Clearly, the restricted map \( \phi|_A : A \to D \) is a homomorphism. To show that \( \phi \) is a homomorphism, we also should check that \( \phi \) preserves the defining relations of \( g \).
For any \( m, s \in \mathbb{Z} \times \mathbb{Z}_+ \), we have that

\[
\begin{align*}
[\phi(1 \cdot t^{m+e_1} \partial_1), \phi(1 \cdot t^s \partial_2)] &= [t^{m+e_1} \partial_1 \otimes 1, t^s \partial_2 \otimes 1] + \sum_{i=0}^{m_2} \binom{m_2}{i} [t_1^{m_1} t_2^i t^s \partial_2] \otimes X_1(m - ie_2), \\
&+ \sum_{j=0}^{s_2} \binom{s_2}{j} [t_1^{m_1+1} t_2^{m_2} \partial_1, t_1^{s_1} t_2^j] \otimes X_2(s - je_2) \\
&+ \sum_{j=0}^{s_2} \binom{s_2}{j} \left[ t_1^{m_1+1} t_2^{s_2} \partial_1, t_1^{s_1} t_2^j \right] \otimes [X_1(m - ie_2), X_2(s - je_2)] \\
&= [t^{m+e_1} \partial_1 \otimes 1, t^s \partial_2 \otimes 1] - \sum_{i=0}^{m_2} \binom{m_2}{i} t_1^{m_1+s_1} t_2^{s_2+i} \otimes X_1(m - ie_2), \\
&+ \sum_{j=0}^{s_2} \binom{s_2}{j} s_1 t_1^{m_1+s_1} t_2^{s_2+j} \otimes X_2(s - je_2) \\
&+ \sum_{i=0}^{m_2} \sum_{j=0}^{s_2} \binom{m_2}{i} \binom{s_2}{j} \left[ t_1^{m_1+s_1} t_2^{i+j} \otimes (s_1 X_2(m + s - (i + j)e_2) - (m_2 - i) X_1(m - (i + j)e_2) \right) \\
&+ \sum_{i=0}^{m_2} \sum_{j=0}^{s_2} \binom{m_2}{i} \binom{s_2}{j} \sum_{j=0}^{s_2} t_1^{m_1+s_1} t_2^{i+j} \otimes (s_1 X_2(m + s - (i + j)e_2) - (m_2 - i) X_1(m + s - (i + j + 1)e_2) \right) \\
&= [t^{m+e_1} \partial_1 \otimes 1, t^s \partial_2 \otimes 1] + \sum_{i=0}^{s_2+m_2} \binom{s_2+m_2}{i} t_1^{m_1+s_1} t_2^i \otimes X_2(s + m - ie_2) \\
&+ m_2 \sum_{i=0}^{s_2+m_2-1} \binom{s_2+m_2-1}{i} t_1^{m_1+s_1} t_2^i \otimes X_1(s + m - e_2 - ie_2) \\
&= \phi([1 \cdot t^{m+e_1} \partial_1, \phi(1 \cdot t^s \partial_2)]).
\end{align*}
\]

Similarly, we can check that

\[
\begin{align*}
(4.1) \quad &\phi([1 \cdot t^{m+e_1} \partial_1, \phi(1 \cdot t^s \partial_2)]) = [\phi(1 \cdot t^{m+e_1} \partial_1), \phi(1 \cdot t^s \partial_2)], \\
(4.2) \quad &\phi([1 \cdot t_1^{m_1+1} t_2^{m_2} \partial_2, 1 \cdot t_1^{e_1+1} t_2^{s_2} \partial_2]) = [\phi(1 \cdot t_1^{m_1+1} t_2^{m_2} \partial_2), \phi(1 \cdot t_1^{e_1+1} t_2^{s_2} \partial_2)].
\end{align*}
\]
Finally, we verify the commutation relation between $g$ and $A$. Indeed,
\[
\begin{align*}
[\phi(1 \cdot t^{m+\delta_{k,1}} \partial_k), \phi(t^s \cdot 1)] &= [t_1^{m_1+\delta_{k,1}} t_2^{m_2} \partial_k \otimes 1 + \sum_{i=0}^{m_2} \binom{m_2}{i} t_1^{m_1} t_2^i \otimes X_1(m-i e_2), t^s \otimes 1] \\
&= s_k t^{m+s+\delta_{k,1}-e_k} \otimes 1 = \phi([1 \cdot t^{m+\delta_{k,1}} \partial_k, t^s \cdot 1]).
\end{align*}
\]

This completes the proof of Lemma 4.1.

Lemma 4.2. The linear map $\rho : \mathcal{D} \otimes U(L) \to A \# U(g)$ defined by
\[
\rho(1 \otimes X_k(m)) = \sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} t_1^{-m_1} t_2^i \cdot t_1^{m_1+\delta_{k,1}} t_2^{m_2-i} \partial_k - \delta_{m_2,0} t_1^{\delta_{k,1}} \partial_k,
\]
\[
\rho(\partial_1 \otimes 1) = t_1^{-1} \cdot t_1 \partial_1, \quad \rho(\partial_2 \otimes 1) = 1 \cdot \partial_2, \quad \rho(t_k \otimes 1) = t_k \cdot 1,
\]
is an associative algebra homomorphism, where $m \in \mathbb{Z} \times \mathbb{Z}_+$, $k = 1, 2$. Moreover, $\rho$ is the inverse of $\phi$.

Proof. By the definition of $X_k(m)$ in (2.1) and Lemma 3.2 we can see that $\rho$ preserves the defining relations of $L$. To show that $\rho$ is a homomorphism, we also need to show that $\rho$ preserves the defining relations of $\mathcal{D}$. Indeed,
\[
[\rho(\partial_k \otimes 1), \rho(t_i \otimes 1)] = [t_k^{-\delta_{k,1}} \cdot t_k^{-\delta_{k,1}} \partial_k, t_i \cdot 1] = \delta_{kl} = \rho([\partial_k \otimes 1, t_i \otimes 1]).
\]

By Lemma 3.2, we see that $\rho$ preserves the commutativity relation between $L$ and $\mathcal{D}$.

By the definition, the composition $\varphi \rho$ is identity on $\mathcal{D} \otimes 1$. Let us check that it is also identity on $1 \otimes U(L)$. We compute that
\[
\begin{align*}
\varphi \rho(1 \otimes X_k(m)) &= \phi(\sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} t_1^{-m_1} t_2^i \cdot t_1^{m_1+\delta_{k,1}} t_2^{m_2-i} \partial_k - \delta_{m_2,0} t_1^{\delta_{k,1}} \partial_k) \\
&= \sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} (t_1^{-m_1} t_2^i \otimes 1) \left(t_1^{m_1+\delta_{k,1}} t_2^{m_2-i} \partial_k \otimes 1ight) + \sum_{j=0}^{m_2-i} \binom{m_2-i}{j} t_1^{m_1} t_2^j \otimes X_k(m-(i+j) e_2) - \delta_{m_2,0} t_1^{\delta_{k,1}} \partial_k \otimes 1 \\
&= \sum_{i=0}^{m_2} (-1)^i \binom{m_2}{i} \sum_{j=0}^{m_2-i} \binom{m_2-i}{j} t_2^{i+j} \otimes X_k(m-(i+j)e_2) \\
&= \sum_{i=0}^{m_2} \binom{m_2}{i} t_2^i \otimes X_k(m-le_2) \\
&= 1 \otimes X_k(m).
\end{align*}
\]
Clearly the composition $\rho \phi$ is identity on $A$. We will check its value on $U(\mathfrak{g})$. Explicitly,

$$
\rho \phi(1 \cdot t^{m+\delta_k e_1} \partial_k)
= \rho(t_1^{m_1+\delta_k} t_2^{m_2} \partial_k \otimes 1) + \sum_{i=0}^{m_2} \left( m_2 \atop i \right) \rho(t_1^{m_1 i} t_2^i \otimes X_k (m - i e_2))
= \sum_{i=0}^{m_2} \sum_{j=0}^{m_2-i} (-1)^{j} t_2^{i+j} \cdot t_1^{m_1+\delta_k} t_2^{m_2-i-j} \partial_k
= \sum_{l=0}^{m_2} \sum_{i=0}^{l} (-1)^{l} \left( \sum_{j=0}^{l} \left( m_2 \atop l \right) \right) t_2^{l} \cdot t_1^{m_1+\delta_k} t_2^{m_2-l} \partial_k
= 1 \cdot t^{m+\delta_k e_1} \partial_k.
$$

Therefore, $\rho$ is the inverse of $\phi$. The proof is complete. \qed

Combining Lemma 4.1 and Lemma 4.2, we can establish Theorem 2.3. A $D$-module $M$ is called a weight module if the actions of $t_1 \partial_1$ and $t_2 \partial_2$ are diagonalizable.

**Lemma 4.3.** Any nonzero weight $D$-module $M$ has an irreducible submodule.

**Proof.** Choose a nonzero $v \in M$ such that $t_i \partial_i v = a_i v$, $i = 1, 2$ for some $a = (a_1, a_2) \in \mathbb{C}^2$. Let $I_a$ be the left ideal of $D$ generated by $t_1 \partial_1 - a_1$, $t_2 \partial_2 - a_2$. We can see that $D/I_a \cong t^a \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$. If $a_2 \in \mathbb{Z}$, then the $D$-module $t^a \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ has two irreducible subquotients: $t_1^{a_1} \mathbb{C}[t_1^{\pm 1}, t_2]$, $t_2^{a_2} \mathbb{C}[t_2^{\pm 1}, t_1]$. Otherwise, it is irreducible. Therefore, as a quotient module of $D/I_a$, the submodule $Dv$ of $M$ must has an irreducible submodule. \qed

Let $P$ be a $D$-module and $V$ be a $\mathfrak{gl}_2$-module. Then the tensor product $M(P, V) = P \otimes_{\mathbb{C}} V$ becomes a $\mathfrak{g}$-module (see [29, 23]) with the action

$$
(4.3) \quad t^{m+\delta_k e_1} \partial_k \cdot (g \otimes v) = (t^{m+\delta_k e_1} \partial_k g) \otimes v + m_1 t^{m} g \otimes E_{1k} v + m_2 t^{m-e_2} g \otimes E_{2k} v,
$$

$$
(4.4) \quad t^{m} \cdot (g \otimes v) = (t^{m} g) \otimes v,
$$

where $t^m \in A$, $g \in P$, $v \in V$, $k = 1, 2$. We will show any irreducible uniformly bounded jet modules for $\mathfrak{g}$ is of the form $M(P, V)$.

The following lemma is well known, see Lemma 2.7 in [21].

**Lemma 4.4.** Let $A, B$ be two unital associative algebras and $B$ has a countable basis. If $M$ is an irreducible module over $A \otimes B$ that contains an irreducible $A = A \otimes \mathbb{C}$ submodule $W$, then $M \cong W \otimes V$ for an irreducible $B$-module $V$.

Finally, we will give a clear description of all irreducible jet $\mathfrak{g}$-modules with finite dimensional weight spaces. An $L$-module $V$ is called a weight module if the actions of $X_2(0, 1)$ is diagonalizable. Through the isomorphism $\phi : A\#U(\mathfrak{g}) \rightarrow D \otimes U(L)$ in Theorem 2.3, an
irreducible jet $g$-module $M$ can be viewed an irreducible module over the algebra $D \otimes U(L)$, denoted by $M^\phi$.

**Theorem 4.5.** Let $M$ be an irreducible jet $g$-module with finite dimensional weight spaces. Then $M^\phi \cong P \otimes V$ for some irreducible weight $D$-module $P$, some irreducible weight $L$-module $V$. Moreover, if $M$ is uniformly bounded, then $M \cong M(P,V)$, where $V$ is an irreducible finite dimensional $gl_2$-module.

**Proof.** Let $h_1 = t_1 \partial_1 \otimes 1, h_2 = t_2 \partial_2 \otimes 1 + 1 \otimes X_2(0,1)$. From $\phi(t_1 \partial_1) = h_1, \phi(t_1 \partial_2) = h_2$, we see that $M^\phi$ is a weight module with respect to the actions of $h_1$ and $h_2$. It can be seen that the adjoint actions of $t_2 \partial_2 \otimes 1$ and $1 \otimes X_2(0,1)$ on $D \otimes U(L)$ is diagonalizable, and each weight space of $M^\phi$ is invariant under the actions of $t_2 \partial_2 \otimes 1$ and $1 \otimes X_2(0,1)$. So $M^\phi$ is a weight module over $D$ and $1 \otimes X_2(0,1)$ acts diagonally on $M^\phi$. Then from Lemma 4.3 and Lemma 4.4 there is an irreducible weight $D$-module $P$, some irreducible weight $L$-module $V$ such that $M^\phi \cong P \otimes V$. If $M$ is uniformly bounded, then the $L$-module $V$ is finite dimensional. By Theorem 3.6 $V$ is an irreducible $gl_2$-module. Then we complete the proof. □

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