Partition functions of
NAHE–based free fermionic string models

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Abstract

The heterotic string free fermionic formulation produced a large class of
three generation models, with an underlying SO(10) GUT symmetry which is
broken directly at the string level by Wilson lines. A common subset of bound-
ary condition basis vectors in these models is the NAHE set, which corresponds
to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold of an SO(12) Narain lattice, with \((h_{1,1}, h_{2,1}) = (27, 3)\). Alter-
tnatively, a manifold with the same data is obtained by starting with a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)
orbifold at a generic point on the lattice, with \((h_{1,1}, h_{2,1}) = (51, 3)\), and adding
a freely acting \( \mathbb{Z}_2 \) involution. The equivalence of the two constructions is
proven by examining the relevant partition functions. The explicit realization
of the shift that reproduces the compactification at the free fermionic point is
found. It is shown that other closely related shifts reproduce the same massless
spectrum, but different massive spectrum, thus demonstrating the utility of ex-
tracting information from the full partition function. A freely acting involution
of the type discussed here, enables the use of Wilson lines to break the GUT
symmetry and can be utilized in non-perturbative studies of the free fermionic
models.

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1 Introduction

Grand unification, and its incarnation in the form of heterotic-string unification, is the only extension of the Standard Model that is rooted in the structure of the Standard Model itself. In this context the most realistic string models discovered to date have been constructed in the heterotic string free fermionic formulation. While this may be an accident, it may also reflect on deeper, yet undiscovered, properties of string theory. It is therefore imperative to enhance our understanding of this particular class of models, with the hope that it will shed further light on their properties, and possibly yield deeper insight into the dynamics of string theory.

An important feature of the realistic free fermionic models is their underlying $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold structure. Many of the encouraging phenomenological characteristics of the realistic free fermionic models are rooted in this structure, including the three generations arising from the three twisted sectors, and the canonical SO(10) embedding for the weak hyper-charge. To see more precisely this orbifold correspondence, recall that the free fermionic models are generated by a set of basis vectors which define the transformation properties of the world-sheet fermions as they are transported around loops on the string world sheet. A large set of realistic free fermionic models contains a subset of boundary conditions, the so-called NAHE set, which can be seen to correspond to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactification with the standard embedding of the gauge connection. This underlying free fermionic model contains 24 generations from the twisted sectors, as well as three additional generation/anti-generation pairs from the untwisted sector. At the free fermionic point in the Narain moduli space, both the metric and the antisymmetric background fields are non-trivial, leading to an SO(12) enhanced symmetry group. The action of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twisting on the SO(12) Narain lattice then gives rise to a model with $(h_{11}, h_{21}) = (27, 3)$, matching the data of the free fermionic model. It is noted that this data differs from that of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold at a generic point in the moduli space of $(T^2)_3$, which yields $(h_{11}, h_{21}) = (51, 3)$. We refer to the $(51, 3)$ and $(27, 3)$ $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models as $X_1$ and $X_2$ respectively.

While the free fermionic construction provides most useful tools to analyze the spectrum and superpotential interactions in a given string model, its drawback is that it is formulated at a fixed point in the compactification moduli space. The moduli dependence of physical quantities may therefore only be studied by including world-sheet Thirring interactions, which may be cumbersome. On the other hand, the moduli dependence is more readily extracted by constructing the models in a geometrical, or orbifold, formalism. Another important advantage of the geometrical approach is that it might provide a closer link to the various strong-weak coupling dualities that have been unraveled in the past few years. In the context of the realistic free fermionic models preliminary investigations have been attempted by relating the F-theory compactification on $X_2$ to the studies of F-theory compactification on $X_1$. This study highlighted the potential relevance of Calabi-Yau manifolds, which
possess a bi-section but not a global section and which was further investigated in [5].

The key ingredient in studying the F-theory compactification on the free fermionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold was to connect it to the $X_1$ by a freely acting twist or shift [6]. Under the freely acting shift, pairs of twisted sector fixed points are identified. Hence reducing the total number of fixed points from 48 to 24. It is noted that at the level of the toroidal compactification, the SO(4)$^3$ and SO(12) lattices are continuously connected by varying the parameters of the background metric and antisymmetric tensor. However, this cannot be done while preserving the $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance, because the continuous interpolation cannot change the Euler characteristic. Indeed, part of the geometric moduli are projected out by the orbifold action and, as a result, though the two toroidal models are in the same moduli space, the two $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models are not.

The connection of these studies to the free fermionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model therefore rests on the conjecture that the model constructed with the additional freely acting shift is identical to the model at the free fermionic point in the Narain moduli space, i.e. to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold on the SO(12) lattice. However, the validity of this conjecture is far from obvious. While the massless spectrum and symmetries of the two models match, their massive spectrum may differ.

In this paper we therefore undertake the task of proving this conjecture. This is achieved by constructing the partition function of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold on an SO(12) lattice, and the partition function of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold on a generic ($T_2$)$^3$ lattice. We then show that adding the freely acting shift to the latter and fixing the radii of the compactified dimensions at the self-dual point reproduces the partition function on the SO(12) lattice, hence proving that the models are identical. However, we show that this matching is highly non-trivial and is obtained only for a specific form of the freely acting shifts which affects simultaneously momenta and windings. In contrast freely acting shifts that act only on momenta or winding do not reproduce the partition function of the free fermionic model. Thus, while the spectra of the three models match at the massless level, they in general differ at the massive level. Hence demonstrating the usefulness of gaining additional valuable information from the construction of the partition function. Additionally we discuss in this paper the general structure of the partition functions of NAHE based free fermionic models.

### 2 Realistic free fermionic models - general structure

In this section we recapitulate the main structure of the realistic free fermionic models. The notation and details of the construction of these models are given elsewhere [7, 8, 9, 10, 11, 12, 13]. In the free fermionic formulation of the heterotic string in four dimensions all the world-sheet degrees of freedom required to cancel the conformal anomaly are represented in terms of free fermions propagating on the string world-sheet [14]. In the light-cone gauge the world-sheet field content consists
of two transverse left- and right-moving space-time coordinate bosons, $X_{1,2}^\mu$ and $\bar{X}_{1,2}^\mu$, and their left-moving fermionic superpartners $\psi_{1,2}^\mu$, and additional 62 purely internal Majorana-Weyl fermions, of which 18 are left-moving, $\chi^I$, and 44 are right-moving, $\phi^a$. In the supersymmetric sector the world-sheet supersymmetry is realized non-linearly and the world-sheet supercurrent is given by $T_F = \psi^\mu \partial X_\mu + i \chi^I y^I \omega^I$, ($I = 1, \ldots, 6$).

The $\{\chi^I, y^I, \omega^I\}$ ($I = 1, \ldots, 6$) are 18 real free fermions transforming as the adjoint representation of SU(2)\(^6\). Under parallel transport around a non-contractible loop on the toroidal world-sheet the fermionic fields pick up a phase, $f \longrightarrow -e^{i\pi\alpha(f)} f$, $\alpha(f) \in (-1, +1]$. Each set of specified phases for all world-sheet fermions, around all the non-contractible loops is called the spin structure of the model. Such spin structures are usually given is the form of 64 dimensional boundary condition vectors, with each element of the vector specifying the phase of the corresponding world-sheet fermion. The basis vectors are constrained by string consistency requirements and completely determine the vacuum structure of the model. The physical spectrum is obtained by applying the generalized GSO projections.

The boundary condition basis defining a typical realistic free fermionic heterotic string model is constructed in two stages. The first stage consists of the NAHE set, which is a set of five boundary condition basis vectors, $\{1, S, b_1, b_2, b_3\}$ [1]. The gauge group after imposing the GSO projections induced by the NAHE set is SO(10) \times SO(6)\(^3\) \times E_8 with N = 1 supersymmetry. The space-time vector bosons that generate the gauge group arise from the Neveu-Schwarz sector and from the sector $\xi_2 \equiv 1 + b_1 + b_2 + b_3$. The Neveu-Schwarz sector produces the generators of SO(10) \times SO(6)\(^3\) \times SO(16). The $\xi_2$-sector produces the spinorial 128 of SO(16) and completes the hidden gauge group to E\(_8\). The NAHE set divides the internal worldsheet fermions in the following way: $\bar{\phi}^{1,\cdots,8}$ generate the hidden E\(_8\) gauge group, $\bar{\psi}^{1,\cdots,5}$ generate the SO(10) gauge group, and $\{\bar{y}^{3,\cdots,6}, \bar{\eta}^1\}, \{\bar{y}^1, \bar{y}^2, \bar{\omega}^5, \omega^6, \bar{\eta}^2\}, \{\bar{\omega}^{1,\cdots,4}, \bar{\eta}^3\}$ generate the three horizontal SO(6)\(^3\) symmetries. The left-moving $\{y, \omega\}$ states are divided into $\{y^{3,\cdots,6}\}, \{y^1, y^2, \omega^5, \omega^6\}, \{\omega^{1,\cdots,4}\}$ and $\chi^{12}, \chi^{34}, \chi^{56}$ generate the left-moving N = 2 world-sheet supersymmetry. At the level of the NAHE set the sectors $b_1, b_2$ and $b_3$ produce 48 multiplets, 16 from each, in the 16 representation of SO(10).

The states from the sectors $b_j$ are singlets of the hidden E\(_8\) gauge group and transform under the horizontal SO(6)\(_j\) ($j = 1, 2, 3$) symmetries. This structure is common to all the realistic free fermionic models.

The second stage of the construction consists of adding to the NAHE set three (or four) additional boundary condition basis vectors, typically denoted by $\{\alpha, \beta, \gamma\}$. These additional basis vectors reduce the number of generations to three chiral generations, one from each of the sectors $b_1, b_2$ and $b_3$, and simultaneously break the four dimensional gauge group. The assignment of boundary conditions to $\{\bar{\psi}^{1,\cdots,5}\}$ breaks SO(10) to one of its subgroups SU(5) \times U(1) [4], SO(6) \times SO(4) [4], SU(3) \times SU(2) \times U(1)^2 [6] or SU(3) \times SU(2)^2 \times U(1) [13]. Similarly, the hidden E\(_8\) symmetry is broken to one of its subgroups by the basis vectors which extend the NAHE set. The flavour SO(6)\(^3\) symmetries in the NAHE-based models
are always broken to flavour U(1) symmetries, as the breaking of these symmetries is correlated with the number of chiral generations. Three such U(1) \( j \) symmetries are always obtained in the NAHE based free fermionic models, from the subgroup of the observable \( E_8 \), which is orthogonal to SO(10). These are produced by the world-sheet currents \( \bar{\eta}\eta^* (j = 1, 2, 3) \), which are part of the Cartan sub-algebra of the observable \( E_8 \). Additional unbroken U(1) symmetries, denoted typically by U(1) \( j \) \( (j = 4, 5, ...) \), arise by pairing two real fermions from the sets \( \{ \bar{y}^{1,2,3} \} \), \( \{ \bar{y}^{1,2,3}, \omega^{5,6} \} \) and \( \{ \omega^{1,2,3} \} \). The final observable gauge group depends on the number of such pairings.

The correspondence of the NAHE-based free fermionic models with the orbifold construction is illustrated by extending the NAHE set, \( \{ 1, S, b_1, b_2, b_3 \} \), by one additional boundary condition basis vector \( \xi_1 = (0, \ldots, 0| 1, \ldots, 1, 0, \ldots, 0) \). With a suitable choice of the GSO projection coefficients the model possesses an SO(4) \( ^3 \times E_8 \times U(1)^2 \times E_8 \) gauge group and \( N = 1 \) space-time supersymmetry. The matter fields include 24 generations in the 27 representation of \( E_6 \), eight from each of the sectors \( b_1 \oplus b_1 + \xi_1, b_2 \oplus b_2 + \xi_1 \) and \( b_3 \oplus b_3 + \xi_1 \). Three additional 27 and \( \overline{27} \) pairs are obtained from the Neveu-Schwarz \( \oplus \xi_1 \) sector.

To construct the model in the orbifold formulation one starts with the compactification on a torus with nontrivial background fields \( \xi_1, \xi_2 \). The subset of basis vectors
\[
\{ 1, S, \xi_1, \xi_2 \}
\]
generates a toroidally-compactified model with \( N = 4 \) space-time supersymmetry and SO(12) \( \times E_8 \times E_8 \) gauge group. The same model is obtained in the geometric (bosonic) language by tuning the background fields to the values corresponding to the SO(12) lattice. The metric of the six-dimensional compactified manifold is then the Cartan matrix of SO(12), while the antisymmetric tensor is given by
\[
B_{ij} = \begin{cases} 
G_{ij} & ; i > j, \\
0 & ; i = j, \\
-G_{ij} & ; i < j.
\end{cases}
\] (2.3)

When all the radii of the six-dimensional compactified manifold are fixed at \( R_I = \sqrt{2} \), it is seen that the left- and right-moving momenta \( P^I_{R,L} = [m_i - \frac{1}{2}(\epsilon_{ij} \pm G_{ij})n_j]e_i^* \) reproduce the massless root vectors in the lattice of SO(12). Here \( e^i = \{ e_i^j \} \) are six linearly-independent vielbeins normalised so that \( (e_i)^2 = 2 \). The \( e_i^* \) are dual to the \( e_i \), with \( e_i^* \cdot e_j = \delta_{ij} \).

Adding the two basis vectors \( b_1 \) and \( b_2 \) to the set \( \{ 2, 2 \} \) corresponds to the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold model with standard embedding. Starting from the Narain model with SO(12) \( \times E_8 \times E_8 \) symmetry \( \{ 3 \} \), and applying the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) twist on the internal coordinates, reproduces the spectrum of the free-fermion model with the six-dimensional
basis set \( \{1, S, \xi_1, \xi_2, b_1, b_2\} \). The Euler characteristic of this model is 48 with \( h_{11} = 27 \) and \( h_{21} = 3 \).

It is noted that the effect of the additional basis vector \( \xi_1 \) of eq. (2.1), is to separate the gauge degrees of freedom, spanned by the world-sheet fermions \( \{\bar{\psi}^{1,\ldots,5}, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3, \phi^{1,\ldots,8}\} \), from the internal compactified degrees of freedom \( \{y, \omega | \bar{y}, \bar{\omega}\}^{1,\ldots,6} \). In the realistic free fermionic models this is achieved by the vector \( 2\gamma \), with

\[
2\gamma = (0, \ldots, 0 | 1, \ldots, 1, 0, \ldots, 0),
\]

which breaks the \( E_8 \times E_8 \) symmetry to \( SO(16) \times SO(16) \). The \( Z_2 \times Z_2 \) twist breaks the gauge symmetry to \( \text{SO}(4)^3 \times \text{SO}(10) \times \text{U}(1)^3 \times \text{SO}(16) \). The orbifold still yields a model with 24 generations, eight from each twisted sector, but now the generations are in the chiral 16 representation of \( \text{SO}(10) \), rather than in the 27 of \( E_6 \). The same model can be realized with the set \( \{1, S, \xi_1, \xi_2, b_1, b_2\} \), by projecting out the \( 16 \oplus 16 \) from the \( \xi_1 \)-sector taking

\[
c \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rightarrow -c \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.
\]

This choice also projects out the massless vector bosons in the 128 of \( \text{SO}(16) \) in the hidden-sector \( E_8 \) gauge group, thereby breaking the \( E_6 \times E_8 \) symmetry to \( \text{SO}(10) \times \text{U}(1)^3 \times \text{SO}(16) \). The freedom in (2.5) corresponds to a discrete torsion in the toroidal orbifold model. At the level of the \( N = 4 \) Narain model generated by the set (2.2), we can define two models, \( Z_+ \) and \( Z_- \), depending on the sign of the discrete torsion in eq. (2.5). The first, say \( Z_+ \), produces the \( E_8 \times E_8 \) model, whereas the second, say \( Z_- \), produces the \( \text{SO}(16) \times \text{SO}(16) \) model. However, the \( Z_2 \times Z_2 \) twist acts identically in the two models, and their physical characteristics differ only due to the discrete torsion eq. (2.5). The important aspect, however, is the separation, by the extended NAHE set, of the world-sheet fermionic degrees of freedom corresponding to the space-time gauge degrees of freedom, from those corresponding to the internal compactified dimensions.

This analysis confirms that the \( Z_2 \times Z_2 \) orbifold on the \( \text{SO}(12) \) Narain lattice is indeed at the core of the realistic free fermionic models. However, it differs from the \( Z_2 \times Z_2 \) orbifold on \( T_1^1 \times T_2^2 \times T_3^2 \) with \( (h_{11}, h_{21}) = (51, 3) \). In [3] it was shown that the two models are connected by adding a freely acting twist or shift to the \( X_1 \) model. Let us first start with the compactified \( T_1^1 \times T_2^2 \times T_3^2 \) torus parametrised by three complex coordinates \( z_1, z_2 \) and \( z_3 \), with the identification

\[
z_i = z_i + 1, \quad z_i = z_i + \tau_i,
\]

where \( \tau \) is the complex parameter of each \( T_2 \) torus. With the identification \( z_i \rightarrow -z_i \), a single torus has four fixed points at

\[
z_i = \{0, \frac{1}{2}, \frac{1}{2} \tau, \frac{1}{2} (1 + \tau)\}.
\]
With the two $\mathbb{Z}_2$ twists

$$
\alpha : (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3), \\
\beta : (z_1, z_2, z_3) \rightarrow (z_1, -z_2, -z_3),
$$

(2.8)

there are three twisted sectors in this model, $\alpha$, $\beta$ and $\alpha \beta = \alpha \cdot \beta$, each producing 16 fixed tori, for a total of 48. Adding to the model generated by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist in (2.8), the additional shift

$$
\gamma : (z_1, z_2, z_3) \rightarrow (z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3 + \frac{1}{2})
$$

(2.9)

produces again fixed tori from the three twisted sectors $\alpha$, $\beta$ and $\alpha \beta$. The product of the $\gamma$ shift in (2.9) with any of the twisted sectors does not produce any additional fixed tori. Therefore, this shift acts freely. Under the action of the $\gamma$-shift, the fixed tori from each twisted sector are paired. Therefore, $\gamma$ reduces the total number of fixed tori from the twisted sectors by a factor of $\frac{1}{2}$, with $(h_{11}, h_{21}) = (27, 3)$. This model therefore reproduces the data of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold at the free-fermion point in the Narain moduli space.

3 NAHE-based partition functions

In the previous section we showed that the freely acting shift (2.9), added to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold at a generic point of $T_1^3 \times T_2^3 \times T_3^3$ Eq. (2.8) reproduces the data of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold acting on the SO(12) lattice, which coincides with the lattice at the free fermionic point. However, this observation does not prove that the vacuum which includes the shift is identical to the free fermionic model. While the massless spectrum of the two models may coincide their massive excitations, in general, may differ. To examine the matching of the massive spectra we construct the partition function of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of an SO(12) lattice and subsequently that of the model at a generic point including the shift. In effect since the action of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold in the two cases is identical our problem reduces to proving the existence of a freely acting shift that reproduces the partition function of the SO(12) lattice at the free fermionic point. Then since the action of the shift and the orbifold projections are commuting it follows that the two $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds are identical.

On the compact coordinates there are actually three inequivalent ways in which the shifts can act. In the more familiar case, they simply translate a generic point by half the length of the circle. As usual, the presence of windings in string theory allows shifts on the T-dual circle, or even asymmetric ones, that act both on the circle and on its dual. More concretely, for a circle of length $2\pi R$, one can have the following options [13]:

$$
A_1 : \quad X_{LR} \rightarrow X_{LR} + \frac{1}{2} \pi R,
$$
\[ A_2 : \ X_{L,R} \rightarrow X_{L,R} + \frac{1}{2} \left( \frac{\pi R \pm \pi \alpha'}{R} \right) , \]
\[ A_3 : \ X_{L,R} \rightarrow X_{L,R} \pm \frac{1}{2} \frac{\pi \alpha'}{R} . \] (3.1)

There is, however, a crucial difference among these three choices: while \( A_1 \) and \( A_3 \) shifts can act consistently on any number of coordinates, level-matching requires instead that the \( A_2 \)-shifts act on (mod) four real coordinates.

We can now proceed to deform the free fermionic models to a (connected) generic point in moduli space. As we already noticed, the \( Z_2 \times Z_2 \) orbifold and the shifts are commuting operations, and thus it suffices to find the correct shift that would correspond the the heterotic string on the SO(12) lattice with discrete torsion

\[ Z_- = (V_8 - S_8) \left[ (|O_{12}|^2 + |V_{12}|^2) \left( \bar{O}_{16} \bar{O}_{16} + \bar{C}_{16} \bar{C}_{16} \right) + (|S_{12}|^2 + |C_{12}|^2) \left( \bar{S}_{16} \bar{S}_{16} + \bar{V}_{16} \bar{V}_{16} \right) + (O_{12} \bar{V}_{12} + V_{12} \bar{O}_{12}) \left( \bar{S}_{16} \bar{V}_{16} + \bar{V}_{16} \bar{S}_{16} \right) + (S_{12} \bar{C}_{12} + C_{12} \bar{S}_{12}) \left( \bar{O}_{16} \bar{C}_{16} + \bar{C}_{16} \bar{O}_{16} \right) \right] , \] (3.2)

or without

\[ Z_+ = (V_8 - S_8) \left[ |O_{12}|^2 + |V_{12}|^2 + |S_{12}|^2 + |C_{12}|^2 \right] \left( \bar{O}_{16} + \bar{S}_{16} \right) \left( \bar{O}_{16} + \bar{S}_{16} \right) , \] (3.3)

discrete torsion. Here we have written \( Z_\pm \) in terms of level-one SO(2n) characters (see, for instance, [16])

\[ O_{2n} = \frac{1}{2} \left( \frac{\eta_3^n}{\eta^n} + \frac{\eta_4^n}{\eta^n} \right) , \]
\[ V_{2n} = \frac{1}{2} \left( \frac{\eta_3^n}{\eta^n} - \frac{\eta_4^n}{\eta^n} \right) , \]
\[ S_{2n} = \frac{1}{2} \left( \frac{\eta_2^n}{\eta^n} + i^{-n} \frac{\eta_1^n}{\eta^n} \right) , \]
\[ C_{2n} = \frac{1}{2} \left( \frac{\eta_2^n}{\eta^n} - i^{-n} \frac{\eta_1^n}{\eta^n} \right) . \] (3.4)

These two models can actually be connected by the orbifold

\[ Z_- = Z_+/a \otimes b , \] (3.5)

with

\[ a = (-1)^{F_{L}^{int} + F_{\xi}^{1}} , \]
\[ b = (-1)^{F_{L}^{int} + F_{\xi}^{2}} . \] (3.6)
Therefore, in the following discussion we shall focus on \( Z^+ \), and the corresponding results for \( Z^- \), cumbersome as they may be, can be obtained applying the orbifold projection (3.6). Starting from \( Z^+ \) we can obtain the partition functions of NAHE–based free fermionic models by applying the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold projection (2.8). As the result is somewhat lengthy we do not display it here explicitly.

Instead our problem is to find the shift that when acting on the lattice \( T_1^1 \otimes T_2^2 \otimes T_3^3 \) at a generic point in the moduli space reproduces the \( SO(12) \) lattice when the radii are fixed at the self–dual point \( R = \sqrt{\alpha'} \). Let us consider for simplicity the case of six orthogonal circles or radii \( R_i \). The partition function reads

\[
Z^+ = (V_8 - S_8) \left( \sum_{m,n} \Lambda_{m,n} \right) \otimes^6 (\bar{O}_{16} + \bar{S}_{16}) \left( \bar{O}_{16} + \bar{S}_{16} \right),
\]

where as usual, for each circle,

\[
p_{LR}^i = \frac{m_i}{R_i} \pm \frac{n_i R_i}{\alpha'},
\]

and

\[
\Lambda_{m,n} = \frac{q^\alpha' p^2}{|\eta|^2}.
\]

We can now act with the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) shifts generated by

\[
g : (A_2, A_2, 0),
\]

\[
h : (0, A_2, A_2),
\]

where each \( A_2 \) acts on a complex coordinate. The resulting partition function then reads

\[
Z^+ = \frac{1}{4} (V_8 - S_8) \sum_{m_i,n_i} \left\{ \left[ 1 + (-1)^{m_1+n_2+m_3+n_1+n_2+n_3+n_4} 
\right. \\
\left. + (-1)^{m_1+m_2+m_5+n_1+n_2+n_3+n_4+n_5+n_6} + (-1)^{m_3+m_4+m_6+n_1+n_2+n_3+n_4+n_5+n_6} \right] \right.
\]

\[
\times \left( \Lambda_{m_1,n_1}^{1,\ldots,6} + \Lambda_{m_1,n_1}^{1,\ldots,4} \Lambda_{m_1,n_1}^{5,6} \right)
\]

\[
\times \left( \Lambda_{m_1,n_1}^{1,\ldots,4} \right)
\]

\[
\times \left( \bar{O}_{16} + \bar{S}_{16} \right) \left( \bar{O}_{16} + \bar{S}_{16} \right)
\]

(3.11)

After some tedious algebra, it is then possible to show that, once evaluated at the self-dual radius \( R_i = \sqrt{\alpha'} \), the partition function (3.11) reproduces that at the \( SO(12) \) point (3.3). To this end, it suffices to notice that

\[
\sum_{m,n} \Lambda_{m,n}(R = \sqrt{\alpha'}) = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2,
\]

\[
\sum_{m,n} (-1)^{m+n} \Lambda_{m,n}(R = \sqrt{\alpha'}) = |\chi_0|^2 - |\chi_{\frac{1}{2}}|^2,
\]

9
\[
\sum_{m,n} \Lambda_{m+\frac{1}{2},n+\frac{1}{2}}(R = \sqrt{\alpha'}) = \chi_0 \bar{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{2}} \bar{\chi}_0, \\
\sum_{m,n} (-1)^{m+n} \Lambda_{m+\frac{1}{2},n+\frac{1}{2}}(R = \sqrt{\alpha'}) = \chi_{\frac{1}{2}} \bar{\chi}_0 - \chi_0 \bar{\chi}_{\frac{1}{2}},
\] (3.12)

where
\[
\chi_0 = \sum_\ell q^\ell, \\
\chi_{\frac{1}{2}} = \sum_\ell q^{(\ell+\frac{1}{2})^2},
\] (3.13)

are the two level-one SU(2) characters, while, standard branching rules, decompose the SO(12) characters into products of SU(2) ones. For instance,
\[
O_{12} = \chi_0 \chi_0 \chi_0 \chi_0 \chi_0 + \chi_0 \chi_0 \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} + \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} \chi_{\frac{1}{2}} \chi_0 \chi_0.
\] (3.14)

Let us now consider the shifts given in Eq. (2.9), and similarly the analogous freely acting shift given by \( (A_3, A_3, A_3) \). The \( SO(4) \) lattice takes the form
\[
\Lambda_{SO(4)} = \frac{1}{|\eta|^4} \left( |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 \right) \left( |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 \right)
\] (3.15)

The effect of the free acting shift Eq. (2.9) is to introduce the projection
\[
\Lambda_{\tilde{m},\tilde{n}}(R) \rightarrow (1 + (-1)^{m_1+m_2}) \Lambda_{\tilde{m},\tilde{n}}(R)
\] (3.16)

where \((-1)^{m_1+m_2}\) is taken inside the lattice sum. Fixing the radii at \( R = \sqrt{\alpha'} \) and evaluating the sum it is seen that (3.16) reduces to
\[
\Lambda_{\tilde{m},\tilde{n}}(R) \rightarrow \frac{1}{|\eta|^4} \left( |\chi_0|^2 - |\chi_{\frac{1}{2}}|^2 \right) \left( |\chi_0|^2 - |\chi_{\frac{1}{2}}|^2 \right)
\] (3.17)

Therefore, the shift in eq. (2.9) does reproduce the same number of massless states as the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) at the free fermionic point, but the partition functions of the two models differ! Replacing the freely acting shift by the shift \( (A_3, A_3, A_3) \) along the momenta modes, rather than the winding modes, induces the projection
\[
\Lambda_{\tilde{m},\tilde{n}}(R) \rightarrow (1 + (-1)^{n_1+n_2}) \Lambda_{\tilde{m},\tilde{n}}(R)
\] (3.18)

reproducing again (3.17). Therefore, while each of the these freely acting shifts when acting on the \( X_1 \) manifold does reproduce the data of the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold at the free fermionic point, none of these shifts, in fact, reproduces the \( SO(12) \) lattice which is realized at the free fermionic point.
4 Discussion and conclusions

Despite its innocuous appearance the connection between $X_1$ and $X_2$ by a freely acting shift in fact has profound consequences. First we must realize that any string construction can only offer a limited glimpse on the structure of string vacua that possess some realistic properties. Thus, the free fermionic formulation gave rise to three generation models that were utilised to study issues like Cabibbo mixing and neutrino masses. On the other hand the free fermionic formulation is perhaps not the best suited to study issues that are of a more geometrical character. Now from the Standard Model data we may hypothesize that any realistic string vacuum should possess at least two ingredients. First, it should contain three chiral generations, and second, it should admit their SO(10) embedding. This SO(10) embedding is not realized in the low energy effective field theory limit of the string models, but is broken directly at the string level. The main phenomenological implication of this embedding is that the weak-hypercharge has the canonical GUT embedding.

It has long been argued that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold naturally gives rise to three chiral generations. The reason being that it contains three twisted sectors and each of these sectors produces one chiral generation. The reason that there are exactly three twisted sectors is essentially because we are modding out a three dimensional complex manifold, or a six dimensional real manifold, by $\mathbb{Z}_2$ projections that preserve the holomorphic three form. Thus, metaphorically speaking, the reason being that six divided by two equals three.

However, this argument would hold for any $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of a six dimensional compactified space, and in particular it holds for the $X_1$ manifold. Therefore, we can envision that this manifold can produce, in principle, models with SO(10) gauge symmetry, and three chiral generations from the three twisted sectors. However, the caveat is that this manifold is simply connected and hence the SO(10) symmetry cannot be broken by the Hosotani-Wilson symmetry breaking mechanism [17]. The consequence of adding the freely acting shift (2.9) is that the new manifold $X_2$, while still admitting three twisted sectors is not simply connected and hence allows the breaking of the SO(10) symmetry to one of its subgroups.

Thus, we can regard the utility of the free fermionic machinery as singling out a specific class of compactified manifolds. In this context the freely acting shift has the crucial function of connecting between the simply connected covering manifold to the non-simply connected manifold. Precisely such a construction has been utilised in [18] to construct non-perturbative vacua of heterotic M-theory. To use a simple analogy, we can regard the free fermionic machinery as heavy duty binoculars, enabling us to look for minute details on a mountain but obscuring the gross structures of the mountain ridge. The geometrical insight on the other hand provides such a gross overview, but is perhaps less adequate in extracting detailed properties. However, as the precise point where the detailed properties should be calculated is not yet known, one should regard the phenomenological success of the free fermionic models
as merely highlighting a particular class of compactified spaces. These manifolds then possess the overall structure that accommodates the detailed Standard Model properties. The precise localization of where these properties should be calculated, will require further understanding of the string dynamics. But, if the assertion that the class of relevant manifolds has been singled out proves to be correct, this is already an enormous advance and simplification.

In this letter we demonstrated the equivalence of the partition function of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold on the SO(12) lattice, with the model which is obtained from eq. (3.11). Thus, the free fermionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ is realized for a specific form of the freely acting shift given in eq. (3.10). However, as we discussed, all the models that are obtained from $X_1$ by a freely acting $\mathbb{Z}_2$-shift have $(h_{11}, h_{21}) = (27, 3)$ and hence may be connected by continuous extrapolations. The study of these shifts in themselves may also yield additional information on the vacuum structure of these models and is worthy of exploration. Such study is currently underway and will be reported in future publications.

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