Kinematical versus dynamical contractions of the de Sitter Lie algebras

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Abstract

We present two kinematical Lie algebras contraction processes to improve the Bacry and Lévy-Leblond contractions (H Bacry, et al, 1968 J. Math. Phys., 9, 1605–1614) : (speed-time, speed-space and space-time contraction). For the first one, we introduce kinematical parameters, namely the radius \( r \) of the Universe, the period \( \tau \) of the Universe and the speed of light \( c = r \tau^{-1} \). Next we present them as static, Newtonian and flat limits through the use of the dynamical parameters, namely the mass, \( m \), the energy, \( E_0 \) and the compliance \( C \), all depending on mass as well as length and time. We consider that the second one as the best. To give a little physical taste for each kinematical Lie algebra, we set up the equations of change with respect each group parameter through the use of the Poisson brackets defined by the Kirillov form.

1. Introduction

Group (algebra) contraction is a method which allows to construct a new group (algebra) from an old one. Contraction of Lie groups and Lie algebras started sixty six years ago with nonu and Wigner [1] in 1953, when they were trying to connect Galilean relativity and special relativity. Eight years later, in 1961, Saletan [2] provided a mathematical foundation for the Inonu-Wigner method. Since then, various papers have been produced and the method of contraction has been applied to various Lie groups and Lie algebras [3–10].

The method has also been used by Bacry and Lévy-Leblond [11] to connect the de Sitter Lie algebras to all other kinematical Lie algebras through three kinds of contractions: speed-space contractions, speed-time contractions and space-time contractions. The terminology is related to the fact that Bacry and Lévy-Leblond have, first of all, scaled the velocity-space generators, the velocity-time translation generators and the space-time translation generators by a parameter \( \epsilon \) to obtain, in the limit \( \epsilon \to 0 \), the respective contractions that we prefer to call velocity-space contractions, velocity-time contractions and space-time contractions. The Lévy-Leblond contraction approach has been also extended to supersymmetry [12] and kinematical superalgebras [13–15].

Within the corresponding eleven Lie groups, four of them, namely the Galilei group \( G \) governing the Newtonian physics (Galilean relativity), the Poincaré group \( P \) governing the Einstein physics (special relativity), the Newton-Hooke groups \( NH \) describing Galilean relativity in the presence of a cosmological constant and the de Sitter Lie groups \( dS \) governing the de Sitter relativity of a space-time in expansion or oscillating universe, are well known in physics literature.

Within the remaining five ones, the Para-Poincaré groups \( P_{\alpha} \) and the Static \( S \) are still unknown in physics, but the Para-Galilei group \( G_{\alpha} \) and the Carroll group \( C \) are gaining more interest in recent times.

The Para-Galilei group has been identified as governing a light spring [16].

The Carroll group has been associated to tachyon dynamics [17–19], to Carrollian electromagnetism [20] versus Galilean electromagnetism [21] or to the dynamics of Carroll particles [22] and Carroll strings [23]. The anisotropic Carroll group in two space dimensions (i.e. without rotations) has been identified as the isometry group of gravitational plane waves [24, 25]. The Carroll group has also been used recently in the study of ultra-relativistic gravity [26] and for the generalization of Newton–Cartan gravity [27, 28]. The Carroll group has been
compared to the Galilei group in the study of gravitational waves \([29, 30]\), of confined dynamical systems \([31]\), of gravity \([32]\) and of covariant hydrodynamics \([33]\).

The purpose of this paper is, first of all, to clarify the origin of the names given to the three Lévy-Leblond types of contraction and then improve the Levy-Leblond method further.

The main purpose of this paper is to improve the contraction process conducting from the de Sitter Lie algebra to other kinematical Lie algebras. It presents a contraction process in terms of a new set of dynamical parameters (a mass \(m\), an energy \(E_0\) and a compliance \(C\)) related to the kinematical parameters (a speed \(c\), a radius \(r\) and a period \(\tau\)) by \(rC E m,\) where \(c = \tau r^{-1}\). Note that the kinematic descriptions are associated only with lengths and times, while the dynamic descriptions are associated with the mass as well as with lengths and time.

In section 2 we recall the Inonu-Wigner contraction, while section 3 recalls the Bacry—Lévy-Leblond method and uses it to establish the twelve kinematical Lie algebras as obtained by Ngendakumana et al \([10]\). We end the section by setting the raison d’être of a need of improvement the Lévy-Leblond contraction process. It is the purpose of the section 4. With the subsection 4.1, we revisit the Lévy-Leblond method by replacing the dimensional basis by a dimensioned one by scaling some of the vector basis, according the kind of Lie algebra we want to obtain, by either \(\frac{1}{\tau}, \omega = \frac{1}{\tau} \text{ or } \kappa = \frac{1}{\tau}\). We recover the all the kinematical Lie, except the static Lie algebra, as results of a velocity-space contraction, velocity-time contraction or a space-time contraction. The static Lie algebra is obtained as a velocity-space-time contraction (that Levy-Leblond call the general one) of the de Sitter Lie algebras. The process of this section fails to find the Static Lie algebra as a velocity-space contraction of the Carroll Lie algebra, as a velocity-time contraction of the Galilei Lie algebra or as a space-time contraction of the Para-Galilei Lie algebra (see figure 1 in \([11]\)). With the subsection 4.3 we solve the problem by working with the kinematical parameters which are radius \(r\) of the Universe, related to the cosmological constant by \(r^2 = \frac{3}{\Lambda}\), the period \(\tau\) of the Universe, and the velocity \(c\) of light defined by \(c = \tau r^{-1}\). In doing so, all the kinematical Lie algebras are found by the contraction process which consists in keeping one parameter finite and letting the remaining two tend to infinity, their ratio being kept finite. The results are summarized in the figure 1 in this paper. However, with the method used in subsection 4.3, we have to keep one parameter finite and let the remaining two tend to infinity, their ratio being kept finite.

To not have to take the precaution above we introduce with section 5 the dynamical contractions by first parameterizing the de Sitter Lie algebras by the dynamical parameters mass \(m\), compliance \(C\) (inverse of stiffness or of Hooke constant or of force constant), and energy, \(E_0\). The dynamical parameters and the kinematical parameters are related by \((r^2, \tau^2) = C(E_0, m)\) implying that \(E_0 = mc^2\). The corresponding contraction consist in letting only of the dynamical parameters go to infinity without constraining the remaining ones, contrary to the kinematical contractions process. The three Bacry-Lévy-Leblond contractions, i.e. the velocity-space contraction, the velocity-time contraction and the space-time contraction correspond then respectively to an
infinite energy $E_0$, an infinite mass $m$ and an infinite compliance $C$. They are the Newtonian limit, the static limit and the flat limit of Dyson [34]. This why we claim that the dynamical contraction process is the best one.

Finally in section 6, the Kirillov method is used to establish, for each kinematical Lie algebra, a Poisson-Lie algebra and the equations of change with respect any parameter of the Lie group. Those equations clarify the relationships and differences between the twelve kinematical Lie algebras according the up-down, right-left and frontward-backward contractions (see figure 2). They also permit to split the dual vector space of a kinematical Lie algebra in direct sum of irreducible vector subspaces with respect the operator $\frac{d}{dt}$, $s$ being a parameter of the Lie group.

2. Inonu-Wigner contractions of Lie algebras

2.1. Contraction of Lie algebras

We start with a Lie algebra $(G, \varphi)$ where $G$ is a vector space generated by $X_i$ and $\varphi$ is a skew symmetric mapping $\varphi: G \times G \rightarrow G$ defined by $\varphi(X_i, X_j) = X_k C_{ij}^k$ and satisfying the Jacobi identity

$$\varphi(X_i, \varphi(X_j, X_k)) + \varphi(X_j, \varphi(X_k, X_i)) + \varphi(X_k, \varphi(X_i, X_j)) = 0, \; \forall X_i, X_j, X_k \in G.$$  \hspace{1cm} (1)

The $C_{ij}^k$ are called the structure constants of the Lie algebra $(G, \varphi)$. The Jacobi identity shows that a Lie algebra is non associative algebra.

If the mapping $\psi: G \rightarrow G$ is singular for a certain value $\epsilon_0$ of $\epsilon$ and if the mapping $\varphi'$: $G \times G \rightarrow G$ is defined by

$$\varphi'(X, Y) = \lim_{\epsilon \rightarrow \epsilon_0} \psi^{-1} \varphi(\psi(X), \psi(Y))$$  \hspace{1cm} (2)

then $(G, \varphi')$ is a new Lie algebra called the contraction of the Lie algebra $(G, \varphi)$ [5].

2.2. Inonu-Wigner contractions

The pioneering contraction method is that of Inonu and Wigner [1] which starts with a Lie algebra $G = H + P$ where $H$ is generated by $X_a$, $P$ is generated by $X_i$; the structure of $G$ being a priori given by

$$\varphi(X_a, X_b) = X_{ab} + X_{ba}, \; \varphi(X_a, X_0) = X_{a0} + X_{0a}, \; \varphi(X_0, X_b) = X_{0b} + X_{b0}, \; \varphi(X_0, X_i) = X_{0i} + X_{i0}$$

where $a, b, \epsilon = 1, \ldots, \text{dim}(H)$ and $\alpha, \beta, \gamma = 1, \ldots, \text{dim}(P)$.

The Inonu-Wigner method uses the parameterized change of basis $\psi: (X_a, X_0) \rightarrow (Y_a, Y_0)$ defined by

$$Y_a = X_a, \; Y_0 = \epsilon X_0.$$  \hspace{1cm} (3)

The structure of the Lie algebra $G$ is given in the new basis by

Figure 2. Vertical arrows, horizontal arrows and oblique arrows indicate energy, mass and compliance tending to infinity respectively.
Table 1. The kinematical Lie algebras in term of $\epsilon, r$ and $\tau$.

| Lie symbol | Lie algebra name | $[K_i, H]$ | $[K_i, K_j]$ | $[K_i, P_j]$ | $[P_i, P_j]$ | $[P_i, H]$ |
|------------|-----------------|------------|------------|------------|------------|------------|
| dS \(_\pm\) | de Sitter | $P_i$ | $-\frac{1}{2\sqrt{2}}\epsilon_i^h$ | $\frac{1}{2\sqrt{2}}H_0$ | $\pm\frac{1}{2\sqrt{2}}\epsilon_i^h$ | $\pm\frac{1}{2\sqrt{2}}K_i$ |
| $P$ | Poincare | $P_i$ | $-\frac{1}{2\sqrt{2}}\epsilon_i^h$ | $\frac{1}{2\sqrt{2}}H_0$ | $0$ | $0$ |
| NH \(_\pm\) | Newton—Hooke | $P_i$ | $0$ | $0$ | $0$ | $\pm\frac{1}{2}K_i$ |
| $P_\pm$ | Para Poincare | $0$ | $0$ | $\frac{1}{2}H_0$ | $\pm\frac{1}{2}\epsilon_i^h$ | $\pm\frac{1}{2}K_i$ |
| $G$ | Galilei | $P_i$ | $0$ | $0$ | $0$ | $0$ |
| $G_\pm$ | Para—Galilei | $0$ | $0$ | $0$ | $0$ | $\pm\frac{1}{2}K_i$ |
| $C$ | Carroll | $0$ | $0$ | $\frac{1}{2}H_0$ | $0$ | $0$ |
| $S$ | Static | $0$ | $0$ | $0$ | $0$ | $0$ |

$\varphi(Y_\alpha, Y_\beta) = Y_\epsilon C_{ab}^\epsilon + \epsilon^{-1} Y_\alpha C_{\alpha\beta}^{ab}$  \(\varphi(Y_\alpha, Y_\beta) = \epsilon Y_\epsilon C_{ab}^\epsilon + Y_\alpha C_{\alpha\beta}^{ab}\)

In which condition an Inonu-Wigner contraction is it possible? In the limit $\epsilon \to 0$, the term $\epsilon^{-1} Y_\epsilon C_{\alpha\beta}^{ab}$ diverges. A limit will exist if only if the structure constants $C_{\alpha\beta}^{ab}$ vanish. Hence to get a Inonu-Wigner contraction, the structure of $\mathcal{G}$ in the basis $(X_\alpha, X_\beta)$ must be

$\varphi(X_\alpha, X_\beta) = X_\epsilon C_{ab}^\epsilon$,  \(\varphi(X_\alpha, X_\beta) = X_\epsilon C_{ab}^\epsilon + X_\alpha C_{\alpha\beta}^{ab}\),  \(\varphi(X_\alpha, X_\beta) = X_\epsilon C_{ab}^\epsilon + X_\gamma C_{\gamma\beta}^{ab}\)

\begin{equation}
(3)
\end{equation}

and

$\varphi(Y_\alpha, Y_\beta) = Y_\epsilon C_{ab}^\epsilon$,  \(\varphi(Y_\alpha, Y_\beta) = Y_\alpha C_{\alpha\beta}^{ab}\),  \(\varphi(Y_\alpha, Y_\beta) = y Y_\epsilon C_{ab}^\epsilon + Y_\alpha C_{\alpha\beta}^{ab}\)

\begin{equation}
(4)
\end{equation}

The Lie algebra $(\mathcal{G}, \varphi)$ defined by (4) is a Inonu-Wigner contraction of the mother Lie algebra $(\mathcal{G}_I, \varphi^I)$ with respect to the Lie subalgebra $\mathcal{H}$. It is a semi-direct sum of $(\mathcal{H}, \varphi^I)$ and the abelian Lie algebra $(\mathcal{P}, \varphi^I)$.

This process has been used by Bacry and Levy-Leblond [11]. In the next section we briefly recall the results and point why we need to improve some aspect of that paper.

### 3. Possible kinematical Lie algebras à la Lévy-Leblond

According to Bacry and Lévy-Leblond [11], a kinematical group is a space–time transformation group which keeps laws of physics invariant. Due to the assumptions of space isotropy, space–time homogeneity and existence of inertial transformations, a kinematical group is a ten dimensional Lie group whose Lie algebra is generated by three rotation generators $I_j$ (isotropy of space), three space translation generators $P_i$ (homogeneity of space), a time translation generator $H$ (homogeneity of time) and three inertial transformation generators $K_i$. Following Bacry and Lévy-Leblond [11], Ngendakumana and coauthors [10] have shown that under some mathematical physics assumptions only twelve kinematical Lie algebras exist. Their Lie algebraic structures have in common the Lie brackets defining the adjoint representation of the rotation generators

$[I_i, J_j] = J_k \epsilon_i^{jk}$,  \([I_i, K_j] = K_k \epsilon_i^{jk}\),  \([J_i, P_j] = P_k \epsilon_i^{jk}\),  \([J_i, H] = 0\)

The remaining Lie brackets are given by the table 1 [10]. The ParaPoincaré Lie algebra $\mathcal{P}_{1+}$ which is isomorphic to the Euclidean Lie algebra $\mathcal{E}(4)$ where the ‘translations’ generated by $K_i$ and $H$ form an abelian Lie subalgebra does not appear in the list of kinematical ones by Bacry and Lévy-Leblond [11]. The argument is that the inertial transformations are compact. However they are noncompact and only space translations are compact.

Using the Inonu-Wigner contraction method [1], Bacry and Lévy-Leblond [11] have established that these Lie algebras are approximations of the de Sitter Lie algebras. Their links are summarized by the contractions scheme (see figure 1 on page 1610 of [11]). We will refer to these nomenclature in the next two sections.

The parameter used by Bacry and Levy-Leblond to scale the subalgebra $\mathcal{P}$ is dimensionless. This is certainly due to the fact those authors have set $\epsilon = 1$ and $r = 1$, $c$ being the speed of light while $r$ is the radius of the Universe. The table at page 1608 shows that the generators of the kinematical Lie algebras in question seem to be dimensionless. This physics interpretation behind is then a bit difficult to follow. We propose to improve this situation in the next two sections.
4. Kinematical improvement of the Levy-Leblond approach

4.1. The Lie algebras $\mathcal{O}_{\pm}(5)$

We propose to recover the Bacry-Lévy-Leblond contractions scheme by using the kinematical parameters $r$, $\tau$ and $c$ which are respectively the radius of universe, the period of the Universe and speed of light. We first introduce the de Sitter Lie algebras $dS_{\pm}$ as isomorphic to the pseudo-orthogonal Lie algebras $O_{\pm}(5)$, i.e. that $dS_{\pm}(3) [dS_{\pm}(3)]$ is isomorphic to $O(1, 4) [O(2,3)]$ Lie algebra. The aim of this section is to better clarify velocity-space contractions, velocity-time contractions and space-time contractions of Bacry and Lévy-Leblond [11]. Let $V$ be a five dimensional manifold equipped with the metric

$$ds^2 = \delta_{ij} dx^i dx^j - (dx^c)^2 \pm (dx^3)^2 \equiv \eta_{ab} dx^a dx^b, \quad i, j = 1, 2, 3$$

(5)

where the dimension of the $x^a$ is that of length. The matrix elements $\eta_{ab}$ form the diagonal matrix $\eta_{\pm} = \text{diag}(I_{\pm \times \pm}, -1, \pm 1)$. Let denote by $G_{\pm}$ the group of transformations $x'^a = g_{\pm}^a x^b$ keeping $ds^2$ invariant. It is the group of real square matrices $g$ with order five satisfying $g_{\pm}^a \eta_{\pm} g_{\pm} = \eta_{\pm}$. The Lie group $SO_0(4, 1)$ is the connected component of $G_{\pm}$ while $SO_0(3, 2)$ is the connected component of $G_-$. The Lie algebra $\mathcal{O}_{\pm}(5)$ is the set of the real square matrices $X$ of order 5 satisfying $[X \eta_{\pm} + \eta_{\pm} X] = 0$. We easily verify that $X = J_k \theta^k + A_k \alpha^k + B_k \beta^k + \gamma \Gamma$, $k = 1, 2, 3, 5$ is the dimensionless matrix

$$X = \begin{pmatrix} e_{ij} \theta^k & \alpha^i & \beta^j \\ \alpha_j & 0 & \gamma \\ \mp \beta_j & \pm \gamma & 0 \end{pmatrix}$$

(6)

and that $(J_k, A_k, B_k, \Gamma)$ is a basis of $\mathcal{O}_{\pm}(5)$. The Lie algebra $\mathcal{O}_{\pm}(5)$ structure is defined by the Lie brackets

$$[J_i, J_j] = J_k \theta^k, \quad [J_i, A_j] = A_k \theta^k, \quad [J_i, B_j] = B_k \theta^k, \quad [J_i, \Gamma] = 0$$

(7)

$$[A_i, A_j] = \mp I \theta^k, \quad [A_i, B_j] = \Gamma \delta_{ij}, \quad [A_i, \Gamma] = 0$$

(8)

$$[B_i, B_j] = \pm J_k \theta^k, \quad [B_i, \Gamma] = \mp A_i$$

(9)

The dimensionless generator $A_i$ and $B_i$ play the role of velocity generator and space translation generator in the $i$th direction respectively while the dimensionless $\Gamma$ plays the role of time translations generator.

4.2. Inonu-Wigner approach revisited

4.2.1. Relative space Lie algebras: de Sitter, Newton-Hooke, Poincare and Galilei

Let set $K_i = \frac{1}{c} A_i$, $P_i = \frac{2}{c} B_i$ and $H = \omega \Gamma$ where $c$ is a speed while $\omega = \frac{1}{r}$ is a frequency. The Lie brackets (7), (8) and (10) become

$$[J_i, J_j] = J_k \theta^k, \quad [J_i, K_j] = K_k \theta^k, \quad [J_i, P_j] = P_k \theta^k, \quad [J_i, H] = 0$$

(10)

$$[K_i, K_j] = -\frac{1}{c^2} J_k \theta^k, \quad [K_i, P_j] = \frac{1}{c^2} H \delta_{ij}, \quad [K_i, H] = P_i$$

(11)

$$[P_i, P_j] = \pm \frac{\omega^2}{c^2} J_k \theta^k, \quad [P_i, H] = \pm \omega^2 K_i$$

(12)

They define the de Sitter Lie algebras $dS_{\pm}$ in the basis $(J_i, K_i, P_i, H)$. The general element of the Lie algebra $dS_{\pm}$ is $X = J_i \theta^i + K_i \omega^i + P_i x^i + H t^i$ being dimensionless, the physical dimensions of $\theta^i, x^i$ and $t^i$ being a velocity, a length and a time respectively. Also the general element of the dual of $dS_{\pm}$ is $\alpha = j^i J_i \omega^i + k^i K_i \omega^i + p^i P_i \omega^i + E \omega^i$ where $j^i$ is the $i$th component of the angular momentum, $k^i$ is the $i$th component of the static momentum, $p_i$ is the $i$th component of the linear momentum and $E$ is an energy. In the limit $c \to 0$, the Lie brackets (10), (11) and (12) become

$$[J_i, J_j] = J_k \theta^k, \quad [J_i, K_j] = K_k \theta^k, \quad [J_i, P_j] = P_k \theta^k, \quad [J_i, H] = 0$$

(13)

$$[K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, H] = P_i$$

(14)

$$[P_i, P_j] = 0, \quad [P_i, H] = \pm \omega^2 K_i$$

(15)

They define the Newton-Hooke Lie algebras $\mathcal{N}H_{\pm}$ in the basis $(J_i, K_i, P_i, H)$. In the limit $\omega \to 0$, the Lie brackets (10), (11) and (12) become

$$[J_i, J_j] = J_k \theta^k, \quad [J_i, K_j] = K_k \theta^k, \quad [J_i, P_j] = P_k \theta^k, \quad [J_i, H] = 0$$

(16)

$$[K_i, K_j] = -\frac{1}{c^2} J_k \theta^k, \quad [K_i, P_j] = \frac{1}{c^2} H \delta_{ij}, \quad [K_i, H] = P_i$$

(17)
They define the Poincaré Lie algebra \( \mathcal{P} \) in the basis \((J, K, P, H)\).

As \( \frac{1}{c} \) multiplies \( A_i \) and \( B_i \) while \( \omega \) multiplies \( B_i \) and \( \Gamma \), it follows that the Newton–Hooke lie algebras and the Poincaré Lie algebra are velocity-space contractions and space-time contraction of the de Sitter Lie algebras respectively.

The Lie algebra structure

\[
[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, K_j] = K_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, H] = 0
\]

(19)

\[
[K_i, J_j] = 0, \quad [K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, H] = 0
\]

(20)

\[
[P_i, J_j] = 0, \quad [P_i, P_j] = 0, \quad [P_i, H] = 0
\]

(21)

which defines the Galilei Lie algebra \( \mathcal{G} \) in the basis \((J, K, P, H)\), is obtained from (13), (14) and (15) when \( \omega \to 0 \) or from (16), (17) and (18) when \( \epsilon = \frac{1}{c} \to 0 \). The Galilei Lie algebra is then a velocity-space contraction of the Poincaré Lie algebra and a space-time contraction of the Newton–Hooke Lie algebras.

4.2.2. Relative time Lie algebras: de Sitter, Poincare, Para-Poincare and Carroll

Let set \( K_i = \frac{1}{\kappa} A_i, \quad P_i = \kappa B_i \) and \( M = \frac{\kappa \Gamma}{c^2} \). The Lie brackets (7), (8) and (8) become

\[
[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, K_j] = K_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, M] = 0
\]

(22)

\[
[K_i, J_j] = 0, \quad [K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, M] = 0
\]

(23)

\[
[P_i, J_j] = 0, \quad [P_i, P_j] = 0, \quad [P_i, M] = 0
\]

(24)

They define the de Sitter Lie algebras \( \mathcal{G} \) in the basis \((J, K, P, M)\). The general element of the Lie algebra \( \mathcal{G} \) is

\[
K = J, K, L, M, \quad \epsilon^k_{ij} = \frac{J_i J_j - K_i K_j}{\sqrt{J_i J_i - K_i K_i}}
\]

where \( \epsilon_i^j \) is the Kronecker delta. The Lie algebra structure

\[
[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, K_j] = K_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, M] = 0
\]

(25)

\[
[K_i, J_j] = 0, \quad [K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, M] = 0
\]

(26)

\[
[P_i, J_j] = 0, \quad [P_i, P_j] = 0, \quad [P_i, M] = 0
\]

(27)

They define the Para-Poincare Lie algebras \( \mathcal{N} \) in the basis \((J, K, P, M)\). In the limit \( \kappa \to 0 \) the Lie brackets (22), (23) and (24) become

\[
[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, K_j] = K_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, M] = 0
\]

(28)

\[
[K_i, J_j] = 0, \quad [K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, M] = 0
\]

(29)

\[
[P_i, J_j] = 0, \quad [P_i, P_j] = 0, \quad [P_i, M] = 0
\]

(30)

They define the Poincare Lie algebra \( \mathcal{P} \) in the basis \((J, K, P, M)\). As \( \frac{1}{c} \) multiplies \( A_i \) and \( \Gamma \) while \( \kappa \) multiplies \( B_i \) and \( \Gamma \), it follows that the Para-Poincare lie algebras and the Poincare Lie algebra are velocity-time contractions and space-time contraction of the de Sitter Lie algebras respectively.

The Lie algebra structure

\[
[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, K_j] = K_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, M] = 0
\]

(31)

\[
[K_i, J_j] = 0, \quad [K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, M] = 0
\]

(32)

\[
[P_i, J_j] = 0, \quad [P_i, P_j] = 0, \quad [P_i, M] = 0
\]

(33)

which defines the Carroll Lie algebra \( C \) in the basis \((J, K, P, M)\), is obtained from (25), (26) and (27) when \( \kappa \to 0 \) or from (28), (29) and (30) when \( \epsilon = \frac{1}{c} \to 0 \). The Carroll Lie algebra is then a velocity-time contraction of the Poincaré Lie algebra and a space-time contraction of the Para-Poincare Lie algebras.
4.2.3. Cosmological Lie algebras: de Sitter, Newton-Hooke, Para-Poincare and Para-Galilei

Let set $F_i = \frac{1}{r} A_i$, $P_i = \frac{1}{r} B_i$ and $H = \frac{1}{\Gamma}$ where $r$ is a radius while $\omega = \frac{1}{\tau}$ is a frequency. The Lie brackets (7), (8) and (8) become

$$[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, K_j] = K_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, H] = 0 \quad (34)$$

$$[F_i, F_j] = -\frac{\omega^2}{r^2} J_k \epsilon^k_{ij}, \quad [F_i, P_j] = \frac{1}{r^2} H \delta_{ij}, \quad [F_i, H] = \omega^2 P_i \quad (35)$$

$$[P_i, P_j] = \pm \frac{1}{r^2} J_k \epsilon^k_{ij}, \quad [P_i, H] = \pm F_i \quad (36)$$

They define the de Sitter Lie algebras $dS_n$ in the basis $(J_i, F_i, P_i, H)$. The general element of the Lie algebra $dS_n$ is $X = J_i \theta^i + F_i \zeta^i + P_i \chi^i + H_\tau \theta^i$ being dimensionless, the physical dimensions of $\zeta^i$, $\chi^i$ and $\tau$ being a velocity, a length and a time respectively. Also the general element of the dual of $dS_n$ is $\alpha = J_i \rho^i + F_i \rho^i + P_i \rho^i + E H^\pi$ where $j_i$ is the $ith$ component of the angular momentum, $f_i$ is the $ith$ component of a force, $p_i$ is the $ith$ component of the linear momentum and $E$ is an energy.

In the limit $\omega = \frac{1}{\tau} \rightarrow 0$, the Lie brackets (34), (35) and (36) become

$$[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, F_j] = F_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, H] = 0 \quad (37)$$

$$[F_i, F_j] = 0, \quad [F_i, P_j] = \frac{1}{r^2} H \delta_{ij}, \quad [F_i, H] = 0 \quad (38)$$

$$[P_i, P_j] = \pm \frac{1}{r^2} J_k \epsilon^k_{ij}, \quad [P_i, H] = \pm F_i \quad (39)$$

They define the Para-Poincare Lie algebras $P_n$ in the basis $(J_i, F_i, P_i, H)$.

In the limit $\kappa = \frac{1}{\tau} \rightarrow 0$, the Lie brackets (34), (35) and (36) become

$$[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, F_j] = F_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, H] = 0 \quad (40)$$

$$[F_i, F_j] = 0, \quad [F_i, P_j] = 0, \quad [F_i, H] = \omega^2 P_i \quad (41)$$

$$[P_i, P_j] = 0, \quad [P_i, H] = \pm F_i \quad (42)$$

They define the Newton-Hooke Lie algebra $NH_n$ in the basis $(J_i, F_i, P_i, H)$.

As $\omega$ multiplies $A_i$ and $\Gamma$ while $\kappa = \frac{1}{\tau}$ multiplies $A_i$ and $B_i$, it follows that the Newton-Hooke lie algebras and the Para-Poincare Lie algebras are velocity-space contractions and space-time contraction of the de Sitter Lie algebras respectively.

The Lie algebra structure

$$[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, F_j] = F_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, H] = 0 \quad (43)$$

$$[F_i, F_j] = 0, \quad [F_i, P_j] = 0, \quad [F_i, H] = 0 \quad (44)$$

$$[P_i, P_j] = 0, \quad [P_i, H] = \pm F_i \quad (45)$$

which defines the Para-Galilei Lie algebra $G^\pm$ in the basis $(J_i, F_i, P_i, H)$, is obtained from (37), (38) and (39) when $\kappa = \frac{1}{\tau} \rightarrow 0$ or from (40), (41) and (42) when $\omega = \frac{1}{\tau} \rightarrow 0$. The Para-Galilei Lie algebra is then a velocity-space contraction of the Newton-Hooke Lie algebra and a velocity-space contraction of the Para-Poincare Lie algebras.

4.2.4. The Static Lie algebra

By setting $K_i = \frac{1}{\omega} A_i$, $P_i = \kappa B_i$, and $M = \frac{\kappa}{\omega} \Gamma$, the structure of the de Sitter Lie $dS_n$ becomes

$$[K_i, K_j] = -\frac{\kappa^2}{\omega^2} J_k \epsilon^k_{ij}, \quad [K_i, P_j] = M \delta_{ij}, \quad [K_i, H] = \frac{\kappa^2}{\omega^2} P_i \quad (46)$$

$$[P_i, P_j] = \pm \kappa^2 J_k \epsilon^k_{ij}, \quad [P_i, M] = \pm \kappa^2 K_i \quad (47)$$

In the limit $\kappa \rightarrow 0$, the structure defined by (46), (47) and (48) becomes the structure

$$[J_i, J_j] = J_k \epsilon^k_{ij}, \quad [J_i, K_j] = K_k \epsilon^k_{ij}, \quad [J_i, P_j] = P_k \epsilon^k_{ij}, \quad [J_i, H] = 0 \quad (49)$$

$$[K_i, K_j] = -\frac{\kappa^2}{\omega^2} J_k \epsilon^k_{ij}, \quad [K_i, P_j] = M \delta_{ij}, \quad [K_i, M] = \frac{\kappa^2}{\omega^2} P_i \quad (50)$$

$$[P_i, P_j] = \pm \kappa^2 J_k \epsilon^k_{ij}, \quad [P_i, M] = \pm \kappa^2 K_i \quad (51)$$
defining the Static Lie algebra $S$ in the relative time kinematical Lie algebras. As $\kappa$ multiplies $A$, $B$, and $\Gamma$, the static Lie algebra is a \textit{velocity-space-time} (the general according Levy-Leblond [11]) contraction of the de Sitter Lie algebras.

4.3. Kinematical improvement

Even if the previous section clarifies better the Inonu-Wigner contractions of the de Sitter Lie algebras, there remain the following problem: the structures of relative space Lie algebras, the relative time Lie algebras and the cosmological Lie algebras are defined in different bases. We propose to remove that situation in the next section. We use three kinematical parameters: the speed $c$, the radius $r$ and the period $\tau$. We ignore the relation $c = r\tau^{-1}$ during all the process.

4.3.1. The de Sitter Lie algebras

Let $K_i = \frac{1}{r}A_i$, $P_i = \frac{1}{r}B_i$ and $H = \frac{1}{r}\Gamma$, where $\sigma = \frac{1}{r}$ is a slowness, $\kappa = \frac{1}{r}$ is a curvature while $\omega = \frac{1}{r}$ is a frequency. In the new basis $(J_i, K_i, P_i, H)$ of $O_5(5)$ the matrix $X$ above becomes

$$X = \begin{pmatrix} \frac{\phi^i}{c} & \frac{\psi^i}{r} & 0 & \frac{\kappa^i}{r} \\ \frac{\psi^i}{r} & \frac{\kappa^i}{r} & 0 & \frac{\kappa^i}{r} \\ \frac{\kappa^i}{r} & \frac{\kappa^i}{r} & 0 & 0 \\ \frac{\phi^i}{c} & \frac{\psi^i}{r} & 0 & \frac{\kappa^i}{r} \end{pmatrix}$$

(52)

where $\alpha^i = \frac{\phi^i}{c}$, $\beta^i = \frac{\psi^i}{r}$, $\gamma = \frac{\kappa^i}{r}$. Hence the parameters associated with $K_i, P_i$ and $H$ have velocity, length and time as respective physical dimension. The Lie brackets (7), (8) and (8) become then

$$[J_i, J_j] = J_k \epsilon_{ij}^k, \quad [J_i, K_j] = K_k \epsilon_{ij}^k, \quad [J_i, P_j] = P_k \epsilon_{ij}^k, \quad [J_i, H] = 0$$

(53)

$$[K_i, K_j] = -\frac{1}{c^2} J_k \epsilon_{ij}^k, \quad [K_i, P_j] = \frac{\tau}{c\tau} H \delta_{ij}, \quad [K_i, H] = \frac{\tau}{c\tau} P_j$$

(54)

$$[P_i, P_j] = \pm \frac{1}{r^2} J_k \epsilon_{ij}^k, \quad [P_i, H] = \pm \frac{\psi}{r\tau} K_i$$

(55)

Let us now study the limits of the de Sitter Lie algebras as the constants tend to infinity. Normally the three constants are constrained by $c = r\tau^{-1}$. However, we ignore it for a moment. We use it at the end of the section to show that our way of doing has recovered the results of table 1. It is first of all evident that (53) does not change. We are then only interested in the behavior of (54) and (55).

4.3.2. The Newton-Hooke, Poincaré and Para-Poincaré Lie algebras

In this section we look for the limits of the de Sitter Lie algebras as two of the constants tend to infinity while their ratio is kept finite.

(a) Newton-Hooke Lie algebras.

We verify that the limits of (54) and (55), as the speed $c$ and the radius $r$ tend to infinity while their ratio $\xi = \frac{c}{r}$ and $\tau$ are kept finite, are

$$[K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, H] = \frac{\tau}{c\tau} P_i$$

(56)

$$[P_i, P_j] = 0, \quad [P_i, H] = \pm \frac{\psi}{r\tau} K_i$$

(57)

The Lie brackets (53), (56) and (57) define the Newton-Hooke Lie algebra $\mathcal{NH}_\pm$.

(b) Poincaré Lie algebra.

If the period $\tau$ and the radius $r$ tend to infinity while their ratio $\xi = \frac{c}{r}$ and $c$ are kept finite, the brackets (54) and (55) become

$$[K_i, K_j] = -\frac{1}{c^2} J_k \epsilon_{ij}^k, \quad [K_i, P_j] = \frac{\tau}{rc\tau} H \delta_{ij}, \quad [K_i, H] = \frac{\tau}{c\tau} P_j$$

(58)

$$[P_i, P_j] = 0, \quad [P_i, H] = 0$$

(59)

The Lie brackets (53), (58) and (59) define the Poincaré Lie algebra $\mathcal{P}$.

(c) Para-Poincaré Lie algebras.

Similarly if the speed $c$ and the time $\tau$ tend to infinity while their ratio $\xi = \frac{c}{r}$ and $r$ are kept finite then the Lie brackets (54) and (55) become
\[ [K_i, K_j] = 0, \quad [K_i, P_j] = \frac{\tau}{rc} H \delta_{ij}, \quad [K_i, H] = 0 \]  
(60)

\[ [P_i, P_j] = \pm \frac{1}{r^2} J_k \epsilon^k_{ij}, \quad [P_i, H] = \pm \frac{c}{\tau r} K_i \]  
(61)

The Lie brackets (53), (60) and (61) define the Para-Poincaré Lie algebra \( \mathcal{P}_r \). We then notice that the Newton-Hooke Lie algebras, the Poincaré Lie algebra and the Para-Poincaré Lie algebras are respectively the velocity-space, space-time and velocity-time contractions of the de Sitter Lie algebras as in [11].

4.3.3. The Galilei, Para-Galilei and Carroll Lie algebras

(a) Galilei Lie algebra.

The limit of the Lie brackets (56) and (57) as the radius \( r \) and the period \( \tau \) tend to infinity while \( \frac{c}{r} \) and \( \tau \) are kept finite and the limit (58) and (59) as the radius \( r \) and the speed \( c \) tend to infinity while \( \frac{c}{r} \) and \( \tau \) are kept finite are the same, i.e.

\[ [K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, H] = \frac{r}{c \tau} P_i \]  
(62)

\[ [P_i, P_j] = 0, \quad [P_i, H] = 0 \]  
(63)

The Lie brackets (53), (62) and (63) define the Galilei Lie algebra \( \mathcal{G} \).

(b) Para-Galilei Lie algebras.

The limit of the Lie brackets (56) and (57) as the speed \( c \) and the period \( \tau \) tend to infinity while \( \frac{c}{r} \) and \( r \) are kept finite and the limit (60) and (61) as the radius \( r \) and the speed \( c \) tend to infinity while \( \frac{c}{r} \) and \( \tau \) are kept finite are the same, i.e.

\[ [K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, H] = 0 \]  
(64)

\[ [P_i, P_j] = 0, \quad [P_i, H] = \pm \frac{c}{\tau r} K_i \]  
(65)

The Lie brackets (53), (64) and (65) define the Para-Galilei Lie algebras \( \mathcal{G}_\tau \).

(c) Carroll Lie algebra.

The limit of the Lie brackets (58) and (59) as the speed \( c \) and the period \( \tau \) tend to infinity while \( \frac{c}{r} \) is kept finite and the limit (60) and (61) as the radius \( r \) and the period \( \tau \) tend to infinity while \( \frac{c}{r} \) is kept finite are the same, i.e.

\[ [K_i, K_j] = 0, \quad [K_i, P_j] = \frac{\tau}{rc} H \delta_{ij}, \quad [K_i, H] = 0 \]  
(66)

\[ [P_i, P_j] = 0, \quad [P_i, H] = 0 \]  
(67)

The Lie brackets (53), (66) and (67) define the Carroll Lie algebra \( \mathcal{C} \). Hence the Galilei, the Para-Galilei, the Carroll Lie algebras are respective contractions of the Newton-Hooke or Poincaré Lie algebras, the Newton-Hooke or the Para-Poincaré Lie algebras, the Poincaré or the Para-Poincaré Lie algebras respectively.

4.3.4. The static Lie algebra

The limit of the Lie brackets (62) and (63) as the speed \( c \) and the period \( \tau \) tend to \( \infty \) while \( \frac{c}{r} \) and \( r \) are kept finite, the limit (64) and (65) as the radius \( r \) and the period \( \tau \) tend to \( \infty \) while \( \frac{c}{r} \) and \( \tau \) are kept finite and the limit of the Lie brackets (66) and (67) as the speed \( c \) and the radius \( r \) tend to infinity while \( \frac{c}{r} \) and \( \tau \) are kept finite are the same; i.e.

\[ [K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [K_i, H] = 0 \]  
(68)

\[ [P_i, P_j] = 0, \quad [P_i, H] = 0 \]  
(69)

The Lie brackets (53), (68) and (69) define the Static Lie algebra \( \mathcal{S} \).

When the constraint \( c = r \tau^{-1} \) is taken in account, the Lie brackets in the table 1 are recovered.

These approximations through kinematical parameters are summarized in the following cube (see figure 1). On the cube, the horizontal arrows represent the contractions as \( c, \tau \to \infty, \frac{c}{r}, r \) finite (velocity-time contractions), the vertical arrows represent the contractions as \( c, r \to \infty, \frac{c}{r}, \tau \) and \( \tau \) finite (velocity-space contractions) and the oblique arrows represent the contractions as \( r, \tau \to \infty, \frac{c}{r}, c \) finite (space-time contractions).
5. Dynamical improvement of the Levy-Leblond approach

In the process of the previous section, we were sending two parameters at infinity while keeping finite their ratio and the third one. We introduce in this section the dynamical parameters compliance $C$, mass $m$ and energy $E_0$. These dynamical parameters enter the de Sitter Lie algebras structure by replacing the boost generators $K_i$ by the momentum generators $Q_i = \frac{1}{m}K_i$, $m$ being a mass and by defining the compliance $C$ and the energy $E_0$ respectively by $C = \frac{\epsilon^2}{m}$ and $E_0 = mc^2$. In the contraction process, only one parameter will be sent to infinity without any precaution on the other two. Similarly to the kinematical contraction process, at the end of the dynamical contraction process, the kinematical Lie algebras will distributed on a cube. Two opposite faces with finite versus infinite mass, two opposite faces with a finite versus infinite energy and two faces with a finite versus infinite compliance.

5.1. Three finite parameters Lie algebras: the de Sitter

The de Sitter Lie algebras $dS_4$ are then defined in the basis $(J_0, Q_0, P_0, H)$, by the Lie brackets

\[ [I_i, J_i] = I_k \epsilon^{k}_{ij}, [I_i, Q_j] = Q_k \epsilon^{k}_{ij}, [I_i, P_j] = P_k \epsilon^{k}_{ij}, [J_0, H] = 0 \] (70)

\[ [Q_0, Q_i] = -\frac{1}{mE_0}I_k \epsilon^{k}_{ij}, [Q_0, P_i] = \frac{1}{E_0}H \epsilon_{ij}, [Q_0, H] = \frac{1}{m}P_i \] (71)

\[ [P_0, P_i] = \pm \frac{1}{CE_0}I_k \epsilon^{k}_{ij}, [P_0, H] = \pm \frac{1}{C}Q_i \] (72)

The de Sitter Lie algebras $dS_4$ are then characterized by the three dynamical parameters $m$, $C$ and $E_0$. They are at the edge of the finite mass, finite energy and finite compliance. The de Sitter Lie algebras are then characterized by three finite kinematical parameters: the frequency $\omega = \frac{1}{\sqrt{mC}}$ (time $\tau = \sqrt{mC}$), the speed $c = \frac{\epsilon}{\sqrt{m}}$ (slowness $s = \frac{\epsilon}{\sqrt{m}}$) and the curvature $\kappa = \frac{1}{\sqrt{CE_0}}$ (the radius $r = \sqrt{CE_0}$).

5.2. Two finite parameters Lie algebras: Newton-Hooke, Poincare and Para-Poincare

When one of the three parameters becomes infinite, the structure of the de Sitter Lie algebras gives rise to a Lie algebra characterized by the remaining two. In one dimension of space (case of $O_+(3)$), these algebras are the solvable ones.

5.2.1. Mass-Compliance Lie algebras: Newton-Hooke

When the energy $E_0$ tends to infinity, the structure of the de Sitter Lie algebras given by (70), (71) and (72) becomes

\[ [I_i, J_i] = I_k \epsilon^{k}_{ij}, [I_i, Q_j] = Q_k \epsilon^{k}_{ij}, [I_i, P_j] = P_k \epsilon^{k}_{ij}, [J_0, H] = 0 \] (73)

\[ [Q_0, Q_i] = 0, [Q_0, P_i] = 0, [Q_0, H] = \frac{1}{m}P_i \] (74)

\[ [P_0, P_i] = 0, [P_0, H] = \pm \frac{1}{C}Q_i \] (75)

and defines the Newton-Hooke Lie algebras characterized by the mass $m$ and the compliance $C$ related to the frequency $\omega = \frac{1}{\sqrt{mc}}$ (time $\tau = \sqrt{mc}$). The Newton-Hooke Lie group is then a semi direct product of the direct product of rotations and time translations on the abelian group of impulses-positions.

5.2.2. Mass-Energy Lie algebra: Poincare

When the compliance $C$ tends to infinity, the structure of the de Sitter Lie algebras given by (70), (71) and (72) becomes

\[ [I_i, J_i] = I_k \epsilon^{k}_{ij}, [I_i, Q_j] = Q_k \epsilon^{k}_{ij}, [I_i, P_j] = P_k \epsilon^{k}_{ij}, [J_0, H] = 0 \] (76)

\[ [Q_0, Q_i] = -\frac{1}{mE_0}I_k \epsilon^{k}_{ij}, [Q_0, P_i] = \frac{1}{E_0}H \epsilon_{ij}, [Q_0, H] = \frac{1}{m}P_i \] (77)

\[ [P_0, P_i] = 0, [P_0, H] = 0 \] (78)

and defines the Poincare Lie algebra characterized by the mass $m$ and the energy $E_0$ related to the speed $c = \frac{\epsilon}{\sqrt{E_0}}$ (slowness $s = \frac{\epsilon}{\sqrt{E_0}}$). The Poincare Lie group is then a semi direct product of the direct product of the Lorentz group on the abelian group of space-time translations.
5.2.3. Compliance-Energy Lie algebras: Para-Poincare
When the mass $m$ tends to infinity, the structure of the de Sitter Lie algebras given by (70), (71) and (72) becomes

\[ [J_i, J_j] = J_k \epsilon^{k}_{ij}, \quad [J_i, Q_j] = Q_k \epsilon^{k}_{ij}, \quad [J_i, P_j] = P_k \epsilon^{k}_{ij}, \quad [J_i, H] = 0 \]  
\[ [Q_i, Q_j] = 0, \quad [Q_i, P_j] = \frac{1}{E_0}H \delta_{ij}, \quad [Q_i, H] = 0 \]  
\[ [P_i, P_j] = \pm \frac{1}{CE_0}I_{k}J_{k} \epsilon^{k}_{ij}, \quad [P_i, H] = \pm \frac{1}{C} Q_i \]  
and defines the Para-Poincare Lie algebra characteristic by the compliance $C$ and the energy $E_0$ related to the curvature $\kappa = \frac{1}{\sqrt{E_0}}$ (the radius $r = \sqrt{E_0}$). The Para-Poincare Lie group is then a semi direct product of the Para-Lorentz group on the abelian group of impulses-time translations.

5.3. One finite parameter Lie algebras: Galilei, Para-Galilei and Carroll
When two of the three parameters become infinite, the structure of the de Sitter Lie algebras gives rise to a Lie algebra characterized by the remaining one. In one dimension of space (case of $O_3(3)$), these algebras are the nilpotent ones.

5.3.1. Energy Lie algebra: Carroll
The structure

\[ [J_i, J_j] = J_k \epsilon^{k}_{ij}, \quad [J_i, Q_j] = Q_k \epsilon^{k}_{ij}, \quad [J_i, P_j] = P_k \epsilon^{k}_{ij}, \quad [J_i, H] = 0 \]  
\[ [Q_i, Q_j] = 0, \quad [Q_i, P_j] = \frac{1}{E_0}H \delta_{ij}, \quad [Q_i, H] = 0 \]  
\[ [P_i, P_j] = 0, \quad [P_i, H] = 0 \]  

defining the Carroll Lie algebra is obtained from that of the Poincare Lie algebra defined by (76), (77) and (78) when the mass $m$ tends to infinity or from that of the Para-Poincare Lie algebras defined (88), (86) and (90) when the compliance tends to infinity. The Carroll Lie algebras characterized by a finite energy $E_0$.

5.3.2. Compliance Lie algebra: Para-Galilei
The structure

\[ [J_i, J_j] = J_k \epsilon^{k}_{ij}, \quad [J_i, Q_j] = Q_k \epsilon^{k}_{ij}, \quad [J_i, P_j] = P_k \epsilon^{k}_{ij}, \quad [J_i, H] = 0 \]  
\[ [Q_i, Q_j] = 0, \quad [Q_i, P_j] = 0, \quad [Q_i, H] = 0 \]  
\[ [P_i, P_j] = 0, \quad [P_i, H] = \pm \frac{1}{C} Q_i \]  

defining the Para-Galilei Lie algebras is obtained from that of the Newton-Hooke Lie algebras defined by (73), (74) and (75) when the mass $m$ tends to infinity or from that of the Para-Poincare Lie algebras defined (88), (86) and (90) when the energy tends to infinity. The Para-Galilei Lie algebras characterized by a finite compliance $C$.

5.3.3. Mass Lie algebra: Galilei
The structure

\[ [J_i, J_j] = J_k \epsilon^{k}_{ij}, \quad [J_i, Q_j] = Q_k \epsilon^{k}_{ij}, \quad [J_i, P_j] = P_k \epsilon^{k}_{ij}, \quad [J_i, H] = 0 \]  
\[ [Q_i, Q_j] = 0, \quad [Q_i, P_j] = 0, \quad [Q_i, H] = \frac{1}{m} P_i \]  
\[ [P_i, P_j] = 0, \quad [P_i, H] = 0 \]  

defining the Galilei Lie algebra is obtained from that of the Newton-Hooke Lie algebras defined by (73), (74) and (75) when the compliance $C$ tends to infinity or from that of the Poincare Lie algebras defined (76), (77) and (78) when the energy tends to infinity. The Galilei Lie algebras characterized by a finite mass $m$.

We can say that the mass $m$ is galilean, the compliance $C$ is para-galilean and the energy $E_0$ is carrollian.

5.4. Zero parameters Lie algebra: Static
The zero parameters Lie algebra is the static Lie algebra which is obtained from the Galilei Lie algebra as the mass tends to infinity, from the Carroll Lie algebra as the energy tends to infinity or from the Para-Galilei Lie algebra as the compliance tends to infinity. Its structure is

\[ [J_i, J_j] = J_k \epsilon^{k}_{ij}, \quad [J_i, Q_j] = Q_k \epsilon^{k}_{ij}, \quad [J_i, P_j] = P_k \epsilon^{k}_{ij}, \quad [J_i, H] = 0 \]  

Let the general element of the dual of a kinematical Lie algebra be \( \pi \). We know that the Poisson bracket of two functions \( f, g \) is given by

\[
\{ f, g \} = \pi_\theta(\partial f / \partial \theta_j) \pi_\pi(\partial g / \partial \pi_k)
\]

We note that \( \{ f, g \} \) is nothing else than the Kirillov form \( \mathcal{K}(f, g) \) of the Lie algebra \( \mathcal{G} \) in the coordinate system \( \{ F, \pi \} \) in which the Kirillov form is given by

\[
\mathcal{K}(f, g) = \pi_\theta(\partial f / \partial \theta_j) \pi_\pi(\partial g / \partial \pi_k) - \pi_\theta(\partial g / \partial \theta_j) \pi_\pi(\partial f / \partial \pi_k)
\]

It is the abelian Lie algebra in the case of the dimension one of space (case of \( O(3) \)).

\[ [Q_0, Q_1] = 0, \quad [Q_0, P] = 0, \quad [Q_1, H] = 0 \]

\[ [P_0, P] = 0, \quad [P_0, H] = 0 \]

6. A glance at the physics associated to the kinematical Lie algebras

Let us have a look at the physics associated to the kinematical Lie algebras in function of the three dynamical parameters: compliance \( C \), mass \( m \) and energy \( E_0 \).

6.1. Poisson brackets

We know that the Poisson bracket of two functions defined on the dual \( \mathcal{G}^* \) of any Lie algebra \( \mathcal{G} \) is defined by

\[
\{ f, g \} = K_{ij}(a) \frac{\partial f}{\partial a_j} \frac{\partial g}{\partial a_i}
\]

where \( a_i \) are the coordinates on \( \mathcal{G}^* \) and \( K_{ij}(a) = -a_k C_{ij}^k \) are the matrix elements of the Kirillov form.

Let the general element of the dual of a kinematical Lie algebra be

\[
j_k \theta^k + \pi_k \theta^k + \pi_k \theta^k + EH^k
\]

where \( j_k \) are the components of the angular momentum conjugated to the angle \( \theta^k \), \( \pi_k \) are the components of the linear
momentum conjugate to the space translation \( x^k, q_k \) are the components of the position conjugated to the impulse \( p^k \) while \( E \) is an energy conjugated to the time translation \( t \).

From (70), (71) and (72) follows that the Kirillov matrix associated to the de Sitter groups is

\[
K = \begin{pmatrix}
-\hbar \epsilon_{ij}^k - q_k \epsilon_{ij}^k & -m \epsilon_{ij}^k & 0 \\
q_k \epsilon_{ij}^k & \frac{\hbar}{mE_0} \epsilon_{ij}^k & -\frac{E}{E_0} \epsilon_{ij}^k & -\frac{m}{E} \epsilon_{ij}^k \\
m \epsilon_{ij}^k & \frac{\hbar}{mE_0} \epsilon_{ij}^k & \frac{E}{E_0} \epsilon_{ij}^k & \frac{m}{E} \epsilon_{ij}^k \\
0 & \frac{\hbar}{mE_0} \epsilon_{ij}^k & \frac{E}{E_0} \epsilon_{ij}^k & \frac{m}{E} \epsilon_{ij}^k
\end{pmatrix}
\]

(95)

From the Lie brackets (see previous section) defining the kinematical Lie algebras in function of \( m, E_0 \) and \( C \) follows then that the kinematical Poisson-Lie algebras are defined by the common Poisson brackets

\[
\{ j_i, j_j \} = -\hbar \epsilon_{ij}^k, \quad \{ j_i, q_j \} = -q_k \epsilon_{ij}^k, \quad \{ j_i, \pi_j \} = -m \epsilon_{ij}^k, \quad \{ j_i, E \} = 0
\]

(96)

and the other Poisson brackets given by the table 4.

### 6.2. Equations of Change

We rewrite (94) as \( [f, g] = X_f(g) \) where the vector field \( X_f \) is defined by

\[
X_f = K_0(a) \frac{\partial f}{\partial a_i} \frac{\partial}{\partial a_j}
\]

(97)

and verifies \( [X_f, X_g] = X_{[f,g]} \). It is known that the mapping \( \rho \) defined by \( \rho(f) = X_f \) is a realization of the Lie algebra \( \mathcal{G} \).

Use of the structure of de Sitter Lie algebras \( dS_n \) defined by the Lie brackets (70), (71) and (72) permits to verify that the de Sitter is the realized by the vector fields

\[
X_{\hbar} = -\hbar \epsilon_{ij}^k \frac{\partial}{\partial j_i} - q_k \epsilon_{ij}^k \frac{\partial}{\partial q_j} - m \epsilon_{ij}^k \frac{\partial}{\partial \pi_j}
\]

(98)

\[
X_{q_k} = q_k \epsilon_{ij}^k \frac{\partial}{\partial j_i} + \frac{\hbar}{mE_0} \epsilon_{ij}^k \frac{\partial}{\partial q_j} - \frac{E}{E_0} \frac{\partial}{\partial \pi_j} - \frac{m}{E} \frac{\partial}{\partial E}
\]

(99)

\[
X_{\pi_j} = m \epsilon_{ij}^k \frac{\partial}{\partial q_j} + \frac{E}{E_0} \frac{\partial}{\partial \pi_j} + \frac{m}{E} \frac{\partial}{\partial E}
\]

(100)

\[
X_E = \frac{\pi_j}{m} \frac{\partial}{\partial q_j} \pm \frac{q_k}{C} \frac{\partial}{\partial \pi_j}
\]

(101)

The realizations of the other kinematical Lie algebras are obtained from (98) to (101) through the dynamical contraction process defined in the previous section. No need to make them explicit here.

If \( X_f \) is the generating function of the one parameter diffeomorphism \( \Phi_0 : \mathcal{G}^* \rightarrow \mathcal{G}^* \) and if \( X_\lambda \) is the generating function of the one parameter diffeomorphism \( \Phi_\lambda : \mathcal{G}^* \rightarrow \mathcal{G}^* \), then the equation of change of the function \( g \) with respect to \( s \) is \( \frac{dg}{ds} = X_f(g) \) while the equation of change of \( f \) with respect \( \lambda \) is \( \frac{df}{d\lambda} = X_\lambda(f) \).
For each parameter $s$, let define $F_s$ as $F_s = \{ f \in \mathcal{G}^*: \frac{df}{dx} = 0 \}$ and let $V_s = \mathcal{G}^*/F_s$ be the variables submanifold of $\mathcal{G}^*$. The equations of change describe how the coordinates on $V_s$ change with respect the ad hoc parameter.

Note that as $\Phi_0 \circ \Phi_1 = \Phi_{01}$, the change parameters must be additive. It is true for the longitudinal angle. We show in the appendix that it is also for the momentum parameter $p^i$, the space translation $x^i$ and the time translation parameter $t$ appearing in the $dS_5$ Lie algebra element $X = J_0 \theta^k + Q_0 p^k + P_0 x^k + H t$. We use the one spatial Poincaré Para-Poincaré and Newton-Hooke, Lie algebras to respectively associate a non additive boost to momentum, a non additive force to space translation and to time translation a non additive damping like coefficient.

In the maximal case of the de Sitter case, it follows from (98) to (101) that the equations of change with

- the angle $\theta^i$ are

\[
\frac{dj_i}{d\theta^i} = \dot{j}_k \epsilon^{ki} \quad \frac{dq_k}{d\theta^i} = q_k \epsilon^{ki} \quad \frac{d\pi_i}{d\theta^i} = \pi_i \epsilon^{ki} \quad \frac{dE}{d\theta^i} = 0
\]  

(102)

meaning that $j_k$, $q_k$, $\pi_i$, and $E$ are constant with respect $\theta^i$.

It follows that $J_0 = \{ j_k, q_k, \pi_i, E \}$ is quadridimensional while $V_{\theta^i} = \{ j_k, q_k, \pi_i \}$ is 6-dimensional, a direct sum $V_{\theta^i} = V_{\theta^i}(j_k) \oplus V_{\theta^i}(q_k) \oplus V_{\theta^i}(\pi_i)$, $k = i$ is 2-dimensional irreducible subspaces under the operator $\frac{d}{d\theta^i}$.

- the impulse $p^i$ are

\[
\frac{dp_i}{dt} = \dot{q}_k \epsilon^{ki} \quad \frac{dq_k}{dt} = \frac{j_k}{mE_0} \epsilon^{ki} \quad \frac{d\pi_i}{dt} = -\frac{E}{E_0} \dot{\pi}_i \quad \frac{dE}{dt} = -\frac{\pi_i}{m}
\]  

(103)

meaning that $j_k$, $q_k$, are constant with respect $p^i$. We then have that $V_{p^i} = \{ j_k, q_k \}$ is 2 dimensional and that $V_{p^i} = \{ j_k, q_k, \pi_i, E \}$, $k = i$ is 8-dimensional a direct sum $V_{p^i} = V_{p^i}(j_k) \oplus V_{p^i}(q_k) \oplus V_{p^i}(\pi_i)$, $k = i$, two 4 dimensional irreducible subspaces $\frac{d}{dp^i}$.

- the space translation $x^i$ are

\[
\frac{dx_i}{d\theta^i} = \pi_i \epsilon^{ki} \quad \frac{dx_i}{dt} = \frac{E}{E_0} \epsilon^{ki} \quad \frac{d\pi_i}{dt} = \pm \frac{j_k}{CE_0} \epsilon^{ki} \quad \frac{dE}{dt} = \pm \frac{\pi_i}{C}
\]  

(104)

meaning that $j_k$, $\pi_i$, $E$ are constant with respect $x^i$. We then have that $V_{x^i} = \{ j_k, \pi_i \}$ is 2 dimensional and that $V_{x^i} = \{ j_k, \pi_i, q_k, E \}$, $k = i$ is 8-dimensional a direct sum $V_{x^i} = V_{x^i}(j_k) \oplus V_{x^i}(\pi_i) \oplus V_{x^i}(q_k) \oplus V_{x^i}(E)$, $k = i$, two 4 dimensional irreducible subspaces $\frac{d}{dx^i}$.

- the time translation $t$ are

\[
\frac{dt}{dt} = 0 \quad \frac{d\pi_i}{dt} = \frac{\pi_i}{m} \quad \frac{dE}{dt} = \pm \frac{j_k}{C}
\]  

(105)

meaning that $j_k$, $E$, are constant with respect $t$. We then have that $V_t = \{ q_k, \pi_i \}$, a 6 dimensional irreducible space $\frac{d}{dt}$.

The equations of change corresponding to all kinematical Lie algebra are given in the table 5. In this table we illustrate the equations change with respect the longitude angle $\varphi = \theta^i$, the momentum up $p^i = p^i$, the altitude $z = x^3$ and the time $t$. Also the greek indices take the value 1 and 2 while the latin indices take the values 1, 2 and 3. The equations of the form $\frac{dx^i}{d\theta^i} = 0$ do not appear in the table. Moreover the first column of the table contains the dimensions of $V_{\theta^i}$, the second one indicates the first order differential equations, the third one the second order differential equation when available, finally the last one shows the corresponding kinematical Lie algebras paired as mother-daughter in the parental relations (see figure 2) up-down for the variations with respect time (energy $E_0$ is absent in the equations), front-backward for the variations with respect the altitude $z$ (mass $m$ is absent in the equations) and right-left for the variations with respect the momentum up $p$ (compliance $C$ is absent in the equations). We also notice that $V_{\theta^i} = V_{\theta^i}(j_k) \oplus V_{\theta^i}(q_k) \oplus V_{\theta^i}(\pi_i)$ (i.e. $6 = 2 + 2 + 2$ as sum of dimensions) under the differential rotation operator $\frac{d}{d\theta^i}$. Note that $V_{\theta^i}(j_k)$ means that the components of $j$ form an irreducible entity under the differential operator $\frac{d}{d\theta^i}$. Also $V_{\theta^i}$ is irreducible under the differential time operator $\frac{d}{dt}$ in the de Sitter and Newton-Hooke case, that $V_{\theta^i} = V_{\theta^i}(j_k, \pi_i) \oplus V_{\theta^i}(q_k, E)$ (i.e. $6 = 4 + 2$) is irreducible under the differential altitude operator $\frac{d}{d\theta^i}$ in the de Sitter and Para-Poincaré cases, and finally that
Table 5. Equations of change.

| dimV | 1st order DE | 2nd order DE | Lie algebras |
|------|--------------|--------------|--------------|
| 6    | $\frac{d\alpha_s}{d\theta} = \alpha_s e_{\mu\nu}, \alpha_{0s} = \lambda_s, q_{0s}, \pi_0$ | $\frac{d^2\alpha_s}{d\theta^2} = -\alpha_s$ | All |
| 6    | $d\phi/d\tau = \lambda_s, ±\frac{\pi_0}{\tau_0}$ | $d\phi/d\theta = \pm \frac{\pi_0}{\tau_0}$ | ds, NH, P |
| 6    | $\frac{d\phi_0}{d\theta} = \pi_0, e_{\mu\nu}, (\text{cst velocity})$ | $\frac{d^2\phi_0}{d\theta^2} = \pi_0$ | P, G |
| 6    | $\frac{d\phi}{d\theta} = \pi_0, e_{\mu\nu}, (\text{cst force})$ | $\frac{d^2\phi}{d\theta^2} = 0$ | No one |

V_p = V_p(j_x, q_x) + V_p(\pi, E) (i.e. 6 = 4 + 2) is irreducible under the differential momentum up operator $d\phi/d\theta$ in the de Sitter and Poincaré cases. The reader can also verify that the three dimensional manifolds are irreducible under the operator $d\phi/d\theta$. They are direct sums (i.e. 3 = 2 + 1) under the operators $d\phi_0/d\theta$ and $d\phi/d\theta$. Finally the two dimensional ones are irreducible and correspond to the pairs containing the static Lie algebra as a daughter. Note also that all the ten coordinates on $G^*$ are constant in the Carroll and static Lie algebras cases.

7. Conclusion

In this paper we have shown how to obtain straightally all kinematical Lie algebras from the de Sitter Lie algebras through contraction using dynamical parameters mass m, compliance C and energy E_0. We consider that it is the best kinematical Lie algebra contraction process. We had also a little glance at the physics associated to each kinematical Lie algebra. We noticed that V_z = V_z(j_x) + V_z(q_x) + V_z(\pi_x) (i.e. 6 = 2 + 2 + 2) under the differential momentum operator $d\phi/d\theta$, where V_z(j_x) means the components of j form an irreducible entity under the differential operator $d\phi/d\theta$. Also V_z is irreducible under the differential time operator $d\phi/d\tau$ in the de Sitter and Newton-Hooke case, that V_z = V_z(j_x) + V_z(q_x) (i.e. 6 = 4 + 2) under the differential time operator $d\phi/d\theta$ in the de Sitter and Para-Poincaré cases and finally that V_p = V_p(j_x, q_x) + V_p(\pi, E) (i.e. 6 = 4 + 2) under the differential momentum operator $d\phi/d\theta$ in the de Sitter and Poincaré cases.

We notice from table 4 that positions do not commute in the de Sitter and Poincaré cases, that linear momenta do not commute in de Sitter and Para-Poicare cases and that the uncertainty—like relation \{π_0, q_x\} = $\frac{\pi_0}{\xi_0} \xi_0$ occurs in finite energy (relative time) groups.

Appendix

This appendix serves to show how non additive parameters such as Lorentz boost are obtained from additive ones such as momentum. We use the brackets of table 2.

A.1. From momentum to boost

The Poincaré Lie algebra in one space dimension is defined by the Lie brackets

$$[Q, P] = \frac{1}{E_0}H, \ [Q, H] = \frac{1}{m}P, \ [P, H] = 0$$ (106)

where Q generates momenta, P generates space translations while P generates time translations. We verify that $exp(x_0^4P + t_0^4H) = A \exp(x_0^4P + t_0^4H)(\exp(x_0^4P + t_0^4H))$ gives the Poincare space-time transformations
\[ x' = \cosh \left( \frac{p}{\sqrt{mE_0}} \right) x_0 + \sqrt{\frac{E_0}{m}} \sinh \left( \frac{p}{\sqrt{mE_0}} \right) t_0 + x, \quad t'_0 = \frac{m}{E_0} \sinh \left( \frac{p}{\sqrt{mE_0}} \right) x_0 + \cosh \left( \frac{p}{\sqrt{mE_0}} \right) t_0 + t \]

where \( p \) is an additive momentum.

If we define the boost by \( v = \frac{E_0}{m} \tanh \left( \frac{p}{\sqrt{mE_0}} \right) \), then we recover the corresponding non additive boosts composition law \( v'' = \frac{v + v'}{1 + \frac{p v}{m}} \). It is the usual Lorentz one when \( E_0 = mc^2 \). Similarly if slowness is defined by

\[ s = \frac{m}{E_0} \tanh \left( \frac{p}{\sqrt{mE_0}} \right), \]

then the slowness composition law is \( s'' = \frac{s + s'}{1 + \frac{p s}{m}} \).

The limits of (107) are given in the table

| \( E_0 \to \infty \) | \( m \to \infty \) |
|-------------------|-------------------|
| \( x'_0 = x_0 + \frac{p}{m} t_0 + x, \quad t'_0 = t_0 + t \) | \( x'_0 = x_0 + x, \quad t'_0 = t_0 + \frac{p}{E_0} x_0 + t \) |

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where \( v = \frac{p}{m} \) is a Galilean boost and \( s = \frac{p}{E_0} \) is a Carrollian slowness.

A.2. From space translation to force

The Lie algebra of one spatial Para-Poincaré Lie algebra \( P_{\pm} \) is defined by the Lie brackets

\[ [Q, P] = \frac{1}{E_0} H, \quad [[Q, H], H] = 0, \quad [P, H] = \pm \frac{1}{C} Q. \] (109)

We verify that \( \exp(p_0 Q + t_0 H) = Ad_{\exp(pQ+xP+dH)}(\exp(p_0 Q + t_0 H)) \) gives the Para-Poincare momentum-time transformations

\[ p'_0 = \cosh \left( \frac{x}{\sqrt{CE_0}} \right) p_0 - \sqrt{\frac{E_0}{C}} \sinh \left( \frac{x}{\sqrt{CE_0}} \right) t_0 + p, \quad t'_0 = - \frac{C}{E_0} \sinh \left( \frac{x}{\sqrt{CE_0}} \right) p_0 + \cosh \left( \frac{x}{\sqrt{CE_0}} \right) t_0 + t \] (110)

in the \( P \) case and

\[ p'_0 = \cos \left( \frac{x}{\sqrt{CE_0}} \right) p_0 + \sqrt{\frac{E_0}{C}} \sin \left( \frac{x}{\sqrt{CE_0}} \right) t_0 + p, \quad t'_0 = - \frac{C}{E_0} \sin \left( \frac{x}{\sqrt{CE_0}} \right) p_0 + \cos \left( \frac{x}{\sqrt{CE_0}} \right) t_0 + t \] (111)

in the \( P \) case. The additive parameter \( x \) is non compact (compact) in the \( P \) (\( P \)) case.

If \( f = \frac{E_0}{C} \tanh \left( \frac{x}{\sqrt{CE_0}} \right) \) \( f = \frac{E_0}{C} \tan \left( \frac{x}{\sqrt{CE_0}} \right) \) is a force for \( P \) (\( P \)) while \( \phi = \frac{C}{E_0} \tanh \left( \frac{x}{\sqrt{CE_0}} \right) \)

\( \phi = \frac{C}{E_0} \tan \left( \frac{x}{\sqrt{CE_0}} \right) \) is an inverse of force for \( P \) (\( P \)), then we get the non additive composition laws

\[ f'' = \frac{f + f'}{1 + \frac{p f}{C}} \quad \text{and} \quad \phi'' = \frac{\phi + \phi'}{1 + \frac{p \phi}{C}}, \]

for the \( P_{\pm} \) case. Moreover we have that

\[ E_0 \to \infty \quad C \to \infty \]

\[ p'_0 = p_0 \pm \frac{x}{C} t_0 + p, \quad t'_0 = t_0 + t \quad p'_0 = p_0 + p, \quad t'_0 = t_0 - \frac{x}{E_0} p_0 + t \] (112)

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where \( f = \pm \frac{x}{C} \) is a force for the Para-Galilei case \( P_{\pm} \) an force and \( \phi = - \frac{x}{E_0} \) is a Carrollian inverse of force.

A.3. From time translation to dampinglike coefficient

The Lie algebra of one spatial Newton-Hooke Lie algebra \( NH_{\pm} \) is defined by the Lie brackets

\[ [Q, P] = 0, \quad [Q, H] = \frac{1}{m} P, \quad [P, H] = \pm \frac{1}{C} Q \] (113)

We verify that \( \exp(p_0 Q + x_0 P) = Ad_{\exp(pQ+xP+dH)}(\exp(p_0 Q + x_0 P)) \) gives the Newton-Hooke momentum-space transformations
\[ p_i' = \cosh\left(\frac{t}{\sqrt{mC}}\right) p_i - \sqrt{\frac{m}{C}} \sinh\left(\frac{t}{\sqrt{mC}}\right) x_0 + p, \ x_i' = -\frac{C}{m} \sinh\left(\frac{t}{\sqrt{mC}}\right) p_0 + \cosh\left(\frac{t}{\sqrt{mC}}\right) x_0 + x \]

in the \(NH_+\) case and

\[ p_i' = \cos\left(\frac{t}{\sqrt{mC}}\right) p_i + \sqrt{\frac{m}{C}} \sin\left(\frac{t}{\sqrt{mC}}\right) x_0 + p, \ x_i' = -\frac{C}{m} \sin\left(\frac{t}{\sqrt{mC}}\right) p_0 + \cos\left(\frac{t}{\sqrt{mC}}\right) x_0 + x \]

in the \(NH_-\) case. The additive parameter \(t\) is non compact (compact) in the \(NH_+(NH_-)\) case. Moreover the dampinglike coefficient \(b = \sqrt{\frac{m}{C}} \tanh\left(\frac{t}{\sqrt{mC}}\right) (b = -\sqrt{\frac{m}{C}} \tanh\left(\frac{t}{\sqrt{mC}}\right))\) for the \(NH_+(NH_-)\) whose the dimension is \(MT^{-1}\) and \(\beta = \frac{C}{m} \tanh\left(\frac{t}{\sqrt{mC}}\right)\) and an inverse of a dampinglike \(\beta = \frac{C}{m} \tanh\left(\frac{t}{\sqrt{mC}}\right)\) for the \(NH_-(NH_-)\) whose the dimension is \(M^{-1}T\) satisfy the non additive composition laws \(b'' = \frac{b + b'}{1 + \frac{b b'}{m}}\) and \(\beta'' = \frac{\beta + \beta'}{1 + \frac{\beta \beta'}{m}}\). for the \(NH_-\) cases.

We finally have

\[
\begin{align*}
    p_i' &= p_i + \frac{t}{\sqrt{mC}} x_0 + p, \ x_i' = x_0 + x \\
    p_0' &= p_0 + \frac{t}{\sqrt{mC}} x_0 + x \\
    p_i' &= p_i + \frac{t}{\sqrt{mC}} x_0 - \frac{1}{\sqrt{mC}} p_0 + x
\end{align*}
\]

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