Singular connection and Riemann theta function

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Abstract

We prove the Chern-Weil formula for $SU(n+1)$-singular connections over the complement of an embedded oriented surface in smooth four manifolds. The expression of the representation of a number as a sum of nonvanishing squares is given in terms of the representations of a number as a sum of squares. Using the number theory result, we study the irreducible $SU(n+1)$-representations of the fundamental group of the complement of an embedded oriented surface in smooth four manifolds.

Keywords: Singular connection, Chern-Weil formula, Riemann theta function, Representation.

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1 Introduction

We study $SU(n+1)$-singular connections over $X \setminus \Sigma$ in this paper, where $X$ is a smooth closed oriented 4-manifold and $\Sigma$ is a closed embedded surface. In [6], S. Wang first started to understand the topological information from singular connections. Later, Kronheimer and Mrowka [3] studied the Donaldson invariants under the change of $SU(2)$-singular connections. The paper [3] turns out to be a crucial step for analyzing the structure of the Donaldson invariants and for recent development in the Seiberg-Witten theory.

In §2, we describe the $SU(n+1)$-singular connection space over $X \setminus \Sigma$. The $SU(n+1)$-singular Chern-Weil formula is given in Proposition 2.2. In order to study the irreducible $SU(n+1)$-representations, we need to study the sum of nonvanishing squares. This is one of well-known number theory problems. Jacobi initially studied the representation $r_k(n)$ of a number $n$ as a sum of $k$-squares via Riemann theta function as a generating function. For a topological reason, we would like to understand the representation $R_k(n)$ of a number $n$ as a sum of nonvanishing $k$-squares. In general it is difficult to calculate both $r_k(n)$ and $R_k(n)$. We prove a nice relation between $r_k(n)$ and $R_k(n)$ in Proposition 3.1. Up to the author’s knowledge, it has not known before how to express the number $R_k(n)$ (see [2]). Similar relation for the representation of a number by a quadratic form is also obtained.

In the last section, we use those number theory criterions to study $SU(n+1)$-singular flat connections. This gives an interesting interaction between $R_n(N)$ in the number theory and the irreducible $SU(n+1)$-representations of $\pi_1(X \setminus \Sigma)$ in topology (see Proposition 4.2).
2 $SU(n+1)$ singular connection and Chern-Weil formula

(i) Singular connections over $X \setminus \Sigma$

Let $X$ be a smooth closed oriented 4-manifold and let $\Sigma$ be a closed embedded surface. We will assume that both $X$ and $\Sigma$ are connected, and $\Sigma$ to be orientable or oriented for simplifying our discussions. Denote $tN$ be a closed tubular neighborhood of $\Sigma$. Identify $tN$ diffeomorphically to the unit disk bundle of the normal bundle. Let $Y$ be the boundary of $tN$, which has the circle bundle structure over $\Sigma$ by this diffeomorphism. Let $i\eta$ be a connection 1-form for the circle bundle, so $\eta$ is an $S^1$-invariant 1-form on $Y$ which coincides with the 1-form $d\theta$ on each circle fiber. Using $(r, \theta)$ polar coordinates in some local trivialization of the disk bundle, we have that $dr \wedge d\theta$ is the correct orientation for the normal plane. By radial projection, $\eta$ can be extended to $tN \setminus \Sigma$.

We will work on the structure group $SU(n+1)$ for $n \geq 1$. So a connection $A$ on $X \setminus \Sigma$ which is represented on each normal plane to $\Sigma$ by a connection matrix looks like

\[
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_n
\end{pmatrix} d\theta + \text{(lower terms)}, \quad \sum_{i=0}^{n} \alpha_i \equiv 0 \pmod{1}. \tag{2.1}
\]

The size of the connection matrix is $o(r^{-1})$, so $A$ is singular along the surface $\Sigma$.

For every $SU(n+1)$-singular connection, one can associate with holonomy as in \[4, 6\]. Let $P \rightarrow X \setminus \Sigma$ be a vector bundle with structure group $SU(n+1)$ for $n \geq 1$. To define the holonomy around $\Sigma$ of a connection $A$ on $P$, for any point $\sigma \in \Sigma$ and real number $0 < r < 1$, let $S^1_\sigma(r)$ be a circle with center $\sigma$ and radius $r$ on the normal plane of $tN$ over $\sigma$. An element $h(A; \sigma, r) \in SU(n+1)$ can be obtained by parallel transport of a frame of $P$ along $S^1_\sigma(r)$ via the connection $A$. Although $h(A; \sigma, r)$ depends on the choice of a frame, its conjugacy class $[h(A; \sigma, r)]$ in $SU(n+1)$ does not (c.f. \[4, 6\]). If for all $\sigma \in \Sigma$, the limit $h_A = \lim_{r \to 0^+} [h(A; \sigma, r)]$ is independent of $\sigma$ and $tN$, we call it the holonomy of $A$ along $\Sigma$.

The holonomy of this connection on the positively oriented small circles of radius $r$ is approximately

\[
\exp 2\pi i \begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_n
\end{pmatrix}, \quad \sum_{i=0}^{n} \alpha_i \equiv 0 \pmod{1}. \tag{2.2}
\]

Since only the conjugacy class of the holonomy has any invariant meaning, we may suppose that $\alpha_i$ lies in the interval $[0,1)$, therefore the matrices (2.2) modulo the permutation group $S_{n+1}$ run through each conjugacy class just once. When $\alpha_i = 0$
for all $0 \leq i \leq n$, the holonomy is trivial, and if the phrase "lower terms" makes sense, we have ordinary connections on $X$. Also when $\alpha_i \in \{0, \frac{1}{n}, \cdots, \frac{n-1}{n}\}$ for all $i$, the holonomy is in the center of $SU(n+1)$ ($n$-th root of unity), the associated $SU(n+1)/Z_n$-bundle has trivial holonomy; and with this twist we can consider these as $PSU(n+1)$-connections on $X$.

Conjugacy classes in $SU(n+1)$ can be characterized by parameters $\alpha_i$ with

$$\alpha = (\alpha_i)_{0 \leq i \leq n} \in [0, 1)^{n+1}/(\alpha_0 + \alpha_1 + \cdots + \alpha_n \equiv 0 \pmod{1}).$$

Note that any permutation of $(\alpha_i)$ gives the same conjugacy class. Hence we can stay on the region for conjugacy classes by making $\alpha_i$ satisfying the following

$$1 > \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \geq 0, \text{ and } \alpha_0 + \alpha_1 + \cdots + \alpha_n \in \{0, 1, \cdots, n\}.$$  (2.3)

The region of conjugacy classes of $SU(n+1)$ can be identified with the quotient space \( \{ z_i \in S^1, \prod_{i=0}^n z_i = 1 \} / S_{n+1} \) of the maximal torus of $SU(n+1)$ under the Weyl group action. When $n = 1$ ($SU(2)$ case), $1 > \alpha_0 \geq \alpha_1 \geq 0, \alpha_0 + \alpha_1 = 0, \alpha_0 + \alpha_1 = 1$. The equation $\alpha_0 + \alpha_1 = 0$ always gives the identity conjugacy class. So $1 > \alpha_0 \geq \alpha_1 \geq 0$ and $\alpha_0 + \alpha_1 = 1$ describe the conjugacy classes of $SU(2)$ as in [3] with $\alpha_i' = \alpha_0 + 1/2, \alpha_1' = \alpha_1 + 1/2$. For $\alpha_i = 0 \geq \alpha_{i+1} \geq \cdots \alpha_n \geq 0$, the conjugacy classes can be viewed as conjugacy classes in $SU(i)$ or $U(i)$ in $SU(n+1)$.

The matrix-valued 1-form given on $X \setminus \Sigma$ by the expression

$$i \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} \eta,$$  (2.4)

has the asymptotic behavior of (2.1), but is only defined locally. To make an $SU(n+1)$ connection on $X \setminus \Sigma$ which has this form near $\Sigma$, start with $SU(n+1)$ bundle $\mathcal{P}$ over $X$ and choose a $C^\infty$ decomposition of $\mathcal{P}$ on $N$ as

$$\mathcal{P}|_N = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n,$$

compatible with the hermitian metric. Since we will work on the modeled connection $A^\alpha$, the decomposition of $\mathcal{P}|_N$ gives a natural model. Although $\mathcal{P}|_N$ is trivial, $\mathcal{L}_i$ may not be. We define topological invariants in this situation:

$$\begin{cases} k = c_2(\mathcal{P})[X] \\ l_i = -c_1(\mathcal{L}_i)[\Sigma], \quad \sum_{i=0}^n l_i = 0. \end{cases}$$  (2.5)

Choose any smooth $SU(n+1)$ connection $\mathcal{A}^\alpha$ on $\mathcal{P}$ which respects to the decomposition over $N$, so we have

$$\mathcal{A}^\alpha|_N = \begin{pmatrix} b_0 & & & \\ & b_1 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}, \quad \sum_{i=0}^n b_i = 0,$$  (2.6)
where \( b_i \) is a smooth connection in \( L_i \). Let the model connection \( A^\alpha \) on \( P = \mathcal{P}_{X \setminus \Sigma} \) be the following:

\[
A^\alpha = A^0 + i \beta(r) \left( \begin{array}{c} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{array} \right) \eta, \tag{2.7}
\]

where \( \beta \) is a smooth cutoff function equal to 1 in \([0, \frac{2}{3}]\) and equal to 0 for \( r \geq \frac{1}{2} \).

In terms of trivialization compatible with the decomposition, the second term in \( A^\alpha \) is an element of \( \Omega^1_{N \setminus \Sigma}(AdP) \) which can be extended to all of \( X \setminus \Sigma \). The curvature \( F(A^\alpha) \) extends to a smooth 2-form with values in \( Ad\mathcal{P} \) on the whole \( X \), since \( id \eta \) is smooth on \( tN \), \( i \eta \) is the pullback to \( tN \) of the curvature form of the circle bundle \( Y \). It can be thought as a smooth 2-form on the surface \( \Sigma \).

The connection \( A^\alpha \) in (2.7) defines a connection on \( X \setminus \Sigma \). The holonomy \( h_{A^\alpha} \) around small linking circles is asymptotically equal to (2.2). We now define an affine space of connections modeled on \( A^\alpha \) by choosing \( p > 2 \) and denoting

\[
\mathcal{A}^{\alpha,p}_1 = \{ A^\alpha + a \mid \| a \|_{L^p(X \setminus \Sigma)} + \| \nabla A^\alpha a \|_{L^p(X \setminus \Sigma)} < \infty \}.
\]

Similarly a gauge group

\[
\mathcal{G}^{\alpha,p}_2 = \{ g \in AutP \mid \| g \|_{L^p(X \setminus \Sigma)} + \| \nabla A^\alpha g \|_{L^p(X \setminus \Sigma)} + \| \nabla^2 A^\alpha g \|_{L^p(X \setminus \Sigma)} < \infty \}.
\]

The \( L^p \) space is defined by using the measure inherited from any smooth measure on \( X \).

**Proposition 2.1**

(i) The space \( \mathcal{A}^{\alpha,p}_1 \) and \( \mathcal{G}^{\alpha,p}_2 \) are independent of the choices of \( A^0 \) and the connection 1-form \( \eta \).

(ii) The space \( \mathcal{G}^{\alpha,p}_2 \) is a Banach Lie group which acts smoothly on \( \mathcal{A}^{\alpha,p}_1 \) and is independent of \( \alpha \). The stabilizer of \( A \) is \( Z_n \) or \( H \) (\( Z_n \subset H \subset SU(n+1) \)) according as \( A \) is irreducible or reducible respectively.

**Proof:** The proof is same as in the Proposition 2.4 in [3] (see also Chapter 3 [6]). \( \blacksquare \)

(ii) The Chern-Weil formula for \( \mathcal{A}^{\alpha,p}_1 \)

By the same token of the Proposition 3.7 in [3], we have that the equivalence class of the norm \( L^p_{k,A^\alpha} \) is independent of \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \), so the gauge group \( \mathcal{G}^{\alpha,p}_{k+1} = \mathcal{G}^{\alpha}_{k+1} \) is independent of \( \alpha \) as a parameter in the definition of the model connection \( A^\alpha \). So the space \( \mathcal{A}^{\alpha,p}_1 \) is a Banach space which is independent of \( \alpha \), and \( a \in L^1_{k}(\Omega(X \setminus \Sigma), AdP)) \) if \( L^1_{k}(\Omega(X \setminus \Sigma), AdP)) \) is a Banach space which is independent of \( \alpha \), and \( a \) is in \( L^1_{k}(\Omega(X \setminus \Sigma), AdP)) \), then the diagonal component \( D(a) \) of \( a \) is in \( L^p_{1} \), and \( (a - D(a)) \) is in \( L^p_{1} \), \( r^{-1}(a - D(a)) \) is in \( L^p \).

**Proposition 2.2**

For all \( A \in \mathcal{A}^{\alpha,p}_1 \), the following Chern-Weil formula holds.

\[
\frac{1}{8\pi^2} \int_{X \setminus \Sigma} \text{tr}(F_A \wedge F_A) = k + \sum_{i=0}^{n} \alpha_i l_i - \frac{1}{2} \sum_{i=0}^{n} \alpha_i^2) \Sigma \cdot \Sigma. \tag{2.8}
\]
Therefore by (2.11) and (2.10),

\[ su \]

Observe that on the Lie algebra \( \mathfrak{su}(n) \) of skew adjoint matrices \( tr(M^2) = -|M|^2 \).

Hence by (2.10) and (2.11) the Chern-Weil formula for the modeled connection \( A^\alpha \) is

\[
\frac{1}{8\pi^2} \int_{X \setminus \Sigma} tr F_{A^\alpha} \wedge F_{A^\alpha} = \frac{1}{2} \sum_{i=0}^{n} (-l_i^2 + 2\alpha_i l_i - \alpha_i^2 \Sigma \cdot \Sigma).
\]
Note that $c_2(E) = \sum_{i<j} c_1(L_i) \cdot c_1(L_j) = \sum_{i<j} l_i l_j$. Also from $c_1(E) = \sum_{i=0}^n l_i = 0$, we have

$$0 = \left( \sum_{i=0}^n l_i \right)^2 = \sum_{i=0}^n l_i^2 + 2 \sum_{i<j} l_i l_j,$$

i.e. $c_2(E) = -\frac{1}{2} \sum_{i=0}^n l_i^2$. By (2.13) we have

$$\frac{1}{8\pi^2} \int_{X \setminus \Sigma} tr F_A^a \wedge F_A^a = k + \sum_{i=0}^n \alpha_i \cdot l_i - \frac{1}{2} \left( \sum_{i=0}^n \alpha_i^2 \right) (\Sigma \cdot \Sigma). \quad (2.14)$$

Although this calculation is global, it has an interpretation locally on $tN$. Let $Y_\varepsilon \subset tN$ be the 3-manifold circle bundle over $\Sigma$ given by radius $r = \varepsilon$. The Chern-Simons integral is given by the following:

$$cs_\varepsilon(A^a) = \frac{1}{8\pi^2} \int_{Y_\varepsilon} tr (dA^a \wedge A^a + \frac{2}{3} A^a \wedge A^a \wedge A^a).$$

The integral $cs_\varepsilon(A^a)$ depends only on the homotopy class of the trivialization of the bundle on $Y_\varepsilon$ with respect to which the connection matrix $A^a$ is computed (see [4]). Since there is a distinguished trivialization on $Y_\varepsilon$ which extends to $tN$, we have the Chern-Simons $cs_\varepsilon$ defined as a real number. Let $X_\varepsilon$ be the complement of $\varepsilon$-neighborhood of $\Sigma$ with boundary $Y_\varepsilon$. Thus

$$\frac{1}{8\pi^2} \int_{X_\varepsilon} tr F_A \wedge F_A = k + cs_\varepsilon(A^a).$$

By (2.14) from the reducible connection, we have

$$\lim_{\varepsilon \to 0} cs_\varepsilon(A^a) = \sum_{i=0}^n \alpha_i \cdot l_i - \frac{1}{2} \left( \sum_{i=0}^n \alpha_i^2 \right) \Sigma \cdot \Sigma.$$

So the Chern-Weil formula holds whenever $A$ is a connection which is smooth and reducible near to $\Sigma$ by applying the above local statement. Since such connections are dense in $A^{1,p}_{1,A^a}$ and the curvature integral is a continuous function of $A$ in the $L^p_{1,A^a}$-topology, the result follows.

**Remarks:**

1. When $\alpha_i = 0$ for all $i$, Proposition 2.2 is the usual Chern-Weil formula.

2. When $n = 1$, we have the $SU(2)$-situation. Proposition 2.2 for $\alpha_i = \alpha_i' + 1/2 (i = 0, 1)$ and $l_0 + l_1 = 0$ case is

$$\frac{1}{8\pi^2} \int_{X \setminus \Sigma} tr F_A \wedge F_A = k + 2(\alpha_0')l_0 - (\alpha_0')^2 (\Sigma \cdot \Sigma),$$

which is the Proposition 5.7 in [3]. So our formula extends their formula to the $SU(n + 1)$ group.
Corollary 2.3  Let $a$ be the restriction of $A \in A^\alpha_{1-p}$ on the boundary of $X \setminus \Sigma$. Then the Chern-Simons invariant takes the value

$$cs(a) \equiv \sum_{i=0}^{n} \alpha_i l_i - \frac{1}{2}(\sum_{i=0}^{n} \alpha_i^2) \Sigma \cdot \Sigma \quad (\text{mod } 1). \quad (2.15)$$

The proof follows directly from the proof of Proposition 2.2. The Chern-Weil formula gives the charge for singular $SU(n+1)$-connections over $X \setminus \Sigma$. We study the maximum and minimum of the charge over the conjugacy holonomy region.

Corollary 2.4  For $\Sigma \cdot \Sigma \neq 0$, the charge takes its maximum and minimum by comparing $k$ with the following values

$$k + \sum_{i=0}^{n} \alpha_i l_i - \frac{1}{2}(\sum_{i=0}^{n} \alpha_i^2) \Sigma \cdot \Sigma,$$

$$k + \sum_{i=0}^{j-1} \frac{l_i^2}{2\Sigma \cdot \Sigma} - \frac{j^2 \Sigma \cdot \Sigma}{2(n+1)}, \quad j = 1, 2, \ldots, n;$$

$$k + \sum_{i=0}^{j-1} \frac{l_i^2}{2\Sigma \cdot \Sigma} - \frac{m^2 \Sigma \cdot \Sigma}{2j} + \frac{s_j}{j}(m - \frac{sj}{2\Sigma \cdot \Sigma}), \quad j = 2, 3, \ldots, n; m = 1, \ldots, j - 1;$$

where $s_j = \sum_{i=0}^{j-1} l_i$ (not necessary zero).

Proof: First of all, we find out the extreme values inside the region. By the method of Lagrange multiplier,

$$f(\alpha_0, \ldots, \alpha_n) = k + \sum_{i=0}^{n} \alpha_i l_i - \frac{1}{2}(\sum_{i=0}^{n} \alpha_i^2) \Sigma \cdot \Sigma,$$

with constraints $\sum_{i=0}^{n} \alpha_i = j (j = 1, \ldots, n)$, so we have the critical point

$$\alpha_i = \frac{l_i}{\Sigma \cdot \Sigma} + \frac{j}{n+1}, \quad 0 \leq i \leq n,$$

and its corresponding charge is

$$k + \sum_{i=0}^{j-1} \frac{l_i^2}{2\Sigma \cdot \Sigma} - \frac{j^2 \Sigma \cdot \Sigma}{2(n+1)}, \quad j = 1, \ldots, n.$$
with constraints $\sum_{i=0}^{j-1} \alpha_i = m$ for $m = 1, \cdots, j - 1$. The critical point is
\[
\alpha_i = \frac{l_i}{\Sigma \cdot \Sigma} + \frac{m}{j} - \frac{\sum_{i=0}^{j-1} l_i}{j \Sigma \cdot \Sigma}, \quad i = 0, 1, \cdots, j - 1,
\]
and its corresponding charge is, by a straightforward calculation,
\[
k + \frac{\sum_{i=0}^{j-1} l_i^2}{2 \Sigma \cdot \Sigma} - \frac{m^2 \Sigma \cdot \Sigma}{2j} + \frac{s_j}{j} (m - \frac{s_j}{2 \Sigma \cdot \Sigma}),
\]
where $s_j = \sum_{i=0}^{j-1} l_i$. So the result follows by comparing these extreme values.

3 Riemann theta function

In this section, the needed number theory criteria are shown. The Riemann theta function enters our picture naturally from the representation of a number as a sum of $k$-squares, or by a quadratic form. It is one of well-known classic problems in number theory. Jacobi initially studied this problem by using Riemann theta function as a generating function; in particular, Siegel generalized vastly to several complex variables (see [5]). At the moment we are only interested in the relation between topology and number theory (see §4). Further investigation along this line will be discussed elsewhere.

(i) The representation of a number as a sum of nonvanishing squares.

Let $N$ be an integer, $N \geq 1$, with the representation
\[
N = N_1^2 + N_2^2 + \cdots + N_n^2,
\]
where the $(N_i)_{1 \leq i \leq n}$’s are integers including zero. Let $r_n(N)$ denote the number of representations of $N$ as the sum of $n$ squares.

1. For $n = 2$, Jacobi derived an identity from the generating function
\[
\theta_3(0, z) = \sum_{n=-\infty}^{\infty} q^n, \quad q = e^{\pi i z}, \text{ with } Imz > 0.
\]

Jacobi identity is
\[
\{\theta_3(0, z)\}^2 = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{2n-1}}{1 - q^{2n-1}},
\]
which gives the result that
\[
r_2(N) = 4 \sum_{\text{odd, } d | N} (-1)^{d/2} (d-1) = 4\{d_1(N) - d_3(N)\},
\]
where $d_1(N)$ and $d_3(N)$ are the numbers of the divisors of $N$ of the form $4m + 1$ and $4m + 3$ respectively.
2. For \( n = 3 \), Legendre proved that a number \( N \) is the sum of three squares if and only if \( N \neq 4^a(8b+7), a \geq 0, b \geq 0 \). For all \( N \), \( r_3(4^aN) = r_3(N) \). The function \( r_3(N) \) has been evaluated by Dirichlet as a finite sum involving symbols of quadratic reciprocity. We give the following formula for \( r_3(N) \) (see [2]):

\[
    r_3(N) = \frac{G_N}{\pi} \sqrt{NL(1, \chi)} ,
\]

where

\[
    G_N = \begin{cases} 
    0 & N \equiv 0, 4, 7 \pmod{8} \\
    16 & N \equiv 3 \pmod{8} \\
    24 & N \equiv 1, 2, 5, 6 \pmod{8} 
    \end{cases}
\]

and \( L(s, \chi) = \sum_{m=1}^{\infty} \chi(m) m^{-s} \) with \( \chi(m) = (-4N/m), (r/N) \) the Jacobi symbol.

3. For \( n \geq 4 \), Lagrange (1770) proved that every positive integer can be represented as the sum of four squares, hence also as the sum of five or more squares. In particular for \( n = 4 \),

\[
    r_4(N) = 8 \sum_{d|N, d \neq 4} d,
\]

where the summation is over those positive divisors of \( N \), which are not divisible by 4.

For our purpose, we need to get the representation \( N \) as the sum of nonvanishing integer squares. Let \( \mathcal{R}_n(N) \) denote the number of representations of \( N \) as the sum of nonvanishing squares. For example, \( r_1(3) = r_2(3) = 0, r_3(3) = 8, r_4(3) = 32 \), but \( \mathcal{R}_4(3) = 0 \) (see [3.3]). Note that \( r_1(N) = \mathcal{R}_1(N) \) for all \( N \). The following proposition gives the relation between \( r_n(N) \) and \( \mathcal{R}_n(N) \).

**Proposition 3.1** If \( N \) is an integer (\( N \geq 1 \)), then

\[
    \mathcal{R}_n(N) = \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} r_i(N). \tag{3.1}
\]

**Proof:** If \( q = e^{\pi iz} \) with \( \text{Im} z > 0 \), then by definition we have

\[
    \{\theta_3(0, z)\}^n = (\sum_{l=-\infty}^{\infty} q^{l^2})^n = \sum_{N=0}^{\infty} r_n(N)q^N, \quad r_n(0) = 1.
\]

The generating function for \( \mathcal{R}_n(N) \) is \( \theta_3(0, z) - 1 \), so

\[
    \{\theta_3(0, z) - 1\}^n = (\sum_{l \neq 0} q^{l^2})^n = \sum_{N=1}^{\infty} \mathcal{R}_n(N)q^N.
\]

On the other hand the binomial formula gives

\[
    \{\theta_3(0, z) - 1\}^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \{\theta_3(0, z)\}^i.
\]
For the constant coefficient, it corresponds to \( i = 0 \), i.e. \((1 - 1)^n = 0\). Therefore by comparing the coefficients of \( q^N (N \geq 1) \), we have the desired relation.

Although we know that for each \( n \geq 5 \) all but a finite set of integers are sums of exactly \( n \) nonvanishing squares (see [2] Chapter 6), Proposition 3.1 gives the precise relation among the numbers of representations of \( N \) as sums of squares and sums of nonvanishing squares.

(ii) The representation of a number by a quadratic form

Let \( (a_{pq}) \) be a real, symmetric, \( n \times n \) matrix, and let the associated quadratic form \( Q(x) = \sum_{p,q=1}^{n} a_{pq}x_p x_q \) be positive definite. It is well-known that the multiple series
\[
\sum_{i_1, \ldots, i_n = -\infty}^{\infty} e^{\pi i z (i_1, \ldots, i_n)}
\]
converges absolutely and uniformly in every compact set in the upper half-plane \( \text{Im} z > 0 \). The theta function associated to \( Q \) is defined to be
\[
\theta(z, Q) = \sum_{i_1, \ldots, i_n = -\infty}^{\infty} e^{\pi i z Q(i_1, \ldots, i_n)}.
\]

In case \( a_{pq} = \delta_{pq} \) is the identity matrix, then the \( \theta(z, Id) \) reduces to \( \{\theta_3(0, z)\}^n \). In our application later, we have the matrix even, i.e. \( a_{pp} \) are even. Then the definition of \( \theta(z, Q) \) yields
\[
\theta(z + 1, Q) = \theta(z, Q).
\]

In the next section we will consider the particular even matrix:

\[
(a_{pq}) = \begin{pmatrix}
2 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 2
\end{pmatrix}_{n \times n}.
\] (3.2)

Its determinant is \( n + 1 \).

**Theorem 3.2** Let \( (a_{pq}) \) be a symmetric, \( n \times n \) matrix of integers, where \( a_{pp} \) are all even for \( p = 1, 2, \ldots, n \), and the associated quadratic form \( Q(x) \) be positive definite with determinant \( D \). Let \( Q^{-1} \) be the inverse form of \( Q \). Then we have
\[
\theta(z + 1, Q) = \theta(z, Q), \quad \theta(-\frac{1}{z}, Q) = (\sqrt{\frac{z}{i}})^n D^{-\frac{1}{2}} \theta(z, Q^{-1}),
\]
for all complex \( z \) with \( \text{Im} z > 0 \).

From the above relations, one can derive the formula for \( \theta(\frac{-a+i}{cz+d}, Q) \), with \( a, b, c, d \) are integers and \( ad - bc = 1 \), since the modular group is generated by the two transformations \( A : z \to z + 1 \), and \( B : z \to -\frac{1}{z} \) (see [3]).
Let \( r_Q(N) \) (or \( R_Q(N) \)) denote the number of (or all nonzero) solutions \( x_1, \ldots, x_n \), with \( x_i \) integral for every \( i \), such that \( 1 \leq i \leq n \) of the equation
\[
\sum_{p,q=1}^{n} a_{pq} x_p x_q = 2N.
\]

Let \( (a_{pq})_i \) be the \( (n-1) \times (n-1) \) matrix by deleting \( i \)-th row and \( i \)-th column of the matrix \( (a_{pq}) \). Denote the corresponding quadratic form be \( Q_i \). Clearly \( Q_i \) is an even, symmetric, positive definite form. Similarly \( Q_{i_1 \cdots i_j} (N) \) is the quadratic form with \( x_{i_1} = x_{i_2} = 0 \), etc. The following lemma gives the relation among \( r_Q(N) \), \( r_{Q_{i_1 \cdots i_j}}(N) \)(\( j = 1, 2, \ldots, n-1 \)) and \( R_Q(N) \).

**Proposition 3.3** For the even quadratic form \( Q \), we have the relation
\[
R_Q(N) = r_Q(N) - \sum_{i=1}^{n} r_{Q_i}(N) + \sum_{1 \leq i_1 < i_2 \leq n} r_{Q_{i_1 i_2}}(N) - \cdots - (-1)^{n-1} \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} r_{Q_{i_1 \cdots i_{n-1}}}(N). \tag{3.3}
\]

**Proof:** Since \( Q(x) \) is an even form and \( \theta(z, Q) \) is holomorphic for \( \text{Im} z > 0 \), so an expansion of \( \theta(z, Q) \) in power of \( e^{2\pi i z} \) is given by
\[
\theta(z, Q) = 1 + \sum_{N=1}^{\infty} r_Q(N) e^{2\pi i N z}, \quad \text{Im} z > 0.
\]

There is another way to write the expansion of \( \theta(z, Q) \) as
\[
\theta(z, Q) = 1 + \sum_{N=1}^{\infty} R_Q(N) e^{2\pi i N z} + \sum_{i=1}^{n} \left( \sum_{x_i=0} \sum_{x_i=x_j=0} e^{\pi i z Q_i(x)} - \sum_{x_i=x_j=0} e^{\pi i z Q_i(x)} \right) + \cdots
\]
\[
= 1 + \sum_{N=1}^{\infty} R_Q(N) e^{2\pi i N z} + \sum_{i=1}^{n} \left( \sum_{N=1}^{\infty} r_Q(N) e^{2\pi i N z} \right) - \sum_{i=1}^{n} \left( \sum_{N=1}^{\infty} r_{Q_{i_1 i_2}}(N) e^{2\pi i N z} \right) + \cdots \tag{3.4}
\]

Hence the relation follows by comparing the coefficients of \( e^{2\pi i N z} \). \( \square \)

In particular, if we take the matrix \( Q = 2I_d \), then
\[
r_Q(N) = r_n(N), \quad \text{and} \quad R_Q(N) = R_n(N).
\]
So the above lemma gives Proposition [3.1]. It is clear from our discussion that there are general relations between the number of solutions and the number of nonvanishing solutions via the recursive formula in the theta function.

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4 Unitary representation of $\pi_1(X \setminus \Sigma)$

In this section, we will use previous results to derive the nontrivial $SU(n + 1)$-representations of $\pi_1(X \setminus \Sigma)$.

**Lemma 4.1** Let $A$ be a flat $\alpha$-twisted $SU(n + 1)$ connection. Then the holonomy parameter $(\alpha_i)_{0 \leq i \leq n}$, the instanton number $k$ and monopole numbers $l_i$ are related by

\[
\begin{align*}
  l_i &= \alpha_i(\Sigma \cdot \Sigma), \quad \text{for} \quad 0 \leq i \leq n, \\
  k &= -\sum_{i=0}^{n} l_i^2 / 2\Sigma \cdot \Sigma.
\end{align*}
\]

(4.1)

If $\Sigma \cdot \Sigma = 0$, then $k = 0$ and $l_i = 0$ for all $i$.

Proof: The flat $\alpha$-twisted connection $A$ is one of the model connection corresponding to some integers $k, l_i$. Since the bundle is flat, we have

\[
w = \text{diag}(c_1(L_i) + \alpha_i P.D(\Sigma))_{0 \leq i \leq n},
\]

where the 2-form $w$ is in the proof of Proposition 2.2. Thus for each $i$, $c_1(L_i) + \alpha_i P.D(\Sigma) = 0$; the equality $l_i = \alpha_i(\Sigma \cdot \Sigma)$ follows from integrating over $\Sigma$. If $\Sigma \cdot \Sigma = 0$, $l_i = 0$ for all $i$ as well.

On the other hand, if $A$ is flat, the Chern-Weil formula gives

\[
0 = \frac{1}{8\pi^2} \int_{X \setminus \Sigma} \text{tr} F_A \wedge F_A
\]

\[
= k + \sum_{i=0}^{n} \alpha_i l_i - \frac{1}{2} \left( \sum_{i=0}^{n} \alpha_i^2 \right) \Sigma \cdot \Sigma
\]

\[
= k + \frac{\sum_{i=0}^{n} l_i^2}{2\Sigma \cdot \Sigma}
\]

If $\Sigma \cdot \Sigma = 0$, we have $k = 0$ from the second equality. We obtain the formula (4.2).

**Remark:** Note that for $SU(n + 1)$-flat bundles $|l_i| < |\Sigma \cdot \Sigma|$ due to $0 \leq \alpha_i < 1$ from (4.1). All $\alpha_i$’s are rational. Using $\sum_{i=0}^{n} l_i = 0, l_0 = -\sum_{i=1}^{n} l_i$ then we have

\[
k = -\frac{2\sum_{i=1}^{n} l_i^2 + \sum_{i \neq j} l_i l_j}{2\Sigma \cdot \Sigma}.
\]

(4.3)

**Proposition 4.2** For a simply connected $X$ and an embedded oriented surface $\Sigma$ with $\Sigma \cdot \Sigma \neq 0$,

1. If $\sum_{i,j=1}^{n} l_i l_j = 0$, $\Sigma \cdot \Sigma$ is not a divisor of any $N(< n(\Sigma \cdot \Sigma)^2)$ with $\mathcal{R}_n(N) \neq 0$;

2. In general $\Sigma \cdot \Sigma$ is not a divisor of any $N(< n(n+1)/2(\Sigma \cdot \Sigma)^2)$ with $\mathcal{R}_Q(N) \neq 0$ for $Q$ as (3.2);

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then $\pi_1(X \setminus \Sigma)$ has no irreducible representation in $SU(n + 1)$.

Proof: Suppose there were an irreducible representation $\rho : \pi_1(X \setminus \Sigma) \to SU(n + 1)$ ($n \geq 1$). The image of $\rho$ does not contain in any proper subgroup of $SU(n + 1)$. Denote $A$ be the corresponding flat connection on $X \setminus \Sigma$. By Seifert-Ven Kempf theorem, we have

$$
\begin{array}{ccc}
\pi_1(Y_\varepsilon) & \to & \pi_1(X \setminus \Sigma) \\
\downarrow & & \downarrow \\
\pi_1(N_\varepsilon) & \to & \pi_1(X) = \{1\}.
\end{array}
$$

So the holonomy on $\pi_1(X \setminus \Sigma)$ is same as on $\pi_1(Y_\varepsilon)$. The space $Y_\varepsilon$ is the $S^1$-bundle over $\Sigma$, the homotopy exact sequence of the fibration $S^1 \to Y_\varepsilon \to \Sigma$ yields

$$
\{1\} \to \pi_1(S^1) \to \pi_1(Y_\varepsilon) \to \pi_1(\Sigma) \to \{1\}.
$$

In other words, $\pi_1(Y_\varepsilon)$ is a central extension of $\pi_1(\Sigma)$. Let $\gamma$ be a generator of $\pi_1(S^1)$.

Since the conjugacy class $[\gamma]$ generates $\pi_1(X \setminus \Sigma)$ and $\rho$ is an irreducible representation, so the holonomy of $\rho_A$ is not in $Z_n$ and other proper subgroups.

Therefore $\rho_A : \pi_1(X \setminus \Sigma) \to SU(n + 1)/Z_n = PSU(n + 1)$.

$$
\pi_1(Y_\varepsilon) = \{a_i, b_i, \gamma| \prod_{i}[a_i, b_i] = \gamma^m, [\gamma^m, a_i] = 1, [\gamma^m, b_i] = 1\},
$$

where $m = |\Sigma \cdot \Sigma|$ the absolute value of $\Sigma \cdot \Sigma$. So the representation $\rho_A(\gamma)^m = \prod_i[\rho_A(a_i), \rho_A(b_i)]$, and $[\rho_A(\gamma)^m, \rho_A(a_i)] = 1, [\rho_A(\gamma)^m, \rho_A(b_i)] = 1$. This derives that the matrix $\rho_A(\gamma)$ must be a diagonal matrix:

$$
\rho_A(\gamma) = exp2\pi i \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \end{pmatrix},
$$

for $\alpha_i$’s in the domain (2.3). Now we can take the flat connection $A$ as our model connection corresponding to $\alpha = (\alpha_0, \cdots, \alpha_n)$ and instanton number $k$, monopole numbers $l_i$. By Lemma 4.1 and (4.3), we have

$$
\begin{align*}
\ell_i &= \alpha_i \Sigma \cdot \Sigma \\
k &= -2 \frac{\sum_{i=1}^{n} l_i^2 + \sum_{i \neq j} l_i l_j}{2 \Sigma \cdot \Sigma}.
\end{align*}
$$

(4.4)

If $\sum_{i,j=1}^{n} l_i l_j = 0$, $k = -\frac{\sum_{i=1}^{n} l_i^2}{2 \Sigma \cdot \Sigma}$ (an integer), then for (1) (same argument for (2)) the resulting number on the right hand (4.4) is not an integer by the very definition of $R_n(N)$ ($R_Q(N)$) in §3 with $N = \sum_{i=1}^{n} l_i^2$ ($N = \sum_{i=1}^{n} l_i^2 + \frac{1}{2} \sum_{i,j=1}^{n} l_i l_j$). Note that $l_i = \alpha_i \Sigma \cdot \Sigma < \Sigma \cdot \Sigma$, thus the range for $N = \sum_{i=1}^{n} l_i^2$ is $< \frac{n(n+1)}{2} (\Sigma \cdot \Sigma)^2$.

If $\Sigma \cdot \Sigma = \pm 1$, then any number $N < \frac{n(n+1)}{2} (\pm 1)^2$ has $R_n(N) = 0$ ($R_Q(N) = 0$). So at least one of $l_i = 0$ for $N = \sum_{i=1}^{n} l_i^2$ ($N = \sum_{i=1}^{n} l_i^2 + \frac{1}{2} \sum_{i,j=1}^{n} l_i l_j$), i.e. the corresponding $\alpha_i = 0$. Hence the induced image of $\rho_A$ is a proper subgroup.
of $SU(n+1)$. So $\pi_1(X \setminus \Sigma)$ has no irreducible $SU(n+1)$-representations.

Remarks:

1. We need to use the definition of $\mathcal{R}_n(N)$ to cover the case of $\Sigma \cdot \Sigma = \pm 1$. For $n = 1$ Proposition 1.2 is the Corollary 5.8 in [3].

2. The condition $\sum_{i,j=1}^n l_i l_j = 0$ is different from $\sum_{i,j=0}^n l_i l_j = 0$. The later one with $\sum_{i=0}^n l_i = 0$ will imply all $l_i = 0$; we have all $l_i \neq 0$, otherwise it will reduce to $SU(m)$ or $U(m)(m < n + 1)$. So we may take $n + 1$ as minimum number of $l_i \neq 0$.

3. One can (inductively) apply Proposition 4.2 to $\mathcal{R}_k(N)$ for non representations of $\pi_1(X \setminus \Sigma)$ in an rank $k$ subgroup of $SU(n+1)$.

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