NON-CONNECTED TORIC HILBERT SCHEMES

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Abstract. We construct small (50 and 26 points, respectively) point sets in dimension 5 whose graphs of triangulations are not connected. These examples improve our construction in J. Amer. Math. Soc. 13:3 (2000), 611–637 not only in size, but also in that the associated toric Hilbert schemes are not connected either, a question left open in that article. Additionally, the point sets can easily be put into convex position, providing examples of 5-dimensional polytopes with non-connected graph of triangulations.

Introduction

The graph of triangulations of a finite point set \( A \subset \mathbb{R}^d \) has as vertices all the triangulations of \( A \) and as edges certain natural local modifications of them, analogous to the bistellar flips considered in combinatorial topology \([4]\). See the precise definition in Section \( \S 1 \). This graph is interesting in several contexts:

In Geometric Combinatorics, its study is a special case of the Generalized Baues problem, which includes monotone paths on polytopes, zonotopal tilings, or the extension space of realizable oriented matroids as other cases. See the survey \([16]\), or \([3, 20]\).

In Computational Geometry, triangulations are a standard tool and flips have been often proposed as a method to explore the space of all possible triangulations or search for optimal ones (e.g., the Delaunay one). See \([6, 9, 12]\) or the survey \([2]\).

In Algebraic Geometry, lattice polytopes and triangulations of them are closely related to toric varieties \([7, 10, 22]\). In particular, the graph of triangulations of a point set \( A \) is connected if and only if the Chow variety \([11]\) of \( A \) is connected (see Corollary 4.9 in \([17]\)). The same question for the toric Hilbert scheme \([14, 15, 21]\) of the point set is not clear. Even though Sturmfels and Haiman \([8]\) have recently constructed a natural morphism from the toric Hilbert scheme to the toric Chow variety, this morphism is in general not surjective. In particular, it remained open until the present.

Date: April 2002, revised November 2003. Dedicated to Bernd Sturmfels on his 40th birthday.

1991 Mathematics Subject Classification. Primary 52B11; Secondary 52B20.

Key words and phrases. Triangulation, geometric bistellar flip, polyhedral subdivision, toric variety, toric Hilbert scheme.

This result was obtained in the fall of 2001, while I was a visiting professor in the Department of Mathematics, U.C. Davis, supported by U. C. Davis, M.S.R.I. and the spanish government. I am also partially supported by grant BFM2001–1153 of the Spanish Dirección General de Enseñanza Superior e Investigación Científica.
construction whether non-connected toric Hilbert schemes existed. The solution we present here uses the result, due to Maclagan and Thomas [14], that at least the subgraph induced by unimodular triangulations of $A$ has a faithful relation to the toric Hilbert scheme. See Section 1.4.

More details on these interrelations can also be found in [17], where we constructed the first example of a point set whose graph of triangulations is not connected, in dimension 6 with 324 points. Here we show much smaller examples. The essential new idea is that we construct triangulations with sub-structures which cannot be destroyed by flips, rather than triangulations without flips. This gives more flexibility to the construction, and it also saves us several technicalities in the proofs.

The paper is organized as follows: in Section 1 we give the precise definition of the graph of triangulations and the main ingredients needed to prove that the graph is not connected in our examples, summarized in Theorem 1.5 and its Corollary 1.6. This section describes also the connection between the graph of triangulations and the toric Hilbert scheme.

The other two sections describe our two point sets $A_{50}$ and $A_{26}$ and prove that their graphs of triangulations and their toric Hilbert schemes are not connected. These point sets have 50 and 26 points respectively, both in dimension 5. The reason why we include both, and not only the smaller one, is that $A_{50}$ is simpler to describe and it has stronger properties which make it easier to prove what we need. More precisely,

$$A_{50} = (A \cup \{O\}) \times \{0, 1\}$$

where $A$ and $O$ are, respectively, the set of vertices and the centroid of the regular 24-cell in $\mathbb{R}^4$. The point set $A_{26}$ is related to the regular cross-polytope, but not in such a direct way.

Our main results (Theorem 2.1, Theorem 3.4 and Propositions 2.5 and 3.6) can be summarized as:

**Theorem 1.** The 5-dimensional point sets $A_{26}$ and $A_{50}$, with 26 and 50 points respectively, have the following properties:

1. Their graphs of triangulations are not connected.
2. Their toric Hilbert schemes are not connected (for $A_{26}$, the scheme is considered with respect to a non-homogeneous grading).
3. The graphs of triangulations remain non-connected under suitable perturbations of the point sets into convex position. In particular, there is a 5-dimensional polytope with 26 vertices whose graph of triangulations is not connected.

Concerning part (3), a lifting construction mentioned in [17] already allowed us to obtain polytopes with a non-connected space of triangulation, but only after a drastic explosion to dimension 234.

Thanks to symmetries, we can even be more precise: the graphs of triangulations (and the toric Hilbert schemes) of $A_{26}$ and $A_{50}$ have at least 17 and 13 connected components respectively. In $A_{50}$, each of these 13 components contains at least $3^{48}$ unimodular triangulations. The corresponding connected components in the toric Hilbert scheme have dimension at least 96, in contrast to the fact that the coherent irreducible component has dimension $50 - 5 - 1 = 44$ (Corollary 2.4).
The point set $A_{50}$ has an additional feature: it equals the set of lattice points in a lattice polytope $Q$. Point sets of this type play an even more important role in toric geometry than general lattice point sets. Actually, because the construction of [17] did not have this property the following assertion of V. Alexeev [1, p.705, question 6] was not completely true: “A recent example of F. Santos shows that the moduli space $M_Q$ is in general not connected, where $Q$ is a lattice polytope”. The assertion is true, however, for the polytope $Q = \text{conv}(A_{50})$:

**Theorem 2.** The moduli space of all stable semiabelic toric pairs of type bounded by the product $Q$ of a segment and a 24-cell (w.r.t. the lattice generated by the vertex set of $Q$) has at least 13 connected components, each of dimension at least 96 and each with at least $3^{48}$ torus fixed points.

*Proof.* It follows from Corollary 4.9 of [17] and the discussion preceding it that the moduli space in the statement has the same number of connected components as the graph of triangulations of the set of lattice points in $Q$.

The dimension bound follows the ideas detailed in the proof of Corollary 2.4 for the toric Hilbert scheme. There, we show that in any of the 13 connected components of the graph of triangulations of $A_{50}$ there are triangulations refining a certain subdivision $S_0$ whose only minimal non-simplicial cells are 48 different octahedra. We then argue on the state polytope of a certain $A_{50}$-graded ideal whose existence is guaranteed by our Lemma 1.10.

For the present statement the situation is even simpler: Theorem 2.13.3 in [1] says that the generalized secondary polytope (cf. [1] Section 2.12 or [17] Section 4) of any subdivision of $A_{50}$ is the image of a certain stratum in the moduli space of the statement. For our subdivision $S_0$, the generalized secondary polytope is 96-dimensional and has $3^{48}$ vertices: it is a product of 48 triangles. □

It is even possible to obtain a lattice point set that simultaneously satisfies all the properties stated above. It is obtained from $A_{50}$ via the so-called reoriented Lawrence construction introduced in [19, Section 4.4].

Even if the present constructions improve considerably those in [17], they still leave open some of the problems mentioned there:

- Can the graphs of triangulations be non-connected for point sets of dimensions 3 and 4? Dimension 3 is specially important for Computational Geometry and Engineering applications. In dimension 2 the graph of triangulations is easily proved to be connected.
- Can the graphs of triangulations be non-connected for point sets in general position? General position (i.e., no $d+2$ points lie in an affine hyperplane) is interesting in applied areas. Also, a disconnected graph of triangulations for a point set in general position would imply that the refinement poset of subdivisions of either the whole point set or a proper subset of it is disconnected too. Connectivity of this poset is still open. Its study is sometimes referred to as the generalized Baues problem for triangulations. See [16] or the introduction to [17] for more details.
- Can the graphs of triangulations be non-connected for Lawrence polytopes? This would provide realizable oriented matroids with
non-connected extension space. See more information on this, for example, in the introduction to [17].

1. Triangulations and Flips.

1.1. Triangulations and flips of point sets. A triangulation of a finite point set $A \subset \mathbb{R}^d$ is a geometric simplicial complex with vertex set contained in $A$ and which covers the convex hull of $A$.

Geometric bistellar flips are local operations which transform one triangulation of $A$ into another. Essentially, they correspond to switching between the two triangulations of a minimal affinely dependent subset of $A$. More precisely: Every minimal dependent subset $Z \subset A$ can be divided in a unique way in two parts $Z^+$ and $Z^-$ whose convex hulls intersect. We call the ordered pair $(Z^+, Z^-)$ a circuit of $A$. (This deviates slightly from the standard oriented matroid terminology, in which $Z$ is a circuit and $(Z^+, Z^-)$ an oriented circuit). The only two triangulations of $Z$ are:

$$T_+(Z) := \{ S \subseteq Z : Z^+ \nsubseteq S \}, \quad T_-(Z) := \{ S \subseteq Z : Z^- \nsubseteq S \}.$$

**Definition 1.1.** Let $T$ be a triangulation of $A$ and let $(Z^+, Z^-) \subseteq A$ be a circuit of $A$. Suppose that the following conditions are satisfied:

(i) The triangulation $T_+(Z)$ is a sub-complex of $T$.

(ii) All the maximal simplices of $T_+(Z)$ have the same link $L$ in $T$. In particular, $T_+(Z) \ast L$ is a sub-complex of $T$. Here $A \ast B := \{ S \cup T : S \in A, T \in B \}$ is the join of two simplicial complexes.

Then, we can obtain a new triangulation $T'$ of $A$ replacing the sub-complex $T_+(Z) \ast L$ of $T$ by the complex $T_-(Z) \ast L$. This operation is called a **geometric bistellar operation** or **geometric bistellar flip** (or a flip, for short) supported on the circuit $(Z^+, Z^-)$.

This definition is literally taken from [17]. It originally comes from [7, Chapter 7], where it is called a **modification**. See also [13, p.287]. The following arguments will help convince the reader that it is the right concept of minimal or elementary change between triangulations:

(1) There is a certain subset of all triangulations of $A$, called regular or coherent, which are in bijection with the vertex set of a polytope of dimension $|A| - d - 1$ (the secondary polytope of $A$). The edges of this polytope are in bijection with flips between regular triangulations [7].

(2) Triangulations are the minimal elements in the poset of polyhedral subdivisions of $A$, with the partial order given by refinement. Flips are the “next to minimal” elements, in the following well-defined sense: any subdivision whose only proper refinements are triangulations has exactly two proper refinements and they are triangulations related by a flip. Conversely, every flip arises in this way [20, Corollary 4.5 and Proposition 5.3].

1.2. Locally acyclic orientations. Let $A$ be a point set in $\mathbb{R}^d$. Let $I = [0, 1] \subset \mathbb{R}$. If $T$ is a triangulation of $A$, we abbreviate as $T \times I$ the polyhedral subdivision of $A \times \{0, 1\}$ into prisms $\sigma \times I$, $\sigma \in T$. We are interested in studying the triangulations of $A \times \{0, 1\}$ that refine a given such subdivision.
For the refining process we need to understand triangulations of the prism $\Delta^d \times I$, where $\Delta^d$ is a simplex of dimension $d$. The following description appears in [5] and [7]; see also [18, Section 3]. It can be rephrased as saying that all triangulations of $\Delta^d \times I$ are staircase triangulations.

**Proposition 1.2.** Let the vertices of $\Delta^d \times I$ be labeled $\{a_1, \ldots, a_{d+1}, b_1, \ldots, b_{d+1}\}$ so that the $a_i$’s are the vertices of the facet $\Delta^d \times \{0\}$ and each $b_i$ is the vertex corresponding to $a_i$ in the opposite facet $\Delta^d \times \{1\}$. Then:

1. There is a bijection between triangulations of $\Delta^d \times I$ and linear orderings (permutations) of the numbers $\{1, \ldots, d+1\}$. The triangulation corresponding to the ordering $(s_1, \ldots, s_{d+1})$ has the following $d+1$ maximal simplices:
   \[
   \{a_{s_1}, \ldots, a_{s_i}, b_{s_i}, \ldots, b_{s_{d+1}}\} : \quad i = 1, \ldots, d+1
   \]

2. Two triangulations of $\Delta^d \times I$ differ by a bistellar flip if and only if the corresponding orderings differ by a transposition of a pair of consecutive elements.

In particular, we get the following result, where a **locally acyclic orientation** of the 1-skeleton of a simplicial complex is an orientation of all its edges which is acyclic on every simplex.

**Proposition 1.3.** The triangulations of $A \times \{0,1\}$ which can be obtained by refining the product $T \times I$ are in bijection with the locally acyclic orientations of the 1-skeleton of $T$. Flips between such triangulations correspond to reversal of single edges.

**Proof.** A locally acyclic orientation induces a linear ordering on every simplex $\sigma$ of $T$: $i < j$ iff there is an arrow from $i$ to $j$. We can use this to triangulate $\sigma \times I$. The triangulations so obtained agree on common faces of any two prisms because the linear orderings agree on the intersection of any two simplices. Conversely, a refinement of $T \times I$ triangulates in particular each prism $\sigma \times I$, hence it induces a linear ordering on the vertices of every simplex $\sigma$ of $T$. \qed

The triangulation of $A \times \{0,1\}$ obtained from a certain locally acyclic orientation of $T$ is characterized by the following property: for every edge $\{v, w\}$ of $T$, directed from $v$ to $w$, the triangulation uses the diagonal $\{(v,0), (w,1)\}$ in the quadrilateral $\{(v,0), (w,0), (v,1), (w,1)\}$ of $T \times I$.

We will be specially interested in locally acyclic orientations of $T$ without reversible edges, meaning that every single-edge reversal creates a cycle in a simplex. The smallest one we know of has 11 vertices and dimension 3. A slightly bigger example, with 15 vertices, can be obtained as a Schl"{a}gel diagram of the boundary of the example considered in Remark 3.4 of [17].

Unfortunately, Proposition 1.3 does not imply that locally acyclic orientations of $T$ produce refinements of $T \times I$ without flips. They only produce refinements of $T \times I$ none of whose flip neighbors refine $T \times I$.

### 1.3. Freezing sub-complexes in triangulations

All we have mentioned so far is essentially present in [15]. The new ingredient in this paper is that we focus on restrictions of triangulations to sub-complexes of faces. Let $F$ be a face of the polytope $\text{conv}(A)$. It is obvious that every triangulation of
A restricts to a triangulation of $F \cap A$. Our next statement says that the same happens for flips:

**Proposition 1.4.** If $F$ is a face of $\text{conv}(A)$ and $T$ and $T'$ are triangulations of $A$ differing by a flip, then $T$ and $T'$ restricted to $F$ either coincide or differ by a flip on a circuit contained in $F$.

**Proof.** The only simplices of a triangulation that are removed by a flip are those containing the negative part of the circuit $Z$ in which the flip is supported. For $T$ restricted to $F$ to be affected by the flip it is necessary that $Z^- \subseteq F$ and, being a face, then $Z^+ \subseteq F$ too. Hence, the circuit is contained in $F$ and the flip restricts to a flip in $F$. □

More generally, let $K$ be a simplicial sub-complex of the face complex of $\text{conv}(A)$. With the word *simplicial* we do not only mean that every $F \in K$ is a simplex, but also that $F \cap A$ is affinely independent for every $F \in K$. That is, that $F \cap A$ is the vertex set of $F$. By a triangulation of $K \times I$ we mean any geometric simplicial complex $T$ with vertex set contained in $A \times \{0,1\}$ satisfying that: (1) every simplex of $T$ is contained in one of the products $F \times I$, $F \in K$; and (2) $\bigcup_{\sigma \in T} \text{conv}(\sigma) = \bigcup_{F \in K} F \times I$.

The following result has essentially the same proof as Proposition 1.3, taking into account Proposition 1.4 and the following observation: for every face $F$ in $K$, $F \times I$ is a face of $\text{conv}(A) \times I$. In particular, every triangulation of $K \times I$ induces a triangulation of $F \times I$.

**Theorem 1.5.** Let $K$ be a simplicial subcomplex of the face complex of $\text{conv}(A)$, for a finite point set $A$. Triangulations of $K \times I$ are in bijection with locally acyclic orientations of the 1-skeleton of $K$. Flips between triangulations correspond to locally acyclic orientations differing on the reversal of a single edge. □

**Corollary 1.6.** In the above conditions, let $A'$ be any point set in $\mathbb{R}^{d+1}$ containing $A \times \{0,1\}$ such that for every $F \in K$ the following two conditions hold:

- $F \times I$ is still a face of $\text{conv}(A')$, and
- $F \times I$ contains no point of $A'$ other than its vertex set.

Then, “restriction to $K \times I$” induces a simplicial (in particular, continuous) map from the graph of triangulations of $A'$ to the graph of locally acyclic orientations of $K$. (Edges in the latter are single-edge reversals).

In particular, if $K$ has a locally acyclic orientation without reversible edges and the corresponding triangulation of $K \times I$ can be extended to a triangulation of $A'$, then the graph of triangulations of $A'$ is not connected.

**Proof.** The conditions on $A'$ imply that restriction of triangulations of $A'$ to $K \times I$ is a well-defined operation and, by Theorem 1.5, the restricted triangulations can be considered elements in the graph of locally acyclic orientations of $K$. The map is simplicial by Proposition 1.4.

The last sentence in the statement holds because a triangulation extending (the triangulation of $K \times I$ corresponding to) a locally acyclic orientation without reversible edges will not be connected by flips to a triangulation extending any other locally acyclic orientation. For example, one extending
any globally acyclic orientation of edges of $K$, which clearly has reversible edges and can be extended to a lexicographic triangulation of $A'$.

1.4. Unimodular triangulations and the toric Hilbert scheme. Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{Q}^d$ be a rational point set. We transform the point configuration $A$ into a vector configuration $A' = \{a_1', \ldots, a_n'\} \subset \mathbb{Q}^{d+1}$, by choosing a positive integer $l_i$ for each $i = 1, \ldots, n$ and letting $a_i = (l_ia_i, l_i) \in \mathbb{Z}^{d+1}$. We assume, without loss of generality, that $A$ has only integer entries.

The standard choice of scaling factors $l_i$ is the homogeneous one, with $l_i = 1$ for every $i$, but in Section 3 we need a different one.

As detailed in [22, Chapter 10], we use $A$ to define a $(d-1)$-dimensional multi-grading on the polynomial ring $k[x_1, \ldots, x_n]$, assigning multi-degree $a_i$ to the variable $x_i$. Ideals $I \subset k[x_1, \ldots, x_n]$ that are homogeneous with respect to this grading have a well-defined Hilbert function $N_A \rightarrow \mathbb{N}$, $\mathbf{b} \mapsto \dim_k I_{\mathbf{b}}$ where $\mathbb{N}$ is the set of non-negative integers, $N_A$ is the semigroup of non-negative integer combinations of $A$ and for each $\mathbf{b} \in N_A$, $I_{\mathbf{b}}$ is the part of degree $\mathbf{b}$ of $I$.

The most natural $A$-homogeneous ideal is the toric ideal $I_A$ of $A$, generated by the binomials

$$\{x_1^{\lambda_1} \cdots x_n^{\lambda_n} - x_1^{\mu_1} \cdots x_n^{\mu_n} : \lambda, \mu \in \mathbb{N}^n, \sum_{i=1}^n (\lambda_i - \mu_i) a_i = 0\}.$$

For every $\mathbf{b} \in N_A$, $(I_A)_\mathbf{b}$ has codimension one in $(k[x_1, \ldots, x_n])_\mathbf{b}$. This characterizes the Hilbert function of $I_{\mathbf{b}}$.

**Definition 1.7.** An $A$-homogeneous ideal $I \subset k[x_1, \ldots, x_n]$ is called $A$-graded if it has the same Hilbert function as the toric ideal.

Most of the literature on $A$-graded ideals, starting with [22], assume $A$ (hence $A'$) to have non-negative entries. In our context this is not important: if $A$ has negative entries, then $A$ can be mapped to a non-negative configuration by a unimodular transformation, and the concept of $A$-graded ideal is invariant under such transformations. It is not invariant, however, under change of choice of the scalars $l_i$.

$A$-graded ideals include all the initial ideals (more generally, all the toric deformations) of $I_A$. The toric Hilbert scheme, as introduced by Peeva and Stillman [15], is the set of all $A$-graded ideals with a suitable algebraic structure defined by some determinantal equations. (An equivalent description via binomial equations appeared in [23, Section 6]). See also [14, 21].

Sturmfels [22, Theorem 10.10] proved that every $A$-graded ideal $I$ has canonically associated a polyhedral subdivision $S_I$ of $A$. Observe that subdivisions of $A$, in the affine setting used in this paper, coincide with subdivisions of $A$ in the linear setting used in [22, Chapter 10]. If $I$ is monomial, then $S_I$ is a triangulation, whose simplices are the standard monomials in $k[x_1, \ldots, x_n]/\text{Rad}(I)$. In other words, $S_I$ is the triangulation whose Stanley-Reisner ideal equals the radical of $I$. 

This produces a map $\Phi$ from the toric Hilbert scheme of $\mathcal{A}$ to the poset of all polyhedral subdivisions of $\mathcal{A}$. (This map factors via the morphism to the Chow variety constructed in [8], followed by the natural map from the Chow variety to the poset of subdivisions). Maclagan and Thomas [14] go further and construct a graph of monomial $\mathcal{A}$-graded ideals (mono-$\mathcal{A}$-GIs for short) by suitably defining a concept of flip between mono-$\mathcal{A}$-GIs with the following properties:

**Proposition 1.8.**  
(1) The toric Hilbert scheme is connected if and only if the graph of mono-$\mathcal{A}$-GIs is connected. 
(2) The triangulations of $\mathcal{A}$ corresponding to neighboring mono-$\mathcal{A}$-GIs either coincide or differ by a geometric bistellar flip.  

This does not imply that a non-connected graph of triangulations provides a non-connected toric Hilbert scheme, because the map $\Phi$ is in general not surjective [22, Example 10.13], [15]. However, Maclagan and Thomas make the observation, based on [22, Theorem 10.14], that the image of $\Phi$ contains all the unimodular triangulations of $\mathcal{A}$. A unimodular triangulation of $\mathcal{A}$ is one in which every maximal simplex is a basis for the lattice spanned by $\mathcal{A}$.

**Corollary 1.9.** The toric Hilbert scheme has at least as many connected components as connected components of the graph of triangulations contain unimodular triangulations.  

To prove the dimension bound of Corollary 2.4 we need the following generalization of (one direction of) Theorem 10.14 of [22]:

**Lemma 1.10.** Let $S$ be a subdivision of $\mathcal{A}$ with the property that every cell $\sigma \in S$ can be covered by unimodular simplices with vertex set contained in $\sigma$. Then, there is an $\mathcal{A}$-graded ideal of the form $\cap_{\sigma \in S} J_\sigma$, where each $J_\sigma$ is torus isomorphic to the toric ideal of the set $\sigma$. 

In particular, $S$ is in the image of the map $\Phi$.  

**Proof.** The proof follows the ideas in [22, Theorem 10.14]. For each cell $\sigma \in S$, let $I_\sigma$ denote the usual toric ideal of the vertex set of $\sigma$ and define 

$$J_\sigma = I_\sigma + \langle x_j : j \notin \sigma \rangle.$$  

We claim that $I_S := \cap_{\sigma \in S} J_\sigma$ is indeed $\mathcal{A}$-graded. 

Let $b \in \mathbb{N} \mathcal{A}$, and let $\sigma$ be a cell of $S$ whose cone contains $b$. The claim follows from the following three assertions: (1) all monomials of degree $b$ with support not in $\sigma$ are in $I_S$, because none of them has support contained in a cell of $S$; hence they are in the monomial part of every component of $I_S$. (2) all monomials of degree $b$ with support in $\sigma$ are equal modulo $I_S$, because $I_S$ contains $I_\sigma$. (3) there are monomials of degree $b$ not in $I_S$: by our covering hypothesis, there is a unimodular simplex $\tau$ contained in $\sigma$ and whose cone contains $b$. Consider the unique positive combination $\sum_{i \in \tau} \lambda_i a_i$ that gives $b$. This combination is integer since $\tau$ is unimodular. The corresponding monomial $\Pi_{i \in \sigma} x_i^{\lambda_i}$ is clearly not in $I_\sigma + \langle x_j : j \notin \sigma \rangle = J_\sigma$, hence not in $I_S$. 

That $\Phi(I_S) = S$ is essentially the definition of $\Phi$; see [22, Theorem 10.10].
2. A construction with 50 points, based on the 24-cell

The 24-cell is one of the six regular 4-dimensional polytopes. Its 24 vertices are:

- The 8 points $\pm 2e_i$.
- The 16 points $(\pm 1, \pm 1, \pm 1, \pm 1)$.

Let $A$ consist of these 24 vertices. Let $O$ be the origin and let $K$ be the 2-skeleton of the 24-cell, which consists of 96 triangles and 96 edges. Let

$$A_{50} := (A \cup \{O\}) \times \{0, 1\}.$$ 

**Theorem 2.1.** The 5-dimensional point configuration with 50 elements $A_{50}$ has a non-connected space of triangulations and its homogenized version $A_{50} = A_{50} \times \{1\}$ has a non-connected toric Hilbert scheme.

**Proof.** We show below that the 2-skeleton $K$ of the 24-cell can be given a locally acyclic orientation with no reversible edges (Lemma 2.2) and that the triangulation of $K \times I$ corresponding to this orientation can be extended to a unimodular triangulation $T'$ of $A_{50}$ (Lemma 2.3). Since $A_{50}$ satisfies the conditions required for $A'$ in Corollary 1.6 its graph of triangulations is not connected. Unimodularity of $T'$ implies that the toric Hilbert scheme of $A_{50} := A_{50} \times \{1\} \subset \mathbb{R}^6$ is not connected either, via Corollary 1.9. □

In the rest of this section we fill in the details in this proof and prove more precise quantitative results, and that the graph of triangulations of $A_{50}$ remains not connected when its only two non-vertices $(O, 0)$ and $(O, 1)$ are moved into convex position.

The facets of the 24-cell are 24 octahedra. One of them is the octahedron with vertices $(2, 0, 0, 0), (0, 2, 0, 0), (1, 1, 1, 1), (1, 1, -1, -1), (1, 1, 1, -1)$, and $(1, 1, -1, 1)$. We denote it $F_{1, 1, 0, 0}$. From this we get three other octahedra $F_{-1, 1, 0, 0}, F_{1, -1, 0, 0}$ and $F_{1, 1, 0, 0}$ by the rotation of order 4 on the first two coordinates. And from these four we get the rest of the octahedra by permuting coordinates.

The subindices in each octahedron give the coordinates of its centroid. The 24-cell is self-polar: These 24 centroids are the vertices of another regular (and smaller) 24-cell. Our coordinates are chosen to highlight the symmetries of the 24-cell that we are interested in.

We orient the edges in $F_{1, 1, 0, 0}$ with a source at $(2, 0, 0, 0)$, a sink at $(0, 2, 0, 0)$ and a cycle of length 4 on the equatorial square $(1, 1, 1, 1) \rightarrow (1, 1, -1, 1) \rightarrow (1, 1, 1, -1) \rightarrow (1, 1, 1, -1)$. See Figure 1. Observe that among the 12 edges of this octahedron only the equatorial four can be reversed without creating a local cycle. We let the other 84 edges of the 24-cell be oriented by the action of the affine group $G$ of order 32 generated by the exchange of first two and last two coordinates and the rotation of order 4 on the plane of the first two coordinates (or of the last two coordinates).

**Lemma 2.2.** This orientation of the 1-skeleton of $K$ is well-defined, locally acyclic, and has no reversible edges.

**Proof.** The orientation is well-defined because edges of $F_{1, 1, 0, 0}$ in the same orbit under $G$ receive orientations compatible with the action. Indeed, the
Figure 1. A locally acyclic orientation of the 1-skeleton of the octahedron $F_{1,1,0,0}$, with only 4 reversible edges.

only symmetries in $G$ sending some edge of $F_{1,1,0,0}$ to another are the rotations on the last two coordinates (rotation around the vertical axis, in Figure 1) and they are compatible with the orientation given.

Under $G$, there are two orbits of octahedra in the 24-cell, 3 orbits of triangles and edges, and two orbits of vertices. The following are representatives of the three orbits of triangles, with their orientations indicated:

- $(2, 0, 0, 0) \rightarrow (1, 1, 1, -1) \rightarrow (1, 1, 1, 1)$,
- $(1, 1, 1, -1) \rightarrow (1, 1, 1, 1) \rightarrow (0, 2, 0, 0)$, and
- $(1, 1, 1, -1) \rightarrow (0, 0, 2, 0) \rightarrow (1, 1, 1, 1)$.

This proves that the orientation is locally acyclic. The three orbits of edges have as representatives $(2, 0, 0, 0) \rightarrow (1, 1, 1, 1)$, $(1, 1, 1, -1) \rightarrow (0, 2, 0, 0)$, and $(1, 1, 1, -1) \rightarrow (1, 1, 1, 1)$. That is, the non-reversible edges of the three stated representatives of triangles. This proves that the orientation has no reversible edges.

Actually, the symmetry group of the locally acyclic orientation we are using is larger than $G$. We leave it to the interested reader to check that it is an orientation-preserving group of order 96 and acts transitively over the 96 edges and over the 96 triangles in the 24-cell. It is generated by $G$ and, for example, the following orthogonal transformation:

$$
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix}
\mapsto
\frac{1}{2}
\begin{pmatrix}
  x_1 - x_2 + x_3 + x_4 \\
  x_1 + x_2 - x_3 + x_4 \\
  x_1 + x_2 + x_3 - x_4 \\
  -x_1 + x_2 + x_3 + x_4
\end{pmatrix}
$$

Lemma 2.3. The triangulation $T$ of $K \times I$ corresponding via Theorem 1.5 to this locally acyclic orientation of $K$ can be extended to a unimodular triangulation $T'$ of $A_{50}$.

In the above statement, a triangulation of a lattice point set $B \subset \mathbb{R}^d$ is called unimodular if every maximal simplex is an affine basis for the affine lattice spanned by $B$. Equivalently, if it is unimodular as a triangulation of the vector set $B$ obtained from $B$ with a homogeneous choice of scaling lengths.
Proof. Let $T_{1,1,0,0}$ be the triangulation of the octahedron $F_{1,1,0,0}$ that uses the axis $\{(2,0,0,0), (0,2,0,0)\}$, oriented from $(2,0,0,0)$ to $(0,2,0,0)$. This triangulation extends in a locally acyclic way the orientation of edges of $F_{1,1,0,0}$.

Let $T_0$ be the triangulation of $A \cup \{O\}$ obtained by first replicating $T_{1,1,0,0}$ to the other octahedra by the action of $G$ and then coning the triangulated boundary of the 24-cell to the centroid $O$. We also extend the orientation to all the boundary via $G$, and orient all the edges incident to $O$ with a source at $O$. This gives a locally acyclic orientation of the 1-skeleton of $T_A$ and, hence, a triangulation $T'$ of $A_{50}$ which refines $T_0 \times I$. Since the orientation of $T_0$ extends the one we had in $K$, $T'$ extends what we had in $K \times I$.

Only unimodularity remains to be checked. Since a prism over a unimodular simplex is totally unimodular (meaning that all its triangulations are unimodular), every refinement of $T_0 \times I$ will be unimodular as long as $T_0$ itself is unimodular. By symmetry, we only need to check that one of the 96 maximal simplices of $T_0$ is unimodular. For example, take the 4-simplex with vertex set $(0,0,0,0), (2,0,0,0), (0,2,0,0), (1,1,1,1), (1,1,1,-1)$. This is unimodular (in the sub-lattice spanned by $A \cup \{0\}$) because

$$\begin{vmatrix}
0 & 2 & 0 & 1 & 1 \\
0 & 0 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & -1 \\
1 & 1 & 1 & 1 & 1
\end{vmatrix} = -8$$

and the sub-lattice spanned by $A \cup \{O\}$, given by the conditions “all coordinates have the same parity”, has index 8 in the integer lattice. \(\square\)

That finishes the proof of Theorem 2.1. We now prove the quantitative results mentioned after Theorem 1 and that the graph of triangulations of $A_{50}$ remains unimodular in a convex-position version of the point set.

Corollary 2.4. The graph of triangulations of $A_{50}$ and the toric Hilbert scheme of its homogenized version $A_{50} = A_{50} \times \{1\}$ have at least 13 connected components, each with at least $3^{48}$ unimodular triangulations/monomial ideals. In the toric Hilbert scheme, each of these 13 connected components has dimension at least 96.

Proof. Observe in Figure 1 that, out of the 48 symmetries of the octahedron $F_{1,1,0,0}$, only the four generated by the rotation around the vertical axis leave our locally acyclic orientation invariant. Hence, there are 12 ways of constructing a locally acyclic orientation with exactly the same properties. (This agrees with our claim that our locally acyclic orientation has 96 symmetries, since the symmetry group of the 24-cell has $24 \times 48 = 12 \times 96$ elements). Each of these 12 orientations is a different isolated element in the graph of locally acyclic orientations of $K$, hence it produces a different connected component in the graph of triangulations of $A_{50}$. To these 12 we have to add the regular component of the graph, obtained for example by starting with a globally acyclic orientation of $K$.

To prove the number $3^{48}$, observe the following: let $F$ be one of the 24 octahedra in the 24-cell. Let $v$ and $w$ be the source and sink of its orientation.
In the triangulation $T_0$ of the proof of Lemma 2.3, $F$ is triangulated into four 3-simplices which are joined to the origin $O$. In particular, the two flips of the triangulation of $F$ are still flips in $T_0$.

Moreover, these four 4-simplices of $T_0$ have the same source $O$ and sink $w$. Hence, the copies of them in $(A \cup O) \times \{0\}$ are joined to the origin $O$, and the copies of them in $(A \cup O) \times \{1\}$ are joined to the same vertex of $(A \cup O) \times \{0\}$ (namely $(O, 0)$). This implies that the two flips in $F$ produce four flips in $T'$: Two with supporting circuit contained in the octahedron $F \times \{0\}$ and another two with supporting circuit in $F \times \{1\}$. Moreover, flips in different octahedra, out of the total of 48, are independent. Hence, from $T'$ we can at least reach the $3^{48}$ (including $T'$) triangulations obtained by these flips, which are all unimodular.

For the dimension bound, let $S_0$ be the subdivision of $A_{50}$ obtained from $T_0$ by making each of the 48 octahedra (and the two vertices coned to each of them) a single cell. Any refinement of this subdivision is unimodular, because the three triangulations of each octahedron are unimodular. By Lemma 1.10 there is an $A_{50}$-graded ideal $I_0$ associated to that subdivision. Since $I_0$ is the sum of 48 independent copies of the toric ideal of an octahedron, the state polytope of $I_0$ (cf. [22]) is the product of 48 copies of a triangle, hence it has dimension 96. And the toric variety of the state polytope is immersed (modulo normalization) in the toric Hilbert scheme because every toric deformation of an $A$-graded ideal is $A$-graded as well.

We can even be more precise; the subgraph induced by the $3^{48}$ triangulations/ideals mentioned in the statement of Corollary 2.4 is the 1-skeleton of the polytope $(\Delta^2)^{48}$, where $\Delta^2$ is a triangle.

**Proposition 2.5.** Let $A_{50}'$ be the point set obtained from $A_{50}$ by moving the points $(O, 0)$ and $(O, 1)$ to $(O, \alpha)$ and $(O, \beta)$, for any $\alpha < 0$ and $\beta > 1$. Then, $A_{50}'$ is in convex position and its graph of triangulations still has at least 13 components, each with at least $3^{48}$ triangulations.

**Proof.** $A_{50}'$ is still a point set containing $A \times \{0, 1\}$, and $K$ satisfies the two hypotheses in Corollary 1.6. The only thing to prove is that our triangulation $T$ of $K \times I$ extends to a triangulation of $A_{50}'$. Actually, the following is true: the same triangulation $T'$ of Lemma 2.3 considered in $A_{50}'$ with the substitution $(O, 0) \mapsto (O, \alpha)$ and $(O, 1) \mapsto (O, \beta)$ is still a triangulation of $A_{50}'$. This holds because the facets $A \times \{0\}$ and $A \times \{1\}$ of $A_{50}$ are centrally triangulated in $T'$ and the perturbed points $(O, \alpha)$ and $(O, \beta)$ lie beyond these facets of $A_{50}$ (with the standard meaning of beyond in polytope theory: a point lies beyond a facet $F$ of the polytope $P$ if only that facet is visible from the point).

**Remark 2.6.** We cannot say anything about the toric Hilbert scheme of $A_{50}'$, even if we assume $\alpha$ and $\beta$ to be rational or even integer, because our triangulations are no longer unimodular in $A_{50}'$. 


3. A construction with 26 points, based on the cross-polytope

In this section we let $A \subset \mathbb{R}^4$ denote the vertex set of a regular cross-polytope, with centroid $O$:

$A := \{ \pm e_1, \pm e_2, \pm e_3, \pm e_4 \} \subset \mathbb{R}^4$, \quad \quad O = (0, 0, 0, 0) \in \mathbb{R}^4$.

The faces of the cross-polytope are 16 tetrahedra, 32 triangles, 24 edges and 8 vertices. Our complex $K$ will contain all the 24 edges, but only 24 of the 32 triangles. The point set $A_{26} \subset \mathbb{R}^4$ will contain the 18 points in

$(A \cup \{O\}) \times \{0, 1\}$

plus 8 points of the form $(p, 1/2)$ where $p$ (essentially) runs over the centroids of the eight triangles missing in $K$. See the definitions of $K$ and $A$ below.

We will show that $K$ can be given a locally acyclic orientation with no reversible edges, and that this orientation can be extended to a triangulation of $A_{26}$. Also, that this triangulation is unimodular when considered in a vector configuration $A_{26}$ obtained from $A_{26}$ as in Section 1.4 but with a non-homogeneous choice of scaling factors. These facts imply that neither the graph of triangulations of $A_{26}$ nor the toric Hilbert scheme of $A_{26}$ are connected.

Our construction is symmetric under the group $G$ of order six generated by the central symmetry and the rotation which cyclically permutes the first three coordinates:

$$(x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_1, x_4).$$

This group produces 4 orbits of edges, with six members each, and 6 orbits of triangles, one with two elements and five with six elements. We show them as columns in Tables 1 and 2. To save space we write $\overline{e}_i$ meaning $-e_i$.

| $\{e_1, e_2\}$ | $\{\overline{e}_2, e_1\}$ | $\{e_4, e_1\}$ | $\{e_1, e_4\}$ |
|----------------|----------------|----------------|----------------|
| $\{e_2, e_3\}$ | $\{\overline{e}_3, e_2\}$ | $\{e_4, e_2\}$ | $\{\overline{e}_2, e_4\}$ |
| $\{e_3, e_1\}$ | $\{\overline{e}_1, e_3\}$ | $\{e_4, e_3\}$ | $\{\overline{e}_3, e_4\}$ |
| $\{\overline{e}_1, \overline{e}_2\}$ | $\{e_2, \overline{e}_1\}$ | $\{\overline{e}_4, \overline{e}_1\}$ | $\{e_1, \overline{e}_4\}$ |
| $\{\overline{e}_2, \overline{e}_3\}$ | $\{e_3, \overline{e}_2\}$ | $\{\overline{e}_4, \overline{e}_2\}$ | $\{e_2, \overline{e}_4\}$ |
| $\{\overline{e}_3, \overline{e}_1\}$ | $\{e_1, \overline{e}_3\}$ | $\{\overline{e}_4, \overline{e}_3\}$ | $\{e_3, \overline{e}_4\}$ |

Table 1. The 24 edges of the cross-polytope, divided into four orbits.

| $\{e_1, \overline{e}_4, \overline{e}_2\}$ | $\{e_1, \overline{e}_3, e_2\}$ | $\{\overline{e}_4, e_4, e_2\}$ | $\{e_4, e_1, e_2\}$ | $\{e_1, e_2, \overline{e}_4\}$ |
|----------------|----------------|----------------|----------------|----------------|
| $\{e_2, \overline{e}_4, \overline{e}_3\}$ | $\{e_2, \overline{e}_3, e_3\}$ | $\{\overline{e}_1, e_4, e_3\}$ | $\{e_4, e_2, e_3\}$ | $\{e_2, e_3, \overline{e}_4\}$ |
| $\{e_1, e_2, e_3\}$ | $\{e_3, \overline{e}_4, \overline{e}_1\}$ | $\{e_3, \overline{e}_2, e_1\}$ | $\{\overline{e}_2, e_4, e_1\}$ | $\{e_4, e_3, e_1\}$ | $\{e_3, e_1, \overline{e}_4\}$ |
| $\{\overline{e}_1, \overline{e}_2, e_3\}$ | $\{e_1, \overline{e}_4, e_2\}$ | $\{\overline{e}_3, e_4, \overline{e}_2\}$ | $\{e_3, e_4, \overline{e}_2\}$ | $\{\overline{e}_4, \overline{e}_1, \overline{e}_2\}$ | $\{e_4, \overline{e}_3, \overline{e}_2\}$ |
| $\{\overline{e}_2, e_4, e_3\}$ | $\{e_3, \overline{e}_2, e_1\}$ | $\{e_1, \overline{e}_4, e_3\}$ | $\{\overline{e}_4, \overline{e}_2, e_3\}$ | $\{e_2, \overline{e}_3, e_4\}$ |
| $\{\overline{e}_3, e_4, e_1\}$ | $\{e_2, \overline{e}_4, e_1\}$ | $\{e_2, \overline{e}_3, e_1\}$ | $\{\overline{e}_4, \overline{e}_3, e_1\}$ | $\{e_2, \overline{e}_4, e_1\}$ |

Table 2. The 32 triangles of the cross-polytope, divided into six orbits.
We orient edges in the way implicit in Table 1. That is, each edge is oriented from the first vertex listed to the second (this is clearly compatible with the action of $G$). Figure 2 may help understand the construction. It shows the star of $e_4$ in the boundary of the cross-polytope, which is an octahedron centrally triangulated into eight tetrahedra. This is half of the boundary of the cross-polytope; the other half is obtained by central symmetry and is not drawn. The edges in the figure have been oriented, and the reader can verify that only five of the triangles in the figure have cyclic orientations; namely, those belonging to the first two orbits of triangles in Table 2. Triangles in the last four orbits receive the acyclic orientation corresponding to the ordering in which their vertices are listed in the table.

![Diagram of a locally acyclic orientation](image)

**Figure 2.** A locally acyclic orientation, without reversible edges, of a sub-complex of the 2-skeleton of a cross-polytope.

Let $K$ be the union of the last four orbits of triangles in Table 2.

**Proposition 3.1.** The orientation given to the 1-skeleton is locally acyclic (in $K$) and has no reversible edges.

**Proof.** Check that indeed the last four orbits of triangles in Table 2 receive an acyclic orientation and that the non-reversible edges of these four orbits lie respectively in the four orbits of edges (only one representative triangle of each orbit needs to be checked). □

As in Section 2, the symmetry group of this locally acyclic orientation is actually larger than $G$. Let us denote it $\tilde{G}$. Apart from $G$, it contains for example the simultaneous rotation of order 4 in the planes of the first two and last two coordinates:

$$\rho(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3).$$

**Proposition 3.2.** $\tilde{G}$ is an orientation-preserving group of order 24 generated by $G$ and the above transformation $\rho$. It acts transitively over the 24 edges and 24 triangles of $K$, and over the eight triangles not in $K$.

**Proof.** Since $\rho$ sends the edges $\{e_1, e_2\}$, $\{e_2, e_3\}$ and $\{e_3, e_1\}$ of the first $G$-orbit in Table 1 to edges in the other three $G$-orbits, respectively, the group generated by $\rho$ and $G$ (and hence $\tilde{G}$) acts transitively over the 24 edges of the 24-cell. Moreover, $\tilde{G}$ acts with trivial stabilizer on every edge, because every
The functional saying that $(1 / x)$ is not in $\mathcal{F}$ (to check this consider any single triangle of $\mathcal{K}$, for example $\{-e_3, e_4, e_2\}$).

This proves that $\tilde{G}$ is generated by $G$ and $\rho$, that it has order 24, and that it acts transitively over edges. It also acts transitively on triangles of $\mathcal{K}$ because every edge is the non-reversible edge of a different triangle. It acts transitively on the remaining eight triangles because $\rho$ sends the member $\{-e_1, -e_2, -e_3\}$ of the first $G$-orbit to the member $\{e_1, -e_2, -e_4\}$ of the second $G$-orbit.

Let $T$ denote the triangulation of $\mathcal{K} \times I$ corresponding to the locally acyclic orientation of the 1-skeleton of $\mathcal{K}$ we have described. It is clear that $T$ cannot be extended to a triangulation of $(\mathcal{A} \cup \{O\}) \times \{0, 1\}$, because the prism $\{e_1, e_2, e_3\} \times \{0, 1\}$, for example, has its three quadrilateral faces triangulated in a non-extendable way. The same holds for the other 7 triangles not in $\mathcal{K}$.

This is why we need to define the following point set $A_{26}$ with 26 elements, the first 18 of which are $(\mathcal{A} \cup \{O\}) \times \{0, 1\}$. Let $A_{26}$ consist of:

- The 8 points $\pm a_i := (\pm e_i, 0)$, $i = 1, 2, 3, 4$.
- The 8 points $\pm b_i := (\pm e_i, 1)$, $i = 1, 2, 3, 4$.
- The two points $a_0 := (0, 0, 0, 0, 0)$ and $b_0 := (0, 0, 0, 0, 1)$.
- The following eight points:

  \[
  c_{+,+,+,0} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}\right), \quad c_{-,+,+,0} = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}\right), \\
  c_{+,+,0,-} = \left(\frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}\right), \quad c_{-,+,0,+} = \left(-\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}\right), \\
  c_{-,0,+,-} = \left(-\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \quad c_{+,0,-,+} = \left(\frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \\
  c_{0,+,-,+} = \left(0, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0\right), \quad c_{0,-,+,+} = \left(0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).
  \]

Observe that $A_{26}$ is compatible with the action of $\tilde{G}$ (times the trivial group acting on the fifth coordinate).

Recall that a point $p$ is said to lie beyond a certain facet $F$ of a polytope $\mathcal{P}$ if it lies outside $\mathcal{P}$ and $F$ is the only facet visible from $p$. We generalize this and say that $p$ lies beyond a face $F$ if the facets visible from $p$ are precisely those containing $F$. We do not allow $p$ to lie in any facet-defining hyperplane.

**Lemma 3.3.**

1. Each point $c_{+,+,+,+}$ lies beyond a face $\sigma \times I$ of $\text{conv}(\mathcal{A}) \times I$, where $\sigma$ is one of the 8 triangles of $\text{conv}(\mathcal{A})$ missing in $\mathcal{K}$.

2. For each triangle $\sigma \in \mathcal{K}$, $\sigma \times I$ is still a face in $\text{conv}(A_{26})$ and contains no point of $A_{26}$ other than its vertex set.

**Proof.** By symmetry, we need to prove part (2) only for one triangle in $\mathcal{K}$ and part (1) only for one not in $\mathcal{K}$.

For part (1), let $\sigma = \text{conv}(\{e_1, e_2, e_3\})$. The statement is equivalent to saying that $(1/2, 1/2, 1/2, 0)$ lies beyond $\sigma$ in $\text{conv}(\mathcal{A})$. This is easy to check: the facet-defining half-spaces of $\text{conv}(\mathcal{A})$ are those of the form $\pm x_1 \pm x_2 \pm x_3 \pm x_4 \leq 1$, and the only ones not containing $(1/2, 1/2, 1/2, 0)$ are $+x_1 + x_2 + x_3 + x_4 \leq 1$ and $+x_1 + x_2 + x_3 - x_4 \leq 1$.

For part (2), let $\sigma = \{e_4, e_1, e_2\}$. We leave it to the reader to check that the functional

\[
x_1 + x_2 - x_3/2 + x_4
\]
on $A_{26}$ is maximized exactly on the six vertices of $\sigma \times \{0,1\}$. □

Let $A_{26} \subset \mathbb{R}^6$ be the vector configuration obtained from $A_{26}$ with scaling factor equal to 1 for the points $a_*$ and $b_*$ and equal to 2 for the points $c_{*,*,*,*}$. In other words, let $A$ consist of:

- The 8 vectors $\pm a_i := (\pm e_i, 0, 1), i = 1, 2, 3, 4$.
- The 8 vectors $\pm b_i := (\pm e_i, 1, 1), i = 1, 2, 3, 4$.
- The two vectors $a_0 := (0, 0, 0, 0, 1, 1)$ and $b_0 := (0, 0, 0, 1, 1)$.
- The following eight vectors:
  
  $c_{+,+,+,0} = (1, 1, 1, 0, 1, 2), \quad c_{-,+,+,0} = (-1, -1, -1, 0, 1, 2),$
  
  $c_{+,+,-,0} = (1, -1, 0, -1, 1, 2), \quad c_{-,+,+,-} = (-1, 1, 0, 1, 1, 2),$
  
  $c_{-,0,+,0} = (-1, 0, 1, -1, 1, 2), \quad c_{+,0,+,0} = (1, 0, -1, 1, 1, 2),$
  
  $c_{0,+,+,0} = (0, 1, -1, -1, 1, 2), \quad c_{0,-,+,+} = (0, -1, 1, 1, 1, 2).$

**Theorem 3.4.** (1) The triangulation $T$ of $K \times I$ can be extended to a triangulation of $A_{26}$.

(2) This extended triangulation is unimodular when considered in $A_{26}$.

Before proving this, let us show its implications:

**Corollary 3.5.** The graph of triangulations of $A_{26}$ and the toric Hilbert scheme of $A_{26}$ have at least 17 connected components.

**Proof.** Condition (2) in Lemma 3.3 says that we can apply Corollary 3.4 to $K$ and $A_{26}$. By that corollary, the triangulation extending $T$ cannot be connected by flips to any triangulation with a different restriction to $K \times I$, which implies that the graph of triangulations of $A_{26}$ (and, by unimodularity, the toric Hilbert scheme of $A_{26}$) is not connected.

The number 17 comes from the fact that the symmetry group of our locally acyclic orientation has order 24 (Proposition 3.2) versus the 16 × 24 symmetries of the cross-polytope: There are 15 other equivalent ways of doing our construction, and any regular triangulation of $A_{26}$ provides a 17th connected component. □

**Proof of Theorem 3.4.** Let us consider the extended complex $K' = K \ast O$. The orientation in $K$ can be extended trivially to a locally acyclic orientation of $K'$ by letting $O$ be a global source. We extend $T$ to $K' \times I$ using this orientation. This is compatible with the central triangulations of the two facets $(A \cup \{O\}) \times \{0\}$ and $(A \cup \{O\}) \times \{1\}$. Hence, we have a triangulation, that we denote $T'$, of

$$(K' \times I) \cup (\text{conv}(A) \times \{0,1\}).$$

We claim that the complement of this polyhedral complex in conv$(A_{26})$ consists of eight convex regions, each homeomorphic to a 5-dimensional closed half-space and with one of the eight points $c_{*,*,*,*}$ on its boundary. This claim implies that we can extend $T'$ to $A_{26}$ by coning each point $c_{*,*,*,*}$ to the part of $T'$ visible from it.

To prove the claim, observe that the complement of $K$ in the boundary of conv$(A)$ consists of eight regions (homeomorphic to 3-balls), each being the union of the two tetrahedra incident to one of the points $c_{*,*,*,*}$. Hence,
the complement of $\mathcal{K}'$ in $\text{conv}(A)$ consists of eight regions $R_{*,*,*,*}$, one for each point $c_{*,*,*}$ defined in the following way for $c_{+,+,+}$ (with the obvious generalization to the other seven regions). $R_{+,+,+}$ equals the closed region

$$\text{conv}(O, e_1, e_2, e_3, e_4) \cup \text{conv}(O, e_1, e_2, e_3, -e_4).$$

minus the part of its boundary incident to $O$. In particular, $R_{+,+,+}$ is convex and homeomorphic to a 4-dimensional half-space.

By Lemma 3.3 the boundary of $\text{conv}(A_{26})$ can be considered a stellar subdivision of the boundary of $\text{conv}(A \times I)$, obtained by pulling the centroids of certain faces out to the points $c_{*,*,*}$. In other words, the complement of $(K \times I) \cup (\text{conv}(A) \times \{0, 1\})$ in $\text{conv}(A_{26})$ consists of eight regions, each being the open region $R_{*,*,*,*}$ times the open segment $(0, 1)$ except it has been slightly pulled out from its interior point $c_{*,*,*}$. This finishes the proof of the claim.

Summing up, we have proved that $T$ extends to the triangulation of $A_{26}$ obtained by letting $\tilde{G}$ (times the trivial group in the fifth coordinate) act over the twenty-eight 5-simplices listed in Table 3. All the simplices are meant to contain $c_{+,+,+}$, which we omit.

The first six groups of four simplices in the list come from the six tetrahedra

$$\{O, e_4, e_1, e_2\}, \quad \{O, e_4, e_2, e_3\}, \quad \{O, e_4, e_3, e_1\},$$

$$\{O, e_1, e_2, -e_4\}, \quad \{O, e_2, e_3, -e_4\} \quad \text{and} \quad \{O, e_3, e_1, -e_4\},$$

which appear in $\mathcal{K}'$ with their vertices ordered in this way. The last four simplices come from the central triangulations of $\text{conv}(A) \times \{0\}$ and $\text{conv}(A) \times \{1\}$.

---

**Table 3.** The triangulation of $A_{26}$. 

| Region Description | Region Description | Region Description |
|-------------------|-------------------|-------------------|
| $\{a_0, a_4, a_1, a_2, b_2\}$ | $\{a_0, a_4, a_2, a_3, b_3\}$ | $\{a_0, a_4, a_3, a_1, b_1\}$ |
| $\{a_0, a_4, a_1, b_1, b_2\}$ | $\{a_0, a_4, a_2, b_2, b_3\}$ | $\{a_0, a_4, a_3, b_3, b_1\}$ |
| $\{a_0, a_4, b_4, b_1, b_2\}$ | $\{a_0, b_0, a_4, b_2, b_3\}$ | $\{a_0, b_0, b_4, b_3, b_1\}$ |
| $\{a_0, b_0, b_1, b_2, -b_4\}$ | $\{a_0, a_1, a_2, a_3, a_4\}$ | $\{b_0, b_1, b_2, b_3, -b_4\}$ |
| $\{a_0, a_1, a_2, a_3, -a_4\}$ | $\{a_0, a_1, a_2, a_3, -a_4\}$ | $\{a_0, a_3, a_1, -a_4, -b_4\}$ |
| $\{a_0, a_3, a_1, -a_4, -b_4\}$ | $\{a_0, a_2, a_4, -a_4, -b_4\}$ | $\{a_0, a_2, a_4, -a_4, -b_4\}$ |
| $\{a_0, a_2, a_3, b_3, -b_4\}$ | $\{a_0, a_2, a_3, b_3, -b_4\}$ | $\{a_0, a_2, a_3, b_3, -b_4\}$ |
| $\{a_0, a_2, a_3, b_3, -b_4\}$ | $\{a_0, b_0, a_2, b_3, -b_4\}$ | $\{a_0, b_0, a_2, b_3, -b_4\}$ |
| $\{a_0, b_0, a_2, b_3, -b_4\}$ | $\{a_0, b_0, b_2, b_3, -b_4\}$ | $\{a_0, b_0, b_2, b_3, -b_4\}$ |
| $\{a_0, b_0, b_2, b_3, -b_4\}$ | $\{a_0, b_0, b_2, b_3, -b_4\}$ | $\{a_0, b_0, b_2, b_3, -b_4\}$ |
It is easy to check, even by hand, that each of the simplices listed above is unimodular when regarded in $A$. For example, consider the first one:

$$
\begin{pmatrix}
    a_0 & a_1 & a_2 & b_2 & c_{+,+,+,0} \\
    0  & 0  & 1  & 0  & 1 \\
    0  & 0  & 0  & 1  & 1 \\
    0  & 0  & 0  & 0  & 1 \\
    0  & 1  & 0  & 0  & 0 \\
    0  & 0  & 0  & 0  & 1 \\
    1  & 1  & 1  & 1  & 2 
\end{pmatrix}
$$

Its determinant is clearly $\pm 1$ because only the highlighted entries produce a non-zero summand in its expansion. The same occurs with the next 25 simplices in Table 3. The last two simplices in the list are related to the previous-to-last two by the unimodular transformation $(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1, x_2, x_3, x_4, x_6 - x_5, x_6)$.

Observe that only the two points $a_0$ and $b_0$ in $A_{26}$ are not vertices. As we did with $A_{50}$ we can pull them out to become vertices, keeping the triangulation we have constructed:

**Proposition 3.6.** Let $A'_{26}$ be the point set obtained from $A_{26}$ by moving the points $a_0 = (O, 0)$ and $b_0 = (O, 1)$ to $(O, \alpha)$ and $(O, \beta)$, where $\alpha \in (-\epsilon, 0)$ and $\beta \in (1, \epsilon)$ for a sufficiently small positive $\epsilon$.

Then, $A'_{26}$ is in convex position and its graph of triangulations has at least 17 connected components.

**Proof.** The only difference with the proof of Proposition 2.5 is that now the points $(O, \alpha)$ and $(O, \beta)$ only lie beyond the facets $A \times \{0\}$ and $A \times \{1\}$ for a sufficiently small perturbation. □

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