On Bandlimited Spatiotemporal Field Sampling with Location and Time Unaware Mobile Sensors

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Abstract

Sampling of a spatiotemporal field for environmental sensing is of interest. Traditionally, a few fixed stations or sampling locations aid in the reconstruction of the spatial field. Recently, there has been an interest in mobile sensing and location-unaware sensing of spatial fields. In this work, the general class of fields governed by a constant coefficient linear partial differential equations is considered. This class models many physical fields including temperature, pollution, and diffusion fields. The analysis is presented in one dimension for a first exposition. It is assumed that a mobile sensing unit is available to sample the spatiotemporal field of interest at unknown locations and unknown times – both generated by independent and unknown renewal processes. Based on measurement-noise affected samples, a spatial field reconstruction algorithm is proposed. It is shown that if the initial condition on the field is bandlimited, then the mean squared error between the original and the estimated field decreases as $O(1/n)$, where $n$ is the average sampling density of the mobile sensor. The field reconstruction algorithm is universal, that is, it does not require the knowledge of unknown sampling locations’ or unknown time instants’ statistical distributions.

Index Terms

Additive white noise, partial differential equations, nonuniform sampling, signal reconstruction, signal sampling, time-varying fields

I. INTRODUCTION

Sampling of smooth spatiotemporally varying fields is a problem that has been addressed in literature for multiple reasons. Often the aim has been to estimate the sources in a diffusion field while some papers have addressed the problem of estimating the field from the samples. Classical approaches towards this study have generally involved samples from distributed sensor networks corrupted with measurement noise and often assume that the time instants of the measurements are precisely known or have a control over them. Sampling using a mobile sensor has been a problem that has received attention of late. This problem also has been well studied in literature for temporally
fixed fields with samples from precisely known sampling locations to unknown locations. Also, certain works do
consider a time variation of field and have addressed generally using known sampling locations and known time
stamps.

Extensive research has been done in sampling fields which can be modelled using the diffusion equation. In fact,
almost all models studying spatiotemporal fields assume a model of the field which is evolving according to the
standard diffusion equation. However, to the best knowledge of the authors, there has been no work in sampling
fields governed by any linear partial differential equation (PDE) with constant coefficients. It is important to note
here that, all PDEs are not good models for physical fields. Thus, PDEs that are under consideration here are the
ones which can possibly model a physical field. An important criteria here is that the energy has to be finite and
typically decreasing with time due to finite support considerations and inherent "diffusive" nature. This be will
quantified in a later section of this work. Furthermore, sampling of fields varying with time has generally been
studied in specific, constrained environments like uniform sampling, (in space or time or both), or non uniform
sampling with precisely known spatial locations and time stamps, or unknown locations of either a very slowly
varying fields or at known time instants. The primary motivation of this work is to analyze sampling in a highly
generalized setup of any physical field. Such scenarios, are rather common in real world. An example is sampling of
a pollution field using an inexpensive device. Often adding precision of knowledge of location and time to sampling
system leads to a considerable increase in cost and hence often inexpensive devices is used which can record the
location or the time stamps of the samples. Modelling realistic scenarios of sampling fields without the knowledge
of spatial locations or time instants is another important aspect of motivation behind this work.

This work considers a very general model of a smooth field with finite support that is evolving according a
known, linear partial differential equation with constant coefficients. For mathematical tractability, field is assumed
to be one dimensional but is evolving with time according to the known PDE. The smoothness of the field is
modelled by is spatial bandlimitedness. However, the field need not be bandlimited in time, which in fact, is often
the case. It is important to note here that if a field and its certain number of temporal derivatives are known to be
spatially bandlimited at \( t = 0 \), then it can be concluded that the field will be always be bandlimited if it evolves
according to the given PDE. The number of the temporal derivatives depends on the degree of the time derivative
in the PDE and result has been proven in this work (Appendix A). Also, it is considered that location and time
stamps of all samples are unknown and are realizations of two independent unknown renewal processes. Also, the
samples are assumed to be corrupted with measurement noise which is independent of all other processes and is
assumed to have zero mean and a finite variance. There are no other assumptions about the nature of the noise or
its statistics. The primary way of reducing error in this setup would be oversampling, like any other setup involving
spatial sampling with unknown locations. However, it is important to note that the oversampling is only in the
spatial domain and not in time domain. The number of samples will solely be governed by spatial sampling density.

The main result of this paper is that if a spatially bandlimited field evolving according to a constant coefficient
linear PDE, is sampled such that sampling locations and instants are unknown and are obtained from unknown
renewal processes that are independent, then the mean square error between the estimated field from the noisy
samples and the original field at \( t = 0 \) decreases as \( O(1/n) \), where \( n \) is the average sampling density, that is, the
expected number of samples over the support of the field.

Prior Art: Estimating a spatiotemporally varying field or sources in a diffusive field has been a problem addressed in literature. Classically, the problem involved estimating sources in a diffusive field from distributed sensor networks, which often requires solving an inverse problem, i.e., inferring certain characteristics of the field like its distribution at any instant, from a small number of samples of the field. What makes such problems difficult is the fact that these inverse problems involving diffusion equation are known to be severely ill-conditioned \[28\]. Several works have thus looked at different forms to regularize the problem. Nehorai et al. \[1\] invoked spatial sparsity of sources and studied the detection and the localization of a single vapor-emitting source using a maximum likelihood estimator. Two different approaches to the reconstruction of a sparse source distributions, one involving spatial super-resolution[2] and other on an adaptive spatiotemporal sampling scheme[3], were introduced by Lu and Vetterli. Another method[4] using Prony’s method has been proposed. Ranieri et al.[5] employed compressed sensing on a discrete grid to estimate the field which has been extended to real line[6]. Apart from the standard diffusion PDE, the Poisson PDE has also been studied and solutions to that using finite elements has been also proposed[7], [8]. The scenario when spatial sparsity is not realistic has also been studied[9]. Sampling using a mobile sensor has been a topic of recent interest[10], [11]. Estimating fields using mobile sampling has been a well-studied problem. This problem reduces to the classical sampling and interpolation problem (as described in [12], [13], [14]), if the samples are collected on precisely known locations in absence of any measurement noise. A more generic version with precisely known locations, in presence of noise, both measurement and quantization, has also been addressed (refer [15] - [20]). Sampling and reconstruction of bandlimited signals from samples taken at unknown locations has also been studied using a number of variations of sampling models (see [21] - [26]). This work is different from all previous works in the following ways: (i) The field is considered to be evolving with a known constant linear PDE, which need not be the diffusion equation. It can be any linear PDE as long as it is a feasible model for a physical field. (ii) Both the sampling locations and the timestamps of the samples are unknown, unlike previous works where either fields are considered to be temporally fixed or time stamps are assumed to be known.

Notation: The spatiotemporally varying field will be denoted as \( g(x, t) \). The gradient of \( g \) is defined as \( \nabla g = \)
\[
\left[ \frac{\partial}{\partial x} g(x,t), \frac{\partial}{\partial t} g(x,t) \right].
\]

\( n \) denotes the average sampling density, while \( M \) is the random variable which denotes the number of samples taken over the support of the field. All vectors will be denoted in bold. The \( L^\infty \) norm of a vector \( x \) will be denoted by \( ||x||_\infty \). The expectation operator will be denoted by \( E[.] \). The expectation is over all the random variables within the arguments. The trace of a matrix \( A \) will be denoted by \( \text{tr}(A) \). The set of natural number, integers, reals and complex numbers will be denoted by \( \mathbb{N}, \mathbb{Z}, \mathbb{R} \) and \( \mathbb{C} \) respectively. Also, \( j = \sqrt{-1} \).

II. FIELD, SAMPLING AND NOISE MODEL, DISTORTION CRITERIA

A. Field Model

The field is considered to be spatially smooth over a finite support, one dimensional in space and evolving with time according to a Partial Differential Equation. The PDE is linear with constant coefficients and is assumed to be known and given by

\[
\sum_{i=0}^{m} p_i \frac{\partial^i}{\partial t^i} g(x,t) = \sum_{i=0}^{m'} q_i \frac{\partial^i}{\partial x^i} g(x,t) \tag{1}
\]

where, \( \frac{\partial^0}{\partial y^0} f(y) = f(y) \). This will be represented throughout the paper in the terms of polynomials. Define two polynomials as

\[
p(z) = \sum_{i=0}^{m} p_i z^i ; \quad q(z) = \sum_{i=0}^{m'} q_i z^i \tag{2}
\]

where the coefficients are same as in the differential equation. If for notation purposes, \( \left( \frac{\partial}{\partial z} \right)^t = \left( \frac{\partial^t}{\partial z^t} \right) \), then the original equation can be written as

\[
p \left( \frac{\partial}{\partial t} \right) g(x,t) = q \left( \frac{\partial}{\partial x} \right) g(x,t) \tag{3}
\]

To incorporate the smoothness of the field, the field is assumed to be bandlimited. It is important to note we need to ensure that the field is bandlimited as it evolves with time. Intuitively speaking, this condition should hold true if we have know the field is spatially bandlimited at \( t = 0 \), because time evolution is unlikely to affect the spatial bandlimitedness. Formally speaking, if the degree of the polynomial \( p \) is \( m \), then if \( m-1 \) partial derivatives of \( g(x,t) \) along with \( g(x,t) \) are spatially bandlimited then the function \( g(x,t) \) will be spatially bandlimited \( \forall t \geq 0 \) if the field evolves according to the given PDE. A detailed proof of this has been given in Appendix A. We will assume that such a condition holds for the field under consideration. This helps us ensure that field will always be bandlimited. Since the field considered has the finite support, assumed WLOG as \([0,1]\), it can be represented as

\[
g(x,t) = \sum_{k=-b}^{b} a_k(t) \exp(j2\pi kx) ; \quad a_k(t) = \int_{-\infty}^{\infty} g(x,t) \exp(-j2\pi kx) dx \tag{4}
\]

Also the field is assumed to be bounded. That is, \( |g(x,t)| \leq 1 \ \forall (x,t) \).
B. Distortion Criteria

To measure the distortion, we will use a simple mean squared error between the estimated field and the actual field. All measurements will be considered for \( t = 0 \) i.e., the mean square error will be considered between the estimated field at \( t = 0 \) and the actual field at \( t = 0 \). Let the estimated field be \( \hat{G}(x,t) \) and its Fourier coefficients be \( \hat{A}[k] \), then the distortion criterion is defined as

\[
D[\hat{G},g] = E\left[ \int_0^1 |\hat{G}(x,t) - g(x,t)|^2 dt \right]_{t=0} \\
= E\left[ \sum_{k=-b}^b |\hat{A}_k(t) - a_k(t)|^2 \right]_{t=0} \\
= \sum_{k=-b}^b E\left[ |\hat{A}_k(0) - a_k(0)|^2 \right]
\]

C. Sampling Model

The sampling model is in this case a renewal process based sampling model, similar to the one in [27]. Let \( X_1, X_2, \ldots \) denote the intersample distances and \( N_1, N_2, \ldots \) denote the intersample time intervals. The sampling model employed assumes the spatial and the temporal separations as realizations of two independent renewal processes. In other words, \( X_1, X_2, \ldots \) are i.i.d. random variables having a common distribution \( X > 0 \) and \( N_1, N_2, \ldots \) are also i.i.d. random variables having a single common distribution \( N > 0 \), such that \( X_i \) and \( N_j \) are independent random variables for all values of \( i, j \in \mathbb{N} \). Using these intersample distances, the sampling locations, \( S_n \), are given by \( S_n = \sum_{i=1}^n X_i \). The sampling is done over the interval \([0, 1]\), the support of the field, and \( M \) is the random number of samples that lie in the interval i.e. it is defined such that, \( S_M \leq 1 \) and \( S_{M+1} > 1 \). Thus \( M \) is a well defined measurable random variable[30]. Note that, the number of samples in the interval only depend on the spatial sampling density and not on the temporal counterpart.

For the purpose of ease of analysis and tractability, the support of the distributions of \( X \) and \( N \) are considered to be finite and inversely proportional to the sampling density. Hence, it is assumed that

\[
0 < X \leq \frac{\lambda}{n} \quad \text{and} \quad E[Y] = \frac{1}{n} \\
0 < N \leq \frac{\mu}{n} \quad \text{and} \quad E[N] = \frac{1}{n}
\]

(6)

where \( \lambda, \mu > 1 \) are parameters that characterize the support of the distributions. Both are finite numbers, independent of the average sampling density \( n \) and much smaller than \( n \) i.e., \( \lambda, \mu \ll n \). These would be important factors that govern the constant of proportionality in the expected error of the estimate. Furthermore, \( \lambda \) is an important factor also to determine the threshold on the minimum number of samples. Applying Wald’s identity[30], on \( S_{M+1} \),

\[
E[M+1]E[X] = E[S_{M+1}] \\
\implies (E[M] + 1) \frac{1}{n} = E[S_{M+1}] \\
\implies E[M] = nE[S_{M+1}] - 1
\]

(7)
By definition, $S_{M+1} > 1$ and $S_M \leq 1$. Since $S_{M+1} = S_M + X_{M+1}$, therefore, $S_{M+1} \leq 1 + X_{M+1} \leq 1 + \frac{\lambda}{n}$. Use these inequalities with equation (7), to obtain,

$$n - 1 < \mathbb{E}[M] \leq n + \lambda - 1 \quad (8)$$

Also, the bound on each $X_i \leq \frac{\lambda}{n}$, along with $S_{M+1} > 1$ gives,

$$(M + 1)\frac{\lambda}{n} > 1 \implies M > \frac{n}{\lambda} - 1 \quad (9)$$

Note that all the bounds on the number of samples, a random variable, is characterized solely in terms of $\lambda$, the parameter that defines support for the spatial renewal process. There is no involvement of the temporal renewal process as expected. The only assumption on the sampling model with respect to time is that it has been assumed that all samples are collected within some time $T_0$, which is known. That is to say, $T_M \leq T_0$ and $T_{M+1} > T_0$. Note that this value is variable. This value is important in general to be known in sampling scenarios especially in case of time varying fields because if it is too large then the field has likely decayed to a very small value which can lead to erroneous results. Thus, the knowledge of this is assumed. Since $nN \leq \mu \ll n$, we expect that $T_0 \ll M$.

### D. Measurement Noise Model

It will be assumed that the obtained samples have been corrupted by additive noise that is independent both of the samples and of both the renewal processes, the one governing the spatial sampling as well as the one related to the temporal sampling. For simplicity, the noise is considered to be varying only spatially. That is, at all time instants, the distribution of the noise remains the same, which is assumed to be unknown in this work. Hence, $W(x, t) \equiv W(x)$. Thus, the samples obtained would be sampled versions of $g(x, t) + W(x, t)$, where $W(x, t) \equiv W(x)$ is the noise. Also, since the measurement noise is independent, that is for any set of measurements at distinct points $s_1, s_2, s_3, \ldots, s_n$, the samples $W(s_1), W(s_2), W(s_3), \ldots W(s_n)$ would be independent and identically distributed random variables. Note that the sampling instants have not been considered because of the distribution being temporally static. Thus the sampled version of the measurement noise has been assumed to be a discrete-time white noise process. It is essential to note that the distribution of the noise is also unknown. The only statistics known about that noise in addition to the above are that the noise is zero mean and has a finite variance, $\sigma^2$.

### III. Field Estimation from the Obtained Samples

This section will mainly deal with the estimation of the field from the samples whose locations and time stamps come from two unknown independent renewal processes. Before that, it is essential to analyse the development of the field under the differential equation. Combining the equations \[\text{(1)}\] and \[\text{(1)}\], we can write,
\[
\sum_{i=0}^{m} p_i \frac{\partial^i A}{\partial t^i} \left( \sum_{k=-b}^{b} a_k(t) \exp(j2\pi kx) \right) = \sum_{i=0}^{n} q_i \frac{\partial^i x}{\partial t^i} \left( \sum_{k=-b}^{b} a_k(t) \exp(j2\pi kx) \right)
\]

\[
\implies \sum_{k=-b}^{b} \left( \sum_{i=0}^{m} p_i \frac{\partial^i a_k(t)}{\partial t^i} \right) \exp(j2\pi kx) = \sum_{k=-b}^{b} a_k(t) \left( \sum_{i=0}^{n} q_i (j2\pi k)^i \right) \exp(j2\pi kx)
\]

\[
\implies \sum_{i=0}^{b} \left( \sum_{i=0}^{m} p_i \frac{\partial^i a_k(t)}{\partial t^i} \right) \exp(j2\pi kx) = \sum_{k=-b}^{b} a_k(t) q(j2\pi k) \exp(j2\pi kx)
\]

\[
\implies (a) \sum_{i=0}^{b} \left( \sum_{i=0}^{m} p_i \frac{\partial^i a_k(t)}{\partial t^i} \right) - q(j2\pi k) a_k(t) = 0 \quad \forall k = -b, \ldots, -1, 0, 1 \ldots, b
\]

where (a) follows from \[2\] and (b) uses the orthogonality property for the Fourier basis. This gives us a differential equation for each \(a_k(t)\). To solve for \(a_k(t)\), the general method is adopted and the solution is assumed to be of the form \(e^{rt}\). For each \(k\), this leads to the polynomial equation,

\[
\sum_{i=0}^{m} p_i \frac{\partial^i A e^{rt}}{\partial t^i} - q(j2\pi k) A e^{rt} = 0
\]

\[
\implies \left( \sum_{i=0}^{m} p_i r^i - q(j2\pi k) \right) A e^{rt} = 0
\]

\[
\implies p(r) - q(j2\pi k) = 0
\]

The solution for \(a_k(t)\) is of the form \(A e^{rt}\), where \(r\) is the root of the above polynomial and \(A\) is a constant independent of \(t\). Let the roots of the above polynomial be \(r_1(k), r_2(k), \ldots, r_m(k)\). Note that the roots of the polynomial are indexed by \(k\) as well, implying there is a set of \(m\) roots for each value of \(k\). It is essential here to realise that if the field is a physically feasible one, then \(\Re(r_i(k)) \leq 0\); \(i = 1, 2, \ldots, m; k = -b, \ldots, -1, 0, 1 \ldots, b\), that is all roots have a non positive real part. Generally for all physical fields it has to be strictly less than 0, but we are allowing the possibility of sustained oscillating (in time) fields. Furthermore for simplicity of analysis, all of the roots \(r_1(k), r_2(k), \ldots, r_m(k)\) are considered to be distinct for a given \(k\).\[3\] However, it is possible that \(r_i(k_1) = r_j(k_2)\) for some \(i, j, k_1 \neq k_2\). This assumption is realistic enough as generally for physical fields, \(m\) is generally very small thus the chance of repeated roots is lesser. The condition is similar to the one obtained in control theory, where we want the poles of the closed loop system to lie in the left half plane. Thus, we can use criteria like the Routh Hurwitz condition, to ensure the roots have negative real parts in our case.

In fact, the solution for \(a_k(t)\) can thus be written as a linear combination of these roots. Thus \(a_k(t) = \sum_{i=1}^{m} a_{ki}(0) \exp(r_i(k)t)\). The coefficients have been represented so to maintain consistency of representation of \(a_k(t)\) as a function of time. Also \(a_{ki}(0)\) are finite constants independent of everything else. Let \(\alpha_k = \max_i |a_{ki}(0)|\).

\[1\] If there is a repeated root \(r\), then the solution will be of the form \(e^{rt}\) and also \(te^{rt}\), which will make the problem very complicated. Such cases can also be treated in a similar manner that has been described in the paper. To be very specific, if the repeated root is 0, it can be easily taken into the given framework by combining all the repeated terms with it. This is because if \(r = 0\), then \(te^{rt} = t\), which diverges and hence cannot be the solution for a physical field. Such nuances have been omitted to simplify the description of the process.
\[
\left| \frac{\partial}{\partial t} g(x,t) \right| = \left| \frac{\partial}{\partial t} \sum_{i=1}^{m} a_{ki}(0) \exp(r_i(k)t) \exp(j2\pi kx) \right|
\]
\[
= \left| \sum_{i=1}^{m} a_{ki}(0) r_i(k) \exp(r_i(k)t) \exp(j2\pi kx) \right|
\]
\[
\leq \sum_{i=1}^{m} \left| a_{ki}(0) r_i(k) \exp(r_i(k)t) \exp(j2\pi kx) \right|
\]
\[
\leq \sum_{i=1}^{m} |a_{ki}(0)||r_i(k)|
\]
\[
\leq m\alpha_k R
\]
(12)

\[
\left| \frac{\partial}{\partial x} g(x,t) \right| = \left| \frac{\partial}{\partial x} \sum_{k=-b}^{b} a_k(t) \exp(j2\pi kx) \right|
\]
\[
= \left| \sum_{k=-b}^{b} a_k(t) j2\pi k \exp(j2\pi kx) \right|
\]
\[
\leq \sum_{i=1}^{m} \left| a_{ki}(0) j2\pi k \exp(r_i(k)t) \exp(j2\pi kx) \right|
\]
\[
\leq \sum_{i=1}^{m} 2|\pi| |a_{ki}(0)|
\]
\[
\leq m\alpha_k 2|\pi|
\]
(13)

The third step follows from triangle inequality, fourth step uses the fact that \(\Re(r_i(k)) \leq 0\) and the bound on \(r_i(k)\) uses the Rouche’s theorem. The value of \(R\) can be expressed in terms of \(|p_i|\)'s, which are finite and so is the upper bound.

Using the above expression for \(a_k(t)\) and using it in equation (4) to obtain the value at \((x,t) = (S_n, T_n)\), we get,

\[
g(S_n, T_n) = \sum_{k=-b}^{b} \sum_{i=1}^{m} a_{ki}(0) \exp(r_i(k)T_n) \exp(j2\pi kS_n)
\]
(14)

The above equation can be written in a vector notation form. For this purpose, define the following.

\[
e_k(x,t) = [e_k^1(x,t), e_k^2(x,t), e_k^3(x,t), \ldots, e_k^m(x,t)], \text{ where } e_k^i(x,t) = \exp(r_i(k)t + j2\pi kx)
\]
\[
a_k = [a_{k1}(0), a_{k2}(0), a_{k3}(0), \ldots, a_{km}(0)]
\]
\[
a = [a_{-b}, \ldots, a_{-1}, a_0, a_1, \ldots, a_b]^T
\]
\[
e(x,t) = [e_{-b}(x,t), \ldots, e_{-1}(x,t), e_0(x,t), e_1(x,t), \ldots, e_b(x,t)]^H
\]
(15)

\[
\Rightarrow ||e_k(x,t)||^2 = \sum_{i=1}^{m} |\exp(r_i(k)t + j2\pi kx)|^2 \leq \sum_{i=1}^{m} 1 \leq m
\]
\[
\Rightarrow ||e(x,t)||^2 = \sum_{k=-b}^{b} ||e_k(x,t)||^2 \leq \sum_{k=-b}^{b} m = m(2b + 1)
\]
Therefore, on using equation (15), equation (14) can be rewritten as,

\[ g(S_n, T_n) = e^{H}(S_n, T_n) \mathbf{a} \]  

(16)

Recall that the sampling locations \((S_n)\) and their respective time stamps \((T_n)\) are given by,

\[
S_1 = X_1, \quad S_2 = X_1 + X_2, \ldots, S_n = \sum_{i=1}^{n} X_i \\
T_1 = N_1, \quad T_2 = N_1 + N_2, \ldots, T_n = \sum_{i=1}^{n} N_i
\]

where \(X_1, X_2, \ldots, X_M\) and \(N_1, N_2, \ldots, N_M\) are all unknown and both \(X_i\)'s and \(N_i\)'s are drawn from independent distributions. The obtained samples are value of the field at these locations and instants, that have been corrupted with noise. The observed values are, thus, \(g(S_i, T_i) + W(S_i, T_i)\) \(i = 1, 2, 3, \ldots, M\). Define two vectors

\[
g = [g_1, g_2, \ldots, g_M]^T \quad \text{and} \quad \mathbf{w} = [w_1, w_2, \ldots, w_M]^T
\]

(17)

where, \(g_i = g(S_i, T_i)\) and \(w_i = W(S_i, T_i)\) for \(i = 1, 2, 3, \ldots, M\).

The motivation behind this is to continue to matrix vector notation and hence all the samples have been stacked up to form a single vector. The vector that would be obtained on stacking up the samples would be \(\mathbf{g}_s = \mathbf{g} + \mathbf{w}\).

Combining equation (15) and (17), we can write,

\[
\begin{pmatrix}
e^{H}(S_1, T_1) \\
e^{H}(S_2, T_2) \\
\vdots \\
e^{H}(S_M, T_M)
\end{pmatrix} \mathbf{a} = \mathbf{Y} \mathbf{a} ; \quad \text{where} \quad \mathbf{Y} =
\begin{pmatrix}
e^{H}(S_1, T_1) \\
e^{H}(S_2, T_2) \\
\vdots \\
e^{H}(S_M, T_M)
\end{pmatrix}
\]

(18)

The main idea behind the reconstruction of the field would be that the sampling location and time instants are "near" to the locations and time instants, had we sampled uniformly both in time and space for \(M\) points. Thus, to incorporate the same into the formulation, define the following.

\[
\mathbf{g}_0 = [g_{u1}, g_{u2}, \ldots, g_{uM}]^T ; \quad \text{where} \quad g_{ui} = g\left(\frac{i}{M}, \frac{iT_0}{M}\right)
\]

\[
\mathbf{Y}_0 =
\begin{pmatrix}
e^{H}(s_1, t_1) \\
e^{H}(s_2, t_2) \\
\vdots \\
e^{H}(s_M, t_M)
\end{pmatrix} ; \quad \text{where} \quad (s_i, t_i) = \left(\frac{i}{M}, \frac{iT_0}{M}\right)
\]

(19)

\[ \implies \mathbf{g}_0 = \mathbf{Y}_0 \mathbf{a} ; \quad \text{where} \quad \mathbf{g} \in \mathbb{C}^{M \times 1}, \mathbf{Y}_0 \in \mathbb{C}^{M \times (2b+1)}, \mathbf{a} \in \mathbb{C}^{m(2b+1) \times 1}
\]

Note that \(\mathbf{Y}_0\) has Vandermonde structure.,[31].

Now, since we expect that the sampling locations are "near" to the grid points, we can estimate the Fourier coefficients by assuming that samples have been obtained by multiplying the Fourier coefficient vector by \(\mathbf{Y}_0\) instead of \(\mathbf{Y}\). The best estimate of the Fourier coefficients, \(\hat{\mathbf{a}}\), thus would be

\[
\hat{\mathbf{a}} = \arg \min_{\mathbf{b}} ||\mathbf{g}_s - \mathbf{Y}_0 \mathbf{b}||^2
\]

(20)
It is important to note here that instead of \( g \), we have used \( g_s \) since that is the best knowledge we have about \( g \).

Since the main way to achieve to estimate the field relies on oversampling, the sampling density will be generally very large and thus, \( n > m(2b+1) \), making this problem a standard least squares estimation problem. The solution to this problem is well known and uses the pseudo-inverse of the matrix. Therefore,

\[
\hat{a} = (Y_0^H Y_0)^{-1} Y_0^H g_s \\
a = (Y_0^H Y_0)^{-1} Y_0^H g_0
\]

The second equation is obtained in a similar manner. However, it is important to realize at this point that the first equation is a least-square estimate because of the unknown locations and noise while the second equation is an exact solution. Having defined all the above quantities, we can go ahead and estimate the error using the distortion criteria mentioned in the above section.

\[
\begin{align*}
\sum_{k=-b}^{b} E \left[ |\hat{A}_k(0) - a_k(0)|^2 \right] &= \sum_{k=-b}^{b} E \left[ \sum_{i=1}^{m} \hat{A}_{ki}(0) - \sum_{i=1}^{m} a_{ki}(0) |^2 \right] \\
&= \sum_{k=-b}^{b} E \left[ \sum_{i=1}^{m} (\hat{A}_{ki}(0) - a_{ki}(0)) |^2 \right] \\
&\leq \sum_{k=-b}^{b} E \left[ m \sum_{i=1}^{m} |\hat{A}_{ki}(0) - a_{ki}(0)|^2 \right] \\
&= m \sum_{k=-b}^{b} E \left[ |\hat{a}_k - a_k|^2 \right] \\
&\leq m \sum_{k=-b}^{b} E \left[ |\hat{a} - a|^2 \right] \\
&= m(2b+1) E \left[ |\hat{a} - a|^2 \right]
\end{align*}
\]

This is the estimate for the Fourier coefficients of the field and distortion criteria expressed in that estimate. We will establish the bound on the estimation error as the main result in this work.

**Theorem 1.** Let \( \hat{a} \) and \( a \) be defined in equation (21). Under the sampling model discussed and the corruption by the measurement noise, the following result holds

\[
E \left[ ||\hat{a} - a||^2 \right] \leq \frac{C'}{n}
\]

where \( n \) is the sampling density and \( C' \) is a positive constant independent of \( n \). It depends on the bandwidth, \( b \) of the signal, the support parameters of the renewal processes, \( \lambda \) and \( \mu \), the coefficients of the PDE and the noise variance, \( \sigma^2 \). The dependence on \( b, \lambda, \mu \) and \( \sigma^2 \) is such that if these constant would increase, the proportionality constant would increase, worsening the bound. The dependence on the coefficients of the PDE is in a very non linear way through the roots of the equations whose almost all coefficients are determined by these values. Correspondingly, the distortion error can be bounded as \( \frac{m(2b+1)C'}{n} \).
Proof. Thus, now we will upper bound $E \left[ |\hat{a} - a| \right]^2$ to obtain a bound on distortion. Letting, $A = (Y_0^H Y_0)^{-1} Y_0^H$, we can write,

$$E \left[ |\hat{a} - a| \right]^2 = E \left[ \left| (Y_0^H Y_0)^{-1} Y_0^H g_s - a \right| \right]^2$$

$$= E \left[ \left| (Y_0^H Y_0)^{-1} Y_0^H (g + w) - (Y_0^H Y_0)^{-1} Y_0^H g_0 \right| \right]^2$$

$$= E \left[ ||A(g + w - g_0)||^2 \right]$$

$$\leq 2E \left[ ||A(g - g_0)||^2 \right] + 2E \left[ ||Aw||^2 \right]$$

$$\leq 2E \left[ \lambda_{\text{max}}^A ||g - g_0||^2 \right] + 2E \left[ ||Aw||^2 \right]$$

where second step follows from equation (21) and definition of $g_s$, the fourth step from Cauchy Schwartz inequality and $\lambda_{\text{max}}^A$ is the largest eigenvalue of $A^H A$. Both the terms, along with the bound on $\lambda_{\text{max}}^A$, in the last step will be analyzed separately to obtain the bound on the error. Now,

$$||g - g_0||^2 = \sum_{i=1}^{M} |g(S_i, T_i) - g(s_i, t_i)|^2$$

$$\leq \sum_{i=1}^{M} ||\nabla g||_{\infty} \left( |S_i - s_i|^2 + |T_i - t_i|^2 \right)$$

$$\leq C_0 \left\{ \sum_{i=1}^{M} \left| S_i - \frac{i}{M} \right|^2 + \sum_{i=1}^{M} \left| T_i - \frac{iT_0}{M} \right|^2 \right\}$$

The second step follows from the fact that for a smooth function $h$, $|h(x_1) - h(x_2)| \leq ||\nabla h||_{\infty} ||x_1 - x_2||$ and third step uses the fact that $||\nabla g||$ is upper bounded. This follows from the definition of $\nabla g$ and equations (13) and (12).

$$\lambda_{\text{max}}^A \overset{(a)}{=} \text{tr}(A^H A)$$

$$= \text{tr}(AA^H)$$

$$= \text{tr}((Y_0^H Y_0)^{-1} Y_0^H Y_0 (Y_0^H Y_0)^{-1})$$

$$= \text{tr}((Y_0^H Y_0)^{-1})$$

(a) follows from the fact that trace of a matrix is the sum of its eigenvalues and since $A^H A$ is symmetric, all its eigenvalues will be non negative therefore, the sum will be greater than the largest eigenvalue. For the second term, note that using the assumptions on the noise model, $E[w] = 0$ and $E[ww^T] = \sigma^2 I$, where $I$ is the identity matrix.
\[ E[||Aw||^2] = E[w^T(A^H A w)] \]

\[ \overset{(a)}{=} E[\text{tr}(w^T(A^H A w))] \]

\[ \overset{(b)}{=} E[\text{tr}((A^H A w)w^T)] \]

\[ \overset{(c)}{=} \text{tr}(E[A^H Aww^T]) \]

\[ \overset{(d)}{=} \text{tr}(E[A^H A E[ww^T]]) \]

\[ = \text{tr}(E[A^H A]E[ww^T]) \]

\[ = \text{tr}(E[A^H A]E[ww^T]) \]

\[ = \text{tr}(E[A^H A]E[ww^T]) \]

\[ = \text{tr}(E[ww^T]E[A^H A]) \]

\[ = \text{tr}(E[ww^T]E[A^H A]) \]

\[ = \sigma^2 E[\text{tr}((Y_0^H Y_0)^{-1})] \]

where,

(a) uses the fact that \(|Aw|^2\) is scalar hence, it equals its trace,

(b) follows from \(\text{tr}(AB) = \text{tr}(BA)\)

(c) uses linearity of expectation and the trace operator

(d) is a result of independence of noise and sampling

where the second step follows from the property mentioned above, third from the noise model and last one from equation 25. Thus, from equation 23, 25 and 26, it is clear that characterizing the bound on \(\text{tr}((Y_0^H Y_0)^{-1})\) is required and will also suffice for the purpose.

Let \(\lambda_1, \lambda_2, \ldots, \lambda_{m(2b+1)}\) be eigenvalues of \(Y_0^H Y_0\). Since the matrix is symmetric, \(\lambda_i \geq 0 \ i = 1, 2, \ldots, m(2b+1)\). Using the property of eigenvalues, the eigenvalues of \((Y_0^H Y_0)^{-1}\), will be \(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_{m(2b+1)}}\). Let \(\lambda_{\text{max}}\) and \(\lambda_{\text{min}}\) be the maximum and minimum eigenvalues of \(Y_0^H Y_0\). Therefore, \(\frac{1}{\lambda_{\text{min}}}\) and \(\frac{1}{\lambda_{\text{max}}}\) will be the maximum and minimum eigenvalues of \((Y_0^H Y_0)^{-1}\). Applying the Polya-Szego’s inequality 32 on the sequence formed by eigenvalues of \(Y_0^H Y_0\) and those of \((Y_0^H Y_0)^{-1}\), we can write

\[ \sum_i \frac{\lambda_i}{\lambda_{\text{max}}} \sum_i \frac{1}{\lambda_i}(1/\lambda_i) \leq \frac{1}{4} \left( \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} + \frac{\lambda_{\text{min}}}{\lambda_{\text{max}}} \right)^2 \]

\[ \Rightarrow \text{tr}(Y_0^H Y_0) \text{tr}((Y_0^H Y_0)^{-1}) \leq \frac{m^2(2b+1)^2}{4} \left( \kappa + \frac{1}{\kappa} \right)^2 \]  \hspace{1cm} (27)

\[ \Rightarrow \text{tr}((Y_0^H Y_0)^{-1}) \leq \frac{m^2(2b+1)^2}{4 \text{tr}(Y_0^H Y_0)} \left( \kappa + \frac{1}{\kappa} \right)^2 \]

where \(\kappa\) is the condition number of the matrix \(Y_0^H Y_0\). Let us analyse the trace of the matrix \(Y_0^H Y_0\).
\[ \text{tr}(Y_0^*H Y_0) = \text{tr} \left( [e(s_1, t_1), e(s_2, t_2), \ldots, e(s_M, t_M)] \begin{bmatrix} e^H(s_1, t_1) \\ e^H(s_2, t_2) \\ \vdots \\ e^H(s_M, t_M) \end{bmatrix} \right) \]

\[ = ||e(s_1, t_1)||^2 + ||e(s_2, t_2)||^2 + \cdots + ||e(s_M, t_M)||^2 \]

\[ = \sum_{i=1}^{M} \sum_{k=-b}^{b} ||e_k(s_i, t_i)||^2 \]

\[ = \sum_{i=1}^{M} \sum_{k=-b}^{b} \sum_{j=1}^{m} |e_k(s_i, t_i)|^2 \]

\[ = \sum_{i=1}^{M} \sum_{k=-b}^{b} \sum_{j=1}^{m} \exp(2 \Re(r_j(k)) t_i) \]

\[ = \sum_{j=1}^{m} \sum_{k=-b}^{b} \sum_{i=1}^{M} \exp \left( 2 \Re(r_j(k)) \frac{i T_0}{M} \right) \] (28)

Both (a) and (b) follow directly from equation 15. For a given value of \( j \) and \( k \), the sum \( \sum_{i=1}^{M} \exp \left( 2 \Re(r_j(k)) \frac{i T_0}{M} \right) \) can be considered as a Riemann sum approximation of the integral \( \int_{0}^{T_0} \exp(2 \Re(r_j(k)) t) dt \). The only difference here that the terms in the sum are not multiplied by the interval difference, which in this case is the same for all intervals and hence can be taken out as a scalar. It is interesting to note here that the value of this integral can be used as a bound on the value of the sum because of that fact that \( \exp(2 \Re(r_j(k)) t) \) is decreasing since \( \Re(r_j(k)) \leq 0 \ \forall j, k. \)

\[ \int_{0}^{T_0} \exp(2 \Re(r_j(k)) t) dt = \sum_{i=1}^{M} \int_{\frac{i-1}{M} T_0}^{\frac{i}{M} T_0} \exp(2 \Re(r_j(k)) t) dt \]

\[ \leq \sum_{i=1}^{M} \int_{\frac{i-1}{M} T_0}^{\frac{i}{M} T_0} \exp \left( 2 \Re(r_j(k)) \frac{(i - 1) T_0}{M} \right) dt \]

\[ = \sum_{i=1}^{M} \exp \left( 2 \Re(r_j(k)) \frac{i T_0}{M} \right) \exp \left( -2 \Re(r_j(k)) \frac{T_0}{M} \right) \int_{\frac{i-1}{M} T_0}^{\frac{i}{M} T_0} dt \] (29)

\[ = \exp \left( -2 \Re(r_j(k)) \frac{T_0}{M} \right) \frac{T_0}{M} \sum_{i=1}^{M} \exp \left( 2 \Re(r_j(k)) \frac{i T_0}{M} \right) \]

\[ \leq \exp \left( -2 \Re(r_j(k)) \frac{T_0}{M} \right) \frac{T_0}{M} \sum_{i=1}^{M} \exp \left( 2 \Re(r_j(k)) \frac{i T_0}{M} \right) \]

where the last step uses that fact that \( T_0 \leq M \).

\[ \Rightarrow \ exp \left( 2 \Re(r_j(k)) \frac{T_0}{M} \right) \geq M \exp(2 \Re(r_j(k))) \int_{0}^{T_0} \exp(2 \Re(r_j(k)) t) dt = M C_{jk} \] (30)
where $C_{jk} = \exp(2\Re(r_j(k))) \int_0^{T_0} \exp(2\Re(r_j(k))t)dt$ is a finite constant for each $j, k$. Using equation (30) in equation (28), we get
\[
\text{tr}(Y_0^H Y_0) \geq m \sum_{j=1}^m b \sum_{k=-b}^b MC_{jk} = MC_3
\] (31)
where $C_3$ is a finite deterministic constant independent of $n$, given by
\[
C_3 = \sum_{j=1}^m b \sum_{k=-b}^b C_{jk}.
\]

Since $\kappa \geq 1 \exists K > 0$, such that the term \( (\kappa + \frac{1}{\kappa})^2 \leq K \). Combining this results with ones obtained in 27 and 31
\[
\text{tr}((Y_0^H Y_0)^{-1}) \leq \frac{m^2(2b + 1)^2}{4MC_3} K = \frac{C_t}{M}
\]
\[
\implies E\left[\text{tr}((Y_0^H Y_0)^{-1})\right] \leq \frac{C_t \lambda}{n} \leq \frac{C_t \lambda}{n - \lambda}
\] (32)

Substituting the results obtained above in equations 25 and 26 and combining them with the equations 23 and 24 we get,
\[
E\left[||\hat{a} - a||^2\right] \leq 2E\left[\frac{C_t}{M} \left\{ \sum_{i=1}^M \left| S_i - \frac{i}{M} \right|^2 + \sum_{i=1}^M \left| T_i - \frac{i}{M} \right|^2 \right\} \right] + \frac{2C_t \lambda}{n - \lambda}
\] (33)

From Appendix B, it is noted that,
\[
E\left[\frac{1}{M} \sum_{i=1}^M \left| S_i - \frac{i}{M} \right|^2\right] \leq \frac{C_S}{n}\quad \text{and}\quad E\left[\frac{1}{M} \sum_{i=1}^M \left| T_i - \frac{iT_0}{M} \right|^2\right] \leq \frac{C_T}{n}
\]

Therefore, we can conclude that
\[
E\left[||\hat{a} - a||^2\right] \leq \frac{C_t C_S}{n} + \frac{C_t C_T}{n} + \frac{C_t \lambda}{n - \lambda} \leq \frac{C'}{n}
\] (34)

This completes the proof. \(\square\)

IV. SIMULATIONS

This section presents the results of extensive simulations and explanations to the same. The simulations have been presented in Fig. . The simulations help ease out verification of the results obtained of different PDEs and analyze the effect of sampling density.

Firstly, for the purpose of analysis, a field $g(x, t)$ with $b = 3$ is considered and its Fourier coefficients have been generated using independent trials of a Uniform distribution over $[-1, 1]$ for all real and imaginary parts separately. It is ensured that the field is real by using conjugate symmetry. Finally, the field is scaled to have $|g(x)| \leq 1$. 

Three differential equations have been considered for the purpose and Fourier coefficients have been reused. The simulations have been carried out using the following PDEs

$$\frac{\partial}{\partial t} g(x, t) = 0.01 \frac{\partial^2}{\partial x^2} g(x, t)$$

$$\frac{\partial^2}{\partial t^2} g(x, t) + 3 \frac{\partial}{\partial t} g(x, t) = 0.01 \frac{\partial^2}{\partial x^2} g(x, t)$$

$$\frac{\partial^2}{\partial t^2} g(x, t) + 3 \frac{\partial}{\partial t} g(x, t) = 0.01 \left( \frac{\partial^2}{\partial x^2} g(x, t) - 0.125 \frac{\partial^4}{\partial x^4} g(x, t) \right)$$

(35)

The same two sets of Fourier coefficients were used in last two equations. Note that the others are conjugate of these to ensure real field.

$$a_1[0] = 0.3002; \quad a_2[0] = 0.2445;$$

$$a_1[1] = -0.0413 + j0.0216; \quad a_2[1] = -0.0357 + j0.0478;$$

$$a_1[2] = 0.0871 + j0.0343; \quad a_2[2] = 0.0978 + j0.0729;$$

$$a_1[3] = -0.1679 - j0.0586; \quad a_2[3] = -0.1796 - j0.0756;$$

(36)

The Fourier coefficients used in the first equation are

$$a[0] = 0.11; \quad a[1] = 0.023 - j0.076; \quad a[2] = 0.069 + j0.0551; \quad a[3] = 0.2 + j0.0821;$$

(37)

Fig. 2. The three different figures show the variation of error with the number of samples for different PDEs. The first one is the corresponding to first PDE in equation (35), the middle one corresponds to the second equation and the last one to the third equation in (35), which is the standard diffusion equation. The variation in the distortion is clearly of $O(1/n)$ as denoted by the slopes in the plots. However, the error is slightly large because of the large condition number of the matrix $Y_0$ giving issues regarding numerical stability.

The Fig. 2 shows the mean square error of the estimate for different PDEs. The plots are shown for the different PDEs as they have been listed in the equation (35). The slopes of the lines obtained are $-1.0019, -1.0110$ and $-1.0086$ which confirms the $O(1/n)$ decrease.

V. CONCLUSIONS

Mobile sampling of fields which are modelled by more general constant coefficient linear partial differential equations was addressed. Additionally, this work addressed an extremely general and challenging problem, in which
a spatiotemporal field under physical PDE law is sampled using an inexpensive mobile sensor such that both, the locations of the samples and the timestamps of the samples are unknown. Moreover, the locations and timestamps of the samples are assumed to be realizations of two independent unknown renewal processes. Furthermore, the field samples were affected by measurement noise. It was shown that the mean square error between the original and the estimated signal decreases as $O(1/n)$, where $n$ is the average sampling density of the mobile sensor.

APPENDIX A

This appendix mainly deals with proving the bandlimitedness of the field for all instants. It states that for a field evolving according to the equation (3), if it is known that the field and its $m-1$, temporal derivatives are bandlimited at $t=0$, then the field will always remain bandlimited. Here $m$ is the degree of the polynomial $p$. A major part of this proof will be similar to the approach in Section III and has been reproduced here for ease of understanding. Since the field is assumed to have a finite support, the field can be written as

$$g(x,t) = \sum_{k=\infty}^{\infty} a_k(t) \exp(j2\pi kx) \quad ; \quad a_k(t) = \int_{-\infty}^{\infty} g(x,t) \exp(-j2\pi kx) dx$$

(38)

Substituting this in the equation (1), we can write,

$$\sum_{i=0}^{m} p_i \frac{\partial^i}{\partial t^i} \left( \sum_{k=-\infty}^{\infty} a_k(t) \exp(j2\pi kx) \right) = \sum_{i=0}^{n} q_i \frac{\partial^i}{\partial x^i} \left( \sum_{k=-\infty}^{\infty} a_k(t) \exp(j2\pi kx) \right)$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} \left( \sum_{i=0}^{m} p_i \frac{\partial^i}{\partial t^i} a_k(t) \right) \exp(j2\pi kx) = \sum_{k=-\infty}^{\infty} a_k(t) \left( \sum_{i=0}^{n} q_i (j2\pi k)^i \right) \exp(j2\pi kx)$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} a_k(t) \left( \sum_{i=0}^{n} q_i (j2\pi k)^i \right) \exp(j2\pi kx)$$

(39)

$$\Rightarrow \sum_{i=0}^{m} p_i \frac{\partial^i a_k(t)}{\partial t^i} - q(j2\pi k)a_k(t) = 0 \quad \forall k \in \mathbb{Z}$$

where (a) follows from (1) and (b) uses the orthogonality property for the Fourier basis. This gives us a set of $2b+1$ equations for each $a_k(t)$. To solve for $a_k(t)$, the general method is adopted and the solution is assumed to be of the form $e^{rt}$. For each $k \in \mathbb{Z}$, this leads to the polynomial equation,

$$\sum_{i=0}^{m} p_i \frac{\partial^i A e^{rt}}{\partial t^i} - q(j2\pi k) A e^{rt} = 0$$

$$\Rightarrow \left( \sum_{i=0}^{m} p_i r^i - q(j2\pi k) \right) A e^{rt} = 0$$

(40)

$$\Rightarrow p(r) - q(j2\pi k) = 0$$

The solution for $a_k(t)$ is a of the form $A e^{rt}$, where $r$ is the root of the above polynomial and $A$ is a constant independent of $t$. Let the roots of the above polynomial be $r_1(k), r_2(k), \ldots r_m(k)$. Note that the roots of the polynomial are indexed by $k$ as well, implying there is a set of $m$ roots for each value of $k$. Furthermore for simplicity of analysis, all of the roots $r_1(k), r_2(k), \ldots r_m(k)$ are considered to be distinct for a given $k$ as in Section III. However, it is possible that $r_i(k_1) = r_j(k_2)$ for some $i, j, k_1 \neq k_2$. In fact, the solution for $a_k(t)$ can

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thus be written as a linear combination of these roots. Thus \( a_k(t) = \sum_{i=1}^{m} a_{ki}(0) \exp(r_i(k)t) \). The coefficients have been chosen to maintain consistency of represent as a function of time. Now, we know that the field is bandlimited and so are its \( m - 1 \) partial derivatives at \( t = 0 \). This can be written as \( \forall i = 0, 1, 2 \ldots m - 1 \)

\[
\left. \frac{\partial^i a_k(t)}{\partial t^i} \right|_{t=0} = \begin{cases} 
  c_{ki} & \text{if } |k| \leq b \\
  0 & \text{otherwise}
\end{cases} 
\]

(41)

where \( c_{ki} \)'s are real constants. Since \( a_k(t) = \sum_{i=1}^{m} a_{ki}(0) \exp(r_i(k)t) \), therefore,

\[
\frac{\partial^i a_k(t)}{\partial t^j} = \sum_{i=1}^{m} a_{ki}(0)r_i^j(k) \exp(r_i(k)t) \forall j \geq 0
\]

(42)

Consider \( k' \) in the range \(|k| > b\). Then for \( k' \) and \( \forall i = 0, 1, 2 \ldots m \), we have \( \left. \frac{\partial^i a_{k'}(t)}{\partial t^i} \right|_{t=0} = 0 \). Then for \( k' \) using equation (42) we can combine all the equations for all \( i \) and the resulting expression can be written in matrix form as,

\[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
r_1(k') & r_2(k') & \ldots & r_m(k') \\
r_1^2(k') & r_2^2(k') & \ldots & r_m^2(k') \\
\vdots & \vdots & \ddots & \vdots \\
r_1^m(k') & r_2^m(k') & \ldots & r_m^m(k')
\end{bmatrix}
\begin{bmatrix}
a_{k'1} \\
a_{k'2} \\
\vdots \\
a_{k'm}
\end{bmatrix} = 0
\]

(43)

Since the roots are all distinct as assumed, therefore the matrix on the left is a Vandermonde matrix and is always invertible [31]. This means that \( a_{k'i} = 0 \forall i = 1, 2, 3 \ldots m \) and all \(|k'| > b\). This implies that the field is bandlimited, i.e., \( a_k(t) \equiv 0 \) for all \(|k| > b\).

**APPENDIX B**

This section primarily deals with establishing upper bound on the terms \( \mathbb{E}\left[ \frac{1}{M} \sum_{i=1}^{M} \left| S_i - \frac{i}{M} \right|^2 \right] \) and \( \mathbb{E}\left[ \frac{1}{M} \sum_{i=1}^{M} \left| T_i - \frac{iT_0}{M} \right|^2 \right] \). The renewal based sampling model for the spatial terms is the same that has been considered in [27]. Moreover, the bound on the term \( \mathbb{E}\left[ \frac{1}{M} \sum_{i=1}^{M} \left| S_i - \frac{i}{M} \right|^2 \right] \) has been elaborately derived there [27, Appendix A]. Using that we have the bound,

\[
\mathbb{E}\left[ \frac{1}{M} \sum_{i=1}^{M} \left| S_i - \frac{i}{M} \right|^2 \right] \leq \frac{C_S}{n}
\]

(44)

for some \( C_S > 0 \) and independent of \( n \). The proof for the other term follows in a similar manner. Define,

\[
y_m := \mathbb{E}\left[ \left( N_1 - \frac{T_0}{M} \right)^2 | M = m \right] \\
z_m := \mathbb{E}\left[ \left( N_1 - \frac{T_0}{M} \right) \left( N_2 - \frac{T_0}{M} \right) | M = m \right]
\]

(45)

From equation (31) in [27], we have
Consider the expression,

\[
E \left[ \frac{1}{M} \sum_{i=1}^{M} \left| T_i - \frac{iT_0}{M} \right|^2 \right] | M = m = \frac{m + 1}{2} y_m + \frac{m^2 - 1}{3} z_m \tag{46}
\]

where the second step is obtained from evaluating and rearranging the expression along the exchangeability of \( N_i \)'s. Since we know that, \( T_M \leq T_0 \) and \( T_{M+1} > T_0 \), define

\[
J_M = T_0 - T_M \implies J_M < T_{M+1} - T_M \leq \frac{\mu}{n} \tag{48}
\]

Also \( J_M^2 = (T_M - T_0)^2 \implies E[(T_M - T_0)^2 | M = m] = E[J_M^2 | M = m] \leq \frac{\mu^2}{n^2} \). Using the above two results, we can conclude that, \( my_m + m(m - 1)z_m = E[J_M^2 | M = m] \). Combining this with (46), we can write,

\[
E \left[ \frac{1}{M} \sum_{i=1}^{M} \left| T_i - \frac{iT_0}{M} \right|^2 \right] | M = m = \frac{m + 1}{2} y_m + \frac{m^2 - 1}{3m(m - 1)} \left( -my_m + E[J_M^2 | M = m] \right)
\]

\[
= \frac{m + 1}{2} y_m + \frac{m^2 + 1}{3m} E[J_M^2 | M = m]
\]

\[
\leq \frac{m + 1}{2} y_m + \frac{2 \mu^2}{3 n^2} \tag{49}
\]

This is exactly the same result as obtained for \( E \left[ \frac{1}{M} \sum_{i=1}^{M} \left| S_i - \frac{i}{M} \right|^2 \right] \) in [27]. Using the same steps for the expression in \( S_i \), we can conclude,

\[
E \left[ \frac{1}{M} \sum_{i=1}^{M} \left| T_i - \frac{iT_0}{M} \right|^2 \right] | M = m \leq \frac{C_T}{n} \tag{50}
\]

An important thing to note here is that even if \( J_M \leq \frac{K \mu}{n} \) for some positive constant \( K \), the result will hold. Interestingly, \( K \), can be \( O(\sqrt{n}) \), and still the result will hold. The idea is that the difference between \( T_0 \) and \( T_M \) should of \( O(1/\sqrt{n}) \), that is \( T_0 \) cannot be simply any number larger than \( T_M \). It has to be a reasonably accurate estimation of the time taken. It is important to know this to help decide the construction of \( Y_0 \) because the entire idea is based on the assumption that the samples are "near" the grid points. But to determine the grid points, we must have the knowledge of the support of the function which has to be finite.

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