Research Article

A New Model of Discrete-Continuous Bivariate Distribution with Applications to Medical Data

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The bivariate Poisson exponential-exponential distribution is an important lifetime distribution in medical data analysis. In this article, the conditionals, probability mass function (pmf), Poisson exponential and probability density function (pdf), and exponential distribution are used for creating bivariate distribution which is called bivariate Poisson exponential-exponential conditional (BPEEC) distribution. Some properties of the BPEEC model are obtained such as the normalized constant, conditional densities, regression functions, and product moment. Moreover, the maximum likelihood and pseudolikelihood methods are used to estimate the BPEEC parameters based on complete data. Finally, two data sets of real bivariate data are analyzed to compare the methods of estimation. In addition, a comparison between the BPEEC model with the bivariate exponential conditionals (BEC) and bivariate Poisson exponential conditionals (BPEC) is considered.

1. Introduction

The construction of bivariate distributions based on specifying the marginals or the conditionals received great attention from researchers since the beginning of the nineties [1–4]. Some conferences held about this technique, as the one was held in New York (Fisher and Sen [5]) under the title “The collected works of Wassily Hoeffding,” and in Prague (Benes and Stepan [6]) under the title “Distributions with given marginals and moment problems.” It is often easier to visualize conditional pdf (pmf) or properties of \( f_{XY}(x|y) \) and \( f_{YX}(y|x) \) then \( f_x(x)\) and \( f_y(y) \), or \( f_{XY}(x,y) \) [7] (p.1, [8]) rather than the joint distribution. In this sense, studying the bivariate distributions when both conditionals belong to discrete or continuous distributions has received special attention, as Arnold et al. [7, 9], Johnson et al. [10], and Castillo and Galambos [11–13] see also Arnold [14], Arnold et al. [15] Kottas et al. [16], Gharib et al. [17], and Mohammed et al. [4, 18]. But what is new in this paper is that one distribution is discrete and the other is continuous in the considered bivariate distributions. A similar type of class was derived by Sarabia et al. [19] where the conditional distributions of those bivariate distributions were Poisson (discrete) and gamma (continuous). The utilization of this type of class in bonus-malus systems (Sarabia et al. [19]), medical applications as analysis of HIV infection (Nazife Sultanoglu et al. [20]), and
fractional modeling for improving the scholastic performance of students with optimal control Abdullahi (Yusuf et al. [21]).

In this paper, we will be interested in studying an interesting trend in constructing a set of bivariate distributions with discrete and continuous conditional distributions as exponential and exponential Poisson distributions, respectively.

The univariate discrete Poisson exponential distribution has the following pmf (Fazal and Bashir [22]):

\[ P(X = x) = \frac{\lambda^x}{(1 + \lambda)^x}, \quad x = 1, 2, \ldots, \lambda > 0. \quad (1) \]

i.e., \( X \sim \text{PE(\lambda)} \). Moreover, the univariate continuous exponential distribution has the following pdf (Fazal and Bashir [23]):

\[ f_y(y) = \beta y^{-\beta}, \quad y > 0, \quad \beta > 0. \quad (2) \]

i.e., \( Y \sim E(\beta) \).

2. Bivariate Poisson Exponential-Exponential Conditionals (BPEEC) Class

Suppose that the conditional distributions \( X \mid Y \) and \( Y \mid X \) are, respectively,

\[ f_{X|Y}(x \mid y) = \frac{\lambda(y)}{(1 + \lambda(y))^{x+1}}, \quad \text{i.e.} \quad X \mid Y = y \sim \text{Poisson exponential(\lambda(y))}. \]

\[ f_{Y|X}(y \mid x) = \beta(x) \text{Exp}(-\beta(x)y), \quad \text{i.e.} \quad Y \mid X = x \sim \text{Exponential(\beta(x))}, \]

where \( \lambda(y) \) and \( \beta(x) \) are some positive functions. According to these conditional distributions, the \( f_{X,Y}(x, y) \) is

\[ \left( \frac{\lambda(y)}{(1 + \lambda(y))^{x+1}} \right) f_Y(y) = \beta(x) \text{Exp}(-\beta(x)y) f_X(x), \quad (4) \]

where \( f_X(x) \) and \( f_Y(y) \) are, respectively, the marginal distributions of \( X \) and \( Y \). Then,

\[ \log [\lambda(y) f_Y(y)] - (x + 1) \log (1 + \lambda(y)) = \log [\beta(x) f_X(x)] - \beta(x) y. \quad (5) \]

Considering

\[ h(y) = \log [\lambda(y) f_Y(y)], \quad (6) \]

\[ g(x) = \log [\beta(x) f_X(x)]. \quad (7) \]

By substituting equations (6) and (7) into equation (5) we obtain equation

\[ g(x) - \beta(x) y - h(y) + (x + 1) \log (1 + \lambda(y)) = 0. \quad (8) \]

Equation (8) is functional equation, which is a special of \( \sum_{k=1}^{\infty} f_k(x) g_k(y) = 0 \), whose general solution is given by Aczel [24], (p. 161), as

\[ \beta(x) = \alpha_2 - \alpha_3 (x + 1), \quad \log[1 + \lambda(y)] = -\alpha_1 - \alpha_3 y. \quad (9) \]

substituting (9) into (5) results in

\[ f_{X,Y}(x, y) = [N(A)]^{-1} \text{Exp}[\alpha_1 (x + 1) - (\alpha_2 - \alpha_3 (x + 1))y], \quad \alpha_1 > 0, \alpha_2, \alpha_3 \in \mathbb{R}, \] \( x = 0, 1, 2, \ldots, y > 0, \]

where \([N(A)]^{-1} = \text{Exp}[-\alpha_0], A = (\alpha_1, \alpha_2, \alpha_3)\), is the normalizing constant.

The joint distribution \( f_{X,Y}(x, y) \) in equation (10) describes the new model of BPEEC distribution that has \( \alpha_1, \alpha_2 \) intensity parameters for \( X \) \( Y \) and \( \alpha_3 \) dependence parameter, where \( \alpha_3 = 0 \) coincide with independence between \( X \) \( Y \).

3. Properties of BPEEC Class

The general properties of BPEEC class are studied in this part.

3.1. Normalizing Constant. The normalizing constant \([N(A)]^{-1}\) of the discrete-continuous BPEEC class given in equation (10) is

\[ N(A) = \sum_{x=0}^{\infty} \int_{0}^{\infty} \text{Exp}[\alpha_1 (x + 1) - \alpha_2 y + \alpha_3 (x + 1)y] dy \]

\[ = \sum_{x=0}^{\infty} e^{\alpha_1 (x+1)} \]

The previous expression could be written in a new form:

\[ \frac{1}{\alpha_2 - \alpha_3 (x + 1)} = \frac{1}{\alpha_2} \left( 1 - \frac{\alpha_3}{\alpha_2} (x + 1) \right)^{-1} \]

\[ = \frac{1}{\alpha_2} \sum_{k=0}^{\infty} \left( \frac{\alpha_3}{\alpha_2} (x + 1) \right)^k. \quad (12) \]

Therefore,

\[ N(A) = \frac{1}{\alpha_2} \sum_{x=0}^{\infty} \left( \frac{\alpha_3}{\alpha_2} (x + 1) \right)^k \sum_{x=0}^{\infty} (x + 1)^k \text{Exp}[\alpha_1 (x + 1)] \]

\[ = \frac{1}{\alpha_2} \sum_{x=0}^{\infty} \left( \frac{\alpha_3}{\alpha_2} (x + 1) \right)^k \frac{d^k}{d\alpha_1^k} \left( 1 - \text{Exp}[\alpha_1] \right). \quad (13) \]

3.2. Conditional Distributions and Regression Functions. A specific form of the \( f_{X|Y}(x \mid y) \) and \( f_{Y|X}(y \mid x) \) for the new
The regression functions of $f_{X|Y}(x | y)$ and $f_{Y|X}(y | x)$, are, respectively, defined as
\begin{align*}
f_{X|Y}(x | y) &= \frac{\exp[-\alpha_1 - \alpha_3 y] - 1}{\exp[-\alpha_1 - \alpha_3 y]^{x+1}}, \quad x = 0, 1, 2, \cdots, y > 0, \\
f_{Y|X}(y | x) &= \frac{(\alpha_2 - \alpha_5 (x + 1))}{\exp[(\alpha_2 - \alpha_5 (x + 1)) y]}, \quad x = 0, 1, 2, \cdots, y > 0,
\end{align*}
\hspace{1cm} (14) \hspace{1cm} (15)
given that,
\begin{align*}
X|Y = y &\sim \text{Poisson Exponential}(\exp[-\alpha_1 - \alpha_3 y] - 1), \\
Y|X = x &\sim \exp(\alpha_2 - \alpha_3 (x + 1)).
\end{align*}
\hspace{1cm} (16)
Both $f_{X|Y}(x | y)$ and $f_{Y|X}(y | x)$ given in equations (14) and (15) are satisfying the compatibility conditions stated by Arnold et al. [7], for the existing BPEEC class in equation (10).

The regression functions of $f_{X|Y}(x | y)$ and $f_{Y|X}(y | x)$, are, respectively, defined as
\begin{align*}
E(X|Y = y) &= \frac{1}{\exp[-\alpha_1 - \alpha_3 y] - 1}, \\
E(Y|X = x) &= \frac{1}{\alpha_2 - \alpha_3 (x + 1)}.
\end{align*}
\hspace{1cm} (17)
These functions are nonlinear, and we noticed that $E(X | Y = y)$ and $E(Y | X = x)$ are decreasing if $\theta_3 < 0$ and increasing if $\theta_3 > 0$. Figures 1 and 2 demonstrate that.

3.3. Marginals and Moments. We substitute (9) into (6) and (7) to get the following marginal functions
\begin{align*}
f_X(x) &= \frac{[N(A)]^{-1}}{\alpha_2 - \alpha_3 (x + 1)} \exp[\alpha_1 (x + 1)], \\
f_Y(y) &= \frac{[N(A)]^{-1}}{\exp[\alpha_1 (x + 1)] - 1} \exp[-\alpha_2 y].
\end{align*}
\hspace{1cm} (18)
For $(1 + x) < \theta_2/\theta_3$, the product-moment of BPEEC distribution is
\begin{align*}
E(XY) &= \frac{[N(A)]^{-1}}{} \exp[2\alpha_1] \text{HypergeometricPFQ}[\{2, 2 - \alpha_5/\alpha_3, 2 - \alpha_5/\alpha_3\}, \{3 - \alpha_2/\alpha_3, 3 - \alpha_2/\alpha_3\}, \exp[\alpha_1]] \\
&= \frac{[N(A)]^{-1}}{} \exp[2\alpha_1] \frac{(\alpha_2 - 2\alpha_3) - \alpha_2 y + 1}{(\alpha_2 - 2\alpha_3)^2}, \\
&= \frac{[N(A)]^{-1}}{} \exp[2\alpha_1] \frac{(\alpha_2 - 2\alpha_3) - \alpha_2 y + 1}{(\alpha_2 - 2\alpha_3)^2}.
\end{align*}
\hspace{1cm} (19)
where HypergeometricPFQ \([\{a_1, \ldots, a_p\}, \{b_1, \ldots, b_q\}, z]\) is the generalized hypergeometric function \(pFq(a; b; z)\).

### 4. Parameter Estimation

In this section, the maximum likelihood estimation (MLE) and maximum pseudolikelihood estimator (MPLE) are used to estimate \(\alpha_1, \alpha_2\) and \(\alpha_3\) of BPEEC class.

#### 4.1. The Maximum Likelihood Estimation

Suppose that \((x_i, y_i), (i = 1, 2, \ldots, n)\) are observed values from the BPEEC distribution with \(f_{X,Y}(x, y)\) given in equation (10), then the logarithm of the likelihood function is

\[
\log(\mathcal{L}(A)) = \sum_{i=1}^{n} \left( (x_i + 1) \log(y_i) - x_i y_i \right).
\]

The estimates of \(\alpha_1, \alpha_2, \alpha_3\) are obtained by differentiating \(\log(\mathcal{L}(A))\) with respect to each parameter. This results in the following likelihood equations:

\[
\begin{align*}
\frac{\partial \log(\mathcal{L}(A))}{\partial \alpha_1} &= -n \frac{\partial \mathcal{L}(A)}{\partial \alpha_1} + \sum_{i=1}^{n} (x_i + 1), \\
\frac{\partial \log(\mathcal{L}(A))}{\partial \alpha_2} &= -n \frac{\partial \mathcal{L}(A)}{\partial \alpha_2} - \sum_{i=1}^{n} y_i, \\
\frac{\partial \log(\mathcal{L}(A))}{\partial \alpha_3} &= -n \frac{\partial \mathcal{L}(A)}{\partial \alpha_3} + \sum_{i=1}^{n} (x_i + 1) y_i.
\end{align*}
\]

#### 4.2. Maximum Pseudolikelihood Estimator

The normal approximation of the MLE can also be used to create asymptotic confidence intervals (CIs) for \(\alpha_i, i = 1, 2, 3\) when the sample size is large. A two-sided (1-\(\alpha\)) 100% CI for \(\alpha_i, i = 1, 2, 3\) are defined as \((\hat{\alpha}_i \pm z_{\alpha/2} \sqrt{\text{Var}(\hat{\alpha}_i)})\), where \(\text{Var}(\hat{\alpha}_i), i = 1, 2, 3\) are the asymptotic variances of \(\hat{\alpha}_i\).

#### 5. Applications

5.1. Seizure Data

The bivariate data set in Table 1 has been obtained from Johnson and Davis [29]. This data represents the number of seizures observed in the first week and the second week for 30 patients after admission to the hospital.

| Patient | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---------|---|---|---|---|---|---|---|---|---|----|
| Week-1(X) | 5 | 1 | 1 | 3 | 3 | 0 | 1 | 4 | 0 | 3 |
| Week-2(Y) | 0 | 2 | 4 | 2 | 1 | 0 | 0 | 0 | 0 | 2 |
| Patient | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| Week-1(X) | 3 | 3 | 1 | 3 | 0 | 1 | 1 | 0 | 1 | 2 |
| Week-2(Y) | 0 | 2 | 0 | 2 | 0 | 0 | 3 | 2 | 1 | 1 |
| Patient | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| Week-1(X) | 0 | 2 | 1 | 1 | 0 | 2 | 0 | 1 | 6 | 3 |
| Week-2(Y) | 0 | 1 | 4 | 0 | 0 | 2 | 0 | 1 | 0 | 0 |

The bivariate data set in Table 1 has been obtained from Johnson and Davis [29]. This data represents the number of seizures observed in the first week and the second week for 30 patients after admission to the hospital.
Table 2: Kidney infection data for 38 patients.

| Patient | X | Y |
|---------|---|---|
| 1       | 8 | 23 |
| 2       | 22 | 447 |
| 3       | 30 | 24 |
| 4       | 7 | 511 |
| 5       | 53 | 15 |
| 6       | 16 | 13 |
| 7       | 28 | 318 |
| 8       | 12 | 245 |
| 9       | 9 | 30 |
| 10      | 196 | 154 |

Table 3: Parameter’s estimation of the BPEC.

| True values | MLE | MSE | MPLE | MSE |
|-------------|-----|-----|------|-----|
| α₁ = 0.5    | 0.0921 | 0.3507 | 0.3293 | 0.0291 |
| α₂ = 0.6    | 0.5426 | 0.0033 | 0.6074 | 0.0001 |
| α₃ = 0.5    | -0.1529 | 0.4264 | 0.5758 | 0.0057 |

Table 4: Parameters’ estimation of the BPEC.

| True values | MLE | MSE | MPLE | MSE |
|-------------|-----|-----|------|-----|
| θ₁ = 0.8    | 0.6255 | 0.0304 | 0.5651 | 0.0042 |
| θ₂ = 0.3    | 0.0115 | 0.0832 | -0.3497 | 0.4221 |
| θ₃ = 0.5    | 0.2999 | 0.0400 | 0.3997 | 0.0101 |

The joint pmf of BPEC(θ₁, θ₂, θ₃) distribution is Mohammed et. al [18])

\[
P_{X,Y}(x,y) = |N(\theta_1, \theta_2, \theta_3)|^{-1} \exp [\theta_1 y - \theta_2 x y + \theta_3 x],
\]

where \(N(\theta_1, \theta_2, \theta_3)\)^{-1} is the normalizing constant. The conditionals \(Y \mid X\) and \(X \mid Y\) are Poisson exponential distributions.

The joint pdf of BEC(λ₁, λ₂, λ₃) distribution is Arnold and Strauss [2])

\[
f_{X,Y}(x,y) = |N(\lambda_1, \lambda_2, \lambda_3)|^{-1} \exp (-\lambda_1 y - \lambda_2 x + \lambda_3 xy),
\]

where \(N(\lambda_1, \lambda_2, \lambda_3)\)^{-1} is the normalizing constant. The conditionals \(Y \mid X\) and \(X \mid Y\) are exponential distributions.

5.2. Kidney Infection Data. The bivariate data set in Table 2 represents the infection for kidney patients and has been

Table 5: Log-likelihood, AIC, and BIC of BPEEC, BPEC, and BEC distributions.

| Model Parameter | MLE | Log-likelihood | AIC | BIC |
|-----------------|-----|----------------|-----|-----|
| α₁              | 0.0921 | -44.647 | -50.647 | -49.749 |
| α₂              | 0.54266 | -44.647 | -50.647 | -49.749 |
| α₃              | -0.15297 | -5.3262 | -10.1895 | -10.0997 |
| θ₁              | -0.40667 | -95.895 | -101.895 | -100.997 |
| θ₂              | 0.04571 | 17 | -6.9812 | -6.8764 |
| θ₃              | 0.56715 | -76.485 | -82.485 | -81.5876 |

Table 6: Log-likelihood, AIC, and BIC of BPEEC, BPEC, and BEC distributions.

| Model Parameter | MLE | Log-likelihood | AIC | BIC |
|-----------------|-----|----------------|-----|-----|
| α₁              | 0.62556 | 122711 | 122704.8 | 122705.3 |
| α₂              | -0.30913 | 4065.1 | 4059.1 | 4059.6 |
| α₃              | 0.29990 | 0.01144 | -113 | -113.357 |
| θ₁              | 0.00510 | -444.99 | -450.99 | -450.45 |
| θ₂              | 0.00078 | 0.00007 | -444.99 | -450.99 | -450.45 |

obtained from Gilchrist and Aisbett [30]. Let X and Y be the first and second recurrence times, respectively.

From the obtained results of previous cases, the AIC and BIC of the BPEC model are more than the corresponding of the BPEC and BEC models which means that the BPEC model is a better fit for the given data. The approximated 95% two-sided CI of the parameters α₁, α₂, and α₃ are given, respectively, as [0.0287, 0.1555], [-0.0419, 1.1271], and [-0.44599, 0.14019] for seizure data, whilst for the Kidney infection data are [0.02087, 0.16351], [-0.11518, 1.2005], and [-0.48284, 0.1769]. Tables 3 and 4 present the estimated parameters of the BPEEC model and its mean square error (MSE). The BPPEC model is more appropriate as we can see in Tables 5 and 6 as compared to the other models.

6. Conclusion

In this article, a bivariate Poisson exponential-exponential distribution (BPEEC) is introduced by specified conditional pmf and pdf distributions as Poisson exponential and exponential distributions, respectively. In addition, we obtained some properties such as conditional, marginal distributions, and moments. The MLE and MPLE of α₁, α₂, and α₃ for
BPEEC distribution are present. By analyzing the results obtained, we get the following:

(i) From Tables 3 and 4, the results of MPLE are better than MLE since the MPLE method does not contain a normalizing constant

(ii) From Tables 5 and 6, the model selection AIC and BIC of discrete-continuous BPEEC distribution are better than the discrete BPEC and continuous BEC distributions

We will apply and investigate the effectiveness of the proposed BPEEC model in censored experiments either on simulation studies or in different real-world scenarios.

Data Availability

All data are available in the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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