Lanczos’s equation to replace
Dirac’s equation?

Andre Gsponer and Jean-Pierre Hurni

Independent Scientific Research Institute
Box 30, CH-1211 Geneva-12, Switzerland.

Published in J.D. Brown et al., eds., Proceedings of the
Cornelius Lanczos International Centenary Conference
Raleigh, North Carolina, December 12–17, 1993
(ISBN 0-89871-339-0, SIAM, Philadelphia, 1994) 509–512.

Abstract

Lanczos’s quaternionic interpretation of Dirac’s equation provides a uni-
fied description for all elementary particles of spin 0, \( \frac{1}{2} \), 1, and \( \frac{3}{2} \). The
Lagrangian formulation given by Einstein and Mayer in 1933 predicts two
main classes of solutions. (1) Point like partons which come in two families,
quarks and leptons. The correct fractional or integral electric and baryonic
charges, and zero mass for the neutrino and the u-quark, are set by eigenvalue
equations. The electro-weak interaction of the partons is the same as with the
Standard model, with the same two free parameters: \( e \) and \( \sin^2 \theta \). There is
no need for a Higgs symmetry breaking mechanism. (2) Extended hadrons
for which there is no simple eigenvalue equation for the mass. The strong
interaction is essentially non-local. The pion mass and pion-nucleon cou-
pling constant determine to first order the nucleon size, mass and anomalous
magnetic moment.

Since 1928, Dirac’s relativistic wave-equation has been rewritten in various
forms in order to facilitate its interpretation. For example, in 1930, Sauter [1] and
Proca [2] rewrote it using Clifford numbers. However, the most direct road was
taken by Lanczos in 1929 [3]. He showed how to derive Dirac’s equation from the
coupled biquaternion system:

\[
\nabla A = mB, \quad \nabla B = mA. \tag{1}
\]
As Lanczos’s dissertation was an attempt to formulate classical electrodynamics as a quaternion field theory \[4\], he knew that Maxwell’s equations could be written:

$$\nabla A = mB, \quad \nabla B = 0.$$  \hspace{1cm} (2)

Hence, (1) can be seen as Maxwell’s equations with feed-back. This was a very important idea, and precisely the one that lead Proca in 1936 to discover the correct equation for the massive spin one particle \[5\]. The concept was easily generalized, and Kemmer finally wrote down the wave-equations of the (pseudo-)scalar and (pseudo-)vector particles in the form we still use today \[6\]. Later, Gürsey \[7\] showed that Proca’s and Kemmer’s equations are just degenerated cases of Lanczos’s.

In fact, Lanczos’s equation also admits spin $\frac{3}{2}$ solutions. These are apparently devoid of the problems which plague the usual formulations of spin $\frac{3}{2}$. Thus, (1) provides a unified description of all elementary particles, showing that in their fundamental state their spin must be 0, $\frac{1}{2}$, 1, or $\frac{3}{2}$.

In the standard spinor or two-component formalisms, Dirac’s equation can be written as follows:

$$\partial L = mR, \quad \partial R = mL.$$  \hspace{1cm} (3)

Here $L$ and $R$ are the left- and right-handed components of Dirac’s four-component spinor. The main difference between (1) and (3) is that $A$ and $B$ are biquaternions with four complex components while $L$ and $R$ have only two. This is the ‘doubling’ problem that puzzled Lanczos for many years. It often reappears, e.g., in relativistic Hamiltonian dynamics \[8\] or general relativity \[9\], whenever Dirac’s equation emerges in a form or another.

To get Dirac’s spin $\frac{1}{2}$ field, Lanczos made the following superposition:

$$D = A\sigma + B\ast\sigma.$$  \hspace{1cm} (4)

Here $\sigma$ is an idempotent\[4\] which has the effect of projecting out half of $A$, which added to another half of the complex conjugated of $B$, gives a Lorentz covariant superposition that obeys the following wave equation:

$$\nabla D = mD\ast\dot{e}_3.$$  \hspace{1cm} (5)

This equation, to be called the *Dirac-Lanczos equation*, is equivalent to Dirac’s equation. It will be rediscovered by other people, in particular by Gürsey \[7\] and Hestenes \[10\].

---

\[1\] We assume $\sigma = \frac{1}{2}(1 + i\vec{e}_3)$ where $\vec{e}_3$ is the third quaternion unit.
An important observation is that one can go from (1) to (3) by simply requiring that \(A\) and \(B\) are singular quaternions \[11\]. However, Dirac’s system (3) does not only involve half as many components as Lanczos’s system (1), it also incorporates the ingredients that make fermions essentially different from bosons, a feature that is embodied in the Lanczos-Dirac equation (5). In effect, studying the time-reversal transformation, one finds that \(T^2 = -1\), which means that (5) implies the Pauli exclusion principle and Fermi statistics \[12\, p.\, 41\].

The origin of the difference between fermions and bosons is to be found in the superposition (4) which involves a complex conjugation and singles out a unique, but arbitrary, direction in space, i.e., \(\vec{e}_3\) when \(\sigma = \frac{1}{2}(1 + \imath \vec{e}_3)\). Then, when the operators
\[
P = ( \, )_{\, \, (t - \bar{t})}^\dagger \quad T = ( \, )_{\, \, (t - \bar{t})}^\dagger \imath \vec{e}_2, \quad C = ( \, ) \imath \vec{e}_2,
\]
are applied to plane waves such as \(D = D_0 \exp[\vec{e}_3(\bar{t} - \vec{p} \cdot \vec{x})]\), one sees that Lanczos’s theory necessarily implies the Stueckelberg-Feynman interpretation of antiparticles. Indeed, since \(\vec{e}_2\) anticommutes with \(\vec{e}_3\), the effect of \(C\) is to reverse the sign of the argument in the exponential. Moreover, these three involutions satisfy the CPT theorem, i.e., \(CPT = 1 \, (\, )_{\, \, (t - \bar{t}, -\bar{x})}\), and finally \(T^2 = -1\) as required for spin \(\frac{1}{2}\).

The four rest-frame solutions, spin up-down, particle-antiparticle are: \(D_0 = 1, \vec{e}_1, \imath \vec{e}_3, \text{ and } \imath \vec{e}_2\). Operating by \(I = ( \, ) \imath \vec{e}_1\), one obtains another quartet of solutions which together with the first one make the spin \(\frac{3}{2}\) solutions.

However, the superposition (4) is not the only one leading to a spin \(\frac{1}{2}\) field obeying equation (5). As shown by Gürsey in 1957, if (4) represents a proton, the neutron is then \[13\]:
\[
N = (A\vec{\sigma} - B^*\vec{\sigma})\imath \vec{e}_1. \tag{7}
\]
In particular, Nishijima’s operator \(\exp[\sigma i \varphi(t, \vec{x})]\), \[14\], which leaves the \(N\) field invariant, shows that \(D\) is electrically charged while \(N\) is neutral. Hence, Lanczos’s doubling is nothing but isospin. Gürsey’s articles \[13, 15\] had a tremendous impact \[16\] to \[22\]. They demonstrated that ‘internal’ symmetries such as isospin are explicit and trivial in Lanczos’s formulation (1), while only space-time symmetries are explicit in (5).

When he wrote his 1929 papers on Dirac’s equation, Lanczos was with Einstein in Berlin. In 1933, Einstein and Mayer (using semi-vectors, a formalism allied to quaternions) derived a spin \(\frac{1}{2}\) field equation (in fact, a generalized form of Lanczos’s equation) predicting that particles would come in doublets of different masses \[23\]. The idea was that the most general Lagrangian for quaternionic fields, to be called the \(\text{Einstein-Mayer-Lanczos (EML) Lagrangian}\), should have the form:
\[
L = \mathcal{S}[A^+ \nabla A + B^+ \nabla B - (A^+ BE^+ + B^+ AE) + (\ldots)^+] \tag{8}
\]
The field equations are then
\[ \nabla A = BE^+, \quad \nabla B = AE, \] (9)
which reduces to (1) when \( E = m \). In the general case, the second order equations for \( A \) or \( B \) become eigenvalue equations for the mass. The generalization (9) is obtained by the substitution \( Am \rightarrow AE \) in the Lagrangian leading to (1). Therefore, mass-generation is linked to a maximally parity violating field.

There are two basic conserved currents, the probability current \( C \), and the barycharge current \( S \):
\[ C = AA^+ + BB^+, \quad S = AE A^+ + EB E^+. \] (10)
Keeping \( E \) constant, \( C \) is invariant in any non-Abelian unitary gauge transformation, \( SU(2) \otimes U(1) \), of \( A \) or \( B \). On the other hand, \( S \) is only invariant for Abelian gauge transformations which also commute with \( E \): this is the general Nishijima group [14] which contains the electric and baryonic gauge groups.

Of special interest are the cases in which \( E \) is also a global gauge field. The first such gauge is when \( E \) is idempotent. One solution of (9) is then massive and the other one massless [24]. The most general local gauge transformations compatible with (8) are elements of the unitary Nishijima group \( U_N(1,\mathbb{C}) \) combined with one non-Abelian gauge transformation which operates on \( E \) and \( A \), or \( E \) and \( B \), exclusively. This leads directly to the Standard model of electro-weak interactions.

A remarkable property of the Nishijima group is that it has exactly two non-trivial decompositions in a product \( U(1,\mathbb{Q}) \otimes U(1) \), such that the electric charge is quantized. The doublet has charges \((0, -1)\) in one, and \((+\frac{2}{3}, -\frac{1}{3})\) in the other, \( U(1,\mathbb{Q}) \) being the discrete ring \( R_4 \) or \( R_6 \), respectively. Hence, electron charge quantization necessarily implies the existence of quark states of fractional electric charge, with both the neutrino and the u-quark massless.

The other fundamental case is when \( E \) is real: \( E \) describes a nucleonic field, non-locally coupled to a pseudoscalar eta-pion field. The chiral \( SU(2) \otimes SU(2) \) symmetry of low-energy strong interaction is explicit because one can perform independent isotopic rotations on the \( A \) and \( B \) fields independently. From this stage, by various transformations and additions, one can easily get the sigma-model [19], Heisenberg’s [21] or Nambu-Jona-Lasinio’s [22] non-linear theories, the Skyrme model [25], etc.

Lanczos’s and Einstein-Mayer’s theories provide a unifying framework for electro-weak and low-energy strong interactions. The gauge invariance of the EML Lagrangian leads to a surprising formal proximity of the W-bosons in the
electro-weak sector with the pions in the strong interaction sector. Of course, differences between electro-weak and strong interactions are considerable. But, leaving the details aside, any comparison should involve ‘$e$’ on one side, and the pseudoscalar pion-nucleon coupling constant ‘$g$’ on the other. That this is the case is well known. For example, the masses of nucleons and electrons are such that:

$$\frac{g^2}{e^2} \approx \frac{M}{m}. \quad (11)$$

What remains to be studied, is whether all of this is just an approximation to some more fundamental theory, or whether the quark model and QCD will come out as high energy approximations in a biquaternion field theory of elementary particles, an idea suggested by Lanczos for electrons and protons in his PhD thesis of 1919.

**Acknowledgments**

We wish to express our gratitude to Professor J.A. Wheeler, Professor W.R. Davis, and Professor G. Marx for their kind encouragement.

**References**

1. F. Sauter, Z. f. Phys. 63 (1930) 803–814, ibid , 64 (1930) 295–303.
2. A. Proca, J. Phys. Radium, 1 (1930) 235–248.
3. C. Lanczos, Z. f. Phys. 57 (1929) 447–473, 474–483, 484–493. Reprinted and translated in W.R. Davis et al., eds., Cornelius Lanczos Collected Published Papers With Commentaries (North Carolina State University, Raleigh, 1998) pages 2-1132 to 2-1225. [arXiv:physics/0508012](arXiv:physics/0508012) [arXiv:physics/0508002](arXiv:physics/0508002) [arXiv:physics/0508009](arXiv:physics/0508009)
4. C. Lanczos, doctoral dissertation, (Budapest, 1919) 80 pp. Reprinted in W.R. Davis et al., eds., ibid, pages A-1 to A-82. [arXiv:math-ph/0408079](arXiv:math-ph/0408079).
5. A. Proca, J. Phys. Radium 7 (1936) 347–353.
6. N. Kemmer, Proc. Roy. Soc. A166 (1938) 127–153.
7. F. Gürsey, PhD thesis, (University of London, 1950) 204 pp. F. Gürsey, Phys. Rev. 77 (1950) 844. In his PhD thesis Gürsey also generalized Lanczos’s equation to curved space-time.
8. C. Lanczos, Z. f. Phys. 81 (1933) 703–732. Reprinted and translated in W.R. Davis et al., eds., *ibid* pages 2-1294 to 2-1353.

9. C. Lanczos, Rev. Mod. Phys. 34 (1962) 379–389. Reprinted in W.R. Davis et al., eds., *ibid* pages 2-1896 to 2-1906.

10. D. Hestenes, J. Math. Phys. 8 (1967) 798–808, 809–812.

11. J. Blaton, Z. f. Phys. 95 (1935) 337–354.

12. R.P. Feynman, *The reason for antiparticles*, in Elementary Particles and the Laws of Physics (Cambridge University Press, 1987) 1-59.

13. F. Gürsey, Nuovo Cim. 7 (1958) 411–415. See also E.J. Schremp, Phys. Rev. 99 (1955) 1603.

14. K. Nishijima, Nuovo Cim. 5 (1957) 1349–1354.

15. F. Gürsey, Nuovo Cim. 16 (1960) 230–240.

16. W. Pauli and W. Heisenberg, *On the isospin group in the theory of the elementary particles*, preprint, March 1958. Published in W. Heisenberg, Collected Works, Series A / Part III (Springer Verlag, Berlin, 1993) 336–351.

17. G. Marx, Nucl. Phys. 9 (1958/59) 337–346.

18. W. Pauli and B. Touschek, Supl. Nuovo Cim. 14 (1959) 205–211.

19. M. Gell-Mann and M. Levy, Nuovo Cim. 16 (1960) 705–725.

20. Y. Nambu, Phys. Rev. Lett. 4 (1960) 380–382.

21. H.-P. Dür et al., Z. Naturforschung 14a (1959) 441–485, *ibid*, 14a (1959) 327–345. See also [16] p. 325–336.

22. Y. Nambu and G. Jona-Lasinio, Phys. Rev. 124 (1961) 246–254.

23. A. Einstein and W. Mayer, Proc. Roy. Acad. Amsterdam 36 (1933) 497–516, 615–619.

24. V. Bargmann, Helv. Phys. Acta 7 (1934) 57–82.

25. H. Y. Cheung and F. Gürsey, Mod. Phys. Lett. A5 (1990) 1685–1691.
Errata

This electronic version corrects a number of typographical errors that plagued the published paper version.

The restriction of the Nishijima group to $U(1, \mathbb{Q}) \otimes U(1)$ is not sufficient to set the correct fractional or integral electric and baryonic charges of the quarks and leptons. The statement in the abstract “The correct fractional or integral electric and baryonic charges, and zero mass for the neutrino and the u-quark, are set by eigenvalue equations” is thus wrong.

Moreover, the spin 3/2 interpretation of Lanczos’s equation is not as simple as suggested in the text: see arXiv:math-ph/0210055.