On the small time asymptotics of scalar stochastic conservation laws

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ABSTRACT
In this paper, we established a small time large deviation principles for scalar stochastic conservation laws driven by multiplicative noise. The doubling variables method plays a key role.

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1. Introduction
In this paper, we investigate the small time asymptotics of the first-order scalar conservation laws with stochastic forcing. Precisely, fix any $T > 0$ and let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]}, \{(\beta_k(t))_{t\in[0,T]}\}_{k\in\mathbb{N}})$ be a stochastic basis. Without loss of generality, here the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is assumed to be complete and $\{(\beta_k(t))_{t\in[0,T]}\}_{k\in\mathbb{N}}$ are one-dimensional i.i.d. real-valued $\{\mathcal{F}_t\}_{t\in[0,T]}$-Wiener processes. The symbol $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$. For any fixed $N \in \mathbb{N}$, let $\mathbb{T}^N \subset \mathbb{R}^N$ be the $N$-dimensional torus with the periodic length to be 1. We are concerned with the following Cauchy problem for the scalar conservation laws with stochastic forcing

$$
\begin{align*}
\begin{cases}
\mathrm{d}u + \text{div}(A(u)) \, \mathrm{d}t = \Phi(u) \, \mathrm{d}W(t) & \text{in } \mathbb{T}^N \times (0, T), \\
u(\cdot, 0) = \eta(\cdot) & \text{on } \mathbb{T}^N, \quad s
\end{cases}
\end{align*}
$$

where $u : (\omega, x, t) \in \Omega \times \mathbb{T}^N \times [0, T] \mapsto u(\omega, x, t) := u(x, t) \in \mathbb{R}$ is a random field, the flux function $A : \mathbb{R} \rightarrow \mathbb{R}^N$ and the coefficient $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ are measurable and fulfill certain conditions (see Section 2), and $W$ is a cylindrical Wiener process defined on a given (separable) Hilbert space $U$ with the form $W(t) = \sum_{k \geq 1} \beta_k(t) e_k, t \in [0, T]$, where $\{e_k\}_{k \geq 1}$ is an orthonormal base of the Hilbert space $U$. Moreover, the initial value $\eta \in L^\infty(\mathbb{T}^N)$ is a deterministic function.

When $\Phi \equiv 0$, the system (1) is reduced to the deterministic scalar conservation law, which is fundamental to our understanding of the space–time evolution laws of interesting physical quantities. For more background on this model, we refer the readers to the monograph [1], the work of Ammar et al. [2] and references therein. As we know, the Cauchy problem for the deterministic first-order partial differential equations (1) does not admit any (global) smooth solutions, but there exist infinitely
many weak solutions to the deterministic Cauchy problem. To solve the problem of non-uniqueness, an additional entropy condition was added to identify the physical weak solution. Under this condition, the notion of entropy solutions for the deterministic first-order scalar conservation laws was introduced by Kružkov [3,4]. The kinetic formulation of weak entropy solution of the Cauchy problem for a general multi-dimensional scalar conservation law (also called the kinetic system) was derived by Lions et al. [5]. The authors of Ref. [5] also discussed the relationship between entropy solutions and the kinetic system.

Adding a stochastic forcing (i.e. a noise) to this physical model is quite natural as it either represents an external random perturbation or gives a remedy for lack of (empirical) knowledge of certain involved physical parameters. Along with the great successful developments of deterministic scalar conservation laws, the random situation has also been developed rapidly. For example, in Ref. [6], Kim studied the Cauchy problem for the scalar stochastic conservation laws (1) driven by additive noise. Later, these results were extended to the multi-dimensional Dirichlet problem with additive noise by Vallet and Wittbold [7]. The authors of Ref. [7] succeed to show the existence and uniqueness of the stochastic entropy solutions by utilizing the vanishing viscosity method, Young measure techniques and Kružkov doubling variables technique. Concerning the multiplicative noise, for the Cauchy problem over the whole spatial space, Feng and Nualart [8] introduced a notion of strong entropy solutions to prove the uniqueness of the entropy solution. Moreover, the authors in Ref. [8] established the existence of strong entropy solutions in one-dimensional case using the vanishing viscosity and compensated compactness method. Recently, Debussche and Vovelle [9] proved the existence and uniqueness of kinetic solution to the Cauchy problem for (1) in any dimension by utilizing a kinetic formulation developed by Lions et al. for deterministic first-order scalar conservation laws [5]. Due to the equivalence between kinetic formulation and entropy solution, the existence and uniqueness of the entropy solutions were obtained in Ref. [9]. It is worth mentioning that Ref. [9] is the starting point of the present paper. In addition, the long-time behavior of the first-order scalar conservation laws has also attracted a lot of interests. For example, Debussche and Vovelle established the existence and uniqueness of invariant measures of scalar conservation laws driven by additive stochastic forcing [10]. Concretely, for sub-cubic fluxes, the authors of Ref. [10] show the existence of an invariant measure, and for sub-quadratic fluxes, they proved the uniqueness of the invariant measure. Recently, combining techniques used in the context of kinetic solutions as well as new results on large deviations, Dong et al. [11] established Freidlin–Wentzell’s type large deviation principles (LDPs) for the kinetic solution to the scalar stochastic conservative laws.

The purpose of this paper is to investigate the small time LDP of the kinetic solution to the scalar stochastic conservation laws, which describes the behaviors of the solution at a very small time. Specifically, we focus on the limiting behavior of the kinetic solution to the scalar stochastic conservation laws in a time interval $[0, t]$ as $t$ goes to zero. An important motivation for such a problem comes from Varadhan identity

$$\lim_{t \to 0} 2t \log \mathbb{P}(u(0) \in B, u(t) \in C) = -d^2(B, C),$$

where $u$ is the kinetic solution to the scalar stochastic conservation laws and $d$ is an appropriate Riemann distance associated with the diffusion generated by $u$.

The mathematical study of the small time LDP for finite dimensional processes was initiated by Varadhan [12]. Since then, the cases for the infinite dimensional diffusion processes were extensively studied (see Refs. [13–17] and the references therein). On the other hand, many researchers have also studied the small time LDP for infinite dimensional stochastic partial differential equations. For instance, Xu and Zhang [18] established the small time LDP of 2D Navier–Stokes equations in the state space $C([0, T]; H)$. Dong and Zhang [19] proved the small time LDP of 3D stochastic primitive equations in the state space $C([0, T]; H^1)$. In this paper, we will prove that the small time LDP of the kinetic solution to the scalar stochastic conservation laws holds in the space $L^1((0, T]; L^1(\mathbb{T}^N))$. To our knowledge, the present paper is the first work towards proving the small time LDP directly
for the kinetic solution to the scalar stochastic conservation laws. Due to the fact that the kinetic solutions are living in a rather irregular space, we will use the doubling variables method as in the work of Debussche and Vovelle [9]. As an important part of the proof, we need to make high-order estimates of martingale terms and error terms carefully. In particular, we mention that when applying Burkholder–Davis–Gundy inequality to the martingale term, the constant $\varepsilon p$ appears in front of the righthand side of (42) which results in this term not converging to 0 after taking $\varepsilon = 1/p$ at the end of the proof. To solve this problem, we force the constant $\varepsilon p$ to be the exponent of $e$ through applying Gronwall inequality to (47) which is derived by taking advantage of the characteristics of the equation and using identities (44). Thus, the term $r$ defined by (57) converges to 0, which implies the desired result of small time LDP (for details, see the proof of Proposition 4.2).

The rest of the paper is organized as follows. In Section 2, we recall the mathematical formulation of scalar stochastic conservation laws. In Section 3, we introduce the small time asymptotics and state our main result. Section 4 is devoted to the proof of exponential equivalence.

2. Framework

In the following, we will follow closely the framework of Ref. [9] to introduce some notations. Let $\| \cdot \|_{L^p}$ denote the norm of usual Lebesgue space $L^p(\mathbb{T}^N)$ for $p \in [1, \infty]$. In particular, set $H = L^2(\mathbb{T}^N)$ with the corresponding norm $\| \cdot \|_H$. $C_b$ represents the space of bounded, continuous functions and $C^1_b$ stands for the space of bounded, continuously differentiable functions having bounded first-order derivative. Define the function $f(x, t, \xi) := I_{u(x,t) > \xi}$, which is the characteristic function of the subgraph of $u$. We write $f := I_{u > \xi}$ for short. Moreover, denote by the brackets $\langle \cdot, \cdot \rangle$ the duality between $C^\infty_c(\mathbb{T}^N \times \mathbb{R})$ and the space of distributions over $\mathbb{T}^N \times \mathbb{R}$. In what follows, with a slight abuse of the notation $\langle \cdot, \cdot \rangle$, we denote the following integral by

$$\langle F, G \rangle := \int_{\mathbb{T}^N} \int_{\mathbb{R}} F(x, \xi) G(x, \xi) \, dx \, d\xi, \quad F \in L^p(\mathbb{T}^N \times \mathbb{R}), \quad G \in L^q(\mathbb{T}^N \times \mathbb{R}),$$

where $1 \leq p \leq +\infty$, $q := p/(p - 1)$ is the conjugate exponent of $p$. In particular, when $p = 1$, we set $q = \infty$ by convention. For a measure $m$ on the Borel measurable space $\mathbb{T}^N \times [0, T] \times \mathbb{R}$, the shorthand $m(\phi)$ is defined by

$$m(\phi) := \langle m, \phi \rangle([0, T]) := \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \phi(x, t, \xi) \, dm(x, t, \xi), \quad \phi \in C_b(\mathbb{T}^N \times [0, T] \times \mathbb{R}).$$

In the sequel, the notation $a \lesssim b$ for $a, b \in \mathbb{R}$ means that $a \leq Db$ for some constant $D > 0$ independent of any parameters.

2.1. Hypotheses

For the flux function $A$ and the coefficient $\Phi$ of (1), we assume the following.

**Hypothesis H** The flux function $A$ belongs to $C^2(\mathbb{R}; \mathbb{R}^N)$ and its derivative $a := A'$ is polynomial growth with degree $q_0 > 1$. That is, there exists a constant $N(q_0) \geq 0$ such that

$$|a(\xi)| \leq N(q_0)(1 + |\xi|^{q_0}), \quad |a(\xi) - a(\zeta)| \leq \Upsilon(\xi, \zeta)|\xi - \zeta|, \quad (2)$$

where $\Upsilon(\xi, \zeta) := N(q_0)(1 + |\xi|^{q_0-1} + |\xi|^{q_0-1})$.

For each $u \in \mathbb{R}$, the map $\Phi(u) : U \to H$ is defined by $\Phi(u)e_k = g_k(\cdot, u)$, where $(e_k)_{k \geq 1}$ is the orthonormal base in the Hilbert space $U$ and each $g_k(\cdot, u)$ is a regular function on $\mathbb{T}^N$. More precisely, we assume that $g_k \in C(\mathbb{T}^N \times \mathbb{R})$ satisfies the
following bounds:

\[ |g_k(x, u)| \leq C^0_k(1 + |u|), \quad \sum_{k \geq 1} |C^0_k|^2 \leq \frac{D_0}{2}, \]

\[ |g_k(x, u) - g_k(y, v)| \leq C^1_k(|x - y| + |u - v|), \quad \sum_{k \geq 1} |C^1_k|^2 \leq \frac{D_1}{2}, \]

for \( x, y \in \mathbb{T}^N, u, v \in \mathbb{R} \), where \( C^0_k, C^1_k, D_0, D_1 \) are positive constants.

From (3) and (4), we deduce that

\[ G^2(x, u) := \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0(1 + |u|^2), \]

\[ \sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq D_1 \left(|x - y|^2 + |u - v|^2\right). \]

Based on the above notations, Equation (1) can be rewritten as

\[
\begin{aligned}
\left\{
\begin{array}{ll}
\mathrm{du}(t, x) + \operatorname{div}A(u(t, x)) \, dt = \sum_{k \geq 1} g_k(x, u(t, x)) \, d\beta_k(t) & \text{in } \mathbb{T}^N \times (0, T], \\
u(\cdot, 0) = \eta(\cdot) & \text{on } \mathbb{T}^N.
\end{array}
\right.
\end{aligned}
\]  

\[ (\omega, t) \in \Omega \times [0, T] \rightarrow (m, \phi)([0, t]) := \int_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \phi(x, \xi) \, dm(x, s, \xi) \in \mathbb{R} \]

is predictable.

**Remark 2.1:** For any \( \phi \in C_b(\mathbb{T}^N \times \mathbb{R}) \) and kinetic measure \( m \), define \( A_t := (m, \phi)([0, t]), \) then a.s., \( t \mapsto A_t \) is a right continuous function of finite variation. Moreover, the function \( A \) has left limits in any \( t \in (0, T] \). We write \( A_t^- = \lim_{s \uparrow t} A_s \) and set \( A_0^- = 0 \). As a result, \( A_t^- = (m, \phi)([0, t]) \), which is càglàd (left continuous with right limits).

**Definition 2.2 (Kinetic solution):** Let \( \eta \in L^\infty(\mathbb{T}^N) \). A measurable function \( u : \mathbb{T}^N \times [0, T] \times \Omega \rightarrow \mathbb{R} \) is called a kinetic solution to (1) with initial datum \( \eta \) if
(1) \((u(t))_{t \in [0,T]}\) is predictable,
(2) for any \(p \geq 1\), there exists \(C_p \geq 0\) such that
\[
\mathbb{E} \left( \text{ess sup}_{0 \leq t \leq T} \|u(t)\|_{L^p(T\Omega)}^p \right) \leq C_p,
\]
(3) there exists a kinetic measure \(m\) such that \(f := I_{u > \xi}\) satisfies: for all \(\varphi \in C^1_c(T^N \times [0,T] \times \mathbb{R})\),
\[
\int_0^T \langle f(t), \partial_t \varphi(t) \rangle \, dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle \, dt \\
= - \sum_{k \geq 1} \int_0^T \int_{T^N} g_k(x, u(t,x)) \varphi(x, t, u(x,t)) \, dx \, d\beta_k(t) \\
- \frac{1}{2} \sum_{k \geq 1} \int_0^T \int_{T^N} \partial_{\xi} \varphi(x, t, u(x,t)) G^2(x, u(x,t)) \, dx \, dt + m(\partial_{\xi} \varphi), \text{ a.s.},
\]
where \(f_0 = I_{\eta > \xi}, u(t) = u(\cdot, t, \cdot)\) and \(G^2 = \sum_{k=1}^{\infty} |g_k|^2\).

Let \(u\) be a kinetic solution to (1) and \(f = I_{u > \xi}\). We use \(\overline{f} := 1 - f\) to denote its conjugate function. Define \(\Lambda_f := f - I_{0 > \xi}\), which can be viewed as a correction to \(f\). Note that \(\Lambda_f\) is integrable on \(T^N \times [0,T] \times \Omega\) if \(u\) is. In addition, it is shown in Ref. [9] that almost surely, the function \(f = I_{u > \xi}\) admits left and right weak limits at any point \(t \in [0,T]\).

**Proposition 2.1 ([9], Left and right weak limits):** Let \(f = I_{u > \xi}\) satisfy (10) with initial value \(f_0 = I_{\eta > \xi}\). Then \(f\) admits, almost surely, left and right limits, respectively, at every point \(t \in [0,T]\). More precisely, for any \(t \in [0,T]\), there exist kinetic functions \(f^{t \pm}\) on \(\Omega \times T^N \times \mathbb{R}\) such that \(\mathbb{P}-\text{a.s.}\)
\[
\langle f(t-r), \varphi \rangle \to \langle f^{t-}, \varphi \rangle, \quad \langle f(t+r), \varphi \rangle \to \langle f^{t+}, \varphi \rangle
\]
as \(r \to 0\) for all \(\varphi \in C^1_c(T^N \times \mathbb{R})\). Moreover, almost surely,
\[
\langle f^{t+} - f^{t-}, \varphi \rangle = - \int_{T^N \times [0,T] \times \mathbb{R}} \partial_{\xi} \varphi(x, \xi) I_{[t]}(s) \, dm(x, s, \xi).
\]
In particular, almost surely, the set of \(t \in [0,T]\) fulfilling \(f^{t+} \neq f^{t-}\) is countable.

For the above function \(f = I_{u > \xi}\), define \(f^{\pm}(t) = f^{t \pm}, t \in [0,T]\). Since we are dealing with the filtration associated to Brownian motion, both \(f^{+}\) and \(f^{-}\) are clearly predictable as well. Also \(f = f^{+} = f^{-}\) almost everywhere in time and we can take any of them in an integral with respect to the Lebesgue measure or in a stochastic integral. However, if the integral is with respect to a measure, typically a kinetic measure in this article, the integral is not well-defined for \(f\) and may differ if one chooses \(f^{+}\) or \(f^{-}\).

At the end of this part, we mention that with the aid of Proposition 2.1, the following result was verified by Debussche and Vovelle [9].

**Lemma 2.1:** The weak form (10) satisfied by \(f = I_{u > \xi}\) can be strengthened to be weak only with \(x\) and \(\xi\). Concretely, for all \(t \in [0,T]\) and \(\varphi \in C^1_c(T^N \times \mathbb{R})\),
\[
\langle f^{+}(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle \, ds
\]
\[
+ \sum_{k \geq 1} \int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) \, d\nu_{x,s}(\xi) \, dx \, d\beta_k(s) \\
+ \frac{1}{2} \int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, \xi) G^2(x, \xi) \, d\nu_{x,s}(\xi) \, dx \, ds - \langle m, \partial_\xi \varphi \rangle([0, t]), \quad \text{a.s.},
\]
with \( \nu_{x,s} = -\partial_\xi f = \delta_{u(x,s)=\xi} \) and we set \( f^+(T) = f(T) \).

**Remark 2.2:** By making modifications, we have for all \( t \in (0, T] \) and \( \varphi \in C^1_c(\mathbb{T}^N \times \mathbb{R}) \),

\[
\langle f^-(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_{0}^{t} \langle f(s), a(\xi) \cdot \nabla \varphi \rangle \, ds \\
+ \sum_{k \geq 1} \int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) \, d\nu_{x,s}(\xi) \, dx \, d\beta_k(s) \\
+ \frac{1}{2} \int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, \xi) G^2(x, \xi) \, d\nu_{x,s}(\xi) \, dx \, ds - \langle m, \partial_\xi \varphi \rangle([0, t]), \quad \text{a.s.}
\]
and we set \( f^-(0) = f_0 \).

The following result was shown by Theorem 24 in Ref. [9].

**Theorem 2.2 ([9], Existence and uniqueness):** Let \( \eta \in L^\infty(\mathbb{T}^N) \). Assume Hypothesis H holds, then there is a unique kinetic solution \( u \) to Equation (1) with initial datum \( \eta \).

Moreover, by Corollary 16 in Ref. [9], it follows that

**Corollary 2.3 (Continuity in time):** Let \( \eta \in L^\infty(\mathbb{T}^N) \). Assume Hypothesis H is in force, then for every \( p \in [1, +\infty) \), the kinetic solution \( u \) to (1) with initial datum \( \eta \) has almost sure continuous trajectories in \( L^p(\mathbb{T}^N) \).

### 3. Small time asymptotics and statement of our main result

In the rest part, we take \( T = 1 \). Let \( 0 < \varepsilon \leq 1 \), by the scaling property of the Brownian motion, it is readily to deduce that \( u(\varepsilon t) \) coincides in law with the solution of the following equation:

\[
u^\varepsilon(t,x) + \varepsilon \int_{0}^{t} \text{div}(A(u^\varepsilon(s))) \, ds = \eta(x) + \sqrt{\varepsilon} \int_{0}^{t} \sum_{k \geq 1} g_k(x, u^\varepsilon(s,x)) \, d\beta_k(s).
\]

By Theorem 2.2, there is a unique kinetic solution \( u^\varepsilon \). Applying Sections 6 and 7 in Ref. [20] with \( A = 0 \), we obtain for any \( p \geq 1 \),

\[
sup_{0 < \varepsilon \leq 1} \mathbb{E} \sup_{0 \leq s \leq 1} \| u^\varepsilon(s) \|_{L^p(\mathbb{T}^N)}^p < \infty.
\]

Using similar methods as Theorem 5.1 in [7] and Lemma 3.1 in [24], for any \( p \geq 2 \), under Hypothesis H, we get

\[
\lim_{M \to \infty} \sup_{0 < \varepsilon \leq 1} \varepsilon \log \mathbb{P}\left( \mathbb{E} \sup_{0 \leq s \leq 1} \| u^\varepsilon(s) \|_{L^p(\mathbb{T}^N)}^p > M \right) = -\infty.
\]

By Lemma 2.1, there exists a kinetic measure \( m^\varepsilon \) such that \( f^\varepsilon(x, t, \xi) := I_{u^\varepsilon(x,t)>\xi} \) satisfies that for all \( t \in [0, 1) \) and \( \varphi \in C^1_c(\mathbb{T}^N \times \mathbb{R}) \),

\[
\langle f^\varepsilon(t), \varphi \rangle = \langle f^\varepsilon(t), \varphi \rangle + \int_{0}^{t} \langle f_1(s), a(\xi) \cdot \nabla \varphi \rangle \, ds
\]
\[
+ \sqrt{\varepsilon} \sum_{k \geq 1} \int_{0}^{t} \int_{\mathbb{T}^{N}} g_{k}(x, \xi) \varphi(x, \xi) \, d\nu_{X, \xi}(\xi) \, dx \, d\beta_{k}(s) \\
+ \frac{\varepsilon}{2} \int_{0}^{t} \int_{\mathbb{T}^{N}} \partial_{\xi} \varphi(x, \xi) C^{2}(x, \xi) \, d\nu_{X, \xi}(\xi) \, dx \, ds - \langle m_{t}^{\varepsilon}, \partial_{\xi} \varphi \rangle([0, t]), \quad \text{a.s., (16)}
\]

where \( \nu_{X, \xi}(\xi) = -\partial_{\xi} f_{1}(x, s, \xi) = \delta_{\nu_{X, (x, s)} = \xi} \) and we set \( f_{1}^{+}(1) = f_{1}(1) \).

For \( h \in L^{2}([0, 1]; U) \) with the form \( h(t) = \sum_{k \geq 1} h_{k}(t) \epsilon_{k} \), consider the following deterministic equation:

\[
\begin{cases}
\, d u^{h}(t, x) = \sum_{k \geq 1} g_{k}(x, u^{h}(t, x)) h_{k}(t) \, dt, \\
\, u^{h}(0) = \eta.
\end{cases}
\]

Applying Theorem 5.1 and Theorem 5.3 in Ref. [11] with \( A = 0 \), there exists a unique kinetic solution \( u_{\eta}^{h} \) in the space \( L^{1}([0, 1]; L^{1}(\mathbb{T}^{N})) \). Define

\[
R(h) = \frac{1}{2} \sum_{k \geq 1} \int_{0}^{1} |h_{k}(t)|^{2} \, dt.
\]

For \( \varrho \in L^{1}([0, 1]; L^{1}(\mathbb{T}^{N})) \), define

\[
\mathcal{L}_{\varrho} = \left\{ h \in L^{2}([0, 1]; U) : \varrho(\cdot) = u_{\eta}^{h}(\cdot) \right\}.
\]

Set

\[
I(\varrho) = \begin{cases}
\inf_{h \in \mathcal{L}_{\varrho}} R(h) & \text{if } \mathcal{L}_{\varrho} \neq \emptyset, \\
+\infty & \text{if } \mathcal{L}_{\varrho} = \emptyset.
\end{cases}
\]

(17)

For any initial value \( \eta \in L^{\infty}(\mathbb{T}^{N}) \), let \( u_{\eta}^{h} \) be the kinetic solution of (13). Denote by \( \mu_{\eta}^{h} \) the law of \( u_{\eta}^{h} \) on the space \( L^{1}([0, 1]; L^{1}(\mathbb{T}^{N})) \). The main result of this article reads as follows.

**Theorem 3.1:** Let the initial value \( \eta \in L^{\infty}(\mathbb{T}^{N}) \). Assume Hypothesis \( H \) is in force, then \( \{\mu_{\eta}^{h}, \varepsilon > 0\} \) satisfies a large deviation principle with the rate function \( I(\cdot) \) defined by (16), that is,

(i) For any closed subset \( F \subset L^{1}([0, 1]; L^{1}(\mathbb{T}^{N})) \),

\[
\lim \sup_{\varepsilon \to 0} \varepsilon \log \mu_{\eta}^{h}(F) \leq -\inf_{\varrho \in F} I(\varrho).
\]

(ii) For any open subset \( G \subset L^{1}([0, 1]; L^{1}(\mathbb{T}^{N})) \),

\[
\lim \inf_{\varepsilon \to 0} \varepsilon \log \mu_{\eta}^{h}(G) \geq -\inf_{\varrho \in G} I(\varrho).
\]

**Proof:** Applying Theorem 24 in Ref. [9] with \( A = 0 \), we know that there exists a unique kinetic solution \( v_{\eta}^{h} \) to the following stochastic equation:

\[
v_{\eta}^{h}(t, x) = \eta(x) + \sqrt{\varepsilon} \int_{0}^{t} \sum_{k \geq 1} g_{k}(x, v_{\eta}^{h}(s, x)) \, d\beta_{k}(s).
\]

Let \( \vartheta_{\eta}^{h} \) be the law of \( v_{\eta}^{h}(\cdot) \) on \( L^{1}([0, 1]; L^{1}(\mathbb{T}^{N})) \). According to Theorem 4.2 in Ref. [11] with \( A = 0 \), it follows that \( \vartheta_{\eta}^{h} \) satisfies a large deviation principle with the rate function \( I(\cdot) \). Based on
Theorem 4.2.13 in Ref. [21], it suffices to show that two families of the probability measures $\mu^\varepsilon_n$ and $\theta^\varepsilon_n$ are exponentially equivalent, that is, for any $t > 0$,

$$
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left( \| u^\varepsilon_n - v^\varepsilon_n \|_{L^1([0,1];L^1(\mathbb{T}^N))} > t \right) = -\infty. \tag{19}
$$

In Section 4, we devote to proving (19) using doubling of variables method.

From now on, for the sake of simplicity, we denote by $u^\varepsilon = u^\varepsilon_n$ and $v^\varepsilon = v^\varepsilon_n$ when the initial value is not emphasized.

4. Proof of (18)

Recall that $v^\varepsilon$ is the unique kinetic solution to (18). Applying Sections 6 and 7 in Ref. [20] with $A = 0$, $B = 0$, we obtain that, for any $p \geq 1$,

$$
\sup_{0 < \varepsilon \leq 1} \mathbb{E} \sup_{0 \leq s \leq 1} \| v^\varepsilon(s) \|^p_{L^p(\mathbb{T}^N)} < \infty. \tag{20}
$$

Using similar methods as Theorem 5.1 in [7] and Lemma 3.1 in [24], we can obtain a stronger result, that is, for any $p \geq 2$, under Hypothesis $H$,

$$
\lim_{M \to \infty} \sup_{0 < \varepsilon \leq 1} \varepsilon \log \mathbb{P}\left( \esssup_{0 \leq s \leq 1} \| v^\varepsilon(s) \|^p_{L^p(\mathbb{T}^N)} > M \right) = -\infty. \tag{21}
$$

Moreover, by Lemma 2.1, there exists a kinetic measure $m_2^\varepsilon$ such that $f_2(x, t, \xi) := I_{v^\varepsilon(x,t) > \xi}$ satisfies that for all $t \in [0, 1)$ and $\varphi \in C^1(\mathbb{T}^N \times \mathbb{R})$,

$$
\langle f_2^+(t), \varphi \rangle = \langle f_{2,0}, \varphi \rangle + \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) \, d\nu^2_{x,\xi}(\xi) \, dx \, d\beta_k(s)
$$

$$
+ \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, \xi) \mathcal{G}^2(x, \xi) \, d\nu^2_{x,\xi}(\xi) \, dx \, ds
$$

$$
- \langle m_2^\varepsilon, \partial_\xi \varphi \rangle([0, t]), \quad a.s., \tag{22}
$$

where $f_{2,0} = I_{\eta > \xi}$, $v^\varepsilon_{x,\xi} = -\partial_\xi f_2 = \delta_{v^\varepsilon(x,t) = \xi}$ and we set $f_2^+(1) = f_2(1)$.

Following the idea of Proposition 13 in Ref. [9] and by utilizing the doubling of variables method, we have the following result relating $u^\varepsilon$ and $v^\varepsilon$.

**Proposition 4.1:** Assume Hypothesis $H$ is in place. Let $u^\varepsilon$ and $v^\varepsilon$ be the kinetic solution of (13) and (18), respectively. Then, for all $0 < t < 1$, and non-negative test functions $\rho \in C^\infty(\mathbb{T}^N)$, $\psi \in C^\infty_c(\mathbb{R})$, the corresponding functions $f_1(x, t, \xi) := I_{u^\varepsilon(x,t) > \xi}$ and $f_2(y, t, \xi) := I_{v^\varepsilon(y,t) > \xi}$ satisfy

$$
\int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x - y) \psi(\xi - \zeta) (f_{1,1}^+(x, s, \xi, \eta) f_{2,1}^+(y, s, \zeta) + f_{1,2}^+(x, s, \xi) f_{2,1}^+(y, s, \zeta)) \, d\xi \, d\zeta \, dx \, dy
$$

$$
\leq \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x - y) \psi(\xi - \zeta) (f_{1,0}(x, \xi, \eta) f_{2,0}(y, \zeta) + f_{1,0}(x, \xi) f_{2,0}(y, \zeta)) \, d\xi \, d\zeta \, dx \, dy
$$

+ I(t) + J(t) + K(t), \quad a.s., \tag{23}
$$

where $I(t) = \varepsilon \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} (f_{1,j} f_{2} + f_{1} f_{2,0})(\eta(\xi) \cdot \nabla_\xi) \alpha \, d\xi \, dx \, ds$,
\[ J(t) = \varepsilon \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \alpha \sum_{k \geq 1} |g_k(x, \xi) - g_k(y, \zeta)|^2 \, dv_{\xi, s}^{1,\varepsilon} \otimes v_{\zeta, s}^{2,\varepsilon}(\xi, \zeta) \, dx \, dy \, ds, \]
\[ K(t) = 2\sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{T}^N} (g_k(x, \xi) - g_k(y, \zeta)) \rho(x - y) \chi_1(\xi, \zeta) \, dv_{\xi, s}^{1,\varepsilon} \otimes v_{\zeta, s}^{2,\varepsilon}(\xi, \zeta) \, dx \, dy \, d\beta_k(s), \]

with \( f_{1,0}(x, \xi) = I_{\eta(x) > \xi}, f_{2,0}(y, \xi) = I_{\eta(y) > \zeta} \), \( \alpha = \rho(x - y)\psi(x - \zeta) \), \( v_{\xi, s}^{1,\varepsilon} = -\partial_\xi f_1(s, x, \xi) = \delta_{\psi(x,s)=\xi}, v_{\zeta, s}^{2,\varepsilon} = \partial_\zeta f_2(s, y, \zeta) = \delta_{\psi(y,s)=\zeta} \) and \( \chi_1(\xi, \zeta) = \int_{-\infty}^\xi \psi(\xi' - \zeta) \, d\xi' = \int_\zeta^\xi \psi(y) \, dy \).

**Proof:** Let \( \varphi_1 \in C^1_c(\mathbb{T}^N \times \mathbb{R}_x) \) and \( \varphi_2 \in C^1_c(\mathbb{T}^N \times \mathbb{R}_\zeta) \). For all \( t \in (0, 1) \), according to (16), it yields
\[ \langle f_1^+(t), \varphi_1 \rangle = \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) + F_1(t), \quad a.s., \]
with
\[ \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) = \langle f_{1,0}, \varphi_1 \rangle \delta_0([0, t]) + \varepsilon \int_0^t \langle f_1(s), a(\xi) \cdot \nabla_\xi \varphi_1(s) \rangle \, ds + \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi_1(x, \xi) G^2(x, \xi) \, dv_{\xi, s}^{1,\varepsilon}(\xi) \, dx \, ds - \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) \]

and
\[ F_1(t) = \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi_1(x, \xi) \, dv_{\xi, s}^{1,\varepsilon}(\xi) \, dx \, d\beta_k(s). \]

Similarly, by utilizing (21), we have
\[ \langle \tilde{f}_2^+(t), \varphi_2 \rangle = \langle \tilde{m}_2^*, \partial_\xi \varphi_2 \rangle([0, t]) + \tilde{F}_2(t), \]
where
\[ \langle \tilde{m}_2^*, \partial_\xi \varphi_2 \rangle([0, t]) = \langle \tilde{f}_{2,0}, \varphi_2 \rangle \delta_0([0, t]) - \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\zeta \varphi_2(y, \zeta) G^2(y, \zeta) \, dv_{\zeta, s}^{2,\varepsilon}(\zeta) \, dy \, ds + \langle m_2^*, \partial_\xi \varphi_2 \rangle([0, t]) \]

and
\[ \tilde{F}_2(t) = -\sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(y, \zeta) \varphi_2(y, \zeta) \, dv_{\zeta, s}^{2,\varepsilon}(\zeta) \, dy \, d\beta_k(s). \]

Clearly, \( F_1(t) \) and \( \tilde{F}_2(t) \) are continuous martingales, \( t \mapsto \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) \) and \( t \mapsto \langle \tilde{m}_2^*, \partial_\xi \varphi_2 \rangle([0, t]) \) are functions of finite variation. Moreover, it is shown in Remark 12 of Ref. [9] that \( \langle m_1^*, \partial_\xi \varphi_1 \rangle([0]) = \langle f_{1,0}, \varphi_1 \rangle \) and \( \langle \tilde{m}_2^*, \partial_\xi \varphi_2 \rangle([0]) = \langle \tilde{f}_{2,0}, \varphi_2 \rangle \).

Denote the duality distribution over \( \mathbb{T}^N \times \mathbb{R}_x \times \mathbb{T}_\nu \times \mathbb{R}_\zeta \) by \( \langle \cdot, \cdot \rangle \). Let \( \alpha(x, \xi, y, \zeta) = \varphi_1(x, \xi) \varphi_2(y, \zeta) \). Applying Itô formula to \( F_1(t) \tilde{F}_2(t) \), it yields
\[ F_1(t) \tilde{F}_2(t) = \int_0^t F_1(s) \, d\tilde{F}_2(s) + \int_0^t \tilde{F}_2(s) \, dF_1(s) + [F_1, \tilde{F}_2]_t, \]
where \([F_1, \tilde{F}_2]_t\) is the quadratic variation of \( F_1 \) and \( \tilde{F}_2 \) at time \( t \). Moreover, according to Proposition (4.5) on P6 in Ref. [22] and by using integration by parts for \( \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) \langle \tilde{m}_2^*, \partial_\xi \varphi_2 \rangle([0, t]) \), it yields that
\[ \langle m_1^*, \partial_\xi \varphi_1 \rangle([0, t]) \langle \tilde{m}_2^*, \partial_\xi \varphi_2 \rangle([0, t]) = \langle m_1^*, \partial_\xi \varphi_1 \rangle([0]) \langle \tilde{m}_2^*, \partial_\xi \varphi_2 \rangle([0]) \]
Since \( \tilde{F}_2 \) is continuous, we have
\[
\langle m_1^x, \partial_\xi \varphi_1 \rangle ([0, t]) \tilde{F}_2(t) = \int_0^t \langle m_1^x, \partial_\xi \varphi_1 \rangle ([0, s]) \, d \tilde{F}_2(s) + \int_0^t \tilde{F}_2(s) \langle m_1^x, \partial_\xi \varphi_1 \rangle (ds),
\]
and we have the similar formula for \( \langle \tilde{m}_2^x, \partial_\xi \varphi_2 \rangle ([0, t]) F_2(t) \).

Based on the above formulas and using (12), we obtain that
\[
\langle f_1^+(t), \varphi_1 \rangle (\tilde{F}_2^+(t), \varphi_2) = \langle \langle f_1^+(t) \tilde{f}_2^+(t), \alpha \rangle \rangle
\]
satisfies
\[
\langle \langle f_1^+(t) \tilde{f}_2^+(t), \alpha \rangle \rangle = \langle \langle f_1, \tilde{f}_2, 0, \alpha \rangle \rangle + \varepsilon \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 f_2 (a(\xi) - \nabla_x) \alpha \, d\xi \, d\zeta \, dx \, dy \, ds
- \frac{\varepsilon}{2} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(s, x, \xi) \partial_\alpha G^2(y, \zeta) \, d\xi \, d\zeta \, dy \, ds
+ \frac{\varepsilon}{2} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \tilde{f}_2(s, y, \zeta) \partial_\alpha G^2(x, \xi) \, d\xi \, d\zeta \, dx \, dy \, ds
- \varepsilon \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} G_{1,2}(x, y, \xi, \zeta) \alpha \, dv_{X,2}^{1,\varepsilon} \otimes v_{Y,2}^{2,\varepsilon} (\xi, \zeta) \, dx \, dy \, ds
+ \int_{(0, t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^-(s, x, \xi) \partial_\xi \alpha \, dm_2^x (y, \zeta, s) \, d\xi \, dx
- \int_{(0, t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \tilde{f}_2^+(s, y, \zeta) \partial_\xi \alpha \, dm_1^x (x, \xi, s) \, d\xi \, dy
- \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(s, x, \xi) g_k(y, \zeta) \alpha \, dv_{Y,2}^{2,\varepsilon} (\zeta) \, dy \, d\xi \, d\beta_k (s)
+ \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \tilde{f}_2(s, y, \zeta) g_k(x, \xi) \alpha \, dv_{X,2}^{1,\varepsilon} (\xi) \, dx \, d\xi \, d\beta_k (s)
=: \langle \langle f_1, \tilde{f}_2, 0, \alpha \rangle \rangle + \sum_{i=1}^{8} I_i(t), \quad \text{a.s.,}
\]
where \( G^2(x, \xi) = \sum_{k \geq 1} |g_k(x, \xi)|^2 \) and \( G_{1,2}(x, y, \xi, \zeta) = \sum_{k \geq 1} g_k(x, \xi) g_k(y, \zeta) \).

Similarly, we have
\[
\langle \langle f_1^+(t) \tilde{f}_2^+(t), \alpha \rangle \rangle = \langle \langle f_1, \tilde{f}_2, 0, \alpha \rangle \rangle + \varepsilon \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 f_2 (a(\xi) - \nabla_x) \alpha \, d\xi \, d\zeta \, dx \, dy \, ds
+ \frac{\varepsilon}{2} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(s, x, \xi) \partial_\alpha G^2(y, \zeta) \, d\xi \, d\zeta \, dy \, ds
- \frac{\varepsilon}{2} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_2(s, y, \zeta) \partial_\alpha G^2(x, \xi) \, d\xi \, d\zeta \, dx \, dy \, ds
\]
Noting that $C^1(T^N_x \times \mathbb{R}_k) \otimes C^1(T^N_y \times \mathbb{R}_\xi)$ is dense in $C^1_c(T^N_x \times \mathbb{R}_k \times T^N_y \times \mathbb{R}_\xi)$ and the assumption that $\alpha$ is compactly supported can be relaxed, thanks to (8), (14) and (19). By truncation, we can take $\alpha \in C^\infty_c(T^N_x \times \mathbb{R}_k \times T^N_y \times \mathbb{R}_\xi)$ compactly supported in a neighborhood of the diagonal

$$\{(x, \xi, x, \xi); x \in T^N_x, \xi \in \mathbb{R}\},$$

with the form $\alpha = \rho(x - y)\psi(\xi - \zeta)$, which implies the following remarkable identities

$$(\nabla_x + \nabla_y)\alpha = 0, \quad (\partial_\xi + \partial_\zeta)\alpha = 0. \quad (26)$$

From now on, we devote to making estimates of $I_i, \tilde{I}_i$, for $i = 1, \ldots, 8$. Clearly, it holds that

$$I_1(t) + \tilde{I}_1(t) = \varepsilon \int_0^t \int_{T^N_x} \int_{\mathbb{R}^2} (\tilde{f}_1 \tilde{f}_2 + \bar{f}_1 \bar{f}_2)(a(\xi) \cdot \nabla_x)\alpha \, d\xi \, d\zeta \, dx \, dy \, ds$$

$$=: I(t).$$

In view of (24), it holds that

$$I_5 = -\int_{(0,t]} \int_{T^N_x} \int_{\mathbb{R}^2} f_1^- (s, x, \xi) \partial_\xi \alpha \, dm^\xi_2 (y, \xi, s) \, d\xi \, dx$$

$$= -\int_{(0,t]} \int_{T^N_x} \int_{\mathbb{R}^2} \alpha \, dm^\xi_2 (y, \xi, s) \, dv_{x,s}^{1,\varepsilon,-}(\xi) \leq 0, \quad a.s.,$$

and

$$I_6 = \int_{(0,t]} \int_{T^N_x} \int_{\mathbb{R}^2} f_2^+ (y, s, \xi, \zeta) \partial_\zeta \alpha \, dm^\zeta_1 (x, \xi, s) \, d\zeta \, dy$$

$$= -\int_{(0,t]} \int_{T^N_x} \int_{\mathbb{R}^2} \alpha \, dm^\zeta_1 (x, \xi, s) \, dv_{y,s}^{2,\varepsilon,+}(\zeta) \leq 0, \quad a.s.$$

By the same method as above, we deduce that $\tilde{I}_5 + \tilde{I}_6 \leq 0$, a.s.

Moreover, it is readily to deduce that

$$I_2 + I_3 + I_4 = \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4$$
\[
\frac{\varepsilon}{2} \int_0^t \int_{(\mathbb{T}_N)^2} \int_{\mathbb{R}^2} \alpha (G^2(x, \xi) + G^2(y, \zeta) - 2G_{1,2}(x, y, \xi, \zeta)) \, dv_{\chi,s} \otimes v_{\chi,s} (\xi, \zeta) \, dx \, dy \, ds
\]

hence,
\[
\sum_{i=2}^4 (I_i + \tilde{I}_i) = \frac{\varepsilon}{2} \int_0^t \int_{(\mathbb{T}_N)^2} \int_{\mathbb{R}^2} \alpha \sum_{k \geq 1} |g_k(x, \xi) - g_k(y, \zeta)|^2 \, dv_{\chi,s} \otimes v_{\chi,s} (\xi, \zeta) \, dx \, dy \, ds =: J(t).
\]

Define \( \chi_1(\xi, \zeta) = \int_{-\infty}^{\xi} \psi (\xi' - \zeta) \, d\xi' \), then
\[
I_7(t) = -\frac{\sqrt{\varepsilon}}{2} \sum_{k \geq 1} \int_0^t \int_{(\mathbb{T}_N)^2} \int_{\mathbb{R}^2} f_1(s, x, \xi) g_k(y, \zeta) \rho(x - y) \partial_x \chi_1(\xi, \zeta) \, dv_{\chi,s} (\xi, \zeta) \, dx \, dy \, ds \tilde{\nu}_k(s)
\]

\[
= -\frac{\sqrt{\varepsilon}}{2} \sum_{k \geq 1} \int_0^t \int_{(\mathbb{T}_N)^2} \int_{\mathbb{R}^2} g_k(y, \zeta) \rho(x - y) \chi_1(\xi, \zeta) \, dv_{\chi,s} \otimes v_{\chi,s} (\xi, \zeta) \, dx \, dy \, ds \tilde{\nu}_k(s).
\]

Define \( \chi_2(\xi, \zeta) = \int_{\zeta}^{+\infty} \psi (\xi - \zeta') \, d\zeta' \), then
\[
I_8(t) = -\frac{\sqrt{\varepsilon}}{2} \sum_{k \geq 1} \int_0^t \int_{(\mathbb{T}_N)^2} \int_{\mathbb{R}^2} f_2(s, y, \zeta) g_k(x, \xi) \rho(x - y) \partial_x \chi_2(\xi, \zeta) \, dv_{\chi,s} (\xi, \zeta) \, dx \, dy \, ds \tilde{\nu}_k(s)
\]

\[
= \frac{\sqrt{\varepsilon}}{2} \sum_{k \geq 1} \int_0^t \int_{(\mathbb{T}_N)^2} \int_{\mathbb{R}^2} g_k(x, \xi) \rho(x - y) \chi_2(\xi, \zeta) \, dv_{\chi,s} \otimes v_{\chi,s} (\xi, \zeta) \, dx \, dy \, ds \tilde{\nu}_k(s).
\]

Since \( \chi_1(\xi, \zeta) = \chi_2(\xi, \zeta) = \int_{-\infty}^{\xi} \psi (y) \, dy \), we get
\[
I_7(t) + I_8(t) = \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \int_{(\mathbb{T}_N)^2} \int_{\mathbb{R}^2} (g_k(x, \xi) \\
- g_k(y, \zeta) \rho(x - y) \chi_1(\xi, \zeta) \, dv_{\chi,s} \otimes v_{\chi,s} (\xi, \zeta) \, dx \, dy \, ds \tilde{\nu}_k(s).
\]

Similarly, we deduce that
\[
\tilde{I}_7(t) + \tilde{I}_8(t) = \sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \int_{(\mathbb{T}_N)^2} \int_{\mathbb{R}^2} (g_k(x, \xi) \\
- g_k(y, \zeta) \rho(x - y) \chi_1(\xi, \zeta) \, dv_{\chi,s} \otimes v_{\chi,s} (\xi, \zeta) \, dx \, dy \, ds \tilde{\nu}_k(s).
\]

Thus, it yields
\[
\sum_{i=7}^8 (I_i + \tilde{I}_i) = 2\sqrt{\varepsilon} \sum_{k \geq 1} \int_0^t \int_{(\mathbb{T}_N)^2} \int_{\mathbb{R}^2} (g_k(x, \xi) \\
- g_k(y, \zeta) \rho(x - y) \chi_1(\xi, \zeta) \, dv_{\chi,s} \otimes v_{\chi,s} (\xi, \zeta) \, dx \, dy \, ds \tilde{\nu}_k(s)
\]

\[=: K(t).\]
Combining all the previous estimates, it follows that
\[
\int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x - y) \psi(\xi - \xi)(f_1^+(x, t, \xi)\tilde{f}_2^+(y, t, \xi) + \tilde{f}_1^+(x, t, \xi)f_2^+(y, t, \xi)) \, d\xi \, d\xi \, dx \, dy \\
\leq \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x - y) \psi(\xi - \xi)(f_{1,0}(x, \xi)f_{2,0}(y, \xi) + \tilde{f}_{1,0}(x, \xi)f_{2,0}(y, \xi)) \, d\xi \, d\xi \, dx \, dy \\
+ I(t) + J(t) + K(t), \quad a.s.
\]  
(27)
Taking \( t_n \uparrow t \), we have that (25) holds for \( f_i^+(t_n) \) and let \( n \to \infty \), we get that (25) holds for \( f_i^- (t) \). We complete the proof.

Now, we are ready to proceed with the proof of (18), which implies the main result of Theorem 3.1.

**Proposition 4.2:** For any \( \iota > 0 \), it holds that
\[
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \left( \| u^\varepsilon - v^\varepsilon \|_{L^1([0,1];L^1(\mathbb{T}^N))} > \iota \right) = -\infty.
\]  
(28)

**Proof:** Let \( \rho, \psi \) be approximations to the identity on \( \mathbb{T}^N \) and \( \mathbb{R} \), respectively. That is, let \( \rho \in C^\infty(\mathbb{T}^N), \psi \in C^\infty(\mathbb{R}) \) be symmetric non-negative functions such as \( \int_{\mathbb{T}^N} \rho = 1, \int_{\mathbb{R}} \psi = 1 \) and \( \text{supp} \psi \subset (-1, 1) \). We define
\[
\rho_\gamma(x) = \frac{1}{\gamma^N} \rho \left( \frac{x}{\gamma} \right), \quad \psi_\delta(\xi) = \frac{1}{\delta} \psi \left( \frac{\xi}{\delta} \right).
\]

Letting \( \rho := \rho_\gamma(x - y) \) and \( \psi := \psi_\delta(\xi - \xi) \) in Proposition 4.1, we get from (21) that
\[
\int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\gamma(x - y) \psi_\delta(\xi - \xi)(f_1^+(x, t, \xi)\tilde{f}_2^+(y, t, \xi) + \tilde{f}_1^+(x, t, \xi)f_2^+(y, t, \xi)) \, d\xi \, d\xi \, dx \, dy \\
\leq \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\gamma(x - y) \psi_\delta(\xi - \xi)(f_{1,0}(x, \xi)f_{2,0}(y, \xi) + \tilde{f}_{1,0}(x, \xi)f_{2,0}(y, \xi)) \, d\xi \, d\xi \, dx \, dy \\
+ \tilde{I}(t) + \tilde{J}(t) + \tilde{K}(t), \quad a.s.,
\]
where \( \tilde{I}, \tilde{J}, \tilde{K} \) are the corresponding \( I, J, K \) in the statement of Proposition 4.1 with \( \rho, \psi \) replaced by \( \rho_\gamma, \psi_\delta \), respectively. For simplicity, we still denote by \( \chi_1(\xi, \xi) \) with \( \psi \) replaced by \( \psi_\delta \).

For any \( t \in [0, 1] \), define the error term
\[
\mathcal{E}_t(\gamma, \delta) \\
:= \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} (f_1^+(x, t, \xi)\tilde{f}_2^+(y, t, \xi) + \tilde{f}_1^+(x, t, \xi)f_2^+(y, t, \xi))\rho_\gamma(x - y)\psi_\delta(\xi - \xi) \, dx \, dy \, d\xi \, d\xi \\
- \int_{\mathbb{T}^N} \int_{\mathbb{R}} (f_{1,0}(x, \xi)f_{2,0}(y, \xi) + \tilde{f}_{1,0}(x, \xi)f_{2,0}(y, \xi)) \psi_\delta(\xi - \xi) \, dx \, d\xi.
\]  
(29)

By utilizing \( \int_{\mathbb{R}} \psi_\delta(\xi - \xi) \, d\xi = 1, \int_{-\varepsilon}^{\varepsilon} \psi_\delta(\xi - \xi) \, d\xi = 1/2 \) and \( \int_{(\mathbb{T}^N)^2} \rho_\gamma(x - y) \, dx \, dy \leq 1 \), we deduce that
\[
\left| \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\gamma(x - y)f_1^+(x, t, \xi)\tilde{f}_2^+(y, t, \xi) \, d\xi \, dx \, dy \\
- \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \tilde{f}_1^+(x, t, \xi)f_2^+(y, t, \xi) \rho_\gamma(x - y)\psi_\delta(\xi - \xi) \, dx \, dy \, d\xi \right|
\]
Similarly,

\[
\int_{(TN)^2} \int_{\mathbb{R}} \rho_{\gamma}(x - y) \int_{\mathbb{R}} I_{u_\gamma \pm \delta(x,t) > \xi} \int_{\mathbb{R}} \psi_{\delta}(\xi - \zeta)(I_{u_\gamma \pm \delta(y,t) \leq \xi} - I_{u_\gamma \pm \delta(y,t) \leq \zeta}) \, d\zeta \, d\xi \, dx \, dy \\
\leq \frac{1}{2} \int_{(TN)^2} \int_{\mathbb{R}} \rho_{\gamma}(x - y) I_{u_\gamma \pm \delta(x,t) > \xi} \int_{\mathbb{R}} \psi_{\delta}(\xi - \zeta) I_{\zeta < v_\gamma \pm \delta(y,t) \leq \xi} \, d\zeta \, d\xi \, dx \, dy \\
+ \frac{1}{2} \int_{(TN)^2} \int_{\mathbb{R}} \rho_{\gamma}(x - y) I_{u_\gamma \pm \delta(x,t) > \xi} \int_{\mathbb{R}} \psi_{\delta}(\xi - \zeta) I_{\zeta < v_\gamma \pm \delta(y,t) \leq \xi} \, d\zeta \, d\xi \, dx \, dy \\
\leq \frac{1}{2} \int_{(TN)^2} \rho_{\gamma}(x - y) I_{u_\gamma \pm \delta(x,t) > v_\gamma \pm \delta(y,t)} \int_{v_\gamma \pm \delta(y,t)} \min(u_\gamma \pm \delta(x,t), v_\gamma \pm \delta(y,t)) \, d\xi \, dx \, dy \\
+ \frac{1}{2} \int_{(TN)^2} \rho_{\gamma}(x - y) I_{u_\gamma \pm \delta(x,t) > v_\gamma \pm \delta(y,t) - \delta} \int_{v_\gamma \pm \delta(y,t) - \delta} \min(u_\gamma \pm \delta(x,t), v_\gamma \pm \delta(y,t)) \, d\xi \, dx \, dy \\
= \frac{\delta}{2} \int_{(TN)^2} \rho_{\gamma}(x - y) I_{u_\gamma \pm \delta(x,t) > v_\gamma \pm \delta(y,t) + \delta} \, dx \, dy \\
+ \frac{1}{2} \int_{(TN)^2} \rho_{\gamma}(x - y) I_{u_\gamma \pm \delta(x,t) < u_\gamma \pm \delta(x,t)} \int_{v_\gamma \pm \delta(y,t) + \delta} \rho_{\gamma}(x - y) \psi_{\delta}(\xi - \zeta) \, dx \, dy \, d\xi \, d\zeta \\
\leq 2\delta, \quad a.s. \quad (31)
\]

Moreover, it follows that

\[
\int_{(TN)^2} \int_{\mathbb{R}} \rho_{\gamma}(x - y) f_1^\pm(x, t, \xi) f_2^\pm(y, t, \xi) \, dx \, dy \\
- \int_{(TN)^2} \int_{\mathbb{R}^2} \tilde{f}_1(x, t, \xi) f_2^\pm(y, t, \xi) \rho_{\gamma}(x - y) \psi_{\delta}(\xi - \zeta) \, dx \, dy \, d\xi \, d\zeta \leq 2\delta, \quad a.s. \quad (32)
\]
In view of the integrability of \( \Lambda_{t_2^+} \), it yields that for a countable sequence \( \gamma_n \downarrow 0 \), (30) holds a.s. for all \( n \); hence, passing to the limit \( n \rightarrow \infty \), we get

\[
\lim_{n \rightarrow \infty} \int_{(T^n)^2} \int_{\mathbb{R}} \rho_\gamma (x - y) f_{1+} (x, t, \xi) f_{2+} (y, t, \xi) \ dx \ dy
\]

\[
- \int_{T^n} \int_{\mathbb{R}} \tilde{f}_{1+} (x, t, \xi) f_{2+} (x, t, \xi) \ dx \ d\xi = 0, \ a.s.
\] (33)

Similarly, it holds that

\[
\lim_{n \rightarrow \infty} \int_{(T^n)^2} \int_{\mathbb{R}} \rho_\gamma (x - y) f_{1+} (x, t, \xi) f_{2+} (y, t, \xi) \ dx \ d\xi = 0, \ a.s.
\] (34)

By a similar argument, passing to the limit \( \delta \rightarrow 0 \), it follows from (28) to (32) that

\[
\lim_{n \rightarrow \infty} \mathcal{E}_t (\gamma_n, \delta_n) = 0, \ a.s.
\]

Without confusion, from now on, we write

\[
\lim_{\gamma, \delta \rightarrow 0} \mathcal{E}_t (\gamma, \delta) = 0, \ a.s.
\] (35)

In particular, when \( t = 0 \), it holds that

\[
\lim_{\gamma, \delta \rightarrow 0} \mathcal{E}_0 (\gamma, \delta) = 0.
\] (36)

In the following, we aim to make estimates of \( \tilde{I}(t) \), \( \tilde{J}(t) \) and \( \tilde{K}(t) \). We start with the estimation of \( \tilde{I}(t) \).

For any \( M > 0 \), define stopping times

\[
\tau_{1+}^{1+} := \inf \{ t \geq 0 : \text{ess sup} \ |u^\epsilon(s)|^{q_0+1}_{L^{q_0+1}(T^n)} > M \}, \quad \tau_{2+}^{1+} := \inf \{ t \geq 0 : \text{ess sup} \ |v^\epsilon(s)|^{q_0+1}_{L^{q_0+1}(T^n)} > M \}.
\]

Let \( \tau := \tau_{1+}^{1+} \wedge \tau_{2+}^{1+} \). Note that for any \( 0 \leq t \leq 1 \),

\[
\tilde{I}(t \wedge \tau) = \epsilon \int_0^{t \wedge \tau} \int_{(T^n)^2} \int_{\mathbb{R}^2} f_{1+} f_{2+} (\alpha) \ dx \ d\xi \ ds
\]

\[
+ \epsilon \int_0^{t \wedge \tau} \int_{(T^n)^2} \int_{\mathbb{R}^2} \tilde{f}_{1+} f_{2+} (\alpha) \ dx \ d\xi \ ds
\]

\[
=: \tilde{I}_1(t) + \tilde{I}_2(t).
\]

By Hypothesis H, we know that \( a(\cdot) \) is polynomial growth with degree \( q_0 \), then \( |a(\xi)| \leq \mathcal{N} (q_0) (1 + |\xi|^{q_0}) \) with \( \mathcal{N} (q_0) < \infty \). As a result, it yields

\[
|\tilde{I}_1(t \wedge \tau)| \leq \epsilon \mathcal{N} (q_0) \int_0^{t \wedge \tau} \int_{(T^n)^2} \int_{\mathbb{R}^2} f_{1+} f_{2+} (1 + |\xi|^{q_0}) \psi_{b} (\xi - \zeta) |\nabla_{\xi} \rho_\gamma (x - y)| \ d\xi \ ds, \ a.s.
\]

Define

\[
\Gamma (\xi, \zeta) = \int_{-\infty}^{\xi} \int_{\xi}^{\xi} (1 + |\xi'|^{q_0}) \psi_{b} (\xi' - \zeta') \ d\xi' d\xi',
\]
then
\[
|\tilde{I}_1(t)| \leq \varepsilon N(q_0) \int_0^{t \wedge T} \int_{\mathbb{T}^N} |\nabla_x \rho_y(x - y)| \int_{\mathbb{T}^2} \Gamma(\xi, \zeta) \, dv_{x,s}^{1,\varepsilon} \otimes v_{y,s}^{2,\varepsilon}(\xi, \zeta) \, dx \, dy \, ds, \quad a.s.
\]

Clearly, it yields
\[
\Gamma(\xi, \zeta) \leq \int_\xi^\infty \int_{|\xi''| < \delta, \xi'' < \xi, -\xi'} (1 + |\xi''|^q_0 + |\xi'|^{q_0}) \psi_\delta(\xi'') \, d\xi'' \, d\zeta'
\]
\[
\leq \int_\xi^{\xi + \delta} (1 + |\delta|^{q_0} + |\xi'|^{q_0}) \left( \int_{\mathbb{R}} \psi_\delta(\xi'') \, d\xi'' \right) \, d\zeta'
\]
\[
\leq \int_\xi^{\xi + \delta} (1 + |\delta|^{q_0} + |\xi'|^{q_0}) \, d\zeta'
\]
\[
\leq C(q_0)(1 + |\xi|^q_0 + 1 + |\xi|^{q_0} + 1 + |\delta|^{q_0} + 1).
\]

Then, we deduce that
\[
|\tilde{I}_1(t)|
\]
\[
\leq \varepsilon N(q_0) C(q_0) \int_0^{t \wedge T} \int_{\mathbb{T}^N} |\nabla_x \rho_y(x - y)| \int_{\mathbb{R}^2} (1 + |\xi|^q_0 + 1 + |\xi|^{q_0} + 1 + |\delta|^{q_0} + 1) \, d\xi \, dx \, dy \, ds
\]
\[
+ |\delta|^{q_0} \int_{\mathbb{R}^2} v_{x,s}^{1,\varepsilon} \otimes v_{y,s}^{2,\varepsilon}(\xi, \zeta) \, d\xi \, dx \, dy \, ds
\]
\[
\leq \varepsilon \gamma^{-1} N(q_0) C(q_0)(1 + |\delta|^{q_0} + 1 + 4 \varepsilon \gamma^{-1} N(q_0) C(q_0) M, \quad a.s.
\]

For \( \tilde{I}_2(t) \), we have the same estimation as \( \tilde{I}_1(t) \). Hence, we conclude that
\[
|\tilde{I}(t)| \leq 2 \varepsilon \gamma^{-1} N(q_0) C(q_0)(1 + |\delta|^{q_0} + 1 + 4 \varepsilon \gamma^{-1} N(q_0) C(q_0) M, \quad a.s. \quad (37)
\]

By (6) in Hypothesis H, we arrive at
\[
\tilde{J}(t) = \varepsilon \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \alpha \sum_{k \geq 1} |g_k(x, \xi) - g_k(y, \zeta)|^2 \, dv_{x,s}^{1,\varepsilon} \otimes v_{y,s}^{2,\varepsilon}(\xi, \zeta) \, dx \, dy \, ds
\]
\[
\leq \varepsilon D_1 \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \rho_y(x - y)|x - y|^2 \, \int_{\mathbb{R}^2} \psi_\delta(\xi - \zeta) \, dv_{x,s}^{1,\varepsilon} \otimes v_{y,s}^{2,\varepsilon}(\xi, \zeta) \, dx \, dy \, ds
\]
\[
+ \varepsilon D_1 \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \rho_y(x - y) \psi_\delta(\xi - \zeta) |\xi - \zeta|^2 \, dv_{x,s}^{1,\varepsilon} \otimes v_{y,s}^{2,\varepsilon}(\xi, \zeta) \, dx \, dy \, ds
\]
\[
=: \tilde{J}_1(t) + \tilde{J}_2(t).
\]

Note that
\[
\int_{\mathbb{R}^2} \psi_\delta(\xi, \zeta) \, dv_{x,s}^{1,\varepsilon} \otimes v_{y,s}^{2,\varepsilon}(\xi, \zeta) \leq \delta^{-1}, \quad a.s.,
\]
\[
\int_{\mathbb{T}^N} \rho_y(x - y)|x - y|^2 \, dx \, dy \leq \gamma^2.
\]

It follows that
\[
\tilde{J}_1(t) \leq \varepsilon D_1 \delta^{-1} \gamma^2, \quad a.s. \quad (38)
\]
Referring to (35) in Ref. [9], it yields
\[
\mathcal{J}_2 \leq \varepsilon \delta D_1 \int_0^t \int_{\mathbb{T}^N_t} \int_{|\xi - \zeta| \leq \delta} \rho_\gamma(x - y) \psi_\delta(\xi - \zeta) |\bar{\zeta}|^{-1} \, dx \, dy \, ds
\]
\[
\leq \varepsilon \delta D_1 C_\psi, \quad \text{a.s.,}
\]
where \( C_\psi := \sup_{\xi \in \mathbb{R}} \| \psi(\xi) \| \). In view of (36) and (37), we arrive at
\[
\mathcal{J}(t) \leq \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta, \quad \text{a.s.}
\]
Combining all the above estimates, we conclude that
\[
\int_{\mathbb{T}^N_t} \int_{\mathbb{R}^2} \rho_\gamma(x - y) \psi_\delta(\xi - \zeta) (\bar{f}_1 \pm (x, t, \xi) \bar{f}_2 \pm (y, t, \xi) + \bar{f}_1 \pm (x, t, \xi) \bar{f}_2 \pm (y, t, \xi)) \, dx \, dy \, ds
\]
\[
x \leq \int_{\mathbb{T}^N_t} \int_{\mathbb{R}^2} \rho_\gamma(x - y) \psi_\delta(\xi - \zeta) (\bar{f}_{1,0} (x, \xi) \bar{f}_{2,0} (y, \xi) + \bar{f}_{1,0} (x, \xi) \bar{f}_{2,0} (y, \xi)) \, dx \, dy \, ds
\]
\[
+ 2 \varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) (1 + |\delta|^{q_0+1}) + \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta
\]
\[
+ 2 \varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) \left( \text{ess sup}_{0 \leq s \leq t} \| u^\varphi \|_{L_0^{q_0+1}(\mathbb{T}^N_t)} \right) + \text{ess sup}_{0 \leq s \leq t} \| v^\varphi \|_{L_0^{q_0+1}(\mathbb{T}^N_t)} + \bar{K}(t), \quad \text{a.s.}
\]
For any \( s \in (0, 1) \), denote by
\[
R(s) := \int_{\mathbb{T}^N_t} \int_{\mathbb{R}^2} \rho_\gamma(x - y) \psi_\delta(\xi - \zeta) (\bar{f}_1 \pm (x, s, \xi) \bar{f}_2 \pm (y, s, \xi) + \bar{f}_1 \pm (x, s, \xi) \bar{f}_2 \pm (y, s, \xi)) \, dx \, dy \, ds.
\]
Then, we deduce from (38) that
\[
\text{ess sup}_{0 \leq s \leq t} \| R(s) \| \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} (\bar{f}_{1,0} \bar{f}_{2,0} + \bar{f}_{1,0} \bar{f}_{2,0}) \, dx \, dy + \mathcal{E}_0(\gamma, \delta)
\]
\[
+ 2 \varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) (1 + |\delta|^{q_0+1}) + \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta
\]
\[
+ 4 \varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) M + \sup_{0 \leq s \leq t} \| \bar{K}(s) \|, \quad \text{a.s.,}
\]
where \( \lim_{\gamma, \delta \to 0} \mathcal{E}_0(\gamma, \delta) = 0 \).

Furthermore, by Hölder inequality, it gives that
\[
\left( \mathbb{E} \left[ \text{ess sup}_{0 \leq s \leq t, \tau} \| R(s) \| \right] \right)^{1/p} \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} (\bar{f}_{1,0} \bar{f}_{2,0} + \bar{f}_{1,0} \bar{f}_{2,0}) \, dx \, dy + \mathcal{E}_0(\gamma, \delta)
\]
\[
+ 2 \varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) (1 + |\delta|^{q_0+1}) + \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta
\]
\[
+ 4 \varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) M + \left( \mathbb{E} \left[ \sup_{s \in [0,t] \cap \mathbb{T}} \| \bar{K}(s) \| \right] \right)^{1/p}.
\]
To estimate the stochastic integral term, we will use the following remarkable result from Refs. [23,24] that there exists a universal constant \( C_0 \) such that for any \( p \geq 2 \) and for any continuous martingale \( M_t \) with \( M_0 = 0 \),
\[
\mathbb{E}(|M_t|^p) \leq C_0 \frac{p}{2} \mathbb{E}(M_t)^{p/2},
\]
where \( M^* = \sup_{s \in [0, T]} |M_s| \).

Utilizing (41), we derive that

\[
E \left| \sup_{s \in [0, T]} |\bar{K}(s)| \right|^p
\]

\[
= e^{p/2} E \left| \sup_{s \in [0, T]} \int_0^s \int_{\mathbb{R}^2} \chi_1(\xi, \zeta) \rho_\gamma(x - y)(g_k(x, \xi) - g_k(y, \zeta)) \, d\nu^1_{x,r} \otimes d\nu^2_{y,r} (\xi, \zeta) \, dx \, dy \, d\beta_k(r) \right|^p
\]

\[
\leq e^{p/2} p^{p/2} C_0^{p/2} D_2^{p/2} E \left[ \int_0^{t \wedge \tau} \int_{\mathbb{R}^2} |x - y| + |\xi - \zeta| |\rho_\gamma(x - y)\chi_1(\xi, \zeta) \, d\nu^1_{x,r} \otimes d\nu^2_{y,r} (\xi, \zeta) \, dx \, dy |^2 \, dr \right]^{p/2}. \tag{43}
\]

Recall (4) in Hypothesis H, it gives

\[
|g_k(x, \xi) - g_k(y, \zeta)| \leq C_k^1(|x - y| + |\xi - \zeta|), \quad \sum_{k \geq 1} |C_k^1|^2 \leq \frac{D_1}{2} := D_2,
\]

hence, by (42), we deduce that

\[
E \left| \sup_{s \in [0, T]} |\bar{K}(s)| \right|^p
\]

\[
\leq e^{p/2} p^{p/2} C_0^{p/2} D_2^{p/2} E \left[ \int_0^{t \wedge \tau} \int_{\mathbb{R}^2} |x - y| + |\xi - \zeta| |\rho_\gamma(x - y)\chi_1(\xi, \zeta) \, d\nu^1_{x,r} \otimes d\nu^2_{y,r} (\xi, \zeta) \, dx \, dy |^2 \, dr \right]^{p/2}. \tag{44}
\]

Since \( \chi_1(\xi, \zeta) \leq 1 \), it yields

\[
\int_{\mathbb{R}^2} |x - y| |\rho_\gamma(x - y)\chi_1(\xi, \zeta) \, d\nu^1_{x,r} \otimes d\nu^2_{y,r} (\xi, \zeta) \, dx \, dy \leq \gamma, \quad a.s.
\]

Taking into account that \( \nu_{\gamma,r}^1(\xi) = \delta_{u_\gamma(x,r)=\xi}, \nu_{\gamma,r}^2(\xi) = \delta_{v_\gamma(y,r)=\xi} \), and by Corollary 2.3, it follows that

\[
E \left| \sup_{s \in [0, T]} |\bar{K}(s)| \right|^p
\]

\[
\leq e^{p/2} p^{p/2} C_0^{p/2} D_2^{p/2} E \left[ \int_0^{t \wedge \tau} \gamma + \int_{\mathbb{R}^2} |u_{\gamma,r}^\pm - v_{\gamma,r}^\pm| |\rho_\gamma(x - y) \, dx \, dy |^2 \, dr \right]^{p/2}
\]

\[
= e^{p/2} p^{p/2} C_0^{p/2} D_2^{p/2} E \left[ \int_0^{t \wedge \tau} \gamma + \int_{\mathbb{R}^2} |u_{\gamma,r}^\pm - v_{\gamma,r}^\pm| |\rho_\gamma(x - y) \, dx \, dy |^2 \, dr \right]^{p/2}. \tag{44}
\]

With the help of the following identities

\[
\int_{\mathbb{R}} I_{\gamma,r > \xi} I_{\nu_{\gamma,r} > \xi} \, d\xi = (u_{\gamma,r}^\pm - v_{\gamma,r}^\pm)^+ \quad \text{and} \quad \int_{\mathbb{R}} I_{\gamma,r > \xi} I_{\nu_{\gamma,r} > \xi} \, d\xi = (u_{\gamma,r}^\pm - v_{\gamma,r}^\pm)^-,
\]

we deduce that

\[
\int_{\mathbb{R}^2} |u_{\gamma,r}^\pm(x, r) - v_{\gamma,r}^\pm(y, r)| \rho_\gamma(x - y) \, dx \, dy
\]
where we have used (28) and (29). Combining (43) and (45), we deduce that for $0 < t < 1$, it holds that

\[
\mathbb{E}\left(\mathbb{E}\left[\left|\mathbb{K}(s)\right|^p\right]^{1/p}\right) \leq \varepsilon^p/p^2 C_0^{p/2} D_2^{p/2} \mathbb{E}\left[\int_0^{t\wedge\tau} \left|\gamma + 4\delta + R(r)\right|^2 \, dr\right]^{p/2}
\]

\[
\leq \varepsilon^p/p^2 C_0^{p/2} D_2^{p/2} 2^{p/2} \mathbb{E}\left[\int_0^{t\wedge\tau} \left|\gamma + 4\delta\right|^2 \, dr + \int_0^{t\wedge\tau} |R(r)|^2 \, dr\right]^{p/2}
\]

\[
\leq \varepsilon^p/p^2 C_0^{p/2} D_2^{p/2} 2^p |\gamma + 4\delta|^p + \varepsilon^p/p^2 C_0^{p/2} D_2^{p/2} 2^p \mathbb{E}\left(\int_0^{t\wedge\tau} R^2(r) \, dr\right)^{p/2}.
\]

(47)

Then, it follows from (39) and (46) that

\[
\left(\mathbb{E}\left[\mathbb{E}\left[\left|\mathbb{K}(s)\right|^p\right]^{1/p}\right]^{1/p}\right) \leq \int_\mathbb{R} \int_{\mathbb{R}^2} (f_1,0,\bar{f}_1,0,2,0) \, d\xi \, dx + \mathcal{E}_0(\gamma, \delta) + 2\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) (1 + |\delta| q_0^{q_0+1})
\]

\[+ \varepsilon D_1 \gamma^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta + 4\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) M + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta| + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} \left(\mathbb{E}\left(\int_0^{t\wedge\tau} R^2(r) \, dr\right)^{p/2}\right)^{1/p}.
\]

For any $p \geq 2$, by Minkowski’s integral inequality, it holds that

\[
\left(\mathbb{E}\left[\int_0^{t\wedge\tau} R^2(r) \, dr\right]^{p/2}\right)^{1/p} = \left[\mathbb{E}\left(\int_0^{t\wedge\tau} R^2(r) \, dr\right)^{p/2}\right]^{1/2}
\]

\[
\leq \left[\int_0^{t\wedge\tau} \left(\mathbb{E} R^p(r)\right)^{2/p} \, dr\right]^{1/2}
\]

\[
\leq \left[\int_0^{t\wedge\tau} \left(\mathbb{E} \mathbb{E}\left[\left|\mathbb{K}(s)\right|^p\right]^{1/p}\right)^{2/p} \, dr\right]^{1/2}.
\]

Thus, we reach

\[
\left(\mathbb{E}\left[\mathbb{E}\left[\left|\mathbb{K}(s)\right|^p\right]^{1/p}\right]^{1/p}\right)^{2/p} \leq D^2 \int_\mathbb{R} \int_{\mathbb{R}^2} (f_1,0,\bar{f}_1,0,2,0) \, d\xi \, dx + \mathcal{E}_0(\gamma, \delta) + 2\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) (1 + |\delta| q_0^{q_0+1})
\]

\[+ \varepsilon D_1 \gamma^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta + 4\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) M + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta| + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} \left(\mathbb{E}\left(\int_0^{t\wedge\tau} R^2(r) \, dr\right)^{p/2}\right)^{1/p}.
\]
\[ + \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta + 4\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) M + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta|^2 \]
\[ + D^2 e^{D^2 p C_0 D_2} \int_0^t \left( \mathbb{E} \left[ \text{ess sup } R(s) \right]^p \right)^{2/p} \, \text{dr}, \] (48)

where \( \mathcal{D} \) is defined in Section 2. Let \( G(t) := (\mathbb{E} \text{ess sup } R(s)^p)^{2/p} \), applying Gronwall inequality to (47), we get

\[ G(t) \leq D^2 e^{D^2 p C_0 D_2} \left[ \int_{T_R} \int_{R} \left( f_{1,0} + \tilde{f}_{1,0} \right) \, dx \, d\xi + \mathcal{E}_0(\gamma, \delta) + 2\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0)(1 + |\delta|^{q_0+1}) \right. \]
\[ + \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta + 4\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) M + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta|^2 \]. (49)

which implies that

\[ \left( \mathbb{E} \left[ \text{ess sup } R(s)^p \right] \right)^{1/p} \leq e^{D^2 p C_0 D_2} \left[ \int_{T_R} \int_{R} \left( f_{1,0} + \tilde{f}_{1,0} \right) \, dx \, d\xi + \mathcal{E}_0(\gamma, \delta) + 2\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0)(1 + |\delta|^{q_0+1}) \right. \]
\[ + \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta + 4\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) M + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta|^2 \]. (50)

Recall the definition of \( R(s) \), it holds that

\[ \left( \mathbb{E} \left[ \text{ess sup } \int_{(T_R)^2} \int_{(R)^2} \rho(\xi - \xi') \psi(\xi - \xi) f_{1,0} + \tilde{f}_{1,0} (s, y, \xi) \right] \right)^{1/p} \]
\[ \leq e^{D^2 p C_0 D_2} \left[ \int_{T_R} \int_{R} \left( f_{1,0} + \tilde{f}_{1,0} \right) \, dx \, d\xi + \mathcal{E}_0(\gamma, \delta) + 2\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0)(1 + |\delta|^{q_0+1}) \right. \]
\[ + \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta + 4\varepsilon \gamma^{-1} \mathcal{N}(q_0) C(q_0) M + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta|^2 \]. (51)

Applying the same procedure to \( f_{2,0} \) and \( \tilde{f}_{2,0} \) (in this case, \( A = 0 \) and \( f_{2,0} \int_{(T_R)^2} \int_{(R)^2} \rho(\xi - \xi') \psi(\xi - \xi) f_{2,0} + \tilde{f}_{2,0} \, dx \, d\xi = 0 \), we obtain

\[ \left( \mathbb{E} \left[ \text{ess sup } \int_{(T_R)^2} \int_{(R)^2} \rho(\xi - \xi') \psi(\xi - \xi) f_{2,0} + \tilde{f}_{2,0} (s, y, \xi) \right] \right)^{1/p} \]
\[ \leq e^{D^2 p C_0 D_2} \left[ \mathcal{E}_0(\gamma, \delta) + \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta|^2 \right]. \]
For the sake of convenience, denote by

\[ Q(s) := \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R})^2} \rho_\gamma(x - y) \psi_\gamma(\xi - \zeta) \left(f_1^+(s, x, \xi) f_2^+(s, y, \zeta) + f_1^-(s, x, \xi) f_2^-(s, y, \zeta)\right) \, d\xi \, dx \, dy, \]

then, it yields

\[
\left( \mathbb{E} \left| \text{ess sup}_{0 \leq s \leq 1} |Q(s)|^p \right|^{1/p} \right) \lesssim e^{D^2 \varepsilon C_0 D_2} \times \left[ \mathcal{E}_0(\gamma, \delta) + \varepsilon D_1 \delta^{-1} \gamma^2 + \varepsilon D_1 C_\psi \delta + \varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta| \right]. \tag{52}
\]

On the other hand, from (27), it follows that

\[
\left( \mathbb{E} \left| \text{ess sup}_{0 \leq s \leq 1} \int_{\mathbb{T}^N} \int_{\mathbb{R}} \left(f_1^+(s, x, \xi) f_2^+(s, y, \zeta) + f_1^-(s, x, \xi) f_2^-(s, y, \zeta)\right) \, d\xi \, dx \right|^p \right)^{1/p} \lesssim \left( \mathbb{E} \left| \text{ess sup}_{0 \leq s \leq 1} |\mathcal{E}_s(\gamma, \delta)|^p \right| \right)^{1/p} + \left( \mathbb{E} \left| \text{ess sup}_{0 \leq s \leq 1} |R(s)|^p \right| \right)^{1/p}. \tag{53}
\]

In the following, we devote to making estimates of \( (\mathbb{E} \left| \text{ess sup}_{0 \leq s \leq 1} |\mathcal{E}_s(\gamma, \delta)|^p \right|^{1/p} \). For any \( s \in (0, 1) \), we have

\[
\mathcal{E}_s(\gamma, \delta) = \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R})^2} \left(f_1^+(s, x, \xi) f_2^+(y, s, \zeta) + f_1^-(s, x, \xi) f_2^-(y, s, \zeta)\right) \rho_\gamma(x - y) \psi_\delta(\xi - \zeta) \, dx \, dy \, d\xi \, d\zeta
\]

\[
- \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R})^2} \left(f_1^+(s, x, \xi) \bar{f}_2^+(s, x, \xi) + \bar{f}_1^+(s, x, \xi) f_2^-(s, x, \xi)\right) \, d\xi \, dx
\]

\[
= \left[ \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R})^2} \rho_\gamma(x - y) \left(f_1^+(s, x, \xi) f_2^+(y, s, \zeta) + f_1^-(s, x, \xi) f_2^-(y, s, \zeta)\right) \, dx \, dy \, d\xi \right]
\]

\[
- \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R})^2} \left(f_1^+(s, x, \xi) \bar{f}_2^+(s, x, \xi) + \bar{f}_1^+(s, x, \xi) f_2^-(s, x, \xi)\right) \, d\xi \, dx
\]

\[
= H_1 + H_2.
\]

By (28) and (29), it gives

\[
|H_2| \leq 4\delta, \quad \text{a.s.} \tag{54}
\]

Moreover, it is easy to deduce that

\[
|H_1| \leq \left| \int_{(\mathbb{T}^N)^2} \rho_\gamma(x - y) \int_{\mathbb{R}} I_{\tilde{f}_r^+(s, x, \xi) > \xi} \left( I_{\tilde{f}_r^+(s, x, \xi) \leq \xi} - I_{\tilde{f}_r^+(y, s, \zeta) \leq \xi} \right) \, d\xi \, dx \right|
\]

\[
+ \left| \int_{(\mathbb{T}^N)^2} \rho_\gamma(x - y) \int_{\mathbb{R}} I_{\tilde{f}_r^+(s, x, \xi) \leq \xi} \left( I_{\tilde{f}_r^+(s, x, \xi) > \xi} - I_{\tilde{f}_r^+(y, s, \zeta) > \xi} \right) \, d\xi \, dx \right|
\]
Utilizing (28) and (29) again, it follows that
\[
\int_{(TN)^2} \rho_y(x - y)|v^{\pm,\pm}(x, s) - v^{\pm,\pm}(y, s)| \, dx \, dy
\]
\[
= \int_{(TN)^2} \int_{\mathbb{R}} \rho_y(x - y)(f_2^{\pm}(x, s, \xi)\tilde{f}_2^{\pm}(y, s, \xi) + f_1^{\pm}(x, s, \xi)f_2^{\pm}(y, s, \xi)) \, d\xi \, dx \, dy
\]
\[
\leq \int_{(TN)^2} \int_{\mathbb{R}} \rho_y(x - y)\psi_1(\xi - \zeta)(f_2^{\pm}(x, s, \xi)\tilde{f}_2^{\pm}(y, s, \xi) + f_1^{\pm}(x, s, \xi)f_2^{\pm}(y, s, \xi)) \, d\xi \, d\zeta \, dx \, dy + 4\delta
\]
\[
= Q(s) + 4\delta, \quad a.s.
\]
Then,
\[
|H_1| \leq 2Q(s) + 8\delta, \quad a.s. \tag{55}
\]
Collecting (52) and (53), it yields
\[
|E_\delta(\gamma, \delta)| \leq 2Q(s) + 12\delta, \quad a.s.,
\]
hence, by (50), we deduce that
\[
\left(\mathbb{E}|\text{ess sup}_{0 \leq s \leq 1} |E_\delta(\gamma, \delta)||^p\right)^{1/p} \leq \left(\mathbb{E}|\text{ess sup}_{0 \leq s \leq 1} Q(s)|^p\right)^{1/p} + \delta
\]
\[
\leq e^{D^2pC_0D_2} \left[ E_0(\gamma, \delta) + \epsilon D_1 \delta^{-1} \gamma^2 + \epsilon D_1 C_\psi \delta + \epsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta| \right] + \delta. \tag{56}
\]
Combining (48) and (54), we deduce from (51) that
\[
\left(\mathbb{E}\left|\text{ess sup}_{0 \leq s \leq 1 \wedge T} \int_{TN} |(f_1^{\pm}(s, x, \xi)\tilde{f}_2^{\pm}(s, x, \xi) + f_1^{\pm}(s, x, \xi)f_2^{\pm}(s, x, \xi))| \, d\xi \, dx\right|^p\right)^{1/p} \leq e^{D^2pC_0D_2} \left[ \int_{TN} |(f_1, 0, 0, 0) + \tilde{f}_1, 0, f_2, 0)| \, d\xi \, dx + 2E_0(\gamma, \delta) + 2\epsilon \gamma^{-1} N(q_0)C(q_0)(1 + |\delta|^{q_0+1})
\right.
\]
\[
+ 2\epsilon D_1 \delta^{-1} \gamma^2 + 4\epsilon D_1 C_\psi \delta + 2\epsilon \gamma^{-1} N(q_0)C(q_0)M + 2\epsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta| \right] + \delta.
\]
Note that we have \(f_1^{\pm} = I_{U^{\pm,\pm} > \xi} \) and \(f_2^{\pm} = I_{U^{\pm,\pm} > \xi} \) with initial data \(f_1, 0 = I_{\eta > \xi} \) and \(f_2, 0 = I_{\eta > \xi} \), respectively. With the help of (43), we deduce that
\[
\left(\mathbb{E}\left|\text{ess sup}_{0 \leq s \leq 1 \wedge T} \|U^{\pm,\pm}(s) - v^{\pm,\pm}(s)\|_{L^1(TN)}\right|^p\right)^{1/p} \leq r(\epsilon, p, \gamma, \delta), \tag{57}
\]
where
\[
r(\epsilon, p, \gamma, \delta) := e^{D^2 p C_0 D_2} \left[ \int_{TN} |(f_1, 0, 0, 0) + \tilde{f}_1, 0, f_2, 0)| \, d\xi \, dx + 2E_0(\gamma, \delta) + 2\epsilon \gamma^{-1} N(q_0)C(q_0)(1 + |\delta|^{q_0+1})
\right.
\]
\[
+ 2\epsilon D_1 \delta^{-1} \gamma^2 + 4\epsilon D_1 C_\psi \delta + 2\epsilon \gamma^{-1} N(q_0)C(q_0)M + 2\epsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta| \right] + \delta.
\]
+ 2\varepsilon D_1 \delta^{-1} \gamma^2 + 2\varepsilon D_1 C_\psi \delta + 4\varepsilon \gamma^{-1} N(q_0)C(q_0)M + 2\varepsilon^{1/2} p^{1/2} C_0^{1/2} D_2^{1/2} |\gamma + 4\delta| \right] + \varepsilon. \quad (58)

Taking

\delta = \gamma = \varepsilon^{1/2}

and letting \( p = 1/\varepsilon \), by (34) and (40), we have

\[
 r(\varepsilon, p, \gamma, \delta) = e^{D^2 C_0 D_2} \left[ 2\varepsilon^2(\gamma, \delta) + 2N(q_0)C(q_0)\varepsilon^{1/2}(1 + \varepsilon(q_0 + 1/2)) + 2D_1 \varepsilon^{3/2} + 2D_1 C_\psi \varepsilon^{3/2} + 4\varepsilon^{1/2} N(q_0)C(q_0)M + 10\varepsilon^{1/2} C_0^{1/2} D_2^{1/2} \right] + \varepsilon^{1/2}
\to 0, \quad \text{as} \quad \varepsilon \to 0.
\]

Therefore, we deduce from (55) that

\[
 \left( \mathbb{E} \left[ \text{ess sup}_{0 \leq s \leq 1 \wedge \tau} \| u^{\varepsilon, \pm}(s) - v^{\varepsilon, \pm}(s) \|_{L^1(T^N)}^p \right] \right)^{1/p} \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]

By using Chebyshev inequality and (55), for any \( \ell > 0 \), we deduce that

\[
 \varepsilon \log \mathbb{P} \left( \| u^{\varepsilon} - v^{\varepsilon} \|_{L^1([0,1 \wedge \tau]; L^1(T^N))} > \ell \right)
\leq \varepsilon \log \left( \mathbb{E} \left( \| u^{\varepsilon, \pm} - v^{\varepsilon, \pm} \|_{L^1([0,1 \wedge \tau]; L^1(T^N))}^p \right) / \ell^p \right)
\leq - \log \ell + \log \left( \mathbb{E} \left[ \text{ess sup}_{0 \leq s \leq 1 \wedge \tau} \| u^{\varepsilon, \pm}(s) - v^{\varepsilon, \pm}(s) \|_{L^1(T^N)}^p \right] \right)^{1/p}
\to -\infty, \quad \text{as} \quad \varepsilon \to 0.
\]

Note that

\[
 \mathbb{P} \left( \| u^{\varepsilon} - v^{\varepsilon} \|_{L^1([0,1 \wedge \tau]; L^1(T^N))} > \ell \right)
\leq \mathbb{P} \left( \| u^{\varepsilon} - v^{\varepsilon} \|_{L^1([0,1 \wedge \tau]; L^1(T^N))} > \ell, \text{ess sup}_{0 \leq t \leq 1} \| u^{\varepsilon}(t) \|_{L^{q_0+1}(T^N)}^{q_0+1} \leq M, \text{ess sup}_{0 \leq t \leq 1} \| v^{\varepsilon}(t) \|_{L^{q_0+1}(T^N)}^{q_0+1} \leq M \right)
\]

\[
+ \mathbb{P} \left( \text{ess sup}_{0 \leq t \leq 1} \| u^{\varepsilon}(t) \|_{L^{q_0+1}(T^N)}^{q_0+1} > M \right) + \mathbb{P} \left( \text{ess sup}_{0 \leq t \leq 1} \| v^{\varepsilon}(t) \|_{L^{q_0+1}(T^N)}^{q_0+1} > M \right)
\leq \mathbb{P} \left( \| u^{\varepsilon} - v^{\varepsilon} \|_{L^1([0,1 \wedge \tau]; L^1(T^N))} > \ell \right) + \mathbb{P} \left( \text{ess sup}_{0 \leq t \leq 1} \| u^{\varepsilon}(t) \|_{L^{q_0+1}(T^N)}^{q_0+1} > M \right)
\]

\[
+ \mathbb{P} \left( \text{ess sup}_{0 \leq t \leq 1} \| v^{\varepsilon}(t) \|_{L^{q_0+1}(T^N)}^{q_0+1} > M \right).
\]

Due to (3.3) and (4.2), for any \( R > 0 \), there exits a constant \( M \) such that

\[
 \sup_{0 < \varepsilon \leq 1} \varepsilon \log \mathbb{P} \left( \text{ess sup}_{0 \leq t \leq 1} \| u^{\varepsilon}(t) \|_{L^{q_0+1}(T^N)}^{q_0+1} > M \right) \leq -R,
\]
\[ \sup_{0 \leq \varepsilon \leq 1} \varepsilon \log \mathbb{P} \left( \operatorname{esssup}_{0 \leq t \leq 1} \| v^\varepsilon(t) \|_{L^{q_0+1}(\mathbb{T}^d)}^{q_0+1} > M \right) \leq -R. \]

For such \( M \), (4.40) holds, then for the above \( R > 0 \), there exists a \( \varepsilon_0 \) such that for any \( \varepsilon \) satisfying \( 0 < \varepsilon \leq \varepsilon_0 \),

\[ \mathbb{P} \left( \| u^\varepsilon - v^\varepsilon \|_{L^1([0,1] \wedge \tau; L^1(\mathbb{T}^d))} > \iota \right) \leq e^{-R \varepsilon}. \]

Thus, for any \( R > 0 \), there exists a constant \( \varepsilon_0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \),

\[ \mathbb{P} \left( \| u^\varepsilon - v^\varepsilon \|_{L^1([0,1] \wedge \tau; L^1(\mathbb{T}^d))} > \iota \right) \leq 3e^{-R \varepsilon}. \]

By the arbitrary of \( R \), we complete the proof. ■

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