D-dimensional metrics with D-3 symmetries

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Abstract

Hidden symmetry transformations of D-dimensional vacuum metrics with D − 3 commuting Killing vectors are studied. We solve directly the Einstein equations in the Maison formulation under additional assumptions. We relate the 4-dimensional Reissner-Nordström solution to a particular case of the 5-dimensional Gross-Perry metric.

1 Introduction

Stationary vacuum Einstein equations admit the symmetry group SL(2, R) [1, 2]. This symmetry was generalized to an action of the group SL(D − 2, R) in a class of D-dimensional vacuum metrics with D − 3 commuting Killing vectors [3]. In the case D = 5 this group contains the group SO(1, 2) which preserves asymptotical flatness of a metric [4]. For instance, using this action one can reproduce the Myers-Perry solution from the Schwarzschild-Tangherlini metric [4].

In this paper we investigate in detail the action of the SL(D − 2, R) symmetry group when integral submanifolds of the Killing vectors are spacelike or timelike. We identify relevant parameters of this action and we discuss corresponding changes of the metric signature. We solve directly the Einstein equations in D = 5 assuming two commuting Killing vectors and additional symmetries which are not isometries. We give an example of the SL(3, R) symmetry transformation which generates the Reissner-Nordström 4-dimensional solution, with a dyonic electromagnetic field, from the 5-dimensional Gross-Perry metric [6] of the Euclidean signature.

In the considered class of solutions there are near horizon metrics of extremal black holes [5] and metrics obtained by Clément (see [7] and references therein).
2 Generation method for reduced vacuum Einstein equations

Let $M$ be $(n + 2)$-dimensional manifold with metric $g$ admitting $n - 1$ commuting Killing vectors which define a non-null integrable distribution. In special coordinates $x^i$, $i = 1, \ldots, n - 1$, and $x^n$, $a = n, n + 1, n + 2$, the Killing vectors are $\partial_i$ and the metric takes the form

$$g = g_{ij}(dx^i + A^i)(dx^j + A^j) + \tau^{-1}\tilde{g}_{ab}dx^a dx^b,$$

(1)

where $\tau = |\det g_{ij}|$, $A^i = A^i_a dx^a$ and functions $g_{ij}$, $A^i_a$, $\tilde{g}_{ab}$ do not depend on coordinates $x^i$. The vacuum Einstein equations for metrics (1) are equivalent to the following equations

$$d(\chi^{-1} * d\chi) = 0,$$

(2)

$$\tilde{R}_{ab} = \frac{1}{4} \text{Tr}(\chi^{-1} \chi_{,a} \chi^{-1} \chi_{,b})$$

(3)

for $\tilde{g}_{ab}$ and $n \times n$ symmetric matrix $\chi$ constrained by the conditions

$$\det \chi = \pm 1, \quad \varepsilon_{nn} < 0,$$

(4)

where $\varepsilon = \text{sgn}(|\det g_{ij}|)$. Here $\tilde{R}_{ab}$ is the Ricci tensor of the metric $\tilde{g} = \tilde{g}_{ab} dx^a dx^b$ and $*$ denotes the Hodge dualization with respect to this metric. The matrix $\chi$ is related to components of (1) via the equations

$$\chi = \begin{pmatrix}
g_{ij} - \frac{\varepsilon}{\tau} V_i V_j & \frac{1}{\tau} V_i \\
\frac{1}{\tau} V_j & -\frac{\varepsilon}{\tau}
\end{pmatrix},$$

(5)

$$dV_i = \tau g_{ij} * dA^j.$$  

(6)

Note that equation (5) is integrable by virtue of (2) and that

$$\text{sgn}(|\det g_{ij}|) = -\varepsilon \det \chi.$$  

(7)

Equations (2)-(4) are preserved by the transformation

$$\chi \mapsto \chi' = \varepsilon' S^T \chi S$$

(8)

where $S \in SL(n, \mathbb{R})$ is a constant matrix and value of $\varepsilon' = \pm 1$ is fixed by the condition

$$\varepsilon' \varepsilon (S^T \chi S)_{nn} < 0.$$  

(9)

Transformations given by (8)-(9) can be used to obtain new vacuum metrics from known ones. They generalize the Ehlers transformation [1] for stationary 4-dimensional metrics. In dimension 5 they contain the group $SO(1, 2)$ preserving asymptotical flatness of metrics [4].
3 Relevant parameters and change of signature

Most of parameters in $S$ (symmetry) do not change the seed metric in a nontrivial way. Any matrix $S$ with $S^n_n \neq 0$ can be uniquely decomposed into a product of three matrices $S = S_0HT$, where

$$S_0 = \begin{pmatrix} \delta^i_j & \alpha^i \\ 0 & 1 \end{pmatrix},$$

$$H = \begin{pmatrix} \beta^i_j & 0 \\ 0 & (\det \beta^i_j)^{-1} \end{pmatrix}, \quad T = \begin{pmatrix} \delta^i_j & 0 \\ \gamma_j & 1 \end{pmatrix}.$$

The matrix $T$ yields translations of $V_i$ by constants $-\varepsilon \gamma_i$. Its action on a seed metric is trivial. The matrix $H$ corresponds to a linear transformation of $x^i$ and $A^i$ combined with a multiplication of the full metric $g$ by $(\det \beta^i_j)^{-2}$. Thus, modulo coordinate transformations, $H$ is a homothety and can be replaced by

$$H_0 = \begin{pmatrix} \beta \delta^i_j & 0 \\ 0 & \beta^{1-n} \end{pmatrix}$$

with an appropriate constant $\beta$. The only nontrivial action is that of the matrix $S_0$ (true symmetry),

$$S_0^T \chi S_0 = \begin{pmatrix} g_{ij} - \varepsilon V_i V_j \\ \alpha_j - \varepsilon V_j (\alpha^k V_k - \varepsilon) \end{pmatrix},$$

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_0^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $S^n_n = 0$ then $S$ is equivalent, modulo $H$ and $T$, to one of the matrices $S_l$, $l = 1 \ldots n - 1$, given by

$$S^n_i = \begin{cases} 1 & \text{if } i = n - l \\ 0 & \text{if } i > n - l \end{cases}, \quad S^n_j = \begin{cases} -1 & \text{if } j = n - l \\ 0 & \text{if } j \neq n - l \end{cases}.$$

Note that $S_l$ contains $n - l - 1$ free parameters (components $S^n_i$ for $i < n - l$). Summarizing, without loss of generality, symmetry transformations (8) can be reduced to the action of one of matrices $S_0, S_l$ composed with $H_0$, the latter equivalent to a homothety of $g$.

Transformations (8) can change signature of $g_{ij}$, and hence the signature of (1). Let the initial signature of $g_{ij}$ be $(p, q)$, where $p$ denotes the multiplicity of the value $+1$. If $\varepsilon' > 0$ then transformation (8) preserves this
signature. If \( \varepsilon' < 0 \) then the signature of the final metric \( g_{ij} \) is \( (q + \varepsilon, p - \varepsilon) \). Note that if \( (p, \varepsilon) = (0, 1) \) or \( (q, \varepsilon) = (0, -1) \) then \( \varepsilon' > 0 \). For instance, if we start with a 5-dimensional metric of the Lorentz signature \( (-++++) \) then the transformed metric has the same signature or the Euclidean one. Transformation (13) adds two parameters to the seed metric. (They become dependent, \( \alpha_2 = \frac{1}{2}\alpha_1^2 \), if the asymptotical flatness is to be preserved [4].) Transformations \( S_l \) take the form

\[
S_1 = \begin{pmatrix}
1 & 0 & \alpha \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix},
\]

(15)

\[
S_2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

(16)

4 Solutions with 2-dimensional space of constant curvature

Let us assume that \( \chi = \chi(z) \) depends only on one coordinate \( z \) and metric \( \tilde{g} \) has the following form

\[
\tilde{g} = dz^2 + f(z)g^{(k)},
\]

(17)

where \( g^{(k)} \) is a 2-dimensional metric of constant curvature \( k = 0, \pm 1 \) and signature ++ or +−,

\[
g^{(k)} = \frac{4(d\tilde{x}^2 + \varepsilon d\tilde{y}^2)}{(1 + k(\tilde{x}^2 + \varepsilon \tilde{y}^2))^2}.
\]

(18)

Note that \( z \) can be shifted by a constant and metric (17) can be multiplied by another constant since this transformation can be compensated by a change of coordinates \( x^i \). Thus, \( \tilde{g} \) can be simplified by means of transformations

\[
z \mapsto cz + c_0, \quad f \mapsto c^{-2}f, \quad c, c_0 = \text{const}.
\]

(19)

The Ricci tensor of \( \tilde{g} \) reads

\[
\text{Ricci}(\tilde{g}) = \frac{(f_z)^2 - 2ff_{zz}z^2 + \left( k - \frac{f_{zz}}{2} \right) g^{(k)}}{2f^2}.
\]

(20)
It follows from (3) that
\[ f = k z^2 + a z + b, \quad a, b = \text{const} \] (21)
and
\[ \text{Tr} \left( \chi^{-1} \chi_z \right)^2 = 2a^2 - 8kb. \] (22)
A double integration of (2) gives
\[ \chi = \chi_0 \exp \left( (w + w_0)C \right), \] (23)
where \( \chi_0, C \) are constant matrices such that
\[ \chi_0 = \chi_0^T, \quad \chi_0 C = C^T \chi_0, \quad \text{Tr} \ C = 0, \quad \text{Tr} \ C^2 = 2a^2 - 8kb \] (24)
and \( w(z) \) is a particular solution of
\[ w_z = \pm f^{-1}. \] (25)
The constant \( w_0 \) and the sign in (25) can be arbitrarily chosen.

One can classify functions \( f \) and \( w \) by putting them into a canonical form. First, let us note that by virtue of (19) one can transform \( f \) into one of the following expressions labelled by \( k \) and a new index \( k' \)
\[ f^{(k,k')} = \begin{cases} z & \text{if } k = k' = 0 \\ kz^2 + k' & \text{if } k^2 + k'^2 \neq 0. \end{cases} \] (26)
Using (26) we find particular solutions of (25). They are presented in the table

| \( w(z) \) | conditions | \( \text{Tr} \ C^2 \) |
|---|---|---|
| i) \( \log |z| \) | \( k = k' = 0 \) | 2 |
| ii) \( \text{arctanh} \ z \) | \( k' = -k \neq 0, \ z^2 < 1 \) | 8 |
| iii) \( \text{arccoth} \ z \) | \( k' = -k \neq 0, \ z^2 > 1 \) | 8 |
| iv) \( z \) | \( k = 0, \ k' \neq 0 \) | 0 |
| v) \( 1/z \) | \( k \neq 0, \ k' = 0 \) | 0 |
| vi) \( \text{arccot} \ z \) | \( k' = k \neq 0 \) | -8 |

The symmetry (8) induces transformations \( C \mapsto S^{-1}CS \) and \( \chi_0 \mapsto S^T \chi_0 S \) which allow us to reduce matrices \( C \) and \( \chi_0 \) to simpler forms. First, we put matrix \( C \) into a canonical Jordan form. Then, we find \( \chi_0 \) satisfying (24) and we simplify it by means of matrices which commute with \( C \). Below we present results of this procedure in the case of 5-dimensional metrics. Then
\( n = 2 \) and there are four canonical forms of \( C \) and \( \chi \), in which \( \alpha, \beta = \text{const} \) and \( \varepsilon, \varepsilon_i = \pm 1 \).

\[
C = \begin{pmatrix}
\alpha & \beta & 0 \\
-\beta & \alpha & 0 \\
0 & 0 & -2\alpha
\end{pmatrix}, \quad \chi = \begin{pmatrix}
-e^{\alpha w} \sin(\beta w) & e^{\alpha w} \cos(\beta w) & 0 \\
e^{\alpha w} \cos(\beta w) & e^{\alpha w} \sin(\beta w) & 0 \\
0 & 0 & -e^{-2\alpha w}
\end{pmatrix},
\]

\( \text{Tr} \ C^2 = 6\alpha^2 - 2\beta^2, \ \beta \neq 0 \).

(28)

\[
C = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & -\alpha - \beta
\end{pmatrix}, \quad \chi = \begin{pmatrix}
\varepsilon_1 e^{\alpha w} & 0 & 0 \\
0 & \varepsilon_2 e^{\beta w} & 0 \\
0 & 0 & -e^{-(\alpha+\beta) w}
\end{pmatrix},
\]

(29)

\( \text{Tr} \ C^2 = 2(\alpha^2 + \alpha\beta + \beta^2) \)

\[
C = \begin{pmatrix}
\alpha & 1 & 0 \\
0 & \beta & 0 \\
0 & 0 & -2\alpha
\end{pmatrix}, \quad \chi = \begin{pmatrix}
0 & \varepsilon_1 e^{\alpha w} & 0 \\
\varepsilon_1 e^{\alpha w} & \varepsilon_1 w e^{\alpha w} & 0 \\
0 & 0 & -e^{-2\alpha w}
\end{pmatrix},
\]

(30)

\( \text{Tr} \ C^2 = 6\alpha^2 \)

\[
C = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad \chi = -\varepsilon \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & w \\
1 & w & \frac{1}{2}w^2
\end{pmatrix}, \quad \text{Tr} \ C^2 = 0 \)
\]

(31)

Note that case (28), with parameters in an appropriate range, can be merged with any function from table (27), cases (29) and (30) admit functions i)-v) while (31) is only compatible with functions iv)-v).

5 Example

As an example we consider metrics related to (29) with \( k = -k' \neq 0 \). In this case we obtain

\[
g = \varepsilon_1 \left| \frac{z + 1}{z - 1} \right|^{\alpha/2} (dx^1)^2 + \varepsilon_2 \left| \frac{z + 1}{z - 1} \right|^{\beta/2} (dx^2)^2 \\
\quad + \left| \frac{z + 1}{z - 1} \right|^{-(\alpha+\beta)/2} (dz^2 + k(z^2 - 1)g^{(k)}), \)
\]

(32)

where \( \alpha^2 + \alpha\beta + \beta^2 = 4 \). If \( \varepsilon_1 = \varepsilon_2 = \varepsilon = k = 1 \), \( \alpha = -\beta = -2 \) and we introduce a multiplicative constant in \( g \) we obtain the Euclidean Gross-Perry solution

\[
g = \left( \frac{\rho - q}{\rho + q} \right)^2 (dx^1)^2 + \left( \frac{\rho + q}{\rho - q} \right)^2 (dx^2)^2 + \frac{1}{4} \left( 1 - \frac{q^2}{\rho^2} \right)^2 (d\rho^2 + \rho^2 d\Omega^2), \)
\]

(33)
where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), \( q = \text{const} \) and \( \rho = q(z + \sqrt{z^2 - 1}) \). The metric can be transformed by means of with

\[
S = -\frac{1}{2q} \begin{pmatrix}
\sqrt{2(m^2 - q^2)} & m + q & m - q \\
-\sqrt{2(m^2 - q^2)} & -m + q & -m - q \\
\sqrt{2(m^2 - q^2)} & \sqrt{2(m^2 - q^2)} & \sqrt{2(m^2 - q^2)}
\end{pmatrix}
\]

(34)

For \( \varepsilon' = -1 \) this transformation yields the following 5-dimensional vacuum solution

\[
g = \left( dx^1 - \sqrt{2}Q \cos \theta d\phi + \frac{\sqrt{2}Q}{r} dt \right)^2 - \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega^2,
\]

(35)

where \( t = x^2, \rho = r - m + \sqrt{r^2 - 2mr + Q^2} \) and \( q = \sqrt{m^2 - Q^2} \). Within the Kaluza-Klein approach the latter metric decomposes into the 4-dimensional Reissner-Nordström solution

\[
g' = -\left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega^2
\]

(36)

and electromagnetic field given by

\[
A = -\sqrt{2}Q \cos \theta d\phi + \frac{\sqrt{2}Q}{r} dt.
\]

(37)

This field represents a magnetic and electric monopoles of the same strength and placed at the same point.

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