Supergravity corrections to \((g-2)_\mu\) in differential renormalization

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Abstract

The method of differential renormalization is extended to the calculation of the one–loop graviton and gravitino corrections to \((g-2)_\mu\) in unbroken supergravity. Rewriting the singular contributions of all the diagrams in terms of only one singular function, \(U(1)\) gauge invariance and supersymmetry are preserved. We compare this calculation with previous ones which made use of momentum space regularization (renormalization) methods.
1 Introduction

The method of differential renormalization (DR) has appeared recently [1]. It is a method of renormalization in coordinate space that yields directly finite amplitudes and leaves the space–time dimension unchanged. Thus it is expected to be especially suited for supersymmetric theories, where the dimension of the space-time appears to be crucial for preserving the symmetry. DR has already been successfully applied in several contexts: the Wess–Zumino model [2], lower–dimensional [3] and non–abelian [4] gauge theories, two–loop QED [5], chiral models [6], non–relativistic anyon models [7] and curved space–time [8]. Other formal aspects of the method have been developed in [9, 10] and different versions of DR can be found in [11]. It is the purpose here to push even further the method and tackle a relatively complex problem such as the computation of $(g - 2)_l$ in supergravity (SUGRA). This is the first time that DR is applied to the calculation of a physical observable and to the calculation of gravitational corrections. As we will show our results preserve supersymmetry (SUSY) and abelian gauge invariance.

Although SUSY is not an exact symmetry of Nature, it is believed that any fundamental theory must be originally supersymmetric. It is also known that when SUSY is made local it naturally includes gravity [12]. The resulting theory is non–renormalizable but it is constrained by the symmetries. One of the few finite calculations in gravity is the one–loop correction to the anomalous magnetic moment of the lepton $(g - 2)_l$ [13]. In SUGRA not only is $(g - 2)_l$ finite but SUSY requires it to be zero [14]. Therefore, the anomalous magnetic moment of the lepton besides being an observable, is also an ideal arena to check theoretical implications and to perform consistency tests of the methods of regularization in SUGRA.

In a supersymmetric theory $(g - 2)_l$ vanishes because no such term appears in the Lagrangian of a chiral supermultiplet [14]. (This has been generalized to a set of sum rules valid for any charged supersymmetric multiplet [15, 16].) Hence, as long as SUSY is preserved, all quantum corrections must cancel order by order. Ferrara and Remiddi also proved explicitly that in global SUSY the one–loop QED corrections, order $e^3$, do cancel. In this case two diagrams contribute (see Fig. 1), one exchanging a photon and the lepton, and the other a photino and the corresponding sleptons, and both are finite.

The one–loop gravitational corrections are of order $\kappa^2 = 8\pi G_N$, resulting from a graviton or gravitino exchange. Using dimensional regularization [17], Berends and Gastmans calculated the five diagrams where a graviton is exchanged (see Fig. 2) [13].
Figure 1: Diagrams of order $e^3$ contributing to $(g - 2)\ell$ in superQED.

Figure 2: Diagrams of order $e\kappa^2$ contributing to $(g - 2)\ell$ in SUGRA. A graviton is exchanged in diagrams D1-D5 and a gravitino in D6-D10.

All five diagrams are infinite but their sum is finite. The finiteness of $(g - 2)\ell$ in a non–renormalizable theory such as gravity seemed miraculous. Del Aguila et al. [18] and Bellucci et al. [19] checked that when gravitation is embedded in a supersymmetric theory (unbroken), the contributions from the graviton and the gravitino cancel, as required by SUSY. Bellucci et al. also traced back to an effective chiral symmetry in the gravitino sector the finiteness of the gravitino contribution and then of the graviton contribution, if their sum has to vanish. Dimensional regularization does not yield a vanishing value for $(g - 2)\ell$. This is not so surprising for dimensional regularization is known to break SUSY. A (one–loop) SUSY preserving method such as dimensional reduction [21] is then required in order to obtain such a cancellation and this was shown to be the case. However, whether the finite contributions to $(g - 2)\ell$ of the graviton and gravitino sectors are well–defined quantities for some unknown reason (symmetry)
is a question which remained unanswered. The lack of manageable, alternative regularization methods preserving SUSY has prevented to address this question although the answer is expected to be negative as hinted in [13, 21, 13].

In this paper we use DR to calculate the one–loop corrections to $(g−2)_l$ in unbroken SUGRA. We find that in DR the individual contributions of the graviton and gravitino sectors, although finite and opposite, are different from previous results. Thus, although SUSY is preserved, these two contributions are not well–defined separately, only their sum is.

The method of DR is based on the observation that in coordinate space singularities arise when points in the quantum fields coincide. The idea is to write singular expressions as derivatives of less singular functions. One solves some differential equations and considers the solutions as a definition of the amplitudes in the sense of distributions, i.e., the derivatives are understood to act on test functions. In fact the method gives a definition of the expressions at the ill–defined points. The constants that naturally appear in the differential equations play the role of renormalization scales. The resulting finite amplitudes satisfy the renormalization group equations. While retaining the spirit of the method, we have developed some aspects that may be considered unsatisfactory in the original version. For instance, gauge invariance had to be checked at each step of the computation and the renormalization scales chosen accordingly. In this paper we show that it is possible to preserve gauge invariance (no term proportional to the photon momentum is generated at one loop) and supersymmetry ($(g−2)_l$ has no one–loop corrections) if related singularities are treated always in the same way throughout the computations. Only one scale for each type of singularity is then needed. In particular, for the computation of $(g−2)_l$ only the scale corresponding to logarithmic singularities appears. Obviously it cancels out when all diagrams are summed up. These new features of the DR approach will be more extensively discussed in a simpler context in a forthcoming publication [22].

The plan of the paper is as follows. In Section 3 the lagrangian describing a minimal superQED–SUGRA theory is presented. Section 4 is devoted to describing the constrained differential renormalization procedure which we will follow. Section 4 contains the detailed calculation of one diagram, where the main techniques used in the paper are applied, whereas in Section 5 we discuss briefly the calculation of the remaining diagrams contributing to $(g−2)_l$ and give the final results. We finish with the conclusions in Section 6. Three Appendices gather the Feynman rules in Euclidean space and other technical details.
2 The SUGRA Lagrangian

The coupling of matter to supergravity has been extensively studied in the literature [23]. We are interested here only in first-order gravitational corrections. We shall use the lagrangian of superQED–SUGRA obtained by imposing canonical kinetic terms (minimal Kähler potential and \( f \) function) and expanding the curved metric around the Minkowski one (\( \eta_{\alpha\beta} \)). The interaction lagrangian in Minkowski space reads

\[
\mathcal{L}_{ee\gamma+\bar{e}e\gamma+e\bar{e}\gamma} = -e \bar{\Psi} A \Psi - \{ i e A^\mu \phi_L^\dagger_\mu \phi_L \}
- e \sqrt{2} (\lambda \phi_L^\dagger_\mu \phi_L + \text{h.c.}) + (L \leftrightarrow R) ,
\]

\[
\mathcal{L}_{eeg+ee\bar{g}} = -\frac{k}{4} h^{\alpha\beta} [i \bar{\Psi} (\gamma_\alpha \partial_\beta + \gamma_\beta \partial_\alpha) \Psi + \text{h.c.}]
- 2 \bar{e} \bar{L} (\gamma_\alpha A_\beta + \gamma_\beta A_\alpha) \Psi ,
\]

\[
\mathcal{L}_{ee\tilde{g}+e\bar{e}\tilde{g}} = -\frac{k}{\sqrt{2}} [\tilde{\chi}^\nu L (i \not\partial - m) \phi_L^\dagger_\nu \Psi
+ e \tilde{\chi}^\nu L A^\alpha_\beta L \gamma_\nu \Psi + \text{h.c.}] + (L \leftrightarrow R) ,
\]

\[
\mathcal{L}_{\gamma\gamma g+\gamma\tilde{g}} = \kappa [h^{\alpha\beta} (F_{\alpha\beta} F^\mu_\mu - \frac{1}{4} \eta_{\alpha\beta} F_{\mu\nu} F^{\mu\nu})
+ (\tilde{\lambda} \gamma^\nu [\not\partial, A] \chi_\nu + \text{h.c.})] ,
\]

where \( \not\partial \equiv \not\partial - \not\partial \), with \( \not\partial \) acting on the function on the left, and \( \mathcal{P}_{L,R} = \frac{1}{2} (1 \pm \gamma_5) \) are the chiral projectors. The kinetic terms are the canonical ones, with the photon in the Feynman gauge and the graviton in the de Donder gauge. We use the following notation for particles and their corresponding fields:

lepton \( \rightarrow e \), \( \Psi \), slepton \( \rightarrow \tilde{e}_{L,R} \), \( \phi_{L,R} \),
photoin \( \rightarrow \gamma \), \( A_\mu \), photino \( \rightarrow \tilde{\gamma} \), \( \lambda \),
graviton \( \rightarrow g \), \( h_{\mu\nu} \), gravitino \( \rightarrow \tilde{g} \), \( \chi_\mu \).

The lagrangian is written in Minkowski space for easier comparison with previous work. However we shall work in Euclidean space as is usual in DR. The corresponding Feynman rules are collected in Appendix A.

3 Constrained differential renormalization

The \( e^3 \) and \( e\kappa^2 \) corrections to the lepton-lepton-photon vertex in SUGRA are given by the diagrams in Figs. 1 and 2 respectively. Their expressions contain many different singular pieces. To obtain renormalized expressions, each singular piece must be substituted by a regular one according to the DR prescription. It is crucial, however, to perform all such renormalizations in a consistent way if the symmetries are to be
preserved. In the literature this is taken care of by imposing certain relations among
the renormalization scales introduced in every DR replacement. For instance, Ward
identities in QED require that the ‘logarithmic’ and ‘quadratic’ renormalization scales
in the one–loop vacuum polarization be equal, while the scales in the fermion selfenergy,
\( M_\Sigma \), and the vertex correction, \( M_V \), must satisfy the equality \( \log \frac{M_\Sigma}{M_V} = \frac{1}{4} \).

Our approach here is to proceed in such a way that symmetries are automatically
preserved. This can be accomplished if all singular pieces are written in terms of
a minimal set of singular functions, which are renormalized afterwards. This ensures
that related singularities are treated in an identical way, no matter in which diagram or
position they appear. Within this scheme, differential renormalization should not break
the symmetries of the bare theory. In Ref. [22] it is shown, for instance, that with this
approach the vertex Ward identity in spinorial QED and scalar QED is automatically
satisfied to the one–loop level. Here, the final result will be directly compatible with
both supersymmetry and \( U(1) \) gauge invariance.

This ‘constrained’ differential renormalization relates diagrams of different topol-
ygy. This is done ‘separating’ the points of diagrams with a smaller number of propa-
gators. Let us classify the diagrams in Figs. 1 and 2 in two classes according to their
topology:

i) **Triangular diagrams**: they are products of three propagators with at most four
derivatives. Translation invariance and a systematic use of the Leibnitz rule allow
to express them in terms of a set of functions defined as

\[
T[O] \equiv \Delta(x)\Delta(y)[O^x \Delta(x - y)],
\]

where \( x \equiv x_1 - x_3, y \equiv x_2 - x_3 \) and \( x_i \) are the coordinates of the triangle
vertices. \( \Delta(x) \) is a scalar propagator and \( O^x \) is a differential operator acting on
\( x \) of order \( \leq 4 \). Each of these propagators can be either massless or massive.
(In the following, with the exception of Eq. (3.6) below which is analogous to
Eq. (3.1), but for bubble diagrams, we reserve the notation \( \Delta(x) \) for massless
propagators.) In our calculation we find T–functions with one massless and two
massive propagators,

\[
T_1[O] \equiv \Delta_m(x)\Delta_m(y)[O^x \Delta(x - y)],
\]

and T–functions with two massless and one massive propagators,

\[
T_2[O] \equiv \Delta(x)\Delta(y)[O^x \Delta_m(x - y)],
\]

where
where \( \Delta(x) = \frac{\pi}{2} \frac{1}{x^2} \) and \( \Delta_m(x) = \frac{1}{4\pi^2} \frac{mK_1(mx)}{x} \) fulfill the propagator equations

\[
\Box^x \Delta(x) = -\delta(x), \tag{3.4}
\]

\[
(\Box^x - m^2) \Delta_m(x) = -\delta(x), \tag{3.5}
\]

and \( K_1 \) are modified Bessel functions \(^{[24]}\). The mass structure does not affect the singular behaviour. Hence, as far as renormalization is concerned, \( T_1 \) and \( T_2 \) can be considered identical (although some care is needed when both massless and massive DR identities are used, as discussed in Section \[^5\].

**ii) Bubble diagrams:** they are products of two propagators and a delta function, and contain at most two derivatives. They can be written in terms of a set of functions whose general expression, omitting the index for any massive propagator, is

\[
B_y[O] \equiv \delta(y) \Delta(x) O^x \Delta(x), \tag{3.6}
\]

and analogously for \( B_x \), with \( x \leftrightarrow y \). \( O^x \) acts on \( x \) and is now of order \( \leq 2 \). In our calculation B–functions always contain one massive and one massless propagators. Therefore the notation

\[
B_y[O] \equiv \delta(y) \Delta(x) O^x \Delta_m(x), \tag{3.7}
\]

\[
B_x[O] \equiv \delta(x) \Delta(y) O^y \Delta_m(y), \tag{3.8}
\]

can be used in what follows.

Although the structure of B–functions and T–functions is apparently different —and \textit{a priori} they could be renormalized differently—, they are in fact related. B–functions can be expressed in terms of T–functions using the propagator equality \(^{[3,8]}\). This substitution ‘separates’ points and allows to express a bubble diagram as a triangular one. In this way one is able to express all diagrams (and then all singularities) in terms of T–functions only. The procedure is described in detail in Section \[^4\]. There it is also proven that all the relevant singular T–functions can be written using one single singular function: \( T[\Box] \). Therefore just one renormalization will be eventually required and only one arbitrary scale parameter will have to be introduced. Although the full correction to \((g - 2)_l\) is finite and does not depend on the renormalization scale, we calculate the contribution of each diagram separately in order to compare with previous results.
4 A detailed example

In this Section we present in detail the evaluation of the contribution to \((g - 2)_l\) of diagram D6 in Fig. 2. Using the Feynman rules in Appendix A, this vertex correction reads (all indices, including the \(\gamma\)-matrix ones, are in Euclidean space)

\[
V^{(6)}_{\mu}(x_1, x_2, x_3) = 2i\kappa^2 \frac{\gamma_{\alpha} \left( \partial^{x_1} - m \right) \mathcal{P}_R \Delta_m(x_1 - x_3) \partial^\mu_{\mu} \Delta_m(x_3 - x_2) \leftrightarrow^{x_3}}{4} \times \left[ \gamma_{\beta} \partial^{x_2 - x_1} \gamma_{\alpha} \Delta(x_2 - x_1) \mathcal{P}_L(\partial - m) \gamma_{\beta} \right].
\]

(4.1)

The factor 2 comes from the fact that the scalar particle propagating in the diagram can be either \(\tilde{e}_L\) or \(\tilde{e}_R\). Translation invariance allows to write \(V^{(6)}_{\mu}\) as a function of \(x = x_1 - x_3\) and \(y = x_2 - x_3\) only:

\[
V^{(6)}_{\mu}(x_1, x_2, x_3) = V^{(6)}_{\mu}(x, y) = \frac{i\kappa^2}{2} \left[ \partial^x \Delta(x - y) \right] \times \left[ \gamma_{\alpha} \left( \partial^y - m \right) \mathcal{P}_R \gamma_{\tau} \gamma_{\alpha} \times (\partial^y + m) \gamma_{\beta} \right] \left( \Delta_m(x) \Delta_m(y) \right) \right],
\]

(4.2)

where \(\partial^\rho = \partial^\rho_\mu - \partial^\rho_\beta\). The terms containing a \(\gamma_5\) do not contribute to \((g - 2)_l\) and hereafter will be ignored. As explained in Section 3 all vertex corrections can be written in terms of derivatives of T–functions. Using the Leibnitz rule and the properties of the \(\gamma\)-matrices, we find

\[
V^{(6)}_{\mu}(x, y) = i\kappa^2 \left\{ (m^2 - 2\partial^x \cdot \partial^y) \partial^\mu \gamma_{\beta} - 2m \partial^\mu \partial^\beta \right\} T_1[\partial_{\beta}] + (2\partial^y - \partial^y) \gamma_{\beta} \delta_{\alpha \mu} - 4m \partial^\mu \delta_{\alpha \mu} + 2\partial^\mu \partial^\beta \gamma_{\alpha} - 2m \partial^\mu \delta_{\alpha \beta} \right\} T_1[\partial_{\alpha} \partial_{\beta}] + (4\partial^\beta \gamma_{\alpha} \delta_{\mu \gamma} - 4m \delta_{\alpha \beta} \delta_{\mu \gamma} + 2\partial^\mu \gamma_{\beta} \delta_{\alpha \gamma} \right\} T_1[\partial_{\alpha} \partial_{\beta} \partial_{\gamma}] + (4\gamma_{\beta} T_1[\partial_{\beta} \partial_{\mu} \square].
\]

(4.3)

The external derivatives must be understood in the sense of distribution theory according to the DR prescription: they act ‘on the left’ over test (wave) functions. Of the four T–functions in Eq. 13, only \(T_1[\partial_{\beta}]\) is regular (i.e., a tempered distribution). The other three are singular at \(x = y = 0\) and must be renormalized (regularized). This renormalization is done in three steps.

First, the singular T–functions are split according to their tensor structure into trace and traceless parts:

\[
T_1[\partial_{\alpha} \partial_{\beta}] = T_1[\partial_{\alpha} \partial_{\beta}] - \frac{1}{4} \delta_{\alpha \beta} \square + \frac{1}{4} \delta_{\alpha \beta} T_1[\square],
\]

(4.4)
\[ T_1[\partial_\alpha \partial_\beta \partial_\gamma] = T_1[\partial_\alpha \partial_\beta \partial_\gamma] - \frac{1}{6}(\delta_{\alpha \beta} \partial_\gamma + \delta_{\alpha \gamma} \partial_\beta + \delta_{\beta \gamma} \partial_\alpha)\Box \]

\[ + \frac{1}{6}(\delta_{\alpha \beta} T_1[\partial_\gamma \Box] + \delta_{\alpha \gamma} T_1[\partial_\beta \Box] + \delta_{\beta \gamma} T_1[\partial_\alpha \Box]) \]  

(4.5)

\[ T_1[\partial_\alpha \partial_\beta \Box] = T_1[\partial_\alpha \partial_\beta] - \frac{1}{4}\delta_{\alpha \beta} T_1[\Box] + \frac{1}{4}\delta_{\alpha \beta} T_1[\Box] \]  

(4.6)

The traceless parts are two orders less singular. Thus, \( T_1[\partial_\alpha \partial_\beta] - \frac{1}{4}\delta_{\alpha \beta} \Box \) and \( T_1[\partial_\alpha \partial_\beta \partial_\gamma - \frac{1}{6}(\delta_{\alpha \beta} \partial_\gamma + \delta_{\alpha \gamma} \partial_\beta + \delta_{\beta \gamma} \partial_\alpha)\Box] \) are finite and \( T_1[(\partial_\alpha \partial_\beta - \frac{1}{4}\delta_{\alpha \beta} \Box)\Box] \) is 'logarithmically' singular. \( T_1[\Box] \) and \( T_1[\Box \Box] \) remain 'logarithmically' and 'quadratically' singular, respectively. \( T_1[\partial_\alpha \Box] \), which seems to be 'linearly' singular, is in fact 'logarithmically' singular because it can be written as a function of \( T_1[\Box] \). Indeed, using the symmetry of \( T \)-functions under \( x \leftrightarrow -y \) interchange (see Appendix B),

\[ T_1[\partial_\alpha \Box] = -\frac{1}{2}\partial_\alpha T_1[\Box] \]  

(4.7)

Second, all singular \( T \)-functions are expressed as functions of \( T_1[\Box] \) and \( T_1[\Box \Box] \) only. Indeed, the other singular \( T \)-function left, \( T_1[(\partial_\alpha \partial_\beta - \frac{1}{4}\delta_{\alpha \beta} \Box)\Box] \), can be written

\[ T_1[(\partial_\alpha \partial_\beta - \frac{1}{4}\delta_{\alpha \beta} \Box)\Box] = \]

\[ \left[ \frac{1}{3}(\partial_\alpha^x \partial_\beta^y + \partial_\alpha^y \partial_\beta^x) - \frac{1}{6}(\partial_\alpha^x \partial_\beta^y + \partial_\alpha^y \partial_\beta^x) + \frac{1}{12}\delta_{\alpha \beta}(\partial^x \cdot \partial^y - \Box^x - \Box^y)\right] T_1[\Box] \]

\[ + \frac{1}{4}(4\pi^2)^2 \left[ (\partial_\alpha^x + \partial_\beta^y)(\partial_\alpha^x + \partial_\beta^y) - \frac{1}{4}\delta_{\alpha \beta}(\partial_\rho^x + \partial_\rho^y)(\partial_\rho^x + \partial_\rho^y) \right] \]

\[ \times [\delta(x - y)(\frac{m^3 K_0(mx) K_1(mx)}{x} + m^4(K_0^2(mx) - K_1^2(mx)))] \]  

(4.8)

as is proven in Appendix B. Only the first term in the r.h.s. of Eq. (4.8) is singular.

Third, \( T_1[\Box] \) and \( T_1[\Box \Box] \) are renormalized. The singularity of \( T_1[\Box] \) goes as the inverse of the distance to the origin to the fourth power and is easily renormalized. Inserting the propagator equation (B.4), one obtains

\[ T_1[\Box] = -[\Delta_m(x) \Delta_m(y)] \delta(x - y) \]

\[ = -\frac{1}{(4\pi^2)^2} \frac{m^2 K_0^2(mx)}{x^2} \delta(x - y) \]  

(4.9)

and following Ref. [10] the renormalized \( T \)-function is

\[ T_1^R[\Box] = -\frac{1}{(4\pi^2)^2} \delta(x - y) \left[ \frac{1}{2} \Box - 4m^2 \right] \frac{m^3 K_0(mx) K_1(mx)}{x} + \pi^2 \log \frac{\tilde{M}^2}{m^2} \delta(x) \]  

(4.10)

where \( \tilde{M} = 2M/\gamma_E \) is an arbitrary scale. The renormalized \( T_1[\Box \Box] \) can be found in Ref. [22]. However it does not contribute to \( (g - 2)_l \) because the corresponding term in Eq. (1.3) is proportional to \( \gamma_\mu \). Hence, the renormalization of this correction to \( (g - 2)_l \) reduces to renormalizing \( T_1[\Box] \) as above.
Putting everything together we can now evaluate the contribution of D6 to the anomalous magnetic moment of the lepton. We substitute Eqs. (4.4 – 4.8) and (4.10) into Eq. (4.3) ignoring the $T_1[\square \square]$ term. Then we Fourier transform the renormalized vertex correction and put the lepton and the photon on their mass–shells. The part proportional to $p_\mu - p'_\mu$, where $p$ and $p'$ are the incoming fermion momenta, gives the $(g - 2)_l$ contribution of this diagram. The Fourier transforms of the regular terms entering in $(g - 2)_l$ are given in Appendix C. They add to

$$
(g - 2)_l^{D6} = \frac{\kappa^2 m^2}{4\pi^2} \left( \frac{4}{3} \log \frac{m^2}{M^2} + \frac{19}{18} \right).
$$

(4.11)

In other diagrams the proliferation of terms makes these manipulations rather cumbersome. However, a symbolic program developed for this purpose greatly simplifies the calculation.

## 5 The complete calculation

In this Section we describe the main aspects of the computation of $(g - 2)_l$ for all the diagrams in Figs. [1] and [2].

**Diagrams DA, DB, D1 and D6.** The first two diagrams (Fig. [1]) are of order $e^3$, and the rest of order $e\kappa^2$. In the example above we obtained the contribution of D6 to $(g - 2)_l$ and showed how to treat expressions containing $T_1$–functions. In diagram D1 the same set of functions appears, so the procedure is completely analogous. In particular, quadratically singular terms are again proportional to $\gamma_\mu$ and do not contribute to $(g - 2)_l$. In fact, the same occurs for all diagrams because the quadratically singular functions are scalars containing the maximum number of derivatives (four in T’s and two in B’s), so the only Lorentz vectors left are Dirac gammas. Diagrams DA and DB are even simpler, for no singular terms contribute to $(g - 2)_l$ once the trace–traceless splitting has been performed.

**Diagrams D4, D5, D9 and D10.** These diagrams contain two massless and one massive propagators, and are written in terms of $T_2$–functions. Here three derivatives appear at most, so there are no quadratically singular terms (not even in the $\gamma_\mu$ part). All $T_2$–functions are reduced to $T_2[\square]$ plus regular terms in a similar way as we reduced $T_1$–functions. Then the equation (3.5) is used to write this singular function as

$$
T_2[\square] = m^2 T_2[1] - \frac{1}{(4\pi^2)^2} \frac{1}{x^4} \delta(x - y),
$$

(5.1)

which together with the DR identity [1]...
This renormalization is the same as the one in Eq. (4.10), so it involves the same mass scale \( M \). Indeed, if the function in Eq. (4.9) is expanded in the mass parameter, we obtain

\[
\frac{1}{x^4} \bigg|_R = -\frac{1}{4} \log \frac{x^2 M^2}{x^2} \quad (5.2)
\]

allow to define the renormalized function:

\[
T_2^R[\Box] = m^2 T_2[1] + \frac{1}{4(4\pi^2)^2} \log \frac{x^2 M^2}{x^2} \delta(x - y) .
\quad (5.3)
\]

This renormalization is the same as the one in Eq. (4.10), so it involves the same mass scale \( M \). Indeed, if the function in Eq. (4.9) is expanded in the mass parameter, we obtain

\[
\frac{m^2 K_2(m x)}{x^2} = \frac{1}{x^4} + \mathcal{R}(m, x) , \quad (5.4)
\]

where \( \mathcal{R}(m, x) \) is of order \( m^2 \) and regular, and the r.h.s. of Eq. (5.4) can be also renormalized with the identity (5.2). In Eq. (4.10) we used a massive DR identity instead, in order to obtain more compact expressions, but we were careful to define the renormalized expression in such a way that it agrees with the one obtained from massless renormalization of the expansion in Eq. (5.4). Hence the scales in Eqs. (4.10) and (5.3) are the same.

**Diagrams D2, D3, D6 and D7.** They can be written in terms of the functions \( B_y (D2, D6) \) and \( B_x (D3, D7) \). As discussed in Section 3, one must express \( B \)-functions in terms of \( T \)-functions before renormalizing. This can be achieved using the equation (5.5) to ‘separate’ the delta function into a propagator, thus obtaining a triangular structure:

\[
B_y[1] = \delta(y) \Delta(x) \Delta_m(x)
\]

\[
= \delta(y) \Delta(x - y) \Delta_m(x)
\]

\[
= -\Box^y \Delta_m(y) \Delta(x - y) \Delta_m(x) + m^2 \Delta_m(y) \Delta(x - y) \Delta_m(x)
\]

\[
= -(\Box^y - m^2) T_1[1] - 2 \partial^y T_1[\partial_x] - T_1[\Box] . \quad (5.5)
\]

Analogously we obtain:

\[
B_y[\partial_\mu] = -\partial^x(\Box^y - m^2) T_1[1] + (\Box^y - m^2) T_1[\partial_x] - 2 \partial^x \partial^y T_1[\partial_x]
\]

\[
- \partial^x T_1[\Box] + 2 \partial^x T_1[\partial_x, \partial_\sigma] + T_1[\partial_\mu, \Box] , \quad (5.6)
\]

\[
B_y[\partial_\mu, \partial_\nu] = -\partial^x \partial^x(\Box^y - m^2) T_1[1] + \partial^x(\Box^y - m^2) T_1[\partial_\nu] + \partial^x(\Box^y - m^2) T_1[\partial_\mu]
\]

\[
- 2 \partial^x \partial^x \partial^y T_1[\partial_\sigma] - \partial^x \partial^x T_1[\Box] - (\Box^y - m^2) T_1[\partial_\nu, \partial_\sigma]
\]

\[
+ 2 \partial^x \partial^y T_1[\partial_\mu, \partial_\nu] + 2 \partial^x \partial^y T_1[\partial_\mu, \partial_\sigma] + \partial^x T_1[\partial_x, \Box] + \partial^x T_1[\partial_\mu, \Box]
\]

\[
- 2 \partial^y T_1[\partial_\mu, \partial_\nu, \partial_\sigma] - T_1[\partial_\nu, \partial_\mu, \Box] . \quad (5.7)
\]

and similar formulae for \( B_x \). Had we naively renormalized the \( B \)-functions independently, we would have a priori no control over the relation between the renormalization
scales of triangular diagrams $M_T$ and of bubble diagrams $M_B$, and the choice $M_T = M_B$
would break SUSY.

Once all the relevant terms in a diagram have been renormalized, one only has to extract from the regular
expressions the contribution to $(g - 2)_l$. This is conveniently done by performing a Fourier transform and
taking the appropriate on–shell limits, as described in Appendix C.

Let us summarize the steps we have followed to evaluate $(g - 2)_l$:

1. All diagrams are written in terms of T– and B–functions.
2. B–functions are rewritten in terms of T–functions.
3. The singular parts of the T–functions are identified and renormalized.
4. Finally, the contribution to $(g - 2)_l$ is extracted Fourier transforming, putting the external leptons on–shell and taking the $q^2 \to 0$ limit.

The explicit calculations have been carried out by hand and checked with a dedicated symbolic program. The necessary equalities are collected in the Appendices.

The resulting contributions of the superQED diagrams are

\[
DA \rightarrow \left(\frac{g - 2}{2}\right)^A = \frac{\alpha}{2\pi}, \quad (5.8)
\]
\[
DB \rightarrow \left(\frac{g - 2}{2}\right)^B = -\frac{\alpha}{2\pi}. \quad (5.9)
\]

Note that they are finite, scale independent, and, of course, agree with previous results.

The graviton and gravitino corrections to $(g - 2)_l$ are given in Table 1, together with the dimensional reduction and dimensional regularization results. In next Section we comment on the different contributions and compare them.

At this point, as the total graviton and gravitino contributions are finite, one may wonder if it is possible to add all the contributions of the different diagrams to start with and work with non–singular expressions. Actually this requires treating all the diagrams in appropriate manner. Writing all the contributions in terms of T–functions our procedure allows to obtain a compact result free of singularities. We find for the part of the vertex amplitude contributing to $(g - 2)_l$

\[
\frac{-ie\alpha^2}{2} \left\{ \left[ -2m^2 \partial_{\mu} \gamma_{\beta} + 4m^3 \delta_{\mu\beta} - 8m \partial_{\mu} \partial_{\beta} \right] T_1[\partial_{\beta}] \\
- 8m \partial_{\mu} T_1[\square] + 8m^3 \partial_{\mu} T_2[1] \\
+ 8m \partial_{\mu} \partial_{\beta} T_2[\partial_{\beta}] + 8m \partial_{\mu} T_2[\square] \right\}. \quad (5.10)
\]
This expression is regular because the only singular terms are those proportional to \( T_1 [\Box] \) and \( T_2 [\Box] \), which have the same singular parts:

\[
T_2[\Box] = T_1[\Box] - (\Delta(x) - \Delta_m(x))\Delta_m(x)\delta(x - y) \\
- (\Delta(x) - \Delta_m(x))\Delta(x)\delta(x - y) + m^2 T_2[1],
\]

(5.11)

and appear with opposite sign.

Another important remark, further discussed in Appendix C, is the infrared divergent behaviour of the Fourier transforms of the \( T_2 \)–functions in the static limit \( q^2 \to 0 \). It can be shown, however, that these divergencies cancel out in the sum Eq. (5.10). In fact we have checked that all diagrams, with the external leptons on–shell, are well–behaved in this limit.

We have also verified the absence of gauge non–invariant terms, proportional to the photon momentum \( q_\mu \), when the leptons are on their mass shell. This corresponds in coordinate space to a symmetry under the \( x \leftrightarrow -y \) interchange, which is preserved throughout the calculation.

### 6 Conclusions

Our explicit results are summarized in Table 1. The contributions to \( (g - 2)_l \) of the graviton and gravitino sectors in differential renormalization are separately finite and independent of the scale \( \tilde{M} \). In fact, in our procedure the logarithms of \( \tilde{M} \) keep track of the singularities and their sum is shown to vanish. Notice that these logarithms are proportional to the infinities of the regularization methods in momentum space, as expected at one loop. Obviously, finite parts may be different and are responsible for the preservation or the breaking of the symmetries in the problem. In differential renormalization the total contribution cancels out as dictated by supersymmetry. In the dimensional reduction scheme the total contribution also vanishes. This is not the case in dimensional regularization as this scheme is known to break supersymmetry.

Note that the values of the \( (g - 2)_l \) contribution of the graviton and gravitino sectors are different in differential renormalization \((1, -1)\) and in dimensional reduction \((-1/2, 1/2)\), although both schemes preserve supersymmetry. These contributions appear to be regularization dependent. This ambiguity is related to the presence of linear singularities (divergences) at one loop in SUGRA. All that can be said is that global supersymmetry ensures a vanishing value of \( (g - 2)_l \) and a cancellation of the ambiguities between the graviton and gravitino sectors. In Ref. [21] yet another finite
| Diagram | Differential Renormalization | Dimensional Reduction | Dimensional Regularization |
|---------|------------------------------|-----------------------|---------------------------|
| D1      | $-\frac{1}{6} \log \left( \frac{M^2}{m^2} \right) - \frac{25}{18}$ | $\frac{1}{3} \frac{1}{n-4} - \frac{29}{18}$ | $\frac{1}{3} \frac{1}{n-4} - \frac{61}{36}$ |
| D2+D3   | $-\frac{11}{6} \log \left( \frac{M^2}{m^2} \right) - \frac{11}{18}$ | $\frac{11}{3} \frac{1}{n-4} - \frac{35}{9}$ | $\frac{11}{3} \frac{1}{n-4} - \frac{32}{9}$ |
| D4+D5   | $2 \log \left( \frac{M^2}{m^2} \right) + 1$ | $-4 \frac{1}{n-4} + 6$ | $-4 \frac{1}{n-4} + 7$ |
| Graviton (D1+D2+D3+D4+D5) | $-1$ | $1/2$ | $7/4$ |
| D6      | $-\frac{4}{3} \log \left( \frac{M^2}{m^2} \right) + \frac{19}{18}$ | $\frac{8}{3} \frac{1}{n-4} - \frac{47}{18}$ | $\frac{8}{3} \frac{1}{n-4} - \frac{55}{18}$ |
| D7+D8   | $-\frac{2}{3} \log \left( \frac{M^2}{m^2} \right) + \frac{17}{18}$ | $\frac{4}{3} \frac{1}{n-4} - \frac{4}{9}$ | $\frac{4}{3} \frac{1}{n-4} - \frac{13}{9}$ |
| D9+D10  | $2 \log \left( \frac{M^2}{m^2} \right) - 1$ | $-4 \frac{1}{n-4} + 2$ | $-4 \frac{1}{n-4} + 4$ |
| Gravitino (D6+D7+D8+D9+D10) | $1$ | $-1/2$ | $-1/2$ |
| **TOTAL** | | | |
| (Graviton+Gravitino) | $0$ | $0$ | $5/4$ |

Table 1: Contributions of the diagrams in Fig. 2 to $\left( \frac{4-\alpha}{2} \right)_l$ in units of $\frac{G_N m^2}{\pi}$, obtained with DR, dimensional reduction and dimensional regularization.
value for the graviton contribution to \((g - 2)_{\mu} - 13/12(Gm^2/\pi)\), was obtained using source theory techniques. The situation is that in four different schemes the graviton contribution is different. The Pauli term is allowed in the gravity lagrangian and its one–loop corrections are scheme dependent in such a non–renormalizable theory.

An important point developed in our work is the exact maintenance of symmetries within the differential renormalization scheme. Direct renormalization with different scales requires enforcing the relevant symmetries (gauge and supersymmetry) with the corresponding Ward identities at the end of the calculation. In our approach all expressions have been handled in a symmetric way. We have reduced all singularities to a minimal set of independent functions. This was done relating diagrams of different topology with the technique of point separation. This procedure ensures that singularities are consistently treated. Then we obtain a result which is supersymmetric and compatible with U(1) gauge invariance.

Acknowledgements

F.A. thanks R. Stora for discussions. The symbolic program MATHEMATICA has been extensively used. The figures have been produced with the Feyndiag package. This work has been supported by CICYT contract number AEN96–1672, Junta de Andalucia 1201, and European Network Contracts CHRX-CT-92-0004 and ERBCHB1CT941777 (R.M.T.).
In this Appendix we give the Feynman rules (Figs. 3 and 4) for the lagrangian (2.1 – 2.4). They are written in Euclidean space with the convention \( \{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta} \).

\[
\begin{align*}
-ie\gamma_\mu & \\
\bar{e} & \gamma_\mu e \\
\bar{e} & e
\end{align*}
\]

\[
\begin{align*}
-ie\delta_{\alpha\mu} \gamma_\beta & \\
\bar{e} & \gamma_\mu e \\
\bar{e} & e
\end{align*}
\]

\[
\begin{align*}
ie(\partial_\mu^\bar{e} - \partial_\mu^e) & \\
\bar{e}_L & \gamma_\mu \bar{e}_L \\
\bar{e}_R & \bar{e}_R
\end{align*}
\]

\[
\begin{align*}
\frac{i\kappa}{\sqrt{2}} P_{L,R}(\bar{e}^\pm + m)\gamma_\alpha & \\
\bar{e}_L & \gamma_\mu \bar{e}_L \\
\bar{e}_R & \bar{e}_R
\end{align*}
\]

\[
\begin{align*}
\pm \gamma_\alpha (\partial_\beta^\bar{e} - \partial_\beta^e) & \\
\bar{e}_L & \gamma_\mu \bar{e}_L \\
\bar{e}_R & \bar{e}_R
\end{align*}
\]

\[
\begin{align*}
-\frac{\kappa}{2} \gamma_\alpha (\partial_\beta^\bar{e} - \partial_\beta^e) & \\
\bar{e}_L & \gamma_\mu \bar{e}_L \\
\bar{e}_R & \bar{e}_R
\end{align*}
\]

\[
\begin{align*}
\frac{\kappa}{2} \gamma_\mu \gamma_\alpha & \\
\bar{e}_L & \gamma_\mu \bar{e}_L \\
\bar{e}_R & \bar{e}_R
\end{align*}
\]

\[
\begin{align*}
-2\kappa [\partial_\alpha^\gamma \partial_\mu^\gamma \delta_{\mu\nu} - \partial_\alpha^\gamma \partial_\mu^\gamma \delta_{\beta\nu} & - \partial_\alpha^\gamma \partial_\mu^\gamma \delta_{\alpha\mu} + \partial_\alpha^\gamma \partial_\mu^\gamma \delta_{\alpha\mu} \delta_{\beta\nu} ] \\
\bar{e}_L & \gamma_\nu \bar{e}_L \\
\bar{e}_R & \bar{e}_R
\end{align*}
\]

Figure 3: Feynman rules for vertices. \( P_{R,L} = \frac{1}{2}(1 \pm \gamma_5) \) are the chiral projectors. All derivatives are with respect to the vertex space-time point. The superscripts indicate the field they are acting on. The rules for diagrams with opposite charge and fermion number arrows are obtained from these by the transformation \( FR \rightarrow \gamma_5 FR^\dagger \gamma_5 \) [25].
\[ y \xrightarrow{e} x \quad -(\partial^x - m)\Delta_m(x - y) \]

\[ y \xrightarrow{\tilde{e}} x \quad \Delta_m(x - y) \]

\[ y \xrightarrow{\gamma} x \quad \delta_{\mu\nu}\Delta(x - y) \]

\[ y \xrightarrow{\tilde{\gamma}} x \quad -\partial^x \Delta(x - y) \]

\[ y \xrightarrow{g} x \quad \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\rho\sigma})\Delta(x - y) \]

\[ y \xrightarrow{\tilde{g}} x \quad \frac{1}{2}\gamma_{\nu} \partial^x \gamma_{\mu}\Delta(x - y) \]

Figure 4: Feynman rules for propagators. \( \Delta(x) = \frac{1}{4\pi^2} \frac{1}{2^2} \) and \( \Delta_m(x) = \frac{1}{4\pi^2} \frac{mK_1(mx)}{x} \) are the massless and massive scalar propagators. Here the superscript on the derivatives refers to the space-time point.

Appendix B

In this Appendix we show how \( T_1[\partial_\alpha \Box] \) and \( T_1[(\partial_\alpha \partial_\beta - \frac{1}{2}\delta_{\alpha\beta} \Box)\Box] \) can be expressed in terms of the logarithmically singular function \( T_1[\Box] \). Let us first deal with the linearly singular \( T_1[\partial_\alpha \Box] \):

\[
T_1[\partial_\alpha \Box] = \Delta_m(x)\Delta_m(y)\partial_\alpha^x \Box \Delta(x - y)
= -\Delta_m(x)\Delta_m(y)\partial_\alpha^x \delta(x - y)
= -\partial_\alpha^x [\Delta_m(x)\Delta_m(y)\delta(x - y)] + [(\partial_\alpha^x \Delta_m(x))\Delta_m(y)]\delta(x - y)
= \partial_\alpha^x T_1[\Box] + [(\partial_\alpha^x \Delta_m(x))\Delta_m(x)]\delta(x - y) .
\]  

(B.1)
On the other hand,

\[ T_1[\partial_\alpha \Box] = -\Delta_m(x)\Delta_m(y)\partial_\alpha^y \Box^x \Delta(x - y) \]
\[ = -\partial_\alpha^y T_1[\Box] - [\Delta_m(x)(\partial_\alpha^y \Delta_m(x))]\delta(x - y) \tag{B.2} \]

So, combining both expressions

\[ T_1[\partial_\alpha \Box] = \frac{1}{2}(\partial_\alpha^x - \partial_\alpha^y)T_1[\Box] \tag{B.3} \]

With the ‘integration by parts’ prescription the linear singularity has been reduced to a logarithmic one. For \( T_1[(\partial_\alpha^x - \frac{1}{4}\delta_{\alpha\beta}\Box)\Box] \) we have

\[ T_1[(\partial_\alpha^x - \frac{1}{4}\delta_{\alpha\beta}\Box)\Box] = -\frac{1}{2}(\partial_\alpha^x \partial_\beta^x + \partial_\alpha^y \partial_\beta^y - \frac{1}{4}\delta_{\alpha\beta}(\Box^x + \Box^y))\left[\Delta_m(x)^2\delta(x - y)\right] \]
\[ + \left[\delta_{\alpha\beta} \Delta_m(x) \partial_\beta^x \Delta_m(x) - \frac{1}{4}\delta_{\alpha\beta} \partial_\beta^x \Delta_m(x) \partial_\beta^y \Delta_m(x)\right] \]
\[ \times \delta(x - y) \tag{B.4} \]

where the previous result has been used. Using recurrence relations among modified Bessel functions, the second term in the r.h.s. can be written

\[ \frac{1}{(4\pi^2)^2} \left( \frac{x_\alpha x_\beta}{x^2} - \frac{1}{4}\delta_{\alpha\beta} \right) \frac{m^4K_3^2(mx)}{x^2} \delta(x - y) \]
\[ = \frac{1}{(4\pi^2)^2} \frac{1}{6} \delta(x - y)\left[(\partial_\alpha^x \partial_\beta^x - \frac{1}{4}\delta_{\alpha\beta}\Box)\left(\frac{m^2K_1^2(mx)}{x^2} + \frac{m^3K_0(mx)K_1(mx)}{x} \right) \right. \]
\[ + m^4(K_0^2(mx) - K_1^2(mx))] \]
\[ = \frac{1}{(4\pi^2)^2} \frac{1}{6} \left[(\partial_\alpha^x + \partial_\alpha^y)(\partial_\beta^x + \partial_\beta^y) - \frac{1}{4}\delta_{\alpha\beta}(\partial_\rho^x + \partial_\rho^y)(\partial_\rho^x + \partial_\rho^y)\right] \]
\[ \times \delta(x - y)\left(\frac{m^2K_1^2(mx)}{x^2} + \frac{m^3K_0(mx)K_1(mx)}{x} \right) \]
\[ + m^4(K_0^2(mx) - K_1^2(mx))] \tag{B.5} \]

where the identity

\[ [\partial_\rho^x f(x)]\delta(x - y) = (\partial_\rho^x + \partial_\rho^y)[f(x)\delta(x - y)] \tag{B.6} \]

has been used to obtain the last equality. Only the term with \( \frac{m^2K_1^2(mx)}{x^2} \) in Eq. (B.5) is singular and equal to

\[ -\frac{1}{6}\left[(\partial_\alpha^x + \partial_\alpha^y)(\partial_\beta^x + \partial_\beta^y) - \frac{1}{4}\delta_{\alpha\beta}(\partial_\rho^x + \partial_\rho^y)(\partial_\rho^x + \partial_\rho^y)\right]T_1[\Box] \tag{B.7} \]

Thus we finally obtain

\[ T_1[(\partial_\alpha \partial_\beta - \frac{1}{4}\delta_{\alpha\beta}\Box)\Box] = \]
\[
\left[ \frac{1}{3} (\partial^x_\alpha \partial^y_\beta + \partial^y_\alpha \partial^y_\beta) - \frac{1}{6} (\partial^x_\alpha \partial^y_\beta + \partial^y_\alpha \partial^x_\beta) + \frac{1}{12} \delta_{\alpha\beta} (\partial^x \cdot \partial^y - \Box^x - \Box^y) \right] T_1[\Box] \\
+ \frac{1}{6} \frac{1}{(4\pi^2)^2} \left[ (\partial^x_\alpha + \partial^y_\beta)(\partial^y_\alpha + \partial^y_\beta) - \frac{1}{4} \delta_{\alpha\beta} (\partial^x + \partial^y)(\partial^x + \partial^y) \right] \\
\times \left[ \delta(x - y) \left( \frac{m^3 K_0(mx) K_1(mx)}{x} + m^4 (K_0^2(mx) - K_1^2(mx)) \right) \right].
\]

\textbf{Appendix C}

In order to extract the contribution to the anomalous magnetic moment of a given graph from its full renormalized expression, external fields must be on–shell. External derivatives in 1PI graphs act directly by parts on the external fields, so the Dirac or Maxwell equations can be used straightforwardly. When internal derivatives are present, however, the situation is more involved due to the noncommutative character of the derivatives. A simple procedure to follow is to perform a Fourier transform of the whole graph and deal with momenta, which are commuting objects. Notice that all Fourier transforms are ultraviolet convergent since the singular pieces have already been renormalized. On–shell conditions can then be readily imposed and the \((g - 2)_\mu\) contribution identified.

The Fourier transform of a distribution \(f(x, y)\) is

\[
\hat{f}(p, p') = \int d^4x \ d^4y \ e^{ip \cdot x} e^{ip' \cdot y} f(x, y),
\]

where \(p\) and \(p'\) are the incoming momenta of the external leptons and \(q = p + p'\) is the outgoing momentum of the external photon. External derivatives yield

\[
\partial^x_\mu \rightarrow -ip_\mu; \ \partial^y_\mu \rightarrow -ip'_\mu
\]

and we are left with the Fourier transforms of \(T\)–functions. When computing them we distinguish the ‘originally regular’ ones from the ‘renormalized’ ones (which, of course, are also regular, but involve different kinds of functions). The originally regular \(T\)–functions are products of three propagators with some inner derivatives. Their Fourier transform is a convolution of momentum space propagators with the corresponding internal momenta. For \(T_1\)–functions:

\[
\hat{T}_1[\mathcal{O}(\partial)] = I[\mathcal{O}(\partial \to ik)],
\]

where

\[
I[f(k)] = \int \frac{d^4k}{(2\pi)^4} \hat{\Delta}_m(p + k) \hat{\Delta}_m(p' - k) \hat{\Delta}(k),
\]

\[
\hat{\Delta}(k) = \frac{1}{k^2}, \ \hat{\Delta}_m(k) = \frac{1}{k^2 + m^2}.
\]
Thus, \( e.g. \),

\[
\hat{T}_1[\partial_\alpha \partial_\beta - \frac{1}{4} \delta_{\alpha\beta} \Box] = - I[k_\alpha k_\beta - \frac{1}{4} k^2 \delta_{\alpha\beta}],
\]

(C.14)

The integrals \( I[f(k)] \) appear in one–loop momentum space calculations and can be evaluated with standard techniques such as Passarino-Veltman reduction and Feynman parametrization. In our case, since we are interested in the static limit

\[
p^2 = p'^2 = -m^2, \quad q^2 \rightarrow 0,
\]

(C.15)

it is convenient to impose these conditions before computing the integrals, which then become much simpler \[26\].

Formally, the Fourier transform of originally regular \( T_2 \)-functions is identical:

\[
\hat{T}_2[\mathcal{O}(\partial)] = J[\mathcal{O}(\partial \rightarrow i k)],
\]

(C.16)

where

\[
J[f(k)] \equiv \int \frac{d^4k}{(2\pi)^4} \hat{\Delta}(p + k) \hat{\Delta}(p' - k) \hat{\Delta}_m(k).
\]

(C.17)

However the presence of two massless propagators makes the static limit of these integrals infrared divergent\(^1\), so one must keep \( q^2 \neq 0 \). It is gratifying to note that all the infrared divergencies cancel out in the \((g - 2)\) part of each diagram, rendering the limit \( q^2 \rightarrow 0 \) finite.

Renormalized expressions of \( T \)-functions always contain a delta function, \( \delta(x - y) \). Their Fourier transforms become integrals over one (four-dimensional) variable:

\[
\int d^4x \, d^4y \, e^{ip \cdot x} e^{ip' \cdot y} [g(x) \delta(x - y)] = \int d^4x \, e^{iq \cdot x} g(x),
\]

(C.18)

with \( q_\mu = p_\mu + p'_\mu \). Hence we only need the Fourier transforms of certain functions of one variable containing logarithms or modified Bessel functions. Derivatives, if present, are trivially pulled out yielding momenta \( q \). The transform of \( \log M^2 x^2 / x^2 \) was given in \[1\]:

\[
\int d^4x \, e^{iq \cdot x} \log M^2 x^2 / x^2 = - \frac{4\pi^2}{q^2} \log \frac{q^2}{M^2}.
\]

(C.19)

Note that Eq. \( (C.19) \) is divergent when \( q^2 \rightarrow 0 \). In the computation of \((g - 2)_t\), infrared divergencies coming from originally regular pieces and from renormalized pieces cancel separately. To obtain the Fourier transforms of expressions with modified Bessel functions, the following recurrence relations in four dimensions prove useful:

\[
\frac{\Box K_1(x)}{x} = \frac{K_1(x)}{x} - 4\pi^2 \delta(x),
\]

(C.20)

\(^1\)\( T_1[1] \) is also infrared divergent when \( q^2 \rightarrow 0 \), but it does not appear in the \((g - 2)_t\) parts.
\[ \Box K_2(x) = K_0(x) - 8\pi^2 \delta(x), \]  
\[ \Box K_0(x) = K_0(x) - 2 \frac{K_1(x)}{x}, \]  
\[ \Box K_1^2(x) = 2(K_0^2(x) + K_1^2(x)) - 4\pi^2 \delta(x), \]  
\[ \Box(K_1^2(x) - K_0^2(x)) = 4 K_0(x) K_1(x) - 4\pi^2 \delta(x). \]  

All these expressions are tempered distributions. The transforms we need are

\[ \int d^4x e^{iq \cdot x} \frac{m K_1(mx)}{x} = 4\pi^2 \Delta_m(q), \]  
\[ \int d^4x e^{iq \cdot x} K_0(mx) = 8\pi^2 [\Delta_m(q)]^2, \]  
\[ \int d^4x e^{iq \cdot x} mK_0(mx) K_1(mx) = 32\pi^4 \int \frac{d^4k}{(2\pi)^4} \Delta_m(q - k) [\Delta_m(q)]^2 
\quad q^2 \to 0 \quad \frac{\pi^2}{m^2}, \]  
\[ \int d^4x e^{iq \cdot x} m^2 (K_1^2(mx) - K_0^2(mx)) = -128\pi^4 \frac{m^2}{q^2} \int \frac{d^4k}{(2\pi)^4} \Delta_m(q - k) [\Delta_m(q)]^2 
\quad + \frac{4\pi^2}{q^2} 
\quad q^2 \to 0 \quad \frac{2\pi^2}{3 m^2}. \]

Finally, the momentum space expressions allow for direct use of the Dirac equation \((\not p \to -im, \ not p' \to im)\), and the \((g - 2)\) parts are easily recognized, for they are proportional to \(p_\mu - p'_\mu\).

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