An improved approximation algorithm for ATSP\(^1\)

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Abstract

We revisit the constant-factor approximation algorithm for the asymmetric traveling salesman problem by Svensson, Tarnawski, and Végh \([25]\). We improve on each part of this algorithm. We avoid the reduction to irreducible instances and thus obtain a simpler and much better reduction to vertebrate pairs. We also show that a slight variant of their algorithm for vertebrate pairs has a much smaller approximation ratio. Overall we improve the approximation ratio from 506 to \(22 + \varepsilon\) for any \(\varepsilon > 0\). This also improves the upper bound on the integrality ratio from 319 to 22.

1 Introduction

The asymmetric traveling salesman problem (ATSP) is one of the most fundamental and challenging combinatorial optimization problems. Given a finite set of cities with pairwise non-negative distances, we ask for a shortest tour that visits all cities and returns to the starting point.

The first non-trivial approximation algorithm was due to Frieze, Galbiati, and Maffioli \([8]\). Their \(\log_2(n)\)-approximation ratio, where \(n\) is the number of cities, was improved to \(0.99\log_2(n)\) by Bläser \([4]\), to \(0.842\log_2(n)\) by Kaplan, Lewenstein, Shafrir, and Sviridenko \([11]\), and to \(\frac{3}{5}\log_2(n)\) by Feige and Singh \([7]\). Then a \(O(\log(n)/\log(\log(n)))\)-approximation algorithm was discovered by Asadpour, Goemans, Mądry, Oveis Gharan, and Saberi \([3]\), and this inspired further work on the traveling salesman problem. Major progress towards a constant-factor approximation algorithm was made by Svensson \([23]\): he devised such an algorithm for the special case in which the distances are given by an unweighted digraph. This was extended to two different edge weights by Svensson, Tarnawski, and Végh \([24]\).

In a recent breakthrough, Svensson, Tarnawski, and Végh \([25]\) devised the first constant-factor approximation algorithm for the general ATSP. In their STOC 2018 paper, they showed an approximation ratio of 5500. Later they optimized their analysis and obtained an approximation ratio of 506.

Since this algorithm is analyzed with respect to the natural linear programming relaxation, it also yields a constant upper bound on the integrality ratio. In fact, Svensson, Tarnawski, and Végh showed an upper bound of 319 on the integrality ratio, but their algorithm that computes such

\(^1\)An extended abstract of this paper appeared in the proceedings of STOC 2020.
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Figure 1: The sequence of reductions of Svensson, Tarnawski, and Végh [25] (top) and our algorithm (bottom).

a solution does not have polynomial running time. Before [25], the best known upper bound was \((\log(\log(n)))^{O(1)}\) [2]. The strongest known lower bound on the integrality ratio is 2 [5].

We describe a polynomial-time algorithm that computes a tour of length at most \(22 + \varepsilon\) times the LP value for any given ATSP instance. Hence, the integrality ratio is at most 22. Via the reductions of [7] and [14], our result also implies stronger upper bounds for the path version, where the start and end of the tour are given and distinct.

## 2 Outline

An instance of ATSP can be described as a strongly connected digraph \(G = (V, E)\) and a cost (or length) function \(c : E \to \mathbb{R}_{\geq 0}\). We look for a minimum-cost closed walk in \(G\) that visits every vertex at least once. Such a closed walk may use (and then has to pay) edges several times. A tour is a multi-set \(F\) of edges such that \((V, E)\) is connected and Eulerian, i.e. every vertex has the same number of entering and leaving edges. Since such a graph admits an Eulerian walk (a closed walk that visits every vertex at least once and every vertex exactly once), an equivalent formulation of ATSP asks for a tour \(F\) with \(c(F)\) minimum.

The algorithm by Svensson, Tarnawski, and Végh [25] proceeds through a sequence of reductions, which we follow with some modifications (see Figure 1). First they show that it suffices to consider so-called laminarly-weighted instances. We strengthen this reduction to what we call strongly laminar instances (Section 3). In contrast to the following reductions this causes no loss in the approximation ratio. In a strongly laminar instance the cost of an edge \(e\) is given by the cost of entering or leaving sets in a laminar family \(L\) each of whose elements induces a strongly connected subgraph. More precisely,

\[
c(e) = \sum_{L \in \mathcal{L}, e \in \delta(L)} y_L
\]

for some positive weights \(y_L (L \in \mathcal{L})\) and all \(e \in E\).

In Section 4 we reduce strongly laminar instances to even more structured instances called vertebrate pairs. In a vertebrate pair we already have a given subtour, called backbone, that visits not necessarily all vertices but all non-singleton elements of the laminar family \(L\). In contrast to the reduction to strongly laminar instances, the reduction to vertebrate pairs causes some loss in the approximation ratio. While Svensson, Tarnawski, and Végh also reduce to vertebrate pairs, they first reduce to what they call an irreducible instance as an intermediate step before reducing to vertebrate pairs. We show that this intermediate step is not necessary. This leads to a simpler algorithm. Moreover, the loss in the approximation ratio in this step is much smaller. In fact, a significant part of the improvement of the overall approximation ratio is due to our new reduction
to vertebrate pairs.

Finally, in Section 5 and Section 6 we explain how to compute good solutions for vertebrate pairs. The main algorithmic framework, essentially due to Svensson [23], follows on a very high level the cycle cover approach by Frieze, Galbiati and Maffioli [8]. It maintains an Eulerian subgraph $H$ which initially consists of the backbone only. In each iteration it computes an Eulerian set $F$ of edges that connects every connected component of $H$, except possibly the backbone, to another connected component. However, in order to achieve a constant-factor approximation for ATSP we need additional properties and will not always add all edges of $F$ to $H$.

In Section 5 we explain a sub-routine that computes the edge set $F$ in every iteration of Svensson’s algorithm. The problem solved by the sub-routine, which we call Subtour Cover, can be viewed as the analogue of the cycle cover problem that is solved in every iteration of the $\log_2(n)$-approximation algorithm by Frieze, Galbiati and Maffioli [8]. It is very similar to what Svensson, Tarnawski, and Végh call Subtour Partition Cover and Eulerian Partition Cover and Svensson [23] calls Local Connectivity ATSP. Svensson, Tarnawski, and Végh compute a solution for Subtour Cover by rounding a circulation in a certain flow network, which is constructed from the LP solution using a so-called witness flow. By using a special witness flow with certain minimality properties our Subtour Cover solution will obey stronger bounds.

In Section 6 we then explain how to compute solutions for vertebrate pairs using the algorithm for Subtour Cover as a sub-routine. The essential idea is due to Svensson [23], who considered node-weighted instances, and was later adapted to vertebrate pairs in [25]. In this part we make two improvements compared to the algorithm in [25].

The more important change is the following. Svensson’s algorithm uses a potential function to measure progress, and in each of [23] and [25] two different potential functions are considered. One potential function is used to obtain an exponential time algorithm that yields an upper bound on the integrality ratio of the linear programming relaxation, and the other potential function is used to obtain a polynomial-time algorithm. This leads to different upper bounds on the integrality ratio of the LP and the approximation ratio of the algorithm. We show in Section 6 that we can make this discrepancy arbitrarily small by a slightly different choice of the potential function for the polynomial-time algorithm. This leads to a better approximation ratio. Moreover, the analysis of the polynomial-time algorithm then immediately implies the best upper bound we know on the integrality ratio and there is no need anymore to consider two different potential functions.

The second change compared to the algorithm in [25] is that we include an idea that Svensson [23] used for node-weighted instances. This leads to another small improvement of the approximation guarantee.

Overall, we obtain for every $\varepsilon > 0$ a polynomial-time $(22 + \varepsilon)$-approximation algorithm for ATSP. The algorithm computes a solution of cost at most $22 + \varepsilon$ times the cost of an optimum solution to the classic linear programming relaxation (ATSP LP), which we describe next.

3 Reducing to strongly laminar instances

As in the Svensson–Tarnawski–Végh algorithm, we begin by solving the classic linear programming relaxation:

$$\min c(x)$$

s.t. $x(\delta^-(v)) - x(\delta^+(v)) = 0$ for $v \in V$

$$x(\delta(U)) \geq 2$$ for $\emptyset \neq U \subset V$

$$x_e \geq 0$$ for $e \in E$,  \hspace{1cm} (ATSP LP)
where \( c(x) := \sum_{e \in E} c(e)x_e \), \( x(F) := \sum_{e \in F} x_e \) for \( F \subseteq E \), \( \delta^-(v) \) and \( \delta^+(v) \) denote the sets of edges entering and leaving \( v \), respectively, and \( \delta(U) \) denotes the set of edges with exactly one endpoint in \( U \). We also solve the dual LP:

\[
\max \sum_{\emptyset \neq U \subseteq V} 2y_U \\
\text{s.t. } a_w - a_v + \sum_{U \subseteq \delta(v)} y_U \leq c(e) \quad \text{for } e = (v, w) \in E \\
\quad \quad \quad \quad \quad \quad y_U \geq 0 \quad \text{for } \emptyset \neq U \subseteq V,
\]

where the variables \( a_v \) (\( v \in V \)) are unbounded. A family \( \mathcal{L} \) of subsets of \( V \) is called laminar if for any \( A, B \in \mathcal{L} \) we have \( A \subseteq B \), \( B \subseteq A \), or \( A \cap B = \emptyset \). Such a family has at most \( 2|V| \) elements. The following is well-known (see e.g. [25]).

**Lemma 1.** Let \( (G, c) \) be an instance of ATSP. Then we can compute in polynomial time an optimum solution \( x \) to (ATSP LP) and an optimum solution \( (a, y) \) to (ATSP DUAL), such that \( y \) has laminar support, i.e. \( \mathcal{L} := \{ U : y_U > 0 \} \) is a laminar family.

We will now obtain LP solutions with more structure. By \( G[U] = (U, E[U]) \) we denote the subgraph of \( G = (V, E) \) induced by the vertex set \( U \).

**Definition 2.** Let \( (G, c) \) be an instance of ATSP. Moreover, let \( (a, y) \) be a dual LP solution, i.e. a solution to (ATSP DUAL). We say that \( y \) or \( (a, y) \) has strongly laminar support if

- \( \mathcal{L} := \{ U : y_U > 0 \} \) is a laminar family, and
- for every set \( U \in \mathcal{L} \), the graph \( G[U] \) is strongly connected.

The following lemma allows us to assume that our optimum dual solution has strongly laminar support.

**Lemma 3.** Let \( (G, c) \) be an instance of ATSP. Moreover, let \( x \) be an optimum solution to (ATSP LP) and \( (a, y) \) an optimum solution to (ATSP DUAL) with laminar support. Then we can compute in polynomial time \( (a', y') \) such that

- \( (a', y') \) is an optimum solution of (ATSP DUAL), and
- \( (a', y') \) has strongly laminar support.

**Proof.** As long as there is a set \( U \) with \( y_U > 0 \), but \( G[U] \) is not strongly connected, we do the following. Let \( U \) be a minimal set with \( y_U > 0 \) and such that \( G[U] \) is not strongly connected. Moreover, let \( S \) be the vertex set of the first strongly connected component of \( G[U] \) in a topological order. Then we have \( \delta^-(S) \subseteq \delta^-(U) \).

Define a dual solution \( (a', y') \) as follows. We set \( y'_U := 0 \), \( y'_S := y_S + y_U \), and \( y'_W := y_W \) for other sets \( W \). Moreover, \( a'_v := a_v - y_U \) for \( v \in U \setminus S \) and \( a'_v := a_v \) for all other vertices \( v \). The only edges \( e = (v, w) \) for which \( a'_w - a'_v + \sum_{U \subseteq \delta(v)} y'_U > a_w - a_v + \sum_{U \subseteq \delta(v)} y_U \), are edges from \( U \setminus S \) to \( S \). However, such edges do not exist by choice of \( S \). Hence, \( (a', y') \) is a feasible dual solution. Since \( \sum_{\emptyset \neq U \subseteq V} 2y'_U = \sum_{\emptyset \neq U \subseteq V} 2y_U \), it is also optimal.

We now show that the support of \( y' \) is laminar. Suppose there is a set \( W \) in the support of \( y' \) that crosses \( S \). Then \( W \) must be in the support of \( y \) and hence a subset of \( U \) because the support of \( y \) is laminar. By the minimal choice of \( U \), \( G[W] \) is strongly connected. But this implies that \( G \) contains an edge from \( W \setminus S \) to \( W \cap S \), contradicting \( \delta^-(S) \subseteq \delta^-(U) \).

We now decreased the number of sets \( U \) in the support for which \( G[U] \) is not strongly connected. After iterating this at most \( 2|V| \) times the dual solution has the desired properties. \( \square \)
As Svensson, Tarnawski, and Végh [25], we next show that we may assume the dual variables \( a_v \) to be 0 for all \( v \in V \). This leads to the following definition.

**Definition 4.** A strongly laminar ATSP instance is a quadruple \((G, \mathcal{L}, x, y)\), where

(i) \( G = (V, E) \) is a strongly connected digraph;

(ii) \( \mathcal{L} \) is a laminar family of subsets of \( V \) such that \( G[U] \) is strongly connected for all \( U \in \mathcal{L} \);

(iii) \( x \) is a feasible solution to (ATSP LP) such that \( x(\delta(U)) = 2 \) for all \( U \in \mathcal{L} \) and \( x_e > 0 \) for all \( e \in E \);

(iv) \( y : \mathcal{L} \to \mathbb{R}_{>0} \).

This induces the ATSP instance \((G, c)\), where \( c \) is the induced weight function defined by \( c(e) := \sum_{U \in \mathcal{L}, e \in \delta(U)} y_U \) for all \( e \in E \).

By complementary slackness, \( x \) and \((0, y)\) are optimum solutions of (ATSP LP) and (ATSP DUAL) for \((G, c)\). For a strongly laminar instance \( \mathcal{I} \) we denote by \( \text{lp}(\mathcal{I}) = c(x) \) the value of these LPs. We now prove that for ATSP it is sufficient to consider strongly laminar instances.

**Theorem 5.** Let \( \alpha \geq 1 \). If there is a polynomial-time algorithm that computes for every strongly laminar ATSP instance \((G, \mathcal{L}, x, y)\) a solution of cost at most \( \alpha \cdot c(x) \), then there is a polynomial-time algorithm that computes for every instance of ATSP a solution of cost at most \( \alpha \times \text{lp}(\mathcal{I}) \).

**Proof.** Let \((G, c)\) be an arbitrary instance. We apply Lemma 1 to compute an optimum solution \( x \) of (ATSP LP) and an optimum solution \((a, y)\) of (ATSP DUAL) such that the support of \( y \) is a laminar family \( \mathcal{L} \).

Now let \( E' \) be the support of \( x \) and define \( G' := (V, E') \). Let \( x' \) be the vector \( x \) restricted to its support \( E' \). Then apply Lemma 3 to \((G', x', y)\). We obtain an optimum dual solution \((a', y')\) to (ATSP DUAL) with strongly laminar support \( \mathcal{L}' \). By complementary slackness we have \( x'(\delta(U)) = 2 \) for all \( U \in \mathcal{L}' \) with \( y'_U > 0 \).

Then the induced weight function of the strongly laminar ATSP instance \((G', \mathcal{L}', x', y')\) is given by \( c'(e) = \sum_{S \in \mathcal{L}', e \in \delta(S)} y'_S = c(e) + a_e - a_w \) for all \( e = (v, w) \in E' \) (by complementary slackness). Because every tour in \( G' \) is Eulerian, it has the same cost with respect to \( c \) and with respect to \( c' \). Moreover, \( c(x) = c'(x') \) and \((0, y')\) is an optimum dual solution for \((G', c')\). Hence also the LP values are the same and thus the theorem follows.

One advantage of this structure is that we always have nice paths, which are defined as follows.

**Definition 6.** Let \( G = (V, E) \) be a directed graph and let \( \mathcal{L} \) be a laminar family. Let \( v, w \in V \) and let \( \tilde{U} \) be the minimal set in \( \mathcal{L} \cup \{V\} \) with \( v, w \in \tilde{U} \). A v-w-path is nice if it is in \( G[\tilde{U}] \) and it enters and leaves every set \( U \in \mathcal{L} \) at most once.

**Lemma 7.** Let \( G = (V, E) \) be a strongly connected directed graph and let \( \mathcal{L} \) be a laminar family such that \( G[U] \) is strongly connected for every \( U \in \mathcal{L} \). Then for any \( v, w \in V \) we can find a nice v-w-path in polynomial time.

**Proof.** Let \( P \) be a path from \( v \) to \( w \) in \( G[\tilde{U}] \). Now repeat the following until \( P \) enters and leaves every set in \( \mathcal{L} \) at most once. Let \( U \) be a maximal set with \( U \in \mathcal{L} \) that \( P \) enters or leaves more than once. Let \( v' \) be the first vertex that \( P \) visits in \( U \) and let \( w' \) be the last vertex that \( P \) visits in \( U \). Since \( G[U] \) is strongly connected, we can replace the \( v'-w' \)-subpath of \( P \) by a path in \( G[U] \). After at most \( |\mathcal{L}| < 2|V| \) iterations, \( P \) is a nice v-w-path. \( \square \)
Let $L \geq 2 := \{ L \in L : |L| \geq 2 \}$ be the family of all non-singleton elements of $L$. In this section we show how to reduce ATSP to the case where we have already a given subtour $B$, called backbone, that visits all elements of $L \geq 2$; see Figure 2. By a subtour we mean a connected Eulerian multi-subgraph of $G$. We call a strongly laminar ATSP instance together with a given backbone a vertebrate pair.

Definition 8. A vertebrate pair consists of

- a strongly laminar ATSP instance $I = (G, L, x, y)$ and
- a connected Eulerian multi-subgraph $B$ of $G$ (the backbone) such that $V(B) \cap L \neq \emptyset$ for all $L \in L \geq 2$.

Let $\kappa, \eta \geq 0$. A $(\kappa, \eta)$-algorithm for vertebrate pairs is an algorithm that computes, for any given vertebrate pair $(I, B)$, a multi-set $F$ of edges such that $E(B) \cup F$ is a tour and

$$c(F) \leq \kappa \cdot \text{LP}(I) + \eta \cdot \sum_{v \in V \setminus V(B) : \{v\} \in \mathcal{L}} 2y_{\{v\}}.$$  \hfill (1)

Note that this definition is slightly different to the one in [25] (where $G[L]$ was not required to be strongly connected for $L \in \mathcal{L}$), but this will not be relevant.

In this section we will show that a $(\kappa, \eta)$-algorithm for vertebrate pairs (for any constants $\kappa$ and $\eta$) implies a $(3\kappa + \eta + 2)$-approximation algorithm for ATSP. This is the reason for using the bound in (1); we would get a worse overall approximation guarantee if we just worked with the weaker inequality $c(F) \leq (\kappa + \eta) \cdot \text{LP}(I)$.

Let $(G, L, x, y)$ be a strongly laminar ATSP instance and $c$ the induced cost function. In the following we fix for every $u, v \in V$ a nice $u$-$v$-path $P_{u,v}$. Such paths can be computed in polynomial time by Lemma 7.

Lemma 9. Let $W \in \mathcal{L} \cup \{V\}$ and let $u, v \in W$. Then

$$c(E(P_{u,v})) = \sum_{L \in \mathcal{L} : L \subseteq W, V(P_{u,v}) \neq \emptyset} 2y_L - \sum_{L \in \mathcal{L} : u \in L \subseteq W} y_L - \sum_{L \in \mathcal{L} : v \in L \subseteq W} y_L.$$

Proof. Since the path $P_{u,v}$ is nice, it is contained in $G[W]$. Moreover, it leaves every set $L \in \mathcal{L}$ at most once and enters every set $L \in \mathcal{L}$ at most once. A set $L \in \mathcal{L}$ with $u \in L$ is never entered by $P_{u,v}$ and a set $L \in \mathcal{L}$ with $w \in L$ is never left by $P_{u,v}$. \qed
We define
\[ \text{value}(W) := \sum_{L \in \mathcal{L} : v \subseteq W} 2y_L, \]
and
\[ D_W(u, v) := \sum_{L \in \mathcal{L} : u \subseteq L \subset W} y_L + \sum_{L \in \mathcal{L} : v \subseteq L \subset W} y_L + c(E(P_{u,v})) \]
for \( u, v \in W \). Note that \( D_W(u, v) \leq \text{value}(W) \) by Lemma 9. We write
\[ D_W := \max\{D_W(u, v) : u, v \in W\}. \]

The intuitive meaning of \( D_W \) in the analysis of our reduction to vertebrate pairs is the following. On the one hand, it can be useful if \( D_W \) is small: if we enter the set \( W \) at some vertex \( s \in W \) and leave it at some other vertex \( t \in W \), we can always find a cheap \( s-t \)-walk inside \( G[W] \). On the other hand, if \( D_W \) is large, we can find a nice path inside \( W \) that visits many sets \( L \in \mathcal{L} \) (or more precisely, sets of high weight in the dual solution \( y \)).

The reduction to vertebrate pairs is via a recursive algorithm. For a given set \( W \in \mathcal{L} \cup \{V\} \) it constructs a tour in \( G[W] \). See Figure 3 for an illustration.

**Algorithm 1:** Recursive algorithm to reduce to vertebrate pairs.

**Input:** a strongly laminar ATSP instance \( I = (G, \mathcal{L}, x, y) \) with \( G = (V, E) \),
a set \( W \in \mathcal{L} \cup \{V\} \), and
a \((\kappa, \eta)\)-algorithm \( A \) for vertebrate pairs (for some constants \( \kappa, \eta \geq 0 \))

**Output:** a tour \( F \) in \( G[W] \)

1. If \( W \neq V \), contract \( V \setminus W \) into a single vertex \( v_W \) and redefine \( y_W := \frac{D_W}{2} \).
2. **Construct a vertebrate pair:** Let \( u^*, v^* \in W \) such that \( D_W(u^*, v^*) = D_W \). Let \( B \) be the multi-graph corresponding to the closed walk that results from appending \( P_{u^*,v^*} \) and \( P_{v^*,u^*} \). Let \( \mathcal{L}_B \) be the set of all maximal sets \( L \in \mathcal{L} \) with \( L \subseteq W \) and \( V(B) \cap L = \emptyset \). Contract every set \( L \in \mathcal{L}_B \) to a single vertex \( v_L \) and set \( y_L := y_L + \frac{D_W}{2} \). Let \( G' \) be the resulting graph. Let \( \mathcal{L}' \) be the laminar family of subsets of \( V(G') \) that contains singletons \( \{v_L\} \) for \( L \in \mathcal{L}_B \) and all the sets arising from \( L \in \mathcal{L} \) with \( L \subseteq W \) and \( L \cap V(B) \neq \emptyset \). Let \( I' = (G', \mathcal{L}', x, y) \) be the resulting strongly laminar instance.
3. **Compute a solution for the vertebrate pair:** Apply the given algorithm \( A \) to the vertebrate pair \((I', B)\). Let \( F' \) be the resulting Eulerian edge set.
4. **Lift the solution to a subtour:** Fix an Eulerian walk in every connected component of \( F' \). Now uncontract every \( L \in \mathcal{L}_B \). Whenever an Eulerian walk passes through \( v_L \), we get two edges \((u', u) \in \delta^- (L) \) and \((v, v') \in \delta^+ (L) \). To connect \( u \) and \( v \) within \( L \), add the path \( P_{u,v} \).

Moreover, if \( W \neq V \) do the following. Whenever an Eulerian walk passes through \( v_W \) using the edges \((u, v_W) \) and \((v_W, v) \), replace them by the path \( P_{u,v} \).
5. **Recurse to complete to a tour of the original instance:** For every set \( L \in \mathcal{L}_B \), apply Algorithm 1 recursively to obtain a tour \( F_L \) in \( G[L] \). Let \( F'' \) be the union of \( F' \) and all these tours \( F_L \) for \( L \in \mathcal{L}_B \).
6. Return \( F := F'' \cup E(B) \).

First, we observe that Algorithm 1 indeed returns a tour in \( G[W] \).
Figure 3: Illustration of Algorithm 1. The ellipses show the laminar family $\mathcal{L}$. Picture (a) shows the set $W$ (orange), the subtour $B$ (blue), and the elements of $\mathcal{L}_B$ (red). The subtour $B$ is the union of the paths $P_{u^*,v^*}$ and $P_{v^*,u^*}$. Picture (b1) shows the resulting vertebrate pair instance as constructed in step 2 of Algorithm 1. The vertices resulting from the contraction of elements of $\mathcal{L}_B$ are shown in red and the vertex $v_{\bar{W}}$ that results from the contraction of $V \setminus W$ is shown in orange. Picture (b2) shows in green a possible solution to this vertebrate pair. Picture (c) illustrates step 4 of Algorithm 1: the green edges are those that arise from the vertebrate pair solution from Picture (b2) by undoing the contraction of the sets in $\mathcal{L}_B$. The red edges are the paths that we add to connect within $L \in \mathcal{L}_B$ when uncontracting $L$. The orange edges show the $u$-$v$-path in $G[W]$ that we add to replace the edges $(u,v_{\bar{W}})$ and $(v_{\bar{W}},v)$ in the vertebrate pair solution from Picture (b2).
Lemma 10. \(\kappa, \eta \geq 1\). Suppose we have a polynomial-time \((\kappa, \eta)\)-algorithm \(A\) for vertebrate pairs. Then Algorithm 1 has polynomial runtime and returns a tour in \(G[W]\) for every strongly laminar ATSP instance \(I = (G, L, x, y)\) and every \(W \in L \cup \{V\}\).

Proof. We apply induction on \(|W|\). For \(|W| = 1\), the algorithm returns \(F = \emptyset\). Now let \(|W| > 1\). At the end of step 3, we have that \(F^t\) is Eulerian and \(F^t \cup E(B)\) is a tour in the instance \(I^t\). In step 4, the set \(F^t\) remains Eulerian and \(F^t \cup E(B)\) remains connected. Moreover, let \(F^t \cup E(B)\) visits all sets in \(L_B\), i.e. we have \(F^t \cap \delta(L) \neq \emptyset\) for all \(L \in L_B\). The subtour \(F^t \cup E(B)\) also visits all vertices in \(W\) that are not contained in any set \(L \in L_B\), i.e. for these vertices \(v\) we have \(\delta(v) \cap (F^t \cup E(B)) \neq \emptyset\). After step 4, we have \(F^t \subseteq E[W]\). We conclude that the graph \((W, F^t \cup E(B))\) is connected and Eulerian; here we applied the induction hypothesis to the sets \(L \in L_B\).

To see that the runtime of the algorithm is polynomially bounded we observe that there are in total at most \(|L| + 1 \leq 2|V|\) recursive calls of the algorithm because \(L \cup \{V\}\) is a laminar family. \(\square\)

Next we observe that our backbone \(B\) visits many sets \(L \in L\) inside \(W\) if \(D_W\) is large. This is because the path \(P_{u^*, v^*}\) enters and leaves each set in \(L\) at most once and thus \(D_W\) can only be large if this path visits sets in \(\{L \in L : L \subseteq W\}\) of large total weight.

Lemma 11. Let \(I = (G, L, x, y)\) be a strongly laminar ATSP instance, and let \(W \in L \cup \{V\}\). Moreover, let \(B\) be as in step 2 of Algorithm 1. Then

\[
\sum_{L \in L_B} (2y_L + value(L)) \leq value(W) - D_W. \tag{2}
\]

Proof. By Lemma 9 and the choice of \(u^*\) and \(v^*\) we get

\[
value(W) - \sum_{L \in L_B} (2y_L + value(L)) = \sum_{L \in L : L \subseteq W, L \cap V(B) \neq \emptyset} 2y_L \\
\geq \sum_{L \in L : L \subseteq W, L \cap V(P_{u^*, v^*}) \neq \emptyset} 2y_L \\
= c(E(P_{u^*, v^*})) + \sum_{L \in L : u^* \in L \subseteq W} y_U + \sum_{L \in L : v^* \in L \subseteq W} y_U \\
= D_W(u^*, v^*) \\
= D_W. \quad \square
\]

Now we analyze the cost of the tour \(F\) in \(G[W]\) computed by Algorithm 1.

Lemma 12. \(\kappa, \eta \geq 0\). Suppose we have a \((\kappa, \eta)\)-algorithm \(A\) for vertebrate pairs. Let \(I = (G, L, x, y)\) be a strongly laminar ATSP instance, \(c\) the induced cost function, and \(W \in L \cup \{V\}\). Then the tour \(F\) in \(G[W]\) returned by Algorithm 1 has cost at most

\[
c(F) \leq (2\kappa + 2) \cdot value(W) + (\kappa + \eta) \cdot (value(W) - D_W).
\]

Proof. By induction on \(|W|\). The statement is trivial for \(|W| = 1\) since then \(c(F) = 0\) (because \(F \subseteq E[W] = \emptyset\)). Let now \(|W| \geq 2\). By definition of \(D_W\), we have

\[
c(E(B)) = c(E(P_{u^*, v^*})) + c(E(P_{v^*, u^*})) \leq 2D_W. \tag{3}
\]
We now analyze the cost of $F'$ in step 3 of Algorithm 1. Since $F'$ is the output of a $(\kappa, \eta)$-algorithm applied to the vertebrate pair $(I', B)$, we have $c(F') \leq \kappa \cdot \text{LP}(I') + \eta \cdot \sum_{L \in \mathcal{L}_B} 2y_{\{v_L\}}$. Using $\sum_{L \in \mathcal{L}_B} 2y_{\{v_L\}} = \sum_{L \in \mathcal{L}_B} (2y_L + D_L)$ and

$$
\text{LP}(I') \leq D_W + \sum_{L \in \mathcal{L}, L \subseteq W, L \cap V(B) \neq \emptyset} 2y_L + \sum_{L \in \mathcal{L}_B} 2y_{\{v_L\}}
$$

(where we used that we set $y_W := \frac{D_W}{2}$ in step 1 of the algorithm if $W \neq V$), this implies

$$
c(F') \leq \kappa \cdot D_W + \sum_{L \in \mathcal{L}, L \subseteq W, L \cap V(B) \neq \emptyset} \kappa \cdot 2y_L + \sum_{L \in \mathcal{L}_B} (\kappa + \eta) \cdot (2y_L + D_L)
$$

(4)

at the end of step 3. As in [25], the lifting and all the amendments of $F'$ in step 4 do not increase the cost of $F'$ by Lemma 9 and the choice of the values $y_{\{v_L\}}$ in step 2 and $y_W$ in step 1. (Here we use that whenever a Eulerian walk passes through $v_W$, we leave and enter $W$.)

To bound the cost increase in step 5 we apply the induction hypothesis. Adding the edges resulting from a single recursive call of Algorithm 1 in step 5 for some $L \in \mathcal{L}_B$ increases the cost by at most $c(F_L) \leq (2\kappa + 2) \cdot \text{value}(L) + (\kappa + \eta) \cdot \text{value}(L) - D_L$. Using (4), we obtain the following bound:

$$
c(F'') \leq \kappa \cdot D_W + \sum_{L \in \mathcal{L}, L \subseteq W, L \cap V(B) \neq \emptyset} \kappa \cdot 2y_L
$$

$$
+ \sum_{L \in \mathcal{L}_B} ((2\kappa + 2) \cdot \text{value}(L) + (\kappa + \eta) \cdot (2y_L + \text{value}(L)))
$$

$$
\leq \kappa \cdot D_W + \kappa \cdot \text{value}(W)
$$

$$
+ \sum_{L \in \mathcal{L}_B} ((\kappa + 2) \cdot \text{value}(L) + (\kappa + \eta) \cdot (2y_L + \text{value}(L)))
$$

$$
\leq \kappa \cdot D_W + \kappa \cdot \text{value}(W)
$$

$$
+ (\kappa + 2) \cdot (\text{value}(W) - D_W) + (\kappa + \eta) \cdot (\text{value}(W) - D_W)
$$

$$
= (2\kappa + 2) \cdot \text{value}(W) - D_W + (\kappa + \eta) \cdot (\text{value}(W) - D_W),
$$

where we used the definition of $\mathcal{L}_B$ for the second inequality and Lemma 11 for the third inequality; note that the elements of $\mathcal{L}_B$ are pairwise disjoint. Together with (3) this implies the claimed bound on $c(F)$.

Now we prove the main result of this section.

**Theorem 13.** Let $\kappa, \eta \geq 0$. Suppose we have a polynomial-time $(\kappa, \eta)$-algorithm for vertebrate pairs. Then there is a polynomial-time algorithm that computes a solution of cost at most

$$
3\kappa + \eta + 2
$$

times the value of (ATSP LP) for any given ATSP instance.

**Proof.** By Theorem 5 it suffices to show that there is a polynomial-time algorithm that computes a solution of cost at most $(3\kappa + \eta + 2) \cdot \text{LP}(I)$ for any given strongly laminar ATSP instance $I$. Given such an instance, we apply Algorithm 1 to $W = V$. By Lemma 10 and Lemma 12, this algorithm computes in polynomial time a tour of cost at most

$$
c(F) \leq (2\kappa + 2) \cdot \text{value}(V) + (\kappa + \eta) \cdot (\text{value}(V) - D_V)
$$

$$
= (2\kappa + 2) \cdot \text{LP}(I) + (\kappa + \eta) \cdot (\text{LP}(I) - D_V)
$$

$$
\leq (3\kappa + \eta + 2) \cdot \text{LP}(I).\quad \square
$$
In the following we will present a \((2, 14 + \varepsilon)\)-algorithm for vertebrate pairs, improving on the \((2, 37 + \varepsilon)\)-algorithm by Svensson, Tarnawski, and Végh [25]. Using their vertebrate pair algorithm, Theorem 13 immediately implies a \((45 + \varepsilon)\)-approximation algorithm for ATSP.

**Remark 14**

One could achieve a slightly better overall approximation ratio for ATSP by the following modifications. Change Algorithm 1 and generalize the notion of vertebrate pairs as follows. First, in the definition of a vertebrate pair allow that the backbone is not necessarily Eulerian but could also be an \(s-t\)-path for some \(s, t \in V\). In this case the solution for the vertebrate pair would again be an Eulerian multi-set \(F\) of edges such that \((V, E(B) \cup F)\) is connected; then \(E(B) \cup F\) is an \(s-t\)-tour. The algorithm for vertebrate pairs that we will describe in later sections extends to this more general version.

Then fix a constant \(\delta \in [0, 1]\) depending on \(\kappa\) and \(\eta\) and change step 5 of Algorithm 1 as follows. If in step 4 we added a \(u-v\)-path \(P_{u,v}\) in \(G[L]\) with \(D_L(u, v) \geq (1 - \delta) \cdot D_L\) for a set \(L \in \mathcal{L}_B\), then we use this path as a backbone in the recursive call of Algorithm 1 instead of constructing a new backbone. This saves the cost \(2D_L\) of the backbone in the recursive call, but we also pay some additional cost. Because the total \(y\)-weight of the sets in \(\mathcal{L}\) visited by \(P\) is not \(D_L\) (as with the old choice of the backbone) but slightly less, we obtain a worse bound in Lemma 11. If for a set \(L \in \mathcal{L}_B\) we did not add a path in \(G[L]\) of length at least \((1 - \delta) \cdot D_L\) in step 4, then we do not change the recursive call of Algorithm 1 in step 5. In this case we gain because the bound on the cost of the edges that we added in step 4 is not tight.

Optimizing \(\delta\) depending on \(\kappa\) and \(\eta\) leads to an improvement of the overall approximation ratio. However, the improvement is small. We will later show that there is a polynomial-time \((2, 14 + \varepsilon)\)-algorithm for vertebrate pairs for any fixed \(\varepsilon > 0\). For \(\kappa = 2\) and \(\eta > 14\) the improvement is less than 0.2, and it is less than 1 for any \(\kappa\) and \(\eta\).

### 5 Computing subtour covers

Very roughly, the algorithm that Svensson, Tarnawski, and Végh [25] use for vertebrate pairs follows the cycle cover approach by Frieze, Galbiati and Maffioli [8]. The algorithm by Frieze, Galbiati and Maffioli always maintains an Eulerian (multi-)set \(H\) of edges and repeatedly computes another Eulerian (multi-)set \(F\) of edges that enters and leaves every connected component of \((V, H)\) at least once. Then it adds the edges of \(F\) to \(H\) and iterates until \((V, H)\) is connected.

In order to achieve a constant approximation ratio, the algorithm for vertebrate pairs and its analysis are much more involved. The main algorithm is essentially due to Svensson [23], and we describe an improved version of this algorithm in Section 6.

In this section we discuss a sub-routine called by Svensson’s algorithm. The sub-routine we present here is an improved version of an algorithm by Svensson, Tarnawski, and Végh [25]. It computes solutions to the Subtour Cover problem, which we define below. One can view the Subtour Cover problem as the analogue of the cycle cover problem that is solved in every iteration of the algorithm by Frieze, Galbiati and Maffioli. However, we do not only require that the multi-set \(F\) of edges that we compute is Eulerian and enters and leaves every connected component of \((V, H)\), but require in addition that every component of \((V, F)\) that crosses the boundary of a set \(L \in \mathcal{L}_{\geq 2}\) is connected to the backbone \(B\). Recall that \(\mathcal{L}_{\geq 2} = \{L \in \mathcal{L} : |L| \geq 2\}\).

**Definition 15.** An instance of Subtour Cover consists of a vertebrate pair \((I, B)\) with \(I = (G, \mathcal{L}, x, y)\) and a multi-set \(H\) of \(E[V \setminus V(B)]\) such that
• $(V, H)$ is Eulerian, and
• $H \cap \delta(L) = \emptyset$ for all $L \in \mathcal{L}_{\geq 2}$.

A solution to such an instance $(I, B, H)$ is a multi-set $F$ of edges such that the following three conditions are fulfilled:

(i) $(V, F)$ is Eulerian.

(ii) $\delta(W) \cap F \neq \emptyset$ for all vertex sets $W$ of connected components of $(V \setminus V(B), H)$.

(iii) If for a connected component $D$ of $(V, F)$ there is a set $L \in \mathcal{L}_{\geq 2}$ with $E(D) \cap \delta(L) \neq \emptyset$, then $V(D) \cap V(B) \neq \emptyset$.

Subtour Cover is very similar to the notions of Subtour Partition Cover from [25] and Local Connectivity ATSP from [23]. The difference between instances of Subtour Cover and Subtour Partition Cover is that we require that $H \cap \delta(L) = \emptyset$ for all $L \in \mathcal{L}_{\geq 2}$ in Definition 15. Moreover, a solution for Subtour Partition Cover is not required to fulfill condition (iii). However, the instances to which Svensson, Tarnawski, and Végh apply their algorithm for Subtour Partition Cover also fulfill the definition of Subtour Cover and the solutions computed by this algorithm also fulfill condition (iii). We include these properties explicitly in Definition 15 because we will exploit them for some improvement in Svensson’s algorithm (see Section 6).

For the analysis of Svensson’s algorithm for vertebrate pairs it is not sufficient to have only a bound on the total cost of a solution to Subtour Cover. In this section we explain an algorithm that computes solutions to Subtour Cover that fulfill certain “local” cost bounds. More precisely, the goal of this section is to show the following theorem, where we write $y_v := y_{\{v\}}$ if $\{v\} \in \mathcal{L}$ and $y_v := 0$ otherwise.

**Theorem 16.** There is a polynomial-time algorithm for Subtour Cover that computes for every instance $(I, B, H)$ a solution $F$ such that

$$c(F) \leq 2 \cdot \text{LP}(I) + \sum_{v \in V \setminus V(B)} 2y_v,$$

and for every connected component $D$ of $(V, F)$ with $V(D) \cap V(B) = \emptyset$ we have

$$c(E(D)) \leq 3 \cdot \sum_{v \in V(D)} 2y_v.$$  

Svensson, Tarnawski, and Végh [25] proved a similar statement, but instead of (6) they showed the weaker bound $c(E(D)) \leq 4 \cdot \sum_{v \in V(D)} 2y_v$.

The reason why we need bounds on the cost of single connected components rather than the total Subtour Cover solution is the following. When Svensson’s algorithm computes a solution $F$ to Subtour Cover, it does not include all edges of $F$ in the tour that it computes but only those edges that are part of some carefully selected connected components of $(V, F)$.

In the rest of this section we prove Theorem 16. We first give a brief outline.

### 5.1 Outline

Let $W_1, \ldots, W_k$ be the vertex sets of the connected components of $(V \setminus V(B), H)$. To find a solution $F$ that fulfills the properties (i) and (ii) we would like to find an integral circulation $x^*$ in $G$ that satisfies $x^*(\delta(W_i)) \geq 2$ for $i = 1, \ldots, k$. Note that $x$ is a fractional circulation with this
property. However, if we include the constraints $x^*(\delta(W_i)) \geq 2$ in the linear program describing a minimum-cost circulation problem, we will in general not obtain an integral optimum solution. Svensson [23] suggested the following. We can introduce new vertices $a_i$ for $i = 1, \ldots, k$ and reroute one unit of flow going through the set $W_i$ through the new vertex $a_i$. Then we can add constraints $x^*(\delta^-(a_i)) = 1$ to our flow problem and maintain integrality. After solving the minimum-cost circulation problem, we can map the one unit of flow through $a_i$ back to some flow entering and leaving $W_i$ (with some small additional cost).

The bound (5) is obtained by minimizing the total cost of the circulation. The most difficult properties to achieve are (iii) and (6). If we have (iii), it is relatively easy to obtain a bound of a similar form as (6) (with some other constant): we can add constraints of the form $x^*(\delta^-(v)) \leq \lceil x(\delta^-(v)) \rceil$ to our minimum-cost circulation problem. Because of (iii) and the definition of the induced cost function $c$, this implies a bound similar to (6).

To achieve property (iii), Svensson, Tarnawski, and Végh [25] introduced the concept of the split graph. This graph contains two copies of every vertex of the original graph $G$. Every Eulerian edge set in the split graph can be projected to an Eulerian edge set in the original graph $G$. The crucial property of the split graph is that every cycle that contains an edge corresponding to $e \in \delta_G(L)$ for some $L \in \mathcal{L}_{\geq 2}$ also contains a copy of a backbone vertex $v \in V(B)$. Therefore, if we round a circulation in the split graph (and then project the solution back to $G$), we will automatically fulfill property (iii).

While every circulation in the split graph can be projected to a circulation in the original graph $G$, we cannot lift any arbitrary circulation in $G$ to a circulation in the split graph. However, Svensson, Tarnawski, and Végh [25] showed that this is possible for every solution $x$ to (ATSP LP). For this, they use a so-called witness flow. We will choose the witness-flow with a certain minimality condition to achieve the bound (6), improving on the Subtour Cover algorithm from [25]. To obtain the improved bound we also choose the flow that is rerouted through the auxiliary vertices $a_i$ more carefully.

Because we cannot lift an arbitrary circulation in $G$ to a circulation in the split graph $G^{01}$ of $G$, we proceed in the following order. First, we lift the circulation $x$ to a circulation $z$ in the split graph $G^{01}$. Then we add the auxiliary vertices $a_i$ to $G$ and add the two corresponding copies $a_i^0$ and $a_i^1$ to the split graph $G^{01}$. In the resulting split graph $\bar{G}^{01}$ we reroute flow through the new

Figure 4: Overview of the different graphs and circulations occurring in the proof of Theorem 16. The integral circulation $\bar{x}^*$ corresponds to an Eulerian (multi) edge set $\bar{F}$ in $\bar{G}$.
auxiliary vertices $a_0^i, a_1^i$ and round our fractional circulation to an integral one. See Figure 4.

We now explain our algorithm in detail.

5.2 The split graph

In this section we explain the concept of the split graph due to Svensson, Tarnawski, and Végh (in earlier versions of [25]). This is an important tool for achieving property (iii) of a solution to Subtour Cover. This property will also be crucial in the proof of (6). For defining the split graph, we number the non-singleton elements of our laminar family $L$ as follows. Number $L \geq 2 \cup \{ V \} = \{ L_1, \ldots, L_r \text{max} \}$ such that $|V| = |L_1| \geq \cdots \geq |L_r \text{max}| \geq 2$. Let $r(v) := \max \{ i : v \in L_i \}$, and call an edge $e = (v, w) \in E$ forward if $r(v) < r(w)$, backward if $r(v) > r(w)$, and neutral if $r(v) = r(w)$. See Figure 5.

We will need the following simple observation about cycles in $G$. A cycle is a connected digraph in which every vertex has in-degree and out-degree exactly 1.

Lemma 17. Let $C$ be the edge set of a cycle. If there exists a set $L \in L_{\geq 2}$ with $C \cap \delta(L) \neq \emptyset$, then $C$ contains a forward edge and a backward edge.

Proof. Because $C$ is Eulerian there exists an edge $e = (v, w) \in C \cap \delta^+(L)$. By the choice of the numbering $L_1, \ldots, L_r \text{max}$, we have $L_{r(v)} \subseteq L$ and hence $w \notin L_{r(w)}$. Therefore, the cycle with edge set $C$ contains vertices $v, w$ with $r(v) \neq r(w)$. Hence, $C$ contains both a forward and a backward edge. □

Next we define the split graph $G^{01}$ of $G$ and extend the cost function $c$ to it.

- For every vertex $v \in V$ it contains two vertices $v^0$ and $v^1$ (on the lower and upper level).
- For every $v \in V$ it contains an edge $e^0_v = (v^1, v^0)$ with $c(e^0_v) = 0$.
- For every $v \in V(B)$ it also contains an edge $e^1_v = (v^0, v^1)$ with $c(e^1_v) = 0$.
- For every forward edge $e = (v, w) \in E$, the split graph contains an edge $e^0 = (v^0, w^0)$ with $c(e^0) = c(e)$. 

Figure 5: The laminar family $L \cup \{ V \} = \{ L_1, \ldots, L_{11} \}$. In this example, the set $L_2 \setminus (L_6 \cup L_4)$ is the set of all vertices $v$ with $r(v) = 2$; it is shown in blue.
• For every backward edge \( e = (v, w) \in E \), the split graph contains an edge \( e^1 = (v^1, w^1) \) with \( c(e^1) = c(e) \).

• For every neutral edge \( e = (v, w) \in E \), the split graph contains edges \( e^0 = (v^0, w^0) \) and \( e^1 = (v^1, w^1) \) with \( c(e^0) = c(e^1) = c(e) \).

We write \( V^0 := \{ v^0 : v \in V \} \) and call \( G^{01}[V^0] \) the lower level of the split graph \( G^{01} \). Similarly, we write \( V^1 := \{ v^1 : v \in V \} \) and call \( G^{01}[V^1] \) the upper level of \( G^{01} \). For a set \( W \subseteq V \) let \( W^{01} := \{ v^j : v \in W, j \in \{0, 1\} \} \) be the vertex set of \( G^{01} \) that corresponds to \( W \).

For any subgraph of \( G^{01} \) we obtain a subgraph of \( G \) (its image) by replacing both \( v^0 \) and \( v^1 \) by \( v \) and removing loops. Then, obviously, the image of a cycle is an Eulerian graph. The next lemma shows how we can use the split graph to achieve property (iii) of a solution to Subtour Cover.

**Lemma 18.** If the image of a cycle in \( G^{01} \) contains an edge \( e \in \delta(L) \) for some \( L \in \mathcal{L}_{\geq 2} \), it also contains a vertex of \( B \).

**Proof.** Let \( C^{01} \) be a cycle in \( G^{01} \) such that its image \( C \) (an Eulerian subgraph of \( G \)) contains an edge \( e \in \delta(L) \) for some \( L \in \mathcal{L}_{\geq 2} \). By Lemma 17, \( C \) contains a forward edge and a backward edge. Therefore \( C^{01} \) visits both levels of \( G^{01} \) and thus contains an edge \( e^1_v \) for some \( v \in V(B) \). \( \square \)

### 5.3 Witness flows

We now want to map \( x \) to a circulation \( z \) in the split graph \( G^{01} \). To this end, we define a flow \( f \leq x \), which we will call a witness flow. In the construction of \( z \), we will map the witness flow \( f \) to the lower level of \( G^{01} \) and map the remaining flow \( x - f \) to the upper level of \( G^{01} \). See Figure 6.

**Definition 19** (witness flow). Let \( x' \) be a circulation in \( G \). Then we call a flow \( f' \) in \( G \) a witness flow (for \( x' \)) if

(a) \( f'(e) = 0 \) for every backward edge \( e \);

(b) \( f'(e) = x'(e) \) for every forward edge \( e \);

(c) \( 0 \leq f'(e) \leq x'(e) \) for every neutral edge \( e \); and

(d) \( f'(\delta^+(v)) \geq f'(\delta^-(v)) \) for all \( v \in V \setminus V(B) \).

The concept of witness flow was introduced in [25]. We now show that the pairs \( (x', f') \) where \( f' \) is a witness flow for the circulation \( x' \) in \( G \), correspond to circulations in the split graph \( G^{01} \).

**Lemma 20.** Let \( z' \) be a circulation in \( G^{01} \). Define \( \pi(z') := (x', f') \) where \( x', f' \) are flows in \( G \) defined by

- \( x'(e) := z'(e^0) + z'(e^1) \), and

- \( f'(e) := z'(e^0) \),

where we set \( z'(e^1) := 0 \) for forward edges \( e \) and \( z'(e^0) := 0 \) for backward edges \( e \). Then \( x' \) is a circulation in \( G \) with \( c(x') = c(z') \) and \( f' \) is a witness flow for \( x' \).

**Proof.** (a) holds because for a backward edge \( e \), the graph \( G^{01} \) does not contain an edge \( e^0 \). Similarly, (b) holds because for a forward edge \( e \), the graph \( G^{01} \) does not contain an edge \( e^1 \). Property (c) is obvious by construction and (d) holds because for \( v \in V \setminus V(B) \) the split graph does not contain an edge \( e^1_v \). \( \square \)
Figure 6: An example of the construction of the circulation $z$ in $G^{01}$. Picture (a) shows the laminar family $\mathcal{L}_{\geq 2} = \{L_2, L_3, L_4\}$ and in blue the backbone $B$. Picture (b) shows a solution $x$ to (ATSP LP) where we have $x_e = \frac{1}{2}$ for all edges; a witness flow $f$ is shown in red. The vertices in $V(B)$ are shown as squares. Every cycle crossing the boundary of a set $L \in \mathcal{L}_{\geq 2}$ contains both a green and a red edge. Picture (c) shows the resulting circulation $z$ in $G^{01}$, where we have $z_e > 0$ for every thick edge $e$ and and $z_e = 0$ for all thin edges. The green vertices are those on the upper level of the split graph; the red vertices are those on the lower level. The flow $x - f$ is mapped to the upper level (green) and the flow $f$ is mapped to the lower level (red).
Having a circulation \( x' \) in \( G \) and a witness flow \( f' \) for \( x' \), we can map \( x' \) to a circulation \( z' \) in \( G^{01} \) with \( \pi(z') = (x', f') \) as follows:

- For every edge \( e^0 \) of the lower level of \( G^{01} \) we set \( z'(e^0) = f'(e) \).
- For every edge \( e^1 \) of the upper level of \( G^{01} \) we set \( z'(e^1) = x'(e) - f'(e) \).
- For every edge \( e^1_v \) (for \( v \in V(B) \)) we set \( z(e^1_v) = \max\{0, f'((\delta^-)(v)) - f'((\delta^+)(v))\} \).
- For every edge \( e^1_u \) (for \( v \in V \)) we set \( z(e^1_u) = \max\{0, f'((\delta^+)(v)) - f'((\delta^-)(v))\} \).

Notice that \( x'(e) = z'(e^0) \) for every forward edge \( e \) and \( x'(e) = z'(e^1) \) for every backward edge \( e \). Moreover, \( x'(e) = z'(e^0) + z'(e^1) \) for every neutral edge \( e \). Furthermore, \( z' \) indeed defines a circulation in \( G^{01} \) because \( f'((\delta^+)(v)) \geq f'((\delta^-)(v)) \) for all \( v \in V \setminus V(B) \).

The following was already proved in [25]. Here we give a simpler proof.

**Lemma 21.** Let \( (I, B) \) be a vertebrate pair, with \( I = (G, \mathcal{L}, x, y) \). Then there exists a witness flow \( f \) for \( x \).

**Proof.** Consider \( G' \), which arises from \( G \) by adding a new vertex \( a \) and edges \((a, v)\) for all \( v \in V \) and edges \((v, a)\) for all \( v \in V(B) \). Set \( l(e') = 0 \) and \( u(e') = \infty \) for the new edges. Moreover, for \( e \in E \) set the lower bound \( l(e) \) and the upper bound \( u(e) \) according to the requirements from Definition 19, i.e. set \( u(e) = x(e) \) if \( e \) is a forward or neutral edge and \( u(e) = 0 \) otherwise and set \( l(e) = x(e) \) if \( e \) is a forward edge and \( l(e) = 0 \) otherwise.

Then we are looking for a circulation \( f' \) in \( G' \) with \( l \leq f' \leq u \). By Hoffman’s circulation theorem, this exists if

\[
  l((\delta^-)(U)) \leq u((\delta^+)(U)) \tag{7}
\]

for all \( U \subseteq V \cup \{a\} \). We show that this is indeed true. Suppose not, and let \( U \) be a minimal set violating (7). Since (7) obviously holds whenever \( a \in U \) or \( B \cap U \neq \emptyset \), we have \( U \subseteq V \setminus V(B) \). Let \( i \) be the largest index so that \( U \cap L_i \neq \emptyset \). See Figure 7.

**Case 1:** \( U \setminus L_i \neq \emptyset \).

Then (by the minimality of \( U \)) we have \( l((\delta^-)(U \cap L_i)) \leq u((\delta^+)(U \cap L_i)) \) and \( l((\delta^-)(U \setminus L_i)) \leq u((\delta^+)(U \setminus L_i)) \). Since all edges from \( U \setminus L_i \) to \( U \cap L_i \) are forward edges and all edges from \( U \cap L_i \) to \( U \setminus L_i \) are backward edges, we get

\[
  l((\delta^-)(U)) + x((\delta^+(U \setminus L_i) \cap (\delta^-)(U \cap L_i))) = l((\delta^-)(U \cap L_i)) + l((\delta^-)(U \setminus L_i)) \leq u((\delta^+(U \cap L_i)) + u((\delta^+(U \setminus L_i))
\]

\[
  = u((\delta^+(U)) + x((\delta^+(U \setminus L_i) \cap (\delta^-)(U \cap L_i)))
\]

and hence (7), which is a contradiction to the choice of \( U \).

**Case 2:** \( U \subseteq L_i \).

Then \( r(u) = i \) for all \( u \in U \) and \( r(w) \geq i \) for all \( w \in L_i \). Hence \( l((\delta^-)(U)) \leq x((\delta^-)(L_i) \cap (\delta^-)(U)) \) because we have \( l(e) > 0 \) only for forward edges and all edges in \( (\delta^-)(U) \setminus (\delta^-)(L_i) \) are neutral or backward edges. Moreover, edges in \( (\delta^+(U) \setminus (\delta^+(L_i)) \) are not backward edges, implying \( x((\delta^+(U) \setminus (\delta^+(L_i)))) = u((\delta^+(U)) \setminus (\delta^+(L_i))) \leq u((\delta^+(U))) \). Therefore,

\[
  l((\delta^-)(U)) \leq x((\delta^-)(L_i)) \cap (\delta^-)(U))
  = x((\delta^-)(L_i)) + x((\delta^+(U) \setminus (\delta^+(L_i)))) - x((\delta^-)(L_i) \setminus U)) \leq x((\delta^-)(L_i)) + u((\delta^+(U)) - x((\delta^-)(L_i) \setminus U)).
\]
Since \( L_i \setminus U \neq \emptyset \) (because \( L_i \cap V(B) \neq \emptyset = U \cap V(B) \)), we have \( x(\delta^-(L_i \setminus U)) \geq 1 \). Moreover, \( L_i \in \mathcal{L} \cup \{V\} \) implies \( x(\delta(L_i)) \in \{0, 2\} \) and hence \( x(\delta^-(L_i)) \leq 1 \). Hence (7) follows, which is again a contradiction.

Working with an arbitrary witness flow \( f \) is sufficient to obtain a constant-factor approximation for ATSP and this is essentially what Svensson, Tarnawski, and Végh did. However, to obtain a better approximation ratio we will not work with an arbitrary witness flow \( f \), but will choose \( f \) with some additional properties. Recall that \( W_1, \ldots, W_k \) are the vertex sets of the connected components of \( (V \setminus V(B), H) \).

**Lemma 22.** We can compute in polynomial time a witness flow \( f \) for \( x \) such that

1. the support of \( f \) is acyclic, and
2. \( \sum_{i=1}^{k} f(\delta(W_i)) \leq \sum_{i=1}^{k} f'(\delta(W_i)) \) for every witness flow \( f' \) for \( x \).

**Proof.** We first compute a witness flow \( \tilde{f} \) by minimizing \( \sum_{i=1}^{k} f(\delta(W_i)) \) subject to the constraints (a) – (d) from Definition 19. This linear program is feasible by Lemma 21. Then the flow \( \tilde{f} \) fulfills property (f).

To compute the flow \( f \) we minimize \( \sum_{e \in E} f(e) \) subject to the constraints (a) – (d) and \( f(e) \leq \tilde{f}(e) \) for all \( e \in E \). This linear program is feasible because \( \tilde{f} \) is a feasible solution. Then \( f \) is a witness flow for \( x \) with \( \sum_{i=1}^{k} f(\delta(W_i)) \leq \sum_{i=1}^{k} \tilde{f}(\delta(W_i)) \). Since the flow \( \tilde{f} \) fulfills property (f), the same holds for the flow \( f \).

Suppose \( f \) does not fulfill (e), i.e. \( f \) is not acyclic. Then there is a cycle \( C \subseteq E \) with \( f(e) > 0 \) for all \( e \in C \). As \( f \) fulfills (a), the set \( C \) does not contain any backward edge. This implies that \( C \) also contains no forward edge because \( C \) is a cycle. Let \( \varepsilon := \min_{e \in C} f(e) \). For \( e \in E \) we set \( f'(e) := f(e) - \varepsilon \leq \tilde{f}(e) \) if \( e \in C \) and \( f'(e) := f(e) \leq \tilde{f}(e) \) otherwise. Because \( C \) contains neither forward nor backward edges, \( f' \) fulfills (a) and (b). By the choice of \( \varepsilon \), we have \( f'(e) \geq 0 \) for all \( e \in E \), implying (c). Finally, \( f'(\delta^+(v)) - f'(\delta^-(v)) = f(\delta^+(v)) - f(\delta^-(v)) \geq 0 \) for all \( v \in V \setminus V(B) \), where we used that \( C \) is a cycle and \( f \) fulfills (d). This shows that \( f' \) is a witness flow and \( f'(e) \leq \tilde{f}(e) \) for all \( e \in E \), but \( \sum_{e \in E} f'_e < \sum_{e \in E} f_e \), a contradiction to the choice of \( f \).

**5.4 Rerouting and rounding**

Recall that the sets \( W_1, \ldots, W_k \) are the vertex sets of the connected components of \( (V \setminus V(B), H) \). Thus they are pairwise disjoint subsets of \( V \setminus V(B) \).
Lemma 23. Let $i \in \{1, \ldots, k\}$ and $v, w \in W_i$. Then $r(v) = r(w)$.

Proof. For all $L \in \mathcal{L}_{\geq 2}$ we have $H \cap \delta(L) = \emptyset$ and therefore $W_i \subseteq L$ or $W_i \cap L = \emptyset$. This implies $r(v) = \max\{j : v \in L_j\} = \max\{j : w \in L_j\} = r(w)$. 

We will now work with a flow $f$ as in Lemma 22. Let $G_f$ denote the residual graph of the flow $f$ and the graph $G$ with edge capacities $\chi$. So for every edge $e = (v, w) \in E$ with $f(e) < x(e)$, the residual graph contains an edge $(v, w)$ with residual capacity $u_f((v, w)) = x(e) - f(e)$. For every edge $e = (v, w) \in E$ with $f(e) > 0$ the residual graph contains an edge $(w, v)$ with residual capacity $u_f((w, v)) = f(e)$. Parallel edges can arise.

We will transform the graph $G$ into another graph $\bar{G}$. The circulation $z$ in $G^{01}$ will be transformed into a circulation $\bar{z}$ in the split graph $G^{01}$ of $\bar{G}$. We construct $\bar{G}$ from $G$ by doing the following for $i = 1, \ldots, k$.

We add an auxiliary vertex $a_i$ to $G$ and set $r(a_i) := r(v)$ for $v \in W_i$; this is well-defined by Lemma 23. Let $\bar{W}_i$ be the vertex set of the first strongly connected component of $G_f[W_i]$ in some topological order. For every edge $(v, w) \in \delta^-(\bar{W}_i)$ we add an edge $(v, a_i)$ of the same cost. Similarly, for every edge $(v, w) \in \delta^+(\bar{W}_i)$ we add an edge $(a_i, w)$ of the same cost. Note that then a new edge is a forward/backward/neutral edge if and only if its corresponding edge in $G$ is forward/backward/neutral. Then the split graph $\bar{G}^{01}$ of $\bar{G}$ contains new vertices $a_i^0, a_i^1$, connected by an edge $e_i^{a_i} = (a_i^1, a_i^0)$ of cost zero. Let $\bar{G}$ the graph resulting from $G$ by the modifications described above and let $\bar{G}^{01}$ be its split graph.

We will now reroute some of the flow $z$ going through $\bar{W}_i$ such that it goes through one of the new vertices $a_i^0, a_i^1$. See Figure 8. We need the following lemma, where $\chi^F \in \mathbb{Z}_{\geq 0}^E$ denotes the incidence vector of $F$ for any multi-subset $F$ of $E$.

Lemma 24. Let $G'$ be a directed graph and $z'$ a circulation in $G'$. Let $U \subseteq V(G')$ with $z'(\delta(U)) \geq 2$. Then we can compute in polynomial time a multiset $P$ of paths in $G'[U]$ and for every $P \in P$ starting in $s \in U$ and ending in $t \in U$

- a weight $\lambda_P > 0$,
- an edge $e^\text{in}_P = (s', s) \in \delta^-(U)$, and
- an edge $e^\text{out}_P = (t, t') \in \delta^+(U)$,

such that $\sum_{P \in P} \lambda_P = 1$ and

$$\sum_{P \in P} \lambda_P \cdot \left(\chi^{e^\text{in}_P} + \chi^{E(P)} + \chi^{e^\text{out}_P}\right) \leq z'.$$

Proof. We contract $V(G') \setminus U$ to a vertex $v_{\text{outside}}$. Then $z'(\delta(v_{\text{outside}})) = z'(\delta(U)) \geq 2$. Because $z'$ remains a circulation, we can compute in polynomial time a set $\mathcal{C}$ of cycles containing $v_{\text{outside}}$ and weights $\lambda_C > 0$ for $C \in \mathcal{C}$ with $\sum_{C \in \mathcal{C}} \lambda_C = 1$ such that

$$\sum_{C \in \mathcal{C}} \lambda_C \cdot \chi^{E(C)} \leq z'.$$

After undoing the contraction, every cycle $C$ results in an edge $e^{\text{in}} = (s', s) \in \delta^-(U)$, an edge $e^{\text{out}} = (t, t') \in \delta^+(U)$, and an $s$-$t$-path $P$ in $G'[U]$. 

We construct a circulation $\bar{z}$ in $\bar{G}^{01}$ from $z$ by doing the following for $i = 1, \ldots, k$. We apply Lemma 24 to the vertex set $U = \bar{W}_i^{01}$. We partition the resulting set $P$ into sets $P^0$ and $P^1$ such
that $\mathcal{P}^0$ contains the paths $P \in \mathcal{P}$ for which $e^i_P$ is contained in the lower level of the split graph and $\mathcal{P}^1$ contains the paths $P \in \mathcal{P}$ for which $e^o_P$ is contained in the upper level of the split graph. Since $\sum_{P \in \mathcal{P}} \lambda_P = 1$, we have $\sum_{P \in \mathcal{P}^q} \lambda_P \geq \frac{1}{q}$ for some $q \in \{0, 1\}$. We can choose values $0 \leq \lambda'_P \leq \lambda_P$ such that $\sum_{P \in \mathcal{P}^q} \lambda'_P = \frac{1}{2}$. To obtain $\bar{z}$ from $z$, we do the following for every $P \in \mathcal{P}^q$:

- We decrease the flow on $e^i_P$ and increase the flow on its corresponding edge in $\delta^-(a^0_i)$ by $\lambda'_P$.
- We decrease the flow on every edge $e \in E(P)$ by $\lambda'_P$.
- Let $p = 0$ if $e^o_P$ is contained in the lower level of the split graph and $p = 1$ otherwise. We decrease the flow on $e^o_P$ and increase the flow on its corresponding edge in $\delta^+(a^0_i)$ by $\lambda'_P$.
- Because $W_i \cap V(B) = \emptyset$, the path $P$ contains no edge from the lower to the upper level; hence $p \leq q$. If $p < q$, i.e. $q = 1$ and $p = 0$, we increase the flow on $e^+_a$ by $\lambda'_P$.

Note that we maintain a circulation in the split graph $\bar{G}^0$.

Let $\bar{z}$ be the circulation in $\bar{G}^0$ resulting from $z$. Note that $c(\bar{z}) \leq c(z)$. Moreover, $\bar{z}$ is a circulation such that for every $i \in \{1, \ldots, k\}$ we have $\bar{z}(\delta^-(a^0_i)) = \frac{1}{2}$ or $\bar{z}(\delta^-(a^1_i)) = \frac{1}{2}$. Because we could only reroute $\frac{1}{2}$ unit of flow through $a^0_i$ or $a^1_i$, we consider the circulation $2\bar{z}$.

We round $2\bar{z}$ to an integral circulation: by Corollary 12.2b of [18], we can find in polynomial time an integral circulation $\bar{z}^*$ in $\bar{G}^0$ with

(A) $0 \leq \bar{z}^*(e) \leq \lfloor 2\bar{z}(e) \rfloor$ for all $e \in E(\bar{G}^0)$,
(B) $c(\bar{z}^*) \leq c(2\bar{z})$,
(C) $\bar{z}^*(\delta^-(v^1)) \leq \lfloor 2\bar{z}(\delta^-(v^1)) \rfloor$ for all $v \in V$, and
(D) for every $i \in \{1, \ldots, k\}$ we have $\bar{z}^*(\delta^-(a^0_i)) = 1$ or $\bar{z}^*(\delta^-(a^1_i)) = 1$.

Let $(\bar{x}, \bar{f}) := \pi(\bar{z})$ and $(\bar{x}^*, \bar{f}^*) := \pi(\bar{z}^*)$. Let $\bar{F} \subseteq E(\bar{G})$ be the multi-set of edges with $\chi^i_{\bar{F}} = \bar{x}^*$; see Figure 8 (c). Then $\bar{F}$ is Eulerian because $\bar{x}^*$ is a circulation.

We now show several properties of $\bar{F}$, before we show how to map $\bar{F}$ to a solution $F$ for Subtour Cover in $G$ (in Section 5.5). First we observe

$$c(\bar{F}) = c(\bar{x}^*) = c(\bar{z}^*) \leq 2 \cdot c(\bar{z}) = 2 \cdot c(z) = 2 \cdot \text{LP}(I).$$

The following lemma will be used in the proof of property $(ii)$ of Definition 15.

**Lemma 25.** Let $i \in \{1, \ldots, k\}$. Then $|\delta^i_{\bar{F}}(a^i_i)| = 1$.

**Proof.** We have

$$|\delta^i_{\bar{F}}(a^i_i)| = \bar{z}^*(\delta^-(a^i_i)) = \bar{x}^*(\delta^-(a^i_i)) + \bar{z}^*(\delta^-(a^0_i) \setminus \{e^0_{a^i_i}\})$$

By property (D), we have $\bar{z}^*(\delta^-(a^0_i)) = 1$ or $\bar{z}^*(\delta^-(a^1_i)) = 1$. Moreover, by property (A), the support of the integral flow $\bar{z}^*$ is contained in the support of the flow $\bar{z}$. If we have $\bar{z}^*(\delta^-(a^1_i)) = 1$, then we have by construction of $\bar{z}$ that $\bar{z}^*(e) \leq \lfloor 2\bar{z}(e) \rfloor = 0$ for all $e \in \delta^-(a^0_i) \setminus \{e^0_{a^i_i}\}$, implying $|\delta^i_{\bar{F}}(a^i_i)| = 1$. Otherwise, we have $\bar{z}^*(\delta^-(a^0_i)) = 1$ and by construction of $\bar{z}$ we have $\bar{z}(\delta^-(a^1_i)) = 0$ and $\bar{z}(e^0_{a^i_i}) = 0$. Therefore, by property (A) we have $\bar{z}^*(\delta^-(a^1_i)) = 0$ and $\bar{z}^*(e^0_{a^i_i}) = 0$.

The proof of the following lemma is where we use our choice of $f$ as in Lemma 22. Here, an arbitrary witness flow is not sufficient. See Figure 9 (a) – (b) for an illustration.

**Lemma 26.** The flows $\bar{f}$ and $\bar{f}^*$ have acyclic support.
Figure 8: Example of the construction of the solution $F$ from the witness flow $f$. On all pictures, a set $W_i$ (blue with white interior) and the subset $\hat{W}_i$ (blue and filled) is shown. The pictures show only edges with at least one endpoint in $W_i$. Picture (a) shows (parts of) a possible solution $x$ to (ATSP LP) (green and red) and a witness flow $\hat{f}$ (red). The edges drawn with a single line have value $\frac{1}{4}$, the edges drawn with a double line have value $\frac{1}{2}$. Pictures (b1) and (b2) show two possible circulations $\bar{x}$ in $\bar{G}$ that could result from rerouting flow through $a_i$ (blue); the witness flow $\hat{f}$ is shown in red. Picture (c) shows in orange an possible integral flow $\bar{x}^*$ in $\bar{G}$ that could result if we rerouted flow through $a_i$ as in (b2); The orange edges are elements of the edge set $\bar{F}$ with $\chi^{\bar{F}} = \bar{x}^*$. Picture (d) shows the result of mapping $\bar{F}$ back to $G$. In blue the path $P_i$ in $G[W_i]$ is shown; it completes the orange edges to a circulation.
Figure 9: Illustration of the proof of Lemma 26 and the reason why choosing an arbitrary flow $f$ as in Lemma 21 is not sufficient. Three sets $W_i$ are shown in blue with white interior; pictures (a)–(c) also show their subsets $\hat{W}_i$ (blue and filled). Picture (a) shows (parts of) a flow $f$ as in Lemma 21 (red); the thick edges show forward edges. This flow $f$ will not be chosen by our algorithm; it does not minimize $\sum_{i=1}^{k} f(\delta(W_i))$. Picture (b) shows what would happen if we chose this flow anyway. We see a possible result of rerouting this flow through the vertices $a_i \in V(\bar{G})$ (shown in blue). In this example, the support of $f$ contains a cycle $\bar{C}$. Picture (c) shows a corresponding closed walk $C$ in the residual graph $G_f$. The blue edges show paths inside the sets $\hat{W}_i$; these exist because $G_f[\hat{W}_i]$ is strongly connected. Picture (d) shows the flow resulting from $f$ by augmenting along $C$. The augmentation decreased $\sum_{i=1}^{k} f(\delta(W_i))$, but did not change the flow on forward edges.
Proof. Since the support of \( \tilde{f}^* \) is contained in the support of \( \tilde{f} \) by (A), it suffices to show that \( \tilde{f} \) has acyclic support. Suppose the support of \( \tilde{f} \) contains a cycle \( \bar{C} \). Then there exists \( i \in \{1, \ldots, k\} \) such that \( a_i \in V(\bar{C}) \) because otherwise \( \bar{C} \) is contained in the support of \( f \) (which is acyclic). Let \( \bar{e} = (a_i, v) \in E(\bar{C}) \) and let \( e = (u, v) \in \delta^+(\bar{W}_i) \) be the edge of \( G \) corresponding to \( \bar{e} \). Then \( f(e) > 0 \) and hence the residual graph \( G_f \) contains an edge \( (v, u) \in \delta^{-}_G(\bar{W}_i) \). Therefore \( v \notin \bar{W}_i \) since \( \bar{W}_i \) is the vertex set of the first strongly connected component of \( G_f[\bar{W}_i] \). This shows \( E(\bar{C}) \cap \delta(\bar{W}_i \cup \{a_i\}) \neq \emptyset \).

We claim that we can map \( \bar{C} \) to a closed walk \( C \) in the residual graph \( G_f \). See Figure 9 (b) – (c). We first map every edge of the cycle \( \bar{C} \) to its corresponding edge in \( G \). Notice that the resulting edge set \( F \) is not necessarily a cycle: if \( a_i \in V(\bar{C}) \) for some \( i \in \{1, \ldots, k\} \), then \( F \) contains an edge entering \( \bar{W}_i \) and an edge leaving \( \bar{W}_i \), but might be disconnected in between.

We have \( f(e) > 0 \) for every edge \( e \in F \). Thus, by reversing all edges in \( F \) we obtain edges in \( G_f \) (with positive residual capacity \( u_f \)). Moreover, we can complete this edge set to a closed walk \( C \) in \( G_f \) (with positive residual capacity \( u_f \)) by adding only edges of \( G_f[\bar{W}_i] \) for \( i \in \{1, \ldots, k\} \); this is possible because for every \( i \in \{1, \ldots, k\} \), the subgraph \( G_f[\bar{W}_i] \) is strongly connected by the choice of \( \bar{W}_i \). We found a closed walk \( C \) in \( G_f \). Let \( i \in \{1, \ldots, k\} \) such that \( E(\bar{C}) \cap \delta(\bar{W}_i \cup \{a_i\}) \neq \emptyset \). Then \( E(\bar{C}) \cap \delta(\bar{W}_i) \neq \emptyset \).

Also note that \( r(v) \geq r(w) \) for all \( (v, w) \in E(C) \): every edge \( (v, w) \in E(G_f) \) of \( C \) has a corresponding edge \( (w, v) \in E(G) \) with \( f(e) > 0 \) or it has both endpoints in the same set \( \bar{W}_i \). In the first case, we can conclude that \( (w, v) \) is not a backward edge and hence \( r(w) \leq r(v) \). In the latter case, \( r(v) = r(w) \) by Lemma 23. Since \( C \) is a closed walk we conclude that \( r(v) = r(w) \) for all \( v, w \in V(C) \).

This shows that augmenting \( f \) along the closed walk \( C \) changes flow only on neutral edges. We augment by some sufficiently small but positive amount and maintain a witness flow. We claim that this augmentation decreases \( \sum_{i=1}^{k} f(\delta(\bar{W}_i)) \), which contradicts our choice of \( f \). See Figure 9 (d). The only edges of \( C \) contained in a cut \( \delta(\bar{W}_i) \) for some \( i \in \{1, \ldots, k\} \) result from mapping the edges of the cycle \( \bar{C} \) in \( G \) to \( G_f \) and reversing them; for these edges the augmentation decreases the flow value. The other edges that we added to \( C \) are contained in some \( G_f[\bar{W}_i] \) for \( i \in \{1, \ldots, k\} \) and hence they do not cross the boundary of any set \( W_i \). Therefore, augmenting \( f \) along \( C \) decreases the flow value on all edges in \( E(C) \cap (\delta(W_1) \cup \cdots \cup \delta(W_k)) \) and we have already shown that this set is nonempty. \[ \square \]

Lemma 27. Let \( \bar{D} \) be a connected component of \( (V, \bar{F}) \) with \( V(\bar{D}) \cap V(B) = \emptyset \). Then \( \tilde{f}^*(E(\bar{D})) = 0 \).

Proof. Because \( \tilde{f}^* \) is a witness flow, we have \( \tilde{f}^*(\delta^- (v)) \leq \tilde{f}^*(\delta^+ (v)) \) for every \( v \in V(\bar{D}) \). Since
\[
\tilde{f}^*(E(\bar{D})) = \sum_{v \in V(\bar{D})} \tilde{f}^*(\delta^- (v)) \leq \sum_{v \in V(\bar{D})} \tilde{f}^*(\delta^+ (v)) = \tilde{f}^*(E(\bar{D})),
\]
we have \( \tilde{f}^*(\delta^- (v)) = \tilde{f}^*(\delta^+ (v)) \) for every \( v \in V(\bar{D}) \). In other words, \( \tilde{f}^* \) restricted to \( E(\bar{D}) \) is a circulation. Because the support of \( \tilde{f}^* \) is is acyclic by Lemma 26, this implies \( \tilde{f}^*(E(\bar{D})) = 0 \). \[ \square \]

Lemma 28. Let \( i \in \{1, \ldots, k\} \). Then \( \tilde{F} \cap \delta(W_i \cup \{a_i\}) \neq \emptyset \).

Proof. By Lemma 25 there exists an edge \( \bar{e} = (v, a_i) \in \bar{F} \). If \( v \notin \bar{W}_i \), we have \( \bar{e} \notin \tilde{F} \cap \delta(W_i \cup \{a_i\}) \). Otherwise, the edge \( e \) of \( G \) that corresponds to \( \bar{e} \) fulfills \( e \in E[\bar{W}_i] \cap \delta^- (\bar{W}_i) \). Therefore, we have \( f(e) = x(e) \) as otherwise also the residual graph \( G_f \) contained \( e \), contradicting the choice of \( \bar{W}_i \). This implies
\[
\tilde{z}^*(\bar{e}^\dagger) \leq [2\tilde{z}(\bar{e}^\dagger)] \leq [2z(e^\dagger)] = [2(x(e) - f(e))] = 0.
\]
But then
\[ \tilde{f}^*(\bar{e}) = \tilde{z}^*(\bar{e}^0) = \tilde{z}^*(\bar{e}^0) + \tilde{z}^*(\bar{e}^1) = \bar{a}^*(\bar{e}) \geq 1, \]
because \( \bar{e} \in \bar{F} \). By Lemma 27, this implies that the connected component \( \bar{D} \) of \( (V(\bar{G}), \bar{F}) \) that contains \( a_i \) also contains a vertex \( w \in V(B) \). Since \( W_i \cap V(B) = \emptyset \), this completes the proof.

In the proof of property (iii) of Definition 15 we will use the following observation.

**Lemma 29.** Let \( \bar{D} \) be a connected component of \( (V, \bar{F}) \) with \( V(\bar{D}) \cap V(B) = \emptyset \). Then \( E(\bar{D}) \) contains no forward edge.

**Proof.** By Lemma 27 we have \( \bar{f}^*(E(\bar{D})) = 0 \). Since \( \bar{f}^* \) is a witness flow for \( \bar{x}^* = \bar{\chi}^\bar{F} \), this implies that \( E(\bar{D}) \) contains no forward edge. \( \square \)

The following lemma will be used in the proof of (6) of Theorem 16.

**Lemma 30.** Let \( \bar{D} \) be a connected component of \( (V, \bar{F}) \) with \( V(\bar{D}) \cap V(B) = \emptyset \). Then for every vertex \( v \in V(\bar{D}) \setminus \{a_1, \ldots, a_k\} \) with \( y_v > 0 \) we have \( |\delta^-_\bar{F}(v)| \leq 2 \).

**Proof.** For every vertex \( v \in V(\bar{D}) \) we have
\[
|\delta^-_\bar{F}(v)| = \bar{x}^*(\delta^-(v)) = \tilde{z}^*(\delta^-(v^1)) + \tilde{z}^*(\delta^-(v^0) \setminus \{e^1_v\})
= \tilde{z}^*(\delta^-(v^1)) + \bar{f}^*(\delta^-(v))
= \tilde{z}^*(\delta^-(v^1)),
\]
where we used Lemma 27. For all \( v \in V(\bar{D}) \setminus \{a_1, \ldots, a_k\} \) with \( y_v > 0 \) we have \( \{v\} \in \mathcal{L} \) and hence \( x(\delta^-(v)) = 1 \) by property (iii) of Definition 4. Therefore, by (C) we get
\[
|\delta^-_\bar{F}(v)| = \tilde{z}^*(\delta^-(v^1)) \leq [2\tilde{z}(\delta^-(v^1))] \leq [2x(\delta^-(v))] = 2.
\]
\( \square \)

### 5.5 Mapping back to \( G \)

We now transform \( \bar{F} \) into a solution \( F \) of the Subtour Cover problem in \( G \). See Figure 8 (c)–(d). By Lemma 25, every vertex \( a_i \) for \( i \in \{1, \ldots, k\} \) has exactly one incoming edge in \( \bar{F} \) and because \( \bar{F} \) is Eulerian, \( a_i \) also has exactly one outgoing edge. We replace all the edges in \( \delta^-_\bar{F}(a_i) \) for \( i \in \{1, \ldots, k\} \) by their corresponding edges in \( G \). For every \( i \in \{1, \ldots, k\} \) we added one edge \( (v, s) \in \delta^-(W_i) \) and an edge \( (t, w) \in \delta^+(W_i) \); to obtain an Eulerian edge set we add an \( s\)-\( t \)-path \( P_i \) in \( G[W_i] \). Such a path exists because \( G[W_i] \) is strongly connected. Let \( F \) be the resulting Eulerian multi-set of edges in \( G \). Note that if two vertices \( a, b \in V(G) \) are in the same connected component of \( (V(\bar{G}), \bar{F}) \), then they are also in the same connected component of \( (V(G), F) \).

**Lemma 31.** Let \( i \in \{1, \ldots, k\} \) and \( L \in \mathcal{L}_{\geq 2} \). Then \( E(P_i) \cap \delta(L) = \emptyset \).

**Proof.** We have \( H \cap \delta(L) = \emptyset \) and the sets \( W_1, \ldots, W_k \) are the vertex sets of the connected components of \( (V \setminus V(B), H) \). Now \( E(P_i) \subseteq E[W_i] \) implies \( E(P_i) \cap \delta(L) = \emptyset \). \( \square \)

We claim that \( F \) is a solution to the Subtour Cover problem and fulfills (5) and (6) of Theorem 16. Property (i) of a solution of the Subtour Cover problem (Definition 15) holds because \( F \) is Eulerian. Property (ii) follows from Lemma 28.
We now show property \((iii)\). Let \(D\) be a connected component of \((V, F)\) with \(E(D) \cap \delta(L) \neq \emptyset\) for some \(L \in \mathcal{L}_{\geq 2}\). Because \(D\) is Eulerian it then contains a cycle \(C\) with \(E(C) \cap \delta(L) \neq \emptyset\). But then \(E(C) \subseteq E(D)\) contains a forward edge by Lemma 17. By Lemma 23, the edges of the paths \(P_i\) for \(i \in \{1, \ldots, k\}\) are neutral edges. Hence, the forward edge in \(C\) was already present in \(F\) and thus Lemma 29 implies \(V(D) \cap V(B) \neq \emptyset\). This shows that \(F\) is a solution to the Subtour Cover problem. It remains to show \((5)\) and \((6)\).

Lemma 31 implies
\[
c(E(P_i)) = \sum_{v \in V(P_i)} |E(P_i) \cap \delta(v)| \cdot y_v \leq \sum_{v \in W_i} 2y_v.
\]
Moreover, the sets \(W_i\) for \(i \in \{1, \ldots, k\}\) are pairwise disjoint. Using also \(V(B) \cap W_i = \emptyset\) for \(i \in \{1, \ldots, k\}\), we obtain \(\sum_{i=1}^k c(E(P_i)) \leq \sum_{v \in V \setminus V(B)} 2y_v\). Together with \((8)\), this implies \((5)\).

Finally, we prove \((6)\). Let \(D\) be a connected component of \((V, F)\) with \(V(D) \cap V(B) = \emptyset\). By property \((iii)\), \(c(E(D)) = \sum_{v \in V(D)} |F \cap \delta^-(v)| \cdot 2y_v\). Because the sets \(W_1, \ldots, W_k\) are pairwise disjoint, we have \(|F \cap \delta^-(v)| \leq |\bar{F} \cap \delta^-(v)| + 1\) for every vertex \(v \in V(D)\). By Lemma 30, this implies \(|F \cap \delta^-(v)| \leq 3\) for every vertex \(v \in V(D)\) with \(y_v > 0\). This shows \((6)\) and concludes the proof of Theorem 16.

6 Algorithm for vertebrate pairs

In this section we present an algorithm for vertebrate pairs. This algorithm is essentially due to Svensson [23] who used it for node-weighted ATSP instances. Later Svensson, Tarnawski, and Végh [25] adapted the algorithm to work with vertebrate pairs. Here, we present an improved variant of their algorithm.

As a subroutine we will use the Subtour Cover algorithm from Theorem 16. In order to exhibit the dependence of the approximation guarantee of the algorithm on the subroutine we introduce the notion of an \((\alpha, \kappa, \beta)\)-algorithm for Subtour Cover. Theorem 16 yields a \((3, 2, 1)\)-algorithm for Subtour Cover.

**Definition 32.** Let \(\alpha, \kappa, \beta \geq 0\). An \((\alpha, \kappa, \beta)\)-algorithm for Subtour Cover is a polynomial-time algorithm that computes a solution \(F\) for every instance \((\mathcal{I}, B, H)\) such that
\[
c(F) \leq \kappa \cdot \text{LP}(\mathcal{I}) + \beta \cdot \sum_{v \in V \setminus V(B)} 2y_v,
\]
and for every connected component \(D\) of \((V, F)\) with \(V(D) \cap V(B) = \emptyset\) we have
\[
c(E(D)) \leq \alpha \cdot \sum_{v \in V(D)} 2y_v.
\]

Let \(\alpha, \kappa, \beta \geq 0\) such that there is an \((\alpha, \kappa, \beta)\)-algorithm \(A\) for Subtour Cover and let \(\varepsilon > 0\) be a fixed constant. The goal of this section is to show that there is a polynomial-time \((\kappa, 4\alpha + \beta + 1 + \varepsilon)\)-algorithm for vertebrate pairs.

6.1 Outline

Let \((\mathcal{I}, B)\) be a vertebrate pair. Svensson’s algorithm is initialized with an Eulerian multi-set \(\bar{H} \subseteq E[V \setminus V(B)]\) with \(\bar{H} \cap \delta(L) = \emptyset\) for all \(L \in \mathcal{L}_{\geq 2}\), and then computes either a “better” initialization \(\bar{H}'\) or extends \(\bar{H}\) to a solution \(H\) of the given vertebrate pair \((\mathcal{I}, B)\).
The initialization $\bar{H}$ of the algorithm will always be light (see Definition 33). To define what a light edge set is, we introduce a function $\ell : V \to \mathbb{R}_{\geq 0}$. For $v \in V$ we set

$$
\ell(v) := \begin{cases} 
(1 + \epsilon') \cdot 2\alpha \cdot 2y_v + \epsilon' \cdot \sum_{u \in V \setminus V(B)} 2y_u & \text{if } v \in V \setminus V(B) \\
\kappa \cdot \text{LP}((I) + \beta \sum_{u \in V \setminus V(B)} 2y_u) & \text{if } v \in V(B),
\end{cases}
$$

where $\epsilon' := \frac{\epsilon}{3 + 4\alpha + \frac{\epsilon'}{2\alpha}}$.

**Definition 33.** Let $\bar{H}$ be a (multi-)subset of $E$. We call $\bar{H}$ light if $c(E(D)) \leq \ell(V(D))$ for every connected component $D$ of $(V, \bar{H})$.

Note that for $v \in V \setminus V(B)$ the first term of the definition of $\ell(v)$ is proportional to the corresponding dual variable $y_v$. We need the additional term $\frac{\epsilon}{3 + 4\alpha + \frac{\epsilon'}{2\alpha}} \cdot \sum_{u \in V \setminus V(B)} 2y_u$ to guarantee that $\ell(v)$ cannot be too close to zero; see the proof of Lemma 40. For vertices in $V(B)$ we will only need that $\ell(V(B)) = \kappa \cdot \text{LP}((I) + \beta \sum_{u \in V \setminus V(B)} 2y_u$.

To measure what a “better” initialization for Svensson’s algorithm is, we introduce a potential function $\Phi$. For a multi-subset $\bar{H}$ of $E[V \setminus V(B)]$ such that the connected components of $(V \setminus V(B), \bar{H})$ have vertex sets $\bar{W}_1, \ldots, \bar{W}_k$, we write

$$
\Phi(\bar{H}) = \sum_{i=1}^{k} \ell(\bar{W}_i)^{1+p},
$$

where $p := \log_{1+\epsilon'}(\frac{2\epsilon'}{\epsilon'})$. Svensson [23] and Svensson, Tarnawski, and Végh [25] used $p = 1$. Our choice of the potential function $\Phi$ will lead to an improved approximation ratio.

The following lemma states the result of Svensson’s algorithm.

**Lemma 34.** Let $\alpha, \kappa, \beta \geq 0$ such that there is an $(\alpha, \kappa, \beta)$-algorithm for Subtour Cover and let $\epsilon > 0$ be a fixed constant. Then there exists a constant $C > 0$ such that the following holds.

Given a vertebrate pair $(I, B)$ with $I = (G, L, x, y)$ and a light Eulerian multi-subset $\bar{H}$ of $E[V \setminus V(B)]$ with $\bar{H} \cap \delta(L) = \emptyset$ for all $L \in \mathcal{L}_{\geq 2}$, we can compute in polynomial time

(a) a solution $H$ for the vertebrate pair $(I, B)$ such that

$$
c(H) \leq \ell(V(B)) + (2 + \frac{1}{C\kappa}) \cdot \ell(V \setminus V(B)),
$$

or

(b) a light Eulerian multi-subset $\bar{H}'$ of $E[V \setminus V(B)]$ such that $\bar{H}' \cap \delta(L) = \emptyset$ for all $L \in \mathcal{L}_{\geq 2}$ and

$$
\Phi(\bar{H}') - \Phi(\bar{H}) \geq \left(\frac{1}{C\kappa} \cdot \ell(V \setminus V(B))\right)^{1+p}.
$$

From Lemma 34 we can derive the main result of this section.

**Theorem 35.** Let $\alpha, \kappa, \beta \geq 0$ such that there is an $(\alpha, \kappa, \beta)$-algorithm for Subtour Cover and let $\epsilon > 0$ be a fixed constant.

Then there is a polynomial-time $(\kappa, 4\alpha + \beta + 1 + \epsilon)$-algorithm for vertebrate pairs.
Proof. Define $\varepsilon' := \frac{\varepsilon}{3+4\alpha+\frac{1}{2\alpha}}$, $p$, $\ell$ and $\Phi$ as above. We start with $\tilde{H} = \emptyset$ and apply Lemma 34. If we obtain a set $\tilde{H}'$ as in Lemma 34 (b), we set $\tilde{H} := \tilde{H}'$ and iterate, i.e. we apply Lemma 34 again until we obtain a set $H$ as in Lemma 34 (a). Since $0 \leq \Phi(\tilde{H}) \leq \ell(V \setminus V(B))^{1+p}$, we need at most $(C \cdot n)^{1+p}$ iterations. At the end, the algorithm guaranteed by Lemma 34 returns a solution $H$ for the vertebrate pair $(I, B)$ such that

$$c(H) \leq \ell(V(B)) + (2 + \frac{1}{2\alpha}) \cdot \ell(V \setminus V(B))$$

$$\leq \kappa \cdot \text{LP}(I) + \left(\beta + \left(2 + \frac{1}{2\alpha}\right) \cdot (1 + \varepsilon') \cdot 2\alpha + \varepsilon'\right) \cdot \sum_{v \in V \setminus V(B)} 2y_v$$

$$= \kappa \cdot \text{LP}(I) + \left(\beta + 4\alpha + 1 + 4\alpha \cdot \varepsilon' + \varepsilon' + (2 + \frac{1}{2\alpha}) \cdot \varepsilon'\right) \cdot \sum_{v \in V \setminus V(B)} 2y_v$$

$$= \kappa \cdot \text{LP}(I) + (4\alpha + \beta + 1 + \varepsilon) \cdot \sum_{v \in V \setminus V(B)} 2y_v.$$

$$\square$$

### 6.2 Basic properties of the function $\ell$ and algorithm $A$

In this section we describe the key properties of the function $\ell$ and our given $(\alpha, \kappa, \beta)$-algorithm $A$ for Subtour Cover.

**Lemma 36.** Let $A$ be an $(\alpha, \kappa, \beta)$-algorithm for Subtour Cover. Let $F$ be the output of $A$ applied to an instance $(I, B, H)$.

(i) For every connected component $D$ of $(V, F)$ with $V(D) \cap V(B) = \emptyset$ we have

$$c(E(D)) \leq \frac{1}{2(1 + \varepsilon')} \cdot \ell(V(D)).$$

(ii) Let the graph $D_B$ be the union of all connected components $D$ of $(V, F)$ with $V(D) \cap V(B) \neq \emptyset$. Then

$$c(E(D_B)) \leq \ell(V(B)).$$

**Proof.** The claimed properties follow directly from the definition of $\ell$ and the definition of an $(\alpha, \kappa, \beta)$-algorithm for Subtour Cover: property (i) follows from (10) and property (ii) follows from (9). \(\square\)

The next lemma will be needed to show that Svensson’s algorithm makes sufficient progress when finding “a better initialization”.

**Lemma 37.** There exists a constant $C > 0$ such that for every vertex $v \in V \setminus V(B)$ we have

$$\ell(v) \geq \frac{1}{C \cdot n} \cdot \ell(V \setminus V(B)).$$

**Proof.** We have

$$\ell(V \setminus V(B)) \leq (1 + \varepsilon') \cdot 2\alpha \cdot \sum_{u \in V \setminus V(B)} 2y_u + \varepsilon' \cdot \sum_{u \in V \setminus V(B)} 2y_u$$

$$\leq ((1 + \varepsilon') \cdot 2\alpha + \varepsilon') \cdot \sum_{u \in V \setminus V(B)} 2y_u.$$
Therefore, for every vertex \( v \in V \setminus V(B) \) we have
\[
\ell(v) \geq \frac{\epsilon'}{n} \cdot \sum_{u \in V \setminus V(B)} 2y_u \geq \frac{\epsilon'}{(1 + \epsilon') \cdot 2\alpha + \epsilon'} \cdot n \cdot \ell(V \setminus V(B)),
\]
which completes the proof because \( \alpha \) and \( \epsilon' \) are constants. \( \square \)

The following property of \( \ell \) is not crucial for obtaining a constant-factor approximation, but allows us to obtain a better approximation ratio.

**Lemma 38.** For every cycle \( C \) in \( G[V \setminus V(B)] \) with \( E(C) \cap \delta(L) = \emptyset \) for all \( L \in \mathcal{L}_{\geq 2} \), we have
\[
c(E(C)) \leq \frac{1}{2\alpha(1 + \epsilon')} \cdot \ell(V(C)).
\]

**Proof.** Let \( C \) be a cycle with \( E(C) \cap \delta(L) = \emptyset \) for all \( L \in \mathcal{L}_{\geq 2} \). Then we have
\[
c(E(C)) = \sum_{v \in V(C)} 2y_v \leq \frac{1}{(1 + \epsilon')2\alpha} \cdot \ell(V(C)). \quad \square
\]

In the following sections we will only use Lemma 36, Lemma 37 and Lemma 38 and we will not use the precise definition of \( \ell \) anymore.

### 6.3 Finding a better initialization

In this section we discuss how Svensson’s algorithm finds in certain cases a better initialization \( \tilde{H}' \).

We will need the following well-known statement about the knapsack problem.

**Lemma 39.** Suppose we are given a finite set \( I \) of items and for every item \( j \in I \) a weight \( w_j > 0 \) and a profit \( p_j \geq 0 \). Moreover, let \( \bar{w} < \sum_{j \in I} w_j \) be a given weight limit.

Then we can compute in polynomial time a set \( J \subseteq I \) such that

- \( \sum_{j \in J} w_j \leq \bar{w} \), and
- \( \sum_{j \in J} p_j \geq \bar{w} \cdot \max_{j \in I} p_j - \sum_{j \in I} p_j \cdot \frac{\bar{w}}{\sum_{j \in I} w_j} \cdot \sum_{j \in I} p_j \)

**Proof.** We run the following greedy algorithm. Sort the items by nonincreasing ratio \( \frac{p_j}{w_j} \). Consider the items in this order and, starting with \( J = \emptyset \), add items to the set \( J \) as long as \( \sum_{j \in J} w_j \leq \bar{w} \). Then adding the next item to \( J \) would result in a set \( J' \) with \( \sum_{j \in J'} w_j > \bar{w} \). By the sorting of the items,
\[
\sum_{j \in J'} p_j = \sum_{j \in J'} p_j \cdot \sum_{j \in J'} w_j \geq \sum_{j \in I} p_j \cdot \sum_{j \in J'} w_j > \sum_{j \in I} p_j \cdot \sum_{j \in I} \frac{\bar{w}}{\sum_{j \in I} w_j} \cdot \bar{w},
\]
Because \( J' \setminus J \) contains only one element, this implies
\[
\sum_{j \in J} p_j \geq \frac{\sum_{j \in I} p_j}{\sum_{j \in I} w_j} \cdot \bar{w} - \max_{j \in I} p_j = \frac{\bar{w}}{\sum_{j \in I} w_j} \cdot \sum_{j \in I} p_j - \max_{j \in I} p_j.
\]

Let \( \tilde{H} \) be a light Eulerian multi-subset of \( E[V \setminus V(B)] \) with \( \tilde{H} \cap \delta(L) = \emptyset \) for all \( L \in \mathcal{L}_{\geq 2} \). Let \( \tilde{W}_0 = V(B) \) and let \( \tilde{W}_1, \ldots, \tilde{W}_k \) be the vertex sets of the connected components of \( (V \setminus V(B), \tilde{H}) \), ordered so that \( \ell(\tilde{W}_1) \geq \cdots \geq \ell(\tilde{W}_k) \). For a connected multi-subgraph \( D \) of \( G \) we define the index of \( D \) to be
\[
\text{ind}(D) := \min\{j \in \{0, \ldots, k\} : V(D) \cap \tilde{W}_j \neq \emptyset\}.
\]

The following is the main lemma that we will use to find a better initialization \( \tilde{H}' \).
Figure 10: Illustration of the proof of Lemma 40. The gray and blue rectangles show the partition of $V \setminus V(B)$ into $\tilde{W}_1, \ldots, \tilde{W}_7$. In red we see the vertex set $V(D)$ of the given connected graph $D$. The rectangles with blue boundary show the sets $\tilde{W}_i$ with $i \in I$. In this example $I = \{2, 3, 4, 6\}$. The filled areas show vertex sets of connected components of $(V \setminus V(B), \tilde{H}')$. In this example we have $J = \{2, 3\}$. The connected components $\tilde{H}[\tilde{W}_3], \tilde{H}[\tilde{W}_5], \text{and } \tilde{H}[\tilde{W}_7]$ remain unchanged and we get a new component $D^*$ with vertex set $V(D) \cup \tilde{W}_2 \cup \tilde{W}_3$; we also get singleton components (without edges) for all vertices in $\tilde{W}_4 \setminus V(D)$ and $\tilde{W}_6 \setminus V(D)$.

**Lemma 40.** Let $D$ be a subtour, i.e., a connected and Eulerian multi-subgraph of $G$, such that $V(D) \cap V(B) = \emptyset$, $E(D) \cap \delta(L) = \emptyset$ for all $L \in L_{\geq 2}$, and such that

$$c(E(D)) \leq \frac{2}{2 + \varepsilon'} \cdot \ell(V(D)),$$

and

$$\ell(V(D)) > (1 + \varepsilon') \cdot \ell(\tilde{W}_{\text{ind}(D)}).$$

Then we can compute in polynomial time a light Eulerian multi-subset $\tilde{H}'$ of $E[V \setminus V(B)]$ such that $\tilde{H}' \cap \delta(L) = \emptyset$ for all $L \in L_{\geq 2}$ and (12) holds.

**Proof.** Let $I := \{j \in \{0, \ldots, k\} : V(D) \cap \tilde{W}_j \neq \emptyset\}$ and $i := \min I = \text{ind}(D)$. We have $i > 0$ because $V(D) \cap V(B) = \emptyset$. We will compute a subset $J$ of $I$ and replace the components $\tilde{H}[\tilde{W}_j]$ for $j \in I$ by one new component that is the union of $E(D)$ and all $\tilde{H}[\tilde{W}_j]$ with $j \in J$. More precisely, we set

$$\tilde{H}' := \bigcup_{h \in \{1, \ldots, k\} \setminus I} \tilde{H}[\tilde{W}_h] \cup E(D) \cup \bigcup_{j \in J} \tilde{H}[\tilde{W}_j].$$

See Figure 10. Let $D^*$ be the connected component of $(V, \tilde{H}')$ with edge set

$$E(D) \cup \bigcup_{j \in J} \tilde{H}[\tilde{W}_j].$$

We will choose $J$ such that

$$\sum_{j \in J} \ell(\tilde{W}_j \cap V(D)) \leq \frac{\varepsilon'}{2 + \varepsilon'} \cdot \ell(V(D)).$$

(15)
We first show that then \( c(E(D^*)) \leq \ell(V(D^*)) \), which implies that \( \tilde{H}' \) is light. Indeed, using (13) in the first inequality and (15) in the last inequality,

\[
c(E(D^*)) \leq \frac{2}{2 + \varepsilon'} \cdot \ell(V(D)) + \sum_{j \in J} \ell(\tilde{W}_j)
\]

\[
= \frac{2}{2 + \varepsilon'} \cdot \ell(V(D)) + \sum_{j \in J} \ell(\tilde{W}_j \setminus V(D)) + \sum_{j \in J} \ell(\tilde{W}_j \cap V(D))
\]

\[
\leq \frac{2}{2 + \varepsilon'} \cdot \ell(V(D)) + \sum_{j \in J} \ell(\tilde{W}_j \setminus V(D)) + \frac{\varepsilon'}{2 + \varepsilon'} \cdot \ell(V(D))
\]

\[
= \ell(V(D^*)).
\]

We conclude the proof by showing that we can choose \( J \) such that (15) and (12) hold. To this end, we would like to make the new component, spanning \( V(D) \cup \bigcup_{j \in J} \tilde{W}_j \), as large as possible. More precisely, we want to maximize \( \sum_{j \in J} \ell(\tilde{W}_j \setminus V(D)) \) subject to (15). This is a knapsack problem: the items are indexed by \( I \), and item \( j \in I \) has weight \( w_j = \ell(\tilde{W}_j \cap V(D)) \) and profit \( p_j = \ell(\tilde{W}_j \setminus V(D)) \). Since \( \sum_{j \in J} \ell(\tilde{W}_j \cap V(D)) = \ell(V(D)) \), the weight limit \( \bar{w} = \frac{\varepsilon'}{2 + \varepsilon'} \cdot \ell(V(D)) \) is an \( \frac{\varepsilon'}{2 + \varepsilon'} \) fraction of the total weight of all items. Since any item \( j \in I \) has profit at most \( \ell(\tilde{W}_j \setminus V(D)) \leq \ell(\tilde{W}_j) \leq \ell(\tilde{W}_i) \), Lemma 39 yields a set \( J \) with (15) and

\[
\sum_{j \in J} \ell(\tilde{W}_j \setminus V(D)) \geq \frac{\varepsilon'}{2 + \varepsilon'} \sum_{j \in I} \ell(\tilde{W}_j \setminus V(D)) - \ell(\tilde{W}_i).
\]

Finally we show (12). Using (14) in both strict inequalities, \( (1 + \varepsilon')^p = \frac{2 + \varepsilon'}{\varepsilon'} \) in the second equation, and \( \ell(\tilde{W}_i) \geq \ell(\tilde{W}_j) \) for all \( j \in I \) in the last inequality, we obtain

\[
\ell(V(D^*))^{1+p} = \left( \ell(V(D)) + \sum_{j \in J} \ell(\tilde{W}_j \setminus V(D)) \right)^{1+p}
\]

\[
\geq \ell(V(D)) \cdot \left( \ell(V(D)) + \frac{\varepsilon'}{2 + \varepsilon'} \sum_{j \in I} \ell(\tilde{W}_j \setminus V(D)) - \ell(\tilde{W}_i) \right)
\]

\[
> \left( (1 + \varepsilon') \cdot \ell(\tilde{W}_i) \right)^p \cdot \left( \frac{2}{2 + \varepsilon'} \cdot \ell(V(D)) + \frac{\varepsilon'}{2 + \varepsilon'} \ell(V(D)) + \frac{\varepsilon'}{2 + \varepsilon'} \sum_{j \in I} \ell(\tilde{W}_j \setminus V(D)) - \ell(\tilde{W}_i) \right)
\]

\[
> \left( (1 + \varepsilon') \cdot \ell(\tilde{W}_i) \right)^p \cdot \left( \frac{\varepsilon'}{2 + \varepsilon'} \cdot \ell(\tilde{W}_i) + \frac{\varepsilon'}{2 + \varepsilon'} \ell(V(D)) + \frac{\varepsilon'}{2 + \varepsilon'} \sum_{j \in I} \ell(\tilde{W}_j \setminus V(D)) \right)
\]

\[
= \frac{2 + \varepsilon'}{\varepsilon'} \cdot \ell(\tilde{W}_i)^p \cdot \left( \frac{\varepsilon'}{2 + \varepsilon'} \cdot \ell(\tilde{W}_i) + \frac{\varepsilon'}{2 + \varepsilon'} \sum_{j \in I} \ell(\tilde{W}_j) \right)
\]

\[
\geq \ell(\tilde{W}_i)^{1+p} + \sum_{j \in I} \ell(\tilde{W}_j)^{1+p}.
\]

and hence

\[
\Phi(\tilde{H}') - \Phi(\tilde{H}) \geq \ell(V(D^*))^{1+p} - \sum_{j \in I} \ell(\tilde{W}_j)^{1+p} > \ell(\tilde{W}_i)^{1+p}.
\]
Figure 11: Illustration of Lemma 41 and Lemma 42. Here the filled ellipses show the partition \( \tilde{W}_0, \ldots, \tilde{W}_{10} \) of \( V \). The curves show a possible solution \( F \) to some instance \((I, H)\) of Subtour Cover; the set \( H \) is not shown here. In red we see a subgraph \( D \) as in Lemma 41: here the red curves are the graph \( F_1 \) and \( D \) is the union of \( F_1 \) and \( \tilde{H}[\tilde{W}_1] \). In blue we see a subgraph \( D \) as in Lemma 42: here \( D \) is a single connected component of \( F \) and in this example we have \( \text{ind}(D) = 2 \).

Since \( \tilde{W}_i \) contains at least one vertex, by Lemma 37, \( \ell(\tilde{W}_i) \geq \frac{1}{C_m} \cdot \ell(V \setminus V(B)) \) for the constant \( C \) from Lemma 37.

The two different ways how we obtain \( D \) during Svensson’s algorithm are described by Lemma 41 and Lemma 42. See also Figure 11.

**Lemma 41.** Let \( A \) be an \((\alpha, \kappa, \beta)\)-algorithm for Subtour Cover. Let \( F \) be the output of \( A \) applied to an instance \((I, B, H)\). For \( i \in \{0, \ldots, k\} \) let the graph \( F_i \) be the union of the connected components \( D' \) of \((V, F)\) with \( \text{ind}(D') = i \).

Suppose we have \( c(E(F_i)) > \ell(\tilde{W}_i) \) for some \( i \in \{0, \ldots, k\} \). Then the union

\[
D := (\tilde{W}_i \cup V(F_i), \tilde{H}[\tilde{W}_i] \cup E(F_i))
\]

of \( \tilde{H}[\tilde{W}_i] \) and \( F_i \) fulfills the conditions of Lemma 40, i.e. \( D \) is a connected Eulerian multi-subgraph of \( G \) with \( V(D) \cap V(B) = \emptyset \), \( E(D) \cap \delta(L) = \emptyset \) for all \( L \in \mathcal{L} \geq 2 \), (13) and (14).

**Proof.** Let \( i \in \{0, \ldots, k\} \) such that \( c(E(F_i)) > \ell(\tilde{W}_i) \). Note that \( i > 0 \) because \( c(E(F_0)) \leq \ell(\tilde{W}_0) \) by Lemma 36 (ii). This implies \( V(D) \cap V(B) = \emptyset \) and \( E(D) \cap \delta(L) = \emptyset \) for all \( L \in \mathcal{L} \geq 2 \). Moreover, we have

\[
\frac{1}{2 + 2\varepsilon'} \cdot \ell(V(D)) \geq \frac{1}{2 + 2\varepsilon'} \cdot \ell(V(F_i)) \geq c(E(F_i)) > \ell(\tilde{W}_i),
\]

where the second inequality holds by Lemma 36 (i). This shows (14) and implies

\[
c(E(D)) = c(\tilde{H}[\tilde{W}_i] \cup E(F_i)) \leq \ell(\tilde{W}_i) + c(E(F_i)) \leq \frac{2}{2 + 2\varepsilon'} \cdot \ell(V(D)).
\]

Therefore also (13) holds. \( \square \)
Lemma 42. Let \( A \) be an \((\alpha, \kappa, \beta)\)-algorithm for Subtour Cover. Let \( F \) be the output of \( A \) applied to an instance \((I, B, H)\). Suppose \((V, F)\) has a connected component \(D\) with \(\text{ind}(D) > 0\) and
\[
\ell(V(D)) > (1 + \varepsilon') \cdot \ell(\tilde{W}_{\text{ind}(D)}).
\]
Then \(D\) fulfills the conditions of Lemma 40, i.e. \(D\) is a connected Eulerian multi-subgraph of \(G\) with \(V(D) \cap V(B) = \emptyset\), \(E(D) \cap \delta(L) = \emptyset\) for all \(L \in \mathcal{L}_{\geq 2}\), (13) and (14).

Proof. We have (14) by assumption. Moreover, \(V(D) \cap V(B) = \emptyset\) and \(E(D) \cap \delta(L) = \emptyset\) for all \(L \in \mathcal{L}_{\geq 2}\) because \(\text{ind}(D) > 0\). Since \(D\) is a connected component of \((V, F)\) that does not intersect the backbone, Lemma 36 (i) implies
\[
c(E(D)) \leq \frac{1}{2 + 2\varepsilon'} \cdot \ell(V(D)) \leq \frac{2}{2 + \varepsilon'} \cdot \ell(V(D)),
\]
implying (13). \(\square\)

6.4 Svensson’s algorithm

In this section we prove Lemma 34. To this end we consider Algorithm 2, essentially due to Svensson [23]. We maintain an Eulerian edge set \(H\) which is initialized with \(H = \tilde{H}\). Then we iterate the following steps. First, we call the given algorithm for Subtour Cover, then we try to find an improved initialization \(\tilde{H}'\) as discussed in the previous section, and finally, if we could not find a better initialization, we extend the set \(H\). The careful update of \(H\) in step 3 of Algorithm 2 is illustrated in Figure 12.

In addition to our definition of \(\Phi\), the other main difference to the version of this algorithm in [25] are the properties of \(\mathcal{C}\) in step (3c). This is inspired by a remark in [23]. In order to make this work for vertebrate pairs, we exploit our slightly stronger definition of the Subtour Cover problem (see the proof of Lemma 45).

To implement step (3c), consider each edge \(e = (v, w) \in \delta^+(V(Z))\) and compute a shortest \(w-v\)-path \(P\) in \((V \setminus V(B), E[V \setminus V(B)] \setminus (\cup_{L \in \mathcal{L}_{\geq 2}} \delta(L)))\) and check if \(c(e) + c(E(P)) \leq \frac{1}{2\alpha} \cdot \ell(\tilde{W}_{\text{ind}(Z)})\).

Note that adding \(E(\mathcal{C})\) to \(X\) in step (3c) decreases the number of connected components of \((V, H \cup F \cup X)\), and adding edges to \(H\) in step (3d) decreases the number of connected components of \((V, H)\). Thus the procedure terminates after a polynomial number of steps.
Algorithm 2: Svensson’s Algorithm

**Input:** a vertebrate pair $(I, B)$ with $I = (G, L, x, y)$, a light Eulerian multi-subset $\tilde{H} \subseteq E[V \setminus V(B)]$ with $\tilde{H} \cap \delta(L) = \emptyset$ for all $L \in \mathcal{L}_{\geq 2}$, $\alpha, \kappa, \beta \geq 0$, $\varepsilon > 0$, and an $(\alpha, \kappa, \beta)$-algorithm $A$ for Subtour Cover

**Output:** either $\tilde{H}'$ as in Lemma 34 (b) or $H$ as in Lemma 34 (a)

Let $\tilde{W}_0 := V(B)$ and let $\tilde{W}_1, \ldots, \tilde{W}_k$ be the vertex sets of the connected components of $(V \setminus V(B), \tilde{H})$ such that $\ell(\tilde{W}_1) \geq \ell(\tilde{W}_2) \geq \cdots \geq \ell(\tilde{W}_k)$.

Set $H := \tilde{H}$.

While $(V, E(B) \cup H)$ is not connected, repeat the following:

1. **Compute a solution to Subtour Cover:**
   (1a) Apply $A$ to the Subtour Cover instance $(I, B, H)$ to obtain a solution $F'$.
   (1b) Let $F$ result from $F'$ by deleting all edges of connected components of $(V, F')$ whose vertex sets are contained in a connected component of $(V, E(B) \cup H)$.

2. **Try to find a better initialization $\tilde{H}'$:**
   For $i \in \{0, \ldots, k\}$ let the graph $F_i$ be the union of the connected components $D'$ of $(V, F)$ with $\text{ind}(D') = i$.
   (2a) If for some $i \in \{0, \ldots, k\}$ we have $c(E(F_i)) > \ell(\tilde{W}_i)$, apply Lemma 40 to $D = (\tilde{W}_i \cup V(F_i), \tilde{H}[\tilde{W}_i] \cup E(F_i))$ to obtain an edge set $\tilde{H}'$. Then return $\tilde{H}'$.
   (2b) If $(V, F)$ has a connected component $D$ with $\ell(V(D)) > (1 + \varepsilon') \cdot \ell(\tilde{W}_{\text{ind}(D)})$ and $\text{ind}(D) > 0$, apply Lemma 40 to obtain an edge set $\tilde{H}'$. Then return $\tilde{H}'$.

3. **Extend $H$:**
   (3a) Set $X := \emptyset$.
   (3b) Select the connected component $Z$ of $(V, E(B) \cup H \cup F \cup X)$ for which $\text{ind}(Z)$ is largest.
   (3c) If there is a cycle $C$ in $G[V \setminus V(B)]$ with
      - $E(C) \cap \delta(V(Z)) \neq \emptyset$,
      - $E(C) \cap \delta(L) = \emptyset$ for all $L \in \mathcal{L}_{\geq 2}$, and
      - $c(E(C)) \leq \frac{1}{2\alpha} \cdot \ell(\tilde{W}_{\text{ind}(Z)})$,
     then add $E(C)$ to $X$ and go to step (3b).
   (3d) Add the edges of $(V, F \cup X)[V(Z)]$ to $H$.

Return $H$. 

33
Figure 12: An illustration of step 3 in the first iteration of Svensson’s algorithm. The edge set $F$ is shown in red. First the component $Z$ with vertex set $\tilde{W}_7 \cup \tilde{W}_8$ is considered, with $\text{ind}(Z) = 7$. We may find the blue cycle $C$ with $c(E(C)) \leq \frac{1}{2\alpha} \ell(\tilde{W}_7)$. After adding $E(C)$ to $X$, the component $Z$ with vertex set $\tilde{W}_3 \cup \tilde{W}_5$ is considered next, with $\text{ind}(Z) = 3$. Then we may find the green cycle $C'$ with $c(E(C')) \leq \frac{1}{2\alpha} \ell(\tilde{W}_3)$. Then $E(C')$ is added to $X$, and now $(V, H \cup F \cup X)$ has three connected components. The component $Z$ with vertex set $\tilde{W}_2 \cup \tilde{W}_3 \cup \tilde{W}_4 \cup \tilde{W}_5 \cup \tilde{W}_9$ is considered next. Suppose there is no cycle $C''$ connecting it to the rest and with $c(E(C'')) \leq \frac{1}{2\alpha} \ell(\tilde{W}_2)$. Then the edges drawn as solid curves are added to $H$, concluding the first iteration.

The following observation implies that $(I, B, H)$ is indeed an instance of Subtour Cover in step (1a) of Algorithm 2.

**Lemma 43.** As long as $(V, E(B) \cup H)$ is not connected in Algorithm 2, $H$ is an Eulerian multi-subset of $E[V \setminus V(B)]$ with $H \cap \delta(L) = \emptyset$ for all $L \in \mathcal{L}_{\geq 2}$.

*Proof.* At the beginning of the algorithm we set $H := \bar{H}$ and thus $H$ is a multi-subset of $E[V \setminus V(B)]$ with $H \cap \delta(L) = \emptyset$ for all $L \in \mathcal{L}_{\geq 2}$. The cycles that we find in step (3c) neither contain a vertex from the backbone nor an edge in $\delta(L)$ for any $L \in \mathcal{L}_{\geq 2}$ by construction. Moreover, by the definition of the Subtour Cover problem (Definition 15), we have $F_i \cap \delta(L) = \emptyset$ for every $i \in \{1, \ldots, k\}$ and $L \in \mathcal{L}_{\geq 2}$. By the choice of $Z$ in step 3 of Algorithm 2, the component $Z$ contains edges from $F_0$ only if $(V, E(B) \cup H \cup F \cup X)$ is connected, and in this case $(V, E(B) \cup H)$ becomes connected when the edges in $F \cup X$ are added to $H$. $\blacksquare$

Also notice that step (1b) maintains all properties required for the output of an $(\alpha, \kappa, \beta)$-algorithm for Subtour Cover. Hence, the computation of $F$ in step 1 (including both step (1a) and step (1b)) is an $(\alpha, \kappa, \beta)$-algorithm for Subtour Cover. Therefore, we can apply Lemma 41 for step (2a) and Lemma 42 for step (2b) to show that the application of Lemma 40 is indeed possible.

We conclude that if Algorithm 2 returns a (multi-)set $\bar{H}'$ in step 2, then $\bar{H}'$ is a multi-set as in Lemma 34 (b).

Now suppose the algorithm does not terminate in step 2. Since $H$ remains Eulerian throughout the algorithm and $(V, E(B) \cup H)$ is connected at the end of Algorithm 2, the returned edge set $H$ is a solution for the vertebrate pair $(I, B)$. It remains to show the upper bound (11) on the cost of $H$. Initially we have $c(H) = c(\bar{H}) \leq \ell(V \setminus V(B))$. We bound the cost of the $X$-edges and the cost of the $F$-edges added to $H$ separately.

**Lemma 44.** The total cost of all $X$-edges that are added to $H$ is at most $\frac{1}{2\alpha} \cdot \ell(V \setminus V(B))$.

*Proof.* A cycle $C$ that is selected in step (3c) and will later be added to $H$ connects $Z$ with another connected component $Y$ with $\text{ind}(Y) < \text{ind}(Z)$. We say that it marks $\text{ind}(Z)$. It has cost at most $\frac{1}{2\alpha} \cdot \ell(\tilde{W}_{\text{ind}(Z)})$. No cycle added later can mark $\text{ind}(Z)$ because the new connected component of
Let \( F \) be the connected component of \( \tilde{W} \) in iteration \( t \).

We have 

\[
(V, H \cup F \cup X) \text{ containing } Y \cup Z \text{ will have smaller index by the choice of } Z. \text{ Hence the total cost of the added cycles is at most } \frac{1}{2\alpha} \cdot \sum_{i=1}^{k} \ell(W_i) = \frac{1}{2\alpha} \cdot \ell(V \setminus V(B)). \]

\( \square \)

**Lemma 45.** The total cost of all \( F \)-edges that are added to \( H \) is at most \( \ell(V) \).

**Proof.** Let \( Z_t \) denote \( Z \) at the end of iteration \( t \) of the while-loop. Let \( F^t_i \) be the graph \( F_i \) in iteration \( t \) if the set of edges of \( F_i \) is nonempty and is added to \( H \) at the end of this iteration, and let \( F^t_i = \emptyset \) otherwise.

For \( i = 0, \ldots, t \) the total cost of \( F^t_i \) is \( c(E(F^t_i)) \leq \ell(\tilde{W}_i) \) by step (2a). We claim that for any \( i \), at most one of the \( F^t_i \) is nonempty. Then summing over all \( i \) and \( t \) concludes the proof.

Suppose there are \( t_1 < t_2 \) such that \( F^{t_1}_i \neq \emptyset \) and \( F^{t_2}_i \neq \emptyset \). We have \( i > 0 \) because otherwise the algorithm would terminate after iteration \( t_1 \) by the choice of \( Z_t \). Then \( V(F^{t_1}_i) \subseteq V(Z^{t_1}) \) and thus \( \tilde{W}_i \subseteq V(Z^{t_1}) \). See Figure 13. Moreover, \( F^{t_2}_i \) contains a vertex of \( \tilde{W}_i \) and is not completely contained in \( Z^{t_1} \) by step (1b) of the algorithm. Thus, \( F^{t_2}_i \) contains a cycle \( C \) with \( E(C) \cap \delta(V(Z^{t_1})) \neq \emptyset \). We have \( V(F^{t_2}_i) \cap V(B) = \emptyset \) (since \( i > 0 \)). This implies that \( C \) is a cycle in \( G[V \setminus V(B)] \) and \( E(C) \cap \delta(L) \subseteq E(F^{t_2}_i) \cap \delta(L) = \emptyset \) for all \( L \in \mathcal{L}_{\geq 2} \) because \( F \) is a solution to Subtour Cover.

If \( c(E(C)) \leq \frac{1}{2\alpha} \cdot \ell(\tilde{W}_{\text{ind}(Z^{t_1})}) \), due to step (3c), this is a contradiction to reaching step (3d) in iteration \( t_1 \) and adding \( Z_t \) there. Otherwise, let \( D \) be the connected component of \( F^{t_2}_i \) containing \( C \). Note that \( \text{ind}(D) = i \geq \text{ind}(Z^{t_1}) \).

Since \( C \) is a cycle with \( E(C) \cap \delta(L) = \emptyset \) for all \( L \in \mathcal{L}_{\geq 2} \), we can apply Lemma 38 to obtain

\[
\frac{1}{1 + \varepsilon'} \cdot \frac{1}{2\alpha} \cdot \ell(V(C)) \geq c(E(C)) > \frac{1}{2\alpha} \cdot \ell(\tilde{W}_{\text{ind}(Z^{t_1})}) \geq \frac{1}{2\alpha} \cdot \ell(\tilde{W}_{\text{ind}(D)}).
\]

This shows

\[
\ell(V(D)) \geq \ell(V(C)) > (1 + \varepsilon') \cdot \ell(\tilde{W}_{\text{ind}(D)}).
\]

Due to step (2b), this is a contradiction to reaching step (3d) in iteration \( t_2 \) and adding \( F^{t_2}_i \) there.

\( \square \)

Using \( c(\tilde{H}) \leq \ell(V \setminus V(B)) \), Lemma 44, and Lemma 45, we conclude that the cost of the returned edge set \( H \) is at most \( \ell(V(B)) + (2 + \frac{1}{2\alpha}) \cdot \ell(V \setminus V(B)) \). This concludes the proof of Lemma 34.

### 7 The main result

We can now combine the results of the previous sections and obtain the following.
Theorem 46. For every $\varepsilon > 0$ there is a polynomial-time algorithm that computes for every instance $(G, c)$ of ATSP a solution of cost at most $22 + \varepsilon$ times the cost of an optimum solution to (ATSP LP).

Proof. Theorem 16 yields a $(3, 2, 1)$-algorithm for Subtour Cover and by Theorem 35 this implies that there is a polynomial-time $(2, 14 + \varepsilon)$-algorithm for vertebrate pairs. Using Theorem 13 we then obtain a polynomial-time algorithm that finds a solution of cost at most $(22 + \varepsilon) \cdot \text{LP}(I)$ for every ATSP instance $I$.

As a consequence of Theorem 46 we obtain the following.

Corollary 47. The integrality ratio of (ATSP LP) is at most 22.

Proof. Suppose there is an instance $I$ of ATSP where $\frac{\text{OPT}(I)}{\text{LP}(I)} > 22$. Then there exists $\varepsilon > 0$ such that $\frac{\text{OPT}(I)}{\text{LP}(I)} > 22 + \varepsilon$. By Theorem 46 we can compute an integral solution for $I$ with cost at most $(22 + \varepsilon) \cdot \text{LP}(I) < \text{OPT}(I)$, a contradiction.

Using the observation from Remark 14, one could slightly improve Theorem 46 and Corollary 47, but the improvement would be less than 1.

Using the black-box reductions of [7] and [14], our results immediately imply:

Corollary 48. There is a $(44 + \varepsilon)$-approximation algorithm for the path version of ATSP. The integrality ratio of its classic LP relaxation is at most 85.

Using our new algorithm for ATSP not only as a black-box, one can achieve even better bounds [26]: there is a $(43 + \varepsilon)$-approximation algorithm for the path version of ATSP and the integrality ratio of its classic LP relaxation is at most 43. Moreover, our improved version of Svensson’s algorithm yields a $(13 + \varepsilon)$-approximation algorithm for the special case of unit weights, improving on Svensson’s [23] factor 27. See [26] for details.

Although we have reduced the upper bounds on the integrality ratios substantially and proved matching approximation ratios, the remaining gaps to the known lower bound of 2 on the integrality ratios [5, 14], let alone to the inapproximability lower bound of $\frac{23}{22}$ [13], are much larger than for Symmetric TSP and Symmetric Path TSP, for which approximation ratios of $\frac{3}{2} - 10^{-36}$ [12, 28] have been obtained, improving on [6, 22, 10, 1, 19, 29, 9, 21, 27, 30]. (For the unit weight special cases of Symmetric TSP and Symmetric Path TSP the best known approximation ratios are $\frac{7}{5}$ and $\frac{7}{5} + \varepsilon$ [20, 28], improving on [1, 17, 15, 16].) Improving the upper bounds further remains interesting.

Acknowledgment

We thank the three anonymous reviewers for their careful reading and useful remarks.

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