Anomalous 1D fluctuations of a simple 2D random walk in a large deviation regime

Sergei Nechaev$^{1,2}$, Kirill Polovnikov$^{3,4}$, Senya Shlosman$^{4,5}$, Alexander Valov$^6$, and Alexander Vladimirov$^5$

$^1$ Interdisciplinary Scientific Center Poncelet (CNRS UMI 2615), 119002 Moscow, Russia
$^2$ P.N. Lebedev Physical Institute RAS, 119991 Moscow, Russia
$^3$ Physics Department, Lomonosov Moscow State University, 119992 Moscow, Russia
$^4$ Skolkovo Institute of Science and Technology, 143005 Skolkovo, Russia
$^5$ Institute of Information Transmission Problems RAS, 127051 Moscow, Russia
$^6$ N.N. Semenov Institute of Chemical Physics RAS, 119991 Moscow, Russia

In this work we address the following question: could a simple two-dimensional random walk, being pushed to an improbable "large deviation regime", possess a non-Gaussian statistics with fluctuations typical for the Kardar-Parisi-Zhang (KPZ) one-dimensional strongly correlated process? The answer is positive and we provide two examples of 2D systems in which imposed external constraints force the underlying stationary stochastic process to stay into a very untypical regime with anomalous statistics. The first example deals with the fluctuations of a 2D random walk above a semicircle or a triangle in a regime of "stretched" trajectories. In the second example we consider a 2D biased random walk along a channel with inaccessible voids of circular and triangular shapes. In both examples, the typical span of the trajectory above the top of the semicircle shares KPZ scaling with the critical exponent $\nu = \frac{1}{3}$, while is constant above the tip of the triangle. We propose heuristic derivation of critical exponents and justify these scaling conjectures by explicit computations. For practical purposes, our results demonstrate that the geometry of voids in a channel might have a crucial impact on the width of the boundary layer in which the laminar flow lines fluctuate.

I. INTRODUCTION

Intensive investigation of extremal problems of correlated random variables in statistical mechanics has gradually lead mathematicians, and then, physicists, to understanding that the Gaussian distribution is not as ubiquitous in nature, as it was supposed over the centuries, and shares its omnipresence (at least in one dimension) with another distribution, known as the Tracy-Widom (TW) law. The necessary (though not sufficient) feature of the TW distribution is the second moment scaling exponent, $\nu = \frac{1}{3}$, the so-called Kardar-Parisi-Zhang (KPZ) critical exponent. For the first time the KPZ exponent $\nu$ has appeared in the seminal paper [1] (see [2] for review) in the context of the non-equilibrium one-dimensional directed stochastic growth process, where the theoretical analysis of growth was focused mainly on statistical properties of the enveloping surface developing in time.

Nowadays one has accumulated many examples of one-dimensional statistical systems of seemingly different physical nature, whose fluctuations are controlled by the KPZ exponent $\nu = \frac{1}{3}$, contrary to the exponent $\nu = \frac{1}{2}$ typical for the distribution of independent random variables. Among such examples it is worth mentioning the restricted solid-on-solid [3] and...
Eden [4] models, molecular beam epitaxy [5], polynuclear growth [6–10], several ramifications of the ballistic deposition [11–14], alignment of random sequences [15], traffic models of TASEP type [16], (1+1)D vicious walks [17], and the 1D directed polymer in the random environment [18]. Recently, this list is replenished by the one-dimensional modes describing the fluctuational statistics of cold atoms [19].

The question addressed here sounds as follows: what could be a “minimal” two-dimensional model demonstrating the one-dimensional KPZ scaling behavior with the critical exponent $\nu = \frac{1}{3}$ for the fluctuations? Below we describe the corresponding stationary stochastic process in terms of the restricted simple two-dimensional random walk. To realize the desired behavior, the 2D random walk should be pushed to an atypical fluctuational regime (i.e. to the “large deviation” region in the configurational space), where it demonstrates the 1D KPZ scaling.

The system under our investigation, is the 2D generalization of the (1+1)D model proposed by H. Spohn and P. Ferrari in [20] of 1D directed random walk above the semicircle. The motivation of the authors of the work [20] to study such a model is as follows. The fluctuations of a top line in a bunch of $N$ one-dimensional directed ”vicious walks” glued at their extremities (ensemble of world lines of free fermions in 1D) are governed by the so-called Tracy-Widom distribution with the KPZ critical exponent $\nu = \frac{1}{3}$ for the fluctuations. Proceeding as in [21], define the averaged position of the top line and look at its fluctuations near the mean displacement. In such a description all vicious walks lying below the top line play a role of a ”mean field” of a ”bulk”, which pushes the top line to some ”atypical” equilibrium position, around which it fluctuates. Replacing the effect of ”internal” (or ”bulk”) vicious walks by the semicircle, one arrives at the model proposed in [20] where the 1D directed random walk stays always above the semicircle, whose interior is inaccessible for the path. In [20] authors confirmed that fluctuations of the directed random walk above the top of the semicircle have the 1D KPZ statistics.

Finding the 1D KPZ-type behavior in a 2D restricted simple random walk goes far beyond the pure academic interest. Two important relevant applications should be mentioned. First, by this model we provide an explicit example of the two-dimensional statistical system which, being pushed to the large-deviation region, mimics the behavior of the one-dimensional strongly correlated stochastic process. Second, our study deals with the manifestations of a 1D KPZ-type scaling in the localization phenomena of two-dimensional constrained disordered systems. To provide some speculations behind this idea, recall that the density of states, $r(E)$, of the one-dimensional Anderson model (the tight-binding model with the randomness on the main diagonal) at $E \to 0$, has the asymptotics known as the ”Lifshitz singularity”

$$r(E) \sim e^{-a/\sqrt{E}}$$  \quad (1)$$

where $E$ is the energy of the system and $a$ is some positive constant (see [22, 23] for more details). Consider the grand canonical ensemble, in which $E$ is controlled only in average by the conjugated Legendre variable, $N$. In this case the density of states, $r(E)$, gets converted to $r(N)$ via the Laplace transform:

$$r(N) = \int_0^\infty r(E) e^{-NE} dE \bigg|_{N \gg 1} \sim \varphi(N) e^{-bN^{1/3}}$$  \quad (2)$$

where $b = \left(\frac{3a}{2}\right)^{2/3}$. At $N \gg 1$ we pay attention to the exponential asymptotics only and neglect the power-law corrections such as $\varphi(N) \sim N^{-5/6}$. Correspondingly, the density
of states, $r(E)$, can be restored back from $r(N)$ via the inverse Laplace transform. It is known [24] that the leading exponential asymptotic (2), has appeared in the literature under various names, like "stretched exponent", "Griffiths singularity", "Balagurov-Waks trapping exponent", but in all cases this is nothing else as the Laplace-transformed Lifshitz tail of the one-dimensional disordered systems possessing Anderson localization. Thus, it seems that the KPZ-type behavior with the critical exponent $\nu = \frac{1}{3}$ can also be regarded as an incarnation of a specific "optimal fluctuation" for the one-dimensional Anderson localization. If so, finding in some 2D system a behavior typical for 1D localization, seems to be a challenging problem of connecting localization in constrained 2D and 1D systems.

The paper is organized as follows. In Section II we formulate the model of a 2D stretched random walk above the semicircle and the triangle and provide heuristic arguments for the averaged span of paths above the tip of the figures, supplied by numeric simulations. In Section III consider the statistics of biased 2D random walks above the forbidden voids of different shapes, we summarise the obtained results and speculate about possible applications. In Appendices A and B we describe the details of explicit computations of the averaged span of the 2D path above the semicircle and the triangle.

II. STATISTICS OF STRETCHED TWO-DIMENSIONAL SIMPLE RANDOM WALK THAT EVADES SEMICIRCLE AND TRIANGLE

A. The model

We begin with the lattice version of the model. Consider the $N$-step symmetric random walk, $r_n = \{x_n, y_n\}$, on a two-dimensional square grid in discrete time $n$ ($n = 1, 2, ..., N$). The walk begins at the point $A$, terminates after $N$ steps at the point $B$, and satisfies two requirements: (i) for any $n$ one has $y_n \geq 0$, and (ii) the random walk evades the semicircle of the diameter $2R$, or the rectangular triangle of the base $2R$, remaining always outside of the shapes shown in Fig. 1. Let us note that the requirement (i) is not crucial and can be easily relaxed.

The main question to be asked, is as follows. Fix the number of steps, $N$, of the two-dimensional path, as $N = cR$, where $c$ is some positive constant ($c > \pi$), and denote by $d$ the "span" of the particular path configuration, $r_n$, above the point $O$, located at the tip of the semicircle, or of the triangle, as is depicted in Fig. 1. We are interested in finding the critical exponent $\nu$ in the scaling dependence $\langle d(R) \rangle \sim R^\nu$ as $R \to \infty$ for the semicircle and for the triangle. In physical terms, the condition $N = cR$ implies that we are interested in the statistical behavior of "stretched" trajectories, forced to stay near the semicircle, such that their lengths (measured in number of steps) do not essentially deviate from $\pi R$. It means, that we are interested in the statistics of 2D random paths pushed to an atypical ("large deviation") region of the phase space, where we could expect non-Gaussian fluctuations of trajectories.

Below we provide qualitative scaling estimates for the expectations of the mean span of 2D paths above the semicircle (the model "S") and the triangle (the model "T"). In Appendices A ad B we re-derive the dependencies in question using the method of conformal transforms for the model "S" and straightforwardly solving the boundary problem in the open wedge for the model "T".
FIG. 1: Simple two-dimensional random walk on a square grid, that evades: (a) the semicircle of radius $R$, and (b) the rectangular triangle of base $2R$.

B. Heuristic arguments: semicircle

The stretched path likes to go along the straight line as much as possible, and gets curved only if curving cannot be avoided. A path which has to travel a distance $x_s$, is localized within a strip around the corresponding straight segment of typical width ("span") $y_s \sim \sqrt{x}$. If the path is forced to travel a distance $x_s$ along some curve, and the curve itself fits the strip, then the curving is ignored. Consider a path which has to follow a circle of radius $R$. Note that the arc of that circle of length $x_s$ fits a strip of width $x_s/2R$. Therefore the duration of the arc, the curving of which can be ignored, is

$$x_s^2/R \leq \sqrt{x_s}$$

That puts a limit to $x_s$: it has to be at most $R^{2/3}$. On a shorter distances the stretched path can be considered as uncurved. The fluctuations around an arc of the length $R^{2/3}$ are of the order of $\sqrt{R^{2/3}} = R^{1/3}$. Beyond the length $x_s = R^{2/3}$ the arc itself deviates considerably from a straight segment, and therefore the estimate $\sqrt{x_s}$ for the fluctuations around it is no longer applicable.

To add some geometric flavor to the above arguments, consider Fig. 2a, and denote by $y_s$ an average span of the path in the vertical direction above the point $O$ of the semicircle, and by $x_s$ — the typical size of the horizontal segment, along which the semicircle can be considered as a nearly flat. We divide the path in three parts: $AA'$, $A'B'$ and $B'B$. The parts $AA'$ and $BB'$ of the trajectory run above essentially curved domains, while the part $A'B'$ lies above the domain which is mainly flat. Schematically this is shown in Fig. 2b: in the limit $y_s \ll R$, the horizontal segment $LM$ linearly approximates the corresponding arc of the circle. Our goal is to estimate $x_s$ and to provide self-consistent scaling arguments for fluctuations $y_s(R) \sim R^\nu$ of the stretched path above the point $O$ of the semicircle.

From the triangle $KLM$ we have:

$$|LM| = \sqrt{R^2 - |KM|^2} = \sqrt{R^2 - (R - y_s)^2}\bigg|_{y_s \ll R} \approx \sqrt{2Ry_s}$$
FIG. 2: (a) Two-dimensional random walk evading the semicircle. The part $A'B'$ lies above the essentially flat region of the semicircle. The figure (b) provides the auxiliary geometric construction for Eq. (4).

Since $|LM| \equiv x_s$, the condition $y_s \ll R$ implicitly implies the relation

$$x_s \sim \sqrt{Ry_s}$$

Consider now the two-dimensional random walk which starts at the point $L$ and terminates anywhere at the segment $MN$ ($|MN| = y_s$). Since the horizontal support, $KM$, of the path is flat, the span of the trajectory in the vertical direction is the same as for an ordinary random walk. Using the condition that the path is stretched ($N = cR$) and its length is proportional to the length of the segment $|LM| = x_s$, we can straightforwardly estimate the typical span, $y_s$, as

$$y_s \sim \sqrt{x_s}$$

(6)

It should be noted that (5) is insensitive to the specific way of stretching. Eq (6) remains unchanged even if we introduce the asymmetry in random jumps along $x$–axis keeping the symmetry of jumps in $y$ direction. In that case we end up again with the stretched path along $x$ with the scaling provided by (6). Substituting the scaling (6) into (4), we get for the semicircle (the model "S"):

$$x_s \sim \sqrt{R \sqrt{x_s}}$$

(7)

From the first equation of (7) we get for the semicircle:

$$x_s \sim R^{2/3}; \quad y_s \sim \sqrt{x_s} \sim R^{1/3}$$

(8)

which implies that $\nu = \frac{1}{3}$. The method of conformal transforms supports presented scaling estimates for the model "S" – see the Appendix A.

We expect that our heuristic arguments can be extended to random walks and random membranes above the surface of any prescribed curvature in any dimension. For example, the critical exponent $\nu$ for the fluctuations of the random walk above the curve $\Gamma$: $y = x^\gamma$ in 2D should be understood as follows. Define the characteristic length scale, $R$, and rewrite the equation for $\Gamma$ in dimensionless units:

$$\frac{y}{R} = \left( \frac{x}{R} \right)^\gamma$$

(9)
At $\gamma = 2$ we return to (5). As in the former case, Eq. (9) should be equipped by (6). Solving these equations selfconsistently, we get the following scaling dependence for the fluctuations above the curve $\Gamma$:

$$x \sim R^{2(\gamma-1)/(2\gamma-1)}; \quad y \sim R^{(\gamma-1)/(2\gamma-1)}$$

At $\gamma \to \infty$ we return to the fluctuations with the standard Gaussian exponent, $y \sim R^{1/2}$.

In the work [20] authors provide prediction for fluctuations of (1+1)D trajectories above different algebraic curves and propose the answer which apparently differs from the one obtained in [10].

C. Heuristic arguments: triangle

To estimate the fluctuations of the path of $N$ steps ($N = cR$) stretched above the triangle of base $2R$, the above arguments for the semicircle need to be modified since the curvature of the boundary of the triangle is non-analytic being concentrated at one single point $O$ at the tip of the figure – see Fig. 3a. To proceed, some auxiliary construction is of use, it is schematically shown in Fig. 3b and is zoomed in Fig. 3b.

![Diagram](image)

**FIG. 3:** (a) Two-dimensional random walk evading the semicircle, and (b) the magnified part of the system near the tip of the triangle. The points $A'$ and $B'$ are respectively the points of the first entry by the random walk into the wedge above the point $O$ and the last exit from it.

We split the full trajectory between points $A$ and $B$ into three parts: the part of $N_1$ steps running between point $A$ and first entry to the point $A'$, the part of $M$ steps running in the wedge $A'OB'$ between point $A'$ and first entry to the point $B'$, and the part $N_2$ running between $B$ and first entry to the point $B'$. The parts $N_1$ and $N_2$ lie above the flat boundaries of the triangle $AOB$, while the part $A'B'$ is located in the vicinity of the tip of the triangle. The partition function, $Z_N$, of the full $N$-step path with the extremities at $A$ and $B$ can be written as follows:

$$Z_N = \sum_{N_1, M, N_2} U_{N_1}(m_1) W_M(m_1, m_2) U_{N_2}(m_2)$$

where $m_1$ and $m_2$ are the positions of the points $A'$ and $B'$ on the edges of the wedge (see Fig. 3b), $U_{N_1}(m_1)$, $W_M(m_1, m_2)$, $U_{N_2}(m_2)$ are, correspondingly, the partition functions of
parts $AA'$, $A'B'$ and $B'B$, and the prime in (11) means that the summation runs under the condition $N_1 + M + N_2 = N$. Our goal is to estimate the typical length $M$ of the subpath inside the wedge between the points $A'$ and $B'$. Below we show that $M = \text{const}$ which immediately leads to the conclusion that $y_t = \text{const}$.

To proceed with the estimation of $M$, it is convenient to use the canonical formulation of the problem. Let us define the generating function $Z(s) = \sum_{N=0}^{\infty} Z_N s^N$ of the grand canonical ensemble, and introduce the variable $\beta = -\ln s$, which has the sense of an "energy" attributed to each step of the trajectory. Now we can rewrite (11) as follows

$$ Z(\beta) = \sum_{m_1,m_2} U(\beta,m_1) W(\beta,m_1,m_2) U(\beta,m_2) \quad (12) $$

The partition functions $U$ and $W$ differ by boundary conditions and could be explicitly computed, but for rough estimate of $y_t$, we do not need their exact expressions; some qualitative arguments would be sufficient. First of all note that values of $m_1$ and $m_2$ have nearly flat distribution within the intervals: $m_1 \in [0, \sqrt{N_1}]$ and $m_2 \in [0, \sqrt{N_2}]$. The maximal contribution to the canonical partition function $Z(\beta)$ are given by the values of $\beta$ such that $\beta N \sim 1$, i.e. $\beta \sim N^{-1}$. Since by the order of magnitude for stretched paths one has $\{N_1, N_2\} \sim N$, in the canonical ensemble the values $m_1, m_2$ fluctuate nearly uniformly within the interval

$$ \{m_1, m_2\} \in \left[0, \frac{1}{\sqrt{\beta}}\right] \quad (13) $$

Thus, any values of $m_1$ and $m_2$ satisfying the condition (13) provide nearly equal contribution to the partition function $Z(\beta)$ in (12).

Now we should account for the contribution of $W(\beta,m_1,m_2)$ to (12). Note, that each step of the path of length $M$ between points $A'$ and $B'$ carries the energy $\beta > 0$. To maximize the corresponding contribution to $W(\beta,m_1,m_2)$, one should make the corresponding length $M$ between $A'$ and $B'$ as small as possible, since we loose the energy $\beta M$ for $M$ steps. Combining this requirement for $W(\beta)$ with the weak sensitivity of $U(\beta,m_1)$ and $U(\beta,m_2)$ to the positions of points $m_1$ and $m_2$ (see the condition (13)), we conclude that the optimal $M$ for $\beta > 0$ is reached when $M$ is as small as possible and is independent on $R$, i.e. $M = \text{const}$. That immediately implies the following estimate of the span $y_t$ (for $N = cR$ and $R \gg 1$):

$$ y_t = \text{const} \quad (14) $$

The same conclusion follows from the solution of the boundary problem in the open wedge for the model "T" – see the Appendix B. Note, that putting $\gamma = 1$ into (10), we arrive at the same conclusion of independence of the span of fluctuations of stretched path above the tip of the triangle on $R$.

D. Numerics

We have computed numerically the mean height of the 2D ensemble of trajectories above the top of the semicircle and the triangle. The extremities of trajectories are located at points $A$ and $B$ in the geometry shown in the Fig. 1a,b and the number of steps is $N = cR$. The values of $R$ and $c$ are chosen as follows: $R = \{10, 20, 40, 60, 100, 200, 300, 400\}$ and
$c = \{5, 10, 20, 50\}$. The results of simulations in doubly-logarithmic scale are presented in Fig. 4. The physical meaning of the constant $c$ is the effective "stretching" of the path: the less $c$, the more stretched the path.

![Graph A](image1.png)

**Fig. 4:** The mean deviation of the path of $N$ steps above the semicircle (a) and the triangle (b) for different values of the parameter $c$, which controls "stretching" of the path (the less $c$ the more stretched the path).

As one sees from Fig. 4a, all stretched paths above the semicircle demonstrate the scaling $\langle d \rangle \sim R^\nu$ with the exponent $\nu$ close to $\frac{1}{3}$. For less stretched paths (larger values of $c$) the deviation from the scaling with $\nu = \frac{1}{3}$ can be noted. The fluctuations above the tip of the triangle are almost independent on $R$ (i.e. the exponent $\nu$ is very small), which is consistent with the theoretical arguments. The speculations about possible physical consequences of the difference between fluctuations of stretched random trajectories above the semicircle and above the triangle are provided in the discussion.

### III. DISCUSSION

We have considered an example of simple two-dimensional systems in which imposed external constraints push the underlying stochastic processes to the "improbable" (i.e. large deviation) regime which possess the anomalous statistics. Specifically, we deal with the fluctuations of a two-dimensional random walk above a circle in a special regime of "elongated" trajectories. We propose the simple scaling consideration justified by the conformal approach (see Appendix A for details), which demonstrates the emergence of a one-dimensional Kardar-Parisi-Zhang scaling with the critical exponent $\nu = \frac{1}{3}$. It is worth highlighting two important points:

- Imposing constraints on a conformational space, which cut off a tiny region of available ensemble of trajectories, we may push the sub-ensemble of random walks into the atypical large deviation regime possessing anomalous fluctuations, which might have some signature of the statistics of correlated random variables;
Imposing constraints on a conformational space, we may effectively reduce the dimension: by "stretching" the 2D random paths above the semicircle, we force the system to display the 1D KPZ scaling with $\nu = \frac{1}{3}$.

As a further development of the problem of 2D random walk statistics above the semicircle and triangle, we considered an ensemble of 2D random walks with a horizontal drift in presence of forbidden voids of different shapes shown in Fig. 5. The setting of this model is slightly different from the one discussed above. Namely, all the trajectories start from the point that is to the left from the semicircle of the triangle, however the terminal point is not fixed and could be everywhere. Instead of controlling the lengths of the path, $N$, we have fixed the value of the horizontal drift, $\varepsilon$. Thus, the coordinates of the tadpole of a growing path can be as follows:

$$
(x_N, y_N) \rightarrow \begin{cases} 
(x_N - 1, y_N) & \text{with probability } \frac{1}{4} - \varepsilon \\
(x_N + 1, y_N) & \text{with probability } \frac{1}{4} + \frac{\varepsilon}{3} \\
(x_N, y_N + 1) & \text{with probability } \frac{1}{4} + \frac{\varepsilon}{3} \\
(x_N, y_N - 1) & \text{with probability } \frac{1}{4} + \frac{\varepsilon}{3}
\end{cases}
$$

at $\varepsilon = 0$ we return to the symmetric two-dimensional random walk.

We have studied numerically the average fluctuations of the trajectories with the drift $\varepsilon$ ($\varepsilon \geq 0$) above the top of the semicircle and the tip of the triangle. The corresponding results are presented in Fig. 6 for $\varepsilon = \frac{3}{28}$, for which the quotient of forward to backward jump rates is equal to 2. In the case of a semicircle, the KPZ scaling $\langle d(R) \rangle \sim R^{1/3}$ holds, while for the case of the triangle the fluctuations do not depend on its size, $R$, and the behavior $\langle d(R) \rangle \text{ const}$ is clearly seen. Thus, the statistics of the biased 2D random walks in presence of excluded voids of semicircular and triangular shapes reproduces the features of elongated 2D random walks above the same shapes discussed at length of the Section II.
FIG. 6: Mean deviations of open random paths shown in Fig. 5 for $\epsilon = \frac{3}{25}$: (a) above the top of the semicircle; (b) above the tip of the triangle.

For the biased random walk in the channels with excluded voids of various shapes, the combination of the drift and the geometry pushes the laminar flow lines into a very improbable (large deviation) regime in which the extreme value statistics, typical for 1D systems with spatial correlations, is realized. Our results demonstrate that the geometry has a crucial impact on the width of the boundary layer in which the diffusive mixing of laminar flow lines takes place. We could speculate that such effect might be important in some technical applications of rheology of viscous liquids, for instance, cooling of laminar flows of liquids in channels with periodically displaced excluded voids of different shapes (like shown in Fig. 5). Namely, the heat transfer through walls depends not only on the total contact surface of the flow with the wall, but also on a width of a mixing skin layer: the bigger the mixing layer near the boundary, the better the cooling. However as we have seen throughout the paper, the mixing layer width is shape-dependent, which might influence the "optimal" channel geometry for cooling of viscous liquid flows.

Let us conclude with the following remarks. In this paper we were concerned about the KPZ critical exponent only. The derivation of the critical KPZ exponent via the conformal approach in Appendix A allows one to conjecture that the KPZ exponent could be derived by the methods of conformal field theory (CFT). This idea rhymes with the subject of the paper [29] where by studying the $k$-point correlation function of non-intersecting 2D random walks, it was shown that elongating the bunch of 2D loop-erased random walks, one can pass from the partition function of the logarithmic CFT with the central charge $c = -2$ (for non-stretched paths) to the partition function of vicious random walks described by the random matrix theory (for stretched paths).

Acknowledgments

We are grateful to V. Avetisov, A. Gorsky, A. Grosberg, and M. Tamm for numerous fruitful discussions and remarks. The work of S.N. is partially supported by the RFBR grant
Appendix A: 2D stretched random walks above the semicircle

The distribution of the two-dimensional random walk on a lattice depicted in Fig. 1a in the limit $N \to \infty$, $a \to 0$ (where $a$ is the lattice spacing) under the constraint $Na = t$, converges to the two-dimensional Brownian motion of time $t$ with diffusion coefficient $D = \frac{1}{4a^2}$, that evades the semicircle of radius $R$.

Consider the plane $(x, y)$ as the complex plane $z = x + iy$ and using the conformal invariance of the Laplacian $\Delta_z = \partial_z^2$, where $\bar{z} = x - iy$, perform the conformal mapping $w(z)$ of the upper halfplane $\text{Im} z > 0$ to the part of the new ”covering” plane $w = u + iv$ via the function $w(z) = \ln z$ as it is shown in Fig. 7. Thus, we get

$$\begin{cases} u = \ln |z| = \ln \sqrt{x^2 + y^2} \\ v = \text{arg} z = \arctan \frac{y}{x} \end{cases}$$  \hspace{1cm} (A1)$$

The probability distribution, $P(w, t)$, for the diffusion process that evades the semicircle in the plane $z$ acquires a simple form in the $w$–plane,

$$\partial_t P(w, t) = D|z'(w)|^{-2} \partial_{ww}^2 P(w, t)$$  \hspace{1cm} (A2)$$

with properly chosen boundary and initial conditions, and where $\partial_{ww}^2 = |z'(w)|^2 \partial_z^2$ is the conformally transformed Laplacian in the plane $w$. Explicitly, substituting (A1) into (A2), we get

$$\begin{cases} \partial_t P(u, v, t) = De^{-2u} (\partial_{uu}^2 + \partial_{vv}^2) P(u, v, t) \\ P(u = R, v, t) = P(u \to \infty, v, t) = \partial_v P(u, v = -\pi, t) = \partial_v P(u, v = 0, t) = 0 \hspace{1cm} (A3) \\ P(u, v, t = 0) = \delta(u - u_0)\delta(v - \pi) \end{cases}$$

which is the boundary problem in the geometry shown in Fig. 7b.

Up to now (A3) formulates a problem on the propagator of a two-dimensional Brownian path winding around the circle in the complex plane [25], for which the exact solution can be expressed in terms of the MacDonald function [26]. However, we are interested in atypical trajectories that are stretched to the semicircle. In other words, we prohibit deviations far from the surface of the semicircle in the plane $z$ (and, hence, from the line $u = \ln R$ in the plane $w$), i.e.

$$u = \ln |z| = \ln (R + r) \sim \ln R + \frac{r}{R}; \hspace{1cm} 0 < \frac{r}{R} \ll 1$$  \hspace{1cm} (A4)$$
 Seeking for the particular solution in the form $P_n(u, v, t) = T_n(t)P_n(u, v)$, we get a system of equations where we have explicitly implied the condition $0 < \frac{r}{R} \ll 1$:

$$\begin{cases}
\nu_n \left(1 + 2\frac{r}{R}\right) P_n(r, v) + D \left(\frac{\partial^2_{rr} + \frac{\partial^2_{vv}}{R^2}}{}\right) P_n(r, v) = 0 \\
P_n(r = 0, v) = P_n(r \to \infty, v) = \partial_v P_n(r, v = -\pi) = \partial_v P_n(r, v = 0, t) = 0 \\
\partial_t T_n(t) + \nu_n T_n(t) = 0
\end{cases} \quad (A5)$$

where $\nu_n$ are eigenvalues corresponding to the temporal evolution to be determined below. Separating radial $r$ and angular $v$ variables $P_n(r, v) = \sum_k P_{kn}(r, v) = \sum_k Q_{kn}(r)V_k(v)$, we get two Sturm-Liouville problems:

$$\begin{cases}
\partial^2_{rr} Q_{kn}(r) - \left(\frac{k^2}{R^2} - \frac{\nu_n}{D} \left(1 + 2\frac{r}{R}\right)\right) Q_{kn}(r) = 0 \\
Q_{kn}(r = 0) = Q_{kn}(r \to \infty) = 0
\end{cases} \quad (A6)$$

and

$$\begin{cases}
\partial^2_{vv} V_k(v) + \lambda_k V_k(v) = 0 \\
\partial_v V_k(v = -\pi) = \partial_v V_k(v = 0) = 0
\end{cases} \quad (A7)$$

where $\lambda_k = k^2$, $k \in \mathbb{Z}$ and cosine eigenvectors $V_k(v) = A_k \cos(kv)$ with amplitudes to be specified by initial conditions.

Up to a change of variables, $y = y_{kn}(r) \equiv \frac{k^2}{R^2} - \frac{\nu_n}{D} \left(1 + 2\frac{r}{R}\right)$, (A5) is a stationary Shrödinger-type equation in a linear "compressive" potential $U(y) = \kappa y$ with $\kappa > 0$, which keeps the trajectory close the surface $y = y(r = 0)$ of the semicircle

$$\begin{cases}
\partial^2_{yy} Q_{kn}(y) - \kappa \kappa_n Q_{kn}(y) = 0 \\
\kappa_n = \left(\frac{DR}{2\nu_n}\right)^2
\end{cases} \quad (A8)$$
From physical point of view it is already clear that \( y(r = 0) \) should vanish, as long as action of the potential vanishes in infinitesimal vicinity of the disk surface. The general solution to (A8) is known to be expressed as a linear combination of Airy functions of the first and the second kind [27]

\[
Q_{kn}(y) \propto C_1 \text{Ai}(\kappa_n^{1/3}y) + C_2 \text{Bi}(\kappa_n^{1/3}y) \quad (A9)
\]
where constants \( C_1, C_2 \) are to be chosen to satisfy the boundary conditions (A9). Clearly, the second boundary condition at \( r \to \infty \) (i.e. at \( y \to -\infty \)) is met automatically due to the oscillatory power-law decay of Airy functions at negative arguments

\[
x \to \infty : \quad \text{Ai}(-x) \sim \frac{\sin \left( \frac{2}{3}x^{3/2} + \frac{x}{4} \right)}{\sqrt{\pi}x^{1/4}}, \quad \text{Bi}(-x) \sim \frac{\cos \left( \frac{2}{3}x^{3/2} + \frac{x}{4} \right)}{\sqrt{\pi}x^{1/4}} \quad (A10)
\]

The condition at \( r = 0 \) yields an asymptotic in \( n \) behaviour of the discrete set \( \nu_n \) after analyses of zeroes of the Airy functions. Using (A10), one may obtain the following behavior of (A9) for \( |y| \gg \kappa^{-1/3}_n \)

\[
Q_{kn}(|y|) \sim \frac{1}{\sqrt{\pi}|y|^{1/4} \kappa_n^{1/12}} \cos \left( \frac{2}{3}|y|^{3/2} \kappa_n^{1/2} + \varphi \right) \quad (A11)
\]
where \( \tan (\varphi - \pi/4) = -C_1/C_2 \). From (A11) it is straightforward to see that the boundary condition at \( y(0) \) is met when \( \varphi = \frac{\pi}{2} \), and the following expressions for spectra \( \nu_n, \kappa_n \) and variable \( y(r) \) hold

\[
\nu_n \sim D \left( \frac{n}{R} \right)^2, \quad \kappa_n \sim \left( \frac{R^3}{n^2} \right)^2, \quad y(r) = y_n(r) \sim -\frac{2r}{R} \left( \frac{n}{R} \right)^2 \quad (A12)
\]

Therefore, the particular solution \( Q_{kn} \equiv Q_n \) at large \( n \gg (R^3/r)^{1/2} \) up to the norm coefficients reads

\[
Q_n(r) \sim \text{Ai} \left( -\frac{2r}{R} \frac{n^{3/2}}{n^2} \right) - \text{Bi} \left( -\frac{2r}{R} \frac{n^{3/2}}{n^2} \right) \sim \left( \frac{R}{2} \right) \left( \frac{R}{2} \right)^{3/2} \left( \frac{n}{R} \right) \quad (A13)
\]

At \( v = -\frac{\pi}{2} \) only even modes survive in (A7). Thus behavior of the general solution on top of the semicircle can be evaluated by switching to the integral representation

\[
P \left( r, v = -\frac{\pi}{2}, t \right) \propto \int_0^\infty C_n e^{-\nu_n t} Q_n(r) \, dn \quad (A14)
\]
where constants \( C_n \) are determined by the initial condition at \( t = 0 \). After applying (A12) and (A13) the integral (A14) can be approximated by the value of the integrand at \( n^* \propto r^{3/2} R^{1/2} (Dt)^{-1} \) which corresponds to the saddle point of the expression under the exponent. Therefore, the probability of large fluctuation \( r^2/Dt \gg R/r \gg 1 \) above the semicircle is given by

\[
P \left( r, v = -\frac{\pi}{2}, t \right) \sim \exp \left( -\frac{r^3}{DtR} \right) \quad (A15)
\]
which coincides with the left-tail asymptotic of the Tracy-Widom distribution (see for example, [28]). Calculating the first moment of (A15), one immediately obtains the typical KPZ fluctuations above the semicircle

\[
\langle r \rangle \sim t^{1/3} \quad (A16)
\]
Appendix B: 2D stretched random walks above the triangle

The statistics of the random path above the triangle we consider in polar coordinates centered at the cusp \(O\) of the triangle – see Fig. 1b. The random walk is free in the outer sector \(AOB\) with the angle \(\frac{3\pi}{2}\) and zero boundary conditions at the sides \(OA\) and \(OB\). Seeking for the solution of corresponding diffusion equation in the form \(P(r, v, t) = T(t)P(r, v)\), we have:

\[
\begin{align*}
\frac{\nu}{D} P(r, v) + \left( \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} + \frac{\partial^2}{r^2} \frac{\partial}{\partial v} \right) P(r, v) &= 0 \\
\mathcal{P}(r = 0, v) &= \mathcal{P}(r \to \infty, v) = \mathcal{P} \left( r, \frac{-\pi}{4} \right) = \mathcal{P} \left( r, \frac{5\pi}{4} \right) = 0 \\
\partial_t T(t) + \nu T(t) &= 0
\end{align*}
\]

(B1)

Separating variables, we can write \(\mathcal{P}(r, v) = Q(r)V(v)\) and get a set of coupled eigenvalue problems for the "angular", \(v\), and "radial", \(r\), coordinates. The solution of the "angular" boundary problem

\[
\begin{align*}
\left( \frac{\partial^2}{\partial v^2} V(v) + k^2 V(v) \right) &= 0 \\
V \left( \frac{-\pi}{4} \right) &= V \left( \frac{5\pi}{4} \right) = 0
\end{align*}
\]

(B2)

reads

\[
V_k = \frac{\cos \left( k v + \frac{\pi k}{4} - \frac{\pi}{2} \right)}{\cos \left( \frac{\pi k}{4} \right)}; \quad k = \frac{2n}{3} \quad \text{for} \quad n = \{i \mid i \in \mathbb{N} \text{ and } i \neq 3(2j + 1) \forall j \in \mathbb{N} \}
\]

(B3)

while the solution of the "radial" boundary problem

\[
\begin{align*}
\left( \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} + \left( \frac{\nu}{D} - \frac{k^2}{r^2} \right) \right) Q(r) &= 0 \\
Q(r = 0) &= Q(r \to \infty) = 0
\end{align*}
\]

(B4)

has the following generic form

\[
P(r, v, t) = \int_0^\infty \sum_n \alpha_{2n/3} Q_{2n/3}(r, v) \cos \left( \frac{2n\nu}{3} + \frac{\pi n}{6} - \frac{\pi}{2} \right) e^{-\nu t} d\nu
\]

(B5)

where

\[
\left\{ \begin{array}{l}
\alpha_i = \frac{\sqrt{D} \int_0^\infty \delta (r - R) J_i \left( \sqrt{\frac{\nu}{D}} r \right) \delta (v - v_0) V_i (v) d v d r}{\sqrt{D} || J_i (x) ||^2 || V_i (v) ||^2} = c_i \sqrt{\frac{\nu}{D}} J_i \left( \sqrt{\frac{\nu}{D}} R \right) \\
Q_i (r, v) = J_i \left( \sqrt{\frac{\nu}{D}} r \right)
\end{array} \right\}
\]

(B6)
The sum in (B5) can be rewritten as

\[ P(r, v, t) = \int_0^\infty \sum_n c_{2n/3} V_{2n/3}(v) \sqrt{\frac{v}{D}} J_{2n/3} \left( \sqrt{\frac{v}{D} R} \right) J_{2n/3} \left( \sqrt{\frac{v}{D} r} \right) e^{-\nu t} d\nu \]  

(B7)

To evaluate the integral (B7), let us pass to the new variable, \( \nu = (cR)^{\gamma} \). Now, (B7) can be rewritten as

\[ \int_0^\infty \sqrt{\frac{v}{D}} J_k \left( \sqrt{\frac{v}{D} R} \right) J_k \left( \sqrt{\frac{v}{D} r} \right) e^{-\nu t} d\nu = \ln(cR) \sqrt{\frac{v}{D}} \int_{-\infty}^{\infty} (cR)^{3\gamma/2} J_k \left( \frac{(cR)^{\gamma/2+r}}{c\sqrt{D}} \right) e^{-(cR)^{\gamma+1}} d\gamma \rightarrow \]

(B8)

where we have split the integration over \( \gamma \) at \( R \gg 1 \) into two parts. The corresponding integrals in (B8) can be approximated as:

\[ \int_{-2}^{-1} \ldots d\gamma \sim \left( \frac{r}{\sqrt{R}} \right)^k \quad \text{and} \quad \int_{-\infty}^{-2} \ldots d\gamma \sim \left( \frac{r}{R} \right)^k \]  

(B9)

Taking into account that the path stays near the triangle boundary, i.e. \( R \gg r \), we find in the vicinity of the tip the following behavior of the function \( P(r, v, t) \):

\[ P \left( r, \frac{\pi}{2}, t \right) \sim f(t, R, v_0) r^{2/3} \]  

(B10)

where \( f(...) \) is some function which depends on \( t \) and \( R \) but does not depend on \( r \). Thus,

\[ \langle r \rangle = \frac{\int r P \left( r, \frac{\pi}{2}, t \right) dr}{\int P \left( r, \frac{\pi}{2}, t \right) dr} \sim \text{const} \]  

(B11)

which means that the fluctuations above the tip of the triangle of the stretched trajectory are bounded and do not depend on the size of the triangle, \( R \). This result supports the simple scaling consideration provided in the Section II.

[1] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. 56 889 (1986)
[2] T. Halpin-Healy and Y.-C. Zhang, Physics Reports 254 215 (1995)
[3] J.M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. 62 2289 (1989)
[4] F. Family and T. Vicsek, J. Phys. A: Math. Gen. 18 L75 (1985)
[5] M.A. Herman and H. Sitter, *Molecular Beam Epitaxy: Fundamentals and Current* (Springer: Berlin, 1996)
[6] P. Meakin, *Fractals, scaling, and growth far from equilibrium* (Cambridge University Press: Cambridge, 1998)
[7] M. Prähöfer and H. Spohn, Phys. Rev. Lett. 84 4882 (2000)
[8] M. Prähöfer, H. Spohn, J. Stat. Phys. 108 1071 (2002)
[9] J. Baik and E.M. Rains, J. Stat. Phys. 100 523 (2000)
[10] K. Johansson, Comm. Math. Phys. 242 277 (2003)
[11] B.B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982)
[12] P. Meakin, P. Ramanlal, L. M. Sander, and R. C. Ball, Phys. Rev. A 34 5091 (1986)
[13] J. Krug and P. Meakin, Phys. Rev. A 40 2064 (1989)
[14] D. Blomker, S. Maier-Paape, and T. Wanner, Interfaces and Free Boundaries 3 465 (2001)
[15] S.N. Majumdar and S. Nechaev, Phys. Rev. E 72 020901(R) (2005); S.N. Majumdar, K. Mallick, and S. Nechaev, Phys. Rev. E 77 011110 (2008)
[16] B. Derrida, M. R. Evans, V. Hakim, and V. Pasquier, J. Phys. A: Math. Gen. 26 1493 (1993)
[17] G. Schehr, S.N. Majumdar, A. Comtet, and J. Randon-Furling, Phys. Rev. Lett. 101 150601 (2008)
[18] V. Dotsenko, Europhys. Lett. 90 20003 (2010); V. Dotsenko, J. Stat. Mech. P11014 (2012)
[19] D.S. Dean, P. Le Doussal, S.N. Majumdar, and G. Schehr, J. Stat. Mech.: Theory and Experiment, 063301 (2017)
[20] P.L. Ferrari and H. Spohn, Annals of Probability, 33 1302 (2005)
[21] P.L. Ferrari, M. Prähöfer, and H. Spohn, Stochastic Growth in One Dimension and Gaussian Multi-Matrix Models, In proceedings of the 14th International Congress on Mathematical Physics (ICMP 2003), World Scientific (Ed. J.-C. Zambrini) 404 (2006)
[22] I. M. Lifshitz, Sov. Phys. JETP, 26 462 (1968)
[23] I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur, *Introduction to the theory of disordered systems* (Wiley-Interscience: 1988)
[24] Th. M. Nieuwenhuizen, Phys. Rev. Lett. 62 357 (1989)
[25] K. Ito, H.P. McKean, *Diffusion Processes and Their Sample Paths* (Berlin, New York: Springer-Verlag, 1965)
[26] A. Grosberg and H. Frisch, J. Phys. A: Math. Gen., 37 3071 (2004)
[27] O. Vallée, M. Soares, *Airy functions and applications to physics* (London: Imperial College Press, 2004)
[28] G. Schehr, S. N. Majumdar, A. Comtet, and P. J. Forrester, J. Stat. Phys. 150 491-530 (2013)
[29] A. Gorsky, S. Nechaev, V.S. Poghosyan, and V.B. Priezzhev, Nucl. Phys. B 870 [FS] 55 (2013)