Geometric tracking control for a nonholonomic system: a spherical robot

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Abstract: This paper presents tracking control laws for two different objectives of a nonholonomic system - a spherical robot - using a geometric approach. The first control law addresses orientation tracking using a modified trace potential function. The second law addresses contact position tracking using a right transport map for the angular velocity error. A special case of this is position and reduced orientation stabilization. Both control laws are coordinate free. The performance of the feedback control laws are demonstrated through simulations.

Keywords: Differential geometry, feedback stabilization, spherical robot.

1. INTRODUCTION

The problem of tracking of nonholonomic systems is a challenging one in control theory. Applications include robotics, rolling and locomotive mechanisms. A better understanding of the system’s intrinsic properties simplify the control synthesis. Geometric control theory plays an important role in accomplishing such design strategies, see (Isidori, 1995), (Zenkov et al., 1999), (Ostrowski, 1996). In this paper we study the tracking problem of one such nonholonomic system - a spherical robot.

A spherical mobile robot is a spherical shell actuated by a driving mechanism mounted inside to make the shell roll. In this paper we consider the driving actuators as three rotors. So the robot has three input degrees of freedom (rotors) which are used to control two translation and three rotational degrees of freedom (shell). Several modeling approaches and motion planning algorithms have been proposed for the spherical robot to achieve desired orientation and position, see (Joshi and Banavar, 2009), (Svinin et al., 2013), (Mukherjee et al., 1999), (Bicchi et al., 1996), (Zhan et al., 2008), that are solely based on coordinate dependent approach like quaternions and Euler parametrizations. Geometric control addressed the development of control laws for systems evolving on manifolds in a coordinate free setting. Recently, Schneider (2002) has derived the dynamic model of the Chaplygin’s sphere using geometric mechanics and presented orientation stabilization of a Chaplygin’s sphere with a rotor by the controlled Lagrangian matching condition. In (Karimpour et al., 2012), (Muralidharan and Mahindrakar, 2015), the authors address the control methods based on quaternions and stereographic projection respectively. In Karimpour et al. (2012), the authors applied backstepping to achieve position stabilization and tracking by expressing attitude in quaternion representation. In (Shen et al., 2008), the authors propose motion planning algorithms using symmetric products on manifold (Lie group) to achieve position convergence with arbitrary orientation and vice versa. As spherical robot is a nonholonomic system, it fails to satisfy a necessary condition for asymptotic stabilization on $SO(3) \times R^2$ by a continuous feedback law, see Brockett (Brockett, 1983). Due to this negative result, point-to-point stabilization of position and orientation of spherical robot through continuous state-feedback is not possible. In (Muralidharan and Mahindrakar, 2015), the authors consider stereographic projection map and design smooth kinematic control law to achieve position tracking and position with reduced attitude stabilization.

The contribution of this paper is to present two geometric control laws to achieve two different objectives: 1) tracking of a desired orientation trajectory; 2) contact position trajectory tracking asymptotically. The intermediate result of contact position tracking law is position and reduced attitude stabilization. The use of the transport map for velocity error on $SO(3)$ gives a better and complete understanding of the nonholonomic constraint in case of position tracking. For orientation tracking, a potential function which is the trace of the relative orientation and desired orientation, is constructed. The stability result is derived using Lyapunov direct method (Nijmeijer and van der Schaft, 1990) which is recently restated in (Bullo, 1999) to achieve asymptotic stability. While the notion of transport map in velocity error has been considered (see, (Lewis and Bullo, 2005) for tracking of fully actuated and (Lee et al., 2010) for underactuated systems), this is the first instance where such a treatment is considered in the presence of a nonholonomic constraint and with underactuation.

The model is derived using Lagrangian reduction defined on a symmetry group. The well developed theory on geometric nonholonomic mechanics is presented in (Marsden and Ratiu, 1994), (Holm et al., 1998), (Cendra et al., 1998), (Bloch et al., 1996), (Bloch, 2003). By symmetry we can study the dynamics of a mechanical system on a reduced space and the reduced equations are in the Euler-Poincaré form. Due to nonholonomic constraints the system may or may not have full symmetry as in the case of the rigid body with gravitational field, for example, a heavy top; and the Euler-Poincaré equation will depend
on an advection term (Schneider, 2002). In this paper we follow this modelling tool and derive the reduced equations of motion. The paper is organized as follows: In Section 2 we present the description and modelling of the spherical robot using Lagrangian reduction theory. In Section 3 we formulate the control problem for orientation tracking and then position tracking and axis stabilization. We identify this stabilization as position and axis stabilization. Section 4 follows with the concluding remarks on the above control strategies.

### 2. DESCRIPTION OF A SPHERICAL ROBOT

Consider a spherical mobile robot with internal rotors which can roll without slipping on a flat surface under a uniform gravitational field. All the three rotors are placed along three mutually orthogonal axes of the sphere-body frame, as shown in Fig. (1). To balance the mass symmetrically, the rotor is placed on one side and a dead weight is placed on the diametrically opposite side. All the rotors and dead weights are placed such that the center of mass of the rotors coincides with the geometric center of the sphere. Let the sphere body coordinate frame be located with its origin at the center of the sphere. Let $\mathbf{x} \in \mathbb{R}^3$ be the position of the center of the sphere in an inertial frame, and let $\mathbf{R}_s$ be $SO(3)$ be the rotation matrix which maps from the sphere body coordinate frame to the inertial coordinate frame. The relative motion of three rotors with respect to the sphere body frame is given by generalized shape coordinates $\theta_i \in S^1$, where $i = 1, 2, 3$. Hence, the configuration space is $Q = \mathbb{R}^2 \times SO(3) \times Q_s$, where $Q_s = S^1 \times S^1 \times S^1$. The following notation is adopted here:

- $(e_1, e_2, e_3)$ - Unit vectors in inertial frame,
- $\omega_a^I, \omega_a^s$ - Angular velocity of the sphere in inertial frame and sphere frame respectively; $\dot{\theta}_i$ - Angular velocity of the $i^{th}$ rotor,
- $m_s, m_i$ - Mass of the sphere and $i^{th}$ rotors; $\mathbb{I}^s = \text{diag}(I_s, I_s, I_s)$ - Inertia matrix of the sphere without rotors about its center of mass in sphere frame and $J^I$ - Moment of inertia of the rotors about the three principal axes, i.e. $J^1 = \text{diag}(J_a, J_a, J_a); J^2 = \text{diag}(J_b, J_a, J_a); J^3 = \text{diag}(J_b, J_b, J_a)$ with $J_a = 2J_b$.

The Lagrangian of the system consists only of kinetic energy and is given as

$$L = \frac{1}{2} \left( m_T |\dot{\mathbf{x}}|^2 + \frac{1}{2} \sum_{i=1}^3 (I_i \omega_i^I + J_a (\omega_i + \dot{\theta}_i))^2 \right)$$

where $\omega_i^I = [\omega_1, \omega_2, \omega_3]$, $m_T = (m_s + \sum_{i=1}^3 m_i)$ and $I_i = I_s + 2J_b$. The Lagrangian is now expressed as

$$L = \frac{1}{2} m_T |\dot{\mathbf{x}}|^2 + \frac{1}{2} \omega_a^s \mathbb{I} \omega_a^s + \frac{1}{2} (\dot{\Theta} + \omega_a^s)^T \mathbb{J} (\dot{\Theta} + \omega_a^s)$$

(1)

where $\mathbb{J} = \text{diag}(J_a, J_a, J_a)$, $\mathbb{I} = \text{diag}(I_1, I_2, I_3)$, and $\dot{\Theta} = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$. The rolling without slipping assumption on the robot yields a nonholonomic constraint given as

$$\dot{\mathbf{x}} - \omega_a^I \times \mathbf{r}_{\mathbf{c}} = 0.$$  

(2)

#### 2.1 Dynamics of the spherical robot

The configuration space $Q$ is a smooth manifold, $TQ$ is the velocity space called the tangent bundle and a smooth distribution $\mathcal{D} \subset TQ$ defines the constraints, the set of admissible velocities. With the Lagrangian $L$ defined in (1) and distribution $\mathcal{D} \subset TQ$ satisfying the constraint (2), let $G = SO(3) \times \mathbb{R}^3$ be a group with its Lie algebra $\mathfrak{g} = \mathfrak{so}(3) \times \mathbb{R}^3$, where $\mathfrak{so}(3)$ is the Lie algebra of $SO(3)$. The group action of $G$ on $Q$ is given by $\Phi: G \times Q \to Q$; $\Phi(R_s, b, \theta) = (R_s R_b \hat{R}_a, R_b b, \theta)$. It is seen that the Lagrangian $L$ and distribution $\mathcal{D}$ is invariant with respect to the subgroup $G_{e_3}$ of $G$ given as $G_{e_3} = \{(R_s, b) \in G | R_s^T e_3 = e_3\} = SO(2) \times \mathbb{R}^2$. When the Lagrangian $L$ and the distribution $\mathcal{D}$ are invariant under the action of the subgroup group $G_{e_3}$, the system is reduced to the space $TQ/G_{e_3}$ and the Lagrangian is termed as the reduced Lagrangian $l$. Define

$$\dot{Y} = R_s^T \dot{x} \quad \Gamma \triangleq R_s^T e_3,$$

(3)

where $\dot{Y}$ is the velocity of the contact point in the sphere frame and $\Gamma$ is called an advection variable (Gajbeaye and Banavar, 2012). The reduced Lagrangian $l: TQ/G_{e_3} \to \mathbb{R}$

$$l = \frac{1}{2} m_T |\dot{Y}|^2 + \frac{1}{2} \omega_a^s \mathbb{I} \omega_a^s + \frac{1}{2} (\dot{\Theta} \cdot \mathbb{J} \dot{\Theta} + 2 \omega_a^s \cdot \mathbb{J} \dot{\omega}_a^s)$$

(4)

and the rolling constraint is now expressed in the sphere body coordinate frame as $\dot{Y} = \omega_a^s \mathbb{I} \Theta$, where $\dot{\Theta}$, $\Gamma \in \mathbb{R}^3$ and $\omega_a^s = R_s^T \dot{R}_a \in \mathfrak{so}(3)$ is the (left-invariant) sphere-body angular velocity. Substituting $\dot{Y}$ in $l$, the system is reduced to the quotient space $\mathcal{D}/G_{e_3}$ given by the reduced-constraint Lagrangian $l_c$ as

$$l_c = \frac{1}{2} \omega_a^s \cdot (-m_T \dot{\mathbb{I}} \mathbb{I} + I_s \omega_a^s) + \frac{1}{2} (\dot{\Theta} \cdot \mathbb{J} \dot{\Theta} + 2 \omega_a^s \cdot \mathbb{J} \dot{\omega}_a^s)$$

(4)

Due to subgroup symmetry, there is an advection dynamic and differentiating (3) it is calculated as

$$\dot{\Gamma} = -\omega_a^s \times \Gamma.$$  

(5)

The dynamics is calculated using the intermediate theorem given by (Schneider, 2002). The equations of motion is given by the Euler-Poincaré equation for the group variable $R_s$ and the Euler-Lagrange equation for the shape variable $\Theta$. Let $\Pi_s = \frac{\partial L}{\partial \dot{\omega}_a^s}$ be the angular momentum of the sphere(momentum conjugate to $\omega_a^s$) and $\Pi_\theta$ be the angular momentum of the $i^{th}$ rotor, the dynamics is given as,

$$\dot{\Pi}_s = \Pi_s \times \omega_a^s; \quad \dot{\Pi}_\theta = u,$$

(6)
Recasting the dynamic equation (6) as,
\[
M(\Gamma)\dot{\omega}^s_\ast = (I^\ast \omega^s_\ast + J\dot{\Theta}) \times \omega^s_\ast - u,
\]
\[
J\dot{\omega}^s_\ast + J\dot{\Theta} = u.
\]
where \(M(\Gamma) = I^s - m_T r_T p_T^T p_T\) and using the solution \(\omega^s_\ast\) of the equation (7), we can find the curve \(R_s(t)\) by solving the reconstruction equation
\[
\dot{R}_s(t) = R_s(t)\Sigma^s_\ast \text{ with } R_s(0) = R_{so}.
\]
Hence, equations (2), (7) and (5), together with the reconstruction equation (8), give the complete dynamics of the spherical robot. If \(w^s = 0\), it is easily seen that any configuration is an equilibrium and hence the equilibrium manifold is the whole configuration manifold \(Q\). By expressing the control in terms of the gradient of a potential function (or error function), the equilibrium can be changed to any desired point. A similar procedure is followed in the next two sections to achieve tracking.

Remarks on controllability

The controllability for the three rotor case has been analysed by Svinin et al. (2013), Joshi and Banavar (2009) in the literature. One can use fiber configuration controllability definition to check the controllability. This controllability has been studied for the Chaplygin’s sphere with rotors in Shen et al. (2008), Karimpour et al. (2012). We will mention this result and then design stabilization/tracking control laws for our system. To check the controllability, equation (7) is cast in an affine-control form as
\[
\dot{q} = f(q) + g(q)u
\]
where, \(q = (\omega^s_\ast, \dot{\Theta})\), the drift vector field \(f = 0\) and the control vector fields \(g = [g_1, g_2, g_3]\) are expressed as
\[
g_i = \begin{cases} 
-\left(J^s - \frac{m_T r_T p_T^T p_T}{\Delta} - J\right)^{-1} \Delta \frac{\partial}{\partial \hat{e}_i} 
\end{cases}
\]
where \(\Delta = J - \frac{m_T r_T p_T^T p_T}{\Delta} - J\). If \(v = \Delta u\) is an equivalent control input, then \(v = (v_1, v_2, v_3)\) is a transformed control input; then the control vector fields on \(SO(3) \times Q_s\) are written as
\[
g_i = \begin{cases} 
-\left(-A_1(\Gamma)^{T} \right) \hat{e}_i 
\end{cases}
\]
where \(A_1\) is the \(i^{th}\) column of \(A = (I^s - m_T r_T p_T^T p_T)^{-1}J\). Let \(\pi_s = R_s P_s\), then from (7), which is the Euler-Poincaré form, we see
\[
\frac{d}{dt}(R_s P_s) = 0.
\]
that is, the inertial momentum \(\pi_s\) is conserved. Suppose that the system is initially at equilibrium, then \(P_s = 0\) and therefore \(\omega_s = -(I^s - m_T r_T p_T^T p_T)^{-1}J\dot{\Theta} = -A(\Gamma)^{T} \Theta, \) where \(A(\Gamma)\) is called as mechanical connection Schneider (2002).
From (3), \(\dot{Y} = r A(\Gamma)^{T} \Theta,\) then the control vector fields for the complete configuration \(SO(3) \times Q_s \times \mathbb{R}^2\) are expressed as
\[
\begin{cases} 
\dot{g}_i = \begin{cases} 
-\left( -A(\Gamma)^{T} \right) \hat{e}_i 
\end{cases} \\
\end{cases}
\]
\[
\frac{d}{dt}(R_s P_s) = 0.
\]
The iterative Lie brackets \([g_1, g_2, g_3, [g_1, g_2], [g_1, g_3]]\) span the tangent space of \(SO(3) \times \mathbb{R}^2\) (termed as the fiber configuration) at any configuration. Hence, the system is fiber configuration accessible at any configuration and therefore fiber configuration controllable.

3. ORIENTATION TRAJECTORY TRACKING

The control objective here is to design a feedback control law which tracks a desired orientation trajectory \(R_d(t)\). The rotational system dynamics described by (7) and (8) can be expressed in the standard control form with \(q = (R_s, \omega^s_\ast)\) as
\[
\dot{q} = f(q) + g(q)u
\]
where
\[
M^{-1}\left((I^s \omega^s_\ast + J\dot{\Theta}) \times \omega^s_\ast\right)
\]
\[
g = [g_1, g_2, g_3] = \begin{bmatrix} 0 \end{bmatrix},
\]
for notational simplification we write \(M(\Gamma) = M\) and \(b_i\)’s are the columns of \(M^{-1}\). We now define a scalar valued potential function to achieve this objective and then prove the stability of the system. Subsequently, we add a damping term to get asymptotic convergence to the equilibrium. Let \(V : Q \rightarrow \mathbb{R}\) be an error function about \(R_d(t) \in SO(3)\) constructed by a modified trace function as
\[
V(R_s) = \text{trace}(K_p (I_{3 \times 3} - R_d^T R_s)),
\]
where \(K_p = \text{diag}(\lambda_1, \lambda_2, \lambda_3)\) with \(\lambda_1, \lambda_2, \lambda_3 > 0\) and \(\lambda_1 \neq \lambda_2 \neq \lambda_3\). The modified trace function was first employ by (Chillingworth et al., 1982) for the purpose of feedback stabilization. Define \(\Sigma^s = R_d^T R_s\) and set \(R_s = R_d R_s\) and the error in angular velocity as \(\omega_e = \omega^s_\ast - R_d^T \omega^s_\ast\). Taking time derivative of \(V\),
\[
\frac{dV}{dt} = \text{trace}(K_p (R_d^T R_s R_d - R_d^T R_d))
\]
\[
= \text{trace}(K_p R_s (R_d^T \Sigma^s_\ast - \Sigma^s_\ast R_d)) = -\text{trace}(K_p R_s \omega_e)
\]
\[
= \frac{1}{2} \text{trace}([\text{skew}(K_p R_d^T R_s) + \text{sym}(K_p R_d^T R_s)] \omega_e)
\]
\[
= \frac{1}{2} \text{trace}([\text{skew}(K_p R_d^T R_s) \omega_e])
\]
and from the equality \(\text{trace}(\Sigma^s_\ast) = -2r \cdot y,\) where \(r : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)\) is a hat map and \((\cdot)^\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3\) is a breve map (inverse of hat map), it follows that
\[
V = \text{skew}(K_p R_d^T R_s) \omega_e, \omega_e = (\sum_{i=1}^{3} \lambda_i R_d^T \hat{e}_i \times R_d^T \hat{e}_i), \omega_e = dV \cdot \omega_e,
\]
where \(dV\) can also be termed as differential of \(V\) with respect to \(R_s\).

We compute the feedforward (FF) control term which tracks the desired velocity and add the proportional-derivative (PD) term to stabilize/track the orientation asymptotically. The velocity error has geometric interpretation since \(\Sigma^s_\ast\) is Lie algebraic element. Since \(\dot{R}_s\) and \(\dot{R}_d\) are the two velocities taking values in different tangent spaces, to define the error velocity we need to compare tangent vectors in the same tangent space. This can be achieved by the transport map \(\tau\) as defined in ([Lewis and Bullo, 2005], §11). If \(\dot{R}_s \in T_{R_s} SO(3)\) and \(\dot{R}_d \in T_{R_d} SO(3)\) are the two vectors at the points \(R_s\) and \(R_d\) respectively, then a right transport map \(\tau(R_s, R_d)\) transforms \(\dot{R}_d\) into a vector at \(T_{R_s} SO(3)\) and the error is expressed as
\[
\dot{R}_s - \tau(R_s, R_d)(\dot{R}_d) = \dot{R}_s - \dot{R}_d(R_d^T R_s),
\]
\[
= R_s R_d^T R_s R_d - (R_d R_d^T R_d)(R_d R_d^T R_s),
\]
\[
= R_s \omega^s_\ast - R_s (R_d^T R_d) \Sigma^s_\ast (R_d^T R_s) = R_s [\omega^s_\ast - (R_d^T \omega^s_\ast)^\vee] = R_s \omega_e
\]
Now, the feedforward control term is calculated by taking the covariant derivative of the transport map along \(\omega^s_\ast\).
Associated with a Riemannian manifold is the notion of the affine connection $\nabla$ that defines the covariant derivative. For details on Riemannian manifolds and affine differential geometry one can refer to (Do Carmo, 2009), (Lewis and Bullo, 2005). For a given affine connection $\nabla$, two vector fields $X_\xi, X_\eta$ with $\xi, \eta \in \mathfrak{g}$ its Lie algebra, the covariant derivative is defined as $\nabla_X Y$. If $X_\xi, X_\eta$ are left-invariant vector fields on $Q$, then the covariant derivative is defined as

$$\nabla_X Y = \frac{d}{dt} X_t + \frac{\partial}{\partial t} Y |_{t=0},$$

where $\nabla_X Y$ is a bilinear map defined as

$$\nabla_X Y = M^{-1} \left( \frac{1}{2} \xi \times M \eta + \frac{1}{2} \eta \times M \xi \right).$$

In our case $X_\xi = \dot{R}_s$, $X_\eta = \tau (R_s, \dot{R}_d)$, and $g = \mathfrak{so}(3)$ with $\xi = \omega_s^a$ and $\eta = (R_d^T \omega_d^a)^c$. With this the $f_{FF}$ is calculated as

$$f_{FF} = M \left( \frac{1}{2} M^{-1} \left( \omega_s^a \times M R_d^T \omega_d^a \right) + M \omega_s^a \times M R_d^T \omega_d^a \right).$$

Theorem 1. Under the feedback torque $u(R_s) = dV(R_s) - f_{FF}$ the closed loop system (13) is Lyapunov stable about $(R_d, \omega_d^a)$.

Proof: Define the function $H \colon TQ \to \mathbb{R}$

$$H(R_s, \omega_s^a) = V(R_s) + \frac{1}{2} \| e_{\omega_s} \|_M^2 = V(R_s) + \frac{1}{2} \mathcal{G}(I) (e_{\omega_s}, e_{\omega_s}),$$

where $\mathcal{G}(I) = M$ is the Riemannian metric on $Q$. Since $V$ is an error function and $M > 0$, it follows that the function $H$ is locally positive definite around $(R_d, \omega_d^a)$. It follows

$$\frac{d}{dt} H(x, \omega_s^a) = \nabla \mathcal{G}(I) (e_{\omega_s}, \nabla \omega_s^a e_{\omega_s}).$$

Then the Lyapunov function about $(R_d, \omega_d^a)$ is stable in the sense of Lyapunov for system (13).

The next step is to introduce damping or the dissipative term $u_{diss}$ to achieve asymptotic stability. Introducing damping to the control by defining $u = (dV + f_{FF}) + u_{diss}$ where $u_{diss} = [u_{diss}^1, u_{diss}^2, u_{diss}^3]^T$. Then the closed loop control system becomes

$$\dot{q} = F_c(q) + g(q) u_{diss},$$

where $g = [g_1, g_2, g_3]$ and $F_c = \left[ M^{-1} \left( I^* \omega_s^a \times \omega_s^a + J \hat{\omega} \times \omega_s^a + (dV + f_{FF}) \right) \right]$, $g_t = [0, b_0]$.

Lemma 1: The control system (19) is locally controllable on $SO(3) \times \mathbb{R}^3$.

Proof: The proof is given in Appendix.

We now prove asymptotic stability of our system about the desired equilibrium. To prove this we use the stability result stated in [Bullo, 1999, theorem 1].

Theorem 2. Consider the system (19) with input torque $u_{diss}$. Let $H$ be described in (17). If $\mathcal{L}_F, H = 0$ and $u_{diss} = -\mathcal{L}_g H$ is the dissipative input, then the closed loop system asymptotically stabilizes $(R_d, \omega_d^a)$.

Proof: Consider the Lyapunov function $H$ as defined in (17), Computing the rate of $H$ we get

$$\frac{dH}{dt} = \frac{\partial H}{\partial q} \dot{q} = \mathcal{L}_F \dot{q} + \mathcal{L}_g H u_{diss},$$

From (18), we see that, $\mathcal{L}_F, H = 0$ which implies

$$\dot{H} = \mathcal{L}_g H u_{diss}. \tag{20}$$

Defining $u_{diss} = -\mathcal{L}_g H \dot{H} = -(\mathcal{L}_g H)^2$. We know that $u_{diss} = [u_{diss}^1, u_{diss}^2, u_{diss}^3]^T$ then calculating $u_{diss}$ as

$$u_{diss} = -\left[ \frac{\partial H}{\partial q} \right] = \left( \frac{\partial H}{\partial \omega_s} \right)^T \left( \frac{\partial \dot{H}}{\partial \omega_s} \right) \left[ \begin{array}{c} 0 \\ -b_0 \end{array} \right] = M e_{\omega_s} \cdot b_i,$$

where $i = 1, 2, 3$. From this the dissipative control is calculated as

$$u_{diss} = [\mathcal{L}_g H \mathcal{L}_g H \mathcal{L}_g H]^T = M e_{\omega_s} \cdot (M^{-1})^T = K_v e_{\omega_s},$$

where $M = M^T$ is a symmetric positive definite matrix and $K_v$ is a positive constant. Substituting the value of $\mathcal{L}_g H$ in (20) we get $\dot{H} = -K_v (e_{\omega_s})^2 \leq 0$. Since, system is locally controllable from Lemma 1 and $H$ is negative semidefinite we conclude from Theorem 1 of [Bullo, 1999] that the point $(R_d, \omega_d^a)$ is locally asymptotically stable.

Corollary 3. If $\omega_s^a = 0$ implies $f_{FF} = 0$, then the system (13) with control $u = dV + K_v \omega_s^a$ is local asymptotically stable about $(R_d, 0)$.

Infact the system is local exponential stable about $(R_d, 0)$. To check exponential stability, we compute the second variation of $H$. If the second variation is positive definite about the equilibrium point we say that the equilibrium is exponentially stable. From (17) $H = T + V$ where $T = (1/2) \omega_s^a \cdot M \omega_s^a$ and for $T$ being a kinetic energy, yields

$$\delta^2 T(q) > 0 \forall q.$$ The second variation of the error functions $V$ is calculated as follows; let $\eta = R_d^T R_s e_{\omega_s}$, then

$$\delta V(R_s) = \delta (\mathcal{G}(I)) \left( e_{\omega_s}, \nabla \omega_s^a e_{\omega_s} \right) = \delta \mathcal{G}(I) \left( e_{\omega_s}, \nabla \omega_s^a e_{\omega_s} \right),$$

Thus, $H$ is a Lyapunov function about $(R_d, \omega_d^a)$ and therefore $(R_d, \omega_d^a)$ is stable in the sense of Lyapunov for system (13).

Simulation: We choose the model parameters as: $m_\omega = 1 kg; m_i = 0.672 kg; r_v = 0.176 m; I^* = \text{diag}(0.0153, 0.0153, 0.0153) kg \cdot m^2; J = \text{diag}(0.672, 0.672, 0.672) kg \cdot cm^2$; and control parameters as $K_p = \text{diag}(2, 8, 1)$ and $K_v = 0.5$. 

Choosing rotation matrix \( R_s = \exp(\alpha \hat{e}_3) \exp(\beta \hat{e}_3) \exp(\gamma \hat{e}_3) \), the simulations are carried out for both attitude tracking and stabilization by three rotors with the following control law:

\[
 u = - \left( \sum_{i=1}^{3} \lambda_i R_i^T \hat{e}_i \times R_i^T \hat{e}_i \right) + K_v e_\omega + f_{FF}.
\]

Keeping the desired orientation trajectory as \( R_d(t) = \exp(2\pi(1 - \cos \pi t) \hat{e}_2) \), then Fig. (2(a)) and (2(b)) shows the error in angular velocity \( e_\omega \) of the sphere and the error norm of \( R_s \), indicating asymptotic convergence to the desired trajectory. The error norm is calculated as

\[
 E_R = (3 - \text{trace}(K_p R_d^T R_s))^2.
\]

For stabilization, setting the desired orientation as \( R_d = R(\pi/9) R(\pi/18) R(\pi/3) \) and initial angular velocity as \( \omega_s(0) = (12.5, 7.1) \). Then Fig. (3(a)) shows the torque applied at the internal rotors. The angular velocity of the three rotors is shown in Fig (3(b)) converges to the initial momentum.

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**4. CONTACT POINT TRACKING AND AXIS STABILIZATION**

In this section we derive a control law based on a configuration error function which ensures contact position tracking by tracking the angular velocity. The control objective is to design a control law which aligns \( \omega_s^\ast \) to a desired angular velocity and stabilizes/tracks the contact position on the plane asymptotically. Suppose \( \hat{x}_d = R_d \omega_s^\ast \times r e_3 \) is the desired contact point position, where \( R_d \) is a desired orientation. Note that given the nonholonomic constraint, \( \hat{x}_d \) gets determined by \( \omega_s^\ast = R_d \omega_s^\ast \) which eventually determine \( \hat{x}_d \). Let \( V_1 \) be a potential function given by

\[
 V_1 = \frac{1}{2} [\hat{x} - \hat{x}_d]^2.
\]

Taking the time derivative of \( V \) along the system’s trajectory,

\[
 V_1 = (\hat{x} - \hat{x}_d) \cdot (\hat{x} - \hat{x}_d) = (\hat{x} - \hat{x}_d) \cdot [(R_d \omega_s^\ast \times r e_3) - (R_d \omega_s^\ast \times r e_3)],
\]

\[
 = -r e_3 \times (\hat{x} - \hat{x}_d) \cdot R_s (\omega_s^\ast - R_d^T R_s \omega_s^\ast),
\]

\[
 = -r R_s^T [e_3 \times (\hat{x} - \hat{x}_d)] \cdot (\omega_s^\ast - R_d^T R_s \omega_s^\ast),
\]

Set \( R_s = R_d^T R_s \) and define the error in angular velocity as \( e_\omega = \omega_s^\ast - R_d^T \omega_s^\ast \). The proportional-derivative (PD) control term is given as \( f_{PD} = k_p r R_s^T [e_3 \times (\hat{x} - \hat{x}_d)] - k_d e_\omega \), where \( k_p \) and \( k_d \) are positive definite matrices. With this PD control and \( (16) \), the nonlinear controller is given as

\[
 u = - (f_{PD} + f_{FF}).
\]

**Theorem 4.** The system \( (7) \) with control input \( (23) \), given by

\[
 \omega_s^\ast = M^{-1}(I \omega_s^\ast + \dot{\theta}) \times \omega_s^\ast + M^{-1}(f_{PD} + f_{FF}),
\]

is locally asymptotically stable at \( (\hat{x}_d, \omega_s^\ast) \).

**Proof:** Define a candidate error function

\[
 H(\hat{x}, \omega_s^\ast) = V_1 + \frac{1}{2} \| e_\omega \|_M^2 = V_1 + \frac{1}{2} G(I)(e_\omega, e_\omega),
\]

The time derivative of the Lyapunov function is

\[
 \frac{d}{dt} H(\hat{x}, \omega_s^\ast) = \frac{d}{dt} V_1 + G(I)(e_\omega, \nabla_\omega^\ast e_\omega),
\]

\[
 = V_1 + G(I)(e_\omega, \nabla_\omega^\ast e_\omega) - r R_s^T [e_3 \times (\hat{x} - \hat{x}_d)] \cdot e_\omega + \{e_\omega, f_{PD} + f_{FF}\} - \{e_\omega, f_{FF}\},
\]

\[
 = -k_d e_\omega \cdot e_\omega \leq 0.
\]

Let \( \Omega \leq \{(\hat{x}, \omega_s^\ast)|H(\hat{x}, \omega_s^\ast) \leq \epsilon \} \), where \( \epsilon > 0 \) and since \( H(\hat{x}, \omega_s^\ast) \leq 0 \) along the trajectories are bounded and contained within \( \Omega_\epsilon \). Define \( N \) to be the set of all points of \( \Omega_\epsilon \) satisfying \( \dot{H} = 0 \). From \( (25) \), we have \( N = \{(\hat{x}, \omega_s^\ast) \in \Omega_\epsilon \| e_\omega = 0 \} \). As \( e_\omega = 0 \) implies \( \omega_s^\ast = R_d^T \omega_s^\ast \) which yields the dynamics, \( \hat{x} = \hat{x}_d \) and \( \omega_s^\ast = -r M^{-1} R_s^T [\hat{e}_3 \times (\hat{x} - \hat{x}_d)] + \frac{d}{dt}(R_s^T \omega_s^\ast) \). Since the robot rolls on a horizontal plane, at any point \( (\hat{x} - \hat{x}_d) = 0 \). So the only possibility of \( \hat{e}_3 = 0 \) to happen is when \( x = x_d \). Hence, the largest invariant set will be the set \( N_1 = \{(\hat{x}, \omega_s^\ast)|\omega_s^\ast = R_d^T \omega_s^\ast \times \hat{x}_d \in \Omega_\epsilon \} \). And from LaSalle’s invariance principle, the trajectories in \( \Omega_\epsilon \) converge to \( N_1 \) as \( t \to \infty \), i.e., to the equilibrium \( (\hat{x}_d, \omega_s^\ast) \).

**Position and reduced attitude stabilization:** For position stabilization there are two cases: 1) when \( x_d = 0 \) & \( R_d \hat{e}_3 = 0 \Rightarrow \omega_s^\ast = 0 \); and 2) when \( x_d = 0 \) & \( R_d \hat{e}_3 = \alpha \hat{e}_3 \), where \( \alpha \) is any scalar. Case 1 is immediate. When \( \omega_s^\ast = 0 \) the control law \( u = f_{PD} \) and the robot converges to the origin asymptotically. In case 2, \( R_d \hat{e}_3 = \alpha \hat{e}_3 \) implies that the robot’s final contact position is the origin and the angular velocity is about the \( z \)-axis. The control law in this case is expressed as

\[
 u = - k_p r R_s^T (\hat{e}_3 \times \hat{x}) + k_d (\omega_s^\ast - \alpha R_d^T \hat{e}_3) - \frac{1}{2} (\omega_s^\ast \times \alpha MR_s^T \hat{e}_3)
\]

\[
 + \frac{1}{2} (\hat{e}_3 \times \hat{x}) - M(\omega_s^\ast \times R_d^T \hat{e}_3).
\]

The first term in the control law is responsible for the contact position stabilization and remaining terms will orient the sphere such that the angular velocity tracks \( \hat{e}_3 \). Such is a case of reduced attitude stabilization where stabilizing \( R_s \) up to a rotation about \( \hat{e}_3 \) is equivalent to...
stabilizing the angular velocity direction of the axis $R^T e_3$ (Bullo et al., 1995). Thus, we can restate the attitude as $R_s \in S^2$ and conclude that the control law (23) gives the contact point and reduced attitude stabilization in terms of the points in $R^2 \times S^2$.

**Simulation:** We take the model parameters as in section (3) with initial orientation $R_{s_0} = \exp(\frac{\pi}{4} e_1)$ and starting point on the horizontal plane as $(x_0, y_0) = (4, 2)$ units. Setting the desired orientation $R_d = \exp(\frac{\pi}{4} e_3)$ and $\omega_d = \dot{e}_3$ which satisfy $R^T_d \omega_d = \dot{e}_3$, Fig. 4(a)) and Fig. 5(a) shows that as the angular velocity achieve $\omega_d$ asymptotically, the sphere attains the desired Line-of-sight that is $R^T_d \dot{e}_3 = (\Gamma_1, \Gamma_2, \Gamma_3)$ converges to $\dot{e}_3 = (0, 0, 1)$. The initial oscillations in $R^T_d \dot{e}_3$ plot are due to the sphere rotating in the spiral type motion on plane and then asymptotic converges to $(0, 0, 1)$. The position on $xy$ plane is illustrated in Fig. (4(b)). To illustrate contact point trajectory tracking, we choose $x_d$ to track line and circle, as shown in Fig. 7, (a) and (b). Keeping $\omega_d = 0$, then Fig. 7(a) and (b) shows the angular velocity of the sphere and the phase plane of position. Setting $x_d = (r_0 \sin(t), r_0 \cos(t))$ which yields $\omega_d = (-\sin(t), -\cos(t))$. Fig. 8 shows the spherical robot follows the desired circular trajectory. Setting $\omega_d = (0.2, 0.3)$ and $x_d = (0.2t + 0.4, 0.3t + 0.6)$ then the sphere will rotate at constant speed shown in Fig. 9 and follows the line given by $x_d$.

5. **DISCUSSION**

In conclusion, we say that both the control strategies derived using the geometric approach, without parametrization, illustrate a more general philosophy on the control design, preserving the mechanical notions of the system.
it will remain zero at final time. And if we keep \( \omega^*_k = \dot{\epsilon}_3 \)
we get the \( \epsilon_k(0) = \epsilon_k(T) = 0 \) for all time, and the result is contact position and a reduced attitude stabilization.

**APPENDIX**

**Proof of Lemma 1:** In this section we will compute the Lie brackets of \( F_i \) and \( g_i \), where \( i = 1, 2, 3 \). Given two vector field \( X, Y \in TQ \) the Lie derivative (bracket) of \( Y \) along \( X \) is \( \{X,Y\} \equiv \frac{d}{dt}_{|_{t=0}} T\Phi_t(Y) \), where \( \Phi \) is the flow of \( X \) and \( \Phi_t(Y) \) is pull-back of a vector field \( Y \). From system (19), the Lie bracket of \( F_i \) and \( g_i \) will be

\[
[F_i,g_i](q) = -[g_i,F_i](q) = -\frac{d}{dt}_{|_{t=0}}(DF_i \Phi_t(q))^{-1} F_i(\Phi_t(q))
\]

where \( \Phi_t \) is the flow of \( g_i \). The control vector field \( g_i(q) = (0, M^{-1} \dot{\epsilon}_i) \) then flow of \( g_i \) is given as \( \Phi_t^{g_i}(q) = (R_\omega^* s + t M^{-1} \dot{\epsilon}_i) \) and \( DF_i \Phi_t(q) \) is the identity map on the manifold \( Q \).

\[
[g_i,F_i](q) = \frac{d}{dt}_{|_{t=0}}(DF_i \Phi_t(q))^{-1} F_i(\Phi_t(q))
\]

Similarly, \( [g_2,F_3](q) \) and \( [g_3,F_1](q) \) are calculated and given as

\[
[g_2,F_3](q) = \left[ \frac{R_\omega(s)}{s} \right] \quad \text{and} \quad [g_3,F_1](q) = \left[ \frac{R_\omega(s)}{s} \right]
\]

where \( s \) denotes some functions we are not interested in. The vectors \( g_1,g_2,g_3 \in T_pQ \) are linearly independent since \( \{\epsilon_1, \epsilon_2, \epsilon_3\} \) are linearly independent. To see the linear independence, we write all the six vectors as

\[
\alpha_i M^{-1} \dot{\epsilon}_i + \beta_i R_\omega(M^{-1} \dot{\epsilon}_i) = 0.
\]

(26)

Now, for these vectors to be linearly independent, all the scalars \( \alpha_i \)'s and \( \beta_i \)'s equal to zero. From the values of \( g_i \) and \( [g_i,F_i] \), for (26) to hold that, it follows that

\[
\alpha_i M^{-1} \dot{\epsilon}_i = 0 \quad \beta_i R_\omega(M^{-1} \dot{\epsilon}_i) = 0.
\]

(27)

for all \( i \). Since \( \{g_1,g_2,g_3\} \) is linearly independent, (27) will hold only when \( \alpha_1 = 0 \). And \( (R_\omega(M^{-1} \dot{\epsilon}_1)^c, R_\omega(M^{-1} \dot{\epsilon}_2)^c, R_\omega(M^{-1} \dot{\epsilon}_3)^c) \) are linearly independent when \( \beta_1 = 0 \) to satisfy (26). Hence, the set \( \{g_1,g_2,g_3,[f,g_1],[f,g_2],[f,g_3]\} \) are linearly independent on \( Q = SO(3) \times \mathbb{R}^3 \) of dimensional six and spans the tangent space of the configuration space at any configuration. Therefore, the system is locally controllable.

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