Fermionic Mapping For Eigenvalue Correlation Functions Of Weakly Non-Hermitian Symplectic Ensemble

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Abstract

The eigenvalues of an arbitrary quaternionic matrix have a joint probability distribution function first derived by Ginibre. We derive the j.p.d. for the weakly non-Hermitian version of this problem and then show that there exists a mapping of this system onto a fermionic field theory. This mapping is used to integrate over the positions of the eigenvalues and obtain eigenvalue density as well as all higher correlation functions for both the strongly and weakly non-Hermitian cases.

I. INTRODUCTION

Several ensembles of non-Hermitian matrices were given by Ginibre [1]. These are the ensembles of matrices with arbitrary real, complex, or quaternionic entries. Ginibre gave joint probability distributions for the eigenvalues for the complex and quaternionic cases, and succeeded in obtaining correlation functions in the complex case, while correlation functions for the quaternionic case were found later [3]. The purpose of this paper is to extend the correlation functions for the quaternionic problem to the weakly non-Hermitian case [7], as well as to introduce a fermionic mapping to simplify the computation of these correlation functions. Further, the mapping also permits us to derive the 4-point, and higher, correlation functions, that were only conjectured before.

Although Ginibre’s ensembles are interesting in themselves, they are also closely connected with the chiral random matrix ensembles that appear in QCD [5] and some condensed matter systems [6]. Knowing the eigenvalue correlation functions in the non-Hermitian ensemble, one can easily determine correlation functions in the corresponding chiral ensemble. Further, the weakly non-Hermitian [7] versions of these ensembles are of interest in open quantum systems; there exists a study using supersymmetric techniques of the eigenvalue distribution in the weakly non-Hermitian version of the ensemble considered in this paper [2].

One interesting feature to observe in the two-level correlation function is the crossover from a non-monotonic correlation function with algebraic tails in the limit of very weak non-Hermiticity to a monotonically decaying correlation function with Gaussian tails in the limit of strong non-Hermiticity.
Another interesting property that the symplectic non-Hermitian ensemble exhibits is a depletion of the eigenvalue density near the real axis; this could be guessed at by looking at the joint probability distribution (j.p.d.) derived originally by Ginibre (given in equation (2) below). The depletion was also found numerically [4].

Consider an arbitrary $N$-by-$N$ matrix of quaternions. This is equivalent to a $2N$-by-$2N$ matrix $M$ with complex entries. Let $M$ be chosen from an ensemble of such matrices with Gaussian weight

$$P(M) = e^{-\frac{1}{2} \text{Tr}(M^dM)} dM$$

This defines the strongly non-Hermitian ensemble. The eigenvalues of $M$ come in complex conjugate pairs; for every eigenvalue $z = x + iy$, there is an eigenvalue $\bar{z} = x - iy$. Let the matrix $M$ have eigenvalues $z_i, \bar{z}_i, i = 1...N$. Then the j.p.d. of the eigenvalues is given, up to a constant factor, by

$$\frac{1}{N!} \prod_i e^{-\frac{1}{2} |z_i - \bar{z}_i|^2} \prod_{i<j} (z_i - z_j)(z_i - \bar{z}_j)(\bar{z}_i - z_j)(\bar{z}_i - \bar{z}_j) \prod_i dz_i d\bar{z}_i$$

where $dz d\bar{z} = 2dz dy$.

In section II, a fermionic mapping is introduced to write equation (2) as a correlation function in a fermionic field theory. The mapping is then used to calculate the eigenvalue density. In section III, we introduce the weakly non-Hermitian ensemble and calculate eigenvalue density for that ensemble. In section IV, multi-eigenvalue correlation functions are calculated for both strongly and weakly non-Hermitian cases. The calculations in section III and IV are simple extensions of the calculation given in section II. For this reason, the calculation in section II is given in the most detail, while the other calculations are sketched.

For the strongly non-Hermitian case, the main results are equations (20, 26) for the eigenvalue density, and equation (44) for the two-level correlation function. For the weakly non-Hermitian case, the main results are equation (42) for the eigenvalue density and equations (48, 49) for the two-level correlation function.

II. FERMIONIC MAPPING AND GREEN’S FUNCTION

In this section we develop the fermionic mapping for the j.p.d. of the strongly non-Hermitian ensemble. First, we write equation (2) as a correlation function in a fermionic field theory. Then, for convenience we shift to radial coordinates, making a conformal transformation. Finally, we integrate over all but one of the $z_i$ to obtain the eigenvalue density. The integral over the $z_i$ is done inside the correlation function; only after doing the integral is the correlation function evaluated. This amounts to commuting the order of doing the integral and evaluating the correlation function, and is the essential trick used in this section. In section IV we will demonstrate how to obtain multi-level correlation functions by a simple extension of the procedure of this section.
A. Fermionic Mapping

First, let us show that equation (2) can be written as a correlation function in a two-dimensional fermionic field theory. A similar fermionic mapping was demonstrated previously for the Hermitian orthogonal and symplectic ensembles [8]. Let the field $\psi(x)$ have the action

$$S = \frac{1}{2} \int d\tau dz \psi^\dagger(z) \bar{\psi}(z)$$

(3)

Note that we are using only one chirality of fermionic field. Consider a correlation function of this field, such as

$$\langle \prod_{i=1}^{2N} \psi(a_i) \psi^\dagger(b_i) \rangle$$

(4)

This correlation function is equal to

$$\frac{\left( \prod_{i<j} (a_j - a_i) \right) \left( \prod_{i<j} (b_j - b_i) \right)}{\left( \prod_{i,j} (a_j - b_i) \right)}$$

(5)

Let us consider a specific choice of $b_j$, with $b_j = L e^{2\pi i j/2N}$. In the limit $L \to \infty$, we find that equation (5) reduces to

$$L^{-2N^2} \prod_{i<j} (a_j - a_i)$$

(6)

Comparing this to equation (2) we realize that equation (2) can be written as

$$\frac{1}{N!} \lim_{L \to \infty} L^{2N^2+N} \langle \prod_{j=1}^{2N} \psi^\dagger(b_j) \rangle \left( \prod_{j=i}^{N} U(\zeta_j, z_j) \psi(\zeta_j) \psi(z_j) d\zeta_j dz_j \right)$$

(7)

where $b_j = L e^{2\pi i j/2N}$, and $U(\zeta, z) = e^{-\zeta\bar{z}}(z - \bar{z})$. When integrating over $z_i$, the limit $L \to \infty$ must be taken before doing the integral over $z_i$.

Now we will make a conformal transformation to radial coordinates. Write $z = e^w$ and $\bar{z} = e^{\bar{w}}$. Let $w = t + i\theta$. The action for the fermionic field is unchanged under this transformation, but we must change equation (7) as the field $\psi$ has non-vanishing scaling dimension and conformal spin. Equation (7) gets replaced by

$$\frac{1}{N!} \lim_{L \to \infty} L^{2N^2/2} \langle \prod_{j=1}^{2N} \psi^\dagger(v_j) \rangle \left( \prod_{j=1}^{N} e^{t_j} U(e^{w_j}, e^{w_j}) \psi(w_j) \psi(\bar{w}_j) dw_j d\bar{w}_j \right)$$

(8)

where $v_j = \log b_i L + 2\pi i j/2N$.

Now, we will introduce Fourier transforms for the creation and annihilation operators. We will write $\psi(w_i) = \sum_k e^{kw_i} a(k)$ and $\psi^\dagger = \sum_k e^{-kw_i} a^\dagger(k)$. In the limit $L \to \infty$, the only
states involved in equation (8) are those with \( k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{2N-1}{2} \). If there are excitations in states with higher \( k \), they will vanish in the large \( L \) limit.

Then, we can rewrite equation (8), up to factors of order unity, as

\[
\frac{1}{N!} \langle \left( \prod_{k=1/2}^{2N-1/2} a^\dagger(k) \right) \prod_{j=1}^{N} \left( \sum_{k,k'} a(k)a(k')e^{w_j k}e^{\bar{w}_j k'} e^{j \omega j} U(e^{w_j}, e^{w_j})\right) \rangle
\] (9)

In equation (9), consider integrating over \( w_j, \bar{w}_j \) for some given set of \( j = 1 \ldots M \). We will do the integral inside the correlation function. The integral

\[
\int \prod_{j=1}^{M} \left( \sum_{k,k'} a(k)a(k') e^{w_j k}e^{\bar{w}_j k'} e^{j \omega j} U(e^{w_j}, e^{w_j})\right) d\bar{w}_j d w_j
\] (10)

is equal to

\[ O^M \] (11)

where the operator \( O \) is defined by

\[ O = \sum_k a(k)a(k+1)4\pi(k+1/2)! \] (12)

Therefore, if we integrate over all eigenvalues in equation (9), we obtain

\[
Z = \frac{1}{N!} \langle \left( \prod_{k=1/2}^{2N-1/2} a^\dagger(k) \right) O^N \rangle = \prod_k \left( 4\pi(k+1/2)! \right)
\] (13)

where the product extends over \( k = 1/2, 5/2, 9/2, \ldots, 2N-3/2 \).

**B. Eigenvalue Density**

To calculate the density of eigenvalues, we must integrate over all, except one, of the coordinate pairs \( \bar{w}_j, w_i \) in equation (9). Using equation (11), and normalizing with equation (13), we wish to compute

\[
U(e^{\bar{w}}, e^{w}) e^t \sum_{m,m'} e^{mw}e^{m'\bar{w}} \langle \left( \prod_{k=1/2}^{2N-1/2} a^\dagger(k) \right) O^{N-1} a(m)a(m') \rangle d\bar{w} d w
\]

\[
\frac{1}{(N-1)!Z}
\] (14)

It may be verified that the correlation function appearing in the sum of equation (14) is nonvanishing only if either \( m - 1/2 \) is even, \( m' - 1/2 \) is odd, and \( m < m' \), or if \( m' - 1/2 \) is even, \( m - 1/2 \) is odd, and \( m' < m \). In the first case, with \( m < m' \), the contribution to equation (14) is

\[
\frac{1}{4\pi} \prod_k \left( \frac{1}{k+1/2} \right) \prod_l (l + 1/2)! e^{-\bar{w}z} \sqrt{\bar{w}z} (z - \bar{z}) z^m \bar{z}^{m'} d\bar{w} d w
\] (15)
where the product over \( k \) extends over
\[
k = 1/2, 5/2, 9/2, \ldots
\] (16)
and the product over \( l \) extends over
\[
l = 1/2, 5/2, 9/2 \ldots m - 2, m + 1, m + 3, \ldots, m' - 2, m' + 1, m' + 3, \ldots, 2N - 1/2
\] (17)
This is equal to
\[
\frac{1}{4\pi (m - 1/2)!!(m' - 1/2)!!} e^{-\sqrt{\pi z}(z - \bar{z})} z^m \bar{z}^{m'} d\bar{w} dw
\] (18)
In the second case, with \( m' < m \), the result is
\[
-\frac{1}{4\pi (m - 1/2)!!(m' - 1/2)!!} e^{-\sqrt{\pi z}(z - \bar{z})} z^m \bar{z}^{m'} d\bar{w} dw
\] (19)
We can obtain the eigenvalue density \( \rho(\bar{z}, z) \) by adding equations (18, 19) and summing over \( m, m' \). Shifting \( m \) and \( m' \) by one-half, and changing from \( d\bar{w} dw \) to \( dz dw \), we find that the final result for the eigenvalue density \( \rho(\bar{z}, z) dz dw \) is
\[
\rho(\bar{z}, z) dz dw = \frac{1}{4\pi} e^{-\sqrt{\pi z}(z - \bar{z})} G(\bar{z}, z) dz dw
\] (20)
where the Green’s function \( G(\bar{z}, z) \) is given by
\[
G(\bar{z}, z) = \sum_{m < m': m=0,2,4,\ldots; m'=1,3,5,\ldots} \frac{1}{m!!m'!!} (z^m \bar{z}^{m'} - \bar{z}^m z^{m'})
\] (21)

C. Discussion

Let us now look at the properties of equation (24). We will discuss in turn the normalization of the density; the way the density depends on \( x \) and \( y \) separately, where \( z = x + iy \); an integral representation for the density; the circular law; and the depletion of density near the real axis.

First, consider the normalization of the single particle density. It is automatic from the above derivation that the eigenvalue density is properly normalized, although one must be careful about defining the normalization depending on whether one is counting total number of eigenvalues or total number of pairs of eigenvalues. The normalization is defined such that \( \int \rho(\bar{z}, z) dz dw = N \).

Next, writing \( z = x + iy \) and \( \bar{z} = x - iy \), one can show by differentiating the power series in equation (21) that \( \partial_x G(x + iy, x - iy) = \frac{\partial^2}{\partial^2 y} G(x + iy, x - iy) = 2x G(x + iy, x - iy) \), for large \( N \). This implies that
\[
G(x + iy, x - iy) = e^{x^2} f(y)
\] (22)
and therefore $\rho = \frac{1}{4\pi} 2y e^{-y^2} f(y)$, for some function $f$, so the interesting properties of the eigenvalue density are contained in $f(y)$. Later we will discuss the properties of $f(y)$ for small $y$, and show that there is a depletion of the density of eigenvalues near the real axis.

Using equations (20, 21, 22), we can derive an integral representation for $\rho$. We can use equation (22) to write

$$G(\zbar, z) = e^{z\zbar} G(\zbar - z, 0)$$  \hspace{1cm} (23)

Then equation (21) implies that

$$G(\zbar - z, 0) = \sum_{m=1,3,5,...} \frac{1}{m!!} (\zbar - z)^m$$  \hspace{1cm} (24)

This is equal to

$$\int_0^\infty i \sin\left(\frac{\zbar - z}{i} t\right) e^{-t^2/2} \, dt$$  \hspace{1cm} (25)

Using this integral representation in equation (20) we find that

$$\rho(\zbar, z) = \frac{1}{4\pi} 2y \int_0^\infty \sin(2yt) e^{-t^2/2} \, dt$$  \hspace{1cm} (26)

Let us now consider the circular law. For large $y$, equation (26) reduces to

$$\rho(\zbar, z) \, d\zbar \, dz \rightarrow \frac{1}{4\pi} d\zbar \, dz$$  \hspace{1cm} (27)

So, the density tends to a constant for large $y$. However, this integral representation is valid only for $N$ infinite; for finite $N$, the density tends to a constant only within a disc of radius $\sqrt{2N}$, and vanishes outside the disc. This is the well-known circular law [1,9]. The vanishing of the density outside the disc is easy to see from the power series representation. For finite $N$ the highest power of $(\zbar z)$ appearing in equation (21) is roughly $2N$ and so $\rho$ will be exponentially small for $(\zbar z) > 2N$.

Note that the total density in the disc is correct. The area of a disc of radius $R$ is $2\pi R^2$, where we are using the measure $d\zbar \, dz = 2dx \, dy$. The density is $\frac{1}{4\pi}$. So, the number of particles in a disc of radius $\sqrt{2N}$ is indeed $N$, as desired.

For small $y$, we find that $\rho$ is reduced below the expected result. Such a reduction was found numerically before [4]. In the figure, we graph the eigenvalue density as a function of $y$, for $x = 0$, for a system of 100 particles.

### III. WEAKLY NON-HERMITIAN CASE

Now we will consider the weakly non-Hermitian version of the ensemble given above. In the weakly non-Hermitian random matrix ensemble, we again consider an arbitrary $N$-by-$N$ matrix of quaternions, $H$, but use a different Gaussian weight. Let $H = H_h + H_a$, where
the $H_h$ is Hermitian and $H_a$ is anti-Hermitian. Then, we chose the matrix $H$ with Gaussian weight

$$e^{-\frac{N}{2}\text{Tr}(H_h^\dagger H_h) - \frac{N^2}{2}\text{Tr}(H_a^\dagger H_a)}$$

(28)

where $a$ is some constant. In the large $a$ limit, this reduces to the Gaussian Symplectic Ensemble. For finite $a$, the weight in equation (28) is chosen to make sure that the imaginary part of the eigenvalues scales with $N$ in the same way as the level spacing.

If the matrix $H$ is chosen with weight given by equation (28), then the j.p.d. of equation (2) gets replaced by

$$\frac{1}{N!} \prod_i e^{-Nz_i^2 - N^2ay_i^2} |z_i - \overline{z}_i|^2 \prod_{i<j} (z_i - z_j)(z_i - \overline{z}_j)(\overline{z}_i - z_j)(\overline{z}_i - \overline{z}_j) \prod_i d\overline{z}_i \, dz_i$$

(29)

The only difference in the weakly non-Hermitian case is that the Gaussian function of eigenvalue position $e^{-\overline{z}_iz_i}$ is replaced by $e^{-x_i^2 - Nay_i^2}$.

We have not found equation (29) previously in the literature. This equation can be derived most easily as follows: write an $N$-by-$N$ matrix of quaternions, $H$, as

$$H = X^{-1}TX$$

(30)

where $X$ is a quaternion matrix such that $X^{-1} = X^\dagger$, and $T$ is an upper triangular matrix of quaternions. This procedure is a Schur decomposition, and is possible since the field of quaternions, like the field of complex numbers, is algebraically closed.

The eigenvalues, $z_i$, can be obtained from the diagonal elements of $T$; each diagonal element of $T$ is a quaternion, which is associated with a pair of complex conjugate eigenvalues $z_i, \overline{z}_i$. If a given diagonal element of $T$ is $T_i = A + Bi + Cj + Dk$, then $z_i = A \pm i\sqrt{B^2 + C^2 + D^2}$.

The Jacobian associated with this change of variables is $\prod_{i<j} (z_i - z_j)(z_i - \overline{z}_j)$. Further,

$$e^{-\frac{N}{2}\text{Tr}(H_h^\dagger H_h) - \frac{N^2}{2}\text{Tr}(H_a^\dagger H_a)} = e^{-\frac{N}{2}\text{Tr}(T_h^\dagger T_h) - \frac{N^2}{2}\text{Tr}(T_a^\dagger T_a)}$$

(31)

where $T_h, T_a$ are Hermitian and anti-Hermitian parts of $T$. The integral over the elements of $T$ above the diagonal can be done trivially as this integral is Gaussian. The integral over the diagonal elements of $T$ includes a Gaussian factor and a factor from the Jacobian. This integral is exactly the integral over the j.p.d. of equation (29).

Given equation (29), we could follow the procedure of the previous section. However, we would run into some difficulties which are purely technical. The problem is that, while in the strongly non-Hermitian case the eigenvalue density is independent of $x$, it is not independent of $x$ in the weakly non-Hermitian case. This makes the power series expansion very awkward. We will find it convenient to change to a different geometry, given in equation (32) below, for the weakly non-Hermitian case. Let me again stress that the reason for choosing a different geometry is purely technical, to simplify the math.

An analogous simplification is often used in the Hermitian ensembles. For example, consider the Gaussian Symplectic Ensemble in the large $N$ limit. The eigenvalue density is
a function of energy, but if one appropriately scales all energies by the local level spacing, it is simpler to obtain correlation functions from the Circular Symplectic Ensemble [10].

Let us introduce new coordinates, $z = \phi + ir$ and $\bar{z} = \phi - ir$, where $\phi$ is periodic with period $2\pi$. Let us replace equation (32) by

$$\frac{1}{N!} \prod_i e^{-aN^2v_i^2}(e^{iz_i} - e^{iz_j})^2 \prod_{i<j} \frac{(e^{iz_i} - e^{iz_j})(e^{iz_i} - e^{iz_j})}{e^{-2i\phi_i - 2i\phi_j}} \prod_i dz_i dz_i$$

This describes a system of $N$ pairs of levels, with average level spacing $2\pi/N$. The imaginary part of the level is of order $1/N$, so it is of order the level spacing. For large $a$ this reduces to the Circular Symplectic Ensemble. For finite $a$ and large $N$, we expect that the ensemble of equation (32) reproduces the behavior of the ensemble of equation (29) within a small neighborhood of some given energy, just as the Circular Symplectic Ensemble reproduces the results of the Gaussian Symplectic Ensemble within a neighborhood of a given energy.

The next step is to write equation (32) as a correlation function in a fermionic field theory. We will introduce $N$ creation operators at $r = +\infty$ and $N$ creation operators at $r = -\infty$. We find that the desired correlation function is

$$\frac{1}{N!} \lim_{L \to \infty} e^{N^2L} \langle \left( \prod_{j=1}^N \psi^+(b_j) \right) \left( \prod_{j=1}^N \psi^+(c_j) \right) \left( \prod_j U(r_j) \psi(\bar{z}_j) \psi(z_j) d\bar{z}_j dz_j \right) \rangle$$

where $b_j = \frac{2\pi j}{N} + iL$ and $c_j = \frac{2\pi j}{N} - iL$ and $U(r_j) = e^{-aN^2v_j^2}(r^*_j - e^{r_j})$. For large $N$, we can write $U(r_j) = e^{-aN^2v_j^2/2}r_j$.

Now, we will introduce Fourier modes for the creation and annihilation operators, writing $\psi(z) = \sum_k e^{ikz}a(k)$ and $\psi^+(z) = \sum_k e^{-ikz}a(k)$. In the limit $L \to \infty$, the only states involved in equation (33) are those with $k = -N + 1/2, -N + 3/2, \ldots, N - 1/2$. If there are excitations in states with higher $k$, they will vanish in the large $L$ limit. Then, equation (33) can be written as

$$\frac{1}{N!} \lim_{L \to \infty} \langle \left( \prod_{k=-N+1/2}^{-N-1/2} a^+(k) \right) \left( \sum_{k,k'} a(k)a(k')e^{i\bar{z}_j}e^{ik'z_j}U(r_j)d\bar{z}_j dz_j \right) \rangle$$

As in the previous section, we will integrate over some set of $z_j$, for $j = 1 \ldots M$, inside the correlation function. The integral

$$\int \prod_{j=1}^M \left( \sum_{k,k'} a(k)a(k')e^{i\bar{z}_j}e^{ik'z_j}U(r_j)d\bar{z}_j dz_j \right)$$

is equal to

$$O_w^M$$

where the operator $O_w$ is defined by

$$O_w = 8 \left( \frac{\pi}{aN^2} \right)^{3/2} \sum_k \left( ke^{i\alpha}a(k)a(-k) \right)$$
So, if we integrate over all coordinates $\zeta, z$ in equation (34), we obtain

$$Z = \frac{1}{N!} \left( \prod_{k=-N+1/2}^{N-1/2} a^\dagger(k) \right)^N = \prod_{k=1/2}^{N-1/2} \left( 16 \left( \frac{\pi}{aN^2} \right)^{3/2} ke^{\xi^2} \right)$$

(38)

To obtain the eigenvalue density, we must integrate over all but one of the coordinates in equation (34). Using equation (36), and normalizing with equation (38), we obtain

$$U(r) \sum_{m,m'} e^{i m z} e^{i m' z} \left( \prod_{k=-N+1/2}^{N-1/2} a^\dagger(k) \right)^{N-1} a(m) a(m') \right) d\zeta dz \frac{(N-1)!Z}{N-1)!Z}$$

(39)

The correlation function in equation (39) is non-vanishing only if $m = -m'$. Equation (39) is equal to

$$U(r) \sum_{m=-N+1/2}^{N-1/2} e^{i m(z-z')} G_w(m) d\zeta dz \frac{(N-1)!Z}{N-1)!Z}$$

(40)

where

$$G_w(m) = \frac{1}{16m} \left( \frac{\pi}{aN^2} \right)^{-3/2} e^{-\frac{m^2}{aN^2}}$$

(41)

In the large $N$ limit, we can simplify equation (41) by introducing scaled coordinates. Let us introduce $k = m/N$ and let us also scale $z$ by a factor of $N$ so that $\phi$ now runs from 0 to $2\pi N$. Then we can replace the sum by an integral and obtain

$$\rho(\phi, r) d\phi dr = \frac{1}{4} \left( \frac{\pi}{a} \right)^{-3/2} e^{-ar^2} G_w(\zeta, z) d\phi dr$$

(42)

where

$$G_w(\zeta, z) = \int_{-1}^{1} e^{ik(\zeta-z)} e^{-k^2/a} \frac{1}{k} dk$$

(43)

As in the previous section, the proper normalization of the above result is automatic from the derivation. It is possible to show that equation (42) is equivalent to equation (26) in the limit of very small $a$. The qualitative feature of a depletion of eigenvalues near the real axis is the same for weak and strong non-Hermiticity. Equation (42) may be compared to the results of the SUSY calculation [2], and found to agree, with some differences in notation between the two calculations.

IV. MULTI-POINT CORRELATION FUNCTIONS

The calculation of multi-level correlation functions is quite easy. In equations (34), we must integrate over all except for two, three, or more, of the coordinate pairs $\zeta_i, \zeta_j$. Since the system is a non-interacting fermion system, the multi-point Green’s functions can be expressed very simply in terms of the Green’s function (21), using Wick’s theorem. This permits the two-point correlation function to be easily generalized to a multi-point correlation function, as conjectured previously [3]. We will not show this in detail, but simply sketch the results, first for the strongly non-Hermitian case and then for the weakly non-Hermitian case.
A. Strongly Non-Hermitian Case

First let us examine the strongly non-Hermitian case, generalizing the results of section II. Consider the two-level correlation function, the probability to find a pair of levels at position $\zbar, z$ given that there is another pair at position $\zbar', z'$. Then, the two-level correlation function is

$$
\left(\frac{1}{4\pi}\right)^2 e^{-|z-z'|^2} (z - \zbar)(z' - \zbar') \left( G(\zbar, z)G(\zbar', z') - G(\zbar, z')G(\zbar', z) + G(\zbar, z)G(z', z) \right) d\zbar\, dz\, d\zbar'\, dz'
$$

(44)

This is just an application of Wick’s theorem.

Let us consider the behavior of equation (44) in the limit when both $z$ and $z'$ are far from the real axis. Without loss of generality, assume that $\text{Re}(z') > \text{Re}(z)$. Then, use the integral representation of the Green’s function to rewrite $e^{-|z-z'|^2}G(\zbar, z)G(\zbar', z)$ as

$$
e^{-|z-z'|^2} \left( \int_0^\infty e^{(\zbar-z)t}e^{-t^2/2}dt - \frac{1}{2} e^{(\zbar-z')^2/2} \right) \left( \frac{1}{2} e^{(\zbar-z)^2/2} - \int_{-\infty}^0 e^{(\zbar-z)t}e^{-t^2/2}dt \right)
$$

(45)

We can find a similar representation for $G(\zbar, z')G(\zbar', z)$. Now, in the limit with $z$ and $z'$ both far from the real axis, then either $\text{Im}(z-z')$ is large or $\text{Im}(z-\zbar)$ is large. In the first case, equation (44) is exponentially small because of the factor of $e^{-|z-z'|^2}$. In the second case, the integrals over $t$ can be performed in this limit, while $e^{(\zbar-z')^2/2}$ and $e^{(\zbar-z)^2/2}$ are small. The integral, $\int_0^\infty e^{(\zbar-z)t}e^{-t^2/2}dt$, is equal to $\frac{1}{\zbar-z}$, for large $\text{Im}(z'-\zbar)$; here we rely on the fact that $\text{Re}(z'-\zbar) > 0$.

So, up to exponentially small terms, equation (13) is equal to

$$e^{-|z-z'|^2} \frac{1}{\zbar-z'} \frac{1}{\zbar'-z}
$$

(46)

Also, in this limit, if equation (46) is not exponentially small, then $\frac{1}{\zbar-z'} \frac{1}{\zbar'-z} = \frac{1}{\zbar-z} \frac{1}{\zbar'-z'}$. Inserting this result, and similar results for $G(\zbar, z')G(z', z)$, back into equation (44) we find that, for both $z$ and $z'$ far from the real axis, the two-level correlation function is equal to

$$
\left(\frac{1}{4\pi}\right)^2 \left( 1 - e^{-|z-z'|^2} - e^{-|\zbar-\zbar'|^2} \right) d\zbar\, dz\, d\zbar'\, dz'
$$

(47)

This is essentially the same as the two-level correlation function found in the complex non-Hermitian case [4].

For $z$ and $z'$ near the real axis, I have examined the behavior of equation (13) numerically. If $\text{Im}(z) = \text{Im}(z')$, then the correlation function is a monotonically decaying function of $\text{Re}(z-z')$, with no signs of any oscillation. The correlation function is exponentially small if both $z$ and $z'$ are near the real axis.
B. Weakly Non-Hermitian Case

Now let us consider the two-level correlation function in the weakly non-Hermitian case, the probability to find one pair of levels \( z, z \) given that there is another pair at \( z', z' \). As before, we must integrate over all except for two of the eigenvalue coordinates. Using the scaled coordinates, we find that the two-level correlation function is given by

\[
\frac{1}{16} \left( \frac{\pi}{a} \right)^{-3} e^{-ar^2-ar'^2} rr' \left( G_w(z, z)G_w(z', z') - G_w(z, z')G_w(z', z) + G_w(z, z)G_w(z', z') \right) d\phi dr d\phi' dr'
\]

As in the previous subsection, this is just an application of Wick’s theorem.

To examine the behavior of the two-level correlation function, let us integrate over \( r, r' \) in equation (48), to be left with a function of \( \phi - \phi' \). The result is

\[
\frac{1}{4\pi^2} - \frac{1}{32\pi^2} \int_{-1}^{1} \frac{(k+k')^2}{kk'} e^{-\frac{(k-k')^2}{2(2a)}} e^{ik(\phi-\phi')} e^{ik'(\phi'-\phi')} dk dk'
\]

In the limit \( a \to \infty \), the integral over \( k, k' \) in the above expression can be performed to yield

\[
\frac{1}{4\pi^2} - \frac{1}{4\pi^2} \int_{-2}^{2} \left( 1 - \frac{|k|}{2} - \frac{|k|}{4} \log(|k| - 1) \right) e^{ik(\phi-\phi')} dk
\]

Equation (50) is the known result for the correlation function in the Circular Symplectic Ensemble. It is a non-monotonic function, algebraically decaying for large \( \phi \). For sufficiently small \( a \), equation (49) will describe a monotonically decaying function of \( \phi' - \phi \), but for fixed, non-vanishing \( a \), the function will always decay algebraically for large \( \phi \).

V. CONCLUSION

In conclusion, we have given a simple fermionic mapping for determining the correlation functions of the non-Hermitian symplectic ensemble. Although the eigenvalue density was found previously using SUSY, the present derivation is simpler and can be more easily extended to the two-level correlation function. The two-level correlation function in the strongly non-Hermitian case was found to be similar to that for the ensemble of arbitrary complex matrices. In the weakly non-Hermitian case, the two-level correlation function exhibits an interesting crossover as a function of \( a \).
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FIG. 1. Eigenvalue Density