Gravitational collapse of a minimally coupled massless scalar field

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We study here the evolution of a massless scalar field in a spacetime, developing from a regular initial spacelike surface. The Einstein equations and regularity and boundary conditions governing the same are specified. Both homogeneous and inhomogeneous collapse models are considered and we analyze when the occurrence of singularity will be simultaneous or otherwise. In the inhomogeneous collapse case, we characterize a wide family of black hole solutions arising in scalar field collapse. We also discuss the possibility of existence of classes of non-singular models where collapse could almost freeze if suitable conditions are satisfied.

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I. INTRODUCTION

The formation of spacetime singularities in gravitational collapse and formation of black holes is an issue of great importance in gravitation physics, which has been investigated in much detail in Einstein’s theory. The occurrence of singularities offers the regime where gravity is extreme, and where the quantum gravity effects would be important. As is known, dynamical evolution of matter fields in a spacetime generically yields a singularity, provided reasonable physical conditions are satisfied such as the causality, a suitable energy condition ensuring the positivity of energy density, and formation of trapped surfaces.

The case of gravitational collapse of a massless scalar field is of particular interest in both collapse situations as well as cosmological scenarios. In cosmology, special importance is attached to the evolution of a scalar field, which has attracted a great deal of attention in past decades. This is because one would like to know the behaviour for fundamental matter fields towards understanding the transition from matter dominated regime to dark energy domination (see e.g. [1] and references therein). Scalar fields are of much interest in view of the inflationary scenarios that govern the early universe dynamics because such a field can act as an ‘effective’ cosmological constant in driving the inflation [2]. In gravitational collapse studies, the nature of singularity for massless scalar fields has been examined and a number of numerical and analytical works have been done in recent years.
on spherical collapse models \([3, 10]\), from the perspective of the cosmic censorship hypothesis. The massless free scalar field has been studied for the static case also in some detail (see e.g. \([11, 12, 13, 14, 15, 16]\)). However, the dynamical case is more important to understand because that may give insights into phenomena such as gravitational collapse and the cosmic censorship hypothesis, other than the early universe and cosmological considerations.

In the present study, we first develop here a mathematical structure in a general manner to deal with the evolution of massless scalar fields in a spacetime. This would be applicable to problems either in cosmology or for gravitational collapse final states in a spherically symmetric spacetime. We then examine the classes of collapsing models where the singularity in future forms simultaneously as the collapse develops, even when the density of the field could be inhomogeneous. We study here both homogeneous as well as inhomogeneous collapse models. We then characterize here a wide class of black hole models forming as collapse final state. Apart from the spherical symmetry, we do not assume here any further constraints on the spacetime, such as the presence of a homothetic Killing vector, homogeneous or shearfree nature for the fluid, or such other conditions.

We note here the classes of models where our considerations would apply. The analysis in this paper has been done using a comoving coordinate system. For massless scalar fields, such a coordinate system would break down when the gradient of the scalar field becomes null. So the conclusions based on the present analysis would be valid for those classes of models and solutions for which the gradient of the scalar field is timelike throughout the dynamical evolution of the field. We note that the homogeneous and isotropic Friedman-Roberson-Walker solutions with a massless scalar field as the matter content is such an example, where this condition is satisfied. Massless scalar field solutions with inhomogeneous perturbations around a homogeneous background would also be in this class. In fact, as we shall argue later, this class where the gradient of the scalar field always remains timelike, would include a large number of physically relevant situations. On the other hand, our analysis does not apply to the case where the gradient of the scalar field changes its sign, becoming null and then spacelike, from its original nature of being timelike. However, as we have argued here, the present class is of sufficient physical interest in its own right to carry out an analysis of the same as far as the dynamical evolutions of the scalar fields are concerned. A related important point that we note is, the analysis given here would also hold and describe the evolution of stiff fluids in a spacetime in a general manner. This is because, a massless scalar field with a timelike gradient, which is minimally coupled to gravity, has an exact correspondence with a stiff fluid minimally coupled to gravity.
The massless scalar field \( \phi(x^a) \) on a spacetime \((M,g_{ab})\) is described by the Lagrangian,

\[
\mathcal{L} = -\frac{1}{2} \phi_{;a} \phi_{;b} g^{ab}.
\] (1)

The corresponding Euler-Lagrange equation is \( \phi_{;ab} g^{ab} = 0 \), and the energy-momentum tensor for the scalar field, as calculated from the above Lagrangian is,

\[
T_{ab} = \phi_{;a} \phi_{;b} - \frac{1}{2} g_{ab} \left( \phi_{;c} \phi_{;d} g^{cd} \right).
\] (2)

A massless scalar field is a \textit{Type I} matter field \[17\], \textit{i.e.}, it admits one timelike and three spacelike eigenvectors. At each point \( q \in M \), we can then express the tensor \( T_{ab} \) in terms of an orthonormal basis \((E_0, E_1, E_2, E_3)\), where \( E_0 \) is a timelike eigenvector with an eigenvalue \( \rho \) and \( E_\alpha \) \((\alpha = 1, 2, 3)\) are three spacelike eigenvectors with eigenvalues \( p_\alpha \). The eigenvalue \( \rho \) represents the energy density of the scalar field as measured by an observer whose world line at \( q \) has an unit tangent vector \( E_0 \) and the eigenvalues \( p_\alpha \) represent the principal pressure in three spacelike directions \( E_\alpha \). We now choose the spherically symmetric coordinates \((t,r,\theta,\phi)\) along the eigenvectors \((E_0, E_\alpha)\), such that the reference frame is \textit{comoving}. As discussed in \[18\], the general spherically symmetric metric in comoving coordinates can be written as,

\[
d s^2 = -e^{2\nu(t,r)} dt^2 + e^{2\psi(t,r)} dr^2 + R^2(t,r) d\Omega^2,
\] (3)

where \( d\Omega^2 \) is the metric on a unit 2-sphere and we have used the two gauge freedoms of two variables, namely, \( t' = f(t,r) \) and \( r' = g(t,r) \), to make the \( g_{tr} \) term in metric and the radial velocity of the matter field to vanish. We note that we still have two scaling freedoms of one variable in \( t \) and \( r \).

In general, we have \( \phi = \phi(t,r) \), but from equation \[2\] it is easily seen that in the comoving reference frame \[3\], we must have \( \phi(t,r) = \phi(t) \) or \( \phi(t,r) = \phi(r) \), because the energy-momentum tensor is diagonal. As we would like to investigate here the dynamic behaviour of the scalar field, we consider the former option. In this comoving frame the components of the energy-momentum tensor are,

\[
T^t_t = T^r_r = T^\theta_\theta = T^\phi_\phi = \frac{1}{2} e^{-2\nu(t,r)} \dot{\phi}^2.
\] (4)

Thus, we see that in the comoving frame the massless scalar field behaves like a \textit{stiff} isentropic perfect fluid with the equation of state

\[
p(t,r) = \rho(t,r) = e^{-2\nu(t,r)} \dot{\phi}^2 / 2.
\] (5)
We can easily see that for any real valued function $\phi(t)$, all energy conditions are satisfied by the matter field. We note that if we consider the field $\phi = \phi(t)$, then the weak energy condition guarantees that $\phi_{,\mu}$ is either timelike or null always in general. However, in a comoving frame, the gradient of the scalar field remains timelike throughout the collapse, which is a property of the comoving frame by definition.

In a physically reasonable collapse situation, the energy density of the matter field and the magnitude of the Ricci scalar are expected to increase with time. For a massless scalar field we have $| \phi_{,\mu} \phi^{,\mu} | = 2 \rho$. In such cases, if collapse progresses from a regular spacelike hypersurface where the gradient of the scalar field is timelike, then the density is non-zero initially and it can only increase afterwards. This would mean that throughout the collapse evolution of the scalar field from the given initial regular surface, the gradient of the scalar field would always remain timelike. So in this case of a collapsing scalar field, we can use the comoving coordinate system without any loss of generality and without a concern on a possible breakdown of the coordinate system.

It is important to note that we are not specifying here the class of initial data from which one can have collapse situations where the density would always increase. But given such a solution, our analysis would hold. Therefore, the comoving system can be used to study the collapse of a massless scalar field, where the density either increases or does not decrease with time. In this paper, by a collapsing scalar field, we would mean only such cases.

The correspondence here with a stiff fluid can be seen from (2). The energy momentum tensor for the perfect fluid is

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab},$$

where $u^\mu$ is the velocity vector. For stiff fluid, $p = \rho$.

Since $\phi_{,\mu}$ is timelike, $\phi_{,\mu} \phi^{,\mu} = -| \phi_{,\mu} \phi^{,\mu} |$. Defining $u_\mu = \frac{\phi_{,\mu}}{|\phi_{,\mu} \phi^{,\mu}|}$, the energy momentum tensor for the massless scalar field can be expressed as, $T_{ab} = (| \phi_{,\mu} \phi^{,\mu} |) u_a u_b + \frac{1}{2} g_{ab} (| \phi_{,\mu} \phi^{,\mu} |)$.

Denoting $| \phi_{,\mu} \phi^{,\mu} | = \rho = p$, this expression for the energy momentum tensor is the same as that for the stiff fluid. The unit velocity vector, $u^\mu = (\dot{\phi}(t))^{-1}$ In a comoving coordinate system, we can choose $u^\mu = (1, 0, 0, 0)$, which for the massless scalar field would be equivalent to choosing $\dot{\phi}(t) = 1$. If $\dot{\phi}(t)$ does not diverge, such a choice is always possible.
II. EINSTEIN EQUATIONS, REGULARITY AND BOUNDARY CONDITIONS

The dynamic evolution of the initial data, as specified on a spacelike surface of constant time is determined by the Einstein equations. For the metric \([3]\), using the definitions

\[
G(t, r) = e^{-2\psi(R')}^2, \quad H(t, r) = e^{-2\nu(\dot{R})^2},
\]

and

\[
F = R(1 - G + H),
\]

the independent Einstein equations for the massless scalar field (in the units \(8\pi G = c = 1\)) are then given by,

\[
F' = \frac{1}{2} e^{-2\nu} \dot{\phi}^2 R^2 R',
\]

\[
\dot{F} = -\frac{1}{2} e^{-2\nu} \dot{\phi}^2 R^2 \dot{R},
\]

\[
\partial_t \left( R^2 e^{\psi - \nu} \dot{\phi} \right) = 0,
\]

\[
R' \dot{G} - 2 R \nu' G = 0.
\]

Here \(\cdot'\) denotes the partial derivative with respect to the coordinate \(r\) and \(\cdot\) with respect to \(t\). The function \(F = F(t, r)\) has an interpretation of the mass function for the collapsing cloud, and it gives the total mass in a shell of comoving radius \(r\) on any spacelike slice \(t = \text{const.}\). The energy conditions imply \(F \geq 0\).

The function \(R(t, r)\) is the area radius of a shell labeled \(r\) at an epoch \(t\). For the sake of definiteness let us consider the situation of a collapsing cloud, and we have \(\dot{R} < 0\) as we are considering the collapsing branch of the solutions. If \(\dot{R}\) changes sign then that corresponds to a bounce or dispersal of the field during evolution. We use the scaling freedom for the radial coordinate \(r\) to write \(R = r\) at the initial epoch \(t = t_i\), and in order to distinguish the regular center of the cloud at \(r = 0\) from the genuine spacetime singularity at the termination of collapse, where the area radius \(R = 0\) in both cases, we introduce a function \(v(t, r)\) as defined by

\[
v(t, r) \equiv \frac{R}{r}.
\]

We then have,

\[
R(t, r) = rv(t, r), \quad v(t_i, r) = 1, \quad v(t_s(r), r) = 0
\]
with $\dot{v} < 0$. The time $t = t_s(r)$ here corresponds to the shell-focusing singularity at $R = 0$, where the matter shell labeled a comoving radius of constant $r$ collapses to a vanishing physical radius $R$ on reaching the genuine spacetime singularity.

We note that equation (11) is the Klein-Gordon equation for the scalar field, which is a part here of the Einstein equations via the Bianchi identities. We can integrate this equation to get

$$R^2 e^{\psi - \nu \phi} = r^2 f(r), \quad (15)$$

where $f(r)$ is an arbitrary function of integration. We can now eliminate the function $\dot{\phi}(t)$ from equations (9) and (10) to get

$$\frac{F'}{R'} = -\frac{\dot{F}}{R} = \frac{1}{2} \frac{r^4 f^2(r) G}{R^2 R'^2}. \quad (16)$$

We now have four Einstein equations, namely (8), (12) and (16), and four unknown functions of two variables, $\psi, \nu, R$ and $F$. Solution of these equations, subject to the initial data and energy conditions, would determine the time evolution of the system.

Our purpose now is to construct the classes of solutions to the Einstein field equations, which give the dynamical scalar field collapse evolutions, given the initial data at an initial time $t = t_i$. We define the suitably differentiable functions $\mathcal{M}(r, v)$ and $A(r, v)$ as below

$$\mathcal{M}(r, v) \equiv \frac{F(t, r)}{r^3}, \quad (17)$$

$$A(r, v) \equiv \frac{\nu'}{R'}. \quad (18)$$

We note that Einstein equations (9) and (10), imply that $F$ must behave necessarily as $r^3$ closer to the regular center $r = 0$ of the cloud, in order to preserve the regularity of initial data and to preserve the finiteness of matter density at all regular epochs of evolution. Hence, as $\mathcal{M}(r, v)$ is a general, at least $C^2$ function, the equation (17) is not really any ansatz or a special choice, but a fully generic class of mass profiles for the collapsing cloud, consistent with and as allowed by the regularity conditions.

Also, in order to be specific, we construct only classes of collapse evolutions which admit no shell-crossing singularities in the spacetime where $R' = 0$. We therefore consider only the genuine singularity at $R = 0$ where the physical radii of the collapsing shells vanish, and not the cases when nearby shells of matter may cross, giving rise to a density singularity which need not be gravitationally strong. Therefore the function $A(r, v)$ is well-defined for all non-singular epochs. Now using equation (18) in equation (12) we get, as a class of solutions of Einstein’s equations

$$G(r, v) = b(r)e^{2\nu A(r, v)}. \quad (19)$$
Here $b(r)$ is another arbitrary function of the shell radius $r$. The regularity condition on the velocity function $\dot{v}$ at the center of the cloud implies that the form of $b(r)$ has to be,

\[ b(r) = 1 + r^2 b_0(r). \tag{20} \]

A comparison with the Lemaitre-Tolman-Bondi dust collapse models \[19\] implies that we can interpret $b_0(r)$ as the energy distribution function for the collapsing shells.

We emphasize that the functions $M(r,v)$ and $A(r,v)$ are not independent here in the case of scalar field collapse. Because, using \[19\] in the second part of \[16\] gives the required relation between them as

\[ 2rA(r,v) = \ln \left[ \frac{-2M(r,v)_{,v}v^2(v + rv')^2}{q(r)} \right]. \tag{21} \]

where $q(r) = f^2(r)[1 + r^2b_0(r)]$. Now to determine the function $M(r,v)$, we use \[17\] in the first part of \[16\] to get the required first order equation

\[ 3M(r,v) + rM(r,v)_{,r} + Q(r,v)M(r,v)_{,v} = 0. \tag{22} \]

where $Q(r,v) = (2rv' + v)$. The above equation has a general solution of the form

\[ \mathcal{F}(X,Y) = 0, \tag{23} \]

where $X(r,v,M(r,v))$ and $Y(r,v,M(r,v))$ are the solutions of the system of equations,

\[ \frac{dM(r,v)}{3M(r,v)} = \frac{dr}{r} = \frac{dv}{Q}. \tag{24} \]

Amongst all the classes of solutions $M(r,v)$ of above, those are to be considered which obey the energy conditions and required regularity conditions for the collapse.

To specify the boundary conditions for the cloud, solving \[22\] at $r = 0$, we get

\[ \lim_{r \to 0} M(r,v) = \frac{m_0}{v^3}, \tag{25} \]

where $m_0$ is a constant and from \[21\] we see that the regularity at the center of the cloud requires $6m_0 = q(0)$. Along the singularity curve $v = 0$ the density diverges. To solve for the function $v(t,r)$, we use the equation of motion \[8\], and defining a function $h(r,v)$ as

\[ h(r,v) = [e^{2rA(r,v)} - 1]/r^2, \tag{26} \]

we get,

\[ \sqrt{\dot{v}v} = -G(r,v) \tag{27} \]
where,

\[ G(r, v) = e^{\nu(r,v)} \sqrt{vb_0e^{2rA} + vh(r,v) + M(r,v)}. \]  

(28)

The negative sign in the right hand side of equation (27) corresponds to a collapse scenario where we have \( \dot{R} < 0 \). Also in the \((r,v)\) plane, the function \( \nu(r,v) \) is related to the function \( A(r,v) \) by the relation

\[ \nu(r,v),_r + \nu(r,v),_v v' = A(r,v),_v(v + rv') \]  

(29)

Given the functions \( A(r,v),_v \) and \( v' \) in terms of \( r \) and \( v \), this is again a quasi-linear first order partial differential equation like (22), with a similar general solution.

Now we have derived the relations between all the functions in the \((r,v)\) plane. The solutions to this system describes the evolving scalar fields which we discuss, and then we consider the nature of initial data required to completely specify the collapsing model.

To see the existence of solutions to the complete system evolving from an initial spacelike slice \( t = t_i \), consider the partial derivatives of the function \( v(t,r) \). We get \( v' \) from (22)

\[ v' = -\frac{3\mathcal{M}(r,v) + r\mathcal{M}(r,v),_r + v\mathcal{M}(r,v),_v}{2r\mathcal{M}(r,v),_v}. \]  

(30)

and from (27) we get \( \dot{v} \).

\[ \dot{v} = -\frac{G(r,v)}{\sqrt{v}}. \]  

(31)

These give expressions of \( \dot{v} \) and \( v' \) in terms of \( v, r, \mathcal{M}(r,v) \) and its derivatives. As we have already seen from (21), we can write the function \( A(r,v) \) and hence \( \nu(r,v) \) in terms of \( \mathcal{M}(r,v) \) and it’s derivatives, \( r, v \) and \( v' \). Thus, to get a solution of \( v(t,r) \), which would in turn provide the complete dynamical solution to the collapsing system, we need the Pfaffian differential equation, \( v' dr + \dot{v} dt - dv = 0 \), to be integrable. The integrability condition is given by,

\[ \dot{v}v,_{_v} = \dot{v},_r + v' \dot{v},_v \]  

(32)

This integrability condition gives the required second order equation for the mass function \( \mathcal{M}(r,v) \). Any solution of the above equation would by default solve the quasilinear equation (22), via (30). Also, this would then uniquely solve for functions \( A(r,v) \) and hence \( \nu(r,v) \) via (21). The solution set of this integrability condition is non-empty as \( \mathcal{M}(r,v) = m_0/v^3 \) solves the whole system with \( v = v(t), \nu = \nu(t) \) and \( A = A(r) \) to give a FRW interior.

For each solution \( M \) of (32), we would have a particular Pfaffian equation, which would be integrable. We know that given one integrating factor for a Pfaffian differential equation, we can
find infinity of them. So for any such $M$, if we have one such integrating factor of the corresponding Pfaffian equation, then there would be an infinite number of such integrating factors. In general, they would correspond to an infinite number of solutions of that Pfaffian equation. Among the solutions that exist, we choose only those which obey the required boundary conditions.

We now consider the independent initial profiles required at the epoch $t = t_i$ or $v = 1$ to evolve the system via Einstein equations. From (9) we see that providing the function $M(r, 1)$, which is the same as $F(t_i, r)/r^3$, would determine the function $\nu(t_i, r)$ (upto a multiplicative constant $\phi(t_i)$). This would then determine the function $A(r, v)|_{v=1}$ using (18) or (29). Also from (22), we see that $M(r, 1)$ specifies $M(r, v)|_{v=1}$, which in turn determines $A(r, 1)$ using (21) and $h(r, 1)$. Hence we see that specifying the functions $M(r, 1)$ (interpreted as the initial density profile) and $q(r) = f^2(r)[1 + r^2b_0(r)]$ (which specifies the initial velocity profile) completely determine the evolution of the system.

Assured of the existence of solutions, integrating equation (27) with respect to $v$, we can write the solution of $v(t, r)$ in an integral form,

$$t(v, r) = \int_1^v \frac{\sqrt{vdv}}{G(r, v)}.$$

(33)

Note that the variable $r$ is treated as a constant in the above equation. The above gives the time taken for a shell labeled $r$ to reach a particular epoch $v$ from the initial epoch $v = 1$.

One could sketch the iterative process by which the initial Cauchy data evolves. As we noted, given the density and energy profile we can find all the other functions at the initial epoch $v = 1$. Then we use (33) to find the functional form of $v(t, r)$ for $v = 1 - \epsilon$ where $\epsilon$ is an infinitesimally small positive number. As we easily see, this functional form would be

$$v(\Delta t, r) = 1 - j(r)\Delta t,$$

(34)

where

$$j(r) = e^{\nu_0(r)} \sqrt{b_0 e^{2rA_0(r)} + h_0(r) + M_0(r)}.$$

(35)

Here the subscript 0 denotes the initial data and $\Delta t$ denotes the infinitesimal evolution of the spacelike initial slice in $(t, r)$ plane. We now calculate the function $v'$ as

$$v' = j'(r)\Delta t = j'(r)(1 - v)/j(r).$$

(36)

Using this in (22) gives $M(r, v)$ at $v = 1 - \epsilon$ with respect to the given initial data $M_0(r)$ and boundary conditions. Using the form of $M(r, v)$ and $v'$ we then calculate the function $A(r, 1 - \epsilon)$
using \( \nu(r, 1 - \epsilon) \) using (29), and finally \( h(r, 1 - \epsilon) \). We then plug all these functions in again for the next iteration, until we reach \( v = 0 \).

Now we see that the time taken for a shell labeled \( r \) to reach the spacetime singularity at \( R = 0 \) (which is the singularity curve), is given as

\[
t_s(r) = \int_0^1 \frac{\sqrt{v} dv}{G(r, v)}.
\]  

(37)

In a physically realistic gravitational collapse situation such as collapse of a massive matter cloud which continually collapses, one would focus only on those classes of solutions where \( t_s(r) \) is finite and sufficiently regular. This means that the cloud collapses in a finite amount of time.

### III. COLLAPSE OF THE SCALAR FIELD TO A SIMULTANEOUS SINGULARITY

We now consider the case when the singularity occurring in the spacetime as a result of collapse is simultaneous. This would be the case, for example, when the density is same at every point in space at any given time, i.e. the collapse is homogeneous. In this case, \( \rho = \rho(t) \). Since \( \rho = e^{-2\nu \dot{\phi}(t)^2} \), therefore we have \( \nu = \nu(t) \) only. We rescale \( t \) to make \( e^{2\nu} = 1 \). The spacetime singularity occurs when the physical radius goes to zero, i.e. \( v = 0 \). In the case of the density being homogeneous, this implies that the singularity curve \( t_s(r) \) is independent of \( r \).

The expression for \( t_s(r) \) in general is,

\[
t_s(r) = \int_0^1 \frac{\sqrt{v}}{[\frac{v}{r^2}(G - 1) + M]^{1/2}} dv.
\]  

(38)

Here the \( r \)-independence of \( t_s(r) \) then implies that the integrand on the right hand side is a function of \( v \) only. The time coordinate can be expressed in general as

\[
t = \int \frac{\sqrt{v}}{[\frac{v}{r^2}(G - 1) + M]^{1/2}} dv + h_1(r)
\]  

(39)

where \( h_1(r) \) is an arbitrary function of \( r \). As the integral is a function of \( v \) only, so the initial condition which is \( v = 1 \) at the time \( t = t_i \), implies that \( h_1 \) is a constant. Therefore from (39), it is seen that

\[
v' = 0.
\]  

(40)

From the Einstein equations, we get the relation

\[
e^{-2\nu \dot{\phi}(t)^2} = \frac{2M_v}{v^2}
\]  

(41)
Since the left hand side of the equation is a function of $t$ only, the right side must be a function of $v(t)$ only. This implies $M = M(v)$ only. So (22) gives

$$ M = \frac{m_0}{v^3} \tag{42} $$

From (18) we get,

$$ A_r = 0 \tag{43} $$

This implies $A = A(r)$. This is consistent with the form of $G$ we get from the other equation,

$$ G = -\frac{2v^2Mv(v + rv')^2}{f^2(r)} \tag{44} $$

Putting $M$ in the last equation, we get $G = \frac{6m_0}{f^2(r)}$. Since $t_s(r) \neq 0$, the integrand in its expression must be finite at $r = 0$. This implies,

$$ \frac{1}{r^2} \left( \frac{6m_0}{f^2(r)} - 1 \right) = f_1(r) \tag{45} $$

where $f_1(0)$ is finite. In this case, since $t_s(r)$ is a constant, $f_1(r)$ is a constant also. So we can write

$$ f^2(r) = \frac{6m_0}{1 + cr^2} \tag{46} $$

where $c$ is a constant. So in this case we have $e^{2\psi} = \frac{v^2}{1 + cr^2}$. From (39) we get,

$$ t = -\int \frac{\sqrt{v}}{(cr + \frac{m_0}{v^2})^{\frac{3}{2}}} dv \tag{47} $$

The metric then is given as,

$$ ds^2 = dt^2 - v(t)^2 \left( \frac{dr^2}{1 + cr^2} + r^2d\Omega^2 \right) \tag{48} $$

In this case, $t_s(r)$ is finite and $\dot{\phi}(t)$ blows up at the singularity. This can be seen from (41) as it reduces to $\dot{\phi}(t)^2 = \frac{6m_0}{v^6}$. We also note that $c = 0, \pm 1$, gives all the possible solutions in this case, which correspond to the Friedmann-Robertson-Walker models. When $c = 0$, we have $v(t) = (K_1 - 3\sqrt{m_0}t)^{\frac{2}{3}}$, where $K_1$ is some positive constant.

We note here that we considered above the case when the density $\rho = \rho(t)$, which leads to a simultaneous singularity, and the FRW class of models. However, even when the density is not homogeneous, a simultaneous singularity can result as we shall discuss below.

**IV. COLLAPSE OF A SCALAR FIELD OF INHOMOGENEOUS DENSITY**

In this case, the singularity that results as the collapse endstate need not be simultaneous in general, and in general we have $\rho = \rho(r, t)$. Before proceeding further, we discuss the regularity conditions that are required for a physically reasonable collapse of the scalar field.
A. Regularity conditions

First, we note that at the center when \( r \) goes to zero, we must have

\[
\lim_{r \to 0} (rv') = 0 \tag{49}
\]

Because, when this condition is violated, \( v \) becomes divergent as \( r \) goes to zero at the center. For example, let \( \lim_{r \to 0} (rv') = c_2 \), where \( c_2 \) has some non-zero value. Then \( \lim_{r \to 0} v \) goes as \( \text{mod } lnr \). If \( \lim_{r \to 0} (rv') \) is divergent in \( r \), then the divergence of \( v \) will only become more severe. But divergence of \( v \) means that \( R' \) becomes arbitrarily large for the comoving shells near the center. This is clearly unphysical and is to be ruled out.

The second condition comes from one of the Einstein equations. In the limit of going to the center \( \lim r \to 0 \), using (49) in (22), we get

\[
3M + rM_v + vM_{vv} = 0 \tag{50}
\]

Here \( \lim_{r \to 0} (rM_v) \) can behave in three possible ways, which are, i) \( \lim_{r \to 0} (rM_v) = 0 \), (ii) \( \lim_{r \to 0} (rM_v) \) has some finite non-zero value, and (iii) \( \lim_{r \to 0} (rM_v) \) can be divergent.

Now we point out that only the first option is possible for a physically reasonable collapse model. In the case of (ii) occurring, we have \( \lim_{r \to 0} M = c_0(lnr) + h_2(v) \). Now we can argue that \( h_2(v) \) is not divergent. Otherwise, \( M_{vv} \) will also contain a logarithmic divergence. But for \( v \neq 0 \), this implies a divergence of density, which is not physical. So we cannot consider that case. For the other case, one can see from (38), that the integrand becomes divergent when \( r \) goes to zero. Therefore \( t_s(0) = 0 \), which means that the singularity is present even at the initial epoch. So (ii) is ruled out. Similarly, one can also rule out the possibility (iii), proceeding in the same way as for the case (ii). In this case also \( M \) has \( r \)-dependent divergence, which makes \( t_s(0) = 0 \). Therefore, (i) is the only possibility that is allowed.

In such a case then, (50) reduces to \( 3M + vM_{vv} = 0 \) in the \( \lim r \to 0 \). This implies \( \lim_{r \to 0} M(r,v) = m_0/v^3 \), for \( v \neq 0 \).

B. Black hole formation in scalar field collapse

In the following, we consider and characterize a wide class of black hole models that arise in the gravitational collapse of a massless scalar field, and which satisfy the regularity conditions
above. It is shown that for a large class of collapse scenarios the field collapses to a simultaneous singularity.

We note that the second condition that we adopted above can be stated as

$$M(0, v) = \frac{m_0}{v^3}$$

for all $v \geq 0$, i.e. $1 \geq v \geq 0$. At this point, it is useful to note that $\phi(t)$ is a free function and therefore can be chosen to be any function of $t$ subject to its being regular. However, once the regularity conditions have been imposed, then two solutions (obeying the same regularity conditions), with two different functional forms of $\dot{\phi}(t)$ need not be diffeomorphic to each other in general.

We note that any $r = \text{constant}$ curve is timelike, and the tangent vector of this curve is,

$$\tau^\mu = dx^\mu / ds$$

with components,

$$\tau^\mu = (\frac{dx^0}{ds}, 0, 0, 0)$$

The proper time along this curve is then given by $\tau = \int \tau^\mu \tau_\mu \, ds = \int ds$. Since $ds^2 = e^{2\nu} \, dt^2$, in this case we get $\tau = \int e^\nu \, dt$, and we have,

$$\tau[v(t_f), r] = \int_{v(t_i)}^{v(t_f)} e^{\nu} \, dt$$

So $\tau(v, r)$ is the proper time along any particular shell of comoving radius $r$ to reach the value $v$, starting from the initial epoch $v = 1$.

Subject to the above mentioned conditions, some general results can now be proved about the nature of singularities and black hole formation in gravitational collapse of scalar fields. We thus consider the classes of models where $\dot{\nu} \leq 0$ throughout the evolution. Further, we take that $\nu' \geq 0$ at $r = 0$, and at all other values of $r$. As we show here, this last condition would imply that this class of solutions would admit no non-simultaneous singularities as collapse endstate. Such a condition corresponds to the situation when the central shell at $r = 0$ arrives at the spacetime singularity earlier in time as compared to other shells with greater values of $r$. This is related to avoiding the shell-crossing singularities $R' = 0$ within the cloud which are not generally considered to be physically genuine. We shall show that, in such a case, the matter field then collapses to either a simultaneous singularity in a finite coordinate time, or that both $t_s(r)$ and the proper time $\tau(v, r)$ must diverge along any timelike curve $r = \text{constant}$ as the system evolves under the conditions we shall state below.
**Proposition 1:** If \( \dot{\phi}(t) \) is divergent at some instant \( t_1 \), then there is a simultaneous singularity at the time \( t = t_1 \).

**Proof:** The density \( \rho = \frac{1}{2} e^{-2\nu} \dot{\phi}(t)^2 \). If \( \dot{\phi}(t) \) is divergent, at \( t = t_1 \), then there are two possibilities. The first possibility is, the density remains finite and there is no singularity at \( t = t_1 \) in the case if we have \( e^{-2\nu(r,t_1)} = 0 \). However, this is not allowed because it means \( e^{2\nu(r,t_1)} = \infty \), i.e. the metric component is blowing up at regular spacetime points. As this is not allowed by regularity conditions, the only other possibility that remains is the density must diverge at \( t = t_1 \), so the singularity is simultaneous at the epoch \( t = t_1 \).

Therefore, it follows that the existence of a non-simultaneous singularity curve implies that the function \( \dot{\phi}(t) \) must remain finite at all regular epochs in the spacetime. For a stiff fluid collapse, this means that if \( \lim u^0 \to 0 \) as \( \lim t \to t_s \), there must be a simultaneous singularity in the limit of \( t \) going to \( t_s \).

**Proposition 2:** If \( t_s(r) \) is not constant and if \( v' \geq 0 \) everywhere in the spacetime, then \( t_s(r) \) must be divergent.

**Proof:** First we note that from (44) we get,

\[
G(r,v) = -\frac{v^2 (vM_{rv} - 3M - rM_r)^2}{2f^2(r)M_{rv}}
\]

Therefore,

\[
t_s(r) = \int_0^1 \frac{\sqrt{v}}{e^{\nu}} \frac{\sqrt{v}}{(-\frac{v^2 (vM_{rv} - 3M - rM_r)^2}{2f^2(r)M_{rv}} - 1) + M}^{\frac{1}{2}} dv
\]

We note that \( e^{\nu(0,v)} \neq 0 \) when \( v \neq 0 \) by the regularity conditions. This implies that the other term in the denominator of the integrand must be finite at \( r = 0 \), otherwise \( t_s(0) = 0 \), or the singularity will be present at the initial epoch itself, which violates the regularity of collapse from non-singular initial data. We note that \( M(0,v) = m_0/v^3 \) for all values of \( v \) such that \( 1 \geq v \geq 0 \) as stated above, which is finite whenever \( v \neq 0 \). Therefore the quantity,

\[
\lim_{r \to 0} \frac{1}{r^2} \left[ \frac{-v^2 (vM_{rv} - 3M - rM_r)^2}{2f^2(r)M_{rv}} - 1 \right] = \lim_{r \to 0} X
\]

must be finite. This finiteness condition and (54) together imply that \( M(r,v) \) must be of the form, \( M(r,v) = \frac{mv}{v^3} + r^n g(r,v) \), where \( n \geq 2 \). This can be clearly seen by the direct substitution of this
form for $M$ in $t_s(r)$ (see also Appendix A). Now we can write,
\[
\lim_{r \to 0} X = \lim_{r \to 0} \left[ \frac{1}{r^2} \left( \frac{6m_0}{f^2(r)} - 1 \right) + 2r^{n-2} \frac{v^3}{f^2(r)} ((n+3)g + g_r) + \frac{r^{2n-2}((n+3)g + g_r + vg_v)^2}{2f^2(r)(3m_0 - v^4n^2g)} \right] 
\]

Now \( \frac{1}{r^2} \left( \frac{6m_0}{f^2(r)} - 1 \right) \equiv f_1(r) \), where $f_1(0)$ is a finite quantity in the limit of $r \to 0$ because $X$ cannot diverge. So we get,
\[
f_2(r) = \frac{6m_0}{1 + r^2f_1(r)} 
\]

When $n > 2$, we can see from above that \( \lim_{r \to 0} X = f_1(0) \), and when $n = 2$, \( \lim_{r \to 0} X = f_1(0) + \frac{v^3}{f_2(0)}g_0(v) = f_1(0) + \frac{v^3}{3m_0}g_0(v) \), where $g_0(v) = (n+3)g(0, v) + g_r(0, v)$.

From this, it follows therefore that in the case $n > 2$,
\[
\lim_{r \to 0} \lim_{v \to 0} vX(r, v) + M(r, v) = \frac{m_0}{v^3} 
\]

When $n = 2$, $O[g_0(v)] > O(\frac{1}{v^3})$ which violates the continuity of $\tau_s(r)$ and hence is ruled out (see Appendix B). Therefore we have,
\[
\lim_{r \to 0} \lim_{v \to 0} vX(r, v) + M(r, v) = \frac{c_1}{v^3} 
\]

where $c_1$ is some constant, which is the result in general for $n \geq 2$.

We now note that from (41), we have $e^{\nu(0, v)} = \frac{v}{m_0}$. This can always be written as
\[
\lim_{v \to 0} e^{\nu(0, v)} = v^3 f_3(v) 
\]

The divergence of $t_s(0)$ if it is there can come from the range where $v$ is close to zero. We denote that part of the integral by $t_{sd}(0)$. So we can write,
\[
t_{sd}(0) = \int_0^\epsilon \frac{\sqrt{v}}{v f_3(v) \left( \frac{c_2}{v^3} \right)^2} dv 
\]

Here $c_2 = m_0$ or $c_2 = c_1$, and $\epsilon$ is a small quantity. This can be written as,
\[
t_{sd}(0) = \frac{1}{\sqrt{c_2}} \int_0^\epsilon \frac{1}{v f_3(v)} dv 
\]

Now, if the singularity is non-simultaneous then $\dot{\phi}(t)$ remains finite always. Therefore, $f_3(v)$ is also finite. So we get the result that $t_s(0)$ is divergent. Since $v' \geq 0$, it follows that $t_s(r)$ is also divergent. This proves the required result.

We note that if the singularity is simultaneous, then $\dot{\phi}(t)$ blows up when $v(0,t) = 0$ by Prop. 1. Then $f_3(v)$ must be divergent and then $t_s(0)$ has to be finite. For a collapsing massless scalar
field, if the mass function $M(r,v)$ and all the metric functions are at least $C^2$ near the central shell as they should be, and the singularity curve is non-simultaneous and an increasing function of time near the center, then the time taken for the central shell to reach the singularity diverges logarithmically, as we have shown above.

As an illustrative example, we can consider the class of solutions within an $\epsilon$-ball around the central shell, where we can ignore the contributions of terms higher than $r^2$ even very close to $v \to 0$. We note that this may not be possible for all classes of solutions, as in the Einstein equations there are terms of the form $r^n/v^m$ ($m, n > 0$) and these may have non-zero limit in the vicinity of the singularity, that is $v = 0$.

Since the mass function is $C^2$ near the center, without any loss of generality we can Taylor expand the function around $r = 0$ as,

$$M(r,v) = \frac{m_0}{v^3} + M_2(v)r^2 + \cdots \quad (64)$$

We note that we get the zeroth order term by solving (22) at $r = 0$. Also that the first order term should vanish follows directly from the no force condition at the center of the cloud. The function $M_2(v)$ is an unknown function which solves (22) near the center. Using (64) we can write (30) as

$$v' = \frac{r}{6m_0} [v^5 M_2(v)]_v \left[ 1 + \frac{M_2(v)_v v^4}{3m_0} r^2 + \cdots \right]. \quad (65)$$

Let us consider $b_0(r) = 0$ and $f(r) = f(0) = 6m_0$. Then putting the Taylor expanded forms of different functions in the (21) we get

$$e^{2rA} = 1 + \left( \frac{5v^3 M_2(v)}{3m_0} \right) r^2 + O(r^6) \quad (66)$$

Also using (67) we have

$$h(r,v) = \left( \frac{5v^3 M_2(v)}{3m_0} \right) + O(r^4). \quad (67)$$

From the above equations we get the behaviour of the function $\nu$ near the center (since $\nu' = A_v R'$) as

$$\nu \approx \frac{5v}{12m_0} [v^3 M_2(v)]_v r^2 + O(r^4) + C(t), \quad (68)$$

where $C(t)$ is the constant of integration. Comparing with the density of the central shell we get $C(t) = \ln(v^3/m_0)$. Therefore we can write,

$$e^\nu \approx \frac{v^3}{m_0} \left[ 1 + \frac{5v}{12m_0} [v^3 M_2(v)]_v r^2 + O(r^4) \right] \quad (69)$$
Finally using all the above equations and neglecting terms higher than the order \( r \) (since we are only concerned with an \( \epsilon \)-ball around the center), we get,

\[
\dot{v} = -\frac{v^3}{m_0} \sqrt{\frac{5v^3 M_2(v)}{3m_0}} + \frac{m_0}{v^4} \quad ; \quad v' = \frac{r}{6m_0} [v^5 M_2(v)], \quad (70)
\]

From (62) we see that the integrability condition is trivially satisfied for \( r = 0 \). However, upto the first order correction we get the following condition,

\[
[(v^5 M_2(v))_{,v}]^2 = C \frac{v^6}{m_0^2} \left[ \frac{5v^3 M_2(v)}{3m_0} + m_0/v^4 \right] \quad (71)
\]

The above is a complicated non-linear equation for the function \( M_2(v) \). Let us consider that near the central singularity \( v(0, t) = 0 \), the function has a leading order power law behaviour \( M_2(v) \approx v^\alpha \). Then using the above equation and comparing powers on both sides, we see that only consistent solution can be obtained with \( \alpha = -3 \). In other words the behaviour of the function \( M_2(v) \) near the central singularity is of the order of \( 1/v^3 \). Using this in the singularity curve expression (37) we see that the time taken for the central shell to become singular diverges logarithmically.

Returning to the class of collapse models under consideration, we shall now show that for this class non-simultaneous singularity cannot occur. First we define,

\[
\tau_0(t) = \int_{t_i}^t e^{\nu(0,t)} dt \quad (72)
\]

This is the proper time along the \( r = 0 \) shell. Now let us assume that \( \tau_0(t=0) = \tau_{0s} \) is finite. From (72) we get,

\[
\frac{d\tau_0}{dt} = e^{\nu(0,t)} \quad (73)
\]

We can then show the following result.

**Proposition 3** If \( v' > 0 \) at \( \tau_0 = \tau_{0s} \), then for any \( r_2 > 0 \), \( \tau_{r_2} (\tau_{0s}) \) is divergent.

**Proof:** One can make the following transformation which is allowed by the comoving coordinates, given by, \( x^\mu : (t, r, \theta, \phi) \rightarrow x'^\mu = (\tau, r, \theta, \phi) \). In the new coordinate system, the line element is then written as,

\[
ds^2 = e^{2[\nu(r, \tau)-\nu(0, \tau)]} d\tau_0^2 - e^{2\psi} dr^2 - R^2 d\Omega^2 \quad (74)
\]

The new coordinates cover the manifold upto \( \tau_0 = \tau_{0s} \) only. Since, \( v' > 0 \) at \( \tau_0 = \tau_{0s} \), \( (r_2, \tau_{0s}) \) is a regular spacetime point. The proper time along the \( r = const. \) curve from \( \tau_0 = \tau_{0s} \) to \( \tau_0 = \tau_{0s} \) is
given by, \( \tau_2(\tau_0) = \int_{\tau_0}^{\tau_2} \sqrt{g_{00}(r, \tau_0)} d\tau_0 = \int_{\tau_0}^{\tau_2} e^{\nu(r, \tau_0) - \nu(0, \tau_0)} d\tau_0 \).

Now \( v(0, t) = v_0(t) \) is a function of \( r \) only. Therefore \( dt = \frac{1}{\dot{v}(0, v_0)} dv_0 = \frac{1}{\dot{v}(0, v_0)} dv_0 \). This implies,

\[
d\tau_0 = \frac{e^{\nu(0, v_0)}}{\dot{v}(0, v_0)} dv_0
\]

(75)

Using this, we get

\[
\tau_2(\tau_0) = \int_{0}^{\tau_0} \frac{e^{\nu[r_2, v_0]}}{\dot{v}(0, v_0)} dv_0
\]

(76)

Here it is important to note that \( e^{\nu[r_2, v_0]} = e^{\nu[r_2, v(r_2, v_0)]} \) is finite and non-zero at \( v_0 = 0 \), because \( v(r_2, 0) > 0 \).

\[\vdash \text{From the proof of Prop. 2, we have } \lim_{v_0 \to 0} \dot{v}(0, v_0) = -c_1 v_0, \text{ where } c_1 > 0. \text{ Since } \dot{v} \text{ goes to zero as } v_0 \to 0, \text{ a divergence may be present in } \tau_2. \text{ So we consider that part of the integral where } v_0 < < 1, \text{ which is given by, } \tau_2(\tau_0) = \frac{1}{c_1} \int_{0}^{\epsilon} \frac{e^{\nu[r_2, v_0]}}{v_0} dv_0, \text{ where } \epsilon < < 1. \text{ This gives } \tau_2(\tau_0) = \int_{0}^{\epsilon} \frac{1}{c_1} dv_0, \text{ which is a divergent quantity. It follows that } \tau_2(\tau_0) \text{ is divergent.} \]

The proposition above proves that any non-simultaneous singularity cannot develop for this class of collapse models. That implies for this class only a spacelike singularity can form as collapse endstate. This characterizes a wide class of black hole formation models from a massless scalar field collapse, or for a stiff fluid collapse.

We now also indicate here the possibility of existence of a class of non-singular solutions in evolving scalar fields, or stiff fluids scenario, by proving another proposition as below.

**Proposition 4:** If \( \nu'(r, t) \geq b \) where \( b > 0 \), for \( r_1 \leq r \leq r_2 \) for some \( r_1, r_2 > 0 \), and \( t \in (t_i, \infty) \), then we must have \( \tau(v_1, r) > k \) for all \( k > 0 \) for all \( r \), for some \( v_1 > 0 \).

**Proof:** For \( r \geq r_2 \), we have \( \int_{0}^{r} \nu'(r, t) dr = \int_{0}^{r_1} \nu' dr + \int_{r_1}^{r_2} \nu' dr + \int_{r_2}^{r} \nu' dr \).

This implies that, \( \int_{0}^{r} \nu'(r, t) dr \geq b(r_2 - r_1) \) for \( r \geq r_2 \).

\[
\int_{0}^{r} \nu'(r, t) dr = \nu(r, t) - \nu(0, t)
\]

Denoting \( b(r_2 - r_1) \) by \( c \),

\[
v(r, t) \geq \nu(0, t) + c; \text{ for } r \geq r_2.
\]

\[
\therefore \nu(0, t) \geq 0 \text{ for } t \in (t_i, \infty), \text{ therefore } \nu(0, t) \geq c \text{ for } t \in (t_i, \infty) \text{ for } r \geq r_2.
\]

For \( r \geq r_2 \), the proper time elapsed during the comoving time interval \( (t_i, \infty) \), \( \tau(v_1, r) = \int_{t_i}^{\infty} \nu' dt \), for some \( v_1 \geq c \).

By the regularity condition, \( e^{\nu(r, v)} \) has a positive lower bound in the range \( v \in (c, 1) \).
This implies that, $\tau(v_1, r) > k\forall k > 0$ for $r \geq r_2$ and for some $v_1 \geq c$.

Since $\tau(v_1, r)$ is continuous in each $t = constant$ hypersurface, and $r_2$ can be any non-zero number however small, there must exist some $v > 0$ such that $\tau(v, r) > k\forall k > 0$ for $r < r_2$. So $\tau(v, r) > k\forall k > 0$ for $r$ for some $v > 0$. This proves the result.

Thus all the classes of solutions that satisfy the conditions assumed as above would be singularity free. If there are no such solutions, then that would indicate the possibility of bouncing models within the framework above. We note that since singularities would not form in this class of models, there would not be any formation of trapped surfaces also. We discuss this in some detail below.

For spherically symmetric spacetimes, the equation of the apparent horizon is given by

$$g^{\mu\nu} R_{,\mu} R_{,\nu} = 0 \quad (77)$$

In this case, this implies $G - H = 0$. Using (8), one can rewrite the condition for formation of apparent horizon as $F = R$ or as

$$M = \frac{v}{r^2} \quad (78)$$

**Proposition 5:** For the non-singular class of models discussed in Proposition 3, the apparent horizon does not form in any finite coordinate time.

**Proof:** To prove this, we first prove another result.

**Lemma A:** We must have $\mod \dot{v} < \delta, \forall \delta > 0$, in a total time interval, which is infinite.

**Proof:** We know, $t_s(r) = \infty, \forall r$. Now we have,

$$v(r, t_s(r)) - v(r, t_i) = \int_{t_i}^{t_s(r)} \dot{v}dt. \quad (79)$$

This can be written as $1 = \int_{t_i}^{\infty} (-\dot{v})dt = \int_{t_1}^{t_2} (-\dot{v})dt + \int_{t_2}^{t_3} (-\dot{v})dt + \ldots$ = $I_1 + I_2 + \ldots$, where $(t_1 - t_i), (t_2 - t_1), \ldots$ are all finite.

This implies that, given any $\delta_1 > 0$, there exist infinite number of integrals $I_n$s such that, $I_n < \delta_1$. In each of these integrals, $\mod \dot{v} < \delta, \forall \delta > 0$; i.e. $\mod \text{(change of } v) < \text{any arbitrarily small amount}$. Since there are infinite number of such integrals, $\mod \dot{v} < \delta, \forall \delta > 0$, in a total time interval that is infinite. (Proved)

Since $\mod \dot{v} < \delta, \forall \delta > 0$, for a total time interval which is infinite, there must exist a time instant $t_1$ at which this holds, such that, $t_1 < t_{s0}$ and $t_1 > l$ for any given $l > 0$. 

\[ t_s(r) \geq t_{s0}, \forall r, \text{ so for some } \nu > 0, v(r, t_1) > 0 \text{ for } r > 0. \] By the regularity condition, at \( t = t_1, \) \( e^{\nu(r, t_1)} > 0. \) From Einstein equations we have \( \sqrt{\nu} \dot{\nu} = -e^\nu [\frac{\nu}{r^2} (G - 1) + M] \). Therefore at \( t = t_1, \)
\[ \left[ \frac{\nu}{r^2} (G - 1) + M \right] < \delta, \forall \delta > 0. \tag{80} \]

By the energy condition, \( \rho = -\frac{M}{r^2} > 0. \) This implies \( M < 0. \) From (44), \( G > 0. \)

Let \( v(t_1, r) = v_f(r). \) Then the inequality can be written as \( M < \delta + (1 - G) \frac{v_f}{r^2}, \forall \delta > 0. \) But we know, \( (1 - G) \frac{v_f}{r^2} < \frac{v_f}{r^2}. \) This implies
\[ M(r, v_f) < \frac{v_f}{r^2}. \tag{81} \]

Let \( v_e(r) \) be the value of \( v(r) \) at any time earlier than \( t_1. \)
\[ \frac{v_f(r)}{r^2} \leq \frac{v_e(r)}{r^2} \text{ (Since } \dot{v} \leq 0). \] Also \( M(v_f, r) > M(v_e, r). \) These imply \( M(v_e, r) < \frac{v_f(r)}{r^2} \leq \frac{v_e(r)}{r^2} \) or \( M(v_e, r) < \frac{v_e(r)}{r^2}. \)

Therefore the equation of apparent horizon (78) is not satisfied for \( r \neq 0 \) and when \( t_i \leq t \leq t_1. \) So there is no apparent horizon and consequently no trapped surface formation in that time range.

Since there always exists some \( t_1 \) such that \( t_1 \geq l \) for any given \( l, \) for \( r \neq 0, \) trapped surface does not form in any finite time.

For \( r = 0, \) \( M = m_0 \frac{v^3}{r^3}. \) Putting this in (78), \( m_0 r^2 = \frac{v^4}{r}. \) This is satisfied when \( v = 0, \) i.e. at \( t = t_{s0}. \) Since \( t_{s0} = \infty, \) apparent horizon does not form at the \( r = 0 \) shell at any finite time. So for such a solution of the Einstein equations, the apparent horizon or trapped surfaces do not form at any finite coordinate time. (Proved)

In Proposition 4, it has already been shown that an infinite coordinate time interval correspond to infinite proper time interval along any \( r = \text{constant} \) world line. This implies that for this class of models, no apparent horizon or trapped surface would form at any finite proper time.

C. Models collapsing to form non-simultaneous singularity

Now we discuss briefly the other classes of models for which the singularity can be non-simultaneous. Toward such a purpose, we must clearly relax the condition \( v' \geq 0, \) thus allowing \( v' \) to be either positive or negative. All the other regularity conditions still remain the same.

In the earlier class of models, the divergence of \( t_s(r) \) was responsible for the fact that there could only be simultaneous singularity. The divergence of \( M \) as \( v \to 0 \) caused the divergence of
when evaluating the integral near \( v \) of \( r \) at \( t \), this gives
\[
\lim_{v \to 0} M(r, v) = r^2 g_2(r, v),
\]
where \( n \geq 2 \), \( g_1(r, 0) \) blows up and \( g_2(r, 0) \) is finite. In this case \( M(r, 0) \) would remain finite except at \( r = 0 \).

Here \( t_s(0) \) will be divergent as in the previous case, because \( M(0, v) = \frac{m_0}{v^3} \). But in this case, \( t_s(r) \) would be finite for any \( r > 0 \). To show this we first show that there is no divergence in \( t_s(r) \) when evaluating the integral near \( v = 0 \). We know that \( \lim_{v \to 0} M(r, v) = r^2 g_2(r, v) \), where \( g_2 \) is a regular function. Also, \( \lim_{v \to 0} M(r, v) = r^2 g_2(r, v) \), where
\[
\lim_{v \to 0} v' = -\frac{5g_2(r, 0) + r g_2(r, 0), r}{2r g_2(r, v), v \mid_{v=0}}
\]
(83)

From (44), it is seen that \( \lim_{v \to 0} G \sim v^2 \) and from (41), \( \lim_{v \to 0} e^{\nu(r, v)} \sim v \). Putting them all in (27), we have \( \lim_{v \to 0} \dot{v} \sim -\sqrt{v} \). This means that there would be no divergence in the \( t_s(r) \) integral for any value \( r > 0 \) coming from the part near \( v = 0 \). So \( t_s(r) \) is finite for \( r > 0 \). We note that since \( t_s(0) = \infty \) and \( t_s(r) \) for \( r > 0 \) is finite, \( v' \geq 0 \) is not satisfied. The proper time for the \( r = 0 \) shell to reach the singularity is, however, finite as we can show. The proper time, \( \tau_{s0} = \int_{t_s}^{t_{s0}} e^{\nu} dt \) can be written in terms of an integral of \( v \) also. Then \( \tau_{s0} = \int_1^0 e^{\nu} dv \). If we consider the part of the integral where \( v << 1 \), it is given by \( \int_0^\epsilon e^{\nu} dv \), where \( \epsilon << 1 \). This goes to zero. So the proper time \( \tau_{s0} \) is finite. Thus for the above generic form of \( M \), a non-simultaneous singularity may occur. Such a singularity may also be spacelike or timelike in this case.

V. APPENDIX A: GENERAL FORM OF \( M \)

Since \( M(0, v) = \frac{m_0}{v^3} \), \( M \) can in general be written in the form,
\[
M(r, v) = \frac{m_0}{v^3} + f_2(r) g(r, v)
\]
(84)
where \( g(o, v) \neq 0 \) and \( f_2(0) = 0 \). Then \( M_v = -\frac{3m_0}{v^4} + f_2(r) g_v \) and \( M_r = f'_2(r) g(r, v) + f_2(r) g_r \).

\( v' = \frac{v^4(3f_2(r) g + rf_2'(r) g + rf_2(r) g_v + v f_2(r) g_v)}{2r(3m_0 - f_2(r)v^4 g_v)} \)
(85)
This gives
\[
R' = v + rv' = v[1 + \frac{v^3(3f_2(r) g + rf_2'(r) g + rf_2(r) g_v + v f_2(r) g_v)}{2(3m_0 - f_2(r)v^4 g_v)}]
\]
(86)
From (44), we have

$$G = \frac{2}{f^2(r)} \left[ 3m_0 + v^3(3f_2(r)g + rf_2'(r)g + rf_2(r)g r + vf_2(r)g w) + \frac{v^6}{4} \left( \frac{3f_2(r)g + rf_2'(r)g + rf_2(r)g r + vf_2(r)g w}{3m_0 - f_2(r)v^4g_w} \right)^2 \right]$$

(87)

For regular epochs (i.e. \( v \neq 0 \)), \( M \) and its derivatives are bounded. In the integrand of \( t_s(r) \), we have in the denominator, a factor \( \frac{1}{r^2}(G - 1) + M \frac{1}{r^2} \), which must be finite when \( r \) goes to zero. Since \( M(0,v) \) is finite, \( \frac{1}{r^2}(G - 1) \) must be finite. This means \( \lim r \to 0 \frac{1}{r^2}(G - 1) \) is finite. We can write the quantity \( \frac{1}{r^2}(G - 1) \) as,

$$\frac{1}{r^2}(G - 1) = \left( \frac{6m_0}{r^2f^2(r)} - \frac{1}{r^2} \right) + \frac{2v^3}{r^2f^2(r)} (3f_2(r)g + rf_2'(r)g + rf_2(r)g r + vf_2(r)g w) + \frac{v^6}{2r^2f^2(r)} \left( \frac{3f_2(r)g + rf_2'(r)g + rf_2(r)g r + vf_2(r)g w}{3m_0 - f_2(r)v^4g_w} \right)^2$$

(88)

Now the first term in the bracket and the coefficients of \( v^3 \) and \( v^6 \) have to be zero separately. This gives,

$$f^2(r) = \frac{6m_0}{1 + r^2f_1(r)}$$

(89)

The coefficient of \( v^3 \) is \( \frac{1}{6m_0} (f_1(r) + \frac{1}{r^2}(3f_2(r)g + rf_2'(r)g + rf_2(r)g r + vf_2(r)g w) \). This implies that \( \lim r \to 0 \frac{f_2(r)}{r^2} \) and \( \lim r \to 0 \frac{rf_2'(r)}{r^2} \) are bounded. This implies that as \( r \) goes to zero, \( f_2(r) \) goes to zero atleast as fast as \( r^2 \).

VI. APPENDIX B: RULING OUT SOLUTIONS FOR WHICH \( O[g_o(v)] > O(\frac{1}{v^r}) \)

In this case, \( \lim_{v \to 0} \frac{v}{r^2}(G - 1) + M \frac{v}{r} = \frac{v^4}{3m_0}g_0(v) \). Let, \( \lim_{v \to 0} g_o(v) = \frac{c_3}{v^\delta} \), where \( \delta > 0 \) and \( c_3 \) is some constant. Using this and (61), we get, \( t_{sd}(0) = \int_0^\infty \frac{v^3}{v^3f_3(v)(v^3 + c_3)} dv \). This can be written as \( t_{sd}(0) = \frac{1}{f_3(0)} \left[ \frac{v^3}{v^3} \right]_0^\infty \). This is vanishingly small. So \( t_s(0) \) is finite.

However, for \( r > 0 \), \( \lim_{v \to 0} \frac{v}{r^2}(G - 1) + M \frac{v}{r} = \frac{v^4}{3m_0}g_0(v) \) and \( \lim_{v \to 0} e^{\nu(r,v)} = v^{5+\frac{5}{2}}f_3(r,v) \). Putting them together, we get, \( t_{sd}(r) = \int_0^\infty \frac{v^3}{v^3f_3(v)(v^3 + c_3)} dv \). This implies that \( t_s(r) \) is divergent when \( r > 0 \).

Now there are the following possibilities. Firstly, the singularity can be non-simultaneous, i.e, \( v' > 0 \) at the epoch when some \( r > 0 \) shell reaches the singularity. In this case, since \( t_s(r) = \infty \) for \( r > 0 \), arguing in the same way as in Prop.3; it can be shown that \( \tau_s(r) \) will be divergent for some value of \( r \). This means that the singularity curve \( \tau_s(r) \) is discontinuous. So this possibility is ruled out.

There is also another possibility that all the shells \( r > 0 \) hit the singularity at the same time, i.e. \( v' = 0 \) for \( r > 0 \) at that epoch. Since \( v' = 0 \) at the epoch of singularity of some \( r > 0 \) shell,
all $r > 0$ shells reach the singularity at the same epoch. The proper time taken along a shell with a very small value of $r$ would be close to the proper time taken along the $r = 0$ shell to reach the singularity. If the $\tau_s(r)$ is continuous, then it follows that at $\tau_0 = \tau_{s0}$, all the shells reach the singularity. But this contradicts the fact that $t_s(0) < t_s(r)$ for $r > 0$. So this second possibility is ruled out.

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