Model building by coset space dimensional reduction in ten-dimensions with direct product gauge symmetry

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We investigate ten-dimensional gauge theories whose extra six-dimensional space is a compact coset space, $S/R$, and gauge group is a direct product of two Lie groups. We list up candidates of the gauge group and embeddings of $R$ into them. After dimensional reduction of the coset space, we find fermion and scalar representations of $G_{\text{GUT}} \times U(1)$ with $G_{\text{GUT}} = SU(5)$, $SO(10)$ and $E_6$ which accommodate all of the standard model particles. We also discuss possibilities to generate distinct Yukawa couplings among the generations using representations with a different dimension for $G_{\text{GUT}} = SO(10)$ and $E_6$ models.

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I. INTRODUCTION

The Standard Model (SM) has been successful in describing phenomenology of the elementary particle physics up to the energy of order TeV. Not only did it explain experimental results but it also gave us deeper insights that gauge symmetry governs the interactions among the particles and its spontaneous breaking rises particle masses. Despite of its success, the SM is not a satisfactory model because the choice of the gauge groups and the contents of the particles are the inputs of the model, and all parameters in Higgs and Yukawa sector, which are responsible for the masses, are not predictable. Grand unification addresses the former points by unifying the gauge symmetries into single gauge group and fermions into larger representations. But it requires new scalars to break the grand unification symmetry in the same manner as the SM, resulting in the introduction of more free parameters than those in the SM. Therefore a plausible framework for the physics beyond the SM will be an unification of Higgs and the gauge bosons.

Coset Space Dimensional Reduction (CSDR) scheme is one of the attractive approaches in this regard [1, 2, 3, 5, 6, 7]. This scheme introduces a compact extra dimensional space which has the structure of a coset of Lie groups, $S/R$. The Higgs field and the gauge field of the SM are merged into a gauge field of a gauge group $G$ in the higher-dimensional spacetime. The SM fermions are unified into a representation of this gauge group. The particle contents surviving in four dimensional theory are determined by the identification of the gauge transformation as a rotation within the extra-dimensional space. The four dimensional gauge symmetries are determined by embedding of $R$ into $G$. Since the Higgs originates from extra dimensional components of the gauge field, the Higgs and Yukawa sectors in four dimensional Lagrangian are uniquely determined. Furthermore, as shown in Ref. [8, 9, 10, 11], it is possible to obtain chiral fermions when total dimension, $D$, of the spacetime is even. The chiral fermions can be obtained even from (pseudo)real representations in $D = 8n + 2$ ($D = 8n + 6$) [8,11].

The case $D = 10$ is the most interesting because the superstring theory, which is a candidate of a unified theory including gravity, suggests this world exists in ten-dimensional spacetime. Thus CSDR models of $D = 10$ can bridge the superstring theory and the SM. In this spirit, many works have been done in ten dimension, but no realistic model has emerged yet [8,12,13,14,15,16,17,18,19]. A major obstacle to build realistic models is the difficulty to obtain all the SM fermions. One of the critical reasons of this difficulty is the smallness of SO(6) spinor representation. Another reason is the small degree of freedom in embedding $R$ into $G$. These facts strongly restrict the fermion representations surviving in four dimensions.

In this paper, we introduce a new freedom to the embedding of $R$ into $G$ by allowing $G$ to be a direct product of two Lie groups in order to overcome the latter difficulty. We have more candidates for $G$ and the embeddings.
of \( R \) into them, providing more possibilities to obtain the SM fermions. Furthermore, one of the gauge groups can be responsible to the four-dimensional gauge symmetry while the other can be identified with a family symmetry \cite{20, 21, 22, 23}, which generates a flavour structure in the Yukawa couplings. Thus, it is worthwhile to study the CSDR scheme with direct product gauge groups in ten dimensions. We exhaustively search for fermion contents in the SM and the Grand Unified Theories (GUTs) with SU(5), SO(10) and \( E_6 \), limiting the dimension of a fermion representation less than 1025.

This paper is organized as follows. In section 2, we briefly recapitulate the scheme of the coset space dimensional reduction (CSDR) for the case with a gauge group of ten-dimensional gauge theory which has direct product structure, and the construction of the four-dimensional theory by the scheme. In section 3, we obtain the combinations of the coset space \( S/R \) and the gauge group \( G \) of the ten-dimensional theory. We first obtain the phenomenologically plausible coset space \( S/R \) and then we restrict the possible gauge group \( G \) for each \( S/R \). In section 4, we exhaustively list the viable models in four-dimensions. Section 5 is devoted to summary and discussions.

## II. CSDR SCHEME WITH DIRECT PRODUCT GAUGE GROUP

In this section, we briefly recapitulate the scheme of the coset space dimensional reduction in ten dimensions with a direct product gauge group \cite{3}.

We begin with a gauge theory defined on a ten-dimensional spacetime \( M^{10} \) with a gauge group \( G = G_1 \times G_2 \) where \( G_1 \) and \( G_2 \) are simple Lie groups. Here \( M^{10} \) is a direct product of a four-dimensional spacetime \( M^4 \) and a compact coset space \( S/R \), where \( S \) is a compact Lie group and \( R \) is a Lie subgroup of \( S \). The dimension of the coset space \( S/R \) is thus \( 6 \equiv 10 - 4 \), implying \( \dim S - \dim R = 6 \). This structure of extra-dimensional space requires the group \( R \) to be embedded into the group SO(6), which is a subgroup of the Lorentz group SO(1, 9). Let us denote the coordinates of \( M^{10} \) by \( X^\mu = (x^\alpha, y^\alpha) \), where \( x^\alpha \) and \( y^\alpha \) are coordinates of \( M^4 \) and \( S/R \), respectively. The spacetime index \( \mu \) runs over \( \mu \in \{0, 1, 2, 3\} \) and \( \alpha \in \{4, 5, \cdots, 9\} \). We introduce, in this theory, a gauge field \( A_M(x, y) = (A_\mu(x, y), A_\alpha(x, y)) \), which belongs to the adjoint representation of the gauge group \( G \), and fermions \( \psi(x, y) \), which lies in a representation \( F \) of \( G \).

The extra-dimensional space \( S/R \) admits \( S \) as an isometric transformation group. We impose on \( A_M(X) \) and \( \psi(X) \) the following symmetry under this transformation in order to carry out the dimensional reduction \cite{2, 24, 25, 26, 27, 28}.

Consider a coordinate transformation which acts trivially on \( x \) and gives rise to a \( S \)-transformation on \( y \) as \( (x, y) \rightarrow (x, sy) \), where \( s \in S \). We require that the transformation of \( A_M(X) \) and \( \psi(X) \) under this coordinate transformation should be compensated by a gauge transformation. This symmetry makes the ten-dimensional Lagrangian invariant under the \( S \)-transformation and therefore independent of the coordinate \( y \) of \( S/R \). The dimensional reduction is then carried out by integrating over the coordinate \( y \) to obtain the four-dimensional Lagrangian. The four-dimensional theory consists of the gauge fields \( A_\mu \), fermions \( \psi \), and in addition the scalar fields originated from \( A_\alpha \). The gauge group reduces to a subgroup \( H \) of the original gauge group \( G \).

The gauge symmetry and particle contents of the four-dimensional theory are substantially constrained by the CSDR scheme. We provide below the prescriptions to identify the four-dimensional gauge group \( H \) and its representations for the particle contents.

First, the gauge group of the four-dimensional theory \( H \) is easily identified as

\[
H = C_G(R),
\]

where \( C_G(R) \) denotes the centralizer of \( R \) in \( G = G_1 \times G_2 \) \cite{2}. Thus the four dimensional gauge group \( H \) is determined by the embedding of \( R \) into \( G \). We then assume that \( R \) has also direct product structure \( R = R_1 \times R_2 \) so that we can embed them into \( G_1 \) and \( G_2 \). Here, \( R_1 \) and \( R_2 \) are not necessarily simple. We also assume that four dimensional gauge groups \( H \) is obtained from only \( G_1 \) up to U(1) factors. This assumption ensures the coupling unification if \( H \) is the gauge group of the SM. These conditions imply

\[
\begin{align*}
G &= G_1 \times G_2, \\
R &= R_1 \times R_2, \\
G_1 &\supset H \times R_1, \\
G_2 &\supset R_2,
\end{align*}
\]

up to U(1) factors.

Secondly, the representations of \( H \) for the scalar fields are specified by the following prescription. Let us decompose the adjoint representation of \( S \) according to the embedding \( S \supset R_1 \times R_2 \) as,

\[
\text{adj } S = (\text{adj } R_1, 1) + (1, \text{adj } R_2) + \sum_s (r_{1,s}, r_{2,s}),
\]

where \( s \) runs over the representations of \( R_1 \times R_2 \).
where \( r_{1s} \) and \( r_{2s} \) are representations of \( R_1 \) and \( R_2 \), respectively. We then decompose the adjoint representation of \( G_1 \) and \( G_2 \) according to the embeddings \( G_1 \supset H \times R_1 \) and \( G_2 \supset R_2 \), respectively:

\[
\text{adj } G_1 = (\text{adj } H, 1) + (1, \text{adj } R_1) + \sum_g (h_g, r_{1g}),
\]

\[
\text{adj } G_2 = \text{adj } R_2 + \sum_g r_{2g},
\]

where \( r_{1g} \)s and \( r_{2g} \)s denote representations of \( R_1 \) and \( R_2 \), and \( h_g \)s denote representations of \( H \). The decomposition of \( \text{adj } G \) thus becomes

\[
\text{adj } G = (\text{adj } G_1, 1) + (1, \text{adj } G_2)
= (\text{adj } H, 1) + (1, \text{adj } R_1, 1) + (1, 1, \text{adj } R_2)
+ \sum_g (h_g, r_{1g}, 1) + \sum_g (1, 1, r_{2g}).
\]

The representation of the scalar fields are \( h_g \)s whose corresponding \((r_{1g}, 1)\)s in the decomposition Eq. (7) are contained also in the set \( \{(r_{1s}, r_{2s})\} \) in Eq. (9). Note that the trivial representation \( 1 \)s also remain in four-dimensions if corresponding \((1, r_{2g})\)s of Eq. (9) are also contained in the set \( \{(r_{1s}, r_{2s})\} \) in Eq. (4).

Thirdly, the representation of \( H \) for the fermion fields is determined as follows [29]. Let the group \( R \) be embedded into the Lorentz group \( SO(6) \) in such a way that the vector representation \( 6 \) of \( SO(6) \) is decomposed as \( 6 = \sum_g (r_{1s}, r_{2s}) \), where \( r_{1s} \) and \( r_{2s} \) are the representations obtained in the decomposition Eq. (6). This embedding specifies a decomposition of the Weyl spinor representations \( 4\bar{4} \) of \( SO(6) \) under \( SO(6) \supset R_1 \times R_2 \) as

\[
4 = \sum_i (\sigma_{1i}, \sigma_{2i}) \quad \bar{4} = \sum_i (\sigma_{1i}, \overline{\sigma}_{2i}),
\]

where \( \sigma_{1i}(\overline{\sigma}_{2i}) \)s and \( \sigma_{2i}(\overline{\sigma}_{2i}) \)s are irreducible representations of \( R_1 \) and \( R_2 \). We then decompose the \( SO(1, 9) \) Weyl spinor \( 16 \) according to \((SU(2) \times SU(2))((\approx SO(1, 3)) \times SO(6)) \) as

\[
16 = (2, 1, 4) + (1, 2, 4),
\]

where \((2, 1)\) and \((1, 2)\) representations of \( SU(2) \times SU(2) \) correspond to left- and right-handed spinors, respectively. We now decompose a representation \( F \) of the gauge group \( G \). We take \( F_1 \) and \( F_2 \) to be a representation of \( G_1 \) and \( G_2 \) for the fermions in ten-dimensional spacetime. Decompositions of \( F_1 \) and \( F_2 \) are

\[
F_1 = \sum_f (h_f, r_{1f}),
\]

\[
F_2 = \sum_f r_{2f},
\]

under \( G_1 \supset H \times R_1 \) and \( G_2 \supset R_2 \). Therefore the decomposition of \( F \) becomes

\[
F = \sum_f (h_f, r_{1f}, r_{2f}).
\]

The representations for the left-handed(right-handed) fermions are \( h_f \)s whose corresponding \((r_{1f}, r_{2f})\)s are found in \( \{(\sigma_{1i}, \sigma_{2i})\}, \{(\sigma_{1i}, \overline{\sigma}_{2i})\} \) obtained in Eq. (10). Note that a phenomenologically acceptable model needs chiral fermions in the four dimensions as the SM does. The chiral fermions are obtained most straightforwardly when we introduce a complex representation of \( G \) as \( F \) [8, 9, 10, 11]. More interesting is the possibility to obtain them if \( F \) is real representation, provided rank \( S = \text{rank } R \) [30]. A pair of Weyl fermions appears in a same representation in this case, and one of the pair is eliminated by imposing the Majorana condition on the Weyl fermions [8, 11]. We thus apply the CSDR scheme to complex or real representations of gauge group \( G \) for fermions.

Coset space \( S/R \) of our interest should satisfy rank \( S = \text{rank } R \) to generate chiral fermions in four dimensions [30]. This condition limits the possible \( S/R \) to the coset spaces collected in Table I [3]. The \( R \) of coset (i) in Table I with subscript “max” indicates that this is the maximal regular subgroup of the \( S \). There, the correspondence between the subgroup of \( R \) and the subgroup of \( S \) is clarified by the brackets in \( R \). For example, the coset space (iv) suggests direct product of \( Sp(4)/SU(2) \times SU(2) \) and \( SU(2)/U(1) \).
TABLE I: A complete list of six-dimensional coset spaces $S/R$ with rank $S = rank \, R$ [8]. The brackets in $R$ clarifies the correspondence between the subgroup of $R$ and the subgroup of $S$. The factor of $R$ with subscript “max” indicates that this factor is a maximal regular subgroup of $S$.

| No. | $S/R$ |
|-----|-------|
| (i) | Sp(4)/[SU(2) x U(1)]_{max} |
| (ii) | Sp(4)/[SU(2) x U(1)]_{non-max} |
| (iii) | SU(4)/SU(3) x U(1) |
| (iv) | Sp(4)/[SU(2)/[SU(2) x SU(2)] x U(1)] |
| (v) | G(2)/SU(3) |
| (vi) | SO(7)/SO(6) |
| (vii) | SU(3)/U(1) x U(1) |
| (viii) | SU(3)/[SU(2)/[SU(2) x U(1)] x U(1)] |
| (ix) | (SU(2)/U(1))^3 |

Here we mention the effect of gravity. When we include the effect of gravity and consider dynamics of an extra-space we would find the difficulty to obtain stable extra space. This is the common difficulty of extra-dimensional models and some works have been done on this point. For example it is discussed in terms of radion fields which are the scalar fields originated from higher-dimensional components of metric after compactification [31],[32]. The effect of gravity to CSDR scheme is also discussed in [4], [5]. Although we agree that the effect of gravity is important, we do not discuss about the effect of gravity in this letter since it is beyond the scope of this letter.

III. CANDIDATES OF THE COSET SPACE $S/R$ AND THE GAUGE GROUP $G$

In this section we obtain the combinations of the coset space $S/R$ and the gauge group $G$ of the ten-dimensional theory. We first obtain the coset space $S/R$ and then we restrict the possible gauge group $G$ for each $S/R$.

We select the coset space $S/R$ from the ones listed in Table I by the following two criteria. First, $R$ should be a direct product of subgroups $R_1$ and $R_2$ to have new freedom to embedding of $R$ into $G$. This criterion excludes the candidates of $S/R$ (v) and (vi) in Table I.

Secondly, the four-dimensional gauge group obtained by Eq. (1) should be that of the SM or a GUT with at most one extra U(1) gauge group, i.e. the SM-like gauge group $G_{SM}(\times U(1))$, where $G_{SM} \equiv SU(3) \times SU(2) \times U(1)$, or a GUT-like gauge group $G_{GUT}(\times U(1))$, where $G_{GUT}$ is either SU(5), SO(10) or E$_6$. This criterion excludes the candidates (vii) – (ix) in Table I by the following reasons.

1. We note that the U(1)s in $R$ are also parts of its centralizer, i.e. a part of $H$. We thus exclude the candidate (ix) since we consider the $H$s which have at most two U(1) factors.

2. Similarly, as long as we consider the GUT-like and $G_{SM}$ gauge groups, we do not need to consider the candidates (vii) and (viii).

3. The candidates (vii) and (viii) do not allow $H = G_{SM} \times U(1)$ either for the following reason. The hypercharge of the SM should be reproduced by a certain linear combination of two U(1)s in $R$, which should be matched to the spinor representation of SO(6). The dimension of the SO(6) spinor representation is four, and thus no more than four different values of U(1) charges are available. On the other hand the fermion content of the SM has five different values of U(1) charges. Hence, this case never reproduces the hypercharges of the SM fermions.

4. Due to the above three reasons the candidates (i) – (iv) allow neither $G_{SM}$ nor $G_{SM}$ as $H$.

To summarize, the possible model requires coset space $S/R$ listed in (i) – (iv) of Table I with either $H = G_{SM} \times U(1)$ or $H = G_{GUT} \times U(1)$. In Table II we show the embedding of $R$ in SO(6) for these coset spaces. The representations of $r_a$ in Eq. (6) and $\sigma_i$ in Eq. (10) are listed in the columns of “Branches of 6” and “Branches of 4”, respectively. The embedding of $R$ into higher dimensional gauge group $G = G_1 \times G_2$ is listed in Table III. These embeddings are straightforwardly obtained by decomposing gauge group $G$ to its regular subgroup which contains an $R$-subgroup of $G$. A detailed discussion about the embeddings is summarized in [9]. For each embedding of $R$, the candidates of $G$ are summarized in Table IV. Note that all the candidates of $G$ in Table IV are subgroup of SO(32) or E$_8 \times E_8$ which are required by superstring theory.
TABLE II: The decompositions of the vector representation $6$ and the spinor representation $4$ of $SO(6)$ under $R_s$ which are listed as (i) –(iv) in Table I. The representations of $r_s$ in Eq. (6) and $\sigma_1$ in Eq. (10) are listed in the columns of “Branches of $6$” and “Branches of $4$”, respectively.

| $S/R$          | Branches of $6$ | Branches of $4$ |
|----------------|-----------------|-----------------|
| (i) SU(2)(U(1))| $3(2), 3(-2)$   | $1(3), 3(-1)$   |
| (ii) SU(2)(U(1))| $1(2), 1(-2), 2(1), 2(-1)$ | $2(1), 1(0), 1(-2)$ |
| (iii) SU(3)(U(1))| $3(-1), 3(1)$  | $1(-1), 3(1)$  |
| (iv) (SU(2),SU(2))(U(1))| $(2, 2)(0), (1, 1)(2), (1, 1)(-2)$ | $(2, 1)(1), (1, 2)(-1)$ |

TABLE III: The embedding of $R$ into $G = G_1 \times G_2$ for the coset spaces (i) and (ii).

| $S/R$          | Branches of $6$ | Branches of $4$ |
|----------------|-----------------|-----------------|
| (i) Sp(4)/[SU(2) $\times$ U(1)]$_{\text{max}}$ and (ii) Sp(4)/[SU(2) $\times$ U(1)]$_{\text{inim-max}}$ |
| (a) $G_1 \supset (G_{\text{SM or GUT}}) \times SU(2), G_2 \supset U(1)$ |
| (b) $G_1 \supset (G_{\text{SM or GUT}}) \times U(1), G_2 \supset SU(2)$ |

The representation $F_1$ of $G_1$ for the fermions should be either complex or real but not pseudoreal, since the fermions of pseudoreal representation do not allow the Majorana condition when $D = 10$ and induce doubled fermion contents after the dimensional reduction $\mathbb{R}^8$. Table X lists the candidate groups $G_1$ and their complex and real representations whose dimension is no more than 1024. The representations in this table are the candidates of $F_1$. The groups SU(7) and SO(13) are not listed here since they do not lead to the four-dimensional gauge group of our interest for any of $S/R$ and embedding of $R$ in Table III

The representation $F_2$ of $G_2$ has to be real as well as $F_1$ to impose the Majorana condition. Without this condition, $F_2$ can be any representation. We limited ourselves to the case dim $F = \dim F_1 \times \dim F_2 < 1025$ since larger representations yield numerous higher dimensional representations of fermion under the $G_{\text{SM}} \times U(1)$ and $G_{\text{GUT}} \times U(1)$.

IV. RESULTS

Now we are ready to investigate the representations for fermions and scalars in four dimensions. We first note that we need a $R_2$ singlet in SO(6) vector to obtain the Higgs candidate $h_g$ (cf. Eq. (11) and the discussion below). We can thus exclude the candidates (i) and (iii) of $S/R$ in Table I (cf. Table I). In Tables XI, XII, XIII, we list the possible candidates of $G_1$, $G_2$, $(F_1, F_2)$, and the corresponding representations of four-dimensional scalars and fermions for each $H$, which is either $G_{\text{SM}} \times U(1)$, $SU(5) \times U(1)$, $SO(10) \times U(1)$, or $E_6 \times U(1)$. The representations of four dimensional fermions are classified into A, B, and C. The representations of class A are the standard representations; 5 and 10 for SU(5), 16 for SO(10), and 27 for $E_6$, which lead to the SM fermions after GUT breaking. The representations of class B lead to both of the SM fermions and non-SM fermions after GUT breaking. The representations of class C lead only to non-SM fermions after GUT breaking.

A. $H = G_{\text{SM}} \times U(1)$

We investigate all combinations of $S/R$, $G_1$ and $G_2$ in Table XI, XII which provide $H = G_{\text{SM}} \times U(1)$ in four dimensions. We obtain a representation which is identified as the SM Higgs-doublet in four dimensions from the following cases.

TABLE IV: The embedding of $R$ into $G = G_1 \times G_2$ for the coset space (iii).

| $S/R$          | Branches of $6$ | Branches of $4$ |
|----------------|-----------------|-----------------|
| (iii) SU(4)/SU(3) $\times$ U(1) |
| (a) $G_1 \supset (G_{\text{SM or GUT}}) \times SU(3), G_2 \supset U(1)$ |
| (b) $G_1 \supset (G_{\text{SM or GUT}}) \times U(1), G_2 \supset SU(3)$ |
TABLE V: The embedding of $R$ into $G = G_1 \times G_2$ for the coset space (iv).

| (iv) $Sp(4) \times SU(2)/[SU(2) \times SU(2)] \times U(1)$ | (i)-(a) and (ii)-(a) | (i)-(b) and (ii)-(b) |
|----------------------------------------------------------|----------------------|----------------------|
| $G_1 \supset (G_{SM} \text{ or } G_{GUT}) \times SU(2)$, $G_2 \supset SU(2) \times U(1)$ | $G_1 = SO(10), SO(11), Sp(10)$ | $G_1 = SU(6), SO(10), SO(11), Sp(10)$ |
| $G_2 = SU(2), U(1)$ | $G_2 = SU(2)$ | $G_2 = SU(2)$ |

TABLE VI: The candidates of the gauge groups $G_1$ and $G_2$ for each of the coset space (i) and (ii) in Table I. The top row indicates the assigned number of $S/R$ in Table III and embedding of $R$ assigned in Table III. The leftmost column indicates $H$.

| (i)-(a) and (ii)-(a) | (i)-(b) and (ii)-(b) |
|----------------------|----------------------|
| SU(3) × SU(2) × U(1) × U(1) | G1 = SO(10), SO(11), Sp(10) |
| G2 = SU(2), U(1) | G2 = SU(2), U(1) |
| SU(5) × U(1) | G1 = No candidate |
| G2 = SU(2), U(1) | G2 = SU(2), U(1) |
| SO(10) × U(1) | G1 = SO(13) |
| G2 = SU(2), U(1) | G2 = SU(2), U(1) |
| E6 × U(1) | G1 = No candidate |
| G2 = SU(2), U(1) | G2 = SU(2), U(1) |

TABLE VII: The allowed candidates of the gauge groups $G_1$ and $G_2$ for the coset space (iii) in Table I. The top row indicates the assigned number of $S/R$ in Table III and embedding of $R$ assigned in Table III. The leftmost column indicates $H$.

| (iii)-(a) | (iii)-(b) |
|----------------------|----------------------|
| SU(3) × SU(2) × U(1) × U(1) | G1 = E6 |
| G2 = SU(2), U(1) | G2 = SU(2), U(1) |
| SU(5) × U(1) | G1 = No candidate |
| G2 = SU(2), U(1) | G2 = SU(2), U(1) |
| SO(10) × U(1) | G1 = No candidate |
| G2 = SU(2), U(1) | G2 = SU(2), U(1) |
| E6 × U(1) | G1 = E6 |
| G2 = SU(2), U(1) | G2 = SU(2), U(1) |

TABLE VIII: The allowed candidates of the gauge groups $G_1$ and $G_2$ for the coset space (iv) in Table I. The top row indicates the assigned number of $S/R$ in Table III and embedding of $R$ assigned in Table III. The leftmost column indicates $H$.

| (iv)-(a) | (iv)-(b) |
|----------------------|----------------------|
| SU(3) × SU(2) × U(1) × U(1) | G1 = SO(10), SO(11), Sp(10) |
| G2 = SU(3), Sp(4), G2 | G2 = SU(2), U(1) |
| SU(5) × U(1) | G1 = No candidate |
| G2 = SU(3), Sp(4), G2 | G2 = SU(2), U(1) |
| SO(10) × U(1) | G1 = SO(13) |
| G2 = SU(3), Sp(4), G2 | G2 = SU(2), U(1) |
| E6 × U(1) | G1 = No candidate |
| G2 = SU(3), Sp(4), G2 | G2 = SU(2), U(1) |
TABLE IX: The allowed candidates of the gauge groups $G_1$ and $G_2$ for the coset space (iv) in Table I. The top row indicates the assigned number of $S/R$ in Table I and embedding of $R$ assigned in Table V. The leftmost column indicates $H$.

| $H$ | (iv)-(c) | (iv)-(d) |
|-----|----------|----------|
| SU(3) $\times$ SU(2) $\times$ U(1) $\times$ U(1) | $G_1 = SU(6), SO(10), SO(11), Sp(10)$, $G_2 = G_2, Sp(4)$ | $G_1 = SU(7), SO(12), SO(13), Sp(12), E_6$, $G_2 = SU(2)$ |
| SU(5) $\times$ U(1) | $G_1 = SU(6), SO(10), SO(11), Sp(10)$, $G_2 = G_2, Sp(4)$ | $G_1 = SU(7), SO(13)$, $Sp(12), E_6$, $G_2 = SU(2)$ |
| SO(10) $\times$ U(1) | $G_1 = SO(12), SO(13)$, $E_6$, $G_2 = G_2, Sp(4)$ | $G_1 = SO(14), SO(15)$, $E_7$, $G_2 = SU(2)$ |
| $E_6 \times$ U(1) | $G_1 = E_7$, $G_2 = G_2, Sp(4)$ | $G_1 = E_8$, $G_2 = SU(2)$ |

TABLE X: The complex or real representations of the possible gauge groups $^{33}$. The groups SU(7) and SO(13) are not listed here since they do not lead to the four-dimensional gauge group of our interest for any of $S/R$ and embedding of $R$ in Table III–V.

| Group | Complex representations | Real representations |
|-------|-------------------------|----------------------|
| SU(6) | 6, 15, 21, 56, 70, 84, 105, 105', 120, 126, 210, 210', 252, 280, 315, 336, 384, 420, 462, 490, 504, 560, 700, 720, 792, 840, 840', 840'', 896, · · · | 35, 175, 189, 405, · · · |
| SO(11) | 11, 55, 65, 165, 275, 320, 330, 429, 462, 935, · · · | 12, 66, 77, 220, 352, 462, 495, 560, 792, · · · |
| SO(12) | 12, 66, 77, 220, 352, 462, 495, 560, 792, · · · | 14, 91, 104, 364, 546, 896, · · · |
| SO(14) | 64, 832, · · · | 14, 91, 104, 364, 546, 896, · · · |
| SO(15) | 15, 105, 119, 128, 455, 665, · · · | 15, 105, 119, 128, 455, 665, · · · |
| $F_4$ | 26, 52, 273, 324, · · · | 26, 52, 273, 324, · · · |
| $E_6$ | 27, 351, 351', · · · | 78, 650, · · · |
| $E_7$ | 133 · · · | 133 · · · |
| $E_8$ | 248 · · · | 248 · · · |

1. $R$ embedded as (ii)-(b), $G_1 = SU(6)$ and $G_2 = SU(2)$.
2. $R$ embedded as (ii)-(b), $G_1 = SO(11)$ and $G_2 = SU(2)$.
3. $R$ embedded as (iv)-(c), $G_1 = SU(6)$ and $G_2 = G_2$.
4. $R$ embedded as (iv)-(c), $G_1 = SU(6)$ and $G_2 = Sp(4)$.
5. $R$ embedded as (iv)-(c), $G_1 = SO(11)$ and $G_2 = G_2$.
6. $R$ embedded as (iv)-(c), $G_1 = SO(11)$ and $G_2 = Sp(4)$.
7. $R$ embedded as (iv)-(d), $G_1 = Sp(12)$ and $G_2 = SU(2)$.
8. $R$ embedded as (iv)-(d), $G_1 = E_6$ and $G_2 = SU(2)$.

Any of these cases does not reproduce a whole generation of the SM fermions. Therefore we cannot obtain the SM in four dimensions. The difficulty in obtaining the SM is ultimately due to the smallness of the dimension of SO(6) spinor representation.
B. $H = SU(5) \times U(1)$

We investigate the case of $H = SU(5) \times U(1)$ and summarize the result in Table XI. We obtain the representation $\mathbf{5}$ which corresponds to the Higgs scalar in the following cases.

1. $R$ embedded as (ii)-(b), $G_1 = SU(6)$ and $G_2 = SU(2)$.
2. $R$ embedded as (ii)-(b), $G_1 = SO(11)$ and $G_2 = SU(2)$.
3. $R$ embedded as (iv)-(c), $G_1 = SU(6)$ and $G_2 = Sp(4)$.
4. $R$ embedded as (iv)-(c), $G_1 = SO(11)$ and $G_2 = Sp(4)$.
5. $R$ embedded as (iv)-(d), $G_1 = E_6$ and $G_2 = SU(2)$.

As for the fermions, we see that the standard representations of SU(5) GUT are not obtained at all for the cases 3, 4, and 5, while they are obtained by combining two representations of $F$ in the cases 1 and 2. For the example of case 1, we can choose $(\mathbf{70}, 2)$ and $(\mathbf{280}, 1)$ to obtain all the standard representations, $\bar{\mathbf{5}}$ and $\mathbf{10}$, in four dimensions, along with the extra fermions of class B and C.

C. $H = SO(10) \times U(1)$

We investigate all the combinations of $S/R$, $G_1$, and $G_2$ for $H = SO(10) \times U(1)$. We obtain the representation $\mathbf{10}$ which corresponds to the Higgs scalar in the following cases.

1. $R$ embedded as (ii)-(b), $G_1 = SO(12)$ and $G_2 = SU(2)$.
2. $R$ embedded as (ii)-(b), $G_1 = E_6$ and $G_2 = SU(2)$.
3. $R$ embedded as (iv)-(b), $G_1 = SO(14)$ and $G_2 = SU(2)$.
4. $R$ embedded as (iv)-(b), $G_1 = SO(14)$ and $G_2 = U(1)$.
5. $R$ embedded as (iv)-(b), $G_1 = SO(15)$ and $G_2 = SU(2)$.
6. $R$ embedded as (iv)-(b), $G_1 = SO(15)$ and $G_2 = U(1)$.
7. $R$ embedded as (iv)-(c), $G_1 = SO(12)$ and $G_2 = G_2$.
8. $R$ embedded as (iv)-(c), $G_1 = SO(12)$ and $G_2 = Sp(4)$.
9. $R$ embedded as (iv)-(c), $G_1 = E_6$ and $G_2 = SU(2)$
10. $R$ embedded as (iv)-(c), $G_1 = E_6$ and $G_2 = G_2$.
11. $R$ embedded as (iv)-(d), $G_1 = SO(15)$ and $G_2 = SU(2)$.
12. $R$ embedded as (iv)-(d), $G_1 = E_7$ and $G_2 = SU(2)$.

We further obtain the standard representations of the fermions which lead to all the SM fermions of one generation in the cases $\mathbf{4}, 6, 8, 11$ and $\mathbf{12}$ (see Table XII).

The case $\mathbf{3}$ with $F = \mathbf{832}(1)$ is intriguing since we obtain two $\mathbf{16}$s and two $\mathbf{144}$s, each of which leads to a complete set of the SM fermions of one generation. We thus obtain four generations of fermions which can accommodate the known three generations. Furthermore these representations can form three distinct types of Yukawa coupling: $\mathbf{16} \times \mathbf{16} \times \mathbf{10}$, $\mathbf{144} \times \mathbf{16} \times \mathbf{10}$, and $\mathbf{144} \times \mathbf{144} \times \mathbf{10}$. These couplings may explain the origin of the Yukawa couplings distinguishing the the generations and the mixing among them.
D. \( H = E_6 \times U(1) \)

The results for \( H = E_6 \times U(1) \) are listed in Table XIII. We obtain representation 27 which corresponds to the Higgs scalar in the following cases.

1. \( R \) embedded as (ii)-(b), \( G_1 = E_7 \) and \( G_2 = SU(2) \).
2. \( R \) embedded as (iv)-(c), \( G_1 = E_7 \) and \( G_2 = G_2 \).
3. \( R \) embedded as (iv)-(d), \( G_1 = E_8 \) and \( G_2 = SU(2) \).

The standard representations of fermion 27, which provide all the SM fermions of one generation, are obtained in cases 1 and 3.

Case 1 with \( F = (133,1) \) is interesting since the structure of the SM with three generations may be explained. The Yukawa coupling of this model needs to be in the form 27(-2) x 27(2) x 78(0). The fermion representation 27 + 78 of \( E_6 \) contains three generations of \( 5 + 10 \) in terms of its SU(5) subgroup, giving the origin of the known three generations. Indeed, this fermion content is analyzed in, for example, nonlinear sigma models giving a family unification based on a broken \( E_7 \) symmetry, under which a reproduction of the observed mixing structure among the three generations of fermions has been attempted.
TABLE XI: The models for $H=SU(5) \times U(1)$ which include the SM Higgs-doublet and one generation of the SM fermions in four dimensions. The fermions in four dimensions are classified into A, B, and C. The fermion-As contain only the SM fermions; fermion- Bs contain both the SM fermions and extra fermions; fermion-Cs contain only extra fermions.

| $S/R = Sp(4)/[SU(2) \times U(1)]$, $G_1 \supset SU(5) \times U(1)$, $G_2 \supset SU(2)$ | Scalars | Fermions-A | B | C |
|---|---|---|---|---|
| **SU(6) SU(2)** | | | | |
| $(56, 2)$ | $(5, 6), (5, -6)$ | $10(-3)$ | $15(-3)$, $35(-3)$ | |
| $(70, 2)$ | $(5, 6), (5, -6)$ | $10(-3)$ | $15(-3), 40(-3)$ | |
| $(280, 1)$ | $(5, 6), (5, -6)$ | $(5, -6)$ | $70(-6)$, $24(0), 45(-6), 126(0)$ | |
| | | | $24(0), 126(0)$ | |
| $(405, 1)$ | $(5, 6), (5, -6)$ | $(5, -6)$ | $70(-6)$, $1(0), 24(0), 200(0)$ | |
| | | | | $5(0), 70(6), 1(0), 24(0), 200(0)$ |
| $(840, 1)$ | $(5, 6), (5, -6)$ | $45(0)$ | $280(0), 126(0), 224(0)$ | |
| | | | | $105(6), 126(0), 224(0)$ |
| **SO(11) SU(2)** | | | | |
| $(11, 1)$ | $(5, 2), (5, -2)$ | $5(-2)$ | $1(0)$ | |
| $(55, 1)$ | $(5, 2), (5, -2)$ | $5(-2)$ | $1(0), 24(0)$ | |
| $(65, 1)$ | $(5, 2), (5, -2)$ | $5(-2)$ | $1(0), 24(0)$ | |
| $(165, 1)$ | $(5, 2), (5, -2)$ | $5(-2)$ | $45(-2)$, $1(0), 24(0)$ | |
| $(275, 1)$ | $(6, 2), (5, -2)$ | $5(-2)$ | $70(-2)$, $1(0), 24(0)$ | |
| $(320, 2)$ | $(5, 2), (5, -2)$ | $10(-1), 10(-1)$, $15(-1), 40(-1)$ | | |
| $(330, 1)$ | $(5, 2), (5, -2)$ | $5(-2)$ | $45(-2)$, $1(0), 24(0), 75(0)$ | |
| $(429, 1)$ | $(5, 2), (5, -2)$ | $5(-2)$, $5(-2)$, $45(-2), 70(-2)$ | $1(0), 24(0), 24(0)$ | |
| $(462, 1)$ | $(5, 2), (5, -2)$ | $5(-2)$ | $45(-2), 50(-2)$, $1(0), 24(0), 75(0)$ | |
| $(935, 1)$ | $(5, 2), (5, -2)$ | $5(-2)$ | $70(-2)$, $1(0), 24(0), 200(0)$ | |
TABLE XII: The models for $H = \text{SO}(10) \times U(1)$ which include the SM Higgs and one generation of the SM fermions in four-dimensions. The fermions in four-dimensions are classified into A, B, and C where fermion-As are $\text{16}$ representation of SO(10); fermion-Bs contain both the SM fermions and extra-fermions; fermion-Cs contain only extra-fermions. We can obtain two types of results for fermions from one combination of $(G_1, G_2, F)$ since we have a freedom to change the overall sign of U(1) charges which appear in the $R$-decomposition of SO(6) vector and spinor.

| $S/R$ | $G_1$ | $G_2$ | $\{F_1, F_2\}$ | Scalars | Fermions- $\text{A}$ | $\text{B}$ | $\text{C}$ |
|-------|-------|-------|-----------------|---------|------------------|-------|-------|
| $\text{SO}(10)$ | $\text{SU}(2)$ | $(12, 1)$ | $10(2), 10(-2)$ | $10(0), 1(2)$ | $10(0), 1(-2)$ | $10(0), 1(2)$ | $10(0), 1(-2)$ |
| $\text{SU}(2)$ | $(66, 1)$ | $10(2), 10(-2)$ | $10(2), 45(0)$ | $10(2), 45(0)$ | $10(2), 45(0)$ | $10(2), 45(0)$ | $10(2), 45(0)$ |
| $\text{SU}(2)$ | $(77, 1)$ | $10(2), 10(-2)$ | $10(2), 54(0)$ | $10(2), 54(0)$ | $10(2), 54(0)$ | $10(2), 54(0)$ | $10(2), 54(0)$ |
| $\text{SU}(2)$ | $(220, 1)$ | $10(2), 10(-2)$ | $45(2), 10(0), 120(0)$ | $45(2), 10(0), 120(0)$ | $45(2), 10(0), 120(0)$ | $45(2), 10(0), 120(0)$ | $45(2), 10(0), 120(0)$ |
| $\text{SU}(2)$ | $(352, 1)$ | $10(2), 10(-2)$ | $54(2), 10(0), 210(0)$ | $54(2), 10(0), 210(0)$ | $54(2), 10(0), 210(0)$ | $54(2), 10(0), 210(0)$ | $54(2), 10(0), 210(0)$ |
| $\text{SU}(2)$ | $(462, 1)$ | $10(2), 10(-2)$ | $126(2), 210(0)$ | $126(2), 210(0)$ | $126(2), 210(0)$ | $126(2), 210(0)$ | $126(2), 210(0)$ |
| $\text{SU}(2)$ | $(495, 1)$ | $10(2), 10(-2)$ | $120(2), 45(0), 210(0)$ | $120(2), 45(0), 210(0)$ | $120(2), 45(0), 210(0)$ | $120(2), 45(0), 210(0)$ | $120(2), 45(0), 210(0)$ |
| $\text{SU}(2)$ | $(560, 1)$ | $10(2), 10(-2)$ | $126(2), 54(0), 210(0)$ | $126(2), 54(0), 210(0)$ | $126(2), 54(0), 210(0)$ | $126(2), 54(0), 210(0)$ | $126(2), 54(0), 210(0)$ |
| $\text{SU}(2)$ | $(792, 1)$ | $10(2), 10(-2)$ | $210(2), 120(0), 126(0), 210(0)$ | $210(2), 120(0), 126(0), 210(0)$ | $210(2), 120(0), 126(0), 210(0)$ | $210(2), 120(0), 126(0), 210(0)$ | $210(2), 120(0), 126(0), 210(0)$ |
| $\text{SU}(2)$ | $(78, 1)$ | $16(-3), 16(3)$ | $16(-3)$ | $16(-3)$ | $16(-3)$ | $16(-3)$ | $16(-3)$ |
| $\text{SU}(2)$ | $(650, 1)$ | $16(-3), 16(3)$ | $16(3)$ | $16(3)$ | $16(3)$ | $16(3)$ | $16(3)$ |
| $\text{SU}(2)$ | $(133, 1)$ | $16(-3), 16(3)$ | $16(-3), 16(3)$ | $16(-3), 16(3)$ | $16(-3), 16(3)$ | $16(-3), 16(3)$ | $16(-3), 16(3)$ |
| $\text{SO}(10)$ | $\text{SU}(2)$ | $(64, 2)$ | $10(0)$ | $10(0), 1(2)$ | $10(0), 1(-2)$ | $10(0), 1(2)$ | $10(0), 1(-2)$ |
| $\text{SU}(2)$ | $(64, 1)$ | $10(0)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ |
| $\text{SU}(2)$ | $832(1)$ | $10(0)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ |
| $\text{SO}(10)$ | $\text{SU}(2)$ | $(128, 2)$ | $10(0), 1(0)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ |
| $\text{SU}(2)$ | $(128, 1)$ | $10(0), 1(0)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ | $16(1), 16(1)$ |
| $\text{SU}(2)$ | $(133, 1)$ | $10(2), 10(-2)$ | $16(1)$ | $16(1)$ | $16(1)$ | $16(1)$ | $16(1)$ |
| $\text{SU}(2)$ | $(128, 1)$ | $10(2), 10(-2)$ | $16(1)$ | $16(1)$ | $16(1)$ | $16(1)$ | $16(1)$ |
V. SUMMARY AND DISCUSSIONS

We studied the ten-dimensional gauge theories whose extra six-dimensional spacetime is a coset space of Lie groups. We focused on the case where the gauge group is a direct product of two simple Lie groups, and searched for models which lead to phenomenologically promising four-dimensional models after applying the coset space dimensional reduction.

We first limited the possible coset space $S/R$ to four types listed in (i) – (iv) of Table II by requiring that $R$ should be factored as $R = R_1 \times R_2$. All of these four types have a $U(1)$ factor in $R$, but this $U(1)$ can never be identified as the hypercharge symmetry of the SM. We thus needed to introduce an extra $U(1)$ in the four-dimensional gauge group $H$, and searched for SM-like models or GUT-like ones. The former is the case where $H = SU(3) \times SU(2) \times U(1) \times U(1)$, while the latter is where $H = SU(5) \times U(1)$, $H = SO(10) \times U(1)$, and $H = E_6 \times U(1)$. We also require that the induced four-dimensional model should include the particle contents appropriate for the SM particles. We then found the candidates of the gauge group $G = G_1 \times G_2$ of the ten-dimensional theory and the representations for fermions.

For each of the obtained candidates, we made the complete lists of representations of the scalars and the fermions that constitute the corresponding four-dimensional theory. The results are summarized as follows.

1. No ten-dimensional model was found to induce the promising model with $H = SU(3) \times SU(2) \times U(1) \times U(1)$ in the four-dimensional spacetime.

2. The models which induce a $SU(5) \times U(1)$ gauge theory in four-dimensional spacetime were found when $S/R = Sp(4)/SU(2) \times U(1)$. Possible gauge group is either $SU(6) \times SU(2)$ or $SO(11) \times SU(2)$, and each case has several choices of the representation for the ten-dimensional fermions as listed in Table XI. Many of fermion representations generate either $5$ or $10$ of the $SU(5)$ after the dimensional reduction. None of them, however, generates both from a single representation, and we thus need at least two fermion representations in the ten-dimensional model as well as in four-dimensional one.

3. The models which induce a $SO(10) \times U(1)$ gauge theory in four dimensions were found for the three possible choices of $S/R$, and each choice allows a number of gauge groups as listed in Table XIII.

4. The models which induce an $E_6 \times U(1)$ gauge theory in four dimensions were found when $S/R = Sp(4)/SU(2) \times U(1)$, $G = E_7 \times SU(2)$ and $S/R = Sp(4)/SU(2)/[SU(2) \times SU(2)] \times U(1)$, $G = E_8 \times SU(2)$, as listed in Table XIV.

The fermion representations in four-dimensional theories obtained from the candidate models mentioned above are not limited to the standard ones, i.e. $5$ and $10$ for $H = SU(5) \times U(1)$, $16$ for $H = SO(10) \times U(1)$, and $27$ for $H = E_6 \times U(1)$. Some of these extra representations can accommodate the SM particles as well and thus can take part in the further model building. The following two models are found to be of particular interest.

1. $H = SO(10) \times U(1)$, $S/R = Sp(4)/SU(2)/[SU(2) \times SU(2)] \times U(1)$, $G = SO(14) \times U(1)$, and $F = 832(1)$ (see Table XII). In this case, the fermions in four-dimensional theory include two $16$s and two $144$s. Since both can include a complete set of the SM fermions of a generation, this case has four generations of fermions and thus can accommodate the known three generations. Besides, this case allows three distinct types of Yukawa coupling: $16 \times 16 \times 10$, $144 \times 16 \times 10$, and $144 \times 144 \times 10$. These three types of couplings can admit the different Yukawa couplings, giving rise to the distinction of generations. Hence this model may possibly introduce the mixing among the generations.

2. $H = E_6 \times U(1)$, $S/R = Sp(4)/SU(2)$, $G = E_7 \times SU(2)$, and $F = (133, 1)$ (see Table XIII). The Yukawa coupling of this model is necessarily of the form $27(-2) \times 27(2) \times 78(0)$. The fermion representation $27 + 78$ of $E_6$ contains three generations of $5 + 10$ in terms of its $SU(5)$ subgroup, giving the origin of the known three generations. Indeed, this fermion content is analyzed in, for example, nonlinear sigma models giving a family unification based on a broken $E_7$ symmetry [53], under which a reproduction of the observed mixing structure among the three generations of fermions has been attempted [52].

We leave further analysis for the future study as well as building phenomenological models based on the models mentioned above.

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TABLE XIII: The models for $H = E_6 \times U(1)$ which include the SM Higgs and one generation of the SM fermions in four dimensions. The fermions in four dimensions are classified into A, B and C where fermion-A is the $27$ representation of $E_6$; fermion-Bs contain both the SM fermions and extra fermions; fermion-Cs contain only extra-fermions. We can obtain two types of results for fermions from one combination of $(G_1, G_2, F)$ since we have a freedom to change the overall sign of $U(1)$ charges which appear in $R$-decomposition of $SO(6)$ vector and spinor.

| $G_1$ | $G_2$ | $(F_1, F_2)$ | Scalars | Fermions-A | B | C |
|-------|-------|--------------|---------|------------|---|---|
| $E_7$ | SU(2) | (133, 1)     | $27(2), 2\bar{7}(-2)$ | $27(2)$ | $78(0)$ | 1(0) |

$S/R = Sp(4)/SU(2) \times U(1), \quad G_1 \supset E_6 \times U(1), \quad G_2 \supset SU(2)$

$S/R = Sp(4) \times SU(2)/[SU(2) \times SU(2)] \times U(1), \quad G_1 \supset E_6 \times SU(2) \times U(1), \quad G_2 \supset SU(2)$

| $G_1$ | $G_2$ | $(F_1, F_2)$ | Scalars | Fermions-A | B | C |
|-------|-------|--------------|---------|------------|---|---|
| $E_8$ | SU(2) | F(248, 1)    | $27(-2), 2\bar{7}(2)$ | $27(1)$ | $27(-1)$ |
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