Generalized integrable hierarchies and Combescure symmetry transformations

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Abstract

Unifying hierarchies of integrable equations are discussed. They are constructed via generalized Hirota identity. It is shown that the Combescure transformations, known for a long time for the Darboux system and having a simple geometrical meaning, are in fact the symmetry transformations of generalized integrable hierarchies. Generalized equation written in terms of invariants of Combescure transformations are the usual integrable equations and their modified partners. The KP-mKP, DS-mDS hierarchies and Darboux system are considered.

1 Introduction

The Sato approach (see e.g. [1]-[3]) and the $\bar{\partial}$-dressing method (see e.g. [4]-[7]) are two powerful tools to construct and analyze the hierarchies of integrable equations. A bridge between these seemingly different approaches has been established by the observation that the Hirota bilinear identity can be derived from the $\bar{\partial}$-equation [7], [8]. An approach which combines the characteristic features of both methods, namely, the Hirota bilinear identity from the Sato approach and the analytic properties of solutions from the $\bar{\partial}$-dressing method, has been discussed in [9], [10].

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A connection between wave functions with different normalizations was one of interesting open problems of the \( \partial \)-dressing method. In [9] it was shown that such a connection is given by the Combescure transformation.

The Combescure transformation was introduced last century within the study of the transformation properties of surfaces (see e.g. [11], [12]). It is a transformation of surface such that the tangent vector at a given point of the surface remains parallel. The Combescure transformation is essentially different from the well-known Bäcklund and Darboux transformations. The Combescure transformation plays an important role in the theory of the systems of hydrodynamical type [13]. It is also of great interest for the theory of (2+1)-dimensional integrable systems [14].

In the present paper we discuss an approach to integrable hierarchies based on the generalized Hirota bilinear identity for the wave function \( \psi(\lambda, \mu) \) with simple analytic properties (Cauchy-Baker-Akhiezer function). We derive generalized integrable hierarchies in terms of the function \( \psi(\lambda, \mu) \). Such a generalized equation contains usual integrable equations, their modified partners and corresponding linear problems. Compact form of the generalized equations in terms of the \( \tau \)-function is derived.

It is shown that the generalized equations possess the symmetries given by the Combescure transformations. The invariants of these symmetry transformations are found. The generalized equations written in terms of these invariants coincide with the usual equations or their modified versions. The Darboux transformation and its connection with the Combescure transformation for the generalized hierarchies is also discussed. The Kadomtsev-Petviashvili (KP) and modified KP (mKP) equations, the Davey-Stewartson (DS) and modified Davey-Stewartson (mDS) equations and the matrix Darboux-Zakharov-Manakov (DZM) system are considered.

## 2 Generalized Hirota identity.

We start with the generalized Hirota bilinear identity

\[
\int_{\partial G} \chi(\nu, \mu; g_1 g_2^{-1}(\nu) \chi(\lambda, \nu; g_2) dv = 0
\] (1)

Here \( \chi(\lambda, \mu; g) \) is a matrix function of two complex variables \( \lambda, \mu \in G \) and a functional of the group element \( g \) defining the dynamics (will be specified later), \( G \) is some set of domains of the complex plane, the integration goes over the boundary of \( G \). By definition, the function \( \chi(\lambda, \mu) \) possesses the
The following analytical properties:

\[ \bar{\partial}_\lambda \chi(\lambda, \mu) = 2 \pi i \delta(\lambda - \mu), \quad -\bar{\partial}_\mu \chi(\lambda, \mu) = 2 \pi i \delta(\lambda - \mu), \]

where \( \delta(\lambda - \mu) \) is a \( \delta \)-function, or, in other words, \( \chi \to (\lambda - \mu)^{-1} \) as \( \lambda \to \mu \) and \( \chi(\lambda, \mu) \) is analytic function of both variables \( \lambda, \mu \) for \( \lambda \neq \mu \).

The formula (1) is a basic tool of our construction.

In another form, more similar to standard Hirota bilinear identity, the identity (1) can be written as

\[ \int_{\partial G} \bar{\psi}(\nu, \mu; g_1) \psi(\lambda, \nu; g_2) d\nu = 0, \]

where

\[ \psi(\lambda, \mu; g) = g^{-1}(\mu) \chi(\lambda, \mu; g) g(\lambda). \]

We will suggest in what follows that we are able to solve the equation (1) somehow. Particularly we will bear in mind that the solutions for equation (1) can be obtained using the \( \bar{\partial} \)-dressing method [9]. In frame of algebro-geometric technique the function \( \psi(\lambda, \mu) \) corresponds to Cauchy-Baker-Akhiezer kernel on the Riemann surface (see [15]). The general setting of the problem of solving (1) requires some modification of Segal-Wilson Grassmannian approach [3].

Let us consider two linear spaces \( W(g) \) and \( \tilde{W}(g) \) defined by the function \( \chi(\lambda, \mu) \) (satisfying (1)) via equations connected with the identity (1)

\[ \int_{\partial G} f(\nu; g) \chi(\lambda, \nu; g) d\nu = 0, \]  
\[ \int_{\partial G} \chi(\nu, \mu; g) h(\nu; g) d\nu = 0, \]

here \( f(\lambda) \in W, h(\lambda) \in \tilde{W} \); \( f(\lambda), h(\lambda) \) are defined in \( \tilde{G} \).

It follows from the definition of linear spaces \( W, \tilde{W} \) that

\[ f(\lambda) = 2 \pi i \int_G \eta(\nu) \chi(\lambda, \nu) d\nu \wedge d\nu, \quad \eta(\nu) = \left( \frac{\partial}{\partial \nu} f(\nu) \right), \]
\[ h(\mu) = -2 \pi i \int_G \chi(\nu, \mu) \bar{\eta}(\nu) d\nu \wedge d\nu, \quad \bar{\eta}(\nu) = \left( \frac{\partial}{\partial \nu} h(\nu) \right). \]

These formulae in some sense provide an expansion of the functions \( f, h \) in terms of the basic function \( \chi(\lambda, \mu) \). The formulae (6) readily imply that linear spaces \( W, \tilde{W} \) are transversal to the space of holomorphic functions in \( \tilde{G} \) (transversality property).
From the other point of view, these formulae define a map of the space
of functions (distributions) on $G \eta, \tilde{\eta}$ to the spaces $W, \tilde{W}$. We will call $\eta (\tilde{\eta})$ a normalization of the corresponding function belonging to $W, \tilde{W}$.

The dynamics of linear spaces $W, \tilde{W}$ looks very simple

$$W(g) = W_0 g^{-1}; \quad \tilde{W}(g) = g\tilde{W}_0.$$  (7)

Here $W_0 = W(g = 1), \tilde{W}_0 = \tilde{W}(g = 1)$ (the formulae (7) follow from identity (1) and the formulae (6)).

A dependence of the function $\chi(\lambda, \mu)$ on dynamical variables is hidden
in the function $g(\lambda)$. We will consider here only the case of continuous
variables, for which

$$g_i = \exp(K_i x_i); \quad \frac{\partial}{\partial x_i} g = K_i g$$  (8)

Here $K_i(\lambda)$ are commuting matrix meromorphic functions.

To introduce a dependence on several variables (may be of different type),
one should consider a product of corresponding functions $g(\lambda)$ (all of them
commute).

Let $G$ be a unit disk and $x_\nu$ a variable corresponding to $K_\nu(\lambda) = \frac{A_\nu}{\lambda - \nu}$.
Differentiating the identity (1) over $x_\nu$, one obtains

$$\frac{A_\nu}{\lambda - \nu} \frac{\partial}{\partial x_\nu} \chi(\lambda, \mu, x_\nu) - \frac{\partial}{\partial x_\nu} \chi(\lambda, \mu, x_\nu) \frac{A_\nu}{\mu - \nu} = \chi(\nu, \mu) A_\nu \chi(\lambda, \nu),$$

or, in terms of the $\psi$ function (3),

$$\frac{\partial}{\partial x_\nu} \psi(\lambda, \mu, x_\nu) = \psi(\nu, \mu, x_\nu) A_\nu \psi(\lambda, \nu, x_\nu).$$  (9)

This formula allows one to construct the basic function $\chi(\lambda, \mu)$ using only
two functions with ‘canonical’ normalization, the Baker-Akhiezer function
$\psi(\lambda, \nu, x_\nu)$ and the dual Baker-Akhiezer function $\psi(\nu, \mu, x_\nu)$ corresponding
to some fixed point $\nu$.

3 The matrix DZM system

The matrix DZM system is our first example. In this case the construction
and all formulae are very simple and transparent.
To derive the DZM system of equations, we take a set of three identical unit disks with the center at $\lambda = 0$ $D_i$, $1 \leq i \leq 3$, as $G$. The functions $K_i(\lambda)$, $1 \leq i \leq 3$, are chosen in the form

$$K_i(\lambda) = \frac{A_i}{\lambda}, \quad \lambda \in D_i;$$

$$K_i(\lambda) = 0, \quad \lambda \notin D_i.$$

where $A_i$, $A_j$, $A_k$ are commuting matrices.

It appears that the Hirota bilinear identity in differential form (9) contains enough information to derive equations for the rotation coefficients, DZM equations and linear problem for the DZM equations. Indeed, evaluating the set of three relations (9) for independent variables $x_i$, $x_j$, $x_k$ at the set of points $\lambda, \mu \in \{0_i, 0_j, 0_k\}$ one easily obtains the relations

$$\partial_i \psi(\lambda, \mu) = \psi(0_i, \mu) A_i \psi(\lambda, 0_i),$$

$$\partial_i \psi(\lambda, 0_j) = \psi(0_i, 0_j) A_i \psi(\lambda, 0_i),$$

$$\partial_i \psi(0_j, \mu) = \psi(0_i, \mu) A_i \psi(0_j, 0_i),$$

$$\partial_i \psi(0_j, 0_k) = \psi(0_j, 0_i) A_i \psi(0_k, 0_i),$$

where $\partial_i = \frac{\partial}{\partial x_i}$.

Let us now take into account that all the equations containing $\lambda, \mu$ can be integrated over the boundary of $G$ with some matrix weight functions $\rho(\lambda)$, $\tilde{\rho}(\mu)$ (note that they are connected with the normalization functions defined by (6)) without changing the structure of equations. So we will write the equations in terms of wave functions independent of spectral parameters

$$\partial_i \Phi = \tilde{f}_i f_i, \quad \text{(10)}$$

$$\partial_i f_j = \beta_{ji} f_i, \quad \text{(11)}$$

$$\partial_i \tilde{f}_j = \bar{f}_i \beta_{ij}, \quad \text{(12)}$$

$$\partial_i \beta_{jk} = \bar{\beta}_{ji} \beta_{ik}. \quad \text{(13)}$$

Here

$$\Phi = \iint \tilde{\rho}(\mu) \psi(\lambda, \mu) \rho(\lambda) d\lambda d\mu,$$

$$f_i = (A_i)^{1/2} \int \psi(\lambda, 0_i) \rho(\lambda) d\lambda,$$

$$\tilde{f}_i = \int \tilde{\rho}(\mu) \psi(0_i, \mu) (A_i)^{1/2} d\mu,$$

$$\beta_{ij} = (A_j)^{1/2} \psi(0_j, 0_i) (A_i)^{1/2}. \quad \text{(14)}$$
The system of equations (10)-(13) implies that
\[
\partial_i \partial_j \tilde{f}_k = ((\partial_j \tilde{f}_i) \tilde{f}_k^{-1}) \partial_i \tilde{f}_k + ((\partial_i \tilde{f}_j) \tilde{f}_k^{-1}) \partial_j \tilde{f}_k,
\]
(15)
and
\[
\partial_i \partial_j \Phi = ((\partial_j \tilde{f}_i) \tilde{f}_k^{-1}) \partial_i \Phi + ((\partial_i \tilde{f}_j) \tilde{f}_k^{-1}) \partial_j \Phi
\]
(16)
\[
\partial_i \partial_j f_k = (\partial_i f_k)(f_i^{-1} \partial_j f_i) + (\partial_j f_k)(f_j^{-1} \partial_i f_j),
\]
(17)
and
\[
\partial_i \partial_j \Phi = \partial_i \Phi f_i^{-1}(\partial_j f_i) + \partial_j \Phi f_j^{-1}(\partial_i f_j).
\]
(18)
The system (15) is just the matrix DZM equation derived in [4]. The system (17) is its dual partner. So the solution for the DZM equations is, in fact, given by the dual wave functions, i.e. the wave functions for the linear equations (12), while the compatibility conditions for these equations give the equations for rotation coefficients (13). Solutions of the dual DZM system are given by the wave functions for the linear system (11).

One always has a freedom to choose the dual wave function (or, in other words, the freedom to choose the weight function \(\tilde{\rho}(\lambda)\)), keeping the rotation coefficients invariant. This freedom is described by the Combescure symmetry transformation between the solutions of the DZM system of equations
\[
(f_i')^{-1} \partial_i \tilde{f}_j = \tilde{f}_i^{-1} \partial_i \tilde{f}_j
\]
(19)
The equations (19) just literally reflect the invariance of the rotation coefficients.

Similarly for the dual DZM system
\[
(\partial_i f_j')(f_i')^{-1} = (\partial_i f_j)f_i^{-1}
\]
(20)
In fact, the function \(\Phi\) is a wave function for two linear problems (with different potentials), corresponding to the DZM system and dual DZM system. A general Combescure transformation changes solutions for both the original system and the dual system (i.e. both functions \(\rho(\lambda), \tilde{\rho}(\mu)\)). It is also possible to consider two special subgroups of the Combescure symmetry transformations group. These two subgroups correspond to the change of only one weight function, \(\rho(\lambda)\) or \(\tilde{\rho}(\mu)\); we will call transformations of this type right or left Combescure transformations respectively. The invariants for the right (left) Combescure transformations are the solutions of the dual (or the original) DZM system. Another form of these invariants is
\[
(f_i')\partial_i \Phi' = \tilde{f}_i^{-1} \partial_i \Phi,
\]
(21)
\[
(\partial_i \Phi')(f_i')^{-1} = (\partial_i \Phi)f_i^{-1}.
\]
(22)
These equations are important for the connection with the systems of hydrodynamical type [13].

4 The KP-mKP hierarchy

The KP-mKP hierarchy is generated by

$$g(x, \lambda) = \exp \left( \sum_{i=1}^{\infty} x_i \lambda^{-i} \right),$$

(23)

which in this case is a unit disk. Let us take

$$g_1 g_2^{-1} = \exp \left( \sum_{i=1}^{\infty} (x_i - x_i^') \lambda^{-i} \right) = \exp \left( - \sum_{i=1}^{\infty} \frac{\epsilon i}{i \lambda^i} \right) = (1 - \frac{\epsilon}{\lambda}).$$

Substituting this function to the Hirota bilinear identity (1), we get

$$\left( 1 - \frac{\epsilon}{\mu} \right) \chi(\lambda, \mu, x) - \left( 1 - \frac{\epsilon}{\lambda} \right) \chi(\lambda, \mu, x') = \epsilon \chi(\lambda, 0, x') \chi(0, \mu, x),$$

(24)

or, in terms of the function $\psi(\lambda, \mu)$

$$\psi(\lambda, \mu, x') - \psi(\lambda, \mu, x) = \epsilon \psi(\lambda, 0, x') \psi(0, \mu, x); \quad x_i' - x_i = \frac{1}{i} \epsilon,$$

(25)

This equation is a finite form of the whole KP-mKP hierarchy. Indeed, the expansion of this relation over $\epsilon$ generates the KP-mKP hierarchies (and dual hierarchies) and linear problems for them.

Let us take the first three equations given by the expansion of (25) over $\epsilon$

$$\epsilon: \quad \psi(\lambda, \mu, x)_x = \psi(\lambda, 0, x) \psi(0, \mu, x),$$

(26)

$$\epsilon^2: \quad \psi(\lambda, \mu, x)_y = \psi(\lambda, 0, x)_x \psi(0, \mu, x) - \psi(\lambda, 0, x) \psi(0, \mu, x)_x,$$

(27)

$$\epsilon^3: \quad \psi(\lambda, \mu, x)_t = \frac{1}{4} \psi(\lambda, \mu, x)_{xxx} - \frac{3}{4} \psi(\lambda, 0, x)_x \psi(0, \mu, x)_x + \frac{3}{4} \psi(\lambda, 0, x)_y \psi(0, \mu, x) - \psi(\lambda, 0, x) \psi(0, \mu, x)_y$$

(28)

In the order $\epsilon^2$ the equation (25) gives rise equivalently to the equations

$$\psi(\lambda, \mu, x)_y - \psi(\lambda, \mu, x)_{xx} = -2 \psi(\lambda, 0, x) \psi(0, \mu, x)_x,$$

(29)

$$\psi(\lambda, \mu, x)_y + \psi(\lambda, \mu, x)_{xx} = 2 \psi(\lambda, 0, x)_x \psi(0, \mu, x).$$

(30)
Evaluating the first equation at \( \mu = 0 \), the second at \( \lambda = 0 \) and integrating them with the weight functions \( \rho(\lambda) \) \((\tilde{\rho}(\mu))\), one gets (see (14))

\[
\begin{align*}
  f(x)y - f(x)_{xx} &= u(x)f(x), \\
  \tilde{f}(x)y + \tilde{f}(x)_{xx} &= -u(x)(\tilde{f}(x)),
\end{align*}
\]

where \( u(x) = -2\psi(0,0)_x \).

In a similar manner, one obtains from (26)-(28) the equations

\[
\begin{align*}
  &f_t - f_{xxx} = \frac{3}{2}u f_x + \frac{3}{4}(u_x + \partial^{-1}_x u_y)f, \\
  &\tilde{f}_t - \tilde{f}_{xxx} = \frac{3}{2}u \tilde{f}_x + \frac{3}{4}(u_x - \partial^{-1}_x u_y)\tilde{f}.
\end{align*}
\]

Both the linear system (31), (33) for the wave function \( f \) and the linear system (32), (34) for the wave function \( \tilde{f} \) give rise to the same KP equation

\[
u_t = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x + \frac{3}{4} \partial^{-1}_x u_{yy}.
\]

To derive linear problems for the mKP and dual mKP equations, we will integrate equations (26), (29), (30) and (28) with two weight functions \( \rho(\lambda), \tilde{\rho}(\mu) \) (see (14))

\[
\begin{align*}
  \Phi(x)_x &= f(x)\tilde{f}(x), \\
  \Phi(x)_y - \Phi(x)_{xx} &= -2f(x)\tilde{f}(x)_x, \\
  \Phi(x)_y + \Phi(x)_{xx} &= 2f(x)_x\tilde{f}(x), \\
  \Phi(x)_t - \Phi(x)_{xxx} &= -\frac{3}{2}f(x)_x\tilde{f}(x)_x - \frac{3}{4}(f(x)\tilde{f}(x)_y - f(x)_y\tilde{f}(x)).
\end{align*}
\]

Using the first equation to exclude \( f \) from the second (and \( \tilde{f} \) from the third), we obtain

\[
\begin{align*}
  \Phi(x)_y - \Phi(x)_{xx} &= v(x)\Phi(x)_x, \\
  \Phi(x)_y + \Phi(x)_{xx} &= -\tilde{v}(x)\Phi(x)_x,
\end{align*}
\]

where \( v = -\frac{2\tilde{f}(x)_x}{f(x)} \), \( \tilde{v} = 2\frac{\tilde{f}(x)_x}{f(x)} \).

Similarly, one gets from (28)

\[
\begin{align*}
  \Phi(x)_t - \Phi(x)_{xxx} &= \frac{3}{2}v(x)\Phi(x)_{xx} + \frac{3}{4}(v_x + v^2 + \partial^{-1}_x v_y)\Phi_x, \\
  \Phi(x)_t - \Phi(x)_{xxx} &= \frac{3}{2}\tilde{v}(x)\Phi(x)_{xx} + \frac{3}{4}(\tilde{v}_x + v^2 - \partial^{-1}_x \tilde{v}_y)\Phi_x.
\end{align*}
\]
The system (40), (42) gives rise to the mKP equation

\[ v_t = v_{xxx} + \frac{3}{4} v^2 v_x + 3v_x \partial_x^{-1} v_y + 3\partial_x^{-1} v_{yy}, \]  

while the system (41), (43) leads to the dual mKP equation, which is obtained from (44) by the substitution \( v \to \tilde{v}, t \to -t, y \to -y, x \to -x. \)

So the function \( \Phi \) is simultaneously a wave function for the mKP and dual mKP L-operators with different potentials, defined by the dual KP (KP) wave functions.

Using the equation (28) and relations (40) and (41), it is possible to obtain an equation for the function \( \Phi(x) \)

\[ \Phi_t - \frac{1}{4} \Phi_{xxx} + \frac{3}{8} \Phi_y^2 - \frac{\Phi_{xx}^2}{\Phi_x} + \frac{3}{4} \Phi_x W_y = 0, \quad W_x = \frac{\Phi_y}{\Phi_x}. \]  

This equation first arose in Painleve analysis of the KP equation as a singularity manifold equation [16].

The higher analogues of equations (26)-(28) provide us, with the use of relations (40), (41), with the higher analogues of equation (45). The compact form of the hierarchy of equations for \( \Phi \) can be obtained from the basic finite relation (25). Integrating both parts of equation (25) with the weights \( \rho(\lambda) \) and \( \tilde{\rho}(\mu) \), one gets

\[ \Phi(x') - \Phi(x) - \epsilon f(x')\tilde{f}(x) = 0. \]  

Differentiating (46) with respect to \( x_1 \), dividing the result by \( f(x')\tilde{f}(x) \) and using (40), (41), one gets

\[ \frac{\Phi_x'(x') - \Phi_x(x)}{\Phi(x') - \Phi(x)} = \frac{\Phi_y'(x') - \Phi_y(x) - \Phi_{xx}(x)}{2\Phi_x'(x')}. \]  

It is also possible to obtain the finite form of KP hierarchy in terms of the \( \tau \)-function. Substituting the expression of the function \( \chi(\lambda, \mu) \) through the \( \tau \)-function

\[ \chi(\lambda, \mu) = \frac{\tau(g(\nu) \times (\frac{\nu-\lambda}{\nu-\mu}))}{\tau(g(\nu))(\lambda - \mu)} \]  

(see e.g. [10], [17]; \( \tau \)-function is a functional of the function \( g(\nu) \) or, in other words, a function of \( x \)) into equation (24), one gets

\[ \lambda(\mu - \epsilon)\tau(x^{(5)})\tau(x^{(0)}) + \mu(\epsilon - \lambda)\tau(x^{(4)})\tau(x^{(1)}) + \epsilon(\lambda - \mu)\tau(x^{(3)})\tau(x^{(2)}) = 0, \]  

(49)
The expansion of (49) in $\epsilon$, $\lambda$, $\mu$ gives the KP hierarchy in the form of Hirota bilinear equations.

Equation (49) is equivalent to one of the addition formulae for the $\tau$-function found in [1].

5 Combescure transformations for the KP-mKP hierarchy

Let us consider now the symmetries of the equations derived above.

Since $\rho(\lambda)$ and $\tilde{\rho}(\mu)$ are arbitrary functions, the equation (45) and the hierarchy (47) possess the symmetry transformation $\Phi(\rho(\lambda), \tilde{\rho}(\mu)) \to \Phi(\rho'(\lambda), \tilde{\rho}'(\mu))$.

The Combescure transformation can be characterized in terms of the corresponding invariants. The simplest of these invariants for the mKP equation is just the potential of the KP equation $L$-operator expressed through the wave function

\[ u = \frac{f(x)y - f(x)xx}{f(x)}, \quad (50) \]

or, in terms of the solution for the mKP (dual mKP) equation

\[ v' + v'_{xx} - \frac{1}{2}(v')^2 = v_y + v_{xx} - \frac{1}{2}(v^2)_x, \quad (52) \]
\[ \tilde{v}' - \tilde{v}'_{xx} - \frac{1}{2}((\tilde{v}')^2) = \tilde{v}_y - \tilde{v}_{xx} - \frac{1}{2}((\tilde{v})^2)_x. \quad (53) \]

The solutions of the mKP equations are transformed only by a subgroup of the Combescure symmetry group corresponding to the change of the weight function $\tilde{\rho}(\mu)$ (left subgroup) and they are invariant under the action of the subgroup corresponding to $\rho(\lambda)$ (vice versa for the dual mKP).
All the hierarchy of the Combescure transformation invariants is given by the expansion over \( \epsilon \) near the point \( x \) of the relation (25) rewritten in the form

\[
\frac{\partial}{\partial \epsilon} \left( \frac{f(x') - \tilde{f}(x)}{\epsilon f(x)} \right) = -\frac{1}{2} \frac{\partial}{\partial \epsilon} \partial_{x'}^{-1} u(x'), \quad x'_i - x_i = \frac{1}{i} \epsilon^i; \\
\frac{\partial}{\partial \epsilon} \left( \frac{f(x) - f(x')}{\epsilon f(x)} \right) = \frac{1}{2} \frac{\partial}{\partial \epsilon} \partial_{x'}^{-1} u(x'), \quad x'_i - x_i = -\frac{1}{i} \epsilon^i.
\]

(54)

(55)

The expansion of the left part of these relations gives the Combescure transformation invariants in terms of the wave functions \( \tilde{f}, f \). To express them in terms of mKP equation (dual mKP equation) solution, one should use the formulae

\[
v = -2 \frac{\tilde{f}_x}{f}, \quad \tilde{f} = \exp(-\frac{1}{2} \partial_x^{-1} v);
\]

(56)

\[
\tilde{v} = 2 \frac{f_x}{f}, \quad f = \exp(\frac{1}{2} \partial_x^{-1} \tilde{v}).
\]

(57)

It is also possible to consider special Combescure transformations keeping invariant the KP equation (dual KP equation) wave functions (i.e., solutions for the dual mKP (mKP) equations). The first invariants of this type are

\[
\frac{\Phi'(x)}{f'(x)} = \frac{\Phi_x(x)}{f(x)};
\]

(58)

\[
\frac{\Phi'(x)}{f'(x)} = \frac{\Phi_x(x)}{f(x)}.
\]

(59)

All the hierarchy of the invariants of this type is generated by the expansion of the left part of the following relations over \( \epsilon \)

\[
\frac{\Phi(x') - \Phi(x)}{f(x)} = \epsilon f(x'), \quad x'_i - x_i = \frac{1}{i} \epsilon^i;
\]

(60)

\[
\frac{\Phi(x) - \Phi(x')}{f(x)} = \epsilon \tilde{f}(x'), \quad x'_i - x_i = -\frac{1}{i} \epsilon^i.
\]

(61)

Now let us consider the equation (45) and all the hierarchy given by the relation (17). This equation admits the Combescure group of symmetry transformations \( \Phi(\rho(\lambda), \tilde{\rho}(\mu)) \rightarrow \Phi' = \Phi(\rho'(\lambda), \tilde{\rho}'(\mu)) \) consisting of two subgroups (right and left Combescure transformations). These subgroups have the following invariants

\[
v = \frac{\Phi_y - \Phi_{xx}}{\Phi_x}.
\]

(62)
and
\[ v = \frac{\Phi_y + \Phi_{xx}}{\Phi_x}. \]  

(63)

From (40), (41) it follows that they just obey the mKP and dual mKP equation respectively. The invariant for the full Combescure transformation can be obtained by the substitution of the expression for \( v \) via \( \Phi \) (62) to the formula (52). It reads
\[ u = \partial_x^{-1} \left( \frac{\Phi_y}{\Phi_x} \right) y - \frac{\Phi_{xxx}}{\Phi_x} + \frac{\Phi^2 - \Phi_x^2}{2\Phi_x^2}. \]

(64)

From (31), (32), (50), (51), (52), (53) it follows that \( u \) solves the KP equation.

So there is an interesting connection between equation (45), mKP-dual mKP equations and KP equation. Equation (45) is the unifying equation. It possesses a Combescure symmetry transformations group. After the factorization of equation (45) with respect to one of the subgroups (right or left), one gets the mKP or dual mKP equation in terms of the invariants for the subgroup (62), (63). The factorization of equation (45) with respect to the full Combescure transformations group gives rise to the KP equation in terms of the invariant of group (64).

In other words, the invariant of equation (45) under the full Combescure group is described by the KP equation, while the invariants under the action of its right and left subgroups are described by the mKP or dual mKP equations.

Thus the generalized hierarchy (47) plays a central role in the theory of the KP and mKP hierarchies.

Using the results of the paper [10], it is possible to get the formulae for the Darboux type transformation for the equation (45) in terms of its special solution \( \psi(\lambda, \mu) \). Indeed, a Darboux type transformation corresponds to
\[ g_d = \frac{\nu - b}{\nu - a}, \quad a, b \in D \]

(65)

(in fact \( a, b \) may also belong to regions not connected with \( D \), this case requires some additional definitions). The action of \( g_d \) (65) on the function \( \chi(\lambda, \mu; g) \) is given by the formula (see [10])
\[ \chi(\lambda, \mu; g \times g_d) = g_d^{-1}(\lambda)g_d(\mu) \frac{\det \left( \begin{array}{cc} \chi(\lambda, \mu; g) & \chi(\lambda, a; g) \\ \chi(b, \mu; g) & \chi(b, a; g) \end{array} \right)}{\chi(b, a; g)}. \]

(66)
In terms of the function $\psi(\lambda, \mu)$ we get

$$
\psi(\lambda, \mu; g \times g_d) = g_d^{-1}(\lambda)g_d(\mu)g(\lambda)g^{-1}(\mu) \times 
\det \begin{pmatrix}
  g^{-1}(\lambda)\psi(\lambda, \mu; g)g(\mu) & g^{-1}(\lambda)\psi(\lambda, a; g)g(a) \\
  g^{-1}(b)\psi(b, \mu; g)g(\mu) & g^{-1}(b)\psi(b, a; g)g(a)
\end{pmatrix}
\frac{g^{-1}(b)\psi(b, a; g)g(a)}
$$

(67)

where $g$ is given by

$$
g(x, \lambda) = \exp \left( \sum_{i=1}^{\infty} x_i \lambda^{-i} \right).
$$

The formula (67) determines a Darboux type transformation for equation (45) in terms of the function $\psi(\lambda, \mu)$. We would like to take notice of the fact that the functions $\psi(\lambda, a; g)$, $\psi(\lambda, \mu; g)$, $\psi(b, \mu; g)$ are connected with the function $\psi(\lambda, \mu; g)$ by the Combescure transformation (left, right and their combination). So the formula (67) demonstrates an intriguing connection between the Darboux and the Combescure transformations for equation (45).

6 Davey-Stewartson - modified Davey-Stewartson hierarchy. The Ishimori equation

Now we will consider the 2-component extension of the KP-mKP hierarchy. We take a set of two identical unit disks with the center at $\lambda = 0$ $D_+$, $D_-$ as $G$. The functions $\lambda_+(\lambda)$, $\lambda_-(\lambda)$ are chosen in the form

$$
\lambda_+(\lambda) = \lambda, \quad \lambda \in D_+;
\lambda_-(\lambda) = \lambda, \quad \lambda \in D_-;
\lambda_+(\lambda) = 0, \quad \lambda \in D_-;
\lambda_-(\lambda) = 0, \quad \lambda \in D_+.
$$

The DS-mDS hierarchy is generated by

$$
g(x, \lambda) = \exp \left( \sum_{i=1}^{\infty} \left( x_i^+ \lambda_+^{-i} + x_i^- \lambda_-^{-i} \right) \right).
$$

Let us take

$$
g_1 g_2^{-1} = \exp \left( \sum_{i=1}^{\infty} \left( \frac{\epsilon_i^+}{i \lambda_+^i} + \frac{\epsilon_i^-}{i \lambda_-^i} \right) \right) = \left( 1 - \frac{\epsilon_+}{\lambda_+} \right) \left( 1 - \frac{\epsilon_-}{\lambda_-} \right).
$$
Substituting this function to the Hirota bilinear identity (1), we get
\[ \psi(\lambda, \mu, x') - \psi(\lambda, \mu, x) = \epsilon_+ \psi(\lambda, 0_+, x')\psi(0_+, \mu, x) + \epsilon_- \psi(\lambda, 0_-, x')\psi(0_-, \mu, x); \]
\[ (x_i^+)' - x_i^+ = \frac{1}{i} \epsilon_i^+; \quad (x_i^-)' - x_i^- = \frac{1}{i} \epsilon_i^- . \] (68)

The expansion of this relation over \( \epsilon_+, \epsilon_- \) generates the DS-mDS hierarchies (and dual hierarchies) and linear problems for them.

The DS equation in the usual form is written in terms of the variables
\[ \xi = \frac{1}{2}(x + y) = x^+_1, \quad \eta = \frac{1}{2}(y - x) = x^-_1, \quad t = -\frac{i}{2}(x^+_2 - x^-_2) . \] The DS hierarchy in the form (68) incorporates also the modified Veselov-Novikov hierarchy.

In the standard DS coordinates one gets from (68)
\[ \psi(\lambda, \mu, x)_\xi = \psi(\lambda, 0_+, x)\psi(0_+, \mu, x), \] (69)
\[ \psi(\lambda, \mu, x)_\eta = \psi(\lambda, 0_-, x)\psi(0_-, \mu, x), \] (70)
\[ iv(\lambda, \mu, x)_t = \frac{1}{2}(\psi(\lambda, 0_+, x)_\xi \psi(0_+, \mu, x) - \psi(\lambda, 0_+, x)\psi(0_+, \mu, x)_\xi - \psi(\lambda, 0_-, x)_\eta \psi(0_-, \mu, x) + \psi(\lambda, 0_-, x)\psi(0_-, \mu, x)_\eta) . \] (71)

Just as in the DZM system case, from (69), (70) one obtains the DS and dual DS spatial linear problems
\[ \partial_\eta f_- = uf_+ \quad \partial_\eta \tilde{f}_- = v\tilde{f}_+ \]
\[ \partial_\xi f_+ = v f_- \quad \partial_\xi \tilde{f}_+ = u \tilde{f}_- \] (72)

Here \( v = \psi(0_-, 0_+) \), \( u = \psi(0_+, 0_-) \).

Similarly to the KP equation case, (71) gives a time linear problem for the DS equation
\[ if_{+t} - \frac{1}{2} f_{+\xi \xi} + \frac{1}{2} f_{+\eta \eta} = (\partial_{\eta}^{-1}(uv)\xi)f_+ - v_\eta f_-, \]
\[ if_{-t} - \frac{1}{2} f_{-\xi \xi} + \frac{1}{2} f_{-\eta \eta} = -(\partial_{\xi}^{-1}(uv)\eta)f_- + u_\xi f_+ . \] (73)

The compatibility condition for the equations (72), (73) gives the DS equation (in fact it is even easier to obtain it directly from (68)-(71))
\[ iv_t - \frac{1}{2} v_{\xi \xi} - \frac{1}{2} v_{\eta \eta} = -((\partial_{\xi}^{-1}(uv)\eta) + (\partial_{\eta}^{-1}(uv)\xi))v, \]
\[ iu_t + \frac{1}{2} u_{\xi \xi} + \frac{1}{2} u_{\eta \eta} = ((\partial_{\eta}^{-1}(uv)\xi) + (\partial_{\xi}^{-1}(uv)\eta))u. \] (74)
The spatial and time linear problems for the mDS - dual mDS case read
\[ \Phi_{\eta \xi} = U_\xi \Phi_\eta + V_\eta \Phi_\xi, \quad (75) \]
\[ i \Phi_t + \frac{1}{2} \Phi_{\eta \eta} - \frac{1}{2} \Phi_{\xi \xi} = V_\eta \Phi_\eta - U_\xi \Phi_\xi, \quad (76) \]
and
\[ \Phi_{\xi \eta} = \tilde{U}_\xi \Phi_\eta + \tilde{V}_\eta \Phi_\xi, \quad (77) \]
\[ i \Phi_t - \frac{1}{2} \Phi_{\eta \eta} + \frac{1}{2} \Phi_{\xi \xi} = -\tilde{V}_\eta \Phi_\eta + \tilde{U}_\xi \Phi_\xi. \quad (78) \]
Here \( V = \log \tilde{f}_+, \, U = \log \tilde{f}_-, \, \tilde{V} = \log f_+, \, \tilde{U} = \log f_- \). The compatibility condition for the equations (75), (76) gives the equations \[ \begin{aligned}
(iU_t - \frac{1}{2} U_{\xi \xi} - \frac{1}{2} U_{\eta \eta} - \frac{1}{2} U_\eta^2 + U_\eta V_\eta)_{\eta} + (U_\eta V_\xi)_{\xi} &= 0, \\
(iV_t + \frac{1}{2} V_{\xi \xi} + \frac{1}{2} V_{\eta \eta} - \frac{1}{2} V_\eta^2 + U_\xi V_\xi)_{\xi} + (U_\eta V_\xi)_{\eta} &= 0. \quad (79)
\end{aligned} \]
which can be treated as the modified DS equation. This system and its connection with the DS equation has been analyzed in \[ \text{[17]} \]. The dual modified DS equation can be obtained from the (79) by the substitution \( V \rightarrow \tilde{V}, \, U \rightarrow \tilde{U}, \, t \rightarrow -t, \, \xi \rightarrow -\xi, \, \eta \rightarrow -\eta. \)

Solutions for this system are given in terms of dual DS wave functions (DS wave functions) Thus there is a Combescure transformations group acting on the space of solutions. The simplest Combescure invariants are
\[ \frac{\partial_\eta \tilde{f}_+}{\tilde{f}_+} = V_\eta \exp(V - U), \quad (80) \]
\[ \frac{\partial_\xi \tilde{f}_+}{\tilde{f}_-} = U_\xi \exp(U - V). \quad (81) \]
The hierarchy of the Combescure transformation invariants is generated by the relations
\[ \left( \frac{\tilde{f}_+(x') - \tilde{f}_+(x)}{\tilde{f}_-(x)} \right) = \epsilon_+ u(x'), \quad (82) \]
\( (x_+^i)' - x_+^i = \frac{1}{t} e_+^i; \quad x_+^i - (x_-^i)' = 0; \)
\[ \left( \frac{\tilde{f}_-(x') - \tilde{f}_-(x)}{\tilde{f}_+(x)} \right) = \epsilon_- v(x'), \quad (83) \]
\( (x_-^i)' - x_-^i = \frac{1}{t} e_-^i; \quad (x_+^i)' - x_+^i = 0. \)
If one takes a pair of wave functions $\tilde{f}_+, \tilde{f}_-$ and $\tilde{f}'_+, \tilde{f}'_-$, the matrix

$$
\Psi = \begin{pmatrix} \tilde{f}_+ & \tilde{f}'_+ \\ \tilde{f}_- & \tilde{f}'_- \end{pmatrix}
$$

is connected with the solution of the Ishimori equation by the formula (see e.g. [17])

$$
S_1\sigma_1 + S_2\sigma_2 + S_3\sigma_3 = -\Psi^{-1}\sigma_3\Psi
$$

(for real $S$ some reduction conditions should be satisfied). In principle it could be possible to express Combescure invariants for mDS equation in terms of solution for the Ishimori equation and thus obtain Combescure invariants for the Ishimori equation, but it is unclear in this case whether the Combescure transformation survives under reduction conditions.

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