Painlevé-II approach to binary black hole merger dynamics: universality from integrability

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The binary black hole merger waveform is both simple and universal. Adopting an effective asymptotic description of the dynamics, we aim at accounting for such universality in terms of underlying (effective) integrable structures. More specifically, under a “wave-mean flow” perspective, we propose that fast degrees of freedom corresponding to the observed waveform would be subject to effective linear dynamics, propagating on a slowly evolving background subject to (effective) non-linear integrable dynamics. The Painlevé property of the latter would be implemented in terms of the so-called Painlevé-II transcendent, providing a structural link between i) orbital (in particular, EMRI) dynamics in the inspiral phase, ii) self-similar solutions of non-linear dispersive Korteweg-de Vries-like equations (namely, the ‘modified Korteweg-de Vries’ equation) through the merger and iii) the matching with the isospectral features of black hole quasi-normal modes in late ringdown dynamics. Moreover, the Painlevé-II equation provides also a ‘non-linear turning point’ problem, extending the linear discussion in the recently introduced Airy approach to binary black hole merger waveforms. Under the proposed integrability perspective, the simplicity and universality of the binary black hole merger waveform would be accounted to by the ‘hidden symmetries’ of the underlying integrable (effective) dynamics. In the spirit of asymptotic reasoning, and considering Ward’s conjecture linking integrability and self-dual Yang-Mills structures, it is tantalizing to question if such universal patterns would reflect the actual full integrability of a (self-dual) sector of general relativity, ultimately responsible for the binary black hole waveform patterns.

I. BINARY BLACK HOLE MERGERS: UNIVERSALITY AND SIMPLICITY

The gravitational binary black hole (BBH) merger waveform is remarkable simple. This is confirmed by numerical simulations of black hole mergers within general relativity [1], and also by observations of the emitted gravitational radiation [2]. The inspiral regime can be very complicated especially if one begins with a complicated initial configuration with high eccentricity, spins, etc. However, by the time we reach the merger, gravitational radiation emission has served to circularize the orbit and eventually the complexity of the initial configuration is lost. This argument suggests that the merger waveform is simple and in fact also universal.

Such simplicity is consistent with existing treatments of BBH dynamics, but a clear cut identification of the mechanism(s) behind it is still missing. In a companion article [3], a catastrophe-theory model based on the structural stability of fold caustics — and of the diffraction patterns upon such caustics— has been proposed to effectively account for such universality and simplicity of the BBH merger waveform. Such a model leads, in particular, to a phenomenological BBH waveform proposal based on a reparameterized Airy function, in what might be referred to in [3] as a ‘post-Airy expansion’ approach. Thus, the Airy function serves as a leading order approximation to the merger waveform, and more accurate approximations can be obtained perturbatively from this starting point.

The goal here is to advance in the understanding of this problem from a different perspective. Whereas the Airy model focused on structural features related to the phase function of the propagating wave and was largely (but not completely) independent of the details of the underlying gravitational dynamics, the present work aims at targeting the specific structural properties of the BBH gravitational system responsible of the universality and simplicity of the BBH merger. In other words, whereas the Airy model was explicitly meant as an effective treatment aiming at simplifying the functional modelling of the waveform with data analysis purposes, the present discussion aims at unveiling the underlying fundamental mechanisms and structures behind simplicity and universality in BBH dynamics.

Such effective and fundamental approaches are not in contraposition, but are indeed complementary. In particular, at a methodological level, they share an asymptotic reasoning strategy [4] aiming at filtering the overload of details possibly encumbering structural features in complete approaches. In such an asymptotic approach a theory is typically described in a certain limit of a small (or large) parameter, explicitly sacrificing precision and exactness by eliminating details, in order to make underlying patterns explicitly apparent in the appropriate range of the asymptotic parameter, often entailing a gain in the mathematical tractability and, more importantly in the present context, identifying at the asymptotic regime the mathematical structure actually present in the complete dynamics. In sum, both the effective approach [3] and the first-principles here adopt an asymptotic reasoning methodology, but with a focus on different levels of description.

A. From Airy waveforms to Painlevé-II dynamics

In [3] a fold-caustic diffraction model on the phase of the waveform led to an Airy-like model for the BBH waveform,
Specifically, dwelling in a geometric optics setting and adopting the language of catastrophe theory, caustics permit to account for the topological structure underlying the abrupt illumination-darkness transition in a detector. Then, taking a first step from geometric optics to wave theory, diffraction on a caustic regularizes the caustic divergence and provides universal diffraction pattern only depending on the topology of the caustic. Building on these elements, we proposed in [3] a simple model for the BBH merger waveform based on the assumption of a fold-caustic structure in-built in the phase-delay function describing the gravitation wave propagation, identifying the Airy function as the reference functional form to capture the qualitative features of the waveform merger. Being based on a first order correction to geometric optics, such a treatment is explicitly a high-frequency one. It was only in the asymptotic reasoning spirit that such an Airy-like pattern was claimed in [3] to extend to all frequencies.

In this article we provide a more robust justification of such extension of the Airy-like model to arbitrary BBH waveform frequencies. Building on the solid catastrophe-theory identification of the Airy function in the high-frequency limit, we address the extension of the phenomenological model to all frequencies by adopting the following simple assumption: the BBH waveform smoothly (and linearly) transitions from an oscillatory regime to a damped regime. Adopting the language of (semi-)classical treatments of dynamics, such dynamical transitions from oscillatory to damped regimes define a so-called ‘turning point’ problem. The Airy equation

\[
\frac{d^2 u}{d\tau^2} - \tau u = 0 ,
\]

provides precisely the universal leading-order of linear turning points. Here \(\tau\) is a reparameterized time parameter. In section II.A we recover the full functional dependence of BBH merger waveform derived in caustic diffraction model, through a ‘turning point’ ordinary differential equation (ODE) reasoning not restricted to high frequencies.

The methodology described above can indeed be seen as another instance of asymptotic reasoning in which, instead of simply extending the high-frequency result to all frequencies, we have rather identified the fundamental analytical structure underlying the Airy function, namely the Airy equation, and promoted it as the guiding element to extend the phenomenological Airy description of the BBH waveform to all frequencies. This more abstract version of asymptotic reasoning, in which it is the underlying fundamental structure and not the waveform functional form itself, leads to the prototypical argument that we shall follow.

Actually, the ODE ‘turning point’ change of perspective has profound implications. Indeed, the path to the Airy equation (3) relies strongly on the assumption of linearity of waveform, supported by accumulated insights [5,6,7]. But we can refine this notion. Let us consider that:

i) The effectively linearity of the propagating wave occurs on an effective background whose dynamics are indeed non-linear. This provides an effective separation between fast (linear propagating wave) and slow (non-linear background) degrees of freedom.

ii) The actual ‘turning point’ dynamics happens at the level of nonlinear background dynamics and, only at a second stage, are imprinted in the propagating linear wave.

In this perspective, the Airy equation (9) would represent the linear imprint in the propagating wave of the underlying background ‘turning point’ dynamics. The question of a non-linear version of ‘turning point’ dynamics is naturally posed.

If the Airy equation (9) represents the “archetype” of linear turning problem, the so-called Painlevé-II equation, namely

\[
\frac{d^2 w}{d\tau^2} - \tau w - 2w^3 - \alpha = 0 ,
\]

with \(\alpha\) an arbitrary constant, can be seen as an archetype of “non-linear turning point” [10]. In the same way that Eq. (9) defines (under the appropriate asymptotic conditions) the Airy special function, Eq. (2) defines a new transcendental special function \(w(\tau)\) known as the Painlevé-II transcendental (note that, in full generality, \(\tau\) must be seen as a complex variable). This equation reduces upon linearization (in the case \(\alpha = 0\)) to the Airy one, and the Painlevé-II transcendental has asymptotic behaviour controlled indeed by the Airy function (see section III.A for a better assessment of this statement). This special function, together with the other five Painlevé transcendents (see e.g. [11]), plays a prominent role in mathematical physics, in particular in connection with integrable systems. This immediately prompts a new perspective on universality and simplicity of BBH mergers. Specifically, once adopted an non-linear ODE ‘turning point’ perspective to the BBH merger dynamics, the following question is posed: could non-linear Painlevé transcendental, and more specifically Painlevé-II, account for the observed particular features of the BBH dynamics?

The previous question is, admittedly, a bold one, and can be considered only as a conjecture for now. The remarkable fact is that it is indeed supported by existing results. A first instance is presented in a notable article by Rajeev [12], in the setting of the damped orbital motion of a charged particle in a Coulomb potential, subject to radiation reaction. Specifically, Rajeev shows that the Landau-Lifshitz equations for a charged particle moving in a Coulomb potential can be solved exactly in terms of the Painlevé-II equation with \(\alpha = 0\)

\[
\frac{d^2 w}{d\tau^2} - \tau w - 2w^3 = 0 .
\]

Moving now to general relativity itself, consider the extreme mass ratio case where we have a small compact object orbiting around a large black hole. The motion of the small object can be viewed as a sequence of slowly evolving geodesics, suffering the effects of radiation damping, and losing energy and angular momentum in the form of gravitational radiation. However, when it gets sufficiently close to the large black hole, the motion transitions to a plunge. This picture was suggested by Thorne and Ori in Ref. [13]. In the inspiral regime, the motion is governed by an effective potential \(V(\tau, L)\) with \(\tau\) being a radial coordinate and \(L\) the angular momentum. This potential has a minimum in the inspiral regime but as \(L\) decreases, the minimum changes to a saddle point and then disappears. Thorne and Ori showed that in this transition regime,
the dynamics is governed by an equation that, as noticed by Compère and Küchler [14,15], can be rewritten as

\[
\frac{d^2u}{dx^2} - \tau - 6u^2 = 0 ,
\]

namely the Painlevé-I equation. The relevant Painlevé-I transcendent in the plunge problem is, in particular, the so-called \textit{tritronquée} solution to Painlevé-I [16], a function associated with universality in critical phenomena of certain partial differential equations, a remarkable feature by itself. On the other hand, it can be shown that the Painlevé-I equation (4) can be obtained as a (singular) limit of Painlevé-II (5). In view of these elements, it becomes plausible that the orbital dynamics is ultimately governed by the second Painlevé equation. This will be discussed in section 1.1.3.

A natural criticism to the previous discussion refers to its restriction to the extreme mass ratio inspiral (EMRI) limit of the BBH dynamics. This is indeed the case and we claim that it is precisely the focus on this asymptotic limit of vanishing mass ratio, that allows to identify —as a new instance of asymptotic reasoning—the relevant underlying pattern in the full dynamics, namely: \textit{the fundamental structural role of Painlevé transcendent}—and in particular of Painlevé-II—in BBH dynamics, supports the study of BBH mergers in the setting of integrable or quasi-integrable systems.

**B. Universal wave patterns: from ODEs to PDEs**

Painlevé transcendent are naturally introduced in an ODE setting, as in the “turning point” problems discussed above. However, their structural role transcends ODEs, extending to very different problems threaded by the notion of integrability. In our particular BBH problem, if the inspiral phase (at least, up to the plunge) admits natural treatments in terms of ODEs, the situation for the merger and ringdown phases is less clear and a partial differential equation (PDE) treatment could be better adapted. Remarkably, Painlevé transcendent provide a structural link to an important class of non-linear integrable PDEs with a key role in the integrability properties of BHs.

1. Universal wave patterns in dispersive non-linear PDEs

Before tackling the discussion of Painlevé transcendent in BBH mergers, let us make an interlude into dispersive non-linear hydrodynamics, specifically in the context of dispersive shock waves [17].

A remarkable universality feature in this setting is that \textit{all shocks look the same} [18]. Specifically, let us consider the Burgers’ equation with viscosity term regularization

\[
\partial_t u + u \partial_x u = \epsilon \partial_{xx}^2 u .
\]

Inviscid Burgers equation ($\epsilon = 0$) with initial data $u_0(x) = u(0, t)$ develops a shock propagating along a curve $x = x_c(t)$ depending on $u_0(x)$. If we denote $u_-(t) = \lim_{x \to -\infty} u(x, t)$, $u_+(t) = \lim_{x \to +\infty} u(x, t)$, and we zoom around the “critical” shock position $x_c(t)$ in the solution to the regularized equation (5), one finds the solution is (e.g. [19,20])

\[
u(\xi, t) = \lim_{\epsilon \to 0} u(x_c(t) + \epsilon \xi, t)
= \bar{u}(t) - \frac{\Delta u(t)}{2} \tanh \left( \frac{(\xi - \xi_0(t)) \Delta u(t)}{4} \right),
\]

where $u_0(t) = (u_-(t) + u_+(t))/2$, $\Delta u(t) = u_+(t) - u_-(t)$ and $\xi_0(t)$ is a function that can be fixed at higher orders. The remarkable facts are that, in “appropriate” coordinates:

i) The form is fully independent of the initial data $u_0(x)$.

ii) Such form is given by a universal special function, specifically the (elementary) special function $\tanh(x)$.

Initial data enter through $x_c(t), u_-(t), u_+(t)$ and $\xi_0(t)$ in the reparametrization of the function describing the shock.

One could think that such wave pattern universality is a very peculiar property of shocks, but actually it is not. Indeed, such universality is a generic feature associated with a PDE-type of critical behaviour [19] occurring at transitional dynamical wave patterns in integrable dispersive equations [21].

2. Painlevé-II and universal BBH mergers

At this point we come back to the Painlevé-II transcendent, identified as a key structure of BBH dynamics in the ODE inspiral phase, but now from a (merger) PDE perspective.

In the light of discussion above on universal wave patterns, we explore the possibility that the observed universality of BBH merger dynamics could respond to a PDE-type of critical behaviour. More specifically, we consider if non-linear background BBH dynamics can be encoded in an effective PDE—alogous role to Burgers’ Eq. (5)—such that the BBH merger transient—analogue to the Burgers’ shock transient—can be understood in terms of a PDE-type critical behaviour with universal dynamics encoded in the Painlevé-II transcendent—special function analogous to $\tanh(x)$ in Eq. (6).

In section 1.1.3 we will explore this possibility under two radical assumptions: i) \textit{dispersive nature} of the effective background non-linear dynamics, and ii) \textit{self-similar character} of the solution at the (“critical”) merger transition. The first assumption is directed motivated by the Airy-BBH model in [3] and the fact that Airy function controls as the generic first-order behaviour of all dispersive systems. Ultimately, such dispersive character should be justified in terms of an (asymptotic) coarse-grained PDE description where integration over small scales would result on an effective dispersion. Regarding the self-similar assumption, it is motivated by the assumed critical behaviour at the merger transient [23].

Regarding the non-linear dispersive Ansatz, a natural “universal” PDE model is the Korteweg-de Vries (KdV) equation

\[
\partial_t u + 6u \partial_x u + \partial_{xx}^2 u = 0.
\]

More specifically, KdV-type equations give the leading-order asymptotic equations of weakly dispersive and weakly non-linear PDE dynamics. In this setting, and in order to incorporate the second (self-similarity) assumption in connection
with the Painlevé-II transcendent, the natural candidate is the so-called modified-Korteweg-de Vries (m-KdV) equation by considering integrability from a larger scope as a guiding system determined by Painlevé-II. We conclude in section V to the BBH merger dynamics in terms of a (quasi-)integrable form. This configures a self-contained bottom-up approach principle to match our 'bottom-up' approach to BBH dynamics, cast in two complementary realizations: a) the ODE controlling the inspiral and transition to the plunge (in its Painlevé-I limit), and b) the universal PDE wave pattern in the merger transient to the ringdown. Painleve-II is therefore the key guideline to approach BBH mergers as (quasi-)integrable systems.

The plan of the article is the following. In section II we revisit the Airy-function approach to BBH merger waveform, taking an ‘turning point’ ODE approach that permits to extend results in [3] to all frequencies. Once in the ODE setting, in section III we start by reviewing of Painlevé transcendents and Painlevé-Rajeev’s solution of radiation-reaction charged particle orbital motion in terms of Painlevé-II transcendent in an inverse scattering transform (IST) scheme [24, 25]. This scattering perspective connects with the mathematical framework of the late ringdown phase, naturally described in a (direct) scattering theory setting [26]. In section IV has a more heuristic flavor, in which a separation of fast and slow degrees is proposed, in such a way that slow quasi-circular orbits, leading to the identification of Painlevé-II as well as, in a gravitational setting, the emergence of Painlevé-I in the transition to the plunge of EMRIs. Then we extend Rajeev’s discussion to the gravitational radiation reaction setting, in the setting of quasi-circular orbits, leading to the identification of Painlevé-II as a fundamental structure in BBH merger dynamics. Section [27] has a more heuristic flavor, in which a separation of fast and slow degrees is proposed, in such a way that slow background BBH dynamics is approached in terms of a non-linear dispersive integrable equation solvable in terms of the Painlevé-II transcendent through an inverse scattering transform. This configures a self-contained bottom-up approach to the BBH merger dynamics in terms of a (quasi-)integrable system determined by Painlevé-II. We conclude in section V by considering integrability from a larger scope as a guiding principle to match our ‘bottom-up’ approach to BBH dynamics and waveform with a ‘top-down’ scheme to be developed.

II. BBH ‘TURNING POINT’ MODELS: FROM AIRY TO PAINLEVÉ-II

A. Linear oscillation-to-damped transition: Airy case

In ref. [3] we adopted an asymptotic reasoning approach to extend the the Airy model for the BBH waveform merger, based on the diffraction on caustics and valid for high frequencies, to all frequencies. Let us take here rather an “agnostonic” perspective regarding the underlying mechanism driving the BBH merger dynamics and focus on the general structural fact that this dynamics describe a dynamical transition from an oscillatory regime of the system to a regime with damped sinusoids in a continuous and smooth way.

In semiclassical treatments of dynamics such transitions are related to so-called “turning points”. The Airy equation

$$\frac{d^2 u}{d\tau^2} - \tau u = 0,$$

is the archetype of a linear turning point. We focus here on a more general class of linear ODEs describing, in an effective way, a turning point problem. Namely, we consider equations of the form [27]

$$\epsilon^2 \frac{d^2 u}{d\tau^2} - (\tau \phi(\tau) - \epsilon \psi(\tau, \epsilon)) u = 0, \quad \phi(0) \neq 0,$$

where \( \epsilon \) is a small parameter and \( \phi(\tau) \) and \( \psi(\tau, \epsilon) \) are given analytic functions, encoding in an effective manner the physical features of the turning point problem. The Airy equation is recovered by parametrizing \( \tau \to \tau/\epsilon \) and setting \( \phi(\tau) = \epsilon \psi(\tau, \epsilon) = 0 \). We perform a first order reduction of (10) by introducing the variable

$$Y = \begin{pmatrix} u \\ e^{-\tau} \end{pmatrix},$$

where \( \dot{u} = \frac{du}{d\tau} \), so that Eq. (10) writes

$$\epsilon \dot{Y} = \begin{pmatrix} 0 & 1 \\ \tau \phi(\tau) + \epsilon \psi(\tau, \epsilon) & 0 \end{pmatrix} Y \equiv A(\tau, \epsilon) Y,$$

this expression defining the matrix \( A(\tau, \epsilon) \). We note that in the Airy case

$$A_{\text{Airy}}(\tau) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Solutions to Eq. (12) can indeed be expressed in terms of Airy functions solutions to Eq. (9). Specifically, Theorem 29.1 in [27] states that, given a \( 2 \times 2 \) matrix \( A(\tau, \epsilon) \) as defined in (12), holomorphic in a neighborhood of \((0, 0)\) and such that

i) \( A(0, 0) \) is similar to the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

ii) It holds

$$\frac{d}{d\tau} \det A(\tau, 0) \mid_{\tau=0} \neq 0,$$

then solutions to the ’turning point’ problem of the form

$$\epsilon \dot{Y}(\tau) = A(\tau, \epsilon) Y(\tau),$$

can be written as

$$Y(\tau) = P(\tau)e^{\int_0^\tau A(\tau, 0) d\tilde{\tau}} Y(0),$$

where \( P(\tau) \) depends on initial data.
where \( t = \Phi(\tau) \), with \( \Phi(\tau) \) holomorphic at \( \tau = 0 \) with \( \Phi(0) = 0 \) and \( \Phi(0) \neq 0 \), and \( P(\tau) \) is a holomorphic matrix function (with inverse also holomorphic) at \( \tau = 0 \), and \( \tilde{Y}(t) \) satisfies the “instantaneous” Airy equation
\[
\epsilon \frac{d}{dt} \tilde{Y}(\tau) = \tilde{A}(t, \epsilon) \tilde{Y}(t) ,
\] (18)
with
\[
\tilde{A}(t, 0) = A_{\text{Airy}}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (19)
This shows that solutions to these linear turning point problems are generically linear combinations of the Airy function \( \text{Ai}(t) \) and its derivative \( \text{Ai}'(t) \), with appropriate modulations and argument reparametrizations depending on the specific details of the problem, effectively captured by functions \( \phi \) and \( \psi \). This matches precisely the structure found in ref. [3], but without the restriction to high frequencies.

The dynamical transition from oscillating to damped behavior is captured by the asymptotic behavior of the Airy function (for real argument), that we recall here for later comparison
\[
\text{Ai}(t) \sim \frac{1}{\sqrt{\pi t}} \sin \left( \frac{2}{3} t^{3} + \frac{\pi}{4} \right) \quad (t \to -\infty)
\]
\[
\text{Ai}(t) \sim \frac{1}{2\sqrt{\pi t}} e^{-\frac{2}{3} t^{3}} \quad (t \to +\infty).
\] (20)
Note that the overexponential damping at \( t \to +\infty \) precludes this linear turning point problem from accounting for the late ringdown exponential decay. Another mechanism is needed for the transition to the ringdown.

### B. Non-linear Airy: Painlevé-II 'turning point' model

As commented in \([1\alpha]\), the Painlevé-II equation \([2\alpha]\) and, more specifically, the \( \alpha = 0 \) case \([29]\), provides the archetype of non-linear turning point equation that reduces to the Airy discussed above case upon linearization. A key question is whether the non-linear equation \([69]\) has indeed solutions bounded for all real \( t \). Then, it is crucial to understand how to connect the behaviour of this solution at \( t \to -\infty \) to that at \( t \to +\infty \), to proceed to the “connection problem” in a non-linear analogue to the WKB “connection formulae” derived from \([29]\) in the linear Airy case. This requires a control of the asymptotics of this particular Painlevé-II transcendental.

Following \([1\alpha]\), the asymptotics of the solution to \([69]\) with boundary condition
\[
\lim_{t \to \infty} w(t) = 0
\] (21)
is
\[
w(t) \sim \frac{d}{|t|^{3}} \sin \left( \frac{2}{3} |t|^{3} - \frac{4}{3} d^{2} \ln(|t|) - \theta_{0} \right) \quad (t \to -\infty)
\]
\[
w(t) \sim \gamma \text{Ai}(t) \sim \frac{\gamma}{2\sqrt{\pi t}} e^{-\frac{2}{3} t^{3}} \quad (t \to +\infty)
\] (22)
where, given the constant \( \gamma \) with \( |\gamma| < 1 \) (for \( \gamma \geq 1 \) the solution to \([69]\) diverges at some intermediate point), the value of amplitude \( d \) and the phase shift \( \theta_{0} \) are given (“connection formulae”) by
\[
d^{2} = -\frac{1}{\pi} \ln(1 - \gamma^{2})
\]
\[
\theta_{0}(\gamma) = \frac{3}{2} d^{2} \ln 2 + \arg \left( \Gamma \left( 1 - i d^{2} \right) \right) - \frac{\pi}{4}.
\] (23)

It is interesting to compare the Airy asymptotics with Painlevé-II ones in \([20]\) and \([23]\). The non-linear turning point introduces both an asymptotique amplitude and phase shift with respect to the Airy case. In particular, the phase shift \( \theta_{0} \) corrects the standard \( e^{\pi/4} / WKB \) shift, something of potential relevance in the analysis of the BBH merger waveform.

### III. PAINLEVÉ TRANSCENDENTS IN DAMPED BINARY EMRI DYNAMICS

#### A. Painlevé transcendents

Apart from providing, as discussed in the previous section, a non-linear generalization of the turning-point problem defined by the linear Airy equation \([9]\), solutions to the Painlevé-II equation \([2\alpha]\) displays remarkable and fascinating mathematical properties shared with a finite number of other ‘Painlevé transcendents’, namely solutions to the following six Painlevé equations (respectively \( P_{1} - P_{VI} \))
\[
d^{2}w
\]
\[
\frac{d^{2}w}{dz^{2}} = 6w^{2} + z
\]
\[
\frac{d^{2}w}{dz^{2}} = 2w^{3} + zw + \alpha
\]
\[
\frac{d^{2}w}{dz^{2}} = \frac{1}{w} \left( \frac{dw}{dz} \right)^{2} - \frac{1}{z} \frac{dw}{dz} + \frac{\alpha w^{2} + \beta}{z} + \gamma w^{3} + \frac{\delta}{w}
\]
\[
\frac{d^{2}w}{dz^{2}} = \frac{1}{2w} \left( \frac{dw}{dz} \right)^{2} + \frac{3}{2} w^{3} + 4zw^{2} + 2(z^{2} - \alpha)w + \frac{\beta}{w}
\]
\[
\frac{d^{2}w}{dz^{2}} = \left( \frac{1}{2w} + \frac{1}{w - 1} \right) \left( \frac{dw}{dz} \right)^{2} - \frac{1}{w - 1} \frac{1}{w} \frac{dw}{dz}
\]
\[
+ \frac{\gamma w}{z} + \frac{\delta w(w + 1)}{w - 1}
\]
\[
\frac{d^{2}w}{dz^{2}} = \frac{1}{2} \left( \frac{1}{w} + \frac{1}{w - 1} + \frac{1}{w - z} \right) \left( \frac{dw}{dz} \right)^{2} - \left( \frac{1}{z} \frac{1}{w - 1} + \frac{1}{w - z} \right) \frac{dw}{dz}
\]
\[
+ \frac{w(w - 1)(w - z)}{z^{2}(z - 1)^{2}} \left( \alpha \frac{\beta w^{2} + \gamma(z - 1)}{(w - 1)w(z - 1)} + \frac{\delta z(z - 1)}{(w - z)^{2}} \right)
\]
where \( \alpha, \beta, \gamma \) and \( \delta \) are constants. A non-linear ODE is said to possess the so-called ‘Painlevé property’ \([3\alpha]\] if the only ‘movable’ singularity is poles. A singularity of a solution of a given ODE is ‘movable’ if it is not a priori fixed by the equation itself, so its location is fixed by initial data. The remarkable fact is that the Painlevé property \([32]\] severely constrains the form of the equation, so it can be reduced to one of fifty types of equations \([1\alpha]\) all of them integrable.
in terms of known functions except six: precisely the Painlevé equations $P_1 - P_{VI}$ in [24], which are irreducible to classical special functions and whose solutions define the Painlevé transcendents. The latter can be seen as a new class of nonlinear special functions [11].

The relevance of equations satisfying the Painlevé properties is that they can (often) be linearized or solved exactly. Specifically, for Painlevé equations can be studied by using the so-called isomonodromy method (see references in [11]) and, in this sense, they are understood as integrable equations. This simplicity and relation to integrability will be the key point in our later developments. For a recent review of the rich mathematical structures underlying the Painlevé equations, see reference [35].

B. Charged particle in Coulomb potential with radiation damping: Painlevé-II

The first contact between damped orbital dynamics and the Painlevé transcendent is presented in Rajeev’s remarkable article [12], in the setting of the orbital motion of a charged particle in an electrostatic field can be written in terms of the Painlevé-II equation. The Landau-Lifschitz equation sof motion of a radiating charged particle in an electrostatic field can be written in the non-relativistic limit as

$$\frac{d}{dt}(\vec{v} + \tau \nabla U) + \nabla U = 0 ,$$  \hspace{1cm} (25)

with $U = -\frac{k}{r}$ with $k = q/m$.

Let us revisit Rajeev’s argument in [12] by slightly extending its discussion in order to better seize the similarities and differences between the electromagnetic and gravitational cases. Let us start considering the equation

$$\frac{d}{dt}(\vec{v} + \tau \nabla W) + \nabla U = 0 ,$$  \hspace{1cm} (26)

where $\vec{v} = \frac{d\vec{r}}{dt}$, $U = U(r)$ and $W = W(r)$ are radial potentials and $\tau$ is a constant damping time parameter. This equation should be understood as the leading (linear) order in an asymptotic expansion in $\tau$ of the radiation-reaction dynamics. That is, higher-order powers in $\tau$ are neglected in the analysis. In particular, the non-relativistic Landau-Lifshitz equations for a charged particle in an electrostatic field in [12] are recovered for $U(r) = W(r) = \frac{k}{r}$.

In a first stage, rewriting (26) as

$$\frac{d}{dt}(\vec{v} + \tau \nabla W_r) + \vec{r}U = 0 ,$$  \hspace{1cm} (27)

and taking the vector (wedge) product of the equation with $\vec{r}$, we can write

$$\frac{d\vec{L}}{dt} = -\tau \frac{W_r}{r} \vec{L} ,$$  \hspace{1cm} (28)

where $\vec{L} = \vec{r} \wedge \vec{v}$ is the angular momentum. In particular, we note that in this approximation the orbital plane does not change.

In a second stage we use [28] to rewrite [29] as (use standard identities (8-10) in [12], and the notation $L = |\vec{L}|$)

$$\frac{d}{dt} v_r + \tau W_r L^2 - \frac{L^2}{r^3} - U_r ,$$  \hspace{1cm} (29)

Motivated by Rajeev’s electrostatic case and the gravitational setting discussed below, let us consider

$$U(r) = \frac{\alpha}{r^n} , \hspace{0.5cm} W(r) = \frac{\beta}{r^m} .$$  \hspace{1cm} (30)

Then, the last equation in (29) becomes

$$\frac{dL}{dt} = \frac{m \beta \tau}{r^{m+1}} \vec{L} ,$$  \hspace{1cm} (31)

and we can write

$$L^2 = \frac{1}{2 \beta m \tau} \left( \frac{d}{dt} r^{m-1} L^2 \right) - (m-1) L^2 r^{m-2} \frac{\beta}{r} ,$$  \hspace{1cm} (32)

so that the second equation in (29) writes

$$\frac{d}{dt} \left( v_r - \frac{m \beta \tau}{r^{m+1}} - \frac{1}{2 \beta m \tau} r^{m-1} L^2 \right) = \frac{n \alpha}{r^n} - \Delta ,$$  \hspace{1cm} (33)

where

$$\Delta = \frac{m-1}{2 \beta m \tau} r^{m-2} L^2 \frac{\beta}{r} .$$  \hspace{1cm} (34)

Defining $z$ as

$$z = v_r - \frac{m \beta \tau}{r^{m+1}} - \frac{1}{2 \beta m \tau} r^{m-1} L^2 ,$$  \hspace{1cm} (35)

we can therefore rewrite the dynamical equations as

$$\frac{dL}{dz} = \frac{\beta m \tau}{r^{m+2}} L ,$$  \hspace{1cm} (36)

$$\frac{dz}{dt} = \frac{n \alpha}{r^n} - \Delta ,$$  \hspace{1cm} (37)

At this point, the remarkable result of Rajeev is recovered by applying this to the electrostatic case, that is, by choosing $\alpha = \beta = k$ and $m = n = 1$. On the one hand, $\Delta = 0$ and eliminating the derivative in time, the electrostatic case can be written as

$$\frac{dL}{dz} = \tau L ,$$  \hspace{1cm} (38)

$$\frac{dz}{dz} = \frac{r^2 L^2}{2k^2 \tau} + \frac{r^2}{k} z + \tau .$$  \hspace{1cm} (39)

Finally, this leads to

$$\frac{d^2 L}{dz^2} = -\frac{1}{2k^2 \tau} L^3 - \frac{\tau}{k} z L ,$$  \hspace{1cm} (40)

namely the Painlevé-II equation, up to rescaling of $L$ and $z$. We note that no approximations have been made, once (29) has been adopted.
C. EMRI transition to plunge: Painlevé-I dynamics

After discussing the electrostatic case, we proceed to the gravitational setting by considering the limit of extreme mass ratio limit of binaries (EMRIs). Before adapting Rajeev’s argument to a gravitational setting, let us comment on other remarkable result on the connection between Painlevé transcendent and radiation damping orbital problems, namely the identification by Compère and Küchler \[14, 15\] of the role of Painlevé-I in the plunge equations originally derived by Ori and Thorne in \[13\].

Specifically, in \[13\] describes the dynamics of this system in a ‘transition regime’ near the innermost stable circular orbit (ISCO), between the ‘adiabatic inspiral regime’ and the ‘plunge regime’. The radial dynamics is then modelled by the geodesic equation corrected with a radial self-force, in a first stage. However, in a second stage the radial self-force is dropped under the heuristic justification of only involving a shift in the parameters of the effective potential. Adopting some appropriate dimensionless coordinates \((X, T)\), respectively encoding the radial distance to the ISCO and the proper time, the dynamics is (cf. Eq. (3.22) in \[13\])

\[
\frac{d^2 X}{dT^2} = -X^2 - T .
\]  

with asymptotic In a remarkable work, Compère and Küchler \[14, 15\] revisit the problem with two fundamental contributions: i) justifying that Eq. (39) actually holds as such even when all effects are systematically taken into account, and ii) identifying (39) as the Painlevé-I equation.

Indeed, the latter point connecting to Painlevé-I is justified through a rescaling of \(X\) and \(T\), namely introducing

\[
X = \alpha w , \quad T = \beta t ,
\]

the choice

\[
\alpha = -6^{\frac{1}{2}} , \quad \beta = 6^{\frac{1}{2}},
\]

leads to

\[
\frac{d^2 w}{dt^2} = 6w^2 + t ,
\]

namely the Painlevé-I equation in \[4\].

However, this is not enough to identify the specific Painlevé-I transcendent, relevant in this orbital transition to plunge setting. This involves choosing a particular solution to \[39\] under the pertinent boundary conditions. In this sense, Ori and Thorne look for the unique solution matching smoothly the adiabatic inspiral, concluding that the correct asymptotics are

\[
X \sim -\sqrt{T} , \quad T \to -\infty .
\]

The solution should be regular (in particular having poles) for all \(T < 0\). Remarkably this, together with the asymptotics \[43\] fixes the solution to Eq. (39), the relevant Painlevé-I transcendent is the so-called Boutroux’s tritronquée solution.

In particular it was proved in \[36\] that such tritronquée has no poles for negative values of \(T\), the first pole happening at \(t_0 \sim 2.3841687\) with monotonically decreasing behavior in \((-\infty, t_0)\) \[57\]. As a non-trivial check of the consistency with the solution found in \[13\], compare the described behavior of the tritronquée solution with the one of the numerical solution in Fig. 3 of \[13\], in particular diverging at \(T_{\text{plunge}}\)

\[
T_{\text{plunge}} = \beta t_0 = 6^{\frac{1}{2}} t_0 \sim 6^{\frac{1}{2}} \cdot 2.3841687 \sim 3.41167 \]  

(44)

(45)

(46)

using Eqs. (40) and (41) that refines the value in \[13\], namely \(T_{\text{plunge}} \sim 3.412\).

In sum, we identify here that the relevant Painlevé-I transcendent in the Painlevé-I equation appearing in the orbital transition to plunge at the ISCO \[13, 15\] is Boutroux’s tritronquée solution, known to be associated with universality properties \[16\] in the setting of critical behavior of Hamiltonian perturbations of nonlinear hyperbolic PDEs \[38\]. This offers a tantalizing avenue to connect the ODE orbital EMRI problem with universality properties of an underlying effective PDE dynamics describing finite mass-ratio binaries.

D. EMRI quasi-circular orbits: Painlevé-II dynamics

We explore now the adaptation of Rajeev’s argument to a gravitational setting. The idea in the present discussion is to make a first exploration of the EMRI problem by generalizing Rajeev’s by taking Eq. (26) as the starting point. This is a constraining condition and more systematic and general studies exploring systematically the Painlevé property of EMRI ODEs are needed, our discussion here only representing a first step but already illustrating the strategy.

In this spirit, a natural gravitational counterpart of the Landau-Lifshitz equations in the electrostatic setting consisting in considering the leading order in the Post-Newtonian approximation to a gravitational setting, let us comment on other remarkable result on the connection between Painlevé transcendent and radiation damping orbital problems, namely the identification by Compère and Küchler \[14, 15\] revisiting the problem with two fundamental contributions: i) justifying that Eq. (39) actually holds as such even when all effects are systematically taken into account, and ii) identifying (39) as the Painlevé-I equation.

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leads to

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\frac{d^2 w}{dt^2} = 6w^2 + t ,
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However, this is not enough to identify the specific Painlevé-I transcendent, relevant in this orbital transition to plunge setting. This involves choosing a particular solution to (39) under the pertinent boundary conditions. In this sense, Ori and Thorne look for the unique solution matching smoothly the adiabatic inspiral, concluding that the correct asymptotics are

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\]

The solution should be regular (in particular having poles) for all \(T < 0\). Remarkably this, together with the asymptotics (43) fixes the solution to Eq. (39), the relevant Painlevé-I transcendent is the so-called Boutroux’s tritronquée solution.

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T_{\text{plunge}} = \beta t_0 = 6^{\frac{1}{2}} t_0 \sim 6^{\frac{1}{2}} \cdot 2.3841687 \sim 3.41167 \]  

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(45)

(46)

using Eqs. (40) and (41) that refines the value in \[13\], namely \(T_{\text{plunge}} \sim 3.412\).

In sum, we identify here that the relevant Painlevé-I transcendent in the Painlevé-I equation appearing in the orbital transition to plunge at the ISCO \[13, 15\] is Boutroux’s tritronquée solution, known to be associated with universality properties \[16\] in the setting of critical behavior of Hamiltonian perturbations of nonlinear hyperbolic PDEs \[38\]. This offers a tantalizing avenue to connect the ODE orbital EMRI problem with universality properties of an underlying effective PDE dynamics describing finite mass-ratio binaries.
steps performed in the electrostatic case, in this case one does not get to a Painlevé-II equation as in Eq. (38). The responsible is the term $\Delta$ in the second equation in (46). So, in this particular approximation to the EMRI problem, Painlevé-II is not reated to generic orbits.

However, there is a particularly interesting dynamical setting in which this term $\Delta$ can be neglected. Indeed, looking at (44), for quasi-circular orbits $\dot{r} \sim 0$ we have $\Delta \sim 0$ (note that it is crucial to have $m = 2$ in the potential $W(r)$ for this approximation not to depend on $r$, only in $\dot{r}$). Moreover, this is indeed self-consistent with the quasi-circular assumption enforced to pass from Eqs. (45) to Eq. (26). Further reaching studies will need to start from more generic forms of PPN radiation reaction equations (in the same sense that Compère and Küchler’s work [14, 15] systematically extends that of Ori and Thorne [13]) and (probably) look for the fulfilment of the Painlevé property. We leave this for future work.

Restricting ourselves in to such quasi-circular dynamical scenarios in (26), with (46), we can write

$$\frac{dL}{dz} = \frac{2M\tau}{r^2} L$$

$$\frac{dr}{dz} = \frac{r^3}{4M^3\tau} L^2 + \frac{r^2}{M} + \frac{2M\tau}{r}, \quad (47)$$

and, from this, we obtain

$$\frac{d^2L}{dz^2} = -\frac{1}{M^2} L^3 - 4 \left(\frac{\tau}{r_o}\right) zL - \frac{4M^2}{r^2} \left(\frac{\tau}{r_o}\right)^2 L.$$ \( (48) \)

There are two key differences with respect to the electrostatic case in (38). The first one is that the coefficients depend on $r$, and therefore are not constants, as required in the Painlevé-II equation. This however is a mild obstruction precisely in the quasi-circular regime we are considering. Indeed, in that regime we can approximate $r \sim r_o$ during the appropriate timescales dictated by the radiation-reaction process. The second one refers to the last term in the equation, absent in the Painlevé equation. However, this term is a second-order term in $\tau$, that we have neglected from the very beginning. Therefore in the setting we are discussing, this term must also be neglected (48). Under these approximations we get

$$\frac{d^2L}{dz^2} = -\frac{1}{M^2} L^3 - 4 \left(\frac{\tau}{r_o}\right) zL.$$ \( (49) \)

that, again, can be reduced to the Painlevé-II equation under the appropriate scalings.

This discussion partially generalizes Rajeev’s argument to the gravitational case and makes conceptual contact with the Painlevé-I result in [14, 15]. The most important outcome is the illustration of the role of Painlevé transcendents in the EMRI gravitational binary problem. In the asymptotic reasoning spirit in [4], we claim that such structures play also a structural role, though recast in another form more akin to a PDE discussion, in the full BBH problem.

IV. BINARY BLACK HOLE DYNAMICS: AN INTEGRABILITY ANSÄTZ

In the previous section the Painlevé-II transcendent has been identified, in the extreme mass ratio limit, as a relevant underlying structure in BBH dynamics. Consequently, the Painlevé-II will have an imprint in the BBH waveform, but its actual identification is done at the level of BBH orbital dynamics. Specifically, from a methodological perspective, the approach to BBHs above separates: i) an orbital non-linear system described (in the proper asymptotic limit) by an integrable ODE, ii) the (linear) waveform emitted by such system.

In this section we approach the problem from a PDE perspective. Instead of considering the full Einstein equation system—as done, e.g., in numerical BBH simulations—we adopt an asymptotic approach aiming at identifying and focusing on the underlying mechanism(s). Rather than starting from Einstein equations and proceeding in a ‘top-down’ scheme, the effective approach we adopt here is a ‘bottom-up’ one in which, in particular, we follow the methodology in the ODE treatment above, separating the dynamics of the background and that of wave propagating on it.

A. Fast and slow degrees of freedom: linear waves over integrable dynamical backgrounds

Observational evidence, supported by numerical simulations of the full non-linear Einstein equations as well as the success of effective semi-analytic treatments well beyond their natural regimes of application, strongly indicate that waveform dynamics in the BBH problem displays ingredients with effectively linear character. On the other hand, the ODE treatment of the previous section suggests a non-linear dynamics with integrability playing a structural role. This suggests a methodological approach in which we effectively separate dynamics with different time scales, into “fast” and “slow” degrees of freedom. Specifically, we consider:

i) Fast degrees of freedom subject to linear dynamics, corresponding to the propagating and observed waveform.

ii) Slow degrees of freedom subject to non-linear (integrable) dynamics providing the background for waves.

We propose to address the effective BBH dynamics in terms of the coupling of such separated fast and slow dynamics [41].

1. A simplified one-dimensional effective toy model

To fix ideas, in the following we will focus on a highly simplified model in which spatial dynamics are effectively one-dimensional [42]. Denoting formally by $\phi$ the fast linear degrees of freedom, and by $\nu$ the slow (background) degrees of freedom, we consider a PDE dynamical system in which the
fast dynamical degrees of freedom satisfy a linear wave equation, propagating on an effective potential \( V(x, t; u) \)
\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V(x, t; u) \right) \phi = S(x, t; u),
\]  
(50)
where the effective potential \( V \) is determined by the background slow degrees of freedom \( u \), and that satisfy a dynamical equation
\[
\partial_t u = F(u, u_x, u_{xx}, \ldots). 
\]  
(51)
Note that the modelling of the fast degrees of freedom include a dynamical forcing \( S \) in terms of a source possibly depending of the slow degrees of freedoms. This will be crucial in the waveform model.

2. A dispersive Ansatz on background integrable non-linear dynamics.

The role of the Airy function in the BBH waveform model in [3] suggests that the underlying mechanism behind the simplicity and universality in BBH mergers includes a dispersive ingredient. With this motivation, and also as methodological choice to constraint and guide the choice of possible models, we make the hypothesis that the background dynamics is effectively (weakly) dispersive.

This is a strong assumption in the setting of general relativistic gravitational dynamics. Such hypothesis can be motivated in an asymptotic “coarse-grained” description that averages over different background characteristic scales, the latter inducing a qualitatively different scattering behaviour for waves of different wavelength. However, the ultimate validity of such dispersive hypothesis can be justified only in terms of the obtained results. As we will see below, this allows in particular to make explicit contact with the Painlevé-II transcendent identified in the inspiral phase.

In particular, motivated by the identification of Painlevé-II at EMRI BBH dynamics, and using the connection of Painlevé transcendents with integrable systems, we bestow a structuring role to the notion of integrability. It provides a concrete guiding hypothesis aiming at accounting for the observed simplicity and universality in the BBH dynamic providing a methodological guideline unifying the treatment along the different regimes in the entire BBH evolution, we grant a structuring role to the notion of integrability. Specifically, in our fast-slow dynamical coupling scheme, we make the hypothesis of integrability for the non-linear dynamics for the slowly background, over which the fast degrees propagate.

B. Non-linear background dynamics: KdV-type models

Once in a dispersive and non-linear setting, we make the following fundamental remark (cf. e.g. [43]): KdV-type equations provide “universal” models for weakly dispersive and weakly non-linear wave systems.

Specifically, in the KdV equation introduced in (7)
\[
\partial_t u + 6u \partial_x u + \partial_{xxx}^2 u = 0,
\]  
(52)
the advection term accounts for non-linearity, as in the Burgers’ equation, whereas the third derivative term corresponds to dispersion. In spite of the crucial role of non-linearity, it is instructive to start with the linearized case.

1. Linear case: Airy

Let us consider the linearized KdV equation
\[
\partial_t u + \partial_{xxx}^3 u = 0, 
\]  
(53)
and its solutions satisfying \( u \to 0 \) when \( x \to \infty \), with initial data \( u_0(x) = u(t = 0, x) \) satisfying (note we only consider here real solutions)
\[
\int_{-\infty}^{\infty} (u_0(x))^2 dx < \infty. 
\]  
(54)
Specifically, following [44] (see also [43]), we consider its asymptotic solutions corresponding to the regions: i) \( x/t < 0 \), ii) \( x/t > 0 \), and iii) \( x/t \to 0 \).

First we write the (formal) solution of (53) in terms of the Fourier transform \( \hat{u_0}(k) \) of \( u_0(x) \)
\[
u(x, t) = \frac{1}{2\pi} \int \hat{u_0}e^{ikx - \omega(k)t) dx} = \frac{1}{2\pi} \int \hat{u_0}e^{(kx + k^3 t)} (55)
\]  

where the employed relation dispersion \( \omega(k) = -k^3 \), followed by inserting the first expression into (53).

a. Region \( x/t < 0 \). Here a stationary phase method is employed. Writing
\[
u(t, x) = \frac{1}{2\pi} \int \hat{u_0}e^{i\phi(k)t) dx},
\]  
(56)
with \( \phi(k) = k(x/t) + k^3 \), for given \( x \) and \( t \), we determine the stationary points \( k_0 \) from the vanishing of \( \phi(k) \), that is
\[
\phi'(k_0) = \frac{x}{t} + 3(k_0)^2 = 0
\]  
(57)
\[
(k_0)_{\pm} = \pm \sqrt{\frac{x}{3t}} = \pm \sqrt{\frac{x}{3t}}.
\]  
(58)
Using then \( \phi''(k) \), expanding to the second order, and summing over the stationary points in the standard form of the stationary phase expansion, we find
\[
u(t, x) \sim \frac{\hat{u_0}(k_0)_{+}}{\sqrt{2\pi/|\phi''(k_0)_{+}|}} e^{i\phi(k_0)_{+}t + i\mu_{+} \pi/4}
\]  
(59)
\[
+ \frac{\hat{u_0}(k_0)_{-}}{\sqrt{2\pi/|\phi''(k_0)_{-}|}} e^{i\phi(k_0)_{-}t + i\mu_{-} \pi/4},
\]  
(60)
with \( \mu_{\pm} = \text{sgn}(\phi''((k_0)_{\pm}) = \pm 1 \). Inserting the expressions for \( \phi((k_0)_{\pm}) \) and \( \phi''((k_0)_{\pm}) \) and writing \( u_0(k) = \hat{u_0}(k)e^{i\phi(k)} \), we get
\[
u(t, x) \sim \frac{\hat{u_0}(\sqrt{\frac{x}{3t}})}{\sqrt{2\pi/|\phi''(\sqrt{\frac{x}{3t}})|}} \cos \left( 2 \frac{|x|^{3/2}}{3t} - \frac{\pi}{4} - \phi_0 \left( \sqrt{\frac{x}{3t}} \right) \right) \]  
(61)

b. Region $x/t > 0$. In this regime a related asymptotic method, namely the steepest descent, is better adapted to extract the dominant asymptotic behaviour. The solution is now written as a complex integral

$$u(t, x) = \frac{1}{2\pi i} \oint_C \hat{u}_0 e^{i\phi(k)} dk,$$  \hspace{1cm} (60)

where $\phi(k)$ is complex-valued. After finding the saddle points satisfying $\phi'(k_0) = 0$, the contour $C$, initially $]-\infty, \infty[$, is then deformed in the complex plane using Cauchy's theorem so that $\text{Im}(k)$ is constant and given by $\text{Im}(k_0)$. This defines the steepest descent path $C'$. Implementing this scheme (see details in [44]) and choosing the only compatible saddle point, one finally gets

$$u(t, x) \sim \frac{\hat{u}_0(i\sqrt{\frac{x}{t}})}{2\sqrt{\pi t} \left(\frac{3x}{t^2}\right)^{1/4}} e^{-2\left(\frac{x}{t}\right)^{3/2}},$$  \hspace{1cm} (61)

c. Transient region $x/t \to 0$. Looking at Eqs. (59) and (61) in the respective $x/t < 0$ and $x/t > 0$ regions, we notice the particular combination of $x$ and $t$ entering in the asymptotic expressions of $u(t, x)$. This suggests to look for a so-called self-similar solution at the transient regime, namely an Ansatz for the functional form of $u(t, x)$ as

$$u(t, x) \sim \frac{1}{(3t)^{1/3}} f \left(\frac{x}{(3t)^{1/3}}\right).$$  \hspace{1cm} (62)

Injecting (62) into (53) we get an ODE for $f$ in its natural variable $\eta = x/(3t)^{1/3}$. Imposing that the ODE does not depend on $t$ fixes $q = 1/3$. In this linear setting $p$ is not fixed at this level and we must resort to the connection with the other asymptotic regimes above to render $p = 1/3$. As a result, one gets the Airy equation as the ODE satisfied by the function $f$

$$\frac{df}{d\eta^2} - nf = 0.$$  \hspace{1cm} (63)

This permits to write the solution to (53) in the transient region $x/t \to 0$ in terms of the Airy function, namely

$$u(t, x) \sim \frac{1}{(3t)^{1/3}} \text{Ai} \left(\frac{x}{(3t)^{1/3}}\right).$$  \hspace{1cm} (64)

This expression provides a first contact with universality, with the remarkable fact of involving a structural connection with the notion of self-similarity in our PDE context. In particular, the Airy function pattern provides a universal profile of the transient at $x/t \to 0$ between the oscillatory behaviour at $x/t < 0$ and the damped one at $x/t > 0$. The Airy function is the universal special function playing, for linear 'turning point' problems, the analogous role played by the $	anh(x)$ function for shocks, through Eq. (6). That is, paraphrasing the statement for Burgers' equation in this context: all linear turning points transitions look the same.

A more systematic (an improved) manner of obtaining an asymptotic solution in this regime makes use of the integral representation of the Airy function

$$\text{Ai}(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(s\eta + s^3/2)} ds.$$  \hspace{1cm} (65)

Then, starting from the Fourier representation of the solution (56), making the variable change $k = s/(3t)^{1/3}$ and expanding $\hat{u}_0(k) = \hat{u}_0(s/(3t)^{1/3})$ to the first order in $s$ in Taylor series around $s = 0$, one can finally write

$$u(t, x) \sim \frac{\hat{u}_0(0) \text{Ai}(\eta)}{(3t)^{1/3}} - \frac{\hat{u}'_0(0) \text{Ai}'(\eta)}{(3t)^{2/3}},$$  \hspace{1cm} (66)

where both terms are indeed self-similar solutions of the linear KdV equation.

d. Uniform asymptotic expansion. Once the asymptotic behaviours in each $x/t$ region have been identified, a matched asymptotic expansion can be employed to join the different regimes. This actually permits to fix $p = 1/3$ in Eq. (62), in this discussion of linear (53). Remarkably, using the asymptotic expressions of the Airy function one can indeed write an asymptotic expression valid in the three regimes, namely

$$u(t, x) \sim \frac{\hat{u}_0(0) + \hat{u}_0(-k_0)}{2} \left(\frac{x}{(3t)^{1/3}}\right) \text{Ai} \left(\frac{x}{(3t)^{1/3}}\right) - \frac{\hat{u}_0(0) - \hat{u}_0(-k_0)}{2i k_0} \left(\frac{x}{(3t)^{1/3}}\right) \text{Ai}' \left(\frac{x}{(3t)^{1/3}}\right),$$  \hspace{1cm} (67)

with $k_0 = \pm \sqrt{-x/(3t)}$. The global expression (67) represents a slowly varying similarity solutions, namely a linear combination of self-similar solutions (with $f$ given, respectively, by $\text{Ai}$ and $\text{Ai}'$) modulated by slowly varying factors determined by the initial data, through its Fourier transform $\hat{u}_0(k)$.

This expression is structurally similar to the one we have found for the diffraction patterns on a caustic in [3]. This is not an accident, actually a uniform asymptotic expansion [43–48] can be employed for deriving expression (67), following exactly the same steps implemented in determining the universal patterns for caustics (cf. [3]). Such uniform asymptotic expansion remains valid in the regimes in which (real or complex) “critical points” $(k_0)_{\pm} = \pm \sqrt{-x/(3t)}$ are separated and also when they degenerate at the “caustic”. In particular, in our present problem this identifies a (fold) catastrophe in our PDE problem at the transient regime $x/t \to 0$.

On the other hand, the physical role of this solution in terms of the Airy function and its derivative is very different to the one discussed in [3]: whereas in the caustic discussion in [3] the Airy solution accounts for the (linearly) propagating wave field (fast degrees of freedom) according detected at the observer emplacement, Eq. (67) describes (in the linearised approximation of Eq. (IV B 1)) the dynamics of the slow degrees providing the “effective potential” on which the fast degrees of freedom are scattered. We develop further this point below, in the discussion of non-linear turning points in terms of the Painlevé-II trascendent.

2. Non-linear case: Painlevé-II

After considering the linear case (53) of the KdV equation (52), we move to the actual non-linear case. From the linear case, we highlight three structural elements:

i) From Fourier analysis to the inverse scattering transform (IST). In the linear setting, the starting point for the
study of the asymptotics of \( u(t, x) \) in the different dynamical regimes is the Fourier expression \( \tilde{u} \) in terms of initial data. However, in contrast with the linear case, no use of the Fourier transform can be made in the non-linear setting. Remarkably, the IST scheme, provides a solution to the problem that can indeed be dubbed as a non-linear Fourier transform. Most importantly at a structural level, such IST scheme actually characterizes what we mean by integrability at the PDE level in terms of the so-called Painlevé test.

ii) Self-similar solution and universality. The transient asymptotics of the solution are captured by a self-similar solution, that implements universality through a universal pattern. In this spirit, we will enforce the solution \( u(x, t) \) to have a self-similar character.

iii) Matching conditions and uniform asymptotics. The self-similar solution should provide also the key to match asymptotics corresponding to oscillation and damped behaviours, as the Airy function does in the linear case. That is, the self-similar solution should implement a non-linear ‘turning point’ matching providing a uniform asymptotic description through the merger.

In particular, the combination of the second and third points suggests to focus, rather than on the standard non-linear KdV equation \( u_t - 6u^2 u_x + \frac{1}{2} u_{xxx} = 0 \), on the modified one m-KdV in Eq. (30), namely

\[
\begin{align*}
\frac{d^2 f}{dy^2} - \eta f - 2f^3 &= 0, \\
\text{namely the Painlevé-II equation in (24) with } \alpha = 0. \\
\text{This leads } &\text{ to the non-linear analogue of solution (63).}
\end{align*}
\]

An analogue of the more general Eq. (55) in the linear case, matching the uniform asymptotic expansion \( \tilde{u} \), can also be derived \( \tilde{u} \) in the present non-linear setting \( \tilde{u} \), that, given the asymptotics described in section IIIB, directly generalises the Airy-like solution \( \tilde{u} \) to the non-linear case.

As anticipated at the end of section IV, formal contact has been made with the Painlevé-II structures found in the orbital dynamics of damped binary EMRI dynamics discussed in section III. But, as in the ODE orbital case — and as discussion after Eq. (64) — this refers to the ‘slow degrees of freedom’. Therefore, the dynamics determined by \( \tilde{u} \) does not directly describe the BBH waveform, but rather the background dynamics generating such waveform. Indeed, in order to make contact with the waveform dynamics and connect with the model presented in \( \tilde{u} \), a further step is needed.

C. Heuristics on the slow-fast BBH dynamics coupling: \( P_{11} \)-driven wave equation

In this section we discuss some heuristic elements meant as a basis for the development of an approximate (asymptotic \( \tilde{u} \)) scheme for the PDE treatment of the BBH problem. We organise the discussion in term of the following points

i) Dual-frame approach: irrotational fast dynamics over corotating slow dynamics. As discussed above, slow degrees of freedom serve as a background over which fast degrees of freedom propagate. During the inspiral phase the inner wave zone presents an approximately stationary behaviour \( \tilde{u} \). In this setting, a corotating frame seems adapted for describing the slow dynamics \( \tilde{u} \). On the other hand, the fast degrees of freedom are more naturally described in a coordinate system that is irrotational with respect to the far wave zone. In other words, coordinates in \( \tilde{u} \) must be understood in an irrotational frame, whereas coordinates in \( \tilde{u} \) should be understood rather in a corotating system.

ii) Integrability, self-similar solutions, universality and IST for background dynamics: the mKdV model. The dynamics describing the slow degrees of freedom are modelled by a universal self-similar solution to an integrable dispersive non-linear PDE. In particular, such a self-similar solution is structurally related to a Painlevé transcendent, specifically to \( P_{11} \) in our BBH setting. Such reducibility of the non-linear PDE to a Painlevé equation is intimately related, through the so-called Painlevé test, to the integrability of the PDE in the sense of being solvable by IST. This provides, in particular, an infinite set of conserved quantities that constraint and simplify the dynamics. A simple model for this scheme is provided by the mKdV equation \( \tilde{u} \) whose universal self-similar solution is given the expression \( \tilde{u} \).
terms of $P_{11}$ and its derivative. This is the PDE counterpart of the ODE $P_{11}$ terms in the EMRI BBH problem of section III for the orbital (background) dynamics. The imprint of $P_{11}$ into the waveform is realised through the coupling to the fast degrees of freedom.

iii) $P_{11}$-pattern in inspiral and merger waveform: background dynamics as a forcing term. The key mechanism to imprint the Painlevé-II-like pattern into the waveform, therefore providing a $P_{11}$ generalization of the Airy model in [3], is the forcing by the source term $S(x, t; u)$ in Eq. (50). Indeed, ignoring initial transients, such linear forced system is controlled by the driving terms $S$. Such driving term is determined by two ingredients: a) the background solution $u(t, x)$, namely shaped by the $P_{11}$ transcendent, b) a modulation given by the rotation of the (slow-dynamics) corotating frame with respect to the (fat-dynamics) irrotational one. In particular, in an approximation in which we neglect the “potential term” $V(x; u)$ in the homogeneous (left hand side) part of (50), we can write:

$$\phi(t, x) = \int G(t, x; t', x') S(x', t'; u(t', x')) \quad (72)$$

where $G(t, x; t', x') \sim \delta(t' - (t - |x - x'|))$.

The interest of such an expression is that the waveform is now described by $\phi$, that captures the pattern in the forcing term and transport it along the characteristics. When including back the $V(x; u)$ term in (50), resonant phenomena can occur when the source $S(x, t; u)$ triggers the resonant frequencies associated with the potential [54]. In summary, the inspiral and merger transient waveform is controlled by a forcing mechanism driven by the background integrable dynamics, leading to a “modulated-P11” model for the waveform that makes contact and extend the Airy-waveform model in [3].

iv) Merger transition to ringdown: waveform dynamics from background potential. The separation between fast and slow degrees of freedom makes only sense as long as the associated timescales are not commensurate. This is no longer the case in the merger. At this point a “phase transition” occurs that, as discussed in [3], does not happen at the peak of the merger waveform, but a bit later associated with the $t = 0$ value in the $P_{11}$ or Airy ODE, respectively Eqs. (3) and (9). At this point the slow background dynamics freezes, the forcing term $S(x, t; u)$ “decouples” (actually disappears) and the waveform dynamics is determined by the potential $V(x; u)$, so the late ringdown is controlled by the resonant properties of $V(x; u)$. The latter is constructed from a time-independent (solitonic) solution of the integrable background dynamics, so integrability underlies the associated scattering theory.

V. INTEGRABILITY: A UNIFYING PRINCIPLE FOR BACKGROUND BBH NON-LINEAR DYNAMICS.

Integrability is proposed in the previous sections as a structural guideline to account for the simplicity and universality features in the BBH merger waveform and, more generally, BBH dynamics. Such a proposal is supported by the finding of the role played by Painlevé transcendents in the (ODE) binary dynamics in the inspiral phase and, on the other hand, by the integrability features present in the description of the BH scattering in the late merged BH, revealing the sound link to KdV structures responsible of features such as BH quasi-normal mode isospectrality [26, 57, 59]. Between these two phases, in the previous section we have proposed that integrability indeed extends through the merger, discussing a toy-model built on a fast-slow separation of gravitational degrees in which the slow non-linear dynamics would be integrable [60], with self-similarity and dispersion as key ingredients to account for universality and connect with Painlevé transcendents. In this section, we aim at abstracting some of the relevant structural elements that would permit to go beyond the particular (ad hoc) KdV-like models discussed above.

A. Integrability and inverse scattering transform: a “bottom-up” approach to BBHs

Ideally, one would like to start from Einstein equations and, by an appropriate reduction process [61], arrive to an integrable PDE system approximating (background) BBH dynamics. Although performing such a “top-down” approach would be the ultimate objective, here we focus on a (more humble) “bottom-up” approach, limiting ourselves to indicate some elements to explore in the “top-down” setting.

1. Integrability, Inverse Scattering Transform and Lax pairs

Although there is no universal definition of integrability for (non-linear) PDEs, a standard characterisation relies on the possibility of transforming the non-linear PDE into an equivalent linear problem, namely the so-called inverse scattering transform (IST). The key ingredient of this method is the possibility of writing the non-linear equation [51] in a Lax representation, namely finding operators $L$ ad $A$, depending on $u$, $u_x$, $u_{xx}$, ..., such that [51] is equivalent to the equation:

$$L_t = [L, A], \quad (73)$$

In general, the operators $L$ and $A$ depend on a spectral parameter $\lambda$ [62]. We focus here on the case that this parameter $L$ appears as an eigenvalue of $L$, that is:

$$L\psi = \lambda \psi, \quad (74)$$

If we write the evolution of the eigenfunctions as:

$$\psi_t = A\psi, \quad (75)$$

then Eq. (73) is equivalent to the time independence of $\lambda$.

$$\dot{\lambda} = 0 \quad (76)$$
In other words, equation (51) (or its Lax pair representation (73)), can be seen as compatibility condition between the two linear operators in (74) and (75).

The crucial consequence of the Lax pair rewriting (73) is that it permits recast the resolution of the initial value problem of (51) in terms of the direct and inverse scattering problems of the operator $L$, understood as the operator describing the (linear) scattering of a wavefunction $\psi$. In the realisation in the KdV case, $L$ is explicitly the one-dimensional Schrödinger operator (see e.g. [63])

$$L = -\frac{d^2}{dx^2} + u(x,t). \quad (77)$$

In particular, on the one hand, the direct scattering problem permits to determine scattering data, that we formally denote as $\{a(\lambda), b(\lambda), \ldots\}$, in terms of the scattering operator $L$ (namely the potential $u$ in the KdV case, at at given fixed time) when appropriate boundary conditions are imposed on $\psi$. On the hand, the inverse scattering problem permits to retrieve $L$ (i.e. $u(x,t = t_0)$) from scattering data $\{a(\lambda), b(\lambda), \ldots\}$, namely using the Gelfand-Levitan-Marchenko (GLM) equations [24, 25]. Using these elements, the inverse scattering transform (IST) (cf. [43, 64, 65]) proceeds according to the following scheme:

i) Direct scattering: from initial data $u(x,t = 0)$ to scattering data. From the initial data $u(x,t = 0)$ of (51) we determine the operator $L(t=0)$ at time $t=0$. For instance, in the KdV case (77) this fixes the potential. From $L(0)$ one then determines the scattering data at $t = 0$: $\{a(\lambda, t = 0), b(\lambda, t = 0), \ldots\}$.

ii) Evolution of scattering data. Using equation the isospectral time evolution Eq. (73) one then determines the evolution $\{a(\lambda, t), b(\lambda, t), \ldots\}$ of the scattering data. The isospectrality property (76) underlies the existence of an infinite number of conserved quantities, a key feature of the integrability.

iii) Inverse scattering transform: from scattering data to solution $u(x,t)$. From the evolved scattering data, the GLM equations of inverse scattering permit to retrieve $L(t)$, namely the evolved potential $u(x,t)$ in the KdV case. This involves the resolution of a linear integral equation (namely a Riemann-Hilbert boundary problem).

This scheme can be assimilated to a non-linear equivalent of the Fourier approach to solve a linear PDE: indeed steps i) and ii) above would correspond to the calculation of the Fourier transform of the initial data, whereas step ii) would correspond to the inverse Fourier transform to obtain the evolved solution.

2. Painlevé test and self-similar solutions

How can we determine a priori if an equation (51) is integrable? Although there is no full algorithm to address this question for PDEs, the so-called Painlevé test [68, 69] indicates that a PDE that is reduced through a similarity transformation to a Painlevé ODE (more generally, an ODE satisfying the Painlevé property discussed in section III A) is probably solvable through a IST. The test provides actually only necessary conditions [63].

This remark is particularly important in our present case of BBH dynamics. We remind that the main outcome in section III was the identification of Painlevé transcendent (namely $P_1$ and $P_II$) as relevant ODEs for the description of some key structural features of EMRI BBH orbital dynamics. The effort in section IV was to find a link with such Painlevé transcendents from a PDE perspective, this leading us to self-similar solution of certain dispersive non-linear PDEs. In this BBH setting, it is natural to look at the Painlevé test from the opposite perspective (justified, since the Painlevé test is not a sufficient condition, but rather a necessary one): an integrable PDE in the sense of being solvable by IST will admit a reduction to an ODE with the Painlevé property, possibly a Painlevé transcendent.

Our BBH interest on reductions to Painlevé ODEs justifies our focus on PDEs integrable by IST. This is in particular the case of the toy-model in section V B 1 namely the modified-Korteweg-de Vries in Eq. (68), indeed IST integrable and leading to $P_II$ [10]. But, in addition, the associated self-similar solutions to the corresponding PDEs have an interest on their own in our quest for universal features, since their simple behaviour under appropriate re-scalings is a feature pointing to the kind of simplicity and universality akin to the one happening critical phenomena [22].

The Painlevé test algorithm can be summarised [63, 68, 69] as:

i) Find the (Lie-)point symmetries of the studied PDE (this fixes the similarity transformations).

ii) Construct the associated ODE to the group-invariant solutions (self-similar solutions).

iii) Check the Painlevé property in the resulting ODEs associated to self-similar solutions.

B. Towards a top-down integrability approach to BBHs

Understanding and characterising (aspects of) gravity at a fundamental level in terms of integrable systems is a challenging problem that has been addressed from different avenues (see e.g. [70] for a review). In spite of the lack of comprehensive theory, partial results revealing the important role of integrability in gravity are available. We comment on some aspects of potential relevance in the present context.

1. Darboux covariance in the direct scattering problem

Linearised Einstein equations in the context of black hole scattering theory provide a first neat connection with integrability. Specifically, as revealed by Chandrasekhar [26],

\[ L = -\frac{d^2}{dx^2} + u(x,t). \quad (77) \]
odd and even-parity effective potentials in the stationary spherically symmetric case (Schwarzschild and Reissner-Nordström) present the same transmission and reflection coefficients and are isospectral in the associated quasi-normal mode frequencies. These spectral features can be understood in terms of conserved quantities of the KdV equation, that provides an isospectrality condition to be satisfied by the respective effective potentials. A related but alternative approach to this isospectrality in terms of Darboux transformations is presented [57, 71] (see also [72] for an approach in terms of “intertwining operators”). However, it is in the recent work [58, 59] where the connection to integrability, specifically through inverse scattering theory, becomes apparent, as well as shedding light on the link between the KdV equation and Darboux transformations (see also [74–78] for other hints on integrability and hidden symmetries in this perturbative setting). Specifically, in [58] an infinite branch of new admissible of odd/even effective potentials (with their associated master functions) is identified, all related by Darboux transformations preserving the spectral properties, leading to the notion of “Darboux covariance”. On the other hand, the role of the KdV equation as defining a flow in the Darboux-related effective potentials is elucidated, making full use of the Lax pair representation of the KdV equation and, eventually, extending the universality of the infinite KdV hierarchy of conserved quantities to the (infinite) branch of Darboux-related potentials. This work is a most important one in our present context, in particular regarding the integrability properties of the effective potentials in the equation [50] satisfied by the linear degrees of freedom.

2. Symmetry reductions of Einstein equations and integrability

Staying at the level of the full non-linear Einstein equations (in contrast with the linearised version discussed above), the enforcement of symmetries provides another avenue towards integrability. The archetype of this is the Ernst equation, that can indeed be solved through an inverse scattering method [72]. In our present BBH setting, let us comment on two avenues exploiting this symmetry reduction procedure and, finally, a third avenue making a link such symmetry reductions and self-dual Yang-Mills equations. The latter leads to the called Ward’s conjecture, that could provide a characterization of complete integrability in term of twistor spaces potentially richer and beyond the scope of the inverse scattering method. Specifically:

i) Ernst equation. Einstein equations under a stationary and axisymmetric assumption can be written as an non-linear elliptic equation on a complex potential $E$. Specifically, applying a projection formalism due to Ehlers [60] and Geroch [81], the potential $E$ lives in the two-dimensional space obtained by taking the quotient along the orbits of the two existing Killing vector fields. The resulting equation on $E$ is the so-called Ernst equation, that turns out to be completely integrable [62].

Specifically, such equation can be reduced to a linear system through the inverse scattering method, presenting a Lax pair representation and the Painlevé property discussed in section [VA1].

Such integrability has been extraordinarily useful in the construction of stationary axisymmetric solutions in astrophysical settings(e.g. [83–85]). This property is intimately linked to the existence of an infinite-dimensional group generated by the two commuting Killing vectors and acting as a symmetry on the space of solutions, namely the Geroch group. Interestingly, these notions have been explored in the BBH context, specifically in the study of stationary binary inspirals [51–53]. As commented in section [IVC] when sketching the dual-frame approach, these ideas could be of much interest in the construction of integrable effective non-linear dynamics of the slow degrees of freedom, built on a corotating frame.

ii) Binaries and helical Killing vectors. A most interesting construction for the dynamics of the slow degrees in our present BBH setting is provided by Klein’s helical Killing model for quasi-circular binaries introduced in [84, 87]. In this case, a single helical Killing vector (see e.g. [88] for a discussion of helical Killing vectors) is imposed, instead of the two infinitesimal isometries leading to the Ernst equation discussed above. Applying again Ehlers & Geroch’s projection formalism, now only along one Killing vector, the four-dimensional Einstein equations reduce to three-dimensional gravity coupled to “effective matter” described by a $SL(2, R)/SO(1, 1)$ sigma model. The latter is determined by generalised Ernst-like equation (again for a complex potential, but living in the a three-dimensional quotient space). Even though the resulting system is not integrable as the standard (axisymmetric-stationary) Ernst equation is, it provides a most interesting starting point to explore the application of asymptotic methods leading to effective (quasi-)integrable dynamics for the background slow degrees of freedom.

iii) Ward conjecture and BBH integrability. A most remarkably connection between self-dual Yang Mills equations and integrable equations opens the possibility of an avenue to the use of twistorial techniques to address our BBH problem. More specifically, in a first step, a connection between the Ernst equation above in stationary axisymmetric spacetimes and certain self-dual Yang mills equations was identified by L. Witten in [89]. In a second step, this was extended by Ward [90] to a more systematic and extensive treatment of the stationary axisymmetric case using twistor techniques (see also [91]). This twistor insight into integrability in the key stationary axisymmetric situation in general relativity is already of major interest. However, this connection upgrades to a truly fundamental general connection in the perspective of Ward’s conjecture [92], according to which all integrable (or solvable) differential equation systems can be obtained from reductions of self-dual Yang Mills equations. Ref. [92] raises a
critical view on the use of the inverse scattering method to characterize integrability, since such method has limited applicability. On the contrary, the emphasis is placed in the satisfaction of the “Painlevé property” (solutions in terms of meromorphic functions) and the fact that integrable equations do appear as “consistency conditions” of an overdetermined system of linear PDEs (point shared with the inverse scattering method). Self-dual Yang Mills equations would then play the role of a sort of master equations, from which all integrable equations could be derived (see also [63, 93–95]). A most remarkable fact in our gravity setting is then provided by the connection unveiled in [94] between self-dual Einstein vacuum equations and the self-dual Yang-Mills equations. Under the light of Ward’s conjecture, this suggests to approach the identification of the integrable PDE describing the effective dynamics of the slow background degrees of motion in terms of self-dual Einstein vacuum equations and to incorporate the use of twistor concepts and tools to address the universality and simplicity of the BBH merger waveform.

Acknowledgments. We would like to acknowledge Oleg Lisovyy and Peter Miller for the key references that triggered our interest on Painlevé transcendent in the BBH problem. We thank Abhay Ashtekar, Ivan Booth and Andrey Shoom for discussions at the early stage of this project, at the “Focus Session: Dynamical Horizons, Binary Coalescences, Simulations and Waveform” (State College, July 2018). We thank Victor Aldaya, Carlos Barceló, Beatrice Bongà, A. Coutant, Christian Klein, Pavao Marques, O. Meneses-Rojas, Alex Nielsen, Ariadna Ribes Metidieri, Dhruv Sharma and Nikola Stoi lov for discussions. Special thanks to Carlos F. Sopuerta and Mikhail Semenov Tian-Shansky for enlightening discussions on the overall project.

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Instances of such a paradigm involving the linear propagation of perturbations on a background solution to non-linear integral equations have been explored in other physical settings as, e.g., in the propagation of (linear) perturbations on the background of topological-soliton skyrmions in the context of nuclear physics (we thank M. Semenov-Tian-Shansky for bringing our attention to this type of models).

The use of asymptotic tools [44, 101] permits partial implementations of this program in the fluid dynamics context, namely in the setting of dispersive non-linear hydrodynamics.

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