NOTES ON STABLE TEICHMÜLLER QUASIGEODESICS

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Abstract. In this note, we prove that for a cobounded, Lipschitz path \( \gamma : I \to \mathcal{T} \) if the pull back bundle \( \mathcal{H}_\gamma \) over \( I \) is a strongly relatively hyperbolic metric space then there exists a geodesic \( \xi \) in \( \mathcal{T} \) such that \( \gamma(I) \) and \( \xi \) are close to each other.

Suppose \( S_{g,n} \) is a surface of genus \( g \) with \( n \) punctures such that its Euler characteristic \( \chi(S_{g,n}) < 0 \). Consider the Teichmüller space \( \mathcal{T} = \text{Teich}(S_{g,n}) \) of \( S_{g,n} \), there is a smooth fiber bundle \( \mathcal{S} \to \mathcal{T} \) over \( \mathcal{T} \), whose fiber \( \mathcal{S}_\sigma \) over \( \sigma \in \mathcal{T} \) is \( S_{g,n} \) with metric \( \sigma \). Let \( \mathcal{H} \) be the universal cover of \( \mathcal{S} \), then the universal covering \( \mathcal{H} \to \mathcal{S} \) defines a smooth fiber bundle \( \mathcal{H} \to \mathcal{T} \) whose fiber \( \mathcal{H}_\sigma \) over \( \sigma \in \mathcal{T} \) is isometric to the hyperbolic plane \( \mathbb{H}^2 \). The purpose of this note is to prove that for a \( \mathcal{B} \)-cobounded, Lipschitz path \( \gamma : I \to \mathcal{T} \), where \( \mathcal{B} \) is a compact subset of \( \mathcal{T} \), if the pull back bundle \( \mathcal{H}_\gamma \) over \( I \) is a strongly relatively hyperbolic metric space then there exists a geodesic \( \xi \) in \( \mathcal{T} \) such that the Hausdorff distance between \( \gamma(I) \) and \( \xi \) is bounded. This is a straightforward generalization of a result due to Mosher, Theorem 1.1 of [9], where the statement was proven for closed surfaces admitting hyperbolic metrics with the assumption that \( \mathcal{H}_\gamma \) is a hyperbolic metric space.

1. Relative Hyperbolicity

Let \( X \) be a path metric space. A collection of closed subsets \( \mathcal{D} = \{D_\alpha\} \) of \( X \) will be said to be uniformly separated if there exists \( \epsilon > 0 \) such that \( d(D_1, D_2) \geq \epsilon \) for all distinct \( D_1, D_2 \in \mathcal{D} \).

Definition 1.1. (Farb [4]) The electric space (or coned-off space) \( \mathcal{E}(X, \mathcal{D}) \) corresponding to the pair \((X, \mathcal{D})\) is a metric space which consists of \( X \) and a collection of vertices \( v_\alpha \) (one for each \( D_\alpha \in \mathcal{D} \)) such that each point of \( D_\alpha \) is joined to (coned off at) \( v_\alpha \) by an edge of length \( \frac{1}{2} \). \( X \) is said to be weakly hyperbolic relative to the collection \( \mathcal{D} \) if \( \mathcal{E}(X, \mathcal{D}) \) is a hyperbolic metric space.

For a path \( \gamma \subset X \), there is an induced path \( \hat{\gamma} \) in \( \mathcal{E}(X, \mathcal{D}) \) obtained by coning the portions of \( \gamma \) lying in sets \( D \in \mathcal{D} \). If \( \hat{\gamma} \) is a geodesic (resp. \( P \)-quasigeodesic) in \( \mathcal{E}(X, \mathcal{D}) \), \( \gamma \) is called a relative geodesic (resp. relative \( P \)-quasigeodesic).

Definition 1.2. [2] Relative geodesics (resp. \( P \)-quasigeodesics) in \((X, \mathcal{D})\) are said to satisfy bounded region penetration properties if there exists \( K = K(P) > 0 \) such that for any two relative geodesics (resp. \( P \)-quasigeodesics without backtracking) \( \beta, \gamma \) joining \( x, y \in X \) following two properties are satisfied:

1) if precisely one of \( \{\beta, \gamma\} \) meets a set \( D_\alpha \), then the length (measured in the intrinsic path-metric on \( D_\alpha \)) from the first (entry) point to the last (exit) point (of the relevant path) is at most \( K \),

2) if both \( \{\beta, \gamma\} \) meet some \( D_\alpha \) then the length (measured in the intrinsic path-metric on \( D_\alpha \)) from the entry point of \( \beta \) to that of \( \gamma \) is at most \( K \); similarly for exit points.

Definition 1.3. (Farb [2]) \( X \) is said to be hyperbolic relative to the uniformly separated collection \( \mathcal{D} \) if \( X \) is weakly hyperbolic relative to \( \mathcal{D} \) and relative \( P \) quasigeodesics without backtracking satisfy the bounded region penetration properties.

Gromov’s definition of relative hyperbolicity :

Definition 1.4. [7] For any geodesic metric space \((D, d)\), the hyperbolic cone (analog of a horoball) \( D^h \) is the metric space \( D \times [0, \infty) = D^h \) equipped with the path metric \( d_h \) obtained from two pieces
of data
1) \(d_{h,t}(x,t), (y,t)) = 2^{-t}d_D(x,y)\), where \(d_{h,t}\) is the induced path metric on \(D_t = D \times \{t\}\). Paths joining \((x,t), (y,t)\) and lying on \(D_t = D \times \{t\}\) are called horizontal paths.
2) \(d_h((x,t), (x,s)) = |t-s| \) for all \(x \in D\) and for all \(t,s \in [0, \infty)\), and the corresponding paths are called vertical paths.
3) for all \(x, y \in D^h\), \(d_h(x,y)\) is the path metric induced by the collection of horizontal and vertical paths.

**Definition 1.5.** [7] Let \(\delta \geq 0\). Let \(X\) be a geodesic metric space and \(\mathcal{D}\) be a collection of mutually disjoint uniformly separated subsets of \(X\). \(X\) is said to be \(\delta\)-hyperbolic relative to \(\mathcal{D}\) in the sense of Gromov, if the quotient space \(\mathcal{G}(X, \mathcal{D})\), obtained by attaching the hyperbolic cones \(D^h\) to \(D \in \mathcal{D}\) via the identification \((x,0) \sim x\) for all \(x \in D\), is a \(\delta\)-hyperbolic metric space. \(X\) is said to be hyperbolic relative to \(\mathcal{D}\) in the sense of Gromov if \(\mathcal{G}(X, \mathcal{D})\) is a \(\delta\)-hyperbolic metric space for some \(\delta \geq 0\).

**Theorem 1.6.** (Bowditch [1]) Let \(X\) be a geodesic metric space and \(\mathcal{D}\) be a collection of mutually disjoint uniformly separated subsets of \(X\). \(X\) is hyperbolic relative to the collection \(\mathcal{D}\) of uniformly separated subsets of \(X\) in the sense of Farb if and only if \(X\) is hyperbolic relative to the collection \(\mathcal{D}\) of uniformly separated subsets of \(X\) in the sense of Gromov.

## 2. Main Theorem

Suppose \(p_1, \ldots, p_n\) are the punctures of \(S_{g,n}\), then each Teichmüller metric \(\sigma\) on \(S_{g,n}\) corresponds to collections \(D_{\sigma}(p_1), \ldots, D_{\sigma}(p_n)\) of horodisks in the fiber \(H_\sigma\) of the bundle \(H \to T\) satisfying the following properties:

1. Let \(D_{\sigma}(p_i) = \{D_{\sigma}(p_i, \alpha) : \alpha \in \Lambda\}\), then for each \(i\) and \(\alpha\) there exists a sub-bundle \(D(p_i, \alpha) = D_{\sigma}(p_i, \alpha) \to T\) such that the fiber over \(\sigma \in T\) is \(D_{\sigma}(p_i, \alpha)\).
2. each \(D_{\sigma}(p_i)\) is invariant under the action of \(\pi_1(S_{g,n})\).
3. elements of \(D_{\sigma}(p_1) \cup \ldots \cup D_{\sigma}(p_n)\) are disjoint with each other.

For each path \(\gamma : I \to T\), \(1 \leq i \leq n\) and \(\alpha \in \Lambda\), there exists a pull back bundle \(D_{\gamma}(p_i, \alpha) = I\) such that the fiber over \(t \in I\) is \(D_{\gamma,t}(p_i, \alpha)\). Let \(D_{\gamma} = \{D_{\gamma,p_i, \alpha} : 1 \leq i \leq n, \alpha \in \Lambda\}\). Consider a subset \(B\) of the moduli space \(M = T/MCG(S_{g,n})\), a path \(\gamma : I \to T\) is said to be \(B\)-cobounded, if the image of \(\gamma\) under the projection \(T \to M\) lies in \(B\). We prove the following theorem:

**Theorem 2.1.** Let \(I\) be a closed, connected interval of \(\mathbb{R}\). For a compact subset \(B\) of the moduli space \(M = T/MCG(S_{g,n})\) and for every \(\rho \geq 1, \delta \geq 0\) there exists \(P \geq 0\) such that the following holds:

If \(\gamma : I \to T\) is \(B\)-cobounded and \(\rho\)-Lipschitz path, and if \(H_\gamma\) is strongly \(\delta\)-hyperbolic relative to the collection \(D_{\gamma}\), then there exists a geodesic \(\xi : I \to T\) joining end points of \(\gamma\) such that the Hausdorff distance between \(\gamma(I)\) and \(\xi(I)\) is at most \(P\).

Note that the fibers \(H_\sigma = \mathbb{H}^2 \times \sigma\) of \(H \to T\) are (uniformly) strongly hyperbolic relative to the collections \(D_\sigma = \{D_\sigma(p_i, \alpha) : 1 \leq i \leq n, \alpha \in \Lambda\}\) of horodisks. Hence the coned-off spaces \(\mathcal{E}(H_\sigma, D_\sigma), \sigma \in T\), are (uniformly) hyperbolic metric spaces. Thus for a path \(\gamma : I \to T\), there exists a bundle \(PH_\gamma \to I\) of coned-off hyperbolic metric spaces with fiber \(\mathcal{E}(H_{\gamma,I}, D_{\gamma,I})\). \(PH_\gamma\) is also obtained by partially electrocuting each element \(D_{\gamma,p_i, \alpha}\) of \(D_\gamma\) to a hyperbolic space \(L_{\gamma,p_i, \alpha}\), where \(L_{\gamma,p_i, \alpha}\) is the locus of cone points obtained by coning \(D_{\gamma,I,p_i, \alpha}\) for all \(t \in I\). By Lemma 2.8 of [6], if \(H_\gamma\) is strongly hyperbolic relative to the collection \(D_{\gamma}\), then \(PH_\gamma\) is a hyperbolic metric space.

**Definition 2.2.** Given \(\kappa > 1\), a natural number \(n, A \geq 0\), a sequence of positive numbers \(\{r_j : j \in J\}\), where \(J\) is a subinterval of set of integers \(Z\), is said to satisfy \((\kappa, n, A)\)-flaring property if \(j-n, j+n \in J\) and if \(r_j > A\) then \(\max\{r_{j-n}, r_{j+n}\} \geq \kappa r_j\).

A path \(\alpha : J \to PH_\gamma\), where \(J \subset I\), is said to be \(\lambda\)-quasivertical if it is \(\lambda\)-Lipschitz and also a section. Let \(d_\sigma\) denote the metric of the coned-off space \(\mathcal{E}(H_\sigma, D_\sigma)\). Since \(PH_\gamma\) is a hyperbolic space, so we have the following flaring properties:
Proposition 2.3. (Theorem 4.7 of [6]) With the notations as above, given \( \lambda \geq 1 \) there exist \( \kappa > 1 \), an integer \( n \geq 1 \) and a number \( A > 0 \) such that the following holds: Let \( \alpha, \beta : I \to \mathcal{PH}_\gamma \) be two \( \lambda \)-quasivertical paths, then the sequence \( s_j = d_{\gamma(j)}(\alpha(j), \beta(j)) \), where \( j \in J \cap \mathbb{Z} \), satisfies \((\kappa, n, A)\)-flaring property.

We refer to [3] for the definitions of measured foliation \( \mathcal{MF} \) and measured geodesic lamination \( \mathcal{MGL} \) of general hyperbolic surfaces. For each \( \mu \in \mathcal{MF} \), let \( \mu_\gamma \) denote the measured geodesic lamination on the hyperbolic surface \( S_{\gamma(t)} = \mathcal{H}_{\gamma(t)}/\pi_1(S_{g,n}) \). Let \( S^b_{\gamma(t)} \) denote the ‘thick part’ of \( S_{\gamma(t)} \) i.e. \( S^b_{\gamma(t)} \) is obtained from \( S_{\gamma(t)} \) by deleting the images of interior of horodisks under the projection \( \mathcal{H}_{\gamma(t)} \to S_{\gamma(t)} \). Now each \( \mu \in \mathcal{MF} \) induce a geodesic lamination \( \mu^t_b(\subset \mu_i) \) on \( S^b_{\gamma(t)} \).

A connection path of the sub-bundle \( S^b_i \to I \) is a piecewise smooth section of the projection map which is everywhere tangent to the connection on the bundle \( S^b_i \to I \). The connection map \( h_{st} : S^b_{\gamma(s)} \to S^b_{\gamma(t)} \) is defined by moving points of \( S_{\gamma(s)} \) to \( S_{\gamma(t)} \) along connection paths. In [4], it was proved that connection maps \( h_{st} \) are bilipschitz maps. For \( \mu \in \mathcal{MF} \) and \( \sigma \in \mathcal{T} \), the length of \( \mu \) with respect to \( \sigma \) is defined by \( \text{len}_\sigma(\mu) = \int d\mu \). From proposition 2.3, it follows that for any leaf segment \( l_\sigma \) of \( \mu_\sigma \), the sequence of lengths \( \text{len}_{\gamma(s)}(h_{s,s+i}(l_s)) \) satisfies the flaring property.

As a consequence, we have the following theorem:

Theorem 2.4. (Lemma 3.6 of [9]) For a compact subset \( B \) of the moduli space \( \mathcal{M} \) and for every \( \rho \geq 1 \), there exist constants \( L \geq 1, \kappa > 1, n \in \mathbb{Z}_+ \) such that the following holds: Let \( \gamma : I \to \mathcal{T} \) be a \( B \)-cobounded and \( \rho \)-Lipschitz path, for any \( \mu \in \mathcal{M} \), the sequence \( i \to \text{len}_{\gamma(i)}(\mu^i) \), \( (i \in I \cap \mathbb{Z}) \), satisfies the \( L \)-Lipschitz, \((\kappa, n, 0)\)-flaring property.

For \( \mu \in \mathcal{MF} \), we say \( \mu \) is realized at \( p \), where \( p \) is a finite number or \( p \in \{-\infty, +\infty\} \), if \( \text{len}_{\gamma(i)}(\mu) \) achieves minimum at \( p \).

Proposition 2.5. (Proposition 3.12 of [9]) For each \( k \in I \cap \mathbb{Z} \), there exists \( \mu \in \mathcal{MF} \) which is finitely realized. If \( I \) is infinite, for each infinite end \( \pm \infty \) of \( I \) there exists \( \mu_\pm \in \mathcal{MF} \) which is realized at \( \pm \infty \) respectively.

Now for a compact subset \( B \subset \mathcal{M} \), numbers \( \rho \geq 1, \delta \geq 0, \eta > 0 \), consider \( \Gamma_{\beta, \rho, \delta, \eta} \) to be the set of all triples \((\gamma, \mu_-, \mu_+)\) with the following properties (see [9]):

1. \( \gamma : I \to \mathcal{T} \) is \( B \)-cobounded, \( \rho \)-Lipschitz path, such that \( \mathcal{H}_\gamma \) is \( \delta \)-hyperbolic relative to \( \mathcal{D}_\gamma \),
2. \( 0 \in I \), and each \( \mu_\pm \in \mathcal{MF} \) is normalized to have length 1 in the hyperbolic structure \( \gamma(0) \),
3. the lamination \( \mu_+ \) is realized in \( S_\gamma \) near the right end in the following way:
   (a) If \( I \) is right infinite, then \( \mu_+ \) is realized at \( +\infty \),
   (b) If \( I \) is right finite, with right end point \( M \), then there exists a minimum of length sequence \( \text{len}_{\gamma(i)}(\mu_+) \) lying in the interval \([M - \eta, M]) \).

The lamination \( \mu_- \) is realized similarly in \( S_\gamma \) near the left end.

Let \( \mathcal{A} \subset \mathcal{T} \) be a compact set such that each \( (\gamma, \mu_-, \mu_+) \in \Gamma_{\beta, \rho, \delta, \eta} \), may be translated by the action of \( \text{MCG}(S_{g,n}) \) so that \( \gamma(0) \in \mathcal{A} \). If \( \gamma_i \) converges to \( \gamma \), then in the Gromov-Hausdorff topology, \( \mathcal{H}_{\gamma_i} \) converges to \( \mathcal{H}_\gamma \) and \( \mathcal{D}_{\gamma_i} \) converges to \( \mathcal{D}_\gamma \). Hence, \( \mathcal{G}(\mathcal{H}_{\gamma_i}, \mathcal{D}_{\gamma_i}) \) converges to \( \mathcal{G}(\mathcal{H}_\gamma, \mathcal{D}_\gamma) \) in the Gromov-Hausdorff topology. The Gromov-Hausdorff limit of a sequence of \( \delta \)-hyperbolic spaces is \( \delta \)-hyperbolic ([5]). Therefore, if \( \mathcal{H}_{\gamma_i} \) are \( \delta \)-hyperbolic relative to \( \mathcal{D}_{\gamma_i} \), for all \( i \), then \( \mathcal{H}_\gamma \) is also \( \delta \)-hyperbolic relative to \( \mathcal{D}_\gamma \). This justifies the set \( \mathcal{A}_{\beta, \rho, \delta, \eta} = \{ (\gamma, \mu_-, \mu_+) \in \Gamma_{\beta, \rho, \delta, \eta} : \gamma(0) \in \mathcal{A} \} \) is compact.

Proposition 2.6. [9] The action of \( \text{MCG}(S_{g,n}) \) on \( \Gamma_{\beta, \rho, \delta, \eta} \) is cocompact.

Proof of Theorem 2.1
For \( (\gamma, \mu_-, \mu_+) \in \Gamma_{\beta, \rho, \delta, \eta} \), let \( a_-(t) = \frac{1}{\text{len}_{\gamma(i)}(\mu_-)} \) and \( a_+(t) = \frac{1}{\text{len}_{\gamma(i)}(\mu_+)} \). Then \( \mu_- \), \( \mu_+ \) fills \( S_{g,n} \).

(See [9]), therefore \( \mu_- \), \( \mu_+ \) defines a conformal structure \( \sigma(\mu_-, \mu_+) \) on \( S_{g,n} \). Consider the map \( \xi(t) = \sigma(a_-(t)\mu_- a_+(t)\mu_+), t \in I \), then the image of the map \( \xi : I \to \mathcal{T} \) is a geodesic in \( \mathcal{T} \) joining \( \mu_- \) and \( \mu_+ \). For \( i \in I \cap \mathbb{Z} \), define \( \gamma(s) = \gamma(s + i) \), then the triple \( (\gamma', a_-(t)\mu_- a_+(t)\mu_+) \) lies in a translate of the compact set \( \mathcal{A}_{\beta, \rho, \delta, \eta} \) by an element of \( \text{MCG}(S_{g,n}) \). The map taking \( (\alpha, \lambda_-, \lambda_+) \in \Gamma_{\beta, \rho, \delta, \eta} \) to \((\alpha(0), \sigma(\lambda_-, \lambda_+)) \in \mathcal{T} \times \mathcal{T} \) is \( \text{MCG}(S_{g,n}) \) equivariant and continuous and
hence has $\text{MCG}(S_g, a)$ cocompact image. Therefore, the Teichmuller distance $d_T$ between $\gamma(i)$ and $\sigma(a_-(i)\mu_-, a_+(i)\mu_+)) = \xi(i)$ is bounded. Now for $t \in I$, there exists $i \in I \cap \mathbb{Z}$ such that $|t - i| \leq 1$. As $\gamma$ is $\rho$-Lipschitz, therefore $d_T(\gamma(t), \gamma(i)) \leq \rho$. Also, there exists $L > 0$ such that $d_T(\xi(t), \xi(i)) \leq L$ (See [9]). Thus, there exists $P > 0$ such that the Hausdorff distance between $\gamma$ and $\xi$ is at most $P$.

3. Application

Consider the following short exact sequence of pair of finitely generated groups:

$$1 \to (\pi_1(S_g, 1), K_1) \to (G, N_G(K_1)) \to (Q, Q) \to 1,$$

where $K_1$ is peripheral subgroup of $\pi_1(S_g, 1)$, $G$ is strongly hyperbolic relative to $N_G(K_1)$ and $Q$ is a subgroup of $\text{MCG}(S_g, 1)$. Let $\Phi : Q \to T$ denote the orbit map, then for any geodesic $\gamma' : I \to Q$, $\gamma = \Phi \circ \gamma' : I \to T$ is a cobounded and Lipschitz path. Since $G$ is strongly hyperbolic relative to $N_G(K_1)$, the bundle $\mathcal{E}(G, K_1)$ over $Q$ is hyperbolic. Hence, $\mathcal{E}(G, K_1) \to Q$ satisfies flaring property. In particular, the sub-bundle $\mathcal{P}H_\gamma \to I$ satisfies the flaring property. Therefore, by the converse of strong combination theorem in [6], $\mathcal{H}_\gamma$ is strongly hyperbolic relative to $\mathcal{D}_\gamma$. Hence, as an application of Theorem 2.1, $Q$ is a convex cocompact subgroup of $\text{MCG}(S_g, 1)$. The converse of this result is also true (see [8]). So, we have the following theorem:

**Theorem 3.1.** [8] Consider the following short exact sequence of pair of finitely generated groups

$$1 \to (\pi_1(S_g, 1), K_1) \to (G, N_G(K_1)) \to (Q, Q) \to 1,$$

where $\pi_1(S_g, 1)$ is strongly hyperbolic relative to $K_1$. $G$ is strongly hyperbolic relative to $N_G(K_1)$ if and only if $Q$ is a convex cocompact subgroup of $\text{MCG}(S_g, 1)$

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