Itô versus Stratonovich in a stochastic cosmological model

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Abstract
In this work, we study a stochastic version of the Friedmann acceleration equation. This model has been proposed in the cosmology literature as a possible explanation of the uncertainty found in the experimental quantification of the Hubble parameter. Its noise has been tacitly interpreted in the Stratonovich sense. Herein, we prove that this interpretation leads to a positive probability of finite-time blowup of the solution, that is, of the Hubble parameter. In contrast, if we just modify the noise interpretation to that of Itô, then the solution globally exists almost surely. Moreover, the expected asymptotic behavior is found under this interpretation too.

Keywords
Stochastic differential equations · Random dynamical systems · Itô vs Stratonovich dilemma · Finite-time blowup vs global existence · Hubble parameter · Cosmology

Mathematics Subject Classification 60H05 · 37H05 · 60H10 · 83F05

1 Introduction

The Hubble parameter is a measure of the rate at which the universe expands. As such, it is a fundamental quantity in cosmology. Despite its importance, or perhaps as a consequence of it, there is currently an uncertainty in its experimental quantification. In reference [13], this uncertainty is theoretically approached by means of a stochastic generalization of the Friedmann acceleration equation; that is, the Hubble parameter.
is considered to be a stochastic process rather than a deterministic function of time. For the sake of completeness, we offer a derivation of the model studied in [13]; to this end, we follow [5,11,13,15]. Let us start considering the Einstein field equations:

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right),$$  \hspace{1cm} (1)

with the Friedmann–Lemaître–Robertson–Walker metric

$$ds^2 = -dt^2 + C^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),$$  \hspace{1cm} (2)

and the energy–momentum stress tensor

$$T_{\mu\nu} = (p + g)\delta^0_\mu \delta^0_\nu + pg_{\mu\nu}. \hspace{1cm} (3)$$

Herein, $G$ denotes the Newton gravitational constant, $g_{\mu\nu}$ the metric tensor, $C(t)$ the scale factor, $k$ the Gaussian curvature, and $p$ the pressure. If $R_{\mu\nu}$ is the Ricci tensor for metric (2), we have $R_{00} = -3 \frac{C''}{C}$, and from (3), $T = T^\lambda_\lambda = T^\nu_\lambda g^{\nu\lambda} = 3p - \rho$, and $T_{00} = \rho + p + pg_{00} = \rho$, where $\rho$ is the mass density; then, from (1), we find

$$-3 \frac{C''}{C} = 4\pi G (\rho + 3p). \hspace{1cm} (4)$$

If we introduce the Hubble parameter $H := \frac{C'}{C}$, this equation becomes the Friedmann acceleration equation

$$\dot{H} = -H^2 - \frac{4\pi G}{3} (\rho + 3p).$$

On the other hand for $\mu = \nu = 1$, we have $R_{11} = \frac{CC'' + 2(C')^2 + 2k}{1 - kr^2}$ and $T_{11} = pg_{11} = \frac{C^2p}{1 - kr^2}$, and from (1) we find

$$CC'' + 2(C')^2 + 2k = 4\pi G (\rho - p)C^2. \hspace{1cm} (5)$$

Now multiply (4) by $C^2/3$ and sum it to (5) to obtain the Friedmann equation

$$(C')^2 + k = \frac{8\pi G}{3} \rho C^2. \hspace{1cm} (6)$$

From (5) and (6), we deduce that $p$ and $\rho$ only depend on $t$. If we moreover restrict ourselves to flat models (i.e., $k = 0$), and assume the proportionality $p = \xi \rho$, with $\xi$ a parameter yet to be specified, we get

$$\dot{H} = -\frac{3}{2} H^2 (1 + \xi).$$
If we were to consider a deterministic model, and furthermore, following [13], a model in which the condition $3p + \rho = 0$ is fulfilled, then we had to conclude $\xi = -\frac{1}{3}$. Then, $C'' = 0$ by (4), and therefore, if $C(0) > 0$ and $C'(0) > 0$, then $H(0) > 0$ and

$$H(t) = \frac{H_0}{1 + H_0 t},$$

where $H_0 := H(0)$ and we have

$$H(t)t \to 1 \quad \text{when} \quad t \to \infty.$$

Some of the properties of this model, such as uniqueness and global existence of the solution, along with its asymptotic behavior, could be expected to hold in a stochastic counterpart of it. It is one of the goals of this work to specify some conditions that guarantee so.

Now, again following [13], if we assume that the condition $3p + \rho = 0$ has only to be fulfilled on average, we could write $\sigma \xi' = \xi + \frac{1}{3}$, where $\xi'$ could be any zero mean stochastic process. For simplicity, and as in [13], consider it is a standard Gaussian white noise to arrive at the model

$$\dot{H} = -\frac{3}{2} H^2 \left( \frac{2}{3} + \sigma \xi' \right),$$

where the noise amplitude $\sigma > 0$ has been introduced for dimensional consistency. This is a stochastic version of the Friedmann acceleration equation that can be slightly simplified by introducing the rescaling $t \to \sigma^2 t$ and $H \to H/\sigma^2$ to find

$$\dot{H} = -\frac{3}{2} H^2 \left( \frac{2}{3} + \xi' \right), \quad (7)$$

where the rescaled time and Hubble parameter are obviously dimensionless (and their dimensional counterparts are trivially recovered by undoing the rescaling). Of course, this model is not precise, and it could only be made so by means of the introduction of a notion of stochastic integration. This type of matter has been studied in much detail in physics [12]. In [13], the authors tacitly (not explicitly) assume the Stratonovich interpretation of noise, known to be the only one able to preserve the classical chain rule from ordinary differential calculus. However, as we will show herein, this choice is somehow problematic. Precisely, it opens the possibility of finite-time blowups for the Hubble parameter, phenomena of difficult (or at least not completely straightforward) physical interpretation, while this problematic disappears when the Itô interpretation of noise is chosen. Before starting with this program, let us emphasize that equation (7) constitutes an effective model. Its noise could be originated as a coarse-grained effect of quantum fluctuations [8] or inhomogeneity of the distribution of matter in the universe [3], or perhaps as a combination of these and other factors. While approaching this question may open the door to improve this model, herein we do not enter into these physical considerations and focus on the mathematical properties of equation (7).
The remainder of this article is as follows. In Sect. 2, we compare the consequences of making precise (7) via either the Stratonovich or the Itô prescription; in particular, the Stratonovich interpretation of noise (the one tacitly assumed in [13]) shows finite-time blowups of the Hubble parameter with positive probability. In contrast, the Itô prescription shows global existence almost surely and the expected asymptotic behavior. In Sect. 3, we draw our main conclusions. Finally, in the Appendix, we give a concise summary on the theory of stochastic dynamical systems to cover some of the results employed in this work.

2 Blowup versus global existence

2.1 Stratonovich equation

Equation (7) can be written in the precise manner:

\[ dH = -H^2 dt - \frac{3}{2} H^2 \circ dW_t, \]  
\[ \text{(8)} \]
if we assume the stochastic integral of Stratonovich. This is tacitly assumed by the authors in [13], who employ the usual Leibnitz–Newton calculus rules with this equation, known to be valid only in the case of Stratonovich. Due to the regularity of its terms, this equation will possess a unique strong solution as long as it remains bounded [16]. If we further follow reference [13] by formally performing the change of variables \[ x = \frac{1}{H}, \] then (8) transforms into

\[ dx = dt + \frac{3}{2} \circ dW_t = dt + \frac{3}{2} dW_t, \]  
\[ \text{(9)} \]
where we have just employed the Stratonovich (or Leibnitz–Newton) calculus rules on the first line and the fact that an additive noise equation has not to be interpreted (at least, in other sense rather than the one provided by the Wiener integral). For this stochastic differential equation (SDE), we have the following result:

**Lemma 2.1** The SDE (9) subject to an initial condition \( x(0) = \frac{1}{H_0} > 0 \) possesses a unique solution which is both strong and global and moreover fulfills

\[ \lim_{t \to \infty} \frac{x(t)}{t} = 1 \quad a.s. \]

**Proof** Existence and uniqueness of a strong and global solution are direct from the standard theory [16]. To find the asymptotic behavior, multiply the SDE by \( 2/3 \) to get

\[ d\left( \frac{2x}{3} \right) = \frac{2}{3} dt + dW_t, \]
which is of the form (15) with $f = 2/3$ (see the “Appendix”). Now, we will apply to this equation the developments in the last part of the Appendix; first, doing the change of variables (16), which in the present case reads

$$y = \frac{2}{3} x - z^*,$$

leads to (17), which now takes the form

$$y(t) = \frac{2}{3} t + \lambda \int_0^t z^* \, d\tau.$$

It defines a random dynamical system [2], and, by Proposition 3.4, it gives rise to a conjugated random dynamical system for equation (9). Subsequently applying Proposition 3.3 yields

$$\lim_{t \to \infty} \frac{y(t)}{t} = \frac{2}{3} \text{ a.s.}$$

The statement follows from undoing the change of variables to recover $x$ and applying Proposition 3.3 once more. □

From this result, one would be tempted to conclude

$$Ht \to 1 \text{ a.s. when } t \to \infty,$$

which is in fact the behavior found for the deterministic model in the Introduction. However, this conclusion would go through undoing the change of variables $H = 1/x$, which is possible only if $x(t)$ does not cross (or even touch) zero. Unfortunately, this is not the case as we show in the next result.

**Lemma 2.2** The stochastic process $x(t)$ that solves SDE (9) subject to $x(0) = 1/H_0 > 0$ crosses the origin in finite time with a probability

$$\exp\left(-\frac{8}{9H_0}\right).$$

**Proof** Note that we have the explicit formula $x(t) = t + \frac{3}{2} W_t + \frac{1}{H_0}$. Therefore, the statement can be seen as a particular case of the first passage time problem for Brownian motion with drift solved in [14] by means of the Girsanov theorem. Consider the stopping time

$$T_a := \inf \left\{ t \geq 0 : \frac{2}{3} t + W_t = a \right\}.$$
for any $a \neq 0$. It possesses the density [14]:

$$
P(T_a \in dt) = \frac{|a|}{\sqrt{2\pi} t^3} \exp\left(-\frac{(a - \frac{2}{3}t)^2}{2t}\right) dt, \ t > 0,
$$

and therefore

$$
P(T_a < \infty) = \exp\left(\frac{2}{3}a - \frac{2}{3}|a|\right).
$$

As in our case $a = -\frac{2}{3H_0}$, we conclude

$$
P\left(T_{-\frac{2}{3H_0}} < \infty\right) = \exp\left(-\frac{8}{9H_0}\right).
$$

\[\square\]

Since $\exp\left(-\frac{8}{9H_0}\right) \in (0, 1)$ for $H_0 \in (0, \infty)$, there is a positive probability that $H(t)$ blows up in finite time. So, in summary, we obtain the following dual behavior for the solution to SDE (8):

$$
H(t)t \to 1 \ as \ t \to \infty \ with \ probability \ 1 - \exp\left(-\frac{8}{9H_0}\right),
$$

$$
H(t) \to \infty \ as \ t \to T_{-\frac{2}{3H_0}} \ with \ probability \ \exp\left(-\frac{8}{9H_0}\right);
$$

i.e., the solution blows up in finite time with a positive probability $\exp\left(-\frac{8}{9H_0}\right)$, so the solution ceases to exist in finite time and therefore the asymptotic behavior is never achieved for these samples. Consequently, we may conclude that this model presents difficulties in its application to cosmology (at least, in the absence of empirical evidence in favor of such divergence).

**Remark 2.3** Despite the divergence, one could be tempted to use the formula for the Hubble parameter

$$
H(t) = (t + 3Wt/2 + 1/H_0)^{-1}
$$

to define a scale factor

$$
C(t) = \exp\left[\int_0^t H(s)ds\right].
$$

If this expression were well defined, this would open the possibility to somehow reformulate the Stratonovich problem to still get an almost surely meaningful scale factor. However, one can apply the mathematical framework proven in [1] to find
that the singular set of such a Hubble parameter would have Hausdorff dimension 1/2 with positive probability and therefore an uncountable number of divergences. In consequence, we cannot make sense of this formal expression for the scale factor by means of standard improper integration.

**Remark 2.4** Another way out of the problem is to impose a reflecting boundary condition at a sufficiently large value for the Hubble parameter. This would of course stop the divergence, but one would need to justify the presence of this boundary on physical terms. In the absence of such a justification, it looks simpler to us to just use the Itô interpretation, which is free of such a need, as we will show in the next section. Perhaps, more hopeful is to use such a problem to build a sequence of processes that approximates the Stratonovich \( H(t) \) as the upper boundary moves toward infinity. The expression for the scale factor is well defined for each of these reflected processes and thus one could build a sequence of scale factors out of the sequence of reflected Hubble parameters. If this sequence converged in some suitable sense to a stochastic process one could, at least mathematically speaking, define the scale factor as the limit. If this program could be carried out successfully, it would lead to a well-defined scale factor constructed from an ill-defined Hubble parameter. Since we are not sure of a precise mathematical implementation and an intuitive physical interpretation of such an approximation procedure, we leave this question open in the present work.

**Remark 2.5** Yet another way out of the problem caused by the finite time divergence is to assume the existence of a multiverse. Each universe in this multiverse would be characterized by a different realization of the noise, and our universe would be one in which the divergence does not happen (since indeed the Stratonovich model is divergence free with a positive probability). Because multiverse physics is at this moment in a speculative stage, we will not discuss more this possibility.

### 2.2 Itô equation

Equation (7) can also be written in the different precise manner:

\[
\text{d}H = -H^2 \text{d}t - \frac{9}{2}H^2 \text{d}W_t,
\]

if we assume the stochastic integral of Itô. As for the Stratonovich model, the terms are so regular that this equation will possess a unique strong solution as long as it remains bounded [16]. If in this case we change variables again \( x = 1/H \), then, by the Itô chain rule, we find

\[
\text{d}x = \left(1 + \frac{9}{4x}\right) \text{d}t + \frac{3}{2} \text{d}W_t.
\]

Note that this equation, again due to the regularity of its terms, will possess a unique strong solution as long as it remains bounded and bounded away from zero [16]. We will show that it is well behaved in the sense we are looking for. In order to analyze it, we need to introduce the following family of stochastic processes.
Definition 2.6 For any integer \( n \geq 2 \), let \( \tilde{W}_t = (W_1^t, \ldots, W_n^t), t \geq 0 \), be a \( n \)-dimensional Wiener process. Then, the process \( R_t = |W_t|, t \geq 0 \), is called a Bessel process of dimension \( n \).

Remark 2.7 A Bessel process of dimension \( n \) satisfies the SDE
\[
dR_t = \frac{n-1}{2R_t} dt + dW_t, \quad R_0 = 0,
\]
where \( W_t \) is the standard one-dimensional Wiener process; see [14].

Theorem 2.8 The SDE (11) subject to an initial condition \( x(0) = 1/H_0 > 0 \) possesses a unique solution which is both strong and global and moreover fulfills
\[
\lim_{t \to \infty} \frac{x(t)}{t} = 1 \quad \text{a.s.}
\]
Furthermore, \( x(t) > 0 \) for all \( t \geq 0 \) a.s.

Proof First of all, change variables \( z = \frac{2}{3}x \) to obtain
\[
dz = \frac{2}{3} dt + \frac{1}{z} dW_t.
\]
Note that the SDE
\[
dy = \frac{1}{y} dt + dW_t,
\]
subject to \( y(0) = 0 \) possesses as unique solution the Bessel process \( y(t) = R_t \) with dimension \( n = 3 \); see Definition 2.6 and Remark 2.7. Then, \( y(t) > 0 \) for all \( t > 0 \) almost surely, see [14], and consequently \( z(t) > 0 \) for all \( t > 0 \) almost surely provided that, for the same realization of \( W_t \), \( z(t) \geq y(t) \) for all \( t > 0 \) almost surely. Note that this should be true at least in some interval \([0, t]\), for some positive stopping time \( t = t(\omega), \) given the continuity of the trajectories of both \( z(t) \) and \( y(t) \), and the fact that \( z(0) = \frac{2}{3H_0} > 0 = y(0) \). Suppose that, on the contrary, there exists a set \( \tilde{\Omega} \subset \Omega \) of positive measure such that for all \( \omega \in \tilde{\Omega} \) there is a random time \( t_\omega > t > 0 \) such that \( z(t_\omega, \omega) < y(t_\omega, \omega) \), while \( z(t, \omega) > 0 \) for all \( t \in [0, t_\omega] \). Moreover, by the continuity of the trajectories of both stochastic processes, and given their initial conditions, there should exist a second random time \( t'_\omega \), with \( 0 < t \leq t'_\omega < t_\omega \), such that the inequality \( z(t, \omega) < y(t, \omega) \) holds for all \( t \in (t'_\omega, t_\omega] \) while \( z(t'_\omega, \omega) = y(t'_\omega, \omega) \). Therefore,
\[
z(t_\omega, \omega) - y(t_\omega, \omega) = \frac{2(t_\omega - t'_\omega)}{3} + \int_{t'_\omega}^{t_\omega} \left( \frac{1}{z(\tau, \omega)} - \frac{1}{y(\tau, \omega)} \right) d\tau > \frac{2(t_\omega - t'_\omega)}{3} > 0,
\]
in contradiction with $z(t_\omega, \omega) < y(t_\omega, \omega)$. This proves $z(t)$ is positive for all times almost surely and, moreover, that it is lower-bounded by the Bessel process with dimension $n = 3$.

To prove global boundedness and the asymptotic behavior, we proceed again by means of comparison arguments. In particular, we claim that

$$z_-(t) \leq z(t) \leq z_+(t),$$  \hspace{1cm} (13)

for all $t \geq 0$ almost surely, where

$$z_-(t) := \frac{2}{3} t + W_t + z(0),$$

and

$$z_+(t) := y(t) + \frac{2}{3} t + z(0),$$

where we, of course, always assume the same realization of $W_t$. Indeed, in the first case we have

$$z(t, \omega) - z_-(t, \omega) = \int_0^t \frac{1}{z(\tau, \omega)} d\tau \geq 0,$$

almost surely, due to the almost sure positivity of $z(t)$, while in the second case we find

$$z_+(t, \omega) - z(t, \omega) = \int_0^t \left( \frac{1}{y(\tau, \omega)} - \frac{1}{z(\tau, \omega)} \right) d\tau \geq 0,$$

almost surely since $z(t) \geq y(t)$ almost surely. Now, since $\frac{W_t}{t} \to 0$ as $t \to \infty$ almost surely by Proposition 3.3, then

$$\frac{y(t)}{t} = \sqrt{\left( \frac{W_{t1}^1}{t} \right)^2 + \left( \frac{W_{t2}^2}{t} \right)^2 + \left( \frac{W_{t3}^3}{t} \right)^2} \to 0 \text{ a.s.}$$

as $t \to \infty$ by Definition 2.6. Finally, from (13) we conclude

$$\lim_{t \to \infty} \frac{z(t)}{t} = \frac{2}{3} \text{ a.s.},$$

and the statement follows from undoing the change of variables $x = \frac{3}{2} z$. \hfill \Box

From Theorem 2.8, we conclude that the change of variables $x(t) = 1/H(t)$ is well defined for all $t \geq 0$ almost surely, so we can invert it to conclude

$$H(t)t \to 1 \text{ as } t \to \infty \text{ with probability 1.}$$
3 Conclusions

In this work, we have studied a stochastic version of the Friedmann acceleration equation. This SDE was introduced in [13] as a possible theoretical explanation of the uncertainty observed in the experimental quantification of the Hubble parameter. In the original model, the Stratonovich interpretation of noise was tacitly chosen, as the authors analyzed this equation using the standard chain rule from ordinary calculus. Herein, we have shown that this Stratonovich equation presents finite time blowups with a positive probability; in particular, this probability is

$$\exp\left(-\frac{8}{9H_0}\right).$$

On the other hand, we have shown that if the interpretation is changed to that of Itô, then the solution exists globally in time almost surely and moreover we have the asymptotic behavior

$$H(t) \to 1 \text{ as } t \to \infty \text{ with probability 1},$$

or, in other words, we have for almost every sample the universal decay rate $H(t) \approx 1/t$, with universal amplitude, for long times. This coincides with both the deterministic behavior (which of course takes place surely) and the long time dynamics of the Stratonovich SDE in those cases in which the solution does not blow up in finite time. Yet another advantage of the Itô interpretation is that it admits the explicit bounds

$$\frac{1}{\frac{3}{2}y(t) + t + \frac{1}{H_0}} \leq H(t) \leq \frac{1}{\max\left\{\frac{3}{2}W_t + t + \frac{1}{H_0}, \frac{3}{2}y(t)\right\}}$$

for all times $t \geq 0$ almost surely, where $y(t)$ is the Bessel process with dimension $n = 3$ that solves equation (12); this is a direct consequence of the proof of Theorem 2.8.

Although it seems that the Stratonovich interpretation is more accepted for applications in physics, we have herein highlighted an example in the field of cosmology in which the Itô interpretation looks more advantageous. This is not the only case, as other problems in statistical mechanics are somehow similar in this respect [6,9,10]. In these previous cases, however, the uniqueness of solution was lost in the Stratonovich equation, while preserved in the Itô one. In the present case, it is the (global in time) existence of solution what is lost for Stratonovich and preserved for Itô. Another difference between these previous statistical mechanical examples and the current cosmological one is that the deficiency of the Stratonovich equation was of a more fundamental character there, as there is a Stratonovich equation equivalent to equation (10) due to the smoothness of its diffusion; this did not happen in the other cases because the equations under study there did not present regular enough diffusion terms. Anyway, in the problem we have analyzed herein, it seems much simpler to employ the Itô interpretation rather than trying to circumvent it via an equivalent Stratonovich formulation. In summary, we have illustrated one example in cosmology for which Itô
versus Stratonovich means global existence versus finite-time blowup. Of course, this does not mean that physical arguments are not important in the derivation of stochastic differential models. They are fundamental, and, in fact, we have used them in the derivation of the models considered herein. However, the mathematical consistency of a physically derived model must be also checked. In those cases in which basic properties such as existence and uniqueness of solution fail, either there is a physical reason for this failure, or the model needs to be rectified.

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Appendix: Stochastic dynamical systems

In this section, we provide a summary of the mathematical framework we use, that of random and stochastic dynamical systems. To this end, we follow [2,4,7]. We start introducing the definition of random dynamical system. We use the notation $\mathbb{R}^+ := [0, \infty)$.

Definition 3.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ a map $\theta : \mathbb{R} \times \Omega \mapsto \Omega$, satisfying

1. $\theta_0(\omega) = \omega$, $\forall \omega \in \Omega$.
2. $\theta_{t+s}(\omega) = \theta_t \circ \theta_s(\omega)$, $\forall t, s \in \mathbb{R}_+$.
3. $(t, \omega) \mapsto \theta_t(\omega)$ is measurable in $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$, and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$ (i.e., $\mathbb{P}(\theta_t(\omega)) = \mathbb{P}(\omega)$).

Then, $(\Omega, \mathcal{F}, \mathbb{P}; \theta)$ is called a driving dynamical system and $\theta$ is called a driver.

Definition 3.2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random dynamical system (RDS) over $\mathbb{R}^d$ ($d = 1, 2, 3, \cdots$) consists of a pair $(\theta, \varphi)$ with $\theta$ a driver and a cocycle mapping $\varphi : \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying:

1. $\varphi(0, \omega, x) = x$, $\forall \omega \in \Omega, x \in \mathbb{R}^d$.
2. $\varphi(t+s, \omega, x) = \varphi(t, \theta_s(\omega), \varphi(s, \omega, x))$, $\forall t, s \in \mathbb{R}_+, \omega \in \Omega, x \in \mathbb{R}^d$.
3. $(t, \omega, x) \mapsto \varphi(t, \omega, x)$ is measurable.
4. $x \mapsto \varphi(t, \omega, x)$ is continuous $\forall t \in \mathbb{R}_+$ a.s.

Henceforth, we fix $d = 1$ and take the sample space $\Omega = C_0(\mathbb{R}, \mathbb{R})$, the space of continuous functions that depend on a real variable, take values on the reals, and
vanish at the origin; let $\mathcal{F}$ be its Borel sigma-field. Moreover, we take $\mathbb{P}$ to be the Wiener measure, $\theta$ the Wiener shift

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t),$$

and $\omega(t)$ two-sided Brownian motion, with the usual definition (two independent Wiener processes, respectively, with positive and negative time, glued at the origin). In this setting, consider the one-dimensional linear stochastic differential equation (SDE)

$$dz(t) = -\lambda z(t) dt + dW_t,$$

for some $\lambda > 0$ and $W_t(\omega) = \omega(t)$. Note that this is an Orstein–Uhlenbeck SDE.

Then, the following result holds [4].

**Proposition 3.3** Let $\lambda > 0$. Then, there exists a $\{\theta_t\}_{t \in \mathbb{R}}$-invariant (in the sense of (3) in Definition 3.1) subset $\overline{\Omega} \in \mathcal{F}$ of $\Omega = C_0(\mathbb{R}, \mathbb{R})$ with unit measure such that for all $\omega \in \overline{\Omega}$:

1. $\lim_{t \to \pm \infty} \frac{\omega(t)}{t} = 0$.
2. The random variable given by $z^*(\omega) := -\lambda \int_{-\infty}^{0} e^{\lambda \tau} \omega(\tau) d\tau$ is well defined, and the map

$$(t, \omega) \mapsto z^*(\theta_t \omega) = -\lambda \int_{-\infty}^{0} e^{\lambda \tau} \theta_t(\omega(\tau)) d\tau = -\lambda \int_{-\infty}^{0} e^{\lambda \tau} \omega(t + \tau) d\tau + \omega(t),$$

is a stationary solution of (14), with continuous trajectories satisfying:

(a) $\lim_{t \to \pm \infty} \frac{|z^*(\theta_t \omega)|}{t} = 0$,

(b) $\lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} z^*(\theta_t \omega) d\tau = 0$,

(c) $\lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} |z^*(\theta_t \omega)| d\tau < \infty$.

**Proof** We just give a sketch of the proof and refer the reader to the references for more details.

(1) follows from the law of iterated logarithms (see for instance [14]):

$$\mathbb{P} \left\{ \limsup_{t \to \infty} \frac{\omega(t)}{\sqrt{2t \log \log t}} = 1 \right\} = \mathbb{P} \left\{ \limsup_{t \to \infty} \frac{\omega(t)}{\sqrt{2t \log \log t}} = 1 \right\} = 1.$$

(2) is a consequence of the fact that the process $(\omega, t) \mapsto \int_{-\infty}^{t} e^{-\lambda(t-\tau)} dW_\tau(\omega)$ is a solution to (14) with $z(0) = 0$ and of integration by parts. The stationarity of this solution follows from the invariance of the Wiener measure $dW_t$ with respect to the flow $\{\theta_t\}_t$. The other statements follow from the ergodic theorem and the Burkholder inequality (see for instance [14]).

$\square$
At this point, we will employ the theory of conjugated RDSs to SDEs with additive noise. To that end, we will need the following result [4].

**Proposition 3.4** Let \((\theta, \varphi)\) be a RDS on \(\mathbb{R}\). Suppose that the mapping \(T : \Omega \times \mathbb{R} \mapsto \mathbb{R}\) possesses the following properties:

1. The mapping \(T(\omega, \cdot)\) is a homeomorphism over \(\mathbb{R}\) for every \(\omega \in \Omega\).
2. The mappings \(T(\cdot, x)\) and \(T^{-1}(\cdot, x)\) are measurable for every \(x \in \mathbb{R}\).

Then, the mapping \((t, \omega, x) \mapsto \phi(t, \omega, x) := T^{-1}(\theta_t(\omega), \varphi(t, \omega, T(\omega, x)))\) defines a conjugated random dynamical system.

**Proof** First, note that

\[ T^{-1}(\theta_0(\omega), \varphi(0, \omega, T(\omega, x))) = T^{-1}(\omega, T(\omega, x)) = x. \]

Then, let us see that the property of cocycles is fulfilled:

\[
\begin{align*}
\phi(t + \tau, \omega, x) &= T^{-1}(\theta_{t+\tau}(\omega), \varphi(t + \tau, \omega, T(\omega, x))) \\
&= T^{-1}(\theta_t(\theta_\tau(\omega)), \varphi(t, \theta_\tau(\omega), \varphi(\tau, \omega, T(\omega, x))) \\
&= \phi(t, \theta_\tau(\omega), \phi(\tau, \omega, x)).
\end{align*}
\]

\(\square\)

Now, following [4], we apply this theory to a SDE with additive noise of the form

\[ dx(t, \omega) = f(x)dt + dW_t, \ x(0, \omega) = x_0 \in \mathbb{R}, \] (15)

where we suppose that \(f\) is continuously differentiable with bounded derivative, so that a unique strong solution to (15) exists almost surely (see [16] for milder conditions for existence and uniqueness). Subsequently perform the change of variables

\[ y(t, \omega) = x(t, \omega) - z^*(\theta_t(\omega)), \] (16)

where \(z^*\) is defined as in Proposition 3.3. Then, it holds that

\[ y(0, \omega) = x(0, \omega) - z^*(\omega) = x_0 - z^*(\omega). \]

So we obtain

\[
\begin{align*}
dy(t, \omega) &= dx(t, \omega) - dz^* \\
&= f(x(t, \omega))dt + dW_t(\omega) - (\lambda z^*(\theta_t(\omega)))dt + dW_t(\omega) \\
&= (f(x(t, \omega)) + \lambda z^*(\theta_t(\omega)))dt \\
&= (f(y(t, \omega) + z^*(\theta_t(\omega))) + \lambda z^*(\theta_t(\omega)))dt,
\end{align*}
\]

and thus

\[
\frac{dy}{dt}(t, \omega) = f(y(t, \omega) + z^*(\theta_t(\omega))) + \lambda z^*(\theta_t(\omega)). \] (17)
Equation (17) is a random differential equation, and \( f \) is regular enough so that it generates a RDS \( \varphi \) \cite{Antunovic11}; then, from Proposition 3.4 we obtain a conjugated RDS for equation (15). Indeed, in this case, the map \( T : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is given by
\[
T(\omega, x) := x - z^*(\omega),
\]
which satisfies the hypotheses of Proposition 3.4. Therefore,
\[
\phi(t, \omega, x_0) = T^{-1}(\varphi(t, \omega, T(\omega, x_0))) \\
= T^{-1}(\varphi(t, \omega, x - z^*(\omega))) \\
= y(t, \omega, x_0 - z^*(\omega)) + z^*(\omega) = x(t, \omega, x_0),
\]
is our conjugated RDS for the SDE (15).

References

1. Antunović, T., Burdzy, K., Peres, Y., Ruscher, J.: Isolated zeros for Brownian motion with variable drift. Electron. J. Probab. 16, 1793–1814 (2011)
2. Arnold, L.: Random Dynamical Systems. Springer, Berlin (1998)
3. Buchert, T., Räsänen, S.: Backreaction in late-time cosmology. Annu. Rev. Nucl. Part. Sci. 62, 57–79 (2012)
4. Caraballo, T., Han, X.: Applied Nonautonomous and Random Dynamical Systems. Springer Nature, Cham (2016)
5. Chamizo, F.: Seminario 2001: una odisea en el espacio-tiempo. Available at: http://matematicas.uam.es/~fernando.chamizo/libreria/fich/APseminario02.pdf
6. Correales, Á., Escudero, C.: Itô vs Stratonovich in the presence of absorbing states. J. Math. Phys. 60, 123301 (2019)
7. Crauel, H., Flandoli, F.: Attractors for random dynamical systems. Probab. Theory Relat. Fields 100, 365–393 (1994)
8. Demaerel, T., Maes, C., Struyve, W.: Cosmic acceleration from quantum Friedmann equations. Class. Quantum Grav. 37, 085006 (2020)
9. Escudero, C.: Fluctuation-dissipation relation, Maxwell-Boltzmann statistics, equipartition theorem, and stochastic calculus, arXiv:2101.02543
10. Escudero, C.: Kinetic energy of the Langevin particle. Stud. Appl. Math. 145, 719–738 (2020)
11. Foster, J., Nightingale, J.D.: A Short Course in General Relativity. Springer, New York (2006)
12. Horstemke, W., Lefever, R.: Noise Induced Transitions. Springer, New York (2006)
13. John, M., Sivakumar, C., Joseph, B.: Classical stochastic approach to cosmology revisited. Pramana - J. Phys. 60, 1–10 (2003)
14. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. Springer, New York (1998)
15. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Princeton University Press, Princeton (2017)
16. Øksendal, B.: Stochastic Differential Equations: An Introduction with Applications. Springer, Berlin (2003)

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