Birkhoff decompositions for loop groups with
coefficient algebras

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Abstract
For certain locally convex topological algebras $A \subseteq C(S, \mathbb{C})$ generalizing the decomposable $R$-algebras of Gohberg and Fel’dman, we establish Birkhoff decompositions in $\text{GL}_n(A)$ of the form $f = f_+ D f_-$, where $D(z)$ a diagonal metrix with entries $z^{\kappa_1}, \ldots, z^{\kappa_n}$, and moreover $f_+ \in \text{GL}_n(A)$ is the boundary values of a holomorphic $\text{GL}_n(\mathbb{C})$-valued function on the open unit disk and $f_- \in \text{GL}_n(A)$ the boundary value of a $\text{GL}_n(\mathbb{C})$-valued holomorphic function on the complement of the closed unit disk in the Riemann sphere. Corresponding to the case $\kappa_1 = \cdots = \kappa_n = 0$, analogous decompositions are available in Lie groups $G_A \leq C(S, G)$ with Lie algebra $\mathfrak{g} \otimes_A \mathcal{A}$, for $G$ a finite-dimensional complex Lie group on which holomorphic functions separate points. Birkhoff decompositions in $O(C^\times, G)$ are also obtained. Some of the results remain valid if $G$ is a complex Banach-Lie group on which holomorphic functions separate points.

MSC 2020 subject classification: 22E67 (primary); 22E65, 26E15, 26E20, 46E25, 46E35, 46G20, 46J10, 58D15

Key words: Loop groups; Wiener-Lie groups; Birkhoff decomposition; matrix Riemann problem; Riemann-Hilbert problem; differentiability questions

1 Introduction and statement of the results

The book “Loop Groups” by Pressley and Segal [36] is a classical source on Birkhoff decompositions for loops in $\text{GL}_n(\mathbb{C})$ or in complexifications $K_\mathbb{C} \leq \text{GL}_n(\mathbb{C})$ of compact real Lie groups $K$ (a topic which started in [7]). As far as $\text{GL}_n(\mathbb{C})$ is concerned, their results include (see Remark 1.2 for $K_\mathbb{C}$):

1.1 (Pressley-Segal 1986) Each $f \in C^\infty(S, \text{GL}_n(\mathbb{C}))$ can be written as

$$ f = f_+ D f_- $$

where $D : S \to S^n$ is a continuous group homomorphism, $f_+ \in C^\infty(S, \text{GL}_n(\mathbb{C}))$ admits a continuous extension $f_+^\leq : \hat{S} \to \text{GL}_n(\mathbb{C})$ which is holomorphic on $\hat{S} \setminus \mathbb{D}$, and $f_- \in C^\infty(S, \text{GL}_n(\mathbb{C}))$ admits a continuous extension $f_-^\leq : \hat{S} \setminus \mathbb{D} \to \text{GL}_n(\mathbb{C})$ which is holomorphic on $\hat{S} \setminus \mathbb{D}$. The set $C^\infty(S, \text{GL}_n(\mathbb{C}))^+$ of such $f_+$ and the set $C^\infty(S, \text{GL}_n(\mathbb{C}))^-$ of such $f_-$ with $f_-^\leq(\infty) = 1$ are embedded submanifolds and Lie subgroups of the complex Lie group $C^\infty(S, \text{GL}_n(\mathbb{C}))$. The product map

$$ C^\infty(S, \text{GL}_n(\mathbb{C}))^+ \times C^\infty(S, \text{GL}_n(\mathbb{C}))^- \to C^\infty(S, \text{GL}_n(\mathbb{C}))^0, \quad (f_+, f_-) \mapsto f_+ f_- $$

has dense open image and is a $\mathbb{C}$-analytic diffeomorphism onto the latter.
Here $S := \{ z \in \mathbb{C} : |z| = 1 \}$ is the circle group, $\hat{\mathbb{C}} := \mathbb{C} \cup \{ \infty \}$ the Riemann sphere and $\hat{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ the open unit disk. Moreover, $1 \in \text{GL}_n(\mathbb{C})$ is the unit matrix and $C^\infty(S, \text{GL}_n(\mathbb{C}))_0$ the connected component of the neutral element $z \mapsto 1$ in the Lie group $C^\infty(S, \text{GL}_n(\mathbb{C}))$ of smooth loops in $\text{GL}_n(\mathbb{C})$.

**Remark 1.2** (a) Pressley and Segal [36] get analogues of 1.1 also for $C^\omega$ in place of $C^\infty$.
(b) They also show $C^\infty(S, K_\mathbb{C})^+ \times C^\infty(S, K_\mathbb{C})^\ominus \to C^\infty(S, K_\mathbb{C})_0$ is $\mathbb{C}$-analytic diffeomorphism onto a dense open subset.

The approach of Pressley and Segal is geometric in nature and fairly complicated. The effort is justified as it provides valuable additional information beyond the Birkhoff decompositions, like stratifications and cell decompositions.

In this article, we propose a complementary approach to Birkhoff decompositions, based on two pillars:

- The approach of Goberg-Fel’dman [25] to Birkhoff decompositions for loops in $\text{GL}_n(\mathbb{C})$ via Gelfand theory of commutative Banach algebras;
- Techniques from infinite-dimensional differential calculus/holomorphy and infinite-dimensional Lie theory.

More precisely, we shall generalize the approach by Goberg and Fel’dman, replacing Banach algebras with certain locally convex topological algebras, the continuous inverse algebras. In this article, the algebra multiplication

$$ \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (f, g) \mapsto fg $$

of a topological algebra is always assumed jointly continuous. The following concept goes back to work of Waelbroeck (cf. [12]); see [21] for a recent account with a view towards infinite-dimensional Lie theory.

**1.3** A locally convex, Hausdorff, associative complex topological algebra $\mathcal{A}$ with unit is called a *continuous inverse algebra* (or cia) if the group $\mathcal{A}^\times$ of invertible elements is open in $\mathcal{A}$ and the inversion map $\mathcal{A}^\times \to \mathcal{A}, \, a \mapsto a^{-1}$ is continuous.

We recall from Gohber-Fel’dman [25]:

**1.4** A Banach algebra $\mathcal{A} \subseteq C(S, \mathbb{C})$ is called an *R-algebra* if the set of restrictions $f|_\mathbb{S}$ of all rational functions $f$ with poles off $\mathbb{S}$ is dense in $\mathcal{A}$.

We shall us the following generalization beyond Banach algebras.

**Definition 1.5** A cia $\mathcal{A} \subseteq C(S, \mathbb{C})$ is called of type *R* (or an *R-cia*) if the set of restrictions $f|_\mathbb{S}$ of all rational functions $f$ with poles off $\mathbb{S}$ is dense in $\mathcal{A}$.

Then the inclusion map $\mathcal{A} \to (C(S, \mathbb{C}), \| \cdot \|_\infty)$ is continuous, moreover, $\mathcal{A}^\times = \mathcal{A} \cap C(S, \mathbb{C})^\times$ holds (so-called isospectrality), see Lemma [25].

Mimicking Gohber-Fel’dman, we make the following definitions if $\mathcal{A}$ is an *R-cia*:
**Definition 1.6** We write $A^+ \subseteq A$ for the closure of $\mathbb{C}[z]$ in $A$ and $A^\ominus \subseteq A$ for the closure of $z^{-1}\mathbb{C}[z^{-1}]$ in $A$. An $R$-cia $A$ is called *decomposing* if $A = A^+ \oplus A^\ominus$ as a topological vector space.

**1.7** In the following, we write

$$\hat{f}_k := \int_0^1 e^{2\pi i kt} f(e^{2\pi i t}) \, dt$$

for the $k$th Fourier coefficient of $f \in C(S, \mathbb{C})$, for $k \in \mathbb{Z}$.

As in [25], one shows:

**1.8** If $A \subseteq C(S, \mathbb{C})$ is a decomposing $R$-cia and $f \in A$, then the following properties are equivalent:

(a) $f \in A^+$

(b) $\hat{f}_k = 0$ for all $k < 0$;

(c) $f$ has continuous extension $f^\prec : \overline{D} \to \mathbb{C}$ which is holomorphic on $D$.

Likewise, the following conditions are equivalent:

(d) $f \in A^\ominus$;

(e) $\hat{f}_k = 0$ for all $k \geq 0$;

(f) $f$ has a continuous extension $f^\succ : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ which is holomorphic on $\mathbb{C} \setminus \overline{D}$, with $f^\succ(\infty) = 0$.

We set $A^- := A^\ominus + \mathbb{C}1$ (where $1$ is the constant function $z \mapsto 1$). Then $A^-$ is the closure of $\mathbb{C}[z^{-1}]$ in $A$ and $f \in A$ is in $A^-$ if and only if $\hat{f}_k = 0$ for all $k > 0$.

**1.9** If $A \subseteq C(S, \mathbb{C})$ is an algebra which may not be a decomposing $R$-cia, we define $A^+$ as the set of all $f \in A$ satisfying condition (c) of [18] and $A^\ominus$ as the set of all $f \in A$ satisfying condition (f) of [18]. Then $A^+$ is a unital subalgebra of $A$ and $A^\ominus$ is a non-unital subalgebra of $A$.

The following permanence properties of the class of $R$-cias are easily checked.

**Proposition 1.10**  

(a) If $A_1 \supseteq A_2 \supseteq \cdots$ are $R$-cias and all inclusion maps are continuous unital algebra homomorphisms, then $\bigcap_{n \in \mathbb{N}} A_n = \lim_{\leftarrow} A_n$ is an $R$-cia which is decomposing if each $A_n$ is so.

(b) If $A_1 \subseteq A_2 \subseteq \cdots$ are $R$-algebras and all inclusion maps are continuous unital algebra homomorphisms, then the locally convex direct limit $A = \bigcup_{n \in \mathbb{N}} A_n$ is an $R$-cia which is decomposing if each $A_n$ is so.
Note that, in the situation of (b), the inclusion map $\iota : \mathcal{A} \to C(S, \mathbb{C})$ is continuous on each $\mathcal{A}_n$ and hence continuous on the direct limit locally convex space $\mathcal{A}$, being linear. Since $C(S, \mathbb{C})$ is Hausdorff, so is $\mathcal{A}$. Since all of the inclusions $\mathcal{A}_n \to C(S, \mathbb{C})$ are isospectral,

$$\mathcal{A}^\times = \iota^{-1}(C(S, \mathbb{C})^\times)$$

follows, whence $\mathcal{A}^\times$ is open in $\mathcal{A}$. By [1] or [16], $\mathcal{A}$ is locally $m$-convex in the sense of Michael [31], entailing that the inversion map $\mathcal{A}^\times \to \mathcal{A}$ and the algebra multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are continuous. Hence $\mathcal{A}$ is a cia. Since $\mathbb{C}[z, z^{-1}]$ is dense in each $\mathcal{A}_n$, it is dense in $\mathcal{A}$. All rational functions with poles off $S$ are in $\mathcal{A}_1$ and hence in $\mathcal{A}$, whence $\mathcal{A}$ is an $R$-cia.

**Remark 1.11** If $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ are merely $R$-cias in Proposition 1.10 (b), then the conclusions still hold if we assume, moreover, that the algebra multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous. In fact, $\mathcal{A}$ is locally $m$-convex in this case by [28], and now we can complete the argument as before.

**1.12** Let us compile some topological algebras $\mathcal{A}$ which may be interesting algebras of coefficients for loop groups, and their properties. All of them are subalgebras of $C(S, \mathbb{C})$ and contain the algebra $\mathbb{C}[z, z^{-1}]$ of Laurent polynomials as a dense subalgebra, $\mathbb{C}[z, z^{-1}] \subseteq \mathcal{A} \subseteq C(S, \mathbb{C})$.

(a) The Banach algebra $(C(S, \mathbb{C}), \| \cdot \|_\infty)$ of continuous complex-valued functions on $S$ is an $R$-algebra. It is a classical fact that $C(S, \mathbb{C})$ is not decomposing (see [25]).

(b) The Wiener algebra $\mathcal{W} := \{ f \in C(S, \mathbb{C}) : (\hat{f}_k)_{k \in \mathbb{Z}} \in \ell^1 \}$ is a decomposing $R$-algebra using the norm given by $\| f \|_\mathcal{W} := \| (\hat{f}_k)_{k \in \mathbb{Z}} \|_\ell^1 = \sum_{k \in \mathbb{Z}} |\hat{f}_k|$ (see [25]). Here

$$f^<(z) := \sum_{k=0}^{\infty} \hat{f}_k z^k \quad \text{for } z \in \mathbb{D}$$

for $f \in \mathcal{W}^+$ and

$$f^>(z) := \sum_{k=1}^{\infty} \hat{f}_{-k} z^{-k} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{D}$$

for $f \in \mathcal{W}^\ominus$, with $f^>(\infty) := 0$.

(c) For $m \in [0, \infty[$, the set

$$\mathcal{W}(m) := \left\{ f \in C(S, \mathbb{C}) : \sum_{k \in \mathbb{Z}} |k|^m |\hat{f}_k| < \infty \right\}$$

is a unital subalgebra of $\mathcal{W}$. The norm $f \mapsto \sum_{k \in \mathbb{Z}} \max\{|k|^m, 1\} |\hat{f}_k|$ turns $\mathcal{W}(m)$ into a Banach space and makes the algebra multiplication continuous. Let $L_f(g) := fg$ for $f, g \in \mathcal{W}(m)$. Then $\| f \|_{\mathcal{W}(m)} := \| L_f \|_{\text{op}}$ defines

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an equivalent norm on $W(m)$ which is submultiplicative and turns $W(m)$ into an $R$-algebra which is decomposing. We can think of the weighted Wiener algebras as Sobolev-like spaces.

(d) The Fréchet algebra $C^\infty(S,\mathbb{C})$ of all smooth complex-valued functions on the circle can be regarded as the projective limit

$$C^\infty(S,\mathbb{C}) = \bigcap_{m \in \mathbb{N}_0} W(m) = \lim_{\leftarrow} W(m)$$

as a locally convex space, as the intersection of the weighted $\ell^1$-spaces corresponding to $W(m)$ is the Fréchet space $s$ of rapidly decreasing two-sided sequences, which equals the set of Fourier transforms $(\hat{f}_k)_{k \in \mathbb{Z}}$ of smooth functions $f \in C^\infty(S,\mathbb{C})$. By Proposition 1.10 (a), $C^\infty(S,\mathbb{C})$ is a decomposing $R$-cia.

(e) The Silva space $C^\omega(S,\mathbb{C}) = \lim_{\rightarrow} Ob(A_n,\mathbb{C})$ of real-analytic $\mathbb{C}$-valued functions is a decomposing $R$-cia. Here $Ob(A_n,\mathbb{C})$ denotes the Banach algebra of bounded holomorphic functions on the open annulus $A_n := \{z \in \mathbb{C} : 1 - \frac{1}{n} < |z| < 1 + \frac{1}{n}\}$, endowed with the supremum norm. Actually, it will be useful later to use the following equivalent norm which also makes $Ob(A_n,\mathbb{C})$ a Banach algebra:

$$\|f\|_n := \max\{\|f\|_\infty, \|f|_S\|_W\}$$

where $\| \cdot \|_W$ is the norm of the Wiener algebra.

To establish the asserted properties, note that $C^\omega(S,\mathbb{C})$ is locally $m$-convex by [10] and has an open unit group, as the inclusion map $\iota : C^\omega(S,\mathbb{C}) \rightarrow C(S,\mathbb{C})$ is continuous and linear and

$$\iota^{-1}(C(S,\mathbb{C})^\times) = C^\omega(S,\mathbb{C}),$$

using that $1/f$ is real analytic for each real-analytic function $f : S \rightarrow \mathbb{C}$ such that $0 \not\in f(S)$. The principal and regular parts of the Laurent series of $f \in Ob(A_n,\mathbb{C})$ converge uniformly on $A_{n+1}$, entailing that $\mathbb{C}[z, z^{-1}]$ (and hence also rational functions with poles off $S$) are dense in $C^\omega(S,\mathbb{C})$.

(f) Finally, we shall consider the Fréchet algebra $O(\mathbb{C}^\times, \mathbb{C})$ of all holomorphic functions on the punctured plane, endowed with the topology of compact convergence. This algebra is not a cia as its unit group is not open. Still, it is locally $m$-convex and

$$O(\mathbb{C}^\times, \mathbb{C}) = O(\mathbb{C}^\times, \mathbb{C})^+ \oplus O(\mathbb{C}^\times, \mathbb{C})^\ominus$$

holds as a locally convex space. In fact, as the decomposition of a holomorphic function into its regular part and principal part is unique, the
map in (4) is bijective. As its inverse map is continuous and both sides are Fréchet spaces, the map is an isomorphism of topological vector spaces by the open mapping theorem.

Concerning loops in GLₙ(ℂ), we obtain the following results, whose special cases for R-algebras can be found in [25]. Recall that a locally convex topological vector space is called Mackey complete if every Mackey-Cauchy sequence converges in it (see [29] for an in-depth discussion of this property). Every sequentially complete locally convex space (in which every Cauchy sequence converges) is Mackey complete.

**Proposition 1.13** If A is a decomposing R-cia which is Mackey complete, then every g ∈ A× can be written as

\[ g = g_+(\text{id}_S)^\kappa g_- \]

with \( g_+ \in (A^+)^\times \) and some \( \kappa \in \mathbb{Z} \). Moreover, \( \kappa = \text{ind}_0(g) \) is the winding number of g with respect to the origin 0 ∈ ℂ.

**Proposition 1.14** Let A be an R-algebra. Then each \( g \in \text{GL}_n(A) \) can be written in the form

\[ g = g_+Dg_- \]

where \( g_+ \in \text{GL}_n(A^+) \), \( g_- \in \text{GL}_n(A^-) \) and \( D(z) = \text{diag}(z^{\kappa_1}, \ldots, z^{\kappa_n}) \) with integers \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \). The n-tuple \( (\kappa_1, \ldots, \kappa_n) \) of “partial indices” is unique. The same conclusion holds if A is any Mackey complete, decomposing R-cia and \( \text{GL}_n(A^+) \text{GL}_n(A^-) \) is an identity neighbourhood in \( \text{GL}_n(A) \).

1.15 If \( A \) is a complex locally convex commutative topological algebra and \( g \) a finite-dimensional complex Lie algebra, we endow \( g_A := g \otimes_C A \) with the projective tensor topology (as in 2.5). If \( A \) is a subalgebra of \( C(S, \mathbb{C}) \), we have

\[ g_A = g \otimes_C A \subseteq g \otimes_C C(S, \mathbb{C}) = C(S, g) \]

and we consider \( g_A \) as a subalgebra of \( C(S, g) \). If the inclusion map \( A \to C(S, \mathbb{C}) \) is continuous, then also the inclusion map \( g_A \to C(S, g) \).

1.16 If \( G \) is a complex Lie group modeled on a Banach space (a complex Banach–Lie group), with Lie algebra \( g := T_0G \) and exponential function \( \exp_G : g \to G \), then \( C(S, G) \) can be made a complex Banach–Lie group with Lie algebra \( C(S, g) \), as is well known. Its exponential function is the map

\[ C(S, g) \to C(S, G), \quad f \mapsto \exp_G \circ f. \]

If \( A \) is an R-algebra, then the inclusion map \( g_A \to C(S, g) \) is an injective continuous Lie algebra homomorphism between Banach–Lie algebras. Hence

\[ G_A := \langle \exp_G \circ f : f \in A \rangle \]
can be made a Banach–Lie group with Lie algebra $\mathfrak{g}_A$, see [14]. If $A$ is decomposing, we get analogous subgroups $G_A^+$ and $G_A^\ominus$ in $C(S, G)$ which are Banach-Lie groups with Lie algebras $\mathfrak{g}_A^+: = \mathfrak{g}_A^+ \oplus \mathfrak{g}_A^\ominus$, respectively. The inclusion maps $G_A^+ \to G_A$ and $G_A^\ominus \to G_A$ are complex analytic group homomorphisms.

1.17 If $A$ is a Mackey complete $R$-cia, we get a Lie group $G_A \subseteq C(S, G)$ with Lie algebra $\mathfrak{g}_A$ and Lie groups $G_A^+$ and $G_A^\ominus$ which are subgroups of $C(S, G)$ if $A$ is, moreover, decomposing. The latter have Lie algebras $\mathfrak{g}_A^+ \ominus \mathfrak{g}_A^\ominus$, respectively and properties as described in 1.16 (see 2.11).

Note that $G_A$ is the connected component $C^\infty(S, G)_0$ of $C^\infty(S, G)$ if $A = C^\infty(S, \mathbb{C})$. Likewise, $G_A = C^\infty(S, G)_0$ if $A = C^\infty(S, \mathbb{C})$.

1.18 If $\mathfrak{g}$ is a complex Banach–Lie algebra with norm $\| \cdot \|$, we can consider the Fréchet–Lie algebra $C^\infty(S, \mathfrak{g})$ of smooth $\mathfrak{g}$-valued functions on $S$, endowed with the compact-open $C^\infty$-topology (as in [24]). We can also consider the locally convex topological Lie algebra $C^\omega(S, \mathfrak{g}) = \lim_{\to} \mathfrak{g}_A^\omega(A_n, \mathfrak{g})$ of real-analytic $\mathfrak{g}$-valued functions on $S$, the Wiener–Lie algebra

$$\mathfrak{g}_W := \{ f \in C(S, \mathfrak{g}) : (\hat{f}_k)_{k \in \mathbb{Z}} \in \ell^1(\mathbb{Z}, \mathfrak{g}) \}$$

and the weighted Wiener–Lie algebras

$$\mathfrak{g}_{W(m)} := \left\{ f \in C(S, \mathfrak{g}) : \sum_{k \in \mathbb{Z}} |k|^m \| \hat{f}_k \| < \infty \right\}$$

for $m \in [0, \infty]$, where $\hat{f}_k$ is defined as the $\mathfrak{g}$-valued integral (1). The Wiener–Lie algebras and weighted Wiener–Lie algebras are Banach–Lie algebras. If $\mathfrak{g} = T_e G$ for some complex Banach–Lie group $G$, we define

$$G_{W}, G_{W}^+, G_{W}^\ominus, G_{W(m)}, G_{W(m)}^+, G_{W(m)}^\ominus,$$

as in 1.16.

The inverse function theorem readily implies the following (see Section 5).

**Proposition 1.19** Let $A$ be a decomposing $R$-algebra and $G$ be a finite-dimensional complex Lie group, or $A = W(m)$ with $m \in [0, \infty]$ and $G$ be a complex Banach–Lie group. Assume that $O(G, \mathbb{C})$ separates points on $G$. Then $G_A^+ G_A^\ominus$ is open in $G_A$ and the product map

$$\pi : G_A^+ \times G_A^\ominus \to G_A^+ G_A^\ominus$$

is a $\mathbb{C}$-analytic diffeomorphism.
For a complex Banach–Lie group $G$, we shall use the complex Fréchet–Lie groups $\mathcal{O}(\mathbb{C}^\times, G)$, $\mathcal{O}(\mathbb{C}, G)$, and $\mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, G)$ of all holomorphic $G$-valued functions on $\mathbb{C}^\times$, $\mathbb{C}$, and $\hat{\mathbb{C}} \setminus \{0\}$, respectively, as constructed in [35]. We recall from [35]:

1.20 Let $M$ be a non-compact Riemann surface with finitely generated fundamental group. Let $G$ be a complex Banach–Lie group with neutral element $e$, Lie algebra $\mathfrak{g} := T_e G$ and exponential function $\exp_G : \mathfrak{g} \to G$. Then the group $\mathcal{O}(M, G)$ can made a Fréchet–Lie group with Lie algebra $\mathcal{O}(M, \mathfrak{g})$ and exponential function $\mathcal{O}(M, \mathfrak{g}) \to \mathcal{O}(M, G), \ f \mapsto \exp_G \circ f$,
such that the following exponential law holds: For each complex analatic manifold $N$ modelled on a complex locally convex space, a map $f : N \to \mathcal{O}(M, G)$
is complex analytic if and only if the corresponding mapping $f^\wedge : N \times M \to G, \ (n, m) \mapsto f(n)(m)$
is complex analytic.

Remark 1.21 By construction, $\mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, G)_* := \{ f \in \mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}) : f(\infty) = e \}$ is an embedded complex submanifold of $\mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, G)$ and a Lie subgroup. As a consequence, the exponential law also holds for functions $f : N \to \mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, G)_*$.

The following three theorems are the main results of the article.

Theorem 1.22 For each complex Banach–Lie group $G$ on which $\mathcal{O}(G, \mathbb{C})$ separates points, the map $p : \mathcal{O}(\mathbb{C}, G) \times \mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, G)_* \to \mathcal{O}(\mathbb{C}^\times, G), \ (g_+, g_\Theta) \mapsto g_+|_{\mathbb{C}^\times} g_\Theta|_{\mathbb{C}^\times}$
has open image and is a $\mathbb{C}$-analytic diffeomorphism onto its image.

1.23 Given a complex Banach–Lie group $G$ on $\mathcal{O}(G, \mathbb{C})$ separates points, we let $C^\infty(\mathbb{S}, G)^+$ be the set of all $f \in C^\infty(\mathbb{S}, G)$ admitting a continuous extension $f^\wedge : \overline{\mathbb{D}} \to G$ which is complex analytic on $\mathbb{D}$. We let $C^\infty(\mathbb{S}, G)^\circ$ be the set of all $f \in C^\infty(\mathbb{S}, G)$ admitting a continuous extension $f^\wedge : \hat{\mathbb{C}} \setminus \mathbb{D} \to G$ which is complex analytic on $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and satisfies $f(\infty) = e$. Then $C^\infty(\mathbb{S}, G)^+$ and $C^\infty(\mathbb{S}, G)^\circ$ are subgroups of $C^\infty(\mathbb{S}, G)$. Subgroups $C^\infty(\mathbb{S}, G)^+$ and $C^\infty(\mathbb{S}, G)^\circ$ of $C^\omega(\mathbb{S}, G)$ can be defined along the same lines.

Theorem 1.24 For each complex Banach–Lie group $G$ on which $\mathcal{O}(G, \mathbb{C})$ separates points, the map $\pi : C^\omega(\mathbb{S}, G)^+ \times C^\omega(\mathbb{S}, G)^\circ \to C^\omega(\mathbb{S}, G)$
has open image and is a $\mathbb{C}$-analytic diffeomorphism onto its image.
In the following theorem, Lie subgroups are understood as embedded complex submanifolds.

**Theorem 1.25** Let $G$ be a finite-dimensional complex Lie group which is a Lie subgroup of $\text{GL}_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Then the map

$$\pi : C^\infty(S, G)^+ \times C^\infty(S, G) \supset \rightarrow C^\infty(S, G).$$

has open image and is a $\mathbb{C}$-analytic diffeomorphism onto its image.

Let us compare our approach with [36].

**Remark 1.26** In our approach, we recover all of the known facts from 1.1 and Remark 1.2, except for the density property. The following aspects are new in our approach:

(a) The product map (in the Birkhoff decompositions) is a $\mathbb{C}$-analytic diffeomorphism for $\mathcal{O}(\mathbb{C}^\times, G)$ and $C^\infty(S, G)$;

(b) The theory applies to more general complex Lie groups $G$ (including some Banach–Lie groups), not only to complexifications $K_\mathbb{C}$;

(c) The approach covers further classes of loop groups based on other coefficient algebras.

We mention that our work has points of contact with [26], but differs in details. A broader discussion will be given in a later version of this manuscript.

## 2 Preliminaries and basic facts

We write $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. All topological vector spaces are assumed Hausdorff. If $(E, \| \cdot \|)$ is a normed space, we write $B_r^E(x) := \{y \in E : \|y - x\| < r \}$ and $\overline{B}_r^E(x) := \{y \in E : \|y - x\| \leq r \}$ for $x \in E$ and $r > 0$.

See [19] [24] for the following concept (as well as [8] [33] if $F$ is sequentially complete).

**Definition 2.1** Let $E$ and $F$ be locally convex complex topological vector spaces and $U \subseteq E$ be an open subset. A map $f : U \rightarrow F$ is called complex analytic (or $\mathbb{C}$-analytic) if it is continuous and, for each $x \in U$,

$$f(y) = \sum_{k=0}^{\infty} p_k(y - x)$$

for all $y$ in some $x$-neighbourhood in $U$, where $p_k : E \rightarrow F$ is a continuous complex homogeneous polynomial of degree $k$. 
If $F$ is Mackey complete, then a continuous function $f$ as before is complex analytic if and only if $f$ is holomorphic in the sense that the complex directional derivative
\[ df(x, y) := \lim_{z \to 0} \frac{f(x + zy) - f(x)}{z} \]
(with $z \in \mathbb{C} \setminus \{0\}$ in some 0-neighbourhood) exists for all $x \in U$ and $y \in E$, and $df : U \times E \to F$ is continuous (see, e.g., [24]). We shall then use the words “complex analytic” and “holomorphic” interchangeably.

Following [19, 24] (see already [33] for sequentially complete spaces), we define:

**Definition 2.2** Let $E$ and $F$ be locally convex real topological vector spaces and $U \subseteq E$ be an open subset. A map $f : U \to F$ is called real analytic if it admits a complex-analytic extension $V \to F_{\mathbb{C}}$ with values in the complexification $F_{\mathbb{C}}$ of $F$, defined on an open subset $V \subseteq E_{\mathbb{C}}$ with $U \subseteq V$.

Complex-analytic maps are real analytic (see [24]). Compositions of composable $\mathbb{K}$-analytic maps are $\mathbb{K}$-analytic for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (see [19, 24]). Therefore complex manifolds and complex Lie groups modelled on complex locally convex spaces can be defined as expected.

2.3 A complex-analytic manifold (or simply: complex manifold) modelled on a complex locally convex space $E$ is a Hausdorff topological space $M$, together with a maximal set $\mathcal{A}$ of homeomorphisms $\phi : U_\phi \to V_\phi$ (“charts”) from an open subset $U_\phi \subseteq M$ onto an open subset $V_\phi \subseteq E$ such that $\bigcup_{\phi \in \mathcal{A}} U_\phi = M$ and the chart changes $\phi \circ \psi^{-1}$ are complex analytic for all $\phi, \psi \in \mathcal{A}$.

Complex-analytic maps between complex manifolds are defined as usual, as continuous maps which are complex analytic in charts (see [24] for details).

2.4 A complex Lie group modelled on a complex locally convex space $E$ is a group $G$, endowed with a complex manifold structure modelled on $E$ which turns the group operations into complex analytic maps (see [24] for more details).

When we use the word “complex manifold” or “complex Lie group” in this article, we always mean a manifold or Lie group modelled on a complex locally convex space, which may be infinite-dimensional. We shall say explicitly when a manifold or Lie group is assumed of finite dimension.

2.5 If $E$ is locally convex space and $F$ a finite-dimensional vector space with basis $b_1, \ldots, b_\ell$, give $F \otimes_{\mathbb{C}} E$ the locally convex vector topology making
\[ E^\ell \to F \otimes_{\mathbb{C}} E, \quad (v_1, \ldots, v_\ell) \mapsto \sum_{j=1}^{\ell} b_j \otimes v_j \]
an isomorphism of topological vector spaces. It is independent of the choice of basis. Thus: If $F = F_1 \oplus F_2$, then
\[ F \otimes_{\mathbb{C}} E = (F_1 \otimes_{\mathbb{C}} E) \oplus (F_2 \otimes_{\mathbb{C}} E) \]
as a topological vector space. Notably, $F_1 \otimes_{\mathbb{C}} E$ is closed in $F \otimes_{\mathbb{C}} E$. 
The following properties can be shown as in [25].

**Lemma 2.6** For every $R$-cia $\mathcal{A} \subseteq C(S, \mathbb{C})$, the following holds:

(a) $S \to \mathbb{C}$, $z \mapsto z$ has spectrum $S$;

(b) $\mathbb{C}[z, z^{-1}]$ is dense in $\mathcal{A}$;

(c) The set $\mathcal{M}$ of algebra homomorphisms $\mathcal{A} \to \mathbb{C}$ is homeomorphic to $S$;

(d) Identifying $\mathcal{M}$ with $S$, the Gelfand transform is the inclusion map $\iota: \mathcal{A} \to C(S, \mathbb{C})$. □

See [34, 24] for the following concept (cf. also [39, 20]).

**Definition 2.7** A complex Lie group $G$ modelled on a complex locally convex space is called a **BCH-Lie group** if it has an exponential function which is a local diffeomorphism of complex analytic manifolds at 0.

Then $\exp^{-1}_G(\exp_G(x) \exp_G(y)) = x + y + \frac{1}{2}[x, y] + \cdots$ is given by the BCH-series for small $x, y \in \mathfrak{g}$ (see [24]).

The following fact from [21] is one source of BCH-Lie groups.

**2.8** For every complex continuous inverse algebra $\mathcal{A}$ which is Mackey complete, the unit group $\mathcal{A}^\times$ is a BCH-Lie group.

For example, $\mathcal{A}^{n \times n}$ is a cia for each cia $\mathcal{A}$, and thus $\text{GL}_n(\mathcal{A}) = (\mathcal{A}^{n \times n})^\times$ is a BCH-Lie group if $\mathcal{A}$ is Mackey complete.

**Definition 2.9** A locally convex, complex topological Lie algebra $\mathfrak{g}$ is called **BCH** if the BCH-series converges to a $\mathbb{C}$-analytic function $U \times U \to \mathfrak{g}$ for some open 0-neighbourhood $U \subseteq \mathfrak{g}$ (cf. [39]).

See [24] for the following fact (cf. also [39]).

**Lemma 2.10** (Integration of Lie subalgebras) Let $\mathfrak{g}$ be a BCH-Lie algebra. If there exists an injective, continuous Lie algebra homomorphism $\alpha: \mathfrak{g} \to L(H)$ to the Lie algebra $L(H)$ of some complex BCH-Lie group $H$, then $G := \langle \exp_H(\alpha(\mathfrak{g})) \rangle \leq H$ can be made a BCH-Lie group with $L(G) \cong \mathfrak{g}$.

**2.11** By Ado’s Theorem, every finite-dimensional $\mathbb{C}$-Lie algebra $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathbb{C}^{n \times n}$ for some $n \in \mathbb{N}$. Now $\mathfrak{g} \subseteq \mathbb{C}^{n \times n}$.

For each Mackey-complete $R$-cia $\mathcal{A}$, the Lie subalgebra

$$\mathfrak{g}_\mathcal{A} = \mathfrak{g} \otimes_\mathbb{C} \mathcal{A}$$

of $\mathbb{C}^{n \times n} \otimes_\mathbb{C} \mathcal{A}$ is a closed vector subspace. As $\mathcal{A}^{n \times n} \cong T_1(\text{GL}_n(\mathcal{A}))$ is BCH, also the BCH-series of $\mathfrak{g}_\mathcal{A}$ converges on an open 0-neighbourhood and thus $\mathfrak{g}_\mathcal{A}$ is BCH. Let $G$ be a complex Lie group with Lie algebra $T_e(G) \cong \mathfrak{g}$. The inclusion map

$$\mathfrak{g}_\mathcal{A} = \mathfrak{g} \otimes_\mathbb{C} \mathcal{A} \to \mathfrak{g} \otimes_\mathbb{C} C(S, \mathbb{C}) = C(S, \mathfrak{g})$$
is an injective continuous Lie algebra homomorphism, where \( C(S, g) \cong L(C(S, G)) \) for the BCH-Lie group \( C(S, G) \). Hence:

\[
G_A := \langle \exp_G \circ f : f \in g_A \rangle \leq C(S, G)
\]

can be made a BCH-Lie group with Lie algebra \( L(G_A) \cong g_A \).

If \( A \) is decomposing, we have subgroups

\[
G^+_A := \langle \exp_G \circ f : f \in g \otimes C A^+ \rangle, \quad G^C_A := \langle \exp_G \circ f : f \in g \otimes C A^C \rangle
\]

of \( G_A \) which are Lie groups with Lie algebras \( g^+ := g \otimes C A^+ \) and \( g^C := g \otimes C A^C \), respectively.

### 3 Uniqueness of decompositions

The following well-known fact will be used repeatedly:

#### 3.1 Let \( r < 1 < R \) and \( f : U \to \mathbb{C} \) be a continuous function on the open annulus \( U := \{ z \in \mathbb{C} : r < |z| < R \} \) such that \( f \) is holomorphic on \( U \setminus S \). Then \( f \) is holomorphic.

For the proof, one verifies the hypotheses of Morera’s Theorem for \( z \mapsto f(e^z) \) on \( \{ z \in \mathbb{C} : \ln(r) < \Re(z) < \ln(R) \} \).

**Lemma 3.2** Let \( G \) be a complex Lie group. Let \( f_+, g_+ : \overline{D} \to G \) be continuous functions holomorphic on \( D \) and \( f_-, g_- : \mathbb{C} \setminus D \to G \) be continuous functions holomorphic on \( \mathbb{C} \setminus \overline{D} \), such that

\[
f_+(z)f_-(z) = g_+(z)g_-(z) \quad \text{for all } z \in S \quad \text{and} \quad f_- (\infty) = g_- (\infty).
\]

If holomorphic functions \( G \to \mathbb{C} \) separate points on \( G \), then \( f_+ = g_+ \) and \( f_- = g_- \).

**Proof.** The function

\[
h : \mathbb{C} \to G, \quad z \mapsto \begin{cases} g_+(z)^{-1}f_+(z) & \text{if } z \in \overline{D} \\ g_-(z)f_-(z)^{-1} & \text{if } z \in \mathbb{C} \setminus D \end{cases}
\]

is continuous. Now \( \phi \circ h : \mathbb{C} \to \mathbb{C} \) is continuous for each holomorphic function \( \phi : G \to \mathbb{C} \) and holomorphic on \( \mathbb{C} \setminus S \), whence \( \phi \circ h \) is holomorphic (see Lemma 3.2) and hence constant as \( \mathbb{C} \) is compact. Thus \( h \) is constant, with value \( g_-(\infty)f_-(\infty)^{-1} = e \).

Passing to transposes, Theorems 1.1 and 1.2 in \[25\] Chapter VIII, applied to continuous functions, show the following:

**Lemma 3.3** Let \( A_+, A_- : C(S, GL_n(\mathbb{C})) \) and \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \), as well as \( \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \) be integers. Let \( D(z) \) be the diagonal matrix
diag($z^{\kappa_1}, \ldots, z^{\kappa_n}$) and $\tilde{D}(z) := \text{diag}(z^{\kappa_1}, \ldots, z^{\kappa_n})$ for $z \in \mathbb{C}$. Assume that $A_+$ and $A_+$ admit continuous extensions $\overline{D} \to \text{GL}_n(\mathbb{C})$ (denoted by the same symbol) which are holomorphic on $D$. Assume that $A_-$ and $\tilde{A}_-$ admit continuous extensions $\mathbb{C} \setminus \overline{D} \to \text{GL}_n(\mathbb{C})$ (denoted by the same symbol) which are holomorphic on $\mathbb{C} \setminus \overline{D}$. If

$$A_+(z)D(z)A_-(z) = \tilde{A}_+(z)\tilde{D}(z)\tilde{A}_-$$

for all $z \in \mathbb{D}$, then $\kappa_j = \nu_j$ for all $j \in \{1, \ldots, n\}$. Moreover, there exists a continuous function $C : \mathbb{C} \to \text{GL}_n(\mathbb{C})$, $C(z) = (C_{jk})_{j,k=1}^n$ for $z \in \mathbb{C}$ such that

$$\tilde{A}_+(z) = A_+(z)C(z) \quad \text{for all } z \in \overline{D};$$  

$$\tilde{A}_-(z) = D(z)^{-1}C(z)^{-1}D(z)A_-(z) \quad \text{for all } z \in \mathbb{C} \setminus \overline{D}$$

and the following conditions are satisfied for all $j,k \in \{1, \ldots, n\}$:

(a) $c_{kj} = 0$ if $\kappa_k < \kappa_j$;

(b) $c_{kj}$ is constant if $\kappa_k = \kappa_j$;

(c) $c_{kj}$ is a polynomial of degree $\leq \kappa_k - \kappa_j$, if $\kappa_k > \kappa_j$.

Note that if the matrix elements of $C$ satisfy the conditions (a)–(c) for all $j,k \in \{1, \ldots, n\}$, then also the matrix elements of $z \mapsto C(z)^{-1}$.

### 4 Birkhoff decompositions in $\text{GL}_n(\mathcal{A})$

Let $\mathcal{A}^- := \mathbb{C}1 + A^\oplus$.

**Proof.** (of Proposition 1.13). Holomorphic functional calculus provides an exponential function $\exp$ and a logarithm function for $\mathcal{A}$, and one finds that $\exp$ is a local $C^\infty$-diffeomorphism at 0 (see [21]). The Lie group $A^\times$ being abelian, $\exp : \mathcal{A} \to A^\times$ is a group homomorphism. Hence $\exp(\mathcal{A})$ is an open subgroup of $A^\times$. If $g \in \exp(\mathcal{A})$, then $g = \exp(x)$ with $x \in \mathcal{A} = A^+ \oplus A^\oplus$. Write $x = x_+ + x_\oplus$. Then $g = \exp(x_+ + x_\oplus)$ with $\exp(x_+) \in (A^+)^\times$ and $\exp(x_\oplus) \in (A^-)^\times$. Since $\mathbb{C}[z,z^{-1}] \cap A^\times$ is dense, it remains to factor Laurent polynomials $g$ in this dense set: $g = z^p$ with $p(0) \neq 0$ then

$$g(z) = c \prod_{|w| > 1} (z - w)^{\nu_w} z^{\kappa} \prod_{|w| < 1} (z - w)^{\nu_w}.$$  

The $w$ with $|w| < 1$ are denoted by $g_+(z)$ and the $w$ with $|w| > 1$ are denoted by $g_-(z)$.

**Proof.** (of Proposition 1.14). See Theorem 2.1 in [25, Chapter VIII] for the first assertion. For the final conclusion, note that the proof of the “special case” of the cited theorem (on pages 189–190) applies without changes. In the third line of [25, page 191], we replace the Lemma 5.1 of loc. cit. with our hypothesis that $\text{GL}_n(A^+) \cap \text{GL}_n(A^-)$ is an identity neighbourhood in $\text{GL}_n(\mathcal{A})$.  

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Lemma 4.1 Let \( A \) and \( B \) be decomposing \( R \)-cias such that \( A \subseteq B \). Given \( A \in \text{GL}_n(A) \), assume that

\[
A(z) = A_+(z)D(z)A_-(z) = \tilde{A}_+(z)\tilde{D}(z)\tilde{A}_-(z),
\]

where \( A_+ \in \text{GL}_n(A^+) \), \( A_- \in \text{GL}_n(A^-) \), \( \tilde{A}_+ \in \text{GL}_n(B^+) \), \( \tilde{A}_- \in \text{GL}_n(B^-) \) and \( D(z) = \text{diag}(z^{\kappa_1}, \ldots, z^{\kappa_n}) \), \( \tilde{D}(z) = \text{diag}(z^{\nu_1}, \ldots, z^{\nu_n}) \) with integers \( \kappa_1 \geq \cdots \geq \kappa_n \) and \( \nu_1 \geq \cdots \geq \nu_n \). Then \( D = \tilde{D} \), \( A_+ \in \text{GL}_n(A^+) \), and \( \tilde{A}_- \in \text{GL}_n(A^-) \).

Proof. By Lemma [5.3] we have \( D = \tilde{D} \). Let \( C \) be as in Lemma [5.3]. Since \( C \) and \( z \mapsto C(z)^{-1} \) are polynomial, we have \( C|_S \in A^{\times n} \) and \( C^{-1}|_S \in A^{\times n} \), entailing that \( C \in \text{GL}_n(A) \). We now deduce from [5] and [6] that \( \tilde{A}_+ \in \text{GL}_n(A) \cap \text{GL}_n(B^+) = \text{GL}_n(A^+) \) and \( \tilde{A}_- \in \text{GL}_n(A) \cap \text{GL}_n(B^-) = \text{GL}_n(A^-) \). \( \square \)

Proposition 4.2 Let \((A_j)_{j \in \mathbb{N}}\) be a sequence of Mackey complete, decomposing \( R \)-cias. Let \( n \in \mathbb{N} \) such that \( \text{GL}_n(A_j^+) \) \( \text{GL}_n(A_j^-) \) is an identity neighbourhood in \( \text{GL}_n(A_j) \) for each \( j \in \mathbb{N} \). If \( A_1 \supseteq A_2 \supseteq \cdots \) and all inclusion maps are continuous, then also the decomposing \( R \)-cia \( A := \bigcap_{j \in \mathbb{N}} A_j = \lim \limits_{\longrightarrow} A_j \) has the property that \( \text{GL}_n(A^+) \text{GL}_n(A^-) \) is an identity neighbourhood in \( \text{GL}_n(A) \).

Proof. Let \( A \in \text{GL}_n(A) \). Then \( A \in \text{GL}_n(A_j) \) for each \( j \in \mathbb{N} \). By Proposition [1.4] we have

\[
A = A_{j,+}D_jA_{j,-}
\]

with \( D_j \) (as described in the proposition) taking values in diagonal matrices) and \( A_{j,+} \in \text{GL}_n(A_j^+) \), \( A_{j,-} \in \text{GL}_n(A_j^-) \). By Lemma [4.1] we have

\[
A_+ := A_{1,+} \in \text{GL}_n(A_j^+) \quad \text{and} \quad A_- := A_{1,-} \in \text{GL}_n(A_j^-)
\]

for each \( j \in \mathbb{N} \), whence

\[
A_+ \in \bigcap_{j \in \mathbb{N}} \text{GL}_n(A_j^+) = \text{GL}_n(A^+) \quad \text{and} \quad A_- \in \bigcap_{j \in \mathbb{N}} \text{GL}_n(A_j^-) = \text{GL}_n(A^-).
\]

The inclusion map \( \mathcal{A} \to A_1 \) is a continuous homomorphism of unital algebras, whence the inclusion map \( j: \text{GL}_n(A) \to \text{GL}_n(A_1) \) is a continuous group homomorphism. Hence

\[\Omega := j^{-1}(\text{GL}_n(A_1^+) \text{GL}_n(A^-))\]

is an identity neighborhood in \( \text{GL}_n(A) \). By the preceding discussion, \( \Omega \subseteq \text{GL}_n(A^+) \text{GL}_n(A^-) \). \( \square \)

We recall that parts (a) and (b) of the following proposition can also be found in [35].

Proposition 4.3 Let \( n \in \mathbb{N} \). Consider the following situations:

(a) \( G := C^\infty(S, \text{GL}_n(C)) \cong \text{GL}_n(C^\infty(S, \mathbb{C})) \), \( G^+ := C^\infty(S, \text{GL}_n(C))^+, G^\ominus := C^\infty(S, \text{GL}_n(C))^\ominus \), and \( G^- := C^\infty(S, \text{GL}_n(C))^- \); or:
Then $G^+ G^\ominus$ is open in $G$. Each $g \in G$ can be written as

$$g = g_+ D g_-$$

with $g_+ \in G^+$, $g_- \in G^-$ and $D : S \to \text{GL}_n(C)$, $z \mapsto \text{diag}(z^{\kappa_1}, \ldots, z^{\kappa_n})$ with integers $\kappa_1 \geq \kappa_2 \geq \cdots$. The integers $\kappa_1, \ldots, \kappa_n$ are uniquely determined.

**Proof.** (a) The hypotheses of Proposition 4.2 are satisfied by $C^\infty(S, C) = \bigcap_{m \in \mathbb{N}} W(m)$, whence $G^+ G^- = G^+ G^\ominus$ is an identity neighbourhood in $G$ and hence open, being a $G^+ \times G^\ominus$ orbit if we let $G^+$ act on $G$ by multiplication on the left and $g \in G^\ominus$ multiplication with $g^{-1}$ on the right. The remaining assertions now follow from Proposition 1.14.

(b) The openness of $G^+ G^\ominus$ is a special case of Theorem 1.22 (and we shall not use the current statement before the theorem is proved). Since $\mathcal{A}$, being a Silva space, is complete, the hypotheses of Proposition 1.14 are satisfied.

(c) The proof is very similar to the beginning of the proof of Theorem 1.22. Details will be given in a later version of the preprint. 

\section{5 Decompositions of loops in $G$}

**Proof.** (of Proposition 1.19). We know from Lemma 3.2 that $\pi$ is injective. $\pi$ is a local complex-analytic diffeomorphism (and hence a complex-analytic diffeomorphism) if it is so at $(e, e)$. In fact, $\pi$ is equivariant for the $\mathbb{C}$-analytic $G^+_A \times G^\ominus_A$-actions

$$(g_+, g_\ominus)(x, y) := (g_+ x, yg_\ominus^{-1}) \quad \text{and} \quad (g_+, g_\ominus).z := g_+ z g_\ominus^{-1}$$

on $G^+_A \times G^\ominus_A$ and $G_A$, respectively; the first of these is a transitive action. Hence $\pi$ will be a local complex-analytic diffeomorphism at each point in its domain if it is a local complex-analytic diffeomorphism at $(e, e)$. We have

$$T_{(e, e)}(G^+_A \times G^\ominus_A) \cong T_e G^+_A \times T_e G^\ominus_A = g^+_A \times g^\ominus_A,$$

using this identification, the tangent map $T_{(e, e)} \pi$ is the map

$$g^+_A \times g^\ominus_A \to g_A, \quad (f, g) \mapsto f + g,$$

which is an isomorphism of topological vector spaces. We can now apply the inverse function theorem for complex-analytic mappings between open subsets
of complex Banach spaces, to conclude that $\pi$ is a local complex-analytic diffeomorphism at $(e, e)$. $\square$

**Proof.** (of Theorem 1.22). The restriction map

$$\rho: \mathcal{O}(\mathbb{C}^\times, G)_0 \to C^\infty(S, G), \quad f \mapsto f|_S$$

is a homomorphism of groups. It is $C^\infty$ by a suitable exponential law (as in [3]), since the corresponding map $\rho^\land: \mathcal{O}(\mathbb{C}^\times, G)_0 \times S \to G$ is smooth. The differential $T_e \rho_1$ corresponds to the restriction map $\mathcal{O}(\mathbb{C}^\times, \mathfrak{g}) \to C^\infty(S, \mathfrak{g})$, whence it is complex linear. As a consequence, the smooth group homomorphism $\rho$ is complex analytic. In particular, $\rho$ is continuous, whence it takes $\mathcal{O}(\mathbb{C}^\times, G)_0$ into $C^\infty(S, G)_0$. The exponential map of $C^\infty(S, G)_0$ is

$$\exp_\infty: C^\infty(S, \mathfrak{g}) \to C^\infty(S, G)_0, \quad f \mapsto \exp_G \circ f.$$ 

The exponential map of $G_W$ is

$$\exp_W: \mathfrak{g}_W \to G_W, \quad f \mapsto \exp_G \circ f.$$ 

The inclusion map $\iota: C^\infty(S, \mathfrak{g}) \to \mathfrak{g}_W$ is continuous and linear, hence complex analytic. We have

$$(\exp_W \circ \iota)(f) = \exp_G \circ f = \exp_\infty(f)$$

for each $f \in C^\infty(S, \mathfrak{g})$, whence $\exp_\infty(f) \in G_W$. Inside $C(S, G)$, we therefore have

$$C^\infty(S, G)_0 = \langle \exp_\infty(f): f \in C^\infty(S, \mathfrak{g}) \rangle \subseteq \langle \exp_G \circ f: f \in \mathfrak{g}_W \rangle = G_W.$$ 

We can therefore consider the inclusion map $\lambda: C^\infty(S, G)_0 \to G_W$, which is homomorphism of groups. It is complex analytic on some open identity neighbourhood (and hence complex analytic), as we have a commuting diagramme

\[
\begin{array}{ccc}
C^\infty(S, G)_0 & \overset{\exp_\infty}{\rightarrow} & G_W \\
\uparrow & & \uparrow \\
C^\infty(S, \mathfrak{g}) & \rightarrow & \mathfrak{g}_W \\
\end{array}
\]

in which $\exp_W$ and $\iota$ are complex analytic and $\exp_\infty$ is a local complex analytic diffeomorphism at 0, with $\exp_\infty(0) = e$. By the preceding, the restriction map

$$\lambda \circ \rho: \mathcal{O}(\mathbb{C}^\times, G) \to G_W, \quad f \mapsto f|_S$$

is a complex analytic homomorphism of groups. In particular, $\lambda \circ \rho$ is continuous, whence $\Omega := (\lambda \circ \rho)^{-1}(G_W^+G_W^{-1})$ is an open identity neighbourhood in $\mathcal{O}(\mathbb{C}^\times, G)$. To see that *the image* $\text{im}(\rho)$ is *open*, it suffices to show that $\text{im}(\rho)$ equals the set $\Omega$ just defined. Thus, we need to prove that $\text{im}(\rho)$ contains all $f \in \mathcal{O}(\mathbb{C}^\times, G)$.
such that $g := f_{|S} \in G^+_W G^G_W$. To achieve this, we write $g = g_+ g_\ominus$ with $g_+ \in G^+_W$ and $g_\ominus \in G^G_W$. Then

$$\tilde{f}_+(z) := \begin{cases} g_+^<(z) & \text{if } z \in \mathbb{T}, \\ f(z) g_\ominus^>(z) & \text{if } z \in \mathbb{C} \setminus \mathbb{T} \end{cases}$$

defines a continuous function $\mathbb{C} \to G$ which is complex analytic on $\mathbb{C} \setminus S$ and hence complex analytic on all of $\mathbb{C}$. Likewise,

$$\tilde{f}_\ominus(z) := \begin{cases} g_\ominus^<(z)^{-1} f(z) & \text{if } z \in \mathbb{T}, \\ g_\ominus^>(z) & \text{if } z \in \mathbb{C} \setminus \mathbb{T} \end{cases}$$

defines a continuous function $\hat{\mathbb{C}} \setminus \{0\} \to G$ which is complex analytic on $(\hat{\mathbb{C}} \setminus \{0\}) \setminus S$ and hence complex analytic on all of $\hat{\mathbb{C}} \setminus \{0\}$. Moreover, $\tilde{f}_\ominus(\infty) = g_\ominus^>(\infty) = e$.

Then

$$f = f_+ f_\ominus$$

by the Identity Theorem, as both functions are complex analytic and coincide on $S$. Thus $f = p(f_+, \tilde{f}_\ominus) \in \text{im}(p)$.

By Lemma 3.2, $p$ is injective.

Analyticity of $p^{-1}$, say of $f \mapsto \tilde{f}_+$: By the exponential law, we only need to show that the corresponding map

$$\text{im}(p) \times \mathbb{C} \to G, \quad (f, z) \mapsto \tilde{f}_+(z)$$

of two variables is complex analytic. Actually, complex analyticity of the restriction of the map in (7) to $\text{im}(f) \times \mathbb{D}$, as map is equivariant for natural $\mathbb{C}^\times$-action, $(w.f)(z) := f(wz)$. We have a $\mathbb{C}$-analytic composition

$$\psi : \text{im}(p) \longrightarrow G_W \longrightarrow G_W^+ \longrightarrow \mathcal{O}(\mathbb{D}, G) \quad f \mapsto f_{|S} \mapsto f_+ \mapsto \tilde{f}_+|_{\mathbb{D}}$$

of $\mathbb{C}$-analytic maps. The leftmost map used here is complex analytic as a restriction of $\lambda \circ \rho$. The second map is complex analytic by Proposition 1.19. The final map is a group homomorphism which is complex analytic on some identity neighbourhood (and hence complex analytic) because it is the upper map in the commutative diagramme

$$\begin{array}{c}
G_W^+ \to \mathcal{O}(\mathbb{D}, G) \\
\exp_W^+ \uparrow \quad \text{EXP} \\
g^W_+ \to \mathcal{O}(\mathbb{D}, \mathfrak{g})
\end{array}$$

where $\exp_W^+ : g^W_+ \to G_W^+$, $f \mapsto \exp \circ f$ is a local complex analytic diffeomorphism at 0, the exponential map $\text{EXP} : \mathcal{O}(\mathbb{D}, \mathfrak{g}) \to \mathcal{O}(\mathbb{D}, G)$, $f \mapsto \exp_G \circ f$ is complex analytic and the bottom map is the composition

$$g^W_+ \to \mathcal{O}_b(\mathbb{D}, \mathfrak{g}) \to \mathcal{O}(\mathbb{D}, \mathfrak{g})$$
of the continuous complex linear maps $\mathfrak{g}_W^+ \to \mathcal{O}(\mathbb{D}, \mathfrak{g})$, $f \mapsto f^<$ (noting that $\|f^<\|_\infty \leq \|f\|_W$ as a consequence (2)) and the complex linear inclusion map $\mathcal{O}_b(\mathbb{D}, \mathfrak{g}) \to \mathcal{O}(\mathbb{D}, \mathfrak{g})$ which is continuous as the topology of compact convergence on $\mathcal{O}_b(\mathbb{D}, \mathfrak{g})$ is coarser than the topology of uniform convergence. It remains to observe that (7) is the map

$$\psi^\wedge : \text{im}(p) \times \mathbb{D} \to G, \quad (f, z) \mapsto \psi(f)(z)$$

which is $\mathbb{C}$-analytic by the Exponential Law for $\mathcal{O}(\mathbb{D}, G)$.

Complex analyticity of the second component of $p^{-1}$ can be shown analogously, exploiting that the map $\phi : \mathbb{C} - \{0\} \to \mathbb{C}, \ z \mapsto 1/z$ (with $1/\infty := 0$) is an isomorphism of complex analytic manifolds and the map $\mathcal{O}(\mathbb{C}, G) \to \mathcal{O}(\mathbb{C} - \{0\}, G), \ f \mapsto f \circ \phi$ an isomorphism of complex analytic Lie groups. $\square$

We shall use a result concerning complex analyticity of mappings on open subsets of LB-spaces (taken from [11]).

**Lemma 5.1 (Dahmen’s Theorem)** Let $E_1 \subseteq E_2 \subseteq \cdots$ and $F_1 \subseteq F_2 \subseteq \cdots$ be complex Banach spaces such that all inclusion maps have operator norm $\leq 1$. Assume that the locally convex direct limit topology on $\bigcup_{n \in \mathbb{N}} E_n$ and $F := \bigcup_{n \in \mathbb{N}} F_n$ is Hausdorff. Let $r > 0$. Then a map $f : \bigcup_{n \in \mathbb{N}} B^{E_n}(0) \to F$ is complex analytic if $f|_{B^{E_n}(0)} : B^{E_n}(0) \to F$ is complex analytic and bounded for each $n \in \mathbb{N}$. $\square$

We recall a well-known concept.

**Definition 5.2** If $f : X \to Y$ is a mapping between metric spaces $(X, d_X)$ and $(Y, d_Y)$, let

$$\text{Lip}(f) := \sup \left\{ \frac{d_Y(f(x), f(y))}{d_X(x, y)} : \text{for } x, y \in X \text{ with } x \neq y \right\} \in [0, \infty].$$

Then $f$ is Lipschitz continuous if and only if $\text{Lip}(f) < \infty$ and $\text{Lip}(f)$ is the smallest Lipschitz constant for $f$ in the case. Also the following fact is well known (see, e.g., [24]):

**5.3** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces and $U \subseteq E$ be a convex open subset. If $f : U \to F$ is continuously Fréchet differentiable and $f'(x) \in L(E, F)$ its derivative at $x \in U$, then

$$\text{Lip}(f) = \sup \{\|f'(x)\|_\text{op} : x \in U\}.$$

We shall always assume that the norm $\|\cdot\|_\theta$ on a Banach–Lie algebra $\mathfrak{g}$ is chosen submultiplicative, i.e., that

$$\| [x, y] \|_\theta \leq \|x\|_\theta \|y\|_\theta \quad \text{for all } x, y \in \mathfrak{g}.$$
Lemma 5.5

There exists a norm \( \| \cdot \|_g \) on \( g \), the norm \( \| \cdot \|_W \) on \( W \), on \( \mathfrak{g}_W \) is submultiplicative. In fact, the map

\[
g_W \to \ell^1(\mathbb{Z}, \mathfrak{g}), \quad f \mapsto (\hat{f}_k)_{k \in \mathbb{Z}}
\]
is isometric and an isomorphism of complex Lie algebras, when the norm \( \| \cdot \|_g \) defined via

\[
\|(a_k)_{k \in \mathbb{Z}}\|_{\ell^1, g} := \sum_{k \in \mathbb{Z}} \|a_k\|_g
\]
is used on \( \ell^1(\mathbb{Z}, g) \). The Lie bracket on \( \ell^1(\mathbb{Z}, g) \) is the Cauchy–Lie bracket,

\[
\left((a_k)_{k \in \mathbb{Z}}, (b_k)_{k \in \mathbb{Z}}\right) := (c_k)_{k \in \mathbb{Z}}
\]

with

\[
c_k := \sum_{\ell \in \mathbb{Z}} [a_\ell, b_{k-\ell}].
\]

Setting \( \alpha_k := \|a_k\|_g \) and \( \beta_k := \|b_k\|_g \) for \( k \in \mathbb{Z} \), we have \( \alpha := (\alpha_k)_{k \in \mathbb{Z}} \in \ell^1 \) and \( \beta := (\beta_k)_{k \in \mathbb{Z}} \in \ell^1 \). Now

\[
\|c_k\|_{\ell^1, g} \leq \sum_{\ell \in \mathbb{Z}} \|(a_\ell, b_{k-\ell})\|_g \leq \sum_{\ell \in \mathbb{Z}} \|a_\ell\|_g \|b_{k-\ell}\|_g
\]

\[
= \|\alpha \ast \beta\|_{\ell^1} \leq \|\alpha\|_{\ell^1} \|\beta\|_{\ell^1} = \|(a_k)_{k \in \mathbb{Z}}\|_{\ell^1, g},
\]

showing that \( \| \cdot \|_{\ell^1, g} \) (and hence also \( \| \cdot \|_W \) on \( g_W \)) is submultiplicative. If \( f \in g_W \), then \( f = f_+ + f_- \) with \( f_+ \in g_W^+ \) and \( f_- \in g_W^- \), given by

\[
f_+(z) := \sum_{k=0}^{\infty} \hat{f}_k z^k \quad \text{and} \quad f_-(z) := \sum_{k=1}^{\infty} \hat{f}_{-k} z^{-k}
\]

for \( z \in S \). Since \( f^+_\ominus : \overline{B} \to g \) and \( f^-_\ominus : \overline{C} \setminus \overline{B} \to g \) are given by \([2] \) and \([3] \), respectively (with \( f^-_\ominus(\infty) := 0 \)), we see that

\[
\|f^+_\ominus\|_\infty \leq \|(\hat{f}_k)\|_{\ell^1} \leq \|(\|\hat{f}_k\|_g)\|_{\ell^1} = \|f\|_W \quad \text{(8)}
\]

and

\[
\|f^-_\ominus\|_\infty \leq \|(\hat{f}_{-k})\|_{\ell^1} \leq \|(\|\hat{f}_{-k}\|_g)\|_{\ell^1} = \|f\|_W. \quad \text{(9)}
\]

Lemma 5.5

There exists \( r > 0 \) with the following properties:

(a) For each complex Banach–Lie algebra \( g \) with submultiplicative norm \( \| \cdot \|_g \), the BCH-series converges on \( B^g_\rho(0) \times B^g_\rho(0) \) to a complex-analytic mapping

\[
\mu_\rho : B^g_\rho(0) \times B^g_\rho(0) \to g, \quad (x, y) \mapsto x \ast y.
\]

(b) If we write \( x \ast y = x + y + R_\rho(x, y) \) for \( (x, y) \in B^g_\rho(0) \times B^g_\rho(0) \) in the situation of (a), then

\[
\text{Lip}(R_\rho) \leq \frac{1}{4}.
\]

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We are using the maximum norm \( g \times g \to [0, \infty[, (x, y) \mapsto \max\{\|x\|_g, \|y\|_g\} \) here for the calculation of the Lipschitz constant.

**Proof.** Let \( \alpha_g : g \times g \to g \) be the continuous linear map \((x, y) \mapsto x + y\). By [5.3] the final assertion means that \( \|\mu'_g(x, y) - \alpha_g\|_{\text{op}} \leq \frac{1}{4} \) for all \((x, y) \in B_1^g(0) \times B_2^g(0)\). The assertion can therefore be proved like [12, Lemma 3.5 (a)], where the the constant \( \frac{1}{2} \) is used in place of \( \frac{1}{4} \).

**5.6** If \( g \) is a Banach–Lie algebra with submultiplicative norm \( \| \cdot \|_g \), we endow \( \mathcal{O}_b(A_n, g) \) with the norm \( \| \cdot \|_n \) given by

\[
\| f \|_n := \max\{\| f \|_{\infty}, \| f \|_{\mathcal{W}}\}.
\]

Using the compact-open \( C^\infty\)-topology on the function spaces in the middle, we get continuous complex linear maps

\[
(\mathcal{O}_b(A_n, g), \| \cdot \|_\infty) \to C^\infty(A_n, g) \to C^\infty(S, g) \to g_{\mathcal{W}},
\]

where the first and final mappings are inclusion maps; the map in the middle is the restriction map \( f \mapsto f|_{S} \). Thus, the composition

\[
\rho : (\mathcal{O}_b(A_n, g), \| \cdot \|_\infty) \to g_{\mathcal{W}}, \quad f \mapsto f|_{S}
\]

is a continuous complex linear map, entailing that \( \| \cdot \|_{\mathcal{W}} \circ \rho \) is a continuous norm on \( (\mathcal{O}_b(A_n, g), \| \cdot \|_\infty) \). As a consequence, the norm

\[
\| \cdot \|_n : B(A_n, g) \to [0, \infty[, \quad f \mapsto \max\{\| f \|_{\infty}, \| f \|_{\mathcal{W}}\}
\]

is equivalent to \( \| \cdot \|_\infty \). Since \( \|[f,g](x)||_g = \|[f(x), g(x)]||_g \leq \|f(x)||_g\|g(x)||_g \leq \|f\|_\infty\|g\|_\infty\) and

\[
\|[f,g]|_\mathcal{W}\leq\|f\|_\mathcal{W}\|g\|_\mathcal{W}
\]

by Example [5.4] the norm \( \| \cdot \|_n \) is submultiplicative. We now show that each \( f \in \mathcal{O}_b(A_n, g) \) admits a decomposition

\[
f = f_+ + f_\ominus
\]

with \( f_+, f_\ominus \in \mathcal{O}_b(A_n, g) \) such that \( f_+ \) admits a complex analytic extension \( f_+^z : B_{1+z}^g(0) \to g \) and \( f_- \) admits a complex analytic extension \( f_-^z : \mathbb{C} \setminus B_{1-z}^g(0) \to g \) with \( f_+^z(\infty) = 0 \). In fact, we write \( g := f|_{S} \in g_{\mathcal{W}} \) in the form \( g = g_+ + g_\ominus \) with \( g_+ \in g^+_W \) and \( g_\ominus \in g^-_W \). Then

\[
f_+^z : B_{1+z}^g(0) \to g, \quad z \mapsto \begin{cases} g_+^z(z) & \text{if } z \in \mathbb{D} \\ f(z) - g_\ominus^z(z) & \text{if } z \in B_{1+z}^g(0) \setminus \mathbb{D} \end{cases}
\]

is a continuous function on \( B_{1+z}^g(0) \) which is holomorphic on \( B_{1+z}^g(0) \setminus \mathbb{S} \) and hence holomorphic. Likewise,

\[
f_\ominus^z : \mathbb{C} \setminus B_{1-z}^g(0) \to g, \quad z \mapsto \begin{cases} g_\ominus^z(z) & \text{if } z \in \mathbb{C} \setminus \mathbb{D} \\ f(z) - g_+^z(z) & \text{if } z \in \mathbb{D} \end{cases}
\]
is a continuous function on \( \hat{C} \backslash \hat{B}^{1}_{\frac{r}{n}}(0) \) which is holomorphic on \( (\hat{C} \backslash \hat{B}^{1}_{\frac{r}{n}}(0)) \backslash \mathbb{S} \) and hence holomorphic. We shall verify presently that \( f_{+} := (f^{\circ})_{|A_{n}} \) and \( f^{\circ}_{\ominus} := (f^{\ominus})_{|A_{n}} \) are in \( \mathcal{O}_{b}(A_{n}, g) \); in fact,

\[
\| f_{+} \|_{n} \leq 2 \| f \|_{n} \quad \text{and} \quad \| f^{-} \|_{n} \leq 2 \| f \|_{n}.
\] (10)

By construction, \( f = f_{+} + f^{\circ}_{\ominus} \). We have \( \| f_{+} \|_{n} \leq \| g_{+} \|_{W} \leq \| g \|_{W} \leq \| f \|_{n} \leq 2 \| f \|_{n} \)

for all \( z \in \mathbb{C} \) with \( 1 - \frac{1}{n} \leq |z| \leq 1 \) and

\[
\| f_{+}(z) \|_{g} = \| g_{+}(z) \|_{g} \leq \| g \|_{W} \leq \| f \|_{n} \leq 2 \| f \|_{n}
\]

for all \( z \in \mathbb{C} \) with \( 1 \leq |z| < 1 + \frac{1}{n} \), we see that \( \| f_{+} \|_{\infty} \leq 2 \| f \|_{n} \), whence \( f_{+} \in \mathcal{O}_{b}(A_{n}, g) \). Moreover, \( \| f_{+} \|_{n} = \max \{ \| f_{+} \|_{W}, \| f_{+} \|_{\infty} \} \leq 2 \| f \|_{n} \). The proofs for \( f^{\circ}_{\ominus} \in \mathcal{O}_{b}(A_{n}, g) \) and the second inequality in (10) are analogous.

We shall also use a well-known quantitative version of the inverse function theorem:

**Lemma 5.7** Let \((E, \| \cdot \|_{E})\) be a Banach space, \( x \in E, r > 0 \) and \( f : B^{E}_{r}(x) \rightarrow E \) be a complex-analytic mapping such that \( f'(x) = \text{id}_{E} \) and

\[
\operatorname{Lip}(f - \text{id}_{E}) \leq \frac{1}{2}.
\]

Then \( f(B^{E}_{r}(x)) \) is open in \( E \), the map \( f : B^{E}_{r}(x) \rightarrow f(B^{E}_{r}(x)) \) is a complex analytic diffeomorphism, and

\[
B^{E}_{r/2}(f(x)) \subseteq f(B^{E}_{r}(x)) \subseteq B^{E}_{3r/2}(f(x)).
\] (11)

**Proof.** By \[23\] Lemma 6.1 (a), \( f \) has open image, is a homeomorphism onto its image, and (11) holds. Since \( \| f'(y) - \text{id}_{E} \|_{\text{op}} \leq \operatorname{Lip}(f - \text{id}_{E}) \leq \frac{1}{2} < 1 \), we have \( f'(y) \in \text{GL}(E) \) for each \( y \in B^{E}_{r}(x) \), whence \( f \) is a local complex-analytic diffeomorphism by the classical inverse function theorem for complex-analytic mappings and hence a complex-analytic diffeomorphism. \( \square \)

**Proof.** (of Theorem \[1\],[24\]). For \( n \in \mathbb{N} \), let \( g_{n} := \mathcal{O}_{b}(A_{n}, g) \), endowed with the submultiplicative norm \( \| \cdot \|_{n} \) from above. Let \( r, \mu_{g_{n}} \) and \( R_{g_{n}} \) be a in Lemma 5.5. The continuous complex linear maps

\[
pr^{+}_{n} : g_{n} \rightarrow g_{n}^{+}, \quad f \mapsto f^{+}
\]

and \( pr^{\circ}_{n} : g_{n} \rightarrow g_{n}^{\circ}, \quad f \mapsto f^{\circ} \) have operator norms \( \| pr^{+}_{n} \|_{\text{op}}, \| pr^{\circ}_{n} \|_{\text{op}} \leq 2 \), by (10). Hence

\[
R_{n} : B^{E}_{r/2}(0) \rightarrow g_{n}, \quad x \mapsto R_{g_{n}}(pr^{+}_{n}(x), pr^{\circ}_{n}(x))
\]

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satisfies \( \text{Lip}(R_n) \leq 2 \text{Lip}(R_0) \leq \frac{1}{2} \). For all \( x \in B_{r/2}^g(x) \), we have \( x_+ = pr_n^+(x) \in B_{r/2}^g(0) \) and \( x_\varnothing = \text{pr}_n^\varnothing(x) \in B_{r/2}^g(0) \). Moreover,

\[
\mu_{g_n}(pr_n^+(x), \text{pr}_n^\varnothing(x)) = x + R_n(x),
\]

since \( x_+ \ast x_\varnothing = x_+ + x_\varnothing + R_{g_n}(x_+, x_\varnothing) = x + R_n(x) \). Hence, the complex-analytic map

\[
h_n: B_{r/2}^g(0) \to g_n, \quad x \mapsto \text{pr}_n^+(x) \ast \text{pr}_n^\varnothing(x)
\]
is of the form

\[
h_n = \text{id}_{g_n} + R_n \quad \text{with} \quad \text{Lip}(R_n) \leq \frac{1}{2} \quad \text{and} \quad h_n'(0) = \text{id}_{g_n}.
\]

by Lemma 5.7, \( h_n \) is a complex-analytic diffeomorphism onto its open image \( h_n(B_{r/2}^g(0)) \), and the latter contains \( B_{r/2}^g(0) \) as a subset. Now \( C^\omega(S, g) = \lim_{n \to \infty} g_n \). Identifying holomorphic functions on annuli with their restrictions to \( S \), we simply have \( g_n \subseteq g_{n+1} \) for each \( n \in \mathbb{N} \) and \( C^\omega(S, g) = \bigcup_{n \in \mathbb{N}} g_n \). Using these identifications,

\[
U := \bigcup_{n \in \mathbb{N}} B_{r/2}^g(0) \quad \text{and} \quad V := \bigcup_{n \in \mathbb{N}} B_{r/4}^g(0)
\]
are open zero-neighbourhoods in \( C^\omega(S, g) \). We have a well-defined injective map

\[
h: U \to C^\omega(S, g), \quad f \mapsto h_n(f) \quad \text{if} \quad f \in B_{r/2}^g(0)
\]
which is holomorphic (by Dahmen’s Theorem) since \( h|_{B_{r/2}^g(0)} = h_n \) is holomorphic and bounded for each \( n \in \mathbb{N} \). By continuity of \( h \), the preimage \( W := h^{-1}(V) \) is open in \( U \). Now \( h|_W: W \to V \) is a continuous bijection. The map

\[
g: V \to C^\omega(S, g), \quad f \mapsto h_n^{-1}(f) \quad \text{if} \quad f \in B_{r/4}^g(0)
\]
is holomorphic (by Dahmen’s Theorem), since \( g|_{B_{r/4}^g(0)} = h_n^{-1}|_{B_{r/4}^g(0)} \) is holomorphic and bounded for each \( n \in \mathbb{N} \). By definition, \( g = (f|_W)^{-1} \). Thus \( f|_W: W \to V \) is a complex-analytic diffeomorphism. Abbreviate \( A := C^\omega(S, \mathbb{C}) \). Let \( \exp_+ \) and \( \exp_\varnothing \) be the exponential functions of the BCH-Lie groups \( G_A^+ \) and \( G_A^- \), respectively. We have a commutative diagramme

\[
\begin{array}{ccc}
G_A^+ \times G_A^\varnothing & \xrightarrow{\pi} & C^\omega(S, G) \\
\exp_+ \times \exp_\varnothing & \uparrow & \uparrow \\
G_A^+ \times G_A^\varnothing & \xrightarrow{(\text{pr}_+, \text{pr}_\varnothing)} & C^\omega(S, G) \\
W & \xrightarrow{h|_W} & V,
\end{array}
\]

where the composition of the vertical maps on the left is a local complex-analytic diffeomorphism at 0 and the vertical map on the right-hand side is the exponential function of the Lie group \( C^\omega(S, G) \), which is a complex-analytic function.
The assertion follows. \hfill \Box

**Proof.** (of Theorem 1.25). Abbreviate \( A := C^\infty(S, \mathbb{C}) \). If \( f \in G_W^+G_W^\circ \cap C^\infty(S, G) \), we have

\[
f = g_+ g_\ominus
\]

with certain \( g_+ \in G_W^+ \) and \( g_\ominus \in G_W^\circ \). On the other hand, since \( G \subseteq \text{GL}_n(\mathbb{C}) \), we have

\[
f = f + Df -
\]

for suitable elements \( f \in \text{GL}_n(A^+) \), \( f - \in \text{GL}_n(A^-) \), and \( D(z) = \text{diag}(z^{\kappa_1}, \ldots, z^{\kappa_n}) \) with \( \kappa_1 \geq \cdots \geq \kappa_n \). Since \( g_+ \in \text{GL}_n(A^+) \) and \( g_\ominus \in \text{GL}_n(A^-) \), Proposition 3.3 shows that \( \kappa_1 = \cdots = \kappa_n = 0 \) and \( f_+ = g_+ C, f_- = C^{-1} g_- \) for some constant matrix \( C \in \text{GL}_n(\mathbb{C}) \). Thus \( g_+ \in \text{GL}_n(A^+) \cap G_W^+ \) and \( g_- \in \text{GL}_n(A^-) \cap G_W^\circ \). For \( g \in G_A \) close to \( e \) in \( G_W \) (which defines an identity neighbourhood in \( C^\infty(S, G) \)), one can show that \( g_+ \in C^\infty(S, G)^+ \) and \( g_- \in C^\infty(S, G)^\circ \). The idea is to write \( g_+ = \exp_G \circ h \) with \( h \in C^\infty(S, g) \cap g_W = C^\infty(S, g)^+ \). Then \( g = \exp_G \circ h \in \text{GL}_n(A^+) \subseteq C^\infty(S, G)^+ \). Likewise for \( g_\ominus \). Thus \( G_A^+G_A^\circ \) is an identity neighbourhood in \( C^\infty(S, G) \) and hence an open identity neighbourhood, being an orbit.

Using the projective limit property as a complex manifolds, the complex analyticity of \( \text{pr}^+: G^+G^\circ \to G^+ \) follows from that of the corresponding maps \( G_W^{(m)}G_W^{(m)} \to G_W^{(m)} \), which appear at the bottom in the commutative diagrammes

\[
G^+G^\circ \xrightarrow{\text{pr}^+} G^+
\]

\[
G_W^{(m)}G_W^{(m)} \longrightarrow G_W^{(m)}.
\]

\hfill \Box

6 Loop groups over algebras of germs

We show that algebras of germs around 0 of holomorphic functions or meromorphic functions on punctured disks are of limited use for the construction of Lie groups of loops.

We first discuss the algebra

\[
\mathcal{H} := \lim_{\to} \mathcal{O}(B_{1/n}(0) \setminus \{0\}, \mathbb{C}) \cong \mathbb{C}\{z\} \times \mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, \mathbb{C}),
\]

of germs \([f]\) of holomorphic functions \( f \) on punctured disks around 0 in the complex plane; \( \mathcal{H} \) is endowed with the locally convex direct limit topology (cf. [4]). Then \( \mathcal{H} \) is not a topological algebra (in the sense of this article):

**Lemma 6.1** The algebra multiplication \( m: \mathcal{H} \times \mathcal{H}, ([f], [g]) \mapsto [fg] \) is not continuous.
Proof. Let $\mathbb{C}\{z\} = \lim \mathcal{O}(B_{1/n}^C(0), \mathbb{C})$ be the algebra of germs of holomorphic functions around 0, endowed with the locally convex direct limit topology. Let $\mathbb{C}\{z\}'$ be its topological dual spaces, endowed with the topology of bounded convergence. Write $f_p: \hat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$ for the principal part of a holomorphic function $f: B_{1/n}^C(0) \setminus \{0\} \to \mathbb{C}$ and $f_r: B_{1/n}^C(0) \to \mathbb{C}$ for its regular part. The map

$$\mathcal{H} \to \mathbb{C}\{z\} \times \mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, \mathbb{C}), \quad [f] \mapsto ([f_r], f_p)$$

is well defined and an isomorphism of topological vector spaces (cf. [4]). Moreover, the map

$$\theta: \mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, \mathbb{C})_* \to \mathbb{C}\{z\}'$$

given by $\theta(f)([g]) := \mathrm{res}_0(fg)$ is an isomorphism of topological vector spaces (cf. [4]). The linear map $\mathrm{res}: \mathcal{H} \to \mathbb{C}, \quad [f] \mapsto \mathrm{res}_0(f)$ is continuous, as it is continuous on each step of the direct system. If $m$ was continuous, then also the map

$$h: \mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, \mathbb{C}) \times \mathbb{C}\{z\} \to \mathcal{H}, \quad ([f], [g]) \mapsto [fg]$$

would be continuous and hence also $\mathrm{res}_0 \circ h$. But $\mathrm{res}_0 \circ f = \mathrm{ev} \circ (\mathrm{id} \times \theta)$ using the identity map of $\mathcal{O}(\hat{\mathbb{C}} \setminus \{0\}, \mathbb{C})$ and the evaluation map

$$\mathrm{ev}: \mathbb{C}\{z\}' \times \mathbb{C}\{z\} \to \mathbb{C}, \quad (\lambda, [g]) \mapsto \lambda([g]).$$

Thus $\mathrm{ev}$ would be continuous. But it is well known that $\mathrm{ev}: E' \times E \to \mathbb{C}$ is discontinuous for each non-normable complex locally convex space $E$ (cf. [29]), contradiction.

We record an immediate consequence:

Lemma 6.2 For each finite-dimensional complex Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$ which is not abelian, the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g}_H$ is discontinuous.

Proof. For each $a \in \mathfrak{g} \setminus \{0\}$, the linear map

$$\iota_a: \mathcal{H} \to \mathfrak{g}_H = \mathfrak{g} \otimes \mathbb{C} \mathcal{H}, \quad [f] \mapsto a \otimes [f]$$

is a topological embedding (cf. [2.5]). We have

$$[\iota_x([f]), \iota_y([g])] = [x, y] \otimes [f][g] = \iota_z(m([f], [g]))$$

for all $[f], [g] \in \mathcal{H}$. Hence, if $[\cdot, \cdot]$ was continuous, then also

$$m = \iota_z^{-1} \circ [\cdot, \cdot] \circ (\iota_x \times \iota_y)$$

were continuous, which contradicts Lemma 6.1.

Remark 6.3 The paper [15] claims to obtain complex Lie groups with Lie algebra $\mathfrak{g}_H$, for certain finite-dimensional non-abelian Lie algebras $\mathfrak{g}$. This is false; the Lie bracket on $\mathfrak{g}_H$ being discontinuous, $\mathfrak{g}_H$ cannot be the Lie algebra of any Lie group in the usual sense (as in [33] and the current article). The
construction remains erroneous when complex analytic Lie groups in the sense of convenient differential calculus are considered (as in [29]). In fact, if one wishes the restriction \( \exp_G |_U \) of the exponential map to an open 0-neighbourhood \( U \subseteq g \) to be a \( \mathbb{C} \)-analytic diffeomorphism onto an open set, one usually accomplishes this by choosing \( U \) small (notably, bounded). But then the subset \( \Omega \) of germs \([f] \in \mathcal{H}\) having a representative \( g \in [f] \) with image in \( U \) necessitates that \( g \) has vanishing principal part. Hence \( \Omega \) has empty interior in \( \mathcal{H} \) and cannot be used as the domain of a local parametrization for a Lie group\(^2\) contrary to the attempts in [15].

The algebra \( \mathcal{M} \) of germs \([f]\) of meromorphic functions around 0 in \( \mathbb{C} \) can be turned into a locally convex topological algebra [9]. The inversion map \( \eta: \mathcal{M}^\times \to \mathbb{C} \) is not continuous (see [3]). Moreover, \( \eta \) is not complex analytic in the sense of convenient differential calculus (as in [29]). In fact, \( \mathcal{M} \) is a Silva space, i.e., a locally convex direct limit

\[
\mathcal{M} = \lim_{\rightarrow} E_n
\]

with complex Banach spaces \( E_1 \subseteq E_2 \subseteq \cdots \) such that all inclusion maps are compact operators. Like every Silva space, \( \mathcal{M} \) is compact regular, i.e., each compact subset \( K \subseteq \mathcal{M} \) is contained in \( E_n \) and compact in there for some \( n \in \mathbb{N} \) (see, e.g., [17] or [24]). Hence convenient complex analyticity of \( \eta \) is equivalent to complex analyticity of \( \eta|_{\mathcal{M}^\times \cap E_n} \) for each \( n \in \mathbb{N} \), which in turn is equivalent to complex analyticity of \( f \) (see [24]). But \( f \) is not complex analytic (being discontinuous). As a consequence, \( \text{GL}_n(\mathcal{M}) = (\mathcal{M}^{n\times n})^\times \) fails to be a complex Lie group when considered as an open subset of \( \mathcal{M}^{n\times n} \).

Using coefficients in \( \mathcal{H} \) of \( \mathcal{M} \), Lie groups of loops can be constructed only in limited situation.

**Proposition 6.4** Let \( g \) be a finite-dimensional complex Lie algebra which is nilpotent. Then the Baker-Campbell-Hausdorff multiplication makes \( g_M \) a complex Lie group in the sense of this article, and it makes \( g_H \) a complex Lie group in the sense of convenient differential calculus.

**Proof.** Since \( g_H \) and \( g_M \) are nilpotent, the BCH-series is given by a finite sum and defines a global group structure. The inversion map is \( g \mapsto -g \) and hence complex analytic. Being a linear combination of nested Lie brackets which are conveniently complex analytic (resp., complex analytic), the BCH-multiplication is conveniently complex analytic and complex analytic, respectively.\[\square\]

\(^1\) Only in rare cases, like 1-connected nilpotent groups, we can choose large (\( U = g \) in the latter case).

\(^2\) One could only recover the Lie groups of germs of \( G \)-valued complex analytic functions on the non-punctured plane \( \mathbb{C} \) around 0, as in [22].
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