Schensted-Type correspondence, Plactic Monoid and Jeu de Taquin for type $C_n$

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Abstract

We use Kashiwara’s theory of crystal bases to study the plactic monoid for $U_q(sp_{2n})$. Then we describe the corresponding insertion and sliding algorithms. The sliding algorithm is essentially the symplectic Jeu de Taquin defined by Sheats and our construction gives the proof of its compatibility with plactic relations.

1 Introduction

It is well known that the Schensted bumping algorithm yields a bijection between words $w$ of length $l$ on the ordered alphabet $A_n = \{1 < 2 < \cdots < n\}$ and pairs $(P(w), Q(w))$ of tableaux of the same shape containing $l$ boxes where $P(w)$ is a semi-standard Young tableau on $A_n$ and $Q(w)$ is a standard tableau. This bijection is called the Robinson-Schensted correspondence (see e.g. [5]). Notice that the tableau $P(w)$ may also be constructed from $w$ by using the Schützenberger sliding algorithm.

We can define a relation $\sim$ on the free monoid $A_n^*$ by:

$$w_1 \sim w_2 \iff P(w_1) = P(w_2).$$

Then the quotient $Pl(A_n) := A_n^*/\sim$ can be described as the quotient of $A_n^*$ by Knuth relations:

$$zxy = xyz \quad \text{and} \quad yzx = yxz \quad \text{if} \quad x < y < z,$$
$$xyz = xzy \quad \text{and} \quad yzx = yxz \quad \text{if} \quad x < y.$$

Hence $Pl(A_n)$, which may be identified with the set of standard Young tableaux, becomes a monoid in a natural way. This monoid is called the "plactic monoid" and has been introduced by Lascoux and Schützenberger in order to give an illuminating proof of the Littlewood-Richardson rule for decomposing tensor products of irreducible $gl_n$-modules [11].

There have been attempts to find a Robinson-Schensted type correspondence and plactic relations for the other classical Lie algebras. In [2], Berele has explained an insertion algorithm for $sp_{2n}$ and in [18] Sundaram gives an insertion scheme for $so_{2n+1}$ but it seems difficult to obtain plactic relations from these schemes. More recently Littelmann has used his path model to introduce a plactic algebra for any simple Lie algebra [13]. In the symplectic case, a set of defining relations of the plactic algebra was described independently (without proof) by Lascloux, Leclerc and Thibon [10]. In [17], Sheats has introduced a symplectic Jeu de Taquin analogous to Schützenberger’s sliding algorithm and has conjectured its compatibility with the plactic relations of [10].

The Robinson-Schensted correspondence has a natural interpretation in terms of Kashiwara’s theory of crystal bases [4], [7], [10]. Let $W_n$ denote the vector representation of $gl_n$. By considering each vertex of the crystal graph of $\bigoplus_{t \geq 0} W_n^\otimes t$ as a word on $A_n$, we have for any words $w_1$ and $w_2$:

- $P(w_1) = P(w_2)$ if and only if $w_1$ and $w_2$ occur at the same place in two isomorphic connected components of this graph.
- $Q(w_1) = Q(w_2)$ if and only if $w_1$ and $w_2$ occur in the same connected component of this graph.
Replacing $W_n$ by the vector representation $V_n$ of $sp_{2n}$ whose basis vectors are labelled by

$$C_n = \{1 < \cdots < n - 1 < n < \pi < n - 1 < \cdots < 1\},$$

we can define similarly a relation on $C_n^*$ by: $w_1 \sim_n w_2$ if and only if $w_1$ and $w_2$ have the same position in two isomorphic connected components of the crystal of $G_n = \bigoplus_l V_n^\otimes l$ and set $Pl(C_n) := C_n^*/\sim_n$.

In this article, we undertake a detailed investigation of $Pl(C_n)$ and of the corresponding insertion and sliding algorithms.

We first recall in part 2 the combinatorial notion of symplectic tableau introduced by De Concini (analogous to Young tableaux for type $C$) and how it relates to Kashiwara’s theory of crystal graphs. In part 3 we derive a set of defining relations for $Pl(C_n)$ similar to Knuth relations using the description of the crystal graphs given by Kashiwara and Nakashima [6]. Our plactic relations are those of [4] up to a small mistake that we correct. We will recover Littelmann’s interpretation [1] of plactic relations in terms of crystal isomorphisms but not his explicit description of $Pl(C_n)$ which is not totally exact. In part 4, we describe a column insertion algorithm analogous to the bumping algorithm for type $A$. We note that the insertion procedure is more complicated as for type $A$ and cannot be described as a mere bumping algorithm. Using the notion of oscillating tableaux (analogous to standard tableaux for type $A$ and cannot be described as a mere bumping algorithm. Using the notion of oscillating tableaux (analogous to standard tableaux for type $C$), this algorithm yields the desired Robinson-Schensted correspondence in part 5. Finally we recall in part 6 the definition of the symplectic Jeu de Taquin introduced by Sheats. Then we prove two conjectures of Sheats: applying SJDT slides stays independent of the order in which the inner corners are filled. In particular Sheats Jeu de Taquin can be used to compute the $P$ symbol of a word $w \in C_n^*$.

Similar results for types $B$ and $D$ will be discussed in a forthcoming paper [12].

Note While writing this work, I have been informed that T. H. Baker [1] has obtained independently and by different methods essentially the same insertion schemes as those described in Section 4.

2 Crystal graphs and symplectic tableaux

This section focuses on the notion of symplectic tableaux introduced by Kashiwara and Nakashima to label the vertices of the crystal graphs of a $U_q(sp_{2n})$-module [3]. We relate the symplectic tableaux of Kashiwara and Nakashima to the tableaux used by De Concini in [3] to express the irreducible characters of the classical group $Sp_{2n}$.

2.1 Convention for crystal graphs of $U_q(sp_{2n})$-modules

We adopt Kashiwara’s convention [3] for crystal graphs. The Dynkin diagram of $sp_{2n}$ is labelled by:

$$\frac{1}{0} - \frac{2}{0} - \frac{3}{0} \cdots - \frac{n-2}{0} - \frac{n-1}{0} \iff \frac{n}{0}$$

Accordingly, the crystal graph of the vector representation $V_n$ of $U_q(sp_{2n})$ will be labelled as follows:

$$1 \to 2 \cdots \to n - 1 \to n \to n \to n - 1 \to \cdots \to 2 \to 1$$

Recall that crystal graphs are oriented colored graphs with colors $i \in \{1, \ldots, n\}$. An arrow $a \to b$ means that $f_i(a) = b$ and $e_i(b) = a$ where $e_i$ and $f_i$ are the crystal graph operators (for a review of crystal bases and crystal graphs see [3]). The action of $e_i$ and $f_i$ on the tensor product of two crystal graphs (which is the crystal graph of the tensor product of these representations) is given by:

$$f_i(u \otimes v) = \begin{cases} f_i(u) \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v) \\ u \otimes f_i(v) & \text{if } \varphi_i(u) \leq \varepsilon_i(v) \end{cases}$$

$$e_i(u \otimes v) = \begin{cases} u \otimes e_i(v) & \text{if } \varphi_i(u) < \varepsilon_i(v) \\ e_i(u) \otimes v & \text{if } \varphi_i(u) \geq \varepsilon_i(v) \end{cases}$$
where $\varepsilon_i(u) = \max\{k; e_i^k(u) \neq 0\}$ and $\varphi_i(u) = \max\{k; f_i^k(u) \neq 0\}$. By induction, this allows us to define a crystal graph for the representations $V_n \otimes l$ for any $l$. Each vertex $u_1 \otimes u_2 \otimes \cdots \otimes u_l$ of the crystal graph of $V_n \otimes l$ will be identified with the word $w = u_1 u_2 \cdots u_l$ over the finite totally ordered alphabet

$$C_n = \{1 < \cdots < n < \pi < \cdots < T\}.$$ 

Write $C_n^*$ for the free monoid defined on $C_n$. A barred (resp. unbarred) letter is a letter of $C_n > n$ (resp. $< \pi$). We denote by $d(w) = (d_1, \ldots, d_n)$ the $n$-tuple where $d_i$ is the number of letters $i$ in $w$ minus the number of letters $\bar{i}$, and by $\Sigma(w)$ the length of this word.

Let $G_n$ and $G_{n,l}$ be the crystal graphs of $T(V_n) = \bigoplus_{l} V_n \otimes l$ and $V_n \otimes l$. Then the vertices of $G_n$ are indexed by the words of $C_n$ and those of $G_{n,l}$ by the words of length $l$. $G_{n,l}$ may be decomposed into connected components which are the crystal graphs of the irreducible representations occurring in the decomposition of the representation $V_n \otimes l$. If $w$ is vertex of $G_n$, we denote by $B(w)$ the connected component of $G_n$ containing $w$. In the sequel we call sub-crystal of $G_n$ a union of connected components of $G_n$. Two sub-crystals $\Gamma$ and $\Gamma'$ of $G_n$ are isomorphic when there exists a bijective map $\xi$ from $\Gamma \cup \{0\}$ to $\Gamma' \cup \{0\}$ such that $\xi(0) = 0$ and

$$\xi \circ \bar{e}_i = \bar{e}_i \circ \xi, \xi \circ \bar{f}_i = \bar{f}_i \circ \xi$$ 

for $i = 1, \ldots, n$.

Then we will say that $\xi$ is a crystal isomorphism. We now introduce the coplactic relation.

**Definition 2.1.1** Let $w_1$ and $w_2 \in C_n^*$. We write $w_1 \leftrightarrow w_2$ if and only if $w_1$ and $w_2$ belong to the same connected component of $G_n$.

Note that $w_1 \leftrightarrow w_2$ if and only if $w_2 = \bar{H}(w_1)$ where $\bar{H}$ is a product of Kashiwara’s operators for $U_q(sp_{2n})$. The Lemma below follows immediately from (1).

**Lemma 2.1.2** If $w_1 = u_1 v_1$ and $w_2 = u_2 v_2$ with $\Sigma(u_1) = \Sigma(u_2)$ and $\Sigma(v_1) = \Sigma(v_2)$

$$w_1 \leftrightarrow w_2 \implies \begin{cases} u_1 \leftrightarrow u_2 \\ v_1 \leftrightarrow v_2. \end{cases}$$

Each connected component contains a unique vertex $v^0$ such that $\bar{e}_i(v^0) = 0$ for $i = 1, \ldots, n$. We will call it the vertex of highest weight. Let $\Lambda_1, \ldots, \Lambda_n$ be the fundamental weights of $U_q(sp_{2n})$. The highest weight of the vector representation $V_n$ is equal to $\Lambda_1$. If $v^0$ is the vertex of highest weight of the crystal graph of an irreducible $U_q(sp_{2n})$-module contained in $V_n \otimes l$, the weight $wt(v^0)$ of $v^0$ is the highest weight of this representation. It is given by:

$$wt(v^0) = d_n \Lambda_n + \sum_{i=1}^{n-1} (d_i - d_{i+1}) \Lambda_i$$

where $d(v^0) = (d_1, \ldots, d_n)$. Then two connected components $B$ and $B'$ are isomorphic if and only if their vertices of highest weight have the same weight. Note that in this case the isomorphism between $B$ and $B'$ is unique. This implies the following lemma that we will need in Section 3.

**Lemma 2.1.3** Consider $\Gamma$ be a sub-crystal of $G_n$. Denote by $\Gamma^0$ the set of highest weight vertices of $\Gamma$. Let $\xi$ be a map $\Gamma \cup \{0\} \rightarrow G_n$ such that

(i) $\xi(0) = 0$,

(ii) if $w^0 \in \Gamma^0$, $\xi(w^0)$ is a highest weight vertex, $d(\xi(w^0)) = d(w^0)$ and the restriction of $\xi$ to $\Gamma^0$ is injective,

(iii) for $w \in \Gamma$, and $i = 1, \ldots, n$ such that $\bar{f}_i(w) \neq 0$, $\xi(\bar{f}_i w) = \bar{f}_i \xi(w)$.

Then $\xi$ is a crystal isomorphism from $\Gamma$ to $\xi(\Gamma)$. 

3
It is convenient to parametrize the irreducible $U_q(sp_{2n})$-modules by Young diagrams. Let $\lambda = \sum_{i=1}^{n} \lambda_i \Lambda_i$ with $\lambda_i \in \mathbb{N}$. We associate to $\lambda$ the Young diagram $Y_\lambda$ containing exactly $\lambda_i$ columns of height $i$ and we set $|\lambda| = \sum_{i=0}^{n-1} \lambda_i i$ (i.e. $|\lambda|$ is the number of boxes of $Y_\lambda$). In the sequel we say that $Y_\lambda$ has shape $\lambda$. We write $B(\lambda)$ for the crystal graph of $V(\lambda)$, the irreducible module of highest weight $\lambda$.

The following lemma comes from formula (\ref{formula}).

**Lemma 2.1.4** For any words $w_1$ and $w_2$ in $C_n^*$, the word $w_1 w_2$ is a vertex of highest weight of a connected component of $G_n$ if and only if:

- $w_1$ is a vertex of highest weight (i.e. $\varepsilon_i(w_1) = 0$ for $i = 1, \ldots, n$)
- for any $i = 1, \ldots, n$ we have $\varepsilon_i(w_2) \leq \varphi_i(w_1)$.

We now recall a simple process described by Kashiwara and Nakashima\cite{KashiwaraNakashima} to compute the action of the crystal graphs operators $\tilde{e}_i$ and $\tilde{f}_i$ on a word $w$ (this is a simple consequence of (\ref{formula})). The idea is to consider first the subword $w_i$ of $w$ containing only the letters $i+1, i, i, i+1$. Then we encode in $w_i$ each letter $i+1$ or $i$ by the symbol $+$ and each letter $i$ or $i+1$ by the symbol $\cdot$. Since $\tilde{e}_i(+) = \tilde{f}_i(+) = 0$ the factors of type $++$ may be ignored in $w_i$. So we obtain a subword $w_i^{(1)}$ in which we can ignore all the factors $++$ to construct a new subword $w_i^{(2)}$ etc... Finally we obtain a subword

$$\rho_i(w) = -^r +^s .$$

- if $r > 0$, $\tilde{e}_i(w)$ is obtained by changing the rightmost symbol $-$ of $\rho(w)$ into its corresponding symbol $+$ (i.e. $i+1$ into $i$ and $\overline{i}$ into $\overline{i+1}$) the others letters of $w$ being unchanged. If $r = 0$, $\tilde{e}_i(w) = 0$.
- if $s > 0$, $\tilde{f}_i(w)$ is obtained by changing the leftmost symbol $+$ of $\rho(w)$ into its corresponding symbol $-$ (i.e. $i$ into $i+1$ and $\overline{i+1}$ into $\overline{i}$) the others letters of $w$ being unchanged. If $s = 0$, $\tilde{f}_i(w) = 0$.

**Example 2.1.5** If

$$w = 12\overline{T321211T321T2242}$$

and $i = 1$ we have

$$w_i = 12TT221211T21T222, \quad w_i^{(1)} = TT12T, \quad w_i^{(2)} = TT2$$

$$\rho_i(w) = -^-+$$

$$\tilde{e}_i(w) = 12T322221211T321T2242, \quad \tilde{f}_i(w) = 12T3T221211T321T2242.$$

### 2.2 Admissible and coadmissible columns

A column is a Young diagram $C$ of column shape filled with letters of $C_n$ strictly increasing from top to bottom. The word of $C_n^*$, obtained by reading the letters of $C$ from top to bottom is called the reading of $C$ and denoted by $w(C)$, that is, we write

$$w(C) = x_1 \cdots x_l$$

for $C = \begin{array}{c} x_1 \\ \vdots \\ x_l \end{array}$ with $x_1 < \cdots < x_l$.

The height of $C$ denoted by $h(C)$ is the number of letters in $C$. A word $w$ is a column word if there exists a column $C$ such that $w = w(C)$, i.e. if $w$ is strictly increasing.
Definition 2.2.1 Let $C$ be a column such that $w(C) = x_1 \cdots x_{h(C)}$. Then $C$ is KN-admissible if there is no pair $(z, \overline{z})$ of letters in $C$ such that:

$$z = x_p, \quad \overline{z} = x_q \text{ and } q - p < h(C) - z + 1$$

Remark 2.2.2 The maximal height of a KN-admissible column is $n$. Moreover, $C$ is KN-admissible if and only if, for any $m = 1, \ldots, n$ the number $N(m)$ of letters $x$ in $C$ such that $x \leq m$ or $x \geq \overline{m}$ satisfies $N(m) \leq m$. Moreover if there exists in $C$ a letter $m \leq n$ such that $N(m) > m$ then $C$ contains a pair $(z, \overline{z})$ satisfying $N(z) > z$.

We will need a different version of admissible columns, which goes back to [3]. The equivalence between the two notions is proved in [7].

Definition 2.2.3 Let $C$ be a column and $I = \{z_1 > \cdots > z_r\}$ the set of unbarred letters $z$ such that the pair $(z, \overline{z})$ occurs in $C$. The column $C$ can be split when there exists (see the example below) a set of $r$ unbarred letters $J = \{t_1 > \cdots > t_r\} \subseteq C_n$ such that:

- $t_i$ is the greatest letter of $C_{n}$ satisfying: $t_i < z_1, t_i \notin C$ and $\overline{t_i} \notin C$,
- for $i = 2, \ldots, r$, $t_i$ is the greatest letter of $C_{n}$ satisfying: $t_i < \min(t_{i-1}, z_i)$, $t_i \notin C$ and $\overline{t_i} \notin C$.

In this case we write:
- $rC$ for the column obtained by changing in $C$, $\overline{z_i}$ into $t_i$ for each letter $z_i \in I$ and by reordering if necessary,
- $lC$ for the column obtained by changing in $C$, $z_i$ into $t_i$ for each letter $z_i \in I$ and by reordering if necessary.

Proposition 2.2.4 (Sheats) A column $C$ is KN-admissible if and only if it can be split.

Example 2.2.5 Let $C = \begin{array}{cccc} 7 & 6 & 7 & 4 \end{array}$. Then:

$$I = \{7, 4, 2\}; \quad J = \{5, 3, 1\}; \quad lC = \begin{array}{cccc} 1 & 3 & 5 & 6 \end{array} \text{ and } rC = \begin{array}{cccc} 4 & 7 & 3 & 2 \end{array}.$$

Therefore $C$ can be split. Notice that $C^{\prime} = \begin{array}{cccc} 2 & 4 & 5 & 7 \end{array}$ cannot be split. Indeed we have $I_{C^{\prime}} = \{7, 4, 2\}$. Then $t_1 = 3$, $t_2 = 1$ (with the notation of the above definition) but it is impossible to find $t_3 < 1$ in $C_n$.

An admissible column word is the reading of an admissible column. In Section 5 we will need the notion of KN-coadmissible column. A column $C$ is said to be KN-coadmissible if for each pair $(z, \overline{z})$ in $C$, the number $N^{\ast}(z)$ of letters $x$ in $C$ such that $x \geq z$ or $x \leq \overline{z}$ satisfies

$$N^{\ast}(z) \leq n - z + 1. \quad (3)$$

Let $C$ be a KN-admissible column. Denote by $C^{\ast}$ the column obtained by filling the shape of $C$ (from top to bottom) with the unbarred letters of $lC$ in increasing order followed by the barred letters of $rC$ in increasing order. Then it is easy to prove that $C^{\ast}$ is KN-coadmissible. More precisely the map:

$$\Phi : C \mapsto C^{\ast} \quad (4)$$

is a bijection between the sets of KN-admissible and KN-coadmissible columns of the same height. Starting from a KN-coadmissible column $C^{\ast}$ we can compute the pair $(lC, rC)$ associated to the unique KN-admissible column $C$ such that $C^{\ast} = C$ by reversing the previous algorithm. Then $C$ is the column containing the unbarred letters of $rC$ and the barred letters of $lC$.

Example 2.2.6

If $C =$

$$\begin{array}{c} 1 \\
4 \\
5 \\
4 \\
3 \end{array}$$

then $(lC, rC) = \begin{array}{cc} 1 & 1 \\
2 & 4 \\
5 & 5 \\
4 & 3 \\
3 & 2 \end{array}$ and $C^{\ast} =$

$$\begin{array}{c} 1 \\
2 \\
5 \\
3 \\
2 \end{array}$$
if \( D^* = \begin{pmatrix}
1 & 2 \\
2 & 2 \\
1 & 
\end{pmatrix} \) then \( (ID, rD) = \begin{pmatrix}
1 & 3 \\
2 & 4 \\
3 & 1 \\
\end{pmatrix} \) and \( D = \begin{pmatrix}
3 \\
4 \\
4 \\
3 \\
\end{pmatrix} \).

The original definition of admissible and coadmissible columns due to De Concini and used in \([3]\) and \([17]\) differs from the above one in that De Concini uses the alphabet \( \{ \overline{n-1} < \cdots < 1 < \cdots < n-1 < n \} \) instead of \( C_n \). Up to this change of notation, De Concini coadmissible columns provide a natural labelling of the crystal graphs \( B(\Lambda_p) \) \( p = 1, \ldots, n \). The one-to-one correspondence between De Concini’s admissible columns and KN-admissible columns is explicitly described by \( \Phi \) composed with the permutation \( p \leftrightarrow n-p+1, \overline{p} \leftrightarrow n-p+1 \). In the sequel we systematically translate from De Concini’s convention to Kashiwara’s convention. To simplify the notation we will write admissible and coadmissible for KN-admissible and KN-coadmissible.

### 2.3 Symplectic tableaux

**Definition 2.3.1 (Kashiwara-Nakashima [6]).** Let \( C_1 \) and \( C_2 \) be admissible columns with \( h(C_2) \leq h(C_1) \) and consider the tableau

\[
C_1 C_2 = \begin{pmatrix}
\vdots & \vdots & \ddots & \vdots \\
i_1 & j_1 & \ddots & s \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & j_M \\
\vdots & \ddots & \ddots & \vdots \\
i_N & & & & \\
\end{pmatrix}
\]

For \( 1 \leq a \leq b \leq n \), \( C_1 C_2 \) contains an \((a,b)\)-configuration if there exists \( 1 \leq p \leq q \leq r \leq s \leq M \) such that \( i_p = a, j_q = b, j_r = \overline{b}, j_s = \overline{a} \) or \( i_p = a, i_q = b, i_r = \overline{b}, j_s = \overline{a} \). Then we denote by \( p(a,b) \) the positive integer:

\[
p(a,b) = (s-r) + (q-p).
\]

**Definition 2.3.2 (Kashiwara-Nakashima [6]).** Let \( C_1, C_2, \ldots, C_r \) be admissible columns such that \( h(C_i) \geq h(C_{i+1}) \) for \( i = 1, \ldots, r-1 \). \( T = C_1 C_2 \cdots C_r \) is a symplectic tableau when its rows are weakly increasing from left to right and for \( i = 1, \ldots, r-1 \), \( C_i C_{i+1} \) does not contain an \((a,b)\)-configuration with \( p(a,b) \geq b-a \).

Let \( C_1 \) and \( C_2 \) be two admissible columns. We write:

- \( C_1 \preceq C_2 \) when \( h(C_1) \geq h(C_2) \) and the rows of the tableau \( C_1 C_2 \) are weakly increasing from left to right.
- \( C_1 \preceq C_2 \) when \( rC_1 \preceq lC_2 \).

Note that for any admissible column \( C \), \( lC \leq C \leq rC \), hence \( C_1 \preceq C_2 \Rightarrow C_1 \preceq C_2 \). The next proposition is proved in \([17]\) (Lemma A.4) with the convention of De Concini. It gives a description of the symplectic tableaux which is the translation into Kashiwara’s convention of the description used by Sheats in \([17]\).

**Proposition 2.3.3** Let \( C_1, C_2, \ldots, C_r \) be admissible columns. \( T = C_1 C_2 \cdots C_r \) is a symplectic tableau if and only if \( C_i \preceq C_{i+1} \) for \( i = 1, \ldots, r-1 \).

Let \( T = C_1 C_2 \cdots C_r \) be a symplectic tableau. The shape of \( T \) is the shape of the Young diagram associated to \( T \). The reading of \( T \) is the word \( w(T) \) defined by \( w(T) = w(C_r)w(C_{r-1}) \cdots w(C_1) \) and we set \( d(T) = d(w(T)) \).
Example 2.3.4 $T = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & \end{array}$ is a symplectic tableau and $w(T) = T \overline{3} \overline{7} \overline{2} \overline{4} \overline{1} \overline{4} \overline{3}$.

Kashiwara and Nakashima have proved in [8] the following crucial theorem, which provides a concrete model for the crystal of an irreducible $U_q(sp_{2n})$-module.

**Theorem 2.3.5** Let $\lambda = \sum_{i=1}^n \lambda_i \Lambda_i$ be an integral dominant weight. Then the readings of the symplectic tableaux of shape $\lambda$ are the vertices of a connected component of the crystal graph $G_{n,\lambda}$ isomorphic to $B(\lambda)$. The highest weight vertex of this graph is the reading of the tableau of shape $\lambda$ having only $i$’s on its $i$-th row for $i = 1, 2, ..., n$.

Using Lemma 2.1.4, we deduce immediately the following corollary which will be used repeatedly.

**Corollary 2.3.6** Let $w, w' \in C_n^*$ such that $w \leftrightarrow w'$. Write $w = uw_T v$ and $w' = u'w'_TV$ such that $u, u', v, v', w_T, w'_T \in C_n^*$, $\Sigma(u') = \Sigma(u)$ and $\Sigma(v') = \Sigma(v)$. Then $w_T$ is the reading of a symplectic tableau if and only if $w'_T$ is the reading of a symplectic tableau.

3 A plactic monoid for $U_q(sp_{2n})$

3.1 Symplectic tableau associated to a word

**Definition 3.1.1** Let $w_1$ and $w_2$ be two words. We write $w_1 \sim w_2$ when these two words occur at the same place in the two isomorphic connected components $B(w_1)$ and $B(w_2)$ of the crystal graph $G_n$, that is if there exist $i_1, ..., i_r$ such that $w_1 = \tilde{f}_{i_r} \cdots \tilde{f}_{i_1}(w_1^0)$ and $w_2 = \tilde{f}_{i_r} \cdots \tilde{f}_{i_1}(w_2^0)$, where $w_1^0$ and $w_2^0$ are the vertices of highest weight of $B(w_1)$ and $B(w_2)$.

Then by Theorem 2.3.3 we have

**Proposition 3.1.2** For any word $w$ of $C_n^*$ there exists a unique symplectic tableau $T$ such that $w \sim w(T)$. We shall denote it by $T = P(w)$.

3.2 The monoid $Pl(C_n)$

**Definition 3.2.1** The monoid $Pl(C_n)$ is the quotient of the free monoid $C_n^*$ by the relations:

$R_1 : yxz \equiv yzx$ for $x \leq y < z$ with $z \neq \overline{x}$, and $xyz \equiv xyx$ for $x < y \leq z$ with $z \neq \overline{x}$;

$R_2 : y(x-1)(x-1) \equiv yxx$ and $xxy \equiv (x-1)(x-1)y$ for $1 < x \leq n$ and $x < y \leq \overline{x}$;

$R_3 :$ let $w$ be a non admissible column word such that each strict factor of $w$ is an admissible column word. Write $z$ for the lowest unbarred letter such that the pair $(z, \overline{z})$ occurs in $w$ and $N(z) = z + 1$ (see Remark 2.2.3). Then $w \equiv \overline{w}$ where $\overline{w}$ is the column word obtained by erasing the pair $(z, \overline{z})$ in $w$.

It is clear that $w_1 \equiv w_2$ implies $d(w_1) = d(w_2)$, that is, $\equiv$ is compatible with the grading given by $d$. Notice that $R_1$ contains the Knuth relations for type $A$. In $R_3$ the condition $N(z) > z$ is equivalent to $N(z) = z + 1$. Indeed it is obvious if $z = 1$ and, if $z \neq 1$, $(z, \overline{z})$ is contained in a strict subword of $w$ of length $\Sigma(w) - 1$ which is admissible.

Denote by $\xi$ the crystal isomorphism $B(121) \overset{\xi}{\to} B(112)$. The words occurring in the right hand side of $R_1$ and $R_2$ are the vertices of $B(112)$. Indeed 112 is the reading of a symplectic tableau of shape $\Lambda_2 + \Lambda_1$. So by Theorem 2.3.3, $B(112)$ contains all the readings of the symplectic tableaux of this shape. By Theorem 2.3.3 the connected component of $G_n$ containing 21 is $B(11)$ which contains the words $bc$ with $b \geq c$. Moreover $B(12)$ contains the words $ab \neq \overline{1T}$ with $a < b$. Hence $B(121)$ contains words $abc$ with $a < b$, $b \geq c$, and $ab \neq \overline{1T}$. But $B(121) \cong B(112)$ so these connected components have the same number of vertices. So $B(121)$ contains all the words of the previous type, that is, exactly the words occurring in the left hand side of relations $R_1$ and $R_2$. 


Lemma 3.2.2 \ Let \( w \) be a word occurring in the left hand side of a relation \( R_1 \) or \( R_2 \). Then \( \xi(w) \) is the word occurring in the right hand side of this relation.

**Proof.** The lemma is true for \( w = 121 \). Consider \( w \in B(112) \) verifying the lemma and such that \( \tilde{f}_i(w) \neq 0 \). By induction, it suffices to prove that the words \( v = \tilde{f}_i(w) \) and \( \xi(v) = \tilde{f}_i(\xi(w)) \) are respectively the left and right hand sides of the same relation \( R_1 \) or \( R_2 \). In the sequel we restrict ourselves to the case \( w = yxz \) with \( x \leq y < z \). The case \( w = xzy \) with \( y < x \leq z \) may be treated similarly. Since \( w \) verifies the lemma, one of the two letters \( y \) or \( x \) is not modified when we compute \( \xi(w) \) from \( w \). More precisely, when \( w = yxz \) is not of the form \( x\overline{x}x \), we can write

\[
\xi(w) = yx'z' \text{ with } \begin{cases} x' = x & \text{if } z \neq \overline{x}, \\ x' = (x + 1) & \text{otherwise.} \end{cases}
\]

When \( w = x\overline{x}x \), we have \( \xi(w) = (x - 1)(x - 1)x \) with \( 1 < x \leq n \). Since \( \tilde{f}_i(w) \neq 0 \), \( i = x \). For \( i < n \), we obtain \( v = x\overline{x}(x + 1) \) and \( \xi(v) = \tilde{f}_i(\xi(w)) = (x - 1)(x - 1)(x + 1) \). For \( i = n \), we have \( v = n\overline{n}n \) and \( \xi(v) = \tilde{f}_n(\xi(w)) = (n - 1)(n - 1)n \). In both cases the lemma is verified.

Now suppose that \( w = yxz \) is not of the form \( x\overline{x}x \). Then \( z \neq \overline{m}n \) (otherwise \( w = n\overline{m}n \)). When we apply \( \tilde{f}_i \) to \( w \) one of the three letters \( t \in \{y, x, z\} \) is changed into \( \tilde{f}_i(t) \). If \( v = \tilde{f}_i(y)xx \), then we obtain by (1) \( \varepsilon_i(z) = 0 \) and \( \xi(v) = \tilde{f}_i(\xi(w)) = (x - 1)(x + 1) \). Indeed we have \( \varepsilon_i(z'x') = \varepsilon_i(xz) \) and \( \varepsilon_i(x'z') = \varepsilon_i(yz) \). So \( x < \tilde{f}_i(y) < z \) and \( v \) verifies the lemma. If \( v = \tilde{f}_i(zx) \), we have similarly \( \xi(v) = yf_i(zx') \) and we consider the two following cases:

(i) : \( v = yf_i(z)x \). Then \( \varepsilon_i(x) = 0 \) hence \( z \neq \overline{x} \) and \( \xi(w) = yxz \).

When \( \varphi_i(x) = 0 \), we have \( \varepsilon_i(y) = 0 \) hence \( \tilde{f}_i(z) \neq \overline{x} \). We obtain \( \xi(v) = \tilde{f}_i(\xi(w)) = yf_i(z) \) with \( x \leq y < \tilde{f}_i(z) \) and \( \tilde{f}_i(z) \neq \overline{x} \). So \( v \) verifies the lemma.

When \( \varphi_i(x) = 1 \), we have \( i < n \) because \( z > x \) and \( \varphi_i(z) = 1 \). Therefore \( x < n \) and \( z = x + 1 \). So \( w = y(x + 1)x \), \( \xi(w) = y(x + 1)x \), \( v = yxx \). If \( y \neq x \), we obtain \( \xi(v) = \tilde{f}_i(\xi(w)) = y(x + 1)(x + 1) \). If \( y = x \), we have \( \xi(v) = (x + 1)(x + 1)x \). In both cases the lemma is verified.

(ii) : \( v = yf_i(z)x \). Then \( \varphi_i(z) = 0 \).

When \( z \neq \overline{x} \), we have \( x < n \) (since \( zz \neq \overline{mn} \)) and \( x < y \) (since \( yzx \neq x\overline{x}x \)). So \( i < n \) and \( \xi(w) = y(x + 1)(x + 1) \). We obtain \( v = y\overline{x}(x + 1) \) and \( \xi(v) = \tilde{f}_i(\xi(w)) = y\overline{x}(x + 1) \) with \( x + 1 \leq y < \overline{x} \). So the lemma is verified.

When \( z \neq \overline{x} \), we have \( \xi(w) = yxz \). If \( x = y \), we have by (1) \( \varepsilon_i(z) = 1 \). So \( z = \tilde{f}_i(x) \) (since \( z \neq \overline{x} \)).

We obtain \( w = x\tilde{f}_i(x)x \), \( \xi(w) = xx\tilde{f}_i(x) \), \( v = x\tilde{f}_i(x)\tilde{f}(x) \) and \( \xi(v) = \tilde{f}_i(x)x\tilde{f}(x) \). If \( x < y \), we have \( z \notin \{x + 1, \overline{x} \} \) (since \( z \geq x + 2 \)) so \( \varepsilon_i(z) = 0 \). We obtain \( \xi(v) = y\tilde{f}_i(x)z \) with \( \tilde{f}_i(x) \leq y < z \) and \( z \neq \tilde{f}_i(x) \) (because \( \varphi_i(z) = 0 \)). In both cases the lemma is verified.  ■
If \( w = x_1 \cdots x_{p+1} \) is a non admissible column word of length \( p + 1 \) each strict factor of which is admissible, then \( B(w) = B(12 \cdots p\overline{p}) \). Indeed \( x_1 \cdots x_p \) and \( x_{p+1} \) are admissible column words, hence by Lemmas 2.1.4 and 2.1.2 the highest weight vertex of \( B(w) \) is necessarily \( 12 \cdots p\overline{p} \). Conversely, every vertex \( w \in B(12 \cdots p\overline{p}) \) may be written \( w = z_1 \cdots z_p y \) where \( z_1 \cdots z_p \in B(1 \cdots p) \) and \( z_p y \in B(p\overline{p}) \).

If \( p = 1 \), \( w = 1\overline{1} \) hence we can suppose \( p \geq 2 \). Then the highest weight vertex of \( B(p\overline{p}) \) is \( 12 \) for \( p\overline{p} \) is an admissible column word of two letters. So we have \( z_1 < \cdots < z_p < y \), that is, \( w \) is a column word. The word \( w \notin B(1 \cdots (p + 1)) \) so it is not admissible. If \( v \) is a strict factor of \( w \), it is admissible because it occurs in the same connected component as a strict factor of \( 12 \cdots p\overline{p} \). Finally the words of \( B(12 \cdots p\overline{p}) \) are the non admissible column words of length \( p + 1 \), each strict factor of which is admissible.

Consider \( p > 1 \). For any word \( w \in B(12 \cdots p\overline{p}) \), we set \( \xi_p(w) = \overline{w} \) with the notation of Definition 3.2.1.

**Lemma 3.2.3** The map \( \xi_p \) is a crystal isomorphism from \( B(12 \cdots p\overline{p}) \) to \( B(12 \cdots (p - 1)) \).

**Proof.** We know that \( \xi_p(12 \cdots p\overline{p}) = 12 \cdots (p - 1) \). Hence it suffices to prove that for any \( i = 1, \ldots, n \) and any \( w \in B(12 \cdots p\overline{p}) \), such that \( f_i(w) \neq 0 \), \( \xi_p f_i(w) = f_i \xi_p(w) \). Denote by \((z, \overline{z})\) the pair of letters erased in \( w \) when we compute \( \xi_p(w) \).

Suppose \( i \notin \{z - 1, z\} \). The words \( w \) and \( \xi_p(w) \) differ only by the letters \( z \) and \( \overline{z} \) which do not interfere in the computation of \( f_i \). So it suffices to show that the pair \((z, \overline{z})\) disappears in \( f_i(\overline{w}) \) when the relation \( R_3 \) is applied. We have again \( N(z) = z - 1 \) in \( f_i(\overline{w}) \) because \( i \notin \{z - 1, z\} \). Suppose that \( \overline{f_i}(w) \) contains a pair \((t, \overline{t})\) satisfying \( N(t) = t + 1 \) and \( t < z \). The word \( w \) can not contain the pair \((t, \overline{t})\) otherwise \((z, \overline{z})\) is not the pair of letters which disappears when we apply \( R_3 \). Hence a letter \( t - 1 \) in \( w \) is replaced by \( t \) when we compute \( \overline{f_i}(w) \). Indeed, by (2) the case \( t \in w \) and a letter \( \overline{t} + \overline{t} \) is replaced by \( \overline{t} \) is impossible. Therefore we have \( N(t - 1) = t + 1 \) in \( w \). So \( w \) contains at least a strict factor \( v \) such that \( N(t - 1) = t \) in \( v \). Such a factor is not an admissible column word. So we obtain a contradiction. Hence the pair \((z, \overline{z})\) disappears in \( f_i(\overline{w}) \) when we compute \( \xi_p(w) \).

Suppose \( i = z - 1 \). Then \( z \neq 1 \) and \( w \) must contain a letter \( z - 1 \) or \( z - \overline{1} \). Otherwise, if \( x \) is the greatest unbarred letter \( < z \) such that \( x \in w \) or \( \overline{x} \in w \), \( N(x) = z - 1 > x + 1 \) and the pair \((z, \overline{z})\) does not disappear by applying \( R_3 \). There are three cases to consider: (i) \( z - \overline{1} \) occurs alone in \( w \), or (ii) \( z - 1 \) occurs alone in \( w \), or (iii) the pair \((z - 1, \overline{z} - 1)\) occurs in \( w \). In each case the result is verified by a direct computation. For example, in case (ii), \( w = \cdots (z - 1)z \cdots \overline{2} \cdots \overline{1} \cdots \) we have \( \xi_p(w) = \cdots (z - 1) \cdots \cdots \) (we have only written the letters \( \overline{z - 1}, \overline{z}, z - 1 \) of \( w \) and \( \xi_p(w) \)). So we obtain \( \overline{f_i}(w) = \cdots (z - 1)z \cdots \overline{z - 1} \cdots \). Hence \( \xi_p f_i(w) = \cdots z \cdots = f_i \xi_p(w) \).
Suppose \( i = z \). The letters \( z+1 \) and \( z+1 \) do not occur simultaneously in \( w \) (otherwise the column word obtained by erasing the last letter of \( w \) is not admissible because \( N(z+1) > z+1 \) in this word). Considering the cases: (i) \( z+1 \) occurs in \( w \), (ii) \( z+1 \) occurs in \( w \), (iii) neither \( z+1 \) or \( z+1 \) occurs in \( w \), we obtain the equality \( \xi_p f_i(w) = f \xi_p (w) \) by a direct computation. ■

The following proposition shows the compatibility of the relations \( R_i \) above with crystal graphs operators:

**Proposition 3.2.4** Let \( w_1 \) and \( w_2 \) be words of \( C_n \) such that \( w_1 \equiv w_2 \). Then for \( i = 1, ..., n \):

\[
\tilde{c}_i(w_1) \equiv \tilde{c}_i(w_2), \quad \varepsilon_i(w_1) = \varepsilon_i(w_2), \quad \tilde{f}_i(w_1) \equiv \tilde{f}_i(w_2), \quad \varphi_i(w_1) = \varphi_i(w_2).
\]

**Proof.** By induction we can suppose that \( w_2 \) is obtained from \( w_1 \) by applying only one plactic relation. In this case we write \( w_1 = u \tilde{w}_1 v \) and \( w_2 = u \tilde{w}_2 v \) where \( \tilde{w}_1, \tilde{w}_2 \) are minimal factors of \( w_1 \) and \( w_2 \) such that \( \tilde{w}_1 \equiv \tilde{w}_2 \). Formula (i) implies that it is enough to prove the proposition for \( \tilde{w}_1 \) and \( \tilde{w}_2 \). When \( \tilde{w}_1 \) and \( \tilde{w}_2 \) differ by applying a relation \( R_1 \) or \( R_2 \) the proposition follows from Lemma 3.2.3. When \( \tilde{w}_1 \in B(1 \cdots p) \) it is a consequence of Lemma 3.2.3. ■

**Corollary 3.2.5** If \( w_1 \equiv w_2 \) then \( w_1 \sim w_2 \) (two congruent words occur at the same place in two isomorphic connected components of \( G_n \)).

**Proof.** Let \( w_1^0 \) and \( w_2^0 \) be the highest weight vertices of \( B(w_1) \) and \( B(w_2) \). Then the above proposition and an induction proves that \( w_1^0 \equiv w_2^0 \). So \( d(w_1^0) = d(w_2^0) \) hence \( B(w_1) \) and \( B(w_2) \) are isomorphic. Moreover, by the previous proposition there exists \( i_1, ..., i_r \in \{1, ..., n\} \) such that \( w_1^0 = \tilde{c}_{i_1} \tilde{c}_{i_2} \cdots \tilde{c}_{i_r}(w_1) \) and \( w_2^0 = \tilde{c}_{i_1} \tilde{c}_{i_2} \cdots \tilde{c}_{i_r}(w_2) \). So \( w_1 \) and \( w_2 \) occur at the same place in two isomorphic connected components. ■

We shall now prove the converse

\[
w_1 \sim w_2 \implies w_1 \equiv w_2.
\] (5)

**Proposition 3.2.6** Consider \( w \) a highest weight vertex of \( G_n \). Then \( w \equiv w(P(w)) \) (every highest weight vertex is congruent to a highest weight tableau word).

**Proof.** We proceed by induction on \( |w| \). If \( |w| = 1 \) then \( w = 1 \) and \( P(w) = 1 \). Suppose the proposition true for the highest weight vertices of length \( l \) and consider a highest weight vertex \( w \) of length \( l+1 \). Write \( w = vx \) where \( x \in C_n \) and \( v \) is a word of length \( l \). Then by Lemma 2.1.4 \( v \) is of highest weight so by induction \( v \equiv w(P(v)) \) where \( P(v) \) contains only unbarred letters. Hence \( w \equiv w(P(v)) \). We deduce from the condition \( \varepsilon_i(x) \leq \varphi_i(v) \) of Lemma 2.1.3 that only the two following situations can occur:

1. \( x \) is an unbarred letter and \( P(v) \) contains a column of reading \( 1 \cdots (x-1) \) if \( x > 1 \),
2. \( x = \overline{p} \) and \( P(v) = C_1 \cdots C_r \) contains a column \( C_i \) with \( w(C_i) = 1 \cdots p \) (we suppose \( i \) maximal in the sequel).

1 : If \( x \) is unbarred \( P(w) \) is obtained from \( P(v) \) by adding a box containing \( x \) at the bottom of the leftmost column of reading \( 1 \cdots (x-1) \) if \( x > 1 \), by adding a box containing \( v \) to the right of \( P(v) \) otherwise. We have \( w = v \equiv w(P(v)) \equiv w(P(w)) \) because in \( P(C_n) \), \( x \) commute with all the column words containing \( x \) and only unbarred letters.

2 : If \( x = \overline{p} \). \( P(w) \) is the reading of the tableau obtained by erasing the letter \( p \) on the top of \( C_i \). The word \( w(C_i) \) is of highest weight, so we can write \( w(C_i) = 1 \cdots q \) with \( q \geq p \) and \( vx = w(C_r) \cdots w(C_2)(1 \cdots q \overline{p}) \). Then by using the contraction relation \( 1 \cdots \overline{p} = 1 \cdots \overline{p} \) (where the hat means removal the letter \( p \)) we obtain \( w = w(C_r) \cdots w(C_2)1 \cdots \overline{p} \cdots q \). We have \( P(w) = P(w(C_r) \cdots w(C_2)1 \cdots \overline{p} \cdots q) \). Hence by case 1 and an easy induction \( w \equiv w(P(w)) \). ■

**Corollary 3.2.7** If \( w_1^0 \) and \( w_2^0 \) are two highest weight vertices with the same weight, then \( w_1^0 \equiv w_2^0 \)

**Proof.** \( P(w_1^0) = P(w_2^0) = T \) because \( w_1^0 \) and \( w_2^0 \) have the same weight. Then \( w_1^0 \equiv T \equiv w_2^0 \). ■
Theorem 3.2.8 For any vertices $w_1$ and $w_2$ of $G_n$

$$w_1 \sim w_2 \iff w_1 \equiv w_2.$$  

Proof. Suppose that $w_1 \sim w_2$ and denote by $w_1^0$ and $w_2^0$ the highest weight vertices of $B(w_1)$ and $B(w_2)$. The corollary above shows that $w_1^0 \equiv w_2^0$. Write $w_1 = \tilde{f}_1 \cdots \tilde{f}_i w_1^0$, then we have $w_2 = \tilde{f}_i \cdots \tilde{f}_n w_1^0$. Hence by Proposition 3.2.4, $w_1 \equiv w_2$. The converse was proved in Corollary 3.2.5.

Remark The sufficiency of the relations obtained from the crystal isomorphisms $B(121) \cong B(112)$ and $B(12 \cdots p) \rightarrow B(11 \cdots p)$ to generate $C_n/\sim$ was proved in [13].

4 A bumping algorithm for type $C$

Now we are going to see how the symplectic tableau $P(w)$ may be computed for each vertex $w$ by using an insertion scheme analogous to the bumping algorithm for type $A$. As a first step, we compute $P(w)$ when $w = w(C)x$, where $x$ and $C$ are respectively a letter and an admissible column. This will be called “the insertion of the letter $x$ in the admissible column $C$” and denoted by $x \rightarrow C = P(w)$.

Then we will be able to obtain $P(w)$ when $w = w(T)x$ with $x$ a letter and $T$ a symplectic tableau. This will be called “the insertion of the letter $x$ in the symplectic tableau $T$” and denoted by $x \rightarrow T$. Our construction of $P$ will be recursive, in the sense that if $P(u) = T$ and $x \in C_n$, then $P(ux) = x \rightarrow T$.

4.1 Insertion of a letter in an admissible column

Consider a word $w = w(C)x$, where $x$ and $C$ are respectively a letter and an admissible column of height $p$. Then by Lemma 2.1.4 the highest weight vertex $w^0$ of $B(w)$ verifies:

(i) $w^0 = 1 \cdots p (p + 1)$ or
(ii) $w^0 = w^0 = 1 \cdots p \overline{p}$ or
(iii) $w^0 = 1 \cdots p \overline{1}$

In case (i) $w^0$ is an admissible column word so $w$ is the reading of the admissible column obtained by adding a box filled by $x$ at the bottom of $C$ that is $x \rightarrow C = \begin{array}{c} C \\ \hline x \end{array}$

In case (ii) we have seen in the previous section that the vertex $w$ is a nonadmissible column word all of whose proper factors are admissible. Hence $x \rightarrow C$ is the column of reading $\tilde{w}$ obtained from $w$ by a congruence $R_4$ as described in Definition 3.2.1. In case (iii), the congruence class of $w^0$ consists of the words $w^0 = 12 \cdots i(i + 1)(i + 2) \cdots p$ for $p \geq i > 0$, and the congruence relations on this class define a linear graph with end points $w^0 = w^0$ and $w^0 = 112 \cdots p$ (it describes the commutation of the letter 1 with the column word 12$\cdots$1; with all congruences of type $R_4$ or $R_2$).

By repeated applications of operators $\tilde{f}_i$, this class can be transformed into that of $w$, which will also have a linear graph with all congruences of type $R_1$ or $R_2$, and with a word $w' \in B(w^0)$ at the end opposite to $w$. Then $w'$ is the reading of a symplectic tableau consisting of a column $C'$ of height $p$ and a column $\begin{array}{c} x' \\ \hline \end{array}$ (with $x' \in C_n$). We can write $x \rightarrow C = \begin{array}{c} C' \\ \hline x' \end{array} = P(w)$.

To obtain the congruence class of $w$, one applies a congruence of type $R_1$ or $R_2$ to the final sub-word of length 3 of $w$ (there is only one possibility and it applies from left to right). On the resulting word, one continues with the overlapping sub-word of length 3 one place to the left and so forth until the left subword of length 3 has been operate upon (see Example 4.1.1 below). This insertion can be regarded as the combinatorial description of the crystal isomorphism:

$$B(1 \cdots p1) \rightarrow B(11 \cdots p)$$  

Example 4.1.1 Suppose $w = 35543$. 

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If \( x = 2 \) the word \( 355432 \) is a non admissible column word each strict subword of which is an admissible column word. Then we obtain by applying \( R_3, \tilde{2} \rightarrow 3 \):

\[
\begin{array}{cccc}
3 & 3 & 4 \\
5 & 5 & 3 \\
4 & 3 & 2 \\
3 & 2 & 1 \\
\end{array}
\]

If \( x = 3 \) we obtain by applying at each step \( R_1 \) or \( R_2 \) the new word:

\[
355433 = 355444 = 355554 = 344554 = 434554
\]

This last word is the reading of a symplectic tableau. Hence \( 3 \rightarrow 4 \).

4.2 Insertion of a letter in a symplectic tableau

Let \( T = C_1 \cdots C_r \) be a symplectic tableau with admissible columns \( C_i \) and \( x \) a letter. We have seen in the proof of Proposition 3.2.6 that the vertex of highest weight of the connected component containing \( w(T)x \) may be written \( w^0 x^0 \) where \( w^0 \) is the reading of a highest weight tableau \( T^0 \), and that one of the following conditions is satisfied:

(i) : \( x^0 = p \) and \( 1 \cdots (p - 1) \) is the reading of a column of \( T^0 \)

(ii) : \( x^0 = \bar{p} \) and \( 1 \cdots p \) is the reading of a column of \( T^0 \).

Write \( T^0 = C^0_1 \cdots C^0_r \). In case (i) let \( k \) be minimal such that \( w(C^0_k) = 1 \cdots (p - 1) \). Denote by \( T^{0'} \) the tableau obtained from \( T^0 \) by adding a box containing the letter \( p \) on the top of \( C^0_k \). There exists a unique sequence \( w_0, ..., w_{k-1} \) such that \( w(T^{0'}) \) is the highest weight vertex of \( B(w_{k-1}) \) and such that \( w_{j-1} = w(C_r) \cdots w(C_j) x_{j-1} w(C_{j-1}) \cdots w(C'_1) \) is transformed into the congruent word \( w_j = w(C_r) \cdots w(C_{j+1}) x_j w(C'_j) \cdots w(C'_1) \) where \( x_j \) and \( C'_j \) are determined by \( x_{j-1} \rightarrow C_j = C'_j [x_j] \). The word \( w_{k-1} \) is the reading of a symplectic tableau \( T' \) and is obtained by computing \( k - 1 \) insertions of type “insertion of a letter in an admissible column”. So we have \( x \rightarrow T = T' \). Notice that there is no contraction in this case because no relation of type \( R_3 \) is applied.

In case (ii), \( w(C_1)x \) is a non admissible column word whose strict subwords are admissible column words. Suppose \( w(C_1)x = y_1 \cdots y_s \) and write \( \hat{T} \) for the tableau whose columns are \( C_2, ..., C_r \). Then we have

\[
x \rightarrow T = y_s \rightarrow (y_{s-1} \rightarrow (\cdots y_1 \rightarrow \hat{T}))
\]

that is \( x \rightarrow T \) is obtained by inserting successively the letters of \( \hat{w(C_1)}x \) into the tableau \( \hat{T} \). Indeed there is no contraction during the insertion of the letters \( y_i \) because the highest weight tableau associated to \( x \rightarrow T \) is obtained by inserting the letters \( 1, ..., p-1, p+1, ..., s+1 \) in the highest weight tableau \( C^0_0 \cdots C^0_r \) and no relation \( R_3 \) is required to realize it. So \( y_s \rightarrow (y_{s-1} \rightarrow (\cdots y_1 \rightarrow \hat{T})) \) is well defined and is a symplectic tableau whose reading is congruent to \( w(T)x \). It must be equal to \( x \rightarrow T \).

Example 4.2.1 let \( T = \begin{array}{ccccc} 1 & 3 & 3 & 2 \\
3 & 3 & 2 & 1 \\
\end{array} \) and \( x = \bar{1} \). The column \( C_1 = \begin{array}{ccc} 1 \\
3 \\
2 \\
\end{array} \) is not admissible and

\[
\tilde{C}_1 = \begin{array}{cc} 3 & 2 \\
2 & 1 \\
\end{array}
\]

So we have to insert 3 next 2 in the tableau \( \hat{T} = \begin{array}{ccccc} 3 & 3 & 2 \\
3 & 2 & 1 \\
\end{array} \). An easy computation
shows that the insertion of \( \overline{3} \) gives the tableau
\[
\begin{array}{cccc}
2 & 3 & 2 & 2 \\
3 & 2 & 2 & \\
\hline
3 & 2 & \\
2 & 1
\end{array}
\]
then by inserting \( \overline{2} \) we obtain \( \overline{1} \rightarrow T = \)
\[
\begin{array}{cccc}
2 & 3 & 2 & 2 \\
3 & 2 & 2 & \\
\hline
3 & 3 & 2 & \\
2 & 1
\end{array}
\]

Note that the insertion scheme of the letter \( x \) in the symplectic tableau \( T = C_1 \cdots C_r \) obtained above and that described by Baker in [1] coincide if no contraction appears during \( x \rightarrow C_1 \). When such a contraction appears, Baker computes first a admissible skew tableau \( T' \) (see Definition 5.1.1) from \( T \) by deleting the box lying on the top of \( C_1 \) and by filling the column diagram so obtained with the letters of \( w(C_1)x \). For example, with the tableau \( T \) above and \( x = 1 \) we obtain
\[
T' = \begin{array}{ccc}
3 & 3 & 2 \\
2 & 1
\end{array}
\]
Then the rest of his algorithm may be regarded as a special case of the Symplectic Jeu de Taquin described in Section 6.

4.3 Computation of \( P(w) \) for any word \( w \)

Consider a word \( w \in C_n^* \). Then we have
\[
P(w) = \overline{w} \quad \text{if } w \text{ is a letter},
\]
\[
P(w) = x \rightarrow P(u) \quad \text{if } w = ux \text{ with } u \text{ a word and } x \text{ a letter}.
\]
that is \( P(w) \) may be computed by induction by using the insertion schemes of Subsections 4.1 and 4.2. Notice that when \( w = x_1 \cdots x_l \) is the reading of the symplectic tableau \( T \), \( P(w) = T \) but the equality
\[
T = x_l \rightarrow (x_{l-1} \rightarrow (\cdots \rightarrow x_1))
\]
is not combinatorially trivial.

5 Robinson-Schensted type correspondence for type \( C_n \)

In this section a bijection is established between words \( w \) of length \( l \) on \( C_n \) and pairs \((P(w), Q(w))\) where \( P(w) \) is the symplectic tableau defined in section 2 and \( Q(w) \) is an oscillating tableau. Such a one-to-one correspondence has already been obtained by Berele [3] using King’s definition of symplectic tableaux [8] and an appropriate insertion algorithm. Unfortunately we do not know if this correspondence is compatible with a monoid structure. Our bijection based on the previous insertion algorithm will be different from Berele’s one but compatible with the plactic relations \( R_i \).

We now recall the definition of an oscillating tableau and we give the construction of \( Q(w) \). Next the Robinson-Schensted type correspondence theorem will be proved.

5.1 Oscillating tableau associated to a word \( w \)

**Definition 5.1.1 (Berele)** An oscillating tableau \( Q \) of length \( l \) is a sequence of Young diagrams of shape \((Q_1, \ldots, Q_l)\) such that any two consecutive shapes differ by exactly one box (i.e. \( Q_{k+1}/Q_k = \) or \( Q_k/Q_{k+1} = \).

If, for any \( i = 1, \ldots, l \), no column of \( Q_i \) has height greater than \( n \) we will say that \( Q \) is an \( n \)-oscillating tableau. Let \( w = x_1 \cdots x_l \in C_n^* \). The construction of \( P(w) \) involves the construction of the \( l \) symplectic tableaux \((P_1, \ldots, P_l)\) defined by \( P_i = P(x_{l-i+1} \cdots x_l) \).
Proposition 5.1.3 For any word \( w \in \mathcal{C}_n \), \( Q(w) \) is an \( n \)-oscillating tableau.

Proof. Write \( w = vx \) where \( v \in \mathcal{C}_n \) and \( x \in \mathcal{C}_n \). By induction it suffices to show that the shapes of \( P(w) \) and \( P(v) \) differ by exactly one box. The shape of \( P(w) \) depends only on the connected component containing \( w \). So we can suppose \( w \) of highest weight. Then by Lemma 2.1.4, \( v \) is of highest weight.

Suppose \( x = \overline{1}_n \geq \pi_n \), then its weight is equal to \( \Lambda_{i-1} - \Lambda_{i} \), and if we denote by \( (\lambda_1, ..., \lambda_n) \) the coordinates of the weight of \( v \) on the basis of fundamental weights we have \( \lambda_i > 0 \) (because \( w \) must be of highest weight). By Theorem 2.3.3, we can deduce that during the insertion \( x \rightarrow P(v) \) a column of height \( i \) (corresponding to the weight \( \Lambda_{i} \)) is turned into a column of height \( i - 1 \) (corresponding to the weight \( \Lambda_{i-1} \)). So the shape of \( P(w) \) is obtained by erasing one box from the shape of \( T \).

When \( x = i \leq n \), its weight is equal to \( \Lambda_{i} - \Lambda_{i-1} \) and similar arguments show that the shape of \( P(w) \) is obtained by adding one box to the shape of \( P(v) \).

5.2 Correspondence theorem

Proposition 5.2.1 Let \( w_1 \) and \( w_2 \) be two words of \( \mathcal{C}_n \). Then

\[
\begin{align*}
    w_1 \leftrightarrow w_2 & \iff Q(w_1) = Q(w_2).
\end{align*}
\]

Proof. We proceed by induction on the length \( l \) of the words \( w_1 \) and \( w_2 \). If \( l = 1 \) the result is immediate. If \( w_1 \) and \( w_2 \) have length \( l \geq 1 \), we can write \( w_1 = u_1x_1 \) and \( w_2 = u_2x_2 \) with \( x_1, x_2 \) letters and \( u_1, u_2 \) words of length \( l - 1 \). Let \( \lambda_1^0 = u_1^0x_1^0 \) and \( \lambda_2^0 = u_2^0x_2^0 \) be the highest weight vertices of \( B(w_1) \) and \( B(w_2) \). Write \( Q_1 \) and \( Q_2 \) for the shapes of \( P(w_1) \) and \( P(w_2) \) (that is those of \( P(\lambda_1^0) \) and \( P(\lambda_2^0) \)). We suppose the Proposition true for the words of length \( l - 1 \). First we have:

\[
\begin{align*}
    w_1 \leftrightarrow w_2 & \iff \begin{cases} u_1 \leftrightarrow u_2 \\
Q_1 = Q_2
\end{cases}
\end{align*}
\]

Indeed if \( u_1 \leftrightarrow u_2 \) then \( x_1^0 \leftrightarrow x_2^0 \) follows from Lemma 2.1.2 and we obtain \( Q_1 = Q_2 \) because the readings of \( P(\lambda_1^0) \) and \( P(\lambda_2^0) \) are in the same connected component of \( G_n \). Conversely, \( u_1 \leftrightarrow u_2 \) implies that \( \lambda_1^0 = \lambda_2^0 \) and it follows from the equality \( Q_1 = Q_2 \) that \( \lambda_1 = \lambda_2 \) (because the shape of \( P(\lambda_i^0) \), \( i = 1, 2 \) coincides with the weight \( \lambda_i^0 \)). So \( x_1^0 \leftrightarrow x_2^0 \). This means that \( u_1^0 = u_2^0 \) i.e. \( w_1 \leftrightarrow w_2 \). Finally we obtain by induction:

\[
\begin{align*}
    w_1 \leftrightarrow w_2 & \iff \begin{cases} Q(w_1) = Q(w_2) \\
Q_1 = Q_2
\end{cases}
\end{align*}
\]

The Robinson-Schensted correspondence for type \( C_n \) follows immediately:

Theorem 5.2.2 Let \( \mathcal{C}_{n,l}^* \) and \( \mathcal{O}_l \) be the set of words of length \( l \) on \( C_n \) and the set of pairs \((P,Q)\) where \( P \) is a symplectic tableau and \( Q \) an \( n \)-oscillating tableau of length \( l \) such that \( P \) has shape \( Q_1 \) (\( Q_1 \) is the last shape of \( Q \)). Then the map:

\[
\Psi : \mathcal{C}_{n,l}^* \rightarrow \mathcal{O}_l \quad \quad w \mapsto (P(w), Q(w))
\]

is a bijection.

Proof. By Theorem 3.2.8 and Proposition 5.2.1, we obtain that \( \Psi \) is injective. Consider a \( n \)-oscillating tableau \( Q \) of length \( l \). Set \( x_1 = 1 \) and for \( i = 2, ..., l \), \( x_i = k \) if \( Q_i \) differs from \( Q_{i-1} \) by adding a box in row \( k \) and \( x_i = \overline{k} \) if \( Q_i \) differs from \( Q_{i-1} \) by removing a box in row \( k \). Consider \( w_Q = x_1 \cdots x_l \). Then \( Q(w_Q) = Q \). By Theorem 2.3.3, the image of \( B(w_Q) \) by \( \Psi \) consists in the pairs \((P,Q)\) where \( P \) is a symplectic tableau of shape \( Q_1 \). We deduce immediately that \( \Psi \) is surjective.

In our correspondence, the inverse bumping algorithm is implicit. It is possible to describe it explicitly but it would be rather cumbersome to do.
6 A sliding algorithm for $U_q(sp_{2n})$

This section is concerned with a symplectic Jeu de Taquin (or sliding algorithm) introduced by J. T. Sheats [17] in order to obtain an explicit bijection between King’s and De Concini’s symplectic tableaux. The sliding algorithm for $U_q(sl_n)$ is confluent because its steps stay within the initial plactic congruence class (see [3]). Sheats has conjectured this property for his Jeu de Taquin. Our aim is now to prove this conjecture and thus obtain an alternative way to compute $P(w)$ for any word $w$.

We first recall the ideas of Sheats and translate them into Kashiwara-Nakashima’s conventions. Next we extend his algorithm in order to make it compatible with the relation of contraction $R_3$. Finally we show that this algorithm is also confluent.

6.1 Sheats sliding algorithm

6.1.1 Skew admissible tableaux

Let $\lambda = \sum_{i=1}^{n} \lambda_i \Lambda_i$ and $\mu = \sum_{i=1}^{n} \mu_i \Lambda_i$ be two dominant weights such that $\mu_i \leq \lambda_i$ for $i = 1, ..., n$. A skew tableau of shape $\lambda / \mu$ over $C_n$ is a filling of the skew Young diagram $Y_\lambda / Y_\mu$ by letters of $C_n$ giving columns strictly decreasing from top to bottom.

Definition 6.1.1 A skew tableau over $C_n$ is admissible if its columns are admissible and the rows of its split form (obtained by turning each column $C$ into its split form $(lC, rC)$) are weakly increasing from left to right.

Example 6.1.2 $T = \begin{array}{ccc}
4 & 4 & 3 \\
3 & 3 & 2 \\
2 & 2 & 
\end{array}$ is a admissible skew tableau. Its split form is

\[
\begin{array}{cccc}
4 & 4 & 4 & 4 \\
1 & 3 & 4 & 3 \\
3 & 3 & 2 & 2 \\
2 & 2 & 1 & 
\end{array}
\]

Lemma 6.1.3 The set of readings of the admissible skew tableaux of shape $\lambda / \mu$ is a sub-crystal of $G_n$.

Proof. Let $r < n$. Denote by $U_r$ the subalgebra of $U_q(sp_{2n})$ generated by $\widetilde{e}_i$, $\widetilde{f}_i$, $i = r + 1, ..., n$. Clearly, $U_r \cong U_q(sp_{2(n-r)})$. By restriction to $U_r$, any $U_q(sp_{2n})$-crystal gives a $U_r$-crystal obtained by forgetting the arrows of color $i = 1, ..., r$. In particular, $B(\lambda)$ decomposes into a union of connected $U_r$-crystals whose vertices are labelled by symplectic tableaux of shape $\lambda$. Moreover, for all tableaux of such a connected component, the subtableau consisting of the letters $1, 2, ..., r$ is the same since $\widetilde{e}_i$, $\widetilde{f}_i$ ($i > r$) leaves these letters unchanged. Hence if we consider the set $B(\lambda, \mu) \subset B(\lambda)$ of readings of the symplectic tableaux $T$ of shape $\lambda$ on $C_n$ such that:

(i) the sub tableau of $T$ formed by the letters $i = 1, ..., r$ is the fixed tableau of shape $\mu$ having, for each $i$, only letters $i$ on its $i$-th row, and

(ii) $T$ does not contain any letters in $\{r, ..., \overline{1}\}$,

then we see that $B(\lambda, \mu)$ is stable under $\widetilde{e}_i$, $\widetilde{f}_i$ ($i > r$), hence is a union of $U_r$-crystals. Finally, this set can be identified to the set of readings of all admissible skew tableaux of shape $\lambda / \mu$ over $\{r + 1, ..., n, \overline{n-r}, ..., \overline{1}\}$. Hence shifting the indices by $r$, we have proved that the set of readings of the admissible skew tableaux of shape $\lambda / \mu$ over $\{1, ..., n - r, \overline{n-r}, ..., \overline{1}\}$ is a sub crystal of $G_{n-r}$ and since $r < n$ are arbitrary, we are done. ■

This lemma implies that the readings of the admissible skew tableaux of shape $\lambda / \mu$ are the vertices of a sub-crystal of $G_n$. We denote by $T(\lambda / \mu)$ the set of admissible skew tableaux of shape $(\lambda / \mu)$ and by $U(\lambda / \mu)$ the set of readings of these skew tableaux.
Definition 6.1.4 Consider an admissible skew tableau of shape $\lambda / \mu$. An inner corner is a box of $Y_\mu$ such that the boxes down and to the right are not in $Y_\mu$. An outside corner is a box of $Y_\lambda$ such that the boxes down and to the right are not in $Y_\lambda$.

Definition 6.1.5 A skew tableau is punctured if one of its box contains the symbol $*$ called the puncture.

A punctured column $C$ is admissible if the column $C'$ obtained by ignoring the puncture is admissible. Then the punctured columns $rC$ and $lC'$ are respectively obtained by replacing the letters of $C$ (except the puncture) by the letters of $rC'$ and $lC'$. The split form of $C$ is $lCrC$.

A punctured skew tableau is admissible if its columns are admissible and the rows of its split form (obtained by splitting its columns) are weakly increasing (ignoring the puncture).

Example 6.1.6 $T = \begin{array}{cccc}
4 & 4 & 3 & *\\
3 & 3 & 2 & \\
2 & 2 & & \\
\end{array}$ is a admissible skew punctured tableau of split form $spl(T) = \begin{array}{cccc}
3 & 4 & 4 & 4 \\
1 & 3 & * & 3 \\
3 & 3 & 2 & 2 \\
2 & 2 & 2 & 1 \\
\end{array}$

6.1.2 Elementary step of the sliding algorithm

Let us consider $T$ an admissible punctured skew tableau containing two columns $C_1$ and $C_2$ with the puncture in $C_1$. To apply an elementary step of the sliding algorithm to $T$ we have first to consider the split form of $T$. In this split form we have a configuration of the type:

```
  ...  ...
  *  *  b  b'
  a  a'  ...
  ...
```

where the boxes containing $a, a'$ and $b, b'$ may be empty.

An elementary step of the Symplectic Jeu de Taquin (SJDT) consists of the following transformations:

1. If $a' \leq b$ or the double box $b b'$ is empty, then the doubles boxes $a, a'$ and $*$ are permuted

2. If $a' > b$ or the double box $a, a'$ is empty then:
   
   (i): when $b$ is a barred letter, $b$ slides into $rC_1$ to the box containing $*$ and $D_1 = \Phi(C_1) - \{a\} + \{b\}$ is a co-admissible column (see section 2.2, (4)). Simultaneously the symbol $*$ slides into $lC_2$ to the box containing $b$ and $C'_2 = C_2 - \{b\} + \{\}$. Then we obtain a new punctured skew tableau $C'_1C'_2$ by setting $C'_1 = \Phi^{-1}(D_1)$.

   (ii): when $b$ is an unbarred letter, $b$ slides into $rC_1$ to the box containing $*$ and give a new column $C'_1 = C_1 - \{a\} + \{b\}$. Simultaneously the symbol $*$ slides into $lC_2$ to the box containing $b$ and $D_2 = \Phi(C_2) - \{b\} + \{\}$ is a punctured co-admissible column. Then we obtain a new punctured skew tableau $C'_1C'_2$ by setting $C'_2 = \Phi^{-1}(D_1)$.

Notice that in case 2 (i) the co-admissibility of $D_1$ is not immediate and in case 2 (ii) the column $C'_1$ may be not admissible.

Lemma 6.1.7 We can always apply an elementary step of the SJDT to an admissible punctured skew symplectic tableau (i.e. $D_1$ is a co-admissible column in case 2 (i)).

Proof. See Lemma 9.1 of [17].
Example 6.1.8

For $T_1 = \begin{bmatrix} 2 & 4 \\ 4 & 5 \\ 3 & 1 \\ 1 & \end{bmatrix}$, $spl(T_1) = \begin{bmatrix} * & 4 \\ * & 4 \\ 3 & 1 \\ 1 & \end{bmatrix}$. We are in case 2 (i) and $C'_1C'_2 = \begin{bmatrix} 2 & 4 \\ 5 & 5 \\ 3 & 1 \\ 1 & \end{bmatrix}$.

For $T_2 = \begin{bmatrix} 2 & 2 \\ 3 & 3 \\ 5 & 5 \\ 1 & \end{bmatrix}$, $spl(T_2) = \begin{bmatrix} * & 5 \\ * & 4 \\ 5 & 5 \\ 1 & \end{bmatrix}$. We are in case 2 (ii) and $C'_1C'_2 = \begin{bmatrix} 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 1 & \end{bmatrix}$.

For $T_3 = \begin{bmatrix} 4 & 4 \\ 5 & 4 \\ 3 & \end{bmatrix}$, we obtain $\begin{bmatrix} 4 & 4 \\ 5 & 4 \\ 3 & \end{bmatrix}$.

The punctured skew tableau obtained by computing a step of the SJDT on an admissible skew tableau is not always admissible. In the second example above $C'_1$ is not admissible and in the third the rows of the split form are not increasing (we will see that this last problem does not occur in the complete SJDT algorithm).

6.1.3 Complete symplectic Jeu de Taquin (SJDT)

Let $T$ be an admissible skew tableau and $c$ an inner corner in $T$. In order to apply the complete sliding algorithm let us puncture the corner $c$. We obtain an admissible punctured skew tableau. To see what happens when we apply successively elementary steps of SJDT to this skew tableau, we need to compute the split form for each intermediate punctured tableau. We have seen that a horizontal move of an unbarred letter may give a new non admissible column $C'_1$ such that all the strict subwords of $w(C'_1)$ are admissible. So it is impossible to compute its split form using letters of $C_n$. To overcome this problem, we embed the alphabet $C_n$ into $C'_{n+1} = \{a_1 < 1 < \cdots < n < \pi < \cdots < \uparrow < \underbar{\uparrow}\}$.

To compute the split form of a non admissible column $C$ such that all the strict subwords of $w(C)$ are admissible, we extend the algorithm of Subsection 2.2 by using the new letter $a_1$. For example

$$\begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 & 2 \\ 2 & 3 \\ 4 & 1 \\ \underbar{\uparrow} \end{bmatrix} \text{ in } C'_{n+1}.$$

So all the columns that may be obtained when we apply an elementary step of SJDT to an admissible skew tableau (defined on $C_n$) can be split in $C'_{n+1}$. We will say that a skew punctured tableau is $a_1$-admissible if all its columns can be split in $C'_{n+1}$ and the rows of the obtained split form are weakly increasing.

Theorem 6.1.9 (Sheats [17])

- Elementary steps of SJDT may be applied to $T$ until the puncture $*$ becomes an outside corner.
- All the skew punctured tableaux obtained as steps in the algorithm are $a_1$-admissible. Moreover $\underbar{\uparrow}$ and $a_1$ may only appear simultaneously in the split form of the column containing the inner corner $c$ of $T$ at which the slide started.

Proof. See Proposition 9.2 of [17].

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Example 6.1.10 Suppose \( \text{spl}(T) = \begin{array}{cccc}
  * & * & 2 & 2 \\
  2 & 2 & 3 & 3 \\
  2 & 3 & 3 & 4 & 4 \\
  5 & 5 & 4 & 4 & 1 & 1 \\
  3 & 2 & 1 & 1
\end{array} \). We compute successively the split form of the \( a_1 \)-admissible punctured skew tableaux:

\[
\begin{array}{ccc}
  \begin{array}{ccc}
  2 & 2 & 2 \\
  2 & 2 & 2 \\
  2 & 3 & 3 \\
  5 & 5 & 4 \\
  3 & 2 & 1
\end{array} & \begin{array}{ccc}
  2 & 2 & 2 \\
  2 & 2 & 2 \\
  3 & 3 & 3 \\
  5 & 5 & 4 \\
  3 & 2 & 1
\end{array} & \begin{array}{ccc}
  a_1 & 2 & 2 \\
  2 & 3 & 3 \\
  2 & 3 & 3 \\
  5 & 5 & 4 \\
  3 & 2 & 1
\end{array}
\end{array}
\]

Then we obtain the \( a_1 \)-admissible skew tableau:

\[
\begin{array}{ccc}
  a_1 & 2 & 2 \\
  2 & 3 & 3 \\
  2 & 3 & 3 \\
  5 & 4 & 1 \\
  3 & 1
\end{array}
\]

6.2 Sliding algorithm on \( \mathcal{C}_n \)

Let \( T \) be an admissible skew tableau and \( c \) an inner corner. If we denote by \( T' \) the skew tableau obtained by applying the complete SJDT to \( T \), then \( T' \) may be only \( a_1 \)-admissible (see Theorem 6.1.9). Suppose that, in the split form, \( \overline{a_1} \) and \( a_1 \) occur in the \( k \)-th split column \( lC_k \) of \( T' \). Then the column \( C'_k \) is not admissible and, in order to obtain an admissible skew tableau, we are led to consider the skew tableau \( \widetilde{T}' \) obtained by erasing the top and bottom boxes of \( C'_k \) and filling this new column with the letters of the word \( w(C'_k) \). We denote this new column by \( \widetilde{C}_k \).

Example 6.2.1 Continuing the previous example we obtain:

\[
\widetilde{T} = \begin{array}{ccc}
  2 \\
  3 & 3 & 1 \\
  5 & 1 \\
  2
\end{array}
\]

Using the notations introduced above, we have:

**Proposition 6.2.2** \( \widetilde{T}' \) is an admissible skew tableau and \( w(T') \equiv w(\widetilde{T}') \).

**Proof.** By Theorem 6.1.9, we know that \( T' \) is \( a_1 \)-admissible and \((a_1, \overline{a_1})\) occurs only in \( C'_k \). This implies that there is no box of \( T' \) weakly below and strictly to the right of the box containing \( \overline{a_1} \). So \( \widetilde{T} \) has the shape of a skew tableau. The pair \((a_1, \overline{a_1})\) disappears when we compute \((l\widetilde{C}_k, r\widetilde{C}_k)\). Moreover each letter \( x \neq \overline{a_1} \) of \( rC'_k \) is turned into a letter \( y \leq x \) in \( r\widetilde{C}_k \) and each letter \( t \neq a_1 \) of \( lC'_k \) is turned into a letter \( v \geq t \) in \( l\widetilde{C}_k \). Thus the rows of the split form of \( \widetilde{T}' \) are weakly increasing; \( \widetilde{T}' \) is an admissible skew tableau. Finally, the identity \( w(T') \equiv w(\widetilde{T}') \) is clear because \( w(C'_k) \equiv w(\widetilde{C}_k) \) by \( R_3 \). \( \blacksquare \)

Given an admissible skew tableau \( T \) and \( c \) an inside corner in \( T \) we can apply elementary steps of SJDT to obtain a skew tableau \( T' \). We set:

\[
\text{SJDT}(T, c) = \begin{cases}
  T' & \text{if } T' \text{ is admissible} \\
  \widetilde{T} & \text{if } T' \text{ is only } a\text{-admissible}
\end{cases}
\]

During the algorithm an inner corner is filled or SJDT\((c, T)\) has two boxes less than \( T \). By choosing a new inner corner at each step, we can iterate the procedure \( T \to \text{SJDT}(T, c) \) to construct a symplectic tableau from any admissible skew tableau. Now we are going to establish the
Conjecture 6.2.3 (Sheats) The symplectic tableau obtained by iterating the sliding algorithm is independent of the order in which the inner corners are filled.

6.3 Proof of Sheats’ conjecture

In this section we show that, for any fixed inner corner c, the map \( w(T) \rightarrow w(\text{SJDT}(T, c)) \) defined on \( U_{(\lambda/\mu)} \) is a crystal isomorphism. This result will imply Sheats’ conjecture.

We begin with skew tableaux of two columns. Consider \( p, q, k \) some integers such that \( 0 < k \leq p \leq q \) and denote by \( T_{(q,p)/k} \) the set of admissible skew tableaux of two columns of shape \( \lambda/\mu \) where \( Y_\lambda \) consists in the two columns of height \( q \) and \( Y_\mu \) is the single column of height \( k \). The skew tableaux of \( T_{(q,p)/k} \) have a unique inner corner \( c \). By Lemma 6.1.3, the subset \( U_{(q,p)/k} \) consisting of the readings of the skew tableaux of \( T_{(q,p)/k} \) is a sub-crystal of \( G_n \). Set \( \Psi_{(q,p)/k}(0) = 0 \) and for any \( T \in T_{(q,p)/k} \)

\[
\Psi_{(q,p)/k}(w(T)) = w(\text{SJDT}(T, c)).
\]

Then \( \Psi_{(q,p)/k} \) is a map defined on \( U_{(q,p)/k} \cup \{0\} \). We are going to prove that \( \Psi_{(q,p)/k} \) is in fact a crystal isomorphism from \( U_{(q,p)/k} \) to its image. To obtain this result it suffices to see that \( \Psi_{(q,p)/k} \) verifies the assertions (ii) and (iii) of Lemma 2.1.3. This is the goal of technical Lemmas 6.3.1, 6.3.2, 6.3.3 and 6.3.4. Elementary steps of SJDT are applied on two consecutive columns of skew punctured tableaux.

So by using crystal isomorphisms of type \( \Psi_{(q,p)/k} \) and Lemma 6.3.5, it will be easy to obtain a crystal isomorphism defined on \( U_{(\lambda/\mu)} \) in Theorem 6.3.8.

We will say that a horizontal move occurs in \( T \in T_{(q,p)/k} \) when \( w(T) \notin U_{(q-1,p)/(k-1)} \). It means that we have in \( spl(T) \) a configuration

\[
H(a,b) = \begin{array}{ccc}
& \cdot & b \\
. & a & . \\
. & . & .
\end{array}
\]

with \( a > b \). Notice that \( \Psi_{(q,p)/k}(w(T)) = w(T) \) if no horizontal move occurs in \( T \).

Lemma 6.3.1 Let \( U_{(q,p)/k}^0 \) be the set of highest weight vertices of \( U_{(q,p)/k} \). Then \( \Psi_{(q,p)/k} \) induces a bijection from \( U_{(q,p)/k}^0 \) to \( \Psi_{(q,p)/k}(U_{(q,p)/k}^0) \). Moreover, \( \Psi_{(q,p)/k}(U_{(q,p)/k}^0) \) contains only highest weight vertices and for any \( w^0 \in U_{(q,p)/k}^0 \), \( d(w^0) = d(\Psi_{(q,p)/k}(w^0)) \).

Proof. The assertion \( d(w^0) = d(\Psi_{(q,p)/k}(w^0)) \) for any \( w^0 \in U_{(q,p)/k}^0 \) follows immediately from the definition of SJDT. Consider \( w_1, w_2 \in U_{(q,p)/k}^0 \) such that \( \Psi_{(q,p)/k}(w_1) = \Psi_{(q,p)/k}(w_2) \). Denote respectively by \( C_1C_2 \) and \( D_1D_2 \) the skew tableaux of \( T_{(q,p)/k} \) of reading \( w_1 \) and \( w_2 \). If no horizontal move occurs in \( C_1C_2 \), no horizontal move occurs in \( D_1D_2 \) and \( w_1 = \Psi_{(q,p)/k}(w_1) = \Psi_{(q,p)/k}(w_2) = w_2 \). So we can suppose that a horizontal move occurs in \( C_1C_2 \) and \( D_1D_2 \). By Lemma 6.1.4, \( w(C_2) \) and \( w(D_2) \) are vertices of highest weight. Hence \( D_2 = C_2 \) since \( h(C_2) = h(D_2) = p \). Write \( w(C_2) = 1 \cdots p \). Denote by \( b \) the letter of \( IC_2 \) which slides into \( rC_1 \). Suppose that \( b < p \). Then \( C_1C_2 \) contains a configuration

\[
H(a,b) = \begin{array}{ccc}
& \cdot & b \\
. & a & . \\
. & b+1 & .
\end{array}
\]

with \( a > b \). We have \( a \leq b + 1 \). This implies that \( a = b + 1 \). Then \( w(C_1) \) contains the letter \( b + 1 \). Hence \( w(C_1) \) contains all the letters \( b + 1, b, \ldots, 1 \) by Lemma 2.1.4. We must have \( \varepsilon_i w(C_1) = 0 \) for \( i = b, \ldots, 1 \). Moreover \( C_1 \) is admissible so \( \overrightarrow{C} \notin C_1 \) hence \( b \in rC_1 \). This is incompatible with the sliding of \( b \) in \( rC_1 \). That means that \( p \) is the letter of \( C_2 \) which slide in \( rC_1 \) when we compute \( \Psi_{(q,p)/k}(w_1) \). Similarly \( p \) slides into \( rD_1 \) when we compute \( \Psi_{(q,p)/k}(w_2) \). Set \( C'_1 = C_1 + \{p\} \) and \( D'_1 = D_1 + \{p\} \). If these two columns are admissible we will have \( \Psi_{(q,p)/k}(w_1) = w(C'_1C'_2) \) and \( \Psi_{(q,p)/k}(w_2) = w(D'_1C'_2) \) with \( C'_2 = C_2 - \{p\} \). Indeed the horizontal moves considered are type 2 (ii). Using \( \Psi_{(q,p)/k}(w_1) = \Psi_{(q,p)/k}(w_2) \), we obtain \( D'_1 = C'_1 \) hence \( D'_1 = C_1 \). If \( C'_1 \) is not admissible, \( C'_2 \) is not admissible for \( \Psi_{(q,p)/k}(w_1) = \Psi_{(q,p)/k}(w_2) \). Then we have \( w(C'_1) = w(D'_1) \). Denote respectively
by \((z, \bar{z})\) and \((t, \bar{t})\) the pairs of letters erased in \(w(C'_1)\) and \(w(D'_1)\). Then \(N(z) = z + 1\) in \(C'_1\) and \(N(t) = t + 1\) in \(D'_1\). But \(C'_1\) and \(D'_1\) may differ only by the pairs \((z, \bar{z})\) and \((t, \bar{t})\). Hence \(N(z) = N(t)\), so \(z = t\). We obtain \(C'_1 = D'_1\) and \(C_1 = D_1\).

To prove that \(\Psi_{(q,p)/k}(w_1)\) is of highest weight, it suffices to show that \(w(C'_1,C'_2)\) is of highest weight because when \(C'_1\) is not admissible \(\Psi_{(q,p)/k}(w_1) = w(C'_1,C'_2)\). We have \(C'_2 = 1 \cdots (p - 1)\). Hence by Lemma 2.1.4, \(w(C'_1,C'_2)\) is of highest weight if and only if

\[
\varepsilon_i(w(C'_1)) = 0 \text{ if } i \neq p - 1 \text{ and } \varepsilon_{p-1}(w(C'_1)) \leq 1.
\]

The columns \(C_1\) and \(C'_1\) differ only by the letter \(p\). Then \((3)\) is satisfied for \(i \notin \{p - 1, p\}\). We have \(\varepsilon_{p-1}(w(C'_1)) \leq 1\). For if it were not so, \(\varepsilon_{p-1}(w(C'_1)) = 2\) and \(\varepsilon_{p-1}(w(C_1)) = 1\) because \(C'_1 = C_1 + \{p\}\). Suppose \(\varepsilon_p(w(C'_1)) \neq 0\). If \(\varepsilon_p(w(C_1)) = 0\) then \(C_1\) is of highest weight, we have \(w(C_1) = 1 \cdots (q - k)\) and no horizontal move occurs in \(C_1C_2\). So \(\varepsilon_p(w(C_1)) = 1\). Then, with the notation of \((3)\), \(\rho_p(w(C'_1)) = -\). So \(\rho(w(C'_1)) = +\) because \(C'_1 = C_1 + \{p\}\). Hence \(\varepsilon_p(w(C'_1)) = 0\) and we obtain a contradiction.

**Lemma 6.3.2** Let \(T \in \mathcal{T}_{(q,p)/k}\) and suppose \(\tilde{f}_n w(T) \neq 0\). Then \(\tilde{f}_n \Psi_{(q,p)/k} w(T) = \Psi_{(q,p)/k} w(T)\).

**Proof.** The Lemma is clear if no horizontal move occurs in \(T\). In the sequel we suppose that a horizontal move occurs in \(T\). Let \(T_n \in \mathcal{T}_{(q,p)/k}\) such that \(w(T_n) = \tilde{f}_n w(T)\). Write \(T = C_1C_2\) and \(T_n = D_1D_2\). We know that \(\text{spl}(T)\) contains a configuration \(H_{(a,b)}\) of type \((3)\) such that \(b\) slides into \(rC_1\) when we apply \(\text{SJDT}\) to \(T\). By considering the letters of \(\text{spl}(T_n)\) occurring at the same place as the letters of \(H_{(a,b)}\) we obtain in \(\text{spl}(T_n)\) a configuration \(H_{(a',b')}\). We have

\[
\begin{array}{c}
. \ . \ b \\
. \ . \ a \\
\end{array}
\quad \text{with } a > b \text{ in } \text{spl}(T) \quad \text{and} \quad \begin{array}{c}
. \ . \ b' \\
. \ . \ a' \\
\end{array}
\quad \text{in } \text{spl}(T_n).
\]

If \(a' \leq b'\) we must have \(a = a' = b' = \pi\) and \(b = n\) (\(n\) is the last unbarred letter of \(C_n\)) because the columns of \(\text{spl}(T)\) and \(\text{spl}(T_n)\) differ by at most a change \(n \to \pi\). We have \(\pi \in rC_1\), so \(n \notin C_1\) and \(\pi \notin D_1\). Hence \(C_1 = D_1\) for these two columns may only differ by a change \(n \to \pi\). This case, \(\tilde{f}_n w(C_1C_2) = w(C_1)\tilde{f}_n w(C_2) \neq 0\). Hence \(n \in C_2\) and \(\pi \notin C_2\). By using the notation of \((3)\), we have \(\rho_n(w(C_1C_2)) = +\), which implies that \(\tilde{f}_n w(T) = 0\). Hence we obtain a contradiction. It means that \(a' > b'\). So a horizontal move occurs in \(T_n\). Moreover, the letter \(b'\) slides into \(rD_1\) when we apply \(\text{SJDT}\) to \(T_n\). If this were not so, we would have the configurations

\[
\begin{array}{c}
. \ . \ n \\
. \ . \ n \\
\end{array}
\quad \text{in } T \quad \text{and} \quad \begin{array}{c}
. \ . \ n \\
. \ . \ \pi \\
\end{array}
\quad \text{in } T_n.
\]

Hence \(C_1\) would contain the letter \(n\) which is changed in \(\pi\) when we apply \(\tilde{f}_n\) and \(C_2 \cap \{n, \pi\} = \{n\}\). We obtain a contradiction because we have \(\rho_n(w(C_1C_2)) = -\), hence by \((3)\), it is the letter \(n\) of \(w(C_2)\) that should be changed in \(\pi\) when we apply \(\tilde{f}_n\).

When we compute \(\text{SJDT}\) on \(T\) and \(T_n\), we first execute a sequence of vertical moves until we obtain a puncture \(*\) on the row containing \(b\) and on the row containing \(b'\). Then we apply a horizontal move on these two punctured tableaux. Denote by \(C'_1C'_2\) and \(D'_1D'_2\) the skew punctured tableaux so obtained.

Suppose first that \(D_1 = C_1\) and \(w(D_2) = \tilde{f}_n w(C_2)\). Then \(D_2 = C_2 - \{n\} + \{\pi\}\) and \(lD_2 = lC_2 - \{n\} + \{\pi\}\) because \(\pi \notin C_2\). If \(b = b'\) we can write \(D'_1D'_2 = C'_1D'_2\) and \(D'_2 = C'_2 - \{n\} + \{\pi\}\). Then \(\tilde{f}_n w(C'_1C'_2)\), which is obtained from \(w(C'_1C'_2)\) by turning \(n\) into \(\pi\) in \(w(C'_2)\), is equal to \(w(C'_1D'_2)\). When \(b \neq b'\), we have \(b' = \pi\) and \(b = n\). We can write \(D'_2 = C'_2 - \{n\} + \{\pi\}\) and \(D'_1 = C'_1 - \{n\} + \{\pi\}\). Indeed \(rC_1\), hence \(C_1\), does not contain a letter of \(\{n, \pi\}\). Then \(\tilde{f}_n w(C'_1C'_2)\), which is obtained from \(w(C'_1C'_2)\) by turning \(n\) into \(\pi\) in \(w(C'_1)\), is equal to \(w(D'_1D'_2)\).

Now suppose that \(D_2 = C_2\) and \(\tilde{f}_n w(C_1) = w(D_1)\). We have \(b = b'\) hence \(C'_1 = D'_1\). Notice that \(b \notin \{n, \pi\}\) because \(\pi \in rC_1\) and \(n \in rD_1\). So \(D'_1 = C'_1 - \{n\} + \{\pi\}\). Then \(\tilde{f}_n w(C'_1C'_2)\), which is obtained from \(w(C'_1C'_2)\) by turning \(n\) into \(\pi\) in \(w(C'_1)\) is equal to \(w(D'_1D'_2)\).

The subalgebra of \(U_q(s_{2n})\) generated by the Chevalley generators \(e_i, f_i\) and \(t_i\) for \(i = 1, \ldots, n - 1\) is isomorphic to \(U_q(s_{2n})\). Each \(U_q(s_{2n})\)-module \(M\) is by restriction an \(U_q(s_{2n})\)-module. To obtain its
crystal graph it suffices to erase the arrows of color \( n \) in the crystal graph of \( M \). If \( w \in G_n \), we denote by \( B^A(w) \) the \( U_q(sl_n) \)-connected component of \( G_n \) considered as a \( U_q(sl_n) \)-crystal graph (i.e. without the arrows of color \( n \)).

Set \( U^+_{(q,p)/k} = \{ w \in U_{(q,p)/k}; w \) contains only unbarred letters \( \} \) and \( U^-_{(q,p)/k} = \{ w \in U_{(q,p)/k}; w \) contains only barred letters \( \} \). Then denote respectively by \( \Psi^+_{(q,p)/k} \) and \( \Psi^-_{(q,p)/k} \) the restrictions of the map \( \Psi_{(q,p)/k} \) to \( U^+_{(q,p)/k} \) and \( U^-_{(q,p)/k} \).

**Lemma 6.3.3** \( U^+_{(q,p)/k} \) and \( U^-_{(q,p)/k} \) are \( U_q(sl_n) \)-sub-crystals of \( G_n \) considered as a \( U_q(sl_n) \)-crystal. Moreover, \( \Psi^+_{(q,p)/k} \) and \( \Psi^-_{(q,p)/k} \) are \( U_q(sl_n) \)-crystal isomorphisms from \( U^+_{(q,p)/k} \) and \( U^-_{(q,p)/k} \) to their images.

**Proof.** Barred and unbarred letters play a symmetric role for the action of Kashiwara’s operators \( \overline{e}_i, \overline{f}_i \) \( i = 1, \ldots, n - 1 \). So it suffices to prove the lemma for \( U^+_{(q,p)/k} \) and \( \Psi^+_{(q,p)/k} \). The set \( U^+_{(q,p)/k} \) is stable under the action of \( \overline{e}_i, \overline{f}_i \) \( i = 1, \ldots, n - 1 \) so it is a \( U_q(sl_n) \)-sub-crystal of \( G_n \). Lemma 3.3.1 show that \( \Psi^+_{(q,p)/k} \) induces a bijection from the set \( U^0_{(q,p)/k} \) of highest weight vertices of \( U^+_{(q,p)/k} \) to \( U_{(q,p)/k} \). Moreover if \( w^0 \in U^0_{(q,p)/k} \), \( \Psi_{(q,p)/k}(w^0) \) and \( w^0 \) have the same weight. Then \( \Psi^+_{(q,p)/k} \) is a \( U_q(sl_n) \)-crystal isomorphism if and only if for any \( w \in U^+_{(q,p)/k} \)

\[
\overline{f}_i \circ \Psi^+_{(q,p)/k}(w) = \Psi^+_{(q,p)/k} \circ \overline{f}_i(w) \quad i = 1, \ldots, n - 1.
\]

This equality was proved in Theorem 3.3.1 of [4]. \( \square \)

Consider \( r \) and \( s \) two integers such that \( r + s \leq n \). The vertices \( 1 \cdots r \overline{\pi} \cdots \overline{(n-s+1)} \) and \( \pi \cdots (n-s+1)1 \cdots r \) are highest weight vertices with the same \( U_q(sl_n) \)-weight of the crystal \( G_n \) considered as a \( U_q(sl_n) \)-crystal graph. Denote by \( \Theta_{r,s} \) the \( U_q(sl_n) \)-crystal isomorphism:

\[
B^A(1 \cdots r \overline{\pi} \cdots \overline{(n-s+1)}) \xrightarrow{\psi_{U_q(sl_n)}} B^A(\pi \cdots (n-s+1)1 \cdots r).
\]

Note that the vertices of \( B^A(1 \cdots r \overline{\pi} \cdots \overline{(n-s+1)}) \) are admissible column words of length \( r + s \) with \( r \) unbarred letters and those of \( B^A(\pi \cdots (n-s+1)1 \cdots r) \) are words of the form \( u^- v^+ \) where the factors \( u^- \) and \( v^+ \) contain respectively \( s \) barred and \( r \) unbarred letters. Then we have the following

**Lemma 6.3.4**

1. \( B^A(1 \cdots r \overline{\pi} \cdots \overline{(n-s+1)}) \) consists of the readings of the admissible column words of length \( r + s \) with \( r \) unbarred letters.

2. \( B^A(\pi \cdots (n-s+1)1 \cdots r) \) consists of the words of the form \( u^- v^+ \) such that \( v^+ u^- \) is the reading of a admissible column of height \( r + s \) with \( r \) unbarred letters.

3. Let \( C \) be a column such that \( w(C) \in B^A(1 \cdots r \overline{\pi} \cdots \overline{(n-s+1)}) \). Write \( w(C^*) = v^+ u^- \) where the words \( u^- \) and \( v^+ \) contain respectively barred and unbarred letters. Then

\[
\Theta_{r,s}(w(C)) = u^- v^+.
\]

**Proof.** 1. Follows from the fact that \( w_{r,s} = 1 \cdots \overline{\pi} \cdots \overline{(n-s+1)} \) is the unique admissible column word of length \( r + s \) with \( r \) unbarred letters satisfying \( \varepsilon_i(w_{r,s}) = 0 \) for \( i = 1, \ldots, n - 1 \). In the sequel we set \( w_{r,s} = \pi \cdots (n-s+1)1 \cdots r \).

2. Consider \( w \in B^A(w_{r,s}) \) and suppose that for any pair \( (z, \overline{\pi}) \in w \)

\[
N^*(z) \quad \text{the number of letters} \quad x \quad \text{such} \quad z \leq x \leq \overline{\pi} \quad \text{verifies} \quad N^*(z) \leq n - z + 1.
\]

Let \( i \in \{1, \ldots, n - 1 \} \) such that \( \overline{f}_i(w) \neq 0 \). Then any pair of letters \( (z, \overline{\pi}) \in \overline{f}_i(w) \) verifies \( N^*(z) \leq n - z + 1 \). Otherwise \( \overline{f}_i(w) \) contains a pair \( (t, \overline{\pi}) \) with \( N^*(t) > n - t + 1 \). Then \( t \notin w \) or \( \overline{\pi} \notin w \). By
We have \( i = t \) and \( w \) is obtained from \( \tilde{f}_i(w) \) by turning \( \overline{t} \) into \( \overline{t} + 1 \). Then the number of letters \( x \in w \) such that \( t \leq x \leq \overline{t} + 1 \) is \( n - t + 1 \). Let \( y \) be the smallest letter of \( w \) such that \( t < y \leq n \) and \( (y, \overline{\gamma}) \in \mathcal{W} \). We have \( N^*(y) > n - y + 1 \). Hence we derive a contradiction. It is clear that \( w_{r,s}^* \), verifies \( \mathcal{W} \). Hence by induction all the vertices of \( B^A(w_{r,s}^*) \) verify \( \mathcal{W} \). By \( \mathcal{W} \), it means that these vertices may be written \( u^* v^* \) where \( v^* u^- \) is the reading of a coadmissible column of height \( r + s \) with \( r \) unbarred letters. It follows from \( \mathcal{W} \) that the number of admissible columns of height \( r + s \) with \( r \) unbarred letters is equal to the number of coadmissible columns of height \( r + s \) with \( r \) unbarred letters. So 2 is proved.

For any \( w = w(C) \in B^A(w_{r,s}) \), we set \( w^* = u^* v^* \) where \( w(C) = v^* u^- \). By 1, 2 and \( \mathcal{W} \), we know that the map \( w \to w^* \) is a bijection from \( B^A(w_{r,s}) \) to \( B^A(w_{r,s}^*) \). Note that \( \Theta_{r,s}(w_{r,s}) = w_{r,s}^* \). Hence, to prove that \( \Theta_{r,s}(w) = w^* \), it suffices to show that for any \( i = 1, \ldots, n - 1 \) such that \( \tilde{f}_i(w) \neq 0 \):

\[
(f_i(w))^* = \tilde{f}_i(w^*).
\]

Fix \( i \in \{1, \ldots, n - 1\} \) and set \( E_i = w \cap \{i, i + 1, \overline{i + 1}, \overline{i} \} \), \( F_i = w^* \cap \{i, i + 1, \overline{i + 1}, \overline{i} \} \). Then \( \mathcal{W} \) \( E_i \) is equal to one of the following sets: (i) \( E_i = \{i\} \), (ii) \( E_i = \{i + 1\} \), (iii) \( E_i = \{i, \overline{i + 1}, \overline{i} \} \), (iv) \( E_i = \{i, i + 1, i + 1\} \), (v) \( E_i = \{i + 1, i + 1\} \) or (vi) \( E_i = \{i, i + 1\} \). Then \( \mathcal{W} \) follows by considering for the six cases above, the possible sets \( F_i \). For example in case (i), we have \( f_i(w) = w - \{i\} + \{i + 1\} \). Moreover \( F_i = \{i\} \) or \( F_i = \{i, i + 1, i + 1\} \). Then \( (f_i(w))^* = \tilde{f}_i(w^*) = w^* - \{i\} + \{i + 1\} \) when \( F_i = \{i\} \) and by \( \mathcal{W} \), \( (\tilde{f}_i(w))^* = \tilde{f}_i(w^*) = w^* - \{i + 1\} + \{\overline{i}\} \) when \( F_i = \{i, i + 1, i + 1\} \).

The proof of Proposition \( \ref{prop:crystal_iso} \) relies on the following simple

**Lemma 6.3.5** Let \( \Gamma_i \) and \( \Gamma_i' \) \( i = 1, \ldots, m \) be sub-crystals of \( G_n \) such that there exists a crystal isomorphism \( \xi_i \) from \( \Gamma_i \) to \( \Gamma_i' \) for \( i = 1, \ldots, m \). Denote respectively by \( \Gamma_1 \cdots \Gamma_m \) and \( \Gamma_1' \cdots \Gamma_m' \) the sub-crystals of \( G_n \) whose vertices are the words of the form \( w_1 \cdots w_m \) obtained by concatenating the words \( w_i \in \Gamma_i \) \( i = 1, \ldots, m \) (resp. \( w_i \in \Gamma_i' \) \( i = 1, \ldots, m \)). Then the map

\[
\Gamma_1 \cdots \Gamma_m \to \Gamma_1' \cdots \Gamma_m'
\]

\[
w_1 \cdots w_m \mapsto \xi_1(w_1) \cdots \xi_m(w_m)
\]

is a crystal isomorphism.

**Proof.** This follows immediately from the description \( \mathcal{W} \) of the tensor product of two crystal graphs.

Now we can state the

**Proposition 6.3.6** The map \( \Psi_{(q,p)/k} \) is a crystal isomorphism from \( \mathcal{U}_{(q,p)/k} \) to its image.

**Proof.** By Lemmas \( \ref{lem:crystal_iso} \), \( \ref{lem:crystal_iso} \) and \( \ref{lem:crystal_iso} \), it suffices to prove that:

\[
\tilde{f}_i \circ \Psi_{(q,p)/k}(w) = \Psi_{(q,p)/k} \circ \tilde{f}_i(w) \text{ for } i = 1, \ldots, n - 1.
\]

This will be deduced from Lemmas \( \ref{lem:crystal_iso} \), \( \ref{lem:crystal_iso} \) and \( \ref{lem:crystal_iso} \). Set \( w = w(C_1 C_2) \) with \( C_1 C_2 \in \mathcal{T}_{(q,p)/k} \). If no horizontal move occurs in \( C_1 C_2 \), \( \mathcal{W} \) is immediate. Otherwise, suppose that an unbarred letter of \( lC_1 \) slides into \( rC_1 \) during the horizontal move. Write \( w(C_1) = u_1^+ u_1^- \) and \( w(C_2) = u_2^+ u_2^- \) where the letters of \( u_1^+ \) and \( u_2^+ \) are unbarred and those of \( u_1^- \) and \( u_2^- \) are barred. Denote respectively by \( r_1, r_2 \) the number of unbarred letters of \( u_1^+ \) and \( u_2^+ \). Then \( u_1^- \) and \( u_2^- \) contain respectively \( q - k - r_1 = s_1 \) and \( p - r_2 = s_2 \) barred letters. Set \( \Theta_{r_1,s_1}(u_1^+ u_2^-) = u_2^- u_1^+ \) (see Lemma \( \ref{lem:crystal_iso} \)). We can decompose the computation of \( \Psi_{(q,p)/k}(w(C_1 C_2)) = \Psi_{(q,p)/k}(u_2^- u_1^+) \) into the following three steps (as illustrated by Example \( \ref{ex:crystal_iso} \)):

(i) we calculate the word

\[
v_2^- v_1^+ u_1^- u_1^-,
\]

where
(ii) we apply \( \Psi_{k+r_1,r_2}/k \) to the admissible skew tableau \( v_2^+ u_1^+ \) containing only unbarred letters to obtain

\[
v_2^{-} \Psi_{k+r_1,r_2}/k(v_2^+ u_1^+) u_1^- = v_2^{-} b_2^+ b_1^+ u_1^-
\]

where \( b_1^+ \) and \( b_2^+ \) contain respectively \( r_1 + 1 \) and \( r_2 - 1 \) unbarred letters.

(iii) we apply \( \Theta_{r_2-1,s_2}^{-1} \) to \( v_2^{-} b_2^+ \), getting

\[
\Psi_{(q,p)/k} w(C_1 C_2) = \Theta_{r_2-1,s_2}^{-1}(v_2^{-} b_2^+) b_1^+ u_1^-.
\]

By Lemmas 6.3.3 and 6.3.4, the maps \( \Psi_{k+r_1,r_2}/k \), \( \Theta_{r_2-1,s_2}^{-1} \) and \( \Theta_{r_2,s_2} \) are \( U_q(sl_n) \)-crystal isomorphisms. Hence \( \Psi_{(q,p)/k} \) commutes with the operators \( \tilde{f}_i \), \( i = 1, \ldots, n-1 \). This follows from Lemma 6.3.5 applied with \( m = 2 \), \( \xi_1 = \Theta_{r_2,s_2}^{-1} \), \( \xi_2 = \text{id} \) in step (i); with \( m = 3 \), \( \xi_1 = \xi_3 = \text{id} \), \( \xi_2 = \Psi_{k+r_1,r_2}/k \) in step (ii) and with \( m = 2 \), \( \xi_1 = \Theta_{r_2-1,s_2}^{-1} \), \( \xi_2 = \text{id} \) in step (iii).

When a barred letter of \( lC_2 \) slides into \( rC_1 \), we obtain that \( \Psi_{(q,p)/k} \) commute with the operators \( \tilde{f}_i \), \( i = 1, \ldots, n-1 \) by similar arguments. This time we compute \( \Theta_{r_1,s_1}(u_1^+ u_1^-) \) and we use the restriction \( \Psi_{(c+s_1,s_2)/c} \) with \( e = k + r_1 - r_2 \) instead of \( \Psi_{k+r_1,r_2}/k \).}

**Example 6.3.7** Consider \( C_1 C_2 \) such that \( \text{spl}(C_1 C_2) = \). We have \( w(C_1) = 2351 \) and \( w(C_2) = 2355 \) so \( w(C_1 C_2) = 23552351 \). When we apply \( \text{SJD} \) to \( C_1 C_2 \), the letter \( 4 \) in \( lC_2 \) slides into \( rC_1 \) (see the punctured skew tableau \( T_2 \) of Example 6.1.4). With the notation of the above proof we have \( v_2^+ v_2^- u_1^+ u_1^- = 42342351 \). By considering the slide

\[
\begin{array}{ccc}
2 & 2 & * \\
3 & 3 & 4 \\
* & 4 & 4 \\
\end{array}
\rightarrow
\begin{array}{ccc}
2 & 2 & 2 \\
3 & 3 & 3 \\
4 & 4 & * \\
\end{array}
\]

we obtain \( \Psi_{(3,3)/1}(234 23) = 23234 \). Then \( v_2^+ b_2^+ b_1^- u_1^- = 42342351 \) and \( \Psi_{(3,4)/1} w(C_1 C_2) = 23423451 \).

This last word is the reading of the skew tableau obtained by applying \( \text{SJD} \) on \( C_1 C_2 \).

Note that the sub-crystal \( \Psi_{(q,p)/k}(U_{(q,p)/k}) \) may contain readings of skew tableaux with different shapes. We can now state the

**Theorem 6.3.8** Let \( c \) be a fixed inner corner of the shape \( (\lambda/\mu) \). Denote by \( \Xi(\cdot,c) \) the map defined on \( U_{(\lambda/\mu)} \) by \( \Xi(w,c) = w(\text{SJD}(T_w,c)) \) where \( T_w \) is the admissible skew tableau of \( T_{(\lambda/\mu)} \) reading \( w \). Then \( \Xi(\cdot,c) \) is a crystal isomorphism from \( U_{(\lambda/\mu)} \) to its image.

**Proof.** By Proposition 6.3.6 and Lemma 1.3.3, each step of \( \text{SJD}(T_w,c) \) may be interpreted on the readings of intermediate tableaux as the result of the action of a crystal isomorphism. Hence \( \Xi(\cdot,c) \) is a crystal isomorphism.

**Corollary 6.3.9** Let \( T \) be an admissible skew tableau. Then by applying the \( \text{SJD} \) successively to the inner corners of \( T \) we obtain a symplectic tableau independent of the order in which these inner corners are filled. Moreover this tableau coincides with \( P(w(T)) \).

**Proof.** By the previous theorem, all the possible readings \( w \) of symplectic tableaux obtained from \( T \) by iterating the \( \text{SJD} \) must satisfy \( w \sim w(T) \). Hence \( w = w(P(T)) \) is independent of the order in which the inner corners are filled.

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