Some Inequalities satisfied by Periodical Solutions of Multi-Time Hamilton Equations

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Abstract

The objective of this paper is to find some inequalities satisfied by periodical solutions of multi-time Hamilton systems, when the Hamiltonian is convex. To our knowledge, this subject of first-order field theory is still open.

Section 1 recall well-known facts regarding the equivalence between Euler-Lagrange equations and Hamilton equations and analyses the action that produces multi-time Hamilton equations, emphasizing the role of the polysymplectic structure. Section 2 extends two inequalities of [21] from a cube to parallelepiped and proves two inequalities concerning multiple periodical solutions of multi-time Hamilton equations.

Key words: multi-time Hamilton action, Wirtinger inequality, convex Hamiltonian.

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1 Multi-time Hamilton equations and polysymplectic structure

The paper studies the solutions with multiple periodicity of the Hamilton multi-time equations.

A function $u = (u^1, ..., u^n)$ with many variables $(t^1, ..., t^p)$, is multiple periodical with the period $T = (T^1, ..., T^p) \in \mathbb{R}^p$ if

$$u\left(t^1 + k_1 T^1, ..., t^p + k_p T^p\right) = u\left(t^1, ..., t^p\right),$$

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where $k_1, ..., k_p$ are integers. We consider the function $u$ defined on the parallelepiped $T_0 = [0, T^1] \times [0, T^2] \times ... \times [0, T^p] \subset \mathbb{R}^p$, with values in $\mathbb{R}^n$. We will denote by $T = (T^1, ..., T^p) \in \mathbb{R}^p$. The existence of the weak gradient of the function $u$ assures the multiple periodicity of the function $u$. We use the Hilbert space $H^1_T$ attached to the Sobolev space $W^{1,2}_T$ of the functions $u \in L^2(T_0, \mathbb{R}^n)$ which have a weak gradient $\frac{\partial u}{\partial t} \in L^2(T_0, \mathbb{R}^n)$. The Wirtinger inequality from this paper has a specific form because of the multidimensional character of the definition domain $T_0$. The inequalities from theorems 3 and 4 constitute generalizations of some theorems of [5], from the particular case $p = 1$ to an arbitrary $p$.

The Euclidean structure on $\mathbb{R}^n$ is based on the scalar product $(u, v) = \delta_{ij} u^i v^j$, and the norm $|u| = \sqrt{\delta_{ij} u^i u^j}$. The Hilbert space $H^1_T$ is endowed with the scalar product

$$\langle u, v \rangle = \int_{T_0} \left( \delta_{ij} u^i(t) v^j(t) + \delta_{ij} \frac{\partial u^i}{\partial t^\alpha}(t) \frac{\partial v^j}{\partial t^\beta}(t) \right) dt^1 \wedge ... \wedge dt^p,$$

and the corresponding norm $\sqrt{\langle u, u \rangle} = \|u\|$.

### 1.1 Multi-time Hamilton equations

We consider the multi-time variable $t = (t^1, ..., t^p) \in T_0 \subset \mathbb{R}^p$, the functions $x^i : \mathbb{R}^p \to \mathbb{R}, (t^1, ..., t^p) \to x^i(t^1, ..., t^p)$, $i = 1, ..., n$, and the partial velocities

$x^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}$, $\alpha = 1, ..., p$.

**Definition 1** The PDEs

$$\frac{\partial}{\partial t^\alpha} \frac{\partial L}{\partial x^i_\alpha} = \frac{\partial L}{\partial x^i}, \quad i = 1, ..., n, \quad \alpha = 1, ..., p$$

(second order PDEs system on the n-dimensional space) are called Euler-Lagrange equations for the Lagrangian

$L : \mathbb{R}^{p+n+np} \to \mathbb{R}, \quad (t^\alpha, x^i, x^i_\alpha) \to L(t^\alpha, x^i, x^i_\alpha)$

The Hamilton equations in the multi-time case are obtained using the partial derivatives (polymomenta)

$$p^i_\alpha = \frac{\partial L}{\partial x^i_\alpha} \quad (1)$$

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and the Hamiltonian $H = p_k^i x^k_\alpha - L$. If $L$ satisfies some regularity conditions, then the system (1) defines a $C^1$ bijective transformation $x^i_\alpha \rightarrow p^i_\alpha$, called the Legendre transformation for the multi-time case. By this transformation we have
\[
\frac{\partial H}{\partial p^i_\alpha} = x^i_\alpha + p_j^k \frac{\partial x^j_k}{\partial p^i_\beta} - \frac{\partial L}{\partial x^j_k} \frac{\partial x^j_k}{\partial p^i_\alpha} = x^i_\alpha
\]
and
\[
\frac{\partial H}{\partial x^i_\alpha} = p_k^i \frac{\partial x^j_k}{\partial x^i_\alpha} - \frac{\partial L}{\partial x^i_\alpha} - \frac{\partial L}{\partial x^j_k} \frac{\partial x^j_k}{\partial x^i_\alpha} = -\frac{\partial L}{\partial x^i_\alpha}.
\]
Consequently, the $np + n$ Hamilton equations
\[
\frac{\partial x^i_\alpha}{\partial t^\alpha} = \frac{\partial H}{\partial p^i_\alpha},
\]
\[
\frac{\partial p^i_\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i_\alpha}
\]
(summation after $\alpha$), $i = 1, \ldots, n$, $\alpha = 1, \ldots, p$ are first order PDEs on the space $R^{n+p}$, equivalent to the Euler-Lagrange equations on $R^n$.

There are different point of views to study these equations which appear in first-order field theory (see [1]-[3], [8]-[10], [12]-[22]). In our context, we need of Hilbert-Sobolev space methods for PDEs ([4], [6], [11]).

Let us write the multi-time Hamilton equations in the form
\[
\delta^\alpha_\beta \delta^i_j \frac{\partial p^i_j}{\partial t^\alpha} + \frac{\partial H}{\partial x^j_\beta} = 0,
\]
\[
-\delta^\alpha_\beta \delta^i_j \frac{\partial x^j_i}{\partial t^\alpha} + \frac{\partial H}{\partial p^i_\beta} = 0, i, j = 1, \ldots, n; \alpha, \beta = 1, \ldots, p
\]
or
\[
(\delta \otimes J) \frac{\partial u}{\partial t} = -\nabla H, \quad (2)
\]
where
\[
\delta \otimes J = \begin{pmatrix}
0 & \delta^i_j \\
-\delta^i_j & 0
\end{pmatrix}, \quad \frac{\partial u}{\partial t} = \begin{pmatrix}
\frac{\partial x^j}{\partial t^\alpha} \\
\frac{\partial p^i_\beta}{\partial t^\alpha}
\end{pmatrix}, \quad \nabla H = \begin{pmatrix}
\frac{\partial H}{\partial x^j} \\
\frac{\partial H}{\partial p^i_\beta}
\end{pmatrix}.
\]

The operator $\delta \otimes J$ is a polysymplectic structure acting on $R^{np + np^2}$ with values in $R^{np}$. The $(1,2)$-block $\delta^i_j \delta^i_\beta$ acts linearly by $\delta^i_\beta$ and tracely by $\delta^i_\beta$. 

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The $2,1$-block $-\delta^\alpha_\beta \delta^i_j$ acts linearly both by $\delta^i_j$ and $\delta^\alpha_\beta$. The operator $\delta \otimes J$ induces a multisymplectic PDE operator $(\delta \otimes J) \frac{\partial}{\partial t}$ which work as follows

$$(\delta \otimes J) \frac{\partial}{\partial t} \left( \begin{array}{c} x \\ p \end{array} \right) : \left( \begin{array}{cc} 0 & \delta^\alpha_\beta \delta^i_j \\ -\delta^\alpha_\beta \delta^i_j & 0 \end{array} \right) \frac{\partial}{\partial t^\alpha} \left( \begin{array}{c} x^j \\ p^\beta_i \end{array} \right) = \left( \begin{array}{c} \frac{\partial p^\alpha_i}{\partial t^\alpha} \\ -\frac{\partial x^j}{\partial t^\beta} \end{array} \right).$$

Repeating we obtain the square

$$(\delta \otimes J) \frac{\partial}{\partial t} \left( \begin{array}{c} \text{div} p \\ -\frac{\partial x^i}{\partial t^\alpha} \end{array} \right) : \left( \begin{array}{cc} 0 & \delta^\alpha_\beta \delta^i_j \\ -\delta^\alpha_\beta \delta^i_j & 0 \end{array} \right) \left( \begin{array}{c} \frac{\partial^2 p^\gamma_i}{\partial t^\alpha \partial t^\gamma} \\ -\frac{\partial^2 x^j}{\partial t^\alpha \partial t^\beta} \end{array} \right) = \left( \begin{array}{c} -\Delta x^i \\ -\frac{\partial^2 p^\gamma_i}{\partial t^\alpha \partial t^\gamma} \end{array} \right).$$

### 1.2 The action that produces multi-time Hamilton equations

We consider a Hamiltonian $H : T_0 \times \mathbb{R}^n \times \mathbb{R}^{np} \to \mathbb{R}$, $(t, u) \to H(t, u)$ whose restriction $H(t, \cdot)$ is $\mathcal{C}^1$ and convex.

**Theorem 1** [21] Let $u = (x, p)$. The action $\Psi$, whose Euler-Lagrange equations are the Hamilton equations, is

$$
\Psi(u) = \int_{T_0} \mathcal{L}\left(t, u, \frac{\partial u}{\partial t}\right) dt^1 \wedge ... \wedge dt^p,
$$

$$
\mathcal{L}\left(t, u, \frac{\partial u}{\partial t}\right) = -\frac{1}{2} G \left( \delta \otimes J \frac{\partial u}{\partial t}, u \right) - H(t, u),
$$

where the scalar product is represented by the matrix

$$
G = \left( \begin{array}{cc} \delta^i_j & 0 \\ 0 & \delta^\beta_\alpha \delta^i_j \end{array} \right)
$$

(standard Riemannian metric from $\mathbb{R}^{n+np}$).

**Proof.** Indeed, the Euler-Lagrange equations produced by

$$
\mathcal{L} = -\frac{1}{2} \left( \frac{\partial p^\alpha_i}{\partial t^\alpha} x^i - \frac{\partial x^i}{\partial t^\alpha} p^\alpha_i \right) - H(t, x, p) =
$$
\[
= -\frac{1}{2} \left( \frac{\partial p_i^\alpha}{\partial t^\alpha} - \frac{\partial x^j}{\partial t^\beta} \right) \left( \begin{array}{cc}
\delta_{ij} & o \\
o & \delta_\alpha^\beta \delta_{ij}
\end{array} \right) \left( \begin{array}{c}
x^i \\
p_i^\alpha
\end{array} \right) - H(t, x(t), p(t))
\]

can be rewritten
\[
\frac{1}{2} \frac{\partial p_i^\alpha}{\partial t^\alpha} = -\frac{1}{2} \frac{\partial p_i^\alpha}{\partial t^\alpha} - \frac{\partial H}{\partial x^i}, \text{ i.e., } \frac{\partial p_i^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i}
\]

and
\[
\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p_i^\alpha}.
\]

2 Basic inequalities

2.1 Wirtinger multi-time inequality

In \( L^2(T_0, R^n) \) we use the scalar product

\[
\langle u, v \rangle = \int_{T_0} \left( \delta_{ij} u^i v^j \right) dt^1 \wedge ... \wedge dt^p
\]

and the norm \( \| u \|_{L^2} = \sqrt{\langle u, u \rangle} \). Similarly, in \( L^2(T_0, C^n) \) we use the scalar product

\[
\langle u, v \rangle = \int_{T_0} \left( \delta_{ij} u^i \overline{v^j} \right) dt^1 \wedge ... \wedge dt^p
\]

and the norm \( \| u \|_{L^2} = \sqrt{\langle u, \overline{u} \rangle} \).

Let us extend the Theorem 4.4 from [21] to the parallelipiped \( T_0 \).

**Theorem 2** Any function \( u \) from \( H_1^1 \) with mean zero satisfies the inequality

\[
\int_{T_0} |u(t)|^2 dt^1 \wedge ... \wedge dt^p \leq \left( \max_i \left\{ T_i^1 \right\} \right)^2 \int_{T_0} \left| \frac{\partial u}{\partial t^i} \right|^2 dt^1 \wedge ... \wedge dt^p.
\]

**Proof.** We express the function \( u \) as the sum of a multiple Fourier series

\[
u(t) = \left( u^1(t^1, ..., t^p), ..., u^n(t^1, ..., t^p) \right)
\]
\[
\sum_{j_1^1, \ldots, j_p^1 \in \mathbb{Z}_p} C_{j_1^1, \ldots, j_p^1} e^{i \left( \frac{2\pi}{T_1} j_1^1 t^1 + \ldots + \frac{2\pi}{T_p} j_p^1 t^p \right)},
\]

\[
\ldots, \sum_{j_1^n, \ldots, j_p^n \in \mathbb{Z}_p} C_{j_1^n, \ldots, j_p^n} e^{i \left( \frac{2\pi}{T_1} j_1^n t^1 + \ldots + \frac{2\pi}{T_p} j_p^n t^p \right)}.
\]

We calculate the square of the norm
\[
(u, u) = \int_{T_0} (u(t), \overline{u(t)}) dt^1 \wedge \ldots \wedge dt^p =
\]

\[
\sum_{j_1^1, \ldots, j_p^1 \in \mathbb{Z}_p} \left( C_{j_1^1, \ldots, j_p^1} \right)^2 \int_0^{T_1} e^{i \frac{2\pi}{T_1} (j_1^1 - j_1^1) t^1} dt^1 \ldots \int_0^{T_p} e^{i \frac{2\pi}{T_p} (j_p^1 - j_p^1) t^p} dt^p + \ldots
\]

\[
+ \sum_{j_1^n, \ldots, j_p^n \in \mathbb{Z}_p} \left( C_{j_1^n, \ldots, j_p^n} \right)^2 \int_0^{T_1} e^{i \frac{2\pi}{T_1} (j_1^n - j_1^n) t^1} dt^1 \ldots \int_0^{T_p} e^{i \frac{2\pi}{T_p} (j_p^n - j_p^n) t^p} dt^p
\]

\[
= \sum_{j_1^1, \ldots, j_p^1 \in \mathbb{Z}_p} \left( C_{j_1^1, \ldots, j_p^1} \right)^2 T^1 \ldots T^p + \ldots + \sum_{j_1^n, \ldots, j_p^n \in \mathbb{Z}_p} \left( C_{j_1^n, \ldots, j_p^n} \right)^2 T^1 \ldots T^p.
\]

If we denote
\[
C_{k_1, \ldots, k_p} = \left( C_{k_1^1, \ldots, k_p^1}, \ldots, C_{k_1^n, \ldots, k_p^n} \right)
\]
and
\[
u = \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}_p} C_{k_1, \ldots, k_p} e^{i \frac{2\pi}{T_1} k_1 t_1 + \ldots + \frac{2\pi}{T_p} k_p t_p},
\]

we find
\[
(u, u) = \sum_{(k_1, \ldots, k_p) \in \mathbb{Z}_p} \left| C_{k_1, \ldots, k_p} \right|^2 T^1 \ldots T^p.
\]

Similarly, we consider the scalar product
\[
\left( \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right) = \int_{T_0} \delta_{ij} \delta_{\alpha \beta} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} dt^1 \wedge \ldots \wedge dt^p.
\]
It follows the square of the norm
\[
\left( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial t} \right)
\]

\[
= \sum_{(k_1, ..., k_p) \in \mathbb{Z}^p} T^1 ... T^p \left[ (C_{k_1...k_p}^1)^2 \left( \frac{2\pi k_1}{T^1} \right)^2 + ... + (C_{k_1...k_p}^1) \left( \frac{2\pi k_p}{T^p} \right)^2 \right]
\]

\[
= \sum_{(k_1, ..., k_p) \in \mathbb{Z}^p} T^1 ... T^p \left[ \left| C_{k_1...k_p} \right|^2 \left( \frac{2\pi k_1}{T^1} \right)^2 + ... + \left| C_{k_1...k_p} \right|^2 \left( \frac{2\pi k_p}{T^p} \right)^2 \right]
\]

\[
\geq \frac{4\pi^2}{\\left( \max_i \{ T^i \} \right)^2} \sum_{(k_1, ..., k_p) \in \mathbb{Z}^p} T^1 ... T^p \left| C_{k_1...k_p} \right|^2 \left( k_1^2 + ... + k_p^2 \right)
\]

\[
\geq \frac{4\pi^2}{\\left( \max_i \{ T^i \} \right)^2} \int_{T_0} |u(t)|^2 \, dt^1 \wedge ... \wedge dt^p.
\]

Consequently
\[
\int_{T_0} |u(t)|^2 \, dt^1 \wedge ... \wedge dt^p \leq \frac{\\left( \max_i \{ T^i \} \right)^2}{4\pi^2} \int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 \, dt^1 \wedge ... \wedge dt^p,
\]

and this ends the proof.
2.2 An estimate of the quadratic form
\[ \int_{T_0} \left( \delta \otimes J \frac{\partial u}{\partial t}, u(t) \right) dt^1 \wedge ... \wedge dt^p \]

Let us extend the Theorem 4.5 from [21] to the parallelepiped \( T_0 \).

**Theorem 3** For any \( u \in H^1_T \) we have
\[ \int_{T_0} \left( \delta \otimes J \frac{\partial u}{\partial t}, u(t) \right) dt^1 \wedge ... \wedge dt^p \geq -\sqrt{p} \max_i \left\{ T^i \right\} \left( \int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \]

**Proof.** We denote \( \tilde{u}(t) = u(t) - \int_{T_0} u(t) dt^1 \wedge ... \wedge dt^p \). By using the Cauchy-Schwarz inequality and the multiple periodicity of \( u \) we obtain the inequality
\[ \int_{T_0} \left( \delta \otimes J \frac{\partial u}{\partial t}, \tilde{u}(t) \right) dt^1 \wedge ... \wedge dt^p \]
\[ = \int_{T_0} \left( \delta \otimes J \frac{\partial u}{\partial t}, \tilde{u}(t) + \int_{T_0} u(t) dt^1 \wedge ... \wedge dt^p \right) dt^1 \wedge ... \wedge dt^p \]
\[ = \int_{T_0} \left( \delta \otimes J \frac{\partial u}{\partial t}, \tilde{u}(t) \right) dt^1 \wedge ... \wedge dt^p \]
\[ + \int_{T_0} \left( \delta \otimes J \frac{\partial u}{\partial t}, \int_{T_0} u(t) dt^1 \wedge ... \wedge dt^p \right) dt^1 \wedge ... \wedge dt^p \]
\[ = \int_{T_0} \left( \delta \otimes J \frac{\partial u}{\partial t}, \tilde{u}(t) \right) dt^1 \wedge ... \wedge dt^p \]
\[ \geq - \left( \int_{T_0} \left| \delta \otimes J \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \left( \int_{T_0} \left| \tilde{u}(t) \right|^2 dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \]

From the inequality given by the Theorem 2 we have
\[ \int_{T_0} |\bar{u}(t)|^2 dt^1 \wedge ... \wedge dt^p \leq \left( \frac{\max_i \{ T^i \}}{4\pi^2} \right)^2 \int_{T_0} \left| \frac{\partial \bar{u}}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p. \]

Because \[ \left| \delta \otimes J \frac{\partial u}{\partial t} \right|^2 \leq p \left| \frac{\partial u}{\partial t} \right|^2, \] we obtain

\[ \int_{T_0} \left( \delta \otimes J \frac{\partial u}{\partial t}, u(t) \right) dt^1 \wedge ... \wedge dt^p \]

\[ \geq - \left( \int_{T_0} \left| \delta \otimes J \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \max_i \{ T^i \} \]

\[ \cdot \left( \int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \]

\[ \geq - \sqrt{p} \max_i \{ T^i \} \frac{\max_i \{ T^i \}}{2\pi} \int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p. \]

### 2.3 Inequalities satisfied by periodical solutions of multi-time Hamilton equations

Let us find properties of solutions of \((\delta \otimes J) \frac{\partial u}{\partial t} + \nabla H(t, u(t)) = 0\) a.e. on \(T_0\) satisfying the boundary conditions

\[ u \mid_{S^+} = u \mid_{S^-}, \]

were \(S^+\) and \(S^-\) are opposite sides of the parallelepiped \(T_0\). Practically, we refer to bounds for such solutions.
Theorem 4 We consider the Hamiltonian

$$H : T_0 \times \mathbb{R}^{n+np} \to \mathbb{R}, (t, u) \to H(t, u)$$

like a measurable function in $t$ for any $u \in \mathbb{R}^{n+np}$, and $C^1$ convex in $u$ for any $t \in T_0 = [0, T^1] \times ... \times [0, T^p] \subset \mathbb{R}^p$.

If there exists the constants

$$\alpha \in \left(0, \frac{\pi}{\sqrt{p} \max_i \{T^i\}}\right), \beta \geq 0, \gamma \geq 0, \delta \geq 0$$

such that

$$\delta |u| + \beta \leq H(t, u) \leq \frac{\alpha}{2} |u|^2 + \gamma$$

for all $t \in T_0$ and $u \in \mathbb{R}^{n+np}$, then, any multiple periodical solution $u = (x_i, p_i^\alpha), i = 1, ..., n, \alpha = 1, ..., p$ of the equation

$$\delta \otimes J \frac{\partial u}{\partial t} + \nabla H(t, u(t)) = 0,$$  \hspace{1cm} (3)

verifies the inequalities

$$\int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p \leq \frac{2\alpha (\beta + \gamma) \pi T^1 ... T^p}{\pi - \alpha \max_i \{T^i\} \sqrt{p}}$$ \hspace{1cm} (4)

$$\int_{T_0} |u(t)| dt^1 \wedge ... \wedge dt^p \leq \frac{\pi T^1 ... T^p (\beta + \gamma)}{\delta \left(\pi - \alpha \max_i \{T^i\} \sqrt{p}\right)}.$$ \hspace{1cm} (5)

Proof. From the inequality

$$\delta |u|^2 - \beta \leq H(t, u) \leq \frac{\alpha}{2} |u|^2 + \gamma$$

we obtain

$$-\beta \leq H(t, u) \leq \alpha 2^{-1} |u|^2 + \gamma.$$
By applying [5, Proposition 2.2], considering $F(u) = H(t, u)$, $p = q = 2$, $v = \nabla H(t, u)$ we obtain

\[
\frac{1}{2\alpha} |\nabla H(t, u)|^2 \leq (\nabla H(t, u), u) + \beta + \gamma.
\]

Because $u$ is the solution of the equation (3), we have $\nabla H(t, u) = -\delta \otimes J \frac{\partial u}{\partial t}$ and the previous inequality becomes

\[
\frac{1}{2\alpha} \left| -\delta \otimes J \frac{\partial u}{\partial t} \right|^2 \leq \left( -\delta \otimes J \frac{\partial u}{\partial t}, u \right) + \beta + \gamma. \quad (6)
\]

In the hypothesis' conditions, by integration of the inequality (6) we have

\[
\frac{1}{2\alpha} \int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p
\]

\[
+ \int_{T_0} \left( \delta \otimes J \frac{\partial u}{\partial t}, u \right) dt^1 \wedge ... \wedge dt^p \leq (\beta + \gamma) T^1...T^p.
\]

By using the inequality from Theorem 3, we have

\[
\frac{1}{2\alpha} \int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p
\]

\[
- \frac{\sqrt{p} \max_i \{T^i\}}{2\pi} \int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p \leq (\beta + \gamma) T^1...T^p.
\]

So

\[
\left( \frac{1}{2\alpha} - \frac{\sqrt{p} \max_i \{T^i\}}{2\pi} \right) \int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p
\]

\[
\leq (\beta + \gamma) T^1...T^p
\]

and, as consequence

\[
\int_{T_0} \left| \frac{\partial u}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p \leq \frac{2\pi \alpha (\beta + \gamma) T^1...T^p}{\pi - \alpha \max_i \{T^i\} \sqrt{p}}.
\]
By integration, the inequality
\[ \delta |u| - \beta \leq H(t, u) \]
produces
\[ \delta \int_{T_0} |u(t)| \, dt^1 \wedge \ldots \wedge dt^p - \beta T^1 \ldots T^p \leq \int_{T_0} H(t, u) \, dt^1 \wedge \ldots \wedge dt^p. \]
Because \( H(t, u) \) is convex in \( u \),
\[ H(t, u) - H(t, 0) \leq (\nabla H(t, u(t)), u(t)), \]
we obtain
\[ \int_{T_0} H(t, u(t)) \, dt^1 \wedge \ldots \wedge dt^p \]
\[ \leq \int_{T_0} [H(t, 0) + (\nabla H(t, u(t)), u(t))] \, dt^1 \wedge \ldots \wedge dt^p \]
\[ \leq \gamma T^1 \ldots T^p - \int_{T_0} \left( \delta \otimes \frac{\partial u}{\partial t}, u(t) \right) \, dt^1 \wedge \ldots \wedge dt^p \]
\[ \leq T^1 \ldots T^p + \frac{\sqrt{p} \max_i \{ T^i \} \frac{2 \pi}{T^{1 \ldots T^p}}}{\frac{\left( \delta \otimes \frac{\partial u}{\partial t}, u(t) \right)}{\frac{\partial u}{\partial t}} \, dt^1 \wedge \ldots \wedge dt^p} \]
\[ \leq \gamma T^1 \ldots T^p + \frac{\sqrt{p} \max_i \{ T^i \} \frac{2 \pi}{\left( \delta \otimes \frac{\partial u}{\partial t}, u(t) \right)} \, dt^1 \wedge \ldots \wedge dt^p}{\frac{\partial u}{\partial t}} \]
By consequence
\[ \int_{T_0} |u(t)| \, dt^1 \wedge \ldots \wedge dt^p \]
\[ \leq \frac{1}{\delta} \left( \beta T^1 \ldots T^p + \gamma T^1 \ldots T^p + \frac{\sqrt{p} \max_i \{ T^i \} \frac{2 \pi}{\alpha \max_i \{ T^i \}} \frac{T^1 \ldots T^p}{\sqrt{p}} \right), \]
meaning that
\[ \int_{T_0} |u(t)| \, dt^1 \wedge \ldots \wedge dt^p \leq \frac{\left( \beta + \gamma \right) T^1 \ldots T^p \pi}{\delta \left( \pi - \alpha \max_i \{ T^i \} \sqrt{p} \right)} \]
and the proof ends.

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