DIFFUSIVE PROPAGATION OF ULTRA–HIGH-ENERGY COSMIC RAYS AND THE PROPAGATION THEOREM

R. Aloisio and V. Berezinsky

Istituto Nazionale di Fisica Nucleare, Laboratori Nazionali del Gran Sasso, Strada Statale 17/bis Km 18+910, I-67010 Assergi (AQ), Italy; aloisio@lngs.infn.it, berezinsky@lngs.infn.it

Received 2004 March 4; accepted 2004 April 13

ABSTRACT

We present a detailed analytical study of the propagation of ultra–high-energy (UHE) particles in extragalactic magnetic fields. The crucial parameter that affects the diffuse spectrum is the separation between sources. In the case of a uniform distribution of sources with a separation between them much smaller than all characteristic propagation lengths, the diffuse spectrum of UHE particles has a universal form, independent of the mode of propagation. This statement has the status of theorem. The proof is obtained using the particle number conservation during propagation and also using the kinetic equation for the propagation of UHE particles. This theorem can be also proved with the help of the diffusion equation. In particular, it is shown numerically how the diffuse fluxes converge to this universal spectrum, when the separation between sources diminishes. We study also the analytic solution of the diffusion equation in weak and strong magnetic fields with energy losses taken into account. In the case of strong magnetic fields and for a separation between sources large enough, the GZK cutoff can practically disappear, as it has been found early in numerical simulations. In practice, however, the source luminosities required are too large for this possibility.

Subject heading: cosmic rays

1. INTRODUCTION

The propagation of ultra–high-energy cosmic-ray (UHECR) protons and nuclei with $E \gtrsim 1 \times 10^{19}$ eV in the large-scale intergalactic magnetic field (IMF) remains an open problem, mainly because the knowledge of the IMF is still very poor. The possibilities vary between rectilinear propagation in a weak field and diffusive propagation in a strong magnetic field. The experimental data on IMF and the models of origin of these fields do not allow us at present to choose even between the two extreme propagation regimes mentioned above.

Most reliable observations of the intergalactic magnetic field are based on the Faraday rotation of the polarized radio emission (for the reviews see Kronberg 1994; Vallee 1997; Carilli & Taylor 2002). The upper limit on the Faraday rotation measure (RM) in the extragalactic magnetic field, obtained from the observations of distant quasars, gives an upper limit of $\text{RM} < 5 \text{ rad m}^{-2}$. It implies an upper limit on the extragalactic magnetic field on each assumed scale of coherence length (Kronberg 1994; Vallee 1997; Ryu et al. 1998). For example, according to Blasi et al. (1999) for an inhomogeneous universe, $B_{l} < 4 \text{ nG}$ on a scale of coherence $l_{c} = 50 \text{ Mpc}$.

According to observations of the Faraday rotations, the extragalactic magnetic field is strongest, or order of 1 $\mu\text{G}$, in clusters of galaxies and radio lobes of radio galaxies (Vallee 1997; Kronberg 1994; Carilli & Taylor 2002). The largest scale in both structures reaches $l_{c} \sim 1 \text{ Mpc}$. Most probably various structures of the universe differ dramatically by magnetic fields, with very weak fields in voids and much stronger ones in the filaments. Superclusters seem to be too young for the regular magnetic field to be formed in these structures on a large scale $l_{c} \sim 10 \text{ Mpc}$.

In case of hierarchical magnetic field structures in the universe, UHE protons with $E > 4 \times 10^{19}$ eV can propagate in a quasi-rectilinear regime. Scattering of UHE protons occurs mostly in galaxy clusters, radio lobes, and filaments. Deflections of UHE protons can be large for some directions and small for the others. The universe looks like a leaky, worm-holed box, and correlation with the sources can be observable (see Tinyakov & Tkachev 2001, where correlations of UHECRs with BL Lac objects are found). Such a picture has been suggested by Berezinsky et al. (2002a).

A promising theoretical tool to predict the IMF in large-scale structures is given by magnetohydrodynamic (MHD) simulations. The main uncertainty in these simulations is related to the assumptions concerning the seed magnetic field.

The MHD simulations of Sigl et al. (2003, 2004) favor the hierarchical structure with strong magnetic fields. Assuming an inhomogeneous seed magnetic field generated by cosmic shocks through the Biermann battery mechanism, the authors obtain a $\sim 100 \text{ nG}$ magnetic field in filaments and $\sim 1 \text{ nG}$ in voids. In some cases they consider IMF up to a few microgauss as allowed. In these simulations UHECRs are characterized by large deflection angles, of the order of $20^\circ$, at energies up to $E \sim 10^{20}$ eV (Sigl et al. 2003, 2004). Thus, the scenario that emerges in these simulations seems to exclude the UHECR astronomy. These simulations have some ambiguity related to the choice of magnetic field at the position of the observer (Sigl et al. 2003, 2004). The authors consider two cases: a strong local magnetic field $B \sim 100 \text{ nG}$ and a weak field $B \ll 100 \text{ nG}$. The different assumptions about the local magnetic field strongly affect the conclusions about UHECR spectrum and anisotropy.

The essential step forward in MHD simulations has been made recently. In Dolag et al. (2003) the local universe is simulated with the observed density and velocity field. This eliminates the ambiguity for the local magnetic field, which is found to be weak. The seed magnetic field, used in this simulation, is normalized by the observed magnetic field in rich clusters of galaxies. The results of these constrained simulations indicate the weak magnetic fields in the universe of the order of 0.1 nG in typical filaments and of 0.01 nG in voids. The strong large-scale magnetic field, $B \sim 10^{3} \text{ nG}$, exists in
clusters of galaxies, which, however, occupy an insignificant volume of the universe. The picture that emerges from simulations of Dolag et al. (2003) favors a hierarchical magnetic field structure characterized by weak magnetic fields. UHE protons with $E > 4 \times 10^{19}$ eV can propagate in a quasi-linear regime, with the expected deflection angles being very small, $\leq 1^\circ$. However, until direct observational evidence for this picture becomes available, an alternative case of propagation in strong magnetic fields, with diffusion as an extreme possibility, can be hardly excluded.

This case has been studied in Sigl et al. (1999), Lemoine et al. (1999), Stanev et al. (2000), Harari et al. (2002), Yoshiguchi et al. (2003), and Deligny et al. (2003). Some interesting features found in these calculations are small-angle clustering of UHE particles as observed by Hayashida et al. (1996), Takeda et al. (1999), Uchiori et al. (2000), and Glushkov & Pravdin (2001), as well as absence of the GZK cutoff in the diffusive propagation, when the magnetic field is very strong. Many aspects of diffusion of UHECRs have been studied in numerical simulations by Casse et al. (2002).

We illustrate UHECR propagation in strong magnetic fields by the calculations by Yoshiguchi et al. (2003). The authors performed Monte Carlo simulations for propagation in random magnetic field with the Kolmogorov spectrum of turbulent energy density $\nu_k \propto k^n$, with $n = -5/3$. The basic (largest) coherent scale is chosen as $l_c$ with the mean field $B_0$. Numerically these parameters vary in the range 1–40 Mpc for $l_c$ and 1–100 nG for $B_0$. The sources are taken as galaxies from the Optical Redshift Survey catalog with absolute magnitude $M_B$ brighter than some critical value $M_c$. The calculated quantities are the energy spectrum, anisotropy, and small-angle clustering. The observed small-angle clustering and absence of the GZK cutoff in the AGASA observations can be reproduced in the case of strong magnetic field $B \geq 10$ nG.

Diffusive propagation of extragalactic UHECRs has been studied earlier in the literature. The stationary diffusion from the Virgo Cluster was considered by Wdowczyk & Wolfendale (1979) and Giller et al. (1980), and nonstationary diffusion from a nearby source was studied by Berezinsky et al. (1990b) and Blasi & Olinto (1999) using the Syrovatskii solution (Syrovatskii 1959) of the diffusion equation. In this case the GZK cutoff can be absent. A very similar problem was considered again more recently by Isola et al. (2002).

In this paper we study how propagation influences the diffuse energy spectrum of UHECRs. We shall prove the theorem that if distance between sources is much smaller than all propagation lengths, the spectrum has the same universal form independent of the mode of propagation. For diffusion in magnetic fields we demonstrate how the spectra converge to the universal one when the separation between sources diminishes. Finally, we obtain, with the help of the Syrovatskii solution, the spectra for the strong magnetic field. In this case with large enough separation between sources, the GZK cutoff becomes weak or absent, and we discuss the physical explanation of this phenomenon.

### 2. PROPAGATION THEOREM

Let us consider a case in which identical UHECR sources are distributed uniformly\(^1\) in the space, with $d$ being the separation between sources. We demonstrate that if $d$ is less than all other characteristic lengths of propagation, such as diffusion length $l_d(E)$ and energy attenuation length $l_{at}(E)$ given by

$$l_{at} = \frac{cE}{dE/dt},$$

then the diffuse energy spectrum has a universal (standard) form independent of the mode of particle propagation. In particular, under the conditions specified above, the magnetic field, both weak and strong, does not affect the shape of the energy spectrum.

Explicitly, this theorem can be formulated as follows:

**Theorem.** For a uniform distribution of identical sources with separation much less than the characteristic propagation lengths, the diffuse spectrum of UHECRs has a universal (standard) form, independent of the mode of propagation.

First, we consider the proof based on the conservation of the number of particles (e.g., protons or nuclei) during the propagation. Let $t$ be the age of the universe with the present age taken as $t_0$. The number of particles per unit volume of the present universe is equal to the number of particles injected into this volume during the entire history of the universe, independent of the mode of propagation. The homogeneity of particles needed for this statement is provided by an almost homogeneous distribution of sources. Thus, the comoving space density of particles $n_p(E)$ from uniformly distributed sources with an age-dependent comoving density $n_c(t)$ and age-dependent generation rate by a source, $Q(E, t)$, is given by

$$n_p(E) dE = \int_0^{t_0} dt Q(E, t) n_c(t) dE,$$

where $E_g(E, t)$ is the required generation energy at age $t$, if the observed energy is $E$. Equation (2) does not depend on the way particles propagate.

The homogeneous distribution of particles in the presence of inhomogeneous magnetic fields follows from the Liouville theorem and can be explained in the following way. Suppose we have an observer in the space with a magnetic field. The diffuse flux in any direction is given by the integral $\int_0^{t_0} n_{pg}(E) dE$ over the trajectory of a particle (or antiparticle emitted from the observation point). If $n_g$ is almost homogeneous, the integral depends on the time of propagation and does not depend on the strength and inhomogeneity of the magnetic field.

To find the explicit form of the universal spectrum $n_p(E)$, one needs some additional assumptions. Let us consider the protons as primaries with continuous energy losses due to interaction with the CMB radiation

$$dE/dt = -b(E, t).$$

Here and everywhere below $b(E, t) > 0$. We use the connection between the redshift $z$ and cosmological time $t$ according to the standard cosmology

$$dt = \frac{dz}{H_0(1+z)\sqrt{(1+z)^3\Omega_m + \Omega_\Lambda}},$$

with $H_0$, $\Omega_m$, and $\Omega_\Lambda$ being the Hubble constant, relative cosmological density of matter, and relative density of vacuum energy, respectively.

---

\(^1\) In this paper we distinguish between uniform and homogeneous distribution of the sources: under the latter we assume a continuous and distance-independent distribution.
The generation rate is assumed to be the same for all sources and is taken in the form

\[ Q(E, t) = L_p(1 + z)^\alpha K(\gamma_g)q_{\text{gen}}(E_g), \]

where \( L_p \) is the CR luminosity of the source with \((1 + z)^\alpha \) describing the possible cosmological evolution of luminosity. The normalization factor \( K(\gamma_g) \) is \( \gamma_g - 2 \) if \( \gamma_g > 2 \) and \( 1/\ln(E_{\text{max}}/E_0) \) if \( \gamma_g = 2 \), with \( E_0 \) and \( E_{\text{max}} \) the minimum and maximum generation energies, respectively. The comoving density \( n_s(t) \) of the sources can also contain the evolutionary factor \((1 + z)^\alpha \), with \( \alpha + \beta = m \). Here and everywhere else we assume \( E_0 = 1 \) GeV; all energies \( E \) are measured in GeV and \( L_p \) in GeV s\(^{-1}\).

Then from equation (2) we obtain the normalized universal spectrum as

\[ J_p(E) = \frac{c}{4\pi} L_0 K(\gamma_g) \int_0^{E_{\text{max}}} dz \left( \frac{dt}{dz}\right)^{(1 + z)^\alpha} q_{\text{gen}}(E_g) \frac{dE_g}{dE}, \]

where \( J_p \) is diffuse flux, \( L_0 = L_p n_s \) is the emissivity at \( z = 0 \), \( dt/dz \) is given by equation (4), \( E_p \) is calculated as \( E_p = E_g(E, z) \), and \( dE_g/dE \) is given by (Berezinsky & Grigorieva 1988; Berezinsky et al. 2002b)

\[
\frac{dE_g(z_g)}{dE} = (1 + z_g) \exp \left\{ \frac{1}{H_0} \int_{z_g}^{z_0} dz \frac{(1 + z)^2}{\Omega_m(1 + z)^3 + \Omega_\Lambda} \times \left[ \frac{db(E')}{dE'} \right]_{E'=(1+z)E_g(z)} \right\},
\]

where \( b(E) \) is the proton energy loss at \( z = 0 \).

The generation spectrum \( q_{\text{gen}}(E_g) \) is not known a priori. Three kinds of spectrum might be considered: (1) the power-law spectrum \( q_{\text{gen}}(E_g) = E_g^{-\gamma_g} \) with \( \gamma_g > 2 \) and the normalization factor \( K(\gamma_g) = \gamma_g - 2 \), (2) the power-law spectrum \( q_{\text{gen}}(E_g) = 1/E_g^2 \) with \( K(\gamma_g) = 1/\ln(E_{\text{max}}/E_0) \), and (3) the complex spectrum

\[ q_{\text{gen}}(E_g) = \begin{cases} 1/E_g^2, & E_g \leq E_c, \\ E_g^{-\gamma_g} E_c^{-\gamma_g}, & E_g \geq E_c, \end{cases} \]

with

\[ K(\gamma_g) = \frac{1}{\ln(E_{\text{max}}/E_c) + [1/(\gamma_g - 2)]}. \]

In numerical calculations with the complex spectrum given by equation (8) we use \( \gamma_g = 2.7 \), which gives the best fit to observational data (Berezinsky et al. 2003).

The universal spectrum given by equation (6) with \( q_{\text{gen}} \) from equation (8) and with \( m = 0 \) is shown in Figure 1 in comparison with experimental data of AGASA and HiRes arrays. From these figures one can see the good agreement of the universal spectrum with experimental data of both detectors up to \( 8 \times 10^{19} \) eV. The required emissivities \( L_0 \) are \( L_0 \approx 1.8 \times 10^{46} \) and \( 8.9 \times 10^{45} \) ergs Mpc\(^{-3}\) yr\(^{-1}\) for the AGASA and HiRes data, respectively. The excess of AGASA events at \( E > 1 \times 10^{20} \) eV needs another component of UHECRs.

Under the condition \( d \ll l_{\text{prop}} \) the energy spectrum given by equation (6) is the same (universal) for rectilinear propagation and propagation in weak and strong magnetic fields. If \( l_{\text{prop}} \leq d \) (or \( l_{\text{prop}} \ll d \)), the propagation theorem is not valid any more. The distortion of the universal spectrum occurs at \( E \geq E_{\text{GZK}} \) as a result of several effects: inhomogeneity of distribution of UHECR sources (local enhancement or local absence of the sources), fluctuation in the distribution of sources, and fluctuations of the photopion energy losses over the distance \( d \) and due to the small diffusion length \( l_d < d \) in the case of strong magnetic fields.

3. KINETIC EQUATION AND UNIVERSAL SPECTRUM

We have obtained the universal spectrum above from the very general condition of particle conservation during propagation. In this section we demonstrate that the universal spectrum follows also from the kinetic equation.

Let us consider the kinetic equation for a homogeneous distribution of the sources (see footnote 1). Since in this case the particle distribution is also homogeneous, the diffusion term in the kinetic equation is absent, and it has the form

\[
\frac{\partial n_p(E, t)}{\partial t} = \frac{\partial}{\partial E} \left[ b(E, t)n_p(E, t) \right] + Q_p(E, t),
\]
with $Q_g = n_s Q$. Let us start with a simple stationary equation when $Q_g$, $n_p$, and $b$ do not depend on time:

$$-\frac{\partial}{\partial E} [b(E)n_p(E)] = Q_g(E).$$

The solution of equation (10) is

$$n_p(E) = \frac{1}{b(E)} \int_E^{E_{\text{max}}} dE_g Q_g(E_g) = \int dt \frac{b(E_g)}{b(E)} Q_g(E_g),$$

where $E_g(E, t)$ is the generation energy at time $t$ and we used $dE_g = -b(E_g)dt$. Using $dE = -b(E)dt$ for the same interval $dt$, we obtain $b(E_g)/b(E) = dE_g/dE$ and

$$n_p(E) = \int dt Q_g(E_g(t)) \frac{dE_g}{dE}.$$  

in agreement with equations (2) and (6).

Let us now come back to equation (9) with time-dependent quantities $Q_g$, $n_p$, and $b$, where $b(t)$ includes also adiabatic energy losses due to redshift. Introducing the new quantities, $b(E, t) = b(E)/H(t)$, $Q_g(E, t) = Q_g(E)H(t)$, and $\tau = (1 + z_{\text{max}})/(1 + z)$, where $z_{\text{max}}$ is the maximal redshift in the evolution of sources, we obtain the equation

$$\frac{\partial n_p(E, \tau)}{\partial \tau} = -\frac{\partial b(E, \tau)}{\partial E} n_p(E, \tau) + Q_g(E).$$

Equation (13) may be solved with the help of an integration factor

$$\mu(\tau) = \exp \left[ -\int_0^\tau d\tau' \frac{\partial b(\tau')}{\partial E} \right].$$

Assuming $n_p(E, z_m) = 0$, we obtain

$$n_p(E, \tau) = \frac{1}{\mu(\tau)} \int_0^\tau d\tau' \mu(\tau') \frac{Q_g(E)}{b(E)}.$$  

At this stage the energy losses should be separated into those due to redshift and those due to the interaction with the CMB radiation

$$b(E', z) = E' H_0 \sqrt{\Omega_m (1 + z)^3 + \Omega_\Lambda} + (1 + z)^2 b_{\text{CMB}} |E'(1 + z)|,$$

where $E'$ is an arbitrary energy at epoch $z$.

Coming back to the variable $z$ in equation (13) and inserting there $\partial b(E, z)/\partial E$ from equation (16), we obtain after integration

$$n_p(E) = \frac{1}{H_0} \int_0^{z_{\text{max}}} \frac{dz}{\sqrt{\Omega_m (1 + z)^3 + \Omega_\Lambda}} Q_g(E_g, z)$$

$$\times \exp \left\{ \frac{1}{H_0} \int_0^z \frac{dz' (1 + z')^2}{\sqrt{\Omega_m (1 + z')^3 + \Omega_\Lambda}} \right\}$$

$$\times \left[ \frac{\partial b_0(E', z')}{\partial E'} \right]_{E' = (1 + z)E(z')},$$

which, after introducing the explicit expression for $Q_g$ from equation (5), coincides exactly with equations (6) and (7).

4. DIFFUSION EQUATION AND THE SYROVATSKY SOLUTION

In the previous section we consider the case of a homogeneous distribution of sources and prove that the universal spectrum is a solution of the kinetic equation (9). Now we assume a nonhomogeneous distribution of the sources. Then one should add the propagation term in equation (9). We assume the diffusive propagation in large-scale magnetic fields. For a single source at point $r$, the diffusion equation has the form

$$\frac{\partial n_p(E, r)}{\partial t} - \text{div} [D(E, r) \nabla n_p(E, r)] - \frac{\partial}{\partial E} [b(E)n_p(E, r)] = Q(E, \tau) \delta(r - r_s),$$

where $D(E, r)$ is the diffusion coefficient that depends on the magnetic field structure. We refer the reader to § 5 for a detailed discussion of the relation between the diffusion coefficient and the magnetic field.

In the case in which $D, b$, and $Q$ depend only on energy, the exact analytic solution of equation (18), found by Syrovatskii (1959), is

$$n_p(E, r) = \frac{1}{b(E)} \int_E^{E_0} dE_g Q(E_g) \exp \left\{ -\frac{r^2}{4 \lambda(E, E_g)} \right\} \left[ 4 \pi \lambda(E, E_g) \right]^{3/2},$$

where

$$\lambda(E, E_g) = \int_E^{E_g} \frac{dE_g}{\lambda(E_g)}.$$
Using the lattice-distributed sources and the Syrovatsky solution, one obtains the diffuse flux as
\[
J_{\rho}(E) = \frac{c}{4\pi} \frac{1}{b(E)} \times \sum_{i} \int_{E}^{E_{\text{max}}} dE_g Q(E_g) \exp \left\{ -r_i^2 / \left[ 4\lambda(E, E_g) \right] \right\} \left[ 4\lambda(E, E_g) \right]^{3/2},
\]
(23)
where \( \lambda(E, E_g) \) is given by equation (20) and the summation goes over all sources in the lattice vertices. The maximum energy in equation (23) is given by the smaller of two quantities: the maximum acceleration energy \( E_{\text{max}} \) and the generation energy \( E_g(E, c\theta_0) \) of a proton with present energy \( E \) propagating during a time \( t_0 \). For the energies \( E \) of interest, the former is always smaller than the latter, and we use as \( E_{\text{max}} \) the maximum acceleration energy.

We have to specify now the diffusion coefficient \( D(E) \), which determines \( \lambda(E, E_g) \) in equation (23). Putting off the detailed discussion until § 5, we give here a short description of the diffusion coefficients used in this work.

We assume diffusion in a random magnetic field with the mean value \( B_0 \) on the maximum coherent length \( l_c \). This assumption determines the diffusion coefficient \( D(E) \) at the highest energies when the proton Larmor radius \( r_L(E) \gg l_c \):
\[
D(E) = \frac{1}{3} \frac{c r_L^2(E)}{l_c^3}.
\]
(24)

At “low” energies, when \( r_L(E) \leq l_c \), we consider three cases:

1. Energy-independent diffusion coefficient
   \[
   D = \frac{1}{3} c l_c.
   \]
(25)

2. The Bohm diffusion coefficient, which provides the lowest value of \( D \),
   \[
   D_B(E) = \frac{1}{3} c r_L(E).
   \]
(26)

3. The Kolmogorov diffusion coefficient
   \[
   D_K(E) = \frac{1}{3} c l_c \left[ \frac{r_L(E)}{l_c} \right]^{1/3}.
   \]
(27)

In all three cases we normalized the diffusion coefficients by \( \frac{1}{3} c l_c \) at \( r_L = l_c \) (see § 5). The characteristic energy \( E_c \) of the transition between the high- and low-energy regimes is determined by the condition \( r_L(E) = l_c \) and is
\[
E_c = 0.93 \times 10^{18} \frac{B_0}{1 \text{ nG}} \left( \frac{l_c}{\text{ Mpc}} \right) \text{ eV}.
\]
(28)

One can describe the low- and high-energy diffusion regimes with the help of an interpolation formula for the diffusion length:
\[
l_d(E) = \Lambda_d + \frac{r_L^2(E)}{l_c},
\]
(29)
with \( \Lambda_d = l_c \) for the regime with \( D = \text{const} \), \( \Lambda_d = r_L(l_c/l_c)^{1/3} \) for the Kolmogorov regime. For later use, we formalize the description of these three regimes using
\[
\Lambda_d(E) = l_c (r_L/l_c)^{\alpha},
\]
(30)
with \( \alpha \) equal to 0, 1, and \( \frac{1}{3} \) for the \( D = \text{const} \), Bohm, and Kolmogorov regimes, respectively.

For completeness we give also the numerical expression for the Larmor radius:
\[
r_L(E) = 1.08 \times 10^2 \frac{E}{1 \times 10^{20} \text{ eV}} \frac{1 \text{ nG}}{B} \text{ Mpc}.
\]
(31)

One can see, therefore, the existence of two different propagation lengths that should be compared with the distance \( d \) between sources. These two lengths are \( l_{\text{min}} \) given by equation (1) and \( l_d \) given by equation (29). The former is large even at the highest energies (\( l_{\text{min}} \sim 25 \text{ Mpc} \) at \( E \sim 10^{21} \text{ eV} \)), while the latter can be small enough at the energies of interest \( 10^{19} \text{ eV} \leq E \leq 10^{21} \text{ eV} \). For a representative case \( B_0 = 100 \text{ nG} \) and \( l_c = 1 \text{ Mpc} \). Here and below we use for the total lattice size \( a = 300 \text{ Mpc} \), while the control calculations have been performed up to \( a = 1000 \text{ Mpc} \). Increasing \( a \) does not change the fluxes. The universal spectrum is calculated with time-independent energy losses, as in the Syrovatsky solution, according to equation (6) with \( \mathcal{L}_0 = L_p n_i \simeq 2.8 \times 10^{46} \text{ ergs Mpc}^{-3} \text{ yr}^{-1} \).

One can see that in both cases the diffusion spectrum tends to the universal one as the separation \( d \) between sources diminishes.

5. DIFFUSIVE PROPAGATION

In this section we study the propagation of UHE particles in the intergalactic turbulent magnetic plasma. Turbulent motion can be considered as random pulsations at different scales, described as an ensemble of many waves with large amplitudes and random phases. The energy is assumed to be injected at the largest scale \( l_c \) as a result of the external processes, such as the contraction of large-scale structures, large-scale shocks, etc., and the scale \( l_c \) might be determined by these processes. This scale might coincide with the size of the region filled by the turbulent plasma, in case of small structures. The smallest scale \( l_{\text{min}} \) is determined by the dissipation of turbulent motion into thermal energy.

We consider magnetic turbulence described by the superposition of MHD waves \( A \exp \left[ i(-\omega t + \mathbf{k} \cdot \mathbf{r}) \right] \) with different frequencies \( \omega \) and different wavenumbers \( \mathbf{k} \). The amplitude \( A \) is related to magnetic field \( B \), density \( \rho \), pressure \( P \), velocity \( v \), etc. These waves propagate with the Alfvén velocity \( \mathbf{u}_A = B/(4\pi \rho)^{1/2} \). The kinetic energy of gas in turbulent motion and the magnetic energy in MHD waves are equal at all scales \( \lambda \) (Landau & Lifshitz 1987):
\[
\rho v_A^2 \sim B^2 / 4\pi.
\]
(32)
For a detailed description of magnetic turbulent plasma and its description as an ensemble of MHD waves we refer the reader to Landau & Lifshitz (1987) and Arzimovich & Sagdeev (1979).

For practical calculations of the UHECR propagation, the magnetic field scale spectrum \( \langle B^2 \rangle_k \) as a function of \( k \) is needed. One should clearly distinguish between the magnetic field \( B_k = \langle B \rangle_k \), on the scale \( \lambda = 2\pi/k \), and the Fourier component of the field \( B(k) \) (see the Appendix). The physical magnetic field that determines the diffusion of UHE particles is \( B_j \).

The turbulence spectrum is described by the dependence of the spectral energy density \( w(k) \) on the wavenumber \( k \). It can be expressed using the magnetic energy density \( kw(k) \sim B_j^2 / 8\pi \). Usually the spectra have a power-law form: \( w(k) \propto k^{-m} \).

In many practical cases the observations confirm the Kolmogorov (1941) spectrum of turbulence with \( m = 5/3 \). This spectrum belongs to a class where the initial energy is assumed to be injected at the largest scale \( l_c \) and then is transferred to lower scales \( l \) in nonlinear processes of wave interactions until it dissipates to thermal energy at the lowest scale \( l_{\text{min}} \). The Kolmogorov spectrum can be derived assuming that energy flux from one scale to another does not depend on the scale \( k \) (Arzimovich & Sagdeev 1979), i.e.,

\[
kw(k)/\tau_k = \text{const},
\]

where \( \tau_k \) is the time for energy transfer to the scale \( k \). Since this process is caused by the nonlinear term \((v\nabla)v \sim kv^2\) in the Euler equation, one can estimate \( \tau_k \) as

\[
\tau_k \sim 1/kv_k,
\]

where the turbulent velocity can be estimated from equation \( kw(k) \sim \rho v_k^2 \). Then one immediately obtains \( w(k) \propto k^{-5/3} \), i.e., the Kolmogorov spectrum.

Landau & Lifshitz (1987) argue that while scale-independent energy flux is a reasonable assumption in ordinary hydrodynamics, for MHD waves it is proportional to \( v_k^3 \). In this case one obtains the Kraichnan (1965) spectrum \( w(k) \propto k^{-3/2} \).

In the case of shock waves, the spectrum is (Vainstein et al. 1989) \( w(k) \propto k^2 \).

5.1. Diffusion Coefficient for UHE Protons

Now we proceed with calculations of the diffusion coefficient \( D(E) \) for UHE protons in extragalactic magnetic fields, which are characterized by the spectrum of turbulence \( w(k) \propto k^{-m} \), with the basic scale \( l_c \) and with the magnetic field on this scale \( B_0 \). We use the characteristic energy \( E_c \), defined by equation (28) from the condition \( r_L(E_c) = l_c \), where the Larmor radius \( r_L \) is given by equation (31).

First we obtain the asymptotic expression for \( D(E) \) valid when \( r_L \gg l_c \), i.e., at \( E \gg E_c \). The diffusion length \( l_d \) is determined as the distance at which the average angle of scattering satisfies

\[
\langle \theta^2 \rangle \sim n \left( \frac{l_c}{r_L} \right) ^2 \sim 1,
\]

where the number of scatterings \( n \) is given by \( n \sim l_d/l_c \). It results in the diffusion length \( l_d(E) \approx c l_d(E_c) / E_c \), and thus the diffusion coefficient, \( D(E) = \frac{1}{2} c l_d(E) \), in the asymptotic limit \( r_L \gg l_c \) is given by

\[
D_{\text{asymp}}(E) = 3.6 \times 10^{34} E_{20}^2 \left( \frac{100 \text{ nG}}{B_0} \right) ^2 \left( \frac{1 \text{ Mpc}}{l_c} \right) \text{ cm}^2 \text{ s}^{-1}.
\]

We calculate now the diffusion coefficient in the low-energy limit \( r_L \ll l_c \) (i.e., at \( E \ll E_c \)). We follow the V. Ptuskin method given in Berezinsky et al. (1990a). Let us consider the scattering of UHE protons with \( r_L \ll l_c \) in a magnetic field of MHD wave \( B_j \). The magnetic field on the basic scale \( B_0 \) is considered as constant field for smaller scales \( \lambda \).

Scattering of UHE protons off the MHD waves in the regime \( r_L \ll l_c \) is dominated by resonance scattering (see Lifshitz & Pitaevskii 2001, \S 55 and 61). The condition for resonant scattering is given by \( \omega' = s \omega'_{B_0} \), where \( \omega' \) and \( \omega'_{B_0} = eB' / mc \) are the wave frequency and the gyrofrequency in the system \( K' \) at rest with particle motion along the field
After a Lorentz transformation to the laboratory system, one obtains
\[ \omega - k_{||} v_z = s \omega_B, \] (36)
where \( \omega \) is the wave frequency, \( k_{||} \) is the projection of the wavevector onto the direction of \( B_0 \), \( v_z \) is the projection of particle velocity onto the same direction, and \( \omega_B = e B / \gamma m_c \). For a magnetized plasma \( r_L \ll \lambda_{L} \) (or equivalently \( k_{||} v_z / \omega_B \ll 1 \)) and \( s = \pm 1 \) (Berezinsky et al. 1990a). Thus, from equation (36) one derives the resonant wavenumber
\[ k_{\\Pi}^{res} = \left| \frac{\omega(k) \pm \omega_B}{v_z} \right| \approx \frac{\omega_B}{v_\mu} = \frac{1}{r_{L,\mu}}, \] (37)
where \( \mu = \cos \theta \) and \( r_{L} = v/\omega_B \) is the Larmor radius. In deriving equation (37) we have used \( \omega(k) \ll \omega_B \), which follows from the dispersion relation for Alfvén waves \( \omega(k) = u_A k_{||}/L \) (Landau & Lifshitz 1987), and \( v/u_A > 1 \) for ultrarelativistic particles.

We assume here first the Kolmogorov spectrum normalized at the basic scale \( k_0 = 2 \pi /A_0 \):
\[ k_w(k) = k_0 w_0 (k/k_0)^{-2/3}, \] (38)
where \( w(k) \) is the spectral magnetic energy density normalized as \( k_0 w_0 = B_0^2 /8 \pi \).

We adopt from Berezinsky et al. (1990a, pp. 387–424) the diffusion coefficient \( D_1 \) (in the direction of \( B_0 \)) expressed with the frequencies of particle scattering off the waves:
\[ \nu(\mu, k_{res}) = \frac{1}{2} (\nu^+ + \nu^-) = 2 \pi^2 \omega_B k_{res} \omega(k_{res}) /B_0^2, \]
\[ D = \frac{\nu^2}{4} \int_0^1 d\mu \frac{1 - \mu^2}{\nu(\mu, k)} , \] (39)
where \( \nu^+ \) and \( \nu^- \) correspond to waves propagating along the field and in the opposite direction, respectively. Hereafter the subscript \( \parallel \) is omitted.

Using the formulae above and performing integration over \( \mu \), we obtain
\[ D(E) = \frac{18}{7 \pi (2 \pi)^{2/3}} \frac{\nu_c}{L_c} \left( \frac{\rho}{B_0} \right)^{1/3}. \] (40)

Numerically
\[ D(E) = 2.3 \times 10^{34} E_20^{1/3} \left( \frac{100 \text{ nG}}{B_0} \right)^{1/3} \left( \frac{k}{1 \text{ Mpc}} \right)^{2/3} \text{ cm}^2 \text{ s}^{-1}. \] (41)

\( D(E) \) from equations (41) and (35) match together fairly well: at energy \( E_c \) they differ by 40%.

The calculations for other spectra are similar: in the case of the Kraichnan spectrum \( w(k) \propto k^{-1/2} \), one obtains \( D(E) \propto E^{1/2} \), and for diffusion on shock waves, \( w(k) \propto k^{-2} \), \( D = \text{const} \) follows.

**Diffusion coefficient for static magnetic field.**—This case can be considered as scattering off MHD waves in the limit \( \omega(k) \to 0 \). Assuming the spectrum
\[ k_w(k) = \frac{B_0^2}{8 \pi} \left( \frac{k}{k_0} \right)^{-\alpha}, \] with \( \alpha < 1 \) for convergence of the integral, we obtain, performing the same calculations as above,
\[ D(E) = \frac{2 (2 \pi)^{-\alpha}}{\pi (1-\alpha)(3-\alpha)} \frac{L_c}{T} \left( \frac{\rho}{B_0} \right)^{1-\alpha}. \] (42)

### 5.2. Intergalactic Medium and Formation of Turbulent Spectrum

The intergalactic medium is represented by different structures (clusters of galaxies, superclusters, filaments, and voids), where the medium properties are very much different. In the estimates below we normalize all quantities by the baryon density \( n_b = 2.75 \times 10^{-7} \text{ cm}^{-3} \), corresponding to \( \Omega_b = 0.044 \) from WMAP measurements (Spergel et al. 2003), and by the temperature \( \sim 10^6 \text{ K} \) (see, e.g., the simulation of Davé et al. 2001 for filaments). Hence, we adopt the sound speed
\[ c_s = (\gamma T / n_b)^{1/2} = 1.2 \times 10^7 (T / 10^6 \text{ K})^{1/2} \text{ cm s}^{-1}, \] (43)

the Alfvén velocity
\[ u_A = \frac{B}{\sqrt{4 \pi \rho_b}} = 4.2 \times 10^5 \frac{B}{1 \text{ nG}} \left( \frac{2.75 \times 10^{-7} \text{ cm}^{-3}}{n_b} \right)^{1/2} \text{ cm s}^{-1}, \] (44)

and the Coulomb scattering length for electron-electron and proton-proton scattering
\[ l_{sc} = \frac{T^2}{4 \pi e^4 n_b L_c} = 1.7 \left( \frac{T}{10^6 \text{ K}} \right)^2 \frac{2.75 \times 10^{-7}}{n_b} \text{ kpc}, \] (45)

where \( L \sim 20 \) is the Coulomb logarithm. The short scattering length \( l_{sc} \) in comparison with the basic scale \( l_c \) is one of the conditions needed to provide the turbulent regime.

With the characteristics of the media above, we discuss now whether the equilibrium turbulence spectrum in the gas can be formed during the age of the universe \( t_0 \). We make estimates for the Kolmogorov spectrum.

The relaxation time \( \tau_k \) to the equilibrium Kolmogorov spectrum is given by equation (34) as \( \tau_k \sim 1 / k v_k \), where \( v_k \) is the turbulent velocity on the scale \( k \). Taking \( v_k \) from \( \rho v_k^2 \sim 2 k w(k) \) and using the Kolmogorov spectrum normalized at the basic scale as in equation (38), we obtain
\[ \tau_k = \frac{1}{k_0} \left( \frac{\rho}{2 k_0 w_0} \right)^{1/2} \left( \frac{k}{k_0} \right)^{-2/3}. \] (46)

The longest time is needed for the formation of the spectrum at the largest scale \( k_0 \):
\[ \tau_0 = \frac{1}{k_0} \left( \frac{\rho}{2 k_0 w_0} \right)^{1/2}. \] (47)

Using \( k_0 w_0 = B_0^2 /8 \pi \) and \( v_0 = B_0 / (4 \pi \rho)^{1/2} = u_A \), we obtain
\[ \tau_0 \sim l_c / u_A. \] (48)

Numerically it gives
\[ \tau_0 = 2.4 \times 10^{11} \frac{l_c}{1 \text{ Mpc}} \frac{1 \text{ nG}}{B_0} \left( \frac{n_b}{2.75 \times 10^{-7}} \right)^{1/2} \text{ yr}. \] (49)
Another estimate of the relaxation time $\tau_0$ can be obtained for the sonic turbulence with the Kolmogorov spectrum. The shortest time is given by $\tau_0 = l_c/c_s$, which for a turbulent plasma is the analog of the causality condition. Numerically it gives

$$\tau_0 = 0.84 \times 10^{19} \frac{l_c}{1 \text{ Mpc}} \left(\frac{10^6 \text{ K}}{T}\right)^{1/2} \text{ yr}. \quad (50)$$

For a more accurate estimate one can use the relaxation time for the sonic turbulence from Arzimovich & Sagdeev (1979),

$$\tau_k \sim \frac{n_h T}{k \omega_k} \left(\frac{l_c}{1 \text{ Mpc}}\right). \quad (51)$$

Using the dispersion relation $\omega_k = k(T/m)^{1/2}$ and the Kolmogorov spectrum $\omega_k = k_0 w_0 (k/k_0)^{2/3}$, with $k_0 w_0 \sim n T$, we obtain numerically an estimate close to that given by equation (50).

From the above estimates we can conclude that the Kolmogorov spectrum cannot be reached for MHD turbulence in the voids, where the magnetic field is presumably small, $B_0 \leq 1 \text{ nG}$. However, it can be established in filaments (see eq. [49]) in the case $B_0 > 1 \text{ nG}$ and $l_c \leq 1 \text{ Mpc}$, and it has enough time to be developed in galaxy clusters, where the magnetic field is strong and density of gas is larger than in other structures. For other types of turbulence, e.g., for sonic turbulence, these conditions can be somewhat relaxed.

### 5.3. Some Features of Diffusive Propagation

Inspired by numerical simulations (Yoshiguchi et al. 2003), we study the diffusion in various magnetic field configurations with the basic parameters $(B_0, l_c)$ in the intervals $B_0 = 10-1000 \text{ nG}$ and $l_c = 1-10 \text{ Mpc}$. As a representative configuration we consider $(100 \text{ nG}, 1 \text{ Mpc})$ with $E_c \approx 1 \times 10^{20} \text{ eV}$.

The calculated diffuse energy spectra are characterized by the sets $(B_0, l_c, d)$, where $d$ is the separation between sources. We use the following definitions. The case when at all relevant (observed) energies $d > l_d(E)$ [or $d \gg l_d(E)$] we refer to as diffusion in a strong magnetic field. The case when $d \leq l_d(E)$ corresponds to diffusion in a weak magnetic field.$^2$

The extreme case $d \ll l_d(E)$ results in the universal spectrum.

**Minimal distance to a source.**—For the diffusion approximation to be valid, a source must be at a distance $r > r_{\text{min}}(E)$. In principle, velocity of light does not enter the diffusion equation, but this equation is not valid when the propagation velocity exceeds the light speed $c$. The value of $r_{\text{min}}(E)$ can be estimated from the condition that the diffusion propagation time must be longer than the time of rectilinear propagation,

$$t_{\text{prop}} \sim \frac{r^2}{D(E)} > \frac{r}{c},$$

as

$$r_{\text{min}}(E) \sim \frac{1}{3} l_d(E). \quad (52)$$

For a source at distance $r \leq r_{\text{min}}(E)$ the protons with energy $E$ or higher propagate in a quasi-rectilinear regime. At $E \geq E_c$ one has

$$r_{\text{min}}(E) \approx \frac{1}{3} \left(\frac{E}{E_c}\right)^2 l_c.$$

**Maximal distance to a source.**—As seen from equation (19), the contribution of a source to the flux at energy $E$ becomes negligible at distances $r > r_{\text{max}}(E)$ with

$$r_{\text{max}}(E) = 2 \sqrt{\frac{\lambda(E, E_g^{\text{max}})}{E_g}}, \quad (53)$$

where $E_g^{\text{max}}$ is the maximum generation energy provided by a source.

For the representative case $(B_0, l_c) = (100 \text{ nG}, 1 \text{ Mpc})$, $r_{\text{max}}$ is plotted in Figure 3 as a function of $E$ for different values of $E_g^{\text{max}}$. One can see that in the case of diffusive propagation only nearby sources contribute the UHECR flux.

In Figure 4 we have plotted also

$$r_{\text{eff}}(E) = 2 \sqrt{\frac{\lambda(E, E_g)}{E}}$$

for different (fixed) ratios $E_g/E = 2, 5, \text{ and } 10$. The increase of $r_{\text{eff}}$ with $E$ provides the increase of diffuse flux $J_\mu(E)$ with
6. DIFFUSE FLUXES IN THE DIFFUSION APPROXIMATION

In this section we compute the diffuse flux according to equation (23) as the sum of fluxes from single sources located in the lattice vertices with a separation \( d \) and with a total size \( a \). We assume the complex generation spectrum given by equation (8). For the diffusion coefficient we use \( D = A_B \) of \((\text{generation spectrum given by equation (8)}\) and for two sets calculated spectra are presented in Figure 5.

At small distances \( r \leq r_{\text{min}}(E) \), given by equation (52), the fluxes from the individual sources are calculated in the rectilinear approximation, and the diffuse flux is given by

\[
J_{p}^{\text{rect}}(E) = \frac{L_p K(g)}{4\pi} \sum_i \frac{q_{\text{gen}}[E_{\text{gen}}(E, r_i)]}{r_i} \frac{dE_{\text{gen}}(E, r)}{dE}.
\]

At large distances \( r \geq r_{\text{min}}(E) \) the diffuse flux is given by equation (23) and with explicit normalization has the form

\[
J_{p}^{\text{diff}}(E) = \frac{e}{4\pi} \frac{L_p K(g)}{b(E)} \times \frac{E_{\text{max}}}{E} \frac{E}{E_{\text{gen}}(E)} \exp\left(-r_i^2/[4\lambda(E, E_{\text{gen}})]\right) \left[4\pi\lambda(E, E_{\text{gen}})\right]^{3/2},
\]

where \( E_{\text{max}} \) is the maximum acceleration energy (see § 4). The calculated spectra are presented in Figure 5.

In Figure 5 the diffuse spectra are shown for the complex generation spectrum given by equation (8) and for two sets of \((B_p, l, d)\) equal to \((100 \text{ nG}, 1 \text{ Mpc}, 30 \text{ Mpc})\), shown by a dashed curve, and \((1000 \text{ nG}, 1 \text{ Mpc}, 30 \text{ Mpc})\), shown by a solid curve. Both of them are characterized by the same \( d = 30 \text{ Mpc} \), but in the latter case the magnetic field is much stronger. This case is characterized by \( d > l_p(E) \) at all observable energies and hence corresponds to diffusion in the strong magnetic field (see § 5). From Figure 5 one can observe that GZK cutoff is weak in this case.

The dashed curve in Figure 5 is characterized by \( l_p(E) = (E/E_c)^2 \text{ Mpc} \) at \( E > E_c = 1 \times 10^{20} \text{ eV} \), and this regime of propagation can be described as an intermediate one between that for strong and weak magnetic fields.

The regime with weak GZK cutoff (propagation in a strong magnetic field) requires a much higher luminosity \( L_p \) to fit the observational data, \( L_p = 1.5 \times 10^{47} \text{ ergs s}^{-1} \), while the intermediate case \((100 \text{ nG}, 1 \text{ Mpc}, 30 \text{ Mpc})\) needs only \( L_p = 8.5 \times 10^{46} \text{ ergs s}^{-1} \).

We analyze now the regime of propagation in the strong magnetic field in more detail. First we comment qualitatively why in the diffusion approximation the GZK cutoff in the spectrum might be weak or absent.

The essence of the GZK cutoff consists of much different energy losses above and below \( 3 \times 10^{19} \text{ eV} \). Consider, for example, two protons with energies \( 1 \times 10^{19} \) and \( 1 \times 10^{20} \text{ eV} \), both propagating rectilinearly from a remote single source. The first proton loses little energy: the ratio \( E_p/E \) (with \( E_q \) being the generation energy) is not large, and the flux of these protons is weakly suppressed by the generation spectrum. The energy losses of the second proton are large, \( E_p/E \) is very high, and the flux suppression is dramatic (the GZK cutoff). Now consider the diffusive propagation. A proton with energy \( E \) \( 1 \times 10^{19} \text{ eV} \), because of the \( D(E) \) dependence, travels a much longer time than a proton with \( E \) \( 1 \times 10^{20} \text{ eV} \), and it results in increased ratio \( E_p/E \), making this ratio comparable with that for the second proton. It causes a less steep GZK cutoff or its absence. However, the price for the absence of the GZK cutoff is a very high luminosity of the sources \( L_p \) needed to provide the observed flux at \( E \geq 1 \times 10^{19} \text{ eV} \), e.g., \( L_p \geq 1 \times 10^{47} \text{ ergs s}^{-1} \) for the spectrum shown in Figure 5.

Let us now come over to the quantitative analysis of absence of the GZK cutoff in the strong magnetic fields. Consider the three cases of diffusion regimes at \( E < E_c \) with \( \alpha = 0, \frac{1}{2}, 1 \), i.e., \( D = \text{const}, \text{Kolmogorov}, \text{and Bohm regimes} \), respectively (see eq. [30]). The diffuse spectra calculated for configuration \((1000 \text{ nG}, 1 \text{ Mpc}, 30 \text{ Mpc})\) are displayed in Figure 6.

Why are the three spectra so much different at low energies and the same at high energies?
The quantitative explanation can be given explicitly in terms of \( \lambda(E, E_g) \), which is the basic parameter of the Syrovatsky solution (see eq. [19]).

In Figure 7 we plot the values of

\[
r_{\text{max}}(E, E_g) = 2\sqrt{\lambda(E, E_g)},
\]

which according to equation (19) determines the maximum distance to a source in case the observed energy is \( E \) and generation energy is \( E_g \). The left panel of Figure 7 corresponds to \( D = \text{const} \), while the right panel corresponds to the Bohm diffusion.

From Figure 7 one can see that in the energy interval \((1\text{--}4)\times 10^{19} \text{ eV} \) \( \lambda(E, E_g) \) practically does not depend on \( E_g \). Then from equation (19) one obtains for a single source

\[
n_{p}(r, E) = \frac{Q(E)}{(\gamma_g - 1)(4\pi)^{3/2}} \frac{Q(E)}{\lambda(E)} \exp \left[ -\frac{r^2}{4\lambda(E)} \right],
\]

where \( \beta(E) = E^{-1}dE/dt \).

From equation (56) and values of \( \lambda(E) \) for the two diffusion regimes, one may observe the main effect: strong suppression of flux from nearby sources \((r \sim d \sim 30 \text{ Mpc})\) in the case of the Bohm diffusion and weak suppression in the case of \( D = \text{const} \) diffusion. It occurs because the Bohm diffusion coefficient, \( D_\text{B}(E) = D_\text{B}(E/E_c) \), is small at small \( E < E_c \), and hence \( \lambda(E) \) is also small, which results in exponential suppression of the flux, according to equation (56). In the case \( D(E) = D_0 \), the diffusion coefficient is large, \( \lambda(E) \) is large too (see left panel of Fig. 7), and exponential suppression in equation (56) is much smaller. This quantitative explanation agrees with the qualitative one, given above: large propagation time, i.e., small diffusion coefficient or small \( \lambda \), suppresses the flux at \( E < E_{\text{GZK}} \) to the level of the flux at the highest energies.

Let us come over to the higher energies.

One can see from Figure 7 that at \( E \geq 1\times 10^{20} \text{ eV} \) for \( D = \text{const} \) diffusion and \( E \geq 1\times 10^{19} \text{ eV} \) for the Bohm diffusion \( \lambda(E) \) is small but increases fast with energy up to \( r_{\text{max}}(E, E_g) \sim 30 \text{ Mpc} \) at \( E = E_{\text{max}} \). The exponent in equation (19) grows very fast and all other quantities can be taken out of integral at energy \( E_g = E_{\text{max}} \). After integration one arrives at the analytic expression

\[
n_{p}(r, E) = \frac{Q(E_{\text{max}})}{4\pi r^2} \frac{\sqrt{\lambda_0}}{D_0} \exp \left( -\frac{r^2}{4\lambda_0} \right)
\]

\[
\times 2\sqrt{\pi} \frac{E_{\text{max}}}{E} \frac{\beta(E_{\text{max}})}{\beta(E)} \left( \frac{E_{\text{max}}}{E} \right)^2,
\]

where \( \beta(E) = E^{-1}dE/dt \) and \( \lambda_0 = \lambda(E, E_{\text{max}}) \), which in fact does not depend on \( E \). From equation (57) one can see that \( n_{p}(r, E) \) has universal dependence for all diffusion regimes, provided by \( D(E_{\text{max}}) = D_0(E_{\text{max}}/E_{g})^2 \) for all of them. In asymptotically high energy regimes (not seen in Fig. 6) \( E^3 n_{p}(E) \propto E^2 \).

We found and analyzed above the diffusion regime with weak GZK cutoff for very strong magnetic field \( B_0 = 1000 \text{ nG} \). Can this effect exist for much weaker magnetic field? We have not found such regimes in any realistic cases we studied.

The strong restriction to existence of these regimes is imposed by the upper limit to the distance between sources \( d \). This distance cannot be arbitrarily large. They are limited by the maximum acceleration energy \( E_{\text{max}} \), which we keep here.
reasonably high, $E_{\text{max}} = 1 \times 10^{22}$ eV. For rectilinear propagation from a source at distance $r$ the proton with observed energy $E = 1 \times 10^{20}$ eV must have at $r = 100$ Mpc the generation energy $E_g = 1 \times 10^{22}$ eV, and thus the sharp cutoff at $E = 1 \times 10^{20}$ eV is predicted. For $r = 50$ Mpc the cutoff energy is $E = 3 \times 10^{20}$ eV. Assuming $r \sim d$, one obtains $d \lesssim 50$–$100$ Mpc for $E_{\text{max}} = 1 \times 10^{22}$ eV.

We have performed many calculations with magnetic fields in the range 100–300 nG, with $l_d$ in a range 1–10 Mpc, for all three diffusion regimes $\alpha = 1, \frac{1}{3}, 0$ and with $d$ in a range 30–50 Mpc. In all cases the spectra expose GZK cutoff.

On this basis we conclude that in the case $E_{\text{max}} \lesssim 1 \times 10^{22}$ eV, i.e., for $d \lesssim 50$ Mpc, the diffusion regime with weak GZK cutoff appears only at very strong magnetic fields $B_0 \sim 1000$ nG, and it needs very high source luminosities $L_p \sim 10^{47}$ ergs s$^{-1}$.

7. CONCLUSIONS

We have performed a formal study of the propagation of UHE particles, using an analytic approach. We demonstrated that the distance between sources is a crucial parameter that strongly affects the diffuse energy spectrum.

We have proved that, for a uniform distribution of sources, when the separation between them is much smaller than all characteristic propagation lengths, most notably the diffusion length $l_d(E)$ and the energy attenuation length $l_{\text{att}}(E)$, the diffusion spectrum of UHECRs has a universal form, independent of the mode of propagation. This statement has the status of a theorem and is valid for propagation in strong magnetic fields. The proof is given using particle number conservation during propagation and also using the kinetic equation for the propagation of UHE particles. In particular, the exact solution to the kinetic equation (9) for a homogeneous (i.e., continuous and distance independent) distribution of the sources gives exactly the same spectrum as in the method using particle number conservation. Note that in equation (9) the diffusion term is absent as a result of homogeneous distribution of the sources.

Another proof of the theorem is given using the diffusion equation (18) and its exact solution (Syrovatskiy 1959) for a single source in the case of time-independent energy losses. We calculated the diffuse flux putting the sources at the vertices of a big lattice with size $a$ and with separation $d$ between vertices (sources). The results of the calculations are shown in Figure 2. One can see that when the distance between sources diminishes from 50 to 10 Mpc, the spectra converge to the universal spectrum (solid curves), and at $d = 3$ Mpc they become identical to the universal spectrum. This result is confirmed by analytic calculations. When the separation between sources is small, summation over the sources can be replaced by an integration and the corresponding spectrum given by equation (21) coincides exactly with the universal spectrum.

In this paper we have studied the diffusive propagation of UHE particles in intergalactic space, which is considered as a turbulent magnetic plasma with baryonic gas density $n_b$ and temperature $T$, and with two basic turbulence scales $l_\alpha$, where external energy is injected, and $l_{\text{min}}$, where turbulent energy is dissipated. Turbulent motion is considered to be random pulsations at different scales, described by an ensemble of MHD waves $B_k \exp \left[ i(-\omega t + \mathbf{k} \cdot \mathbf{r}) \right]$ with different frequencies $\omega$ and different wavenumbers $k$. Diffusion arises as a result of resonant scattering of particles in magnetic fields of MHD waves. At “low” energies, when the Larmor radius $r_L$ is much smaller than the basic scale $l_\alpha$, the diffusion coefficient can be calculated, provided that the spectral energy density of the turbulent plasma, $w(k)$, is known as a function of $k$. Such calculations for the Kolmogorov spectrum of turbulence result in the diffusion coefficient given by equation (41).

It is interesting to note that the propagation in static magnetic fields can be calculated by this method as the limiting case, when wave frequency $\omega \to 0$. The diffusion coefficient for static magnetic fields is given by equation (42).

The diffusion coefficient in the high-energy limit, when $r_L \gg l_\alpha$, can be reliably calculated as the process of multiple scattering. The diffusion coefficient for this extreme case is given by equation (35). The diffusion coefficients in the low-energy regime ($r_L \ll l_\alpha$), given by equation (41), and in the high-energy regime ($r_L \gg l_\alpha$), given by equation (35), match each other well.

In practical calculations of the diffuse fluxes we characterize the magnetic configuration by three parameters ($B_0$, $l_\alpha$, $d$), where $B_0$ is the mean magnetic field on the basic scale $l_\alpha$ and $d$ is the separation of sources. We put the sources at the vertices of a lattice of total size $a$, for which we typically used $a = 300$ Mpc. Increasing $a$ does not change the fluxes. The fluxes from the individual sources were found as the Syrovatsky solution given by equation (19), and the diffuse flux was found by summation over all sources, given by equation (23). As a representative case for a strong magnetic field, we considered the configuration ($B_0$, $l_\alpha$) = (100 nG, 1 Mpc).

An important feature of the diffusion model is that the observed diffuse flux is produced by nearby sources (see Figs. 3 and 4). The maximum distance $r_{\text{max}}$ depends strongly on the maximum acceleration energy $E_{\text{max}}$. For example, for $E_{\text{max}} = 1 \times 10^{21}$ eV and ($B_0$, $l_\alpha$) = (100 nG, 1 Mpc), the maximum distance is less than 70 Mpc, while for $E_{\text{max}} = 1 \times 10^{22}$ eV this distance is 200 Mpc. For smaller magnetic fields these distances are larger (see Fig. 3, right panel). The small radius $r_{\text{max}}$ of the region, which provides the dominant contribution to the observed diffuse flux of UHECRs, imposes a constraint on the diffusion models, which will become more severe when/if particles with higher energies are observed. As an example, let us consider the representative configuration (100 nG, 1 Mpc) and highest observed energy $E = 3 \times 10^{20}$ eV. In this case we have $E_g = 1 \times 10^{20}$ eV, $l_d(E) \approx 10$ Mpc, and $l_{\text{att}} = 21$ Mpc. To avoid rectilinear propagation from nearby sources, we should impose the separation $d > l_d(E) \approx 10$ Mpc. For a maximum generation energy $E_{\text{max}} = 1 \times 10^{21}$ eV, the maximum distance is $r_{\text{max}} < 70$ Mpc and is only marginally consistent with $d > 10$ Mpc. The UHECR sources at $r < 70$ Mpc with $d > 10$ Mpc and with maximum acceleration energy $E_{\text{max}} \sim 1 \times 10^{21}$ eV tentatively imply active galactic nuclei, whose luminosities satisfy the energy requirement for the observed UHECR fluxes in the case of weak magnetic fields. However, for the case of a weak GZK cutoff (diffusion in strong magnetic fields and large separation $d$) the required luminosities are very high, $L_p \sim 10^{47}$ ergs s$^{-1}$.

The calculated diffuse energy spectra are shown in Figure 5 for the Bohm diffusion coefficient and for the complex generation energy spectrum given by equation (8). In the case of configuration with a very strong magnetic field (1000 nG, 1 Mpc, 30 Mpc) the spectrum has a weak GZK cutoff (Fig. 5, solid curve). This spectrum agrees reasonably with the AGASA excess but requires very large source luminosity $L_p = 1.5 \times 10^{47}$ ergs s$^{-1}$. When magnetic field diminishes to 100 nG, the GZK cutoff appears (dotted curve), as it should according to the propagation theorem.
A weak GZK cutoff in the case of diffusive propagation is explained by the flux suppression at the $E < E_{GZK}$ cutoff due to longer propagation time from the source (see § 6 for a detailed explanation).

The spectra calculated in this paper in the diffusion approximation are compatible, for the relevant parameters, with the numerical simulations in Yoshiguchi et al. (2003). As an example we present in Figure 8 our spectra from a single source calculated in the diffusion approximation for magnetic configuration $(B_0, l_d) = (100 \text{ nG}, 1 \text{ Mpc})$ with the Kolmogorov spectrum of turbulence for a distance to the source of 10 (left panel) and 30 Mpc (right panel). For the distance 10 Mpc we also plot the spectrum for the rectilinear propagation shown by the dotted curve. The two curves intersect at $E_s \sim 3 \times 10^{20} \text{ eV}$, as they should provided by the condition $l_d(E_s) = r$, where $r$ is the distance to the source. At $E > E_s$ the particles propagate rectilinearly and the flux is given by the dashed curve. This spectrum is compared with the numerical simulations of Yoshiguchi et al. (2003) for the same magnetic field configuration (H. Yoshiguchi 2004, private communication), shown by the crosses. Since the calculations of Yoshiguchi et al. (2003) are not normalized, we equate the fluxes at energies with rectilinear propagation as shown in Figure 8. One can observe considerable disagreement at $E \sim 8 \times 10^{19} \text{ eV}$, where it is possible to suspect a transitional regime between quasi-rectilinear and diffusive propagation in numerical simulations. For the distance to the source $r = 30 \text{ Mpc}$ (right panel) the agreement is much better, probably because at these distances the diffusion regime is reached in the numerical simulations.

The study of this paper is not intended to be a realistic one. We consider the diffusive propagation of UHE particles using an analytic approach with the aim of understanding the basic properties of propagation in strong magnetic fields. A realistic propagation should be studied within the hierarchical model of different magnetic fields in different large-scale structures, as it is done in simulations (Sigl et al. 2003, 2004; Dolag et al. 2003). However, we are able to reach some practical conclusions.

Our study is focused on the spectra of UHECRs. We demonstrated that the crucial parameter is the source separation $d$. For a wide class of magnetic field configurations, when the separation $d$ is smaller than the diffusion length $l_d$, the spectrum is universal, i.e., the same as in the case of rectilinear propagation. For the simulation of Dolag et al. (2003) the spectrum must be universal. The simulations of Sigl et al. (2003, 2004) and Yoshiguchi et al. (2003) include diffusive and intermediate regimes. According to our calculation, in the diffusive regime very strong magnetic fields, $B_0 \sim 1000 \text{ nG}$, and large separation between sources are needed to produce a spectrum with weak (or absent) GZK cutoff. In this case very high source luminosities are required, $L_p > 1 \times 10^{47} \text{ ergs s}^{-1}$. It disfavors diffusion in strong magnetic fields as the explanation for the AGASA excess at high energies.

We acknowledge participation of Askhat Gazizov at an early stage of this work. We are grateful to Vladimir Ptuskin for valuable discussion of diffusive propagation of UHECRs and to Igor Tkachev for early discussion of hierarchical magnetic field structure of the universe and the possibility of quasi-rectilinear propagation of UHE protons. We acknowledge useful discussions with Pasquale Blasi and Yuri Eroshenko. We are grateful to the authors of Yoshiguchi et al. (2003), especially Hiroyuki Yoshiguchi and Katsuhiko Sato, for discussions and calculations made for comparison with our results. We also thank Richard Ford for a critical reading of the manuscript. We thank the transnational access to research infrastructures (TARI) program available through the LNGS TARI grant contract HPRI-CT-2001-00149.

APPENDIX

SPECTRAL ENERGY DENSITY AND SCALE DEPENDENCE OF MAGNETIC FIELDS

We derive here $\langle B^2 \rangle \propto k^{-4}$ from the spectral energy density of a turbulent plasma $w(k) \propto k^{-m}$. The turbulence is assumed to be described as an ensemble of hydromagnetic waves with wavenumbers $k$ and with vanishing mean electric field. The mean magnetic

Fig. 8.—Comparison of the analytic diffusive spectrum with the Monte Carlo simulation by Yoshiguchi et al. (2003). The spectra are given in the case of Kolmogorov diffusion with $(B_0, l_d) = (100 \text{ nG}, 1 \text{ Mpc})$ and for a single source placed at distance 10 (left) and 30 Mpc (right).
field on each scale $\lambda = 2\pi/k$ is $\langle B \rangle_\lambda = B_\lambda$; i.e., on the scale $\lambda$ the magnetic field is locally homogeneous. For the Alfven waves the kinetic energy of turbulent fluid is equal to magnetic energy (Landau & Lifshitz 1987).

Let us write down the Fourier expansion for the wave magnetic field. In the limit $u_A/c \ll 1$, where $u_A$ is the Alfven velocity, $\omega/ku_A \ll 1$ and the magnetic fields can be considered as quasi-static. Here and below we omit the coefficients $(2\pi)^3$, which are inessential for our discussion:

$$B(r, t) = \int d^3k B(k, t)e^{ikr},$$

$$B^*(r, t) = \int d^3k' B^*(k', t)e^{-ikr}.$$  \hspace{1cm} (A1)

The energy of the magnetic field in the normalizing volume $V$ is

$$W = \frac{1}{8\pi} \int d^3r B(r, t)B^*(r, t).$$ \hspace{1cm} (A2)

Putting equation (A1) into equation (A2) and using the definition of the $\delta$-function

$$\int d^3r e^{i(k-k')r} = \delta^3(k-k'),$$ \hspace{1cm} (A3)

we obtain

$$W = \frac{1}{8\pi} \int d^3k |B(k, t)|^2.$$ \hspace{1cm} (A4)

Assuming $B(k, t) = B(k, t)$, we find the energy density of waves and fluid as

$$w = \frac{1}{2V} \int dk k^2 B^2(k).$$ \hspace{1cm} (A5)

The spectral energy density is then

$$w(k) = \frac{1}{2V} k^2 B^2(k),$$ \hspace{1cm} (A6)

and hence for the Fourier component $B^2(k) \propto k^{-s_F}, s_F = m + 2$. Note that the Fourier component $B(k)$ does not have the meaning of the magnetic field strength over the scale $k$; moreover, the dimension of $B(k)$ is different from a magnetic field. The connection between $B_\lambda$, the average field on the scale $\lambda$, and the Fourier component $B(k)$ can be readily found from the relation

$$w = \frac{1}{8\pi} B^2_\lambda = \frac{1}{2V} \int dk k^2 B^2(k),$$

where the integral is to be evaluated over region $\sim \lambda$. It results in

$$B^2_\lambda \sim \frac{4\pi}{V} k^3 B^2(k) = 8\pi kw(k).$$ \hspace{1cm} (A7)

Using the definition $B^2_\lambda \propto k^{-s_I}$, we obtain for the physical field $B_\lambda, s_I = m - 1$, and $s_F - s_I = 3$.

In particular, for the Kolmogorov spectrum we have $s_F = 11/3$ and $s_I = 5/3$, and thus for deflection of UHE particles one should use the wavenumber spectrum of magnetic fields in the form

$$B^2_\lambda \propto k^{-2/3}.$$ \hspace{1cm} (A8)

REFERENCES

Arzimovich, L. A., & Sagdeev, R. Z. 1979, Physics of Plasma for Physicists
(Moscow: Atomizdat)

Berezinsky, V. S., Buleanov, S. V., Dogiel, V. A., Ginzburg, V. L., & Ptuskin, V. S. 1990a, Astrophysics of Cosmic Rays (Amsterdam: North-Holland)

Berezinsky, V. S., Dogiel, V. A., & Grigorieva, S. I. 1990b, A&A, 232, 582

Berezinsky, V. S., Gazizov, A. Z., & Grigorieva, S. I. 2002a, preprint (astro-ph/0210095)

Berezinsky, V. S., Gazizov, A. Z., & Grigorieva, S. I. 2002b, preprint (hep-ph/0204357)

Berezinsky, V. S., & Grigorieva, S. I. 1988, A&A, 199, 1

Blasi, P., Burles, S., & Olinto, A. V. 1999, ApJ, 514, L79

Blasi, P., & Olinto, A. V. 1999, Phys. Rev. D, 59, 023001

Carilli, C. L., & Taylor, G. B. 2002, ARA&A, 40, 319

Casse, F., Lemoine, M., & Pelletier, G. 2002, Phys. Rev. D, 65, 023002

Davé, R., et al. 2001, ApJ, 547, 574

Deligny, O., Letessier-Selvon, A., & Parizot, E. 2003, preprint (astro-ph/0303624)

Dolag, K., Grasso, D., Springel, V., & Tkachev, I. 2003, preprint (astro-ph/0310902)

Giller, M., Wdowczyk, J., & Wolfendale, A. W. 1980, J. Phys. G, 6, 1561

Glushkov, A. V., & Pravdin, M. I. 2001, Astron. Lett., 27, 493
Harari, D., Mollerach, S., & Roulet, E. 2002, JHEP, 0207, 006
Hayashida, N., et al. 1996, Phys. Rev. Lett., 77, 1000 (AGASA Collaboration)
Isola, C., Lemoine, M., & Sigl, G. 2002, Phys. Rev. D, 65, 023004
Kolmogorov, A. N. 1941, Dokl. Akad. Nauk. USSR, 30, 299
Kraichnan, R. H. 1965, Phys. Fluids, 8, 1385
Kronberg, P. P. 1994, Rep. Prog. Phys., 57, 325
Landau, L. D., & Lifshitz, E. M. 1987, Electrodynamics of Continuous Media
   (New York: Pergamon)
Lemoine, M., Sigl, G., & Biermann, P. 1999, preprint (astro-ph/9903124)
Lifshitz, E. M., & Pitaevskii, L. P. 2001, Physical Kinetics (New York: Pergamon)
Ryu, D., Kang, H., & Biermann, P. 1998, A&A, 335, 19
Sigl, G., Lemoine, M., & Biermann, P. 1999, Astropart. Phys., 10, 141
Sigl, G., Miniati, F., & Ensslin, T. A. 2003, Phys. Rev. D, 68, 043002
   ———. 2004, preprint (astro-ph/0401084)
Spergel, D. N., et al. 2003, ApJS, 148, 175 (WMAP Collaboration)
Stanev, T., et al. 2000, Phys. Rev. D, 62, 093005
Syrovatskii, S. I. 1959, Soviet Astron., 3, 22
Takeda, M., et al. 1999, ApJ, 522, 225 (AGASA Collaboration)
Tinyakov, P. G., & Tkachev, I. I. 2001, Soviet Phys.—JETP Lett., 74, 445
Uchii, Y., et al. 2000, Astropart. Phys., 13, 151
Vainstein, S. I., Bykov, A. M., & Toptygin, I. N. 1989, Turbulence, Stream Layers and Shock Waves (Moscow: Nauka)
Vallee, J. P. 1997, Fundam. Cosmic Phys., 19, 1
Wdowczyk, J., & Wolfendale, A. W. 1979, Nature, 281, 356
Yoshiguchi, H., Nagataki, S., Tsubaki, S., & Sato, K. 2003, ApJ, 586, 1211