Matrix factorizations and link homology II

Mikhail Khovanov and Lev Rozansky

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Abstract

To a presentation of an oriented link as the closure of a braid we assign a complex of bigraded vector spaces. The Euler characteristic of this complex (and of its triply-graded cohomology groups) is the HOMFLYPT polynomial of the link. We show that the dimension of each cohomology group is a link invariant.

1 Matrix factorizations with a parameter

In the paper [KR] we constructed, for each $n > 0$, a bigraded cohomology theory of links in $\mathbb{R}^3$ whose Euler characteristic is a certain one-variable specialization $(x^n, q)$ of the HOMFLYPT polynomial [HOMFLY], [PT]. The $n = 0$ specialization is the Alexander polynomial, equal to the Euler characteristic of the knot homology theory discovered by Ozsváth, Rasmussen and Szabó [OS], [R1]. The approach in [KR] fails for $n = 0$, assigning trivial groups to any link.

In this sequel to [KR] we assume that the reader is familiar with that paper. Recall that our construction of link cohomology was based on matrix factorizations with potentials being sums and differences of $x^{n+1}$, for various $x$. When $n = 0$, the category of matrix factorizations (up to chain homotopies) with the potential $\sum \pm x_i$ is trivial. Looking for a remedy, let us add a formal variable $a$ and change the potential from $x$ to $ax$.

Take an oriented arc $c$ as in figure 1, label its ends $x_1$ and $x_2$, and assign the potential $ax_1 - ax_2$ to the arc. Let $R = \mathbb{Q}[a, x_1, x_2]$ and define $C_c$ as the factorization

$$R \xrightarrow{a} R \xrightarrow{x_1 - x_2} R.$$ 

We have $d^2 = ax_1 - ax_2$ and view $C_c$ as an object of the homotopy category of matrix factorizations with the potential $a(x_1 - x_2)$. 
Figure 1: An arc

Figure 2: Wide edge $t$

Make $R$ bigraded by setting

$$\deg(a) = (2, 0), \quad \deg(x_i) = (0, 2). \tag{1}$$

This implies $\deg(d^2) = (2, 2)$ and we select the bigrading of the middle $R$ in the factorization so that $\deg(d) = (1, 1)$:

$$R \xrightarrow{a} R\{-1, 1\} \xrightarrow{x_1-x_2} R, \tag{2}$$

where the bidegree shift by $(n_1, n_2)$ is denoted $\{n_1, n_2\}$.

Next, given a wide edge $t$ as in figure 2 assign variables $x_1, x_2, x_3, x_4$ to the edges next to it. We can write

$$ax_1 + ax_2 - ax_3 - ax_4 = a(x_1 + x_2 - x_3 - x_4) + 0(x_1x_2 - x_3x_4).$$

Define $C_t$ as the tensor product (over $R$) of factorizations

$$R \xrightarrow{a} R\{-1, 1\} \xrightarrow{x_1+x_2-x_3-x_4} R \tag{3}$$

and

$$R \xrightarrow{0} R\{-1, 3\} \xrightarrow{x_1x_2-x_3x_4} R \tag{4}$$

where $R = \mathbb{Q}[a, x_1, x_2, x_3, x_4]$.

Throughout the paper we work with matrix factorizations with potentials $w = a \sum_i \epsilon_i x_i$ where $i$ ranges over some finite set $I$ of integers and $\epsilon_i \in \{1, -1\}$ are "orientations" of $x_i$. The category $mf_w$ has objects $(M, d)$ where $M = \mathbb{Q}[a, x_1, x_2, x_3, x_4]$. 

Figure 3: Lattice of factorization $M$

$M^0 \oplus M^1$ and $M^0, M^1$ are free bigraded $R$-modules (possibly of infinite rank), while $d$ is a generalized differential

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} M^0$$

of bidegree $(1, 1)$ and subject to $d^2 = w$. Here $R$ is the ring of polynomials in $a$ and $x_i$’s with rational coefficients. The bidegrees are given by formula (1). Morphisms in $mf_w$ are bidegree-preserving maps of $R$-modules $M^0 \to N^0$, $M^1 \to N^1$ that commute with $d$.

We found it useful to visualize a matrix factorization as above by decomposing

$$M^0 = \bigoplus_{k,l} M^0_{k,l}, \quad M^1 = \bigoplus_{k,l} M^1_{k,l},$$

as direct sums of vector spaces, one for each bidegree $(k, l)$, and placing them in the nodes of a coordinate lattice, see figure 3. Diagonal arrows denote the differential, horizontal arrows show multiplication by $a$ and vertical arrows—multiplications by $x_i$.

The category $hm f_w$ of matrix factorizations up to chain homotopies has the same objects as $mf_w$ and the $\mathbb{Q}$-vector space of morphisms from $M$ to $N$ is the quotient of the space of morphisms in $mf_w$ by null-homotopic morphisms (the homotopy maps must have bidegree $(-1, -1)$).

If the index set $I$ is empty, then $R = \mathbb{Q}[a]$ and $mf_w$ is equivalent to the category of complexes of free graded $\mathbb{Q}[a]$-modules.
In general, given a planar marked graph $\Gamma$ (as described in [KR, Introduction]), possibly with boundary points, we assign to it a matrix factorization $C(\Gamma)$ which is the tensor product of $C_c$, over all arcs $c$ in $\Gamma$, and $C_t$, over all wide edges $t$ in $\Gamma$. For instance, for the graph in figure 4,

$$C(\Gamma) = C_{t_1} \otimes C_{t_2} \otimes C_{c_1} \otimes C_{c_2}$$

where $c_1, c_2$ are the arcs of $\Gamma$ with endpoints labelled $x_3, x_5$ and $x_7, x_6$, respectively. The tensor product is taken over suitable polynomial rings $\mathbb{Q}[a, x_i]$ so that $C(\Gamma)$ is a finite rank free $\mathbb{Q}[a, x_1, \ldots, x_9]$-module. The potential

$$w = a(x_1 + x_2 - x_7 - x_4 - x_8 - x_9)$$

and we view $C(\Gamma)$ as an object of $mf_w$ (or $hmf_w$) with the ground ring $R$ the polynomial ring $\mathbb{Q}[a, x_1, x_2, x_4, x_7, x_8, x_9]$ in $a$ and external (or boundary) variables. The other variables $x_3, x_5, x_6$ are "internal". Notice that $C(\Gamma)$ has infinite rank as an $R$-module.

When $\Gamma$ has no boundary points, $w = 0$ and $C(\Gamma)$ becomes a 2-periodic complex

$$C^0(\Gamma) \xrightarrow{d} C^1(\Gamma) \xrightarrow{d} C^0(\Gamma)$$

of bigraded $\mathbb{Q}[a]$-modules. Its cohomology, denoted $H(\Gamma)$, is a bigraded $\mathbb{Q}[a]$-module.

If $\Gamma$ is a single circle with one mark (glue together the endpoints of the arc in the figure and place a mark there), the complex is

$$\mathbb{Q}[a, x_1] \xrightarrow{a} \mathbb{Q}[a, x_1] \{-1, 1\} \xrightarrow{0} \mathbb{Q}[a, x_1]$$
(since now $x_2 = x_1$), and $H(\Gamma) \cong \mathbb{Q}[x_1]\{-1, 1\}$.

Consider the diagrams $\Gamma^0, \Gamma^1$ in figure 5. Factorization $C(\Gamma^0)$ is the tensor product of 

$$R \xrightarrow{a} R\{-1, 1\} \xrightarrow{x_1-x_4} R$$

and 

$$R \xrightarrow{a} R\{-1, 1\} \xrightarrow{x_2-x_3} R,$$

where $R = \mathbb{Q}[a, x_1, x_2, x_3, x_4]$. In the product basis, $C(\Gamma^0)$ has the form 

$$R \oplus R\{-1, 1\} \xrightarrow{P_0} R \oplus R\{-1, 1\} \xrightarrow{P_1} R \oplus R\{-2, 2\}$$

with 

$$P_0 = \begin{pmatrix} a & x_3 - x_2 \\ a & x_1 - x_4 \end{pmatrix}, \quad P_1 = \begin{pmatrix} x_1 - x_4 & x_2 - x_3 \\ -a & a \end{pmatrix}.$$

Likewise, $C(\Gamma^1)$ has the presentation 

$$R \oplus R\{-1, 1\} \xrightarrow{Q_0} R \oplus R\{-1, 3\} \xrightarrow{Q_1} R \oplus R\{-2, 4\}$$

with 

$$Q_0 = \begin{pmatrix} a & x_3x_4 - x_1x_2 \\ 0 & x_1 + x_2 - x_3 - x_4 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1x_2 - x_3x_4 \\ 0 & a \end{pmatrix}.$$

A map between $C(\Gamma^0)$ and $C(\Gamma^1)$ can be described by a pair of $2 \times 2$ matrices with coefficients in $R$ that specify the images of the basis vectors of $C^i(\Gamma^0)$ in $C^i(\Gamma^1)$ for $i = 0, 1$. 

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Figure 6: Complex assigned to a crossing

Let $\chi_0 : C(\Gamma^0) \to C(\Gamma^1)$ be given by the pair of matrices

$$U_0^0 = \begin{pmatrix} x_4 - x_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_0^1 = \begin{pmatrix} x_4 & -x_2 \\ -1 & 1 \end{pmatrix}. \quad (5)$$

Our bases in $C(\Gamma^0)$ and $C(\Gamma^1)$ are homogeneous with respect to the bigrading of $R$. It’s easy to see that $\chi_0$ is a homogeneous map of bidegree $(0, 2)$.

Next, define $\chi_1 : C(\Gamma^1) \to C(\Gamma^0)$ by the pair of matrices

$$U_1^0 = \begin{pmatrix} 1 & 0 \\ 0 & x_4 - x_2 \end{pmatrix}, \quad U_1^1 = \begin{pmatrix} 1 & x_2 \\ 1 & x_4 \end{pmatrix}. \quad (6)$$

The map $\chi_1$ is bidegree-preserving.

Given a plane diagram $D$ of a tangle, place at least one mark on each internal edge of the diagram (an edge disjoint from the boundary of $D$), and label the marks and boundary points by $x_1, \ldots, x_m$. To each crossing $p$ of the diagram assign the complex $C_p$ of matrix factorizations as follows. Up to shifts, the complex is the cone of the map $\chi_0$ or $\chi_1$, depending on whether the crossing is positive or negative. The shifts are explained in figure 6.

Thus, if the crossing is positive,

$$C_p = \begin{pmatrix} x_0 \end{pmatrix} : 0 \to C(\Gamma^0) \to C(\Gamma^1) \to 0,

with $C(\Gamma^1)$ positioned in cohomological degree 0. The shift $\{0, 2\}$ makes the differential preserve the bidegree. If the crossing is negative,

$$C_p = \begin{pmatrix} x_0 \end{pmatrix} : 0 \to C(\Gamma^0) \to C(\Gamma^1) \to 0,

with $C(\Gamma^0)$ in cohomological degree 0. The overall bigrading shift by $\{0, -2\}$ is here for the normalization of the Reidemeister move IIa (see later).
Define $C(D)$ as the tensor product of $C_p$, over all crossings $p$ of $D$, and $C_c$, over all arcs $c$. It's a complex built out of matrix factorizations $C(\Gamma)$, over all resolutions $\Gamma$ of $D$. The differential $\partial$ preserves the bigrading of each term $C^j(D)$. We view $C(D)$ as an object of the category $K(hmf_w)$. The latter is the category whose objects are complexes of objects in $hmf_w$ and whose morphisms are homomorphism of complexes modulo null-homotopic morphisms.

Now we specialize to the case when $D$ is a link diagram (has empty boundary). Then each term $C^j(D)$ in the complex $C(D)$ is an object of the homotopy category of bigraded free $\mathbb{Q}[a]$-modules. We’ll see that $C^j(D)$, for any diagram $D$, decomposes as a direct sum of contractible pieces

$$0 \to \mathbb{Q}[a] \to \mathbb{Q}[a] \to 0$$

and the cohomology $H(C^j(D))$, which we denote $CH^j(D)$. Moreover, $a$ acts trivially on $CH^j(D)$, so we can ignore the $\mathbb{Q}[a]$-module structure and think of it as a bigraded $\mathbb{Q}$-vector space,

$$CH^j(D) = \oplus_{k,l} CH^j_{k,l}(D).$$

The bigrading descends from the bigrading on matrix factorizations $C(\Gamma)$.

Thus, to $D$ we assign the complex $CH(D)$ of bigraded $\mathbb{Q}$-vector spaces

$$\cdots \xrightarrow{\partial} CH^j(D) \xrightarrow{\partial} CH^{j+1}(D) \xrightarrow{\partial} \cdots$$

As a $\mathbb{Q}$-vector space, $CH(D)$ is the direct sum of cohomology groups $H(\Gamma)$ of complexes $C(\Gamma)$, over all resolutions of $D$.

The cohomology $H(D) = H(CH(D), \partial)$ of the above complex is triply-graded,

$$H(D) = \oplus_{j,k,l} H^j_{k,l}(D).$$

Of course, for the whole construction to be interesting, $H(D)$ should not depend on the choice of $D$, given $L$.

**Proposition 1** Let $D$ be a marked tangle diagram. Then $C(D)$, as an object of $K(hmf_w)$, does not depend on the number of markings on each edge of $D$.

Proposition 1 is proved in the next section.

Next we run into an obstacle: things seem to work well only if we restrict to diagrams $D$ that come from braids. Let’s say that $D$ is a braid diagram if
Figure 7: A braid diagram

Figure 8: Reidemeister moves

$D$ depicts the link $L$ as the closure of a clockwise oriented braid, see figure 7. We denote the braid by $D$ as well.

To justify the introduction of braid diagrams, partition the Reidemeister moves of links into types I, II, III in the usual way and then separate II into two subtypes, IIa and IIb, depending on orientations, see figure 8. We only consider the type III move with the orientations pointing in the same direction.

**Proposition 2** If diagrams $D_1, D_2$ are related by a Reidemeister move of type I, IIa or III, the complexes $C(D_1)$ and $C(D_2)$ are isomorphic as objects of the category $K(hm_{f_w})$, up to an overall shift in the triple grading.

For a proof see Section 2.
It seems likely to us that $C(D_1)$ and $C(D_2)$ are not isomorphic (up to a shift) if $D_1, D_2$ are the two tangles in the IIb move in figure $\mathbf{K}$ although we did not prove this. To avoid the IIb move, we restrict our consideration to braid diagrams. Closures of braid diagrams $D_1, D_2$ are isotopic as oriented links iff $D_1, D_2$ are related by a chain of Markov moves. A Markov move is one of the following:

(a) conjugation $DD' \leftrightarrow D'D$,

(b) transformations in the braid group:

\[
\begin{align*}
D\sigma_j\sigma_i & \leftrightarrow D\sigma_i\sigma_j \text{ if } |i-j| > 1, \\
D & \leftrightarrow D\sigma_i\sigma_i^{-1}, \\
D & \leftrightarrow D\sigma_i^{-1}\sigma_i, \\
D\sigma_i\sigma_{i+1}\sigma_i & \leftrightarrow D\sigma_{i+1}\sigma_i\sigma_{i+1}.
\end{align*}
\]

(c) transformations $D \leftrightarrow D\sigma_n^{\pm 1}$, for a braid $D$ with $n$ strands.

Notice that we never see the Reidemeister move IIb when dealing with braid diagrams. The propositions stated earlier imply:

**Theorem 1** Given two braid diagrams $D_1, D_2$ of an oriented link $L$, the cohomology groups $H(D_1)$ and $H(D_2)$ are isomorphic as triply-graded vector spaces, up to an overall shift in the grading.

To describe the Euler characteristic of $H(D)$, we consider the function $F$ from braid diagrams to the ring of rational functions in $q$ and $t$ that is uniquely determined by the following properties:

- $F(D_1) = F(D_2)$ if $D_1, D_2$ are conjugate braid presentations (see Markov move (a) above),
- $F(D_1) = F(D_2)$ if $D_1, D_2$ are related by a braid presentation move (Markov moves (b) above),
- $F(D\sigma_n) = F(D)$, for a braid $D$ with $n$ strands,
- $F(D\sigma_n^{-1}) = -t^{-1}q^{-1}F(D)$, for a braid $D$ with $n$ strands,
- For any braid diagram $D$ there is a skein relation

\[
q^{-1}F(D\sigma_i) - qF(D\sigma_i^{-1}) = (q - q^{-1})F(D),
\]

- If $D$ is the one-strand diagram of the unknot, $F(D) = \frac{t^{-1}}{q^{1-q}}$. 

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To see that $F$ is simply a version of the HOMFLYPT polynomial, let $\alpha = -t^{-1}q^{-1}$ and consider

$$\tilde{F}(D) = \sqrt{\alpha}^{\vert D \vert_+ - \vert D \vert_- - s(D) + 1} F(D)$$

where $\vert D \vert_+$, respectively $\vert D \vert_-$, is the number of positive, respectively negative, crossings of $D$, while $s(D)$ is the number of strands of $D$. Our conventions are explained in figure 9. Then $\tilde{F}(D)$ is invariant under all Markov moves of braids and satisfies the HOMFLYPT skein relation

$$q\sqrt{\alpha} \tilde{F}(D\sigma_i^{-1}) - (q\sqrt{\alpha})^{-1} \tilde{F}(D\sigma_i) = (q - q^{-1})\tilde{F}(D).$$

Thus, $\tilde{F}(D)$ equals the HOMFLYPT polynomial of the link $L$, normalized so that

$$\tilde{F}(\text{unknot}) = \frac{\alpha}{1 - q^{-2}}.$$

**Theorem 2** For any braid diagram $D$ the Euler characteristic $\langle D \rangle$ of $H(D)$ equals $F(D)$.

The Euler characteristic

$$\langle D \rangle \overset{\text{def}}{=} \sum_{j,k,l} (-1)^j t^k q^l \dim_{\mathbb{Q}} H_{k,l}^j(D)$$

is a power series in $q$ with coefficients in $\mathbb{Z}[t, t^{-1}]$. The theorem claims that each vector space $H_{k,l}^j(D)$ is finite dimensional and the sum above is a rational function of $t$ and $q$ equal to $F(D)$. See the end of Section 2 for a proof.

Specializing to $q\sqrt{\alpha} = 1$ in the equation (7) nets us the Alexander polynomial. In terms of $t$ and $q$, we are imposing the relation $t = -q$. Homologically, $t$, $q$ and the minus sign correspond to the three grading directions. Hence, suitably collapsing the tri-grading to a bigrading we get a categorification of the Alexander polynomial.
Sergei Gukov, Albert Schwarz and Cumrun Vafa recently conjectured \cite{GSV} that there exist integer-valued link invariants $D_{Q,s,r}$ depending on three integer parameters $Q, s, r$, that can be used to determine ranks of $\text{sl}(n)$ link homology groups as well as the coefficients of the HOMFLYPT polynomial of a link. These invariants should come from the physical theory of the BPS states and should be related to ranks of cohomology groups of suitable moduli spaces. It would be interesting to try relating $D_{Q,s,r}$ to the dimensions of cohomology groups $H^j_{k,l}$. Our normalization of the HOMFLYPT polynomial is similar to the one in \cite{GSV}, both having $q - q^{-1}$ as the denominator of the unknot invariant.

On the other hand, it’s been independently suggested by several people, including Oleg Viro \cite{V}, that there should exist a triply-graded link homology theory with the HOMFLYPT polynomial as the Euler characteristic. The current paper resulted from our search for such a theory and for a combinatorial categorification of the Alexander polynomial.

Triply-graded cohomology theories had previously appeared in the work of Asaeda, Przytycki and Sikora \cite{APS} on categorification of invariants of links in I-bundles over surfaces, and in Audoux and Fiedler \cite{AF}, who introduced a refined Jones polynomial and its categorification, which are only invariant under braid-like isotopies. Restriction to braid-like isotopies appears in our construction as well, but we don’t know how our invariant relates to those of \cite{APS} and \cite{AF}.

To conclude this section, we mention several modifications, potential generalizations and illnesses of the homology theory $H$.

- It’s not natural that we have to restrict to braid diagrams to get a link invariant. In another sign of disfunctionality, the theory does not extend to all cobordisms. For instance, the cohomology groups of the unknot do not have a Frobenius algebra structure over the cohomology ring of the empty link (it’s convenient to define the latter ring to be $\mathbb{Q}[a]$), preventing us from extending the theory even to unknotted cobordisms between unlinks.

- Any field $k$ can be used instead of $\mathbb{Q}$. More generally, we can work over $\mathbb{Z}$, so that the invariant of a closed planar graph $\Gamma$ is a complex of graded free abelian groups, up to chain homotopy equivalence, and the invariant of a link is a complex of complexes as above, up to chain homotopy equivalence. Taking the homology $H(\Gamma, \mathbb{Z})$ of each resolution of $D$ and forming a complex out of them produces a complex $CH(D, \mathbb{Z})$ from a diagram $D$. We then specialize to braid diagrams and take the cohomology of $CH(D, \mathbb{Z})$. The resulting groups $H(D, \mathbb{Z})$ are triply-
graded and, up to isomorphism, do not depend on the choice of braid diagram $D$, given $L$.

- In Section 2 we rewrite the factorizations $C(\Gamma^0), C(\Gamma^1)$ and the maps $\chi_0, \chi_1$ in the form that depends only on $a$ and the differences $x_i - x_j$ of the variables $x_i$. This allows us to pass from the ring $R = \mathbb{Q}[a, x_1, \ldots, x_m]$ to the smaller ring $\overline{R} = \mathbb{Q}[a, x_2 - x_1, \ldots, x_m - x_1]$. The definition of cohomology and the proof of its invariance work over $\overline{R}$ as well, leading to reduced cohomology groups $H(D)$, with the property $H(D) = H(D) \otimes_\mathbb{Q} \mathbb{Q}[x]$. In the reduced theory the unknot has one-dimensional cohomology groups.

- $\text{sl}(n)$ link homology theory (see [KR]) utilized the potential $x^{n+1}$. Soon afterwards Gornik [G] studied a deformation of that theory with the potential $x^{n+1} - (n+1)\beta^n x$. In the $n = 2$ case the deformation was found earlier by Lee [L] and used by Rasmussen in his combinatorial proof of the Milnor conjecture [R2]. The definitions in [KR] can be generalized to the potential $x^{n+1} + a_n x^n + \cdots + a_1 x$ where $a_1, \ldots, a_n$ are formal variables. We hope that this generalization will be invariant under the Reidemeister moves and will turn out to be the "$\text{sl}(n)$-equivariant" version of $\text{sl}(n)$ link homology. The invariant of the empty link should be the ring of polynomials in $a_1, \ldots, a_n$, and naturally isomorphic to the $U(n)$-equivariant cohomology ring of the point. The invariant of the unknot should be the quotient of the polynomial ring $\mathbb{Q}[x, a_1, \ldots, a_n]$ by the relation $x^{n+1} + a_n x^n + \cdots + a_1 x = 0$, isomorphic to the $U(n)$-equivariant cohomology ring of $\mathbb{C}P^n$. A certain version of Bar-Natan link homology [BN], [K] should correspond to the potential $x^3 + ax$.

- For a common generalization of the $U(n)$-equivariant link homology and the theory described here one could try the potential $a_{n+1} x^{n+1} + a_n x^n + \cdots + a_1 x$ with all $a$'s being formal variables. Factorizations $C(\Gamma^0), C(\Gamma^1)$, the maps $\chi_0, \chi_1$ and the complex $C(D)$ can be defined for this potential as well, but we don’t know whether this theory will be invariant under the Reidemeister-Markov moves of braid diagrams.

2 Proofs

1. Product factorizations, graph homology and Koszul complexes.

Given a polynomial ring $R$ and a pair of elements $a_1, b_1 \in R$, we denote by $(a_1, b_1)$ the factorization

$$R \xrightarrow{a_1} R \xrightarrow{b_1} R.$$
Given a finite set of such pairs \((a_i, b_i), 1 \leq i \leq n\), we denote by \((a, b)\) their tensor product (over \(R\)):

\[
(a, b) \overset{\text{def}}{=} \bigotimes_i (a_i, b_i), \quad a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n).
\]

We will also write \((a, b)\) in the column form

\[
\begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2 \\
\vdots & \vdots \\
a_n & b_n
\end{pmatrix}
\]

and call it a Koszul factorization. By an elementary transformation of rows \(i\) and \(j\) we mean a modification

\[
\begin{pmatrix}
a_i & b_i \\
a_j & b_j
\end{pmatrix} \overset{[ij]_\lambda}{\longrightarrow} \begin{pmatrix}
a_i & b_i + \lambda b_j \\
a_j & a_j - \lambda a_i
\end{pmatrix}
\]

for some \(\lambda \in R\). We denote it by \([ij]_\lambda\). All other rows of \((a, b)\) are left unchanged. An elementary transformation takes a Koszul factorization \((a, b)\) to an isomorphic factorization, since we're only changing a basis vector in the free \(R\)-module underlying the factorization \((a, b)\).

Suppose now that \(y\) is one of the generators of the polynomial ring \(R\), so we can write \(R = R'[y]\), and that the potential \(w = \sum a_i b_i\) lies in \(R'\) (in this situation we say that \(y\) is an \textit{internal} variable). Then any factorization \(M\) over \(R\) restricts to (an infinite rank) factorization over \(R'\), which we denote \(M'\). Assume furthermore that one of the rows in \((a, b)\) has the form \((0, y - \mu)\) where \(\mu \in R'\). Denote by \((a', b')\) the factorization over \(R'\) obtained from \((a, b)\) by removing the row \((0, y - \mu)\) and substituting \(\mu\) for \(y\) everywhere in all other rows.

\textbf{Proposition 3} Factorizations \((a', b')\) and \((a, b)'\) are chain homotopy equivalent.

\textit{Proof:} By changing a variable \(y \rightarrow y - \mu\) we reduce to the case \(\mu = 0\). We can write \(a_i = a_i' + ya_i''\) and \(b_i = b_i' + yb_i''\) where \(a_i', b_i' \in R'\). Applying elementary transformations to rows \((0, y)\) and \((a_i, b_i)\) we reduce the latter to \((a_i', b_i')\), while \((0, y)\) is transformed into \((\sum a_i b_i'', y)\). Next, change \((\sum a_i b_i'', y)\) into \((y, \sum a_i b_i'')\) (this shifts factorization \(M\) to \(M(1)\)) and apply elementary transformations to rows \((a_i, b_i')\) and \((y, \sum a_i b_i'')\). The row \((a_i, b_i')\) becomes \((a_i', b_i')\), while the row with \(y\) turns into \((y, \sum (a_i b_i'' + a_i'' b_i'))\). Since the potential does not depend on \(y\), the latter sum is zero. Now shift \((y, 0)\) back to \((0, y)\).
The result is the Koszul factorization, isomorphic to \((a, b)\), with rows \((a'_i, b'_i)\) and \((0, y)\). This factorization is the tensor product of \((a', b')\), as defined above, and \((0, y)\). Therefore, \((a, b)\) is isomorphic, as an \(R\)-factorization, to the total factorization of the bifactorization

\[
(a', b') \otimes_R R'[y] \xrightarrow{0} (a', b') \otimes_R R'[y] \xrightarrow{y} (a', b') \otimes_R R'[y].
\]

As a factorization over the smaller ring \(R'\), it decomposes into a direct sum of contractible factorizations which are the total factorizations of

\[
(a', b') \otimes y^{j+1} \xrightarrow{0} (a', b') \otimes y^j \xrightarrow{y} (a', b') \otimes y^{j+1},
\]

for \(j \geq 0\), and the factorization \((a', b')\). Proposition follows. □

Remark: The second half of the above proof just says that the complex of \(R'\)-modules

\[
0 \rightarrow R'[y] \xrightarrow{y} R'[y] \rightarrow 0
\]

is the direct sum of contractible complexes

\[
0 \rightarrow R'y^j \xrightarrow{y} R'y^{j+1} \rightarrow 0
\]

and the complex \(0 \rightarrow R' \rightarrow 0\).

Suppose we are given a planar marked graph \(\Gamma\), possibly with boundary. To \(\Gamma\) we assigned a Koszul factorization \(C(\Gamma)\) which has a rather special form. Each arc in \(\Gamma\) contributes the row \((a, x_i - x_j)\) to the Koszul matrix of \(C(\Gamma)\), where \(x_i\) and \(x_j\) are the labels at the endpoints of the arc. Each wide edge in \(\Gamma\) contributes two rows

\[
\left(\begin{array}{cc}
  a & x_i + x_j - x_k - x_l \\
  0 & x_i x_j - x_k x_l
\end{array}\right)
\]

to the Koszul matrix, where \(x_i, x_j, x_k, x_l\) are the labels bounding the edge. If \(\Gamma\) has \(m_1\) arcs and \(m_2\) wide edges, the Koszul matrix of \(C(\Gamma)\) will have \(n = m_1 + 2m_2\) rows. Permute these rows so that the first \(m_1 + m_2\) rows have the form \((a, z)\) where \(z\)‘s are some linear functions of \(x_i\)’s. We call these rows linear rows. The last \(m_2\) rows have the form \((0, x_i x_j - x_k x_l)\) for various quadruples of indices \((i, j, k, l)\). Call these quadratic rows.

Apply elementary transformations with \(\lambda = 1\) to the first row paired with every other linear row. In other words, we convert \(b_i\) to \(b_1 + b_2 + \cdots + b_{m_1+m_2}\) and subtract \(a_1 = a\) from \(a_p = a\) for \(p = 2, 3, \ldots, m_1 + m_2\). The Koszul matrix transforms into a matrix with the first row \((a, \sum \epsilon_i x_i)\) where the sum is over all boundary points of \(\Gamma\) and \(\epsilon_i = \pm 1\) depending on the orientation of \(\Gamma\) at that point. All other linear rows acquire the form \((0, z)\), with the same
linear functions $z$ as before. Nothing happens to the quadratic rows. The Koszul matrix now has the form

$$
\begin{pmatrix}
a & b_1 \\
0 & b_2 \\
\vdots & \vdots \\
0 & b_n
\end{pmatrix}
$$

with $b_1 = \sum \epsilon_i x_i$. After this change of basis, every row but the first one has the first term zero. Hence, it comes from a one-term Koszul complex

$$0 \longrightarrow R \xrightarrow{b^0} R \longrightarrow 0$$

by collapsing cohomological grading from $\mathbb{Z}$ to $\mathbb{Z}_2$. Likewise, the tensor product of all rows save the first is a factorization obtained from the Koszul complex of the sequence $(b_2, b_3, \ldots, b_n)$ by collapsing the grading.

Note that our polynomial ring is, in addition, bigraded. Taking all gradings into account, the collapse is from a triple grading to a bigrading (see figure 3). No cyclic components appear in the collapsed grading since the differential has nonzero bidegree $(1, 1)$. Finally, observe that in the new Koszul matrix parameter $a$ appears only once, in the first row.

Next consider the case when $\Gamma$ is closed (has no boundary points). The first row becomes $(a \ 0)$ and the whole factorization comes from the Koszul complex of the sequence $(a, b_2, \ldots, b_n)$ by collapsing its grading. Moreover, $a$ plays a purely decorative role, and, using proposition 3, we can throw out this row simultaneously with removing $a$ from the list of variables, which would then have only $x_i$’s. In other words, the cohomology $H(\Gamma)$ of the factorization $C(\Gamma)$ is isomorphic to the cohomology of the Koszul complex of the sequence $(b_2, b_3, \ldots, b_n)$, with the trigrading collapsed to a bigrading.

Thus, although the 2-periodic complex $C(\Gamma)$ as well as its cohomology $H(\Gamma)$ are $\mathbb{Q}[a]$-modules, $a$ acts trivially on $H(\Gamma)$.

2. Maps $\chi_0, \chi_1$ revisited.

Recall the row operation $[ij]_\lambda$ on a Koszul matrix of a factorization:

$$
\begin{pmatrix}
a_i & b_i \\
a_j & b_j
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
a_i & \lambda b_j \\
a_j - \lambda a_i & b_j
\end{pmatrix}
$$

Denote by $|0\rangle$ and $|1\rangle$ the standard basis vectors in factorizations $(a_i, b_i)$ and $(a_j, b_j)$:

$$
R|0\rangle \xrightarrow{a_i} R|1\rangle \xrightarrow{b_i} R|0\rangle,
$$

$$
R|0\rangle \xrightarrow{a_j} R|1\rangle \xrightarrow{b_j} R|0\rangle.
$$
Let $|00⟩, |01⟩, |10⟩, |11⟩$ be the standard basis vectors in the tensor product factorization $(a_i, b_i) \otimes (a_j, b_j)$. The row operation $[ij]_\lambda$ corresponds to the isomorphism of factorizations

$$(a_i, b_i) \otimes (a_j, b_j) \cong (a_i, b_i + \lambda b_j) \otimes (a_j - \lambda a_i, b_j)$$

which takes the standard basis of the LHS factorization to the basis

$$(|00⟩, |01⟩, |10⟩ > +\lambda |01⟩, |11⟩)$$

of the RHS tensor product.

Denote by $\psi(y)$ the following morphism between two Koszul factorizations:

\[
\begin{array}{ccc}
R & \xrightarrow{x} & R \\
\downarrow 1 & & \downarrow y \\
R & \xrightarrow{xy} & R
\end{array}
\]

\[
\begin{array}{ccc}
R & \xrightarrow{y} & R \\
\downarrow 1 & & \downarrow y \\
R & \xrightarrow{yz} & R
\end{array}
\]

Lemma 1 The following squares are commutative:

\[
\begin{array}{ccc}
\left( \begin{array}{cc}
a_1 & b_1 \\
a_2 & b_2 c_2
\end{array} \right) & \xrightarrow{\text{Id} \otimes \psi(c_2)} & \left( \begin{array}{cc}
a_1 & b_1 \\
a_2 c_2 & b_2
\end{array} \right) \\
\downarrow [12]_\lambda & & \downarrow [12]_{\lambda c_2} \\
\left( \begin{array}{cc}
a_1 & b_1 + \lambda b_2 c_2 \\
a_2 - \lambda a_1 & b_2 c_2
\end{array} \right) & \xrightarrow{\text{Id} \otimes \psi(c_2)} & \left( \begin{array}{cc}
a_1 & b_1 + \lambda b_2 c_2 \\
(a_2 - \lambda a_1) c_2 & b_2
\end{array} \right)
\end{array}
\]

\[
\begin{array}{ccc}
\left( \begin{array}{cc}
a_1 & b_1 \\
a_2 & b_2 c_2
\end{array} \right) & \xrightarrow{\text{Id} \otimes \psi(c_2)} & \left( \begin{array}{cc}
a_1 & b_1 \\
a_2 c_2 & b_2
\end{array} \right) \\
\downarrow [21]_{\lambda c_2} & & \downarrow [21]_\lambda \\
\left( \begin{array}{cc}
a_1 - \lambda c_2 a_2 & b_1 \\
a_2 & c_2 (b_2 + \lambda b_1)
\end{array} \right) & \xrightarrow{\text{Id} \otimes \psi(c_2)} & \left( \begin{array}{cc}
a_1 - \lambda c_2 a_2 & b_1 \\
a_2 c_2 & b_2 + \lambda b_1
\end{array} \right)
\end{array}
\]

Proof: direct computation. □

Denote by $\psi'(y)$ the "opposite" morphism of $\psi(y)$:

\[
\begin{array}{ccc}
R & \xrightarrow{xy} & R \\
\downarrow y & & \downarrow y \\
R & \xrightarrow{z} & R
\end{array}
\]

\[
\begin{array}{ccc}
R & \xrightarrow{y} & R \\
\downarrow 1 & & \downarrow y \\
R & \xrightarrow{yz} & R
\end{array}
\]
The analogue of Lemma 1 holds for $\psi'$ as well (just reverse all horizontal arrows in the commutative diagrams above). We call $\psi$ and $\psi'$ flip morphisms.

Starting with the Koszul matrices for $C(\Gamma^0)$ and $C(\Gamma^1)$ and applying a row transformation to each of them, we get the following equivalent Koszul forms for these factorizations:

\[
C(\Gamma^0) : \begin{pmatrix} a & x_1 - x_4 \\ a & x_2 - x_3 \end{pmatrix} \xrightarrow{[12]} \begin{pmatrix} a & x_1 + x_2 - x_3 - x_4 \\ 0 & x_2 - x_3 \end{pmatrix} \quad (8)
\]

\[
C(\Gamma^1) : \begin{pmatrix} a & x_1 + x_2 - x_3 - x_4 \\ 0 & x_1 x_2 - x_3 x_4 \end{pmatrix} \xrightarrow{[21]} \begin{pmatrix} a & x_1 + x_2 - x_3 - x_4 \\ 0 & (x_2 - x_3)(x_4 - x_2) \end{pmatrix} \quad (9)
\]

The first rows of these new Koszul matrices for $C(\Gamma^0), C(\Gamma^1)$ are identical while the second rows look related. In fact, there is a flip morphism $\psi(x_4-x_2)$ from $(0, (x_2 - x_3)(x_4 - x_2))$ to $(0, x_2 - x_3)$:

$$
\begin{array}{ccc}
R & 0 & R \\
\downarrow & x_4-x_2 & \downarrow \\
R & 0 & R \\
\end{array}
\xrightarrow{(x_2-x_3)(x_4-x_2)}
\begin{array}{ccc}
R & 0 & R \\
\downarrow & x_2-x_3 & \downarrow \\
R & 0 & R \\
\end{array}
$$

and the flip morphism $\psi'(x_4-x_2)$ back. Tensoring these flip morphisms with the identity morphism on the first row, we obtain maps of factorizations

$$
\text{Id} \otimes \psi'(x_4-x_2) : C(\Gamma^0) \to C(\Gamma^1), \quad \text{Id} \otimes \psi(x_4-x_2) : C(\Gamma^1) \to C(\Gamma^0).
$$

**Lemma 2** Maps $\text{Id} \otimes \psi'(x_4-x_2)$ and $\text{Id} \otimes \psi(x_4-x_2)$ are equal to $\chi_0$ and $\chi_1$, respectively.

The proof is a straightforward linear algebra computation. \(\square\)

Therefore, our definition of the complex $C(D)$ of factorizations assigned to a tangle diagram can be rewritten via modified Koszul matrices as above and maps $\psi, \psi'$. We’ll use this alternative presentation in our proof of the invariance of $C(D)$ below. The new definition simplifies the appearance of $C(D)$ by creating more zeros in the Koszul matrices of $C(\Gamma)$ and making the differential easier to describe and understand (at the cost of breaking the "lateral" symmetry $x_1 \leftrightarrow x_2$, $x_3 \leftrightarrow x_4$ of the original Koszul matrices). The differential acts now as the identity on all but $m_2$ rows, where $m_2$ is the number of crossings of $D$.

**3. Markings don’t matter.**

To define the complex $C(D)$ for a tangle diagram $D$, we need to place several marks on $D$: at least one on each internal edge and each circle and
some (possibly none) on each external edge (an edge containing a boundary point). In this subsection we prove proposition 1 that was stated earlier and says that, up to chain homotopy equivalence, \( C(D) \) does not depend on how marks are placed on the edges of \( D \).

**Lemma 3** Factorizations \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic in \( hmf_w \) if \( \Gamma_2 \) is obtained from \( \Gamma_1 \) by removing a mark.

**Proof:** It’s enough to check this property locally. We depicted two such local pairs \((\Gamma_1, \Gamma_2)\) in figure 10 and refer the reader to [KR] for a more detailed treatment. We only check the isomorphism for the top pair in figure 10, other cases are similar. We transform the Koszul matrix of \( C(\Gamma_1) \) as follows:

\[
\begin{pmatrix}
a & x_1 + x_5 - x_3 - x_4 \\
0 & x_1x_5 - x_3x_4 \\
a & x_2 - x_5
\end{pmatrix}
\xrightarrow{[13]_1}
\begin{pmatrix}
a & x_1 + x_2 - x_3 - x_4 \\
0 & x_1x_5 - x_3x_4 \\
0 & x_2 - x_5
\end{pmatrix}
\]

The variable \( x_5 \) is internal. According to proposition 3 with \( y = x_5 \) we can remove the last row of the RHS matrix, substitute \( x_2 \) for \( x_5 \) everywhere else and forget about \( x_5 \). We end up with the Koszul matrix of \( C(\Gamma_2) \).

To show independence of \( D \) on the number and position of marks, we need to check compatibility of the isomorphisms \( C(\Gamma_1) \cong C(\Gamma_2) \) above with maps \( \chi_0, \chi_1 \). We’ll only work through one case and leave the others to an interested reader. Let’s check that complexes of factorizations

\[
0 \longrightarrow C(\Gamma_1^1) \xrightarrow{\chi_1} C(\Gamma_1^0) \longrightarrow 0
\]
and
\[ 0 \longrightarrow C(\Gamma_1^1) \xrightarrow{\chi_1} C(\Gamma_1^0) \longrightarrow 0 \]
are chain homotopy equivalent, for figure 11 diagrams. The first complex, written via Koszul matrices, has the form
\[
\begin{pmatrix}
 a & x_1 + x_5 - x_3 - x_4 \\
 0 & (x_5 - x_3)(x_4 - x_5) \\
 a & x_2 - x_5
\end{pmatrix}
\xrightarrow{\text{Id} \otimes \psi(x_4 - x_5) \otimes \text{Id}}
\begin{pmatrix}
 a & x_1 + x_5 - x_3 - x_4 \\
 0 & x_2 - x_5
\end{pmatrix}
\]
Applying \([13]_1\) simultaneously to both matrices we get an isomorphic complex
\[
\begin{pmatrix}
 a & x_1 + x_2 - x_3 - x_4 \\
 0 & (x_5 - x_3)(x_4 - x_5) \\
 0 & x_2 - x_5
\end{pmatrix}
\xrightarrow{\text{Id} \otimes \psi(x_4 - x_5) \otimes \text{Id}}
\begin{pmatrix}
 a & x_1 + x_2 - x_3 - x_4 \\
 0 & x_2 - x_5
\end{pmatrix}
\]
The only internal variable is \(x_5\). We switch from \(x_5\) to \(x = x_5 - x_1\). We think of \(x\) as an internal variable, while \(a, x_1, x_2, x_3, x_4\) are external. Both matrices have identical bottom rows \((0, x)\) and the differential is the identity on that row. Therefore, we can eliminate \(x\) from the complex, reducing the ground ring to \(Q[a, x_1, x_2, x_3, x_4]\), crossing out the bottom row and setting \(x = 0\). The resulting complex
\[
\begin{pmatrix}
 a & x_1 + x_2 - x_3 - x_4 \\
 0 & (x_2 - x_3)(x_4 - x_2)
\end{pmatrix}
\xrightarrow{\text{Id} \otimes \psi(x_4 - x_2)}
\begin{pmatrix}
 a & x_1 + x_2 - x_3 - x_4 \\
 0 & x_2 - x_3
\end{pmatrix}
\]
4. Invariance under Reidemeister move I.

Consider type IA Reidemeister move as depicted in figure 12. The complex $C(D_2)\{0,2\}$ has the form

$$0 \to C(\Gamma_1) \xrightarrow{\chi_1} C(\Gamma_0) \to 0,$$

see figure 13. In terms of Koszul matrices, the complex is given by

$$
\begin{pmatrix}
a & x_1 - x_4 \\ 0 & 0
\end{pmatrix}
\xrightarrow{id \otimes \psi(x_4-x_2)}
\begin{pmatrix}
a & x_1 - x_4 \\ 0 & 0
\end{pmatrix}
$$

(we set $x_3 = x_2$ in the formulas (8), (9) for $\Gamma^0$ and $\Gamma^1$).

The differential is the identity on the first row, and on the second row given by

$$
\begin{array}{ccc}
R & \xrightarrow{0} & R\{-1,3\} \\
& \downarrow & \downarrow \\
& x_4-x_2 & 1
\end{array}
\xrightarrow{0} R
\begin{array}{ccc}
R & \xrightarrow{0} & R\{-1,1\} \\
& \downarrow & \downarrow \\
& x_4-x_2 & 1
\end{array}
\xrightarrow{0} R
$$

Therefore, the complex splits into a direct sum of a contractible complex and the tensor product of $(a, x_1 - x_4)$ with

$$0 \to R\{-1,3\} \xrightarrow{x_4-x_2} R\{-1,1\} \to 0.$$
Since $x_2$ is an internal variable, we can remove a contractible summand

$$0 \to R\{-1, 3\} \xrightarrow{x_4-x_2} R(x_4-x_2)\{-1, 1\} \to 0$$

from the above complex and reduce the ground ring to $R' = \mathbb{Q}[a, x_1, x_4]$. We get the complex $(a, x_1-x_4)$ shifted by $\{-1, 1\}[-1]$. Thus, $C(D_2)\{0, 2\} \cong C(\Gamma)\{-1, 1\}[-1]$, for $\Gamma$ as in figure 13. Since $\Gamma \cong D_1$, there is an isomorphism $C(D_2)\{1, 1\}[1] \cong C(D_1)$. We record this as

**Proposition 4** Complexes of matrix factorizations $C(D_1)$ and $C(D_2)\{1, 1\}[1]$ are isomorphic as objects of $K(hmf_w)$, with $w = a(x_1-x_4)$.

A similar computation takes care of the Reidemeister move IB:

**Proposition 5** Complexes of matrix factorizations $C(D_1)$ and $C(D_2)$ are isomorphic as objects of $K(hmf_w)$, for $D_1, D_2$ depicted in figure 14.

5. Invariance under Reidemeister move IIa.

Complexes of matrix factorizations $C(D_1)$ and $C(D_2)$, for the diagrams depicted in figure 15, live in the category $K(hmf_w)$ with $w = a(x_1 + x_2 - x_3 - x_4)$, viewed as an element of the ground ring $R = \mathbb{Q}[a, x_1, x_2, x_3, x_4]$. The complex $C(D_2) \cong C(\Gamma_{[1]}$) lies entirely in cohomological degree zero, since $D_2$ has no crossings.
**Proposition 6** Complexes of matrix factorizations $C(D_1)$ and $C(D_2)$ are equivalent as objects of $K(hmf_w)$.

*Proof:* It suffices to show that $f_1$ is an isomorphism (in $hmf_w$) from $C(\Gamma_{00})$ to a direct summand of $C(\Gamma_{10})$, and that there is a decomposition

$$C(\Gamma_{10}) \cong \text{Im}(f) \oplus M$$

with $f_2$ restricting to an isomorphism between $M$ and $C(\Gamma_{11})$. Then $C(D_1)$ would be isomorphic to a direct sum of contractible complexes

$$0 \rightarrow C(\Gamma_{00}) \rightarrow \text{Id} \rightarrow \text{Im}(f) \rightarrow 0,$$

$$0 \rightarrow M \rightarrow C(\Gamma_{11}) \rightarrow 0$$

and the factorization $C(\Gamma_{01})$, isomorphic to $C(D_2)$.

We start by writing down the diagram of factorizations and maps

$$C(\Gamma_{00}) \xrightarrow{f_1} C(\Gamma_{10}) \xrightarrow{f_2} C(\Gamma_{11})$$

and simplify them in $hmf_w$ by removing contractible direct summand fac-
torizations from each of the three terms. The diagram has the form

\[
\begin{pmatrix}
a & x_1 + x_2 - x_5 - x_6 \\
0 & x_2 - x_6 \\
a & x_5 + x_6 - x_3 - x_4 \\
0 & (x_6 - x_4)(x_3 - x_6)
\end{pmatrix}
\xrightarrow{f_1}
\begin{pmatrix}
a & x_1 + x_2 - x_5 - x_6 \\
0 & (x_2 - x_6)(x_5 - x_2) \\
a & x_5 + x_6 - x_3 - x_4 \\
0 & (x_6 - x_4)(x_3 - x_6)
\end{pmatrix}
\xrightarrow{f_2}
\begin{pmatrix}
a & x_1 + x_2 - x_5 - x_6 \\
0 & (x_2 - x_6)(x_5 - x_2) \\
a & x_5 + x_6 - x_3 - x_4 \\
0 & (x_6 - x_4)(x_3 - x_6)
\end{pmatrix}
\xrightarrow{f_3}
\begin{pmatrix}
a & x_1 + x_2 - x_5 - x_6 \\
0 & (x_2 - x_6)(x_5 - x_2) \\
a & x_5 + x_6 - x_3 - x_4 \\
0 & (x_6 - x_4)(x_3 - x_6)
\end{pmatrix}
\]

with

\[f_1 = \text{Id} \otimes \psi'(x_5 - x_2) \otimes \text{Id}^\otimes, \quad f_2 = \text{Id}^\otimes \otimes \psi(x_3 - x_6).\]

Apply row transformation [13] to all three Koszul matrices. The new matrices will have identical first rows \((a, x_1 + x_2 - x_3 - x_4)\) and identical third rows \((0, x_5 + x_6 - x_3 - x_4)\). We remove the third rows and exclude internal variable \(x_5\) substituting \(x_3 + x_4 - x_5\) in its place everywhere else. The diagram becomes the tensor product of the Koszul factorization \((a, x_1 + x_2 - x_3 - x_4)\) and the diagram

\[
\begin{pmatrix}
x_2 - x_6 \\
(x_6 - x_4)(x_3 - x_6)
\end{pmatrix}
\xrightarrow{g_1}
\begin{pmatrix}
(x_2 - x_6)(x_3 + x_4 - x_6 - x_2) \\
(x_6 - x_4)(x_3 - x_6)
\end{pmatrix}
\xrightarrow{g_2}
\begin{pmatrix}
(x_2 - x_6)(x_3 + x_4 - x_6 - x_2) \\
(x_6 - x_4)(x_3 - x_6)
\end{pmatrix}
\]

where

\[g_1 = \psi'(x_3 + x_4 - x_6 - x_2) \otimes \text{Id}, \quad g_2 = \text{Id} \otimes \psi(x_3 - x_5).\]

We omitted the first columns from the Koszul matrices, since their terms are all zeros. The only internal variable left is \(x_6\). The bottom term in the first two factorizations is

\[(x_6 - x_4)(x_3 - x_6) = -x_6^2 + (x_3 + x_4)x_6 - x_3x_4.\]

Let \(R' = \mathbb{Q}[a, x_1, x_2, x_3, x_4]\) be the polynomial ring on all external variables. Currently we’re working over the ring \(R'[x_6]\). We remove the bottom term from the first two factorizations simultaneously reducing to \(R'\), imposing the relation \(x_6^2 = (x_3 + x_4)x_6 - x_3x_4\), and treating multiplication by \(x_6\) as an endomorphism of the free \(R'\)-module \(R'[x_6]/((x_6 - x_4)(x_3 - x_5))\). Likewise, in the rightmost factorization, we remove the bottom row \((x_6 - x_4)\), reduce to
the ground ring $R'$ and impose the relation $x_6 = x_4$. Our diagram simplifies to

\[
\begin{array}{c}
R' \oplus R'x_6 \xrightarrow{x_2-x_6} R' \oplus R'x_6 \\
\downarrow \quad \quad \downarrow \\
R' \oplus R'x_6 \xrightarrow{(x_2-x_4)(x_3-x_2)} R' \oplus R'x_6 \\
\downarrow_{x_6 \rightarrow x_4} \quad \downarrow_{x_6 \rightarrow x_4} \\
R' \xrightarrow{(x_2-x_4)(x_3-x_2)} R'
\end{array}
\]

where, for instance, the bottom row denotes the factorization

\[
R' \xrightarrow{0} R' \xrightarrow{(x_2-x_4)(x_3-x_2)} R'
\]

and the maps $g_1, g_2$ are given by vertical arrows. Stripping off a contractible summand

\[
R' 1 \xrightarrow{1} R'(x_2 - x_6)
\]

from the first factorization, we reduce it to

\[
R'(x_6 + x_2 - x_3 - x_4) \xrightarrow{(x_2-x_4)(x_3-x_3)} R'1.
\]

The middle factorization is a direct sum of two isomorphic (up to grading shift) factorizations

\[
R'1 \xrightarrow{(x_2-x_4)(x_3-x_2)} R'
\]

and

\[
R'(x_6 + x_2 - x_3 - x_4) \xrightarrow{(x_2-x_4)(x_3-x_3)} R'(x_6 + x_2 - x_3 - x_4).
\]

The map $g_1$ takes the top factorization (in its reduced form) isomorphically onto the second summand of the middle factorization. The map $g_2$ restricts to an isomorphism from the first direct summand of the middle factorization to the bottom factorization. Our claim and the proposition follow. □

The invariance under the mirror image of the figure 15 move can be verified similarly.

6. Invariance under Reidemeister move III.

Let factorization $\Upsilon$ be given by the following Koszul matrix:

\[
\begin{pmatrix}
a & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
0 & x_1x_2 + x_1x_3 + x_2x_3 - x_3x_5 - x_4x_6 - x_5x_6 \\
0 & x_1x_2x_3 - x_4x_5x_6
\end{pmatrix}
\]

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The gradings are normalized so that the differential has bidegree (1, 1). For instance, the bottom row denotes the factorization

\[ R \xrightarrow{0} R\{−1, 5\} \xrightarrow{x_1x_2x_3−x_4x_5x_6} R \]

with \( R = \mathbb{Q}[a, x_1, \ldots, x_6] \). The potential is \( w = a(x_1 + x_2 + x_3 - x_4 - x_5 - x_6) \).

**Proposition 7** In \( \text{hmf}_w \) there are isomorphisms

\[
C(\Gamma_1) \cong C(\Gamma_4)\{0, 2\} \oplus \Upsilon, \\
C(\Gamma_3) \cong C(\Gamma_2)\{0, 2\} \oplus \Upsilon,
\]

for \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \) depicted in figure 17.

**Proof:** To prove the first isomorphism, we place labels \( x_7, x_8, x_9 \) (from top to bottom) on the three marks of \( \Gamma_1 \) and write \( C(\Gamma_1) \) in the Koszul form

\[
C(\Gamma_1) = \begin{pmatrix}
    a & x_1 + x_2 - x_8 - x_7 \\
    0 & (x_2 - x_7)(x_8 - x_2) \\
    a & x_7 + x_3 - x_9 - x_6 \\
    0 & (x_3 - x_6)(x_9 - x_3) \\
    a & x_8 + x_9 - x_4 - x_5 \\
    0 & (x_9 - x_5)(x_4 - x_9)
\end{pmatrix}
\]

Applying transformations \([13]_1 \) and \([15]_1 \) we get a matrix with the third and fifth rows

\[
(0, x_7 + x_3 - x_9 - x_6), \quad (0, x_8 + x_9 - x_4 - x_5).
\]
We use these rows to exclude internal variables $x_7$ and $x_8$ and reduce $C(\Gamma_1)$ to the following Koszul form

$$
\begin{pmatrix}
a & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
0 & (x_2 + x_3 - x_6 - x_9)(x_4 + x_5 - x_2 - x_9) \\
0 & (x_3 - x_6)(x_9 - x_3) \\
0 & (x_9 - x_5)(x_4 - x_9)
\end{pmatrix}
$$

Notice that variables $a$ and $x_1$ appear only in the first row. Moreover, Koszul forms of factorizations $Y$ and $C(\Gamma_4)$ have the same first row. Next, we ignore the first row of $C(\Gamma_1)$ and operate on the other three rows. The first column of the Koszul matrix then consists of zeros and we omit it. To simplify the factorization

$$
\begin{pmatrix}
(x_2 + x_3 - x_6 - x_9)(x_4 + x_5 - x_2 - x_9) \\
(x_3 - x_6)(x_9 - x_3) \\
(x_9 - x_5)(x_4 - x_9)
\end{pmatrix}
$$

we use the last term to reduce to at most linear terms in the last remaining internal variable $x_9$. Remove the last row and impose the relation $x_9^2 = (x_4 + x_5)x_9 - x_4x_5$. Modulo this relation and after adding the second row, the first row loses $x_9$ and the matrix becomes

$$
\begin{pmatrix}
(x_3 - x_6)(x_4 + x_5 - x_2 - x_3) + (x_2 - x_5)(x_4 - x_2) \\
(x_3 - x_6)(x_9 - x_3)
\end{pmatrix}
$$

Now $x_9$ appears only in the bottom row, which we can write as

$$R1 \oplus Rx_9 \xrightarrow{(x_3-x_6)(x_9-x_3)} R1 \oplus Rx_9.$$

Changing basis of the free $R$-module on the left hand side from $\{1, x_9\}$ to $\{1, x_9 + x_3 - x_4 - x_5\}$ and of the module on the right to $\{1, x_9 - x_3\}$, we decompose this complex into a direct sum of

$$R(x_9 + x_3 - x_4 - x_5) \xrightarrow{(x_3-x_4)(x_5-x_3)(x_3-x_6)} R1$$

and

$$R1 \xrightarrow{x_3-x_6} R(x_9 - x_3).$$

Adding the other rows, we obtain a decomposition of $C(\Gamma_1)$ into direct sum of factorizations with Koszul matrices

$$
\begin{pmatrix}
a & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
0 & (x_3 - x_6)(x_4 + x_5 - x_2 - x_3) + (x_2 - x_5)(x_4 - x_2) \\
0 & (x_3 - x_4)(x_3 - x_5)(x_3 - x_6)
\end{pmatrix}
$$
Figure 18: Reidemeister move III

and

$$\begin{pmatrix}
    a & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
    0 & (x_3 - x_6)(x_4 + x_5 - x_2 - x_3) + (x_2 - x_5)(x_4 - x_2) \\
    0 & x_3 - x_6
\end{pmatrix},$$

the latter shifted by \{0, 2\} due to the bidegree (0, 2) vector \(x_9 - x_3\) being a generator of the module \(R(x_9 - x_3)\). It is easy to check that the matrices above describe factorizations \(\mathbf{\Upsilon}\) and \(C(\Gamma_4)\), respectively. □

**Proposition 8** Complexes \(C(D_1)\) and \(C(D_2)\), for diagrams depicted in figure 18, are isomorphic in the category \(K(hmf_w)\).

*Proof* is similar to the one in [KR]. The complex \(C(D_1)\) consists of eight factorizations assigned to diagrams depicted in figure 19 (also see figure 6). We ignore the overall shift by \{0, -2\} in the resolution of each crossing, which was needed for the invariance under the Reidemeister move IIa, but does not make any difference for Reidemeister move III. Proposition 7 tells us that

\(C(\Gamma_{111}) \cong \mathbf{\Upsilon} \oplus C(\Gamma_{100})\{0, 2\}\)  
(also observe that \(\Gamma_{100} \cong \Gamma_{001}\)), while our proof of the invariance under the Reidemeister move IIa implies

\(C(\Gamma_{101}) \cong C(\Gamma_{100})\{0, 2\} \oplus C(\Gamma_{100})\).

A computation similar to the one in that proof shows that the map \(\chi_1 : C(\Gamma_{111}) \longrightarrow C(\Gamma_{101})\), when restricted to the direct summand isomorphic to \(C(\Gamma_{100})\{0, 2\}\), is an isomorphism onto a direct summand of \(C(\Gamma_{101})\), while our proof of proposition 6 implies that \(\chi_1 : C(\Gamma_{101}) \longrightarrow C(\Gamma_{001})\) is an isomorphism when restricted to the direct summand \(C(\Gamma_{100})\) of \(C(\Gamma_{101})\).

After removing contractible summands

\(0 \longrightarrow C(\Gamma_{100})\{0, 2\} \xrightarrow{\sim} C(\Gamma_{100})\{0, 2\} \longrightarrow 0\)
Figure 19: Resolution cube of $D_1$
Lemma 4 The complex $C' \cong C(D_1)$ is indecomposable in the category $K(hmf_w)$. 

In other words, we cannot write $C' \cong M \oplus N$ for two nontrivial objects $M, N$ of $K(hmf_w)$. Indeed, invariance under the Reidemeister move IIa tells us that tensoring with a complex of factorizations assigned to a crossing is an invertible functor. Precisely, it’s an invertible functor from the category $K(hmf_v)$ to $K(hmf_u)$ where $v = ax_3 + ax_4 + f(x)$, $u = ax_1 + ax_2 + f(x)$, and $f(x)$ is any polynomial in variables $x$ disjoint from $x_1, \ldots, x_4$. An invertible functor is indecomposable iff the identity functor is. The identity functor, in general, corresponds to the diagram comprised of $n$ parallel lines, compatibly oriented (the diagram of the trivial braid). Its factorization $S$ can be written as the tensor product of $(a, x_i - x_{i+1})$, over $i = 1, \ldots, n$. An easy computation (for instance, as in the proof of the next lemma) shows that the hom space $\text{Hom}_{hmf_w}(S, S)$ of bigrading-preserving factorization homomorphisms up to chain homotopies is one-dimensional. Therefore, $S$ and the identity functor
are indecomposable, for otherwise a projection onto a direct summand would ensure that the above hom space is at least 2-dimensional. □

**Lemma 5** For any arrow \( \Gamma \to \Gamma' \) in figure 20, the space of bidegree-preserving maps \( C(\Gamma) \to C(\Gamma') \) is one-dimensional (over the ground field \( \mathbb{Q} \)) and is generated by \( \chi_1 \).

We can prove the lemma on a case-by-case basis, separately for each arrow. In general, to compute the dimension of \( \text{Hom}_{hmf_w}(M,N) \), for matrix factorizations \( M,N \), with \( N \) of finite rank, we use the isomorphism

\[
\text{EXT}_{hmf_w}(M,N) \cong H(N \otimes_R M_\bullet)
\]

where \( \text{EXT} \) refers to taking ext groups of the pair \( M,N \) in all bidegrees, \( M_\bullet \) is the \( R \)-module dual of \( M \), and \( H \) stands for cohomology. Restricting the left hand side to \( \text{Hom} \) corresponds to taking the bidegree \((0,0)\) summand of the right hand side. The dual of a Koszul factorization \((a_i,b_i)\) is the Koszul factorization \((b_i,-a_i)\), with suitably shifted gradings.

For instance, to determine the dimension of the space \( \text{Hom}_{hmf_w}(C(\Gamma_{110}),C(\Gamma_{100})) \) (the bottom arrow in figure 20) we first write the Koszul matrix of \( C(\Gamma_{110}) \):

\[
\begin{pmatrix}
\{0,0\} & \{1,-1\} & x_1 + x_2 - x_4 - x_7 \\
\{0,0\} & \{1,-1\} & x_7 + x_3 - x_5 - x_6 \\
\{0,0\} & \{1,-3\} & (x_2 - x_4)(x_2 - x_7) \\
\{0,0\} & \{1,-3\} & (x_3 - x_5)(x_3 - x_6)
\end{pmatrix}
\]

Here \( x_7 \) is the variable assigned to the internal mark of \( \Gamma_{110} \). We also added two columns indicating the bidegrees of \( R \). For instance, the second row denotes the factorization

\[
R\{0,0\} \to R\{1,-1\} \xrightarrow{x_7 + x_3 - x_5 - x_6} R\{0,0\}.
\]

After we apply \([12]_1\), the second row becomes \((0, x_7 + x_3 - x_5 - x_6)\); we get rid of it and of the variable \( x_7 \). Thus, \( C(\Gamma_{110}) \) is isomorphic to the factorization assigned to the Koszul matrix

\[
\begin{pmatrix}
\{0,0\} & \{-1,1\} & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
\{0,0\} & \{-1,3\} & (x_2 - x_4)(x_2 + x_3 - x_5 - x_6) \\
\{0,0\} & \{-1,3\} & (x_3 - x_5)(x_3 - x_6)
\end{pmatrix}.
\]
The dual $C(\Gamma_{110})_\bullet$ of $C(\Gamma_{110})$ can be represented by the matrix

$$
\begin{pmatrix}
\{0,0\} & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 & \{1,-1\} & -a \\
\{0,0\} & (x_2 - x_4)(x_2 + x_3 - x_5 - x_6) & \{1,3\} & 0 \\
\{0,0\} & (x_3 - x_5)(x_3 - x_6) & \{1,3\} & 0
\end{pmatrix}
$$

and the tensor product complex $C(\Gamma_{110})_\bullet \otimes C(\Gamma_{100})$ by

$$
\begin{pmatrix}
\{0,0\} & a & \{-1,1\} & x_1 + x_2 - x_4 - x_5 \\
\{0,0\} & 0 & \{-1,3\} & (x_2 - x_4)(x_2 - x_5) \\
\{0,0\} & a & \{-1,1\} & x_3 - x_6 \\
\{0,0\} & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 & \{1,-1\} & -a \\
\{0,0\} & (x_2 - x_4)(x_2 + x_3 - x_5 - x_6) & \{1,3\} & 0 \\
\{0,0\} & (x_3 - x_5)(x_3 - x_6) & \{1,3\} & 0
\end{pmatrix}
$$

where the first three rows describe $C(\Gamma_{100})$. We do transformation [13]_1 and shift rows 4 and 5 by one each. We get

$$
\begin{pmatrix}
\{0,0\} & a & \{-1,1\} & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
\{0,0\} & 0 & \{-1,3\} & (x_2 - x_4)(x_2 - x_5) \\
\{0,0\} & 0 & \{-1,1\} & x_3 - x_6 \\
\{1,-1\} & -a & \{0,0\} & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
\{1,-3\} & 0 & \{0,0\} & (x_2 - x_4)(x_2 + x_3 - x_5 - x_6) \\
\{0,0\} & (x_3 - x_5)(x_3 - x_6) & \{1,3\} & 0
\end{pmatrix}
$$

We apply the transformation [14]_-1, then shift rows 1 and 6 to obtain

$$
\begin{pmatrix}
\{-1,1\} & 0 & \{0,0\} & a \\
\{0,0\} & 0 & \{-1,3\} & (x_2 - x_4)(x_2 - x_5) \\
\{0,0\} & 0 & \{-1,1\} & x_3 - x_6 \\
\{1,-1\} & 0 & \{0,0\} & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
\{1,-3\} & 0 & \{0,0\} & (x_2 - x_4)(x_2 + x_3 - x_5 - x_6) \\
\{1,-3\} & 0 & \{0,0\} & (x_3 - x_5)(x_3 - x_6)
\end{pmatrix}
$$

Cross out row 3 and convert $x_6$ to $x_3$ everywhere else. The matrix reduces to

$$
\begin{pmatrix}
\{-1,1\} & 0 & \{0,0\} & a \\
\{0,0\} & 0 & \{-1,3\} & (x_2 - x_4)(x_2 - x_5) \\
\{1,-1\} & 0 & \{0,0\} & x_1 + x_2 - x_4 - x_5 \\
\{1,-3\} & 0 & \{0,0\} & (x_2 - x_4)(x_2 - x_5) \\
\{1,-3\} & 0 & \{0,0\} & 0
\end{pmatrix}
$$

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We apply [24], then remove rows 1 and 3 simultaneously with getting rid of the variables \(a\) and \(x_5\). The resulting matrix is

\[
\begin{pmatrix}
\{0, 0\} & 0 & \{−1, 3\} & (x_2 − x_4)(x_4 − x_1) \\
\{1, −3\} & 0 & \{0, 0\} & 0 \\
\{1, −3\} & 0 & \{0, 0\} & 0
\end{pmatrix}.
\]

Let now \(R' = \mathbb{Q}[x_1, x_2, x_3, x_4]\). The cohomology of the complex described by this matrix is the tensor product of the quotient \(R'/(x_2 − x_4)(x_4 − x_1)\) and the bigraded vector space 

\[(\mathbb{Q}\{1, -3\} \oplus \mathbb{Q}) \otimes (\mathbb{Q}\{1, -3\} \oplus \mathbb{Q}).\]

The bigraded dimension of \(R'/(x_2 − x_4)(x_4 − x_1)\) has the form \(1 + \alpha\) where \(\alpha \in q^2\mathbb{Z}[q^2]\), while that of the second term is \((1 + tq^{-3})^2\). Therefore, the bigraded dimension of the cohomology of the complex \(C(\Gamma_{110}) \cdot \otimes C(\Gamma_{100})\) has the form

\[(1 + \alpha)(1 + 2tq^{-3} + t^2q^{-6}).\]

Writing it as a polynomial in \(t\) with coefficients being power series in \(q\), we see that the coefficient of the term \(t^0q^0\) equals 1. Therefore, the bidegree \((0, 0)\) summand of the homology is one-dimensional, and the hom space \(\text{Hom}_{hmf_{\omega}}(C(\Gamma_{110}), C(\Gamma_{100}))\) has dimension 1.

To show that \(\chi_1 : C(\Gamma_{110}) \rightarrow C(\Gamma_{100})\) (corresponding to the splitting of the right wide edge of \(\Gamma_{110}\) into two parallel lines) generates this one-dimensional space, it suffices to show that \(\chi_1\) is not null-homotopic. We can write the factorizations and the map in the following Koszul form

\[
\begin{pmatrix}
  a & x_1 + x_2 - x_4 - x_7 \\
  0 & (x_2 - x_4)(x_2 - x_7) \\
  a & x_7 + x_3 - x_5 - x_6 \\
  0 & (x_3 - x_5)(x_3 - x_6)
\end{pmatrix} \xrightarrow{\chi_1} \begin{pmatrix}
  a & x_1 + x_2 - x_4 - x_7 \\
  0 & (x_2 - x_4)(x_2 - x_7) \\
  a & x_7 + x_3 - x_5 - x_6 \\
  0 & x_3 - x_6
\end{pmatrix}
\]

with \(\chi_1 = \text{Id}^{\otimes 3} \otimes \psi(x_3 - x_5)\). Applying row transformation [13] to each matrix and then excluding \(x_7\) we reduce the map to

\[
\begin{pmatrix}
  a & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
  0 & (x_2 - x_4)(x_2 + x_3 - x_5 - x_6) \\
  0 & (x_3 - x_5)(x_3 - x_6)
\end{pmatrix} \xrightarrow{\chi_1} \begin{pmatrix}
  a & x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
  0 & (x_2 - x_4)(x_2 + x_3 - x_5 - x_6) \\
  0 & x_3 - x_6
\end{pmatrix}
\]

with \(\chi_1 = \text{Id}^{\otimes 2} \otimes \psi(x_3 - x_5)\). Turn both \(\Gamma_{110}\) and \(\Gamma_{100}\) into closed diagrams \(\widehat{\Gamma}_{110}\) and \(\widehat{\Gamma}_{100}\) by connecting top endpoints of each diagram with its bottom.
lemma can be reduced to verifying that the hom spaces

\[
\chi : H(\Gamma_{110}) \rightarrow H(\Gamma_{100})
\]

are both one-dimensional and generated by \( \chi \). Actual computations, similar to the one above, are left to a curious reader. For the first of the two hom spaces, by \( \chi \) we mean the composition of \( \chi : C(\Gamma_{111}) \rightarrow C(\Gamma_{110}) \) with the inclusion of \( \Upsilon \) as a direct summand of \( C(\Gamma_{111}) \). □

**Lemma 6** For any arrow \( \Gamma \rightarrow \Gamma' \) in figure 24 factorizations \( C(\Gamma) \) and \( C(\Gamma') \) are not isomorphic in \( hmf \).

**Sketch of proof:** Form the closures \( \Gamma \) and \( \Gamma' \) by connecting top endpoints of each diagram with its bottom endpoints by 3 disjoint arcs. A direct computation shows that complexes \( C(\Gamma) \) and \( C(\Gamma') \) have non-isomorphic cohomology groups (their two-variable Poincare polynomials are different). □

Thus, the complex \( C' \), depicted in figure 24, consists of 6 factorizations and its differential is a sum of 10 maps, one for each arrow of the figure.
Each map is either 0 or a non-zero multiple of the unique (up to rescaling) nontrivial map between the two factorizations. For each arrow \( b : \Gamma \rightarrow \Gamma' \) choose a nontrivial map \( m(b) : C(\Gamma) \rightarrow C(\Gamma') \).

**Lemma 7** For any two composable arrows \( \Gamma \xrightarrow{b_1} \Gamma' \xrightarrow{b_2} \Gamma'' \) the composition \( m(b_2)m(b_1) \) is nontrivial in \( \text{hmf}_w \).

**Proof:** It suffices to check that the composition

\[
\Upsilon \subset C(\Gamma_{111}) \xrightarrow{x_1} C(\Gamma_{110}) \xrightarrow{x_1} C(\Gamma_{100}) \xrightarrow{x_1} C(\Gamma_{000})
\]

is not null-homotopic. Denote the composition of the last 3 maps by \( \chi' \) and the corresponding ”adjoint” composition

\[
C(\Gamma_{000}) \xrightarrow{x_0} C(\Gamma_{100}) \xrightarrow{x_0} C(\Gamma_{110}) \xrightarrow{x_0} C(\Gamma_{111})
\]

by \( \chi'_0 \). We claim that the map \( \chi'_1 \text{pr} \chi'_0 \) is non-zero, where \( \text{pr} \) is the projection from \( C(\Gamma_{111}) \) onto its direct summand \( \Upsilon \). The map \( \chi'_0 \) has degree \((0, 6)\) and the product \( \chi'_1 \chi'_0 \) is equal to the multiplication by \((x_4 - x_2)^2(x_5 - x_3)\) endomorphism of \( C(\Gamma_{000}) \), since the composition \( \chi_1 \chi_0 \) is the multiplication by a suitable linear combination of \( x \)'s. The complementary direct summand of \( C(\Gamma_{111}) \) is isomorphic to \( C(\Gamma_{100})\{0, 2\} \). Denote by \( \tilde{\text{pr}} \) the projection onto this direct summand. Then \( \text{pr} + \tilde{\text{pr}} \) is the identity endomorphism of \( C(\Gamma_{111}) \).

The composition \( \chi'_1 \tilde{\text{pr}} \chi'_0 \) factors though a degree \((0, 2)\) endomorphism of \( C(\Gamma_{110}) \). This endomorphism is a composition

\[
C(\Gamma_{110}) \rightarrow C(\Gamma_{100}) \rightarrow C(\Gamma_{110})
\]

where the first map has degree \((0, 0)\) and the second–degree \((0, 2)\). These maps are, necessarily, rational multiples of \( \chi_0 \) and \( \chi_1 \) (corresponding to the right wide edge of \( \Gamma_{110} \)) and their composition is a rational multiple of the multiplication by \( x_4 - x_5 \). Hence, the composition \( \chi'_1 \tilde{\text{pr}} \chi'_0 \) is a rational multiple of the multiplication by \((x_3 - x_5)^2(x_2 - x_4)\). To show that

\[
\chi'_1 \text{pr} \chi'_0 = \chi'_1 \chi'_0 - \chi'_1 \tilde{\text{pr}} \chi'_0
\]

is not null-homotopic, we observe that the right hand side is the multiplication by

\[
(x_4 - x_2)^2(x_5 - x_3) - \mu(x_3 - x_5)^2(x_2 - x_4)
\]

endomorphism of \( C(\Gamma_{000}) \), for some rational \( \mu \). The image of \( \mathbb{Q}[x_1, \ldots, x_6] \) in the endomorphism ring of \( C(\Gamma_{000}) \) is the quotient ring by relations \( x_1 = x_4, x_2 = x_5 \) and \( x_3 = x_6 \). The polynomial above simplifies to

\[
(x_1 - x_2)^2(x_2 - x_3) - \mu(x_3 - x_2)^2(x_2 - x_1) \neq 0
\]
in $\mathbb{Q}[x_1, x_2, x_3]$. Therefore, the composition $\chi'_1 \text{pr} \chi'_0$ is not null-homotopic, and so is the map $\Upsilon \to C(\Gamma_{000})$ in formula (13). Lemma 7 follows. □

The differential in the complex $C'$ can be written as

$$d = \sum_b \lambda_b m(b),$$

with $\lambda_b \in \mathbb{Q}$, and the sum over all arrows $b$.

**Lemma 8** All coefficients $\lambda_b$ are nonzero rational numbers.

Assume otherwise: $\lambda_b = 0$ for some $b$. Every square in the diagram of $C'$ anticommutes, and from lemma 4 we derive that some other $\lambda$’s would have to be zero. In fact, there will be enough zero maps to split the complex into the direct sum of at least two subcomplexes, each comprised of two or four factorizations in figure 20. Specifically, the complex will either decompose into a direct sum of 3 subcomplexes of the form

$$0 \to C(\Gamma) \xrightarrow{m(b)} C'(\Gamma') \to 0 \quad (14)$$

for some three arrows $b$, or as the direct sum of one subcomplex of type (14) and the complementary summand containing the other four factorizations.

A decomposition of $C'$ into a direct sum of 3 subcomplexes contradicts lemmas 4, 6. To see the impossibility of the decomposition of the second kind, it’s enough to show that the complementary summand cannot be trivial in $\text{hmf}_{\text{w}}$. This summand would consist of four factorizations that sit in the vertices of one of the four squares in figure 20. For instance, it could have the form

$$0 \to C(Y) \to C(\Gamma_{110}) \oplus C(\Gamma_{011}) \to C(\Gamma_{010}) \to 0.$$

Triviality of the summand would imply that its identity map is null-homotopic. In particular, the identity map of the rightmost factorization in the complex would factor through a map to the middle term. This map should have bidegree $(0, 0)$. The following lemma establishes the contradiction.

**Lemma 9** For any arrow $\Gamma \to \Gamma'$ in figure 20 we have

$$\text{Hom}_{\text{hmf}_{\text{w}}}(C(\Gamma'), C(\Gamma)) = 0.$$
Thus, any bidegree zero map is trivial. The lemma can be proved in the same way as lemma □

Lemma □ follows. □

To summarize, we established that the coefficients $\lambda_b$ in the differential for the complex $C'$ are all nonzero. Rescaling, if necessary, we can turn them into 1’s and $-1$’s. Moreover, the complex $C'$ is uniquely determined, up to isomorphism, by the condition that $\lambda_b \neq 0$ for all $b$. We have $C(D_1) \cong C$. Nearly identical arguments show that $C(D_2) \cong C'$ as well. Therefore, $C(D_1) \cong C(D_2)$, and proposition □ follows. □

7. Computing the Euler characteristic.

Given a braid diagram $D$, the complexes $C(D\sigma_i^{\pm 1})$ are, up to shifts, the cones of maps $\chi_0, \chi_1$ between factorizations assigned to diagrams $D$ and $D_{e_i}$ where $e_i$ denotes a wide edge placed between $i$-th and $(i+1)$-st strands of the braid. See Section 1 and figure 6 for details. Recall that $\langle D \rangle$ denotes the Euler characteristic of cohomology $H(D)$. Our definition of $C(D\sigma_i^{\pm 1})$ via cones implies the following relations on the Euler characteristics:

$$\langle D\sigma_i \rangle = \langle D_{e_i} \rangle - q^2 \langle D \rangle,$$
$$\langle D\sigma_i^{-1} \rangle = q^{-2}(\langle D_{e_i} \rangle - \langle D \rangle).$$

Excluding $\langle D_{e_i} \rangle$, we get

$$q^{-1}\langle D\sigma_i \rangle - q^{-1}\langle D\sigma_i^{-1} \rangle = (q^{-1} - q)\langle D \rangle,$$

which is the relation satisfied by the function $F(D)$ defined in Section 1. If $D$ is the one-strand braid diagram of the unknot, $H(D) \cong \mathbb{Q}[x]\{-1, 1\}$ and

$$\langle D \rangle = \frac{t^{-1}}{q^{-1} - q},$$

which is our normalization of $F(D)$. Propositions proved above imply that $\langle D \rangle$ satisfies all other defining properties of $F(D)$. Therefore, $\langle D \rangle = F(D)$ and Theorem 2 follows. □

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Mikhail Khovanov, Department of Mathematics, University of California, Davis, CA 95616, mikhail@math.ucdavis.edu

Lev Rozansky, Department of Mathematics, University of North Carolina Chapel Hill, NC 27599, rozansky@math.unc.edu