On an expansion method for black hole quasinormal modes and Regge poles

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Abstract
We present a new method for determining the frequencies and wavefunctions of black hole quasinormal modes (QNMs) and Regge poles. The key idea is a novel ansatz for the wavefunction, which relates the high-\(l\) wavefunctions to null geodesics which start at infinity and end in perpetual orbit on the photon sphere. Our ansatz leads naturally to the expansion of QNMs in inverse powers of \(L = l + 1/2\) (in 4D), and to the expansion of Regge poles in inverse powers of \(\omega\). The expansions can be taken to high orders. We begin by applying the method to the Schwarzschild spacetime, and validate our results against existing numerical and Wentzel–Kramers–Brillouin methods. Next, we generalize the method to treat static spherically symmetric spacetimes of arbitrary spatial dimension. We confirm that, at lowest order, the real and imaginary components of the QNM frequency are related to the orbital frequency and the Lyapunov exponent for geodesics at the unstable orbit. We apply the method to five spacetimes of current interest, and conclude with a discussion of the advantages and limitations of the new approach, and its practical applications.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
Quasinormal modes (QNMs) are damped resonances which play a key role in black hole dynamics [1–4]. It was noted four decades ago [5] that a perturbed black hole will, after an initial response, emit radiation with well-defined frequencies and rates of damping, in the manner of a bell sounding its last dying pure notes [6]. Physically, a black hole QNM is a decaying resonance which satisfies a pair of causally motivated boundary conditions (typically being ingoing at the event horizon and outgoing at spatial infinity). Mathematically,
‘quasinormal ringing’ emerges from a sum of residues of poles of the Green function in the complex frequency domain [7]. Each pole corresponds to a QNM of a single complex frequency. The real part of the frequency corresponds to the oscillation rate and the (negative) imaginary part corresponds to the damping rate.

In a spherically symmetric spacetime (such as the Schwarzschild spacetime), the QNM frequencies \( \omega_{ln} \) are labelled by two integers: multipole \( l \) and overtone \( n \geq 0 \). The properties of the QNM spectrum depend only on the properties of the field (e.g., spin \( s \)) and the black hole geometry (e.g., mass \( M \), charge \( Q \), angular momentum \( J \)).

Even after years of study, black hole QNMs retain a certain mystique. The frequency spectrum has a beautiful and ornate structure (see, e.g., figure 5 in [2]). The asymptotic properties of the highly damped frequencies of the Schwarzschild hole have inspired speculation that QNMs may be linked to, e.g., Hawking radiation [8], black hole entropy [9] and loop quantum gravity [10] (however, subsequent study of charged, rotating and higher-dimensional black holes has shown that any such links cannot be as simple as originally envisaged [11, 12]). QNMs are also of relevance to holographic principles such as the AdS/CFT conjecture [13, 14]. Here we focus merely on the role that QNMs play in wave propagation on black hole spacetimes, in the weakly damped regime \( l \gtrsim 2, l \gtrsim n \).

It has been known for many years that QNMs are intimately linked to the existence and properties of unstable null orbits [16–18]. In the eikonal regime \( (l \gg n) \), the real part of the complex QNM frequency is related to the angular frequency of the orbit \( \Omega / \omega_1 \). It was recently demonstrated [19] that, in any static spherically symmetric asymptotically flat spacetime, to leading order in \( l \) the QNM frequency is

\[
\lim_{l \to \infty} \text{Re}(\omega_{ln}) = \Omega / l, \quad \lim_{l \to \infty} \text{Im}(\omega_{ln}) = -(n + 1/2)|\lambda|, \tag{1}
\]

where \( \lambda \) is the so-called Lyapunov exponent: the (inverse of the) instability timescale of the unstable orbit [20, 21]. Motivated by this idea, we show here that the QNM frequency for spherically symmetric black holes may be written as an expansion in inverse powers of \( L = l + 1/2 \), as \( \omega_{ln} = \sigma_1^{(1)} L + \sigma_1^{(1)} L^{-1} L^{-1} + \sigma_1^{(2)} L^{-2} + \cdots \). We develop a simple method for determining the expansion coefficients \( \{ \sigma_k^{(n)} \} \) which may be taken to very high order. A key advantage of the method is that it provides expansions (in powers of \( L^{-1} \)) for the radial wavefunctions, which are well defined over all radii \( r > r_h \). It is straightforward to adapt the method to find an expansion of the Regge poles [22, 23] in powers of \( \omega \), as we will show.

This paper is structured as follows. In section 2, we briefly review the existing methods for computing QNM spectra. In section 3, we introduce the expansion-in-\( L \) method by application to the Schwarzschild spacetime, and validate against existing numerical and asymptotic results. In section 4, we adapt the method to find Regge poles of the Schwarzschild spacetime. In section 5, we generalize the QNM analysis to static spherically symmetric spacetimes of arbitrary spatial dimension, and confirm the result (1). In section 6, we apply our method to five spacetimes of physical interest (Reissner–Nordström, Schwarzschild–de Sitter (SdS), Nariai, Schwarzschild–Tangherlini, the canonical acoustic hole). We conclude in section 7 with a discussion of interesting applications of the method and work for the future.

2. Existing methods for determining QNMs

A wide range of numerical methods have been developed for determining QNMs. The following non-exhaustive classification was outlined in [24]: (i) time domain methods, whereby the least-damped frequencies are extracted from long, stable time evolutions [5, 25]; (ii) direct integration in the frequency domain as pioneered in [26]; (iii) inverse potential methods whereby the ‘exact’ potential is replaced with a simple potential whose
spectrum is known [27]; (iv) Wentzel–Kramers–Brillouin (WKB) methods adapted from standard techniques in quantum mechanics [28–31]; (v) phase-integral methods using integration along anti-Stokes lines in the complex \( r \) plane [11, 32, 33]; (vi) continued fraction methods, first employed in the 1930s to find electronic spectrum of the H\(^+\) ion, and adapted to black holes by Leaver [34]. The relative advantages and disadvantages of these methods are discussed in several review articles [1, 2, 4].

For numerical calculations, the continued fraction method (vi) is fast, reliable and accurate, and with certain modifications [35] is robust at large overtones \( n \) and angular momenta \( l \). However, the method itself provides relatively little physical insight. In contrast, WKB methods [28–31] provide insight at the expense of accuracy and applicability. For example, the Bohr–Sommerfeld rule ‘explains’ the link between the low-overtone frequencies and the shape of the peak of the radial potential [28, 29]. For improved accuracy, the WKB method may be extended to sixth order [31] and above; nevertheless, it still breaks down at large overtone \( n \gg l \) or if the potential is complex.

In this work, we develop a new method which bears some passing resemblance to the WKB approach [28–31]. However, instead of matching together independent solutions across a (real) potential barrier [36], we instead seek a single solution that is valid everywhere outside the horizon. It turns out that, once the correct geometrically -motivated ansatz is employed, the wavefunctions and frequencies may be expressed as expansions in the angular momentum parameter:

\[
L = l + 1/2.
\]

3. The expansion of Schwarzschild QNMs

Without further ado, let us introduce the method by applying it to the Schwarzschild spacetime, perturbations of which are governed by the ‘master’ equation

\[
\left[ \frac{d^2}{dr_+^2} + \omega^2 - f(r) \left( \frac{L^2 - 1/4}{r^2} + \frac{2\beta}{r^3} \right) \right] u_{l\omega}(r) = 0,
\]

where \( f(r) = 1 - 2/r \) and \( r_+ = r + 2 \ln(r/2 - 1) \) and \( \beta = 1 - s^2 \) with \( |s| = 0, 1, 2 \) for scalar, electromagnetic (EM) and gravitational perturbations [6], respectively. Note that we have fixed the black hole mass to \( M = 1 \). Solutions which are ‘ingoing’ at the horizon (as \( r_+ \to -\infty \)) satisfy

\[
u_{l\omega}^{in}(r) \sim \begin{cases} A_{l\omega} e^{-i\omega r}, & r_+ \to -\infty, \\ A_{l\omega}^{(in)} e^{-i\omega r} + A_{l\omega}^{(out)} e^{i\omega r}, & r_+ \to +\infty. \end{cases}
\]

Quasinormal modes correspond to the (complex) frequencies \( \omega_{ln} \) at which \( A_{l\omega}^{(in)} = 0 \).

3.1. Method

Let us introduce the following ansatz:

\[
u_{l\omega}(r) = \exp \left( i\omega \int r_+ \left( 1 + \frac{6}{r^2} \right) \frac{1}{2} \left( 1 - \frac{3}{r^2} \right) dr_+ \right) v_{l\omega}(r).
\]

Note that the integrand in the exponent goes as \( 1 + \mathcal{O}(r^{-2}) \) as \( r_+ \to \infty \), and as \( -1 + \mathcal{O}(r^{-2}) \) as \( r_+ \to -\infty \). Hence QNM boundary conditions are automatically satisfied if \( v_{l\omega}(r) \) is well behaved in both limits. The geometric motivation behind this ansatz is explored in section 5.
Substitution of ansatz (5) into (3) leads immediately to
\[
\frac{d}{dr} \left( f(r) \frac{dv}{dr} \right) + \left[ 2i\omega \left( 1 + \frac{6}{r} \right)^{1/2} \left( 1 - \frac{3}{r} \right) \right] \frac{dv}{dr} + \frac{27i\omega^2 - L^2}{r^2} + \frac{27i\omega}{r^3} \left( 1 + \frac{6}{r} \right)^{-1/2} + \frac{1}{4r^2} - \frac{2\beta}{r^3} = 0. \tag{6}
\]

At first glance, (6) looks more complicated than (3), but it has the advantage of being amenable to an expansion in inverse powers of \(L\). To look for the ‘fundamental’ (least-damped) \(n = 0\) mode, let us try the expansion
\[
\omega_{l,n=0} = L\sigma_{l-1} + \sigma_l^{(0)} + L^{-1}\sigma_l^{(1)} + L^{-2}\sigma_l^{(2)} \ldots \tag{7}
\]
\[
u_{l,n}(r) = \exp \left( S_{l-1}^{(0)}(r) + L^{-1}S_{l}^{(0)}(r) + L^{-2}S_{l}^{(1)}(r) + L^{-3}S_{l}^{(2)}(r) + \cdots \right). \tag{8}
\]

Upon substituting into (6), and collecting together like powers of \(L\), we obtain a set of independent equations which may be used to determine the coefficients \(\sigma_k\) and the radial functions \(S_k(r)\) (briefly dropping the superscript for brevity). We find
\[
L^2 : 27\sigma_{-1}^2 - 1 = 0 \quad \Rightarrow \quad \sigma_{-1} = \pm 1/\sqrt{27}, \tag{9}
\]
\[
L^1 : 2i\sigma_{-1} \left( 1 + \frac{6}{r} \right)^{1/2} \left( 1 - \frac{3}{r} \right) S_0^{'} + \frac{54i\sigma_{-1}\sigma_0}{r^2} + \frac{27i\sigma_{-1}}{r^3} \left( 1 + \frac{6}{r} \right)^{-1/2} = 0, \tag{10}
\]
\[
L^0 : 2i \left( 1 + \frac{6}{r} \right)^{1/2} \left( 1 - \frac{3}{r} \right) (\sigma_{-1}S_0^{'} + \sigma_0S_0) + f(S_0^{''}) (S_0^{'} + (S_0)^2) + f'S_0^{'}
\]  
\[+ \frac{27(2\sigma_{-1}\sigma_0 + \sigma_0^2)}{r^2} + \frac{27i\sigma_0}{r^3} \left( 1 + \frac{6}{r} \right)^{-1/2} + \frac{1}{4r^2} - \frac{2\beta}{r^3} = 0, \tag{11}
\]
\[
L^{-1} : \ldots \tag{12}
\]

etc, where ‘\(^\prime\) denotes differentiation with respect to \(r\). The QNM frequencies are found by imposing a continuity condition on \(S_k(r)\) at the unstable null orbit radius \(r = 3\). That is, we demand that \(S_0^{'}\) is continuous and differentiable at \(r = 3\) which implies that
\[
\sigma_0^{(0)} = -\frac{i}{2\sqrt{27}} \tag{13}
\]
and
\[
\frac{dS_0^{(0)}}{dr} = \frac{\sqrt{27}}{2(r + 6)(r - 3)} \left( 1 + \frac{6}{r} \right)^{1/2} - \frac{\sqrt{27}}{r}. \tag{14}
\]

We may continue in this way, iteratively. First, we write down the equation for the \((1 - k)\)th power of \(L\), which contains the unknowns \(\sigma_k^{(0)}\) and \(S_k^{(0)}\). Next we insist that \(S_k^{(0)}\) is regular at \(r = 3\), to fix \(\sigma_k^{(0)}\). Then we rearrange to find \(S_k^{(0)}\), and differentiate once to obtain \(S_k^{(0)}\) which features at the next order. If we wish to construct the wavefunction we also integrate \(S_k^{(0)}\) to obtain \(S_k^{(0)}(r)\) (whereas if we only require the frequency expansion this step may be skipped). Finally, we move to the next order, sending \(k \rightarrow k + 1\), and repeat the steps. It is straightforward to implement the procedure in a symbolic algebra package to extend the expansion to arbitrary order. The lowest-order coefficients for perturbations of spin \(|s| = 0, 1, 2\) of the Schwarzschild spacetime are given in table 1.

The expansion may be
carried to 15th order and beyond with relative ease. For instance, the frequency of the fundamental (s = −2) QNM is

\[ \sqrt{27} \sigma_{k0} = L - 0.5i \left( 1.30092593L^{-1} + 0.20460391iL^{-2} - 0.56376775L^{-3} + 0.25454392iL^{-4} - 0.19978348L^{-5} - 0.05688153iL^{-6} + 0.14611217L^{-7} - 1.16950214L^{-8} + 0.04477903L^{-9} - 3.49565902iL^{-10} - 1.50293894L^{-11} - 6.56185182iL^{-12} - 4.40093979L^{-13} + O(L^{-14}) \] (15)

(expressing rational coefficients to eight decimal places). We note the following: (i) the method provides a neat expansion in inverse powers of L = l + 1/2, and a globally valid (in r) expansion (8) of the QNM function, (ii) odd (even) powers of L in the Schwarzschild QNM frequency expansion (7) have coefficients which are purely real (imaginary), (iii) the series is probably not convergent for low L ≲ 1, rather one expects that equation (15) is an asymptotic expansion. That is, for a given L, series (15) may be formally divergent; nevertheless, excellent approximations to the frequencies may be obtained by truncating at finite order k_0 such that \(|\sigma_{k0}^{(0)}L^{-k_0}| < |\sigma_{r}^{(0)}L^{-k}|, \forall k\).

The method can be extended higher overtones with the ansatz

\[ \psi_{kn}(r) = \left( 1 - \frac{3}{r} \right)^n + \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij}^{(n)} L^{-j} \left( 1 - \frac{3}{r} \right)^{n-j} \exp \left( S_0^{(n)}(r) + L^{-1} S_1^{(n)}(r) + \cdots \right). \] (16)

At the 4th step of the iterative procedure we require continuity of the first n − 1 non-vanishing derivatives at r = 3 to determine the a_{ij}^{(n)}, and nth to determine the correction to the frequency, σ_{i}^{(n)}. When these conditions are imposed, we are left with an explicit equation for S_1^{(n)}(r).

3.2. Results: QNM frequency expansion

Expansion coefficients for the frequency of the ‘fundamental’ mode are listed in table 1, for perturbations of spin 0 (scalar), 1 (electromagnetic) and 2 (gravitational). Expansion coefficients for the n = 1, 2, 3 QNM frequencies of the gravitational field on the Schwarzschild spacetime are given in table 2.

The lowest expansion coefficients for arbitrary spin \( \beta = 1 - s^2 \) and arbitrary overtone number n can be written as

\[ \sqrt{27} \sigma_{k1}^{(n)} = 1, \] (17)
\[ \sqrt{27} \sigma_{k0}^{(n)} = -iN, \] (18)
\[ \sqrt{27} \sigma_{i}^{(n)} = \frac{\beta}{3} = \frac{5N^2}{36} - \frac{115}{432}, \] (19)
Table 2. Series expansion coefficients \( u_k = \sqrt{27} \omega_k^{(n)} \) for gravitational (\( s = -2 \)) Schwarzschild QNMs of higher overtones (\( n = 1, 2, 3 \)) in expansion \( \sqrt{27} \omega_{0n} = \sum_{k=-1}^{\infty} u_k L^{-k} \).

| \( n \) | \( L^1 \) | \( L_0 \) | \( L^{-1} \) | \( L^{-2} \) | \( L^{-3} \) | \( L^{-4} \) | \( L^{-5} \) | \( L^{-6} \) |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

\[ \sqrt{27} \omega_2^{(n)} = -i N \left[ \frac{\beta^2}{9} + \frac{235N^2}{388} - \frac{1415}{15552} \right], \]  
(20)

\[ \sqrt{27} \omega_3^{(n)} = \left[ \frac{\beta^2}{27} + \frac{204N^2 + 211}{388} \beta + \frac{854160N^4 - 1664760N^2 - 776939}{40310784} \right], \]  
(21)

\[ \sqrt{27} \omega_4^{(n)} = i N \left[ \frac{\beta^2}{27} + \frac{1100N^2 - 2719}{46656} \beta + \frac{11273 \beta 136N^4 - 52753 \beta 800N^2 + 66480535}{2902376448} \right], \]  
(22)

where \( N = n + 1/2 \).

3.3. Validation

In Table 3, the results of the expansion in powers of \( L^{-1} \) are compared against the continued-fraction results of [34], and the WKB results of [31], for Schwarzschild gravitational modes with \( l \geq 2, 3 \) and \( n = 0, 1 \). At sixth order (i.e., including all terms in series (7) up to and including \( \omega_5^{(n)} L^{-5} \), resulting in truncation error \( O(L^{-6}) \)), the accuracy of the new method is comparable with the WKB method. It is simple to extend our method to, e.g., 12th order to obtain greater accuracy, as we show in Table 3.

We checked the series expansion coefficients (17)–(22) for general spin \( \beta = 1 - s^2 \) and overtone \( n \) against the WKB results given in equation (3.1) of [30]. We find agreement up to and including coefficient \( \omega_2^{(n)} \) (power \( L^{-2} \)). This is expected, because the truncation error in the WKB results of [30] is of order \( O(L^{-3}) \).

The accuracy of the expansion is examined in Figures 1 and 2. Figure 1 compares the QNM frequencies found via the expansion method with the results of the continued fraction method [34]. As expected, we find good agreement if \( l \gtrsim n \), but not always in the regime \( l \lesssim n \). In general, the expansions of the higher-\( n \) modes are less accurate. This is clearly demonstrated in Figure 2, which shows the difference between the series expansion (taken to orders \( L^{-5} \) and \( L^{-10} \)) and the exact frequencies, for the modes \( n = 0, \ldots, 5 \).

Figure 3 shows the magnitude of the coefficients in the series expansion of the frequencies. It shows that the coefficients \( a_k \) do not seem to tend to zero as \( k \to 0 \), though they do become small at certain intermediate orders. This supports the view that equation (7) is an asymptotic expansion.

3.4. Results: outgoing coefficient \( A^{(\text{out})}_{l0} \) and the wavefunction

A key advantage that the expansion method holds over many other approaches (see Section 2) is that it furnishes us with simple approximations for the wavefunctions (equations (5) and (16)) which are valid at all \( r > 2 \). Here we present the results for the outgoing coefficient \( A^{(\text{out})}_{l0} \) in a compact form, and examine the shape of the wavefunction.
Table 3. Numerical accuracy of the gravitational QNM frequencies. Comparing the results of the new expansion method at sixth and 12th order (with truncation errors of order $O(L^{-6})$ and $O(L^{-12})$, respectively) with numerical results from the continued-fraction method [34] and sixth-order WKB results [31].

|        | $l = 2$, $n = 0$       | $l = 2$, $n = 1$       | $l = 3$, $n = 0$       | $l = 3$, $n = 1$       |
|--------|------------------------|------------------------|------------------------|------------------------|
| Ctd. Frac. | 0.373672 − i0.088962  | 0.346711 − i0.273915  | 0.599444 − i0.092703  | 0.582644 − i0.281298  |
| 12th order | 0.373679 − i0.088934  | 0.346831 − i0.273844  | 0.599443 − i0.092703  | 0.582645 − i0.281297  |
| Sixth order | 0.373642 − i0.088671  | 0.347895 − i0.272766  | 0.599439 − i0.092684  | 0.582713 − i0.281206  |
| WKB (sixth) | 0.3736 − i0.0890     | 0.3463 − i0.2735     | 0.5994 − i0.0927     | 0.5826 − i0.2813     |
The integral appearing in ansatz (5) can be evaluated without much difficulty. Let us define

\[ X(r) = \exp \left( i \omega \int_{r_0}^{r} \left[ (1 + 6/r)^{1/2} (1 - 3/r) \right] f^{-1} \, dr \right) \]

(23)

\[ = (2 + \sqrt{3})^{-6i\omega} \left( \frac{1 + x}{2 - x} \right)^{4i\omega} \exp(i\omega[r_0 - r(1 - x) - \sqrt{27}]), \]

(24)

where

\[ x = \left( 1 + \frac{6}{r} \right)^{1/2}. \]

(25)

The outgoing coefficient \( A^{(\text{out})}_{\ln} \) defined by equation (4) may be computed via

\[ A^{(\text{out})}_{\ln} = (-2)^n \left( \frac{1 + \sum_{k=1}^{n} \sum_{j=1}^{\infty} d_{xk}^{(n)} L^{-j}}{1 + \sum_{k=1}^{n} \sum_{j=1}^{\infty} (-2)^k a_{xk}^{(n)} L^{-j}} \right) \frac{C_{\text{out}}}{C_{\text{in}}} \exp \left( \sum_{k=0}^{\infty} \gamma_k^{(n)} L^{-k} \right), \]

(26)

where we have defined the quantities

\[ C_{\text{in}} \equiv \lim_{r \to 2} X(r)e^{-i\omega r} = \exp(i\omega[6 - \sqrt{27} + 8\ln 2 - 6\ln(2 + \sqrt{3})]) \]

(27)

\[ C_{\text{out}} \equiv \lim_{r \to \infty} X(r)e^{i\omega r} = \exp(i\omega[3 - \sqrt{27} + 4\ln 2 - 6\ln(2 + \sqrt{3})]) \]

(28)

\[ \gamma_k^{(n)} \equiv \lim_{r \to \infty} S_k^{(n)}(r), \]

(29)

and we choose the constant of integration such that \( \lim_{r \to 2} S_k^{(n)}(r) = 0 \). The leading-order function \( S_0^{(n)}(r) \) is independent of the spin of the perturbing field, and may be written compactly as

\[ S_0^{(n)}(r) = \frac{1}{2} \ln(2/x) + 2(n + 1/2) \ln \left( \frac{2 + \sqrt{3}}{x + \sqrt{3}} \right). \]

(30)
Figure 2. Accuracy of expansion of Schwarzschild gravitational QNM frequencies in inverse powers of \( L = l + 1/2 \). The four plots show the difference between the exact QNM frequencies (continued-fraction method) and the approximations from the asymptotic expansion method. The frequency expansion \( \omega_l^{(n)} = \sum_{k=0}^{m} \gamma_{n}^{(k)} L^{-k} \) is truncated at orders \( m = 5 \) (left) and \( m = 10 \) (right). The upper (lower) plots show the accuracy of the real (imaginary) part of the frequency on a logarithmic scale, for overtones \( n = 0, \ldots, 5 \) (dotted lines).

Note \( S_{l}^{(n)}(r) \) is purely real, hence it tells us about the amplitude of the QNM in the large-\( L \) limit. The leading-order coefficient defined in (29) is therefore

\[
\gamma_0^{(n)} = (1/2) \ln 2 + (n + 1/2) \ln[(2 + \sqrt{3})/2].
\]  

Hence, at leading order in \( L \), the outgoing coefficient is simply

\[
A_{l,n=0}^{(\text{out})} \approx (2 + \sqrt{3})^{1/2} e^{-i(3 + 4 \ln 2)} \exp \left( \sum_{k=1}^{\infty} \gamma_{k}^{(0)} L^{-k} \right),
\]  

for all spins.

Calculation of (26) is straightforward for the ‘fundamental’ mode, for which we find

\[
A_{l,n=0}^{(\text{out})} = (2 + \sqrt{3})^{1/2} e^{-i(3 + 4 \ln 2)} \exp \left( \sum_{k=1}^{\infty} \gamma_{k}^{(0)} L^{-k} \right),
\]  

where the first few coefficients for the fundamental mode are

\[
\sqrt{27} \gamma_1^{(0)} = i \left[ 3\beta - \frac{97}{48} \right],
\]  

(34)

\[
\sqrt{27} \gamma_2^{(0)} = \left[ \frac{\beta}{2} - \frac{577}{864} \right],
\]  

(35)
Coefficients of series expansion of 'fundamental' scalar QNM freq. $\omega_{l,0}$

The lower plot shows the same data as the upper plot, using a logarithmic scale. (NB. The coefficients are known as rationals (i.e., exact.).)

\[ \sqrt{27} \gamma_5^{(0)} = \left[ -\beta^2 + \frac{457}{432} \beta - \frac{1013761}{5598720} \right], \]
\[ \sqrt{27} \gamma_4^{(0)} = \left[ -\beta^2 + \frac{20}{27} \beta - \frac{2788571}{12597120} \right]. \]

Higher-order approximations can be found by computing the functions $S^{(n)}_k(r)$ and the coefficients $a^{(n)}_{kj}$. Typical radial profiles of the functions $S^{(0)}_k(r)$ are shown in figure 4. We see that the functions start from zero at $r = 2$ and tend towards a constant as $r \to \infty$, without any significant oscillation between these limits.

Figure 5 shows examples of the QNM wavefunctions. Since the frequency has a negative imaginary part, it is no surprise to find that the wavefunctions diverge as $r \to 2$ and as $r \to \infty$. This makes it rather hard to gain physical insight from examining the radial functions alone. To assess the QNM contribution to physically realistic scenarios, we need to compute the ‘QNM excitation coefficient’ [43]. This quantity is obtained from the residues of the poles in the Green function at QNM frequency. In a forthcoming work [37], we will show how to calculate the residues of the poles, and in particular the key quantity $\partial A^{(n)}_{\omega} / \partial \omega |_{\omega = \omega_{l,0}}$, by combining the expansion method with standard WKB techniques.

An alternative set of physically relevant wavefunctions that do not diverge in the limits $r \to 2$ and $r \to \infty$ are the so-called Regge poles (shown in figure 6), to which we now turn our attention.

4. The expansion of Schwarzschild Regge poles

For a given angular momentum $l$, there are an infinite set of QNMs with frequencies $\omega_n$ defined by $A^{(n)}_{\omega} = 0$. Conversely, for a given frequency $\omega$ there are an infinite set of Regge
Figure 4. Radial profile of the functions $S_k^n(r)$. Note that functions with $k$ even (odd) are purely real (imaginary). The functions for higher overtone $n$ are similar in nature.

Figure 5. Gravitational QNM wavefunctions of lowest overtones ($n = 0, 1$) at $L = l + 1/2 = 4$. Note that the wavefunctions diverge exponentially in both limits, $r_* \rightarrow \pm \infty$.

poles, corresponding to complex angular momenta $\lambda_{\text{out}} = l_{\text{out}} + i/2$, defined by the condition $A^{(\text{in})}_{l_{\text{out}}} = 0$. Regge poles are a key concept in the so-called ‘complex angular momentum’ (CAM) method [38], which has been applied in a variety of contexts. For example, Regge poles received much attention in high-energy physics in the 1960s [39]. In the context of black hole physics, it has been shown that the scattering amplitude for a monochromatic planar wave impinging upon a black hole can be split into a background integral and a sum over Regge poles [40, 41]. The background integral (approximately) corresponds to the ‘classical’ scattering cross section, and the Regge poles correspond to regular oscillations in the intensity-versus-angle plot caused by the interference of parts of the wavefront travelling close to rays passing in opposite senses around the black hole (see [42] for a short review).

Many of the methods used for finding QNMs can also be used to find Regge poles [22, 23]. It is perhaps no surprise to find it is straightforward to expand Regge poles $\lambda_{\text{out}}$ in inverse powers of $\omega$. We simply insert the expansion

$$
\lambda_{\text{out}} = \lambda_0^{(n)} + \lambda_1^{(n)} \omega^{-1} + \lambda_2^{(n)} \omega^{-2} + \cdots
$$

(38)
\[
v_{\omega}(r) = \left[ \left( 1 - \frac{3}{r} \right)^n + \sum_{i=1}^{n} \sum_{j=1}^{\infty} b_{ij}^{(n)} \omega^{-j} \left( 1 - \frac{3}{r} \right)^{n-i} \right] \exp \left( T_{0}^{(n)}(r) + \omega^{-1} T_{1}^{(n)}(r) + \cdots \right) \tag{39}\]

into the differential equation (6), and group like powers of \( \omega \), to obtain a set of equations.

We then solve order by order to determine the coefficients \( \lambda_k^{(n)}, b_{ij}^{(n)} \) and radial functions \( T_k^{(n)} \), as before. We find that the first few expansion coefficients are

\[
\lambda_{-1}^{(n)} = \sqrt{27}, \tag{40}
\]

\[
\lambda_{0}^{(n)} = iN, \tag{41}
\]

\[
\lambda_{1}^{(n)} = \frac{60N^2 - 144\beta + 115}{432\sqrt{27}}, \tag{42}
\]

\[
\lambda_{2}^{(n)} = -iN \left( \frac{1220N^2 - 6912\beta + 5555}{419904} \right), \tag{43}
\]

\[
\lambda_{3}^{(n)} = -\frac{18\beta^2}{6561\sqrt{27}} + \left[ \frac{2316N^2 + 4799\beta}{104976\sqrt{27}} - \frac{2357520N^4 + 1938280N^2 + 2079661}{1088391168\sqrt{27}} \right], \tag{44}
\]

\[
\lambda_{4}^{(n)} = iN \left[ \frac{8\beta^2}{19683} - \frac{5\beta[3716N^2 + 2291]}{17006112} + \frac{144920784N^4 + 1871793480N^2 + 593617841}{2115832430592} \right], \tag{45}
\]

where \( N = n + 1/2 \) and \( \beta = 1 - s^2 \). We have verified that the expansion in inverse powers of \( \omega \) given in equations (40)–(45) is fully consistent with the WKB result recently obtained by Décanini and Folacci ([23], equations (12) and (16)), up to and including order \( \omega^{-2} \) (but not beyond). As in the QNM case (cf section 3.3) this is consistent with the expected truncation error in the WKB results. We may take the expansion to higher orders if desired; for example, the ‘fundamental’ \( n = 0 \) Regge pole for the gravitational perturbation is

\[
\lambda_{\omega,n=0} \approx \sqrt{27\omega} + \frac{i \sqrt{3}}{9} + \frac{281 \sqrt{3}}{1944\omega} - \frac{6649}{209952\omega^2} - \frac{1044601 \sqrt{3}}{1530550068\omega^3} + \frac{264479053824\omega^4}{71361067332161i}
\]

\[
- \frac{257076340516298\omega^5}{166583718925369344\omega^6} + \frac{14390928366797903 \sqrt{3}}{184851431845\sqrt{3}} - \frac{71361067332161i}{14390928366797903 \sqrt{3}}
\]

\[
+ \frac{161919374795459002368\omega^7}{14117571610293670714747 \sqrt{3}} + \cdots. \tag{46}\]

Note that the expansion (46) only works in the regime \( M\omega \gtrsim 0.5 \), but is not appropriate or effective near \( \omega \sim 0 \).

In table 4, the results of the expansion method are validated against the (approximate) WKB and the (exact) continued-fraction results presented in [23]. We find that the high-order expansions are in general more accurate than the WKB results, at large and intermediate \( \omega \). However, at small \( \omega \lesssim 0.5 \) the expansion (38) does not converge well, hence the results are of limited use. The WKB method [23], by contrast, gives results which seem relatively robust down to \( \omega \sim 0.25 \).

Wavefunctions of the lowest overtones \( (n = 0, 1, 2) \) at \( \omega = 3 \) are shown in figure 6. Note that, unlike the QNM wavefunctions (6), they do not diverge in the limits \( r_* \to \pm \infty \).
Let us assume a \( (d+2) \)-dimensional static spherically symmetric spacetime with line element

\[
dx^2 = -f(r) \, dr^2 + g^{-1}(r) \, dr^2 + h(r) \, d\Omega^2_{(d)},
\]

(47)

where \( d\Omega^2_{(d)} \) is the natural line element on \( S^d \) and we assume that \( h(r) > 0 \) everywhere. This includes all the spacetimes mentioned above; indeed for all of them except Nariai we may take \( h(r) = r^2 \) while for Nariai \( h(r) = 1 \).

Null geodesics on this spacetime may be taken to lie in the equatorial plane where the orbital equation may be written as

\[
\frac{f(r)}{g(r)h^2(r)} \left( \frac{dr}{d\phi} \right)^2 = \frac{1}{b^2} - \frac{f(r)}{h(r)}.
\]

(48)

Here the constant \( b = L/E \) may be interpreted as an ‘impact parameter’, where \( L = h(r)\phi \) and \( E = f(r)l \) are constants of the motion and \( \cdot \) denotes differentiation with respect to an affine parameter.
Figure 6. Regge pole wavefunctions of lowest overtones \((n = 0, 1, 2)\) of gravitational perturbations of a Schwarzschild black hole at \(M\omega = 3.0\).

Let us define a new function

\[ k^2(r, b) = \frac{1}{b^2} - \frac{f(r)}{h(r)}. \]

Now assume that there is a critical impact parameter \(b = b_c\), for which \(k^2(r, b_c)\) has a repeated root, so there exists some pair of constants \(\{r_c, b_c\}\) such that

\[ k^2(r_c, b_c) = 0 \quad \text{and} \quad \frac{\partial k^2}{\partial r}(r_c, b_c) = 0. \]

Assuming that the repeated root is a double root, let us then write

\[ k_c(r) = \text{sgn}(r - r_c)\sqrt{k^2(r, b_c)} = (r - r_c)K(r), \]

so \(k_c(r)\) is positive for \(r > r_c\) and negative for \(r < r_c\).

It is conventional to study the stability properties of this orbit by writing the radial equation as

\[ \dot{r}^2 = \frac{L^2 g(r)}{f(r)}k^2(r, b) = V_r(r). \]

Then the period of the circular orbit is given by \(2\pi / \Omega_c\) where

\[ \Omega_c = \frac{\dot{\phi}}{\dot{t}} = \sqrt{\frac{f(r_c)}{h(r_c)}} = \sqrt{\frac{f'(r_c)}{h'(r_c)}} = 1/b_c. \]

and the instability timescales are determined by the Lyapunov exponent [20]:

\[ \lambda = \sqrt{\frac{V''_r(r_c)}{2l^2}} = \sqrt{g(r_c)h(r_c)K(r_c)}. \]
5.2. Wave equations

To motivate our form of the wave equation we start by considering a scalar wave equation with potential \( (\Box - V)\phi = 0 \). Using the fact that the eigenvalues of the Laplacian on \( S^d \) are \(-l(l + d - 1)\) (with multiplicity \( \binom{d+l}{d} - \binom{d+l-2}{d} \)), the wave equation separates to give a radial equation of the form

\[
\sqrt{g}h^{-d/2} \frac{d}{dr} \left( \sqrt{g}h^{d/2} \frac{dR}{dr} \right) + \left[ \omega^2 - \frac{f(r)}{h(r)} \left( L^2 - (d-1)^2/4 \right) - fV(r) \right] R = 0,
\]

where \( L = l + (d-1)/2 \). Introducing \( r_* \) by \( \frac{dr_*}{dr} = 1/\sqrt{g} \) and changing dependent variable to \( R = h^{-d/4}u \), this may be rewritten as

\[
\frac{d^2u}{dr_*^2} + \left[ \omega^2 - \frac{f(r)}{h(r)} \left( L^2 - (d-1)^2/4 \right) - fV_{\text{eff}}(r) \right] u = 0,
\]

where

\[
V_{\text{eff}}(r) = V(r) + h^{-d/4} f^{-1} \frac{d}{dr} \left( \frac{\sqrt{g}h^{d/4}}{\sqrt{f}} \right).
\]

To provide a framework of sufficient generality we shall now study equation (56) but allow \( V_{\text{eff}}(r) \) to contain \( L \) dependence at lower orders than \( L^2 \), so

\[
V_{\text{eff}}(r) = V_{-1}(r)L + V_0(r) + V_1(r)L^{-1} + \cdots.
\]

This form is motivated by, for example, gravitational perturbations of the Reissner–Nordström spacetime which we study below.

Next we introduce the corollary to ansatz (5), namely,

\[
u(r) = \exp \left( i\omega \int_{r_*}^r b_c k_c(r) \, dr \right) v(r),
\]

where \( k_c(r) \) was defined in equation (51). In all the cases we shall assume that we have a horizon as \( r_* \to -\infty \), so \( f(r) \to 0 \) and correspondingly \( b_c k_c(r) \to -1 \). In addition, as \( r_* \to \infty \) we either have an asymptotically flat region or a cosmological horizon, so \( f(r)/h(r) \to 0 \) and \( b_c k_c(r) \to +1 \). Our ansatz thus encapsulates the desired ingoing and outgoing boundary conditions for the wave. Note that our assumptions at this stage have excluded, for example, Schwarzschild-AdS spacetime.

We now substitute ansatz (59) in to the radial equation (56) to obtain

\[
fgv'' + \sqrt{fg}(\sqrt{fg})' + 2ib\, ok_c \nu' + \left[ \left(1 - b_c^2 k_c^2\right)\omega^2 - \frac{f}{h} \left( L^2 - (d-1)^2/4 \right) - fV_{\text{eff}} + i\omega b_c \sqrt{fg} \frac{d k_c}{d r} \right] v = 0.
\]

Once again, we expand \( \omega \) and \( v(r) \) in inverse powers of \( L \), using (7) and (8). Making use of the key property \( 1 - b_c^2 k_c^2(r) = b_c^2 f(r)/h(r) \) we balance at order \( L^2 \) for all \( r \) by taking

\[
\omega_{-1} = \frac{1}{b_c} = \Omega_c.
\]

Next at order \( L \) we require

\[
2i\sqrt{g}(r - r_c) K S_0 + 2\left(1 - b_c^2 (r - r_c)^2 K^2 \right) \Omega_c \omega_0 - fV_{-1} + i\sqrt{g} [K - (r - r_c) K'] = 0.
\]

Setting \( r = r_c \), requiring \( S_0 \) to be regular there, we find

\[
\omega_0 = \frac{f_c V_{-1c}}{2\Omega_c} - i\frac{1}{2} \sqrt{g_c h_c K_c} = \frac{f_c V_{-1c}}{2\Omega_c} - i\frac{1}{2} \lambda_c.
\]
where a subscript $c$ on a function of $r$ denotes that function evaluated at $r = r_c$. Note that, to leading order, the imaginary part of the frequency is set by the Lyapunov exponent $\Lambda F (54)$, as anticipated in equation (1) (see also [19–21]). Substituting back into equation (62), we may write an explicit and manifestly regular equation for $S'_0$:

$$2i\sqrt{fg} KS'_0 = \Delta[F V_{-1}] - i\Delta[\sqrt{fg} K'] = i\sqrt{fg} K' + 2(\omega_0/\Omega_c)(r - r_c) K^2,$$

where we have introduced the notation

$$\Delta[F](r) = \frac{F(r) - F(r_c)}{r - r_c}.$$  (65)

This process may readily be extended to higher orders yielding

$$\omega_{p+1} = \left(-f_{\ell} g_c \left(S''_{pc} + \sum_{i=0}^{p} S'_{pc} S'_{p+i}\right) - i(\omega_{p}/\Omega_c)\sqrt{fg} K_c - \sqrt{fg} K' S'_{pc}\right.$$

$$- \sum_{i=0}^{p} \omega_{i}\omega_{p-i} + f V_{pc} - \frac{f_{\ell}(d-1)^2}{4\hbar_c} \delta_{p0}\right) / (2\Omega_c),$$

and

$$2i\sqrt{fg} KS'_{p+1} = -\Delta \left[ f g \left(S''_{p} + \sum_{i=0}^{p} S'_{pc} S'_{p-i}\right) \right] - \Delta[\sqrt{fg}(\sqrt{fg})' S'_{p}] - i(\omega_{p}/\Omega_c) \Delta[\sqrt{fg} K]$$(66)

and

$$2i\sqrt{fg} KS'_{p+1} = -\Delta \left[ f g \left(S''_{p} + \sum_{i=0}^{p} S'_{pc} S'_{p-i}\right) \right] - \Delta[\sqrt{fg}(\sqrt{fg})' S'_{p}] - i(\omega_{p}/\Omega_c) \Delta[\sqrt{fg} K']$$

$$- \Delta \left[ \frac{f(d-1)^2}{4\hbar_c} \right] \delta_{p0} + \Delta[f V_{p}] - i(\omega_{p}/\Omega_c) \sqrt{fg} K'$$

$$- 2i\sqrt{fg} K \sum_{i=0}^{p} \omega_{i}\omega_{p-i} / \Omega_c \left(2\Omega_c/\omega_{p+1}/\Omega_c + \sum_{i=0}^{p} \omega_{i}\omega_{p-i} / \Omega_c^2\right)(r - r_c) K^2.$$

(67)

### 6. Examples

To further demonstrate the utility of the above approach, we now apply to five cases of current interest.

#### 6.1. Reissner–Nordström black hole

The expansion method may be applied to deduce the QNM frequencies and wavefunctions of the charged (Reissner–Nordström) black hole, with a charge-to-mass ratio $q$. The relevant radial equation is [6, 24, 44, 46]

$$f(r) \frac{d^2 u}{dr^2} + \left[ \omega^2 - f(r) \frac{L^2 - 1/4}{r^2} - \kappa_{|s|} + q^2 \eta_{|s|} \right] u = 0,$$  (68)

where

$$\kappa_{|s|} = \begin{cases} -2 & |s| = 0 \, \text{scalar} \\ 3 - \sqrt{9 + 4q^2(L^2 - 9/4)} & |s| = 1 \, \text{electromagnetic} \\ 3 + \sqrt{9 + 4q^2(L^2 - 9/4)} & |s| = 2 \, \text{gravitational} \end{cases}$$  (69)

and $\eta_{|s|} = [2, 4, 4]$ for $|s| = [0, 1, 2]$ and $f(r) = 1 - 2/r + q^2/r^2$. For $|s| = 1, 2$, we may expand $\kappa_{|s|}$ in inverse powers of $L$, e.g., $\kappa_{|s|} = \pm 2qL + 3 + O(L^{-1})$. The resultant terms at order $L$ in the radial equation mean that the odd (even) powers of $L$ are no longer purely real (imaginary).
Applying the method, we find the ‘fundamental’ \((n = 0)\) gravitational mode is
\[
bc\omega_{l,n=0} = L - \left[ \frac{2q}{\alpha + 3} + \frac{i}{2} \left( \frac{2\alpha}{\alpha + 3} \right)^{1/2} \right] \left( \frac{1 + \alpha}{64(\alpha + 3)} \right)^{\alpha^2/2} \left( \frac{11 - 25\alpha + 84\alpha^2}{64(\alpha + 3)} \right) - \frac{3\sqrt{2q(\alpha^2 - 1)}}{8(3\alpha + \alpha^2)^{3/2}} L \right) + O(L^{-2}),
\]
(70)
where \(b_c = (3 + \alpha)^{3/2}/(2(\alpha + 1))^{1/2}\)
and \(\alpha = \sqrt{9 - 8q^2}\), and \(q\) is the charge-to-mass ratio of the black hole. Note that the formula breaks down, not when the twin horizons merge and disappear at \(q = 1\), but rather when the unstable circular orbit disappears, at \(q = 3/\sqrt{8}\). More accurate numerical results for \(l = 2\) are given in, e.g., [44, 45].

For the critically charged case, \(q = 1\), we obtain
\[
\omega_{l,n=0} = \frac{L}{4} - \frac{2 + \sqrt{2}}{16} - \frac{35}{256L} - \frac{560 - 131\sqrt{2}i}{8192L^2} \left( \frac{5031 - 2096\sqrt{2}i}{131072L^3} - \frac{392512 - 137605\sqrt{2}i}{16777216L^4} \right) + O(L^{-5}).
\]
(71)
The accuracy of the eigenvalues obtained from our series is competitive with the most accurate values in the literature [46]. For example, at tenth order we obtain \(\omega_{l,0} = 0.9657626 - 0.08700129i\) which agrees with the \(l = 4\) mode presented in [46] to the five significant figures given there.

\subsection{6.2. Schwarzschild–de Sitter black hole}

The expansion method may also be applied to non-asymptotically flat spacetimes. For example, in Schwarzschild–de Sitter spacetime, the line element takes the form (47) with \(f(r) = 1 - \frac{r}{\Lambda} - \frac{r^2}{3}\) and \(h(r) = r^2\), where \(0 \leq \Lambda \leq 1/9\). The ‘fundamental’ \((n = 0)\) QNMs of the scalar field in SdS are given by
\[
\sqrt{\frac{27}{1 - 9\Lambda}} \omega_{l,n=0} = L - \frac{1}{2} \frac{7 - 61\eta}{216L} - \frac{137 + 186\eta - 2005\eta^2}{7776L^2} + \frac{5230 + 440043\eta - 1274856\eta^2 + 750851\eta^3}{2519424L^3} + \frac{590983 + 37791548\eta - 243504102\eta^2 + 340616636\eta^3 - 135495065\eta^4}{362797056L^4} + O(L^{-5})
\]
(72)
where \(\eta = 9\Lambda\). It is straightforward to extend to fields of higher spin, or to higher modes. The results compare well with, e.g., the sixth-order WKB results of Zhidenko ([47], table 1).

\subsection{6.3. Nariai spacetime}

Nariai spacetime [52] is the natural metric on the Lorentzian version of \(S^2 \times S^2\), corresponding to \(f(r) = g(r) = 1 - r^2\) and \(h(r) = 1\). It provides a valuable check on our ansatz since the QNM frequencies can be determined exactly; for a scalar field with curvature coupling \(V = V_{\text{eff}} = \xi R = 4\xi\),
\[
\omega_{l,n} = \sqrt{L^2 + 4\xi} - \frac{1}{2} - i(n + \frac{1}{2}).
\]
(73)
Our expansion precisely yields the large \(L\) expansion of this exact result, so at least in this case our expansion is not merely asymptotic.

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6.4. Higher-dimensional black hole

An uncharged spherically symmetric higher-dimensional black hole in \( D = d + 2 \) dimensions (i.e., one dimension of time and \( d + 1 \) spatial dimensions) is known as a Schwarzschild–Tangherlini black hole [48], and its metric is

\[
d\mathbf{s}^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2\,d\Omega_1^2\tag{74},
\]

where \( f(r) = g(r) = 1 - (r_h/r)^{d-1} \), and \( r_h \) is the outer horizon. There is a circular orbit at \( r = r_c \) with a critical impact parameter \( b = b_c \) where

\[
r_c = \left(\frac{d+1}{2}\right)^{1/(d-1)} \, r_h \quad \text{and} \quad b_c = \left(\frac{d+1}{d-1}\right)^{1/2} \, r_c. \tag{75}
\]

The gravitational perturbations are divided into three classes, labelled ‘tensor’, ‘vector’ and ‘scalar’ [49, 50]. The tensor perturbations are the easiest to analyse; they satisfy the radial equation

\[
f(r) \frac{d}{dr} \left( f(r) \frac{du}{dr} \right) + \left[ \omega^2 - f(r) \left( \frac{L^2 - (d-1)^2/4}{r^2} + \frac{f'}{2r} + \frac{d(d-2)f(r)}{4} \right) \right] u = 0, \tag{76}
\]

where recall \( L = l + (d - 1)/2 \). The lowest-order (in \( L \)) terms in the ‘tensor’ QNM expansion are

\[
\omega_{l,n=0} = \frac{L}{b_c} - i \frac{(n+1/2)\sqrt{d-1}}{b_c} + \mathcal{O}(L^{-1}), \tag{77}
\]

where we have used the result \( r_c^2 k'_c(r_c) = b_c \sqrt{d - 1} \). This result is, unsurprisingly, consistent with, e.g., equation (51) in [19] and equations (12) and (13) in [31]. It should be possible to apply the method to obtain higher-order terms, and it may be possible to analyse ‘vector’ and ‘scalar’ perturbations in the same way, although the potential for the ‘scalar’ case is not positive definite, and it has more than one extrema [49–51].

6.5. Canonical acoustic hole

Black hole analogues—systems with some of the key properties of black holes—have received much attention in recent years [53, 54]. One of the simplest models for an analogue in fluid mechanics is the ‘canonical acoustic hole’, first proposed in [55]. An ‘horizon’ forms where the bulk speed of the fluid (inwards) exceeds the speed of sound in the fluid. This raises the interesting possibility that QNM frequencies may one day be measured in the laboratory [56]. The effective geometry of the ‘canonical hole’ is given by the line element (47) with \( f(r) = g(r) = 1 - r_h^d/r^4 \) and \( h(r) = r^2 \) (and \( d = 2 \)). The ‘fundamental’ QNM frequencies are given by

\[
b_c \omega_{l,n=0} = L - i \frac{61}{216L} - i \frac{17}{972L^2} - \frac{532843}{2519424L^3} + i \frac{4802843}{5668704L^4} + \mathcal{O}(L^{-5}), \tag{78}
\]

where \( b_c = 3^{3/4}/2^{1/2} r_h \) and \( L = l + 1/2 \), a result which is consistent with equation (53) in [56]. The accuracy of the QNMs obtained via (78) is comparable or superior to the best estimates in the literature. For example, at \( l = 2 \), equation (78) gives \( \omega r_h \approx 1.4726 - 0.0687 \pm [0.010 + 0.011] \), which should be compared with \( \omega r_h \approx 1.41 - 0.70i \) found using a sixth-order WKB scheme [56]. A detailed study of the QNMs of acoustic holes is currently in progress [57].
7. Conclusion

In this paper, we have introduced a new method for obtaining QNM frequencies and wavefunctions through expansion in inverse powers of the angular momentum \( L \) (where \( L = l + 1/2 \) in four dimensions). We illustrated the method by applying to the Schwarzschild spacetime (section 3), to obtain an expansion of the QNM frequency (equation (7) and equations (17)–(22) that may be taken to very high orders (equation (15), tables 1 and 2). We validated the frequency expansion against existing results (table 3), to show that it is accurate and rapidly convergent (figure 2) in the regime \( l \gtrsim 2, l \gtrsim n \) (but generally inaccurate/poorly convergent outside this regime; see figure 1). In section 4, we showed that the expansion method may also be applied to find Regge poles (equations (40)–(45)). In section 5, we generalized the method to treat static spherically symmetric spacetimes of arbitrary spatial dimension. The method was then applied to a selection of five such spacetimes of interest in section 6.

The expansion method complements the standard WKB approach [28–31]. In addition, we believe the method holds certain key advantages. It provides a simple high-order expansion for QNM (and Regge pole) wavefunctions that is convergent everywhere outside the horizon (in the regime of validity \( l \gtrsim 2, l \gtrsim n \)); it yields the complex coefficient \( A_{\text{out}}(l\omega) \) (equations (32) and (33)) which expresses the ratio of flux at infinity to flux at the horizon, and it provides additional insight into the link between orbiting null geodesics and QNMs.

Let us briefly expand on this last point. The crucial step in the new method is a ‘geometric’ ansatz for the wavefunction, given in equations (5) and (59). In section 5.1, we showed how the ansatz is related to the family of ‘critical’ null geodesics that approach from infinity and end in perpetual orbit on the photon sphere at \( r = r_c \). In some sense, in this work we have conducted an expansion about these ‘critical’ geodesics. In section 5.2, we proved a key result: in the eikonal limit \( l \to \infty \), the QNM frequencies depend only upon the properties of the ‘critical’ null geodesics, confirming the result of [19]. The real part of frequency is \( \text{Re}(\omega_{\text{out}}) = L/b_c \), where \( b_c \) is the critical impact parameter (and \( b_c = 1/\Omega_c \) where \( \Omega_c \) is the frequency of the null orbit at \( r = r_c \)), and the imaginary part is \( \text{Im}(\omega_{\text{out}}) = -(n + 1/2)|\lambda| \), where \( |\lambda| \) is the Lyapunov exponent given in equation (54). Our analysis is valid in general for any perturbation of a static spherically symmetric spacetime that satisfies ‘radiative’ conditions at both boundaries \( r_\ast \to \pm \infty \) (a condition which excludes, for example, Schwarzschild AdS).

In the highly damped regime \( n \gg l, n \gg 0 \), it is well known that QNMs have the asymptotic form \( \omega_n \sim \Re - in\Im \) (see, e.g., [63] for a discussion). Here \( \Re \) and \( \Im \) are real constants, where \( \Re \) depends on the geometry and the spin of the field, and \( \Im \) depends only on the geometry. For single-horizon asymptotically flat black holes it has been shown that \( \Im \) is set by the surface gravity of the horizon [64]; for multi-horizon holes the situation is more complicated [65]. A possible challenge for the future would be to adapt the methods of this paper to investigate the highly damped regime \( n \gg l, n \gg 0 \).

It is hoped that the existence and properties (e.g., mass, angular momentum) of astrophysical black holes may one day be inferred from observations of QNM ringing (in lightly damped modes) at gravitational-wave detectors. We have seen that the relevant part of the QNM frequency spectrum is relatively easy to compute; however, it is much more difficult to obtain an estimate of the degree of ringing excited by a given source. The theoretical framework has been in place for over 20 years [7], yet because of the technical complexity only a few attempts at practical calculations have been made [7, 43, 59–62]. This remains an interesting avenue for further study (and the Regge pole formalism, developed in [22, 23, 38, 40], may provide an alternative route to the same objective), particularly in the present era of precision numerical relativity where linear perturbation theory estimates may be
compared directly with nonlinear exact results. A key ingredient in the recipe for computing the degree of QNM ringing is the so-called QNM excitation factor: $B_{\text{ln}} = A_{\text{ln}}(\omega) / \left(2\omega \partial A_{\text{ln}}(\omega)/\partial \omega \right)$. In a forthcoming paper, we will show how the expansion method can be combined with WKB techniques to find $\partial A_{\text{ln}}(\omega)/\partial \omega$ and to compute excitation factors in the large-$L$ limit. We will also show that the singular structure of the retarded Green function near the null cone may be deduce from the large-$L$ asymptotics of QNM wavefunctions and excitation factors [37, 58].

Further extensions of this work are possible. Firstly, we hope it would be a simple task to use the expansion method to explore the examples in section 6 (or others) in greater depth, or to study the Regge pole frequencies and wavefunctions for these spacetimes. Secondly, there remains the question of how the expansion method could be extended to treat the Schwarzschild-AdS case (of interest in the context of the AdS/CFT conjecture [13]), where ‘reflecting’ boundary conditions at $r_* \rightarrow +\infty$ are imposed. Thirdly, how might the expansion method be applied to the Kerr (rotating black hole) spacetime? Here, an added complication is that the angular separation constant depends on the frequency of the perturbation. However, it is known that a one-parameter ($Q$) family of unstable photon orbits at fixed (Boyer–Lindquist coordinate) $r = r_*(Q)$ exist [66], so a geometrically motivated line of enquiry may pay dividends. So far we have had some limited success in finding the large-$L$ asymptotics of the ‘polar’ ($m = 0$) mode of the scalar field. We hope to report on further progress in this direction in the near future.

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