A NOTE ON THE VILENKIN-FOURIER COEFFICIENTS

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Abstract. The main aim of this paper is to find the estimation for Vilenkin-Fourier coefficients.

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Let $P_+$ denote the set of the positive integers, $P := P_+ \cup \{0\}$.
Let $m := (m_0, m_1, \ldots)$ denote a sequence of the positive integers not less than 2.
Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the additive group of integers modulo $m_k$.
Define the group $G_m$ as the complete direct product of the group $Z_{m_j}$ with the product of the discrete topologies of $Z_{m_j}$'s.
The direct product $\mu$ of the measures
$$\mu_k (\{j\}) := 1/m_k, \quad (j \in Z_{m_k})$$
is the Haar measure on $G_m$ with $\mu (G_m) = 1$.
If $\sup_n m_n < \infty$, then we call $G_m$ a bounded Vilenkin group. If the generating sequence $m$ is not bounded then $G_m$ is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups only.
The elements of $G_m$ represented by sequences
$$x := (x_0, x_1, \ldots, x_j, \ldots), \quad (x_k \in Z_{m_k}).$$
It is easy to give a base for the neighborhood of $G_m$
$$I_0 (x) := G_m,$$
$$I_n (x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}, \quad (x \in G_m, \; n \in P).$$
Denote $I_n := I_n (0)$ for $n \in P$ and $\bar{I}_n := G_m \setminus I_n$.
It is evident
$$\bar{I}_N = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1}. \quad (1)$$
If we define the so-called generalized number system based on \( m \) in the following way:

\[
M_0 := 1, \quad M_{k+1} := m_k M_k, \quad (k \in P),
\]

then every \( n \in P \) can be uniquely expressed as \( n = \sum_{k=0}^{\infty} n_j M_j \), where \( n_j \in \mathbb{Z}_{m_j} \) \( (j \in P) \) and only a finite number of \( n_j \)'s differ from zero. Let \( |n| := \max \{ j \in P; n_j \neq 0 \} \).

Denote by \( L_1(G_m) \) the usual (one dimensional) Lebesque space.

Next, we introduce on \( G_m \) an ortonormal system which is called the Vilenkin system.

At first define the complex valued function \( r_k(x) : G_m \to \mathbb{C} \), the generalized Rademacher functions as

\[
\begin{align*}
r_k(x) &:= \exp \left( \frac{2\pi i x k}{m_k} \right), \quad (i^2 = -1, \ x \in G_m, \ k \in P),
\end{align*}
\]

Now define the Vilenkin system \( \psi := (\psi_n : n \in P) \) on \( G_m \) as:

\[
\psi_n(x) := \prod_{k=0}^{\infty} r_k^n(x), \quad (n \in P).
\]

Specifically, we call this system the Walsh-Paley one if \( m = 2 \).

The Vilenkin system is ortonormal and complete in \( L_2(G_m) \) \([1, 7]\).

Now we introduce analogues of the usual definitions in Fourier-analysis. If \( f \in L_1(G_m) \) we can establish the the Fourier coefficients, the partial sums, the Dirichlet kernels with respect to the Vilenkin system in the usual manner:

\[
\begin{align*}
\hat{f}(k) &:= \int_{G_m} f \overline{\psi_k} d\mu, \ (k \in P), \\
S_n f &:= \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \ (n \in P_+, \ S_0 f := 0), \\
D_n &:= \sum_{k=0}^{n-1} \psi_n, \ (n \in P_+).
\end{align*}
\]

Recall that (see \([1]\))

\[
D_{M_n}(x) = \begin{cases} 
M_n, & \text{if } x \in I_n, \\
0, & \text{if } x \notin I_n.
\end{cases}
\]

and

\[
D_n(x) = \psi_n(x) \left( \sum_{j=0}^{\infty} D_{M_j}(x) \sum_{u=m_j-n_j}^{m_j-1} r_j^u(x) \right).
\]

The norm (or quasinorm) of the space \( L_p(G_m) \) is defined by

\[
\|f\|_p := \left( \int_{G_m} |f|^p d\mu \right)^{1/p} \quad (0 < p < \infty).
\]
The σ-algebra generated by the intervals \( \{ I_n(x) : x \in G_m \} \) will be denoted by \( F_n \ (n \in P) \). The conditional expectation operators relative to \( F_n \ (n \in P) \) are denoted by \( E_n \). Then

\[
E_n f(x) = S_{M_n} f(x) = \sum_{k=0}^{M_n-1} \hat{f}(k) w_k = |I_n(x)|^{-1} \int_{I_n(x)} f(x) d\mu(x),
\]

where \( |I_n(x)| = M_n^{-1} \) denotes the length of \( I_n(x) \).

A sequence \( F = (f^{(n)}, n \in P) \) of functions \( f^{(n)} \in L_1(G) \) is said to be a dyadic martingale if (for details see e.g. [8])

(i) \( f^{(n)} \) is \( F_n \) measurable for all \( n \in P \),

(ii) \( E_n f^{(m)} = f^{(n)} \) for all \( n \leq m \).

The maximal function of a martingale \( f \) is denoted by

\[
f^* = \sup_{n \in P} |f^{(n)}|.
\]

In case \( f \in L_1 \), the maximal functions are also be given by

\[
f^*(x) = \sup_{n \in P} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|.
\]

For \( 0 < p < \infty \) the Hardy martingale spaces \( H_p(G_m) \) consist of all martingales for which

\[
\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.
\]

If \( f \in L_1 \), then it is easy to show that the sequence \( (S_{M_n} f : n \in P) \) is a martingale. If \( f = (f^{(n)}, n \in P) \) is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

\[
\hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x) \psi(x) d\mu(x).
\]

The Vilenkin-Fourier coefficients of \( f \in L_1(G_m) \) are the same as those of the martingale \( (S_{M_n} f : n \in P) \) obtained from \( f \).

A bounded measurable function \( a \) is p-atom, if there exist a interval \( I \), such that

\[
\begin{align*}
(a) & \quad \int_I a d\mu = 0, \\
(b) & \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \\
(c) & \quad \text{supp}(a) \subset I.
\end{align*}
\]

The Hardy martingale spaces \( H_p(G) \) for \( 0 < p \leq 1 \) have an atomic atomic characterization (see [8]):

**Theorem W 1.** A martingale \( f = (f^{(n)}, n \in P) \) is in \( H_p(0 < p \leq 1) \) if and only if there exist a sequence \( (a_k, k \in P) \) of p-atoms and a sequence \( (\mu_k, k \in P) \) of a real numbers such that for every \( n \in P \)
\[ \sum_{k=0}^{\infty} \mu_k S_{M_k} a_k = f^{(n)}, \]

Further, \( \|f\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p} \), where the infimum is taken over all decomposition of \( f \) of the form (6).

Moreover, \( \|f\|_{H_p} \leq \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p} \), where the infimum is taken over all decomposition of \( f \) of the form (6).

It follows that if any operator is uniformly bounded on the p-atom, then it is a bounded operator from the Hardy space \( H_p \) to the space \( L_p \). Moreover, the following theorem is true (see [9]):

**Theorem W2.** Suppose that an operator \( T \) is sublinear and for some \( 0 < p \leq 1 \)

\[ \int a \leq c \|T a\|_{L_p(G_m)} \leq c \|T f\|_{L_p(G_m)}. \]

for every p-atom \( a \), where \( I \) denote the support of the atom. If \( T \) is bounded from \( L_{\infty} \) to \( L_{\infty} \), then

\[ \|T f\|_{L_p(G_m)} \leq c \|T f\|_{L_p(G_m)}. \]

The classical inequality of Hardy type is well known in the trigonometric as well as in the Vilenkin-Fourier analysis. Namely,

\[ \sum_{k=1}^{\infty} \left| \hat{f}(k) \right| k \leq c \|f\|_{H_1}, \]

where the function \( f \) belongs to the Hardy space \( H_1 \) and \( c \) is an absolute constant. This was proved in the trigonometric case by Hardy and Littlewood [5] (see also Coifman and Weiss [1]) and for Walsh system in [6].

Weisz [8, 10] generalized this result for Vilenkin system and proved:

\[ \sum_{k=1}^{\infty} \left| \hat{f}(k) \right| k^{2-p} \leq c \|f\|_{H_p}, \]

for all \( f \in H_p \) \( (0 < p \leq 2) \).

It is also well-known (see [11]) that

\[ \hat{f}(n) \to 0, \text{ when } n \to \infty, \]

for all \( f \in L_1(G_m) \), where \( \hat{f}(n) \) denotes n-th Fourier coefficients of the function \( f \).

The main aim of this paper is to prove that the following theorem is true:

**Theorem 1.** a) Let \( 0 < p < 1 \) and \( f \in H_p(G_m) \). Then there exists an absolute constant \( c_p \), depend only \( p \), such that
\[
\left| \hat{f}(n) \right| \leq c_p n^{1/p-1} \|f\|_{H_p}.
\]

b) Let \(0 < p < 1\) and \(\Phi(n)\) is any nondecreasing, nonnegative function, satisfying condition
\[
\lim_{n \to \infty} \frac{n^{1/p-1}}{\Phi(n)} = \infty,
\]
then there exists a martingale \(f_0 \in H_p(G_m)\), such that
\[
\lim_{n \to \infty} \left| \hat{f}_0(n) \right| = \infty.
\]

**Proof of Theorem 1.** At first we prove that the maximal operator
\[
\tilde{S}_p^* (f) := \sup_{n \in P} \frac{|S_n f|}{(n+1)^{1/p-1}}
\]
is bounded from the Hardy space \(H_p\) to the space \(L_p\) for \(0 < p < 1\). Since the maximal operator \(\tilde{S}_p^*\) is bounded from \(L_\infty\) to \(L_\infty\) by Theorem W 2 we obtain that the proof of theorem will be complete, if we show that
\[
\int_{I_N} \left( \sup_{n \in P} \frac{|S_n a|}{(n+1)^{1/p-1}} \right)^p d\mu \leq c < \infty, \quad \text{when } 0 < p < 1,
\]
for every \(p\)-atom \(a\) \((0 < p \leq 1)\), where \(I\) denote the support of the atom. Let \(a\) be an arbitrary \(p\)-atom with support \(I\) and \(\mu(I) = M_N^{-1}\). We may assume that \(I = I_N\). It is easy to see that \(S_n(a) = 0\) when \(n \leq M_N\). Therefore we can suppose that \(n > M_N\).

Let \(x \in I_t \setminus I_{t+1}\). Combining (2) and (3) we have
\[
|D_n(x)| \leq \sum_{j=0}^{t} n_j D_{M_j}(x) = \sum_{j=0}^{t} n_j M_j \leq c M_t.
\]

Since \(t \in I_N\) and \(x \in I_s \setminus I_{s+1}\), \(s = 0, \ldots, N-1\), we obtain that \(x - t \in I_s \setminus I_{s+1}\). Using (8) we get
\[
|D_n(x-t)| \leq c M_s,
\]
and
\[
\int_{I_N} |D_n(x-t)| d\mu(t) \leq c \frac{M_s}{M_N}.
\]

Hence
\begin{equation}
\frac{|S_n (a)|}{(n + 1)^{1/p - 1}} \leq \int_{I_N} |a (t)| \left| \frac{D_n (x - t)}{(n + 1)^{1/p - 1}} \right| d\mu (t)
\end{equation}
\begin{equation}
\leq \frac{\|a\|_{\infty}}{(n + 1)^{1/p - 1}} \int_{I_N} |D_n (x - t)| d\mu (t)
\leq \frac{M_N^{1/p}}{(n + 1)^{1/p - 1}} \int_{I_N} |D_n (x - t)| d\mu (t)
\leq \frac{c M_N^{1/p}}{M_N^{1/p - 1} M_N} = c M_s.
\end{equation}

Combining (11) and (10) we have
\begin{equation}
\int_{I_N} \left| \tilde{S}_p^s a (x) \right|^p d\mu (x)
= \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \tilde{S}_p^s a (x) \right|^p d\mu (x)
\leq c \sum_{s=0}^{N-1} \frac{M_s^p}{M_s} < c < \infty.
\end{equation}

Now we are ready to prove the main result. Let $0 < p < 1$. Then
\begin{equation}
\left| \hat{f} (n) \right| = |S_{n+1} f - S_n f| \leq 2 \sup_{n \in P} |S_n f|
\end{equation}
and
\begin{equation}
\left( \frac{n + 1}{n+1} \right)^{1/p - 1} \leq 2 \sup_{n \in P} |S_n f|.
\end{equation}

Consequently,
\begin{equation}
\left( \frac{n + 1}{n+1} \right)^{1/p - 1} \leq \left\| \frac{\sup_{n \in P} |S_n f|}{(n + 1)^{1/p - 1}} \right\|_p \leq c_p \|f\|_{H_p}.
\end{equation}

It follows that
\begin{equation}
\left| \hat{f} (n) \right| \leq c_p n^{1/p - 1} \|f\|_{H_p}.
\end{equation}

b) In the proof of second part of theorem we follow the method of Blahota, Gát and Goginava (see [2, 3]).
Let $0 < p < 1$ and $\Phi(n)$ is any nondecreasing, nonnegative function, satisfying condition

\[(12) \quad \lim_{n \to \infty} n^{1/p - 1} \Phi(n) = \infty,\]

then for every $0 < p < 1$, there exists an increasing sequence $\{\alpha_k : k \in P\}$ of the positive integers such that:

\[(13) \quad \sum_{k=0}^{\infty} \left( \frac{\Phi(M_{\alpha_k})}{M_{\alpha_k}^{1/p-1}} \right)^{p/2} < \infty\]

Let

\[f(A)(x) = \sum_{\{k : \alpha_k < A\}} \lambda_k a_k,\]

where

\[\lambda_k = \left( \frac{\Phi(M_{\alpha_k})}{M_{\alpha_k}^{1/p-1}} \right)^{1/2}\]

and

\[a_k(x) := \frac{M_{\alpha_k}^{1/p-1}}{M} \left( D_{M\alpha_k+1}(x) - D_{M\alpha_k}(x) \right).\]

It is easy to show that the martingale $f = (f(1), f(2), ... f(A), ...)$ $\in H_p$.

Indeed, since

\[(14) \quad S_{M\alpha_k} a_k(x) = \begin{cases} a_k(x), & \alpha_k < A, \\ 0, & \alpha_k \geq A, \end{cases}\]

\[\text{supp}(a_k) = I_{\alpha_k},\]

\[\int_{I_{\alpha_k}} a_k d\mu = 0\]

and

\[\|a_k\|_{\infty} \leq \frac{M_{\alpha_k}^{1/p-1}}{M} M_{\alpha_k+1} \leq (M_{\alpha_k})^{1/p} = (\text{supp } a_k)^{-1/p}\]

if we apply Theorem W 1 and (13) we conclude that $f \in H_p$.

It is easy to show that

\[(15) \quad \hat{f}(j) = \begin{cases} \frac{1}{M} M_{\alpha_k}^{(1/p-1)/2} \Phi^{1/2}(M_{\alpha_k}), & \text{if } j \in \{M_{\alpha_k}, ..., M_{\alpha_k+1} - 1\}, k = 0, 1, 2, ... \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{M_{\alpha_k}, ..., M_{\alpha_k+1} - 1\}. \end{cases}\]
It follows that

\[
\lim_{n \to \infty} \frac{\hat{f}(n)}{\Phi(n)} \geq \lim_{k \to \infty} \frac{\hat{f}(M_{\alpha_k})}{\Phi(M_{\alpha_k})} \\
\geq \lim_{k \to \infty} \frac{1}{M \Phi(M_{\alpha_k})} \left( M_{\alpha_k}^{(1/p-1)/2} \Phi^{1/2}(M_{\alpha_k}) \right)^{1/2} \\
\geq \lim_{k \to \infty} \left( \frac{M_{\alpha_k}^{(1/p-1)}}{\Phi(M_{\alpha_k})} \right)^{1/2} = \infty.
\]

Theorem 1 is proved.

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