A new bidirectional generalization of (2+1)-dimensional matrix k-constrained KP hierarchy

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We introduce a new bidirectional generalization of (2+1)-dimensional k-constrained KP hierarchy ((2+1)-BDk-cKPH). This new hierarchy generalizes (2+1)-dimensional k-cKP hierarchy, \((t_A, \tau_B)\) and \((\gamma_A, \sigma_B)\) matrix hierarchies. (2+1)-BDk-cKPH contains a new matrix (1+1)-k-constrained KP hierarchy. Some members of (2+1)-BDk-cKPH are also listed. In particular, it contains matrix generalizations of DS systems, (2+1)-dimensional modified Korteweg-de Vries equation and the Nizhnik equation. (2+1)-BDk-cKPH also includes new matrix (2+1)-dimensional generalizations of the Yajima-Oikawa and Melnikov systems. Binary Darboux Transformation Dressing Method is also proposed for construction of exact solutions for equations from (2+1)-BDk-cKPH. As an example the exact form of multi-soliton solutions for vector generalization of the Davey-Stewartson system is given.

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I. INTRODUCTION

In the modern theory of nonlinear integrable systems, algebraic methods play an important role (see the survey in [1]). Among them there are the Zakharov-Shabat dressing method\(^2^,\) Marchenko’s method\(^5\), and an approach based on the Darboux-Crum-Matveev transformations\(^6,7\). Algebraic methods allow us to omit analytical difficulties that arise in the investigation of corresponding direct and inverse scattering problems for nonlinear equations. A significant contribution to such methods has also been made by the Kioto group\(^8^-^1^2\). In particular, they investigated scalar and matrix hierarchies for nonlinear integrable systems of Kadomtsev-Petviashvili type (KP hierarchy).

The KP hierarchy is of fundamental importance in the theory of integrable systems and shows up in various ways in mathematical physics. Several extensions and generalizations of it have been obtained. For example, the multi-component KP hierarchy contains several physically relevant nonlinear integrable systems, including the Davey-Stewartson equation, the two-dimensional Toda lattice and the three-wave resonant interaction system. There are several equivalent formulations of this hierarchy: matrix pseudo-differential operator (Sato) formulation, \(\tau\)-function approach via matrix Hirota bilinear identities, multi-component free fermion formulation. Another kind of generalization is the so-called “KP equation with self-consistent sources” (KPSCS), discovered by Melnikov\(^1^3^-^1^7\). In\(^1^8^-^2^2\), k-symmetry constraints of the KP hierarchy which have connections with KPSCS were investigated. The resulting k-constrained KP (k-cKP) hierarchy contains physically relevant systems like the nonlinear Schrödinger equation, the Yajima-Oikawa system, a generalization of the Boussinesq equation, and the Melnikov system. Multi-component generalizations of the k-cKP hierarchy were introduced in\(^2^3\). In the papers\(^2^4^-^2^7\) the differential type of the gauge transformation operator was applied to the k-constrained KP hierarchy at first. In\(^2^8\) differential and integral type of the gauge transformation operators were applied to k-cKP hierarchy. In\(^2^9\) the binary Darboux transformations for dressing of the multi-component generalizations of the k-cKP hierarchy were investigated. A modified k-constrained KP (k-cmKP) hierarchy was proposed in\(^2^1,2^3,2^6\). It contains, for example, the vector Chen-Lee-Liu and the modified KdV (mKdV) equation. Multi-component versions of the Kundu-Eckhaus and Gerdjikov-Ivanov\(^3^2,3^3\) equations were also obtained in\(^3^0\), via gauge transformations of the k-cKP, respectively the k-cmKP hierarchy.
Moreover, in\textsuperscript{34,35}, (2+1)-dimensional extensions of the k-cKP hierarchy were introduced and dressing methods via differential transformations were investigated. In\textsuperscript{36,37}, exact solutions for some representatives of the (2+1)-dimensional k-cKP hierarchy were obtained by dressing binary Darboux transformations. This hierarchy was also rediscovered recently in\textsuperscript{38}. Dressing methods via differential transformations for this hierarchy and its modified version were investigated in\textsuperscript{39}. The (2+1)-dimensional k-cKP hierarchy in particular contains the DS-III ($k = 1$), Yajima-Oikawa ($k = 2$) and Melnikov ($k = 3$) hierarchies. The corresponding Lax representations of this hierarchy (see (9)) consists of one differential and one integro-differential operator. Our aim was to generalize the (2+1)-dimensional k-cKP hierarchy (9) to the case of two integro-differential operators in Lax pair. It is essential to call this new hierarchy a bidirectional generalization of (2+1)-dimensional k-cKP hierarchy or simply (2+1)-BDk-cKP hierarchy (see (30)). We will consider this hierarchy in the most general matrix case. Another aim of our investigation is a construction of the Binary Darboux Transformation Dressing Method for (2+1)-BDk-cKP hierarchy (30).

This work is organized as follows. In Section 2 we present a short survey of results on constraints for KP hierarchies and their (2+1)-dimensional generalizations. In Section III we introduce (2+1)-BDk-cKP hierarchy. Members of the obtained hierarchy are also listed there. (2+1)-BDk-cKP hierarchy contains matrix generalizations of Davey-Stewartson hierarchy (first members of it are two different matrix versions of DS-III system: DS-III-a (48) and DS-III-b (49)), new Yajima-Oikawa and Melnikov hierarchies. In Section IV we consider dressing via binary Darboux transformation for the hierarchy constructed in Section III. Exact forms of solutions for some members of this hierarchy are also presented here in terms of Grammians. In the final section, we discuss the obtained results and mention problems for further investigations. Some examples of Lax representations from the bidirectional generalization of (2+1)-dimensional k-constrained modified KP hierarchy ((2+1)-BDk-cmKP hierarchy) are also presented there.

II. $k$-CONSTRAINED KP HIERARCHY AND ITS EXTENSIONS

To make this paper self-contained, we briefly introduce the KP hierarchy\textsuperscript{1}, its k-symmetry constraints (k-cKP hierarchy), and the extension of the k-cKP hierarchy to the (2+1)-
A Lax representation of the KP hierarchy is given by

\[ L_{t_n} = [B_n, L], \quad n \geq 1, \]  

(1)

where \( L = D + U_1 D^{-1} + U_2 D^{-2} + \ldots \) is a scalar pseudodifferential operator, \( t_1 := x, \ D := \frac{\partial}{\partial x} \), and \( B_n := (L^n)_+ := (L^n)_{\geq 0} = D^n + \sum_{i=0}^{n-2} u_i D^i \) is the differential operator part of \( L^n \). The consistency condition (zero-curvature equations), arising from the commutativity of flows (1), is

\[ B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0. \]  

(2)

Let \( B_n^\tau \) denote the formal transpose of \( B_n \), i.e. \( B_n^\tau := (-1)^n D^n + \sum_{i=0}^{n-2} (-1)^i D^i u_i^\top \), where \( ^\top \) denotes the matrix transpose. We will use curly brackets to denote the action of an operator on a function whereas, for example, \( B_n q \) means the composition of the operator \( B_n \) and the operator of multiplication by the function \( q \). The following formula holds for \( B_n q \) and \( B_n \{ q \} \):

\[ B_n \{ q \} = B_n q - (B_n q)_{>0}. \]  

The k-cKP hierarchy \[24,25\] is given by

\[ L_{t_n} = [B_n, L], \]  

(3)

with the k-symmetry reduction

\[ L_k := L^k = B_k + q D^{-1} r. \]  

(4)

The hierarchy given by (3)-(4) admits the Lax representation (here \( k \in \mathbb{N} \) is fixed):

\[ [L_k, M_n] = 0, \quad L_k = B_k + q D^{-1} r, \quad M_n = \partial_{t_n} - B_n. \]  

(5)

Lax equation (5) is equivalent to the following system:

\[ [L_k, M_n]_{\geq 0} = 0, \quad M_n \{ q \} = 0, \quad M_n^\tau \{ r \} = 0. \]  

(6)

Below we will also use the formal adjoint \( B_n^* := B_n^\tau = (-1)^n D^n + \sum_{i=0}^{n-2} (-1)^i D^i u_i^\top \) of \( B_n \), where * denotes the Hermitian conjugation (complex conjugation and transpose). In \[23\], multi-component (vector) generalizations of the k-cKP hierarchy were introduced,

\[ L_k := L^k = B_k + \sum_{i=1}^m \sum_{j=1}^m q_i m_{ij} D^{-1} r_j = B_k + q \mathcal{M}_0 D^{-1} r^\top, \]  

(7)

where \( q = (q_1, \ldots, q_m) \) and \( r = (r_1, \ldots, r_m) \) are vector functions, \( \mathcal{M}_0 = (m_{ij})_{i,j=1}^m \) is a constant \( m \times m \) matrix. \( \mathcal{M}_0 \) can be annihilated by the change of functions: \( q \mathcal{M}_0 \rightarrow q, \)
\( r^\top \rightarrow r^\top \) (or \( q \rightarrow q, \mathcal{M}_0 r^\top \rightarrow r^\top \)). However, we remain this matrix because it plays an important role in Section IV where we consider dressing methods for \((2+1)\text{-BDk-cKP hierarchy (30)}\). We shall note that extensions of hierarchies given by (3) and (7), namely \((2+1)\)-dimensional k-cKP (10) and \((2+1)\text{-BDk-cKP (40), (42)}\) hierarchies are represented analogously to (6) (with more general \( L_k \) and \( M_n \) operators).

For \( k = 1 \), the hierarchy given by (3) and (7) is a multi-component generalization of the AKNS hierarchy. For \( k = 2 \) and \( k = 3 \), one obtains vector generalizations of the Yajima-Oikawa and Melnikov hierarchies, respectively.

In \( 21,30,31 \), a k-constrained modified KP (k-cmKP) hierarchy was introduced and investigated. Its Lax representation has the form

\[
[\tilde{L}_k, \tilde{M}_n] = 0, \quad \tilde{L}_k = \tilde{B}_k + q \mathcal{M}_0 D^{-1} r^\top D, \quad \tilde{M}_n = \partial_{t_n} - \tilde{B}_n, \tag{8}
\]

where \( \tilde{B}_k = D^k + \sum_{j=1}^{k-1} w_j D^j \). For \( k = 1, 2, 3 \), this leads to vector generalizations of the Chen-Lee-Liu, the modified multi-component Yajima-Oikawa and Melnikov hierarchies.

An essential extension of the k-cKP hierarchy is its \((2+1)\)-dimensional generalization \( 34,35 \), given by

\[
L_k = \beta_k \partial_{\tau_k} - B_k - q \mathcal{M}_0 D^{-1} r^\top, \quad M_n = \alpha_n \partial_{t_n} - A_n, \tag{9}
\]

\[
B_k = D^k + \sum_{j=0}^{k-2} u_j D^j, \quad A_n = D^n + \sum_{i=0}^{n-2} v_i D^i, \notag
\]

where \( u_j \) and \( v_i \) are scalar functions, \( q \) and \( r \) are \( m \)-component vector-functions. Lax equation \([L_k, M_n] = 0\) is equivalent to the system:

\[
[L_k, M_n]_{\geq 0} = 0, \quad M_n \{q\} = 0, \quad M_n^r \{r\} = 0. \tag{10}
\]

System (10) can be rewritten as

\[
\alpha_n B_{k,t_n} = \beta_k A_{n,\tau_k} + [A_n, B_k] + ([A_n, q \mathcal{M}_0 D^{-1} r^\top])_{\geq 0}, \quad M_n \{q\} = 0, \quad M_n^r \{r\} = 0.
\]

We list some members of this \((2+1)\)-dimensional generalization of the k-cKP hierarchy:

1. \( k = 0, n = 2 \).

\[
L_0 = \beta_0 \partial_{\tau_0} - q \mathcal{M}_0 D^{-1} r^\top, \quad M_2 = \alpha_2 \partial_{\tau_2} - D^2 - v_0. \tag{11}
\]

The commutator equation \([L_0, M_2] = 0\) is equivalent to the system:

\[
\alpha_2 q_{\tau_2} = q_{xx} + v_0 q, \quad -\alpha_2 r_{\tau_2}^\top = r_{xx}^\top + r^\top v_0, \quad \beta_0 v_{0,\tau_0} = -2(q \mathcal{M}_0 r^\top)_x. \tag{12}
\]
After the reduction $\beta_0 \in \mathbb{R}$, $\alpha_2 \in i\mathbb{R}$, $r = \mathbf{q}$, $\mathcal{M}_0 = \mathcal{M}_0^*$, $v_0 = \bar{v}_0$, the operators $L_1$ and $M_2$ in (11) are skew-Hermitian and Hermitian, respectively, and (12) becomes the DS-III system:

$$\alpha_2 q_{t_2} = q_{xx} + v_0 q, \quad \beta_0 v_{0, \tau_0} = -2(q \mathcal{M}_0 q^*)_x. \quad (13)$$

2. $k = 1, n = 2$. Then (9) has the form

$$L_1 = \beta_1 \partial_{\tau_1} - D - q \mathcal{M}_0 D^{-1} r^\top, \quad M_2 = \alpha_2 \partial_{t_2} - D^2 - v_0, \quad (14)$$

and the equation $[L_1, M_2] = 0$, is equivalent to the system,

$$\alpha_2 q_{t_2} = q_{xx} + v_0 q, \quad \alpha_2 r_{t_2} = -r_{xx} - v_0 r, \quad \beta_1 v_{0, \tau_1} = v_{0, x} - 2(q \mathcal{M}_0 r^*)_x. \quad (15)$$

After the reduction $\beta_1 \in \mathbb{R}$, $\alpha_2 \in i\mathbb{R}$, $r = \mathbf{q}$, $\mathcal{M}_0 = \mathcal{M}_0^*$, $v_0 = \bar{v}_0$, we obtain the DS-III system in the following form:

$$\alpha_2 q_{t_2} = q_{xx} + v_0 q, \quad \beta_1 v_{0, \tau_1} = v_{0, x} - 2(q \mathcal{M}_0 q^*)_x. \quad (16)$$

In Remark 1 we will show that DS-III systems (13) and (16) as well as all the other equations of the hierarchy (9) related to the operators $L_0$ and $L_1$ in (9) are equivalent via linear change of independent variables.

3. $k = 1, n = 3$. Now (9) becomes

$$L_1 = \beta_1 \partial_{\tau_1} - D - q \mathcal{M}_0 D^{-1} r^\top, \quad M_3 = \alpha_3 \partial_{t_3} - D^3 - v_1 D - v_0. \quad (17)$$

After the additional reduction $\alpha_3, \beta_1 \in \mathbb{R}$, $\mathcal{M}_0 = \mathcal{M}_0^*$, $v_1 = \bar{v}_1$, $\bar{v}_0 + v_0 = v_{1x}$, the operators $L_1$, $M_3$ in (17) are skew-Hermitian, and the Lax equation $[L_1, M_3] = 0$ is equivalent to the following $(2+1)$-dimensional generalization of the mKdV system:

$$\alpha_3 q_{t_3} = q_{xxx} + v_1 q_x + v_0 q, \quad \beta_1 v_{0, \tau_1} = v_{0, x} - 3(q \mathcal{M}_0 q^*)_x, \quad \beta_1 v_{1, \tau_1} = v_{1, x} - 3(q \mathcal{M}_0 q^*)_x. \quad (18)$$

This system admits the real version ($\mathcal{M}_0^\top = \mathcal{M}_0^*$, $q^* = q^\top$)

$$\alpha_3 q_{t_3} = q_{xxx} + v_1 q_x + \frac{1}{2} v_{1, x} q, \quad \beta_1 v_{1, \tau_1} = v_{1, x} - 3(q \mathcal{M}_0 q^*)_x. \quad (19)$$
4. $k = 2, n = 2$. (9) takes the form

$$L_2 = \beta_2 \partial_{\tau_2} - D^2 - u_0 - q M_0 D^{-1} r^\top, \quad M_2 = \alpha_2 \partial_{\tau_2} - D^2 - u_0. \quad (20)$$

Under the reduction $\alpha_2, \beta_2 \in i \mathbb{R}, \ r = \bar{q}, \ M_0 = -M_0^\ast, \ u_0 = \bar{u}_0 := 2u,$ the operators $L_2$ and $M_2$ in (20) become Hermitian and the Lax equation $[L_2, M_2] = 0$ is equivalent to the following system:

$$\alpha_2 q_{t_2} = q_{xx} + 2u q, \quad \alpha_2 u_{t_2} = \beta_2 u_{\tau_2} + (q M_0 q^\ast)_{\tau_2}, \quad (21)$$

which is a $(2+1)$-dimensional vector generalization of the Yajima-Oikawa system.

5. $k = 2, n = 3$. Now (9) becomes

$$L_2 = \beta_2 \partial_{\tau_2} - D^2 - 2u_0 - q M_0 D^{-1} r^\top, \quad M_3 = \alpha_3 \partial_{\tau_3} - D^3 - 3u_0 D - \frac{3}{2} \left(u_{0,x} + \beta_2 D^{-1} \{u_{0,\tau_2}\} + q M_0 r^\top\right). \quad (22)$$

In formula (22), $D^{-1} \{u_{0,\tau_2}\}$ denotes indefinite integral of the function $u_{0,\tau_2}$ with respect to $x$. With the additional reduction $\beta_2 \in i \mathbb{R}, \ \alpha_3 \in \mathbb{R}, \ u_0 = \bar{u}_0 := u, \ M_0 = -M_0^\ast$ and $r = \bar{q}$, this is equivalent to the following generalization of the higher Yajima-Oikawa system\textsuperscript{32} (in scalar case ($m = 1$) Manakov LAB-triad for this equation was proposed in\textsuperscript{14}):

$$\alpha_3 q_{t_3} = q_{xxx} + 3u q_x + \frac{3}{2} (u_x + \beta_2 D^{-1} \{u_{\tau_2}\} + q M_0 q^\ast) q, \quad [\alpha_3 u_{t_3} - \frac{1}{4} u_{xxx} - 3u u_x + \frac{3}{4} (q M_0 q_x^\ast - q_x M_0 q^\ast)_{\tau_2} - \frac{3}{4} \beta_2 (q M_0 q^\ast)_{\tau_2}] = \frac{3}{4} \beta_2^2 u_{\tau_2} u_{\tau_2}. \quad (23)$$

6. $k = 3, n = 3$. (9) takes the form

$$L_3 = \beta_3 \partial_{\tau_3} - D^3 - u_1 D - \frac{1}{2} u_{1,x} - q M_0 D^{-1} r^\top, \quad M_3 = \alpha_3 \partial_{\tau_3} - D^3 - u_1 D - \frac{1}{2} u_{1,x}. \quad (24)$$

With the additional reduction $\alpha_3, \beta_3 \in \mathbb{R}, \ u_1 = \bar{u}_1 := u, \ M_0 = M_0^\ast = \bar{M}_0$ and $r = \bar{q} = q$, this is equivalent to the following $(2+1)$-dimensional generalization of the Drinfeld-Sokolov-Wilson equation\textsuperscript{40-42}:

$$\alpha_3 q_{t_3} = q_{xxx} + u q_x + \frac{1}{2} u_x q, \quad \alpha_3 u_{t_3} = \beta_3 u_{\tau_3} + 3(q M_0 q^\ast)_{\tau_2}. \quad (25)$$

The following remark establishes connections between equations represented by Lax pairs $(L_0, M_k)$ and $(L_1, M_k), (L_k, M_k)$ and $(L_1, M_k)$ respectively (see formulae (2)).
Remark 1. The change of variables $\tilde{x} := x - \frac{1}{\beta_0} \tau_0$, $\tau_1 := \frac{\beta_1}{\beta_0} \tau_0$, $\tilde{q}(\tilde{x}, \tau_1) := q(x, \tau_0)$, $\tilde{v}_0(\tilde{x}, \tau_1) := v_0(x, \tau_0)$ maps DS-III equation (13) to another form (16) of it. The same change maps the "higher" equations related to operator $L_0$ to the equations related to $L_1$ (see formulae (9)).

Remark 2. The change of variables $\tilde{x} := x - \frac{1}{\beta_k} \tau_k$, $\tilde{t}_k := t_k + \frac{\alpha_k}{\beta_k} \tau_k$, $\tau_1 := \frac{\beta_1}{\beta_k} \tau_k$, $v_k-2(\tilde{x}, \tilde{t}_k, \tau_1) := u_k-2(x, t_k, \tau_k)$, $\tilde{q}(\tilde{x}, \tilde{t}_k, \tau_1) := q(x, t_k, \tau_k)$, (26)

maps (2+1)-dimensional generalization of Yajima-Oikawa system (21) to DS-III system (16) in the case $k = 2$. In the case $k = 3$ it maps (2+1)-extension of Drinfeld-Sokolov-Wilson equation (27) to the real version of (2+1)-extension of mKdV equation (19). An analogous relation holds between all other equations with Lax pairs $(L_k, M_k)$ and $(L_1, M_k)$ where the operators $L_1$, $L_k$ and $M_k$ are defined by formulae (9).

Thus, for $k = 1$ we have the DS-III hierarchy (its first members are DS-III $(k = 1, n = 2)$ and a special (2+1)-dimensional extension of mKdV $(k = 1, n = 3)$, see (16) and (18). For $k = 2$, $k = 3$, we have (2+1)-dimensional generalizations of the Yajima-Oikawa (in particular, it contains (21) and (23)) and the Melnikov hierarchy, respectively.

III. BIDIRECTIONAL GENERALIZATIONS OF (2+1)-DIMENSIONAL K-CONSTRAINED KP HIERARCHY

In this section we introduce a new generalization of the (2+1)-dimensional k-constrained KP hierarchy given by (9) to the case of two integro-differential operators. One of them (the operator $M_{n,l}$ in (30)) generalizes the corresponding operator $M_n$ (9) and depends on two independent indices $l$ and $n$. It leads to generalization of (2+1)-dimensional k-cKP hierarchy (9) in additional direction $l$ ($l = 1, 2, \ldots$). We do not consider the case $l = 0$ because of the Remark $\S$. For further purposes we will use the following well-known formulae for integral operator $h_1 D^{-1} h_2$ constructed by matrix-valued functions $h_1$ and $h_2$ and the differential operator $A$ with matrix-valued coefficients in the algebra of pseudodifferential operators:

\begin{align*}
Ah_1 D^{-1} h_2 &= (Ah_1 D^{-1} h_2)_{\geq 0} + A\{h_1\} D^{-1} h_2, \quad (27) \\
h_1 D^{-1} h_2 A &= (h_1 D^{-1} h_2 A)_{\geq 0} + h_1 D^{-1} [A^r \{h_2^T\}]^T, \quad (28)
\end{align*}
Consider the following bidirectional generalization of \((2+1)\)-dimensional \(k\)-constrained KP hierarchy \((9)\):

\[
L_k = \beta_k \partial_{\tau_n} - B_k - \mathbf{q} \mathbf{M}_0 D^{-1} \mathbf{r}^\top, \quad B_k = \sum_{j=0}^{k} u_j D^j, \quad u_j = u_j(x, \tau_k, t_n), \quad \beta_k \in \mathbb{C},
\]

\[
M_{n,l} = \alpha_n \partial_{\tau_n} - A_n - c_l \sum_{j=0}^{l} \mathbf{q}(j) \mathbf{M}_0 D^{-1} \mathbf{r}^\top[l - j], \quad l = 1, \ldots
\]

\[
A_n = \sum_{i=0}^{n} v_i D^i, \quad v_i = v_i(x, \tau_k, t_n), \quad \alpha_n \in \mathbb{C},
\]

where \(u_j\) and \(v_i\), \(\mathbf{q}\) and \(\mathbf{r}\) are \(N \times N\) and \(N \times M\) matrix functions respectively; \(\mathbf{q}(j)\) and \(\mathbf{r}(j)\) are matrix functions of the following form: \(\mathbf{q}(j) := (L_k)^j\{\mathbf{q}\}, \quad \mathbf{r}^\top[j] := ((L_k^*)^j\{\mathbf{r}\})^\top\).

The following theorem holds.

**Theorem 1.** The Lax equation \([L_k, M_{n,l}] = 0\) is equivalent to the system:

\[
[L_k, M_{n,l}] \geq 0 = 0, \quad M_{n,l}\{\mathbf{q}\} = c_l (L_k)^{l+1}\{\mathbf{q}\}, \quad M_{n,l}\{\mathbf{r}\} = c_l (L_k^{*})^{l+1}\{\mathbf{r}\}.
\]

Proof. From the equality \([L_k, M_{n,l}] = [L_k, M_{n,l}] \geq 0 + [L_k, M_{n,l}] < 0\) we obtain that the Lax equation \([L_k, M_{n,l}] = 0\) is equivalent to the following one:

\[
[L_k, M_{n,l}] \geq 0 = 0, \quad [L_k, M_{n,l}] < 0 = 0.
\]

Thus, it is sufficient to prove that \([L_k, M_{n,l}] < 0 = 0 \iff (M_{n,l} - c_l (L_k)^{l+1})\{\mathbf{q}\} = (M_{n,l}^{*} - c_l (L_k^{*})^{l+1})\{\mathbf{r}\} = 0\). Using bi-linearity of the commutator and explicit form \((30)\) of the operators \(L_k\) and \(M_{n,l}\) we obtain:

\[
[L_k, M_{n,l}] < 0 = c_l \sum_{j=0}^{l} \mathbf{q}(j) \mathbf{M}_0 D^{-1} \mathbf{r}^\top[l - j], \quad \beta_k \partial_{\tau_k} - B_k < 0 +
\]

\[
+c_l \sum_{j=0}^{l} \mathbf{q} \mathbf{M}_0 D^{-1} \mathbf{r}^\top \mathbf{q}(j) \mathbf{M}_0 D^{-1} \mathbf{r}^\top[l - j] < 0 + [\alpha_n \partial_{\tau_n} - A_n, \mathbf{q} \mathbf{M}_0 D^{-1} \mathbf{r}^\top] < 0.
\]

After direct computations of each of the three items on the right-hand side of formula \((33)\) we obtain:

1.

\[
c_l \sum_{j=0}^{l} \mathbf{q}(j) \mathbf{M}_0 D^{-1} \mathbf{r}^\top[l - j], \quad \beta_k \partial_{\tau_k} - B_k < 0 = -c_l \sum_{j=0}^{l} (\beta_k \mathbf{q}_{\tau_k}(j) - B_k\{\mathbf{q}(j)\}) \cdot
\]

\[
\mathbf{M}_0 D^{-1} \mathbf{r}^\top[l - j] - c_l \sum_{j=0}^{l} \mathbf{q}(j) \mathbf{M}_0 D^{-1} (\beta_k \mathbf{r}^\top_{\tau_k}[l - j] + B_k^* \{\mathbf{r}^\top[l - j]\}).
\]

Equality \((34)\) is a consequence of formulae \((27)-(28)\).
2. From formulae (33)-(36) we have
\[
\begin{align*}
  &c_l \sum_{j=0}^{l} [q \mathcal{M}_0 D^{-1} r^\top, q[j] \mathcal{M}_0 D^{-1} r^\top [l - j]]_{<0} = c_l \sum_{j=0}^{l} q \mathcal{M}_0 D^{-1} \{ r^\top q[j] \} \times \\
  &\times \mathcal{M}_0 D^{-1} r^\top [l - j] - c_l \sum_{j=0}^{l} q \mathcal{M}_0 D^{-1} D^{-1} \{ r^\top q[j] \} \mathcal{M}_0 r^\top [l - j] - \\
  &- c_l \sum_{j=0}^{l} q[j] \mathcal{M}_0 D^{-1} \{ r^\top [l - j] q \} \mathcal{M}_0 D^{-1} r^\top + \\
  &+ c_l \sum_{j=0}^{l} q[j] \mathcal{M}_0 D^{-1} D^{-1} \{ r^\top [l - j] q \} \mathcal{M}_0 r^\top. 
\end{align*}
\]
\[(35)\]

Formula (35) follows from (29).

3. \[
\begin{align*}
  &[\alpha_n \partial_{t_n} - A_n, q \mathcal{M}_0 D^{-1} r^\top]_{<0} = (\alpha_n q_{t_n} - A_n(q)) \mathcal{M}_0 D^{-1} r^\top + \\
  &+ q \mathcal{M}_0 D^{-1} (\alpha_n r_{t_n}^\top + (A_n^\tau \{ q \})^\top). 
\end{align*}
\[(36)\]

The latter equality is obtained via (27)-(28).

From formulae (33)-(36) we have
\[
\begin{align*}
  &\{ L_k, M_{n,l} \}_<0 = c_l \left( \sum_{j=0}^{l} q[j] \mathcal{M}_0 D^{-1} L_k^\top \{ r[l - j] \} - \sum_{j=0}^{l} L_k \{ q[j] \} \mathcal{M}_0 D^{-1} r^\top [l - j] \right) + \\
  &+ M_{n,l} \{ q \} \mathcal{M}_0 D^{-1} r^\top - q \mathcal{M}_0 D^{-1} (M_{n,l}^\tau \{ q \})^\top = c_l \sum_{j=0}^{l} q[j] \mathcal{M}_0 D^{-1} r^\top [l - j + 1] - \\
  &- c_l \sum_{j=0}^{l} q[j + 1] \mathcal{M}_0 D^{-1} r^\top [l - j] + M_{n,l} \{ q \} \mathcal{M}_0 D^{-1} r^\top - q \mathcal{M}_0 D^{-1} (M_{n,l}^\tau \{ q \})^\top = \\
  &= (M_{n,l} - c_l(L_k)^{l+1}) \{ q \} \mathcal{M}_0 D^{-1} r^\top - q \mathcal{M}_0 D^{-1} ((M_{n,l}^\tau - c_l(L_k)^{l+1}) \{ r \})^\top.
\end{align*}
\[(37)\]

From the last equality we obtain the equivalence of the equation \([L_k, M_{n,l}] = 0\) and (31). \(\square\)

New hierarchy (30) consists of several special cases:

1. \(\beta_k = 0, c_l = 0\). Under this assumption we obtain Matrix k-constrained KP hierarchy\(^{24}\).

   We shall also point out that the case \(\beta_k = 0\) and \(c_l \neq 0\) also reduces to Matrix k-constrained KP hierarchy.

2. \(\alpha_n = 0\). In this case we obtain a new \((1+1)\)-dimensional Matrix k-constrained KP hierarchy\(^{22}\). Their members are stationary with respect to \(t_n\) members of \((2+1)\)-BDk-cKP hierarchy (30).

3. \(c_l = 0, N = 1, v_n = u_k = 1, v_{n-1} = u_{k-1} = 0\). In this case we obtain \((2+1)\)-dimensional k-cKP hierarchy given by (9).

4. \(n = 0\). In this case the differential part of \(M_{0,l}\) in (30) is equal to zero: \(A_0 = 0\). We obtain a new generalization of DS-III hierarchy.
5. $c_l = 0$. We obtain $(t_A, \tau_B)$-Matrix KP Hierarchy that was investigated in\cite{44}.

6. If we set $l = 0$ we obtain $(\gamma_A, \sigma_B)$-Matrix KP hierarchy that was investigated in\cite{45}. However, we do not include $l = 0$ in formula (30) because of the following remark.

**Remark 3.** *In the case $l = 0$ we obtain:*

\[ M_{n,0} = \alpha_n \partial_{\tau_n} - A_n - c_0 q M_0 D^{-1} r^\top = (\alpha_n \partial_{\tau_n} - c_0 \beta_k \partial_{\tau_k}) - (A_n - c_0 B_k) + c_0 L_k. \quad (38) \]

*The last summand in the latter formula can be ignored because $L_k$ commutes with itself.*

Thus, we obtained an evolution differential operator again. It means that in the case $l = 0$ the hierarchy given by the operators $M_{n,0}$ and $L_k$ after the change of independent variables $(\alpha_n \partial_{\tau_n} - c_0 \beta_k \partial_{\tau_k} \rightarrow \tilde{\alpha}_n \partial_{\tilde{\tau}_n})$ coincides with hierarchy (9) in the matrix case.

The following corollary follows from Theorem 1:

**Corollary 1.** *The Lax equation $[L_k, \tilde{M}_{n,l}] = 0$, where*

\[ \tilde{M}_{n,l} = M_{n,l} - c_l (L_k)^{l+1}, \]

*and the operators $L_k$ and $M_n$ are defined by (30), is equivalent to the system:*

\[ [L_k, \tilde{M}_{n,l}] \geq 0, \tilde{M}_{n,l}\{q\} = 0, \tilde{M}_{n,l}\{r\} = 0. \quad (40) \]

*Proof. It is evident that the operator $L_k$ commutes with the natural powers of itself: $[L_k, (L_k)^{l+1}] = 0$, $l \in \mathbb{N}$. Thus by bilinearity of the commutator we obtain that $[L_k, \tilde{M}_{n,l}] = 0$ holds if and only if $[L_k, M_{n,l}] = 0$. It remains to use Theorem 1 in order to complete the proof. \qed*

As a result of the hierarchy given by operators (30) and a bilinearity of the commutator we obtain the following essential generalization of (30):

\[ L_k = \beta_k \partial_{\tau_k} - B_k - q M_0 D^{-1} r^\top, \quad B_k = \sum_{j=0}^k u_j D^j, u_j = u_j(x, \tau_k, t_n), \beta_k \in \mathbb{C}, \]
\[ P_{n,m} = \alpha_n \partial_{\tau_n} - A_n - \sum_{l=1}^m c_l \left( \left( \sum_{j=0}^l q[j] M_0 D^{-1} r^\top [l - j] \right) + (L_k)^{l+1} \right), \quad m = 1, 2, \ldots \quad (41) \]
\[ A_n = \sum_{i=0}^n v_i D^i, v_i = v_i(x, \tau_k, t_n), \alpha_n \in \mathbb{C}, \]

and the following corollary that is immediate consequence of Theorem 1:

**Corollary 2.** *The commutator equation $[L_k, P_{n,m}] = 0$ is equivalent to the following system:*

\[ [L_k, P_{n,m}] \geq 0, P_{n,m}\{q\} = 0, P_{n,m}\{r\} = 0. \quad (42) \]
For further convenience we will consider the Lax pairs consisting of the operators $L_k$ (30) and $\tilde{M}_{n,l}$ (39) (the operator $\tilde{M}_{n,l}$ is involved in equations for functions $q$ and $r$; see formulae (40)). Consider examples of equations given by operators $L_k$ (30) and $\tilde{M}_{n,l}$ (39) that can be obtained under certain choice of $(k,n,l)$.

I. $k = 0$.

In this case the operator $L_0$ (30) has the form:

$$L_0 = \beta_0 \partial_{\tau_0} - qM_0D^{-1}r^\top. \tag{43}$$

For further simplicity we use the change of variables $\tau_0 := y$, $\beta := \beta_0$:

$$L_0 = \beta \partial_y - qM_0D^{-1}r^\top. \tag{44}$$

1. $n = 2$, $l = 1$.

$$\tilde{M}_{2,1} = M_{2,1} - c_1(L_0)^2 = \alpha_2 \partial_{t_2} - c(D^2 + v_0) - c_1 \beta^2 \partial_y^2 + 2c_1 \beta qM_0D^{-1}r^\top + 2c_1 \beta qM_0D^{-1}r^\top \partial_y, \quad c \in \mathbb{C}. \tag{45}$$

If $c = 0$ we obtain that $\tilde{M}_{2,1} = \tilde{M}_{0,1}$. We use $c$ in formula (45) in order not to consider separately the case $n = 0$. The commutator equation $[L_0, \tilde{M}_{2,1}] = 0$ is equivalent to the system:

$$\begin{align*}
\alpha_2 q_{t_2} &= c q_{xx} + c_1 \beta^2 q_{yy} + cv_0 q + c_1 qM_0S, \\
-\alpha_2 r_{t_2}^\top &= c r_{xx}^\top + c_1 \beta^2 r_{yy}^\top + cr^\top v_0 + c_1 S M_0 r^\top, \\
\beta v_{0y} &= -2(qM_0 r^\top)_x, \quad S_x = -2\beta(r^\top q)_y. \tag{46}
\end{align*}$$

The Lax pair $L_0$ (44) and $\tilde{M}_{2,1}$ (45) and the corresponding system (46) were investigated in46. Consider additional reductions of pair of the operators $L_0$ (44) and $\tilde{M}_{2,1}$ (45) and system (46). After the reduction $c, c_1 \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\alpha_2 \in i\mathbb{R}$; $r^\top = q^*$, $M_0 = M_0^*$, the operators $L_0$ and $\tilde{M}_{2,1}$ are skew-Hermitian and Hermitian respectively, and (46) takes the form

$$\begin{align*}
\alpha_2 q_{t_2} &= c q_{xx} + c_1 \beta^2 q_{yy} + cv_0 q + c_1 qM_0S, \\
\beta v_{0y} &= -2(qM_0 q^*)_x, \quad S_x = -2\beta(q^*q)_y. \tag{47}
\end{align*}$$

This has the following two interesting subcases:

(a) $c_1 = 0$. Then we have

$$\begin{align*}
\alpha_2 q_{t_2} &= c q_{xx} + cv_0 q, \quad \beta v_{0y} = -2(qM_0 q^*)_x. \tag{48}
\end{align*}$$
(b) \( c = 0 \). Then (47) takes the form

\[
\alpha_2 q_{t_2} = c_1 \beta^2 \mathbf{q}_{yy} + c_1 \mathbf{q} \mathcal{M}_0 S, \quad S_x = -2 \beta (\mathbf{q}^\ast \mathbf{q})_y. \quad (49)
\]

Systems (48) and (49) are two different matrix generalizations of the Davey-Stewartson equation (DS-III-a and DS-III-b). In the scalar case equation (48) was investigated in\(^{24,4}\). The vector version of DS-III (48) and its Lax representation given by operators (44) and (45) in case \( c_1 = 0, N = 1 \) were introduced in\(^{31,34,35}\).

Let us consider (47) in the case where \( \beta = 1, u := \mathbf{q} \) and \( \mathcal{M}_0 := \mu \) are scalars. Then (47) becomes

\[
\alpha_2 u_{t_2} = cu_{xx} + c_1 u_{yy} + vu + \mu c_1 S u, \quad v_{0y} = -2 \mu (|u|^2)_x, \quad S_x = -2(|u|^2)_y. \quad (50)
\]

Setting \( c = c_1 = 1 \) and \( \mu = 1 \), as a consequence of (50) we obtain

\[
\alpha_2 u_{t_2} = u_{xx} + u_{yy} + S_1 u, \quad S_{1,xy} = -2(|u|^2)_{xx} - 2(|u|^2)_{yy}, \quad (51)
\]

where \( S_1 = v_0 + S \). This is the well-known Davey-Stewartson system (DS-I) and (47) is therefore a matrix (noncommutative) generalization. The interest in noncommutative versions of DS systems (in particular, solution generating technique) has also arisen recently in\(^{52,53}\). In Section [IV] we will consider dressing method for matrix generalization of DS system (47) that leads to its exact solutions. In\(^{46}\) DS-II system was also obtained from system (46) and its Lax pair (44), (45) after the change \( x \rightarrow z, y \rightarrow \bar{z} \).

2. \( n = 2, l = 2 \).

\[
\tilde{M}_{2,2} = M_{2,2} - c_2 (L_0)^3 = \alpha_2 t_2 - c_2 \beta^3 \partial_y^3 - D^2 - v_0 + 3 \beta^2 c_2 q_y \mathcal{M}_0 D^{-1} r_y^T - 3 \beta c_2 q \mathcal{M}_0 \partial_y D^{-1} r_y^T q \mathcal{M}_0 D^{-1} r_y^T + 3 \beta c_2 q \mathcal{M}_0 D^{-1} \{ r_y^T q \} y \mathcal{M}_0 D^{-1} r_y^T + 3 c_2 \beta^2 \partial_y q \mathcal{M}_0 D^{-1} r_y^T \partial_y \quad (52)
\]

The commutator equation \([L_0, \tilde{M}_{2,2}] = 0\) is equivalent to the system:

\[
\begin{align*}
\alpha_2 q_{t_2} - q_{xx} - c_2 \beta^3 q_{yy} - v_0 q + 3 c_2 \beta q_y \mathcal{M}_0 S_1 + 3 \beta c_2 q \mathcal{M}_0 S_1 q - 3 c_2 q \mathcal{M}_0 S_2 & \\
-3 c_2 q D^{-1} \{ \mathcal{M}_0 r_y^T q \mathcal{M}_0 S_1 - \mathcal{M}_0 S_1 \mathcal{M}_0 r_y^T \} & = 0, \\
-\alpha_2 t_2 - r_{xx} - \beta^3 c_2 r_{yy} - r^T v_0 + 3 \beta c_2 S_1 \mathcal{M}_0 r_y^T + 3 c_2 S_2 \mathcal{M}_0 r^T & \\
-3 c_2 D^{-1} \{ S_1 \mathcal{M}_0 r^T q - r^T q \mathcal{M}_0 S_1 \} \mathcal{M}_0 r^T & = 0,
\end{align*}
\]

\[
\beta v_{0y} = -2 (q \mathcal{M}_0 r^T)_x, \quad S_{1x} = \beta (r^T q)_y, \quad S_{2x} = \beta^2 (r^T q)_y.
\]
3. \( n = 3, l = 1 \).

In this case the operator \( \tilde{M}_{3,1} \) has the form:

\[
\tilde{M}_{3,1} = M_{3,1} - c_1(L_0)^2 = \alpha_3 \partial_{x_3} - D^3 - v_1 D - v_0 - c_1 \beta^3 \partial_y^3 + 2c_1 \beta q M_0 D^{-1}r_y^\top + 2c_1 \beta q M_0 D^{-1}r^\top \partial_y.
\] (54)

The equation \([L_0, \tilde{M}_{3,1}] = 0\) is equivalent to the system:

\[
\begin{align*}
\alpha_3 q_{x_3} &= q_{xxx} + c_1 \beta^2 q_{yy} + v_1 q_x + v_0 q + c_1 q M_0 S_1, \\
-\alpha_3 r_{x_3} &= -r_{xxx} + c_1 \beta^2 r_{yy} - (r^\top v_1)_x + r^\top v_0 + c_1 S_1 M_0 r^\top, \\
\beta v_{0,y} &= -3(q_x M_0 r^\top)_x + [q M_0 r^\top, v_1], \\
\beta v_{1,y} &= -3(q M_0 r^\top)_y, \quad S_{12} = -2\beta (r^\top q)_y.
\end{align*}
\] (55)

4. \( n = 3, l = 2 \)

\[
\tilde{M}_{3,2} = \alpha_3 \partial_{x_3} - cD^3 - c_2 \beta^3 \partial_y^3 - cv_1 D - cv_0 + 3\beta^2 c_2 q_y M_0 D^{-1}r_y^\top - 3\beta c_2 q M_0 \partial_y D^{-1}r^\top q M_0 D^{-1} r^\top + 3\beta c_2 q M_0 D^{-1} \{r^\top q \}_y M_0 D^{-1} r^\top + 3c_2 \beta^2 \partial_y q M_0 D^{-1} r^\top \partial_y.
\] (56)

The Lax equation \([L_0, \tilde{M}_{3,2}] = 0\) results in the system:

\[
\begin{align*}
\alpha_3 q_{x_3} &= q_{xxx} - c_2 \beta^2 q_{yyy} - cv_1 q_x + 3c_2 \beta q_y M_0 S_1 + 3c_2 \beta q M_0 S_{1y} - cv_0 q - 3c_2 q M_0 S_2 - 3c_2 q D^{-1} \{M_0 r^\top q M_0 S_1 - M_0 S_1 M_0 r^\top q \} = 0, \\
-\alpha_3 r_{x_3} &= -r_{xxx} - \beta c_2 r_{yyy} - cr_{y}^\top v_1 - cr^\top v_{1x} + 3\beta c_2 S_1 M_0 r_y^\top + cr^\top v_0 + 3c_2 S_2 M_0 r^\top - 3c_2 D^{-1} \{S_1 M_0 r^\top q - r^\top q M_0 S_1 \} M_0 r^\top = 0, \\
\beta v_{1,y} &= -3(q M_0 r^\top)_x, \quad S_{12} = \beta (r^\top q)_y, \\
\beta v_{0,y} &= -3(q_x M_0 r^\top)_x + [q M_0 r^\top, v_1], \quad S_{22} = \beta^2 (r_y^\top q)_y.
\end{align*}
\] (57)

System (57) and its Lax pair \( L_0 \) (44) and \( \tilde{M}_{3,2} \) (56) was investigated in\(^{46}\). We list some reductions of this system:

(a) \( \alpha_3, \beta, c, c_2 \in \mathbb{R}, \quad r^\top = q^*, \quad M_0 = M_0^* \). The operators \( L_1 \) and \( M_3 \) are then skew-Hermitian and (57) takes the form

\[
\begin{align*}
\alpha_3 q_{x_3} &= q_{xxx} - c_2 \beta^2 q_{yyy} - cv_1 q_x + 3c_2 \beta q_y M_0 S_1 + 3c_2 \beta q M_0 S_{1y} - cv_0 q - 3c_2 q M_0 S_2 - 3c_2 q D^{-1} \{M_0 q^* q M_0 S_1 - M_0 S_1 M_0 q^* q \} = 0, \\
\beta v_{1,y} &= -3(q M_0 q^*)_x, \quad S_{12} = \beta (q^* q)_y, \\
\beta v_{0,y} &= -3(q_x M_0 q^*)_x + [q M_0 q^*, v_1], \quad S_{22} = \beta^2 (q_y^* q)_y.
\end{align*}
\] (58)
In the scalar case ($N = m = 1$), setting $\mathbb{R} \ni \mu := M_0$, $q(x, y, t_3) := q(x, y, t_3)$, (58) reads
\[
\alpha_3 q_{t_3} - c q_{xxx} - c_2 q_{yyy} + 3 c \mu q_x \int |q|^2 dy + 3 c_2 \mu q_y \int |q|^2 dx + 3 c_2 \mu q \int (\bar{q}q)_y dy + 3 c_2 \mu q \int (\bar{q}q)_x dy = 0. \tag{59}
\]
In the real case $q = \bar{q}$, (58) becomes
\[
\alpha_3 q_{t_3} - c q_{xxx} - c_2 \beta^3 q_{yyy} - c_1 q_x + 3 c_2 \beta q_y M_0 S_1 + 3 c_2 \bar{q} \bar{M}_0 S_1 y - c v_3 q - 3 c_2 q \bar{M}_0 S_2 - 3 c_2 q D^{-1} \{ M_0 q^T q M_0 S_1 - \bar{M}_0 S_1 \bar{M}_0 q^T q \} = 0, \tag{60}
\]
\[
\beta v_{1y} = -3 (q \bar{M}_0 q^T)_x, \quad S_{1x} = \beta (q^T q)_y, \\
\beta v_{0y} = -3 (q_x \bar{M}_0 q^T)_x + [q \bar{M}_0 q^T, v_1], \quad S_{2x} = \beta^2 (q_y q)_y.
\]
In the scalar case ($N = m = 1$), writing $M_0 = \mu$ and $q = q(x, y, t_3) = q(x, y, t_3)$, after setting $y = x$ and $c + c_2 = -1$, $\beta = 1$ (60) takes the form
\[
\alpha_3 q_{t_3} + q_{xxx} - 6 \mu q^2 q_x = 0, \tag{61}
\]
which is the mKdV equation. The systems (58) and (60) are therefore, respectively, complex and real, spatially two-dimensional matrix generalizations of it.

(b) $\beta = 1$, $M_0 r^T = \nu$ with a constant matrix $\nu$. In terms of $u := q \nu$, (57) takes the form
\[
\alpha_3 u_{t_3} - c u_{xxx} - c_2 u_{yyy} + 3 c D \left\{ \left( \int u_x dy \right) u \right\} + 3 c_2 \partial_y \left\{ u \left( \int u_y dx \right) \right\} \\
-c \left( \int [u, v_1] dy \right) u - 3 c_2 u \left( \int [u, S_1] dx \right) = 0, \\
\nu \left( c \int [u, v_1] dy - 3 c_2 \int [S_1, u] dx \right) = 0, \quad v_{1y} = -3 u_x, \quad S_{1x} = u_y. \tag{62}
\]
In the scalar case ($N = 1, m = 1$), this reduces to
\[
\alpha_3 u_{t_3} - c u_{xxx} - c_2 u_{yyy} + 3 c D \left\{ \left( \int u_x dy \right) u \right\} + 3 c_2 \partial_y \left\{ u \left( \int u_y dx \right) \right\} = 0, \tag{63}
\]
which is the Nizhnik equation\ref{58}. The system (62) thus generalizes the latter to the matrix case.

II. $k = 2$

Now we will consider two Lax pairs connected with the operator $L_2$:
\[
L_2 = \beta_2 \partial_{\tau_2} - D^2 - 2 u - q M_0 D^{-1} r^T. \tag{64}
\]
1. $n = 2, l = 1$

$$\tilde{M}_{2,1} = M_{2,1} - c_1(L_2)^2 = \alpha_2 \partial_{t_2} - D^2 - 2u - c_1 \left( \beta_2^2 \partial_{r_2}^2 - 2 \beta_2 \partial_{r_2} D^2 + \partial^4 - 2 \beta_2 u \partial_{r_2} + 4u D^2 + 4u_x D + 2u_{xx} + 2q r \cdot \mathcal{M}_0 r^\top + 2q \mathcal{M}_0 r^\top D + 4u^2 - 2 \beta_2 q \mathcal{M}_0 D^{-1} \partial_{r_2} r^\top \right). \tag{65}$$

Lax equation $[L_2, \tilde{M}_{2,1}] = 0$ is equivalent to the system:

$$\begin{align*}
\alpha_2 u_2 - \beta_2 u_{r_2} &= (q \mathcal{M}_0 r^\top)x - c_1 (L_2 \{q\} \mathcal{M}_0 r^\top + q \mathcal{M}_0 (L_2^\top \{r\})^\top)x, \\
M_{2,1}\{q\} &= 0, M_{2,1}\{r\} = 0. \tag{66}
\end{align*}$$

In vector case ($N = 1$) and under additional reductions $c_1 = 0, \alpha_2, \beta_2 \in i \mathbb{R}, \mathcal{M}_0 = -\mathcal{M}_0^*, u = \bar{u}$ and $q = \bar{r}$ system $(66)$ reduces to $(2+1)$-generalization of Yajima-Oikawa equation $(21)$.

2. $n = 3, l = 1$

$$\begin{align*}
\tilde{M}_{3,1} &= M_{3,1} - c_1(L_2)^2 = \alpha_3 \partial_{t_3} - D^2 - 3u D - \frac{3}{2} \left( u_x + \beta_2 D^{-1} \{u_{r_2}\} + q \mathcal{M}_0 r^\top \right) - c_1 \left( \beta_2^2 \partial_{r_2}^2 - 2 \beta_2 \partial_{r_2} D^2 + D^4 - 2 \beta_2 u_{r_2} - 4 \beta_2 u \partial_{r_2} + 4u D^2 + 4u_x D + 2u_{xx} + 2q r \cdot \mathcal{M}_0 r^\top + 2q \mathcal{M}_0 r^\top D + 4u^2 - 2 \beta_2 q \mathcal{M}_0 D^{-1} \partial_{r_2} r^\top \right) - \partial^4 - 2 \beta_2 u_{r_2} - 4 \beta_2 u \partial_{r_2} + 4u D^2 + 4u_x D + 2u_{xx} + 2q r \cdot \mathcal{M}_0 r^\top + 2q \mathcal{M}_0 r^\top D + 4u^2 - 2 \beta_2 q \mathcal{M}_0 D^{-1} \partial_{r_2} r^\top \right). \tag{67}
\end{align*}$$

The equation $[L_2, \tilde{M}_{3,1}] = 0$ is equivalent to the following system:

$$\begin{align*}
\alpha_3 u_3 - \frac{1}{4} u_{xxx} - 3u u_x - \frac{3}{4} \beta_2 (q \mathcal{M}_0 r^\top)_{r_2} + &+ \frac{3}{4} (q \mathcal{M}_0 r^\top - q \mathcal{M}_0 r^\top)_x + \frac{3}{4} [u_x + \beta_2 D^{-1} \{u_{r_2}\} + q \mathcal{M}_0 r^\top] - \left[ q \mathcal{M}_0 r^\top, u \right] - c_1 L_2 \{q\} \mathcal{M}_0 r^\top - c_1 q \mathcal{M}_0 (L_2^\top \{r\})^\top = \frac{3}{4} \beta_2 D^{-1} \{u_{r_2}\}, \\
\tilde{M}_{3,1}\{q\} &= 0, \tilde{M}_{3,1}\{r\} = 0. \tag{68}
\end{align*}$$

In the vector case ($N = 1$) under additional reductions $\alpha_3 \in \mathbb{R}, \beta_2 \in i \mathbb{R}, \mathcal{M}_0 = -\mathcal{M}_0^*, q = \bar{r}$ and $u = \bar{u}$ system $(68)$ reduces to $(2+1)$-generalization of higher Yajima-Oikawa system $(23)$.

**Remark 4.** After putting $\alpha_2 = 0$ and $\alpha_3 = 0$ in formulae $(65)$-$(68)$ we obtain examples of a new $(1+1)$-dimensional Matrix k-cKP hierarchy$(33)$ (see case 2 after the proof of Theorem $(31)$).
IV. DRESSING METHODS FOR EXTENSIONS OF (2+1)-DIMENSIONAL K-CONSTRAINED KP HIERARCHY

In this section our aim is to consider hierarchy of equations given by the Lax pair (30). We suppose that the operators $L_k$ and $M_{n,l}$ in (30) satisfy the commutator equation $[L_k, M_{n,l}] = 0$. At first we recall some results from 49. Let $N \times K$-matrix functions $\varphi$ and $\psi$ be solutions of linear problems:

\[
L_k \{ \varphi \} = \varphi \Lambda, \quad L_k^\tau \{ \psi \} = \psi \tilde{\Lambda}, \quad \Lambda, \tilde{\Lambda} \in \text{Mat}_{K \times K}(\mathbb{C}).
\]  

(69)

Introduce binary Darboux transformation (BDT) in the following way:

\[
W = I - \varphi \left( C + D^{-1} \{ \psi^\top \varphi \} \right)^{-1} D^{-1} \psi^\top,
\]

(70)

where $C$ is a $K \times K$-constant nondegenerate matrix. The inverse operator $W^{-1}$ has the form:

\[
W^{-1} = I + \varphi D^{-1} \left( C + D^{-1} \{ \psi^\top \varphi \} \right)^{-1} \psi^\top.
\]

(71)

The following theorem is proven in 49.

**Theorem 2.** The operator $\hat{L}_k := WL_k W^{-1}$ obtained from $L_k$ in (30) via BDT (70) has the form

\[
\hat{L}_k := WL_k W^{-1} = \beta_k \partial_{\tau_k} - \hat{B}_k - \hat{q} \mathcal{M}_0 D^{-1} \tilde{r}^\top + \Phi \mathcal{M}_1 D^{-1} \Psi^\top, \quad \hat{B}_k = \sum_{j=0}^{k} \hat{u}_j D^j,
\]

(72)

where

\[
\mathcal{M}_1 = C \Lambda - \tilde{\Lambda}^\top C, \quad \Phi = \varphi \Delta^{-1}, \quad \Psi = \psi \Delta^{-1,\top}, \quad \Delta = C + D^{-1} \{ \psi^\top \varphi \},
\]

\[
\hat{q} = W \{ q \}, \quad \tilde{r} = W^{-1,\tau} \{ r \}.
\]

(73)

$\hat{u}_j$ are $N \times N$-matrix coefficients depending on functions $\varphi$, $\psi$ and $u_j$. In particular,

\[
\hat{u}_n = u_n, \quad \hat{u}_{n-1} = u_{n-1} + \left[ u_{n, \varphi} \left( C + D^{-1} \{ \psi^\top \varphi \} \right)^{-1} \psi^\top \right].
\]

(74)

Exact forms of all coefficients $\hat{u}_j$ are given in 49.

The following corollary follows from Theorem 2

**Corollary 3.** The functions $\Phi = \varphi \Delta^{-1} = W \{ \varphi \} C^{-1}$ and $\Psi = \psi \Delta^{-1,\top} = W^{-1,\tau} \{ \psi \} C^\tau,^{-1}$ satisfy the equations

\[
\hat{L}_k \{ \Phi \} = \Phi C \Lambda C^{-1}, \quad \hat{L}_k^\tau \{ \Psi \} = \Psi C^\tau \tilde{\Lambda} C^\tau,^{-1}.
\]

(75)
For further purposes we will need the following lemmas.

**Lemma 1.** Let $\mathcal{M}_{l+1}$ be a matrix of the form

$$
\mathcal{M}_{l+1} = C A^{l+1} - (\hat{A}^T)^{l+1} C, \ l \in \mathbb{N}.
$$

The following formula holds:

$$
\mathcal{M}_{l+1} = \sum_{s=0}^{l} CA^{s} C^{-1} \mathcal{M}_{l} C^{-1} (\hat{A}^T)^{l-s} C.
$$

**Lemma 2.** The following formula

$$
\Phi \mathcal{M}_{l+1} D^{-1} \Psi^T = \sum_{s=0}^{l} \Phi[s] \mathcal{M}_{l} D^{-1} \Psi^T[l - s],
$$

holds, where

$$
\Phi[j] := (\hat{\Phi}_k)^j \{\Phi\}, \ \Psi[j] := (\hat{\Phi}_k)^j \{\Psi\}.
$$

**Proof.** Lemma 2 is a consequence of Corollary 1 and formula (77) of Lemma 1. Namely, the following relations hold:

$$
\Phi \mathcal{M}_{l+1} D^{-1} \Psi^T = \sum_{s=0}^{l} \Phi CA^{s} C^{-1} \mathcal{M}_{l} C^{-1} D^{-1} (\hat{A}^T)^{l-s} C \Psi^T = \sum_{s=0}^{l} \Phi[s] \mathcal{M}_{l} D^{-1} \Psi^T[l - s].
$$

Now we assume that the functions $\varphi$ and $\psi$ in addition to equations (69) satisfy the equations:

$$
M_{n,l}\{\varphi\} = c_l \varphi A_l^{l+1} = c_l L_k^{l+1}\{\varphi\}, \ M_{n,l}\{\psi\} = c_l \psi \tilde{A}^{l+1} = c_l (L_k^\tau)^{l+1}\{\psi\}.
$$

Problems (80) can be rewritten via the operator $\tilde{M}_{n,l}$ (39) as:

$$
\tilde{M}_{n,l}\{\varphi\} = 0, \ \tilde{M}_{n,l}\{\psi\} = 0.
$$

The following theorem for the operators $M_{n,l}$ (30) and $\tilde{M}_{n,l}$ (39) holds:

**Theorem 3.** Let $N \times K$ -matrix functions $\varphi$, $\psi$ be solutions of problems (69) and (80). The transformed operator $\tilde{M}_{n,l} := WM_{n,l}W^{-1}$ obtained via BDT $W$ (70) has the form:

$$
\tilde{M}_{n,l} := WM_{n,l}W^{-1} = \alpha_n \partial_l - \hat{A}_n - c_l \sum_{j=0}^{l} \hat{q}[j] \mathcal{M}_{l+1} D^{-1} \hat{r}^T[l - j] + c_l \sum_{s=0}^{l} \Phi[s] \mathcal{M}_{l} D^{-1} \Psi^T[l - s], \ \hat{A}_n = \sum_{i=0}^{n} \hat{v}_i D^i,
$$

(82)
where the matrix $\mathcal{M}_n$ and the functions $\hat{q}$, $\hat{r}$, $\Phi[s]$, $\Psi[l-s]$ are defined by formulae (73), (79) and $\hat{q}[j]$, $r[j]$ have the form

$$\hat{q}[j] = (\hat{L}_k^j)\{\hat{q}\}, \quad \hat{r}[j] = (\hat{L}_k^j)^\tau\{\hat{r}\},$$

(83)

$\hat{v}_i$ are $N \times N$-matrix coefficients that depend on the functions $\varphi$, $\psi$ and $v_i$. The transformed operator $\hat{M}_{n,l} = W\hat{M}_{n,l}W^{-1}$ has the form:

$$\hat{M}_{n,l} = W\hat{M}_{n,l}W^{-1} = \check{M}_{n,l} - c_l(\check{L}_k)^{l+1},$$

(84)

where $\check{L}_k$ is given by (72).

**Proof.** We shall rewrite the operator $M_{n,l}$ (30) in the form

$$M_{n,l} = \alpha_n \partial_{\alpha_n} - \sum_{i=0}^n \hat{v}_i D^i - c_q \hat{q}m_0 D^{-1}r^T,$$

(85)

where $\check{M}_0$ is an $m(l+1) \times m(l+1)$- block-diagonal matrix with entries of $\mathcal{M}_0$ at the diagonal; $\check{q} := (\check{q}[0], \check{q}[1], \ldots, \check{q}[l])$, $\check{r} := (\check{r}[l], \check{r}[l-1], \ldots, \check{r}[0])$. Using Theorem 2 we obtain that

$$\hat{M}_{n,l} = \alpha_n \partial_{\alpha_n} - \sum_{i=0}^n \hat{v}_i D^i - c_q \hat{q}m_0 D^{-1}r^T + \Phi \mathcal{M}_{l+1} D^{-1} \Psi^T,$$

(86)

where $\hat{q} = W\{\check{q}\}$, $\hat{r} = W^{-1}\{\check{r}\}$. Using the exact form of $\check{q}$ and $\check{r}$ we have

$$\check{q} = W\{\check{q}\} = (W\{\check{q}[0]\}, \ldots, W\{\check{q}[l]\}),$$

$$\check{r} = W^{-1}\{\check{r}\} = (W^{-1}\{\check{r}[l]\}, \ldots, W^{-1}\{\check{r}[0]\}).$$

(87)

We observe that

$$W\{\check{q}[i]\} = WL^i\{\check{q}\} = WL^iW^{-1}\{W\{\check{q}\}\} = \check{L}^i\{\check{q}\} =: \check{q}[i].$$

(88)

It can be shown analogously that $W^{-1}\tau\{\check{r}[i]\} = \check{L}^{\tau,i}\{W^{-1}\tau\{\check{r}\}\} = \check{L}^{\tau,i}\{\check{r}\} =: \check{r}[i]$. Thus we have:

$$\hat{q}m_0 D^{-1}r^T = \sum_{j=0}^l \hat{q}[j]m_0 D^{-1}r^T[l-j],$$

(89)

For the last item in (86) from Lemma 2 we have:

$$\Phi \mathcal{M}_{l+1} D^{-1} \Psi^T = \sum_{s=0}^l \Phi[s]m_1 D^{-1} \Psi^T[l-s].$$

(90)

Using formulae (86), (89), (90) we obtain that the operator $\hat{M}_{n,l}$ has form (82). The exact form of the operator $\hat{M}_{n,l}$ follows from formula (82) and Theorem 2.

\[\square\]
From Theorem 3 we obtain the following corollary.

**Corollary 4.** Assume that functions $\varphi$ and $\psi$ satisfy problems (69) and (80). Then the functions $\Phi = W\{\varphi\}C^{-1}$ and $\Psi = W^{-1,\tau}\\{\psi\}C^{\tau,-1}$ (see formulae (73)) satisfy the equations:

$$\hat{M}_{n,l}\{\Phi\} = \hat{M}_{n,l}\{\Phi\} - c_l(\hat{L}_k)^{l+1}\{\Phi\} = 0, \quad \hat{M}_{n,l}^{\tau}\{\Psi\} = \hat{M}_{n,l}^{\tau}\{\Psi\} - c_l(\hat{L}_k^{\tau})^{l+1}\{\Psi\} = 0, \quad (91)$$

where the operators $\hat{L}_k$, $\hat{M}_{n,l}$ and $\hat{M}_{n,l}^{\tau}$ are defined by (72), (82) and (84).

As it was shown in Sections II-III the most interesting systems arise from the $(2+1)$-dimensional $k$-cKP hierarchy (9) and its extension (30) after a Hermitian conjugation reduction. Our aim is to show that under additional restrictions Binary Darboux Transformation $W$ (70) preserves this reduction. We shall point out that a differential dressing operator that was used in $^{34,35,39}$ does not satisfy such a property. It imposes nontrivial constraints on the dressing operator (see, for example, $^{50}$). The following proposition holds.

**Proposition 1.**

1. Let $\psi = \bar{\varphi}$ and $C = C^*$ in the dressing operator $W$ (70). Then the operator $W$ is unitary ($W^* = W^{-1}$).

2. Let the operator $L_k$ (30) be Hermitian (skew-Hermitian) and $M_{n,l}$ (30) be Hermitian (skew-Hermitian). Then the operator $\hat{L}_k = WL_kW^{-1}$ (see (72)) transformed via the unitary operator $W$ is Hermitian (skew-Hermitian) and $\hat{M}_{n,l} := WM_{n,l}W^{-1}$ (84) is Hermitian (skew-Hermitian).

3. Assume that the conditions of items 1 and 2 hold. Let $\hat{\Lambda} = \bar{\Lambda}$ in the case of Hermitian $L_k$ ($\hat{\Lambda} = -\bar{\Lambda}$ in skew-Hermitian case). We shall also assume that the function $\varphi$ satisfies the corresponding equations in formulae (69) and (80). Then $M_1 = -M_1^*$ ($M_1 = M_1^*$) and $\Psi = \bar{\Phi}$ (see formulae (73)).

**Proof.** By using formulae (70) and (71) it is easy to check that $W$ is unitary (in the case $\psi = \varphi$, $C = C^*$). Assume that $L_k = L_k^*$ (in the case of skew-Hermitian $L_k$ analogous considerations can be applied). Then we have $(\hat{L}_k)^* = W^{-1,\tau}L_k^{\tau}W^* = WL_kW^{-1} = \hat{L}_k$. Analogously we can show that $W$ maintains Hermitian (skew-Hermitian) property for $M_{n,l}$.

The formulae from item 3 can be checked by direct calculations.

Now we list several examples of dressing for Lax pairs and the corresponding equations from Section III.
Consider dressing methods for equations connected with the operator \( L_0 \). Assume that \( \varphi \) and \( \psi \) are \( N \times K \)-matrix functions that satisfy the equations

\[
L_0\{\varphi\} = \varphi \Lambda, \quad L_0^T\{\psi\} = \psi \tilde{\Lambda}, \quad L_0 := \beta \partial_y.
\]  

(92)

By Theorem 2 we obtain that the dressed operator \( \hat{L}_0 \) via BDT \( W \) has the form

\[
\hat{L}_0 = W L_0 W^{-1} = \beta \partial_y + \Phi M_1 D^{-1} \Psi^T.
\]

(93)

1. \( n = 2, \ l = 1 \). Assume that \( N \times K \)-matrix functions \( \varphi \) and \( \psi \) in addition to equations (92) also satisfy the equations

\[
M_2\{\varphi\} = c_1 \varphi \Lambda^2 = c_1 L_0^2\{\varphi\}, \quad M_2^T\{\psi\} = c_1 \psi \tilde{\Lambda}^2 = c_1 (L_0^T)^2\{\psi\}, \quad M_2 := \alpha_2 \partial_{y_2} - D^2. \quad (94)
\]

By Theorem 3 we obtain that the transformed operator \( \hat{M}_2 \) has the form

\[
\hat{M}_2 = W M_2 W^{-1} = \alpha_2 \partial_{y_2} - D^2 - \hat{v}_0 + \hat{L}_0\{\Phi\} M_1 D^{-1} \hat{\Psi}^T + \Phi M_1 D^{-1}\{\hat{L}_0\{\Psi}\}\hat{\Psi}^T. \quad (95)
\]

By direct calculations it can be obtained that \( \hat{v}_0 = 2(\varphi \Delta^{-1} \psi^T)_x, \Delta = C + D^{-1}\{\psi^T \varphi\} \).

It can be easily checked that

\[
\beta(\varphi \Delta^{-1} \psi^T)_y = \beta \varphi_y \Delta^{-1} \psi^T - \beta \varphi \Delta^{-1} D^{-1}\{\psi^T \varphi\}_y \Delta^{-1} \psi^T + \beta \varphi \Delta^{-1} \psi^T_y =
\]

\[
= \varphi \Lambda \Delta^{-1} \psi^T - \beta \varphi \Delta^{-1} D^{-1}\{\psi^T \varphi\}_y \Delta^{-1} \psi^T - \varphi \Delta^{-1} \tilde{\Lambda}^T \psi^T =
\]

\[
= \varphi \Delta^{-1}(\lambda A + D^{-1}\{\psi^T \varphi\} \lambda) \Delta^{-1} \psi^T - \beta \varphi \Delta^{-1} D^{-1}\{\psi^T \varphi\}_y \Delta^{-1} \psi^T + 
\]

\[
+ \varphi \Delta^{-1}(-\tilde{\Lambda}^T C - \tilde{\Lambda}^T D^{-1}\{\psi^T \varphi\}) \Delta^{-1} \psi^T =
\]

\[
= \varphi \Delta^{-1}(\lambda A + \beta D^{-1}\{\psi^T \varphi\}) \Delta^{-1} \psi^T - \beta \varphi \Delta^{-1} D^{-1}\{\psi^T \varphi\}_y \Delta^{-1} \psi^T + 
\]

\[
+ \varphi \Delta^{-1}(-\tilde{\Lambda}^T C + \beta D^{-1}\{\psi^T \varphi\}) \Delta^{-1} \psi^T = \Phi M_1 \Psi^T. \quad (96)
\]

From the latter formula we obtain that

\[
\beta \hat{v}_{0y} = 2\beta(\varphi \Delta^{-1} \psi^T)_xy = 2(\Phi M_1 \Psi^T)_x. \quad (97)
\]

From Corollary 4 we see that the functions \( \Phi = \varphi \Delta^{-1} \) and \( \Psi = \psi \Delta^T \Delta^{-1} \) where \( \Delta = C + D^{-1}\{\psi^T \varphi\} \) (see formulae (73)) satisfy equations (91). After the change \( q := \Phi, \ r := \Psi, \ M_0 := -M_1, \ v_0 := \hat{v}_0 \) from formulae (91) and (97) we obtain that \( N \times K \)-matrix functions \( q, r \), an \( N \times N \)-matrix function \( v_0 \) and a \( K \times K \)-matrix \( M_0 \) satisfy equations (46) in the case \( c = 1 \). In the case of additional reductions in formulae (92)-(94): \( \alpha_2 \in i\mathbb{R}, \beta \in \mathbb{R}, c_1 \in \mathbb{R}, \tilde{\Lambda} = -\Lambda, \psi = \varphi \) and \( C = C^* \) in gauge transformation.
operator \( W \) from Proposition 1, we obtain that the functions \( q := \Phi \) and \( v_0 = \hat{v}_0 = 2(\varphi \Delta^{-1} \varphi^*)_x \) satisfy matrix DS system (47) in the case \( c = 1 \).

Consider more precisely the exact solutions of a vector version \((N = 1)\) of DS system (47) in the case \( c = 1 \). Assume that the \( K \times K \)-matrix \( \Lambda \) in (92) is diagonal: \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_K) \), \( \lambda_j \in \mathbb{C} \). Let us fix arbitrary natural numbers \( K_1, \ldots, K_m \). Denote by \( K \) the number \( K = K_1 + \ldots + K_m \). Our aim is to present the exact form of solution of \( m \)-component DS system (47) \((q = (q_1, \ldots q_m))\) Assume that a \( K \times K \)-matrix \( C \) has the form:

\[
C = \text{diag}(C_{K_1}, \ldots, C_{K_m}), \quad C_{K_s} = \left( \frac{1}{\mu_s \lambda_j + \lambda_i} \right)_{i,j=1}^{K_s}, \quad \mu_s = \pm 1, \quad s = 1, m. \tag{98}
\]

I.e., the matrix \( C \) has diagonal blocks \( C_{K_s} \) of dimension \( K_s \) on its diagonal. \( \mu_s, s = 1, m \), are numbers each equal to 1 or \(-1\). It can be checked by direct calculations that the matrix \( M_1 := CA + \Lambda^*C \) has the form:

\[
M_1 = \begin{pmatrix}
\mu_1 E_{K_1} & 0_{K_1,K_2} & 0_{K_1,K_3} & \ldots & 0_{K_1,K_{m-1}} & 0_{K_1,K_m} \\
0_{K_2,K_1} & \mu_2 E_{K_2} & 0_{K_2,K_3} & \ldots & 0_{K_2,K_{m-1}} & 0_{K_2,K_m} \\
0_{K_3,K_1} & 0_{K_3,K_2} & \mu_3 E_{K_3} & \ldots & 0_{K_3,K_{m-1}} & 0_{K_3,K_m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0_{K_{m-1},K_1} & 0_{K_{m-1},K_2} & 0_{K_{m-1},K_3} & \ldots & \mu_{m-1} E_{K_{m-1}} & 0_{K_{m-1},K_m} \\
0_{K_m,K_1} & 0_{K_m,K_2} & 0_{K_m,K_3} & \ldots & 0_{K_m,K_{m-1}} & \mu_m E_{K_m}
\end{pmatrix}, \tag{99}
\]

where by \( E_{K_s} \) we denote the \( K_s \times K_s \)-square matrix consisting of 1. I.e., \( E_{K_s} = 1_{K_s}^T 1_{K_s} \), where \( 1_{K_s} = (1, \ldots, 1) \) is \( 1 \times K_s \)-vector consisting of 1. \( 0_{K_i,K_j} \) is the \( K_i \times K_j \)-matrix consisting of zeros. The matrix \( M_1 \) can be rewritten as \( M_1 = \text{diag}(\mu_1 1_{K_1}^T, \ldots, \mu_m 1_{K_m}^T) \). It can be noticed that the matrix \( M_1 \) admits the factorization:

\[
M_1 = P \sigma P^*, \quad P = (e_{K_1}, \ldots, e_{K_m}), \quad \sigma = \text{diag}(\mu_1, \ldots, \mu_m), \tag{100}
\]

In formula (100) by \( 0_{K_j} = (0, \ldots, 0) \) we denote the \( 1 \times K_j \)-vector consisting of zeros. \( e_{K_j} \) denotes the \( 1 \times K \)-vector consisting of row vectors \( 0_{K_j} \) and \( 1_{K_j} \).

**Example 1.** Consider the case \( m = 2, K_1 = 1, K_2 = 2 \). Then formulae (42)–(100)
for the matrix $\mathcal{M}_1$ become

$$
\mathcal{M}_1 = \begin{pmatrix}
\mu_1 & 0 & 0 \\
0 & \mu_2 & \mu_2 \\
0 & \mu_2 & \mu_2
\end{pmatrix}, \quad \mathcal{M}_1 = P\sigma P^* = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & \mu_2 \\
1 & 0 & 1
\end{pmatrix}.
$$

(101)

We put $\beta = 1$ in (92) and choose solution of systems (92) and (94) in the following form:

$$
\varphi = (\varphi_1, \ldots, \varphi_K), \quad \varphi_j = \exp \left\{ \left( \frac{\nu_j^2 + c_1 \lambda_j^2}{\alpha_2} \right) t_2 + \nu_j x + \lambda_j y \right\}, \quad \lambda_j, \nu_j \in \mathbb{C}.
$$

(102)

Using Corollary 4 and formula (97) we see that the functions $\Phi, v_0 = 2(\varphi \Delta^{-1} \varphi^*)_x, \quad \hat{S} = -2D^{-1}\{\Phi^*\Phi\}_y = 2(\Delta^{-1})_y$ and $\mathcal{M}_1 = P\sigma P^*$ satisfy the matrix DS-system (47):

$$
\alpha_2 \Phi_{t_2} = \Phi_{xx} + c_1 \Phi_{yy} + v_0 \Phi - c_1 \Phi P\sigma P^* \hat{S}, \quad \alpha_2 \in i\mathbb{R},
$$

$$
v_{0y} = 2(\Phi P\sigma P^* \Phi^*)_x, \quad \hat{S}_x = -2(\Phi^*\Phi)_y.
$$

(103)

Define functions $\mathbf{q}, v_0$ and $S$ in the following way:

$$
\mathbf{q} = \Phi P = \varphi \Delta^{-1} P = \varphi (C + D^{-1}\{\varphi^*\varphi\})^{-1} P, \quad v_0 = 2(\varphi \Delta^{-1} \varphi^*)_x, \quad S = 2P^*(\Delta^{-1})_y P,
$$

(104)

where the matrices $C$ and $P$ are defined by (98) and (100). The integral $D^{-1}$ with respect to $x$ in the formula $D^{-1}\{\varphi^*\varphi\}$ (104) is realized in the form

$$
D^{-1}\{\varphi^*\varphi\} = \left( \frac{\varphi_i \varphi_j}{\nu_j + \nu_i} \right)_{i,j=1}^K.
$$

(105)

Then from equation (103) it follows that functions (104) satisfy the $m$-component DS system

$$
\alpha_2 \mathbf{q}_{t_2} = \mathbf{q}_{xx} + c_1 \mathbf{q}_{yy} + v_0 \mathbf{q} - c_1 \mathbf{q}\sigma S, \quad \alpha_2 \in i\mathbb{R},
$$

$$
v_{0y} = 2(\mathbf{q}\sigma \mathbf{q}^*)_x, \quad S_x = -2(\mathbf{q}^*\mathbf{q})_y,
$$

(106)

that can be rewritten as:

$$
\alpha_2 \mathbf{q}_{i,t_2} = \mathbf{q}_{i,xx} + c_1 \mathbf{q}_{i,yy} + v_0 \mathbf{q}_i - c_1 \sum_{j=1}^m \mu_j q_j S_{ji}, \quad i = 1, m, \quad \alpha_2 \in i\mathbb{R},
$$

$$
v_{0y} = 2(\sum_{j=1}^m \mu_j |q_j|^2)_x, \quad S_{ji,x} = -2(\bar{q}_j q_i)_y, \quad i, j = 1, m.
$$

(107)
Remark 5. By the well-known formulae from the matrix theory solutions (104) can be rewritten as

\[ q_j = \Phi e_{K_j} = \sum_{i=K_{j-1}+1}^{K_j} \Phi_i = \varphi \Delta^{-1} e_{K_j} = -\frac{\det \begin{pmatrix} \Delta & e_{K_j} \\ \varphi & 0 \end{pmatrix}}{\det \Delta}, \quad j = 1, m; \quad (108) \]

\[ v_0 = 2(\varphi \Delta^{-1} \varphi^*)_x = -2 \left( \begin{array}{c} \det \begin{pmatrix} \Delta & \varphi^* \\ \varphi & 0 \end{pmatrix} \\ \det \Delta \end{array} \right)_x, \quad (109) \]

\[ S_{ij} = 2(P_{i})^*(\Delta^{-1})_y P_j = -2 \left( \begin{array}{c} \det \begin{pmatrix} \Delta & P_j \\ (P_{i})^* & 0 \end{pmatrix} \\ \det \Delta \end{array} \right)_y, \quad i, j = 1, m. \quad (110) \]

In formulae (108)–(110) we set \( K_0 := 0 \). By \( S_{ij} \), \( i, j = 1, m \), we denote the elements of the corresponding matrix-function \( S \). By \( q_j \) we denote the elements of the vector-function \( q = (q_1, \ldots, q_m) \). \( P_i \) denotes the \( i \)-th vector-column of the matrix \( P \).

In the case \( m = 1 \) system (107) reduces to the following one:

\[ \alpha_2 q_{t2} = q_{xx} + c_1 q_{yy} + v_0 q - c_1 \mu_1 q S, \quad \alpha_2 \in i\mathbb{R}, \]

\[ v_{0y} = 2(\mu_1 |q|^2)_x, \quad S_x = -2(|q|^2)_y, \quad (111) \]

and the formulae from Remark 5 take the form:

\[ q = \varphi \Delta^{-1} 1_K^T = -\frac{\det \begin{pmatrix} \Delta & 1_K^T \\ \varphi & 0 \end{pmatrix}}{\det \Delta}, \quad S = 2(1_K \Delta^{-1} 1_K^T)_y = -2 \left( \begin{array}{c} \det \begin{pmatrix} \Delta & 1_K^T \\ 1_K & 0 \end{pmatrix} \\ \det \Delta \end{array} \right)_y, \]

\[ v_0 = 2(\varphi \Delta^{-1} \varphi^*)_x = -2 \left( \begin{array}{c} \det \begin{pmatrix} \Delta & \varphi^* \\ \varphi & 0 \end{pmatrix} \\ \det \Delta \end{array} \right)_x. \]
and represent a $K$-soliton solution of the scalar DS system. The latter in the case $K = 1$ takes the form:

$$
q = \frac{\exp(\theta)}{\Delta}, \quad S = -2 \frac{\text{Re}(\lambda_1) \exp(2\text{Re}(\theta))}{\text{Re}(\nu_1) \Delta^2}, \quad v_0 = 2\mu_1 \frac{\text{Re}(\nu_1) \exp(2\text{Re}(\theta))}{\text{Re}(\lambda_1) \Delta^2},
$$

(112)

where $\Delta = \frac{\mu_1}{2\text{Re}(\lambda_1)} + \frac{1}{2\text{Re}(\nu_1)} \exp(2\text{Re}(\theta))$ and $\theta = \left(\frac{\nu_1^2 + c_1 \lambda_1^2}{\alpha_2}\right) t_2 + \nu_1 x + \lambda_1 y$. We shall point out that DS equation (111) consists of two special cases:

(a) $\mu_1 = 1$. In this case formulae (108)-(110) from Remark 5 and (112) represent regular solutions of DS-equation (111).

(b) $\mu_1 = -1$. In this case Remark 5 (formulae (108)-(110)) and formula (112) give us singular solutions of DS-equation (111).

In a similar way dressing technique has been elaborated for several multicomponent integrable systems from the k-cKP hierarchy in\cite{51}.

**Remark 6.** It should be also pointed out that besides soliton solutions DS equation (111) has also an important class of solutions known as dromions. Dromions were obtained at first in\cite{57} via Bäcklund transformations and nonlinear superposition formulae. They were also recovered by the Hirota direct method\cite{58} and Darboux Transformations\cite{59}.

In paper\cite{53} dromions were constructed for matrix DS equation (47). However, the Lax pair for DS equation used in\cite{57,59} involves non-stationary Dirac operator and differs from the integro-differential Lax pair (44) for DS equation that we considered.

2. $n = 3$, $l = 2$. Assume that in addition to equations (92) the functions $\varphi$ and $\psi$ satisfy the equations:

$$
M_3\{\varphi\} = c_2 \varphi \Lambda^3 = c_2 L_0^3\{\varphi\}, \quad M_3\{\psi\} = c_2 \psi \tilde{\Lambda}^3 = c_2 (L_0^3)^3\{\psi\},
$$

$$
M_3 := \alpha_3 \partial_{t_3} - D^3.
$$

(113)

As it was done in the case $n = 2$, $l = 1$, we obtained that the transformed operator $\tilde{M}_3$ via BDT $W$ (70) has the form

$$
\tilde{M}_3 = WM_3 W^{-1} = \alpha_3 \partial_{t_3} - D^3 - \hat{v}_1 D - \hat{v}_0 + c_2 \sum_{i=0}^{2} \hat{L}_0^i\{\Phi\} M_0 D^{-1}((\hat{L}_0^i)^{2-i}\{\Psi\})^\top,
$$

(114)

where $\hat{v}_1 = 3(\varphi \Delta^{-1}\psi^\top)_x$, $\hat{v}_0 = 3\varphi \Delta^{-1}\psi^\top (\varphi \Delta^{-1}\psi^\top)_x - 3(\varphi_x \Delta^{-1}\psi^\top)_x$. After the change $\textbf{q} := \Phi$, $\textbf{r} := \Psi$, $\mathcal{M}_0 := -\mathcal{M}_1$, $v_1 := \hat{v}_1$, $v_0 := \hat{v}_0$ it can be shown that the functions $\textbf{q}$, $\textbf{r}$, $v_1$, $v_0$ with the matrix $\mathcal{M}_0$ satisfy equations (57) in case $c = 1$.25
Analogously, dressing methods can be done for the other Lax pairs from Section III. As an example we consider equation (68) with Lax pair (67):

3. $k = 2, n = 3, l = 1$. Assume that $N \times K$-matrix functions $\varphi$ and $\psi$ satisfy the equations:

$$
L_2\{\varphi\} = \varphi \Lambda, \quad L_2\{\psi\} = \psi \tilde{\Lambda}, \quad L_2 := \beta_2 \partial_{\tau_2} - D^2,
$$

$$
M_3\{\varphi\} = c_1 \varphi \Lambda^2, \quad M_3\{\psi\} = c_1 \psi \tilde{\Lambda}^2, \quad M_3 := \alpha_3 \partial_{\tau_3} - D^2.
$$

Using Theorem 2 we obtain that the dressed operator $\hat{L}_2$ has the form:

$$
\hat{L}_2 = \beta_2 \partial_{\tau_2} - D^2 - 2\hat{u} + \Phi M_1 D^{-1} \Psi^T,
$$

where the functions $\Phi$ and $\Psi$ are defined by (73); $\hat{u} = (\varphi \Delta^{-1} \psi^T)_x$. Using Theorems 2 and 3, their corollaries and direct calculations it can be checked that the functions $\hat{u} = (\varphi \Delta^{-1} \psi^T)_x$, $\varpi := \Phi$, $\varrho := \Psi$ and the matrix $M_0 := -M_1$ satisfy equations (68).

We note that in the case $c_1 = 0$, $N = 1$ we obtain solutions for the (2+1)-dimensional generalization of higher Yajima-Oikawa equation:

$$
\alpha_3 u_{t_3} - \frac{1}{4} u_{xxx} - 3uu_x - \frac{3}{4} \beta_2 \left(q M_0 r^T\right)_{x_2} + \frac{3}{4} \left(q M_0 r^T - q_x M_0 r^T\right)_{x_2} - \frac{3}{4} \beta_2^2 D^{-1} \{u_{\tau_2}\},
$$

$$
\alpha_3 q_{t_3} - q_{xxx} - 3u q_x - \frac{3}{2} \left(u_x + \beta_2 D^{-1} \{u_{\tau_2}\} + q M_0 r^T\right) q = 0,
$$

$$
\alpha_3 r_{t_3} - r_{xxx} - 3(u r)_{x_2} + \frac{3}{2} \left(u_x + \beta_2 D^{-1} \{u_{\tau_2}\} + q M_0 r^T\right) r = 0,
$$

(117)

that was investigated as a member of (2+1)-dimensional k-cKP in\textsuperscript{34,35}. Its solutions via differential gauge operators were considered in\textsuperscript{38,39}. It should be mentioned that the solution generating technique via Binary Darboux Transformation $W$ (70) presented for matrix DS-system in item 1 can also be elaborated for system (117) in the case of Hermitian conjugation reduction ($u$ is a real-valued function, $r = \bar{q}$, $\alpha_3 \in \mathbb{R}$, $\beta_2 \in i\mathbb{R}$, $M_0^* = -M_0$).

We also point out that some of the matrix equations presented in items 1-3 and their solutions were also investigated recently\textsuperscript{52,53}.

V. CONCLUSIONS

In this paper we proposed a new (2+1)-BDk-cKP hierarchy (30) that generalizes (2+1)-dimensional KP hierarchy (13) which was introduced in\textsuperscript{34,35} and rediscovered in\textsuperscript{38}. For some
members of (2+1)-dimensional k-cKP hierarchy (9) (e.g. (21) and (23)), solutions were obtained via the binary Darboux transformation dressing method. Dressing methods for (2+1)-dimensional extension of the k-cKP and modified k-cKP hierarchy via differential transformations were elaborated recently in. New (2+1)-BDk-cKP hierarchy (30) extends hierarchy (9). It also contains integrable systems and their matrix generalizations that do not belong to (2+1)-dimensional k-cKP hierarchy (30). (2+1)-BDk-cKP hierarchy also extends matrix KP hierarchies such as (t_A, τ_B)-hierarchy and (γ_A, σ_B)-hierarchy. Some members of new hierarchy (30) such as Davey-Stewartson systems (DS-I,DS-II,DS-III), matrix generalizations of (2+1)-dimensional extensions of the mKdV and their representations via integro-differential operators were considered recently in. Dressing methods via binary Darboux transformations presented in Section IV give us an opportunity to construct the exact solutions for equations that are contained in (2+1)-BDk-cKP hierarchy (30). The most interesting systems obtained from hierarchy (30) arise after a Hermitian conjugation reduction. It is much more suitable to use BDT for dressing methods in this situation rather than a differential operator. Elaborated dressing technique method provides us also with a possibility to investigate exact solutions of matrix equations that (2+1)-BDk-cKP hierarchy (30) contains. We shall note that the interest to noncommutative equations has also arisen recently. Thus one of problems for further investigation is the generalization of (2+1)-BDk-cKP hierarchy to the case of noncommutative algebras (namely, the elements q, r, u_j and v_i belong to some noncommutative ring). It is also very important to point out that initial operators that we have chosen for dressing in (92), (94), (113), (115) are differential. However, the statements of the theorems in Section IV (in particular Theorem 2 and 3) concern initial integro-differential operators. The case of initial (undressed) integro-differential operators should also be elaborated. We recently obtained that for (1+1)-dimensional integrable systems it leads to classes of solutions that do not tend to zero at infinities (e.g. finite density solutions). Generalizations of (2+1)-dimensional k-cKP hierarchy (30) also provides us with problems for further investigations. In particular, using (2+1)-BDk-cKP hierarchy it becomes possible to investigate generalizations of modified (2+1)-extended k-cKP hierarchy (modified (2+1)-BDk-cKP hierarchy) and elaborate dressing methods for it. Some members of this hierarchy were recently considered in. In particular, we investigated Lax integro-differential representations for the Nizhnik equation and (2+1)-dimensional extension of the Chen-Lee-Liu equation. Representations for some of those systems in
the algebra of purely differential operators with matrix coefficients can be found in\textsuperscript{60}. Using (2+1)-BDk-cKP hierarchy we also intend to generalize (2+1)-extended Harry-Dym hierarchy presented in\textsuperscript{61}. We recently obtained multi-component equation with the following Lax pair contained in (2+1)-BDk-cmKP hierarchy (bidirectional generalization of (2+1)-dimensional k-constrained modified KP hierarchy):

\begin{equation}
L_0 = \partial_y - q M_0 D^{-1} r^\top D, \quad \alpha \in \mathbb{C},
M_3 = \alpha_3 \partial_{t_3} + c_0 D^3 - c_2 \partial_y^3 - 3c_0 v_1 D^2 - 3c_0 v_3 D + 3c_2 q_y M_0 D^{-1} \partial_y r^\top D +
+c_2 q M_0 D^{-1} \partial_y r^\top \partial_y D - 3c_2 q M_0 \partial_y D^{-1} r^\top D q M_0 D^{-1} r^\top D +
+3c_2 q M_0 D^{-1} \{ r^\top q_x \} y D^{-1} M_0 r^\top D, \quad c_0, c_2 \in \mathbb{R}.
\end{equation}

where \( q \) and \( r \) are 1 \times m-vector functions, \( u, v_1 \) and \( v_3 \) are scalar functions, \( M_0 \) is an \( m \times m \)-constant matrix. If we impose additional reduction in (118):

\begin{equation}
q = (\alpha_1 q_1, \alpha_1 q_2, \alpha_2 D^{-1} \{ u \}, \alpha_2), \quad r = q, \quad M_0 = \text{diag}(I_2, \sigma),
\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_1, \alpha_2 \in \mathbb{R},
\end{equation}

where \( I_2 \) is the identity matrix of dimension 2 \times 2; \( q_1, q_2 \) and \( u \) are scalar functions. The system that is equivalent to the Lax equation \([L_0, M_3] = 0\) generalizes (2+1)-extension of modified KdV equation that was investigated in\textsuperscript{37,62} (reduction \( c_2 = 0, \alpha_2 = 0 \) in (118)-(119)) and the Nizhnik equation\textsuperscript{63} (reduction \( \alpha_1 = 0 \) in (119)).

The following remark shows connections between constrained KP hierarchies and Lax pairs with recursion operators.

\textbf{Remark 7.} It should be also mentioned that the majority of Lax representations with recursion operators (the representations connected with bi-hamiltonian pairs of the corresponding integrable systems) involve integro-differential operators\textsuperscript{63}. A part of such representations is contained in the constrained KP hierarchies (presented in Section III and IV) with additional (not only a Hermitian conjugation reduction) specific reductions.

From Remark\textsuperscript{7} it follows that one of the essential problems for further investigation is an adaptation of the dressing method (Theorem\textsuperscript{2} and\textsuperscript{3} from Section IV) for Lax representations with the recursion operators. The work in this direction is in progress.

\textbf{Remark 8.} It should be also pointed out that the interesting problem consists in the possibility of obtaining other classes of exact solutions (e.g., lumps and dromions; see Remark\textsuperscript{6}) for nonlinear equations from the proposed (2+1)-BDk-cKP hierarchy.
We plan to consider the above mentioned problem (Remark 8) in a separate paper.

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VII. APPENDIX. PROOF OF LEMMA 1.

Proof. We will use the following recurrent formulae that can easily be checked by direct calculation:

\[
\mathcal{M}_2 = C \Lambda C^{-1} \mathcal{M}_1 + \mathcal{M}_1 C \tilde{\Lambda}^\top C^{-1}, \tag{120}
\]

\[
\mathcal{M}_{l+1} = C \Lambda C^{-1} \mathcal{M}_l + \mathcal{M}_l C^{-1} \tilde{\Lambda}^\top C - C \Lambda C^{-1} \mathcal{M}_{l-1} C^{-1} \tilde{\Lambda}^\top C. \tag{121}
\]

At first we will prove the following formula:

\[
\mathcal{M}_{l+1} = \sum_{s=0}^{k} C \Lambda^s C^{-1} \mathcal{M}_{l-k+1} C^{-1} (\tilde{\Lambda}^\top)^{k-s} C - \sum_{s=1}^{k} C \Lambda^s C^{-1} \mathcal{M}_{l-k} C^{-1} (\tilde{\Lambda}^\top)^{k-s+1} C, \quad k \leq l-2, \tag{122}
\]
via induction by $k$. Assume that (122) holds for some $k < l - 2$. We shall show that then this formula holds for $k + 1 < l - 1$ using (121):

\[
\mathcal{M}_{t+1} = \sum_{s=0}^{k} CA^sC^{-1} \mathcal{M}_{t-k+1}C^{-1}(\tilde{A}^T)^{k-s}C - \sum_{s=1}^{k} CA^sC^{-1} \mathcal{M}_{t-k}C^{-1}(\tilde{A}^T)^{k-s+1}C = \\
= \sum_{s=0}^{k} CA^{s+1}C^{-1} \mathcal{M}_{t-k}C^{-1}(\tilde{A}^T)^{k-s}C + \sum_{s=0}^{k} CA^sC^{-1} \mathcal{M}_{t-k}C^{-1}(\tilde{A}^T)^{k-s+1}C - \\
- \sum_{s=0}^{k} CA^{s+1}C^{-1} \mathcal{M}_{t-k-1}C^{-1}(\tilde{A}^T)^{k-s+1}C - \sum_{s=0}^{k} CA^sC^{-1} \mathcal{M}_{t-k}C^{-1}(\tilde{A}^T)^{k+1-s}C = \\
= \sum_{i=1}^{k+1} CA^iC^{-1} \mathcal{M}_{t-k-i}C^{-1}(\tilde{A}^T)^{k+1-i}C + \mathcal{M}_{t-k}C^{-1}(\tilde{A}^T)^{k+1}C - \\
- \sum_{i=1}^{k+1} CA^iC^{-1} \mathcal{M}_{t-k-i}C^{-1}(\tilde{A}^T)^{k+1-i}C + \mathcal{M}_{t-k}C^{-1}(\tilde{A}^T)^{k+1}C - \\
- \sum_{i=1}^{k+1} CA^iC^{-1} \mathcal{M}_{t-k-i}C^{-1}(\tilde{A}^T)^{k+1-i}C
\]

(123)

Thus, we have proven (122). After the substitution of $k = l - 2$ in (77) and using (120) we obtain:

\[
\mathcal{M}_{t+1} = \sum_{s=0}^{l-1} CA^sC^{-1} \mathcal{M}_2C^{-1}(\tilde{A}^T)^{l-1-s} + \sum_{s=1}^{l-1} CA^sC^{-1} \mathcal{M}_1C^{-1}(\tilde{A}^T)^{l-s}C = \\
+ \sum_{s=0}^{l-1} CA^{s+1}C^{-1} \mathcal{M}_1C^{-1}(\tilde{A}^T)^{l-1-s} + \sum_{s=0}^{l-1} CA^sC^{-1} \mathcal{M}_1C^{-1}(\tilde{A}^T)^{l-s}C - \\
- \sum_{s=0}^{l-1} CA^{s+1}C^{-1} \mathcal{M}_1C^{-1}(\tilde{A}^T)^{l-s}C = \sum_{s=0}^{l-1} CA^sC^{-1} \mathcal{M}_1C^{-1}(\tilde{A}^T)^{l-s}C.
\]

(124)

This finishes the proof of formula (77) and Lemma 1.

\[\square\]

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