Neutron matter at next-to-next-to-next-to-leading order in chiral effective field theory

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Neutron matter presents a unique system for chiral effective field theory (EFT), because all many-body forces among neutrons are predicted to next-to-next-to-next-to-leading order (N$^3$LO). We present the first complete N$^3$LO calculation of the neutron matter energy. This includes the subleading three-nucleon (3N) forces for the first time and all leading four-nucleon (4N) forces. We find relatively large contributions from N$^3$LO 3N forces. Our results provide constraints for neutron-rich matter in astrophysics with controlled theoretical uncertainties.

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The physics of neutron matter ranges from universal properties at low densities to the structure of extreme neutron-rich nuclei and the densest matter we know to exist in neutron stars. For these extreme conditions, controlled calculations with theoretical error estimates are essential. Chiral EFT provides such a systematic expansion for nuclear forces [1]. This is particularly exciting for neutron matter and neutron-rich systems, because all three- and four-nucleon forces are predicted to N$^3$LO [2].

Neutron matter based on chiral EFT has been studied using lattice simulations [3] at low densities, $n \lesssim n_0/10$ (with saturation density $n_0 = 0.16$ fm$^{-3}$), and following an in-medium chiral perturbation theory approach [4, 5], where low-energy couplings are adjusted to empirical nuclear matter properties. In addition, the renormalization group (RG) has been used to evolve chiral EFT interactions to low momenta [6], which has enabled perturbative calculations for nucleonic matter [7]. While these constrain the properties of neutron-rich matter to a much higher degree than is reflected in neutron star modeling [8], the dominant uncertainties are due to 3N forces, which were included only to N$^2$LO. A consistent inclusion of higher-order many-body forces is therefore key.

Here we present the first calculations at nuclear densities based directly on chiral EFT interactions without RG evolution. To this end, we studied the perturbative convergence of chiral two-nucleon (NN) potentials for neutron matter in detail, and found that the available N$^2$LO and N$^3$LO potentials with lower cutoffs $\Lambda = 450 - 500$ MeV are perturbative. This is supported by small Weinberg eigenvalues at low energies indicating the perturbative convergence in the particle-particle channel [6]. In neutron matter, it comes as a result of effective range effects [6], which weaken NN interactions at higher momenta, combined with weaker tensor forces among neutrons, and with limited phase space at finite density due to Pauli blocking [10].

At the NN level we use the N$^2$LO and N$^3$LO potentials developed by Epelbaum, Glöckle and Meißner (EGM) [11] with $\Lambda/\tilde{\Lambda} = 450/500$ and $450/700$ MeV ($\Lambda/\tilde{\Lambda}$ denotes the cutoff in the Lippmann-Schwinger equation and in the two-pion-exchange spectral-function regularization, respectively). We also use the $\Lambda = 500$ MeV N$^3$LO NN potential of Entem and Machleidt (EM) [12], which is most commonly used in nuclear structure calculations. The larger $\Lambda = 550 - 600$ MeV NN potentials of EGM and EM have been found to be nonperturbative [13] and are therefore not included. Moreover, the LO NN contact couplings in the 600/600 and 600/700 EGM potentials break Wigner symmetry perturbatively (at the interaction level), with a repulsive spin-independent $C_S$.

FIG. 1. (Color online) Neutron matter energy per particle as a function of density including NN, 3N and 4N forces at N$^3$LO. The three overlapping bands are labeled by the different NN potentials and include uncertainty estimates due to the many-body calculation, the low-energy $c_i$ constants and by varying the 3N/4N cutoffs (see text for details). For comparison, results are shown at low densities (see also the inset) from NLO lattice [8] and Quantum Monte Carlo (QMC) simulations [22], and at nuclear densities from variational (APR; the different points are with/without boost corrections) [23] and Auxiliary Field Diffusion MC calculations (GCR) [24] based on adjusted nuclear force models.
and an unnaturally large spin-dependent \( C_T \sim C_S \), leading to unexpectedly large \( C_T \)-dependent 3N forces.

In this Letter, we include for the first time all N\(^3\)LO 3N and 4N forces, which have been derived only recently \[14–17\], in addition to the N\(^2\)LO 3N forces. Figure 1 shows our complete N\(^3\)LO calculation of the neutron matter energy as our main result, where the bands include estimates of the theoretical uncertainties due to the many-body calculation and in the many-body forces.

For neutrons, only the two-pion-exchange 3N forces contribute at N\(^2\)LO \[2\]. For the corresponding low-energy constants \( c_1 \) and \( c_3 \), we take the range of values from a high-order analysis \[18\], at N\(^2\)LO: \( c_1 = -(0.37 - 0.81) \text{ GeV}^{-1} \) and \( c_3 = -(2.71 - 3.40) \text{ GeV}^{-1} \) (which includes the \( c_i \) values in the EGM and EM NN potentials), and when the N\(^2\)LO 3N forces are included in an N\(^3\)LO calculation: \( c_1 = -(0.75 - 1.13) \text{ GeV}^{-1} \) and \( c_3 = -(4.77 - 5.51) \text{ GeV}^{-1} \). It has been shown \[2\] that the N\(^2\)LO 3N force contributions in neutron matter can be to a good approximation calculated at the Hartree-Fock level. In this first calculation, we therefore evaluate the N\(^3\)LO 3N and 4N force contributions to the energy per particle \( E/N \) at the Hartree-Fock level. The A-body contributions are then given by

\[
\frac{E}{N} = \frac{1}{n!} \sum_{\sigma_1, \ldots, \sigma_n} \int \frac{dk_1}{(2\pi)^3} \cdots \int \frac{dk_n}{(2\pi)^3} f_R n_{k_1} \cdots n_{k_n} \times \langle 1 \ldots A | A_A \sum_{i_1 \neq \ldots \neq i_A = 1} A(i_1, \ldots, i_A) | 1 \ldots A \rangle, \tag{1}
\]

with short-hand notation \( i \equiv k_i \sigma_i \). \( A_A \) denotes the A-body antisymmetrizer and \( n_{k_i} = \delta(k_F - k_i) \) the Fermi-Dirac distributions at zero temperature. We use a Jacobi-momenta regulator; in terms of \( k_i \) given by \( f_R = \exp[-((k_i^2 + \ldots + k_{A-1}^2 - k_A) - k_{A-1} - k_A)/\Lambda^2)^2 \exp] \) with \( \Lambda = 4 \) and N\(^3\)/4N cutoff \( \Lambda = 2 - 2.5 \text{ fm}^{-1} \). For the nucleon and pion mass, we use \( m = 938.92 \text{ MeV} \) and \( m_{\pi} = 138.04 \text{ MeV} \), and for the axial coupling \( g_A = 1.29 \) and the pion decay constant \( f_\pi = 92.4 \text{ MeV} \).

Chiral 3N forces at N\(^3\)LO can be grouped into

\[
V^{3\text{NLO}} = V^{2\pi} + V^{2\pi -1,1} + V^{\text{ring}} + V^{2\pi-\text{cont}} + V^{1/m}, \tag{2}
\]

where we take the long-range parts, the subleading two-pion-exchange, the two-pion–one-pion-exchange and the pion-ring 3N forces, from Ref. \[15\], and the short-range parts, the two-pion-exchange–contact and relativistic 1/m-corrections 3N forces from Ref. \[16\]. In Fig. 2 we give the individual Hartree-Fock contributions to the neutron matter energy. The evaluation is aided because parts of the different 3N force topologies vanish for neutrons, and the results have been checked by two independent calculations. The details of the calculation will be presented in a future paper. At the Hartree-Fock level, the 3N/4N contributions change by < 5% if the cutoff is taken to infinity (i.e., \( f_R = 1 \)), but we will also include N\(^3\)LO 3N forces beyond Hartree-Fock. This requires a consistently used regulator. Estimates of the theoretical uncertainty are provided by varying the 3N/4N cutoff.

The two-pion-exchange 3N forces at N\(^3\)LO can be largely written as shifts of the low-energy constants, \( \delta c_2 = -0.13 \text{ GeV}^{-1} \) and \( \delta c_3 = 0.89 \text{ GeV}^{-1} \) \[15\] of the N\(^2\)LO 3N forces, plus a smaller contribution. The resulting energy of about \( -1.5 \text{ MeV} \) per particle at saturation density \( n_0 \) in Fig. 2 is \( \sim 1/3 \) of the N\(^2\)LO 3N energy, as expected based on the chiral EFT power counting. In contrast, the two-pion–one-pion-exchange 3N force contributions, which include 14 diagrams, are relatively large with \( -3.6 \text{ MeV} \) per particle at \( n_0 \). The short-range parts of N\(^3\)LO 3N forces depend on the momentum-independent NN contacts, \( C_T \) and \( C_S \), which we take consistently from the N\(^2\)LO EM/EGM potential used. The contributions from the two-pion-exchange–contact 3N forces include 11 diagrams and depend only on \( C_T \). The resulting energy ranges from \( -2.8 \) to \( +1.3 \text{ MeV} \) at \( n_0 \) depending on the NN potential used. These larger 3N results at N\(^3\)LO are consistent with contributions from the large \( c_i \) constants at N\(^2\)LO exactly in these three topologies \[18\]. This shows that higher-order many-body forces still need to be investigated and that a chiral EFT with explicit \( \Delta \) excitations may be more efficient, since this would capture these effects already at N\(^3\)LO. Finally, the relativistic-corrections 3N forces depend also on \( \beta_k \) and \( \beta_0 \) \[12\] and contribute at the few hundred keV level.

The 4N force contributions in Fig. 2 are an order of magnitude smaller than those from the N\(^3\)LO 3N forces and of similar size as the 3N relativistic corrections. We follow the 4N force notation \( V^a \) through \( V^n \) of Ref. \[17\], and include the direct and all 23 exchange terms. Due to the spin-isospin structure, only 3 topologies contribute to neutron matter: the three-pion-exchange 4N forces \( V^a \) and \( V^n \), and the pion-pion-interaction 4N forces \( V^f \). The 4N forces \( V^k \) and \( V^\pi \) involving the contact \( C_T \) vanish in neutron matter due to their spin structure. We find a total 4N force contribution of \( -174 \pm 10 \text{ keV} \) per particle at \( n_0 \). The \( V^\pi \) and \( V^f \) energies largely cancel \[19\], and their sum agrees with the very small \( -20 \text{ keV} \) per particle at \( n_0 \) of Ref. \[20\], which considered these two parts.

Since diagrams beyond Hartree-Fock involving NN interactions and N\(^3\)LO 3N forces (in particular with the larger \( c_i \) at N\(^3\)LO 3N and without RG evolution) provide non-negligible contributions \[2\], we include all such diagrams to second order, as well as particle-particle diagrams to third order, which is technically possible based on Ref. \[7\]. In addition to using NN potentials with different cutoffs and varying the 3N/4N cutoffs, we include estimates of the theoretical uncertainties of the \( c_i \) constants and in the convergence of the many-body calculation. The latter is probed by studying the sensitiv-
FIG. 2. (Color online) Energy per particle versus density for all individual NLO 3N and 4N force contributions to neutron matter at the Hartree-Fock level. The bands are obtained by varying the 3N/4N cutoff $\Lambda = 2 - 2.5 \text{fm}^{-1}$. For the two-pion-exchange–contact and the relativistic-corrections 3N forces, the different bands correspond to the different NN contacts, $C_T$ and $C_S$, determined consistently for the N3LO EM/EGM potentials. The inset diagram illustrates the 3N/4N force topology.

As we find relatively large contributions from N3LO 3N forces, it is important to study the EFT convergence from N2LO to N3LO. This is shown in Fig. 3 for the EGM potentials (N2LO is not available for EM), where the N3LO results are found to overlap with the N2LO band across a $\pm 1.5 \text{MeV}$ range around 17 MeV at saturation density. As expected from the net-attractive N3LO 3N contributions in Fig. 2, the N3LO band yields lower energies. For the N2LO band, we have estimated the theoretical uncertainties in the same way, and the neutron matter energy ranges from 15.5 – 21.4 MeV per particle at $n_0$. The theoretical uncertainty is reduced from N2LO to N3LO to 14.1 – 18.4 MeV, but not by a factor $\sim 1/3$ based on the power counting estimate. This reflects the...
large $c^{i}_{3}$ 3N contributions at $N^{4}$LO, and is similar to the convergence pattern observed in chiral NN potentials [1].

The neutron matter energy in Fig. 1 is in very good agreement with NLO lattice results [3] and Quantum Monte Carlo simulations [22] at very low densities (see also the inset) and approximately reproduces the scaling $\sim 0.5 \frac{365}{10^{m}}$, which we attribute to effective-range effects combined with low cutoffs [3]. At nuclear densities, we compare our $N^{4}$3LO results with variational calculations based on phenomenological potentials (APR) [23], which are within the $N^{3}$LO band, but do not provide theoretical uncertainties. In addition, we compare the density dependence with results from Auxiliary Field Diffusion MC calculations (GCR) [24] based on nuclear force models adjusted to an energy difference of 32 MeV between neutron matter and the empirical saturation point. The density dependence is similar to the $N^{3}$LO band, but the GCR results are higher below 0.05 fm$^{-3}$.

The $N^{3}$LO band provides key constraints for the nuclear equation of state and for astrophysics. Figure 1 shows, following Ref. [23], the allowed range for the symmetry energy $S_{\nu}$ and its density dependence $L = 3n_{0}\partial_{n}S_{\nu}(n_{0})$ (for details on the determination of $S_{\nu}$ and $L$ see Ref. [8]). Compared to the results from RG-evolved chiral interactions with 3N forces at $N^{2}$LO only [8], we find the same correlation (with the same slope), but not as tight due to the additional density dependences at $N^{3}$LO. The $N^{3}$LO ranges for $S_{\nu}$ and $L$ are $S_{\nu} = 28.9 - 34.9$ MeV and $L = 43.0 - 66.6$ MeV. The two neutron-matter bands in Fig. 1 are complementary, because the RG evolution in Hebler et al. [8] improves the many-body convergence, while the band presented in this work is the first consistent $N^{4}$LO calculation. The predicted $N^{3}$LO range, as well as that of Hebler et al. [8], are in agreement with constraints obtained from energy-density functionals for nuclear masses [25] and from the $^{208}$Pb dipole polarizability [26]. In the future, the $N^{3}$LO band can be narrowed further by a higher-order many-body calculation with $N^{4}$LO 3N forces and by taking into account $\Delta$ excitations (explicitly or through large $c^{i}_{3}$ contributions at $N^{4}$LO [8]). Combined with the heaviest $2M_{\odot}$ neutron star [28] and a general extension to high densities [8], our $N^{3}$LO energy band leads to a radius range of $R = 9.7 - 13.9$ km for a typical $1.4M_{\odot}$ neutron star, in remarkable agreement with Ref. [8]. For an alternative determination using in-medium chiral perturbation theory for all densities see Ref. [8].

We have presented the first complete $N^{3}$LO calculation of the neutron matter energy, including NN, 3N and 4N forces, with the first application of $N^{3}$LO 3N forces to many-body systems. The significant contributions from $N^{3}$LO 3N forces show that their inclusion will also be very important for nuclear structure and reactions. Our results provide constraints for the nuclear equation of state and for neutron-rich matter in astrophysics, and highlight the exciting role neutron matter and neutron-rich systems play in chiral EFT, where all many-neutron forces are predicted. The large contributions from $N^{3}$LO 3N forces signal the importance of $\Delta$ contributions at nuclear densities.

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FIG. 4. (Color online) Range for the symmetry energy $S_{\nu}$ and its density dependence $L$ obtained at $N^{3}$LO (this work) versus including 3N forces at $N^{2}$LO (Hebler et al. [8]). For comparison [23], we show constraints obtained from energy-density functionals for nuclear masses (Kortelainen et al. [26]) and from the $^{208}$Pb dipole polarizability (Tamii et al. [27]).

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