ON THE CENTRALIZER OF A BALANCED NILPOTENT SECTION

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ABSTRACT. Let $G$ be a split reductive algebraic group defined over a complete discrete valuation ring $\mathcal{O}$, with residue field $\mathbb{F}$ and fraction field $\mathbb{K}$, where the fiber $G_{\mathbb{F}}$ is geometrically standard. A balanced nilpotent section $x \in \text{Lie}(G)$ can roughly be thought of as an $\mathcal{O}$-point in a $\mathbb{K}$ nilpotent orbit such that the corresponding orbits over $\mathbb{K}$ and $\mathbb{F}$ have the same Bala–Carter label. In this paper, we will establish a number of results on the structure of the centralizer $G_x \subseteq G$ of $x$. This includes a proof that $G_x$ is a smooth group scheme, and that the component groups of the geometric fibers $G_{x,\mathbb{K}}$ and $G_{x,\mathbb{F}}$ are isomorphic.

1. Introduction

Let $p$ be a prime, and let $\mathcal{O}$ be a complete discrete-valuation-ring (DVR) with uniformizer $\omega$, fraction field $\mathbb{K}$, and perfect residue field $\mathbb{F}$ of characteristic $p$. For an $\mathcal{O}$-scheme $X$, and an $\mathcal{O}$-algebra $A$, we use the notation $X_A := X \times_{\text{Spec}(\mathcal{O})} \text{Spec}(A)$.

Let $G$ be a split reductive algebraic group scheme over $\text{Spec}(\mathcal{O})$, with a (lower) Borel subgroup $B$ and a maximal torus $T$. We will assume that $G_{\mathbb{K}}$ and $G_{\mathbb{F}}$ are geometrically standard (cf. [Mc3, §3.1]).

Example 1.1. If $G = \text{SL}_n$, then $G_{\mathbb{F}}$ is geometrically standard provided $p \nmid n$.

Let $g = \text{Lie}(G)$ be the Lie algebra of $G$, regarded as a scheme, and let $g_\mathcal{O} := g(\mathcal{O})$ denote the $\mathcal{O}$-points of $g$, which gives a lattice $g_\mathcal{O} \subseteq g_{\mathbb{K}}$. We will refer to the elements of $g_\mathcal{O}$ as sections, since such an element corresponds to a map $x : \text{Spec}(\mathcal{O}) \to g$.

For each section $x \in g_\mathcal{O}$, let $x_{\mathbb{K}} \in g_{\mathbb{K}}$ and $x_{\mathbb{F}} \in g_{\mathbb{F}}$ denote the values of this map at the generic point and closed point of $\text{Spec}(\mathcal{O})$ respectively.

The adjoint action gives $g_\mathcal{O}$ the structure of a rational $G$-module (equivalently an $\mathcal{O}[G]$-comodule). For any $x \in g_\mathcal{O}$, let $G^x_{\mathbb{K}} \subseteq G$ denote the scheme-theoretic centralizer of $x$, and let $G^x_{\mathbb{K},A} \subseteq G_A$ and $G^x_{\mathbb{F}} \subseteq G_{\mathbb{F}}$ denote the scheme-theoretic centralizers of $x_{\mathbb{K}}$ and $x_{\mathbb{F}}$ (see [J1, I.2.12(1)] for the definition). In fact, it can be deduced from the definition that for any $\mathcal{O}$-algebra $A$,

$$G^x_{\mathbb{K},A} \cong (G^x)_{\mathbb{A}}.$$

To improve notation, we will often omit the parenthesis on the right hand side, and take $G^x_{\mathbb{K}} \subseteq G_{\mathbb{K}}$ to mean the centralizer of $x_{\mathbb{K}} \in g_{\mathbb{K}}$.

Definition 1.2. We say that a section $x$ is balanced if $G^x_{\mathbb{K}}$ and $G^x_{\mathbb{F}}$ are smooth group schemes and $\dim G^x_{\mathbb{K}} = \dim G^x_{\mathbb{F}}$ (cf. [Mc3]).

1To simplify notation, we will often write $G^x_{\mathbb{K}}$ and $G^x_{\mathbb{F}}$ in place of $G^x_{\mathbb{K},A}$ and $G^x_{\mathbb{F}}$ respectively.
Remark 1.3. The definition of balanced in particular implies that the scheme-theoretic centralizers of $x_K$ and $x_F$ are actually reduced, since smooth implies reduced. Thus, over the algebraic closures $\overline{K}$ and $\overline{F}$, the base changes $G_{\overline{K}}$ and $G_{\overline{F}}$ coincide with the “classical” centralizers, which are defined only on the geometric points as in [LS].

Remark 1.4. It follows from [Mc3, Theorem 4.5.2 and Corollary 7.3.2] and the proof of [Mc3, Corollary 9.2.2], that the orbits of $x_K$ and $x_F$ have the same Bala–Carter label.

It is worth mentioning that balanced nilpotent sections exist for every $F$-orbit by [Mc3, Theorem 4.5.2]. Thus, since there exist only finitely many nilpotent $F$-orbits, it is possible to enlarge $\mathcal{O}$ by a finite extension so that for every $F$-nilpotent orbit $C_F$, there exists a balanced section $x \in \mathfrak{g}_0$ such that $x_F \in C_F$ (the corresponding extension of $\mathbb{F}$ is also finite, and thus will remain perfect).

1.1. Smoothness of $G^x$. By a smooth morphism $f : X \to Y$ of schemes, we will mean a morphism which satisfies the definition given in [St, Tag 01V8]. We have included a proof of the following lemma due to the lack of a proper reference.

Lemma 1.5. A morphism $f : X \to Y$ between schemes of finite-type is smooth if and only if

1. $f$ is flat,
2. for every geometric point $\overline{y} \to Y$, the fiber product $X \times_Y \overline{y}$ is a smooth variety.

Proof. By first applying [St, Tag 01V4], we can reduce down to checking smoothness at every fiber $X \times_Y y$ for any point $\{y\} \to Y$. By definition, $\{y\} = \text{Spec}(k)$ for some field $k$, and so by [Ht, Theorem III.10.2], $X \times_Y \text{Spec}(k)$ is smooth if and only if $X \otimes_Y \text{Spec}(\overline{k})$ is smooth. Combining these two results gives the lemma. □

We will prove the following theorem.

Theorem 1.6. If $x \in \mathfrak{g}_0$ is a balanced nilpotent section, then $G^x \to \text{Spec}(\mathcal{O})$ is a smooth morphism.

Remark 1.7. We already know that the morphism $G^x \to \text{Spec}(\mathcal{O})$ is finite-type with geometric fibers $G_{\overline{K}}^x$ and $G_{\overline{F}}^x$, which are smooth varieties by Remark 1.3, so according to Lemma 1.5, it suffices to show that $\mathcal{O}[G^x]$ is a flat $\mathcal{O}$-module. However, since $\mathcal{O}$ is a DVR, this is equivalent to proving that $\mathcal{O}[G^x]$ is torsion-free over $\mathcal{O}$. This will be proven in §3.3.

1.2. Results on component groups. Let $A(x_{\overline{K}}) = G_{\overline{K}}^x(\overline{K})/(G_{\overline{K}}^x)^\circ(\overline{K})$ and $A(x_{\overline{F}}) = G_{\overline{F}}^x(\overline{F})/(G_{\overline{F}}^x)^\circ(\overline{F})$ denote the (discrete) component groups of the geometric fibers $G_{\overline{K}}^x$ and $G_{\overline{F}}^x$ respectively. In the case where $G$ is simple and of adjoint type, it follows from Remark 1.4 and [MS] that there is an isomorphism of groups $A(x_{\overline{K}}) \cong A(x_{\overline{F}})$. We will extend this result to arbitrary split reductive groups $G$ with geometrically standard fiber $G_{\overline{F}}$.

Theorem 1.8. If $x \in \mathfrak{g}_0$ is a balanced nilpotent section, then there is a group isomorphism $A(x_{\overline{K}}) \cong A(x_{\overline{F}})$.

Remark 1.9. This will be proven in §3.4.
We will also prove in Theorem 3.12, that it always possible to enlarge \( O \) so that the identity components for the fibers can be lifted to a normal subgroup scheme \((G^x)^O \leq G^x\), where the quotient \( G^x/(G^x)^O \) is an affine group scheme whose geometric fibers are \( A(x_{\overline{\mathbb{F}}}^x) \) and \( A(x_{\overline{\mathbb{F}}}^x) \).

1.3. Centralizers for the \( G \times \mathbb{G}_m \) action. In §4, we consider the centralizers for a certain action of \( G \times \mathbb{G}_m \) on \( g \). For instance, we will show that if \( x \in g_0 \) is balanced for the \( G \) action, then the centralizer \((G \times \mathbb{G}_m)^x \subseteq G \times \mathbb{G}_m\) is also smooth, and there exists an isomorphism of component groups for the geometric fibers (see Theorem 4.4). An application of this will be given in Proposition 4.7, which can be used to relate the representation theory for the reductive quotients of the \( K \) and \( \mathbb{F} \) centralizers by considering the representation theory of \((G \times \mathbb{G}_m)^x\).

1.4. Additional comments. The main source of motivation for this project originated from the author’s work with P. Achar and S. Riche on the modular Lustig–Vogan bijection in [AHR1], where the structure and representation theory of \( G^x \) plays a crucial role.

It should also be mentioned that the smoothness of \( G^x \) result has been verified in a number of cases, and by very different arguments, in [AHR1] and [B].

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2. The torsion-free subgroup scheme

Fix a balanced nilpotent section \( x \in g_0 \) and let \( I_{tor} = \ker(\mathcal{O}[G^x] \to K \otimes \mathcal{O}[G^x] \cong K[G^x]) \).

denote the ideal consisting of all the \( \omega \)-torsion elements of \( \mathcal{O}[G^x] \), then \( V(I_{tor}) \) is a closed torsion-free subscheme of \( G^x \).

2.1. We will begin by establishing some general properties of this subscheme.

**Lemma 2.1.** Let \( M \) an arbitrary \( \mathcal{O} \)-module, and let \( N \leq M \) be any free finite-rank \( \mathcal{O} \)-submodule, then the natural morphism

\[
N \otimes_{\mathcal{O}} N \to M \otimes_{\mathcal{O}} M
\]

is injective.

**Proof.** Since the functor \(- \otimes_{\mathcal{O}} K : \mathcal{O}\text{-mod} \to K\text{-mod}\) is exact, the induced map

\[
N_K \to M_K
\]

is also injective, so we can regard \( N_K \) as a submodule of \( M_K \). There is a commutative diagram

\[
\begin{array}{ccc}
(N \otimes_{\mathcal{O}} N) \otimes K & \to & N \otimes_{\mathcal{O}} N_K \\
\downarrow & & \downarrow \\
(M \otimes_{\mathcal{O}} M) \otimes K & \to & M \otimes_{\mathcal{O}} M_K
\end{array}
\]

where the “\( \sim \)” symbols denote natural isomorphisms. The rightmost arrow is the canonical injection

\[
N_K \otimes_K N_K \to M_K \otimes_K M_K,
\]

so in particular, \( v \otimes 1 \) is injective.
We will denote by $H$ the closed subscheme $V$ gives the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & S \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S \otimes K
\end{array}
$$

The exactness of the top row implies exactness of the bottom row since $- \otimes K$ is exact. However, the injectivity of $v \otimes 1$ implies $S \otimes K = 0$. Therefore, $S \subseteq N \otimes N$ is a torsion submodule of the free finite-rank $O$-module $N \otimes N$, and hence, $S = 0$. □

We will also require the following lemma.

**Lemma 2.2.** Let $H$ be any affine algebraic group scheme over Spec $O$, then the closed subscheme $V(I_{tor}) \subseteq H$ is actually a (torsion-free) subgroup scheme, which we will denote by $H_{tf}$. 

**Proof.** By [J1, I.2.4(6)] it suffices to show

$$(2.1) \quad \Delta(I_{tor}) \subseteq I_{tor} \otimes O[H] + O[H] \otimes I_{tor},$$

$\varepsilon(I_{tor}) = 0$ and $\sigma(I_{tor}) \subseteq I_{tor}$ where $\Delta$, $\varepsilon$ and $\sigma$ are the comultiplication, counit and antipode for $O[H]$ respectively.

Clearly $I_{tor} \subseteq I_1$, since $O[H]/I_1 \cong O$ is torsion-free, where $I_1 = \ker(\varepsilon)$. Also, since $\sigma : O[H] \rightarrow O[H]$ is an $O$-linear map, it must preserve torsion. So the second and third identities are verified and we are left to verify (2.1).

Let $f \in I_{tor}$ be arbitrary, then $\Delta(f)$ is also torsion since $\Delta$ is $O$-linear. Now suppose,

$$\Delta(f) = \sum_{i=1}^{m} f_i \otimes h_i,$$

and let $M \leq O[H]$ be the $O$-submodule generated by $f_i, h_i$, $1 \leq i \leq m$. Since $M$ is finitely-generated and $O$ is a DVR, then there exists a decomposition $M = M_{tor} \oplus M_{free}$. Let $\{e_1, \ldots, e_r\}$ be a basis for $M_{free}$, then for $i = 1, \ldots, m$,

$$f_i = \left( \sum_{j=1}^{m} a_{i,j} e_j \right) + \varphi_i, \quad h_i = \left( \sum_{j=1}^{m} b_{i,j} e_j \right) + \psi_i,$$

where $a_{i,j}, b_{i,j} \in O$ and $\varphi_i, \psi_i \in M_{tor}$. Let $\alpha_i = \sum_{j=1}^{m} a_{i,j} e_j$ and $\beta_i = \sum_{j=1}^{m} b_{i,j} e_j$, so that

$$f_i \otimes h_i = \alpha_i \otimes \beta_i + (\alpha_i \otimes \varphi_i + \varphi_i \otimes \beta_i + \varphi_i \otimes \psi_i).$$

Clearly, $f_i \otimes h_i - \alpha_i \otimes \beta_i \in I_{tor} \otimes O[G^*] + O[G^*] \otimes I_{tor}$, and is, in particular, torsion.

On the other hand, the $\alpha_i \otimes \beta_i$ terms are in the image of the natural map

$$M_{free} \otimes M_{free} \longrightarrow O[H] \otimes O[H].$$

By Lemma 2.1, this map is injective, and thus $M_{free} \otimes M_{free}$ can be regarded as a free finite-rank $O$-submodule of $O[H] \otimes O[H]$. It follows that

$$\sum_{i=1}^{m} \alpha_i \otimes \beta_i = \Delta(f) - \sum_{i=1}^{m} (\alpha_i \otimes \varphi_i + \varphi_i \otimes \beta_i + \varphi_i \otimes \psi_i).$$
is both torsion and lies in the free submodule $M_{\text{free}} \otimes M_{\text{free}}$, and therefore must be zero.

**Remark 2.3.** The property that $V(I_{\text{tor}})$ is a subgroup scheme, also appears to follow from a more general property, stated at the beginning of §3.1 in [GM].

2.2. In this subsection, it will be proven that $G^x_{\text{iff}}$ is smooth, and that $G^x_{\text{iff}} \times \mathbb{F}$ contains the identity component $(G^x_{\text{iff}})^0$ (see [Mi, Definition 13.12] for the definition of the identity component over a general field). By Remark 1.3, the identity component of $G^x_{\text{iff}}$ is also reduced. Our strategy will be to compare the distribution algebras of the $\mathbb{K}$ and $\mathbb{F}$ centralizers.

Let us first recall that the $\mathbb{K}$ and $\mathbb{F}$ centralizers admit the Levi decompositions

$$G^x_{\mathbb{K}} = G^x_{\mathbb{K}, \text{red}} \ltimes G^x_{\mathbb{K}, \text{unip}}, \quad G^x_{\mathbb{F}} = G^x_{\mathbb{F}, \text{red}} \ltimes G^x_{\mathbb{F}, \text{unip}}$$

(cf. [Mc1, Corollary 29]). By [Mc3, Corollary 9.2.2], the $\mathbb{K}$ and $\mathbb{F}$ reductive quotients also have the same identity component. Moreover, by [Mc1, Theorem 28], the unipotent radicals $G^x_{\mathbb{K}, \text{unip}}$ and $G^x_{\mathbb{F}, \text{unip}}$ are both split, and have the same rank (i.e. as schemes, they are both isomorphic to affine spaces of the same dimension). Thus, there is an isomorphism of graded vector spaces

$$\dim \text{Dist}(G^x_{\mathbb{K}, \text{unip}}) = \dim \text{Dist}(A^d_{\mathbb{K}}), \quad \dim \text{Dist}(G^x_{\mathbb{F}, \text{unip}}) = \dim \text{Dist}(A^d_{\mathbb{F}}),$$

where $d = \dim G^x_{\mathbb{K}, \text{unip}} = \dim G^x_{\mathbb{F}, \text{unip}}$. These distributions are explicitly calculated in [J1, I.7.3].

**Lemma 2.4.** For all $n \geq 0$,

$$\dim \text{Dist}_n(G^x_{\mathbb{K}}) = \dim \text{Dist}_n(G^x_{\mathbb{F}}).$$

**Proof.** Apply (2.2), and observe that since $G^x_{\mathbb{K}, \text{red}}$ and $G^x_{\mathbb{F}, \text{red}}$ have the same root-datum, then the classification results from [J1, II.1] imply the existence of a Kostant $\mathcal{O}$-form $\mathcal{U}_0 \subseteq \text{Dist}(G^x_{\mathbb{K}, \text{red}})$ which satisfies $\mathbb{F} \otimes \mathcal{O} \mathcal{U}_0 \cong \text{Dist}(G^x_{\mathbb{F}, \text{red}})$. So in particular,

$$\dim \text{Dist}_n(G^x_{\mathbb{K}, \text{red}}) = \dim \text{Dist}_n(G^x_{\mathbb{F}, \text{red}})$$

for all $n \geq 0$.

The result now follows by tensoring both sides with the respective unipotent distribution algebras and applying (2.3). □

**Definition 2.5.** By [J1, I.7.4], an affine scheme $X$ over $\mathcal{O}$ is said to be infinitesimally flat at $z \in X(\mathcal{O})$ provided $\mathcal{O}[X]/I^n_{z, 1}$ are flat $\mathcal{O}$-modules for all $n \geq 0$, where $I_{z, 1}$ is the ideal corresponding to $z$.

The infinitesimal flatness property is necessary in order for distribution algebras to work “nicely” for an $\mathcal{O}$-group scheme.

**Lemma 2.6.** Both $G^x$ and $G^x_{\text{iff}}$ are infinitesimally flat at $1 \in G^x_{\text{iff}}(\mathcal{O}) = G^x(\mathcal{O})$, and $I_{\text{tor}} \subseteq \cap_{n \geq 1} I^n_{1}$. 

**Proof.** Let $I_{1, \mathbb{K}}$ and $I_{1, \mathbb{F}}$ be the augmentation ideals for the corresponding fibers. From the map

$$\mathcal{O}[G^x] \rightarrow \mathbb{F} \otimes \mathcal{O}[G^x] \cong \mathcal{O}[G^x]/\mathcal{O}[G^x],$$

we can see that $I_{1, \mathbb{F}} = I_{1, \mathbb{O}}[G^x]/\mathcal{O}[G^x]$, and more generally,

$$I^n_{1, \mathbb{F}} = I^n_{1, \mathbb{O}}[G^x]/\mathcal{O}[G^x].$$
This gives

$$\mathbb{F}[G^x]/\mathcal{I}_1^n \simeq \Omega[G^x]/\mathcal{I}_1^n.$$ 

On the other hand,

$$\mathbb{F} \otimes_{\mathbb{F}} (\Omega[G^x]/\mathcal{I}_1^n) \simeq \Omega[G^x]/\mathcal{I}_1^n \simeq \Omega[G^x]/\mathbb{F}[G^x]/\mathcal{I}_1^n.$$ 

So there exist morphisms

$$\mathbb{K}[G^x]/\mathcal{I}_1^n \otimes \mathbb{K} \otimes_{\mathbb{F}} (\Omega[G^x]/\mathcal{I}_1^n) \to \mathbb{F} \otimes_{\mathbb{F}} (\Omega[G^x]/\mathcal{I}_1^n) \simeq \mathbb{F}[G^x]/\mathcal{I}_1^n.$$ 

By Lemma 2.4, the dimensions of the left and right side are equal, so that \(\Omega[G^x]/\mathcal{I}_1^n\) are torsion-free, and hence flat, for all \(n\). Therefore, \(G^x\) is infinitesimally flat at 1. Finally, since \(\mathcal{I}_1 \subseteq \bigcap_{n \geq 1} \mathcal{I}_1^n\), it also follows that \(G^x_{\mathbb{F}}\) is infinitesimally flat at 1. \(\square\)

The preceding lemma also implies that \(\text{Dist}_n(G^x_{\mathbb{F}}) \cong \text{Dist}_n(G^x)\) for all \(n \geq 0\). The infinitesimal flatness for both groups allows us to apply [J1, I.7.4(1)], which gives

$$\text{dist}_n(G^x_{\mathbb{F}}) \cong \mathbb{F} \otimes \mathbb{F} \text{dist}_n(G^x_{\mathbb{F}}) \cong \mathbb{F} \otimes \mathbb{F} \text{dist}_n(G^x) \cong \text{dist}_n(G^x_{\mathbb{F}}),$$

for all \(n \geq 0\). In particular, \(\text{Dist}(G^x_{\mathbb{F}}) \cong \text{Dist}(G^x_{\mathbb{F}})\) as filtered algebras.

We may now prove the main result of this section.

**Proposition 2.7.** The group scheme \(G^x_{\mathbb{F}}\) is smooth and \((G^x_{\mathbb{F}})^o \subseteq G^x_{\mathbb{F}}\).

**Proof.** By construction, \(G^x_{\mathbb{F}} \subseteq G^x\), and thus by [J1, I.7.17(7)], \(\text{Dist}(G^x_{\mathbb{F}}) \subseteq \text{Dist}(G^x)\) as filtered algebras. So \(\text{Dist}_n(G^x_{\mathbb{F}}) \subseteq \text{Dist}_n(G^x)\) for all \(n \geq 0\). However, by (2.4), it follows that \(\text{dim}_n \text{Dist}_n(G^x_{\mathbb{F}}) = \text{dim}_n \text{Dist}_n(G^x)\), and hence, \(\text{Dist}_n(G^x_{\mathbb{F}}) = \text{Dist}_n(G^x)\) for all \(n \geq 0\). Therefore,

$$\text{Dist}(G^x_{\mathbb{F}}) = \text{Dist}(G^x).$$

Let us now base change to \(\mathbb{F}^\circ\) where we note that (2.5) also holds over \(\mathbb{F}^\circ\) by [J1, I.7.4(1)]. The definition of the identity component implies \(1_{G^x_{\mathbb{F}}} \in (G^x_{\mathbb{F}})^o\), and so it follows that \(\text{Dist}((G^x_{\mathbb{F}})^o) = \text{Dist}(G^x_{\mathbb{F}})\). Now since \((G^x_{\mathbb{F}})^o\) is an irreducible subgroup scheme of \(G^x_{\mathbb{F}}\), then by (2.5) and [J1, I.7.17(7)], \((G^x_{\mathbb{F}})^o \subseteq G^x_{\mathbb{F}}\), proving the second statement of the proposition.

To prove the first statement, we first recall that by [J1, I.7.1(2)],

$$\text{dim}_{\mathbb{F}} \text{Dist}_n^+(G^x_{\mathbb{F}}) = \text{dim}_{\mathbb{F}} \text{Dist}_n^+(G^x_{\mathbb{F}})$$

for all \(n \geq 1\). Thus, by [J1, I.7.7], the Lie algebras of \(G^x_{\mathbb{F}}\) and \(G^x_{\mathbb{F}}\) have the same dimension. On the other hand,

$$(G^x_{\mathbb{F}})^o \subseteq G^x_{\mathbb{F}} \subseteq G^x$$

implies \(\text{dim} G^x_{\mathbb{F}} = \text{dim} G^x_{\mathbb{F}}\), and hence, \(G^x_{\mathbb{F}}\) is smooth by [J1, I.7.17(1)]. Likewise, \(G^x_{\mathbb{F}}\) is automatically smooth by [J1, I.7.17(2)]. Thus, by Lemma 1.5, \(G^x_{\mathbb{F}}\) will be smooth provided \(\hat{\Omega}[G^x_{\mathbb{F}}]\) is torsion-free. However, this property holds from the definition of \(G^x_{\mathbb{F}}\), and so we are done. \(\square\)

3. Smoothness and Component Groups of Centralizers

To simplify our arguments, we will assume throughout this section that \(\mathcal{O}\) is such that \(\mathbb{F} = \mathbb{F}^\circ\) is algebraically closed, unless specified otherwise.
3.1. Diagonalizable group schemes. For any commutative group \( \Lambda \) and a commutative ring \( k \), the diagonalizable group scheme over \( k \) associated to \( \Lambda \), denoted \( \text{Diag}(\Lambda) \), is defined by setting \( k[\text{Diag}(\Lambda)] := k[\Lambda] \), where \( k[\Lambda] \) is the group algebra for \( \Lambda \) with the usual Hopf algebra structure (cf. [J1, I.2.5]).

**Lemma 3.1.** Let \( k \) be a commutative integral ring, and suppose \( X = \text{Diag}(\Lambda_X) \), \( Y = \text{Diag}(\Lambda_Y) \) and \( Z = \text{Diag}(\Lambda_Z) \) are diagonalizable group schemes over \( k \) with morphisms \( X \xleftarrow{\varphi} Z \) and \( Y \xrightarrow{\psi} Z \), then

\[
X \times_Z Y \cong \text{Diag}(\Lambda),
\]

where \( \Lambda \) is a pushout induced by two uniquely determined group homomorphisms \( \Lambda_Z \xleftarrow{\varphi'} \Lambda_X \) and \( \Lambda_Z \xrightarrow{\psi'} \Lambda_Y \).

**Proof.** Since \( k \) is integral, then the isomorphism [J1, I.2.5(2)] ensures that the morphisms \( \varphi \) and \( \psi \) are induced by unique group homomorphisms \( \Lambda_Z \xleftarrow{\varphi'} \Lambda_X \) and \( \Lambda_Z \xrightarrow{\psi'} \Lambda_Y \). More precisely, if we identify \( k[X] = k[\Lambda_X] \), \( k[Y] = k[\Lambda_Y] \) and \( k[Z] = k[\Lambda_Z] \), where the Hopf algebras on the right are the group algebras for \( \Lambda_X \), \( \Lambda_Y \) and \( \Lambda_Z \) respectively, then the comorphisms \( \varphi^* \) and \( \psi^* \) are the group algebra homomorphisms induced by \( \varphi' \) and \( \psi' \) respectively.

Now let \( \Lambda \) be the pushout of \( \varphi' \) and \( \psi' \), then

\[
\Lambda = \frac{\Lambda_X \times \Lambda_Y}{\langle (\varphi'(\lambda_Z), 1)(1, \psi'(\lambda_Z)^{-1}) \mid \lambda_Z \in \Lambda_Z \rangle},
\]

where we use multiplicative notation to denote the group structure. Let us employ the natural identification \( \Lambda' \subseteq k[\Lambda'] \) for an abelian group \( \Lambda' \) (i.e. identifying \( \Lambda' \) with the “group-like” elements of \( k[\Lambda'] \)), so that by definition, \( \Lambda' \) gives a \( k \)-basis of \( k[\Lambda'] \). We will also make use of the isomorphism of algebras

\[
k[\Lambda_X] \otimes_k k[\Lambda_Y] \cong k[\Lambda_X \times \Lambda_Y],
\]

induced by sending \( \lambda_X \otimes_k \lambda_Y \mapsto (\lambda_X, \lambda_Y) \) for any \( \lambda_X \in \Lambda_X \) and \( \lambda_Y \in \Lambda_Y \).

Now let us note that the tensor product \( k[\Lambda_X] \otimes_k k[\Lambda_Y] \) coincides with the pushout along \( \varphi^* \) and \( \psi^* \) in the category of commutative \( k \)-algebras. By definition \( k[\Lambda_X] \otimes_k k[\Lambda_Y] \cong k[\Lambda_X \times \Lambda_Y]/I \), where \( I \) is the \( k \)-submodule generated by \( f \varphi^*(h) \otimes g - f \otimes \psi^*(h)g \) for all \( f \in k[X], g \in k[Y] \) and \( h \in k[Z] \). In particular, \( I \) is spanned by

\[
\lambda_X \varphi'(\lambda_Z) \otimes \lambda_Y - \lambda_X \otimes \psi'(\lambda_Z) \lambda_Y,
\]

for all \( \lambda_X, \lambda_Y \) and \( \lambda_Z \).

Observe now that

\[
\lambda_X \varphi'(\lambda_Z) \otimes \lambda_Y = \lambda_X \otimes \psi'(\lambda_Z) \lambda_Y \iff \varphi'(\lambda_Z) \otimes \psi'(\lambda_Z)^{-1} = 1 \otimes 1
\]

\[
\iff (\varphi'(\lambda_Z), \psi'(\lambda_Z)^{-1}) = 1
\]

for \( \lambda_X, \lambda_Y \) and \( \lambda_Z \), where the second “ \( \iff \) ” arises from (3.2). From this, we can see that \( k[\Lambda_X] \otimes_k k[\Lambda_Y] \) can be equivalently obtained from \( k[\Lambda_X \times \Lambda_Y] \) by imposing the relation \( (\varphi'(\lambda_Z), \psi'(\lambda_Z)^{-1}) = 1 \) for all \( \lambda_Z \in \Lambda_Z \). Comparing with the description in (3.1) gives the group algebra \( k[\Lambda] \). Therefore, we are done.

Let \( k \) be any commutative ring, then we say that an affine \( k \)-scheme \( X \) is constant, provided the associated \( k \)-functor

\[
X : \{ k \text{-algebras} \} \to \{ \text{sets} \}
\]
is a constant functor. If the cardinality $|X(k)| = r < \infty$, then there exists an algebra isomorphism $k[X] \cong k^{r}$, where the right hand side is an $r$-fold direct product with pointwise multiplication. An affine group scheme is called constant if it is constant as a scheme.

**Lemma 3.2.** Let $\Lambda$ be a finite abelian group with $r = |\Lambda|$. If $p \nmid r$, then $\text{Diag}(\Lambda)$ is a constant $\mathbb{O}$-group scheme (i.e. $\mathbb{O}[\Lambda] \cong \mathbb{O}^{r}$ as an $\mathbb{O}$-algebra).

**Proof.** Since $\Lambda$ is a finite abelian group, then

$$\Lambda \cong \mathbb{Z}/n_{1}\mathbb{Z} \times \mathbb{Z}/n_{2}\mathbb{Z} \times \cdots \times \mathbb{Z}/n_{t}\mathbb{Z},$$

where the $n_{1}, \ldots, n_{t} \in \mathbb{Z}$ may have repeated multiplicities. In particular,

$$\mathbb{O}[\Lambda] \cong \mathbb{O}[\mathbb{Z}/n_{1}\mathbb{Z}] \otimes \mathbb{O}[\mathbb{Z}/n_{2}\mathbb{Z}] \otimes \cdots \otimes \mathbb{O}[\mathbb{Z}/n_{t}\mathbb{Z}].$$

Thus, $\text{Diag}(\Lambda)$ is constant if $\text{Diag}(\mathbb{Z}/n_{i}\mathbb{Z})$ is constant for all $i$. The condition $p \nmid r$ clearly implies $p \nmid n_{i}$ for all $i$. Therefore, without loss of generality, we may assume $\Lambda$ is cyclic.

Supposing now that $\Lambda = \mathbb{Z}/r\mathbb{Z}$, gives

$$\mathbb{O}[\Lambda] = \mathbb{O}[z]/(z^{r} - 1).$$

Our assumption on $\mathbb{O}$ at the beginning of this section, implies that $\mathbb{O}$ (and hence $\mathbb{K}$) contain all roots of unity which are co-prime to $p$. Thus,

$$z^{r} - 1 = (z - \zeta_{1})(z - \zeta_{2}) \cdots (z - \zeta_{r}),$$

where $\zeta_{1}, \ldots, \zeta_{r} \in \mathbb{O}^{*} \subset \mathbb{K}^{*}$ are the primitive $r$-th roots of unity. We now define a ring homomorphism

$$\varphi : \mathbb{O}[\Lambda] \to \mathbb{O}^{r}$$

$$f \mapsto (f(\zeta_{1}), f(\zeta_{2}), \ldots, f(\zeta_{r})).$$

The result will follow if we can prove that $\varphi$ is an isomorphism.

It suffices to prove that $\varphi$ is surjective as an $\mathbb{O}$-module homomorphism. For $i = 1, \ldots, r$, set $f_{i} = \prod_{j \neq i} (z - \zeta_{j})$. Then $f(\zeta_{i}) = 0$ for all $j \neq i$ and

$$f_{i}(\zeta_{i}) = \prod_{j \neq i}(\zeta_{i} - \zeta_{j}).$$

Our assumption $p \nmid r$ implies that the reductions $\overline{\zeta_{1}}, \overline{\zeta_{2}}, \ldots, \overline{\zeta_{r}} \in \mathbb{F} = \mathbb{O}/(\omega)$ are all distinct (where $\omega \in \mathbb{O}$ is the uniformizer). In particular, for $i \neq j$, $\overline{\zeta_{i}} - \overline{\zeta_{j}} \neq 0$, and hence, $\omega \not\mid (\zeta_{i} - \zeta_{j})$ so that $(\zeta_{i} - \zeta_{j})$ is invertible in $\mathbb{O}$. Thus, $f_{i}(\zeta_{i}) \in \mathbb{O}$ is also invertible for all $i$.

Finally, if $\epsilon_{i} \in \mathbb{O}^{r}$ denotes the $i$-th coordinate function for $i = 1, \ldots, r$, then $\varphi(f_{i}) = f_{i}(\zeta_{i})\epsilon_{i}$ for all $i$. The invertibility of the $f_{i}(\zeta_{i})$ imply that the $f_{i}(\zeta_{i})\epsilon_{i}$ form a basis for the free $\mathbb{O}$-module $\mathbb{O}^{r}$, and therefore, $\varphi$ is surjective. \hfill $\square$

Following the convention from [Mc3, p. 5], we say that a subgroup scheme $S \subseteq H$ of an $\mathbb{O}$-group scheme $H$ is a **maximal torus** if the subgroup schemes $S_{k} \subseteq H_{k}$ and $S_{\mathbb{F}} \subseteq H_{\mathbb{F}}$ are both maximal tori.

**Lemma 3.3.** Let $C_{\mathbb{F}}$ be a nilpotent orbit, then there exists a balanced nilpotent section $x \in g_{\mathbb{O}}$ such that $x_{\mathbb{F}} \in C_{\mathbb{F}}$ and $S_{\mathbb{F}} = (T_{\mathbb{F}} \cap G_{\mathbb{F}})^{0}$ is a maximal torus for $G_{\mathbb{F}}$. 

Proof. By [Mc3, Theorem 1.2.1(a)], any \( x_\mathfrak{F} \in C_\mathfrak{F} \) lifts to some balanced nilpotent section \( x \in \mathfrak{g}_\mathfrak{O} \). Suppose now that \( y_\mathfrak{F} \in C_\mathfrak{F} \) is arbitrary, and let \( y \in \mathfrak{g}_\mathfrak{O} \), be a corresponding balanced nilpotent section which lifts \( y_\mathfrak{F} \). If \( S'_\mathfrak{O} \subseteq G^r_\mathfrak{O} \) is a maximal torus for the centralizer, then there must exist a maximal torus \( T'_\mathfrak{F} \) of \( G_\mathfrak{F} \) such that \( S'_\mathfrak{F} \subseteq T'_\mathfrak{F} \). Now since the maximal tori for \( G_\mathfrak{F} \) are conjugate, then there exists \( g \in G_\mathfrak{F}(\mathbb{F}) \) such that \( T_\mathfrak{F} = gT'_\mathfrak{F}g^{-1} \). Let \( x_\mathfrak{F} = g^{-1}y_\mathfrak{F}g \) and fix a balanced lift \( x \in \mathfrak{g}_\mathfrak{O} \), then \( x_\mathfrak{F} \in C_\mathfrak{F} \) and \( S_\mathfrak{F} = g^{-1}S'_\mathfrak{F}g \subseteq T_\mathfrak{F} \), must also be a maximal torus in \( G^r_\mathfrak{O} \).

It will follow that \( S_\mathfrak{F} = (T_\mathfrak{F} \cap G^r_\mathfrak{F})^0 \) provided \( T_\mathfrak{F} \cap G^r_\mathfrak{F} \) is reduced. To see why this is true, let us first fix the root space decomposition

\[
\mathfrak{g}_\mathfrak{F} = t_\mathfrak{F} \oplus \bigoplus_{\alpha \in \Phi \subset \mathfrak{X}} (\mathfrak{g}_\mathfrak{F})_\alpha,
\]

so that the structure of \( \mathfrak{g}_\mathfrak{F} \) as a \( T_\mathfrak{F} \)-module is given by the comodule map

\[
\Delta_{\mathfrak{g}_\mathfrak{F}} : \mathfrak{g}_\mathfrak{F} \to \mathbb{F}[T] \otimes \mathfrak{g}_\mathfrak{F}
\]

\[
v \mapsto \sum_{\mu \in \Phi \cup \{0\}} \mu(t) \otimes v_\mu,
\]

where \( v_\mu \) denotes the projection onto the \( \mu \) weight space of \( \mathfrak{g}_\mathfrak{F} \). The centralizer of \( x_\mathfrak{F} \) in \( T_\mathfrak{F} \) is then determined by the abelian group

\[
\mathfrak{X}^{x_\mathfrak{F}} = \mathfrak{X}/\langle \mu \in \mathfrak{X} \mid (x_\mathfrak{F})_\mu \neq 0 \rangle,
\]

(i.e. \( T \cap G^r_\mathfrak{F} \cong \text{Diag}(\mathfrak{X}^{x_\mathfrak{F}}) \)). From the properties of diagonalizable group schemes, this is reduced if and only if \( \mathfrak{X}^{x_\mathfrak{F}} \) contains no \( p \)-torsion. However, the latter property follows immediately from [He, Definition 2.11 and Theorem 5.2]. □

The following proposition will enable us to relate the torus characters between \( G^r_\mathfrak{K} \) and \( G^r_\mathfrak{F} \) representations.

**Proposition 3.4.** Let \( x \in \mathfrak{g}_\mathfrak{O} \) and \( S_\mathfrak{F} \subseteq G^r_\mathfrak{F} \) be as in Lemma 3.3, then \( S_\mathfrak{F} \) lifts to a split torus \( S \subseteq G^r_\mathfrak{F} \) such that \( S_\mathfrak{K} \subseteq G^r_\mathfrak{K} \) is a maximal torus which is split.

**Proof.** First observe that since \( S_\mathfrak{F} \subseteq G^r_\mathfrak{F} \) is irreducible, and \( \text{Dist}(G^r_\mathfrak{F}) = \text{Dist}(G^r_\mathfrak{K}) \) by (2.5), then from [J1, L7.17(7)], it follows that \( S_\mathfrak{F} \subseteq G^r_{\mathfrak{K}, \mathfrak{F}} \). The fact that \( G^r_{\mathfrak{K}, \mathfrak{F}} \) is smooth by Proposition 2.7 now allows us to apply [Mc3, Theorem 2.1.1(c)] to lift the embedding \( \varphi : G^r_\mathfrak{m} \hookrightarrow G^r_{\mathfrak{K}, \mathfrak{F}} \) (corresponding to the inclusion \( S_\mathfrak{F} \subseteq G^r_{\mathfrak{K}, \mathfrak{F}} \), to a map \( \psi : G^r_\mathfrak{m} \to G^r_{\mathfrak{K}, \mathfrak{F}} \subseteq G^r_\mathfrak{F} \). Moreover, by the same reasoning as in the proof of [AHR2, Lemma 4.1], it can be verified that \( \psi \) is actually a closed embedding.

Thus, if we set \( S = \psi(G^r_\mathfrak{m}) \subseteq G^r_\mathfrak{F} \), where \( S \cong G^r_\mathfrak{m} \), then \( S_\mathfrak{K} = \psi(G^r_\mathfrak{m}) \subseteq G^r_\mathfrak{K} \) is a split rank \( r \)-torus contained in \( G^r_\mathfrak{K} \). Finally, since the maximal tori for \( G^r_\mathfrak{K} \) and \( G^r_\mathfrak{F} \) have the same rank by [Mc3, Corollary 9.2.2], then \( S_\mathfrak{K} \) must also be maximal. □

**Remark 3.5.** In particular, \( S_\mathfrak{K} \subseteq (G^r_\mathfrak{K})^0 \) and \( S_\mathfrak{F} \subseteq (G^r_\mathfrak{F})^0 \) are maximal tori for the connected components of the geometric fibers.

In order to reduce our component group calculations to the case of diagonalizable group schemes, we will need the following proposition.

**Proposition 3.6.** Let \( Z = Z(G) \) denote the center of \( G \), let \( Z' \subseteq Z \) be any diagonalizable subgroup scheme, and let \( S \subseteq G \) be any split \( O \)-torus, then \( S \cap Z' \subseteq G \) is diagonalizable.
Proof. First recall that the center $Z$ is diagonalizable (see [J1, II.1.6, II.1.8]). The strategy of the proof will be to first construct a split $O$-torus $T' \subseteq G$ with $S \subseteq T'$ and $Z' \subseteq T'$, this will allow the intersection of $S \cap Z'$ to be taken inside of $T'$. The result will then follow from Lemma 3.1.

To construct $T'$, first set $H = C_G(S)$, where $C_G(S)$ denotes the centralizer of $S$ in $G$. By [Mc3, Proposition 2.2.1], $H$ is a smooth reductive group scheme over $O$ with connected fibers. Now let $T'_W \subseteq H_\mathbb{F}$ be a maximal split torus (of rank-$r$) for $H$ (recall that $\mathbb{F} = \mathbb{F}$ by our assumption, so such a torus exists). As in the proof of Proposition 3.4, $T'_W$ can be lifted to a rank-$r$ split torus of $H$. Let $T'$ denote this lift. Thus, $T'_W \subseteq H_\mathbb{F}$ is a rank-$r$ split torus of $H_\mathbb{F}$. By [Mc3, Corollary 2.1.2], the maximal tori for $H_\mathbb{F}$ and $H_\mathbb{K}$ must have the same dimension, and hence the same rank if they are split. Therefore, $T'_W$ is maximal in $H_\mathbb{F}$.

Let $Z(H) \subseteq H$ denote the center of $H$. By definition, $S \subseteq Z(H)$ and $Z \subseteq Z(H)$ (and hence $Z' \subseteq Z(H)$). For $k \in \{\mathbb{K}, \mathbb{F}\}$,

\[
Z(H)_k = Z(H_k) \subseteq T'_k,
\]

where the containment on the right holds because $T'_W \subseteq H_\mathbb{K}$ is a maximal torus over a field and $Z(H)_k$ is the center. So in particular, $S_k \subseteq T'_k$.

To prove $S \subseteq T'$, first consider the inclusion

$\varphi : S \cap T' \hookrightarrow S$.

and observe that $S \subseteq T'$ if and only if $\varphi$ is an isomorphism. Since base-change commutes with taking fiber products, and thus commutes with taking intersections, the base-changes of $\varphi$ to $k$,

$\varphi_k : (S \cap T')_k \rightarrow S_k$,

coincide with the inclusion $S_k \cap T'_k \hookrightarrow S_k$. Thus, by (3.3), $\varphi_k$ is an isomorphism (i.e. is surjective). On the level of algebras, $O[S \cap T'] = O[S] \otimes_{O[H]} O[T']$ where the comorphism $\varphi^*$ is surjective, so it suffices to show that $\varphi^*_k$ is injective (as an $O$-module morphism). Base changing to $\mathbb{K}$ induces a commutative diagram

\[
\begin{array}{ccc}
O[S] & \xrightarrow{\varphi^*} & O[S] \otimes_{O[H]} O[T'] \\
\downarrow \psi & & \downarrow \psi' \\
\mathbb{K}[S] & \xrightarrow{\varphi^*_k \sim} & \mathbb{K}[S] \otimes_{\mathbb{K}[H]} \mathbb{K}[T'].
\end{array}
\]

The diagonalizability of $S$ implies that $O[S]$ is torsion-free, and hence $\psi = 1 \otimes id$ must be injective. Thus $\varphi_k \circ \psi$ is injective, and by commutativity, $\psi' \circ \varphi^*$ is injective. It then follows that $\varphi^*$ must be injective, and is therefore an isomorphism.

By a similar argument, we can show that the map $Z' \cap T' \rightarrow Z'$ is an isomorphism (using the fact that $Z'$ is diagonalizable), and thus $Z' \subseteq T'$.

We have established that $S \subseteq T'$ and $Z' \subseteq T'$. This allows us to identify

$S \cap Z' = S \times_{T'} Z'$.

Thus, it follows from Lemma 3.1 that $S \cap Z'$ is diagonalizable. \hfill \Box

3.2. Component groups of the geometric fibers. We will now prove that the component groups $A(x_\mathbb{K})$ and $A(x_\mathbb{F})$ of the geometric fibers, have the same cardinality.
Lemma 3.7. If $G$ is a simple group (with $G_F$ geometrically standard), then $|A(x_F)| = |A(x_F)|$.

Proof. Let $k \in \{ \mathbb{K}, \mathbb{F} \}$ and denote $H = G^z$ for a balanced nilpotent section $x \in g_0$ and $S \subseteq H$ as in Proposition 3.4, so that $S_K$ is a maximal torus for $H_K$ by Remark 3.5.

Let $G' = G_{\text{ad}}$ and $H' = G'_{\text{ad}}$, then the arguments in the proof of [LS, Lemma 2.33] can be used to show that there exists an isogeny

$$1 \to Z_K \to H_K \to H'_K \to 1,$$

obtained by restricting the “covering space” isogeny

$$1 \to Z_K \to G_K \to G'_{\text{ad}} \to 1,$$

down to $H_K$. (In particular, $Z_K \subseteq G_K$ is the center of $G_K$.)

Now since $H_K$ gets mapped onto $H'_K$, there exists a sequence

$$1 \to Z_K H_K \to A(x_K) \to A'(x_K) \to 1,$$

where $A'(x_K) = H'_K / H'^{(1)}_K$. We can also identify $Z_K H_K / H'^{(1)}_K \cong Z_K / Z_K \cap H'^{(1)}_K$. By Remark 1.4 and [MS, Theorem 36], $|A'(x_K)| = |A'(x_F)|$. Thus, by (3.4), it suffices to show $|Z_K \cap H'^{(1)}_K| = |Z_K \cap H'^{(1)}_K|$. However, since $S_K$ is a maximal torus, and $Z_K \cap H'^{(1)}_K$ is central, then it follows that

$$Z_K \cap H'^{(1)}_K \subseteq S_K.$$

This gives, $Z_K \cap H'^{(1)}_K = Z_K \cap S_K$. Observe that the intersection on the right hand side actually arises from an intersection of $G$-group schemes since $Z_K$ and $S_K$ are the (geometric) fibers of the subgroup schemes $Z \subseteq G$ and $S \subseteq G$. In other words,

$$(Z \cap S)_K \subseteq Z_K \cap S_K.$$

By Proposition 3.6, $Z \cap S$ is a diagonalizable subgroup scheme of $Z$. Now observe $Z \cong \text{Diag}(\Lambda)$ with $\Lambda = X/\mathbb{Z}\Phi$. The geometrically standard assumption implies that $p \nmid |X/\mathbb{Z}\Phi|$. In particular, identifying $Z \cap S = \text{Diag}(\Lambda')$ gives $p \nmid |\Lambda'|$ since $\Lambda'$ is a quotient of $\Lambda$. Thus, Lemma 3.2 now implies that $Z \cap S$ constant, and hence $|(Z \cap S)_K| = |(Z \cap S)_K|$. Therefore, we are done.

Now we consider the case of semisimple groups.

Lemma 3.8. Let $G$ be a semisimple group (with $G_F$ geometrically standard), then $|A(x_F)| = |A(x_F)|$.

Proof. In this proof $k$ will denote either $\mathbb{K}$ or $\mathbb{F}$.

First suppose that

$$G = G_1 \times \cdots \times G_r,$$

where the $G_i$ are simple (and hence geometrically standard over $\mathbb{F}$ since $G$ is geometrically standard of $\mathbb{F}$ by [Mc3, §3.1 (S2)]). Now note that since the factors of $G$ act independently on the corresponding factors of $g_0$, then for any nilpotent section

$$x = (x_1, \ldots, x_r) \in (g_1)_0 \times \cdots \times (g_r)_0,$$
the centralizer $G^x$ satisfies

$$G^x = G_1^x \times \cdots \times G_r^x,$$

where $G_i^x \subseteq G_i$ is the centralizer of $x_i$. To simplify notation, set $H_i = G_i^x$ for all $i$, and set $H = G^x$. Thus,

$$H_k = (H_1)_k \times \cdots \times (H_r)_k,$$

and hence, $H_k$ is smooth if and only if $(H_i)_k$ is smooth for all $i$. This implies that $x \in g_0$ is a balanced for $G$ if and only if for all $i, x_i$ is balanced for $G_i$. Thus, if we assume that $x$ is balanced, then from the identity $H_k^0 = (H_1)_k^0 \times \cdots \times (H_r)_k^0$, it follows that

$$A(x_k) = A((x_1)_k) \times \cdots \times A((x_r)_k).$$

Therefore, $|A(x_k)| = |A(x)|$ by Lemma 3.7.

More generally, if $G$ is an arbitrary semisimple group, then there exists an isogeny

$$(3.5) \quad 1 \to Z' \to \prod_{i=1}^r G_i \to G \to 1,$$

for some central, diagonalizable subgroup scheme $Z'$ (see [J1, II.1.6]). Since $G_i^x$ is geometrically standard, then $Z'_k$ is both smooth by [He, Definition 2.11 and Theorem 5.2], and finite since $G_i^x$ is a finite product of quasi-simple group schemes. In particular, $Z'$ is constant.

Let $G' = \prod_{i=1}^r G_i$, and let $x' = (x'_1, \ldots, x'_r) \in g'_0$ be a balanced nilpotent section, and let $x = d\pi(x') \in g_0$. Let $H'$ denote the centralizer of $x'$, and let $H$ denote the centralizer of $x$. By the same argument as in [LS, Lemma 2.33], there exists an isogeny

$$(3.6) \quad 1 \to Z'_k \to H'_k \to H_k \to 1,$$

induced by base changing (3.5) to $\overline{k}$, and restricting down to the centralizer $H_k$. Now since $Z'_k$ is smooth, and $H'_k$ is smooth since $x'$ is balanced, then $H_k$ must also be smooth. It follows that the section $x \in g_0$ is also balanced.

Observe now that the isogeny sends $H'_k$ onto $H_k^0$, and thus induces a sequence

$$1 \to \frac{Z'_k H'_k}{H'_k} \to A'(x_k) \to A(x_k) \to 1,$$

where we identify

$$\frac{Z'_k H'_k}{H'_k} \cong \frac{Z_k H_k^0}{H_k^0}.$$

Without loss of generality, we can assume that $x' \in g'_0$ also satisfies the conditions of Lemma 3.3. Applying Proposition 3.4 to $x'$ and $G'$ now provides a split, maximal torus $S' \subseteq H'$. As in the proof of Lemma 3.7, the proof of this lemma will follow by showing that $Z' \cap S'$ is both diagonalizable and constant. The former property holds by Proposition 3.6, and the latter property can be deduced from Lemma 3.2 since $Z'$ is constant and diagonalizable, and hence any diagonalizable subgroup scheme of $Z'$ must also be constant.

The preceding argument can also be extended to arbitrary reductive groups $G$ with $G_F$ geometrically standard.

**Proposition 3.9.** Let $G$ be a reductive group (with $G_F$ geometrically standard), then $|A(x_k)| = |A(x)|$. 


Proof. In this proof \( k \) will denote either \( \mathbb{K} \) or \( F \).

Let \( G' = [G, G] \), then by [J1, II.1.18] there exists an isogeny
\[
(3.7) \quad 1 \to T_1 \cap T_2 \to G' \times T_2 \xrightarrow{\pi} G \to 1,
\]
where \( T_1 \) and \( T_2 \) are tori and \( T_1 \cap T_2 \subseteq Z(G' \times T_2) \) is central, diagonalizable and finite. Let \( Z' = T_1 \cap T_2 \). The arguments appearing immediately below (3.5) in the proof of Lemma 3.8, also imply that \( Z' \) is diagonalizable and constant.

Let \( h_0 \) denote the Lie algebra of \( T_2 \), and note that there must exist a balanced nilpotent section \( x \in g_0 \) which satisfies the conditions of Lemma 3.3, and is of the form \( x = d\pi(x') \) for some balanced nilpotent section \( x' \in g_0 \times h_0 \), which also satisfies the conditions of Lemma 3.3. (This again follows from the arguments appearing immediately below (3.6).)

Let \( H' \subseteq G' \times T_2 \) be the centralizer of \( x' \), and let \( H \subseteq G \) be the centralizer of \( x \). Again, as in [LS, Lemma 2.33], there is an isogeny
\[
1 \to Z'_{\overline{\mathbb{K}}} \to H_{\overline{\mathbb{K}}} \to H_{\mathbb{K}} \to 1,
\]
obtained from (3.7) by base-changing to \( \overline{\mathbb{K}} \), and restricting down to \( H_{\mathbb{K}} \). Now just as in the proof of Lemma 3.8, it suffices to show that the finite groups
\[
\frac{Z'_{\overline{\mathbb{K}}} H_{\overline{\mathbb{K}}}}{H_{\mathbb{K}}} \cong \frac{Z'_{\mathbb{K}}}{{Z'_{\overline{\mathbb{K}}}} \cap H_{\mathbb{K}}}, \quad \frac{Z_{\overline{\mathbb{K}}} H_{\overline{\mathbb{K}}}}{H_{\mathbb{K}}} \cong \frac{Z_{\mathbb{K}}}{{Z'_{\overline{\mathbb{K}}}} \cap H_{\mathbb{K}}}
\]
have the same order. However, this follows from the argument given in the last paragraph of the proof of Lemma 3.8. \( \square \)

Remark 3.10. In Theorem 3.12, it will be proven that the component groups are actually isomorphic (not just of the same order). The isomorphism of the component groups for general reductive \( G \) with geometrically standard fiber \( G_{\mathbb{F}} \) was originally claimed in [Mc2, Theorem B], however an error was later found in the proof.

3.3. Proof of Theorem 1.6. In this subsection, we will establish the smoothness of \( G_{\mathbb{F}} \). The key step will be to show that \( G_{\mathbb{F}, \mathbb{K}} \) contains all the connected components of \( G_{\mathbb{F}} \). (We are still maintaining our assumption that \( \mathbb{F} = \overline{\mathbb{F}} \).)

Label the connected components of \( G_{\mathbb{F}} \) by \((G_{\mathbb{F}})^i\) for \( i = 0, \ldots, m - 1 \), where \( m = |A(\overline{\mathbb{F}})| \) and \((G_{\mathbb{F}})^0 := (G_{\mathbb{F}})^{\circ} \) is the identity component subgroup scheme. Then the decomposition of \( G_{\mathbb{F}} \) into its connected components induces the decomposition
\[
\mathbb{F}[G_{\mathbb{F}}^\circ] = \mathbb{F}[(G_{\mathbb{F}})^i] \times \mathbb{F}[(G_{\mathbb{F}})^{i+1}] \times \cdots \times \mathbb{F}[(G_{\mathbb{F}})^{m-1}],
\]
where \( \mathbb{F}[(G_{\mathbb{F}})^i] \cong \mathbb{F}[(G_{\mathbb{F}})^{\circ}] \) for all \( i \geq 1 \) (as algebras). The fact that \( G_{\mathbb{F}, \mathbb{K}} \) is smooth implies
\[
\mathbb{F} \otimes \mathbb{Q}[G_{\mathbb{F}, \mathbb{K}}] = \mathbb{F}[(G_{\mathbb{F}})^i] \times \mathbb{F}[(G_{\mathbb{F}})^{i+1}] \times \cdots \times \mathbb{F}[(G_{\mathbb{F}})^{m-1}],
\]
where \( 1 \leq k \leq m \). In particular, \( \mathbb{F} \otimes \mathbb{Q}[G_{\mathbb{F}, \mathbb{K}}] \subseteq \mathbb{F}[G_{\mathbb{F}}^\circ] \) and \( G_{\mathbb{F}, \mathbb{K}} \) has precisely \( k \) connected components. Our goal is to show \( k = m \).

If necessary, let us enlarge \( \mathbb{Q} \), and consequently \( \mathbb{K} \), by a finite extension\(^2\) (the assumption \( \mathbb{F} = \mathbb{F} \) implies that \( \mathbb{F} \) remains unchanged), so that by Proposition 3.9
\[
(3.8) \quad \mathbb{K}[G_{\mathbb{K}}^\circ] = \mathbb{K}[(G_{\mathbb{K}})^i] \times \mathbb{K}[(G_{\mathbb{K}})^{i+1}] \times \cdots \times \mathbb{K}[(G_{\mathbb{K}})^{m-1}].
\]
\(^2\)The integral closure of a (complete) DVR in a finite algebraic extension is a finitely-generated (complete) DVR (cf. [Se, Chap. 1, Proposition 8, and Chap. 2, Proposition 3]).
This decomposition is given by a set of orthogonal idempotents \( \epsilon_0, \epsilon_1, \ldots, \epsilon_{m-1} \in \mathbb{K}[G_{K}^x] \) with \( \epsilon_i^2 = \epsilon_i \) for \( i \geq 0 \), \( \epsilon_i \epsilon_j = 0 \) for \( i \neq j \) and \( \epsilon_0 + \cdots + \epsilon_{m-1} = 1 \). Now
\[
\epsilon_i = \frac{h_i}{\omega^{n_i}}
\]
where \( h_i \in \mathcal{O}[G_{K}^x] \) and \( n_i \geq 0 \) for all \( i \). Assume that each \( n_i \) is chosen minimally so that \( h_i \notin \omega^{n_i} \mathcal{O}[G_{K}^x] \). Thus,
\[
(3.9) \quad \epsilon_i^2 = \epsilon_i \iff h_i^2 = \omega^{n_i} h_i.
\]
On the other hand, \( h_i \notin \omega^{n_i} \mathcal{O}[G_{K}^x] \) implies that \( h_i^2 \neq 0 \in \mathcal{O}[G_{K}^x]/\omega^{n_i} \mathcal{O}[G_{K}^x] \cong \mathbb{F}[G_{K}^x] \).
But by (3.9), \( h_i^2 = 0 \) if \( n_i > 0 \) (which contradicts the fact that \( \mathbb{F}[G_{K}^x] \) has no nonzero nilpotent elements since it is reduced), so \( n_i = 0 \) for all \( i \), and therefore, \( \epsilon_i \in \mathcal{O}[G_{K}^x] \) for all \( i \).

This gives an internal decomposition
\[
\mathcal{O}[G_{K}^x] = A_0 \times A_1 \times \cdots \times A_{m-1},
\]
where \( A_i = \epsilon_i \mathcal{O}[G_{K}^x] \). Tensoring with \( \mathbb{F} \) gives an internal algebra decomposition
\[
\mathbb{F} \otimes \mathcal{O}[G_{K}^x] = \mathbb{F} \otimes A_0 \times \mathbb{F} \otimes A_1 \times \cdots \times \mathbb{F} \otimes A_{m-1},
\]
since for all \( i \), \( \overline{\epsilon_i} \neq 0 \) and \( \overline{\epsilon_i^2} = \overline{\epsilon_i} \), also \( \overline{\epsilon_i \epsilon_j} = 0 \) for \( i \neq j \) and \( \overline{\epsilon_0 + \cdots + \epsilon_{m-1}} = 1 \). Thus, \( G_{K,f}^x \) has at least \( m \) connected components, and hence \( k = m \). In particular, \( G_{K,f}^x = G_{f}^x \), and hence, \( G_{K}^x = G^x \) so \( G^x \) is smooth in this case.

Remark 3.11. We have just shown that Theorem 1.6 will hold if \( \mathcal{O} \) is enlarged by a DVR \( \mathcal{O} \subseteq \mathcal{O}' \) where the residue field of \( \mathcal{O}' \) is algebraically closed and the fraction field of \( \mathcal{O}' \) is large enough so that (3.8) holds.

Proof of Theorem 1.6. For the proof, we return to the setup from §1, where the only condition on \( \mathcal{O} \) is that it is complete and \( \mathbb{F} = \mathcal{O}/\omega \).

Let \( \overline{\mathcal{O}} \) denote the completion of the maximal unramified extension of \( \mathcal{O} \). By definition, \( \omega \) is also the uniformizer for \( \mathcal{O} \) and \( \overline{\mathcal{O}}/\omega \overline{\mathcal{O}} = \mathbb{F} \). It also possible to enlarge \( \overline{\mathcal{O}} \) by a finite purely ramified integral extension \( \overline{\mathcal{O}} \supseteq \mathcal{O}' \) (cf. [Se, §1.4]), such that (3.8) holds over the fraction field \( K' \) of \( \mathcal{O}' \), where we note \( \mathcal{O}' \) must have the same residue field as \( \overline{\mathcal{O}} \). By Remark 3.11, the base change \( G_{K,f}^x \) is smooth, and in particular,
\[
\mathcal{O}'[G_{K,f}^x] = \mathcal{O}' \otimes \mathcal{O}[G^x]
\]
is torsion-free. Observe that \( \mathcal{O}' \) is torsion-free as an \( \mathcal{O} \)-module, since \( \mathcal{O} \) and \( \mathcal{O}' \) are integral domains and \( \mathcal{O} \subseteq \mathcal{O}' \). The fact that \( \mathcal{O} \) is a DVR now implies \( \mathcal{O}' \) is also a flat \( \mathcal{O} \)-module (since these properties are equivalent for discrete valuation rings). Thus, the functor \( \mathcal{O}' \otimes - \) is exact, and the natural map
\[
(3.10) \quad \mathcal{O}[G^x] \xrightarrow{1 \otimes \text{id}} \mathcal{O}' \otimes \mathcal{O}[G^x]
\]
is injective. To see why (3.10) is injective, first let \( 0 \neq f \in \mathcal{O}[G^x] \), be arbitrary. There is a commutative diagram
\[
\begin{array}{ccc}
\mathcal{O}f & \longrightarrow & \mathcal{O}[G^x] \\
\downarrow & & \downarrow 1 \otimes \text{id} \\
\mathcal{O}' \otimes \mathcal{O}f & \longrightarrow & \mathcal{O}' \otimes \mathcal{O}[G^x],
\end{array}
\]
where the bottom map is injective by exactness of \( \mathcal{O}' \otimes - \).
Now suppose that $f$ is torsion, then since $\mathcal{O}f$ is cyclic and non-zero, we must have $\mathcal{O}f \cong \mathcal{O}/\omega^k$ for some $k \geq 1$. If $\omega'$ is the uniformizer of $\mathcal{O}'$, then $(\omega')^r = \omega$ for some $r \geq 1$. Thus

$$\mathcal{O}' \otimes \mathcal{O}f \cong \mathcal{O}'/\omega' \cong \mathcal{O}'/(\omega')^r,$$

where $rk \geq 1$, and so must be non-zero. The injectivity of the bottom map in the preceding diagram implies $\mathcal{O}'(1 \otimes f) \cong \mathcal{O}'/(\omega')^r$, therefore $1 \otimes f$ must also be non-zero, and torsion. But this is not possible since $\mathcal{O}' \otimes \mathcal{O}[G^x]$ is torsion-free.

This allows us to identify identify $\mathcal{O}[G^x] \subseteq \mathcal{O}' \otimes \mathcal{O}[G^x]$, which implies that $\mathcal{O}[G^x]$ must be torsion-free, since $\mathcal{O}' \otimes \mathcal{O}[G^x]$ is torsion-free and any non-zero $\mathcal{O}$-submodule of a torsion-free module is torsion-free. \hfill \Box

3.4. The component group scheme. We will continue to maintain our assumption that $\mathbb{F} = \overline{\mathbb{F}}$.

By [Mi, Definition 13.12], for any field $k$ over $\mathcal{O}$, there exists a component group scheme, denoted $A_k(x)$. This is defined to be the spectrum of the largest étale subalgebra of $k[G^x_k]$ (cf. [Mi, 13b]). Moreover, the coordinate algebra $k[A_k(x)]$ is naturally a Hopf subalgebra of $k[G^x_k]$. The inclusion $k[A_k(x)] \hookrightarrow k[G^x_k]$, induces a surjective group scheme homomorphism $G^x \twoheadrightarrow A_k(x)$. The connected component of $G^x_k$, denoted $(G^x_k)^0$, can now be defined as the normal subgroup scheme given by the kernel of this morphism.

It follows from [Mi, Proposition 13.18], that for any field extension $k' \supseteq k$,

$$A_{k'}(x) \cong A_k(x)_{k'}, \quad \text{and} \quad (G^x_{k'})^0 \cong (G^x_k)^0.$$

In particular, $(A_k(x)_{\overline{\mathbb{F}}})(\mathbb{K}) \cong A(x_{\overline{\mathbb{F}}})$ and $(A_{\mathbb{F}}(x))(\mathbb{F}) \cong A(x_{\mathbb{F}})$, where the discrete groups on the right-hand-side were considered in §3.2.

In §3.3, it was shown that if $x \in \mathfrak{g}_0$ is a balanced nilpotent section, and $\mathbb{K}$ satisfies (3.8), then there exist orthogonal idempotents $\epsilon_0, \ldots, \epsilon_{m-1} \in \mathcal{O}[G^x]$ so that

$$(3.11) \quad \mathcal{O}[G^x] = \epsilon_0 \mathcal{O}[(G^x)^0] \times \epsilon_1 \mathcal{O}[(G^x)^1] \times \cdots \times \epsilon_{m-1} \mathcal{O}[(G^x)^{m-1}],$$

where $m = |A(x)| = A(x)_{\mathbb{F}}|$. Therefore, the $\mathcal{O}$-scheme $G^x$ has a decomposition

$$(3.12) \quad G^x = (G^x)^0 \sqcup (G^x)^1 \sqcup \ldots \sqcup (G^x)^{m-1},$$

where $(G^x)^i = \text{Spec}(\epsilon_i \mathcal{O}[G^x])$ and $(G^x)^0_{\mathbb{F}}$ and $(G^x)^1_{\mathbb{F}}$ give the complete set of connected components for $G^x_{\mathbb{F}}$ and $G^x_{\overline{\mathbb{F}}}$ respectively.

Theorem 3.12. Let $x \in \mathfrak{g}_0$ be a balanced nilpotent section, and suppose that $\mathcal{O}$ is such that $\mathbb{F} = \overline{\mathbb{F}}$ and $\mathbb{K}$ satisfies (3.8). If we let $A(x)$ be the constant scheme defined by the subalgebra

$$(3.13) \quad \mathcal{O}[A(x)] := \sum_{i=0}^{m-1} \mathcal{O} \epsilon_i \subseteq \mathcal{O}[G^x],$$

where $\epsilon_0, \ldots, \epsilon_{m-1} \in \mathcal{O}[G^x]$ are as in (3.11), then

1. for any field $k$ over $\mathcal{O}$, $A(x)_k \cong A_k(x)$,
2. $\mathcal{O}[A(x)]$ is a Hopf subalgebra of $\mathcal{O}[G^x]$,
3. $(G^x)^0$ is the kernel of the induced homomorphism $G^x \twoheadrightarrow A(x)$, so that $A(x) \cong G^x/(G^x)^0$ and $(G^x)^0 \trianglelefteq G^x$ is a normal subgroup scheme.
Proof. We begin by proving (1). Let us first consider the case where \( k = \mathbb{K} \), then it suffices to show that \( \mathbb{K}[A(x)_{\mathbb{K}}] = \mathbb{K} \otimes \mathbb{O}[A(x)] \subseteq \mathbb{K}[G^*_k] \) is the largest \( \acute{e} \)tale subalgebra of \( \mathbb{K}[G^*_k] \). To see why this is the case, let \( R \subseteq k[G^*_k] \) be the maximal \( \acute{e} \)tale subalgebra, so that by definition, \( R \supseteq \mathbb{K}[A(x)_{\mathbb{K}}] \) (such a subalgebra always exists). By [Mi, Proposition 13.8], \( \mathbb{K} \otimes R \subseteq \mathbb{K}[G^*_k] \) also gives the maximal \( \acute{e} \)tale subalgebra. Now, [Mi, Corollary 13.9] and [Mi, Lemma 13.4] imply

\[
\dim_{\mathbb{K}} R = \dim_{\mathbb{K}}(\mathbb{K} \otimes R) = |\{\text{connected components of } G^*_k\}| = m,
\]

where the rightmost equality follows from assumption (3.8). Thus, \( \dim_{\mathbb{K}} R = \dim_{\mathbb{K}} \mathbb{K}[A_k(x)] \), and therefore, \( R = \mathbb{K}[A_k(x)] \). The same argument also show that \( \mathbb{F}[A(x)_{\mathbb{F}}] \) is the maximal \( \acute{e} \)tale subalgebra of \( \mathbb{F}[G^*_k] \). Finally, suppose \( k \) is any field over \( \mathbb{O} \). Then either \( k \supseteq \mathbb{K} \) or \( k \supseteq \mathbb{F} \), and in both cases [Mi, Proposition 13.8] implies that \( k[A(x)_k] \) is the maximal \( \acute{e} \)tale subalgebra of \( k[G^*_k] \). So we have verified (1).

Now we will verify (2). Let \( \Delta, \sigma \) and \( \varepsilon \) denote the coproduct, antipode and counit for \( \mathbb{O}[G^2] \) respectively. It suffices to show

\[
\Delta(\mathbb{O}[A(x)]) \subseteq \mathbb{O}[A(x)] \otimes \mathbb{O}[A(x)],
\]

\[
\sigma(\mathbb{O}[A(x)]) \subseteq \mathbb{O}[A(x)] \quad \text{and} \quad \varepsilon(\mathbb{O}[A(x)]) = \mathbb{O}.
\]

The third identity can be verified immediately by observing that \( \varepsilon(\mathbb{O}[A(x)]) \subseteq \mathbb{O} \) is an \( \mathbb{O} \)-subalgebra of \( \mathbb{O} \), which implies equality since \( \mathbb{O} \) cannot have any proper \( \mathbb{O} \)-subalgebras.

To verify the first two identities, we first note that for an \( \mathbb{O} \)-algebra \( k \), the Hopf algebra structure of \( k[G^2_k] = k \otimes \mathbb{O}[G^2] \) is given by \( \Delta_k := \text{id}_k \otimes \Delta, \sigma_k := \text{id}_k \otimes \varepsilon, \) \( \sigma_k := \text{id}_k \otimes \sigma \). So, in particular, for \( k = \mathbb{K} \), the first two identities must hold for \( \mathbb{K} \otimes \mathbb{O}[A(x)] \) by (1). To verify the second identity, it suffices to show \( S(\varepsilon_i) \in \mathbb{O}[A(x)] \) for all \( i \) since \( S \) is \( \mathbb{O} \)-linear. However, the fact that \( \mathbb{O}[A(x)] \) is torsion-free, allows us to identify \( \mathbb{O}[A(x)] \subset \mathbb{K} \otimes \mathbb{O}[A(x)] \), so that \( \sigma = (\sigma_k^\mathbb{K})\mathbb{O}[A(x)] \). Thus, if \( i \) is arbitrary, then

\[
\sigma(\varepsilon_i) = \sigma(\varepsilon_i) = a_0 \varepsilon_0 + a_1 \varepsilon_1 + \cdots + a_{m-1} \varepsilon_{m-1} \in (\mathbb{K} \otimes \mathbb{O}[A(x)]) \cap \sigma(\mathbb{O}[A(x)])
\]

for some \( a_0, \ldots, a_{m-1} \in \mathbb{K} \). But since \( \sigma \) is a morphism of \( \mathbb{O} \)-algebras, and the \( \varepsilon_0, \ldots, \varepsilon_{m-1} \) are pairwise orthogonal idempotents, then

\[
\sigma(\varepsilon_i)^2 = \sigma(\varepsilon_i) = \sigma(\varepsilon_i)
\]

implies

\[
a_0^2 \varepsilon_0 + a_1^2 \varepsilon_1 + \cdots + a_{m-1}^2 \varepsilon_{m-1} = a_0 \varepsilon_0 + a_1 \varepsilon_1 + \cdots + a_{m-1} \varepsilon_{m-1},
\]

so \( a_i^2 = a_i \), and hence, \( a_i \in \{0, 1\} \subset \mathbb{O} \) for all \( i \). So the second identity is verified.

Similarly, if \( i \) is arbitrary, then

\[
\Delta(\varepsilon_i) = \Delta_k(\varepsilon_i) = \sum_{(j,k)} a_{jk} \varepsilon_j \otimes \varepsilon_k \in (\mathbb{K} \otimes \mathbb{O}[A(x)] \otimes \mathbb{K} \otimes \mathbb{O}[A(x)]) \cap \Delta(\mathbb{O}[A(x)])
\]

for some \( a_{jk} \in \mathbb{K} \). Now observe set of \( \varepsilon_i \otimes \varepsilon_k \) gives a linearly independent set of pairwise orthogonal idempotents for \( \mathbb{O}[G^2] \otimes \mathbb{O}[G^2] \), and that \( \Delta \) is a morphism algebras. Thus,

\[
\Delta(\varepsilon_i)^2 = \Delta(\varepsilon_i) = \Delta(\varepsilon_i)
\]

implies

\[
\sum_{(j,k)} a_{jk}^2 \varepsilon_j \otimes \varepsilon_k = \sum_{(j,k)} a_{jk} \varepsilon_j \otimes \varepsilon_k,
\]
so that $a_{jk} = a_{jk}$, which forces $a_{jk} \in \{0, 1\} \subset \mathcal{O}$. Therefore, $\mathcal{O}[A(x)]$ is a Hopf subalgebra of $\mathcal{O}[G^x]$.

So the inclusion $\mathcal{O}[A(x)] \hookrightarrow \mathcal{O}[G^x]$ of Hopf algebras, induces a surjective map of group schemes

$$G^x \rightarrow A(x).$$

Finally, let $H$ be the kernel of this homomorphism. From the definition of the kernel of a group scheme homomorphism, we have

$$\mathcal{O}[H] = \mathcal{O}[G^x]/(\epsilon_1 + \cdots + \epsilon_{m-1})\mathcal{O}[G^x] \cong \mathcal{O}[(G^x)^o].$$

Thus, $H = (G^x)^o$, and in particular, $(G^x)^o$ is a normal subgroup scheme of $G^x$.

Now we can prove Theorem 1.8.

**Proof of Theorem 1.8.** Let us now return to our hypothesis on $\mathcal{O}$, $k$ and $\mathbb{F}$ from §1. Just as in the proof of Theorem 1.6, we begin by replacing $\mathcal{O}$ with the completion of its maximal unramified extension, which we denote by $\overline{\mathcal{O}}$. Now the residue field of $\overline{\mathcal{O}}$ is the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$, and let $K'' \supset k$ be the fraction field of $\overline{\mathcal{O}}$. Also let $\mathcal{O}'$ be a finite integral extension of $\overline{\mathcal{O}}$ (with fraction field $K' \supset K'' \supset k$) such that the hypothesis of Theorem 3.12 is satisfied.

Applying Theorem 3.12 to this setup, immediately implies $A(x_{\overline{\mathbb{F}}}) \cong A(x_{\overline{\mathbb{F}}})$, where $x_{\overline{\mathbb{F}}}$ denotes the algebraic closure of $x_{\overline{\mathbb{F}}}$. Finally, $A_{\overline{\mathbb{F}}}$ denote the component group scheme, and noting that $k \subseteq \overline{\mathbb{F}}$, then it follows from [Mi, Proposition 13.8] that $A(x_{\overline{\mathbb{F}}}) \cong A(x_{\overline{\mathbb{F}}})$ as groups. Therefore, $A(x_{\overline{\mathbb{F}}}) \cong A(x_{\overline{\mathbb{F}}}).$}

4. Centralizers for the $G \times \mathbb{G}_m$ action

4.1. Smoothness in the graded case. For simplicity, we will introduce the notation $\mathcal{G} = G \times \mathbb{G}_m$. The scheme $\mathcal{g}$ is also equipped with a $\mathcal{G}$ action, where $\mathbb{G}_m$ acts by the **cohomological action**, $t \cdot x = t^{-2}x$ for $t \in \mathbb{G}_m(k)$ and $x \in \mathcal{g}(k)$ and any $\mathcal{O}$-algebra $k$. This gives $\mathcal{O}[\mathcal{g}]$ a non-negative, even grading which is generated in degree 2. Now, for any balanced nilpotent $x \in \mathcal{g}_{\mathcal{O}},^3$ consider the centralizer $G^x$, as well as the centralizers for the base-changes $G^x_k$ and $G^x_{\mathbb{F}}$.

Suppose now that $x \in \mathcal{g}_\mathcal{O}$ is a balanced nilpotent section, then by [Mc3, Theorem 1.2.1], there exists an integral **associated cocharacter**

$$\phi_x : \mathbb{G}_m \rightarrow G^x,$$

such that $\phi_{x,k}$ and $\phi_{x,\mathbb{F}}$ are the associated cocharacters arising from the Jacobson-Morozov triples for $x_k$ and $x_{\mathbb{F}}$ respectively.

Let $\text{Int} : (G^x)^{op} \rightarrow \text{Aut}(G^x)$, with $\text{Int}_h(g) = hgh^{-1}$ for $h, g \in G^x(k)$ be a morphism of group schemes. An action of $\mathbb{G}_m$ on $G^x$ can then be given by

$$t \cdot g = \phi_x(t^{-1})g\phi_t(t) = \text{Int}_{\phi_x(t^{-1})}(g),$$

for $g \in G^x(k)$ and any $\mathcal{O}$-algebra $k$. Let

$$\mathbb{G}_m \ltimes_{\phi_x} G^x,$$

be the semi-direct product formed from this action.

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3Recall that $\mathcal{g}_\mathcal{O} := \mathcal{g}(\mathcal{O})$. 

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3Recall that $\mathcal{g}_\mathcal{O} := \mathcal{g}(\mathcal{O})$. 

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Proposition 4.1. There is an isomorphism of group schemes
\[
G_m \rtimes_{\phi_x} G^x \simto G^x \subseteq G \times G_m,
\]
given by \( t \ltimes g \mapsto (g\phi_x(t^{-1}), t^{-1}) \) for any \( g \in G(k) \), \( t \in G_m(k) \) and any \( \mathcal{O} \)-algebra \( k \).

Proof. Let us first note that this map is well-defined and natural for any \( \mathcal{O} \)-algebra \( k \). Now, letting \( k \) be arbitrary and identifying \( x \in g(k) \) with its image under the morphism \( g(\mathcal{O}) \to g(k) \), we observe that \( (g, t) \in G^x(k) \subseteq (G \times G_m)(k) \) if and only if
\[
x = (g, t) \cdot x = \text{Ad}(1, t) \circ \text{Ad}(g, 1)x = t^{-2} \text{Ad}_g x,
\]
or equivalently, \( \text{Ad}_g x = t^2 x \). Now since \( \phi_x \) is an associated cocharacter, then \( x \in g_2 \) with respect to its induced grading on \( g \) (equivalently \( \text{Ad}_{\phi_x(t)}x = t^2 x \) for any \( t \in G_m(k) \)). Thus,
\[
\text{Ad}_g x = \text{Ad}_{\phi_x(t)}x \iff \text{Ad}_{g\phi_x(t^{-1})}x = x,
\]
and so \( g\phi_x(t^{-1}) \in G^x(k) \). Therefore,
\[
(g, t) \in G^x(k) \iff (g, t) = (h\phi_x(t), t) \text{ for } h = g\phi_x(t^{-1}) \in G^x(k).
\]
Therefore, this map is canonically a group isomorphism for every \( \mathcal{O} \)-algebra \( k \), and hence is an isomorphism of group schemes. \( \square \)

Corollary 4.2. The scheme-theoretic centralizers \( G^x_K \) and \( G^x_F \) are smooth.

Now recall that if \( k \) is a field over \( \mathcal{O} \), then there exists an additional Levi-decomposition of \( G^x_K \). Let \( G^x_{k, \text{red}} \times G_m \) act on \( G^x_{k, \text{unip}} \) by
\[
(g, t) \cdot u = g\phi_x(t^{-1})u\phi_x(t)g^{-1},
\]
for \( (g, t) \in (G^x_{k, \text{red}} \times G_m)(k') \), \( u \in G^x_{k, \text{unip}}(k') \) and any \( k \)-algebra \( k' \). Let \( (G^x_{k, \text{red}} \times G_m) \ltimes G^x_{k, \text{unip}} \) be the semi-direct product formed from this action.

Corollary 4.3. For \( k \in \{K, F\} \), there is an isomorphism
\[
(G^x_{k, \text{red}} \times G_m) \ltimes G^x_{k, \text{unip}} \simto G^x_K
\]
given by \( (g, t) \ltimes u \mapsto (g\phi_x(t^{-1})u, t^{-1}) \) for \( (g, t) \ltimes u \in ((G^x_{k, \text{red}} \times G_m) \ltimes G^x_{k, \text{unip}})(k') \) and any \( k \)-algebra \( k' \). In particular, \( G^x_{k, \text{red}} \cong G^x_{k, \text{red}} \times G_m \) and \( G^x_{k, \text{unip}} \cong G^x_{k, \text{unip}} \).

Proof. This follows by observing that for any \( k \)-algebra \( k' \), this map is canonically equivalent to the one in Proposition 4.1, and is therefore an isomorphism. \( \square \)

The following theorem is now immediate.

Theorem 4.4. Let \( x \in g_o \) be a balanced nilpotent section, and let \( k \in \{K, F\} \), then the morphism \( G^x \to \text{Spec}(\mathcal{O}) \) is smooth and \( G^x_{k}/(G^x_{k})^0(K) \cong A(x_K) \).

Furthermore, if the conditions of Theorem 3.12 are satisfied, then the morphism \( G^x \to A(x) \) lifts to a group scheme homomorphism \( G^x \to A(x) \), where the base changes \( A(x)_k \) also give the component group schemes for \( G^x_k \) (i.e. there are group scheme isomorphisms \( G^x_K/(G^x_K)^0 \cong G^x_{K}/(G^x_{K})^0 \).
4.2. Integral lattices. For a balanced nilpotent section \( x \in \mathfrak{g}_0 \), let \( H \in \{ G^x, G^x \} \), then the flatness of \( H \) now implies that for any finite-dimensional \( H_k \)-module \( V \), there exists an \( H \)-stable \( \mathbb{G} \)-lattice \( M \subset V \) (cf. [J1, I.10.4]).

**Lemma 4.5.** Let \( k \in \{ \mathbb{K}, \mathbb{F} \} \). If \( N \) is a \( G^x_k \) = \( \mathbb{G}_m \times_{\phi_x} G^x_k \)-module such that \( (t \times g) \cdot n = (1 \times g) \cdot n \) for any \( t \times g \in (\mathbb{G}_m \times_{\phi_x} G^x_k)(k') \), \( n \in N(k') \) and any \( k \)-algebra \( k' \), then \( G^x_k_{\text{unip}} \) acts trivially on \( N \).

**Proof.** If we assume the hypothesis, then since \( G^x_k \), \( G^x_m \) and \( G^x_k \) are all reduced by Corollary 4.2, it suffices to show that \( G^x_k_{\text{unip}}(k) \) acts trivially on \( N(\mathbb{F}) = N \otimes_k \mathbb{F} \).

So without loss of generality, assume that \( k = \mathbb{K} \), so that it suffices to work with the geometric points.

The restriction of \( N \) to \( G^x_k \) corresponds to a group variety homomorphism

\[
\psi : G^x_k \rightarrow GL(N),
\]

and our goal is to show \( \psi(G^x_k_{\text{unip}}) = \{1\} \). However, since

\[
\text{Lie}(G^x_k_{\text{unip}}) = d\psi(\text{Lie} G^x_k_{\text{unip}}),
\]

then it suffices to show \( d\psi(\text{Lie} G^x_k_{\text{unip}}) = 0 \).

The fact that \( N \) is a \( (\mathbb{G}_m \times_{\phi_x} G^x_k) \)-module is equivalent to saying that \( \psi \) is \( \mathbb{G}_m \)-equivariant, where \( \mathbb{G}_m \) acts via \( (4.1) \) (e.g. by \( \text{Int}_{\phi_x(t^{-1})} \)), and that \( \mathbb{G}_m \) acts trivially by the hypothesis. Moreover, there are induced actions of \( \mathbb{G}_m \) on the respective Lie algebras (given by the adjoint action), so that

\[
d\psi : \mathfrak{g}^x_k \rightarrow \mathfrak{gl}(N)
\]

is also \( \mathbb{G}_m \)-equivariant. Equivalent the respective Lie algebras are given gradings

\[
\mathfrak{g}^x_k = \bigoplus_{k \in \mathbb{Z}} (\mathfrak{g}^x_k)_k, \quad \mathfrak{gl}(N) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{gl}(N)_k,
\]

where the \( \mathbb{G}_m \)-equivariance of \( d\psi \) induces a decomposition \( d\psi = \bigoplus_{k \in \mathbb{Z}} d\psi_k \) such that

\[
d\psi_k : (\mathfrak{g}^x_k)_k \rightarrow \mathfrak{gl}(N)_k.
\]

Observe now that \( \mathfrak{gl}(N) = \mathfrak{gl}(N)_0 \) and \( d\psi_k = 0 \) for all \( k \neq 0 \), since the action of \( \mathbb{G}_m \) on \( \mathfrak{gl}(N) \), and hence on \( \mathfrak{gl}(N) \), is trivial. By (4.1), the action of any \( t \in \mathbb{G}_m \) is given by the automorphism

\[
d(\text{Int}_{\phi_x}(t^{-1})) = \text{Ad}_{\phi_x(t^{-1})} : \mathfrak{g}^x_k \rightarrow \mathfrak{g}^x_k,
\]

and thus by [J2, Proposition 5.10],

\[
\text{Lie}(G^x_k_{\text{red}}) = (\mathfrak{g}^x_k)_0, \quad \text{Lie}(G^x_k_{\text{unip}}) = \bigoplus_{k < 0} (\mathfrak{g}^x_k)_k,
\]

with \( (\mathfrak{g}^x_k)_k = 0 \) for all \( k > 0 \). Finally, we see that since \( d\psi_k = 0 \) for all \( k \neq 0 \), then

\[
d\psi(\text{Lie} G^x_k_{\text{unip}}) = 0.
\]

□

**Remark 4.6.** The reason for the sign difference between the \( \mathbb{Z} \)-grading on \( \mathfrak{g}^x_k \) in the preceding proof, and the \( \mathbb{Z} \)-grading in [J2, Proposition 5.10] is due to the fact that the grading here is induced from the action (4.1), which gives \( \text{Ad}_{\phi_x(t^{-1})} \), while the cited proposition is with respect to the inverse action given by \( \text{Ad}_{\phi_x(t)} \), for \( t \in \mathbb{G}_m \).

**Proposition 4.7.** If \( V \) is any \( G^x_k \)-module which factors through \( G^x_k_{\text{red}} \), then there exists a \( G^x \)-stable \( \mathbb{G} \)-lattice \( M \subset V \) such that \( M_y \) factors through \( G^x_k_{\text{red}} \) (i.e. \( G^x_{k,y} \) acts trivially on \( M_y \)).
Proof. By Corollary 4.3, $V$ can be lifted to a module for $G^x_K$ which has trivial $\mathbb{G}_m$ and $G^x_{\mathbb{G}_{\text{unip}}}$ actions. Comparing with Proposition 4.1, we observe that the restriction of $V$ to

$$(\mathbb{G}_m \ltimes_{\phi_*} 1)_K \subseteq (\mathbb{G}_m \ltimes_{\phi_*} G^x_K)_K$$

is also trivial. Now Theorem 4.4 ensures that $G^x$ is smooth (and hence flat over $\mathcal{O}$), therefore there must exist a $G^x$-stable $\mathcal{O}$-lattice $M \subset V$. Now Theorem 4.4 ensures that $G^x$ is smooth (and hence flat over $\mathcal{O}$), therefore there must exist a $G^x$-stable $\mathcal{O}$-lattice $M \subset V$. And moreover,

$$V = M \otimes K = \bigoplus_{k \in \mathbb{Z}} (M_k \otimes \mathbb{K}),$$

where $V_k = M_k \otimes \mathbb{K}$ is the grading arising from the $(\mathbb{G}_m \ltimes_{\phi_*} 1)_K$ module structure. Since this module structure is trivial, it must be the case that $V = V_0$, which, by the fact that $M$ is free, implies $M_k = 0$ for all $k \neq 0$. Finally, by base-changing to $\mathbb{F}$, it can be observed that the $(\mathbb{G}_m \ltimes_{\phi_*} \mathbb{G}_m)_\mathbb{F}$ module structure on $M_\mathbb{F}$ is given by

$$M_\mathbb{F} = M \otimes \mathbb{F} = \bigoplus_{k \in \mathbb{Z}} (M_k \otimes \mathbb{F}) = M_0 \otimes \mathbb{F}.$$ 

This implies that $M_\mathbb{F}$ satisfies the conditions of Lemma 4.5, and therefore, it factors through $G^x_{\mathbb{F}, \text{red}}$.

References

[AHR1] P. Achar, W. Hardesty, and S. Riche, Integral exotic sheaves and the modular Lusztig–Vogan bijection, in preparation.

[AHR2] P. Achar, W. Hardesty, and S. Riche, Representation theory of disconnected reductive groups, in preparation.

[B] J. Booher, Geometric Deformations of Orthogonal and Symplectic Galois Representations, Ph.D. thesis, Stanford University, 2016.

[GM] P. Gille, L. Moret-Bailly, Actions algébriques de groupes arithmétiques, Torsors, étale homotopy and applications to rational points, 231-249, London Math. Soc. Lecture Note Ser., 405, Cambridge Univ. Press, Cambridge, 2013.

[Ht] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, no. 52, Springer-Verlag, New York, 1977.

[He] S. Herpel, On the smoothness of centralizers in reductive groups, Trans. Amer. Math. Soc. 365.7 (2013), 3753–3774.

[J1] J. C. Jantzen, Representations of algebraic groups, second edition, Mathematical surveys and monographs 107, Amer. Math. Soc., 2003.

[J2] J. C. Jantzen, Nilpotent orbits in representation theory, in Lie theory, 1–211, Progr. Math. 228, Birkhäuser Boston, 2004.

[LS] M. Liebeck, G. Seitz, Unipotent and nilpotent classes in simple algebraic groups and Lie algebras, Mathematical Surveys and Monographs 180, American Mathematical Society, 2012.

[Mc1] G. McNinch, Nilpotent orbits over ground fields of good characteristic, Math. Ann. 329 (2004), no. 1, 49-85.

[Mc2] G. McNinch, The centralizer of a nilpotent section, Nagoya J. Math 190 (2008), pp. 129-181.

[Mc3] G. McNinch, On the nilpotent orbits of a reductive group over a local field, preprint, 2016.

[MS] G. McNinch and E. Sommers, Component groups of unipotent centralizers in good characteristic, J. Algebra 260 (2003), pp. 323-337.

[Mi] J. S. Milne, Algebraic Groups, Lie Groups, and their Arithmetic Subgroups, 2011, Available at www.jmilne.org/math/.

[Se] J-P. Serre, Local fields, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg.
[St] The Stacks Project Authors, *Stacks project*, http://stacks.math.columbia.edu, 2018.

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