Asymptotic single-particle states in quantum field theories with small departures from Lorentz symmetry are investigated. To this end, one-loop radiative corrections for a sample Lorentz-violating Lagrangian contained in the Standard-Model Extension (SME) are studied. It is found that the spinor kinetic operator is modified in momentum space by Lorentz-violating operators not present in the original Lagrangian. It is demonstrated how both the standard renormalization procedure as well as the Lehmann–Symanzik–Zimmermann reduction formalism need to be adapted as a consequence of this result.

PACS numbers: 11.30.Cp, 11.10.Gh, 11.15.Bt, 13.40.Ks

I. INTRODUCTION

Current understanding of physics at the fundamental level is based on two distinct theories: general relativity (GR) and the Standard Model (SM) of particle physics. It is commonly believed that these two theories arise as the low-energy limit of an underlying Planck-scale framework that consistently merges gravity and quantum mechanics. Since direct measurements at this scale are presently impractical, experimental research in this field relies largely on ultrahigh-precision searches for Planck-suppressed effects at attainable energies.

One possible effect in this context is a minute breakdown of Lorentz invariance. Lorentz symmetry is a fundamental feature of both GR and the SM, so that any observed deviation from this symmetry would imply new physics. A number of theoretical approaches to physics beyond the SM, such as strings [1], non-commutative field theories [2], cosmologically varying scalar fields [3], quantum gravity [4], random-dynamics models [5], multiverses [6], brane-world scenarios [7], and massive gravity [8], are believed to allow for small violations of Lorentz invariance at low energies. Searches for such violations are also motivated by the apparent fundamental character of Lorentz symmetry. Consequently, Lorentz invariance ought to be supported as firmly as possible by experimental evidence.

It is natural to expect that Lorentz-violating effects can be described within effective field theory, at least at currently attainable energies [9]. The framework generally adopted in this context is the Standard-Model Extension (SME) [10, 11], which contains both GR and the SM as limiting cases. The additional Lagrangian terms present in the SME include all operators for Lorentz violation that are scalars under coordinate changes. The SME has constituted the basis for the analysis of numerous experimental searches for Lorentz breakdown [12].

Paralleling the conventional Lorentz-symmetric case, perturbative quantum-field analyses within the SME also rely on a few key theoretical concepts. Some of these, such as canonical quantization [13, 14] and renormalization [16–18], have previously been studied and generalized to the SME. Another such core concept concerns the treatment of external states. They span the asymptotic Hilbert space, so their determination is of fundamental importance for perturbation theory. For example, explicit S-matrix calculations require a separate, independent determination of the external legs up to the desired order. This special status of external-leg physics is highlighted by the usual Feynman rules: external-leg corrections cannot be incorporated into the diagram for a scattering process; the rules for S-matrix calculations specifically call for “amputed” diagrams. The usual treatment of radiative corrections to external legs involves sophisticated theoretical concepts like the Källén–Lehmann representation [19] and the Lehmann–Symanzik–Zimmermann (LSZ) reduction formalism [20]. Although a number of prior investigations have considered radiative corrections from various other perspectives [21, 22], we are unaware of any dedicated study to generalize the Lorentz-invariant external-leg treatment to Lorentz-violating field theories.

A second need for a proper understanding of the asymptotic Hilbert space in the presence of Lorentz breakdown derives from its phenomenological importance: external-state effects govern the physics of free particles and are therefore also crucial for numerous Lorentz tests. Examples include various kinematical threshold effects in cosmic rays [23], photon birefringence and dispersion [24, 25], collider kinematics and interferometry [26–29], and neutrino propagation [30]. Paralleling the conventional Lorentz-symmetric case, all
previous analyses have been performed under the tacit assumption that the physics of free particles is determined by the quadratic pieces of the corresponding Lagrangian [31]. However, this approach disregards the self-interactions of the particle, although such effects are always present, even for asymptotic states. Consequently, they need to be considered, e.g., in any scattering process beyond tree level. In a conventional renormalizable quantum field theory (QFT), Lorentz symmetry implies that the quadratic Lagrangian can only acquire a mass shift and a field-strength factor, both of which can be treated by renormalization of existing quantities. The external QFT legs are then identical in structure to the quadratic-Lagrangian solutions, which therefore indeed describe the propagation of free particles correctly. A non-perturbative rigorous justification for this feature is given by the aforementioned LSZ reduction formalism [20]. However, in the presence of Lorentz violation a similar line of reasoning fails, and the question regarding the determination of free-particle properties arises.

The present work is intended to initiate a theoretical investigation of these issues. In particular, we demonstrate that in the absence of Lorentz symmetry the external legs in perturbative quantum field theory exhibit a different structure than the plane-wave solutions arising from the quadratic Lagrangian. This result is in accordance with a recent work [32] in which a generalization of the Källén–Lehmann representation for the propagator was derived for a field-theoretic model with fermions that are coupled to the same Lorentz-violating SME coefficients as the ones we are considering in this work. In fact, we will use the results obtained in Ref. [32] to extract consistently the one-particle poles defining the one-particle pole is extracted, and the dispersion relation corrections to the fermion propagator are evaluated, the Lorentz-violating case, and the derivation of a general formula for the corresponding spinor wave-function renormalization factor. In Sec. IV, the one-loop radiative corrections to the fermion propagator are evaluated, the one-particle pole is extracted, and the dispersion relation as well as the spinor wave-function renormalization factor are obtained. Section V extends the LSZ formalism to the Lorentz-violating case and establishes the associated Feynman expansion of the scattering matrix. In Sec. VI, the formalism developed in this paper is applied to the example of Coulomb scattering. Our summary and an outlook are contained in Sec. VII. Supplementary material is collected in various appendices.

II. MODEL BASICS AND SCOPE

Our model is based on the bare gauge-invariant flat-spacetime Lagrange density for single-flavor quantum electrodynamics (QED) within the minimal SME:

\[
\mathcal{L}_{\text{SME}} = \frac{i}{2} \bar{\psi} \gamma^\mu B^\mu \psi - \frac{1}{4} F_{\mu
u} F^{\mu
u} - \frac{1}{4} (k_B^B)_{\mu\rho\sigma} F^{\mu\rho\sigma} + (k_{AF}^B)_{\alpha\mu} F^{\mu\nu} F_{\nu\alpha} \tag{1}
\]

The label \(B\) denotes bare quantities, \(\psi\) is a Dirac spinor and \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) a gauge-field strength. We have also implemented the conventional notation for the U(1)-covariant derivative \(D_\mu = \partial_\mu + i e B_\mu\) and for the dual field-strength tensor \(F_{\mu\nu} = \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}\). The Lorentz-violating effects are contained in the quantities \((k_B^B)_{\mu\rho\sigma}\) and \((k_{AF}^B)_{\alpha\mu}\) as well as in the the generalized gamma matrices \(\Gamma^\mu_B\) and the generalized mass matrix \(M_B\). The latter are given by the explicit expressions

\[
\Gamma^\mu_B = \gamma^\mu + e B^\nu \gamma^\nu + i f_B^\mu + \frac{1}{2} \lambda_B^\mu \sigma_{\lambda\nu} + e B^\mu, \\
M_B = m_B + g_B^\mu \gamma^\mu + \frac{1}{4} H^\mu_B \sigma_{\mu\nu} \tag{2}
\]

The nondynamical spacetime constant quantities \((k_B^B)_{\mu\rho\sigma}\), \((k_{AF}^B)_{\alpha\mu}\), \(a_B^\mu\), \(b_B^\mu\), \(c_B^\mu\), \(d_B^\mu\), \(f_B^\mu\), \(g_B^\mu\), and \(H^\mu_B\) control the type and extent of Lorentz and CPT breakdown. It has been shown that this flat-spacetime Lagrangian is multiplicatively renormalizable at one-loop order [16], and that this renormalizability property is maintained in curved spacetimes [18].

The complete one-loop structure of Lagrangian (1) would be of interest, but lies beyond the scope of this work. Our present goal is rather to initiate the study of finite radiative corrections in the presence of Lorentz violation by highlighting several theoretical issues that can arise within this context. For such illustrative purposes, it seems appropriate to simplify the model (1) such that tractability, phenomenological importance, and theoretical relevance are optimized. Considerations along these lines are presented next.

A key simplification is setting to zero all Lorentz-violating coefficients, with the exception of \(e_B^\mu\) and \((k_B^B)_{\mu\rho\sigma}\). We may also take the \(e_B^\mu\) coefficient to be symmetric because its antisymmetric piece can be removed from the Lagrangian by a field redefinition at leading order [10]. Moreover, we will choose \(k_B^B\) to be of the form

\[
(k_B^B)_{\mu\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho} \tilde{k}_B^{\sigma} - \eta_{\mu\sigma} \tilde{k}_B^{\rho} - \eta_{\rho\sigma} \tilde{k}_B^{\mu} + \eta_{\mu\sigma} \tilde{k}_B^{\rho} + \eta_{\rho\sigma} \tilde{k}_B^{\mu}) \tag{3}
\]

where \(\tilde{k}_B^\mu\) is taken as symmetric, traceless, and given by

\[
\tilde{k}_B^\mu = (k_F^B)^{\mu\alpha\nu} \alpha \tag{4}
\]
investigations in electron–photon scattering [29], and as-
resonance cavities [35], kinematical threshold studies at
lar, measurements have been performed in the context of
ence 2
order in any fermion–photon system; only their differ-
cients are observationally indistinguishable at leading
otypes field redefinitions and reinterpretations [33]. In the
independent interpretation of our model.

First, we note that Eqs. (3) and (4) are incompatible in
space-time dimensions $d \neq 4$. When dimensional regular-
ization is employed, it might then appear that this could
lead to interpretational difficulties with previously deter-
dined $(k^B_{\mu})_{\nu \rho \sigma}$ and $c_{\mu \nu}$ counterterms [16] in the context
of minimal subtraction, affect finite radiative corrections,
or may even be associated with trace anomalies. How-
ever, it turns out that such spurious issues can be avoided
altogether by considering a model with $k^B_{\mu \nu}$ in its own
right rather than as the limit of the full $(k^B_{\mu})_{\nu \rho \sigma}$ La-
grangian. Throughout this work, we follow this latter,
independent interpretation of our model.

Second, the identification of observables in Lorentz-
violating field theories requires special care due various
kinds of field redefinitions and reinterpretations [33]. In
the present case, it turns out that the $c_{\mu \nu}$ and $k^B_{\mu \nu}$ coef-
cients are observationally indistinguishable at leading
order in any fermion–photon system; only their differ-
ence $2c_{\mu \nu} - k^B_{\mu \nu}$ can be measured within the context of
Lagrangian (1). This feature has previously been dis-
cussed from various perspectives [24, 27, 34]. For exam-
ple, suitable coordinate rescalings can eliminate the $c_{\mu \nu}$
coefficient in favor of $k^B_{\mu \nu}$, or vice versa. Such coordinate
redefinitions can be exploited to simplify calculations. In
what follows, however, we will for the most part avoid the
choice of a particular coordinate scaling by keeping both
$c_{\mu \nu}$ and $k^B_{\mu \nu}$ nonzero. This will provide an independent
partial test of our results, since coordinate–scalar expres-
sions for physically observable radiative effects should
only depend on $2c_{\mu \nu} - k^B_{\mu \nu}$.

Third, we also note that various experimental investiga-
tions have sought to constrain $2c_{\mu \nu} - k^B_{\mu \nu}$. In particu-
lar, measurements have been performed in the context of
resonance cavities [35], kinematical threshold studies at
colliders [27], synchrotron radiation [28], Compton-edge
investigations in electron–photon scattering [29], and astro-
physical observations [36]. Through these investiga-
tions, all components of $2c_{\mu \nu} - k^B_{\mu \nu}$ are currently obey-
ing bounds at the levels of $10^{-13} \ldots 10^{-17}$. At present,
$2c_{\mu \nu} - k^B_{\mu \nu}$ nevertheless remains the parameter combi-
nation in Lagrangian (1) with the weakest experimental lim-
its providing additional phenomenological justification
for dropping the other SME coefficients from our analysis.
We finally mention that further constraints on $2c_{\mu \nu} - k^B_{\mu \nu}$
may, for example, also be determined with spectroscopic
studies of hydrogen [37] and ultrahigh-energy photon-
shower measurements [38].

Paralleling the conventional case, perturbative cal-
culations within the present model are conveniently per-
formed by fixing a gauge and allowing for the need to reg-
ularize infrared divergences. For the general description
of massive Lorentz-violating photons, a modified Stueck-
elberg procedure has recently been developed [39]. It es-
entially consists of amending any Lorentz-violating QED
Lagrange density by

$$
\Delta \mathcal{L} = -\frac{1}{4} \xi^{-1} (\partial_{\mu} \eta^\mu B) (\partial_{\nu} i c_{\mu \nu} A^B_\nu - m_B) \psi_B
+ \frac{1}{2} m^2 B_{\mu \nu} B^{\mu \nu} + \frac{1}{2} \xi (\partial_{\mu} \eta^\mu B) (\partial_{\nu} i c_{\mu \nu} A^B_\nu - m_B) \psi_B,
$$

(5)

where $\xi$ denotes a gauge parameter and $m_\gamma$ parametrizes
the gap in the photon dispersion relation. The tensorial
structure $\eta^\mu B \eta^\nu B + \delta \eta^\mu B$ can involve small, but
otherwise arbitrary Lorentz-breaking contributions $\delta \eta^\mu B$.
Note that both the $\xi$ and the $m_\gamma$ term need to contain the
same tensor $\eta^\mu B$ [39].

The choice of $\delta \eta^\mu B$ in the present context needs to be
compatible with the specific purpose of the $m_\gamma$ term as
a regulator: no Lorentz violation in addition to $c$ and $\tilde{k}$
should be introduced. One obvious possibility would be
the Lorentz-symmetric choice $\delta \eta^\mu B = 0$. Another possi-
bility is to match the Lorentz-violating structure of the
effective kinetic term of the photon. At leading order in $c$
and $\tilde{k}$, this kinetic term depends on the combination
$\eta^\mu B + \tilde{k}_{\mu \nu}$. If radiative corrections are included, the $c_{\mu \nu}$
coefficient may also appear, but only in the combination
$(2c_{\mu \nu} - \tilde{k}_{\mu \nu})$, as discussed above. These considerations
suggest the following choice for $\delta \eta^\mu B$:

$$
\delta \eta^\mu B = \tilde{k}_{\mu \nu} + f (2c_{\mu \nu} - \tilde{k}_{\mu \nu})
$$

(6)

Here, $f$ is a free multiplicative coefficient that may for
example be chosen to match the radiative corrections to
free-photon propagation. In the present work, it will be
convenient to select the choice (6). Since our primary
focus is the fermion two-point function, we will set $f = 0$.

Altogether, the above considerations lead to the follow-
ing bare Lagrange density [54]:

$$
\mathcal{L} = \psi_B \left[ i (\gamma^\mu c_{\mu \nu} + \eta^\mu B \eta^\nu B) (\partial_{\mu} + i e B_{\mu}) - m_B \right] \psi_B
- \frac{1}{4} \eta^\mu B \eta^\nu B F_{\mu \nu}^B F_{\rho \sigma}^B
- \frac{1}{2} \xi (\partial_{\mu} \eta^\mu B) (\partial_{\nu} i c_{\mu \nu} A^B_\nu - m_B) \psi_B
+ \frac{m^2}{2} B_{\mu \nu} B^{\mu \nu} + \frac{1}{2} \xi (\partial_{\mu} \eta^\mu B) (\partial_{\nu} i c_{\mu \nu} A^B_\nu - m_B) \psi_B.
$$

(7)

The next step is to define finite fields and couplings. To
this end, we employ the universal multiplicative renormal-
ization procedure with its Lorentz-violating generalization
established in Ref. [16]:

$$
\psi_B = \sqrt{Z_p} \psi, \quad A^B_\mu = \sqrt{Z_A} A^\mu,
\quad m_B = Z_m m, \quad e_B = Z_e e,
\quad c_{\mu \nu} = (Z_c)_{\alpha \beta} c_{\alpha \beta} \equiv (Z_c)_{\mu \nu},
\quad \tilde{k}_{\mu \nu} = (Z_k)_{\alpha \beta} \tilde{k}_{\alpha \beta} \equiv (Z_k)_{\mu \nu}.
$$

(8)

Adopting Feynman gauge ($\xi = 1$), working in $4 - \epsilon$
dimensions, and employing minimal subtraction, we have
at one-loop order [16]:
\[
Z_m = 1 - \frac{3e^2}{8\pi^2\epsilon}, \quad Z_e = 1 + \frac{e^2}{12\pi^2\epsilon}, \quad (9)
\]
\[
Z_\psi = 1 - \frac{e^2}{8\pi^2\epsilon}, \quad Z_A = 1 - \frac{e^2}{6\pi^2\epsilon}, \quad (10)
\]
\[
(Z_c)^{\mu\nu} = \delta^{\mu\nu} - \frac{e^2}{6\pi^2\epsilon}(2\delta^{\mu\nu} - k^{\mu\nu}), \quad (11)
\]
\[
(Z_k \tilde{k})^{\mu\nu} = \tilde{k}^{\mu\nu} + \frac{e^2}{6\pi^2\epsilon}(\tilde{k}^{\mu\nu} - 2k^{\mu\nu}). \quad (12)
\]

We remark that these expressions are compatible with the usual QED Ward–Takahashi identity \(Z_e \sqrt{Z_A} = 1\). In terms of the above physical couplings and fields, our model Lagrange density reads
\[
\mathcal{L}_{e\tilde{k}} = Z_\psi \psi \bar{\psi}
\left[
\frac{i}{2} (\gamma^{\mu} + (Z_c)^{\mu\nu} \gamma_{\nu}) + \left(\partial_{\mu} + iZ_{\psi} \sqrt{Z_A} e A_{\mu}\right) - Z_{m} m \right] \psi
\]
\[
- \frac{Z_A}{4} \left( (\eta^{\mu\nu} + (Z_k \tilde{k})^{\mu\nu}) (\eta^{\alpha\beta} + (Z_k \tilde{k})^{\alpha\beta}) F_{\mu\alpha} F_{\nu\beta} \right)
\]
\[
- \frac{Z_A}{2\epsilon} \left( (\eta^{\mu\nu} + (Z_k \tilde{k})^{\mu\nu}) \partial_{\mu} A_{\nu} \right)^2
\]
\[
+ \frac{Z_A m^2}{2} A_{\mu} \left( \eta^{\mu\nu} + (Z_k \tilde{k})^{\mu\nu} \right) A_{\nu}.
\]

III. THE FERMION TWO-POINT FUNCTION

Our main objective being the treatment of external fermion states in the context of Lorentz-violating field theory, we will turn in this section to the general procedure of determining the on-shell limit of the two-point function in our model (13). We are interested in particular in the Lorentz-breaking radiative corrections to this limit. In Lorentz-symmetric field theory, one extracts from the general off-shell two-point function the one-particle pole and its residue, which respectively determine the asymptotic single-particle solutions and the wave-function renormalization coefficient. Lorentz invariance strongly restricts the form of the different types of terms that can occur in the fermion two-point function. In the presence of the Lorentz-violating parameters \(e^{\mu\nu}\) and \(\tilde{k}^{\mu\nu}\), this procedure has to be generalized because more general terms can (and do) occur, the only fundamental restriction being that they are observer Lorentz scalars. What makes this generalization particularly non-trivial is the fact that some of the terms involving gamma matrices become noncommuting, an effect that does not occur in the usual Lorentz-symmetric case.

In a recent study [32], a generalization of the Källén–Lehmann spectral representation was derived for scalar and fermion field theories in the presence of a Lorentz-violating background of the type considered in this work. We will use the form derived for the one-particle pole in that work as a guiding principle for extracting the one-particle fermion pole in the present analysis.

We begin our general discussion of the two-point function with a few general remarks about the choice of perturbation scheme. An actual perturbative calculation of this function will have to wait until the next section.) To set up perturbation theory for calculating the radiative corrections to the two-point function we have to make a suitable choice of a zeroth-order system with known solutions, such that the remaining piece can be considered a small perturbation relative to this zeroth-order system. While usually one takes as the zeroth-order system the full quadratic part of the action, in the case at hand there are at least two reasonable choices one might consider.

In the first scheme, one defines as a basis the renormalized quadratic Lagrangian of the conventional Lorentz-symmetric case. All Lorentz-violating contributions to the Lagrangian are then taken as perturbations. One can, for example, use the minimal-subtraction scheme to define the counterterms (either Lorentz-symmetric or Lorentz-violating).

In the second scheme, one defines as a basis the full renormalized quadratic Lagrangian, including the Lorentz-violating part. The perturbations are then just the non-quadratic contributions to the Lagrangian. For the latter, it is convenient to join the corresponding Lorentz-symmetric and Lorentz-violating vertices in the same diagram. One can do this also for the counterterms.

A key difference between the two schemes concerns their kinematical features, which is best explained by an example. Consider the one-loop fermion self energy. In the conventional Lorentz-symmetric case, energy–momentum kinematics prohibits the internal photon and fermion lines from going on-shell simultaneously for physical incoming momenta. Let us next look at the leading-order Lorentz-violating generalization of this process in the above two schemes with focus on the special case with a photon \(k\) coefficient only.

In the first scheme, this process involves the conventional diagram plus another version of the diagram with a \(k\) insertion on the photon line represented by a box in Fig. 1. Both of these diagrams exhibit the same Lorentz-symmetric propagators and the same Lorentz-symmetric dispersion relation for external momenta. The energy–momentum kinematics is therefore unchanged relative to the conventional case. In particular, the internal photon and fermion propagators cannot go on-shell simultaneously for physical incoming momenta at this order in perturbation theory.
In the second scheme, there is a single diagram analogous to the conventional one, but with the usual photon propagator replaced by a Lorentz-violating propagator containing $\tilde{k}$, represented by a double photon line in Fig. 2. At this order, the dispersion relation of the incoming momentum remains Lorentz symmetric, but the $\tilde{k}$ propagator modifies the internal kinematics of the self-energy diagram. In particular, the internal photon and fermion lines can now go on-shell simultaneously for physical incoming momenta in certain regimes. This is evident from the well-established result [44] that photons that are slowed down by certain values of $\tilde{k}$ lead to vacuum Cherenkov radiation at ultrahigh energies: a real free fermion can now emit a real photon. The non-zero cross section for this effect is then directly related to imaginary contributions to the fermion self-energy via the optical theorem. Note that imaginary contributions are absent in the first scheme.

Such Cherenkov instabilities are relatively rare: they do not occur for all Lorentz-violating coefficients and are in any case only present at ultrahigh energies in our model. They correspond exactly to the two-particle regime discussed in Ref. [32], where the Källén–Lehmann representation of the fermion two-point function was analyzed in the presence of a $\epsilon^{\mu\nu}$-type Lorentz-violating perturbation. As was shown there, for ultrahigh momenta the two-point function can pass from a stable one-particle regime into an unstable two-particle regime and provoke a Cherenkov-type decay. In the case at hand, the latter consists of the fermion plus a photon.

In this work, we are primarily interested in the properties of external states before such rare instabilities lead to their decay. We therefore omit imaginary contributions to the fermion self-energy. If needed, the physics of such instabilities may still be included subsequently, for example via cut diagrams using Cutkosky’s rules [45]. For the above purpose, which disregards instabilities, the two schemes become equivalent—a fact we have verified explicitly at one-loop order. For definiteness, we present our analysis in the second scheme. It has the advantage of being more economical, as the number of diagrams is considerably reduced. In particular, the diagrams in this scheme are in one-to-one correspondence with the diagrams in the Lorentz-symmetric case. Moreover, the prescription for calculating the corresponding amplitude is more involved due to the Lorentz-violating coefficients, we have found that the full calculation is easier to carry out in the second scheme and is also less error-prone.

Explicitly, the second scheme implies that our model Lagrange density (13) is split into the following three pieces:

$$L^{\epsilon\tilde{k}} = L_0 + L_1 + L_2 ,$$

where

$$L_0 = \bar{\psi} \left[ i (\gamma^\mu + \epsilon^{\mu\nu} \gamma_\nu) \partial_\mu - m \psi - \frac{1}{4} \left( \eta^{\mu\nu} + \tilde{k}^{\mu\nu} \right) F_{\mu\nu} F_{\nu\mu} \right.$$

$$- \frac{1}{2\xi} \left( \partial_\mu A^\mu + \tilde{k}^{\mu\nu} \partial_\mu A_\nu \right)^2 + \frac{m^2}{2} A_\mu \left( \eta^{\mu\nu} + \tilde{k}^{\mu\nu} \right) A_\nu ,$$

$$L_1 = \bar{\psi} \left[ -e (\gamma^\mu + \epsilon^{\mu\nu} \gamma_\nu) A_\mu \right] \psi ,$$

and

$$L_2 = \bar{\psi} \left[ \left( (Z_\psi - 1) \eta^{\mu\nu} + Z_\psi (Z_\psi c)^{\mu\nu} - \epsilon^{\mu\nu} \right) i\gamma_\nu \left( \partial_\mu + ie A_\mu \right) - (Z_\psi Z_{m - 1}) \right] \psi$$

$$- \frac{1}{4} \left[ \left( Z_A - 1 \right) \eta^{\mu\nu} \eta^{\alpha\beta} + 2 \left( Z_A (Z_k \tilde{k})^{\mu\nu} - \tilde{k}^{\mu\nu} \right) \eta^{\alpha\beta} \right. $$

$$+ \left. \left( Z_A (Z_k \tilde{k})^{\mu\nu} (Z_k \tilde{k})^{\alpha\beta} - \tilde{k}^{\mu\nu} \tilde{k}^{\alpha\beta} \right) \right]$$

$$\times \left( F_{\mu\alpha} F_{\nu\beta} + \frac{2}{\xi} (\partial_\mu A_\nu) (\partial_\alpha A_\beta) \right)$$

$$+ \frac{m^2}{2} A_\mu \left[ \left( Z_A - 1 \right) \eta^{\mu\nu} + Z_A (Z_k \tilde{k})^{\mu\nu} - \tilde{k}^{\mu\nu} \right] A_\nu .$$

The corresponding Feynman rules, which are collected in Appendix A, now facilitate an order-by-order calculation of our model’s two-point function

$$\Gamma^{(2)} (p) = \Gamma^o p_\mu - m - \Sigma(p^2) .$$

Paralleling the conventional perturbative determination of this function, $\Sigma$ denotes the contribution of the one-particle irreducible Feynman diagrams. Before proceeding with an actual one-loop calculation, it is instructive to determine the general structure of $\Sigma$. We may decompose this quantity as

$$\Sigma = \Sigma_{\text{LI}} (p) + \Sigma_{\text{LV}} (p^2, c^2, \tilde{k}^2) + \delta (p^\mu, \epsilon^{\mu\nu}, \tilde{k}^{\mu\nu}) ,$$

where we have defined $c^2 = (\epsilon^{\mu\nu} \gamma_\mu p_\nu)$ and $\tilde{k}^2 = \tilde{k}^{\mu\nu} \gamma_\mu p_\nu$. In this decomposition, $\Sigma_{\text{LI}} (p)$ denotes the Lorentz-symmetric contributions equivalent to the conventional diagrams; it can thus only be a function of $\tilde{p}$, as usual:

$$\Sigma_{\text{LI}} (p) = f_0 (p^2) m + f_1 (p^2) \tilde{p} .$$
where both $f_0(p^2)$ and $f_1(p^2)$ are understood to depend on the fine-structure constant $\alpha = e^2/4\pi$ and the square of the 4-momentum $p^2$. The remaining two terms involve deviations from Lorentz symmetry, so we will describe them in more detail.

The second term $\Sigma_{LV}(p^2, c_{\mu}, k_{\mu})$ contains all those Lorentz-violating terms with a gamma-matrix structure that is already present in the fermion Lagrange density (i.e., a Lorentz-breaking symmetric traceless 2-tensor contracted with a gamma matrix and a single momentum factor). For example, $\Sigma_{LV}$ includes the counterterms for $e^{\mu\nu}$ and $\tilde{k}^{\mu\nu}$ together with the corresponding (regulated) infinities they cancel to yield an ultraviolet finite expression:

$$\Sigma_{LV}(p^2, c_{\mu}, k_{\mu}) = f_2^c(p^2) c_{\mu} + f_2^k(p^2) k_{\mu},$$

where $f_2^c(p^2)$ and $f_2^k(p^2)$ depend only on $\alpha$ and $p^2$ since we are working at leading order in Lorentz violation. Explicit expressions for $f_2^c(p^2)$ and $f_2^k(p^2)$ can in principle be determined within perturbation theory to any given order in $\alpha$. Below we will calculate $f_2^c(p^2)$ and $f_2^k(p^2)$ at one loop, i.e., at $O(\alpha)$. Initially, $f_2^c(p^2)$ and $f_2^k(p^2)$ may also depend on an infrared regulator and an arbitrary mass scale introduced by the chosen ultraviolet regularization procedure. But a consistent treatment of infrared effects and the renormalization conditions should remove free parameters from $\Sigma_{LV}(p^2, c_{\mu}, k_{\mu})$.

The remaining term $\delta \Sigma(p^\mu, c^{\mu\nu}, k^{\mu\nu})$ contains novel Lorentz-breaking structures that are not already present in the original Lagrange density (13). Like the second term, it must involve combinations of $p^{\mu}$ factors, $\gamma$ matrices, and—since we are working at linear order in Lorentz violation—a single $c^{\mu\nu}$ or $\tilde{k}^{\mu\nu}$ coefficient. Up to factors consisting of powers of $p^2$, a multitude of terms can be constructed that satisfy these requirements. They are

$$c^{\mu\nu} p_{\mu} p_{\nu}, \quad \bar{p} c^{\mu\nu} p_{\mu} p_{\nu}, \quad \gamma^5 c^{\mu\nu} p_{\mu} p_{\nu}, \quad \gamma^5 \bar{p} c^{\mu\nu} p_{\mu} p_{\nu},$$

$$\gamma^5 c^{\mu\nu} \gamma_{\mu} p_{\nu}, \quad \sigma^{\mu\nu} \gamma_{\mu} p_{\nu}, \quad \gamma^5 \sigma^{\mu\nu} \gamma_{\mu} p_{\nu},$$

as well as an additional seven terms with $c^{\mu\nu}$ replaced by $k^{\mu\nu}$ [41].

The above list (22) can be constrained further by noting that electromagnetic interactions preserve $\gamma$, $P$, and $T$. Quantum corrections linear in $c^{\mu\nu}$ and $k^{\mu\nu}$ must exhibit the same discrete symmetries as the original Lorentz-violating operators. This fact together with our scope set out earlier (i.e., omitting instabilities and thus nonhermitian expressions) leaves only the first two terms in the list (22) and their $k^{\mu\nu}$ analogues [43]. For this reason, $\delta \Sigma(p^\mu, c^{\mu\nu}, k^{\mu\nu})$ can only depend on $c_{\mu} \equiv c^{\mu\nu} p_{\mu} p_{\nu}$ and $k_{\mu} \equiv k^{\mu\nu} p_{\mu} p_{\nu}$, and we may write

$$\delta \Sigma(p^\mu, c_{\mu}, k_{\mu}) = f_3^c(p^2) \frac{c_{\mu}}{m^2} + f_3^k(p^2) \frac{k_{\mu}}{m^2} \bar{p} p_{\mu} + f_4^k(p^2) \frac{k_{\mu}}{m^2} \bar{p} k_{\mu} p_{\nu} + f_4^c(p^2) \frac{c_{\mu}}{m^2} \bar{p} c_{\mu} p_{\nu}.$$

Here, we have introduced the dimensionless functions $f_3^c(p^2)$, $f_3^k(p^2)$, $f_4^c(p^2)$, and $f_4^k(p^2)$, which can in principle be calculated within perturbation theory to any given order in $\alpha$. These functions may initially still contain infrared regulators, which can presumably be removed by a soft-photon treatment. Disregarding the aforementioned possibility of high-energy nonhermitian contributions, Eqs. (19), (20), (21), and (23) determine the full off-shell structure of the fermion two-point function in our $ck$ model (13) at all orders in $\alpha$ and at linear order in Lorentz violation.

Before deriving explicit expressions for the scalar functions appearing in the corrections (20), (21), and (23)—a task to which we will turn in Sec. IV—it is instructive to construct a general procedure for extracting the on-shell external-leg physics determined by the structure of these corrections. The standard method in the Lorentz-invariant case is to consider $\Gamma^{(2)}(p)$ as a function of $\bar{p}$, and to determine the value of $\bar{p} = m_{\text{phys}}$ for which the two-point function vanishes. Laurent-expanding the inverse of the latter around $\bar{p} = m_{\text{phys}}$ yields the coefficient of the simple pole (residue), which determines the field-strength renormalization.

Unfortunately, for the Lorentz-violating expression (19) the situation is more complicated: $\Gamma^{(2)}(p)$ depends not only on $\bar{p}$, but also on $c_{\mu}$, $k_{\mu}$, as well as on the Lorentz-violating coordinate scalars $c_{\mu}$ and $k_{\mu}$. Consequently, the propagator can be expected to have a pole in terms of a value of some functional combination of all those quantities, rather than $\bar{p}$ only. Of course, this is already the case before introducing loop effects, as $\Gamma^{(2)}(p)$ depends on $c_{\mu}$ as well as $\bar{p}$.

Let us briefly review the way the propagator pole is extracted in the Lorentz-invariant case. Displaying the $4 \times 4$ identity matrix $\mathbb{I}$ for clarity, one can then write $\Gamma^{(2)}(p)$ as a function of $\bar{p}$ only

$$\Gamma^{(2)}_{L,I}(p) = A(p^2) \bar{p} + B(p^2) \mathbb{I} \equiv A(\bar{p}) \bar{p} + B(\bar{p}) \mathbb{I} \quad (24)$$

and determine the physical mass $m_{\text{phys}}$ by requiring

$$\Gamma^{(2)}_{L,I}(p) \bigg|_{\bar{p} = m_{\text{phys}}} = 0. \quad (25)$$

One can then expand

$$\Gamma^{(2)}_{L,I}(p) = Z_R^{-1}(\bar{p} - m_{\text{phys}}) + \Sigma_2(\bar{p})(\bar{p} - m_{\text{phys}})^2, \quad (26)$$

where $Z_R$ is the wave-function renormalization and $\Sigma_2(\bar{p})$ a (finite) function.

How can this procedure be generalized to the Lorentz-violating case? The obstacle arises due to the form of the last term in Eq. (26): the inclusion of structures that involve $c_{\mu}$ and $k_{\mu}$ becomes ambiguous, as these expressions do not commute with $\bar{p}$. Nevertheless, it is possible to make a meaningful generalization. First of all, we write the momentum-space two-point function in the most gen-
eral possible form

\[ \Gamma^{(2)}(p) = A(p^2, (c, k)_p^p)c_p^p + C(p^2, (c, k)_p^p)c_p^p \\
+ K(p^2, (c, k)_p^p)k_p^p - M(p^2, (c, k)_p^p). \tag{27} \]

Here, the expressions \( A, C, K, \) and \( M \) are functions of \( p^2 \) and \((c, k)_p^p, \) where the latter are quantities of the form

\[ (t_1 \cdot t_2 \cdots \cdot t_n)_p^p, \quad n \geq 1, \tag{28} \]

with \( t_1^{\mu \nu} \) equaling either \( e^{\mu \nu} \) or \( \bar{k}^{\mu \nu}. \)

Following Ref. [32], it should be possible to extract from the full fermion propagator a pole of the form

\[ \bar{P}(p) = \bar{p} - \bar{m}((c, k)_p^p) + \bar{x}_c((c, k)_p^p)c_p^p + \bar{x}_k((c, k)_p^p)\bar{k}_p^p \\
+ \bar{x}_{cc}((c, k)_p^p)(c^2)_p^p + \bar{x}_{ck}((c, k)_p^p)(c \cdot \bar{k})_p^p + \ldots, \tag{29} \]

where the ellipsis stands for terms proportional to

\[ (t_1 \cdot t_2 \cdots \cdot t_n)_p^p, \tag{30} \]

with \( t_1^{\mu \nu} \) again equal to either \( e^{\mu \nu} \) or \( \bar{k}^{\mu \nu} \) (cf. definition (28) above). Note that the coefficient functions \( \bar{x}_{c(k)} \) and \( \bar{m} \) only depend on the quantities (28), not on \( p^2. \)

In this paper, we will be working only to first order in Lorentz violation. Consequently, (29) reduces to

\[ \bar{P}(p) = \bar{p} - \bar{m} + \bar{x}c_p^p + \bar{y}\bar{k}_p^p, \tag{31} \]

where we can take

\[ \bar{x} \equiv \bar{x}_c = x_0, \quad \bar{y} \equiv \bar{x}_k = y_0, \tag{32} \]

and

\[ \bar{m} \approx m_0 + m_{10}c_p^p + m_{01}\bar{k}_p^p. \tag{33} \]

We should expect to be able to write down a generalization of relation (26) with \( \Gamma^{(2)}(p) \) being equal to a sum of a term linear in \( \bar{P}(p) \) and a rest term of quadratic order. This task is complicated by the presence of the quantities \( c_p^p \) and \( \bar{k}_p^p, \) which do not commute with \( \bar{p}. \) But it turns out that a self-consistent generalization of Eq. (26) to the Lorentz-violating case is given by

\[ \Gamma^{(2)}(p) = Z_R^{-1}((c, k)_p^p)\bar{P}(p) \\
+ \bar{P}(p)\Sigma_2(\bar{p}, c_p^p, \bar{k}_p^p, (c, k)_p^p)\bar{P}(p), \tag{34} \]

where \( \Sigma_2(\bar{p}, c_p^p, \bar{k}_p^p, (c, k)_p^p) \equiv \Sigma_2(p) \) is some Lorentz-violating expression involving the momentum. Note the order of the noncommuting functions \( \bar{P}(p) \) and \( \Sigma_2(p) \) in the second term. The function \( Z_R((c, k)_p^p) \) is the generalization of the wave-function renormalization constant in the Lorentz-violating case. Its form is in accordance with the results in Ref. [32]. Indeed, Eq. (34) implies that

\[ \Gamma^{(2)}(p)^{-1} = \]

\[ = Z_R((c, k)_p^p)\left[ \bar{P}(p)\left(1 - Z_R((c, k)_p^p)\Sigma_2(p)\bar{P}(p)\right)\right]^{-1} \\
= Z_R((c, k)_p^p)\left[1 + Z_R((c, k)_p^p)\Sigma_2(p)\bar{P}(p) + \ldots \right]\bar{P}(p)^{-1} \\
= Z_R((c, k)_p^p)\left[\bar{P}(p)^{-1} + Z_R((c, k)_p^p)\Sigma_2(p) \bar{P}(p) + \ldots \right] \\
= Z_R((c, k)_p^p)\bar{P}(p)^{-1} + \text{finite}, \tag{35} \]

so that \( \bar{P}(p)^{-1} \) and \( \Gamma^{(2)}(p)^{-1} \) have the same pole structure.

To determine the coefficient functions \( \bar{x}, \bar{y}, \) and \( \bar{m} \) in Eq. (31), we write

\[ \bar{p} = \bar{P} - \bar{x}c_p^p - \bar{y}\bar{k}_p^p + \bar{m}, \tag{36} \]

which leads to

\[ p^2 = \bar{p}p = \bar{P} \bar{P} + 2\bar{m} \bar{P} + \bar{\beta}, \tag{37} \]

where

\[ \bar{\beta} = \bar{m}^2 - 2\bar{x}c_p^p - 2\bar{y}\bar{k}_p^p - \bar{x}^2(c^2)_p^p - \bar{y}^2(\bar{k}^2)_p^p - 2\bar{x}\bar{y}(c \cdot \bar{k})_p^p \\
\approx \bar{m}^2 - 2\bar{x}c_p^p - 2\bar{y}\bar{k}_p^p. \tag{38} \]

We can now rewrite

\[ \Gamma^{(2)}(p) = \frac{1}{2}\{A, \bar{P} - \bar{x}c_p^p - \bar{y}\bar{k}_p^p + \bar{m}\} \\
+ \frac{1}{2}\{C, c_p^p\} + \frac{1}{2}\{K, \bar{k}_p^p\} - M \tag{39} \]

(note that the coefficient functions \( A, C, \) and \( K \) commute with the gamma matrices) and evaluate the anticommutators. In the above relation, we consider \( A \equiv A(\bar{P} \bar{P} + 2\bar{m} \bar{P} + \bar{\beta}, (c, k)_p^p) \) with analogous definitions for \( C, K, \) and \( M. \) Using that

\[ \{\bar{P}, c_p^p\} = 2c_p^p + 2\bar{x}(c^2)_p^p + 2\bar{y}(c \cdot \bar{k})_p^p - 2\bar{m}c_p^p \]

\[ \approx 2c_p^p - 2\bar{m}c_p^p \tag{40} \]

and

\[ \{\bar{P}, \bar{k}_p^p\} \approx 2\bar{k}_p^p - 2\bar{m}\bar{k}_p^p, \tag{41} \]

the anticommutators in Eq. (39) can be worked out. It then follows that

\[ \Gamma^{(2)}(p) \approx \left\{ \bar{m} A - M + (C - \bar{x} A)c_p^p + (K - \bar{y} A)\bar{k}_p^p \right. \\
+ \bar{P}\left[A + 2\bar{m}(\partial \bar{t} A - \partial \bar{t} M) \right. \\
+ 2(\partial \bar{t} C - \bar{x} \partial \bar{t} A)c_p^p + 2(\partial \bar{t} K - \bar{y} \partial \bar{t} A)\bar{k}_p^p \right] \\
+ \bar{P}[\ldots] \bar{P}, \tag{42} \]

where \( Z_R((c, k)_p^p) \) is the generalization of the wave-function renormalization constant in the Lorentz-violating case. Its form is in accordance with the results in Ref. [32]. Indeed, Eq. (34) implies that
where the first argument of the coefficient functions (originally $p^2$) should be taken equal to $\beta$. We see from Eq. (37) that this amounts to taking $p^2$ on the mass shell defined by $P = 0$. The symbol $\partial_1$ indicates a partial derivative with respect to the first argument.

It is apparent that $\Gamma^{(2)}(p)$ in Eq. (42) takes the expected form of Eq. (34), provided the relations

\[ \bar{m} A(\beta, (c, \bar{k})_p^2) = M(\beta, (c, \bar{k})_p^2), \]
\[ \bar{x} A(\beta, (c, \bar{k})_p^2) = C(\beta, (c, \bar{k})_p^2), \]
\[ \bar{y} A(\beta, (c, \bar{k})_p^2) = K(\beta, (c, \bar{k})_p^2) \]

are satisfied. Moreover, in Eq. (42) we can identify the expression in square brackets multiplying $\bar{y}$ with the inverse of the wave-function renormalization $Z_R((c, \bar{k})_p^2)$.

Equations (43)–(45) determine the three coefficient functions $\bar{x}$, $\bar{y}$, and $\bar{m}$. Note that these are implicit relations, as $\beta$, which is an argument of the functions $A$, $M$, $C$, and $K$, depends itself on $\bar{x}$, $\bar{y}$, and $\bar{m}$.

IV. ONE-LOOP CALCULATION OF THE MODIFIED PROPAGATOR

An interesting question concerns the determination of the functions $f_0(p^2)$, $f_1(p^2)$, $f_2(p^2)$, $f_3(p^2)$, $f_3'(p^2)$, $f_4(p^2)$, and $f_4'(p^2)$ perturbatively at leading order in the fine-structure constant $\alpha$. To this end, we will adopt the perturbative scheme based on the expressions (14)–(17), in which the propagators are built from the full quadratic Lagrange density, including the Lorentz-violating parts. The corresponding Feynman rules are presented in Appendix A, and the only loop diagram involved in the fermion self-energy is:

\[
-i \Sigma_{\text{loop}}(p) = \begin{array}{c}
\text{Diagram}
\end{array}
\]

This diagram together with the corresponding counterterm contributions gives the conventional Lorentz-symmetric $\mathcal{O}(\alpha)$ results

\[ f_0(p^2) = \frac{\alpha}{\pi} \left[ -\frac{1}{2} - \gamma_E - \int_0^1 dy \ln \left( \frac{\Delta}{4\pi \mu^2} \right) \right], \]
\[ f_1(p^2) = \frac{\alpha}{4\pi} \left[ 1 + \gamma_E + 2 \int_0^1 dy (1 - y) \ln \left( \frac{\Delta}{4\pi \mu^2} \right) \right] \]

Here,

\[ \Delta = -y(1 - y)p^2 + y(m^2 - m_\gamma^2) + m_\gamma^2, \]
\[ \gamma_E = 0.57721 \ldots \] denotes the Euler–Mascheroni constant, $m_\gamma$ a fictitious photon mass introduced as an infrared regulator, and the arbitrary mass scale $\mu$ is a remnant from dimensional regularization. The integrations over $y$ can be performed requiring $p^2$ to be close to the mass shell: $(m - m_\gamma)^2 < p^2 < (m + m_\gamma)^2$. They yield the infrared finite limits on the (conventional) mass shell

\[ f_0(m^2) = \frac{\alpha}{\pi} \left[ \frac{3}{2} - \gamma_E - \ln \left( \frac{m^2}{4\pi \mu^2} \right) \right], \]
\[ f_1(m^2) = \frac{\alpha}{4\pi} \left[ 1 - \frac{1}{2} + \frac{\gamma_E}{4} + \frac{1}{4} \ln \left( \frac{m^2}{4\pi \mu^2} \right) \right]. \]

For the on-shell values of their derivatives we find

\[ f'_0(m^2) = \frac{\alpha}{\pi m^2} \left[ \ln \left( \frac{m}{m_\gamma} \right) - 1 \right], \]
\[ f'_1(m^2) = \frac{\alpha}{\pi m^2} \left[ -\frac{1}{2} \ln \left( \frac{m}{m_\gamma} \right) + \frac{3}{4} \right]. \]

An explicit calculation also yields ultraviolet-finite expressions for the remaining, Lorentz-violating contributions to $\Sigma$. For the functions $f_i'(p^2)$ we obtain:

\[ f_2'(p^2) = \frac{\alpha}{2\pi} \left[ \frac{1}{6} - \frac{5\gamma_E}{6} \right], \]
\[ f_3'(p^2) = \frac{2\alpha m^2}{\pi} \int_0^1 dy (1 - y)(1 + 2y) \ln \left( \frac{\Delta}{4\pi \mu^2} \right), \]
\[ f_4'(p^2) = -\frac{\alpha m^2}{\pi} \int_0^1 dy (1 - y)^3 \]

While $f_2'(p^2)$ is infrared finite, this is not the case for $f_3'(p^2)$ and $f_4'(p^2)$: the latter both diverge for $p^2 = m^2$ when taking $m_\gamma \to 0$:

\[ f_2'(m^2) = \frac{\alpha}{\pi} \left[ \frac{10}{9} - \frac{5\gamma_E}{12} - \frac{5}{12} \ln \left( \frac{m^2}{4\pi \mu^2} \right) \right], \]
\[ f_3'(m^2) = \frac{\alpha}{\pi} \left[ 2 \ln \left( \frac{m}{m_\gamma} \right) - 3 \right], \]
\[ f_4'(m^2) = \frac{\alpha}{\pi} \left[ -\ln \left( \frac{m}{m_\gamma} \right) + \frac{11}{6} \right]. \]

Below, we will also need the derivatives of these functions at their on-shell value $p^2 = m^2$. To this effect, we use that

\[ \frac{d}{dp^2} \ln \Delta = -\frac{y(1 - y)}{\Delta}, \quad \frac{d}{dp^2} \frac{1}{\Delta} = \frac{y(1 - y)}{\Delta^2}. \]

Applying these relations to (54)–(56), and evaluating the integrals at their on-shell value one finds

\[ f_2''(m^2) = \frac{\alpha}{2\pi m^2} \left[ \ln \left( \frac{m}{m_\gamma} \right) - \frac{5}{6} \right], \]
\[ f_3''(m^2) = \frac{\alpha}{\pi m^2} \left[ \pi \frac{m}{2m_\gamma} + 6 \ln \left( \frac{m}{m_\gamma} \right) + 7 \right], \]
\[ f_4''(m^2) = \frac{\alpha}{\pi m^2} \left[ -\frac{\pi m}{4m_\gamma} + 4 \ln \left( \frac{m}{m_\gamma} \right) - \frac{35}{6} \right]. \]
Note that $f_2'(p^2)$ and $f_3'(p^2)$ have linear, rather than logarithmic, infrared divergences.

For the coefficients $f_i^k(p^2)$ one finds:

$$f_2^k(p^2) = \frac{\alpha}{\pi} \left( \frac{1}{12} + \frac{\gamma_E}{3} + \frac{1}{4} \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right),$$

$$f_3^k(p^2) = \frac{\alpha m^2}{\pi} \int_0^1 dy \frac{y^2(1-y)}{\Delta},$$

$$f_4^k(p^2) = -\frac{\alpha m^2}{2\pi} \int_0^1 dy \frac{y^2(1-y)^2}{\Delta}.$$  

All three coefficients $f_i^k(p^2)$ are infrared finite on the conventional mass shell:

$$f_2^k(m^2) = \frac{\alpha}{\pi} \left( \frac{29}{36} + \frac{\gamma_E}{3} + \frac{1}{3} \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right),$$

$$f_3^k(m^2) = \frac{\alpha}{2\pi},$$

$$f_4^k(m^2) = -\frac{\alpha}{6\pi}.$$  

Applying the relations (60) to Eqs. (64)–(66) and evaluating the integrals at $p^2 = m^2$ one finds the infrared-divergent expressions

$$f_2^{k'}(m^2) = \frac{\alpha}{\pi m^2} \left( -\frac{1}{2} \ln \left( \frac{m}{m_\gamma} \right) + \frac{7}{12} \right),$$

$$f_3^{k'}(m^2) = \frac{\alpha}{\pi m^2} \left( -\ln \left( \frac{m}{m_\gamma} \right) - 2 \right),$$

$$f_4^{k'}(m^2) = \frac{\alpha}{\pi m^2} \left( -\frac{1}{2} \ln \left( \frac{m}{m_\gamma} \right) + \frac{7}{6} \right).$$  

Let us now see how the formalism developed in Sec. III can be used to incorporate the radiative corrections derived above. First, let us express the quantities $A, C, K,$ and $M$ in Eq. (27) in terms of the coefficients $f_i^k$ and $f_i^{k'}$ calculated above. Explicitly, we have

$$A = 1 - f_1(p^2) - f_2^k(p^2) \frac{\epsilon p}{m^2} - f_3^k(p^2) \frac{\tilde{k} p}{m^2} + \ldots,$$

$$C = 1 - f_2^k(p^2) + \ldots,$$

$$K = -f_2^k(p^2) + \ldots,$$

$$M = m(1 + f_0(p^2)) + f_3^k(p^2) \frac{\epsilon p}{m} + f_4^k(p^2) \frac{\tilde{k} p}{m} + \ldots.$$  

Using the expansions (32) and (33) it follows, by taking Eq. (43) to zeroth order in Lorentz violation, that

$$m(1 - f_0(m_0^2)) = m(1 + f_0(m_0^2)).$$

As the functions $f_i$ are $O(\alpha)$, we obtain

$$m_0 \approx m(1 + f_0(m_0^2) + f_1(m_0^2))$$

$$\approx m \left( 1 + \frac{\alpha}{\pi} \left( \frac{3\gamma_E}{4} - \frac{3}{4} \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right) \right).$$  

Taking relations (44) and (45) to zeroth order in Lorentz violation, it follows that

$$x_0 \approx 1 + f_1(m^2) - f_2^k(m^2)$$

$$= 1 + 2\frac{\alpha}{\pi} \left( \frac{29}{36} + \frac{\gamma_E}{3} + \frac{1}{3} \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right),$$

$$y_0 \approx -f_2^k(m^2)$$

$$= -\alpha \frac{29}{36} + \frac{\gamma_E}{3} + \frac{1}{3} \ln \left( \frac{m^2}{4\pi\mu^2} \right).$$  

We continue by extracting $m_{10}$ and $m_{01}$ from Eq. (44), taken to first order in Lorentz violation. We now have to expand

$$f_1(\beta) = f_1(m^2) - 2c_p \alpha + O(\alpha),$$

$$f_1(m^2) = f_1(m^2) - 2c_p f_1'(m^2) + O(\alpha),$$

so that

$$A(\tilde{\beta},(c,k)p) = 1 - f_1(m^2) + c_p \left( 2f_1'(m^2) - f_3'(m^2) \right) - \frac{\tilde{k} p f_2^k(m^2)}{m^2} + O(\alpha^2,(c,k)^2),$$

and

$$M(\tilde{\beta},(c,k)p) = m \left[ 1 + f_0(m^2) + \frac{\tilde{k} p f_2^k(m^2)}{m^2} + c_p \left( f_3^k(m^2) - 2f_0'(m^2) \right) \right].$$

Expanding $\tilde{m} = A(\tilde{\beta},\ldots)^{-1} M(\tilde{\beta},\ldots)$ and taking the coefficients of $c_p$ and $\tilde{k} p$ it follows that

$$m_{10} = -2m(f_0'(m^2) + f_1'(m^2)) + \frac{1}{m} (f_3'(m^2) + f_4'(m^2))$$

$$= -\frac{2\alpha}{3\pi m},$$

$$m_{01} = \frac{1}{m} (f_3^k(m^2) + f_4^k(m^2))$$

$$= \frac{\alpha}{3\pi m}.$$  

These results allow us to evaluate the explicit form of the wave-function renormalization $Z_R((c,k)p)$, which is the
inverse of the coefficient of $\tilde{P}$ in Eq. (42):

$$Z_{\alpha}^{-1}((c, k)^p_p) =$$

$$= \left[ A + 2m(m \partial_1 A - \partial_1 M) + 2(\partial_1 C - x \partial_1 A )c_p^\mu + 2(\partial_1 K - y \partial_1 A )\tilde{p}_{\rho} \right] p^\rho + \tilde{p}_{\beta}$$

$$= 1 - f_1(m^2) - 2m^2(f_2^\mu(m^2) + f_3^\mu(m^2))$$

$$+ c_p^\mu \left[ f_2^\mu(m^2) + 4m^2(f_2'^\mu(m^2) + f_2''^\mu(m^2)) - \frac{1}{m^2} f_2^\mu(m^2) \right]$$

$$+ \tilde{p}_{p} - 2(f_2^\mu(m^2) + f_2'^\mu(m^2) + f_2''^\mu(m^2))$$

$$- \frac{1}{m^2} f_2^\mu(m^2)$$

$$\approx 1 - \frac{2\alpha}{\pi} \left[ \ln \left( \frac{m}{m_\gamma} \right) + \frac{\gamma E}{4} + \frac{1}{4} \ln \left( \frac{m^2}{4\pi^2} \right) \right]$$

$$- \frac{2\alpha}{\pi} \left[ 2(c_p^\mu - \tilde{p}_p) \right].$$

In order to evaluate the expressions $\partial_1 A(\beta)$ and $\partial_1 C(\beta)$ in Eq. (87) to the required order it is necessary to Taylor-expand the functions $f_1^\mu(\beta)$ and $f_2^\mu(\beta)$ just like in Eq. (81). The Lorentz-symmetric part of $Z_{\alpha}((c, k)^p_p)$ is identical to the usual (one-loop) wave-function renormalization constant for the fermion in QED. In particular, it has a logarithmic infrared divergence. The linear and logarithmic infrared divergences that are present in various terms proportional to $c_p^\mu$ and $\tilde{p}_p$ in Eq. (87) cancel in the final expression Eq. (88). As is well known, the Lorentz-symmetric infrared divergences cancel in scattering cross sections when we take into account the contributions of soft-photon emission from the corresponding external legs. In Sec. VI, we will check that these contributions do not introduce additional infrared divergences proportional to $c_p^\mu$ or $\tilde{p}_p$.

We are now in a position to determine the radiative corrections to the fermion dispersion relation. It is easy to check that the dispersion relation that follows from $\tilde{P}(p) = 0$ is (to first order in the Lorentz-violating coefficients and $\alpha$)

$$0 = p^2 + 2\tilde{p}_{\rho} + 2\tilde{y}_{\rho} - m^2$$

$$= p^2 + 2(x_0 - m m_{10})c_p^\mu + 2(y_0 - m m_{01})\tilde{p}_{\rho} - m^2$$

$$= p^2 + 2c_p^\mu - m^2 \left( 1 - \frac{2\alpha}{\pi} \left[ \frac{1}{4} \ln \left( \frac{m^2}{4\pi^2} \right) \right] \right)$$

$$+ \frac{2\alpha}{\pi} \left[ \frac{1}{3} \right] \frac{\gamma E}{4} + \frac{1}{3} \ln \left( \frac{m^2}{4\pi^2} \right) \left[ 2c_p^\mu - \tilde{p}_p \right].$$

It is gratifying to note that the $O(\alpha)$ Lorentz-violating radiative corrections to the dispersion relation turn out to be proportional to the combination $(2c_p^\mu - \tilde{p}_p)$, in accordance with the general expectation (see Sec. II).

At this point it is useful to give a physical interpretation of the various quantities that were calculated in this section. From Eqs. (31)–(33), (78)–(80), (84), and (85), we see that the loop-corrected (Dirac-)operator $\tilde{P}(p)$ can be written, to first order in $c^\mu\nu$, $\tilde{k}^\mu\nu$, and in the fine-structure constant $\alpha$ as

$$\phi + (c_{\text{phys}})^p_p - \frac{\alpha}{3\pi m} \left[ 2(c_{\text{phys}})^p_p - (\tilde{k}_{\text{phys}})^p_p \right],$$

where

$$m_{\text{phys}} = m_0 \left[ 1 + \frac{\alpha}{\pi} \left( \frac{3}{4} \gamma E - \frac{3}{4} \ln \left( \frac{m^2}{4\pi^2} \right) \right) \right]$$

is the usual loop-corrected mass in the minimal-subtraction scheme, and

$$(c_{\text{phys}})^{\mu\nu} = c^\mu_{\nu} + \frac{\alpha}{\pi} \left[ -\frac{29}{36} + \frac{\gamma E}{3} + \frac{1}{3} \ln \left( \frac{m^2}{4\pi^2} \right) \right] (2\tilde{c}^\mu_{\nu} - \tilde{k}^\mu_{\nu}).$$

Equation (92) expresses the radiatively corrected, physical value of the Lorentz-violating parameter $c^\mu\nu$ in terms of its tree-level value. Note that both $c^\mu\nu$ and the mass scale $\mu$ are unphysical, renormalization-scheme-dependent quantities, unlike $(c_{\text{phys}})^{\mu\nu}$, which is in principle measurable. This situation is completely analogous to relation (91) expressing $m_{\text{phys}}$ in terms of $m$ and $\mu$.

In Eq. (90), we have anticipated the loop-corrected version $(\tilde{k}_{\text{phys}})^{\mu\nu}$ of the parameter $\tilde{k}^\mu\nu$. While we do not compute this expression in this work, it will of course differ from the tree-level value by an expression of order $\alpha$, so that it can be substituted for the latter in Eq. (90), as we disregard terms of order $\alpha^2$.

The last term in Eq. (90) is a radiative correction in the form of an operator that is of a different form than any of the terms present in the tree-level Dirac-operator

$$\tilde{P}_{\text{tree}}(p) = \phi + c_p^\mu - m.$$

This might seem surprising, as such a situation never arises in the Lorentz-symmetric case. However, it is fully compatible with the general form of the one-particle fermion pole of the Källén–Lehmann representation derived in Ref. [32]. This ($\mu$-independent) correction constitutes one of the central results of this work.

It is pleasing to note that the procedure outlined in this section deals nicely with the potential infrared divergences that are present in some of the coefficients $f_i$ and their derivatives. We see that the dispersion relation (89) is infrared finite. One can also readily verify that, at least up to the order considered (linear in $c, k$ and in $\alpha$), this is in fact the case for all coefficients $A(\bar{\beta}, (c, k)^p_p)$, $M(\bar{\beta}, (c, k)^p_p)$, $C(\bar{\beta}, (c, k)^p_p)$, and $K(\bar{\beta}, (c, k)^p_p)$, and hence for the coefficients $\bar{x}, \bar{y}$, and $\bar{m}$. Note the cancellation of infrared divergences in the expression multiplying $c_p^\mu$, both in Eq. (82) and Eq. (83). We conjecture that infrared divergences will continue to cancel at any order in the perturbative expansion.
Passing the momenta to derivatives, it follows that the
Lagrangian describing the on-shell free fermion field ac-
quires higher spacetime derivatives. This is the case for
either the Lagrangian derived from the original proper
two-point function or for the one derived from the op-
erator \( P(p) \). Often it might be possible to avoid working
directly with the physical field and employ the bare field
instead treating the higher-dimensional Lorentz-violating
terms perturbatively. However, this becomes problematic
or impossible if we consider on-shell external states, as
in the derivation of the Lehmann–Symanzik–Zimmermann
reduction formula, which computes scattering amplitudes
for on-shell external physical states. In the next section,
we will analyze how this situation generalizes to loop-
corrected Lorentz-violating Lagrangians.

V. EXTERNAL STATES IN FEYNMAN
DIAGRAMS AND THE LSZ FORMULA

How do the Lorentz-violating radiative corrections
parametrized by the coefficient functions \( \tilde{x} \), \( \tilde{y} \), and \( \tilde{m} \)
in Eq. (31) contribute to S-matrix elements? Let us reflect
a moment on how we can determine the latter.

In the quantum-field description of scattering experi-
ments, it is presupposed that the Fock space of physical
states is generated from a unique vacuum by free fields
\( \psi_{in}(x) \) and \( \tilde{\psi}_{in}(x) \) (here we will only consider fermions
in the asymptotic states and ignore the possibility of
photons). One assumes that the coupling terms in the
equations of motion are affected by some adiabatic cut-
off function equal to unity at finite times and vanishing
smoothly as \( |t| \to \infty \), and the particles in the initial and
final states have become well separated. Then, according
to the usual adiabatic hypothesis the interacting fields
\( \psi(x) \) and \( \tilde{\psi}(x) \) are presumed to satisfy, in a weak sense,

\[
\psi(x) \to Z^{1/2} \psi_{in}(x) \quad \text{as} \quad t \to -\infty \quad (94)
\]

(and similarly for \( \tilde{\psi}(x) \)) for some normalization constant
\( Z \) that should be smaller than one, in order to account
for the fact that the content of the state \( \psi(x)|0\rangle \) is not ex-
hausted by the matrix elements with one-particle states,
while \( \psi_{in}|0\rangle \) is.

In Sec. IV, we saw that for the Lorentz-violating model
we are considering the normalization constant analogous
to the constant \( Z \) in Eq. (94) is not only Lorentz viol-
ating, but becomes dependent on the momentum of the
external particle: \( Z \to Z_R(p) \), see Eqs. (87) and (88).
To see how this will affect the usual treatment of external
states in scattering amplitudes, let us begin by looking at
the free field \( \psi_{in}(x) \) (the out-field \( \psi_{out}(x) \)
will be analogous). Consider the spinor wave functions
\( \psi_{in}^{\ast}(\tilde{p}) \) and \( \psi_{in}^{\ast}(\tilde{p}) \) of the physical field \( \psi_{in}(x) \). They are
modified with respect to the Lorentz-invariant situation.
While in the latter case we have (\( p - m \))\( \psi_{in}^{\ast}(\tilde{p}) = 0 \) (with
\( p^0 = \omega_p > 0 \)) and (\( p - m \))\( \psi_{in}^{\ast}(\tilde{p}) = 0 \) (with \( p^0 = -\omega_p < 0 \)),
the spinors now satisfy

\[
\tilde{P}(p)\psi_{in}^{\ast}(\tilde{p}) = 0, \quad (p^0 > 0) \quad (95)
\]

for the positive-energy solutions and

\[
\tilde{P}(p)\psi_{in}^{\ast}(\tilde{p}) = 0, \quad (p^0 < 0) \quad (96)
\]

for the negative-energy solutions corresponding to a given
3-momentum. Thus, we conclude that our model’s ex-
ternal spinors, unlike in the Lorentz-invariant case, are
modified by the one-loop radiative-correction terms cal-
culated in the previous section.

Note also that we get the Lorentz-violating multiplica-
tive contribution \( Z_R((c,k)^2) \) (wave-function renormal-
ization) to the S-matrix for every external fermion that
factors out of the fermion propagator pole (see Eq. (35)).

Let us analyze in more detail the new equations of
motion for the spinors, Eqs. (95) and (96). To first order
in radiative corrections and in Lorentz-violating param-
eters, we can use the approximations of Eqs. (32) and (33),
which yield

\[
\left( \tilde{p} + x_0 c_\nu + y_0 \tilde{k}_\nu - m_0 - m_{10} c_\nu - m_{01} \tilde{k}_\nu \right) \psi_{in}^{\ast}(\tilde{p}) = 0 . \quad (97)
\]

For fixed 3-momentum \( \tilde{p} \), the value of \( p^0 \) in Eq. (97) is
determined as the positive root of the dispersion rela-
tion (89). Every term in the dispersion relation is of
even order in the 4-momentum, so that when \( (p^0, \tilde{p}) \),
where \( p^0 > 0 \), satisfies the dispersion relation (89), so
does \( (-p^0, -\tilde{p}) \). The latter solution is taken to corre-
spond to \( \psi_{in}^{\ast}(\tilde{p}) \), an anti-fermion with momentum \( \tilde{p} \)
and energy \( p^0 \). Thus,

\[
\left( \tilde{p} + x_0 c_\nu + y_0 \tilde{k}_\nu + m_0 + m_{10} c_\nu + m_{01} \tilde{k}_\nu \right) \psi_{in}^{\ast}(\tilde{p}) = 0 , \quad (98)
\]

where \( p^0 \) takes the same value as in Eq. (97). The fact
that a fermion and an anti-fermion with the same
momentum have equal energy is a consequence of CPT in-
variance, which is unbroken by the \( c_{\mu\nu} \) (and by the \( k_{\mu\nu} \))
coefficients.

On the other hand, in general there are Lorentz-
violating quadratic terms in the dispersion relation (89)
that mix \( p^0 \) and \( \tilde{p} \). As a consequence, the fact that \( (p^0, \tilde{p}) \)
(with \( p^0 > 0 \)) satisfies Eq. (89) does not imply the same
for \( (-p^0, \tilde{p}) \). This expresses the fact that the \( c_{\mu\nu} \) (and
\( k_{\mu\nu} \)) violate parity. Another useful observation is that
the dispersion relation (89) is not sensitive to the spin
label \( s \). Note that spin-dependence does play a role
for some of the other types of SME coefficients, but we
will not consider them in this work.

We see from Eqs. (97) and (98) that the equation of
motion has terms quadratic in the momentum due to the
presence of \( c_{\rho} \) and \( k_{\rho} \). These terms make a rigor-
as analysis of the equation of motion for the external
fermion field and a quantization of the latter along the
lines of Appendix B problematic. For instance, they
likely introduce spurious non-physical solutions. For
this reason, we use the zeroth-order dispersion relation
\( p^0 = \sqrt{p^2 + m^2} \equiv \omega_p \) to substitute for \( c_{\rho} \),

\[
c_{\rho} \to c_{\rho 0} \omega_p^2 - 2c_{\rho 0} p^0 p^0 + c_{\rho 1} p^4 \quad (99)
\]
and similarly for $\tilde{k}_s^\alpha$. For simplicity, we will suppress the $\tilde{k}$ terms in the following. Equation (97) becomes
\[
(\Gamma^\mu p_\mu - m_0 - m_{10}c^{00}_p\omega_p^2 + 2m_{10}c^{0i}_p p^i - m_{10}c^{ij}_p p^j)u^s_{in,1}(p\bar{\psi}) = 0, \quad (100)
\]
with
\[
\Gamma^\mu = \gamma^\mu + x_0 e^{\mu\nu} \gamma^\nu. \quad (101)
\]
The spinor $u^s_{in,1}(p\bar{\psi})$ satisfying Eq. (100) differs from the original one $u^s_{in}(p\bar{\psi})$ by terms of second order (and higher) in the Lorentz-violating coefficients. We remark that this higher-order difference between the spinors permits a self-consistent treatment of the asymptotic Hilbert space in terms of $u^s_{in,1}(p\bar{\psi})$, while also allowing us to switch back to the original spinors $u^s_{in}(p\bar{\psi})$ at a later point in the calculation.

With these considerations, we can proceed as in Appendix B. The equation of motion can be written in the form of an eigenvalue equation:
\[
\tilde{\Gamma}^0 (p\bar{\psi})^{-1} \left[ \Gamma^i p^i + m_0 + m_{10}c^{00}_p\omega_p^2 + m_{10}c^{ij}_p p^i p^j \right] u^s_{in,1}(p\bar{\psi}) = p^0 u^s_{in,1}(p\bar{\psi}), \quad (102)
\]
where
\[
\tilde{\Gamma}^0 (p\bar{\psi}) = \Gamma^0 + 2m_{10}c^{0i}_p p^i. \quad (103)
\]
The operator acting on the left-hand side of Eq. (102) on the spinor is hermitian with respect to the inner product
\[
(u_1 | u_2) \equiv \bar{u}_1 \tilde{\Gamma}^0 (p\bar{\psi}) u_2, \quad (104)
\]
which is different from that in Eq. (B13). Consequently, it has real eigenvalues, with the corresponding eigen-spinors forming an orthonormal basis in spinor space \(\{u^s_{in,1}(p\bar{\psi}), v^s_{in,1}(p\bar{\psi}), v^s_{in,1}(-p\bar{\psi}), v^s_{in,1}(-(-p\bar{\psi})\}\) satisfying the relations
\[
\bar{u}^r_{in,1}(p\bar{\psi})\tilde{\Gamma}^0 (p\bar{\psi}) u^s_{in,1}(p\bar{\psi}) = \frac{\omega^2}{m} \delta_{rs},
\]
\[
\bar{v}^r_{in,1}(-p\bar{\psi})\tilde{\Gamma}^0 (p\bar{\psi}) v^s_{in,1}(-p\bar{\psi}) = \frac{\omega^2}{m} \delta_{rs}. \quad (105)
\]

As well as
\[
\sum_{s=1}^{2} [u^s_{in,1}(p\bar{\psi}) u^s_{in,1}(p\bar{\psi}) + v^s_{in,1}(-p\bar{\psi}) v^s_{in,1}(-p\bar{\psi})] \tilde{\Gamma}^0 (p\bar{\psi}) = \frac{\omega^2}{m} \mathbb{I}, \quad (106)
\]
in analogy to Eqs. (B14) and (B15). Incidentally, note the hermiticity relation $\tilde{\Gamma}^0 (p\bar{\psi})^\dagger = \gamma^0 \tilde{\Gamma}^0 (p\bar{\psi})\gamma^0$ for $\tilde{\Gamma}^0 (p\bar{\psi})$.

The free field has the Fourier decomposition
\[
\psi^s_{in}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{m}{\omega_p} \sum_{s=1}^{2} [b_s^{in}(p\bar{\psi}) u^s_{in,1}(p\bar{\psi}) e^{-ip\cdot x} + d_s^{in}(p\bar{\psi}) v^s_{in,1}(p\bar{\psi}) e^{ip\cdot x}]. \quad (107)
\]
From Eq. (100) we see that it satisfies the (linearized) equation of motion
\[
[i\Gamma^\mu \partial_\mu - m_0 + m_{10}c^{00}(\nabla^2 - m^2) - 2m_{10}c^{0i}\partial^i + m_{10}c^{ij}\partial^i\partial^j] \psi^s_{in}(x) = 0. \quad (108)
\]

The creation and annihilation operators can be expressed by the following projections
\[
b_s^{in\dagger}(p\bar{\psi}) = \int d^3x e^{-ip\cdot x} \bar{\psi}^s_{in}(x) \tilde{\Gamma}^0 (p\bar{\psi}) u^s_{in,1}(p\bar{\psi}), \quad (109)
\]
\[
d_s^{in\dagger}(p\bar{\psi}) = \int d^3x e^{-ip\cdot x} \bar{v}^s_{in,1}(p\bar{\psi}) \tilde{\Gamma}^0 (-p\bar{\psi}) \psi^s_{in}(x), \quad (110)
\]
and their hermitian conjugates. We remind the reader that the zeroth components of the momentum in the plane-wave exponentials in Eqs. (109) and (110) depend on the corresponding mode: $p^0 u^s_{in}$ and $p^0 v^s_{in}$.

The results derived in Appendix B for the free-field quantization in the presence of Lorentz violation hold analogously for the field $\psi^s_{in}(x)$. Note in particular the Feynman propagator
\[
(0|T\psi^s_{in}(x)\bar{\psi}^s_{in}(y)|0) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot (x-y)}}{\Gamma^\mu p_\mu - m_0 - m_{10}(c^{00}_p\omega_p^2 - 2c^{0i}_p p^i + c^{ij}_p p^i p^j) + i\epsilon} \approx \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot (x-y)}}{P(p) + i\epsilon}. \quad (111)
\]

From the discussion at the beginning of this section, we can now give a more precise formulation of the adiabatic hypothesis for the interacting field $\psi(x)$ and the free field $\psi^s_{in}(x)$. Comparing Eq. (111) with the on-shell limit of Eq. (35) it follows that instead of relation (94) we now
have, in the limit \( x_0 \to -\infty \):

\[
\psi(x) \to \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{m}{\omega_p} \Phi^\dagger_R((c, \vec{k})_p) \sum_{s=1}^{2} [b^{in\dagger}_s(\vec{p})u^{s*}_m(\vec{p})e^{-ipx} + d^{in\dagger}_s(\vec{p})v^{s*}_m(\vec{p})e^{ipx}].
\]

(112)

For large negative times, Eq. (109) can thus be written in terms of the interacting field:

\[
b^{in\dagger}_s(\vec{p}) = \int d^3x \ e^{-ipx} \tilde{\psi}(x) \tilde{\Phi}^0(\vec{p})u^{s*}_m(\vec{p}) \Phi^\dagger_R((c, \vec{k})_p),
\]

and similarly for \( d^{in\dagger}_s \). In the same way we can express the out-oscillators in terms of the interacting field for large positive times.

We will now use the above results to derive the LSZ reduction formula for the Lorentz-violating case. For this expression to exist, it is important that the Cauchy initial-value problem of the field theory be well defined.

Fortunately, the above procedure leading to a spinor equation of motion linear in the zeroth component of the 4-momentum is exactly what is needed for a consistent derivation of the LSZ formula, as we will see below.

We begin by defining a smearing procedure for the definite-momentum creation and annihilation operators, so that the states created by the smeared operators can be localized in position space. For example,

\[
b^{out\dagger}_q,s = \int d^3\vec{p} \ f_q(\vec{p})b^{in}_q(\vec{p}),
\]

with analogous definitions for the various other creation and annihilation operators. Here, we have abbreviated \( d^2\vec{p} \equiv (m \ d^3\vec{p})/\omega_p \), and \( f_q(\vec{p}) \sim \exp[-(\vec{p}-\vec{q})^2/4\sigma^2] \), describing the creation of a particle localized in 3-momentum space near \( \vec{q} \) and localized in 3-position space near the origin. In the Schrödinger picture, a state created by this operator evolves in time. Applying this smearing to \( b^{in\dagger}_q \) given in the form of Eq. (113) yields:

\[
b^{in\dagger}_q,s = \int d^3\vec{p} \ f_q(\vec{p}) \int d^3x \ e^{-ipx} \tilde{\psi}(x) \tilde{\Phi}^0(\vec{p})u^{s*}_m(\vec{p}) \Phi^\dagger_R((c, \vec{k})_p).
\]

(114)

We can now use the equation of motion (102) for \( u^s_m(\vec{p}) \) to express \( \tilde{\Phi}^0(\vec{p}) \) in the last equation in terms of \( \vec{p} \). We then trade the \( p^i \) components for partial derivatives acting to the right on the exponential. By performing partial integrations they can be converted to partial derivatives acting to the left, which yields

\[
b^{in\dagger}_q,s - b^{out\dagger}_q,s =
\]

\[
= i \int d^3\vec{p} \ f_q(\vec{p}) \int d^4x \ \tilde{\psi}(x) \left[ i\tilde{\partial}_0 \tilde{\Phi}^0(i\vec{\nabla}) + i\Gamma^i \tilde{\partial}_i + m_0 + m_{10} \left( e^{00}(m^2-\vec{\nabla}^2) - e^{ij}\tilde{\partial}_i \tilde{\partial}_j \right) \right] u^{s*}_m(\vec{p}) e^{-ipx} \Phi^\dagger_R((c, \vec{k})_p)
\]

\[
= i \int d^3\vec{p} \ f_q(\vec{p}) \int d^4x \ \tilde{\psi}(x) \left[ i\Gamma^\mu \tilde{\partial}_\mu + m_0 + m_{10} \left( e^{\mu\nu} \tilde{\partial}_\mu \tilde{\partial}_\nu - e^{00}(\vec{\nabla}^2 + m^2) \right) \right] u^{s*}_m(\vec{p}) e^{-ipx} \Phi^\dagger_R((c, \vec{k})_p)
\]

\[
\approx -i \int d^3\vec{p} \ f_q(\vec{p}) \int d^4x \ \tilde{\psi}(x) \tilde{P}(-i\partial_0)u^{s*}_m(\vec{p}) e^{-ipx} \Phi^\dagger_R((c, \vec{k})_p). \]

(115)

The last identity is valid to first order in Lorentz violation, on the physical mass shell (i.e., any spurious, unphysical solutions of \( \tilde{P}(p) = 0 \) far from the mass shell should be disregarded). Similarly, if we start with an antifermion in the initial state:

\[
d^{in\dagger}_q,s - d^{out\dagger}_q,s \approx i \int d^3\vec{p} \ f_q(\vec{p}) \int d^4x \ \tilde{\bar{\psi}}(x) e^{-ipx} \tilde{P}(i\partial_0) \psi(x) \Phi^\dagger_R((c, \vec{k})_p). \]

(116)

\[
\langle f | i \rangle = \langle \text{out} | \cdots \ b^{out\dagger}_{q_1} \cdots b^{out\dagger}_{q_2} \cdots | \text{in} \rangle \]

(117)
can be expressed, using Eqs. (116) and (117) and their hermitian conjugates, as the LSZ reduction formula

\[
\langle f| i \rangle = \int d^4x_1 \cdots d^4y' \cdots \exp \left[ -i(p \cdot x + \cdots + p' \cdot x' + \cdots - q \cdot y - \cdots - q' \cdot y' - \cdots) \right]
\]

\[
\times \cdots (-i) Z_R^{-\frac{1}{2}} ((c, \vec{k})_{q_i}^n) \tilde{u}_\text{in}(q_i) \tilde{P}(i \tilde{q}_i) \cdots \times Z_R^{-\frac{1}{2}} ((c, \vec{k})_{p_i}^n) \tilde{v}_\text{in}(p_i) \tilde{P}(i \tilde{p}_i)
\]

\[
\times \langle 0| [\cdots \tilde{\psi}(y_1) \cdots \tilde{\psi}(x_1) \cdots] | 0 \rangle
\]

\[
\times \tilde{P}(-i \tilde{q}_i) \tilde{u}_\text{in}(\tilde{p}_i)(-i) \tilde{Z}_R^{-\frac{1}{2}} ((c, \vec{k})_{p_i}^n) \cdots \tilde{P}(-i \tilde{p}_i) \tilde{v}_\text{in}(\tilde{q}_i) i \tilde{Z}_R^{-\frac{1}{2}} ((c, \vec{k})_{q_i}^n) \cdots + \text{disconnected terms.}
\]

As in the derivation for the Lorentz-invariant case (see, e.g., Ref. [46]), the introduction of the time-ordered product in Eq. (119) is necessary so that the field operators are in the convenient order with respect to the in- and out-vacua. In deriving Eq. (119), we have taken the momentum distributions \( f_q(\vec{p}) \) to the delta-function limit

\[
f_q(\vec{p}) \to \delta^3(\vec{p} - q). \quad (120)
\]

In practical calculations it is most useful to express the scattering amplitude in terms of truncated Green’s functions. Using the definition

\[
\tilde{G}_{2n}(p'_1, \ldots, p'_n; p_1, \ldots, p_n) = \int \prod_{i=1}^n d^4z'_i d^4z_i \exp \left[ i \sum_{i=1}^n (p'_i \cdot z'_i + p_i \cdot z_i) \right] T \left[ \prod_{i=1}^n \tilde{\psi}(z'_i) \prod_{j=1}^n \tilde{\psi}(z_j) \right] \quad (121)
\]

in Eq. (119), the connected scattering amplitude can be expressed as

\[
\langle f| i \rangle_c = \cdots (-i) Z_R^{-\frac{1}{2}} ((c, \vec{k})_{q_i}^n) \tilde{u}_\text{in}(q_i) \tilde{P}(q_i) \cdots \times Z_R^{-\frac{1}{2}} ((c, \vec{k})_{p_i}^n) \tilde{v}_\text{in}(p_i) \tilde{P}(p_i) \tilde{G}^{(2n)}(-q'_1, \ldots, p_1, \ldots, -q_1, \ldots, p'_1, \ldots)
\]

\[
\times \tilde{P}(-p_1) \tilde{u}_\text{in}(p_1)(-i) \tilde{Z}_R^{-\frac{1}{2}} ((c, \vec{k})_{p_i}^n) \cdots \tilde{P}(q_1) \tilde{v}_\text{in}(q_1) i \tilde{Z}_R^{-\frac{1}{2}} ((c, \vec{k})_{q_i}^n) \cdots. \quad (122)
\]

If we now introduce the truncated Green’s functions

\[
(2\pi)^4 \delta \left( \sum p_i + p'_i \right) G_{\text{trunc}}^{(2n)}(p'_1, \ldots, p'_n; p_1, \ldots, p_n) = \prod_{i=1}^n \left[ \Gamma^{(2)}(p'_i) \Gamma^{(2)}(p_i) \right] \tilde{G}_{2n}(p'_1, \ldots, p'_n; p_1, \ldots, p_n), \quad (123)
\]

in which all external legs are multiplied by the inverses of the corresponding complete propagators, it follows from Eqs. (42), (122), and (121) that

\[
\langle f| i \rangle_c = (2\pi)^4 \delta^4 \left( \sum p_i + \sum p'_i - \sum q_i - \sum q'_i \right) \cdots (-i) Z_R^{-\frac{1}{2}} ((c, \vec{k})_{q_i}^n) \tilde{u}_\text{in}(q_i) \cdots \times Z_R^{-\frac{1}{2}} ((c, \vec{k})_{p_i}^n) \tilde{v}_\text{in}(p_i)
\]

\[
\times G_{\text{trunc}}^{(2n)}(-q'_1, \ldots, p_1, \ldots, -q_1, \ldots, p'_1, \ldots) \tilde{u}_\text{in}(p_1)(-i) \tilde{Z}_R^{-\frac{1}{2}} ((c, \vec{k})_{p_i}^n) \cdots \tilde{v}_\text{in}(q'_1) i \tilde{Z}_R^{-\frac{1}{2}} ((c, \vec{k})_{q_i}^n) \cdots. \quad (124)
\]

Note that \( G_{\text{trunc}}^{(2n)} \) carries 2n Dirac indices (that are contracted with the spinors), which are suppressed here for readability.

Formula (124) embodies the Feynman rules for the scattering amplitude, incorporating:

- a momentum-conserving delta-function;
- the amputated Green’s function;
- a momentum-dependent wave-function renormalization factor \( \pm i Z_R^2 ((c, \vec{k})_{p_i}^n) \) for every external leg;
- a Dirac spinor for every external leg:
  - \( u_{\text{in}}(\vec{p}) \) for an incoming fermion;
  - \( \bar{u}_{\text{in}}(\vec{p}) \) for an outgoing fermion;
  - \( v_{\text{out}}(\vec{p}) \) for an outgoing anti-fermion;
  - \( \bar{v}_{\text{out}}(\vec{p}) \) for an incoming anti-fermion.
We will end this section with a derivation of some explicit formulas for the spinors \( u^s_{in}(p) \) and \( v^s_{in}(p) \) satisfying Eqs. (95) and (96). The most convenient way to achieve this is to take them proportional to the usual Lorentz-invariant spinor functions \( u^s_{L1}(p) \) and \( v^s_{L1}(p) \), but then not calculated for the real, physical momentum \( p \), but for a redefined momentum value \( \tilde{p} \) satisfying
\[
\tilde{p} - m_0 \propto \tilde{P}(p) .
\]
Thus
\[
u_{in}(p) = C_{u}(p) u^s_{L1}(p) \tag{126}
\]
\[
u_{in}(p) = C_{v}(p) v^s_{L1}(p) \tag{127}
\]
where \( C_{u}(p) \) and \( C_{v}(p) \) are normalization constants to be determined below. One easily checks that
\[
\tilde{p}^\mu = \left( 1 - \frac{m_0}{m_0} \frac{c_p}{c_p - \frac{m_0}{m_0} 4\pi m^2} \left( 2k^p - \tilde{k}^p \right) \right) p^\mu + \left( x_0 c_{\mu\nu} + y_0 \tilde{k}^\mu p_\nu \right)
\]
\[
\tilde{p}^\mu = \left( 1 + \frac{\alpha}{3! m^2} \left( 2k^p - \tilde{k}^p \right) \right) p^\mu + (c_{\text{phys}})_{\mu\nu} p_\nu
\]
satisfies Eq. (125) to first order in the Lorentz-violating parameters and obeys the dispersion relation \( \tilde{p}^2 = m_0^2 \).

Let us work out the normalization constants \( C_{u}(\tilde{p}) \) and \( C_{v}(\tilde{p}) \), in accordance with Eq. (105). Consider the case of \( C_{u}(\tilde{p}) \) first. For \( u^s_{L1}(\tilde{p}) \) we have the usual relations
\[
n_{L1}(\tilde{p}) \gamma^\mu u^s_{L1}(\tilde{p}) = \frac{\tilde{p}^\mu}{m_0} \delta_{rs} \tag{129}
\]
\[
n_{L1}(\tilde{p}) u^s_{L1}(\tilde{p}) = \delta_{rs} \tag{130}
\]
Demanding now that \( u^s_{in}(p) \) satisfies the normalization condition (105) it follows that
\[
|C_{u}(\tilde{p})|^2 n_{L1}(\tilde{p}) n_{L1}(\tilde{p}) = \frac{\omega_p}{m} \delta_{rs} .
\]
Using Eqs. (129) and (130) and working to first order in Lorentz violation one obtains the following expression for the normalization constant:
\[
|C_{u}(\tilde{p})|^2 = \frac{m}{\omega_p m_0} \left[ \tilde{p}^\mu + \left( c_{\text{phys}} \right)_{0\nu} \tilde{p}_\nu + \frac{2\alpha}{3!} \left( 2c_{0i} - \tilde{k}_{0i} \tilde{p}^i \right) \right]
\]
\[
\frac{m}{\omega_p m_0} \left[ \frac{1}{\omega_{\tilde{p}}} \left( c_{\text{phys}} \right)_{0\nu} \tilde{p}_\nu + \frac{2\alpha}{3!} \left( 2c_{0i} - \tilde{k}_{0i} \tilde{p}^i \right) \right] .
\]
In the last equation, we defined \( \omega_{\tilde{p}} \equiv \tilde{p}^0 = \sqrt{\tilde{p}^2 + m_0^2} \). Note that the same analysis can be done for the \( v^s_{in}(p) \) spinors. The normalization constant turns out to be the same as for the \( u \) spinors, so that we can safely suppress the \( u \) and \( v \) indices:
\[
C_{u}(\tilde{p}) = C_{u}(\tilde{p}) \equiv C(\tilde{p}) .
\]
As an additional simplification, the normalization constants are also independent of the spin index \( s \). This allows us to determine spin-sum formulas. They follow directly from the usual expressions for the Lorentz-invariant case:
\[
\sum_{\sigma} u^s_{in}(\tilde{p}) u^s_{in}(\tilde{p}) = |C(\tilde{p})|^2 \frac{\tilde{p}^2 + m_0}{2m_0} \tag{134}
\]
\[
\sum_{\sigma} v^s_{in}(\tilde{p}) v^s_{in}(\tilde{p}) = |C(\tilde{p})|^2 \frac{\tilde{p}^2 - m_0}{2m_0} \tag{135}
\]
In Eqs. (134) and (135), it is understood that \( \tilde{p} \equiv \tilde{p}^0 = \tilde{p}^0 \) (see Eqs. (97) and (98)).

VI. SAMPLE CALCULATION: INFRARED DIVERGENCES IN COULOMB SCATTERING

It is instructive to apply the techniques described above to a particular case. We will do this for the Coulomb (or rather Mott) scattering of a fermion off a stationary charge. For simplicity, we will assume that only the Lorentz-violating parameter \( \tilde{k}^\mu \) is nonzero.

Let us review quickly the Lorentz-invariant case. We have for the scattering amplitude at tree level
\[
S_{fi} = \frac{iZ}{\sqrt{\gamma_i}} \int d^4x \bar{u}^s(p_f)\mathcal{A}(x)e^{i(p_f - p_i)\cdot x} u^s(p_i) .
\]
Here, we have normalized the states in a finite volume \( V \). For the Coulomb problem, we can take \( \mathcal{A} = 0 \) and \( A_0 = Ze/4\pi|\vec{x}| \), so
\[
S_{fi} = \frac{iZ}{\sqrt{\gamma_i}} \frac{m}{\sqrt{E_f E_i}} 2\pi \delta(E_f - E_i)
\]
\[
\times \int d^3x \frac{e^{-i\vec{q}\cdot\vec{x}}}{|\vec{x}|} \bar{u}^s(p_f) \gamma^0 u^s(p_i) .
\]
We can now pass to the cross section by squaring the absolute value of Eq. (137), multiplying by the number of possible final states \( V^3 p_f / (2\pi)^3 \) and dividing by the incident flux \( |\bar{u}^s|/V \) and the time interval \( T \). Note that for large time intervals \( T \), one can take \( |2\pi \delta(E_f - E_i)|^2 \equiv T^2 \pi \delta(E_f - E_i) \). It then follows that
\[
\frac{d^3\sigma_{fi}}{dp_f} = \int \frac{4Z^2 \alpha^2 m^2}{|\vec{p}_f| E_f |\vec{q}|} \delta(E_f - E_i)
\]
\[
\times |\bar{u}^s(p_f) \gamma^0 u^s(p_i)|^2 p_f^2 d\Omega_f .
\]
Using now that
\[
|\vec{p}_f| = |\vec{p}_f| = p_f \quad \text{and} \quad p_f d\Omega_f = E_f dE_f ,
\]
it follows that
\[
\frac{d^3\sigma_{fi}}{dp_f} = \frac{4Z^2 \alpha^2 m^2}{|\vec{q}|} |\bar{u}^s(p_f) \gamma^0 u^s(p_i)|^2 d\Omega_f .
\]
If we do not observe the final polarization, we must sum over \( s \), while for an unpolarized incident wave we average
over the initial polarizations \( r \). With the usual formulas for the spin sums one obtains

\[
\frac{d\sigma_{\mu\nu}}{d\Omega} \big|_{\text{unpol}} = \frac{4Z^2\alpha^2m^2}{|\vec{q}|^4} \frac{1}{2} \text{tr} \left( \gamma^\alpha \beta_1 + m \gamma^\alpha \beta_1 + m \right) \left( \gamma^\beta \frac{\vec{q}}{2m} - \frac{\gamma^\beta \vec{k}}{2m} \right) \left( \gamma^\gamma \frac{\vec{u}}{2m} - \frac{\gamma^\gamma \vec{l}}{2m} \right). 
\]

(141)

When turning on Lorentz violation, various adaptations have to be made to the formulas (137)–(141) at tree level:

1. The Maxwell equations become Lorentz violating:

\[
\Box A^\mu = j^\mu \quad \Rightarrow \quad \Box \tilde{\eta}^{\mu\nu} A_\nu = j^\mu, 
\]

(142)

where \( \tilde{\eta}^{\mu\nu} = \eta^{\mu\nu} + \tilde{k}^{\mu\nu} \) and \( \Box = \partial_\alpha \tilde{\eta}^{\alpha\beta} \partial_\beta \). This means that the Fourier transform of the Coulomb potential becomes

\[
\tilde{A}_\mu = Ze_\gamma^{\mu\nu} \delta(q^0) = Ze_\gamma^{\mu\nu} \delta(q^0) + \tilde{A}_\mu^{(1)}.
\]

(143)

2. The incident velocity is now given by the group velocity \( v^\alpha = \partial E/\partial p^\alpha \), which is fixed by the dispersion relation (89). However, note that, as in this example, we choose \( c^{\mu\nu} = 0 \), there is no Lorentz-violating effect at tree level.

3. The dispersion relation (89) also implies Lorentz-violating modifications to the integration-variable transformation from \( dp_f \) to \( dE_f \) implied by Eq. (139). However, also here there is no effect at tree level because we take \( c^{\mu\nu} = 0 \). Incidentally, note that the factors \( \sqrt{E_i} \) and \( \sqrt{E_f} \) in the denominator of Eq. (137) remain equal to their Lorentz-invariant form \( \sqrt{m^2} \) (\( \omega \equiv \sqrt{p^2 + m^2} \)).

4. The spinors are modified according to the relations (126)–(128) and (132). In the unpolarized cross section (141), we have to use the modified spin sum (134) or (135), as appropriate. Again, there is no effect at tree level as we take \( c^{\mu\nu} = 0 \).

It is straightforward but tedious to adapt the formulas for the tree-level cross sections (140) and (141) to the Lorentz-violating case accordingly. Rather than doing this explicitly, we will move our attention to radiative corrections. The diagrams shown in Fig. 3 contribute to one-loop order. Note that the fermion self-energy diagrams are taken into account implicitly in the order \( \alpha \) corrections to the external spinors in formulas (126)–(132), as well as by the inclusion of the wave-function renormalization \( Z_R^{1/2} \) for each external fermion leg.

Instead of carrying out the full calculation of the one-loop diagrams, we will just concentrate on the infrared-divergent contributions to the scattering amplitude. We will then show that they indeed cancel in the experimental cross section, just as in the Lorentz-invariant case.

Paralleling the usual Lorentz-invariant case, the vacuum-polarization diagram is infrared finite, so that we only have to consider the vertex-correction diagram. With the modified photon propagator (recall that we have chosen \( c^{\mu\nu} = 0 \)) the amplitude for the vertex correction is given by

\[
-iq\Gamma^\mu(p',p) = (-i)^3 \mu^\nu \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\alpha i(p' - k + m)\gamma^\nu i(k - p + m)\gamma^\beta (-i)(\eta_{\alpha\beta} - \tilde{k}_{\alpha\beta})}{((p' - k)^2 - m^2 + i\epsilon)((p - k)^2 - m^2 + i\epsilon)(k^2 + 2k^2 - m_\gamma^2 + i\epsilon)}.
\]

(144)

The the integral on the right-hand-side of Eq. (144) is infrared divergent. This divergence arises from the residue of the photon propagator. In order to isolate it, we can use the fermion on-shell relations \( p^2 - m^2 = O(\alpha) \approx 0 \) and \( (p')^2 - m^2 = O(\alpha) \approx 0 \). It is then straightforward to show that

\[
\tilde{u}(p',\sigma')\gamma^\alpha(p' + m)\gamma^\nu(p + m)\gamma^\beta u(p,\sigma)(\eta_{\alpha\beta} - \tilde{k}_{\alpha\beta}) = 4\tilde{u}(p',\sigma')\gamma^\mu u(p,\sigma)(p' \cdot k - \tilde{k}_{\mu})(p' \cdot \tilde{k}_\nu)
\]

(145)

by using the (approximate) equations of motion for the spinors. Moreover, we can drop terms linear in \( k \) in the numerator and quadratic in \( k \) in the fermion pole factors in the denominator. It then follows that

\[
\tilde{u}(p',\sigma')\Gamma^\mu(p',p)u(p,\sigma) \approx -iq^2\mu^\nu\tilde{u}(p',\sigma')\gamma^\mu u(p,\sigma) \int \frac{d^4k}{(2\pi)^4} \frac{p' \cdot k - \tilde{k}_\mu}{k^2 + 2k^2 - m_\gamma^2 + i\epsilon (p' \cdot k)}.
\]

(146)

We can evaluate the integral (146) by using the identity

\[
\frac{1}{k^2 + 2k^2 - m_\gamma^2 + i\epsilon} = \frac{1}{k^2 + 2k^2 - m_\gamma^2} - i\pi\delta(k^2 + 2k^2 - m_\gamma^2),
\]

(147)
where \( P \) denotes the principle-value part. Omitting the infrared-finite contribution from the principal value we find

\[
\bar{u}(p', \sigma') \Gamma^\mu(p', p) u(p, \sigma) = -\frac{\alpha}{4\pi^2} \bar{u}(p', \sigma') \gamma^\mu u(p, \sigma) \int d^4k \delta(k^2 + 2\hat{k}_k^2 - m_\gamma^2) \frac{p' \cdot p - \bar{k}_p'}{(p' \cdot k)(p \cdot k)}. \tag{148}
\]

It follows that the one-loop contributions to the elastic Coulomb scattering amplitude are obtained by substituting \( \mathcal{M}_0 \equiv \bar{u}(p_f, \sigma_f) \gamma^\mu u(p_i, \sigma_i) \) in Eq. (140) by

\[
\mathcal{M} = \mathcal{M}_0 \left\{ 1 - \frac{\alpha}{4\pi^2} \int d^4k \delta(k^2 + 2\hat{k}_k^2 - m_\gamma^2) \frac{p' \cdot p - \bar{k}_p'}{(p' \cdot k)(p \cdot k)} \right\} Z_R^{1/2}(p') Z_R^{1/2}(p) + \cdots. \tag{149}
\]

where the ellipsis indicates infrared-finite contributions. For the elastic cross section, this implies

\[
\frac{d\sigma^{el}}{d\Omega} = \frac{d\sigma^0}{d\Omega} \left\{ 1 - \frac{\alpha}{2\pi^2} \int d^4k \delta(k^2 + 2\hat{k}_k^2 - m_\gamma^2) \frac{p' \cdot p - \bar{k}_p'}{(p' \cdot k)(p \cdot k)} \right\} Z_R(p') Z_R(p) + \cdots. \tag{150}
\]

The term proportional to \( \alpha \) as well as the factors \( Z_R \) in expression (150) are infrared-divergent in the limit \( m_\gamma \to 0 \).

In the Lorentz-invariant case, infrared-divergences are canceled if one incorporates the fact that some final states that include soft photons are experimentally indistinguishable. We will now proceed to check that this continues to hold true in the case at hand.

Following the usual procedure, we consider final states that include one soft photon with energy smaller than the detector resolution \( \Delta E \). The amplitude for this process is

\[
\mathcal{M} = q_0 \mathcal{M}_0 \left[ \frac{2p' \cdot \epsilon_r}{2p' \cdot k} + \frac{2p \cdot \epsilon_r}{-2p \cdot k} \right], \tag{151}
\]

where \( \mathcal{M}_0 \) is the amplitude for elastic scattering without photon emission. To get the cross section for this process, we have to include also the final-state volume element for the photon. As shown in Appendix C, a consistent way to do this is to take

\[
\frac{d^3k}{(k_0^+ + k_0^-)(2\pi)^3} \tag{152}
\]

rather than the conventional factor \( d^3k/(2\omega_k(2\pi)^3) \). Here, \( k_0^\pm(\hat{k}) \) are the absolute values of the solutions for \( k_0 \) to the modified dispersion relation.

Combining Eqs. (151) and (152) and employing formula (C25) one then finds for the cross section for this (soft-bremsstrahlung) process

\[
\frac{d^4k}{\omega_{k^0} \omega_{k^1} (2\pi)^4} \int_{\omega_{k^1} \leq \Delta E} d^4k \delta^4(k^2 + \hat{k}_k^2 - m_\gamma^2) \left[ \sum_{\lambda=1}^{3} \left( \frac{2p' \cdot \epsilon_r^{(\lambda)}(k)}{2p' \cdot k} + \frac{2p \cdot \epsilon_r^{(\lambda)}(k)}{-2p \cdot k} \right) \right]^2. \tag{153}
\]

Using the polarization-sum formula (C24), Eq. (153) reduces to

\[
\frac{d\sigma^{el}}{d\Omega} \approx \frac{d\sigma^0}{d\Omega} \frac{\alpha}{(2\pi)^2} \int_{\omega_{k^1} \leq \Delta E} d^4k \delta^4(k^2 + \hat{k}_k^2 - m_\gamma^2) \left[ \frac{(p')^2 - \hat{k}_p'}{(p' \cdot k)^2} - \frac{p^2 - \hat{k}_p^2}{(p \cdot k)^2} + \frac{2p' \cdot p - 2\hat{k}_p'}{(p' \cdot k)(p \cdot k)} \right]. \tag{154}
\]

Comparing the third term inside brackets in Eq. (154) with Eq. (150) we see that both contain integrals over
the same \( p, p' \) term, with opposite sign, so that the corresponding infrared divergences cancel.

To verify that the same happens for the \( p, p \) and the \( p', p' \) terms in Eq. (154) and the \( Z_R(p')Z_R(p) \) factors in Eq. (150), we evaluate the \( k \) integral over, say, the \( p, p \) term in Eq. (154). To this end, we perform a change of variables \( k^\mu \rightarrow \tilde{k}^\mu \) such that

\[
\tilde{k}^2 = k^2 + \frac{k^2}{k}. \tag{155}
\]

To first order, this means that \( \tilde{k}^\mu = k^\mu + \frac{k^2}{k} k^\nu k_\nu \). To lowest order, the measure \( d^4k \) is invariant under this transformation, as

\[
\frac{\partial \tilde{k}}{\partial k} \approx \frac{\delta^\mu_\nu + \frac{1}{2} \tilde{k}^\mu k_\nu}{k} \approx 1 + \frac{1}{2} \tilde{k}^\mu = 1. \tag{156}
\]

It follows that

\[
\int_{\omega \leq \Delta E} d^4k \delta(k^2 + \tilde{k}^2 - m^2) p^2 - \tilde{k}^2 \approx \int_{\omega \leq \Delta E} d^4k \delta(k^2 - m^2) \frac{p^2}{(k \cdot p)^2}, \tag{157}
\]

where we have defined \( \tilde{p}^\mu = (\gamma^{\mu\nu} - \frac{1}{2} k^{\mu\nu}) p_\nu \). Note that strictly speaking, the upper limit of the transformed energy \( \tilde{\omega} \) is modified by the transformation, but the result is an effect of higher order in \( \tilde{k} \) and can be ignored. The resulting integration is a standard one with result [46]

\[
4\pi \ln \left( \frac{\Delta E}{m_\gamma} \right) + \cdots, \tag{158}
\]

where the ellipsis indicates terms that are finite as \( m_\gamma \rightarrow 0 \). Substitution in Eq. (154) gives (for the second term)

\[
-\frac{d\sigma}{d\Omega} \frac{\alpha}{\pi} \ln \left( \frac{\Delta E}{m_\gamma} \right) + \text{finite}. \tag{159}
\]

It follows that the infrared divergence in Eq. (159) indeed cancels the corresponding one in the elastic cross section (150) arising from the multiplicative renormalization function \( Z_R(p) \) that was evaluated in Eq. (88):

\[
Z_R((c, k)^p) = 1 + \frac{\alpha}{\pi} \ln \left( \frac{m}{m_\gamma} \right) + \text{finite}. \tag{160}
\]

VII. SUMMARY AND OUTLOOK

Perturbative Lorentz-invariant quantum field theory rests on a few core field-theoretic techniques. One of these concerns the order-by-order determination of the asymptotic Hilbert space, and thus the calculation of quantum corrections to the external states. Such effects govern the propagation of free particles and are indispensable for scattering amplitudes. The present work for the first time has addressed the issue how to generalize this core technique to Lorentz-violating quantum-field theories. To illustrate the salient features of this generalization, we have focused on a particular sector of the SME. We expect, though, that our reasoning can also be applied to other Lorentz-violating field theories with a more complex structure and a wider variety of coefficients.

Specifically, we considered the SME’s single-flavor QED sector in the presence of the \( \epsilon^{\mu\nu} \) and the non-birefringent piece of the \( k_\mu k_\nu \) coefficients. We found that the presence of these Lorentz-breaking terms in the Lagrangian has some profound consequences for the radiative corrections to the pole structure of the external states. In particular, the Dirac equation satisfied by the external-state spinors turns out to be modified by Lorentz-violating operators not present in the Lagrangian, a feature that is unknown in usual Lorentz-symmetric field theories. Our analysis also shows that the wave-function renormalization will typically contain Lorentz-breaking coefficients contracted with momenta. We note that this is in contrast to the usual one-loop QED result, where \( Z_\psi \) is a momentum-independent constant. Momentum dependence of the wave-function renormalization is known to occur in certain other contexts [48].

We have limited our present study primarily to theoretical techniques for determining quantum corrections to external states in Lorentz-violating backgrounds. However, our results indicate that such corrections may have profound phenomenological implications for Lorentz tests, which can be seen as follows. The new, radiatively induced term exhibits two powers of the momentum, whereas the existing terms contain only up to a single power. The correction term should therefore grow faster with the momentum than the existing terms. This opens the possibility—at least in principle—that the Lorentz-violating radiative corrections become larger than the original tree-level Lorentz violation. Note that this does not necessarily signal a breakdown of perturbation theory because the perturbation Hamiltonian also includes the conventional electromagnetic interaction, which is comparatively much larger.

In our simple model, the radiative-correction term of size \( \sim \alpha \pi m(2k_\mu^p - k_\mu^p) \) can reach the size of the tree-level contribution \( c_\mu^p \) when \( \alpha p \sim m \). It follows that for Lorentz tests involving free electrons with energies \( \gtrsim 100 \text{ MeV} \), radiative Lorentz-breaking corrections may not always be negligible relative to the tree-level Lorentz violation. Note that electrons in such an energy range are routinely employed in various Lorentz tests. We remark, however, that in our particular model the resulting fermion eigenenergies are free of this effect. This may be a special property of our model as both the tree-level Lorentz violation and the induced correction have the same C, P, and T properties. Nevertheless, the model discussed in this work could still exhibit other observables, such as ones involving the one-loop eigenspinors, in which the Lorentz-breaking correction dominates the tree-level effects. In the context of more general models involving
the parity-odd weak interaction, the radiatively induced terms are likely to display a greater variety of structures since they do not necessarily have to share the same C, P, and T properties of the tree-level Lorentz violation. Then, even the eigenenergies may show the effects mentioned above.

Another immediate consequence of our result concerns multimetric theories [49], such as recently proposed bimetric models [50, 51]. The basic idea in models of this type is that different fields experience different effective metrics. But our analysis shows that the concept of two metrics is difficult to maintain in a quantum theory: beyond tree level, radiative corrections to particle propagation typically induce higher-order terms incompatible with an effective-metric interpretation. This difficulty by itself does not affect the consistency of such models; it rather illustrates, for example, that the trajectory of the particle is not a geodesic with respect to some metric.

To see this more explicitly, consider first the free electromagnetic field in our model. Inspection of Eqs. (6) and (7) establishes that \( \hat{\eta}^{\mu \nu} = \eta^{\mu \nu} + \hat{k}^{\mu \nu} \) can be interpreted as the effective (inverse) metric that governs photon propagation at tree level. Similarly, comparison of our fermion kinetic term \( \frac{1}{2i} \bar{\psi} (\partial^\mu + \xi^\mu) \gamma^\nu \partial_\nu \psi \) with that in general coordinates \( \frac{1}{2} i \bar{e}_\psi e_{\psi}^* \gamma^\mu \partial_\mu \psi \) reveals that we may interpret \( \delta^\mu_\alpha + c^\mu_\alpha \) as the vierbein \( e^\mu_\alpha \) [55]. It is then apparent that the fermion propagation is controlled by the (inverse) effective metric \( (g_f)^{\mu \nu} = e^\mu_\alpha e^\nu_\beta \eta^{\alpha \beta} = \eta^{\mu \nu} + 2\epsilon^{\mu \nu} + \mathcal{O}(c^2) \) at tree level. We see that in the absence of quantum corrections our Lorentz-violating QED extension can indeed be interpreted as a bimetric model in the flat-spacetime limit. We remark in passing that this is consistent with our earlier discussion that only \( 2\epsilon^{\mu \nu} - \hat{k}^{\mu \nu} \) is observable. On the other hand, our analysis has shown that the leading radiative corrections to the free-fermion propagation—displayed in Eq. (90)—are determined by a term of the form \( \bar{\psi} (2\epsilon^{\mu \nu} - \hat{k}^{\mu \nu}) \partial_\mu \partial_\nu \psi \). But such a term precludes an interpretation of the fermion’s propagation as being governed by an effective metric.

On a more practical level, we applied our formalism to Coulomb scattering for the case \( \epsilon^{\mu \nu} = 0, \hat{k}^{\mu \nu} \neq 0 \). We showed that, just like in the usual Lorentz-symmetric case, infrared divergences cancel when soft-photon emission is taken into account in the final fermion states. It should be stressed that this result involves a non-trivial cancellation between various infrared-divergent Lorentz-violating terms.

Our study also demonstrates how to extract the S-matrix of a process with external fermion states in the presence of Lorentz-breaking coefficients, generalizing the usual LSZ reduction formula to the Lorentz-violating case.

In this work, we have only considered Lorentz-violating radiative effects on fermion external states. In particular, we have left unaddressed the effects of the Lorentz breakdown on vacuum polarization. The main reason for this omission is that a Källén–Lehmann representation for the photon propagator in the presence of the \( k_P^{\alpha \beta} \) coefficient (and in particular its birefringent part) has not yet been developed, which makes a proper extraction of the photon pole(s) unclear. We expect to come back to this issue in the future.

Acknowledgments

We thank Gregory Adkins, Alan Kostelecký, Jacob Noordmans, and Keri Vos for discussions. R.P. and M.C. wish to thank the Physics Department of Indiana University for the kind hospitality during several stages of this work. This work has been supported in part by the Portuguese Fundação para a Ciência e a Tecnologia, by the Mexican RedFAE, by FONDECYT Grant No. 11121633, by the Indiana University Center for Spacetime Symmetries, by the IU Collaborative Research and Creative Activity Fund of the Office of the Vice President for Research, and by the IU Collaborative Research Grants program.

Appendix A: Feynman rules for the second perturbation scheme

This appendix presents the Feynman rules needed for perturbative calculations in our model. In Sec. III, we briefly discussed two natural schemes for setting up perturbation theory, each entailing different decompositions of our model’s full Lagrangian and the ensuing different sets of Feynman rules for each scheme. For convenience, we selected as the zeroth-order system the full renormalized quadratic Lagrangian (including quadratic Lorentz-violating pieces) and to treat the non-quadratic terms as a perturbation. The Feynman rules for this choice, i.e., for decomposition corresponding to Eqs. (15), (16), and (17), are:

\[
\begin{align*}
\text{photon} & : \quad p \quad = \quad \frac{i(p + c^\mu_p + m)}{p^2 + 2c^\rho_p - m^2}, \quad (A1) \\
\text{fermion} & : \quad \mu \quad \leftrightarrow \quad \nu \quad = \quad -i(\eta^{\mu \nu} - k^{\mu \nu}) \quad \frac{q^2 + k^2_q - m^2_\gamma}{q^2 + k^2_q - m^2_\gamma}, \quad (A2) \\
\text{vertex} & : \quad \mu \quad \leftrightarrow \quad \nu \quad = \quad -ie(\gamma^\mu + \epsilon^{\mu \nu} \gamma_\nu), \quad (A3)
\end{align*}
\]

where we have selected \( \xi = 1 \) Feynman gauge. Counterterm expressions, which are not displayed here, have been taken from Ref. [16].
Appendix B: Quantization of the Dirac field in the presence of Lorentz violation

In this appendix, we carry out the explicit quantization of the Dirac field in the presence of Lorentz violation. One possible approach is to use a field redefinition \( \psi = A \chi \) that transforms the terms with time derivatives to the standard Dirac form [13, 47]. Alternatively, one may use an unconventional Dirac form in spinor space to bypass the hermiticity issues associated with unconventional time derivatives [52]. The method presented below is based on the latter approach; it is more direct; it maintains spinor coordinate covariance, and it introduces explicit creation and annihilation operators for the particle modes corresponding to the physical field \( \psi \) as well as an expression for the Hamiltonian in terms of these. These features are particularly suitable for the quantization of the spinor equation of motion (108).

We start with the free-fermion Lagrange density

\[
\mathcal{L}_f = \bar{\psi}(i \Gamma^\mu \partial_\mu - M)\psi, \tag{B1}
\]

with

\[
\Gamma^\mu = \gamma^\mu + c^{\mu\nu} \gamma_\nu + d^{\mu\nu} \gamma_\nu \gamma_5 + i f^{\mu\nu\sigma} \gamma_\sigma + \frac{1}{2} g^{\lambda\mu\nu} \sigma_{\lambda\nu} + e^\mu, \tag{B2}
\]

\[
M = m + a^\mu \gamma_\mu + b^\mu \gamma_\mu \gamma_5 + \frac{1}{2} H^{\mu\nu} \sigma_{\mu\nu}. \tag{B3}
\]

The Lorentz-violating background is taken as real-valued, so that \( \Gamma^\mu = \gamma^0 \Gamma^\mu \gamma^0 \) and \( M = \gamma^0 M \gamma^0 \). The canonical momentum is given by

\[
\pi = \frac{\partial \mathcal{L}_f}{\partial \dot{\psi}} = i \bar{\psi} \Gamma^0, \tag{B4}
\]

and the canonical Hamiltonian becomes

\[
H = \int d^3 x \left[ \pi \dot{\psi} - \mathcal{L}_f \right] = \int d^3 x \ \bar{\psi} (-i \Gamma \cdot \nabla + M) \psi
\]

\[
= \int d^3 x \ \bar{\psi} (\Gamma^0)^{-1} (-\Gamma \cdot \nabla - i M) \psi. \tag{B5}
\]

The matrix \( \Gamma^0 \) is indeed invertible for perturbatively small Lorentz violation [13]. From this Hamiltonian, one can recover the equation of motion

\[
\dot{\psi}(x) = \frac{\delta H}{\delta \pi(x)} = (\Gamma^0)^{-1} (-\Gamma \cdot \nabla - i M) \psi. \tag{B6}
\]

Next, we expand the solutions of the equation of motion for \( \psi \) in Fourier modes:

\[
\psi(x) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \ \frac{m}{\omega_p} \sum_{s=1,2} \left[ b_s(\vec{p}) u^s(\vec{p}) e^{-ip \cdot x}
\right.
\]

\[
+ d_s^+(\vec{p}) \bar{v}^s(\vec{p}) e^{ip \cdot x} \right], \tag{B7}
\]

where \( \omega_p \equiv + (m^2 + \vec{p}^2)^{1/2} \) denotes the conventional fermion energies, and \( p^0 \) takes the absolute value of the respective (four) eigenvalues of \( (\Gamma^0)^{-1}(\Gamma^\mu \pi^\mu + M) \). The latter are, in general, all different. But for small enough Lorentz-violating parameters, two of them (which we will denote \( p_{a,s}^0 \)) are positive, and two \( (-p_{a,s}^0) \) are negative [13]. Below, we will show that these eigenvalues are, in fact, real. The corresponding eigenvectors are \( u^s(\vec{p}) \) and \( v^s(\vec{p}) \), which satisfy

\[
(\Gamma^0 p_a - M) u^s(\vec{p}) = 0, \quad (\Gamma^0 p_a + M) v^s(\vec{p}) = 0. \tag{B8}
\]

Here, Latin superscripts \( r, s, \ldots \) from the middle of the alphabet label the spin-type state. To quantize, we replace the Poisson brackets with \( i \times \) anti-commutators:

\[
[\bar{\psi}_a(\vec{x}, t), \psi_b(\vec{y}, t)]_+ = [\bar{\psi}_a(\vec{x}, t), -i \pi_c(\vec{y}, t)]_+ (\Gamma^0)^{-1}_{ab} = (\Gamma^0)^{-1}_{ab} \delta^3(\vec{x} - \vec{y}). \tag{B9}
\]

and

\[
[\bar{\psi}_a(\vec{x}, t), \psi_b(\vec{y}, t)]_+ = [\bar{\psi}_a(\vec{x}, t), \psi_b(\vec{y}, t)]_+ = 0, \tag{B10}
\]

where Latin subscripts \( a, b, c, \ldots \) from the beginning of the alphabet denote spinor components. We now proceed to check that the anticommutation relations (B9) and (B10) correspond to taking for the oscillators the following nonzero anticommutators:

\[
[b_r(\vec{p}), b_s^*(\vec{q})]_+ = [d_r(\vec{p}), d_s^*(\vec{q})]_+ = (2\pi)^3 \frac{\omega_p}{m} \pi^3(\vec{q} - \vec{p}) \delta_{rs}. \tag{B11}
\]

With these relations, we find from Eq. (B7)

\[
[\bar{\psi}_a(\vec{x}, t), \bar{\psi}_b(\vec{y}, t)]_+ = \int \frac{d^3 \vec{p}}{(2\pi)^3} \ \frac{m}{\omega_p} \sum_s [u^s_a(\vec{p}) \bar{u}^s_b(\vec{p})
\]

\[
+ v^s_a(\vec{y}, \vec{p}) v^s_b(\vec{y}, \vec{p}) e^{+i\vec{p} \cdot (\vec{x} - \vec{y})}. \tag{B12}
\]

To see that the matrix \( (\Gamma^0)^{-1} (\Gamma^\mu \pi^\mu + M) \) has real eigenvalues, we note that

\[
\langle u|v \rangle \equiv \bar{u} \Gamma^0 v \tag{B13}
\]

represents an (unconventional) inner product on spinor space. In this expression, \( u \) and \( v \) are arbitrary spinors with the usual notation \( u \equiv u^\gamma \gamma^0 \). The definition (B13) clearly satisfies the appropriate linearity conditions of an inner product. To establish positive definiteness, note first that \( \bar{u} \Gamma^0 u \) is always real since \( \Gamma^0 \) differs from \( \mathbb{1} \) only by SME corrections, which implies positive definiteness for sufficiently small Lorentz violation [13]. The matrix \( (\Gamma^0)^{-1} (\Gamma^\mu \pi^\mu + M) \) turns out to be hermitian with respect to the modified inner product (B13), which shows that this matrix indeed possesses real eigenvalues, as claimed. Moreover, the presumed small size of the SME corrections implies that these eigenvalues are perturbations around the conventional ones, so two eigenenergies are positive and two negative [13]. The four corresponding eigenspinors are orthogonal and can be normalized in the usual way:

\[
\bar{u}^\gamma(\vec{p}) \Gamma^0 u^s(\vec{p}) = \bar{v}^\gamma(-\vec{p}) \Gamma^0 v^s(-\vec{p}) = \frac{\omega_p}{m} \delta_{rs}. \tag{B14}
\]
As \( \{ u^1(\vec{p}), u^2(\vec{p}), v^1(\vec{p}), v^2(\vec{p}) \} \) forms a complete set of eigenspinors of the operator \((\Gamma^0)^{-1}(\vec{\Gamma} \cdot \vec{p} + M)\), we have the completeness relation
\[
\sum_{s=1}^2 \left[ u_s^a(\vec{p}) u_s^a(\vec{p}) + v_s^a(\vec{p}) v_s^a(\vec{p}) \right] \Gamma^0_{cb} = \frac{\omega_p}{m} \delta_{ab}. \tag{B15}
\]
We can now use this completeness relation in Eq. (B12), recovering the result of Eq. (B9).

As a further application, let us derive an expression for the Hamiltonian in terms of the oscillators. The quantum-field version of Eq. (B5) is given by
\[
H = \int d^3 x : \bar{\psi}(-i \vec{\Gamma} \cdot \vec{\nabla} + M) \psi :. \tag{B16}
\]
With the Fourier decomposition (B7) we find, using the equations of motion (B8) for \( u^s(\vec{p}) \) and \( v^s(\vec{p}) \), an explicitly positive-definite expression for the Hamiltonian:
\[
H = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{m}{\omega_p} \sum_{s=1}^2 \left[ E^s_a b^s_a(\vec{p}) b_s(\vec{p}) + E^s_a d^s_a(\vec{p}) d_s(\vec{p}) \right], \tag{B17}
\]
where \( E^s_a \) and \( E^a_s \) are the (positive) energies corresponding to the solutions of Eq. (B8). We also note the useful expressions
\[
b^s_a(\vec{p}) = \int d^3 x \ e^{-i\vec{p} \cdot \vec{x}} \bar{\psi}(x) \Gamma^0 u^s(\vec{p}), \tag{B18}
\]
\[
d^s_a(\vec{p}) = \int d^3 x \ e^{-i\vec{p} \cdot \vec{x}} \bar{\psi}(x) \Gamma^0 \psi(x), \tag{B19}
\]
and their hermitian conjugates.

The time-ordered product is defined in the usual way [53] and satisfies
\[
T \psi_a(x) \bar{\psi}_b(y) = \langle 0 | T \psi_a(x) \bar{\psi}_b(y) | 0 \rangle + : \psi_a(x) \bar{\psi}_b(y) :, \tag{B20}
\]
where
\[
\langle 0 | T \psi_a(x) \bar{\psi}_b(y) | 0 \rangle \equiv iS(x - y)_{ab} \tag{B21}
\]
is the modified Feynman propagator:
\[
S(x - y) = \int \frac{dk}{(2\pi)^4} \frac{e^{-i\vec{k} \cdot (x - y)}}{\Gamma^0 k_\mu - M + i\epsilon}. \tag{B22}
\]
Indeed, one can verify that
\[
(i\Gamma^\mu \partial_\mu - M) \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = i\delta^4(x - y). \tag{B23}
\]
Here, we used that \( \psi(x) \) satisfies the modified Dirac equation (B6) and the anticommutator relation (B9).

Appendix C: Canonical quantization of the radiation field with Lorentz-violating parameter \( \tilde{k}^{\mu\nu} \)

Our starting point is the following Lorentz-violating Stueckelberg Lagrange density in \( \xi = 1 \) Feynman gauge:
\[
\mathcal{L}_\gamma = -\frac{1}{4} \eta^{\alpha\beta} \eta^{\gamma\delta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{2} (\partial_\mu \tilde{\eta}^{\mu\nu} A_\nu)^2 + \frac{1}{2} m^2_A \tilde{\eta}^{\mu\nu} A_\nu, \tag{C1}
\]
where \( \tilde{\eta}^{\mu\nu} = \eta^{\mu\nu} + \tilde{k}^{\mu\nu} \), with \( \tilde{k}^{\mu\nu} = \tilde{k}^{\nu\mu} \) and \( \tilde{k}^{\mu\mu} = 0 \). The quantization of such photon models has recently been studied by various authors [15]. Here, we summarize the main results and tailor the presentation to the case at hand.

We begin by finding the canonical momenta
\[
\Pi^\alpha = \frac{\partial L}{\partial (\partial_\beta A_\mu)} = \tilde{\eta}^{\alpha\beta} F_{\alpha\beta} - \tilde{\eta}^{\beta\mu} (\partial_\mu \tilde{\eta}^{\nu\mu} A_\nu), \tag{C2}
\]
and impose the fundamental equal-time commutation relations
\[
[A_\mu(t, \vec{x}, A_\nu(t, \vec{y})] = [\Pi^\mu(t, \vec{x}), \Pi^\nu(t, \vec{y})] = 0, \tag{C3}
\]
\[
[A_\mu(t, \vec{x}), \Pi^\nu(t, \vec{y})] = i\delta^\nu_\mu \delta^3(\vec{x} - \vec{y}). \tag{C4}
\]
From Eq. (C3) it follows that the spatial derivatives of \( A_\mu \) commute at equal times. Using Eqs. (C4) and (C2), one then deduces that
\[
[A_\mu(t, \vec{x}), \Pi^\nu(t, \vec{y})] = i(\tilde{\eta}^{00})^{-1} \tilde{\eta}_{\mu\nu} \delta^3(\vec{x} - \vec{y}), \tag{C5}
\]
where \( \tilde{\eta}_{\mu\nu} \) is defined as the inverse of \( \tilde{\eta}^{\mu\nu} \):
\[
\tilde{\eta}^{\mu\nu} \tilde{\eta}_{\alpha\nu} = \delta^\mu_\alpha, \quad \tilde{\eta}_{\mu\nu} \approx \eta_{\mu\nu} - \tilde{k}_{\mu\nu}. \tag{C6}
\]
The equation of motion following from (C1) is
\[
(\partial_\mu \tilde{\eta}^{\alpha\beta} \partial_\beta + m_A^2) A_\mu = 0, \tag{C7}
\]
which implies that the dispersion relation is the same for all four modes. In other words, our model is strictly free of any birefringence.

Consider now the vacuum expectation value of the time-ordered product
\[
\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle. \tag{C8}
\]
Acting on it with the kinetic operator we find
\[
(\partial_\mu \tilde{\eta}^{\alpha\beta} \partial_\beta + m_A^2) \langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle = \langle 0 | T \left[ (\partial_\mu \tilde{\eta}^{\alpha\beta} \partial_\beta + m_A^2) A_\mu(x) \right] A_\nu(y) | 0 \rangle \]
\[
+ \delta(x_0 - y_0) \langle 0 | [\tilde{\eta}^{\alpha\beta} \partial_\beta A_\mu(x) A_\nu(y)] | 0 \rangle \]
\[
= \delta(x_0 - y_0) \delta^{00} \langle 0 | [\tilde{A}_\mu(x), A_\nu(y)] | 0 \rangle \]
\[
= i\delta^4(x - y) \tilde{\eta}_{\mu\nu}, \tag{C9}
\]
where we have used relation (C5). We infer that \( \langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle \) must indeed be equal to the (modified) Feynman propagator:
\[
\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle = -i \int \frac{d^3 k}{(2\pi)^4} \frac{e^{-i\vec{k} \cdot (x - y)}}{k_\alpha \tilde{\eta}^{\alpha\beta} k_\beta - m_A^2 + i\epsilon} \tilde{\eta}_{\mu\nu}, \tag{C10}
\]
paralleling the Lorentz-invariant case.

It is useful to cast Eq. (C10) in an alternative form. To this end, we write
\[
\frac{1}{k_\alpha \eta^{\alpha \beta} k_\beta - m^2 + i\epsilon} = \frac{1}{k_{0+} + k_{0-}} \left( \frac{1}{k_0 - k_{0+} + i\epsilon} - \frac{1}{k_0 + k_{0-} - i\epsilon} \right),
\]
(C11)

where \( \pm k_{0\pm}(\vec{k}) \) are the two roots of the dispersion relation
\[
k_\alpha \eta^{\alpha \beta} k_\beta - m^2 = 0,
\]
(C12)

which follows from Eq. (C7). Here, \( k_{0\pm}(\vec{k}) \) are both taken positive, so that the roots of the dispersion relation have alternate signs, as in the Lorentz-invariant case. (This is justified, for example, in concordant frames, in which we may take \( |\vec{k}^{\mu\nu}| \ll 1 \) on experimental grounds.) Note that, generally, \( k_{0+}(\vec{k}) \neq k_{0-}(\vec{k}) \), but
\[
k_{0\pm}(\vec{k}) = k_{0\mp}(\vec{k})
\]
(C13)

follows because Eq. (C12) is even in the components of the momentum. Using Eq. (C11), one derives easily that the Feynman propagator (C10) can be represented as
\[
\langle 0| T A_\mu(x) A_\nu(y) |0\rangle = -\bar{\eta}_{\mu\nu}(\bar{\eta}^{00})^{-1} \int \frac{d^3k}{(2\pi)^3 k_{0+} + k_{0-}} \left[ \theta(x^0 - y^0)e^{-i(k_{0+} x_0 - \vec{k} \cdot \vec{x})} + \theta(y^0 - x^0)e^{i(k_{0+} x_0 - \vec{k} \cdot \vec{x})} \right]. \tag{C14}
\]

Note that in both terms of Eq. (C14) only the positive root for \( k_0 \) appears in the exponentials. It is also worthwhile pointing out the factor \( k_{0+} + k_{0-} \) (which, unlike the individual roots \( k_{0\pm} \), is an even function of \( \vec{k} \)) that appears in the denominator, replacing the usual factor \( 2k_0 \).

Let us now try to represent the dynamical system described above by the simple mode expansion
\[
A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 N(k)} \sum_{\lambda=0}^3 \left[ a^{(\lambda)}_\mu(k)e^{(\lambda)}_\mu(k) - i k^\mu x \right], \tag{C15}
\]
where \( k^\mu = (k_{0+}, \vec{k}) \) satisfies the dispersion relation (C12), and \( N(k) \) is a (yet to be determined) function. Next, we posit creation and annihilation operators satisfying the nonzero commutation relations
\[
[a^{(\lambda)}_\mu(k), a^{(\lambda')}_{\mu'}(k')'] = -(2\pi)^3 \eta^{\lambda\lambda'} M(\vec{k}) \delta^3(\vec{k} - \vec{k}'). \tag{C16}
\]

Here, the normalization \( M(\vec{k}) \) is to be chosen later. With Eqs. (C15) and (C16) at hand, the time-ordered product \( \langle 0| T A_\mu(x) A_\nu(y) |0\rangle \) can be expressed as
\[
\langle 0| T A_\mu(x) A_\nu(y) |0\rangle = -\int \frac{d^3k}{(2\pi)^3 N(k)^2} \sum_{\lambda,\lambda'} \epsilon^{(\lambda)}_\mu(k) \epsilon^{(\lambda')}_{\mu'}(k) \eta_{\lambda\lambda'} \left[ \theta(x^0 - y^0)e^{-i k^\mu(x-y)} + \theta(y^0 - x^0)e^{i k^\mu(x-y)} \right]. \tag{C17}
\]

Comparing Eq. (C14) with our earlier form of the Feynman propagator (C17), we deduce
\[
\frac{M(k)}{N(k)^2} \sum_{\lambda,\lambda'} \epsilon^{(\lambda)}_\mu(k) \epsilon^{(\lambda')}_{\mu'}(k) \eta_{\lambda\lambda'} = \frac{1}{k_{0+} + k_{0-}}(\bar{\eta}^{00})^{-1} \bar{\eta}_{\mu\nu}. \tag{C18}
\]

While there are various ways to satisfy Eq. (C18), we will take
\[
N(k) = M(k) = k_{0+} + k_{0-} \tag{C19}
\]
\[
\sum_{\lambda,\lambda'} \epsilon^{(\lambda)}_\mu(k) \epsilon^{(\lambda')}_{\mu'}(k) \eta_{\lambda\lambda'} = (\bar{\eta}^{00})^{-1} \bar{\eta}_{\mu\nu} \tag{C20}
\]
in what follows. It is a non-trivial but straightforward exercise to verify that with the choices (C19) and (C20) the equal-time commutators (C3) and (C4) are correctly represented. As an aside, we note that the usual relation \([A_\mu(t, \vec{x}), A_\nu(t, \vec{y})] = 0\) is no longer valid.

Equation (C20) implies the normalization condition
\[
\epsilon^{(\lambda)}_\mu(k) \epsilon^{(\lambda')}_{\mu'}(k) \bar{\eta}^{\mu\nu} = (\bar{\eta}^{00})^{-1} \eta^{\lambda\lambda'}. \tag{C21}
\]
It is convenient to select the time-like, unphysical polarization mode as
\[
\epsilon^{(0)}_\mu(k) = \frac{k_\mu}{m \sqrt{\bar{\eta}^{00}}}. \tag{C22}
\]
in accordance with the spin sum (C21). The three transverse, physical polarization modes \( \epsilon^{(\lambda)}_\mu(k) \) for \( \lambda = 1, 2, 3 \) are orthonormal, space-like vectors, orthogonal to \( k^\mu \).
with respect to the effective metric $\tilde{\eta}^{\mu\nu}$. This choice of the transverse modes corresponds to defining the physical states satisfying
\[
(\text{phys}|\partial_\mu \tilde{\eta}^{\mu\nu} A_\nu|\text{phys}) = 0. \tag{C23}
\]
It follows from Eqs. (C20) and (C22) that the physical-polarization sum becomes
\[
\tilde{\eta}^{00} \sum_{\lambda = 1}^{3} \epsilon^{(\lambda)}_\mu(k) \epsilon^{(\lambda)}_\nu(k) = -\tilde{\eta}_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}. \tag{C24}
\]
Finally, we note that with the choice (C19) the 3-momentum measure appearing in Eq. (C15) satisfies the property
\[
\int \frac{d^4k}{k_{0+} + k_{0-}} f(k_0, \vec{k}) = \frac{i\tilde{\eta}^{00}}{2} \int d^4k \delta^4(k_\alpha \tilde{\eta}_{\alpha\beta} k_\beta - m^2) f(k_0, \vec{k}), \tag{C25}
\]
where $f(k^\mu)$ is an arbitrary even function of $k^\mu$.

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The Feynman rules for the insertion in Fig. 1 and for the photon propagator in Fig. 2 are irrelevant at this stage.

In a recent paper all higher-dimensional Lorentz-violating operators that can appear in the kinetic term for the Dirac fermion have been classified [42]. The operators in Eq. (22) involve coefficients of the operators $\tilde{m}, \epsilon^\mu_\nu, \tilde{m}_5, \tilde{d}^\mu_\nu, \tilde{d}^\nu_\mu$ and $\tilde{H}_{\mu\nu}$ defined in Ref. [42], respectively.

Beyond linear order in Lorentz violation and when additional interactions are considered (e.g., parity-violating electroweak physics) the general structure of $\delta \Sigma$ becomes more intricate than that given in Eq. (23).

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In the conventional case, the time-ordered product is coordinate independent because the Dirac field anticommutates outside the lightcone. In the present context involving Lorentz violation, certain SME-parameter combinations allow the anticommutator function to be supported also in small spacelike regions in the close vicinity of the light cone. Extremely large boosts of the coordinate system may then spoil coordinate independence of the time-ordered product.

Strictly speaking, we should include the action for the ghost fields [39]. As in this paper we will be mainly interested in external physical states, we will disregard them in this work.

Note that our $c$ coefficient is spacetime constant, so that with our interpretation the spin connection vanishes. Note also that $c$ is traceless, so $c \equiv \det e^\mu_\nu = 1 + O(c^2)$. 

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