Chiral transition in the probe approximation from an Einstein-Maxwell-dilaton gravity model

Hardik Bohra^a,b, David Dudal^c,d†, Ali Hajilou^c,f‡, Subhash Mahapatra^a§

^a Department of Physics and Astronomy, National Institute of Technology Rourkela, Rourkela - 769008, India
^b Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA
^c KU Leuven Campus Kortrijk–Kulak, Department of Physics, Etienne Sabbelaan 53 bus 7657, 8500 Kortrijk, Belgium
^d Ghent University, Department of Physics and Astronomy, Krijgslaan 281-S9, 9000 Gent, Belgium
^e Department of Physics, Shahid Beheshti University G.C., Evin, Tehran 19839, Iran
^f School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), Tehran 19395-5531, Iran

Abstract

We refine an earlier introduced 5-dimensional gravity solution capable of holographically capturing several qualitative aspects of (lattice) QCD in a strong magnetic background such as the anisotropic behaviour of the string tension, inverse catalysis at the level of the deconfinement transition or sensitivity of the entanglement entropy to the latter. Here, we consistently modify our solution of the considered Einstein-Maxwell-dilaton system to not only overcome an unphysical flattening at large distances in the quark-antiquark potential plaguing earlier work, but also to encapsulate inverse catalysis for the chiral transition in the probe approximation. This brings our dynamical holographic QCD model yet again closer to a stage at which it can be used to predict magnetic QCD quantities not directly computable via lattice techniques.

1 Introduction

Subjecting QCD to extreme external conditions such as temperature, density and/or electromagnetic fields is not only a matter of formal and theoretically challenging studies, but of direct possible relevance for current particle accelerator driven research programs [1–3], early universe physics [4,5], dense neutron stars [6], gravitational waves physics [7], etc.

Next to the necessary high temperature conditions to liberate quarks from their permanent confinement, (non-central) relativistic heavy ion collisions might also create during the short-lived quark-gluon plasma stage [8,9], a strong magnetic background [10–15], another player affecting the QCD phase diagram [16–18].

Unfortunately, understanding the QCD phase structure under the aforementioned circumstances remains challenging [19]. Analytical approaches are hard because of the strong coupling, and there are always some modelling or truncation artifacts, etc. The simulation-based approach of lattice QCD is a powerful ally, but due to the inherently Euclidean nature of the Monte Carlo setup, the effects of a chemical potential or transport coefficients—related to out-of-equilibrium physics, are conceptually difficult to access because of the infamous sign-problem. The modern tensor network paradigm does not suffer from this particular conceptual drawback, but as of now seems to be computationally limited to lower-dimensional gauge theories [20].

Another option—the one we will follow here—is applying the gauge-gravity correspondence rooted in [21–23], which has become a key player in the field of theoretical studies of the strongly coupled quark-gluon plasma. A key modification of the original AdS/CFT vocabulary is that QCD requires a mass scale/confinement, not available in a conformal setting. This means the original AdS gravity background is untenable and needs to be replaced by more involved backgrounds reflecting the fundamental QCD scale, allowing at least for confinement and a massive spectrum built from the (almost) massless original degrees of freedom. Relevant examples of such AdS/QCD theories are [24–26], some examples of the role of anisotropy, as brought in by a magnetic field, in dual gauge theories can be found in e.g. [27–38].

In this work, we will mainly focus on the inverse magnetic catalysis of chiral symmetry breaking, something which was first perceived as unexpected and counterintuitive: indeed, the earlier papers [39, 40] gave support to a magnetic field induced catalysis of the chiral condensate, from which naively a larger chiral restoration temperature could be guessed. Nonetheless, the opposite behaviour received convincing lattice evidence from [41], see also [42, 43]. Likewise, the deconfinement transition temperature also follows this inverse catalysis behaviour. An incomplete list on (inverse) magnetic catalysis motivated works is [44–62], with holographic contributions being, for example, [63–87].

This paper continues the study laid out in [88]. To model QCD in a magnetic background, we will rely on an exact solution of the Einstein-Maxwell-dilaton (EMD) gravity system, obtainable via the potential reconstruction method.
In this paper we will not be interested in studying the effect of a chemical potential, so we will set it equal to zero after this subsection.

For our current qualitative purposes, we will keep using the 5-dimensional magnetic field. Notice this is more than just $B$.

We remind here that this magnetic field $f$ in [88], complete solutions can be expressed in terms of two arbitrary functions, i.e. the gauge coupling function $A_1(z)$ and electromagnetic field tensors $F_{(1)MN}$.

We end with a short outlook to further research in Sect. 4.

## 2 Survey of the gravitational background

### 2.1 Magnetized Einstein-Maxwell-dilaton gravity

In order to study the effect of a magnetic field and chemical potential on some features of QCD in the context of holographic QCD (AdS/QCD) models, we rely on a gravity background with two Maxwell fields, a first one dual to the chemical potential (or better said, the neutral baryon number current), and a second one dual to the electromagnetic current in the boundary field theory. In such $U(1) \times U(1)$ setup, mesons are charge neutral, so we cannot directly couple electromagnetism to the theory. We will employ the second gauge field just to introduce a (constant) magnetic field i.e., we have no interest in the associated mesonic fluctuations.

So, the background that we have utilized is five-dimensional EMD gravity [88],

$$S_{EM} = -\frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ -R - \frac{f_1(\phi)}{4} F_{(1)MN} F^{MN} - \frac{f_2(\phi)}{4} F_{(2)MN} F^{MN} - \frac{1}{2} \partial_M \phi \partial^M \phi - V(\phi) \right], \quad (2.1)$$

where $G_5$ is the Newton constant in five dimensions, $R$ is the Ricci scalar, $\phi$ is the dilaton field, $F_{(1)MN}$ and $F_{(2)MN}$ are the field strength tensors for the two $U(1)$ gauge fields, $f_1(\phi)$ and $f_2(\phi)$ are the gauge kinetic functions that act as coupling between gauge fields and dilaton field and $V(\phi)$ is the potential of the dilaton field (for more details about this action see [88]).

To obtain the on-shell solutions, the following Ansätze have been considered for the metric field $g_{MN}$, dilaton field $\phi$ and electromagnetic field tensors $F_{(i)MN}$:

$$ds^2 = \frac{L^2 S(z)}{z^2} \left[ -g(z) dt^2 + \frac{dz^2}{g(z)} + dy_2^2 + e^{B^2 z^2} \left( dy_1^2 + dy_3^2 \right) \right],$$

$$\phi = \phi(z), \quad A_{(1)M} = A_1(z) \delta_M, \quad F_{(2)MN} = B dy_2 \wedge dy_3, \quad (2.2)$$

where $L$ is the AdS length scale, $S(z)$ is the scale factor, and $g(z)$ is the blackening function. Here, $z$ is the radial coordinate with $z = 0$ at the AdS boundary. This coordinate $z$ runs from the boundary to the horizon at $z = z_h$ for the black hole case or to $z = \infty$ for the thermal-AdS case. Also, the magnetic field $B$ is located parallel to the $y_1$-direction. We remind here that this magnetic field $B$ (mass dimension 1) is actually the 5-dimensional one, which needs a rescaling via the AdS length $L$ to get the physical, 4-dimensional, boundary magnetic field $B$ (mass dimension 2) [63], see also [100].

For our current qualitative purposes, we will keep using the 5-dimensional magnetic field. Notice this is more than just working in units $L = 1$, there is still an undetermined dimensionless ratio between the two magnetic fields.

Utilizing the above Ansätze eq. (2.2), imposing suitable boundary conditions and following the procedure outlined in [88], complete solutions can be expressed in terms of two arbitrary functions, i.e. the gauge coupling function $f_1(z)$ and the scale function $S(z)$. Doing so, the solution for $A_1(z)$ is

$$A_1(z) = \frac{\mu \xi_e^{-B^2 z^2}}{f_1(\xi) \sqrt{S(\xi)}} \int_{z_h}^{z} d\xi \frac{\xi e^{-B^2 \xi^2}}{f_1(\xi) \sqrt{S(\xi)}}, \quad (2.3)$$

and the solution for gauge coupling function $f_2(z)$

$$f_2(z) = -\frac{e^{2B^2 z^2} L^2 S(z)}{z} \left[ g(z) \left( 4B^2 z^2 + 3S(z) \right) - 4 \right] + 2y'(z), \quad (2.4)$$

while for the potential $V(z)$

$$V(z) = \frac{g(z)}{L^2} \left( -\frac{9B^2 z^2 S(z)}{2S(z)} - \frac{10B^2 z^2}{S(z)} - \frac{3z^2 S(z)^2}{S(z)^3} + \frac{12z S(z)}{S(z)^2} + \frac{z^2 \phi'(z)^2}{2S(z)} - \frac{12}{S(z)} \right),$$

$$-\frac{4}{2L^4S(z)^2} + \frac{g'(z)}{L^2} \left( -\frac{B^2 z^2}{2S(z)} - \frac{3z^2 S(z)}{S(z)^3} + \frac{3z}{S(z)} \right). \quad (2.5)$$

1In this paper we will not be interested in studying the effect of a chemical potential, so we will set it equal to zero after this subsection.

2We set $L = 1$ in the numerical calculations.
can write the solutions for $g(z)$ and $\phi(z)$ more explicitly as

$$g(z) = 1 + \int_0^z d\xi \xi^4 e^{-B^2z^2-3A(\xi)} \left[ K_3 + \frac{\tilde{\mu}^2}{2cL^2} e^{\xi^2} \right],$$

with

$$K_3 = -\frac{1 + \frac{\tilde{\mu}^2}{2cL^2} \int_0^{z_b} d\xi \xi^4 e^{-B^2z^2-3A(\xi)}}{\int_0^{z_b} d\xi \xi^4 e^{-B^2z^2-3A(\xi)}}.$$

$$\phi(z) = \int dz \sqrt{-\frac{2}{z} \left( 3zA'(z) - 3zA'(z)^2 + 6A'(z) + 2B^4z^3 + 2B^2z \right) + K_5}$$

where we can fix the constant $K_3$ by demanding that $\phi|_{z=0} = 0$, $A(z)$ is scale factor, and $c$ is a constant that can be fixed as $c = 1.16$ GeV$^2$, see (88,98,99). Also, to obtain the critical Hawking-Page (deconfinement) transition temperature $T_{\text{crit}}$ as a function of magnetic field, we need the black hole temperature and entropy that are

$$T = -\frac{3}{4\pi} \frac{e^{-3A(z_b)-B^2z_b^2}}{4\pi} \left[ K_3 + \frac{\tilde{\mu}^2}{2cL^2} e^{z_b^2} \right],
\qquad S = \frac{e^{B^2z_b^2+3A(z_b)}}{4z_b^2}.$$

It is important to stress that the eqs. (2.3), (2.4), (2.5), (2.6) and (2.7) are a complete solution for magnetized EMD gravity, just depending on the form factor choice $A(z)$. The consistency of this potential reconstruction approach was thoroughly addressed in our previous paper (88), including the compatibility of the on-shell potential with the Gubser stability criterion (107), next to the almost independence on the external parameters $z_b$ (or $T$), $B$ and $\mu$ of the (on-shell) potential $V(z)$. In the rest of the paper, we will first shortly reconsider our original choice for the form factor, as used in (88), and then introduce a slightly modified form factor to overcome the unphysical string breaking. Afterwards, we discuss the chiral phase transition in Sect. 3.

### 2.2 The original form factor $A(z) = A_1(z) = -az^2$

Similar to (88,98) the first case for scale factor that we have used is $A(z) = A_1(z) = -az^2$ where $a$ can be fixed by matching to the deconfinement temperature obtained from lattice QCD at $B = 0$, yielding $a = 0.15$ GeV$^2$. Utilizing this form factor, the dilaton field $\phi(z)$ is

$$\phi(z) = \left( \frac{9a - B^2}{\sqrt{6a^2 - B^4}} \right) \log \left( \sqrt{6a^2 - B^4} \sqrt{6a^2 z^2 + 9a - B^4 z^2 - B^4} \right) + z \sqrt{6a^2 z^2 + 9a - B^4 (B^2 z^2 + 1)} - \left( \frac{9a - B^2}{\sqrt{6a^2 - B^4}} \right) \log \left( \sqrt{6a^2 - B^4} \sqrt{6a^2 - B^4} \right).$$

From eq. (2.9) we see that there is maximal value of the magnetic field for which our system is physical. Indeed, the dilaton field should be real-valued and this will be satisfied only if $B^4 \leq B_c^4 = 6a^2$. So, when we are working with this form factor $A_1(z)$, the largest attainable value of magnetic field is $B_c \simeq 0.61$ GeV.

In our previous paper (88), we already showed that the magnetic field induced an inverse magnetic catalysis behaviour for the deconfinement transition temperature, the situation is summarized in Fig. [1].

![Figure 1: Deconfinement transition temperature in terms of magnetic field for the case $A_1(z) = -az^2$. Here we set $\mu = 0$. In units GeV.](image)

### 2.3 A new form factor: $A(z) = A_2(z) = -az^2 - d B^2 z^5$

We will now construct a new scale factor that respects two important conditions:

1. a real-valued dilaton field, preferably also for larger values of the magnetic field.
2. the free energy of a connected string attached to a boundary quark-antiquark pair should be smaller than the disconnected one\(^\text{3}\) to assure a confined quark-antiquark pair in the dual boundary theory, at least when the metric is

\(^3\)The connected solution is U-shape configuration starting at the quark-antiquark pair living on the boundary ($z = 0$) and extends into the bulk. On the other hand, the disconnected solution is a configuration of two lines extending from the boundary to the horizon (88,98).
of the thermal-AdS type (no black hole), irrespective of the separation length between quark and antiquark. In [88], we found that the linearly increasing interquark potential becomes flattened at large separation when the scale factor $A_1(z)$ is employed.

Here, we introduce the new scale factor $A(z) = A_2(z) = -a z^2 - B z^3$, still with the parameter $a$ as before, i.e. $a = 0.15 \text{ GeV}^2$ as this was determined via comparison with $B = 0$ lattice data. For what concerns the value of the extra parameter $d$, for every positive value and as long as the magnetic field is not too large (the concrete maximal value depending on $d$), the dilaton field is real, then satisfying the first condition.

More restrictions will come from the second condition, let us therefore look at the disconnected free energy of the quark-antiquark [88-08],

$$F_{\text{discon}} = \frac{L^2}{\pi \ell_s^2} \int_{-\infty}^{\infty} dz \frac{e^{2A(z)}}{z^2}$$

where $\epsilon$ corresponds to a UV cut-off in boundary theory [108] and the $A_z(z) = A_2(z) + \sqrt{2} \phi(z)$ the form factor converted to the string frame, with $\phi(z)$ given in eq. (2.7). $\ell_s$ is the open string length, which is related to the open string constant as $T_s = 1/2\pi \ell_s^2$. For the new form factor $A_2(z)$, the disconnected free energy always diverges. Indeed, we expand $\phi(z)$ and $A_z(z)$ in the IR (viz. at large $z$) and then plug this into eq. (2.10),

$$F_{\text{discon}} = \frac{L^2}{\pi \ell_s^2} \int_{-\infty}^{\infty} dz \frac{e^{2A(z)}}{z^2} = \frac{L^2}{\pi \ell_s^2} \int_{-\infty}^{\infty} dz \left[ z^4 + \frac{2B^2 z^5}{15d} + \ldots \right],$$

where at $z = \infty$ clearly the disconnected free energy will diverge. So, the ($\epsilon$-regularized) connected free energy will always be smaller than the disconnected one, implying that the quark-antiquark pair will always enjoy confinement.

Let us investigate now in a bit more detail the eventual choice of $d > 0$, inasmuch as the influence $d$ has on the deconfinement phase transition. We set out the deconfinement transition temperature $T_{\text{crit}}$ in terms of the magnetic field $B$ for different values of $d > 0$ in Fig. 3. We observe that inverse magnetic catalysis is persistent just for sufficiently small values of $d$, i.e. $d \leq 0.013 \text{ GeV}^2$. We will from now on consider $d = 0.013 \text{ GeV}^2$ to have the largest attainable value for the magnetic field, being $B_c \simeq 1.02 \text{ GeV}$. The corresponding confinement-deconfinement phase diagram is displayed in Fig. 4 to be compared with Fig. 3, which is the $d = 0$ case.

![Figure 2: Deconfinement transition temperature in terms of magnetic field for the case $A(z) = -a z^2 - B z^3$. Here we set $\mu = 0$. In units GeV.](image1)

![Figure 3: Deconfinement transition temperature in terms of magnetic field for the case $A(z) = -a z^2 - B z^3$. Here we set $\mu = 0$. In units GeV.](image2)

Finally, we delve a bit deeper into the free energy of the quark-antiquark pair, see Figs. 4 and 5 from which it is clear that the flattening obtained in [88] is now avoided. We have determined the string tensions for both parallel and perpendicular orientation of the Wilson loop [89] shown in Figs. 4 and 5. Our results for the associated QCD string tensions with this new form factor $A_2(z)$ are compatible with lattice results for smaller values of the magnetic field that were reported in [109] [110], see also [111] [112] for other approaches: a weaker confinement for the parallel orientation and a stronger one for the perpendicular case. At larger $B$, we also find that the perpendicular string tension starts to decrease again, something not really visible from [109] [110]. This being said, unlike these lattice works, we do not have $(2 + 1)$ dynamical quark flavours in our model.

### 3 Chiral phase transition

In this section, we will investigate the chiral sector of the dual boundary theory and, in particular, investigate the behaviour of chiral condensate and the corresponding chiral critical temperature as a function of the magnetic field. The holographic action relevant for investigating the chiral properties of the boundary QCD theory will be taken as [63][113].

$$S_{\text{chiral}} = \frac{N_c}{16\pi^2} \int d^4 x \sqrt{g} \text{Tr} \left[ |DX|^2 - m_0|X|^2 - \frac{F_2^2(\phi)}{3} (F_L^2 + F_R^2) \right].$$

Here, $X$ is a $N_f \times N_f$ matrix-valued field which is in the bifundamental representation of $SU(N_f)_L \times SU(N_f)_R$. In the AdS/CFT terminology, the field $X^{\alpha \beta}$ is dual to the quark field operator $\langle \bar{q}^\alpha \psi^\beta \rangle$, with $\alpha, \beta$ being the flavour indices,\footnote{That is, the $q, \bar{q}$ pair is either oriented parallel or perpendicular to the applied magnetic field.}
The connected free energy $F_{\text{con}}^{\parallel}$ as a function of separation length $\ell^\parallel$ in the thermal-AdS background for the case when the Wilson loop is parallel to $\vec{B}$. Here $\mu = 0$, and red, green, blue, brown and orange curves correspond to $B = 0, 0.2, 0.4, 0.6$ and 0.8 respectively. In units GeV.

The connected free energy $F_{\text{con}}^{\perp}$ as a function of separation length $\ell^\perp$ in the thermal-AdS background for the case when the Wilson loop is perpendicular to $\vec{B}$. Here $\mu = 0$, and red, green, blue, brown and orange curves correspond to $B = 0, 0.2, 0.4, 0.6$ and 0.8 respectively. In units GeV.

Figure 7: The string tension in the perpendicular direction $\sigma^\perp$ as a function of $B$ in the thermal-AdS background with $\mu = 0$. In units GeV.

Figure 4: The connected free energy $F_{\text{con}}^{\parallel}$ as a function of separation length $\ell^\parallel$ in the thermal-AdS background for the case when the Wilson loop is parallel to $\vec{B}$. Here $\mu = 0$, and red, green, blue, brown and orange curves correspond to $B = 0, 0.2, 0.4, 0.6$ and 0.8 respectively. In units GeV.

Figure 5: The string tension in the parallel direction $\sigma^\parallel$ as a function of $B$ in the thermal-AdS background with $\mu = 0$. In units GeV.

and it is associated with the chiral symmetry breaking on the dual boundary side. $m_X^2$ is the mass of the field $X$ and in this work we will consider $m_X^2L^2 = -3$. $F_{\text{L,R}}$ are the field strength tensors for the two (left and right) gauge fields $A_{L,R}$. The covariant derivative of the chiral field is defined as $D_{\mu}X = \partial_{\mu}X - iA_{\mu,L}X + iXA_{R,\mu}$, which makes the above “chiral action” invariant under $SU(N_f)_L \times SU(N_f)_R$ gauge transformations.

A word about the coupling constants. The comparison between the gauge sectors of (3.1) and (2.1) in principle implicitly fixes the Newton constant $G_N$. In future work, we will see how our choice of $f_2(\phi)$ (or better said self-consistently determined solution (2.4)) is related to the QCD OPE result for the vector current correlation function, the standard way to fix by hand this gauge coupling constant [25]. For the time being, in the present paper we borrowed the prefactor of $X$-sector from [114], where it was matched upon the QCD OPE result for the scalar meson correlator.

For simplicity, following [63, 113], we will work in the approximation of degenerate flavours and consider the field $X$ to be proportional to the identity matrix in flavour space, i.e. $X(z,x^\mu) = X_0(z)1_{N_f}e^{i\alpha(z,x^\mu)}$, where $X_0(z)$ is the component independent of the boundary directions and $\pi(z,x^\mu)$ represents the chiral field. In this approximation, our main quantity of interest—the chiral condensate—becomes proportional to the quark field operator $\langle \bar{\psi}\psi \rangle$. The condensate, therefore, can be extracted by solving the $X$-field equation of motion.

It is important to point out one major difference between the chiral action (3.1) considered here and the action generally considered in the soft wall AdS/QCD models. In particular, in soft wall models, a multiplicative dilaton factor $e^{-\varphi}$ is commonly included in the chiral action in an ad-hoc way, this to get well defined QCD-like properties holographically.\footnote{In these soft wall models, the factor $e^{-\varphi}$ was introduced merely based on the “analogy” with probe matter coming from branes. At the practical level, the dilaton prefactor $e^{-\varphi}$ was considered to “smoothly cut-off” the standard AdS geometry to ensure confinement dynamics.}

The main reason for this is that the usual background geometry does not contain the back reaction of the dilaton field and, therefore, the thermal-AdS geometry does not really correspond to the confined phase. Moreover, in soft wall models, the confinement-deconfinement phase transition exists only because of this additional dilaton term. Now, in our model, the back reaction of the dilaton field is introduced consistently from the beginning, and, correspondingly, we have a genuine confinement-deconfinement phase transition. Therefore, we do not need to include the dilaton field in the chiral action of our model in an ad-hoc manner, as the self-consistent background geometry already takes care of it. Further, as we will discuss in Appendix A, including such extra $e^{-\varphi}$ factor would even introduce additional UV divergences in the chiral condensate for $(B \neq 0, T \neq 0)$ than those already present for $(B = 0, T = 0)$, which is certainly not the case in real QCD [71]. This suggests from yet another angle this ad-hoc factor has no place in our current construction, indicating
once again the self-contained nature of our model.

Before we move on to investigate the quark condensate for the different form factors case by case, it is important to point out that there are effectively two ways by which the magnetic field enters in the chiral action: (i) through the background metric, and (ii) via the covariant derivative of \( X \). The latter contribution, however, vanishes identically as the magnetic field is introduced into the (diagonal) vector part of the flavour gauge group \( A_L = A_R \), for which

\[
D_\mu X = \partial_\mu X.
\]  

(3.2)

Therefore, the only way the magnetic field can influence the quark condensate and chiral critical temperature is through its explicit presence in the background metric. The sheer importance of this observation can even be more appreciated from the recent work \[115\] where, also in the probe brane approximation whilst using a tachyon condensation-based description of chiral symmetry breaking in the \( B = 0 \) background metric, magnetic catalysis rather than its inverse version was found.

To be more precise, our EMD model is mimicking QCD in a magnetic background, but we should not forget that we are using a dual (i.e. probe) version of quenched QCD, that is, no dynamical quarks. As gluons can only couple to the magnetic field through the charged quarks, strictly speaking there should be no dependence of gluon-dominated quantum physics (as confinement-deconfinement, dually encoded in the gravity background) on the magnetic field in a quenched approximation. Constructing a fully dual background, including the flavour brane back reaction, is however highly nontrivial and leads to more intricate, \( B \)-dependent modeling, such as the V-QCD based ones \[78, 79, 110\]. Needless to say, as there is no top-down derivation of “standard” QCD from string theory, there is certainly none for magnetized QCD. So, one is always condemned to some level of modelling in QCD-like features \[85, 117\].

At the same time, the black hole metric also allows investigating the temperature-dependent profile of the quark condensate.

Now, a word about the numerical procedure for extracting the chiral information from eq. (3.1) is in order. Using eq. (3.2), the equation of motion of the \( X \)-field is given by

\[
X''_0(z) + X'_0(z) \left( -\frac{3}{z^3} + 2B^2z + \frac{g'(z)}{g(z)} + 3A'(z) \right) + \frac{3c^2A(z)X_0(z)}{z^2g(z)} = 0.
\]  

(3.3)

The solution to this equation of motion will depend on the confined-deconfined background geometries as well as the form factor \( A(z) \). Unfortunately, to the best of our knowledge, the equation is not solvable analytically even for the simplest form factor. However, it can be straightforwardly solved numerically. We employ two different numerical shooting techniques. With the first method, we numerically integrate eq. (3.3) from the horizon to the asymptotic boundary and then extract the boundary information using the standard gauge-gravity dictionary. In particular, according to this dictionary, the leading term of the boundary expansion of \( X \) starts with the bare quark mass \( m_q \) (set by hand) as the lowest order coefficient, whereas the sub-leading term contains the information about the chiral condensate. Therefore, by fixing the bare quark mass by hand, we can integrate eq. (3.3) and numerically obtain the chiral condensate. With the second method, we do the opposite and shoot from the AdS boundary until a normalization solution for the chiral condensate is found, see Appendix B. Needless to say, both these numerical techniques render the same answer.

### 3.1 Using the form factor \( A(z) = A_1(z) = -az^2 \)

Let us first evaluate the chiral condensate for the simplest case \( A(z) = A_1(z) = -az^2 \). Since the analytic results for the background metric are explicitly known for \( A_1(z) = -az^2 \), this case will, therefore, allow us to showcase the numerical routine through which we can extract results for the chiral condensate. The results for more complicated form factors can be obtained analogously.

Let us first consider the ultraviolet, near boundary expansion of field \( X \). This is needed for the calculations of physical observables in the dual field theory side. The near boundary expansion of \( X \) is given by \[4\]

\[
X(z) = m_q z + \sigma z^3 + m_q n z^3 \ln \sqrt{\alpha z} + O(z^4).
\]  

(3.4)

where \( m_q \) is the bare quark mass and \( n = 6a - B^2 \). The coefficient \( \sigma \) is related to the actual quark condensate \( \langle \bar{\psi}\psi \rangle \) in the following way (see Appendix A for more details),

\[
\langle \bar{\psi}\psi \rangle_{B,T} - \langle \bar{\psi}\psi \rangle_{B=0,T=0} = \frac{N_c}{2\pi^2} \left[ \sigma(B,T) - \sigma(B=0,T=0) \right] + \frac{N_c m_q}{8\pi^2} B^2
\]  

(3.5)

where \( N_c = 3 \) is the number of colours. The coefficient \( \sigma \) is a temperature and magnetic field dependent quantity \[5\] which gives a temperature and magnetic field dependent profile for the quark condensate \( \langle \bar{\psi}\psi \rangle \). Here, we will use the thermal-AdS background to calculate \( \sigma(B = 0, T = 0) \), whereas the black hole background will be used to calculate \( \sigma(B,T) \). It should be noted both \( \langle \bar{\psi}\psi \rangle_{B,T} \) and \( \langle \bar{\psi}\psi \rangle_{B=0,T=0} \) contain UV divergences, however, the structure of divergences are the same for both \( \langle \bar{\psi}\psi \rangle_{B,T} \) and \( \langle \bar{\psi}\psi \rangle_{B=0,T=0} \). Correspondingly, they cancel out in the difference \( \langle \bar{\psi}\psi \rangle_{B,T} - \langle \bar{\psi}\psi \rangle_{B=0,T=0} \), giving us a finite result. Therefore, eq. (3.5) provides a renormalized quark condensate information, and we will use this equation to investigate the chiral condensate and the chiral critical temperature. This renormalization procedure is reminiscent to the renormalization procedure generally adopted in lattice studies \[11, 12\]. In particular, the left hand side of eq. (3.5) is completely analogous to the subtracted definition of the \( B \)-dependent chiral condensate on the lattice.

To obtain \( \langle \bar{\psi}\psi \rangle_{B,T} \) we further need \( \langle \bar{\psi}\psi \rangle_{B=0,T=0} \). Unfortunately, \( \langle \bar{\psi}\psi \rangle_{B=0,T=0} \) cannot be consistently obtained from the gravity input alone and it needs to be considered as an external input. Here, we take input from \( N_f = 1 \) lattice QCD and consider \( \langle \bar{\psi}\psi \rangle_{B=0,T=0} = 0.0194 \text{ GeV}^3 \) \[18\].

\[6\] This expansion is valid for both confined and deconfined geometries.

\[7\] The coefficient \( \sigma \) here should not be confused with the string tension.
Let us now briefly discuss the infrared (near horizon) expansion of $X$. Near the horizon, the field $X$ is considered to be smooth and we can assume the following Taylor expansion,

$$X(z) = A_0 + B_0(z - z_h) + O(z - z_h)^2.$$  

(3.6)

On substituting eq. (3.6) into eq. (3.3) and expanding around the horizon, we get

$$B_0 = \left( \frac{3e^{0A(z_h) + B^2z_h^2} \int_0^1 \xi^3 e^{-3A(z) - Bz^2} \, d\xi}{z_h^2} - \frac{1}{z_h} \right) A_0.$$  

(3.7)

Therefore, near the horizon $X$ behaves as

$$X = A_0 \left[ 1 + \frac{e^{-5a_3z_h^2} \left( 2e^{0a_3z_h^2} (B^2 - 3a_3)^2 z_h^4 + e^{0a_3z_h^2} (3(B^2 - 3a_3) z_h^2 + 3) - 3e^{B^2z_h^2} \right)(z - z_h) + O(z - z_h)^2 \right].$$  

(3.8)

Hence, for a fixed $a$, $z_h$ and $B$, there is one independent parameter $X(z_h)$ at the horizon, viz. $A_0$. This independent parameter can be used to construct an initial value problem for the $X$-field. The chiral condensate can then be obtained by numerically integrating the $X$-equation of motion from the horizon to the boundary and imposing the boundary condition (3.4). In particular, integrating out from the horizon to boundary gives a map $X(z_h) \rightarrow \sigma$, and this map reduces to a one-parameter family of solutions for each value of $m_q, a, z_h$ and $B$.

Our numerical results for the thermal profile of the quark condensate for different values of $B$ and a particular value of quark mass $m_q = 1.0$ GeV are shown in Fig. 8. We observe that the magnitude of the chiral condensate decreases with the magnetic field. This is a first indication that our model exhibits inverse magnetic catalysis behaviour in the chiral sector as well. Indeed, the chiral critical temperature $T_{\text{chiral}}^{\text{crit}}$, defined as the temperature at which the condensate goes to zero—decreases with the magnetic field. The complete dependence of $T_{\text{chiral}}^{\text{crit}}$ on $B$ is shown in Fig. 9. Overall, our holographic results for $T_{\text{chiral}}^{\text{crit}}$ qualitatively agree with state-of-the-art lattice results [41, 42], albeit with a slightly larger magnitude. That larger magnitude of $T_{\text{chiral}}^{\text{crit}}$ might seem problematic at first sight, however, it non-trivially depends on $m_q$ and, as we will see shortly, one can always consider different values of $m_q$ to tune its magnitude to some extent.

Note also that since the black hole geometry is thermodynamically stable only for $T > T_{\text{crit}}(B)$, we should in principle only study the thermal profile of the quark condensate only above $T_{\text{crit}}(B)$, i.e. in the deconfined phase. In the confined phase (dual to thermal-AdS), the temperature does not appear in the geometry itself. Therefore, in our model, the chiral condensate would be a temperature independent constant all the way up to $T_{\text{crit}}(B)$ and then follow the above shown condensate pattern for $T > T_{\text{crit}}(B)$. A troublesome side effect of this is that the chiral condensate would exhibit a discontinuous jump at $T_{\text{crit}}(B)$, which sounds problematic. In particular, the discontinuous jump of the chiral condensate at $T_{\text{crit}}(B)$ is in contrast with lattice findings, where no such jump is observed. Unfortunately, this is an inherent property of most holographic QCD models (arising because of implicit $N \rightarrow \infty$ approximation) and our model is no exception to that. This was already established in [113].

Further, we thoroughly investigate the behaviour of chiral condensate for various values of quark masses $m_q$. Our results are shown in Fig. 10. We consider the quark mass ranging from MeV to GeV, and for all masses, the magnitude of the chiral condensate is found to be weakened by the magnetic field. The weakening, however, is more pronounced for larger mass. Similarly, the chiral critical temperature is found to be decreasing with magnetic field for all masses. Again, the decrement is more pronounced for larger masses. The complete phase diagram with the dependence of $T_{\text{chiral}}^{\text{crit}}$ on $B$ and $m_q$ is shown in Fig. 11. Our whole analysis suggests that, irrespective of the values of $m_q$, the inverse magnetic catalysis phenomenon is a robust feature of our model.

Although the dual boundary theory exhibits an inverse magnetic catalysis phenomenon in the chiral sector for all $m_q$, however, a few subtleties arise for larger values of $m_q$. In particular, depending upon the value of $m_q$, there exists a maximum magnetic field beyond which the condensate becomes negative at all temperatures. For example, with $m_q =
Figure 10: Quark condensate $\langle \bar{\psi} \psi \rangle$ as a function of temperature $T$ for various values of the quark mass $m_q$ for the case $A_1(z) = -az^2$. The upper left, upper right, lower left and lower right figures correspond to $m_q = 0.01, 0.1, 2.0$ and $10.0$ respectively. In units GeV.

Figure 11: Variation of the chiral critical temperature with respect to the magnetic field $B$ for different quark masses for the case $A_1(z) = -az^2$. Here red, green, blue, brown and orange curves correspond to $m_q = 0.01, 0.10, 1.0, 2.0$ and $10.0$ respectively. In units GeV.

10.0 GeV, the condensate remains negative at all temperatures beyond $B \gtrsim 0.12$ GeV. Therefore, for these large magnetic fields, we cannot really define a chiral transition temperature in the usual sense, i.e. the temperature at which the condensate goes to zero. For this situation, we can use other physical observations and intuition to get some information about $T_{c^{\text{chiral}}}$. For instance, a sign change in the condensate would certainly indicate a transition to a different phase. Since the condensate remains positive in the confined metric case whereas it attains a negative value in the deconfined case for large $B$, this suggests that for these larger values of $B$ the deconfinement behaviour has already been set in and that the chiral transition temperature is less than or equal to the deconfinement temperature. If we accept this definition, then it would imply that the chiral symmetry is restored before, or at least at, the deconfinement temperature, a result which qualitatively agrees with lattice findings, where the (cross-over) transition between (de)confined and chirally broken/symmetric phases occurs within the same temperature window.
3.2 Using the form factor $A(z) = A_2(z) = -a z^2 - d B^2 z^5$

Let us now discuss the chiral sector of the dual boundary theory using the second form factor $A(z) = A_2(z)$. Most of the numerical routine and procedure are similar to what we alluded to in the above subsection, and can be straightforwardly generalised to the $A_2(z)$ case. The near boundary expansion of the chiral field $X$ remains the same (cf. eq. (3.4)) whereas the near horizon expansion changes accordingly (eq. (3.6)). Since most of the chiral analysis is similar to the previous case, therefore, we can be rather brief here.

Our results for the chiral condensate for $A_2(z)$ for various values of $m_q$ are shown in Fig. 12. As the allowed range of the magnetic field decreases for $A_2(z)$, we can now probe the chiral condensate for slightly larger magnetic field values. The chiral condensate displays a non-monotonic behaviour with respect to the temperature. This non-monotonic nature, however, is more pronounced for larger $m_q$ and $B$. Again, irrespective of the value of $m_q$, the magnitude of the chiral condensate decreases with the magnetic field, again giving indications for the inverse magnetic catalysis in the chiral sector. We further calculated the chiral critical temperature and found it to be a decreasing function of the magnetic field, thereby explicitly confirming the inverse magnetic catalysis behaviour. The complete phase diagram with the dependence of $T_{c \text{chiral}}$ on $B$ and $m_q$ is shown in Fig. 13. The magnitude of $T_{c \text{chiral}}$ again decreases with $m_q$. Importantly, the dual boundary theory exhibits inverse magnetic catalysis behaviour for all values of $m_q$.

![Figure 12: Quark condensate $\langle \bar{\psi} \psi \rangle$ as a function of temperature $T$ for various values of the quark mass $m_q$ for the case $A_2(z) = -a z^2 - d B^2 z^5$. The upper left, upper right, lower left and lower right figures correspond to $m_q = 0.01, 0.1, 1.0$ and $10.0$ respectively. In units GeV.](image)

4 Outlook

We have continued the research set out in [88] and constructed an Einstein-Maxwell-dilaton gravity model that not only captures the inverse magnetic catalysis for the deconfinement transition, but also the same phenomenon affecting the chiral transition. As a byproduct, we also investigated the anisotropy in the string tension, showing that linear confinement is always realized within the range of validity of our model. These findings are in qualitative agreement with other studies, in particular those coming from lattice simulations.

This being said, we should point out at least one subtle issue which needs further discussion. As discussed above, for both form factors $A_1(z)$ and $A_2(z)$, the chiral critical temperature depends non-trivially on the bare quark mass $m_q$. Notably, $T_{c \text{chiral}}$ decreases as $m_q$ increases when $B$ is kept fixed. Since $m_q$ is an independent boundary parameter, one may also try to manipulate it to get a reasonable $T_{c \text{chiral}}$. In particular, by taking larger $m_q$ one can try reducing $T_{c \text{chiral}}$ to get it near to the deconfinement transition temperature, i.e. to get closer to the lattice QCD counterpart. We find that the $m_q$-value needed for this purpose is unappealingly large (of the order of $10^4$ GeV) compared to the observed quark masses. We have investigated a few other relatively simple form factors as well and find similar results in all these cases. It thus appears that the larger magnitude of $T_{c \text{chiral}}$, compared to lattice results, is a generic feature of our type of models. In any case, no matter what $m_q$ we choose, our model does exhibit the inverse magnetic catalysis behaviour in the chiral
Figure 13: The variation of the chiral critical temperature with respect to magnetic field $B$ for different quark masses for the case $A_2(z) = -az^2 - dB^2z^2$. Here red, green, blue and brown curves correspond to $m_q = 0.01, 0.10, 1.0$ and 10.0 respectively. In units GeV.

sector, indicating the novelty of our model. To improve it, we plan on including a potential $V_C(\chi)$ for the chiral $X$-field, following earlier efforts as in [20, 21], the latter without magnetic field though. This will make the chiral condensate a truly dynamical feature, as for now, the crucial parameter $\sigma$ entering the chiral condensate (A.6) is directly proportional to the bare quark mass $m_q$, as can be rapidly proven from the boundary expansion (5.4).

A subtle point in such construction will also be the correct identification of the chiral condensate by extending the analysis in our Appendix A, as we have shown, see also [63].

Another issue worthy of our attention will be the proper identification of the boundary magnetic field in terms of the bulk one. To do this properly relative to QCD, we should use $N_f = 2$ flavours in the chiral sector and mimic a magnetic field by slightly gauging the unbroken diagonal sector of the underlying $U(1)_b \times SU(2)_Y \times SU(2)_C$ model [24, 119], rather than the current simplified $U(1)_b \times U(1)$ version. $U(1)_b$ refers to the baryon number current, $SU(2)_Y$ to the (unbroken) flavour symmetries and $SU(2)_C$ to the (broken) chiral symmetries, with $SU(2)_L \times SU(2)_R \simeq SU(2)_W \times SU(2)_C$. This also implies that there will be a more direct link between the chiral and EMD-action, also requiring a proper study of the coupling prefactors of both parts of the action in relation to QCD OPE results.

In principle, we should also try to include the back reaction of the chiral field $X$ into the Einstein equations of motions, which would correspond the unquenching of our setup. Although this might sound as an ambitious step, it might be possible via a generalization of the potential reconstruction method, in combination with a phenomenological profile for the potential $V_C(\chi)$ as proposed in e.g. [20].

Once all (or at least most) of the above is achieved, we can aim at studying magnetic field dependent QCD observables that are not accessible via lattice simulations, such as various transport properties.

We will report on these and other topics in the near future.

Acknowledgments

A.H. would like to thank a scholarship that has been awarded by the Ministry of Science, Research and Technology (Department of Scholarship and Students’ Affairs Abroad) of the Islamic Republic of Iran which made the initial stages of this research at KU Leuven–Kulak possible. The work of S.M. is supported by the Department of Science and Technology, Government of India under the Grant Agreement number IFA17-PH207 (INSPIRE Faculty Award).

A Appendix A: the chiral condensate

In this Appendix, we derive the relation between the coefficient $\sigma$ and the chiral condensate $\langle \bar{\psi}\psi \rangle$, following [63]. From the Lagrangian $\mathcal{L} = \bar{\psi}(\gamma^\mu\partial_\mu - m_q)\psi$, the $\langle \bar{\psi}\psi \rangle$ condensate can be obtained by differentiating the partition function $W = \ln Z$ with respect to $m_q$ as,

$$\frac{1}{Z} \frac{dZ}{dm_q} = \frac{\int [D\psi D\bar{\psi}] \left( \int d^4x \bar{\psi}\psi \right) e^{-\int d^4x \mathcal{L}}}{\int [D\psi D\bar{\psi}] e^{-\int d^4x \mathcal{L}}}.$$  \hspace{1cm} (A.1)

Here we restrict ourselves to the single quark flavour sector. Using the gauge-gravity duality and equating the bulk and the boundary partition functions, i.e. $Z = e^{-S_{chiral}}$, we obtain

$$V_4 \langle \bar{\psi}\psi \rangle = -\frac{d}{dm_q} \left( \frac{N_c}{16\pi^2} \int d^4x \sqrt{-g} \left[ \partial_\mu X \partial^\mu X + m_q^2 X^2 \right] \right)$$  \hspace{1cm} (A.2)

where $V_4$ is the volume of the four-dimensional boundary spacetime. Using the $X$-equation of motion and restricting to a homogeneous condensate $X(z, x^0) = X(z)$, we can further simplify the above expression to

$$\langle \bar{\psi}\psi \rangle = -\frac{N_c}{16\pi^2} \frac{d}{dm_q} \left( \sqrt{-g} g^{zz} X(z) X'(z) \bigg|_{z=0} \right) = \frac{N_c}{16\pi^2} \frac{d}{dm_q} \left( \sqrt{-g} g^{zz} X(z) X'(z) \bigg|_{z=0} \right).$$  \hspace{1cm} (A.3)
Here we have considered the black hole background and have used the fact that \( g^{zz}(z_h) = 0 \). For the thermal-AdS background, the upper limit \( z = z_h \) in the above equation will get replaced by \( z = \infty \). Substituting the near boundary expansion of the \( X \)-field,

\[
X(z) = m_q z + \sigma z^3 + m_q n z^3 \ln \sqrt{\alpha} + O(z^4)
\]

(A.4)

into eq. (A.3) and simplifying, we get

\[
\langle \bar{\psi} \psi \rangle = \frac{N_c}{8\pi^2} \left( \frac{m_q}{\epsilon^2} + 4m_q n \log \sqrt{\alpha} + 4\sigma + B^2 m_q + 5m_q n - 3m_q \right).
\]

(A.5)

We have again introduced a UV cut-off \( z = \epsilon \). We clearly need to renormalize the chiral condensate to proceed any further. Notice that UV divergent terms in the condensate are independent of magnetic field and temperature. If we take the difference between the \( (B \neq 0, T \neq 0) \) and \( (B = 0, T = 0) \) condensates, then the divergent terms cancel out. Indeed, in real QCD, \( B \neq 0 \), no additional UV divergences arise at finite temperature and/or magnetic field than those already present at \( B = 0, T = 0 \), and this is what we are also seeing in our holographic model. Finally, we arrive at our desired relationship,

\[
\langle \bar{\psi} \psi \rangle_{B,T} - \langle \bar{\psi} \psi \rangle_{B=0,T=0} = \frac{N_c}{2\pi^2} \left[ \sigma(B, T) - \sigma(B = 0, T = 0) \right] + \frac{N_c B^2 m_q}{8\pi^2}.
\]

(A.6)

The above holographic renormalization procedure for the chiral condensate in a magnetic field background is thus completely analogous to the one generally used during lattice studies \([41, 42]\).

At this point, we further like to emphasize the reason for not taking into account an additional dilaton factor in the chiral action (3.1). In particular, as mentioned before, in soft wall models one generally includes a prefactor \( e^{-\phi} \) in real QCD, \([41]\), no additional UV divergences arise at finite temperature and/or magnetic field than those already shown in eq. (A.5). In this case, as opposed to the renormalization scheme generally adopted in lattice QCD, we would not be able to properly renormalize \( \langle \bar{\psi} \psi \rangle_{B,T} \) by subtracting \( \langle \bar{\psi} \psi \rangle_{B=0,T=0} \) from it, which is certainly not desirable.

## Appendix B: a few words about the shooting method

Here we briefly describe how to calculate the “chiral parameter” \( \sigma \) via a shooting method. We start from the confining metric and must integrate the equation of motion from 0 to infinity. In practice, infinity is replaced by a large number, for our purposes \( z = 10 \) suffices. To solve the ODE (3.2) from the boundary \( z = 0 \) to \( z \to \infty \), we use the analytical series solution around \( z = 0 \) to set initial values for both \( X(\epsilon) \) and \( X'(\epsilon) \) at \( 0 < \epsilon \ll 1 \). Thence, utilizing the near boundary expansion of \( X \), eq. (3.4), we have

\[
\begin{align*}
X(\epsilon) &= m_q \epsilon + \sigma \epsilon^3 + m_q n \epsilon^3 \ln \sqrt{\alpha}, \\
X'(\epsilon) &= 3nm_q \epsilon^2 \ln(\sqrt{\alpha}) + (nm_q + 3\sigma) \epsilon^2 + m_q.
\end{align*}
\]

(B.1)

Here, \( \sigma \) enters as the shooting parameter. Its value can be fixed by imposing that the numerical solution for \( X(z) \) is normalizable and thus vanishing at infinity. This is achieved as follows:

1. Find initial values \( \sigma_1 \) and \( \sigma_2 \), for which \( X_{\sigma_1}[10] < 0 \) and \( X_{\sigma_2}[10] > 0 \).
2. Define \( \sigma_3 = \frac{\sigma_1 + \sigma_2}{2} \).
3. If \( X_{\sigma_3}[10] < 0 \), then we know the \( \sigma \) that we are looking for is located in between \( \sigma_2 \) and \( \sigma_3 \).
4. Redefine \( \sigma_1 = \sigma_3 \), and keep \( \sigma_2 \). Otherwise, if \( X_{\sigma_3}[10] > 0 \), \( \sigma_2 = \sigma_3 \) and keep \( \sigma_1 \).
5. Repeat until \( |\sigma_1 - \sigma_2| < \text{preset tolerance level} \).

Evidently, this method is based on the intermediate value theorem.

## References

[1] K. H. Ackermann et al. [STAR Collaboration], Phys. Rev. Lett. 86, 402 (2001); K. Aamodt et al. [ALICE Collaboration], Phys. Rev. Lett. 105, 252302 (2010); A. Adare et al. [PHENIX Collaboration], Phys. Rev. Lett. 98, 172301 (2007); A. Adare et al. [PHENIX Collaboration], Phys. Rev. C 84, 044905 (2011); B. Abelev et al. [ALICE Collaboration], Phys. Rev. Lett. 111, 102301 (2013); F. Scardina, S. K. Das, S. Plumari, J. I. Bellone and V. Greco, Nucl. Part. Phys. Proc. 276-278, 329 (2016)

[2] B. Bannier [PHENIX Collaboration], Nucl. Phys. A 931, 1189 (2014)

[3] D. Lohner [ALICE Collaboration], J. Phys. Conf. Ser. 446, 012028 (2013)

[4] D. Grasso and H. R. Rubinstein, Phys. Rept. 348, 163 (2001)

[5] T. Vachaspati, Phys. Lett. B 265, 258 (1991)

[6] R. C. Duncan and C. Thompson, Astrophys. J. 392, L9 (1992)

[7] C. Ecker, M. Järvinen, G. Nijs and W. van der Schee, Phys. Rev. D 101 no.10, 103006 (2020)

[8] L. McLerran and V. Skokov, Nucl. Phys. A 929, 184 (2014)

[9] K. Tuchin, Phys. Rev. C 88, no.2, 024911 (2013)
[109] C. Bonati, M. D’Elia, M. Mariti, M. Mesiti, F. Negro and F. Sanfilippo, Phys. Rev. D 89, no. 11, 114502 (2014)
[110] C. Bonati, M. D’Elia, M. Mariti, M. Mesiti, F. Negro, A. Rucci and F. Sanfilippo, Phys. Rev. D 94, no. 9, 094007 (2016)
[111] Y. A. Simonov and M. A. Trusov, Phys. Lett. B 747, 48 (2015)
[112] M. N. Chernodub, Mod. Phys. Lett. A 29, 1450162 (2014)
[113] P. Colangelo, F. Giannuzzi, S. Nicotri and V. Tangorra, Eur. Phys. J. C 72, 2096 (2012)
[114] P. Colangelo, F. De Fazio, F. Giannuzzi, F. Jugeau and S. Nicotri, Phys. Rev. D 78, 055009 (2008)
[115] A. Ballon-Bayona, J. P. Shock and D. Zoakos, [arXiv:2005.00500 [hep-th]]
[116] M. Jarvinen and E. Kiritsis, JHEP 03, 002 (2012)
[117] S. S. Gubser and A. Nellore, Phys. Rev. D 78, 086007 (2008)
[118] T. DeGrand, PoS LAT2006, 005 (2006)
[119] N. Callebaut and D. Dudal, JHEP 01, 055 (2014)
[120] T. Gherghetta, J. I. Kapusta and T. M. Kelley, Phys. Rev. D 79, 076003 (2009)
[121] X. Chen, D. Li, D. Hou and M. Huang, JHEP 03, 073 (2020)