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PARABOLIC REGULARITY AND DIRICHLET BOUNDARY VALUE PROBLEMS

MARTIN DINDOŠ AND LUKE DYER

Abstract. We study the relationship between the Regularity and Dirichlet boundary value problems for parabolic equations of the form $Lu = \text{div}(A\nabla u) - u_t = 0$ in Lip$(1, 1/2)$ time-varying cylinders, where the coefficient matrix $A = [a_{ij}(X, t)]$ is uniformly elliptic and bounded.

We show that if the Regularity problem $(R)_p$ for the equation $Lu = 0$ is solvable for some $1 < p < \infty$ then the Dirichlet problem $(D^*)_{p'}$ for the adjoint equation $L^*v = 0$ is also solvable, where $p' = p/(p - 1)$. This result is an analogue of the result established in the elliptic case by Kenig and Pipher [KP93]. In the parabolic settings in the special case of the heat equation in slightly smoother domains this has been established by Hofmann and Lewis [HL96] and Nyström [Nys06] for scalar parabolic systems. In comparison, our result is abstract with no assumption on the coefficients beyond the ellipticity condition and is valid in more general class of domains.

1. Introduction

We are interested in the relationship between the solvability of the Regularity and the Dirichlet boundary value problems for parabolic operators

$$L = \text{div}(A\nabla \cdot) - \partial_t$$

on Lip$(1, 1/2)$ cylinders $\Omega$. These domains are bounded and Lipschitz in spatial variables, unbounded and Lip$_{1/2}$ in time. Furthermore, we assume that the matrix $A(X, t)$ satisfies an ellipticity condition, and its coefficients are bounded and measurable.

Our main result proves that if the Regularity problem $(R)_p$ for the operator $L$ on the domain $\Omega$ is solvable for some $1 < p < \infty$ ($(R)_p$ has boundary data in a Sobolev space $L^p_{1,1/2}(\partial\Omega)$, which is a space of functions with spatial derivatives and a half-time derivative in $L^p$) then the Dirichlet problem $(D^*)_{p'}$ ($(D^*)_{p'}$ has boundary data in $L^{p'}(\partial\Omega))$ for the adjoint operator

$$L^* = \text{div}(A^*\nabla \cdot) + \partial_t$$

is also solvable on the domain $\Omega$.

Observe that $L^*$ is a backward in time parabolic operator. This however does not cause any issues as by the change of variables of $v(X, t) = u(X, -t)$ and $\tilde{A}(X, -t) = A(X, t)$ we see that $L^*u = 0$ on $\Omega$ is equivalent to

$$\tilde{L}v = \text{div}(\tilde{A}^*\nabla v) - v_t = 0 \quad \text{on } \tilde{\Omega},$$

where $\tilde{\Omega}$ is the reflection of $\Omega$ in the $t$ variable i.e. $\tilde{\Omega} = \{(X, -t) : (X, t) \in \Omega\}$. Hence, the solvability of the $L^*$ Dirichlet problem for the operator $L^*$ on $\Omega$ is equivalent to the solvability of the $\tilde{L}$ Dirichlet problem for the operator $\tilde{L}$ on $\tilde{\Omega}$. Here $\tilde{L}v = 0$ is the usual forward in time parabolic PDE.

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The study of the heat equation in non-smooth domains, or more generally of parabolic operators with non-smooth coefficients, has historically followed the development of the elliptic theory with some delay due to new challenges presented by the parabolic term.

[Dah77] showed that, in a Lipschitz domain, the harmonic measure and surface measure are mutually absolutely continuous, and that the elliptic Dirichlet problem is solvable with data in $L^2$ with respect to surface measure. R. Hunt then asked whether Dahlberg’s result held for the heat equation in domains whose boundaries are given locally as functions $\phi(x,t)$, Lipschitz in the spatial variable. It was conjectured that the correct regularity of $\phi(x,t)$ in the time variable $t$ should be a Hölder condition of order $1/2$ in $t$ (denoted $\text{Lip}_{1/2}$ in $t$). It turns out that under this assumption the parabolic measure associated with the equation (1.3) is doubling [Nys97].

This is the class of domains we work on. It is worth pointing out however that in order to answer R. Hunt’s question positively one has to consider more regular domains. This follows from the counterexample of [KW88] where it was shown that under just the $\text{Lip}(1,1/2)$ condition on the domain $\Omega$ the associated caloric measure (that is the measure associated with the operator $\partial_t - \Delta$) might not be mutually absolutely continuous with the natural surface measure. The issue was resolved in [LM95] where it was established that mutual absolute continuity of caloric measure and a certain parabolic analogue of surface measure holds when $\phi$ has $1/2$ of a time derivative in the parabolic BMO$(\mathbb{R}^n)$ space, which is a slightly stronger condition than $\text{Lip}_{1/2}$. [HL96] subsequently showed that this condition was sharp. In particular in this paper the authors has solved the $L^2$ Dirichlet problem for the heat equation in graph domains of Lewis-Murray type. A related class of localised domains in which parabolic boundary value problems are solvable was considered in [Riv14] as well as in [DH18,DPP17]. The paper [DH18] has established $L^p$ solvability for parabolic Dirichlet problem under assumption that the coefficients satisfy certain natural small Carleson condition which also appears for elliptic PDEs. The second paper [DPP17] finds sufficient and necessary condition for the parabolic measure to be $A_\infty$ with respect to the parabolic analogue of the surface measure.

Our result is motivated by the analogous result in the elliptic setting by [KP93] where, amongst other relationships, they show that $(R_p)$ implied $(D^*)_p$ for elliptic operators $\text{div}(A\nabla \cdot)$ in bounded Lipschitz domains. This has been observed previously for some specific parabolic PDEs (such as the heat equation and constant coefficient systems [HL96, p. 418; Nys06] respectively). [Nys06] also shows that no duality can be expected between Dirichlet and Neumann boundary value problems in non-smooth time-varying domains.

In our result we remove any restrictions on the coefficients of the scalar elliptic operator (beyond the ellipticity hypothesis) and establish the result on the largest reasonable class of domains. It is worth pointing out that due to the roughness of the coefficients and of the boundary of these domains the usual techniques (such as layer potentials and Fourier methods) are not available.

**Theorem 1.1.** Let $\Omega$ be a $\text{Lip}(1,1/2)$ cylinder, as in definition 2.2, with character $(\ell,N,C_0)$. Let $A(X,t)$ be bounded, measurable and elliptic, that is

$$\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(X,t)\xi_i \xi_j \leq \Lambda|\xi|^2$$

(1.2)
for all $\xi \in \mathbb{R}^n$ and a.e. $X \in \mathbb{R}^n$, $t \in \mathbb{R}$. Let the Regularity problem $(R)_p$ be solvable for the equation
\[\begin{align*}
  u_t &= \text{div}(A\nabla u) \quad \text{in } \Omega \subset \mathbb{R}^{n+1}, \\
  u &= f \quad \text{on } \partial \Omega,
\end{align*}\] (1.3)
for some $1 < p < \infty$. Then the Dirichlet problem $(D^*)_{p'}$ is solvable for the adjoint equation
\[\begin{align*}
  -u_t &= \text{div}(A^*\nabla u) \quad \text{in } \Omega \subset \mathbb{R}^{n+1}, \\
  u &= f \quad \text{on } \partial \Omega,
\end{align*}\] (1.4)
where $p' = p/(p - 1)$.

The paper is organized as follows. In section 2 we introduce Lip$(1,1/2)$ cylinders, a suitable local pullback transformation, parabolic non-tangential maximal functions, and the $L^p$ parabolic Sobolev space on $\mathbb{R}^n$ and domains. In section 3 we state and prove some basic results for parabolic equations and some lemmas needed for the proof of Theorem 1.1. In section 4 we prove our main result Theorem 1.1.

The outline of the proof of the main result is as follows. The $L^{p'}$ solvability of the Dirichlet problem is equivalent to establishing certain reverse-Hölder inequality (c.f. (2.27)). Here $K^{V^r}_{\sigma}$ is the Radon-Nykodim derivative $d\omega^{V^r}_{\sigma} / d\sigma$ and for this reason we seek estimates for the ratio $\frac{V^r(\Delta_s(P))}{\sigma(\Delta_s(P))}$ for all small $s << r$. The assumption that the Regularity problem $(R)_p$ is solvable is used to construct a nonnegative solution $0 \leq u \leq 1$ in $\Omega$ with boundary value $f$ such that supp $f \subset \Delta_4r$, $f = 0$ on $\Delta_r$, $\|N(\nabla u)\|_{L^p(\Omega)} \leq r^{(n+1)/p-1}$ and $u \approx 1$ at the cork-screw point $V^{-r}_{\sigma}$. This idea originates from [KP93]. Where we depart significantly from the elliptic paper [KP93] is in the use of the Hardy-Littlewood maximal function in a certain estimate (c.f. (4.9)) which then carries on in the rest of the proof.

Further difficulties are caused by the time irreversibility of parabolic equations and the non-commutativity of taking the adjoint and the pullback mapping. We get around these problems using lemmas developed in [Nys97], the maximum principle, and a new Carleson type estimate.

2. Preliminaries

Here and throughout we consistently use $\nabla u$ to denote the gradient in the spatial variables, $u_t$ or $\partial_t u$ the gradient in the time variable and use $Du = (\nabla u, \partial_t u)$ for the full gradient of $u$.

2.1. Parabolic measure. It is well known by the Perron-Wiener-Brelot method [Ekl79] that the parabolic PDE (1.3) with continuous boundary data is uniquely solvable (c.f. remark 3.7) and that there exists a unique measure $\omega^{(X,t)}$, called the parabolic measure, such that
\[u(X,t) = \int_{\partial \Omega} f(y,s) d\omega^{(X,t)}(y,s)\] (2.1)
for all continuous data $f$. Under the assumptions of definition 2.2 this measure is doubling [Nys97]. As $\omega^{(X,t)}$ is a Borel measure, it follows that we can use (2.1) to extend the solvability of (1.3) to a class of bounded Borel measurable functions $f$.

2.2. Lip$(1,1/2)$ Cylinders. In this subsection we recall the class of Lip$(1,1/2)$ time-varying cylinders in [Nys97] whose boundaries are given locally as functions $\phi(x,t)$, Lipschitz in the spatial variable and Lip$_{1/2}$ in the time variable. At each time $\tau \in \mathbb{R}$ the set of points in $\Omega$ with fixed time $t = \tau$, that is $\Omega_\tau = \{(X,\tau) \in \Omega\}$, will be a non-empty bounded Lipschitz domain in $\mathbb{R}^n$. We start with few preliminary definitions, motivated by the standard definition of a Lipschitz domain.
Definition 2.1. \( Z \subset \mathbb{R}^n \times \mathbb{R} \) is an \( \ell \)-cylinder of diameter \( d \) if there exists a coordinate system \( (x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \) obtained from the original coordinate system only by translation in spatial and time variables and rotation in the spatial variables such that

\[
Z = \{(x_0, x, t) : |x| \leq d, |t| \leq d^2, -(\ell + 1)d \leq x_0 \leq (\ell + 1)d \}
\]

and for \( s > 0 \)

\[
sZ := \{(x_0, x, t) : |x| < sd, |t| \leq s^2d^2, -(\ell + 1)sd \leq x_0 \leq (\ell + 1)sd \}.
\]

Definition 2.2. \( \Omega \subset \mathbb{R}^n \times \mathbb{R} \) is a Lip(1, 1/2) cylinder with character \((\ell, N, C_0)\) if there exists a positive scale \( r_0 \) such that for any time \( \tau \in \mathbb{R} \) there are at most \( N \) \( \ell \)-cylinders \( \{Z_j\}_{j=1}^N \) of diameter \( d \), with \( \frac{r_0}{C_0} \leq d \leq C_0r_0 \), satisfying the following:

1. \( \partial Z_j \cap \partial \Omega \) is the graph \( \{x_0 = \phi_j(x, t)\} \) of a function \( \phi_j : Q_{sd} \rightarrow \mathbb{R} \), with \( Q_{sd} \subset \mathbb{R}^n \), such that

\[
|\phi_j(x, t) - \phi_j(y, s)| \leq \ell \left(|x - y| + |t - s|^{1/2}\right) \quad \text{and} \quad \phi_j(0, 0) = 0. \tag{2.2}
\]

2. \( \partial \Omega \cap \{|t - \tau| \leq d^2\} = \bigcup_j (Z_j \cap \partial \Omega). \)

3. In the coordinate system \((x_0, x, t)\) of the \( \ell \)-cylinder \( Z_j \)

\[
Z_j \cap \Omega \supset \left\{(x_0, x, t) \in \Omega : |x| < d, |t| < d^2, \delta(x_0, x, t) := \text{dist}((x_0, x, t), \partial \Omega) \leq \frac{d}{2}\right\}.
\]

Here and throughout \( \text{dist} \) is the parabolic distance \( \text{dist}((X, t), (Y, \tau)) = |X - Y| + |t - \tau|^{1/2} \).

The parabolic norm \( \| (X, t) \| \) on \( \mathbb{R}^n \times \mathbb{R} \) is defined as the unique positive solution \( \rho \) to the following equation

\[
\frac{|X|^2}{\rho^2} + \frac{t^2}{\rho^2} = 1. \tag{2.3}
\]

One can easily show that \( \| (X, t) \| \sim |X| + |t|^{1/2} \) and that this norm scales correctly according to the parabolic nature of the PDE.

Remark 2.3. It follows from this definition that for each \( \tau \in \mathbb{R} \) the time-slice \( \Omega_\tau = \Omega \cap \{t = \tau\} \) of a Lip(1, 1/2) cylinder \( \Omega \subset \mathbb{R}^n \times \mathbb{R} \) is a bounded Lipschitz domain in \( \mathbb{R}^n \) with character \((\ell, N, C_0)\). Therefore, the Lipschitz domains \( \Omega_\tau \) for all \( \tau \in \mathbb{R} \) have all uniformly bounded diameter That is

\[
\inf_{\tau \in \mathbb{R}} \text{diam} (\Omega_\tau) \sim r_0 \sim \sup_{\tau \in \mathbb{R}} \text{diam} (\Omega_\tau),
\]

where \( r_0 \) is the scale from Definition 2.2 and the implied constants in the estimate only depend on \( N \) and \( C_0 \). In particular, if \( \mathcal{O} \subset \mathbb{R}^n \) is a bounded Lipschitz domain then the parabolic cylinder \( \Omega = \mathcal{O} \times \mathbb{R} \) is an example of a domain satisfying Definition 2.2.

Definition 2.4 (Pullback transformation). Let \( \Omega \) be a Lip(1, 1/2) cylinder with character \((\ell, N, C_0)\) then we define the pullback transformation \( \rho_j : Q_{sd} \rightarrow \partial \Omega \cap 8Z_j \), with \( Q_{sd} \subset \mathbb{R}^{n+1} \), to be

\[
\rho_j(x_0, x, t) = (\phi_j(x, t), x, t).
\]

This mapping transforms a set on the upper half space into a subset of \( \partial \Omega \). By item 2 of definition 2.2, \( \partial \Omega \cap \{|t - \tau| \leq d^2\} \) can be fully described by at most \( N \) pullback transformations \( \rho_j \).

Remark 2.5. By multiplying \( \phi_j \) with a suitable cut off function we may assume \( \phi_j \) and \( \rho_j \) are defined on \( \mathbb{R}^n \) with comparable Lip(1, 1/2) norms and all the axioms of definition 2.2 hold with \( 8Z_j \) replaced by \( 4Z_j \).
Definition 2.6. Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ be a Lip$(1, 1/2)$ cylinder with character $(\ell, N, C_0)$. We define the measure $\sigma$ on sets $A \subset \partial \Omega$ to be
\[
\sigma(A) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(A \cap \{(X, t) \in \partial \Omega\}) \, dt,
\]
where $\mathcal{H}^{n-1}$ is the $n - 1$ dimensional Hausdorff measure on the Lipschitz boundary $\partial \Omega$.

We consider solvability of the $L^p$ Dirichlet and $L^p$ regularity boundary value problems with respect to the measure $\sigma$. The measure $\sigma$ may not be comparable to the usual surface measure on $\partial \Omega$: in the $t$-direction the functions $\phi_j$ from Definition 2.2 are only Lip$_{1/2}$ and hence the standard surface measure might not be locally finite. However, our definition assures that for any $A \subset 8Z_j$, where $Z_j$ is an $\ell$-cylinder, we have
\[
\mathcal{H}^n(A) \sim \sigma(\rho_j(A)),
\]
where the constants in (2.5), by which these measures are comparable, only depend on the $\ell$ of the character $(\ell, N, C_0)$ of the domain $\Omega$. If $\Omega$ has a smoother boundary, such as Lipschitz (in all variables) or better, then the measure $\sigma$ is comparable to the usual $n$-dimensional Hausdorff measure $\mathcal{H}^n$. In particular, this holds for a parabolic cylinder $\Omega = \mathcal{O} \times \mathbb{R}$.

We will adopt the convention of $(X, t)$ referring to points inside $\Omega$ and $(x, t)$ referring to points on $\partial \Omega$.

Definition 2.7.

\[
B_r(X, t) = \{(Z, \tau) \in \mathbb{R}^n \times \mathbb{R} : \text{dist}[(X, t), (Z, \tau)] < r\},
\]
\[
Q_r(X, t) = \{(Z, \tau) \in \mathbb{R}^n \times \mathbb{R} : |x_i - z_i| < r \text{ for all } 0 \leq i \leq n - 1, |t - \tau|^{1/2} < r\},
\]
\[
\Psi_r(Y, s) = \{(Z, \tau) \in \mathbb{R}^n \times \mathbb{R} : |y_0 - z_0| < (\ell + 1)r, |y_i - z_i| < r, |t - \tau|^{1/2} < r\},
\]
\[
\Delta_r = \partial \Omega \cap B_r(Y, s), \quad T(\Delta_r) = \Omega \cap B_r(Y, s),
\]
\[
\delta(X, t) = \inf_{(Y, s) \in \partial \Omega} \text{dist}[(X, t), (Y, s)].
\]

Note that $\Psi_r(Y, s)$ is just a cube elongated in the $x_0$ direction so that the graph of $\phi_j$ won’t escape the box in the $x_0$ direction.

Definition 2.8 (Corkscrew points). Let $\Omega$ be a Lip$(1, 1/2)$ cylinder from definition 2.2 and $r_0 > 0$ be the scale defined there. For any surface ball $\Delta_r = \Delta_r(Y, s) \subset \partial \Omega$ with $0 < r \lesssim r_0$ we say that a point $(X, t) \in \Omega$ is a corkscrew point of the ball $\Delta_r$ if
\[
t = s + 2r^2 \quad \text{and} \quad \delta(X, t) \sim r \sim \text{dist}[(X, t), (Y, s)].
\]

That is the point $(X, t)$ is an interior point of $\Omega$ of distance to the ball $\Delta_r$ and the boundary $\partial \Omega$ of order $r$. The point $(X, t)$ lies at the time of order $r^2$ further than the times for the ball $\Delta_r$. Finally, the implied constants in the definition above only depend on the domain $\Omega$ but not on $r$ and the point $(Y, s)$.

Each ball of radius $0 < r \lesssim r_0$ has infinitely many corkscrew points; for each ball we choose one and denote it by $V(\Delta_r)$ or if there is no confusion to which ball the corkscrew point belongs just $V_r$.

Remark 2.9. Given the fact that the time slices $\Omega_t$ of the domain $\Omega$ are of approximately diameter $r_0$ the corkscrew points do not exist for balls of sizes $r \gg r_0$. 
2.3. Parabolic Non-tangential Cones and Maximal Functions. We proceed with the definition of parabolic non-tangential cones. We define the cones in a (local) coordinate system where \( \Omega = \{(x_0, x, t) : x_0 > \phi(x, t)\} \). In particular this also applies to the upper half-space \( U = \{(x_0, x, t) : x_0 > 0\} \). We note here, that a different choice of coordinates (naturally) leads to different sets of cones, but the particular choice of non-tangential cones is not important as it only changes constants in the estimates for the non-tangential maximal function defined using these cones. However the norms defined using different sets of non-tangential cones are comparable.

For a constant \( a > 0 \), we define the parabolic non-tangential cone at a point \((x_0, x, t) \in \partial \Omega\) as follows

\[
\Gamma_a(x_0, x, t) = \{(y, y, s) \in \Omega : |y - x| + |s - t|^{1/2} < a(y_0 - x_0), \ x_0 < y_0 \}.
\]

We occasionally truncate the cone \( \Gamma \) at the height \( r \)

\[
\Gamma_r^a(x_0, x, t) = \{(y, y, s) \in \Omega : |y - x| + |s - t|^{1/2} < a(y_0 - x_0), \ x_0 < y_0 < x_0 + r \}.
\]

**Definition 2.10** (non-tangential maximal function). For a function \( u : \Omega \to \mathbb{R} \), the non-tangential maximal function \( N_a(u) : \partial \Omega \to \mathbb{R} \) and its truncated version at a height \( r \) are defined as

\[
N_a(u)(x_0, x, t) = \sup_{(y, y, s) \in \Gamma_a(x_0, x, t)} |u(y_0, y, s)|,
\]

\[
N_r^a(u)(x_0, x, t) = \sup_{(y, y, s) \in \Gamma_r^a(x_0, x, t)} |u(y_0, y, s)| \quad \text{for } (x_0, x, t) \in \partial \Omega.
\]

We also define the following \( L^p \) variant of the non-tangential maximal function

\[
\tilde{N}_r^p(u)(x_0, x, t) = \sup_{(Y, s) \in \Gamma_r^a(x_0, x, t)} \left( \int_{B(y, |Y|/2, Y, s)} |u(Z, \tau)|^p \, dZ \, d\tau \right)^{1/p}.
\]

2.4. Parabolic Sobolev Space on \( \partial \Omega \). When considering the appropriate function space for our boundary data we want it to have the same homogeneity as the PDE. As a rule of thumb, one derivative in time behaves like two derivatives in space and so the correct order of our time derivative should be \( 1/2 \) if we impose data with one derivative in spatial variables. This problem has been studied previously in [HL96, HL99, Nys06], who have followed [FJ68] in defining the homogeneous parabolic Sobolev space \( L^p_{1,1/2} \) in the following way.

**Definition 2.11.** The homogeneous parabolic Sobolev space \( L^p_{1,1/2}(\mathbb{R}^n) \), for \( 1 < p < \infty \), is defined to consist of an equivalence class of functions \( f \) with distributional derivatives satisfying \( \|f\|_{L^p_{1,1/2}(\mathbb{R}^n)} < \infty \), where

\[
\|f\|_{L^p_{1,1/2}(\mathbb{R}^n)} = \|Df\|_{L^p(\mathbb{R}^n)}
\]

and

\[
(\mathcal{D}f)\varphi(\xi, \tau) := \|\varphi(\xi, \tau)\|f(\xi, \tau).
\]

We also define the inhomogeneous parabolic Sobolev space \( L^p_{1,1/2}(\mathbb{R}^n) \) as an equivalence class of functions \( f \) with distributional derivatives satisfying \( \|f\|_{L^p_{1,1/2}(\mathbb{R}^n)} < \infty \), where

\[
\|f\|_{L^p_{1,1/2}(\mathbb{R}^n)} = \|Df\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}.
\]

**Remark 2.12.** In the definition above we consider \((x, t) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \). This is due to the fact that the boundary \( \partial \Omega \) is a lower dimensional set (recall that \( \Omega \subset \mathbb{R}^n \times \mathbb{R} \)) and our aim is to define the parabolic Sobolev space on \( \partial \Omega \).
In addition, following [FR67], we define a parabolic half-order time derivative by
\[
(\mathbb{D}_n f)^\wedge(\xi, \tau) := \frac{\tau}{|\xi|^{1/2}} \hat{f}(\xi, \tau).
\] (2.11)

If \(0 < \alpha \leq 2\), then for \(g \in C^\infty_c(\mathbb{R})\) the one-dimensional fractional differentiation operators \(\mathbb{D}_n\) are defined by
\[
(\mathbb{D}_n g)^\wedge(\xi, \tau) := |\tau|^\alpha \hat{g}(\tau).
\] (2.12)

It is also well known that if \(0 < \alpha < 1\) then
\[
D_\alpha g(s) = c \int_{\mathbb{R}} \frac{g(s) - g(\tau)}{|s - \tau|^{1 + \alpha}} \, d\tau
\] whenever \(s \in \mathbb{R}\). If \(h(x, t) \in C^\infty_c(\mathbb{R}^n)\) then by \(D_\alpha^t h : \mathbb{R}^n \to \mathbb{R}\) we mean the function \(D_\alpha h(x, \cdot)\) defined a.e. for each fixed \(x \in \mathbb{R}^n\). We now establish connections between \(\mathbb{D}_n, \mathbb{D}_n\) and \(D_\alpha^t\).

First, by parabolic singular integral theory [FR66,FR67] we have that
\[
\|Df\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathbb{D}_n f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}.
\] (2.14)

Our aim is to establish
\[
\|Df\|_{L^p(\mathbb{R}^n)} \lesssim \|D_\alpha^t f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}.
\] (2.15)

When \(p = 2\) this is a simple consequence of the Plancherel’s theorem. For general \(1 < p < \infty\) we have the following:

**Theorem 2.13.** Let \(f : \mathbb{R}^n \to \mathbb{R}\) and \(1 < p < \infty\) then
\[
\|D_\alpha^t f\|_{L^p(\mathbb{R}^n)} \lesssim \|Df\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}.
\] (2.16)

Therefore
\[
\|f\|_{L^p_{1/2}(\mathbb{R}^n)} = \|Df\|_{L^p(\mathbb{R}^n)} \sim \|D_\alpha^t f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}
\] for \(1 < p < \infty\) and so
\[
\|f\|_{L^p_{1/2}(\mathbb{R}^n)} \sim \|D_\alpha^t f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}.
\]

The proof uses the same approach as [HL96, Section 7] to obtain \(L^p\) bounds instead of their mixed BMO and \(L^\infty\) bounds.

**Proof.** To see (2.17), by approximation we may assume that \(f \in C^\infty_c(\mathbb{R}^n)\) and also that \(f(0) = 0\) by replacing \(f\) by \(f - f(0)\) and noting that \(\mathbb{D}_n\) and \(D_\alpha^t\) map constants to the 0 element. Let
\[
m(\xi, \tau) = \frac{\tau}{|\tau|^{1/2}||\xi, \tau||}
\]
then we have
\[
(\mathbb{D}_n f)^\wedge(\xi, \tau) = \hat{f}(\xi, \tau) \left(m(\xi, \tau)|\tau|^{1/2}\right)
\]
for \((\xi, \tau) \in \mathbb{R}^n\), where \(^\wedge\) denotes the Fourier transform on \(\mathbb{R}^n\).

This multiplier \(m\) is not smooth enough to apply standard multiplier theorems so as in [HL96] we use a smooth cut off function \(\eta\) to split this multiplier in two. Let \(\eta \in C^\infty_c(\mathbb{R})\) be an even function with \(\eta = 1\) on \((-3/2, -1/2), (1/2, 3/2)\), supported in \((-2, -1/4), (1/4, 2)\) and choose \(\eta\) such that \(|D^k \eta| \leq 2^k\) for \(0 \leq k \leq n + 4\). Let
\[
m^+(\xi, \tau) = m(\xi, \tau) \eta \left(\frac{\tau}{||\xi, \tau||^2}\right)
\]
and
\[
m^{++}(\xi, \tau) = |\tau|^{1/2}m(\xi, \tau)||\xi, \tau|| (1 - \eta) \left(\frac{\tau}{||\xi, \tau||^2}\right)
\]
then
\[
(\mathbb{D}_n f)^\wedge(\xi, \tau) = \hat{f}(\xi, \tau) \left(m^+(\xi, \tau)|\tau|^{1/2} + |\xi|^2 ||\xi, \tau|| m^{++}(\xi, \tau)\right).
\]
Let \( m_j^{++}(\xi, \tau) = \frac{\eta_j}{||\xi, \tau||} m^{++}(\xi, \tau) \) for \( 0 \leq j \leq n - 1 \) then we show there exists singular integral operators \( T_{m_j^{++}} \) and \( T_{m^+} \) corresponding to \( m_j^{++} \) and \( m^+ \) respectively such that

\[
\mathbb{D}_n f = c T_{m^+}(D_{1/2} f) + c \sum_{j=0}^{n-1} T_{m_j^{++}}(\partial_x f). \tag{2.18}
\]

All we have to show is that \( T_{m^+} \) and \( T_{m_j^{++}} \) exist and map \( L^p \) into \( L^p \) for \( 1 < p < \infty \).

First we consider \( m^+ \), which is infinitely differentiable away from the origin. It is not hard to show that if \( \gamma \) is a multi-index and \( a \) a non-negative integer then

\[
|\partial_\xi^a \partial_\tau^b m^+(\xi, \tau)| \lesssim ||(\xi, \tau)||^{-(|\gamma| + 2a)}, \tag{2.19}
\]

for \( 1 \leq a + |\gamma| \leq n + 4 \), and that \( |m^+(\xi, \tau)| \lesssim 1 \). By singular integral with mixed homogeneity theory [FR66, p. 28] we have that \( T_{m^+} \) exists and is bounded on \( L^p \) for \( 1 < p < \infty \).

Similarly considering \( m_j^{++} \), by [HL96, (7.10)–(7.11)] we have

\[
|\partial_\xi^a \partial_\tau^b m_j^{++}(\xi, \tau)| \lesssim |\tau|^{1/2-a} ||(\xi, \tau)||^{-(|\gamma|+1)}, \tag{2.20}
\]

for \( 0 \leq a + |\gamma| \leq n + 4 \) and that the support of \( m_j^{++} \) is contained in

\[
\{(\xi, \tau) : 0 \leq |\tau| \leq ||(\xi, \tau)||^2/2\}. \tag{2.21}
\]

Using these \( |m_j^{++}(\xi, \tau)| \lesssim 1 \) and by the same argument as before \( T_{m_j^{++}} \) exists and is bounded on \( L^p \) for \( 1 < p < \infty \). The proof of (2.16) goes along similar lines as above for (2.17).

So far we have only studied this parabolic Sobolev space \( L^p_{1,1/2} \) on \( \mathbb{R}^n \) however our aim is to work on the boundary \( \partial \Omega \) where \( \Omega \) is a Lip(1,1/2) cylinder. Recall that other authors [Bro89, Bro90, HL99, Mit01, Nys06, CRLS18] have only considered either Lipschitz cylinders or graph domains and so they only needed to control the homogeneous norm. Because we are considering an infinite time-varying cylinder made from a local collection of graphs \( \phi_j \) we require to have an additional control over the \( L^p \) norm of \( f \) to control terms that arise from taking a smooth partition of unity. This leads us to give the following definition.

**Definition 2.14 (Parabolic Sobolev spaces on Lip(1,1/2) cylinders).** Let \( 1 < p < \infty \) and \( \Omega \) be a Lip(1,1/2) cylinder, as in definition 2.2, with pullback mappings \( \rho_j \). Let \( \eta_j \) be a smooth partition of unity of \( \partial \Omega \) with the following properties:

1. \( 0 \leq \eta_j \leq 1 \),
2. \( \sum \eta_j = 1 \),
3. the \( \eta_j \) have bounded overlap: i.e. for each fixed \( (x, t) \) \#\( \{ j : \eta_j(x, t) > 0 \} \leq M \) and
4. \( \text{supp} \eta_j \subset B_j(x_j, t_j) \) with \( r_j \sim r_0 \), where \( r_0 \) is from definition 2.2.

We then define the \( L^p_{1,1/2} \) norm on \( \partial \Omega \) as

\[
\|f\|_{L^p_{1,1/2}(\partial \Omega)} = \left( \sum_j \left( \|\mathbb{D} ((f \eta_j) \circ \rho_j)\|_{L^p_{1,p}(\mathbb{R}^n)} + \|(f \eta_j) \circ \rho_j\|_{L^p_{1,p}(\mathbb{R}^n)} \right)^{1/p} \right) \tag{2.22}
\]

By the relationship in Theorem 2.13 this is equivalent to

\[
\|f\|_{L^p_{1,1/2}(\partial \Omega)} \sim \sum_j \left( \|\nabla ((f \eta_j) \circ \rho_j)\|_{L^p_{1,p}(\mathbb{R}^n)} + \|D_{1/2} ((f \eta_j) \circ \rho_j)\|_{L^p_{1,p}(\mathbb{R}^n)} \right) \tag{2.23}
\]

\[
+ \|(f \eta_j) \circ \rho_j\|_{L^p_{1,1}(\mathbb{R}^n)}.
\]
It can be shown that when $\partial \Omega = \mathbb{R}^n$ the norm defined here is equivalent to the one given in Definition 2.11.

2.5. $L^p$ Regularity and $L^p$ Dirichlet Boundary Value Problems. We are now in the position to define the $L^p$ regularity and $L^p$ Dirichlet problems.

**Definition 2.15** ([Ar68]). We say that $u$ is a weak solution to a parabolic operator of the form (1.3) in $\Omega$ if, for $u, \nabla u \in L^2_{\text{loc}}(\Omega)$, we have

$$\sup_t \|u(\cdot, t)\|_{L^2_{\text{loc}}(\Omega)} < \infty \text{ and } \int_{\Omega} (-u \psi_t + A \nabla u \cdot \nabla \psi) \, dX \, dt = 0$$

for all $\psi \in C^\infty_c(\Omega)$. A weak solution to the adjoint operator (1.4) is defined similarly.

**Definition 2.16.** In light of section 2.4, following [Bro89, Bro90, HL96, HL99] we say the $L^p$ Regularity problem for the equation (1.3) is solvable if the unique solution $u$ of this equation in $\Omega$ with boundary data $f \in C_0(\partial \Omega) \cap L^p_{1,1/2}(\partial \Omega, d\sigma)$ satisfies the following non-tangential maximal function estimate

$$\|\tilde{\nabla} u\|_{L^p(\partial \Omega, d\sigma)} \lesssim \|f\|_{L^p_{1,1/2}(\partial \Omega, d\sigma)}, \tag{2.24}$$

with the implied constants depending only on the ellipticity constants, $n, p$ and triple $(\ell, N, C_0)$ of definition 2.2. Here $\tilde{\nabla}$ denotes the $L^2$ based nontangential maximal function. When (2.24) holds we say that the equation (1.3) has the property $(R)_p$ in $\Omega$.

Here the use of the $L^2$ based non-tangential maximum function is natural since $\nabla u \in L^2_{\text{loc}}(\Omega)$. In general, better smoothness of the gradient cannot be expected unless we assume more smoothness of the coefficients of the parabolic operator.

**Remark 2.17.** Some authors [Bro87, Mit01, Nys06, CRLS18] also require

$$\|\tilde{\nabla} H(D^1/2 u)\|_{L^p} \lesssim \|f\|_{L^p_{1,1/2}}$$

or

$$\|\tilde{\nabla} (HD^1/2 u)\|_{L^p} \lesssim \|f\|_{L^p_{1,1/2}},$$

where $H$ is the Hilbert transform in the time variable. For our result we do not assume this, hence our notion of solvability is slightly weaker than that of the authors above. It follows therefore that the $(R)_p$ solvability in the sense of [Bro87, Mit01, Nys06, CRLS18] implies solvability in the sense of definition 2.15.

**Definition 2.18.** We say the $L^p$ Dirichlet problem for the equation (1.3) is solvable if the unique solution $u$ of this equation in $\Omega$ with boundary data $f \in C_0(\partial \Omega) \cap L^p(\partial \Omega, d\sigma)$ satisfies the following non-tangential maximal function estimate

$$\|N(u)\|_{L^p(\partial \Omega, d\sigma)} \lesssim \|f\|_{L^p(\partial \Omega, d\sigma)}, \tag{2.25}$$

with the implied constant depending only on the ellipticity constants, $n, p$ and triple $(\ell, N, C_0)$ of definition 2.2. When (2.25) holds we say that the equation (1.3) has the property $(D)_p$ in $\Omega$. The property $(D^*)_p$ for the adjoint equation (1.4) is defined analogously and is equivalent to solvability of the $L^p$ Dirichlet problem for the equation (1.1) in the domain $\tilde{\Omega}$.

**Remark 2.19.** It is well known that the $L^p$ solvability of the Dirichlet problem for some $1 < p < \infty$ is equivalent to the parabolic measure $\omega$ belonging to a “parabolic $A_\infty$” class with respect to the measure $\sigma$ on the surface $\partial \Omega$, [Nys97, Theorem 6.2]. More specifically, the property $(D)_p^*$ is equivalent to $\omega \in B_p(\sigma)$.

We now recall the definition of parabolic $A_\infty$ and $B_p$.

**Definition 2.20** ($A_\infty$ and $B_p$). Let $\Omega$ be a Lip(1,1/2) cylinder from definition 2.2. For a ball $\Delta_d$ with radius $d \leq \sup \text{diam}(\Omega_\tau)$ we denote its corkscrew point by $V_d$. [git]
We say that the parabolic measure $\omega^{\psi e}$ of (1.3) is in $A_{\infty}(\Delta_d)$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any ball $\Delta \subset \Delta_d$ and subset $E \subset \Delta$ we have

$$\omega^{\psi e}(E) < \delta \ implies \ \frac{\sigma(E)}{\sigma(\Delta)} < \varepsilon. \quad (2.26)$$

The measure $\omega$ is in $A_{\infty}$ if $\omega^{\psi e}$ belongs to $A_{\infty}(\Delta_d)$ for all $\Delta_d$. If $A_{\infty}$ holds then $\omega^{\psi e}$ and $\sigma$ are mutually absolutely continuous and hence one can write $d\omega^{\psi e} = K^{\psi e} \, d\sigma$.

For $p \in (1, \infty)$ we say that $\omega$ belongs to the reverse-Hölder class $B_p(\sigma)$ if for all $\Delta_d$ the kernel $K^{\psi e}$ satisfies the reverse Hölder inequality

$$\left( \sigma(\Delta)^{-1} \int_\Delta (K^{\psi e})^p \, d\sigma \right)^{1/p} \lesssim \sigma(\Delta)^{-1} \int_\Delta K^{\psi e} \, d\sigma, \quad (2.27)$$

for all balls $\Delta \subset \Delta_d$.

**Remark 2.21.** $A_{\infty} = \bigcup_{p>1} B_p$.

3. Basic Results and Interior Estimates

**Lemma 3.1.** (Poincaré type inequality, [Zie89, Cor. 4.5.3]). If $u \in W^{1,p}(E)$ and $p > 1$ then

$$\|u\|_{L^p(E)} \leq C(B_{1,p}(N))^{-1/p} \|D\,u\|_{L^p(E)}, \quad (3.1)$$

where $B_{1,p}(E)$ is the Bessel capacity of the set $E$. $N$ is the set where $u$ vanishes, i.e. $N = \{x : u(x) = 0\}$, and $p^* = \frac{np}{n-p}$ if $p < n$, $1 \leq p^* < \infty$ if $p = n$ and $p^* = \infty$ if $p > n$.

In our work we use this for the case where $E$ is a time slice of $T(\Delta_r)$.

**Corollary 3.2.** Let $u \in W^{1,p}(T(\Delta_r))$, where $u = 0$ on $\Delta_e|_{t'}$ for a fixed time $t'$. Let $p > 1$ then there is a constant $C$ independent of $r$ such that

$$\|u\|_{L^p(T(\Delta_r))} \leq C \|\nabla u\|_{L^p(T(\Delta_r))}. \quad (3.2)$$

**Proof.** The case for $r = 1$ follows from the positivity of $B_{1,p}(\Delta_1|_{t'})$ [Zie89, §2.6]. Lemma 3.1, and Hölder’s inequality. For a general $r$ apply the substitution $v(x) := u(rx)$ then $v \in W^{1,p}(T(\Delta_r))$ and applying the $r = 1$ case and a change of variables gives the general result. \(\square\)

We now recall some foundational estimates needed to prove the main theorem.

**Lemma 3.3.** (A Caccioppoli inequality, see [Aro68]). Let $A$ satisfy (1.2) and suppose that $u$ is a weak solution of (1.3) or (1.4) in $Q_{4r}(X,t)$ with $0 < r < \delta(X,t)/8$. Then there exists a constant $C = C(\lambda, A, n)$ such that

$$r^2 \left( \sup_{Q_{4r}(X,t)} u \right)^2 \leq C \sup_{t-r^2 \leq s \leq t+r^2} \int_{Q_r(X)} u^2(Y,s) \, dY + C \int_{Q_r(X,t)} |\nabla u|^2 \, dY \, ds \leq C^2 \int_{Q_{2r}(X,t)} u^2(Y,s) \, dY \, ds. \quad (3.3)$$

Lemmas 3.4 and 3.5 in [HL01] give us the following estimates for weak solutions of (1.3) or (1.4).

**Lemma 3.4.** (Interior Hölder continuity). Let $A$ satisfy (1.2) and suppose that $u$ is a weak solution of (1.3) or (1.4) in $Q_{4r}(X,t)$ with $0 < r < \delta(X,t)/8$. Then for any $(Y,s), (Z,\tau) \in Q_{2r}(X,t)$

$$|u(Y,s) - u(Z,\tau)| \leq C \left( \frac{\|Y,s\| - (Z,\tau)}{r} \right)^{\alpha} \sup_{Q_{4r}(X,t)} |u|, \quad (3.4)$$

See [Zie89] for a definition of Bessel capacity.
where \( C = C(\lambda, \Lambda, n), \alpha = \alpha(\lambda, \Lambda, n), \) and \( 0 < \alpha < 1 \).

**Lemma 3.5** (Harnack inequality). Let \( A \) satisfy (1.2) and suppose that \( u \) is a weak non-negative solution of (1.3) in \( Q_{2r}(X, t) \), with \( 0 < r < \delta(X, t)/8 \). Suppose that \((Y, s), (Z, \tau) \in Q_{2r}(X, t)\) then there exists \( C = C(\lambda, \Lambda, n) \) such that, for \( \tau < s \),

\[
  u(Z, \tau) \leq u(Y, s) \exp \left[ C \left( \frac{|Y - Z|^2}{|s - \tau|} + 1 \right) \right].
\]

If \( u \geq 0 \) is a weak solution of (1.4) then this inequality holds when \( \tau > s \).

We state a version of the maximum principle from [DH18] that is a modification of Lemma 3.38 from [HL01].

**Lemma 3.6** (Maximum Principle). Let \( A \) satisfy (1.2), \( \Omega \) be a Lip(1,1/2) cylinder and let \( u, v \) be bounded continuous weak solutions to (1.3) in \( \Omega \). If \( |u|, |v| \to 0 \) uniformly as \( t \to -\infty \) and

\[
  \limsup_{(Y, s) \to (X, t)} (u - v)(Y, s) \leq 0
\]

for all \((X, t) \in \partial \Omega\), then \( u \leq v \) in \( \Omega \).

**Remark 3.7** ([DH18]). The proof of Lemma 3.38 from [HL01] works given the assumption that \( |u|, |v| \to 0 \) uniformly as \( t \to -\infty \). Even with this additional assumption, the lemma as stated is sufficient for our purposes. We shall mostly use it when \( u \leq v \) on the boundary of \( \Omega \cap \{ t \geq \tau \} \) for a given time \( \tau \). Obviously then the assumption that \( |u|, |v| \to 0 \) uniformly as \( t \to -\infty \) is not necessary. Another case when the Lemma as stated here applies is when \( u|_{\partial \Omega}, v|_{\partial \Omega} \in C_0(\partial \Omega), \) where \( C_0(\partial \Omega) \) denotes the class of continuous functions decaying to zero as \( t \to \pm \infty \). This class is dense in any \( L^p(\partial \Omega, \sigma) \), \( p < \infty \) allowing us to consider an extension of the solution operator from \( C_0(\partial \Omega) \) to \( L^p \).

The following Carleson type estimate was proved for Lipschitz cylinders in [Sal81] and extended to Lip(1,1/2) cylinders in [Nys97, Lemma 2.4].

**Lemma 3.8** (Carleson type estimate, [Nys97]). Let \( \Omega \) be a Lip(1,1/2) cylinder from definition 2.2 with character \((\ell, N, C_0)\) and \( A \) satisfy (1.2). Let \( u \) be a non-negative weak solution of (1.3) or the adjoint (1.4) in \( \Psi_{2r}(Y, s) \) with \( (Y, s) \in \partial \Omega \) and \( 0 < r < r_0/2 \). Let \( u \) vanish continuously on \( \Psi_{2r}(Y, s) \cap \partial \Omega \), then there exists \( C = C(\ell, \lambda, \Lambda, n) \) such that for \((X, t) \in \Psi_r(Y, s)\)

\[
  u(X, t) \leq C u(V_r^\pm),
\]

where the plus sign is taken when \( u \) is a weak solution of (1.3) and the minus sign is taken when \( u \) is a weak solution of the adjoint (1.4). Here \( V_r^+ \) is the usual (forward in time) corkscrew point of \( \Delta_r(Y, s) \), while \( V_r^- \) is a backward-time corkscrew point of \( \Delta_r(Y, s) \) (i.e. a point at time \( s - 2r^2 \)).

**Lemma 3.9** (Parabolic doubling, corkscrew point, see [Nys97] for more general statements in time-varying domains). Let \( \Omega \) be a Lip(1,1/2) cylinder from definition 2.2 with character \((\ell, N, C_0)\). Let \( \Delta_{2r} \subset \Delta_d \) be surface balls, and \( V_{2r} \) and \( V_d \) be their corkscrew points. Let \( A \) satisfy (1.2) and \( \omega^{V_d} \) be the parabolic measure of (1.3). Then there exists \( C = C(\lambda, \Lambda, \alpha, \ell) \) such that

\[
\begin{align*}
(1) & \quad \omega^{V_d}(\Delta_d) \geq C \\
(2) & \quad \omega^{V_d}(\Delta_{2r}) \leq C \omega^{V_d}(\Delta_r) \quad \text{(doubling)} \\
(3) & \quad \text{If } E \subset \Delta_{2r} \text{ is a Borel set then} \\
& \quad \omega^{V_d}(E) \sim \frac{\omega^{V_d}(E)}{\omega^{V_d}(\Delta_{2r})}.
\end{align*}
\]
The next lemma shows that the parabolic measure of different corkscrew points of large balls are comparable.

Lemma 3.10 (Change of corkscrew point). Let $\Omega$ be a Lip$(1,1/2)$ cylinder. Let $\Delta_r(Y,s)$ be a surface ball with $r \sim \sup \text{diam} \Omega_r$ and $(Y,s) \in \partial \Omega$, and let $V_r$ and $V'_r$ be two corkscrew points of $\Delta_r(Y,s)$ both later in time than $s + (2r)^2$. Let $\omega^{V_r}$ be the parabolic measure of (1.3), $A$ satisfy (1.2) and $E \subset \Delta_r(Y,s)$ be a Borel set then

$$\omega^{V_r}(E) \sim \omega^{V'_r}(E).$$

(3.4)

The same result holds with the adjoint parabolic measure $\omega^{*V_r}$, and $V_r$ and $V'_r$ are corkscrew points earlier in time than $s - (2r)^2$.

Proof. The idea of this proof is to view $\omega^{V_r}(E)$ as $u(V_r)$, where $u$ is the solution of (1.3) with boundary data $\chi_E$ ($\chi$ is the usual indicator function). We then set up to apply the maximum principle to an appropriately chosen domain $\partial \Omega \cap \{t \geq s'\}$.

Let $(Y',s') \in \partial \Omega$ be such that $\Delta_{r/2}(Y',s')$ is a surface ball later in time than $\Delta_r(Y,s)$ and, $E$ and $\Delta_{r/2}(Y',s')$ are disjoint. Therefore the boundary data is 0 on $\Delta_{r/2}(Y',s')$. We can apply Lemma 3.8 to control $u$ in $\Psi_{r/4}(Y',s')$ by $u(V^+)$, where $V^+$ is a corkscrew point of $\Delta_{r/2}(Y',s')$ and at a time earlier than $s' + (2r)^2$.

Since $r \sim \text{diam} \Omega_r$, using Harnack chains, the Harnack inequality (Lemma 3.5) and by varying $Y'$ we can uniformly control $u$ at the time $s'$ by $u(V_r)$, that is we have $u(X,s') \lesssim u(V_r)$ for all $(X,s') \in \Omega_r$. It follows by the maximum principle in remark 3.7 applied to the domain to $\partial \Omega \cap \{t \geq s'\}$ that $u(X,t) \lesssim u(V_r)$ for all $(X,t) \in \Omega \cap \{t \geq s'\}$. In particular, $u(V_r) \lesssim u(V_r)$ and therefore $\omega^{V^+}(E) \lesssim \omega^{V^+}(E)$. Exchanging the roles of $V_r$ and $V'_r$ gives the other inequality. \qed

We use the following properties of the Green’s function. The existence of the Green’s functions $G$ and $G^*$ in $\Omega$ for (1.3), (1.4), respectively is well known and follows from Hölder continuity and a Perron-Wiener-Brelot style argument.

Lemma 3.11 ([Fri64]). Let $\Omega$ be a Lip$(1,1/2)$ cylinder and $A$ satisfy (1.2) then the Green’s function $G$ for (1.3) has the following properties.

1. $G(X,t,Y,s) = 0$ for $s \gtrless t$, $(X,t), (Y,s) \in \Omega$.
2. For fixed $(Y,s) \in \Omega$, $G(\cdot,Y,s)$ is a solution to (1.3) in $\Omega \setminus \{(Y,s)\}$.
3. For fixed $(X,t) \in \Omega$, $G(X,t,\cdot)$ is a solution to (1.4), the adjoint equation in $\Omega \setminus \{(X,t)\}$.
4. If $(X,t), (Y,s) \in \Omega$ then $G(X,t,\cdot)$ and $G(\cdot,Y,s)$ extend continuously to $\overline{\Omega}$ provided both functions are defined to be zero on $\partial \Omega$.

The following lemma is a consequence of [Nys97]. We state it for the adjoint equation (1.4) in $\Omega$ as we apply the lemma in this context. This lemma was originally stated in Lipschitz cylinders in [FGS86, Theorem 1.4], [FS97, Theorem 4] and was extended to the domains in question by [Nys97].

Lemma 3.12. Let $\Omega$ be a Lip$(1,1/2)$ cylinder, $A$ satisfy (1.2), $G^*$ be Green’s function and $\omega^*$ be the parabolic measure associated to (1.4). Let $\Delta_r \subset \Delta_d$ be the surface balls on $\partial \Omega$ such that $\Delta_{2r} \subset \Delta_d$ and $d \gtrsim \frac{r}{Cn}$. Then there exists constants depending on $n, \lambda$ and $\Lambda$ and character of the domain $\Omega$ such that

$$r^n G^*(-(\Delta_d), V^-(\Delta_r)) \sim \omega^*(-(\Delta_d), \Delta_r).$$

(3.5)

Here $V^-(\Delta_r)$ and $V^-(\Delta_d)$ are backward in time corkscrew points as in Lemma 3.8.
4. Proof of Theorem 1.1

This proof uses some of the ideas from Kenig and Pipher’s [KP93] proof in the elliptic setting. However, the time irreversibility of parabolic equations and the non-commutativity of taking the adjoint and the pullback mapping introduce additional difficulties. We get around these problems using lemmas developed in [Nys97], the maximum principle, a different Carleson type estimate from [CFMS81, Theorem 1.1], approaching some estimates from an integral instead of a pointwise point of view, and using the Hardy-Littlewood maximal function.

Assume that \((R)_p\) holds for (1.3) and let \(\omega^*\) be the parabolic measure associated to the adjoint equation (1.4). By remark 2.19 to show that \((D^*)_p\) holds we need to show that \(\omega^* \ll \sigma\), where \(\sigma\) is the measure on \(\partial \Omega\) in definition 2.6, and \(\omega^*\) belongs to the reverse Hölder class \(B_p(\delta \sigma)\), see definition 2.20. We split the argument into several smaller steps.

**Step 1: Preliminaries**

We claim that proving (2.27) for surface balls that fit inside a cylinder \(2Z_j\) is sufficient, since then we can use a covering argument to show that (2.27) holds for all balls with the correct scaling.

Hence from now on we may assume that \(\Delta_d\) is a surface ball on \(\partial \Omega\) with \(d \leq \frac{r}{2}\); that is \(\Delta_d\) lies completely inside an \(\ell\)-cylinder \(2Z_j\). Note that since (2.5) holds in \(2Z_j\) so we can replace \(\sigma\) by \(\mathcal{H}^n\).

Consider now the pullback transformation \(\rho_j\) from definition 2.4. Pulling back using this map we may straighten the boundary \(\partial \Omega\) near the ball \(\Delta_d\) and think about the surface ball \(\Delta_d\) as being a ball on \(\partial U\), where \(U\) is the upper half-space. Consider \(\Delta_r(Y,a) \subset \Delta_d \subset \partial U\) to be a surface ball such that \(4r < d\). Note that if we shall omit below the point where \(\Delta_r\) is centred; whenever we write \(\Delta_{3r}, \Delta_{3r}, \ldots\), we shall implicitly understand that these balls are centered at the same point \((Y,a)\).

Inspired by [KP93], consider a non-negative \(C_c^\infty\) function \(f\) on \(\partial U\) with the following properties: \(f = 0\) on \(\Delta_{r}, f = 1\) on \(\Delta_{3r}\setminus\Delta_{2r}\) and \(f = 0\) on \(\partial U\setminus\Delta_{4r}\) with \(|\nabla f| \lesssim 1/r\) and \(|\partial_t f| \lesssim 1/r^2\). Here we note that \(\Delta_{4r} \subset \Delta_{d}\) due to the assumption we have made earlier. By Theorem 2.13 and interpolation we have

\[
\int_{\partial \Omega} |\nabla_T f|^p \, d\mathcal{H}^n \lesssim r^{n+1-p},
\]

\[
\int_{\partial \Omega} |D_{\alpha} f|^p \, d\mathcal{H}^n \lesssim \|D_{\alpha} f\|_{L^p}^p + \|\nabla f\|_{L^p}^p \lesssim \|\partial_t f\|_{L^p}^{p/2} \|f\|_{L^p}^p + \|\nabla f\|_{L^p}^p \lesssim r^{n+1-p}. \tag{4.1}
\]

By Sobolev embedding, since \(f \in C_c^\infty(\Delta_d)\), for a fixed time \(t\), \(\int_{\mathbb{R}^{n-1}} |f(x,t)|^p \, dx \lesssim \int_{\mathbb{R}^{n-1}} |\nabla_T f(x,t)|^p \, dx\). Here and in the following estimate the implied constant will depend on \(d\). Integrating the previous estimate in time gives

\[
\int_{\partial \Omega} |f|^p \, d\mathcal{H}^n \lesssim \int_{\partial \Omega} |\nabla_T f|^p \, d\mathcal{H}^n \lesssim r^{n+1-p}. \tag{4.2}
\]

It follows that \(f^u = f \circ \rho_j^{-1}\) is \(\Delta_d\) supported boundary data on \(\partial \Omega\) with \(L^p_{1,1/2}(\partial \Omega, d\sigma)\) norm comparable to \(r^{(n+1)/p-1}\).

We now use our assumption of solvability of the regularity problem. Let \(u\) be the solution of (1.3) in \(\Omega\) with boundary data \(f^u\). Since we assume \((R)_p\) solvability for the equation (1.3), it follows that we have for \(u\) the following estimate

\[
\|	ilde{N}(\nabla u)\|_{L^p(\partial \Omega)} \lesssim r^{(n+1)/p-1}. \tag{4.3}
\]

Let \(s \ll r\) (we are going to take limit \(s \to 0^+\)) and let \(P \in \partial \Omega\) be a point on the boundary such that \(\Delta_{10s}(P) \subset \Delta_r\).
We can estimate the value \( u \) by the infimum of \( t = \inf T(G) \) for a fixed time \( T \). We may then use interior Harnack's inequality to conclude that \( u \) is controlled by \( G \) on the boundary of \( T(\Delta_{\delta/2}) \). As functions on both sides of this inequality solve the same parabolic PDE, we may then conclude by the maximum principle Lemma 3.6 that
\[
G(X, t, V_{d}) \lesssim u(X, t)G(V_{s}^{*}, V_{d}^{*}) \quad \text{for all} \quad (X, t) \in T(\Delta_{\delta/2}).
\] (4.5)

Observe that on \( \Delta_{\delta/2} \) we have that \( 0 = G(\cdot, V_{d}^{*}) \leq u(\cdot)G(V_{s}^{*}, V_{d}^{*}) \) so we are left to show that \( G^{(d)}(X) \lesssim G^{(s)}(V_{s}) \) and \( u \sim 1 \) on \( \partial T(\Delta_{\delta/2}) \). To complete the argument given above.

**Step 3.a:** \( G(X, t, V_{d}^{*}) \lesssim G(V_{s}^{*}, V_{d}^{*}) \) on \( \partial T(\Delta_{\delta/2}) \).<ref>
Here we use that \( T(\Delta_{\delta/2}) \) is later than \( V_{d}^{*} \) in time, i.e. \( T(\Delta_{\delta/2}) \subset \{ t > a - (9r)^{2} \} \). For points \( (X, t) \) in \( \partial T(\Delta_{\delta/2}) \) away from \( \partial \Omega \) we can just apply the interior Harnack inequality to conclude that \( G(X, t, V_{d}^{*}) \lesssim G(V_{s}^{*}, V_{d}^{*}) \). For points \( (X, t) \) near \( \partial \Omega \) we can apply Lemma 3.8, to obtain \( G(X, t, V_{d}^{*}) \lesssim G(V_{s}^{*}(Z, \tau), V_{d}^{*}) \), where \( (Z, \tau) \) is any point in \( \Delta_{s/2} \). Since \( V_{d}^{*} \) is at an earlier time than \( V_{s}^{*}(Z, \tau) \), we can again apply the Harnack inequality, Lemma 3.5, to obtain \( G(X, t, V_{d}^{*}) \lesssim G(V_{s}^{*}, V_{d}^{*}) \) for \( (X, t) \in T(\Delta_{s}(Z, \tau)) \). From this the claim above follows.

**Step 3.b:** \( u \sim 1 \) on \( \partial T(\Delta_{\delta/2}) \).<ref>
As before, near to \( \partial \Omega \) we apply Lemma 3.8 to the function \( 1 - u \). This gives us that \( u(X, t) \sim 1 \) for \( (X, t) \in \Psi_{r/4}(Z, \tau) \), where \( (Z, \tau) \in \partial \Delta_{\delta/2} \). Away from \( \partial \Omega \) we can then use interior Harnack’s inequality to conclude that \( u \sim 1 \) at a later time on \( \partial T(\Delta_{\delta/2}) \).<ref>

**Step 3.c:** Combining the estimates<ref>
We may now conclude that (4.5) holds. Since \( V_{s}^{*} \in T(\Delta_{\delta/2}) \) we have that \( G(V_{s}^{*}, V_{d}^{*}) \lesssim u(V_{s}^{*})G(V_{s}^{*}, V_{d}^{*}) \). Combining this with an earlier estimate (4.4) we have now established
\[
\frac{\omega^{V_{d}^{*}}(\Delta(P))}{\omega^{V_{d}^{*}}(\Delta_{r})} \lesssim \frac{s^{n}u(V_{s}^{*})}{r^{n}}. \tag{4.6}
\]

**Step 4:** Applying the Poincaré type inequality to the spacial variables, Corollary 3.2, for a fixed time \( t = t' \) we have for any \( q > 1 \)
\[
\left( \frac{1}{T(\Delta(P))} |u(X, t)|^{q} dX \right)^{1/q} \lesssim s \left( \frac{1}{T(\Delta_{s}(P))} |\nabla u(X, t)|^{q} dX \right)^{1/q}.
\]
We now average in time over \( (a' - s^{2}, a' + s^{2}) \). This gives us
\[
\left( \frac{1}{T(\Delta(P))} |u(X, t)|^{q} dX dt \right)^{1/q} \lesssim s \left( \frac{1}{T(\Delta_{s}(P))} |\nabla u(x, t)|^{q} dX dt \right)^{1/q}. \tag{4.7}
\]
We can estimate the value \( u(V_{s}^{*}) \) using the Harnack inequality as being dominated by the infimum of \( u \) over the ball \( Q_{s/8}(V_{s}^{*} + (0, s^{2}/4^{2})) \) (the centre of this ball is \( V_{s}^{*} \) shifted by \( s/4 \) forward in time). Using (4.7) it follows that
Therefore by (4.6)
\[
\frac{\omega^s V^-_a (\Delta_a(P))}{\omega^s V^-_a (\Delta_r)} \lesssim \frac{s^n}{r^n} u \lesssim \frac{s^{n+1}}{r^n} \left( \int_{T(\Delta_{12a}(P))} |\nabla u(X,t)|^q dX dt \right)^{1/q}.
\]

**Step 5:** We would like to bound the righthand side of the expression above by \( \tilde{N}_2(\nabla u)(P) \), the \( L^2 \) based non-tangential maximal function. This is easy to do in the elliptic setting but it is not clear whether it is possible to do in our setting due to the time irreversibility of the parabolic PDE. Instead we clearly have the following bound

\[
\left( \int_{T(\Delta_{12a}(P))} |\nabla u(X,t)|^q dX dt \right)^{1/q} \lesssim \left( M \left( \tilde{N}_q(\nabla u)^q \right)(P) \right)^{1/q},
\]

where \( M \) is the parabolic version of the Hardy-Littlewood maximal function defined using parabolic surface balls. This is a major departure in our argument from the elliptic proof; we still have not chosen \( q > 1 \) which we shall do below.

Combining this estimate with (4.8) we have

\[
\frac{\omega^s V^-_a (\Delta_a(P))}{\sigma(\Delta_a(P))} \lesssim \frac{\omega^s V^-_a (\Delta_r)}{\sigma(\Delta_r)} \left( M \left( \tilde{N}_q(\nabla u)^q \right)(P) \right)^{1/q},
\]

where as before \( s < r/10 \) and \( P \) is such that \( \Delta_{10a}(P) \subseteq \Delta_r \). In particular this estimate holds for all \( P \in \Delta_{r/2} \).

**Step 6:** Proving the \( B_p \) condition.

To show the property \( (D^*)_p \) we need to show that \( K^-_a = dw^s \sigma/d\sigma \) belongs to the reverse Hölder class \( B_p(d\sigma) \), c.f. (2.27). To do this we take the same approach as [KP93]. Let

\[
h^-_a (P) := \sup_{s \in (0, r/10]} \frac{\omega^s V^-_a (\Delta_s(P))}{\sigma(\Delta_s(P))}.
\]

Clearly, we have \( K^-_a (P) \leq h^-_a (P) \) for \( P \in \Delta_{r/2} \). By (4.10) we have

\[
h^-_a (P) \leq \frac{\omega^s V^-_a (\Delta_r)}{r^n} \left( M \left( \tilde{N}_q(\nabla u)^q \right)(P) \right)^{1/q}.
\]

Since \( f \mapsto (M(|f|^q))^{1/q} \) is \( L^p \) bounded for \( 1 < q < p \) and \( \tilde{N}_q(f) \leq \tilde{N}_2(f) \) for \( 0 < q \leq 2 \) we choose \( q \in (1, \min\{2, p\}) \) to conclude that

\[
\|K^-_a\|_{L^p(d\sigma)} \leq \|h^-_a\|_{L^p(d\sigma)} \lesssim \frac{\omega^s V^-_a (\Delta_r)}{r^n} \left( \left\| \tilde{N}_q(\nabla u) \right\|_{L^p(d\sigma)} \right) \lesssim \frac{\omega^s V^-_a (\Delta_r)}{r^n} \left\| \tilde{N}_2(\nabla u) \right\|_{L^p(d\sigma)} \lesssim \frac{\omega^s V^-_a (\Delta_r)}{r^n} \|f\|_{L^p_{1,1/2}(d\sigma)} < \infty.
\]

Therefore, \( K^-_a, h^-_a \in L^p(d\sigma) \) and so \( \omega^s V^-_a \ll \sigma \).
Using (4.11) and (4.3) the weight $K_{V}^{1/p}$ satisfies the $B_{p}$ condition (2.27) for the
ball $\Delta_{r/2}$

$$
\left( \frac{1}{\sigma(\Delta_{r/2})} \int_{\Delta_{r/2}} \left( K_{V}^{1/p} \right)^{p} \, d\sigma \right)^{1/p} \lesssim \frac{\omega_{V}^{1/p}(\Delta_{r})}{\sigma(\Delta_{r})} \left( \frac{1}{\sigma(\Delta_{r/2})} \right)^{1/p} \lesssim \frac{\omega_{V}^{1/p}(\Delta_{r})}{\sigma(\Delta_{r/2})}.
$$

(4.12)

Recall that $\Delta_{r}$ was an arbitrary surface ball $\Delta_{r} \subset \Delta_{d}$ with $4r \leq d \leq \frac{2r}{C_{0}}$. One can then use Lemma 3.10 to see the above reverse Hölder inequality holds for all balls up to size $d$. It follows by remark 2.19 that the $L^{p}$ Dirichlet problem for the adjoint PDE (1.4) is solvable in $\Omega$. \hfill \square

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