MAXIMAL POLYNOMIAL MODULATIONS
OF SINGULAR INTEGRALS

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Abstract. Let $K$ be a standard Hölder continuous Calderón–Zygmund kernel on $\mathbb{R}^d$ whose truncations define $L^2$ bounded operators. We show that the maximal operator obtained by modulating $K$ by polynomial phases of a fixed degree is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. This extends Sjölin’s multidimensional Carleson theorem and Lie’s polynomial Carleson theorem.

1. Introduction

This article continues a line of research in time-frequency analysis started with Carleson’s theorem on pointwise almost everywhere convergence of Fourier series of $L^2$ functions on $\mathbb{R}/\mathbb{Z}$ [Car66]. In view of Stein’s maximal principle [Ste61], this result is equivalent to the weak type $(2, 2)$ bound for an associated maximal operator, called the Carleson operator. Two essentially different approaches to $L^p$ bounds for this operator were introduced in [Fef73; LT00].

There are several possible analogues of Carleson’s theorem in higher dimensions, depending on the chosen generalization of the interval multipliers. One direction concerns Bochner–Riesz summation, which becomes necessary because the ball multiplier is unbounded on any $L^p(\mathbb{R}^d)$ space for $d \geq 2$ and $p \neq 2$ [Fef71]. Bochner–Riesz summation is embedded in a network of open problems centered around the so-called local smoothing conjecture. We refer to [BHS21] for a recent survey of this topic and remark that this conjecture has been solved in dimension $d = 2$ in [GWZ20].

We are concerned with a different direction, where the Hilbert transform appearing in connection with Fourier summation is replaced by a more general singular integral. A first result of this kind, in which the singular integral is maximally modulated by plane waves, appeared in [Sjö71]. Polynomial modulations of singular integrals were studied in [RS87] in connection with analysis on nilpotent Lie groups. This led to Stein’s conjecture that maximal polynomial modulations of singular integrals define $L^p$ bounded operators, backed by a proof of concept result in [Ste95] involving maximal quadratic modulations. A more general result, involving maximal modulations by polynomials without linear terms, was obtained by more flexible methods in [SW01]. Decisive progress on Stein’s conjecture was made in [Lie09], where boundedness of the Hilbert transform maximally modulated by polynomial phases involving both linear and quadratic terms was proved. That result was later extended to polynomials of an arbitrary fixed degree and the full range of $L^p$ spaces in [Lie20]. In this article, drawing most heavily on the ideas introduced in [Fef73; SW01; Lie20], we prove Stein’s conjecture in dimensions $d > 1$.

Our result is most conveniently formulated as a uniform estimate for truncated singular integrals. We begin with the necessary notation. Let $K$ be a $\tau$-Hölder continuous Calderón–Zygmund (CZ) kernel on $\mathbb{R}^d$, where $\tau > 0$ and $d \geq 1$, that is, a function $K : \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d | x \neq y\} \to \mathbb{C}$ such that

$$|K(x, y)| \lesssim |x - y|^{-d} \quad \text{if } x \neq y \quad \text{and}$$

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \lesssim \frac{|x - x'|^{\tau}}{|x - y|^{d + \tau}} \quad \text{if } |x - x'| < \frac{1}{2}|x - y|.$$ 

Suppose that the associated truncated integral operators

$$T_R^R f(x) := \int_{R < |x - y| < R} K(x, y) f(y) dy$$
are bounded on $L^2(\mathbb{R}^d)$ uniformly in $0 < R < R < \infty$ (this condition follows from boundedness of a CZ operator associated to $K$ on $L^2(\mathbb{R}^d)$). We define the associated maximally polynomially modulated, maximally truncated singular integral operator by

\begin{equation}
Tf(x) := \sup_{Q \in \mathcal{Q}_d} \sup \int_{\mathbb{R}^d} K(x, y) e^{2\pi i Q(y)} f(y) dy \quad (1.3)
\end{equation}

for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, where $d \geq 1$ and $Q_d$ denotes the set of all polynomials in $d$ variables with real coefficients and degree at most $d$.

**Theorem 1.4.** The operator (1.3) is bounded on $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$.

Theorem 1.4 extends several previous results.

1. Carleson’s theorem [Car66; Hun68] is the case $d = 1$, $K(x, y) = 1/(x-y)$ (alternative proofs are due to C. Fefferman [Fef73] and Lacey and Thiele [LT00]).
2. Sjölin’s multidimensional Carleson theorem [Sjö71; PS00] is the translation invariant case $K(x, y) = K(x-y)$ with $d = 1$ (see also [PT03; GTT04] for an alternative proof using methods from [LT00]).
3. Ricci and Stein’s oscillatory singular integrals [RS87] arise if $\sup_Q$ is replaced by $Q = Q_x$ that itself depends polynomially on $x$ (see also [KL20] for sparse bounds).
4. Stein and Wainger [Ste95; SW01] restricted the supremum over $Q$ in such a way as to eliminate modulation invariance by linear phases.
5. V. Lie [Lie09; Lie20] proved the general polynomial case $d \geq 1$ with $K(x, y) = 1/(x-y)$ in dimension $d = 1$.
6. A non-translation invariant extension of Carleson’s theorem was considered in [Saw10].

By the extrapolation argument introduced in [BT13] (see Appendix B for details), Theorem 1.4 is a consequence of the following localized $L^2$ estimates.

**Theorem 1.5.** Let $0 \leq \alpha < 1/2$ and $0 \leq \nu, \kappa < \infty$. Let $F, G \subset \mathbb{R}^d$ be measurable subsets and $\tilde{F} := \{ M1_F > \kappa \}$, $\tilde{G} := \{ M1_G > \nu \}$, where $M$ denotes the Hardy–Littlewood maximal operator. Then

\begin{align}
\|T\|_{2 \to 2} & \lesssim 1, \quad (1.6) \\
\|1_G T1_{\mathbb{R}^d \setminus G}\|_{2 \to 2} & \lesssim \alpha \nu, \quad (1.7) \\
\|1_{\mathbb{R}^d \setminus \tilde{F}} T1_F\|_{2 \to 2} & \lesssim \alpha \kappa. \quad (1.8)
\end{align}

The estimate (1.6) is a special case of both (1.7) and (1.8), but we formulate and prove it separately because it is the easiest case.

The estimate (1.7) is used in the range $2 < p < \infty$. It is also possible to reduce Theorem 1.4 in this range to the case $p = 2$. Indeed, it can be shown using known techniques that, for any $1 \leq p < \infty$, an unweighted weak type $(p, p)$ estimate for the operator (1.3) implies that this operator can be dominated by sparse operators with $L^p$ means (see [Bel7, Theorem 4.3.2] and [Ler16] for the shortest available proof of this implication). This in turn implies strong type $(\tilde{p}, \tilde{p})$ estimates (even vector-valued [CDO17] and with a certain class of Muckenhoupt weights) for all $p < \tilde{p} < \infty$. The observation that weighted estimates for maximally modulated singular integrals can be obtained using unweighted estimates as a black box by essentially the same argument as without the modulations goes back to [HY74] and was expounded in [GMS05; DL14; Bel18; Kar19].

Since the above discussion shows that the strength of Theorem 1.4 decreases with $p$, it is unsurprising that (1.7) can be obtained by a minor variation of the proof of (1.6). Nevertheless, we hope that the simplicity of this localized estimate can motivate the more difficult localization argument required to prove the estimate (1.8), that is used in the range $1 < p < 2$.

The following ingredients of our proof have appeared in previous works.
(1) The overall structure of the argument (in particular the decomposition into trees, the selection algorithm in Section 3.2, the single tree estimate, and the splitting into rows) is due to C. Fefferman [Fef73].

(2) The discretization of the space of polynomials has the same properties (parts 1 and 2 of Lemma 2.12) as in [Lie09; Lie20].

(3) The iteration of the Fefferman selection algorithm between stopping times as in Lemma 3.3 and the associated spatial orthogonality argument in Section 5.6 have been introduced in [Lie20]. This is the main tool that allows to obtain $L^2 \to L^2$ estimates directly (without interpolation with $L^p$, $p < 2$). Earlier arguments, starting with [Fef73], only use $L^\infty$-forests of generation $k = 0$ (see Definition 3.17).

(4) The selection of and estimates for antichains and boundary parts of trees in Proposition 3.22 and Proposition 4.6, respectively, are adapted from [Lie20].

(5) The extrapolation of localized $L^2$ estimates to $L^p$ estimates was found by Bateman in connection with the directional Hilbert transform [BT13].

(6) For the usual Carleson operator (case $d = d = 1$), the localized estimates in Theorem 1.5 are contained in [BM21, estimate (76) in arxiv version 2] (more generally, that article also deals with the $r$-variational Carleson operator, in which case the range of $\alpha$ also depends on the variational exponent $r$). A different approach to localization can be found in [DDU18].

The following elements are new in this context.

(1) In Lemma 3.3, we use a single stopping time for all densities. This helps to ensure that all trees in the decomposition (3.29) are convex (unlike the version of the argument from [Lie20] explained in [Dem15]). Also, we consider all dyadic scales at once rather than splitting them in congruence classes modulo a large integer. This is crucial for general CZ kernels (that do not satisfy a cancellation condition), since removing some scales from a general CZ operator can destroy its $L^2$ boundedness.

(2) Our tiles are nested both in space and in frequency (part 3 of Lemma 2.12), similarly to [Fef73] and differently from [Lie09; Lie20]. This is achieved using a variant of the Christ grid cubes construction and simplifies the combinatorics of tiles.

(3) We estimate oscillatory integrals using a single scale van der Corput type estimate (Lemma A.1, adapted from [SW01]). This allows us to substantially reduce the regularity hypothesis on the kernel $K$ compared to the previous works in which this issue was raised [Sjö71; Roo19].

(4) We use the $L^2(\mathbb{R}^d)$ boundedness of truncated operators associated to $K$ as a black box. This hypothesis can be verified for example using a $T(b)$ theorem.

(5) We apply the extrapolation idea from [BT13] in the context of a Fefferman type argument for the Carleson operator. The required localized estimate is obtained by an argument that resembles the single tree estimate in [LT00]. Specifically, in Lemma 5.6 we obtain sharp decay and in Proposition 4.13 almost sharp decay in both localization parameters.

It appears plausible that our proof should also work for CZ kernels adapted to an anisotropic group of dilations (see [Roo19] for a recent result in this setting) using a discretization based on Christ grid cubes [Chr90] also in space.

Two different approaches to $L^p$ estimates for the (polynomial) Carleson operator in the range $1 < p < 2$ appear in [Lie20; Lie17] and in [Lie13]. Our approach is closer to the latter, and it seems possible to obtain Lorentz space estimates near $L^1$ combining our arguments with the ideas in [Lie13]. However, I have not been able to recover the best known estimates for the Carleson operator in this way.

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We denote elements of $\mathcal{Q}$. 

2.1. Discretization

Modifying the notation used in the introduction, we denote by $\mathcal{Q}$ the vector space of all real polynomials in $d$ variables of degree at most $d$ modulo $+ \mathbb{R}$. That is, we identify two polynomials if and only if their difference is constant. This identification is justified by the fact that the absolute value of the integral in (1.3) does not depend on the constant term of $Q$. Notice that $Q(x) - Q(x') \in \mathbb{R}$ is well-defined for $Q \in \mathcal{Q}$ and $x, x' \in \mathbb{R}^d$.

Let $D = D(d, d)$ be a large integer to be chosen later. Let $\psi$ be a smooth function supported on the interval $[1/(4D), 1/2]$ such that $\sum_{s \in \mathbb{Z}} \psi(D^{-s} \cdot) = 1$ on $(0, \infty)$. Then the kernel can be decomposed as

$$K(x, y) = \sum_{s \in \mathbb{Z}} K_s(x, y) \text{ with } K_s(x, y) := K(x, y) \psi(D^{-s}|x - y|).$$

The functions $K_s$ are supported on the sets $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid D^{s - 1}/4 < |x - y| < D^s/2\}$ and satisfy

$$|K_s(x, y)| \lesssim D^{-ds} \text{ for all } x, y \in \mathbb{R}^d,$$

$$|K_s(x, y) - K_s(x', y)| + |K_s(y, x) - K_s(y, x')| \lesssim \frac{|x - x'|^r}{D^{(d + r)s}} \text{ for all } x, x', y \in \mathbb{R}^d. \quad (2.2)$$

We can replace the maximal operator (1.3) by the smoothly truncated operator

$$Tf(x) := \sup_{Q \in \mathcal{Q}, |Q| \leq \sigma(x)} \left| \sum_{s \in \mathcal{Q}(x)} \int K_s(x, y)e(Q(y))f(y)dy \right|, \quad (2.3)$$

where $e(t) = e^{2\pi it}$ denotes the standard character on $\mathbb{R}$, at the cost of an error term that is controlled by the Hardy–Littlewood maximal operator $M$ (the required localized estimates for $M$ are easy, see Lemma B.2).

Since the absolute value of the integral in (2.3) is a continuous function of $Q$, we may restrict $\sigma, \sigma, Q$ to a finite set as long as we prove estimates that do not depend on this finite set. After these preliminary reductions, we can linearize the supremum in (2.3) and replace that operator by

$$Tf(x) := \sum_{s \in \mathcal{Q}(x)} \int K_s(x, y)e(Q_x(x) - Q_x(y))f(y)dy, \quad (2.4)$$

where $\sigma, \sigma : \mathbb{R}^d \to \mathbb{Z}, Q : \mathbb{R}^d \to \mathcal{Q}$ are measurable functions with finite range. Let $s_{min} := \min_{x \in \mathbb{R}^d} \sigma(x) > -\infty$ and $s_{max} := \max_{x \in \mathbb{R}^d} \sigma(x) < +\infty$. All stopping time constructions will start at the largest scale $s_{max}$ and terminate after finitely many steps at the smallest scale $s_{min}$.

2.1. Tiles. The grid of $D$-adic cubes in $\mathbb{R}^d$ will be denoted by

$$\mathcal{D} := \bigcup_{s \in \mathbb{Z}} \mathcal{D}_s, \quad \mathcal{D}_s := \left\{ \prod_{i=1}^d [D^s a_i, D^s (a_i + 1)] \mid a_1, \ldots, a_d \in \mathbb{Z} \right\}. \quad (2.4)$$

We denote elements of $\mathcal{D}$ by the letters $I, J$ and call them grid cubes. The unique integer $s = s(I)$ such that $I \in \mathcal{D}_s$ will be called the scale of a grid cube. The parent of a grid cube $I$ is the unique grid cube $\bar{I} \supset I$ with $s(\bar{I}) = s(I) + 1$. The side length of...
a cube $I$ is denoted by $\ell(I)$. If $I$ is a cube and $a > 0$, then $aI$ denotes the concentric cube with side length $a\ell(I)$.

For every bounded subset $I \subset \mathbb{R}^d$, we define a norm on $\mathcal{Q}$ by

$$\|Q\|_I := \sup_{x,x' \in I} |Q(x) - Q(x')|, \quad Q \in \mathcal{Q}. \quad (2.5)$$

**Lemma 2.6.** If $Q \in \mathcal{Q}$ and $B(x,r) \subset B(x,R) \subset \mathbb{R}^d$, then

$$\|Q\|_{B(x,r)} \lesssim d (R/r)^d \|Q\|_{B(x,r)}, \quad (2.7)$$

$$\|Q\|_{B(x,r)} \lesssim d (r/R) \|Q\|_{B(x,R)}. \quad (2.8)$$

**Proof.** By translation, we may assume $x = 0$, and we choose a representative for the congruence class modulo $+\mathbb{R}$ with $Q(0) = 0$.

To show (2.7), suppose by scaling that $r = 1$ and $\|Q\|_{B(x,r)} = 1$. The coefficients of $Q$ can now be recovered from its values on the unit ball using a multivariate Lagrange interpolation formula, see e.g. [Nic72, Theorem 3.1]. In particular, these coefficients are $O_{d,d}(1)$, and the conclusion follows.

Similarly, to show (2.8), suppose by scaling that $R = 1$ and $\|Q\|_{B(x,R)} = 1$. Then the coefficients of $Q$ are again $O_{d,d}(1)$, and the conclusion follows. \qed

**Corollary 2.9.** If $D$ is sufficiently large, then, for every $I \in \mathcal{D}$ and $Q \in \mathcal{Q}$, we have

$$\|Q\|_I \geq 10^d \|Q\|_I. \quad (2.10)$$

We choose $D$ so large that (2.10) holds.

**Definition 2.11.** A pair $p$ consists of a spatial cube $I_p \in \mathcal{D}$ and a Borel measurable subset $\mathcal{Q}(p) \subset \mathcal{Q}$ that will be called the associated uncertainty region. Abusing the notation, we will say that $Q \in p$ if and only if $Q \in \mathcal{Q}(p)$. Also, $s(p) := s(I_p)$.

**Lemma 2.12.** There exist collections of pairs $\Psi_I$, indexed by the grid cubes $I \in \mathcal{D}$ with $s_{\text{min}} \leq s(I) \leq s_{\text{max}}$, such that

1. To each $p \in \Psi_I$ is associated a central polynomial $Q_p \in \mathcal{Q}$ such that

$$B_I(Q_p, 0.2) \subset \mathcal{Q}(p) \subset B_I(Q_p, 1), \quad (2.13)$$

where $B_I(Q, r)$ denotes the ball with center $Q$ and radius $r$ with respect to the norm (2.5),

2. for each grid cube $I \in \mathcal{D}$, the uncertainty regions $\{Q(p) \mid p \in \Psi_I\}$ form a disjoint cover of $\mathcal{Q}$, and

3. if $I \subseteq I'$, $p \in \Psi_I$, $p' \in \Psi_{I'}$, then either $Q(p) \cap Q(p') = \emptyset$ or $Q(p) \supseteq Q(p')$.

This is similar to the construction of Christ grid cubes, but easier, because we can start at a smallest scale, and we do not need a small boundary property.

The requirement (2.13) on the uncertainty regions $\mathcal{Q}(p)$ is dictated by Lemma A.1. The uncertainty regions used in [Lie09; Lie20] in the case $d = 1$ also satisfy (2.13), up to multiplicative constants. However, it is convenient not to prescribe the exact shape of the uncertainty regions, in order to obtain the nestedness property (3).

**Proof.** For each $I \in \mathcal{D}$, choose a maximal $0.7$-separated subset $Q_I \subset \mathcal{Q}$ with respect to the $I$-norm (2.5). Then the balls $B_I(Q, 0.3)$, $Q \in Q_I$, are disjoint, and the balls $B_I(Q, 0.7)$, $Q \in Q_I$, cover $\mathcal{Q}$. Hence, there exists a partition $\mathcal{Q} = \bigcup_{Q \in Q_I} \tilde{Q}(I,Q)$ such that $B_I(Q, 0.3) \subset \tilde{Q}(I,Q) \subset B_I(Q, 0.7)$. We fix such a partition for each $I \in \mathcal{D}$.

Next, we construct nested partitions $\mathcal{Q} = \bigcup_{Q \in Q_I} \mathcal{Q}(I,Q)$ for $I \in \mathcal{D}_s$ in decreasing order of the scale $s$, starting with $s = s_{\text{max}}$. For $I \in \mathcal{D}_{s_{\text{max}}}$ and $Q \in Q_I$, we let

$$\tilde{Q}(I,Q) := \hat{Q}(I,Q).$$

Suppose now that $\mathcal{Q}(I', Q)$ have been constructed for some $I' \in \mathcal{D}$. For a grid cube $I \in \mathcal{D}$ with $I = I'$ and $Q \in Q_I$, we define

$$\mathcal{Q}(I, Q) := \bigcup_{Q' \in Q_I \cap \tilde{Q}(I,Q)} \mathcal{Q}(I', Q').$$

By downward induction on $s$, we will show that, for every $I \in \mathcal{D}_s$ and $Q \in Q_I$, we have

$$B_I(Q, 0.2) \subset \mathcal{Q}(I, Q) \subset B_I(Q, 1). \quad (2.14)$$
For $s = s_{\text{max}}$, this holds by construction, so suppose that $s < s_{\text{max}}$ and that the claim is known with $I$ replaced by $I' := \hat{I}$. For every $Q' \in Q_{I'} \setminus \hat{Q}(I, Q)$ and $\hat{Q} \in Q(I', Q')$, we have

$$\|Q - \hat{Q}\|_I \geq \|Q - Q'\|_I - \|Q' - \hat{Q}\|_I \geq 0.3 - 10^{-4}\|Q' - \hat{Q}\|_{I'} \geq 0.3 - 10^{-4} \cdot 1 \geq 0.2,$$

where we used (2.10) and the inductive hypothesis. This shows the first inclusion in (2.14).

For every $Q' \in Q_{I'} \cap \hat{Q}(I, Q)$ and $\hat{Q} \in Q(I', Q')$, we have

$$\|Q - \hat{Q}\|_I \leq \|Q - Q'\|_I + \|Q' - \hat{Q}\|_I \leq 0.7 + 10^{-4}\|Q' - \hat{Q}\|_{I'} \leq 0.7 + 10^{-4} \cdot 1 \leq 1,$$

where we again used (2.10) and the inductive hypothesis. This shows the second inclusion in (2.14).

Finally, the required collections of pairs will be defined by

$$\mathcal{P}_I := \{(I, Q(I, Q)) \mid Q \in Q_I\}.$$

We have verified (2.13) in (2.14), and the remaining properties easily follow from the construction.

**Definition 2.15.** Fixing a choice of collections of pairs from Lemma 2.12, we write

$$\mathcal{P} := \bigcup_{s=s_{\text{min}}}^{s_{\text{max}}} \bigcup_{I \in D_s} \mathcal{P}_I.$$

The pairs in the set $\mathcal{P}$ are called *tiles*.

For a pair $p$, let

$$E(p) := \{x \in I_p \mid Q_x \in Q(p) \land \sigma(x) \leq s(p) \leq \bar{\sigma}(x)\},$$

$$\bar{E}(p) := \{x \in I_p \mid Q_x \in Q(p) \land s(p) \leq \bar{\sigma}(x)\}.$$

The need for the latter notion becomes apparent in the tree estimate, see (5.13).

For every tile $p \in \mathcal{P}$, we define the corresponding operator

$$T_p f(x) := 1_{E(p)}(x) \int e(Q_x(x) - Q_x(y))K_s(p)(x, y)f(y)dy.$$

The tile operators and their adjoints have the support properties

$$\text{supp } T_p f \subseteq I_p, \quad \text{supp } T_p^* g \subseteq I^*_p := 2I_p$$

for any $f, g \in L^1_{\text{loc}}(\mathbb{R}^d)$. For a collection of tiles $\mathcal{E} \subset \mathcal{P}$, we write $T_{\mathcal{E}} := \sum_{p \in \mathcal{E}} T_p$. Then the linearized operator (2.4) can be written as $T_{\mathcal{P}}$.

### 2.2. General notation.

The characteristic function of a set $I$, as well as the corresponding multiplication operator, is denoted by $1_I$. The *Hardy–Littlewood maximal operator* is given by

$$Mf(x) := \sup_{x \in I} \frac{1}{|I|} \int_I |f|,$$

the latter supremum being taken over all (not necessarily grid) cubes containing $x$.

For $1 < q < \infty$, the $q$-*maximal operator* is given by

$$M_q f := (\|M|f|^q\|)^{1/q}.$$

Parameters $\epsilon, \eta$ (standing for small numbers) and $C$ (standing for large numbers) are allowed to change from line to line, but may only depend on $d, d, \tau$, and the implicit constants related to $K$, unless an additional dependence is indicated by a subscript.

For $A, B > 0$, we write $A \lesssim B$ (resp. $A \gtrsim B$) in place of $A < CB$ (resp. $A > CB$).

If the constant $C = C_\delta$ depends on some quantity $\delta$, then we may write $A \lesssim_\delta B$.

The operator norm on $L^2(\mathbb{R}^d)$ is denoted by $\|T\|_{2 \to 2} := \sup_{\|f\|_2 \leq 1} \|Tf\|_2$. 
3. Tree selection algorithm

3.1. Spatial decomposition. We begin with a simplified version of V. Lie’s stopping time construction from [Lie20].

Definition 3.1. Let \( p, p' \) be pairs. We say that
\[
\begin{align*}
\mathbf{p} < \mathbf{p}' & : \iff I_p \subseteq I_{p'} \text{ and } Q(p') \subseteq Q(p), \\
\mathbf{p} \leq \mathbf{p}' & : \iff I_p \subseteq I_{p'} \text{ and } Q(p') \subseteq Q(p).
\end{align*}
\]

The relations \( < \) and \( \leq \) are transitive, similarly to [Fef73] and differently from [Lie20].

Definition 3.2. A stopping collection is a subset \( F \subset D \) of the form \( F = \bigcup_{k \geq 0} F_k \), where each \( F_k \) is a collection of pairwise disjoint cubes such that, for each \( F \in F_{k+1} \), there exists \( F' \in F_k \) with \( F' \supseteq F \) (\( F' \) is called the stopping parent of \( F \)). The collection of stopping children of \( F \in F_k \) is \( \text{ch}_F(F) := \{ F' \in F_{k+1} \mid F' \supseteq F \} \). More generally, the collection of stopping children of \( I \in D \) is \( \text{ch}_F(I) := \{ F \in F \text{ maximal} \mid F \subseteq I \} \). We denote by \( \mathcal{F}^m \) the set of children of \( m \)-th generation, that is,
\[
\mathcal{F}^0(I) := \{ I \}, \quad \mathcal{F}^{m+1}(I) := \bigcup_{I' \in \text{ch}_F(I)} \text{ch}(I').
\]

Lemma 3.3. There exists a stopping collection \( F \) with the following properties.

1. \( F_0 = D_{\text{smax}} \).
2. For each \( F \in F \), we have
\[
\sum_{F' \in \text{ch}(F)} |F'| \leq D^{-10d}|F|.
\]
3. For each \( k \geq 0 \), the set of grid cubes
\[
\hat{C}_k := \{ I \in D \mid \exists F \in F_k : I \subseteq F \}
\]
satisfies
\[
I \in \hat{C}_k, I', I'' \in D, I'' \supseteq 3I, s(I') < s(I) \implies I' \in \hat{C}_k.
\]
4. For \( k \geq 0 \), define the set of grid cubes
\[
C_k := \hat{C}_k \setminus \hat{C}_{k+1}
\]
and the corresponding set of tiles
\[
\mathfrak{P}_k := \{ p \in \mathfrak{P} \mid I_p \in C_k \}.
\]
Then, for every \( n \geq 1 \), the set of tiles
\[
\mathfrak{M}_{n,k} := \{ p \in \mathfrak{P} \text{ maximal w.r.t. } "\ll" \mid |\mathcal{E}(p)|/|I_p| \geq 2^{-n} \}
\]
satisfies
\[
\| \sum_{p \in \mathfrak{M}_{n,k}} 1_{I_p} \|_\infty \lesssim 2^n \log(n + 1).
\]

The stopping property (3) can be informally stated by saying that each stopping cube is completely surrounded by stopping cubes of the same generation \( k \) and similar (up to \( \pm 1 \)) scale. This is very useful for handling tail estimates.

Proof. We start with \( F_0 := D_{\text{smax}} \) being the set of all cubes of the maximal spatial scale, this is part 1 of the conclusion. Part 3 holds with \( k = 0 \), since \( \hat{C}_0 = D \).

Let now \( k \geq 0 \) and suppose that \( F_k \) has been constructed already. Let
\[
\tilde{M}_{n,k} := \{ p \in \mathfrak{P} \text{ maximal w.r.t. } "\ll" \mid |\mathcal{E}(p)|/|I_p| \geq 2^{-n} \text{ and } I_p \in \hat{C}_k \}.
\]
Since the sets \( \mathcal{E}(p) \) corresponding to \( p \in \tilde{M}_{n,k} \) are pairwise disjoint, we have the Carleson packing condition
\[
\sum_{p \in \tilde{M}_{n,k} : I_p \subseteq J} |I_p| \leq 2^n \sum_{p \in \tilde{M}_{n,k} : I_p \subseteq J} |\mathcal{E}(p)| \leq 2^n |J| \text{ for every } J \in D.
\]
Let $C$ be a large constant to be chosen later, and for $F \in \mathcal{F}_k$ let
\begin{equation}
B(F) := \bigcup_{n \geq 1} \left\{ \sum_{p \in \mathbb{M}_{n,k} : I_p \subseteq F} 1_{I_p} \geq C2^n \log(n+1) \right\}.
\end{equation}

By the John–Nirenberg inequality, we obtain
\[
|B(F)| \lesssim \sum_{n \geq 1} e^{-cC2^n \log(n+1)} |F| \lesssim \left( \sum_{n \geq 1} (n+1)^{-cC} \right) |F|.
\]

The numerical constant on the right-hand side can be made arbitrarily small by taking $C$ sufficiently large. Let $\mathcal{J}(F) \subset \mathcal{D}$ be the set of grid cubes contained in $B(F)$, and let $\mathcal{J}'(F) \subset \mathcal{D}$ be the minimal set containing $\mathcal{J}(F)$ and satisfying the analog of (3.6), namely
\[
I \in \mathcal{J}'(F), I' \in \mathcal{D}, I' \subseteq 3I, s(I') < s(I) \implies I' \in \mathcal{J}'(F).
\]

Let $\mathcal{F}_{k+1}$ be the set of maximal cubes in $\tilde{\mathcal{C}}_{k+1} = \bigcup_{F \in \mathcal{F}_k} \mathcal{J}'(F)$. Then Part 3 of the conclusion with $k$ replaced by $k+1$ holds by construction. Part 4 of the conclusion holds for the given $k$, because $\mathbb{M}_{n,k} \subseteq \mathbb{M}_{n,k}$, and we removed all tiles whose spatial cubes are contained in the sets (3.12), where the overlap is large.

Let us now verify part 2 of the conclusion for $F \in \mathcal{F}_k$. By disjointness of the maximal cubes, we have
\[
\sum_{F' \in \text{ch}(F)} |F'| \leq \sum_{\tilde{F} \in \mathcal{F}_k} |F \cap \bigcup \mathcal{J}'(\tilde{F})| \\
\leq \sum_{\tilde{F} \in \mathcal{F}_k : F \cap \bigcup \mathcal{J}'(\tilde{F}) \neq \emptyset} |\bigcup \mathcal{J}'(\tilde{F})| \\
\lesssim \sum_{\tilde{F} \in \mathcal{F}_k : F \cap \bigcup \mathcal{J}'(\tilde{F}) \neq \emptyset} |B(\tilde{F})| \\
\lesssim c \sum_{\tilde{F} \in \mathcal{F}_k : F \cap \bigcup \mathcal{J}'(\tilde{F}) \neq \emptyset} |\tilde{F}|,
\]
where the constant $c > 0$ can be made arbitrarily small by choosing a suitably large $C$. Moreover, if $\tilde{F} \in \mathcal{F}_k$ is such that $F \cap \bigcup \mathcal{J}'(\tilde{F}) \neq \emptyset$, then $\text{dist}(\tilde{F}, F) \lesssim \sup_{F' \in \mathcal{J}'(\tilde{F})} \ell(F')$, and, choosing $C$ sufficiently large, we may assume $\text{dist}(\tilde{F}, F) \leq \ell(\tilde{F})$. It follows that $s(\tilde{F}) \leq s(F) + 1$, since otherwise $3\tilde{F} \supset \tilde{F}$ and $s(\tilde{F}) > s(\tilde{F})$, so that $\tilde{F} \in \mathcal{C}_k$ by the inductive hypothesis (3.6), contradicting $F \in \mathcal{F}_k$. Therefore, the sum over $\tilde{F}$ in the above display is $\lesssim |F|$.

3.2. **Forest selection.** A set of tiles $\mathcal{A} \subset \mathcal{P}$ is called an antichain if no two tiles in $\mathcal{A}$ are related by “$<$” (this is the standard order theoretic term for a concept already used in [Fef73] under a different name). A set of tiles $\mathcal{C} \subset \mathcal{P}$ is called convex if $p_1, p_2 \in \mathcal{C}, p \in \mathcal{P}, p_1 < p < p_2 \implies p \in \mathcal{C}$.
We call a subset $\mathcal{D} \subset \mathcal{C}$ of a convex set $\mathcal{C} \subset \mathcal{P}$ a down subset if $p < p'$ with $p \in \mathcal{C}$ and $p' \in \mathcal{D}$ implies $p \in \mathcal{D}$. Unions of down subsets are again down subsets. Both down subsets and their relative complements are convex.

For $a \geq 1$ and a tile $p$, we will write $ap$ for the pair $(I_p, B_{I_p}(Q_p, a))$. Counterintuitively, for $a' \geq a \geq 1$ and a tile $p$, we have $a'p \leq ap$; this notational inconsistency cannot be avoided without breaking the convention used in all time-frequency analysis literature starting with [Fef73].

**Definition 3.13.** A tree (of generation $k$) is a convex collection of tiles $\mathcal{T} \subset \mathcal{P}_k$, together with a top tile $p_0 = \text{top}\mathcal{T} \in \mathcal{P}_k$ such that for all $p \in \mathcal{T}$ we have $4p < p_0$.
To each tree, we associate the central polynomial $Q_\mathcal{T} = Q_{\text{top}\mathcal{T}}$ and the spatial cube $I_\mathcal{T} = I_{\text{top}\mathcal{T}}$.

**Definition 3.14.** For $p \in \mathcal{P}$ and $Q \in \mathcal{Q}$, we write
\[
\Delta(p, Q) := \|Q_p - Q\|_p + 1.
\]
Definition 3.15. Two trees $T_1$ and $T_2$ are called $\Delta$-separated if

(1) $p \in T_1 \land I_p \subseteq I_{T_2} \implies \Delta (p, Q_{T_2}) > \Delta$ and

(2) $p \in T_2 \land I_p \subseteq I_{T_1} \implies \Delta (p, Q_{T_1}) > \Delta$.

Remark 3.16. If $I_{T_1} \cap I_{T_2} = \emptyset$, then $T_1$ and $T_2$ are $\Delta$-separated for any $\Delta$.

Definition 3.17. Let $n, k \in \mathbb{N}$. An $L^\infty$-forest of level $n$ and generation $k$ is a disjoint union $\mathcal{F} = \bigcup_j \mathcal{F}_j$ of $2^{kn}$-separated trees $\mathcal{F}_j \subset \mathcal{P}_k$ (with a large constant $C$ to be chosen later) such that

\begin{equation}
    \|\sum_j 1_{\mathcal{F}_j}\|_\infty \lesssim 2^n \log(n + 1).
\end{equation}

Definition 3.19. We define the maximal density of a tile $p \in \mathcal{P}$ by

\begin{equation}
    \overline{\text{dens}}_k(p) := \sup_{\lambda \geq 2} \lambda^{-\dim Q} \sup_{p' \in \mathcal{P}_k : \lambda p \leq \lambda p'} \frac{|E(\lambda p')|}{|I_{p'}|},
\end{equation}

where $\dim Q$ is the dimension of the vector space $Q$. We also write $\overline{\text{dens}}_k(\mathcal{S}) = \sup_{p \in \mathcal{S}} \overline{\text{dens}}_k(p)$ for sets of tiles $\mathcal{S} \subset \mathcal{P}$. The subset of “heavy” tiles is defined by

\begin{equation}
    \mathcal{H}_{n,k} := \{ p \in \mathcal{P}_k \mid \overline{\text{dens}}_k(p) > C_0 2^{-n}\},
\end{equation}

where $C_0 = C_0(d, d) > 1$ is a sufficiently large constant to be chosen later.

The maximal density is monotonic in the sense that if $p_1 \leq p_2$ are in $\mathcal{P}_k$, then $\overline{\text{dens}}_k(p_1) \geq \overline{\text{dens}}_k(p_2)$. Indeed, in this case by (2.10) we have $\lambda p_1 \leq \lambda p_2$ for every $\lambda \geq 2$, and the claim follows by transitivity of $\leq$. It follows that each set $\mathcal{H}_{n,k} \subset \mathcal{P}_k$ is a down subset, and in particular convex.

Proposition 3.22. For every $n \geq 1$ and every $k \geq 0$, the set $\mathcal{H}_{n,k}$ can be represented as the disjoint union of $O(n^2)$ antichains and $O(n)$ $L^\infty$-forests of level $n$ and generation $k$.

Proof. We would like to avoid the $\lambda$-dilates in Definition 3.19. To this end, we consider the down subset of $\mathcal{P}_k$

$$
\mathcal{C}_{n,k} := \{ p \in \mathcal{P}_k \mid \exists m \in \mathcal{M}_{n,k} : 2p < 100m \}.
$$

We claim that the remaining set of tiles $\mathcal{H}_{n,k} \setminus \mathcal{C}_{n,k}$ can be partitioned into at most $n$ antichains. Indeed, otherwise there exists a chain $p_0 \leq \cdots \leq p_n$ inside $\mathcal{H}_{n,k} \setminus \mathcal{C}_{n,k}$. By definition (3.20), there exists $\lambda \geq 2$ and a tile $p' \in \mathcal{P}_k$ such that $\lambda p_n \leq \lambda p'$ and

\begin{equation}
    |E(\lambda p')|/|I_{p'}| > C_0 2^{-n} \lambda^{\dim Q}.
\end{equation}

By the John ellipsoid theorem applied to the unit ball of the norm $\|\cdot\|_{I_{p'}}$, the set $Q(\lambda p')$ can be covered by $O(\lambda^{\dim Q})$ uncertainty regions of the form $Q(p'')$, where $p'' \in \mathcal{P}_k$ are tiles with $I_{p''} = I_{p'}$ and $|Q_{p''} - Q_{p'}|/|I_{p'}| \leq \lambda + 1$. For at least one such tile, we have $|E(p'')| \gtrsim C_0 2^{-n}|I_{p'}|$, so that $|E(p'')| > 2^{-n}|I_{p'}|$ provided that $C_0$ in (3.21) is sufficiently large. By definition (3.9), there exists $m \in \mathcal{M}_{n,k}$ with $p'' \leq m$.

From (3.23), we obtain

$$
\lambda \leq \lambda^{\dim Q} < 2^n |E(\lambda p')|/|I_{p'}| \leq 2^n;
$$

and it follows from (2.10) that, for all $Q \in Q(100m)$, we have

\begin{align*}
    &\|Q_{p_0} - Q\|_{I_{p_0}} \
    \leq &\|Q_{p_0} - Q_{p_n}\|_{I_{p_0}} + \|Q_{p_n} - Q_{p'}\|_{I_{p_0}} + \|Q_{p'} - Q_{p''}\|_{I_{p_0}} \
    &+ \|Q_{p''} - Q_m\|_{I_{p_0}} + \|Q_m - Q\|_{I_{p_0}} \
    \leq &1 + 10^{-kn}(\|Q_{p_n} - Q_{p'}\|_{I_{p_n}} + \|Q_{p'} - Q_{p''}\|_{I_{p'}} \
    &+ \|Q_{p''} - Q_m\|_{I_{p'}} + \|Q_m - Q\|_{I_{m}}) \
    \leq &1 + 10^{-kn}(\lambda + (\lambda + 1) + 1 + 100) \leq 2.
\end{align*}

Hence $2p_0 \leq 100m$, contradicting the choice $p_0 \notin \mathcal{C}_{n,k}$.
We want to show that $\mathcal{C}_{n,k}$ can be decomposed into $O(n)$ $L^\infty$-forests and $O(n^2)$ antichains; then since $\mathcal{H}_{n,k}$ is convex the same will hold for $\mathcal{H}_{n,k} \cap \mathcal{C}_{n,k}$. Let

$$\mathcal{B}(p) := \{ m \in \mathcal{M}_{n,k} \mid 100p \leq m \}, \quad p \in \mathcal{C}_{n,k}.$$ 

In view of (3.10), we have $1 \leq |B(p)| \lesssim 2^n \log(n+1)$ for every $p \in \mathcal{C}_{n,k}$. Let

$$\mathcal{C}_{n,k,j} := \{ p \in \mathcal{C}_{n,k} \mid 2^j \leq |\mathcal{B}(p)| < 2^{j+1} \}.$$ 

For the remaining part of the proof, fix $j \geq 0$ such that $2^j \lesssim 2^n \log(n+1)$. It suffices to show that $\mathcal{C}_{n,k,j}$ can be written as the union of an $L^\infty$-forest and $O(n)$ antichains.

First we verify that the set $\mathcal{C}_{n,k,j}$ is convex. Indeed, if $p_1 < p < p_2$ with $p_1, p_2 \in \mathcal{C}_{n,k,j}$ and $p \in \mathcal{C}_{n,k}$, then $100p_1 < 100p < 100p_2$, so that $\mathcal{B}(p_1) \supseteq \mathcal{B}(p) \supseteq \mathcal{B}(p_2)$, so that $p \in \mathcal{C}_{n,k,j}$.

Let $\mathcal{U} \subseteq \mathcal{C}_{n,k,j}$ be the set of tiles $u$ such that there is no $p \in \mathcal{C}_{n,k,j}$ with $I_u \subseteq I_p$ and $Q(100u) \cap Q(100p) \neq \emptyset$. These are our candidates for being tree tops.

In order to verify the counting function estimate (3.18), we will show that for every $x \in \mathbb{R}^d$ the set $\mathcal{U}(x) := \{ u \in \mathcal{U} \mid x \in I_u \}$ has cardinality $O(2^{-j}2^n \log(n+1))$. The family $\mathcal{U}(x)$ can be subdivided into $O(1)$ families, denoted by $\mathcal{U}'(x)$, in each of which the sets $Q(100u), u \in \mathcal{U}'(x)$, are disjoint (just make this decomposition at each scale independently). In particular, the sets $\mathcal{B}(u), u \in \mathcal{U}'(x)$, are pairwise disjoint. These sets have cardinality at least $2^j$, and their union has cardinality at most $2^n \log(n+1)$ by (3.10). This implies $|U'(x)| \lesssim 2^{-j}2^n \log(n+1)$.

Let

$$\mathcal{D}(u) := \{ p \in \mathcal{C}_{n,k,j} \mid 2p < u \}, \quad u \in \mathcal{U}.$$ 

We will show that

$$\mathcal{A}'_j := \mathcal{C}_{n,k,j} \setminus \bigcup_{u \in \mathcal{U}} \mathcal{D}(u)$$

is an antichain. Suppose that, on the contrary, there exist $p, p_1 \in \mathcal{A}'_j$ with $p < p_1$.

We claim that, in this case, for every $l = 1, 2, \ldots$, there exists a sequence of tiles $p_1, \ldots, p_l \in \mathcal{C}_{n,k,j}$ with

$$2p < 200p_1 < \cdots < 200p_l.$$ 

This will produce a contradiction, because the spatial cubes of these tiles are in $C_k$, and therefore have bounded scale. For $l = 1$, the claim follows from (2.10). Suppose now that the claim is known for some $l \geq 1$. If $p_l \in \mathcal{U}$, then $p \in \mathcal{D}(p_l)$, and this is a contradiction. Otherwise, by definition of $\mathcal{U}$, there exists a tile $p_{l+1} \in \mathcal{C}_{n,k,j}$ such that $I_p \subseteq I_{p_{l+1}}$ and $Q(100p_l) \cap Q(100p_{l+1}) \neq \emptyset$. It follows from (2.10) that $Q(200p_l) \supseteq Q(200p_{l+1})$, hence $200p_l < 200p_{l+1}$. This proves the finish of the proof of the claim and of the fact that $\mathcal{A}'_j$ is an antichain.

Let $\mathcal{U}' := \{ u \in \mathcal{U} \mid \mathcal{D}(u) \neq \emptyset \}$ and introduce on this set the relation

$$u \propto u' : \iff \exists p \in \mathcal{D}(u) \text{ with } 10p \leq u'.$$

We claim that

$$u \propto u' \implies I_u = I_{u'} \text{ and } Q(100u) \cap Q(100u') \neq \emptyset.$$ 

Proof of the claim (3.25). Let $u, u' \in \mathcal{U}'$ with $u \propto u'$. By definition, there exists $p \in \mathcal{C}_{n,k,j}$ with $2p < u$ and $10p \leq u'$.

First we notice that it suffices to show that

$$Q(100u) \cap Q(100u') \neq \emptyset.$$ 

Indeed, the spatial cubes $I_u, I_{u'}$ both contain $I_p$, so, unless they coincide, they are strictly nested, contradicting $u, u' \in \mathcal{U}$.

Now we make a case distinction. If $I_p = I_{u'}$, then $100u' < 2p < u$, and (3.26) follows.

In the case $I_p \subseteq I_{u'}$, we deduce from (2.10) that $100p < 100u'$ and $100p < 100u$. If (3.26) does not hold, then the sets $\mathcal{B}(u)$ and $\mathcal{B}(u')$ are disjoint. On the other hand, $\mathcal{B}(p) \supseteq \mathcal{B}(u) \cup \mathcal{B}(u')$, so that $|\mathcal{B}(p)| \geq |\mathcal{B}(u)| + |\mathcal{B}(u')| \geq 2 \cdot 2^j$, a contradiction to $p \in \mathcal{C}_{n,k,j}$. This establishes (3.26).\hfill $\square$

\footnote{1This counting argument is due to C. Fefferman \cite[569]{Fefferman}.}
Next, we verify that \( \propto \) is an equivalence relation. Let \( u, u', u'' \in \mathcal{U}' \) be such that \( I_u = I_{u'} = I_u'' \), \( \mathcal{Q}(100u) \cap \mathcal{Q}(100u') \neq \emptyset \), and \( \mathcal{Q}(100u') \cap \mathcal{Q}(100u'') \neq \emptyset \). For all, and since \( \mathcal{D}(u) \neq \emptyset \) in particular for some, \( p \in \mathcal{D}(u) \) we have \( 2p < u \). By (2.10) this implies \( 4p < 1000u \), and it follows that

\[
(3.27) \quad 4p < u''.
\]

Using (3.25) and the fact that (3.27) implies \( u \propto u'' \), we deduce transitivity, symmetry, and reflexivity of the relation \( \propto \).

Let \( \mathcal{U} \subseteq \mathcal{U}' \) be a set of representatives for equivalence classes modulo \( \propto \), and let

\[
\mathcal{T}(v) := \cup_{v \in \mathcal{U}} \mathcal{D}(u), \quad v \in \mathcal{U}.
\]

Each \( \mathcal{T}(v) \) is a union of down subsets \( \mathcal{D}(u) \subseteq \mathcal{C}_{n,k,j} \), and therefore convex. It follows from (3.27) that each \( \mathcal{T}(v) \) is a tree with top \( v \). It follows from (3.24) that these trees satisfy the separation condition

\[
(3.28) \quad \forall v \neq v' \quad \forall p \in \mathcal{T}(v) \quad 10p \not\subseteq v'.
\]

In order to upgrade the condition (3.28) to \( 2^{CN} \)-separateness, it suffices to remove the bottom \( O(n) \) layers of tiles.\(^2\) More precisely, for \( l = 1, \ldots, Cn \), let \( \mathcal{A}_{n,k,j,l} \) be the set of minimal tiles in \( \cup_{p < l} \mathcal{A}_{n,k,j,l} \). Then each \( \mathcal{A}_{n,k,j,l} \) is an antichain, and the total number of antichains \( \mathcal{A}_{n,k,j,l} \) for all \( j,l \) is \( O(n^2) \). Each \( \mathcal{T}'(v) := \mathcal{T}(v) \setminus \cup_{p} \mathcal{A}_{n,k,j,l} \) is still a convex set, hence a tree with top \( v \). Moreover, it follows from (3.28) that tiles in distinct trees \( \mathcal{T}(v) \) are not comparable. Therefore, for every \( p \in \mathcal{T}'(v) \), there exist tiles \( p_1 < \cdots < p_{Cn} < p \in \mathcal{T}(v) \). If \( I_p \subseteq I_{v'} \) for some \( v' \neq v \), then, using (2.10) and (3.28) for the tile \( p_1 \), we obtain

\[
\|Q_p - Q_{v'}\|_{I_p} \geq (10^4)^n \|Q_p - Q_{v'}\|_{I_{p_1}} \geq (10^4)^n 9,
\]

and this implies \( 10^{4CN} \)-separateness.

\( \square \)

The trees supplied by Proposition 3.22 at different levels \( n \) need not be disjoint. We will now make them disjoint. Let \( \mathcal{T}_{n,k,j,l}' \) be the trees and \( \mathcal{A}_{n,k,j,l}' \) the antichains provided by Proposition 3.22 at level \( n \geq 1 \) and generation \( k \). For \( n = 1 \), define

\[
\mathcal{T}_{n,k,j,l} := \mathcal{T}_{n,k,j,l}', \quad \mathcal{A}_{n,k,j} := \mathcal{A}_{n,k,j}'.
\]

For \( n > 1 \), define

\[
\mathcal{T}_{n,k,j,l} := \mathcal{T}_{n,k,j,l}' \setminus \mathcal{F}_{n-1,k}, \quad \mathcal{A}_{n,k,j} := \mathcal{A}_{n,k,j}' \setminus \mathcal{F}_{n-1,k}.
\]

Since we remove down subsets, the sets \( \mathcal{T}_{n,k,j,l} \) are still (convex) trees.

These sets have the following properties.

1. The set of all tiles can be decomposed as the disjoint union

\[
(3.29) \quad \mathcal{P} = \bigcup_{n=1}^{\infty} \bigcup_{k \in \mathbb{N}} \bigcup_{j \leq n} \bigcup_{l \leq n^2} \mathcal{T}_{n,k,j,l} \cup \mathcal{A}_{n,k,j}.
\]

2. Each \( \mathcal{A}_{n,k,j} \) is an antichain.

3. Each \( \mathcal{T}_{n,k,j,l} \) is a tree.

4. Each \( \mathcal{F}_{n,k,j} := \cup_{l} \mathcal{T}_{n,k,j,l} \) is an \( L^\infty \)-forest of level \( n \) and generation \( k \).

5. \( \mathrm{dens}_k(\mathcal{F}_{n,k,j}) \leq 2^{-n} \).

6. \( \mathrm{dens}_k(\mathcal{A}_{n,k,j}) \leq 2^{-n} \).

4. Estimates for error terms

In this section, we consider error terms coming from antichains and boundary parts of trees. These terms are morally easier to handle than the main terms, in the sense that they are controlled by positive operators (after a suitable \( TT^* \) argument).

\( \text{\footnotesize{\(^2\)In order to perform this step, C. Fefferman used tiles with "central" frequency intervals, see }[\text{Fe073}, \text{Section 5}]. \text{In order to avoid this restriction and the associated averaging argument, V. Lie has introduced a separation condition similar to (3.28), see }[\text{Lie09}, \text{Proposition 2, hypothesis 2}].} \)
4.1. The basic $TT^*$ argument.

Lemma 4.1. Let $p_1, p_2 \in \mathcal{P}$ with $|I_{p_1}| \leq |I_{p_2}|$. Then

$$\left| \int T_{p_1}^* g_1 T_{p_2} g_2 \right| \lesssim \frac{\Delta(p_1, Q_{p_2})^{-\frac{\rho}{\delta}}}{|I_{p_2}|} \int_{E(p_1)} |g_1| \int_{E(p_2)} |g_2|.$$  

(4.2)

**Proof.** We may assume $I_{p_1}^* \cap I_{p_2}^* \neq \emptyset$, since otherwise the left-hand side of the conclusion vanishes. Expanding the left-hand side of (4.2), we obtain

$$\left| \int \int e(-Q_{x_1}(x_1) + Q_{x_1}(y)) K_{s(p_1)}(x_1, y)(1_{E(p_1)} g_1)(x_1)dx_1 
\quad \cdot \int e(-Q_{x_2}(x_2) + Q_{x_2}(y)) K_{s(p_2)}(x_2, y)(1_{E(p_2)} g_2)(x_2)dx_2dy \right|$$

$$\leq \int_{E(p_1)} \int_{E(p_2)} \int e((Q_{x_1} - Q_{x_2})(y) - Q_{x_1}(x_1) + Q_{x_2}(x_2))
\quad \cdot K_{s(p_1)}(x_1, y) K_{s(p_2)}(x_2, y)dy \left|g_1(x_1)g_2(x_2)\right|dx_2dx_1.$$  

By Lemma A.1 applied to the cube $I_{p_1}^*$, the integral inside the absolute value is bounded by

$$\left(\|Q_{x_1} - Q_{x_2}\|_{I_{p_1}^*} + 1\right)^{-\tau/d/|I_{p_2}|},$$

and the conclusion follows, since

$$\|Q_{x_1} - Q_{x_2}\|_{I_{p_1}^*} \geq \|Q_{p_1} - Q_{p_2}\|_{I_{p_1}^*} - \|Q_{p_1} - Q_{x_1}\|_{I_{p_1}^*} - \|Q_{p_2} - Q_{x_2}\|_{I_{p_1}^*}
\geq \|Q_{p_1} - Q_{p_2}\|_{I_{p_1}^*} - \|Q_{p_1} - Q_{x_1}\|_{I_{p_1}^*} - \|Q_{p_2} - Q_{x_2}\|_{CI_{p_2}}
\geq \Delta(p_1, Q_{p_2}) - 1 - C - \|Q_{p_2} - Q_{x_2}\|_{I_{p_2}}
\geq \Delta(p_1, Q_{p_2}) - C.$$  

4.2. Antichains and boundary parts of trees. A separate treatment of boundary parts of trees was introduced in [Lie20] and allows to preserve the sharp spatial support of adjoint tree operators $T_{p}^*$ throughout the main argument in Section 5, while avoiding exceptional sets in [Fel73].

**Lemma 4.3.** There exists $\epsilon = \epsilon(d, d) > 0$ such that, for every $0 \leq \eta \leq 1$, every $1 \leq \rho \leq \infty$, every antichain $A \subseteq \mathcal{P}_k$, and every $Q \in \mathcal{Q}$, we have

$$\left| \sum_{p \in A} \Delta(p, Q)^{-\eta}1_{E(p)} \right|_{\rho} \lesssim \text{dens}_k(A)^{\epsilon\eta/|\cup_{p \in A} I_p|^{1/\rho}}.$$  

(4.4)

**Proof.** Since the sets $E(p)$, $p \in A$, are disjoint, the claimed estimate clearly holds for $\rho = \infty$. Hence, by Hölder’s inequality, it suffices to consider $\rho = 1$. Let also $\delta = \text{dens}_k(A)$. We have to show

$$\sum_{p \in A} \Delta(p, Q)^{-\eta}|E(p)| \lesssim \delta^{\eta|S|}, \quad S = \cup_{p \in A} I_p.$$  

Let $\epsilon > 0$ be a small number to be chosen later and split the summation in two parts. For those $p \in A$ with $\Delta(p, Q) \geq \delta^{-\epsilon}$, the estimate clearly holds, because the sets $E(p)$ are pairwise disjoint.

Let $A' = \{p \in A \mid \Delta(p, Q) < \delta^{-\epsilon}\}$, and consider the collection $\mathcal{L}$ of the maximal grid cubes $L \in \mathcal{D}$ such that $L \subseteq I_p$ for some $p \in A'$ and $I_p \not\subseteq L$ for all $p \in A'$. The collection $\mathcal{L}$ is a disjoint cover of the set $\cup_{p \in A'} I_p$. Fix $L \in \mathcal{L}$; we will show that

$$\sum_{p \in A'} |E(p) \cap L| \lesssim \delta^{1-\epsilon \dim \mathcal{Q}|L|}.$$  

The conclusion will follow with $\epsilon = 1/(\dim \mathcal{Q} + 1).

By construction, $\hat{L} \in \mathcal{C}_{\delta}$ and there exists a tile $p_L \in A'$ with $I_{p_L} \subseteq \hat{L}$. If $I_{p_L} = \hat{L}$, let $p_L' := p_L$, otherwise let $p_L'$ be the unique tile with $I_{p_L'} = \hat{L}$ and $Q \in \mathcal{Q}(p_L')$. In both cases, with $\lambda = C\delta^{-\epsilon}$ for a sufficiently large constant $C$, the tile $p_L'$ satisfies

1. $\lambda p_L \leq \lambda p_L'$ and
(2) for every \( p \in \mathfrak{A}' \) with \( L \cap I_p \neq \emptyset \), we have \( \lambda p'_L \leq p \).

In view of disjointness of \( E(p) \)'s, this implies

\[
\sum_{p \in \mathfrak{A}'} |E(p) \cap L| \leq |E(\lambda p'_L)| \leq \lambda^{\dim Q} |I_{p'_L}| \overline{\text{dens}}_k(p_L) \lesssim \delta^{1-\epsilon} \dim Q |L|. \]

\( \square \)

For a tree \( \mathfrak{T} \), the boundary component is defined by

\[
\text{bd}(\mathfrak{T}) := \{ p \in \mathfrak{T} \mid I_p^* \not\subseteq I_\mathfrak{T} \}.
\]

Notice that \( \text{bd}(\mathfrak{T}) \) is an up-set: if \( p \in \text{bd}(\mathfrak{T}) \), \( p' \in \mathfrak{T} \), \( p \leq p' \), then \( I_p^* \supseteq I_{p'}^* \), so that also \( p' \in \text{bd}(\mathfrak{T}) \). In particular, \( \mathfrak{T} \setminus \text{bd}(\mathfrak{T}) \) is still a (convex) tree.

**Proposition 4.6.** Fix \( n, j \), and let either \( G = \cup_k \cup_l \text{bd}(\mathfrak{T}_{n,k,j,l}) \) or \( G = \cup_k \mathfrak{A}_{n,k,j} \). Then

\[
\|T_G\|_{2-\epsilon} \lesssim 2^{-\epsilon n}.
\]

**Proof.** For \( p \in \mathcal{P} \), let \( \text{gen}(p) \) denote the unique natural number such that \( p \in \mathcal{P}_{\text{gen}(p)} \). For \( p \in G \), let

\[
\mathcal{D}(p') := \{ p \in G \mid s(p) \leq s(p') \wedge \text{gen}(p) \geq \text{gen}(p') \wedge I_p^* \cap I_{p'}^* = \emptyset \}
\]

Then \( I_p \subset 5 I_{p'} \) for \( p \in \mathcal{D}(p') \).

We claim that, for every \( p, p' \in G \) with \( I_p^* \cup I_{p'}^* \neq \emptyset \), at least one of the relations \( p \in \mathcal{D}(p') \) or \( p' \in \mathcal{D}(p) \) holds. Indeed, otherwise we may assume \( s(p') < s(p) \) and \( \text{gen}(p') < \text{gen}(p) \). Then \( I_p^* \subset 3 I_p \), and, since \( I_p \in \mathcal{C}_{\text{gen}(p)} \), it follows from (3.6) that \( I_p^* \subset \mathcal{C}_{\text{gen}(p)} \). But then \( \text{gen}(p') \geq \text{gen}(p) \), a contradiction.

Using the above claim and Lemma 4.1, we obtain

\[
\int |T_{G_{p'}}|^2 \leq 2 \sum_{p' \in G} \sum_{p \in \mathcal{D}(p')} \int T_{p'}^* g T_{p'}^* g \lesssim \sum_{p' \in G} \sum_{p \in \mathcal{D}(p')} \int E(p') |g||p \Delta(p,Q_{p'})^{-\tau/d} \|E(p')|g||I_{p'}|^{\frac{1}{\tau}}.
\]

By Hölder’s inequality with exponent \( 1 < q < 2 \), this is

\[
\leq \sum_{p' \in G} \int E(p') |g||g^{q'} \left( \frac{\sum_{p \in \mathcal{D}(p')} \Delta(p,Q_{p'})^{-\tau/d} 1_{E(p')}}{|I_{p'}|^{\frac{1}{\tau}} g'} \right)^{\frac{1}{q'}}
\]

First we will show that the last fraction is \( O(2^{-\epsilon n}) \) uniformly in \( p' \). Let \( k' := \text{gen}(p') \), so that \( \mathcal{D}(p') \subset \cup_{k > k'} \mathfrak{P}_k \).

We begin by estimating the spatial support of \( \mathcal{D}(p') \cap \mathfrak{P}_k \). If \( F \in \mathcal{F}_{k'+1} \) and \( F \cap 5 I_{p'} \neq \emptyset \), then \( s(F) \leq s(p') \), since otherwise an ancestor of \( I_{p'} \) would have been included in \( \mathcal{F}_{k'+1} \) by part (3) of Lemma 3.3. Therefore, by (3.4) for \( k > k' \), we have

\[
\|F \cap 5 I_{p'}| \lesssim \left| \bigcup_{F' \in \mathcal{F}_{k'}: F' \cap 5 I_{p'} \neq \emptyset} F' \right| \lesssim e^{k'-k} \left| \bigcup_{F' \in \mathcal{F}_{k'+1}: F' \cap 5 I_{p'} \neq \emptyset} F' \right| \lesssim e^{k'-k} |I_{p'}|,
\]

and the same estimate also clearly holds for \( k = k' \).

Next, we decompose \( \mathcal{D}(p') \) into antichains. Consider first the case \( G = \cup_{k,l} \text{bd}(\mathfrak{T}_{n,k,j,l}) \). For \( k \geq k' \) and \( m \geq 0 \), let

\[
\mathfrak{A}_{k,m} := \bigcup_l \{ p \in \mathcal{D}(p') \cap \text{bd}(\mathfrak{T}_{n,k,j,l}) \mid s(p) = s(k,l) - m \},
\]

where

\[
s(k,l) := \begin{cases} \min(s(\text{top}\mathfrak{T}_{n,k'},l), s(p')) & \text{if } k = k', \\ s(\text{top}\mathfrak{T}_{n,k,j,l}) & \text{if } k > k'. \end{cases}
\]
The sets \( A_{k,m} \) are pairwise disjoint antichains and partition \( D(p') = \bigcup_{k \geq k', m \geq 0} A_{k,m} \). We have

\[
\left| \bigcup_{p \in A_{k',m}} I_p \right| \leq \sum_l \left| 5I_{p'} \cap \bigcup_{p \in \text{bd}(I_{n,k',j,l}) : s(p) = s(k',l) - m} I_p \right|
\]

\[
\leq \sum_l \left| 5I_{p'} \cap \{ x \in I_{n,k',j,l} \mid \text{dist}(x, \mathbb{R}^d \setminus I_{n,k',j,l}) < CD^{s(k',l) - m} \} \right|
\]

\[
\lesssim D^{-m} \sum_l \left| 5I_{p'} \cap I_{n,k',j,l} \right|
\]

\[
\lesssim D^{-m} 2^n \log(n + 1) |I_{p'}|,
\]

where we have used (3.18) in the last step. Analogously, using (4.9) for \( k > k' \), we obtain

\[
\left| \bigcup_{p \in A_{k,m}} I_p \right| \leq \sum_{l : I_{n,k,j,l} \cap 5I_{p'} \neq \emptyset} \left| \bigcup_{p \in \text{bd}(I_{n,k,j,l}) : s(p) = s(k,j) - m} I_p \right|
\]

\[
\leq \sum_{l : I_{n,k,j,l} \cap 5I_{p'} \neq \emptyset} \left| \{ x \in I_{n,k,j,l} \mid \text{dist}(x, \mathbb{R}^d \setminus I_{n,k,j,l}) < CD^{s(k,j) - m} \} \right|
\]

\[
\lesssim D^{-m} \sum_{F \in F_k : F \cap 5I_{p'} \neq \emptyset} |F| 2^n \log(n + 1)
\]

\[
\lesssim e^{k' - k} D^{-m} 2^n \log(n + 1) |I_{p'}|.
\]

Combining this with a trivial estimate coming from (4.9), we obtain

\[
(4.10) \quad \left| \bigcup_{p \in A_{k,m}} I_p \right| \lesssim e^{k' - k} \min(1, C2^n \log(n + 1) D^{-m}) |I_{p'}|.
\]

In the case \( S = \bigcup_k A_{n,k,j} \), we define \( A_{k,0} := A_{n,k,j} \cap D(p') \) and \( A_{k,m} := \emptyset \) for \( m > 0 \). The estimate (4.10) also holds in this case.

Using Lemma 4.3 with \( \rho = q' \) and \( 0 \leq \eta \leq 1 \) and (4.10), it follows that

\[
(4.11) \quad \frac{\| \sum_{p \in D(p')} \Delta(p, Q_p')^{-\eta} \mathbf{1}_{E(p)} \|_\rho}{|I_{p'}|^{1/\rho}} \leq \sum_{k \geq k', m \geq 0} \frac{\| \sum_{p \in A_{k,m}} \Delta(p, Q_p')^{-\eta} \mathbf{1}_{E(p)} \|_\rho}{|I_{p'}|^{1/\rho}}
\]

\[
\lesssim 2^{-\epsilon \eta m/\rho} \sum_{k \geq k', m \geq 0} \frac{|\bigcup_{p \in A_{k,m}} I_p|^{1/\rho}}{|I_{p'}|^{1/\rho}}
\]

\[
\lesssim 2^{-\epsilon \eta m/\rho} \sum_{k \geq k', m \geq 0} e^{(k' - k)/\rho} \min(1, C2^n \log(n + 1) D^{-m})^{1/\rho}
\]

\[
\lesssim_{\rho} 2^{-\epsilon \eta m/\rho} \sum_{k \geq k'} e^{(k' - k)/\rho} \lesssim_{\rho} 2^{-\epsilon \eta m/\rho} \rho n.
\]

Using the estimate (4.11) with \( \eta = \tau/d \) in the last factor of (4.8), we obtain the claimed exponential decay in \( n \).

In order to conclude, it now suffices to show

\[
\sum_{p \in S} \int_{E(p)} |g|\, (g)_{5I,p,q} \lesssim n \|g\|_2^2,
\]

where \((g)_{5I,p,q} := (|I|)^{-1} \int_{5I} |g|^q \, dz \). Similarly to the estimate (4.11) with \( \eta = 0 \), we obtain the Carleson packing condition

\[
(4.12) \quad \| \sum_{p \in S : I_p \subseteq J} \mathbf{1}_{E(p)} \|_\rho \lesssim_{\rho} n |J|^{1/\rho}, \quad 1 \leq \rho < \infty.
\]
Let $S \subset D$ be the stopping time associated to the average $(g)_{5I,q}$, that is, $\text{ch}_S(I)$ are the maximal cubes $J \subset I$ with $(g)_{5I,q} > C(g)_{5I,q}$ for some large constant $C$. Since the $q$-maximal operator (2.19) has weak type $(q, q)$, the family $S$ is sparse in the sense that there exist pairwise disjoint subsets $\mathcal{E}(I) \subset I \in S$ with $|\mathcal{E}(I)| \gtrsim |I|$ (one can take $\mathcal{E}(I) = I \setminus \cup_{J \in \text{ch}_S(I)} J$). Then

$$
\sum_{p \in \mathcal{S}} \int_{E(p)} |g| (g)_{5I,p,q} \lesssim \sum_{I \in S} (g)_{5I,q} \int |g| \sum_{p \in \mathcal{S}, I_p \subset I} \mathbf{1}_{E(p)}
$$

by Hölder \leq \sum_{I \in S} (g)_{5I,q} |I| (g)_{I,q} \left( |I|^{-1} \int I \left( \sum_{p \in \mathcal{S}, I_p \subset I} 1_{E(p)} \right)^{q'} \right)^{1/q'}

by (4.12) and sparseness \lesssim n \sum_{I \in S} (g)_{5I,q} |\mathcal{E}(I)|(g)_{I,q}

by disjointness \lesssim n \int (M_q g)^2 \lesssim n \|g\|_2^2,

where we have used the strong type $(2, 2)$ inequality for $M_q$, $q < 2$, in the last step.

4.3. Localization. In order to handle exponents $p \neq 2$, we localize the operator $T_{\mathcal{S}}$.

**Proposition 4.13.** Let $\mathcal{S}$ be as in Proposition 4.6. Let $F, G \subset \mathbb{R}^d$ be such that

$$
|I_p \cap G| \lesssim \nu |I_p| \text{ and } |5I_p \cap F| \lesssim \kappa |I_p|
$$

for every $p \in \mathcal{S}$. Then, for every $0 \leq \alpha < 1/2$, we have

$$
\|1_G T_{\mathcal{S}} 1_F\|_{2 \to 2} \lesssim_\alpha \nu^{\alpha} \kappa^{2 - \alpha}.
$$

**Proof.** Taking a geometric average with (4.7), it suffices to show

$$
\|1_G T_{\mathcal{S}} 1_F\|_{2 \to 2} \lesssim n \nu^{\rho} \kappa^{\alpha}.
$$

To this end, we replace (4.4) by the estimate

$$
\left\| \sum_{p \in \mathcal{S}} 1_{E(p) \cap G} \right\|_{\rho} \lesssim \left| \cup_{p \in \mathcal{S}} I_p \cap G \right|^{1/\rho} \lesssim \nu^{1/\rho} \left| \cup_{p \in \mathcal{S}} I_p \right|^{1/\rho}
$$

for all anticubes $\mathcal{A} \subset \mathcal{S}$. Following the proof of the Carleson packing condition (4.12), we obtain

$$
\left\| \sum_{p \in \mathcal{S}, I_p \subset I} 1_{E(p) \cap G} \right\|_{\rho} \lesssim_\rho n \nu^{1/\rho} |J|^{1/\rho}, \quad 1 \leq \rho < \infty.
$$

Fix functions $f, g$ with $\text{supp} f \subset F$ and $\text{supp} g \subset G$. Consider the stopping time $\mathcal{S} \subset \{I_p \mid p \in \mathcal{S} \}$ associated to the average $(f)_{5I,1}$ and let $\mathcal{E}(I) \subset I \in \mathcal{S}$ be pairwise disjoint subsets with $|\mathcal{E}(I)| \gtrsim |I|$. With $\alpha = 1/q'$, we obtain

$$
\int |g T_{\mathcal{S}} f| \lesssim \sum_{p \in \mathcal{S}} (f)_{5I_p,1} \int |g|
$$

$$
\lesssim \sum_{I \in S} (f)_{5I} \sum_{p \in \mathcal{S}, I_p \subset I} 1_{E(p)} |g|
$$

$$
\lesssim \sum_{I \in S} (f)_{5I,q} (1_F)_{5I,q} (g)_{I,q} \left( |I|^{-1} \int I \left( \sum_{p \in \mathcal{S}, I_p \subset I} 1_{E(p) \cap G} \right)^{q'} \right)^{1/q'}
$$

$$
\lesssim n \kappa^{1/q'} \nu^{1/q'} \sum_{I \in S} (f)_{5I,q} |\mathcal{E}(I)|(g)_{I,q}
$$

$$
\lesssim n \kappa^{1/q'} \nu^{1/q'} \int (M_q f)(M_q g)
$$

$$
\lesssim n \kappa^{1/q'} \nu^{1/q'} \|M_q f\|_2 \|M_q g\|_2
$$

$$
\lesssim n \kappa^{1/q'} \nu^{1/q'} \|f\|_2 \|g\|_2.
$$

□
5. Estimates for trees and forests

In this section, we consider the bulk of tiles that are organized into trees. The contribution of each tree will be estimated by a maximally truncated operator associated to the kernel $K$.

5.1. Cotlar’s inequality. We call a subset $\sigma \subseteq \mathbb{Z}$ convex if it is order convex, that is, $s_1 < s < s_2$ and $s_1, s_2 \in \sigma$ implies $s \in \sigma$. For a measurable function $\sigma$ that maps $\mathbb{R}^d$ to the set of finite convex subsets of $\mathbb{Z}$, we consider the associated truncated singular integral operator

$$\sigma f(x) := \sum_{s \in \sigma(x)} \int K_s(x, y)f(y)dy.$$  

An inspection of the proof of Cotlar’s inequality, see e.g. [Ste93, Section I.7.3], shows that the non-tangentially maximally truncated operator

$$Nf(x) := \sup_{\sigma} \sup_{|x-x'| \leq \min \sigma(x)} |T_\sigma f(x')|$$

is bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$ (more precisely, the proof of Cotlar’s inequality in the above reference shows that this holds if the constant $C$ is sufficiently small, see also [Ler16, Lemma 3.2]; one can subsequently pass to larger values of $C$, see e.g. [Ste93, Section II.2.5.1]). We refer to this fact as the non-tangential Cotlar inequality.

We will use truncated singular integral operators with sets of scales given by trees.

**Definition 5.3.** For a tree $\mathfrak{T}$, we define

$$\sigma(\mathfrak{T}, x) := \{ s(p) \mid p \in \mathfrak{T}, x \in E(p) \},$$

$$\sigma(\mathfrak{T}, x) := \max \sigma(x),$$

$$\sigma(\mathfrak{T}, x) := \min \sigma(x).$$

We will omit the argument $\mathfrak{T}$ if it is clear from the context. By construction of the set of all tiles $\mathfrak{P}$, the set $\sigma(\mathfrak{T}, x)$ is convex in $\mathbb{Z}$ for every tree $\mathfrak{T}$ and every $x \in \mathbb{R}^d$.

5.2. Tree estimate.

**Definition 5.4.** For a non-empty finite collection of tiles $\mathfrak{G} \subseteq \mathfrak{P}$,

1. let $\mathcal{J}(\mathfrak{G}) \subset \mathcal{D}$ be the collection of the maximal grid cubes $J$ such that $100DJ$ does not contain $I_p$ for any $p \in \mathfrak{G}$, and
2. let $\mathcal{L}(\mathfrak{G}) \subset \mathcal{D}$ be the collection of the maximal grid cubes $L$ such that $L \subseteq I_p$ for some $p \in \mathfrak{G}$ and $I_p \not\subseteq L$ for all $p \in \mathfrak{G}$.

For a collection of pairwise disjoint grid cubes $\mathcal{J} \subset \mathcal{D}$, we define the projection operator

$$P_Jf := \sum_{J \subseteq \mathcal{J}} 1_J|J|^{-1} \int_J f.$$  

For later use, we note that the scales of adjacent cubes in $\mathcal{J}(\mathfrak{G})$ differ at most by 1, in the sense that if $J, J' \in \mathcal{J}$ and $\text{dist}(J, J') \leq 10 \max(\ell(J), \ell(J'))$, then $|s(J) - s(J')| \leq 1$. Indeed, if $J, J' \in \mathcal{J}$, $s(J) \leq s(J') - 2$, and $\text{dist}(J, J') \leq 10 \ell(J')$, then $100DJ \subset 100DJ'$ does not contain any $I_p$, $p \in \mathfrak{G}$, contradicting maximality of $J$.

**Lemma 5.6** (Tree estimate). Let $\mathfrak{T} \subseteq \mathfrak{P}$ be a tree, $\mathcal{J} := \mathcal{J}(\mathfrak{T})$, and $\mathcal{L} := \mathcal{L}(\mathfrak{T})$. Then, for every $1 < p < \infty$, $f \in L^p(\mathbb{R}^d)$, and $g \in L^p(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} g \mathfrak{T}_T f \leq \|P_Jf\|_p\|P_Lg\|_{p'}.$$  

**Proof.** The conclusion (5.7) will follow from the estimate

$$\sup_{x \in L} \epsilon(Q_T) T_n \epsilon(Q_T)f(x) \leq C \inf_{x \in L} (M + S) P_Jf(x) + \inf_{x \in L} |T_n P_Jf(x)|,$$

where
(1) \( L \in \mathcal{L} \) is arbitrary,
(2) \( Q_\Xi \) denotes the central polynomial of \( \Xi \) (notice that the left-hand side is well-defined in the sense that it does not depend on the choice of the constant term of \( Q_\Xi \)),
(3) the operator \( S \), while depending on \( \Xi \), is bounded on \( L^p(\mathbb{R}^d) \) for \( 1 < p < \infty \) with constants independent of \( \Xi \), and
(4) the non-tangentially maximally truncated singular integral \( T_N \), defined in (5.2), is bounded on \( L^p(\mathbb{R}^d) \) by Cotlar’s inequality.

Let \( \sigma = \sigma(\Xi) \) be as in Definition 5.3 and fix \( x \in L \in \mathcal{L} \). By definition,

\[
|e(Q_\Xi)T_\Xi e(Q_\Xi)f(x)| = \left| \sum_{s \in \sigma(x)} \int e(-Q_\Xi(x) + Q_s(x)) - Q_s(y))K_s(x, y)f(y)dy \right|
\leq \sum_{s \in \sigma(x)} \left| \int |e(Q_\Xi(y) - Q_s(y)) - 1||K_s(x, y)||f(y)dy \right|
+ |T_\sigma P_\sigma f(x)| + |T_\sigma(1 - P_\sigma)f(x)| =: A(x) + B(x) + C(x).
\]

The term \( B(x) \) is a truncated singular integral, and is dominated by \( \inf I_p T_N P_\sigma f \).

We turn to \( A(x) \). If \( K_s(x, y) \neq 0 \), then \( |x - y| \lesssim D^s \), and in this case

\[
|e(Q_\Xi(y) - Q_s(y)) - 1| \leq \|Q_s - Q_\Xi\|_{B(x, CD^s)} \lesssim D^{s-\pi(x)} \|Q_s - Q_\Xi\|_{B(x, CD^s)} \lesssim D^{s-\pi(x)},
\]

where we have used Lemma 2.6. For \( x \in L \in \mathcal{L} \), we have \( s(L) \leq \sigma(x) - 1 \), and it follows that

\[
A(x) \lesssim D^{-\pi(x)} \sum_{s \in \sigma(x)} D^{s(1-d)} \int_{B(x, 0.5D^s)} |f|(y)dy.
\]

Since the collection \( J \) is a partition of \( \mathbb{R}^d \), this can be estimated by

\[
A(x) \lesssim D^{-\pi(x)} \sum_{s \in \sigma(x)} D^{s(1-d)} \sum_{J \in J : J \cap B(x, 0.5D^s) \neq \emptyset} \int |f|(y)dy.
\]

The expression on the right hand side does not change upon replacing \( |f| \) by \( P_\sigma |f| \).

Moreover,

\[
(5.9) \quad I_p^x \cap J \neq \emptyset \quad \text{with} \quad p \in \Xi \quad \text{and} \quad J \in J \implies J \subset 3I_p.
\]

Hence, the sum over \( J \in J \) is in fact restricted to cubes contained in \( B(x, CD^s) \), so that

\[
A(x) \lesssim D^{-\pi(x)} \sum_{s \in \sigma(x)} D^{s(1-d)} \int_{B(x, CD^s)} P_\sigma |f|(y)dy
\lesssim D^{-\pi(x)} \sum_{s \in \sigma(x)} D^s \inf_L MP_{\sigma} |f| \lesssim \inf_L MP_{\sigma} |f|.
\]

It remains to treat \( C(x) \). Using (5.9), we estimate

\[
|T_\sigma(1 - P_\sigma)f(x)| = \left| \sum_{p \in \Xi} 1_{E(p)}(x) \int K_{s(p)}(x, y)((1 - P_\sigma)f)(y)dy \right|
\leq \sum_{p \in \Xi} 1_{E(p)}(x) \sum_{J \in J : J \subset 3I_p} \sup_{y, y' \in J} |K_{s(p)}(x, y) - K_{s(p)}(x, y')| \|f\|
\lesssim \sum_{I \in \mathcal{H}} 1_I(x) \sum_{J \in J : J \subset 3I} D^{-(d+\tau)s(I)} \diam(J)^{\tau} \int_{J} P_\sigma |f|,
\]
where $\mathcal{H} = \{I_p \mid p \in \mathfrak{T}\}$. The right-hand side of this inequality is constant on each $L \in \mathcal{L}$. Hence, we obtain (5.8) with

$$Sf(x) := \sum_{I \in \mathcal{D}} 1_I(x) \sum_{J \in \mathcal{J} : J \subseteq 3I} D^{-(d+\tau)(s(I))} \operatorname{diam}(J)^\tau \int_J f.$$  

It remains to obtain an $L^p$ estimate for the operator $S$. We have

$$\left| \int gSf \right| \lesssim \sum_{J \in \mathcal{J}} \int_J |f| Mg \sum_{I \in \mathcal{D} : J \subseteq 3I} D^{(s(J) - s(I))} \lesssim \sum_{J \in \mathcal{J}} \int_J |f| Mg \lesssim |f|_p \|Mg\|_{p'}.$$  

By the Hardy–Littlewood maximal inequality and duality, it follows that $\|S\|_{p \to p} \lesssim 1$ for $1 < p < \infty$.

**Corollary 5.10.** Let $\mathfrak{T} \subseteq \mathfrak{F}_k$ be a tree. Let also $F \subseteq \mathbb{R}^d$ and $\kappa > 0$ be such that

$$I_p \not\subseteq \{M1_F > \kappa\} \text{ for all } p \in \mathfrak{T}.$$  

Then, for every $1 < p < \infty$ and $f \in L^p(\mathbb{R}^d)$, we have

$$\|T_{\mathfrak{T}}1_F f\|_p \lesssim \kappa^{1/p'} \|\text{dens}_k(\mathfrak{T})\|^{1/p} \|f\|_p.$$  

Notice that the hypothesis (5.11) holds with $\kappa = 1$ and $F = \mathbb{R}^d$ for every tree $\mathfrak{T}$.

**Proof.** Fix $L \in \mathcal{L} := \mathcal{L}(\mathfrak{T})$. By construction, $\hat{L} \in \mathcal{C}_k$, and there exists a tile $p_L \in \mathfrak{T}$ with $I_{p_L} \subseteq \hat{L}$. If $I_{p_L} = \hat{L}$, let $p_L' := p_L$; otherwise let $p_L' \in \mathfrak{F}_k$ be the unique tile with $I_{p_L'} = \hat{L}$ and $Q_{\mathfrak{T}} \in \mathfrak{Q}(p_L')$. In both cases, the tile $p_L'$ satisfies

1. $10p_L \leq 10p_L'$, and
2. for every $p \in \mathfrak{T}$ with $L \cap I_p \neq \emptyset$ we have $10p_L' \leq p$.

It follows that the spatial support

$$E(L) := L \cap \bigcup_{p \in \mathfrak{T}} E(p) = L \cap \bigcup_{p \in \mathfrak{T} : I_p \supseteq L} E(p)$$

satisfies

$$|E(L)| \leq |E(10p_L')| \leq 10^d \|\text{dens}_k(\mathfrak{T})\| \lesssim |\text{dens}_k(\mathfrak{T})|L|.$$  

It also follows from the hypothesis (5.11) that

$$|F \cap J| \lesssim \kappa |J|$$

for all $J \in \mathcal{J} := \mathcal{J}(\mathfrak{T})$. Using Lemma 5.6, Hölder’s inequality, and the estimates (5.13) and (5.14), we obtain

$$\left| \int_{\mathbb{R}^d} gT_{\mathfrak{T}}1_F f \right| = \left| \int_{\mathbb{R}^d} \sum_{L \in \mathcal{L}} \sum_{J \in \mathcal{J}} 1_E(L) gT_{\mathfrak{T}}1_F f \right|$$

$$\lesssim \|P_L\| \left| \left( \sum_{L \in \mathcal{L}} \frac{|L| (|L|^{-1} \int_L 1_{E(L)} |g|)^{\kappa'}}{|J|} \right)^{1/p'} \left( \sum_{J \in \mathcal{J}} |J| (|J|^{-1} \int_J 1_{F} |f|)^{\kappa'} \right) \right|^{1/p}$$

$$\leq \left( \sum_{L \in \mathcal{L}} |L| (|L|^{-1} \int_L |g|^p') (|L|^{-1} \int_L 1_{E(L)}^{p'} |f|^{p'/p'}) \right)^{1/p} \cdot \left( \sum_{J \in \mathcal{J}} |J| (|J|^{-1} \int_J |f|^p) (|J|^{-1} \int_J 1_{F}^{p'} |f|^{p'/p'}) \right)^{1/p}.$$
\[
\lesssim \frac{\text{dens}_k(\mathfrak{T})^{1/p} K^{1/p'}}{(\sum_{j \in J} \int \lvert \mathfrak{T} \rvert^{1/p'})^{1/p} \left(\sum_{j \in J} \int \mathfrak{T}^{1/p'} \right)^{1/p}} \leq \frac{\text{dens}_k(\mathfrak{T})^{1/p} K^{1/p'}}{\lVert g \rVert_{p'} \lVert f \rVert_p}. \]

5.3. Separated trees.

Definition 5.15. A tree \( \mathfrak{T} \) is called normal if for every \( p \in \mathfrak{T} \) we have \( I_p^* \subset I_\mathfrak{T} \).

For a normal tree \( \mathfrak{T} \), we have \( \text{supp} T^*_\mathfrak{T} g \subseteq I_\mathfrak{T} \) for every function \( g \).

Lemma 5.16. There exists \( \epsilon = \epsilon(d, \tau) > 0 \) such that, for any two \( \Delta \)-separated normal trees \( \mathfrak{T}_1, \mathfrak{T}_2 \), we have

\[
\left\lVert \int_{\mathbb{R}^d} T^*_\mathfrak{T}_1 g_1 T^*_\mathfrak{T}_2 g_2 \right\rVert \lesssim \Delta^{-\epsilon} \prod_{j=1,2} \lVert T^*_\mathfrak{T}_j g_j \rVert_{L^2(I_{\mathfrak{T}_1} \cap I_{\mathfrak{T}_2})}.
\]

Proof. The estimate clearly holds without decay in \( \Delta \), so it suffices to consider \( \Delta \gg 1 \).

Without loss of generality, assume \( I_0 := I_{\mathfrak{T}_1} \subseteq I_{\mathfrak{T}_2} \) and \( \mathfrak{T}_1 \neq \emptyset \).

Recall that \( Q_{\mathfrak{T}} \) denotes the central polynomial of a tree \( \mathfrak{T} \) and let \( Q := Q_{\mathfrak{T}_1} - Q_{\mathfrak{T}_2} \).

Let \( 0 < \eta < 1 \) be chosen later, and let \( \mathfrak{G} := \{ p \in (\mathfrak{T}_1 \cup \mathfrak{T}_2) \mid \|Q\|_{I_p} \geq \Delta^{1-\eta} \} \). It follows from the definition of \( \Delta \)-separation that

\[
p \in (\mathfrak{T}_1 \cup \mathfrak{T}_2) \cap I_p \subseteq I_0 \implies \|Q\|_{I_p} \geq \Delta - 5,
\]

and the same still holds in the case \( I_p \supset I_0 \) by monotonicity of the norms \((2.5)\). Therefore, for sufficiently large \( \Delta \), we may assume

\[
p \in (\mathfrak{T}_1 \cup \mathfrak{T}_2) \setminus \mathfrak{G} \implies I_p \cap I_0 = \emptyset,
\]

and in particular \( \mathfrak{T}_1 \subset \mathfrak{G} \).

Let \( \mathcal{J} := \{ J \in \mathcal{J}(\mathfrak{G}) \mid J \subseteq I_0 \} \). This is a partition of \( I_0 \). Since the scales of adjacent cubes in this partition differ at most by \( 1 \), there exists an adapted partition of unity \( 1_{I_0} = \sum_{J \in \mathcal{J}} \chi_J \), where each \( \chi_J : I_0 \to [0, 1] \) is a smooth function supported on \((1 + 1/D)J\) with \( \lVert \nabla \chi_J \rVert \lesssim \ell(J)^{-1} \). We extend each \( \chi_J \) to be zero on \( \mathbb{R}^d \setminus I_0 \); it will not matter that these extended functions are not necessarily continuous.

We claim that

\[
\Delta_J := \|Q\|_J \gtrsim \Delta^{1-\eta} \quad \text{for all } J \in \mathcal{J}.
\]

Indeed, by definition there exists \( p \in \mathfrak{G} \) with \( 100D \Delta \supseteq I_p \), and, by Lemma 2.6, we obtain

\[
\|Q\|_J \gtrsim \|Q\|_{100D \Delta} \gtrsim \|Q\|_{I_p} \gtrsim \Delta^{1-\eta}.
\]

In order to prepare the application of Lemma A.1, we need to estimate local moduli of continuity of \( T^*_\mathfrak{T} g \) for a tree \( \mathfrak{T} \). For every \( p \in \mathfrak{T} \) and \( y, y' \in I_p^* \), using \((2.1), (2.2),\) and Lemma 2.6, we obtain

\[
\left| c(Q_{\mathfrak{T}}(0) - Q_{\mathfrak{T}}(y)) T^*_p g(y) - c(Q_{\mathfrak{T}}(0) - Q_{\mathfrak{T}}(y')) T^*_p g(y') \right|
\]

\[
= \int \left( c(-Q_{\mathfrak{T}}(x) + Q_{\mathfrak{T}}(y) - Q_{\mathfrak{T}}(0)) K_{s(p)}(x, y) \right.
\]

\[
- c(-Q_{\mathfrak{T}}(x) + Q_{\mathfrak{T}}(y') - Q_{\mathfrak{T}}(0)) K_{s(p)}(x, y') \left| dx \right|
\]

\[
\leq \int_{E(p)} |g(x)| \left| c(-Q_{\mathfrak{T}}(y') + Q_{\mathfrak{T}}(y) - Q_{\mathfrak{T}}(0)) K_{s(p)}(x, y) - K_{s(p)}(x, y') \right| dx
\]

\[
\leq \int_{E(p)} |g(x)| \left( c(-Q_{\mathfrak{T}}(y') + Q_{\mathfrak{T}}(y) - Q_{\mathfrak{T}}(0)) + 1 \right) K_{s(p)}(x, y) dx
\]

\[
+ \left| K_{s(p)}(x, y) - K_{s(p)}(x, y') \right| dx
\]

\[
\lesssim \int_{E(p)} |g(x)| \left( \|Q_{\mathfrak{T}} - Q_{\mathfrak{T}}\|_{I_p^*} \frac{|y - y'|}{D^{s(p)}} D^{-s(p)} d + \frac{|y - y'|}{D^{s(p)}} \right) \frac{D^{-s(p)} d}{D^{s(p)}} dx
\]

\[
\lesssim \frac{D^{-s(p)} d}{D^{s(p)}} \int_{E(p)} |g(x)| dx.
\]
Let $J \in \mathcal{D}$ be such that for every $p \in \mathcal{T}$ we have $I^*_p \cap (1 + 1/D)J \neq \emptyset \implies s(p) \geq s(J)$. Then, for every $y, y' \in (1 + 1/D)J$, we obtain

\begin{align}
(5.20) \quad & \left| e(Q\chi(0) - Q\chi(y))T_{\mathcal{T}}^s g(y) - e(Q\chi(0) - Q\chi(y'))T_{\mathcal{T}}^s g(y') \right| \\
& \leq \sum_{p \in \mathcal{T}: I^*_p \cap (1 + 1/D)J \neq \emptyset} \left| e(Q\chi(0) - Q\chi(y))T_{\mathcal{T}}^s g(y) - e(Q\chi(0) - Q\chi(y'))T_{\mathcal{T}}^s g(y') \right| \\
& \lesssim \sum_{s \geq s(J)} \sum_{p \in \mathcal{T}: I^*_p \cap (1 + 1/D)J \neq \emptyset, s(p) = s} \frac{|y - y'|}{D^s} \int \sup_{E(p)} |g(x)dx| \\
& \lesssim \left( \frac{|y - y'|}{D^{s(J)}} \right) \inf_M M g.
\end{align}

The estimate (5.20) implies in particular

\begin{align}
(5.21) \quad & \sup_{y \in (1 + 1/D)J} |e(Q\chi(0) - Q\chi(y))T_{\mathcal{T}}^s g(y)| \leq \inf_{y \in J} |T_{\mathcal{T}}^s g(y)| + C \inf_{y \in J} M g(y).
\end{align}

We claim that, for an absolute constant $s_0$, we have

\begin{align}
(5.22) \quad & p \in \mathcal{T}_2 \setminus \mathcal{G}, J \in \mathcal{J}, I^*_p \cap J \neq \emptyset \implies s(p) \leq s(J) + s_0.
\end{align}

**Proof of Claim (5.22).** Let $s_0 > 1$ be chosen later and suppose $s(p) > s(J) + s_0$. By definition, there exists $p' \in \mathcal{G}$ with $I^*_{p'} \subseteq 10D^2J$. On the other hand, $10I^*_p \supseteq D^{s_0 - 1}J$. Using Lemma 2.6, we obtain

\[ \Delta^{1 - \eta} \|Q\|_{I^*_p} \gtrsim \|Q\|_{10I^*_p} \gtrsim D^{s_0} \|Q\|_{10D^2J} \gtrsim D^{s_0} \|Q\|_{I^*_p} \gtrsim D^{s_0} \Delta^{1 - \eta}, \]

so that $1 > cD^{s_0}$ for some absolute constant $c > 0$. This is a contradiction if $s_0$ is sufficiently large. \hfill \Box

Using (5.18) and (5.22) for every $J \in \mathcal{J}$, we obtain

\begin{align}
(5.23) \quad & \sup_{y \in (1 + 1/D)J} |T_{\mathcal{T}_2 \cap \mathcal{G}}^s g_2(y)| \leq \sup_{y \in J} \sum_{s = s(J)} \sum_{p \in \mathcal{T}, s(p) = s} |T_{\mathcal{T}}^s g_2(y)| \lesssim (s_0 + 1) \inf_{y \in J} M g_2.
\end{align}

Using (5.21) together with this fact, we obtain

\begin{align}
(5.24) \quad & \sup_{y \in (1 + 1/D)J} |T_{\mathcal{T}_2 \cap \mathcal{G}}^s g_2(y)| \leq \inf_{y \in J} |T_{\mathcal{T}_2 \cap \mathcal{G}}^s g_2(y)| + C \inf_{y \in J} M g_2(y) \\
& \leq \inf_{y \in J} |T_{\mathcal{T}_2}^s g_2(y)| + \sup_{y \in J} |T_{\mathcal{T}_2 \cap \mathcal{G}}^s g_2(y)| + C \inf_{y \in J} M g_2(y) \\
& \leq \inf_{y \in J} |T_{\mathcal{T}_2}^s g_2(y)| + C \inf_{y \in J} M g_2(y)
\end{align}

for $J \in \mathcal{J}$. Using (5.20) with $\mathcal{T} = \mathcal{T}_1$ and $\mathcal{T} = \mathcal{T}_2 \cap \mathcal{G}$, (5.21) with $\mathcal{T} = \mathcal{T}_1$, and (5.23), we obtain the estimate

\[ |h_J(y) - h_J(y')| \lesssim \left( \frac{|y - y'|}{L(J)} \right) \prod_{j=1,2} \left( \inf_{y \in J} |T_{\mathcal{T}_j}^s g_j| + \inf_{y \in J} M g_j \right) \]

for $y, y' \in I_0 \cap (1 + 1/D)J$ and the functions

\[ h_J(y) := \chi_J(y) \left( e(Q\chi_1(0) - Q\chi_1(y))T_{\mathcal{T}_1}^s g_1(y) \right) \cdot \left( e(Q\chi_2(0) - Q\chi_2(y))T_{\mathcal{T}_2}^s \cap \mathcal{G} g_2(y) \right). \]

Moreover, since the function $T_{\mathcal{T}_1}^s g_1$ is continuous and vanishes outside $I_0$, while $h_J$ vanishes outside $(1 + 1/D)J$, the same Hölder type estimate continues to hold for all $y, y' \in \mathbb{R}^d$. Using (5.19) and Lemma A.1, this allows us to estimate

\[ \left| \int_{\mathbb{R}^d} T_{\mathcal{T}_1}^s g_1 T_{\mathcal{T}_2}^s \cap \mathcal{G} g_2 \right| \leq \sum_j \left| \int e(Q(y) - Q(0))h_J(y)dy \right| \]
\[
\lesssim \sum_j \Delta_j^{-\tau/d} |J| \prod_{j=1,2} \inf \left( |T^s_{\xi_j} g_j| + Mg_j \right)
\]
\[
\lesssim \Delta^{-(1-\eta)/d} \int_{I_0} \prod_{j=1,2} \left( |T^s_{\xi_j} g_j| + Mg_j \right)
\]
\[
\leq \Delta^{-(1-\eta)/d} \prod_{j=1,2} \| |T^s_{\xi_j} g_j| + Mg_j \|_{L^2(I_0)}
\]

It remains to consider the contribution of \( \mathcal{T}_2 \setminus \mathcal{S} \). Let \( J' := \{ J \in \mathcal{J}(\mathcal{T}_1) \mid J \subset I_0 \} \). We claim that, for some \( s_\Delta \) with \( D^{s_\Delta} \sim \Delta^{\eta/d} \), we have
\[
(5.24) \quad p \in \mathcal{T}_2 \setminus \mathcal{S}, J \in J', I_p \cap J \neq \emptyset \implies s(p) \leq s(J) - s_\Delta.
\]
Indeed, if \( s(p) > s(J) - s_\Delta \), then \( CD^{s_\Delta} I_p \supset 100D^J \supset I_p' \) for some \( p' \in \mathcal{T}_1 \), and, by Lemma 2.6, we obtain
\[
\Delta^{1-\eta} > \|Q\|_{I_p} \gtrsim D^{-s_\Delta} \|Q\|_{CD^{s_\Delta} I_p} \gtrsim D^{-s_\Delta} \|Q\|_{100D^J} \gtrsim D^{-s_\Delta} \|Q\|_{I_p'} \gtrsim D^{-s_\Delta}(\Delta - 5),
\]
so that \( 1 > cD^{-s_\Delta} \Delta^{\eta} \). This is a contradiction if the proportionality constants in \( D^{s_\Delta} \sim \Delta^{\eta/d} \) are chosen appropriately. Using Lemma 5.6 and (5.24), we obtain
\[
\left| \int_{\mathbb{R}^d} T^s_{\xi_1} g_1 T^s_{\xi_2} g_2 \right| \lesssim \|g_1\|_2 \|P_{J'} |T^s_{\xi_2} g_2|\|_2
\]
\[
\lesssim \|g_1\|_2 \sum_{s \geq s_\Delta} \left( \sum_{J \in J'} |J|^{-1} \left( \sum_{p \in \mathcal{T}_2} |s(J) - s_\Delta| / |s(J) - s_\Delta| \right)^{1/2} \right)
\]
\[
\lesssim \|g_1\|_2 \sum_{s \geq s_\Delta} \left( \sum_{J \in J'} |J|^{-1} \left( \sum_{I \in D_{s(J) - s_\Delta} I_{J \cap I_0 = 0, 2I \cap J \neq 0}} 1_{2I} \right)^{1/2} \right)
\]
\[
\lesssim \|g_1\|_2 \sum_{s \geq s_\Delta} \left( \sum_{J \in J'} \left( \int |Mg_2|^2 \right) \left( \sum_{I \in D_{s(J) - s_\Delta} I_{J \cap I_0 = 0, 2I \cap J \neq 0}} 1_{2I} \right)^{1/2} \right)
\]
\[
\lesssim \|g_1\|_2 \sum_{s \geq s_\Delta} \left( \sum_{J \in J'} \left( \int |Mg_2|^2 \right) \frac{D^{s(J) - s + s(J)(d-1)}}{D^{s(J)d}} \right)^{1/2}
\]
\[
\lesssim \|g_1\|_2 \sum_{s \geq s_\Delta} D^{-s/2} \|Mg_2\|_{L^2(I_0)}
\]
\[
\lesssim \Delta^{-(\eta/2d)} \|g_1\|_2 \|Mg_2\|_{L^2(I_0)}.
\]
Choosing \( \eta = 2\tau/(2\tau + 1) \) and observing that \( \|g_1\|_2 \leq \|Mg_1\|_{L^2(I_0)} \), we obtain the claim (5.17) with \( \epsilon = \tau/(d(2\tau + 1)) \).

5.4. Rows.

**Definition 5.25.** A row is a union of normal trees with tops that have pairwise disjoint spatial cubes.

**Lemma 5.26 (Row estimate).** Let \( \mathcal{R}_1, \mathcal{R}_2 \) be rows such that the trees in \( \mathcal{R}_1 \) are \( \Delta \)-separated from the trees in \( \mathcal{R}_2 \). Then, for any \( g_1, g_2 \in L^2(\mathbb{R}^d) \), we have
\[
\left| \int T^s_{\mathcal{R}_1} \mathcal{T}_{\xi_1} T^s_{\mathcal{T}_{\xi_2}} g_2 \right| \lesssim \Delta^{-\epsilon} \|g_1\|_2 \|g_2\|_2.
\]

**Proof.** The operators \( S_{\xi g} := |T^s_{\xi} g| + Mg \) are bounded on \( L^2(\mathbb{R}^d) \) uniformly in \( \xi \) by Lemma 5.6 and the Hardy–Littlewood maximal inequality. Using Lemma 5.16, we estimate
\[
\left| \int T^s_{\mathcal{R}_1} \mathcal{T}_{\xi_1} T^s_{\mathcal{T}_{\xi_2}} g_2 \right| \lesssim \sum_{\xi_1, \xi_2} \left| \int T^s_{\xi_1} \mathcal{T}_{\xi_1} T^s_{\mathcal{T}_{\xi_2}} g_2 \right|
\]
\[
= \sum_{\xi_1, \xi_2} \left| \int T^s_{\xi_1} (1_{\xi_1} g_1) T^s_{\xi_2} (1_{\xi_2} g_2) \right|
\]
\[ \Delta^{-\epsilon} \sum_{T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2} \prod_{j=1,2} \|S_{T_j} 1_{T_j} g_j\|_{L^2(I_{T_1} \cap I_{T_2})} \]
\[ \leq \Delta^{-\epsilon} \prod_{j=1,2} \left( \sum_{T_j \in \mathcal{T}_j} \|S_{T_j} 1_{T_j} g_j\|_{L^2(I_{T_j})}^2 \right)^{1/2} \]
\[ \leq \Delta^{-\epsilon} \prod_{j=1,2} \left( \sum_{T_j \in \mathcal{T}_j} \|1_{I_{T_j}} g_j\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} \]
\[ \leq \Delta^{-\epsilon} \|g_1\|_2 \|g_2\|_2. \]

5.5. **Forest estimate.** Recall our decomposition (3.29) of the set of all tiles. In view of Proposition 4.6, it remains to estimate the contribution of the normal trees

\[ \mathcal{N}_{n,k,j,l} := \mathcal{I}_{n,k,j,l} \setminus \text{bd}(\mathcal{I}_{n,k,j,l}) \]

These sets are indeed (convex) trees, since \(\text{bd}(\mathcal{I})\) are up-sets (recall the definition (4.5)).

**Proposition 5.27.** Let \(\mathcal{F}'_{n,k,j} := \cup_l \mathcal{N}_{n,k,j,l}\). Then

\[ \|T_{\mathcal{F}_{n,k,j}}\|_{2 \to 2} \lesssim 2^{-n/2}. \quad \text{(5.28)} \]

Assuming in addition (5.11) for all \(p \in \mathcal{F}'_{n,k,j}\), we obtain

\[ \|T_{\mathcal{F}_{n,k,j}}1_F\|_{2 \to 2} \lesssim \kappa^\alpha 2^{-n\epsilon} \quad \text{for any } 0 \leq \alpha < 1/2. \quad \text{(5.29)} \]

**Proof.** We subdivide \(\mathcal{F}'_{n,k,j}\) into rows by the following procedure: for each \(m \geq 0\), let inductively \(\mathcal{R}_{n,k,m} = \bigcup_{l \in L(k,m)} \mathcal{N}_{n,k,j,l}\) be the union of a maximal set of trees whose spatial cubes are disjoint and maximal among those that have not been selected yet. This procedure terminates after \(O(2^n \log(n + 1))\) steps, because the tree top cubes have overlap bounded by \(O(2^n \log(n + 1))\). Applying Corollary 5.10 with the set \(F\) and with the set \(R\) replaced by \(\mathbb{R}^d\) to each tree, we obtain

\[ \|T_{\mathcal{N}_{n,k,j,l}}1_F\|_{2 \to 2} \lesssim \kappa^{1/2} 2^{-n/2}, \quad \|T_{\mathcal{N}_{n,k,j,l}}\|_{2 \to 2} \lesssim 2^{-n/2}. \]

Using normality of the trees and disjointness of their top cubes, we obtain

\[ \|T_{\mathcal{R}_{n,k,m}}1_F\|_{2 \to 2} \lesssim \kappa^{1/2} 2^{-n/2}, \quad \|T_{\mathcal{R}_{n,k,m}}\|_{2 \to 2} \lesssim 2^{-n/2}. \quad \text{(5.30)} \]

Using the fact that

\[ T_{\mathcal{R}_{n,k,m}} T_{\mathcal{R}_{n,k,m'}} = 0 \quad \text{for } m \neq m' \]

due to disjointness of \(E(p)\) for tiles that belong to separated trees, as well as Lemma 5.26 and an orthogonality argument, we obtain (5.28).

Using (5.31) and (5.30) gives

\[ \|T_{\mathcal{F}'_{n,k,j}}1_F\|_2 = \left( \sum_{m \leq 2^n \log(n + 1)} \|T_{\mathcal{R}_{n,k,m}}1_F\|_2^2 \right)^{1/2} \]
\[ \lesssim \left( \sum_{m \leq 2^n \log(n + 1)} (\kappa^{1/2} 2^{-n/2} \|f\|_2^2)^2 \right)^{1/2} \]
\[ \lesssim \kappa^{1/2} 2^{-n/2} \|f\|_2^2 (2^n \log(n + 1))^{1/2} \]
\[ \lesssim \kappa^{1/2} (\log(n + 1))^{1/2} \|f\|_2. \]

Taking a geometric average with (5.28), we obtain (5.29). \(\square\)
5.6. Orthogonality between stopping generations.

Lemma 5.32. Let $\mathcal{T} \subset \mathcal{P}_k$ be a tree and $k' > k$. Then
\[
\| T_{\mathcal{T}} T F_{k'} \|_{2 \to 2} \lesssim e^{-(k' - k)},
\]
where $F_{k'} = \cup_{F \in F_{k'}} F$.

Proof. Let $J := J(\mathcal{T})$ and $J \in J$, so that $100D \hat{J} \supseteq I_p$ for some $p \in \mathcal{T}$.
Let $F' \in F_{k+1}$ be such that $J \cap F' \neq \emptyset$. Suppose that $s(F') \geq s(J) + 4$. Then
\[ (1 + \frac{1}{2}) E' \supseteq 100D \hat{J} \supseteq I_p \]
and $s(F') > s(p)$. By part 3 of Lemma 3.3, this implies $I \in F_{k+1}$ for some $I \supseteq I_p$, contradicting $I_p \in \mathcal{C}_k$.
Therefore, we must have $s(F') \leq s(J) + 3$, and it follows that
\[
\sum_{F' \in F_{k+1} : J \cap F' \neq \emptyset} |F'| \lesssim |J|.
\]
Hence,
\[
|J \cap F_{k'}| \leq \sum_{F' \in F_{k+1} : J \cap F' \neq \emptyset} |F' \cap F_{k'}| \lesssim \sum_{F' \in F_{k+1} : J \cap F' \neq \emptyset} e^{-(k' - k) - 1} |F'| \lesssim e^{-(k' - k)} |J|.
\]
This implies $\| P_J T F_{k'} \|_{2 \to 2} \lesssim e^{-(k' - k)}$, and the claim follows from Lemma 5.6. □

Proposition 5.33. For any measurable subset $F' \subset \mathbb{R}^d$, we have
\begin{align*}
(5.34) & \quad \| T_{\mathcal{G}_{n,k,j}}^* T_{\mathcal{G}_{n,k',j}}^* \|_{2 \to 2} \lesssim 10^n e^{-|k - k'|}, \\
(5.35) & \quad \| T_{\mathcal{G}_{n,k,j}}^* F T_{\mathcal{G}_{n,k',j}}^* \|_{2 \to 2} \lesssim 10^n e^{-|k - k'|}.
\end{align*}

Proof. Let $\mathcal{R}_{n,k,m}$ be the rows defined in the proof of Proposition 5.27. It suffices to show
\begin{align*}
(5.36) & \quad \| T_{\mathcal{R}_{n,k,m}}^* T_{\mathcal{R}_{n,k',m}}^* \|_{2 \to 2} \lesssim e^{-|k - k'|}, \\
(5.37) & \quad \| T_{\mathcal{R}_{n,k,m}}^* T_{\mathcal{G}_{n,k',m}}^* \|_{2 \to 2} \lesssim e^{-|k - k'|}.
\end{align*}

Without loss of generality we may assume $k' \geq k$. We will use the fact that
\[ T_{\mathcal{R}_{n,k',m'}} = F_{k'} T_{\mathcal{G}_{n,k',m'}} = T_{\mathcal{R}_{n,k',m'}} F_{k'} \]
with $F_{k'} = \cup_{F \in F_{k'}} F$ (the last equality uses normality of the trees).

Using (5.30), we estimate
\[
LHS(5.36) = \| T_{\mathcal{R}_{n,k,m}}^* T_{\mathcal{G}_{n,k',m'}} \|_{2 \to 2} \lesssim \| T_{\mathcal{R}_{n,k,m}}^* F_{k'} \|_{2 \to 2} \| T_{\mathcal{G}_{n,k',m'}} \|_{2 \to 2} \lesssim \| F_{k'} \|_{2 \to 2} \| T_{\mathcal{R}_{n,k',m'}} \|_{2 \to 2}.
\]

As a consequence of (3.4), we have
\[
\| P_{\mathcal{L}(\mathcal{R}_{n,k,j}, i)} T_{\mathcal{G}_{n,k',j}} \|_{2 \to 2} \lesssim e^{-|k - k'|},
\]
and (5.36) follows from Lemma 5.6. Similarly,
\[
LHS(5.37) = \| T_{\mathcal{R}_{n,k,m}}^* F_{k'} T_{\mathcal{G}_{n,k',m'}}^* \|_{2 \to 2} \lesssim \| T_{\mathcal{R}_{n,k,m}}^* F \|_{2 \to 2} \| T_{\mathcal{G}_{n,k',m'}}^* \|_{2 \to 2} \lesssim \| T_{\mathcal{R}_{n,k,m}}^* F_{k'} \|_{2 \to 2} \lesssim e^{-|k - k'|}
\]
by Lemma 5.32. □
6. Proof of Theorem 1.5

As previously mentioned in Section 2, in view of Lemma B.2, we may replace the operator (1.3) by (2.3), which in turn can be replaced by $T_p$.

**Proof of (1.6).** Using the decomposition (3.29), we split

$$
\|T_p\|_{2 \to 2} \leq \sum_{n=1}^{\infty} \sum_{j=1}^{Cn^2} \left( \| \sum_{k \in \mathbb{N}} T_{\tilde{n},n,k,j} \|_{2 \to 2} + \| \sum_{k \in \mathbb{N}} T_{\tilde{n},n,k,j} \|_{2 \to 2} + \| \sum_{k \in \mathbb{N}} T_{\tilde{\tau}(n,k,j)} \|_{2 \to 2} \right).
$$

The contribution of the last two summands is estimated by Proposition 4.6. In the first summand, we split the summation over $k$ in congruence classes modulo $Cn$ and use Propositions 5.27, 5.33, and the Cotlar–Stein Lemma (see e.g. [Ste93, Section VII.2]).

In the remaining part of the proof, we may assume $0 < \nu, \kappa < 1$. Indeed, the cases $\nu = 0$ and $\kappa = 0$ are trivial, and in the cases $\nu \geq 1$ or $\kappa \geq 1$ the respective estimates (1.7) and (1.8) follow from (1.6).

**Proof of (1.7).** Let $\mathcal{P}_G := \{ p \in \mathcal{P} \mid I_p \subseteq \tilde{G} \}$, then $T_p 1_{\mathbb{R}^d \setminus \tilde{G}} = 0$ if $p \in \mathcal{P}_G$. Hence

$$
\| 1_G T_p 1_{\mathbb{R}^d \setminus \tilde{G}} \|_{2 \to 2} = \| 1_G T_p \psi_{\mathcal{P}_G} 1_{\mathbb{R}^d \setminus \tilde{G}} \|_{2 \to 2} \leq \| 1_G T_p \psi_{\mathcal{P}_G} \|_{2 \to 2}.
$$

In order to estimate the latter quantity, we run the proof of (1.6) with $\mathcal{P}$ replaced by $\mathcal{P}_G$ and (formally) $\sigma(x) = -\infty$ for $x \in \mathbb{R}^d \setminus \tilde{G}$.

The main change is that all tiles now have density $2^{\nu} \preceq \nu$. This yields the required improvement in the estimate for the main term. In the error terms, we use Proposition 4.13 with $F = \mathbb{R}^d$. The hypothesis (4.14) is satisfied, because we have removed all tiles whose spatial cubes are contained in $\tilde{G}$. \hfill \Box

**Proof of (1.8).** Let $\mathcal{P}_F := \{ p \in \mathcal{P} \mid I_p \subseteq \tilde{F} \}$, then $1_{\mathbb{R}^d \setminus \tilde{F}} T_p = 0$ if $p \in \mathcal{P}_F$. Hence

$$
\| 1_{\mathbb{R}^d \setminus \tilde{F}} T_p 1_F \|_{2 \to 2} = \| 1_{\mathbb{R}^d \setminus \tilde{F}} T_p \psi_{\mathcal{P}_F} 1_F \|_{2 \to 2} \leq \| T_p \psi_{\mathcal{P}_F} 1_F \|_{2 \to 2}.
$$

In order to estimate the latter term, we again run the proof of (1.6) with $\mathcal{P}$ replaced by $\mathcal{P}_F$. In particular, we split

$$
\| T_p 1_F \|_{2 \to 2} \leq \sum_{n=1}^{\infty} \sum_{j=1}^{Cn^2} \left( \| \sum_{k \in \mathbb{N}} T_{\tilde{n},n,k,j} 1_F \|_{2 \to 2} + \| \sum_{k \in \mathbb{N}} T_{\tilde{\tau}(n,k,j)} 1_F \|_{2 \to 2} \right).
$$

The contribution of the last two terms is taken care of by Proposition 4.13 with $G = \mathbb{R}^d$. In the estimate for the main term, we use $(5.29)$ in place of $(5.28)$ and split the summation over $k$ in congruence classes modulo $[Cn(|\log \kappa| + 1)]$. \hfill \Box

**Appendix A. A van der Corput type oscillatory integral estimate**

We will use the following van der Corput type estimate for oscillatory integrals in $\mathbb{R}^d$ that refines [SW01, Proposition 2.1].

**Lemma A.1.** Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be a measurable function with $\text{supp} \, \psi \subset J$ for a cube $J$. Then, for every $Q \in \mathcal{Q}_d$, we have

$$
| \int_{\mathbb{R}^d} e(Q(x)) \psi(x) dx | \leq \sup_{|y| < \Delta^{-1/d} |J|} | \int_{\mathbb{R}^d} | \psi(x) - \psi(x - y) | dx |, \quad \Delta = \| Q + \mathbb{R} \|_J + 1.
$$

**Remark A.2.** The supremum over $|y| < \Delta^{-1/d} |J|$ above can be replaced by an average.
By duality between $G$ and $\phi$, we may assume $\ell(J) \sim 1$ and $J \subset B(0,1/2)$. Let $\beta$ denote the right-hand side of the conclusion. If $\Delta \lesssim 1$, then $\|\psi\|_1 \lesssim \beta$, so the result is only non-trivial if $\Delta \gg 1$. In this case, we replace $\psi$ on the left-hand side by $\tilde{\psi} := \phi \ast \psi$, where $\phi = \Delta^{d/d} \phi_0(\Delta^{1/d})$ and $\phi_0$ is a smooth positive bump function with integral 1 supported on the unit ball. The error term is controlled by

$$\int |\psi - \tilde{\psi}|(x)dx = \int \int (\psi(x) - \psi(x - y))\phi(y)dy|dx$$

$$\leq \int \phi(y) \int |\psi(x) - \psi(x - y)|dx dy$$

$$\lesssim \beta.$$  

Moreover, supp $\tilde{\psi} \subset B(0,1)$ and

$$\int |\partial_i \tilde{\psi}(x)|dx = \int \int |\psi(x-y)|\partial_i \phi(y)dy|dx$$

$$= \int \int (\psi(x) - \psi(x - y))\partial_i \phi(y)dy|dx$$

$$\leq \int \int |\psi(x) - \psi(x - y)||\partial_i \phi(y)|dydx$$

$$\lesssim \Delta^{d/d+1} \int \int_{B(0,\Delta^{-1/d})} |\psi(x) - \psi(x - y)|dydx$$

$$\lesssim \Delta^{1/d} \beta$$

for every $i = 1, \ldots, d$. The result now follows from the proof of [SW01, Proposition 2.1] applied to $\tilde{\psi}$. Notice that the one-dimensional van der Corput estimate (Corollary on p. 334 of [Ste93]) used in that proof only requires an estimate on the integral of $\nabla \tilde{\psi}$.

\[\square\]

**Appendix B. The extrapolation argument**

Theorem 1.4 is deduced from Theorem 1.5 using Bateman’s extrapolation argument that first appeared in [BT13] (see also [DS15, Theorem 1.1] and [Di +18, Theorem 2.27] for an abstract formulation of this argument). In order to keep our exposition self-contained, we present this argument in the case needed here.

**Lemma B.1.** Let $1 < p, q < \infty$, $(X, \mu)$ be a $\sigma$-finite measure space, and $g : X \to \mathbb{C}$ be a measurable function. Suppose that there exists $A < \infty$ such that, for every measurable subset $G \subset X$ with $0 < \mu(G) < \infty$, there exists a measurable subset $\tilde{G} \subset X$ with $\mu(\tilde{G}) \leq \mu(G)/2$ such that $\|g1_{G\setminus\tilde{G}}\|_{L_q(X)} \leq A\mu(G)^{1/p-1/q}$. Then $\|g\|_{L_p,\infty(X)} \lesssim A$.

**Proof.** Let $G_0 \subset X$ with $\mu(G_0) < \infty$ be given. For $n \in \{1, 2, \ldots\}$, define inductively $G_{n+1} := \tilde{G}_n$, so that $\mu(G_n) \leq 2^{-n}\mu(G_0)$. Then

$$\int_{G_0} |g|d\mu = \sum_{n=0}^{\infty} \int_X |g|1_{G_n \setminus G_{n+1}} |d\mu$$

$$\leq \sum_{n=0}^{\infty} \|g1_{G_n \setminus G_{n+1}}\|_{L_q(X)} \|1_{G_n}\|_{L_{q',1}}$$

$$\lesssim A \sum_{n=0}^{\infty} \mu(G_n)^{1/q-1/p}\mu(G_n)^{1/q'}$$

$$\leq A \sum_{n=0}^{\infty} (2^{-n}\mu(G_0))^{1/p'}$$

$$\lesssim A\mu(G_0)^{1/p'}.$$

By duality between $L_p,\infty$ and $L_{p',1}$, this implies the claim. \[\square\]
Proof of Theorem 1.4 assuming Theorem 1.5. By standard real interpolation theory [BL76], it suffices to show that $T$ is a bounded operator from $L^{p,1}(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ for every $1 < p < \infty$. To see this, let $F \subset \mathbb{R}^d$ be a measurable subset with $0 < |F| < \infty$, $f : \mathbb{R}^d \to \mathbb{C}$ a measurable function with $|f| \leq 1_F$, and $g := T f$. If $G \subset \mathbb{R}^d$ is a measurable subset with $0 < |G| < \infty$, then, for a sufficiently large absolute constant $C$, the set $\tilde{G} := \{ M 1_F > C |G|^{-1} \}$ satisfies $|\tilde{G}| \leq |G|/2$. On the other hand, by (1.8), for every $0 \leq \alpha < 1/2$, we have

$$
\| g 1_{G^c \tilde{G}} \|_{L^{2,\infty}} \leq \| g 1_{\mathbb{R}^d \setminus \tilde{G}} \|_2 \lesssim (|G|^{-1})^\alpha \| f \|_2 \leq |F|^{\alpha+1/2} |G|^{-\alpha}.
$$

Using this with $\alpha = 1/p - 1/q$ for $1 < p \leq 2 = q$, we see that the hypothesis of Lemma B.1 holds for the function $g$ with $A \lesssim_p |F|^{1/p}$. Hence, by Lemma B.1, we obtain $\| g \|_{L^{p,\infty}} \lesssim |F|^{1/p}$. This shows that $T$ is a bounded operator from $L^{p,1}$ to $L^{p,\infty}$.

In the case $2 < p < \infty$, we can run the above argument for the adjoint operator $T^*$ in place of $T$, using the estimate (1.7) in place of (1.8).

Finally, in Section 6, we have used a localized estimate for the Hardy–Littlewood maximal operator. We include the short proof.

Lemma B.2. Let $0 \leq \alpha < 1/2$ and $0 < \nu \leq 1$. Let $G \subset \mathbb{R}^d$ be a measurable subset and $\tilde{G} := \{ M 1_G > \nu \}$. Then

$$
\| 1_{G^c \tilde{G}} M 1_{\mathbb{R}^d \setminus \tilde{G}} \|_{2 \to 2} \lesssim \nu^\alpha \| 1_{\mathbb{R}^d \setminus \tilde{G}} M 1_G \|_{2 \to 2} \lesssim \nu^\alpha.
$$

Proof. By the Fefferman–Stein maximal inequality [FS71], we have

$$
\| 1_G M 1_{\mathbb{R}^d \setminus \tilde{G}} f \|_{1,\infty} = \| M 1_{\mathbb{R}^d \setminus \tilde{G}} f \|_{L^{1,\infty}(1_G)} \lesssim \| 1_{\mathbb{R}^d \setminus \tilde{G}} f \|_{L^1(1_G)} \lesssim \nu \| f \|_1.
$$

Interpolating with the trivial $L^{\infty}$ estimate, we obtain the first claim.

Let now $q = 1/\alpha$. Then, by Hölder’s inequality,

$$
M 1_G f \leq (M_q 1_G)(M_{q'} f) = (M 1_G)\alpha(M_q f).
$$

Hence,

$$
\| 1_{\mathbb{R}^d \setminus \tilde{G}} M 1_{\mathbb{R}^d \setminus \tilde{G}} f \|_2 \leq \| 1_{\mathbb{R}^d \setminus \tilde{G}} (M 1_G)\alpha M_{q'} f \|_2 \leq \nu^\alpha \| M_{q'} f \|_2 \lesssim \nu^\alpha \| f \|_2,
$$

where we have used the fact that $M_{q'}$ is bounded on $L^2$ provided that $q' < 2$. 

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