INVARINATIVES OF SELF-INTERSECTED AND INVERSIVE N-PERIODICS IN THE ELLIPTIC BILLIARD

RONALDO GARCIA AND DAN REZNIK

ABSTRACT. We study more invariants in the elliptic billiard, including those manifested by self-intersected orbits and inversive polygons. We also derive expressions for some entries in “Eighty New Invariants of N-Periodics in the Elliptic Billiard” (2020), arXiv:2004.12497.

Keywords invariant, elliptic, billiard.

MSC 51M04 and 37D50 and 51N20 and 51N35 and 68T20

1. INTRODUCTION

The two classic invariants of N-periodics in the elliptic billiard are perimeter $L$ and Joachimsthal’s constant $J$. The former is a direct consequence of integrability whereas the latter is equivalent to stating that all trajectory segments are tangent to a confocal caustic [20]; see Figure 1.

It turns out constant $L$ and $J$ have often surprising cartesian manifestations. In [18] the following experimentally-found invariants were introduced: (i) the sum of N-periodic angle cosines was shown to be invariant; (ii) the ratio of outer (aka. tangential) polygon (see Figure 2) to the N-periodic’s is invariant if $N$ is odd. These were soon elegantly proved using tools of algebraic and differential geometry [2, 4, 6]. For the $N = 3$ case some invariants have been explicitly derived [11].

Video.

Figure 1. Elliptic Billiard (black) 4- and 5-periodics (blue). Every trajectory vertex $P_i$ (resp. segment $P_iP_{i+1}$) is bisected by the local normal $\hat{n}_i$ (resp. tangent to the confocal caustic, brown). A second, equi-perimeter member of each family is also shown (dashed blue). Video.
Figure 2. The N-periodic (blue) is associated with an outer (green) and an inner (red) polygons. The former’s sides are tangent to the billiard (black) at each N-periodic vertex; the latter’s vertices are the tangency points of N-periodics sides to the confocal caustic (brown). Video

Figure 3. The inversive polygon (pink) has vertices at inversions of the $P_i$ (blue polygon) with respect to a unit circle (dashed black) centered on $f_1$. It turns out its perimeter is also invariant over the family as is the sum of its spoke lengths (pink lines) Video.

Experimentation with additional polygons such as pedals, antipedals, etc., derived from N-periodics has added to our list of invariants (we are now at some 80 entries) [17]. A particular prolific object has been the focus-inversive polygon, shown in see Figure 3. Its vertices are the inversions of N-periodic vertices with respect to a unit-circle centered on a focus. Though the inversive family is inscribed in Pascal’s Limaçon, a surprising experimental observation is that like N-periodics, the inversive perimeter and sum of cosines are also invariant for all $N$ [17, k803,k804].

Referring to Figures 7 and 14, here we also explore properties of self-intersected N-periodics which exist for all $N > 3$. Namely, if an invariant works for a simple orbit, will it also work in the self-intersected case?
Main contributions:

- Section 2: Focus-inversive 3-periodics are a constant-perimeter family inscribed in a rigidly moving ellipse, i.e., a moving elliptic billiard.
- Section 3: The vertices of self-intersected 4-periodics and their outer polygon are concyclic with the foci on two distinct variable-radius circles.
- Section 4: for N=3,4,6,8, we derive expressions for selected invariants listed in [17] including a few self-intersecting cases.
- In Appendix B.4 (resp. B.6) we show that the caustic semi-axes for N=5 (N=7) periodics, simple or self-intersected, are roots of degree-6 (resp. degree-12) polynomials. There are two valid cases for N = 5 and 3 for N = 7; see Figure 4.

Since many phenomena are best understood in motion, a link is provided in the caption of most figures to an animation. Table 2 in Section 5 compiles all videos mentioned and a few extra ones.

In Appendix A we review some classical quantities for the Billiard. In Appendix B we provide explicit expressions for vertices and caustics for N = 3,4,5,6,7,8 in both simple and self-intersected configurations. In Appendix C we list all symbols used.

2. N=3 INVERSIVE FAMILY IS NEW BILLIARD FAMILY

Throughout this article we assume the elliptic billiard is the ellipse:

\[ f(x, y) = \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1, \quad a > b > 0. \]

A generic N > 3 focus-inversive polygon is illustrated in Figure 3. This is a polygon whose vertices are inversions of the N-periodic vertices wrt a circle of radius \( \rho \) centered on \( f_1 \).

It is well known the inversion of an ellipse wrt to a focus is Pascal’s Limaçon [9]. Therefore, for any N:

**Remark 1.** The focus-inversive family is inscribed in Pascal’s Limaçon.

Figure 5 illustrates the N = 3 focus-inversive triangle. Recall two well-known triangle centers: the Gergonne point \( X_7 \) is the perspector of the incircle and the
Mittenpunkt $X_9$ is the point of concurrence of lines drawn from the excenters through the medians [13].

**Proposition 1.** Over the $3$-periodic family, the Gergonne point $X_7$ of its focus-inversive triangles is stationary on the billiard’s major axis.

**Proposition 2.** Over the $3$-periodic family, the Mittenpunkt $X_9$ of its focus-inversive triangles moves along a circle whose center lies on the billiard’s major axis.

Note: we omit explicit expressions for the two previous results since they would occupy several pages.

**Proposition 3.** The $X_9$-centered circumellipse $C^\dagger$ of the $N = 3$ inversive family has invariant semi-axes $a^\dagger$ and $b^\dagger$ given by:
\[ a^\dagger = \rho k_1 \sqrt{a c k_2 + \delta (2 a^2 - b^2 - \delta)} \]
\[ b^\dagger = \rho k_1 \sqrt{-a c k_2 + \delta (2 a^2 - b^2 - \delta)} \]
\[ k_1 = \frac{c \sqrt{2}}{k_3} \sqrt{(8 a^4 + 4 a^2 b^2 + 2 b^4) \delta + 8 a^6 + 3 a^2 b^4 + 2 b^6} \]
\[ k_2 = \sqrt{(-4 a^2 + 2 b^2) \delta + 5 a^4 - 5 a^2 b^2 + 2 b^4} \]
\[ k_3 = 2 a b^2 \left( (2 a^2 - b^2) \delta + 2 a^4 - 2 a^2 b^2 - b^4 \right) \]

Note: \( C^\dagger \) is the rigidly-moving circumbilliard [16] of the inversive polygon.

**Proposition 4.** The perimeter \( L^\dagger \) of the \( N=3 \) focus-inversive family is invariant and given by:

\[ L^\dagger = \rho \frac{\sqrt{(8 a^4 + 4 a^2 b^2 + 2 b^4) \delta + 8 a^6 + 3 a^2 b^4 + 2 b^6}}{a^2 b^2} \]

In turn, the above entail:

**Theorem 1.** With respect to a reference system centered on \( \bar{X}_9 \) and oriented along the semi-axes of \( C^\dagger \), the inversive \( N = 3 \) family is a 3-periodic billiard family of \( C^\dagger \).

Experimentally we have observed:

**Conjecture 1.** The perimeter of the focus-inversive polygon is invariant for any \( N \).

3. **Vertices \( N=4 \) Self-Intersected are Concyclic with Foci**

The family of non-self-intersected (simple) billiard 4-periodics is comprised of parallelograms [7]. In this section consider self-intersected 4-periodics whose caustic is a confocal hyperbola; see Figure 7. We start deriving simple facts about them and then proceed to certain elegant properties.

**Proposition 5.** The perimeter \( L \) of the self-intersected 4-periodic is given by:

\[ L = \frac{4a^2}{c}, \quad \text{with} \quad c^2 = a^2 - b^2. \]

**Proof.** Since perimeter is constant, use as the \( N = 4 \) candidate the centrally-symmetric one, Figure 6 (right). Its upper-right vertex \( P_1 = (x_1, y_1) \) is such that it reflects a vertical ray toward \(-P_1\), and this yields:

\[ P_1 = (x_1, y_1) = \left[ \frac{a \sqrt{a^2 - 2b^2}}{bc}, \frac{b}{c} \right] \]

Since \( P_2 = -P_1 \) its perimeter is \( L = 2(|2P_1| + 2y_1) \) and this can be simplified to (2), invariant over the family. \( \square \)

with \( a/b \geq \sqrt{2} \). At \( a/b = \sqrt{2} \) the family is a straight line from top to bottom vertex of the EB, Figure 6 (left).
Observation 1. At $a/b = \sqrt{1+\sqrt{2}} \approx 1.55377$ the two self-intersecting segments of the bowtie do so at right-angles.

Observation 2. At $a/b \approx 1.55529$ the perimeter of the bowtie equal that of the EB.

Referring to Figure 7:

Proposition 6. The $N = 4$ self-intersected family has zero signed orbit area and zero sum of signed cosines, i.e., both are invariant. The same two facts are true for its outer polygon.

Proof. This stems from the fact all self-intersected 4-periodics are symmetric with respect to the EB’s minor axis. □

Referring to Figure 7, as in Appendix B.2, let vertex $P_1$ of the self-intersected 4-periodic be parametrized as $P_1(u) = [au, b\sqrt{1-u^2}]$, with $|u| \leq \frac{a}{c}\sqrt{a^2 - 2b^2}$. Then:

Theorem 2. The four vertices of the self-intersected 4-periodic (resp. outer polygon) are concyclic with the two foci of the elliptic billiard, on a circle $C$ of variable radius $R$ (resp. $R'$) whose center $C$ (resp. $C'$) lies on the y axis. These are given by:

$$C = \left[0, \frac{c^2u^2 - a^2 + 2b^2}{2b\sqrt{1-u^2}}\right]$$
$$R = \frac{a^2 - c^2u^2}{2b\sqrt{1-u^2}}$$
$$C' = \left[0, -\frac{2bc\sqrt{1-u^2}}{a^2 + (u^2 - 2)c^2}\right]$$
$$R' = \frac{c(c^2u^2 - a^2)}{a^2 - 2c^2 + c^2u^2}$$

Corollary 1. The half harmonic mean of $R^2$ and $R'^2$ is invariant and equal to $c^2 = a^2 - b^2$, i.e., $1/R^2 + 1/R'^2 = 1/c^2$.

Note: the above Pythagorean relation implies that the polygon whose vertices are a focus, and the inversion of $C, C', O$ (center of billiard) with respect to a unit circle centered on said focus, is a rectangle of sides $1/R$ and $1/R'$ and diagonal $1/c$.

Let $P^1$ (resp. $Q^1$) denote the inversive polygon of 4-periodics (resp. its outer polygon) wrt a unit circle $C^1$ centered on one focus. From properties of inversion:
Figure 7. The vertices of the self-intersected 4-periodic (blue) are concyclic with the billiard foci on a circle (dashed blue) centered on $C$. The inversive polygon (pink segment) with respect to a unit circle $C^\dagger$ (dashed black) centered on the left focus degenerates to a segment along the radical axis of the two circles. The vertices of the outer polygon (green) are also concyclic with the foci on a distinct circle (dashed green) centered on $C'$. Therefore the outer’s inversive polygon (dotted pink) is also a segment along the radical axis of this circle with $C^\dagger$. Note the two radical axes are dynamically perpendicular. Video 1, Video 2

**Corollary 2.** $P^\dagger$ (resp. $Q^\dagger$) has four collinear vertices, i.e., it degenerates to a segment along the radical axis of $C^\dagger$ and $C$ (resp. $C'$).

**Proposition 7.** The two said radical axes are perpendicular.

**Proof.** It is enough to check that the vectors $C - [-c, 0]$ and $C' - [-c, 0]$ are orthogonal. Observe that when $u^2 = (a^2 - 2b^2)/c = (2c^2 - a^2)/c$ the outer polygon is contained in the horizontal axis.

4. **Deriving Invariants for $N=3,4,5,6,8$**

In this section, we derive expressions for selected invariants introduced in [17], specifically for “low-N” cases, e.g., $N=3,4,5,6,8$. In that publication, each invariant is identified by a 3-digit code, e.g., $k_{101}$, $k_{102}$, etc. Table 1 lists the invariants considered herein. The quantities involved are defined next.

Let $\theta_i$ denote the $i$th $N$-periodic angle. Let $A$ the signed area of an $N$-periodic. Referring to Figure 2, singly-primed quantities (e.g., $\theta'_i$, $A'_i$, etc.), etc., always refer to the outer polygon: its sides are tangent to the EB at the $P_i$. Likewise, doubly-primed quantities ($\theta''_i$, $A''_i$, etc.) refer to the inner polygon: its vertices lie at the touchpoints of $N$-periodic sides with the caustic. More details on said quantities appear in Appendix A.

Recall $k_{101} = JL - N$, as introduced in [18, 4].
Referring to Figure 3, the \textit{f}_1-\textit{inversive polygon} has vertices at inversions of the \( P_i \) with respect to a unit circle centered on \( f_1 \). Quantities such as \( L_{\overset{*}{1}} \), \( A_{\overset{*}{1}} \), etc., refer to perimeter, area, etc. of said polygon.

\textbf{A word about our proof method.} We omit most proofs as they have been produced by a consistent process, namely: (i) using the expressions in Appendix B, find the vertices an axis-symmetric \( N \)-periodic, i.e., whose first vertex \( P_1 = (a,0) \); (ii) obtain a symbolic expression for the invariant of interest, (iii) simplify it (both human intervention and CAS), and finally (iv) verify its validity numerically over several \( N \)-periodic configurations and elliptic billiard aspect ratios \( a/b \).

\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
code & invariant & valid \( N \) & derived & proofs \\
\hline
\kappa_{101} & \sum \cos \theta_i & all & n/a & [2, 4] \\
\kappa_{102} & \prod \cos \theta'_i & all & 3,4,5,5_i,6_i,6_{ii} & [2, 4] \\
\kappa_{103} & A'/A & odd & 3,5,5_i & [2, 6] \\
\kappa_{104} & \sum \cos(2\theta'_i) & all & 3,4,5,5_i,6_i,8 & [1] \\
\kappa_{105} & \prod \sin(\theta_i/2) & odd & 3,5,5_i & [1] \\
\kappa_{106} & A'/A & even & 4,6,6_i,6_{ii} & [6] \\
\kappa_{110} & A_{\overset{*}{1}}A'' & even & 4,6,6_i,6_{ii} & ? \\
\kappa_{119} & \sum \kappa_i^{2/3} & all & 3,4,6 & [2, 19] \\
\hline
\kappa_{802,a} & \sum 1/d_{1,i} & all & 3,4,6 & [2] \\
\kappa_{803} & L_{1} & all & 3,4,6 & ? \\
\kappa_{804} & \sum \cos \theta_{1,i}^* & \neq 4 & 3 & ? \\
\kappa_{805,a} & A_{\overset{*}{1}}A_{1} & \equiv 0 \text{ (mod } 4) & 4,8 & ? \\
\kappa_{806} & A/A_{\overset{*}{1}} & \equiv 2 \text{ (mod } 4) & 6 & ? \\
\kappa_{807} & A_{1}A_{\overset{*}{1}} & odd & 3 & ? \\
\hline
\end{tabular}
\end{center}
\caption{List of selected invariants taken from [17] as well as the low-\( N \) cases (column “derived”) for expressions are derived herein. \( * \) co-discovered with P. Roitman. A closed-form expression for \( k_{119} \) was derived by H. Stachel; see (5).}
\end{table}

4.1. \textbf{Invariants for \( N=3 \).} Let \( \delta = \sqrt{a^4 - a^2b^2 + b^4} \). For \( N = 3 \) explicit expressions for \( J \) and \( L \) have been derived [11]:

\[
J = \sqrt{\frac{2\delta - a^2 - b^2}{c^2}} \tag{3}
\]

\[
L = 2(\delta + a^2 + b^2)J
\]

When \( a = b, J = \sqrt{3}/2 \) and when \( a/b \to \infty, J \to 0 \).

\textbf{Proposition 8.} For \( N = 3 \), \( k_{102} = (JL)/4 - 1 \).

\textit{Proof.} We’ve shown \( \sum_{i=1}^{3} \cos \theta_i = JL - 3 \) is invariant for the \( N = 3 \) family [11]. For any triangle \( \sum_{i=1}^{3} \cos \theta_i = 1 + r/R \) [21], so it follows that \( r/R = JL - 4 \) is also invariant. Let \( r_h, R_h \) be the Orthic Triangle’s Inradius and Circumradius. The relation \( r_h/R_h = 4 \prod_{i=1}^{3} |\cos \theta_i| \) is well-known [21, Orthic Triangle]. Since a
triangle is the Orthic of its Excentral Triangle, we can write \( r/R = 4 \prod_{i=1}^{3} \cos \theta_i' \), where \( \theta_i' \) are the Excentral angles which are always acute [21] (absolute value can be dropped), yielding the claim. □

**Proposition 9.** For \( N = 3 \), \( k_{103} = k_{109} = 2/(k_{101} - 1) = 2/(JL - 4) \).

**Proof.** Given a triangle \( A' \) (resp. \( A'' \)) refers to the area of the Excentral (resp. Extouch) triangles. The ratios \( A'/A \) and \( A/A'' \) are equal. Actually, \( A'/A = A/A'' = (s_1s_2s_3)/(r^2L) \), where \( s_i \) are the sides, \( L \) the perimeter, and \( r \) the Inradius [21, Excentral,Extouch]. Also known is that \( A'/A = 2R/r \) [12]. Since \( r/R = \sum_{i=1}^{3} \cos \theta_i - 1 = k_{101} - 1 \) [21, Inradius], the result follows. □

**Proposition 10.** For \( N = 3 \), \( k_{104} = -k_{101} \) and is given by:

\[
k_{104} = \frac{(a^2 + b^2)(a^2 + b^2 - 2\delta)}{c^4} = 3 - JL
\]

**Proposition 11.** For \( N = 3 \), \( k_{105} = (JL)/4 - 1 = k_{102} \).

**Proof.** Let \( r, R \) be a triangle’s Inradius and Circumradius. The identity \( r/R = 4\prod_{i=1}^{3} \sin(\theta_i)/2 \) holds for any triangle [21, Inradius], which with Proposition 8. This completes the proof. □

**Proposition 12.** For \( N = 3 \), \( k_{119} \) is given by:

\[
(k_{119})^3 = \frac{2J^3L}{(JL - 4)^2}, \quad k_{119} = \frac{a^2 + b^2 + \delta}{(ab)^4}
\]

**Proof.** Use the expressions for \( L, J \) in (3). □

**Proposition 13.** For \( N = 3 \), \( k_{802,a} \) is given by:

\[
k_{802,a} = \frac{a^2 + b^2 + \delta}{ab^2} = \frac{J\sqrt{2}\sqrt{JL + \sqrt{9 - 2JL}}}{JL - 4}
\]

Remark 2. For \( N = 3 \), \( k_{803} \) was given in Proposition 4.

**Proposition 14.** For \( N = 3 \), \( k_{804} \) is given by:

\[
k_{804} = \frac{\delta(a^2 + c^2 - \delta)}{a^2c^2}
\]

**Proposition 15.** For \( N = 3 \), \( k_{807} \) is given by:

\[
k_{807} = \frac{\rho^8}{S_0a^6b^4} \left[ (a^4 + 2a^2b^2 + 4b^4)\delta + a^6 + (3/2)a^4b^2 + 4b^6 \right]
\]

\( r \) is the radius of the inversion circle, included above for unit consistency. By default \( r = 1 \).
4.2. Invariants for N=4.

Proposition 16. For $N = 4$, $k_{102} = 0$.

Proof. Non-self-intersecting 4-periodics are parallelograms [7] whose outer polygon is a rectangle inscribed in Monge’s Orthoptic Circle [18]. This finishes the proof. □

Proposition 17. For $N = 4$, $k_{104} = -4$.

Proof. As in Proposition 16, outer polygon is a rectangle. □

Proposition 18. For $N = 4$, $k_{106} = 8a^2b^2$.

Proposition 19. For $N = 4$, $k_{110}$ is given by $\frac{2a^4b^4}{(a^2+b^2)^4}$

Let $\kappa = (ab)^{-2/3}$ is the affine curvature of the ellipse and $r_m = \sqrt{a^2 + b^2}$ the radius of Monge’s orthoptic circle [21].

Proposition 20. For $N = 4$, $k_{119} = \frac{2(a^2+b^2)}{(ab)^{2}} = 2(\kappa r_m)^2$

Proposition 21. For $N = 4$, $k_{s02,a} = \frac{2(a^2+b^2)}{a^2+b^2}$

Proposition 22. For $N = 4$, $k_{s03} = \frac{4\sqrt{\kappa^2+2\kappa}}{\kappa^2}$

Observation 3. Experimentally, $k_{s04}$ is invariant for all non-intersecting N-periodics, except for $N = 4$.

Proposition 23. For $N = 4$, $k_{s05,a} = 4$

Note: see Section 3 for a treatment of N=4 self-intersected geometry.

4.3. Invariants for N=5. As seen in Appendix B, the vertices of 5-periodics can only be obtained via an implicitly-defined caustic. I.e., we first numerically obtain the caustic semi-axes and then compute a axis-symmetric polygon tangent to it. Note that both simple and self-intersected 5-periodics possess an elliptic confocal caustic; see Figure 8.

Proposition 24. For $N = 5$ simple (resp. self-intersected), $k_{102}$ is given by the largest negative (resp. positive) real root of the following sextic:

\[ k_{102} : 1024c^{20}x^6 + 2048(a^4 + a^3b - ab^3 + b^4)(a^4 - a^2b + ab^2 + b^4)c^{12}x^5 \]
\[ + 256(4a^{12} - a^{10}b^2 + 32a^8b^4 - 22a^6b^6 + 32a^4b^8 - a^2b^{10} + 4b^{12})c^8x^4 \]
\[ - 64a^2b^2(4a^{12} - 7a^{10}b^2 + 38c^8b^4 - 126a^6b^6 + 38a^4b^8 - 27a^2b^{10} + 4b^{12})c^4x^3 \]
\[ - 16a^8b^8(7a^4 - 96c^4b^2 + 114a^4b^4 - 96a^2b^6 + 7b^8)x^2 \]
\[ - 8a^8b^8(7a^4 + 30a^2b^2 + 7b^4)x - a^{10}b^{10} = 0 \]

Proposition 25. For $N = 5$ simple (resp. self-intersected), $k_{103}$ is given by the smallest (resp. largest) real root greater than 1 of the following sextic:

\[ k_{103} : a^6b^6x^6 - 2b^2a^2(4a^8 - a^6b^2 - a^2b^6 + 4b^8)x^5 \]
\[ - b^2a^2(4a^8 + 19a^6b^2 - 62a^4b^4 + 19a^2b^6 + 4b^8)x^4 \]
\[ + 12b^2a^2(a^4 + b^4)c^4x^3 + (4a^8 + 19a^6b^2 + 66a^4b^4 + 19a^2b^6 + 4b^8)c^4x^2 \]
\[ + (2a^8 + 12a^6b^2 + 36a^4b^4 + 12a^2b^6 + 2b^8)c^4x - c^{12} \]
**Proposition 26.** For $N = 5$ simple (resp. self-intersected), $k_{104}$ is given by the only negative (resp. smallest largest) real root of the following sextic:

$$c^{12}x^6 - 2(a^4 + 10a^2b^2 + b^4)c^8x^5 - (37a^4 - 6a^2b^2 + 37b^4)c^8x^4$$
$$+ 4(5a^8 + 92a^6b^2 + 62a^4b^4 + 92a^2b^6 + 5b^8)c^4x^3$$
$$+ (423a^{12} - 354a^{10}b^2 + 2713a^8b^4 - 4796a^6b^6 + 2713a^4b^8 - 354a^2b^{10} + 423b^{12})x^2$$
$$+ (270a^{12} + 740a^{10}b^2 - 3630a^8b^4 + 7160a^6b^6 - 3630a^4b^8 + 740a^2b^{10} + 270b^{12})x$$
$$- 675a^{12} - 850a^{10}b^2 + 1075a^8b^4 - 3900a^6b^6 + 1075a^4b^8 - 850a^2b^{10} - 675b^{12} = 0$$

**Proposition 27.** For $N = 5$ simple (resp. self-intersected), $k_{105}$ is given by the largest positive real root (resp. the symmetric value of the largest negative root) of the following sextic:

$$2^{10}c^{20}x^6 + 2^{10}(2a^{12} + a^8b^4 + 26a^8b^4 + 70a^6b^6 + 26a^2b^{10} + 2b^{12})c^8x^5$$
$$+ 2^8(4a^{12} + 30a^{10}b^2 + 71a^8b^4 + 350a^6b^6 + 71a^4b^8 + 30a^2b^{10} + 4b^{12})c^4x^3$$
$$+ 2^6a^2b^6(4a^{12} + 9a^{10}b^2 - 318a^8b^4 - 126a^6b^6 + 318a^4b^8 + 9a^2b^{10} + 4b^{12})c^4x^3$$
$$- 2^6a^2b^6(8a^{16} - 53a^{14}b^2 + 253a^{12}b^4 - 1041a^{10}b^6 + 1650a^8b^8 - 1041a^6b^{10} + 253a^4b^{12} - 53a^2b^{14} + 8b^{16})x^2$$
$$- 2^4a^2b^6(16a^{16} - 12a^{14}b^2 + 5a^{12}b^4 + a^{10}b^6 + 2a^8b^8 + a^6b^{10} + 5a^4b^{12} - 12a^2b^{14} + 16b^{16})x$$
$$- a^{10}b^{10} = 0$$

**4.4. Invariants for $N=6$.** Referring to Figure 2 (right):

**Proposition 28.** For $N = 6$, $k_{102} = a^2b^2/(4a + b)^4 = (JL - 4)^2/64$.

**Proposition 29.** For $N = 6$, $k_{104} = k_{101} = JL - 6$.

**Proposition 30.** For $N = 6$, $k_{106}$ is given by:
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Figure 9. Type I self-intersected 6-periodic (blue) and its doubled-up configuration (dashed red), both tangent to a hyperbolic confocal caustic (brown). Its asymptotes (dashed black) pass through the center of the EB. Also shown are the outer (green) and inner (dark red) polygons. Video

\[ k_{106} = \frac{4b^2(2a+b)a^2(a+2b)}{(a+b)^2} = -\frac{(JL-12)(JL-4)^2}{16J^4} \]

Proposition 31. For \( N = 6 \), \( k_{110} \) is given by:

\[ k_{110} = \frac{4a^3b^3(2a+b)^2(a+2b)^2}{(a+b)^6} = -\frac{(JL-12)^2(JL-4)^3}{256J^4} \]

Proposition 32. For \( N = 6 \), \( k_{119} = \frac{2^{5}J^3L^3}{(JL-4)^4} \)

Proposition 33. For \( N = 6 \), \( k_{802,a} = \frac{2(a^2+ab+b^2)}{ab^2} = \frac{4J^2L(1+\sqrt{JL-3})}{(JL-4)^2} \)

Proposition 34. For \( N = 6 \), \( k_{803} = 2\rho^2(2a^2+2ab-b^2)/(ab^2) \).

Proposition 35. For \( N = 6 \), \( k_{806} = 4\rho^{-4}a^3b^4/((2a-b)(a+b)^2) \).

Referring to Figure 9:

4.5. \( N=6 \) self-intersecting. Take a regular hexagon. There are 60 orderings with which to connect its vertices (modulo rotations and chirality); see [21, Pascal Lines].

It turns out only two \( N = 6 \) topologies can produce closed trajectories, both with two self-intersections. These will be referred to as type I and type II, and are depicted in Figures 9, and 10, respectively.

Proposition 36. For \( N = 6 \) type I, \( k_{102} = a^2b^2/(4(a-b)^4) = (JL-4)^2/64 \)

Proposition 37. For \( N = 6 \) type II,

\[ k_{102} = a^2(a-c)^2/(4c^4) = (JL-8)^2(JL-4)^2/1024 \]

Proposition 38. For \( N = 6 \) type I, \( k_{104} = -\frac{2(a^2-4ab+b^2)}{(a-b)^2} = JL-6 = k_{101} \).
Figure 10. Self-intersected 6-periodic (type II) shown both at one of its doubled-up configurations (dashed red) and in general position (blue). Segments are tangent to a hyperbolic confocal caustic (brown) whose asymptotes (dashed black) pass through the center of the EB. Also shown is the outer polygon (green) which in this case is always simple. Video

Note that $k_{104} = k_{101} = JL - 6$ for both $N = 6$ simple and type I. However:

**Proposition 39.** For $N = 6$ type II,

$$k_{104} = \frac{2(a^2 - ac + c^2)(a^2 - ac - c^2)}{c^4} = \frac{(J^2L^2 - 12JL + 16)(J^2L^2 - 12JL + 48)}{128}$$

**Proposition 40.** For $N = 6$ type I, $k_{106} = 4a^2b^2(a - 2b)(2a - b)/(a - b)^2 = -(JL - 12)(JL - 4)^2/(16J^4)$

**Proposition 41.** For $N = 6$ type I,

$$k_{110} = \frac{-4a^3b^3(a - 2b)^2(2a - b)^2}{(a - b)^6} = \frac{(JL - 12)^2(JL - 4)^3}{2^8 J^4}$$

**Proposition 42.** For $N = 6$ type II, both $A$ and $A'$ vanish, and therefore $k_{106} = 0$ and $k_{110} = 0$.

**Observation 4.** Experimentally, $k_{804}$ is invariant for $N = 6$ simple, and type I. However, it is variable for $N = 6$ type II.

4.6. **Invariants for $N=8$.** Referring to Figure 11:

**Proposition 43.** For $N = 8$, $k_{102}$ is given by $(1/2^{12})(JL - 4)^2(JL - 12)^2$.

Referring to Figure 11:

**Proposition 44.** For $N = 8$, $k_{104} = 0$.

**Proof.** Using the CAS, we checked that $k_{104}$ vanishes for an 8-periodic in the “horizontal” position, i.e., $P_1 = (a, 0)$. Since $k_{104}$ is invariant [2], this completes the proof. □
4.7. **N=8 Self-Intersected.** There are 3 types of self-intersected 8-periodics [5], here called type I, II, and III. These correspond to trajectories with turning numbers of 0, 2, and 3, respectively. These are depicted in Figures 12, 13, and 14.

**Observation 5.** The signed area of $N = 8$ type I is zero.

Referring to Figure 14, the following is related to the Poncelet Grid [14] and the Hexagramma Mysticum [3]:

**Observation 6.** The outer polygon to $N=8$ type III is inscribed in an ellipse.

5. **Videos**

Animations illustrating some of the above phenomena are listed on Table 2.
Figure 13. Four positions of a type II self-intersecting 8-periodic (blue) in an $a/b = 1.2$ elliptic billiard, at four different locations of a starting vertex (red). In general position, these have turning number 2. Also shown is confocal hyperbolic caustic (brown). Video

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APPENDIX A. BILLIARD RECAP

Joachimsthal’s Integral expresses that every trajectory segment is tangent to a confocal caustic [20]. Equivalently, a positive quantity $J$ remains invariant at every bounce point $P_i = (x_i, y_i)$:

$$J = \frac{1}{2} \nabla f_i \cdot \hat{v} = \frac{1}{2} |\nabla f_i| \cos \alpha$$

where $\hat{v}$ is the unit incoming (or outgoing) velocity vector, and:

$$\nabla f_i = 2 \left( \frac{x_i}{a^2}, \frac{y_i}{b^2} \right).$$

Hellmuth Stachel contributed [19] an elegant expression for Joahmsthal’s constant $J$ in terms of EB semiaxes $a, b$ and the major semiaxes $a''$ of the caustic:

$$J = \frac{\sqrt{a^2 - a''^2}}{ab}$$
Let $\kappa_i$ denote the curvature of the EB at $P_i$ given by [21, Ellipse]:

$$\kappa = \frac{1}{a^2b^2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^{-3/2}$$

The signed area of a polygon is given by the following sum of cross-products [15]:

$$A = \frac{1}{2} \sum_{i=1}^{N} (P_{i+1} - P_i) \times (P_i - P_{i+1})$$

Let $d_{j,i}$ be the distance $|P_i - f_j|$. The inversion $P^\dagger_{j,i}$ of vertex $P_i$ with respect to a circle of radius $\rho$ centered on $f_j$ is given by:

$$P^\dagger_{j,i} = f_j + \left( \frac{\rho}{d_{j,i}} \right)^2 (P_i - f_j)$$

The following closed-form expression for $k_{119}$ for all $N$ was contributed by H. Stachel [19]:

$$\sum_{i=1}^{N} \kappa_i^{2/3} = L/[2J(ab)^{4/3}]$$

Figure 14. Self-intersecting 8-periodic of type III (blue) and its doubled-up configuration (dashed red) in an $a/b = 1.1$ billiard ellipse (shown rotated by 90° to save space). The turning number is 3. The confocal caustic is an ellipse (brown). Also shown is the outer polygon (green) whose vertices are inscribed in an axis-aligned, concentric ellipse (dashed green), a result related to [3]. Video
Table 2. Videos illustrating some concepts in the article. The last column provides a YouTube code.

APPENDIX B. VERTICES & CAUSTICS $N=3,4,5,6,8$

B.1. $N=3$ Vertices & Caustic. Let $P_i = (x_i, y_i)/q_i$, $i = 1, 2, 3$, denote the 3-periodic vertices, given by [10]:

$q_1 = 1$

$x_2 = -b^4 \left((a^2 + b^2) k_1 - a^2\right) x_1^3 - 2a^4 b^2 k_2 x_1^2 y_1$
$+ a^4 \left((a^2 - 3b^2) k_1 + b^2\right) x_1 y_1^2 - 2a^6 k_2 y_1^3$

$y_2 = 2b^6 k_2 x_1^3 + b^4 \left((b^2 - 3a^2) k_1 + a^2\right) x_1^2 y_1$
$+ 2a^2 b^4 k_2 x_1 y_1^2 - a^4 \left((a^2 + b^2) k_1 - b^2\right) y_1^3$

$q_2 = b^4 \left(a^2 - c^2 k_1\right) x_1^2 + a^4 \left(b^2 + c^2 k_1\right) y_1^2 - 2a^2 b^2 c^2 k_2 x_1 y_1$

$x_3 = b^4 \left(a^2 - (b^2 + a^2)\right) k_1 x_1^3 + 2a^4 b^2 k_2 x_1^2 y_1$
$+ a^4 \left(k_1 (a^2 - 3b^2) + b^2\right) x_1 y_1^2 + 2a^6 k_2 y_1^3$

$y_3 = -2b^6 k_2 x_1^3 + b^4 \left(a^2 + (b^2 - 3a^2) k_1\right) x_1^2 y_1$
$- 2a^2 b^4 k_2 x_1 y_1^2 + a^4 \left(b^2 - (b^2 + a^2) k_1\right) y_1^3$,

$q_3 = b^4 \left(a^2 - c^2 k_1\right) x_1^2 + a^4 \left(b^2 + c^2 k_1\right) y_1^2 + 2a^2 b^2 c^2 k_2 x_1 y_1$.

where:
\[ k_1 = \frac{d_1^2 \delta_1^2}{d_2^2} = \cos^2 \alpha, \]

\[ k_2 = \frac{\delta_1 d_1^2}{d_2^2} \sqrt{d_2 - d_1^2 \delta_1^2} = \sin \alpha \cos \alpha \]

\[ c^2 = a^2 - b^2, \quad d_1 = (a b/c)^2, \quad d_2 = b^4 x_1^2 + a^4 y_1^2 \]

\[ \delta = \sqrt{a^4 + b^4 - a^2 b^2}, \quad \delta_1 = \sqrt{2 \delta - a^2 - b^2} \]

where \( \alpha \), though not used here, is the angle of segment \( P_1 P_2 \) (and \( P_1 P_3 \)) with respect to the normal at \( P_1 \).

The caustic is the ellipse:

\[ \frac{x^2}{a''^2} + \frac{y^2}{b''^2} - 1 = 0, \quad a'' = \frac{a(\delta - b^2)}{a^2 - b^2}, \quad b'' = \frac{b(a^2 - \delta)}{a^2 - b^2} \]

B.2. N=4 Vertices & Caustic.

B.3. Simple. The vertices of the 4-periodic orbit are given by:

\[ P_1 = (x_1, y_1), \quad P_2 = \left( -\frac{a^4 y_1}{\sqrt{b^4 x_1^2 + a^2 y_1^2}}, \frac{b^4 x_1}{\sqrt{b^4 x_1^2 + a^2 y_1^2}} \right) \]

\[ P_3 = -P_1, \quad P_4 = -P_2 \]

The caustic is the ellipse:

\[ \frac{x^2}{a''^2} + \frac{y^2}{b''^2} - 1 = 0, \quad a'' = \frac{a^2}{\sqrt{a^2 + b^2}}, \quad b'' = \frac{b^2}{\sqrt{a^2 + b^2}} \]

The area and its bounds are given by:

\[ A = \frac{2(b^4 x_1^2 + a^4 y_1^2)}{\sqrt{b^4 x_1^2 + a^2 y_1^2}}, \quad \frac{4a^2 b^2}{a^2 + b^2} \leq A \leq 2ab \]

The minimum (resp. maximum) area is achieved when the orbit is a rectangle with \( P_1 = (x_1, b^2 x_1/a^2) \) (resp. rhombus with \( P_1 = (a, 0) \)).

The perimeter is given by:

\[ L = 4 \sqrt{a^2 + b^2} \]

The exit angle \( \alpha \) required to close the trajectory from a departing position \( (x_1, y_1) \) on the billiard boundary is given by:

\[ \cos \alpha = \frac{a^2 b}{\sqrt{a^2 + b^2} \sqrt{a^4 - c^2 x_1^2}} \]
B.3.1. Self-intersected. When \(a/b > \sqrt{2}\), the vertices of the 4-periodic self-intersecting orbit are given by:

\[P_1 = \left[au, b\sqrt{1-u^2}\right], \quad P_2 = \left[-a\sqrt{a^2 - 2b^2} - c^2u^2, -\frac{b^3}{c^2}\right],\]
\[P_3 = \left[-au, b\sqrt{1-u^2}\right], \quad P_4 = \left[a\sqrt{a^2 - 2b^2} - c^2u^2, -\frac{b^3}{c^2}\right],\]

where \(|u| \leq \frac{a}{c^2}\sqrt{a^2 - 2b^2}\).

The confocal hyperbolic caustic is given by:

\[x^2 - \frac{y^2}{b'^2} = 1, \quad a'' = \frac{a\sqrt{a^2 - 2b^2}}{c}, \quad b'' = \frac{b^2}{c}\]

The four intersections of the caustic and the EB are given by:

\[\left[\pm \frac{a^2\sqrt{a^2 - 2b^2}}{c^2}, \pm \frac{b^3}{c^2}\right]\]

The exit angle \(\alpha\) required to close the trajectory from a departing position \((x_1, y_1)\) on the billiard boundary is given by:

\[\cos \alpha = \frac{a^2b}{c\sqrt{a^2 - c^2x_1^2}}\]

The perimeter of the orbit is \(L = 4a^2/c\).

B.4. \textbf{N=5 Vertices \\& Caustic.} Let \(a > b\) be the semi-axes of the elliptic billiard. Consider a 5-periodic with vertices \(P_i, i = 1, ..., 5\) where \(P_1\) is at \((a, 0)\), i.e., the orbit is “horizontal”.

**Proposition 45.** The major semiaxis length \(a''\) of the caustic for \(N = 5\) non-intersecting (resp. intersecting, i.e., pentagram) is given by the root of the largest (resp. smallest) real root \(x \in (0, a)\) of the following bi-sextic polynomial:

\[P_5(x) = c^{12}x^{12} - 2c^4a^2 (3a^8 - 9a^6b^2 + 31a^4b^4 + a^2b^6 + 6b^8) x^{10} + c^4a^4 (15a^8 - 30a^6b^2 + 191a^4b^4 + 16a^2b^6 + 16b^8) x^8 - 4c^4a^{10} (5a^8 - 5a^6b^2 + 66b^4) x^6 + a^{12} (15a^8 - 30a^6b^2 + 191a^4b^4 - 368a^2b^6 + 208b^8) x^4 - 2a^{14} (3a^8 - 3a^6b^2 + 22a^4b^4 - 48a^2b^6 + 32b^8) x^2 + a^{24}\]

**Proof.** The polynomial \(P_5\) is exactly the Cayley condition for the existence of 5-periodic orbits, see [8]. For \(c = 0\) the roots are \(a'_5 = (\sqrt{5} - 1)a/4\) and \(a'_5 = (\sqrt{5} + 1)a/4\) and corresponds to the regular case. For \(b = 0\) the roots are coincident in given by \(x = a\). By analytic continuation, for \(c \in (0, a)\), the two roots are in the interval \((0, a)\). \(\square\)
For $N = 5$ non-intersecting, the abcissae of vertices $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$ are given by the smallest positive solution (resp. unique negative) of the following equations:

$$x_2 : \ c^6 x_2^6 - 2a \left(2a^2 - b^2\right) c^4 x_2^5 + a^2 \left(5a^2 + 4b^2\right) c^4 x_2^4 - 8a^5b^2c^2x_2^3 - a^8 \left(5a^2 - 9b^2\right) x_2^3 + 2a^9 \left(2a^2 - b^2\right) x_2 - a^{12} = 0$$

$$x_3 : \ c^6 x_3^6 - 2ab^2c^2 \left(3a^2 + b^2\right) x_3^5 - a^2c^2 \left(3a^4 - 3a^2b^2 + 4b^4\right) x_3^4 + 12a^5b^2c^2x_3^3 + a^6 \left(3a^4 - 3a^2b^2 + 4b^4\right) x_3^3 - 2a^7b^2 \left(3a^2 - 4b^2\right) x_3 - a^{12} = 0$$

The Joachimstall invariant $J$ of the simple orbit is given by the small positive root of

$$4096c^{12}J^{12} + 2048(3a^2 + b^2)(a^2 + 3b^2)(a^2 + b^2)c^4J^{10} - 256(29a^4 + 54a^2b^2 + 29b^4)c^4J^8 + 2304(a^2 + b^2)c^4J^6 - 16(3a^2 - 4ab - 3b^2)(3a^2 + 4ab - 3b^2)J^4 - 40(a^2 + b^2)J^2 + 5 = 0$$

The perimeter of the simple orbit is given by

$$L = \frac{p}{q}$$

$$p = (1024 \ (a^2 + b^2) c^4 b^2 J^7 - 256 c^4 b^2 J^5 - 64 \ (a^2 + b^2) b^2 J^3 + 16 J b^2) \sqrt{1 - 4a^2 J^2} - 1024 c^2 (5a^4 + 2a^2b^2 + b^4) b^2 J^7 + 256 c^2 (3a^2 + b^2) b^2 J^5 + 64 c^2 b^2 J^3 + 16 J b^2 q = 256 c^2 J^8 - 256 c^2 (a^2 + b^2) J^6 + 32 c^2 (3a^2 + 5b^2) J^4 - 16 c^2 J^2 + 1$$

B.5. N=6 Vertices & Caustic.

B.5.1. Simple. Vertices $P_i, i = 2, ..., 6$ with $P_1 = (a, 0)$ are given by:

$$P_4 = [-a, 0] \quad P_2 = [k_x, k_y], \quad P_5 = -P_2 \quad P_3 = [-k_x, k_y], \quad P_6 = -P_3 \quad k_x = \frac{a^2}{a + b}, \quad k_y = \frac{b\sqrt{b(2a + b)}}{a + b}$$

The confocal, elliptic caustic is given by:

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \quad a'' = \frac{a\sqrt{a(a + 2b)}}{a + b}, \quad b'' = \frac{b\sqrt{b(2a + b)}}{a + b}$$

The perimeter is given by:

$$L = \frac{4(a^2 + ab + b^2)}{a + b}$$
B.5.2. Self-Intersected (type I). This orbit only exists for \( a > 2b \). Vertices \( P_i, i = 2, \ldots, 6 \) with \( P_1 = (0, b) \) are given by:

\[
P_4 = [0, -b] \\
P_2 = [k_x, k_y], \quad P_5 = -P_2 \\
P_3 = [k_x, -k_y], \quad P_6 = -P_3 \\
k_x = \frac{a \sqrt{a(a - 2b)}}{b - a}, \quad k_y = \frac{b^2}{b - a}
\]

The confocal, hyperbolic caustic is given by:

\[
\frac{x^2}{a''^2} - \frac{y^2}{b''^2} = 1, \quad a'' = \frac{a^{3/2} \sqrt{a - 2b}}{a - b}, \quad b'' = \frac{b^{3/2} \sqrt{2a - b}}{a - b}
\]

The 4 intersections \((x_{int}, y_{int})\) of the caustic with the EB are given by:

\[
[x_{int}, y_{int}] = \left[ \pm \sqrt{\frac{a^3(a - 2b)}{(a - b)^3(a + b)}}, \pm \sqrt{\frac{b^3(b - 2a)}{(b - a)^3(a + b)}} \right]
\]

The perimeter is given by:

\[
L = 4(a^2 - ab + b^2)
\]

B.5.3. Self-Intersected (type II). This orbit only exists for \( a > \frac{2b\sqrt{3}}{3} \). Vertices \( P_i, i = 2, \ldots, 6 \) with \( P_1 = (0, b) \) are given by:

\[
P_4 = [0, -b] \\
P_2 = [k_x, k_y], \quad P_3 = -P_2 \\
P_5 = [k_x, -k_y], \quad P_6 = -P_3 \\
k_x = -\frac{a^{3/2} \sqrt{2c - a}}{c}, \quad k_y = \frac{(c - a)b}{c}
\]

The confocal hyperbolic caustic is given by:

\[
\frac{x^2}{a'''^2} - \frac{y^2}{b'''^2} = 1, \quad a'''^2 = \frac{a^3 (3ac - 2b^2)}{c(3a^2 + b^2)}, \quad b'''^2 = \frac{b^3 (a^2 + c)}{c(3a^2 + b^2)}
\]

The 4 intersections \((x_{int}, y_{int})\) of the caustic with the EB are given by:

\[
[x_{int}, y_{int}] = \left[ \pm \sqrt{\frac{a (3ac - 2b^2)}{c(3a^2 + b^2)}}, \pm \sqrt{\frac{b^2 (3a^2 (a - c) + c^3)}{c(3a^2 + b^2)}} \right]
\]

The perimeter is given by:

\[
L = 4(a + c)\sqrt{2a/c - 1}
\]
B.6. **N=7 Caustic.** Referring to Figure 4, there are three types of 7-periodics: (i) non-intersecting, (ii) self-intersecting type I, i.e., with turning number 2, (iii) self-intersecting type II.

**Proposition 46.** The caustic semi-azis for non-intersecting 7-periodics (resp. self-intersecting type I, and type II self-intersecting) are given by the smallest (resp. second and third smallest) root of the following degree-12 polynomial:

\[
c^{12} x_1^{12} - 4 (a^2 + b^2) c^6 a (3 a^2 + b^2) b^2 x_1^{11}
- 2 c^8 a^2 (3 a^6 - 6 a^4 b^2 + 13 a^2 b^4 - 2 b^6) x_1^{10} + (60 a^4 + 60 b^2 a^2 + 8 b^4) c^6 a^3 x_1^9
+ a^6 c^2 (15 a^8 - 45 a^6 b^2 + 125 a^4 b^4 - 143 a^2 b^6 + 112 b^8) x_1^8 - 8 a^7 b^2 c^2 (15 a^6 - 20 a^4 b^2 - 7 a^2 b^4 + 8 b^6) x_1^7
- 4 a^8 c^2 (5 a^8 - 10 a^6 b^2 + 35 a^4 b^4 - 30 a^2 b^6 + 36 b^8) x_1^6 + 8 a^9 b^2 c^2 (15 a^6 - 25 a^4 b^2 - 2 a^2 b^4 + 4 b^6) x_1^5
+ a^{10} c^2 (15 a^8 - 15 a^6 b^2 + 80 a^4 b^4 - 32 a^2 b^6 + 64 b^8) x_1^4 - 4 a^{11} b^2 (15 a^4 - 45 b^2 a^2 + 32 b^4) x_1^3
- 2 a^{12} (3 a^6 - 3 a^4 b^2 + 10 a^2 b^4 - 8 b^6) x_1^2 + 4 a^{13} b^2 (3 a^2 - 4 b^2) (a^2 - 2 b^2) x_1 + a^{24} = 0
\]

It can shown the first two smallest (resp. third smallest) roots of the above polynomial are negative (resp. positive), and all have absolute values within (0, a).

For \(a = b\) the polynomial equation above is given by \(a^3 + 4 a^2 x_1 - 4 a x_1 - 2 x_1^3 = 0\) with roots \(-0.90096888680a\), \(-0.2225209340a\), \(0.6234898025a\).

B.7. **N=8 Vertices & Caustic.**

B.7.1. **Simple.** Vertices \(P_i, i = 1, \ldots, 8\) with \(P_1 = [a, 0]\) are given by:

\[
\begin{align*}
P_1 &= [a, 0], & P_5 = [-a, 0], & P_3 = [0, b], & P_7 = [0, -b] \\
P_{2,4,6,8} &= [\pm ax, \pm b \sqrt{1 - x^2}], \text{where } x \text{ is a positive root of:} \\
c^4 x^4 - 2 a^2 c^2 x^3 + 2 a^2 b^2 x^2 + 2 a^2 c^2 x - a^4 &= 0
\end{align*}
\]

B.7.2. **Self-Intersected (type I and type II).** These have hyperbolic confocal caustics; see Figures 12, 13. Let \(P_1 = (x_1, y_1)\) be at the intersection of the hyperbolic caustic with the billiard for each case. For type I (resp. type II) \(x_1\) is given by the smallest (resp. largest) positive root \(x_1 \in (0, a)\) of the following degree-8 polynomial:

\[
x_1 : c^{16} x_1^8 - 4 a^4 c^8 (3 a^6 - 4 a^4 b^2 + a^2 b^4 - 2 b^6) x_1^6 + 2 a^8 c^6 (3 a^6 - 15 a^4 b^2 - 4 b^6) x_1^4
- 4 a^{16} c^4 (a^2 - 6 b^2) x_1^2 + a^{20} (a^4 - 8 a^2 b^2 + 8 b^4) = 0
\]

B.7.3. **Self-Intersected (type III).** Let \(P_1 = (x_1, y_1)\) and \(P_2 = (x_1, -y_1)\) be two consecutive vertices in the doubled-up type III 8-periodic (dashed red in Figure 14) connected by a vertical line (the figure is rotated, therefore this line will appear horizontal). \(x_1^3\) is given by the smallest positive root of the following quartic polynomial on \(\omega\):

\[
x_1 : (a^4 + 6 a^2 b^2 + b^4) c^4 x_1^8 - 4 a^4 (a^2 + 5 b^2) c^4 x_1^6 + 2 a^6 (3 a^6 + 6 a^4 b^2 - 21 a^2 b^4 + 16 b^6) x_1^4
- 4 a^8 (a^6 + 4 a^2 b^4 - 4 a^2 b^4 + 4 b^6) x_1^2 + a^{16} = 0
\]

\[
x_1 : \alpha^{16} + (\alpha^2 - 1)^2 (\alpha^4 + 6 a^2 + 1) \omega^4 - 4 (\alpha^2 - 1)^2 (\alpha^2 + 5) \alpha^4 \omega^3 +
2 (3 (a^4 + 2 a^2 - 7) a^2 + 16) \alpha^6 \omega^2 - 4 (a^6 + a^2 + 4 a^4) \alpha^8 \omega = 0
\]
where $\alpha = a/b$.

**Appendix C. Table of Symbols**

| symbol     | meaning                                      |
|------------|----------------------------------------------|
| $O, N$     | center of billiard and vertex count          |
| $L, J$     | inv. perimeter and Joachimsthal’s constant   |
| $a, b$     | billiard major, minor semi-axes              |
| $a''$, $b''$ | caustic major, minor semi-axes             |
| $f_1, f_2$ | foci                                        |
| $P_i$, $P_i'$, $P_i''$ | $N$-periodic, outer, inner polygon vertices |
| $d_{j,i}$  | distance $|P_i - f_j|$                      |
| $\kappa_i$ | ellipse curvature at $P_i$                  |
| $\theta_i$, $\theta_i'$ | $N$-periodic, outer polygon angles |
| $A, A'$, $A''$ | $N$-periodic, outer, inner areas |
| $\rho$     | radius of the inversion circle              |
| $P_{j,i}$  | vertices of the inversive polygon wrt $f_j$  |
| $L_{j,i}$, $A_{j,i}$ | perimeter, area of inversive polygon wrt $f_j$ |
| $\theta_{j,i}$ | ith angle of inversive polygon wrt $f_j$      |

Table 3. Symbols used in the invariants. Note $i \in [1, N]$ and $j = 1, 2$.

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