Critical Current of the Spin-Triplet Superconducting Phase in Sr$_2$RuO$_4$

Hae-Young Kee$^1$, Yong Baek Kim$^1$, and K. Maki$^2$

$^1$Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada
$^2$Department of Physics, University of Southern California, Los Angeles, CA 90089

(Dated: March 22, 2022)

There have been two different proposals for the spin-triplet order parameter of the superconducting phase in Sr$_2$RuO$_4$; an $f$-wave order parameter and the multigap model where some of the bands have the line node. In an effort to propose an experiment that can distinguish two cases, we study the behavior of the supercurrent and compute the critical current for these order parameters when the sample is a thin film with the thickness $d \ll \xi$ where $\xi$ is the coherence length. It is found that the supercurrent behaves very differently in two models. This will serve as a sharp test for the identification of the correct order parameter.

PACS numbers: 74.25.Sv, 74.20.Rp, 74.25.Fy

Introduction: The order parameter of the superconducting phase in Sr$_2$RuO$_4$ has been a subject of intensive research since its discovery in 1994 [1, 2]. In particular, the field has been stimulated by the prospect of identifying the simplest electronic version of spin-triplet superconductor. The early proposal made by Rice and Sigrist suggested the following spin-triplet order parameter with $p$-wave symmetry [3, 4].

$$\hat{\Delta}(k) = \Delta \hat{d} (k_x \pm ik_y),$$

(1)

where $\Delta$ is the magnitude of the order parameter and $\hat{d}$ is a unit vector perpendicular to the spin of the condensed pair [2]. This order parameter breaks time-reversal symmetry and has a full gap on the Fermi surface. Indeed the flat $^{17}$O Knight shift across $T_c$ [4], for the magnetic field along the $ab$ plane, and the observation of spontaneous magnetic moment in $\mu$SR [5] are consistent with the spin-triplet order parameter and broken time-reversal symmetry, respectively. This encouraged theoretical investigations of the effects of the characteristic collective modes (the spin waves and clapping mode) and the topological defects with the order parameter of Eq.1 [3, 4, 6, 7, 8, 9, 10].

The experiments performed later on cleaner samples, however, reveal the existence of the nodal structure in the order parameter [21, 22]. In an effort to resolve this issue, the following order parameter with the horizontal node was proposed as a strong candidate [17].

$$\hat{\Delta}(k) = \Delta \hat{d} (k_x \pm ik_y) \cos(ck_z),$$

(2)

where $c$ is the lattice constant in the $c$-axis. It was shown that the magneto-thermal conductivity would provide useful information about the validity of this order parameter [23, 24]. Such experiments were indeed carried out [25, 26], and the results are consistent with Eq.2, even though the data cannot exclude the possible presence of small amount of $p$-wave mixture.

An alternative model was suggested by Zhitomirsky and Rice [27, 28]. Their multigap model utilizes the fact that there exist three bands; $\alpha$, $\beta$, and $\gamma$ [4]. Here the dominant $p$-wave order parameter with Eq.1 resides on the active $\gamma$ band while the proximity effect leads to the following $f$-wave-like (it will be called $f'$-wave hereafter) order parameter with the line nodes in the $\alpha$ and $\beta$ bands [27].

$$\hat{\Delta}(k) = \Delta \hat{d} (k_x \pm ik_y) \cos(ck_z/2).$$

(3)

The observation of a double transition in a recent specific heat measurement near $H_{c2}$ was interpreted in terms of this multigap model [29]. On the other hand, the specific heat and the magnetic penetration depth for low temperatures ($T < T_c$) and low field ($H < H_{c2}$) appear to be consistent with the $f$-wave order parameter of Eq.2.

Thus it is not clear at the moment which order parameter is the correct description of the superconducting phase; this is the fundamental issue for Sr$_2$RuO$_4$. Previously it was suggested that the angle dependence of the magneto-thermal conductivity can be used to distinguish different order parameters [30]. The Raman spectra can be also used to detect different contributions from the clapping mode in the $f$-wave and the multigap models [31, 32]. These proposals, however, rely on the quantitative, albeit detectable, difference in two models. Thus it is still desirable to have a sharp test that leads to the qualitatively different outcome in two cases.

In this paper, we provide such a proposal; the study of the critical current in two models for the spin-triplet superconductor. In particular, we investigated the behavior of the supercurrent for the $f$-wave and $f'$-wave order parameters. It is found that the supercurrent in the $ab$-plane has the same form in both cases and has significant...
contributions from the quasiparticles. On the other hand, if the current flows along the c-axis, the supercurrent in the $f'$-wave is relatively immune to the quasiparticles at small current while that of the $f$-wave still acquires a large contribution from the quasiparticles. Notice that the supercurrent in the multigap model would be dominated by the $p$-wave component on the $\gamma$-band and its behavior would be the same as the case of $s$-wave superconductor where the quasiparticle contribution is almost absent at small current $[33]$. Combining all these, it can be seen that the supercurrent in the $f$-wave case is much more affected by the quasiparticles. Thus the study of the supercurrent provides a clear mean to distinguish two different proposals for the order parameter of the superconducting phase in $\text{Sr}_2\text{RuO}_4$.

More specifically, we will consider a thin film with the sample thickness $d \ll \xi$, where $\xi$ is the coherent length. Under this condition, the magnitude of the superconducting gap, $\Delta$, and the supercurrent will be uniform across the system $[33]$. When a uniform current flows, the Cooper-pair acquires a center of mass momentum of $q_s$. If there were no contribution from the quasiparticles, the supercurrent would be simply proportional to $q_s$. The quasiparticles, however, change this behavior via the shifted quasiparticle dispersion, $E_k$, in the presence of the superflow $[33]$

$$E_k = E_0^k + v_k \cdot q_s, \quad (4)$$

where $E_0^k = \sqrt{\xi_k^2 + \Delta^2}$ and $v_k = \partial \xi_k / \partial k$ is the quasiparticle velocity. The total current, $j_s$, as a function of $s = v_F q_s$, rises linearly at small $s$ and has a maximum at some value of $s$, then drops as $s$ is further increased. When $dj_s / ds < 0$, the system is unstable; thus the critical current is determined by the value of $s$ where $dj_s / ds = 0$ $[33]$. The detailed behavior of the supercurrent and the value of the critical current in each case are discussed below.

**f-wave order parameter:** We will consider two different cases; the current in the $ab$-plane and along the $c$-axis.

**a. the current in the $ab$-plane**

The gap function for the order parameter in the presence of a uniform current can be written as

$$\Delta(k) = -T \sum_{\omega_n} \sum_p V(k, p) \text{Tr} [\rho_1 \sigma_1 G(i\omega_n, p)] \quad (5)$$

where $\Delta(k) = \Delta(f(k))$, $V(k, p) = V(f(k))f(p)^*$, and $f(k)$ represents the momentum dependence of the order parameter. Here the single particle Green’s function $G(i\omega_n, k)$ is given by

$$G^{-1}(i\omega_n, k) = i\omega_n - v_k \cdot q_s + \xi_k \rho_3 + \Delta(k) \rho_1 \sigma_1, \quad (6)$$

where $\xi_k = (k_x^2 + k_y^2)/2m - t \cos(ck_z) - \mu$ is the single particle dispersion.

The uniform current reduces the amplitude of the order parameter and the current dependence of the amplitude at $T = 0$ can be obtained from the following equation derived from Eq.4 (the lattice constant $c = 1$ for simplicity).

$$\ln \left[ \frac{\Delta(0)}{\Delta(s)} \right] = \frac{8}{\pi^2} \int_0^{\pi} dk_z \int_0^{\pi} d\phi |f|^2 \text{Re} \arccosh \left[ \frac{s \sin \phi}{\Delta(s) |f|} \right]$$

where $s = v_F q_s$, $\phi$ is the angle between the direction of the current $q_s$ and the quasiparticle velocity $v_k$. $\Delta(s)$ and $\Delta(0)$ represent the amplitude of the order parameter in the presence and absence of the current, respectively.

**FIG. 1:** The supercurrent as a function of $s/\Delta(0)$ with the current in the $ab$-plane for both of the $f$-wave and $f'$-wave, (b) along the $c$-axis for the $f$-wave, (c) along the $c$-axis for the $f'$-wave. The unit of the supercurrent is $(en/mv_F)\Delta(0)$ and $(en/m)(v_F/v_s^2)\Delta(0)$ for the current in the plane and along the $c$-axis, respectively.
In the case of the \( f \)-wave order parameter, \( f(k) = e^{\pm i\varphi} \cos(ck_z) \), a straightforward computation at \( T = 0 \) leads to

\[
\ln \left[ \frac{\Delta(0)}{\Delta(s)} \right] = \frac{2}{\pi} \left[ \arcsin y \left( \ln y - \frac{1}{2} \right) - \frac{y}{2} \sqrt{1 - y^2} - \int_0^{\arcsin y} \frac{d\phi}{\sin \phi} \ln(\sin \phi) \right]
\]

for \( y = s/\Delta(s) < 1 \). When \( y > 1 \), there is no solution for the gap equation. Now the contribution from the quasiparticles to the current can be computed from \( ^{33} \)

\[
j_{qp} = cT \sum_{\omega_n} \sum_k \text{Tr}[v_k G(i\omega_n, k)],
\]

Taking into account this, the net supercurrent for the \( f \)-wave order parameter is obtained as

\[
j_s = \frac{en}{m} q_s \left[ 1 - \frac{8}{\pi^2 y} \int_0^{\pi/2} dk_z \int_0^{\pi/2} d\phi \cos \phi \Re \sqrt{(y \cos \phi)^2 - (\cos k_z)^2} \right]
\]

\[
= \frac{en}{m} q_s \left[ 1 - \frac{1}{\pi y} \left( \sqrt{1 - y^2} - \frac{1 - 2y^2}{y} \arcsin y \right) \right],
\]

where \( n = k_F^2/2\pi \) is the density of electrons in the plane. The supercurrent as a function of \( s/\Delta(0) \) can be obtained from Eq\(^8\) and Eq\(^10\) (the result is plotted in Fig. 1(a)). The critical current occurs at \( s = 0.74\Delta(0) \) and the value of the critical current is \( j_c = 0.465(en/mv_F)\Delta(0) \).

**b. the current along the c-axis**

When the current is along the \( c \)-axis, the gap equation for the \( f \)-wave order parameter at \( T = 0 \) is now given by

\[
\ln \left[ \frac{\Delta(0)}{\Delta(s)} \right] = \frac{4}{\pi} \int_0^{\pi/2} dk_z (\cos k_z)^2 \Re \arccosh \left( \frac{s \sin k_z}{\Delta \cos k_z} \right) = \arcsinh y - \frac{y}{\sqrt{1 + y^2}},
\]

where \( s = v_{Fc}q_s \) with \( v_{Fc} = tc \) and \( v_k = (tc) \sin k_z \hat{q}_s \) is used. The net supercurrent is also found as

\[
j_s = \frac{en}{m} \left( \frac{v_{Fc}}{v_F} \right)^2 q_s \left[ 1 - \frac{4}{\pi y} \int_0^{\pi/2} dk_z \sin k_z \Re \sqrt{(y \sin k_z)^2 - (\cos k_z)^2} \right]
\]

\[
= \frac{en}{m} \left( \frac{v_{Fc}}{v_F} \right)^2 q_s \left[ 1 - \frac{y}{\sqrt{1 + y^2}} \right].
\]

Notice that \( v_{Fc} \) and \( v_F \) are the velocities along the \( c \)-direction and in the \( ab \)-plane, respectively. The supercurrent computed from Eq\(^11\) and Eq\(^12\) is plotted in Fig. 1(b); the critical current occurs at \( s = 0.61\Delta(0) \) and the value of the critical current is \( j_c = 0.277(en/mv_F)(v_{Fc}/v_F)^2\Delta(0) \).

**\( f' \)-wave order parameter**: When the current is in the \( ab \)-plane, the gap equation and the expression for the supercurrent turns out to be the same as those of the \( f \)-wave case. Thus the supercurrent in this case is given by Fig. 1(a). On the other hand, when the current is along the \( c \)-axis, both of the gap equation and the supercurrent for the \( f' \)-wave order parameter are quite different from the \( f \)-wave counterpart. The gap equation now has the following form with \( s = v_{Fc}q_s \).

\[
\ln \left[ \frac{\Delta(0)}{\Delta(s)} \right] = \frac{4}{\pi} \int_0^{\pi/2} dk_z \left[ \cos \left( \frac{k_z}{2} \right) \right]^2 \Re \arccosh \left( \frac{s \sin k_z}{\Delta \cos \left( \frac{k_z}{2} \right)} \right)
\]

\[
= \ln(2y) - \frac{1}{2} \left( 1 - \frac{1}{4y^2} \right) \theta \left( y - \frac{1}{2} \right),
\]

where \( \theta(x) = 1 \) for \( x > 0 \) and is zero if \( x < 0 \). The supercurrent is given by

\[
j_s = \frac{en}{m} \left( \frac{v_{Fc}}{v_F} \right)^2 q_s \left[ 1 - \frac{2}{\pi y} \int_0^{\pi} dk_z \sin k_z \Re \sqrt{(y \sin k_z)^2 - (\cos k_z)^2} \right]
\]

\[
= \frac{en}{m} \left( \frac{v_{Fc}}{v_F} \right)^2 q_s \left[ 1 - \left( 1 - \frac{1}{4y^2} \right)^2 \theta \left( y - \frac{1}{2} \right) \right].
\]
The supercurrent as a function of $s/\Delta(0)$ is plotted in Fig. 1(c). Notice that the quasiparticle contribution to the current does not enter for $s/\Delta < 1/2$, so the supercurrent is proportional to $q_s$ in this regime. The critical current is found at $s = 0.56\Delta(0)$ and the value of the critical current is $j_c = 0.531(\pi n/m)(v_F/v_F^2)\Delta(0)$.

**Summary and Conclusion:** We investigated the behavior of the supercurrent and obtained the critical current in the cases of the $f$-wave and $f'$-wave order parameters defined in Eq.2 and Eq.3. In the superconductor with the full gap, the quasiparticle contribution to the supercurrent would enter only when the current exceeds some value. If the order parameter has nodes, however, the quasiparticle contribution may affect the supercurrent even in the small current limit. When the current flows in the $ab$-plane, the supercurrent in the $f$-wave and $f'$-wave cases is indeed affected by quasiparticles even for small current and it has the same form in both cases. On the other hand, if the current is along the $c$-axis, the supercurrent in the $f'$-wave case behaves similarly to the case with the full gap; in contrast, the quasiparticle contribution still enters at small current in the $f$-wave case.

Now the discussion about different models for the superconducting order parameter in Sr$_2$RuO$_4$ is in order.

1) When the current is in the $ab$-plane, the supercurrent in the multigap model would be mainly determined by the dominant $p$-wave component on the $\gamma$-band and its behavior is similar to the case of $s$-wave superconductor; the small contribution from the $f'$-wave component is subdominant so that the supercurrent is not much affected by the quasiparticles at small current. Thus, the critical current in the multigap model would be much bigger than that of the $f$-wave model where the supercurrent acquires a significant quasiparticle contribution.

2) If the current is along the $c$-axis, even the supercurrent from the $f'$-wave component (as well as the $p$-wave component) in the multigap model does not acquire the quasiparticle contribution at small current and as a result the supercurrent is even less affected by the quasiparticles. On the other hand, the supercurrent in the $f$-wave model is still very much affected so that the critical current is again much smaller.

The qualitative difference in the behavior of the supercurrent in two models can be used to discriminate one of the leading candidates for the order parameter of the superconducting phase in Sr$_2$RuO$_4$.

**Acknowledgments:** HYK and YBK thank Aspen Center for Physics for its hospitality during the summer workshop in 2003. This work was supported by the Natural Sciences and Engineering Research Council of Canada, Canadian Institute for Advanced Research, Canada Research Chair Program, and the Alfred P. Sloan Fellowship (HYK and YBK).

[1] Y. Maeno et al, Nature **372**, 532 (1994).

---

[2] For a recent review on the subject, see A. P. MacKenzie and Y. Maeno, Rev. Mod. Phys. **75**, 657 (2003).

[3] T. M. Rice and M. Sigrist, J. Phys. Condens. Matter **7**, L643 (1995).

[4] Y. Maeno, T. M. Rice, and M. Sigrist, Physics Today, **54**, 42 (2001).

[5] The Superfluid Phases of Helium 3, D. Vollhardt and P. Wolffe (Taylor & Francis, New York, 1990).

[6] K. Ishida et al, Nature **396**, 658 (1998), J. A. Duffy et al, Phys. Rev. Lett. **85**, 5412 (2000).

[7] G. M. Luke et al, Nature **394**, 558 (1998).

[8] L. Tewordt, Phys. Rev. Lett. **83**, 1007 (1999).

[9] H.-Y. Kee, Y. B. Kim, and K. Maki, Phys. Rev. B **61**, 3584 (2000).

[10] H.-Y. Kee, Y. B. Kim, and K. Maki, Phys. Rev. B **62**, 5877 (2000); S. Higashitani and K. Nagai, Phys. Rev. B **62**, 3042 (2000).

[11] H.-Y. Kee, Y. B. Kim, and K. Maki, Phys. Rev. B **62**, R9275 (2000).

[12] S. Nishizaki, Y. Maeno and Z. Q. Mao, J. Phys. Soc. Jpn. **69**, 572 (2000).

[13] I. Bonalde, et al, Phys. Rev. Lett. **85**, 4775 (2000).

[14] K. Ishida et al, Phys. Rev. Lett. **84**, 5387 (2000).

[15] C. Lupien et al, Phys. Rev. Lett. **86**, 5986 (2001); This experiment excludes all the models with the vertical nodes.

[16] Y. Hasegawa, et al, J. Phys. Soc. Jpn. **69**, 336 (2000).

[17] H. Won and K. Maki, Europhys. Lett. **52**, 427 (2000).

[18] T. Dahm, K. Maki, and H. Won, cond-mat/0006301

[19] K. Miyake and O. Narikiyo, Phys. Rev. Lett. **83**, 1423 (1999).

[20] M. J. Graf and A. V. Balatsky, Phys. Rev. B **62**, 9697 (2000).

[21] M. Sigrist, et al, Physica C **317**, 134 (1999).

[22] M. Sato and M. Kohmoto, J. Phys. Soc. Jpn. **69**, 3505 (2000).

[23] H. Won and K. Maki, cond-mat/0004105

[24] H. Won and K. Maki, Current Appl. Phys. **1**, 219 (2001).

[25] M. A. Tanatar et al, Phys. Rev. Lett. **86**, 2649 (2001).

[26] K. Izawa, et al, Phys. Rev. Lett. **86**, 2653 (2001).

[27] M. E. Zhitomirsky, and T. M. Rice, Phys. Rev. Lett. **87**, 057001 (2001).

[28] J. F. Annett, G. Litak, B. L. Gyoeffy, and K. I. Wysokin,ski, Phys. Rev. B **66**, 134514 (2002).

[29] K. Deguchi, et al, cond-mat/0210537

[30] H. Won and K. Maki, Physics B **312-313**, 44 (2002).

[31] H.-Y. Kee, K. Maki, and C. H. Chung, Phys. Rev. B **67**, 180504 (2003).

[32] B. Dora et al, Europhys. Lett. **62**, 426 (2003).

[33] See, for example, the article by K. Maki, Gapless Superconductivity, in Superconductivity, edited by R. D. Parks (Marcel Dekker, 1969); and references therein.