STOCHASTICALLY POSITIVE STRUCTURES ON WEYL ALGEBRAS. THE CASE OF QUASI-FREE STATES.

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Abstract. We consider quasi-free stochastically positive ground and thermal states on Weyl algebras in Euclidean time formulation. In particular, we obtain a new derivation of a general form of thermal quasi-free state and give conditions when such state is stochastically positive i.e. when it defines periodic stochastic process with respect to Euclidean time, so called thermal process. Then we show that thermal process completely determines modular structure canonically associated with quasi-free thermal state on Weyl algebra. We discuss a variety of examples connected with free quantum field theories on globally hyperbolic stationary space-times and models of quantum statistical mechanics.

I Introduction

It is well known that the idea of analytic continuation to the imaginary time (Euclidean) variables is very fruitful for study several quantum systems, especially Wightman quantum field theories on a flat space-time. From general properties of Wightman distributions follows the possibility of analytic continuation to the Euclidean region, what gives an alternative, purely Euclidean, description in terms of so called Schwinger functions [1, 2, 3, 4]. In the context of bosonic fields, we obtain commutative objects, which are much easier to analyse [5]. Particularly nice situation occurs in the case of free scalar fields. As was noticed by Nelson [6] (see also [7]), the corresponding Euclidean structure is given by a Gaussian Markov generalized random field. This observation and the general reconstruction theorem [8], gave powerful input for the development of the constructive quantum field theory on a flat space-time ([9, 10] and references therein).

One of the main objectives of the present paper is to develop systematically the rigorous Euclidean formalism in the context of equilibrium states given by the KMS condition. As the interesting application of general results, we consider Euclidean approach to quantum field theories at non-zero temperature on globally hyperbolic stationary space-times ([11, 12]). Some aspects of Euclidean formalism in this context, were discussed in the literature at various levels of mathematical rigour (see e.g. [13, 14, 15, 16]). But to our knowledge, still there is the lack of a systematic and mathematically rigorous approach to these problems. In the context of quantum statistical mechanics, Euclidean approach developed in [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] also appears to be very useful. In particular, for a class of

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quantum systems with so called stochastic positivity, which are described by some stochastic processes, methods of classical statistical mechanics can be applied to study several questions like the existence and properties of the limiting thermal states, phase transitions, etc. ([20, 21, 24, 25, 26, 27]).

In the present paper a general conceptual framework for stochastically positive structures on Weyl algebras is introduced (Section II). As a starting point for further development, we analyse in details the case of quasi-free states (Section III). Although basic structural results characterizing quasi-free thermal states have been obtained long time ago [28, 29, 30], we present new (in our opinion simpler) proof of the theorem giving general form of a quasi-free state satisfying the KMS condition (see Theorem 3.1 and 3.7). Section 3.4 contains complete characterizations of stochastically positive quasi-free KMS states and corresponding periodic stochastic processes. The important problem of the equivalence of the modular structure given by a stochastic process (Theorem 3.3) with the canonical modular structure associated with a quasi-free KMS state is solved in full generality in Section 3.5. This result shows that in the case of quasi-free stochastically positive states, all relevant informations about KMS structure are contained in the commutative sector given by a thermal stochastic process. Similar results are true also for the ground state case (Section 3.7). Let us emphasize that it gives new possibilities for the description of KMS structures in the case when interaction is present. Having described quasi-free systems in terms of stochastic processes, one may perturb them with multiplicative-like functionals, thereby creating some new non-Gaussian thermal process. Furthermore, given such a process, we can reproduce its KMS structure. Some results into this direction were obtained in [24] for gentle perturbations of the free Bose gas in the noncritical region of densities and in [25] for the critical case. Essentially equivalent (via the Feynman-Kac formula) type of perturbations which can be studied using this method are central perturbations of a quasi-free state, by which we mean the perturbations of the the form ”$H_\omega + V$”, where $H_\omega$ is the generator of the free evolution in the thermal representation, and $V$ is an operator affiliated with the von Neumann algebra $\pi_\omega(\mathfrak{A})''$, for apropriate abelian subalgebra $\mathfrak{A}$ of the Weyl algebra. That kind of perturbations will be discussed in the second part of the present work. Section IV contains several examples to which our general arguments can be applied. In particular case of the scalar quantum field theory on a globally hyperbolic stationary space-time, we give a general result on the existence of Markov thermal process, which determines the whole modular structure of the theory. Similar arguments are also valid for ground state structure. In the case of a static space-time, we obtain much more explicite description of the arising process and we are able to discuss its continuity properties. In particular, in the case of Rindler wedge (Example 4.1.2) we show stochastic positivity of the KMS structure arising in the context of Bisognano-Wichmann theorem [31]. These results can be used to develop a rigorous constructive approach to non-linear quantum field theories, by adopting the tools worked out in constructive quantum field theory on a flat space-time. A detailed studies of the perturbed Euclidean quantum fields on stationary globally hyperbolic space-times will be presented in our forthcoming publication. In this section we discuss also the models of quantum statistical mechanics, including the model of nonrelativistic Bose matter and infinite harmonic crystal, and describe the corresponding stochastic processes.
II Weyl Algebra. Vacuum and Temperature Green Functions.

2.1 Weyl algebra. Let $\mathcal{D}$ be a real vector space with a locally convex topology $\tau$. If $\sigma$ is a $\tau$-continuous symplectic form on $\mathcal{D}$ (i.e., $\sigma$ is a bilinear, antisymmetric and nondegenerate mapping from $\mathcal{D} \times \mathcal{D}$ into $\mathbb{R}$), then $((\mathcal{D}, \tau), \sigma)$ is called a vector symplectic space. Let $W_f$ be the real function on $\mathcal{D}$ defined by

$$W_f(g) = \begin{cases} 1 & \text{if: } f = g \\ 0 & \text{if: } f \neq g \end{cases}$$

With the product

$$W_f W_g = e^{-i\sigma(f,g)/2} W_{f+g}$$

and involution

$$W_f^* = W_{-f}$$

the complex algebra generated by $W_f, f \in \mathcal{D}$ becomes a *-algebra $\mathfrak{W}_0(\mathcal{D}, \sigma)$. We define $\mathfrak{W}(\mathcal{D}, \sigma)$ (the Weyl algebra over $((\mathcal{D}, \sigma))$ in the following way. Let

$$|| \sum_{k=1}^N z_k W_{f_k} ||_1 = \sum_{k=1}^N |z_k|$$

$|| \cdot ||_1$ is a *-norm on $\mathfrak{W}_0(\mathcal{D}, \sigma)$ and let the completion $\overline{\mathfrak{W}(\mathcal{D}, \sigma)}^{|| \cdot ||_1}$ is a *-Banach algebra with unit. Let $\mathcal{F}$ be the set of states on $\overline{\mathfrak{W}(\mathcal{D}, \sigma)}^{|| \cdot ||_1}$, then we define a $C^*$-algebra norm as follows [32]:

$$||W|| = \sup_{\rho \in \mathcal{F}} \sqrt{\rho(W^*W)}$$

$\mathfrak{W}(\mathcal{D}, \sigma)$ is the completion of $\overline{\mathfrak{W}_0(\mathcal{D}, \sigma)}^{|| \cdot ||_1}$ with respect to this norm.

A state $\omega$ on $\mathfrak{W}(\mathcal{D}, \sigma)$ is called regular if it is $\tau$-continuous i.e. the function

$$f \rightarrow \omega(W_f)$$

is $\tau$-continuous.

Remark. Usually, the weaker form of continuity is assumed, namely the map

$$\mathbb{R} \ni t \rightarrow \omega(W_{tf})$$

should be continuous, for every $f \in \mathcal{D}$.

The set of all regular states on $\mathfrak{W}(\mathcal{D}, \sigma)$ will be denoted by $E(\mathfrak{W})$. For a given $\omega \in E(\mathfrak{W})$, $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ will be the corresponding GNS representation.

Let $\{T_t\}_{t \in \mathbb{R}}$ be a one-parameter group of $\tau$-continuous symplectic mappings from $\mathcal{D}$ onto $\mathcal{D}$. In the following, $(\mathcal{D}, \sigma, \{T_t\})$ will be called a linear dynamical system. If we put

$$\alpha_t(W_f) = W_{T_t f}$$

we obtain a one-parameter group of *-automorphisms of the Weyl algebra $\mathfrak{W}(\mathcal{D}, \sigma)$. For a given linear dynamical system $(\mathcal{D}, \sigma, \{T_t\})$, let $E^\alpha(\mathfrak{W})$ be the set of all regular $\alpha_t$-invariant states on $\mathfrak{W}(\mathcal{D}, \sigma)$. If $\omega \in E^\alpha(\mathfrak{W})$, then on the GNS space $\mathfrak{H}_\omega$, the evolution $\omega$ is represented by a unitary group $U_\omega(t)$ with a self-adjoint generator $H$. 


Definition 2.1.

1. \( \omega \in E^\alpha(\mathcal{M}) \) is a ground state iff \( H_\omega \geq 0 \).

2. Let \( 0 < \beta < \infty \) be given. A faithfull state \( \omega \in E^\alpha(\mathcal{M}) \) is \( \alpha_\beta \)-KMS state at the inverse temperature \( \beta \) iff in the corresponding GNS representation the unitary group \( e^{itH_\omega} \) defines a weakly continuous group of \( * \)-automorphisms of \( \pi_\omega(\mathcal{M}(\mathcal{D}; \sigma))'' \) such that

\[
\langle e^{-(\beta/2)H_\omega} \pi_\omega(W_f) \Omega_\omega, e^{-(\beta/2)H_\omega} \pi_\omega(W_g) \Omega_\omega \rangle = \langle \pi_\omega(W_g)^* \Omega_\omega, \pi_\omega(W_f)^* \Omega_\omega \rangle
\]

for all \( f, g \in \mathcal{D} \). In this case \( \Omega_\omega \) is cyclic and separating for \( \pi_\omega(\mathcal{M}(\mathcal{D}; \sigma))'' \) and the modular operator associated with \( \Omega_\omega \) coincides with \( e^{-(\beta/2)H_\omega} \).

2.2 Euclidean Green Functions. Let \( \omega \in E^\alpha(\mathcal{M}) \). We define the following multi-time Green functions

\[
G(f_1, \ldots, f_n; t_1, \ldots, t_n) = \omega(\alpha_{t_1}(W_{f_1}) \cdots \alpha_{t_n}(W_{f_n}))
\]

\[
= \langle \Omega_\omega, U_{t_1}(W_{f_1})U_{t_2}(-t_1) \cdots U_{t_n}(W_{f_n})U_{-t_n}(W_{f_n})U_{-t_n}(\Omega_\omega) \rangle
\]

If \( \omega_\infty \in E^\alpha(\mathcal{M}) \) is a ground state, then it can be shown that \( G(f_1, \ldots, f_n; t_1, \ldots, t_n) \) can be analytically continued to the tubular region

\[
\mathcal{T}_n^\infty = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : -\infty < \text{Im } z_1 < \text{Im } z_2 < \ldots < \text{Im } z_n < \infty \}
\]

Let us define the Euclidean region \( \mathcal{E}_n^\infty \subset \overline{\mathcal{T}_n^\infty} \)

\[
\mathcal{E}_n^\infty = \{(s_1, \ldots, s_n) \in \mathbb{R}^n : s_1 \leq \ldots \leq s_n \}
\]

The restriction of analytically continued Green functions to \( \mathcal{E}_n^\infty \) will be called Euclidean Green functions corresponding to the ground state \( \omega \) and denoted by

\[
G^E(f_1, \ldots, f_n; s_1, \ldots, s_n)
\]

By linearity and continuity, we extend them to the Weyl algebra and obtain

\[
G^E(W_1, \ldots, W_n; s_1, \ldots, s_n), \quad W_1, \ldots, W_n \in \mathcal{M}(\mathcal{D}; \sigma)
\]

Similarly, let \( \omega \in E^\alpha(\mathcal{M}) \) be the strongly \( \beta \)-KMS state. By the theorem of Araki [33], the Green functions \( G(f_1, \ldots, f_n; t_1, \ldots, t_n) \) can be analytically continued to the functions holomorphic in the tube

\[
\mathcal{T}_n^\beta = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : -\beta/2 < \text{Im } z_1 < \cdots < \text{Im } z_n < \beta/2 \}
\]

and continuous on the boundary of \( \mathcal{T}_n^\beta \). The restrictions of holomorphic functions to the Euclidean region defined by

\[
\mathcal{E}_n^\beta = \{(s_1, \ldots, s_n) : -\beta/2 \leq s_1 \leq \cdots \leq s_n \leq \beta/2 \}
\]

will be called Euclidean Green functions corresponding to the KMS state \( \omega \), and denoted by \( G^E(f_1, \ldots, f_n; s_1, \ldots, s_n) \) as in the ground state case. Again by linearity and continuity, we extend them to the Weyl algebra and obtain

\[
G^E(W_1, \ldots, W_n; s_1, \ldots, s_n), \quad W_1, \ldots, W_n \in \mathcal{M}(\mathcal{D}; \sigma)
\]
and continuity, Euclidean Green functions defined by KMS state can be extended to the Weyl algebra $\mathfrak{W}(\mathcal{D},\sigma)$.

The properties of Euclidean Green functions corresponding to the KMS state $\omega$ were studied in [23]. Similarly we can obtain the properties of ground state Euclidean Green functions. In the formulation of this properties, the following notation will be used: for $W_1,\ldots,W_k \in \mathfrak{W}(\mathcal{D},\sigma)$

$$W_k = (W_1,\ldots,W_k) \quad \text{and} \quad W_k^* = (W_k^*,\ldots,W_1^*)$$

for $f_1,\ldots,f_k \in \mathcal{D}$, $\underline{f}^k = (f_1,\ldots,f_k)$. Similarly

$$\underline{s}^k = (s_1,\ldots,s_k) \quad \text{and} \quad \underline{s}^k = (-s_k,\ldots,-s_1)$$

Moreover

$$G^E(\underline{f}^k;\underline{s}^k) = G^E(f_1,\ldots,f_n;s_1,\ldots,s_n)$$

$$G^E(\underline{W}^k;\underline{s}^k) = G^E(W_1,\ldots,W_k;s_1,\ldots,s_k)$$

and

$$W_k^{-1}(i) = (W_1,\ldots,W_{i-1},W_iW_{i+1},W_{i+2},\ldots,W_k)$$

$$W_k^{-1}([i]) = (W_1,\ldots,W_{i-1},W_{i+1},\ldots,W_k)$$

$$\underline{s}^{-1}(i) = (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_k)$$

For the ground state $\omega_\infty \in E^\alpha(\mathfrak{W})$ the set of Euclidean Green functions $G^E_\infty = \{G^E(\underline{W}^k;\underline{s}^k)\}$ has the following properties:

(E1) $\infty$

(1) for each $W_k$ the map

$$\underline{s}^k \rightarrow G^E(\underline{W}^k;\underline{s}^k)$$

is continuous,

(2) for a fixed $\underline{s}^k \in \mathcal{E}_k^\infty$, the map

$$\underline{W}^k \rightarrow G^E(\underline{W}^k;\underline{s}^k)$$

is multilinear and bounded i.e.

$$|G^E(\underline{W}^k;\underline{s}^k)| \leq \prod_{i=1}^k ||W_i||$$

Moreover, the map

$$\underline{f}^k \rightarrow G^E(\underline{f}^k;\underline{s}^k)$$

is $\tau$-continuous,

(3) for any $s \in [0,\infty)$

$$G^E(W_k^{s_k};\underline{s}^k) = G^E(W_k^{s_k+1};\underline{s}^k)$$
(4) for any $W^k$ and $s^k \in \mathcal{E}_k^\infty$ such that $s_i = s_{i+1}$, $1 \leq i \leq k-1$

$$G^E(W^k; s^k) = G^E(W^{k-1}(i); s^{k-1}([i+1]))$$

(5) for any $W^k$ such that $W_i = I: 1 \leq i \leq k$

$$G^E(W^k; s^k) = G^E(W^{k-1}(i); s^{k-1}([i]))$$

(6)

$$G^E(I; 0) = 1$$

\textbf{(EG2)}_\infty \text{ (OS-positivity)}

For every terminating sequences\n
$$\left(W_0^0, W_1^1, \ldots, W_k^k, \ldots\right), \quad \left(s_0^0, s_1^1, \ldots, s_k^k, \ldots\right), \quad s^k \in \mathcal{E}_k^\infty, s^k \in \mathcal{E}_k^\infty : s_1 \geq 0$$

we have\n
$$\sum_{k,l} G^E(W^k, W^l; s^k, s^l) \geq 0$$

\textbf{(EG3)}_\infty

For every terminating sequences\n
$$\left(W_0^0, W_1^1, \ldots, W_k^k, \ldots\right), \quad \left(s_0^0, s_1^1, \ldots, s_k^k, \ldots\right), \quad s^k \in \mathcal{E}_k^\infty, s^k \in \mathcal{E}_k^\infty : s_1 \geq 0$$

and every $W \in \mathcal{M}(\mathcal{D}, \sigma)$ we have\n
$$\sum_{k,l} G^E(W^{k*}, W^*, W, W^l; s^{k*}, s^l, 0, 0, s^l) \leq ||W||^2 \sum_{k,l} G^E(W^{k*}, W^l; s^{k*}, s^l)$$

Similarly, for $\beta$ - KMS state $\omega$ the set $\mathcal{G}^E = \{G^E(W^k; s^k)\}$ of Euclidean Green functions has the following properties:

\textbf{(EG1)}_{\beta}

(1) for each $W^k$ the map

$$s^k \rightarrow G^E(W^k; s^k)$$

is continuous,

(2) for a fixed $s^k \in \mathcal{E}_k^{\beta}$ the map

$$W^k \rightarrow G^E(W^k; s^k)$$

is multilinear, and bounded

$$|G^E(W^k; s^k)| \leq \prod_{i=1}^{k} ||W_i||$$

Moreover, the map

$$f^k \rightarrow G^E(f^k, s^k)$$
is $\tau$-continuous,

(3) for any $\tilde{s}^k \in \mathcal{E}_k^\beta$ and $s \in [-\beta/2, \beta/2]$ such that $s_k + s \leq \beta/2$ the Euclidean Green functions are locally shift invariant, i.e.

$$G^E(W^k; \tilde{s}^k + s) = G^E(W^k; \tilde{s}^k)$$

(4) for any $W^k$ and $\tilde{s}^k \in \mathcal{E}_k^\beta$ such that $s_i = s_{i+1}$, $1 \leq i \leq k - 1$

$$G^E(W^k; \tilde{s}^k) = G^E(W^{k-1}(i); \tilde{s}^{k-1}([i + 1]))$$

(5) for any $W^k$ such that $W_i = 1 : 1 \leq i \leq k$

$$G^E(W^k; \tilde{s}^k) = G^E(W^{k-1}([i]); \tilde{s}^{k-1}([i]))$$

(6)

$$G^E(\mathbb{I}; 0) = 1$$

\(\textbf{(EG2)}_\beta\) (OS-positivity)

For every terminating sequences

$$(W^0, W^1, \ldots, W^k, \ldots), \quad (\tilde{s}^0, \tilde{s}^1, \ldots, \tilde{s}^k, \ldots), \quad \tilde{s}^k \in \mathcal{E}_k^{\beta,+} = \{s^k \in \mathcal{E}_k^\beta : s_1 \geq 0\}$$

we have

$$\sum_{k,l} G^E(W^{k*}, W^l; \tilde{s}^{k*}, \tilde{s}^l) \geq 0$$

\(\textbf{(EG3)}_\beta\)

For every terminating sequences

$$(W^0, W^1, \ldots, W^k, \ldots), \quad (\tilde{s}^0, \tilde{s}^1, \ldots, \tilde{s}^k, \ldots), \quad \tilde{s}^k \in \mathcal{E}_k^{\beta,+} = \{s^k \in \mathcal{E}_k^\beta : s_1 \geq 0\}$$

and every $W \in \mathfrak{M}(\mathcal{D}, \sigma)$ we have

$$\sum_{k,l} G^E(W^{k*}, W^*, W, W^l; \tilde{s}^{k*}, 0, 0, \tilde{s}^l) \leq ||W||^2 \sum_{k,l} G^E(W^{k*}, W^l; \tilde{s}^{k*}, \tilde{s}^l)$$

\(\textbf{(EG4)}_\beta\) (Weak form of KMS condition)

Let us define for $0 \leq s_1 \leq \cdots \leq s_n \leq \beta$

$$\hat{G}^E(W_0, \ldots, W_n; s_1, \ldots, s_n) := G^E(W_0, \ldots, W_n; -\beta/2, s_1 - \beta/2, \ldots, s_n - \beta/2)$$

Then for each $n$

$$\hat{G}^E(W_0, \ldots, W_n; s_1, \ldots, s_n)$$

$$= \hat{G}^E(W_n, W_0, W_1, \ldots, W_n-1; \beta - s_n, \beta - s_n + s_1, \ldots, \beta - s_n + s_{n-1})$$

On the other hand, starting with Euclidean Green functions, and proceeding similarly as in [23] (see also [18]), we can obtain the following reconstruction theorems.

Thus for ground state case we have:
Theorem 2.1. Let $G^E_\infty$ be an abstract set of Euclidean Green functions on the Weyl algebra $\mathcal{W}(\mathcal{D}, \sigma)$ with properties (EG1)$_\infty$–(EG3)$_\infty$. Then there exist:

1. the Hilbert space $\mathcal{H}_\infty$,
2. the vector $\Omega_\infty \in \mathcal{H}_\infty$,
3. the $*$-representation $\pi_\infty$ of $\mathcal{W}(\mathcal{D}, \sigma)$ in $\mathcal{B}(\mathcal{H}_\infty)$,
4. the weakly continuous one-parameter group of unitary operators $U_\infty(t) = e^{itH_\infty}$ with $H_\infty \geq 0$

such that

$$G^E(f_k; s_k) = \langle \Omega_\infty, e^{-s_1H_\infty}\pi_\infty(W_{f_1}) \cdots e^{-(s_k-s_{k-1})H_\infty}\pi_\infty(W_{f_k})\Omega_\infty \rangle$$

Moreover, the vector $\Omega_\infty$ is cyclic for the algebra $\mathcal{M}_0^\infty$ generated by operators of the form $e^{it_1H_\infty}\pi_\infty(W_{f_1})e^{-it_1H_\infty} \cdots e^{it_nH_\infty}\pi_\infty(W_{f_n})e^{-it_nH_\infty}$; $t_1, \ldots, t_n \in \mathbb{R}$

and the linear space spanned by vectors

$$e^{-s_1H_\infty}\pi_\infty(W_{f_1}) \cdots e^{-(s_k-s_{k-1})H_\infty}\pi_\infty(W_{f_k})\Omega_\infty$$

is dense in $\mathcal{H}_\infty$. Additionally, $\alpha^\infty_t(W) = e^{itH_\infty}We^{-itH_\infty}$ is $\sigma$-weakly continuous group of automorphisms of $\mathcal{M}^\infty = (\mathcal{M}_0^\infty)^\prime\prime$.

And similarly, for temperature case:

Theorem 2.2. Let $G^E_\beta$ be an abstract set of Euclidean Green functions on the Weyl algebra $\mathcal{W}(\mathcal{D}, \sigma)$ with properties (EG1)$_\beta$–(EG4)$_\beta$. Then there exist:

1. the Hilbert space $\mathcal{H}_\beta$,
2. the vector $\Omega_\beta \in \mathcal{H}_\beta$,
3. the $*$-representation $\pi_\beta$ of $\mathcal{W}(\mathcal{D}, \sigma)$ in $\mathcal{B}(\mathcal{H}_\beta)$,
4. the weakly continuous one-parameter group of unitary operators $U_\beta(t) = e^{itH_\beta}$

such that

$$G^E(f_k; s_k) = \langle \Omega_\beta, e^{-s_1H_\beta}\pi_\beta(W_{f_1}) \cdots e^{-(s_k-s_{k-1})H_\beta}\pi_\beta(W_{f_k})\Omega_\beta \rangle$$

Moreover, the vector $\Omega_\beta$ is cyclic for the algebra $\mathcal{M}_0^\beta$ generated by operators of the form

$$e^{it_1H_\beta}\pi_\beta(W_{f_1})e^{-it_1H_\beta} \cdots e^{it_nH_\beta}\pi_\beta(W_{f_n})e^{-it_nH_\beta}; \quad t_1, \ldots, t_n \in \mathbb{R}$$

and the state $\omega(M) = \langle \Omega_\beta, M\Omega_\beta \rangle$ is $\beta$-KMS state on $\mathcal{M}_\beta = (\mathcal{M}_0^\beta)^\prime\prime$ with respect to the unitary group $e^{itH_\beta}$.

In the following, the $W^*-\beta$-KMS system constructed in the Theorem 2.2 will be denoted as

$$\mathbb{C} = (\mathcal{H}_\beta, \Omega_\beta, \pi_\beta, e^{itH_\beta}, \mathcal{M}_\beta)$$
2.3 Stochastic positivity. For a general KMS state $\omega$ on an abstract C*-algebra $\mathfrak{A}$, Klein and Landau [18] discussed the problem of construction of a stochastic process corresponding to $\omega$. As they showed, such a process can be constructed using some abelian sub-C*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ on which Euclidean Green functions corresponding to $\omega$ are positive in some special sense. In such case, the process has values in the spectrum of the abelian algebra $\mathfrak{B}$. To study the existence of stochastic process in the case of Weyl algebra $W(D, \sigma)$, we consider abelian subalgebras of $W(D, \sigma)$ defined in terms of so called *abelian splitting* of the symplectic space $(D, \sigma)$.

**Definition 2.2.** A pair $(D_+, D_-)$ of linear subspaces of the symplectic space $(D, \sigma)$ is called an *abelian splitting* if $D = D_+ + D_-$ and $\sigma(D_\pm, D_\pm) = 0$. For the abelian splitting $(D_+, D_-)$, let $W_+$ and $W_-$ be the abelian subalgebras of the Weyl algebra $W(D, \sigma)$ generated by $W_f$ with $f \in D_+$ and $D_-$ respectively.

Now the desired positivity condition is the following:

**Definition 2.3.**

(1) The set of Euclidean Green functions $G_E^{\beta}$ on the Weyl algebra $W(D, \sigma)$ with the given abelian splitting is $W_\pm$-stochastically positive if for every positive elements $W_1, \ldots, W_n \in W_\pm$ and every $(s_1, \ldots, s_n) \in \mathcal{E}_n^\beta$

$$G_E(W_1, \ldots, W_n; s_1, \ldots, s_n) \geq 0$$

(2) In the case of ground state system $G_E^{\infty}$ the definition is analogous.

In the following discussion we will use the notions of thermal and ground state processes with values in some Hausdorff topological space $V$.

**Definition 2.4.**

(1) Let $V$ be a Hausdorff space. A stochastic process $\xi_t$ taking values in $V$ will be called *thermal process* (with the inverse temperature $\beta$) if:
- TP1$\beta$: $\xi_t$ is stochastically continuous and faithful i.e. for any $f \in C_b(V); f > 0$ we have $\mathbb{E}f(\xi_t) \neq 0$,
- TP2$\beta$: $\xi_t$ is periodic with period $\beta$ i.e. for all $t \in \mathbb{R}, \xi_t = \xi_{t+\beta}$ in law (if $K_\beta$ is the circle with length $\beta$ parametrized as $[-\beta/2, \beta/2]$ with end points glued, then $\xi_t$ can be indexed by $t \in K_\beta$),
- TP3$\beta$: $\xi_t$ is OS-positive on $K_\beta$ i.e. for every $F \in C_b(V^n)$ and $t_1, \ldots, t_n \in [0, \beta/2]$

$$\mathbb{E}F(\xi_{-t_1}, \ldots, \xi_{-t_n})F(\xi_{t_1}, \ldots, \xi_{t_n}) \geq 0$$

- TP4$\beta$: $\xi_t$ is shift invariant on $K_\beta$ i.e. $\xi_{t+\tau} = \xi_t$ for all $t, \tau \in K_\beta$, and $t + \tau$ is defined modulo $\beta$,
- TP5$\beta$: $\xi_t$ is reflection invariant i.e. for every $F \in C_b(V^n)$ and $t_1, \ldots, t_n \in K_\beta$

$$\mathbb{E}F(\xi_{t_1}, \ldots, \xi_{t_n}) = \mathbb{E}F(\xi_{-t_1}, \ldots, \xi_{-t_n})$$

(2) Similarly, the process $\xi_t$ is called *ground state process* if it is stochastically continuous and faithful, shift and reflection invariant on $\mathbb{R}$ and OS-positive on $\mathbb{R}$.
Let $\xi_t$ be the thermal process with values in $\mathcal{D}_+^*$, where $\mathcal{D}_+^*$ is an algebraic dual to $\mathcal{D}_+$. Define

$$G^\xi(f^k; s^k) = \mathbb{E}(\prod_{l=1}^k e^{i\langle \xi_{s_l}, f_l \rangle})$$

where $f^k \in \mathcal{D}_+^k$ and $s^k \in K_\beta^k$. It can be shown that $\mathbb{G}^\xi = \{G^\xi(f^k; s^k)\}$ satisfy conditions (EG1)$_\beta$–(EG4)$_\beta$ for the abelian algebra $\mathfrak{W}_+$. Thus from Theorem 2.2, there exists the unique (up to the unitary equivalence) $W^* - \beta$-KMS system

$$\mathbb{C}^\xi = (\mathfrak{F}_\xi, \Omega_\xi, \pi_\xi, e^{itH_\xi}, \mathfrak{M}_\xi)$$

such that for every $f^k \in \mathcal{D}_+^k$ and $s^k \in K_\beta^k$

$$G^\xi(f^k; s^k) = \langle \Omega_\xi, e^{-s_1H_\xi} \pi_\xi(W_{f_1}) \cdots e^{-(s_k-s_{k-1})H_\xi} \pi_\xi(W_{f_k}) \Omega_\xi \rangle$$

Now we show the converse:

**Theorem 2.3.**

1. Let $(\mathcal{D}_+, \mathcal{D}_-)$ be the abelian splitting of the symplectic space $(\mathcal{D}, \sigma)$. If $\mathbb{G}_\beta^E$ is the set of $\mathfrak{W}_+$-stochastically positive Euclidean Green functions on $\mathfrak{W}(\mathcal{D}, \sigma)$, there exists the unique (up to the stochastic equivalence) thermal process $\xi_t^\beta$ with values in $\mathcal{D}_+^*$ such that for every $f_1, \ldots, f_k \in \mathcal{D}_+$ and $s^k \in K_\beta^k$

$$G^E(f^k; s^k) = \mathbb{E}(\prod_{l=1}^k e^{i\langle \xi_{s_l}, f_l \rangle}) \quad (2.1)$$

2. Similarly, if $\mathbb{G}_\infty^E$ is the set of $\mathfrak{W}_+$-stochastically positive ground state Euclidean Green functions, there exists the unique (up to the stochastic equivalence) ground state process $\xi_t^\infty$ with values in $\mathcal{D}_+^*$ such that for every $f_1, \ldots, f_k \in \mathcal{D}_+$ and $-\infty < s_1 \leq \cdots \leq s_k < \infty$

$$G^E(f^k; s^k) = \mathbb{E}(\prod_{l=1}^k e^{i\langle \xi_{s_l}^\infty, f_l \rangle}) \quad (2.2)$$

**Proof.** Let $\mathbb{C}^+$ be the $W^* - \beta$-KMS system generated by the set $\mathbb{G}^+$ of Euclidean Green functions $G^E(f^k; s^k)$ restricted to $f^k \in \mathcal{D}_+^k$. Then using the result of [18], we conclude that there exists the thermal process $\xi_t^{KL}$ with values in the spectrum of von Neumann algebra $(\pi_+^{(\mathfrak{W}_+)^\prime\prime}(\mathfrak{W}_+))$ (where $\pi_+$ denotes the representation defined by the set $\mathbb{G}^+$) and such that

$$G^E(f^k; s^k) = \int_{Q_\beta} \prod_{l=1}^k \pi^{(\mathfrak{W}_+)(q(s_l))}d\mu^{KL}(q)$$

where $Q_\beta$ is the path space of the process $\xi^{KL}$ and $\pi_+^{(\mathfrak{W}_+)}$ is the Gelfand transformation of $\pi_+(\mathfrak{W}_+)$. 


For fixed \( f^k \) and \( s^k \) define the map
\[
\alpha \rightarrow \Gamma_{f^k, s^k}(\alpha)
\]
where \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k \), by
\[
\Gamma_{f^k, s^k}(\alpha) = G^E(\alpha_1 f_1, \ldots, \alpha_k f_k; s_1, \ldots, s_k)
\]
\( \Gamma \) is positive definite on \( \mathbb{R}^k \) since for every \( c_l \in \mathbb{C}, \alpha^l \in \mathbb{R}^k; l = 1, \ldots, M \)
\[
\sum_{l,m=1}^{M} c_l \overline{c}_m \Gamma_{f^k, s^k}(\alpha^l - \alpha^m) = \int |\sum_{l=1}^{M} \prod_{i=1}^{k} \overline{W}_{\alpha^l_i f_i}(q(s_i))|^2 d\mu^{KL}(q) \geq 0
\]
By the Bochner theorem, there exists Borel probability measure \( d\nu_{f^k, s^k} \) on \( \mathbb{R}^k \) such that
\[
\Gamma_{f^k, s^k}(\alpha) = \int e^{i\langle \alpha, \gamma \rangle} d\nu_{f^k, s^k}
\]
The system of finite dimensional measures \( d\nu_{f^k, s^k} \) forms a compatible system of measures on the cylindrical sets of \( D^k_{+} K_\beta \) as it follows from (EG1) \((5)\). Thus, by the theorem of Kolmogorov, there exists unique up to the stochastic equivalence stochastic process \( \xi^\beta_t \) with values in \( D^*_+ \) such that (2.1) is satisfied. Similar arguments can be used in the ground state case.

Remarks.

(1) It is worth to stress that the process corresponding to stochastically positive Euclidean Green functions on the Weyl algebra \( \mathfrak{W}(D, \sigma) \) can be realized as a process with values not in an abstract spectrum of abelian algebra, but in the dual space to \( D_+ \). In particular situations, the space of values of the process \( \xi^\beta_t \) can be localized in much smaller subspace of \( D^*_+ \) (see i.e. Section IV below and [24, 25]).

(2) In order to reconstruct all relevant informations about KMS structure (or ground state structure) from the stochastic process, it is important to take the abelian sub-algebra in such a way that the \( W^* \)-systems \( \mathbb{C} \) and \( C^\xi \) (or analogous in the ground state case) are unitarily equivalent. It is interesting that in the quasi-free case discussed in the next section, abelian subalgebras defined by some natural splitting always have this property.

### III Quasi-free stochastically positive states.

#### 3.1 Quasi-free states.
To discuss the properties of quasi-free stochastically positive systems, assume that \( D \) has a structure of complex pre-Hilbert space with a scalar product \( \langle \cdot, \cdot \rangle \) and \( \sigma(f,g) = \text{Im}(f,g) \). Moreover, assume that \( T_t \) is a group of unitary operators with respect to \( \langle \cdot, \cdot \rangle \) which leaves \( D \) invariant and has a self-adjoint generator \( h \). Let \( \mathbb{B}(f,g) \) be a positive, sesquilinear form on \( D \) which is \( T_t \)-invariant and such that
\[
|\sigma(f,g)|^2 \leq \mathbb{B}(f,f) \mathbb{B}(g,g)
\]
Then \( \omega \) defined by
\[
\omega(W_f) = e^{-\frac{1}{4}B(f,f)}
\] (3.1)
and extended by linearity and continuity to the whole \( \mathcal{W}(\mathcal{D},\sigma) \) is a state [34], so called gauge invariant quasi-free state on the Weyl algebra \( \mathcal{W}(\mathcal{D},\sigma) \).

For quasi-free states, two point Green functions
\[
G^{(2)}(f, g : t) = \omega(W_f W_{T_t g})
\]
can be calculated, and have the form
\[
G^{(2)}(f, g; t) = e^{-\frac{1}{4}B(f,f) - \frac{1}{4}B(g,g)} e^{\frac{1}{2}F(f, g; t)}
\] (3.2)
where
\[
F(f, g; t) = \Re B(f, T_t g) + i \sigma(f, T_t g)
\]

3.2 KMS quasi-free states. Now we consider a quasi-free state \( \omega \) which is \( \alpha_t \)- KMS state at the inverse temperature \( \beta \) for \( \alpha_t \) defined by \( T_t = e^{it \hbar} \). Then for every \( f, g \in \mathcal{D} \) there exists a function \( \Phi(f, g; z) \) analytic in the strip \( 0 < \Im z < \beta \) and continuous on the boundary, such that
\[
\Phi(f, g; t + i0) = F(f, g; t)
\]
and
\[
\Phi(f, g; t + i\beta) = F(g, f; -t)
\]
Then
\[
G^{(2)}(f, g; t) = e^{-\frac{1}{4}(B(f,f) + B(g,g))} e^{-\frac{1}{2}\Phi(f, g; t+i0)}
\]
and
\[
G^{(2)}(g, f; -t) = e^{-\frac{1}{4}(B(f,f) + B(g,g))} e^{-\frac{1}{2}\Phi(f, g; t+i\beta)}
\]
To find the explicite form of \( \mathbb{B}(f, g) \) let us first consider the case when \( \mathcal{D} \) is a Hilbert space and \( \mathbb{B}(f, g) \) is bounded on \( \mathcal{D} \), hence defined by some bounded operator \( B \) such that \( B \geq 1 \).

**Theorem 3.1.** Suppose that \( \omega \) defined by (3.1) is an \( \alpha_t \)- KMS state at the inverse temperature \( \beta \). Then

1. there exists \( \varepsilon > 0 \) such that \( \hbar \geq \varepsilon \),
2. \[
B = \frac{\mathbb{I} + e^{-\beta \hbar}}{\mathbb{I} - e^{-\beta \hbar}}
\]

**Proof.** Since \( \omega \) is \( \alpha_t \)- KMS state, it is \( \alpha_t \)-invariant. Thus \( B \) commutes with \( T_t = e^{it \hbar} \), hence \( B \) commutes with spectral projectors \( E(\lambda) \) of \( \hbar \). First we show that 0 is not an eigenvalue of \( \hbar \). Suppose that there exists \( f_0 \in \mathcal{D} \) such that \( \hbar f_0 = 0 \). Then for every \( t \in \mathbb{R} \), \( T_t f_0 = f_0 \). Because
\[
F(f, f_0; t) = \Re \mathbb{B}(f, T_t f_0) + i \Im \langle f, T_t f_0 \rangle = F(f, f_0; 0)
\]
\( \Phi(f, f_0, z) \) is identically constant. Hence
\[
F(f, f_0; 0) = F(f_0, f_0; 0)
\]
On the other hand \( F(f, f_0; 0) = F(f_0, f; 0) \), what implies \( \sigma(f, f_0) = 0 \). But \( \sigma \) is nondegenerate, so \( f_0 = 0 \). Because \( \Phi(f, f; z) \) is the analytic continuation of \( F(f, f; t) \) and \( F(f, f; -t) \), so [22]

\[
\mathcal{F}(\mathbb{T}_F)(-p) = e^{-\beta p} \mathcal{F}(\mathbb{T}_F)(p)
\]

where \( \mathbb{T}_F \) is a distribution from \( S' \) given by

\[
<\mathbb{T}_F, \varphi> = \int_{-\infty}^{\infty} F(f, f; t) \varphi(t) \, dt, \quad \varphi \in S
\]

and \( \mathcal{F} \) is the Fourier transform in \( S' \). Because

\[
F(f, f; t) = \text{Re} \langle f, Be^{it} f \rangle + i \text{Im} \langle f, e^{it} f \rangle
\]

\[
= \text{Re} \int_{\mathbb{R}\setminus\{0\}} e^{it\lambda} \, d\langle Bf, E(\lambda)f \rangle + i \text{Im} \int_{\mathbb{R}\setminus\{0\}} e^{it\lambda} \, d\langle f, E(\lambda)f \rangle
\]

\[
= \frac{1}{2} \int_{\mathbb{R}\setminus\{0\}} e^{it\lambda} \, d\langle (B + I)f, E(\lambda)f \rangle + \frac{1}{2} \int_{\mathbb{R}\setminus\{0\}} e^{-it\lambda} \, d\langle (B - I)f, E(\lambda)f \rangle
\]

for any \( \varphi \in S \) we have

\[
<\mathcal{F}(\mathbb{T}_F), \varphi_{\text{inv}}> = \frac{1}{2} \int_{-\infty}^{\infty} dt \int_{\mathbb{R}\setminus\{0\}} d\langle (B + I)f, E(\lambda)f \rangle e^{it\lambda} \mathcal{F}(\varphi_{\text{inv}})(t)
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} dt \int_{\mathbb{R}\setminus\{0\}} d\langle (B - I)f, E(\lambda)f \rangle e^{-it\lambda} \mathcal{F}(\varphi_{\text{inv}})(t)
\]

\[
= \frac{\sqrt{2\pi}}{2} \int_{\mathbb{R}\setminus\{0\}} d\langle (B + I)f, E(\lambda)f \rangle \varphi(-\lambda) + \frac{\sqrt{2\pi}}{2} \int_{\mathbb{R}\setminus\{0\}} d\langle (B - I)f, E(\lambda)f \rangle \varphi(\lambda)
\]

On the other hand it is equal to

\[
<e^{-\beta} \mathcal{F}(\mathbb{T}_F), \varphi>
\]

\[
= \frac{\sqrt{2\pi}}{2} \int_{\mathbb{R}\setminus\{0\}} d\langle (B + I)f, E(\lambda)f \rangle e^{-\beta \lambda} \varphi(\lambda) + \frac{\sqrt{2\pi}}{2} \int_{\mathbb{R}\setminus\{0\}} d\langle (B - I)f, E(\lambda)f \rangle e^{\beta \lambda} \varphi(-\lambda)
\]

Because \( \text{Ker} \ h = 0 \)

\[
h = \int_{0^-}^\infty \lambda dE(\lambda) + \int_0^\infty \lambda dE(\lambda)
\]
Assume that $E((−∞,0)) \neq 0$ and take $f \neq 0$, $f \in \widetilde{D} = E((−∞,0))\mathcal{D}$. Suppose also that $\varphi \geq 0$ and $\text{supp } \varphi = [0,\infty)$. Then

$$\int_{-\infty}^{0} d\langle (B + \mathbb{1})f, E(\lambda)f \rangle \varphi(-\lambda) = \int_{-\infty}^{0} d\langle (B - \mathbb{1})f, E(\lambda)f \rangle e^{\beta\lambda} \varphi(-\lambda)$$

thus

$$\langle f, (B + \mathbb{1})\varphi(h^-)f \rangle = \langle f, (B - \mathbb{1})e^{-\beta h^-} \varphi(h^-)f \rangle$$

where

$$h^- = \int_{-\infty}^{0} (-\lambda)dE(\lambda)$$

Because $B : \widetilde{D} \to \widetilde{D}$ so by polarization we obtain that

$$(B + \mathbb{1})\varphi(h^-) = (B - \mathbb{1})e^{-\beta h^-} \varphi(h^-)$$

But $\varphi$ is arbitrary, so

$$B|\widetilde{D} = \frac{\mathbb{1} + e^{-\beta h^-}}{e^{-\beta h^-} - \mathbb{1}} < -\mathbb{1}$$

Thus we get the contradiction. Hence $h > 0$ and using similar arguments we show that

$$B = \frac{\mathbb{1} + e^{-\beta h}}{\mathbb{1} - e^{-\beta h}}$$

Suppose now that for every $n \in \mathbb{N}$ the projector $E((0, \frac{1}{n}))$ is nonzero. Take $f_n \in E((0, \frac{1}{n}))\mathcal{D}$ such that $\|f_n\| = 1$. Then

$$\|B^{1/2}f_n\|^2 = \int_{0}^{1/n} a(\lambda) d\rho_n(\lambda)$$

where $a(\lambda) = (1 + e^{-\beta \lambda})^{-1}(1 - e^{-\beta \lambda})$ and $d\rho_n(\lambda) = d\langle f_n, E(\lambda)f_n \rangle$. But for every $n \in \mathbb{N}$ the measure $d\rho_n$ is normed to 1 on $(0, \frac{1}{n})$, so

$$\|B^{1/2}f_n\|^2 \geq a(\frac{1}{n}) \to \infty$$

It contradicts the assumption that $B$ is bounded.

### 3.3 Analytic continuation.

Starting with the functions $\Phi(f, g; z)$ we can define two point Euclidean Green function in terms of

$$S(f, g; s) = \Phi(f, g; is)$$

for $s \in [0, \beta]$. Then

$$G^{E,2}(f, g; s) = e^{-\frac{i}{2}(B(f, f) + B(g, g))} e^{-\frac{i}{2}s}S(f, g; s)$$

(3.4)
Proposition 3.1. $S(f, g; s)$ defined by (3.3) satisfy:

1. For every $f, g \in D$ and $s \in [0, \beta]$
   \[ S(f, g; s) = S(g, f; \beta - s) \]

2. For every $f \in D$ the mapping $s \to S(f, f; s)$ is OS-positive i.e. for every sequences $\{s_k\}, s_k \in [0, \beta/2]; \{c_k\}, c_k \in \mathbb{C}$
   \[ \sum_{k,l} \overline{c_k}c_l S(f, f; s_k + s_l) \geq 0 \]

more generally, for every terminating sequences $\{f_k\}, f_k \in D; \{s_k\}, s_k \in [0, \beta/2] \text{ and } \{c_k\}, c_k \in \mathbb{C}$
   \[ \sum_{k,l} \overline{c_k}c_l S(f_k, f_l; s_k + s_l) \geq 0 \]

Proof. The function $S(f, g; s)$ can be represented by the following integral formula [22]

\[ S(f, g; s) = \int_{-\infty}^{\infty} [\Psi(\rho, s)F(f, g; \rho) + \Psi(\rho, \beta - s)F(g, f; -\rho)]d\rho \quad (3.5) \]

with the kernel

\[ \Psi(\rho, s) = \frac{1}{2\beta} \sin \frac{\pi s}{\beta} (\cosh \frac{\pi \rho}{\beta} - \cos \frac{\pi s}{\beta})^{-1} \]

From this formula, we see that $S(f, g; s) = S(g, f; \beta - s)$. On the other hand, the function $t \to F(f, f; t)$ is positive definite since

\[ F(f, g; t) = F_+(f, g; t) + F_-(f, g; t) \]

where

\[ F_+(f, g; t) = \frac{1}{2} \langle f, (B + \mathbb{I})e^{it\mathbb{I}}g \rangle \quad F_-(f, g; t) = \frac{1}{2} \langle g, (B - \mathbb{I})e^{-it\mathbb{I}}f \rangle \]

and the functions $F_+(f, f; t)$, $F_-(f, f; t)$ are obviously positive definite. Thus $S(f, f; s)$ is OS-positive as a Laplace transform of positive measure. By polarization

\[ \sum_{k,l} \overline{c_k}c_l F_\pm(f_k, f_l; t_k - t_l) \geq 0 \]

and

\[ F_\pm(f, g; t) = \int_{-\infty}^{\infty} e^{it\lambda} d\mu_\pm_{f,g} (\lambda) \]

for complex-valued measure $\mu_\pm_{f,g}$. Thus

\[ \sum_{k,l} \overline{c_k}c_l F(f_k, f_l; t_k - t_l) \geq 0 \]
and
\[ F(f, g; t) = \int_{-\infty}^{\infty} e^{it\lambda} d\mu_{f,g}(\lambda) \]
for \( \mu_{f,g} = \mu_{f,g}^+ + \mu_{f,g}^- \).
Hence the matrix of measures \((\mu_{f,k} f_l)\) is positive definite, and the matrix
\[ (S(f_k, f_l; s_k + s_l)) = \left( \int_{-\infty}^{\infty} e^{-(s_k+s_l)\lambda} d\mu_{f_k,f_l}(\lambda) \right) \]
is also positive-definite.

**Corollary 3.1.**

\[ G^{E,2}(f, g; s) \]
satisfies:
1. \[ G^{E,2}(f, g; s) = G^{E,2}(g, f; \beta - s) \quad s \in [0, \beta] \]
2. \[ \sum_{k,l} \tau_k c_l G^{E,2}(-f_k, f_l; s_k + s_l) \geq 0 \quad s_k \in [0, \beta/2] \]

**Remark.** Multi-time Euclidean Green functions corresponding to the quasi-free KMS state can be obtained as follows. By a direct computations we show that
\[ G(f_1, \ldots, f_n; t_1, \ldots, t_n) = \]
\[ = \omega(W_{T_1 f_1} \cdots W_{T_n f_n}) = \prod_{l=1}^{n} e^{\frac{n-2}{4} \mathbb{B}(f_l, f_l)} \prod_{(j_n, k_n)} G^{(2)}(f_{j_n}, f_{k_n}; t_{k_n} - t_{j_n}) \]
where the second product is taken over all pairs of different indices \((j_n, k_n)\) such that \( j_n, k_n \in \{1, \ldots, n\}, t_{j_n} < t_{k_n} \). Analytic continuation of the right-hand side of this expression gives after restriction to Euclidean points the formula
\[ G^{E}(f_1, \ldots, f_n; s_1, \ldots, s_n) = \prod_{l=1}^{n} e^{\frac{n-2}{4} \mathbb{B}(f_l, f_l)} \prod_{(j_n, k_n)} G^{E,2}(f_{j_n}, f_{k_n}; s_{k_n} - s_{j_n}) \]
which can be rewritten in the following way
\[ G^{E}(f_1, \ldots, f_n; s_1, \ldots, s_n) = \prod_{k=1}^{n} e^{-\frac{1}{2} \mathbb{B}(f_k, f_k)} \prod_{(j_k, l_k)} e^{-\frac{1}{2} S(f_{j_k}, f_{l_k}; s_{l_k} - s_{j_k})} \]
It can be checked that \( G^{E}(f_1, \ldots, f_n; s_1, \ldots, s_n) \) and satisfy all conditions of Theorem 2.1.
3.4 Stochastic positivity and thermal process. Now we can pass to the question of stochastic positivity of quasi-free Euclidean Green functions. This property can be naturally formulated in terms of the functions $S(f, g; s)$.

Let $C$ be an abstract complex conjugation on $\mathcal{D}$ i.e. $C$ is antiunitary and $C^2 = 1$. The mapping $C$ naturally defines the abelian splitting $(\mathcal{D}_+, \mathcal{D}_-)$ of $\mathcal{D}$:

$$\mathcal{D}_+ = \{ f \in \mathcal{D} : Cf = f \} , \quad \mathcal{D}_- = \{ f \in \mathcal{D} : Cf = -f \}$$

**Lemma 3.1.** The functions $S(f, g; s)$ restricted to $\mathcal{D}_+$ satisfy

$$S(f, g; s) = S(g, f; s), \quad s \in [0, \beta]$$

if $h$ is $C$-real i.e. $C$ leaves the domain $D(h)$ invariant and

$$Chf = hCf$$

for every $f \in D(h)$

**Proof.** By analytic continuation, the property (3.6) is equivalent to

$$F(f, g; t) = F(g, f; t)$$

for $f, g \in \mathcal{D}_+$ and $t \in \mathbb{R}$. So we have to show that

$$\text{Re} \langle f, (T_t - T_{-t})Bg \rangle = 0$$

(3.7)

and

$$\text{Im} \langle f, (T_t + T_{-t})g \rangle = 0$$

(3.8)

for all $f, g \in \mathcal{D}_+$. Because $h$ is $C$-real, $C$ commutes with spectral projectors of $h$ and

$$CT_t = T_{-t}C$$

thus

$$T_t + T_{-t} : \mathcal{D}_+ \to \mathcal{D}_+$$

$$T_t - T_{-t} : \mathcal{D}_+ \to \mathcal{D}_-$$

Hence (3.8) is satisfied. $B$ as a real function of $h$ is $C$-real too and so $B : \mathcal{D}_+ \to \mathcal{D}_+$. But

$$\text{Re} \langle f_+, g_- \rangle = 0$$

for all $f_+ \in \mathcal{D}_+, g_- \in \mathcal{D}_-$ so (3.7) follows.

Combining this results with that of Proposition 3.1 we get

$$S(f, g; s) = S(f, g; \beta - s) \quad s \in [0, \beta]$$

This allows to extend the function $S(f, g; s)$ (for the fixed $f, g \in \mathcal{D}_+$) to the periodic function of $s$ with the period $\beta$, defined for all $s \in \mathbb{R}$. The extended functions will also be denoted by $S(f, g; s)$. 

Proposition 3.2. For every fixed \( f, g \in D_+ \) there exists a finite, real-valued measure \( \nu_{f,g} \) on \([0, \infty)\) such that

\[
S(f, g; s) = \int_{0}^{\infty} (e^{-sp} + e^{-(\beta-s)p}) \, d\nu_{f,g} \tag{3.9}
\]

Proof. For a fixed \( f \in D_+ \), the function \( S(f, f; s) \) is OS-positive, thus by the Widder theorem there exists a finite measure \( \tilde{\nu}_{f,f} \) on \( \mathbb{R} \) such that

\[
S(f, f; s) = \int_{-\infty}^{\infty} e^{-sp} \, d\tilde{\nu}_{f,f}
\]

for \( s \in [0, \beta] \). Since \( S(f, f; s) = S(f, f; \beta - s) \)

\[
d\tilde{\nu}_{f,f}(-p) = e^{-\beta p} d\tilde{\nu}_{f,f}(p)
\]

Thus there exists a unique finite measure \( \nu_{f,f} \) with support on \([0, \infty)\) such that

\[
S(f, f; s) = \int_{0}^{\infty} (e^{-sp} + e^{-(\beta-s)p}) \, d\nu_{f,f}
\]

By polarization, this gives the integral representation of \( S(f, g; s) \) in terms of real-valued measure \( \nu_{f,g} \).

Proposition 3.3. \( S(f, g; s) \) is positive definite i.e.

\[
\sum_{k,l=1}^{n} \overline{c}_k c_l S(f_k, f_l; s_k - s_l) \geq 0 \tag{3.10}
\]

for all terminating sequences \( f_1, \ldots, f_n \in D_+ \); \( s_1, \ldots, s_n \in \mathbb{R} \) and \( c_1, \ldots, c_n \in \mathbb{C} \).

Proof. Since

\[
e^{-sp} + e^{-(\beta-s)p} = \sum_{n \in \mathbb{Z}} c_n(p) e^{i2\pi ns/\beta} \tag{3.11}
\]

for all \( s \in [0, \beta] \), where

\[
c_n(p) = ((p\beta)^2 + (2\pi n)^2)^{-1} (2\beta p(1 - e^{-\beta p})) \geq 0
\]

the right-hand side of (3.11) defines a periodic function for all \( s \in \mathbb{R} \) ([35]). Combining this with (3.9) we get

\[
S(f, f; s) = \sum_{n \in \mathbb{Z}} e^{i2\pi ns/\beta} \int_{0}^{\infty} c_n(p) \, d\nu_{f,f}(p)
\]

so for a fixed \( f \in D_+ \), \( S(f, f; s) \) is a real-valued positive-definite function as a function of \( s \). Thus, it is a Fourier transform of a positive measure \( \nu_{f,f} \) on \( \mathbb{R} \).
Using the polarization formula for $S(f, g; s)$ we obtain that $S(f, g; s)$ is a Fourier transform of real-valued measure $m_{f,g}$ with finite variation. Let 

$$m = (m_{f_k, f_l})$$

be $n \times n$ matrix of such measures. We show that $m$ is positive-definite i.e. for every Borel subset $E \subset \mathbb{R}$, the matrix $(m_{f_k, f_l}(E))$ is positive-definite. Let $\varphi \in \mathcal{S}$ be non-negative. For $g_k = r_k f_k$, $r_k \in \mathbb{R}$ we have

$$\sum_{k,l} r_k r_l m_{f_k, f_l}(\varphi) = \sum_{k,l} m_{g_k, g_l}(\varphi) = m((\sum_k g_k), (\sum_k g_k))(\varphi) \geq 0$$

It implies that the matrix

$$(S(f, f; s_k - s_l)) = \left( \int_{-\infty}^{\infty} e^{i(s_k - s_l)p} dm_{f_k, f_l}(p) \right)$$

is also positive-definite. To show this, notice that the integral on the right-hand side of (3.12) can be approximated by the sum

$$\sum_{j=0}^{N} e^{i(s_k - s_l)p_j} m_{f_k, f_l}(E_j)$$

Because $(e^{i(s_k - s_l)p_j})$ and $(m_{f_k, f_l}(E_j))$ are positive-definite matrices, from Schur lemma the matrix

$$\left( e^{i(s_k - s_l)p_j} m_{f_k, f_l}(E_j) \right)$$

is positive-definite for each j, so the matrix (3.12) is positive-definite. Thus $S(f_k, f_l; s_k - f_l)$ is real-valued positive-definite matrix in real vector space. It follows that it is also positive-definite as a matrix in complex vector space.

**Theorem 3.2.** $S(f, g; s)$ defines an operator-valued covariance function $R_\beta(s)$ of a periodic Gaussian OS-positive stochastic process indexed by $\mathcal{D}_+$. Thus $R_\beta(s)$ is an operator-valued positive-definite function on $\mathcal{D}_+$ which is periodic and OS-positive. Moreover

$$S(f, g; s) = \langle f, R_\beta(s)g \rangle$$

**Proof.** From positive-definiteness of $S(f, g; s)$ we have

$$|S(f, g; s)|^2 \leq S(f, f; 0) S(g, g; 0) \leq ||B||^2 ||f||^2 ||g||^2$$

Since $S(f, g; s)$ is bilinear and symmetric, there exists a bounded and positive operator $R_\beta(s)$ on $\mathcal{D}_+$ such that

$$S(f, g; s) = \langle f, R_\beta(s)g \rangle$$

Moreover, the function

$$s \to R_\beta(s)$$

is positive-definite, OS-positive and weakly periodic.
Corollary 3.2. Let $\xi_\beta^s$ be the Gaussian process indexed by $\mathcal{D}_+$ defined by

$$E(\langle \xi_\beta^s, f >) = 0$$

and

$$E(\langle \xi_\beta^s, f_1 >\langle \xi_\beta^s, f_2 >) = \frac{1}{2} \langle f_1, R_\beta(s_2 - s_1)f_2 \rangle$$

then, for $f_1, f_2 \in \mathcal{D}_+$

$$G_{E.2}(f_1, f_2; s_2 - s_1) = E(e^{i\langle \xi_\beta^s, f_1 >}e^{i\langle \xi_\beta^s, f_2 >})$$

Similarly, for $f_1, \ldots, f_n \in \mathcal{D}_+$

$$G_{E}(f_1, \ldots, f_n; s_1, \ldots, s_n) = E(e^{i\sum_{k=1}^n \langle \xi_\beta^s, f_k >})$$

Proposition 3.4. If $f \in \mathcal{D}_+$ is such that $m(f) = \int_0^\infty p \, d\nu_{f,f}(p) < \infty$, then the coordinate process $\xi_f^s := \langle \xi_\beta^s, f >$ has a version (denoted by the same symbol) with H"older continuous paths. More precisely, for any $0 < \gamma < 1/2$ there exists an integrable random variable $d(f, \gamma)$ such that

$$|\xi_f^s - \xi_f^{s'}| \leq d(f, \gamma)|s - s'|^\gamma$$

with probability one.

Proof. Since

$$|S(f, f; h) - S(f, f; 0)| = |\int_0^\infty ((e^{-hp} - 1) + (e^{-\beta h}p - e^{-\beta p})) \, d\nu_{f,f}(p)|$$

$$\leq 2|h|\int_0^\infty p \, d\nu_{f,f}(p) = 2m(f)|h|$$

it follows that

$$|E e^{z|\xi_{s+h}^f - \xi_s^f|} - E e^{z|\xi_{s}^f - \xi_s^f|} + E e^{-|z|(|\xi_{s+h}^f - \xi_s^f| - |\xi_{s}^f - \xi_s^f|)}|$$

$$\leq 2e^{2|z||S(f,f;0) - S(f,f;h)|} \leq 2e^{2|z||h|m(f)}$$

On the other hand, by the Cauchy integral formula

$$|\xi_{s+h}^f - \xi_s^f|^n = \frac{n!}{2\pi i} \int_{C_r} e^{z|\xi_{s+h}^f - \xi_s^f|} \frac{dz}{z^{n+1}}$$

Thus

$$E |\xi_{s+h}^f - \xi_s^f|^n \leq 2n! \frac{e^{2r^2|h|m(f)}}{r^n}$$

Now taking $r = |h|^{-1/2}$ we obtain

$$E |\xi_{s+h}^f - \xi_s^f|^n \leq c_n |h|^{n/2}$$

The assertion follows by the application of the Kolmogorov continuity test.
Definition 3.1. A periodic stochastic process $\xi_s$ indexed by $D_+$ has the \textit{two-sided Markov property on the circle} $K_\beta$ iff for all $r, s \in K_\beta$

$$E_{[s,r]}E_{[r,s]} = E_{[r,s]}E_{[r,s]}$$

where for $I \subset K_\beta$, $E_I$ denotes conditional expectation with respect to the $\sigma$-algebra generated by $\{\xi_s, s \in I\}$ and $\{r, s\} = \{s, r\}$ is the set consisting of the two elements $r, s$.

Proposition 3.5. $\xi_s^\beta$ has a version with the two-sided Markov property on the circle $K_\beta$.

\textit{Proof.} The covariance operator $R_\beta(s)$ has the form

$$R_\beta(s) = \frac{e^{-sh} + e^{-(\beta-s)h}}{1 - e^{-\beta h}}$$

Let us introduce the new scalar product in $D_+$ given by

$$\langle f, g \rangle_\beta := \langle f, (1 - e^{-\beta h})^{-1}g \rangle$$

The norms $\| \cdot \|$ and $\| \cdot \|_\beta$ are obviously equivalent. Let $\tilde{\xi}_s$ be the Gaussian process indexed by $(D_+, \langle \cdot, \cdot \rangle_\beta)$ with zero mean and covariance operator

$$\tilde{R}_\beta(s) = e^{-sh} + e^{-(\beta-s)h}$$

$\tilde{\xi}_s$ is stochastically equivalent to $\xi_s^\beta$ and by the result of [35] it satisfies two-sided Markov property on the circle $K_\beta$.

3.5 KMS structure generated by thermal process. Let $\xi_s^\beta$ be a Gaussian process constructed above and let $(Q, \Sigma, \mu)$ be its underlying probability space. Since the process is stationary, $u(t)$ defined by

$$u(t)(e^{i\langle \xi_{s_1}^\beta, f_1 \rangle} \cdots e^{i\langle \xi_{s_n}^\beta, f_n \rangle}) = e^{i\langle \xi_{s_1+t}^\beta, f_1 \rangle} \cdots e^{i\langle \xi_{s_n+t}^\beta, f_n \rangle}$$

extends to a one parameter group of unitary operators on $L^2(Q, \Sigma, \mu)$. By periodicity, $u(\beta) = I$. Since the process is symmetric, $\Theta$ defined by

$$\Theta(e^{i\langle \xi_{s_1}^\beta, f_1 \rangle} \cdots e^{i\langle \xi_{s_n}^\beta, f_n \rangle}) = e^{i\langle \xi_{-s_1}^\beta, f_1 \rangle} \cdots e^{i\langle \xi_{-s_n}^\beta, f_n \rangle}$$

extends to an unitary operator on $L^2(Q, \Sigma, \mu)$ such that $\Theta^2 = I$. Finally, since the process is OS-positive

$$\langle \Theta F, F \rangle_{L^2} \geq 0$$

for all $F \in L^2(Q, \Sigma_{[0,\beta/2]}, \mu)$ where for $S \subset \mathbb{R}$, $\Sigma_S$ denotes the $\sigma$-algebra generated by $\{\xi_s^\beta, s \in S\}$. 
Theorem 3.3. Let $\xi_\beta^s$ be a Gaussian, periodic (with period $\beta$), OS-positive stochastic process indexed by $D_+$. Then there exist a Hilbert space $H_\xi$ with a unit vector $\Omega_\xi$, a weakly continuous one parameter group of unitary operators $U_\xi(t) = e^{itH_\xi}$ and a von Neumann algebra $M_\xi$ of operators acting on $H_\xi$ such that $\Omega_\xi$ is cyclic and separating for $M_\xi$ and $\alpha^\xi_t(M) = e^{itH_\xi} M e^{-itH_\xi}$ is the modular automorphisms group associated with $\Omega_\xi$.

Proof. On the space $L^2(Q, \Sigma_{[0,\beta/2]}, \mu)$ define a sesquilinear form by

$$\langle F, G \rangle = \langle \Theta F, G \rangle_{L^2}$$

By OS-positivity, it is positive semi-definite. Let

$$N = \{ F \in L^2(Q, \Sigma_{[0,\beta/2]}, \mu) : \langle F, F \rangle = 0 \}$$

Then

$$D = L^2(Q, \Sigma_{[0,\beta/2]}, \mu)/N$$

is a pre-Hilbert space with respect to the inner product

$$\langle [F], [G] \rangle = \langle F, G \rangle$$

where $[F]$ denotes the class containing $F$. $H_\xi$ is defined as a Hilbert space completion of $D$ and $\Omega_\xi = [1]$. Let $D_t$ be the linear space generated by vectors $[F]$ for $F \in L^2(Q, \Sigma_{[0,\beta/2-t]}, \mu)$, $t \in [0, \beta/2]$. For every $t \in [0, \beta/2]$ we can define the linear operator $p(t)$ with domain $D_t$ by

$$p(t)[F] = [u(t)F]$$

and we can show that $(p(t), D_t)$ form a symmetric local semigroup ([36]). Hence there exists a unique self-adjoint operator $H_\xi$ on $H_\xi$ such that $p(t) = e^{-tH_\xi}$. $U_\xi(t)$ is defined by $U_\xi(t) = e^{itH_\xi}$. Let now $F_0 \in L^\infty(Q, \Sigma_0, \mu)$. Then

$$\pi_0(F_0)[F] = [F_0F]$$

defines a bounded operator on $H_\xi$ and

$$M_0 = \{ \pi_0(F_0) : F_0 \in L^\infty(Q, \Sigma_0, \mu) \}$$

is a von Neumann algebra of operators on $H_\xi$. Let $M_\xi$ be the von Neumann algebra generated by elements

$$e^{it_1H_\xi}B_1 e^{-it_1H_\xi} \ldots e^{it_nH_\xi}B_n e^{-it_nH_\xi}$$

with $t_j \in \mathbb{R}$, $B_j \in M_0$. Then $\Omega_\xi$ is cyclic and separating for $M_\xi$ ([23]). Using the properties of the process $\xi_\beta^s$ we can now define the modular conjugation and modular group corresponding to $\Omega_\xi$. Let

$$\Theta_{\beta/4} = u(\beta/4)\Theta u(-\beta/4)$$

and

$$L[F] = [\Theta F]$$
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Then
\[ \langle J_\xi[F], J_\xi[G] \rangle = \langle \Theta F, \Theta G \rangle_{L^2} = \langle G, F \rangle_{L^2} = \langle [G], [F] \rangle \]
since \( \Theta/F \) commutes with \( \Theta \) by periodicity. Hence \( J_\xi \) can be extended to an
antiunitary operator on \( \mathcal{H}_\xi \) such that \( J_\xi^2 = I \). Computing the action of \( J_\xi \) on \( \mathcal{M}_\xi \Omega_\xi \), \( M \in \mathcal{M}_\xi \) we can show that
\[ J_\xi M^* \Omega_\xi = e^{-\beta/2} H_\xi M \Omega_\xi \]
Now since \( \mathcal{M}_\xi \Omega_\xi \) is a core for \( e^{-\beta/2} H_\xi \) ([18]), \( J_\xi \) defined above is the modular
conjugation operator and the corresponding modular operator \( \Delta_\xi \) can be identified
with \( e^{-\beta/2} H_\xi \). For more details see [18].

Let \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\) be the GNS representation defined by quasi-free KMS state \( \omega \). Then \( \Omega_\omega \) is cyclic and separating for \( \pi_\omega(\mathcal{M}(D, \sigma))'' \). Let \( J_\omega \) and \( \Delta_\omega \) be the
corresponding modular conjugation and modular operator. We are going to show
that in the case of quasi-free state the modular structure constructed from the
process \( \xi_{s}^\beta \) is unitarily equivalent to the canonical modular structure defined by
KMS state \( \omega \). Thus all relevant informations about KMS structure are con-
tained in the (commutative) stochastic process \( \xi_{s}^\beta \). To obtain this result we need the
following property of the dynamics \( e^{it h} \).

**Theorem 3.4.** Let \( h \) be \( C \)-real on \( D \) and let \( E(\{0\}) = 0 \), where \( dE(\lambda) \) is the
spectral measure of \( h \). Then the family
\[ \{ \sum_{k=1}^{n} e^{it_k h} f_k : n \in \mathbb{N}, t_k \in \mathbb{R}, f_k \in D_+ \} \]
is dense in \( D \).

**Proof.** Because \( C \) commutes with spectral projectors of \( h \) and 0 is not an eigenvalue
of \( h \), so
\[ h = h_1 \oplus h_2, \quad D = D_1 \oplus D_2 \quad \text{and} \quad D_+ = D_{+,1} \oplus D_{+,2} \]
where
\[ D_1 = E((0, \infty)) D, \quad D_2 = E((\infty, 0)) D, \quad D_{+,1} = D_+ \cap D_1, \quad D_{+,2} = D_+ \cap D_2 \]
Hence it is enough to consider a positive operator \( h \) such that \( \ker h = \{0\} \). Because
\( D \) is separable, \( h \) can be written as
\[ h = \bigoplus_{l=1}^{\infty} h_l, \quad D = \bigoplus_{l=1}^{\infty} D_l \]
where each \( h_l \) is a positive operator with a simple spectrum. Moreover, each gen-
erating vector \( g_l \) for a Hilbert space \( D_l \) can be taken from \( D_+ \) and normalized.
Indeed, let us take arbitrary \( g_1 \in D_+ \) with \( ||g_1|| = 1 \) and define
\[ D_l = \bigcap_{k=1}^{\infty} E(\sigma) g_1, \quad \sigma \in \mathbb{B}(0, \infty) \]
Because $C$ commutes with all $E(\sigma)$, so $C : \mathcal{D}_1 \rightarrow \mathcal{D}_1$ and hence $C : \mathcal{D}^\perp_1 \rightarrow \mathcal{D}^\perp_1$. In the second step we take $g_2 \in \mathcal{D}^\perp_1 \cap \mathcal{D}^+_1$, $\|g_2\| = 1$ and define

$$\mathcal{D}_2 = \overline{\text{Lin}_C \{E(\sigma)g_2\}}$$

It is clear that $\mathcal{D}^\perp_1 \perp \mathcal{D}_2$. The total decomposition follows from Kuratowski-Zorn lemma. Because every $\mathcal{D}_l$ reduces $h$, $h = \bigoplus_{l=1}^\infty h_l$ where $h_l : D(h) \cap \mathcal{D}_l \rightarrow \mathcal{D}_l$. Moreover, each $h_l$ is $C|\mathcal{D}_l$-real.

Let us assume that we have proved that for every $n \in \mathbb{N}$ the set

$$\{ \sum_{k=1}^m e^{it_k h} f_n^k : m \in \mathbb{N}, t_k \in \mathbb{R}, f_n^k \in \mathcal{D}^+_n \}$$

is dense in $\mathcal{D}_n$. Then for any $\varepsilon > 0$ and for any $f \in \mathcal{D}$ we can find $n_0 \in \mathbb{N}$ such that

$$\|\bigoplus_{n=1}^{n_0} f_n - f\| < \varepsilon/3$$

and for these $f_n \in \mathcal{D}_n$ there are $f_n^k \in \mathcal{D}^+_n$ such that

$$\|f_n - \sum_{k=1}^m e^{it_k (n) h} f_n^k\| \leq \varepsilon/2^n$$

It follows that

$$\|f - \bigoplus_{n=1}^{n_0} \sum_{k=1}^m e^{it_k (n) h} f_n^k\| \leq \|f - \bigoplus_{n=1}^{n_0} f_n\| + \left(\sum_{n=1}^{n_0} \|f_n - \bigoplus_{n=1}^{n_0} \sum_{k=1}^m e^{it_k (n) h} f_n^k\|\right)^{1/2} \leq \varepsilon/3 + \left[\sum_{n=1}^{n_0} (\varepsilon^2/4^n)^{1/2}\right] < \varepsilon$$

Because every $f_n^k$ is an element of $\mathcal{D}_+$ and we can write

$$e^{it_k (n) h} f_n^k = e^{it_k (n) h} f_n^k$$

we obtain

$$\|f - \sum_{n=1}^{n_0} \sum_{k=1}^m e^{it_k (n) h} f_n^k\| < \varepsilon$$

Hence to finish the proof we need only the following Lemma.

**Lemma 3.2.** Suppose that $h$ is $C$-real and positive on $\mathcal{D}$. If $\text{Ker} h = \{0\}$ and $h$ is simple with a generating vector $g \in \mathcal{D}_+$, $\|g\| = 1$, then the family

$$\{ \sum e^{it_k h} f_k : n \in \mathbb{N}, t_k \in \mathbb{R}, f_k \in \mathcal{D}_+ \}$$
is dense in $\mathcal{D}$.

**Proof.** By the spectral theorem, $h$ is unitarily isomorphic to the multiplication operator $\hat{x}$ in $L^2(\mathbb{R}_+, \mathcal{B}, \rho)$, where $\rho(\sigma) = \langle g, E(\sigma)g \rangle$. It means that the unitary operator $U$ given by

$$Ua = \int_0^\infty a(\lambda) \, dE(\lambda)g$$

maps $D(\hat{x})$ onto $D(h)$ and $U^*hU = \hat{x}$. Because $C$ commutes with $E(\sigma)$ and $Cg = g$, we also have that $U(L^2_r) = D_+$ where

$$L^2_r = \{ a \in L^2(\mathbb{R}_+, \mathcal{B}, \rho) : \bar{a} = a \}$$

Thus for $a_k \in L^2_r$

$$U\left( \sum_{k=1}^n e^{it_k \hat{x}} a_k \right) = \sum_{k=1}^n e^{it_k U\hat{x} U^*} (Ua_k) = \sum_{k=1}^n e^{it_k h} f_k$$

where $f_k = Ua_k$. So it is enough to prove the statement for $\hat{x}$.

Let $K$ be an arbitrary compact set $K \subset (0, \infty)$. Let us consider the family of continuous, real valued functions on $K$ given by

$$\{ a : a = \sum_{k=1}^n \sin(t_k \hat{x})a_k, n \in \mathbb{N}, a_k \in C_r(K) \}$$

It is clear that this family is an algebra which separates points in $K$ and does not vanish identically on some point in $K$. By Stone-Weierstrass theorem this family is dense in $C_r(K)$. Now let $f \in C(K)$. Then $f = f_1 + if_2, f_1, f_2 \in C_r(K)$. At first we find a finite sum $\sum_{k=1}^n \sin(t_k \hat{x})a_k, a_k \in C_r(K)$ such that

$$||f_2 - \sum_{k=1}^n \sin(t_k \hat{x})a_k||_{\sup} < \varepsilon$$

Next put $t_0 = 0$, and $a_0 = f_1 - \sum_{k=1}^n \cos(t_k \hat{x})a_k$. Then

$$||f - \sum_{k=0}^n e^{it_k \hat{x}} a_k||_{\sup} = ||f_2 - \sum_{k=1}^n \sin(t_k \hat{x})a_k||_{\sup} < \varepsilon$$

Because $\rho$ is a probability measure on $\mathbb{R}_+$, we have also that

$$||f - \sum_{k=0}^n e^{it_k \hat{x}} a_k||_{L^2} < \varepsilon$$

But $\mathbb{R}_+$ is locally compact and $\rho$ is a Borel measure, so for every $\varphi \in L^2(\mathbb{R}_+, \mathcal{B}, \rho)$ there is a compact set $K \subset \mathbb{R}_+$ and a function $f \in C(K)$ such that $||\varphi - f||_{L^2} < \varepsilon$. Thus the proof of Lemma is finished.
**Theorem 3.5.** For quasi-free KMS state $\omega$ defined on the Weyl algebra $\mathfrak{W}(\mathcal{D}, \sigma)$ the canonical modular structure $(\pi_\omega(\mathfrak{W}(\mathcal{D}, \sigma))'', \Delta_\omega, J_\omega)$ is unitarily equivalent to the modular structure $(\mathfrak{M}_\xi, \Delta_\xi, J_\xi)$ constructed from the stochastic process $\xi^\beta$.

**Proof.** Let $w(f)$ denote the following elements of $L^\infty(Q, \Sigma_0, \mu)$

$$w(f) = e^{i\xi^\beta_0 \cdot f}$$

On a dense set of vectors in $\mathfrak{H}_\xi$ generated by elements

$$e^{it_1 H_\xi} \pi_0(w(f_1))e^{-it_1 H_\xi} \cdots e^{it_n H_\xi} \pi_0(w(f_n))e^{-it_n H_\xi} \Omega_\xi, \quad t_1, \ldots, t_n \in \mathbb{R}, f_1, \ldots, f_n \in \mathcal{D}_+$$

we define a map $V$ by

$$V \left( e^{it_1 H_\xi} \pi_0(w(f_1))e^{-it_1 H_\xi} \cdots e^{it_n H_\xi} \pi_0(w(f_n))e^{-it_n H_\xi} \Omega_\xi \right) = e^{it_1 H_\omega} \pi_\omega(W_{f_1})e^{-it_1 H_\omega} \cdots e^{it_n H_\omega} \pi_\omega(W_{f_n})e^{-it_n H_\omega} \Omega_\omega$$

Since the functions

$$\phi_\xi(t_1, \ldots, t_n) = \langle \Omega_\xi, e^{it_1 H_\xi} \pi_0(w(f_1))e^{-it_1 H_\xi} \cdots e^{it_n H_\xi} \pi_0(w(f_n))e^{-it_n H_\xi} \Omega_\xi \rangle$$

and

$$\phi_\omega(t_1, \ldots, t_n) = \langle \Omega_\omega, e^{it_1 H_\omega} \pi_\omega(W_{f_1})e^{-it_1 H_\omega} \cdots e^{it_n H_\omega} \pi_\omega(W_{f_n})e^{-it_n H_\omega} \Omega_\omega \rangle$$

can be analytically continued to $\mathfrak{H}_n^\beta$ and coincide in Euclidean points, they are equal for all $t_1, \ldots, t_n \in \mathbb{R}$. Thus $V$ is an isometry with dense domain in $\mathfrak{H}_\xi$. But

$$e^{it_1 H_\omega} \pi_\omega(W_{f_1})e^{-it_1 H_\omega} \cdots e^{it_n H_\omega} \pi_\omega(W_{f_n})e^{-it_n H_\omega} = \pi_\omega(W_{\sum_{k=1}^n e^{it_k h} f_k}) \prod_{1 \leq k < l \leq n} e^{-\frac{1}{2} \sigma(f_k, e^{it_1 - t_k} h f_1)}$$

hence the range of $V$ is dense in $\mathfrak{H}_\omega$ by Theorem 3.4 and continuity of $f \rightarrow \omega(W_f)$ with respect to the Hilbert space norm on $\mathcal{D}$. Moreover $V \Omega_\xi = \Omega_\omega$. By direct computation we show that

$$V \pi_0(w(f)) V^* = \pi_\omega(W_f), \quad f \in \mathcal{D}_+; \quad V e^{it H_\xi} V^* = e^{it H_\omega}$$

and

$$V \left( \prod_{k=1}^m e^{it_k H_\xi} \pi_0(w(f_k)) e^{-it_k H_\xi} \right) V^* = \prod_{k=1}^m e^{it_k H_\omega} \pi_\omega(W_{f_k}) e^{-it_k H_\omega}$$

Thus

$$V \mathfrak{M}_\xi V^* = \pi_\omega(\mathfrak{W}(\mathcal{D}, \sigma))''$$

and

$$V J_\xi V^* = J_\omega, \quad V \Delta_\xi V^* = \Delta_\omega$$
3.6 General case. If the operator $h$ is only bounded from below, that is

$$h \geq \mu I, \quad \mu < 0$$

then we can still obtain a quasi-free KMS state on the whole algebra $M(D, \sigma)$ if we replace $h$ by $\tilde{h} = h - (\mu - 1)I$ for example. Then the operator $B$ will be equal to

$$B = \frac{I + ze^{-\beta h}}{I - ze^{-\beta h}}$$

where $z = e^{\beta(\mu - 1)}$ and $B$ will be bounded. But at the same time the dynamics $\alpha_t$ have to be replaced by $\tilde{\alpha}_t$ given by

$$\tilde{\alpha}_t(W_f) = W_{e^{it \tilde{h}}} = W_{z^{-it/\beta}T_t f}$$

In some cases it is necessary to consider the operator $h$ itself. Then we have to restrict $M(D, \sigma)$ to a Weyl algebra over a suitable subspace of the Hilbert space $D$.

**Theorem 3.6.** Suppose that $h \geq 0$ and $\text{Ker } h = 0$. Let $B = (I + e^{-\beta h})^{-1}(I + e^{-\beta h})$ and

$$\mathcal{B}(f, g) = (B^{1/2} f, B^{1/2} g) \quad \text{with} \quad D(\mathcal{B}) = D(B^{1/2})$$

Let us define

$$\omega(W_f) = e^{-\frac{1}{2} \mathcal{B}(f,f)} \quad \text{for} \quad f \in D(\mathcal{B})$$

and extend it onto the Weyl algebra $M(D(\mathcal{B}), \sigma)$. Then $\omega$ is an $\alpha_t$-KMS state at the inverse temperature $\beta$ for the dynamics $\alpha_t$ given by $T_t = e^{it h}$.

**Proof.** First let us check that $\alpha_t$ maps $M(D(\mathcal{B}), \sigma)$ into itself. But since $T_t$ commutes with spectral projectors of $h$

$$T_t : D(B^{1/2}) \to D(B^{1/2})$$

hence $\alpha_t$ leaves $M(D(\mathcal{B}), \sigma)$ invariant. Now consider two point Green function $G^{(2)}(f, g; t)$ given by (3.2) for $f, g \in D(\mathcal{B})$. It is clear that this function is continuous and bounded. Hence to show the existence of analytic function interpolating between $G^{(2)}(f, g; t)$ and $G^{(2)}(g, f; -t)$ in the strip $\mathbb{R} \times i[0, \beta]$ it is enough to check that

$$\mathcal{F}(\mathbb{T}_F(f,g)) = e^{-\beta p} \mathcal{F}(\mathbb{T}_F(g,f))$$

For $f = g$ it can be done by direct calculations similarly as in the proof of Theorem 3.1. The general result follows from the polarization formula. It is clear that such analytic function exists for any

$$a_n = \sum_{k=1}^n c_k W_{f_k}, \quad b_n = \sum_{k=1}^n d_k W_{g_k}$$

in $M(D(\mathcal{B}), \sigma)$. Now for arbitrary $a, b \in M(D(\mathcal{B}), \sigma)$

$$G^{(2)}(a, b; t) = \lim_{n \to \infty} G^{(2)}(a_n, b_n; t)$$

and this function is continuous and bounded, so

$$\mathbb{T}_{G^{(2)}(a,b)} = \lim_{n \to \infty} \mathbb{T}_{G^{(2)}(a_n,b_n)}$$

in $\mathcal{S}'$. But since $\mathcal{F}$ is a homeomorphism of $\mathcal{S}'$

$$\mathcal{F}(\mathbb{T}_{G^{(2)}(a,b)})(-p) = e^{-p\beta} \mathcal{F}(\mathbb{T}_{G^{(2)}(b,a)})(p)$$

**Remark.** In the above case the state $\omega$ is not $D$-continuous in the Hilbert space topology.

Conversely, we show that among quasi-free gauge invariant states this is the only possibility.
Theorem 3.7. Suppose that $\mathbb{B}$ is positive, closed sesquilinear form on $D$ with domain $D(\mathbb{B})$. Let $\mathbf{h} \geq 0$ and $\text{Ker} \mathbf{h} = 0$. If $e^{it\mathbf{h}} : D(\mathbb{B}) \to D(\mathbb{B})$ and a state $\omega$ determined by $\mathbb{B}$ by formula (4.1) is an $\alpha_1$-KMS state at inverse temperature $\beta$, defined on the Weyl algebra $\mathfrak{W}(D(\mathbb{B}), \sigma)$, then

$$\mathbb{B}(f, g) = \langle B^{1/2}f, B^{1/2}g \rangle \quad \text{where} \quad B = \frac{1 + e^{-\beta \mathbf{h}}}{1 - e^{-\beta \mathbf{h}}}$$

Proof. Let $f \in D(\mathbb{B})$. Then

$$\int_0^\infty d\langle (B + I)f, E(\lambda)f \rangle \varphi(-\lambda) + \int_0^\infty d\langle (B - I)f, E(\lambda)f \rangle \varphi(\lambda) =$$

$$\int_0^\infty d\langle (B + I)f, E(\lambda)f \rangle e^{-\beta \lambda} \varphi(\lambda) + \int_0^\infty d\langle (B - I)f, E(\lambda)f \rangle e^{\beta \lambda} \varphi(-\lambda)$$

where $E(\lambda)$ are spectral projectors of $\mathbf{h}$. Suppose that $\varphi > 0$ for every $x \in \mathbb{R}$. Then

$$\langle (B - I)f, \varphi(h)f \rangle = \langle (B + I)f, e^{-\beta \mathbf{h}} \varphi(h)f \rangle$$

By polarization we get

$$\varphi(h)(B - I)f = \varphi(h)e^{-\beta \mathbf{h}}(B + I)f$$

But $\varphi(h)$ is injective, hence

$$(B - I)f = e^{-\beta \mathbf{h}}(B + I)f$$

So

$$(I - e^{-\beta \mathbf{h}})Bf = (I + e^{-\beta \mathbf{h}})f$$

It implies that $f \in D((I - e^{-\beta \mathbf{h}})^{-1}(I + e^{-\beta \mathbf{h}}))$ and

$$\frac{I + e^{-\beta \mathbf{h}}}{I - e^{-\beta \mathbf{h}}}f = Bf$$

Thus

$$B \subset \frac{I + e^{-\beta \mathbf{h}}}{I - e^{-\beta \mathbf{h}}}$$

Because both of them are self-adjoint, $D(B) = D((I + e^{-\beta \mathbf{h}})^{-1}(I - e^{-\beta \mathbf{h}}))$ and

$$B = \frac{I + e^{-\beta \mathbf{h}}}{I - e^{-\beta \mathbf{h}}}$$

Remark. The results of Sections 3.3 and 3.4 can also be extended to this case. In particular, the functions $S(f, g; s)$ restricted to $D(\mathbb{B})_+ = \{f \in D(\mathbb{B}) : Cf = f\}$ define periodic Gaussian OS-positive stochastic process indexed by $D(\mathbb{B})_+$, where $D(\mathbb{B})_+$ is the real Hilbert space with respect to the inner product

$$\langle f, g \rangle_B = \langle B^{1/2}f, B^{1/2}g \rangle$$

Similarly, since $e^{it\mathbf{h}}$ is unitary with respect to the inner product $\langle \cdot, \cdot \rangle_B$, Theorem 3.5 can be applied to this case. Moreover, the mapping

$$f \to \omega(W_f)$$

is continuous with respect to the norm $\| \cdot \|_B = \langle \cdot, \cdot \rangle_B^{1/2}$. Thus also in this case the process $\xi^B$ determines the modular structure associated with the state $\omega$. Similarly, one can show that $\xi^B$ has a version with two-sided Markov property on $K$. 

3.7 Ground state process. Now consider a quasi-free state $\omega_0$ which is a ground state with respect to the evolution $\alpha_t$ defined by $T_t = e^{i t h}$. Then, for every $f, g \in \mathcal{D}$ there exists a function $\Phi_0(f, g; z)$, analytic and bounded for $\text{Im} \, z > 0$, continuous on $\text{Im} \, z \geq 0$ and such that

$$\Phi_0(f, g; t) = F(f, g; t)$$

for all $t \in \mathbb{R}$. We know that for ground state case $\text{supp} \, F(\mathbb{T}_F) \subseteq [0, \infty)$. Similar arguments as used in the proof of Theorem 3.1 lead to the conclusion that for $\omega_0$,

$$F(f, g; t) = \langle f, e^{it h} g \rangle$$

and $h \geq 0$. By analytic continuation to the Euclidean region we obtain the functions

$$S_0(f, g; s) = \Phi_0(f, g; is)$$  \hspace{1cm} (3.13)

defined for $s \geq 0$. Two point Euclidean Green functions are now given by

$$G_0^{E, 2}(f, g; s) = e^{-\frac{i}{4}(\Phi(f, f) + \Phi(g, g) - \frac{1}{2} S_0(f, g; s))}$$

**Proposition 3.6.** Let $S_0(f, g; s)$ defined by (3.13) satisfy:

1. For every $f \in \mathcal{D}$ and $s \in [0, \infty)$, $S(f, f; s)$ is OS-positive.
2. More generally, for every terminating sequences $f_k \in \mathcal{D}, s_k \in [0, \infty), c_k \in \mathbb{C}$

$$\sum_{k, l} c_k c_l S_0(f_k, f_l; s_k + s_l) \geq 0$$

**Proof.** Since the function

$$t \to F(f, f; t) = \langle f, e^{it h} f \rangle$$

is positive definite, $S_0(f, f; s)$ is OS-positive as a Laplace transform of positive measure. The general positivity condition follows by polarization.

Similarly as in non zero temperature case, the existence of complex conjugation $C$ defining the abelian splitting $(\mathcal{D}_+, \mathcal{D}_-)$ of $\mathcal{D}$, commuting with $h$ is necessary for stochastic positivity of $S_0(f, g; s)$ restricted to $\mathcal{D}_+$. Adopting the arguments from Section 3.3 to the ground state case we can also prove the following Proposition.

**Proposition 3.7.** Let $h$ be $C$-real. Then functions $S_0(f, g; s)$ restricted to $\mathcal{D}_+$ satisfy:

1. $S_0(f, g; s) = S_0(g, f; s), \quad s \in [0, \infty)$
2. $S_0(f, g; s)$ can be extended to the function of $s \in \mathbb{R}$ (denoted by the same symbol) such that for all terminating sequences $f_1, \ldots, f_n \in \mathcal{D}_+; s_1, \ldots, s_n \in \mathbb{R}$ and $c_1, \ldots, c_n \in \mathbb{C}$

$$\sum_{k, l=1}^n \overline{c_k c_l} S_0(f_k, f_l; s_k - s_l) \geq 0$$

3. $S_0(f, g; s)$ defines an operator valued covariance function $R_\infty(s)$ of Gaussian OS-positive stochastic process indexed by $\mathcal{D}_+$.

Let $\xi^\infty_t$ be the Gaussian process indexed by $\mathcal{D}_+$ defined by

$$\mathbb{E}(< \xi^\infty_t, f >) = 0; \quad \mathbb{E}(< \xi^\infty_{t_1}, f_1 > < \xi^\infty_{t_2}, f_2 >) = \frac{1}{2} \langle f_1, R_\infty(t_2 - t_1) f_2 \rangle$$

If in addition $\text{Ker} \, h = \{0\}$, then as in the KMS state case, the process $\xi^\infty_t$ corresponding to the ground state, completely determines the ground state structure.
Theorem 3.8. Let $\xi_t^\infty$ be a Gaussian, OS-positive stochastic process indexed by $D_+ \times \mathbb{R}$. Then there exist a Hilbert space $\mathcal{H}_\xi^\infty$ with a unit vector $\Omega_\xi^\infty$, a weakly continuous one parameter group of unitary operators $U_{\xi}(t) = e^{itH_\xi^\infty}$ with $H_\xi^\infty \geq 0$, $\ker H_\xi^\infty = \{0\}$ and a von Neumann algebra $\mathcal{M}_\xi^\infty$ of operators acting on $\mathcal{H}_\xi^\infty$ such that $\Omega_\xi^\infty$ is cyclic for $\mathcal{M}_\xi^\infty$. Moreover, the canonical ground state structure $(\pi_\omega(\mathcal{M}(D,\sigma))'' , e^{itH_\omega^\infty}, \Omega_\omega^\infty)$ reconstructed from the state $\omega_0$ is unitarily equivalent to the ground state structure $(\mathcal{M}_\xi^\infty , e^{itH_\xi^\infty}, \Omega_\xi^\infty)$.

Remark. In the case $h > 0$, there is the following relation between covariance $R_\beta$ and $R_\infty$. If $\beta \to \infty$, then

$$R_\beta(s) \to R_\infty(s)$$

weakly. On the other hand

$$R_\beta(s) = \sum_{n \in \mathbb{Z}} R_\infty(s + n\beta)$$

where the series on the right hand side is weakly convergent.

IV Examples.

4.1 Ground states and KMS states for quantum fields on a stationary space-time. Let $(\mathcal{M}, g)$ be a stationary space-time i.e. $(\mathcal{M}, g)$ is space and time orientable with a global time-like Killing vector field $X$. Thus, $(\mathcal{M}, g)$ can be always realized as $(\mathbb{R} \times C, g)$ where $(C, \hat{g})$ is a Riemannian 3-manifold and

$$g = \begin{pmatrix}
a^2 - b^i b_i & -b_i \\
-b_i & -\hat{g}_{ij}
\end{pmatrix}$$

with a scalar field $a$ (laps field) and a vector field $b$ (shift field) satisfying

$$a > 0, \quad a^2 - \hat{g}(b, b) > 0$$

and with the Killing vector field

$$X := \frac{\partial}{\partial t} = aN(C) + b$$

where $N(C)$ is a unit future-pointing normal vector field to $C$. If $(\mathcal{M}, g)$ is globally hiperbolic, then $C$ can be chosen to be a global Cauchy surface. On $(\mathcal{M}, g)$ we consider the covariant Klein Gordon equation

$$(g^{\mu\nu} \nabla_\mu \nabla_\nu + m^2 + V) \varphi = 0$$

Given some Cauchy surface $C$, let

$$D(C) = C^\infty_0(C) + C^\infty_0(C)$$

be the space of real smooth Cauchy data of compact support. Then, by the Leray’s theorem [37], the Cauchy data $\Phi \in D(C)$ given by

$$\Phi = \begin{pmatrix} f \\ \end{pmatrix}$$
define a unique solution $\varphi$ of the Klein Gordon equation having compact support on every other Cauchy surface and such that

$$\varphi|_C = f, \quad N(C)\varphi|_C = p$$

Thus, we may view time evolution as a one-parameter group

$$T_t : D(C) \to D(C)$$

Moreover, $T_t$ preserves the symplectic form\(^\ast\)

$$\hat{\sigma}(\Phi_1, \Phi_2) = \int_C (f_1 p_2 - p_1 f_2) \, d\eta(C)$$

where $\eta(C)$ is the Riemannian volume element on $(C, \hat{g})$ and

$$\frac{d}{dt} T_t \Phi|_{t=0} = -\hat{h} \Phi$$

with $\hat{h} = -gA$ and

$$A = \begin{pmatrix} -\partial^i a \partial_i + a(m^2 - \Delta(C) + V) & -(\nabla_i b^i + b^i \partial_i) \\ b^i \partial_i & a \end{pmatrix}; \quad g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\Delta(C)$ is the Laplace-Beltrami operator on $(C, \hat{g})$ and $\nabla_i$ is the covariant derivative on $(C, \hat{g})$.

In order to apply the results of Section III we need a Hilbert space $(D, \langle \cdot, \cdot \rangle_D)$ containing $D(C)$, such that $\hat{\sigma}(\cdot, \cdot) = \text{Im} \langle \cdot, \cdot \rangle_D$ and $T_t = e^{it\hat{h}}$, where $\hat{h}$ is self-adjoint on $D$. The result of Kay [38] shows that such a one-particle Hilbert space structure exists under the following (mass gap) assumptions:

1. $V$ is stationary and $\inf V(x) + m^2 > 0$
2. $\inf a > 0$ on $C$.
3. $\inf (a - b^i b_i/a) > 0$ on $C$.

The above assumptions imply also that the generator $\hat{h}$ has a bounded inverse.

**Proposition 4.1.** Let $(D, e^{it\hat{h}})$ be the one-particle Hilbert space structure corresponding to the Klein Gordon equation on a globally hyperbolic stationary spacetime $(\mathcal{M}, g)$. There exists a complex conjugation $C$ on $D$ such that $\hat{h}$ is $C$-real.

**Proof.** To show the existence of complex conjugation $C$ we quote basic steps of the Kay’s construction. Since

$$\langle \Phi, A\Phi \rangle_{L^2+L^2} > \varepsilon \|\Phi\|_{L^2+L^2}^2$$

for some $\varepsilon > 0$, we define the $A$-norm

$$\|\Phi\|_A^2 = \langle \Phi, A\Phi \rangle_{L^2+L^2}$$

and $D_A$ as the completion of $D(C)$ in this norm. Then, it can be shown that $\hat{h}$ is essentially skew-adjoint and $(\hat{h})^{-1}$ exists and is bounded. Since $\hat{\sigma}$ is continuous in $A$-norm, it can be extended to $\hat{\sigma}'$ on $D_A$ and

$$\hat{\sigma}'(\Phi, \Psi) = \langle \Phi, (\hat{h})^{-1}\Psi \rangle$$
Let $D^C_A$ be the natural complexification of the real Hilbert space $D_A$. We change the inner product in $D^C_A$ by modifying the complex structure on $D^C_A$. To do this we introduce a unitary operator $I$ satisfying $I^2 = -1$ and $\tilde{\sigma}'(\Phi, I\Phi) > 0$. Since $i\hbar$ is self-adjoint on $D^C_A$ we can define

$$I = i(P_+ - P_-) = |\hbar|^{-1}i\hbar$$

where $P_+, P_-$ are projectors onto positive and negative parts of its spectrum. Now we define the new inner product by

$$\langle \Phi, \Psi \rangle_D = \tilde{\sigma}'(\Phi, I\Psi) + i \tilde{\sigma}'(\Phi, \Psi)$$

Let $D$ be the Hilbert space completion of $D^C_A$ with respect to $\langle , \rangle_D$. From the construction of $D$ we see that $T_t$ is the unitary group with respect to the inner product $\langle , \rangle$ and the generator $h$ of it is strictly positive. The complex conjugation $C$ in $D$ is defined as follows. Let $C_0$ be the complex conjugation in $D^C_A$ with respect to the complex structure given by multiplication by $i$. Then

$$C = C_0(P_+ - P_-)$$

is the complex conjugation in $D$ such that $h$ is $C$-real.

Applying the results of Section III we have:

**Corollary 4.1.**

1. Let $\omega$ be the KMS state at the inverse temperature $\beta$, defined on $\mathfrak{M}(D, \sigma)$ as in Theorem 3.1 and let $\xi^\beta_t$ be the corresponding Gaussian Markov thermal process indexed by $D_+$ (where $D_+$ is defined by the complex conjugation $C$). The modular structure defined by $\omega$ is unitarily equivalent to the modular structure reconstructed from $\xi^\beta_t$ as in Theorem 3.5.

2. Similarly, let $\omega_0$ be the ground state on $\mathfrak{M}(D, \sigma)$ and let $\xi^\infty_t$ be the corresponding ground state process. The ground state structure defined by $\omega_0$ is unitarily equivalent to the ground state structure reconstructed from the process $\xi^\infty_t$ as in Theorem 3.8.

In the case of a static space-time, we can realize $(\mathcal{M}, g)$ as above, but with the vector field $b = 0$. Then the matrix $A$ becomes diagonal

$$A = \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}$$

with

$$A = -(\partial^i a) \partial_i + a(m^2 - \Delta(C) + V$$

In this case the one-particle structure can be described in more explicit way. As was shown by Kay [38], $D$ can be identified with $L^2(\mathcal{C}, d\eta)$ and the space $D(\mathcal{C})$ of Cauchy data is mapped into $L^2(\mathcal{C}, d\eta)$ by

$$(f, p) \mapsto Yf + iY^* p$$
where

\[ Y = (a^{1/2}Aa^{1/2})^{-1/4}a^{1/2} \]

and \( \overline{\mathcal{B}} \) means closure of operator \( B \). The one-particle hamiltonian \( h \) is now given by

\[ h = (a^{1/2}Aa^{1/2})^{1/2} \]

and the complex conjugation \( C \) is the natural one in \( L^2(\mathcal{C},d\eta) \). Then \( \mathcal{D}_+ \) is defined as Hilbert space completion in \( L^2(\mathcal{C},d\eta) \) of the linear space generated by vectors

\[ Yf; \quad f \in C^\infty_0(\mathcal{C}) \]

and similarly, \( \mathcal{D}_- \) is the completion in \( L^2(\mathcal{C},d\eta) \) of linear space generated by

\[ Y^{-1}p; \quad p \in C^\infty_0(\mathcal{C}) \]

From the general results of Section III it follows that there exist thermal (respectively ground state) processes indexed by \( \mathcal{D}_+ \) and \( \mathcal{D}_- \). The former we call field thermal (respectively ground state) process and denote by \( \xi^\beta_t \) (respectively \( \xi^\infty_0 \)) and the later we call momentum thermal (respectively ground state) process and denote by \( \pi^\beta_t \) (respectively \( \pi^\infty_0 \)). It is enough to consider the covariances of these processes for the indexes of the form \( Yf \) or \( Y^{-1}p \) with \( f,p \in C^\infty_0(\mathcal{C}) \). Thus for thermal processes we obtain

\[
E(\langle \xi^\beta_t, f \rangle \langle \xi^\beta_0, g \rangle) = \int_{\mathcal{C} \times \mathcal{C}} (Yf)(x)R^\beta(t,s;x,y)(Yg)(y)d\eta(x)d\eta(y)
\]

where

\[
R^\beta(t,s;x,y) = \frac{e^{-|t-s|h} + e^{-(\beta-|t-s|)h}}{1 - e^{-\beta h}}(x,y)
\]

for \( |t-s| \leq \beta \). Similarly, for the ground state processes

\[
E(\langle \xi^\infty_t, f \rangle \langle \xi^\infty_0, g \rangle) = \int_{\mathcal{C} \times \mathcal{C}} (Yf)(x)R^\infty(t,s;x,y)(Yg)(y)d\eta(x)d\eta(y)
\]

where

\[
R^\infty(t,s;x,y) = e^{-|t-s|h}(x,y)
\]

Let \( dE_h(\lambda) \) be the spectral resolution of the operator \( h \) in the Hilbert space \( L^2(\mathcal{C},d\eta) \). Then

\[
E(\langle \xi^\beta_s, f \rangle \langle \xi^\beta_0, g \rangle) = \int_0^\infty \frac{e^{-t\lambda} + e^{-(\beta-t)\lambda}}{\lambda(1 - e^{-\beta\lambda})}d\langle \pi^{1/2} f, E_h(\lambda)\pi^{1/2} g \rangle
\]
Therefore, if \( f \) is such that \( \bar{\alpha}^{1/2} f \in L^2(\mathcal{C}, d\eta) \) then

\[
m(f) = \int_{0}^{\infty} \frac{e^{-t\lambda} + e^{-(\beta-t)\lambda}}{1 - e^{-\beta\lambda}} d\langle \bar{\alpha}^{1/2} f, E_{\lambda}(\lambda) \bar{\alpha}^{1/2} g \rangle < \text{Const}||\bar{\alpha}||_{L^2(\mathcal{C}, d\eta)}^2
\]

and we can apply Proposition 3.6 to conclude that for every such \( f \) there exists a version of the coordinate process \( <\xi^\beta_t, f> \) with Hölder continuous paths. Similar arguments work also for the ground state (field) process and momentum processes. Thus we obtain

**Corollary 4.2.** Let \((\mathcal{M}, g)\) be a static, globally hyperbolic space-time with the lapse field \( a \). Then

1. For any \( f \in C^\infty(0, R) \) such that \( \bar{\alpha}^{1/2} f \in L^2(\mathcal{C}, d\eta) \) and \( \beta \in (0, \infty) \) there exists an integrable random variable \( d(f, \gamma) \) such that

\[
|<\xi^\beta_t, f> - <\xi^\beta_s, f>| \leq d(f, \gamma)|t-s|^\gamma
\]

with probability one.

2. For any \( f \in C^\infty(0, R) \) such that \( \bar{\alpha}^{1/2} f \in H_{1/2}(\mathcal{C}, d\eta) \) where

\[
H_{1/2}(\mathcal{C}, d\eta) = \text{metric completion of } C^\infty_{0,R}(\mathcal{C}) \text{ with respect to (Sobolev -like) norm}
\]

\[
||f||^2_{1/2} = \int_{\mathcal{C} \times \mathcal{C}} (h^{1/2}f)(x)(h^{1/2}f)(y)d\eta(x)d\eta(y)
\]

the coordinate process \( \langle \pi^\beta_t, f \rangle \) for \( \beta \in (0, \infty) \) has a version with Hölder continuous paths similarly as in (1).

**Example 4.1.1**

Let \( \mathcal{M}_d \) be a flat \( d \)-dimensional Minkowski space-time with the metric tensor

\[
(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & -I_{d-1} \end{pmatrix}
\]

Taking

\[
\mathcal{C} = \{(x^0, x) \in \mathcal{M}_d : x^0 = 0\}
\]

and \( V = 0 \), we obtain the following covariances of thermal and ground state (field) processes:

\[
\mathbb{E}(<\xi^\beta_t, f> <\xi^\beta_s, g>) = \int_{\mathbb{R}^{d-1}} \frac{dp}{\sqrt{p^2 + m^2}} \hat{f}(p) \hat{g}(p) e^{-|t-s|\sqrt{p^2 + m^2} + e^{-(\beta-|t-s|)\sqrt{p^2 + m^2}}}
\]

for \( |t-s| \leq \beta \) and

\[
\mathbb{E}(<\xi^\infty_t, f> <\xi^\infty_s, g>) = \int_{\mathbb{R}^{d-1}} \frac{dp}{\sqrt{p^2 + m^2}} \hat{f}(p) \hat{g}(p) e^{-|t-s|\sqrt{p^2 + m^2}}
\]
where $f, g \in C_0^\infty(\mathbb{R}^{d-1})$. And similarly for momentum processes:

$$\mathbb{E}(\pi_t^\beta, f \rangle \langle \pi_s^\beta, g >) = \int_{\mathbb{R}^{d-1}} dp \sqrt{p^2 + m^2} \bar{f}(p) \bar{g}(p) \frac{e^{-|t-s|\sqrt{p^2 + m^2}} + e^{-(\beta-|t-s|)\sqrt{p^2 + m^2}}}{1 - e^{-\beta\sqrt{p^2 + m^2}}}$$

$$\mathbb{E}(\pi_t^\infty, f \rangle \langle \pi_s^\infty, g >) = \int_{\mathbb{R}^{d-1}} dp \sqrt{p^2 + m^2} \bar{f}(p) \bar{g}(p) e^{-|t-s|\sqrt{p^2 + m^2}}$$

The law of the process $\xi_t^\beta$ is given by the Gaussian random field $\mu_0^\beta$ indexed by $S(K_\beta \times \mathbb{R}^{d-1})$ and defined by

$$\mathbb{E}^{\mu_0^\beta}(\varphi, f >) = 0 \quad \text{and} \quad \mathbb{E}^{\mu_0^\beta}(\varphi, f \rangle \langle \psi, g >) = \langle f, (-\Delta_0^\beta + m^2)^{-1}g \rangle$$

where $-\Delta_0^\beta$ denotes $d$-dimensional Laplace operator with periodic boundary conditions in time direction ($-\Delta_0^\infty$ is defined as Friedrichs extension of $-\Delta$ in $L^2(\mathbb{R}^d)$).

The law of the momentum process $\pi_t^\beta$ is given by the Gaussian random field $\nu_0^\beta$ indexed by $S(K_\beta \times \mathbb{R}^{d-1})$ and defined by

$$\mathbb{E}^{\nu_0^\beta}(\psi, f >) = 0 \quad \text{and} \quad \mathbb{E}^{\nu_0^\beta}(\psi, f \rangle \langle \psi, g >) = \langle f, (-\Delta_{d-1}^\beta + m^2)(-\Delta_0^\beta + m^2)^{-1}g \rangle$$

In the present case we are able to analyse the properties of continuity of the processes in more details.

**Proposition 4.2.** Let $\rho \in L^2(\mathbb{R}^{d-1}) \cap C(\mathbb{R}^{d-1}), \alpha > (d-2)/2$. Then the field process $\xi_t^\beta$ has a version realized in the space

$$\bigcap_{\alpha > (d-2)/2} H_{-\alpha}^\rho(\mathbb{R}^{d-1})$$

where

$$H_{-\alpha}^\rho(\mathbb{R}^{d-1}) = \text{metric completion of } C_0^\infty(\mathbb{R}^{d-1}) \text{ in the norm}$$

$$||f||_{-\alpha}^\rho = \left[ \int \frac{|\hat{f}(p)|^2}{(p^2 + m^2)^\alpha} \rho^2(p) dp \right]^{1/2}$$

and such that for any $\alpha > (d-1)/2$:

1. if $\beta < \infty$, there exists a Borel measurable function $\Theta_\beta : S'(K_\beta \times \mathbb{R}^{d-1}) \to \mathbb{R}_+$ such that for all $t, s \in K_\beta$ such that $|t-s| < 1$

$$||\xi_t^\beta - \xi_s^\beta||_{-\alpha}^\rho \leq \Theta_\beta \left( \frac{|t-s|}{\ln(1/|t-s|)} \right)^{1/2}$$

2. if $\beta = \infty$, then for any $n \in \mathbb{N}$ there exists a Borel measurable function $\Theta_n : S'(\mathbb{R}^d) \to \mathbb{R}_+$ such that for any $t, s \in [-n, n]$ such that $|t-s| < e^{-1}$

$$||\xi_t^\infty - \xi_s^\infty||_{-\alpha}^\rho \leq \Theta_n \left( \frac{|t-s|}{\ln(1/|t-s|)} \right)^{1/2}$$
In the case of momentum process $\pi_t^\infty$, let $\rho$ be as above. Then the process $\pi_t^\infty$ is realized in the space

$$\bigcap_{\alpha > d/2} H_{-\alpha}^\rho(\mathbb{R}^{d-1})$$

Moreover, for any $\alpha > (d + 1)/2$, $\pi_t^\beta$ has a version with Hölder continuous (in $H_{-\alpha}^\rho(\mathbb{R}^{d-1})$) paths and obeys inequalities similar to (1) and (2).

**Sketch of the proof.** Let

$$C_0^\beta(x, y) = (-\Delta_d + m^2)^{-1}(x, y)$$

be the covariance of the Gaussian random field $\mu_0^\beta$. Let $\{\chi_\epsilon\}$ be the net of functions satisfying: $\chi_\epsilon \in C_0^\infty(\mathbb{R}^{d-1})$, $\chi_\epsilon > 0$, $\int_{\mathbb{R}^{d-1}} \chi_\epsilon(x) \, dx = 1$, i.e. $\chi_\epsilon \to \delta$ in $\mathcal{D}'(\mathbb{R}^{d-1})$. Define

$$C_{0, \epsilon}^\beta = C_0^\beta \ast (\chi_\epsilon \otimes \chi_\epsilon)$$

and let $\mu_{0, \epsilon}^\beta$ be the corresponding Gaussian measure on $(S'(K_{\beta} \times \mathbb{R}^{d-1}), \mathcal{B})$, where $\mathcal{B}$ is the Borel $\sigma$-algebra of sets in $S'(K_{\beta} \times \mathbb{R}^{d-1})$. For every $t > 0$ and $\mu_{0, \epsilon}^\beta$-almost every $\varphi \in S'(K_{\beta} \times \mathbb{R}^{d-1})$ we can define the map

$$x \to X^\epsilon_t(x) = \varphi(t, x)$$

Moreover, it can be shown that $X^\epsilon_t(x)$ takes values in the space $H_{-\alpha}^\rho(\mathbb{R}^{d-1})$ [39]. Suitable modification of $X^\epsilon_t$ gives rise to $L^2(S'(K_{\beta} \times \mathbb{R}^{d-1}), \mu_{0}^\beta|_{H_{-\alpha}^\rho(\mathbb{R}^{d-1})})$ stochastic process and its $L^2$ limit is easily seen to be the process $\xi^\beta_t$. Thus, there exists a version of $\xi^\beta_t$ (denoted by the same symbol) which is $L^2(S'(K_{\beta} \times \mathbb{R}^{d-1}), \mu_{0}^\beta|_{H_{-\alpha}^\rho(\mathbb{R}^{d-1})})$ stochastic process and which takes values in $H_{-\alpha}^\rho(\mathbb{R}^{d-1})$ for all $\alpha > (d - 2)/2$.

Notice that for $\alpha > (d - 1)/2$

$$\sup_{|t - s| \leq r} \left( \mathbb{E}(||\xi^\beta_t - \xi^\beta_s||_{-\alpha}^2) \right)^{1/2} \leq \text{Const} |t - s|^{1/2}$$

and

$$\lim_{t \to s} \mathbb{E}(||\xi^\beta_t - \xi^\beta_s||_{-\alpha}^2) = 0$$

Thus we can apply the continuity criterion due to Preston and Garsia [40, 41] as formulated in Theorem 5.1 in [42]. This leads to the Hölder continuity of $\xi^\beta_t$ in $H_{-\alpha}^\rho(\mathbb{R}^{d-1})$ for all $\alpha > (d - 1)/2$ and $\rho$ as above. Similarly we can prove the analogous properties of the momentum processes.

**Remark.** For more refined continuity properties and additional references to other continuity properties of the process $\xi_t^\infty$ we refer to [42]. Another aspects of the process $\xi_t^\beta$ are discussed in [43, 35]. It is interesting to note the essential difference between Markov properties of the field and momentum processes. For example, if $\beta = \infty$, then the law of $\xi_t^\infty$ fulfills so called sharp Markov property [6], while the law of $\pi_t^\infty$ satisfies another kind of Markov property discussed in [44].
Example 4.1.2

Let 

\[ W_R = \{ (T, X) \in M_2 : T^2 - X^2 < 0, X > 0 \} \]

be the right wedge in two-dimensional Minkowski space-time. In the hyperbolic coordinates \((\tau, x)\) defined by

\[ T = e^x \sinh \tau, \quad X = e^x \cosh \tau \]

the action of the Lorentz boost \(\Lambda(t)\) becomes

\[ \Lambda(t)(\tau, x) = (\tau + t, x) \]

Let 

\[ C = \{ (\tau, x) : \tau = 0 \} = \mathbb{R} \]

Then, on

\[ \hat{D}_C = \{ (f, \hat{p}) : f \in C_0^\infty(\mathcal{C}), \hat{p} = e^x p, p \in C_0^\infty(\mathcal{C}) \} \]

the symplectic form

\[ \hat{\sigma}((f, \hat{p}), (f', \hat{p}')) = \int_C (f \hat{p}' - \hat{p} f') \, dx \]

is invariant under the induced action of

\[ \hat{\Lambda}(t) = e^{-t\mathcal{L}}, \quad \text{where} \quad \mathcal{L} = \begin{pmatrix} 0 & -1 \\ A & 0 \end{pmatrix} \quad \text{and} \quad A = -\frac{\partial^2}{\partial x^2} + e^{2x} m^2 \]

Although in this case the mass gap conditions is not fulfilled, a one-particle Hilbert space can also be constructed [45]. \(D\) can be identified with \(L^2(\mathbb{R}, dx)\) and \(\hbar = A^{1/2}\). The space \(\hat{D}(\mathcal{C})\) of Cauchy data is mapped in \(L^2(\mathbb{R}, dx)\) by

\[ (f, \hat{p}) \rightarrow A^{-1/4} f + iA^{1/4} \hat{p} \]

As was shown in Section 3.6 we can define thermal state (for any \(\beta > 0\)) on the Weyl algebra over \(D(B^{1/2}) \subset L^2(\mathbb{R}, dx)\), where

\[ B = \frac{I + e^{-\beta \hbar}}{I - e^{-\beta \hbar}} \]

Using the natural complex conjugation on \(L^2(\mathbb{R}, dx)\) we obtain thermal processes indexed by \(D(B^{1/2})_\pm\). On a suitable domain in \(C_0^\infty(\mathbb{R})\) we can compute the covariance of the field process, denoted now by \(l^\beta_\tau\), for the indexes of the form \(A^{-1/4} f, f \in C_0^\infty(\mathbb{R})\), and we obtain

\[ \mathbb{E}(\langle l^\beta_\tau, f \rangle \langle l^\beta_\tau, g \rangle) = \langle f, \frac{e^{-|\tau - \tau'|A^{1/2}} + e^{-(\beta - |\tau - \tau'|)A^{1/2}}}{A^{1/2}(A^{1/2} - A^{1/2}/\beta)} g \rangle L^2(\mathbb{R}) \]
Let $\varphi$ be a free scalar massive quantum field on $\mathcal{M}_2$. It is well known that for $f$ real, the field operator $\varphi(f)$ is essentially self-adjoint on $D\varphi$ defined by

$$D\varphi = \mathrm{lh}\{\varphi(f_1) \cdots \varphi(f_n)\Omega_0 : f_1, \ldots, f_n \in C_0^\infty(\mathcal{M}_2), n \in \mathbb{N}\}$$

where $\Omega_0$ is the Fock vacuum. Let for any open $G \subset \mathcal{M}_2$, let $\mathcal{R}_G$ be the von Neumann algebra generated by

$$e(f) = e^{i\varphi(f)} \quad \text{with} \quad \text{supp} \ f \subset G$$

If $G \subset \mathcal{M}_2$ is such that its causal complement is non empty, then by the Reeh-Schlieder theorem, $\Omega_0$ is cyclic and separating for $\mathcal{R}_G$. In particular, we can put $G = W_R$ and using the Bisognano-Wichmann theorem [31] we conclude that $(\mathcal{R}_{W_R}, \Omega_0, \Lambda(\tau))$ forms $W^*$-KMS system at $\beta = 2\pi$. On the other hand, for any $f \in C_0^\infty(\mathbb{R})$, $t \in \mathbb{R}$, the operator $\varphi(\delta_t \otimes f)$ is also essentially self-adjoint on $D\varphi$. Thus we can define the abelian von Neumann algebra $\mathfrak{A}_R$ generated by

$$e^{i\varphi(\delta_t \otimes f)} \quad \text{with} \quad f = \overline{f} \quad \text{supp} \ f \subset \{(x, 0) : x > 0\}$$

By the direct computation we also obtain

$$\langle \Omega_0, e^{i\varphi(\delta_0 \otimes f)} \Lambda(\tau) e^{i\varphi(\delta_0 \otimes g)} \Omega_0 \rangle = \mathbb{E}(e^{i<\ell_0^{2\pi}f>} e^{i<\ell_0^{2\pi}g>})$$

for $\tau \in [0, 2\pi], \text{supp} \ f, \text{supp} \ g \subset \{(x, 0) : x > 0\}$. Moreover, it is easy to show that the modular structure reconstructed from the thermal (boost) process $\ell_0^{2\pi}$ is unitarily equivalent to $(\mathcal{R}_{W_R}, \Omega_0, \Lambda(\tau))$.

**Remark.** Similar construction of the thermal process can be done in the case of exterior Schwarzschild right wedge region in the Kruskal space-time $\mathcal{M} \simeq \mathbb{R}^2 \times \Omega$ where $\Omega$ is some $d-2$-dimensional Riemann manifold with the metric $dt_\Omega^2$. The metric on $\mathcal{M}$ is given by

$$ds^2 = 32M^3r^{-1}e^{-r/(2M)}(dT^2 - dX^2) - r^2dt_\Omega^2$$

where $M$ is the mass of the black hole and $r$ is the Schwarzschild radius defined by

$$T^2 - X^2 = (1 - r/(2M))e^{r/(2M)}$$

The exterior Schwarzschild right wedge is defined as

$$\mathcal{R} = \{(T, X, \zeta) : X > |T|, r > 2M, \zeta \in \Omega\}$$

The corresponding thermal process has the covariance defined as above, but in terms of the operator $A$ given by

$$A = -\frac{\partial^2}{\partial x^2} + (1 - 2M/r)(2M/r^3 - \Delta_\Omega/r^2 + m^2)$$

where $\Delta_\Omega$ is the Laplace operator on $\Omega$. Detailed analysis of this process together with the construction corresponding to the de Sitter case will be discussed in a separate paper.
4.2 Nonrelativistic Bose matter. Let $\mathcal{D} = L^2(\mathbb{R}^d)$ and $T_t = e^{it\mathbf{h}}$ where $\mathbf{h} \geq 0$, $\text{Ker} \mathbf{h} = \{0\}$, commutes with the natural complex conjugation on $L^2(\mathbb{R}^d)$. Then there exist thermal (and ground state) processes indexed by a suitable subspaces of $L^2_h(\mathbb{R}^d)$. In particular, then the covariance operator of the thermal process $\xi^\beta$ is given by

$$R^\beta(\tau) = \frac{e^{-|\tau|\mathbf{h}} + e^{-(\beta-|\tau|)\mathbf{h}}}{1 - e^{-\beta \mathbf{h}}}$$

for $|\tau| \leq \beta$. As in the previous examples the modular structure defined by the canonical KMS state at the inverse temperature $\beta$ coincides with the modular structure obtained from the thermal process $\xi^\beta$. If additionally $\inf \sigma(\mathbf{h}) > 0$, it is easy to see that for any $f \in L^2_h(\mathbb{R}^d)$ the assumption of Proposition 3.4 is satisfied, hence each coordinate process $<\xi^\beta, f>$ has a version with H"older continuous paths.

The case when the operator $\mathbf{h}$ (called a kinetic energy) is of the form $\mathbf{h} = \mathcal{E}(-i\nabla)$ where $\mathcal{E}$ is some locally bounded, measurable function satisfying $\mathcal{E}(p) \geq 0$, is of particular interest to physics. The choice $\mathcal{E}(p) = p^2 + \mu; \mu > 0$ corresponds to the standard kinetic energy [24], whereas $\mathcal{E}(p) = \sqrt{p^2 + m^2}$ corresponds to the semirelativistic kinetic energy. For a given $\mathcal{E} \in B_0(\mathbb{R}^d)$ where

$$B_0(\mathbb{R}^d) = \{\text{the set of locally bounded, measurable functions on } \mathbb{R}^d \text{ satisfying } \inf \mathcal{E}(p) = \epsilon > 0\}$$

we denote

$$\hat{C}^\beta(p) = \frac{1 + e^{-\beta \mathcal{E}(p)}}{1 - e^{-\beta \mathcal{E}(p)}}, \quad \hat{S}^\beta(\tau, p) = \frac{e^{-|\tau|\mathcal{E}(p)} + e^{-(\beta-|\tau|)\mathcal{E}(p)}}{1 - e^{-\beta \mathcal{E}(p)}}$$

Since

$$\hat{C}^\beta(p) = 1 + \frac{2e^{-\beta \mathcal{E}(p)}}{1 - e^{-\beta \mathcal{E}(p)}}$$

the corresponding expression in $x$-space has a $\delta$ singularity at $x = 0$. If we assume that $\mathcal{E} \in C(\mathbb{R}^d)$ and $\mathcal{E}(p) \sim |p|^\eta$ for $|p| \to \infty, \eta > 0$ we obtain

$$C^\beta(x) = \delta(x) + \mathcal{R}^\beta(x)$$

where $\mathcal{R}^\beta$ is continuous and

$$\mathcal{R}^\beta \in \bigcap_{p \geq 1} L^p(\mathbb{R}^d)$$

The kernel $\hat{S}^\beta(\tau, p)$ has the similar properties.

Remark. The technical difficulties arise only in the case when the set

$$\mathcal{N}_\mathcal{E} = \{p \in \mathbb{R}^d : \mathcal{E}(p) = 0\}$$

is non-empty. In that case, we have to restric the index space of the process $\xi^\beta$. For example, if $\mathcal{E}$ is continuous and

$$\mathcal{E}(p) \sim |p|^\eta \quad \text{as} \quad |p| \to 0$$
we can take as the index space \( C_\eta^\infty(\mathbb{R}^d) \) defined as follows

\[
C_\eta^\infty(\mathbb{R}^d) = \{ f \in S(\mathbb{R}^d) : \frac{\partial^{|i|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} f|_{x=0} = 0, \text{ for all } |i| \leq \eta^* \}
\]

where \( \eta^* \geq d - 1 - \nu \).

Using the similar arguments as in the proof of Proposition 4.2, we obtain the following:

**Proposition 4.3.**

(1) Let \( \mathcal{E} \in \mathcal{B}_0(\mathbb{R}^d) \). Then for any \( \alpha > d/2 \), there exists a version of the process \( \xi_\tau^\beta \) with values in the space

\[
H_{-\alpha}(\mathbb{R}^d) = \{ f \in S'(\mathbb{R}^d) : \int_{\mathbb{R}^d} \frac{|\hat{f}(p)|^2}{(p^2 + 1)^\alpha} dp < \infty \}
\]

If in addition \( \mathcal{E}(p) \sim |p|^\eta \) for \( |p| \to \infty \), then for any \( \alpha > (d + \eta)/2 \) there exists a version of \( \xi_\tau^\beta \) such that with probability one

\[
||\xi_\tau^\beta - \xi_{\tau'}^\beta||_{2,\alpha} \leq \Theta_\beta \left( \frac{|\tau - \tau'|}{\ln(1/|\tau - \tau'|)} \right)^{1/2}
\]

for sufficiently small \( |\tau - \tau'| \) and some integrable random variable \( \Theta_\beta \).

(2) Let \( \mathcal{E} \) be such that \( \mathcal{E}(p) \sim |p|^\eta \) as \( |p| \to +0 \) and \( \mathcal{E}(p) \sim |p|^\nu \) as \( |p| \to \infty \). Let

\[
H_{-\alpha,\eta}(\mathbb{R}^d) = \{ \text{metric completion of } C_0^\infty(\mathbb{R}^d) \text{ with the norm} \}
\]

\[
||f||_{2,\alpha,\eta}^2 = \int_{\mathbb{R}^d} \frac{|p|^\eta|f(p)|^2}{(p^2 + 1)^\alpha} dp
\]

Then for any \( \alpha > (d + \eta)/2 \) there exists a version of the process \( \xi_\tau^\beta \) with values in \( H_{-\alpha,\eta}(\mathbb{R}^d) \) and for any \( \alpha > (d + \eta + \nu)/2 \) there exists a version of \( \xi_\tau^\beta \) with Hölder continuous paths as in (1).

**Remark.** The case of critical (standard) Bose gas defined by the thermal state \( \omega_{cr} \) on the Weyl algebra over \( S(\mathbb{R}^d) \) given by

\[
\omega_{cr}(W_f) = \exp c|\hat{f}(0)|^2 \exp \left[ -\frac{1}{4} \int_{\mathbb{R}^d} \frac{1}{1 - e^{-\beta p^2}} |\hat{f}(p)|^2 dp \right]
\]

where \( c > 0 \) is a constant and \( d \geq 3 \), was discussed in [25]. Here we quote only some continuity results.

Let \( d \geq 3 \) and \( \xi_\tau^{\beta,cr} \) be the thermal process indexed by \( S(\mathbb{R}^d) \) with the covariance given by the density

\[
S^{\beta,cr}(\tau, p) = c\delta(p) + \frac{e^{-\beta|\tau|^2} + e^{-(\beta - |\tau|)p^2}}{1 - e^{-\beta p^2}}
\]

For any \( \alpha > d/2 \) there exists a version of \( \xi_\tau^{\beta,cr} \) with values in \( H_{-\alpha}(\mathbb{R}^d) \) and for any \( \alpha > (d + 1)/2 \) there exists a version of \( \xi_\tau^{\beta,cr} \) with Hölder continuous paths.
4.3 Quantum lattice models. Let \( \mathbb{L} \) be a countable set with elements denoted by \( j \). Assume that for any \( j \in \mathbb{L} \) there is a separable Hilbert space \( \mathcal{D}_j \) with an unitary group \( T_{j,t} = e^{it\mathbf{h}_j} \), where \( \mathbf{h}_j \) is self-adjoint and \( \sigma(\mathbf{h}_j) \subset [\epsilon, \infty), \epsilon > 0 \). Assume also that there is a complex conjugation \( \mathbf{C}_j \) on \( \mathcal{D}_j \) such that \( \mathbf{h}_j \) is \( \mathbf{C}_j \)-real. Let us define

\[
\mathcal{D} = \bigoplus_{j \in \mathbb{L}} \mathcal{D}_j, \quad T_t = \bigoplus_{j \in \mathbb{L}} T_{j,t}, \quad C = \bigoplus_{j \in \mathbb{L}} C_j
\]

On \( \mathcal{D} \) we define a symplectic form

\[
\sigma(\varphi, \psi) = \sum_{j \in \mathbb{L}} \text{Im} \langle \varphi_j, \psi_j \rangle_{\mathcal{D}_j}
\]

where \( \varphi = (\varphi_j)_{j \in \mathbb{L}} \in \mathcal{D} \). Consider a state \( \omega \) defined on the Weyl algebra \( \mathfrak{W}(\mathcal{D}, \sigma) \) by

\[
\omega(W_\varphi) = \prod_{j \in \mathbb{L}} \exp \left[ -\frac{1}{4} \langle \varphi_j, \text{coth} \frac{\beta}{2} \mathbf{h}_j \varphi_j \rangle_{\mathcal{D}_j} \right]
\]

(4.1)

The state (4.1) is \( \alpha_t \)-KMS state (for \( \alpha_t \) defined by \( T_t \)) at the inverse temperature \( \beta \). The corresponding thermal process indexed by \( \mathcal{D}_+ = \bigoplus_{j \in \mathbb{L}} \mathcal{D}_{+,j} \) will be denoted by \( \xi_\tau^{\beta,\mathbb{L}} \). The process \( \xi_\tau^{\beta,\mathbb{L}} \) is a direct sum of the site processes \( \xi_\tau^{\beta,j} \), indexed by \( \mathcal{D}_{+,j} \) and its covariance is given by

\[
\mathbb{E} \langle \xi_\tau^{\beta,j}, \varphi \rangle \langle \xi_\tau^{\beta,j}, \psi \rangle = \sum_{j \in \mathbb{L}} \mathbb{E} \langle \xi_\tau^{\beta,j}, \varphi_j \rangle \langle \xi_\tau^{\beta,j}, \psi_j \rangle
\]

(4.2)

From Proposition 3.4 and equation (4.2) follows that for \( \varphi = (\varphi_j)_{j \in \mathbb{L}} \), such that

\[
m(\varphi) = \sum_{j \in \mathbb{L}} m_j(\varphi_j) < \infty
\]

where \( m_j(\varphi_j) \) is the moment of the measure corresponding to the covariance of \( \xi_\tau^{\beta,j} \) (see Proposition 3.4), there exists a Hölder continuous version of the coordinate process

\[
\langle \xi_\tau^{\beta,\mathbb{L}}, \varphi \rangle = \sum_{j \in \mathbb{L}} \langle \xi_\tau^{\beta,j}, \varphi_j \rangle.
\]

Let \( A \) be a self-adjoint, non-negative operator on \( \mathcal{D} \) such that \( \bigoplus_{j \in \mathbb{L}} \mathbf{h}_j + A \) is essentially self-adjoint on \( D(\bigoplus_{j \in \mathbb{L}} \mathbf{h}_j) \). Define \( \mathbf{h}_A = \bigoplus_{j \in \mathbb{L}} \mathbf{h}_j + A \). The state

\[
\omega_A(W_\varphi) = \exp \left[ -\frac{1}{4} \langle \varphi, \frac{1}{\Pi + e^{-\beta A}} \frac{1}{\Pi - e^{-\beta A}} \varphi \rangle \right]
\]

is a KMS state with respect to the evolution defined by \( T_{A} = e^{i\mathbf{h}_A} \). If \( \mathbf{h}_A \) is \( C \)-real, there exists a thermal process \( \xi_\tau^{\beta,A} \) indexed by \( \mathcal{D}_+ = \bigoplus_{j \in \mathbb{L}} \mathcal{D}_{+,j} \), with the covariance

\[
\mathbb{E} \langle \xi_\tau^{\beta,A}, \varphi \rangle \langle \xi_\tau^{\beta,A}, \psi \rangle = \langle \varphi, \frac{e^{-|\tau - \tau'| \mathbf{h}_A} + e^{-(\beta - |\tau - \tau'|) \mathbf{h}_A}}{\Pi - e^{-\beta A}} \psi \rangle
\]

The law of this process is given by a centered Gaussian random field \( \mu_\xi^{\beta} \) with covariance

\[
\mathbb{E}^{\mu_\xi^{\beta}} \langle \varphi, \varphi \otimes g \rangle = \langle \varphi, \psi, g \rangle (\frac{d}{dt^2})_{\text{per}} + \mathbf{h}_A
\]

where \( (-\frac{d^2}{dt^2})_{\text{per}} \) is a periodized version of the operator \( -\frac{d^2}{dt^2} \). Similar conclusion is true for the ground state process.
4.3.1 Harmonic crystal model.

Let \( \mathbb{L} = \mathbb{Z}^d \) be a lattice consisting of points in \( \mathbb{R}^d \) with all coordinates integer. To each \( j \in \mathbb{Z}^d \) we associate a copy of the complex plane \( \mathbb{C} \) with a natural complex conjugation \( z \rightarrow \overline{z} \) and a copy of an unitary group \( u_j(t) = e^{\frac{i}{2}t} \). By the direct sum construction we obtain the Hilbert space \( \mathcal{D} = l^2(\mathbb{Z}^d) \) with the complex conjugation \( C(z)_j \in \mathbb{Z}^d = (\overline{z})_j \in \mathbb{Z}^d \) and the unitary group \( T_t = \bigoplus_{j \in \mathbb{Z}^d} u_j(t) \). Defining the KMS state by (4.1) and proceeding as above, we obtain a thermal process \( \xi_{\beta,\mathbb{Z}^d} \) which is a direct sum of the periodized Ornstein-Uhlenbeck processes \( \xi_{\tau,j} \) with covariances

\[
\mathbb{E} \xi_{\tau,j} \xi_{0,j'} = \frac{1}{2} \frac{e^{-|\tau|/2} + e^{-(\beta-|\tau|)/2}}{1 - e^{-\beta/2}} \delta_{jj'}
\]

where \( |\tau| \leq \beta \).

Let now \( A \) be a self-adjoint and non-negative operator on \( l^2(\mathbb{Z}^d) \) with real matrix coefficients. Since \( A \) is \( \mathbb{C} \)-real, \( h_A = \frac{1}{2} \mathbb{I} + A \) is also \( \mathbb{C} \)-real, and we obtain a thermal process \( \xi_{\tau,A} \) indexed by \( l_0^2(\mathbb{Z}^d) \). Its covariance is given by

\[
\mathbb{E} \langle \xi_{\tau,A}^\beta, (z_j) \rangle \langle \xi_{0,A}^\beta, (z'_j) \rangle = \langle (z_j), \frac{e^{-|\tau| h_A} + e^{-(\beta-|\tau|) h_A}}{\mathbb{I} - e^{-\beta h_A}} (z'_j) \rangle_{l^2(\mathbb{Z}^d)}
\]

In the Gibbs state approach discussed in [46, 47, 48, 49], the process \( \xi_{\tau,A}^\beta \) can be obtained as a perturbation of the process \( \xi_{\tau,\mathbb{Z}^d}^\beta \). The perturbation is defined as follows. Let

\[
d\mu_0^\beta = \bigotimes_{j \in \mathbb{Z}^d} d\mu_0^\beta,j
\]

where \( d\mu_0^\beta,j \) is the measure on trajectories of the periodic Ornstein-Uhlenbeck process \( \xi_{\tau,j}^\beta \). Then

\[
d\mu_{\tau,A}^\beta((\omega_j)_{j \in \mathbb{Z}^d}) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{Z_\Lambda} \exp \left[ - \int_{-\beta/2}^{\beta/2} \sum_{j,k \in \Lambda} A_{jk} \omega_j(\tau) \omega_k(\tau) d\tau \right] \cdot d\mu_0^\beta((\omega_j)_{j \in \mathbb{Z}^d})
\]

\[
Z_\Lambda = \int d\mu_0^\beta((\omega_j)_{j \in \mathbb{Z}^d}) \exp \left[ - \int_{-\beta/2}^{\beta/2} \sum_{j,k \in \Lambda} A_{jk} \omega_j(\tau) \omega_k(\tau) d\tau \right]
\]

where \( \Lambda \) is a finite subset of \( \mathbb{Z}^d \) and the limit is taken in the sense of DLR equations ([50, 51]). Clearly it defines a measure on trajectories of the process \( \xi_{\tau,A}^\beta \). In fact, the set of limiting Gibbs measures can contain also some non-translation invariant solutions [52], which differ from choosen above only by non-zero means. This is connected with the nontrivial kernel of the operator \( \frac{d^2}{dt^2} + h_A \). Using this and general results of Section 3.5, we obtain the following

**Proposition 4.4.** *The modular structure of the KMS state corresponding to the harmonic crystal model with the Fock space Hamiltonian \( d\Gamma(h_A) \) formally given by*

\[
d\Gamma(h_A) = \sum \left( -\frac{1}{2} \frac{d^2}{dx_j^2} + \frac{1}{2} x_j^2 \right) + \sum A_{jk} x_j x_k
\]
where \((A_{jk}) = A \geq 0\) is determined by the thermal process \(\xi_t^{\beta,A}\) with the covariance (4.3).

**Remark.** The result that the modular structure of the harmonic crystal model is stochastically determined, seems to be new. For another aspects of this model, we refer to [53]. Gibbsian perturbations of the abelian sector were discussed recently in [48, 49].

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