UPPER TAIL LARGE DEVIATIONS OF THE CYCLE COUNTS IN ERDŐS-RÉNYI GRAPHS IN THE FULL LOCALIZED REGIME

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Abstract. For a $\Delta$-regular graph $H$ the problem of determining the upper tail large deviation for the number of copies of $H$ in $G(n, p)$, an Erdős-Rényi graph on $n$ vertices with edge probability $p$, has generated significant interests. For $p = o(1)$ and $np^{\Delta/2} \gg (\log n)^{1/(v_H - 2)}$, where $v_H$ is the number of vertices in $H$, the upper tail large deviation event is believed to occur due to the presence of localized structures. In this regime the large deviation of the event that the number of copies of $H$ in $G(n, p)$ exceeds its expectation by a constant factor is predicted to hold at a speed $n^2 p^{\Delta} \log(1/p)$ and the rate function is conjectured to be given by the solution of a mean-field variational problem. After a series of developments in recent years covering progressively broader ranges of $p$, when $H$ is a clique of a fixed size this conjecture was recently settled by Harel, Mousset, and Samotij [12] for the entire localized regime. In this paper we resolve the same for cycles of any given length.

1. Introduction

Let $G_n$ be a random graph on $n$ vertices and $H$ be a fixed graph. In recent years the study of large deviations for the number of copies of $H$ in $G_n$ has received a paramount interest. Consider the simplest non-trivial set up: $G_n = G(n, p)$ is the Erdős-Rényi graph on $n$ vertices with edge connectivity probability $p = p(n) \in (0, 1)$, and $H = K_3$ is a triangle. After a series of works [1, 6, 8, 11, 18], the large deviations bounds for the upper tail of triangle counts in $G(n, p)$ for all $p \ll 1$ satisfying $np \gg \log n$ is established by Harel, Mousset, and Samotij [12]. In this regime the large deviation event is due to the presence of localized structures in $G(n, p)$. Whereas, in the complement regime, i.e. $1 \ll np \ll \log n$, as shown in [12], the large deviation is given by the large deviation of a Poisson random variable with appropriate mean. Thus these two regimes can be termed as the localized regime and the Poisson regime, respectively.

Moving to more general subgraph counts it was established in the series of works mentioned above that for any $\Delta$-regular graph $H$, the upper tail large deviation occurs due to the presence of localized structures for $1 \gg p \gg n^{-\alpha_H}$ for successively improved values of $\alpha_H \in (0, 1)$.

On the other hand, it was shown in [12] that for any $\Delta$-regular graph $H$ the Poisson regime is characterized by the threshold $1 \ll np^{\Delta/2} \ll (\log n)^{1/(v_H - 2)}$ and the exponent $1/(v_H - 2)$ is optimal (see [12, Section 8] for a discussion on the optimality), where $v_H$ is the number of vertices in $H$. It naturally leads to the conjecture that for any $\Delta$-regular connected graph for the entire regime $(\log n)^{1/(v_H - 2)} \ll np^{\Delta/2} \ll n$ the large deviation for the upper tail of the number of copies of $H$ in $G(n, p)$ is due to the presence of localized structures, where the speed is predicted to be $n^2 p^{\Delta} \log(1/p)$ with the rate function to be given by a mean-field variational problem as in [6].

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\textsuperscript{1}For two sequences of positive reals $\{a_n\}$ and $\{b_n\}$ we write $b_n = o(a_n)$, $a_n \gg b_n$, and $b_n \ll a_n$ to denote $\limsup_{n \to \infty} b_n/a_n = 0$. 

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When $H$ is a clique this conjecture was proved in [12]. For general regular graphs the best known result in this direction is again due to [12], where the authors derived the upper tail large deviations for $np^{\Delta/2} \gg (\log n)^{\Delta^2/2}$ for all $\Delta$-regular non-bipartite graphs $H$, and for $p^{\Delta/2} \geq n^{-1/2-o(1)}$, where the $o(1)$ term decays to zero at any arbitrary rate as $n \to \infty$, when $H$ is a $\Delta$-regular bipartite graph.

The goal of this paper is to establish this conjecture for cycles of any fixed length. In particular, for even cycles $C_{2\ell}$ (henceforth for any $\ell \geq 3$ we write $C_\ell$ to denote the cycle graph with $\ell$ vertices) we improve the lower bound of $p \geq n^{-1/2-o(1)}$ in [12] to $p \gg n^{-1}(\log n)^{1/(2\ell-2)}$ and for odd cycles $C_{2\ell+1}$ we improve the result of [12] to obtain the correct power of $\log n$.

To state our main result we need to introduce some notation. For any graph $G$ we write $V(G)$ and $E(G)$ to denote its vertex and edge sets, respectively. Next we define the notion of labelled copies of a given graph in another graph.

**Definition 1.1.** Given graphs $G$ and $H$ we write $N(H,G)$ to denote the number of labelled copies of $H$ in $G$. That is,

$$N(H,G) := \sum_{\varphi : V(H) \to V(G)} \prod_{(x,y) \in E(H)} a_{\varphi(x),\varphi(y)}^G,$$

where the sum is over all injective maps $\varphi$ from $V(H)$ to $V(G)$, and $(a_{i,j}^G)_{i,j \in V(G)}$ is the adjacency matrix of the graph $G$.

Next for any graph $H$ we denote its independence polynomial by $P_H(\cdot)$. That is,

$$P_H(x) := \sum_{k \geq 0} i_H(k)x^k,$$

where $i_H(k)$ is the number of $k$-element independent subsets of $H$ and set $i_H(0) := 1$. We further denote $\theta_H$ to be unique positive solution to $P_H(\theta) = 1 + \delta$. When $H = C_\ell$ for some $\ell \geq 3$, for brevity we write $\theta_\ell$ instead of $\theta_{C_\ell}$.

Let us also note that for any graph $H$ we have $\mathbb{E}[N(H,G)] = (1 + o(1))n^{v_H}e(G)$, where $v_H$ and $e(G)$ denote the number of vertices and edges of $H$, respectively. Thus for any graph $H$ and $\delta > 0$ the upper tail event can be written as

$$\text{UT}(H, \delta) := \left\{ N(H,G(n,p)) \geq (1 + \delta)n^{v_H}e(G) \right\}.$$

We now state the main result of this paper.

**Theorem 1.2.** For any $\ell \geq 3$ and $\delta > 0$,

$$\lim_{n \to \infty} \frac{\log \mathbb{P}(\text{UT}(C_\ell, \delta))}{n^2p^2 \log(1/p)} = \begin{cases} \min\{\theta_\ell, \frac{1}{2}\delta^2\} & \text{if } n^{1/2} \ll np \ll n, \\ \frac{1}{2}\delta^2 & \text{if } (\log n)^{1/(\ell-2)} \ll np \ll n^{1/2}. \end{cases}$$

We point to the reader that the upper tail large deviations of unlabelled copies of $H$ in $G(n,p)$ have been considered in [12]. As the number of labelled and unlabelled copies only differ from each other by a factor $|\text{Aut}(H)|$, the number of automorphisms of $H$, the large deviation speed as well as the rate function are identical in these two cases.

**Remark 1.3.** Most of the steps in the proof of Theorem 1.2 hold for any $\Delta$-regular connected graph. Only Lemmas 4.4 and 5.4 use explicitly that for cycle graphs one has $\Delta = 2$. Proving analogues of these two lemmas accommodating $\Delta \geq 3$ would resolve the conjecture for the upper

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$^2P_H$ being a polynomial with nonnegative coefficients (hence increasing on $[0,\infty)$) with $P_H(0) = 1$, the existence of a unique solution is guaranteed.
tail large deviations for regular graphs in full generality. At this moment the case of irregular
graphs is not well understood and it requires new ideas, for example, see [10, 13, 20].

Remark 1.4. A related problem of interest is to study upper tail large deviations of the homomor-
phism count. The homomorphism count of $H$ in $G$, denoted by $\text{Hom}(H,G)$, is defined to be sum in
(1.1) when $\varphi$ varies over all maps from $V(H)$ to $V(G)$. It follows from [1, 8, 12] that the upper tail
large deviations for $N(C_\ell, G(n,p))$ and $\text{Hom}(C_\ell, G(n,p))$ are the same for $n^{-1/2}(\log n)^2 \ll p \ll 1$.
The same phenomenon should hold for a wider range of $p$.

A natural way to derive the upper tail large deviations of $\text{Hom}(C_\ell, G(n,p))$ from that of labelled
copies of subgraphs in $G(n,p)$ is to write the former as a sum of $N(H_\ast, G(n,p))$, where $H_\ast$ is a
quotient subgraph of $C_\ell$ (see [17, Chapter 5] for more details on this representation), and derive
the upper tail large deviations of $N(H_\ast, G(n,p))$ for each such $H_\ast$. As the quotient graphs of $C_\ell$
involves star graphs this route would in particular need an understanding of the upper tail large
deviations for such irregular graphs. The best known result in this direction is due to [21] where
the speed of the large deviations is identified.

Remark 1.5. It is immediate to note that for any graph $G$ and $t \geq 2$,
$$\text{Hom}(C_{2t}, G) = \text{tr}[\text{Adj}(G)^{2t}] = \sum_{i=1}^{n} \lambda_i^{2t},$$
where $\text{Adj}(G)$ is the adjacency matrix of $G$, $\text{tr}(\cdot)$ denotes the trace, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$
are the eigenvalues of $\text{Adj}(G)$ arranged in a non-increasing order. So upper tail large deviations of
$\text{Hom}(C_{2t}, G(n,p))$ for all $t \geq 2$ would yield the same for the top eigenvalue of $\text{Adj}(G(n,p))$. Using
results of [8] this has been achieved in [2] for $p \gg n^{-1/2}$. Extending the same for a sparser regime
would need an understanding of the upper tail large deviations of $\text{Hom}(C_{2t}, G(n,p))$. We postpone
it to a future work.

Remark 1.6. Note that Theorem 1.2 does not discuss the nature of the large deviations when $p \sim
n^{-1/2}$.\footnote{For two sequences of positive reals $\{a_n\}$ and $\{b_n\}$ we write $a_n \sim b_n$ if $0 < \liminf_{n \to \infty} a_n/b_n \leq \limsup_{n \to \infty} a_n/b_n < \infty$.} It follows from [12] that for such $p$ the large deviation speed continues to be $n^2 p^2 \log(1/p)$. The rate function turns out to be the limit of an $n$-dependent constrained optimization problem
declared over the space of graphs on $n$ vertices (see (2.2) below). We refer the reader to Remark 2.5
for a description of the rate function for $C_4$.

Remark 1.7. It is of interest to study the typical structure of $G(n,p)$ conditioned on the upper
tail event that $N(H, G(n,p))$ exceeds its expectation by a constant factor. When $H$ is a clique graph
it has been shown in [12, Theorem 1.8] that conditioned on the upper tail event the Erdős-Rényi
graph typically has either a clique-like or a hub-like structure. In the setting of cycle graphs one can
modify the proof of Theorem 1.2 appropriately and proceed similarly along the lines of the proofs
of [12, Propositions 6.4 and 6.6] to deduce the same. To retain the clarity of the proof of Theorem
1.2 this extension is not carried out in detail.

Remark 1.8. Much less is known about the large deviations of subgraph counts in random graph
models beyond $G(n,p)$. This has been studied in the context of random $d_n$-regular graphs [4] and
random hypergraphs [16] when the (hyper-)graphs are not too sparse. It is worthwhile to investigate
whether the ideas of [12] and this paper can be adapted to these problems to treat sparser regimes.
1.1. Background and related results. As alluded to already, the study of upper tail large deviations of subgraph counts in an Erdős-Rényi graph has a long and rich history. It can be traced back to the work of Janson and Ruciński [14] where the problem is described as the infamous upper tail problem. When $H = K_3$ is a triangle it was shown in [13, 15] that for any $p \geq \log n/n$ and $\delta > 0$ one has the bounds
\[
\exp(-c_1(\delta)n^2p^2\log(1/p)) \leq \mathbb{P}(\text{UT}(K_3, \delta)) \leq \exp(-c_2(\delta)n^2p^2),
\]
for some constants $0 < c_1(\delta), c_2(\delta) < \infty$, where we recall the definition of $\text{UT}(\cdot , \cdot)$ from (1.2).

About a decade later the discrepancy between the exponents of the upper and lower bounds in (1.4) was resolved in [5, 9] by showing that the exponent in the upper bound can be tightened to $c_2(\delta)n^2p^2\log(1/p)$. These left open the problem of determining the asymptotic dependence of the constants $c_1(\delta)$ and $c_2(\delta)$ in $\delta$ and showing that they differ from each other only by $o(1)$.

For $p \in (0, 1)$ fixed this problem has been resolved by Chatterjee and Varadhan [7] where a key ingredient was Szemerédi’s regularity lemma [22] for dense graphs. Due to the poor quantitative bound of Szemerédi’s regularity lemma the same approach cannot be adopted when $p \sim n^{-c}$ for some $c > 0$. Recently there was a series of breakthroughs in this area. Chatterjee and Dembo [6] developed a general framework to treat the upper tail large deviation of any nonlinear smooth function $f(\cdot)$ of i.i.d. Ber($p$) random variables, where $f(\cdot)$ is of low-complexity characterized by the existence of a net of small cardinality of the image of the unit hypercube under the map $\nabla f$, the gradient of $f$. This when applied to the problem of the upper tail of $N(K_3, G(n, p))$ yields large deviations for $n^{-1/42}(\log n)^{11/14} \ll p \ll 1$.

Eldan in [11] derived upper tail large deviations of a nonlinear Lipschitz function $f(\cdot)$ of i.i.d. Ber($p$) variables when the Gaussian width of the image of the discrete unit hypercube under the gradient map $\nabla f$ is small. This improved the range of $p$ where the large deviations of $\text{UT}(K_3, \cdot)$ could be established to $n^{-1/18}(\log n) \ll p \ll 1$. It was further improved by Augeri [1], and Cook and Dembo [8] to show that the large deviations of the upper tail events $\text{UT}(C_\ell, \cdot)$, for any $\ell \geq 3$ (including in particular the case of $K_3$), is due to the presence of localized structures in the regime $n^{-1/2}(\log n)^2 \ll p \ll 1$. The work of Augeri [1] is an advancement of [6] where one can now consider non-smooth convex Lipschitz functions and it provides a cleaner error bound suitable to use for a wider sparse regime. Whereas the key to [8] is the derivation of a new quantitative version of Szemerédi’s regularity lemma and the counting lemma tailored for the sparse regime.

Let us add that the approaches used in [1, 8] require approximating symmetric square matrices of dimension $n$ with entries in $[0, 1]$ in the Hilbert-Schmidt norm. This is done by using standard nets for the large eigenvalues of such matrices and the eigenvectors corresponding to those large eigenvalues. As seen from Theorem 1.2 for $p \ll (\log n)^{-1/2}n^{-1/2}$ the speed of the large deviations for $\text{UT}(C_\ell, \cdot)$ is $o(n)$. Observe that log of the cardinality of any net of constant mesh-size of a unit vector in dimension $n$ is at least of order $n$. Thus for $p \ll (\log n)^{-1/2}n^{-1/2}$ using a standard net even for one eigenvector will be too expensive to deduce the large deviation. Therefore, to be able to use the machinery of [1, 8] for that sparser regime of $p$ one needs to a-priori show that the eigenvectors corresponding to large eigenvalues of $\text{Adj}(G(n, p))$ are localized with respect to some appropriately chosen collection of basis vectors with sufficiently large probability.

The recentmost breakthrough in the context of upper tail large deviations is due to Harel, Mousset, and Samotij [12] where a novel idea is put forward. Their general approach can be described as follows: Using an adaptation of the classical moment argument of [13] it is shown that the probability of $\text{UT}(H, \cdot)$ can be bounded by that of the existence of a subgraph $G_*$ of $G(n, p)$ such that it does not have too many edges and has an adequate number of copies of subgraphs of $H$. Next by peeling off edges from $G_*$, without losing too many copies of subgraphs of $H$, one
obtains a subgraph \( G \subset G_* \) such that each edge in \( G \) participates in a large number of copies of subgraphs of \( H \). Following [12] we term any such graph \( G \) to be a core graph (see Definition 2.3 for a precise formulation). Thus the probability of \( UT(H, \cdot) \) is bounded by that of the existence of a core subgraph of \( G(n, p) \) (up to a multiplicative factor of \( 1 + o(1) \)).

This then leaves the task of finding a bound on \( \mathcal{N}_e \), the number of core graphs with a given number of edges \( e \). If
\[
\mathcal{N}_e \leq \exp(e \log(1/p) \cdot o(1)),
\]
then \( \mathcal{N}_e \) being sufficiently small compared to \( (1/p)^e \), the inverse of the probability of observing any graph with \( e \) edges, one can take a union bound to derive that the probability of \( UT(H, \cdot) \) is bounded by \( \exp\left(-(1 - o(1)) \cdot e_0 \log(1/p)\right) \), where \( e_0 \) is the minimum number of edges a core graph must possess. This gives the desired upper bound on the probability of \( UT(H, \cdot) \) (see (2.2)).

In [12], this general scheme is successfully employed for cliques to derive the upper tail large deviations in the entire localized regime (and also the upper tail large deviations of \( k \)-term arithmetic progressions). For \( \Delta \)-regular bipartite graphs (e.g. \( C_4 \)) this scheme could only derive the upper tail large deviations when \( p \geq n^{-1/\Delta - o(1)} \).

The obstacle of extending the above for a sparser regime stems from the fact that the bound (1.5) breaks down for bipartite graphs when \( p \leq n^{-1/\Delta - o(1)} \). Indeed, as already noted in [12, Section 10], for any \( C > 0 \) the number of labelled copies of \( K_{2, Cn^2p^2} \) in the complete graph on \( n \) vertices exceeds the RHS of (1.5). For \( C \) sufficiently large the graph \( K_{2, Cn^2p^2} \) becomes a core graph, for \( H = C_4 \), and hence one cannot proceed as in [12].

To tackle this obstacle and establish the upper tail large deviation behavior for the entire localized regime we introduce a couple of new key ideas.

1. We show that for any core graph \( G \) with \( e(G) = O(n^2p^2) \) one can extract a bipartite subgraph \( G_b \) of it which has a block path structure, as shown in Figure 1 below. Using combinatorial arguments we show that \( G_b \) and \( G \setminus G_b \) are individually entropically stable. In this context, by entropic stability we broadly means that if \( G_b \) and \( G \setminus G_b \) are individually assumed to contain adequate numbers of copies of \( C_\ell \) then there exist appropriate lower bounds on their edges suitable for the union bound yielding the correct large deviation probability.

In [12] an upper tail event is termed to be entropically stable if (1.5) holds for all \( e \). As already discussed above that it does not hold for non-bipartite graphs for \( p \leq n^{-1/2 - o(1)} \) and moreover it can be easily seen that bounds weaker than (1.5) suffices for the union bound to work, so we adopt the above notion of entropic stability that is somewhat different, weaker, and broader than the one in [12].

Next, continuing the description of the key ideas of the proof, the block path structure of Figure 1 also allows us to choose \( G_b \) in such a way so that almost all copies of \( C_\ell \) in \( G \) must either be contained in \( G_b \) or \( G \setminus G_b \). This, in turn implies that showing entropic stability of \( G_b \) and \( G \setminus G_b \) separately guarantees the same for the whole graph \( G \).

2. To bound the number of core graphs with \( e(G) \geq n^2p^2 \) we focus at its subgraph induced by the edges with at least one end point having a low degree. If \( np \geq (\log n)^\ell \), this subgraph can be shown to be bipartite and then using a simple combinatorial argument we show that this case is entropically non-viable (or equivalently entropically sub-optimal), i.e. the probability is much smaller than the large deviation probability.

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For two sequences of positive reals \( \{a_n\} \) and \( \{b_n\} \) hereafter we use the standard notation \( b_n = O(a_n) \) to denote that \( \limsup_{n \to \infty} b_n/a_n < \infty \).
However for $np \leq (\log n)^\ell$ the bipartite structure described above ceases to exist. For example consider the graph $\mathcal{K}_{2,Cn^2p^2}$ where each vertex of degree two in $K_{2,Cn^2p^2}$ is replaced by an edge. The graph $\mathcal{K}_{2,Cn^2p^2}$ is indeed a core graph for $C_6$. To tackle this additional difficulty and cover the full localized regime we use a chaining-type argument.

For a more elaborate description of these two key ingredients we refer the reader to Section 2.

Let us add that during the chaining argument the condition $np \gg (\log n)^{1/(\ell-2)}$ is used only to argue that $\tilde{N}_{1,1}(C_\ell, G)$, the number of copies $C_\ell$ in $G$ that uses only those edges for which both of their end points are of low degree, is negligible compared to $N(C_\ell, G)$. If $np \sim (\log n)^{1/(\ell-2)}$ and $e(G) = O(n^2p^2 \log(1/p))$ then this is no longer true. Furthermore in the Poisson regime the large deviation is expected to be driven by $\tilde{N}_{1,1}(C_\ell, G)$. Therefore we believe that with some additional efforts the ideas of this paper may be used to show that the localized and the Poisson behaviors coexist when $np \sim (\log n)^{1/(\ell-2)}$.

Let us also make the following remark: In [12] the localized nature of the upper tail large deviation event for non-bipartite $\Delta$-regular graphs $H$ is derived in [12] when $np^{\Delta/2} \gg (\log n)^{\Delta v^2_H}$. The sub-optimality in the exponent of $\log n$ is most likely due to the absence of the chaining procedure which as will be seen below is crucial to treat the entire localized regime.

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2. Preliminaries and proof outline

In this section we describe the idea behind the proof of Theorem 1.2 in some detail and introduce relevant definitions and notation. The upper bound in (1.3) (i.e. the LHS upper bounded by the RHS) or equivalently the lower bound on the large deviations probability essentially follows by planting a clique of appropriate size. Hence the main work is the derivation of the lower bound in (1.3). To derive the desired lower bound we show that the logarithm of the upper tail probability for the cycles is bounded above by the solution of an $n$-dependent variational problem and then from [3] it follows that the limit of that solution is the negative of the RHS of (1.3). We now state the variational problem. It needs some notation.

For $x \in [0,1]$ we define the binary entropy

$$I_p(x) := x \log \frac{x}{p} + (1-x) \log \frac{1-x}{1-p},$$

and for a vector $x := (x_1, x_2, \ldots, x_N) \in [0,1]^N$, with $N \in \mathbb{N}$, we let

$$I_p(x) := \sum_{i=1}^{N} I_p(x_i).$$

By setting $N := \binom{n}{2}$, and identifying $\{(i,j) : 1 \leq i < j \leq n\}$ with $[N] := \{1,2,\ldots,N\}$, any vector $x$ of length $N$ can be associated to a unique weighted simple graph, denoted hereafter by $G[x]$. Similar to an unweighted graph one can define homomorphism count $\text{Hom}(H, G[x])$ for such weighted graphs. It has been established in [1, 6, 8, 11] that given a graph of fixed size $H$ if
$p \gg n^{-\alpha_H}$, for certain $\alpha_H \in (0, 1)$, the logarithm of the probability of the upper tail event $\text{UT}(H, \delta)$ equals, upon excluding smaller order terms,

\begin{equation}
\varphi_{n,H}(\delta) := \inf \left\{ \int p(x) : \text{Hom}(H, G[x]) \geq (1 + \delta)n^{\alpha_H}p^e(H) \right\},
\end{equation}

where we remind the reader that the notation $v_H$ and $e(H)$ denote the number of vertices and edges of $H$, respectively.

Let $X := (a_{ij})_{i,j=1}^n$ be the adjacency matrix of $G(n, p)$. So $a_{ij}$ are i.i.d. Ber($p$). Let us introduce one more notation: for any graph $G \subset K_n$, the complete graph on $n$ vertices, by a slight abuse of notation we write

$$\mathbb{E}_G[N(H, G(n, p))] := \mathbb{E}[N(H, G(n, p)) | a_{ij} = 1, (i, j) \in E(G)].$$

Equipped with the above notation and upon restricting $x \in \{p, 1\}^N$ one can note that the variational problem (2.1) transforms to the following variational problem:

\begin{equation}
\Phi_{n,H}(\delta) := \min \left\{ e(G) \log(1/p) : G \subset K_n, \mathbb{E}_G[N(H, G(n, p))] \geq (1 + \delta)n^{\alpha_H}p^e(H) \right\}.
\end{equation}

When $H = C_\ell$, for some $\ell \geq 3$ for brevity we write $\Phi_{n,\ell}(\cdot)$ instead of $\Phi_{n,C_\ell}(\cdot)$ and this will be the variational problem determining the log of the probability of $\text{UT}(C_\ell, \delta)$ up to the leading order.

Having stated the variational problem we proceed to describe the idea in showing that the log of the probability of $\text{UT}(C_\ell, \cdot)$ is upper bounded by $\Phi_{n,\ell}(\cdot)$ up to a factor of $1 + o(1)$. The initial part of the proof proceeds as in [12]. Indeed, using [12] we show that the probability of $\text{UT}(C_\ell, \delta)$ is bounded by that of the existence of subgraphs in $G(n, p)$ which are near-optimizers of the variational problem in (2.2). More precisely, such subgraphs are defined as follows.

**Definition 2.1** (Pre-seed graph). Fix $\varepsilon > 0$ sufficiently small and integer $\ell \geq 3$. Let $\tilde{C} := \tilde{C}(\delta, \varepsilon)$ be a sufficiently large constant. A graph $G \subset K_n$ is said to be a pre-seed graph if the followings hold:

- (PS1) $\mathbb{E}_G[N(C_\ell, G(n, p))] \geq (1 + \delta)(1 - \varepsilon))n^\ell p^\ell$.
- (PS2) $e(G) \leq \tilde{C}n^2p^2 \log(1/p)$.

The choice of the constant $\tilde{C}$ will be made precise in Section 3 during the course of the proof of Theorem 1.2. Hereafter, we fix a sufficiently small but arbitrary $\varepsilon > 0$. With that choice of $\varepsilon$ we will show that the lower bound in (1.3) holds with an additional factor $(1 - f(\varepsilon))$ on its RHS, for some function $f(\cdot)$ satisfying $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. Thus sending $\varepsilon$ to zero afterwards would yield (1.3).

Since the case $p \geq n^{-1/2-o(1)}$ is treated in [12] we prove Theorem 1.2 in the complement region. The upper bound $p \leq n^{-1/2-o(1)}$ allows us to consider only those subgraphs for which the property (PS1) of Definition 2.1 can be replaced by a simpler condition as stated below.

**Definition 2.2** (Seed graph). Let $\varepsilon, \delta$, and $\tilde{C}$ be as in Definition 2.1. A graph $G \subset K_n$ is said to be a seed graph if the followings hold:

- (S1) $N(C_\ell, G) \geq \delta(1 - 2\varepsilon)n^\ell p^\ell$.
- (S2) $e(G) \leq \tilde{C}n^2p^2 \log(1/p)$.

We remark that the pre-seed graphs of Definition 2.1 is termed as seed graphs in [12]. Since the assumption $p \leq n^{-1/2-o(1)}$ allows us to obtain a simpler description of seed graphs of [12] we have chosen to deviate from the terminology of [12].

Next we recall that the proof of Theorem 1.2 will eventually use a union bound. To reduce the cardinality of the set of subgraphs of $K_n$ to be considered under the union bound we then show that any seed graph must have a subgraph containing most of its copies of $C_\ell$ such that each of the
Definition 2.3 (Core graph). With \( \epsilon, \ell, \) and \( \bar{C} \) as in Definition 2.1 we define a graph \( G \subset K_n \) to be a core graph if

\[
\begin{align*}
(C1) \quad & N(C_\ell, G) \geq \delta (1 - 3\epsilon) n^\ell \ell!, \\
(C2) \quad & e(G) \leq \bar{C} n^2 \ell! \log(1/p),
\end{align*}
\]

and

\[
(C3) \quad \min_{e \in E(G)} N(C_\ell, G, e) \geq \delta \epsilon n^\ell \ell! / (\bar{C} n^2 \ell! \log(1/p)),
\]

where for an \( e \in E(G) \) the notation \( N(C_\ell, G, e) \) denotes the number of labelled copies of \( C_\ell \) in \( G \) that contain the edge \( e \).

As mentioned in Section 1 the cardinality of the set of core subgraphs of \( K_n \) is too large to apply a union bound. One of the difficulties, as noted in \cite[Section 10]{Basak2021}, is due to the bound on the number of copies of \( K_{2, \bar{C} n^2 \ell!} \) in \( K_n \). It can be checked that for \( K_{2, \bar{C} n^2 \ell!} \) to be a core graph one needs \( C \geq \bar{C}_\delta := \frac{1}{2} \delta^{2/\ell} \). One can further show that for any \( C \geq \bar{C}_\delta \) the log of the probability of the existence of a labelled copy of \( K_{2, \bar{C} n^2 \ell!} \) in \( G(n, p) \) equals to that of \( UT(C_\ell, \delta) \) upon excluding a negligible factor. In other words the set of all labelled copies of \( K_{2, \bar{C} n^2 \ell!} \) that are core graphs although do not satisfy (1.5) but are still entropically stable according to the weaker notion that is adopted in this paper. So one can potentially hope to overcome the obstacle stated in \cite[Section 10]{Basak2021}. On the other hand we also note that a disjoint union of \( K_{2, \bar{C} n^2 \ell!} \), with \( C \geq \bar{C}_\delta \), and a clique on \( \lceil \delta^{2/\ell}/np \rceil \) vertices, for any \( \delta \geq 0 \) such that \( \delta + \delta \geq \delta \) is also a core graph. Thus to carry out this scheme one should also be able to show the entropic stability of these graphs.

These two observations are the motivation behind the next step in the proof of Theorem 1.2. Indeed, we show that if the number of edges in a core graph \( G \) is rather small, namely \( e(G) = O(n^2 \ell!) \), then there exists a bipartite subgraph \( G_b \) of \( G \) so that \( G_b \) and \( G \setminus G_b \) are individually entropically stable and almost none of the copies of \( C_\ell \) use edges from both \( G_b \) and \( G \setminus G_b \). Heuristically, the reader may view this decomposition as a separation of copies of \( K_{2, \bar{C}_\delta \ell!} \) and \( K_{\lceil \delta^{2/\ell}/np \rceil} \) that may be present in a core graph.

To implement this idea we run a second stage peeling procedure on a core graph with \( O(n^2 \ell!) \) edges to obtain a further subgraph of it so that every edge of that subgraph participates in an even larger number of copies of \( C_\ell \). These subgraphs of \( K_n \) will be termed as strong-core graphs.

Definition 2.4 (Strong-core graph). Let \( \epsilon \) and \( \ell \) be as in Definition 2.2, and \( \bar{C}_* := \bar{C}_*(\delta) < \infty \) be a large constant, depending only on \( \delta \). We define a graph \( G \subset K_n \) to be a strong-core graph if

\[
\begin{align*}
(SC1) \quad & N(C_\ell, G) \geq \delta (1 - 6\epsilon) n^\ell \ell!, \\
(SC2) \quad & e(G) \leq \bar{C}_* n^2 \ell!,
\end{align*}
\]

and

\[
(SC3) \quad \min_{e \in E(G)} N(C_\ell, G, e) \geq (\delta \epsilon / \bar{C}_*) \cdot (np)^{\ell - 2}.
\]

Note the difference in the lower bounds in (C3) and (SC3) in Definitions 2.3 and 2.4, respectively. We will see below that any \( \bar{C}_* = \bar{C}_* \delta^{2/\ell} \) with \( C_* \geq 32 \) will suffice for the proof of Theorem 1.2.

The upper bound on the number of copies given by (SC2) and the lower bound in (SC3) of Definition 2.4 allows us to deduce that the product of the degrees of the end points of most of the edges of a strong-core graph \( G \) satisfies a tight upper and lower bound (see Lemma 4.2). This in turn helps us to show that there exists a bipartite subgraph \( G_* \subset G \) with vertices \( \mathcal{V} \cup \mathcal{V} ' \) that have a block path structure as shown in Figure 1, where \( \mathcal{V} := \{ V_i \}_{i=1}^{C_3} \) and \( \mathcal{V} ' := \{ \bar{V}_i \}_{i=1}^{C_3} \) are the two
partite sets (i.e. the maximal independent sets), and $C_3$ is some large constant depending on $\ell, \delta$, and $\varepsilon$. The block path structure of Figure 1 is useful in extracting a further subgraph $G_b \subset G_*$ such that barring a small fraction, all other copies of $C_\ell$ in $G$ is contained in either $G_b$ or in $G\setminus G_b$. Using combinatorial arguments we then deduce that $G_b$ and $G\setminus G_b$ are individually entropically stable. Hence one now has that the set of all core graphs are entropically stable. This argument is carried out in the proof of Proposition 3.3 and it can be found in Section 4.

To complete the proof of the lower bound in (1.3) it remains to establish the entropic stability of core graphs with at least $\bar{C}_s n^2 p^2$ edges. If we additionally assume that $np \geq (\log n)^\ell$ then one has a lower bound on the product of the degrees of the end points of any edge in a core graph (see Lemma 3.6) which in turn implies that the subgraph of $G$ induced by edges are that incident to at least one vertex of low degree must be bipartite. Note that by definition the minimum degree in a core graph must be at least two. The last two facts together with the upper bound $p \leq n^{1/2}$ then allows one to derive a bound on the number of such graphs adequate for a union bound showing their entropic sub-optimality.

As explained in Section 1 one does not have the above bipartite structure in a core graph when $np \leq (\log n)^\ell$. To derive the entropic stability in this case we carry out a delicate chaining type argument as follows:

**Step 1.** We consider core graphs $G$ with $e(G) \leq \bar{C} n^2 p^2 \log(1/p)/(\log \log n)^2$. We break this event into three sub-events:

- If $\bar{e}$, the number of edges in $G$ that are incident to at least one vertex of high degree, is large, then using a simple combinatorial argument one can show that this scenario is entropically non-viable. This is the content of Lemma 3.7.
- If $N_{1,1}(C_\ell, G)$, the number of labelled copies of $C_\ell$ in $G$ that use at least one edge whose both end points are of low degree, is large then using another combinatorial argument (see Proposition 5.2) we find a lower bound on $\bar{e}$ which shows that this case too is entropically sub-optimal. This argument is carried out in the proof of Lemma 3.8.
- On the remaining sub-event by first removing the edges with both end points of low degree and then performing another peeling procedure one can procure a strong-core subgraph of $G$. As we already know that the set of all strong-core graphs are indeed entropically stable this sub-event is entropically stable, too.
Step 2. It remains to consider core graphs $G$ with $e(G) \geq \tilde{C}n^2p^2 \log(1/p)/(\log \log n)^2$. We break the range of the number of edges into dyadic intervals $\{J_j\}_{j=1}^{L_n}$, for $L_n = O(\log \log n)$. Let $G$ be such that $e(G) \in J_{j_0}$ for some $j_0 \in [L_n]$.

If $\mathcal{N}_{1,1}(C_\ell, G)$ is large, where now the threshold for $\mathcal{N}_{1,1}(C_\ell, G)$ to be large depends dyadically on $j_0$ as well, using Proposition 5.2 again we deduce that this scenario is sub-optimal.

On the complement event, i.e. if $\mathcal{N}_{1,1}(C_\ell, G)$ is small the most natural strategy would be to focus at the subgraph $G_0 \subset G$ which is the 2-core of the subgraph obtained from $G$ after removal of all the edges of $G$ with both end points of small degree. As $\mathcal{N}_{1,1}(C_\ell, G)$ is small this peeling procedure loses only a small fraction of the number of copies of $C_\ell$ in $G$. If after this peeling procedure one still has that $e(G_0) \in J_{j_0}$ using the fact that $e(G \setminus G_0)$ is small and another combinatorial argument we establish that it is entropically stable. This is contained in Lemma 3.11.
Finally, if \(e(G_0) \in \cup_{j=0}^{J_\infty} J_j\) then we iterate the whole procedure as above. This process continues until we procure a subgraph \(G_0\) for which the number of edges is at most \(\tilde{C} n^2 p^2 \log(1/p)/(\log \log n)^2\). As we know from Step 1 above that such scenario is entropically stable it concludes the chaining procedure and thus the outline of the proof of Theorem 1.2 is now complete.

We remind the reader that our Theorem 1.2 does not discuss the upper tail large deviations when \(p \sim n^{-1/2}\). As already mentioned in Remark 1.6, thanks to [12], one only needs to identify the rate function for such \(p\). In the remark below we provide a short outline of the derivation of the rate function for \(C_4\).

**Remark 2.5.** To describe the rate function for \(p \sim n^{-1/2}\) and \(H = C_4\) let us introduce a few notation. For any \(U \subset V(G)\) we write \(G[U]\) to be the graph spanned by the vertices in \(U\) and \(G[U, U]\) to be the bipartite subgraph of \(G\) induced by the two disjoint subsets of vertices \(U\) and \(\bar{U} := V(G) \setminus U\). We let \(N_u(K_{1,2}, G)\) to denote the number of labelled copies of \(K_{1,2}\) in \(G\) for which the center vertex is in \(U\) and the leaf vertices are in \(\bar{U}\).

It can be checked that the near-optimizers of the variational problem (2.2) are the graphs \(G\) for which there exists a partition of \(V(G) = V_1 \cup V_2\) such that

\[
N(C_4, G[V_1]) \geq \delta(1 - \varepsilon) \cdot x \cdot n^4 p^4
\]

and

\[
\mu(\kappa) := N(C_4, G[V_1, V_2]) + 4\kappa N_{V_2}(K_{1,2}, G) \geq \delta(1 - \varepsilon) \cdot (1 - x) \cdot n^4 p^4,
\]

for some \(x \in [0, 1]\), and any \(\varepsilon > 0\) sufficiently small, where \(\kappa := \lim_{n \to \infty} np^2\). Thus the near minimizers are the graphs with the minimal number of edges satisfying (2.3)-(2.4).

To identify such graphs we note the following: If \(\deg_G(v)\) denotes the degree of vertex \(v\) in graph \(G\) then \(N(C_4, G[V_1, V_2])\) is maximized if the differences between pairs \(\deg_G(v_i)\) and \(\deg_G(v_j)\) are minimized for every pair of vertices \(v_i, v_j \in V_1\). On the other hand, \(N_{V_2}(K_{1,2}, G)\) is maximized when \(\deg_G(v)\) are set to be the maximum value subject to the natural constraint that \(\deg_G(v) \leq |V_2|\), where \(|\cdot|\) denotes the cardinality of a set. Due to convexity it can be further deduced that the maximizer for \(\mu(\kappa)\) has to be one of the two maximizers described above. However, which one of them will dominate depends on \(\kappa\).

Using this observation and minimizing over \(x \in [0, 1]\) one can find the near minimizers and hence also the rate function. For a general \(C_\ell\) this picture is more intricate. We refrain from fleshing out the detail for the general case. This may be considered elsewhere.

**Outline of the rest of the paper.** In Section 3 we provide a proof of Theorem 1.2 assuming that we have the necessary bounds on the probabilities of various sub-events of the event that \(G(n, p)\) contains a core graph. The proofs of these bounds being combinatorial in nature are pushed to Sections 4 and 5. In Section 4.1 we provide a short proof showing the entropic stability of strong-core graphs for cycles of odd length. While Sections 4.2 and 4.3 are devoted in deriving bounds that enable us in proving entropic stability of \(G_b\) and \(G \setminus G_b\), respectively. Combining results from these two sections we then, in Section 4.4, finish the proof of the entropic stability of strong-core graphs in the case of cycles of even length.

In Section 5.1 we treat core graphs with large number of edges when \(np \geq (\log n)^7\). In the case of \(np \leq (\log n)^7\), as already mentioned, the proof splits into two further cases. In Section 5.2 we derive bounds showing entropic stability of core graphs with many but not too many edges. Whereas in Section 5.3 we obtain necessary combinatorial results suitable for carrying out the chaining argument described above. Finally, in Appendix A we derive bounds on the product of the degrees of the end points of most of the edges in a strong-core graph that plays an important role in showing that such graphs are entropically stable.
3. Proof of Theorem 1.2

In this section we provide the proof of our main result Theorem 1.2. As already mentioned, we will only focus on the case \( np \leq n^{1/2-o(1)} \), where \( o(1) \) is an appropriately chosen term decaying to zero as \( n \to \infty \) since the other case is proved in [12]. First let us state the result showing the upper bound in (1.3).

**Proposition 3.1.** Let \( \delta > 0 \) and \( \ell \geq 3 \) be fixed. For \( p = p_n \in (0,1) \) such that \( p \ll n^{-1/2} \) we have

\[
\limsup_{n \to \infty} - \frac{\log \mathbb{P} \left( N(C_\ell, \mathbb{G}(n,p)) \geq (1+\delta)n^\ell p^\ell \right)}{n^2 p^2 \log(1/p)} \leq \frac{1}{2} \delta^2/\ell.
\]

The proof of Proposition 3.1 is standard. We include it for completeness.

**Proof of Proposition 3.1.** We aim to apply [12, Lemma 3.5]. As the edges in \( \mathbb{G}(n,p) \) are independent and each edge occurs with probability \( p \) we find that for any \( G \subset K_n \),

\[
(3.1) \quad \mathbb{E}_G[N(C_\ell, \mathbb{G}(n,p))] - \mathbb{E}[N(C_\ell, \mathbb{G}(n,p))] = \sum_{\emptyset \neq H \subset C_\ell} N(H, G) \cdot (1-p^\varepsilon(H)) \cdot n^{\ell-\varepsilon_H} p^{\ell-\varepsilon(H)},
\]

where the sum is taken over all subgraphs \( H \) of \( C_\ell \) with no isolated vertices, and as before \( \varepsilon_H \) and \( \varepsilon(H) \) denote the number of vertices and edges of \( H \), respectively. This, in particular, implies that

\[
(3.2) \quad \mathbb{E}_G[N(C_\ell, \mathbb{G}(n,p))] - \mathbb{E}[N(C_\ell, \mathbb{G}(n,p))] \geq N(C_\ell, G) \cdot (1-p^\ell).
\]

We also observe that any clique on \( m \) vertices contains \( (m) = m(m-1) \cdots (m-\ell + 1) \) labelled copies of \( C_\ell \). As \( \mathbb{E}[N(C_\ell, \mathbb{G}(n,p))] = n^\ell p^\ell (1+o(1)) \) taking \( G \) to be the clique on \( [(\delta + 2\delta)/\ell np] \) vertices in (3.2), and recalling the definition of the variational problem \( \Phi_{n,\ell}(\delta) \) from (2.2), we therefore deduce that for any \( \delta > 0 \) and \( n \) sufficiently large,

\[
\Phi_{n,\ell}(\delta + \varepsilon) \leq \frac{1}{2} (\delta + 3\varepsilon)^2 n^2 p^2 \log(1/p),
\]

where we also used the fact that \( p = o(1) \). Now we apply [12, Lemma 3.5] to deduce

\[
\limsup_{n \to \infty} - \frac{\log \mathbb{P} \left( N(C_\ell, \mathbb{G}(n,p)) \geq (1+\delta)n^\ell p^\ell \right)}{n^2 p^2 \log(1/p)} \leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\Phi_{n,\ell}(\delta + \varepsilon)}{n^2 p^2 \log(1/p)} = \frac{1}{2} \delta^2/\ell,
\]

where the rightmost equality is due to [3, remark 8.3]. This completes the proof. \( \square \)

Let us now move to the proof of the upper bound which takes up the rest of the paper. As outlined in Section 2 the initial step in this direction is to show that the probability of \( UT(C_\ell, \delta) \) can be bounded above by that of existence of a core graph as defined in Definition 2.3.

**Lemma 3.2.** Let \( \ell \geq 3 \) and \( \delta > 0 \) be fixed. If \( np^2 \ll (\log n)^{-\ell} \) then for every \( \varepsilon > 0 \) and all large \( n \) we have

\[
(3.3) \quad \mathbb{P}(N(C_\ell, \mathbb{G}(n,p)) \geq (1+\delta)n^\ell p^\ell) \leq (1+\varepsilon)\mathbb{P}(\mathbb{G}(n,p) \text{ contains a core graph}).
\]

In [12, Lemma 3.6, Lemma 3.7], a similar statement was established for a slightly different definition of core graphs (that corresponds to replacing our seed graphs by pre-seed graphs). The proof of Lemma 3.2 will follow their proof together with using the bound \( p \leq n^{-1/2-o(1)} \) to move from pre-seed graphs to seed graphs.

**Proof of Lemma 3.2.** We first claim that

\[
(3.4) \quad \mathbb{P}(N(C_\ell, \mathbb{G}(n,p)) \geq (1+\delta)n^\ell p^\ell) \leq (1+\varepsilon)\mathbb{P}(\mathbb{G}(n,p) \text{ contains a pre-seed graph}).
\]
To this end, we apply [12, Lemma 3.6] with \( N(C_\ell, G(n, p)) \) and \( \bar{C}n^2p^2 \log(1/p) \) taking the roles of \( X \) and \( \ell \) there, respectively. Applying [12, Lemma 3.6] with these choices from the definition of pre-seed graphs one has that

\[
\mathbb{P}\left( \left\{ N(C_\ell, G(n, p)) \geq (1 + \delta)n^{\ell/p}\right\} \cap \{ G(n, p) \text{ contains a pre-seed graph} \}^c \right) 
\leq \left( \frac{1 + \delta - \varepsilon}{1 + \delta} \right)^{\bar{C}n^2p^2 \log(1/p)}.
\]

On the other hand, as a clique on \( m \) vertices contains \( (m)_\ell \) labelled copies of \( C_\ell \) it is immediate from Definition 2.1 and (3.2) that, for all large \( n \),

\[
\mathbb{P}(G(n, p) \text{ contains a pre-seed graph}) \geq \mathbb{P}(G(n, p) \text{ contains a clique on } \lceil (\delta (1 - \varepsilon/2))^{1/\ell} np \rceil \text{ vertices}) 
\geq \exp \left( - \frac{1}{2} \left( \delta \left( \frac{1 - \varepsilon}{4} \right) \right)^{2/\ell} n^2p^2 \log(1/p) \right).
\]

Therefore, for \( \bar{C} \) sufficiently large, depending only on \( \delta \) and \( \varepsilon \), we deduce from (3.5)-(3.6) that the LHS of (3.5) is at most

\[
\varepsilon \cdot \mathbb{P}(G(n, p) \text{ contains a pre-seed graph}).
\]

This proves the claim (3.4).

We now proceed to prove that if \( np^2 \ll (\log n)^{-\ell} \) then for all sufficiently large \( n \) the existence of a pre-seed subgraph of \( G(n, p) \) guarantees the existence of a seed subgraph. Once we have a seed graph \( G \) then we peel off its edges iteratively that participate in strictly less than \( \delta \varepsilon n^{\ell/p} / (\bar{C}n^2p^2 \log(1/p)) \) labelled copies of \( C_\ell \) to produce a subgraph \( G_0 \subset G \) so that

\[
\min_{e \in E(G_0)} N(C_\ell, G_0, e) \geq \delta \varepsilon n^{\ell/p} / (\bar{C}n^2p^2 \log(1/p)).
\]

Note that by triangle inequality it follows that this peeling procedure loses at most \( \delta \varepsilon n^{\ell/p} \) labelled copies of \( C_\ell \) in \( G \). Thus \( G_0 \) satisfies the condition (C1) of Definition 2.3. The condition (C2) is automatic. Therefore \( G_0 \) is indeed a core graph.\(^6\) This then yields the desired conclusion.

So it now remains to show that the existence of a pre-seed graph implies the same for a seed graph. Turning to this task we begin by noting that any proper subgraph \( H_\star \subset C_\ell \) (without any isolated vertices) must be a disjoint union of paths \( \{ P_{k_i}\}_{i=1}^m \), for some \( m \in \mathbb{N} \), where \( P_k \) denotes the path of length \( k \), i.e. \( v_{P_k} = k + 1 \) and \( e(P_k) = k \). We claim that

\[
N(P_k, G) \leq (2e(G))^{\lceil \frac{k+1}{2} \rceil}.
\]

To see this we note that once we fix the odd numbered edges of \( P_k \), and the last the even numbered edge, when \( k \) is even, the remaining edges are automatically fixed. The choice of these edges is precisely bounded by the RHS of (3.7), where the factor two appears because of the choice of the orientation of an edge.\(^7\) This proves (3.7).

Next note that, as \( H_\star \) is a disjoint union of paths we have

\[
\sum_{i=1}^m k_i = e(H_\star) = v(H_\star) - m.
\]

\(^6\)This peeling procedure is similar to that in [12, Lemma 3.7]. We refer the reader to there for more details on this step.

\(^7\)One can also deduce (3.7) by using [12, Theorem 5.7].
Since \( m \) can be at most \( \ell \) we also have that
\[
(3.9) \quad v(H_v) + m \leq 2\ell.
\]
From Definition 2.1 for any pre-seed graph \( G \) we have the bound \( e(G) \leq C n^2 p^2 \log(1/p) \). Hence equipped with (3.7) and using the fact that \( 2\left\lceil \frac{k+1}{2} \right\rceil \leq k + 2 \) for all \( k \geq 1 \) and any pre-seed graph \( G \) we derive that
\[
N(H_v, G) = \prod_{i=1}^{m} N(P_{k_i}, G) = O(1) \cdot (np)^{\sum_{i=1}^{m} \left\lceil \frac{k_i}{2} \right\rceil} \log(1/p) \sum_{i=1}^{m} \left\lceil \frac{k_i}{2} \right\rceil
\]
\[
\leq O(1) \cdot (np)^{e(H_v)} \cdot (np)^{2m} \cdot (\log(1/p))^\ell = O(n^{v(H_v)} p^{e(H_v)}) \cdot (np)^2 m \cdot (\log(1/p))^\ell,
\]
where in the penultimate step we have used (3.8) and (3.9), and in the final step we have used the second equality of (3.8).
As \( m \geq 1 \) for any \( H_v \subseteq C_\ell \), we deduce from above that for \( np^2 \ll (\log n)^{-\ell} \)
\[
N(H_v, G) = o(n^{v(H_v)} p^{e(H_v)}).
\]
This in turn implies that
\[
\sum_{\forall \not\in H_v \subseteq C_\ell} N(H_v, G) \cdot n^{\ell - v(H_v)} p^{\ell - e(H_v)} = o(n^\ell p^\ell).
\]
As \( \mathbb{E}[N(C_\ell, G(n, p))] = n^\ell p^\ell (1 + o(1)) \) it is now immediate from Definition 2.1 and (3.1) that for any pre-seed graph \( G \) one must have that
\[
N(C_\ell, G) \geq N(C_\ell, G) \cdot (1 - p^\ell) \geq \delta (1 - 2\varepsilon) n^{\ell} p^\ell,
\]
for all large \( n \). Thus \( G \) is indeed a seed graph as well. The proof of the lemma is now complete. \( \square \)

Equipped with Lemma 3.2 the proof of upper bound of the log-probability of \( \text{UT}(C_\ell, \delta) \) now splits into two parts.

**Proposition 3.3.** Fix \( \delta > 0 \) and \( \ell \geq 3 \). Then, for any \( p \in (0, 1) \) satisfying \( 1 \ll np \leq n^{1/2} \), and \( \varepsilon > 0 \) sufficiently small, we have that
\[
\lim_{n \to \infty} \inf \frac{-\log \mathbb{P}(\exists G \subseteq G(n, p) : G \text{ is a strong-core graph})}{n^2 p^2 \log(1/p)} \geq \frac{1}{2} \delta^2 (1 - f_\ell(\varepsilon)),
\]
for some nonnegative function \( f_\ell(\cdot) \) such that \( \lim_{\varepsilon \to 0} f_\ell(\varepsilon) = 0 \).

**Proposition 3.4.** Let \( \delta, \ell, \varepsilon, \) and \( f_\ell(\cdot) \) be as in Proposition 3.3. Let \( p \in (0, 1) \) be such that for \( \log n )^{1/(\ell-2)} \ll np \leq n^{1/2} \). Then
\[
\lim_{n \to \infty} \inf \frac{-\log \mathbb{P}(\exists G \subseteq G(n, p) : G \text{ is a core graph with } e(G) \geq \bar{C}_* n^2 p^2)}{n^2 p^2 \log(1/p)} \geq \frac{1}{2} \delta^2 (1 - f_\ell(\varepsilon)).
\]

Let us now complete the proof of Theorem 1.2 using Propositions 3.3 and 3.4.

**Proof of Theorem 1.2 using Propositions 3.3-3.4.** Recalling Proposition 3.1 we note that it only remains to prove the lower bound in (1.3). For brevity let us also write
\[
\mathcal{C}_1 := \{ \exists G \subseteq G(n, p) : G \text{ is a core graph with } e(G) \leq \bar{C}_* n^2 p^2 \}
\]
and
\[
\mathcal{C}_2 := \{ \exists G \subseteq G(n, p) : G \text{ is a core graph with } e(G) \geq \bar{C}_* n^2 p^2 \}.
\]
Using Lemma 3.2 we have that
\[
\limsup_{n \to \infty} \frac{\log \mathbb{P}(\text{UT}(C_\ell, \delta))}{n^2 p^2 \log(1/p)} = \limsup_{n \to \infty} \frac{\log \mathbb{P}(\exists G \subset \mathcal{G}(n, p) : G \text{ is a core graph})}{n^2 p^2 \log(1/p)} \leq \max \left\{ \limsup_{n \to \infty} \frac{\log \mathbb{P}(\mathcal{E}_1)}{n^2 p^2 \log(1/p)}, \limsup_{n \to \infty} \frac{\log \mathbb{P}(\mathcal{E}_2)}{n^2 p^2 \log(1/p)} \right\}.
\] (3.10)

We claim that any core graph $G$ with $e(G) \leq C_* n^2 p^2$ contains a strong-core subgraph $G' \subset G$. To see this for any such core graph $G$ we iteratively remove edges from $E(G)$ that participate in less than $(\delta \varepsilon)/C_* \cdot (np)^{\ell-2}$ copies of $C_\ell$. This ensures that subgraph $G'$ obtained at the end of this peeling procedure have the desired lower bound (SC3) of Definition 2.4. The upper bound on $e(G')$ is automatic and (SC1) follows from triangle inequality. This proves that $G'$ is a strong-core graph. Hence
\[
\mathbb{P}(\mathcal{E}_1) \leq \mathbb{P}(\exists G \subset \mathcal{G}(n, p) : G \text{ is a strong-core graph}).
\]

Therefore, continuing from (3.10), and using Propositions 3.3-3.4, we deduce that
\[
\limsup_{n \to \infty} \frac{\log \mathbb{P}(\text{UT}(C_\ell, \delta))}{n^2 p^2 \log(1/p)} \leq \frac{1}{2} \delta^2 (1 - f_\ell(\varepsilon)),
\]
for any sufficiently small $\varepsilon > 0$. Sending $\varepsilon$ to zero the proof completes. \qed

The rest of this paper will be devoted to proving Propositions 3.3 and 3.4. The proof of Proposition 3.3 is deferred to Section 4. To prove Proposition 3.4 we treat two regimes $np \geq (\log n)^\ell$ and $np \leq (\log n)^\ell$ separately. First let us consider the easier case $np \geq (\log n)^\ell$.

### 3.1. Proposition 3.4 in large $p$ regime

As already outlined in Section 2 the key here is to derive that the subgraph of a core graph $G$ induced by the edges that are adjacent to vertices of low degree is a bipartite graph. To make this idea precise let us consider the following set of low degree vertices
\[
\mathcal{W} := \mathcal{W}(G) : = \{ v \in V(G) : \deg_G(v) \leq D \},
\]
where
\[
D := D(\varepsilon) := \lceil 32/\varepsilon \rceil.
\]

Further let $G_{\mathcal{W}} \subset G$ be the subgraph induced by edges adjacent to vertices in $\mathcal{W}$. For $v \in V(G)$ we write $\deg_G(v)$ to denote the degree of vertex $v$ in graph $G$. Finally, for $e \geq e_* := \tilde{C}_* n^2 p^2$ we let
\[
\mathcal{A}_e := \{ G \subset \mathcal{G}(n, p) : G \text{ is a core graph, } G_{\mathcal{W}} \text{ is bipartite, and } e(G) = e \}.
\]

Equipped with the above set of notation let us state the lemma that yields the entropic stability of the set of all graphs $G$ with a large number of edges for which $G_{\mathcal{W}}$ is bipartite.

**Lemma 3.5.** Fix $\delta > 0$ and $\ell \geq 3$. Let $np \geq (\log n)^\ell$. Then for any $\varepsilon \in (0, \frac{1}{8})$,
\[
\limsup_{n \to \infty} \frac{\log \mathbb{P}(\cup_{e \geq e_*} \mathcal{A}_e)}{n^2 p^2 \log(1/p)} \geq \frac{1}{16} \tilde{C}_*.
\]

Lemma 3.5 follows from an easy combinatorial argument bounding the number of potential graphs participating in the event $\cup_{e \geq e_*} \mathcal{A}_e$. Its proof is postponed to Section 5.1. To complete the proof of Proposition 3.4 we need the following lower bound on the product of the degrees of the end points of edges in core graphs which will show that for such graphs $G_{\mathcal{W}}$ is indeed bipartite.
Lemma 3.6. Let \( G \) be a core graph. If \( np \gg (\log n)^{\ell/2} \), then for every edge \( e = (u,v) \in E(G) \)
\begin{equation}
\deg_G(u) \cdot \deg_G(v) \geq \frac{\tilde{c}_0(\varepsilon) \cdot e(G)}{(\log n)^{\ell}},
\end{equation}
for some constant \( \tilde{c}_0(\varepsilon) > 0 \).

Bounds same as above have been derived in [12] (see Claim 7.5 there). We include a short outline of the proof of Lemma 3.6 in Appendix A for reader’s convenience. The lower bound on \( p \) in Lemma 3.6 is added because otherwise the lower bound (3.13) would become useless for the graphs for which we will apply this result.

Proof of Proposition 3.4 for \( np \geq (\log n)^{\ell} \). Let \( w \in \mathcal{W} \subset V(G) \) and \( (w, w') \in E(G) \), for some \( w' \in V(G) \), where \( G \) is a core graph. As \( np \geq (\log n)^{\ell} \), Lemma 3.6 implies that
\begin{equation}
\deg_G(w') \geq \frac{1}{D} \cdot \deg_G(w) \cdot \deg_G(w') \geq \frac{1}{D} \cdot \frac{\tilde{c}_0(\varepsilon) \cdot e(G)}{(\log n)^{\ell}} \geq \frac{1}{2D} \cdot (\delta(1-3\varepsilon))^\frac{\ell}{2} \cdot \frac{\tilde{c}_0(\varepsilon) \cdot n^2p^2}{(\log n)^{\ell}} \geq 2D,
\end{equation}
for all large \( n \), where the penultimate step follows from the fact that \( G \) being a core graph must possess at least \( \delta(1-3\varepsilon)^{\ell/2}p^\ell \) copies of \( C_\ell \) and hence the lower bound on \( e(G) \) follows from that fact that \( N(C_\ell, G) \leq (2e(G))^{\ell/2} \) (see [12, Lemma 5.5]).

The lower bound (3.14) in particular implies that the subgraph \( G_\mathcal{W} \) is bipartite. Therefore
\[ 
\mathbb{P}(\exists G \subset G(n,p) : G \text{ is a core graph with } e(G) \geq \tilde{C}_*n^2p^2) \leq \mathbb{P}(\cup_{\varepsilon \geq \varepsilon_0} G \notin \mathcal{A}).
\]
The proof now completes upon using Lemma 3.5 and setting \( \tilde{C}_* \geq 32\varepsilon^{\ell/\ell} \).

3.2. Proposition 3.4 in small \( p \) regime. We begin by introducing the following set of notation. First we split the set of all core graphs with at least \( \tilde{C}_*n^2p^2 \) edges into two subsets:

\begin{align*}
\text{Core}_1 := \{ \exists G \subset G(n,p) : G \text{ is a core graph with } \tilde{C}_*n^2p^2 \leq e(G) \leq (\log \log n)^{-2}\tilde{C}_*n^2p^2 \log(1/p) \}
\end{align*}

and

\begin{align*}
\text{Core}_2 := \{ \exists G \subset G(n,p) : G \text{ is a core graph with } (\log \log n)^{-2}\tilde{C}_*n^2p^2 \log(1/p) \leq e(G) \leq \tilde{C}_*n^2p^2 \log(1/p) \}.
\end{align*}

As already discussed earlier that obtaining desired probability bounds on \( \text{Core}_1 \) and \( \text{Core}_2 \) requires two different arguments. First let us proceed to show that the set of core graphs participating in the event \( \text{Core}_1 \) are entropically stable. This demands a further subdivision of \( \text{Core}_1 \). We let

\begin{align*}
\text{Core}_{1,1} := \{ \exists G \subset G(n,p) : G \text{ is a core graph with } \tilde{C}_*n^2p^2 \leq e_{1,2}(G) + e_{2,2}(G) \leq e(G) \leq (\log \log n)^{-2}\tilde{C}_*n^2p^2 \log(1/p) \},
\end{align*}

where
\begin{equation}
\begin{align*}
e_{1,2}(G) &:= |E_{1,2}(G)|, & e_{2,2}(G) &:= |E_{2,2}(G)|, \\
E_{1,2}(G) &:= \{ e = (u_1, u_2) \in E(G) : \text{ one of } u_1 \text{ and } u_2 \text{ is in } \mathcal{W}(G) \}, \\
E_{2,2}(G) &:= \{ e = (u_1, u_2) \in E(G) : u_1, u_2 \notin \mathcal{W}(G) \},
\end{align*}
\end{equation}

and we recall the definition of \( \mathcal{W}(G) \) from (3.11). Setting
\begin{equation}
E_{1,1} := E_{1,1}(G) := \{ e = (u_1, u_2) \in E(G) : u_1, u_2 \in \mathcal{W}(G) \},
\end{equation}

we define $N_{1,1}(C_\ell, G)$ be the number of labelled copies of $C_\ell$ in $G$ that use at least one edge from $E_{1,1}(G)$. In words $N_{1,1}(C_\ell, G)$ is the number of copies of $C_\ell$ in $G$ that uses at least an edge with both end points of low degree. Now define

$$\text{Core}_{1,2} := \left\{ \exists G \subset G(n, p) : G \text{ is a core graph with } N_{1,1}(C_\ell, G) \geq \varepsilon \delta n^\ell p^\ell \right\}.$$}

The next two results yield upper bounds on the probabilities of $\text{Core}_{1,1}$ and $\text{Core}_{1,2}$. For a later use during the chaining procedure we will in fact bound probabilities of events that are somewhat larger than $\text{Core}_{1,1}$ and $\text{Core}_{1,2}$. Let us define these events. We set $d_{\min}(G) := \min_{v \in V(G)} \deg_G(v)$. Then denote

$$(3.19) \quad \bar{\text{Core}}_{1,1} := \left\{ \exists G \subset G(n, p) : d_{\min}(G) \geq 2 \text{ and } C_\ell n^2 p^2 \leq e_{1,2}(G) + e_{2,2}(G) \leq e(G) \leq C n^2 p^2 \log(1/p)/(\log \log n)^2 \right\}$$

and

$$(3.20) \quad \bar{\text{Core}}_{1,2} := \left\{ \exists G \subset G(n, p) : d_{\min}(G) \geq 2, \ N_{1,1}(C_\ell, G) \geq \varepsilon \delta n^\ell p^\ell, \text{ and } e(G) \leq \bar{C} n^2 p^2 \log(1/p)/(\log \log n)^2 \right\}.$$}

Note the differences between $\bar{\text{Core}}_{1,i}$ for $\text{Core}_{1,i}$, for $i = 1, 2$, is that in the former we do not require the graph $G$ to be a core graph. It requires a mild condition that the minimum degree should be at least two. This mild requirement will suffice to obtain the desired probability bounds. We now state the results.

**Lemma 3.7.** Fix $\delta, \varepsilon > 0$, and $\ell \geq 3$. Let $p \in (0, 1)$ be such that $np \leq (\log n)^\ell$. Then

$$\limsup_{n \to \infty} \log \mathbb{P}(\bar{\text{Core}}_{1,1}) = -16\delta^2/\ell.$$}

**Lemma 3.8.** Let $\delta, \varepsilon$, and $\ell$ be as in Lemma 3.7. If $p \in (0, 1)$ such that $np \gg (\log n)^{1/(\ell-2)}$ then we have

$$\limsup_{n \to \infty} \log \mathbb{P}(\bar{\text{Core}}_{1,2}) = -\infty.$$}

The proofs of these two results being combinatorial in nature are moved to Sections 5.2 and 5.3, respectively. Combining Lemmas 3.7 and 3.8 we now have the following upper bound on the probability of $\text{Core}_1$ establishing its entropic stability.

**Proposition 3.9.** Fix $\delta, \varepsilon > 0$, and $\ell \geq 3$. For $\varepsilon$ sufficiently small and $(\log n)^{1/(\ell-2)} \ll np \leq (\log n)^\ell$ we have

$$\limsup_{n \to \infty} \log \mathbb{P}(\text{Core}_1) = -\frac{1}{2} \delta^2 \left( 1 - f_\ell(\varepsilon) \right),$$}

where $f_\ell(\cdot)$ is as in Proposition 3.3.

**Proof.** We begin by claiming that

$$\text{Core}_1 \setminus (\text{Core}_{1,1} \cup \text{Core}_{1,2}) \subset \left\{ \exists G \subset G(n, p) : G \text{ is a strong-core graph} \right\}.$$}
To see this we consider a graph $G'$ for which
\begin{equation}
(3.23) \quad \bar{e}(G') := e_{1,2}(G') + e_{2,2}(G') \leq \bar{C} n^2 p^2 \quad \text{and} \quad \mathcal{N}_{1,1}(C_{i}, G') \leq \varepsilon \delta n^i p^i.
\end{equation}
Let $G_0$ be the 2-core of the subgraph $G'_0$ obtained from $G'$ by removing all edges in $E_{1,1}(G')$. Since $G_0$ is a 2-core of $G'_0$ it is straightforward to note that
\[ N(C_{i}, G_0) = N(C_{i}, G'_0) \geq N(C_{i}, G') - \delta \varepsilon n^i p^i \geq \delta (1 - 4 \varepsilon) n^i p^i, \]
where the penultimate step is due to (3.23) and the last step is due to the fact that $G'$ is a core graph. Since $e(G_0) \leq \bar{e}(G') \leq \bar{C} n^2 p^2$ we now run a peeling procedure as in the proof of Lemma 3.2 to extract a further subgraph $G$ which is a strong-core graph. Since
\[ \text{Core}_{1}(\text{Core}_{1,1} \cup \text{Core}_{1,2}) \subset \{ \exists G \subset G(n, p) : G \text{ is a core graph and obeys (3.23)} \}, \]
we deduce (3.22). Therefore, as
\[ \text{Core}_{1,1} \subset \text{Core}_{1,1} \quad \text{and} \quad \text{Core}_{1,2} \subset \text{Core}_{1,2}, \]
using Lemmas 3.7-3.8, (3.22), and Proposition 3.3 we now arrive at (3.21). This completes the proof.

In the next section we derive the entropic stability of core graphs with at least $\bar{C} n^2 p^2 \log(1/p) \cdot (\log \log n)^{-2}$ edges, i.e. we find an appropriate bound on the probability of Core$_2$.

3.3. Core graphs with larger number of edges. Finding a suitable bound on the probability of Core$_2$ requires a chaining-type argument. To run the chaining procedure effectively we need a few more notation. Recall that we need to consider core graphs $G$ for which
\[ (\log \log n)^{-2} \bar{C} n^2 p^2 \log(1/p) \leq e(G) \leq \bar{C} n^2 p^2 \log(1/p). \]
We divide this range into dyadic scales. Set $L_n := \lfloor 2 \log_2(\log \log n) \rfloor + 1$. For $j = 1, 2, \ldots, L_n$, define
\[ J_j := \left\{ G \subset K_n : e(G) \in \left( 2^{-j} \bar{C} n^2 p^2 \log(1/p), 2^{-(j-1)} \bar{C} n^2 p^2 \log(1/p) \right) \right\}, \]
and
\[ J_{L_n+1} := \left\{ G \subset K_n : e(G) \leq 2^{-L_n} \bar{C} n^2 p^2 \log(1/p) \right\}. \]
It will be clear from below that during the chaining argument we may end up with graphs with no a-priori lower bound on its edges. Therefore in $J_{L_n+1}$ we do not impose any lower bound on the number of edges. The partition $\{ J_j \}_{j=1}^{L_n+1}$ naturally yields a partition of Core$_2$. For $j \in [L_n+1]$ we define
\[ \text{Core}_{2,j} := \{ \exists G \subset G(n, p) : G \text{ is a core graph and } G \in J_j \}. \]
Let us also define the following sequence of events: for $j \in [L_n + 1]$ we let
\[ \widetilde{\text{Core}}_{2,j} := \left\{ \exists G \subset G(n, p) : N(C_{i}, G) \geq (1 - 3 \varepsilon - s_j \varepsilon) \delta n^i p^i \text{ and } G \in J_j \right\}, \]
where for brevity we write $s_j := \sum_{i=1}^{j-1} 2^{-i}$. The difference in Core$_{2,j}$ and $\widetilde{\text{Core}}_{2,j}$ lies in the fact that the former event requires $N(C_{i}, G) \geq (1 - 3 \varepsilon) \delta n^i p^i$, whereas the latter requires a slightly weaker lower bound on $N(C_{i}, G)$. Furthermore the latter one does not need to obey (C3) of Definition 2.3. Therefore
\begin{equation}
(3.24) \quad \text{Core}_{2,j} \subset \widetilde{\text{Core}}_{2,j} \quad \text{for } j \in [L_n + 1].
\end{equation}
The rational behind defining the events $\{ \widetilde{\text{Core}}_{2,j} \}$ is as follows: During the chaining argument we need to iteratively run the peeling procedure already described in the proof Lemma 3.2. This results in loosing a small fraction of the number of copies of $C_{i}$ in the graph with which we start
the chaining argument. Hence one not only requires to bound the probabilities of \( \{ \text{Core}_{2,j} \} \) but also those of \( \{ \widetilde{\text{Core}}_{2,j} \} \). Bounding probabilities of \( \widetilde{\text{Core}}_{2,j} \) requires a further subdivision of it. We define

\[
(3.25) \quad \text{Core}_{2,j,\alpha} := \left\{ \exists G \subseteq G(n, p) : N(C_{\ell, G}) \geq (1 - 3\varepsilon - s_j\varepsilon)\delta n^\ell p^\ell, N_{1,1}(C_{\ell, G}) \geq 2^{-j}\varepsilon\delta n^\ell p^\ell, \text{ and } G \in J_j \right\}.
\]

Thus \( \text{Core}_{2,j,\alpha} \) can be considered to be the subset of \( \widetilde{\text{Core}}_{2,j} \) for which \( N_{1,1}(C_{\ell, G}) \) is large. Note that this threshold for \( N_{1,1}(C_{\ell, G}) \) to be considered to be large depends on \( j \). This will be crucial for our proof.

Next for a graph \( G \) we denote \( \varpi_2(G) \) to be the 2-core of the subgraph of \( G \) obtained by removing the edges in \( E_{1,1}(G) \) (recall (3.18)). We then denote

\[
(3.26) \quad \text{Core}_{2,j,\beta} := \left\{ \exists G \subseteq G(n, p) : \varpi_2(G), G \in J_j, \text{ and } N(C_{\ell, G}) \geq (1 - 3\varepsilon - s_j\varepsilon)\delta n^\ell p^\ell \right\}.
\]

The following two lemmas yield bound on the probabilities of \( \text{Core}_{2,j,\alpha} \) and \( \text{Core}_{2,j,\beta} \), respectively.

**Lemma 3.10.** Fix \( \delta, \varepsilon > 0 \), and \( \ell \geq 3 \). If \( p \leq n^{1/2} \) then we have

\[
\limsup_{n \to \infty} \frac{\max_{j \in [L_n]} \{ \log P(\text{Core}_{2,j,\alpha}) \}}{n^2p^2 \log(1/p)} = -\infty.
\]

**Lemma 3.11.** Let \( \delta, \varepsilon, \) and \( \ell \) be as in Lemma 3.10. If \( p \in (0, 1) \) is such that \( (\log n)^{1/(\ell - 2)} \ll np \leq (\log n)^{\ell} \) then we have

\[
\limsup_{n \to \infty} \frac{\max_{j \in [L_n]} \{ \log P(\text{Core}_{2,j,\beta}) \}}{n^2p^2 \log(1/p)} = -\infty.
\]

Proofs of Lemmas 3.10 and 3.11 are deferred to Sections 5.3 and 5.2, respectively. Building on Lemmas 3.10-3.11 we now have the result yielding a desired bound on the probability of \( \text{Core}_2 \).

**Proposition 3.12.** Fix \( \delta, \varepsilon > 0 \), and \( \ell \geq 3 \). For \( \varepsilon \) sufficiently small and \( p \in (0, 1) \) such that \( (\log n)^{1/(\ell - 2)} \ll np \leq (\log n)^{\ell} \) we have

\[
(3.27) \quad \limsup_{n \to \infty} \frac{\log P(\text{Core}_2)}{n^2p^2 \log(1/p)} \leq \frac{1}{2} \delta^2(1 - f_\ell(\varepsilon)),
\]

where \( f_\ell(\cdot) \) is as in Proposition 3.3.

**Proof.** We claim that for any \( j \in [L_n] \)

\[
(3.28) \quad P(\text{Core}_{2,j}) \leq P(\widetilde{\text{Core}}_{2,j}) \leq 2 \exp \left( -\frac{\delta^2 n^2p^2 \log(1/p)}{2} \right) + \sum_{j' = j+1}^{L_n+1} P(\widetilde{\text{Core}}_{2,j'})
\]

The first inequality is immediate from (3.24). To see the second inequality, we observe that if \( G \in J_j \) is graph such that

\[
(3.29) \quad N(C_{\ell, G}) \geq (1 - 3\varepsilon - s_j\varepsilon)\delta n^\ell p^\ell, \quad N_{1,1}(C_{\ell, G}) \leq 2^{-j}\varepsilon\delta n^\ell p^\ell \quad \text{and} \quad \varpi_2(G) \notin J_j
\]

then, as \( \varpi_2(G) \subseteq G \), we have that \( \varpi_2(G) \in \cup_{j' = j+1}^{L_n+1} J_{j'} \). Furthermore, denoting \( G' \) to be the subgraph of \( G \) obtained by removing edges in \( E_{1,1}(G) \) we note that

\[
N(C_{\ell, \varpi_2(G)}) = N(C_{\ell, G'}) \geq N(C_{\ell, G}) - N_{1,1}(C_{\ell, G}) \geq (1 - 3\varepsilon - s_j\varepsilon)\delta n^\ell p^\ell,
\]
for any $j' = j + 1, \ldots, L_n + 1$, where the final inequality is a consequence of (3.29). Thus the last two observations together imply that

$$\widetilde{\text{Core}_{2,j}} \setminus \left( \text{Core}_{2,j,\alpha} \cup \text{Core}_{2,j,\beta} \right) \subset \cup_{j' = j+1}^{L_n + 1} \widetilde{\text{Core}_{2,j'}}.$$

Now Lemmas 3.10-3.11 together with the union bound yield the second inequality of (3.28). To complete the proof of the proposition we use (3.28) with $j = 1$ to derive that

$$\mathbb{P}(\text{Core}_2) \leq \sum_{j=1}^{L_n+1} \mathbb{P}(\text{Core}_{2,j}) \leq 2 \exp \left( -\delta^2 n^2 p^2 \log(1/p) \right) + 2 \sum_{j=2}^{L_n+1} \mathbb{P}(\widetilde{\text{Core}_{2,j'}}).$$

Repeating the same procedure as above iteratively with $j = 2, 3, \ldots, L_n$, we arrive at the bound

$$\mathbb{P}(\text{Core}_2) \leq 2^{L_n+2} \left[ \exp \left( -\delta^2 n^2 p^2 \log(1/p) \right) + \mathbb{P}(\widetilde{\text{Core}_{2,L_n+1}}) \right].$$

So it now remains to evaluate the probability of $\widetilde{\text{Core}_{2,L_n+1}}$. To evaluate the same we recall that the number copies of $C_\ell$ in any graph $G$ and its 2-core are the same. Therefore recalling the definitions of $\text{Core}_{\ell,i}$, $i = 1, 2$, from (3.19)-(3.20), we derive that

$$\widetilde{\text{Core}_{2,L_n+1}} \setminus (\text{Core}_{1,1} \cup \text{Core}_{1,2})$$

$$\subset \left\{ \exists G \subset G(n,p) : N(C_\ell, G) \geq (1 - 3\varepsilon - s_{L_n, \varepsilon}) \delta n^p \ell, N_{\ell,1}(C_\ell, G) \leq \varepsilon \delta n^p \ell, \right.$$  

$$d_{min}(G) \geq 2, \text{ and } e_{1,2}(G) + e_{2,2}(G) \leq C_n n^2 p^2 \right\}$$

$$\subset \{ \exists G \subset G(n,p) : G \text{ is a strong-core graph} \},$$

where the last step follows by proceeding similarly as in the proof of (3.22). We omit the details.

Hence, applying Lemmas 3.7-3.8, and Proposition 3.3 we derive that

$$\limsup_{n \to \infty} \frac{\log \mathbb{P}(\text{Core}_{2,L_n+1})}{n^2 p^2 \log(1/p)} \leq -\frac{1}{2} \delta^2 (1 - f_\ell(\varepsilon)).$$

As $L_n = O(\log \log n) \ll (\log n)^{1/(\ell-2)} \ll np$ combining (3.30)-(3.31) we obtain (3.27). This completes the proof of the proposition. \qed

4. Strong-core graphs are entropically stable

In this section we prove Proposition 3.3. As outlined in Section 2 the proof relies on the fact that for any strong-core graph, except possibly a few “bad” edges, the product of the degrees of the two end points of any of its “good” edges satisfies a strong upper and lower bound. Below we provide a precise formulation of good and bad edges of a strong-core graph.

**Definition 4.1.** Fix $C_0 < \infty$ and let $G_{\text{high}} \subset G$ be the subgraph spanned by the edges $e = (u, v) \in E(G)$ for which

$$\deg_G(u) \cdot \deg_G(v) \geq C_0 n^2 p^2. \quad (4.1)$$

Set $G_{\text{low}} := G \setminus G_{\text{high}}$, i.e. $G_{\text{low}}$ is spanned by the edges for which (4.1) does not hold. Furthermore, we write $G_{\text{bad}} \subset G$ to denote the subgraph induced by the edges $e \in E(G)$ for which every copy of $C_\ell$ passing through it uses at least one edge belonging to $G_{\text{high}}$. 


In the following lemma we show that, if \( C_0 \) in (4.1) is chosen to be sufficiently large, then the number of edges in \( G_{\text{bad}} \) is only a small desired fraction of that in \( G \), and moreover the number of labelled copies of \( C_\ell \) in \( G_{\text{low}} \) is almost same as that in \( G \). Furthermore, we will establish a lower bound on the product of the degrees of any pair of adjacent vertices. These facts together will imply that one can work with \( G_{\text{low}} \) instead of \( G \) for which one has tight upper and lower bounds on the products of the degrees of the end points of any edge.

**Lemma 4.2.** Let \( G \) be a strong-core graph and \( p \in (0,1) \). Then, for any \( \varepsilon > 0 \), there exist \( 0 < c_0(\varepsilon), C_0(\varepsilon) < \infty \), such that the followings hold:

(a) For every edge \( e = (u,v) \in E(G) \)

\[
\deg_G(u) \cdot \deg_G(v) \geq c_0(\varepsilon)n^2p^2.
\]

(b) Let \( G_{\text{high}} := G_{\text{high}}(\varepsilon) \subset G \) be the subgraph spanned by the edges \( e \in E(G) \) for which (4.1) holds with \( C_0 = C_0(\varepsilon) \). Having defined \( G_{\text{high}} \), we let \( G_{\text{low}} := G_{\text{low}}(\varepsilon) \) and \( G_{\text{bad}} := G_{\text{bad}}(\varepsilon) \) to be as in Definition 4.1. Then

\[
N(C_\ell, G_{\text{low}}) \geq (1 - \varepsilon)N(C_\ell, G),
\]

and

\[
e(G_{\text{high}}) \leq \varepsilon e(G_{\text{bad}}) \leq \varepsilon e(G).
\]

Note that lower bound in Lemma 4.2(a) is similar to that in Lemma 3.6 while the former is sharper. This is due to the stronger bounds on \( e(G) \) and on \( N(C_\ell, G, e) \) in a strong-core graph \( G \). Bounds analogous to Lemma 4.2(b) have also been derived in [12] for core graphs. Repeating a same line of argument and using the bounds (SC2)-(SC3) of Definition 2.4 one can deduce Lemma 4.2. We include its proof in Appendix A for completeness.

Equipped with Lemma 4.2 we now proceed to the proof of Proposition 3.3.

**4.1. Proposition 3.3 for cycles of odd length.** When the length of the cycle is odd we provide an alternate shorter proof than the one outlined in Section 2. Using the fact that the length of the cycle is odd we show that the tight upper and lower bounds on the product of the degree of any two adjacent vertices in \( G_{\text{low}} \) translates to a lower bound on the degree of the non-isolated vertices of \( G \setminus G_{\text{bad}} \). This together with the upper bound on \( e(G_{\text{bad}}) \) yields a bound on the number of strong-core graphs with a given number of edges, which in turn produces an effective bound on the probability of the existence of a strong-core graph with that many edges. Below we carry out the details.

**Proof of Proposition 3.3 for \( C_{2t+1} \).** We begin by claiming that for every \( v \in V(G \setminus G_{\text{bad}}) \)

\[
(4.2) \quad \deg_G(v) \geq c_\star(\varepsilon)np,
\]

for some \( c_\star(\varepsilon) > 0 \). To prove this we note that any such vertex \( v \) must participate in at least one labelled copy \( C \) of \( C_{2t+1} \) that is contained in \( G_{\text{low}} \). Let \( V(C) := \{v_1, v_2, \ldots, v_{2t+1}\} \). For ease of writing, without loss of generality, let us also assume that the vertices are labelled so that \( v = v_1 \), and \( (v_i, v_{i+1}) \in E(G_{\text{low}}) \) for \( i \in [2t] \) and \( (v_{2t+1}, v_1) \in E(G_{\text{low}}) \). It follows that

\[
(4.3) \quad 2 \log \deg_G(v) = \sum_{i=1}^{(t+1)} \log (\deg_G(v_{2i-1}) \cdot \deg_G(v_{2i})) - \sum_{i=1}^{t} \log (\deg_G(v_{2i}) \cdot \deg_G(v_{2i+1})),
\]

\[8\text{A similar argument has appeared in [12, Claim 7.7] for core graphs when } np \gg (\log n)^{\Delta^2} \text{ and } H \text{ is a } \Delta\text{-regular graph.} \]
where for ease of writing we set \( v_{2t+2} = v_1 \). Using the upper and lower bounds on the product of the degrees of the two end points of an edge in \( G_{\text{low}} \) derived in Lemma 4.2, and setting
\[ c_*(\varepsilon) := \left( \min \left\{ c_0(\varepsilon), C_0(\varepsilon)^{-1} \right\} \right)^{t/2} \]
we now immediately arrive at (4.2). Using this lower bound on the degree of the non-isolated vertices of \( G \setminus G_{\text{bad}} \) we therefore obtain that
\[
(4.4) \quad |V(G \setminus G_{\text{bad}})| \leq \frac{2e(G)}{c_*(\varepsilon)np}.
\]
Equipped with (4.4) we next bound the number of strong-core graphs with \( e(G) = e \) as follows:

1. Choose the non-isolated vertices of \( G \setminus G_{\text{bad}} \).
2. Choose \( e(G_{\text{bad}}) \) edges arbitrarily out of all possible edges of the complete graph on \( n \) vertices to construct \( G_{\text{bad}} \).
3. Choose \( e - e(G_{\text{bad}}) \) edges out of \( \binom{|V(G \setminus G_{\text{bad}})|}{2} \) possible choices to construct \( G \setminus G_{\text{bad}} \).
4. Finally take a union over \( e(G_{\text{bad}}) \) in the allowable range \([0, \varepsilon]\).

To implement steps (1)-(4) we need some bounds for which we make the following observations: For any graph \( G \) one has \( N(C_{2t+1}, G) \leq (2e(G))^{(2t+1)/2} \) (see [12, Lemma 5.5]). Since for any strong-core graph we have \( N(C_{2t+1}, G) \geq \delta (1 - 6\varepsilon)n^{2t+1}p^{2t+1} \) it immediately implies that the minimum number of edges of a strong-core graph, denoted hereafter by \( \hat{e} \), satisfies the lower bound
\[
(4.5) \quad \hat{e} \geq \hat{e}_0(\delta(1 - 6\varepsilon)),
\]
where for any \( \delta_0 > 0 \) we set \( \hat{e}_0(\delta_0) := e_0(\delta_0, 2t + 1) \) and
\[
(4.6) \quad e_0(\delta_0, \ell) := \frac{1}{2} \delta_0^3 n^2 p^2.
\]
That is, \( e_0(\delta_0, \ell) \) is the minimum number of edges a graph must possess to have at least \( \delta_0 n^\ell p^\ell \) labelled copies of \( C_\ell \).

So for any \( e \geq \hat{e} \), upon shrinking \( c_*(\varepsilon) \), if necessary, one also has that
\[
(4.7) \quad e \leq \frac{1}{3} \cdot \frac{4e^2}{c_*(\varepsilon) n^2 p^2}.
\]

Now denote \( I_e \) to be set of strong-core graphs with \( e \) edges. Following the steps (1)-(4) to bound the cardinality of \( I_e \), upon applying (4.4), (4.7), and Stirling’s approximation it yields that there exists some constant \( C < \infty \), depending on \( \delta \) and \( \varepsilon \), such that
\[
(4.8) \quad |I_e| \leq \sum_{e \leq e} \left( \frac{n^2}{2e/(c_*(\varepsilon)np)} \right)^e \cdot \left( \frac{4e^2/(c_*(\varepsilon)np)^2}{e - e} \right)^e \leq \sum_{e \leq e} \left( \frac{ec_*(\varepsilon)n^2p^{2}}{2e} \right)^e \cdot \left( \frac{4e^2}{c_*(\varepsilon) n^2 p^2} \right)^e \leq e \cdot \left( \frac{1}{p} \right)^{6e} \cdot C^e \leq \left( \frac{1}{p} \right)^{7e},
\]
for any \( e \geq \hat{e} \) and all large \( n \). The second step in (4.8) follows from Stirling’s approximation, the fact that for nonnegative integers \( y \leq x \) the binomial coefficient \( \binom{x}{y} \) is increasing for \( y \leq \lfloor x/2 \rfloor \), and (4.7). Whereas, the third step uses that \( e \leq C_* n^2 p^2 \) and \( p \leq n^{-1/2} \). Finally to obtain the last inequality above we recall that \( p = o(1) \).
Equipped with (4.8) we now take a union bound over $e \in [\bar{e}, \bar{C}, n^2 p^2]$ to find that
\[
\log \mathbb{P}(\exists \text{ a strong-core graph}) \leq \log \left( \sum_{e=\bar{e}}^{C, n^2 p^2} \mathbb{P}(G \subset G(n, p) : G \in \mathcal{I}_e) \right) = \log \left( \sum_{e=\bar{e}}^{C, n^2 p^2} p^{e(1-7\varepsilon)} \right)
\leq \log 2 - (1 - 7\varepsilon) \cdot \bar{e} \cdot \log(1/p)
\leq \log 2 - \frac{1}{2} \delta^{2t+1} \cdot (1 - f_{2t+1}(\varepsilon)) \cdot n^2 p^2 \log(1/p),
\]
for all large $n$, where $f_{2t+1}(\varepsilon) := 1 - (1 - 6\varepsilon) \frac{2t}{2t+1} \cdot (1 - 7\varepsilon)$. Dividing both sides by $n^2 p^2 \log(1/p)$ and then sending $n$ to infinity the proof completes for $C_{2t+1}$. □

When the length of the cycle is even we lose the identity (4.3). Therefore one cannot repeat the above argument. In fact, as already mentioned earlier, one can have strong-core graphs with many of its vertices having small degrees. Thus one indeed needs to follow the route outlined in Section 2. Recall from there that we split a strong-core graph $G$ into two subgraphs: a bipartite subgraph $G_b$ and $G \setminus G_b$.

Upon assuming a lower bound on the number of copies of $C_2t$ in $G_b$ we next derive a lower bound on the difference on the number of edges of $G_b$ and its number of vertices of low degree. This bound would lead to establishing that $G_b$ is entropically stable. This is the content of the following section.

4.2. Lower bound on the difference of the number of edges and vertices. To prove Proposition 3.3 we will need to be able to use the tight upper and lower bounds on the product of the degrees of Lemma 4.2. Therefore, similar to the last section, we need to work with $G_{low}$ again. It will be verified below that any bipartite subgraph $G$ of it must satisfy the following property.

**Assumption 4.3.** Let $\bar{G}$ be a bipartite graph with two parts $U_1$ and $U_2$, i.e. $E(\bar{G}) \subset U_1 \times U_2$ for some disjoint set of vertices $U_1$ and $U_2$. Assume that there exists $U_{1,1} \subset U_1$ such that
\[
\min_{u \in U_{1,1}} \deg_{\bar{G}}(u) \geq 2
\]
and
\[
|U_1 \setminus U_{1,1}| \leq \varepsilon_2 e(\bar{G}),
\]
for some $\varepsilon_2 \in (0, 1)$.

Under the above assumption we find a lower bound $e(\bar{G}) - |U_1|$ which will be used later in showing the entropic stability of $G_b$.

**Lemma 4.4.** Fix $\delta_1 > 0$. Let $\bar{G}$ be a bipartite graph with parts $U_1$ and $U_2$ satisfying Assumption 4.3. Let $t \geq 2$. If
\[
N(C_{2t}, \bar{G}) \geq \delta_1 n^{2t} p^{2t}
\]
then
\[
e(\bar{G}) - |U_1| \geq \frac{1}{2} \delta_1 \cdot (1 - \varepsilon_2)^{\frac{1}{2}} n^2 p^2.
\]

**Proof.** Since $\bar{G}$ is a bipartite graph with its partite sets being $U_1$ and $U_2$, any labelled copy of $C_{2t}$ in $\bar{G}$ must have either its odd or its even indexed vertices in $U_1$. 
Given any $u = \{u_1, u_2, \ldots, u_t\} \in U_1^t$, a set of $t$ distinct vertices from $U_1$ we denote $N(C_{2t}, \bar{G}, u)$ to be the number of copies of $C_{2t}$ where the vertex $u_i$ gets mapped to the $(2i-1)^{th}$ vertex of $C_{2t}$, for $i \in [t]$. Let $C$ be one such copy of $C_{2t}$ with $v_1, v_2, \ldots, v_t \in U_2$ being the remaining vertices of $C$ so that $v_i$ is mapped to the $(2i)^{th}$ vertex of $C_{2t}$. So $v_i$ is a common neighbor of $u_i$ and $u_{i+1}$, for $i \in [t]$, where for ease of writing we set $u_{t+1} = u_1$. Since $\{v_i\}_{i=1}^{t-1}$ are all distinct we note that having chosen $\{v_i\}_{i=1}^{t-1}$ the number of choices of $v_t$ is bounded above by $\deg_G(u_t) - 1$. Therefore iterating this argument we deduce that

$$\sum_{u \in U_1^t} N(C_{2t}, \bar{G}, u) \leq \sum_{u \in U_1^t} \deg_G(u_1) \cdot \prod_{i=2}^t (\deg_G(u_i) - 1) \leq \left(\sum_{u \in U_1} \deg_G(u)\right) \cdot \left(\sum_{u \in U_1} (\deg_G(u) - 1)\right)^{t-1} \tag{4.12}$$

where the last equality is a consequence of the fact that $\bar{G}$ is a bipartite graph. Furthermore, using (4.9)-(4.10) we see that

$$2|U_1| \leq \sum_{u \in U_1} \deg_G(u) + |U_1 \setminus U_{1,1}| \leq (1 + \varepsilon_2)e(\bar{G})$$

and therefore

$$e(\bar{G}) \leq \frac{2}{1 - \varepsilon_2} (e(\bar{G}) - |U_1|).$$

Plugging this bound in (4.12) we find that

$$N(C_{2t}, \bar{G}) = 2 \sum_{u \in U_1^t} N(C_{2t}, \bar{G}, u) \leq \frac{4}{1 - \varepsilon_2} (e(\bar{G}) - |U_1|)^t.$$

Since $t \geq 2$, the proof now finishes by using the lower bound on $N(C_{2t}, \bar{G})$. \qed

In the next section we derive a bound on the number of core graphs (and hence also for strong-core graphs) in terms of its number of edges and the number of vertices of small degree. This combinatorial lemma will be used to derive the entropic stability of $G \setminus G_b$.

4.3. Bound on the number of core graphs. The following is main result of this section.

**Lemma 4.5.** Fix $\varepsilon \in (0, 1)$, $\ell \geq 3$, and integers $D := D(\varepsilon)$ and $D := D(\varepsilon)$ such that $D \geq D \geq 32/\varepsilon$. Let

$$\mathcal{V}_1 := \{v \in V(G) : \deg_G(v) \leq D\}$$

and set $\overline{V}_1 := V(G) \setminus \mathcal{V}_1$. Further let $N_0(e, v, D)$ be the number of core graphs with $e(\bar{G}) = e$ and $|\mathcal{V}_1| = v$. Then, for any $p \leq n^{-1/2}$ and all large $n$,

$$N_0(e, v, D) \leq \frac{n}{v} \cdot \exp(\varepsilon e \log(1/p)).$$

**Proof.** We split the proof into two parts. First let us consider the easier case $np = O((\log n)^\ell)$, and then we consider the case $np = \Omega((\log n)^\ell)$.

Since $\min_{v \in \overline{V}_1} \deg_G(v) \geq D \geq D$ we have that

$$|\overline{V}_1| \leq 2e/D,$$

For two sets of positive reals $\{a_n\}$ and $\{b_n\}$ the notation $a_n = \Omega(b_n)$ means $\lim \inf_{n \to \infty} a_n/b_n > 0$. \hfill $\square$
where $e(G) = e$. Thus, using the lower bound on $D$, we find that the number of ways to choose the vertices in $\overline{V}_1$ can be bounded by

$$\sum_{u=0}^{\min\{2e/D,n\}} n^u \leq n^{4e/D} \leq n^{4e} \leq p^{-\frac{4e}{9}},$$

(4.13)

where in the last step we use the fact that $p \leq n^{-1/2}$. For ease of writing let us denote $G_1 \subset G$ be the subgraph induced by the edges in $E(G)$ that are incident to some vertex in $V_1$.

Next we note that the number of ways to choose the edges of $G$ that are not adjacent to any vertex in $V_1$ (and hence both end points must be in $\overline{V}_1$) can be trivially bounded by

$$|\overline{V}_1|^{2e} \leq \exp(O(\log \log n) \cdot e) \leq \exp\left(\frac{\varepsilon}{8} e \log(1/p)\right),$$

(4.14)

for all large $n$, where we have used the fact that $np = O((\log n)^{\ell})$ and $|V(G)| \leq e = O(n^2 p^2 \log(1/p))$. Now we need to bound the number of ways to choose the edges of $G_1$, which we denote by $e_1$. It is easy to see that, applying Stirling’s approximation, this can be bounded by

$$\left(\frac{|V_1| \cdot |V(G)|}{e_1}\right) \leq (2e|V(G)|)^{e_1} \leq \exp\left(\frac{\varepsilon}{8} e \log(1/p)\right),$$

(4.15)

for all large $n$, where we have used the fact that $|V_1| \leq 2e_1$. Finally the number of ways to choose the vertices in $V_1$ such that $|V_1| = v$ is bounded by $\binom{n}{v}$. Therefore, combining the bounds in (4.13)-(4.15) we derive that the number of core graphs with $|V_1| = v$, $e(G_1) = e_1$ and $e(G) = e$ is bounded by

$$\binom{n}{v} \cdot \exp\left(\frac{3\varepsilon}{4} e \log(1/p)\right).$$

Since

$$\log(e_1) \leq \log(e) \leq \frac{\varepsilon}{4} e \log(1/p),$$

for all large $n$, finally taking an union bound over the ranges of $e_1$ we derive the desired upper bound on $N_0(e, v, D)$.

Next we consider the regime $np = \Omega((\log n)^{\ell})$. As we have already seen in (4.13) that the choices of the number of vertices of $\overline{V}_1$ can be adequately bounded for the entire regime $p \leq n^{-1/2}$. However, the arguments used to derive the bounds (4.14)-(4.15) becomes ineffective when $np$ is a fractional power of $n$. In this regime, we use the bound derived in Lemma 3.6 to deduce that any two arbitrary vertices cannot be connected. This significantly reduces the cardinality of the possible edge set.

Turning to implement this idea we split the vertices in $\overline{V}_1$ as follows: For $k = 1, 2, \ldots, k_*$, we consider the following nested sequence of sets of vertices

$$W_{k_*} \subset W_{k_*-1} \subset \cdots \subset W_2 \subset W_1,$$

where

$$W_k := \{v \in \overline{V}_1: \deg_G(v) \geq 2^{k-1}D\},$$

and

$$k_* := \left\lfloor \frac{\log(n/D)}{\log 2} \right\rfloor.$$

For ease of writing we set $W_{k_*+1} := \emptyset$. Let $u \in W_{i_0-1} \setminus W_{i_0}$ for some $i_0 \geq 2$. This implies that $\deg_G(u) \leq 2^{i_0-1}D$. Therefore, if $(u, v) \in E(G)$ for some $v \in V(G)$ then using Lemma 3.6 we have
that
\begin{equation}
\deg_G(u) \cdot \deg_G(v) \geq \frac{\bar{c}_0(\varepsilon) \cdot e(G)}{(\log n)^\ell}.
\end{equation}

We claim that the above implies that \( v \in W_{j_0} \), where \( j_0 := j_0(i_0) \) is the smallest integer satisfying
\begin{equation}
2^{i_0+j_0-1} \geq \frac{\bar{c}_0(\varepsilon) \cdot e(G)}{(\log n)^\ell}.
\end{equation}

If not, then
\[
\deg_G(u) \cdot \deg_G(v) \leq 2^{i_0+j_0-2} \cdot \frac{\bar{c}_0(\varepsilon) \cdot e(G)}{(\log n)^\ell} < \frac{\bar{c}_0(\varepsilon) \cdot e(G)}{(\log n)^\ell},
\]
where the last step is a consequence of the definition of \( j_0(i_0) \). However, this contradicts (4.16).

A similar argument also shows that if \( u \in V_1 = \{ v \in V(G) : \deg_G(v) \leq \mathcal{D} \} \) then any of its adjacent vertices must be in \( W_{j_*} \), where \( j_* \) is the smallest integer satisfying
\begin{equation}
2^{j_*} \cdot \mathcal{D}^2 \geq \frac{\bar{c}_0(\varepsilon) \cdot e(G)}{(\log n)^\ell}.
\end{equation}

Note that the number of edges in a core graph must be \( \Omega(n^2 \rho^2) \) (this is again a consequence of [12, Lemma 5.5]). Therefore \( j_* \gg 1 \). This in particular implies that any vertex in \( V_1 \) cannot connect to another vertex in \( V_1 \). Furthermore, it is easy to note that for any \( i \geq 1 \),
\begin{equation}
|W_i| \leq (4/\mathcal{D}) \cdot 2^{-i} \cdot e(G).
\end{equation}

Equipped with these observations we bound the number of core graphs with \( |V_1| = \nu \) and \( e(G) = e \) as follows:

1. Choose the vertices in \( V_1 \).
2. Choose the vertices in \( \{W_{i_0-1} \setminus W_{i_0}\}_{i_0=2}^{k_*+1} \).
3. Choose \( e \) edges from the set of possible edges
\[
\mathfrak{M} := \{(u,v) : u \in V_1, v \in W_{j_*}\} \cup \bigcup_{i_0=2}^{k_*+1} \{(u,v) : u \in W_{i_0-1} \setminus W_{i_0}, v \in W_{j_0(i_0)}\}.
\]

Let us find a bound on the cardinality of \( \mathfrak{M} \). Using the bound \( |V_1| \leq \nu \), and (4.18)-(4.19) we find that
\[
|\{(u,v) : u \in V_1, v \in W_{j_*}\}| \leq \nu \cdot |W_{j_*}| \leq \frac{4\mathcal{D}(\log n)^\ell}{\bar{c}_0(\varepsilon)} \cdot \nu.
\]

Similarly using (4.17) and (4.19), for any \( i_0 \geq 2 \) we deduce that
\[
|\{(u,v) : u \in W_{i_0-1} \setminus W_{i_0}, v \in W_{j_0(i_0)}\}| \leq \frac{8(\log n)^\ell}{\bar{c}_0(\varepsilon)} \cdot \nu.
\]

Thus
\begin{equation}
|\mathfrak{M}| \leq (k_*+1) \cdot \frac{8\mathcal{D}(\log n)^\ell}{\bar{c}_0(\varepsilon)} \cdot \nu \leq \frac{8\mathcal{D}(\log n)^{\ell+1}}{\log 2 \cdot \bar{c}_0(\varepsilon)} \cdot \nu.
\end{equation}

Next we aim to obtain a bound on the number of choices of the vertices belonging to \( \{W_{i_0-1} \setminus W_{i_0}\}_{i_0=2}^{k_*+1} \).

Using (4.19) we find that for any \( i_0 = 2, 3, \ldots, k_*+1 \)
\[
\nu_i := |W_{i_0-1} \setminus W_{i_0}| \leq \frac{8}{\mathcal{D}} \cdot 2^{-i} \cdot \nu.
\]
Thus the number of ways to choose the vertices \( \{W_{i_0-1}\setminus W_{i_0}\}_{i_0=2}^{k_1+1} \) is bounded by

\[
(4.21) \quad \prod_{i_0=2}^{k_1+1} \left( \sum_{w_{i_0}=0}^{((8/D)2^{-i_0} - e)} n^{w_{i_0}} \right) \leq \prod_{i_0=2}^{k_1+1} n^{(16/D)2^{-i_0} - e} \leq n^{8e/D} \leq p^{-\frac{e}{2}},
\]

where in the last step we again use the fact that \( p \leq n^{-1/2} \) and also the lower bound on \( D \geq D \).

Therefore proceeding as in steps (1)-(3) and applying (4.20)-(4.21) we now derive hat

\[
\mathcal{N}_0(e, v, D) \leq \left( \frac{n}{v} \right) \cdot p^{-\frac{e}{2}} \cdot \left( \frac{|M|}{e} \right) \leq \left( \frac{n}{v} \right) \cdot p^{-\frac{e}{2}} \cdot \left( \frac{16eD(\log n)^{t+1} - e}{c_0(\varepsilon) \cdot \log 2} \right) \leq \left( \frac{n}{v} \right) \cdot p^{-\frac{e}{2}},
\]

for all large \( n \), where in the last step we once again use \( p \leq n^{-1/2} \). This completes the proof of the lemma. \( \square \)

Finally we in the following section, upon combining the result of this and previous section we prove the entropic stability of the strong-core graphs for cycles of even length.

4.4. **Proposition of 3.3 for cycles of even length.** Before going to the proof of Proposition 3.3 we remind the reader that we would like to choose \( G_b \) in such a way so that almost all copies of \( C_3 \) in \( G \) are either completely contained in \( G_b \) or in \( G \setminus G_b \). This necessitates the following decomposition of the vertices of the subgraph \( G_{\text{low}} := G_{\text{low}}(\varepsilon) \).

** Decomposition of the vertex set. ** Let \( t \geq 2 \) and \( D := D(\varepsilon) \) be as in Lemma 4.5. Set

\[
(4.22) \quad C_3 := C_3(\varepsilon) := (t - 1) \left( \left\lceil \frac{(2\bar{C}_3)\ell}{\delta\varepsilon} \right\rceil + 2 \right).
\]

Now for \( i \in [C_3] \) define

\[
V_i := \{ v \in V(G_{\text{low}}) : D_{i-1} + 1 \leq \deg_G(v) \leq D_i \},
\]

where

\[
D_i := D_i(\varepsilon) := D \cdot \left( \frac{C_0(\varepsilon)}{c_0(\varepsilon)} \right)^{i-1},
\]

with \( C_0(\varepsilon) \) and \( c_0(\varepsilon) \) as in Lemma 4.2, and we set \( D_0 := 0 \). Denote

\[
\bar{V}_1 := \{ v \in V(G_{\text{low}}) : (u, v) \in E(G_{\text{low}}) \text{ for some } u \in V_1 \}.
\]

For \( i = 2, 3, \ldots, C_3 \), we then iteratively define

\[
\bar{V}_i := \{ v \in V(G_{\text{low}}) : (u, v) \in E(G_{\text{low}}) \text{ for some } u \in V_i \}\setminus \bar{V}_{i-1},
\]

and let

\[
V_i := V(G_{\text{low}}) \setminus \left( \bigcup_{i=1}^{C_3} V_i \cup \bar{V}_i \right).
\]

For ease of writing, for \( i \in [C_3] \) let us also denote \( G_{i,g} \) to be subgraph spanned by the edges that are incident to some vertex in \( \bigcup_{j=1}^{i} V_j \), and \( \bar{G}_{i,g} \) to be its complement graph when \( G_{i,g} \) is viewed as a subgraph of \( G_{\text{low}} \).

Equipped with the above notation we now proceed to the proof of Proposition 3.3 for \( C_{2t} \).
Proof of Proposition 3.3 for $C_{2t}$. We begin the proof by first identifying the subgraph $G_b$ having the desired property mentioned above. To this end, for any strong-core graph $G$, as $e(G) \leq C_* n^2 t^2 p^2$, it follows from [12, Lemma 5.5] that

\begin{equation}
N(C_{2t}, G) \leq (2C_*)^t \cdot n^{2t} p^{2t}.
\end{equation}

Therefore there exists $i_* \in [C_3 - t + 1]$ such that

\begin{equation}
N(C_{2t}, G_{i_*+(t-1),g}) - N(C_{2t}, G_{i_*g}) \leq \varepsilon n^{2t} p^{2t}.
\end{equation}

Otherwise, as $G_{C_3g} \subseteq G$ and $t \geq 2$,

\begin{equation}
N(C_{2t}, G) \geq \left[ \frac{C_3}{(t-1)-1} \sum_{i=1}^{C_3/(t-1)-1} N(C_{2t}, G_{(t-1)(i+1),g}) - N(C_{2t}, G_{(t-1)(i-1)+1,g}) \right] + N(C_{2t}, G_{1g})
\end{equation}

\begin{equation}
\geq \left( \frac{C_3}{t-1} - 1 \right) \varepsilon n^{2t} p^{2t} > (2C_*)^t n^{2t} p^{2t},
\end{equation}

yielding a contradiction to (4.23), where the last inequality follows by recalling the definition of $C_3$ (see (4.22) above). Next we make the other observation:

**Claim 4.6.** Fix $i \in [C_3 - t + 1]$. Any copy of $C_{2t}$ in $G_{low}$ that uses an edge of $G_{i,g}$ must be contained in $G_{i+(t-1),g}$.

Equipped with Claim 4.6 we note that any copy of $C_{2t}$ that uses edges of both $G_{i_*,g}$ and $G_{i_*g}$ must be contained in $G_{i_*(t-1),g}$ but not in $G_{i_*g}$. Hence by (4.24) the number of such cycles is at most $\varepsilon n^{2t} p^t$. We note that any labelled copy of $C_{2t}$ in $G_{low}$ must be either contained in $G_{i_*,g}$, or $G_{i_*g}$, or must use edges of both $G_{i_*,g}$ and $G_{i_*g}$. Therefore from Lemma 4.2(b) it now follows that

\begin{equation}
N(C_{2t}, G_{i_*,g}) + N(C_{2t}, G_{i_*g}) \geq N(C_{2t}, G_{low}) - \varepsilon n^{2t} p^{2t}
\end{equation}

\begin{equation}
\geq (1 - \varepsilon)N(C_{2t}, G) - \varepsilon n^{2t} p^{2t} \geq \delta(1 - 8\varepsilon)n^{2t} p^{2t},
\end{equation}

where in the last step we use the fact that $G$ is a strong-core graph.

Thus setting $G_b = G_{i_*,g}$ we indeed have that almost all the copies of $C_t$ in $G$ are either contained in $G_b$ in $G \setminus G_b$. However, we note that this splitting procedure is not identical for all strong-core graphs, which is captured by the existence of some $i_\star \in [C_3]$ that may very well vary for different graphs. Nevertheless, as we will see below this indeterminacy in the parameter $i_\star$ results in another union bound. This additional bound turns out to be harmless for our purpose.

Before proceeding further let us prove Claim 4.6. Turning to do this we fix an edge $e = (u, \bar{u}) \in E(G_{i_*,g}) \subseteq E(G_{low})$ for some $i \in [C_3 - t + 1]$. Without loss of generality assume that $u \in \bigcup_{j=1}^{t} V_j$.

From definition of the set $V_j$ we have that $deg_G(u) \leq D_{i}$ and therefore from Lemma 4.2(a) it follows that

\begin{equation}
\text{deg}_G(\bar{u}) \geq \left( \frac{C_0(\varepsilon)}{c_0(\varepsilon)} \right) / D_{i} \cdot n^{2} p^{2}.
\end{equation}

Let $w$ be any vertex adjacent to $\bar{u}$ in $G_{low}$. Using Lemma 4.2(b) we deduce that

\begin{equation}
\text{deg}_G(w) \leq \left( \frac{C_0(\varepsilon)}{c_0(\varepsilon)} \right) \cdot D_{i} = D_{i+1},
\end{equation}

which in particular implies that $w \in \bigcup_{j=1}^{t+1} V_j$. This shows that any edge adjacent to some edge in $G_{i_*,g}$ must be in $G_{i_*g}$.

Now consider a labelled copy $C$ of $C_{2t}$ that uses an edge of $G_{i_*g}$. For ease of writing, let us index the edges of $C$ as $\{e_1, e_2, \ldots, e_{2t}\}$ so that for $i \in [2t - 1]$ the edge $e_i$ is adjacent to $e_{i+1}$, and $e_{2t}$ is
adjacent to $e_1$. Since $C$ uses an edge of $G_{i,s}$, without loss of generality, we may further assume that $e_1 = (u, v)$ and $e_2 = (u, u)$ for some $u \in \cup_{j=1}^{t+1} V_j$. Therefore, upon using the observation from the paragraph above we find that $e_2, e_{2t-1} \in E(G_{i+1,s})$. As the number of edges in $C_{2t}$ is $2t$, proceeding iteratively we find that all edges of $C$ must be contained in $E(G_{i+(t-1),s})$. This proves Claim 4.6.

Returning to the proof of the proposition we observe that (4.25) implies that

$$\{ \exists \text{ a strong-core graph} \} \subset \bigcup C_t = 1 \bigcup e_{0}(\delta(1-6\varepsilon)) \{ \exists G \subset \mathcal{G}(n, p) : G \in \mathcal{I}_{i,e} \},$$

where

$$\mathcal{I}_{i,e} := \left\{ G : G \text{ is a strong-core graph with } e(G) = e \right\}$$

$$N(C_{2t}, G_{i,s}) + N(C_{2t}, \bar{G}_{i,s}) \geq \delta(1 - 8\varepsilon)n^{2t^2}p^{2t^2},$$

where $\bar{e}_0(\delta_0) := e_0(\delta_0, 2t)$ for $\delta_0 > 0$, and we recall from (4.6) that $e_0(\delta_0, 2t)$ is the minimum number of edges that a graph must possess to have $\delta_0 n^{2t^2}p^{2t}$ labelled copies of $C_{2t}$. Thus, to find an upper bound on the probability of the LHS of (4.27) it suffices to prove the same for $\mathcal{I}_{i,e}$, and then take a union bound over the allowable range of $i_s$ and $e$.

We split $\mathcal{I}_{i,e}$ further into two subsets: $N(C_{2t}, G_{i,s})$ is small and $N(C_{2t}, \bar{G}_{i,s})$ is large. Let us first consider the case when $N(C_{2t}, G_{i,s})$ is small.

**Case 1.** $N(C_{2t}, G_{i,s}) \leq \varepsilon n^{2t^2}p^{2t}$.

In this case as $N(C_{2t}, G_{i,s})$ is small the graph $G_{i,s}$ can potentially be close to an empty graph. So we cannot use the entropic stability of it. We need to rely on the entropic stability of $G_{i,s}$.

Turning to make this idea precise we fix $e_\# \leq e$ and for ease of writing denote

$$\mathcal{I}_{i,e}^{(1)} := \left\{ G : G \in \mathcal{I}_{i,e}, e(G) = e_\#, \text{ and } N(C_{2t}, G_{i,s}) \leq \varepsilon n^{2t^2}p^{2t} \right\}.$$

We aim to derive a bound on the probability of the event $\{ \exists G \subset \mathcal{G}(n, p) : G \in \mathcal{I}_{i,e}^{(1)} \}$. To achieve this goal we will apply Lemma 4.5 with

$$\mathcal{V}_i = \left( \bigcup_{j=1}^{t+1} V_j \right) \bigcup \{ v \in V(G_{\text{high}}) : \text{deg}_{G}(v) \leq D_i \},$$

and

$$\mathcal{D} = D_i.$$

Before applying that lemma we need to make several observations.

From the definition of $\mathcal{V}_i$ it follows that $\mathcal{V}_i \setminus (\bigcup_{j=1}^{t+1} V_j) \subset V(G_{\text{high}})$. Thus

$$|\mathcal{V}_i| \leq |V(G_{\text{high}})| + \left( \bigcup_{j=1}^{t+1} V_j \right) \setminus V(G_{\text{bad}}) \leq |V(G_{\text{bad}})|,$$

where we recall the definition of $G_{\text{bad}}$ from Definition 4.1. We next note that if $v \notin V(G_{\text{bad}})$ then there exists at least one copy $C_{2t}$ passing through $v$ contained in $G_{\text{low}}$. This, in particular implies that $\text{deg}_{G_{i,s}}(v) \geq 2$ for all $v \in (\bigcup_{j=1}^{t+1} V_j) \setminus V(G_{\text{bad}})$. From (4.26), as $np \gg 1$, we also observe that $G_{i,s}$ is a bipartite graph with one part $\bigcup_{j=1}^{t+1} V_j$. Hence using Lemma 4.2(b), as

$$|V(G_{\text{high}})| \leq |V(G_{\text{bad}})| \leq 2\varepsilon(e(G) + e(G_{i,s})),$$

from (4.30) we derive that

$$2|\mathcal{V}_i| \leq 2 \left( |V(G_{\text{high}})| + \left( \bigcup_{j=1}^{t+1} V_j \right) \setminus V(G_{\text{bad}}) \right) \leq 8\varepsilon(e(G) + e(G_{i,s})).$$
We now apply Lemma 4.5 to find that the cardinality of the set of graphs belonging to $\mathcal{I}_{i,e,e_\#}^{(1)}$ with $|\mathcal{V}_1| = v$ is bounded by

$$n^2 p^{e - v - \varepsilon e} \leq p^{2 \varepsilon e} \leq p^{e_\#} \cdot p^{-9\varepsilon e},$$

for all large $n$, where in the first step we used the fact that $p \leq n^{-1/2}$, and in the last step we used (4.31) and the fact that for any $G \in \mathcal{I}_{i,e,e_\#}^{(1)}$ one has $e(G_{i,e}) = e_\#$.

As the probability of observing any graph with $e$ edges is $p^e$ taking an union over $v \leq 2e \leq 2C_n n^2 p^2$ we conclude that

$$\mathbb{P}(G \subset G(n,p) : G \in \mathcal{I}_{i,e,e_\#}^{(1)}) \leq 2C_p n^2 p^2 \cdot \exp (-\log(1/p) \{(1 - 9\varepsilon)(e - e_\#) - 9\varepsilon e_\#\}).$$

From the definition of $\mathcal{I}_{i,e,e_\#}^{(1)}$ it further follows that

$$N(C_{2t}, G_{i,e}) \geq \delta (1 - 9\varepsilon)n^2p^2t$$

for any $G \in \mathcal{I}_{i,e,e_\#}^{(1)}$. Thus

$$e - e_\# = e(G) - e(G_{i,e}) \geq e(G_{i,e}) \geq \frac{1}{2} \delta t (1 - 9\varepsilon) \frac{1}{2} n^2 p^2,$$

where the final lower bound is again a consequence of [12, Lemma 5.5]. Furthermore, as $G$ is a strong-core graph,

$$e_\# \leq e \leq C_n n^2 p^2.$$

Hence summing both sides of (4.32) over the allowable range of $e_\#$ and $e$, given by (4.33)-(4.34), we derive that

$$\mathbb{P} \left( \bigcup_{e,e_\#} \left\{ \exists G \subset G(n,p) : G \in \mathcal{I}_{i,e,e_\#}^{(1)} \right\} \right) \leq 2(C_p n^2 p^2)^3 \cdot \exp \left(-\log(1/p) \left\{ \frac{1}{2} \delta t (1 - 9\varepsilon)(1 - 6\varepsilon) \frac{1}{2} n^2 p^2 - 9\varepsilon \cdot C_n n^2 p^2 \right\} \right) \leq \exp \left(-\log(1/p) \cdot \frac{1}{2} \delta t (1 - \eta^{(1)}(\varepsilon)) n^2 p^2 \right),$$

for all large $n$, where $\eta^{(1)}(\varepsilon)$ is some function with the property $\lim_{\varepsilon \to 0} \eta^{(1)}(\varepsilon) = 0$.

This gives the desired bound when $N(C_{2t}, G_{i,e})$ is small. Next we consider the other case.

**Case 2.** $N(C_{2t}, G_{i,e}) > \varepsilon \delta n^2 p^2 t$.

In this case we need to use the entropic stability of both $G_b$ and $G \setminus G_b$. As before, let us set

$$\mathcal{I}_{i,e,e_\#}^{(2)} := \{ G : G \in \mathcal{I}_{i,e,e_\#}, e(G_{i,e}) = e_\#, \text{ and } N(C_{2t}, G_{i,e}) \geq \varepsilon \delta n^2 p^2 t \}.$$

To carry out the argument effectively we need to discretize the range of $N(C_{2t}, G_{i,e})$. To this end, denote

$$S := \{ \varepsilon, 2\varepsilon, 3\varepsilon, \ldots, s_0 \varepsilon \},$$

where $s_0 := |(1 - 9\varepsilon)/\varepsilon|$. For $\eta \in S$ let

$$\mathcal{I}_{i,e,e_\#}^{(2)} := \{ G : G \in \mathcal{I}_{i,e,e_\#}, e(G_{i,e}) = e_\#, \text{ and } N(C_{2t}, G_{i,e}) \in [\eta \delta n^2 p^2 t, (\eta + \varepsilon) \delta n^2 p^2 t] \}.$$

Finally let

$$\hat{\mathcal{I}}_{i,e,e_\#}^{(2)} := \{ G : G \in \mathcal{I}_{i,e,e_\#}, e(G_{i,e}) = e_\#, \text{ and } N(C_{2t}, G_{i,e}) \geq (1 - 9\varepsilon) \delta n^2 p^2 t \}.$$
Let us proceed to bound the cardinality of $\mathcal{I}^{(2)}_{i, e, e_\#; \eta}$. This will be done by applying Lemma 4.5. To apply that lemma we need several estimates.

Since $i_* \leq C_3$ recalling the definition of the sets $\{V_j\}$ and recalling that $G_{i_*, g}$ is a bipartite graph we find that

$$D \cdot \left(\frac{C_0(\varepsilon)}{c_0(\varepsilon)}\right)^{C_3} \cdot |\cup_{j=1}^{i_*} V_j| \geq e(G_{i_*, g}).$$

If $G \in \mathcal{I}^{(2)}_{i, e, e_\#; \eta}$ then $N(C_{2t}, G_{i_*, g}) \geq \eta \delta n^{2t}p^{2t}$ which in turn, by yet another application of [12, Lemma 5.5], implies that

$$e(G_{i_*, g}) \geq \frac{1}{2} \eta^2 \delta^4 n^2 p^2 \geq \frac{1}{2} \eta \delta^4 n^2 p^2.$$  \hspace{1cm} (4.36)

Thus

$$v_# := |\cup_{j=1}^{i_*} V_j| \geq enp,$$  \hspace{1cm} (4.37)

for all large $n$. Furthermore, we recall that for $V_1$ as in (4.28) the set of vertices $V_1 \setminus (\cup_{j=1}^{i_*} V_j) \subset V(G_{\text{high}})$. Thus using Lemma 4.2(b) we also have that

$$0 \leq |V_1| - |\cup_{j=1}^{i_*} V_j| \leq 2\varepsilon e(G).$$  \hspace{1cm} (4.38)

Using the lower bound on $v_#$ we now apply Lemma 4.5 with $V_1$ and $\mathcal{D}$ as in (4.28)-(4.29) to find that the number of graphs in $\mathcal{I}^{(2)}_{i, e, e_\#; \eta}$ with $|V_1| = v$ and $|\cup_{j=1}^{i_*} V_j| = v_#$ is bounded by

$$\left(\frac{n}{v}\right)^{p-\varepsilon e} \leq \left(\frac{en}{v}\right)^{p-\varepsilon e} \leq \left(\frac{1}{p}\right)^{v+\varepsilon e} \leq \left(\frac{1}{p}\right)^{v_# + 3\varepsilon e},$$

where the first step is due to Stirling’s approximation, and the second step is due to the fact that $v \geq v_#$ and (4.37). While the last inequality is due to (4.38). Thus

$$\mathbb{P}\left(\exists G \subset G(n, p) : G \in \mathcal{I}^{(2)}_{i, e, e_\#; \eta} \text{ such that } |V_1| = v \text{ and } |\cup_{j=1}^{i_*} V_j| = v_# \right) \leq \exp\left(-\log(1/p) \cdot \{(e_# - v_#) + (1 - 3\varepsilon)(e - e_#) - 3\varepsilon e_#\}\right).$$

To simplify the RHS we need lower bounds on $(e_# - v_#)$ and $(e - e_#)$. Here we will use the lower bound on $N(C_{2t}, G_{i_*, g})$ and Lemma 4.4.

To this end, we remind the reader that we already noted above that the graph $G_{i_*, g}$ is bipartite graph with one part $\cup_{j=1}^{i_*} V_j$. We also recall that for any $v \in \cup_{j=1}^{i_*} V_j \setminus V(G_{\text{bad}})$ its degree $\text{deg}_{G_{i_*, g}}(v) \geq 2$. By Lemma 4.2(b) again we have that

$$|V(G_{\text{bad}})| \leq \varepsilon e(G) \leq \varepsilon \cdot C_{\eta^2} n^2 p^2 \leq K_0 \varepsilon' \cdot \frac{1}{2} \varepsilon \delta^4 n^2 p^2 \leq K_0 \varepsilon' \cdot e(G_{i_*, g}),$$

for some large constant $K_0$ and $\varepsilon' = \varepsilon^{1 - \frac{1}{5}}$, where the first inequality is due to the fact that $G$ is a strong-core graph, and the last step follows from (4.36). Therefore $G_{i_*, g}$ satisfies Assumption 4.3 with $\varepsilon_2 = K_0 \varepsilon'$. Hence applying Lemma 4.4 with $\delta_1 = \eta \delta$ and $\varepsilon_2$ as above we find that

$$e_# - v_# \geq \frac{1}{2} \eta \delta \cdot (1 - \varepsilon_2) \cdot n^2 p^2.$$  \hspace{1cm} (4.40)

On the other hand the assumption $N(C_{2t}, G_{i_*, g}) \leq (\eta + \varepsilon) \delta n^{2t} p^{2t}$ together with (4.25) implies that

$$N(C_{2t}, G_{i_*, g}) \geq (1 - 9\varepsilon - \eta) \delta n^t p^t.$$
This in turn yields the lower bound
\begin{equation}
(4.41) \quad e - e_\# \geq \frac{1}{2} \delta t^\frac{1}{t} \cdot (1 - 9\varepsilon - \eta) \frac{1}{t} n^2 p^2.
\end{equation}

Since for any $x \in [0, 1]$ and $t \geq 1$ one has $x^\frac{1}{t} + (1 - x)^\frac{1}{t} \geq 1$, we deduce from (4.40) and (4.41) that any $\eta \in \mathcal{S}$ we have
\[
(e_\# - v_\#) + (1 - 3\varepsilon)(e - e_\#) \geq \frac{1}{2} \delta t^\frac{1}{t} \cdot (1 - \varepsilon_2) \frac{1}{t} \cdot (1 - 3\varepsilon) \cdot (1 - 9\varepsilon) \frac{1}{t} \cdot n^2 p^2.
\]

Plugging this bound in the RHS of (4.39) and then taking a union over $\eta \in \mathcal{S}$, and $v, v_\#, e$ and $e_\#$ over their respective allowable ranges we derive that
\begin{equation}
(4.42) \quad \mathbb{P} \left( \bigcup_{e = e_0(\delta(1 - 6\varepsilon))} \bigcup \left\{ \exists G \subset \mathbb{G}(n, p) : G \in \mathbb{T}^{(2)}_{i, e, e_\#} \right\} \right)
\leq 4(\tilde{C}_n p^2)^4 \cdot |\mathcal{S}| \cdot \exp \left( - \log(1/p) \left\{ \frac{1}{2} \delta t^\frac{1}{t} (1 - \varepsilon_2) \frac{1}{t} (1 - 3\varepsilon)(1 - 9\varepsilon) \frac{1}{t} n^2 p^2 - 3\varepsilon \cdot \tilde{C}_n p^2 \right\} \right)
\leq \exp \left( - \log(1/p) \cdot \frac{1}{2} \delta t^\frac{1}{t} (1 - f^{(2)}_{2t}(\varepsilon)) n^2 p^2 \right),
\end{equation}
for all large $n$, where $f^{(2)}_{2t}(\cdot)$ is some other function satisfying $\lim_{\varepsilon \downarrow 0} f^{(2)}_{2t}(\varepsilon) = 0$.

Next we need to obtain a bound for the case $G \in \mathbb{T}^{(2)}_{i, e, e_\#}$. Observe that only the lower bound on $N(C_{2t}, \mathbb{G}_{i, e})$ was used in deriving (4.39) and hence the same argument gives (ignoring the $(1 - 3\varepsilon)(e - e_\#)$ term)

\begin{equation}
(4.43) \quad \mathbb{P} \left( \exists G \subset \mathbb{G}(n, p) : G \in \mathbb{T}^{(2)}_{i, e, e_\#} \text{ such that } |\mathcal{V}_1| = v \text{ and } |\bigcup_{j=1}^{i_\#} V_j| = v_\# \right)
\leq \exp \left( - \log(1/p) \cdot \left\{ (e_\# - v_\#) - 3\varepsilon e_\# \right\} \right).
\end{equation}

Also, as in the derivation of (4.40), using Lemma 4.4 with $\delta_1 = (1 - 9\varepsilon)\delta$ we get
\begin{equation}
(4.44) \quad e_\# - v_\# \geq \frac{1}{2} (1 - 9\varepsilon)^\frac{1}{t} \delta t^\frac{1}{t} \cdot (1 - \varepsilon_2) \frac{1}{t} n^2 p^2.
\end{equation}

Taking a union bound over $v, v_\#, e$ and $e_\#$ over their respective allowable ranges we derive as in (4.42) that
\begin{equation}
(4.45) \quad \mathbb{P} \left( \bigcup_{e = e_0(\delta(1 - 6\varepsilon))} \bigcup \left\{ \exists G \subset \mathbb{G}(n, p) : G \in \mathbb{T}^{(2)}_{i, e, e_\#} \right\} \right)
\leq 4(\tilde{C}_n p^2)^4 \cdot \exp \left( - \log(1/p) \left\{ \frac{1}{2} \delta t^\frac{1}{t} (1 - \varepsilon_2) \frac{1}{t} (1 - 9\varepsilon) \frac{1}{t} n^2 p^2 - 3\varepsilon \cdot \tilde{C}_n p^2 \right\} \right)
\leq \exp \left( - \log(1/p) \cdot \frac{1}{2} \delta t^\frac{1}{t} (1 - f^{(3)}_{2t}(\varepsilon)) n^2 p^2 \right),
\end{equation}
for all large $n$, where $f^{(3)}_{2t}(\cdot)$ is another function satisfying $\lim_{\varepsilon \downarrow 0} f^{(3)}_{2t}(\varepsilon) = 0$.

As
\[
\left\{ \exists G \subset \mathbb{G}(n, p) : \mathbb{T}^{(2)}_{i, e, e_\#} \right\} \subset \left\{ \exists G \subset \mathbb{G}(n, p) : G \in \mathbb{T}^{(2)}_{i, e, e_\#} \right\} \bigcup_{\eta \in \mathcal{S}} \left\{ \exists G \subset \mathbb{G}(n, p) : G \in \mathbb{T}^{(2)}_{i, e, e_\#, \eta} \right\},
\]
equipped with (4.35), (4.42), and (4.45) we then take another union over \( i_* \in [C_3] \), set
\[
f_{2t}(\varepsilon) := \max\{f_{2t}^{(1)}(\varepsilon), f_{2t}^{(2)}(\varepsilon), f_{2t}^{(3)}(\varepsilon)\} + \varepsilon,
\]
and use (4.27) to derive the desired bound on the existence of a strong-core graph. This finally finishes the proof of the proposition. \( \square \)

5. Entropic Stability of Core Graphs with Many Edges

In this section our objective is to prove Proposition 3.4. We recall that the proof of Proposition 3.4 splits into two cases: \( np \geq (\log n)^{\ell} \) and \( np \leq (\log n)^{\ell} \). First we consider the easier case of large \( p \).

5.1. Core Graphs with Many Edges in the Large \( p \) Regime. We recall from Section 3.1 that in this case the proof of Proposition 3.4 follows once we show have Lemma 3.5. In the remainder of this section we prove Lemma 3.5.

Proof of Lemma 3.5. We begin by reminding ourselves of the definitions of \( \mathcal{W}(G) \), \( E_{1,2}(G) \), \( E_{2,2}(G) \), \( e_{1,2}(G) \), and \( e_{2,2}(G) \) (see (3.11), and (3.15)-(3.17)). Recall that \( G_{\mathcal{W}} \) is the subgraph of \( G \) induced by edges that are incident to some vertex in \( \mathcal{W} \). Since by assumption \( G_{\mathcal{W}} \) is bipartite we have that \( e(G) = e_{1,2}(G) + e_{2,2}(G) \).

Equipped with this observation we now proceed as follows: we fix \( \mathfrak{h} := \{w, e_{1,2}, e_{2,2}\} \) with \( e_{1,2} + e_{2,2} = e \geq e_* \) and let
\[
\mathcal{A}_{\mathfrak{h}} := \{G \subset G(n, p) : G \text{ is a core graph with } (|\mathcal{W}(G)|, e_{1,2}(G), e_{2,2}(G)) = \mathfrak{h}\}.
\]

We bound the probability of \( \mathcal{A}_{\mathfrak{h}} \) for each fixed choice of \( \mathfrak{h} \) and then take a union bound over the allowable range of \( \mathfrak{h} \).

Observe that for \( \mathcal{A}_{\mathfrak{h}} \) to be non-empty the following constraint needs to be satisfied:
\[
(5.1) \quad 2w \leq e_{1,2} \leq Dw.
\]

Since \( G_{\mathcal{W}} \) is bipartite the upper bound is immediate as the maximal degree among the vertices in \( \mathcal{W}(G) \) is at most \( D \). On the other hand \( G \) being a core graph each edge must participate in at least one copy \( C_\ell \). This yields that the minimum degree of the vertices in \( G \) is at least two which in turn implies the lower bound in (5.1).

We now split the proof into two cases: (1) \( e_{2,2} \geq e_{1,2} \), and (2) \( e_{2,2} \leq e_{1,2} \).

Case 1. \( e_{2,2} \geq e_{1,2} \).

We invoke Lemma 4.5 to upper bound the total number of core graphs with \((|\mathcal{W}|, e_{1,2}(G), e_{2,2}(G)) = \mathfrak{h}\) by
\[
\left( \frac{n}{w} \right) \left( \frac{1}{p} \right)^{\varepsilon} \leq n^w \cdot \left( \frac{1}{p} \right)^{\varepsilon} \leq \left( \frac{1}{p} \right)^{\varepsilon + e_{1,2}}
\]
where we have used that \( p \leq n^{-1/2} \) and the lower bound from (5.1). Thus
\[
(5.2) \quad \mathbb{P}(\mathcal{A}_{\mathfrak{h}}) \leq \left( \frac{1}{p} \right)^{\varepsilon + e_{1,2}} \cdot p^{e_{1,2} + e_{2,2}} \leq p^{e_{2,2} - \varepsilon} \leq p^{(\frac{1}{2} - \varepsilon)e},
\]
where the last inequality follows upon noting that \( e_{2,2} \geq \frac{1}{2}e \).
Case 2. $e_{2,2} \leq e_{1,2}$.

In this case using the upper bound in (5.1) and the fact that $e_{1,2} \geq \frac{1}{2} e \geq \frac{1}{2} e_*$ we find that
\[
\frac{n}{w} \leq e_{1,2} \leq \frac{2nD}{C_* n^2 p^2} \leq \frac{1}{e^p}.
\]
for all large $n$. Therefore applying Lemma 4.5 again and using Stirling’s approximation we deduce that the number of core graphs with $(|W|, e_{1,2}(G), e_{2,2}(G)) = h$ is bounded by
\[
\left( \frac{en}{w} \right)^w \cdot \left( \frac{1}{p} \right)^{\varepsilon e} \leq \left( \frac{1}{p} \right)^{\varepsilon e + w}.
\]
Now using the lower bound (5.1) and the fact that $e_{1,2} \geq \frac{1}{2} e$ we derive from above that
\[
\Pr(\mathcal{A}_{e,h}) \leq \left( \frac{1}{p} \right)^{\varepsilon e + w} \cdot p^{e_{1,2}} \leq p^{\frac{1}{2} e_{1,2} - \varepsilon e} \leq p^{(1 - \varepsilon)e}.
\]
To complete the proof we first sum over all possible choices of $h$ for a given $e$. Observe that the total number of choices for each of $w$ and $e_{1,2}$ are trivially upper bounded by $e$. Thus, using (5.2)-(5.3) and applying an union bound we find
\[
\Pr(\mathcal{A}_e) = \Pr(\cup_{h} \mathcal{A}_{e,h}) \leq e^2 p^{(1 - \varepsilon)e}.
\]
Now summing the over all $e_* \leq e \leq C n^2 p^2 \log(1/p)$ we derive that
\[
\Pr(\cup_{e \geq e_*} \mathcal{A}_e) \leq 2(C n^2 p^2 \log(1/p))^2 \cdot p^{(1 - \varepsilon)e_*} \leq \exp \left( - \frac{1}{16} C_* n^2 p^2 \log(1/p) \right),
\]
for all large $n$, where in the last step we have used the facts $n^2 p^2 \log(1/p) \gg 1$ and $\varepsilon \leq \frac{1}{5}$. This completes the proof of the lemma.

We now turn to the case of $np \leq (\log n)^*$. We remind the reader that in this case for a core graph $G$ the subgraph $G_W$ need not be a bipartite graph. Nevertheless, as we show below in the next section if we assume that $e_{1,2}(G) + e_{2,2}(G)$ is sufficiently large then the set of those core graphs are entropically stable.

5.2. Entropic stability for graphs with large $e_{1,2}(G) + e_{2,2}(G)$. In this section we prove Lemmas 3.7 and 3.11. Both proofs will use an argument analogous to the proof of Lemma 4.5. However, we remind the reader these lemmas find bounds on the probabilities of certain events that may involve graphs which are no longer core graphs. Therefore we cannot directly apply Lemma 4.5. To this end, we have the following general lemma. Its proof is similar in nature to that of Lemma 4.5. Before stating the lemma we introduce a few more notation.

For every graph $G$ we denote $W_P(G)$ to be the subset of vertices satisfying some property $P_G$. That is, $P_G : V(G) \mapsto \{0, 1\}$ is a map and $W_P(G) := \{v \in V(G) : P_G(v) = 1\}$. Set $W_P(G) := V(G) \setminus W_P(G)$. Thus $\{W_P(G), W_P(G)\}$ is some partition of the vertices of $G$ which may be determined by some properties of the graph $G$. Denote
\[
\tilde{e}_{1,1}(G) := \{ (u, v) \in E(G) : u, v \in W_P(G) \},
\]
\[
\tilde{e}_{1,2}(G) := \{ (u, v) \in E(G) : u \in W_P(G), v \in W_P(G) \},
\]
and
\[
\tilde{e}_{2,2}(G) := \{ (u, v) \in E(G) : u, v \in W_P(G) \}.
\]
We now state the relevant lemma.
Lemma 5.1. Fix non-negative integers $\bar{e}_{1,1}, \bar{e}_{1,2}, \bar{e}_{2,2}, \bar{w}$, and $\bar{\varepsilon} > 0$. Set $e := (\bar{e}_{1,1}, \bar{e}_{1,2}, \bar{e}_{2,2})$. Let $\mathcal{N}_\#(e, \bar{w})$ be the number of graphs $G$ with $|W_P(G)| = \bar{w}$, such that $\bar{e} := \bar{e}_{1,1} + \bar{e}_{1,2} + \bar{e}_{2,2} \leq Cn^2p^2 \log(1/p)$, and

\begin{align}
(5.4) \quad &\bar{e}_{1,1}(G) = \bar{e}_{1,1}, \quad \bar{e}_{1,2}(G) = \bar{e}_{1,2}, \quad \text{and} \quad \bar{e}_{2,2}(G) = \bar{e}_{2,2}, \\
(5.5) \quad &|W_P(G)| \leq \bar{e}(\bar{e}_{1,2}(G) + \bar{e}_{2,2}(G)).
\end{align}

If $np \leq (\log n)^{\ell}$ then

$$\mathcal{N}_\#(e, \bar{w}) \leq \exp((\log 1/p) \{\bar{w} + \bar{\varepsilon}(\bar{e}_{1,2} + \bar{e}_{2,2})\}) + K(\log \log n \cdot \bar{e})$$

for all large $n$, where $K$ is some absolute constant.

Proof. The proof uses simple combinatorial bounds. For ease of writing let us denote $\bar{w}_1 := |W_P(G)|$. We write $\mathcal{N}_\#(e, \bar{w}, \bar{w}_1)$ to denote the number of graphs with $|W_P(G)| = \bar{w}$, $|W_P(G)| = \bar{w}_1$, $\bar{e} \leq Cn^2p^2 \log(1/p)$, and satisfies (5.4)-(5.5). First we find a bound on $\mathcal{N}_\#(e, \bar{w}, \bar{w}_1)$ and then take a union over the allowable range of $\bar{w}_1$ to derive a bound on $\mathcal{N}_\#(e, \bar{w})$.

To this end, consider any $\bar{w}_1 \leq \bar{e}(\bar{e}_{1,2} + \bar{e}_{2,2})$ (note this bound is imposed by (5.5)). The vertices in $W_P(G)$ and $W_P(G)$, and hence all the vertices in $G$ can be chosen in at most $n^{\bar{w}_1}$ ways. Once these vertices are chosen the number of ways to specify edges is at most $(\bar{w} + \bar{w}_1)^{2\bar{e}}$.

As $np \leq (\log n)^{\ell}$ and $\bar{e} \leq Cn^2p^2 \log(1/p)$ we also get that $\bar{e} + \bar{w}_1 \leq 2\bar{e} \leq (\log n)^{2\ell} + 2$ for all large $n$. This implies that $(\bar{w} + \bar{w}_1)^{2\bar{e}} \leq \exp(K' \log \log n \cdot \bar{e})$ for some $K' > 0$ and all $n$ sufficiently large. Combining these estimates we obtain that

$$\mathcal{N}_\#(e, \bar{w}, \bar{w}_1) \leq n^{\bar{w}_1} \exp(K' \log \log n \cdot \bar{e}) \leq \exp((\log 1/p) (\bar{w} + \bar{w}_1) + 2K' \log \log n \cdot \bar{e}),$$

where in the last step we again used the facts $np \leq (\log n)^{\ell}$ and $\bar{w} + \bar{w}_1 \leq 2\bar{e}$. Finally taking a union over all $\bar{w}_1 \leq \bar{e}(\bar{e}_{1,2} + \bar{e}_{2,2})$ we arrive at the desired result. \hfill \Box

Using Lemma 5.1 we now prove Lemma 3.7.

Proof of Lemma 3.7. Fix a vector $\bar{e} := (e_{1,1}, e_{1,2}, e_{2,2})$ and $\bar{w}$. Set $e_{1,1}(G) := |E_{1,1}(G)|$. Let $\mathcal{N}(\bar{e}, \bar{w})$ denote the number of possible graphs satisfying the hypothesis of the event Core$_{1,1}$ (recall its definition from (3.19)), i.e., the set of graphs $G$ with

\begin{enumerate}
  \item $e_{1,1}(G) = e_{1,1}, \quad e_{1,2}(G) = e_{1,2}, \quad e_{2,2}(G) = e_{2,2}, \quad |W(G)| = \bar{w},$
  \item $e_{1,1} + e_{1,2} + e_{2,2} = e \leq (\log \log n)^{-2}Cn^2p^2 \log(1/p),$
  \item $e_{1,2} + e_{2,2} \geq C\ast n^2p^2$ and $d_{\min}(G) \geq 2$.
\end{enumerate}

We refer the reader to (3.11), and (3.15)-(3.17) to recall the definitions of $W(G)$, $e_{1,2}(G)$, and $e_{2,2}(G)$. Now set $P_G : V(G) \mapsto \{0, 1\}$ to be

$$P_G(v) := \begin{cases} 1 & \text{if } \deg_G(v) \leq D, \\ 0 & \text{otherwise,} \end{cases}$$

where $D$ is as in (3.12). With this choice of $P_G$ we have $W(G) = W_P(G)$. Thus any edge incident to some vertex in $\overline{W_P(G)}$ must either be in $E_{1,2}(G)$ or be in $E_{2,2}(G)$. Therefore, using the fact any vertex in $V(G) \setminus W(G)$ has degree at least $D$, we have

$$|W_P(G)| \leq (2/D) \cdot (e_{1,2}(G) + e_{2,2}(G)) \leq (\varepsilon/16) \cdot (e_{1,2}(G) + e_{2,2}(G)),$$

indicating that (5.5) holds with $\bar{\varepsilon} = \varepsilon/16$. Hence, we can now apply Lemma 5.1 with $P_G$ as in (5.6) and $\bar{\varepsilon}$ as above. To obtain a usable bound we further note that any edge adjacent to some vertex in
\(W(G)\) must be in \(E_{1,1}(G) \cup E_{1,2}(G)\). Moreover, it follows from the definition that both end points of an edge in \(E_{1,2}(G)\) cannot be in \(W(G)\). As \(d_{\text{min}}(G) \geq 2\) we deduce that
\[
(5.7) \quad |W(G)| \leq \sum_{v \in W(G)} \deg_G(v) \leq 2|E_{1,1}(G)| + |E_{1,2}(G)|.
\]

By Lemma 5.1 we have that
\[
\mathcal{N}(\tilde{e}, \mathbf{w}) \leq \text{exp}\left(\log(1/p) \{ \mathbf{w} + (\varepsilon/16) \cdot (e_{1,2} + e_{2,2}) \} + K \log \log n \cdot e\right)
\]
\[
\leq \text{exp}\left(\log(1/p) \left\{ \frac{1}{2} e_{1,2} + (\varepsilon/16) \cdot (e_{1,2} + e_{2,2}) \right\} + K \log \log n \cdot e\right),
\]
where we the last step is a consequence of (5.7). Furthermore, as \(e_{1,2} + e_{2,2} \geq \bar{C} n^2 p^2\), \(\bar{C} \leq n^2 p^2 \log(1/p)\), and \(np \leq (\log n)^\ell\) we have
\[
K \log \log n \cdot e \leq (\varepsilon/16) \cdot (e_{1,2} + e_{2,2}) \cdot \log(1/p),
\]
for all large \(n\). Thus, using (5.8) and the above inequality we derive
\[
(5.9) \quad \mathbb{P}(\text{Core}_{1,1}) \leq \sum_{\tilde{e}, \mathbf{w}} \mathcal{N}(\tilde{e}, \mathbf{w}) \cdot p^{e_{1,1} + e_{1,2} + e_{2,2}}
\]
\[
\leq \sum_{\tilde{e}, \mathbf{w}} \exp\left(-\log(1/p) \left\{ \frac{1}{2} e_{1,2} + e_{2,2} - (\varepsilon/8) \cdot (e_{1,2} + e_{2,2}) \right\} \right),
\]
for all large \(n\), where the sums over \(\tilde{e}\) and \(\mathbf{w}\) are to be taken over their allowable respective ranges.

Since \(\mathbf{w} \leq 2e \leq 2\bar{C} n^2 p^2 \log(1/p)/(\log \log n)^2\) and \(e_{1,2} + e_{2,2} \geq \bar{C} n^2 p^2\) it is now immediate from (5.9) that
\[
(5.10) \quad \mathbb{P}(\text{Core}_{1,1}) \leq \exp\left(-\frac{\bar{C}}{8} n^2 p^2 \log(1/p)\right),
\]
for all large \(n\). Recalling that \(\bar{C} \geq 32\delta^2\) we obtain the desired probability upper bound. This completes the proof. \(\Box\)

Next, using Lemma 5.1 again we prove Lemma 3.11.

**Proof of Lemma 3.11.** Fix any \(j \in [L_0]\) and let us recall the definition of \(\text{Core}_{2,j,\beta}\) from (3.26). We see that any graph \(G_0\) satisfying the hypothesis of \(\text{Core}_{2,j,\beta}\) must be contained in the set
\[
\Omega := \{ G_0 : \exists G' := G'(G_0) \ni G_0 \text{ such that } G_0, G' \in J_j \text{ and } \varpi(G') = G_0 \},
\]
where we recall that \(\varpi(G')\) is the 2-core of the subgraph of \(G'\) obtained upon removing the edges in \(E_{1,1}(G') = \{ (u, v) \in E(G') : \deg_G(u) \leq D \}\), where \(D\) is as in (3.12). For a \(G_0 \in \Omega\) there may be more than one \(G' \in J_j\) such that \(G' \in J_j\) and \(\varpi(G') = G_0\). Choose any one of them arbitrarily and fix it for the rest of the proof of this lemma.

Our goal would be to bound the cardinality of \(\Omega\). To this end, set \(\mathcal{P}_{G_0} : V(G_0) \mapsto \{0, 1\}\) to be
\[
\mathcal{P}_{G_0}(v) := \begin{cases} 0 & \text{if } \deg_G(v) \leq D, \\ 1 & \text{otherwise.} \end{cases}
\]

With this choice of \(\mathcal{P}_{G_0}\) we now apply Lemma 5.1. Since \(G_0, G' \in J_j\) we find that \(e(G_0) \geq 1/2 e(G')\). Therefore, as \(D \geq 32/\varepsilon\) we derive that
\[
|\mathcal{W}(G_0)| \leq |\{ v \in V(G) : \deg_G(v) \geq D \}| \leq (\varepsilon/16) \cdot e(G') \leq (\varepsilon/8) \cdot e(G_0).
\]
Thus (5.5) is satisfied with \( \varepsilon = \varepsilon / 8 \). Hence denoting \( \mathcal{N}(e, w) \) to be the number of graphs \( G_0 \in \Omega \) with \( |W_p(G_0)| = w \) and satisfying (5.4)-(5.5), and applying Lemma 5.1 we deduce that
\[
\mathcal{N}(e, w) \leq \exp \left( \log(1/p) \left\{ w + (\varepsilon/8) \cdot (e_{1,2} + \bar{e}_{2,2}) \right\} + K(\log \log n) \cdot \bar{e} \right)
\]
(5.11)
where the last step is due to the fact as \( \pi_2(G') = G_0 \), it follows from the definition of \( P_{G_0} \) and \( E_{1,1}(G') \) that \( \bar{e}_{1,1} = \bar{e}_{1,1}(G_0) = 0 \), and thus \( \bar{e} = e_{1,2} + \bar{e}_{2,2} \).

Equipped with (5.11) we observe that
\[
P(C_{2,j,3}) \leq \sum_{e,w} \mathcal{N}(e, w) \cdot p^{e_{1,2} + \bar{e}_{2,2}} \leq \sum_{e,w} \exp \left( -\log(1/p) \left\{ \frac{e_{1,2}}{2} + \bar{e}_{2,2} - (\varepsilon/4) \cdot (e_{1,2} + \bar{e}_{2,2}) \right\} \right),
\]
where the sum is over allowable ranges of \( w \) and \( e \), and the last step follows from the fact that for any \( G_0 \in \Omega \) we have \( d_{\min}(G_0) \geq 2 \) and thus \( w \leq \frac{1}{2} e_{1,2} \).

Finally using that
\[
\bar{w} \leq 2(e_{1,2} + \bar{e}_{2,2}) \leq 2\bar{e} \leq 2Cn^2p^2 \log(1/p),
\]
and the lower bound
\[
(1/2) \cdot (e_{1,2} + \bar{e}_{2,2}) \geq \frac{1}{2} Cn^2p^2 \log(1/p)/(\log \log n)^2,
\]
induced by the fact \( G_0 \in \mathcal{J}_j \), one evaluates the above sum to obtain a desired bound. We omit further details. This completes the proof of the lemma. \( \square \)

5.3. Graphs with large \( N_{1,1} \). Let us begin this section recalling that for a graph \( G \) the notation \( N_{1,1}(C_\ell, G) \) denotes the number of copies of \( C_\ell \) that uses at least one edge from \( E_{1,1}(G) \), where \( E_{1,1}(G) \) is as in (3.18). In this section our goal is prove Lemmas 3.8 and 3.10. Similar to Section 5.2 here we will also rely on Lemma 5.1. Note that to complete the proofs of Lemmas 3.7 and 3.11 we needed lower bounds on \( e_{1,2}(G) + e_{2,2}(G) \). Here we will show that such bounds follow once we assume a lower bound on \( N_{1,1}(C_\ell, G) \). Thus the following is the main result of this section.

**Proposition 5.2.** Fix \( \tau > 0 \) and an integer \( \ell \geq 3 \). Let \( G \) be a graph with
\[
e(G) \leq c_n \cdot \tilde{C}n^2p^2 \log(1/p),
\]
for some \( \frac{1}{2}(\log \log n)^{-2} \leq c_n \leq 1 \). Assume
\[
N_{1,1}(C_\ell, G) \geq \frac{c_n}{2} \cdot \tau n^\ell p^\ell.
\]
If \( np \gg (\log n)^{1/(\ell-2)} \) then we have
\[
e_{1,2}(G) + e_{2,2}(G) \geq (n^2p^2)^{1+\frac{1}{\ell-1}},
\]
for all large \( n \).

The proof of Proposition 5.2 follows from the following two lemmas. Before stating the lemmas, for convenience in writing, let us introduce a couple more notation. We write \( \tilde{N}_{1,1}(C_\ell, G) \) to denote the number of labelled copies of \( C_\ell \) consisting of only edges belonging to \( E_{1,1}(G) \) and set
\[
\tilde{N}_{1,1}(C_\ell, G) := N_{1,1}(G, C_\ell) - \tilde{N}_{1,1}(G, C_\ell).
\]
That is, \( \tilde{N}_{1,1}(C_\ell, G) \) is the number of labeled copies of \( C_\ell \) in \( G \) that uses at least one edge from \( E_{1,1}(G) \) and at least one belonging to \( E_{1,2}(G) \cup E_{2,2}(G) \).
Lemma 5.3. Let $\tau, \ell$, and $c_n$ be as in Proposition 5.2. Suppose $G$ be a graph satisfying (5.12). If $np \gg (\log n)^{1/(\ell-2)}$ then
\[
\tilde{N}_{1,1}(C_\ell, G) \leq c_n \cdot \frac{\tau}{4} n^\ell p^\ell,
\]
for all large $n$.

Lemma 5.4. For any $\ell \geq 3$ and a graph $G$ we have
\[
(5.14) \quad \tilde{N}_{1,1}(C_\ell, G) \leq \ell 2^{\ell+1} D^\ell \cdot (2(e_{1,2}(G) + e_{2,2}(G)))^{\lfloor (\ell-1)/2 \rfloor}.
\]

Using Lemmas 5.3 and 5.4 let us now prove Proposition 5.2.

Proof of Proposition 5.2. Since $N_{1,1}(C_\ell, G) \geq (c_n/2) \cdot \tau n^\ell p^\ell$ by Lemma 5.3 we deduce that
\[
N_{1,1}(C_\ell, G) \geq c_n \cdot \frac{\tau}{4} n^\ell p^\ell.
\]
Therefore Lemma 5.4 now implies that there exists some constant $\tilde{c} > 0$, depending on $\tau$, so that
\[
e_{1,2}(G) + e_{2,2}(G) \geq \frac{1}{2} \cdot \left( c_n \cdot \frac{\tau}{4} \cdot D^{-\ell} \ell^{-1} 2^{-(\ell+1)} \right)^{1/(\ell-1)/2} \cdot (n^2 p^2)^{\frac{1}{\ell-1}}
\geq \tilde{c} \cdot (\log \log n)^{\frac{-1}{\ell-2}} \cdot (n^2 p^2)^{1+\frac{1}{\ell-1}} \geq (n^2 p^2)^{1+\frac{1}{\ell(\ell-1)}},
\]
for all large $n$, where the penultimate step uses the fact that $c_n \geq \frac{1}{2} (\log \log n)^{-2}$ and the last step uses that
\[
(\log \log n)^{4/(\ell-2)} \ll (\log n)^{\frac{1}{(\ell-1)(\ell-1)}} \ll (np)^{\frac{1}{\ell(\ell-1)}}.
\]
This completes the proof. \(\square\)

We now proceed to the proof of Lemma 5.3. The proof is straightforward.

Proof of Lemma 5.3. We construct a labelled copy of $C_\ell$ contributing to $\tilde{N}_{1,1}(C_\ell, G)$ as follows: Let an edge $e_0 \in E_{1,1}(G)$ and take one of the endpoints to be the first vertex of a copy of $C_\ell$ whereas take the other one to be the $\ell$-th vertex. Clearly this can be done in $2|E_{1,1}(G)|$ ways. Having chosen this edge, we choose the remaining $(\ell - 2)$ vertices sequentially so that the remaining edges are in $E_{1,1}(G)$ to construct a copy of $C_\ell$ that uses only edges in $E_{1,1}(G)$. Since each endpoint of edges in $E_{1,1}(G)$ has degree upper bounded by $D$ each of the remaining $(\ell - 2)$ vertices can be chosen at most in $D$ ways. This shows that
\[
\tilde{N}_{1,1}(C_\ell, G) \leq 2|E_{1,1}(G)| \cdot D^{\ell-2} \leq 2e(G) \cdot D^{\ell-2} = c_n \cdot o(n^\ell p^\ell),
\]
where in the last step we used that $e(G) \leq c_n \cdot C n^2 p^2 \log(1/p)$, and the fact that $np \gg (\log n)^{1/(\ell-2)}$ implies $n^2 p^2 \log(1/p) \ll n^\ell p^\ell$. This completes the proof of the lemma. \(\square\)

We need a few combinatorial definitions before proving Lemma 5.4. For any labelled copy $H$ of either the $\ell$-cycle or the path of length $\ell$ in $G$, we associate it with an element $s(H) := (s_1, s_2, \ldots, s_\ell)$ of $\{0, 1\}^\ell$ as follows. We set $s_i = 0$ if the $i$-th edge of $H$ (according to the labelling) is in $E_{1,1}(G)$ and $s_i = 1$ otherwise. For any fixed $s \in \{0, 1\}^\ell$ and $v_1, v_2 \in V(G)$ we write $N(P_\ell, G, s, v_1, v_2)$ to denote the number of labelled copies $H$ of $P_\ell$, the path of length $\ell$, such that $s(H) = s$ with the starting and the ending vertices being $v_1$ and $v_2$ respectively.

We have the following counting lemma which will be key in proving Lemma 5.4.

Lemma 5.5. For any integer $\ell \geq 3$, $s \in \{0, 1\}^\ell$, and $v_1, v_2 \in V(G)$ we have
\[
N(P_\ell, G, s, v_1, v_2) \leq D^\ell (2(e_{1,2}(G) + e_{2,2}(G)))^{\lfloor \ell/2 \rfloor}.
\]
Proof. The proof is done by an induction argument. Clearly for \( \ell = 1 \), there can be at most one copy of \( P_1 \) with the fixed starting and ending points.

For \( \ell = 2 \), let us consider different possible choices of \( s \). Clearly if \( s = 00 \), since the starting vertex is fixed, then there are at most \( D \) possibilities for each of the edges and hence \( \mathcal{N}(P_2, G, s, v_1, v_2) \leq D^2 \). If \( s \in \{10,01,11\} \) then, as the starting and the ending vertices are fixed, there are at most \( 2\bar{e}(G) \) choices for the edge corresponding to the 1, where for ease in writing we use the shorthand \( \bar{e}(G) := e_{1,2}(G) + e_{2,2}(G) \). Having chosen this edge, because the two leaf vertices are fixed, there are at most one choices for the remaining edge. This gives the desired bound for \( \ell = 2 \), any \( s \in \{0,1\}^2 \) and \( v_1, v_2 \in V(G) \).

Let us suppose that the statement of the lemma is true for all \( \ell = 1, 2, \ldots, t - 1 \), any \( s \in \{0,1\}^{\ell} \), and \( v_1, v_2 \in V(G) \). We now establish the lemma for \( \ell = t \geq 3 \), any \( s \in \{0,1\}^{t} \), and \( v_1, v_2 \in V(G) \). Write \( s = s_1 s_2 s' \), where \( s_1 \) and \( s_2 \) are the first two digits of \( s \) and \( s' \) is the remaining substring of length \( t - 2 \).

If \( s_1 = 0 \) then there are at most \( D \) choices for the first edge \( e = (v_1, v'_1) \), and for each such choice there are at most \( \mathcal{N}(P_{t-1}, G, s_2 s', v'_1, v_2) \) choices for the remaining edges. Hence by induction hypothesis we obtain

\[
\mathcal{N}(P_t, G, s, v_1, v_2) \leq D \cdot D^{t-1}(2\bar{e}(G))^\lfloor (t-1)/2 \rfloor \leq D^t(2\bar{e}(G))^\lfloor t/2 \rfloor.
\]

If \( s_1 = 1 \) we need to consider two cases depending on whether \( s_2 = 0 \) or 1. If \( s_1s_2 = 10 \), arguing as above, we see that there are at most \( 2\bar{e}(G)D \) many choices for the first two edges \( e_1 = (v_1, v'_1) \) and \( e_2 = (v'_1, v''_1) \). For each of these choices there at most \( \mathcal{N}(P_{t-2}, G, s', v''_1, v_2) \) choices for the remaining edges. So by induction hypothesis in this case we derive

\[
\mathcal{N}(P_t, G, s, v_1, v_2) \leq (2\bar{e}(G)D) \cdot D^{t-2}(2\bar{e}(G))^\lfloor (t-2)/2 \rfloor \leq D^t(2\bar{e}(G))^\lfloor t/2 \rfloor.
\]

If \( s_1s_2 = 11 \) we first choose the second edge \( e_2 = (v'_1, v''_1) \), where there are at most \( 2\bar{e}(G) \) many choices. Given this choice, as the first vertex is fixed at \( v_1 \) there are at most one choice for the first edge \( e_1 = (v_1, v'_1) \). Now there are at most a total of \( \mathcal{N}(P_{t-2}, G, s', v''_1, v_2) \) choices for the third edge onwards. Hence, by induction hypothesis we get

\[
\mathcal{N}(P_t, G, s, v_1, v_2) \leq (2\bar{e}(G)) \cdot D^{t-2}(2\bar{e}(G))^\lfloor (t-2)/2 \rfloor \leq D^t(2\bar{e}(G))^\lfloor t/2 \rfloor.
\]

This completes the induction and the proof of the lemma. \( \square \)

We are now ready to prove Lemma 5.4.

Proof of Lemma 5.4. Let \( \mathcal{N}^*(C_\ell, G) \) denote the number of labelled copies of \( C_\ell \) in \( G \) such that the first edge (according to the labelling) belongs to \( E_{1,2}(G) \cup E_{2,2}(G) \) and the second edge belongs to \( E_{1,1}(G) \). Since \( \mathcal{N}_{1,1}(C_\ell, G) \) counts the number of labelled copies of \( C_\ell \) in \( G \) that have at least one edge in \( E_{1,1}(G) \) and at least one belonging to \( E_{1,2}(G) \cup E_{2,2}(G) \), every such labelled copy must contain an edge in \( E_{1,1}(G) \) that is adjacent to some edge \( E_{1,2}(G) \cup E_{2,2}(G) \) also contained in that copy. Therefore we find that

\[
\mathcal{N}_{1,1}(C_\ell, G) \leq 2\ell \mathcal{N}^*(C_\ell, G),
\]

where the factor \( \ell \) is due to the choice of the location of the edge belonging to \( E_{1,2}(G) \cup E_{2,2}(G) \) that is adjacent to the edge in \( E_{1,1}(G) \) and the factor two is due to the orientation of that edge. So it suffices to prove

\[
\mathcal{N}^*(C_\ell, G) \leq 2\ell D^\ell \cdot (2\bar{e})^\lfloor (\ell-1)/2 \rfloor.
\]

Recall the \( \ell \)-bit string \( s(H) \) associated with every labelled copy \( H \) of \( C_\ell \) in \( G \). Clearly for any labelled copy that is counted in \( \mathcal{N}^*(C_\ell, G) \) the string \( s(H) \) must start with the substring 10. For any such \( s \), for ease of explanation, we further introduce the notation \( \mathcal{N}^*(C_\ell, G, s) \) to denote the
number of labelled copies $H$ of $C_\ell$ that are counted in $\mathcal{N}^*(C_\ell, G)$ such that $s(H) = s$. Equipped with this notation it now suffices to prove that for any $\ell$-bit string $s$ that starts with 10 we have

$$\mathcal{N}^*(C_\ell, G, s) \leq D^\ell \cdot (2\bar{e})^{[(\ell - 1)/2]}.$$  

Fix an $s$ as above and let $s'$ be the substring of $s$ that ends with the second 1 of $s$. Observe that by our construction the last edge of any cycle contributing to $\mathcal{N}^*(C_\ell, G)$ must belong to $E_{1,2}(G) \cup E_{2,2}(G)$ because the first edge also belonging to $E_{1,2}(G) \cup E_{2,2}(G)$ can have only one of its end point adjacent to an edge in $E_{1,1}(G)$. Hence for any $\ell$-bit starting $s$ starting with 10 containing only one 1 we trivially have $\mathcal{N}^*(C_\ell, G, s) = 0$. Thus such strings can be safely ignored and the substring $s'$ is well defined. Let us now write $s = s's''$. Note that by definition the length of the substring $s'$ is at least three.

If $s''$ is an empty string we get

$$\mathcal{N}^*(C_\ell, G, s) \leq (2\bar{e})D^{\ell - 1},$$

as there are at most $2\bar{e}$ many choices for the first edge whereas each subsequent edge must start at a small degree vertex and hence the total number of choices is upper bounded by $D$. This completes the proof of \((5.15)\) in the case $s''$ is empty.

If $s''$ is non-empty we let $3 \leq t < \ell$ to be the length of $s'$. Note that arguing as before there are at most $(2\bar{e})D^{t-1}$ many choices for the first $t$ edges, and for each such choice the total number of choices for the remaining $(\ell - t)$ edges is upper bounded by $\mathcal{N}(P_{t-1}, G, s'', v_t, v_1)$, where $v_t$ is the end point of the $t$-th edge and $v_1$ is the appropriate end point of the first edge. This observation together with Lemma 5.5 now yields that

$$\mathcal{N}^*(C_\ell, G, s) \leq (2\bar{e})D^{t-1} \cdot D^{\ell - t}(2\bar{e})^{[(\ell - t)/2]}$$

Noting that $t \geq 3$ implies $1 + [(\ell - t)/2] \leq [(\ell - 1)/2]$ the proof completes.

We now provide the proofs of Lemmas 3.8 and 3.10.

**Proof of Lemma 3.8.** This proof is an easy consequence of Lemma 5.1 and Proposition 5.2. Fix $\tilde{\mathbf{e}} := (e_{1,1}, e_{1,2}, e_{2,2})$, a non-negative integer $\mathbf{w}$, and let $\mathcal{M}_\#(\tilde{\mathbf{e}}, \mathbf{w})$ be the number of graphs with

$$\mathcal{N}(e_{1,1}(G), e_{1,2}(G), e_{2,2}(G)) = \tilde{\mathbf{e}}, \quad \mathcal{W}(G) = \mathbf{w},$$

and satisfying the hypothesis of the event $\overline{\text{Core}_{1,2}}$, where we refer the reader to \((3.11)\), and \((3.15)-(3.17)\) to recall the definitions of $\mathcal{W}(G)$, $e_{1,2}(G)$, and $e_{2,2}(G)$, and recall that $e_{1,1}(G) = |E_{1,1}(G)|$. Before applying Lemma 5.1 we note that any graph $G$ satisfying the hypothesis of $\overline{\text{Core}_{1,2}}$ must also satisfy the inequality $\mathcal{N}(C_{1,1}(G), G) \geq \varepsilon\delta n^{\gamma}p^\ell$. Therefore applying Proposition 5.2 with $\tau = \varepsilon\delta$ and $c_n = 1$, as $np \gg (\log n)^{1/(\ell - 2)}$, yields that for any such graph $G$ \((5.17)\)

$$e_{1,2}(G) + e_{2,2}(G) \geq n^2p^2 \cdot (\log n)^\gamma,$$

for some $\gamma > 0$. Thus setting $\mathcal{P}_G$ as in \((5.6)\), applying Lemma 5.1, and proceeding as in the steps leading to \((5.8)\) we find that

$$\mathcal{M}_\#(\tilde{\mathbf{e}}, \mathbf{w}) \leq \exp \left( \log(1/p) \left\{ e_{1,1} + \frac{1}{2}e_{1,2} + (\varepsilon/16) \cdot (e_{1,2} + e_{2,2}) \right\} + K\log \log n \cdot e \right)$$

$$\leq \exp \left( \log(1/p) \left\{ e_{1,1} + \frac{1}{2}e_{1,2} + (\varepsilon/8) \cdot (e_{1,2} + e_{2,2}) \right\} \right),$$

where the last step is a consequence of \((5.17)\) and the fact that $e(G) \leq \tilde{C}n^2p^2\log(1/p)$. Having obtained \((5.18)\) we now again proceed similar to the steps leading to \((5.10)\) and use the lower bound
(5.17) to derive the desired upper bound on the probability of \(\overline{\text{Core}}_{1,2}\). To avoid repetition we omit further details. This finishes the proof of this lemma.

Next we prove Lemma 3.10. It is quite similar to that of Lemma 3.8.

*Proof of Lemma 3.10.* Fix \(j \in [L_n]\). As before we fix \(\widehat{e} := (e_{1,1}, e_{1,2}, e_{2,2})\) and \(w\), and set \(\mathcal{N}(\widehat{e}, w)\) to be the set of the graphs satisfying (5.16) and the hypothesis of the event \(\overline{\text{Core}}_{2,j,\alpha}\). Once again we apply Proposition 5.2. This time we apply it with \(\tau = \varepsilon \delta\) and \(c_n = 2^{-(j-1)}\) to see that (5.17) continues to hold, uniformly for any \(j \in [L_n]\).

Note that moving from a graph \(G\) to its 2-core does not change the number of copies of \(C_\ell\) in it. So without loss of generality we may also assume that any graph \(G\) satisfying the hypothesis of \(\overline{\text{Core}}_{2,j,\alpha}\) must have \(d_{\min}(G) \geq 2\). Therefore arguing exactly same as above we notice that (5.18) holds for \(\mathcal{N}(\widehat{e}, w)\) as well, uniformly for all \(j \in [L_n]\). Thus repeating the same arguments as in the proof of Lemma 3.8 the desired bounds is derived. Further details are omitted.

\[\square\]

**Appendix A. Proofs of Lemmas 3.6 and 4.2**

Let us remind the reader that Lemmas 3.6 and 4.2 provide a lower bound on the product of the degrees of adjacent vertices of any (strong)-core graph. Furthermore, Lemma 4.2 also shows that for any strong-core graph there exists a large subgraph, containing most of the copies of \(C_\ell\) of the whole graph, such that the end points of all of its edges satisfy a tight upper and lower bound on the product of their degrees. This observation was crucial to the proof of the fact that the strong-core graphs are entropically stable.

*Proof of Lemma 4.2.* First let us prove the lower bound on the product of the degrees. To this end, from [12, Lemma 5.13], for any edge \(e = (u, v) \in E(G)\) we have that

\[N(C_\ell, G, e) \leq 4 \ell \cdot (2e_G)^{\frac{\ell}{2} - \frac{3}{2}} \cdot (4 \text{deg}_G(u) \cdot \text{deg}_G(v))^{\frac{1}{2}}.
\]

Since for any strong-core graph \(G\) and an edge \(e \in E(G)\) we have that

\[N(C_\ell, G, e) \geq (\varepsilon / \tilde{C}_*) \cdot (np)^{\ell - 2},
\]

it now follows from above that

\[\text{deg}_G(u) \cdot \text{deg}_G(v) \geq \frac{1}{4} \cdot \left(\frac{\varepsilon}{\tilde{C}_*}\right)^2 \cdot \left(\frac{np}{16 \ell^2 (2e_G)^{\ell - 3}}\right) \geq c_0(\varepsilon)n^2p^2,
\]

where the last step follows upon choosing \(c_0(\varepsilon)\) sufficiently small and the fact that \(e(G) \leq \tilde{C}_*n^2p^2\). This completes the proof of part (a).

Turning to proof part (b) let us fix some \(C_0 < \infty\) and let \(G_{\text{high}}\) be as in Definition 4.1. We claim that

\[e(G_{\text{high}}) \leq \frac{5e(G)^2}{C_0n^2p^2}.
\]

To see the above claim we recall that

\[N(P_3, G) \leq (2e(G))^2.
\]
On the other hand it is easy to see that

$$N(P_3, G) \geq 2 \sum_{(u, v) \in E(G)} (\deg_G(u) - 1) \cdot (\deg_G(v) - 2)$$

$$\geq 2 \sum_{(u, v) \in E(G)} \deg_G(u) \cdot \deg_G(v) - 3 \sum_{v \in V(G)} (\deg_G(v))^2 \geq 2C_0 n^2 p^2 \cdot e(G_{\text{high}}) - 6e(G)^2,$$

where the last step follows from the lower bound (4.1). Combining the upper and lower bounds on $N(P_3, G)$ the inequality (A.3) is now immediate.

Using (A.3) we next proceed to find the desired upper bound on $e(G_{\text{bad}})$. Recalling the definitions of $G_{\text{bad}}$ and $G_{\text{low}}$ we see that for any edge $e \in E(G_{\text{bad}})$ none of the copies of $C_\ell$ passing through is contained in $G_{\text{low}}$. As the lower bound (A.1) holds for every $e \in E(G)$ we deduce that

$$e(G_{\text{bad}}) \cdot (\varepsilon/\bar{C}_\ast) (np)^{\ell-2} \leq \sum_{e \in E(G_{\text{bad}})} N(C_\ell, G, e) \leq \sum_{e \in E(G_{\text{low}})} N(C_\ell, G, e).$$

From [12, Lemma 5.15] we have that

$$\sum_{e \in E(G_{\text{high}})} N(C_\ell, G, e) \leq \ell \cdot (2e(G))^{\ell/2} \cdot \sqrt{\frac{\varepsilon(G_{\text{high}})}{e(G)}} \leq \frac{3\ell 2^{\ell/2}}{\sqrt{C_0 np}} \cdot e(G)^{\ell+1/2}$$

$$\leq \frac{3\ell 2^{\ell/2} \bar{C}_\ast^{(\ell-1)/2}}{\sqrt{C_0}} \cdot (np)^{\ell-2} \cdot e(G),$$

where the penultimate inequality follows from (A.3) and the last step follows from the fact that $e(G) \leq \bar{C}_\ast n^2 p^2$. As, by definition, $G_{\text{high}} \subset G_{\text{bad}}$, now (A.4) together with (A.5) yield that

$$e(G_{\text{high}}) \leq e(G_{\text{bad}}) \leq \varepsilon e(G),$$

upon choosing $C_0$ sufficiently large. Finally noting that

$$N(C_\ell, G) - N(C_\ell, G_{\text{low}}) \leq \sum_{e \in E(G_{\text{high}})} N(C_\ell, G, e)$$

the lower bound

$$N(C_\ell, G_{\text{low}}) \geq (1 - \varepsilon) N(C_\ell, G)$$

follows from (A.5) upon using the upper bound on $e(G)$, the fact that $N(C_\ell, G) \geq \delta (1 - 6\varepsilon) n^\ell p^\ell$, and enlarging $C_0$ if necessary. This completes the proof of the lemma. $\square$

The proof of Lemma 3.6 follows from a same line reasoning as that in Lemma 4.2. Indeed, replacing (A.1) by the lower bound

$$N(C_\ell, G, e) \geq \varepsilon n^\ell p^\ell/(\bar{C}n^2 p^2 \log(1/p))$$

which holds for any edge $e$ in a core graph, arguing similarly as in (A.2), and using the upper bound $e(G) \leq \bar{C} n^2 p^2 \log(1/p)$ Lemma 3.6 follows. We omit further details.

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