LONGTIME EXISTENCE OF KÄHLER RICCI FLOW AND HOLOMORPHIC SECTIONAL CURVATURE

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ABSTRACT. In this work, we obtain a existence criteria for the longtime Kähler-Ricci flow solution. Using the existence result, we generalize a result by Wu-Yau on the existence of Kähler Einstein metric to the case with possibly unbounded curvature. Moreover, the Kähler Einstein metric with negative scalar curvature must be unique up to scaling.

1. INTRODUCTION

In this work we will discuss the existence of Kähler-Einstein metric on a complete noncompact Kähler manifold in terms of upper bound of holomorphic sectional curvature. In [31], Wu and Yau proved that if a compact complex manifold supports a Kähler metric with negative holomorphic sectional curvature, then it also supports a Kähler-Einstein metric with negative scalar curvature, under an additional assumption that the manifold is projective. Later Tosatti and Yang [29] were able to remove the assumption of projectivity. Using Kähler-Ricci flow, Normura [15] recovers the result by proving that under the assumption that the holomorphic sectional curvature is bounded above by a negative constant, the metric can be deformed under the normalized Kähler-Ricci flow to a Kähler-Einstein metric with negative scalar curvature. In case that the holomorphic sectional curvature is quasi-negative, namely it is non-positive and is negative somewhere, Diverio-Trapani [5] and Wu-Yau [32] then the canonical bundle is ample.

For the noncompact case, it was proved by Wu and Yau [30] if a noncompact complex manifold supports a Kähler metric with holomorphic sectional curvature bounded between two negative constants, then it also supports a Kähler-Einstein metric with negative scalar curvature. It is well-known that if the holomorphic sectional curvature is bounded then the curvature is bounded. Hence one may consider using Kähler-Ricci flow to obtain the same result. This has been done successfully by the fourth author of this work.

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In this paper, we first will give a rather general condition for a normalized Kähler-Ricci flow to converge to a Kähler-Einstein metric. We prove the following:

**Theorem 1.1.** Suppose there is a complete noncompact Hermitian metric $h$ on a complex manifold $M^n$ compatible with the complex structure $J$ such that

\[
H_h + \frac{2n}{n+1} |\hat{\nabla}_\delta \hat{T}|_h \leq -k
\]

for some $k > 0$. Then any longtime solution of normalized Kähler-Ricci flow $g(t)$ will converge uniformly in $C^\infty$ to a Kähler Einstein metric $g_\infty = -\text{Ric}(g_\infty)$. In particular, there is no Ricci flat Kähler metric on $M$ compatible with the same complex structure $J$.

Here $\hat{\nabla}$ is the derivative with respect to the Chern connection of $h$. See [28] for more details on the Chern connection, its torsion and curvature. Note that the Kähler-Einstein metric is unique, see [30, p.30] for example.

By the theorem, the question is to obtain longtime solution to the Kähler-Ricci flow. We have:

**Theorem 1.2.** Let $(M^n, g_0)$ be a complete Kähler manifold and $h$ is a fixed complete Hermitian metric on $M$ such that the following holds.

(i) There exists smooth exhaustion $\rho \geq 1$ such that

\[
\limsup_{\rho \to \infty} \left( \frac{|\partial \rho|_h}{\rho} + \frac{\sqrt{-1} \partial \bar{\partial} \rho|_h}{\rho} \right) = 0;
\]

(ii) and the holomorphic sectional curvature of $h$ and torsion of $h$ satisfy

\[
H_h + \frac{2n}{n+1} |\hat{\nabla}_\delta \hat{T}|_h \leq -k
\]

with $k \geq 0$

(iii) $\exists \alpha > 1$ such that on $M$, $\alpha^{-1} g_0 \leq h \leq \alpha g_0$, $|\hat{T}|_h \leq \alpha$;

(iv) $\limsup_{\rho \to \infty} \rho^{-1} |\hat{\nabla} g_0|_h = 0$;

Then there is $\beta(n, \alpha) > 0$ such that the Kähler Ricci flow has a complete solution $g(t)$ on $M \times [0, +\infty)$ with $g(0) = g_0$ and satisfies

\[
\beta h \leq g(t)
\]

on $M \times [0, +\infty)$.

It is known that if $M$ has bounded curvature, then it will support an exhaustion function $\rho$ with bounded gradient and Hessian [23, 26]. Hence $h$ is uniformly equivalent to a complete manifold with bounded curvature then condition (i) in the theorem is also satisfied. See also a recent result in [10]. Hence condition (i) is more general than the condition that the curvature is bounded.
Combining Theorems 1.1 and 1.2, we conclude that if \((M^n, g_0)\) is a complete Kähler metric, then the normalized Kähler-Ricci flow will converge to the Kähler-Einstein metric with negative scalar curvature in the following cases:

(a) The holomorphic sectional curvature is bounded above by \(-k\) for some \(k > 0\) and \(g_0\) support an exhaustion function with bounded gradient and bounded complex Hessian.
(b) There exists a complete Hermitian metric \(h\) so that \(g_0, h\) satisfy the conditions in Theorem 1.2 with \(k > 0\).
(c) \(g_0\) satisfy a Sobolev inequality and the curvature is bounded in some \(L^p\) sense so that the holomorphic sectional curvature is bounded above by \(-k\) for some \(k > 0\). (See more precise statement in Corollary 5.2.)

In case \(g_0\) has bounded curvature so that the holomorphic sectional curvature is bounded above by \(-k < 0\) for some constant \(k\), then the conditions in (c) will also be satisfied. (a)–(c) are some generalizations to Wu-Yau’s result \([30]\).

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2. Short time existence lemma

As a approximation, it is more convenient to consider the case when \(g_0\) is only Hermitian metric but not necessary Kähler. Let \((M^n, g_0)\) be a complete noncompact Hermitian manifold with complex dimension \(n\). In the following, connection and curvature will be refered to the Chern connection and curvature with respect to the Chern connection. When the torsion vanishes, it coincides with the Levi-Civita connection. For basic facts on Chern connection and curvature of Hermitian manifolds see \([28]\) for example. In this section, we want to discuss the existence of the Chern-Ricci flow:

\[
\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -R_{ij}; \\
g(0) &= g_0. 
\end{aligned}
\]

where \(R_{ij} = -\partial_i \partial_j \log \det(g(t))\) is the Chern-Ricci curvature of \(g(t)\). This equation is equivalent to the following parabolic complex Monge-Ampère equation:

\[
\begin{aligned}
\frac{\partial}{\partial t} \psi &= \log \frac{(\omega_0 - t \text{Ric}(\omega_0) + \sqrt{-1} \partial \bar{\partial} \psi)^n}{\omega_0^n}, \\
\psi(0) &= 0.
\end{aligned}
\]

More precisely, if \(g(t)\) is a solution to (2.1), let

\[
\begin{aligned}
\psi(x, t) &= \int_0^t \log \left( \frac{\omega^n(x, s)}{\omega_0^n(x)} \right) ds.
\end{aligned}
\]

where \(\omega(t)\) and \(\omega_0\) are the associated \((1,1)\) forms of \(g(t), g_0\) respectively. Then \(\psi\) satisfies (2.2). One can see that \(\omega(t) = \omega_0 - t \text{Ric}(\omega_0) + \sqrt{-1} \partial \bar{\partial} \psi\). Conversely,
if \( \psi \) is a smooth solution to \((2.2)\) so that \( \omega_0 - t\text{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0 \), then \( \omega(t) \) defined by the above relation satisfies \((2.1)\). We will say that \( \psi \) is the solution of \((2.2)\) corresponding to the solution \( g(t) \) of \((2.1)\).

In [13], it has been shown that when \((M, g_0)\) has bounded curvature of infinity order, the Monge-Ampère equation \((2.2)\) and hence the Chern-Ricci flow equation \((2.1)\) has a short time solution on \( M \).

**Lemma 2.1** (see [2, 13]). Let \((M^n, g_0)\) be a complete noncompact Hermitian metric. Suppose \( g_0 \) has bounded geometry of infinite order, then \((2.1)\) has a solution \( g(t) \) on \( M \times [0, S] \) for some \( S > 0 \) and there is a constant \( C > 0 \) such that \( C^{-1}g_0 \leq g(t) \leq Cg_0 \).

### 3. A-priori estimate for the Chern Ricci flow

Let \((M_n, g)\) be a Hermitian manifold. Under a local holomorphic coordinate system \((z_1, \ldots, z_n)\), the torsion tensor of \( g \) is defined by

\[
T_{ijkl} = \partial_i g_{jk} - \partial_j g_{ik}.
\]

Let \( T^k_{ij} = g^{kl}T_{ijkl} \), then \( T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} \) where \( \Gamma^k_{ij} \) is the Chern connection. \( T^k_{ij} \) is usually called the torsion. Here we use \( \Gamma_{ijk} \) to denote the torsion. The advantage is that it is invariant under the Chern-Ricci flow. The curvature tensor of the Chern connection has components

\[
R_{ijkl} = -\frac{\partial^2 g_{kl}}{\partial z_i \partial \bar{z}_j} + g^{qp} \frac{\partial g_{kp}}{\partial z_i} \frac{\partial g_{ql}}{\partial \bar{z}_j}.
\]

If the torsion tensor \( T = 0 \), then \( g \) is Kähler. It can be checked easily that for \( X, Y \in T^{1,0}M \), \( R(X, \bar{X}, Y, \bar{Y}) \) is real-valued. We introduce the following curvature condition.

**Definition 3.1.** We say that \((M, g)\) has holomorphic sectional curvature bounded above by \( \kappa \) if for any \( p \in M \), \( X \in T_p^{1,0}M \),

\[
R(X, \bar{X}, X, \bar{X}) \leq \kappa |X|^4.
\]

For notational convenience, we denote it to be \( H_g \leq \kappa \).

Let \( g(t) \) be a solution of the Chern-Ricci flow with initial metric \( g(0) = g_0 \) and \( h \) is another Hermitian metric on \( M \). Now we wish to obtain some a priori estimates for \( g(t) \). First we list some evolution equations which is related to the Chern-Ricci flow.

**Lemma 3.1**. The evolution equation of \( \Lambda = Tr_g h = g^{ij}h_{ij} \) is given by

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \Lambda = (I) + (II) + (III).
\]
where

\[ (I) = -h_{ij}g^{pq}g^{pr}q_jP^k_{ri} + 2Re \left[ g^{ij}g^{pq}h_{kji} \right] ; \]

\[ (II) = g^{lk}g^{ij}p_lh_{jki}(T_0)_{ijp} + \hat{T}_{qsh}h^{rs} - (T_0)_{iqs}g^{rs} \]

\[ + g^{ij}k^lgh_{jki} \hat{\nabla}_p(T_0)_{qij} + \hat{\nabla}_i(T_0)_{pq} ; \]

\[ (III) = g^{ij}p^qR_{piqj} . \]

In particular, \( (I) \leq h_{pp}h_{qq}h^{k}g^{xy}g^{cd}g^{ij}h_{kij}(T_0)_{xia}(T_0)_{yjk} \). Moreover, we have the following evolution equation of the quantity \( \log tr_{gh} \).

\[ (3.2) \quad \left( \frac{\partial}{\partial t} - \Delta \right) \log \Lambda = (IV) + \Lambda^{-1} [ (II) + (III) ] \]

where

\[ (IV) \leq \Lambda^{-1}h_{pq}h_{rs}h^{k}g^{xy}g^{cd}g^{ij}h_{kij}(T_0)_{xia}(T_0)_{yjk} \]

\[ + 2\Lambda^{-2}Re \left[ h_{pq}g^{xy}g^{cd}g^{ij}h_{kij}(T_0)_{xia}(T_0)_{yjk} \right] . \]

Here \( \Delta F = g^{ij}\partial_i\partial_j F \) for function \( F \in C^2(M) \). \( T_0 \) and \( \hat{T} \) are the torsion of metric \( g_0 \) and \( h \) respectively.

Proof.

\[ \partial_t tr_{gh} = g^{ij}p^qh_{ij}R_{pq} . \]

Denote \( P^k_{ij} = \hat{P}^k_{ij} - P^k_{ij} \).

\[ \Delta tr_{gh} = g^{ij}g^{pq}\nabla_q\nabla_p h_{ij} \]

\[ = g^{ij}g^{pq}\nabla_q \left( P^k_{ij}h_{kj} \right) \]

\[ = g^{ij}g^{pq} \left( R^k_{pqi} - \hat{R}^k_{pqi} \right) h_{kj} + P^k_{iqj}R^j_{ki} . \]

Using the fact that \( \partial \theta_0 = \partial \theta \), we have

\[ R_{pqi} = R_{ilpq} - \nabla_p(T_0)_{qii} - \nabla_iT_{pqi} \]

\[ = R_{ilpq} - \nabla_i(T_0)_{qii} - \nabla_i(T_0)_{qip} . \]

Hence,

\[ g^{ij}g^{pq}h_{kj}R^k_{pqi} = g^{ij}g^{kl}g^{pq}h_{kj}R_{pqk} \]

\[ = g^{ij}g^{kl}g^{pq}h_{kj} \left( R_{ilpq} - \nabla_p(T_0)_{qii} - \nabla_i(T_0)_{qip} \right) \]

\[ = g^{ij}g^{kl}h_{kj}R_{ilpq} - g^{ij}g^{kl}g^{pq}h_{kj} \left( \nabla_p(T_0)_{qii} + \nabla_i(T_0)_{qip} \right) . \]
Therefore,

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_g h
= -h_{kl}g^{ij} g^{pq} \psi^k_{pi} \overline{\psi}^l_{qj} + g^{ij} g^{pq} \tilde{R}_{pqi\bar{j}} + g^{ij} g^{k\bar{j}} g^{pq} h_{ki} \left[ \nabla_p(T_0)q_{\bar{l}i} + \nabla_l(T_0)p_{\bar{q}i} \right]
= -h_{kl}g^{ij} g^{pq} \psi^k_{pi} \overline{\psi}^l_{qj} + g^{ij} g^{k\bar{j}} g^{pq} h_{ki} \left[ \psi_{pq}(T_0)_{i\bar{q}r} + \psi_{\bar{l}q}(T_0)_{ip} \right]
+ g^{ij} g^{k\bar{j}} g^{pq} h_{ki} \left[ \hat{\nabla}_p(T_0)q_{\bar{l}i} + \hat{\nabla}_l(T_0)p_{\bar{q}i} \right] + g^{ij} g^{pq} \tilde{R}_{pqi\bar{j}}
= (I) + (II) + (III),
\]

where

\[
(I) = -h_{kl}g^{ij} g^{pq} \psi^k_{pi} \overline{\psi}^l_{qj} + 2 \text{Re} \left[ g^{ij} g^{k\bar{j}} g^{pq} h_{ki} \psi_{\bar{l}q}(T_0)_{ip} \right];
\]

\[
(II) = g^{ik} g^{j\bar{k}} g^{ap} h_{jk}(T_0)_{i\bar{q}r} \left[ \tilde{T}_{q\bar{r}i} h^{\bar{s}r} - (T_0)_{q\bar{r}i} g^{r\bar{s}} \right]
+ g^{ij} g^{k\bar{j}} g^{pq} h_{ki} \left[ \hat{\nabla}_p(T_0)q_{\bar{l}i} + \hat{\nabla}_l(T_0)p_{\bar{q}i} \right];
\]

\[
(III) = g^{ij} g^{pq} \tilde{R}_{pqi\bar{j}}.
\]

Thus,

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \log \Lambda = \Lambda^{-1} \left( \frac{\partial}{\partial t} - \Delta \right) \Lambda + \Lambda^{-2} g^{ij} \partial_i \Lambda \partial_j \Lambda
= \frac{1}{\Lambda} \left[ (I) + \frac{1}{\text{tr}_g h} |\partial \Lambda|^2 \right] + \frac{1}{\Lambda} [(II) + (III)].
\]

In the special case where $g_0$ is Kähler, it was shown by Yau [36] that the first bracket term is nonpositive. In the Hermitian case, we follow a generalization of this argument in [28]. We consider the following nonegative term.

\[
K = h_{kl}g^{ij} g^{pq} \left( \psi^k_{pi} - \frac{\delta^k_i}{\text{tr}_g h} \partial_p \Lambda + C^{k}_{pi} \right) \left( \psi^l_{\bar{q}j} - \frac{\delta^l_{\bar{q}}}{\text{tr}_g h} \partial_q \Lambda + C^{\bar{q}l}_{\bar{q}j} \right)
= h_{kl}g^{ij} g^{pq} \psi^k_{pi} \psi^l_{\bar{q}j} - \frac{1}{\text{tr}_g h} |\partial \text{tr}_g h|^2 + h_{kl}g^{ij} g^{pq} C^{k}_{pi} \left( \psi^l_{\bar{q}j} - \frac{\delta^l_{\bar{q}}}{\text{tr}_g h} \partial_q \Lambda \right)
+ h_{kl}g^{ij} g^{pq} C^{\bar{q}l}_{\bar{q}j} \left( \psi^k_{pi} - \frac{\delta^k_i}{\text{tr}_g h} \partial_p \Lambda \right) + h_{kl}g^{ij} g^{pq} C^{k}_{pi} C^{\bar{q}l}_{\bar{q}j}.
\]
Hence,

\[(I) + \frac{1}{\Lambda} |\partial \Lambda|^2 = -K + 2Re \left[ g^{ij} g^{kl} g_{kq} h_{ij} (T_0)_{ipq} \right] + 2Re \left[ h_{kil} g^{ij} g^{pq} C^k_p \tilde{\psi}_{lj} \right] + h_{kij} g^{ij} g^{pq} C^k_p C^l_q - 2\Lambda^{-1} Re \left[ h_{kij} g^{ij} g^{pq} C^k_p \partial_q \Lambda \right] = -K + h_{kij} g^{ij} g^{pq} C^k_p C^l_q - 2\Lambda^{-1} Re \left[ h_{kij} g^{ij} g^{pq} C^k_p \partial_q \Lambda \right] + 2Re \left[ \tilde{\psi}_{i \bar{j}} g^{pq} g^{ij} (C^k_p h_{kl} + g^{k \rho} h_{\rho \sigma} (T_0))_{k\bar{l}} \right].

Therefore, if we choose the tensor $C$ to be

$C^q_{pi} = -g^{k \rho} h^{q \sigma} (T_0)_{k\bar{l}}$

then the last term vanished. Hence, we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \log \Lambda = \Lambda^{-1} \left( \frac{\partial}{\partial t} - \Delta \right) \Lambda + \Lambda^{-2} g^{ij} \partial_i \Lambda \partial_j \Lambda = (IV) + \Lambda^{-1} (II) + \Lambda^{-1} (III),
\]

where

\[(IV) \leq \Lambda^{-1} h_{\rho \sigma} h_{\bar{c} q} h^{k \bar{a}} g^{s \bar{e}} g^{t \bar{d}} g^{ij} g^{pq} (T_0)_{si \bar{a}} (T_0)_{\bar{d} \bar{j} k} + 2\Lambda^{-2} Re \left[ h_{\rho \sigma} g^{q \bar{p}} g^{i \bar{l}} g^{j} g^{pq} (T_0)_{ai \bar{l}} \partial_q \Lambda \right].
\]

The inequality on $(I)$ follows the same line as in the inequality of $A_1$ by considering a simpler quantity

$K = h_{kij} g^{ij} g^{pq} (\tilde{\psi}_{ki}^{\rho} - C^k_p) (\tilde{\psi}_{ij}^{\rho} - C^l_q).$

\[\square\]

We have the following form of Royden’s Lemma [20] which relates the holomorphic sectional curvature with a bisectional curvature quantity.

In the following, $|\hat{\nabla}_g \hat{T}|_h(x)$ at a point $x$ is defined as:

\[(3.3) \quad |\hat{\nabla}_g \hat{T}|_h = \max |\hat{\nabla}_i \hat{T}_{jik}|\]

where the maximum is taken over all unitary frames $e_i$ of $h$ at $x$. Define $|\hat{\nabla}_g T_0|_h$ similarly.

**Lemma 3.2.** Let $(M, h)$ be a Hermitian manifold and $g$ is another metric on $M$. Suppose that the holomorphic sectional curvature of $h$ at $x$ is bounded above by $\kappa(x)$. Suppose $\kappa(x) \leq \kappa_0$. Then we have

\[g^{ij} g^{kl} \hat{R}_{ijkl} \leq \left( \frac{n + 1}{2n} \kappa + |\hat{\nabla}_g \hat{T}|_h \right) (tr g \, h)^2 + \frac{1}{2} \kappa_0 \left[ -\frac{1}{n} (tr g \, h)^2 + g^{ij} g^{kl} h_{ij} h_{kl} \right].\]
By the “Kähler” identity for the Chern curvature, e.g. see [28]. We have

\( g^{ij}g^{kl} \hat{R}_{ijkl} + g^{ij}g^{kl} \hat{R}_{iklj} \leq \kappa (tr_g h)^2 + \kappa g^{ij}g^{kl}h_{kj}h_{il}. \)

By the”Kähler” identity for the Chern curvature, e.g. see [28]. We have

\[
(3.4) \quad g^{ij}g^{kl} \hat{R}_{ijkl} = g^{ij}g^{kl}(\hat{R}_{iklj} + \hat{\nabla}_i \hat{T}_{jlk}) \\
= \frac{1}{2} g^{ij}g^{kl}(\hat{R}_{iklj} + \hat{R}_{iklj} + \hat{\nabla}_i \hat{T}_{jlk}) \\
\leq \frac{\kappa}{2} (tr_g h)^2 + g^{ij}g^{kl}\left(\kappa h_{kj}h_{il} + \hat{\nabla}_i \hat{T}_{jlk}\right) \\
\leq \frac{1}{2}(\kappa(x) - \kappa_0)\left[ (tr_g h)^2 + g^{ij}g^{kl}h_{kj}h_{il}\right] + (tr_g h)^2 |\hat{\nabla}_\partial \hat{T}|_h \\
+ \frac{1}{2}\kappa_0 \left[ (tr_g h)^2 + g^{ij}g^{kl}h_{kj}h_{il}\right] \\
\leq \frac{1}{2}(\kappa(x) - \kappa_0)(1 + \frac{1}{n})(tr_g h)^2 + (tr_g h)^2 |\hat{\nabla}_\partial \hat{T}|_h \\
+ \frac{1}{2}\kappa_0 \left[ (tr_g h)^2 + g^{ij}g^{kl}h_{kj}h_{il}\right] \\
= \left(\frac{n+1}{2n}\kappa + |\hat{\nabla}_\partial \hat{T}|_h\right)(tr_g h)^2 + \frac{1}{2}\kappa_0 \left[ \frac{1}{n}(tr_g h)^2 + g^{ij}g^{kl}h_{kj}h_{il}\right]
\]

\[ \square \]

**Corollary 3.1.** With the same assumptions and notation as in Lemma [3.3]. Suppose the holomorphic sectional curvature of \( h \) is bounded above by \( \kappa(x) \) at \( x \) and suppose \( \frac{n+1}{2n}\kappa(x) + |\hat{\nabla}_\partial \hat{T}|_h(x) \leq \kappa_0 \) for some \( \kappa_0 \geq 0 \) for all \( x \). Then

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \Lambda \leq c(n) \left( \Lambda^4 |T_0|^2 + \Lambda^3 (|T_0|_h|\hat{T}|_h + |\hat{\nabla}_\partial T_0|_h) + \Lambda^2 \kappa_0 \right)
\]

for some constant \( c(n) > 0 \) depending only on \( n \).

To get a \( C^0 \) estimate, it is useful to consider the Chern scalar curvature of \( g(t) \) which gives us information on the derivatives of the volume form.

**Lemma 3.3.** Under the evolution of metrics

\[ \partial_t g = -\text{Ric} \]

the Chern scalar curvature \( R \) satisfies

\[
\left( \frac{\partial}{\partial t} - \Delta \right) R = |\text{Ric}|^2 \geq \frac{1}{n} R^2.
\]
Lemma 3.5. Suppose is a solution to the Chern Ricci flow on \( M_{\alpha>1} \) outside \((\rho, \bar{\rho})\) with \( \lambda_0 \delta_{ij} \). Then it follows immediately by Cauchy inequality. \( \square \)

We have the following maximum principle.

**Lemma 3.4.** Let \((M^n, h)\) be a complete noncompact Hermitian manifold satisfying condition: (a1) There exists a smooth positive real exhaustion function \( \rho \) such that \( |\partial \rho|^2_h + |\sqrt{-1} \partial \bar{\partial} \rho|_h \leq C_1 \). Suppose \( g_0 \) is another Hermitian metric uniformly equivalent to \( h \) and \( g(t) \) is a solution to the Chern-Ricci flow with initial metric \( g(0) = g_0 \) on \( M \times [0, S) \). Assume for any \( 0 < S_1 < S \), there is \( C_2 > 0 \) such that

\[
C_2^{-1} h \leq g(t)
\]

for \( 0 \leq t \leq S_1 \). Let \( f \) be a smooth function on \( M \times [0, S) \) which is bounded from above such that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) f \leq 0
\]

on \( \{f > 0\} \). Suppose \( f \leq 0 \) at \( t = 0 \), then \( f \leq 0 \) on \( M \times [0, S) \).

**Proof.** For any \( \epsilon > 0 \), if \( \sup_{M \times [0, T]} (f - \epsilon \rho - 2 \epsilon C_1 C_2 t) > 0 \), then there is \((x_0, t_0)\) with \( t_0 > 0 \) such that \( f - \epsilon \rho - 2 \epsilon C_1 C_2 t \leq 0 \) on \( M \times [0, t_0] \) and \( f - \epsilon \rho - 2 \epsilon C_1 C_2 t = 0 \) at \((x_0, t_0)\). In particular, \( f(x_0, t_0) > 0 \). Hence at \((x_0, t_0)\), we have

\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta \right) (f - \epsilon \rho - 2 \epsilon C_1 C_2 t) < 0,
\]

which is impossible. Since \( \epsilon \) is arbitrary, this completes the proof. \( \square \)

Next we give a local estimate on the Chern scalar curvature’s lower bound.

**Lemma 3.5.** Suppose \( h \) is a fixed hermitian metric satisfying (a1) and \( g(t) \) is a solution to the Chern Ricci flow on \( M \times [0, S) \) with \( g(t) \geq \alpha^{-1} h \) for some \( \alpha > 1 \). Then for any \( 0 < r_1 < r_2 \), there exists \( C_n > 0 \) such that for any \( x \in M \) with \( \rho(x) < r_1 \) and \( t \in [0, S) \), we have

\[
R(x, t) \geq -\max \left\{ C_n \alpha [(r_2 - r_1)^{-2} + 1], \sup_{\rho(y) < r_2} R_-(y, 0) \right\}.
\]

**Proof.** Let \( \phi \) be a cutoff function on \( \mathbb{R} \) such that \( \phi \equiv 1 \) on \((\infty, 1]\), vanishes outside \((-\infty, 2]\) and satisfies \( \phi^{-1} |\phi'|^2 \leq 100 \) and \( \phi'' \geq -100 \phi \). Define

\[
\Phi(x, t) = \phi \left( \frac{\rho(x) + r_2 - 2r_1}{r_2 - r_1} \right).
\]
When the function $\Phi R$ achieves its local minimum at $(x_0, t_0)$ in which we may assume $R(x_0, t_0) < 0$ and $t_0 > 0$, it satisfies the following.

$$
0 \geq \left( \frac{\partial}{\partial t} - \Delta \right) (\Phi R) \\
= \Phi \left( \frac{\partial}{\partial t} - \Delta \right) R - R \Delta \Phi - 2 \text{Re} \left( g^{ij} \partial_i \Phi \partial_j R \right) \\
\geq \frac{1}{n} \Phi R^2 - R \left[ \frac{\phi''}{(r_2 - r_1)^2} |\partial \rho|^2 + \frac{\phi'}{r_2 - r_1} \Delta \rho - 2 \frac{(\phi')^2}{(r_2 - r_1)^2} \phi |\partial \rho|^2 \right] \\
\geq \frac{1}{n} \Phi R^2 + C_n \Lambda R [(r_2 - r_1)^{-2} + 1].
$$

Hence, at its minimum point $(x_0, t_0)$,

$$
\Phi R \geq -C_n \Lambda [(r_2 - r_1)^{-2} + 1].
$$

The conclusion follows by the minimum principle. $\square$

### 4. Existence of the Chern-Ricci Flow

In this section, we will discuss the existence of the Chern-Ricci flow starting from a Hermitian metric with holomorphic sectional curvature bounded from above. We will give a estimate on the existence time. More generally, we will consider initial metric which is uniformly equivalent to a Hermitian metric with holomorphic sectional curvature bounded from above.

Recall the following definition of bounded geometry:

**Definition 4.1.** Let $(M^n, g)$ be a complete Hermitian manifold. Let $k \geq 1$ be an integer and $0 < \alpha < 1$. $g$ is said to have bounded geometry of order $k + \alpha$ if there are positive numbers $r, \kappa_1, \kappa_2$ such that at every $p \in M$ there is a neighborhood $U_p$ of $p$, and local biholomorphism $\xi_p$ from $D(r)$ onto $U_p$ with $\xi_p(0) = p$ satisfying the following properties:

(i) the pull back metric $\xi_p^*(g)$ satisfies:

$$
\kappa_1 g_e \leq \xi_p^*(g) \leq \kappa_2 g_e
$$

where $g_e$ is the standard metric on $\mathbb{C}^n$;

(ii) the components $g_{ij}$ of $\xi_p^*(g)$ in the natural coordinate of $D(r) \subset \mathbb{C}^n$ are uniformly bounded in the standard $C^{k+\alpha}$ norm in $D(r)$ independent of $p$.

$(M, g)$ is said to have bounded geometry of infinity order if instead of (ii) we have for any $k$, the $k$-th derivatives of $g_{ij}$ in $D(r)$ are bounded by a constant independent of $p$. $g$ is said to have bounded curvature of infinite order on a compact set $\Omega$ if (i) and (ii) are true for all $k$ for all $p \in \Omega$.

**Lemma 4.1.** Let $(M^n, g_0)$ be a Hermitian metric with bounded geometry of infinite order. Suppose $g_0$ is uniformly equivalent to a Hermitian metric $h$ with
holomorphic sectional curvature and torsion satisfying: $H_h(x)$ bounded above by $\kappa(x)$ and $\frac{n+1}{2n}\kappa(x) + |\nabla_\bar{\partial}\hat{T}|_h(x) \leq \kappa_0$ for some $\kappa_0 \geq 0$ for all $x$, so that
\[ \alpha^{-1}h \leq g_0 \leq \alpha h, \]

Then the Chern-Ricci flow has a solution $g(t)$ with $g(0) = g_0$ on $M \times [0, S]$ with the following properties:

(i) There is a constant $c = c(n) > 0$ so that
\[ S \geq \frac{1}{3c(n\alpha + 1)^3} =: S_1 \]

where
\[ s = \sup_M \left( |T_0|^2 + |T_0| |\hat{T}|_h + |\nabla_\bar{\partial}T_0|_h + \kappa_0 \right) \]

where $T_0$ is the torsion of $g_0$, $\hat{T}$ is the torsion of $h$ and $\nabla$ is the derivative of $h$ with respect to the Chern connection; and

(ii) $g(t)$ is uniformly equivalent to $h$ with
\[ \text{tr}_g h \leq \left( \frac{1}{(n\alpha + 1)^{-3} - 3c_2s} \right)^\frac{1}{3} \]
on $M \times [0, S_1]$.

Proof. By [13, Theorem 4.2], there is a maximal $S > 0$ such that the Chern-Ricci flow has a solution $g(t)$ with $g(0) = g_0$ on $M \times [0, S)$ so that $g(t)$ is uniformly equivalent to $g_0$ on $[0, S')$ for all $S' < S$. Let $\Lambda = \text{tr}_{g(t)} h$. By Corollary 3.1

\[ \left( \frac{\partial}{\partial t} - \Delta \right) \Lambda \leq c_1 \left( \Lambda^4 |T_0|^2 + \Lambda^3 (|T_0| |\hat{T}|_h + |\hat{T}_0|) + \kappa_0 \right) \leq c_1 (\Lambda + 1)^4 s \]

Here and below $c_i$ will denote a positive constants depending only on $n$. Let
\[ v(t) = \left( \frac{1}{(n\alpha + 1)^{-3} - 3c_2s} \right)^\frac{1}{3} \]

Then $v(t)$ is defined on $[0, S_1]$ with $S_1 = 1/ [3c_2(n\alpha + 1)^3s]$, with
\[ \frac{dv}{dt} = c_2sv^4 \]

and $v(0) \geq (\Lambda + 1)|_{t=0}$. Suppose $S < S_1$. Since $\Lambda$ and $v$ are bounded on $[0, S')$ for all $0 < S' < S$, by Lemma 3.4 as in the proof of [13, Theorem 4.2], one can conclude that
\[ \Lambda \leq v(t) - 1 \]
on $M \times [0, S)$. In particular,
\[ h \leq c_3(v(t) - 1)g(t) \]
If $S < S_1$, then $v(t) \leq C_1 < \infty$ on $[0, S]$ for some $C_1$. Hence $\Lambda \leq C_1$ on $M \times [0, S)$.

On the other hand, since $g_0$ has bounded geometry of infinite order, by Lemma 3.5 we conclude that $R(x, t) \geq -C_2$ on $M \times [0, S)$ for some $C_2$. Since

$$\frac{\partial}{\partial t} \left( \log \frac{\det(g(t))}{\det(h)} \right) = -R \leq C_2,$$

we conclude that $\det(g(t)) \leq C_3 \det(h)$. Together with (4.1), we conclude that $C_3^{-1} g_0 \leq g(t) \leq C_3 g_0$

on $M \times [0, S)$ for some $C_3 > 0$. Here we have used the fact that $g_0$ is uniformly equivalent to $h$. Using the fact that $g_0$ has bounded geometry of infinite order and by the local estimates of [22], $g(t)$ can be extended to be a solution of the Chern-Ricci flow which is uniformly equivalent to $g_0$ beyond $S$. Hence we have $S \geq S_1$. This proves (i).

(ii) Follows from (4.1). □

Let $(M^n, h)$ be a complete noncompact Hermitian manifold and let $g_0$ be another Hermitian metric satisfying the following:

(a) There exists smooth exhaustion $\rho \geq 1$, and constant $\beta > 0$ such that

$$|\bar{\partial} \partial \rho|_h + |\sqrt{-1} \partial \bar{\partial} \rho|_h \leq \beta \rho$$

if $\rho$ is large enough.

(b) The holomorphic sectional curvature at $x$ is bounded from above by $\kappa(x)$, and the torsion of $h$ is such that

$$\frac{n + 1}{2n} \kappa + |\hat{\nabla} \hat{T}|_h \leq \kappa_0$$

for some $\kappa_0 \geq 0$.

**Theorem 4.1.** Let $(M^n, h)$ be a complete Hermitian metrics as above. Suppose $g_0$ is another Hermitian metric satisfying:

(i) $\alpha^{-1} g_0 \leq h \leq \alpha g_0$ and $|\bar{T}|_h \leq \alpha$ for some $\alpha > 0$;

(ii) There is $\beta > 0$, $|T_0|_h^2 + |\bar{T}|_h|T_0|_h + |\hat{\nabla} \hat{g}(T_0)|_h \leq \beta$; and

(iii) $|\hat{\nabla} g_0|_h \leq \beta \rho$ for $\rho$ large enough.

There exist constants $c_1$ depending only on $n$ and $c_2$ depending only on $n$, $\alpha$ such that there is a solution $g(t)$ for the Chern-Ricci flow on $M \times [0, S)$ with $g(0) = g_0$, where

$$S = \frac{1}{2c_1(n\alpha + 1)^3 s}$$

and $s = \kappa_0 + c_2 \beta (1 + \beta)$. Moreover,

$$tr g \leq v(t) - 1$$
where
\[ v(t) = \left( \frac{1}{(n\alpha + 1)^{3/2} - 3c_1st} \right)^{1/3}. \]
on $M \times [0, S]$.

We want to apply Lemma 4.1. However, in general it is not true that $g_0$ has bounded geometry of all order, we cannot apply Lemma 2.1 directly to obtain a solution of the Chern-Ricci flow. We now proceed as in [13, 12] to construct a Hermitian approximation.

Let $\tau \in (0, \frac{1}{2})$, $f : [0, 1) \to [0, \infty)$ be the function:
\begin{equation}
(4.2) \quad f(s) = \begin{cases} 
0, & s \in [0, 1 - \tau]; \\
-\log \left[ 1 - \left( \frac{s - 1 + \tau}{\tau} \right)^2 \right], & s \in (1 - \tau, 1).
\end{cases}
\end{equation}

Let $\varphi \geq 0$ be a smooth function on $\mathbb{R}$ such that $\varphi(s) = 0$ if $s \leq 1 - \tau + \tau^2$,
\begin{equation}
(4.3) \quad \varphi(s) = \begin{cases} 
0, & s \in [0, 1 - \tau + \tau^2]; \\
1, & s \in (1 - \tau + 2\tau^2, 1).
\end{cases}
\end{equation}

such that $\frac{2}{\tau^2} \geq \varphi' \geq 0$. Define
\[ \mathfrak{F}(s) := \int_0^s \varphi(\tau)f'(\tau)d\tau. \]

From [12], we have:

**Lemma 4.2.** Suppose $0 < \tau < \frac{1}{2}$. Then the function $\mathfrak{F} \geq 0$ defined above is smooth and satisfies the following:

(i) $\mathfrak{F}(s) = 0$ for $0 \leq s \leq 1 - \tau + \tau^2$.
(ii) $\mathfrak{F}' \geq 0$ and for any $k \geq 1$, $\exp(-k\mathfrak{F})\mathfrak{F}^{(k)}$ is uniformly bounded.
(iii) For any $1 - 2\tau < s < 1$, there is $\tau > 0$ with $0 < s - \tau < s + \tau < 1$ such that
\[ 1 \leq \exp(\mathfrak{F}(s + \tau) - \mathfrak{F}(s - \tau)) \leq (1 + c_2\tau); \quad \tau \exp(\mathfrak{F}(s_0 - \tau)) \geq c_3\tau^2 \]
for some absolute constants $c_2 > 0, c_3 > 0$.

For any $\rho_0 > 0$, let $U_{\rho_0}$ be the component of $\{x \mid \rho(x) < \rho_0\}$ which contains a fixed point and $\rho$ is the positive exhaustion function mentioned above. Hence $U_{\rho_0}$ will exhaust $M$ as $\rho_0 \to \infty$.

Let $\rho_i > 1$ be a sequence increasing to $+\infty$, let $F^{(i)}(x) = \mathfrak{F}(\rho(x)/\rho_i)$. Let $g_{0,i} = e^{2F^{(i)}}g_0$. In the following, $F^{(i)}$ will be denoted simply by $F$ if there is no confusion arisen.

Then $(U_{\rho_i}, g_i)$ is a complete Hermitian metric, (e.g. see [6]) and $g_{i,0} = g_0$ on $\{\rho(x) < (1 - \kappa + \kappa^2)\rho_0\}$. Moreover, the new manifold has a very nice structure.
Lemma 4.3 ([13]). For each $\rho_i > 1$ sufficiently large, $(U_{\rho_i}, g_{0,i})$ has bounded geometry of infinite order.

In the following, we will estimate the torsion and the holomorphic sectional curvature after performing conformal change.

Lemma 4.4. Let $g_0$ and $h$ be as in Theorem 4.1. For $i \to \infty$, let $g_{0,i}$ be as in Lemma 4.3 and $h_i = e^{2F} h$ for the corresponding $F$. Let $T_{0,i}$ be the torsion of $g_{0,i}$. Then there is a constant $c(n, \alpha)$ depending only on $n$ and $\alpha$ so that as $i \to \infty$, there

(i) $|T_{0,i}|_{h_i}^2 \leq c_1 \beta$, where

(ii) $|T_{0,i}|_{h_i} |\hat{T}^{(i)}|_{h_i} \leq c_1 \beta$;

(iii) $|\hat{\nabla}_\partial^{(i)} T_{0,i}|_{h_i} \leq c_1 (1 + \beta)$, where $\hat{\nabla}^{(i)}$ is derivative with respect to the Chern connection of $h_i$;

(iv) $\frac{n+1}{2n} \kappa_i(x) + |\hat{\nabla}_\partial^{(i)} T_i|_{h_i}(x) \leq \kappa_0 + c_1 (1 + \beta)$

where $\kappa_i(x)$ is the upper bound of holomorphic sectional curvature of $h_i$ at $x$ and $T_i$ is the torsion of $h_i$.

Proof. In the following, $c_i$ will denote a positive constant depending only on $n, \alpha$. Note that we still have $\alpha^{-1} g_{0,i} \leq h_i \leq \alpha g_{0,i}$. For notational convenience, we will denote $g = g_{0,i}$, $\bar{g} = g_0$, $\hat{h} = h_i$ in the proof.

(i)

\[ (T_{0,i})_{pki} = \partial_p (e^{2F} g_{kq}) - \partial_k (e^{2F} g_{pqi}) = 2e^{2F} (F_p g_{kq} - F_k g_{pq}) + e^{2F} (T_g)_{pki} \]

\[ = 2(F_p \bar{g}_{kq} - F_k \bar{g}_{pq}) + e^{2F} (T_g)_{pki} \]

\[ = 2\rho_0^{-1} \tilde{\mathcal{S}}' (\rho_p \bar{g}_{kq} - \rho_k \bar{g}_{pq}) + e^{2F} (T_g)_{pki}. \]

Hence

\[ |T_{0,i}|_{h_i}^2 \leq c_1 \beta. \]

This proves (i). The proof of (ii) is similar.

(iii)

\[ \hat{\nabla}_l^{(i)} (T_{0,i})_{pki} = \hat{\nabla}_l^{(i)} (2(F_p g_{kq} - F_k g_{pq}) + e^{2F} (T_g)_{pki}) \]

\[ = 2(F_p g_{kq} - F_k g_{pq}) + 2 \left( F_p \hat{\nabla}_l^{(i)} g_{kq} - F_k \hat{\nabla}_l^{(i)} g_{pq} \right) + 2e^{2F} F_i (T_g)_{pki} + e^{2F} \hat{\nabla}_l^{(i)} (T_g)_{pki} \]

\[ = 2\rho_0^{-1} \tilde{\mathcal{S}}'(\rho_p g_{kq} - \rho_k g_{pq}) + 2\rho_0^{-1} \tilde{\mathcal{S}}''(\rho_p \rho_l g_{kq} - \rho_k \rho_l g_{pq}) \]

\[ + 2\tilde{\mathcal{S}}' \rho_0^{-1} \left( \rho_p \hat{\nabla}_l^{(i)} g_{kq} - \rho_k \hat{\nabla}_l^{(i)} g_{pq} \right) \]

\[ + 2e^{2F} \rho_0^{-1} \tilde{\mathcal{S}}' \rho_l (T_g)_{pki} + e^{2F} \hat{\nabla}_l (T_g)_{pki}. \]
Using the fact that
\[ (\hat{\Gamma}^{(i)} - \hat{\Gamma})^t_{pq} = 2F_p\delta^t_q = 2\rho_0^{-1}\mathcal{G}^t_{pq}\]
and hence
\[ \hat{\nabla}^{(i)}_t g_{k\bar{q}} = -2F_tg_{k\bar{q}} + (\hat{\nabla}^{(i)}_t - \hat{\nabla}_t)g_{k\bar{q}} + e^{2F}\hat{\nabla}_tg_{k\bar{q}}\]
(4.6)
\[ = -2\rho_0^{-1}\mathcal{G}^t_{k\bar{q}} + 2\rho_0^{-1}\mathcal{G}^t_{t}g_{k\bar{q}} + e^{2F}\hat{\nabla}_tg_{k\bar{q}}\]
We may further infer that (iii) is true.

Now we examine the holomorphic sectional curvature after conformal change. Let \( e_1 \in T^{1,0}U_R \) be such that \( |e_1|_{h_i} = 1, \ |e_1|_h = e^{-F} \). Let \( \kappa(x) \) be the upper bound of the holomorphic sectional curvature of \( h \) at \( x \)
\[ \hat{R}_{1111} = -\partial_1\partial_1(e^{2F}h_{11}) + e^{-2F}h^{pl}\partial_1(e^{2F}h_{11})\partial_1(e^{2F}h_{p1})\]
\[ = -\partial_1(e^{2F}\partial_1h_{11} + 2e^{2F}h_{11}F_1)\]
\[ + e^{-2F}h^{pl}(e^{2F}\partial_1h_{11} + 2\hat{h}_{11}F_1)\]
(4.7)
\[ = e^{2F}\hat{R}_{1111} - 2\hat{h}_{11}F_1\]
\[ \leq e^{-2F}\kappa - 2F_1\]
\[ \leq e^{-2F}\kappa + c_3(\beta + \beta^2).\]
Estimate \( |\hat{\nabla}^{(i)}_t T_{1i}|_{h_i} \) in a similar way as above, we may conclude that
\[ \frac{n + 1}{2n}\kappa_i(x) + |\hat{\nabla}^{(i)}_t T_{1i}|_{h_i} \leq e^{-2F}\kappa_0 + c_3\beta(1 + \beta).\]
From this (iv) is true.

Now we are able to construct a solution of the Chern Ricci flow on \( M \).

**Proof of Theorem 4.4.** For each sufficiently large \( \rho_i \), \( (U_{\rho_i}, g_{0,i}) \) has bounded geometry by Lemma 4.3. By Lemma 4.3 using the notation in the lemma, we have for any \( \epsilon > 0 \):
\[ \frac{n + 1}{2n}\kappa_i(x) + |\hat{\nabla}^{(i)}_t T_{1i}|_{h_i} \leq \kappa_0 + c(\beta + (1 + \epsilon^{-1})\beta^2) + \epsilon\gamma =: \kappa_{0,i}.\]
\[ s_i = \sup_M \left( |T_{0,i}|_{h_i}^2 + |T_{0,i}|_{h_i}|\hat{T}_{1i}|_{h_i} + |\hat{\nabla}^{(i)}_t T_{0,i}|_{h_i} + \kappa_{0,i} \right)\]
Then
\[ s_i \leq \kappa_0 + c_1\beta(1 + \beta) =: s.\]
By Lemma 4.4 there is a solution \( g_i(t) \) on \( U_{\rho_i} \times [0, S) \) with initial metric \( g_{0,i} \) where
\[ S = \frac{1}{3c_1(n\alpha + 1)^3s}\]
for some constant \( c_1 = c_1(n) \). Moreover, \( g_i \) is uniformly equivalent to \( g_{0,i} \) and
(4.8)
\[ \text{tr}_{g_i} h_i \leq v(t) - 1\]
on $U_{\rho_i} \times [0, S)$ where
\[ v(t) = \left( \frac{1}{(n\alpha + 1)^{-3} - 3c_1 \delta t} \right)^{\frac{1}{3}}. \]

Fix any compact subset $K \subset M$. Then for sufficiently large $i$, $g_i(t)$ is a solution of the Chern Ricci flow defined on $U_{\rho_i} \supset U_{2r} \supset U_r \supset K$ for some large $r > 0$. By Lemma 3.5, for any $(x, t) \in K \times [0, S]$,
\[ R_{g_i(t)} \geq -\max \left\{ C_n \beta^{-1}[(r_i - r)^{-2} + 1], \sup_{\rho(y) < 2r} R_-(y, 0) \right\} \]
where we have used the fact that $g_i(0) = g_0$ on $U_r$ for sufficiently large $\rho_i$. In particular, it is bounded from below uniformly. Since
\[ \frac{\partial}{\partial t} \left( \log \frac{\det g_i(t)}{\det h} \right) = -R_{g_i(t)} \leq C(n, K, \alpha). \]
Therefore, on $K \times [0, S]$,
\[ \alpha h \leq g(t) \leq e^{Ct \beta^{-n}} h. \]
By the local estimate of the Chern Ricci flow [22], for any $k \in \mathbb{N}$, there is $C(n, k, g_0, h, \beta)$ such that for any $(x, t) \in K \times [0, S]$,
\[ |\nabla^k g_i(t)|_h \leq C(n, k, g_0, h, \beta). \]
By taking diagonal subsequence and using Arzelà-Ascoli theorem, we may obtain a limiting solution of $g(t)$ defined on $M \times [0, S)$. The conclusion on $tr_g h$ follows from [14.8]. This completes the proof of the theorem. \[\square\]

Next we want to apply Theorem 4.1 to obtain longtime solution for Kähler-Ricci flow.

**Theorem 4.2.** Let $(M, g_0)$ be a complete Kähler manifold and $h$ is a fixed complete Hermitian metric on $M$ such that the following holds.

(i) There exists smooth exhaustion $\rho \geq 1$ such that
\[ \limsup_{\rho \to \infty} \left( \frac{|\partial \rho|_h}{\rho} + \frac{|\sqrt{-1} \partial \bar{\partial} \rho|_h}{\rho} \right) = 0; \]

(ii) and the holomorphic sectional curvature of $h$ and torsion of $h$ satisfy
\[ H_h + \frac{2n}{n + 1} |\hat{\nabla}_g \hat{T}|_h \leq 0. \]

(iii) $\exists \alpha > 1$ such that on $M$, $\alpha^{-1} g_0 \leq h \leq \alpha g_0$, $|\hat{T}|_h \leq \alpha$;

(iv) $\limsup_{\rho \to \infty} \rho^{-1} |\hat{\nabla} g_0|_h = 0$;

Then there is $\beta(n, \alpha) > 0$ such that the Kähler Ricci flow has a complete solution $g(t)$ on $M \times [0, +\infty)$ with $g(0) = g_0$ and satisfies
\[ \beta h \leq g(t) \]
on $M \times [0, +\infty)$. 
Proof. By Theorem 4.1 and the assumptions, one can apply the theorem with \( \beta \) arbitrarily small. Hence one can find solution \( g_i(t) \) to the Chern-Ricci flow with \( g_i(0) = g_0 \) on \( M \times [0, T_i] \) with \( T_i \to \infty \). Moreover, \( \operatorname{tr}_g h \leq c(n, \alpha) \). Using the local estimate of scalar curvature in Lemma 3.5 as in the proof of Theorem 4.1 the results follow. \( \square \)

5. Existence of the Kähler Einstein metric

In this section, we discuss the existence the Kähler Einstein metric on \( M \) via the Kähler-Ricci flow. We have the following:

**Theorem 5.1.** Suppose there is a complete Hermitian metric \( h \) on \( M \) compatible with the complex structure \( J \) such that

\[
H_h + \frac{2n}{n+1} |\nabla \bar{T}|_h \leq -k
\]

for some \( k > 0 \). Then any longtime solution of normalized Kähler-Ricci flow \( g(t) \) will converge uniformly in \( C^\infty \) to a Kähler Einstein metric \( g_\infty = -\operatorname{Ric}(g_\infty) \). In particular, there is no Ricci flat Kähler metric on \( M \) compatible with the same complex structure \( J \).

Combining with Theorem 4.2, we have the following results which are some generalization of the result by Wu-Yau [30]:

**Corollary 5.1.** Let \((M^n, g_0)\) be a complete noncompact Kähler manifold with holomorphic sectional curvature bounded above by a negative constant. Suppose \( M \) supports an exhaustion function with uniformly bounded gradient and uniformly bounded Hessian, which is the case if \( M \) has bounded curvature. Then \( M^n \) support a unique Kähler Einstein metric with negative scalar curvature.

**Corollary 5.2.** Let \((M^n, g_0)\) be a complete noncompact Kähler manifold with complex dimension \( n \). Suppose there is \( K_1, r, A_0, r > 0, p > n \) such that for all \( x \in M, f \in C_0^\infty(B_{g_0}(x, 4r)) \),

\[
\begin{align*}
H_{g_0} &\leq \kappa_0 < 0; \\
\int_{B_{g_0}(x, r)} |\operatorname{Rm}(g_0)|_p^p d\mu_{g_0} &\leq K_1; \\
\int_{B_{g_0}(x, 4r)} |f|^{\frac{2n}{n-1}} d\mu_{g_0} &\leq A_0 r^2 \int_{B_{g_0}(x, 4r)} |\nabla f|^2 d\mu_{g_0}.
\end{align*}
\]

Then \( M \) supports a unique Kähler Einstein metric with negative scalar curvature.

Proof. By [33], there is a complete short time solution \( g(t) \) to the Ricci flow with \( g(t) = g_0 \) such that \( |\operatorname{Rm}(g(t))| \leq Ct^{-a} \) for some \( 0 < a < 1 \). By [11], \( g(t) \) is Kähler for \( t > 0 \). On the other hand for fixed \( g(t) \), there is an exhaustion function \( \rho \) with uniformly bounded gradient and uniformly bounded Hessian. Since \( g(t) \) is uniformly equivalent to \( g_0 \), \( \rho \) is also an exhaustion function \( \rho \) with uniformly bounded gradient and uniformly bounded Hessian with respect to \( g_0 \). By Corollary 5.1 the result follows. \( \square \)
We also want to discuss metrics which are uniformly equivalent to $g_0$ as in the previous corollaries. The following is an immediate consequence of Theorem 4.2 and Theorem 5.1.

**Corollary 5.3.** Let $(M^n, h)$ be a complete noncompact Hermitian manifold with bounded torsion and satisfies the conditions in Corollary 5.1: The holomorphic sectional curvature of $h$ is bounded above by a negative constant, and $(M, h)$ support an exhaustion function with uniformly bounded gradient and uniformly bounded Hessian. Let $g_0$ be a Kähler metric which is uniformly equivalent to $h$ so that $|\hat{\nabla}g_0|_h$ is bounded, where $\hat{\nabla}$ is the derivative with respect to the Chern connection of $h$. Then $M$ supports a unique Kähler-Einstein metric with negative scalar curvature.

Let us prove Theorem 5.1. Here we do not have a good exhaustion function for $h$. However, the distance function $d(x, t)$ in a Ricci flow behaves well. Using the idea in the work of Chen [3], we have the following:

**Lemma 5.1.** Let $(M^m, g(t))$ be a complete noncompact solution to the Ricci flow on $M \times [0, T]$ with $0 < T < \infty$, where $m \geq 2$ is the real dimension of $M$. Let $Q$ be a smooth function so that

$$\left( \frac{\partial}{\partial t} - \Delta \right) Q \leq -\alpha Q^2 + \beta$$

for some $\alpha, \beta > 0$ at the point where $Q > 0$. Then

$$tQ(x, t) \leq \frac{1 + \sqrt{1 + 4\alpha \beta T^2}}{2\alpha}$$

on $M \times (0, T]$.

**Proof.** Let $x_0 \in M$, and let $r_0 > 0$ be small enough so that:

$$\text{Ric}(x, t) \leq (m - 1)r_0^{-2}$$

for $x \in B_t(x_0, r_0)$, $t \in [0, T]$. By [19] (see also [3]), we then have

$$\left( \frac{\partial}{\partial t} - \Delta \right) d_t(x, x_0) \geq -\frac{5(m - 1)}{3}r_0^{-1}$$

whenever $d_t(x, x_0) \geq r_0$ in the sense of barrier, where $d_t(x, x_0)$ is the distance function from $x_0$ with respect to $g(t)$. In the following, argue as in [11], we may assume that $d_t(x, x_0)$ to be smooth when applying maximum principle. We consider the function

$$u(x, t) = t\varphi \left( \frac{1}{Ar_0^2} \left[ d_t(x, x_0) + \frac{5(m - 1)t}{3r_0} \right] \right) Q(x, t)$$

$A$ is sufficiently large so that $Ar_0 >> \frac{5(m - 1)t}{3r_0}$, and $\varphi$ is a fixed smooth non-negative non-increasing function such that $\varphi \equiv 1$ on $(-\infty, \frac{1}{2}]$, vanishes outside
and satisfies $|2(\frac{\varphi'}{\varphi})^2| + |\varphi''| \leq c_1$ for some absolute constant. Note that $u$ also depends on $A$. However,

$$u(x_0, t) = tQ(x_0, t).$$

if $Ar_0 \geq \frac{10(m-1)T}{3r_0}$.

If $u \leq 0$, then we are done. Suppose the function $u > 0$ somewhere, then there exists $(x_1, t_1)$ with $0 < t_1 \leq T$ so that $u$ attains its maximum at $(x_1, t_1)$. We have at $(x_1, t_1)$ we have

$$0 \leq \left( \frac{\partial}{\partial t} - \Delta \right) u; \quad \nabla Q = -\frac{\nabla \varphi}{\varphi} Q.$$

Suppose $d_{t_1}(x_1, x_0) < r_0$, then $u(x, t) = tQ(x, t)$ near $x_1, t_1$ provided $Ar_0$ is large enough. Then we have at $(x_1, t_1)$

$$0 \leq \left( \frac{\partial}{\partial t} - \Delta \right) u = t_1 \left( \frac{\partial}{\partial t} - \Delta \right) Q + Q \leq -\alpha t_1 Q^2 + \beta t_1 + Q$$

and so

$$0 \leq -\alpha u^2 + u + \beta T^2$$

which implies

$$(5.4) \quad u(x_0, t) \leq u(x_1, t_1) \leq \frac{1 + \sqrt{1 + 4\alpha \beta T^2}}{2\alpha}.$$ 

for $t \in [0, T]$.

Suppose $d_{t_1}(x_1, x_0) \geq r_0$, then at $(x_1, t_1)$,

$$0 \leq \left( \frac{\partial}{\partial t} - \Delta \right) u = Qt \left( \frac{\partial}{\partial t} - \Delta \right) \varphi + \varphi \left( \frac{\partial}{\partial t} - \Delta \right) (Qt) - 2t \langle \nabla \varphi, \nabla Q \rangle$$

$$\leq Qt \varphi' \frac{1}{Ar_0} \left[ \left( \frac{\partial}{\partial t} - \Delta \right) d_t(x, p) + \frac{5}{3}(m-1)r_0^{-1} \right]$$

$$+ |\varphi''| \frac{1}{(Ar_0)^2} tQ + \varphi (-\alpha tQ^2 + \beta t + Q) + 2tQ \frac{1}{(Ar_0)^2} \cdot \left( \frac{\varphi'}{\varphi} \right)^2$$

$$\leq -\alpha t \varphi Q^2 + \varphi Q + \beta t \varphi + c_1 Qt \frac{1}{(Ar_0)^2}.$$

Multiply both the inequality by $t\phi = t_1 \phi$, we have

$$0 \leq -\alpha u^2 + \left( 1 + \frac{c_1 T}{Ar_0^2} \right) u + \beta T^2.$$
Hence we have

\begin{equation}
\frac{1}{\alpha} \left( 1 + \frac{2 T}{Ar_0^2} + \sqrt{1 + \frac{2 T}{Ar_0^2}} \right)^2 + 4 \alpha \beta T^2
\end{equation}

for \(0 < t \leq T\). Let \(A \to \infty\) together with (5.5) and the fact that \(x_0\) is any point in \(M\), we conclude the lemma is true. \(\square\)

As an application of the lemma, we prove that complete noncompact Kähler Einstein metrics with negative scalar curvature is unique up to scaling. Here we do not assume the curvature is bounded, see also [30]. Namely, we have the following: Moreover, the Kähler Einstein metric with negative scalar curvature must be unique up to scaling.

**Theorem 5.2.** Suppose \(\omega_1\) and \(\omega_2\) are complete noncompact Kähler Einstein metrics on \(M\) with \(\text{Ric}(\omega_i) = -\omega_i\) for \(i = 1, 2\). Then \(\omega_1 = \omega_2\) on \(M\).

**Proof.** Let \(\tilde{\omega}_1(t) = (t + 1)\omega_1\) and \(\tilde{\omega}_2(t) = (t + 1)\omega_2\). Then both \(\tilde{\omega}_1, \tilde{\omega}_2\) are solutions to the Kähler-Ricci flow on \(M \times [0, +\infty)\). Define \(F(x, t) = F(x)\) to be the function

\[F(x, t) = \log \left( \frac{\tilde{\omega}_2^n}{\tilde{\omega}_1^n} \right)^{\frac{1}{n}} = \log \left[ \frac{\omega_2^n}{\omega_1^n} \right]^{\frac{1}{n}}\]

which is independent of \(t\). The function \(F\) is independent of \(t > 0\) but we treat it as a function over the Kähler-Ricci flow. Let \(\Delta\) be the Laplacian of \(\tilde{\omega}_1\). Then on \(M \times (0, \infty)\), On \([0, 1]\), it satisfies

\[\left( \frac{\partial}{\partial t} - \Delta \right) F = \frac{1}{t + 1} \left( 1 - \frac{1}{n} \text{tr}_{\omega_1} \omega_2 \right) \leq \frac{1}{t + 1} \left( 1 - e^F \right) \leq -\frac{1}{4} F^2\]

whenever \(F > 0\) on \(M \times [0, 1]\).

Apply Lemma [5.1] on \(M \times [0, 1]\), \(tF \leq 8\). In particular, \(F(x)\) is bounded from above uniformly on \(M\). By interchanging \(\omega_1\) and \(\omega_2\), we conclude that \(F\) is a bounded function on \(M\). Let \(\Delta_1\) be the Laplacian of \(\omega_1\), we have as above

\[\Delta_1 F \leq 1 - e^F.\]

By the generalized maximum principle [11], we conclude that \(F \leq 0\). Interchanging the roles of \(\omega_1\) and \(\omega_2\), we can prove similarly that \(F \geq 0\). Hence \(F = 0\). So \(\partial \bar{\partial} F = 0\) and \(\omega_1 = \omega_2\) because they are Kähler-Einstein. \(\square\)

Now we are ready to prove Theorem [5.1]
Proof of Theorem 5.1. Let $g(t)$ and $h$ as in the Theorem. Let $\Lambda = \text{tr}_g h$. By Corollary 3.1, we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \Lambda \leq -c_1 k \Lambda^2,
\]
for some $c_1$ depending only on $n$. By Lemma 5.1 with $\beta = 0$, we conclude that (5.6)
\[
\Lambda(x,t) \leq \frac{1}{2c_1 k t}
\]
on $M \times (0, \infty)$.

On the other hand, let $R(x,t)$ be the scalar curvature of $g(t)$ at $x$ and let $R_-(x,t)$ be its negative part. For any $\epsilon > 0$, let $f = \frac{1}{2} \left((R^2 + \epsilon^2)^{\frac{1}{2}} - R \right)$. Note that if $\epsilon \to 0$, then $f \to R_-$. Using the fact that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) R \geq \frac{1}{n} R^2,
\]
direct computations show that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) f \leq -\frac{1}{n} f (f - 2c_1 \epsilon) \leq \frac{1}{n} (f - c_1 \epsilon)^2 + c_2 \epsilon
\]
for some absolute constant $c_1 > 0$ and $c_2 > 0$ depending only on $n$. By Lemma 5.1 we conclude that
\[
t(f - c_1 \epsilon) \leq \frac{n}{2} \left( 1 + \sqrt{1 + \frac{4c_2 \epsilon}{n}} \right).
\]
on $M \times (0, \infty)$. Let $\epsilon \to 0$, we conclude that (5.7) $t R(x,t) \geq -n$.

Since
\[
\frac{\partial}{\partial t} \log \left( \frac{\det g(t)}{\det h} \right) = -R \leq \frac{n}{t}
\]
we conclude for any bounded open set $\Omega$, there is a constant $C_1$ depending only on $\Omega, g(1), h, k, n$ such that
\[
\frac{\det g(t)}{\det h} \leq C_1 t^n
\]
on $\Omega \times [1, \infty)$. Combining this with (5.6), we conclude that
\[
C_2^{-1} th \leq g \leq C_2 th
\]
on $\Omega \times [1, \infty)$ for some constant $C_2 > 0$ depending only on $\Omega, g(1), h, k, n$.

Consider the normalized metric
\[
\tilde{g}(x,s) = e^{-s} g(x, e^s).
\]
Then we have (5.8)
\[
\frac{\partial}{\partial s} \tilde{g} = -\text{Ric}(\tilde{g}) - \tilde{g}
\]
on $M \times [0, \infty)$, and
\begin{equation}
C_2^{-1} h \leq \tilde{g}(s) \leq C_2 h
\end{equation}
on $\Omega \times [0, \infty)$.

In the following, let $\omega(t)$, $\tilde{\omega}(s)$ be the Kähler forms of $g(t)$, $\tilde{g}(s)$ respectively. By [25, Theorem 2.17], we conclude that for any bounded open set in $M$ and $\ell \geq 0$, there is a constant $C_3$ depending only on $\Omega, g(1), h, k, n.$ and $\ell$ such that
\begin{equation}
||\tilde{\omega}(s)||_{C^\ell(\Omega, \tilde{g}_0)} \leq C_3.
\end{equation}

On the other hand, let $\tilde{\phi}(x, s) = e^{-\int_0^s \log \left(\frac{\tilde{\omega}(\tau)}{\tilde{\omega}(0)}\right) d\tau}.$ Then
\begin{equation}
\tilde{\omega}(s) = e^{-s} \tilde{\omega}(0) - (1 - e^{-s}) \text{Ric}(\tilde{\omega}(0)) + \sqrt{-1} \partial \bar{\partial} \tilde{\phi}(s).
\end{equation}

Moreover,
\begin{equation}
\begin{cases}
\frac{\partial}{\partial s} \tilde{\phi} = \log \left(\frac{\tilde{\omega}^n}{(\tilde{\omega}(0))^{n}}\right) - \tilde{\phi} & \text{in } M \times [0, \infty); \\
\tilde{\phi}(0) = 0 & \text{on } M.
\end{cases}
\end{equation}

Denote $\partial_s \tilde{\phi}$ by $\tilde{\phi}'$ etc., and let $\tilde{R}$ be the scalar curvature of $\tilde{g}$, then
\begin{equation}
\tilde{\phi}'' + \tilde{\phi}' = -\tilde{R} - n
\end{equation}
\begin{align*}
&= -e^s R(g(e^s)) - n \\
&= -e^s (R(g(e^s)) + e^{-s} n) \\
&\leq 0
\end{align*}

by (5.7). Hence $\tilde{\phi}' + \tilde{\phi}$ is non-increasing and $\tilde{\phi}' + \tilde{\phi} \leq 0$ because $\tilde{\phi}' + \tilde{\phi} = 0$ at $s = 0$. On the other hand, by (5.10) and (5.12), we conclude that for any bounded open set $\Omega$, there exists $s_i \to \infty$ such that
\begin{equation}
(\tilde{\phi}' + \tilde{\phi})(s_i)
\end{equation}
converge uniformly in $C^\infty$ norm in $\Omega$. By the monotonicity of $\tilde{\phi}' + \tilde{\phi}$, we conclude that $\tilde{\phi}' + \tilde{\phi}$ converges in $C^\infty$ norm in $\Omega$ to some function.

By (5.13), we have
\begin{equation}
(e^s \tilde{\phi})' \leq 0,
\end{equation}
and so $\tilde{\phi}' \leq 0$ because $\tilde{\phi}' = 0$ at $s = 0$. Combine this with (5.10) and (5.11), we conclude that $\tilde{\phi}$ also converge in $C^\infty$ norm to some function $\phi_\infty$. Hence $\tilde{\phi}'$ also converge in $C^\infty$ norm to some function. However, by (5.9) we conclude that $\phi$ is bounded from below. This implies that $\tilde{\phi}' \to 0$ as $s \to \infty$. Moreover, $\tilde{\omega}(s) \to \tilde{\omega}_\infty$ in $C^\infty$ norm in $\Omega$ as $s \to \infty$ with
\begin{equation}
\tilde{\omega}_\infty = -\text{Ric}(\tilde{\omega}(0)) + \sqrt{-1} \partial \bar{\partial} \tilde{\phi}_\infty.
\end{equation}
Note that $\tilde{\omega}_\infty$ is a Kähler form of a Kähler metric by (5.9). Moreover,

$$\tilde{\phi}_\infty = \log \left( \frac{\tilde{\omega}^n}{(\tilde{\omega}(0))^n} \right).$$

Taking $\partial \bar{\partial}$ to both sides, we conclude that

$$\text{Ric}(\tilde{\omega}_\infty) = -\tilde{\omega}_\infty.$$

Suppose $\bar{\omega}$ is a Ricci flat metric compatible with the same complex structure of $h$. Then $\omega(t) = \bar{\omega}$ is a steady solution of the Kähler-Ricci flow. By the convergence of normalized Kähler-Ricci flow, $t^{-1}\omega(t)$ converges to a Kähler Einstein metric on $M$ which is impossible since $t^{-1}\omega(t) \equiv t^{-1}\bar{\omega}$ converges to a zero tensor on $M$. This completes the proof. \hfill \Box

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