SELF-SIMILAR BLOW UP FOR ENERGY SUPERCRITICAL
SEMILINEAR WAVE EQUATION

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ABSTRACT. We analyse the energy supercritical semilinear wave equation
\[ \Phi_{tt} - \Delta \Phi - |\Phi|^{p-1}\Phi = 0 \]
in \( \mathbb{R}^d \) space. We first prove in a suitable regime of parameters the existence of a
countable family of self similar profiles which bifurcate from the soliton solution.
We then prove the non radial finite codimensional stability of these profiles by
adapting the functional setting of [15].

Keywords: Semi-linear wave equation, Self-similar solution, Blow up, Focusing,
Energy super-critical, Finite codimensional stability

1. INTRODUCTION

1.1. Setting of the problem. In this paper, we consider the energy supercritical
semilinear wave equation with focusing power nonlinearity
\[ \Phi_{tt} - \Delta \Phi - |\Phi|^{p-1}\Phi = 0, \quad \Phi = \Phi(t,x), \quad x \in \mathbb{R}^d, \]
that is \( d \geq 3 \) and the critical scaling is above 1 i.e.
\[ s_c := \frac{d}{2} - \frac{2}{p-1} > 1. \]
We will focus on the cases where \( 1 < s_c < \frac{d}{2} \). Note that this is equivalent to
\[ d = 3 \text{ or } d \geq 4 \text{ and } 1 + \frac{4}{d-2} < p < 1 + \frac{4}{d-3}. \] (1.2)
We aim at describing so called "type I" self similar solutions as generated by exact
profiles
\[ \Phi(t,r) = (T-t)^{-\alpha} u(\rho), \quad \rho := |y|, \quad y := \frac{x}{T-t} \] (1.3)
solutions to (1.1), that is equivalently in the radially symmetric setting
\[ (1 - \rho^2)u'' + \left[ \frac{d-1}{\rho} - 2(1 + \alpha)\rho \right] u' - \alpha(1 + \alpha)u + |u|^{p-1}u = 0 \] (1.4)
with suitable "self similar" boundary conditions at \( +\infty \). A family of smooth self similar
solutions to (1.4) for the range (1.2) is constructed in [7] using ode oscillation techniques. These solutions cannot be in the natural energy space, and a full dynamical
stability analysis is required to show that these profiles indeed are the asymptotic
profile of blow up solutions emerging from smooth well localized initial data. This
dynamical stability analysis in the super critical regime is initiated in the pioneering
works by Donninger and Shorkuber for wave maps [8], see also [5] and references
therein, who studied the finite codimensional stability in the renormalized light cone.
1.2. **Statements of the result.** The aim of this paper is twofold. First we aim at proposing an alternative to the method developed in [7] for the construction of self-similar profiles, and we construct self-similar profiles by directly "gluing" soliton solutions to asymptotic self-similar tails as in [6]. The advantage of this method is its robustness as it can be applied to many more complicated problems, see e.g. [2], and also allows for a full description of the profile in space, but it describes only one end of the branch of solutions and cannot address the description of the full family as in [7]. The statement is the following.

**Theorem 1** (Existence and asymptotes of excited self-similar solutions). There exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), there exists a smooth radially symmetric self-similar solution to equation (1.1) such that \( \Lambda u_n \) vanishes exactly \( n \) times on \((0, \infty)\). Moreover, for \( \rho_0 \) small enough,

(i) Behaviour at infinity:
\[
\lim_{n \to \infty} \sup_{\rho \geq \rho_0} (1 + \rho^\alpha)|u_n(\rho) - u_\infty(\rho)| = 0
\]

(ii) Behaviour at the origin: There exists \( \mu_n \to 0 \) such that
\[
\lim_{n \to \infty} \sup_{\rho \leq \rho_0} \left| u_n(\rho) - \mu_n^{-\alpha}Q\left(\frac{\rho}{\mu_n}\right)\right| = 0
\]
where \( Q \) is the unique non trivial radially symmetric solution to
\[
\Delta Q + Q^p = 0, \quad Q(\rho) = b_\infty \rho^{-\alpha} + O(\rho^{1-\frac{d}{2}}). \tag{1.5}
\]

Our second main task is to propose a robust functional framework, adapting [15], to prove the global in space finite codimensional stability of these profiles against possibly non-radial perturbations.

**Theorem 2** (Finite codimensional stability). Let \( u_n \) be the self-similar profiles constructed in Theorem 1 with corresponding initial data \((\Phi(0), \partial_t \Phi(0)) = P_n\) for
\[
P_n := \left(\frac{1}{T^\alpha} u_n\left(\frac{r}{T}\right), \frac{1}{T^{\alpha+1}} \Lambda u_n\left(\frac{r}{T}\right)\right).
\]

(i) Radial perturbation. Let
\[
d = 3, \quad 5 < p \quad \text{or} \quad d = 4, \quad 4 < p < 5. \tag{1.6}
\]
For \( T \ll 1 \), there exists a finite codimensional manifold of initial data \((\Phi(0), \partial_t \Phi(0)) \in \cap_{m \geq 0} H^m_{rad}(\mathbb{R}^d, \mathbb{R}^2)\) such that in the neighbourhood of \( P_n \) with respect to a topology specified in Proposition 7.1, the corresponding solution \( \Phi \) gives rise to a Type I blow up at time \( T \) at the center of symmetry i.e. as \( t \to T \),
\[
\|\Phi(t)\|_{L^\infty} \sim (T - t)^{-\alpha}.
\]

(ii) Non-radial perturbation. Let
\[
d = 3, \quad 6 < p. \tag{1.7}
\]
For \( T \ll 1 \), there exists a finite codimensional manifold of initial data \((\Phi(0), \partial_t \Phi(0)) \in \cap_{m \geq 0} H^m(\mathbb{R}^d, \mathbb{R}^2)\) such that in the neighbourhood of \( P_n \) with respect to a topology specified in Proposition 7.1, the corresponding solution \( \Phi \) gives rise to a Type I blow up at time \( T \) at the center of symmetry.
As in [15], a key step in the analysis is to realize the linearized operator close to a self similar profile as a compact perturbation of a maximal accretive operator in a global in space homogeneous Sobolev space with super critical regularity. Using sufficient regularity and propagating additional weighted energy estimates than allows to close non linear terms. Hence the counting of the exact number of instability is reduced to an explicit spectral problem.

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2. Construction of exterior solutions

Our aim in this section is to construct a family of outer solutions to the self similar equation (1.4). The key is that the outer spectral problem, including the singularity through the renormalized light cone $\rho = 1$, is explicit.

We introduce relevant notations for this section. We define the generator of scaling operator $\Lambda$:

$$\Lambda = \alpha + y \cdot \nabla.$$ 

Introduce the linearized operator $\mathcal{L}_\infty$ for (1.4) near $u = u_\infty$:

$$\mathcal{L}_\infty = (1 - \rho^2) \frac{d^2}{d\rho^2} + \left[ \frac{d - 1}{\rho} - 2(1 + \alpha)\rho \right] \frac{d}{d\rho} - \alpha(1 + \alpha) + p\alpha(1 - \alpha)\rho^{-2}. \quad (2.1)$$

Also, let

$$\omega = \sqrt{pb_\infty^{p-1} - \frac{(d - 2)^2}{4}}.$$ 

Note that $\omega \in \mathbb{R}$ if and only if

$$1 + \frac{4}{p - 2} < p < p_{\text{JL}} := \begin{cases} \infty & \text{if } d \leq 10, \\ 1 + \frac{4}{d - 4 - 2\sqrt{d - 1}} & \text{if } d \geq 11 \end{cases}$$

with sufficient condition being $1 < s_c < \frac{3}{2}$. $p_{\text{JL}}$ is known as the Joseph-Lundgren exponent. We denote by $\,_{2}F_{1}$ the Gauss hypergeometric functions:

$$\,_{2}F_{1}(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} \quad (2.2)$$

where $(a)_n = a(a+1) \cdots (a+n-1)$.

2.1. Fundamental solutions and exterior resolvent.

**Lemma 2.1** (Fundamental solutions).

$$\psi_1^L = \Re \left[ \rho^{1-\frac{d}{2}+i\omega} \,_{2}F_{1}\left( \frac{1-s_c+i\omega}{2}, \frac{2-s_c+i\omega}{2}, 1+i\omega, \rho^2 \right) \right] \quad (2.3)$$

$$\psi_2^L = \Im \left[ \rho^{1-\frac{d}{2}+i\omega} \,_{2}F_{1}\left( \frac{1-s_c+i\omega}{2}, \frac{2-s_c+i\omega}{2}, 1+i\omega, \rho^2 \right) \right]$$
forms a basis of solutions to $\mathcal{L}_\infty(\psi) = 0$ in $\rho \in (0, 1)$.

\[
\psi_1^R = \rho^{-\alpha-1} \binom{2 - s_c - i\omega}{2 \rho^{-2}} \binom{2 - s_c + i\omega}{2 \rho^{-2}} F_1(\gamma, 2 - \gamma, \frac{1 - s_c - i\omega}{2 \rho^{-2}}, \frac{1 - s_c + i\omega}{2 \rho^{-2}})
\]

\[
\psi_2^R = \rho^{-\alpha} \binom{2 - s_c - i\omega}{2 \rho^{-2}} \binom{2 - s_c + i\omega}{2 \rho^{-2}} F_1(\gamma, 2 - \gamma, \frac{1 - s_c - i\omega}{2 \rho^{-2}}, \frac{1 - s_c + i\omega}{2 \rho^{-2}})
\]

forms a basis of solutions to $\mathcal{L}_\infty(\psi) = 0$ in $\rho \in (1, \infty)$.

**Proof.** For $\rho \in (0, 1)$, consider solutions of the form $\psi = \rho^\gamma \sum_{n=0}^\infty a_n \rho^n$ for $(a_n)_{n=0}^\infty$ a bounded sequence in $\mathbb{R}$ with $a_0 \neq 0$ so the sum is absolutely convergent in $(0, 1)$. Then

\[
\mathcal{L}_\infty(\psi) = [\gamma(\gamma + d - 2) + \rho \alpha(1 - \alpha)]a_0 \rho^{\gamma-2} + [(\gamma + 1)(\gamma + d - 1) + \rho \alpha(1 - \alpha)]a_1 \rho^{\gamma-1}
\]

\[
+ \sum_{n=0}^\infty \{[(\gamma + n + 2)(\gamma + n + d) + \rho \alpha(1 - \alpha)]a_{n+2} - [(\gamma + n)(\gamma + n + 2\alpha) + \alpha(1 + \alpha)]a_n\} \rho^{\gamma+n}
\]

Equating first two terms to 0, we infer $\gamma = 1 - \frac{d}{2} \pm i\omega$ and $a_1 = 0$. Equating higher order terms to 0,

\[
a_{n+2} = \frac{(\gamma + n + \alpha)(\gamma + n + 1 + \alpha)}{(\gamma + n + \frac{d}{2} + 1 + i\omega)(\gamma + n + \frac{d}{2} + 1 - i\omega)} a_n.
\]

The cases $\gamma = -\frac{1}{2} + i\omega$ and $-\frac{1}{2} - i\omega$ give rise to complex conjugate solutions. Thus, real and imaginary parts of the complex solution

\[
\rho^{1-\frac{d}{2} + i\omega} F_1\left(1 - \frac{s_c + i\omega}{2}, \frac{2 - s_c + i\omega}{2}, \frac{1 + i\omega}{2}, \rho^{-2}\right)
\]

yields two linearly independent real solutions. In the region $(1, \infty)$, consider solutions of the form $\psi = \rho^{-\gamma} \sum_{n=0}^\infty a_n \rho^{-n}$ and proceed as in the region $(0, 1)$.

**Proposition 2.2.** Let $f \in C^m([0, T], \mathbb{R}^n)$, $A \in C^m([0, T], \mathbb{R}^{n \times n})$ for an $m \geq 1$, $m > \max_{\lambda_k \in \sigma(A(0))} \text{Re}(\lambda_k)$ and $1 \leq l \leq m$,

\[
\sigma(A(0)) \cap \{l, l + 1, \cdots\} = \emptyset.
\]

For $u_0^n, \ldots, u_0^{(l-1)} \in \mathbb{R}^m$ such that

\[
(kI - A(0))u_0^{(k)} = f^{(k)}(0) + \sum_{j=0}^{k-1} \binom{k}{j} A^{(k-j)}(0)u_0^{(j)}, \quad k = 0, \cdots, l - 1
\]

holds, there exists a unique solution $u \in C^m([0, T], \mathbb{R}^m)$ of the problem

\[
tu'(t) = A(t)u(t) + f(t), \quad 0 < t \leq T, \quad u^{(j)}(0) = u_0^{(j)}, \quad j = 0, \cdots, l - 1.
\]

**Proof.** See Theorem 1.11 from [17].

**Lemma 2.3.** There exists unique $\psi_1 \in C^1((0, \infty))$ solution to $\mathcal{L}_\infty(\psi) = 0$ with $\psi(1) = 1$ and $\psi_1$ is smooth.
Proof. We write $L_\infty(\psi) = 0$ in the form required by Proposition 2.2 so for $(\Psi_1, \Psi_2) = (\psi, \partial_\rho \psi)$,
\[
\begin{cases}
(\rho - 1)\partial_\rho \Psi_1 = (\rho - 1)\Psi_2 \\
(\rho - 1)\partial_\rho \Psi_2 = \frac{1}{\rho^2} \{ [\rho a (1 - \alpha) \rho^{-2} - \alpha (1 + \alpha)] \Psi_1 + \frac{d - 1}{\rho} - 2(1 + \alpha) \rho \Psi_2 \}.
\end{cases}
\]
Hence, we can write
\[
(\rho - 1)\partial_\rho \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = A(\rho) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},
\]
for $A$ smooth in $(0, \infty)$. Then since $\sigma(A(0)) = \{ s_c - \frac{d}{2}, 0 \}$, by Proposition 2.2, we infer for $a \in \mathbb{R}$, there exists unique $\psi_a \in C^1((0, \infty))$ solving $L_\infty(\psi_a) = 0$ with
\[
(\psi_a(0), \psi_a'(0)) = (a, 0)
\]
and in fact, $\psi_a \in C^\infty((0, \infty))$ so done by setting $a = 1$. □

For $\rho_0 > 0$, define the spaces of functions on which we invert our linearized operator $L_\infty$.
\[
X_{\rho_0} = \left\{ w : (\rho_0, \infty) \to \mathbb{R} \mid \|w\|_{X_{\rho_0}} := \sup_{\rho_0 \leq \rho \leq 1} \rho^{d-1}|w| + \sup_{\rho \geq 1} \rho^{\alpha + 1}|w| < \infty \right\},
\]
\[
Y_{\rho_0} = \left\{ w : (\rho_0, \infty) \to \mathbb{R} \mid \|w\|_{Y_{\rho_0}} := \int_{\rho_0}^1 \rho^{d-1}|1 - \rho|^{\frac{d}{2} - s_c}|\psi| \, d\rho + \int_{\rho_0}^\infty \frac{d-1}{\rho^2}|1 - \rho|^{\frac{d}{2} - s_c}|w| \, d\rho < \infty \right\}.
\]
For convenience, denote
\[
L_\infty = P(\rho) \frac{d^2}{d\rho^2} + S(\rho) \frac{d}{d\rho} + R(\rho).
\]

**Proposition 2.4** (Exterior resolvent). (i) Basis of fundamental solutions: There exists $\psi_2$ given by
\[
\psi_2 := \begin{cases}
c_1 \psi_1 \quad &\text{if } \rho \in (0, 1) \\
c_2 \psi_1 &\text{if } \rho \in (1, \infty),
\end{cases}
\]
for some $c_i \in \mathbb{R}$ which is linearly independent of $\psi_1$ such that the Wronskian is given by
\[
W := \psi_1' \psi_2 - \psi_2' \psi_1 = \rho^{1-d} |1 - \rho^{\delta_i} - \frac{d}{2}|.
\]
These have asymptotic behaviours:
\[
\psi_1 \propto \rho^{1-\frac{d}{2}} \sin(\omega \log \rho + \delta_i) + O_{\rho \to 0}(\rho^{3-\frac{d}{2}})
\]
for some $\delta_i \in \mathbb{R}$.

(ii) Continuity of the resolvent: There exists a bounded linear operator $T : Y_{\rho_0} \to X_{\rho_0}$ such that $L_\infty \circ T = \text{id}_{Y_{\rho_0}}$ given by
\[
T(f) = \psi_1 \int_{\rho}^\infty \frac{f \psi_2}{(1 - r^2)W} \, dr - \psi_2 \int_{1}^{\rho} \frac{f \psi_1}{(1 - r^2)W} \, dr
\]
with $\|T\|_{L(Y_{\rho_0}, X_{\rho_0})} \leq 1$ for all $\rho_0 > 0$. Furthermore,
\[
\Lambda \psi_1 \propto \rho^{-\alpha - 2}(1 + O_{\rho \to \infty}(\rho^{-2}))
\]
Proof. (i): Since $L_\infty(\psi^L) = 0$ and $L_\infty(\psi^R) = 0$, $PW' + SW = 0$ in $(0, \infty) \setminus \{1\}$. Then $W \propto \rho^{1-d}|1 - \rho^2|^{\frac{3}{2} - \frac{s_c}{2}}$ in $(0, 1)$ and $(1, \infty)$. Since $\psi^L$ and $\psi_1$ are linearly independent, there exists $c_1 \in \mathbb{R}$ such that $W = \rho^{1-d}|1 - \rho^2|^{\frac{3}{2} - \frac{s_c}{2}}$. Similarly we choose $c_2$. The asymptotic behaviours follow from the definitions (2.3).

(ii): Integrals in (2.9) are well-defined since

$$
\psi_1 \sim \begin{cases} O_{\rho \to 1}(1) & \text{if } \rho \geq 1 \\
\rho^{-\alpha} O_{\rho \to \infty}(\rho^{-\alpha}) & \text{if } \rho < 1
\end{cases}, \quad \psi_2 \sim \begin{cases} O_{\rho \to 1}(1) & \text{if } \rho \geq 1 \\
(1 - \rho^2)W & \text{if } \rho < 1
\end{cases}.
$$

where we have used the asymptotic behaviour for the hypergeometric functions at $\rho = 1$ form p. 559, [1]. Direct computation yields

$$
L_\infty\left[ \psi_1 \left( a_1 + \int_\rho^\infty \frac{f \psi_2}{(1 - r^2)W} dr \right) - \psi_2 \left( a_2 + \int_1^\rho \frac{f \psi_1}{(1 - r^2)W} dr \right) \right] = f.
$$

Since $T : Y_{\rho_0} \to X_{\rho_1}$, we choose $a_1 = 0$. By requiring $T(f)$ to be differentiable at $\rho = 1$, $a_2 = 0$. It suffices to prove that $T$ is bounded. For all $\rho \geq 1$,

$$
\rho^{1+\alpha}|T(f)(\rho)| \lesssim \rho^{1+\alpha} \left( |\psi_1| \int_\rho^\infty \left| \frac{f \psi_2}{(1 - r^2)W} \right| dr + |\psi_2| \int_1^\rho \left| \frac{f \psi_1}{(1 - r^2)W} \right| dr \right)
$$

$$
\lesssim \sup_{\rho \geq 1} \left( \rho \int_\rho^\infty r^{\frac{d}{2} - 2}(r - 1)^{\frac{1}{2} - s_c} |f| dr + \int_1^\rho r^{\frac{d-1}{2}} (r - 1)^{\frac{1}{2} - s_c} |f| dr \right) \lesssim \|f\|_{Y_{\rho_0}}.
$$

For all $\rho_0 \leq \rho \leq 1$,

$$
\rho^{\frac{1}{2}}|T(f)(\rho)| \lesssim \rho^{\frac{1}{2} - 1} \left( |T(f)(1)| + |\psi_1| \int_\rho^1 \left| \frac{f \psi_2}{(1 - r^2)W} \right| dr + |\psi_2| \int_\rho^1 \left| \frac{f \psi_1}{(1 - r^2)W} \right| dr \right)
$$

$$
\lesssim \|f\|_{Y_{\rho_0}} + \sup_{\rho_0 \leq \rho \leq 1} \int_\rho^1 r^{\frac{d}{2}} (s - 1)^{\frac{1}{2} - s_c} |f| ds \lesssim \|f\|_{Y_{\rho_0}}
$$

where in the final inequality, we used $\psi_1 = \mathcal{O}(\rho^{1 - \frac{d}{2}})$ and $\frac{1}{(1 - \rho^2)W} = \mathcal{O}(\rho^{d-1})$ as $\rho \to 0$. Thus, $||T(f)||_{X_{\rho_0}} \lesssim \|f\|_{Y_{\rho_0}}$. We now prove (2.10). Set $\tilde{\psi}_1 \propto \rho^{-\alpha} + \tilde{\psi}_1$ for $\tilde{\psi}_1 = \mathcal{O}(\rho^{-\alpha - 2})$ as $\rho \to \infty$. Then since $L_\infty(\psi_1) = 0$ and $L_\infty(u_\infty) \propto \rho^{-\alpha - 2}$,

$$
L_\infty(\tilde{\psi}_1) \propto -L_\infty(u_\infty) \propto \rho^{-\alpha - 2}.
$$

In view of (2.9),

$$
\tilde{\psi}_1 \propto \psi_1 \left( b_1 + \int_\rho^\infty \frac{r^{-\alpha - 2} \psi_2}{(1 - r^2)W} dr \right) - \psi_2 \left( b_2 + \int_\rho^\infty \frac{r^{-\alpha - 2} \psi_1}{(1 - r^2)W} dr \right).
$$

Since $\tilde{\psi}_1 = \mathcal{O}(\rho^{-\alpha - 2})$ as $\rho \to \infty$, $a_1 = a_2 = 0$. Then from asymptotic behaviours of $\Lambda \psi_1$, $\Lambda \psi_2$ as $\rho \to \infty$,

$$
\Lambda \tilde{\psi}_1 \propto (\Lambda \psi_1) \int_\rho^\infty \frac{r^{-\alpha - 2} \psi_2}{(1 - r^2)W} dr - (\Lambda \psi_2) \int_\rho^\infty \frac{r^{-\alpha - 2} \psi_1}{(1 - r^2)W} dr
$$

$$
\propto \rho^{-\alpha - 2} (1 + \mathcal{O}_{\rho \to \infty}(\rho^{-2})).
$$

Hence, claim follows from $\Lambda \psi_1 = \Lambda \tilde{\psi}_1$. \qed
2.2. Exterior solutions.

**Lemma 2.5** (Non-linear bounds). For \( w \in X_{\rho_0} \) and \( \varepsilon > 0 \), define

\[
G[\psi_1, \varepsilon]w = (\psi_1 + w)^2 \int_0^1 (1 - s)(u_\infty + s\varepsilon(\psi_1 + w))^{p-2} \, ds \quad \text{:=} \quad A[\psi_1, \varepsilon]w.
\]

Then for all \( \varepsilon \ll \rho_0^{-\frac{s}{2}} \) and \( w_1, w_2 \in B_{X_{\rho_0}} \),

\[
\|G[\psi_1, \varepsilon]w_1\|_{Y_{\rho_0}} \lesssim \rho_0^{\frac{1-s}{2}}, \quad \|G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2\|_{Y_{\rho_0}} \lesssim \rho_0^{1-s/2}\|w_1 - w_2\|_{X_{\rho_0}}.
\]

**Proof.** Note that for all \( \rho \geq 1 \),

\[
|\psi_1(\rho)| + |w_1(\rho)| \lesssim |u_\infty(\rho)|.
\]

Since \( \psi_1 = \mathcal{O}(\rho^{-\alpha}) \) as \( \rho \to \infty \) and \( \varepsilon \lesssim 1 \),

\[
|G[\psi_1, \varepsilon]w_1(\rho)| \lesssim (|\psi_1| + |w_1|)^2 (|u_\infty| + \varepsilon(|\psi_1| + |w_1|))^{p-2}
\]

\[
\lesssim \rho^{-2\alpha} \left( 1 + \sup_{r \geq 1} r^{\alpha+1} |w_1| \right)^2 |u_\infty(\rho)|^{p-2}
\]

\[
\lesssim \rho^{-\alpha}(1 + \|w_1\|_{X_{\rho_0}})^2 \lesssim \rho^{-\alpha/2}.
\]

so

\[
\int_1^{\infty} \int_0^1 \rho^{\frac{d-1}{2}} |1 - \rho^{\frac{1-s}{2}}|G[\psi_1, \varepsilon]w_1| \, d\rho \lesssim \int_1^{\infty} \rho^{\frac{s-1}{2}} |1 - \rho|^{\frac{1-s}{2}} \, d\rho \lesssim 1.
\]

Note that since \( \psi_1 = \mathcal{O}(\rho^{-\frac{d}{2}}) \) as \( \rho \to 0 \), for all \( \rho_0 \leq \rho \leq 1 \),

\[
|\psi_1(\rho)| + |w_1(\rho)| \lesssim \rho^{1-\frac{d}{2}} \lesssim \rho^{1-s/2}|u_\infty(\rho)|.
\]

Then since \( \varepsilon \ll \rho_0^{\frac{s-1}{2}} \),

\[
|G[\psi_1, \varepsilon]w_1| \lesssim \rho^{2-d} \left( 1 + \sup_{\rho \leq r \leq 1} r^{\frac{d}{2}} |w_1| \right)^2 |u_\infty(\rho)|^{p-2}
\]

\[
\lesssim \rho^{\alpha-d}(1 + \|w_1\|_{X_{\rho_0}})^2 \lesssim \rho^{\alpha-d}.
\]

Then

\[
\int_{\rho_0}^{1} \rho^{\frac{d}{2}} (1 - \rho)^{\frac{1-s}{2}}|G[\psi_1, \varepsilon]w_1| \, d\rho \lesssim \int_{\rho_0}^{1} \rho^{-\frac{s-1}{2}} (1 - \rho)^{\frac{1-s}{2}} \, d\rho \lesssim \rho_0^{1-s/2}.
\]

Hence, the first bound in (2.12) holds. For the contraction estimate, note that

\[
|G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| \leq |Aw_1 - Aw_2| |Bw_1| + |Aw_2| |Bw_1 - Bw_2|
\]

\[
\lesssim |2\psi_1 + w_1 + w_2| |w_1 - w_2| |u_{\infty} + \varepsilon(\psi_1 + |w_1|)|^{p-2} + \varepsilon |w_1 - w_2| (\psi_1 + w_2)^2 I_{w_1, w_2}
\]

where

\[
I_{w_1, w_2} := \int_0^1 \varepsilon^{-1} \partial_w B[\psi_1, \varepsilon]w|_{w_2 + \sigma(w_1 - w_2)} d\sigma
\]

\[
= \int_0^1 s(1 - s) \int_0^1 (u_\infty + s\varepsilon(\psi_1 + w_2) + \sigma s\varepsilon(w_1 - w_2))^{p-3} d\sigma ds
\]

\[
\lesssim |u_{\infty} + \varepsilon(\psi_1 + |w_1| + |w_2|)|^{p-3} \lesssim u_\infty^{p-3}.
\]
where the final inequality follows since $\varepsilon \ll \rho_0^{s_c-1}$. Then
\[
|G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| \lesssim \left( (|\psi_1| + |w_1| + |w_2|) u_\infty |p-2| + \varepsilon (|\psi_1| + |w_2|)^2 |u_\infty|^{p-3}|w_1 - w_2| \right).
\]
Since $\psi_1 = O(\rho^{-\alpha})$ as $\rho \to \infty$, for all $\rho \geq 1$,
\[
\int_1^\infty \rho^{\frac{d-1}{2}} (1-\rho)^{\frac{d}{2}-s_c} |G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| \, d\rho \lesssim \int_1^\infty \rho^{s_c-\frac{d}{2}} (1-\rho)^{\frac{d}{2}-s_c} \, d\rho \|w_1 - w_2\|_{X_{\rho_0}}.
\]
Since $\psi_1 = O(\rho^{1-\frac{d}{2}})$ as $\rho \to 0$, for all $\rho_0 \leq \rho \leq 1$,
\[
|G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| \lesssim \left( \rho^{2(1-\frac{d}{2})-\alpha(p-2)} + \varepsilon \rho^{2(1-\frac{d}{2})-\alpha(p-3)} \right) \sup_{\rho_0 \leq r \leq 1} r^{\frac{d}{2}-1} |w_1 - w_2| \lesssim \rho^{\alpha-d} \|w_1 - w_2\|_{X_{\rho_0}}.
\]
Then by our choice of $\varepsilon$,
\[
\int_{\rho_0}^1 \rho^{\frac{d}{2}} (1-\rho)^{\frac{d}{2}-s_c} |G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| \, d\rho \lesssim \rho_0^{1-s_c} \|w_1 - w_2\|_{X_{\rho_0}}.
\]
Hence, the second bound in (2.12) holds. \qed

**Proposition 2.6 (Exterior solutions).** For all $0 < \varepsilon \ll \rho_0^{s_c-1}$, there exists a smooth solution to (1.4) of the form
\[
u = u_\infty + \varepsilon (\psi_1 + w)
\]
with
\[
\|w\|_{X_{\rho_0}} \lesssim \varepsilon \rho_0^{1-s_c}, \quad \|Au\|_{X_{\rho_0}} \lesssim \varepsilon \rho_0^{1-s_c}.
\]
Furthermore,
\[
w|_{\varepsilon=0} = 0, \quad \|\partial_{\varepsilon} w|_{\varepsilon=0}\|_{X_{\rho_0}} \lesssim \rho_0^{1-s_c}.
\]

**Proof.** $u = u_\infty + \varepsilon v > 0$ solves (1.4) if and only if
\[
L_{\infty}(v) = \varepsilon^{-1} [u_{\infty}^p + pu_{\infty}^{p-1} \varepsilon v - (u_\infty + \varepsilon v)^p] = -p(p-1)\varepsilon v^2 \int_0^1 (1-s)(u_\infty + s\varepsilon v)^{p-2} \, ds.
\]
We further decompose $v = \psi_1 + w$. Since $L_\infty(\psi_1) = 0$,
\[
w = -p(p-1)\varepsilon v^2 \int_0^1 (1-s)(u_\infty + s\varepsilon v)^{p-2} \, ds.
\]
Lemma 2.5 together with Proposition 2.4 states precisely that for $\varepsilon \ll \rho_0^{s_c-1}$,
\[
-p(p-1)\varepsilon \mathcal{T} \circ G[\psi_1, \varepsilon] : B_{X_{\rho_0}} \to B_{X_{\rho_0}}
\]
is a contraction map. From the Banach fixed point theorem, there exists a unique solution $w$ to (2.14) with $\|w\|_{X_{\rho_0}} \lesssim \varepsilon \rho_0^{1-s_c}$. Clearly, $w$ is smooth in $(0, \infty) \setminus \{1\}$. In view of (2.14), $w \in C^1((0, \infty))$ so $u \in C^1((0, \infty))$. Writing (1.4) in the form required by Proposition 2.2, for $(\Psi_1, \Psi_2) = (u, u')$,
\[
\begin{aligned}
(\rho - 1)\partial_\rho \Psi_1 &= (\rho - 1)\Psi_2 \\
(\rho - 1)\partial_\rho \Psi_2 &= \frac{1}{1+\rho} \{ -\alpha(\alpha + 1)\Psi_1 + [(d-1)\rho^{-1} - 2(\alpha + 1)\rho]\Psi_2 + u^p \}.
\end{aligned}
\]
Hence,
\[
(\rho - 1)\partial_\rho \left( \begin{array}{c}
\Psi_1 \\
\Psi_2
\end{array} \right) = A(\rho) \left( \begin{array}{c}
\Psi_1 \\
\Psi_2
\end{array} \right) + \frac{1}{\rho + 1} \left( \begin{array}{c}
1 \\
u^p
\end{array} \right).
\]
where $A$ is smooth in $(0, \infty)$ and

$$A(1) = \frac{1}{2} \begin{pmatrix} 0 & 2 \\ -\alpha(\alpha + 1) & 2s_c - 3 \end{pmatrix}$$

with $\sigma(A(1)) = \{s_c - \frac{3}{2}, 0\}$. By Proposition 2.2, since $u \in C^1((0, \infty))$, $(u, u') \in C^1(0, \infty)$ so $u \in C^2((0, \infty))$. Iterating this, we conclude that $u$ is smooth.

Applying $\Lambda$ to (2.14), we infer

$$\Lambda w = -p(p - 1)\varepsilon \left[ (\Lambda \psi_1) \int_{\rho}^{\infty} \frac{G[\psi_1, \varepsilon](w)\psi_2}{(1 - r^2)W} \, dr - (\Lambda \psi_2) \int_{1}^{\rho} \frac{G[\psi_1, \varepsilon](w)\psi_1}{(1 - r^2)W} \, dr \right].$$

Hence, by considering the asymptotes of $\Lambda \psi_i$ and proceeding as in the proof of Proposition 2.4, we infer

$$\|\Lambda w\|_{X_{\rho_0}} \lesssim \varepsilon \|G[\psi_1, \varepsilon]w\|_{Y_{\rho_0}} \lesssim \varepsilon \rho_0^{-s_c}.$$

In view of (2.14) $w|_{\varepsilon = 0} = 0$. Differentiating (2.14) in $\varepsilon$,

$$\partial_\varepsilon w|_{\varepsilon = 0} = -p(p - 1) \left( T \circ G[\psi_1, 0]w|_{\varepsilon = 0} + \varepsilon T(\partial_\varepsilon G[\psi_1, \varepsilon]w)|_{\varepsilon = 0} \right)$$

so by continuity of the resolvent and the asymptotic behaviour of $\psi_1$ as $\rho \to 0$ and $p \to \infty$,

$$\|\partial_\varepsilon w|_{\varepsilon = 0}\|_{X_{\rho_0}} \lesssim \int_{\rho_0}^{\infty} \rho^{-s_c + \frac{d}{2}}(1 - \rho)^{\frac{d}{2} - s_c} \, d\rho \lesssim \rho_0^{-s_c}.$$  \(\square\)

### 3. Construction of interior solutions

In this section, we construct inner solutions to the self similar equation (1.4) which are perturbations of a rescaled soliton.

Let us introduce some notations for this section. We let the linearized operator $\mathcal{H}_\infty$ for (1.4) near $u = \lambda^{-\alpha}Q(\frac{x}{\lambda})$ be

$$\mathcal{H}_\infty = -\Delta - pQ^{p-1} = -\frac{d^2}{d\rho^2} - \frac{d - 1}{\rho} \frac{d}{d\rho} - pQ^{p-1}.  \quad (3.1)$$

In this section, assume $\rho_1 \geq 1$ and define

$$\tilde{X}_{\rho_1} = \left\{ w : (0, \rho_1) \to \mathbb{R} \mid \|w\|_{\tilde{X}_{\rho_1}} := \sup_{0 \leq \rho \leq \rho_1} (1 + \rho)^{-\frac{d}{2} - 3}(|w| + \rho|w'| + \rho^2|w''|) < \infty \right\}$$

$$\tilde{Y}_{\rho_1} = \left\{ w : (0, \rho_1) \to \mathbb{R} \mid \|w\|_{\tilde{Y}_{\rho_1}} := \sup_{0 \leq \rho \leq \rho_1} (1 + \rho)^{-\frac{d}{2} - 1}|w| < \infty \right\}.  \quad (3.2)$$

**Proposition 3.1** (Interior resolvent). (i) Basis of fundamental solutions: $\mathcal{H}_\infty(\Lambda Q) = 0$ and there exists $\varphi$ linearly independent to $\Lambda Q$ solving $\mathcal{H}_\infty(\varphi) = 0$ such that the Wronskian

$$W := (\Lambda Q)'\varphi - \varphi'\Lambda Q = \rho^{1-d}.$$
Then the asymptotic behaviours hold:
\[ \Lambda Q, \varphi \propto \rho^{1-\frac{d}{2}} \sin(\omega \log \rho + \delta_*) + O_{\rho \to \infty}(\rho^{2-d+\alpha}) \]  
(3.3)
for some \( \delta_{\Lambda Q}, \delta_\varphi \in \mathbb{R} \).

(ii) Continuity of the resolvent: There exists a bounded linear operator \( S : \tilde{Y}_{\rho_1} \to \tilde{X}_{\rho_1} \) such that \( \mathcal{H}_\infty \circ S = \text{id}_{\tilde{Y}_{\rho_1}} \) given by
\[
S(f) = \Lambda Q \int_0^\rho f \varphi r^{d-1} \, dr - \varphi \int_0^\rho f \Lambda Q r^{d-1} \, dr
\]
(3.4)
with \( \|S\|_{\mathcal{L}(\tilde{Y}_{\rho_1}, \tilde{X}_{\rho_1})} \leq 1 \) for all \( \rho_1 > 0 \).

**Proof.** (i): See Lemma 2.3 from [6].

(ii): In Lemma 2.3 from [6], it is proved that
\[
\sup_{0 \leq \rho \leq \rho_1} (1 + \rho)\frac{d}{2} \rho^2 |S(f)| \leq \|S\|_{\tilde{Y}_{\rho_1}} \leq \|f\|_{\tilde{Y}_{\rho_1}}
\]
so it suffices to bound \( (1 + \rho)\frac{d}{2} \rho^2 |S(f)^\prime| \). Note that
\[
(\Lambda Q)^\prime \varphi - \varphi^\prime \Lambda Q = \rho^{1-d}.
\]
Let \( R_0 > 0 \) such that \( \Lambda Q > 0 \) in \([0,R_0]\). Then solving for \( \varphi \),
\[
\varphi = -\Lambda Q \int_0^{R_0} \frac{dr}{(\Lambda Q)^2 r^{d-1}}.
\]
Differentiating twice,
\[
|\varphi''| = \left| \frac{d-1}{\Lambda Q \rho^d} - (\Lambda Q)^\prime \right| \int_0^{R_0} \frac{dr}{(\Lambda Q)^2 r^{d-1}} \leq \rho^{-d}
\]
as \( \rho \to 0 \) since \( \Lambda Q \), \( (\Lambda Q)^\prime \) are bounded near \( \rho = 0 \) and \( \mathcal{H}_\infty(\Lambda Q) = 0 \) so that \( (\Lambda Q)^\prime = O_{\rho \to 0}(\rho^{-2}) \). Thus, for all \( 0 \leq \rho \leq 1 \),
\[
|\rho^2 S(f)^\prime| = \rho^2 \left| (\Lambda Q)^\prime \int_0^\rho f \varphi r^{d-1} \, dr - \varphi^\prime \int_0^\rho f \Lambda Q r^{d-1} \, dr + f \right|
\]
\[
\leq \rho^2 \left[ \rho^{-2} \int_0^\rho r \, dr + \rho^{-d} \int_0^\rho r^{d-1} \, dr + 1 \right] \sup_{0 \leq \rho \leq 1} |f| \lesssim \|f\|_{\tilde{Y}_{\rho_1}}.
\]
Since \( \Lambda Q, \rho(\Lambda Q)^\prime = O_{\rho \to \infty}(\rho^{1-\frac{d}{2}}) \) and \( \mathcal{H}_\infty(\Lambda Q) = 0 \), \( \rho^2 (\Lambda Q)^\prime = O_{\rho \to \infty}(\rho^{1-\frac{d}{2}}) \) and similarly, \( \rho^2 \varphi'' = O_{\rho \to \infty}(\rho^{1-\frac{d}{2}}) \). Thus, for all \( 1 \leq \rho \leq \rho_1 \),
\[
|(1 + \rho)^\frac{d}{2} \rho^2 S(f)^\prime| = (1 + \rho)^\frac{d}{2} \rho^2 \left| (\Lambda Q)^\prime \int_0^\rho f \varphi r^{d-1} \, dr - \varphi^\prime \int_0^\rho f \Lambda Q r^{d-1} \, dr + f \right|
\]
\[
\lesssim (1 + \rho)^{-2} \int_0^\rho |f|(1 + r)^\frac{d}{2} \, dr + (1 + \rho)^\frac{d}{2} |f| \lesssim \|f\|_{\tilde{Y}_{\rho_1}}.
\]
Thus, \( \|S(f)\|_{\tilde{X}_{\rho_1}} \lesssim \|f\|_{\tilde{Y}_{\rho_1}} \). \( \Box \)
Lemma 3.2 (Non-linear bounds). For \( w \in \tilde{X}_{\rho_1} \) and \( \lambda > 0 \), define

\[
F[Q, \lambda]w = p(p-1)\lambda^2 w^2 \int_0^1 (1-s)(Q + \lambda^2 sw)^{p-2} ds - F(Q + \lambda^2 w).
\]  

(3.5)

where

\[
F = \rho^2 \frac{d^2}{d\rho^2} + 2(1+\alpha)\rho \frac{d}{d\rho} + \alpha(1+\alpha).
\]

Then there exists \( C > 0 \) such that for all \( \lambda \ll \rho_1 \) and \( \|w_1\|_{\tilde{X}_{\rho_1}}, \|w_1\|_{\tilde{X}_{\rho_1}} \leq C \),

\[
\|F[Q, \lambda]w_1\|_{\tilde{X}_{\rho_1}} \leq C\|S\|^{-1}_{L(\tilde{Y}_{\rho_1}, \tilde{X}_{\rho_1})}, \quad \|F[Q, \lambda]w_1 - F[Q, \lambda]w_2\|_{\tilde{X}_{\rho_1}} \leq \rho_1^2 \lambda^2 \|w_1 - w_2\|_{\tilde{X}_{\rho_1}}
\]  

(3.6)

Proof. We first bound \( F(Q) \). In view of (1.5),

\[
\rho^2 Q^{p-1} = b_{\infty}^{p-1} + O_{\rho \to \infty}(\rho^{1-s_c}).
\]

Then in view of (3.3), since \( Q'' + \frac{d}{d\rho} Q' + Q^p = 0 \), we infer

\[
F(Q) = -\rho^2 Q^p + (3 - 2s_c)\rho Q' + \alpha(1+\alpha)Q
\]

\[
= (b_{\infty}^{p-1} - \rho^2 Q^{p-1})Q + (3 - 2s_c)\Lambda Q = O_{\rho \to \infty}(\rho^{-\frac{d}{2}}).
\]

Note also that for all \( 0 \leq \rho \leq \rho_1 \),

\[
|w_1(\rho)| \lesssim (1 + \rho_1)^{3-\frac{d}{2}} \|w_1\|_{\tilde{X}_{\rho_1}} \lesssim (1 + \rho_1)^2 |Q(\rho)| \|w_1\|_{\tilde{X}_{\rho_1}}
\]

so by our choice of \( \lambda \),

\[
\lambda^2 |w_1(\rho)| \lesssim |Q(\rho)| \|w_1\|_{\tilde{X}_{\rho_1}}.
\]

With these estimates, for all \( 0 \leq \rho \leq \rho_1 \),

\[
|F[Q, \lambda]w_1| \lesssim \lambda^2 \|w_1\|_{\tilde{X}_{\rho_1}}^2((Q + \lambda^2 |w_1|)^{p-2} + |F(Q)| + \lambda^2 |F(w_1)|)
\]

\[
\lesssim \lambda^2 (1+\rho)^{6-d-\alpha(p-2)}(\|w_1\|_{\tilde{X}_{\rho_1}}^2 + \|w_1\|_{\tilde{X}_{\rho_1}}^p) + (1+\rho)^{1-\frac{d}{2}} + \lambda^2 (1+\rho)^{3-\frac{d}{2}} \|w_1\|_{\tilde{X}_{\rho_1}}
\]

\[
\lesssim [\rho_1^{3-s_c} \lambda^2 (\|w_1\|_{\tilde{X}_{\rho_1}}^2 + \|w_1\|_{\tilde{X}_{\rho_1}}^p) + 1 + \rho_1^2 \lambda^2](1+\rho)^{1-\frac{d}{2}}
\]

\[
\lesssim [1 + \rho_1^2 \lambda^2 (1+\rho_1)\|w_1\|_{\tilde{X}_{\rho_1}}^p](1+\rho)^{1-\frac{d}{2}}
\]

where we have used that \( s_c < 1 \) in the last inequality. Choose \( C > 0 \) such that

\[
|F[Q, \lambda]w_1| \leq \frac{C}{2} \|S\|^{-1}_{L(\tilde{Y}_{\rho_1}, \tilde{X}_{\rho_1})}[1 + \rho_1^2 \lambda^2 (\|w_1\|_{\tilde{X}_{\rho_1}} + \|w_1\|_{\tilde{X}_{\rho_1}}^p)](1+\rho)^{1-\frac{d}{2}}.
\]

Then for \( \rho_1 \lambda \ll 1 \) and \( \|w_1\|_{\tilde{X}_{\rho_1}} \leq C \),

\[
|F[Q, \lambda]w_1| \leq C \|S\|^{-1}_{L(\tilde{Y}_{\rho_1}, \tilde{X}_{\rho_1})}(1+\rho)^{1-\frac{d}{2}}.
\]

Hence, the first bound in (3.6) holds.

\[
|F[Q, \lambda]w_1 - F[Q, \lambda]w_2| \leq |\tilde{A}w_1 - \tilde{A}w_2| |\tilde{B}w_1| + |\tilde{A}w_2| |\tilde{B}w_1 - \tilde{B}w_2| + \lambda^2 |F(w_1 - w_2)|
\]

\[
\lesssim \lambda^2 \|w_1 + w_2\| \|w_1 - w_2\|((Q + \lambda^2 |w|)^{p-2} + \lambda^4 \|w_1 - w_2\| |w_2|^2 \|w_1 - w_2\| + \lambda^2 (1+\rho)^{3-\frac{d}{2}} \|w_1 - w_2\|_{\tilde{X}_{\rho_1}}
\]
where
\[ \tilde{I}_{w_1,w_2} := \left| \int_0^1 \lambda^{-2} \partial_w \tilde{B} [Q, \lambda] w_{\left| w_2 + \sigma (w_1 - w_2) \right|} \ d\sigma \right| \]
\[ = \left| \int_0^1 s (1 - s) \int_0^1 (Q + s \lambda^2 w_2 + \sigma \sigma 2 \lambda^2 (w_1 - w_2))^{p-3} d\sigma ds \right| \]
\[ \lesssim \| Q \| + \lambda^2 (|w_1| + |w_2|))^{p-3} \lesssim (1 + \rho)^{-\alpha (p-3)}. \]

Thus,
\[ | F [Q, \lambda] w_1 - F [Q, \lambda] w_2 | \]
\[ \lesssim \| \lambda^2 (1 + \rho)^{6-d-\alpha} + \lambda^4 (1 + \rho)^{9-\frac{4d}{p+3}-(p-3)\alpha} + \lambda^2 (1 + \rho)^{3-\frac{4}{p+3}} \| w_1 - w_2 \|_{\tilde{X}_{\rho_1}} \]
\[ \lesssim (\rho_1^{3-s_c} \lambda^2 + \rho_1^{\frac{6-d}{2} - \alpha} + \rho_1^2 \lambda^2) (1 + \rho)^{1-\frac{4}{p+3}} \| w_1 - w_2 \|_{\tilde{X}_{\rho_1}} \lesssim \rho_1^2 \lambda^2 (1 + \rho)^{1-\frac{4}{p+3}} \| w_1 - w_2 \|_{\tilde{X}_{\rho_1}} \]

where again, we have used that \( s_c > 1 \). Hence the second bound in (3.6) holds. \( \square \)

**Proposition 3.3** (Interior solutions). For all \( 0 \leq \rho_0 \ll 1, 0 < \lambda \leq \rho_0 \), there exists a solution to (1.4) on \( 0 \leq \rho \leq \rho_0 \) of the form
\[ u = \lambda^{-\alpha} (Q + \lambda^2 w) \left( \frac{\rho}{\lambda} \right) \]
with \( \| w \|_{\tilde{X}_{\rho_1}} \lesssim 1 \) where \( \rho_1 = \frac{\rho_0}{\lambda} \geq 1 \).

**Proof.** \( u = \lambda^{-\alpha} (Q + \lambda^2 w) \left( \frac{\rho}{\lambda} \right) \) solves (1.4) if and only if
\[ \mathcal{H}_\infty (w) = \lambda^{-2} [(Q + \lambda^2 w)^p - Q^p - \rho Q^{p-1} \lambda^2 w] - F (Q + \lambda^2 w) = F [Q, \lambda] w. \] (3.7)

Lemma 3.2 together with Proposition 3.1 states precisely that for \( \rho_1 \lambda = \rho_0 \ll 1 \),
\[ \mathcal{S} \circ F [Q, \lambda] : B_{\tilde{X}_{\rho_1}} (C) := \{ w \in \tilde{X}_{\rho_1} \mid \| w \|_{\tilde{X}_{\rho_1}} \leq C \} \rightarrow B_{\tilde{X}_{\rho_1}} (C) \]
is a contraction map. Thus, Banach fixed point theorem applies and yields a unique solution \( w \) to (3.7) with \( \| w \|_{\tilde{X}_{\rho_1}} \leq C \). \( \square \)

4. The matching

We are now in position to "glue" inner and outer solutions to produce exact solutions to (1.1).

**Proposition 4.1** (Existence of a countable number of smooth self-similar profiles). There exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), there exists a smooth solution \( u_n \) to (1.1) such that \( \Delta u_n \) vanishes exactly \( n \) times.

**Proof.** We will present a sketch of the proof which adapts Proposition 2.5 from [6].
First, we make some definitions. Since
\[ \psi_1 \propto \rho^{1-\frac{4}{p+3}} \sin (\omega \log \rho + \delta_1) + \mathcal{O}_{\rho \to 0} (\rho^{3-\frac{4}{p+3}}) \]
\[ \Lambda \psi_1 \propto \rho^{1-\frac{4}{p+3}} [(\alpha + 1 - \frac{4}{p+3}) \sin (\omega \log \rho + \delta_1) + \omega \cos (\omega \log \rho + \delta_1)] + \mathcal{O}_{\rho \to 0} (\rho^{3-\frac{4}{p+3}}), \] (4.1)
we can choose \( 0 < \rho_0 \ll 1 \) such that
\[ \psi_1 (\rho_0) \gtrsim \rho_0^{1-\frac{4}{p+3}}, \quad \Lambda \psi_1 (\rho_0) \gtrsim \rho_0^{1-\frac{4}{p+3}}. \]
and Proposition 2.6 and Proposition 3.3 apply. Let
\[ u_{\text{ext}}[\varepsilon] = u_{\infty} + \varepsilon \psi_1 + \varepsilon w_{\text{ext}} \]
\[ u_{\text{int}}[\lambda] = \lambda^{-\alpha}(Q + \lambda^2 w_{\text{int}}) \left( \frac{\rho}{\lambda} \right) \]
be solutions to (1.4) in the regions \([\rho_0, \infty)\) and \([0, \rho_0]\) respectively. Define
\[ \mathcal{I}[\rho_0](\varepsilon, \lambda) = u_{\text{int}}[\varepsilon](\rho_0) - u_{\text{ext}}[\lambda](\rho_0). \]
Then
\[ \partial_\varepsilon \mathcal{I}[\rho_0](\varepsilon, \lambda) = \partial_\varepsilon u_{\text{ext}}[\varepsilon](\rho_0) = \psi_1(\rho_0) + w_{\text{ext}}(\rho_0) + \varepsilon \partial_\varepsilon w(\rho_0). \]
In view of Proposition 2.6, since \(\psi_1(\rho_0) \neq 0\),
\[ \partial_\varepsilon \mathcal{I}[\rho_0](0, 0) = \psi_1(\rho_0) \neq 0. \]
From the asymptotic behaviour of \(Q\) as \(\rho \to \infty\), as \(\lambda \to 0\),
\[ \left| \lambda^{-\alpha}(Q - u_{\infty} + \lambda^2 w_{\text{int}}) \left( \frac{\rho_0}{\lambda} \right) \right| \lesssim \lambda^{-\alpha} \left[ \left( \frac{\rho_0}{\lambda} \right)^{1-\frac{d}{2}} + \lambda^2 \left( \frac{\rho_0}{\lambda} \right)^{3-\frac{d}{2}} \right] \lesssim \lambda^{s_c - 1} \rho_0^{1-\frac{d}{2}} (1 + \rho_0^2) \to 0 \]
Since \(u\) is self-similar, this implies
\[ \mathcal{I}[\rho_0](0, 0) = u_{\infty}(\rho_0) - \lim_{\lambda \to 0} \lambda^{-\alpha} u_{\infty} \left( \frac{\rho_0}{\lambda} \right) = 0. \]
Applying the implicit function theorem to
\[ \hat{I}[\rho_0](\varepsilon, \mu) := \mathcal{I}(\varepsilon, \mu^{\frac{1}{s_c - 1}}) \]
which is \(C^1\), there exists \(\lambda_0 > 0\) and \(\bar{\varepsilon} \in C^1([0, \lambda_0^{s_c - 1}])\) such that \(\hat{I}(\bar{\varepsilon}(\mu), \mu) = 0\) i.e. for \(\varepsilon(\lambda) := \bar{\varepsilon}(\lambda^{s_c - 1}), \mathcal{I}(\varepsilon(\lambda), \lambda) = 0\) and \(\varepsilon \in C^{s_c - 1}([0, \lambda_0])\). Hence,
\[ u_{\text{ext}}[\varepsilon(\lambda)](\rho_0) = u_{\text{int}}[\lambda](\rho_0) \]
on \([0, \lambda_0)\) i.e.
\[ \varepsilon(\lambda)(\psi_1(\rho_0) + w_{\text{ext}}(\rho_0)) = \lambda^{-\alpha}(Q - u_{\infty} + \lambda^2 w_{\text{int}}) \left( \frac{\rho_0}{\lambda} \right). \] (4.2)
By the definition of \(\rho_0\) and from the bounds on \(w_{\text{ext}}\) and \(w_{\text{int}}\) in Propositions 2.6 and 3.3, we infer
\[ \text{LHS} \propto \varepsilon(\lambda) \rho_0^{1-\frac{d}{2}} [1 + \mathcal{O}(\rho_0^2 + \varepsilon(\lambda) \rho_0^{s_c - 1})], \quad \text{RHS} \lesssim \lambda^{s_c - 1} \rho_0^{1-\frac{d}{2}} [1 + \mathcal{O}(\rho_0^2)] \]
as \(\rho_0 \to 0\), so as \(\lambda \to 0\),
\[ |\varepsilon(\lambda)| \lesssim \lambda^{s_c - 1}. \]
It then follows from (4.2) and (2.13) that
\[ \varepsilon(\lambda) = \psi_1^{-1}(\rho_0) \lambda^{-\alpha}(Q - u_{\infty}) \left( \frac{\rho_0}{\lambda} \right) + \mathcal{O}[\lambda^{s_c - 1}(\rho_0^2 + \lambda^{s_c - 1} \rho_0^{1-s_c})]. \] (4.3)
Consider now the spatial derivative
\[ \mathcal{I}'[\rho_0](\lambda) = \varepsilon(\lambda)(\psi_1'(\rho_0) + w_{\text{ext}}'(\rho_0)) - \lambda^{1-\alpha}(Q' - u_{\infty}' + \lambda^2 w_{\text{int}}') \left( \frac{\rho_0}{\lambda} \right). \]
From the bound on $\varepsilon(\lambda)$ above and the bound on $w'_{\text{ext}}$ and $w'_{\text{int}}$ in Propositions 2.6 and 3.3, we infer

$$T'[\rho_{0}](\lambda) = \varepsilon(\lambda)\psi'_{1}(\rho_{0}) - \lambda^{-1-\alpha}(Q' - u'_{\infty})\left(\frac{\rho_{0}}{\lambda}\right) + \mathcal{O}[\lambda^{s_{c}-1}(\rho_{0}\frac{2-\delta}{2} + \lambda^{s_{c}-1}\rho_{0}^{1-d+\alpha})]$$

$$= \frac{\lambda^{s_{c}-1}}{\rho_{0}^{\frac{d-1}{2}}\psi_{1}(\rho_{0})}\left[\left(\frac{\rho_{0}}{\lambda}\right)^{\frac{d-1}{2}}(Q - u_{\infty})\left(\frac{\rho_{0}}{\lambda}\right)\psi'_{1}(\rho_{0}) - \left(\frac{\rho_{0}}{\lambda}\right)^{\frac{d}{2}}(Q' - u'_{\infty})\left(\frac{\rho_{0}}{\lambda}\right)\psi_{1}(\rho_{0})\right]$$

$$+ \mathcal{O}[\lambda^{s_{c}-1}(\rho_{0}\frac{2-\delta}{2} + \lambda^{s_{c}-1}\rho_{0}^{1-d+\alpha})].$$

From the asymptotic behaviours (4.1) for $\psi_{1}$ and

$$Q(\rho) - u_{\infty}(\rho) \approx \rho^{1-\frac{d}{2}}\sin(\omega \log \rho + \delta_{2}) + \mathcal{O}(\rho^{\frac{1}{2} - s_{c}})$$

$$Q'(\rho) - u'_{\infty}(\rho) \approx \rho^{-\frac{d}{2}}[(\frac{d}{2} - 1)\sin(\omega \log \rho + \delta_{2}) + \omega \cos(\omega \log \rho + \delta_{2})] + \mathcal{O}(\rho^{\frac{1}{2} - s_{c}}),$$

it follows that

$$\frac{\rho_{0}^{\frac{d-1}{2}}}{\lambda^{s_{c}-1}}T'[\rho_{0}](\lambda) \approx \rho_{0}^{\frac{d}{2}}|\sin(\omega \log \rho_{0} - \omega \log \lambda + \delta_{2})\cos(\omega \log \rho + \delta_{1}) - \cos(\omega \log \rho_{0} - \omega \log \lambda + \delta_{2})\sin(\omega \log \rho_{0} + \delta_{1})| + \mathcal{O}(\rho_{0}^{2-\frac{d}{2}} + \lambda^{s_{c}-1}\rho_{0}^{1-d+\alpha})$$

$$= \rho_{0}^{-\frac{d}{2}}\sin(-\omega \log \lambda + \delta_{2} - \delta_{1}) + \mathcal{O}(\rho_{0}^{2-\frac{d}{2}} + \lambda^{s_{c}-1}\rho_{0}^{1-d+\alpha}).$$

Thus,

$$T'[\rho_{0}](\lambda) \approx \lambda^{s_{c}-1}\left[\frac{\sin(-\omega \log \lambda + \delta_{2} - \delta_{1})}{\rho_{0}^{d-1}\psi_{1}(\rho_{0})} + \mathcal{O}(\rho_{0}^{2-\frac{d}{2}} + \lambda^{s_{c}-1}\rho_{0}^{1-d+\alpha})\right]. \quad (4.5)$$

Let

$$\lambda_{n,\pm} = \exp\left[\frac{(n + \frac{1}{2})\pi + \delta_{2} - \delta_{1}}{\omega}\right], \quad \lambda_{n,-} = \exp\left[\frac{-(n - \frac{1}{2})\pi + \delta_{2} - \delta_{1}}{\omega}\right].$$

Since $\lambda_{n,\pm} \to 0$ as $n \to \infty$, for $n \gg 1$, $0 < \cdots < \lambda_{n,+} < \lambda_{n,-} \leq \lambda_{0}$. Then,

$$T'[\rho_{0}](\lambda_{n,\pm}) \propto (\pm(-1)^{n}\lambda_{n,\pm}^{-1})\left[\frac{1}{\rho_{0}^{d-1}\psi_{1}(\rho_{0})} + \mathcal{O}(\rho_{0}^{2-\frac{d}{2}} + \lambda_{n,\pm}^{s_{c}-1}\rho_{0}^{1-d+\alpha})\right]$$

For $\rho_{0} \ll 1$, and $n \gg 1$,

$$T'[\rho_{0}](\lambda_{n,-})T'[\rho_{0}](\lambda_{n,+}) < 0.$$  

Since $\lambda \mapsto T'[\rho_{0}](\lambda)$ is continuous, it follows from intermediate value theorem that for all $n \geq N \gg 1$, there exists $\lambda_{n,+} < \mu_{n} < \lambda_{n,-}$ such that $T'[\rho_{0}](\mu_{n}) = 0$ i.e.

$$u_{\text{ext}}[\varepsilon(\mu_{n})](\rho_{0}) = u_{\text{int}}[\mu_{n}](\rho_{0}), \quad u'_{\text{ext}}[\varepsilon(\mu_{n})](\rho_{0}) = u'_{\text{int}}[\mu_{n}](\rho_{0}).$$

Hence, the function

$$u_{n}(\rho) := \begin{cases} u_{\text{int}}[\mu_{n}](\rho) & 0 \leq \rho \leq \rho_{0}, \\ u_{\text{ext}}[\varepsilon(\mu_{n})](\rho) & \rho_{0} < \rho \end{cases}$$

is a smooth solution to (1.4) in $[0, \infty)$ for all $n \geq N$. The remaining part of the proof is devoted to counting the number of zeroes of $\Lambda u_{n}$. See Proposition 2.5 from [6] for the detailed argument.
Corollary 4.2. Let $u_n$ be the solution to (1.4) constructed in Proposition 4.1. For $\rho_0 \ll 1$,

(i) Convergence to $u_\infty$ as $n \to \infty$:

$$\lim_{n \to \infty} \sup_{\rho \geq \rho_0} (1 + \rho^\alpha)|u_n(\rho) - u_\infty(\rho)| = 0. \quad (4.6)$$

(ii) Convergence to $Q$ at the origin: There exists $\mu_n \to 0$ such that

$$\lim_{n \to \infty} \sup_{\rho \leq \rho_0} \left| u_n(\rho) - \mu_n^\alpha Q\left(\frac{\rho}{\mu_n}\right) \right| = 0. \quad (4.7)$$

(iii) Last zeros: Let

$$\rho_{0,n} := \max\{\rho | \Lambda u_n(\rho) = 0, \rho < \rho_0\}, \quad \rho_{\Lambda Q,n} := \max\{\rho | \Lambda Q(\rho) = 0, \rho < \frac{\rho_0}{\mu_n}\}.$$ 

Then

$$\rho_{0,n} = \mu_n \rho_{\Lambda Q,n}[1 + O_{\rho_0 \to 0}(\rho_0^2)].$$

Furthermore, for $n \geq N$,

$$e^{-\frac{3\pi}{2} \frac{\rho_0}{\mu_n}} \rho_0 < \rho_{0,n} < \rho_0.$$ 

Proof. Choose $\rho_0 \ll 1$ as in the proof of Proposition 4.1.

(i) In view of (4.1) and (2.13), we infer

$$\sup_{\rho \geq \rho_0} (1 + \rho^\alpha)|u_n(\rho) - u_\infty(\rho)| = \sup_{\rho \geq \rho_0} (1 + \rho^\alpha)|\varepsilon(\mu_n)(\psi_1(\rho) + w_{ext}(\rho))|$$

$$\lesssim \varepsilon(\mu_n) \left[ \sup_{\rho_0 \leq \rho \leq 1} (|\psi_1(\rho)| + |w_{ext}(\rho)|) \right. + \left. \sup_{\rho \geq 1} \rho^\alpha(|\psi_1(\rho)| + |w_{ext}(\rho)|) \right]$$

$$\lesssim \varepsilon(\mu_n) \rho_0^{1 - \frac{d}{2}}.$$ 

Since $\varepsilon(\mu_n) \to 0$ as $n \to \infty$, result follows.

(ii) In view of Proposition 3.3, we infer

$$\sup_{\rho \leq \rho_0} \left| u_n(\rho) - \mu_n^{-\alpha}Q\left(\frac{\rho}{\mu_n}\right) \right| \leq \mu_n^{2 - \alpha} \sup_{\rho \leq \rho_0} \left| w_{int}\left(\frac{\rho}{\mu_n}\right) \right| \lesssim \mu_n^{s_c - 1}.$$ 

Since $\mu_n \to 0$ as $n \to \infty$, result follows.

(iii) In view of (3.3),

$$\Lambda Q\left( e^{-\frac{3\pi}{2} \frac{\rho_0}{\mu_n}} \frac{\rho_0}{\mu_n} \right) \Lambda Q\left( \frac{\rho_0}{\mu_n} \right) < 0$$

so by intermediate value theorem, there exists a zero of $\Lambda Q$ in the interval $\left[ e^{-\frac{3\pi}{2} \frac{\rho_0}{\mu_n}}, \frac{\rho_0}{\mu_n} \right]$. In particular,

$$e^{-\frac{3\pi}{2} \frac{\rho_0}{\mu_n}} \rho_0 \leq \rho_{\Lambda Q,n} \leq \frac{\rho_0}{\mu_n}. \quad (4.8)$$

Also, if

$$e^{-\frac{3\pi}{2} \frac{\rho_0}{\mu_n}} \leq \rho \leq \rho_0,$$
then \( \frac{n}{\mu} \gg 1 \) for \( n \geq N \gg 1 \). Thus, from (3.3) and Proposition 3.3 since

\[
\sup_{0 \leq \rho \leq \frac{n}{\mu}} (1 + \rho)\frac{\mu}{\mu} |\Lambda w_{\text{int}}| \lesssim 1,
\]

it follows that

\[
\Lambda u_n(\rho) = \mu_n^{-\alpha}(\Lambda Q + \mu_n^2 \Lambda w_{\text{int}}) \left( \frac{\rho}{\mu_n} \right)
\]

\[
\propto \mu_n^{-\frac{1}{2}} \rho^{-\frac{5}{4}} \left[ \sin(\omega \log \rho - \omega \log \mu_n + \delta_2) + O_{\rho \to 0}(\rho_0^2) \right].
\]

Thus,

\[
|\omega \log \rho_{0,n} - \omega \log \mu_n - \omega \log \rho_{\Lambda Q,n}| \lesssim \rho_0^2.
\]

Hence,

\[
\rho_{0,n} = \mu_n \rho_{\Lambda Q,n} e^{O(\rho_0^2)} = \mu_n \rho_{\Lambda Q,n} [1 + O_{\rho_0 \to 0}(\rho_0^2)].
\]

Furthermore, since (4.8) holds, we deduce

\[
e^{-\frac{2\pi}{\omega}} \rho_0 < \rho_{0,n} < \rho_0.
\]

\[\square\]

**Remark 1.** Statements of Proposition 4.1 and Corollary 4.2 yields Theorem 1.

### 5. Dissipativity of Linearized Operator

We now start the study of the dynamical stability of self similar profiles. Our aim in this section is to realize the linearized operator as a compact perturbation of a maximal accretive operator in a *global in space* Sobolev norm. From now on, we assume (1.6).

Let us introduce some notations. The similarity transformation variables:

\[
\tilde{\Psi}(s, y) = (T - t)^{\alpha} \Phi(t, x), \quad s = -\log(T - t).
\]

maps the wave equation (1.1) onto

\[
\partial_s^2 \tilde{\Psi} = -2y \cdot \nabla \partial_s \tilde{\Psi} - (2 + 2\alpha) \partial_s \tilde{\Psi} + \sum_{i,j} (\delta_{ij} - y_i y_j) \partial_{\rho_i \rho_j} \tilde{\Psi} - 2(1 + \alpha) y \cdot \nabla \tilde{\Psi} - \alpha(1 + \alpha) \tilde{\Psi} + |\tilde{\Psi}|^{p-1} \tilde{\Psi}.
\]

We write above as a system of linearized equations near \( u_n \) for \( \Psi = \tilde{\Psi} - u_n \) and \( \Omega = -\partial_s \Psi - \Lambda \Psi \):

\[
\partial_s X = \mathcal{M} X + G, \quad X = \begin{pmatrix} \Psi \\ \Omega \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ -|\tilde{\Psi}|^{p-1} \tilde{\Psi} + u_p + pu_p^{-1} \Psi \end{pmatrix}
\]

and

\[
\mathcal{M} = - \begin{pmatrix} \Lambda & 1 \\ \Delta + pu_p^{-1} & \Lambda + 1 \end{pmatrix}.
\]

Denote

\[
\Psi_j = \Delta^j \Psi, \quad \Omega_j = \Delta^j \Omega
\]

and

\[
\nabla^j = \begin{cases} \Delta^j & j = 2i \\ \nabla \Delta^i & j = 2i + 1 \end{cases}.
\]
Lemma 5.1 (Commuting with derivatives). For \( k \in \mathbb{N} \), there holds
\[
\Delta^k \mathcal{M} X = \mathcal{M}_k \Delta^k X + \tilde{\mathcal{M}}_k X
\]
where
\[
\mathcal{M}_k = - \begin{pmatrix} \Lambda + 2k & 1 \\ \Delta & \Lambda + 2k + 1 \end{pmatrix},
\]
and \( \tilde{\mathcal{M}}_k \) satisfies the pointwise bound
\[
|\tilde{\mathcal{M}}_k X| \lesssim_k \left( \sum_{j=0}^{2k} \langle \rho \rangle^{-2+j-2k} |\nabla^j \Psi| \right).
\]

Proof. Direct computation yields the following formulae
\[
[\Delta^k, V] = \sum_{|\alpha|+|\beta|=2k, |\beta| \leq 2k-1} c_{k,\alpha,\beta} \partial^\alpha V \partial^\beta, \quad [\Delta^k, \Lambda] = 2k \Delta^k.
\]
Hence, by Lemma A.1, since \( \partial^k (\rho_{n}^{p-1}) = O(\rho^{-2-k}) \) as \( \rho \to \infty \) for all \( k \),
\[
\Delta^k (\Delta + pu_n^{p-1}) \Psi = \Delta \Psi_k + O \left( \sum_{j=0}^{2k} \langle \rho \rangle^{-2+j-2k} |\nabla^j \Psi| \right)
\]
and
\[
\Delta^k \Lambda \Omega = (\Lambda + 2k) \Omega_k, \quad \Delta^k (\Lambda + 1) \Omega = (\Lambda + 2k + 1) \Omega_k.
\]

5.1. Subcoercivity. Let us introduce some notations. For \( \gamma > 0 \), define the weighted \( L^2 \) space \( L^2_\gamma \) as the completion of \( C_c^\infty(\mathbb{R}^d) \) with respect to the norm induced by the inner product
\[
(\Psi, \tilde{\Psi})_\gamma = \int_{\mathbb{R}^d} \Psi \tilde{\Psi} \langle \rho \rangle^{-2\gamma} dy
\]
where \( \langle \cdot \rangle \) denotes the Japanese bracket. For \( k \in \mathbb{N} \) and \( 2k + 1 < \gamma \leq 2k + 2 \), define the inner product
\[
\langle \langle \Psi, \tilde{\Psi} \rangle \rangle = - (\Delta \Psi_k, \tilde{\Psi}_k) + (\Psi, \tilde{\Psi})_\gamma
\]
where
\[
(\Psi, \tilde{\Psi}) = \int_{\mathbb{R}^d} \Psi \tilde{\Psi} dy.
\]
Then denote by \( \mathbb{H}_\Psi \) the completion of \( C_c^\infty(\mathbb{R}^d) \) with respect to the norm induced by the inner product \( \langle \langle \cdot, \cdot \rangle \rangle \).

Remark 2. It is a classical consequence of Rellich-Kondrachov theorem and Hardy’s inequality that since \( \gamma > 2k + 1 \), the embedding \( \iota : \mathbb{H}_\Psi \hookrightarrow L^2_\gamma \) is compact.

Proof. An improved Hardy’s inequality in Theorem 1.7 from [4] states that for all \( \alpha \in \mathbb{R} \) and \( f \in C_c^\infty(\mathbb{R}^d \setminus B_1(0)) \),
\[
\int_{\mathbb{R}^d} \frac{|f|^2}{|y|^{2+\alpha}} dy \lesssim \int_{\mathbb{R}^d} \frac{|\nabla f|^2}{|y|^\alpha} dy.
\]
Also an improved Hardy-Rellich inequality in Remark 1.10 from [4] states that for all \( \beta \in \mathbb{R} \) and \( f \in C_\infty^0(\mathbb{R}^d \setminus B_1(0)) \)
\[
\int_{\mathbb{R}^d} \frac{|f|^2}{|y|^{4+\beta}} \, dy \lesssim \int_{\mathbb{R}^d} \frac{|\Delta f|^2}{|y|^2} \, dy.
\]
By repeatedly applying these inequalities, we infer for all \( \Psi \in H_\Psi \),
\[
\|\Psi\|_{L^2_{2k+1}(\mathbb{R}^d \setminus B_1(0))} \lesssim \int_{\mathbb{R}^d \setminus B_1(0)} \frac{|\Psi|^2}{|y|^{2(2k+1)}} \, dy \lesssim \int_{\mathbb{R}^d \setminus B_1(0)} \frac{|\Delta \Psi|^2}{|y|^{2(2k-1)}} \, dy
\]
\[
\lesssim \cdots \lesssim \int_{\mathbb{R}^d \setminus B_1(0)} \frac{|\Lambda_k \Psi|^2}{|y|^2} \, dy \lesssim \int_{\mathbb{R}^d \setminus B_1(0)} |\nabla^{2k+1} \Psi|^2 \, dy \lesssim \|\Psi\|^2_{H_\Psi}.
\]
By Rellich-Kondrachov theorem, the embedding
\[
\iota : H_\Psi \hookrightarrow L^2_{loc,\gamma} := \{ \Psi \mid \chi \Psi \in L^2_\gamma \text{ for all } \chi \in C_\infty(\mathbb{R}^d) \}
\]
is compact. Combining the two and using smallness of \( \langle \rho \rangle^{-\gamma-2k+1} \) for large \( \rho \), result follows.

**Lemma 5.2** (Subcoercivity estimate). There exist \( c > 0 \) and \( 0 < \mu_n \) with \( \lim_{n \to \infty} \mu_n = \infty \) and \( \Pi_n \in H_\Psi \), \( c_n > 0 \) such that for all \( n \geq 0 \), \( \Psi \in H_\Psi \),
\[
\langle \langle \Psi, \Psi \rangle \rangle \geq \mu_n \sum_{j=0}^{2k} \int_{\mathbb{R}^d} |\nabla^j \Psi|^2 \langle \rho \rangle^{-2\gamma - 2k + 1} \, dy - c_n \sum_{i=1}^{n} (\Psi, \Pi_i)^2.
\]

**Proof.** Given \( T \in L^2_\gamma \), the antilinear map \( h \mapsto \langle (T, h) \rangle \) is continuous on \( H_\Psi \) since
\[
\langle \langle \Psi, \Psi \rangle \rangle \gamma \leq \langle \langle \Psi, \Psi \rangle \rangle.
\]
By Riesz, there exists a unique \( L(T) \in H_\Psi \) such that
\[
\forall h \in H_\Psi, \quad \langle \langle L(T), h \rangle \rangle = \langle (T, h) \rangle \gamma
\]
and \( L : L^2_\gamma \to H_\Psi \) is a bounded linear map. By Remark 2, the map \( \iota \circ L : L^2_\gamma \to L^2_\gamma \) is compact. If \( \Psi_i = L(T_i) \), \( i = 1, 2 \), then
\[
(L(T_1), T_2)_{\gamma} = \langle \Psi_1, T_2 \rangle_{\gamma} = \langle \langle \Psi_1, L(T_2) \rangle \rangle = \langle \langle \Psi_1, \Psi_2 \rangle \rangle.
\]
Similarly,
\[
(T_1, L(T_2))_{\gamma} = \langle \langle \Psi_1, \Psi_2 \rangle \rangle = \langle (L(T_1), T_2) \rangle_{\gamma}
\]
i.e. \( L \) is self-adjoint on \( L^2_\gamma \). Since \( L > 0 \) from (5.7), there exists an \( L^2_\gamma \)-orthonormal eigenbasis \( (\Pi_{n,i})_{1 \leq i \leq I(n)} \) of \( L \) with positive eigenvalues \( \lambda_n \to 0 \). The eigenvalue equation implies \( \Pi_{n,i} \in H_\Psi \). Let
\[
A_n = \{ \Psi \in H_\Psi \mid (\Psi, \Psi)_{\gamma} = 1, (\Psi, \Pi_{j,i})_{\gamma} = 0, 1 \leq i \leq I(j), 1 \leq j \leq n \}
\]
and consider the minimization problem
\[
I_n = \inf_{\Psi \in A_n} \langle \langle \Psi, \Psi \rangle \rangle,
\]
whose infimum is attained at \( \Psi \in A_n \) since \( L \) is compact. Also, by a standard Lagrange multiplier argument,
\[
\forall h \in H_\Psi, \quad \langle \langle \Psi, h \rangle \rangle = \sum_{j=1}^{n} \sum_{i=1}^{I(j)} \beta_{i,j} (\Pi_{j,i}, h)_{\gamma} + \beta(\Psi, h)_{\gamma}.
\]
Set $h = \Pi_j, i$ and since $\Pi_j, i$ is an eigenvector of $L$, we infer $\beta_{j, i} = 0$ and in view of (5.7), $L(\Psi) = \beta^{-1}\Psi$. Together with the orthogonality conditions, $\beta^{-1} \leq \lambda_{n+1}$. Hence

$$I_n = \langle \langle \Psi, \Psi \rangle \rangle = \beta(\Psi, \Psi) \geq \frac{1}{\lambda_{n+1}}.$$  \hspace{1cm} (5.8)

For all $\varepsilon > 0, k \geq 1$, from Gagliardo-Nirenberg interpolation inequality with weight (see [14]) together with Young’s inequality, we infer

$$2 \sum_{j=0}^{2k} \int_{\mathbb{R}^d} |\nabla_j \Psi|^2(\rho)^{-2\gamma \frac{2k+1}{2k+1+1}} dy \leq \varepsilon \int_{\mathbb{R}^d} |\nabla^{2k+1} \Psi|^2 dy + c_{\varepsilon, k} \int_{\mathbb{R}^d} |\Psi|^2(\rho)^{-2\gamma} dy.$$

Together with (5.8), we have that for all $\Psi$ satisfying orthogonality condition of $\mathcal{A}_n$ and $\delta > 0$,

$$2 \sum_{j=0}^{2k} \int_{\mathbb{R}^d} |\nabla_j \Psi|^2(\rho)^{-2\gamma \frac{2k+1}{2k+1+1}} dy \leq (\varepsilon + c_{\varepsilon, k} \lambda_{n+1}) \langle \langle \Psi, \Psi \rangle \rangle.$$

Choosing $\varepsilon_n \to 0$ such that $c_{\varepsilon_n, k} \lambda_{n+1} \leq \varepsilon_n$ yields (5.6). \hspace{1cm} \Box

5.2. Dissipativity. We now turn to the fundamental dissipativity property.

Hilbert space. Let $\mathbb{H}_{2k}$ be the completion of $C_c^\infty(\mathbb{R}^d, \mathbb{R}^2)$ with respect to the norm associated to inner product:

$$\langle X, \tilde{X} \rangle = -(\Delta \Psi_k, \tilde{\Psi}_k) + (\Omega_k, \tilde{\Omega}_k) + (\Psi, \tilde{\Psi})_\gamma + (\Omega, \tilde{\Omega})_\gamma =: \langle X, \tilde{X} \rangle_1 \hspace{1cm} \hspace{1cm} := \langle X, \tilde{X} \rangle_2.$$

Further, define the domain of $\mathcal{M}$

$$D(\mathcal{M}) = \{ X \in \mathbb{H}_{2k} \mid \mathcal{M}X \in \mathbb{H}_{2k} \}$$

which is a Banach space equipped with the graph norm

$$\|X\|_{D(\mathcal{M})} = \|X\|_{\mathbb{H}_{2k}} + \|\mathcal{M}X\|_{\mathbb{H}_{2k}}.$$

Spherical harmonics. Denote by $\Delta_{S^{d-1}}$ the Laplace-Beltrami operator defined on a unit sphere $S^{d-1}$. Then we can write

$$\Delta = \frac{\partial^2}{\partial^2 \rho} + \frac{d-1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \Delta_{S^{d-1}} =: \mathcal{L} + \rho^{-2} \Delta_{S^{d-1}}.$$

Denote by $Y_l^{(m)}$ the orthonormal $\Delta_{S^{d-1}}$-eigenbasis (e.g. spherical harmonics if $d = 3$) of $L^2(S^{d-1})$ with discrete eigenvalues $-\lambda_m = -m(m + d - 2)$ for $m \geq 0$. Define the space of test functions

$$\mathcal{D} = \left\{ X = \sum_{l, m} X_{l, m}(\rho) Y_l^{(m)} \text{ a finite sum and smooth} \right\}.$$

Note then, that $\mathcal{D}$ is dense in $\mathbb{H}_{2k}$.
Dissipativity. We will first prove dissipativity in the space of test functions
\[ \mathcal{D}_R = \left\{ X \in C^\infty(\mathbb{R}^d, \mathbb{R}^2) \mid \sup_{m=0}^{2k+3} \sum_{\mathbb{R}^d} \rho^{a+R+m} |\nabla^m X| \leq \infty, (-\mathcal{M} + R)X \in \mathcal{D} \right\} \]
and argue that the result extends to \( \mathbb{H}_{2k} \).

**Proposition 5.3** (Maximal dissipativity). There exists \( c > 0 \), such that for all \( k \in \mathbb{N} \), there exists \( (X_i)_{1 \leq i \leq N} \in \mathbb{H}_{2k} \) such that for the finite rank projection operator
\[ \mathcal{P} = \sum_{i=1}^N \langle \cdot, X_i \rangle X_i, \]
the modified operator
\[ \tilde{\mathcal{M}} = \mathcal{M} - \mathcal{P} \]
is dissipative:
\[ \forall X \in D(\mathcal{M}), \quad \langle -\tilde{\mathcal{M}} X, X \rangle \geq c k \langle X, X \rangle \quad (5.9) \]
and is maximal:
\[ \forall R > 0, \quad F \in \mathbb{H}_{2k}, \quad \exists X \in D(\mathcal{M}) \quad \text{such that} \quad (-\tilde{\mathcal{M}} + R)X = F. \quad (5.10) \]

**Proof. Step 1** (Dissipativity on dense subset): We claim the bound (5.9) for \( X \in \mathcal{D}_R \) for \( R \) sufficiently large so integrating by parts is justified. Integrate by parts the principal part:
\[ \langle -\mathcal{M} X, X \rangle_1 = (\Delta^{k+1}(\mathcal{M}X) \Psi, \Psi_k) - (\Delta^{k}(\mathcal{M}X) \Omega_k) \]
\[ = \int_{\mathbb{R}^d} \nabla((\Lambda + 2k)\Psi_k + \Omega_k) \cdot \nabla \Psi_k + (\Delta \Psi_k - (\tilde{\mathcal{M}}_k X) \Omega_k + (1 + 2k + \Lambda) \Omega_k) \Omega_k \, dy \]
\[ = \int_{\mathbb{R}^d} \nabla((\Lambda + 2k)\Psi_k) \cdot \nabla \Psi_k + (1 + 2k + \Lambda) \Omega_k \Omega_k - (\tilde{\mathcal{M}}_k X) \Omega_k \, dy \]
\[ = (s_c + 2k + 1) \left[(\nabla \Psi_k, \nabla \Psi_k) + (\Omega_k, \Omega_k)\right] - \int_{\mathbb{R}^d} (\tilde{\mathcal{M}}_k X) \Omega_k \, dy \]
where in the last equality, we have used the Pohozaev identity. In view of (5.5) and by Young’s inequality, we infer
\[ \left| \int_{\mathbb{R}^d} (\tilde{\mathcal{M}}_k X) \Omega_k \, dy \right| \leq \varepsilon \int_{\mathbb{R}^d} |\Omega_k|^2 \, dy + c_{\varepsilon, k} \sum_{j=0}^{2k} \int_{\mathbb{R}^d} |\nabla^j \Psi|^2 \langle \rho \rangle^{-4j-4k} \, dy. \]
Taking \( \varepsilon > 0 \) small, it follows that
\[ \langle -\mathcal{M} X, X \rangle_1 \geq c_k \left[ (\nabla \Psi_k, \nabla \Psi_k) + (\Omega_k, \Omega_k) \right] - C_k \sum_{j=0}^{2k} \int_{\mathbb{R}^d} |\nabla^j \Psi|^2 \langle \rho \rangle^{-4j-4k} \, dy. \]
We also lower bound the non-principal part:
\[ \langle -\mathcal{M} X, X \rangle_2 = -\int_{\mathbb{R}^d} (\mathcal{M}X \Psi + (\mathcal{M}X) \Omega) \, dy \]
\[ \geq -\int_{\mathbb{R}^d} \left[ |\Psi|^2 + |y \cdot \nabla \Psi|^2 + |\Delta \Psi|^2 + |\Omega|^2 + |y \cdot \nabla \Omega|^2 \right] \langle \rho \rangle^{-2} \, dy. \]
since \( M \) is first order in \( \Omega \) and second order in \( \Psi \). Adding these, we infer

\[
\langle -MX, X \rangle \geq c k \left[ (\nabla \Psi_k, \nabla \Psi_k) + (\Omega_k, \Omega_k) \right] - \tilde{C}_k \left[ \sum_{j=0}^{2k} \int_{\mathbb{R}^d} |\nabla^j \Psi|^2 (\rho)^{-2\gamma} 2^{k+1-j} \frac{\gamma - 1}{k+1} dy + \sum_{j=0}^{2k-1} \int_{\mathbb{R}^d} |\nabla^j \Omega|^2 (\rho)^{-2\gamma} 2^{k-j} \frac{\gamma - 1}{k+1} dy \right]
\]

since \( 2k + 1 < \gamma \leq 2k + 2 \). We conclude using (5.6) and its analogue for \( \Omega \):

\[
\langle -MX, X \rangle \geq c k \langle X, X \rangle - C \sum_{i=1}^{N} \left[ (\Psi, \Pi_i)_{\gamma} + (\Omega, \Sigma_i)_{\gamma} \right]
\]

for \((\Pi_i)\) as in Lemma 5.2 and some \((\Sigma_i) \in \mathbb{H}_\Omega\). Since the linear form

\[
X = (\Psi, \Omega) \mapsto \sqrt{C}(\Psi, \Pi)_\gamma
\]

is continuous on \( \mathbb{H}_{2k} \), by Riesz theorem, there exists \( X_i \in \mathbb{H}_{2k} \) such that

\[
\forall X \in \mathbb{H}_{2k}, \quad \langle X, X_i \rangle = (\Psi, \Pi)_\gamma
\]

and similarly for \( \Sigma_i \). Hence, the claim follows.

**Step 2** (ODE formulation of maximality): Next, we claim that for all \( R \) sufficiently large,

\[
\forall F \in \mathcal{D}, \quad \exists! X \in \mathbb{H}_{2k} \text{ such that } (-M + R)X = F. \tag{5.11}
\]

Furthermore, we claim that \( X \in \mathcal{D}_R \). Note that this is equivalent to

\[
\begin{cases}
(\Lambda + R)\Psi + \Omega = F_{\Psi} \\
(\Delta + pu_n^{p-1})\Psi + (\Lambda + R + 1)\Omega = F_{\Omega}.
\end{cases} \tag{5.12}
\]

Solving for \( \Psi \), we have

\[
[\Delta - (\Lambda + R + 1)(\Lambda + R) + pu_n^{p-1}]\Psi = \underbrace{F_{\Omega} - (\Lambda + R + 1)F_{\Psi}}_{:= H}.
\tag{5.13}
\]

Since \( \Lambda \) commutes with \( \Delta_{d-1} \), we can write

\[
F = \sum_{l,m} F_{l,m} Y^{(l,m)}; \quad H = \sum_{l,m} H_{l,m} Y^{(l,m)}
\]

as a finite sum where \( H_{l,m} \in C_{c,\text{rad}}(\mathbb{R}^d) \). Then the solution is of the form

\[
\Psi = \sum_{l,m} Y^{(l,m)} \Psi_{l,m}; \quad [\mathcal{L} - \rho^{-2}\lambda_m - (\Lambda + R + 1)(\Lambda + R) + pu_n^{p-1}]\Psi_{l,m}(\rho) = H_{l,m}(\rho). \tag{5.14}
\]

By Lemma B.2, it follows that for all \( R \) sufficiently large and \( F_{l,m} \in C_{c,\text{rad}}(\mathbb{R}^d, \mathbb{R}^2) \) such that \( F_{l,m}(\rho)Y^{(l,m)} \) is smooth at the origin, there exists unique \( \Psi_{l,m} \in H_{\text{rad}}^{2k+1}(\mathbb{R}^d) \) solution to (5.14). Hence, there exists a unique \( \Omega_{l,m} \in H_{\text{rad}}^{2k}(\mathbb{R}^d) \) given by first equation of (5.12) so that \( X_{l,m} = (\Psi_{l,m}, \Omega_{l,m}) \in \mathbb{H}_{2k} \) smooth. Thus, we have (5.11) and \( X_{l,m}(\rho)Y^{(l,m)} \) is smooth. Also, from the decay properties of each \( X_{l,m} \) proved in Lemma B.2, we infer \( X \in \mathcal{D}_R \).

Now, we extend these results from \( \mathcal{D}_R \) to \( D(M) \). Claim that for \( R \) large, \( \mathcal{D}_R \subset D(M) \).
is dense. For $X \in D(\mathcal{M})$, we have $X, MX \in \mathbb{H}_{2k}$ so there exists a sequence $(Y_n) \in \mathcal{D}$ such that

$$Y_n \to (-\mathcal{M} + R)X \quad \text{in } \mathbb{H}_{2k}.$$ 

By (5.11) and Lemma B.2, there exists unique $X_n \in \mathbb{H}_{2k}$ smooth solution to

$$(-\mathcal{M} + R)X_n = Y_n \to (-\mathcal{M} + R)X, \quad X_n \in \mathbb{H}_{2k}.$$ 

It suffices to prove the $X_n \to X$ in $\mathbb{H}_{2k}$. Recall that for $R$ sufficiently large all integration by parts used to prove (5.9) is justified. Then since $X_n \in \mathcal{D}_R$, (5.9) holds for $X_n - X_m$ i.e.

$$\langle Y_n - Y_m, X_n - X_m \rangle = \langle (-\mathcal{M} + R)(X_n - X_m), X_n - X_m \rangle \equiv \langle (-\mathcal{M} + \mathcal{P})(X_n - X_m), X_n - X_m \rangle - \langle \mathcal{P}(X_n - X_m), X_n - X_m \rangle + R\|X_n - X_m\|_{\mathbb{H}_{2k}}^2 \geq R\|X_n - X_m\|_{\mathbb{H}_{2k}}^2 - \langle \mathcal{P}(X_n - X_m), X_n - X_m \rangle.$$ 

Since $\mathcal{P}$ is a bounded operator, we infer for $R$ large,

$$\frac{R}{2}\|X_n - X_m\|_{\mathbb{H}_{2k}} \leq \|Y_n - Y_m\|_{\mathbb{H}_{2k}}.$$ 

In view of the convergence of $(Y_n)$ in $\mathbb{H}_{2k}$, we deduce that $(X_n)$ is a Cauchy sequence hence convergent in $\mathbb{H}_{2k}$ to say, $\tilde{X}$. Then $	ilde{X} - X \in \mathbb{H}_{2k}$ and

$$(-\mathcal{M} + R)(\tilde{X} - X) = 0$$

as distributions. By the uniqueness statement in (5.11), it follows that $\tilde{X} = X$ i.e.

$$X_n \to X, \quad MX_n \to MX \quad \text{in } \mathbb{H}_{2k} \iff X_n \to X \quad \text{in } D(M).$$

Hence, $\mathcal{D}_R$ is dense in $D(\mathcal{M})$ as claimed.

**Step 3** (Conclusion): Since (5.9) holds for all $X \in \mathcal{D}_R$, by density of $\mathcal{D}_R$, we have dissipativity i.e. (5.9) holds for all $X \in D(\mathcal{M})$. It remains to prove (5.10). Let $F \in \mathbb{H}_{2k}$. There exists $(F_n) \in \mathcal{D}$ such that

$$F_n \to F \quad \text{in } \mathbb{H}_{2k}.$$ 

By (5.11), there exists $X_n \in \mathbb{H}_{2k}$ solution to

$$(-\mathcal{M} + R)X_n = F_n.$$ 

Using (5.9) and arguing as in the proof of density, we infer for $R$ large,

$$\frac{R}{2}\|X_n - X_m\|_{\mathbb{H}_{2k}} \leq \|F_n - F_m\|_{\mathbb{H}_{2k}}$$

so $X_n$ has a limit say, $X \in \mathbb{H}_{2k}$. Since $F_n$ converges to $F$ in $\mathbb{H}_{2k}$ and $D(\mathcal{M})$ is a Banach space, we infer

$$(-\mathcal{M} + R)X = F, \quad X \in D(M).$$

Thus we have shown that for $R$ large,

$$\forall F \in \mathbb{H}_{2k}, \quad \exists X \in D(\mathcal{M}) \quad \text{such that} \quad (-\mathcal{M} + R)X = F. \quad \text{(5.15)}$$

Now we prove this for $\tilde{M}$. Let $F \in \mathbb{H}_{2k}$. Since $\mathcal{P}$ is bounded, for $R$ large, by (5.9), for $X$ as in (5.15),

$$\langle F, X \rangle = \langle (-\mathcal{M} + R)X, X \rangle = \langle (-\tilde{M} - \mathcal{P} + R)X, X \rangle \geq \frac{R}{2}\|X\|_{\mathbb{H}_{2k}}^2.$$
Thus, for all $F \in \mathbb{H}_{2k}$, solution $X$ to (5.15) is unique i.e. $(-\mathcal{M} + R)^{-1}$ is well-defined on $\mathbb{H}_{2k}$ with
\[
\|(-\mathcal{M} + R)^{-1}\| \lesssim \frac{1}{R}.
\]
Hence,
\[
-\tilde{\mathcal{M}} + R = -\mathcal{M} + \mathcal{P} + R = (-\mathcal{M} + R)[\text{id} + (-\mathcal{M} + R)^{-1}\mathcal{P}]
\]
is invertible on $\mathbb{H}_{2k}$ for $R$ large which yields (5.10). An elementary induction argument ensures that (5.10) holds for all $R > 0$ (see Proposition 3.14 from [9]).

**Remark 3.** Conclusion of Proposition 5.3 holds for $\mathbb{H}_{2k}^\text{rad} = C^\infty_c(\mathbb{R}^d, \mathbb{R}^2)$ in place of $\mathbb{H}_{2k}$ i.e. maximal dissipativity holds for radial functions.

### 6. Growth bounds for the dissipative operators

In this section, we recall some classical facts on growth bounds for compact perturbations of maximal accretive operators.

In this section, $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a Hilbert space and $A$ a closed operator defined on a dense domain $D(A)$. Define the adjoint operator $A^*$ on the domain
\[
D(A^*) = \{ X \in \mathcal{H} \mid Y \in D(A) \mapsto \langle X, AY \rangle \text{ extends to an element of } \mathcal{H}^* \}
\]
to be $X \mapsto A^*X$ the unique element of $\mathcal{H}$ given by Riesz theorem such that
\[
\forall Y \in D(A), \quad \langle A^*X, Y \rangle = \langle X, AY \rangle.
\]
Denote by
\[
\Lambda_\nu(A) = \{ \lambda \in \sigma(A) \mid \langle \lambda \rangle \geq \nu \}, \quad V_\nu(A) = \bigoplus_{\lambda \in \Lambda_\nu(A)} \ker(A - \lambda).
\]

**Lemma 6.1** (Perturbative exponential decay). Let $T_0$ and $T$ be the strongly continuous semigroup generated by a maximal dissipative operator $A_0$ and $A = A_0 + K$ where $K$ is a compact operator on $\mathcal{H}$. Then for all $\nu > 0$, the following holds:

(i) The set $\Lambda_\nu(A)$ is finite and each eigenvalue $\lambda \in \Lambda_\nu(A)$ has finite algebraic multiplicity $k_\lambda$.

We have $\Lambda_\nu(A) = \overline{\Lambda_\nu(A^*)}$ and $\dim V_\nu(A^*) = \dim V_\nu(A)$. The direct sum decomposition
\[
\mathcal{H} = V_\nu(A) \bigoplus V_\nu^\perp(A^*)
\]
is preserved by $T(s)$ and there holds
\[
\forall X \in V_\nu^\perp(A^*), \quad \| T(s)X \| \leq M_\nu e^{\nu s} \| X \|.
\]

(iii) The restriction of $A$ to $V_\nu(A)$ is given by a direct sum of Jordan blocks. Each block corresponds to an invariant subspace $J_\lambda$ and the semigroup $T$ restricted to $J_\lambda$ is
given by
\[
T(s)|_{J_\lambda} = \begin{pmatrix}
e^{\lambda s} & s e^{\lambda s} & \cdots & s^{m_\lambda - 1} e^{\lambda s} \\
0 & e^{\lambda s} & \cdots & s^{m_\lambda - 2} e^{\lambda s} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{\lambda s}
\end{pmatrix}
\]
where \(m_\lambda\) is the geometric multiplicity of the eigenvalue \(\lambda\).

Proof. See Lemma 3.9 of [15]. \(\square\)

Corollary 6.2 (Exponential decay modulo finitely many instabilities). Let \(\nu > 0\), \(T_0\), \(T\) be the strongly continuous semigroup generated by a maximal dissipative operator \(A_0\) and \(A = A_0 - \nu + K\) respectively where \(K\) is a compact operator on Hilbert space \(H\). Then \(\Lambda_0(A)\) is finite and let

\[
H = U \bigoplus V
\]
where \(U\) and \(V\) are invariant subspaces for \(A\) and \(V\) is the image of the spectral projection of \(A\) for the set \(\lambda_\nu(A)\). Then there exists \(C, \delta > 0\) such that

\[
\forall X \in U, \quad \|T(s)X\| \leq Ce^{-\frac{\delta}{2}s}\|X\|.
\]

Proof. We apply Lemma 6.1 to \(\tilde{A} = A_0 + K\) which generates the semigroup \(\tilde{T}\). Note that \(\Lambda_\nu(\tilde{A})\) is finite and \(\Lambda_0(A) \subset \Lambda_\nu(\tilde{A})\). Let

\[
H = U_\nu \bigoplus V_\nu
\]
be the invariant decomposition of \(\tilde{A}\) associated to the set \(\Lambda_\nu\). Then \(U_\nu \subset U\) and

\[
U = U_\nu \bigoplus O_\nu
\]
where \(O_\nu\) is the image of the spectral projection of \(A\) associated with the set \(\Lambda_\nu(\tilde{A}) \setminus \Lambda_0(A)\). Then by Lemma 6.1,

\[
\forall X \in U_\nu, \quad \|T(s)X\| = e^{-\nu s}\|\tilde{T}(s)X\| \leq M_\nu e^{-\frac{3\nu}{4}s}\|X\|.
\]
Now for \(X \in U\), since \(U_\nu\) is invariant under \(T\) and we have exponential decay on \(U_\nu\), assume \(X \in O_\nu\). \(O_\nu\) is an invariant subspace of \(A\) generated by the eigenvalues \(\lambda\) such that \(-\frac{3\nu}{4} \leq \Re(\lambda) < 0\). Then for

\[
\delta = \inf \left\{ \Re(\lambda) \mid 0 < \Re(\lambda) \leq \frac{3\nu}{4} \right\}
\]

Lemma 6.1 implies that

\[
\|T(s)X\|_{O_\nu} \leq \sup_{\Re(\lambda) < 0} e^{\lambda s} s^{m_\lambda - 1}\|X\| \leq e^{-\frac{\delta}{2}s}\|X\|.
\]

\(\square\)

Corollary 6.3. Let \(A, \delta, U\) and \(V\) as in Corollary 6.2. For \(c, s_0 > 0\), let \(F(s) \in V\) such that

\[
\|F\| \leq e^{-\frac{\delta}{2}(1+c)s}.
\]

If \(X(s)\) solves

\[
\frac{dX(s)}{ds} = AX(s) + F(s), \quad X(s_0) = x \in V
\]
for some \( \|x\| \leq e^{-\frac{\delta}{2}(1+\frac{c}{2})s_0} \), then

\[
\|X(s)\| \leq e^{-\frac{\delta}{2}s}, \quad s_0 \leq s \leq s_0 + \Gamma_{A,s_0}
\]

where \( \Gamma_{A,s_0} \) can be made arbitrarily large by a choice of \( s_0 \). Moreover, there exists \( x \in V, \|x\| \leq e^{-\frac{\delta}{2}(1+\frac{c}{2})s_0} \) such that for all \( s \geq s_0 \),

\[
\|X(s)\| \leq e^{-\frac{\delta}{2}(1+\frac{c}{2})s}.
\]  

**Proof.** By Lemma 6.1, the subspace \( V \) can be further decomposed into invariant subspaces on which \( A \) is represented by Jordan blocks. Therefore, without loss of generality, assume that \( V \) is irreducible and for \( \Re(\Lambda) \geq 0 \),

\[
A = \lambda + N, \quad e^{sN} = \begin{pmatrix}
1 & t & \ldots & t^{m_\lambda-1} \\
0 & 1 & \ldots & t^{m_\lambda-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}.
\]

Then from the growth bound on the Jordan block, we infer, for all \( s_0 \leq s \leq s_0 + \Gamma \) that

\[
\|X(s)\| = \left\| e^{(s-s_0)A} x + \int_{s_0}^{s} e^{(s-\tau)A} F(\tau) \, d\tau \right\|
\]

\[
\lesssim \Gamma^{m_\lambda-1} e^{\Re(\Lambda)\Gamma} e^{-\frac{\delta}{2}(1+\frac{c}{2})s_0} + \int_{s_0}^{s} |\tau - s_0|^{m_\lambda-1} e^{\Re(\Lambda)(s-\tau)} e^{-\frac{\delta}{2}(1+c)\tau} \, d\tau
\]

\[
\lesssim \Gamma^{m_\lambda-1} e^{\Re(\Lambda)\Gamma} e^{-\frac{\delta}{2}(1+\frac{c}{2})s_0}.
\]

Hence (6.1) follows by choosing \( \Gamma \) such that

\[
\Gamma^{m_\lambda-1} e^{\Re(\Lambda)\Gamma} e^{-\frac{\delta}{2}(1+\frac{c}{2})s_0} \leq e^{-\frac{\delta}{2}(s_0 + \Gamma)},
\]

a sufficient condition being

\[
\Gamma \leq \frac{s_0}{2} \left[ \frac{c \delta}{2 \Re(\Lambda) + \delta} \right].
\]

Now consider

\[
Y(s) = e^{-sN} e^{\frac{\delta}{2}(1+\frac{c}{2})s} X(s), \quad \tilde{F}(s) = e^{-sN} e^{\frac{\delta}{2}(1+\frac{c}{2})s} F(s).
\]

Then since \( N \) and \( A \) commute,

\[
\frac{dY(s)}{ds} = \left[ \lambda + \frac{\delta}{2} \left( 1 + \frac{3c}{4} \right) \right] Y(s) + \tilde{F}(s), \quad Y(s_0) = y.
\]

For \( s_0 \) sufficiently large, for all \( s \geq s_0 \),

\[
\|\tilde{F}(s)\| \leq \left[ \Re(\Lambda) + \frac{\delta}{4} \left( 1 + \frac{3c}{4} \right) \right] e^{-\frac{\delta}{4}s}.
\]

We now run a standard Brouwer type argument for \( Y \). For \( \|y\| \leq 1 \), define the exit time

\[
s^* = \inf\{s \geq s_0 \mid \|Y(s)\| \geq 1\}.
\]
If $s^* = \infty$ for some $\|y\| \leq 1$, then we’re done. Otherwise, the map $\Phi : B = \{\|y\| \leq 1\} \to S = \{\|y\| = 1\}$ given by $\Phi(y) = Y(s^*)$ is well-defined. Note that $\Phi|_S = \text{id}_S$ and $\Phi$ is continuous since

$$
\frac{d\|Y\|^2}{ds}(s^*) = 2\Re(\lambda) + \frac{\delta}{2} \left( 1 + \frac{3\epsilon}{4} \right) + 2\Re(\tilde{F}(s^*), Y(s^*)) \geq \frac{\delta}{4} \left( 1 + \frac{3\epsilon}{4} \right) > 0
$$

i.e. the outgoing condition is met. This is a contradiction by Brouwer fixed point theorem. Thus, there exists $x$ such that for all $s \geq s_0$,

$$
\|e^{-sN}X(s)\| \leq e^{-\frac{\delta}{2}(1+\frac{3\epsilon}{4})s}.
$$

Since $e^{-sN}$ is invertible with inverse $e^{sN}$ bounded by $s^{m\lambda-1}$, result follows immediately. \(\square\)

7. Finite codimensional stability

We are now in position to prove non linear finite codimensional stability of the self similar profiles for the full problem.

In this section, we assume (1.6) holds and set $k = 1$ for radial problem and assume (1.7) holds and set $k = 2$ for non-radial problem so that in either case, $\mathcal{H}^{2k+1}(\mathbb{R}^d)$ is an algebra which we shall later use in the proof of Theorem 2. Also, set $\gamma = \frac{2k + 5}{4}$. For convenience, we write $\mathbb{H} = \mathbb{H}_2^\text{rad}$ in radial case and $\mathbb{H} = \mathbb{H}_4$ in the non-radial case. Recall from Proposition 5.3 and Remark 3 that $\mathcal{M} - \mathcal{P} + ck$ is maximal dissipative so Corollary 6.2 applies:

$$\Lambda_0(\mathcal{M}) = \{ \lambda \in \sigma(\mathcal{M}) \mid \Re(\lambda) \geq 0 \}$$

is a finite set with an associated finite dimensional invariant subspace $V$. Consider the invariant decomposition

$$\mathbb{H} = U \bigoplus V$$

and let $P$ be the associated projection on $V$. We denote by $\mathcal{N}$ the nilpotent part of the matrix representing $\mathcal{M}$ on $V$. Let $\delta > 0$ such that the conclusions of Corollary 6.2 and 6.3 hold.

We produce a finite energy initial value by dampening the tail of the self-similar profiles on $|x| \geq 1$: for some large constant $n_p$, let $\eta : \mathbb{R}_+ \to \mathbb{R}$ be a smooth function

$$\eta(r) = \begin{cases} 
1 & r \leq 1 \\
2 & r \geq 2 \end{cases}$$

and define the dampened profile

$$u_n^D(s, \rho) = \eta(e^{-s}\rho)u_n(\rho).$$

We introduce the perturbation variables $(\Psi^D, \Omega^D)$:

$$\tilde{\Psi} = \Psi + u_n = \Psi^D + \eta(e^{-s}\rho)u_n, \quad \Omega + \Lambda u_n = \Omega^D + \eta(e^{-s}\rho)\Lambda u_n.$$
Then the wave equation (5.1) yields
\[
\begin{align*}
\partial_t \Psi^D &= -\Lambda \Psi^D - \Omega^D \\
\partial_t \Omega^D &= -\Delta \Psi - (\Lambda + 1)\Omega^D - \Psi\rho.
\end{align*}
\tag{7.1}
\]

7.1. Bootstrap bound and proof of Theorem 2. The heart of the proof of Theorem 2 is the following bootstrap proposition.

**Proposition 7.1** (Bootstrap). Assume either

(i) Radial perturbation. (1.6) holds, \(k = 1\) and \(\mathcal{H}_2 = \mathbb{H}^{rad}_2\) or

(ii) Non-radial perturbation. (1.7) holds, \(k = 2\) and \(\mathcal{H}_2 = \mathbb{H}_4\)

and \(\gamma = 2k + \frac{5}{4}\). Given \(c \ll 1\) and \(s_0 \gg 1\) to be chosen in the proof, consider \(X(s_0) \in \mathcal{H}\) such that
\[
\|(I - P)X(s_0)\|_\mathcal{H} \leq e^{-\frac{s}{2}s_0}, \quad \|PX(s_0)\|_\mathcal{H} \leq e^{-\frac{s}{2}(1+\frac{4}{3})s_0}
\tag{7.2}
\]
and for all \(0 \leq j \leq 2k + 1\),
\[
\left\|\left(\rho\right)^{j+1} \nabla_j \Psi^D(s_0)\right\|_{L^\infty(\mathbb{R}^d)} + \left\|\left(\rho\right)^{j+1} \nabla_j^{-1} \Omega^D(s_0)\right\|_{L^\infty(\mathbb{R}^d)} \leq 1
\tag{7.3}
\]
where we have used the convention \(\nabla^{-1} = 0\).

Define the exit time \(s^*\) to be the maximal time such that the following bootstrap bounds hold on \(s \in [s_0, s^*]\):
\[
\|e^{\Lambda N}PX(s)\|_\mathcal{H} \leq e^{-\frac{s}{2}(1+\frac{4}{3})s},
\tag{7.4}
\]
for all \(0 \leq j \leq k\),
\[
\left\|\left(\rho\right)^{j} \frac{\Psi}{\rho} \nabla_j \Psi^D(s)\right\|_{L^\infty(|y| \geq 1)} + \left\|\left(\rho\right)^{j} \frac{\Psi}{\rho} \nabla_j^{-1} \Omega^D(s)\right\|_{L^\infty(|y| \geq 1)} \leq 1
\tag{7.5}
\]
for all \(0 \leq j \leq 2k + 1\),
\[
I_j(s) := e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-sp})^{2n_p+1} \left( |\nabla^j \Psi^D(s)|^2 + |\nabla^{-1} \Omega^D(s)|^2 \right) dy \leq 1
\tag{7.6}
\]
where
\[
\xi(r) = \eta(r)^{-\frac{1}{\Delta r}} = \begin{cases} 1 & r \leq 1 \\ r & r \geq 2 \end{cases}
\]
and for \(\frac{\delta}{1+c} < \delta_0 < \delta\),
\[
\|X(s)\|_\mathcal{H} \leq e^{-\frac{s}{2}s_0}.
\tag{7.7}
\]
Then the bootstrap bounds (7.5), (7.6) and (7.7) can be strictly improved in \(s \in [s_0, s^*]\). Equivalently, if \(s^* < \infty\), then equality holds for (7.4) at \(s = s^*\). Furthermore, the following non-linear bound holds:
\[
\forall s \in [s_0, s^*], \quad \|G(s)\|_\mathcal{H} \leq e^{-\frac{s}{2}(1+c)s}.
\tag{7.8}
\]
Let us assume Proposition 7.1 and conclude the proof of Theorem 2.

**proof of Theorem 2.** We distinguish the radial and non radial statements.

**Proof of Theorem 2 (i):** Radial case. Assume Proposition 7.1 (i) holds. Let $s_0$ be as in Proposition 7.1. Note that the bootstrap bounds (7.6) and (7.7) imply
\[
\int_{\mathbb{R}^d} |\Psi|^2 dy \leq \int_{|y| \leq 1} |\Psi|^2 dy + \int_{|y| \geq 1} \rho^{-2s_0+2n+1}|\Psi|^2 dy < \infty
\]
and
\[
\int_{\mathbb{R}^d} |\nabla^{2k+1}\Psi|^2 dy < \infty.
\]

Then
\[
\|\tilde{\Psi}\|_{H^{2k+1,1}(\mathbb{R}^d)}^2 \leq \|u_n^D\|_{L^2(\mathbb{R}^d)}^2 + \|\Psi^D\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla^{2k+1}u_n\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla^{2k+1}\Psi\|_{L^2(\mathbb{R}^d)}^2 < \infty.
\]

Similarly for $\Omega^D$. Thus, we infer
\[
\|u_n + \Psi\|_{H^{2k+1,1}(\mathbb{R}^d)} + \|\Lambda u_n - \Omega\|_{H^{2k}(\mathbb{R}^d)} \leq C(s)
\]
for $s \in [s_0, s^\ast]$ so it follows that
\[
\|\Phi\|_{H^{s,c}(\mathbb{R}^d)} + \|\partial_t\Phi\|_{H^{s-1,c}(\mathbb{R}^d)} \leq C(t)
\]
so the bootstrap time is strictly smaller than the life time provided by the standard Cauchy theory as in [12] and [10].

We now conclude from the Brouwer fixed point argument. Note that for all initial data satisfying (7.2) and (7.3) in the space
\[
H = \left\{ (\Psi^D, \Omega^D) \in H^{2k+1,1}_{\text{rad}} \times H^{2k}_{\text{rad}} \bigg| \sum_{j=0}^{2k+1} \|\rho^{j+\alpha+np+1} \nabla^j \Psi^D\|_{L^\infty} + \|\rho^{j+\alpha+np+1} \nabla^j \Omega^D\|_{L^\infty} < \infty \right\},
\]
the non-linear bound (7.8) and (7.4) have been shown to hold on $[s_0, s^\ast]$. Then by Corollary 6.3, $s^\ast \geq s_0 + \Gamma$ for $\Gamma$ large. Moreover, as explained in the proof of Corollary 6.3, after a choice of projection of initial data on the subspace of unstable nodes $PX(s_0)$, the solution can be immediately propagated to any time $t < T$. This choice is dictated by Corollary 6.3.

**Proof of Theorem 2 (ii):** Non radial case. Assume Proposition 7.1 (ii) holds. Then the proof is identical to the proof of Theorem 2 (i) with our initial data lying in the space
\[
H = \left\{ (\Psi^D, \Omega^D) \in H^{2k+1} \times H^{2k} \bigg| \sum_{j=0}^{2k+1} \|\rho^{j+\alpha+np+1} \nabla^j \Psi^D\|_{L^\infty} + \|\rho^{j+\alpha+np+1} \nabla^j \Omega^D\|_{L^\infty} < \infty \right\}.
\]

The rest of this section is devoted to the proof of the bootstrap Proposition 7.1.
7.2. Weighted Sobolev bounds in the radial case. In order to control non linear terms we need weighted Sobolev bounds related to the energy norms of Proposition 7.1. We start with the radial case.

Lemma 7.2 (Radial Sobolev). Let $(\Psi^D, \Omega^D)$ be radial function such that the right hand side of the bound (7.9) is finite. Then, for $0 \leq j \leq k$,

\[
\left\| \frac{\rho^{\frac{j}{4p}} \nabla_j \Psi^D(s)}{u_n^D} \right\|_{L^\infty(|y| \geq 2)} \lesssim |\nabla^j \Psi^D(2)| + \left[ (I_j(s) + I_{j+1}(s)) e^s \right]^{\frac{1}{2}},
\]

\[
\left\| \frac{\rho^{j-\frac{1}{4p}} \nabla^j \Omega^D(s)}{u_n^D} \right\|_{L^\infty(|y| \geq 2)} \lesssim |\nabla^{j-1} \Omega^D(2)| + \left[ (I_j(s) + I_{j+1}(s)) e^s \right]^{\frac{1}{2}}.
\]

Proof. Note that for $0 < r < R$ and $u \in C^1((r, R))$, $\rho_0, \rho \in (r, R)$ and $\lambda > 0$,

\[
|u(\rho)| \leq |u(\rho_0)| + \left( \int_r^R \tau^{-1-2\lambda} d\tau \right)^{\frac{1}{2}} \left( \int_r^R \tau^{1+2\lambda} |\partial_\rho u|^2 d\tau \right)^{\frac{1}{2}}.
\]

Then for all $2 \leq \rho \leq e^s$, since $u_n(\rho) = u_n^D(\rho)$, we apply (7.9) with $\lambda = \frac{1}{4p}$ and $u = \rho^{j+\alpha-\frac{1}{4p}} \nabla^j \Psi^D$ to infer

\[
\| \rho^{j-\frac{1}{4p}} \nabla^j \Psi^D \|_{L^\infty(2 \leq \rho \leq e^s)} \lesssim_n \| \rho^{j+\alpha-\frac{1}{4p}} \nabla^j \Psi^D \|_{L^\infty(2 \leq \rho \leq e^s)}
\]

\[
\leq |\nabla^j \Psi^D(2)| + \left( \int_2^{e^s} \rho^{-1-\frac{1}{4p}} d\rho \right)^{\frac{1}{2}} \left( \int_2^{e^s} \rho^{2j+2\alpha-1} (|\nabla^j \Psi^D|^2 + \rho^2 |\nabla^j+1 \Psi^D|^2) d\rho \right)^{\frac{1}{2}}
\]

\[
\leq |\nabla^j \Psi^D(2)| + \left[ (I_j(s) + I_{j+1}(s)) e^s \right]^{\frac{1}{2}}.
\]

Similarly, for $\rho \geq e^s$, set $\lambda = \frac{1}{2} + \frac{1}{4p}$ and $u = \rho^{j+\alpha-1-\frac{1}{4p}} \xi (e^{-s} \rho)^{n_p + 1} \nabla^j \Psi^D$ in (7.9) to infer

\[
e^{-s} \| \rho^{j-\frac{1}{4p}} \nabla^j \Psi^D \|_{L^\infty(\rho \geq e^s)} \lesssim_n \| \rho^{j+\alpha-1-\frac{1}{4p}} \xi (e^{-s} \rho)^{n_p + 1} \nabla^j \Psi^D \|_{L^\infty(\rho \geq e^s)}
\]

\[
\leq e^{-s} \left( \int_{e^s}^{\infty} \rho^{-2-\frac{1}{4p}} d\rho \right)^{\frac{1}{2}} \left( \int_{e^s}^{\infty} \rho^{2j+2\alpha-1} \xi (e^{-s} \rho)^{2n_p+1} (|\nabla^j \Psi^D|^2 + \rho^2 |\nabla^j+1 \Psi^D|^2) d\rho \right)^{\frac{1}{2}}
\]

\[
\leq e^{-s} \left( |\nabla^j \Psi^D(1)| + (1 + e^{-\frac{1}{4p}}) \left[ (I_j(s) + I_{j+1}(s)) e^s \right]^{\frac{1}{2}} \right)
\]

\[
\lesssim e^{-s} \left( |\nabla^j \Psi^D(1)| + \left[ (I_j(s) + I_{j+1}(s)) e^s \right]^{\frac{1}{2}} \right).
\]

Proof for $\Omega^D$ is identical. \qed
7.3. Weighted Sobolev bounds in the non radial case. In this section, we set $d = 3$, $k = 2$ and $\gamma = \frac{21}{4}$. We write $\mathbb{H} = \mathbb{H}_4$.

**Lemma 7.3 (Non-radial Sobolev).** Let $(\Psi^D, \Omega^D)$ be such that the right hand side of the bound (7.10) is finite. Then, for $0 \leq j \leq 2$,

$$
\left\| \frac{\partial^j}{\partial r^j} \frac{\partial}{\partial \varphi} \Psi^D(s) \right\|_{L^\infty(|\varphi| \geq 1)} \lesssim \left\| \nabla^j \Psi^D \right\|_{L^\infty(|\varphi| = 2)} + \left( \sum_{i=0}^5 I_i(s) e^s \right)^{\frac{1}{2}},
$$

$$
\left\| \frac{\partial^j}{\partial r^j} \frac{\partial}{\partial \varphi} \Omega^D(s) \right\|_{L^\infty(|\varphi| \geq 1)} \lesssim \left\| \nabla^j \Omega^D \right\|_{L^\infty(|\varphi| = 2)} + \left( \sum_{i=0}^5 I_i(s) e^s \right)^{\frac{1}{2}}.
$$

(7.10)

**Proof.** Step 1 (General bound): We claim that given $i \in \mathbb{N}$ and $\beta \in \mathbb{R}$ and for all $f \in C_{c,\text{rad}}(\mathbb{R}^3 \setminus \{0\})$,

$$
\int_{\mathbb{R}^3} r^\beta \left| \nabla^i f(r) Y^{(l,m)}(\theta, \varphi) \right|^2 \, dx = (1 + o_{m \to \infty}(1)) \sum_{j=0}^i \binom{i}{j} \lambda_{m}^{i-j} \int_0^\infty r^{2+2\beta+2(j-i)} |f(j)|^2 \, dr.
$$

(7.11)

We proceed by induction on $i$. Claim for $i = 1, 2$ is proved in Lemma 2.1 from [4]. If claim holds for $i = 2k - 1, 2k$, then by replacing $f$ in (7.11) by $(\mathcal{L} - r^{-2}\lambda_m)f$ we infer

$$
\int_{\mathbb{R}^3} r^\beta \left| \nabla^{i+2} f(r) Y^{(l,m)}(\theta, \varphi) \right|^2 \, dx
$$

$$
= (1 + o_{m \to \infty}(1)) \sum_{j=0}^i \binom{i}{j} \lambda_{m}^{i-j} \int_0^\infty r^{2+2\beta+2(j-i)} \left| \partial_r \right|^2 \left( |f(j+1)|^2 + 2\lambda_m r^{-2} |f(j)|^2 \right) \, dr + o_{m \to \infty}(S_{i+2,m}[f])
$$

where the last equality holds since

$$
-\lambda_{m}^{i-j} \int_0^\infty r^{\beta+2(j-i)} f(i+2) f(j) \, dr = \lambda_{m}^{i-j} \int_0^\infty r^{\beta+2(i+j)} \left| f(j) \right|^2 \, dr + o_{m \to \infty}(S_{i+2,m}[f])
$$

which can easily be checked by integrating by parts

$$
\int_0^\infty r^{\beta+2(j-i)} \partial_r^2 |f(j)|^2 \, dr.
$$

Hence, the result follows for $i + 2$. This concludes the proof of our claim (7.11).

Step 2 (Interior Bound): From the claim, we have that for $M$ large, for all $f \in C_{c,\text{rad}}(\mathbb{R}^3 \setminus \{0\})$ and $m \geq M$,

$$
\sum_{j=0}^i \lambda_{m}^{i-j} \int_0^\infty \rho^{2j+2\alpha-1} |f(j)|^2 \, d\rho \lesssim_i \int_{\mathbb{R}^3} \rho^{2i+2\alpha-3} |\nabla^i f(\rho) Y^{(l,m)}|^2 \, dx.
$$
Also, by induction on $i$, we have

$$
\sum_{j=0}^{i} \int_{0}^{\infty} \rho^{2j+2\alpha-1} |f(j)|^2 \, d\rho \lesssim_{i,m} \sum_{j=0}^{i} \int_{\mathbb{R}^3} \rho^{2j+2\alpha-3} \left| \nabla^j f(\rho) Y^{(l,m)} \right|^2 \, dx.
$$

(7.12)

Thus, (7.12) holds for all $m \in \mathbb{N}$ with some universal constant independent of $m$. By density, we can drop the assumption that $f$ is smooth. Thus, by writing

$$
\Psi^D(y) = \sum_{l,m} \Psi^D_{l,m}(\rho) Y^{(l,m)}(\theta, \varphi),
$$

and multiplying $\Psi^D_{l,m}(\rho)$ by a cut-off $\chi_s \in C^\infty_{\text{rad}}(\mathbb{R}^3)$ such that for $\eta \in C^\infty(\mathbb{R})$ with

$$
\eta(\rho) = \begin{cases} 
0 & \rho \leq 1 \\
1 & \rho \geq 2
\end{cases},
\chi_s(y) = \begin{cases} 
\eta(|y|) & |y| \leq e^s \\
1 - \eta(e^{-s}|y|) & |y| \geq e^s
\end{cases},
$$

and apply (7.12) to $f(\rho) = \chi_s \Psi^D_{l,m}(\rho)$, we infer,

$$
\sum_{j=0}^{5} \lambda^{5-j}_m \int_{2}^{e^s} r^{2j+2\alpha-1} \left| \partial^j \chi_s \Psi^D_{l,m} \right|^2 \, d\rho \\
\leq \sum_{j=0}^{5} \lambda^{5-j}_m \int_{0}^{\infty} r^{2j+2\alpha-1} \left| \partial^j (\chi_s \Psi^D_{l,m}) \right|^2 \, d\rho \\
\lesssim \sum_{j=0}^{5} \int_{\mathbb{R}^3} \rho^{2j-2s} \left| \nabla^j (\chi_s \Psi^D_{l,m}(\rho) Y^{(l,m)}(\theta, \varphi)) \right|^2 \, dy \\
\lesssim \sum_{j=0}^{5} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s})^{2n+1} \left| \nabla^j (\Psi^D_{l,m}(\rho) Y^{(l,m)}(\theta, \varphi)) \right|^2 \, dy
$$

where in the last inequality we have used that for all $e^s \leq \rho \leq e^{2s},$

$$
|\partial^j \chi_s(\rho)| \lesssim_j e^{-js} \lesssim \rho^{-j}.
$$

Since the universal constant here does not depend on $m$, we sum over $l$ and $m$ to infer

$$
\sum_{l,m} \sum_{j=0}^{5} \lambda^{5-j}_m \int_{2}^{e^s} r^{2j+2\alpha-1} \left| \partial^j \Psi^D_{l,m} \right|^2 \, d\rho \lesssim \sum_{j=0}^{5} I_j(s) e^s.
$$

Note the universal $L^\infty$ bound for spherical harmonics which one can find in [16] states that

$$
\|Y^{(l,m)}(\theta, \varphi)\|_{L^\infty(S^2)} \lesssim \lambda_m^{\frac{1}{2}}.
$$
Thus, we infer for $2 \leq |y| \leq e^s$,

$$\left| \frac{\rho^{1-\frac{4s}{5}} \nabla \Psi^D(y)}{u_n^D} \right| \lesssim \| \Psi^D \|_{L^\infty(|y|=2)} + \sum_{l,m} \| Y^{(l,m)} \|_{L^\infty(S^2)} \int_2^{e^s} |\partial_\rho (\rho^{\alpha - \frac{1}{4s}} \Psi^D_{l,m})| \, d\rho$$

$$\lesssim \| \Psi^D \|_{L^\infty(|y|=2)} + \sum_{l,m} \frac{1}{\lambda_{l,m}^2} \left( \int_2^{e^s} \rho^{-\frac{1}{5s}} \, d\rho \right)^{\frac{1}{2}} \left( \int_2^{e^s} \rho^{2\alpha - 1} (|\Psi^D_{l,m}|^2 + \rho^2 |\partial_\rho \Psi^D_{l,m}|^2) \, d\rho \right)^{\frac{1}{2}}$$

$$\lesssim \| \Psi^D \|_{L^\infty(|y|=2)} + \left( \sum_{l,m} \lambda_{l,m}^{-\frac{3}{2}} \int_2^{e^s} \rho^{2\alpha - 1} (|\Psi^D_{l,m}|^2 + \rho^2 |\partial_\rho \Psi^D_{l,m}|^2) \, d\rho \right)^{\frac{1}{2}}$$

Next, we bound the derivatives of $\Psi^D$. Explicit calculation of the derivatives of $Y^{(l,m)}$ yields

$$\| \partial_\theta Y^{(l,m)} \|_{L^\infty(S^2)} + \| \partial_\varphi Y^{(l,m)} \|_{L^\infty(S^2)} \lesssim \lambda_{l,m}^{\frac{3}{2}}.$$
For second order derivatives of $\Psi^D$, we have for $2 \leq |y| \leq e^s$,
\[
\left\| \frac{\partial^{-\frac{1}{4}+i_0} \Delta \Psi^D(y)}{u_m^D} \right\| \leq \| \Delta \Psi^D \|_{L^\infty(|y|=e^s)} + \sum_{l,m} \frac{\lambda^4}{s^2} \int_{e^s}^{e^s} \| \partial_{\rho} (\rho^{\alpha+2-\frac{1}{4}} (\mathcal{L} - \rho^{-2} \lambda_m) \Psi_{l,m}^D) \| \, d\rho \\
\leq \| \Delta \Psi^D \|_{L^\infty(|y|=e^s)} + \left( \sum_{l,m} \lambda^{-\frac{3}{2}} \right) \frac{1}{2} \left( \sum_{l,m} \lambda_m^2 \int_{e^s}^{e^s} \rho^{2\alpha+1} (||\partial_{\rho} \Psi_{l,m}^D||^2 + \rho^2 |\partial_{\rho}^2 \Psi_{l,m}^D|^2 + \rho^4 |\partial_{\rho}^3 \Psi_{l,m}^D|^2) \, d\rho \\
+ \sum_{l,m} \lambda_m \int_{e^s}^{e^s} \rho^{2\alpha-1} (||\Psi_{l,m}^D||^2 + \rho^2 |\partial_{\rho} \Psi_{l,m}^D|^2) \, d\rho \right)^{\frac{1}{2}} \leq \| \Delta \Psi^D \|_{L^\infty(|y|=e^s)} + \left( \sum_{l=0}^{5} I_l(s)e^s \right)^{\frac{1}{2}}.
\]

**Step 3 (Exterior Bound):** From the claim in **Step 1**, we infer the bound
\[
\sum_{j=0}^{\infty} \lambda^{m-j} \int_{e^s}^{e^s} \rho^{2j+2\alpha+2n_p} |f^{(j)}|^2 \, d\rho \lesssim \sum_{j=0}^{\infty} \int_{\mathbb{R}^3} \rho^{2j+2\alpha+2n_p-2} |\nabla f(\rho) Y^{(l,m)}|^2 \, dx
\]
with some universal constant independent of $m$. Using the same $\eta$ and decomposition of $\Psi^D$ as in **Step 2** and apply the above bound with $f(\rho) = \chi_s \Psi_{l,m}^D(\rho)$ for a cut-off $\chi_s(y) = \eta(2e^s - |y|)$, we infer
\[
\sum_{j=0}^{5} \lambda^{5-j} \int_{e^s}^{e^s} \rho^{2j+2\alpha-1} \xi(e^s, \rho)^{2n_p+1} |\partial_{\rho}^j \Psi_{l,m}^D|^2 \, d\rho \\
\lesssim \sum_{j=0}^{5} \int_{|y| \geq 1} \rho^{2j-2\alpha} \xi(e^s, \rho)^{2n_p+1} |\nabla^j (\Psi_{l,m}^D(\rho) Y^{(l,m)}(\theta, \varphi))|^2 \, dy
\]
Thus, as in **Step 2**, we infer
\[
\sum_{j=0}^{5} \sum_{l,m} \lambda^{5-j} \int_{e^s}^{e^s} \rho^{2j+2\alpha-1} \xi(e^s, \rho)^{2n_p+1} |\partial_{\rho}^j \Psi_{l,m}^D|^2 \, d\rho \lesssim \sum_{j=0}^{5} I_j(s)e^s.
\]
Thus, we infer for $|y| \geq e^s$,
\[
\left\| \frac{\partial^{-\frac{1}{4}+i_0} \Delta \Psi^D(y)}{u_m^D} \right\| \leq \left\| \frac{\partial^{-\frac{1}{4}+i_0} \Delta \Psi^D(y)}{u_m^D} \right\|_{L^\infty(|y|=e^s)} + \sum_{l,m} \frac{\lambda^4}{s^2} \int_{e^s}^{e^s} e^{-n_p s} |\partial_{\rho} (\rho^{\alpha+2-\frac{1}{4}} \Psi_{l,m}^D)| \, d\rho.
\]
Since
\[
\sum_{l,m} \int_{e^s}^{e^s} e^{-n_p s} |\partial_{\rho} (\rho^{\alpha+2-\frac{1}{4}} \Psi_{l,m}^D)| \, d\rho \\
\lesssim \sum_{l,m} \frac{\lambda^{5-j}}{s^2} \left( \int_{e^s}^{e^s} e^s \rho^{-2-\frac{1}{4}} \, d\rho \right)^{\frac{1}{2}} \left( \int_{e^s}^{e^s} \rho^{2\alpha-1} \xi(e^s, \rho)^{2n_p+1} (||\Psi_{l,m}^D||^2 + \rho^2 |\partial_{\rho} \Psi_{l,m}^D|^2) \, d\rho \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \sum_{l,m} \lambda^{-\frac{3}{2}} \right)^{\frac{1}{2}} \left( \sum_{l,m} \lambda_m^2 \int_{e^s}^{e^s} \rho^{2\alpha-1} \xi(e^s, \rho)^{2n_p+1} (||\Psi_{l,m}^D||^2 + \rho^2 |\partial_{\rho} \Psi_{l,m}^D|^2) \, d\rho \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \sum_{l=0}^{5} I_l(s)e^s \right)^{\frac{1}{2}},
\]
combining with the interior bound, we infer (7.10) for $\Psi^D$. As in Step 2, we can bound the derivatives of $\Psi^D$ in the region $|y| \geq e^s$. This concludes the proof of (7.10).

□

7.4. Proof of Proposition 7.1. We are in position to prove Proposition 7.1.

**proof of Proposition 7.1.** **Step 1** (Energy estimates): We claim the energy estimate

$$
\frac{dI_j}{ds} + I_j \lesssim n e^{-Cs}
$$

holds for some $C > 1$ for all $0 \leq j \leq 2k + 1$ so in particular, by the choice of initial value (7.3),

$$
I_j(s) \leq e^{-(s-s_0)}I_j(s_0) + C_n e^{-Cs_0-s} \leq e^{-s}
$$

for $s_0$ sufficiently large and $C_n$ a universal large constant. We prove this by induction on $j$. The base case $j = 0$ is considered later.

**Case 1** ($1 \leq j \leq 2k + 1$): Suppose claim holds for $< j$ cases. Denote by $I^j_\Psi$, $I^j_\Omega$ the weighted $L^2$-norm of $\Psi^D$ and $\Omega^D$ in $I_j$. For the $\Psi^D$ component, we infer

$$
\frac{dI^j_\Psi}{ds} + I^j_\Psi = e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \left[ -\rho \frac{\partial}{\partial \rho} \xi(e^{-s} \rho)^{2n+1} |\nabla^j \Psi^D|^2 + 2\xi(e^{-s} \rho)^{2n+1} \nabla^j \Psi^D \cdot \partial_s \nabla^j \Psi^D \right] dy
$$

$$
\leq 2e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n+1} \left[ (j + \Lambda + \partial_s) \nabla^j \Psi^D \right] \cdot \nabla^j \Psi^D dy
$$

where we integrate by parts for the last inequality and note that the boundary terms are non-positive. By the commutation relations

$$
[\Delta^k, \Lambda] = 2k \Delta^k, \quad [\partial_\rho \Delta^k, \Lambda] = (2k + 1) \partial_\rho \Delta^k
$$

and (7.1), we infer

$$
\frac{dI^j_\Psi}{ds} + I^j_\Psi \leq 2e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n+1} \nabla^j (\Lambda + \partial_s) \Psi^D \cdot \nabla^j \Psi^D dy
$$

$$
= -2e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n+1} \nabla^j \Omega^D \cdot \nabla^j \Psi^D dy
$$

Similarly, for $\Omega^D$ component, it follows from the above commutation relation and (7.1) that

$$
\frac{dI^j_\Omega}{ds} + I^j_\Omega \leq 2e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n+1} \left[ (j + \Lambda + \partial_s) \nabla^{j-1} \Omega^D \right] \cdot \nabla^{j-1} \Omega^D dy
$$

$$
= 2e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n+1} \nabla^{j-1} (-\Delta \bar{\Psi} - \bar{\Psi}\partial_\rho) \cdot \nabla^{j-1} \Omega^D dy.
$$
Integrate by parts the first term and using the asymptotic behaviour of \( u_n \), we infer

\[
e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n_p+1}(- \nabla j+1 \tilde{\Psi}) \cdot \nabla j^{-1} \Omega^D dy
\leq e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n_p+1} \left[ \nabla j \Psi^D \cdot \nabla j \Omega^D + C_n \rho^{-j} \int u_n^D | \nabla j^{-1} \Omega^D | dy \right] dy \tag{7.14}
+ e^{-s} \int_{|y| \geq 1} \nabla \left[ \rho^{2j-2s} \xi(e^{-s} \rho)^{2n_p+1} \right] \cdot \nabla j \Psi^D \nabla j^{-1} \Omega^D dy.
\]

From the bootstrap bound (7.7), we infer for \( \varepsilon < \frac{\delta_0}{4\gamma} \) small, the bound for the last term above

\[
e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n_p+1} \nabla j \Psi^D \left| \nabla j^{-1} \Omega^D \right| dy
\leq e^{-s-\varepsilon} \int_{|y| \geq e^x} \rho^{2j+2s-1} \xi(e^{-s} \rho)^{2n_p+1} \left| \nabla j \Psi^D \right| \left| \nabla j^{-1} \Omega^D \right| dy
+ e^{-s} \int_{1 \leq |y| \leq e^x} \rho^{2j-2s-1} \left| \nabla j \Psi^D \right| \left| \nabla j^{-1} \Omega^D \right| dy
\leq e^{-\varepsilon s} \left( I_j^\Psi \right)^{\frac{1}{2}} + e^{-s+4\gamma \varepsilon s} \left( \int_{|y| \leq e^x} \left| \nabla j \Psi^D \right| | \nabla \left| j^{-1} \Omega^D \right| dy \right)^{\frac{1}{2}} \leq \|X\|_H
\leq e^{-\varepsilon s} I_j + e^{1+4\gamma \varepsilon - \delta_0} s \leq e^{-\varepsilon s} I_j + e^{-C^s}
\]

for some \( C > 1 \). Thus, we infer the bound for (7.14):

\[
e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n_p+1}(- \nabla j+1 \tilde{\Psi}) \cdot \nabla j^{-1} \Omega^D dy
\leq e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n_p+1} \nabla j \Psi^D \cdot \nabla j \Omega^D dy
+ C_n \left[ e^{-\frac{1}{4}} \left( \int_0^\infty \rho^{-3} \xi(e^{-s} \rho) d\rho \right) \left( I_j^\Omega \right)^{\frac{1}{2}} + e^{-\varepsilon s} I_j + e^{-C^s} \right]
\leq e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n_p+1} \nabla j \Psi^D \cdot \nabla j \Omega^D dy + C_n (e^{-\varepsilon s} I_j + e^{-C^s})
\]

Next, we prove the bound for the term with \( \tilde{\Psi}^p \). By the bootstrap bound (7.5) together with the asymptotic behaviour of \( u_n^D \), it holds for \( 0 \leq l \leq k \) that

\[
\left\| \frac{\rho^{-\frac{l}{4p}} \nabla \tilde{\Psi}^p(s)}{u_n^D} \right\| \leq_{n,k} 1,
\]
we infer for all $\rho \geq 1$,

$$\left| \nabla^{j-1} \tilde{\Psi}^p \right| \lesssim_k \sum_{i=1}^{j-1} \left| \tilde{\Psi} \right|^{p-i} \sum_{|\alpha|=j-1, \alpha > 0} \left| \nabla^{\alpha_1} \tilde{\Psi} \right| \cdots \left| \nabla^{\alpha_i} \tilde{\Psi} \right|$$

$$\lesssim_k \sum_{l=1}^{j-1} \left| \nabla^l \tilde{\Psi} \right| \sum_{i=1}^{j-1} \left| \tilde{\Psi} \right|^{p-i} \sum_{|\alpha|=j-l-1, \alpha > 0, ||\alpha||_\infty \leq k} \left| \nabla^{\alpha_1} \tilde{\Psi} \right| \cdots \left| \nabla^{\alpha_l} \tilde{\Psi} \right|$$

$$\lesssim_{n,k} \sum_{l=0}^{j-1} \rho^{-j+l+1+\frac{1}{2} \frac{l-1}{p-1}} \left( u_n^D \right)^{p-1} \left| \nabla^l \tilde{\Psi} \right| \leq \sum_{l=0}^{j-1} \rho^{-j+l-\frac{3}{4}} \left| \nabla^l \tilde{\Psi} \right|$$

where we have used that $p \geq 2k$ to bound $\tilde{\Psi}^{p-i}$. Thus, we infer

$$e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n+p+1} \nabla^{j-1} \tilde{\Psi}^p \cdot \nabla^{j-1} \Omega^D \ dy$$

$$\lesssim_{n,k} e^{-s} \sum_{l=0}^{j-1} \int_{|y| \geq 1} \rho^{j+l-2s} \xi(e^{-s} \rho)^{2n+p+1} \left| \nabla^l \tilde{\Psi} \right| \left| \nabla^{j-1} \Omega^D \right| \ dy$$

$$\leq e^{-s} \sum_{l=0}^{j-1} \int_{|y| \geq 1} \rho^{j+l-2s} \xi(e^{-s} \rho)^{2n+p+1} \left| \nabla^l u_n^D \right| \left| \nabla^{j-1} \Omega^D \right| \ dy$$

$$+ e^{-s-\frac{\delta}{4}} \sum_{l=0}^{j-1} \int_{|y| \geq e^{cs}} \rho^{j+l-2s} \xi(e^{-s} \rho)^{2n+p+1} \left| \nabla^l \Psi \right| \left| \nabla^{j-1} \Omega^D \right| \ dy$$

Thus, from the bootstrap bound (7.7), we infer for $\varepsilon < \frac{\delta_0}{2\gamma}$, the bound

$$e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n+p+1} \nabla^{j-1} \tilde{\Psi}^p \cdot \nabla^{j-1} \Omega^D \ dy$$

$$\leq \sum_{l=0}^{j-1} \left[ e^{-\frac{\delta}{4}} \left( \int_{|y| \geq 1} \rho^{-\frac{\gamma}{4}} \xi(e^{-s} \rho)^{2n+p+1} \ dy \right)^{\frac{1}{2}} + e^{-\frac{\delta}{4}} \left( I^\rho_{l_i} \right)^{\frac{1}{2}} \right]$$

$$+ e^{-s-4\varepsilon} \int_{|y| \leq e^{cs}} \left| \nabla^l \Psi \right|^2 \left( \rho^{-2\gamma+2l} \ dy \right)^{\frac{1}{2}} \left( \int_{|y| \leq e^{cs}} \left| \nabla^{j-1} \Omega \right|^2 \left( \rho^{-2\gamma+2l-2(j-1)} \ dy \right) \right)^{\frac{1}{2}}$$

$$\leq e^{-\frac{\delta}{4}} + e^{-\frac{\delta}{4}} I_j + \sum_{l=0}^{j-1} e^{-\frac{\delta}{4}} I_l + e^{(-1+4\varepsilon-\frac{\delta}{4})} \leq e^{-\frac{\delta}{4}} I_j + e^{-C_s}$$

where the last inequality follows from induction hypothesis for $l < j$. Thus,

$$\frac{dI^\rho_j}{ds} + l_j \leq 2e^{-s} \int_{|y| \geq 1} \rho^{2j-2s} \xi(e^{-s} \rho)^{2n+p+1} \nabla^{j} \Psi^D \cdot \nabla^{j} \Omega^D \ dy + C_n \left[ e^{-C_s} + e^{-\frac{\delta}{4}} I_j \right].$$
Hence, by adding the bounds for $I_j^\Psi$ and $I_j^\Omega$, we obtain the overall bound
\[
\frac{dI_j}{ds} + (1 - e^{-\frac{s}{2}})I_j \lesssim_n e^{-Cs}. \tag{7.15}
\]
Using the pointwise bound (7.3) we infer
\[
I_j(s_0) \leq e^{-2s_0}.
\]
Choose $s_0$ such that $e^{-\frac{s_0}{2}} < \frac{s_0}{2}$. It then follows that
\[
I_j(s) \leq e^{-(1-e^{-\frac{s-s_0}{2}})I_j(s)} + C_ne^{-(1-e^{-\frac{s-s_0}{2}})s} \lesssim_n e^{-(1-e^{-\frac{s-s_0}{2}})s}.
\]
Plug this back into (7.15) so the claim (7.13) follows.

**Case 2** ($j = 0$): Note that $I_0 = I_0^\Psi$. As in Case 1,
\[
\frac{dI_0}{ds} + I_0 \leq -2e^{-s} \int_{|y| \geq 1} \rho^{-2s_0} e^{-s} e^{-\rho^2} |\Omega^D| \Psi_d \ dy.
\]
We bound $\Omega^D$ using (7.5) to infer
\[
\frac{dI_0}{ds} + I_0 \lesssim_n e^{-s} \int_{|y| \geq 1} \rho^{-2s_0-\alpha-1} \frac{1}{4p} e^{-s} e^{-\rho^2} |\Psi_d| \ dy \lesssim e^{-(\frac{s}{2} + \frac{1}{4p})s} \leq e^{-\frac{s}{2}}.
\]
Hence, the claim.

**Step 2** (Improved bounds for (7.5) and (7.6)): Given $d_0 \ll 1$, we claim that these quantities can be bounded by $d_0$ in $s \in [s_0, s^*]$.

**Improved bound for the weighted Sobolev norm**: It follows from the energy estimate (7.13) and the choice of initial value (7.3) that for all $0 \leq j \leq 2k + 1$,
\[
I_j(s) \leq e^{-(s-s_0)} I_j(s_0) + C_n e^{-Cs_0-s} \leq d_0 e^{-s} \tag{7.16}
\]
for $s_0$ sufficiently large.

**Improved pointwise bound**: Let $0 \leq j \leq k$. Since $X$ is radially symmetric, we write $(\Psi, \Omega) = (\tilde{\Psi}(\rho), \tilde{\Omega}(\rho))$ and so on. By Sobolev embedding and (7.7), we infer for large $s_0$ that
\[
\|\nabla^j \Psi_d\|_{L^\infty(|y| \leq 2)} + \|\nabla^{j-1} \Omega_d\|_{L^\infty(|y| \leq 2)} \ll d_0.
\]
Then, by Lemma 7.2, we infer for $0 \leq j \leq k$,
\[
\left\| \rho^{-\frac{1}{4p}} \nabla^j \Psi_d \right\|_{L^\infty(|y| \geq 2)} + \left\| \rho^{-\frac{1}{4p}} \nabla^{j-1} \Omega_d \right\|_{L^\infty(|y| \geq 2)} \lesssim \|\nabla^j \Psi_d\|_{L^\infty(|y| = 2)} + \|\nabla^{j-1} \Omega_d\|_{L^\infty(|y| = 2)} + \sum_{l=0}^{2k+1} I_l(s) e^s \lesssim d_0, \tag{7.17}
\]
where the last inequality follows from (7.16). Note that for the non-radial perturbation, one can apply Lemma 7.3 instead of Lemma 7.2.
**Step 3** (Improved $\| \cdot \|_H$ bound and non-linear bound): Recall that

$$G_\Omega = -(\Psi + u_n)^p + u_n^p + pu_n^{p-1}\Psi = -p(p-1)\Psi^2 \int_0^1 (1-\tau)(u_n + \tau\Psi)^{p-2} d\tau.$$  

We claim that by choosing $s_0$ sufficiently large and $c > 0$ small,

$$\forall s \in [s_0, s^*], \quad \|G(s)\|_H \leq \|X(s)\|^1+c.$$  

(7.18)

Let $\rho \geq 1$. Then

$$|\Delta^k G_\Omega| \lesssim \sum_{i=0}^{2k} |\nabla^{2k-i}\Psi| \sum_{j=0}^{\min(i,k)} |\nabla^j\Psi| \int_0^1 |\nabla^{i-j}(u_n + \tau\Psi)^{p-2}| d\tau.$$  

By the $L^\infty$ bound (7.5) which implies $|\nabla^j\Psi| \lesssim_n (\rho)^{-j-\alpha-\frac{1}{p'}}$ for $j \leq k$ and the asymptotic behaviour of $u_n$, we infer

$$G_{i,j}[\Psi] \lesssim |\nabla^j\Psi| \sum_{m=0}^{i-j} \sup_{0 \leq \tau \leq 1} (u_n + \tau\Psi)^{p-m-2} \sum_{|\alpha|=i-j, \alpha > 0} \prod_{l=1}^m (|\nabla^\alpha u_n| + |\nabla^\alpha\Psi|)$$

$$\lesssim n \rho^{-(-\alpha-j+\frac{1}{p'})} (\rho^{-2} + \rho^{-2\gamma}(\rho^{-1}\Psi) \lesssim \rho^{-\frac{2}{p'}}.$$  

Note that the bound for $(u_n + \tau\Psi)^{p-m-2}$ holds since $p > 2k+2$ so that the exponents $p - m - 2 > 0$.

$$|\Delta^k G_\Omega| \lesssim_n \sum_{i=0}^{2k} \rho^{-i-\frac{2}{p'}} |\nabla^{2k-i}\Psi|$$

Also, by (7.5)

$$|G_\Omega| \lesssim_n \rho^{(p-1)(-\alpha+\frac{1}{p'})} |\Psi| = \rho^{-\frac{2}{p'}} |\Psi|.$$  

Then for $R \geq 1$, since $\gamma = 2k + \frac{5}{4}$

$$\int_{|y| \geq R} (|\Delta G_\Omega|^2 + |G_\Omega|^2 (\rho)^{-2\gamma}) dy \lesssim_n \sum_{i=0}^{2k} \int_{|y| \geq R} \rho^{-2i-\frac{2}{p'}} |\nabla^{2k-i}\Psi|^2 dy$$

$$\lesssim_n \int_{|y| \geq R} \rho^{-\frac{2}{p'}} |\nabla^{2k+1}\Psi|^2 + \rho^{-2\gamma} |\Psi|^2 dy \leq R^{-1} \|X\|^2_H.$$  

(7.19)

where we have used the weighted Gagliardo-Nirenberg interpolation inequality.

Now we consider the region $0 \leq \rho \leq R$. Note that $H^{2k+1}(\mathbb{R}^d)$ is an algebra since $2k+1 > \frac{d}{2}$, hence so is $H^{2k+1}(B_1(0))$. Note also that for $\varphi \in C^\infty_c(\mathbb{R}^d)$,

$$\int_{B_R(0)} |\varphi(x)|^2 dx = R^d \int_{B_1(0)} |\varphi(Rx)|^2 dx, \quad \int_{B_R(0)} |\Delta^k \varphi(x)|^2 dx = R^{4k+d} \int_{B_1(0)} |\Delta^k \varphi(Rx)|^2 dx$$

and since $R \geq 1$, it follows that

$$\|\varphi\|_{H^{2k+1}(B_R(0))}^2 \lesssim R^{4k+d} \|\varphi(R \cdot)\|^2_{H^{2k+1}(B_1(0))}, \quad \|\varphi(R \cdot)\|^2_{H^{2k+1}(B_1(0))} \leq R^{-d} \|\varphi\|^2_{H^{2k+1}(B_R(0))}.$$
Note also that the $L^\infty$ bound (7.5) implies $|\nabla^j \Psi| \lesssim_n (\rho)^{-j-\alpha + \frac{1}{4}}$ for all $s \in [s_0, s^*]$, so we infer

$$
\sum_{j=0}^{2k} \sup_{0 \leq \tau \leq 1} \|\nabla^j (u_n + \tau \Psi)^{p-2}\|_{L^\infty(B_R(0))} \lesssim_n 1.
$$

Then it follows that

$$
\int_{|y| \leq R} (|\Delta^k G_\Omega|^2 + |G_\Omega|^2 (\rho)^{-2\gamma}) \, dy \leq \|G_\Omega\|_{H^{2k}(B_R(0))}^2
$$

$$
\lesssim \left( \sum_{j=0}^{2k} \sup_{0 \leq \tau \leq 1} \|\nabla^j (u_n + \tau \Psi)^{p-2}\|_{L^\infty(B_R(0))} \right)^2 \|\Psi\|_{H^{2k+1}(B_R(0))}^2
$$

$$
\lesssim_n R^{4k+d} \|\Psi(\cdot)\|_{H^{2k+1}(B_1(0))}^2 \lesssim R^{4k+d} \|\Psi(\cdot)\|_{H^{2k+1}(B_1(0))}^4
$$

$$
\lesssim_n R^{4k-d} \|\Psi\|_{H^{2k+1}(B_R(0))}^4 \leq R^{4\gamma+4k-d} \|X\|_{L^4}^4 =: R^M \|X\|_{L^4}^4.
$$

Set $R = \|X\|_{H^{1+M}}^{-2}$ and add (7.19) with (7.20) so the claim (7.18) follows by choosing $c(\gamma) < \frac{1}{1+M}$.

By the decay estimate in Corollary 6.2,

$$
\|(I - P)X(s)\|_{H^{1}} \lesssim e^{-\frac{s}{4} (s-s_0)} \|X(s_0)\|_{H} + \int_{s_0}^{s} e^{-\frac{s}{4} (s-\tau)} \|G(\tau)\|_{H} \, d\tau
$$

$$
\lesssim e^{-\frac{s}{4} \delta} \left[ e^{\frac{s}{4} \delta_0} \|X\|_{H} + \int_{s_0}^{s} e^{(\frac{s}{4} - \delta (1+c)) \tau} \, d\tau \right] \lesssim e^{-\frac{s}{4} \delta}
$$

since $\frac{\delta}{1+c} < \delta_0$. This, together with (7.4), we infer

$$
\|X(s)\|_{H} \lesssim e^{-\frac{s}{4} \delta}.
$$

This proves an improved bound for (7.7). Then, by (7.18), the non-linear bound (7.8) follows. □

**APPENDIX A. Bound on self-similar profiles**

**Lemma A.1.** By induction on $k$. Let $u_n$ be the self-similar profiles constructed in Proposition 4.1. For all $k \in \mathbb{N}$, as $\rho \to \infty$,

$$
\partial^k_\rho u_n = O(\rho^{-\alpha-k}), \quad \partial^k_\rho (u_n^{p-1}) = O(\rho^{-2-k}). \tag{A.1}
$$

**Proof.** In view of (2.13), taking $\varepsilon \ll 1$ we infer

$$
u_n = O(\rho^{-\alpha}), \quad u'_n = O(\rho^{-\alpha-1})
$$

and $u_n \geq 0$ for all $\rho$ sufficiently large. It follows immediately that

$$
u_n^{p-1} = O(\rho^{-2}), \quad (u_n^{p-1})' = (p-1)u_n^{p-1}u'_n = O(\rho^{-3}).$$

In view of (1.4), we infer
\[ |u_n^{(k)}| \lesssim_{k, \rho} \partial^{k-2} \frac{1}{\rho^2} \left( \rho u_n' + u_n + u_n^p \right) \]
for all \( \rho > \rho_0 \) and \( k \geq 2 \). Suppose lemma holds for some \( k \geq 2 \). Then by hypothesis, for all \( \rho > \rho_0 \),
\[ |u_n^{(k+1)}| \lesssim \sum_{j=0}^{k} \rho^{-j-2} u_n^{(k-j)} + \sum_{j=0}^{k-1} \rho^{-j-2} \sum_{i=0}^{k-j-1} u_n^{(i)} (u_n^{p-1})^{(k-j-i-1)} \lesssim \rho^{-\alpha-k-1}. \]
Furthermore, by hypothesis and bound on \( u_n^{(k+1)} \), we infer
\[ |(u_n^{p-1})^{(k+1)}| \lesssim \sum_{j=0}^{k+1} u_n^{p-k+j-2} \sum_{|\alpha|=k+1, \alpha > 0} u_n^{(\alpha_1)} \cdots u_n^{(\alpha_j)} \lesssim \rho^{-3-k} \]
and this concludes the proof by induction. \( \square \)

**Appendix B. Maximaliy of \( \hat{M} \)**

In this section, we consider the problem (5.14). Given \( H \) such that \( H(\rho)Y^{(l,m)} \in C^\infty_c(\mathbb{R}^d) \), we seek solution to
\[ [\mathcal{L} - \rho^{-2} \lambda_m - (\Lambda + R + 1)(\Lambda + R) + p u_n^{p-1}] \Psi = H. \] (B.1)

**Lemma B.1.** Let \( H \in C^\infty([0, \infty)) \). Then for \( R \) sufficiently large, there exists a unique solution \( \Psi \in C^1([0, \infty)) \) to (B.1). Furthermore, if \( H(\rho)Y^{(l,m)} \in C^\infty_c(\mathbb{R}^d) \), then \( \Psi(\rho)Y^{(l,m)} \) is smooth on \( \mathbb{R}^d \).

**Proof.** In the first part of the proof, we use Proposition 2.2 to prove local existence at singular points \( \rho = 0 \) and 1. Set \( (\Psi_1, \Psi_2) = (\rho^{m+d-2} \Psi, \partial_\rho (\rho^{m+d-2} \Psi)) \). Writing (B.1) in the form required in Proposition 2.2,
\[
\begin{cases}
\rho \partial_\rho \Psi_1 = \rho \Psi_2 \\
\rho \partial_\rho \Psi_2 = \frac{\rho}{1-\rho^2} \left\{ [\xi(m, d, \alpha, R) - pu_n^{p-1}] \Psi_1 + \frac{2m+d-3}{\rho \rho^2} + \eta(m, d, \alpha, R) \rho \right\} \Psi_2 + \rho^{m+d-2} H
\end{cases}
\]
where
\[ \xi = (m + d - \alpha - R - 2)(m + d - \alpha - R - 3), \quad \eta = -2(m + d - \alpha - R - 3). \]
Hence,
\[ \rho \partial_\rho \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = A(\rho) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} + \frac{\rho^{m+d-1}}{1-\rho^2} \begin{pmatrix} 0 \\ H \end{pmatrix} \]
where \( A \) is smooth in \([0, 1)\),
\[ A(0) = \begin{pmatrix} 0 & 0 \\ 0 & 2m + d - 3 \end{pmatrix} \]
with \( \sigma(A(0)) = \{0, 2m + d - 3\} \). Thus, by Proposition 2.2 with \( l = 2m + d - 2 \), we infer for all \( a, b \in \mathbb{R} \), there exists unique smooth solution \( \Psi_{a,b} \in C^\infty([0, 1)) \) to (B.1) such that
\[ (\Psi_1(0), \Psi_1'(0), \cdots, \Psi_1^{(2m+d-3)}(0), \Psi_1^{(2m+d-2)}(0)) = (a, 0, \cdots, 0, b). \]
In other words, we can write
\[ \Psi_{a,b} = \Psi_0 + a \psi_1 + b \psi_2, \]
where \( \psi_1, \psi_2 \) are the linearly independent solutions to the homogenous problem for (B.1) in \([0, 1]\) with appropriate initial values.

Similarly, for \((\tilde{\Psi}_1, \tilde{\Psi}_2) = (\Psi, \partial_r \Psi)\), write (B.1) as
\[
\begin{cases}
(r - 1) \partial_r \tilde{\Psi}_1 = (r - 1) \tilde{\Psi}_2 \\
(r - 1) \partial_r \tilde{\Psi}_2 = \frac{1}{1 + r} \left( \left[ - (\alpha + R)(\alpha + R + 1) + pu_{n}^{\rho - 1} - \frac{\lambda_m}{\rho^2} \right] \tilde{\Psi}_1 + \left[ \frac{d - 1}{\rho} - 2(\alpha + R + 1) \right] \rho \tilde{\Psi}_2 - H \right).
\end{cases}
\]
Hence,
\[
(r - 1) \partial_r \left( \frac{\tilde{\Psi}_1}{\tilde{\Psi}_2} \right) = B(r) \left( \frac{\tilde{\Psi}_1}{\tilde{\Psi}_2} \right) + \frac{1}{r + 1} \left( 0 \right)
\]
where \( B \) is smooth in \((0, \infty)\),
\[
B(1) = \frac{1}{2} \begin{pmatrix}
0 & 0 \\
- (\alpha + R)(\alpha + R + 1) - \lambda_m + pu_{n}^{\rho - 1}(1) & 2s_c - 2R - 3
\end{pmatrix}
\]
with \( \sigma(B(1)) = \{s_c - R - \frac{3}{2}, 0\} \). Thus, by Proposition 2.2, for all \( b \in \mathbb{R} \), there exists a unique smooth solution \( \tilde{\Psi}_b \in C^\infty((0, \infty)) \) to (B.1) with
\[
(\tilde{\Psi}_c(1), \tilde{\Psi}_c'(1)) = \left( 2c, -\left( \alpha + R + 1 + \frac{pu_{n}^{\rho - 1}(1) - \lambda_m}{s_c + R + \frac{3}{2}} \right) c + \frac{H(1)}{-s_c + R + \frac{3}{2}} \right).
\]
We can write
\[
\tilde{\Psi}_c = \tilde{\Psi}_0 + c \tilde{\psi}, \quad (\tilde{\psi}(1), \tilde{\psi}'(1)) = \left( 2, - (\alpha + R + 1) - \frac{pu_{n}^{\rho - 1}(1) - \lambda_m}{s_c + R + \frac{3}{2}} \right)
\]
where \( \tilde{\psi} \) is the unique solution to the homogeneous problem for (B.1) in \((0, \infty)\) with the given initial values.

Next, we claim that for \( R \) sufficiently large and for all \( m \geq 0 \), the homogeneous problem for (B.1) with \( H = 0 \) has a unique \( C^1 \) solution \( \Psi \equiv 0 \) on \([0, 1]\). Suppose otherwise i.e. there is \( R \) arbitrarily large and \( m \geq 0 \) such that there exists \( \Psi_{l,m} \neq 0 \) smooth in \([0, 1]\) such that (B.1) holds with \( H = 0 \) and \( \Psi = \Psi_{l,m}(\rho)Y_{l,m} \) is smooth at the origin. Extend uniquely the homogeneous solution \( \Psi_{l,m} \) to \([1, \infty)\). Then, using the fixed point argument as in the proof of Lemma B.2 we infer
\[
\sum_{j=0}^{2k+3} \sup_{\rho \geq 1} \rho^{\alpha + R + j} |\partial_r^j \Psi_{l,m}| < \infty
\]
and therefore, \((\Psi, - (\Lambda + R) \Psi) \in \mathcal{D}_R \) and
\[
\langle \mathcal{M} X, X \rangle = R(X, X).
\]
By dissipativity of \( \mathcal{M} \) for \( X \in \mathcal{D}_R \) proved in Step 1 of the proof of Proposition 5.3, we infer for all \( X \in \mathcal{D}_R \)
\[
\langle \mathcal{M} X, X \rangle \leq C \langle X, X \rangle
\]
for some $C$ independent of $R$ and this is a contradiction so we have our claim. This yields the uniqueness result.

Choose $R$ sufficiently large so the claim holds. Since $\{\rho^{2-m-d}\psi_1, \rho^{2-m-d}\psi_2\}$ is a basis of solutions to the homogeneous problem in $(0,1)$, there exists $A, B \in \mathbb{R}$ such that

$$\tilde{\psi} = \rho^{2-m-d}(A\psi_1 + B\psi_2)$$

in $(0,1)$. If $A = 0$, then $\tilde{\psi} \in C^\infty([0,1])$ contradicting the claim above. Since $\{\rho^{2-m-d}\psi_1, \rho^{2-m-d}\psi_2\}$ is a basis of solutions to the homogeneous problem in $(0,1)$, there exists $a, b \in \mathbb{R}$ such that

$$\rho^{2-m-d}\psi_{a,b} = \tilde{\psi}_0$$

Then,

$$\Psi = \tilde{\psi}_0 - \frac{a}{A}\tilde{\psi} = \rho^{2-m-d}\left(\Psi_{a,b} - a\psi_1 - \frac{aB}{A}\psi_2\right)$$

is smooth at $\rho = 0$ and $1$. Thus, we have the existence and uniqueness of $C^1([0,\infty))$ solution. Furthermore, if $H(\rho)Y^{(l,m)}$ is smooth i.e. $H = O_{\rho \to 0}(\rho^m)$ and $H^{(2k+m+1)}(0) = 0$ for $k \in \mathbb{N}_{\geq 0}$, then $\Psi^{(2k+m+1)}(0) = 0$ for $k \in \mathbb{N}_{\geq 0}$. Thus, $\Psi(\rho)Y^{(l,m)}$ is smooth. \(\square\)

**Lemma B.2.** For $H$ such that $H(\rho)Y^{(l,m)} \in C^\infty_c(\mathbb{R}^d)$, let $\Psi$ be the unique $C^1$ solution to (B.1) found in Lemma B.1. Then for $R$ sufficiently large, $\Psi \in H^{2k+1}_{rad}(\mathbb{R}^d)$.

**Proof.** Using the fixed point argument, we prove the existence of $C^{2k+1}$ solution $\Psi$ to (B.1) in $\{\rho \geq \rho_0\}$ for $\rho_0$ sufficiently large with sufficiently rapid decay as $\rho \to \infty$ so that $\Psi \in H^{2k+1}_{rad}(\{\rho \geq \rho_0\})$. Then by uniqueness of solution, we argue that this solution is indeed what we found in Lemma B.1.

Consider the homogeneous problem for (B.1) without the $p\mu_{2n-1}$ potential term:

$$\{(1 - \rho^2)\partial_\rho^2 + [(d - 1)\rho^{-1} - 2(\alpha + R + 1)\rho]\partial_\rho - \lambda_m\rho^{-2} - (\alpha + R)(\alpha + R + 1)\} \varphi = 0$$

$$:= \mathcal{L}_R$$

(B.2)

in $[1,\infty)$. Computation similar to Lemma 2.1 yields a pair of linearly independent solutions

$$\varphi_1 = \rho^{-\alpha - R - m - 1}2F_1\left(\frac{\alpha + R + m - d + 3}{2}, \frac{\alpha + R + 1}{2}, \frac{3}{2}, \rho^{-2}\right)$$

$$\varphi_2 = \rho^{-\alpha - R}2F_1\left(\frac{\alpha + R - m}{2}, \frac{\alpha + R + m - d + 2}{2}, \frac{1}{2}, \rho^{-2}\right)$$

(B.3)

with the Wronskian

$$W := \varphi'_1\varphi_2 - \varphi'_2\varphi_1 \propto \rho^{1-d}|1 - \rho^2|^{-R-\frac{3}{2}}.$$
Define the spaces
\[
\tilde{X}_{\rho_0} = \left\{ w \in C^{2k+3}(\rho_0, \infty) \left| \|w\|_{\tilde{X}_{\rho_0}} := \sum_{j=0}^{2k+3} \sup_{\rho \geq \rho_0} \rho^{\alpha+j} |\partial_{\rho}^j w| \right. \right\},
\]
\[
\tilde{Y}_{\rho_0} = \left\{ w \in C^{2k+3}(\rho_0, \infty) \left| \|w\|_{\tilde{Y}_{\rho_0}} := \sum_{j=0}^{2k+3} \sup_{\rho \geq \rho_0} \rho^{\alpha+j+2} |\partial_{\rho}^j w| \right. \right\}.
\]

Claim that for \(\rho_0 > 1\), the resolvent map \(T_R : \tilde{Y}_{\rho_0} \to \tilde{X}_{\rho_0}\) given by
\[
T_R(f) = \varphi_1 \int_{\rho_0}^{\rho} \frac{f \varphi_2}{(1 - r^2)W} \, dr - \varphi_2 \int_{\rho_0}^{\rho} \frac{f \varphi_1}{(1 - r^2)W} \, dr
\]
is well-defined and bounded with \(L_R \circ T_R = \text{id}_{\tilde{Y}_{\rho_0}}\). Note that
\[
\partial_{\rho}^j T_R(f) = \varphi_1 \int_{\rho_0}^{\rho} \frac{f \varphi_2}{(1 - r^2)W} \, dr - \varphi_2 \int_{\rho_0}^{\rho} \frac{f \varphi_1}{(1 - r^2)W} \, dr
\]
+ \sum_{i=0}^{j-2} \partial_{\rho}^i \left[ f \frac{(\varphi_1^{j-i-1}) \varphi_2 - \varphi_2^{j-i-1} \varphi_1}{(1 - \rho^2)W} \right].
\]

In view of (B.3) and the asymptotic expansion of the fundamental solutions, we infer
\[
\partial_{\rho}^j \left[ \frac{(\varphi_1^{j-i-1}) \varphi_2 - \varphi_2^{j-i-1} \varphi_1}{(1 - \rho^2)W} \right] = O_{\rho \to \infty}(\rho^{i-j-1}).
\]

Then for all \(\rho \geq \rho_0\) and \(0 \leq j \leq 2k + 3\),
\[
\rho^{\alpha+j} |\partial_{\rho}^j T_R(f)| \lesssim \left( \rho^{-1} \int_{\rho_0}^{\rho} \rho^{-2} \, dp \sup_{r \geq \rho_0} r^{\alpha+r+2} |f| + \left( \int_{\rho_0}^{\rho} \rho^{-3} \, dp \sup_{r \geq \rho_0} r^{\alpha+r+2} |f| \right) \right.
\]
+ \sum_{i=0}^{j-2} \rho^{j-i-2} \sup_{r \geq \rho_0} r^{\alpha+r+i+2} |\partial_{\rho}^i f| \lesssim \rho_0^{-2} \|f\|_{\tilde{Y}_{\rho_0}}.
\]

Thus, \(T_R\) is a bounded map with operator norm \(\|T_R\| \lesssim \rho_0^{-2}\) as claimed. Now we solve the fixed point problem:
\[
\Psi = c_1 \varphi_1 + c_2 \varphi_2 + T_R[H - u_{n-1}^p \Psi] \quad \text{B.4}
\]
for \(c_1, c_2\) such that the \(\Psi(\rho_0), \Psi'(\rho_0)\) agree with the corresponding values in Lemma B.1. Note that \(\varphi_1, \varphi_2 \in \tilde{X}_{\rho_0}, H \in C^\infty([0, \infty))\). By Lemma A.1, \(\partial_{\rho}^p(u_{n-1}^p) = O(\rho^{-m-2})\) as \(\rho \to \infty\) so we infer
\[
\|u_{n-1}^p \psi \|_{\tilde{Y}_{\rho_0}} \lesssim \|\psi\|_{\tilde{X}_{\rho_0}}
\]
and hence, \(H - u_{n-1}^p \Psi \in \tilde{Y}_{\rho_0}\) so indeed \(G_R : \tilde{X}_{\rho_0} \to \tilde{X}_{\rho_0}\). For \(\rho_0\) sufficiently large, \(G_R\) is a contraction map since for all \(\Psi_1, \Psi_2 \in \tilde{X}_{\rho_0}\),
\[
\|G_R(\Psi_1) - G_R(\Psi_2)\|_{\tilde{X}_{\rho_0}} \lesssim \|T_R\| \|u_{n-1}^p (\Psi_1 - \Psi_2)\|_{\tilde{Y}_{\rho_0}} \lesssim \rho_0^{-2} \|\Psi_1 - \Psi_2\|_{\tilde{X}_{\rho_0}}.
\]

Thus, it follows from the Banach fixed point theorem that there exists a unique \(\Psi \in \tilde{X}_{\rho_0}\) such that (B.4) holds. Taking \(R > s_c, \tilde{X}_{\rho_0}\) continuously embeds in \(H^{2k+1}_\text{rad}(\{\rho \geq \rho_0\})\) so \(\Psi \in H^{2k+1}_\text{rad}(\{\rho \geq \rho_0\})\). Also, by uniqueness of solution to an ODE at ordinary point, this is indeed the solution we found in Lemma B.1. \(\Box\)
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