THE CURRENT AND THE CHARGE NOISE OF A SINGLE-ELECTRON TRANSISTOR IN THE REGIME OF LARGE CHARGE FLUCTUATIONS OUT OF EQUILIBRIUM

Y. UTSUMI, H. IMAMURA, M. HAYASHI, AND H. EBISAWA

Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan
E-mail: utsumi@cmt.is.tohoku.ac.jp

By using the Schwinger-Keldysh approach, we evaluate the current noise and the charge noise of the single-electron transistor (SET) in the regime of large charge fluctuations caused by large tunneling conductance. Our result interpolates between previous theories; the “orthodox” theory and the “co-tunneling theory”. We find that the life-time broadening effect suppresses the Fano factor below the value estimated by the previous theories. We also show that the large tunnel conductance does not reduce the energy sensitivity so much. Our results demonstrate quantitatively that SET electrometer can be used as the high-sensitivity and high-speed device for quantum measurements.

1 Introduction

A single electron transistor (SET) electrometer is an important device for the single shot measurement of a charge qubit realized in the ultrasmall Josephson junction systems. For present day experiment, the dominant mechanism of the decoherence of the charge qubit is the $1/f$ back ground charge noise, which is expected to be reduced in the high-frequency regime with technical improvements. When the $1/f$-noise is suppressed, the back-action of the measurement, i.e. the intrinsic charge fluctuation of SET electrometer, becomes important. Especially for a high-speed SET, whose tunneling conductance is relatively large, the charge fluctuation related to the higher-order tunneling process is expected to be important. So far, the noise has been investigated by using the master equation with Markovian approximation, namely “orthodox” theory. Beyond the orthodox theory, the quantum fluctuation effect has been investigated in the Coulomb blockade (CB) regime within the second order perturbation theory, “co-tunneling theory”. Though the semi-quantitative estimation at the threshold has been performed a decade ago, there is no quantitative estimation which covers both the sequential tunneling (ST) regime and CB regime.

Besides the practical application, the noise in the regime of large quantum fluctuations of charge itself is interesting from the point of view of the strongly correlated system out of equilibrium. There have been much development on this topic. The renormalization of the conductance and charging energy was predicted theoretically and confirmed experimentally. The life-time broadening effect at finite bias voltage was also predicted. However how the noise is modified by the renormalization effect or the life-time broadening effect has not been clarified.

In this paper, we derive noise expressions which cover both ST regime and CB regime. We adopt the modern style of the Keldysh formalism, Schwinger-Keldysh approach, which enables one to calculate any order moment by functional derivative of the generating functional. We also calculate the energy sensitivity...
quantitatively by using our expressions.

2 Model and Calculations

Figure 1(a) shows an equivalent circuit of a SET. A normal metal island exchanges electrons with a left (right) lead via a small tunnel junction characterized by a tunnel matrix element $T_{L(R)}$ and a capacitor $C_{L(R)}$. The island is also coupled to a gate via a capacitor $C_G$. In the following discussion, we limit ourselves to the symmetric case, $C_L = C_R$ and $T_L = T_R$. We use the two-state model to describe the strong Coulomb interaction. We begin with the effective action of SET in the drone.

![Diagram of SET transistor and closed time-path](image1)

Figure 1. (a) The equivalent circuit of a SET transistor. (b) The closed time-path going from $-\infty$ to $\infty$ $(C_+)$, going back to $-\infty$ $(C_-)$, connecting the imaginary time path $C_\tau$ and closing at $t = -\infty - i\hbar \beta$.

In the limit of large number of transverse channels, the effective action for $c$ and $d$-field is given by

$$S = \int_C dt \{ c(t)^* (i\hbar \partial_t - h(t)) c(t) + i\hbar d(t)^* \partial_t d(t) \} + \int_C d1d2 \{ \sigma_+ (1) \alpha (1, 2) \sigma_- (2) \},$$

(1)

where $C$ is the closed time-path (Fig. 1(b)). The variable $\sigma_\pm$ related to the spin-$1/2$ operator is written by two Grassmann variables $c$ and $d$ as $\sigma_+ = c^* \phi$, where $\phi = d^* + d$. Here $\alpha = \sum_{r=L,R} \alpha_r$ and $\alpha_r (1, 2)$ is a particle-hole Green function (GF) which is proportional to the phase factor $e^{i\psi_r (\varphi (1) - \varphi (2))}$, where $\varphi$ is the phase difference between the left and the right lead. $\kappa_L = -\kappa_R = 1/2$ characterizes the voltage drop between the lead and the island. The auxiliary source fields, i.e. the phase difference $\varphi$ and the scalar potential for $c$-field $h$ are introduced to calculate the average current, the average charge and noise by the functional derivative technique. It is noticed that the degree of freedom of $\varphi (h)$ is duplicated, i.e., we can define $\varphi_+ (h_+)$ and $\varphi_- (h_-)$ on the forward $C_+$ and the backward branch $C_-$, respectively. The practical form of the particle-hole GF in the **physical representation** is written as

$$\tilde{\alpha}_r = \begin{pmatrix} 0 & \alpha^A_r \\ \alpha^R_r & 0 \end{pmatrix}, \quad \alpha^A_r (\varepsilon) = -i\pi \alpha_0 (\delta \varepsilon^2), \quad \alpha^R_r (\varepsilon) = 2\alpha^R_0 (\varepsilon) \coth \left( \frac{\delta \varepsilon^2}{2T} \right),$$

(2)
in the energy space. Here \( \rho(\varepsilon) = \varepsilon \) with Lorentzian cut-off at the charging energy \( E_C \) and \( \delta\varepsilon^\tau = \varepsilon - \varepsilon_\text{F} \). The dimensionless junction conductance \( \alpha_0^0 \) is defined with the resistance of junction \( r \) \( R \) as \( \frac{R}{(2\pi)^2} r \). \( \alpha \) is zero on \( C \) because the tunneling Hamiltonian is adiabatically switched on at remote past and off at distant future. The generating functional for the connected Green function \( W \) defined as 

\[ -i\hbar \ln \int D[c',d',d] \exp(iS/h), \]

is calculated by performing the perturbation series expansion in powers of \( \alpha_0 \). We propose the approximate generating functional including the effect of infinite order tunneling process.  

\[ W = -i\hbar \text{Tr}[\ln G_c^{-1}], \quad G_c^{-1}(t,t') = g_c^{-1}(t,t') - \sum_{r=L,R} \Sigma_r(t,t'), \quad (3) \]

where the trace is performed over \( C \). Here \( g_c^{-1}(t,t') = (i\hbar \partial_t - h(t)) \delta(t,t') \) and the self-energy is given by \( \Sigma_r(t,t') = -i\hbar g_\alpha(t',t) \alpha_r(t') \) where \( g_\phi^{-1}(t,t) = i\hbar \partial_t \delta(t,t')/2 \). Here, \( \delta \)-function is defined on \( C \) and \( g_\alpha \) and \( g_\phi \) satisfy the anti-periodic boundary conditions: \( g_\alpha(t,-\infty \in C) = -g_\alpha(t, -i\hbar \beta - \infty), \) etc. 

Once an approximate generating functional is obtained, one can calculate any frequency noise, \( S_{II}(\varepsilon) = \langle \delta I(t)\delta I(t') \rangle , \quad (\delta I = I - \langle I \rangle), \) etc. In the path integral representation, the noise is calculated by the second derivative of the generating functional as 

\[ S_{II}(t,t') = \frac{1}{i\hbar} \frac{2e^2 \delta^2 W}{\delta \varepsilon(\tau) \delta \varphi(\tau')} |_{\varphi=\Delta=0} , \quad S_{QQ}(t,t') = \frac{-2ie^2 \hbar \delta^2 W}{\delta \varphi(\tau) \delta \varphi(\tau')} |_{\varphi=\Delta=0}. \quad (4) \]

In the following discussions, we limit ourselves to the discussions on the zero frequency noise, \( S_{II} = \int dt' S_{II}(t,t'), \) etc. \( S_{II} = \sum_{r,r'=L,R} \kappa_r \kappa_r' S_{II,r,r'}, \) is calculated as 

\[ S_{II,r,r'} = (e^2/h) \int \text{d} \varepsilon \text{Tr} \left[ \Sigma_r \Sigma_r' \alpha_r \alpha_r' + \Sigma_r \tau^1 \Sigma_r' \tau^1 \Sigma_r' \Sigma_r - \Sigma_r \tau^1 \Sigma_r' \tau^1 \Sigma_r' \right]. \]

\( \tau^1 \) is the Pauli matrix and we omit the argument \( \varepsilon \). GFs denoted with tilde are those in the physical representation. By paying attention to such conditions as 

\[ \int \text{d} \varepsilon \Sigma_r^{R(A)} \Sigma_r^{R(A)} = 0, \]

we obtain the expression for \( S_{II}(R\beta/2) \) written with the Fermi function \( f^- \) and \( f^+ = 1 - f^- \) as 

\[ \int \text{d} \varepsilon [T^F(\varepsilon) \{ f^- (\delta \varepsilon^L) f^+ (\delta \varepsilon^R) + f^+ (\delta \varepsilon^L) f^- (\delta \varepsilon^R) \} - T^F(\varepsilon) \{ f^- (\delta \varepsilon^L) - f^- (\delta \varepsilon^R) \}]^2, \quad (5) \]

where the effective transmission probability \( T^F = -\langle \alpha_0^0 \alpha_0^0 / \alpha^R \rangle 2i \text{Im} G_c^R \) includes the inelastic scattering process. Here \( G_c^R(\varepsilon) = 1/(\varepsilon - \Delta_0 - \Sigma_c^R(\varepsilon)), \) and \( \Sigma_c^R(\varepsilon) = \alpha_0^R(\varepsilon) \{ 2 \text{Re} \left( \frac{\delta C}{2\pi \varepsilon} \right) - \psi(1 + \frac{E_C}{2\pi \varepsilon}) - \psi(\frac{E_C}{2\pi \varepsilon}) \} + \alpha_0^R(\varepsilon)/2 \). \( \Sigma_c^R(\varepsilon) \) has the same form as the noise expression of a point contact without Coulomb
interaction. This result is anticipated, because the tunneling current is expressed as the Landauer formula with the effective transmission probability. However, to our knowledge, there is no literature which derived it microscopically. The charge noise is expressed with off-diagonal components of the particle-hole GF in the single time representation as

\[ S_{QQ} = e^2 \hbar^2 \int \frac{d\varepsilon}{4\hbar} \text{Tr} \left[ \tilde{G}_c \tilde{G}_c \right] = -\frac{e^4 R_K}{2\pi^2} \int d\varepsilon \left| G^R_c(\varepsilon) \right|^4 \alpha^{-+}(\varepsilon) \alpha^{+-}(\varepsilon). \]  

Here we note that our approximation satisfies the minimum required properties. It is known that the gauge invariance of the generating functional leads to the charge conservation. In our system, the invariance of Eq. (3) under the transformation \( \varphi_r \rightarrow \varphi_r + \delta\psi, h \rightarrow h - \hbar(\partial_t \psi) \), where \( \delta\psi \) is defined on \( C \), leads to the relation,

\[ \partial_t \partial_{t'} S_{QQ}(t, t') = \sum_{r, r'} S_{II}(t, t'). \]

Moreover, one can easily check that Eq. (5) satisfies the fluctuation-dissipation theorem at \( V = 0 \).

### 3 Results and Discussions

![Graphs](image-url)

Figure 2. (a) The excitation energy dependence of the current noise at 0K and \( eV / E_C = 0.4 \) normalized by \( I_\text{c} = G_0 V / 2 \), where \( G_0 \) is the series junction conductance. The solid and dotted lines show our results Eq. (5) for \( \alpha_0 = 0.05 \) and \( 10^{-5} \), respectively. The dotted and dashed lines are results of the co-tunneling theory. (b) The excitation energy dependence of the Fano factor for \( \alpha_0 = 0.1 \) (solid line), \( 0.05 \) (dashed line) and \( 10^{-5} \) (dotted line).

Figure 2 (a) shows the excitation energy dependence of the normalized zero-frequency current noise at finite bias voltage and zero temperature. When \( \alpha_0 \) is small, our result (the dot-dashed line) reproduce the orthodox theory. Moreover, our results are consistent with the co-tunneling theory (dotted and dashed lines) in the regime \( |\Delta_0/eV| \gg 0.5 \). As \( \alpha_0 \) becomes large (the solid line), the higher order tunneling process enhances the current noise around the threshold bias voltage \( |\Delta_0/eV| = 0.5 \). However, around \( \Delta_0 = 0 \), the current noise is suppressed due to the lifetime broadening. The life-time broadening effect is related to the dissipation process which is the leak of an electron from the island while another electron tunnels into the island.
Figure 2 (b) shows the excitation energy dependence of the Fano factor for various $\alpha_0$. The Fano factor is the measure how the noise deviates from the Poissonian behavior. It is known that in ST regime $|\Delta_0/eV| < 0.5$ the Fano factor is suppressed by the Coulomb interaction and that in CB regime $|\Delta_0/eV| > 0.5$ the Fano factor reaches 1 which means that co-tunneling events behave like Poissonian process. Our result reproduces this behavior when $\alpha_0$ is small (the dotted line). As $\alpha_0$ increases, the Fano factor is suppressed over ST regime and in CB regime near the threshold bias voltage (the dashed line and the solid line). In CB regime, the suppression is mainly caused by the enhancement of the tunneling probability due to the higher order tunneling process. On the other hand, around $\Delta_0 = 0$, the Fano factor suppression is caused by the dissipation, i.e. the dissipation regulates tunneling events.

Above discussions, we consider the condition $eV \gg T_K$, where $T_K$ is the Kondo temperature. In this regime the renormalization effect is negligible. In the opposite regime, the charge noise is suppressed due to the renormalization of charge.

Next we discuss on the performance of SET electrometer. The energy sensitivity is defined by the product of the charge noise and the charge sensitivity as $\epsilon = (\hbar/2)\sqrt{S_{QQ}S_{II}/|\partial I/\partial \Delta_0|}$, which is not allowed to be smaller than $\hbar/2$. Figure 3 (a) shows the excitation energy dependence of the energy sensitivity at 0 K and $eV/E_C = 0.4$ for $\alpha_0 = 10^{-3}$, $10^{-2}$ and 0.1 (solid lines). Three dashed lines show results of the orthodox theory with corresponding parameters. The dotted line shows the result of the co-tunneling theory, which is independent of $\alpha_0$. At the threshold, the orthodox theory and the co-tunneling theory predict $\epsilon = 0$ and $\epsilon \to \infty$, respectively. Our results interpolate between two theories with satisfying $\epsilon > \hbar/2$. This fact is considered as an evidence to justify our approximation. For the typical value of $E_C = 0.1$ meV ($C \sim 800$ aF), the time constants of SET $(R_T C)^{-1} = 4\pi \alpha_0 E_C / \hbar$ for $\alpha_0 = 10^{-3}$, $10^{-2}$ and 0.1 are 1.91, 19.1 and 191 GHz, respectively. As seen in Fig. 3 (a), the energy sensitivity is at worst $\sim \hbar$ at the
threshold which is the usual optimum working point. Thus our result demonstrate quantitatively that the SET electrometer can be operated in the high-frequency regime without reduction of the sensitivity as performed in an experiment\cite{16}. Figure 3 (b) shows the slope of the excitation energy dependence of the average current for $\alpha_0 = 10^{-3}$ (dotted line), $10^{-2}$ (dot-dashed line) and 0.1 (solid line). As $\alpha_0$ becomes large, the structure is smeared. Our results show that the large $\alpha_0$ does not reduce the energy sensitivity so much, however it makes difficult to obtain the sharp onset of the current.

4 Summary

In conclusion, we have evaluated the noise and the energy sensitivity in the regime of large quantum fluctuations out of equilibrium. We have reformulated and extended RTA in a charge conserving way. Our approximation is justified by the fact that the current noise satisfies the fluctuation-dissipation theorem and that the energy sensitivity does not exceed the quantum limit. Our approximation consistent with the orthodox theory in the limit $\alpha_0 \to 0$, and the co-tunneling theory in CB regime. We found that the dissipation, i.e. the life-time broadening effect, regulates tunneling events and suppresses the Fano factor. We have also shown that large $\alpha_0$ does not reduce the energy sensitivity so much at the threshold bias voltage. Our results demonstrate quantitatively that SET electrometer can be used as the high-sensitivity and high-speed device for quantum measurements.

Acknowledgments

We would like to thank Y. Isawa, J. Martinek and Yu. V. Nazarov for variable discussions and comments. This work was supported by a Grant-in-Aid for Scientific Research (C), No. 14540321 from MEXT. H.I. was supported by MEXT, Grant-in-Aid for Encouragement of Young Scientists, No. 13740197.

References

1. M. H. Devoret and R. J. Schoelkopf, *Nature* **406**, 1039 (2000).
2. Y. Nakamura, Yu. A. Pashkin and J. S. Tsai, *Nature* **398**, 786 (1999).
3. Y. Nakamura et al., *Phys. Rev. Lett.* **88**, 047901 (2002).
4. A. N. Korotkov et al., in *Single-Electron Tunneling and Mesoscopic Devices*, eds. H. Koch and H. Lübbig (Springer-Verlag, Berlin, 1992), p. 45.
5. A. N. Korotkov, *Phys. Rev. B* **49**, 10381 (1994).
6. D. V. Averin, in *Macroscopic Quantum Coherence and Quantum Computing*, eds. D. V. Averin et al., (Kluwer, New York, 2001), p. 399.
7. E. V. Sukhorukov, G. Burkard and D. Loss, *Phys. Rev. B* **63**, 125315 (2001).
8. G. Falci and G. Schön and G. T. Zimanyi, *Phys. Rev. Lett.* **74**, 3257 (1995).
9. H. Schoeller and G. Schön, *Phys. Rev. B* **50**, 18436 (1994).
10. P. Joyez et al., *Phys. Rev. Lett.* **79**, 1349 (1997).
11. K.-C. Chou, Z.-B. Su, B.-L. Hao and L. Yu, *Phys. Rep.* **118**, 1 (1985).
12. A. Kamenev and A. Andreev, *Phys. Rev. B* **60**, 2218 (1999).
13. Y. Utsumi, H. Imamura, M. Hayashi and H. Ebisawa, Phys. Rev. B 66, 024513 (2002) : Physica C 367, 237 (2002).
14. Y. Isawa and H. Horii, J. Phy. Soc. Jpn. 69, 655 (2000).
15. G. B. Lesovik, JETP Lett. 49, 592 (1989).
16. R. J. Schoelkopf et al., Science 280, 1238 (1998).