SPACELIKE GRAPHS WITH PRESCRIBED MEAN CURVATURE ON EXTERIOR DOMAINS IN THE MINKOWSKI SPACETIME

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Abstract. We consider a Dirichlet problem for the mean curvature operator in the Minkowski spacetime, obtaining a necessary and sufficient condition for the existence of a spacelike solution, with prescribed mean curvature, which is the graph of a function defined on a domain equal to the complement in $\mathbb{R}^n$ of the union of a finite number of bounded Lipschitz domains. The mean curvature $H = H(x, t)$ is assumed to have absolute value controlled from above by a locally bounded, $L^p$-function, $p \in [1, 2n/(n+2)]$, $n \geq 3$.

1. Introduction and statement of the results

Due to their importance in general relativity, spacelike hypersurfaces with constant mean curvature or, more generally, with prescribed mean curvature $H$, have been extensively studied since Lichnerowicz’s paper [34] (cf. [35]). Maximal spacelike hypersurfaces ($H = 0$) have also attracted the interest of researchers because of their similarity with minimal hypersurfaces in the Riemannian setting. In fact, they are critical points of the area functional and, under some curvature assumptions on the ambient spacetime, locally maximize it among all nearby spacelike hypersurfaces having the same boundary (cf. [22]). Two very relevant examples in this perspective are the so-called Calabi-Bernstein problem in the Minkowski spacetime [24, 26] and its counterpart with constant mean curvature hypersurfaces [41]. Since then a lot of related papers have been appearing; just to recall a few, we refer here to the new proofs of the Calabi-Bernstein theorem for surfaces given in [39, 2], as well as to related Calabi-Bernstein type results in different ambient spaces [27, 3, 1, 23, 25] and to some existence results in Minkowski spacetime for entire or radial spacelike graphs under different growth conditions on the mean curvature [13, 4, 5, 38, 37].

The Dirichlet problem on a bounded open set for spacelike hypersurfaces, described as the graph of a function, with prescribed mean curvature was

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studied in the Minkowski spacetime [9], in spacetimes conformal to an orthogonal splitting [29] and for some cosmological spacetimes [7]. A fairly general case was considered in [8]. Recently, non-smooth critical point theory [40] has been used in [11] to obtain existence and multiplicity of spacelike solutions in the Minkowski spacetime for the Dirichlet problem with homogeneous boundary data on a $C^2$ domain, when the mean curvature is a function depending on a parameter. We would like to emphasize that the result in [9] does not require assumptions on the regularity of the boundary. Namely, boundary values are considered according to the definition given in [9, p. 133] which allows one dealing with quite general open bounded subsets of $\mathbb{R}^n$ (for instance bounded domains with just continuous boundaries are admitted).

Among other results, in [9] it is also proved that if a variational solution, with mean curvature not depending on the time coordinate $t$, contains a segment of light ray, then it contains the ray extended to the boundary or to infinity. This property will be fundamental in the proof of our main result, Theorem 1.7, dealing with solutions vanishing at infinity of the Dirichlet problem for spacelike hypersurfaces on an unbounded open subset of $\mathbb{R}^n$. Theorem 1.7 can be considered as an extension of [9, Theorem 4.1] to exterior domains. A solution is indeed obtained by a standard minimization argument (Proposition 3.2) applied to functional $I$ in (5) which, differently from the area functional, is well-defined on functions with square integrable gradient on the exterior domain (see Section 3). The boundary conditions are as usual encoded in the functional setting adopted (see Section 2). As we need an extendibility property (Lemma 2.7), we reinforce a bit the regularity of the domain w.r.t. [9], by considering a Lipschitz boundary. Functional $I$ is then defined on a convex, closed subset (Proposition 2.9) of an affine subspace of the homogeneous Sobolev space of locally integrable functions with square integrable partial derivatives which share the same trace on the boundary of the exterior domain. The tangent space of this affine manifold is the Hilbert space obtained as the completion, w.r.t. the $L^2$-norm of the gradient, of the space of test functions on the exterior domain. This fact allows us to recover some embedding properties (Lemma 2.5) in a similar variational setting exploited in [14] to find solutions of a Born-Infeld equation in $\mathbb{R}^n$ (see the end of this introduction for more details).

Let $(\mathbb{L}^{n+1}, \langle \cdot, \cdot \rangle)$ be the $(n+1)$-dimensional Minkowski spacetime with the following sign convention: $\langle (\tau, v), (\tau, v) \rangle = -\tau^2 + |v|^2$ for $\tau \in \mathbb{R}, v \in \mathbb{R}^n$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. A smooth immersion $\phi: \Sigma \to \mathbb{L}^{n+1}$ of an $n$-dimensional connected manifold $\Sigma$ is a spacelike hypersurface if the metric induced by $\phi$ is a Riemannian metric on $\Sigma$. Let $A$ be the shape operator of $\Sigma$ and $H := -\frac{1}{n} \text{tr} A$ its mean curvature. Assume that $\Sigma$ is an open subset of $\mathbb{R}^n$ and $\phi(x) = (x, u(x)), x \in \Sigma$; when $u$ is at least of class
$C^2$, the mean curvature of this hypersurface is then equal to
\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) = nH(x, u),
\]
where $\text{div}(\cdot)$ is the divergence operator in $\mathbb{R}^n$.

Hereafter by a domain in $\mathbb{R}^n$ we mean an open and connected subset. Given a domain $\Omega$ and a $C^1$ function $u$ on it, its graph $t = u(x_1, \ldots, x_n) = u(x)$ defines a $C^1$ spacelike hypersurface in $\mathbb{L}^{n+1}$ if and only if the Euclidean gradient of $u$ satisfies $|\nabla u(x)| < 1$, for all $x \in \Omega$. In this case, we will say that $u$ is a spacelike function and its graph is then called a spacelike hypersurface.

Instead, a locally Lipschitz function $u$ is said weakly spacelike if $|\nabla u| \leq 1$, for a.e. $x \in \Omega$.

We also need the following definition.

**Definition 1.1.** Let $V \subset \mathbb{R}^n$; a function $\psi: V \to \mathbb{R}$ is said spacelike displacing if its graph is an acausal set, namely no couple of its points can be joined by a timelike or lightlike segment. This is equivalent to $|\psi(x) - \psi(y)| < |x - y|$, for all $x, y \in V$, $x \neq y$. Moreover, $\psi: V \to \mathbb{R}$ is said spacelike displacing in $W \subset \mathbb{R}^n$ if $|\psi(x) - \psi(y)| < |x - y|$, for all $x, y \in V$, $x \neq y$, with the inner points of the line segment $xy$ contained in $W$.

**Remark 1.2.** Notice that in $[9]$ $C^1$ spacelike functions are named strictly spacelike, while spacelike displacing functions in an open subset $V$ of $\mathbb{R}^n$ (and defined on the same $V$) are called spacelike (although the graph of a spacelike displacing function can have degenerate tangent spaces). Furthermore, note that if $V$ is open, the graph $G_\psi$ of a continuous spacelike displacing function in $V$, $\psi: V \to \mathbb{R}$, is spacelike in the usual sense for $C^0$ hypersurfaces, i.e., for each $p \in G_\psi$ there exists a neighbourhood $U$ in $\mathbb{L}^{n+1}$ such that $G_\psi \cap U$ is acausal and edgeless in $U$ (cf., e.g., [28, p. 213], [10, Definition 14.28]). Nevertheless, if $V$ is not convex, the graph of a spacelike displacing $\psi: V \to \mathbb{R}$ in $V$ can be not acausal.

We recall the definition of a Lipschitz domain.

**Definition 1.3.** An open subset $U \subset \mathbb{R}^n$ is said Lipschitz if for each $p \in \partial U$ there exist an open ball $B(p, r) \subset \mathbb{R}^n$ and a Lipschitz function $f: B(p, r) \to \mathbb{R}$ such that $B(p, r) \cap U = f^{-1}((0, +\infty))$.

**Remark 1.4.** If $U$ is a Lipschitz open subset, then $\partial U = \partial \overline{U}$ (see, e.g., [33, Remark 9.59]). For further use, notice that if $U \subset \mathbb{R}^n$ is Lipschitz, then $\partial U = \partial (\mathbb{R}^n \setminus \overline{U})$.

We deal with Lipschitz exterior domains of $\mathbb{R}^n$. More precisely, we require the following assumption.

**Assumption 1.5.** We consider an exterior domain in $\mathbb{R}^n$, $n \geq 3$, defined by means of a finite collection of bounded Lipschitz domains $\Omega_i$, such that
\( \Omega_i \cap \Omega_j = \emptyset \) for all \( i, j \in \{1, \ldots, m\}, i \neq j, m \geq 1 \). Let
\[
\Omega := \bigcup_{i=1}^{m} \Omega_i \quad \text{and} \quad \Omega_c := \mathbb{R}^n \setminus \overline{\Omega}.
\]

Notice that by Remark 1.4 \( \partial \Omega_c = \partial \Omega \).

Next let us set our problem. We consider the Dirichlet problem
\[
\begin{cases}
\text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) = nH(x, u) \quad \text{in} \ \Omega_c \\
u = \varphi \quad \text{on} \ \partial \Omega \\
\lim_{|x| \to +\infty} u(x) = 0
\end{cases}
\tag{1}
\]
where \( \varphi : \partial \Omega \to \mathbb{R} \) and \( H : \Omega_c \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function\(^1\) satisfying:

(H) there exists \( h \in L^s(\Omega_c) \cap L^\infty_{\text{loc}}(\Omega_c), s \in \left[1, \frac{2n}{n+2}\right] \), such that
\[
n|H(x, t)| \leq h(x) \quad \text{for a.e.} \ x \in \Omega_c \text{ and all } t \in \mathbb{R}.
\]

In order to introduce our functional framework (see Section 2 for more details and remarks), let us recall that \( \tilde{W}^{1,2}(\Omega_c) \) is the homogeneous Sobolev space of distributions on \( \Omega_c \) with partial derivatives in \( L^2(\Omega_c) \). Some useful properties of such a space can be found, for example, in [33, §11].

Let us finally introduce the space \( \mathcal{X} \) of admissible functions in the variational setting for problem (1):
\[
\mathcal{X} := \tilde{W}^{1,2}(\Omega_c) \cap L^{2^*}(\Omega_c) \cap \{u \in C^{0,1}_{\text{loc}}(\Omega_c) : \|\nabla u\|_{\infty} \leq 1\};
\tag{2}
\]
where \( C^{0,1}_{\text{loc}}(\Omega_c) \) denotes the space of locally Lipschitz functions on \( \Omega_c \) and as usual \( 2^* = \frac{2n}{n-2} \) is the Sobolev critical exponent.

Now we give the definition of weak solution of (1) (see also Remark 2.6).

**Definition 1.6.** A function \( u : \Omega_c \to \mathbb{R} \) is called a weak solution of (1) if it belongs to \( \mathcal{X} \), \( \varphi \) is the trace of \( u \) and
\[
\int_{\Omega_c} \frac{\nabla u \cdot \nabla v}{\sqrt{1 - |\nabla u|^2}} \ dx + n \int_{\Omega_c} H(x, u)v \ dx = 0, \quad \text{for all} \ v \in C^{\infty}_c(\Omega_c).
\tag{3}
\]

We state our main result.

**Theorem 1.7.** Let \( \Omega_c \) satisfy Assumption 1.5 and \( H : \Omega_c \times \mathbb{R} \to \mathbb{R} \) be such that (H) holds. Then, there exists a spacelike\(^2\) weak solution of (1) if and only if \( \varphi : \partial \Omega \to \mathbb{R} \) is the trace of a function \( w \in \mathcal{X} \) and moreover \( \varphi \) is spacelike displacing in \( \Omega_c \) when \( \Omega \) is not convex.

**Remark 1.8.** Let us emphasize some points about Theorem 1.7.

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\(^1\)\( H(\cdot, t) \) is measurable in \( \Omega_c \) for all \( t \in \mathbb{R} \) and \( H(x, \cdot) \) is continuous in \( \mathbb{R} \) for a.e. \( x \in \Omega_c \).

\(^2\)In the terminology of [9] such a solution is strictly spacelike, cf. Remark 1.2.
(1) The boundary condition, \( u = \varphi \) on \( \partial \Omega \), in (1) is meant in a trace sense, but as we will show in the next section a weak solution \( u \) can indeed be continuously Lipschitz extended to \( \partial \Omega \) (Lemma 2.7). Therefore, \( u|_{\partial \Omega_c} = \varphi \), hence \( \varphi \) is a posteriori Lipschitz continuous on \( \partial \Omega \).

(2) The limit at infinity in (1) is intended in the classical sense.

(3) As shown in [9, p. 148] (see also [16, p. 5]), as \( H \) is locally bounded, by elliptic regularity theory, a spacelike weak solution \( u \) belongs to \( W^{2,2}_{\text{loc}}(\Omega_c) \) and \( \nabla u \) is locally Hölder. Moreover, if \( H \in C^{k,\alpha}(\Omega_c \times \mathbb{R}) \), \( k \in \mathbb{N} \cup \{\infty\} \), then \( u \in C^{k+2,\alpha}(\Omega_c) \).

(4) If \( H \) does not depend on \( t \), our statement holds just assuming that \( H \in L^s(\Omega_c) \cap L^\infty_{\text{loc}}(\Omega_c) \). In particular, it holds for \( H = 0 \) and, in such a case, gives the existence of a maximal hypersurface on the exterior domain \( \Omega_c \). Previous existence (and uniqueness, with respect to a given asymptotic profile) results for the Dirichlet problem of maximal graphs on an exterior domain of the Minkowski spacetime have been recently obtained in [30]. Existence and multiplicity results for radial solutions outside a ball, with homogeneous boundary condition, have been obtained in [42] for a separable-variables \( H \) which is also radial in the \( x \) variable. We are not aware of other results for the Dirichlet problem in an exterior domain when \( H \neq 0 \).

Since when \( \varphi \) is \((1-\epsilon)\)-Lipschitz continuous on \( \partial \Omega \), for a \( \epsilon > 0 \), we can ensure that there exists \( u \in \mathcal{X} \) such that \( u|_{\partial \Omega} = \varphi \), we have the following:

**Corollary 1.9.** Under the assumptions on \( \Omega_c \) and \( H \) in Theorem 1.7, let \( \varphi : \partial \Omega \to \mathbb{R} \) be a \((1-\epsilon)\)-Lipschitz continuous. Then, there exists a spacelike weak solution of (1).

A further relevant physical motivation to the problem under study is given by the differential operator

\[
\mathcal{Q}(u) = \text{div}\left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right)
\]

which naturally appears in the Born-Infeld theory. Almost a century ago, Born and Infeld introduced a new electromagnetic theory in a series of papers [17, 19, 18, 20] as an alternative to the classical Maxwell theory. Such a theory was proposed as a nonlinear model of electrodynamics having the notable feature of being a fine answer to the well-known infinity energy problem (the electromagnetic field generated by a point charge has finite energy in Born-Infeld theory).

In last years many authors have been focusing their attention on problems related to \( \mathcal{Q} \) in the whole \( \mathbb{R}^n \), \( n \geq 1 \). In particular, some results for

\[
-\text{div}\left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) = \rho, \quad \text{in } \mathbb{R}^n
\]
can be found in [31, 14, 32, 16, 12, 15], under different assumptions on \( \rho \). Here \( \rho \) can be considered as an assigned charges source. We also refer to [6], where the Born-Infeld equation is coupled with the nonlinear Schrödinger one. Therefore, Theorem 1.7 can also be seen as an existence result for the Born-Infeld problem on an exterior domain with assigned boundary conditions.

2. Functional setting

Let us denote by \( dx \) the Lebesgue measure on \( \mathbb{R}^n \) and, unless differently specified, by \( \| \cdot \|_r \) the \( L^r \)-norm in the Lebesgue space \( L^r(\Omega_c) \), \( 1 \leq r \leq +\infty \), where \( \Omega_c \) is defined in Assumption 1.5. By \( \| \nabla \cdot \|_r \) we mean \( \| \nabla \cdot \|_r \).

Let \( V \subset \mathbb{R}^n \) be open and unbounded and let us consider the homogeneous Sobolev space \( D^{1,2}(V) \) defined as the completion of \( C^\infty_c(V) \), the space of smooth functions with compact support in \( V \), with respect to the \( L^2 \)-norm of the gradient. Moreover, let us denote by \( \mathcal{D}'(V) \) the space of distribution on \( V \).

The following properties hold for \( D^{1,2}(\Omega_c) \):

**Proposition 2.1.**

1. \( D^{1,2}(\Omega_c) \hookrightarrow L^2(\Omega_c) \); moreover \( D^{1,2}(\Omega_c) \hookrightarrow \mathcal{D}'(\Omega_c) \) and the partial distributional derivatives of its elements are represented by functions in \( L^2(\Omega_c) \);
2. all \( u \in D^{1,2}(\Omega_c) \) vanish at infinity, i.e., \( \text{meas}\{x \in \Omega_c : |u(x)| > t\} < +\infty \), for all \( t > 0 \);
3. for all \( u \in D^{1,2}(\Omega_c) \), the trace \( \text{Tr}(u) \) on \( \partial \Omega_c \) is well-defined and equal to 0.

**Proof.** (1) Since \( n \geq 3 \), by the Sobolev embedding theorem, \( D^{1,2}(\Omega_c) \) is continuously embedded in \( L^2(\Omega_c) \) (see [36, §15.1]). As a consequence, every \( u \in D^{1,2}(\Omega_c) \) can be identified with a distribution on \( \Omega_c \) and, if \( (u_k)_k \subset D^{1,2}(\Omega_c) \) is such that \( \| \nabla u_k - \nabla u \|_2 \to 0 \), then it converges to \( u \) in distributional sense. Let now \( (\varphi_k)_k \subset C^\infty_c(\Omega_c) \) be a Cauchy sequence representing \( u \in D^{1,2}(\Omega_c) \); then \( (\varphi_k)_k \) is Cauchy in \( L^2(\Omega_c) \) and for all \( \varphi \in C^\infty_c(\Omega_c) \), \( i \in \{1, \ldots, n\} \), we have

\[
(\partial_i u, \varphi) := -\int_{\Omega_c} u \partial_i \varphi \, dx = -\lim_k \int_{\Omega_c} \varphi_k \partial_i \varphi \, dx = \lim_k \int_{\Omega_c} \partial_i \varphi_k \varphi \, dx = \int_{\Omega_c} \psi_i \varphi \, dx,
\]

where \( \psi_i \in L^2(\Omega_c) \) is the element representing the Cauchy sequence \( \partial_i \varphi_k \) in \( L^2(\Omega_c) \).

(2) Notice that \( u \notin L^2(\Omega_c) \) if, by contradiction, there exists \( t > 0 \) such that \( \text{meas}\{x \in \Omega_c : |u(x)| > t\} = +\infty \).

(3) Take \( R > 0 \) such that \( \overline{\Omega} \subset B(0, R) \) and consider \( \nu \), the restriction to \( \Omega_c \) of a smooth function assuming values in the interval \([0, 1]\) and equal to
1 on $B(0, R)$, with compact support on $B(0, 2R)$ and such that $|\nabla u| < 1$. Let $(u_k)_k \subset C^\infty_c(\Omega_c)$ be a sequence converging to $u$ in $D^{1,2}(\Omega_c)$. Then $vu_k \to vu$ both in $D^{1,2}(\Omega_c)$ and $L^2^*(\Omega_c)$, thus $vu_k \to vu$ in the $H^1$-norm on $B(0, 2R) \setminus \Omega$, which implies that $vu \in H^1_0(B(0, 2R) \setminus \Omega)$. As $\partial \Omega_c$ is Lipschitz, $\text{Tr}(vu) = 0$ (cf., e.g., [33, Theorem 18.7]) and since $v = 1$ on $\partial \Omega_c$, we get that $\text{Tr}(u)_{|\partial \Omega} = 0$. 

Actually, the above inclusions and properties characterize $D^{1,2}(\Omega_c)$. In the following we give some useful properties of the homogeneous Sobolev space $\dot{W}^{1,2}(\Omega_c)$ and a relation between it and $D^{1,2}(\Omega_c)$ which is crucial to our purposes.

Remark 2.2. Since $\Omega_c$ is a Lipschitz domain, the trace operator is well-defined on $\dot{W}^{1,2}(\Omega_c)$. Indeed, for any $R > 0$ such that $\overline{\Omega} \subset B(0, R)$ we have $\dot{W}^{1,2}(B(0, R) \setminus \overline{\Omega}) = W^{1,2}(B(0, R) \setminus \overline{\Omega})$ (see [36, Corollary 1.1.11]) and then $\text{Tr}(u) \in L^2(\partial \Omega_c)$, for all $u \in \dot{W}^{1,2}(\Omega_c)$. However, the trace operator is evidently not bounded in $\dot{W}^{1,2}(\Omega_c)$. Anyway, if $\|\nabla u_k - \nabla u\|_2 \to 0$ and $\|u_k - u\|_2 \to 0$, then $\|\text{Tr}(u_k) - \text{Tr}(u)\|_{L^2(\partial \Omega_c)} \to 0$. Moreover, if $u \in C^0(\overline{\Omega_c})$, then $\text{Tr}(u) = u_{|\partial \Omega}$.

Proposition 2.3. The space $D^{1,2}(\Omega_c)$ is given by

$$D^{1,2}(\Omega_c) = \{ u \in \dot{W}^{1,2}(\Omega_c) : u \in L^2^*(\Omega_c), \text{Tr}(u) = 0 \}.$$ 

Proof. The first inclusion has already been shown in Proposition 2.1. For the other one, let $u \in \dot{W}^{1,2}(\Omega_c) \cap L^2^*(\Omega_c)$ be such that $\text{Tr}(u) = 0$. As $u \in L^2^*(\Omega_c)$ and $|\nabla u| \in L^2(\Omega_c)$, for any $\epsilon > 0$ there exists $R > 0$ such that $\overline{\Omega} \subset B(0, R)$, $\|u\|_{L^2^*(\mathbb{R}^n \setminus B(0, R))} < \epsilon$ and $\|\nabla u\|_{L^2(\mathbb{R}^n \setminus B(0, R))} < \epsilon$. Let us take $v$ as in the proof of Proposition 2.1, with the further requirement that $|\nabla v| \leq 1/R$ on $A_R := B(0, 2R) \setminus B(0, R)$, so that $vu \in H^1_0(B(0, 2R) \setminus \overline{\Omega})$. Let $(u_k)_k \subset C^\infty_c(B(0, 2R) \setminus \overline{\Omega})$ converge to $vu$ in the $H^1$-norm. Then, taking into account that $\|u\|_{L^2^*(A_R)} < \epsilon$ and $\|\nabla u_k\|_{L^2(A_R)} \to \|\nabla(vu)\|_{L^2(A_R)}$, trivially extending $u_k$ on $\mathbb{R}^n \setminus B(0, 2R)$, we get that $u_k \to u$ in $D^{1,2}(\Omega_c)$.

Let us now introduce the following subset of $D^{1,2}(\Omega_c)$:

$$\mathcal{X}_0 := D^{1,2}(\Omega_c) \cap \{ u \in C^{0,1}_{\text{loc}}(\Omega_c) : \|\nabla u\|_\infty \leq 1 \}.$$ 

Let $j(u) : \mathbb{R}^n \to \mathbb{R}$, $u \in \mathcal{X}_0$, be defined as follows:

$$j(u)(x) = \begin{cases} 
 u(x) & \text{if } x \in \Omega_c \\
 0 & \text{otherwise}. 
\end{cases}$$

Notice that a Lipschitz domain according to Definition 1.3 is called $C^{0,1}$ domain in [36]; $C^{0,1}$ bounded domains satisfy the cone property required in [36, Corollary 1.1.11] (see [36, Remark 1, p. 15]).
Lemma 2.4. The map \( j \) is an isometry from \( \mathcal{X}_0 \) (as a subset of the Hilbert space \( D^{1,2}(\Omega_c) \)) into the subset \( D^{1,2}(\mathbb{R}^n) \cap \{ u \in C^{0,1}(\mathbb{R}^n) : \| \nabla u \|_{L^\infty(\mathbb{R}^n)} \leq 1 \} \) of \( D^{1,2}(\mathbb{R}^n) \).

Proof. Let \( u \in \mathcal{X}_0 \); it is clear that \( j(u) \in D^{1,2}(\mathbb{R}^n) \). Hence, \( j(u) \in L^{2^*}(\mathbb{R}^n) \) and \( |\nabla j(u)| \in L^{2}(\mathbb{R}^n) \); moreover, \( |\nabla j(u)(x)| \leq 1 \) for a.e. \( x \in \mathbb{R}^n \). By Morrey’s embedding theorem and we deduce that \( j(u) \in L^{\infty}(\mathbb{R}^n) \), hence \( j(u) \in W^{1,\infty}(\mathbb{R}^n) \) and therefore it is Lipschitz.

As a consequence of Lemma 2.4, \( \mathcal{X}_0 \) can be identified with a subset of \( D^{1,2}(\mathbb{R}^n) \cap \{ u \in C^{0,1}(\mathbb{R}^n) : \| \nabla u \|_{L^\infty(\mathbb{R}^n)} \leq 1 \} \) and then by [14, Lemma 2.1] we immediately get:

**Lemma 2.5.** As a subset of the Hilbert space \( D^{1,2}(\Omega_c) \), \( \mathcal{X}_0 \) satisfies the following properties:

1. it is continuously embedded in \( W^{1,p}(\Omega_c) \), for all \( p \in [2^*,+\infty) \);
2. it is continuously embedded in \( L^{\infty}(\Omega_c) \);
3. all \( u \in \mathcal{X}_0 \) satisfy \( \lim_{|x| \to \infty} u(x) = 0 \);
4. it is a convex and weakly closed subset of \( D^{1,2}(\Omega_c) \);
5. any bounded sequence \( \{u_k\} \subset \mathcal{X}_0 \) admits a subsequence weakly converging to some \( u \in \mathcal{X}_0 \) and uniformly on compact subsets of \( \Omega_c \).

Remark 2.6. Let us observe that if \( u \in \mathcal{X} \) (cf. (2)) is a weak solution according to Definition 1.6, then the identity in (3) also holds for any \( u \in \mathcal{X}_0 \) by means of a convolution argument and [14, Lemma 2.10].

Lemma 2.7. Let \( \Omega_c \) satisfy Assumption 1.5. Then, every \( u \in C^{0,1} loc(\Omega_c) \) such that \( |\nabla u| \in L^{\infty}(\Omega_c) \) can be extended to a Lipschitz function on \( \Omega_c \).

Proof. Let us take \( R > 0 \) such that \( \overline{\Omega} \subset B(0,R) \) and consider the bounded open set \( V_R := B(0,R) \setminus \overline{\Omega} \). Let us show that \( u \) is bounded on \( V_R \). Clearly \( V_R \) is a Lipschitz domain and then it is the union of a finite number of Lipschitz domains \( V_j \) starshaped with respect to balls \( B(y_j,R_j) \) contained in \( V_R \) (see [36, Lemma 1, p. 15]).

Now assume by contradiction that \( u|_{V_R} \) is not bounded. Then there exists a sequence \( \{x_k\} \subset V_R \) such that \( |u(x_k)| \to +\infty \). Let \( j_k \) be one of the indices \( j \) such that \( x_k \in V_{j_k} \). Then the segment \( y_{j_k}x_k \) is contained in \( V_{j_k} \). Hence,

\[
|u(x_k)| \leq |u(y_{j_k})| + \| \nabla u \|_{L^\infty}|x_k - y_{j_k}| \leq M + 2R\| \nabla u \|_{L^\infty},
\]

where \( M := \max_{j} |u(y_j)| \), a contradiction.

Thus, \( u|_{V_R} \in W^{1,\infty}(V_R) \) and, since \( V_R \) has the extendibility property (see [33, Th. 13.17]4), it admits an extension \( \tilde{u} \in W^{1,\infty}(\mathbb{R}^n) \). Then

\[
\tilde{u}(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in B(0,R) \\ u(x) & \text{otherwise} \end{cases} \quad (4)
\]

4We point out that the theorem can be applied since \( \partial \Omega = \partial \Omega_c \) is bounded, hence by [33, p. 424] being Lipschitz is equivalent to be uniformly Lipschitz.
is an extension of \( u \) to \( \mathbb{R}^n \) such that \( |\nabla \bar{u}| \in L^\infty(\mathbb{R}^n) \); then it is Lipschitz on \( \mathbb{R}^n \) (see, e.g. [33, Ex. 11.50-(i)]) and \( \bar{u}_{|\Omega_c} \) is a Lipschitz extension of \( u \).

Now we are ready to show that \( X \) is included in \( W^{1,p}(\Omega_c) \), for all \( p \in [2^*, +\infty] \).

**Lemma 2.8.** Let \( u \in X \); then \( u \in W^{1,p}(\Omega_c) \), for all \( p \in [2^*, +\infty] \) and moreover \( \lim_{|x|\to\infty} u(x) = 0 \).

**Proof.** As in the proof of Lemma 2.7, \( u \) admits a Lipschitz extension \( \tilde{u} \) to \( \mathbb{R}^n \). Since \( \Omega \) is bounded, we get that

\[
\tilde{u} \in W^{1,2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap C^{0,1}(\mathbb{R}^n) = D^{1,2}(\mathbb{R}^n) \cap C^{0,1}(\mathbb{R}^n).
\]

we deduce that \( \tilde{u} \in W^{1,p}(\mathbb{R}^n) \) for all \( p \in [2^*, +\infty] \) and \( \lim_{|x|\to\infty} \tilde{u}(x) = 0 \); plainly, analogous properties hold for \( u \).

Finally, let \( \varphi \in L^2(\partial\Omega) \) be such that there exists \( w \in X \) with \( \text{Tr}(w) = \varphi \). By Lemma 2.7 we have that \( \varphi = w|_{\partial\Omega} \) (in particular \( \varphi \) must be Lipschitz on \( \partial\Omega \)). Let us set

\[
X_\varphi := \{ u \in X : \text{Tr}(u) = \varphi \}.
\]

Then by Proposition 2.3 for every \( w \in X_\varphi \) we get

\[
X_\varphi = \{ w \} + X_0.
\]

**Proposition 2.9.** \( X_\varphi \) is convex and weakly closed as a topological subset of the semi-normed vector space \( \dot{W}^{1,2}(\Omega_c) \). Moreover, if \( (u_k)_k \) is a bounded sequence in \( X_\varphi \), then up to a subsequence it weakly converges to some \( u \in X_\varphi \) and uniformly converges on compact subsets of \( \Omega_c \).

**Proof.** Fix any \( w \in X_\varphi \); since \( X_\varphi \) is the translation by \( w \) of a convex and weakly closed subset of \( D^{1,2}(\Omega_c) \hookrightarrow \dot{W}^{1,2}(\Omega_c) \) and the trace operator is linear, we get that \( X_\varphi \) is convex and weakly closed in \( \dot{W}^{1,2}(\Omega_c) \).

If \( (u_k)_k \) is a bounded sequence in \( X \), then \( (u_k - w)_k \) is bounded in \( D^{1,2}(\Omega_c) \), hence there exists a weakly converging subsequence to a function \( v \in D^{1,2}(\Omega_c) \). This implies that \( u_k \to w + v \in X_\varphi \). Moreover, by Lemma 2.5–(2), \( (u_k - w)_k \) is bounded in \( L^\infty(\Omega_c) \) and then, by Ascoli-Arzelà theorem, for any given compact set \( K \subset \Omega_c \) it admits a subsequence uniformly converging to \( v \) on \( K \).

3. Proofs of the main results

Let \( \varphi \in L^2(\partial\Omega) \) be such that there exists \( w \in X \) with \( \text{Tr}(w) = \varphi \) and consider the functional

\[
\mathcal{I}(u) = \int_{\Omega_c} \left( 1 - \sqrt{1 - |\nabla u|^2} \right) \, dx + \int_{\Omega_c} G(x, u) \, dx
\]

with \( G(x, t) := n \int_0^t H(x, s) \, ds \). Recalling that \( \frac{1}{2} t \leq 1 - \sqrt{1 - t} \leq t \), for all \( t \in [0, 1] \), by assumption (H) \( \mathcal{I} \) is well-defined on \( X_\varphi \) (see also the proof of
the first part of Lemma 3.1). Notice also that every spacelike critical point $u$ of $\mathcal{I}$ weakly satisfies (in the sense of Definition 1.6) the first equation in (1). Thus, a spacelike weak solution of (1) can be found if $\mathcal{I}$ has a spacelike minimizer on $\mathcal{X}_ϕ$. In order to prove this last statement we need the following lemmas. Let us set

$$\mathcal{I}_0(u) := \int_{\Omega_c} \left( 1 - \sqrt{1 - |\nabla u|^2} \right) dx \quad \text{and} \quad \mathcal{G}(u) := \int_{\Omega_c} G(x,u) \, dx.$$  

**Lemma 3.1.** Under the assumption on $\Omega_c$ and $H$ in Theorem 1.7, assume also that $\varphi = \text{Tr}(w)$ for some $w \in \mathcal{X}$. Then $\mathcal{G}$ is well-defined on $\mathcal{X}_ϕ$ and sequentially weakly continuous.

**Proof.** Let us denote by $s'$ the conjugate exponent of $s$. As $u \in \mathcal{X}_ϕ \subset \mathcal{X}$, by Lemma 2.8 we get that $u \in W^{1,p}(\Omega_c)$ for all $p \in [2^*,+\infty]$. Then for all $s \in [1, \frac{2n}{n+2}]$, $s' \geq 2^*$ and

$$|\mathcal{G}(u)| \leq \|h\|_s \|u\|_{s'} < +\infty.$$  

Let us now show that $\mathcal{G}$ is sequentially weakly continuous on $\mathcal{X}_ϕ$. Assume that $(u_k)_k \subset \mathcal{X}_ϕ$ weakly converges to $u \in W^{1,2}(\Omega_c)$. By Proposition 2.9, $u \in \mathcal{X}_ϕ$; moreover $(u_k - w)_k \subset \mathcal{X}_0$ and then it is bounded in $D^{1,2}(\Omega_c)$. Since $h \in L^s(\Omega_c)$, for a given $\epsilon > 0$ there exists $R > 0$ such that $\overline{\Omega} \subset B(0, R)$ and

$$\|h\|_{L^s(\mathbb{R}^n \setminus B(0,R))} < \epsilon;$$

therefore by Lemma 2.5–(1), there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^n \setminus B(0,R)} |G(x,u_k)| \, dx \leq \|h\|_{L^s(\mathbb{R}^n \setminus B(0,R))} \|u_k\|_{s'}$$

$$\leq \|h\|_{L^s(\mathbb{R}^n \setminus B(0,R))} (\|u_k\|_{s'} + \|w\|_{s'}) \leq C_1 \epsilon \quad (6)$$

and

$$\int_{\mathbb{R}^n \setminus B(0,R)} |G(x,u)| \, dx \leq C_1 \epsilon. \quad (7)$$

Notice that, again by assumption (H), taking the bounded subset $B(0,R) \setminus \overline{\Omega}$, it results in particular $h \in L^1(B(0,R) \setminus \overline{\Omega})$ and then by Lemma 2.5–(2), for a $C_2 > 0$ we get

$$|G(x,u_k)| \leq h(x) \|u_k\|_{\infty} \leq h(x)(\|u_k - w\|_{\infty} + \|w\|_{\infty}) \leq C_2 h(x),$$

for all $k \in \mathbb{N}$, a.e. in $B(0,R) \setminus \overline{\Omega}$. By Proposition 2.9 the bounded sequence $(u_k)_k$ uniformly converges on compact subsets in $\overline{\Omega}$, up to a subsequence. Hence, $(G(x,u_k))_k$ converges to $G(x,u)$ a.e. in $B(0,R) \setminus \overline{\Omega}$ and by the Lebesgue’s dominated convergence theorem it follows that

$$\int_{B(0,R) \setminus \overline{\Omega}} G(x,u_k) \, dx \to \int_{B(0,R) \setminus \overline{\Omega}} G(x,u) \, dx, \quad \text{as} \quad k \to +\infty.$$  

As the last convergence actually holds for the whole sequence, being $\epsilon$ arbitrary, by (6) and (7), we are done. \qed
Let us now show that $I$ has a global minimum point on $X_\varphi$.

**Proposition 3.2.** Under the assumption on $\Omega$ and $H$ in Theorem 1.7, the functional $I$ possesses at least a minimizer in $X_\varphi$.

**Proof.** By assumption (H), Lemma 2.8 and (1)-(2) of Lemma 2.5, denoting by $s'$ the conjugate exponent of $s$, for all $u \in X_\varphi$ we have:

$$I(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \|h\|_s\|u\|_{s'}$$

$$\geq \frac{1}{2}\|\nabla u\|_2^2 - \|h\|_s\|u - w\|_{s'} - \|h\|_s\|w\|_{s'}$$

$$\geq \frac{1}{2}\|\nabla u\|_2^2 - C_1\|h\|_s\|\nabla u - \nabla w\|_2 - \|h\|_s\|w\|_{s'}$$

$$\geq \frac{1}{2}\|\nabla u\|_2^2 - C_1\|h\|_s\|\nabla u\|_2 - C_1\|h\|_s\|\nabla w\|_2 - \|h\|_s\|w\|_{s'}$$

hence $I$ is coercive. By Proposition 2.9, in order to get the existence of a minimizer, we just need to prove that $I$ is sequentially weakly lower semi-continuous. Actually this holds because the first term $I_0$ of $I$ is convex and strongly continuous (cf. [14, Lemma 2.2] for details), thus it is sequentially weakly lower semi-continuous in the semi–normed space $W^{1,2}(\Omega_c)$ and $G$ is sequentially weakly continuous by Lemma 3.1.

The lemma below is based on [21] and shows that a minimizer $u$ of $I$ is a minimizer also for the analogous functional corresponding to $I$ with $H^*(x) = nH(x, u(x))$ replacing $nH$.

**Lemma 3.3.** Under the assumptions of Lemma 3.1, assume also that $u$ is a minimizer of $I$ on $X_\varphi$ and set $H^*(x) := nH(x, u(x))$. Then $u$ is also a minimizer of the functional $I^*: X_\varphi \to \mathbb{R}$, where

$$I^*(v) := \int_{\Omega_c} \left(1 - \sqrt{1 - |\nabla v|^2}\right) dx + \int_{\Omega_c} H^*(x)v(x) dx.$$

**Proof.** Let $v \in X_\varphi$; then $u_\lambda := u + \lambda(v - u) \in X_\varphi$, for all $\lambda \in [0, 1]$. Hence for all $\lambda \in (0, 1]$

$$I_0(u_\lambda) + G(u_\lambda) \geq I_0(u) + G(u)$$

and being $I_0$ convex we obtain

$$I_0(v) - I_0(u) - \frac{1}{\lambda}(G(u_\lambda) - G(u)) \geq 0.$$  \hspace{1cm} (8)

Let $\sigma(x) \in [0, 1]$ be such that

$$\frac{1}{\lambda}(G(u_\lambda) - G(u)) = n \int_{\Omega_c} H(x, u + \sigma(x)\lambda(v - u))(v - u) \, dx;$$

hence, by assumption (H) and Lebesgue’s dominated convergence theorem (recall that $v - u \in X_0 \hookrightarrow L^\infty(\Omega_c)$), we get

$$n \int_{\Omega_c} H(x, u + \sigma(x)\lambda(v - u))(v - u) \, dx \longrightarrow \int_{\Omega_c} H^*(x)(v - u) \, dx,$$
as \( \lambda \to 0 \), which by \((8)\) implies
\[
\mathcal{I}^*(v) - \mathcal{I}^*(u) \geq 0.
\]
\[\square\]

**Remark 3.4.** Let \( \bar{u} \in \mathcal{X}_\varphi \) be a minimizer of \( \mathcal{I} \) found in Proposition 3.2. By Lemma 3.3 all the conclusions of \([14, \text{Proposition 2.7}]\), suitable modified, hold and, in particular, we have that
\[
\text{meas}\{x \in \Omega_c : |\nabla \bar{u}| = 1\} = 0
\]
and
\[
\frac{|\nabla \bar{u}|^2}{\sqrt{1 - |\nabla \bar{u}|^2}} \in L^1(\Omega_c).
\]
However, this is still not enough to conclude that \( \bar{u} \) is a weak solution of problem \((1)\) in the sense of Definition 1.6.

Finally, we point out that, by \([9, \text{Proposition 1.1}]\), the minimizer is unique if \( H \) is non-decreasing in \( t \).

We can now conclude the proof of our main result.

**Proof of Theorem 1.7.** As a first step we want to prove that if \( \varphi \) is the trace of a function in \( \mathcal{X}_\varphi \), then each minimizer \( \bar{u} \) of \( \mathcal{I} \) in \( \mathcal{X}_\varphi \) is not only weakly spacelike, but spacelike as well, and so it is a weak solution of \((1)\).

By Lemma 3.3 \( \bar{u} \) is a minimizer for the functional \( \mathcal{I}^* \) and, since \( H \) is locally bounded by \([9, \text{Theorem 3.2}]\), we infer that any segment of a lightlike geodesic possibly contained in the graph of \( \bar{u} \) can be extended until it reaches a point on the graph of \( \varphi \). This fact and the local estimates in the proof of \([9, \text{Theorem 4.1}]\) imply that the subset \( K \subset \Omega_c \) where a minimizer could be non-regular is precisely given by the points which are the projections on \( \Omega_c \) of light rays and lightlike segments (i.e., resp., lines or half-lines and segments whose tangent vectors are lightlike) contained in the graph \( G_{\bar{u}} \), such that, respectively, no point or at least one point or both endpoints belong to \( \partial \Omega_c \). Thus, if \( \Omega \) is convex, \( K \) might contain only lines or half-lines (without, potentially, their points in the boundary) which are projections of light rays in \( G_{\bar{u}} \), but this is incompatible with the fact that \( \lim_{|x| \to \infty} \bar{u}(x) = 0 \) (recall Lemma 2.8). If \( \Omega \) is not convex and there exists a segment \( \overline{xy} \) contained in \( K \) whose endpoints \( x, y \) belong to \( \partial \Omega \), we would have \( |\varphi(x) - \varphi(y)| = |\bar{u}(x) - \bar{u}(y)| = |x - y| \), in contradiction with the spacelike displacing assumption on \( \varphi \). Therefore, \( K = \emptyset \) and \( \bar{u} \) is spacelike.

Let us now prove the reverse implication of the theorem. We observe that, according to Definition 1.6, any spacelike weak solution \( \bar{u} \) of \((1)\) is locally 1–Lipschitz, then by Lemma 2.7 \( \varphi = \bar{u}|_{\partial \Omega} \) and if \( \Omega \) is not convex and there exists a segment \( \overline{xy} \) in \( \Omega_c \) connecting two points \( x, y \in \partial \Omega \), then \( |\varphi(x) - \varphi(y)| = |\bar{u}(x) - \bar{u}(y)| < |x - y| \), i.e., \( \varphi \) is spacelike displacing in \( \Omega_c \).
\[\square\]
If \( \varphi \) is \((1-\epsilon)-\)Lipschitz, we can show that it is the trace of a function in \( X_\varphi \).

**Proof of Corollary 1.9.** We can extend \( \varphi \) to a bounded \((1-\epsilon)-\)Lipschitz function \( \psi \) on \( \mathbb{R}^n \) such that \( \min \psi = \min \varphi \) and \( \max \psi = \max \varphi \) (see, e.g., [33, p. 243]). Let us take \( R > 0 \) such that \( \overline{\Omega} \subset B(0, R) \), \( \epsilon R > \| \varphi \|_\infty \) and a smooth function \( v : \mathbb{R}^n \to [0, 1] \) with compact support in \( B(0, 2R) \), equal to 1 on \( B(0, R) \) and such that \( |\nabla v| \leq 1/R \). As \( |\nabla (v \psi)| \leq \frac{\| \varphi \|_\infty + 1 - \epsilon}{R} \), the restriction of \( v \psi \) to \( \Omega_\epsilon \) belongs to \( X_\varphi \). Therefore \( \varphi \) is the trace of a function in \( X_\varphi \) and we can conclude by Theorem 1.7. \( \square \)

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