Godunov Variables in Relativistic Fluid Dynamics

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Abstract

This note presents Godunov variables and 4-potentials for the relativistic Euler equations of barotropic fluids. The associated additional conservation/production law has different interpretations for different fluids. In particular it refers to entropy in the case of thermobarotropic fluids, and to matter in the case of isentropic fluids. The paper also presents an explicit formula for the generating function of the Euler equations in the case of ideal gases. It pursues ideas on symmetric hyperbolicity going back to Godunov (cf. also Lax and Friedrichs as well as Boillat) that were elaborated as Ruggeri and Strumia’s theory of convex covariant density systems.

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1 Convex covariant density systems

In famous papers of 1954, 1957, and 1961, Friedrichs presented the notion of symmetric hyperbolic systems of first-order partial differential equations and started their mathematical theory [9]. Lax defined the concept of hyperbolic systems of conservation laws [15] and started their theory, and Godunov identified a class of systems of conservation laws that are symmetric hyperbolic when written in appropriate natural variables [12]. All three steps were strongly motivated from physics, which in each case provides many important examples. After related contributions by Lax and Friedrichs [10] and Boillat [1], Ruggeri and Strumia formulated in 1981 a covariant version of first-order symmetric hyperbolicity as a theory of “convex covariant density systems” [18], notably for application exactly to relativistic fluid dynamics.

Definition 1. One calls \( \Upsilon = (\Upsilon_0, \ldots, \Upsilon_N) \) Godunov variables and a vector \( X^\beta(\Upsilon) \) a 4-potential for a covariant causal system of conservation laws

\[
\frac{\partial}{\partial x^\beta} F^{a\beta} = 0, \quad a = 0, \ldots, N, \tag{1.1}
\]

if

\[
F^{a\beta} = \frac{\partial X^\beta(\Upsilon)}{\partial \Upsilon_a} \tag{1.2}
\]

and

\[
\left( \frac{\partial^2 X^\beta(\Upsilon)}{\partial \Upsilon_a \partial \Upsilon_g} T^\beta \right)_{a,g=0,\ldots,N} \text{ is definite for all } T^\beta \text{ with } T^\beta T_\beta < 0. \tag{1.3}
\]

Corollary 1. In the situation of Definition 1, equations (1.1) can be written as a quasilinear symmetric system

\[
A^{a\beta g}(\Upsilon) \frac{\partial \Upsilon_g}{\partial x^\beta} = 0, \tag{1.4}
\]

which has every time-like vector \( T^\beta \) as a direction of hyperbolicity. All smooth solutions of (1.1) satisfy the additional conservation law

\[
\frac{\partial}{\partial x^\beta} \left( X^\beta(\Upsilon) - F^{a\beta}(\Upsilon) \Upsilon_a \right) = 0. \tag{1.5}
\]

Note that symmetry and hyperbolicity follow with

\[
A^{a\beta g} = \frac{\partial^2 X^\beta}{\partial \Upsilon_a \partial \Upsilon_g}, \quad a, g = 0, \ldots, N,
\]

and equation (1.3) is a consequence of

\[
\frac{\partial X^\beta}{\partial x^\beta} = \frac{\partial X^\beta}{\partial \Upsilon_a} \frac{\partial \Upsilon_a}{\partial x^\beta} = F^{a\beta} \frac{\partial \Upsilon_a}{\partial x^\beta} = \frac{\partial}{\partial x^\beta} \left( F^{a\beta} \Upsilon_a \right). \tag{1.6}
\]
The contents of Definition 1 and Corollary 1 closely follow \[9, 15, 12, 10, 1, 18\].

The relativistic dynamics of perfect fluids is governed by the conservation equations for energy-momentum,

\[ \frac{\partial}{\partial x^\beta} (T^{\alpha\beta}) = 0, \]  

and matter,

\[ \frac{\partial}{\partial x^\beta} (nU^\beta) = 0, \]

where

\[ T^{\alpha\beta} = \left( \rho + p \right) U^\alpha U^\beta + pg^{\alpha\beta} \]

and \( nU^\beta \) are the energy-momentum tensor and the matter current. Here \( U^\alpha \) denotes the 4-velocity of the fluid, and the fluid itself is specified by its specific internal energy \( e \) as a function of matter density \( n \) and specific entropy \( \sigma \), from which internal energy \( \rho \) and pressure \( p \) derive,

\[ e = e(n, \sigma), \quad \rho = \rho(n, \sigma) = ne(n, \sigma), \quad p = p(n, \sigma) = n^2 e_n(n, \sigma). \]

The five partial differential equations (1.7), (1.8) constitute a system for ‘five fields’: the 4-velocity (three degrees of freedom, as it is constrained by unitarity, \( U^\alpha U_\alpha = -1 \)) and two thermodynamic variables (such as \( n \) and \( \sigma \)).

Ruggeri and Strumia have shown in \[18\] that for very general fluids \[1.10\], very similarly to the non-relativistic situation (cf. \[12\]), the quantities

\[ \psi_\alpha = U^\alpha \theta, \quad \alpha = 0, 1, 2, 3, \quad \text{and} \quad \psi_4 = \frac{\mu}{\theta} \]

are Godunov variables for the system (1.7), (1.8), where

\[ \theta = e_\sigma(n, \sigma) \quad \text{and} \quad \mu = \frac{\rho + p}{n} - \theta \sigma \]

denote temperature and chemical potential, and the additional conservation law\(^2\) is that for the entropy,

\[ \frac{\partial}{\partial x^\beta} (n\sigma U^\beta) = 0. \]

The purposes of this note are to give related particular treatments for (a) barotropic fluids, (b) isentropic fluids, and to (c) derive an explicit formula for the 4-potential in the case of ideal gases.

\(^1\)The only difference from \[18\] is the fact that in consistency with causality, we require the definiteness in \[1.3\] (and obtain hyperbolicity) for all time-like \( T_\beta \) instead of some.

\(^2\)Cf. Section 3 for solutions with shock waves.
2 Godunov variables for barotropic and isentropic fluids

A perfect fluid is barotropic and causal if there is a one-to-one relation between internal energy and pressure,

\[ p = \hat{\rho}(p), \quad (2.1) \]

which satisfies

\[ \hat{\rho}'(p) \geq 1. \quad (2.2) \]

For barotropic fluids, the conservation laws (1.7) for energy-momentum are a self-consistent system that determines \( \rho \) and \( p \) (for given initial data) without reference to \( n, \theta \) or \( s \); this may be called a four-field theory, as now only one thermodynamic variable (which can be taken to be \( \rho \) or \( p \)) is needed in addition to \( U^\alpha \). It is probably due also to this nice reduction in the number of variables that barotropic fluids are widely used in physics (cf. [13, 22, 3, 20]).

Remarkably, these four-field theories permit an independent analogue of the abovementioned results for five-field theories.

**Theorem 1.** (i) To every causal barotropic fluid \((2.1), (2.2)\), there exist essentially unique\(^3\) functions \( f = f(p) \) and \( X(\Upsilon) = \tilde{X}(f) \) such that the quantities

\[ \Upsilon_\alpha = \frac{U_\alpha}{f}, \]

are Godunov variables and

\[ X^\beta(\Upsilon) = \frac{\partial X(\Upsilon)}{\partial \Upsilon_\beta} = f^3 \tilde{X}'(f) \Upsilon^\beta, \quad f = \left(-\Upsilon_\alpha \Upsilon^\alpha\right)^{-1/2} \]

is a 4-potential which together represent \((1.7)\) in the form \((1.1), (1.2)\) with \((1.3)\).

(ii) The corresponding additional conservation law \((1.5)\) is given by

\[ \frac{\partial}{\partial x^\beta} (\nu U^\beta) = 0, \quad (2.3) \]

with

\[ \nu = \nu(p) \equiv \frac{\hat{\rho}(p) + p}{f(p)} = \frac{1}{f'(p)}. \quad (2.4) \]

**Proof.** (i) Consider a momentarily arbitrary function of the form

\[ X(\Upsilon) = \tilde{X}(f) \quad \text{with} \quad f = \left(-\Upsilon_\alpha \Upsilon^\alpha\right)^{-1/2}. \]

\(^3f \) is unique up to a positive multiplicative constant. Once \( f \) is normalized, say by \( f(1) = 1 \), \( X \) is determined up to an additive constant.
As 
\[ df = f^3 \Upsilon^\alpha d \Upsilon_\alpha, \]
we have 
\[ X^\beta(Y) = \frac{\partial X}{\partial \Upsilon_\beta} = \pi(f) \Upsilon^\beta \quad \text{with} \quad \pi(f) \equiv f^3 \dot{X}'(f) \]
and 
\[ \frac{\partial^2 X}{\partial \Upsilon_\alpha \partial \Upsilon_\beta} = f^3 \pi'(f) \Upsilon^\alpha \Upsilon^\beta + \pi(f) g^{\alpha\beta} = f \pi'(f) U^\alpha U^\beta + \pi(f) g^{\alpha\beta} \]
Now, equation (1.2), here
\[ T^{\alpha\beta} = \frac{\partial^2 X}{\partial \Upsilon_\alpha \partial \Upsilon_\beta}, \quad (2.5) \]
is equivalent to
\[ \pi(f(p)) = p \quad \text{and} \quad f(p) \pi'(f(p)) = \dot{\rho}(p) + p, \]
and this is equivalent to
\[ \frac{f'(p)}{f(p)} = \frac{1}{\dot{\rho}(p) + p} \quad \text{and} \quad \pi = f^{-1}. \quad (2.6) \]
From \( \pi \), one determines \( \dot{X} \) as
\[ \dot{X}(f) = \int (1/f)^3 \pi(f) df. \]
(iii) This follows from 
\[ X^\beta(Y) - T^{\alpha\beta}(Y) \Upsilon_\alpha = \pi'(f) U^\beta. \]
\[ \square \]
Among the barotropic fluids, isentropic fluids are characterized by the property that their specific internal energy, internal energy, and pressure do not depend on an entropy, but solely on the matter density \( n \),
\[ e = e(n) > 0, \quad \rho = \rho(n) = n e(n) > 0, \quad p = p(n) = n^2 e'(n) > 0. \quad (2.7) \]
For isentropic fluids, also the specific enthalpy \( h \) is a function of \( n \) alone,
\[ h = \frac{\rho + p}{n} \equiv h(n). \quad (2.8) \]
**Lemma 1.** For isentropic fluids \((2.7)\), the index \( f \) can be chosen as the specific enthalpy,
\[ f(p(n)) = h(n). \quad (2.9) \]
Proof. From

\[ p'(n) = 2ne'(n) + n^2e''(n) = nh'(n) \]

one obtains

\[ \frac{dh}{dp} = \frac{h'(n)}{p'(n)} = \frac{1}{n} = \frac{h}{\rho + p}, \]

which means that \( f \) defined through \( f(p(n)) = h(n) \) satisfies the differential equation (2.6).

\[ \square \]

**Remark 1.** The quantity \( f \) identified by Theorem 1 is the index of a barotropic fluid that Lichnerowicz has introduced in [16] as

\[ f = f(p) \equiv \exp \int \frac{dp}{\rho(p) + p}. \]

Lichnerowicz worked not with the Godunov variable \( U^\alpha/f \), but with the ‘dynamic’ velocity \( fU^\alpha \).

**Remark 2.** The integrability of the 4-potential, here the existence of \( X(\Upsilon) = \hat{X}(f) \) such that \( X^\beta(\Upsilon) = \partial X(\Upsilon)/\partial \Upsilon^\beta \), can be viewed as a consequence of the symmetry of the energy-momentum tensor \( T^{\alpha\beta} \) [11]. On the other hand, already the naturally needed isotropy of the mapping \( \Upsilon^\beta \mapsto X^\beta \) induces the form \( X^\beta = \hat{X}(\Upsilon_\alpha \Upsilon^\alpha) \Upsilon^\beta \), thus the integrability, and one may conversely re-understand the symmetry of \( T^{\alpha\beta} \) as a consequence of the isotropicity requirement for fundamental equations.

**Remark 3.** For isentropic fluids, Eq. (2.6) reads

\[
\pi(h(n)) = p(n) \quad \text{and} \quad \pi'(h(n)) = n,
\]

which implies

\[ \rho(n) + \pi(h) = nh, \]

i.e., density and pressure are Legendre conjugate, with matter density and enthalpy,

\[ n = \pi'(h) \quad \text{and} \quad h = \rho'(n), \]

as dual variables. For instance, the massless \((m = 0)\) isentropic \( \gamma \)-law gases \( e(n) = \frac{1}{\gamma} n^{\gamma-1} \) have

\[ \rho(n) = \frac{1}{\gamma} n^\gamma \quad \text{and} \quad \pi(h) = \frac{1}{\delta} h^\delta \quad \text{with} \quad \gamma + \delta = \gamma \delta, \]

and a beautiful example is given by the limiting ‘stiff’ fluid, that corresponds to \( \gamma = \delta = 2 \), the fixed point of the Legendre transform.\footnote{For the stiff fluid, cf. [16] [3] [6].}
3 The additional law of conservation/production

The purpose of this section is to discuss the interpretation of the additional conservation law \([2.3]\). To illustrate that there is a spectrum of possibilities, we begin with two particular cases. The considerations and results of Section 2 hold in particular for fluids of the form:

\[
e(n, \sigma) = n^{\gamma - 1} r(\sigma), \quad 1 < \gamma < 2,
\]

which includes ideal gases of vanishing or negligible particle mass, corresponding to

\[
r(\sigma) = k \exp(\sigma/c_v)
\]

as well as “double \(\gamma\)-law” gases, for which

\[
r(\sigma) = k \sigma^{\gamma}.
\]

Lemma 2. (i) For double \(\gamma\)-law gases \([3.3]\) the index \(f\) is a constant multiple of the temperature \(\theta\) and the quantity \(\nu\) introduced in Theorem \([1]\) is a constant multiple of the entropy density \(n\sigma\); the additional conservation law \([1.5]\) is that of entropy, \([1.12]\). (ii) These three assertions are wrong for the massless ideal gases \([3.2]\).

Proof. (i) The product form \([3.1]\) implies

\[
f \sim n^{\gamma - 1} (r(\sigma))^{1 - 1/\gamma}, \quad \nu \sim n(r(\sigma))^{1/\gamma},
\]

and in case of \([3.3]\)

\[
r(\sigma)^{1/\gamma} \sim \sigma \quad \text{and} \quad \theta = e_\sigma(n, \sigma) \sim n^{\gamma - 1} r(\sigma)^{1 - 1/\gamma}.
\]

(ii) Both relations in \([3.5]\) are wrong in case of \([3.2]\). \(\square\)

Easy as it is to obtain now, the following general result seems particularly interesting.

Theorem 2. For isentropic fluids \([2.7]\), the quantity \(\nu\) can be chosen as the particle number density,

\[
\nu(p(n)) = n,
\]

and the additional conservation law \([2.3]\) is that of matter, \([1.8]\).

Proof. In view of \([2.9]\), equation \([2.4]\) implies

\[
\nu = \frac{\rho + p}{h} = n.
\]

\(\square\)

5Note that for any barotropic fluid in product form \(e(n, \sigma) = ˇe(n) r(\sigma)\) with non-constant \(r\), ˇ\(e(n)\) must be a power law.

6These gases have \(\rho = kn^{\gamma} \sigma^{\gamma}\). An example is pure radiation, \(\gamma = 4/3\).
In other words, isentropic fluids have the property that the conservation of matter, (1.8), is implied by that of energy-momentum, (1.7).

**Remark 4.** Hawking and Ellis\(^7\) point out that for arbitrary barotropic fluids, one can “introduce” a “conserved quantity” and an “internal energy”\(^7\) (p. 70, l. 6,7 from below, \(\rho\) and \(\epsilon\) in their notation). Identical (though not derived there) with our \(\nu\), in particular this conserved quantity is thus *not* always the matter density\(^8\), but deviates from it by a generically non-constant factor.

We now turn to the fact that the additional conservation law “for” \(\nu\) indeed holds only for smooth solutions of the original system (which here is the four-field theory \(1.7\) by itself). For the five-field context, this phenomenon is well-known for the entropy law, which is replaced, in the presence of shock waves, by the inequality

\[
\frac{\partial}{\partial x^\beta} (n \sigma U^\beta) > 0,
\]

that expresses the *second law of thermodynamics*. (See \[14, 18\], in analogy to the non-relativistic setting, cf., e.g., \[5\].)

**Theorem 3.** Assume that for the Euler equations \((1.7), (1.9)\) for a barotropic fluid \((2.1), (2.2)\), the acoustic mode is genuinely nonlinear. Then any Lax shock solving \((1.7)\) satisfies the ‘production law’

\[
\frac{\partial}{\partial x^\beta} (\nu U^\beta) > 0
\]

in the sense of distributions, with \(\nu\) from \((2.4)\).

On shock waves in relativistic fluid dynamics, some results can be found in \[21, 14, 20, 19, 7, 8\]. For the mathematical notions of shock wave and genuine nonlinearity, see \[15\]. For relativistic barotropic fluids, genuine nonlinearity is equivalent to the condition

\[
(\rho + \hat{p}(\rho))\hat{p}''(\rho) + 2(1 - \hat{p}'(\rho))\hat{p}'(\rho) > 0,
\]

where \(\hat{p} = \hat{p}^{-1}\), holding for all \(\rho > 0\); cf. \[3, 2, 20\].

Theorem 3 is an immediate corollary of “Statement II” in Section 6 of \[18\]. One just has to note that our development in the previous section of the present paper implies that what Ruggeri and Strumia denote by \(\eta\) is, in the present case of application, the ‘production’ \((\partial/\partial x^\beta)(\nu U^\beta)\) at the shock wave!

Theorems 2 and 3 readily yield

\(^{7}\)While these authors do not distinguish between ‘barotropic’ and ‘isentropic’ (at least at the time when they wrote \[13\]), we here stick to the above definitions of the two notions.

\(^{8}\)The authors didn’t claim it was ... .

\(^{9}\)This list is by far not exhaustive.
Corollary 2. Assume that for the Euler equations (1.7),(1.9) for a causal isentropic fluid (2.7), (2.2), the acoustic mode is genuinely nonlinear. Then any Lax shock solving (1.7) is accompanied by strictly positive ‘matter production’,
\[
\frac{\partial}{\partial x^\beta} (nU^\beta) > 0 \tag{3.9}
\]
in the sense of distributions.

Remark 5. Shock waves\textsuperscript{10} are a phenomenon of dissipation, and the production laws (3.6), (3.7), (3.9) hold also when the conservation laws (1.7) are augmented by proper dissipation terms. This will be carried out elsewhere.

4 Ideal gases

In this section we return to the context originally studied in particular by Ruggeri and Strumia in [18] and show an explicit formula for a particular example, namely the ideal gas.

Assume that \( \rho = ne(n, \sigma), p = n^2e_n(n, \sigma) \) where
\[
e(n, \sigma) = m + kn^{\gamma - 1}\exp(\sigma/c_v) \tag{4.1}
\]
with \( 1 < \gamma \leq 2 \), and write
\[
T^{4\beta} \equiv N^\beta \equiv nU^\beta.
\]

Theorem 4. A function \( X(\psi_\alpha, \psi) = \hat{X}(\theta, \psi) \) can be explicitly determined such that when written in the variables
\[
\psi_\alpha = \frac{U_\alpha}{\theta} \quad \text{and} \quad \psi_4 = \psi = \frac{\rho + p}{\theta n} - \sigma,
\]
the equations
\[
\frac{\partial}{\partial x^\beta} T^{a\beta} = 0, \quad a = 0, \ldots, 4,
\]
are symmetric hyperbolic, with
\[
T^{a\beta} = \frac{\partial^2 X}{\partial \psi_a \partial \psi_\beta}. \tag{4.2}
\]

Proof. Equation (4.2) yields
\[
T^{a\beta} = \theta^2 \frac{\partial \hat{X}(\theta, \psi)}{\partial \theta} \psi^a \psi^{\beta} + \hat{X}(\theta, \psi) g^{a\beta}, \quad N^\beta = \frac{\partial \hat{X}(\theta, \psi)}{\partial \psi} \psi^{\beta}.
\]

\textsuperscript{10}though, like here, often understandable from their properties as ingredients of weak solutions to first-order systems of conservation laws without explicit reference to the dissipation mechanism(s) (cf. [5]).
The ideal gas equation of state leads to
\[
\hat{X}(\theta, \psi) = (\gamma - 1) c_v \theta \left( \frac{k}{c_v \theta} \right)^{\frac{1}{\gamma - 1}} \exp \left( \frac{1}{c_v (1 - \gamma)} \frac{m}{\theta} \right) \exp \left( \frac{1}{c_v (1 - \gamma)} (-\psi + \gamma c_v) \right)
\]
\[
= \hat{k} \theta^{1/(1-1/\gamma)} \exp \left( \frac{1}{c_v (\gamma - 1)} (\psi - \frac{m}{\theta} - \gamma c_v) \right).
\]

The symmetric hyperbolicity follows from the fact, mentioned already in Section 2, that \(\psi_a, a = 0, \ldots, 4\), are the Godunov variables [18].

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