Four Vignettes on Apparent Size

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Abstract. Problems in optimization and geometric probability are discussed, all connected with angles subtended at an observer’s eye by an object at a distance. Several of these remain unsolved.

A well-known exercise in calculus asks how far an observer should stand from a wall so as to maximize the apparent size of an object (a painting) on the wall [1, 2, 3]. Assume that the floor is represented by the positive \( x \)-axis and that the object is modeled by a subinterval \([r-1, r+1]\) of the positive \( y \)-axis (hence \( r > 1 \)). The height of the observer is negligible. By the Law of Cosines, the angle \( \Omega(r, x) \) subtended at his/her eye by the line segment is

\[
\Omega = \arccos \left( \frac{-4 + [x^2 + (r-1)^2] + [x^2 + (r+1)^2]}{\sqrt{x^2 + (r-1)^2} \sqrt{x^2 + (r+1)^2}} \right)
\]

and this is largest when

\[x = x_{\text{max}} = \sqrt{r^2 - 1}, \quad \Omega_{\text{max}} = \arccos \left( \frac{\sqrt{r^2 - 1}}{r} \right).\]

For future reference, we note that \( x_{\text{max}} \sim r \) and, for fixed \( x \),

\[\Omega(r, x) \sim \frac{2x}{r^2}\]

as \( r \to \infty \). The preceding serves as inspiration for the remainder of this paper: two vignettes on optimization and two on geometric probability (Table 1).

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Four Vignettes on Apparent Size

Table 1: Starting from upper left and moving clockwise: a railroad advance warning sign, a rectangular billboard, a circular deadbolt with keyhole, and a carpenter’s steel framing square.

1. Railroad Advance Warning Sign

How far should an observer (a motorist) stand from an object (a circular road sign) so as to maximize its apparent size? Assume that the road is represented by the positive $x$-axis and that the object is modeled by a disk of radius 1 in the $yz$-plane, centered at a fixed distance $r$ from the origin (again $r > 1$). The height of the observer is negligible (Figure 1). “Apparent size” is here quantified as the solid angle $\Omega(r, x)$ subtended by the disk at the observer’s eye. We wish to determine the vehicle location $x = x_{\text{max}}$ for which $\Omega(r, x)$ is maximized, i.e., the sign has greatest visual impact on the motorist (occupying the largest field of view).

This problem was solved in [4] via a Fourier-Legendre series approximation for $\Omega$. An exact formula, however, was mentioned by Maxwell [5] in 1873, explicitly given by Tallqvist [6] in 1931, and subsequently rediscovered several times [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18]:

\[
\Omega(r, x) = \pi (1 + \text{sgn}(1 - r)) - \frac{2x}{\sqrt{(1 + r)^2 + x^2}} \cdot \left( K \left[ \frac{4r}{(1 + r)^2 + x^2} \right] + \frac{1 - r}{1 + r} \Pi \left[ \frac{4r}{(r + 1)^2}, \frac{4r}{(1 + r)^2 + x^2} \right] \right)
\]

where

\[
1 + \text{sgn}(1 - r) = \begin{cases} 
2 & \text{if } r < 1, \\
1 & \text{if } r = 1, \\
0 & \text{if } r > 1;
\end{cases}
\]
Figure 1: Observer is at \((x, 0, 0)\); object in \(yz\)-plane is separated by \(r\) from origin.

\[
K[m] = \int_{0}^{1} \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - m t^2}}
\]

is the complete elliptic integral of the first kind; and

\[
\Pi[n, m] = \int_{0}^{1} \frac{dt}{(1 - n t^2) \sqrt{1 - t^2} \sqrt{1 - m t^2}}
\]

is the complete elliptic integral of the third kind (Figure 2). This formula enables precise asymptotics that could not be deduced in [4]. We have

\[
x_{\text{max}} \sim \frac{1}{\sqrt{2}} r - \frac{7\sqrt{2}}{24}
\]

as \(r \to \infty\), which implies that

\[
x_{\text{max}} \approx (0.707...) r - (0.412...)
\]

for large \(r\) and which is an improvement on \(x_{\text{max}} \approx (0.7)r - (0.1)\) given earlier. We also have, for fixed \(x > 0\),

\[
\Omega(r, x) \sim \frac{\pi x}{r^3}
\]
which plays a role in Koopman’s “inverse cube law of detection” from search theory [19, 20, 21].

2. Square versus Rectangular Billboards

The position of the circular sign in the preceding vignette was unspecified: it sufficed to require only that its center be at distance $r$ from the origin. Here, for a rectangular sign, we must further stipulate that its center possess $yz$-coordinates

$$\left( \frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}} \right)$$

that is, it lies on the diagonal line $z = y$. Moreover, given that the rectangle has length $\ell \geq 1$ and width $1/\ell$, its vertices are assumed to be at

$$(y_1, z_2) = \left( \frac{r}{\sqrt{2}} - \frac{\ell}{2}, \frac{r}{\sqrt{2}} + \frac{1}{2\ell} \right), \quad (y_2, z_2) = \left( \frac{r}{\sqrt{2}} + \frac{\ell}{2}, \frac{r}{\sqrt{2}} + \frac{1}{2\ell} \right),$$

$$(y_1, z_1) = \left( \frac{r}{\sqrt{2}} - \frac{\ell}{2}, \frac{r}{\sqrt{2}} - \frac{1}{2\ell} \right), \quad (y_2, z_1) = \left( \frac{r}{\sqrt{2}} + \frac{\ell}{2}, \frac{r}{\sqrt{2}} - \frac{1}{2\ell} \right).$$
Letting the observer be at \((x_0, y_0, z_0) = (x, 0, 0)\), we have \[22, 23, 24, 25, 26\]

\[
\Omega = \arctan\left[\frac{(y_2 - y_0)(z_2 - z_0)}{x_0 \sqrt{x_0^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2}}\right] - \arctan\left[\frac{(y_1 - y_0)(z_2 - z_0)}{x_0 \sqrt{x_0^2 + (y_1 - y_0)^2 + (z_2 - z_0)^2}}\right]
\]

\[
- \arctan\left[\frac{(y_2 - y_0)(z_1 - z_0)}{x_0 \sqrt{x_0^2 + (y_2 - y_0)^2 + (z_1 - z_0)^2}}\right] + \arctan\left[\frac{(y_1 - y_0)(z_1 - z_0)}{x_0 \sqrt{x_0^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}}\right]
\]

and consequently

- if \(x = 1\) and \(r = 1\), then \(\ell_{\text{max}} = 1\) (a square)
- if \(x = 1\) and \(r = 9/8\), then \(\ell_{\text{max}} = 1.0687058269964383331915489\ldots\)
- if \(x = 1\) and \(r = 5/4\), then \(\ell_{\text{max}} = 1.6697745993679378292165263\ldots\)

This may be surprising to some readers: the rectangle of largest solid angle (with fixed observer and fixed center) need not be a square \[3, 4\]. The asymmetry (due to off-center location of observer relative to object) is responsible for this phenomena.

For \(r > 1.309987792\), the inequality \(|r/\sqrt{2} - \ell_{\text{max}}/2| > 0\) is no longer satisfied, i.e., the billboard spills into the street. To talk of asymptotics as \(r \to \infty\), we would perhaps wish to alter the position of the rectangle center.

### 3. Circular Deadbolt with Keyhole

The initial setting here is the \(xy\)-plane. Let the observer be at coordinate \(\theta\) on the unit circle, centered at the origin. Let the object be the subinterval \([-1/2, 1/2]\) of the \(y\)-axis. By the Law of Cosines, the angle \(\Omega(\theta)\) subtended at the observer’s eye by the line segment is

\[
\Omega = \arccos\left(-1 + \frac{\cos(\theta)^2 + (\sin(\theta) - \frac{1}{2})^2 + \frac{3}{4}}{2 \sqrt{\cos(\theta)^2 + (\sin(\theta) - \frac{1}{2})^2}}\right)
\]

\[
= \arccos\left(-\frac{3}{4} \sqrt{\frac{5}{4} - \sin(\theta)\sqrt{\frac{5}{4} + \sin(\theta)}}\right) = \arccos\left(-\frac{3}{\sqrt{25 - 16 \sin^2(\theta)}}\right)
\]

\[
= \arctan\left(\frac{4 \sqrt{1 - \sin(\theta)^2}}{3}\right) = \arctan\left(\frac{4}{3} \left|\cos(\theta)\right|\right).
\]

If \(\theta \in [0, 2\pi)\) is uniformly distributed on the circle, then the first two moments of \(\Omega\) are

\[
E(\Omega) = \frac{1}{3\pi} \left[\pi^2 - 6 \ln(2)^2 - 3 \text{Li}_2 \left(\frac{1}{4}\right)\right] = 0.6561351817594584154390278\ldots,
\]
\[ \mathbb{E}(\Omega^2) = \text{Li}_2\left(\frac{1}{4}\right) - \text{Li}_2\left(-\frac{1}{4}\right) = 0.5035529367689960607402666... \]

where

\[ \text{Li}_2(\xi) = \sum_{k=1}^{\infty} \frac{\xi^k}{k^2} = -\int_0^\xi \frac{\ln(1-t)}{t}dt \]

is the dilogarithm function [27]. Also, the probability density function of \( \Omega(\theta) \) is

\[-\frac{6}{\pi} \frac{1}{\cos(\omega)^2 \sqrt{16 - 9 \tan(\omega)^2}}, \quad 0 < \omega < 2 \arctan\left(\frac{1}{2}\right) \]

via standard techniques [28].

The final setting here is \( xyz \) space. Let the observer be at coordinates \((\theta, \phi)\) on the unit sphere, centered at the origin. Let the object be the subinterval \([-1/2, 1/2]\) of the \( y \)-axis. Note that

\[
\cos(\theta)^2 \sin(\varphi)^2 + \left(\sin(\theta) \sin(\varphi) \mp \frac{1}{4}\right)^2 + \cos(\varphi)^2 = \sin(\varphi)^2 \mp \sin(\theta) \sin(\varphi) + \frac{1}{4} + \cos(\varphi)^2 = \frac{5}{4} \mp \sin(\theta) \sin(\varphi).
\]

By the Law of Cosines, the angle \( \Omega(\theta, \varphi) \) subtended at the observer’s eye by the line segment is

\[
\Omega = \arccos \left( \frac{\frac{3}{4}}{\sqrt{\frac{5}{4} - \sin(\theta) \sin(\varphi) \sqrt{\frac{5}{4} + \sin(\theta) \sin(\varphi)}}} \right) = \arccos \left( \frac{3}{\sqrt{25 - 16 \sin(\theta)^2 \sin(\varphi)^2}} \right) = \arctan \left( \frac{4 \sqrt{1 - \sin(\theta)^2 \sin(\varphi)^2}}{3} \right).
\]

If \((\theta, \varphi) \in [0, 2\pi) \times [0, \pi]\) is uniformly distributed on the sphere, then the first two moments of \( \Omega \) are

\[
\mathbb{E}(\Omega) = \frac{\pi}{4} = 0.7853981633974483096156608..., \quad \mathbb{E}(\Omega^2) = \int_0^1 \frac{\eta \arctan \left[\frac{(4/3)\eta}{\sqrt{1 - \eta^2}}\right]^2}{\sqrt{1 - \eta^2}}d\eta = 0.6472381347206737507335484...
\]

and the probability density function of \( \Omega(\theta) \) is

\[-\frac{9}{4} \frac{\tan(\omega)}{\cos(\omega)^2 \sqrt{16 - 9 \tan(\omega)^2}}, \quad 0 < \omega < \arctan\left(\frac{4}{3}\right).\]

A closed-form expression for \( \mathbb{E}(\Omega^2) \) remains open. Substituting the line segment by a disk of diameter 1 in the \( xy \)-plane, centered at the origin, also gives an unsolved problem.
4. Sphere around Framing Square

So far, we have addressed apparent size of lengths (of line segments) and areas (of planar regions). We conclude with apparent magnitude of angles (at the intersection of two lines).

Again, the setting is \( \mathbb{R}^3 \) space. Let the observer be at coordinates \((\theta, \varphi)\) on the unit sphere, centered at the origin. Call this point \( A \). Let \( B = (1, 0, 0) \), \( C = (0, 1, 0) \) and \( D = (0, 0, 0) \), that is, a fixed angle of \( \pi/2 \) in the base plane. The apparent magnitude of \( \pi/2 \) relative to \( A \) is the (dihedral) angle \( \alpha \) between normal vectors \( A \times B \) and \( A \times C \) to the triangular faces \( ADB \), \( ADC \) respectively:

\[
\alpha = \arccos \left( \frac{(A \times B) \cdot (A \times C)}{\|A \times B\| \|A \times C\|} \right).
\]

This formula is useful for simulation purposes. It’s best, however, to employ the spherical triangle \( ABC \) and to recognize that the side \( a \) opposite angle \( \alpha \) is the constant \( \pi/2 \). From section 1.3 of [29], the conditional density for angle \( \alpha \), given \( a = \pi/2 \), is

\[
-\frac{1}{\pi} \ln \left| \cos(\alpha) \right| \sin(\alpha)^2, \quad 0 < \alpha < \pi.
\]

In particular, a singularity exists at \( \alpha = \pi/2 \) and

\[
E \left( \alpha \bigg| a = \frac{\pi}{2} \right) = \frac{\pi}{2}, \quad E \left( \alpha^2 \bigg| a = \frac{\pi}{2} \right) = \frac{\pi^2}{4} + \ln(2)^2 = \frac{\pi^2}{4} + 0.480453\ldots
\]

Let us change the vector \( C \) from \((1, 0, 0)\) to \((1/2, \sqrt{3}/2, 0)\). The side \( a \) opposite angle \( \alpha \) is now the constant \( \pi/3 \). It is known that [30]

\[
E \left( \alpha \bigg| a = \frac{\pi}{3} \right) = \frac{\pi}{3}
\]

but exact evaluation of

\[
E \left( \alpha^2 \bigg| a = \frac{\pi}{3} \right) = \frac{\pi^2}{12} + \int_0^{\pi/3} \int_0^{\pi/3} \arctan \left[ \frac{(1/\sqrt{3}) \sin(y) + \cos(x) \cos(y)}{\sin(x)} \right]^2 \frac{\sin(y)}{2\pi} \, dx \, dy
\]

\[
= \frac{\pi^2}{9} + 0.398812\ldots
\]

appears to be difficult.

Interested readers should examine [31] for an alternative derivation of the conditional mean \( E(\alpha | a) \), as well as a higher dimensional analog involving tetrahedra on \( \mathbb{S}^3 \) (rather than triangles on \( \mathbb{S}^2 \)). No formula for the corresponding mean square \( E(\alpha^2 | a) \) is known here.
An expression for the solid angle subtended by a polygon can be used to approximate solid angles for arbitrary piecewise smooth closed curves \[32\] [33]. Our treatment of apparent size is based on Euclid (a visual cone of rays emanating from the eye); this system of thought is arguably inconsistent with the Renaissance theory of linear perspective \[34\] [35] [36] [37] [38]. We welcome correspondence on this topic, as an opportunity for deeper understanding.

5. Addendum

The “inverse cube law” mentioned at the end of the first vignette appears in other settings. We discuss this not with regard to spherical projection (definition of solid angle) but instead with respect to planar projection (image formed within a camera). Apparent size is quantified differently than before.

Consider an observer at \((-1, 0, 1)\) in \(xyz\)-space, surveying the integer lattice \(\{(m, n, 0) : m \geq 1\}\) in the half \(xy\)-plane; according to this model, the observer does not see a uniform grid, but rather its projection into the \(yz\)-plane (Figure 3). The formula \[39\]

\[
(x, y, 0) \mapsto \left(0, \frac{y}{1 + x}, -\frac{x}{1 + x}\right)
\]

allows us to calculate the size of the \(k\)th trapezoid in the (green) center strip. We find

\[
\text{area} = \frac{3 + 2k}{(1 + k)^2(2 + k)^2} \sim \frac{2}{k^3}
\]
as \(k \to \infty\).

Consider instead an observer at \((-\sqrt{2}, 0, 1)\) in \(xyz\)-space, surveying the lattice \(\left\{\left(\frac{m+n}{\sqrt{2}}, -\frac{m+n}{\sqrt{2}}, 0\right) : m \geq 1 \text{ and } n \geq 1\right\}\) in the quarter \(xy\)-plane; the observer here sees a contrasting \(yz\)-projection (Figure 4). The formula

\[
(x, y, 0) \mapsto \left(0, \frac{\sqrt{2}(-x + y)}{2 + x + y}, \frac{x + y}{2 + x + y}\right)
\]
gives us the size of the \(k\)th quadrilateral in the (gold) right strip. We find here

\[
\text{area} = \frac{4\sqrt{2}}{(3 + k)(4 + k)(5 + k)} \sim \frac{4\sqrt{2}}{k^3}
\]
as \(k \to \infty\).

References

[1] H. Dörrie, 100 Great Problems of Elementary Mathematics: Their History and Solution, Dover, 1965, pp. 369–370; MR0183598.
Figure 3: One-point linear perspective: if we drew infinitely many gridsquares, they would converge at the “vanishing point” \((y, z) = (0, 1)\).

Figure 4: Two-point linear perspective: the “vanishing points” are at \((y, z) = (\mp\sqrt{2}, 1)\).
B. Letson and M. Schwartz, The Regiomontanus problem, *Math. Mag.* 90 (2017) 259–266; MR3706845.

A. Tan, The best viewing distance for highway signboards, *Math. Spectrum* 17 (1984/85) 33–35; summary in *College Math. J.* 16 (1985) 426.

I. S. Jones, The solid angle subtended by a circular disc with application to roadsigns, *Int. J. Math. Educ. Sci. Technol.* 27 (1996) 667–674.

J. C. Maxwell, *A Treatise on Electricity and Magnetism*, v. 2, 3rd ed., Dover, 1954, p. 331.

Hj. Tallqvist, Tafeln zur scheinbaren Grösse des Kreises, *Soc. Sci. Fennica, Commentationes Phys.-Math.*, v. 24, n. 5 (1931) 1–36.

M. W. Garrett, Solid angle subtended by a circular aperture, *Rev. Sci. Instrum.* 25 (1954) 1208–1211; cites Hj. Tallqvist.

A. V. Masket, Solid angle contour integrals, series, and tables, *Rev. Sci. Instrum.* 28 (1957) 191–197; cites P. A. Macklin.

M. Naito, A method of calculating the solid angle subtended by a circular aperture, *J. Phys. Soc. Japan* 12 (1957) 1122–1129; cites Hj. Tallqvist.

M. M. Agrest, M. Z. Maksimov and A. K. Khmelevskii, Determination of the solid angle subtended by a circular target at a point source, *Zhur. Tekh. Fiziki* 28 (1958) 1345–1347; Engl. transl. in *Soviet Phys.-Tech. Phys.* 3 (1958) 1249–1251.

F. Paxton, Solid angle calculation for a circular disk, *Rev. Sci. Instrum.* 30 (1959) 254–258.

A. V. Masket, Solid angle subtended by a circular disk, *Rev. Sci. Instrum.* 30 (1959) 950.

V. I. Lozgachev, The determination of solid angles, *Zhur. Tekh. Fiziki* 30 (1960) 1109–1114; Engl. transl. in *Soviet Phys.-Tech. Phys.* 5 (1961) 1039–1044.

R. E. Rothe, The solid angle at a point subtended by a circle, *J. Franklin Instit.* 287 (1969) 515–521.

V. A. Shelyuto, Exact analytic results for the solid angle in systems with axial symmetry, *Z. Angew. Math. Phys.* 40 (1989) 608–612; MR1008927.
[16] M. J. Prata, Solid angle subtended by a cylindrical detector at a point source in terms of elliptic integrals, *Radiat. Phys. Chem.* 67 (2003) 599–603; arXiv:math-ph/0211061.

[17] D. M. Timus, M. J. Prata, S. L. Kalla, M. I. Abbas, F. Oner and E. Galiano, Some further analytical results on the solid angle subtended at a point by a circular disk using elliptic integrals, *Nucl. Instrum. Methods Phys. Res. A* 580 (2007) 149–152.

[18] J. T. Conway, Analytical solution for the solid angle subtended at any point by an ellipse via a point source radiation vector potential, *Nucl. Instrum. Methods Phys. Res. A* 614 (2010) 17–27.

[19] B. O. Koopman, The theory of search. II. Target detection, *Operations Research* 4 (1956) 503–531; MR0090467.

[20] L. D. Stone, *Theory of Optimal Search*, Academic Press, 1975, pp. 22–29; MR0439145.

[21] L. D. Stone, J. O. Royset and A. R. Washburn, *Optimal Search for Moving Targets*, Springer-Verlag, 2016, pp. 13–19; MR3468906.

[22] F. S. Crawford, Solid angle subtended by a finite rectangular counter *Rev. Sci. Instrum.* 24 (1953) 552–553.

[23] A. Khadjavi, Calculation of solid angle subtended by rectangular apertures, *J. Optical Soc. Amer.* 58 (1968) 1417–1418.

[24] H. Gotoh and H. Yagi, Solid angle subtended by a rectangular slit, *Nucl. Instrum. Methods* 96 (1971) 485–486.

[25] J. Cook, Solid angle subtended by two rectangles, *Nucl. Instrum. Methods* 178 (1980) 561–564.

[26] R. J. Mathar, Solid angle of a rectangular plate, unpublished note (2014), http://www2.mpia-hd.mpg.de/~mathar/public/.

[27] S. R. Finch, Apéry’s constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 40–53; MR2003519.

[28] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, 1965, pp. 125–127, 201–205; MR0176501.

[29] S. R. Finch, Correlation between angle and side, arXiv:1012.0781.
[30] Y. Maeda and H. Maehara, Observing an angle from various viewpoints, *Discrete and Computational Geometry, Japanese Conference* (JCDCG 2002), ed. J. Akiyama and M. Kano, Lect. Notes in Comp. Sci. 2866, Springer-Verlag, 2003, pp. 200–203; MR2097746.

[31] Y. Maeda, Observing a solid angle from various viewpoints, *Yokohama Math. J.* 54 (2008) 155–160; MR2488281.

[32] P. Olivier and D. Gagnon, Mathematical modeling of the solid angle function, part I: approximation in homogeneous medium, *Optical Engin.* 32 (1993) 2261–2265.

[33] J. S. Asvestas and D. C. Englund, Computing the solid angle subtended by a planar figure, *Optical Engin.* 33 (1994) 4055–4059.

[34] C. D. Brownson, Euclid’s Optics and its compatibility with linear perspective, *Arch. Hist. Exact Sci.* 24 (1981) 165–194; MR0632568.

[35] W. R. Knorr, On the principle of linear perspective in Euclid’s Optics, *Centaurus* 34 (1991) 193–210; MR1166071.

[36] W. R. Knorr, When circles don’t look like circles: an optical theorem in Euclid and Pappus, *Arch. Hist. Exact Sci.* 44 (1992) 287–329; MR1192853.

[37] A. Jones, Pappus’ notes to Euclid’s Optics, *Ancient & Medieval Traditions in the Exact Sciences*, Proc. 1998 Stanford conf., CSLI Publ., 2000, pp. 49–58; MR1827652.

[38] L. Catastini and F. Ghione, The geometry of sight: from Euclid’s Optics to Renaissance perspective, *Matematica, Arte, Tecnologia, Cinema*, ed. M. Emmer and M. Manaresi, Springer-Verlag Italia, 2002, pp. 53–66; MR2000794.

[39] V. S. Nalwa, The basis of computer vision, *Mathematical Aspects of Artificial Intelligence*, Proc. 1996 Orlando, FL conf., ed. F. Hoffman, Proc. Sympos. Appl. Math. 55, Amer. Math. Soc., 1998, pp. 139–174; MR1619608.

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