The optimal strategy for symmetric rendezvous search on $K_3$

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Abstract

In the symmetric rendezvous search game played on $K_n$ (the completely connected graph on $n$ vertices) two players are initially placed at two distinct vertices (called locations). The game is played in discrete steps and at each step each player can either stay where he is or move to a different location. The players share no common labelling of the locations. They wish to minimize the expected number of steps until they first meet. Rendezvous search games of this type were first proposed by Steve Alpern in 1976. They are simple to describe, and have received considerable attention in the popular press as they model problems that are familiar in real life. They are notoriously difficult to analyse. Our solution of the symmetric rendezvous game on $K_3$ makes this the first interesting game of its type to be solved. It establishes the 20 year old conjecture that the Anderson-Weber strategy is optimal.

Keywords: rendezvous search, search games, semidefinite programming

1 Symmetric rendezvous search on $K_3$

In the symmetric rendezvous search game played on $K_n$ (the completely connected graph on $n$ vertices) two players are initially placed at two distinct vertices (called locations). The game is played in discrete steps, and at each step each player can either stay where he is or move to another location. The players wish to meet as quickly as possible. They are required to use an identical strategy, which must involve some random moves or else the players will never meet. They have no common labelling of the locations, so a given player must choose the probabilities with which he moves to each of the locations at step $k$ as only a function of where he has been at previous steps.

Let $T$, $w$ and $w_k$ denote respectively the number of the step on which the players meet, the minimum achievable value of $ET$, and the minimum achievable value of $E[\min\{T, k+1\}] = \sum_{i=0}^{k} P(T > i)$. We call $w$ the ‘rendezvous value’ of the game. A long-standing conjecture of Anderson and Weber (1990) is that for symmetric rendezvous search on $K_3$ the rendezvous value is $w = \frac{5}{2}$. This rendezvous value is achieved by a type of strategy which is now commonly known as the Anderson–Weber strategy (AW). For rendezvous search on $K_n$ the AW strategy specifies that in blocks of $n-1$ consecutive steps the players should randomize between staying at their initial location and touring the other $n-1$ locations in random order. On $K_3$ this means that in each successive block of two steps, each player should,

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independently of the other, either stay at his initial location or tour the other two locations in random order, doing these with respective probabilities \( \frac{1}{3} \) and \( \frac{2}{3} \). The rendezvous value with this strategy is \( ET = \frac{5}{2} \).

Rendezvous search problems have a long history. One finds such a problem in the ‘Quo Vadis’ problem of Mosteller (1965) and recently as ‘Aisle Miles’ (O’Hare, 2006). The first formal presentation of our problem is due to Alpern (1976), who states it as his ‘Telephone Problem’. “Imagine that in each of two rooms, there are \( n \) telephones randomly strewn about. They are connected in a pairwise fashion by \( n \) wires. At discrete times \( t = 0, 1, \ldots \), players in each room pick up a phone and say ‘hello’. They wish to minimize the time \( t \) when they first pick up paired phones and can communicate.” The AW strategy was conjectured to be optimal for \( K_3 \) by Anderson and Weber (1990), who proved its optimality for \( K_2 \). Subsequently, there have been proofs that AW is optimal for \( K_3 \) within restricted classes of Markovian strategies, such as those that must repeat in successive blocks of \( k \) steps, where \( k \) is small. See Alpern and Pikounis (2000) (for optimality of AW amongst 2-Markovian strategies for rendezvous on \( K_3 \)), and Fan (2009) (for optimality of AW amongst 4-Markovian strategies for rendezvous on \( K_3 \), and amongst 3-Markovian strategies for rendezvous on \( K_4 \)).

The rest of the paper concerns symmetric rendezvous search on \( K_3 \). In Section 2 we prove that AW is optimal. The symmetric rendezvous game on \( K_3 \) becomes the first interesting game of its type to be fully solved. In Section 3 we discuss the thinking that led to discovery of this proof. Section 4 discusses some generalizations and intriguing open problems.

2 Optimality of the Anderson–Weber strategy

Recall that \( T \) denotes the step on which the players meet. Let us begin by commenting that the AW strategy does not minimize \( P(T > i) \) for all \( i = 1, 2, \ldots \). In particular, the AW strategy produces \( P(T > 4) = 1/9 \). However, one can find a strategy such that \( P(T > 4) = 1/10 \). This is somewhat of a surprise and shows that \( ET = \sum_{i=0}^{\infty} P(T > i) \) cannot be minimized by minimizing each term of the sum simultaneously.

With Junjie Fan, we have gained greater computational experience of the problem and have been motivated to make the conjecture that AW achieves \( w_k \) for all \( k \), i.e., minimizes the truncated sum \( \sum_{i=0}^{k} P(T > i) \). In the following section we prove this is true. Theorem 1 states that \( \{w_k\}_{k=0}^{\infty} = \{1, \frac{5}{7}, \frac{20}{27}, \frac{7}{27}, \frac{65}{27}, \ldots \} \) with \( w_k \to \frac{5}{2} \).

**Theorem 1** The Anderson–Weber strategy is optimal for the symmetric rendezvous search game on \( K_3 \), minimizing \( E[\min\{T, k+1\}] \) to \( w_k \) for all \( k = 1, 2, \ldots \), where

\[
\begin{align*}
w_k &= \begin{cases} 
\frac{5}{2} - \frac{5}{2}^3 - \frac{k+1}{2}, & \text{when } k \text{ is odd}, \\
\frac{5}{2} - \frac{5}{2}^3 - \frac{k}{2}, & \text{when } k \text{ is even}.
\end{cases}
\end{align*}
\]

(1)

Consequently, the minimal achievable value of \( ET \) is \( w = \frac{5}{2} \).

**Proof** Throughout most of the following a subscript \( k \) on a vector means that its length is \( 3^k \). A subscript \( k \) on a matrix means that it is \( 3^k \times 3^k \). Let

\[
B_k = B_1 \otimes B_{k-1}, \quad \text{where } B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.
\]
Then \( k \) columns of \( \mathbf{B}_1 \) is an indicator for the event that they do not meet when at the first step player I moves \( i \) positions clockwise from his initial location, and player II moves \( j \) positions clockwise from his initial location. \( B^\top \) contains the indicators for the same event, but when player II starts two positions clockwise of player I. Let \( 1_k \) denote the length \( 3^k \) vector of 1s. Since the starting position of player II is randomly chosen, the problem of minimizing the probability of not having met after the first step is that of minimizing

\[
p^\top \left( \frac{1}{2}(B_1 + B_1^\top) \right)p,
\]

over \( p \in \Delta_k \), where \( \Delta_k = \{ p : p \in \mathbb{R}^{3k}, p \geq 0 \text{ and } 1_k^\top p = 1 \} \). Similarly, the 9 rows and 9 columns of \( \mathbf{B}_2 \) can be labelled as \( 0, \ldots, 8 \) (base 10), and also \( 00, 01, 02, 10, 11, 12, 20, 21, 22 \) (base 3). The base 3 labelling is helpful, for we may understand \( B_2(i_1 i_2, j_1 j_2) \) as an indicator for the event that the players do not meet when at his first and second steps player I moves to locations that are respectively \( i_1 \) and \( i_2 \) positions clockwise from his initial position, and player II moves to locations that are respectively \( j_1 \) and \( j_2 \) positions clockwise from his initial position. The problem of minimizing the probability that they have not met after \( k \) steps is that of minimizing

\[
p^\top \left( \frac{1}{2}(B_k + B_k^\top) \right)p.
\]

In this manner we can also formulate the problem of minimizing \( E[\min\{T, k + 1\}] \). Let \( J_k \) be the \( 3^k \times 3^k \) matrix that is all 1s and let

\[
M_1 = J_1 + B_1 \\
M_k = J_k + B_1 \otimes M_{k-1} \\
\quad = J_k + B_1 \otimes J_{k-1} + \cdots + B_{k-1} \otimes J_1 + B_k.
\]

Then

\[
w_k = \min_{p \in \Delta_k} \left\{ p^\top M_k p \right\} = \min_{p \in \Delta_k} \left\{ \frac{1}{2} p^\top (M_k + M_k^\top) p \right\}.
\]

It is a difficult problem to find the minimizing \( p \), because \( \frac{1}{2}(M_k + M_k^\top) \) is not positive semidefinite once \( k \geq 2 \). The quadratic form \( p^\top \left( \frac{1}{2}(M_k + M_k^\top) \right)p \) has many local minima that are not global minima. For example, the strategy which randomizes equally over the 3 locations at each step, taking \( p^\top = 1/3^k \) \( = (1, 1, \ldots, 1)/3^k \), is a local minimum of this quadratic form.

Consider, for example, \( k = 2 \). To show that \( w_2 = 2 \) we must minimize \( p^\top (M_2 + M_2^\top) p \). However, the eigenvalues of \( \frac{1}{2}(M_2 + M_2^\top) \) are \( \{19, \frac{5}{7}, \frac{5}{7}, 1, 1, 1, 1, -\frac{1}{2}, -\frac{1}{2} \} \), so this matrix is not positive semidefinite. In general, the minimization over \( x \) of a quadratic form such as \( x^\top Ax \) is \( NP \)-hard if \( A \) is not positive semidefinite. An alternative approach might be to try to show that \( \frac{1}{2}(M_2 + M_2^\top) - 2J_2 \) is a copositive matrix. For general \( k \), we would wish to show that \( x^\top \left( \frac{1}{2}(M_k + M_k^\top) - w_k J_k \right)x \geq 0 \) for all \( x \geq 0 \), where \( \{w_k\}_{k=1}^\infty = \{\frac{5}{3}, 2, \frac{20}{9}, \frac{7}{3}, \frac{65}{27}, \ldots \} \) are the

1 Readers familiar with the problem will be aware that it might make a difference whether or not the players are equipped with a common notion of clockwise. We assume that they are. However, since we show that the \( AW \) strategy cannot be bettered and this strategy makes no use of the clockwise information, \( AW \) is also optimal if the players do not have a common notion of clockwise.
values obtained by the Anderson–Weber strategy. However, to check copositivity numerically is also $\mathcal{NP}$-hard.

The key idea in this proof is to exhibit a matrix $H_k$ such that $M_k \preceq H_k \succeq 0$, where $H_k$ is positive semidefinite (denoted $H_k \succeq 0$) and $p^\top H_k p$ is minimized over $p \in \Delta_k$ to $w_k$. Since $p$ is nonnegative we must have $p^\top M_k p \succeq p^\top H_k p \succeq w_k$ for all $p$. For example, we may take

$$M_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 3 & 2 & 3 & 3 & 1 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 3 & 2 & 3 & 3 & 2 \\ 1 & 1 & 1 & 2 & 3 & 2 & 3 & 3 & 2 \\ 3 & 3 & 2 & 1 & 1 & 1 & 3 & 3 & 2 \\ 2 & 3 & 3 & 1 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 1 & 1 & 3 & 2 & 3 \end{pmatrix} \succeq H_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 & 3 & 3 & 0 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 3 & 2 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 1 & 0 & 3 & 3 \\ 2 & 3 & 3 & 0 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 0 & 1 & 3 & 2 & 3 \end{pmatrix} .$$

where $\frac{1}{2}(H_2 + H_2^\top)$ is positive semidefinite, with eigenvalues $\{18, 3, 3, \frac{3}{2}, \frac{3}{2}, 0, 0, 0, 0\}$. The minimum value of $p^\top H_2 p$ is 2.

We restrict our search for $H_k$ to matrices of a special form. For $i = 0, \ldots, 3^k - 1$ we write $i_{\text{base3}} = i_1 \cdots i_k$ (always keeping $k$ digits, including leading 0s when $i \leq 3^k - 1$); so $i_1, \ldots, i_k \in \{0, 1, 2\}$. We define

$$P_i = P_{i_1} \cdots P_{i_k} = P_1^{i_1} \otimes \cdots \otimes P_1^{i_k} ,$$

where

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

Note that the subscript is now used for something other than the size of the matrix. It will always be easy for the reader to know the $k$ for which $P_i$ is $3^k \times 3^k$ by context. Observe that $M_k = \sum_i m_k(i) P_i$, where $m_k$ is the first row of $M_k$. This motivates a search for an appropriate $H_k$ amongst those of the form

$$H_k = \sum_{i=0}^{3^k-1} x_k(i) P_i .$$

The condition $M_k \preceq H_k$ is equivalent to $m_k \preceq x_k$. In the example above, $H_2 = \sum_i x_2(i) P_i$, where $x_2 = (3, 3, 2, 3, 2, 1, 1, 0)$, the first row of $H_2$.

Let us observe that the matrices $P_0, \ldots, P_{3^k-1}$ commute with one another and so have a common set of eigenvectors. Also, $P_i^\top = P_{i'}$, where $i'_{\text{base3}} = i'_1 \cdots i'_k$ is obtained from $i_{\text{base3}} = i_1 \cdots i_k$ by letting $i'_j$ be 0, 2, 1 as $i_j$ is 0, 1, 2, respectively.

Let the columns of the matrices $U_k$ and $W_k$ contain the common eigenvectors of the $\frac{1}{2}(P_i + P_i^\top)$ and also of $\frac{1}{2}(M_k + M_k^\top)$. The columns of $W_k$ are eigenvectors with eigenvalues of 0. We shall now argue that the condition $\frac{1}{2}(H_k + H_k^\top) \succeq 0$ is equivalent to $U_k x_k \succeq 0$. To see this, note that the eigenvalues of $\frac{1}{2}(H_k + H_k^\top)$ are the same as the real parts of the eigenvalues of $H_k$. The eigenvectors and eigenvalues of $H_k$ can be computed as follows. Let
\( \omega \) be the cube root of 1 that is \( \omega = -\frac{1}{2} + i\frac{1}{2}\sqrt{3} \). Then

\[
V_k = V_1 \otimes V_{k-1}, \quad \text{where} \quad V_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.
\]

We write \( V_k = U_k + iW_k \), and shall make use of the facts that \( U_k = U_1 \otimes U_{k-1} - W_1 \otimes W_{k-1} \) and \( W_k = U_1 \otimes W_{k-1} + W_1 \otimes U_{k-1} \). It is easily checked that the eigenvectors of \( P_i \) are the columns (and rows) of the symmetric matrix \( V_k \) and that the first row of \( V_k \) is \((1,1,\ldots,1)\). The eigenvalues are also supplied in \( V_k \), because if \( V_k(j) \) denotes the \( j \)th column of \( V_k \) (an eigenvector), we have \( P_iV_k(j) = V_k(i,j)V_k(j) \). Thus the corresponding eigenvalue is \( V_k(i,j) \).

Since \( H_k \) is a sum of the \( P_i \), we also have \( H_kV_k(j) = \sum_i x_i V_k(i,j)V_k(j) \), so the eigenvalue is \( \sum_i x_i V_k(i,j) \), or \( \sum_i V_k(j,i)x_i \) since \( V_k \) is symmetric. Thus the real parts of the eigenvalues of \( H_k \) are the elements of the vector \( U_kx_k \). This is nonnegative if and only if the symmetric matrix \( \frac{1}{2}(H_k + H_k^\top) \) is positive semidefinite.

Recall that \( 1_k \) denotes the length \( 3^k \) vector of 1s. We will show that we may take \( H_k = \sum_i x_k(i)P_i \), where

\[
x_1 = (2,2,1)^\top, \quad x_2 = (3,3,2,3,2,1,1,0)^\top
\]

and that we may choose \( a_k \) so that for \( k \geq 3 \),

\[
x_k = 1_k + (1,0,0)^\top \otimes x_{k-1} + (0,1,0)^\top \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)^\top \otimes 1_{k-3}.
\]

In this construction of \( x_k \) the parameter \( a_k \) is chosen maximally such that \( U_kx_k \geq 0 \) and \( m_k \geq x_k \)

The sum of the components of \( x_k \) is

\[
1_k^\top x_k = 3^k + 1_{k-1}^\top x_{k-1} + 3^{k-2}(3 + a_k).
\]

To prove the theorem we want \( 1_k^\top x_k/3^k = w_k \), where these are the values specified in (1).

This requires the values of the \( a_k \) to be:

\[
a_k = \begin{cases} 
3 - \frac{1}{3^{(k-3)/2}}, & \text{when } k \text{ is odd,} \\
3 - \frac{2}{3^{(k-2)/2}}, & \text{when } k \text{ is even.}
\end{cases}
\]

So

\[
\{a_3, a_4, \ldots, a_{11}, \ldots\} = \{2, \frac{7}{3}, \frac{8}{3}, \frac{25}{9}, \frac{26}{9}, \frac{27}{9}, \frac{28}{9}, \frac{29}{9}, \frac{30}{9}, \frac{241}{27}, \frac{242}{27}, \ldots\}.
\]

Alternatively, the values of \( 3 - a_k \) are \( 1, 2, 3, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \ldots \). For example, with \( a_3 = 2 \) we have

\[
m_3 = (4, 4, 3, 4, 4, 3, 2, 2, 2, 4, 4, 3, 4, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1),
\]

\[
x_3 = (4, 4, 3, 4, 4, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).
\]

Note that \( a_k \) increases monotonically in \( k \), from 2 towards 3. As \( k \to \infty \) we find \( a_k \to 3 \) and \( 1_k^\top x_k/3^k \to \frac{5}{2} \). It remains to prove that with these \( a_k \) we have \( m_k \geq x_k \) and \( U_kx_k \geq 0 \).

\footnote{There are many choices of \( x_2 \) that work. We can also take \( x_2 = (3, 3, 2, 3, 2, 1, 1, 1) \) or \( x_2 = (3, 3, 2, 3, 2, 1, 1, 1) \).}
\( m_k \geq x_k \)

To prove \( m_k \geq x_k \) is easy; we use induction. The base of the induction is \( m_2 = (3, 3, 2, 3, 2, 1, 1, 1) \geq x_2 = (3, 3, 2, 3, 2, 1, 1, 0) \). Assuming \( m_{k-1} \geq x_{k-1} \), we then have

\[
m_k = 1_k + (1, 1, 0)^\top \otimes m_{k-1} \\
\geq 1_k + (1, 0, 0)^\top \otimes m_{k-1} + (0, 1, 0)^\top \otimes (1_{k-1} \\
+ (1, 1, 0)^\top \otimes 1_{k-2} + (1, 1, 0, 1, 0, 0, 0, 0)^\top \otimes 1_{k-3}) \\
= 1_k + (1, 0, 0)^\top \otimes m_{k-1} + (0, 1, 0)^\top \otimes (3, 3, 2, 3, 2, 1, 1, 1)^\top \otimes 1_{k-3} \\
\geq 1_k + (1, 0, 0)^\top \otimes x_{k-1} + (0, 1, 0)^\top \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)^\top \otimes 1_{k-3} \\
= x_k.
\]

\( U_k x_k \geq 0 \)

To prove \( U_k x_k \geq 0 \) is much harder. Indeed, \( U_k x_k \) is barely nonnegative, in the sense that as \( k \to \infty \), \( \frac{5}{9} \) of its components are 0, and \( \frac{2}{9} \) of them are equal to \( \frac{2}{9} \). Thus most of the eigenvalues of \( \frac{1}{2} (H_k + H_k^\top) \) are 0. We do not need this fact, but it is interesting that \( 2 U_k x_k \) is a vector only of integers.

Let \( f_k \) be a vector of length \( 3^k \) in which the first component is 1 and all other components are 0. Using the facts that \( U_k = U_1 \otimes U_{k-1} - W_1 \otimes W_{k-1} = U_3 \otimes U_{k-3} - W_3 \otimes W_{k-3} \) and \( W_k 1_k = 0 \) and \( U_k 1_k = 3^k f_k \), we have

\[
U_2 x_2 = (18, \frac{3}{2}, 3, 0, 0, 3, 0, 0)^\top,
\]

and for \( k \geq 3 \),

\[
U_k x_k = 3^k f_k + (1, 1, 1)^\top \otimes U_{k-1} x_{k-1} \\
+ \left(U_3 \left((0, 1, 0)^\top \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)^\top\right)\right) \otimes U_{k-3} 1_{k-3} \\
= 3^k f_k + (1, 1, 1)^\top \otimes U_{k-1} x_{k-1} + 3^{k-3} r_k \otimes f_{k-3},
\]

where \( r_k \) is

\[
r_k = U_3 \left((0, 1, 0)^\top \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)\right)^\top \\
= \frac{3}{2} (6 + 2a_k, 0, 0, a_k - 1, 0, a_k - 2, a_k - 1, a_k - 2, 0), \tag{6}
\]

\[
- 3 - a_k, 2 - a_k, a_k - 2, -a_k, 0, 0, 1, 2 - a_k, 0, \tag{7}
\]

\[
- 3 - a_k, a_k - 2, 2 - a_k, 1, 0, 2 - a_k, -a_k, 0, 0)^\top. \tag{8}
\]

Note that we make a small departure from our subscripting convention, since \( r_k \) is not of length \( 3^k \), but of length 27. We use the subscript \( k \) to denote that \( r_k \) is a function of \( a_k \).

Using (5)–(8) it is easy to compute the values \( U_k x_k \), for \( k = 2, 3, \ldots \). Notice that there is no need to calculate the \( 3^k \times 3^k \) matrix \( U_k \). Computing \( U_k x_k \) as far as \( k = 15 \), we find that for the values of \( a_k \) conjectured in (4) we do indeed always have \( U_k x_k \geq 0 \). This gives a lower bound on the rendezvous value of \( w \geq w_{15} = 16400/6561 \approx 2.49962 \). It would not be hard to
continue to even larger $k$ (although $U_{15}x_{15}$ is already a vector of length $3^{15} = 14,348,907$). Clearly the method is working. It now remains to prove that $U_k x_k \geq 0$ for all $k$.

Consider the first third of $U_k x_k$. This is found from (3) and (5) to be

$$3^k f_{k-1} + U_{k-1} x_{k-1} + 3^{k-3} \frac{3}{2} (6 + 2a_k, 0, 0, a_k - 1, 0, a_k - 2, a_k - 1, a_k - 2, 0) \otimes f_{k-3}.$$ 

Assuming $U_{k-1} x_{k-1} \geq 0$ as an inductive hypothesis, and using the fact that $a_k \geq 2$, this vector is nonnegative. So this part of $U_k x_k$ is nonnegative.

As for the rest of $U_k x_k$ (the part that can be found from (3) and (7)–(8)), notice that the part that can be found from (3) and (7)–(8), notice that $r_k$ is symmetric, in the sense that $S_3 r_k = r_k$, where

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $S_3 = S_1 \otimes S_1 \otimes S_1$. The matrix $S_k$ transposes $1$s and $2$s. Indeed $S_k P_f = P_f^\top$. Thus the proof is complete if we can show that just the middle third of $U_k x_k$ is nonnegative. Assuming that $U_{k-1} x_{k-1} \geq 0$ and $a_k \geq 2$, there are just 4 components of this middle third that depend on $a_k$ and which might be negative. Let $I_k$ denote a $3^k \times 3^k$ identity matrix. This middle third is found from (3) and (7) and is as follows, where we indicate in bold face terms that might be negative,

$$((0, 1, 0) \otimes I_{k-1}) U_k x_k = U_{k-1} x_{k-1} + 3^{k-3} (-3 - a_k, 2 - a_k, a_k - 2, -a_k, 0, 0, 1, 2 - a_k, 0) \otimes f_{k-3}. $$

The four possibly negative components of the middle third are shown above in bold and are

$$t_{k1} = (0, 1, 0) \otimes (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \otimes f_{k-3}^\top U_k x_k$$

$$= (U_{k-1} x_{k-1})_1 + 3^{k-3} (-3 - a_k)$$

$$t_{k2} = (0, 1, 0) \otimes (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \otimes f_{k-3}^\top U_k x_k$$

$$= (U_{k-1} x_{k-1})_{3^{k-3}+1} + 23^{k-3} (2 - a_k)$$

$$t_{k3} = (0, 1, 0) \otimes (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \otimes f_{k-3}^\top U_k x_k$$

$$= (U_{k-1} x_{k-1})_{3^{k-3}+1} + 3^{k-3} (-a_k)$$

$$t_{k4} = (0, 1, 0) \otimes (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0) \otimes f_{k-3}^\top U_k x_k$$

$$= (U_{k-1} x_{k-1})_{73^{k-3}+1} + 3^{k-3} (2 - a_k)$$

The remainder of the proof is devoted to proving that all these are nonnegative. Consider $t_{k1}$. It is easy to work out a formula for $t_{k1}$, since

$$(U_k x_k)_1 = f_k^\top U_k x_k$$

$$= 3^k + f_k^\top U_{k-1} x_{k-1} + 3^{k-3} \frac{3}{2} (6 + 2a_k)$$

$$= (U_{k-1} x_{k-1})_1 + 43^{k-1} + 3^{k-2} a_k$$

Thus

$$(U_k x_k)_1 = 23^k + \sum_{i=3}^{k} 3^{i-2} a_i,$$ (13)
and
\[ t_{k_1} = \frac{1}{3} 3^k + \sum_{i=3}^{k-1} 3^{i-2} a_i - \frac{1}{2} 3^{k-2} a_k \tag{14} \]

This is nonnegative since \( a_k \leq 3 \).

Amongst the remaining terms, we observe empirically that \( t_{k_2} \geq t_{k_4} \geq t_{k_3} \). It is \( t_{k_3} \) that is the least of the four terms, and which constrains the size of \( a_k \). Let us begin therefore by finding a formula for \( t_{k_3} \). We have
\[
t_{k_3} = (0, 1, 0) \otimes (0, 0, 0, 1, 0, 0, 0, 0, 0) \otimes f_{k-3}^\top U_k x_k
\]
\[
= (0, 0, 0, 1, 0, 0, 0, 0, 0) \otimes f_{k-3}^\top U_{k-1} x_{k-1} - 3^{k-2} \frac{1}{2} a_k
\]
\[
= (0, 1, 0) \otimes f_1^\top \otimes f_{k-3}^\top (3^{k-1} f_{k-1} + (1, 1, 1) \otimes U_{k-2} x_{k-2} + 3^{k-4} r_{k-1} \otimes f_{k-4}) - 3^{k-2} \frac{1}{2} a_k
\]
\[
= (1, 0, 0, 0, 0, 0, 0, 0, 0) \otimes f_{k-4}^\top U_{k-2} x_{k-2} + 3^{k-4} (0, 1, 0) \otimes f_2) r_{k-1} - 3^{k-2} \frac{1}{2} a_k
\]
\[
= (U_{k-2} x_{k-2})_1 - 3^{k-4} \frac{3}{2} (3 + a_{k-1}) - 3^{k-2} \frac{1}{2} a_k
\]
\[
= (U_{k-2} x_{k-2})_1 - 3^{k-4} \frac{3}{2} (3 + a_{k-1}) - 3^{k-2} \frac{1}{2} a_k
\]

This means that \( t_{k_3} \) can be computed from the first component of \( U_{k-2} x_{k-2} \), which we have already found in [13]. So
\[
t_{k_3} = 2 \cdot 3^{k-2} + \sum_{i=3}^{k-2} 3^{i-2} a_i - 3^{k-3} \frac{3}{2} (3 + a_{k-1}) - 3^{k-2} \frac{1}{2} a_k
\]
\[
= \frac{1}{2} 3^{k-1} + \sum_{i=3}^{k-2} 3^{i-2} a_i - \frac{1}{2} 3^{k-3} a_{k-1} - \frac{1}{2} 3^{k-2} a_k . \tag{15}\]

We now put the \( a_k \) to the values specified in [4]. It is easy to check with [14] and [15] that \( t_{k_3} = 0 \) for all \( k \).

It remains only to check that also \( t_{k_2} \geq 0 \) and \( t_{k_4} \geq 0 \). We have
\[
t_{k_2} = (0, 1, 0) \otimes (0, 1, 0, 0, 0, 0, 0, 0, 0) \otimes f_{k-3}^\top U_k x_k
\]
\[
= (0, 1, 0, 0, 0, 0, 0, 0, 0) \otimes f_{k-3}^\top U_{k-1} x_{k-1} + 3^{k-2} (1 - \frac{1}{3} a_k)
\]
\[
= (1, 0, 0) \otimes (0, 1, 0) \otimes f_{k-3}^\top (3^{k-1} f_{k-1} + (1, 1, 1) \otimes U_{k-2} x_{k-2} + 3^{k-4} r_{k-1} \otimes f_{k-4})
\]
\[
+ 3^{k-2} (1 - \frac{1}{3} a_k)
\]
\[
= (0, 1, 0) \otimes f_{k-3}^\top U_{k-2} x_{k-2} - 3^{k-4} \frac{3}{2} (1 - a_{k-1}) + 3^{k-2} (1 - \frac{1}{3} a_k). \]

We recognize \( (0, 1, 0) \otimes f_{k-3}^\top U_{k-2} x_{k-2} \) to be the first component of the middle third of \( U_{k-2} x_{k-2} \). The recurrence relation for this is
\[
(0, 1, 0) \otimes f_{k-1}^\top U_k x_k = (0, 1, 0) \otimes f_{k-1}^\top (3^k f_k + (1, 1, 1) \otimes U_{k-1} x_{k-1} + 3^{k-3} r_k \otimes f_{k-3})
\]
\[
= f_{k-1}^\top U_{k-1} x_{k-1} - 3^{k-2} \frac{3}{2} (3 + a_k). \]

The right hand side can be computed from [13]. So we now have,
\[
t_{k_2} = 2 \cdot 3^{k-3} + \sum_{i=3}^{k-2} 3^{i-2} a_i - 3^{k-4} \frac{3}{2} (3 + a_{k-2}) - 3^{k-3} \frac{3}{2} (1 - a_{k-1}) + 3^{k-2} (1 - \frac{1}{3} a_k)
\]
\[
= 4 \cdot 3^{k-3} + \sum_{i=3}^{k-2} 3^{i-2} a_i - \frac{1}{2} 3^{k-4} a_{k-2} + \frac{1}{2} 3^{k-3} a_{k-1} - \frac{1}{2} 3^{k-2} a_k . \tag{16}\]
Finally, we establish a formula for $t_{k4}$.

$$
t_{k4} = (0, 1, 0) \otimes (0, 0, 0, 0, 0, 0, 1, 0) \otimes f_{k-3}^T U_k x_k
= (0, 0, 0, 0, 0, 0, 0, 1, 0) \otimes f_{k-3}^T U_{k-1} x_{k-1} + 3^{k-2}(1 - \frac{1}{2}a_k)
= (0, 0, 1) \otimes (0, 1, 0) \otimes f_{k-3}^T (3^{k-1} f_{k-1} + (1, 1, 1)^\top \otimes U_{k-2} x_{k-2} + 3^{k-4} r_{k-1} \otimes f_{k-4})
+ 3^{k-2}(1 - \frac{1}{2}a_k)
= (0, 1, 0) \otimes f_{k-3}^T U_{k-2} x_{k-2} + 3^{k-4} \frac{3}{2} + 3^{k-2}(1 - \frac{1}{2}a_k)
= 5 \cdot 3^{k-3} + \sum_{i=3}^{k-3} 3^{i-2} a_i - \frac{1}{2} 3^{k-4} a_{k-2} - \frac{1}{2} 3^{k-2} a_k.
$$

(18)

Thus we can check the fact that we observed empirically, that $t_{k2} \geq t_{k4} \geq t_{k3}$. We find

$$
t_{k2} - t_{k4} = \frac{1}{2} 3^{k-3}(a_{k-1} - 2),
\quad t_{k4} - t_{k3} = \frac{1}{2} 3^{k-3}(1 - a_k - 2 + a_{k-1}).
$$

Since $a_k$ is at least 2 and $a_k$ is increasing in $k$, both of the above are nonnegative. So $t_{k2}$ and $t_{k4}$ are both at least as great as $t_{k3}$, which we have already shown to be 0. This establishes $U_k x_k \geq 0$ and so the proof is now complete.  

3 On discovery of the proof

The proof begs the question: how did we guess the recursion for $x_k$? Let us restate it here for convenience. With $a_k$ given by (14), the recursion is

$$
x_k = 1_k + (1, 0, 0)^\top \otimes x_{k-1} + (0, 1, 0) \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1) \otimes 1_{k-3}.
$$

(19)

Let us briefly describe the steps and ideas in research that led to (19).

We began by computing lower bounds on $w_k$ by solving the semidefinite programming problem

$$
\text{maximize } \text{trace}(J_k H_k) : H_k \preceq M_k, \ H_k \succeq 0.
$$

(20)

A similar line of approach has been followed concurrently by Han, Du, Vera and Zuluaga (2006). The lower bounds that are obtained by solving (20) turn out to be achieved by the AW strategy and so are useful in proving the Fan–Weber conjecture (that AW minimizes $E[\min\{T, k + 1\}]$) up to $k = 5$. However, they only produce numerical answers, with little guide as to a general form of solution. In fact, since one can only solve the SDPs up to the numerical accuracy of a SDP solver (which, like sedumi, uses interior point methods), such proofs are only approximate. For example, by this method one can only prove that $w_5 \geq 2.40740740$, but not $w_5 = \frac{65}{27} = 2.40740740$.

We tried to find rational solutions so the proofs could be exact. A major breakthrough was to realise that we could compute a common eigenvector set for $P_1, \ldots, P_{3k-1}$ and write $M_k = \sum_i m_k(i) P_i$. We discovered this as we noticed and tried to explain the fact that the real parts of all the eigenvalues of $2M_k$ are integers. (In fact, there is a little-known theorem which says that if a real symmetric matrix has only integer entries then all its rational eigenvalues
must be integers.) This allowed us to recast (20) as the linear program
\[
\text{maximize } \sum_{i=0}^{3^k-1} x(i) : x \leq m_k, \ U_k x \geq 0. \tag{21}
\]

Now we could find exact proofs of the Fan–Weber conjecture as far as \( k = 8 \), where \( U_8 \) is 6561 × 6561. These solutions were found using Mathematica and were in rational numbers, thus providing us with tight proofs of the optimality of \( \mathbf{AW} \) up to \( k = 8 \). They also allowed us to prove the Fan-Weber conjecture for greater values of \( k \) since the number of decision variables in the LP grows as \( 3^k \), whereas in the SDP it grows as \( 3^{2k} \).

It seems very difficult to find a general solution to (21) that will hold for all \( k \). The LP is highly degenerate with many optimal solutions. There are indeed 12 different extreme point solutions to the LP at just \( k = 2 \). No general pattern to the solution emerges as it is solved for progressively larger \( k \). For, say \( k = 4 \), there are many \( H_4 \) that can be used to prove \( w_k = \frac{7}{3} \). We searched amongst the many solutions for ones with some pattern that might be generalized. This proved very difficult. We tried forcing lots of components of the solution vector to be integers, or identical, and looked for solutions in which the solution vector for \( k - 1 \) was embedded within the solution vector for \( k \). We looked at adding other constraints, and constructed some solutions by augmenting the objective function and choosing amongst possible solution by a minimizing a sum of squares penalty.

Another approach to the problem of minimizing \( p^\top M_k p \) over \( p \in \Delta_k \) is to make the identification \( Y = pp^\top \). With this identification, \( Y \) is positive semidefinite, \( \text{trace}(J_k Y) = 1 \), and \( \text{trace}(M_k Y) = \text{trace}(M_k pp^\top) = p^\top M_k p \). This motivates a semidefinite programming relaxation of our problem: minimize \( \text{trace}(M_k Y) \), subject to \( \text{trace}(J_k Y) = 1 \) and \( Y \succeq 0 \). This can be recast as the LP
\[
\text{minimize } y^\top m_k : y^\top U_k y \geq 0, \ 1_k^\top y = 1, \ y \geq 0. \tag{22}
\]

This is nearly the dual of (21) (which is the same, but has an additional constraint of \( y^\top U_k S_k = y^\top U_k ) \).

With (22) in mind, we imagined taking \( y \) as \( \mathbf{AW} \) and worked at trying to guess a full basis in the columns of \( U_k \) that is complementary slack to \( y \) and from which one can then compute a solution to (21). We also explored a number of different LP formulations. All of this was helpful in building up intuition as to how a general solution might possibly be constructed.

Another major breakthrough was to choose to work with the constraint \( x \leq m_k \) in which \( m_k \) is the first row of the nonsymmetric matrix \( M_k \), rather than to use the first row of the symmetric matrix \( \frac{1}{2}(M_k + M_k^\top) \). By not ‘symmetrizing’ \( M_k \) we were able to find solutions with a simpler form, and felt that there was more hope in being able to write the solution vector \( x_k \) in a Kronecker product calculation with the solution vector \( x_{k-1} \). Noticing that all the entries in \( M_k \) are integers, we found that it was possible to find a solution for \( H_k \) in which all the entries in \( H_k \) are integers, as far as \( k = 5 \). It is not known whether this might be possible for even greater \( k \). The \( H_k \) constructed in the proof above have entries that are not integers, although they are of course rational.

Since \( M_k \) is computed by Kronecker products it is natural to look for a solution vector of a form in which \( x_k \) is expressed in terms of \( x_{k-1} \) in some sort formula using Kronecker products. The final breakthrough came in discovering the length 27 vector \((0, 1, 0) \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)\). This was found only after despairing of something simpler. We
expected that if it were possible to find a Kronecker product form solution similar to (19),
then this would use a vector like the above, but of length only 3 or 9. It was only when we
hazarded to try something of length 27 that the final pieces fell in place. The final trick was
to make the formula for obtaining $x_k$ from $x_{k-1}$ not be constant, but depending on $k$, as we
have done with our $a_k$. We were lucky at the end that we could solve the recurrence relations
for $t_{k1}, t_{k2}, t_{k3}, t_{k4}$ and prove $U_k x_k \geq 0$. It all looks so easy with hindsight!

4 Ongoing research

1. One would like to have a direct proof that $w = \frac{5}{2}$, without needing to also find the $w_k$.
   Perhaps an idea for such a proof is pregnant within the proof above.

2. While for many graphs it is possible to use the solution of a semidefinite programming
   problem to obtain a lower bound on the rendezvous value, it is not usually possible to
   recast the SDP as a linear program. A very important feature of the $K_3$ problem is
   that it is so strongly captured within the algebra of a group of rotational symmetry,
   whose permutation matrices are the $P_i$. This continues to be true for rendezvous search
   on $C_n$, in which $n$ locations are arranged around a circle and players have a common
   notion of clockwise. We are presently looking for results in that direction.

3. It is as easy consequence of Theorem 1 that $\text{AW}$ maximizes $E[\beta^T]$ for all $\beta \in (0,1)$.
   This follows from the fact that $\text{AW}$ minimizes $\sum_{i=0}^k P(T > i)$ for all $k$.

4. It will be interesting to explore whether our methods are helpful for rendezvous problems
   on $K_n$ ($n \geq 4$), or on other graphs. It is not hard to compute the optimal $\text{AW}$ strategy
   for $K_n$. See Anderson and Weber (1990). For example, for $n = 4$, an $\text{AW}$ strategy
   achieves $ET \approx 3.4247$, using probabilities of staying and touring of $p \approx 0.3320$ and
   $1 - p \approx 0.6680$, respectively. As $n \to \infty$, an $\text{AW}$ strategy achieves $ET \approx .8289 n$ with
   $p \approx 0.2475$. Interestingly, Fan (2009) has shown that if the rendezvous game is played
   on $K_4$, locations are imagined to be placed around a circle, and players are provided
   with a common notion of clockwise, then there exists a 3-Markovian strategy that is
   better than $\text{AW}$. However, it is open as to whether $\text{AW}$ is optimal when players are
   not provided with such initial information.

5. We conjecture that $\text{AW}$ is optimal in a rendezvous game played on $K_3$ in which players
   may overlook one another with probability $\epsilon$, (that is, they can fail to meet even when
   they are in the same location). This is easily shown to be true for the game on $K_2$. To
   analyse this game on $K_3$ we simply redefine

   \[ B_1 = \begin{pmatrix} 1 & 1 & \epsilon \\ \epsilon & 1 & 1 \\ 1 & \epsilon & 1 \end{pmatrix}, \]

   where $0 < \epsilon < 1$. We can generalize all the ideas in this paper, except that we have not
   been able to guess a construction for the matrix $H_k$. Fan (2009) has observed that not
   only does $\text{AW}$ appear to be optimal, but also that the optimal probability of ‘staying’
   is the same for all $\epsilon$, i.e., $p = 1/3$. However, for games on $K_4, K_5, \ldots$, the optimal value
   of $p$ is decreasing in $\epsilon$. 
6. In the asymmetric version of the rendezvous search game (in which players I and II can adopt different strategies) the rendezvous values for the games on $K_2$ and $K_3$ are 1 and 1.5 respectively (and are achieved by the ‘wait-for-mommy’ strategy). These are exactly 1 less than the rendezvous values of 2 and 2.5 that pertain in the symmetric games (and are achieved by the AW strategy).

A rendezvous search game can also be played on a line. The players start 2 units apart and can move 1 unit left or right at each step. The asymmetric rendezvous value is known to be 3.25 (Alpern and Gal, 1995). In the symmetric game it is known that $4.1820 \leq w \leq 4.2574$. Han, et al. (2006) have conjectured $w = 4.25$. If that is correct then the difference in rendezvous values for asymmetric and symmetric games is again exactly 1.

7. In the symmetric rendezvous search game played on $K_3$ it is of no help to the players to be provided with a common notion of clockwise. Similarly, our experience in studying the symmetric rendezvous search game on the line suggests that it is no help to the players to be provided with a common notion of left and right.

8. No one has yet found a way to prove that the rendezvous value for the symmetric rendezvous search game on $K_n$ is an increasing function of $n$.

Thanks

I warmly thank my Ph.D. student Junjie Fan (Jimmy) for his enthusiasm, many helpful discussions and proof-reading of this paper. By pioneering the use of semidefinite programming as a method of addressing rendezvous search problems, he has been the first in many years to obtain significantly improved lower bounds on $w$.

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