On Progressive Filtration Expansions with a Process; Applications to Insider Trading

Younes Kchia∗ Philip Protter†‡

March 26, 2014

Abstract

In this paper we study progressive filtration expansions with càdlàg processes. Using results from the theory of the weak convergence of σ-fields, we first establish a semimartingale convergence theorem. Then we apply it in a filtration expansion with a process setting and provide sufficient conditions for a semimartingale of the base filtration to remain a semimartingale in the expanded filtration. Applications to the expansion of a Brownian filtration are given. The paper concludes with applications to models of insider trading in financial mathematics.

1 Introduction

One of the key insights of K. Itô when he developed the Itô integral was to restrict the space of integrands to what we now call predictable processes. This allowed the integral to have a type of bounded convergence theorem that N. Wiener was unable to obtain with unrestricted random integrands. The Itô integral has since been extended to general semimartingales. If one tries however to expand (i.e. to enlarge) the filtration, then one is playing with fire, and one may lose the key properties Itô originally obtained with his restriction to predictable processes. In the 1980’s a theory of such filtration expansions was nevertheless successfully developed for two types of expansion: initial expansions and progressive expansions; see for instance [34] and [42], or the more recent partial exposition in [54, Chapter VI]. The initial expansion of a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ with a random variable $\tau$ is the filtration $\mathbb{H}$ obtained as the right-continuous modification of $(\mathcal{F}_t \vee \sigma(\tau))_{t \geq 0}$. The progressive expansion $\mathcal{G}$ is obtained as any right-continuous filtration containing $\mathcal{F}$ and making $\tau$ a stopping time. When referring to the progressive expansion with a random variable in this paper, we mean the smallest such filtration. One is usually interested in the cases where $\mathcal{F}$ semimartingales remain semimartingales in the expanded filtrations and in their decompositions when viewed as semimartingales in the expanded filtrations. The theory of

∗Centre de Mathématiques Appliquées, Ecole Polytechnique, Paris, and Merrill Lynch, London
†Statistics Department, Columbia University, New York, NY, 10027
‡Supported in part by NSF grant DMS-1308483
the expansion of filtrations has proved useful and of continuing interest in abstract probability theory (see for example the papers [3], [7], [16], [25], [26], [40], [44], [48], [50], [52]).

The subject has regained interest recently, due to applications in Mathematical Finance. The work of A. Kyle [47] in 1985 and of K. Back [4], [5] in the early 1990’s laid the foundation for a theory of the modeling of insider trading via a filtration of expansions approach. More recent work in the area includes the papers [2], [6], [8], [9], [13], [18], [27], [28], [32], [33], [55], [57].

In this article we go beyond the simple cases of initial expansion and progressive expansion with a random variable. Instead we consider the (more complicated) case of expansion of a filtration through dynamic enlargement, by adding a stochastic processes as it evolves simultaneously to the evolution of the original process. In order to do this, we begin with simple cases where we add marked point processes, and then we use the theory of the convergence of $\sigma$ fields recently developed by Antonelli, Coquet, Kohatsu-Higa, Mackevicius, Mémín, and Slominski (see [3], [19], [20]) to obtain more sophisticated enlargement possibilities. We combine the convergence results with an extension of an old result of Barlow and Protter [11], finally obtaining the key results, which include the forms of the semimartingale decompositions in the enlarged filtrations. We then apply these results to an example where we enlarge the filtration with another process which is evolving backwards in time. To do this we need to use density estimates inspired by the work of Bally and Talay [10]. Finally we conclude in a section where we develop some applications of our results to the financial theory of the modeling of insider trading. Here we build upon much preliminary work already done in the area.

The techniques developed in this paper require a long preliminary treatment of the convergence of $\sigma$ fields, and to a lesser extent the convergence of filtrations. This delays the key theorems such that they occur rather late in the paper, so perhaps it is wise to indicate that the main results of interest (in the authors’ opinion) are Theorems 6 and 10, which show how one can expand filtrations with processes and have semimartingales remain semimartingales in the enlarged filtrations. The authors also wish to mention here that the example provided in Theorem 12 shows how the hypotheses (perhaps a bit strange at first glance) of Theorem 10 can arise naturally in applications, and it shows the potential utility of the results of this paper. That said, the preliminary results on the weak convergence of $\sigma$ fields has an interest in their own right.

At the suggestion of a referee of a previous version of this paper, we have added a section (Section 6) where we apply our results to models of insider trading. In this section we also develop a concrete example of insider trading via high frequency trading. This builds on an already existing theory, developed by a variety of researchers, and relevant papers include [2], [4], [5], [8], [13], [23], [25], [32], [33], [45], [47], [55], [57], which is by no means an exhaustive list. With the plethora of recent scandals, such an addition seems timely.

1.1 Previous Results

For the initial expansion $\mathbb{H}$ of a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ with a random variable $\tau$, one well-known situation where $\mathbb{F}$ semimartingales remain semimartingales in the expanded
filtration is when Jacob’s criterion is satisfied (see [34] or alternatively [54], Theorem 10, p. 371), and as far as one is concerned by the progressive expansion, filtration $G$, this always holds up to the random time $\tau$ as proved by Jeulin and Yor and holds on all $[0, \infty)$ for honest times (see [42]). In both [44] and [40], this is proved to hold also for random times satisfying Jacob’s criterion. In [44], the authors link the two previous types of expansions and are able to provide similar results for more general types of expansion of filtrations. They extend for instance these results to the multiple time case, without any restrictions on the ordering of the individual times and more importantly to the filtration expanded by a counting process $N^n_t = \sum_{i=1}^{n} X_i 1\{\tau_i \leq t\}$, i.e. the smallest right-continuous filtration containing $F$ and to which the process $N^n$ is adapted.

For a given filtration $F$ and a given càdlàg process $X$, the smallest right-continuous filtration containing $F$ and to which $X$ is adapted will be called the progressive expansion of $F$ with $X$. In this paper we pursue the analysis started in [44] and investigate the stability of the semimartingale property of $F$ semimartingales in progressive expansions of $F$ with càdlàg processes $X$. We apply the results in [44] together with results from the theory of weak convergence of $\sigma$-fields (see [19] and [20]) to obtain a general criterion that guarantees this property, at least for $F$ semimartingales satisfying suitable integrability assumptions. Hoover [30], following remarks by M. Barlow and S. Jacka, introduced the weak convergence of $\sigma$-fields and of filtrations in 1991. The next big step was in 2000 with the seminal paper of Antonelli and Kohatsu-Higa [3]. This was quickly followed by the work of Coquet, Mém in and Mackevicius [20] and by Coquet, Mém in and Slominsky [19]. We will recall fundamental results on the topic but we refer the interested reader to [19] and [20] for details. In these papers, all filtrations are indexed by a compact time interval $[0, T]$. We work within the same framework and assume that a probability space $(\Omega, \mathcal{H}, P)$ and a positive integer $T$ are given. All filtrations considered in this paper are assumed to be completed by the $P$-null sets of $\mathcal{H}$. By the natural filtration of a process $X$, we mean the right-continuous filtration associated to the natural filtration of $X$. The concepts of weak convergence of $\sigma$-fields and of filtrations rely on the topology imposed on the space of càdlàg processes and we use the Skorohod $J_1$ topology as it is done in [19].

1.2 Outline

An outline of this paper is the following. In section 2 we recall basic facts on the weak convergence of $\sigma$-fields and establish fundamental lemmas for subsequent use. The last subsection provides a sufficient condition for the semimartingale property to hold for a given càdlàg adapted process based on the weak convergence of $\sigma$-fields. The sufficient condition we provide at this point is unlikely to hold in a filtration expansion context, however the proof of this result underlines what can go wrong under the more natural assumptions considered in the next section.

Section 3 extends the main theorem in [11] and proves a general result on the convergence of $G^n$ special semimartingales to a $G$ adapted process $X$, where $(G^n)_{n \geq 1}$ and $G$ are filtrations such that $G^n_t$ converges weakly to $G_t$ for each $t \geq 0$. The process $X$ is proved to be a $G$ special semimartingale under sufficient conditions on the regularity of the local marting-
gale and finite variation parts of the $G^n$ semimartingales. This is then applied to the case where the filtrations $G^n$ are obtained by progressively expanding a base filtration $F$ with processes $N^n$ converging in probability to some process $N$. We provide sufficient conditions for an $F$ semimartingale to remain a $G$ semimartingale, where $G$ is the progressive expansion of $F$ with $N$. Section 4 contains a useful little theorem (Theorem 9) concerning a dynamic process expansion obtained through the use of a sequence of sequences of honest times.

Section 5 applies the results obtained in Section 3 to the case where the base filtration $F$ is progressively expanded by a càdlàg process whose increments satisfy a generalized Jacod’s criterion with respect to the filtration $F$ along some sequence of subdivisions whose mesh tends to zero. An application to the expansion of a Brownian filtration with a time reversed diffusion is given through a detailed study, and the canonical decomposition of the Brownian motion in the expanded filtration is provided. Finally, we apply our results to models of insider trading and provide several concrete examples in Section 6.

Acknowledgements:

The authors wish to thank Monique Jeanblanc for pointing out the seminal work of Imkeller [32] and Zwierz [57]. They are also grateful to Jean Jacod, Umut Çetin, and Nizar Touzi for valuable help and advice. The first author wants to thank the hospitality of both Cornell University and Columbia University. The second author wishes to thank INRIA at Sophia-Antipolis and also the Courant Institute of NYU for their hospitality during a Sabbatical leave from Columbia University. Finally the authors would like to thank with enthusiasm one of the referees who was especially helpful with his or her constructive remarks.

2 Weak convergence of $\sigma$-fields and filtrations

2.1 Definitions and fundamental results

Let $\mathbb{D}$ be the space of càdlàg\(^1\) functions from $[0, T]$ into $\mathbb{R}$. Let $\Lambda$ be the set of time changes from $[0, T]$ into $[0, T]$, i.e. the set of all continuous strictly increasing functions $\lambda : [0, T] \to [0, T]$ such that $\lambda(0) = 0$ and $\lambda(T) = T$. We define the Skorohod distance as follows

$$d_S(x, y) = \inf_{\lambda \in \Lambda} \{ ||\lambda - Id||_{\infty} \vee ||x - y \circ \lambda||_{\infty} \}$$

for each $x$ and $y$ in $\mathbb{D}$. Let $(X^n)_{n \geq 1}$ and $X$ be càdlàg processes (i.e. whose paths are in $\mathbb{D}$), indexed by $[0, T]$ and defined on $(\Omega, \mathcal{H}, P)$. We will write $X^n \overset{P}{\to} X$ when $(X^n)_{n \geq 1}$ converges in probability under the Skorohod $J_1$ topology to $X$ i.e. when the sequence of random variables $(d_S(X^n, X))_{n \geq 1}$ converges in probability to zero. We can now introduce the concepts of weak convergence of $\sigma$-fields and of filtrations.

\(^1\)French acronym for right-continuous with left limits
Definition 1 A sequence of σ-fields $\mathcal{A}^n$ converges weakly to a σ-field $\mathcal{A}$ if and only if for all $B \in \mathcal{A}$, $E(1_B \mid \mathcal{A}^n)$ converges in probability to $1_B$. We write $\mathcal{A}^n \xrightarrow{w} \mathcal{A}$.

Definition 2 A sequence of right-continuous filtrations $\mathbb{F}^n$ converges weakly to a filtration $\mathbb{F}$ if and only if for all $B \in \mathcal{F}_T$, the sequence of càdlàg martingales $E(1_B \mid \mathcal{F}^n)$ converges in probability under the Skorohod $J_1$ topology on $\mathbb{D}$ to the martingale $E(1_B \mid \mathcal{F})$. We write $\mathbb{F}^n \xrightarrow{w} \mathbb{F}$.

The following lemmas provide characterizations of the weak convergence of σ-fields and filtrations. We refer to [19] for the proofs.

Lemma 1 A sequence of σ-fields $\mathcal{A}^n$ converges weakly to a σ-field $\mathcal{A}$ if and only if $E(Z \mid \mathcal{A}^n)$ converges in probability to $Z$ for any integrable and $\mathcal{A}$ measurable random variable $Z$.

Lemma 2 A sequence of filtrations $\mathbb{F}^n$ converges weakly to a filtration $\mathbb{F}$ if and only if $E(Z \mid \mathbb{F}^n)$ converges in probability under the Skorohod $J_1$ topology to $E(Z \mid \mathbb{F})$, for any integrable, $\mathcal{F}_T$ measurable random variable $Z$.

The weak convergence of the σ-fields $\mathcal{F}^n_t$ to $\mathcal{F}_t$ for all $t$ does not imply the weak convergence of the filtrations $\mathbb{F}^n$ to $\mathbb{F}$. The reverse implication does not hold either.

Coquet, Mémin and Slominsky provide a characterization of weak convergence of filtrations when the limiting filtration is the natural filtration of some càdlàg process $X$, see Lemma 3 in [19]. We provide a similar result for weak convergence of σ-fields when the limiting σ-field is generated by some càdlàg process $X$.

Lemma 3 Let $X$ be a càdlàg process. Define $\mathcal{A} = \sigma(X_t, 0 \leq t \leq T)$ and let $(\mathcal{A}^n)_{n \geq 1}$ be a sequence of σ-fields. Then $\mathcal{A}^n \xrightarrow{w} \mathcal{A}$ if and only if

$$E(f(X_{t_1}, \ldots, X_{t_k}) \mid \mathcal{A}^n) \xrightarrow{P} f(X_{t_1}, \ldots, X_{t_k})$$

for all $k \in \mathbb{N}$, $t_1, \ldots, t_k$ points of a dense subset $\mathcal{D}$ of $[0, T]$ containing $T$ and for any continuous and bounded function $f : \mathbb{R}^k \to \mathbb{R}$.

Proof. Necessity follows from the definition of the weak convergence of σ-fields. Let us prove the sufficiency. Let $A \in \mathcal{A}$ and $\varepsilon > 0$. There exists $k \in \mathbb{N}$ and $t_1, \ldots, t_k$ in $\mathcal{D}$ such that

$$E(|f(X_{t_1}, \ldots, X_{t_k}) - 1_A|) < \varepsilon.$$

Let $\eta > 0$. We need to show that $P(|E(1_A \mid \mathcal{A}^n) - 1_A| \geq \eta)$ converges to zero.

$$P(|E(1_A \mid \mathcal{A}^n) - 1_A| \geq \eta) \leq P(|E(1_A \mid \mathcal{A}^n) - E(f(X_{t_1}, \ldots, X_{t_k}) \mid \mathcal{A}^n)| \geq \frac{\eta}{3})$$

$$+ P(|E(f(X_{t_1}, \ldots, X_{t_k}) \mid \mathcal{A}^n) - f(X_{t_1}, \ldots, X_{t_k})| \geq \frac{\eta}{3}) + P(|f(X_{t_1}, \ldots, X_{t_k}) - 1_A| \geq \frac{\eta}{3})$$

$$\leq \frac{6}{\eta} E(|f(X_{t_1}, \ldots, X_{t_k}) - 1_A|) + P(|E(f(X_{t_1}, \ldots, X_{t_k}) \mid \mathcal{A}^n) - f(X_{t_1}, \ldots, X_{t_k})| \geq \frac{\eta}{3})$$

$$\leq \frac{6}{\eta} \varepsilon + P(|E(f(X_{t_1}, \ldots, X_{t_k}) \mid \mathcal{A}^n) - f(X_{t_1}, \ldots, X_{t_k})| \geq \frac{\eta}{3})$$

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where the second inequality follows from the Markov inequality. By assumption, there exists $N$ such that for all $n \geq N$,

$$P(|E(f(X_{t_1}, \ldots, X_{t_k}) | A^n) - f(X_{t_1}, \ldots, X_{t_k})| \geq \frac{\eta}{3}) \leq \varepsilon$$

hence $P(|E(1_A | A^n) - 1_A| \geq \eta) \leq (\frac{6}{\eta} + 1)\varepsilon$. ■

In [19], the authors provide cases where the weak convergence of a sequence of natural filtrations of given càdlàg processes is guaranteed. We provide here a similar result for point wise weak convergence of the associated $\sigma$-fields.

**Lemma 4** Let $(X^n)_{n \geq 1}$ be a sequence of càdlàg processes converging in probability to a càdlàg process $X$. Let $\mathbb{F}^n$ and $\mathbb{F}$ be the natural filtrations of $X^n$ and $X$ respectively. Then $\mathcal{F}_t^n \xrightarrow{w} \mathcal{F}_t$ for all $t$ such that $P(\Delta X_t \neq 0) = 0$.

**Proof.** Let $t$ be such that $P(\Delta X_t \neq 0) = 0$. Since $X$ is càdlàg, there exists $k \in \mathbb{N}$, and $t_1, \ldots, t_k \leq t$ such that $P(\Delta X_{t_i} \neq 0) = 0$, for all $1 \leq i \leq k$. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}$ be a continuous and bounded function. By Lemma 3 it suffices to show that

$$E(f(X_{t_1}, \ldots, X_{t_k}) | \mathcal{F}_t^n) \overset{P}{\rightarrow} f(X_{t_1}, \ldots, X_{t_k})$$

An application of Markov’s inequality leads to the following estimate

$$P(|E(f(X_{t_1}, \ldots, X_{t_k}) | \mathcal{F}_t^n) - f(X_{t_1}, \ldots, X_{t_k})| \geq \eta)$$

$$\leq P(|E(f(X_{t_1}, \ldots, X_{t_k}) - f(X_{t_1}^n, \ldots, X_{t_k}^n) | \mathcal{F}_t^n)| \geq \frac{\eta}{2})$$

$$+ P(|E(f(X_{t_1}^n, \ldots, X_{t_k}^n) | \mathcal{F}_t^n) - f(X_{t_1}, \ldots, X_{t_k})| \geq \frac{\eta}{2})$$

$$\leq \frac{4}{\eta} E(|f(X_{t_1}^n, \ldots, X_{t_k}^n) - f(X_{t_1}, \ldots, X_{t_k})|)$$

Since $X^n \overset{P}{\rightarrow} X$ and $P(\Delta X_t \neq 0) = 0$, for all $1 \leq i \leq k$, it follows that

$$(X_{t_1}^n, \ldots, X_{t_k}^n) \overset{P}{\rightarrow} (X_{t_1}, \ldots, X_{t_k})$$

and hence $f(X_{t_1}^n, \ldots, X_{t_k}^n)$ converges in $L^1$ to $f(X_{t_1}, \ldots, X_{t_k})$. This ends the proof of the lemma. ■

For a given càdlàg process $X$, a time $t$ such that $P(\Delta X_t \neq 0) > 0$ will be called a fixed time of discontinuity of $X$, and we will say that $X$ has no fixed times of discontinuity if $P(\Delta X_t \neq 0) = 0$ for all $0 \leq t \leq T$. Lemma 4 can be improved when the sequence $X^n$ is the discretization of the càdlàg process $X$ along some refining sequence of subdivisions $(\pi_n)_{n \geq 1}$ such that each fixed time of discontinuity of $X$ belongs to $\cup_n \pi_n$.

**Lemma 5** Let $X$ be a càdlàg process. Consider a sequence of subdivisions $(\pi_n = \{t^n_k\}, n \geq 1)$ whose mesh tends to zero and let $X_n$ be the discretized process defined by $X_{t_i}^n = X_{t_i}$, for all $t^n_k \leq t < t^n_{k+1}$. Let $\mathbb{F}$ and $\mathbb{F}^n$ be the natural filtrations of $X$ and $X^n$. If each fixed time of discontinuity of $X$ belongs to $\cup_n \pi_n$, then $\mathcal{F}_t^n \xrightarrow{w} \mathcal{F}_t$, for all $t$. 

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Proof. The proof is essentially the same as that of Lemma 4. Now, equation (1) holds because the subdivision contains the discontinuity points of $X$. □

We will also need the two following lemmas from the theory of weak convergence of $\sigma$-fields. The first result is proved in [20] and the second one in [19].

**Lemma 6** Let $(A^n)_{n \geq 1}$ and $(B^n)_{n \geq 1}$ be two sequences of $\sigma$-fields that weakly converge to $A$ and $B$, respectively. Then
\[ A^n \vee B^n \overset{w}{\to} A \vee B \]

**Lemma 7** Let $(A^n)_{n \geq 1}$ and $(B^n)_{n \geq 1}$ be two sequences of $\sigma$-fields such that $A^n \subset B^n$ for all $n$. Let $A$ be a $\sigma$-field. If $A^n \overset{w}{\to} A$ then $B^n \overset{w}{\to} A$.

As pointed out in [19], the results in Lemmas 6 and 7 are not true as far as one is interested in weak convergence of filtrations.

### 2.2 Approximation of a given stopping time

Let $(G^n)_{n \geq 1}$ be a sequence of right-continuous filtrations and let $G$ be a right-continuous filtration such that $G^n_t = G_t$ for all $t$. In order to obtain our filtration expansion results, we need a key theorem that guarantees the $G$ semimartingale property of a limit of $G^n$ semimartingales as in Theorem 3. The following lemma, which permits to approximate any $G$ bounded stopping time $\tau$ by a sequence of $G^n$ stopping times, will be of crucial importance in the proof of Theorem 3, Part (ii). We prove this result using successive approximations in the case where $\tau$ takes a finite number of values and show how this property is inherited by bounded stopping times. We do not study the general case (unbounded stopping times) since we are working on the finite time interval $[0, T]$.

**Lemma 8** Let $(G^n)_{n \geq 1}$ be a sequence of right-continuous filtrations and let $G$ be a right-continuous filtration such that $G^n_t \overset{w}{\to} G_t$ for all $t$. Let $\tau$ be a bounded $G$ stopping time. Then there exists $\phi : \mathbb{N} \to \mathbb{N}$ strictly increasing and a bounded sequence $(\tau_n)_{n \geq 1}$ such that the subsequence $(\tau_{\phi(n)})_{n \geq 1}$ converges in probability to $\tau$ and each $\tau_{\phi(n)}$ is a $G^{\phi(n)}$ stopping time.

**Proof.** Let $\tau$ be a $G$ stopping time bounded by $T$. Then there exists a sequence $\tau_n$ of $G$ stopping times decreasing a.s. to $\tau$ and taking values in $\{\frac{k}{2^n}, k \in \{0, 1, \cdots, 2^n T + 1\}\}$. This is true since the sequence $\tau_n = \frac{2^n \tau + 1}{2^n}$ obviously works. Hence $\tau_n$ takes a finite number of values. We claim that

**Claim.** for each $n$, we can construct a sequence $(\tau_{n,m})_{m \geq 1}$ converging in probability to $\tau_n$, and such that $\tau_{n,m}$ is a $G^m$ stopping time, for each $m$.

Assume we can do so and let $\eta > 0$ and $\varepsilon > 0$. Then for each $n$, $\lim_{m \to \infty} P(|\tau_{n,m} - \tau_n| > \frac{\eta}{2}) = 0$, i.e. for each $n$ there exists $M_n$ such that for all $m \geq M_n$, $P(|\tau_{n,m} - \tau_n| > \frac{\eta}{2}) \leq \frac{\varepsilon}{n}$. Define $\phi(1) = M_1$ and $\phi(n) = \max(M_n, \phi(n-1) + 1)$ by induction. The application $\phi : \mathbb{N} \to \mathbb{N}$ is strictly increasing, and for each $n$, $\tau_{n,\phi(n)}$ is a $G^{\phi(n)}$ stopping time and...
\( P(\{|\tau_n,\phi(n)| - \tau_n| > \frac{\eta}{2}\}) \leq \frac{\varepsilon}{2} \). It follows that
\[
P(\{|\tau_n,\phi(n)| - \tau| > \eta\}) \leq P(\{|\tau_n,\phi(n)| - \tau_n| > \frac{\eta}{2}\}) + P(\{|\tau_n - \tau| > \frac{\eta}{2}\}) \leq \frac{\varepsilon}{2} + P(\{|\tau_n - \tau| > \frac{\eta}{2}\})
\]

Since \( \tau_n \) converges to \( \tau \), there exists some \( n_0 \), such that for all \( n \geq n_0 \), \( P(\{|\tau_n - \tau| > \frac{\eta}{2}\}) \leq \frac{\varepsilon}{2} \).

Hence \( \tau_n,\phi(n) \xrightarrow{P} \tau \). So in order to prove the lemma, it only remains to prove the claim above.

**Proof of the claim.** We drop the index \( n \) and assume that \( \tau \) is a \( \mathcal{G} \) stopping time that takes a finite number of values \( t_1, \cdots, t_M \). Since \( \mathcal{G} \) is right-continuous, \( 1_{\{\tau = t_i\}} \) is \( \mathcal{G}_t \) measurable, and since by assumption, for all \( i \), \( \mathcal{G}^{m} \xrightarrow{w} \mathcal{G}_t \), it follows that for all \( i \)
\[
E(1_{\{\tau = t_i\}} \mid \mathcal{G}^{m}_{t_i}) \xrightarrow{P} 1_{\{\tau = t_i\}}
\]

Now for \( i = 1 \), we can extract a subsequence \( E(1_{\{\tau = t_1\}} \mid \mathcal{G}^{\phi_1(m)}_{t_1}) \) converging to \( 1_{\{\tau = t_1\}} \) a.s. and any sub-subsequence will also converge to \( 1_{\{\tau = t_1\}} \) a.s. Also, \( \mathcal{G}^{\phi_1(m)}_{t_1} \xrightarrow{w} \mathcal{G}_{t_2} \), hence \( E(1_{\{\tau = t_2\}} \mid \mathcal{G}^{\phi_1(m)}_{t_2}) \xrightarrow{P} 1_{\{\tau = t_2\}} \), and we can extract a further subsequence \( E(1_{\{\tau = t_2\}} \mid \mathcal{G}^{\phi_2(m)}_{t_2}) \) that converges a.s. to \( 1_{\{\tau = t_2\}} \). Since we have a finite number of possible values, we can repeat this reasoning up to time \( t_M \). Define then \( \phi = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n \), we get for all \( i \in \{1, \cdots, M\} \),
\[
E(1_{\{\tau = t_i\}} \mid \mathcal{G}^{\phi(m)}_{t_i}) \xrightarrow{a.s.} 1_{\{\tau = t_i\}}
\]

Define \( \tau_m = \min\{i \mid E(1_{\{\tau = t_i\}} \mid \mathcal{G}^{m}_{t_i}) > \frac{1}{2}\} t_i \). Then
\[
\{\tau_m = t_i\} = \{E(1_{\{\tau = t_i\}} \mid \mathcal{G}^{m}_{t_i}) > \frac{1}{2}\} \cap \{\forall t_j < t_i, E(1_{\{\tau = t_j\}} \mid \mathcal{G}^{m}_{t_j}) \leq \frac{1}{2}\}
\]

and hence \( \tau_m \) is a \( \mathcal{G}^m \) stopping time. Also, obviously, \( \tau_{\phi(m)} \xrightarrow{a.s.} \tau \), hence \( \tau_m \xrightarrow{P} \tau \). ■

### 2.3 Weak convergence of σ-fields and the semimartingale property

Assume we are given a sequence of filtrations \((\mathcal{F}^m)_{m \geq 1}\) and define the filtration \( \mathcal{F} = (\bar{\mathcal{F}}_t)_{0 \leq t \leq T} \), where \( \bar{\mathcal{F}}_t = \bigvee_m \mathcal{F}^m_t \). We prove in this section a stability result for \( \mathcal{F} \) semimartingales. More precisely, we prove that if \( X \) is an \( \mathcal{F} \) semimartingale, then it remains an \( \mathcal{F} \) semimartingale for any limiting (in the sense \( \mathcal{F}^m \xrightarrow{w} \mathcal{F}_t \), for all \( t \in [0, T] \)) filtration \( \mathcal{F} \) to which it is adapted.

The crucial tool for proving our first theorem is the Bichteler-Dellacherie characterization of semimartingales (see for example [54]). Recall that if \( \mathcal{H} \) is a filtration, an \( \mathcal{H} \) predictable elementary process \( H \) is a process of the form
\[
H_t(\omega) = \sum_{i=1}^k h_i(\omega) 1_{[t_i, t_{i+1})}(t);
\]
where $0 \leq t_1 \leq \ldots \leq t_{k+1} < \infty$, and each $h_i$ is $\mathcal{H}_{t_i}$ measurable. Moreover, for any adapted càdlàg process $X$ and predictable elementary process $H$ of the above form, we write

$$J_X(H) = \sum_{i=1}^{k} h_i(X_{t_i+1} - X_{t_i})$$

**Theorem 1 (Bichteler-Dellacherie)** Let $X$ be an adapted càdlàg process. Suppose that for every sequence $(H_n)_{n \geq 1}$ of bounded, adapted càdlàg processes that are null outside a fixed interval $[0, N]$ and convergent to zero uniformly in $(\omega, t)$, we have that

$$\lim_{n \to \infty} J_X(H_n) = 0$$

in probability. Then $X$ is an semimartingale.

The converse is true by the Dominated Convergence Theorem for stochastic integrals. We can now state and prove the main theorem of this subsection.

**Theorem 2** Let $(\mathbb{F}^m)_{m \geq 1}$ be a sequence of filtrations. Let $\mathbb{F}$ be a filtration such that for all $t \in [0, T]$, $\mathcal{F}^m_t \stackrel{w}{\to} \mathcal{F}_t$. Define the filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}$, where $\tilde{\mathcal{F}}_t = \bigvee_m \mathcal{F}^m_t$. Let $X$ be an adapted càdlàg process such that $X$ is an $\tilde{\mathbb{F}}$ semimartingale. Then $X$ is an $\mathbb{F}$ semimartingale.

**Proof.** For a fixed $N > 0$, consider a sequence of bounded, adapted càdlàg processes of the form

$$H^n_t = \sum_{i=1}^{k_n} h^n_i 1_{[t^n_i, t^n_{i+1})}(t);$$

null outside the fixed time interval $[0, N]$ and with $h^n_i$ being $\mathcal{F}^m_t$ measurable. Suppose that $H^n$ converges to zero uniformly in $(\omega, t)$. We prove that $J_X(H^n) \overset{P}{\to} 0$.

For each $m$, define the sequence of bounded $\mathbb{F}^m$ predictable elementary processes

$$H^{n,m}_t = \sum_{i=1}^{k_n} E(h^n_i | \mathcal{F}^m_{t^n_i})(1_{[t^n_i, t^n_{i+1})}(t);$$

By assumption, $\mathcal{F}^m_t \overset{w}{\to} \mathcal{F}_t$ for all $0 \leq t \leq T$. Hence for all $n$ and $1 \leq i \leq k_n$, $\mathcal{F}^m_{t^n_i} \overset{w}{\to} \mathcal{F}_{t^n_i}$. Since $h^n_i$ is bounded (hence integrable) and $\mathcal{F}_{t^n_i}$ measurable, it follows from Lemma that $E(h^n_i | \mathcal{F}^m_{t^n_i}) \overset{P}{\to} h^n_i$ and hence $E(h^n_i | \mathcal{F}^m_{t^n_i})(X_{t^n_{i+1}} - X_{t^n_i}) \overset{P}{\to} h^n_i(X_{t^n_{i+1}} - X_{t^n_i})$ for each $n$ and $1 \leq i \leq k_n$ since $(X_{t^n_{i+1}} - X_{t^n_i})$ is finite a.s. Let $\eta > 0$.

$$P\left(\left|J_X(H^{n,m}) - J_X(H^n)\right| > \eta\right) \leq \sum_{i=1}^{k_n} P\left(\left|E(h^n_i | \mathcal{F}^m_{t^n_i}) - h^n_i\right| (X_{t^n_{i+1}} - X_{t^n_i}) > \frac{\eta}{k_n}\right)$$

For each fixed $n$, the right side quantity converges to 0 as $m$ tends to $\infty$. This proves that for each $n$,

$$J_X(H^{n,m}) \overset{P}{\to} J_X(H^n).$$

Let $\delta > 0$ and $\varepsilon > 0$. For each $n$ and $m$,

$$P(|J_X(H^n)| > \delta) \leq P\left(|J_X(H^{n,m}) - J_X(H^n)| > \frac{\delta}{2}\right) + P\left(|J_X(H^{n,m})| > \frac{\delta}{2}\right) \quad (2)$$

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From $J_X(H^{n,m}) \xrightarrow{P} J_X(H^n)$, it follows that for each $n$, there exists $M^n_0$ such that for all $m \geq M^n_0$,

$$P\left(|J_X(H^{n,m}) - J_X(H^n)| > \frac{\delta}{2}\right) \leq \frac{\varepsilon}{2}$$

Hence $P(|J_X(H^n)| > \delta) \leq \frac{\varepsilon}{2} + P(|J_X(H^{n,M^n_0})| > \frac{\delta}{2})$. First $E(h^n_t \mid \mathcal{F}_{t_i}^{M^n_0})$ is bounded, \(\tilde{H}_{t_i}^{n,M^n_0}\) measurable so that $H_{t_i}^{n,M^n_0} = \sum_{i=1}^{k_n} E(h^n_t \mid \mathcal{F}_{t_i}^{M^n_0})1_{[t_i, t_{i+1})}(t)$ is a bounded \(\tilde{F}\) predictable process. Since $H^n$ converges to zero uniformly in $[\omega, t]$, it follows that $h^n_t$ converges to zero uniformly in $[\omega, t]$ so that there exists $n_0$ such that for each $n \geq n_0$, for all $(\omega, i)$, $|h^n_t(\omega)| \leq \varepsilon$. Hence, for all $(\omega, t)$ and $n \geq n_0$

$$|H_{t_i}^{n,M^n_0}(\omega)| \leq \sum_{i=1}^{k_n} E(|h^n_t| \mid \mathcal{F}_{t_i}^{M^n_0})(\omega)1_{[t_i, t_{i+1})}(t) \leq \varepsilon \sum_{i=1}^{k_n} 1_{[t_i, t_{i+1})}(t) \leq \varepsilon$$

Therefore $H_{t_i}^{n,M^n_0}$ is a sequence of bounded \(\tilde{F}\) predictable processes null outside the fixed interval $[0, N]$ that converges uniformly to zero in $[\omega, t]$. Since by assumption $X$ is a \(\tilde{F}\) semimartingale, it follows from the converse of Bichteler-Dellacherie’s theorem that $J_X(H_{t_i}^{n,M^n_0})$ converges to zero in probability, hence, for $n$ large enough, $P(|J_X(H^{n,M^n_0})| > \frac{\delta}{2}) \leq \frac{\varepsilon}{2}$ and

$$P(|J_X(H^n)| > \delta) \leq \varepsilon$$

Applying now Theorem 4 proves that $X$ is an \(\mathbb{F}\) semimartingale. 

Let $X$ be an \(\tilde{F}\) semimartingale. Theorem 2 proves that $X$ remains an \(\mathbb{F}\) semimartingale for any limiting filtration \(\mathbb{F}\) (in the sense $\mathcal{F}^m_t \xrightarrow{w} \mathcal{F}_t$ for all $0 \leq t \leq T$) to which $X$ is adapted. Of course, if $\mathbb{F} \subset \tilde{F}$, Stricker’s theorem already implies that $X$ is an \(\mathbb{F}\) semimartingale. But there is no general link between the filtration $\tilde{F} = \bigvee_m \mathbb{F}^m$ and the limiting filtration \(\mathbb{F}\). A trivial example is given by taking $\tilde{F}$ to be the trivial filtration (it can be seen from Definition 1 that the trivial filtration satisfies $\mathcal{F}^m_t \xrightarrow{w} \mathcal{F}_t$, for all $t$, for any given sequence of filtrations $\mathbb{F}^m$). One can also have $\bigvee_m \mathbb{F}^m \subset \tilde{F}$, as it is the case in the following important example.

**Example 1** Let $X$ be a càdlàg process. Consider a sequence of subdivisions $\{t^n_k\}$ whose mesh tends to zero and let $X^n$ be the discretized process defined by $X^n_t = X^n_{t^n_k}$, for all $t^n_k \leq t < t^n_{k+1}$. Let $\mathbb{F}$ and $\mathbb{F}^n$ be the natural filtrations of $X$ and $X^n$. It is well known that for all $t$, $\mathcal{F}_t \subset \bigvee_n \mathcal{F}^n_t \subset \mathcal{F}_t$. Also, $X^n$ converges a.s. to the process $X$, hence $X^n \xrightarrow{P} X$. Assume now that $X$ has no fixed times of discontinuity. Then Lemma 4 guarantees that $\mathcal{F}^n_t \xrightarrow{w} \mathcal{F}_t$, for all $t$. Moreover, if $\mathbb{F}$ is left-continuous (which is usually the case, and holds for example when $X$ is a càdlàg Hunt Markov process) then $\bigvee_n \mathcal{F}^n_t = \mathcal{F}_t$ for all $t$.

We provide now another example where $\bigvee_n \mathcal{F}^n_t$ is itself a limiting σ-field for $(\mathcal{F}^n_t)_{n \geq 1}$, for each $t$.

**Example 2** Assume that $\mathbb{F}^n$ is a sequence of filtrations such that for all $t$, the sequence of σ-fields $(\mathcal{F}^n_t)_{n \geq 1}$ is increasing for the inclusion. Define $\mathcal{F}_t = \bigvee_n \mathcal{F}^n_t$. Then for each $t$, $\mathcal{F}^n_t \xrightarrow{w} \mathcal{F}_t$. To see this, fix $t$ and let $X$ be an integrable $\mathcal{F}_t$ measurable random variable. Then $M_n = E(X \mid \mathcal{F}^n_t)$ is a closed martingale and the convergence theorem for closed
martingales ensures that $M_n$ converges to $X$ in $L^1$, which implies that $E(X \mid F^n_t)^F \to X$. Lemma 1 allows us to conclude.

Checking in practice that $X$ is an $\mathbb{F}$ semimartingale can be a hard task. In subsequent sections, we replace the strong assumption $X$ is an $\mathbb{F}$ semimartingale by the more natural assumption $X$ is an $\mathbb{F}^n$ semimartingale, for each $n$. Theorem 2 is very instructive since we see from the proof what goes wrong under this new assumption: the change in the order of limits in (2) cannot be justified anymore and extra integrability conditions will be needed. They are introduced in the next section.

This assumption arises naturally in filtration expansion theory in the following way. Assume we are given a base filtration $\mathbb{F}$ and a sequence of processes $N^n$ which converges (in probability for the Skorohod $J_1$ topology) to some process $N$. Let $N^n$ and $N$ be their natural filtrations and $G^n$ (resp. $G$) the smallest right-continuous filtration containing $\mathbb{F}$ and to which $N^n$ (resp. $N$) is adapted. Assume that for each $n$, every $\mathbb{F}$ semimartingale remains a $G^n$ semimartingale. Does this property also hold between $\mathbb{F}$ and $G$? In the next section we answer this question under the assumption of weak convergence of the $\sigma$-fields $G^n_t$ to $G_t$ for each $t$, for a class of $\mathbb{F}$ semimartingales $X$ satisfying some integrability conditions. If moreover $G^n \xrightarrow{w} G$, we are able to provide the $G$ decomposition of such $X$.

3 Filtration expansion with processes

In preparation for treating the expansion of filtrations via processes, we need to establish a general result on the convergence of semimartingales, which is perhaps of interest in its own right.

3.1 Convergence of semimartingales

The following theorem is a generalization of the main result in [11].

**Theorem 3** Let $(G^n_{s \geq 1})$ be a sequence of right-continuous filtrations and let $G$ be a filtration such that $G^n_t \xrightarrow{s} G_t$ for all $t$. Let $(X^n_{s \geq 1})$ be a sequence of $G^n$ semimartingales with canonical decomposition $X^n_s = X^n_0 + M^n_s + A^n_s$. Assume there exists $K > 0$ such that for all $n$,

$$E( \int_0^T |dA^n_s| ) \leq K \quad \text{and} \quad E( \sup_{0 \leq s \leq T} |M^n_s| ) \leq K$$

Then the following holds.

(i) Assume there exists a $G$ adapted process $X$ such that $E(\sup_{0 \leq s \leq T} |X^n_s - X_s|) \to 0$. Then $X$ is a $G$ special semimartingale.

(ii) Moreover, assume $G$ is right-continuous and let $X = M + A$ be the canonical decomposition of $X$. Then $M$ is a $G$ martingale and $\int_0^T |dA_s|$ and $\sup_{0 \leq s \leq T} |M_s|$ are integrable.
Proof. Part (i). The idea of the proof of Part (i) is similar to the one in [11]. First, X is càdlàg since it is the a.s. uniform limit of a subsequence of the càdlàg processes \((X^n)_{n \geq 1}\). Also, since \(||X^n_0 - X_0||_1 \to 0\), we can take w.l.o.g. \(X^n_0 = X_0 = 0\), and we do so. The integrability assumptions guarantee that \(E(\sup_s |X^n_s|) \leq 2K\) and up to replacing K by \(2K\), we assume that \(E(\sup_s |M^n_s|) \leq K\), \(E(\int_0^T |dA^n_s|) \leq K\) and \(E(\sup_s |X^n_s|) \leq K\). Then \(E(\sup_s |X_s|) \leq E(\sup_s |X_s - X^n_s|) + K\) and by taking limits \(E(\sup_s |X_s|) \leq K\).

Let \(H\) be a \(\mathbb{G}\) predictable elementary process of the form \(H_t = \sum_{i=1}^k h_i 1_{[t_i, t_{i+1})}(t)\), where \(h_i\) is a \(\mathcal{G}_t\) measurable random variable such that \(|h_i| \leq 1\) and \(t_1 < \ldots < t_k < T\). Define now \(H^n_t = \sum_{i=1}^k h_i^n 1_{[t_i, t_{i+1})}(t)\), where \(h^n_i = E(h_i | \mathcal{G}^n_t)\). Then \(h^n_i\) is a \(\mathcal{G}^n_t\) measurable random variable satisfying \(|h^n_i| \leq 1\), hence \(H^n\) is a bounded \(\mathbb{G}\) predictable elementary process. It follows that \(H^n \cdot M^n\) is a \(\mathbb{G}\) martingale and for each \(n\),

\[
|E((H^n \cdot X^n)_T)| \leq |E(\int_0^T H^n_s dA^n_s)| \leq E(\int_0^T |dA^n_s|) \leq K
\]

Therefore, for each \(n\),

\[
|E((H \cdot X)_T)| \leq |E((H \cdot X)_T - (H^n \cdot X^n)_T)| + K \quad (3)
\]

Since \(h_i\) is \(\mathcal{G}_t\) measurable and \(\mathcal{G}^n_t \xrightarrow{w} \mathcal{G}_t\) for all \(t\), \(h^n_i \xrightarrow{P} h_i\) for all \(1 \leq i \leq k\). Since the set \(\{1, \ldots, k\}\) is finite, successive extractions allow us to find a subsequence \(\psi(n)\) (independent from \(i\)) such that for each \(1 \leq i \leq k\), \(h^n_i \xrightarrow{a.s.} h_i\). So up to working with the \(\mathbb{G}^{\psi(n)}\) predictable elementary processes \(H^{\psi(n)}\) and the stochastic integrals \(H^{\psi(n)} \cdot X^{\psi(n)}\) in [3], we can assume that \(h^n_i \xrightarrow{a.s.} h_i\), for each \(1 \leq i \leq k\).

Now, \(|E((H \cdot X)_T - (H^n \cdot X^n)_T)| \leq \sum_{i=1}^k E(|h_i Y_i - h^n_i Y^n_i|)\) where \(Y_i = X_{t_{i+1}} - X_{t_i}\) and \(Y^n_i = X^n_{t_{i+1}} - X^n_{t_i}\). Each term in the sum can be bounded as follows.

\[
E(|h_i Y_i - h^n_i Y^n_i|) \leq E(|Y^n_i (h^n_i - h_i)|) + E(|h_i (Y^n_i - Y_i)|)
\]

\[
\leq 2E(\sup_s |X^n_s| |h^n_i - h_i|) + E(|Y^n_i - Y_i|)
\]

\[
\leq 2E(\sup_s |X^n_s - X_s| |h^n_i - h_i|) + 2E(\sup_s |X^n_s| |h^n_i - h_i|) + 2E(\sup_s |X^n_s - X_s|)
\]

\[
\leq 6E(\sup_s |X^n_s - X_s|) + 2E(\sup_s |X^n_s| |h^n_i - h_i|)
\]

Since \(\sup_s |X_s| |h^n_i - h_i|\) converges a.s to zero and that for all \(n\), \(|h^n_i| \leq 1\), hence \(\sup_s |X_s| |h^n_i - h_i| \leq 2 \sup_s |X_s|\) and \(\sup_s |X_s| \in L^1\), the Dominated Convergence Theorem implies that \(E(\sup_s |X_s| |h^n_i - h_i|) \to 0\). Since by assumption \(E(\sup_s |X^n_s - X_s|) \to 0\), it follows that \(|E((H \cdot X)_T - (H^n \cdot X^n)_T)|\) converges to 0. Letting \(n\) tend to infinity in (3) gives \(|E((H \cdot X)_T)| \leq K\). So \(X\) is a \(\mathbb{G}\) quasimartingale, hence a \(\mathbb{G}\) special semimartingale. Therefore \(X\) has a \(\mathbb{G}\) canonical decomposition \(X = M + A\) where \(M\) is a \(\mathbb{G}\) local martingale and \(A\) is a \(\mathbb{G}\) predictable finite variation process.

Part(ii). Let \((\tau_m)_{m \geq 1}\) be a sequence of bounded \(\mathbb{G}\) stopping times that reduces \(M\). Since for all \(t\), \(\mathcal{G}^m_t \xrightarrow{w} \mathcal{G}_t\), it follows from Lemma [3] that for each \(m\) there exist a function \(\phi_m\) strictly increasing and a sequence \((\tau^m_n)_{n \geq 1}\) such that \((\phi_m(\tau^m_n))_{n \geq 1}\) converges in probability.
to $\tau_m$ and $\tau_m^{\phi_m(n)}$ are bounded $\mathbb{G}^{\phi_m(n)}$ stopping times. We can extract a subsequence $(\tau_m^{\phi_m(\psi_m(n))})_{n \geq 1}$ converging a.s. to $\tau_m$. In order to simplify the notation, fix $m \geq 1$ and up to working with $\mathbb{G}^n = \mathbb{G}^{\phi_m(\psi_m(n))}$ instead of $\mathbb{G}^n$ (which satisfies the same assumptions), take $\Phi_m := \phi_m \circ \psi_m$ to be the identity. Let $H$ be a $\mathbb{G}$ elementary predictable process as defined in Part (i). Since $\tau_m$ reduces $M$, $E((H \cdot A)_{\tau_m}) = E((H \cdot X)_{\tau_m})$. We can write

$$E((H \cdot A)_{\tau_m}) = E((H \cdot X)_{\tau_m} - (H^n \cdot X^n)_{\tau_m}) + E((H^n \cdot X^n)_{\tau_m})$$

We start with the third term. Since $H^n \cdot M^n$ is a $\mathbb{G}^n$ martingale and $\tau_m$ is a bounded $\mathbb{G}^n$ stopping time, it follows from Doob’s optional sampling theorem that $E((H^n \cdot X^n)_{\tau_m}) = E((H^n \cdot A^n)_{\tau_m})$, hence $|E((H^n \cdot X^n)_{\tau_m})| \leq E(\int_0^\tau |dA^n|) \leq K$.

We focus now on the first term. Let $Y_s^i = X_s - X_{t_i}$ and $Y_s^{i,n} = X_s^n - X_{t_i}^n$.

$$E_1 := |E((H \cdot X)_{\tau_m} - (H^n \cdot X^n)_{\tau_m})| \leq E\left(\sup_{0 \leq s \leq T} |(H \cdot X)_s - (H^n \cdot X^n)_s|\right)$$

$$\leq E\left(\sum_{i=1}^k \sup_{t_i < s \leq t_{i+1}} |h_i Y_s^i - h_i^n Y_s^{i,n}|\right)$$

$$\leq \sum_{i=1}^k E\left(\sup_{t_i < s \leq t_{i+1}} |Y_s^{i,n}| |h_i^n - h_i| + \sup_{t_i < s \leq t_{i+1}} |h_i||Y_s^{i,n} - Y_s^i|\right)$$

Since $|h_i| \leq 1$, $|Y_s^{i,n}| \leq 2\sup_{0 \leq u \leq T} |X_u^n|$ and $|Y_s^{i,n} - Y_s^i| \leq 2\sup_{0 \leq u \leq T} |X_u^n - X_u|$, it follows that

$$E_1 \leq 2 \sum_{i=1}^k \{E\left(\sup_{0 \leq u \leq T} |X_u^n||h_i^n - h_i|\right) + E\left(\sup_{0 \leq u \leq T} |X_u^n - X_u|\right)\}$$

$$\leq 6kE\left(\sup_{0 \leq u \leq T} |X_u^n - X_u|\right) + 2 \sum_{i=1}^k \{E\left(\sup_{0 \leq u \leq T} |X_u||h_i^n - h_i|\right)\}$$

We study now the second term $E_2 := E((H^n \cdot X^n)_{\tau_m} - (H^n \cdot X^n)_{\tau_m^n})$. Let $0 < \eta < \min_i |t_{i+1} - t_i|$, and define $Y^n := H^n \cdot X^n$. Write now

$$E(|Y^n_{\tau_m} - Y^n_{\tau_m^n}|) = E(|Y^n_{\tau_m} - Y^n_{\tau_m}|1_{|\tau_m - \tau_m^n| \leq \eta}) + E(|Y^n_{\tau_m} - Y^n_{\tau_m^n}|1_{|\tau_m - \tau_m^n| > \eta}) =: e_1 + e_2$$

We study each of the two terms separately. We start with $e_2$.

$$e_2 \leq E\left(|Y^n_{\tau_m} + |Y^n_{\tau_m^n}|1_{|\tau_m - \tau_m^n| > \eta}\right) \leq 2E\left(\sum_{i=1}^k \sup_{t_i < s \leq t_{i+1}} |X_s^n - X_{t_i}^n|1_{|\tau_m - \tau_m^n| > \eta}\right)$$

$$\leq 4kE\left(\sup_{0 \leq u \leq T} |X_u^n|1_{|\tau_m - \tau_m^n| > \eta}\right)$$

$$\leq 4kE\left(\sup_{0 \leq u \leq T} |X_u^n - X_u|\right) + 4kE\left(\sup_{0 \leq u \leq T} |X_u^n|1_{|\tau_m - \tau_m^n| > \eta}\right)$$

We study now $e_1$. On $\{|\tau_m - \tau_m^n| \leq \eta\}$ and since $\eta < \min_i |t_{i+1} - t_i|$, we have $|Y^n_{\tau_m} - Y^n_{\tau_m^n}| \leq 2\sup_{0 \leq s \leq \eta} |X_s^n - X_s^0|$. In fact, one of the two following cases is possible for $\tau_m$ and

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\[ \tau_m. \] Either they are both in the same interval \((t_i, t_{i+1}]\), in which case, \(|Y^n_{\tau_m} - Y^n_{\tau_m^+}| = |h_t^n(X^n_{\tau_m} - X^n_{\tau_m^+})| \leq \sup_{s \leq t \leq s+\eta}|X^n_t - X^n_s|\), or they are in two consecutive intervals. For the second case, take for example \(t_{i-1} < \tau_m \leq t_i < \tau_m \leq t_{i+1}\), then

\[
|Y^n_{\tau_m} - Y^n_{\tau_m^+}| = |h_t^{n-1}(X^n_{\tau_m} - X^n_{\tau_{i-1}}) - h_t^{n-1}(X^n_{\tau_{i-1}} - X^n_{\tau_{i+1}})| \\
= |h_t^n(X^n_{\tau_m} - X^n_{\tau_{i-1}}) - h_t^n(X^n_{\tau_{i-1}} - X^n_{\tau_{i+1}})| \leq |X^n_{\tau_m} - X^n_{\tau_{i-1}}| + |X^n_{\tau_{i-1}} - X^n_{\tau_{i+1}}| \\
\leq 2 \sup_{s \leq t \leq s+\eta}|X^n_t - X^n_s|
\]

The case \(t_{i-1} < \tau_m \leq t_i < \tau_m \leq t_{i+1}\) is similar. Hence

\[
e_1 \leq 2E(\sup_{s \leq t \leq s+\eta}|X^n_t - X^n_s|1_{\{|\tau_m - \tau_m^+| \leq \eta\}}) \leq 2E(\sup_{s \leq t \leq s+\eta}|X^n_t - X^n_s|)
\]

Putting all this together yields for each \(0 < \eta < \min |t_{i+1} - t_i| \) and each \(n\)

\[
|E(H \cdot A)_{\tau_m}| \leq K + 2E(\sup_{s \leq t \leq s+\eta}|X^n_t - X^n_s|) + 2 \sum_{i=1}^{k} E(\sup_{0 \leq s \leq T}|X^n_s||h^n_{i} - h_i|) \\
+ 4kE(\sup_{0 \leq s \leq T}|X^n_s|1_{\{|\tau_m - \tau_m^+| > \eta\}}) + 10kE(\sup_{0 \leq u \leq T}|X^n_u - X_u|)
\]

Getting back to the general case, we obtain for each \(m \geq 1\) and each \(n \geq 1\),

\[
|E(H \cdot A)_{\tau_m}| \leq K + 2E(\sup_{s \leq t \leq s+\eta}|X_t^{\lambda_m(n)} - X_s^{\lambda_m(n)}|) + 2 \sum_{i=1}^{k} E(\sup_{0 \leq s \leq T}|X_s||h_i^{\lambda_m(n)} - h_i|) \\
+ 4kE(\sup_{0 \leq s \leq T}|X_s|1_{\{|\tau_m - \tau_m^+| > \eta\}}) + 10kE(\sup_{0 \leq u \leq T}|X_u^{\lambda_m(n)} - X_u|)
\]

As in the proof of Part (i), successive extractions allow us to find \(\lambda_m(n)\) (independent from \(i\)) such that for all \(1 \leq i \leq k, h_i^{\lambda_m(n)}\) converges a.s. to \(h_i\). Letting \(n\) go to infinity in

\[
|E(H \cdot A)_{\tau_m}| \leq K + 2\lim_{n \to \infty} E(\sup_{s \leq t \leq s+\eta}|X^n_t - X^n_s|)
\]

Let \(\lim_{\eta \to 0} \lim_{n \to \infty} E(\sup_{s \leq t \leq s+\eta}|X^n_t - X^n_s|) = C\). Since \(E(\sup_{s \leq t \leq s+\eta}|X^n_t - X^n_s|) \leq 2E(\sup_u |X^n_u|) \leq 2E(\sup_u |M^n_u| + t_0^T|dA^n_s|) \leq 4K, C < \infty\). Now letting \(\eta\) go to zero yields finally \(|E(H \cdot A)_{\tau_m}| \leq K + 2C, \) for each \(m\). Thus \(E(\int_0^{\tau_m} |dA_u|) \leq K + 2C, \) for each \(m\) and hence \(E(\int_0^{\tau_m} |dA_u|) \leq K + 2C.\)
Now, \( M = X - A = (X - X^n) + M^n + A^n - A \), and so
\[
\sup_{0 \leq s \leq T} |M_s| \leq \sup_{0 \leq s \leq T} |X_s - X^n_s| + \sup_{0 \leq s \leq T} |M^n_s| + \int_0^T |dA^n_s| + \int_0^T |dA_s|
\]
Thus \( E(\sup_{0 \leq s \leq T} |M_s|) \leq 3K + 2C \) and \( M \) is a \( \mathbb{G} \) martingale. 

Once one obtains that \( X \) is a \( \mathbb{G} \) special semimartingale, one can be interested in characterizing the martingale \( M \) and the finite variation predictable process \( A \) in terms of the processes \( M^n \) and \( A^n \). Méméin (Theorem 11 in [49]) achieved this under “extended convergence.” Recall that \((X^n, \mathbb{G}^n)\) converges to \((X, \mathbb{G})\) in the extended sense if for every \( G \in \mathcal{G}_T \), the sequence of càdlàg processes \((X^n_t, E(1_G \mid \mathcal{G}^n_t))_{0 \leq t \leq T}\) converges in probability under the Skorohod \( J_1 \) topology to \((X_t, E(1_G \mid \mathcal{G}_t))_{0 \leq t \leq T}\). The author proves the following theorem. We refer to [49] for a proof. In the theorem below, \( \mathbb{G}^n \) and \( \mathbb{G} \) are right-continuous filtrations.

**Theorem 4** Let \((X^n)_{n \geq 1}\) be a sequence of \( \mathbb{G}^n \) special semimartingales with canonical decompositions \( X^n = M^n + A^n \) where \( M^n \) is a \( \mathbb{G}^n \) martingale and \( A^n \) is a \( \mathbb{G}^n \) predictable finite variation process. We suppose that the sequence \([X^n, X^n]_{\frac{T}{2}}\) is uniformly integrable and that the sequence \((V(A^n)_T)_{n \geq 1}\) (where \( V \) denotes the variation process) of real random variables is tight in \( \mathbb{R} \). Let \( X \) be a \( \mathbb{G} \) quasi-left continuous special semimartingale with a canonical decomposition \( X = M + A \) such that \([X, X]_{\frac{T}{2}} < \infty\).

If the extended convergence \((X^n, \mathbb{G}^n) \to (X, \mathbb{G})\) holds, then \((X^n, M^n, A^n)\) converges in probability under the Skorohod \( J_1 \) topology to \((X, M, A)\).

In a filtration expansion setting, the sequence \( X^n \) is constant and equal to some semimartingale \( X \) of the base filtration. In this case the extended convergence assumption in Theorem 4 reduces to the weak convergence of the filtrations. We can deduce the following corollary from Theorems 3 and 4.

**Corollary 1** Let \((\mathbb{G}^n)_{n \geq 1}\) be a sequence of right-continuous filtrations and let \( \mathbb{G} \) be a filtration such that \( \mathbb{G}^n_t \nrightarrow \mathbb{G}_t \) for all \( t \). Let \( X \) be a stochastic process such that for each \( n \), \( X \) is a \( \mathbb{G}^n \) semimartingale with canonical decomposition \( X = M^n + A^n \) such that there exists \( K > 0 \), \( E(\int_0^T |dA^n_s|) \leq K \) and \( E(\sup_{0 \leq s \leq T} |M^n_s|) \leq K \) for all \( n \).

(i) If \( X \) is \( \mathbb{G} \) adapted, then \( X \) is a \( \mathbb{G} \) special semimartingale.

(ii) Assume moreover that \( \mathbb{G} \) is right-continuous and let \( X = M + A \) be the canonical decomposition of \( X \). Then \( M \) is a \( \mathbb{G} \) martingale and \( \sup_{0 \leq s \leq T} |M_s| \) and \( \int_0^T |dA_s| \) are integrable.

(iii) Furthermore, assume that \( X \) is \( \mathbb{G} \) quasi-left continuous and \( \mathbb{G}^n \nrightarrow \mathbb{G} \). Then \((M^n, A^n)\) converges in probability under the Skorohod \( J_1 \) topology to \((M, A)\).

**Proof.** The sequence \( X^n = X \) clearly satisfies the assumptions of Theorem 3 and the two first claims follow. For the last claim, notice that \([X, X]_T \in L^1 \), so \( \sqrt{[X, X]_T} \in L^1 \) and hence \((\sqrt{[X, X]_T})_{0 \leq t \leq T} \) is a uniformly integrable family of random variables. The tightness of the sequence of random variables \((V(A^n)_T)_{n \geq 1}\) follows from \( E(\int_0^T |dA^n_s|) \leq K \) for any \( n \) and some \( K \) independent from \( n \). ■

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3.2 Applications to filtration expansions

We provide in this subsection a first application to the initial and progressive filtration expansions with a random variable and a general theorem on the progressive expansion with a process. We assume in the sequel that a right-continuous filtration $\mathbb{F}$ is given.

3.2.1 Initial and progressive filtration expansions with a random variable

Assume that $\mathbb{F}$ is the natural filtration of some càdlàg process. Let $\tau$ be a random variable and $\mathbb{H}$ and $\mathbb{G}$ the initial and progressive expansions of $\mathbb{F}$ with $\tau$. In this subsection, the filtration $\mathbb{G}$ is considered only when $\tau$ is non negative. It is proved in [44] that if $\tau$ satisfies Jacod’s criterion i.e. if there exists a $\sigma$-finite measure $\eta$ on $\mathcal{B}(\mathbb{R})$ such that

$$P(\tau \in \cdot \mid \mathcal{F}_t) (\omega) \ll \eta(\cdot) \quad \text{a.s.}$$

then every $\mathbb{F}$ semimartingale remains an $\mathbb{H}$ and $\mathbb{G}$ semimartingale. That it is an $\mathbb{H}$ semimartingale is due to Jacod [34]. That it is also a $\mathbb{G}$ semimartingale follows from Stricker’s theorem. Its $\mathbb{G}$ decomposition is obtained in [44] and this relies on the fact that these two filtrations coincide after $\tau$. We provide now a similar but partial result for a random variable $\tau$ which may not satisfy Jacod’s criterion. Assume there exist a sequence of random times $(\tau_n)_{n \geq 0}$ converging in probability to $\tau$ and let $\mathbb{H}^n$ and $\mathbb{G}^n$ be the initial and progressive expansions of $\mathbb{F}$ with $\tau_n$. The following holds.

**Theorem 5** Let $M$ be an $\mathbb{F}$ martingale such that $\sup_{0 \leq t \leq T} |M_t|$ is integrable. Assume there exists an $\mathbb{H}^n$ predictable finite variation process $A^n$ such that $M - A^n$ is an $\mathbb{H}^n$ martingale. If there exists $K$ such that $E\left(\int_0^T |dA^n_s|\right) \leq K$ for all $n$, then $M$ is an $\mathbb{H}$ and $\mathbb{G}$ semimartingale.

**Proof.** Since $\tau_n$ converges in probability to $\tau$ and $\mathbb{F}$ is the natural filtration of some càdlàg process, we can prove that $\mathcal{H}_t^n \xrightarrow{w} \mathcal{H}_t$ for each $t \in [0, T]$, using the same techniques as in Lemmas 3 and 4. Up to replacing $K$ by $K + E(\sup_{0 \leq t \leq T} |M_t|)$, $M^n = M - A^n$ and $A^n$ satisfy the assumptions of Corollary 1. Therefore $M$ is an $\mathbb{H}$ semimartingale, and a $\mathbb{G}$ semimartingale by Stricker’s theorem.

One case where the first assumption of Theorem 5 is satisfied is when $\tau_n$ satisfies Jacod’s criterion, for each $n \geq 0$. In this case, and if $\mathbb{F}$ is the natural filtration of a Brownian motion $W$, the result above can be made more explicit. Assume for simplicity that the conditional distributions of $\tau_n$ are absolutely continuous w.r.t Lebesgue measure,

$$P(\tau_n \in du \mid \mathcal{F}_t)(\omega) = p^n_t(u, \omega)du$$

where the conditional densities are chosen so that $(u, \omega, t) \rightarrow p^n_t(u, \omega)$ is càdlàg in $t$ and measurable for the optional $\sigma$-field associated with the filtration $\bar{\mathbb{F}}$ given by $\bar{\mathcal{F}}_t = \cap_{u \geq t} \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_u$. From the martingale representation theorem in a Brownian filtration, there exists for each $n$ a family $\{q^n_t(u), u > 0\}$ of $\mathbb{F}$ predictable processes $(q^n_t(u))_{0 \leq t \leq T}$ such that

$$p^n_t(u) = p^n_0(u) + \int_0^t q^n_s(u)dW_s \quad (4)$$
Corollary 2 Assume there exists $K$ such that $E\left( \int_0^T \frac{q^n_{\tau_n}(u)}{p^n_{\tau_n}(u)} \, du \right) \leq K$ for all $n$, then $W$ is a special semimartingale in both $\mathbb{H}$ and $\mathbb{G}$.

Proof. Since $\tau_n$ satisfy Jacod’s criterion, it follows from Theorem 2 in [34] that $W_t - A^n_t$ is an $\mathbb{H}$ local martingale, where $A^n_t = \int_0^t d\langle p^n(u), W \rangle_u \bigg|_{u = \tau_n}$. Now, it follows from [43] that $A^n_t = \int_0^t \frac{q^n_{\tau_n}(u)}{p^n_{\tau_n}(u)} \, du$ and Theorem 5 allows us to conclude. □

Assume the assumptions of Corollary 2 are satisfied and let $W = M + A$ be the $\mathbb{H}$ canonical decomposition of $W$. Let $m$ be a $\mathbb{F}$ predictable process such that $\int_0^t m^2_s \, ds$ is locally integrable and let $M$ be the $\mathbb{F}$ local martingale $M_t = \int_0^t m_s dW_s$. Then $M$ is an $\mathbb{H}$ semimartingale as soon as the process $(\int_0^t m_s dA_s)_{t \geq 0}$ exists as a path-by-path Lebesgue-Stieltjes integral a.s. See [43] for a more comprehensive investigation of this result.

Example 3 In order to emphasize that some assumptions as in Theorem 5 are needed, we provide now a counter-example. Let $\mathbb{F}$ be the natural filtration of some Brownian motion $B$ and choose $\tau$ to be some functional of the Brownian path i.e. $\tau = f((B_s, 0 \leq s \leq 1))$, such that $\sigma(\tau) = F_1$. Then $B$ is not a semimartingale in $\mathbb{H}$ as $(F_1 \vee \sigma(\tau))_{0 \leq t \leq 1}$. Now, define $\tau_n = \tau + \frac{1}{\sqrt{n}}N$, where $N$ is a standard normal random variable independent from $\mathbb{F}$. Then $\tau_n$ converge a.s. to $\tau$ and $P(\tau_n \leq u \mid F_t) = \int_{-\infty}^{u} E(g_n(u - \tau) \mid F_t) \, dv$ where $g_n$ is the probability density function of $\frac{1}{\sqrt{n}}N$, hence $P(\tau_n \in du \mid F_t)(\omega) = E(g_n(u - \tau) \mid F_t)(\omega)(du)$. Therefore, $\tau_n$ satisfies Jacod’s criterion, for each $n$ and $p^n_t(u, \omega) = E(g_n(u - \tau) \mid F_t)(\omega)$. Thus, $B$ is a semimartingale in $\mathbb{H} = (F_t \vee \sigma(\tau_n)_{0 \leq t \leq 1})$ and $\mathbb{H}_t \overset{w}{\to} \mathbb{H}_t$ for each $0 \leq t \leq 1$.

3.2.2 Progressive filtration expansion with a process

Let $(N^n)_{n \geq 1}$ be a sequence of càdlàg processes converging in probability under the Skorohod $J_1$-topology to a càdlàg process $N$ and let $\mathbb{N}^n$ and $\mathbb{N}$ be their natural filtrations. Define the filtrations $\mathbb{G}^{0,n} = \mathbb{F} \vee \mathbb{N}^n$ and $\mathbb{G}^n$ by $\mathbb{G}_t^n = \bigcap_{u>t} \mathbb{G}^{0,n}_u$. Let also $\mathbb{G}^0$ (resp. $\mathbb{G}$) be the smallest (resp. the smallest right-continuous) filtration containing $\mathbb{F}$ and to which $N$ is adapted. The result below is the main theorem of this paper.

Theorem 6 Let $X$ be an $\mathbb{F}$ semimartingale such that for each $n$, $X$ is a $\mathbb{G}^n$ semimartingale with canonical decomposition $X = M^n + A^n$. Assume $E\left( \int_0^T |dA^n_s| \right) \leq K$ and $E(\sup_{0 \leq s \leq T} |M^n_s|) \leq K$ for some $K$ and all $n$. Finally, assume one of the following holds.

- $N$ has no fixed times of discontinuity,
- $N^n$ is a discretization of $N$ along some refining subdivision $(\pi_n)_{n \geq 1}$ such that each fixed time of discontinuity of $N$ belongs to $\cup_n \pi_n$.

Then

(i) $X$ is a $\mathbb{G}^0$ special semimartingale.

(ii) Moreover, if $\mathbb{F}$ is the natural filtration of some càdlàg process then $X$ is a $\mathbb{G}$ special semimartingale with canonical decomposition $X = M + A$ such that $M$ is a $\mathbb{G}$
Therefore, \( N^n \overset{P}{\to} N \) and \( P(\Delta N_t \neq 0) = 0 \) for all \( t \), it follows from Lemma 3 that \( N^n \overset{P}{\to} N \) for all \( t \). The same holds under assumption (ii) using Lemma 4. Lemma 4 then ensures that \( G^0_t \overset{w}{\to} G^0_t \) for all \( t \). Since \( G^0_t \subset \bigcap_{u \geq t} G^0_u = G^0 \), it follows from Lemma 7 that \( G^n = G^0 = G^0 \) for all \( t \). Being an \( \mathcal{F} \) semimartingale, \( X \) is clearly \( G^0 \) adapted. An application of Corollary 4 ends the proof of the first claim. When \( F \) is the natural filtration of some càdlàg process, the same proofs as of Lemmas 4 and 5 guarantee that \( G^n \overset{w}{\to} G^n \) for all \( t \). Since \( G \) is right-continuous, the second and third claims follow from Corollary 4.

We apply this result to expand the filtration \( F \) progressively with a point process. Let \( (\tau_i)_{i \geq 1} \) and \( (X_i)_{i \geq 1} \) be two sequences of random variables such that for each \( n \), the random vector \( (\tau_1, X_1, \ldots, \tau_n, X_n) \) satisfies Jacod’s criterion w.r.t the filtration \( F \). Assume that for all \( t \) and \( P(\tau_i = t) = 0 \) and that one of the following holds:

(i) For all \( i \), \( X_i \) and \( \tau_i \) are independent, \( E|X_i| = \mu \) for some \( \mu \) and \( \sum_{i=1}^{\infty} P(\tau_i \leq T) < \infty \).

(ii) \( E(|X_i^2|) = c \) and \( \sum_{i=1}^{\infty} \sqrt{P(\tau_i \leq T)} < \infty \).

Let \( N^n_t = \sum_{i=1}^{n} X_i 1_{\{\tau_i \leq t\}} \) and \( N_t = \sum_{i=1}^{\infty} X_i 1_{\{\tau_i \leq t\}} \). The assumptions on \( N^n \) and \( N \) as of Theorem 6 are satisfied.

**Lemma 9** Under the assumptions above, \( N_t \in L^1 \) for each \( t \), \( N^n \overset{P}{\to} N \) and \( N \) has no fixed times of discontinuity.

**Proof.** We prove the statement under assumption (i). For each \( t \),

\[
E(|N_t|) \leq \sum_{i=1}^{\infty} E(|X_i| 1_{\{\tau_i \leq t\}}) \leq \mu \sum_{i=1}^{\infty} P(\tau_i \leq t) < \infty
\]

Therefore, \( N_t \in L^1 \). For \( \eta > 0 \) and \( n \) integer, we obtain the following estimate.

\[
P(\sup_{0 \leq s \leq T} |N_t - N^n_t| \geq \eta) = P(\sup_{0 \leq s \leq T} |\sum_{i=n+1}^{\infty} X_i 1_{\{\tau_i \leq t\}}| \geq \eta)
\]

\[
\leq P(\sup_{0 \leq s \leq T} \sum_{i=n+1}^{\infty} |X_i| 1_{\{\tau_i \leq t\}} \geq \eta) = P(\sum_{i=n+1}^{\infty} |X_i| 1_{\{\tau_i \leq t\}} \geq \eta)
\]

\[
\leq \frac{1}{\eta} E(\sum_{i=n+1}^{\infty} |X_i| 1_{\{\tau_i \leq T\}}) = \frac{\mu}{\eta} \sum_{i=n+1}^{\infty} P(\tau_i \leq T) \to 0
\]

This implies \( N^n \overset{P}{\to} N \). Under assumption (ii), the proof is also straightforward and based on Cauchy Schwarz inequalities. Finally, since

\[
P(|\Delta N_t| \neq 0) \leq P(\exists t : \tau_t = t) \leq \sum_{i=1}^{\infty} P(\tau_i = t) = 0
\]
N has no fixed times of discontinuity ■

Since the random vector \((\tau_1, X_1, \ldots, \tau_n, X_n)\) is assumed to satisfy Jacod’s criterion, it follows from [44] that \(F\) semimartingales remain \(G^n\) semimartingales, for each \(n\). Therefore, this property also holds between \(F\) and \(G\) for \(F\) semimartingales whose \(G^n\) canonical decompositions satisfy the regularity assumptions of Theorem [5]. Here \(G\) is the smallest filtration containing \(F\) and to which \(N\) is adapted.

We would like to take a step further and reverse the previous situation. That is instead of starting with a sequence of processes \(N_n\) converging to some process \(N\), and putting assumptions on the semimartingale properties of \(F\) semimartingales w.r.t. the intermediate filtrations \(G^n\) and their decompositions therein, we would like to expand the filtration \(F\) with a given process \(X\) and express all the assumptions in terms of \(X\) and the \(F\) semimartingales considered. We are able to do this for càdlàg processes which satisfy a criterion that can loosely be seen as a localized extension of Jacod’s criterion to processes. The integrability assumptions of Theorem [6] are expressed in terms of \(F_t\)-conditional densities.

Before doing this, we conclude this section by studying the stability of hypothesis \((H)\) with respect to the weak convergence of the σ-fields in a filtration expansion setting.

3.3 The case of hypothesis \((H)\)

Recall that given two nested filtrations \(F \subset G\), we say that hypothesis \((H)\) holds between \(F\) and \(G\) if any square integrable \(F\) martingale remains a \(G\) martingale. Brémaud and Yor proved the next lemma (see [16]).

**Lemma 10** Let \(F \subset G\) two nested filtrations. The following assertions are equivalent.

1. Hypothesis \((H)\) holds between \(F\) and \(G\).
2. For each \(0 \leq t \leq T\), \(F_T\) and \(G_t\) are conditionally independent given \(F_t\).
3. For each \(0 \leq t \leq T\), each \(F \in L^2(F_T)\) and each \(G_t \in L^2(G_t)\),

\[
E(FG_t \mid F_t) = E(F \mid F_t)E(G_t \mid F_t).
\]

Let \(F \subset G\) be two nested right-continuous filtrations and \(G^n\) be a sequence of right-continuous filtrations containing \(F\) and such that \(G^n_T\) converges weakly to \(G_t\) for each \(t\). We mentioned that an \(F\) local martingale that remains a \(G^n\) semimartingale for each \(n\) might still lose its semimartingale property in \(G\) and we provided conditions that prevent this pathological behavior. In this subsection, we prove that this cannot happen in case hypothesis \((H)\) holds between \(F\) and each \(G^n\). One obtains even that hypothesis \((H)\) holds between \(F\) and \(G\).

**Theorem 7** Let \(F\), \(G\) and \((G^n)_{n \geq 1}\) right-continuous filtrations such that \(F \subset G\), \(F \subset G^n\) for each \(n\) and \(G^n_T \xrightarrow{w} G_t\) for each \(t\). Assume that for each \(n\), hypothesis \((H)\) holds between \(F\) and \(G^n\). Then hypothesis \((H)\) holds between \(F\) and \(G\).
Theorem 8 (Jeulin) Let \( M \) be an \( \mathbb{F} \) local martingale. If Assumption 1 holds, then \( M - A^n \) is a \( \mathbb{G}^n \) local martingale, where

\[
A^n_t = \sum_{p=0}^{\infty} \int_0^t 1\{\tau^n_{p+1} < \tau^n_{p} \leq \tau^n_{p+1}\} \frac{1}{Z^n_{s-p-1} - Z^n_{s-p}} d(M, M^{n,p} - M^n, M^{n,p})_s
\]

Assumption 1 (Honest times assumption) For each \( n \geq 1 \), the sequence \( (\tau^n_p)_{p \geq 0} \) is an increasing sequence of \( \mathbb{F} \) honest times such that \( \tau^n_0 = 0 \) and \( \sup_p \tau^n_p = \infty \).

Under Assumption 1 the following holds (see Jeulin [41, Corollary 5.22]).

Proof. We use Lemma 10 and start with the bounded case. Let \( 0 \leq t \leq T, F \in L^2(\mathcal{F}_T) \) and \( G_t \in L^\infty(\mathcal{G}_t) \). For each \( n \), define \( G^n_t = E(G_t | \mathcal{G}^n_t) \). Then \( G^n_t \in L^\infty(\mathcal{G}^n_t) \). Since hypothesis (H) holds between \( \mathbb{F} \) and \( \mathbb{G}^n \), Lemma 10 guarantees that \( E(FG^n_t | \mathcal{F}_t) = E(F | \mathcal{F}_t)E(G^n_t | \mathcal{F}_t) \). But \( \mathcal{F}_t \subset \mathcal{G}^n_t \), hence \( E(G^n_t | \mathcal{F}_t) = E(G_t | \mathcal{G}^n_t) | \mathcal{F}_t) = E(G_t | \mathcal{F}_t) \).

Since \( G^n_t \xrightarrow{w} G_t \), \( FG^n_t \xrightarrow{p} FG_t \). Now \( FG^n_t \) is bounded by a square integrable process (by assumption) so the convergence holds in \( L^1 \) by the Dominated Convergence theorem so that \( E(FG^n_t | \mathcal{F}_t) \xrightarrow{P} E(FG_t | \mathcal{F}_t) \). This proves that \( E(FG_t | \mathcal{F}_t) = E(F | \mathcal{F}_t)E(G_t | \mathcal{F}_t) \).

The general case where \( G_t \in L^2(\mathcal{G}_t) \) follows by applying the bounded case result to the bounded random variables \( G^{(m)}_t = G_t \cap m \). Then for each \( m \),

\[
E(FG^{(m)}_t | \mathcal{F}_t) \xrightarrow{P} E(FG^{(m)}_t | \mathcal{F}_t)
\]

and the Monotone Convergence theorem allows us to conclude. 

4 A Filtration Expansion Result Based on an Assumption Involving Honest Times

This theorem is rather simple, but we include it both for completeness, and also because we need it later for our treatment of Bessel processes (see Section 6.3.1).

Let \( (\varepsilon_n)_{n \geq 0} \) be a sequence of positive real numbers decreasing to zero. We assume that the continuous adapted process \( X \) is increasing to infinity and we define sequences of random times \( (\tau^n_p)_{p \geq 0} \) to be

\[
\tau^n_p = \inf\{t \geq 0, X_t \geq p\varepsilon_n\}
\]

Since \( X \) is increasing to infinity, for \( (\tau^n_p)_{p \geq 0} \) we have that for each \( n \geq 1 \), the sequence \( (\tau^n_p, p \geq 1) \) is strictly increasing to infinity. That is, \( \tau^n_p > \tau^n_{p-1} \) on the set where \( \tau^n_{p-1} < \infty \), and \( \lim_{p \to \infty} \tau^n_p = \infty \).

Define the sequence of processes

\[
X^n_t = \sum_{p=0}^{\infty} 1\{\tau^n_p \leq t < \tau^n_{p+1}\} X^n_{\tau^n_p} = \varepsilon_n \sum_{p=0}^{\infty} p 1\{\tau^n_p \leq t < \tau^n_{p+1}\}
\]

and \( \mathbb{G}^n \) (resp. \( \mathbb{G} \)) the progressive expansion of \( \mathbb{F} \) with \( X^n \) (resp. \( X \)). Let \( \mathbb{G}^{\tau^n} \) be the smallest filtration containing \( \mathbb{F} \) and that makes all \( (\tau^n_p)_{p \geq 1} \) stopping times. We have the containment relation \( \mathbb{G}^n \subset \mathbb{G}^{\tau^n} \). We make now the following assumption, noting that the validity of the assumption will depend on the process \( X \) chosen.

Assumption 1 (Honest times assumption) For each \( n \geq 1 \), the sequence \( (\tau^n_p)_{p \geq 0} \) is an increasing sequence of \( \mathbb{F} \) honest times such that \( \tau^n_0 = 0 \) and \( \sup_p \tau^n_p = \infty \).

Under Assumption 1 the following holds (see Jeulin [41, Corollary 5.22]).
and where \( Z^{n,p} \) is the \( \mathbb{F} \) optional projection of \( \tau_p^n \) and \( M^{n,p} \) is the martingale part in its Doob-Meyer decomposition.

Putting together Theorem \( \mathbb{S} \) above and Theorem \( \mathbb{S} \) we obtain the following result.

**Theorem 9** Suppose Assumption \( \mathbb{S} \) holds. Let \( M \) be an \( \mathbb{F} \) martingale such that \( \sup_{0 \leq s \leq T} |M_s| \) is integrable. If \( E(\int_0^T |dA_s^n|) \leq K \) for some \( K \) and all \( n \geq 1 \), then \( M \) is a \( \mathcal{G} \) semimartingale. Here \( A^n \) is defined in equation \( \mathbb{S} \).

## 5 Filtration expansion with a càdlàg process satisfying a generalized Jacod’s criterion and applications to diffusions

In this section, we assume a càdlàg process \( X \) and a right-continuous filtration \( \mathbb{F} \) are given. We assume throughout this section that our probability space is rich enough to contain non trivial continuous martingales. We study the case where the process \( X \) and the filtration \( \mathbb{F} \) satisfy the following assumption.

**Assumption 2 (Generalized Jacod’s criterion)** There exists a sequence \( (\pi_n)_{n \geq 1} = (\{t^n_i\})_{n \geq 1} \) of subdivisions of \([0,T]\) whose mesh tends to zero and such that for each \( n \), \((X_{t^n_0}, X_{t^n_1} - X_{t^n_0}, \ldots, X_{t^n_n} - X_{t^n_{n-1}}, X_T - X_{t^n_n})\) satisfies Jacod’s criterion, i.e. there exists a \( \sigma \)-finite measure \( \eta_n \) on \( \mathcal{B}(\mathbb{R}^{n+2}) \) such that \( P((X_{t^n_0}, X_{t^n_1} - X_{t^n_0}, \ldots, X_{t^n_n} - X_{t^n_{n-1}}, X_T - X_{t^n_n}) \in \cdot | \mathcal{F}_t)(\omega) \ll \eta_n(\cdot) \) a.s.

Under Assumption \( \mathbb{S} \) the \( \mathcal{F}_t \)-conditional density
\[
p_t^{(n)}(u_0, \ldots, u_{n+1}, \omega) = \frac{P((X_{t^n_0}, X_{t^n_1} - X_{t^n_0}, \ldots, X_{t^n_n} - X_{t^n_{n-1}}, X_T - X_{t^n_n}) \in (du_0, \ldots, du_{n+1}) | \mathcal{F}_t)(\omega)}{\eta_n(du_0, \ldots, du_{n+1})}
\]
exists for each \( n \), and can be chosen so that \( (u_0, \ldots, u_{n+1}, \omega, t) \rightarrow p_t^{(n)}(u_0, \ldots, u_{n+1}, \omega) \) is càdlàg in \( t \) and measurable for the optional \( \sigma \)-field associated with the filtration \( \tilde{\mathcal{F}}_t \) given by \( \mathcal{F}_t = \bigcap_{u \geq T} \mathcal{B}(\mathbb{R}^{n+2}) \otimes \mathcal{F}_u \). For each \( 0 \leq i \leq n \), define
\[
p_{t}^{i,n}(u_0, \ldots, u_i) = \int_{\mathbb{R}^{n+1-i}} p_{t}^{(n)}(u_0, \ldots, u_{n+1}) \eta_n(du_{i+1}, \ldots, du_{n+1})
\]
Let \( M \) be a continuous \( \mathbb{F} \) local martingale. Define
\[
A_{t}^{i,n} = \int_{0}^{t} \frac{d\langle p_{s}^{i,n}(u_0, \ldots, u_i), M \rangle_s}{\eta_n(du_0, \ldots, du_i)} |_{0 \leq k \leq i, u_k = X_{t^n_k} - X_{t^n_{k-1}}}
\]
Finally define
\[
A_{t}^{(n)} = \sum_{i=0}^{n} \int_{t^{i+1} \wedge t^n_{i+1}}^{t} dA_{s}^{i,n}
\]
i.e.
\[
A_{t}^{(n)} = \sum_{i=0}^{n} 1_{t^{i} \leq t < t^{i+1}} \left( \sum_{k=0}^{i-1} \int_{t^{k+1}}^{t^{k+1}} dA_{s}^{k,n} + \int_{t^{i}}^{t} dA_{s}^{i,n} \right)
\]
Of course, on each time interval \( \{t^n_k \leq t < t^n_{k+1}\} \), only one term appears in the outer sum. Let \( \mathcal{G}^0 \) (resp. \( \mathcal{G} \)) be the smallest (resp. the smallest right-continuous) filtration containing \( \mathbb{F} \) and relative to which \( X \) is adapted. The theorem below is the main result of this section.

**Theorem 10** Assume \( X \) and \( \mathbb{F} \) satisfy Assumption 2 and that one of the following holds.

- \( X \) has no fixed times of discontinuity,
- the sequence of subdivisions \( (\pi_n)_{n \geq 1} \) in Assumption 2 is refining and each fixed time of discontinuity of \( X \) belongs to \( \bigcup_n \pi_n \).

Let \( M \) be a continuous \( \mathbb{F} \) martingale such that \( E(\sup_{s \leq T} |M_s|) \leq K \) and \( E(\int_0^T |dA^n_s|) \leq K \) for some \( K \) and all \( n \), with \( A^n \) as in (7). Then

(i) \( M \) is a \( \mathcal{G}^0 \) special semimartingale.

(ii) Moreover, if \( \mathbb{F} \) is the natural filtration of some càdlàg process \( Z \), then \( M \) is a \( \mathcal{G} \) special semimartingale with canonical decomposition \( M = N + A \) such that \( N \) is a \( \mathcal{G} \) martingale and \( \sup_{0 \leq s \leq T} |N_s| \) and \( \int_0^T |dA_s| \) are integrable.

**Proof.** We construct the discretized process \( X^n \) defined by \( X^n_t = X^n_{t_k} \) for all \( t^n_k \leq t < t^n_{k+1} \).

That is

\[
X^n_t = \sum_{i=0}^n X^n_{t_k} \mathbf{1}_{[t^n_k \leq t < t^n_{k+1}]} + X_T \mathbf{1}_{\{t=T\}}
\]

with the convention \( t^n_0 = 0 \) and \( t^n_{n+1} = T \). Let \( \mathcal{G}^n \) be the smallest right-continuous filtration containing \( \mathbb{F} \) and to which \( X^n \) is adapted.

Now, for \( 0 \leq t \leq T \),

\[
X^n_t = \sum_{i=0}^n X^n_{t_k} \mathbf{1}_{[t^n_k \leq t < t^n_{k+1}]} + X_T \mathbf{1}_{\{t=T\}} = \sum_{i=0}^n X^n_{t_k} \mathbf{1}_{[t^n_k \leq t]} - \sum_{i=0}^n X^n_{t_k} \mathbf{1}_{[t^n_{k+1} \leq t]} + X_T \mathbf{1}_{\{t=T\}}
\]

\[
= \sum_{i=1}^n \left(X^n_{t_{i-1}} - X^n_{t_i}\right) \mathbf{1}_{[t^n_i \leq t]} + X_0 \mathbf{1}_{[0 \leq t]} - X^n_0 \mathbf{1}_{[0 \leq t]} + X_T \mathbf{1}_{[t_{n+1} \leq t]}
\]

\[
= X_0 \mathbf{1}_{[0 \leq t]} + \sum_{i=1}^{n+1} \left(X^n_{t_{i-1}} - X^n_{t_i}\right) \mathbf{1}_{[t^n_i \leq t]} = \sum_{i=0}^{n+1} \left(X^n_{t_{i-1}} - X^n_{t_i}\right) \mathbf{1}_{[t^n_i \leq t]}
\]

with the notation \( X^n_{t_{-1}} = 0 \).

For each \( 0 \leq i \leq n+1 \), let \( \mathbb{H}^{i,n} \) be the initial expansion of \( \mathbb{F} \) with \( (X^n_{t_k} - X^n_{t_{k-1}})_{0 \leq k \leq i} \). Since \( (X^n_k - X^n_{k-1})_{0 \leq k \leq i} \) satisfies Jacod’s criterion, it follows that for each \( 0 \leq i \leq n+1 \),

\( M - A^{i,n} \) is an \( \mathbb{H}^{i,n} \) local martingale. Let

\[
\tilde{\mathcal{G}}^n_t = \bigcap_{u \geq t} F_u \vee \sigma((X^n_{t_k} - X^n_{t_{k-1}}) \mathbf{1}_{[t^n_k \leq u]}, i = 0, \ldots, n+1)
\]

Since the times \( t^n_k \) are fixed, \( \mathbb{H}^{i,n} \) is also the initial expansion of \( \mathbb{F} \) with \( (t^n_k, X^n_k - X^n_{k-1})_{0 \leq k \leq i} \) and \( \tilde{\mathcal{G}}^n = \mathcal{G}^n \) using a Monotone Class argument and the fact that \( X^n_{t_k} = X^n_{t_k} \), for all
0 \leq k \leq n + 1$. So it follows from Theorem 8 in [44] that $M - A^{(n)}$ is a $\mathcal{G}^n$ local martingale. An application of Theorem 6 yields the result.

We refrain from stating Theorem 10 in a more general form for clarity but provide two extensions in the remarks below.

(i) Going beyond the continuous case for the $\mathcal{F}$ local martingale $M$ is straightforward. We only need to use Theorem 8 in [44] in its general version rather than its application to the continuous case. However the explicit form of $A^{(n)}$ is much more complicated, which makes it hard to check the integrability assumption of Theorem 10. To be more concrete, one has to replace $A^{(n)}$ in the theorem above by $\tilde{A}^{(n)}$ defined by

$$
\tilde{A}^{(n)}_t = \sum_{i=0}^n \left( \sum_{k=0}^{i-1} \int_{t_n^k}^{t_n^{k+1}} (d\tilde{A}^{i,n}_s + dJ^{i,n}_s) + \int_{t_n^i}^t (d\tilde{A}^{i,n}_s + dJ^{i,n}_s) \right)
$$

where $\tilde{A}^{i,n}$ is the compensator of $M$ in $\mathbb{H}^{i,n}$ as given by Jacod’s theorem (see Theorems VI.10 and VI.11 in [54]) and $J^{i,n}$ is the dual predictable projection of $\Delta M_{[t_n^i, t_n^{i+1}]}$ onto $\mathbb{H}^{i,n}$.

(ii) A careful study of the proof above shows that Assumption 2 is only used to ensure that there exists an $\mathbb{H}^{i,n}$ predictable process $A^{i,n}$ such that $M - A^{i,n}$ is an $\mathbb{H}^{i,n}$ local martingale. Therefore, Theorem 10 will hold whenever this weaker assumption is satisfied.

If the sequence of filtrations $\mathcal{G}^n$ converges weakly to $\mathcal{G}$ then $(M - A^{(n)}, A^{(n)})$ converges in probability under the Skorohod $J_1$ topology to $(N, A)$. Many criteria for this to hold are provided in the literature, see for instance Propositions 3 and 4 in [20]. This holds for example when every $\mathcal{G}$ martingale is continuous and the subdivision $(t_n^n)_{n \geq 1}$ is refining. In this case, for each $0 \leq t \leq T$, $(\mathcal{G}_t^n)_{n \geq 1}$ is increasing and converges weakly to the $\sigma$-field $\mathcal{G}_t$. The following lemma allows us to conclude. See [20] for a proof.

**Lemma 11** Assume that every $\mathcal{G}$ martingale is continuous and that for every $0 \leq t \leq T$, $(\mathcal{G}_t^n)_{n \geq 1}$ increases (or decreases) and converges weakly to $\mathcal{G}_t$. Then $\mathcal{G}^n \xrightarrow{w} \mathcal{G}$.

### 5.1 Application to diffusions

Start with a Brownian filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, $\mathcal{F}_t = \sigma(B_s, s \leq t)$ and consider the stochastic differential equation

$$
dX_t = \sigma(X_t)dB_t + b(X_t)dt
$$

Assume the existence of a unique strong solution $(X_t)_{0 \leq t \leq T}$. Assume in addition that the transition density $\pi(t, x, y)$ exists and is twice continuously differentiable in $x$ and
continuous in \( t \) and \( y \). This is guaranteed for example if \( b \) and \( \sigma \) are infinitely differentiable with bounded derivatives and if the Hörmander condition holds for any \( x \) (see [10]), and we assume that this holds in the sequel. In this case, \( \pi \) is even infinitely differentiable.

We next show how we can expand a filtration dynamically as \( t \) increases, via another stochastic process evolving backwards in time. To this end, define the time reversed process \( Z_t = X_{T-t} \), for all \( 0 \leq t \leq T \). Let \( \mathcal{G} = (\mathcal{G}_t)_{0 \leq t < \frac{T}{2}} \) be the smallest right-continuous filtration containing \( (\mathcal{F}_t)_{0 \leq t \leq \frac{T}{2}} \) and to which \( (Z_t)_{0 \leq t \leq \frac{T}{2}} \) is adapted. We would like to prove that \( B \) remains a special semimartingale in \( \mathcal{G} \) and give its canonical decomposition. That \( B \) is a \( \mathcal{G} \) semimartingale can be obtained using the usual results from the filtration expansion theory. However, our approach allows us to obtain the decomposition, too. We assume (w.l.o.g) that \( T = 1 \). Introduce the reversed Brownian motion \( \tilde{B}_t = B_{1-t} - B_1 \) and the filtration \( \tilde{\mathcal{G}} = (\tilde{\mathcal{G}}_t)_{0 \leq t < \frac{1}{2}} \) defined by

\[
\tilde{\mathcal{G}}_t = \bigcap_{t < u < \frac{1}{2}} \sigma(B_s, \tilde{B}_s, 0 \leq s < u).
\]

**Theorem 11** Both \( B \) and \( \tilde{B} \) are \( \mathcal{G} \) semimartingales.

**Proof.** First, it is well known that \( \tilde{B} \) is a Brownian motion in its own natural filtration and \( \sigma(B_{1-s} - B_1, 0 \leq s < \frac{1}{2}) \) is independent from \( \sigma(B_s, 0 \leq s < \frac{1}{2}) \). Therefore \( (B_t)_{0 \leq t < \frac{1}{2}} \) and \( (\tilde{B}_t)_{0 \leq t < \frac{1}{2}} \) are independent Brownian motions in \( \tilde{\mathcal{G}} \). Now, given our strong assumptions on the coefficients \( b \) and \( \sigma \), \( X_1 \) satisfies Jacod’s criterion with respect to \( \tilde{\mathcal{G}} \). Therefore \( B \) and \( \tilde{B} \) remain semimartingales in \( \tilde{\mathcal{H}} = (\tilde{\mathcal{H}}_t)_{0 \leq t < \frac{1}{2}} \) where \( \tilde{\mathcal{H}}_t = \bigcap_{\frac{1}{2} > u > t} \tilde{\mathcal{G}}_u \vee \sigma(X_1) \). It only remains to prove that \( \mathcal{G} = \tilde{\mathcal{H}} \). For this, use Theorem V.23 in [54] to get that

\[
dX_{1-t} = \sigma(X_{1-t})d\tilde{B}_t + (\sigma'(X_{1-t})\sigma(X_{1-t}) + b(X_{1-t}))dt.
\]

Since \( b + \sigma \sigma' \) and \( \sigma \) are Lipschitz, \( \bigcap_{\frac{1}{2} > u > t} \sigma(X_{1-s}, 0 \leq s < u) = \bigcap_{\frac{1}{2} > u > t} \sigma(\tilde{B}_s, 0 \leq s < u) \vee \sigma(X_1) \) and the result follows. \( \blacksquare \)

We apply now our results to obtain the \( \mathcal{G} \) decomposition. This is the primary result of this article.

**Theorem 12** Assume there exists a nonnegative function \( \phi \) such that \( \int_0^1 \phi(s)ds < \infty \) and for each \( 0 \leq s < t \),

\[
E\left( \frac{1}{\pi} \frac{\partial \pi}{\partial x}(t - s, X_s, X_t) \right) \leq \phi(t - s)
\]

Then the process \( (B_t)_{0 \leq t < \frac{1}{2}} \) is a \( \mathcal{G} \) semimartingale and

\[
B_t - \int_0^t \frac{1}{\pi} \frac{\partial \pi}{\partial x}(1 - 2s, X_s, X_{1-s})ds
\]

is a \( \mathcal{G} \) Brownian motion.

**Proof.** Since the process \( Z_t \) is a càdlàg process with no fixed times of discontinuity, we can apply Theorem 11. First we prove that \( (Z_t)_{0 \leq t < \frac{1}{2}} \) and \( (\mathcal{F}_t)_{0 \leq t < \frac{1}{2}} \) satisfy Assumption 2. Let \( (\pi_n)_{n \geq 1} = (\{t^n_i\})_{n \geq 1} \) be a refining sequence of subdivisions of \( [0, \frac{1}{2}] \) whose
by assumption, it is straightforward to check that

\[ P(Z_{t_0}^n \leq z_0, Z_{t_1}^n - Z_{t_0}^n \leq z_1, \ldots, Z_{t_i}^n - Z_{t_{i-1}}^n \leq z_i \mid \mathcal{F}_t) \]

\[ = P(X_1 \leq z_0, X_1 - X_{1-t_1^n} > -z_1, \ldots, X_{1-t_{i-1}^n} - X_{1-t_i^n} > -z_i \mid \mathcal{F}_t) \]

\[ = E \left( \prod_{k=1}^{i-1} 1_{\{X_{1-t_{k}^n} - X_{1-t_{k+1}^n} \leq -z_{k+1}\}} P(X_{1-t_{k}^n} - z_1 \leq X_1 \leq z_0 \mid \mathcal{F}_{1-t_{k}^n}) \mid \mathcal{F}_t \right) \]

\[ = E \left( \prod_{k=1}^{i-1} 1_{\{X_{1-t_{k}^n} - X_{1-t_{k+1}^n} \leq -z_{k+1}\}} \int_{X_{1-t_{k}^n} - z_1}^\infty 1_{\{u_1 \leq z_0\}} P_{X_{1-t_{k}^n}}(t_{1}^n, u_1) du_1 \mid \mathcal{F}_t \right) \]

\[ = E \left( \prod_{k=1}^{i-1} 1_{\{X_{1-t_{k}^n} - X_{1-t_{k+1}^n} \leq -z_{k+1}\}} \int_{-z_1}^\infty 1_{\{v_1 \leq z_0 - X_{1-t_{k}^n}\}} P_{X_{1-t_{k}^n}}(t_{1}^n, v_1 + X_{1-t_{k}^n}) dv_1 \mid \mathcal{F}_t \right) \]

Repeating the same technique and conditioning successively w.r.t \( \mathcal{F}_{1-t_{2}^n}, \ldots, \mathcal{F}_{1-t_{i}^n} \) gives

\[ P(Z_{t_0}^n \leq z_0, Z_{t_1}^n - Z_{t_0}^n \leq z_1, \ldots, Z_{t_i}^n - Z_{t_{i-1}^n} \leq z_i \mid \mathcal{F}_t) = E \left( \int_{-z_i}^\infty \cdots \int_{-z_i}^\infty \prod_{k=1}^{i} P_{X_{1-t_{k}^n}} + \sum_{j=k+1}^{i} p_{j}^{n,j} v_j (t_{k}^n - t_{k-1}^n, \sum_{l=k}^{i} v_l + X_{1-t_{k}^n}) dv_1 \cdots dv_i \mid \mathcal{F}_t \right) \]

\[ = \int_{-\infty}^\infty \cdots \int_{-z_i}^\infty \prod_{k=1}^{i} P_{u + \sum_{j=k+1}^{i} v_j} v_j (t_{k}^n - t_{k-1}^n, u + \sum_{l=k}^{i} v_l) dv_1 \cdots dv_i du \]

Fubini’s Theorem implies then

\[ P(Z_{t_0}^n \leq z_0, Z_{t_1}^n - Z_{t_0}^n \leq z_1, \ldots, Z_{t_i}^n - Z_{t_{i-1}^n} \leq z_i \mid \mathcal{F}_t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{i} P_{X_{1-t_{k}^n} - t_{k-1}^n} (u + \sum_{l=k}^{i} v_l) dudv_1 \cdots dv_i \]

Since the transition density \( \pi(t, x, y) = P_x(t, y) \) is twice continuously differentiable in \( x \) by assumption, it is straightforward to check that

\[ p_{t}^{i,n}(z_0, \ldots, z_i) = \prod_{k=1}^{i} \pi(t_{k}^n - t_{k-1}^n, \sum_{j=0}^{k} z_j, \sum_{j=0}^{k-1} z_j) \pi(1 - t_{i}^n - t_{i-1}^n, X_{t_i} - \sum_{j=0}^{i} z_j) \]

One then readily obtains

\[ d\langle p_{s}^{i,n}(z_0, \ldots, z_i), B_s \rangle_s = \frac{1}{\pi} \frac{\partial \pi}{\partial x}(1 - t_{i}^n - s, X_{s}, \sum_{j=0}^{i} z_k) p_{s}^{i,n}(z_0, \ldots, z_i) ds \]
Hence by taking the local martingale $M$ in (6) to be $B$, we get

$$A^{i,n}_t = \int_0^t \frac{1}{\pi} \frac{1}{\partial x} (1 - t^n_i - s, X_s, X_{1-t^n_i}) ds$$

Now equation (7) becomes

$$A^{(n)}_t = \sum_{i=0}^n 1\{t^n_i \leq t < t^n_{i+1}\} \left( \sum_{k=0}^{i-1} \int_{t^n_k}^{t^n_{k+1}} \frac{1}{\pi} \frac{1}{\partial x} (1 - t^n_k - s, X_s, X_{1-t^n_k}) ds \right) + \int_{t^n_i}^{t^n_{i+1}} \frac{1}{\pi} \frac{1}{\partial x} (1 - t^n_i - s, X_s, X_{1-t^n_i}) ds$$

In order to apply Theorem 10, it only remains to prove that $E\left( \int_0^1 |dA^{(n)}_s| \right) \leq K$ for some constant $K$ independent from $n$. The finite constant $K = \int_0^1 \phi(s) ds$ works since

$$E\left( \int_0^1 |dA^{(n)}_s| \right) \leq \sum_{k=0}^n \int_{t^n_k}^{t^n_{k+1}} E\left\{ \frac{1}{\pi} \frac{1}{\partial x} (1 - t^n_k - s, X_s, X_{1-t^n_k}) ds \right\} ds \leq \sum_{k=0}^n \int_{t^n_k}^{t^n_{k+1}} \phi(1 - t^n_k - s) ds = \sum_{k=0}^n \int_{1-t^n_k-t^n_{k+1}}^{1-2t^n_k} \phi(s) ds \leq \sum_{k=0}^n \int_{1-2t^n_k}^{1-2t^n_{k+1}} \phi(s) ds = \int_0^1 \phi(s) ds$$

This proves again that $B$ is a $\mathcal{G}$ semimartingale. Now $A^{(n)}$ converges in probability to the process $A$ given by

$$A_t = \int_0^1 \frac{1}{\pi} \frac{1}{\partial x} (1 - 2s, X_s, X_{1-s}) ds$$

Since all $\mathcal{G}$ martingales are continuous, the comment following Theorem 10 ensures that $B-A$ is a $\mathcal{G}$ martingale. Its quadratic variation is $t$, therefore it is a $\mathcal{G}$ Brownian motion.

In the Brownian case, the result in Theorem 10 can also be obtained using the usual theory of initial expansion of filtration. Assume $b = 0$ and $\sigma = 1$, i.e. $Z = B_{1-o}$ and $X = B$.

**Theorem 13** The process $B$ is a $\mathcal{G}$ semimartingale and

$$B_t - \int_0^t \frac{B_{1-s} - B_s}{1 - 2s} ds, \quad 0 \leq t < \frac{1}{2}$$

is a $\mathcal{G}$ Brownian motion.

**Proof.** Introduce the filtration $\mathbb{H}^1 = (\mathcal{H}_t)_{0 \leq t < \frac{1}{2}}$ obtained by initially expanding $\mathbb{F}$ with $B_{1-o}$.

$$\mathcal{H}_t^1 = \bigcap_{u > t} F_u \vee \sigma(B_{1-o})$$

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We know that $B$ remains an $\mathbb{H}^1$ semimartingale and

$$M_t := B_t - \int_0^t \frac{B_s - B_t}{\frac{1}{2} - s} ds, \quad 0 \leq t < \frac{1}{2}$$

is an $\mathbb{H}^1$ Brownian motion. Now expand initially $\mathbb{H}^1$ with the independent $\sigma$-field $\sigma(B_v - B_{\frac{1}{2}}, \frac{1}{2} < v \leq 1)$ to obtain $\mathbb{H}$ i.e.

$$\mathcal{H}_t = \bigcap_{u > t} H^1_u \vee \sigma(B_v - B_{\frac{1}{2}}, \frac{1}{2} < v \leq 1)$$

Obviously $(M_t)_{0 \leq t < \frac{1}{2}}$ remains an $\mathbb{H}$ Brownian motion. But $G_t \subset \mathcal{H}_t$, for all $0 \leq t < \frac{1}{2}$, hence the optional projection of $M$ onto $G$, denoted $^oM$ in the sequel, is again a martingale (see [25]), i.e.

$$^oM_t = B_t - E(\int_0^t \frac{B_s - B_t}{\frac{1}{2} - s} ds \mid \mathcal{G}_t), \quad 0 \leq t < \frac{1}{2}$$

is a $G$ martingale. Also, $N_t := E(\int_0^t \frac{B_s - B_t}{\frac{1}{2} - s} ds \mid \mathcal{G}_t) - \int_0^t E\left(\frac{B_s - B_t}{\frac{1}{2} - s} \mid \mathcal{G}_s\right) ds$ is a $G$ local martingale, see for example [44] for a proof. So

$$B_t = ^oM_t + N_t + \int_0^t E\left(\frac{B_s - B_t}{\frac{1}{2} - s} \mid \mathcal{G}_s\right) ds$$

We prove now the theorem using properties of the Brownian bridge. Recall that for any $0 \leq T_0 < T_1 < \infty$,

$$\mathcal{L}\left((B_t)_{T_0 \leq t \leq T_1} \mid B_s, s \notin [T_0, T_1]\right) = \mathcal{L}\left((B_t)_{T_0 \leq t \leq T_1} \mid B_{T_0}, B_{T_1}\right)$$

and

$$\mathcal{L}\left((B_t)_{T_0 \leq t \leq T_1} \mid B_{T_0} = x, B_{T_1} = y\right) = \mathcal{L}\left(x + \frac{t - T_0}{T_1 - T_0} (y - x) + (Y_{W,T_1-T_0} - T_0) \mid y\right)$$

where $W$ is a generic standard Brownian motion and $Y_{W,T_1-T_0}$ is the standard Brownian bridge on $[0, T_1 - T_0]$. It follows that for all $T_0 \leq t \leq T_1$ and all $x$ and $y$,

$$E(B_t \mid B_{T_0} = x, B_{T_1} = y) = \frac{T_1 - t}{T_1 - T_0} x + \frac{t - T_0}{T_1 - T_0} y$$

(9)

For any $0 \leq s < t < \frac{1}{2}$, if follows from (8) and (9) that

$$E(B_{\frac{1}{2}} \mid B_s) = \frac{1}{2} (B_{1-s} - B_s)$$

Therefore

$$B_t - \int_0^t \frac{B_{1-s} - B_s}{1 - 2s} ds = ^oM_t + N_t$$
is a $G$ local martingale. Since the quadratic variation of the $G$ local martingale $B - A$ is $t$, Levy’s characterization of Brownian motion ends the proof. ■

In the immediately previous proof, the properties of the Brownian bridge allow us to compute explicitly the decomposition of $B$ in $G$. Our method obtains both the semimartingale property and the decomposition simultaneously and generalizes to diffusions, for which the computations as in the proof of Theorem 13 are hard. We provide a shorter proof for Theorem 13 based on Theorem 12. This illustrated that, given Theorem 12, even in the Brownian case our method is shorter, simpler, and more intuitive.

Proof. [Second proof of Theorem 13] In the Brownian case, $\pi(t, x, y) = \frac{1}{\sqrt{2\pi t}} \frac{(y-x)^2}{2t}$. Therefore $\frac{1}{\pi} \frac{\partial \pi}{\partial x}(t, x, y) = \frac{y-x}{t}$. Hence

$$E\left(\frac{1}{\pi} \frac{\partial \pi}{\partial x}(t-s, B_s, B_t)\right) \leq \frac{1}{t-s} E(|B_t - B_s|) = \sqrt{\frac{2}{\pi}} \sqrt{1 - s/t}$$

and $\phi(x) = \frac{\sqrt{2}}{\sqrt{\pi}} x$ is integrable in zero. From the closed formula for the transition density, $A_t = \int_0^t \frac{B_t - B_{t-s}}{2s} ds$. Therefore $B$ is a $G$ semimartingale, and $B - A$ is a $G$ Brownian motion by Theorem 12. ■

This property satisfied by Brownian motion is inherited by diffusions whose parameters $b$ and $\sigma$ satisfy some boundedness assumptions. We add the extra assumptions that $b$ and $\sigma$ are bounded and $k \leq \sigma(x)$ for some $k > 0$. The following holds.

Corollary 3 The process $(B_t)_{0 \leq t < \frac{1}{2}}$ is a $G$ semimartingale and

$$B_t - \int_0^t \frac{1}{\pi} \frac{\partial \pi}{\partial x}(1 - 2s, X_s, X_{t-s}) ds$$

is a $G$ Brownian motion.

Proof. Introduce the following quantities

$$s(x) = \int_0^x \frac{1}{\sigma(y)} dy \quad g = s^{-1} \quad \mu = \frac{b}{\sigma} \circ g - \frac{1}{2} \sigma' \circ g$$

The process $Y_t = s(X_t)$ satisfies the SDE $dY_t = \mu(Y_t) dt + dB_t$. The transition density is known in semi-closed form (see [24]) and given by

$$\pi(t, x, y) = \frac{1}{\sqrt{2\pi t \sigma(y)}} e^{-\frac{(s(y) - s(x))^2}{2t}} U_t(s(x), s(y))$$

where $U_t(x, y) = H_t(x, y) e^{A(x) - A(y)}$, $H_t(x, y) = E(e^{-t \int_0^t h(x+z+y-x) + \sqrt{\sigma(z)} dz})$, $W$ is a Brownian bridge, $A$ a primitive of $\mu$ and $h = \frac{1}{2}(\mu^2 + (\mu')^2)$. It is then straightforward to compute the ratio

$$\frac{1}{\pi} \frac{\partial \pi}{\partial x}(t, x, y) = \frac{1}{\sigma(x)} \left(\frac{s(y) - s(x)}{t} + \frac{1}{U_t(s(x), s(y))} \frac{\partial U_t}{\partial x}(s(x), s(y))\right)$$

$$= \frac{1}{\sigma(x)} \left(\frac{s(y) - s(x)}{t} + \frac{1}{H_t(s(x), s(y))} \frac{\partial H_t}{\partial x}(s(x), s(y)) - \mu(s(x))\right)$$

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From the boundedness assumptions of \( b \) and \( \sigma \) and their derivatives, there exists a constant \( M \) such that 
\[
|\frac{\partial \pi}{\partial x}(t, x, y)| \leq M \left(1 + \frac{|s(y) - s(x)|}{t}ight).
\]
Hence, for \( 0 \leq s < t \)
\[
E \left| \frac{1}{\pi} \frac{\partial \pi}{\partial x}(t - s, X_s, X_t) \right| \leq M \left(1 + E \left| \frac{s(X_t) - s(X_s)}{t - s} \right| \right)
\]
But \( s(X_t) - s(X_s) = Y_t - Y_s = \int_s^t \mu(Y_u) \, du + W_t - W_s \). But \( \mu \) is bounded, hence
\[
E|s(Y_t) - s(Y_s)| \leq ||\mu||_{\infty}|t - s| + E|W_t - W_s| = ||\mu||_{\infty}|t - s| + \sqrt{\frac{2}{\pi} t - s}
\]
This proves the existence of a constant \( C \) such that
\[
E \left| \frac{1}{\pi} \frac{\partial \pi}{\partial x}(t - s, X_s, X_t) \right| \leq C \left(1 + \frac{1}{\sqrt{t - s}} \right)
\]
Since \( \phi(x) = C \left(1 + \frac{1}{\sqrt{x}} \right) \) is integrable in zero, we can apply Theorem 12 and conclude.

6 Applications to insider trading models

We have recently seen evidence of a rather spectacular use of inside information during the trial and conviction of Raj Rajaratnam of the Galleon Group, and the subsequent conviction of his co-conspirator Rajat Gupta, abusing his board membership at Goldman Sachs. Other much reported scandals include those of Martha Stewart (of Martha Stewart Living Omnimedia), and Mark Cuban, the billionaire owner of the Dallas Mavericks (a professional basketball team). There are many stories in the media; one example is given in [29]. The Barclay’s (and other mega banks') manipulation of LIBOR is unfolding as this paper is being written. Therefore it is reasonable to try to understand this recurrent and insidious phenomenon via mathematical modeling.

Insider trading models using stochastic calculus go back to the seminal work of A. Kyle in 1985 [17], with a rigorous treatment developed soon after by K. Back [4, 5]. The ideas are straightforward: If we have a filtration \((\mathcal{F}_t)_{t \geq 0} = \mathbb{F}\) that represents the collective information available to the market, there might arise entities that have extra, “inside” information that is not publicly available. Since the insiders see more observable events than does the market in general, we can model this by using a larger filtration \((\mathcal{G}_t)_{t \geq 0} = \mathbb{G}\) that contains \(\mathbb{F}\). We obtain \(\mathbb{G}\) by carefully expanding \(\mathbb{F}\) in such a way that all \((\Omega, \mathbb{F}, \mathbb{P})\) semimartingales remain semimartingales in the filtered measure space \((\Omega, \mathbb{G}, \mathbb{P})\); often we can also obtain the new semimartingale decomposition in the expanded filtration as well.

To illustrate the basic ideas, let’s assume we are dealing with continuous semimartingales in a Brownian paradigm, with complete markets. We also assume that the spot interest rate is 0. On our underlying space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) we have a nonnegative (continuous) price process \(S\), which has a decomposition \(S = S_0 + M + A\) where \(M\) is a continuous local
martingale, and $A$ is a continuous process with paths of finite variation on compact time sets. Moreover we assume $M_0 = A_0 = 0$ so such a decomposition is unique. Our time interval of interest will be $[0, T]$, so we are working in a finite horizon model. If $S$ satisfies the NFLVR condition, as we will assume it does to avoid uninteresting cases, then we can find a probability measure $Q$ equivalent to $P$ (written $Q \sim P$) such that $S$ itself is a local martingale under $Q$. The measure $Q$ is called the risk neutral measure and in this complete market it determines the fair (no arbitrage) prices of financial derivatives, such as call and put options, via expectation under $Q$. This is all well known, see for example either of [22], [35].

6.1 Constructing the Risk Neutral Measure of the Insider

Recall we have under $P$ that $S$ decomposes as

$$S_t = S_0 + M_t + A_t \quad t \geq 0.$$  (10)

Since the market is assumed to complete, the risk neutral measure $Q$ is unique. Because $M$ is within the Brownian paradigm, we know by martingale representation that $M_t = \int_0^t J_s dB_s$ for some predictable integrand $J$. Also, $S$ satisfies NFLVR, so in this case we must have that $A$ is of the form $A_t = \int_0^t H_s ds$ for some predictable process $H$. We now set

$$dQ/dP = \exp \left( \int_0^T -H_s dB_s - \frac{1}{2} \left( \int_0^T H_s^2 ds \right) \right).$$  (11)

Inspired by standard theory, we let $Z$ be the unique solution of the stochastic exponential equation

$$Z_t = 1 - \int_0^t Z_s \frac{H_s}{J_s} dB_s$$  (12)

and then by Girsanov’s theorem (see, e.g., [54]) we have that

$$N_t = \int_0^t J_s dB_s - \int_0^t \frac{1}{Z_s} d[Z, J \cdot B]_s \text{ is a } Q \text{ local martingale.}$$  (13)

We make the calculation

$$[Z, J \cdot B]_t = [-Z \frac{H}{J} \cdot B, J \cdot B]_t$$
$$= -\int_0^t Z_s \left( \frac{H_s}{J_s} J_s dB_s \right)$$
$$= -\int_0^t Z_s H_s ds$$

Combining this calculation with (13) gives that

$$N_t = \int_0^t J_s dB_s - \int_0^t \frac{1}{Z_s} Z_s H_s ds$$
$$= \int_0^t J_s dB_s + \int_0^t H_s ds = \text{ a } Q \text{ local martingale.}$$
Note that we now have $N = S$ which implies that $S$ is a $Q$ local martingale. Since there is only one such measure that turns $S$ into a local martingale (the market is complete), we see that our seemingly *ad hoc* definition of $d\frac{dQ}{dP}$ that gives us $Q$ is exactly the right one, and the only one, that works to give us the risk neutral measure. As a caveat, we quickly add that in the above analysis, we have implicitly assumed that all stochastic integrals exist, and in particular that dividing by the process $J$ does not cause any problems.

The point of the above calculations is that the Radon-Nikodym density $d\frac{dQ}{dP}$ given in (11) depends on the processes $H$ and $J$. These processes can and usually do change under an expansion of filtrations, and therefore they affect $d\frac{dQ}{dP}$, changing the risk neutral measure. We denote $J^*$ and $H^*$ to be the $\mathcal{G}$ processes in the decomposition of $S$. The risk neutral measure changes from $Q$ to a new measure $Q^*$ for the insider, and it is different than it is for the market. Since derivative prices are expectations under $Q$ for the collective market using $\mathcal{F}$, and they are expectations under $Q^*$ for the insider using $\mathcal{G}$, the result is that the insider has different derivative prices using the filtration $\mathcal{G}$, giving him a potentially tremendous advantage with which he can derive more profits, by knowing when a trade that appears neutral and fair to the market under $Q$ is actually a bargain (or is overpriced) if the price is computed for the insider under $Q^*$.

### 6.2 The Four Questions

**Remark 14** It is also possible that under $\mathcal{G}$ the process $H^*$ does not have almost surely square integrable paths with respect to $dt$, and therefore the second integral on the right side of (11) need not exist a.s.. In this case there is no risk neutral measure $Q^*$, or at least no such measure that is equivalent to $P$, and NFLVR is violated under $\mathcal{G}$. That is, insider models can introduce arbitrage opportunities, even when there is no arbitrage in the original $(\Omega, \mathcal{F}, P)$ model. This idea has been developed for example in the articles [33],[32]. In particular Peter Imkeller, in [32, Theorem 4.1] shows that insider knowledge of the last time $L$ a price process following a recurrent diffusion crosses 0 before a fixed time $T$ is an arbitrage opportunity, and he makes explicit calculations to show that one does indeed lose the square integrability in the drift term under the filtration expansion, so there cannot be a risk neutral measure that is equivalent, and therefore NFLVR is violated. This is a reassuring result, if not a surprising one, because an obvious arbitrage strategy for the insider is to buy the stock immediately after time $L$ if it is above zero, and to sell it short immediately after $L$ if it is below zero. He assumes (for the sake of the calculations) that the recurrent diffusion satisfies a standard stochastic differential equation, with some restrictions on the generality of the coefficients. The choice of the level 0 is of course arbitrary, and we could be dealing with a more general asset than a stock price, so there is no need to insist that it remain nonnegative. A similar event, without obvious arbitrage opportunities, is the second to last time a price process crosses a level, or better, the second to last time it reaches the boundary of a band with upper and lower crossing bounds; however these examples are less amenable to explicit calculations in the spirit of Imkeller.

On the basis of the previous discussion, we are interested in four questions:
The Four Questions

1. Does the risk neutral measure change under an expansion of filtrations?
2. If the risk neutral measure does indeed change, exactly how does it change?
3. When does the risk neutral measure not exist under a filtration expansion, thereby introducing arbitrage opportunities?
4. If the risk neutral measure does not exist as in (3), how might we exploit these arbitrage opportunities?

Remark 15 (Question 1) It seems intuitive that the risk neutral measure should change with an expansion of filtrations. However if the inside information is harmless and irrelevant, then perhaps it should not change at all. This would amount to a martingale in $\mathbb{F}$ remaining a martingale in the expanded filtration $\mathbb{G}$. This is known by the perhaps unfortunate term of “Hypothesis (H).” We already encountered Hypothesis (H) in Subsection 3.3. A trivial example of this phenomenon is expanding $\mathbb{F}$ by adding an independent $\sigma$ field at time 0, and therefore also at every time $t \geq 0$. Nevertheless there are non-trivial and interesting situations where Hypothesis (H) applies; indeed such a situation was treated in [26].

Remark 16 (Question 2) This question is equivalent to determining how the semi-martingale decomposition changes under an expansion of filtrations. This is a very hard question in general. The methods presented in the previous results of this paper do not suggest how to proceed. However in our examples we have seen that in some cases we can make explicit calculations that give the new decompositions. Thus while a general method would be preferable, for now we must content ourselves with a class of examples as the best we can currently do.

Remark 17 (Question 3) This question we can address, and it is perhaps the most interesting in a particular sense: It gives a way for regulators to determine the seriousness of an insider trading crime. While the authors consider any insider trading to be a serious (criminal) problem, the one time example of Martha Stewart might seem less serious than the ongoing procedures of the Galleon Group hedge fund, for example. Martha Stewart was using a changed risk neutral measure due to her inside information, and arguably so too was the Galleon Group. We say “arguably,” because some types of inside information occasionally give the trader a sure profit, as opposed to a highly likely profit. The former would be arbitrage.

One could then ask: If (for example) the Galleon Group hedge fund had (scalable) arbitrage opportunities, why did they not make much higher profits than they actually did? The answer is that had they tried to do so, one or two consequences might have occurred:

(a) Their activity might have revealed to the general market their inside information as is addressed in the “large trader” literature (see for example [21]), and therefore eliminate their insider advantage before they could fully profit from it. Large traders tend to try to hide their trading activities to avoid attacks from front running traders.

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With apologies for this nomenclature, for those who understand; the second author could not resist.
The avoidance of front running dovetails with the activity of ultra high frequency traders, as explained for example in [36].

(b) Even if result (a) does not occur, after the fact regulators might be able to determine, based on their trading behavior, that they were exploiting inside information, and therefore are guilty of a crime.

**Remark 18 (Question 4)** This question was addressed in Remark 14, via the simple example of the last time a price process of a risky asset crosses a certain level before a given fixed time. In the more interesting and complicated example of fractional Brownian motion, it was indirectly addressed, and in an implicit manner, in [17] and for more general processes in [32], and further developed in [12]. Basically, one needs a hedging strategy involving continuous trading in order to exploit the arbitrage opportunities present when a risky asset price follows a fractional Brownian motion, or more generally a class of processes with pathologies similar in nature to fractional Brownian motion. (We are ignoring the alternative approach of Øksendal and his co-authors [31]. See [15] for a discussion of this approach.) We do not address this issue here for the type of expansions given in this paper, even though it is an intrinsically interesting question.

### 6.3 Question 3 and Progressive Expansion

What is interesting about our method as regards models of insider trading is that it allows the expansion of filtration via an ongoing progressive expansion, as more and more information comes available. As we write this paper, the world of banking provides yet another excellent example, mainly the ongoing LIBOR scandal. This works well for our concept of process expansion, because as the fudging of interest rates reported to LIBOR by the banks began and became ongoing, the knowledge of what was going on gradually diffused into the market as time evolved, and what was going on was changing with time. LIBOR is important because other financial products (such as some financial derivatives, adjustable rate mortgages, many kinds of loans) are based on LIBOR, either directly or indirectly.

Our first interesting example is related to the famous Bessel 3 process.

#### 6.3.1 The Case of the Bessel 3 Process

Let $Z$ be a Bessel 3 process, $\mathcal{F}$ its natural filtration, $X_t = \inf_{s > t} Z_s$ and $\mathcal{G}$ the progressive expansion of $\mathcal{F}$ with $X$. This example has been studied in detail by both Jeulin and Pitman using different techniques. Let $B_t = Z_t - \int_0^t \frac{ds}{Z_s}$. It is a classical result that $B$...
is an $F$ Brownian motion. Using Williams' path decomposition for Brownian motion, Pitman \[53\] proves that $B_t - (2X_t - \int_0^t \frac{dZ_t}{2})$ is a $G$ Brownian motion. Using filtration expansion results, Jeulin proves in \[41\] the $G$ semimartingale property of $B$ and provides its decomposition simultaneously. However, his technique is hard to generalise. Our approach allows us to prove the semimartingale property of $B$, albeit without finding the explicit decomposition. Nevertheless it has merit because it can be used in much more general settings as described in Theorem \[9\].

**Theorem 19** The process $B$ is a $G$ special semimartingale.

**Proof.** First $X$ is continuous increasing to infinity since $\lim_{t \to \infty} Z_t = \infty$ (see Lemma 6.20 in Jeulin). Let $\epsilon_n$ be a sequence of positive real numbers decreasing to zero. Therefore $\tau^n_p = \inf\{t, X_t \geq p\epsilon_n\}$ is increasing to infinity, for each $n$. Now, with $Y_t = 2X_t - Z_t$, we have $X_t = \sup_{s \leq t} Y_s$, so that

$$\tau^n_p = \inf\{t, X_t \geq p\epsilon_n\} = \inf\{t, Y_t \geq p\epsilon_n\} = \sup\{t, Z_t = p\epsilon_n\}$$

Therefore $\tau^n_p \geq 1$ satisfies Assumption \[1\] We compute now $A^n$ as of equation (5). It is a classical result that

$$Z^n_{t,p} = P(\tau^n_p > t \mid \mathcal{F}_t) = 1 \wedge \frac{p\epsilon_n}{Z_t}$$

and Tanaka's formula implies that

$$M^n_{t,p} = 1 - p\epsilon_n \int_0^t 1\{Z_s > p\epsilon_n\} dB_s Z^2_s$$

Therefore since on $\{\tau^n_p < s\}$ we have that $Z_s > p\epsilon_n$, we obtain:

$$A^n_t = \sum_{p=0}^\infty \int_0^t 1\{\tau^n_p < s \leq \tau^n_{p+1}\} \frac{d(B, M^{n,p+1} - M^{n,p})_s}{Z^{n,p+1}_s - Z^{n,p}_s}$$

$$= \sum_{p=0}^\infty \int_0^t 1\{\tau^n_p < s \leq \tau^n_{p+1}\} \frac{p\epsilon_n}{Z^2_s} - 1\{Z_s > (p+1)\epsilon_n\} \frac{(p+1)\epsilon_n}{Z^2_s} - \frac{p\epsilon_n}{Z^2_s} 1\{(p+1)\epsilon_n < Z_s\} + 1\{(p+1)\epsilon_n \geq Z_s\} - \frac{p\epsilon_n}{Z^2_s} ds$$

$$= \sum_{p=0}^\infty \int_0^t 1\{\tau^n_p < s \leq \tau^n_{p+1}\} \left(1\{(p+1)\epsilon_n \geq Z_s\} \frac{p\epsilon_n}{Z_s p\epsilon_n + Z^2_s} + 1\{(p+1)\epsilon_n < Z_s\} \frac{1}{Z_s} - 1\{Z_s > (p+1)\epsilon_n\} \frac{(p+1)\epsilon_n}{Z^2_s} \right) ds$$

$$= \sum_{p=0}^\infty \int_0^t 1\{\tau^n_p < s \leq \tau^n_{p+1}\} \left(- \frac{1}{Z_s} + 1\{(p+1)\epsilon_n \geq Z_s\} \frac{1}{Z_s - p\epsilon_n} \right) ds$$

Fubini's theorem implies finally that

$$A^n_t = \int_0^t \frac{ds}{Z_s} - \sum_{p=0}^\infty \int_0^t 1\{\tau^n_p < s\} 1\{(p+1)\epsilon_n \geq Z_s\} \frac{1}{Z_s - p\epsilon_n} ds$$
where we also used $1_{\{s \leq \tau_{n+1}^{p}\}}1_{\{Z_s \leq (p+1)\varepsilon_n\}} = 1_{\{Z_s \leq (p+1)\varepsilon_n\}}$. Now

$$E\left(\sum_{p=0}^{\infty} \int_{0}^{t} 1_{\{\tau_{p}^{n} < s\}} 1_{\{(p+1)\varepsilon_n \geq Z_s\}} \frac{1}{Z_s - p\varepsilon_n} ds\right) = \sum_{p=0}^{\infty} E\left(\int_{0}^{t} 1_{\{\tau_{p}^{n} < s\}} 1_{\{(p+1)\varepsilon_n \geq Z_s\}} \frac{1}{Z_s - p\varepsilon_n} ds\right)$$

$$= \sum_{p=0}^{\infty} E\left(\int_{0}^{t} \frac{1}{Z_s} 1_{\{Z_s > p\varepsilon_n\}} 1_{\{Z_s \leq (p+1)\varepsilon_n\}} ds\right) = E\left(\int_{0}^{t} \frac{1}{Z_s} \sum_{p=0}^{\infty} 1_{\{p\varepsilon_n < Z_s \leq (p+1)\varepsilon_n\}} ds\right) = E\left(\int_{0}^{t} \frac{ds}{Z_s}\right)$$

where the second equality follows because the $\mathcal{F}$ optional projection of $1_{\{\tau_{\cdot}^{n} \leq \cdot\}}$ is $(1 - \frac{p\varepsilon_n}{Z_t})^+$. It remains to use Theorem 9 to conclude. □

We find this example quite interesting, because we see in the proof of Theorem 19 that each process $(A_{n})_{s \geq 0}$ a.s. has paths that are absolutely continuous with respect to $ds$, yet as we see in the limit, thanks to the results of Jeulin and Pitman, that the $\mathcal{G}$ decomposition $B$ is given by

$$B_t = (B_t - (2X_t - \int_{0}^{t} \frac{ds}{Z_s})) + (2X_t - \int_{0}^{t} \frac{ds}{Z_s}). \quad (14)$$

The finite variation term of (14) is $2X_t - \int_{0}^{t} \frac{ds}{Z_s}$. We note that the process $X$ is non decreasing but $dX_s$ has support on a random set which has Lebesgue measure 0 a.s. Due to the presence of a singular term in the decomposition (14) we cannot find an equivalent probability measure that turns $B$ into a $\mathcal{G}$ (local) martingale. So we are in the situation where each approximating term is well behaved, but in the limit the process we are after cannot be transformed into a local martingale and this example shows that we have NFLVR, implying the presence of arbitrage opportunities. Intuitively, this filtration expansion introduces arbitrage opportunities into the market where an insider discovers the extra information $X_t$ progressively and obtains an arbitrage opportunity that is hidden from the rest of the market. Brownian motion is not a good model of a stock price (for example it does not remain positive) but an extension of the above could apply, for example, to a model of what transpired with the Galleon Group, previously mentioned in the introduction to this section (Section 6), and in Section 6.2. See [48] for references concerning this example, and also [56] for a fine analysis of many aspects of the Bessel (3) process and related processes.

6.3.2 The Case of Transient Diffusions

In the preceding section (Section 6.3.1) we showed one individual process remained a semimartingale in the expanded filtration. It is more interesting, and a more powerful result, to show all $\mathcal{F}$ semimartingales remain semimartingales in $\mathcal{G}$, albeit with different decompositions in general. This is known as Hypothesis ($H'$). This might be too much to ask with these rather general filtration expansions, but we can provide a sufficient condition under which a class of semimartingales in $\mathcal{F}$ will remain semimartingales in $\mathcal{G}$. In so doing we extend the results of Section 6.3.1.

Let $R_t$ be a transient diffusion with values in $\mathbb{R}^+$, which has $\{0\}$ as entrance boundary.
Let \( s \) be a scale function for \( R \), which we can choose such that
\[
\lim_{x \to 0} s(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} s(x) = 0
\]

Let \( \mathcal{F} \) be the natural filtration of \( R \). Nikeghbali [51] studied progressive filtration expansions of \( \mathcal{F} \) with last exit times of such diffusions. Define \( X \) to be the remaining infimum of \( R \), i.e. \( X_t = \inf_{s \geq t} R_s \) and \( \mathcal{G} \) the progressive expansion of \( \mathcal{F} \) with \( X \). We provide a sufficient condition for some \( \mathcal{F} \) martingales to remain \( \mathcal{G} \) semimartingales. The process \( M_t = -s(R_t) \), which is well known to be an \( \mathcal{F} \) local martingale, plays a key role. The next theorem is the main result of this section.

**Theorem 20** Let \( N \) be an \( \mathcal{F} \) martingale such that \( \sup_{0 \leq s \leq T} |N_s| \) and \( \int_0^T \frac{|d\langle N, M_s \rangle_s|}{M_s} \) are integrable. Then \( N \) is a \( \mathcal{G} \) semimartingale.

**Proof.** Define the \( \mathcal{F} \) honest random times \( \sigma_y = \sup \{ t, R_t = y \} \). Let \( Y_t = 2X_t - R_t \), then \( X_t = \sup_{s \leq t} Y_s \) and the random times
\[
\tau_p^n := \inf \{ t, X_t \geq p\varepsilon_n \} = \inf \{ t, Y_t \geq p\varepsilon_n \} = \sup \{ t, R_t = p\varepsilon_n \} = \sigma_{p\varepsilon_n}
\]
are \( \mathcal{F} \) honest and \( (\tau_p^n)_{p \geq 1} \) satisfies Assumption [1]. To use Theorem 9 we need to compute \( A^n \) as defined in (5). It is proved in [51] that
\[
P(\sigma_y > t \mid \mathcal{F}_t) = \frac{s(R_t)}{s(y)} \wedge 1 = 1 - \frac{1}{s(y)} \int_0^t 1_{\{R_u > y\}} dM_u + \frac{1}{2s(y)} L^{s(y)}
\]
where \( L^{s(y)} \) is the local time of \( s(R) \) at \( s(y) \). Introduce \( Z_{t \wedge p\varepsilon_n} = P(\tau_p^n > t \mid \mathcal{F}_t) = \frac{s(R_t)}{s(p\varepsilon_n)} \wedge 1 \) and the martingale part in its Doob Meyer decomposition \( M_{t \wedge p\varepsilon_n} = 1 - \frac{1}{s(p\varepsilon_n)} \int_0^t 1_{\{R_u > p\varepsilon_n\}} dM_u \).

Therefore
\[
A_{t \wedge p\varepsilon_n} = \sum_{p=0}^\infty \int_0^t 1_{\{\tau_p^n < s \leq \tau_{p+1}^n\}} \left( \frac{1}{s(p\varepsilon_n)} - 1_{\{R_s \leq (p+1)\varepsilon_n\}} \right) d\langle N, M \rangle_s
\]
where we used that \( R_s > p\varepsilon_n \) on \( \{\tau_p^n < s\} \), and the fact that \( -s \) is positive non increasing by construction. Basic algebraic manipulations give
\[
A_{t \wedge p\varepsilon_n} = \sum_{p=0}^\infty \int_0^t 1_{\{\tau_p^n < s \leq \tau_{p+1}^n\}} \left( \frac{1}{-s(R_s)} + 1_{\{R_s \leq (p+1)\varepsilon_n\}} \right) d\langle N, M \rangle_s
\]
\[
= \int_0^t \frac{d\langle N, M \rangle_s}{M_s} + \sum_{p=0}^\infty \int_0^t 1_{\{\tau_p^n < s \leq \tau_{p+1}^n\}} \left( \frac{1}{s(p\varepsilon_n)} - s(R_s) \right) d\langle N, M \rangle_s
\]
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where the last equality follows from $1_{\{s \leq \tau^{n}_{p} \leq t\}}1_{\{R_{\epsilon} \leq (p+1)\epsilon\}} = 1_{\{R_{\epsilon} \leq (p+1)\epsilon\}}$. Now

$$E\left(\int_{0}^{T} |dA_{s}^{n}| \right) \leq E\left(\int_{0}^{T} \left| \frac{d(N, M)_{s}}{M_{s}} \right| \right)$$

$$+ \sum_{p=0}^{\infty} E\left(\int_{0}^{T} 1_{\{\tau^{n}_{p} \leq s\}}1_{\{R_{\epsilon} \leq (p+1)\epsilon\}} \frac{s(p\epsilon_{n})}{s(R_{s})(s(p\epsilon_{n}) - s(R_{s}))} |d(N, M)_{s}| \right)$$

$$\leq E\left(\int_{0}^{T} \left| \frac{d(N, M)_{s}}{M_{s}} \right| \right) + \sum_{p=0}^{\infty} E\left(\int_{0}^{T} 1_{\{p\epsilon_{n} < R_{\epsilon} \leq (p+1)\epsilon\}} \frac{1}{M_{s}} |d(N, M)_{s}| \right)$$

$$\leq 2E\left(\int_{0}^{T} \frac{1}{M_{s}} d(N, M)_{s} \right)$$

where the second inequality uses that the $\mathbb{F}$ optional projection of $1_{\{\tau^{n}_{p} \leq \cdot\}}$ is given by $(1 - \frac{s(R_{s})}{s(p\epsilon_{n})})^{+}$ and the monotonicity of $s$. Theorem 9 allows to conclude. ■

Unfortunately we do not have any results of Pitman, Jeulin, or even Nikeghbali to help us determine what the $\mathbb{G}$ semimartingale decomposition of $N$ actually is. So we are unable to determine, in general, if such an expansion introduces arbitrage, as it does in the case of the Bessel 3 process, as show in Section 6.3.1 Norris can explicitly calculate the risk neutral measure. Both of these results must await further research, and they are certainly of intrinsic interest. Intuitively, this is an extension of the Bessel 3 example where a singular term appears in the $\mathbb{G}$ decomposition, so we would not be surprised if that pathology occurs much more generally. One can consult [48] for references related to this example.

6.3.3 A Suggestive Example

We study here an example derived from elementary calculations, without using theory. We let $W$ denote a standard Brownian motion, or Wiener process, with natural filtration $\mathbb{F}$ satisfying the usual hypotheses. We also let $V$ be another standard Wiener process, independent of $W$ (and of $\mathbb{F}$). We will expand the filtration $\mathbb{F}$ dynamically in the style of this paper, with the process

$$X_{t} = W_{1} + \varepsilon V_{1-t}$$

We define

$$\mathcal{H}_{t} = \mathcal{F}_{t} \vee \sigma(W_{1} + \varepsilon V_{1-s}, s \leq t) = \mathcal{F}_{t} \vee \sigma(X_{s}, s \leq t).$$

We wish to show the following result, obtained with the help of Jean Jacod, to whom we are grateful.

**Theorem 21** Let $H$ be predictable with $\int_{0}^{1} H_{s}^{2} ds < \infty$ a.s. Define $M_{t} = \int_{0}^{t} H_{s} dW_{s}$, an $\mathbb{F}$ local martingale. If $H$ is a.s. of the order $H_{s} = \frac{1}{1-s}^{1/2+\alpha}$ with $\alpha < \frac{1}{2}$ then $M$ remains a semimartingale in $\mathbb{H}$, and has decomposition

$$A^{\mathbb{H}}_{t} = -\int_{0}^{t} \frac{X_{s} - W_{s}}{H_{s}(1 + \varepsilon^{2})(1 - s)} ds.$$  

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Proof. First we review some standard calculations, which are nevertheless not trivial. Let $0 \leq t \leq T \leq 1$. Then

$$E(W_T|W_1, W_t) = a(W_1 - W_t) + bW_t$$

due to the linearity, and that $\sigma(W_1, W_t) = \sigma(W_1 - W_t, W_t);

$$E(W_T(W_1 - W_t)) = a(1 - t) = T - t \quad \Rightarrow \quad a = \frac{T - t}{1 - t}$$

$$E(W_tW_t) = bt \quad \Rightarrow \quad b = 1,$$

hence

$$E(W_T|\mathcal{F}_t \vee \sigma(W_1)) = W_t + \frac{T - t}{1 - t}(W_1 - W_t).$$

We have that $W$ is a semimartingale for $\mathcal{G}$, where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(W_1)$, with decomposition

$$W = M + A,$$

and

$$A^\mathcal{G}_t = - \int_0^t \frac{W_1 - W_s}{1 - s} ds$$

Note that $\mathcal{F} \subset \mathcal{G} \subset \mathcal{H}$. We use the superscript notation $A^\mathcal{G}$ and $A^\mathcal{H}$ to denote relative to which filtration we are calculating the finite variation process $A$. We make a calculation of the expected total variation, to get (where “Var” denotes total variation, and not variance):

$$E(\text{Var}(A^\mathcal{G}_t)) = \int_0^t \frac{E(|W_1 - W_s|)}{1 - s} ds = \sqrt{\frac{2}{\pi}} \int_0^t \frac{1}{\sqrt{1 - s}} ds < \infty.$$

Now we let $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(W_1 + \varepsilon V_{1-s}, s \leq t)$. Using the above calculations we have

$$E(W_T|\mathcal{H}_t) = E(W_T|W_t, W_1 + \varepsilon V_{1-t})$$

$$= bW_t + a(W_1 - W_t + \varepsilon V_{1-t}),$$

again by linearity, and since

$$E(W_tW_t) = bt = t,$$

we have $b = 1$, and

$$E(W_T(W_1 - W_t + \varepsilon V_{1-t}) = T - t = a \left( (1 - t) + \varepsilon^2 (1 - t) \right) \quad \Rightarrow$$

$$E(W_T|\mathcal{H}_t) = W_t + \frac{T - t}{(1 + \varepsilon^2)(1 - t)}(W_1 - W_t + \varepsilon V_{1-t})$$

$$= W_t + \frac{T - t}{(1 + \varepsilon^2)(1 - t)}(X_t - W_t).$$

Note that

$$X_s - W_s = W_1 - W_s + \varepsilon V_{1-s} \overset{L}{=} \sqrt{1 + \varepsilon^2} W'_{1-s}$$

for another Wiener process $W'$, and where $\overset{L}{=}$ denotes equality in law. In this context we have that our process $A$ becomes

$$A^\mathcal{H}_t = - \int_0^t \frac{X_s - W_s}{(1 + \varepsilon^2)(1 - s)} ds \overset{L}{=} - \int_0^t \frac{W'_{1-s}}{\sqrt{1 + \varepsilon^2}(1 - s)} ds.$$
Next we replace our process $W$ with a local martingale of the form $M_t = \int_0^t H_s dW_s$, as stated in the hypotheses of the theorem. The above reasoning gives us that (in the $\mathcal{G} = (\mathcal{F}_t \lor \sigma(W_1))_{t \geq 0}$ paradigm):

$$A_t^{\mathcal{G}} = -\int_0^t H_s \frac{W_1 - W_s}{1 - s} ds,$$

where of course $\int_0^t H_s^2 ds < \infty$.

Then

$$E(V_{1-t} M_t) = aE(\int_0^t H_s \frac{\sqrt{1-s}}{1 - s} ds) = aE(\int_0^t H_s \frac{1}{\sqrt{1 - s}} ds),$$

but $\int_0^1 H_s^2 ds < \infty$ does not imply that $\int_0^1 |H_s| ds < \infty$. It does indeed imply it for $H_s$ of the form $H_s = \frac{1}{1-s}^{1/2+\alpha}$ with $\alpha < \frac{1}{2}$, but it does not work for $H_s = \frac{1}{\sqrt{1-s} \ln (1-s)}$.

Therefore we obtain the semimartingale property for any predictable process $H$ and the (local) martingale $M_t = \int_0^t H_s dW_s$ as long as $|H|$ is of the order $|H_s| \propto \frac{1}{\sqrt{1-s}}$.

This result follows also from our Theorem 10 and the proof is similar to that of Theorem 12. We sketch it here briefly.

**Proof.** First, we can prove that $(X, \mathcal{F})$ satisfies Assumption 2 and that for each $0 \leq i \leq n$ and $(t^n_i)_{0 \leq i \leq n}$ subdivision of $[0, 1]$, the $\mathcal{F}_t$-conditional density of $(X^n_{t^n_0}, X^n_{t^n_1} - X^n_{t^n_0}, \ldots, X^n_{t^n_n} - X^n_{t^n_{n-1}})$ is given by

$$p^n_{t_i}(x_0, \ldots, x_i) = g\left(\sum_{k=0}^i x_k - W_{t_i}\right) \frac{1}{\sqrt{1-t + \varepsilon^2(1-t^n_i)}} \prod_{k=0}^{i-1} \frac{1}{\sqrt{t^n_{k+1} - t^n_k}} g\left(\frac{-x_{k+1}}{\sqrt{1-t^n_{k+1} - t^n_k}}\right),$$

where $g$ is the gaussian density. Second, with $M = W$, equation (6) translates into

$$A^{i,n}_t = \int_0^t \frac{X^n_{t_i} - W_s}{1 - s + \varepsilon^2(1-t^n_i)} ds$$

and equation (7) becomes

$$A^{(n)}_t = \sum_{i=0}^n 1\{t^n_i \leq t < t^n_{i+1}\} \left(\sum_{k=0}^{i-1} \int_{t^n_k}^{t^n_{k+1}} \frac{X^n_{t_k} - W_s}{1 - s + \varepsilon^2(1-t^n_k)} ds + \int_{t^n_i}^t \frac{X^n_{t_i} - W_s}{1 - s + \varepsilon^2(1-t^n_i)} ds\right)$$

Third, we prove that the total variation of $A^{(n)}$ can be bounded uniformly in $n$:

$$E\left(\int_0^1 |dA^{(n)}_s|\right) \leq \sum_{k=0}^n \int_{t^n_k}^{t^n_{k+1}} E\left|\frac{X^n_{t_k} - W_s}{1 - s + \varepsilon^2(1-t^n_k)}\right| ds \leq \sqrt{\frac{2}{\pi}} \int_0^1 \frac{ds}{(1-s)(1+\varepsilon^2)} < \infty$$

Therefore we can apply Theorem 10 to conclude that $W$ is an $\mathbb{H}$ semimartingale. Finally $A^{(n)}$ converges in probability to the process $A$ given by

$$A_t = \int_0^t \frac{X_s - W_s}{1 - s + \varepsilon^2} ds$$
We can conclude as in the proof of Theorem 12 that $W - A$ is an $H$ Brownian motion.

What this means for insider trading is that since the finite variation terms a.s. has paths that are absolutely continuous with respect to $d[W,W]_t = dt = \text{Lebesgue measure}$, if we have enough integrability satisfied, we can use a Girsanov transformation to construct the risk neutral measure for the insider, as we outlined in Section 6.1.

6.4 A Current Example from Industry

We are including this section in order to please one of the two referees, the Associate Editor, and the Editor of the journal, all of whom insisted that we present an example to show how these methods could be used in a concrete application. We are happy to comply with their wishes.

Recently the attorney general of New York State, Eric Schneiderman, has undertaken an investigation of high frequency trading (HFT). By HFT we mean firms that have co-located around the computers processing trades (in the case of the New York Stock Exchange this is in Mahwah, NJ). See [1]. The claims are that the HFTs effectively have inside information. One way this could happen is a consequence of the analysis presented in a recent paper of Jarrow and Protter [37]. In brief, by the systematic use of IOC (immediate or cancel) orders, the HFT traders are able to construct a representation of the current state of the order book. They are then able to take the liquidity profits via an exploitation both of market orders and also of limit orders. These profits were formerly taken by large institutional traders, at the expense of small and unsophisticated traders. By doing this they can profit in the place of institutional traders, and also at their expense. But one can take the analysis further, and by effectively front running the limit orders of institutional traders they ensure that limit orders are executed almost exclusively at their limits, with the HFT traders pocketing the spread between the market price and the limit order limit price. See [37] for details.

The “inside information” attributed to the HFTs comes primarily from their real time understanding of the limit order book, obtained via the systematic use of the IOC orders technique. If one can see the limit order book, one gains insight into the very near future direction of the stock price. This type of inside information lends itself to the model presented in Section 6.3.3. We now discuss how this works.

We suppose given two independent standard Brownian motions $W$ and $V$ on a space $(\Omega, \mathcal{F}, P, \mathbb{F})$. We assume we have a stock price given by

$$dZ_s = \sigma(s, Z_s) dW_s + b(s, Z_s) ds \text{ with } Z_0 = 1 \quad (18)$$

We suppose we can see the direction of the price via the limit order book (LOB). Obviously this should be impossible in $\mathbb{F}$ since in any event $Z$ is strong Markov with respect to $\mathbb{F}$. However we incorporate this new information via a filtration enlargement, and this destroys the Markov property. Indeed, suppose we can see a future evolution via the LOB. We cannot expect actually to know a future value of the stock, but we can have
a better guess at the future value than a competitive trader who only sees observable events. Therefore it is not unreasonable to assume we can see the future of \( W \) (from which we can infer the future value of \( Z \)), but corrupted by a small amount of noise. We use the terms given in (15) and (16). More precisely, the HFT trader would want to know the information of \( W_1 \) in this example, which would then give him information regarding the future evolution of the stock price \( Z \). He cannot tell with certainty what \( W_1 \) is (since it is in the future), but he can guess at it based on the order book. But the order book information is itself noisy since it is dependent on future human behavior, and – for example – limit orders can be cancelled before execution, and/or have been placed there to deceive, rather than with sincerity. We model this via the inclusion of random noise in the LOB. We use the noise corruption term coming from \( \varepsilon \) times the independent Wiener process \( V \). The multiplication by \( \varepsilon \) represents the idea that while there is noise in the LOB, it is of very small variance. So what the HFT sees at time \( t \) is
\[
X_t = W_1 + \varepsilon V_{1-t} - t
\]
and the more the HFT can see, the better he or she can try to filter out the noise as much as possible. Note that this does not lead to an arbitrage opportunity a fortiori, but what is known as a statistical arbitrage opportunity. This is explained in [37], for example.

To continue the analysis, we apply Theorem 21 to conclude in the larger filtration \( \mathbb{H} \) we have that the \( Z \) of (18) now satisfies
\[
Z_t = 1 + \int_0^t \sigma(s, Z_s) dW_s + \int_0^t b(s, Z_s) ds - \int_0^t \sigma(s, Z_s) \frac{X_s - W_s}{(1 + \varepsilon^2)(1 - s)} ds
\]
(19)

We have that the insiders see a different dynamic evolution than does the rest of the market.

This has a particular consequence when we calculate the risk neutral measures of the general market and that of the insider. Recall that the risk neutral measures (also known as equivalent local martingale probability measures) are typically used to price contingent claims (such as options) in mathematical finance theory.

We begin with the routine computation of the risk neutral measure where \( Z \) is as in (18), and we are working with the filtration \( \mathbb{F} \). To compute the risk neutral measure we first use a Girsanov argument, where we find a change of measure \( Q \) such that \( Z \) becomes a local martingale under \( Q \). To do this, we need to find a process \( H \) such that if \( U \) satisfies the equation
\[
dU_t = U_t H_t dW_t; \quad U_0 = 1
\]
(20)
and if \( Q \) is given by \( dQ = U_1 dP \) then \( Z \) becomes a \((Q, \mathbb{F})\) local martingale. By Girsanov’s theorem (see for example [54]) we have that under \( Q \) the decomposition of \( Z \) is:
\[
Z_t = \left( \int_0^t \sigma(s, Z_s) dW_s - \int_0^t \frac{1}{U_s} d[U, W]_s \right) + \left( \int_0^t b(s, Z_s) ds + \int_0^t \frac{1}{U_s} d[U, W]_s \right)
\]
(21)
\[
= \left( \int_0^t \sigma(s, Z_s) dW_s - \int_0^t H_s ds \right) + \left( \int_0^t b(s, Z_s) ds + \int_0^t H_s ds \right)
\]
(22)
In order to have the drift term become 0, we need to choose \( H_t = -\frac{b(Z_t)}{\sigma(Z_t)} \). However we also need to ensure that \( Q \) is a true probability measure. We can do this using Novikov’s
criterion, if \( E(\exp(\frac{1}{2} \int_0^t \frac{b(s,Z_s)^2}{\sigma(s,Z_s)^2}ds)) < \infty \). This is of course an assumption on the drift coefficient \( b \) of the original equation \( (18) \), which we are free to make when setting up the model. We know that under \( Q \) the process \( Z \) of \( (18) \) satisfies the following equation
\[
dZ_s = \sigma(Z_s) d\beta_s
\]
where \( \beta \) is a Brownian motion under \( Q \) (indeed, from the above analysis we see that \( \beta_t = W_t - \int_0^t (-\frac{b(s,Z_s)}{\sigma(s,Z_s)})ds \)).

The computation of the risk neutral measure \( R \) using the filtration \( \mathbb{H} \) is more delicate. Again we assume \( Z \) follows the equation \( (18) \). By Theorem 21 we have, in \( \mathbb{H} \), \( Z \) satisfies
\[
Z_t = 1 + \int_0^t \sigma(s,Z_s) dW_s + \int_0^t b(s,Z_s) ds - \int_0^t \sigma(s,Z_s) \frac{X_s - W_s}{(1 + \varepsilon^2)(1 - s)} ds.
\]
For equation \( (24) \) to make sense, by Theorem 21 we need to have
\[
\sigma(s,Z_s) \asymp \left( \frac{1}{1 - s} \right)^{1/2 + \alpha} \quad \text{where } \alpha < \frac{1}{2}
\]
This is a serious restriction on the modeling of the stock price, but nonetheless one that we are allowed to make. But now we want to calculate the risk neutral measure. We first proceed via a Girsanov transformation: We let \( dR = UdP \) where \( U \) satisfies an equation of the form \( (20) \). In analogy with our previous (albeit simpler) calculation, we get
\[
H_s = -\left( \frac{b(s,Z_s) - \sigma(s,Z_s) \frac{X_s - W_s}{(1 + \varepsilon^2)(1 - s)}}{\sigma(s,Z_s)} \right)
\]
This is already bad enough, but now we have to verify that \( R \) is a true probability measure (and not a sub probability measure). As before a sufficient condition comes from Novikov’s criterion: It suffices to have that
\[
E(\exp(\int_0^t H_s^2 ds)) < \infty
\]
But we do not in general have that \( (27) \) satisfied, since we have a problem at the pole \( s = 1 \). We can solve this either by imposing a condition on \( \sigma(s, z) \) such that it vanishes at \( s = 1 \), or by not working on the half open time interval \( [0,1) \). Since the former seems unreasonable, we prefer the latter. This can be finessed by a new concept known as local NFLVR, as explained in the thesis of Roseline Bilina-Falafala [14], or in [38].

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