PRODUCTS OF COMPOSITION AND DIFFERENTIATION OPERATORS

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Abstract. We consider products of composition and differentiation operators on the Hardy space. We provide a complete characterization of boundedness and compactness of these operators. Furthermore, we obtain the explicit condition for these operators to be Hilbert-Schmidt operators.

1. PRELIMINARIES

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Hardy space $H^2$ is the Hilbert space of all analytic functions $f$ on $\mathbb{D}$ such that

$$\|f\|^2 = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$ 

It is well known that the Hardy space $H^2$ is a reproducing kernel Hilbert space, with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta,$$

and with kernel functions $K_w^n(z) = \frac{n! z^n}{(1-wz)^{n+1}}$, where $n$ is a non-negative integer and $z, w \in \mathbb{D}$. These kernel functions satisfy $\langle f, K_w^n \rangle = f^n(w)$ for each $f \in H^2$. To simplify notation we write $K_w$ in case $n = 0$. In particular note that $\|K_w\|^2 = K_w(w) = \frac{1}{1-|w|^2}$. Let $\hat{f}(n)$ be the $n$th coefficient of $f$ in its Maclaurin series. Moreover, we have another representation for the norm of $f$ on $H^2$ as follows

$$\|f\|^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.$$ 

The space $H^\infty$ is the Banach space of bounded analytic functions $f$ on $\mathbb{D}$ with $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$.

For $\varphi$ an analytic self-map of $\mathbb{D}$, the composition operator $C_\varphi$ is defined for analytic functions $f$ on $\mathbb{D}$ by $C_\varphi(f) = f \circ \varphi$. It is well known that every composition operator $C_\varphi$ is bounded on $H^2$ (see [2, Corollary 3.7]). For each positive integer $k$, the operator $D^{(k)}$ for any $f \in H^2$ is defined by the rule $D^{(k)}(f) = f^{(k)}$. This operator is called the differentiation operator of order $k$. For convenience, we use the notation $D$ when $k = 1$. The differentiation operators $D^{(k)}$ are unbounded on $H^2$, whereas Ohno [4] found a characterization for $C_\varphi D$ and $D C_\varphi$ to be bounded and compact on $H^2$. The study of operators $C_\varphi D$ and $D C_\varphi$ was initially addressed

2010 Mathematics Subject Classification. 47B38 (Primary), 30H10, 47E99.

Key words and phrases. Composition operator, differentiation operators, boundedness, compactness.

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by Hibschweiler, Portnoy, and Ohno (see [5] and [6]) and has been noticed by many researchers ([3], [4], and [8]). In this paper, we will be considering a slightly broader class of these operators. For each positive integer \( n \), we write \( D_{\varphi,n} \) to denote the operator on \( H^2 \) given by the rule 
\[
D_{\varphi,n}(f) = C_{\varphi} D^{(n)}(f) = f^{(n)} \circ \varphi.
\]

Our main results provide complete characterizations of the boundedness and compactness of operators \( D_{\varphi,n} \) on \( H^2 \) (Theorems 2.1 and 2.2). In addition, we characterize the Hilbert-Schmidt operators \( D_{\varphi,n} \) on \( H^2 \) (Theorem 3.3). In this paper, we use some ideas which are found in [6].

Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \). The Nevanlinna counting function \( N_{\varphi} \) of \( \varphi \) is defined by
\[
N_{\varphi}(w) = \sum_{\varphi(z) = w} \log \left( 1/|z| \right) \quad w \in \mathbb{D} \setminus \{\varphi(0)\}
\]
and \( N_{\varphi}(\varphi(0)) = \infty \). Note that \( N_{\varphi}(w) = 0 \) when \( w \) is not in \( \varphi(\mathbb{D}) \). For each \( f \in H^2 \), by using change of variables formula and Littlewood-Paley Identity, the norm of \( C_{\varphi} f \) is determined as follows:

\[
\|f \circ \varphi\|^2 = |f(\varphi(0))|^2 + 2 \int_{\mathbb{D}} |f'(w)|^2 N_{\varphi}(w) dA(w),
\]
where \( dA \) is the normalized area measure on \( \mathbb{D} \) (see [2, Theorem 2.31]). Moreover, to obtain the lower bound estimate on \( \|D_{\varphi,n}\| \) we need the following well known lemma as follows (see [2, p. 137]):

Suppose that \( \varphi \) is an analytic self-map of \( \mathbb{D} \) and \( f \) is analytic in \( \mathbb{D} \). Assume that \( \Delta \) is any disk not containing \( \{f^{-1}(\varphi(0))\} \) and centered at \( a \). Then

\[
N_{\varphi}(f(a)) \leq \frac{1}{|\Delta|} \int_{\Delta} N_{\varphi}(f(w)) dA(w),
\]
where \( |\Delta| \) is the normalized area measure of \( \Delta \).

2. Boundedness and compactness of \( D_{\varphi,n} \)

The goal of this section is to determine which of these operators \( D_{\varphi,n} \) are bounded and compact.

**Theorem 2.1.** Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( n \) be a positive integer. The operator \( D_{\varphi,n} \) is bounded on \( H^2 \) if and only if

\[
N_{\varphi}(w) = O\left( \left[ \log \left( 1/|w| \right) \right]^{2n+1} \right) \quad (|w| \to 1).
\]
Proof. Suppose that \( D_{\varphi, n} \) is bounded on \( H^2 \). Let \( f(z) = \frac{K_\lambda(z)}{\|K_\lambda\|} = \frac{\sqrt{1-|\lambda|^2}}{1-\lambda \bar{z}} \) for \( \lambda \in \mathbb{D} \). By (2.1), we see that

\[
\|D_{\varphi, n}\|^2 \geq \|D_{\varphi, n}f\|^2 = \|C_{\varphi}\left( n\bar{\lambda}^{n+1} \frac{\sqrt{1-|\lambda|^2}}{1-\lambda \bar{z}} \right) \|^2 = \left\| \frac{n\bar{\lambda}^{n+1} \sqrt{1-|\lambda|^2}}{(1-\lambda \bar{z})^{n+1}} + 2 \right\| \int_{\mathbb{D}} \left( (n+1) \frac{\sqrt{1-|\lambda|^2}}{(1-\lambda \bar{z})^{n+1}} \right)^2 N_\varphi(w) dA(w)
\]

(2.1)

Substitute \( w = \alpha_\lambda(u) = \frac{\lambda \bar{u}}{1-\lambda \bar{u}} \) back into (2.1) and using [7] Theorem 7.26 to obtain

\[
\|D_{\varphi, n}\|^2 \geq \int_{\mathbb{D}} \frac{2((n+1)!)^2 |\lambda|^{2n+2} (1-|\lambda|^2)}{1-\lambda \alpha_\lambda(u)} N_\varphi(\alpha_\lambda(u)) |\alpha'_\lambda(u)|^2 dA(u).
\]

(2.2)

Since \( 1-\lambda \alpha_\lambda(u) = \frac{1-|\lambda|^2}{1-\lambda u} \) and \( \alpha'_\lambda(u) = \frac{|\lambda|^2 - 1}{(1-\lambda u)^2} \), by substituting \( \alpha'_\lambda \) and \( 1-\lambda \alpha_\lambda \) back into (2.2), we see that

\[
\|D_{\varphi, n}\|^2 \geq \int_{\mathbb{D}} \frac{2((n+1)!)^2 |\lambda|^2 2n (1-\lambda u)^2n}{(1-|\lambda|^2)^{2n+1}} N_\varphi(\alpha_\lambda(u)) dA(u).
\]

(2.3)

Because \( |1-\lambda u| \geq \frac{1}{2} \) for any \( u \in \mathbb{D}/2 \), we get from (2.3) that

\[
\|D_{\varphi, n}\|^2 \geq \int_{\mathbb{D}/2} \frac{2((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n} (1-|\lambda|^2)^{2n+1}} N_\varphi(\alpha_\lambda(u)) dA(u).
\]

(2.4)

There exists \( r < 1 \) such that for each \( \lambda \) with \( r < |\lambda| < 1 \), \( \alpha_\lambda^{-1}(\varphi(0)) \notin \mathbb{D}/2 \) because \( |\alpha_\lambda^{-1}(\varphi(0))| = |\alpha_{\varphi(0)}(\lambda)| \) and \( \alpha_{\varphi(0)} \) is an automorphism of \( \mathbb{D} \). By (2.2) and (2.4), we have

\[
\|D_{\varphi, n}\|^2 \geq \frac{2((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n} (1-|\lambda|^2)^{2n+1}} \int_{\mathbb{D}/2} N_\varphi(\alpha_\lambda(u)) dA(u)
\]

\[
\geq \frac{2((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n} (1-|\lambda|^2)^{2n+1}} \cdot \frac{N_\varphi(\alpha_\lambda(0))}{4}
\]

\[
\|D_{\varphi, n}\|^2 \geq \frac{((n+1)!)^2 |\lambda|^{2n+2} N_\varphi(\lambda)}{2^{2n+1} (1-|\lambda|^2)^{2n+1}}
\]

(2.5)

for each \( \lambda \) with \( r < |\lambda| < 1 \). Since \( D_{\varphi, n} \) is bounded, there exists a constant number \( M \) so that

\[
\lim_{|\lambda| \to 1} \frac{((n+1)!)^2 |\lambda|^{2n+2} N_\varphi(\lambda)}{2^{2n+1} (1-|\lambda|^2)^{2n+1}} N_\varphi(\lambda) \leq M.
\]

(2.6)
We know that $\log (1/|\lambda|)$ is comparable to $1 - |\lambda|$ as $|\lambda| \to 1^-$. Note that

$$\lim_{|\lambda| \to 1} \frac{(n + 1)!}{2^{n+1}(1 - |\lambda|^2)^{2n+1}} N_\varphi(\lambda) \lambda^{2n+2} = \lim_{|\lambda| \to 1} \frac{(n + 1)!}{2^{n+1}(1 + |\lambda|)^{2n+1}} \left( \frac{\log (1/|\lambda|)}{1 - |\lambda|} \right)^{2n+1} \frac{N_\varphi(\lambda)}{(\log (1/|\lambda|))^{2n+1}}.$$

(2.7)

By (2.6) and (2.7), we can see that

$$\sup_{R < |\lambda| < 1} N_\varphi(\lambda) \left[ \log (1/|\lambda|) \right]^{2n+1} \leq M.$$

Let $f$ be an arbitrary function in $H^2$. It follows from (1.1) that

$$\|D_{\varphi, n} f\|^2 = \|f^{(n)}(\varphi(0))\|^2 + 2 \int_D |f^{(n+1)}(w)|^2 N_\varphi(w) dA(w) = |f^{(n)}(\varphi(0))|^2 \left( \int_{R_0} |f^{(n+1)}(w)|^2 N_\varphi(w) dA(w) + \int_D |f^{(n+1)}(w)|^2 N_\varphi(w) dA(w) \right).$$

(2.8)

First we estimate the first and the second terms in the right-hand of (2.8). Observe that

$$f^{(n)}(z) = \langle f, K_z^{(n)} \rangle = \int_0^{2\pi} \frac{n! e^{-in\theta} f(e^{i\theta})}{(1 - e^{-i\theta} z)^{n+1}} \frac{d\theta}{2\pi}$$

and hence

$$(2.9) \quad |f^{(n)}(z)| \leq \frac{n!}{(1 - |z|)^{n+1}} \int_0^{2\pi} \frac{|f(e^{i\theta})|}{|1 - e^{-i\theta} z|^{n+1}} \frac{d\theta}{2\pi} \leq \frac{n!}{(1 - |z|)^{n+1}} \|f\|$$

for any $z \in \mathbb{D}$. It follows from (2.9) that

$$|f^{(n)}(\varphi(0))| \leq \frac{n! \|f\|}{(1 - |\varphi(0)|)^{n+1}}.$$

(2.10)

Moreover, we can see that

$$|f^{(n+1)}(z)| = |\langle f, K_z^{(n+1)} \rangle| \leq \frac{(n + 1)!}{(1 - |z|)^{n+2}} \|f\|$$

(2.11)

for any $z \in \mathbb{D}$. Therefore by (2.11), we see that

$$\int_{R_0} |f^{(n+1)}(w)|^2 N_\varphi(w) dA(w) \leq \left( \frac{(n + 1)!}{(1 - R)^{n+2}} \|f\|^2 \int_{R_0} N_\varphi(w) dA(w) \right).$$
Since \( \| \varphi \| = |\varphi(0)|^2 + 2 \int_D N_\varphi(w) dA(w) \) by (1.1), we obtain
\[
(2.12) \quad \int_D N_\varphi(w) dA(w) = \frac{1}{2}(\| \varphi \|^2 - |\varphi(0)|^2) < 1.
\]
From (2.11) and (2.12), we see that
\[
(2.13) \quad \int_{D_{\leq R}} |f^{(n+1)}(w)|^2 N_\varphi(w) dA(w) \leq \left( \frac{(n+1)!}{(1-R)^{n+2}} \right)^2 \| f \|^2.
\]
Now we estimate the third term in the right-hand of (2.13). We have
\[
\int_{D_{\leq R}} |f^{(n+1)}(w)|^2 N_\varphi(w) dA(w)
= \int_{D_{\leq R}} |f^{(n+1)}(w)|^2 \left( \log(1/|w|) \right)^{2n+1} \frac{N_\varphi(w)}{(\log(1/|w|))^{2n+1}} dA(w)
\leq \sup_{R < |w| < 1} N_\varphi(w) \left( \log(1/|w|) \right)^{2n+1} \int_{D_{\leq R}} |f^{(n+1)}(w)|^2 \left( \log(1/|w|) \right)^{2n+1} dA(w)
\leq M \int_{D_{\leq R}} |f^{(n+1)}(w)|^2 \left( \log(1/|w|) \right)^{2n+1} dA(w).
\]
Let \( f(z) = \sum_{m=0}^{\infty} a_m z^m \). We get
\[
\int_{D_{\leq R}} |f^{(n+1)}(w)|^2 \left( \log(1/|w|) \right)^{2n+1} dA(w)
\leq \sum_{m=n+1}^{\infty} m(m-1)...(m-n)a_m (w)^{m-(n+1)} \left( \log(1/|w|) \right)^{2n+1} dA(w)
\leq \sum_{m=n+1}^{\infty} m^2(m-1)^2...|a_m|^2 \left( \int_{D_{\leq R}} |w|^{m-(n+1)} \left( \log(1/|w|) \right)^{2n+1} dA(w) \right)
\leq \sum_{m=n+1}^{\infty} m^2(m-1)^2...|a_m|^2 \left( \int_{D_{\leq R}} (w)^{m-(n+1)} \left( \log(1/|w|) \right)^{2n+1} dA(w) \right)
= \sum_{m=n+1}^{\infty} m^2(m-1)^2...|a_m|^2 \int_0^{2\pi} \int_0^1 |r e^{i\theta}|^{2(m-(n+1))} (\log(1/r))^{2n+1} r dr \frac{d\theta}{\pi}
\leq \sum_{m=n+1}^{\infty} m^2(m-1)^2...|a_m|^2 \int_0^1 (r)^{2(m-(n+1))} (\log(1/r))^{2n+1} 2r dr.
\]
Now substitute \( t = r^2 \) and \( u = \log(1/t) \) to obtain
\[
(2.16) \quad \int_0^1 (r)^{2(m-(n+1))} (\log(1/r))^{2n+1} 2r dr = \int_0^1 t^{(m-(n+1))} \left( \frac{1}{2} \log(1/t) \right)^{2n+1} dt
= (1/2)^{2n+1} \int_0^\infty e^{-u(u-n)} u^{2n+1} du.
\]
By substituting \( x = (m - n)u \) back into (2.16), we have
\[
(1/2)^{2n+1} \int_0^\infty e^{-u(m-n)}u^{2n+1} du = \frac{1}{2^{2n+1}(m-n)^{2n+2}} \int_0^\infty e^{-x^2}x^{2n+1} dx
\]
(2.17)

By (2.14), (2.15), (2.16) and (2.17), we can see that

\[
\int_{D \setminus RD} |f^{(n+1)}(w)|^2 N_\varphi(w) dA(w) \leq M \sum_{m=n+1}^{\infty} m^2(m-1)^2...(m-n)^2|a_m|^2 \frac{\Gamma(2n+2)}{2^{2n+1}(m-n)^{2n+2}}
\]

\[= M \lambda \frac{(2n+1)!}{2^{2n+1}} \sum_{m=n+1}^{\infty} |a_m|^2 \]

(2.18)

where \( \lambda \) is a constant so that \( m^2(m-1)^2...(m-n+1)^2 \leq \lambda \) for each \( m \geq n+1 \) (note that the function \( f(x) = \frac{x^2(x-1)^2...(x-n+1)^2}{(x-n)^2} \) is bounded on \([n+1, +\infty)\)). Then (2.8), (2.10), (2.13) and (2.18) show that \( D_{\varphi,n} \) is bounded.

**Theorem 2.2.** Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( n \) be a positive integer. The operator \( D_{\varphi,n} \) is compact on \( H^2 \) if and only if

\[
N_{\varphi}(w) = o \left( \left[ \log \left( \frac{1}{|w|} \right) \right]^{2n+1} \right) \quad (|w| \to 1).
\]

(2.19)

**Proof.** Let \( h_m(z) = \frac{1-|\lambda_m|^2}{1-\lambda_m z} \) for a sequence \( \{\lambda_m\} \) in \( \mathbb{D} \) so that \( |\lambda_m| \to 1 \) as \( m \to \infty \). Then \( h_m \to 0 \) weakly as \( m \to \infty \) by [2] Theorem 2.17. First suppose that \( D_{\varphi,n} \) is compact. Hence \( \|D_{\varphi,n}h_m\| \to 0 \) as \( m \to \infty \). Therefore (2.5) shows that

\[
limit_{m \to \infty \frac{((n+1)!)^2 |\lambda_m|^{2n+2}}{2^{2n+1}(1-|\lambda_m|^2)^{2n+1}} N_{\varphi}(\lambda_m) = 0.
\]

Since \( \log(1/|\lambda_m|) \) is comparable to \( 1 - |\lambda_m| \) as \( m \to \infty \), the result follows.

Conversely, suppose that (2.19) holds. Let \( \epsilon > 0 \). Then there exists \( R, 0 < R < 1 \), such that

\[
\sup_{R < |w| < 1} N_{\varphi}(w) / \left[ \log(1/|w|) \right]^{2n+1} < \epsilon.
\]

(2.20)

Let \( \{f_m\} \) be any bounded sequence in \( H^2 \). By using the idea which was stated in the proof of [2] Proposition 3.11, we can see that \( \{f_m\} \) is a normal family and there exists a subsequence \( \{f_{m_k}\} \) which converges to some function \( f \in H^2 \) uniformly on all compact subsets of \( \mathbb{D} \). Let \( g_{m_k} = f_{m_k} - f \) for each positive integer \( k \). Note that \( \{g_{m_k}\} \) is a bounded sequence in \( H^2 \) which converges to 0 uniformly on all compact
subsets of $\mathbb{D}$. By (2.8), we obtain

$$
\|D_{\varphi,n}g_{m_k}\|^2 = |g_{m_k}^{(n)}(\varphi(0))|^2 + 2 \int_{\mathbb{D}\setminus\mathbb{R}} |g_{m_k}^{(n+1)}(w)|^2 N_\varphi(w) dA(w)
$$

(2.21)

$$
+ 2 \int_{\mathbb{D}\setminus\mathbb{R}} |g_{m_k}^{(n+1)}(w)|^2 N_\varphi(w) dA(w).
$$

By [1, Theorem 2.1, p. 151], we can choose $k_\varepsilon$ so that

$$
|g_{m_k}^{(n)}(\varphi(0))| < \sqrt{\varepsilon}
$$

(2.22)

and $|g_{m_k}^{(n+1)}| < \sqrt{\varepsilon}$ on $\mathbb{R} \mathbb{D}$ whenever $k > k_\varepsilon$. Substituting $f(z) = z$ into (1.1), we see that

$$
\int_{\mathbb{R} \mathbb{D}} |g_{m_k}^{(n+1)}(w)|^2 N_\varphi(w) dA(w) \leq \varepsilon \int_{\mathbb{R} \mathbb{D}} N_\varphi(w) dA(w)
$$

(2.23)

$$
\leq \frac{\varepsilon}{2} (||\varphi||^2 - |\varphi(0)|^2)
$$

for $k > k_\varepsilon$. On the other hand by (2.20) and the same idea as stated in the proof of (2.13) and (2.18), we see that

$$
\int_{\mathbb{D}\setminus\mathbb{R}} |g_{m_k}^{(n+1)}(w)|^2 N_\varphi(w) dA(w)
$$

$$
\leq \sup_{R < |w| < 1} \frac{N_\varphi(w)}{[\log(1/|w|)]^{2n+1}} \int_{\mathbb{D}\setminus\mathbb{R}} |g_{m_k}^{(n+1)}(w)|^2 [\log(1/|w|)]^{2n+1} dA(w)
$$

(2.24)

$$
\leq C\varepsilon \|g_{m_k}\|,
$$

where $C$ is a constant. Hence we conclude that $\|D_{\varphi,n}g_{m_k}\|$ converges to zero as $k \to \infty$ by (2.21), (2.22), (2.23) and (2.24) and so $D_{\varphi,n}$ is compact.

The preceding theorems lead to characterizations of all bounded and compact operators $D_{\varphi,n}$ when $\varphi$ is a univalent self-map.

**Corollary 2.3.** Let $\varphi$ be a univalent self-map of $\mathbb{D}$ and $n$ be a positive integer. Then the following hold.

(i) $D_{\varphi,n}$ is bounded on $H^2$ if and only if

$$
\sup_{w \in \mathbb{D}} \frac{1 - |w|}{(1 - |\varphi(w)|)^{2n+1}} < \infty
$$

(ii) $D_{\varphi,n}$ is compact on $H^2$ if and only if

$$
\lim_{|w| \to 1} \frac{1 - |w|}{(1 - |\varphi(w)|)^{2n+1}} = 0.
$$

**Proof.** Since $\varphi$ is univalent, we can see that $N_\varphi(w) = \log (1/|z|)$, where $\varphi(z) = w$. We observe that

$$
N_\varphi(w) = \frac{-\log (|z|)}{[\log(1/|w|)]^{2n+1}} = \frac{-\log (|z|)}{(-\log (|\varphi(w)|))^{2n+1}}.
$$

Moreover, we know that $\log (1/|z|)$ is comparable to $1 - |z|$ as $|z| \to 1^-$. Furthermore $|z| \to 1$ as $|\varphi(z)| \to 1$. Therefore the results follow immediately from Theorems 2.1 and 2.2. \qed
3. Hilbert-Schmidt operator $D_{\varphi,n}$

We begin with a few easy observations that help us in the proof of Theorem 3.3. In the proof of the following lemma, we assume that $0^0 = 1$.

**Lemma 3.1.** Let $n$ be a positive integer and $\alpha_k > 0$ for each $0 \leq k \leq n$. Then for $0 \leq x < 1$, the following statements hold.

(a) $\sum_{k=0}^{n} \frac{\alpha_k x^k}{(1-x)^{n+k+1}} \leq \sum_{k=0}^{n} \frac{\alpha_k}{(1-x)^{2n+k+1}}$.

(b) There exists a positive number $\beta$ such that $\sum_{k=0}^{n} \frac{\alpha_k x^k}{(1-x)^{n+k+1}} \geq \beta \frac{1}{(1-x)^{2n+k+1}}$.

**Proof.** (a) We can see that

$$\sum_{k=0}^{n} \frac{\alpha_k x^k}{(1-x)^{n+k+1}} = \sum_{k=0}^{n} \frac{\alpha_k x^k (1-x)^{n-k}}{(1-x)^{2n+1}}.$$

Since $0 \leq x < 1$ and $\alpha_k > 0$, we conclude that $\sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k} \leq \sum_{k=0}^{n} \alpha_k$. Hence the conclusion follows.

(b) We have

$$(1-x)^{2n+1} \sum_{k=0}^{n} \frac{\alpha_k x^k}{(1-x)^{n+k+1}} = \sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k} > 0.$$

Since $\sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k}$ is a continuous function on $[0, 1]$, there exists a positive number $\beta$ such that $\sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k} \geq \beta$. Hence the result follows. \hfill $\Box$

**Lemma 3.2.** Let $n$ be a positive integer. Then

$$\sum_{m=n}^{\infty} \frac{(m(m-1)...(m-n+1))^2}{x^m} = (n!)^2 \sum_{k=0}^{n} \frac{(n+k)!}{(k!)^2 (n-k)!} \frac{x^k}{(1-x)^{n+k+1}}$$

for $0 \leq x < 1$.

**Proof.** See [8, Lemma 1] and the general Leibniz rule. \hfill $\Box$

A Hilbert-Schmidt operator on a separable Hilbert space $H$ is a bounded operator $A$ with finite Hilbert-Schmidt norm $\|A\|_{HS} = (\sum_{n=1}^{\infty} \|Ae_n\|^2)^{1/2}$, where $\{e_n\}$ is an orthonormal basis of $H$. These definitions are independent of the choice of the basis (see [2, Theorem 3.23]).

**Theorem 3.3.** Let $D_{\varphi,n}$ be a bounded operator on $H^2$. Then $D_{\varphi,n}$ is a Hilbert-Schmidt operator on $H^2$ if and only if

$$\lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{(1 - |\varphi(re^{i\theta})|^2)^{2n+1}} < \infty.$$

(3.1)
Proof. Suppose that (3.1) holds. Lemmas 3.1, 3.2 and [7, Theorem 1.27] imply that
\[ \sum_{m=0}^{\infty} \left\| D_{\varphi, n} z^m \right\| = \sum_{m=n}^{\infty} \left\| m(m-1)\ldots(m-n+1)\varphi^{m-n} \right\| \]
\[ = \sum_{m=n}^{\infty} \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \left| m(m-1)\ldots(m-n+1)\varphi^{m-n}(re^{i\theta}) \right|^2 d\theta \]
\[ = \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{m=n}^{\infty} \left| m(m-1)\ldots(m-n+1)\varphi^{m-n}(re^{i\theta}) \right|^2 d\theta \]
\[ = \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{k=0}^{n} \frac{(n!)^2 (n+k)!}{(k!)^2 (n-k)!} \frac{\varphi^{2k}(re^{i\theta})}{(1 - |\varphi(re^{i\theta})|^2)^{n+k+1}} \]
\[ \leq \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\alpha}{(1 - |\varphi(re^{i\theta})|^2)^{2n+1}}, \tag{3.2} \]
where \( \alpha = \sum_{k=0}^{n} \frac{(n!)^2 (n+k)!}{(k!)^2 (n-k)!} \) (note that the interchange of limit and summation is justified by [2, Corollary 2.23] and using Lebesgue’s Monotone Convergence Theorem with counting measure). It follows that \( \sum_{m=0}^{\infty} \left\| D_{\varphi, n} z^m \right\| < \infty \) and so \( D_{\varphi, n} \) is a Hilbert-Schmidt operator on \( H^2 \) by [2, Theorem 3.23].

Conversely, suppose that \( D_{\varphi, n} \) is a Hilbert-Schmidt operator on \( H^2 \). We infer from [2, Theorem 3.23] that
\[ \sum_{m=0}^{\infty} \left\| D_{\varphi, n} z^m \right\|^2 < \infty. \tag{3.3} \]
On the other hand, by the proof of (3.2) and Lemma 3.1 there exists a positive number \( \beta \) such that
\[ \sum_{m=0}^{\infty} \left\| D_{\varphi, n} z^m \right\|^2 = \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{k=0}^{n} \frac{(n!)^2 (n+k)!}{(k!)^2 (n-k)!} \frac{\varphi^{2k}(re^{i\theta})}{(1 - |\varphi(re^{i\theta})|^2)^{n+k+1}} \]
\[ \geq \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\beta}{(1 - |\varphi(re^{i\theta})|^2)^{2n+1}}, \tag{3.4} \]
Hence the result follows from (3.3) and (3.4). \( \square \)

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