MULTI-STEP ITERATIVE ALGORITHM FOR MINIMIZATION AND FIXED POINT PROBLEMS IN P-UNIFORMLY CONVEX METRIC SPACES

Kazeem Olalekan Aremu
School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal
Durban, South Africa

Chinedu Izuchukwu
School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal
Durban, South Africa
and
DSI-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS)
Johannesburg, South Africa

Grace Nnenanya Ogwo and Oluwatosin Temitope Mewomo
School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal
Durban, South Africa

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Abstract. In this paper, we propose and study a multi-step iterative algorithm that comprises of a finite family of asymptotically \( k_i \)-strictly pseudocontractive mappings with respect to \( p \), and a \( p \)-resolvent operator associated with a proper convex and lower semicontinuous function in a \( p \)-uniformly convex metric space. Also, we establish the \( \Delta \)-convergence of the proposed algorithm to a common fixed point of finite family of asymptotically \( k_i \)-strictly pseudocontractive mappings which is also a minimizer of a proper convex and lower semicontinuous function. Furthermore, nontrivial numerical examples of our algorithm are given to show its applicability. Our results complement a host of recent results in literature.

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* Corresponding author: O. T. Mewomo.
1. **Introduction.** Let $D$ be a nonempty subset of a real Banach space $X$. The modulus of convexity of a real Banach space $X$ with $\dim(X) \geq 2$ is the function $\delta_X : [0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) := \inf \left\{ 1 - \left| \frac{x + y}{2} \right| : ||x|| = ||y|| = 1, \epsilon = ||x - y|| \right\}.$$ 

The Banach space $X$ is called uniformly convex [10] (see also [7]), if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$. The notion of $p$-uniformly convex Banach space was introduced in [6] as follows:

For $1 < p < \infty$, a Banach space $X$ is said to be $p$-uniformly convex if there exists a constant $c > 0$ such that $\delta_X(\epsilon) \geq c \epsilon^p \forall \epsilon \in (0,2]$. Equivalently (see [35]), $X$ is called $p$-uniformly convex, if for $2 \leq p < \infty$, there exists $c \geq 1$ such that for any $x, y \in X$,

$$\left| \frac{x + y}{2} \right|^p \leq \frac{1}{2} ||x||^p + \frac{1}{2} ||y||^p - \frac{1}{c^p} \left| \frac{x + y}{2} \right|^p.$$

$P$-uniformly convex Banach spaces are known to be very useful in analysis, precisely in Banach Space Theory, Homology Theory, Degree Theory and Differential Geometry (see [35, 25, 34, 29]).

Let $(X, d)$ be a metric space. $X$ is called a geodesic space if every two points $x, y \in X$ are joined by a geodesic path $c : [0, d(x, y)] \to X$ such that $c(0) = x$ and $c(d(x, y)) = y$. In this case, $c$ is an isometry and the image of $c$ is called a geodesic segment joining $x$ to $y$. The space $X$ is said to be uniquely geodesic if every two points of $X$ are joined by exactly one geodesic segment.

Let $D$ be a nonempty subset of a metric space $X$ and $T : D \to D$ be any nonlinear mapping. A point $x \in X$ is called a fixed point of $T$ if $x = Tx$. We denote by $F(T)$ the set of fixed points of $T$. The mapping $T$ is said to be

(i) $L$-Lipschitzian, if there exists $L > 0$ such that

$$d(Tx, Ty) \leq Ld(x, y) \forall x, y \in D.$$ 

If $L = 1$, then $T$ is nonexpansive,

(ii) uniformly $L$-Lipschitzian, if there exists $L > 0$ such that

$$d(T^n x, T^n y) \leq Ld(x, y) \forall x, y \in D, \ n \geq 1,$$

(iii) asymptotically nonexpansive, if there exists a sequence $\{u_n\}_{n=1}^\infty \subseteq [0, \infty)$, with $\lim_{n \to \infty} u_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + u_n)d(x, y) \forall x, y \in D, \ n \geq 1,$$

(iv) asymptotically $k$-strictly pseudocontractive, if there exist a constant $k \in [0,1)$ and a sequence $\{u_n\}_{n=1}^\infty \subseteq [1, \infty)$, with $\lim_{n \to \infty} u_n = 1$ such that

$$d(T^n x, T^n y)^2 \leq u_n d(x, y)^2 + k(d(x, T^n x) + d(y, T^n y))^2 \forall x, y \in D, \ n \geq 1.$$ 

In 2011, Noar and Siberman [30] introduced the concept of $p$-uniformly convex metric space as follows:

Let $1 < p < \infty$, a metric space $X$ is called $p$-uniformly convex with parameter $c > 0$ if and only if $X$ is a geodesic space and

$$d(v, (1-t)x \oplus ty)^p \leq (1-t)d(v, x)^p + td(v, y)^p - \frac{c}{2} t(1-t)d(x, y)^p, \quad (1)$$
for all \( x, y, v \in X, \ t \in [0, 1] \).

The notion of \( p \)-uniformly convex metric spaces is a natural generalization of the notion of \( p \)-uniformly convex Banach spaces (see \([9, 14, 41, 42]\)). A typical example of \( p \)-uniformly convex metric spaces are \( L_p \) spaces with \( p \geq 2 \), \( \text{CAT}(0) \) spaces (with \( p = 2 \), and parameter \( c = 2 \)) and \( \text{CAT}(k) \) spaces (\( k > 0 \)) with \( \text{diam}(X) < \frac{\pi}{2\sqrt{k}} \) and parameter \( c = (\pi - 2\sqrt{k}\epsilon) \tan(\sqrt{k}\epsilon) \) for any \( 0 < \epsilon \leq \frac{\pi}{2\sqrt{k}} - \text{diam}(X) \) (see \([14, 35, 42]\)).

Let \( X \) be a geodesic space and \( f \) be a proper convex and lower semicontinuous functional defined on \( X \). If there exists a point \( \bar{v} \in X \) such that \( f(\bar{v}) = \min_{v \in X} f(v) \), then \( \bar{v} \) is called a minimizer of \( f \). The set of minimizers of \( f \) is denoted by \( \text{arg min}_{v \in X} f(v) \) and the problem of finding such minimizers is called Minimization Problem (MP).

MP remains one of the most important problems in Optimization theory that has been studied extensively by many authors. The Proximal Point Algorithm (PPA) is one of the most popular and effective methods for solving MPs. The PPA was first introduced by Martinet [28] and was further developed by Rockafellar [33]. The later proved that the PPA converges weakly to a minimizer of a proper convex and lower semicontinuous functional. In 2013, Bacak [3] introduced and studied the PPA in the framework of nonlinear spaces (specifically, the Hadamard space) which he proposed as follows: For arbitrary \( x_1 \in X \), define a sequence iteratively by

\[
x_{n+1} = J^f_{\lambda_n}(x_n),
\]

where \( \lambda_n > 0 \) for all \( n \geq 1 \), and \( J^f_{\lambda_n} : X \to X \) is the Moreau-Yosida resolvent of a proper convex and lower semicontinuous functional defined by

\[
J^f_{\lambda_n}(x) = \arg\min_{v \in X} \left( f(v) + \frac{1}{2\lambda_n} d(v, x)^2 \right).
\]

He established the \( \Delta \)-convergence of (2) with the assumptions that \( f \) has a minimizer in \( X \) and \( \sum_{n=1}^{\infty} \lambda_n = \infty \). Other authors have also studied the PPA in the framework of Hadamard spaces (see for example \([8, 20]\)).

MPs are closely related to other optimization problems such as the monotone inclusion problems \([21]\), equilibrium problems \([23]\) (see also \([19]\)), variational inequality problems \([20]\) and many more. Thus, finding solutions of MPs using the PPA in Hilbert, Banach and Hadamard spaces still remains an active area of research in nonlinear and convex analysis and optimization see \([1, 2, 8, 15, 16, 31, 38, 39, 40, 43]\) and other references therein. In an attempt to introduced and generalized the PPA from these spaces to \( p \)-uniformly convex metric spaces, Choi and Ji [9] introduced the notion of \( p \)-resolvent operator in a \( p \)-uniformly convex metric space as a generalization of the Moreau-Yosida resolvent in a \( \text{CAT}(0) \) space as follows:

\[
J^f_p(x) = \arg\min_{v \in X} \left( f(v) + \frac{1}{2\lambda} d(v, x)^p \right),
\]

where \( f \) is a convex and lower semicontinuous functional not identically \( \infty \) and \( \lambda > 0 \). They studied a PPA involving the \( p \)-resolvent operator (4) as follows:

\[
x_n = J^f_{\lambda_n}(x_{n-1}), \quad n \geq 1,
\]

in a \( p \)-uniformly convex metric space and proved that the sequence (5) converges to the minimizer of \( f \).
Kuwae [24] defined another version of $p$-resolvent operator which is more general than (4) in $p$-uniformly convex metric spaces as follows:

$$J_p^x(x) = \arg \min_{v \in X} \left( f(v) + \frac{1}{p \lambda^{p-1}} d(v, x)^p \right).$$

(6)

He established the unique existence of the $p$-resolvent operator (6) associated to a coercive proper lower semicontinuous functional. In addition, he applied (6) to obtain solutions of initial boundary value problems for $p$-harmonic maps. Very recently, Izuchukwu et. al. [14] adopted (6) to study the Split Minimization Problem (SMP) in $p$-uniformly convex metric spaces. First, they studied some fundamental properties of the $p$-resolvent operator, which includes the unique existence, firmly nonexpansiveness, nonexpansiveness and the monotonicity properties of the $p$-resolvent of a proper convex and lower semicontinuous functional. Then, they proposed a Backward-Backward Algorithm and an Alternating Proximal Algorithm that both converge to a solution of the SMP.

On the other hand, Yildirim and Ozdemir [44] introduced a new multi-step iterative scheme for a finite family of asymptotic non-self mappings in a real uniformly convex Banach space as follows: Let $x_1 \in D \subset X$ and $\{x_n\}$ be a sequence generated by

$$
\begin{aligned}
&x_{n+1} = (1 - \alpha_{m,n})y_{(m-1),n} + \alpha_{m,n}T_{m,n}y_{(m-1),n}, \\
y_{(m-1),n} = (1 - \alpha_{(m-1),n})y_{(m-2),n} + \alpha_{(m-1),n}T_{(m-1),n}y_{(m-2),n}, \\
&\vdots \\
y_{2,n} = (1 - \alpha_{2,n})y_{1,n} + \alpha_{2,n}T_{2,n}y_{1,n}, \\
y_{1,n} = (1 - \alpha_{1,n})y_{0,n} + \alpha_{1,n}T_{1,n}y_{0,n},
\end{aligned}
$$

(7)

where $y_{0,n} = x_n$, $n \geq 1$ and $\{\alpha_{i,n}\}$ is a sequence in $[0, 1]$ for each $i = 1, 2, \ldots, m$. They proved under some suitable assumptions that $\{x_n\}$ weakly and strongly converge to the common fixed point of finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^m$. In [17], Gürsoy et. al. also introduced another multi-step iterative scheme for a single contractive-like operator as follows:

$$
\begin{aligned}
x_0 &\in D, \\
x_{n+1} = (1 - \alpha_n)y_{1,n} + \alpha_nT_1y_{1,n}, \\
y_{i,n} = (1 - \beta_{i,n})y_{(i+1),n} + \beta_{i,n}T_{i+1,n}y_{(i+1),n}, \quad i = 1, 2, \ldots, m - 2, \\
y_{(m-1),n} = (1 - \beta_{(m-1),n})x_{n} + \beta_{(m-1),n}T_{m-1,n}x_{n},
\end{aligned}
$$

(8)

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$. They proved the convergence of (8) to a fixed point of the contractive-like operator and also established some data dependence results for their proposed multi-step iterative scheme. In the same vein, Basarir and Sahin [4] studied (8) in the framework of CAT(0) spaces with $k$-strictly pseudocontractive mappings and they established the convergent results under some suitable conditions. Many authors have also used the multi-step algorithms to approximate fixed points of nonlinear mappings in CAT(0) spaces (see [32, 5, 36, 37] and the references therein).

Motivated by the results of Yildirim and Ozdemir [44], Gürsoy et. al [17] and the current research interest in this direction, we propose and study a multi-step iterative algorithm that comprises of a finite family of asymptotically $k_i$-strictly pseudocontractive mappings with respect to $p$, and a $p$-resolvent operator associated with a proper convex and lower semicontinuous function in a $p$-uniformly convex metric space. Also, we establish the $\Delta$-convergence of the proposed algorithm to a
common fixed point of finite family of asymptotically \( k_i \)-strictly pseudocontractive mappings which is also a minimizer of a proper convex and lower semicontinuous function. Furthermore, nontrivial numerical examples of our algorithm are given to show its applicability. Our results complement the results of Yildirim and Ozdemir [44], Gürsoy et. al. [17] and a host of other recent results in literature.

2. Preliminaries. In this section, we recall some results and definitions that will be needed in the proof of our main results.

Let \( \{x_n\} \) be a bounded sequence in a metric space \( X \) and \( r(\cdot, \{x_n\}) : X \to [0, \infty) \) be a continuous functional defined by \( r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n) \). The asymptotic radius of \( \{x_n\} \) is given by \( r(\{x_n\}) := \inf \{r(x, \{x_n\}) : x \in X\} \), while the asymptotic center of \( \{x_n\} \) is the set \( A(\{x_n\}) = \{ x \in X : r(x, \{x_n\}) = r(\{x_n\}) \} \). A sequence \( \{x_n\} \) in \( X \) is said to be \( \Delta \)-convergent to a point \( x \in X \) if \( A(\{x_n\}) = \{x\} \) for every subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \). In this case, we say that \( x \) is the \( \Delta \)-limit of \( \{x_n\} \) (see [11, 22]). The notion of \( \Delta \)-convergence in metric spaces was introduced and studied by Lim [27], and it is known as analogue of the notion of weak convergence in Banach spaces.

Remark 1. [35, 12] Let \( X \) be a complete \( p \)-uniformly convex metric space. Then,

(i) every bounded sequence in \( X \) has a unique asymptotic center,

(ii) every bounded sequence in \( X \) has a \( \Delta \)-convergent subsequence.

Definition 2.1. Let \( X \) be a complete convex metric space. A nonlinear mapping \( T : X \to X \) is said to be demiclosed at \( 0 \) if for any bounded sequence \( \{x_n\} \) in \( X \) such that \( \Delta - \lim_{n \to \infty} x_n = v \) and \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \), we have that \( v \in F(T) \).

Proposition 1. [14] For \( 1 < p \leq \infty \), let \( X \) be a \( p \)-uniformly convex metric space with parameter \( c > 0 \) and \( f : X \to (-\infty, \infty) \) be a proper convex and lower semicontinuous function. Then, for any \( \lambda > 0 \) and \( x \in X \), there exists a unique point, say \( J^f_\lambda(x) \in X \) such that

\[
J^f_\lambda(x) + \frac{1}{\lambda^{p-1}} d(J^f_\lambda(x), x)^p = \inf_{v \in X} \left( f(v) + d(v, x)^p \right).
\]

Lemma 2.2. [14] For \( 1 < p < \infty \), let \( X \) be a \( p \)-uniformly convex metric space with parameter \( c \geq 2 \) and \( f : X \to (-\infty, \infty) \) be a proper convex and lower semicontinuous function. Then, the \( p \)-resolvent operator \( J^f_\lambda \) of \( f \) is nonexpansive.

Lemma 2.3. [42] For \( 1 < p < \infty \), let \( X \) be a \( p \)-uniformly convex metric space with parameter \( c \geq 2 \) and \( f : X \to (-\infty, \infty) \) be a proper convex and lower semicontinuous function. Let \( J^f_\lambda \) be the \( p \)-resolvent mapping of \( f \) such that \( F(J^f_\lambda) \neq \emptyset \), then for \( \lambda > 0 \), we have the following:

(i) \( v \in F(J^f_\lambda) \) if and only if \( v \) is a minimizer of \( f \);
(ii) \( d(v, J^f_\lambda x)^p + d(J^f_\lambda x, x)^p \leq d(v, x)^p \) for all \( x \in X \) and \( v \in F(J^f_\lambda) \);
(iii) \( d(J^f_\lambda x, x)^p \leq d(J^{\mu} x, x)^p \) for \( \lambda < \mu \) and \( x \in X \).

3. Main results. In this section, we prove some results concerning the modulus of uniform convexity of \( p \)-uniformly convex metric spaces, and establish a \( \Delta \)-convergence theorem for the problem considered in this paper. We begin with the following definition.
Definition 3.1. [26] A convex metric space $X$ is called uniformly convex, if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $a, x, y \in X$, we have that $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq \epsilon r$ imply

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1 - \delta)r. \quad (9)$$

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \epsilon)$ for any given $r > 0$ and $\epsilon \in (0, 2]$, is called the modulus of uniform convexity.

Lemma 3.2. For $1 < p < \infty$, assume that $X$ is a $p$-uniformly convex metric space with parameter $c > 0$, then $X$ is uniformly convex with modulus of uniform convexity

$$\eta(r, \epsilon) := \frac{c\epsilon^p}{8p}.$$

Proof. Let $r > 0$, $\epsilon \in (0, 2]$, $a, x, y \in X$ such that $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq \epsilon r$. Applying (1), we have

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq \sqrt{\frac{1}{2}d(x, a)^p + \frac{1}{2}d(y, a)^p - \frac{c}{8}d(x, y)^p}$$

$$\leq \sqrt{r^p \left(1 - \frac{c\epsilon^p}{8p}\right)r}$$

Therefore, we conclude that the modulus of convexity of $X$ is $\frac{c\epsilon^p}{8p}$. □

If we set $p = 2$ and $c = 2$ in Lemma 3.2, we obtain the modulus of uniform convexity of a CAT(0) space (see [13], Lemma 2.5).

Lemma 3.3. For $1 < p < \infty$, let $X$ be a $p$-uniformly convex metric space with parameter $c > 0$. Then for any $r > 0$, $\epsilon \in (0, 2]$, $\lambda \in (0, 1)$, there exists $\delta := \frac{c\epsilon^p}{8p} \in (0, 1]$ such that for all $v, x, y \in X$, we have that $d(x, v) \leq r$, $d(y, v) \leq r$ and $d(x, y) \geq \epsilon r$ imply

$$d((1 - \lambda)x \oplus \lambda y, v) \leq (1 - 2\lambda(1 - \lambda)\frac{c\epsilon^p}{8p})r.$$

Proof. The proof follows from Definition 3.1 and Lemma 3.2. □

The proof of the following Lemma is similar to the proof of the result in (Lemma 2.5, [18]). However, we give the proof for readers convenience.

Lemma 3.4. For $1 < p < \infty$, let $X$ be a $p$-uniformly convex metric space with parameter $c > 0$. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b_1, b_2]$ for some $b_1, b_2 \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\limsup_{n \to \infty} d(x_n, x) \leq r$ and $\limsup_{n \to \infty} d(y_n, x) \leq r$ and $\lim_{n \to \infty} d((1 - \alpha_n)x_n \oplus \alpha_n y_n, x) = r$, for some $\alpha \geq 0$, then $\lim_{n \to \infty} d(x_n, y_n) = 0$.

Proof. The proof is trivial if $r = 0$. Now, suppose $r > 0$ and assume that $\lim_{n \to \infty} d(x_n, y_n) \neq 0$. If $n_0 \in \mathbb{N}$, then $d(x_n, y_n) \geq \frac{r}{2} > 0$ for some $\gamma > 0$ and for $n \geq n_1$. Since $\limsup_{n \to \infty} d(x_n, x) \leq r$ and $\limsup_{n \to \infty} d(y_n, x) \leq r$, we have that

$$d((1 - \alpha_n)x_n \oplus \alpha_n y_n, x) = r,$$
exist a constant $k$ as in Definition 3.5. Let $X$ be a $p$-uniformly convex metric space. A mapping $T : X \to X$ is said to be asymptotically $k$-strictly pseudocontractive with respect to $p$, if there exist a constant $k \in [0, 1)$ and a sequence $\{u_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim u_n = 1$ such that

$$d(T^n x, T^n y)^p \leq u_n d(x, y)^p + k(d(x, T^n x) + d(y, T^n y))^p \quad \forall \ x, y \in D, \ n \geq 1.$$  

We are now ready to present the main results of this paper.

**Definition 3.5.** Let $X$ be a $p$-uniformly convex metric space. A mapping $T : X \to X$ is said to be asymptotically $k$-strictly pseudocontractive with respect to $p$ if there exist a constant $k \in [0, 1)$ and a sequence $\{u_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim u_n = 1$ such that

$$d(T^n x, T^n y)^p \leq u_n d(x, y)^p + k(d(x, T^n x) + d(y, T^n y))^p \quad \forall \ x, y \in D, \ n \geq 1.$$  

We are now ready to present the main results of this paper.

**Lemma 3.6.** Let $X$ be a $p$-uniformly convex metric space with $1 < p < \infty$ and parameter $c \geq 2$. Let $f : X \to (-\infty, \infty)$ be a proper convex and lower semicontinuous function and let $T_i : X \to X$, $i = 1, 2, \ldots, m$ be a finite family of uniformly $L_i$-Lipschitzian and asymptotically $k_i$-strictly pseudocontractive mappings with respect to $p$, with $k_i \in [0, 1)$, $k_i = \max\{k_i, i = 1, 2, \ldots, m\}$, $k_i \in [0, 1)$, $i = 1, 2, \ldots, m$ and sequence $\{u_{in}\}_{n=1}^{\infty} \subseteq [1, \infty)$. Suppose that $\Gamma := \bigcap_{i=1}^{m} F(T_i) \cap \arg\min_{y \in X} f(y) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by

$$
\begin{align*}
x_{n+1} &= (1 - \alpha_{0,n})y_{1,n} \oplus \alpha_{0,n}T^n_1 y_{1,n}, \\
y_{1,n} &= (1 - \alpha_{1,n})y_{2,n} \oplus \alpha_{1,n}T^n_2 y_{2,n}, \\
\vdots \\
y_{i,n} &= (1 - \alpha_{i,n})y_{i+1,n} \oplus \alpha_{i,n}T^n_{i+1} y_{i+1,n}, \\
\vdots \\
y_{(m-2),n} &= (1 - \alpha_{(m-2),n})y_{(m-1),n} \oplus \alpha_{(m-2),n}T^n_{(m-1)} y_{(m-1),n}, \\
y_{(m-1),n} &= (1 - \alpha_{(m-1),n})z_n \oplus \alpha_{(m-1),n}T^n_m z_n, \\
z_n &= \arg\min_{y \in X} \left[f(y) + \frac{1}{p\lambda_n}d(y, x_n)^p\right] \quad \forall \ n \geq 1,
\end{align*}
$$

where $z_n = y_{mn,n} \forall \ n \geq 1$, and the following conditions are satisfied:

(C1) $0 < a \leq \alpha_{in} \leq 1 - 2k_i$,

(C2) $\sum_{n=1}^{\infty} (\max_{1 \leq i \leq m} u_{in} - 1) < \infty,$
\[(C3) \ L = \max\{L_i, i = 1, 2, \ldots, m\}.\]

Then, let
\[
(a) \ \text{lim}_{n \to \infty} d(x_n, v)^p \exists \forall v \in \Gamma,
\]
\[
(b) \ \text{lim}_{n \to \infty} d(J_{\alpha_n}^{-1}x_n, x_n) = 0 \text{ and}
\]
\[
(c) \ \text{lim}_{n \to \infty} d(T_i x_n, x_n) = 0 \text{ for each } i = 1, 2, \ldots, m.
\]

**Proof.** (a) Let \(v \in \Gamma\), then \(v = T_i(v)\) for each \(i = 1, 2, \ldots, m\) and by (i) in Lemma 2.3, we have that \(v = J_{\alpha_n}^{-1}v\), which implies from (11) and Lemma 2.2 that
\[
d(z_n, v) = d(J_{\alpha_n}^{-1}x_n, J_{\alpha_n}^{-1}v) \leq d(x_n, v). \tag{12}
\]

From (1) and Definition 3.5, we obtain
\[
d(y_{1,n}, v)^p = d((1 - \alpha_{1,n})y_{2,n} \oplus \alpha_{1,n}T_2^n y_{2,n}, v)^p \\
\leq (1 - \alpha_{1,n})d(y_{2,n}, v)^p + \alpha_{1,n}d(T_2^n y_{2,n}, v)^p \\
- \frac{c(1 - \alpha_{1,n})}{2} d(T_2^n y_{2,n}, y_{2,n})^p \\
\leq (1 - \alpha_{1,n})d(y_{2,n}, v)^p + \alpha_{1,n} \left[ u_{2,n}d(y_{2,n}, v)^p + k_2 d(T_2^n y_{2,n}, y_{2,n})^p \right] \\
- \frac{c(1 - \alpha_{1,n})}{2} d(T_2^n y_{2,n}, y_{2,n})^p \\
\leq u_{2,n}(1 - \alpha_{1,n})d(y_{2,n}, v)^p \\
+ \alpha_{1,n} \left[ u_{2,n}d(y_{2,n}, v)^p + k_2 d(T_2^n y_{2,n}, y_{2,n})^p \right] \\
- \frac{c(1 - \alpha_{1,n})}{2} d(T_2^n y_{2,n}, y_{2,n})^p \\
= u_{2,n}d(y_{2,n}, v)^p - \alpha_{1,n} \left[ \frac{c(1 - \alpha_{1,n})}{2} - k_2 \right] d(T_2^n y_{2,n}, y_{2,n})^p. \tag{13}
\]

In a similar way as in (13), we have
\[
d(y_{2,n}, v)^p = d((1 - \alpha_{2,n})y_{3,n} \oplus \alpha_{2,n}T_3^n y_{3,n}, v)^p \\
\leq u_{3,n}d(y_{3,n}, v)^p - \alpha_{2,n} \left[ \frac{c(1 - \alpha_{2,n})}{2} - k_3 \right] d(T_3^n y_{3,n}, y_{3,n})^p. \tag{14}
\]
From (13) and (14), we obtain
\[
d(y_{n,1}, v)^p \leq u_{2,n}u_{3,n}d(y_{3,n}, v)^p - \alpha_{2,n}u_{2,n}\left[\frac{c(1 - \alpha_{2,n})}{2} - k_3\right]d(T^n_{3}y_{3,n}, y_{3,n})^p
\]
\[- \alpha_{1,n}\left[\frac{c(1 - \alpha_{1,n})}{2} - k_2\right]d(T^n_{2}y_{2,n}, y_{2,n})^p
\]
\[= \prod_{i=2}^{3} u_{i,n}d(y_{3,n}, v)^p - \alpha_{n,2}u_{2,n}\left[\frac{c(1 - \alpha_{2,n})}{2} - k_3\right]d(T^n_{3}y_{3,n}, y_{3,n})^p
\]
\[- \alpha_{1,n}\left[\frac{c(1 - \alpha_{1,n})}{2} - k_2\right]d(T^n_{2}y_{2,n}, y_{2,n})^p
\]
\[\leq \prod_{i=2}^{4} u_{i,n}d(y_{4,n}, v)^p - \prod_{i=2}^{3} u_{i,n}\alpha_{3,n}\left[\frac{c(1 - \alpha_{3,n})}{2} - k_3\right]d(T^n_{4}y_{4,n}, y_{4,n})^p
\]
\[- u_{2,n}\alpha_{n,2}\left[\frac{c(1 - \alpha_{2,n})}{2} - k_3\right]d(T^n_{3}y_{3,n}, y_{3,n})^p
\]
\[- \alpha_{1,n}\left[\frac{c(1 - \alpha_{1,n})}{2} - k_2\right]d(T^n_{2}y_{2,n}, y_{2,n})^p
\]
\[
\vdots
\]
\[\leq \prod_{i=0}^{m} u_{(m-i),n}d(z_{n}, v)^p
\]
\[- \prod_{i=0}^{m-1} u_{(m-i),n}\alpha_{(m-1),n}\left[\frac{c(1 - \alpha_{(m-1),n})}{2} - k_m\right]d(T^n_{m}z_{n}, z_{n})^p
\]
\[- \cdots \prod_{i=0}^{2} u_{(m-i),n}\alpha_{3,n}\left[\frac{c(1 - \alpha_{3,n})}{2} - k_3\right]d(T^n_{4}y_{4,n}, y_{4,n})^p
\]
\[- u_{2,n}\alpha_{2,n}\left[\frac{c(1 - \alpha_{2,n})}{2} - k_3\right]d(T^n_{3}y_{3,n}, y_{3,n})^p
\]
\[- \alpha_{1,n}\left[\frac{c(1 - \alpha_{1,n})}{2} - k_2\right]d(T^n_{2}y_{2,n}, y_{2,n})^p
\].

(15)

Again, from (1), (11), (12), (15) and Definition 3.5, we have
\[
d(x_{n+1}, v)^p = d((1 - \alpha_{0,n})y_{1,n} + \alpha_{0,n}T^n_{1}y_{1,n}, v)^p
\]
\[\leq (1 - \alpha_{0,n})d(y_{1,n}, v)^p + \alpha_{0,n}\left[u_{1,n}d(y_{1,n}, v)^p + k_1d(T^n_{1}y_{1,n}, y_{1,n})^p\right]
\]
\[- c\alpha_{0,n}(1 - \alpha_{0,n})d(T^n_{1}y_{1,n}, y_{1,n})^p
\]
\[\leq u_{1,n}d(y_{1,n}, v)^p - \alpha_{0,n}\left[\frac{c(1 - \alpha_{0,n})}{2} - k_1\right]d(T^n_{1}y_{1,n}, y_{1,n})^p
\]
\[\leq \prod_{i=0}^{m} u_{(m-i),n}u_{1,n}d(z_{n}, v)^p
\]
\[- \prod_{i=0}^{m-1} u_{(m-i),n}u_{1,n}\alpha_{(m-1),n}\left[\frac{c(1 - \alpha_{(m-1),n})}{2} - k_m\right]d(T^n_{m}z_{n}, z_{n})^p
\]
\[- \cdots \prod_{i=0}^{2} u_{(m-i),n}u_{1,n}\alpha_{3,n}\left[\frac{c(1 - \alpha_{3,n})}{2} - k_3\right]d(T^n_{4}y_{4,n}, y_{4,n})^p
\].
(b) Without loss of generality, we assume that 
\[ \lim_{n \to \infty} d(T^n y_{3,n}, y_{3,n})^p \]

Thus, taking \( \lim \sup \) of both sides of (12) we have 
\[ \lim_{n \to \infty} d(T^n y_{2,n}, y_{2,n})^p \]

In a similar manner, if we take the \( \lim \inf \) of (17), we have 
\[ \lim_{n \to \infty} d(T^n y_{1,n}, y_{1,n})^p \]

By (ii) in Lemma 2.3, we have that 
\[ \lim_{n \to \infty} d(T^n x_{n}, y_{1,n})^p \]

Thus, from (18) and (20), we obtain that 
\[ \lim_{n \to \infty} d(J^{n-1} y_{n}, x_{n}) = \lim_{n \to \infty} d(z_{n}, x_{n}) = 0. \]

(c) Since \( \lim_{n \to \infty} u_{i,n} = 1, \ i = 1, 2, \ldots, m \), we obtain from (16) and (C1) that 
\[ \lim_{n \to \infty} d(T^n y_{i,n}, y_{i,n})^p = 0, \ i = 1, 2, \ldots, m. \]

By setting \( i = m \) in (23), we have 
\[ \lim_{n \to \infty} d(T^n y_{m,n}, y_{m,n})^p = \lim_{n \to \infty} d(T^n z_{n}, z_{n})^p = 0. \]
From (11) and (24), we have that
\[ d(y_{m-1}, z_n)^p \leq \alpha_{m-1} d(T^n_{m-1} z_n, z_n)^p \rightarrow 0, \text{ as } n \rightarrow \infty. \] (25)

Also, from (11), (23) and (25), we have that
\[ d(y_{m-2}, z_n)^p \leq (1 - \alpha_{m-2}) d(y_{m-1}, z_n)^p + \alpha_{m-2} d(T^n_{m-1} y_{m-1}, z_n)^p \leq (1 - \alpha_{m-2}) d(y_{m-1}, z_n)^p + \alpha_{m-2} [d(T^n_{m-1} y_{m-1}, y_{m-1}) + d(y_{m-1}, z_n)]^p \rightarrow 0, \text{ as } n \rightarrow \infty. \] (26)

Considering the following estimate
\[ d(T^n_{m-1} z_n, v) \leq d(T^n_{m-1} z_n, T^n_{m-1} y_{m-1}) + d(T^n_{m-1} y_{m-1}, y_{m-1}) + d(y_{m-1}, z_n) + d(z_n, v) \leq (1 + L) d(y_{m-1}, z_n) + d(T^n_{m-1} y_{m-1}, y_{m-1}) + d(z_n, v). \]

Then, from (19), (23) and (25), we obtain that
\[ \limsup_{n \rightarrow \infty} d(T^n_{m-1} z_n, v) \leq r^\frac{1}{p}. \]

Again, considering the following estimate
\[ d(T^n_{m-2} z_n, v) \leq d(T^n_{m-2} z_n, T^n_{m-2} y_{m-2}) + d(T^n_{m-2} y_{m-2}, y_{m-1}) + d(y_{m-1}, z_n) + d(z_n, v) \leq (1 + L) d(y_{m-1}, z_n) + d(T^n_{m-2} y_{m-2}, y_{m-1}) + d(z_n, v). \]

Then, from (19), (23) and (26), we obtain that
\[ \limsup_{n \rightarrow \infty} d(T^n_{m-2} z_n, v) \leq r^\frac{1}{p}. \]

Continuing in the same fashion, we have that
\[ \limsup_{n \rightarrow \infty} d(T^n_{m-i} z_n, v) \leq r^\frac{1}{p}, \text{ for } i = 1, 2, \ldots, m - 1, \] (27)

and
\[ \lim_{n \rightarrow \infty} d((1 - \alpha_{m-(i-1)}, n) z_n \oplus \alpha_{m-(i-1)}, n) T^n_{m-i} z_n, v) = r^\frac{1}{p}. \]

Hence by (19), (27) and Lemma 3.4, we obtain that
\[ \lim_{n \rightarrow \infty} d(T^n_{m-i} z_n, z_n) = 0, \text{ for } i = 1, 2, \ldots, m - 1, \] (28)

Now, observe that from Definition 3.5, we obtain that
\[ d(T^n x, T^n y)^p \leq [(u_n)^\frac{1}{p} d(x, y) + k^\frac{1}{p} (d(x, T^n x) + d(y, T^n y))]^p, \]
for each \( x, y \in X \). This implies that
\[ d(T^n x, T^n y) \leq (u_n)^\frac{1}{p} d(x, y) + k^\frac{1}{p} (d(x, T^n x) + d(y, T^n y)). \] (29)
Thus,
\[
d(T_{(m-1)}^n x_n, x_n) \leq d(T_{(m-1)}^n x_n, T_{(m-1)}^n z_n) + d(T_{(m-1)}^n z_n, z_n) + d(z_n, x_n)
\]
\[
\leq [1 + (u_{(m-1)}, n)^{\frac{1}{2}}] d(z_n, x_n) + (k_{m-1})^{\frac{1}{2}} d(x_n, T_{(m-1)}^n x_n)
\]
\[
+ [1 + (k_{m-1})^{\frac{1}{2}}] d(T_{(m-1)}^n z_n, z_n).
\]

Since \(1 - (k_{m-1})^{\frac{1}{2}} > 0\), we obtain from (22) and (28) that
\[
\lim_{n \to \infty} d(T_{(m-1)}^n x_n, x_n) = 0.
\]  
(30)

Again, from (29)
\[
d(T_{(m-2)}^n x_n, x_n) \leq d(T_{(m-2)}^n x_n, T_{(m-2)}^n z_n) + d(T_{(m-2)}^n z_n, z_n) + d(z_n, x_n)
\]
\[
\leq [1 + (u_{(m-2)}, n)^{\frac{1}{2}}] d(z_n, x_n) + (k_{m-2})^{\frac{1}{2}} d(x_n, T_{(m-2)}^n x_n)
\]
\[
+ [1 + (k_{m-2})^{\frac{1}{2}}] d(T_{(m-2)}^n z_n, z_n).
\]

Since \(1 - (k_{m-2})^{\frac{1}{2}} > 0\), we obtain from (22) and (28) that
\[
\lim_{n \to \infty} d(T_{(m-2)}^n x_n, x_n) = 0.
\]  
(31)

Continuing in this manner, we obtain that
\[
\lim_{n \to \infty} d(T_{i}^n x_n, x_n) = 0, \ i = 1, 2, \ldots, m - 1.
\]  
(32)

From (22) and (24), we obtain that
\[
d(T_m^nx_n, x_n) \leq d(T_m^nx_n, T_m^nz_n) + d(T_m^nz_n, z_n) + d(z_n, x_n)
\]
\[
\leq (1 + L)d(x_n, z_n) + d(T_m^nz_n, z_n) \to 0 \text{ as } n \to \infty.
\]  
(33)

Hence, by (32) and (33)
\[
\lim_{n \to \infty} d(T_m^n x_n, x_n) = 0, \ i = 1, 2, \ldots, m.
\]  
(34)

From (11), (22) and (34), we have
\[
d(y_{(m-1)}, n, x_n)^p \leq (1 - \alpha_{(m-1), n}) d(z_n, x_n)^p + \alpha_{(m-1), n} d(T_m^n z_n, x_n)^p
\]
\[
\leq (1 - \alpha_{(m-1), n}) d(z_n, x_n)^p + \alpha_{(m-1), n} [d(T_m^n z_n, T_m^n x_n)
\]
\[
+ d(T_m^n x_n, x_n)]^p
\]
\[
\leq (1 - \alpha_{(m-1), n}) d(z_n, x_n)^p + \alpha_{(m-1), n} [Ld(z_n, x_n)
\]
\[
+ d(T_m^n x_n, x_n)]^p \to 0, \text{ as } n \to \infty.
\]  
(35)

Similarly, we obtain from (11), (23) and (35) that
\[
d(y_{(m-2)}, n, x_n)^p \leq (1 - \alpha_{(m-2), n}) d(y_{(m-1)}, n, x_n)^p
\]
\[
+ \alpha_{(m-2), n} d(T_{m-1} y_{(m-1)}, n, x_n)^p
\]
\[
\leq (1 - \alpha_{(m-2), n}) d(y_{(m-1)}, n, x_n)^p
\]
\[
+ \alpha_{(m-2), n} [d(T_{m-1} y_{(m-1)}, n, y_{(m-1)}, n)
\]
\[
+ d(y_{(m-1)}, n, x_n)]^p \to 0, \text{ as } n \to \infty.
\]  
(36)

Continuing in a similar manner as in (35) and (36), and noting that \(y_{m,n} = z_n\), we obtain that
\[
\lim_{n \to \infty} d(y_{n}, x_n) = 0, \text{ for } i = 1, 2, \ldots, m.
\]  
(37)
Thus, for $i = 1$, we obtain from (23) and (37) that
\[
d(x_n, x_{n+1})^p \leq (1 - \alpha_{0,n})d(y_{1,n}, x_n)^p + \alpha_{0,n}d(T_i^n y_{1,n}, x_n)^p \\
\leq (1 - \alpha_{0,n})d(y_{1,n}, x_n)^p + \alpha_{0,n}d(T_i^n y_{1,n}, y_{1,n}) \\
\quad + d(y_{1,n}, x_n)^p \to 0 \text{ as } n \to \infty.
\] (38)

Since, $T$ is uniformly $L$-Lipchitzian, we obtain that
\[
d(x_n, T_i x_n) \leq d(x_n, T_i^n x_n) + d(T_i^n x_n, T_i x_n) \\
\leq d(x_n, T_i^n x_n) + L_i d(T_i^{n-1} x_n, x_n) \\
\leq d(x_n, T_i^n x_n) + Ld(T_i^{n-1} x_n, T_i^{n-1} x_{n-1}) \\
\quad + Ld(T_i^{n-1} x_{n-1}, x_{n-1}) + Ld(x_{n-1}, x_n) \\
\leq d(x_n, T_i^n x_n) + (L^2 + L)d(x_n, x_{n-1}) + Ld(T_i^{n-1} x_{n-1}, x_{n-1}).
\]

Therefore, utilizing (34) and (38), we obtain that
\[
\lim_{n \to \infty} d(x_n, T_i x_n) = 0, \text{ for } i = 1, 2, \cdots, m. \tag{39}
\]

\[\square\]

**Theorem 3.7.** Let $X$ be a complete $p$-uniformly convex metric space with $1 < p < \infty$ and parameter $c \geq 2$. Let $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function and let $T_i : X \to X$, $i = 1, 2, \cdots, m$ be a finite family of uniformly $L_i$-Lipschitzian and asymptotically $k_i$-strictly pseudocontractive mapping with respect to $p$, with $k \in [0, 1)$, $k = \max\{k_i, i = 1, 2, \cdots, m\}$, $k_i \in [0, 1)$, $i = 1, 2, \cdots, m$ and sequence $\{u_n\}_{n=1}^\infty \subseteq [1, \infty)$. Suppose that $\Gamma := \bigcap_{i=1}^m F(T_i) \cap \arg \min_{y \in X} f(y) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by (11) for which the following conditions are satisfied:

(C1) $0 < \alpha_n \leq 1 - 2k_i$,

(C2) $\sum_{n=1}^\infty \left( \max_{1 \leq i \leq m} u_i - 1 \right) < \infty$,

(C3) $L = \max\{L_i, i = 1, 2, \cdots, m\}$,

(C4) $0 < \lambda \leq \lambda_n^{-1}$ \quad $\forall \ n \geq 1$.

Then, the sequence $\{x_n\}$ $\Delta$-converges to an element of $\Gamma$.

**Proof.** From (22), (C4) and (iii) in Lemma 2.3, we have
\[
d(J_\lambda x_n, x_n) \leq d(J_{\lambda_n^{n-1}} x_n, x_n) \to 0 \text{ as } n \to \infty.
\]

Also, since $\{x_n\}$ is bounded, then it has a unique asymptotic center $A(\{x_n\}) = \{v\}$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $A(\{x_{n_k}\}) = \{v\}$. Then by (39), we have $\lim_{n \to \infty} d(T_i u_{n_k}, u_{n_k}) = 0$, $i = 1, 2, \cdots, m$. Thus, by Remark 1 and the demiclosedness of uniformly $L$-Lipschitzian and asymptotically $k$-strictly pseudocontractive mapping with respect to $p$, we obtain that $u \in \Gamma$. Furthermore, we have from Lemma 3.6(a) that $\lim_{n \to \infty} d(x_n, u)$ exists. Thus, the uniqueness of the asymptotic
Corollary 3. Let \( T \) be a complete \( p \)-uniformly convex metric space with \( 1 < p < \infty \) and parameter \( c \geq 2 \). Let \( f : X \to (-\infty, \infty] \) be a proper convex and lower semicontinuous function and let \( T : X \to X \), be a uniformly \( L \)-Lipschitzian and asymptotically \( k \)-strictly pseudocontractive mapping with respect to \( p \), with \( k \in [0, 1) \), and sequence \( \{u_n\}_{n=1}^{\infty} \subseteq [1, \infty) \). Suppose that \( \Gamma := F(T) \cap \arg \min_{y \in X} f(y) \neq \emptyset \) and for arbitrary \( x_1 \in X \), the sequence \( \{x_n\} \) is generated by

\[
\begin{align*}
    x_{n+1} &= (1 - \alpha_n)z_n + \alpha_n T^n z_n, \\
    z_n &= \arg \min_{y \in X} \left[ f(y) + \frac{1}{p\lambda_n} d(y, x_n)^p \right] \quad \forall \ n \geq 1,
\end{align*}
\]

where

(C1) \( 0 < a \leq \alpha_n \leq 1 - 2k, \)

(C2) \( \sum_{n=1}^{\infty} (u_n - 1) < \infty, \)

(C3) \( 0 < \lambda \leq \lambda_n^{-1}, \quad \forall \ n \geq 1. \)

Then, the sequence \( \{x_n\} \) \( \Delta \)-converges to an element of \( \Gamma \).

By setting \( m = 1 \) in Theorem 3.7, we obtain the following result:

**Corollary 1.** Let \( X \) be a complete \( p \)-uniformly convex metric space with \( 1 < p < \infty \) and parameter \( c \geq 2 \). Let \( f : X \to (-\infty, \infty] \) be a proper convex and lower semicontinuous function and let \( T : X \to X \), be a uniformly \( L \)-Lipschitzian and asymptotically \( k \)-strictly pseudocontractive mapping with respect to \( p \), with \( k \in [0, 1) \), and sequence \( \{u_n\}_{n=1}^{\infty} \subseteq [1, \infty) \). Suppose that \( \Gamma := F(T) \cap \arg \min_{y \in X} f(y) \neq \emptyset \) and for arbitrary \( x_1 \in X \), the sequence \( \{x_n\} \) is generated by

\[
\begin{align*}
    x_{n+1} &= (1 - \alpha_n)z_n + \alpha_n T^n z_n, \\
    z_n &= \arg \min_{y \in X} \left[ f(y) + \frac{1}{p\lambda_n} d(y, x_n)^p \right] \quad \forall \ n \geq 1,
\end{align*}
\]

which implies that \( v = u \). Therefore, \( \{x_n\} \) \( \Delta \)-converges to an element of \( \Gamma \). \( \square \)

Corollary 2. Let \( X \) be a complete \( CAT(0) \) space. Let \( f : X \to (-\infty, \infty] \) be a proper convex and lower semicontinuous function and let \( T_i : X \to X, \ i = 1, 2, \ldots, m \) be a finite family of uniformly \( L_i \)-Lipschitzian and asymptotically \( k \)-strictly pseudocontractive mapping with \( k \in [0, 1) \), \( k = \max\{k_i, \ i = 1, 2, \ldots, m\} \), \( k_i \in [0, 1) \), \( i = 1, 2, \ldots, m \) and sequence \( \{u_{in}\}_{n=1}^{\infty} \subseteq [1, \infty) \). Suppose that \( \Gamma := \bigcap_{i=1}^{m} F(T_i) \cap \arg \min_{y \in X} f(y) \neq \emptyset \) and for arbitrary \( x_1 \in X \), the sequence \( \{x_n\} \) is generated by \( (11) \) for which the following conditions are satisfied:

(C1) \( 0 < a \leq \alpha_{in} \leq 1 - 2k_i, \)

(C2) \( \sum_{n=1}^{\infty} (u_{in} - 1) < \infty, \)

(C3) \( L = \max\{L_i, i = 1, 2, \ldots, m\}, \)

(C4) \( 0 < \lambda \leq \lambda_n, \quad \forall \ n \geq 1. \)

Then, the sequence \( \{x_n\} \) \( \Delta \)-converges to an element of \( \Gamma \).

Corollary 3. Let \( X \) be a complete \( CAT(0) \) space. Let \( f : X \to (-\infty, \infty] \) be a proper convex and lower semicontinuous function and let \( T_i : X \to X, \ i = 1, 2, \ldots, m \) be a
finite family of nonexpansive mappings. Suppose that \( \Gamma := \bigcap_{i=1}^{m} F(T_i) \cap \arg\min_{y \in X} f(y) \neq \emptyset \) and for arbitrary \( x_1 \in X \), the sequence \( \{x_n\} \) is generated by (11) such that \( 0 < a \leq \alpha_n \leq b < 1 \) and \( 0 < \lambda \leq \lambda_n \), \( \forall n \geq 1 \). Then, the sequence \( \{x_n\} \) Δ-converges to an element of \( \Gamma \).

4. Numerical example. In this section, we demonstrate with numerical examples the applicability and efficiency of Algorithm (11). Throughout this section, we take \( \alpha_{i,n} = \frac{n+1}{3n+2} \), \( i = 0, 1 \) and \( \lambda_n = \frac{2n-1}{n} \) for all \( n \geq 1 \). Then, the conditions in Theorem 3.7 are satisfied. Hence, Algorithm (11) becomes

\[
\begin{aligned}
  x_{n+1} &= (1 - \alpha_0,n)y_{1,n} + \alpha_0,nT_1y_{1,n}, \\
  y_{1,n} &= (1 - \alpha_1,n)z_n + \alpha_1,nT_2z_n, \\
  z_n &= \arg\min_{y \in \mathbb{R}^2} \left[ f(y) + \frac{1}{\lambda_n} d(y, x_n)^2 \right] \forall n \geq 1.
\end{aligned}
\]

(41)

In what follows, we formally present the first numerical example of this paper.

Example 4.1. Let \( X = \mathbb{R}^2 \) be endowed with a metric \( d_X : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty) \) defined by

\[
d_X(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2^2 - x_2 - y_2^2 + y_2)^2} \forall x, y \in \mathbb{R}^2.
\]

Then, \( (\mathbb{R}^2, d_X) \) is a complete p-uniformly convex metric space with \( p = 2 \) and parameter \( c = 2 \), and with the geodesic joining \( x \) to \( y \) given by

\[
(1 - t)x \oplus ty = ((1 - t)x_1 + ty_1, (1 - t)x_2 + ty_1)^2 - (1 - t)(x_2^2 - x_2) - t(y_2^2 - y_2)
\]

(see [45, Example 5.2]).

Now define \( T_1 : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( T(x_1, x_2) = (x_1, 2x_1^2 - x_2) \). Then, for all \( x, y \in \mathbb{R}^2 \), we have that

\[
d_X(T_1x, T_1y) = \sqrt{(x_1 - y_1)^2 + (x_2^2 - 2x_1^2 + x_2 - y_2^2 + 2y_1^2 - y_2)^2} = \sqrt{(x_1 - y_1)^2 + (-x_1^2 + x_2 + y_1^2 - y_2)^2} = \sqrt{(x_1 - y_1)^2 + (x_2^2 - x_2 - y_1^2 + y_2)^2} = d_X(x, y),
\]

which implies that \( T_1 \) is nonexpansive in \( (\mathbb{R}^2, d_X) \). Also, define \( T_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( T_2(x_1, x_2) = (-x_1, x_2) \). Then \( T_2 \) is also nonexpansive. Note that \( T_1 \) is not nonexpansive in the classical sense.

Now, define \( f : \mathbb{R}^2 \times \mathbb{R} \) by \( f(x_1, x_2) = 100((x_2 + 1) - (x_1 + 1)^2)^2 + x_1^2 \). Then \( f \) is a proper convex and lower semicontinuous function in \( (\mathbb{R}^2, d_X) \) but not convex in the classical sense (see [45]).

We now consider the following 4 cases for our numerical experiments given in FIGURE 1 above.

Case 1: \( x_1 = (0.5, -0.25)^T \) and Case 2: \( x_1 = (-1.5, -3)^T \),

Case 3: \( x_1 = (0.5, 3)^T \) and Case 4: \( x_1 = (-0.5, 3)^T \).

Next, we consider an example for which the domain is not a simple closed and convex subset of a linear space.
| Iteration number (n) | Errors |
|---------------------|--------|
| 1                   | 0      |
| 2                   | 0.1    |
| 3                   | 0.2    |
| 4                   | 0.3    |
| 5                   | 0.4    |
| 6                   | 0.5    |
| 7                   | 0.6    |

**Figure 1.** Errors vs Iteration numbers(n) for **Example 4.1**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

**Example 4.2.** Let \( Y := \{(x, e^x) : x \in \mathbb{R}\} \) and \( X_n := \{(n, y) : y \geq e^n\} \) for each \( n \in \mathbb{Z}\). Set \( X := Y \cup \bigcup_{n \in \mathbb{Z}} X_n \) and equip it with a metric \( d : X \times X \to [0, \infty) \), defined by (see [8])

\[
d(x, y) = \begin{cases} 
\int_{x_1}^{y_1} ||\dot{\gamma}(t)||_2 dt + |x_2 - e^{x_1}| + |y_2 - e^{y_1}|, & \text{if } x_1 \neq y_1, \\
|x_2 - y_2|, & \text{if } x_1 = y_1,
\end{cases}
\]

(42)

where \( \dot{\gamma} \) is the derivative of the curve \( \gamma : \mathbb{R} \to X \), define by \( \gamma(t) := (t, e^t) \) for each \( t \in \mathbb{R}\). Then \( (X, d) \) is a complete \( p \)-uniformly convex metric space with \( p = 2 \) and parameter \( c = 2 \). Let \( f := ||.||^2_2 : X \to \mathbb{R} \). Then, \( f \) is proper, convex and lower semicontinuous in \( (X, d) \) (see [8, Example 7.1]).
Now, define $T_1, T_2 : X \to X$ by $T_1(x_1, x_2) = (x_1, e^{x_1})$ and $T_2(x_1, x_2) = (-x_1, e^{-x_1})$ for all $x = (x_1, x_2) \in X$. Then, we check that $T_1$ and $T_2$ are nonexpansive mappings. Indeed, for each $x, y \in X$, we have that

$$d(T_1 x, T_1 y) = d((x_1, e^{x_1}), (y_1, e^{y_1}))$$

$$= \begin{cases} \int_{x_1}^{y_1} \|\gamma(t)\|_2 dt + |e^{x_1} - e^{y_1}| + |e^{y_1} - e^{y_1}| & \text{if } x_1 \neq y_1, \\ |e^{x_1} - e^{y_1}| & \text{if } x_1 = y_1, \end{cases}$$

$$= \begin{cases} \int_{x_1}^{y_1} \|\gamma(t)\|_2 dt & \text{if } x_1 \neq y_1, \\ |e^{x_1} - e^{y_1}| & \text{if } x_1 = y_1, \end{cases}$$

$$\leq \begin{cases} \int_{x_1}^{y_1} \|\gamma(t)\|_2 dt + |x_2 - e^{x_1}| + |y_2 - e^{y_1}| & \text{if } x_1 \neq y_1, \\ |x_2 - y_2| & \text{if } x_1 = y_1, \end{cases}$$

$$= d(x, y).$$

Therefore, $T_1$ is nonexpansive. The proof that $T_2$ is also nonexpansive is very similar to that of $T_1$ above. So we shall simply omit it.

We also consider the following 4 cases for our numerical experiments for Example 4.2 given in FIGURE 2 below. 

**Case 1**: $x_1 = (0.5, e^{0.5})^T$ and **Case 2**: $x_1 = (1, 3)^T$.

**Case 3**: $x_1 = (-0.5, e^{0.5})^T$ and **Case 4**: $x_1 = (-2, 0.5)^T$.

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E-mail address: 218081063@stu.ukzn.ac.za
E-mail address: izuchukwu_c@yahoo.com, izuchukwuc@ukzn.ac.za
E-mail address: graceogwo@aims.ac.za
E-mail address: mewomoo@ukzn.ac.za