A Review on Analysis and Development of Mathematical Formulation Interpolants for a 4 Noded Polygon using Wachspress’ Interpolation Function

D Sandeep Kumar and P V Jeyakarthikeyan
Department of Mechanical Engineering, SRM Institute of Science and Technology, Kattankulathur, Chennai, Tamil Nadu, India.
Email: sandeepkumar_sb@srmuiv.edu.in

Abstract. This paper presents a detailed view of a methodology of using Wachspress’ Interpolation Function for the analysis and development of Interpolants by means of a mathematical formulation for a 4 Noded Element i.e. A 4 Noded Quadrilateral Element which has been already addressed by Stephane P.A. Bordas, Sundararajan Natarajan in their research work from where the authors have took have inspiration and worked for advancement. Here a 4 Noded Quadrilateral is taken for analysis and is sub divided into 4 equal Quadrilaterals using Smoothed Finite Element Analysis (SFEM) and the respective coordinates of the discretized element will be same as the local coordinates of the parent element without and coordinate transformation. Once it is done then we form the C Matrix (Shape function Values Matrix) which remains constant. Here we consider the line that connects two nodes and using that we will use to generate the Wachspress’ Interpolants. Later on multiplication of the B Matrix (First Derivative Matrix) with the C Matrix, D Matrix (Material Matrix) and the Area Matrix we can find the Element Stiffness Matrix and later the results will be validated with that of an already existing generic example.

1. Introduction
Finite Element Method also referred as Finite Element Analysis can be said as a tool to solve engineering problems to obtain approximates solutions of boundary value problems using computational techniques. In general this methodology generally works with meshing the domain that is needed to be analyzed with various types of elements i.e. rectangular, triangular or polygon elements with pre-defined shape function equations. The Finite Element process that is carried out consists of a 6 step process which follows as: Discretization, Selection of appropriate Shape functions, Development of finite element equation, Assemble element equation to obtain global equations in the form an equation, Solve for the unknown and Interpretation of the results. To solve such Finite Element Equation a stringent methodology has to be used which in this case is called the Numerical Integration Technique which follows the rule of approximate computation of an integral by means of computational technique and Gaussian Quadrature is the best among them which can be implemented for all the 1-D, 2-D and 3-D problems. Wachspress [1]. Around in the year 1970 proposed a formulation to generate shape functions on arbitrary polygons just on the basis of the geometry. These shape functions are highly unconventional when compared with that of the existing finite element literature but can be used extensively for any element i.e. element with nearly
large number of sides as 200 for development of a biomedical model of an eye through computational techniques. He introduced the philosophy of wedge function through which the shape function generation process occurs through the process of considering the line equations of the element. These formulation can be highly used for problems that needs to be solved with higher accuracy with greater number of sides. This is rather a highly appreciable thing as no such development has been recorded in the last 30 – 40 years with so much accuracy and precision. Gautam Dasgupta [2]. in his research study established a proper methodology to create shape functions. He used a 5 Noded polygon element i.e. pentagon and generated shape functions completely accurate to the existing Gaussian quadrature one. His work has been highly used for this specific research study. Stephane P.A. Bordas and Sundararajan Natarajan [3]. in their study has applied Smoothed Finite Element Analysis along with Wachspress for the development of shape functions for a 4 Noded Quadrilateral element and have also verified the results with the help of numerical tests. William Zeng and G.R.Lui [4]. in their research study have also made appreciable contribution in S-FEM when looked in prospect of a 4 Noded Quadrilateral Element. His methodology of strain smoothening has given higher convergence rates and also showed less sensitivity to mesh distortion. So, taking all these valuable research studies into consideration this study presents a methodology of formulation of Shape Functions for a 4-Noded Quadrilateral Element using Wachspress Methodology.

2. Smoothed Finite Element Method (SFEM) over a 4 Noded Quadrilateral Element
Smoothed Finite Element Method was first established by Liu GR, Dai KY and Nguyen TT [5]. and is a methodology to divide the complete element into a number of smoothing cells to establish a mesh free nodal integration. This in turn provide with Improved Dual Accuracy, Relative Inaccuracy to Mesh Distortion and is relatively softer than Finite Element Method (FEM). Here in this research work a Four Noded Quadrilateral Element is divided into smoothening cells as shown in the below figure:

![Figure 1. A Four Node Element divided into four smoothing cells.](image-url)
The relative values of Shape Functions at different positions of the Quadrilateral is shown in the below chart:

Table 1. Shape Function Values at different sites within an element for above figure 1.

| Size | Node 1 | Node 2 | Node 3 | Node 4 | Description       |
|------|--------|--------|--------|--------|-------------------|
| 1    | 1.0    | 0.0    | 0.0    | 0.0    | Field Node        |
| 2    | 0.0    | 1.0    | 0.0    | 0.0    | Field Node        |
| 3    | 0.0    | 0.0    | 1.0    | 0.0    | Field Node        |
| 4    | 0.0    | 0.0    | 0.0    | 1.0    | Field Node        |
| 5    | 0.5    | 0.5    | 0.0    | 0.0    | Side Midpoint     |
| 6    | 0.0    | 0.5    | 0.5    | 0.0    | Side Midpoint     |
| 7    | 0.0    | 0.0    | 0.5    | 0.5    | Side Midpoint     |
| 8    | 0.5    | 0.0    | 0.0    | 0.5    | Side Midpoint     |
| 9    | 0.25   | 0.25   | 0.25   | 0.25   | Intersection of two bimedians |

3. Wachspress Interpolants over a 4 Noded Quadrilateral Element

Over here we try to establish an approximation on an arbitrary quadrilateral having 4 Nodes named as [1, 2, 3, 4], in anti-clockwise direction. These four nodes forms four sides of the quadrilateral, (1-2, 2-3, 3-4, 4-1) respectively as shown in the below figure.

Figure 2. Four Noded Quadrilateral numbered in anti-clockwise direction.

Now, as said before that Wachspress’ Methodology is based on projective geometry so here we will have line equations of each line which is joined by two corresponding nodes i.e., \( l_1, l_2, l_3 \) and \( l_4 \).

Now, the four line equations established in parametric form is written as:

\[
\begin{align*}
l_1(x, y) &= a_1(x) + b_1(y) + c_1 = 0 \\
l_2(x, y) &= a_2(x) + b_2(y) + c_2 = 0 \\
l_3(x, y) &= a_3(x) + b_3(y) + c_3 = 0 \\
l_4(x, y) &= a_4(x) + b_4(y) + c_4 = 0
\end{align*}
\]  

Where \( a_i, b_i \) and \( c_i \) for \( i = 1, 2, 3 \) and 4 are real constants. Here a philosophy a wedge function has been introduced which vary linearly along the edges that are near to the nodes and vanish along the other nodes [6]. These wedge functions are named as \( w_1, w_2, w_3 \) and \( w_4 \) respectively.
Now, the respective wedge equations are:

\[ w_1(x,y) = k_1 l_2(x,y) l_3(x,y) \]  
\[ w_2(x,y) = k_2 l_3(x,y) l_4(x,y) \]  
\[ w_3(x,y) = k_3 l_4(x,y) l_1(x,y) \]  
\[ w_4(x,y) = k_4 l_1(x,y) l_2(x,y) \]

Where \( w_i \) are wedge functions and \( k_i \) are constants. Now, the Wachspress’ Interpolation Function for respective nodes is stated as follows:

\[ N_i(x,y) = \frac{w_i(x,y)}{\sum w_j(x,y)} \]

Now, using the above formulation we can easily devise the Value of Interpolants on all the four nodes of the quadrilateral.

4. Determination of Wachspress’ Interpolants using Analytical Methodology

Here the overall methodology is divided into three steps which are described below

1. Formulation of the line equation,
2. Obtaining the values of \( k_i \) for all the four nodes individually, and
3. Generation of Wachspress’ Interpolants.

Firstly, we take a line equation and evaluate using two point formulation to find the values of \( a_i, b_i \) and \( c_i \) as shown below

\[ l_i = a_i x + b_i y + c_i = 0 \]

Now using the two point line equation:

\[ \frac{x-x_i}{x_{i+1}-x_i} = \frac{y-y_i}{y_{i+1}-y_i} \]

Where \((x_i,y_i)\) & \((x_{i+1},y_{i+1})\) are coordinates on Nodes respectively. On further solving it, we will get

\[ (y_{i+1} - y_i)x + (x_i - x_{i+1})y + x_i y_i - x_i y_{i+1} + y_i x_{i+1} - x_i y_i \]

On comparing the above equation with (4a) we will obtain the values of \( a, b \) and \( c \):

\[ a_k = y_{i+1} - y_i \]  
\[ b_k = x_i - x_{i+1} \]  
\[ c_k = x_i y_i - x_i y_{i+1} + y_i x_{i+1} - x_i y_i \]

Now, the first step in determination of Wachspress’ Interpolants is to obtain the values of the unknown’s i.e. \( k_i \). For the same let us solve this for a generic quadrilateral
For the feasibility of calculation let us name the numerators as $N_{n_1}$ and so on and denominator as $N_{d_1}$ and so on.

$$N_{n_1}(x, y) = k_1 l_2 l_3$$  \hspace{1cm} (16)

Now for $N_1$ we will consider the values of Node 1 and similarly for $N_2$ and so on.

So, $N_{n_1}(x, y) = 1 \rightarrow (x, y_1)$ \hspace{1cm} (17)

Here the value of $N_{n_1}$ is 1 as it is linear along the edges. So,

$$k_1 = \frac{1}{l_2 l_3}$$

Where $l_2$ and $l_3$ are subsequent line equations.

Similarly,

$$k_2 = \frac{1}{l_3 l_4}$$

$$k_3 = \frac{1}{l_4 l_1}$$

$$k_4 = \frac{1}{l_1 l_2}$$
Now, once we obtain all the respective k values we put those in equation [5, 6, 7 & 8]. To obtain the respective values of \( w_i \).

Once, we obtain the values of \( w_i \) for all the 4 nodes we equate those value in equation number 3 and obtain the values of Wachspress’ Interpolants for all the four nodes i.e. \( N_1, N_2, N_3 \) & \( N_4 \).

5. Flowchart of the above Analytical Method to develop a MATLAB Program
Figure 4. MATLAB Program.

NOTE: - The values of C matrix has been obtained from Table No. 1 which is defined above.

6. Conclusion
Wachspress' Interpolation functions is flexible to use with any element and as the values of shape functions are completely dependent on the shape of the element there is no need use any predefined formulations for interpretation and it can be easily established separately according to the shape of the polygon and even the weights can also be established likewise by means of considering the line equation of two adjacent nodes. In the future scope of this research work numerical analysis has to be performed
on un-distorted and distorted meshes using the modified methodology as stated above to obtain results for comparison and improvement of the existing methodology. So, as a whole this paper provides with the layout of using Wachspress’ Interpolation Function in place of the Gaussian Quadrature Methodology.

Acknowledgements
The authors would like to acknowledge the support by SRM Institute of Science and Technology, Kattankulathur.

7. References
[1] Eugene L amd Wachspress 1973 A Rational Basis for Function Approximation II. Curved Sides J. Inst. Maths Applics 11 83-104
[2] Gautam Dasgupta and M. ASCE 2003 Interpolants within Convex Polygons: Wachspress’ Shape Functions Journal of Aerospace Engineering 16 0893-1321
[3] Bordas S P A and Sundararajan Natarajan 2010 On the approximation in the smoothed finite element method (SFEM) International Journal for Numerical Methods in Engineering. Int. J. Numer. Meth. Engng 81 660-670
[4] William Zeng and G R Lui Smoothed Finite Element Methods (S-FEM): An Overview and Recent Developments Arch. Computat. Methods Eng 9202-3
[5] Liu G, Nguyen Thoi T, Nguyen Xuan H, Dai K, Lam K 2009 On the essence and the evaluation of the shape functions for the smoothed Finite Element Method (SFEM) International Journal for Numerical Methods in Engineering 13 1863-9
[6] Wachspress E L 1975 A Rational Basis for Function Approximation Academic Press Inc. New York