Impact of the rail-pad multi-discrete model upon the prediction of the rail response

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Abstract. Wheel/rail vibration has many technical effects such as wear of the rolling surfaces, rolling noise, settlement of the ballast and subgrade etc. This vibration is depending on the rail pad characteristic and subsequently, it is important to have an accurate overview on the relation between the rail pad characteristic and the level of the wheel/rail vibration. To this end, much theoretical and experimental research has been developed in the past, and for the theoretical approach the track model, in general, and particularly, the rail pad model is of crucial importance. Usually, the rail pad model is discrete model one, neglecting the length of the rail pad. This fact is questionable because the sleepers span is only 4 times the rail pad length. Using the rail pad discrete model, the rail response is overestimated when the frequency of the excitation equals the pinned-pinned resonance frequency. In this paper, a multi-discrete model for the rail pad, consisting in many Kelvin-Voigt parallel systems, is inserted into an analytical model of the track. The track model is reduced to a rail taken as infinite Timoshenko beam, discretely supported via rail pad, sleeper and ballast. The influence of the number of Kelvin-Voigt systems of the rail pad model on the rail response is analysed.

1. Introduction

The railway is one of the most popular transport means which is addressed to both passenger and goods traffic. In our days, the railway knows continuous progress characterised by the development of the high-speed trains network for the passenger traffic and the spectacular increasing of both axle load and speed for the goods traffic.

When train travels along a track, the interaction between the vehicles and track induces vibration both in vehicles and track superstructure. This vibration has many causes, but the track irregularities and the roughness of the rolling surfaces are the most common ones. The wavelength of these irregularities spans over huge range from centimetres to one hundred meters and the frequency interval of the induced vibration extends from 0 to 3-4000 Hz.

The vehicle/track vibration of low frequency affects the dynamic performance of the vehicle, namely, the ride comfort and safety. The high frequency vibration affects the fatigue of the track structure and is the cause of many undesirable effects: wear of the rolling surfaces, including the short-pitch corrugation of the rail [1], rolling noise [2], ballast settlement and deterioration of the track geometry, ground vibration which affects the buildings situated next to the railway area etc.

The mitigation of the high frequency vibration of the track is related by the rail pad features and from this view point it is important to understand how these characteristics influences the level of the
wheel/rail vibration. To this end, much theoretical and experimental research has been developed in the past, and for the theoretical approach the track model, in general, and the rail pad model is of crucial importance.

Usually, the rail pad model is discrete model one, neglecting the length of the rail pad. This fact is questionable because the sleepers span is 4 times only the rail pad length. Using the rail pad discrete model, the rail response is overestimated when the frequency of the excitation equals the pinned-pinned resonance frequency. Many solutions have been proposed: rail with extra-damping [3], three-directional Kelvin-Voigt models [4] and multi-discrete model included in FEM track model.

In this paper, a multi-discrete model for the rail pad, consisting in many Kelvin-Voigt parallel systems, is inserted into an analytical model of the track. Track model is reduced to a rail taken as infinite Timoshenko beam, discretely supported via rail pad, sleeper, ballast block and subgrade. The influence of the number of Kelvin-Voigt systems of the rail pad model on the rail response is analysed, including influence on the spatial periodic feature of the track model.

2. Mechanical model of the track and governing equations

In this section, the track model as depicted in Fig. 1 is considered. In fact, only half-track model is considered based on the symmetric structure of the track. Consequently, we deal with the rail and its equidistant supports including rail pad, sleeper, ballast and subgrade.

Timoshenko beam model is used for the rail, and the sleepers are modelled using SDOF rigid bodies. The viscoelastic feature of the rail pad and ballast are modelled via Kelvin-Voigt systems. It must be underlined that \( n \) Kelvin-Voigt systems are necessary for the rail pad. Parameters for the track model are as follows: for the rail, \( m \) - the mass per length unit, \( E \) – the Young’s modulus, \( G \) – the transversal modulus, \( A \) the area of the cross-section, \( l \) – the area moment of inertia and \( \kappa \) – the share coefficient; for rail supports, \( k_r \) and \( c_r \) – the elastic and damping constants, \( l \) – the rail pad length, \( n \) – the number of Kelvin-Voigt systems for each rail pad, \( M \) – the sleeper mass, \( k_b \) and \( c_b \) – the elastic and damping constants of the ballast, \( d \) – the sleeper bay.

The displacements of the rail are the vertical displacement of its axis \( w(x,t) \) and the cross section rotation \( \theta(x,t) \), where \( x \) stands for the coordinate along the rail and \( t \) is the time. The vertical displacement of the sleeper \( i \) is \( z_i(t) \); \( s_i \) is its position in respect to the reference frame \( Oxz \).

The rail is subjected to a stationary force \( Q(t) \) which act in the \( x_o \) section in respect to the reference frame.
Applying the laws of mechanics, the equations of motion for the track model could be written as follows:

- for the rail

\[
G\frac{\partial^2 w(x,t)}{\partial x^2} - m \frac{\partial^2 w(x,t)}{\partial t^2} = -Q(t)\delta(x-x_0) + 
\]

\[
+ \sum_{i=L}^{n} \sum_{j=1}^{n} \left[ c_r \frac{\partial(w(s_{ij},t)-z_{ij}(t))}{\partial t} + k_r \left(w(s_{ij},t)-z_{ij}(t)\right) \right] \delta(x-x_{ij})
\]

\[
EI \frac{\partial^2 \theta(x,t)}{\partial x^2} + GSo \left( \frac{\partial w(x,t)}{\partial x} - \theta(x,t) \right) - \rho I \frac{\partial^2 \theta(x,t)}{\partial t^2} = 0 ;
\]

- for the \( i \) sleeper

\[
M \dddot{z}_i(t) + (c_r + c_b) \ddot{z}_i(t) + (k_r + k_b) z_i(t) - \sum_{j=1}^{n} \left[ c_r \frac{\partial w(s_{ij},t)}{\partial t} + k_r w(s_{ij},t) \right] = 0 ,
\]

where \( s_{ij} \) is the distance between the \( j \) Kelvin-Voigt system of the \( i \) rail pad and the reference frame.

When the steady-state harmonic behaviour is considered, both applied force and track components have harmonic time-dependence of \( \omega \) angular frequency. Using complex numbers, they can be written as

\[
\bar{w}(x,t) = w(x)e^{i\omega t}, \quad \bar{\theta}(x,t) = \theta(x)e^{i\omega t}, \quad \bar{z}_i(t) = z_i e^{i\omega t}, \quad \bar{Q}(t) = Q e^{i\omega t},
\]

where \( \bar{w}(x,t), \bar{\theta}(x,t), \bar{z}_i(t) \) and \( \bar{Q}(t) \) are the complex quantities associated to the real ones, and \( \bar{w}, \bar{\theta}, \bar{z}_i \) and \( \bar{Q} \) are the amplitudes.

Inserting (4) in Eqs. (1) – (3), it obtains

\[
GSo \left( \frac{d^2 \bar{w}}{dx^2} - \frac{d\bar{\theta}}{dx} \right) + \rho o^2 \bar{w} = -\bar{Q}(x-x_0) + \bar{k}_{ro} \sum_{i=L}^{n} \left( \bar{w}_{ij} - \bar{\bar{z}}_i \right) \delta(x-x_{ij}) ,
\]

\[
EI \frac{d^2 \bar{\theta}}{dx^2} + GSo \left( \frac{d\bar{w}}{dx} - \bar{\theta} \right) + \rho Io^2 \bar{\theta} = 0 ,
\]

\[
(-\omega^2 M + \bar{k}_r + \bar{k}_b) \bar{z}_i - \bar{k}_{ro} \sum_{j=1}^{n} \bar{w}_{ij} = 0 ,
\]

where \( \bar{w}_{ij} = \bar{w}(s_{ij}) \), \( \bar{k}_{r,b} = k_{r,b} + io c_{r,b} \), \( \bar{k}_{ro} = \bar{k}_r / n \).

From Eq. (7), we have

\[
\bar{z}_i = \bar{k} \sum_{j=1}^{n} \bar{w}_{ij} ,
\]

where \( \bar{k} = \frac{\bar{k}_{ro}}{\bar{k}_r + \bar{k}_b - \omega^2 M} \).

Inserting (8) in Eq. (5), the equations of the rail can be rewritten as

\[
Lq = p ,
\]
where it recognises the following matrix operator and vectors

$$
\mathbf{L} = \begin{bmatrix}
GSk \frac{d^2}{dx^2} + m\omega^2 & -GSk \frac{d}{dx} \\
GSk \frac{d}{dx} & El \frac{d^2}{dx^2} - GSk + \rho\omega^2 I
\end{bmatrix},
$$

$$
\mathbf{q} = \begin{bmatrix} \bar{w} \\ \bar{u} \end{bmatrix}^T, \quad \mathbf{p} = \begin{bmatrix} -G\delta(x - x_0) + \bar{K}_r \sum_{i \in Z} \sum_{j=1}^n \left( \bar{w}_{ij} - \bar{K} \sum_{j=1}^n \bar{w}_{ij} \right) \delta(x - s_{ij}) \end{bmatrix}^T.
$$

Eq. (9) can be transformed

$$
\mathbf{D}_T \mathbf{q} = \mathbf{L}^* \mathbf{p},
$$

where \( \mathbf{L}^* \) and \( \mathbf{D}_T \) stand for the matrix operators

$$
\mathbf{L}^* = \begin{bmatrix}
\frac{El}{GSk} \frac{d^2}{dx^2} - 1 + \frac{\rho\omega^2 I}{GSk} & \frac{d}{dx} \\
\left( 1 - \frac{H}{GSk} \right) \frac{d^2}{dx^2} + \frac{m\omega^2}{GSk}
\end{bmatrix}, \quad \mathbf{D}_T = \mathbf{L}^* \mathbf{L} = \mathbf{D}_T \mathbf{I}_2,
$$

with \( \mathbf{I}_2 \) – the 2x2 identity matrix and \( \mathbf{D}_T \) – the differential operator of the Timoshenko beam

$$
\mathbf{D}_T = \frac{El}{dx^4} + \left( \frac{m\omega^2}{GSk} + \rho\omega^2 I - H \right) \frac{d^2}{dx^2} + m\omega^2 \left( \frac{\rho\omega^2 I}{GSk} - 1 \right).
$$

Solution to Eq. (10) can be given in terms of the Green’s function of the \( \mathbf{D}_T \) operator, namely \( \mathbf{G}_T(x, x) \),

$$
\mathbf{q} = \int_{-\infty}^{\infty} \text{diag}(\mathbf{G}_T(x, \xi) \mathbf{G}_T(x, \xi)) \mathbf{L}^*_q \mathbf{p}_\xi d\xi,
$$

where

$$
\mathbf{G}_T(x, \xi) = -\frac{\beta_2 \exp(-\beta_1 |x - \xi|) + i\beta_1 \exp(-i\beta_2 |x - \xi|)}{2\beta_1 \beta_2 (\beta_1^2 + \beta_2^2) El}
$$

with

$$
\beta_{1,2} = \sqrt{\frac{\rho^2 \omega^4}{4El^2} \left( \frac{E}{GSk} + 1 \right)^2 + \frac{m\omega^2}{El} \left( 1 - \frac{\rho\omega^2 I}{GSk} \right)^2 + \frac{\rho\omega^2 I}{2E} \left( \frac{E}{GSk} + 1 \right)}.
$$

We note

$$
\mathbf{G}(x, x') = \begin{bmatrix} G_w(x, x') \\ G_0(x, x') \end{bmatrix} = \int_{-\infty}^{\infty} \mathbf{G}_T(x, \xi) \mathbf{L}^*_q \begin{bmatrix} \delta(\xi - x') \\ 0 \end{bmatrix} d\xi
$$

and then, we have

$$
\begin{bmatrix} G_w(x, x') \\ G_0(x, x') \end{bmatrix} = \begin{bmatrix}
\frac{El}{GSk} \frac{d^2 G_T(x, x')}{d\xi^2} - 1 + \frac{\rho\omega^2 I}{GSk} \\
\frac{GSk \frac{d^2}{d\xi^2}}{dG_T(x, x')} - \frac{GSk \frac{d^2}{d\xi^2} - GSk \frac{d}{d\xi}}{dG_T(x, x')}
\end{bmatrix}.
$$

Finally, the rail displacement is obtained
\[ \bar{w}(x) = -G_w(x, x_0)\overline{Q} + \sum_{i \in I} \sum_{j=1}^{n} \bar{w}_{ij} \left[ G_w(x, s_{ij}) - k \sum_{j=1}^{n} G_w(x, s_{ij}) \right]. \] (14)

Above, the rail displacement corresponding to each Kelvin-Voigt system of the rail pads, \( \bar{w}_{ij} \) is unknown. To calculate this, Eq. (14) is considered for \( x = s_{kl} \):

\[ \bar{w}(s_{kl}) = -G_w(s_{kl}, x_0)\overline{Q} + \sum_{i \in I} \sum_{j=1}^{n} \bar{w}_{ij} \left[ G_w(s_{kl}, s_{ij}) - k \sum_{j=1}^{n} G_w(s_{kl}, s_{ij}) \right] \] (15)

or

\[ \sum_{i \in I} \sum_{j=1}^{n} \bar{w}_{ij} A_{kl,ij} - \bar{w}_{kl} = G_{kl}^0 \overline{Q}, \] (16)

where

\[ A_{kl,ij} = \sum_{n} \bar{w}_{ij} A_{kl,ij} - G_{kl}^0, \]

\[ A_{kl,ij} = G_{kl}^0 \]

From practically view point, only \( N \) sleepers can be considered, with \( N \) sufficiently large to approximate the infinite length of the track. Consequently, Eq. (16) can be written as

\[ AW = B, \] (17)

where

\[ A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1N} \\ A_{21} & A_{22} & \ldots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1N} & A_{2N} & \ldots & A_{NN} \end{bmatrix}, \]

\[ W = [W_1 \ W_2 \ \ldots \ W_N]^T, \]

\[ B = [B_1 \ B_2 \ \ldots \ B_N]^T \]

with

\[ A_{ij} = \begin{bmatrix} A_{11, ij} - \delta_{ij} & A_{12, ij} & \ldots & A_{1N, ij} \\ A_{21, ij} & A_{22, ij} - \delta_{ij} & \ldots & A_{2N, ij} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1, ij} & A_{N2, ij} & \ldots & A_{NN, ij} - \delta_{ij} \end{bmatrix}, \]

\[ W_k = [\bar{w}_{k1} \ \bar{w}_{k2} \ \ldots \ \bar{w}_{kn}]^T, \]

\[ B_k = \overline{Q} \begin{bmatrix} G_{k1}^0 \ G_{k2}^0 \ \ldots \ G_{kn}^0 \end{bmatrix}^T. \]

Finally, when the amplitude of the harmonic force is set 1, the rail displacement becomes the rail receptance.

**3. Numerical application**

In the following lines, the numerical results derived from the above model and method are presented, considering a track with UIC 60 rail and concrete sleepers spaced at 0.6 m.

The parameters for the track model are as follows: \( m = 60 \text{ kg/m}, \rho = 7850 \text{ kg/m}^3, S = 7.69 \times 10^{-3} \text{ m}^2, I = 30.55 \times 10^{-6} \text{ m}^4, E = 210 \text{ GPa}, \mu = 81 \text{ GPa}, \kappa = 0.4, M = 160 \text{ kg}, k_r = 240 \text{ MN/m}, c_r = 55.8 \text{ kNs/m}, l = 0.16 \text{ m}, k_b = 70 \text{ MN/m} \) and \( c_b = 63.5 \text{ kNs/m}. \)
Figure 2. Rail receptance at mid-span: —, \( n = 1 \); - - - , \( n = 8 \).

Figure 3. Rail receptance above sleeper: —, \( n = 1 \); - - - , \( n = 8 \),

Receptance of the rail above sleeper and mid span within 10-1600 Hz is shown in Fig. 2 and Fig. 3, respectively, for \( n = 1 \) (pure rail pad discretely model) and \( n = 8 \) (rail pad multi-discretely model). Diagrams exhibit the typical behaviour of the rail: at low and mid-frequency, the rail behaves like a part of two DOF’s system, and at high frequency, the rail receptance shows at mid span a resonance frequency – the so-called pinned-pinned resonance - due to the first bending mode of the rail which vibrates like a wire fixed by sleepers and, above sleeper, the anti-resonance due to the dynamic stiffness of the sleeper. It should be observed that the number of Kelvin-Voigt systems of the rail pad model influences the numerical results within the pinned-pinned resonance and anti-resonance ranges.

Figure 4. Cross-receptance of the rail: —, \( n = 1 \); - - - - - , \( n = 2 \); - - - - , \( n = 4 \); - - - - - - - , \( n = 8 \).
Cross-receptance of the rail at the pinned-pinned resonance (1070 Hz) is displayed in Fig. 4 for \( n = 1, 2, 4, \) and 8. It observe the huge difference between the discretely model and multi-discretely one, even when the multi-discretely model includes only two Kelvin-Voigt systems. On the other hand, there is no significant difference when the multi-discretely model includes two or eight Kelvin-Voigt systems.

4. Conclusions
In this paper, the influence of the multi-discretely model of the rail pad on the rail response (receptance) is analysed using an analytical model for the track, including an infinite Timoshenko beam for rail, SDOF rigid bodies for sleepers and Kelvin-Voigt systems for both rail pad and ballast models.

The rail pad model has impact upon the rail receptance in two frequency ranges: at pinned-pinned resonance at mid span, and at the anti-resonance above sleeper. At pinned-pinned resonance, the discretely rail pad model overestimates the rail receptance in respect to the multi-discretely model. For instance, the rail receptance is 2.6 times higher in respect to one obtained using the rail pad model with two Kelvin-Voigt systems.

When many Kelvin-Voigt systems are used in the rail pad model, the differences between numerical results are not so high. For instance, the rail receptance given by the rail pad model with two Kelvin-Voigt systems is 1.1 times higher in respect to one calculated via the rail pad model with eight Kelvin-Voigt systems. However, the multi-discretely rail pad model is time consuming and a compromise between accuracy and effectiveness of the calculation seems to be a realistic option.

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6. References
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