GUP Modified Hawking Radiation in Bumblebee Gravity

Sara Kanzi and İzzet Sakallı

Physics Department, Arts and Sciences Faculty,
Eastern Mediterranean University, Famagusta,
North Cyprus via Mersin 10, Turkey.
(09.03.2018; Received)

Abstract

The effect of Lorentz symmetry breaking (LSB) on the Hawking radiation of Schwarzschild-like black hole found in the bumblebee gravity model (SBHBGM) is studied in the framework of quantum gravity. To this end, we consider Hawking radiation spin-0 (bosons) and spin-\(\frac{1}{2}\) particles (fermions), which go in and out through the event horizon of the SBHBGM. We use the modified Klein-Gordon and Dirac equations, which are obtained from the generalized uncertainty principle (GUP) to show how Hawking radiation is affected by the GUP and LSB. In particular, we reveal that, independent of the spin of the emitted particle, GUP causes a change in the Hawking temperature of the SBHBGM. Furthermore, we compute the semi-analytic greybody factors (for both bosons and fermions) of the SBHBGM. Thus, we reveal that LSB is effective on the greybody factor of the SBHBGM such that its redundancy decreases the value of the greybody factor. Our findings are graphically depicted.
I. INTRODUCTION

In spite of their overwhelming successes in describing nature, General Relativity (GR) (i.e., detection of the gravitational waves \[1, 2\] and observation of the shadow of the M87 supermassive black hole (BH) \[3\]) and Standard Model (SM) (i.e., detection of the Higgs boson \[4\]) of particle physics are incomplete theories. While Einstein’s theory of GR successfully describes gravity at a classical level, SM explains particles and the other three fundamental forces (electromagnetic, and the strong and weak nuclear forces) at a quantum level. The unification of GR and SM is a fundamental quest, and this success will necessarily lead us to a deeper understanding of nature. In the search for this unification, some quantum gravity theories (QGTs) have been proposed, but direct tests of their features are beyond the energy scale of the currently available experiments. Because, they will be observed on the Planck scale which is around \(10^{19} (GeV)\). However, it is possible that some signals of the QGT appear at sufficiently low energy scales and their effects can be observed in experiments on
existing energy scales. One of these signals could be related to the LSB\[5\].

The theory of LSB has been under intense research since the proposed SM Extension (SME)\[6–15\], which is an effective field theory that includes the SM, GR, and every possible operator that breaks the Lorentz symmetry. With the SME, further investigations of the LSB can be made in the context of high energy particle physics, nuclear physics, gravitational physics, and astrophysics. The simplest models that contain a vector field which dynamically breaks the Lorentz symmetry are called bumblebee models\[16–20\]. These models, although owning a simpler form, have interesting features such as rotations, boosts, and CPT violations. In a bumblebee gravity model (BGM), potential $V$ is included in the action $S_{BM}$, which evokes a vacuum expectation value (VEV) for the vector field. The potential $V$ is formed as a function of a scalar combination $\mathbb{N}$ of the vector $B_\mu$ and the metric $g_{\mu\nu}$ (plus the other matter fields, if there are any). The potential has a minimum at $\frac{dV}{d\mathbb{N}} = 0$. At the $V_{\text{min}}$, the bumblebee field $B_\mu$ incorporates a vacuum value shown by $\langle B_\mu \rangle = b_\mu$, which is the so-called vacuum vector. In fact, the vacuum vector is nothing but a background vector that gives rise to local (spontaneous) LSB\[21\]. The scalar of the BGM, in general, reads as $\mathbb{N} = (B_\mu B_\mu \pm b^2)$ in which $b$ is a constant having dimensions of mass $(M)$. Thus, the $V_{\text{min}}$ satisfies the condition of $\frac{dV}{d\mathbb{N}} = 0$ for $\mathbb{N} = 0$. Here, $b_\mu$ is spontaneously induced as a timelike vector abiding by $b_\mu b^\mu = -b^2$. For instance, the aether models\[22–25\] are based on a vector field, which is in the Lagrangian density of the system with a non-vanishing VEV. The vector field dynamically selects a preferred frame at each point in the considered space-time and spontaneously breaks the Lorentz invariance. This is a mechanism reminiscent of the breaking of local gauge symmetry described by the Higgs mechanism. In general, the subclass of aether models obeys the following action\[26\]:

$$S_{BM} = \int d^4x \left[ \frac{1}{16\pi G} \left( R + \frac{1}{4} B^\mu B^\nu R_{\mu\nu} \right) - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} - V\mathbb{N} \right], \quad (1)$$

where the parameter $\phi$, having dimensions of $M^{-2}$, denotes the coupling between the Ricci tensor ($R_{\mu\nu}$) and $B^\mu$. $B_{\mu\nu}$ is the bumblebee field strength:

$$B_{\mu\nu} = \nabla_\mu B_\nu - \nabla_\nu B_\mu. \quad (2)$$

As mentioned above, $V$ is the potential of the bumblebee field that drives the breaking of the Lorentz symmetry of the Lagrangian by collapsing onto a non-zero minimum at $\mathbb{N} = 0$ or $B_\mu B^\mu = \mp b^2$. In fact, $B_\mu$ is one of the Lorentz breaking coefficients and it shows a
preferred direction in which the equivalence-principle is locally broken for a certain Lorentz frame. Observations of Lorentz violation can emerge if the particles or fields interact with the bumblebee field \[26\]. It is worth noting that when a smooth quadratic potential is chosen as

\[ V = A R^2, \tag{3} \]

where \( A \) is a dimensionless constant, one gets the Nambu-Goldstone excitations (massless bosons) besides the massive excitations \[26\]. Besides, the linear Lagrange-multiplier potential is given by \( V = \lambda R \). These potentials \((1)\) and \((2)\) present also the breaking of the \(U(1)\) gauge invariance and other implications to the behavior of the matter sector, the photon, and the graviton. For a topical review (from experimental proposals to the test results) of the BGMs, the reader is referred to \[19\] and references therein. Furthermore, the studies using the bumblebee models have gained momentum for the last two decades. The vacuum solutions for the bumblebee field for purely radial, temporal-radial, and temporal-axial Lorentz symmetry breaking were obtained in \[27\]. New spherically static black hole (BH) \[28\] and traversable wormhole \[29\] solutions in the BGM have been recently discovered. Bluhm \[30\] discussed the Higgs mechanism in the BGM. The electrodynamics of the bumblebee fields was studied by \[31\] in which the bumblebee field was considered as a photon field. Propagation velocity of the photon field, along with its possible effects on the accelerator physics and cosmic ray observations, was also investigated. BGMs are also used to limit the likelihood of Lorentz violation in astrophysical objects such as the Sun \[32\]. For other studies demonstrating the physical effects (quasinormal modes, thermodynamics, etc.) of the bumblebee field, the reader may refer to \[33–42\] and references therein.

Hawking’s ground-breaking studies \[43,44\] can be considered as the onset of QGT \[45,46\]. Since then there have been numerous research papers on the subject of Hawking radiation (HR) in the literature (see, for instance, \[47–80\]). Several methods have been developed to calculate the HR of BHs \[81–84\]. In this study, we mainly focus on the quantum gravity effects on the HR of SBHBGM \[28\] in the tunneling paradigm. Although a number of QGTs have been proposed, however, physics literature does not as yet have a complete and consistent QGT. In the absence of a complete quantum description of the HR, we use effective models to describe the quantum gravitational behavior of the BH evaporation. In particular, string theory, loop quantum gravity, and quantum geometry predict the minimal
observable length on the Planck scale [85, 86], which leads to the GUP [87–123]:

\[ \Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \beta(\Delta p)^2 \right], \tag{4} \]

where \( \beta = \frac{\alpha_0}{M_p} \) in which \( M_p = \sqrt{\frac{\hbar c}{G}} \) denotes the Planck mass and \( \alpha_0 \) is the dimensionless parameter, which encodes the quantum gravity effects on the particle dynamics. The upper bound for \( \alpha_0 \) was obtained as \( \alpha_0 < 10^{21} \) [124]. Today, the effects of GUP on BHs have been extensively studied in the literature [125–127]. To amalgamate the GUP with the considered wave equation, the Wentzel-Kramers-Brillouin approximation [128] is generally used. Thus, one can obtain the quantum corrections to the HR of the BH [129, 130].

Since the GUP and LSB effects are high energy modifications of the QGT, it is interesting to investigate their combined effects. To this end, we study the GUP-assisted HR of bosons (spin-0) and fermions’ (spin-\( \frac{1}{2} \)) tunneling [131, 132] from the SBHBGM. Although the SBHBGM looks like the Schwarzschild BH, the differences in the Kretschmann scalars confirm that both BHs are physically different. The effects of spin and Lorentz-violating parameter \( L \) [133, 134] on the quantum corrected HR are analyzed. We also study the problem of low energy greybody factors [135, 136] for the bosons and fermions emitted by the SBHBGM. For this purpose, we implement a method developed by Unruh [137, 138]. It is also worth noting that Lorentz invariant massive gravity can be obtained dynamically from spontaneous symmetry breaking in a topological Poincare gauge theory [139]. Besides, BH radiation in massive gravity (selecting a preferred direction of time) naturally corresponds to violations of the Lorentz symmetry [140–142].

The outline of the paper is as follows: In Sec. 2, we briefly introduce the SBHBGM and discuss some of its basic features. Section 3 is devoted to the computation of GUP-corrected HR of the bosons’ tunneling from the SBHBGM. In Sec. 4, we compute the quantum tunneling rate for the fermions of the SBHBGM using the GUP-modified Dirac equation and derive the modified HR. In the following section, we derive the greybody factor of the SBHBGM. In Sec. 6, we summarize our results. (Throughout the paper, we use geometrized units: \( c = G = 1 \).)
II. SBHBGM SPACETIME

The Lagrangian density of the BGM [143, 144] yields the following extended vacuum Einstein equations

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T^B_{\mu\nu}, \]  

where \( G_{\mu\nu} \) and \( T^B_{\mu\nu} \) are the Einstein and bumblebee energy-momentum tensors, respectively. \( \kappa = 8\pi G_N \) is the gravitational coupling and \( T^B_{\mu\nu} \) is given by

\[ T^B_{\mu\nu} = -B_{\mu\alpha} B^\alpha_{\nu} - \frac{1}{4} B_{\alpha\beta} B^{\alpha\beta} g_{\mu\nu} - V g_{\mu\nu} + 2V' B_{\mu} B_{\nu} + \frac{\xi}{\kappa} \left[ \frac{1}{2} B^\alpha B^\beta R_{\alpha\beta} g_{\mu\nu} - B_{\mu} B^{\alpha} R_{\alpha\nu} 
- B_{\nu} B^{\alpha} R_{\alpha\mu} + \frac{1}{2} \nabla_\alpha \nabla_\mu (B^\alpha B_{\nu}) + \frac{1}{2} \nabla_\alpha \nabla_\nu (B^\alpha B_{\mu}) - \frac{1}{2} \nabla^2 (B_{\mu} B_{\nu}) - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \nabla_\beta (B^\alpha B^\beta) \right], \]  

where \( \xi \) is the real coupling constant (having dimension \( M^{-1} \)) that controls the non-minimal gravity-bumblebee interaction. From now on, the prime symbol shall denote the differentiation with respect to its argument. Meanwhile, there are other generic bumblebee models having non-zero torsion in the literature (see for instance [143]). In Eq. (6), the potential \( V \equiv V (\aleph) \) provides a non-vanishing VEV for \( B_{\mu} \). As it was stated above (see also [145, 146]), the VEV of the bumblebee field is determined when \( V = V' = 0 \). Taking the covariant divergence of the bumblebee Einstein equations [5] and using the contracted Bianchi identities, one gets

\[ \nabla^\mu T^B_{\mu\nu} = 0, \]  

which gives the covariant conservation law for the bumblebee total energy-momentum tensor \( T_{\mu\nu} \). Thus, Eq. (5) reduces to

\[ R_{\mu\nu} = \kappa T^B_{\mu\nu} + \frac{\xi}{4} g_{\mu\nu} \nabla^2 (B_{\alpha} B^\alpha) + \frac{\xi}{2} g_{\mu\nu} \nabla_\alpha \nabla_\beta (B^\alpha B^\beta). \]  

One can immediately see that when the bumblebee field \( B_{\mu} \) vanishes, we recover the ordinary Einstein equations. Recently, the vacuum solution in the BGM induced by the LSB has been derived by Casana et al. [28]. The solution is obtained when the bumblebee field \( B_{\mu} \) remains frozen in its VEV \( b_{\mu} \) [147, 148]. Namely, we have

\[ B_{\mu} = b_{\mu}, \quad \Rightarrow \quad b_{\mu\nu} \equiv \partial_\mu b_\nu - \partial_\nu b_\mu. \]  

6
Thus, the extended Einstein equations are found to be

\[ R_{\mu\nu} + \kappa b_{\mu\alpha} b^\alpha_{\nu} + \frac{\kappa}{4} b_{\alpha\beta} b^{\alpha\beta} g_{\mu\nu} + \xi b_\mu b^\alpha R_{\alpha\nu} + \xi b_\nu b^\alpha R_{\alpha\mu} - \frac{\xi}{2} b^{\alpha\beta} R_{\alpha\beta} g_{\mu\nu} - \]

\[ \frac{\xi}{2} \nabla_\alpha \nabla_{\mu} (b^\alpha b_\nu) - \frac{\xi}{2} \nabla_\alpha \nabla_\nu (b^\alpha b_\mu) + \frac{\xi}{2} \nabla^2 (b_\mu b_\nu) = 0. \]  

(10)

Assuming a spacelike background for \( b_\mu \) as

\[ b_\mu = [0, b_r(r), 0, 0], \]  

(11)

and using the condition \( b^\mu b_\mu = b^2 = \text{constant} \), LSB parameter \( (L) \) is defined as \( L = \xi b^2 \geq 0 \). A spherically symmetric static vacuum solution to Eq. (10) is obtained as follows

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + (1 + L) \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right), \]  

(12)

which we call it SBHBGM solution. This BH solution represents a purely radial Lorentz-violation outside a spherical body characterizing a modified BH solution. In the limit \( L \rightarrow 0 \) \( (b^2 \rightarrow 0) \), one can immediately see that the usual Schwarzschild metric is recovered. For the metric (12), the Kretschmann scalar becomes

\[ K = \frac{4(12M^2 + 4LMr + L^2r^2)}{r^6 (1 + L)^2}, \]  

(13)

which is different than the Kretschmann scalar of the Schwarzschild BH. It means that none of the coordinate transformations link the metric (12) to the usual Schwarzschild BH. When \( r = 2M \), Eq. (12) becomes finite: the coordinate singularity can be removed by applying a proper coordinate transformation. However, in the case of \( r = 0 \), physical singularity cannot be removed. So, we see that the behaviors of the physical \( (r = 0) \) and coordinate \( (r = r_h = 2M : \text{event horizon}) \) singularities do not change in the BGM.

The Hawking temperature of the metric (12) can be computed from Eq. (1), in which the surface gravity is given by

\[ \kappa = \nabla_\mu \chi^\mu \nabla_\nu \chi^\nu, \]  

(14)

where \( \chi^\mu \) is the timelike Killing vector field. Thus, the Hawking temperature of the SBHBGM (12) reads
\[ T_H = \frac{1}{4\pi \sqrt{-g_{tt}g_{rr}}} \left. \frac{dg_{tt}}{dr} \right|_{r=r_h} = \frac{1}{2\pi \sqrt{1 + L}} M \left|_{r=r_h} \right. = \frac{1}{8\pi M \sqrt{1 + L}}. \] \tag{15}

One can easily see from Eq. \ref{15} that the non-zero LSB parameter has the effect of reducing the Hawking temperature of a Schwarzschild BH.

### III. GUP ASSISTED HR OF SBHBGM: BOSONS’ TUNNELING

The generic Klein-Gordon equation within the framework of GUP is given by \cite{145}

\[- (i\hbar)^2 \partial_t \partial_t \Psi = \left[ (i\hbar)^2 \partial_{\mu} \partial_{\mu} + m^2 \right] \left\{ 1 - 2\beta \left[ (i\hbar)^2 \partial_{\mu} \partial_{\mu} + m^2 \right] \right\} \Psi, \tag{16}\]

where \(\beta\) and \(m\) are the GUP parameter and mass of the scalar particle, respectively.

Introducing the following ansatz for the wave function \(\Psi\)

\[ \Psi = \exp \left[ \frac{i}{\hbar} I(t, r, \theta, \varphi) \right], \tag{17}\]

where \(I(t, r, \theta, \varphi)\) is the classically forbidden action for quantum tunneling. Substituting Eq. \ref{17}, together with the metric functions of line-element \ref{12}, into Eq. \ref{16}, we get

\[ (f)^{-1} (\partial_t I)^2 = \left[ \frac{f}{1 + L} (\partial_t I)^2 + \frac{1}{r_h^2} (\partial_{\theta} I)^2 + \frac{1}{r_h^2 \sin^2 \theta} (\partial_{\varphi} I)^2 + m^2 \right] \times \left\{ 1 - 2\beta \left[ \frac{f}{1 + L} (\partial_t I)^2 + \frac{1}{r^2} (\partial_{\theta} I)^2 + \frac{1}{r^2 \sin^2 \theta} (\partial_{\varphi} I)^2 + m^2 \right] \right\}, \tag{18}\]

where

\[ f = 1 - \frac{2M}{r}. \tag{19}\]

It is easy to see that SBHBGM \ref{12} admits two Killing vectors \(<\partial_t, \partial_{\varphi}>\). The existence of these symmetries implies that we can assume a following separable solution for the action

\[ I = -\omega t + R(r) + S(\theta) + J\varphi, \tag{20}\]

where \(\omega\) and \(J\) denote the energy and angular momentum of the radiated particle, respectively. Substituting Eq. \ref{20} into Eq. \ref{18}, we obtain

\[ \frac{\omega^2}{f} = \left[ \frac{f}{1 + L} (\partial_t R)^2 + \frac{1}{r^2} \left( (\partial_{\theta} S)^2 + \frac{J^2}{\sin^2 \theta} \right) + m^2 \right]. \]
\[ \times \left\{ 1 - 2\beta \left[ \frac{f}{1 + L} (\partial_r R)^2 + \frac{1}{r^2} \left( \partial_\theta S \right)^2 + \frac{f^2}{\sin^2 \theta} \right] + m^2 \right\}. \tag{21} \]

We focus only on the radial trajectories in which only the \((r - t)\) sector is considered. Thus, one can set

\[ \frac{1}{r^2} \left( \left( \partial_\theta S \right)^2 + \frac{f^2}{\sin^2 \theta} \right) = e, \tag{22} \]

where \(e\) is a constant. So, Eq. (21) becomes

\[ \left[ \frac{f}{1 + L} (\partial_r R)^2 + e + m^2 \right] \left\{ 1 - 2\beta \left[ \frac{f}{1 + L} (\partial_r R)^2 + e + m^2 \right] \right\} = \frac{\omega^2}{f}, \tag{23} \]

which can be rewritten as a bi-quadratic equation as follows

\[ a(\partial_r R)^4 + b(\partial_r R)^2 + c = 0, \tag{24} \]

where

\[ a = -2\beta \frac{f^2}{(1 + L)^2}, \tag{25} \]

\[ b = \frac{f}{1 + L} \left[ 1 - 4\beta \left( m^2 + e \right) \right], \tag{26} \]

\[ c = e - 2\beta e^2 - 4\beta e m^2 + m^2 - \frac{\omega^2}{f} - 2\beta m^4. \tag{27} \]

Eq. (24) has four roots if \(b^2 - 4ac > 0\). We deduced from our analytical computations that only two roots \((R_{\pm})\) have physical meaning at the event horizon of the SBHBGM. These roots are

\[ R_{\pm} = \pm \int dr \sqrt{\frac{1 + L}{(1 + L)^2 \omega^2 - m^2 f + 2\beta m^4 f}} \left( 1 + 2\beta m^2 \right) = i\pi \omega M \sqrt{1 + L} \left( 1 + 2\beta m^2 \right). \tag{28} \]

It is worth noting that a \(+/-\) sign represents an outgoing/ingoing wave. On the other hand, the integrand of the integral \((28)\) has a pole at \(r = r_h\). Evaluating the integral by using the Cauchy’s integral formula, we obtain the imaginary part of the action as

\[ \text{Im} R_{\pm} \equiv \text{Im} I_{\pm} = \pm \pi \omega M \sqrt{1 + L} \left( 1 + 2\beta m^2 \right). \tag{29} \]

Thus, the tunneling rate of the scalar particles becomes

\[ \Gamma = \frac{P_{\text{emission}}}{P_{\text{absorption}}} = \frac{\exp(-2 \text{Im} I_+)}{\exp(-2 \text{Im} I_-)} = \exp(-4 \text{Im} I_+) = \exp \left[ -4 \omega M \pi \sqrt{1 + L} \left( 1 + 2\beta m^2 \right) \right]. \tag{30} \]
Recalling the expression of the Boltzmann factor

$$
\Gamma = \exp(-\frac{\omega}{T}),
$$

(31)

one can read the modified Hawking temperature ($\tilde{T}_H$) as follows

$$
\tilde{T}_H = \frac{1}{8\pi M\sqrt{1 + L\frac{1 + 2\beta m^2}{1 + 2m^2}}} = \frac{T_H}{(1 + 2\beta m^2)}.
$$

(32)

As can be seen above, after terminating the GUP parameter i.e., $\beta = 0$, one can recover the standard Hawking temperature (15).

IV. GUP-ASSISTED HR OF SBHBGM: FERMIONS’ TUNNELING

In this section, we aim to derive the modified Hawking temperature in the case of radiating fermions. To this end, we consider the Dirac equation, which is given by [146]

$$
\{ i\hbar \gamma^0 \partial_0 + [m + i\hbar \gamma^\mu (\Omega_\mu + \hbar \beta \partial_\mu)] (1 - \beta m^2 + \beta \hbar^2 g_{j k} \partial^j \partial^k) \} \psi = 0,
$$

(33)

where $\psi$ denotes the test spinor field. The $\gamma^\mu$ matrices for the metric (12) are given by

$$
\begin{align*}
\gamma^t &= \frac{1}{\sqrt{f(r)}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\
\gamma^r &= \sqrt{\frac{f(r)}{1 + L}} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \\
\gamma^\theta &= \frac{1}{r} \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \\
\gamma^\phi &= \frac{1}{r \sin \theta} \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix},
\end{align*}
$$

(34)

in which $\sigma^i$'s represent the well-known Pauli matrices [149]. One can easily ignore the terms having $\beta^2$ since $\beta$ is the effect of quantum gravity and it is a relatively very small quantity. For spin-up particles, the wave function can be expressed as [146]

$$
\Psi = \begin{pmatrix} 0 \\ X \\ 0 \\ Y \end{pmatrix} \exp \left( \frac{i}{\hbar} I \right),
$$

(35)

where $X$, $Y$, and $I$ are functions of coordinates ($t, r, \theta, \phi$). $I$ is the action of the emitted fermion. It is worth noting that here we only consider the spin-up case since it is physically same with the spin-down case; the only difference is the sign. Substitution of the wave function in the generalized Dirac equation (33) results in the following coupled equations
\[- iX \frac{1}{\sqrt{f}} \partial_t I - Y(1 - \beta m^2) \left[ \sqrt{\frac{f}{1 + L}} \partial_r I - X m \beta \left[ \frac{f}{1 + L} (\partial_r I)^2 + g_{\theta \theta} (\partial_\theta I)^2 + g_{\phi \phi} (\partial_\phi I)^2 \right] + Y \beta \sqrt{\frac{f}{1 + L}} \partial_r I \left[ \frac{f}{1 + L} (\partial_r I)^2 + g_{\theta \theta} (\partial_\theta I)^2 + g_{\phi \phi} (\partial_\phi I)^2 \right] + X m (1 - \beta m^2) = 0, \quad (36) \]

and

\[ iY \frac{1}{\sqrt{f}} \partial_t I - X (1 - \beta m^2) \left[ \sqrt{\frac{f}{1 + L}} \partial_r I - Y m \beta \left[ \frac{f}{1 + L} (\partial_r I)^2 + g_{\theta \theta} (\partial_\theta I)^2 + g_{\phi \phi} (\partial_\phi I)^2 \right] + X \beta \sqrt{\frac{f}{1 + L}} \partial_r I \left[ \frac{f}{1 + L} (\partial_r I)^2 + g_{\theta \theta} (\partial_\theta I)^2 + g_{\phi \phi} (\partial_\phi I)^2 \right] + Y m (1 - \beta m^2) = 0. \quad (37) \]

Then, one can get the following decoupled equations

\[ X \left\{ -(1 - \beta m^2) \sqrt{g_{\theta \theta}} \partial_\theta I + \beta \sqrt{g_{\theta \theta}} \partial_\theta I \left[ \frac{f}{1 + L} (\partial_r I)^2 + g_{\theta \theta} (\partial_\theta I)^2 + g_{\phi \phi} (\partial_\phi I)^2 \right] 
- i(1 - \beta m^2) \sqrt{g_{\phi \phi}} \partial_\phi I + i \beta \sqrt{g_{\phi \phi}} \partial_\phi I \left[ \frac{f}{1 + L} (\partial_r I)^2 + g_{\theta \theta} (\partial_\theta I)^2 + g_{\phi \phi} (\partial_\phi I)^2 \right] \right\} = 0, \quad (38) \]

and

\[ Y \left\{ -(1 - \beta m^2) \sqrt{g_{\theta \theta}} \partial_\theta I + \beta \sqrt{g_{\theta \theta}} \partial_\theta I \left[ \frac{f}{1 + L} (\partial_r I)^2 + g_{\theta \theta} (\partial_\theta I)^2 + g_{\phi \phi} (\partial_\phi I)^2 \right] 
- i(1 - \beta m^2) \sqrt{g_{\phi \phi}} \partial_\phi I + i \beta \sqrt{g_{\phi \phi}} \partial_\phi I \left[ \frac{f}{1 + L} (\partial_r I)^2 + g_{\theta \theta} (\partial_\theta I)^2 + g_{\phi \phi} (\partial_\phi I)^2 \right] \right\} = 0. \quad (39) \]

By using the fact that SBHBM spacetime has a timelike Killing vector \( \partial_t \), one can obtain the radial action by performing the separation of variables technique:

\[ I = -\omega t + W(r) + \Theta(\theta, \phi), \quad (40) \]

where \( \omega \) is the fermion energy. Substituting Eq. (40) into Eqs. (38) and (39), we find out the identical equations for \( X \) and \( Y \) equations. Thus, we have

\[ \beta \left[ \frac{f}{1 + L} (\partial_r W)^2 + g_{\theta \theta} (\partial_\theta \Theta)^2 + g_{\phi \phi} (\partial_\phi \Theta)^2 \left( \sqrt{g_{\theta \theta}} \partial_\theta \Theta + i \sqrt{g_{\phi \phi}} \partial_\phi \Theta \right) \right] + (1 - \beta m^2) \left( \sqrt{g_{\theta \theta}} \partial_\theta \Theta + i \sqrt{g_{\phi \phi}} \partial_\phi \Theta \right) = 0, \quad (41) \]
or

\[
\left( \sqrt{g^{\theta\theta}} \partial_\theta \Theta + i \sqrt{g^{\varphi\varphi}} \partial_\varphi \Theta \right) \left[ \beta \left( \frac{f}{1+L} (\partial_r W)^2 + g^{\theta\theta}(\partial_\theta \Theta)^2 + g^{\phi\phi}(\partial_\phi \Theta)^2 + m^2 \right) - 1 \right] = 0.
\]

(42)

It is obvious that the expression inside the square brackets can not vanish; thus, one should have

\[
\left( \sqrt{g^{\theta\theta}} \partial_\theta \Theta + i \sqrt{g^{\varphi\varphi}} \partial_\varphi \Theta \right) = 0,
\]

(43)

and the solution of \( \Theta \), therefore, does not contribute to the tunneling rate. The above result helps us to simplify Eqs. (36) and (37) [with ansatz (40)] as follows

\[
X \left\{ \frac{i \omega}{\sqrt{f}} - m \beta \left[ \frac{f}{1+L} (\partial_r W)^2 \right] + m \left( 1 - \beta m^2 \right) \right\} + 
Y \left\{ - (1 - \beta m^2) \sqrt{\frac{f}{1+L}} \partial_r W + \beta \sqrt{\frac{f}{1+L}} \partial_r W \left[ \frac{f}{1+L} (\partial_r W)^2 \right] \right\} = 0,
\]

(44)

and

\[
Y \left\{ \frac{i \omega}{\sqrt{f}} - m \beta \left[ \frac{f}{1+L} (\partial_r W)^2 \right] + m \left( 1 - \beta m^2 \right) \right\} + 
X \left\{ - (1 - \beta m^2) \sqrt{\frac{f}{1+L}} \partial_r W + \beta \sqrt{\frac{f}{1+L}} \partial_r W \left[ \frac{f}{1+L} (\partial_r W)^2 \right] \right\} = 0.
\]

(45)

In the simple way, one can set

\[
XA + YB = 0,
\]

(46)

\[
YA + XB = 0,
\]

(47)

where

\[
A = \frac{i \omega}{\sqrt{f}} - m \left[ \frac{f \beta}{1+L} (\partial_r W)^2 + 1 - \beta m^2 \right],
\]

(48)

and
\[ B = -(1 - \beta m^2) \sqrt{\frac{f}{1 + L}} \partial_r W + \beta \sqrt{\frac{f}{1 + L}} \partial_r W \left[ \frac{f}{1 + L} (\partial_r W)^2 \right]. \] (49)

After making some manipulations, we see that \( A_2 - B_2 = 0 \)

\[
\left( \frac{i\omega}{\sqrt{f}} - m\beta \left[ \frac{f}{1 + L} (\partial_r W)^2 \right] + m(1 - \beta m^2) \right)^2 - \left( -(1 - \beta m^2) \sqrt{\frac{f}{1 + L}} \partial_r W + \beta \sqrt{\frac{f}{1 + L}} \partial_r W \left[ \frac{f}{1 + L} \partial_r^2 W \right] \right)^2 = 0, \] (50)

which yields

\[ L_6 (\partial_r W)^6 + L_4 (\partial_r W)^4 + L_2 (\partial_r W)^2 + L_0 = 0, \] (51)

where

\[ L_6 = \beta^2 f \left( \frac{f}{1 + L} \right)^3, \] (52)

\[ L_4 = \beta \left( \frac{f}{1 + L} \right)^2 f \left( m^2 \beta - 2 \right), \] (53)

\[ L_2 = \frac{f^2}{1 + L} \left( \frac{2i\omega m}{\sqrt{f}} + (1 - \beta m^2) \left( 1 + 2m^2 \beta \right) \right), \] (54)

\[ L_0 = \omega^2 - m^2 f \left( 1 - \beta m^2 \right)^2 - 2i\omega m \sqrt{f} \left( 1 - \beta m^2 \right), \] (55)

ignoring \( O(\beta^2) \) terms, Eq. (51) reduces to

\[ L_4 (\partial_r W)^4 + L_2 (\partial_r W)^2 + L_0 = 0. \] (56)

Therefore, we have

\[ W_\pm = \pm \int dr \sqrt{(1 + L) \left( \omega^2 + m^2 f \right)} \left[ 1 + \beta \left( m^2 + \frac{\omega^2}{f} \right) \right] \]

\[ \approx \pm i\pi \omega r_+ (1 + 2\beta m^2) \] (57)

\[ = \pm 2\pi M \omega \sqrt{1 + L} (1 + 2\beta m^2), \]
in another form

$$\text{Im} W_\pm = 2\pi M \omega \sqrt{1 + L} \left(1 + 2\beta m^2\right).$$  \hspace{1cm} (58)

Recalling Eq. (30), we find the tunneling rate of fermions as follows

$$\Gamma \simeq \exp \left(-4 \text{Im} W_+\right) = \exp \left(8\pi M \omega \sqrt{1 + L} \left(1 + 2\beta m^2\right)\right).$$ \hspace{1cm} (59)

Thus, with the help of the Boltzmann factor (31), we get the GUP-consolidated temperature of the SBHBGM via the emission of the fermions:

$$T = \frac{1}{8\pi M \sqrt{1 + L} \left(1 + 2\beta m^2\right)} = \frac{T_0}{\left(1 + 2\beta m^2\right)},$$ \hspace{1cm} (60)

in which $T_0$ represents the original Hawking temperature (15)

$$T_0 = \frac{1}{8\pi M \sqrt{1 + L}}.$$ \hspace{1cm} (61)

The above result (60) shows that GUP corrected temperature deviates from the standard Hawking temperature.

V. GREYBODY FACTORS OF SBHBGM

In this section, we shall first derive the effective potentials of the scalar and fermion perturbations in the geometry of the SBHBGM. Then, the obtained effective potentials will be used for computing the greybody factors of the SBHBGM. The results will be depicted with some plots and discussed.

A. Scalar Perturbations of SBHBGM

The massless Klein- Gordon equation is given by

$$\Box \Psi = 0,$$ \hspace{1cm} (62)

where the D’Alembert operator is denoted by the box symbol and $\Box = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu)$. For the SBHBGM (12), we have
\[
\sqrt{-g} = r^2 \sin \theta \sqrt{1 + L},
\]

and therefore Eq. (62) reads

\[
\Box \Psi = \frac{1}{f} \partial_t^2 - \frac{1}{r^2(1 + L)} \left( 2rf \partial_r \Psi + r^2 \partial_r f \partial_r \Psi + r^2 f \partial_r^2 \Psi \right) + \frac{1}{r^2 \sin \theta} \left( - \cos \theta \partial_{\theta} \Psi - \sin \theta \partial_{\theta}^2 \Psi \right) - \frac{1}{r^2 \sin^2 \theta} \partial_{\phi}^2 \Psi.
\]

We invoke the following ansatz for the scalar field \( \Psi \) in the above equation:

\[
\Psi = p(r) A(\theta) e^{-i \omega t} e^{im \phi},
\]

so that we have

\[
\Box \Psi = - \frac{\omega^2}{f} - \frac{1}{p(1 + L)} \left[ \frac{2f}{r} p' + f' p' + fp'' \right] - \frac{1}{r^2 A \sin \theta} \left[ \cos \theta A' + \sin \theta A'' - \frac{m^2}{\sin^2 \theta} A \right] = 0.
\]

When one changes the independent variable \( \theta \) to \( \cos^{-1} z \), the angular equation is found to be

\[
(1 - z^2) A'' + 2z A' - \left[ m^2 + \frac{\lambda_s}{1 + L} (1 - z^2) \right] A = 0,
\]

where \( \lambda_s \) denotes the eigenvalue. The above equation is nothing but the Legendre differential equation when one sets

\[
\lambda_s = -l(l + 1)(1 + L).
\]

The radial equation then becomes

\[
p'' + p' \left( \frac{f'}{f} + \frac{2}{r} \right) + \left[ \frac{1 + L}{f^2} \omega^2 + \frac{\lambda_s}{r^2 f} \right] p = 0.
\]

Introducing a new variable \( p = \frac{u}{r} \), we get a Schrödinger-like wave equation

\[
\frac{d u^2}{d r^2} + (\omega^2 - V_{eff}) u = 0,
\]
where $r_*$ is the tortoise coordinate defined by

$$r_* = \sqrt{1 + L \int \frac{dr}{f}}. \quad (69)$$

The effective potential felt by the scalar field then becomes

$$V_{eff} = f \left[ \frac{f'}{(1 + L)r} + \frac{l(l + 1)}{r^2} \right]. \quad (70)$$

[FIG. 1: $V_{eff}$ versus $\frac{r_*}{M}$ graph. The plots are governed by Eq. (70).]

It is obvious from Fig. (1) that the effective potential vanishes both at the event horizon of the SBHBG $T$ and at spatial infinity. This behavior will help us to analytically derive the greybody factor of the scalar field emission from the SBHBG $T$.

**B. Fermion Perturbations of SBHBGM**

In this subsection, we shall employ the Newman-Penrose formalism [150] to find the effective potential of the fermion fields propagating in the geometry of the SBHBGM. Chandrasekar-Dirac equations (CDEs) are given by [151]

$$(D + \varepsilon - \rho) F_1 + (\bar{\delta} + \pi - \alpha) F_2 = \mu^* G_1,$$
\[(\delta + \beta - \tau) F_1 + (\Delta + \mu - \gamma) F_2 = i\mu^* G_2,\]

\[(D + \bar{\rho}) G_2 - (\delta + \bar{\pi} - \bar{\alpha}) = i\mu^* F_2,\]

\[(\Delta + \bar{\rho} - \bar{\tau}) G_1 - (\bar{\delta} + \bar{\beta} - \bar{\gamma}) G_2 = i\mu^* F_1,\]  \hspace{1cm} (71)

where \(F_1, F_2, G_1,\) and \(G_2\) represent the components of the wave functions or the so-called Dirac spinors. \(\varepsilon, \rho, \pi, \alpha, \beta, \tau, \mu,\) and \(\gamma\) are the spin coefficients, and a bar over a quantity denotes complex conjugation. The non-zero spin coefficients are found to be

\[\varepsilon = \gamma = \frac{\sqrt{2} f'}{8 \sqrt{f} \sqrt{1 + L}}, \quad \mu = \rho = -\frac{\sqrt{2} f}{2 r \sqrt{1 + L}} , \quad \beta = -\alpha = -\frac{\sqrt{2} \cot \theta}{4r}.\] \hspace{1cm} (72)

To have separable solutions for the CDEs \((71)\), we introduce the following ansatzes

\[F_1 = f_1(z) A_1(\theta) \exp[i(\omega t + m\phi)],\]

\[G_1 = g_1(z) A_2(\theta) \exp[(\omega t + m\phi)],\]

\[F_2 = f_2(z) A_3(\theta) \exp[i(\omega t + m\phi)],\]

\[G_2 = g_2(z) A_4(\theta) \exp[(\omega t + m\phi)],\] \hspace{1cm} (73)

where \(m\) denotes the azimuthal number and \(\omega\) is the frequency of the spinor fields. Since the directional derivatives \([151]\) are defined by \(D = \ell^a \partial_a, \Delta = n^a \partial_a,\) and \(\delta = m^a \partial_a,\) we have

\[D = \frac{1}{\sqrt{2} f} \partial_t + \frac{f}{2(1 + L)} \partial_r,\]

\[\Delta = \frac{1}{\sqrt{2} f} \partial_t - \frac{f}{2(1 + L)} \partial_r,\]

\[\delta = \frac{1}{r \sqrt{2}} \partial_\theta + \frac{i}{r \sqrt{2} \sin \theta} \partial_\phi,\]

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\[ \delta = \frac{1}{r \sqrt{2}} \partial_{\theta} - \frac{i}{r \sqrt{2} \sin \theta} \partial_{\varphi}. \]  

(74)

After substituting Eqs. (72-74) into the CDEs (71), one can obtain the following set of equations:

\[
\begin{align*}
\left[ \frac{i \omega}{\sqrt{f}} + \frac{r \sqrt{f}}{\sqrt{1 + L}} \partial_{r} + \frac{rf'}{4 \sqrt{f} (1 + L)} + \frac{\sqrt{f}}{\sqrt{1 + L}} \right] f_{1} & \frac{f_{2}}{f_{2}} \frac{A_{3}}{A_{1}} - i \mu r g_{1}A_{2} = 0, \\
\left[ \frac{i \omega}{\sqrt{f}} - \frac{r \sqrt{f}}{\sqrt{1 + L}} \partial_{r} - \frac{rf'}{4 \sqrt{f} (1 + L)} - \frac{\sqrt{f}}{\sqrt{1 + L}} \right] f_{2} & \frac{f_{1}}{f_{1}} \frac{A_{1}}{A_{3}} - i \mu r g_{2}A_{4} = 0,
\end{align*}
\]

(75)

where \( \mu = \sqrt{2} \mu^{*} \). \( \tilde{L} \) and \( \tilde{L}^{\dagger} \) are the angular operators, which are known as the laddering operators:

\[ \tilde{L} = \partial_{\theta} + \frac{m}{\sin \theta} + \frac{\cot \theta}{2}, \quad \tilde{L}^{\dagger} = \partial_{\theta} - \frac{m}{\sin \theta} + \frac{\cot \theta}{2}, \]  

(76)

which lead to the spin-weighted spheroidal harmonics [152, 153] with the following eigenvalue [154, 155]:

\[ \lambda_{f} = - \left( l + \frac{1}{2} \right). \]  

(77)

By considering \( g_{1} = f_{2}, g_{2} = f_{1}, A_{2} = A_{1}, \) and \( A_{4} = A_{3} \), then we reduce the CDEs (75) to two coupled differential equations:

\[
\begin{align*}
\frac{r \sqrt{f}}{\sqrt{1 + L}} \left( \frac{d}{dr} + \frac{i \omega \sqrt{1 + L}}{f} + \frac{f'}{4 f} + \frac{1}{r} \right) g_{2} = (- \lambda_{f} + i \mu r) g_{1}, \\
\frac{r \sqrt{f}}{\sqrt{1 + L}} \left( \frac{d}{dr} - \frac{i \omega \sqrt{1 + L}}{f} + \frac{f'}{4 f} + \frac{1}{r} \right) g_{1} = (- \lambda_{f} - i \mu r) g_{2}.
\end{align*}
\]

(78)

Moreover, if one sets
\[ g_1(r) = \frac{\Psi_1}{r}, \quad \text{and} \quad g_2(r) = \frac{\Psi_2}{r}, \quad (80) \]

and substitute them into Eqs. (78) and (79), after some manipulations, we get:

\[
\frac{r \sqrt{f}}{\sqrt{1+L}} \left( \frac{d}{dr} + \frac{i\omega \sqrt{1+L}}{f} \right) \Psi_2 = \left( -\frac{\lambda_f}{r} + i\mu \right) \Psi_1, \quad (81) \]

\[
\frac{r \sqrt{f}}{\sqrt{1+L}} \left( \frac{d}{dr} - \frac{i\omega \sqrt{1+L}}{f} \right) \Psi_1 = \left( -\frac{\lambda_f}{r} - i\mu \right) \Psi_2. \quad (82) \]

By defining \( \Psi_1 = f^{-\frac{1}{2}} R_1(r) \) and \( \Psi_2 = f^{-\frac{1}{2}} R_2(r) \) and introducing the tortoise coordinate \( (r_*) \) as \( \frac{f}{\sqrt{1+L}} \frac{d}{dr} = \frac{d}{dr_*} \), we obtain

\[
\left( \frac{d}{dr_*} + i\omega \right) R_2(r) = \sqrt{f} \left( -\frac{\lambda_f}{r} + i\mu \right) R_1, \quad (83) \]

\[
\left( \frac{d}{dr_*} - i\omega \right) R_1(r) = \sqrt{f} \left( -\frac{\lambda_f}{r} - i\mu \right) R_2. \quad (84) \]

One can combine the above equations by letting

\[
Z_+ = R_1 + R_2, \quad (85) \]

\[
Z_- = R_2 - R_1. \quad (86) \]

Thus, we end up with the following pair of one dimensional Schrödinger-like wave equations:

\[
\left( \frac{d^2}{dr_*^2} + \omega^2 \right) Z_\pm = V_\pm Z_\pm, \quad (87) \]

where the effective potentials for the Dirac field read

\[
V_\pm = f \left[ \left( -\frac{\lambda_f}{r} \pm i\mu \right)^2 \pm \lambda_f \frac{1}{\sqrt{1+L}} \frac{d}{dr} \left( -\frac{\sqrt{f}}{r} \pm \frac{i\mu \sqrt{f}}{\lambda_f} \right) \right], \quad (88) \]

\[
= f \left[ \left( \frac{2l+1}{2r} \pm i\mu \right)^2 \pm \left( l + \frac{1}{2} \right) \frac{1}{\sqrt{1+L}} \frac{d}{dr} \left( -\frac{\sqrt{f}}{r} \pm \frac{2i\mu \sqrt{f}}{2l+1} \right) \right]. \quad (89) \]
C. Greybody Factor Computations

In general relativity, the greybody factor is one of the most important physical quantities related to the quantum nature of a BH. A high value of the greybody factor indicates a high probability that HR can reach to spatial infinity. Among the many methods (see for example [46] and references therein) for obtaining the greybody factor, here we employ the method of [156], which formulates the general semi-analytic bounds for greybody factors:

\[
\sigma_{\ell}(\omega) \geq \text{sec}^2 h \left( \int_{-\infty}^{+\infty} \varphi \, dr_\ast \right),
\]

where \(\sigma_{\ell}(\omega)\) are the dimensionless greybody factors that depend on the angular momentum quantum number \(\ell\) and frequency \(\omega\) of the emitted particles, and

\[
\varphi = \sqrt{h'^2 + (\omega^2 - V_{\text{eff}} - h^2)^2} / 2h,
\]

in which prime denotes the derivation with respect to \(r\). We have two conditions for the certain positive function \(h\): 1) \(h(r_\ast) > 0\) and 2) \(h(-\infty) = h(\infty) = \omega\) [156]. Without loss of generality, we simply set \(h = \omega\), which reduces the integration of Eq. (90) to

\[
\int_{-\infty}^{+\infty} \varphi \, dr_\ast = \frac{\sqrt{1 + L}}{2\omega} \int_{r_h}^{+\infty} V_{\text{eff}} f(r) \, dr.
\]
For a massless scalar field \( \phi \), considering the effective potential given in Eq. (70), Eq. (90) becomes

\[
\sigma_s^e(\omega) \geq \sec h^2 \left( \frac{\sqrt{1 + L}}{2\omega} \int_{r_h}^{+\infty} \left[ \frac{f'(r)}{(1 + L)r} + \frac{l(l + 1)}{r^2} \right] dr \right). \tag{93}
\]

Taking cognizance of the integral part of Eq. (93):

\[
\frac{1}{2\omega} \int_{-\infty}^{+\infty} \left[ \frac{l(l + 1)}{r^2} f(r) dr + \frac{f'(r)}{r(1 + L)} f(r) dr \right],
\]

\[
= \frac{\sqrt{1 + L}}{2\omega} \left[ l(l + 1) \int_{r_h}^{+\infty} \frac{dr}{r^2} + \int_{r_h}^{+\infty} \frac{dr}{r(1 + L)} \left( \frac{2M}{r^2} \right) \right],
\]

\[
= \frac{\sqrt{1 + L}}{2\omega r_h} \left[ l(l + 1) + \frac{1}{2(1 + L)} \right], \tag{94}
\]

the greybody factor of the SBHBGM due to scalar field radiation yields

\[
\sigma_s^e(\omega) \geq \sec h^2 \left\{ \frac{\sqrt{1 + L}}{2\omega r_h} \left[ l(l + 1) + \frac{1}{2(1 + L)} \right] \right\}. \tag{95}
\]

**FIG. 3:** \( \sigma_s^e(\omega) \) versus \( \omega \) graph. The plots are governed by Eq. (95) with \( M = 1 \).

When one considers the effective potential (88) of the massless Dirac fields:

\[
V_{\pm}|_{\mu=0} = f \left[ \frac{\lambda_f^2}{r^2} \pm \lambda_f \frac{1}{\sqrt{1 + L}} \frac{d}{dr} \left( -\frac{\sqrt{f}}{r} \right) \right], \tag{96}
\]

\[21\]
the integral seen in Eq. (90) can be easily computed. Thus, we find the greybody factor expression of the SBHBGM arising from the fermion radiation:

$$\sigma_f^\ell(\omega) \geq \text{sec}^2 \left( \frac{1}{2\omega} \int_{-\infty}^{+\infty} f \left[ \frac{\lambda_f^2}{r^2} \pm \lambda_f \frac{1}{\sqrt{1+L}} \frac{d}{dr} \left( -\sqrt{f} \right) \right] dr \right).$$  \tag{97}

From now on, without loss of generality, we consider only $V_\omega$. After some manipulation, one can get

$$\sigma_f^\ell(\omega) \geq \text{sec}^2 \left[ \frac{\sqrt{1+L}}{2\omega} \left( \frac{\lambda_f}{r_h} \int_{r_h}^{\infty} \frac{d}{dr} \left( \frac{1}{r^2} - \frac{M}{r^3} \right) dr \right) \right],$$  \tag{98}

which recasts in

$$\sigma_f^\ell(\omega) \geq \text{sec}^2 \left[ \frac{\sqrt{1+L}}{2\omega} \left( \frac{\lambda_f}{r_h} \int_{r_h}^{\infty} \left( 1 + \frac{M}{r^2} \right) \left( 1 - \frac{3M}{r^3} \right) dr \right) \right],$$  \tag{99}

or, in more compact form:

$$\sigma_f^\ell(\omega) \geq \text{sec}^2 \left[ \frac{L + \frac{1}{2}}{4M\omega} \left( L + \frac{1}{2} \pm \frac{1}{4\sqrt{1+L}} \right) \right].$$  \tag{100}

We depict the greybody factors of the SBHBGM arising from the scalar (95) and fermion (100) fields in Figs. (3) and (4), respectively. As is well-known, the greybody factor of the
HR must be < 1 since a BH does not perform a complete black body radiation with a 100% absorption coefficient. Our findings, as shown in Figs. (3) and (4), are in good agreement with the latter remark. Also, it can be seen from these figures that the peak values of the greybody factors decreases with increasing LSB parameter \( L \). In summary, LSB has a greybody factor-reducing effect.

VI. CONCLUSION

In this paper, we studied the quantum thermodynamics \[157\] of the SBHBGM. During this analysis, we had mainly two aims: 1) to obtain the modified Hawking temperature of the SBHBGM, within the framework of GUP, arising from the emission of bosons and fermions; 2) to compute the greybody factors of the scalar and fermion fields from the SBHBGM. To this end, we first derived the effective potentials of the Klein-Gordon and Dirac equations. Next, we used the obtained effective potentials in the greybody expression \[90\]. Then, we illustrated the obtained greybody factors in Figs. (3) and (4). It was clear from those figures that as LSB effect \( L \) increases, the greybody factor decreases: low values of the greybody factor indicate a low probability that HR can reach spatial infinity. In the future, the latter observation might shed light on the LSB effects from both terrestrial experiments and astrophysical observations. Any discovery of the LSB would be an important signal beyond the SM physics \[158\].

In future work, we plan to extend the GUP and greybody factor analysis \[159\] to the various BHs in gravity’s rainbow, which is also a result of quantum gravity \[160\] \[169\]. The deformation of a spacetime owing to the rainbow gravity effect leads to Lorentz violations \[170\] \[171\]. In this way, we hope to achieve new results that will help us to understand the QGT and its effect on the LSB.
Acknowledgements

The authors are grateful to the editor and anonymous referees for their valuable comments and suggestions to improve the paper.

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