Direct proof for the Scalar Product with Bethe eigenstate in Spin chains.

A.A. Ovchinnikov

Institute for Nuclear Research, RAS, Moscow.

Abstract

We present the simple and direct proof of the determinantal formula for the scalar product of Bethe eigenstate with an arbitrary dual state. We briefly review the direct calculation of the general scalar product with the help of the factorizing operator and the construction of the factorizing operator itself. We also comment on the previous determination of the scalar product of Bethe eigenstate with an arbitrary dual state.

1. Introduction.

One of the most important open problems in the theory of quantum integrable models is the calculation of the correlation functions. In the framework of the Algebraic Bethe Ansatz method the problem is the combinatorial complexity of calculations due to the structure of Bethe eigenstates. In spite of these difficulties many important results have been obtained. Among them one can mention the derivation by Korepin the Gaudin formula for the norm of Bethe eigenstates and the calculation of the scalar product of Bethe eigenstate with an arbitrary dual state, which leads straightforwardly to the determinant representation of the formfactors of local operators (see for the case of spin-1/2 chains and for the case of Bose-gas with δ-function interaction). Using the Algebraic Bethe Ansatz and the solution of the quantum inverse scattering problem the authors of ref. obtained the multiple integral representation for the correlation functions found previously using the other methods (see for example and references therein). The concept of factorizing F - matrix introduced recently by Maillet and Sanches de Santos following the concept of Drinfeld’s twists in his theory of Quantum Groups and construction of the Monodromy matrix in the F -basis allows in some respect to simplify the calculations in comparison with the general theory of scalar products developed previously.

At the same time only the very limited number of physical results for the correlation functions (or formfactors) have been obtained from the first principles. So the developing of new calculational methods and better understanding of the underlying mathematical structure for the expressions for the correlators is required. For example it is still not completely clear if it is possible to obtain the determinantal expressions for the correlators without using the auxiliary dual fields.

The aim of the present letter is to present the simple and direct proof for the expression of the scalar product of Bethe eigenstate with an arbitrary dual state. First, to fix the notations

---

1E - mail address: ovch@ms2.inr.ac.ru
and derive the expression for an arbitrary scalar product, we briefly review the results of ref.[14] on the construction of the factorizing operator (Section 2), and simple calculation of the general scalar product (Section 3). Next, we present the new version of the proof given by Slavnov [3] for the scalar with Bethe eigenstate. Finally in Section 4 we present the direct and a simple proof of this formula. Our results could shed some light on the possibility to obtain the determinantal expressions for various correlators.

2. Construction of the factorizing operator.

We consider in this letter the XXX or XXZ spin- 1/2 chains of finite length N. In the present section we diagonalize the operator $A$ and introduce the factorizing operator. Let us fix the notations: the normalization of basic $S$- matrix, the definition of monodromy matrix and write down the Bethe Ansatz equations. For the rational case (XXX- chain) the $S$- matrix has the form

$$ S_{12}(t_1, t_2) = \begin{pmatrix} a(t) & 0 & 0 & 0 \\ 0 & c(t) & b(t) & 0 \\ 0 & b(t) & c(t) & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix}, \quad t = t_1 - t_2. $$

One can choose the normalization $a(t) = 1$ so that the functions $b(t)$ and $c(t)$ become $\tilde{b}(t) = \phi(\eta)/\phi(t + \eta)$, $\tilde{c}(t) = \phi(t)/\phi(t + \eta)$, where $\phi(t) = t$ for the isotropic (XXX) chain and $\phi(t) = \sinh(t)$ for the XXZ- chain. With this normalization the $S$-matrix satisfies the unitarity condition $S_{12}(t_1, t_2)S_{21}(t_2, t_1) = 1$. The monodromy matrix is defined as

$$ T_0(t, \{\xi\}) = S_{10}(\xi_1, t)S_{20}(\xi_2, t)\ldots S_{N0}(\xi_N, t), $$

where $\xi_i$ are the inhomogeneity parameters. We define the operator entries in the auxiliary space (0) as follows:

$$ \langle \beta | T_0 | \alpha \rangle = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}_{\alpha\beta}, \quad \alpha, \beta = (1, 2) = (\uparrow; \downarrow). $$

We denote throughout the paper $(\uparrow; \downarrow) = (1; 0)$ so that the pseudovacuum (quantum reference state) $|0\rangle = |\{00\ldots0\}_N\rangle$. The triangle relation (Yang-Baxter equation) reads:

$$ S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}, \quad R_{00'}T_0T_0' = T_0'T_0R_{00'}; \quad R_{00'} = P_{00'}S_{00'}. $$

The action of the operators on the pseudovacuum is: $A(t)|0\rangle = a(t)|0\rangle$ ($a(t) = \prod_\alpha \tilde{c}(\xi_\alpha - t)$), $D(t)|0\rangle = |0\rangle$, $C(t)|0\rangle = 0$. The Bethe Ansatz equations for the eigenstate of the Hamiltonian $\prod_{i=1}^M B(t_i)|0\rangle$ and the corresponding eigenvalue of the transfer - matrix $Z(t) = A(t) + D(t)$ are

$$ a(t_i) = \prod_{\alpha \neq i} \tilde{c}(t_\alpha - t_i)(\tilde{c}(t_i - t_\alpha))^{-1}, \quad \Lambda(t, \{t_\alpha\}) = a(t) \prod_{\alpha = 1}^M \tilde{c}^{-1}(t_\alpha - t) + \prod_{\alpha = 1}^M \tilde{c}^{-1}(t - t_\alpha), $$

where $\tilde{c}(t) = \exp(t)$. Finally in the following section we diagonalize the operator $A$ and introduce the factorizing operator. Let us fix the notations: the normalization of basic $S$- matrix, the definition of monodromy matrix and write down the Bethe Ansatz equations. For the rational case (XXX- chain) the $S$- matrix has the form $S_{12}(t_1, t_2) = t_1 - t_2 + \eta P_{12}$, where $P_{12}$ is the permutation operator. In general (XXZ) case it can be written as

$$ S_{12}(t_1, t_2) = \begin{pmatrix} a(t) & 0 & 0 & 0 \\ 0 & c(t) & b(t) & 0 \\ 0 & b(t) & c(t) & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix}, \quad t = t_1 - t_2. $$

One can choose the normalization $a(t) = 1$ so that the functions $b(t)$ and $c(t)$ become $\frac{\sinh(t)}{\sinh(t + \eta)}$, $\frac{\sinh(t)}{\sinh(t + \eta)}$, where $\sinh(t) = t$ for the isotropic (XXX) chain and $\sinh(t) = \sinh(t)$ for the XXZ- chain. With this normalization the $S$-matrix satisfies the unitarity condition $S_{12}(t_1, t_2)S_{21}(t_2, t_1) = 1$. The monodromy matrix is defined as

$$ T_0(t, \{\xi\}) = S_{10}(\xi_1, t)S_{20}(\xi_2, t)\ldots S_{N0}(\xi_N, t), $$

where $\xi_i$ are the inhomogeneity parameters. We define the operator entries in the auxiliary space (0) as follows:

$$ \langle \beta | T_0 | \alpha \rangle = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}_{\alpha\beta}, \quad \alpha, \beta = (1, 2) = (\uparrow; \downarrow). $$

We denote throughout the paper $(\uparrow; \downarrow) = (1; 0)$ so that the pseudovacuum (quantum reference state) $|0\rangle = |\{00\ldots0\}_N\rangle$. The triangle relation (Yang-Baxter equation) reads:

$$ S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}, \quad R_{00'}T_0T_0' = T_0'T_0R_{00'}; \quad R_{00'} = P_{00'}S_{00'}. $$

The action of the operators on the pseudovacuum is: $A(t)|0\rangle = a(t)|0\rangle$ ($a(t) = \prod_\alpha \tilde{c}(\xi_\alpha - t)$), $D(t)|0\rangle = |0\rangle$, $C(t)|0\rangle = 0$. The Bethe Ansatz equations for the eigenstate of the Hamiltonian $\prod_{i=1}^M B(t_i)|0\rangle$ and the corresponding eigenvalue of the transfer - matrix $Z(t) = A(t) + D(t)$ are

$$ a(t_i) = \prod_{\alpha \neq i} \tilde{c}(t_\alpha - t_i)(\tilde{c}(t_i - t_\alpha))^{-1}, \quad \Lambda(t, \{t_\alpha\}) = a(t) \prod_{\alpha = 1}^M \tilde{c}^{-1}(t_\alpha - t) + \prod_{\alpha = 1}^M \tilde{c}^{-1}(t - t_\alpha), **
where $t_n$ are the solution of Bethe Ansatz equations.

In order to construct the operator $\hat{O} = \hat{O}_{1..N}$ which diagonalizes the operator $A(t)$ ($\hat{O}^{-1}A\hat{O} = \text{diag}(A)$), let us first construct the eigenfunctions of the operator $A$. One can do it in two different ways. First, note that $A$ is a triangular matrix in a sense that it makes particles (the spin-up - coordinates) move to the right on the lattice $1...N$. Thus its eigenvalues are coincide with its diagonal matrix elements and therefore are characterized by \(|\phi(n_1,...n_M)|\). Clearly, \(|\{00..0\}_M\{11..1\}_M\) is an eigenstate of $A(t)$. Therefore, considering the permutation

$$S_{10}...S_{N0} \rightarrow S_{10}...S_{N0}S_{n_M}0...S_{n_0} \quad (n_1 < n_2 < ... < n_M),$$

we realize that

$$|\phi(n_1,...n_M)⟩ = T_n|n_1, n_2, ... n_M⟩ \quad (1)$$

where

$$T_n = S_{n+1,n}S_{n+2,n}...S_{nn},$$

is an eigenstate of the operator $A(t)$. Note that if we modify the given permutation interchanging $n_i \leftrightarrow n_j$ the operator in the right-hand side of (1) modifies but the state \(|\phi(n)|\) remains the same. For example, for $M = 2$: $T_{n_2}T_{n_1}' = T_{n_1}T_{n_2}S_{n_2,n_1}^{-1}$, where $n_1 < n_2$ and the prime means the absence of the term $S_{n_1,n_2}$ in $T_{n_1}$. Since $S_{n_1,n_2}|n_1, n_2⟩ = |n_1, n_2⟩$ the state remains the same.

The second way - is to consider the state

$$|\phi(n_1,...n_M)⟩ = B(\xi_{n_1})B(\xi_{n_2})...B(\xi_{n_M})|0⟩, \quad (n_i \neq n_j). \quad (2)$$

Using the fundamental commutation relation:

$$A(t)B(q) = \frac{1}{\tilde{c}(q-t)}B(q)A(t) - \frac{\tilde{b}(q-t)}{\tilde{c}(q-t)}B(t)A(q), \quad (3)$$

and the fact that for the pseudovacuum state $A(\xi)|0⟩ = 0$, we find again that \(|\phi(n)|\) \((\{n\} = \{n_1,...n_M\})\) is an eigenstate of $A(t)$ with the eigenvalue $A_{\{n\}t}(t) = \prod_{\alpha \neq n_k} \tilde{c}(\xi_\alpha - t)$.

One can see that the states \(|\phi(n)|\) (3) and (2) are coincide. This can be seen using the identity

$$B(\xi_n) = T_n|0⟩|S_{10}...S_{n-1,0}\rangle_n|1⟩ \quad (4)$$

and ordering the (commuting) operators $B(\xi_n)$ in eq.2 according to the prescription $n_1 < n_2 < ... < n_M$, so that the second operator in the last formula simply creates the particle (spin-up) at the site $n$ with the amplitude equal to unity.

Now we can introduce the diagonalizing operator $\hat{O} = \hat{O}_{1..N}$ (we show later that it is also the factorizing operator). Let us define the operator $\hat{O}$ such that

$$|\phi(n)⟩ = \hat{O}|n⟩ \quad (5)$$
where $\hat{O}(\{n\})$ is given by the equation (4). Clearly the operator $A^F(t) = \hat{O}^{-1}A(t)\hat{O}$ is diagonal. Now we immediately get the simple formula for the matrix elements of the factorizing operator:

$$\hat{O}_{\{m\}\{n\}} = \langle \{m\}|B(\xi_{n_1})B(\xi_{n_2})...B(\xi_{n_M})|0\rangle,$$

where $\{m\} = \{m_1, ..., m_M\}$. From this expression (see (4)) one see that $\hat{O}$ is the triangular matrix (in the same sense as the operator $A((t))$). Let us show that the operator $\hat{O}$ is invertible and construct the inverse operator. To find this operator we define the operator $\hat{\tilde{O}}$ analogously to the previous case as $\langle \hat{\tilde{O}}(\{n\})| = \langle \{n\}|\hat{O}$ where the corresponding dual states and the matrix elements of the operator $\hat{\tilde{O}}$ are:

$$\langle \hat{\tilde{O}}(n_1, ..., n_M)| = \langle 0|C(\xi_{n_1})C(\xi_{n_2})...C(\xi_{n_M})|n\rangle.$$

Let us calculate the following scalar product

$$\langle \hat{\tilde{O}}(\{m\})|\hat{\tilde{O}}(\{n\})\rangle = \langle \{m\}|\hat{O}\hat{\tilde{O}}|\{n\}\rangle = \langle 0|C(\xi_{m_1})C(\xi_{m_2})...C(\xi_{m_M})B(\xi_{n_1})...B(\xi_{n_M})|0\rangle.$$

This scalar product can be calculated using another well known relation for the commutator $[B(q), C(t)]$ following from the Yang-Baxter equation, and using again that for any site $i$, $A(\xi_i)|0\rangle = 0$. Moving the operators $A$ and $D$ to the right, repeating consequently the relation (3) and the analogous relation for $D$ (which differs from eq.(3) by the interchange $t \leftrightarrow q$ in the coefficients between the products of the operators) and using the equations $C(\xi_n)B(\xi_n)|0\rangle = \prod_{\alpha \neq n} \tilde{c}(\xi_n - \xi_\alpha)|0\rangle$, we obtain that the matrix $\hat{f} = \hat{O}\hat{\tilde{O}}$ is diagonal. The corresponding diagonal matrix elements can be found either using the procedure mentioned above or, which is the simplest way, using the representation (4) (and the same for the operator $C(t)$):

$$f(n_1, ..., n_M) = \prod_k \left( \prod_{\alpha \neq n_k} \tilde{c}(\xi_n - \xi_\alpha) \right).$$

Thus we obtained the inverse matrix $\hat{O}^{-1}$:

$$\hat{O}\hat{\tilde{O}} = \hat{f}, \quad \hat{O}^{-1} = \hat{f}^{-1}\hat{O}.$$

Before proceeding with evaluation of the matrix elements of the other operators in the new basis, let us mention some useful properties of the operator $\hat{O}$, and prove that, in fact, it is the factorizing operator in a sense of the definition [3]. First, $\hat{O}$ and $\hat{O}^{-1}$ are the triangular matrices (upper triangular as $A((t))$). Second, the pseudovacuum state is an eigenstate of $\hat{O}$ ($\hat{O}^{-1}$) with the eigenvalue equal to unity. In general we have the following equations for arbitrary number of particles $n$: $\hat{O}|\{00..0\}_{N-n}\{11..1\}_n\rangle = |\{00..0\}_{N-n}\{11..1\}_n\rangle$, the similar formulas for the operator $\hat{O}$ and the same formulas for the inverse operators. From this formulas one can already suspect that $\hat{O}$ is the factorizing operator. Indeed for the particular permutation (4) $\sigma(\{n\})$ the factorizing condition is represented as $\hat{O}(\hat{O}^{\sigma(\{n\})})^{-1} = \hat{O}(\{n\})$. 
where $\hat{O}(\{n\}) = T_{n_1}..T_{n_M}$ and is fulfilled at least for the state $|\{n\}\rangle$ since due to the last formulas $\hat{O}^{\sigma(\{n\})}^{-1} |\{n\}\rangle = |\{n\}\rangle$. The rigorous proof goes as follows. We construct the operator that acting on the state $|\{n\}\rangle$ produces the state $\hat{O}(\{n\})|\{n\}\rangle$. It is easy to see that the operator that fulfills the above requirement is:

$$\hat{O}_{12..N} = \hat{F}_1\hat{F}_2 \ldots \hat{F}_N, \quad \hat{F}_i = (1 - \hat{n}_i) + T_i\hat{n}_i,$$

where $\hat{n}_i$ is the operator of the number of particles (spin up) at the given site $i$. The operators $\hat{F}_i$ entering (9) do not commute and their ordering in eq.(9) is important. To prove the factorizing property of this operator it is sufficient to consider only one particular permutation, say the the permutation $(i, i + 1)$ since all the others can be obtained as a superposition of these ones for different $i$. One can show [9], [10] that

$$\hat{O} = S_{i+1,i} \hat{O}^{(i,i+1)},$$

Evidently, for any transmutation $\sigma \in S_N$ we will obtain only one operator $R^\sigma_{1..N}$ (the operator constructed from $S$-matrices) on the left of the operator $\hat{O}$. Thus $\hat{O}$ is the factorizing operator in a sense of [9].

Let us calculate the matrix elements of $B(t)$ and $C(t)$ - operators in the $F$ - basis: $B^F(t) = FB(t)F^{-1} = \hat{O}^{-1}B(t)\hat{O}$ (and the same for $C(t)$). The general scheme to perform the calculations is to use the formalism developed previously, which leads to the following equation:

$$B(t)B(\xi_{n_1})\ldots B(\xi_{n_M})|0\rangle = \sum_x (B(\xi_x)B(\xi_{n_1})\ldots B(\xi_{n_M})|0\rangle) \phi(x, t, \{n\}),$$

where $\phi(x, t, \{n\})$ - is exactly the matrix element of the operator $B(t)$ in the new basis. To get the single term in the sum in eq.(11), we act by the operator $A(\xi_x)$ ($x \neq n_k$) at both sides of this equation. Using again the property $A(\xi_i)|0\rangle = 0$ and eq.(3) for the left-hand side of (11) we get the single term with $B(\xi_x)$ for the right-hand side, which can be evaluated using eq.(3) and the formula $A(\xi_x)B(\xi_x)|0\rangle = \prod_{\alpha \neq x} \tilde{c}(\xi_\alpha - \xi_x)B(\xi_x)|0\rangle$, which can be proved by direct computations. After the cancellation of similar terms at both sides of eq.(11) we get the matrix element $\phi(x, t, \{n\})$ and finally obtain in the operator form:

$$B^F(t) = \sum_x \sigma_x^+ \tilde{b}(\xi_x - t) \prod_{\alpha \neq x} \left( \tilde{c}(\xi_\alpha - t)(\tilde{c}(\xi_\alpha - \xi_x))^{-1}, \quad \alpha \neq n_k \atop 1, \quad \alpha = n_k \right).$$

With this expression one can prove the equation $\prod_x B^F(\xi_{n_k})|0\rangle = |\{n\}\rangle$ which is consistent with the formulas of the previous section. For the operator $C(t)$ proceeding in a similar way and using the relation for the commutator $[B; C]$, we get after some algebra

$$C^F(t) = \sum_x \sigma_x^- \tilde{b}(\xi_x - t) \prod_{\alpha \neq x} \left( \tilde{c}(\xi_\alpha - t), \quad \alpha \neq n_k \atop (\tilde{c}(\xi_x - \xi_\alpha))^{-1}, \quad \alpha = n_k \right).$$
The operators (12) and (13) are quasilocal i.e. they describe the flipping of the spin on a single site with the amplitude depending on the positions of spins on all the other sites of the chain. The operator $D(t)$ can be found using either the same method or the quantum determinant relation and has a (quasi)bilocal form.

3. Calculation of the scalar products.

In this section we use the developed formalism to obtain the expressions for the correlation functions for the spin chains. We use the factorizing operator and, in particular, the expression for its matrix elements (5) to obtain the expression for the general correlation function (scalar product)

$$S_M(\{\lambda\}, \{t\}) = \langle 0|C(\lambda_1)C(\lambda_2)\ldots C(\lambda_M)B(t_1)B(t_2)\ldots B(t_M)|0\rangle,$$

where $\{\lambda\}$ and $\{t\}$ are two arbitrary sets of parameters (not necessarily satisfying the Bethe Ansatz equations). The correlation function can be represented in the following form

$$\langle 0|\text{Tr}_{0_10_2\ldots 0_{2M}}(\sigma_0^+\ldots \sigma_{0M}^+ \sigma_{0M+1}^- \ldots \sigma_{02M}^- T_0_1 \ldots T_{02M})|0\rangle.$$ 

The auxiliary spaces $0_1, \ldots 0_{2M}$ can be considered as a lattice consisting of $2M$ sites with the corresponding spectral parameters $\lambda_1, \ldots \lambda_M, t_1, \ldots t_M$. Rearranging the basic $S$-matrices entering the product of the monodromy matrices we arrive at the operator

$$\tilde{T}_1 \ldots \tilde{T}_N,$$

where the new monodromy matrices act in the auxiliary space instead of the original quantum space:

$$\tilde{T}_n = S_{n0_1}S_{n0_2}\ldots S_{n0_{2M}}.$$ 

Obviously using this matrices the correlator can be represented as the following matrix element in the new quantum space $0_1, \ldots 0_{2M}$:

$$\langle \{00\ldots 0\}_M | 11..1}_M | \tilde{A}(\xi_1)\ldots \tilde{A}(\xi_N) | \{11..1}_M | 00..0}_M \rangle$$ 

(we use the symmetry $A(t) \leftrightarrow D(t)$, $0 \leftrightarrow 1$ here). Transforming the operators $\tilde{A}(\xi_i)$ to the $F$-basis we find at $\xi_i = 0$:

$$S_M = \sum_{\{n\}} \langle \{00\ldots 0\}_M | 11..1}_M | \hat{O} | \{n\} \rangle \langle \{n\} | \left( \hat{A}^F(0) \right)^N | \{n\} \rangle \langle \{n\} | \hat{O}^{-1} | 11..1}_M | 00..0}_M \rangle.$$ 

The sum is over the states labeled by the positions of $M$ particles $\{n\} = n_1 \ldots n_M$ on a lattice consisting of $2M$ sites with the inhomogeneity parameters $\lambda_1 \ldots \lambda_M, t_1 \ldots t_M$. We use $\hat{O}^{-1} = f^{-1} \hat{O}$ and the representations (5), (6) for the matrix elements in the last formula. Let us denote by $\mu_1, \ldots \mu_M$ the parameters (from the set $(\{\lambda\}, \{t\})$) corresponding to the sites.
the rest of the parameters so that \( \{\lambda\} \cup \{t\} = \{\mu\} \cup \{\nu\} \). The first matrix element in the formula for \( S_M \) is equal to

\[
\langle \{00..0\}_M(11..1)_M|B(\mu_1)\ldots B(\mu_M)|0\rangle = \langle \{11..1\}_M|B'(\mu_1)\ldots B'(\mu_M)|0\rangle,
\]

where \( B'(\mu) \) are the same operators defined on the lattice consisting of \( M \) sites with the inhomogeneity parameters \( t_1 \ldots t_M \). The second matrix element can be reduced to the same expression with the parameters \( \lambda_1 \ldots \lambda_M \) (using the symmetry \( C \leftrightarrow B, \ 0 \leftrightarrow 1 \)). Then using the formula for the matrix elements of the operator \( \hat{f}^{-1} \) (see eq.(8)) we finally obtain the expression:

\[
S_M(\{\lambda\}, \{t\}) = \sum_{n_1 \ldots n_M} \left( \prod_j a(\nu_j) \right) \Phi_M(t, \mu) \Phi_M(\lambda, \mu) \prod_{i,j} \frac{1}{c(\mu_i - \nu_j)},
\]

where we denoted by \( \Phi_M(\xi, t) \) the functions in the right hand side of eq.(14). The determinant representation of this function and its properties are well known (see for example [2], [12], [4], [11]):

\[
\Phi_M(\xi, t) = \frac{\prod_{i,j}(t_i - \xi_j)}{\prod_{i,j}(t_i - t_j)} \frac{\det_{\eta} \left( \frac{\eta}{(t_i - \xi_j)(t_i - \xi_j + \eta)} \right)}{\prod_{i,j}(t_i - \xi_j)}
\]

for the rational case. Here \( \{\xi\} \) are the inhomogeneity parameters and \( \{t\} \) are the arguments of \( B \)-operators (see eq.(14)). Note that in the expression (13) all the dependence on the inhomogeneity parameters \( \xi_i \) is contained only in the functions \( a(\nu_i) \). The functions \( a(\nu) \) in eq.(13) are exactly the functions defined above \( a(\nu) = \prod_\alpha \tilde{c}(\xi_\alpha - \nu) \) while in the rest of this formula due to the definition of the matrices \( \tilde{T}_i \) one should interchange the arguments in the functions \( \tilde{c}^{-1}(\nu_i - \mu_j) \) which is taken into account in eq.(13) (or one could make the replacement \( \eta \rightarrow -\eta \)). Using the properties of the functions \( \Phi_M \) one can represent the general formula (13) in a different way:

\[
\sum_{m=0}^M \sum_{k,n} \left( \prod (a(\lambda_n)a(t_k)) \right) \Phi_m(t_k, \lambda_\beta) \Phi_{M-m}(\lambda_n, t_\alpha) \prod_{\alpha} \frac{1}{\tilde{c}(\lambda_\beta - \lambda_n)} \frac{1}{\tilde{c}(t_\alpha - t_k)} \frac{1}{\tilde{c}(t_\alpha - \lambda_n)} \frac{1}{\tilde{c}(\lambda_\beta - t_k)},
\]

where the sum is over the two sets \( k_1 \ldots k_m \) and \( n_1 \ldots n_{M-m} \). We used the following simplified notations in this formula. We divided the set \( \{t\} \) into two subsets \( \{t_k\} \cup \{t_\alpha\} \) where \( \{t_k\} = (t_k_1 \ldots t_k_m) \in \{\nu\}, \{t_\alpha\} \in \{\mu\} \) and analogously \( \{\lambda\} = (\lambda_n_1 \ldots \lambda_n_m) \in \{\nu\}, \{\lambda_\beta\} \in \{\mu\} \). The products in the last formula are over the indices labeling the elements of the corresponding sets. One can further rewrite this formula using the expressions for the functions \( \Phi_m \) to get the formula which leads to the determinant representation for \( S_M \) after the special dual fields are introduced [11].

Let us derive the scalar product of the Bethe eigenstate with an arbitrary dual state along the lines of ref.[3] starting from the formula (13). The set of the parameters \( \{t\} \) obey the Bethe equations so for each \( a(t_i) \) in the sum (13) one can substitute the function

\[
f(t_i) = \prod_{\alpha \neq i} \frac{\tilde{c}(t_\alpha - t_i)}{\tilde{c}(t_i - t_\alpha)} = \prod_{\alpha \neq i} \frac{\phi(t_\alpha - t_i - \eta)}{\phi(t_\alpha - t_i + \eta)},
\]

where \( \phi(z) \) is the function such that \( \phi(z) = \frac{\tilde{c}(z)}{\tilde{c}(0)} \) and

\[
\tilde{c}(z) = \prod_{\nu} \tilde{c}(\xi_\nu - z).
\]
Then the idea is that one can calculate the sum in (13) for an arbitrary smooth function \( a(\lambda) \), which behaves at least as a constant at infinity, used for the terms \( a(\lambda_j) \) (not necessarily equal to \( a(\lambda) = \prod_{\alpha} \hat{c}(\xi_\alpha - \lambda) \)). In that case the sum has the simple poles in the variables \( \{t\} \) and \( \{\lambda\} \) at the points \( t_i = \lambda_j \) and don’t have any other poles, for example the poles at \( \lambda_i = \lambda_j \), which exist if, according to [3] one considers \( a(\lambda_j) \) as an independent variables. These poles was not considered in [3] (actually only the behaviour of the function in the parameters \( t_i \) was considered - obviously there is no poles at \( t_i = t_j \) if the Bethe Ansatz equations are taken into account). One can use the symmetry of \( S_M \) in \( \{t\} \) and \( \{\lambda\} \) and single out the residue at \( \lambda_1 \to t_1 \) (which is contained in the term with \( \lambda_1 \in \{\nu\}, t_1 \in \{\mu\} \) and vice versa):

\[
S_M (\{\lambda\}, \{t\}, a(\nu_j)) \mid_{\lambda_1 \to t_1} \to \eta \frac{a(\lambda_1) - f(t_1)}{t_1 - \lambda_1} \prod_{\alpha \neq 1} \frac{1}{\hat{c}(t_\alpha - t_1)} \frac{1}{\hat{c}(\lambda_\alpha - t_1)} S_{M-1} (\{\lambda\}', \{t\}', a'(\nu_j)),
\]

where \( \{\lambda\}', \{t\}' \) do not contain \( \lambda_1, t_1 \) and

\[
a'(\nu_j) = a(\nu_j) \frac{\hat{c}(\nu_j - t_1)}{\hat{c}(t_1 - \nu_j)}.\]

The functions \( f(t_i) \) entering the sum \( S_{M-1} \) are also modified so that the terms with \( t_1 \) are absent in their definition. Also the behaviour of \( S_M \sim 1/\lambda_1 \) at \( \lambda_1 \) going to infinity should be taken into account. It can be easily proved that the same recurrence relation at \( \lambda_1 \to t_1 \) is obeyed by the following expression:

\[
S_M (\{\lambda\}, \{t\}) = \frac{1}{\prod_{i<j}(t_i - t_j) \prod_{j<i}(\lambda_i - \lambda_j)} \det_{ij} (M_{ij}(t, \lambda)) \frac{1}{\hat{c}(t_\alpha - \lambda_j)} \left( a(\lambda_j) \prod_{\alpha \neq i} (t_\alpha - \lambda_j + \eta) - \prod_{\alpha \neq i} (t_\alpha - \lambda_j - \eta) \right),
\]

where at the end of calculations one should take the actual form of the function \( a(\lambda) \). So we proved that for an arbitrary function \( a(\lambda) \) the residues of the only existing poles at \( t_i = \lambda_j \) satisfy the the same recursion relation in both formulas. Since the function (in each variable) is completely determined by the positions of (all) poles and the corresponding residues (along with the behaviour at the infinity) the two functions (17) and (13) should coincide for an arbitrary function \( a(\lambda) \). To complete the proof, one should check the formula (14) for \( M = 1, 2 \). Equivalently, one can check the coefficients corresponding to the terms \( \prod_{i=1}^{M} a(\lambda_i) \) and \( \prod_{i=1}^{M} a(t_i) = 1 \) in eq.(17) which are in agreement with (13). The formula (14) is written for the rational case. This formula can also be represented through the Jacobian \( \det_{ij} (\partial/\partial t_i \lambda_j ; \{t_\alpha\}) \). The orthogonality of two different Bethe eigenstates was shown in ref.(3). From the equation (14) taking the limit \( \lambda_i \to t_i \) one can easily obtain the formula for the norm of the Bethe eigenstate [2].

4. Direct proof for the scalar product with Bethe eigenstate.
Here we present the direct and the simple proof of the determinant expression (17) for the scalar product with one Bethe eigenstate. Although the proof given in the previous section is correct the generalization of this method to the expressions for different correlation functions is not straightforward. It is still an open question if it is possible to obtain the determinant expressions for various correlation functions (the simplest example is the emptiness formation probability, which can be represented as a sum very similar to that of eq. (15)) without the auxiliary dual fields [2, 11]. Thus the various direct derivations of the formula (17) are worth of developing. Let us present the direct proof of this formula based on the expression (15).

First, let us rewrite the expression (15) for the case when the parameters $\{t\}$ satisfy the Bethe Ansatz equations. Expressing the functions $a(t_k) = f(t_k)$ and using the explicit form of the function $\Phi_M$ we get:

$$S_M(\lambda, t) = \frac{1}{\prod_{i<j}(t_i - t_j) \prod_{i<k}(\lambda_i - \lambda_j)} \prod_{k,n} (-1)^{P_k} (-1)^{P_n} \prod(a(\lambda_n))$$

$$\det(\lambda; t_k) \det(t_\alpha; \lambda_n) \prod(\lambda_\beta - \lambda_n + \eta)(t_k - t_\alpha + \eta)(t_\alpha - \lambda_n + \eta)(\lambda_\beta - t_k + \eta),$$

where again the products is over all elements of the corresponding sets of parameters. Here the notations are exactly the same as in the previous section. The sign factors $P_k, P_n$ depend on the sets of the coordinates $k_1, \ldots, k_m, n_1, \ldots, n_{M-m}$ in order to obtain the factors in front of the sum from the terms $\prod(k_i - t_\alpha)^{-1}$ and $\prod(\lambda_\beta - \lambda_n)^{-1}$. We also used the simplified notations for the determinants:

$$\det(t, \lambda) = \det_{ij} \left( \frac{\eta}{(t_i - \lambda_j)(t_i - \lambda_j + \eta)} \right).$$

(18)

The order of lines and columns in the determinants $\det(\lambda; t_k), \det(t_\alpha; \lambda_n)$ corresponds to the prescription $k_1 < k_2 < \ldots < k_m$ and the same for the other sets of variables. Another, equivalent way to rewrite the last formula for $S_M$ is:

$$S_M(\lambda, t) = \frac{1}{\prod_{i<j}(t_i - t_j) \prod_{i<k}(\lambda_i - \lambda_j)} \prod_{k,n} (-1)^{P_k} (-1)^{P_n} (-1)^{Mm} \prod(a(\lambda_n))$$

(19)

$$\prod(t - \lambda_n + \eta) \prod(t - \lambda_\beta - \eta) \det(\lambda_\beta; t_k) \det(t_\alpha; \lambda_n) \prod \left( \frac{t_k - t_\alpha + \eta}{t_k - \lambda_n + \eta} \right) \prod \left( \frac{\lambda_\beta - \lambda_n + \eta}{\lambda_\beta - t_\alpha + \eta} \right),$$

where again the products is over all elements of the corresponding sets of parameters. Here again the sign factors $P_k, P_n$ depend on the sets of the coordinates $k_1, \ldots, k_m, n_1, \ldots, n_{M-m}$ and we also used the simplified notations for the determinants (19) and denote for example $\prod(t - \lambda_n + \eta) = \prod_{j} \prod_{i=1}^{M}(t_i - \lambda_{n_j} + \eta)$. In this formula the poles at $t_i = \lambda_j$ are contained in the determinants. If the two last products are dropped out this formula becomes the result of the decomposition of the determinant in eq. (17). Then one can see explicitly that the residues in the poles in the formulas (15) and (17) satisfy the same recurrence relations.

Our proof of the formula (17) is based on the on the expression (13). The idea is to calculate the sum over the set of the coordinates $\{k\}$. For example at the first stage consider
the case $m = M - 1$ which corresponds to the terms containing the single function $a(\lambda_{n_1})$ (the set $\{\lambda_n\}$ contains only one element $\lambda_{n_1}$ and the set $\{t_{\alpha}\}$ - only one element $t_{\alpha_1}$). Then the sum over the set $\{k\}$ in eq. (19) has the following form:

$$a(\lambda_{n_1}) \sum_{\alpha_1} \det(\lambda_{\beta}; t_k) \frac{(-1)^{\alpha_1} \eta}{(t_{\alpha_1} - \lambda_{n_1})(t_{\alpha_1} - \lambda_{n_1} + \eta)} \prod \left( \frac{t_k - t_{\alpha_1} + \eta}{t_k - \lambda_{n_1} + \eta} \right) \prod \left( \frac{\lambda_{\beta} - \lambda_{n_1} + \eta}{\lambda_{\beta} - t_{\alpha_1} + \eta} \right).$$  \hspace{1cm} (20)

This expression can be considered as a first column development of the determinant of some new $M \times M$ matrix, the parameter $\lambda_{n_1}$ being assigned to the first column of this matrix.

To find this matrix let us first consider the determinant

$$\det_{ij}(M_{ij}) = \det_{ij} \left( \frac{1}{(t_i - \lambda_j)(t_i - \lambda_j + \eta)} \right),$$

and modify the first column of the matrix $M_{i1}$ adding a linear combination of the other columns so that the determinant $\det M$ is unchanged:

$$M'_{i1} = M_{i1} + \sum_{x \neq 1} C_x M_{ix}.$$

Let us choose the coefficients as:

$$C_x = - \prod_{\beta \neq 1, x} \frac{(\lambda_1 - \lambda_{\beta})}{(\lambda_x - \lambda_{\beta})} \prod_{\alpha} \frac{(\lambda_x - t_{\alpha} - \eta)}{(\lambda_1 - t_{\alpha} - \eta)}.$$

Notice that this expression does not depend on the index $i$. It is necessary to calculate the following sum:

$$\sum_{x \neq 1} C_x M_{ix} = - \prod_{\beta \neq 1} \frac{(\lambda_1 - \lambda_{\beta})}{(t_i - \lambda_1 + \eta)} \sum_{x \neq 1} f(\lambda_x) \prod_{\beta \neq 1, x} \frac{1}{(\lambda_x - \lambda_{\beta})} \frac{1}{(t_i - \lambda_x)} \frac{1}{(\lambda_1 - \lambda_x)},$$  \hspace{1cm} (21)

where

$$f(\lambda_x) = \prod_{\alpha \neq x} \frac{(t_{\alpha} - \lambda_x + \eta)}{(t_{\alpha} - \lambda_1 + \eta)}.$$

To calculate the sum in eq. (21) consider the integral in the complex plane over the circle of large radius which is equal to zero due to the behaviour of the integrand at infinity:

$$\oint dz \frac{f(z)}{(z - \lambda_1)(z - t_i) \prod_{\beta \neq 1} (z - \lambda_{\beta})} = 0, \hspace{1cm} f(z) = \prod_{\alpha \neq i} \frac{(t_{\alpha} - z + \eta)}{(t_{\alpha} - \lambda_1 + \eta)}.$$

The sum of the residues at $z = \lambda_{\beta}$ is equal to the sum in (21), so it is easily calculated to be

$$\frac{f(\lambda_1)}{(\lambda_1 - t_i) \prod_{\beta \neq 1} (\lambda_1 - \lambda_{\beta})} + \frac{f(t_i)}{(t_i - \lambda_1) \prod_{\beta \neq 1} (t_i - \lambda_{\beta})}.$$

Thus combining all terms we obtain the following expression for the new elements of the first column:

$$M'_{i1} = \frac{1}{(t_i - \lambda_1)(t_i - \lambda_1 + \eta)} \prod_{\beta \neq 1} \frac{(\lambda_1 - \lambda_{\beta})}{(t_i - \lambda_{\beta})} \prod_{\alpha \neq i} \frac{(t_{\alpha} - t_i + \eta)}{(t_{\alpha} - \lambda_1 + \eta)}.$$
Note that there are several ways to choose the coefficients $C_x$ in order to obtain the closed expressions of this type for the first line or column for the matrix $M$.

Now we are ready to come back to the sum $S_1(t)$. Let us define $\lambda'_\beta = \lambda_\beta + \eta$ and take into account the equation $\det(\lambda'; t_k) = \det(t_k; \lambda'_\beta)$. Then we get the following sum:

$$a(\lambda_{n_1}) \sum_{\alpha_1} (-1)^{\alpha_1} \frac{(t_k - t_{\alpha_1} + \eta)}{(t_{\alpha_1} - \lambda_{n_1})(t_{\alpha_1} - \lambda_1 + \eta)} \frac{\eta}{(t_{\alpha_1} - \lambda_{n_1})(t_{\alpha_1} - \lambda_1 + \eta)} \prod (\frac{t_k - t_{\alpha_1} + \eta}{t_{\alpha_1} - \lambda_{n_1}}) \prod (\frac{\lambda'_\beta - \lambda_{n_1}}{\lambda'_\beta - t_{\alpha_1}}).$$

Comparing this sum with the last formula we find that this sum is equal to the determinant

$$a(\lambda_{n_1}) \det(\{t\}; \{\lambda_{n_1}, \lambda'_\beta\}),$$

where the parameter $\lambda_{n_1}$ corresponds to the first column of the new matrix. This determinant in turn can be represented as its first column development:

$$a(\lambda_{n_1}) \sum_{\alpha_1} (-1)^{\alpha_1} \frac{\eta}{(t_{\alpha_1} - \lambda_{n_1})(t_{\alpha_1} - \lambda_1 + \eta)} \det(\lambda'; t_k).$$

Comparing this expression with $S_1(t)$ we see that the two last products in $S_1(t)$ can be removed - the sum over $\alpha_1$ does not change. Now let us come back to the formula (19). Instead of the summation over the set $\{t_k\}$ one can take the sum over the set $\{t_{\alpha}\}$. One can label this set by the coordinates $\alpha_1, \ldots, \alpha_{M-m}$ on the lattice consisting of $M$ sites with the corresponding parameters $t_1, t_2, \ldots, t_M$. We can consider $\alpha_i$ as independent coordinates on the lattice, the only restriction being $\alpha_i \neq \alpha_j$. The determinant $\det(\lambda'; t_{\alpha})$ can be represented as the sum over the permutations of the products of matrix elements corresponding to the pairs of the variables $t_{\alpha_1}, \lambda_{n_1}$. Each term in the sum contains all the variables $t_{\alpha_i}$ and all the variables $\lambda_{n_j}$. For each term one can take the sum consequently over each of the coordinates $\alpha_1, \ldots, \alpha_{M-m}$ with all the other indices fixed, so that only the set $\{k\}$ changes during the summation at each step of this procedure during the summation over each $\alpha_i$. For each $\alpha_i$ one can single out the terms with $\alpha_i$ and the corresponding $n_j$ from the last two products in the formula (19). Then we perform the procedure described above. One can see that for each term in the sum over the permutations (for the determinant $\det(\lambda'; t_{\alpha})$) this procedure removes consequently all terms in the last two products in the formula (19) and we obtain:

$$S_M(\lambda, t) = \prod_{i<j}(t_i - t_j) \prod_{j<i}(\lambda_i - \lambda_j) \sum_{k,n} (-1)^{P_k} (-1)^{P_n} \prod(a(\lambda_n))(-1)^{M_m} \prod(t - \lambda_n + \eta) \prod(t - \lambda_{\beta} - \eta) \det(\lambda_{\beta}; t_k) \det(t_{\alpha}; \lambda_n).$$

This formula is the result of the decomposition of the determinant of the sum of the two matrices in (17). In fact, due to the symmetry of $S_M(\lambda, t)$ in the variables $\lambda_i$, it is sufficient to prove this statement for the set $\{n_0\} = \{1, \ldots, p\}, p = M - m$, and $\{\beta_0\} = \{p + 1, \ldots, M\}$, such that $(-1)^{P_{n_0}} = 1$. Clearly, the factor $(-1)^{P_n}$ makes the expression for $S_M$ symmetric.
with respect to \( \{\lambda\} \). For the term with \( \{n\} = \{n_0\} \) the determinant in eq. (17) is represented as a sum over the permutations \( P \in S_M \) with the sign \((-1)^P\):

\[
S_M(\lambda, t)\big|_{\{n_0\}} = \frac{1}{\prod_{i<j}(t_i - t_j) \prod_{j<i}(\lambda_i - \lambda_j)} \prod a(\lambda_{n_0}) f^+(\lambda_{n_0}) \prod f^-(\lambda_{\beta_0}) (-1)^m \sum_P (-1)^P \prod a(t_{P_{n_0}}, \lambda_{n_0}) \prod a(\lambda_{\beta_0}, t_{P_{\beta_0}}),
\]

where \( f^\pm(\lambda_j) = \prod a(t_\alpha - \lambda_j \pm \eta) \) and \( a(t_i, \lambda_j) \) are the matrix elements (18). Each permutation \( P \) can be represented as \( P = P(\alpha) P(k) P(\{n_0\}, \{\alpha\}) \) where the permutation \( P(\{n_0\}, \{\alpha\}) \) is

\[
(1, 2, \ldots, p, p + 1, \ldots, M) \rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_p, k_1, \ldots, k_m)
\]

(here it is implied \( \alpha_i < \alpha_{i+1}, k_i < k_{i+1} \)) and the permutations \( P(\alpha), P(k) \) corresponding to the permutations of indices \( \alpha_i \) and \( k_i \) among themselves produce the determinants \( \det(t_\alpha; \lambda_{n_0}) \), and \( \det(\lambda_{\beta_0}; t_k) \). The sign of the permutation \( P(\{n_0\}, \{\alpha\}) \) differs from the sign \((-1)^P_k\) by \((-1)^{pm}\) since \((-1)^{P_k}\) corresponds to the permutation \( (k_1, \ldots, k_m, \alpha_1, \ldots, \alpha_p) \). Thus comparing the signs in the equations (23) and (22) we see their equivalence. Thus we reproduce exactly the formula (17).

Acknowledgements

The author is deeply indebted to the stuff of INR Theory Division for the support.

References

[1] L.D.Faddeev, E.K.Sklyanin, L.A.Takhtajan, Theor.Math.Phys.40 (1979) 688.
[2] V.E.Korepin, Commun.Math.Phys. 86 (1982) 391.
[3] N.A.Slavnov, Theor.Math.Phys. 79 (1989) 502.
[4] N.Kitanine, J.M.Maillet, V.Terras, Nucl. Phys. B 554 (1999) 647.
[5] T.Kojima, V.E.Korepin, N.A.Slavnov, Commun.Math.Phys. 189 (1997) 709.
[6] V.E.Korepin, N.A.Slavnov, Int.J.Mod.Phys. B 13 (1999) 2933.
[7] N.Kitanine, J.M.Maillet, V.Terras, math-ph/9907019.
[8] M.Jimbo, T.Miwa, Algebraic analysis of solvable lattice models (AMS,1995).
[9] J.M.Maillet and J.Sanchez de Santos, Preprint (1996), q-alg/9612012.
[10] A.A.Ovchinnikov, Int.J.Mod.Phys.A (to appear), math-ph/0012004.
[11] A.G.Izergin, V.E.Korepin, Commun.Math.Phys. 94 (1984) 67.
    V.E.Korepin, Commun.Math.Phys. 113 (1987) 177.
[12] A.G.Izergin, D.Coker, V.E.Korepin, J.Physics A 25 (1992) 4315.