ON POSITIVITY AND ROOTS IN OPERATOR ALGEBRAS

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Abstract. In earlier papers the second author and Charles Read have introduced and studied a new notion of positivity for operator algebras, with an eye to extending certain $C^*$-algebraic results and theories to more general algebras. The present paper consists of complements to some facts in the just mentioned papers, concerning this notion of positivity. For example we prove a result on the numerical range of products of the roots of commuting operators with numerical range in a sector.

1. Introduction

An operator algebra is a closed subalgebra $A$ of $B(H)$, for a Hilbert space $H$. We will be working over the complex field always, so $H$ is a complex Hilbert space. In operator theory and in the theory of selfadjoint operator algebras ($C^*$-algebras and von Neumann algebras), it is hard to overestimate the importance of the role played by positive elements and their roots. In earlier papers [7, 8, 9, 18] the second author and Charles Read have shown that many of these crucial positivity ideas carry over to more general operator algebras (see also e.g. [1, 6]). These authors introduce and study a new notion of positivity for operator algebras, even in algebras with no nonzero positive elements in the usual sense. This is done with an eye to extending certain $C^*$-algebraic results and theories to more general algebras. A central role is played by the set $\mathcal{F}_A = \{ a \in A : \|1 - a\| \leq 1 \}$, and the cones

$$c_A = \mathbb{R}_+ \mathcal{F}_A,$$

and

$$r_A = \mathcal{F}_A = \{ a \in A : a + a^* \geq 0 \}. $$

Elements of these sets and their roots play the role in many situations of positive elements in a $C^*$-algebra. The present paper consists of complements to some facts in the just mentioned papers, concerning this ‘positivity’. For example, in Section 2, we characterize ‘real completely positive maps’ relative to our cone $r_A$. In Section 3 we clarify a point about the support projection $s(x)$ from [7]. In Section 4 we prove some interesting and surprising facts about ‘roots’ of operators in our ‘positive cone’. These results are used in [9], and should be useful elsewhere. For example we prove that the product of suitable roots of commuting operators in $r_A$ (resp. in $\mathcal{F}_A$) is again in $r_A$ (resp. in $\mathcal{F}_A$). There is an extensive literature on accretive products of matrices and operators (see e.g. [12, 10] for some discussion of the difficulties and basic results here); such product formulae are rare and have extremely important applications. Our result in the accretive case will not be surprising to experts on sectorial operators, however it does not appear to be in the literature. It is related to theorems of R. Bouldin, J. Holbrook, T. Kato, J. P. Williams, and others (see e.g. [10] for references).

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We now state our notation, and some facts. We refer the reader to [3, 7, 8] for additional background on operator algebras, and for some of the details and notation below. We reserve the letter $H$ for a complex Hilbert space, usually the Hilbert space on which our operator algebra is acting, or is completely isometrically represented. We write $X_+$ for the positive operators (in the usual sense) that happen to belong to a subset $X$ of $B(H)$ or of a $C^*$-algebra. We write $\text{oa}(x)$ for the operator algebra generated by $x$ in $A$, namely the smallest closed subalgebra of $A$ containing $x$.

The second dual $A^{**}$ is also an operator algebra with its (unique) Arens product. This is also the product inherited from the von Neumann algebra $B^{**}$ if $A$ is a subalgebra of a $C^*$-algebra $B$. (We use the same symbol $\ast$ for the Banach dual and for the involution or adjoint operator, the reader will have to determine which is meant from the context.) Note that $A$ has a contractive approximate identity (cai) iff $A^{**}$ has an identity $1_{A^{**}}$ of norm 1. In this case we say that $A$ is approximately unital.

We recall that by a theorem due to Ralf Meyer, every operator algebra $A$ has a unitization $A^1$ which is unique up to completely isometric homomorphism (see [3, Section 2.1]). Below 1 always refers to the identity of $A^1$ if $A$ has no identity. If $A$ is a nonunital operator algebra represented (completely) isometrically on a Hilbert space $H$ then one may identify $A^1$ with $A + \mathbb{C}1_H$. For an operator algebra, not necessarily approximately unital, we recall that $r_A$ is the cone of elements with positive ‘real part’. As we just said, $A^1$ is uniquely defined, and can be viewed as $A + \mathbb{C}1_H$. Hence $A^1 + (A^1)^\ast$ is also uniquely defined, by e.g. 1.3.7 in [3]. We define $A + A^\ast$ to be the obvious subspace of $A^1 + (A^1)^\ast$. This is well defined independently of the particular Hilbert space $H$ on which $A$ is represented, as shown at the start of Section 3 in [8]. Thus a statement such as $a + b^\ast \geq 0$ makes sense whenever $a,b \in A$, and is independent of the particular $H$ on which $A$ is represented, and in particular the same is true for $r_A = \{a \in A : a + a^\ast \geq 0\}$. Elements in $r_A$, that is elements in $A$ with $\text{Re}(x) = x + x^\ast \geq 0$, will sometimes be called accretive (although this term is often used in the literature for a slightly different notion).

We recall that $\frac{1}{2}A_A = \{a \in A : \|1 - 2a\| \leq 1\}$. Here 1 is the identity of the unitization $A^1$ if $A$ is nonunital. It is easy to see that $x \in c_A = \mathbb{R}_+ A$ iff there is a positive constant $C$ with $x^*x \leq C(x + x^*)$. These sets (and $r_A$) are studied in earlier work as analogues of the positive cone of a $C^*$-algebra (particularly in [8] and [5, 6]). We showed in [8, Section 3] that $r_A = c_A$, where $c_A = \mathbb{R}_+ A$. It is clear that $c_A \cap A = r_A \cap A = r_A$.

By the numerical range $W(x)$ of an element $x$, we will mean the set of values $\varphi(x)$ for states $\varphi$, while in the literature we quote usually uses the one defined by vector states on $B(H)$. However since the former range is the closure of the latter, as is well known, this will cause no difficulties. For any operator $T \in B(H)$ whose numerical range does not include strictly negative numbers, and for any $\alpha \in [0,1]$, there is a well-defined ‘principal’ root $T^\alpha$, which obeys the usual law $T^\alpha T^\beta = T^{\alpha + \beta}$ if $\alpha + \beta \leq 1$ (see e.g. [15, 13]). Write $S_\psi$ for the sector $\{re^{i\theta} : 0 \leq r, \text{ and } -\psi \leq \theta \leq \psi\}$ where $0 \leq \psi < \pi$. Then $T \mapsto T^\alpha$ is continuous on operators with numerical range in $S_\psi$. Our operators $T$ will be accretive (that is, $\psi \leq \frac{\pi}{2}$), and then these powers obey the usual laws such as $T^{\alpha + \beta} = T^\alpha T^\beta$ for all $\alpha, \beta > 0$, $(T^\alpha)^\beta = T^{\alpha \beta}$ for $\alpha \in (0,1]$ and any $\beta > 0$, and $(T^*\alpha) = (T^\alpha)^\ast$. If $n \in \mathbb{N}$ then $T^\frac{1}{n}$ is the unique $n$th root of $T$ with numerical range in $S_{\frac{\pi}{2n}}$, for any $\alpha \geq 0$. See e.g. [19] Chapter IV,
Section 5] and [11] for all of these facts. We also have $(cT)^{\alpha} = c^{\alpha}T^{\alpha}$ for positive scalars $c$. Also $\alpha \mapsto T^{\alpha}$ is continuous on $(0, \infty)$, for $T$ accretive. For $x \in \tau_A$ we have $x^{\alpha} \in \text{oa}(x)$ if $\alpha > 0$. See [9] Lemma 1.1] for these facts. In particular, $\tau_A$ is closed under taking roots.

2. Positivity and Real Complete Positivity

In [7] Section 8], the second author and Read defined a class of linear maps called OCP or operator completely positive (the precise definition is given below), and proved an extension and Stinespring dilation theorem for them. In particular it was shown that the linear $B(H)$-valued OCP maps on a unital operator space or approximately unital operator algebra $A$ are the restrictions to $A$ of linear completely positive maps in the usual sense on an enveloping $C^*$-algebra. Here we do the same for maps respecting the ‘cone’ of elements with positive real part. If $A$ is a unital operator space in the sense of e.g. 1.3.1 in [3], and [4] (and its sequel by the same authors), notice that $\tau_A = \{ x \in A : x + x^* \geq 0 \}$ also makes sense, and is closed. This is because the operator system $A + A^*$ is well defined independently of the representation of $A$ as a unital operator space on a Hilbert space, by e.g. 1.3.7 in [3]. Clearly $\tau_A$ spans $A$ in this case, since $1 \in \tau_A$, and for any $x \in A$ we have $x + t1 \in \tau_A$ for large enough $t$. If $A \subset B$ is a unital containment of unital operator spaces, then $\tau_A \subset \tau_B$. Finally, $\tau_A = \mathbb{R}_+ \mathfrak{F}_A$. To see this notice that $\mathbb{R}_+ \mathfrak{F}_A \subset \tau_A$ as in the operator algebra case. For the other direction, if $x \in \tau_A$ then

$$\Re(x + \frac{1}{n}) \geq \frac{1}{n} \geq C(x + \frac{1}{n})^* (x + \frac{1}{n})$$

for some positive constant $C$. Hence $x + \frac{1}{n} \in \mathbb{R}_+ \mathfrak{F}_A$, and so $x \in \mathbb{R}_+ \mathfrak{F}_A$.

**Definition 2.1.** A linear completely bounded map $u : A \to B$ between operator algebras (or between unital operator spaces) is real completely positive (RCP) if $u(x) + u(x)^* \geq 0$ whenever $x \in A$ with $x + x^* \geq 0$, and similarly for $x \in M_n(A)$ for all $n \in \mathbb{N}$. In other words, $u_n(\tau_{M_n(A)}) \subseteq \tau_{M_n(B)}$ for all $n \in \mathbb{N}$.

It is clear from properties of $\tau_A$ mentioned earlier, that restrictions of RCP maps to subalgebras, or to unital operator subspaces, are again RCP.

The OCP maps were defined in [7] similarly to Definition 2.1 but requiring the existence of a positive constant $C$ with $u_n(\mathfrak{F}_{M_n(A)}) \subseteq C \mathfrak{F}_{M_n(B)}$ for every $n \in \mathbb{N}$. If $A, B$ are operator algebras or unital operator spaces, and if $T : A \to B$ is OCP then $T$ is RCP. This follows easily from the definitions, and the fact that $\tau_A = \tau_B$. Indeed $T(\tau_A) \subseteq T(\tau_B) \subseteq \tau_B$ if $T$ is OCP, and a similar argument applies at each matrix level. The following is also clear:

**Lemma 2.2.** Restrictions of a linear completely positive map from a $C^*$-algebra into $B(H)$, to a subalgebra or unital subspace, are RCP.

**Lemma 2.3.** If $A$ is a $C^*$-algebra or operator system, then $x \in A_+$ if and only if $zx \in \tau_A$ for all $z \in \mathbb{R}_+ \mathfrak{F}_C$. This is equivalent to: $zx \in \tau_A$ for all $z \in \mathfrak{F}_C$.

**Proof.** We just prove the first ‘iff’, from which the second equivalence is clear.

$(\Rightarrow)$ This is obvious.

$(\Leftarrow)$ (C.f. [7] Lemma 8.5) If $zx \in \tau_A$, then by definition, $\Re(zx) \geq 0$, and hence $\Re(z(x\zeta, \zeta)) \geq 0$, for all $\zeta \in H$ and for all $z \in \mathbb{R}_+ \mathfrak{F}_C$. By calculus this implies that $\langle x\zeta, \zeta \rangle \geq 0$ for all $\zeta \in H$. So $x \in A_+$. \qed
Theorem 2.4. If \( T : A \to B \) is a linear map between \( C^\ast \)-algebras or operator systems then \( T \) is completely positive if and only if \( T \) is RCP.

Proof. This is similar to the proof in [7], but for convenience we give the details. Clearly any completely positive map is RCP. Conversely, if \( T : A \to B \) is RCP and \( x \in \text{Ball}(A) \) then \( \pi(x) \in \mathcal{T}_A \) by Lemma 2.3. Thus \( zT(x) = T(zx) \in \mathcal{T}_B \). So \( T(x) \geq 0 \) by Lemma 2.3. A similar argument applies to matrices. \( \square \)

Theorem 2.5. If \( T : A \to B(H) \) is a linear RCP map on a unital operator space \( A \), then the canonical extension \( \tilde{T} : A + A^\ast \to B(H) : x + y^\ast \to T(x) + T(y)^\ast \) is well defined and completely positive.

Proof. Let \( T : A \to B(H) \) be RCP. Then \( T \) restricted to \( \Delta(A) = A \cap A^\ast \) is RCP, and so it is completely positive and selfadjoint by Theorem 2.4. Define \( \tilde{T}(a + b^\ast) = T(a) + T(b)^\ast \) for \( a, b \in A \). To see that \( \tilde{T} \) is well defined, suppose \( a + b^\ast = x + y^\ast \), for \( a, b, x, y \in A \). Then \( a - x = (y - b)^\ast \in \Delta(A) \), and so

\[
T(a - x) = T((y - b)^\ast) = (T(y) - T(b))^\ast.
\]

Thus, \( T \) is well defined. If \( z = a + b^\ast \) is positive (usual sense), then

\[
z = z^\ast = b + a^\ast = \frac{1}{2}(a + b^\ast + b + a^\ast) = \frac{1}{2}(a + b) + \left(\frac{1}{2}(a + b)\right)^\ast,
\]

and \( \frac{1}{2}(a + b) \in A \). Since \( T \) is RCP, we have

\[
\tilde{T}(z) = T\left(\frac{1}{2}(a + b)\right) + T\left(\frac{1}{2}(a + b)\right)^\ast \geq 0.
\]

So \( \tilde{T} \) is positive, and a similar argument at the matrix levels shows that \( \tilde{T} \) is completely positive. \( \square \)

Theorem 2.6 (Extension and Stinespring Dilation for RCP Maps). If \( T : A \to B(H) \) is a linear map on a unital operator space or on an approximately unital operator algebra, and if \( B \) is a \( C^\ast \)-algebra containing \( A \), then \( T \) is RCP if and only if \( T \) has a completely positive extension \( \tilde{T} : B \to B(H) \). This is equivalent to being able to write \( T \) as the restriction to \( A \) of \( V^\ast \pi(\cdot)V \) for a \( * \)-representation \( \pi : B \to B(K) \), and an operator \( V : H \to K \). Moreover, this can be done with \( \|T\| = \|T\|_{cb} = \|V\|^2 \), and this equals \( \|T(1)\| \) if \( A \) is unital.

Proof. The structure of this proof follows the argument in [7] Theorem 8.9, but using the results we have established above, in particular Theorem 2.6 in place of their \( \mathcal{F}_A \) variants from [7] Section 8. Thus if \( T : A \to B(H) \) is an RCP map on an approximately unital operator algebra, let \( \tilde{T} : A^{**} \to B(H) \) be the canonical weak* continuous extension. Since \( \overline{r_{A^{**}}} = r_{A^{**}} \) (a fact first proved in [8] Section 3), we have

\[
\tilde{T}(r_{A^{**}}) = \tilde{T}(\overline{r_{A^{**}}} \subseteq \overline{T(r_{A^{**}})} \subseteq r_{B(H)}).
\]

Similarly at the matrix levels, so that \( \tilde{T} : A^{**} \to B(H) \) is RCP on the unital operator algebra \( A^{**} \). We now follow the lines of the proof of [7] Theorem 8.9, but using the results established above. \( \square \)

Remark. The last result is connected (see e.g. [11]) to the theory of real states of operator algebras.
Corollary 2.7. Let $T : A \to B(H)$ be a linear map on a unital operator space or a (not necessarily approximately unital) operator algebra $A$. Then $T$ is OCP iff it is RCP.

Proof. If $A$ is a unital operator space or approximately unital operator algebra then this follows from Theorem 2.6 and [7 Theorem 8.9].

If $A$ is a nonunital operator algebra, let $AH$ be the largest approximately unital subalgebra in $A$ as in [8 Section 4], and let $S$ be the restriction of $T$ to $AH$. We have $r_A = r_{AH}$ and $\mathfrak{H} = \mathfrak{H}_{AH}$ by [8 Section 4], so that $T(r_A) \subset r_{B(H)}$ iff $S(r_{AH}) \subset r_{B(H)}$. Similarly for $\mathfrak{H}$ and $\mathfrak{H}_{AH}$. This is true at each matrix level too since $M_n(AH) = M_n(A)H$ by a lemma in [9 Section 2]. Thus $T$ is RCP (resp. OCP) on $A$ iff $S$ is RCP (resp. OCP) on $AH$. This reduces the question to the case of approximately unital operator algebras above.

3. The Support Projection

Note that if $x$ is an element of a subalgebra $A$ of $B(H)$ then there are two natural left support projections for $x$. First there is the left support projection in $A^{**}$, namely the smallest projection $p$ in $A^{**}$ such that $px = x$ (assuming that there is such a $p$, if there is not simply use the identity of the unitization). Second, we have the left support projection in $H$, the smallest projection $P$ in $B(H)$ such that $Px = x$. Note that this is the projection onto $xH$. If the left support projection in $A^{**}$, namely $p$, is in the weak* closure of $xAx$, and if $\pi : A^{**} \to B(H)$ is the natural weak*–continuous homomorphism extending the inclusion map on $A$, then $\pi(p) = P$, the left support projection in $H$. To see this, note that $\pi(p)x = \pi(px) = \pi(x) = x$ in $B(H)$, so that $P \leq \pi(p)$. If $x_t \to p$ weak* with $x_t \in Ax$, then

$$P\pi(p) = \lim_t P x_t = \lim_t x_t = \pi(\lim_t x_t) = \pi(p),$$

so $\pi(p) \leq P$.

Similarly there are two natural right support projections, the second one being the projection onto $H \ominus \ker(x)$. If the left and right support projections of $x$ in $A^{**}$ coincide, then we call this the support projection of $x$, written $s(x)$. If this holds, and if $s(x) \in \pi A x^{**}$, then by the above we can also conclude that the left and right support projections of $x$ in $B(H)$ coincide (and equal $\pi(s(x))$). This will be the case for us below.

The following is a generalization of [7 Lemma 2.5] (which was the case when $x \in \mathfrak{H}_A$).

Proposition 3.1. For any operator algebra $A$, if $x \in A$ with $x + x^* \geq 0$ and $x \neq 0$, then the left support projection of $x$ in $A^{**}$ equals the right support projection, and equals $s(x(1 + x)^{-1})$, where the latter is the support projection studied in [7]. This also is the weak* limit of the net $(x_t)$, and is an open projection in $A^{**}$ in the sense of [2]. If $A$ is a subalgebra of $B(H)$ then the left and right support projection of $x$ in $H$ are also equal.

Proof. The first part follows the lines of the proof of [7 Lemma 2.5]. For example, if $x \in A$ with $x + x^* \geq 0$, then the operator algebra $oa(x)$ generated by $x$ was shown in [8 Section 3] to have $(x_t)$ as a bounded approximate identity. The weak* limit of the latter is the support projection of $x$ by the proof of [7 Lemma 2.5]. To see that the support projection equals $s(x(1 + x)^{-1})$, simply note that $px = x$ iff
\[ px(1 + x)^{-1} = x(1 + x)^{-1}. \] The last assertion follows from the considerations above the Proposition. \[ \square \]

This result has many consequences that are spelled out in \([8, 9]\).

4. Some Properties of Roots in an Operator Algebra

We need a simple fact about the ‘bidisk algebra functional calculus’ \( f \mapsto f(S, T) \), where \( f \) is in the bidisk algebra \( A(\mathbb{D}^2) \), and \( S, T \) are commuting contractions operators. This calculus is essentially the two-variable von Neumann inequality resulting from Ando’s dilation theorem (see e.g. 2.4.13 in \([3]\)). We need the relationship between the bidisk functional calculus, and the ‘disk algebra functional calculus’ \( h \mapsto h(T) \) coming from the usual von Neumann inequality (written as \( u_T \) in 2.4.12 in \([3]\)). There are no doubt more sophisticated variants in the literature (see e.g. \([13]\)) however for the readers convenience we give a short proof.

**Lemma 4.1.** If \( f \in A(\mathbb{D}^2) \), the bidisk algebra, and \( g, h \in A(\mathbb{D}) \), with \( \|g\|_{A(\mathbb{D})} \leq 1 \) and \( \|h\|_{A(\mathbb{D})} \leq 1 \), and if \( S, T \in B(H) \) are commuting contractive operators, then \( f(g(S), h(T)) = (f \circ (g,h))(S,T) \).

**Proof.** Let \( \rho_{g,h} : A(\mathbb{D}^2) \to A(\mathbb{D}^2) : p \mapsto p(g(z), h(w)) \). Then \( \rho_{g,h} \) is a contractive homomorphism. Let \( \theta_{S,T} : A(\mathbb{D}^2) \to B(H) : p \mapsto p(S,T) \) be the bidisk algebra functional calculus, a contractive homomorphism, as is \( \theta_{g(S),h(T)} \). Claim: \( \theta_{S,T} \circ \rho_{g,h} = \theta_{g(S),h(T)} \). This is true since both sides are contractive homomorphism, and they agree on monomials \( z^nw^m \).

Indeed

\[ \theta_{g(S),h(T)}(z^n w^m) = g(S)^n h(T)^m. \]

Viewing \( A(\mathbb{D}^2) \) as the closure of the tensor product of \( A(\mathbb{D}) \) with itself, we have \( \theta_{S,T}(p(z)q(w)) = p(S)q(T) \) if \( p, q \) are polynomials, and also if \( p, q \in A(\mathbb{D}) \) by approximating by polynomials. Thus

\[ \theta_{S,T}(\rho_{g,h}(z^nw^m)) = \theta_{S,T}(g(z)^n h(w)^m) = g(S)^n h(T)^m. \]

From the Claim, if \( f \in A(\mathbb{D}^2) \), then we have

\[ f(g(S), h(T)) = \theta_{g(S),h(T)}(f) = \theta_{S,T}(\rho_{g,h}(f)) = \theta_{S,T}(f \circ (g,h)) = (f \circ (g,h))(S,T) \]

as desired. \[ \square \]

**Lemma 4.2.**  

1. We have \( \{x^2 : x \in \frac{1}{2} \mathbb{S}_C\} = \{xy : x, y \in \frac{1}{2} \mathbb{S}_C\} \), and these coincide with the region \( R \) inside the cardioid given by the polar equation \( r = \frac{1}{2} \cos(\theta) + \frac{1}{2} \) for \( \theta \in [-\pi, \pi] \). Hence if \( x, y \in \frac{1}{2} \mathbb{S}_C \), then \( x^2 y^2 = (xy)^2 \in \frac{1}{4} \mathbb{S}_C \) also.

2. If \( b \in \frac{1}{2} \mathbb{S}_C \), and if \( A \) is an operator algebra and \( a \in \frac{1}{4} \mathbb{S}_A \), then the numerical range of \( ab \) contains no strictly negative numbers, and the unique accretive square root \( (ab)^{\frac{1}{2}} \) is in \( \frac{1}{4} \mathbb{S}_A \).

3. If \( A \) is an operator algebra and \( a, b \in \frac{1}{4} \mathbb{S}_A \) with \( ab = ba \) then \( a^{\frac{1}{2}} b^{\frac{1}{2}} \in \frac{1}{4} \mathbb{S}_A \).

4. If \( A \) is an operator algebra and \( a, b \in \tau_A \) with \( ab = ba \) then \( a^{\frac{1}{2}} b^{\frac{1}{2}} \in \tau_A \).

**Proof.**  

1. The boundary of \( \frac{1}{2} \mathbb{S}_C \) is the circle given by the polar equation \( r = \cos(\theta) \) for \( \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \). So if \( x \in \frac{1}{2} \mathbb{S}_C \), then \( x = re^{i\theta} \) for some \( \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) and some \( 0 \leq r \leq \cos(\theta) \). Thus \( x^2 = r^2 e^{i2\theta} \), where \( r^2 \leq \cos^2(\theta) = \frac{1}{4} \cos(2\theta) + \frac{1}{2} \). Hence \( \{x^2 : x \in \frac{1}{2} \mathbb{S}_C\} \subseteq R \). For the other direction, if \( se^{i\psi} \in R \), that is, \( 0 \leq s \leq \cos(\theta) \), then \( s = r \cos(2\theta) + \frac{1}{2} \) for some \( r \geq 0 \) and \( \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \). Hence \( x = re^{i\theta} \in \frac{1}{2} \mathbb{S}_C \).
\[ \frac{1}{2} \cos(\psi) + \frac{1}{2} \text{ for some } \psi \in [-\pi, \pi], \text{ then } s_{x,y,z}^{\frac{1}{2}} \leq \left( \frac{\cos(\psi) + 1}{2} \right)^{\frac{1}{2}} = \cos(\frac{\psi}{2}). \] 

which shows that \( s_{x,y,z}^{\frac{1}{2}} \in \frac{1}{2} \mathbb{S}_2^r \). So we have shown \( \{ x^2 : x \in \frac{1}{2} \mathbb{S}_2^r \} = R \).

Since clearly \( \{ x^2 : x \in \frac{1}{2} \mathbb{S}_2^r \} \subseteq \{ xy : x, y \in \frac{1}{2} \mathbb{S}_2^r \} \subseteq R \). To this end, suppose \( x = r e^{i\theta} \) and \( y = s e^{i\psi} \) for some \( \theta, \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \), \( 0 \leq r \leq \cos(\theta) \), and \( 0 \leq s \leq \cos(\psi) \). We wish to show that \( xy = r s e^{i(\theta + \psi)} \in R \), or \( 0 \leq rs \leq \frac{1}{2} \cos(\theta + \psi) + \frac{1}{2} \). Since \( rs \leq \cos(\theta) \cos(\psi) \), we will be done if \( \cos(\theta) \cos(\psi) \leq \frac{1}{2} \cos(\theta + \psi) + \frac{1}{2} \). But this is clear since \( \cos(\theta) \cos(\psi) = \frac{1}{2} \cos(\theta + \psi) + \frac{1}{2} \).

(3) Consider the function \( f(z, w) = 1 - 2(\frac{1}{2} - z)(\frac{1}{2} - w)^{\frac{1}{2}} \) on the bidisk \( \mathbb{D} \times \mathbb{D} \). By (1), \( f \) takes values in \( \mathbb{D} \), and it is clearly a member of the bidisk algebra. By the two-variable von Neumann inequality resulting from Ando’s dilation theorem (see e.g. 2.4.13 in [3], we have \( \| f(1 - 2a, 1 - 2b) \| \leq 1 \). We claim that \( f(1 - 2a, 1 - 2b) = 1 - 2a^2 b^{\frac{1}{2}} \). To see this, let \( g(z) = h(z) = ((1 - z)/2)^{\frac{1}{2}} \), then \( g(1 - 2a) = a^{\frac{1}{2}} \) and \( h(1 - 2b) = b^{\frac{1}{2}} \) as in [11 Proposition 2.3]. Letting \( r(z, w) = 1 - 2zw \), we have by Lemma 4.1, with the two-variable von Neumann inequality resulting from Ando’s dilation theorem (see [8, Section 3], and by [13] a unique accretive square root exists. The rest follows from (3), or may be proved directly using a von Neumann inequality/discrepancy functional calculus argument applied to the function \( f(z) = 2(b(z + 1)/2)^{\frac{1}{2}} - 1 \) on the disk.

(4) Suppose that \( a, b \in r_A \) with \( ab = ba \). Then let \( a_1 = ta(1 + ta)^{-1} \) and \( b_1 = tb(1 + tb)^{-1} \) for \( t > 0 \). These are in \( \frac{1}{2} \mathbb{S}_A \) as explained in [3] Section 3, and they commute, by algebra. So \( a_1^* b_1^* \in \frac{1}{2} \mathbb{S}_A \subset r_A \). Therefore

\[ \left( \frac{1}{t} a_1 \right)^{\frac{1}{2}} \left( \frac{1}{t} b_1 \right)^{\frac{1}{2}} = \frac{1}{t} a_1^* b_1^* \in r_A. \]

Taking the limit as \( t \to 0 \), and using the continuity of the roots stated in the introduction, we see that \( a^* b^* \in r_A \). \( \square \)

An application of the last result to noncommutative Urysohn-type lemmas is given in [3].

**Remark.** It is not true in general that \( a^* b^* \in r_A \) for noncommuting \( a, b \in \frac{1}{2} \mathbb{S}_A \).

**Corollary 4.3.** If \( A \) is an operator algebra, \( a_1, \cdots, a_n \) are in \( \frac{1}{2} \mathbb{S}_A \) (resp. in \( r_A \)) with \( a_1 a_j = a_j a_1 \) for all \( i, j \), then \( a_1^* \cdots a_n^* \) is in \( \frac{1}{2} \mathbb{S}_A \) (resp. in \( r_A \)).

**Proof.** We just do the \( \mathbb{S}_A \) case, the other being similar. We first prove this if \( n = 2^k \) by induction on the integer \( k \). Assuming this is true for \( n = 2^k \), write

\[ a_1^* \cdots a_n^* = (a_1^* \cdots a_n^* \cdots a_{n+1}^* \cdots a_{2n}^*)^{\frac{1}{2}}. \]

By the inductive hypothesis \( a_1^* \cdots a_n^* \in \frac{1}{2} \mathbb{S}_A \) and \( a_{n+1}^* \cdots a_{2n}^* \in \frac{1}{2} \mathbb{S}_A \). Hence by Lemma 4.2 we have \( (a_1^* \cdots a_n^*)^{\frac{1}{2}} (a_{n+1}^* \cdots a_{2n}^*)^{\frac{1}{2}} \in \frac{1}{2} \mathbb{S}_A \). This completes the induction step.

To see that the result holds for every \( n \in \mathbb{N} \), we just do the case \( n = 3 \) as an illustration. So suppose that \( x, y, z \in \frac{1}{2} \mathbb{S}_A \). For a large integer \( k \) set \( m = 2^k, p = \)
\[ (x^{\frac{1}{n}})^p (y^{\frac{1}{m}})^p (z^{\frac{1}{r}})^{m-2p} = a_1^{\frac{1}{n}} \cdots a_m^{\frac{1}{r}} \in \frac{1}{2} F A. \]

However \( x^{\frac{1}{m}} \to x^{\frac{1}{m}} \) as \( m \to \infty \) by continuity of powers in the exponent variable (mentioned in the introduction), and similarly \( y^{\frac{1}{n}} \to y^{\frac{1}{n}} \) and \( z^{\frac{1}{r}} \to z^{\frac{1}{r}} \). Since \( \frac{1}{2} F A \) is closed we deduce that \( x^{\frac{1}{m}} y^{\frac{1}{n}} z^{\frac{1}{r}} \in \frac{1}{2} F A. \)

Some ideas used in the following three results came from discussions with Charles Batty (see Acknowledgements below).

**Corollary 4.4.** If \( A \) is an operator algebra, \( n \in \mathbb{N} \), and \( a_1, \cdots, a_n \) are in \( \frac{1}{2} F A \) (resp. \( \tau_A \)), with \( a_i a_j = a_j a_i \) for all \( i, j \), then \( a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n} \) is in \( \frac{1}{2} F A \) (resp. in \( \tau_A \)) for positive \( s_1, \cdots, s_n \) with \( \sum_{k=1}^n s_k \leq 1 \).

**Proof.** Again we just do the \( \tau_A \) case, and \( n = 3 \), the other cases being similar. So suppose that \( x, y, z \in \frac{1}{2} F A \). We may assume that \( 1 \in A \) by passing to \( A^1 \) if necessary. The last result shows that \( x^{\frac{1}{m}} y^{\frac{1}{n}} z^{\frac{1}{r}} \in \frac{1}{2} F A \), for any positive integers \( N, k, m, p, r \) with \( k + m + p + r = N \). Taking limits, using continuity of powers as in the last proof, gives the assertion. \( \square \)

**Lemma 4.5.** If \( x, y \in \tau_A \) with \( xy = yx \), and with the numerical range \( W(x) \subseteq S_\varphi \) and \( W(y) \subseteq S_\psi \), for some \( \varphi, \psi \in [0, \frac{\pi}{2}] \), then \( W(x^{1/2} y^{1/2}) \subseteq S_{(\varphi + \psi)/2} \).

**Proof.** Set \( \alpha = \frac{\varphi}{2} - \varphi \) and \( \beta = \frac{\psi}{2} - \psi \geq 0 \), so that

\[ W(e^{i\alpha} x), W(e^{i\beta} y), W(e^{i(\alpha + \beta)} y) \subseteq S_{\varphi/2}. \]

Then by Corollary 4.4 or Lemma 4.3 (4), we have

\[ W((e^{i\alpha} x)^{1/2}(e^{i\beta} y)^{1/2}) = e^{i(\alpha + \beta)/2} W(x^{1/2} y^{1/2}) \subseteq S_{\varphi/2}, \]

and similarly

\[ e^{-i(\alpha + \beta)/2} W(x^{1/2} y^{1/2}) \subseteq S_{\varphi/2}. \]

The last two displayed equations together imply

\[ W(x^{1/2} y^{1/2}) \subseteq S_{(\varphi - (\alpha + \beta))/2} = S_{(\varphi + \psi)/2} \]

as desired. \( \square \)

The following is a stronger version of the assertion for \( \tau_A \) in Corollary 4.4.

**Corollary 4.6.** Suppose that \( A \) is an operator algebra, \( n \in \mathbb{N} \), and \( a_1, \cdots, a_n \) are in \( A \), with \( a_i a_j = a_j a_i \) for all \( i, j \). Suppose further that \( W(a_k) \) contains no strictly negative numbers for all \( k \), and that \( s_1, \cdots, s_n \) are positive scalars with \( \sum_{k=1}^n s_k \leq 1 \).

(1) Suppose that \( 0 \leq \varphi_k \leq \pi \), and that \( W(a_k) \subseteq S_{\varphi_k} \), when \( 1 \leq k \leq n \). If \( \varphi = \sum_{k=1}^n s_k \varphi_k \), then \( W(a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}) \subseteq S_\varphi \).

(2) There exist angles with \( |\theta_k| \leq \frac{\pi}{2} \), and \( 0 \leq \varphi_k \leq \frac{\pi}{2} \), and \( W(a_k) \subseteq e^{i\theta_k} S_{\varphi_k} \), for \( 1 \leq k \leq n \). In this case we have \( W(a_1^{s_1} \cdots a_n^{s_n}) \subseteq e^{i\theta} S_\varphi \), where \( \theta = \sum_{k=1}^n s_k \theta_k \), and \( \varphi = \sum_{k=1}^n s_k \varphi_k \).
Proof. First, we assume that \( \varphi_k \leq \frac{\pi}{4} \). We just sketch the proof of (1) in this case, since it follows the structure of the proof of Corollary 4.4 and the results leading up to that. Item (1) in the case that \( n = 2^N \) and \( s_k = \frac{1}{2^i} \) is proved by induction on \( N \geq 1 \), similarly to the first paragraph of the proof of Corollary 4.4. The case \( N = 1 \) being Lemma 4.5. The case for general \( n \) and \( s_k = \frac{1}{n} \) then follows similarly to the second paragraph of that proof. Finally the case of (1) for general positive \( s_k \) with \( \sum_{k=1}^{n} s_k \leq 1 \) follows from the case in the last line by the idea in the proof of Corollary 4.4.

Next, we state a general fact about an element \( a \in A \) with numerical range avoiding the strictly negative real axis. Here \( W(a) \), being convex, must lie on a closed half-plane with 0 in its boundary, and hence we can rotate clockwise by an angle \( \theta \) with \( |\theta| \leq \frac{\pi}{4} \), to obtain \( e^{-i\theta}a \) accretive. Claim: \( e^{-i\theta}a^s = e^{-is\theta}a^s \) whenever \( 0 \leq s \leq 1 \). (Note that this is not obvious; even if \( A = C \) one has to beware of simple sounding ‘identities’ about roots, because of the issue of the ‘principal root’. E.g. \( (wz)^{\frac{1}{2}} \neq w^{\frac{1}{2}}z^{\frac{1}{2}} \) for numbers in the third quadrant.) To see this, suppose that \( \theta \geq 0 \) (the negative case is similar). Then \( W(e^{i(\frac{\pi}{2}-\theta)}a) \) lies in the closed upper half plane. By the first few lines of the proof of [14, Theorem 2.8], we know that

\[
a \frac{1}{2} = (e^{i(\frac{\pi}{2}-\theta)}a) \cdot e^{-i(\frac{\pi}{2}-\theta)/n}, \quad n \in \mathbb{N}
\]

and that the numbers in \( W((e^{i(\frac{\pi}{2}-\theta)}a)\frac{1}{2}) \) have argument in \([0, \frac{\pi}{2}]\). So the numbers in \( W(a\frac{1}{2}) = e^{-i(\frac{\pi}{2}-\theta)/n} W((e^{i(\frac{\pi}{2}-\theta)}a)\frac{1}{2}) \) have argument in

\[
[-\frac{1}{n}(\frac{\pi}{2}-\theta), \frac{1}{n}(\frac{\pi}{2}+\theta)] \subset [-\frac{\pi}{n}, \frac{\pi}{n}].
\]

By the uniqueness assertion in [14, Theorem 2.8], these \( n \)th roots are the usual (principal) ones. We also deduce that the numbers in \( W(e^{-i\theta/n}a^{\frac{1}{2}}) = e^{-i\frac{\theta}{n}} W(a^{\frac{1}{2}}) \) have argument in \([-\frac{\pi}{n}, \frac{\pi}{n}]\), and so by the uniqueness assertion in [14, Theorem 2.8] again, \( (e^{-i\theta}a)^{\frac{1}{2}} = e^{-i\frac{\theta}{n}} a^{\frac{1}{2}} \). Raising to the power of a positive integer \( m \leq n \), we obtain the Claim when \( s \) is rational. Hence it holds for all \( s \in [0, 1] \) by the continuity of \( a^s \) in \( s \), which is well known (particularly in the accretive case, which may be ‘rotated’ to give the general case).

Finally, let \( a_1, \ldots, a_n \) be as in the full statement of the corollary. As in the last paragraph, we rotate by an angle \( \theta_k \) with \( |\theta_k| \leq \frac{\pi}{4} \), to obtain \( W(e^{-i\theta_k}a_k) \subset S_{\varphi_k} \), where \( 0 \leq \varphi_k \leq \frac{\pi}{4} \). Applying the case proved in the first paragraph of the proof, if \( s_1, \ldots, s_n \) are positive scalars with \( \sum_{k=1}^{n} s_k \leq 1 \), and if \( \theta = \sum_{k=1}^{n} s_k \theta_k \) and \( \varphi = \sum_{k=1}^{n} s_k \varphi_k \), then \( W(e^{-i\theta}a_1 a_2 \cdots a_n) \subset S_{\sum_{k=1}^{n} s_k \varphi_k} \). (Here we are using the Claim proved in the last paragraph). So \( W(a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}) \subset e^{i\theta} S_{\varphi} \). This proves (2), from which (1) follows as an easy exercise.

\[ \square \]

Remark. A similar proof works for mutually commuting \( a_1, \ldots, a_n \), such that for each \( k \) there is some ray starting at the origin which avoids \( W(a_k) \).

**Proposition 4.7.** If \( A \) is an operator algebra and \( x \in \frac{1}{2}\mathfrak{h}A \) then \((\text{Re}(x^{\frac{1}{2}}))\) is increasing.

Proof. We will prove a little more. Let \( 0 < s < t \leq 1 \), and write

\[
f(z) = ((1-z)/2)^s - ((1-z)/2)^t, \quad z \in \mathbb{C}, |z| \leq 1.
\]

This has positive real part by the easy case of the present result where \( A = \mathbb{C} \). Then apply [19, Proposition 3.1, Chapter IV] to deduce that \( f(1-2x) \) is accretive.
Here \( f(1 - 2x) \) is the ‘disk algebra functional calculus’, arising from von Neumann’s inequality for the contraction \( 1 - 2x \), applied to \( f \). As in [8, Proposition 2.3] we have \( f(1 - 2x) = x^s - x^t \).

\[ \Re(x^s - x^t) \geq 0. \]

Remarks. 1) Proposition 4.7 is false in general for norm 1 elements of \( r_A \). A counterexample is the normalization of the matrix with rows 1 and \( i \), and \( i \) and 0. However it is shown in [9] that for any \( a \in r_A \) there is a positive constant \( c \) with \( \Re((ca)^n)_{n \geq 2} \) increasing.

2) Proposition 4.7 may be used to give the existence of ‘increasing’ approximate identities in separable approximately unital operator algebras [9].

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