ALIGNED CP-SEMIGROUPS

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Abstract. A CP-semigroup is aligned if its set of trivially maximal subordinates is totally ordered by subordination. We prove that aligned spatial $E_0$-semigroups are prime: they have no non-trivial tensor product decompositions up to cocycle conjugacy. As a consequence, we establish the existence of uncountably many non-cocycle conjugate $E_0$-semigroups of type $II_0$ which are prime.

Let $H$ be a Hilbert space, which we will always assume to be separable and infinite-dimensional, and let $\mathfrak{B}(H)$ denote the $*$-algebra of all bounded operators over $H$. A CP-semigroup acting on $\mathfrak{B}(H)$ is a point-$\sigma$-weakly continuous semigroup $\alpha = \{\alpha_t : \mathfrak{B}(H) \to \mathfrak{B}(H)\}_{t \geq 0}$ of normal completely positive contractions such that $\alpha_0 = \text{id}$. When $\alpha_t$ is an endomorphism and $\alpha_t(I) = I$ for all $t \geq 0$, then $\alpha$ is called an $E_0$-semigroup. In the special case when $H = K \otimes L^2(0, \infty)$, a CP-semigroup $\alpha$ acting on $\mathfrak{B}(H)$ is called a CP-flow over $K$ if $\alpha_t(A)S_t = S_tA$ for all $A \in \mathfrak{B}(H)$, $t \geq 0$ where $\{S_t : t \geq 0\}$ is the right shift semigroup. We will say that a CP-semigroup $\beta$ is a subordinate of $\alpha$ if it also acts on $\mathfrak{B}(H)$ and $\alpha_t - \beta_t$ is completely positive. If in addition $\beta$ is a CP-flow, then it is called a flow subordinate of $\alpha$.

We direct the reader to [Arv03] for a general reference on the theory of CP-semigroups and to [Pow03b] for the basic theory of CP-flows.

In this paper we study a class of CP-semigroups which has a set of subordinates which is minimal in a natural sense. We call such CP-semigroups aligned. This class is shown to include examples considered previously in [Pow03a, APP06, Jan10, JMP11]. We prove that aligned $E_0$-semigroups have a notable property: they are prime in the sense that they have no non-trivial tensor product decompositions up to cocycle conjugacy. As a consequence, we establish the existence of uncountably many non-cocycle conjugate $E_0$-semigroups of type $II_0$ which are prime. The previously known non-trivial examples of prime $E_0$-semigroups were obtained independently in [MP09] (type $II_1$), [Tsi08] (type $II_1$) and [Lie09] (type $II_k$ for $k = 1, 2, \ldots$).

Powers introduced in [Pow03a] the concept of $q$-purity which has been valuable for the study of $E_0$-semigroups (see for example [APP06, Jan10]). The notion of $q$-purity was refined in [JMP11]; a CP-flow is $q$-pure if and only if its set of flow subordinates is totally ordered by subordination. It is clear that an $E_0$-semigroup which is in addition a $q$-pure CP-flow must have index 0 or 1, and must be of type $I_1$, $II_0$ or $II_1$. The case of type $I_0$ is excluded because an automorphism group cannot be a CP-flow.

The restriction to flow subordinates in the definition of $q$-pure CP-flows can obscure some useful properties of these CP-semigroups. In order to circumvent this difficulty we consider an alternative concept which is inspired by the notion of $q$-purity.

**Definition 1.** Let $\alpha$ be a CP-semigroup and let $\beta$ be a CP-semigroup subordinate of $\alpha$. We will say that $\beta$ is trivially maximal if the semigroup $e^{kt}\beta_t$ is not a subordinate of $\alpha$ for $k > 0$. 

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We will denote by $\mathcal{S}(\alpha)$ the set of all trivially maximal subordinates of $\alpha$ partially ordered by subordination. We will say $\alpha$ is \textit{aligned} if $\mathcal{S}(\alpha)$ is totally ordered.

Let $\beta$ be a trivially maximal subordinate of $\alpha$. We will denote by $\mathcal{S}(\alpha; \beta)$ the set of all trivially maximal CP-semigroup subordinates of $\alpha$ which dominate $\beta$, partially ordered by subordination. We will say that $\alpha$ is \textit{aligned relative to $\beta$} if $\mathcal{S}(\alpha; \beta)$ is totally ordered.

Notice that if an aligned $E_0$-semigroup is spatial, then it must have index zero.

**Lemma 2.** A unital CP-semigroup is aligned if and only if its minimal dilation is aligned.

**Proof.** Suppose that $\alpha$ is a unital CP-semigroup acting on $\mathcal{H}$ with minimal dilation $\alpha^d$ acting on $\mathcal{B}(\mathcal{H}_1)$, i.e. there exists an isometry $W : \mathcal{H} \to \mathcal{H}_1$ such that

$$\alpha_t(x) = W^* \alpha_t^d(WxW^*)W$$

and $WW^*$ is increasing for $\alpha^d$. In order to prove the statement it suffices to show that there is an order isomorphism between $\mathcal{S}(\alpha)$ and $\mathcal{S}(\alpha^d)$. As proved in [Bha01] or by Theorem 3.5 of [Pow03b], there exists an order isomorphism between the set of CP-semigroup subordinates of $\alpha$ and the set of CP-semigroup subordinates of $\alpha^d$. This isomorphism is described as follows. For every subordinate $\beta$ of $\alpha$ there exists a unique subordinate $\beta'$ of $\alpha^d$ such that

$$\beta_t(x) = W^* \beta'_t(WxW^*)W.$$ (1)

It is clear that if $\beta$ is trivially maximal (with respect to $\alpha$), then $\beta'$ is trivially maximal (with respect to $\alpha^d$). Conversely, suppose that $\beta$ is not trivially maximal, so that there exists $k > 0$ such that $e^{kt}\beta_t$ is a subordinate of $\alpha$. Then there exists $\gamma$ a unique subordinate of $\alpha^d$ such that

$$e^{kt}\beta'_t(x) = W^* \gamma_t(WxW^*)W, \forall x \in \mathcal{B}(\mathcal{H}), t \geq 0.$$ 

Therefore, by dividing by $e^{kt}$ we obtain that $\beta'$ is also the compression of $e^{-kt}\gamma_t$ which is obviously also a subordinate since $k > 0$. By uniqueness of the correspondence, $\beta'_t = e^{-kt}\gamma_t$ for all $t$. Hence $\beta'$ is not trivially maximal.

CP-flow subordinates of a CP-flow are always trivially maximal, therefore if a CP-flow is aligned then it is automatically q-pure. We also note that a CP-flow is q-pure if and only if it is aligned with respect to the subordinate $x \mapsto S_t x S_t^*$. We omit the elementary proof of this fact.

We now show that unital CP-flows are aligned if and only if they are q-pure and induce $E_0$-semigroups of type $I_0$. First let us approach the case when the CP-flow is in a semigroup of endomorphisms.

**Proposition 3.** Let $\alpha$ be an $E_0$-semigroup which is in addition a CP-flow over $\mathcal{R}$. If $\alpha$ has type $I_0$, then $\mathcal{S}(\alpha)$ is precisely the set of flow subordinates of $\alpha$. Therefore, $\alpha$ is aligned if and only if it is type $I_0$ and it is q-pure.

**Proof.** Suppose that $\alpha$ is an $E_0$-semigroup of type $I_0$ which is also a CP-flow over $\mathcal{R}$. Note that every flow subordinate of $\alpha$ is clearly an element of $\mathcal{S}(\alpha)$, as flow subordinates are always trivially maximal. Conversely, let $\beta$ be a trivially maximal CP-semigroup subordinate of $\alpha$. By Theorem 3.4 of [Pow03b] there exists a family of operators $(C(t))_{t \geq 0}$ in $\mathcal{B}(\mathcal{H})$ such that

$$\beta_t(x) = C(t) \alpha_t(x),$$

where the family $(C(t))_{t \geq 0}$ is a contractive positive local cocycle, i.e. $C(t) \in \alpha_t(\mathcal{B}(\mathcal{H}))'$ and $0 \leq C(t) \leq I$ for all $t > 0$, $C(t + s) = C(t)\alpha_t(C(s))$ for all $t, s \geq 0$ and $t \mapsto C(t)$ is strongly continuous for $t \geq 0$ with $C(t) \to I$ as $t \to 0+$.

Let $S_t$ denote as usual the right shift semigroup on $\mathcal{H} = \mathcal{R} \otimes L^2(0, \infty)$ and let $V_t = C(t)S_t$. Note that $V_t$ is strongly continuous and furthermore it is a semigroup: for all $t, r \geq 0$,

$$V_t V_r = C(t)S_tC(r)S_r = C(t)\alpha_t(C(r))S_tS_r = C(t + r)S_{t+r} = V_{t+r}.$$
Moreover, $V_t$ is a unit of $\alpha_t$, since for all $x \in \mathcal{B}(\mathcal{H})$, 

$$\alpha_t(x)V_t = \alpha_t(x)C(t)S_t = C(t)\alpha_t(x)S_t = C(t)S_t x = V_t x.$$ 

Notice, however, that $\alpha$ is type $\Pi_0$, therefore there exists $\kappa \in \mathbb{C}$ such that $V_t = e^{-\kappa t}S_t$ for all $t \geq 0$. Furthermore, since $C(t)$ is a contraction, we have that $V_t$ is a contraction for all $t > 0$, hence we must have $\text{Re}(\kappa) \geq 0$. We now show that in fact $\kappa$ must be a real number. Recall that $\beta_t$ is a CP-semigroup, and

$$0 \leq S^*_t \beta_t(I)S_t = S^*_t C(t)S_t = e^{-\kappa t}I,$$

hence $\kappa$ is real and satisfies $\kappa \geq 0$.

We will now prove that $\kappa = 0$. Let $\gamma_t(x) = e^{\kappa t}\beta_t(x)$. Notice that $S_t$ is an intertwiner semigroup for $\gamma$: for all $t > 0$ and $x \in \mathcal{B}(\mathcal{H})$,

$$\gamma_t(x)S_t = e^{\kappa t} \beta_t(x)S_t = e^{\kappa t} C(t)\alpha_t(x)S_t = e^{\kappa t} C(t)S_t x = S_t x.$$ 

It follows that $\gamma$ is a CP$_\kappa$-flow in the sense of Definition 4.0 of [Pow03b], that is to say, $e^{-\kappa t} \gamma_t$ is a CP-semigroup and $\gamma$ is intertwined by the shift. However, by Theorem 4.15 of [Pow03b], every CP$_\kappa$-flow must be a CP-flow. But a CP-flow is contractive, hence we must have $\gamma_t(I) \leq I$, thus

$$e^{\kappa t} \beta_t(I) = e^{\kappa t} C(t) \leq I$$

for all $t > 0$. Therefore, we have that for all positive $t > 0$, 

$$e^{\kappa t} \beta_t(x) = e^{\kappa t} C(t)\alpha_t(x) \leq \alpha_t(x).$$

Since $\beta$ is trivially maximal, we must have that $\kappa = 0$. Thus we have shown that every element of $\mathcal{S}(\alpha)$ is a CP-flow.

Therefore, if $\alpha$ is $q$-pure and of type $\Pi_0$ then it is aligned. On the other hand, it is clear that if $\alpha$ is an aligned $E_0$-semigroup, then it is type $\Pi_0$ and $q$-pure as discussed before the proposition.

**Theorem 4.** Let $\alpha$ be a unital CP-flow over $\mathcal{A}$. If the minimal dilation of $\alpha$ is type $\Pi_0$, then $\mathcal{S}(\alpha)$ is precisely the set of flow subordinates of $\alpha$. Therefore, $\alpha$ is aligned if and only if it is $q$-pure and its minimal dilation is type $\Pi_0$.

**Proof.** By Lemma 4.50 of [Pow03b], there exists a minimal dilation of $\alpha$ to an $E_0$-semigroup $\alpha^d$ which is a CP-flow over the Hilbert space $\mathcal{H}_1 = \mathcal{A}_1 \otimes L^2(0, \infty)$, i.e. there exists an isometry $W : \mathcal{H} \to \mathcal{H}_1$ such that

$$\alpha_t(x) = W^* \alpha^d_t(W x W^*) W,$$

$WW^*$ is increasing for $\alpha^d$ and if $S^d_t$ denotes the right shift semigroup on the space $\mathcal{H}_1$, then $WS_t = S^d_t W$ for all $t > 0$. We use the order isomorphism established in the proof of Lemma[2] that associates to each subordinate $\beta \in \mathcal{S}(\alpha)$ a unique subordinate $\beta^d \in \mathcal{S}(\alpha^d)$ satisfying (1). If $\alpha^d$ is type $\Pi_0$ and $\beta \in \mathcal{S}(\alpha)$, then it follows from the previous proposition that $\beta^d$ is a CP-flow. Hence for all $t > 0$,

$$\beta_t(x)S_t = W^* \beta^d_t(W x W^*) W S_t = W^* \beta^d_t(W x W^*) S^d_t W = W^* S^d_t W x W^* W = S_t x.$$ 

In other words, $\beta$ is a CP-flow. Thus we proved that all elements of $\mathcal{S}(\alpha)$ are CP-flows. On the other hand, every flow subordinate of $\alpha$ is clearly an element of $\mathcal{S}(\alpha)$.

Thus, if $\alpha$ is a unital $q$-pure CP-flow and its minimal dilation has type $\Pi_0$, then it is aligned. Conversely, it is clear that if $\alpha$ is aligned then its minimal dilation $\alpha^d$ as discussed above is also aligned, thus it has index zero. Since $\alpha^d$ is a CP-flow, it cannot be an automorphism group, hence it has type $\Pi_0$. Moreover, since $\alpha$ is aligned, it is clearly $q$-pure. 

\[ \square \]
**Prime E$_0$-semigroups**

**Definition 5.** An E$_0$-semigroup $\alpha$ is called prime if, whenever $\alpha$ is cocycle conjugate to $\beta \otimes \gamma$ where $\beta$ and $\gamma$ are E$_0$-semigroups, then $\beta$ or $\gamma$ is type I$_0$.

It follows from the complete classification of E$_0$-semigroups of type I in terms of the index, and the additivity of the index with respect to tensoring, that a prime E$_0$-semigroup of type I is cocycle conjugate either to an automorphism group or to the CAR/CCR flow of index 1. It is a corollary of the work on the gauge group of an E$_0$-semigroup by Markiewicz-Powers in [MP09] or Tsirelson in [LT08], that prime E$_0$-semigroups of type II$_1$ exist. And, as it has belatedly come to our attention, earlier Liebscher had proven that prime E$_0$-semigroups of type II$_k$ exist for $k \geq 1$ (see Proposition 4.32 and Note 4.33 in [Lie09]). We establish that there exist uncountably many prime E$_0$-semigroups of type II$_0$.

**Theorem 6.** Aligned spatial E$_0$-semigroups are prime.

*Proof.* We prove the contrapositive. Suppose that $\alpha$ is an E$_0$-semigroup and $\alpha = \beta \otimes \gamma$ where $\beta$ and $\gamma$ are two spatial E$_0$-semigroups neither of which has type I$_0$. Without loss of generality, by applying an appropriate conjugacy, we may assume that both act on $\mathcal{B}(\mathcal{F})$ where $\mathcal{F}$ is infinite-dimensional and separable. Let $U$ and $V$ be normalized units of $\beta$ and $\gamma$, respectively. Let us denote by $\theta^U$ and $\theta^V$ the semigroups given by $\theta^U_t(x) = U_t x U^*_t$ and $\theta^V_t(x) = V_t x V^*_t$, which are E-semigroup subordinates of $\beta$ and $\gamma$, respectively (notice these are not unital since $\beta$ and $\gamma$ are not automorphism groups). Notice that $\sigma^V = \beta \otimes \theta^V$ is also a subordinate of $\alpha$. We show that it is trivially maximal. Suppose that $k \geq 0$ and $e^{kt} \sigma^V_t$ is also a subordinate of $\alpha$. Then we have that for all $x \in \mathcal{B}(\mathcal{F})$,

$$e^{kt}(I \otimes V_t V^*_t) = e^{kt}(\beta_t \otimes \theta^V_t)(I) = e^{kt} \sigma^V_t(I) \leq \alpha_t(I) = I$$

However $\|V_t\| = 1$, hence by taking norms on both sides we conclude $e^{kt} \leq 1$ for all $t > 0$. Thus $k = 0$. Analogously, $\sigma^U = \theta^U \otimes \gamma$ is trivially maximal.

We prove that $\sigma^V$ and $\sigma^U$ are incomparable. Suppose that $\sigma^V \geq \sigma^U$. Notice that for all $x, y \in \mathcal{B}(\mathcal{F})$ and $t > 0$,

$$\sigma^V_t(x \otimes y)(U_t \otimes I) = \alpha_t(x)U_t \otimes \theta^V_t(x) = (U_t \otimes I)(x \otimes \theta^V_t(y))$$

$$\sigma^U_t(x \otimes y)(U_t \otimes I) = \theta^U_t(x)U_t \otimes \gamma_t(x) = (U_t \otimes I)(x \otimes \gamma_t(y))$$

Therefore we have that for all $x, y \in \mathcal{B}(\mathcal{F})$ and $t > 0$,

$$(U_t \otimes I)^* \left[ \sigma^V_t(x \otimes y) - \sigma^U_t(x \otimes y) \right] (U_t \otimes I) = x \otimes [\theta^V_t(y) - \gamma_t(y)]$$

Thus in the special case when $x = I$, we have that the map $y \mapsto I \otimes [\theta^V_t(y) - \gamma_t(y)]$ is completely positive for all $t > 0$, hence $\theta^V \geq \gamma$. However $\theta^V$ is a pure element in the cone of completely positive maps, therefore we have that for every $t > 0$, $\gamma_t$ is a multiple of $\theta^V_t$, and we have a contradiction since $\gamma$ is not type I$_0$. By symmetry, we obtain that $\sigma^U$ and $\sigma^V$ are incomparable as asserted. Thus we proved that $\alpha$ is not aligned. \qed

**Corollary 7.** There exist uncountably many non-cocycle conjugate E$_0$-semigroups of type II$_0$ which are prime.

*Proof.* By Theorem 6 it suffices to exhibit an uncountable family of non-cocycle conjugate aligned E$_0$-semigroups. In order to do so, we consider a class of E$_0$-semigroups arising from boundary weight doubles as in [Jan10]. Let $g(x)$ be a fixed complex-valued measurable function such that $g \not\in L^2(0, \infty)$ yet $(1 - e^{-x})^{1/2} g(x) \in L^2(0, \infty)$, and for each $t > 0$ let $g_t = \chi_{(t, \infty)} g$, which is an element of $L^2(0, \infty)$. Define the weight on $\mathcal{B}(L^2(0, \infty))$ given by $\nu(A) = \lim_{t \to 0^+} (g_t, Ag_t)$. For every $0 < \lambda < 1/2$, let $\mu_\lambda : M_2(\mathbb{C}) \to \mathbb{C}$ be given by

$$\mu_\lambda(X) = \lambda x_{11} + (1 - \lambda) x_{22}.$$
Let us define the boundary weight map from \( M_2(\mathbb{C}) \) to boundary weights on \( \mathcal{B}(\mathcal{C}^2 \otimes \mathcal{L}^2(0, \infty)) \) given by \( \omega(\rho)(A) = \rho(I)\mu_\rho((I \otimes \nu)(A)) \) for all \( \rho \in M_2(\mathbb{C}) \) and \( A \) in the domain of finiteness of \( I \otimes \nu \) (the so called null boundary algebra of definition 4.16 in \[Pow03b\]). Then by Corollary 3.3 of \[Jan10\] and the assumptions on \( g(x) \), we have that \( \omega \) gives rise to an \( \mathcal{E}_0 \)-semigroup of type \( \Pi_0 \). Once one applies Theorem 4.4 of \[JMP11\] to reconcile the earlier definition of \( q \)-purity with the one in this paper, we obtain from Lemma 4.3 and Proposition 5.2 of \[Jan10\] that \( \omega \) gives rise to a \( q \)-pure unital CP-flow. Therefore by Theorem 4 it gives to an aligned \( \mathcal{E}_0 \)-semigroup \( \alpha^\lambda \). Finally, by Theorem 5.4 of \[Jan10\], given \( \lambda, \zeta \in (0, 1/2) \), we have that \( \alpha^\lambda \) is cocycle conjugate to \( \alpha^\zeta \) if and only if \( \lambda = \zeta \). \[\square\]

We should point out that it is possible to obtain a different uncountable family of non-cocycle conjugate \( \mathcal{E}_0 \)-semigroups by using Theorem 3.22 of \[Pow03a\]. For details see the discussion in the end of section III therein. This would be indeed be a different family from the one exhibited above in the sense that, by Corollary 5.5 of \[Jan10\], the \( \mathcal{E}_0 \)-semigroups constituting both families are not cocycle conjugate.

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