New sparse array for non-circular sources with increased degrees of freedom

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Recently, sparse arrays have received considerable attention as they provide larger array aperture and increased degrees-of-freedom (DOFs) compared to uniform linear arrays. These features are essential to enhance the direction-of-arrival estimation performance. However, most of the existing sparse arrays are mainly designed for circular sources and realize limited increment in DOFs for non-circular sources. In this letter, a new sparse array configuration for non-circular sources is presented, which significantly increases the achievable DOFs and improves the direction-of-arrival estimation performance. The proposed geometry comprises two effectively configured uniform linear arrays that exploit the characteristics of non-circular sources and extend the array aperture. For a given number of sensors, its virtual array is a numerously a hole-free uniform linear array. Moreover, the precise sensor locations, achievable DOFs, and optimal distribution of physical sensors are determined analytically by closed-form expressions. Owing to these benefits, the proposed array efficiently resolve multiple sources in under-determined conditions and achieves better direction-of-arrival estimation performance than its counterpart structures. Simulation results validate the superiority of the proposed configuration.

Introduction: Array signal processing is a fundamental and time-honored technology widely used in communication fields [1, 2]. The primary advantages of utilizing sensor arrays include improving the signal quality, mitigating the interference, and spatial selectivity. In this regard, uniform linear arrays (ULAs) are conventionally used [3]. However, they only provide limited degrees-of-freedom (DOFs) and experience severe mutual coupling between array elements. To overcome these limitations, non-uniform linear arrays (NLAs), often known as sparse arrays, were introduced. The NLAs substantially increases the DOFs, and due to larger inter-sensor spacing in NLAs, coupling effects may also be reduced [4].

Minimum redundancy array (MRA) is a well-known sparse array structure that provides maximum array aperture and enhanced DOFs [5]. Although the MRA is an optimum geometry to achieve maximum DOFs, there is no closed-form expression available to determine the antenna locations and DOFs. As a result, it requires exhaustive search algorithms for sensor placement. Such shortcomings lead to computational complexities [6], and thus, limit its application in practice. Lately, the development of NLAs such as nested array [7], co-prime array [8] and maximum inter-element spacing constraint (MISC) array [9] have renewed interest of researchers in this topic as all of them can be constructed by solving the maximum inter-element spacing constraint (MISC) array [9] have renewed interest of researchers in this topic as all of them can be constructed by solving the maximum inter-element spacing constraint (MISC) array [9]. Table 1 illustrates the optimal distribution of sensors in NSANCS that achieves maximum uDOFs, under the constraint of a given number of elements, that is, $M = M_1 + M_2$. Arithmetic mean-geometric mean inequalities can obtain the optimal solutions summarized in Table 1.

Signal model: Consider Z narrowband stationary sources incident on M-element array from different directions $(\theta_1, \theta_2, \ldots, \theta_Z)$ with the corresponding source powers, $p = [\sigma_1^2, \sigma_2^2, \ldots, \sigma_Z^2]^T$, where $(\cdot)^T$ is the transpose. The received signal $x(t)$ at time $t$ can be expressed as:

$$x(t) = A(t) * n(t),$$

where $s_i(t) = [s_1(t), s_2(t), \ldots, s_M(t)]^T$. The components of noise vector $n(t)$ are the additive white gaussian noise, with zero value of mean and covariance of $\sigma_n^2 I$, where $I$ represents the identity matrix. $A = [a(\theta_1), a(\theta_2), \ldots, a(\theta_Z)]$ is the $M \times Z$ steering matrix with $a(\theta_i)$ is the steering vector of $z$-th source, $(z = 1, 2, \ldots, Z)$, which is given by

$$a(\theta_i) = \left[1, \exp(-j2\pi l_1 \sin \theta_i / \lambda), \exp(-j2\pi l_2 \sin \theta_i / \lambda), \ldots, \exp(-j2\pi l_{M-1} \sin \theta_i / \lambda) \right]^T.$$
the corresponding set of physical locations. For the proposed array, \( l_m \in \text{NSANCS} \). The covariance matrix corresponding to \( x(t) \) is given by

\[
R_x = E[x(t)x^H(t)],
\]

where \( E[.] \) is the statistical expectation, \( (\cdot)^H \) shows conjugate transpose.

\[
R_x = AR_xA^H + \sigma_n^2 I.
\]

In practice, the covariance matrix is averaged over \( T \) snapshots

\[
\hat{R}_x = \frac{1}{T} \sum_{t=1}^{T} [x(t)x^H(t)].
\]

Now \( E[x(t)x^*(t)] \) with \( (\cdot)^* \) symbolizing the conjugation indicates the \( p \)-th row and \( q \)-th column of \( R_x \), which is given by

\[
R_{p,q} = \exp\left(-j\frac{2\pi}{\lambda}(l_p - l_q)\right)\sigma_n^2,
\]

where \( \{l_p, l_q\} \in \mathbb{S} \). The difference set can be defined as

\[
\mathbb{L} = \{l_p - l_q | 0 \leq p, q \leq M - 1\}.
\]

The difference set \( \mathbb{L} \) forms the basis of a virtual array, and the elements in \( \mathbb{L} \) signifies the relative location of the virtual array. It is well-known that the EC of CS is zero, and a non-zero value of EC implies that the sources are non-circular, that is, ASK, PAM, BPSK, AM. In contrast to CS, NCS carries more valuable information that can be exploited to extend virtual array and improve the DOA estimation performance.

Since the EC of NCS is not zero

\[
\hat{R}_x = E[x(t)x^T(t)] \neq 0.
\]

By utilizing the non-circular characteristics of sources, the array aperture can be enlarged. Therefore

\[
x_c(t) = \begin{bmatrix} x(t) \\ x^*(t) \end{bmatrix},
\]

\[
x_c(t) = \begin{bmatrix} A \\ A^* \end{bmatrix} s(t) + \begin{bmatrix} n(t) \\ n^*(t) \end{bmatrix}.
\]

The equivalent manifold matrix is represented by \( B = A[A^*]^{-1} \). Hence,

\[
x_c(t) = Bs(t) + \begin{bmatrix} n(t) \\ n^*(t) \end{bmatrix}.
\]

Correspondingly, the extended covariance matrix is given by

\[
R_{x_c} = E[x_c(t)x_c^H(t)].
\]

\[
R_{x_c} = BR_xB^H + \sigma_n^2 I.
\]

Following this, \( BB^H \) extends the array aperture; according to (8), the virtual array manifold of an extended array for NCS can be obtained by

\[
\mathbb{L}_{\text{NCS}} = \{l_p - l_q | 0 \leq p, q \leq M - 1\}.
\]

In general, to realize enhanced uDOFs, the number of continuous virtual locations in the resultant co-array reaches to the maximum. Therefore, the proposed configuration aims to achieve a higher number of continuous virtual sensors as well as produce a hole-free resulting ULA.

**Degrees of freedom:** One of the key benefits of the proposed array is its convenient construction due to the availability of closed-form expressions to obtain not only the exact sensor locations but also uDOFs. We also formulate the closed-form expressions based on an extended virtual array defined in (17) for determining the corresponding uDOFs in nested, improved nested, and MISC arrays. The generalized expression of uDOFs in NSANCS, nested and improved nested arrays for NCS is given by:

When \( M \) is odd, uDOFs

\[
\text{uDOFs} = \frac{M^2 - 1}{2} + M + c (M - 1).
\]

When \( M \) is even, uDOFs

\[
\text{uDOFs} = \frac{M^2 - 2}{2} + M + c (M - 2) + 2,
\]

where \( c \) assumes the values 1, 2, and 2 + 1. Therefore, the proposed configuration aims to achieve a higher number of continuous virtual positions.

Figure 2 shows the geometric distribution of physical and virtual sensors of typical array structures based on the extended array aperture, where \( M = 5 \). It is observed from Figure 2 that the proposed NSANCS configuration achieves a higher number of continuous virtual positions. It is also evident that only the NSANCS array realizes hole-free virtual ULA among all the arrays illustrated in Figure 2. Although SANC also achieves the same number of continuous virtual sensors, its resulting
The SNR is assumed over the range $[1, 5, 15]$ dB with 200 snapshots for seven sources. These sources impinge on the five-element array from different directions, $[-50^\circ, -35^\circ, -20^\circ, -10^\circ, 0^\circ, 35^\circ, 50^\circ]$. It is evident from Figure 4 that the proposed array exhibits better DOA estimation performance over an increasing range of SNR as compared to other array geometries. Likewise, we performed the DOA estimation with reference to the number to snapshots, as shown in Figure 5. We keep the same parameters as in the previous case, except the range of snapshots is considered over a fixed SNR at 0 dB. It is observed from Figure 5 that the DOA estimation accuracy of the proposed array configuration is higher than the other arrays, and owns a smaller RMSE value when the number of snapshots is greater than or equal to 150.

Conclusion: We proposed a new array configuration for NCS that achieves more uDOFs and results in a hole-free virtual ULA. Via utilizing the characteristics of NCS and exploiting extended array aperture, the NSANCS reveals significant improvement in the DOA estimation performance compared to its sparse array counterparts. Unlike MRA and SANC, the NSANCS array is benefited from closed-form expressions to obtain uDOFs for the nested array, improved nested array, and MISC array based on the extended virtual array. Furthermore, the optimal distribution of physical sensors is presented for NSANCS array to maximize its uDOFs capacity. Numerical results validate the effectiveness of the NSANCS configuration.

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