GYSIN FUNCTORS AND THE GROTHENDIECK-WITT CATEGORY, PART I

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Abstract. Fix a field $k$. Consider the motivic stable homotopy category over $k$, and restrict to the full subcategory whose objects are the suspension spectra of separable field extensions of $k$. We give an algebraic description of this category, identifying it with a construction we call the Grothendieck-Witt category. In this first of two papers we develop the general categorical machinery that describes this situation: that of Gysin functors and their associated categories of correspondences. We prove a “recognition theorem” for these correspondence categories, and develop results concerning their structure.

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1. Introduction

Fix a ground field $k$. In this paper we describe a category $\text{GWC}(k)$, called the Grothendieck-Witt category of $k$, whose objects are the finite separable field extensions of $k$. The morphisms are a Grothendieck group of certain kinds of “correspondences” built up from bilinear forms, and there is an intrinsic notion of composition. We then generalize this situation into the theory of what we call Gysin functors and their associated categories of correspondences. We prove several results about the general structure of such categories.

To further explain the ideas and motivation of this paper we take a brief detour into equivariant homotopy theory. Let $G$ be a finite group, and let $G\text{Top}$ be the category of $G$-spaces and equivariant maps. We regard $G\text{Set}$, the category of $G$-sets, as the full subcategory of $G\text{Top}$ consisting of the discrete $G$-spaces. The orbit category $\text{Or}(G)$ of $G$ is the full subcategory of $G\text{Set}$ consisting of the $G$-sets on which $G$ acts transitively. Every object in $\text{Or}(G)$ is isomorphic to a quotient $G/H$, for some subgroup $H$. 
Next consider the stabilization functor $\Sigma^\infty: G\text{Top} \to G\text{Spectra}$ from $G$-spaces to genuine $G$-spectra (the version of $G$-spectra where representation spheres are invertible). When restricted to $G\text{Set}$ this map is an embedding, but it is not full. The full subcategory of $G\text{Spectra}$ whose objects are $\Sigma^\infty \emptyset_+$ for $\emptyset$ a $G$-set is called the **stable category** of $G$-sets, and denoted $G\text{Set}^{st}$. We will actually focus on $G\text{Set}^{st}_\text{fin}$, where we restrict $\emptyset$ to be a finite $G$-set. The full subcategory of $G\text{Set}^{st}_\text{fin}$ consisting of the objects $\Sigma^\infty (G/H)_+$ is called the **stable orbit category**.

There are two common ways of describing $G\text{Set}^{st}_\text{fin}$:

1. Given two finite $G$-sets $\emptyset_1$ and $\emptyset_2$, define a span from $\emptyset_1$ to $\emptyset_2$ to be a diagram

   \[
   \begin{array}{ccc}
   & P & \\
   \emptyset_2 & \downarrow & \emptyset_1 \\
   \end{array}
   \]

   in the category of finite $G$-sets. A map between spans is a map of diagrams that is the identity on $\emptyset_1$ and $\emptyset_2$. This category has a monoidal structure given by disjoint union in the “$P$”-variable. Define $\text{Burn}(\emptyset_1, \emptyset_2)$ to be the Grothendieck group of isomorphism classes of spans from $\emptyset_1$ to $\emptyset_2$, with respect to this disjoint union operation.

   Note that we will sometimes refer to spans as “correspondences”, as that terminology is often used in geometric settings.

   If we have three finite $G$-sets $\emptyset_1$, $\emptyset_2$, and $\emptyset_3$ then we can define a composition of spans via the pullback operation shown in the following diagram:

   \[
   \begin{array}{ccc}
   Q \times_{\emptyset_2} P & \\
   \downarrow & \downarrow & \\
   Q & P & \\
   \emptyset_3 & \emptyset_2 & \emptyset_1 \\
   \end{array}
   \]

   This operation induces a map

   \[ \text{Burn}(\emptyset_2, \emptyset_3) \times \text{Burn}(\emptyset_1, \emptyset_2) \to \text{Burn}(\emptyset_1, \emptyset_3) \]

   which is readily checked to be unital and associative. So we have defined a category $\text{Burn}$ whose objects are the $G$-orbits. This is usually called the **Burnside category** of $G$-sets.

   Here are some things to take note of:

   (a) There is a functor $R: G\text{Set}^\text{st}_\text{fin} \to \text{Burn}$ that is the identity on objects and sends a map $f: \emptyset_1 \to \emptyset_2$ to the span $[\emptyset_2 \leftarrow \emptyset_1 \xrightarrow{id} \emptyset_1]$.

   (b) The category $\text{Burn}$ has a duality anti-automorphism $(-)^*$ which is the identity on objects, and on morphisms sends a span $[\emptyset_2 \leftarrow P \to \emptyset_1]$ to the similar span $[\emptyset_1 \leftarrow P \to \emptyset_2]$ obtained by reversing the order of the maps.

   The duality functor is an isomorphism

   \[ (-)^*: \text{Burn}^{op} \to \text{Burn}. \]

   (c) In particular, setting $I = (-)^* \circ R$ gives a functor $I: G\text{Set}^{op} \to \text{Burn}$. If $f: \emptyset_1 \to \emptyset_2$ then $I(f)$ is the span $[\emptyset_1 \leftarrow \emptyset_1 \xrightarrow{f} \emptyset_2]$.  


If $\mathcal{A}$ is an additive category then additive functors $\mathcal{B}urn^{op} \to \mathcal{A}$ are the same as what are usually called Mackey functors. (One could also identify Mackey functors with additive functors $\mathcal{B}urn \to \mathcal{A}$, since $\mathcal{B}urn$ is self-dual; however, our notation for the $R$ and $I$ maps fits better with the contravariant option).

It is a classical theorem (perhaps a folk theorem) that $\mathcal{B}urn$ is isomorphic to the stable category of finite $G$-sets.

(2) The stable orbit category $Or(G)^{st}$ can also be described in terms of generators and relations. This is the free additive category whose objects are the transitive $G$-sets and whose morphisms are generated by the maps $R_f: O_1 \to O_2$ and $I_f: O_2 \to O_1$ for every map of $G$-sets $f: O_1 \to O_2$. The morphisms satisfy the relations:

(i) $R_{gf} = R_g \circ R_f$;
(ii) $I_{gf} = I_f \circ I_g$;
(iii) Given a pullback diagram of $G$-sets

$$
\begin{array}{ccc}
P & \xrightarrow{f} & O_3 \\
p & \downarrow & \downarrow q \\
O_1 & \xrightarrow{g} & O_2
\end{array}
$$

where the actions on $O_1$, $O_2$, and $O_3$ are transitive, write $P = \coprod_i X_i$ where each $X_i$ is a transitive $G$-set. Then

$$I_g \circ R_q = \sum_i R_{p_i} \circ I_{f_i}$$

where $f_i$ and $p_i$ are the restrictions of $f$ and $p$ to $X_i$.

It is again a classical theorem that this category, defined in terms of generators and relations, is isomorphic to the stable orbit category.

Now let us return to our original setting, where $k$ is a fixed ground field. Keeping the above discussion in mind, the point of this series of papers is to examine the full subcategory of the motivic stable homotopy category over $k$ whose objects are the suspension spectra of fields. This is vaguely analogous to the stable orbit category (although in the case of $G$-spectra the orbits generate the category, whereas field spectra do not generate the category in the motivic setting). Our goal is to give descriptions of this category that are analogs of (1) and (2). To give a sense of this in the first case, the Grothendieck-Witt category of $k$ is defined to be the category $GWC(k)$ whose objects are $\text{Spec} E$ for $E$ a finite, separable field extension of $k$. The morphisms from $\text{Spec} E$ to $\text{Spec} F$ are the Grothendieck-Witt group $GW(F \otimes_k E)$ of quadratic spaces over $F \otimes_k E$ (see Section 2 for details). The definition of composition is a little too cumbersome to be included in this introduction, but it mimics the composition we saw in (1) above.

Morel [Mo] proved that if $k$ is perfect and $F/k$ is a separable field extension then

$$[\Sigma^\infty (\text{Spec} F)^+, S] \cong GW(F)$$

where $S$ is the motivic sphere spectrum and $[-,-]$ denotes maps in the motivic stable homotopy category of smooth $k$-schemes. If $J/k$ is another separable extension
one can then argue that
\[ [\Sigma^\infty (\text{Spec } F)_+, \Sigma^\infty (\text{Spec } J)_+] \cong [\Sigma^\infty (\text{Spec } F)_+ \wedge \Sigma^\infty (\text{Spec } J)_+, S] \]
\[ \cong [\Sigma^\infty (\text{Spec}(F \otimes_k J))_+, S] \]
\[ \cong \text{GW}(F \otimes_k J) \]
where the first isomorphism uses a self-duality \( \Sigma^\infty (\text{Spec } J)_+ \cong \mathcal{I}(\Sigma^\infty (\text{Spec } J)_+, S) \) and the last isomorphism is the aforementioned one of Morel (using that \( F \otimes_k J \) decomposes as a product of separable field extensions of \( k \)). The self-duality is dealt with in the appendix to [H], and in the equivariant context it is in modern times usually couched in the machinery of the Wirthmüller isomorphism (cf. [Ma2], for example).

Accepting the above computation, it remains to compute the composition in the motivic stable homotopy category and relate it to the appropriate pairing of Grothendieck-Witt groups. The present paper exists partly because attempting to do this by ad hoc methods proved unwieldy.

In the narrative we provide here, everything comes down to the existence of transfer maps. Transfer maps coupled with diagonal maps give rise to duality structures, and quite general categorical computations show that any reasonable category with this kind of structure may be described by a “correspondence-like” description of composition.

Let us now explain the results in a bit more detail. Let \( \mathcal{C} \) be a finitary lextensive category (see Section 3.1, but understand that this is basically just a category where coproducts behave nicely with respect to pullbacks). A Gysin functor on \( \mathcal{C} \) is an assignment \( X \mapsto E(X) \) from \( \text{ob}(\mathcal{C}) \) to commutative rings, together with pullback and pushforward maps satisfying certain compatibility properties. Given this situation, one can construct a category of correspondences \( \mathcal{C}_E \) where the object set is \( \text{ob}(\mathcal{C}) \), maps from \( X \) to \( Y \) are the abelian group \( E(Y \times X) \), and composition is obtained by a familiar formula using the pullback and pushforward maps. The category \( \mathcal{C}_E \) is enriched over abelian groups, is closed symmetric monoidal, and has the property that all objects are self-dual.

Now suppose \( \mathcal{H} \) is a closed tensor category (additive category with compatible symmetric monoidal structure) with tensor \( \otimes \) and unit \( S \). Suppose given functors \( R: \mathcal{C} \to \mathcal{H} \) and \( I: \mathcal{C}^{op} \to \mathcal{H} \) satisfying some reasonable hypotheses (see Section 4). For \( f: X \to Y \) in \( \mathcal{C} \) we think of \( RF \) as the “regular” map associated to \( f \) in \( \mathcal{H} \), whereas \( If \) is an associated transfer map. The prototype for this situation is where \( \mathcal{H} \) is the genuine \( G \)-equivariant stable homotopy category, \( \mathcal{C} \) is the category of finite \( G \)-sets, \( R(X) = I(X) = \Sigma^\infty (X_+) \), \( RF \) is the usual map induced by \( f: X \to Y \), and \( If \) is the corresponding transfer map.

Write \( \pi^0 \) for the functor \( \mathcal{C}^{op} \to \text{Ab} \) given by \( \pi^0(X) = \mathcal{H}(RX, S) \). This inherits the structure of a Gysin functor, and we prove the following:

**Theorem 1.1.** Under mild hypotheses, the category of correspondences \( \mathcal{C}_{\pi^0} \) is equivalent to the full subcategory of \( \mathcal{H} \) whose objects lie in the image of \( R \).

That is, we prove that one can reconstruct the appropriate subcategory of \( \mathcal{H} \) as the category of correspondences associated to the Gysin functor \( \pi^0 \). See Theorem 4.16 for a precise version of the above theorem.
The second result of this paper concerns the structure of the category of correspondences $C_E$ for a general Gysin functor $E$. In the Burnside category of a finite group, there are special collections of maps $Rf$ and $Ig$ and every map in the category may be written as a composite $Rf \circ Ig$. There are also rules for rewriting compositions $If \circ Ig$ in the above form. In the case of a general Gysin functor, there are three collections of special maps, elements of which are written $Rf$, $Ig$, and $Da$ where $f$ and $g$ are maps in $C$ and $a \in E(X)$ for some object $X$ in $C$. We prove the following:

**Theorem 1.2.** Every map in $C_E$ can be written as a sum of maps $Rf \circ Da \circ Ig$. Other composites of the $R−D−I$ maps can be rewritten in this form using the rules

(a) $Da \circ Rf = Rf \circ D(f^*a)$,
(b) $If \circ Da = D(f^*a) \circ If$,
(c) $If \circ Ig = Rp \circ Ig$ where $p$ and $q$ are the maps in the pullback diagram

$$
\begin{array}{ccc}
P & \overset{q}{\rightarrow} & A \\
\downarrow{p} & & \downarrow{g} \\
B & \overset{f}{\rightarrow} & C \\
\end{array}
$$

inside the category $C$.

Moreover, for a map $f: X \rightarrow Y$ in $C$ and $a \in E(X)$ one has $Rf \circ Da \circ If = D(f_!(a))$.

The following corollary is really just a reformulation of the theorem:

**Corollary 1.3.** Maps in $C_E(X,Y)$ can be represented by a pair consisting of a span $[Y \leftarrow Z \rightarrow X]$ and an element $a \in E(Z)$: this pair represents $Rf \circ Da \circ Ig$. If a map in $C_E(U,X)$ is represented by $[X \leftarrow Z \rightarrow U, a' \in E(Z')]$ then the composite is represented by the pullback span

$$
\begin{array}{ccc}
P & \overset{q}{\rightarrow} & A \\
\downarrow{p} & & \downarrow{g} \\
B & \overset{f}{\rightarrow} & C \\
\end{array}
$$

and the element $D((s^*a)(t^*a')) \in E(P)$. That is to say,

$$(Rf \circ Da \circ Ig) \circ (Rf' \circ Da' \circ Ig') = R(fs) \circ D((s^*a)(t^*a')) \circ I(g't).$$

Moreover, we have the extra relation

$$
\begin{bmatrix}
Y \overset{f}{\leftarrow} Z \overset{f}{\rightarrow} Y, a \in E(Z)
\end{bmatrix} = \begin{bmatrix}
Y \overset{id}{\leftarrow} Y \overset{id}{\rightarrow} Y, f!(a) \in E(Y)
\end{bmatrix}.
$$

If one assumes the category $C$ to have some basic Galois-type properties (which model the behavior of the category of $G$-sets) then explicit computations become easier. For example, one can prove the following:

**Proposition 1.4.** Assume $C$ is a Galoisien category (see Section 5.12), and let $X$ be an object in $C$ that is Galois. Then in $C_E$ one has

$$
\text{End}_{C_E}(X) = \widetilde{[\text{Aut}_C(X)]}E(X)
$$
where on the right we have the twisted group ring whose elements are finite sums
\[ \sum_i [g_i]a_i \] with \( g_i \in \text{Aut}_C(X) \) and \( a_i \in E(X) \), and the multiplication is determined
by the formula
\[ [g]a \cdot [h]b = [gh](h^*a \cdot b). \]
(Here \([g]a\) corresponds to the element \( Rg \circ Da \)).

The above proposition describes the full subcategory of \( C_E \) consisting of a single
Galois object. In a similar vein, one can explicitly describe the full subcategories
generated by multiple Galois objects. See Section 5.

Although the motivation for this paper comes from a concrete question concerning
motivic homotopy theory, here we only develop the categorical backdrop. In a sequel [D2],
we will explain how this backdrop applies to both the \( G \)-equivariant
and motivic settings.

1.5. Organization of the paper. In Section 2 we write down a complete def-
inition of the Grothendieck-Witt category. In Section 3 we generalize this, by
introducing the notions of a Gysin functor and its associated category of corre-
spondences (a Gysin functor is the same thing as what is called a commutative
Green functor in the group theory literature). Section 4 continues the development
of this machinery and proves the main “reconstruction theorem” (which in this
generality is a simple exercise in category theory).

Section 5 gives a deeper investigation into the structure of correspondence cate-
gories, and serves as a prelude to Section 6 where we work out some basic compu-
tations inside Grothendieck-Witt categories over a field.

1.6. Notation and terminology. The common notation “\( f(x) \)” establishes a
right-to-left trend in symbology: one starts with \( x \) and then applies \( f \) to it. The
common notation \( \text{Hom}(A, B) \) is based on the opposite left-to-right trend. The op-
posing nature of these two notations is one of the most common annoyances in
modern mathematics. Our general philosophy in this paper is that we will always
use the right-to-left convention, except when we write \( \text{Hom}(A, B) \). This has already
appeared in our treatment of the Burnside category, where spans from \( O_1 \) to \( O_2 \)
were drawn with the \( O_1 \) term on the right. That particular convention will have
various incarnations throughout the paper.

The projection map \( X \times Y \times Z \to X \times Z \) will be written \( \pi_{XZ}^{XY} \), and similarly
for other projection maps. If \( f: A \to X \) and \( g: A \to Y \), then it is sometimes useful
to denote the induced map \( A \to X \times Y \) as \( f \times g \). Unfortunately, \( f \times g \) also denotes
the map \( A \times A \to X \times Y \). Usually it is clear from context which one is meant, but
when necessary we will write \( (f \times g)^A_{XY} \) and \( (f \times g)^A_{XY} \) to distinguish them. In all
these conventions, the superscript is the domain and the subscript is the range.

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2. Background on Grothendieck-Witt groups and composition

In this section we recall the definition and basic properties of the Grothendieck-
Witt group. Then we explain how these groups can be assembled to give the
hom-sets in a certain category.
2.1. Grothendieck-Witt groups. Let $R$ be a commutative ring. A quadratic space over $R$ is a pair $(P, b)$ consisting of a finitely-generated, projective $R$-module $P$ together with a map $b: P \otimes_R P \to R$ that is symmetric in the sense that $b(x, y) = b(y, x)$ for all $x, y \in P$. One says that $(P, b)$ is nondegenerate if the adjoint map $P \to \text{Hom}_R(P, R)$ associated to $b$ is an isomorphism of $R$-modules.

Given any maximal ideal $m$ of $R$ there is an induced map $P \otimes_R P \to (P/mP) \otimes_{R/m} (P/mP)$ giving a symmetric bilinear form $b_m$ on the $R/m$-vector space $P/mP$. One readily checks that $(P, b)$ is nondegenerate if and only if $(P/mP, b_m)$ is nondegenerate for every maximal ideal $m$ of $R$. In many cases nondegeneracy is most easily checked using this criterion.

It will be useful for us to sometimes think geometrically. A quadratic space is an algebraic vector bundle on $\text{Spec } R$ equipped with a fibrewise symmetric bilinear form, and it is nondegenerate if the bilinear forms on the closed fibers are all nondegenerate.

Note that there is an evident direct sum operation on quadratic spaces. There is also a tensor product: if $(P, b)$ and $(Q, c)$ are quadratic spaces then $(P \otimes_R Q, b \otimes_R c)$ denotes the projective module $P \otimes_R Q$ equipped with the bilinear form $(P \otimes_R Q) \otimes_R (P \otimes_R Q) \xrightarrow{id \otimes id} P \otimes_R P \otimes_R Q \otimes_R Q \xrightarrow{b \otimes c} R \otimes_R R \xrightarrow{\mu} R$.

It is easy to check that the direct sum and tensor product of nondegenerate quadratic spaces are again nondegenerate.

The Grothendieck-Witt group of $R$, denoted $\text{GW}(R)$, is the Grothendieck group of nondegenerate quadratic spaces with respect to direct sum. It has a ring structure induced by tensor product. If $f: R \to S$ is a map of commutative rings then there is an induced map of rings $f_*: \text{GW}(R) \to \text{GW}(S)$ given by $(P, b) \mapsto (P \otimes_R S, b \otimes_R id_S)$.

It turns out that $\text{GW}(-)$ is also a contravariant functor, but only with respect to certain kinds of maps. We explain this next.

Definition 2.2. A map of commutative rings $R \to S$ is sheer if $S$ is a finitely-generated, projective $S$-module.

When $R \to S$ is sheer there is a trace map $\text{tr}_{S/R}: S \to R$ defined in the evident way: $\text{tr}_{S/R}(s)$ is the trace of the multiplication-by-$s$ map $x \mapsto xs$ on $S$. The map $\text{tr}_{S/R}$ is $R$-linear.

If $f: R \to S$ is sheer then there is a map $f^!: \text{GW}(S) \to \text{GW}(R)$ defined as follows: if $(Q, c)$ is a quadratic space over $S$ then we let $f^!(Q, c)$ be $Q$ regarded as an $R$-module (via restriction of scalars along $f$) equipped with the bilinear pairing $Q \otimes_R Q \xrightarrow{c} S \xrightarrow{\text{tr}_{S/R}} R$.

Note that $f^!$ will usually not be a map of rings.
Remark 2.3. The above material on the Grothendieck-Witt group is standard, and can be found in [S]. The map $f_1$ is sometimes called the Scharlau transfer; one can find it in [S, Chapter 2.5].

2.4. Separable algebras. The following material is classical, but perhaps not as readily accessible in the literature as it could be. See [I], [DI], and [L], though.

Definition 2.5. Let $A \to B$ be a map of commutative rings. We say that $B$ is a separable $A$-algebra if any of the following equivalent conditions is satisfied:

1. $B$ is projective as a $B \otimes_A B$-module,
2. The multiplication map $\mu : B \otimes_A B \to B$ is split in the category of $B \otimes_A B$-modules,
3. There exists an element $\omega \in B \otimes_A B$ such that $\mu(\omega) = 1$ and $(b \otimes 1)\omega = \omega(1 \otimes b)$ for all $b \in B$,
4. The ring map $B \otimes_A B \to B$ is sheer.

The equivalence of the conditions in the above definition is straightforward: clearly (1) $\iff$ (2), and (2) $\iff$ (3) by letting $\omega$ be the image of 1 under the splitting. Note that $B \otimes_A B \to B$ is necessarily surjective, and so $B$ is always cyclic as a $B \otimes_A B$-module (and in particular, finitely-generated). This explains why (1) is equivalent to (4).

If $\omega$ is a class as in (3) of the above definition, then for any $z \in B \otimes_A B$ one has $z.w = (\mu(z) \otimes 1).w = w.(1 \otimes \mu(z))$.

To see this, write $z = \sum a_i \otimes b_i$ and then just compute that

$$z.w = \sum (a_i \otimes b_i).w = \sum (a_i \otimes 1)(1 \otimes b_i).w = \sum (a_i \otimes 1)(b_i \otimes 1).w = \left(\sum a_ib_i\right) \otimes 1).w = (\mu(z) \otimes 1).w.$$ 

In particular, notice that the class $\omega$ from (3) will be unique: if $\omega'$ is another such class then we would have

$$\omega.\omega' = (\mu(\omega) \otimes 1).\omega' = (1 \otimes 1).\omega' = \omega'$$

and likewise $\omega.\omega' = \omega$. Also notice that $\omega$ is idempotent. Consequently, we have the isomorphism of rings

$$B \otimes_A B \cong (B \otimes_A B)/w \times (B \otimes_A B)/(1 - w)$$

given in each component by projection. The second component can be identified with $B$. Indeed, certainly $1 - \omega$ belongs to ker $\mu$. Conversely, if $s \in \ker \mu$ then $s.\omega = (\mu(s) \otimes 1).\omega = 0$, and so $s = s - s.\omega = (1 - \omega)s \in (1 - \omega)$. It follows that $\mu$ induces an isomorphism of rings $(B \otimes_A B)/(1 - \omega) \cong B$.

Remark 2.6. It helps to have some geometric intuition here. When $E \to B$ is a topological covering space, the diagonal $\Delta : E \to E \times_B E$ gives a homeomorphism from $E$ onto a particular component of $E \times_B E$. Similarly, when $A \to B$ is separable then $\Spec B \times_{\Spec A} \Spec B$ splits off the diagonal copy of $\Spec B$ as one connected component. The idempotent $\omega \in B \otimes_A B$ is the algebraic culprit for this splitting.

Remark 2.7. There is another description of $\omega$ that is sometimes useful. Since $B$ is a finitely-generated, projective $B \otimes_A B$-module there is a trace map $\text{tr}_{B/(B \otimes_A B)} : \End(B) \to B \otimes_A B$. The element $\omega$ is simply $\text{tr}_{B/(B \otimes_A B)}(id_B)$. 
When $B$ is a separable $A$-algebra there is a canonical quadratic space over the ring $B \otimes_A B$: it is $B$ itself (with the usual structure of $B \otimes_A B$-module), equipped with the following bilinear form:

$$B \otimes (B \otimes_A B) \rightarrow B \otimes_A B, \quad x \otimes y \rightarrow (xy \otimes 1) \omega.$$  

A moment’s check shows that this is indeed $B \otimes_A B$-bilinear, as required. We will denote this quadratic space as $(B, \mu \cdot \omega)$.

More generally, for any quadratic space $(P, b)$ over $B$ we obtain a quadratic space $(P, b \cdot \omega)$ over $B \otimes_A B$. The underlying module is $P$ (regarded as a $B \otimes_A B$-module, where it is necessarily projective) equipped with the bilinear form

$$P \otimes (B \otimes_A B) \rightarrow B \otimes_A B, \quad x \otimes y \rightarrow (b(x, y) \otimes 1) \omega.$$  

This construction induces a map of groups (not rings)

$$\text{GW}(B) \rightarrow \text{GW}(B \otimes_A B), \quad [P, b] \mapsto [P, b \cdot \omega].$$

Of course this is just the map $\mu^i$ defined at the end of Section 2.1, where $\mu$ is the multiplication $B \otimes_A B \rightarrow B$.

**Remark 2.8.** The significance of the quadratic space $(B, \mu \cdot \omega)$ will become clear in Section 2.14 below. It plays the role of the identity morphism in the Grothendieck-Witt category.

We will shortly restrict ourselves to studying maps $R \rightarrow S$ which are both sheer and separable. Such maps are commonly referred to by another name:

**Proposition 2.9.** Assume that $R$ is Noetherian. Then $R \rightarrow S$ is both sheer and separable if and only if $R \rightarrow S$ is finite and étale.

**Proof.** Suppose $R \rightarrow S$ is sheerly separable. Then $R \rightarrow S$ is automatically finite and flat. Consider the exact sequence

$$0 \rightarrow I \rightarrow S \otimes_R S \xrightarrow{\mu} S \rightarrow 0.$$  

Since $R \rightarrow S$ is separable, this is split as a sequence of $S \otimes_R S$-modules. So there is an $S \otimes_R S$-linear map $\chi: S \otimes_R S \rightarrow I$ splitting the inclusion. Linearity implies that this map sends $I$ into $I^2$, and so surjectivity gives us $I = I^2$. So $\Omega_{S/R} = I/I^2 = 0$. Since $S$ is flat and finite-type over $R$, and $\Omega_{S/R} = 0$, it follows that $R \rightarrow S$ is étale by [Mi, Proposition I.3.5].

Now suppose that $R \rightarrow S$ is finite étale. Since $R \rightarrow S$ is flat, $R$ is Noetherian, and $S$ is finitely-generated, it follows from [Mi Corollary 6.6] that $S$ is projective over $R$. So $R \rightarrow S$ is sheer.

The map $f: S \rightarrow S \otimes_R S$ given by $f(s) = s \otimes 1$ is also étale (geometrically, étale maps are closed under pullback). If $\mu: S \otimes_R S \rightarrow S$ is the multiplication, then $\mu \circ f = id$. Since $f$ and $id$ are étale, so is $\mu$ by [Mi Corollary I.3.6]. Therefore $S$ is flat over $S \otimes_R S$. But $S$ is finite-type over $R$ and $R$ is Noetherian, hence $S$ and $S \otimes_R S$ are both Noetherian as well. Since $S$ is both flat and finitely-generated (in fact, cyclic) over $S \otimes_R S$, it is actually projective by [Mi Corollary 6.6] again. So $R \rightarrow S$ is separable.

**Corollary 2.10.** Let $k$ be a field. A map of commutative rings $k \rightarrow E$ is sheer and separable if and only if there is an isomorphism of $k$-algebras $E \cong E_1 \times E_2 \times \cdots \times E_n$ where each $E_i$ is a separable (in the classical sense) field extension of $k$. 


Proof. By Proposition 2.9 we can replace “sheerly separable” by “finite étale”, and then the result is standard (for example, see [Mi, Proposition I.3.1]). □

Remark 2.11. Suppose we are working in a category that has finite limits. Let \( \mathcal{P} \) be a property of morphisms that is closed under composition and pullback. Say that a morphism \( X \to Y \) has property \( \mathcal{P} \mathcal{P} \) if \( X \to Y \) has \( \mathcal{P} \) and \( \Delta : X \to X \times_Y X \) also has \( \mathcal{P} \). Then it follows by general category theory that property \( \mathcal{P} \mathcal{P} \) is closed under composition and pullback, and has the feature that if composable morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \) are given such that both \( f \) and \( gf \) have \( \mathcal{P} \mathcal{P} \) then so does \( g \). For the proof of the latter, the main ideas can be found in any standard reference dealing with the case where \( \mathcal{P} \) is “étale” (e.g. [Mi, Corollary I.3.6]). In the present context, we can apply this principle to the opposite category of commutative rings, where \( \mathcal{P} \) is “sheer” and \( \mathcal{P} \mathcal{P} \) is therefore “sheerly separable”. So the sheerly separable maps are closed under pullbacks and composition, and have the indicated two-out-of-three property.

Example 2.12. Here are three examples to keep in mind when dealing with these concepts:

(a) If \( R \) and \( S \) are commutative rings then the projection \( R \times S \to R \) is sheerly separable, but not an injection.

(b) If \( R \) is a commutative ring then the map \( R[x] \to R \) sending \( x \mapsto 0 \) is separable but not sheer.

(c) Given any non-separable, finite field extension \( k \hookrightarrow E \), this map is sheer but not separable.

Remark 2.13. The maps we are calling “sheerly separable” are called “strongly separable” in [J], and “projective separable” in [L]. The following two conditions on a map of commutative rings \( R \to S \) are also equivalent to being sheerly separable:

1. \( S \) is separable over \( R \) and \( S \) is projective as an \( R \)-module (but not required to be finitely-generated);

2. \( S \) is a finitely-generated projective module over \( R \) and the trace form \( S \otimes_R S \to R \) (given by \( x \otimes y \mapsto \text{tr}_{S/R}(xy) \)) is nondegenerate.

The proof of these equivalences, or at least a sketch of such, is available in [L, Proposition 6.11]. We will not need either of these characterizations in the present paper.

2.14. The Grothendieck-Witt category of a commutative ring. We next restrict to a somewhat specialized setting. Assume that \( S, T, \) and \( U \) are \( R \)-algebras, but also assume that \( R \to T \) is sheer and separable.

Now suppose given a quadratic space \((Q,c)\) over \( U \otimes_R T \) and another quadratic space \((P,b)\) over \( T \otimes_R S \). In the following diagram, it is readily checked that the “across-the-top-then-down” composite satisfies the appropriate \( T \)-invariance
condition to induce the dotted map:

\[
\begin{array}{c}
\xymatrix{
Q \otimes_R P \otimes_R Q \otimes_R P \ar[r]^{1 \otimes 1} & Q \otimes_R Q \otimes_R P \otimes_R P \ar[r]^{c \otimes b} & (U \otimes_R T) \otimes_R (T \otimes_R S) \\
& U \otimes_R T \otimes_R S \ar[d]^{1 \otimes \tr T/R \otimes 1} & U \otimes_R R \otimes_R S \ar[d] \ar[r]_\cong & U \otimes_R S.
\end{array}
\]

This produces a quadratic space \((Q \otimes T P, c \otimes_T b)\) over the ring \(U \otimes_R S\). It is easy to check that this is nondegenerate if \((P, b)\) and \((Q, c)\) were, and the construction is evidently compatible with direct sums. So we obtain a pairing

\[(2.15) \quad GW(U \otimes_R T) \otimes GW(T \otimes_R S) \xrightarrow{\otimes_T} GW(U \otimes_R S).\]

It is easy to check that these pairings satisfy associativity. They are also unital, with the unit being the canonical element \((T, \mu \cdot \omega)\) in \(GW(T \otimes_R T)\).

If we denote the evident maps as

\[
\begin{align*}
&j_{12}: U \otimes_R T \to U \otimes_R T \otimes_R S, \\
&j_{23}: T \otimes_R S \to U \otimes_R T \otimes_R S, \\
&j_{13}: U \otimes_R S \to U \otimes_R T \otimes_R S
\end{align*}
\]

then the pairing of (2.15) can also be expressed as

\[
\alpha \otimes_T \beta = j_{13}^{-1}((j_{12} \cdot \alpha) \cdot (j_{23} \cdot \beta))
\]

**Definition 2.16.** Let \(R\) be a commutative ring. The **Grothendieck-Witt category** of \(R\) is the category enriched over abelian groups defined as follows:

1. The objects are \(\text{Spec} T\) for \(T\) a sheerly separable \(R\)-algebra,
2. The set of morphisms from \(\text{Spec} T\) to \(\text{Spec} U\) is the additive group \(GW(U \otimes_R T)\);
3. Composition of morphisms is defined by (2.15).

This category will be denoted \(GWC(R)\).

3. **The general theory of Gysin functors**

When studying the Grothendieck-Witt categories \(GWC(R)\), it turns out to be advantageous to investigate the story in greater generality. We do this in the present section. The “Gysin functors” that we introduce here are simply functors with pullback and pushforward maps which are compatible in familiar ways. Certainly such functors have been encountered time and again in the literature, and so it is unlikely that anything in this section is actually “new”. A very early reference is [G], whereas a more recent reference is [B]. In the setting of finite group theory, our Gysin functors are precisely the **commutative Green functors**.

Being unaware of a reference that serves as a perfect source for what we need, we take some time here to develop the theory from first principles. In doing so, we have tried to provide a unity of discussion that justifies this. We stress, though, that much of the material from this section is in [B].

The main things we do here are:
• Give the definition of a Gysin functor and develop the basic properties;
• Observe the existence of a “universal” Gysin functor, called the Burnside functor;
• Observe that any Gysin functor \( E \) on a category \( \mathcal{C} \) gives rise to an associated closed, symmetric monoidal category, denoted \( \mathcal{C}_E \), of “\( E \)-correspondences” between the objects of \( \mathcal{C} \). These symmetric monoidal categories have the properties that all objects are dualizable, and moreover every object is self-dual.

3.1. Gysin functors. Let \( \mathcal{C} \) be a category with finite limits and finite coproducts, with the property that pullbacks distribute over coproducts: that is, given any maps \( A \to X, P_1 \to X, \) and \( P_2 \to X \) the natural map
\[
(A \times_X P_1) \amalg (A \times_X P_2) \to A \times_X (P_1 \amalg P_2)
\]
is an isomorphism. We also assume that for any objects \( A \) and \( B \) in \( \mathcal{C} \) the following diagrams are pullbacks:
\[
\begin{array}{c}
A \xrightarrow{id} A \\
\downarrow \quad \downarrow \\
A \amalg B \xrightarrow{i_0} A \amalg B
\end{array}
\quad
\begin{array}{c}
B \xrightarrow{id} B \\
\downarrow \quad \downarrow \\
B \amalg B \xrightarrow{i_1} A \amalg B
\end{array}
\quad
\begin{array}{c}
\emptyset \xrightarrow{i_1} B \\
\downarrow \quad \downarrow \\
A \amalg B \xrightarrow{i_0} A \amalg B
\end{array}
\quad
\begin{array}{c}
A \xrightarrow{i_0} A \amalg B \\
\downarrow \quad \downarrow \\
A \xrightarrow{i_1} A \amalg B
\end{array}
\]
Such categories are called finitary lextensive [CLW, Corollary 4.9]. Standard examples to keep in mind are the categories \( \text{Set} \) and \( \text{GSet} \) (and more generally, any topos).

Definition 3.2. A **Gysin functor** on \( \mathcal{C} \) is a contravariant functor \( E \) from \( \mathcal{C} \) to \( \text{CommRing} \) together with a covariant functor \( \tilde{E} : \mathcal{C} \to \text{Ab} \) such that \( E(X) = \tilde{E}(X) \) for every object \( X \). If \( f : X \to Y \) is a map we write \( f^* = E(f) \) and \( f! = \tilde{E}(f) \). The maps \( f! \) will be called Gysin maps. For \( a \in E(X) \) and \( b \in E(Y) \) we write
\[
a \otimes b = (\pi^X_Y)^* (a) \cdot (\pi^Y_X)^* (b).
\]
We require the following axioms:

1. [Zero axiom] \( E(\emptyset) = 0 \).
2. [Behavior on sums] For any objects \( X \) and \( Y \), the natural map
\[
i_X^* \times i_Y^* : E(X \amalg Y) \to E(X) \times E(Y)
\]
is an isomorphism of rings. Here \( i_X : X \to X \amalg Y \) and \( i_Y : Y \to X \amalg Y \) are the canonical maps.
3. [Push-product axiom] For any maps \( f : X \to X' \), \( g : Y \to Y' \) and \( a \in E(X) \), \( b \in E(Y) \) one has
\[
(f \times g)^! (a \otimes b) = f_! (a) \otimes g_! (b).
\]
4. [Push-Pull axiom] For every pullback diagram
\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow p \quad \downarrow q \\
C \xrightarrow{g} D
\end{array}
\]
one has \( f \circ p^* = q^* g_! \).
A natural transformation between Gysin functors is a natural transformation of contravariant functors that is also a natural transformation of the covariant piece.

**Remark 3.3.**

(a) The above definition starts with the “internal” multiplications on the abelian groups \( E(X) \) and derives the external pairings \( E(X) \otimes E(Y) \rightarrow E(X \times Y) \).

As usual, the opposite approach can also be taken: we could have written the above definition in terms of external pairings, and then constructed the internal pairings using the diagonal maps. The two approaches are clearly equivalent.

(b) When \( C \) is the category of finite \( G \)-sets, what we have called Gysin functors are more commonly called commutative Green functors; see [B, Chapter 2]. We adopted the term “Gysin functor” due to its brevity.

The following lemmas are useful to record:

**Lemma 3.4.** If \( f \) is an isomorphism in \( C \) then \( f_! = (f^*)^{-1} \) in any Gysin functor.

**Proof.** This follows immediately from the push-pull formula, using the pullback diagram

\[
\begin{array}{ccc}
A & \xrightarrow{id} & A \\
\downarrow{f} & & \downarrow{f} \\
B & \xrightarrow{id} & B
\end{array}
\]

\( \square \)

**Lemma 3.5.** For any objects \( A \) and \( B \), the composition

\[
E(A) \oplus E(B) \xrightarrow{(i_0)_! \oplus (i_1)_!} E(A \amalg B) \xrightarrow{(i_0)^* \times (i_1)^*} E(A) \times E(B)
\]

sends a pair \((x, y)\) to \((x, y)\) (we refrain from calling this the identity only because the domain and target are perhaps not “equal”). Consequently, the pushforward map \((i_0)_! \oplus (i_1)_! : E(A) \oplus E(B) \rightarrow E(A \amalg B)\) is an isomorphism of abelian groups.

**Proof.** Left to the reader. For the first statement use the push-pull axiom applied to the three pullback squares listed in the original introduction of \( C \), together with \( E(\emptyset) = 0 \). The second statement of the lemma then follows directly from Axiom (2) in the definition of Gysin functor. \( \square \)

**Lemma 3.6.** Let \( f : A \rightarrow X \) and \( g : B \rightarrow X \). Then \((f \times_X g)_!(1) = f_!(1) \cdot g_!(1)\).

**Proof.** Use push-pull for the square

\[
\begin{array}{ccc}
A \times_X B & \xrightarrow{f \times g} & A \times B \\
\downarrow{\Delta} & & \downarrow{f \times g} \\
X & \xrightarrow{\Delta} & X \times X
\end{array}
\]

Start with \( 1 \otimes 1 \in E(A \times B) \), and use the push-product axiom. \( \square \)

**Proposition 3.7** (Projection formula). Let \( E \) be a Gysin functor. Then given \( f : X \rightarrow Y \), \( \alpha \in E(X) \), and \( \beta \in E(Y) \) one has

\[
f_!(\alpha \cdot f^!(\beta)) = f_!(\alpha) \cdot \beta.
\]
Proof. Using push-pull applied to $\alpha \otimes \beta \in E(X \times Y)$, the pullback diagram

$$
\begin{array}{ccc}
X & \xrightarrow{id \times f} & X \times Y \\
\downarrow f & & \downarrow (f \times id) \\
Y & \xrightarrow{\Delta} & Y \times Y
\end{array}
$$

implies that $f_!(\alpha \cdot f^* \beta) = \Delta^*(f \times id)_!(\alpha \otimes \beta)$. The push-product axiom finishes the proof. \qed

Example 3.8.

(a) Let $\mathcal{C}$ be the category of sets but with morphisms the maps where all fibers are finite (called quasi-finite maps from now on). Let $E(S) = \text{Hom}(S, \mathbb{Z})$, with the ring operations given by pointwise addition and multiplication. If $f : S \to T$ then $f^*$ is the evident map and $f_! : E(S) \to E(T)$ sends a map $\alpha : S \to \mathbb{Z}$ to the assignment $t \mapsto \sum_{s \in f^{-1}(t)} \alpha(s)$.

(b) Let $G$ be a finite group, and let $\mathcal{C}$ be the category of finite $G$-sets. For $S$ in $\mathcal{C}$ define $\mathcal{A}(S)$ to be the Grothendieck group of maps $X \to S$ (where $X$ is a finite $G$-set), made into a ring via $[X \to S] \cdot [Y \to S] = [X \times_S Y \to S]$. Given $f : S \to T$ one gets maps $f^* : \mathcal{A}(T) \to \mathcal{A}(S)$ by pulling back along $f$, and $f_! : \mathcal{A}(S) \to \mathcal{A}(T)$ by composing with $f$.

(c) Let $\text{Aff}_{sh}$ be the opposite category of commutative rings and sheer maps. For $R$ a commutative ring we write $\text{Spec} R$ for the corresponding object of $\text{Aff}$. Let $K^0(\text{Spec} R)$ be the Grothendieck group of finitely-generated $R$-projectives. For $f : \text{Spec} R \to \text{Spec} S$ we have $f^* : K^0(\text{Spec} S) \to K^0(\text{Spec} R)$ given by $[P] \mapsto [P \otimes_S R]$, and $f_! : K^0(\text{Spec} R) \to K^0(\text{Spec} S)$ given by restriction of scalars (so $[P]_R \mapsto [P]_S$).

(d) Fix a commutative, Noetherian ring $R$, and let $\text{fEt}_R$ be the subcategory of $\text{Aff}$ consisting of objects $\text{Spec} S$ where $R \to S$ is finite étale. Then $\text{Spec} S \mapsto GW(S)$ has the structure of a Gysin functor, as detailed in Section 2.

(e) Let $\mathcal{C}$ be the category of topological spaces, with morphisms the quasi-finite fibrations. Define $E(X) = \text{Hom}(\pi_0(X), \mathbb{Z}) = H^0(X)$. The pullback maps are as expected. For $f : X \to Y$ and $\alpha \in E(X)$ define $f_!(\alpha)$ to be the assignment $[y] \mapsto \sum_{x \in f^{-1}(y)} \alpha([x])$ where $[x]$ and $[y]$ denote the path-components containing $x$ and $y$. This is a Gysin functor (the fibration condition is needed only to show that $f_!$ is well-defined). Note that this Gysin functor has a strong relation to that in (a) above.

(f) The following is not an example of a Gysin functor, but is nevertheless instructive. Let $\mathcal{C}$ be the category of finite sets, and let $\mathcal{P}(X)$ be the powerset of the set $X$; this is not quite a ring, but it does have the intersection operation $\cap$ which we will regard as a multiplication. Given $f : X \to Y$ one has the inverse-image map $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ (which preserves the multiplication) and the image map $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$ (which does not). The axioms of Definition 3.2 are all satisfied, when suitably interpreted. The powerset functor is something like a “non-additive Gysin functor”.

Remark 3.9. Let $\mathcal{C}$ be the category of oriented topological manifolds, and let $E(X) = H^*(X)$. With the usual pullbacks and Gysin morphisms, this is almost (but not quite) a Gysin functor as we defined above. The difficulty is that the push-pull axiom only holds for pullback squares satisfying a suitable transversality
condition. This same problem arises if one uses smooth algebraic varieties and the Chow ring, or if one uses oriented manifolds and complex cobordism. But all of these settings represent appearances in the literature of structure similar to what we consider in the present paper. Especially in the case of cobordism, see the axiomatic treatment in [Q, Section 1]. Prior to [Q, Proposition 1.12] Quillen refers to, but does not give, an axiomatic treatment related to the multiplicative structure; the axioms for a Gysin functor are essentially this.

3.10. The universal Gysin functor. Given an object $X$ in $\mathcal{C}$, define $A_\mathcal{C}(X)$ to be the Grothendieck group of (isomorphism classes of) maps $S \to X$ where 

$$[(S \amalg T) \to X] = [S \to X] + [T \to X].$$

The multiplication $[S \to X] \cdot [T \to X] = [S \times X T \to X]$ is well-defined and makes $A_\mathcal{C}(X)$ into a commutative ring with identity $[id: X \to X]$. We call $A_\mathcal{C}(X)$ the **Burnside ring** of $X$. Note that $A_\mathcal{C}$ has the evident structure of a contravariant functor to rings, as well as that of a covariant structure to abelian groups, generalizing the situation in Example 3.8(b). One readily checks that this is a Gysin functor, called the **Burnside functor** for the category $\mathcal{C}$. When the category $\mathcal{C}$ is understood we abbreviate $A_\mathcal{C}$ to just $A$.

Example 3.11. When $\mathcal{C}$ is the category of finite sets, note that there is a natural isomorphism $A(S) \cong \text{Hom}(S, \mathbb{Z})$, sending the element $[f: M \to S]$ to the assignment $s \mapsto \# f^{-1}(s)$. The Gysin functor given in Example 3.8(a) (restricted to the category of finite sets) is the Burnside functor for this category.

The Burnside functor has the following universal property:

**Proposition 3.12.** If $E$ is a Gysin functor on the category $\mathcal{C}$ then there is a unique map of Gysin functors $A_\mathcal{C} \to E$. It sends $[f: A \to X]$ in $A_\mathcal{C}(X)$ to $f_!(1) \in E(X)$.

**Proof.** An easy exercise. For existence, use the given formula. The fact that $A_\mathcal{C}(X) \to E(X)$ is well-defined follows using Lemma 3.5 (which implies that $(f \amalg g)_!(1) = f_!(1) + g_!(1)$). The fact that it is a ring map follows from Lemma 3.6. Compatibility with pullbacks and pushforwards is trivial. Uniqueness follows from the fact that $[f: A \to X]$ equals $f^! (1)$, the pushforward in the Gysin functor $A$. □

3.13. Categories derived from Gysin functors. Given a Gysin functor $E$ on $\mathcal{C}$ we can define an additive category $\mathcal{C}_E$ as follows. First, the objects of $\mathcal{C}_E$ are the same as the objects of $\mathcal{C}$. Second, for any objects $A$ and $B$ define

$$\mathcal{C}_E(A, B) = E(B \times A).$$

Really what we mean here is that $\mathcal{C}_E(A, B)$ is the underlying abelian group of $E(B \times A)$. Third, define the composition law

$$\mu_{C, B, A}: \mathcal{C}_E(B, C) \otimes \mathcal{C}_E(A, B) \to \mathcal{C}_E(A, C)$$

by

$$\mu_{C, B, A}(\alpha \otimes \beta) = (\pi_{13})((\pi_{12})^*(\alpha) \cdot \pi_{23}^*(\beta))$$

where the $\pi_{rs}$ maps are the evident ones

$$\pi_{12}: A \times B \times C \to A \times B, \quad \pi_{23}: A \times B \times C \to B \times C,$$

$$\pi_{13}: A \times B \times C \to A \times C.$$
We will use the notation
\[ \alpha \circ \beta = \mu_{C;B,A}(\alpha \otimes \beta). \]
Finally, for any object \( A \) define \( i_A \) to be \( \Delta^A(1) \); that is, consider the map
\[ E(A) \xrightarrow{\Delta_A} E(A \times A) \]
and take the image of the unit element of the ring \( E(A) \). Note that \( E(A \times A) \) is a commutative ring and so has a unit element 1, but this is not necessarily equal to \( i_A \). One may check (see Proposition 3.16 below) that this structure makes \( C_E \) into a category.

For lack of a better term, we refer to elements of \( E(B \times A) \) as “\( E \)-correspondences” from \( A \) to \( B \). The category \( C_E \) itself will be referred to as the category of \( E \)-correspondences.

**Remark 3.14.** The construction of the category \( C_E \) is one that appears countless times in the algebraic geometry literature, ultimately going back to Grothendieck. For the category of algebraic varieties over some field \( k \), forming the category of correspondences with respect to the Chow ring functor is the first step in Grothendieck’s attempts to define a category of motives. See for example [M, Section 2].

**Example 3.15.**
(a) Let \( G \) be a finite group, let \( C \) be the category of finite \( G \)-sets, and let \( A \) be the Burnside functor from Example 3.8(b). The category \( C_A \) is precisely the category \( \text{Burn} \) mentioned in Section 1.
(b) Fix a commutative, Noetherian ring \( R \), and let \( C \) be the subcategory of \( \text{Aff} \) consisting of objects \( \text{Spec} \ S \) where \( R \rightarrow S \) is sheer and separable. Then \( C_{GW} \) is the Grothendieck-Witt category over \( R \), defined in Section 2.
(c) Let \( C \) be the category of finite sets, and let \( E \) be the Gysin functor from Example 3.8(a). Then we obtain the category of correspondences \( C_E \). It turns out this category has a familiar model: it is equivalent to the category of finitely-generated, free abelian groups. Proving this is not hard, but it will also fall out of our general “reconstruction theorem” (Theorem 4.16). See Example 4.17.

The following proposition details many (and perhaps too many) useful facts about the category \( C_E \). Recall one piece of notation: maps into products can be unlabelled if there is a self-evident candidate for how the map projects onto each of the factors. For example, if \( f : A \rightarrow B \) then \( A \rightarrow A \times B \) denotes the evident map that is the identity on the first factor and \( f \) on the second.

**Proposition 3.16.** Suppose given a Gysin functor \( E \) on the category \( C \).
(a) The structure described above defines a category \( C_E \) that is enriched over abelian groups, where \( i_A \in C_E(A,A) \) is the identity map on \( A \).
(b) A natural transformation of Gysin functors \( E \rightarrow E' \) induces a functor \( C_E \rightarrow C_{E'} \).
(c) There is a functor \( R : C \rightarrow C_E \) that is the identity on objects and has the property that for \( f : A \rightarrow B \) in \( C \) we have
\[ R_f = (id_B \times f)^*(i_B) \in E(B \times A) = C_E(A,B). \]
One also has \( R_f = (A \rightarrow B \times A)(1) \).
The category \( \mathcal{C}_E \) has an anti-automorphism \((-)^* \) that is the identity on objects, and for \( \alpha \in \mathcal{C}_E(A,B) \) is given by

\[ \alpha^* = \xi^*(\alpha) \]

where \( \xi : A \times B \to B \times A \) is the evident isomorphism. We define \( I : \mathcal{C}^{op} \to \mathcal{C}_E \) to be the identity on objects, and to be given on maps by \( I(f) = (Rf)^* \). We often write \( I_f = I(f) \). If \( f : A \to B \) then \( I_f = (f \times \text{id}_B)^*(\text{id}_B) = (A \to A \times B) ;(1) \).

Remark 3.18. Note that \( E \) and \( E \) that the abelian group \( B \) on the explanation of this, here is some useful notation. If contravariant, extends to a single functor defined on all of \( \mathcal{C}_E \).

The functor \( \mathcal{C}^{op} \to \mathcal{C}_E \) properties of the circle product \( \mathcal{C} \) and \( \mathcal{C}_E \) is immediate from the associativity and unital properties of the circle product in \( \mathcal{C}_E \).

(f) Given \( A \xrightarrow{f} B \xleftarrow{g} C \) in \( \mathcal{C} \) one has \( I_f \circ R_g = (f \times g)^* (\text{id}_C) = (\pi_{AC}^{BA,C}) ;(1) \) in \( \mathcal{C}_E \).

(g) Given \( A \xrightarrow{g} C \) in \( \mathcal{C} \) one has \( R_p \circ I_g = (p \times g)(\text{id}_C) = ((p \times g)(D)_{AC}) ;(1) \) in \( \mathcal{C}_E \).

(h) Given a pullback diagram in \( \mathcal{C} \)

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & W \\
\downarrow{p} & & \downarrow{q} \\
X & \xrightarrow{f} & Y
\end{array}
\]

one has \( R_p \circ I_g = I_f \circ R_q \) in \( \mathcal{C}_E \).

(i) If \( f \) is an isomorphism in \( \mathcal{C} \) then \( R_f = I_f^{-1} = I_{f^{-1}} \).

Proof. This proof is tedious, but completely formal. See Appendix \[ \Box \]

Our next goal is to observe that the the Gysin functor \( E \), which is both co- and contravariant, extends to a single functor defined on all of \( \mathcal{C}_E \). Before embarking on the explanation of this, here is some useful notation. If \( B \) is an object of \( \mathcal{C} \), note that the abelian group \( E(B) \) may be identified with both \( E(B,*) \) and \( E(*,B) \). If \( x \in E(B) \) we write \( x_* \) for \( x \) regarded as an element of \( E(B,*) \) and \( x\) for \( x \) regarded as an element of \( E(*,B) \). This notation makes sense if one remembers our general “right-to-left” notation; e.g., \( x_* \) is \( x \) regarded as a map from the object \( * \).

Define a functor \( E' : \mathcal{C}^{op}_E \to \mathcal{C}^{op} \) as follows. On objects it is the same as \( E' \):

\( E'(A) = E(A) \).

For \( g \in \mathcal{C}_E(A,B) \) define \( E'(g) : E(B) \to E(A) \) by

\[ E'(g)(x) = _* x \circ g = (\pi_A^{BA}) \left[ (\pi_B^{BA})*g \right] \]

The fact that this is a functor is immediate from the associativity and unital properties of the circle product \( \circ \) (the composition product in \( \mathcal{C}_E \)).

Proposition 3.17. The functor \( E' : \mathcal{C}^{op}_E \to \mathcal{C}^{op} \) has the property that \( E'(Rf) = f^* \) and \( E'(I_f) = f_! \) for any map \( f \) in \( \mathcal{C} \).

Proof. Immediate from Proposition 3.16(e), parts (i) and (iii). \[ \Box \]

Remark 3.18. Note that \( E' \) is not a functor from \( \mathcal{C}^{op}_E \) into \( \text{CommRing} \). This would of course be too much to ask, since the transfer maps \( f_! \) do not respect the multiplicative products.
3.19. **Further properties of \( C_E \).** The category \( C_E \) has some extra structure that we have not yet accounted for. The categorical product in \( C \) induces a symmetric monoidal product on \( C_E \): that is, for objects \( X \) and \( Y \) we define
\[
X \otimes Y = X \times_C Y.
\]
We must define \( f \otimes g \) for \( f \in C_E(X, X') \) and \( g \in C_E(Y, Y') \). We do this by
\[
f \otimes g = (t_{X'Y'XY})^*(f \otimes g).
\]
This formula appears self-referential, but the two tensor symbols mean something different: in the second case, we have \( f \in E(X' \times X) \) and \( g \in E(Y' \times Y) \) and \( f \otimes g \) is the element in \( E(X' \times X \times Y' \times Y) \) that was introduced in Definition 3.2. It takes a little work to verify bi-functoriality. The unit object is \( S = * \), the terminal object of \( C \) (note that this is not a terminal object of \( C_E \)). The symmetry isomorphism \( \tau_{XY} \in C_E(X \otimes Y, Y \otimes X) \) is defined to be
\[
\tau_{XY} = R(t_{XY})
\]
where \( t_{XY} : X \times Y \rightarrow Y \times X \) is the canonical isomorphism in \( C \). One must verify that the structure we have defined satisfies the basic commutative diagrams for a symmetric monoidal structure, and we again leave this with simply the remark that it is tedious but not challenging.

We can also define function objects in \( C_E \). For objects \( X \) and \( Y \) define
\[
F(X, Y) = X^* \otimes Y
\]
where \((-)^*\) is the anti-automorphism from Proposition 3.16(d). Of course the object \( X^* \) is exactly equal to \( X \), but we wrote \( X^* \) because this is more compatible with the way the maps work: for \( g : Y \rightarrow Y' \) define \( F(X, g) \) to be the map \( i_{X^*} \otimes g \), and for \( f : X \rightarrow X' \) define \( F(f, Y) \) to be the map \( I_f \otimes i_Y \).

At this point it is useful to recall the notion of dualizability in symmetric monoidal categories. See Appendix A. In this paper we will use the term **tensor category** to signify a symmetric monoidal category that is also enriched over abelian groups, having the property that the tensor product of morphisms is bilinear. A tensor category is **closed** if it is equipped with function objects related to the tensor by the usual adjunction formula, which is required to be linear.

With the above notions in place, we leave the reader to check the following:

**Proposition 3.20.** The above structure makes \( C_E \) into a closed tensor category in which every object is dualizable. Moreover, every object is isomorphic to its own dual.

**Proof.** Tedious, but routine. Perhaps the only thing that needs remark is that the evaluation and co-evaluation morphisms for an object \( X \) are
\[
eve_X = i_X \in E(X \times X) = E(* \times X \times X) = C_E(*, X \times X) = C_E(S, X \otimes X)
\]
and
\[
ev_X = i_X \in E(X \times X) = E(* \times X \times X) = C_E(X \times X, *) = C_E(X \otimes X, S).
\]

The following proposition is easy but important. It will be used implicitly in several later calculations.
Proposition 3.21. Let $f: A \to B$ and $g: X \to Y$ be maps in $C$. Then $R(f \times g) = Rf \otimes Rg$ and $I(f \times g) = If \otimes Ig$.

Proof. Using Proposition 3.16(d) and the definition of tensor product, we have

$$If \otimes Ig = (t^{AXBY}_{ABXY})^*[(A \to AB)!(1) \otimes (X \to XY)!](1)$$

$$= (t^{AXBY}_{ABXY})^*[(A \times X \to A \times B \times X \times Y)!](1)$$

$$= (A \times X \to A \times B \times Y)!$$

The second equality uses the Push-Product Axiom, the third equality uses Push-Pull, and the last equality is Proposition 3.16(d) again. □

We close this section by returning to the functor $E'$ from Proposition 3.17. The following proposition is not needed, but we record it for completeness. The proof is left to the reader.

Proposition 3.22. Let $C$ be a finitary lextensive category, and let $A$ be the Burnside functor for $C$. Let $E$ be a Gysin functor on $C$.

(a) The unit maps $Z \to E(X)$ and pairings $E(X) \otimes E(Y) \to E(X \otimes Y)$ provide $E'$ with the structure of lax symmetric monoidal functor.

(b) The association $E \mapsto E'$ gives a bijection between Gysin functors on $C$ and lax symmetric monoidal functors $C_{op} \to A_{op}$.

4. Gysin schema and the reconstruction theorem

We have seen that given a Gysin functor $E$ on a finitary lextensive category $C$, there is an associated symmetric monoidal category $C_E$ called the category of $E$-correspondences. One could try to run this process in reverse: given a symmetric monoidal category $D$, what do you need to know in order to guarantee that $D$ is the category of $E$-correspondences for an appropriately chosen $E$ and $C$? We might term this the “reconstruction problem”: can $D$ be reconstructed as a category of correspondences? Of course for this to work one must at least require that all objects in $D$ be self-dual.

Unfortunately, in this form the reconstruction problem is a little awkward. The category $C_E$ comes equipped with two distinguished subcategories, one consisting of the forward maps $Rf$ and one consisting of the backward maps $If$. If we are just given a symmetric monoidal category $D$, there is no clear way to separate out analogs of either of these distinguished subcategories.

The way around this problem is to add these special subcategories into the initial data. Then the reconstruction problem becomes solvable, albeit for almost tautological reasons. See Theorem 4.16 below.

4.1. Gysin schema.

Definition 4.2. A Gysin schema consists of the following data:

- A finitary lextensive category $C$, together with an explicit choice $*$ for terminal object and for each objects $X$ and $Y$ of $C$ an explicit choice of product $X \times Y$;
- A tensor category $(D, \otimes, S)$;
- A map of sets $\Theta: \text{ob} C \to \text{ob} D$ and two functors $R: C \to D$ and $I: C_{op} \to D$;
- Isomorphisms $\theta_*: \Theta(*) \to S$ and $\theta_{X,Y}: \Theta(X \times Y) \to (\Theta X) \otimes (\Theta Y)$.
This data is required to satisfy the following axioms:
(1) \( R(X) = \Theta(X) = I(X) \) for all objects \( X \) of \( \mathcal{C} \);
(2) The data \( (R, \theta) \) makes \( R \) into a strong symmetric monoidal functor from \((\mathcal{C}, \times, *)\) to \((\mathcal{D}, \otimes, S)\).
(3) For all maps \( f: A \to X \) and \( g: B \to Y \) in \( \mathcal{C} \), the diagram
\[
I(A \times B) \xleftarrow{I(f \times g)} I(X \times Y) \\
\theta_{A,B} \cong \cong \theta_{X,Y}
\]
is commutative.
(4) For every pullback diagram
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{p} & & \downarrow{q} \\
C & \xrightarrow{g} & D
\end{array}
\]
in \( \mathcal{C} \) one has \( Rf \circ Ip = Iq \circ Rg \).
We will write the Gysin schema as \( \Theta: \mathcal{C} \to \mathcal{D} \), suppressing \( R, I, \) and \( \theta \) from the notation.

**Remark 4.3.** There are a couple of odd features about the above definition. First, the function \( \Theta \) is clearly redundant as it can be recovered from either \( R \) or \( I \). We include \( \Theta \) in the definition because it is often useful to have a notation that does not favor either \( R \) or \( I \). Secondly, conditions (2) and (3) could have been made more symmetric by replacing (3) with the statement that \((I, \theta)\) is strong symmetric monoidal; we leave the equivalence as an exercise. The phrasing from the definition makes applications a little easier, as there is a bit less to verify: in practice one looks for a “nice enough” functor \( R \) that admits transfer maps satisfying (3) and (4).

**Example 4.4.** One readily checks that the following are examples of Gysin schema:
(a) Fix a finite group \( G \), and let \( \mathcal{D} \) be the \( G \)-equivariant stable homotopy category of genuine \( G \)-spectra. Let \( \mathcal{C} \) be the category of finite \( G \)-sets, and let \( R(X) = \Sigma^\infty(X_+) \). The maps \( I(f) \) are the usual transfer maps constructed in stable homotopy theory.
(b) Let \( \mathcal{D} \) be the category of finitely-generated free abelian groups, equipped with the tensor product. Let \( \mathcal{C} \) be the category of finite sets. Let \( R(X) \) be the free abelian group on the set \( X \), with its natural functoriality. If \( f: X \to Y \) then let \( I(f): R(Y) \to R(X) \) send the basis element \([y]\) to \( \sum_{x \in f^{-1}(y)} [x] \).

When \( X \) is an object of \( \mathcal{C} \) we let \( \pi_X \) denote the unique map \( X \to * \), and \( \Delta_X \) denote the diagonal \( X \to X \times X \). The subscripts will usually be suppressed when understood. Note that \( R\pi \) is a map \( RX \to R(*) \), and we have a chosen isomorphism \( R(*) = \Theta(*) \cong S \); so composing these gives a canonical map \( RX \to S \), which we will usually also denote \( R\pi \) by abuse. Similarly, \( R\Delta \) may be regarded as a map \( RX \to RX \otimes RX \). We use these conventions for \( I\pi \) and \( I\Delta \) as well.
4.5. Transfers and duality.

**Proposition 4.6.** Suppose that $\Theta : C \to D$ is a Gysin schema. Then for every object $X$ in $C$, $\Theta X$ is dualizable in $D$. In fact, $\Theta X$ is self-dual with structure maps given by
\[
\begin{align*}
S &\xrightarrow{R\Delta} \Theta(X) \xrightarrow{R\Delta} \Theta(X \times X) \xrightarrow{\cong} \Theta X \otimes \Theta X \\
\Theta X \otimes \Theta X &\xrightarrow{\cong} \Theta(X \times X) \xrightarrow{I\Delta} \Theta X \xrightarrow{R\pi} S.
\end{align*}
\]

**Proof.** The key is the pullback diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times X \\
\downarrow & & \Delta \times \text{id} \\
X \times X & \xrightarrow{\text{id} \times \text{id}} & X \times X \times X,
\end{array}
\]
from which we deduce that $I(\text{id} \otimes \Delta) \circ R(\Delta \otimes \text{id}) = R\Delta \circ I\Delta$. Combining this with axiom (3) from Definition 4.2 gives the first equality below:
\[
(4.7) \quad (\text{id} \otimes I\Delta) \circ (R\Delta \otimes \text{id}) = R\Delta \circ I\Delta = (I\Delta \otimes \text{id}) \circ (\text{id} \otimes R\Delta).
\]
The second equality comes about in the same way, but starting with the reflection of the above pullback square about its central diagonal.

To prove the proposition we must first check that the composition
\[
\Theta X = \Theta X \otimes S \xrightarrow{1 \otimes I\pi} \Theta X \otimes \Theta X \xrightarrow{1 \otimes R\Delta} \Theta X \otimes \Theta X \xrightarrow{I\Delta \otimes 1} \Theta X \otimes \Theta X
\]
\[
\xrightarrow{R\pi \otimes 1} S \otimes \Theta X = \Theta X
\]
equals the identity. But using (4.7) this is equal to
\[
(R\pi \otimes 1) \circ R\Delta \circ I\Delta \circ (1 \otimes \pi) = R((\pi \times 1) \circ \Delta) \circ I((1 \times \pi) \circ \Delta) = R(\pi) \circ I(\pi) = \text{id}.
\]
Note that we have again used axiom (3) of Definition 4.2.

The proof that the composite
\[
\Theta X = S \otimes \Theta X \xrightarrow{I\pi \otimes 1} \Theta X \otimes \Theta X \xrightarrow{R\Delta \otimes 1} \Theta X \otimes \Theta X \xrightarrow{1 \otimes I\Delta} \Theta X \otimes \Theta X
\]
\[
\xrightarrow{1 \otimes R\pi} S \otimes \Theta X = \Theta X
\]
equals the identity is entirely similar. $\square$

The following corollary is also worth recording:

**Corollary 4.8.** Let $\Theta : C \to D$ be a Gysin schema. Then given any map $f : X \to Y$ in $C$, the dual of $Rf : \Theta X \to \Theta Y$ (computed using the duality structures provided by Proposition 4.6) is precisely $If : \Theta Y \to \Theta X$.

**Proof.** The dual of $Rf$ is the following composite:
\[
\Theta Y = \Theta Y \otimes S \xrightarrow{1 \otimes \eta_X} \Theta Y \otimes \Theta X \otimes \Theta X \xrightarrow{1 \otimes Rf \otimes 1} \Theta Y \otimes \Theta Y \otimes \Theta X
\]
\[
\xrightarrow{\epsilon_Y \otimes 1} S \otimes \Theta X = \Theta X.
\]
One unpacks $\eta$ and $\epsilon$ as $\eta_X = R\Delta_X \circ \iota_X$ and $\epsilon_Y = R\pi_Y \circ I\Delta_Y$, and then argues precisely as in the proof of Proposition 4.6 but instead using the pullback diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f \times 1} & Y \times X \\
\downarrow f \times 1 & & \downarrow 1 \times f \times 1 \\
Y \times X & \xrightarrow{\Delta_Y \times 1} & Y \times Y \times X,
\end{array}
\]

The details are left to the reader. \qed

4.9. The canonical Gysin functor for a Gysin schema. Suppose $\Theta: C \to D$ is a Gysin schema. Define $\pi_\Theta: C^{op} \to \mathrm{Ab}$ to be the functor given by

$$
\pi_\Theta(X) = D(\Theta X, S).
$$

Note that the abelian groups $\pi_\Theta(\_)$ also inherit the structure of a covariant functor: given $f: X \to Y$ in $C$ define $f_\pi: \pi_\Theta(X) \to \pi_\Theta(Y)$ by the diagram

\[
\begin{array}{ccc}
\pi_\Theta(X) & \xrightarrow{f_\pi} & \pi_\Theta(Y) \\
\downarrow & & \downarrow \\
D(\Theta X, S) & \xrightarrow{D(f,S)} & D(\Theta Y, S).
\end{array}
\]

Moreover, the abelian groups $\pi_\Theta(X)$ inherit a product: given $a, b \in \pi_\Theta(X)$, define $a \cdot b$ to be the composite

$$
\Theta X \xrightarrow{R_\Delta} \Theta X \otimes \Theta X \xrightarrow{a \otimes b} S \otimes S \cong S.
$$

This gives $\pi_\Theta(X)$ the structure of a commutative ring, and if $f: X \to Y$ is a map in $C$ then $f^*: \pi_\Theta(Y) \to \pi_\Theta(X)$ is a ring homomorphism.

Proposition 4.10. If $\Theta: C \to D$ is a Gysin schema then $\pi_\Theta: C^{op} \to \mathrm{CommRing}$ is a Gysin functor.

Proof. This is simply a matter of chasing through definitions. \qed

Remark 4.11. Recall from Definition 3.2 that if $a \in \pi_\Theta(X)$ and $b \in \pi_\Theta(Y)$ then we have the element $a \otimes b \in \pi_\Theta(X \times Y)$. It is easy to check that this is the map

$$
\Theta(X \times Y) \xrightarrow{\cong} \Theta X \otimes \Theta Y \xrightarrow{a \otimes b} S \otimes S = S.
$$

4.12. Preliminaries on the reconstruction problem. Let $(\mathcal{D}, \otimes, S, F(-,-))$ be a closed, symmetric monoidal category in which every object is dualizable. It turns out all such categories have a description that is somewhat reminiscent of the construction of $\mathcal{C}_E$.

For an object $X$ write $X^* = F(X, S)$, and for $f: X \to Y$ write $f^* = F(f, S)$. Let $e_X: X^* \otimes X \to S$ be the adjoint of the identity map $X^* \to F(X, S)$, and let $c_X: S \to X \otimes X^*$ be the coevaluation map guaranteed by duality (see Appendix A).

Define a new category $\mathcal{D}^{ad}$ as follows. The objects are the same as those in $\mathcal{D}$, and morphisms are given by

$$
\mathcal{D}^{ad}(X, Y) = D(Y^* \otimes X, S).
$$

If $\alpha \in \mathcal{D}^{ad}(X, Y)$ and $\beta \in \mathcal{D}^{ad}(Y, Z)$ then $\beta \circ \alpha$ is given as follows:

\[
\begin{array}{ccc}
Z^* \otimes X & \xrightarrow{\cong} & Z^* \otimes S \otimes X \\
1 \otimes e_Y \otimes 1 & \xrightarrow{\beta \otimes \alpha} & Z^* \otimes Y \otimes Y^* \otimes X \xrightarrow{\beta \otimes \alpha} S \otimes S
\end{array}
\]
One readily checks that this composition is associative, and $ev_X \in \mathcal{D}^{ad}(X, X)$ is a two-sided identity.

There is a functor $\Gamma : \mathcal{D} \to \mathcal{D}^{ad}$ defined as follows. It is the identity on objects, and given $f : X \to Y$ we let $\Gamma f \in \mathcal{D}^{ad}(X, Y) = \mathcal{D}(Y^* \otimes X, S)$ be the composite

$$Y^* \otimes X \xrightarrow{id \otimes f} Y^* \otimes Y \xrightarrow{ev} S.$$ 

The check that this is indeed a functor is best done using the graphical calculus for closed symmetric monoidal categories (see [BS] for an expository account of this). If $f : X \to Y$ and $g : Y \to Z$ are maps in $\mathcal{D}$, then $\Gamma(gf)$ and $(\Gamma g)(\Gamma f)$ are the composite maps represented by the following diagrams:

\begin{center}
\begin{tikzpicture}
    \node (X) at (0,0) {$X$};
    \node (Y) at (0,2) {$Y$};
    \node (S) at (4,0) {$S$};
    \node (g) at (3,1.5) {$g$};
    \node (f) at (1,1.5) {$f$};
    \draw[->] (X) to node [above] {$f$} (Y);
    \draw[->] (Y) to node [right] {$g$} (S);
    \draw[->] (X) to node [below] {$\Gamma(g \circ_f d f)$} (S);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
    \node (X) at (0,0) {$X$};
    \node (Y) at (2,0) {$Y$};
    \node (S) at (4,0) {$S$};
    \node (g) at (3,1.5) {$g$};
    \node (f) at (1,1.5) {$f$};
    \draw[->] (X) to node [above] {$f$} (Y);
    \draw[->] (Y) to node [right] {$g$} (S);
    \draw[->] (X) to node [below] {$\Gamma(g \circ_{D^{ad}} f)$} (S);
\end{tikzpicture}
\end{center}

The graphical calculus clearly shows these composites to be identical in $\mathcal{D}$.

**Proposition 4.13.** The functor $\Gamma : \mathcal{D} \to \mathcal{D}^{ad}$ is an isomorphism of categories.

**Proof.** We only need check that the maps $\mathcal{D}(X, Y) \to \mathcal{D}^{ad}(X, Y) = \mathcal{D}(Y^* \otimes X, S)$ are bijections. There is an evident map in the opposite direction that sends a map $h : Y^* \otimes X \to S$ to the composite

$$X \xrightarrow{c_Y \otimes id_X} Y \otimes Y^* \otimes X \xrightarrow{id_Y \otimes h} Y \otimes S \xrightarrow{id} Y.$$ 

Proving that these assignments are inverses to each other is another exercise in graphical calculus. For example, the composite in one direction sends the map $f : X \to Y$ to the map represented by

\begin{center}
\begin{tikzpicture}
    \node (X) at (0,0) {$X$};
    \node (Y) at (0,2) {$Y$};
    \node (S) at (4,0) {$S$};
    \node (g) at (3,1.5) {$g$};
    \node (f) at (1,1.5) {$f$};
    \draw[->] (X) to node [above] {$f$} (Y);
    \draw[->] (Y) to node [right] {$g$} (S);
    \draw[->] (X) to node [below] {$\Gamma(g \circ_{D^{ad}} f)$} (S);
\end{tikzpicture}
\end{center}

and the graphical calculus shows that this is equal to $f$ in $\mathcal{D}$. The other direction is similarly easy. \qed

Now suppose that $(\mathcal{D}, \otimes, S)$ is a symmetric monoidal category (not necessarily closed) but where all objects are self-dual: assume that for every object $X$ in $\mathcal{D}$ one is supplied maps $\eta_X : S \to X \otimes X$ and $\epsilon_X : X \otimes X \to S$ satisfying the conditions of Definition A.3. In this context one can reproduce the construction of $\mathcal{D}^{ad}$ but without any explicit mention of duals. Specifically, define $\mathcal{D}^{(ad)}$ to be the category with the same objects as $\mathcal{D}$, but where $\mathcal{D}^{(ad)}(X, Y) = \mathcal{D}(Y \otimes X, S)$.
Given \( f \in D^{(ad)}(X,Y) \) and \( g \in D^{(ad)}(Y,Z) \) define \( g \circ f \) to be the composition
\[
\begin{array}{c}
Z \otimes X \xrightarrow{Z \otimes S \otimes X \xrightarrow{1 \otimes \eta_Y \otimes 1}} Z \otimes Y \otimes Y \otimes X \xrightarrow{g \circ f} S \otimes S = S.
\end{array}
\]
Let \( 1_X \in D^{(ad)}(X,X) \) be the map \( \epsilon_X : X \otimes X \to S \). One readily checks that \( D^{(ad)} \) is a category.

**Proposition 4.14.** There is a functor \( \Gamma : D \to D^{(ad)} \) that is the identity of objects and sends a map \( f : X \to Y \) in \( D \) to the composite
\[
\begin{array}{c}
Y \otimes X \xrightarrow{id \otimes f} Y \otimes Y \xrightarrow{\epsilon_Y} S.
\end{array}
\]
The functor \( \Gamma \) is an isomorphism of categories.

**Proof.** A simple exercise. \( \square \)

4.15. **The main reconstruction theorem.** Recall that \( C_{(\pi_\Theta)} \) denotes the category of correspondences associated to the Gysin functor \( \pi_\Theta \).

**Theorem 4.16.** Assume given a Gysin schema \( \Theta : C \to D \). Then there is full and faithful functor of categories \( C_{(\pi_\Theta)} \to D \) that is the identity on objects and sends a map \( f \in C_{(\pi_\Theta)}(X,Y) = D(Y \otimes X,S) \) to the composite
\[
\begin{array}{c}
X \xrightarrow{S \otimes X \xrightarrow{\eta_Y \otimes id}} Y \otimes Y \otimes X \xrightarrow{id \otimes f} Y \otimes S \xrightarrow{R \pi_X} S.
\end{array}
\]

**Proof.** The proof is easier to understand if we first compare \( C_{(\pi_\Theta)} \) to \( D^{(ad)} \). Note that the set of objects of these two categories are identical, and for any objects \( X \) and \( Y \) we have equalities of sets
\[
C_{(\pi_\Theta)}(X,Y) = \pi_\Theta(Y \otimes X) = D(Y \otimes X,S) = D^{(ad)}(X,Y).
\]
The identity element \( i_X \in C_{(\pi_\Theta)}(X,X) = \pi_\Theta(X \otimes X) \) is \( \Delta_1(1) \), which unravelling the definitions equals the composite
\[
\Theta X \otimes \Theta X \xrightarrow{I \Delta} \Theta X \xrightarrow{R \pi} S,
\]
which equals \( \epsilon_X \). This is equal to the identity in \( D^{(ad)} \).

Finally, we must compare the composition rules in \( C_{(\pi_\Theta)} \) and \( D^{(ad)} \). Suppose given \( f \in D(Y \otimes X,S) \) and \( g \in D(Z \otimes Y,S) \). The composition \( g \circ f \) in \( C_{(\pi_\Theta)} \) is given by the composite
\[
\begin{array}{c}
\Theta Z \otimes \Theta X \xrightarrow{\gamma} \Theta(Z \times X) \xrightarrow{1p} \Theta(Z \times Y \times X)
\end{array}
\]
\[
\begin{array}{c}
\Theta(Z \times Y \times X) \otimes \Theta(Z \times Y \times X)
\end{array}
\]
\[
\begin{array}{c}
\Theta Z \otimes \Theta Y \otimes \Theta X \otimes \Theta Z \otimes \Theta Y \otimes \Theta X
\end{array}
\]
\[
\begin{array}{c}
\Theta Z \otimes \Theta Y \otimes S \otimes S \otimes \Theta Y \otimes \Theta X
\end{array}
\]
\[
\begin{array}{c}
S \otimes S \otimes S \otimes S \xrightarrow{g \otimes 1 \otimes f} S
\end{array}
\]
where $p: Z \times Y \times X \to Z \times X$ is the evident projection. The composition $g \circ f$ in $\mathcal{D}^{(ad)}$ is given by the composite

$$
\Theta Z \otimes \Theta Y \otimes \Theta X \xrightarrow{\pi_Y \otimes 1} \Theta Z \otimes \Theta Y \otimes \Theta X \xrightarrow{g \otimes f} S \otimes S \xrightarrow{1 \otimes \eta_Y} S.
$$

A diagram chase shows these two composites to be equal. This is best left to the reader, but main idea is to take the first composite and decompose the diagonal on $Z \times Y \times X$ into the three diagonals on the individual components. The diagonals on $Z$ and $X$ cancel the $\pi_X$ and $\pi_Z$ appearing later, leaving only the diagonal on $Y$. The map $1p$ is equal to $1 \otimes I\pi_Y \otimes 1$, and the $I\pi_Y$ assembles with the $R\Delta_Y$ to make $\eta_Y$.

At this point we have constructed a functor $\mathcal{C}(\pi\Theta) \to \mathcal{D}^{(ad)}$. It is readily seen to induce isomorphisms on the Hom-sets, and so it is an isomorphism of categories. Finally, pair this with Proposition 4.14 to get the desired result. □

Example 4.17. Let $\mathcal{D}$ be the category of finitely-generated free abelian groups, and let $\mathcal{C}$ be the category of finite sets. Let $\Theta: \mathcal{C} \to \mathcal{D}$ be the free abelian group functor, given the structure of a Gysin schema as in Example 4.4. The associated Gysin functor $\pi\Theta$ is precisely the one of Example 3.8(a). By Theorem 4.16 we conclude that $\mathcal{C}(\pi\Theta)$ is isomorphic to the category of finitely-generated free abelian groups.

5. The structure of correspondence categories

Suppose $E: \mathcal{C} \to \text{CommRing}$ is a Gysin functor. Our goal is to better understand how the category of correspondences $\mathcal{C}_E$ relates to the original category $\mathcal{C}$. Given objects $X$ and $Y$ in $\mathcal{C}$, every element $f \in \mathcal{C}(X,Y)$ gives rise to maps $R_f$ and $I_f$ in $\mathcal{C}_E$. In addition, we will see that every element $a \in E(X)$ gives an endomorphism $D_a$ of $X$ in $\mathcal{C}_E$. We will prove that every map in $\mathcal{C}_E$ may be written in the form $R_f \circ D_a \circ I_g$, and we will explain rules for rewriting the composition of two such expressions into the same form.

These results do not give a simple picture for the structure of $\mathcal{C}_E$, but they do give a reasonable prescription for working with these categories in specific examples. In Section 5.12 we explore this in a general “Galoisien” setting (where the category $\mathcal{C}$ has properties formally similar to the category of $G$-sets, $G$ is a finite group).

5.1. The diagonal structure. We will need an extra piece of structure in $\mathcal{C}_E$ coming from the diagonal maps in $\mathcal{C}$. For an object $X$ in $\mathcal{C}$ let $\Delta: X \to X \times X$ be the diagonal. This induces a map of abelian groups

$$
D = \Delta! : E(X) \to E(X \times X) = \mathcal{C}_E(X,X).
$$

The target has two ring structures: it has the generic ring structure that any $E(Z)$ has, and it has the circle product coming from composition in $\mathcal{C}_E$. It is the latter that we wish to consider:

Proposition 5.2. $D: E(X) \to \mathcal{C}_E(X,X)$ is a ring map, and for any $a \in E(X)$ one has $(Da)^* = Da$. 
Proof. Let \( a, b \in E(X) \). We calculate

\[
Db \circ Da = (\pi_{13})(\pi_{23}^*(\Delta b) \cdot \pi_{23}^*(\Delta a))
\]

\[
= (\pi_{13})(\pi_{23}^*(\Delta a))(\pi_{13}^*(\Delta b)) \cdot \pi_{23}^*(\Delta a) \quad \text{(push-pull)}
\]

\[
= \pi_{13}^*(\Delta b) \cdot \pi_{23}^*(\Delta a)
\]

The second equality is from Proposition 3.16(i), and the third equality is because

\[
\Delta!((\Delta^* \pi_1^* b) \cdot a) = \Delta!(\Delta^* \pi_1^* b) \cdot a = D(ba).
\]

The second statement in the proposition is proven by

\[
(Da)^* = t^*(Da) = (t^{-1})^!(Da) = t!(Da) = t!(\Delta a) = \Delta a = Da.
\]

The second equality is from Proposition 3.16(i), and the third equality is because \( t = t^{-1} \).

**Notation 5.3.** We will usually write \( Da \), or if really necessary \( D(a) \), but sometimes we will write \( D_a \) for the same thing.

**Proposition 5.4.** Suppose given \( f : X \to Y \) and \( a \in E(Y) \). Then \( Da \circ R_f = R_f \circ D(f^* a) \) and \( I_f \circ Da = D(f^* a) \circ I_f \).

**Proof.** We compute

\[
Da \circ R_f = \Delta a \circ R_f = (id \times f)^*(\Delta a) = ((f \times id)^ Y_{X \times X})_1(f^* a) = (f \times id)_1 \Delta_!(f^* a) = R_f \circ D(f^* a).
\]

The second and fifth equalities are by Proposition 3.16(e), and the third equality is by push-pull.

To conclude, the second statement in the proposition follows by applying \((-)^*\) to the first and using Proposition 5.2.

**Remark 5.5.** Let \( G = \text{Aut}_E(X) \), with the group structure coming from composition. Note that there is a map \( G^{op} \to \text{Aut}(E(X)) \) given by \( f \mapsto f^* \). Let \( E(X)[\tilde{G}] \) be the twisted group ring defined as follows: it is spanned by elements \( a[f] \) for \( a \in E(X) \) and \( f \in G \), and the multiplication is induced by

\[
a[f] \cdot b[g] = a((f^{-1})^* b)[fg].
\]

Then Proposition 5.4 shows that there is a map of rings

\[
E(X)[\tilde{G}] \longrightarrow C_E(X, X), \quad a[f] \mapsto Da \circ R_f.
\]

In good cases this is an isomorphism: see Proposition 5.15 below.

5.6. **Initial results on the structure of** \( C_E \). If we have maps \( Y \leftarrow f Z \rightarrow g X \) and \( a \in E(Z) \) then \( R_f \circ D_a \circ I_g \) is a morphism from \( X \) to \( Y \) in \( C_E \). We will refer to such an expression as an RDI formula for the composite morphism. Here are some useful facts that relate these RDI formulas in \( C_E \) to pushforwards in \( E \):

**Proposition 5.7.** Suppose given maps \( Y \leftarrow f Z \to g X \) and \( a \in E(Z) \). Then:

(a) \( R_f \circ D_a \circ I_g = ((f \times g)^{(Z,Y)}_{X,Y})(a) \).
(b) \( R_f \circ D_a \circ I_f = D(fa) \).
(c) \( R_f \circ I_f = D(f1) \).

Proof. For (a) we use Proposition 3.16(e) to write
\[
R_f \circ D_a \circ I_g = ((f \times id)_!(id \times g)_!(\Delta_! a)) = ((f \times g)_{Y \times X})_!(a).
\]

Part (b) follows from (a) together with \((f \times f)_{Y \times Y} = \Delta_Y \circ f\). Finally, (c) is just the special case of (b) where we take \(a = 1\) (so \(Da = i_X\)). □

In fact every morphism in \(\mathcal{C}_E\) can be expressed as an \(RDI\) composition. This is actually a triviality, but it is nevertheless important:

**Lemma 5.8.** Every element of \(\mathcal{C}_E(X,Y)\) may be written as \(R_f \circ D_a \circ I_g\) for some object \(Z\), some maps \(f : Z \to Y\), \(g : Z \to X\), and some \(a \in E(Z)\).

Proof. Let \(a \in \mathcal{C}_E(X,Y) = E(Y \times X)\). Set \(Z = Y \times X\). We claim that \(a = R_{\pi_1} \circ D_a \circ I_{\pi_2}\) in \(\mathcal{C}_E\). This is immediate from Proposition 5.7(a). □

Now suppose that we have two maps in \(RDI\) form, and that we wish to compose them; that is, consider a composition of the form
\[
[R_f \circ D_a \circ I_g] \circ [R_{f'} \circ D_{a'} \circ I_{g'}].
\]

There are three rules that allow us to rewrite this in \(RDI\) form once again. We indicate these schematically as:

\[
\begin{align*}
D \circ R & \rightsquigarrow R \circ D \quad \text{[Proposition 5.4]} \\
I \circ D & \rightsquigarrow D \circ I \quad \text{[Proposition 5.4]} \\
I \circ R & \rightsquigarrow R \circ I \quad \text{[Proposition 3.16(h)].}
\end{align*}
\]

To use these in our problem, we start by forming the pullback in the following diagram:

\[
\begin{array}{ccc}
P & \rightarrow & A \\
\downarrow s & & \downarrow f \\
A' & \leftarrow & B \\
\downarrow f' & & \downarrow g' \\
Y & \leftarrow & X.
\end{array}
\]

Then
\[
(5.10) \quad R_f \circ D_a \circ I_g \circ R_{f'} \circ D_{a'} \circ I_{g'} = R_f \circ R_a \circ D_{\alpha} \circ I_{\alpha} \circ D_{\alpha'} \circ I_{\alpha'} = R_{fs} \circ D_{(\alpha \circ \alpha')} \circ I_{\alpha \circ \alpha'}.
\]

The above discussion has proven Corollary 1.3 from the introduction. We also note that we have proven Theorem 1.2 along the way:

**Proof of Theorem 1.2.** Parts (a) and (b) are Proposition 5.4, whereas (c) is Proposition 3.16(h). Part (d) is Proposition 5.7(b). □
5.11. A detailed example of the Burnside functor. Consider the category $\mathcal{C}_A$, where $A$ is the Burnside functor for $C$. Given the role of $A$ as the universal Gysin functor, it is useful to have a particularly good handle on how to work with $\mathcal{C}_A$. Recall that a map in $\mathcal{C}_A$ from $X$ to $Y$ is an element of $A(Y \times X)$, and so is represented by a map $h : S \to Y \times X$ in $C$. It is often useful to represent this data as a span, by writing

$$
\begin{array}{ccc}
S & \xrightarrow{\pi_1 h} & Y \\
\downarrow{\pi_2 h} & & \downarrow{h} \\
X & & X,
\end{array}
$$

The following list gives a “dictionary” for how certain structures are represented in $\mathcal{C}_A$.

(1) $\begin{array}{ccc}
X & \xrightarrow{id} & X \\
\downarrow{f} & & \downarrow{g} \\
A & & B
\end{array} \implies Rf, \quad \begin{array}{ccc}
X & \xrightarrow{id} & X \\
\downarrow{f} & & \downarrow{g} \\
A & & B
\end{array} \implies Ig$

(2) $\begin{array}{ccc}
X & \xrightarrow{g} & B \\
\downarrow{f} & & \downarrow{h} \\
A & & Y
\end{array} \implies Rf \circ Ig$

(3) $\begin{array}{ccc}
X & \xrightarrow{id} & X \\
\downarrow{f} & & \downarrow{g} \\
A & & B
\end{array} \implies i_X$

(4) $\begin{array}{ccc}
X & \xrightarrow{g} & B \\
\downarrow{f} & & \downarrow{h} \\
A & & Y
\end{array} \otimes \begin{array}{ccc}
X' & \xrightarrow{g'} & B' \\
\downarrow{f'} & & \downarrow{h'} \\
A' & & Y'
\end{array} \implies \begin{array}{ccc}
X \times X & \xrightarrow{\pi_1} & X \\
\downarrow{\pi_2} & & \downarrow{\pi_2} \\
X & & X
\end{array}$

(5) unit of $\otimes$ is $S = *$

(6) The adjunction $\text{Hom}(X, F(Y, Z)) \to \text{Hom}(X \otimes Y, Z)$ is

$$
\begin{array}{ccc}
W & \xrightarrow{f \times g} & Y \times Z \\
\downarrow{h} & & \downarrow{g} \\
X & & Z
\end{array} \leftrightarrow \begin{array}{ccc}
W & \xrightarrow{h \times f} & X \times Y \\
\downarrow{f \times g} & & \downarrow{h \times f} \\
X & & Z
\end{array}
$$

(8) The identity $\Delta_X : S \to F(X, X)$ is

$$
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \\
\downarrow{f} & & \downarrow{\Delta} \\
X \times X & & X
\end{array}
$$
The evaluation and coevaluation morphisms are
\[
\begin{array}{c}
X \\
\Delta \\
\downarrow \ \\
X \times X \\
\end{array}
\xrightarrow{\ev_X} 
\begin{array}{c}
X \\
\Delta \\
\downarrow \ \\
X \times X \\
\end{array}
\xleftarrow{\cev_X}
\]

The transposition \( t_{X,Y} : X \otimes Y \to Y \otimes X \) is
\[
\begin{array}{c}
X \times Y \\
\downarrow t \\
Y \times X \\
\downarrow \id \\
X \times Y \\
\end{array}
\]

Given \( a : T \to X \), one has
\[
\begin{array}{c}
T \\
\downarrow a \\
X \\
\end{array}
\xrightarrow{Da}
\begin{array}{c}
T \\
\downarrow a \\
X \\
\end{array}
\]

5.12. Gysin categories in the Galois setting. We now add some extra hypotheses to the category \( C \), all of which are satisfied in the cases of interest. First, say that an object \( X \) in \( C \) is atomic if \( X \neq \emptyset \) and \( X \) is not isomorphic to a coproduct \( A \amalg B \) where both \( A \) and \( B \) are different from the initial object. We will assume that

- If \( X \) is atomic and \( Y \) and \( Z \) are any objects, then the natural map \( C(X,Y) \amalg C(X,Z) \to C(X,Y \amalg Z) \) is a bijection.
- For every atomic object \( X \) in \( C \), the set \( \text{Aut}(X) \) is finite.
- If \( X \neq \emptyset \) then \( C(X,\emptyset) = \emptyset \).

If \( C \) is finitary lextensive and satisfies the above properties, we will say that \( C \) is a Galoisien category.

Example 5.13. Let \( G \) be a finite group, and let \( C \) be the category of finite \( G \)-sets. Then \( C \) is Galoisien, and the atomic objects are the transitive \( G \)-sets.

Let \( Y \) be an object of \( C \), and let \( G(Y) = \text{Aut}(Y) \). There is an evident map
\[
\prod_{\sigma \in G(Y)} Y_\sigma \to Y \times Y
\]
(though \( Y_\sigma \) denotes a copy of \( Y \) labelled by \( \sigma \), where the map \( Y_\sigma \to Y \times Y \) is \( \id \times \sigma \).)

We say that \( Y \) is Galois if the displayed map is an isomorphism. To generalize this somewhat, if \( p : X \to Y \) is a map then let \( G(X/Y) = \{ \alpha \in \text{Aut}(X) \mid p \alpha = p \} \). Say that \( X \to Y \) is Galois if the evident map
\[
\prod_{\sigma \in G(X/Y)} X_\sigma \to X \times_Y X
\]
is an isomorphism.

The results in the following lemma can be proven by elementary category theory:

Lemma 5.14. Suppose that \( X \) and \( Y \) are atomic.

(a) If \( Y \) is Galois then \( C(X,Y) \) is either empty or else it is a \( G(Y) \)-torsor.
(b) If \( Y \) is Galois then every endomorphism of \( Y \) is an isomorphism.
(c) If \( X \) and \( Y \) are Galois and \( f,g : X \to Y \), then the evident map
\[
\prod_{\{\sigma \in G(X) \mid |f \sigma| \neq 0\}} X_\sigma \to \text{pullback}[X \xrightarrow{f} Y \xleftarrow{g} X]
\]
is an isomorphism.

(d) If \( X \) and \( Y \) are Galois then so is every map \( X \to Y \).

(e) Suppose that \( Y \) and \( Z \) are both Galois, and assume given \( f : X \to Y \) and \( g : Z \to Y \). If there exists a map \( u : X \to Z \) such that \( gu = f \), then the evident map
\[
\prod_{\sigma \in G(Z/Y)} X_\sigma \to X \times_Y Z
\]
is an isomorphism.

(f) If \( f : X \to Y \) is a map and \( Y \) is Galois, then \( \prod_{\sigma \in G(Y)} X_\sigma \to X \times Y \) given by \( id \times \sigma f : X_\sigma \to X \times Y \) is an isomorphism.

(g) If \( X \) and \( Y \) are both Galois and \( f : X \to Y \), then for every \( \alpha \in \text{Aut}(X) \) there is a unique \( \alpha_f \in \text{Aut}(Y) \) such that \( f <^\sigma \alpha \). Moreover, the map \( G(X) \to G(Y) \) given by \( \alpha \to \alpha_f \) is a group homomorphism.

Proof. For (a), suppose that \( \mathcal{C}(X,Y) \neq \emptyset \) and let \( f : X \to Y \) be a map. We need to show that the map \( G(Y) \to \mathcal{C}(X,Y) \) given by \( \sigma \to \sigma f \) is a bijection. Let \( g : X \to Y \) be any map, and consider \( f \times g : X \to Y \times Y \). Composing with the isomorphism \( \prod_{G(Y)} Y \to Y \) \( \times Y \), the fact that \( X \) is atomic shows that the resulting map factors through a map \( u : X \to Y_\sigma \), for some \( \sigma \). One then obtains the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y _\sigma \\
\downarrow{f \times g} & & \downarrow{id \times \sigma} \\
Y & \xrightarrow{id} & Y \times Y
\end{array}
\]

which shows that \( u = f \) and \( g = \sigma u = \sigma f \). So the action of \( G(Y) \) on \( \mathcal{C}(X,Y) \) is transitive.

Now suppose that \( \alpha, \beta \in G(Y) \) and \( \alpha f = \beta f \). Then \( f \times \alpha f : X \to Y \times Y \) factors through both \( Y_\alpha \) and \( Y_\beta \) (under the isomorphism \( \prod_{G(Y)} Y \to Y \times Y \)). Therefore it factors through the pullback \( Y_\alpha \to \prod_{G(Y)} Y \leftarrow Y_\beta \). But if \( \alpha \neq \beta \) then this pullback is \( \emptyset \), by our standing hypotheses that \( \mathcal{C} \) is finitary lextensive. Since the map \( f \times \alpha f \) cannot factor through \( \emptyset \), this is a contradiction; so we must have \( \alpha = \beta \).

Part (b) is an immediate consequence of (a). For (c), the pullback in question is isomorphic to the pullback of
\[
Y \xrightarrow{\Delta} Y \times Y \xrightarrow{f \times g} X \times X.
\]

Use the decomposition \( X \times X \cong \prod_{\sigma \in G(X)} X_\sigma \) and the fact that pullbacks distribute over finite coproducts to see that our pullback is isomorphic to
\[
\prod_{\sigma \in G(X)} \text{pullback}[Y \xrightarrow{\Delta} Y \times Y \xrightarrow{f \times g \sigma} X].
\]

Next use the decomposition \( Y \times Y \cong \prod_{\alpha \in G(Y)} Y_\alpha \), together with the fact that \( \Delta : Y \to Y \times Y \) factors through the summand \( Y_{\text{id}} \). Since \( X \) is atomic, we deduce that the pullback inside the above coproduct is either \( \emptyset \) (when \( f \neq g \sigma \)) or \( X \) (when \( f = g \sigma \)). This finishes off part (c).

Part (d) is a direct consequence of (c), applied in the case \( f = g \).
For (e), the existence of \( u \) implies that \( X \times_Y Z \) is isomorphic to the pullback of
\[
X \xrightarrow{u} Z \xleftarrow{\pi_1} Z \times_Y Z.
\]
Next use that \( Z \times_Y Z \cong \bigsqcup_{\sigma \in G(Z/Y)} Z_{\sigma} \) and use the fact that pullbacks distribute
over coproducts.

Part (f) is a special case of (e). Finally, the proof of (g) uses the same techniques
that have been demonstrated in the preceding parts: consider \( f \times f_\alpha : X \to Y \times Y \)
and factor this through some \( Y_\sigma \). Details are left to the reader. \( \square \)

Before proceeding, let us establish some notation. If \( R \) is a ring and \( S \) is a set,
then \( R(S) \) denotes the set of all formal finite sums \( \sum r_is_i \) where \( r_i \in R \) and \( s_i \in S \).
This is the free left \( R \)-module with basis \( S \). Similarly, let \( \langle S \rangle R \) be the set of all
formal finite sums \( \sum s_ir_i \) with \( s_i \in S \) and \( r_i \in R \). When \( R \) is commutative these
are of course isomorphic \( R \)-modules, but the difference in notation will be useful to
us below.

When \( X \) is Galois we can now determine the ring \( C_E(X,X) \) precisely:

**Proposition 5.15.** If \( X \) is Galois then the map \( E(X)[\widehat{\text{Aut}(X)}] \to C_E(X,X) \) from
Remark 5.5 is an isomorphism of rings.

**Proof.** Since \( X \) is Galois, the usual map \( \bigsqcup_{\sigma \in G(X)} X \to X \times X \) is an isomorphism. So
\[
B: \bigoplus_{\sigma \in G(X)} E(X) \to E(X \times X)
\]
is an isomorphism, where on component \( \sigma \) the map \( B \) equals \((id \times \sigma) \circ \Delta_1 \). If \( a \in E(X) \)
then we have a copy of \( a \) in the component of the domain indexed by \( \sigma \). The image of this class in \( E(X \times X) \) is precisely
\[
(id \times \sigma) \circ \Delta_1(a) = D_\sigma \circ I_\sigma = D_\sigma \circ R_{\sigma^{-1}}.
\]
This implies that the map \( E(X)[\widehat{\text{Aut}(X)}] \to C_E(X,X) \) given by \( a.\sigma \mapsto D_\sigma R_{\sigma} \) is
an isomorphism of abelian groups. We already saw in Remark 5.5 that it is a ring
homomorphism, where we give the domain the appropriate structure of twisted
group ring. \( \square \)

**Remark 5.16.** In concrete terms, Proposition 5.15 says that every map in \( C_E(X,X) \)
may be uniquely written as a finite sum of terms \( D_aR_\alpha \) where \( a \in E(X) \)
and \( \alpha \in \text{Aut}_C(X) \). Composition is done according to the rule
\[
D_aR_\alpha \circ D_bR_\beta = D_aD_{(\alpha^{-1})b}R_\alpha R_\beta = D_{a.(\beta^{-1})b}R_{\alpha\beta}
\]
where in the first equality we have used Proposition 5.4 (together with the fact that
\( \alpha \) is an isomorphism). The awkwardness of this formula stems from our representation
of elements of \( C_E(X,X) \) in the form \( D_aR_\alpha \). As we have remarked before, it is
better to use the \( RDI \) system and represent the elements as \( R_\alpha \circ D_a \). If we do
this, then the composition law is
\[
R_\alpha D_a \circ R_\beta D_b = R_{\alpha\beta}D_{(\beta \circ \alpha \circ b)},
\]
which is a little simpler. We will always use this formulation from now on.

We next turn to the case of two objects. Assume that \( f: X \to Y \) is a map
in \( C \), where both \( X \) and \( Y \) are assumed to be atomic and Galois. Our goal is to
describe the full subcategory of \( C_E \) containing \( X \) and \( Y \). If \( f \) is an isomorphism
then this problem reduces to the case of one object, which we handled above. So let us further assume that \( f \) is not an isomorphism. Note that this implies that there cannot exist a map in \( C \) from \( Y \) to \( X \): if there were such a map, then the post- and pre-composites with \( f \) would be isomorphisms by Lemma 5.14(b), and so \( f \) would itself be an isomorphism.

Write \( \text{Aut}(X) = \{a_1, \ldots, a_r\} \) and \( \text{Aut}(Y) = \{\beta_1, \ldots, \beta_s\} \). Note that \( C(X,Y) = \{\beta_1 f, \ldots, \beta_s f\} \) by Lemma 5.14(a), and \( X \times Y \cong \bigsqcup \text{Aut}(Y) \ X \) by Lemma 5.14(f).

Then \( C_E(X,Y) = E(Y \times X) \cong \bigsqcup \text{Aut}(Y) E(X) \), and one can check that the isomorphism is the one that represents each map in \( C_E(X,Y) \) as a sum of maps \( R_{\beta_i} D_a \) where \( a \in E(X) \). A similar analysis works for \( C_E(Y,X) \), and so the full subcategory of \( C_E \) containing \( X \) and \( Y \) may be depicted as follows:

\[
\begin{array}{c}
(X, Y) \\
\text{(R}_{\beta_1, \ldots, \beta_s}) \ E(X) \\
\text{(R}_{\alpha_1, \ldots, \alpha_r}) \ E(Y)
\end{array}
\]

The labels on the arrows depict the abelian group of maps in \( C_E \); e.g., the label on the arrow from \( X \) to \( Y \) depicts \( C_E(X,Y) \). The diagram indicates that every map from \( X \) to \( Y \) may be uniquely written as a sum of terms \( R_{\beta_i} D_a \), where \( a \in E(X) \) (and similarly for other choices of domain and range).

Composition of maps are determined via the \textit{RDI} rules outlined in \textsection 5.9. Here are some examples:

1. \( [X \to Y \to Y] \) compositions. Here one uses
\[
R_{\beta_i} D_a \circ R_{\alpha_j} D_b = R_{\beta_i \alpha_j} D_a D_{\alpha_j^*(a)} D_b = R_{\beta_i \alpha_j} D_{(\alpha_j^* a) b}.
\]

2. \( [Y \to Y \to X] \) compositions. Here one uses that \( \beta_i \) is invertible and so we have
\[
R_{\beta_i} = I_{\beta_i}^{-1}.
\]
\[
D_a I_{\beta_i} \circ R_{\beta_i} D_a = D_a I_{\beta_i} \circ I_{\beta_i}^{-1} D_a = D_a I_{\beta_i} I_{\beta_i}^{-1} D_a = D_a I_{\beta_i}^{-1} \beta_i D_a
\]
\[
= D_a (I_{\beta_i}^{-1} \beta_i) a \cdot I_{\beta_i}^{-1} \beta_i a.
\]

3. \( [Y \to X \to Y] \) compositions. In this case we consider
\[
R_{\beta_i} D_a \circ D_b I_{\beta_i} = R_{\beta_i} \circ D_b \circ I_{\beta_i} = R_{\beta_i} \circ D_{\beta_i^*(b)} \circ I_{\beta_i}
\]
\[
= R_{\beta_i} \circ I_{\beta_i} \circ D_{(\beta_i) a} = R_{\beta_i} \circ R_{\beta_i}^{-1} D_{(\beta_i) a} = R_{\beta_i} \circ R_{\beta_i}^{-1} \beta_i D_a
\]
\[
= R_{\beta_i} \beta_i^{-1} D_{\beta_i^{-1}} (\beta_i f) a.
\]

In the second equality we have used Proposition 5.7(b) and in the third equality we have used Proposition 5.4 (which applies because \( \beta_i \) is invertible).

4. \( [X \to Y \to X] \) compositions. Let \( T = \{ \sigma \in G(X) \mid \beta_j f = \beta_i \sigma f \} \). Observe that by Lemma 5.14(c) since \( X \) and \( Y \) are Galois we have a pullback diagram

\[
\begin{array}{c}
\bigsqcup_{\sigma \in T} X_{\sigma} \to X \\
\downarrow \downarrow \\
X \downarrow \beta_j f \to Y
\end{array}
\]
where the vertical map \( X_\sigma \to X \) is the identity and the horizontal map \( X_\sigma \to X \) is \( \sigma \). We then write
\[
D_a^* I_{\beta,f} \circ R_{\beta,f} D_a = \sum_{\sigma} D_a^* I_\sigma D_a = \sum_{\sigma} I_\sigma D_{(\sigma^{-1})^*(a')} D_a \\
= \sum_{\sigma} D_{\sigma^{-1}} D_{((\sigma^{-1})^*(a'))} D_a.
\]

The second equality is by Proposition \([5.4] \) using that \( \sigma \) is an isomorphism.

(5) [Remaining cases.] The cases that have not been treated so far are all very similar to (1) or (2).

As the reader can see from the above analysis, a complete description of the maps between Galois objects is relatively simple. But the description of compositions becomes unwieldy, although in practice it is a purely mechanical process to work out any given composition.

6. Grothendieck-Witt categories over a field

Let \( k \) be a field of characteristic not equal to 2, and consider the Grothendieck-Witt category \( \text{GWC}(k) \) over \( k \).

Let \( \text{fEt} / k \) be the full subcategory of \( \text{Aff} / \text{Spec} k \) consisting of the objects \( \text{Spec} E \) where \( k \to E \) is finite étale. Let \( \mathcal{A}_{\text{fEt}} \) be the Burnside Gysin functor, and let \( \chi : \mathcal{A}_{\text{fEt}} \to \text{GW} \) be the natural transformation from Proposition \([3.12] \).

The following result is essentially \([\text{De}] \) Appendix B, Theorem 3.1]. We include the proof for completeness. For the proof, recall that if \( a \in E \) then \( (a) \) denotes the quadratic space \( (E, b_a) \) where \( b_a(x, y) = axy \), and \( (a, b) = (a) \oplus (b) \).

**Proposition 6.1.** For any finite separable field extension \( k \to E \), the map \( \chi : \mathcal{A}_{\text{fEt}}(E) \to \text{GW}(E) \) is surjective.

**Proof.** Recall that \( \text{GW}(E) \) is generated as an abelian group by the classes \( (a) \) for \( a \in E^* \). We will show that each of these classes is in the image of \( \chi \).

If \( a \) is not a square in \( E \) then consider the field extension \( E_a = E[x]/(x^2 - a) \). Then \( E_a \) is a separable field extension of \( E \), and \( \chi(E_a) \) is simply \( E_a \) (regarded as an \( E \)-vector space) equipped with the trace form. An easy computation shows this is isomorphic to \( (2, 2a) = (2) + (2a) \). So we have \( (2) + (2a) = \chi(E_a) \).

We claim that \( (2) \in \text{Im} \chi \). If \( 2 \) is a square in \( E \) then this is clear, since \( (2) = (1) \). If \( 2 \) is not a square in \( E \) then we may apply the above analysis with \( a \) replaced by \( 2 \) to find that \( (2) + (4) \in \text{Im} \chi \). Since \( (4) = (1) \in \text{Im} \chi \), we again have \( (2) \in \text{Im} \chi \).

At this point we know that \( (2) + (2a) \in \text{Im} \chi \) and \( (2) \in \text{Im} \chi \), and so \( (2a) \in \text{Im} \chi \). But then \( (4a) = (2) \cdot (2a) \in \text{Im} \chi \). Since \( (4a) = (a) \), we are done. \( \square \)

**Example 6.2.** The map \( \chi \) is usually not an isomorphism. To see this in one example, let \( k = \mathbb{F}_p \) where \( p \) is odd. Then \( \mathcal{A}_{\text{fEt}}(k) \) is a free abelian group on a countably-infinite set of generators, whereas \( \text{GW}(k) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \). In general, it would be interesting to have a set of generators for the kernel of \( \mathcal{A}_{\text{fEt}}(k) \to \text{GW}(k) \) together with some kind of geometric source for them. See Example 6.7 below.

**Remark 6.3.** If \( f : R \to S \) is a sheerly separable map of rings, we have the induced maps \( f_* : \text{GW}(R) \to \text{GW}(S) \) and \( f^* : \text{GW}(S) \to \text{GW}(R) \) from Section \([2] \).

However, for most purposes it is more convenient to use the geometric setting of affine schemes: there we would write \( f^* : \text{GW}((\text{Spec} R) \to \text{GW}((\text{Spec} S) \) and
where each trivial automorphism. Since $GW(\mathbb{C})$ is the full subcategory of $(fEt_k)^{alg}$, we will tend to mix the two notations and write $f^*: GW(R) \rightarrow GW(S)$ and $f_!: GW(S) \rightarrow GW(R)$. In effect, this is basically just dropping the “Spec” and letting it be understood. In practice there is never any confusion here.

Our goal is to be able to analyze pieces of the categories $GWC(k)$ for some explicit choices of $k$. Galois theory gives an equivalence of categories between sheerly separable extensions of $k$ and continuous $Gal(k^{sep}/k)$-sets, and this is a useful tool to exploit.

Fix a finite-dimensional Galois extension $L/k$, and set $G = Gal(L/k)$. Say that a separable $k$-algebra $A$ is $L$-constructible if it is isomorphic to a product $\prod_i A_i$ where each $A_i$ is an algebraic field extension of $k$ that admits an embedding into $L$. For each finite $G$-set $S$, let $\mathcal{F}(S, L)$ be the set of $G$-maps from $S$ to $L$, with ring structure given by pointwise addition and multiplication. Clearly $\mathcal{F}(G/H, L) \cong L^H$ and $\mathcal{F}(S \amalg T, L) \cong \mathcal{F}(S, L) \times \mathcal{F}(T, L)$, hence each $\mathcal{F}(S, L)$ is $L$-constructible. In the opposite direction, given a sheerly separable $k$-algebra $A$ the set of $k$-algebra maps $k$--alg$(A, L)$ inherits an action of $G$. Galois theory says that we have an equivalence of categories

$$\text{finGSet} \cong f\text{Et}_k^{L-con}$$

where the upper arrow is $S \mapsto \mathcal{F}(S, L)$ and the lower arrow is $S \mapsto k$--alg$(A, L)$.

The Grothendieck-Witt functor on $fEt_k$ restricts, via the above Galois equivalence, to a Gysin functor on finite $G$-sets. Let us write

$$GW_L(S) = GW(\mathcal{F}(S, L))$$

for this restricted Gysin functor. Clearly the correspondence category $\text{finGSet}_{GW_L}$ is the full subcategory of $(fEt_k)^{GW}$ whose objects are the $L$-constructible $k$-algebras.

The universality of the Burnside functor gives a natural transformation $\mathcal{A}_G \rightarrow GW_L$, and therefore a functor between correspondence categories

$$\text{finGSet}_{(\mathcal{A}_G)} \rightarrow \text{finGSet}_{GW_L}.$$  

Putting everything together, we have constructed a functor from the Burnside category of $G$ to the Grothendieck-Witt category over $k$.

We now look at several examples:

**Example 6.4.** The category $GWC(\mathbb{R})$ has two objects: Spec $\mathbb{R}$ and Spec $\mathbb{C}$. Let $\pi$: Spec $\mathbb{C} \rightarrow$ Spec $\mathbb{R}$ be the unique map, and $\sigma$: Spec $\mathbb{C} \rightarrow$ Spec $\mathbb{C}$ be the non-trivial automorphism. Since $GW(\mathbb{C}) = \mathbb{Z}$ and $GW(\mathbb{R}) = \mathbb{Z}(\langle 1 \rangle, \langle -1 \rangle)$, the category $GWC(\mathbb{R})$ is readily computed to be as shown in the diagram below. One only needs check that $I_\pi \circ R_\pi = 1 + \sigma$ and $R_\pi \circ I_\pi = (1) + \langle -1 \rangle$.

Similarly, the Burnside category for $\mathbb{Z}/2$ has two objects: $*$ and $\mathbb{Z}/2$. We write $\pi: \mathbb{Z}/2 \rightarrow *$ and $\sigma: \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ for the evident maps. Then $\mathcal{A}_\mathbb{Z}/2(\mathbb{Z}/2) = \mathbb{Z}$ and $\mathcal{A}_\mathbb{Z}/2(\{*, [\mathbb{Z}/2]\})$. Here one computes that $I_\pi \circ R_\pi = 1 + \sigma$ and $R_\pi \circ I_\pi = [\mathbb{Z}/2]$. 


The map from the Burnside category to the Grothendieck-Witt category has the evident behavior (in particular, it sends \([\mathbb{Z}/2]\) to \(\langle 1 \rangle + \langle -1 \rangle\)), and by inspection is an isomorphism.

Before considering our next example we need to recall some facts about finite fields. If \(F\) is a finite field of odd characteristic then \(F^\times\) is cyclic of even order and so \((F^\times)/(F^\times)^2 = \mathbb{Z}/2\). Thus when we partition \(F^\times\) into the squares and the non-squares, any two non-squares are equivalent: if \(a\) and \(b\) are non-squares then \(a = \lambda^2 b\) for some \(\lambda\). A little work shows when \(\text{char}(F) \neq 2\) that \(\text{GW}(F)\) is generated by \(\langle 1 \rangle\) and \((g)\), where \(g \in F^\times\) is any choice of non-square. Moreover, \(2(g) = 2\langle 1 \rangle\) and \(\text{GW}(F) \cong \mathbb{Z} \oplus \mathbb{Z}/2\) with corresponding generators \(\langle 1 \rangle\) and \((g) - \langle 1 \rangle\). See [S] or [D1] Appendix A for details. It is useful to write \(\alpha = (g) - \langle 1 \rangle\).

Note that the calculation of \(\text{GW}(F)\) gives a classification of all non-degenerate quadratic spaces over \(F\): in each dimension there are exactly two, namely \(\langle 1 \rangle\) and \((a-1)(1) + \langle g \rangle = n\langle 1 \rangle + \alpha\). The discriminant of the form, regarded as an element of \(F^\times/(F^\times)^2\), distinguishes the two isomorphism types.

The following lemma calculates the behavior of the Grothendieck-Witt group under a quadratic extension.

**Lemma 6.5.** Let \(q = p^e\) where \(p\) is an odd prime. Fix a non-square \(g \in \mathbb{F}_q\), and fix a non-square \(h \in \mathbb{F}_{q^2}\). If \(j: \mathbb{F}_q \to \mathbb{F}_{q^2}\) is a fixed embedding then the pullback and pushforward maps for \(\text{GW}(-)\) are given by the formulas

\[
\begin{align*}
  j^*(\langle 1 \rangle) &= j^*(\langle g \rangle) = \langle 1 \rangle, \\
  j_!(\langle 1 \rangle) &= \langle 1 \rangle + \langle g \rangle, \\
  j_!(\langle h \rangle) &= 2\langle 1 \rangle.
\end{align*}
\]

Every automorphism of \(\mathbb{F}_q\) induces the identity on \(\text{GW}(\mathbb{F}_q)\) (both via pullback and pushforward).

**Proof.** First note that if \(\alpha\) is an automorphism of \(\mathbb{F}_q\) then \(\alpha\) preserves the property of being a square or non-square; consequently, \(\alpha^*\) is the identity since \(\alpha^*(\langle g \rangle) = \langle g \rangle\). Since \(\alpha_1\) is the inverse of \(\alpha^*\) (Lemma 3.4), this is also the identity. So we have verified the last sentence of the lemma.

Observe that \(\mathbb{F}_{q^2}\) may be identified with the extension \(\mathbb{F}_q[x]/(x^2 - g)\), and we may assume that \(j\) is the evident inclusion of \(\mathbb{F}_q\) (using the previous paragraph). Since \(g = x^2\) in \(\mathbb{F}_{q^2}\) we have \(j^*(\langle g \rangle) = \langle 1 \rangle\).

To compute \(j_!(\langle 1 \rangle)\) we must analyze the trace form on \(\mathbb{F}_{q^2}\). This is represented by the \(2 \times 2\) matrix

\[
\begin{bmatrix}
  \text{tr}(1) & \text{tr}(x) \\
  \text{tr}(x) & \text{tr}(x^2)
\end{bmatrix} = \begin{bmatrix}
  2 & 0 \\
  0 & 2g
\end{bmatrix}.
\]

The discriminant is \(4g\), which is equivalent to \(g\) modulo squares. So \(j_!(\langle 1 \rangle) = \langle 1 \rangle + \langle g \rangle\).
The above work readily generalizes to compute \( j_i(\langle a + bx \rangle) \) for any \( a, b \in \mathbb{F}_q \).
This form is represented by the matrix
\[
\begin{bmatrix}
\text{tr}(a + bx) & \text{tr}(ax + bx^2) \\
\text{tr}(ax + bx^2) & \text{tr}(ax^2 + bx^3)
\end{bmatrix}
= \begin{bmatrix}
2a & 2bg \\
2bg & 2ag
\end{bmatrix}.
\]
The discriminant is \( 4a^2g - 4b^2g^2 = 4g(a^2 - b^2g) \), and so \( j_i(\langle a + bx \rangle) = \langle 1 \rangle + \langle g(a^2 - b^2g) \rangle \).

In a finite field every element can be written as a sum of two squares [S, Lemma 2.3.7], so we can write \( g^{-1} = b^2 + r^2 \) for some \( b, r \in \mathbb{F}_q \). Neither \( b \) nor \( r \) is zero, since \( g \) is not a square. Then
\[ j_i((1 + bx)) = (1) + \langle g(1 - b^2g) \rangle = (1) + \langle g(r^2g) \rangle = (1) + (1) = 2(1). \]
Hence \( (1 + bx) \neq (1) \) (since their images under \( j_i \) are different), and so \( 1 + bx \) is a non-square class; i.e. \( \langle 1 + bx \rangle = \langle h \rangle \) in \( \text{GW}(\mathbb{F}_{q^2}) \). So we have in fact proven that \( j_i((h)) = 2(1) \).

**Proposition 6.6.** Let \( q \) be a power of an odd prime, and consider a field extension \( j : \mathbb{F}_q \to \mathbb{F}_{q^n} \). Let \( g \) and \( g' \) be non-squares in \( \mathbb{F}_q \) and \( \mathbb{F}_{q^n} \), respectively. Then the induced maps \( j^* \) and \( j_i \) are given by
\[
j^*(\langle 1 \rangle) = \langle 1 \rangle, \quad j^*(\langle g \rangle) = \begin{cases} \langle g' \rangle & \text{if } e \text{ is odd}, \\
(1) & \text{if } e \text{ is even}, \end{cases}
j_i(\langle 1 \rangle) = \begin{cases}
e{1} & e \text{ odd}, \\
(e - 1)(1) + \langle g \rangle & e \text{ even}, \end{cases}
j_i(\langle g \rangle) = \begin{cases}
(e - 1)(1) + \langle g \rangle & e \text{ odd}, \\
(1) & e \text{ even}. \end{cases}
\]

These formulas can also be written as:
\[
j^*(\langle 1 \rangle) = \langle 1 \rangle, \quad j^*(\alpha) = \begin{cases} \alpha & e \text{ odd}, \\
0 & e \text{ even}, \end{cases}
j_i(\langle 1 \rangle) = \begin{cases} e(1) & e \text{ odd}, \\
(1) + \alpha & e \text{ even}, \end{cases} \quad j_i(\alpha) = \alpha.
\]

**Proof.** The statement about \( j^* \) is immediate: the extension \( \mathbb{F}_{q^n} \) contains a square root of \( g \) if and only if it contains \( \mathbb{F}_{q^2} \), which happens precisely when \( e \) is even.

To compute \( j_i(\langle 1 \rangle) \) it suffices to analyze the discriminant of the trace form on \( \mathbb{F}_{q^n} \).
A classical computation says this coincides with the discriminant of the minimal polynomial of any primitive element for the extension \( \mathbb{F}_{q^n}/\mathbb{F}_q \). If \( r_1, \ldots, r_n \) are the roots of this minimal polynomial, then this discriminant is \( \Delta = Q^2 \) where
\[ Q = \prod_{i<j}(r_i - r_j). \]
If the roots are indexed appropriately then the Galois group of \( \mathbb{F}_{q^n}/\mathbb{F}_q \) acts by cyclic permutation. It follows that \( Q \) is invariant under the Galois action if and only if \( e \) is odd. So we see that \( \Delta \) is a square in \( \mathbb{F}_q \) if and only if \( e \) is odd. The former condition is equivalent to \( j_i((1)) = e(1) \).

Finally, we analyze \( j_i(\langle g \rangle) \). When \( e \) is odd this is easy, as we can write
\[
j_i(\langle g \rangle) = j_i(j^*(\langle g \rangle) \cdot 1) = \langle g \rangle \cdot j_i((1)) = \langle g \rangle \cdot e(1) = e(g) = (e - 1)(1) + \langle g \rangle.
\]
When $e$ is even the pushforward $\text{GW}(\mathbb{F}_{q^e}) \to \text{GW}(\mathbb{F}_{q^e/2})$ sends $\langle g \rangle$ to $2(1)$ by Lemma 6.5. It follows that $j_t(\langle g \rangle)$ is a multiple of 2, and of course it also has rank $e$. The only such element of $\text{GW}(\mathbb{F}_q)$ is $e(1)$.

**Example 6.7** (The Euler characteristic of a finite field extension). Our goal is to explicitly compute the map $\chi: \mathcal{A}_{\text{Et}}(\mathbb{F}_q) \to \text{GW}(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Given a finite field extension $j: \mathbb{F}_q \to \mathbb{F}_{q^e}$, this is equal to $\chi$.

Proposition 6.6, the Euler characteristic is another name for $j_t(1)$. Using Proposition 6.6 this is equal to $\chi(\mathbb{F}_{q^e}) = e(1) + e_\alpha \in \text{GW}(\mathbb{F}_q)$ where

$$e_\alpha = \begin{cases} 0 & \text{if } e \text{ is odd}, \\ 1 & \text{if } e \text{ is even}. \end{cases}$$

It is an amusing exercise to use the above computation to check the multiplicativity formula

$$\chi(\mathbb{F}_{q^e} \otimes \mathbb{F}_{q^f}) = \chi(\mathbb{F}_{q^e}) \cdot \chi(\mathbb{F}_{q^f}),$$

which is the analog in the present context of the topological formula $\chi(X \times Y) = \chi(X) \times \chi(Y)$.

We can use the above computation to give generators for the kernel of $\chi: \mathcal{A}_{\text{Et}}(\mathbb{F}_q) \to \text{GW}(\mathbb{F}_q)$. If we set $E_n = [\mathbb{F}_{q^n}]$ then by inspection a complete set of relations is

$$E_{n+3} = E_{n+2} + E_{n+1} - E_n \ (n \geq 1), \quad 2E_2 = E_1 + E_3, \quad E_3 = 3E_1.$$ 

It would be interesting to find an explicit geometric explanation for these relations. For example, one might try to produce a degree 4 étale map $f: X \to Y$ of $\mathbb{F}_q$-schemes where $Y$ is $\mathbb{A}^1$-connected and where one fiber of $f$ is $\text{Spec} \mathbb{F}_{q^2} \amalg \text{Spec} \mathbb{F}_{q^2}$ and another fiber is $\text{Spec} \mathbb{F}_{q} \amalg \text{Spec} \mathbb{F}_{q^2}$.

**Example 6.8.** We next explore a small piece of $\text{GWC}(\mathbb{F}_p)$, where $p$ is odd. Specifically, consider the full subcategory whose objects are $\text{Spec} \mathbb{F}_q$ for $q = p^{2^i}$ and $0 \leq i \leq 3$. Set $G = \text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p) = \mathbb{Z}/8$. Let $g_{2^i}$ denote some specific choice of non-square element in $\mathbb{F}_{p^{2^i}}$, and write $\alpha_{2^i} = \langle g_{2^i} \rangle - (1)$. Also write $J_{2^i} = \text{GW}(\mathbb{F}_{p^{2^i}})$; this is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$ with corresponding generators 1 and $\alpha_{2^i}$, subject to the multiplicative relation $\alpha_{2^i}^2 = -2\alpha_{2^i} = 0$. Finally, let $\sigma$ always denote the Frobenius $x \mapsto x^p$ and fix specific embeddings $j_{2^i}: \mathbb{F}_{p^{2^i}} \hookrightarrow \mathbb{F}_{p^{2^i+1}}$ and their induced maps $\pi_{2^i}: \text{Spec} \mathbb{F}_{p^{2^i+1}} \to \text{Spec} \mathbb{F}_{p^{2^i}}$.

The following diagrams show the Burnside category for $\mathbb{Z}/8$ as well as the relevant piece of $\text{GWC}(\mathbb{F}_p)$. Recall that if $R$ is a ring and $S$ is a set then we write $R\langle S \rangle$ and $(S)R$ for the sets of finite sums $\sum r_i s_i$ and $\sum s_i r_i$ where $r_i \in R$, $s_i \in S$. We let $A_{2^i} = A_{\mathbb{Z}/8}(\mathbb{Z}/2^i)$, the Grothendieck ring of $\mathbb{Z}/8$-sets over $\mathbb{Z}/2^i$. So $A_{2^i} = \mathbb{Z}(\mathbb{Z}/2^i, [\mathbb{Z}/2^{i+1}], \ldots, [\mathbb{Z}/8])$. In the $\mathbb{Z}/8$-set context we let $\sigma$ always denote the map $x \mapsto x + 1$. 


Note that we have written $\sigma^i$ instead of $R\sigma^i$. Also, note that $\sigma$ acts trivially on each $J_n$ by Lemma 6.5 and so the endomorphism ring of $\mathbb{F}_p^\infty$ is the group ring $J_n[\mathbb{Z}/n]$. The analogous remark holds in the Burnside category. Finally, note that while the two categories clearly have very similar forms, the map between them is not an isomorphism because $A_4 \not\cong J_4$.

Below we list the main relations in $\text{GWC}(\mathbb{F}_p)$. Recall that $\alpha_n \in J_n$ is the unique element of order 2. We simplify $D_a$ to just $a$, for $a \in J_n$.

\[
\begin{align*}
R\pi_n \circ I\pi_n &= \langle 2 \rangle + \alpha_n \in J_n & I\pi_n \circ R\pi_n &= 1 + \sigma^n \\
\alpha_n \circ R\pi_n &= 0 & I\pi_n \circ \alpha_n &= 0 \\
R\pi_n \circ \alpha_{n+1} \circ I\pi_n &= \alpha_n
\end{align*}
\]

We leave the reader to derive these, as they are simple consequences of using the RDI rules together with the computations in Proposition 6.6. Coupled with the obvious relations that come from the category of fields, e.g. $R\pi_n \circ \sigma^n = R\pi_n$, the above relations allow one to work out all compositions in $\text{GWC}(\mathbb{F}_p)$.

**Example 6.9.** We describe one last example, this time concerning non-Galois extensions. Most of the details will be left to the reader. Write $E_2 = \mathbb{Q}(\sqrt{2})$, $E_\mu = \mathbb{Q}(\mu_3)$ (the cyclotomic field), and $E_{2,\mu} = \mathbb{Q}(\sqrt{2}, \mu_2)$. Note that $[E_2 : \mathbb{Q}] = 3$, $[E_3 : \mathbb{Q}] = 2$, and $[E_{2,\mu} : \mathbb{Q}] = 6$. The extensions $E_\mu/\mathbb{Q}$ and $E_{2,\mu}/E_\mu$ are Galois, but $E_2/\mathbb{Q}$ is not. Let $\pi_i$, $i \in \{0, 1, 2, 3\}$, be the maps of schemes induced by the evident inclusions of fields:

\[
\begin{array}{ccc}
\text{Spec } E_2 & \longrightarrow & \text{Spec } E_{2,\mu} \\
\pi_0 & & \pi_3 \\
\text{Spec } \mathbb{Q} & \leftarrow & \text{Spec } E_\mu.
\end{array}
\]

Finally, write $GW_\mu = \text{GW}(E_\mu)$, and so forth.

Computing in the Grothendieck-Witt category $\text{GWC}(\mathbb{Q})$, maps between $E_\mu$ and $\mathbb{Q}$, or between $E_{2,\mu}$ and $E_\mu$, are handled exactly as the general case discussed at the end of Section 4. For maps from $E_2$ to $E_\mu$, as an abelian group this is $GW_{2,\mu}$ since $E_2 \otimes \mathbb{Q} E_\mu = E_{2,\mu}$. A little thought shows that the maps are all of the form $R\pi_3 \circ D\pi_{2,\mu} \circ I\pi_1$, where $\alpha_{2,\mu} \in GW_{2,\mu}$.

To compute maps from $E_2$ to itself, we start with $E_2 \otimes \mathbb{Q} E_2 \cong E_2 \times E_{2,\mu}$. As an abelian group we then have $\text{GWC}(\mathbb{Q})(E_2, E_2) = GW_2 \oplus GW_{2,\mu}$. The two
summands correspond to elements $Da_2$ for $a_2 \in \text{GW}_2$ and $R\pi_1 \circ Da_2,\mu \circ I\pi_1$ where $a_{2,\mu} \in \text{GW}_{2,\mu}$. The ring structure is determined by the formulas

$$Da_2 \circ Db_2 = D(a_2b_2),$$

$$Da_2 \circ (R\pi_1 \circ Da_2,\mu \circ I\pi_1) = R\pi_1 \circ D(\pi_1^*(a_2) \cdot a_{2,\mu}) \circ I\pi_1$$

$$(R\pi_1 \circ Da_2,\mu \circ I\pi_1) \circ Da_2 = R\pi_1 \circ D(a_{2,\mu} \cdot \pi_1^*(a_2)) \circ I\pi_1$$

$$(R\pi_1 \circ Da_2,\mu \circ I\pi_1) \circ (R\pi_1 \circ Db_2,\mu \circ I\pi_1) = [R\pi_1 \circ D(a_{2,\mu}b_{2,\mu}) \circ I\pi_1] + [R\pi_1 \circ D(\sigma^*(a_{2,\mu})b_{2,\mu}) \circ I\pi_1].$$

These equations all follow from the rules in Theorem 1.2.

To get a sense of the above computation, let us generalize things just a bit. Let $f: R \to S$ be a homomorphism of commutative rings, and let $\sigma: S \to S$ be an automorphism such that $\sigma^2 = id$ and $\sigma f = f$. Define a product on $R \times S$ by

$$(r,s) \cdot (r',s') = (rr', (f^*r)s' + s(f^*r') + \sigma(s)s').$$

Check by brute force that this makes the abelian group $R \times S$ into a ring. Let $\alpha$ be the unique $E_2$-linear automorphism of $E_{2,\mu}$ that has order 2. Applying the above construction to $\pi_1^*: \text{GW}(E_2) \to \text{GW}(E_{2,\mu})$, where $\sigma = \alpha^*$, yields the endomorphism ring of $E_2$ in the Grothendieck-Witt category $\text{GWC}(\mathbb{Q})$.

**Appendix A. Symmetric monoidal categories and duality**

In this section we review some elements from the theory of closed, symmetric monoidal categories. Then we recall the notion of a dualizable object, as well as some standard properties.

A.1. **Basic conventions.** Let $(\mathcal{C}, \otimes, S, F(-,-))$ be a closed symmetric monoidal category. This means $\otimes$ is the monoidal structure, $S$ is the unit, and $X,Y \mapsto F(X,Y)$ is the cotensor.

In this setting there are evident evaluation maps

$$F(A,B) \otimes A \to B$$

defined as the adjoint to the identity on $F(A,B)$. Likewise, there are certain canonical maps

$$F(X,S) \otimes Y \to F(X,Y)$$

and

$$F(A,B) \otimes F(X,Y) \to F(A \otimes X, B \otimes Y)$$

defined to be the adjoints of evident compositions involving symmetry isomorphisms and evaluations. In general, we will use $\psi$ to denote any such canonical map that arises in a general closed symmetric monoidal category. It should always be clear from context exactly what map we mean.

There is one special case where it is useful to have a distinguished name, rather than just the generic “$\psi$”. For any object $X$ in a closed symmetric monoidal category, set $X^* = F(X,S)$. Then we let $ev_X: X^* \otimes X \to S$ be the adjoint of the identity map $X^* \to F(X,S)$.

A.2. **Dualizable objects.** The theory of dualizable objects goes back to Dold and Puppe [DP], but in modern times has been used extensively by May and his collaborators (see [LMS] and [Ma1], for example).
**Definition A.3.** An object $X$ in a symmetric monoidal category is called **dualizable** if there is another object $Y$ together with maps

$$\eta: S \to X \otimes Y, \quad \epsilon: Y \otimes X \to S$$

such that the composite

$$X \xrightarrow{\eta \otimes \text{id}_X} X \otimes Y \otimes X \xrightarrow{\text{id}_X \otimes \epsilon} X \otimes S$$

is $\text{id}_X$ and the composite

$$Y \xrightarrow{\text{id}_Y \otimes \eta} Y \otimes X \otimes Y \xrightarrow{\epsilon \otimes \text{id}_Y} S \otimes Y$$

is $\text{id}_Y$. We say that $Y$ is a d**ual** for $X$, although it is more precise to say that the dual is $(Y, \epsilon, \eta)$ since all three pieces of structure are needed.

**Remark A.4.** If $Y$ is a dual for $X$, then there can be several choices for $\epsilon$ and $\eta$ that serve as structure maps. If one fixes $Y$ and $\epsilon$, however, then there is only one corresponding choice for $\eta$; similarly, if one fixes $Y$ and $\eta$ then there is only one choice for $\epsilon$. This follows by the same argument that shows that a functor can have at most one left (or right) adjoint.

The following result can be pulled out of the proof of [LMS, Theorem III.1.6]:

**Proposition A.5.** In a closed symmetric monoidal category suppose that $X$ is dualizable with dual $(Y, \epsilon, \eta)$. Then the map $\tilde{\epsilon}: Y \to X^*$, adjoint to $\epsilon$, is an isomorphism. Consequently, $X^*$ is also a dual for $X$, with structure maps $\text{ev}_X: X^* \otimes X \to S$ and the composite

$$S \xrightarrow{\eta} X \otimes Y \xrightarrow{\text{id}_Y \otimes \epsilon} X \otimes X^*.$$

**Proof.** The duality axioms imply that the composite

$$\mathcal{C}(W, Y) \to \mathcal{C}(W \otimes X, Y \otimes X) \to \mathcal{C}(W \otimes X, S) = \mathcal{C}(W, X^*)$$

is a bijection, for all objects $W$. One readily checks that this composite is induced by post-composition with the map $\tilde{\epsilon}$ from the statement of the proposition. The Yoneda Lemma then yields that $\tilde{\epsilon}$ is an isomorphism. Finally, one must check that

$$Y \otimes X \xrightarrow{\epsilon \otimes \text{id}_Y} X \otimes X^* \xrightarrow{\text{ev}_X} S$$

equals $\epsilon$, but this is routine. □

If $X$ is dualizable and $\text{ev}_X: X^* \otimes X \to S$ and $\text{cev}_X: S \to X \otimes X^*$ satisfy the conditions of Definition A.3, then we call $\text{cev}_X$ the **coevaluation map** for $X$ (it is uniquely determined, of course). The following two results are standard:

**Proposition A.6.** In a closed symmetric monoidal category, an object $X$ is dualizable if and only if there exists a map $c$ that makes the following diagram commute:

$$\xymatrix{ S \ar[r]^c \ar[d]^{\text{id}_X} & X \otimes X^* \ar[d]^t \\
F(X, X) \ar[r]_{\psi} & X^* \otimes X.}$$

If $c$ exists, it is unique; and moreover, it is precisely the coevaluation map for $X$.

**Proof.** See [LMS, Theorem III.1.6]. The uniqueness of $c$ follows from [LMS, Proposition III.1.3], which shows that the horizontal map $\psi$ is an isomorphism. □
Proposition A.7. If $X$ and $Y$ are dualizable objects in a closed symmetric monoidal category then the following are true:
(a) $X \otimes Y$ and $X^*$ are dualizable;
(b) $\psi: X \rightarrow X^{**}$ is an isomorphism;
(c) $\psi: X^* \otimes Y^* \rightarrow (X \otimes Y)^*$ is an isomorphism.
(d) $cev_X: S \rightarrow X \otimes X^*$ is the composite
\[
S \xrightarrow{S^* \otimes 1_X} (X^* \otimes X)^* \xrightarrow{\cong} X^{**} \otimes X^* \xleftarrow{\psi \otimes 1_X} X \otimes X^*.
\]

Proof. Part (a) is elementary, while parts (b) and (c) are from [LMS, Proposition III.1.3]. For part (d), perhaps the easiest method is to check that $ev_X$ and the given composite satisfy the properties of Definition A.3. To this end, consider the following diagram:
\[
\begin{array}{ccc}
S^* \otimes X & \xrightarrow{ev_X^* \otimes 1_X} & (X^* \otimes X)^* \otimes X \\
& \downarrow{\psi} & \downarrow{\psi} \\
X^{**} & \xrightarrow{1 \otimes ev_X} & X \otimes S
\end{array}
\]
The vertical map labelled $\psi$ is the adjoint to the composite
\[
(X^* \otimes X)^* \otimes X \xrightarrow{1 \otimes \ell} (X^* \otimes X)^* \otimes X^* \otimes X \xrightarrow{ev_X^* \otimes X} S.
\]
We are required to show that the “across-the-top, then down” composition from $S^* \otimes X$ to $X \otimes S$ is the identity (after canonical identifications of the domain and codomain with $X$). But the triangle and the rectangle commute in any closed symmetric monoidal category, by an easy verification (it suffices to check commutativity in the category of finite-dimensional vector spaces over a field, cf. [HHP]). Since $\psi: X \rightarrow X^{**}$ is an isomorphism by (b), this completes the verification.

The second condition from Definition A.3 is checked in a similar manner. The relevant diagram is a little easier:
\[
\begin{array}{ccc}
X^* \otimes S^* & \xrightarrow{1 \otimes ev^*} & X^* \otimes (X^* \otimes X)^* \\
& \xrightarrow{\ell \otimes 1} & X^* \otimes (X^* \otimes X^*) \\
& \leftarrow{\psi} & \leftarrow{\psi} \\
& \downarrow{id} & \downarrow{id} \\
X^* \otimes (X \otimes X^*) & \xrightarrow{ev_X^* \otimes X} & S \otimes X^*
\end{array}
\]
The diagonal map labelled $\psi$ is the adjoint of the composite
\[
X^* \otimes (X^* \otimes X)^* \otimes X \xrightarrow{\ell \otimes 1} (X^* \otimes X)^* \otimes X^* \otimes X \xrightarrow{ev_X^* \otimes X} S.
\]
The “quadrilateral” and “triangle” in the diagram again commute in any closed symmetric monoidal category, and this completes the verification.

Appendix B. Leftover Proofs

Proof of Proposition 3.16 We include details because several steps are a bit hard to remember, and this is the kind of thing one wants to be able to just look up when needed.
For part (a), here is the check that \( i_a \) is a right identity. If \( x \in \mathcal{C}_E(a, b) = E(b \times a) \) then

\[
x \circ i_a = (\pi_{13})_! \left( \pi_{12}^* x \cdot \pi_{23}^* (\Delta^a(1)) \right)
\]

\[
= (\pi_{13})_! \left( \pi_{12}^* x \cdot (id_b \times \Delta^a)_!(1) \right)
\]

\[
= (\pi_{13})_! \left( (id_b \times \Delta^a)_! ((id_b \times \Delta^a)^* \pi_{12}^* x \cdot 1) \right) \quad \text{(projection formula)}
\]

\[
= x.
\]

The last step used that \( \pi_{13} \circ (id_b \times \Delta^a) = id_{b \times a} \) and \( \pi_{12} \circ (id_b \times \Delta^a) = id_{b \times a} \). The verification that \( i_a \) is a left identity is similar.

Write \( \pi_{cba}^{\text{co}} \) for the evident projection map \( c \times b \times a \to c \times a \). Let \( x \in \mathcal{C}_E(a, b) \), \( y \in \mathcal{C}_E(b, c) \), and \( z \in \mathcal{C}_E(c, d) \). The proof of associativity proceeds by analyzing the element

\[
\Omega = (\pi_{da}^{dca})_! \left( (\pi_{dcba}^{dca})^* (z) \cdot (\pi_{cb}^{dca})^* (y) \cdot (\pi_{ba}^{dca})^* (x) \right)
\]

in two different ways. The first proceeds as follows:

\[
\Omega = (\pi_{da}^{dca})_! \left( (\pi_{dcba}^{dca})^* (z) \cdot (\pi_{cb}^{dca})^* (y) \cdot (\pi_{ba}^{dca})^* (x) \right)
\]

\[
= (\pi_{da}^{dca})_! \left[ (\pi_{dcba}^{dca})^* (z) \cdot (\pi_{cb}^{dca})_! \left[ (\pi_{cb}^{dca})^* (y) \cdot (\pi_{ba}^{dca})^* (x) \right] \right] \quad \text{(proj. form.)}
\]

\[
= (\pi_{da}^{dca})_! \left[ (\pi_{dcba}^{dca})^* (z) \cdot (\pi_{cb}^{dca})_! \left[ (\pi_{cb}^{dca})^* (y) \cdot (\pi_{ba}^{dca})^* (x) \right] \right]
\]

\[
= (\pi_{da}^{dca})_! \left[ (\pi_{dcba}^{dca})^* (z) \cdot (\pi_{cb}^{dca})_! \left[ (\pi_{cb}^{dca})^* (y) \cdot (\pi_{ba}^{dca})^* (x) \right] \right] \quad \text{(push-pull)}
\]

\[
= z \cdot (y \cdot x).
\]

The first and third equalities just use functoriality. For example, in the third equality we use that \( \pi_{cba}^{dca} = \pi_{cb}^{dca} \pi_{cb}^{dca} \) and so forth. We leave the reader to perform a similar series of steps to show that \( \Omega = (z \cdot y) \cdot x \). This proves associativity, and so finishes the proof of (a).

Part (b) is obvious.

For (c) we must show that if \( f : a \to b \) and \( g : b \to c \) then \( R_g \circ R_f = R(gf) \). That is, we must check the formula

\[
(\pi_{cb}^{cba})_! \left[ (\pi_{cb}^{cba})^* (id_c \times g)^* (i_c) \cdot (\pi_{ba}^{cba})^* (id_b \times f)^* (i_b) \right] = (id_c \times g f)^* (i_c).
\]

Note that the left side is \( (id_c \times g)^* (i_c) \circ (id_b \times f)^* (i_b) \).

The first step is to use the two pullback squares
to see that \((\pi^{\text{caba}}_{\text{ca}})^*(\text{id}_b \times f)^*(i_b) = (\pi_1 \times f \times \pi_2)_!(1)\) (here we use that \(\pi^*_a\) and \(f^*\) are ring maps and so send 1 to 1). Next we compute that

\[
R_g \circ R_f = (\pi^{\text{caba}}_{\text{ca}})_! \left[ (\pi^{\text{caba}}_{\text{cb}})^*(\pi^{\text{caba}}_{\text{ch}})* (i_a) \cdot (\pi^{\text{caba}}_{\text{bc}})^*(\text{id}_c \times f)^*(i_b) \right]
\]

\[
= (\pi^{\text{caba}}_{\text{ca}})_! \left[ (\pi^{\text{caba}}_{\text{cb}})^*(\pi^{\text{caba}}_{\text{ch}})* (i_a) \cdot (\pi^{\text{caba}}_{\text{bc}})^*(\text{id}_c \times f \times \pi_2)_!(1) \right]
\]

\[
= (\pi^{\text{caba}}_{\text{ca}})_! (\pi_1 \times f \times \pi_2)_! \left[ (\pi_1 \times f \times \pi_2)_!(\pi^{\text{caba}}_{\text{cb}})* (\text{id}_c \times g)^*(i_c) \cdot 1 \right]
\]

\[
= (\text{id}_c \times gf)^*(i_c)
\]

\[
= R(gf).
\]

In the second-to-last equality we have used that \((\pi^{\text{caba}}_{\text{ca}}) \circ (\pi_1 \times f \times \pi_2) = \text{id}_{c \times a}\) and that \((\text{id}_c \times g) (\pi^{\text{caba}}_{\text{ca}}) (\pi_1 \times f \times \pi_2) = \text{id}_c \times gf\).

To prove (d) we must verify that \(i_a^* = i_a\) (for every object \(a\)) and \((g \circ f)^* = f^* \circ g^*\) for every \(f \in C_E(a, b)\) and \(g \in C_E(b, c)\). For the first of these, consider the twist map \(t : a \times a \to a \times a\). Since \(t^2 = \text{id}_{a \times a}\) we have by Lemma 3.4 that \(t_1 = (t^* )^{-1} = t^*\). So

\[
i_a^* = t^*(i_a) = t_1(i_a) = t_1(\Delta_a^!(1)) = (t \circ \Delta^a)_!(1) = \Delta_a^!(1) = i_a.
\]

Write \(t^{a}_{ba}\) for the map \(t : a \times b \to b \times a\), and similarly for other situations. Then

\[
(g \circ f)^* = (\pi^{\text{caba}}_{\text{ca}})_! \left[ (\pi^{\text{caba}}_{\text{cb}})^*(\pi^{\text{caba}}_{\text{ch}})* (g \cdot (\pi^{\text{caba}}_{\text{bc}})^* (f)) \right]
\]

\[
= (\pi^{\text{caba}}_{\text{ca}})_! \left[ (\pi^{\text{caba}}_{\text{cb}})^*(\pi^{\text{caba}}_{\text{ch}})* (g) \cdot (\pi^{\text{caba}}_{\text{bc}})^* (f) \right]
\]

\[
= (\pi^{\text{caba}}_{\text{ca}})_! (\pi^{\text{caba}}_{\text{cb}})^*(\pi^{\text{caba}}_{\text{ch}})* (g) \cdot (\pi^{\text{caba}}_{\text{bc}})^* (f) \cdot [(\pi^{\text{caba}}_{\text{cb}})^*(g) \cdot (\pi^{\text{caba}}_{\text{bc}})^* (f)]
\]

\[
= (\pi^{\text{caba}}_{\text{ca}})_! (\pi^{\text{caba}}_{\text{cb}})^*(g) \cdot (\pi^{\text{caba}}_{\text{bc}})^* (f)
\]

\[
= (\pi^{\text{caba}}_{\text{ca}})_! (\pi^{\text{caba}}_{\text{cb}})^*(t_{ba})^* g \cdot (\pi^{\text{caba}}_{\text{bc}})^* (t_{ba})^* f
\]

\[
= (\pi^{\text{caba}}_{\text{ca}})_! (\pi^{\text{caba}}_{\text{cb}})^*(t_{ba})^* f \cdot (\pi^{\text{caba}}_{\text{bc}})^* (t_{ba})^* g
\]

\[
= f^* \circ g^*.
\]

In the second and fourth equalities we have used Lemma 3.4 but all of the other equalities use only simple functoriality.

To prove the first part of (e) we argue as follows:

\[
\alpha \circ R_f = (\pi^{ZW}_{\text{Y}})^! \left[ (\pi^{ZW}_{\text{Y}})^*(\alpha) \cdot (\pi^{ZW}_{\text{Y}})^*(\text{id}_w \times f)^*(i_w) \right]
\]

\[
= (\pi^{ZW}_{\text{Y}})^! \left[ (\pi^{ZW}_{\text{Y}})^*(\alpha) \cdot (\text{id}_Z \times f \times \text{id}_Y)_!(1) \right]
\]

\[
= (\pi^{ZW}_{\text{Y}})^!(\text{id}_Z \times f \times \text{id}_Y)_! \left[ (\text{id}_Z \times f \times \text{id}_Y)^* (\pi^{ZW}_{\text{Y}})^*(\alpha) \cdot 1 \right]
\]

\[
= \text{id}_Z \left[ (\text{id}_Z \times f)^*(\alpha) \cdot 1 \right]
\]

\[
= (\text{id}_Z \times f)^*(\alpha).
\]
In the second equality we have used the push-pull axiom applied to the pullback diagram
\[
\begin{array}{ccc}
Z \times W \times Y & \xrightarrow{\pi_Z \times f \times id_Y} & W \times Y & \xrightarrow{id_W \times f} & W \times W \\
\downarrow{\pi_Z \times f \times id_Y} & & \downarrow{f \times id_Y} & & \downarrow{\Delta} \\
Z \times Y & \xrightarrow{\pi_2} & Y & \xrightarrow{\Delta} & W.
\end{array}
\]

In the third equality we have used that \(\pi_Z \times f \times id_Y \circ (id_Z \times f \times id_Y) = id_Z \times f\) and \(\pi_Z \times f \times id_Y \circ (id_Z \times f \times id_Y) = id_Z \times f\).

The other parts of (e) are proven by similar arguments. Part (f) follows from (e) using
\[
I_f \circ R_q = I_f \circ i_B \circ R_q = (id \times q)^*(I_f \circ i_B) = (id \times q)^*(f \times id)^*(i_B) = (f \times q)^*(i_B).
\]
The second part of (f) then follows using push-pull applied to the square
\[
\begin{array}{ccc}
A \times_B C & \xrightarrow{\Delta} & B \\
\downarrow{f \times q} & & \downarrow{\Delta} \\
A \times C & \xrightarrow{\Delta} & B \times B.
\end{array}
\]

Part (g) is similar to (f).

For (h), use that \(R_p \circ I_g = ((p \times g)^*_X)_Y(1) = I_f \circ R_q\), by applying (f) and (g) together. Part (i) follows directly from (h) using the pullback diagram
\[
\begin{array}{ccc}
A & \xrightarrow{id} & A \\
\downarrow{id} & & \downarrow{\Delta} \\
A & \xrightarrow{f} & B.
\end{array}
\]

□

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