Calculational HOTT

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Abstract

Based on a loose correspondence between, on one hand, a first order version of intuitionistic logic with preeminence of equality and equivalence over implication, and on the other hand, homotopic equivalence properties of identity, Π, Σ and coproduct types, we formally restate homotopic type theory (HoTT) with equality and homotopic equivalence playing a preeminent role. In addition to this, we exhibit a calculational way of writing effective and elegant formal proofs based on appropriate notations and formats, as well as algebraic identities and inference rules involving the homotopic equivalences with which we restate HoTT.

1 Introduction

As it has been said before “The Curry-Howard isomorphism states an amazing correspondence between systems of formal logic as encountered in proof theory and computational calculi as found in type theory” [13]. The aim of this article is to explore a way to extend the Curry-Howard isomorphism from a would-be higher order version[1] of a so called Calculational logic [8, 10] (whose associated proof calculus is based on the preeminence of logical equivalence and equality over logical implication) to homotopic type theory (HoTT).

In other words, we plan not only to restate HoTT giving equivalence a preeminent role, but endow it with a deduction method based on equational algebraic manipulations that allows for elegant and formal proof constructions.

The ability to effectively prove theorems, by both human and mechanical means, is crucial to formal methods. Formal proofs in mathematics and computer science are being studied because they can be verified by a very simple computer program. An open problem in the Computer Mathematics community is the

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[1] Although an intuitionistic first-order version of calculational logic is well established, to the knowledge of the authors, it is not so for higher-order versions.
feasibility to fully formalize mathematical proofs [3]. Here, feasibility is understood as the capability to generate correct formal mathematics with an effort comparable to that of writing a mathematical paper in, say, \LaTeX. For more than thirty years now, a revolution on the way of reasoning and proving in mathematics (largely unnoticed by pure logicians) has gained a substantial community of enthusiastic practitioners. The \textit{calculational style} of presenting proofs introduced by Dijkstra and Scholten [8] is a deduction method based on formula manipulation. This deduction method has been adopted in some books on theoretical computer science [2, 9, 11, 12] and appeared in papers on set theory, discrete mathematics and combinatorics [1, 4, 6]. It was originally devised as an informal but rigorous and practical theorem-proving discipline, in which use of equational reasoning (understood as mainly based on the preeminence of logical equivalence and equalities) is preferred over the traditional one based on logical implication. This calculational style has led to an extensive array of techniques for elegant proof constructions [7] that we believe, are not only easily formalizable but also machine aided through proof-verifiers. In order to formally express HoTT with equality and equivalence playing a preeminent role, we find inspiration in the Curry-Howard isomorphism based on the facts that, on one hand, HoTT is strongly based on the homotopic character of equality and equivalence, and on the other hand, a calculational version of intuitionistic first order logic is well established [5].

Another goal of this research is an attempt to provide a calculational formalization of theorem proving for the case of homotopy type theory by producing (hopefully) human-readable proofs in the linear formats characteristic of the calculational style. In order to do so, we extend the syntax of type theory introducing an additional judgment that give rise to a conservative extension which facilitates readable proof calculations and introduces an \textit{inhabitation format}, a syntactic tool that imitates the calculational proof format introduced by Dijkstra and Scholten [8]. We will prove some basic equational results where the ability to replace ‘equals for equals’ rendered by equivalences demonstrate the elegance and power of calculational proofs in homotopy type theory.

Before proceeding with our subject matter, we present a brief overview of the main logic principles (algebraic properties, mainly given by equivalences) and notations used to prove theorems using the calculational style.

### 2 Eindhoven quantifier logic and notation

At the \textit{THE} project in Eindhoven, researchers led by E.W. Dijkstra, in the 1970’s, devised a uniform notation for quantification in first order logic and related areas [8]. By $(Qx : T | range \cdot term)$ was meant that quantifier $Q$ binds variable $x$ of type $T$ to be constrained to satisfy formula $range$ within the textual scope delimited by the outer parentheses $(...)$, that expression $term$

\footnote{The original Eindhoven style uses colons as separators; the syntax with $|$ and $\cdot$ is one of the many subsequent notational variations based on their innovation.}
is evaluated for each such \( x \) and that those values then are combined via an associative and commutative operator related to quantifier \( Q \). For brevity, we will refer to Eindhoven quantifiers as \textit{operationals}. For the case of logical operationals (corresponding to the universal and existential quantifiers), the associated operators are respectively, conjunction and disjunction considered as binary boolean operations.

\[
(\forall x : T \mid \text{range} \cdot \text{term}) \text{ means for all } x \text{ in } T \text{ satisfying range we have term,}
\]

\[
(\exists x : T \mid \text{range} \cdot \text{term}) \text{ means for some } x \text{ in } T \text{ satisfying range we have term,}
\]

A general shorthand applying to these notations is that an omitted \(|\text{range}| \) defaults to \( \text{true} \). The following so called \textit{trade rules} translate these logical notations to the usual first order logic formulas.

\textbf{[Trade]} \hspace{1em}
\[
(\forall x : T \mid P \cdot Q) \equiv (\forall x : T \cdot P \Rightarrow Q)
\]
\[
(\exists x : T \mid P \cdot Q) \equiv (\exists x : T \cdot P \land Q)
\]

The following equational rules (i.e. expressed as logical equivalences) correspond to some of the most basic logical axioms and theorems of a calculational version of intuitionistic first order logic.

\textbf{[One-Point]} \hspace{1em}
\[
(\forall x : T \mid x = a \cdot P) \equiv P[a/x]
\]
\[
(\exists x : T \mid x = a \cdot P) \equiv P[a/x]
\]

\textbf{[Equality]} \hspace{1em}
\[
(\forall x, y : T \mid x = y \cdot P) \equiv (\forall x : T \cdot P(x/y))
\]
\[
(\exists x, y : T \mid x = y \cdot P) \equiv (\exists x : T \cdot P[x/y])
\]

\textbf{[Range Split]} \hspace{1em}
\[
(\forall x : T \mid P \lor Q \cdot R) \equiv (\forall x : T \mid P \cdot R) \land (\forall x : T \mid Q \cdot R)
\]
\[
(\exists x : T \mid P \lor Q \cdot R) \equiv (\exists x : T \mid P \cdot R) \lor (\exists x : T \mid Q \cdot R)
\]

\textbf{[Term Split]} \hspace{1em}
\[
(\forall x : T \mid P \cdot Q \land R) \equiv (\forall x : T \mid P \cdot Q) \land (\forall x : T \mid P \cdot R)
\]
\[
(\exists x : T \mid P \cdot Q \land R) \equiv (\exists x : T \mid P \cdot Q) \land (\exists x : T \mid P \cdot R)
\]

\textbf{[Translation]} \hspace{1em}
\[
(\forall x : J \mid P \cdot Q) \equiv (\forall y : K \mid P[f(y)/x] \cdot Q[f(y)/x])
\]
\[
(\exists x : J \mid P \cdot Q) \equiv (\exists y : K \mid P[f(y)/x] \cdot Q[f(y)/x])
\]

where \( f \) is a bijection that maps values of type \( K \) to values of type \( J \).

\textbf{[Congruence]} \hspace{1em}
\[
(\forall x : T \mid P \cdot Q \equiv R) \Rightarrow ((\forall x : T \mid P \cdot Q) \equiv (\forall x : T \mid P \cdot R))
\]
\[
(\forall x : T \mid P \cdot Q \equiv R) \Rightarrow ((\exists x : T \mid P \cdot Q) \equiv (\exists x : T \mid P \cdot R))
\]

\textbf{[Antecedent]} \hspace{1em}
\[
R \Rightarrow (\forall x : T \mid P \cdot Q) \equiv (\forall x : T \mid P \cdot R \Rightarrow Q)
\]
\[
R \Rightarrow (\exists x : T \mid P \cdot Q) \equiv (\exists x : T \mid P \cdot R \Rightarrow Q)
\]

when there are not free occurrences of \( x \) in \( R \).

\(^3\text{\( \lor \) and \( \land \) denote disjunction and conjunction respectively, \( \Rightarrow \) denote implication and \( \equiv \) denotes equivalence. If } E \text{ is a symbolic expression, } E[k/x] \text{ is the expression obtained by replacing every free occurrence of ‘}x\text{’ in } E \text{ by ‘}k\text{’}.\)
3 Extended Syntax of type theory

In this section we present a formulation of Martin-Löf theory defining terms, judgments and rules of inference inductively in the style of natural deduction formalizations. To this formulation, we adjoin an additional judgment yielding (by applying its deriving inference rules) a conservative extension that allows to perform agile and readable proof calculations.

We suppose the reader is familiar with the syntax of Martin-Löf type theories and give an overview of the version appearing in [14].

Contexts

Contexts are finite lists of variable declarations \((x_1 : A_1, ..., x_n : A_n)\), for \(n \geq 0\), where free variables occurring in the \(A_i\)’s belong to \(\{x_1, ..., x_{i-1}\}\) when \(1 \leq i \leq n\). This list may be empty and indicates that the distinct variables \(x_1, ..., x_n\) are assumed to have types \(A_1, ..., A_n\), respectively. We denote contexts with letters \(\Sigma\) and \(\Delta\), which may be juxtaposed to form larger contexts.

The judgment \(\Gamma \; ctx\) formally denotes the fact that \(\Gamma\) is a well-formed context, introduced by the following rules of inference

\[
\frac{}{\quad \text{ctx-EMP}} \quad \frac{x_1 : A_1, ..., x_n : A_n \vdash A_n : U_i \quad (x_1 : A_1, ..., x_n : A_n) \; ctx}{\quad \text{ctx-EMP}}
\]

with a side condition for the rule \(\text{ctx-EMP}\): the variable \(x_n\) must be distinct from the variables \(x_1, ..., x_{n-1}\).

Forms of judgment

We first, consider the three usual basic judgments of type theory.

\[
\begin{align*}
\Gamma \; ctx & \quad \Gamma \vdash a : A & \quad \Gamma \vdash a \equiv_A a' \\
\end{align*}
\]

\(\Gamma \; ctx\) expresses that \(\Gamma\) is a (well-formed) context. \(\Gamma \vdash a : A\) denotes that a term \(a\) has (inhabits) type \(A\) in context \(\Gamma\). \(\Gamma \vdash a \equiv_A a'\) means that \(a\) and \(a'\) are definitionally equal objects of type \(A\) in context \(\Gamma\).

A fourth weaker and derived judgment, the \textit{inhabitation judgment}, will be useful for our purposes:

\[
\begin{align*}
\Gamma \vdash A <: \\
\end{align*}
\]

means that the type \(A\) is inhabited in context \(\Gamma\), that is, for some term \(a\), judgment \(\Gamma \vdash a : A\) holds. This judgment corresponds to a forgetful version of \(\Gamma \vdash a : A\) where the mention of the term \(a\) inhabiting type \(A\) is suppressed.

Since the main inference rule for introducing this judgment is

\[
\begin{align*}
\Gamma \vdash a : A \\
\Gamma \vdash A <: \\
\end{align*}
\]

and its remaining derivating inference rules correspond to forgetful versions of derived inference rules from judgments of the form \(\Gamma \vdash a : A\), this addition only brings forth a conservative extension of the theory.
Structural rules

The following rule expresses that a context holds assumptions, basically by saying that the typing judgments listed in the context may be derived.

\[(x_1 : A_1, \ldots, x_n : A_n) \vdash x_1 : A_1, \ldots, x_{n-1} : A_{n-1} \vdash A_n : U_i \quad \text{VBL}e\]

Although, the following rules corresponding to the principles of \textit{substitution} and \textit{weakening} are derivable by induction on all possible derivations, we state them. The principles corresponding to typing judgments are given by

\[\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash b : B \quad \text{SUBST1} \quad \Gamma \vdash A : U_i \quad \Gamma, \Delta \vdash b : B \quad \text{WKG1}\]

and the rules for the principles of judgmental (definitional) equality are

\[\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash b \equiv B c \quad \text{SUBST2} \quad \Gamma \vdash A : U_i \quad \Gamma, \Delta \vdash b \equiv B c \quad \text{WKG2}\]

The following inference rules express the fact that definitional equality is an equivalence relation preserved by typing.

\[\Gamma \vdash a : A \quad \Gamma \vdash a \equiv_A b \quad \Gamma \vdash b \equiv_A a \quad \Gamma \vdash a \equiv_A c \quad \Gamma \vdash a \equiv_A b \quad \Gamma \vdash a \equiv_A b\]

Besides the inference rule

\[\Gamma \vdash a : A \quad \Gamma \vdash A : \lt ; \quad \text{INHAB}\]

introducing the inhabitation judgment, we present the following derivating inference rules for this judgment.

\[\Gamma \vdash A : \lt ; \quad \Gamma \vdash A \rightarrow B : \lt ; \quad \Gamma \vdash B : \lt ; \quad \text{FAPPL} \quad \Gamma \vdash A \rightarrow B \equiv B \rightarrow C \equiv C \rightarrow \lt ; \quad \Gamma \vdash A \rightarrow C \equiv C \rightarrow \lt ; \quad \text{FCOMP}\]

These rules correspond to forgetful versions of the following rules that are easily derived from the original unextended syntax of type theory.

\[\Gamma \vdash a : A \quad \Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash f(a) : B \quad \Gamma \vdash g : B \rightarrow C \quad \Gamma \vdash g \circ f : A \rightarrow C\]

An additional structural rule applying definitional equality of types to the inhabitation judgment, that we explicitly use, is

\[\Gamma \vdash A : \lt ; \quad \Gamma \vdash A \equiv B \quad \Gamma \vdash B : \lt ; \quad \text{TSUBS}\]
4 Deductive Chains in Type Theory

In classical logic, the task is to derive arbitrary valid formulas from a small set of axiom schema. In type theory, the basic task is to show that certain type can be inhabited from the inhabitation of another types which are related with the first through the inference rules introduced before. This will be done by means of an inhabitation format, a syntactic tool that imitates the calculational proof format introduced by Dijkstra and Scholten [8].

This inhabitation format is a deductive chain that represents the concatenation of premises of inference rules. Each link of the chain is a judgment of the form $A \rightarrow B <\ldots, A \equiv B$ or $a : A$ written vertically, together with an evidence or a statement supporting it, which is written between angular parentheses. Among these links, the first one to consider is a consequence link

\[ B \leftarrow \langle \text{evidence of inhabitation} \rangle A, \]

the second one is an equivalence link

\[ B \equiv \langle \text{evidence of equivalence} \rangle A \]

and the third one is an inhabitation link

\[ A \leftarrow \langle \text{inhabitation statement} \rangle a \]

where $\leftarrow$ is a (vertical) way of stating that $a$ inhabits $A$, namely, that $a : A$. The evidence of inhabitation of a consequence link could be implicit with reference to a previous argument, in such case we will write

\[ B \leftarrow \langle : \text{; statement of inhabitation} \rangle A, \]

or explicit, in which case we will write explicitly the name $f$ of the object and, after a semicolon, its definition:

\[ B \leftarrow \langle : f : \lambda x.f(x) \rangle A. \]
A vertical notation has been chosen in order to chain these links. Inhabitation links are used to close deductive chains if necessary. For every pair of consecutive links in a deductive chain, we have that the bottom type of the first link coincides with the top type of the second link.

One can rewrite the inference rules in terms of links, but will only do so in order to explain how to use and interpret deductive chains. For example, the inference rule (FAPPL) could be written as

\[
\begin{align*}
B & \leftarrow \langle ; \rangle \\
A & \uparrow \langle \rangle \\
a & \underline{B <:}
\end{align*}
\]

The evidence of \( A \to B <: \) could be given explicitly by exhibiting a function \( f \) from \( A \) to \( B \). Therefore, if \( a : A \) then \( f(a) : B \). In the same way, the inference rules (FCOMP) and (Tsubs) could be written as

\[
\begin{align*}
C & \leftarrow \langle ; \rangle \\
B & \equiv \langle \rangle \\
A & \uparrow \langle \rangle \\
\underline{A} & \leftarrow \langle \rangle \\
C & \leftarrow \langle \rangle \\
B & \leftarrow \langle \rangle \\
A & \leftarrow \langle \rangle \\
\underline{a} & \underline{B <:}
\end{align*}
\]

In general, we will suppose that deductive chains will be performed in the presence of a given context \( \Gamma \), from which, certain basic inference rules will be considered implicitly applied.

A deductive chain with consequence links has a representation that looks like this:

\[
\begin{align*}
D & \leftarrow \langle \text{evidence of } C \to D <: \rangle \\
C & \leftarrow \langle \text{evidence of } B \to C <: \rangle \\
B & \leftarrow \langle \text{evidence of } A \to B <: \rangle \\
A
\end{align*}
\]
from which we can conclude that type $A \rightarrow D$ is inhabited. But if the chain is closed by the inhabitation link $a : A$, we can conclude instead, that type $D$ is inhabited: if $f : A \rightarrow B$, $g : B \rightarrow C$, $h : A \rightarrow B$ and $a : A$ then $h(g(f(a))) : D$.

This detailed account of inhabitation give rise to the following chain:

$$
\begin{align*}
D & \leftarrow \{ h \} \\
C & \leftarrow \{ g \} \\
B & \leftarrow \{ f \} \\
A & \leftarrow \{ \text{evidence of inhabitation} \} \\
a. 
\end{align*}
$$

Deductive chains that contain consequence and equivalence links has a representation that looks like this:

$$
\begin{align*}
D & \equiv \{ \text{evidence of } C \equiv D \} \\
C & \leftarrow \{ \text{evidence of } B \rightarrow C < \} \\
B & \equiv \{ \text{evidence of } A \equiv B \} \\
A
\end{align*}
$$

It should be clear that if there exists a deductive chain with source $A$, destination $D$, and context $\Gamma$, then $D < :$ is derivable from $\Gamma \cup \{ A < : \}$.

Before illustrating the use of deduction chains we will introduce some basic types in order to present some consequence links which come with their specifications.

## 5 Basic Types

We will follow the general pattern for introducing new types in Type Theory presented in the HoTT book [14]. The specification of a type consist mainly in four steps: (i) Formation rules, (ii) Construction rules, (iii) Elimination rules, and Computation rules. Here, we express the elimination rules in terms of consequence links.

We will assign a special Greek letter to each induction operator introduced in the respective elimination rule. Namely

| Type | $\Sigma$ | $+$ | $\mathbb{N}$ | $=$ | $\mathbb{O}$ | $\mathbb{I}$ |
|------|----------|-----|------------|----|------------|------------|
| Induction operator | $\sigma$ | $\kappa$ | $\nu$ | $\iota$ | $\theta$ | $\mu$ |
**Π-types.** The dependent function types or Π-types, are the most fundamental basic types and its elimination rule does not provide links for deductive chains.

Given types \( A : \mathcal{U} \) and \( B : A \to \mathcal{U} \) we form the type \( \Pi_{x : A} B(x) : \mathcal{U} \). For \( b : B \) we construct \( \lambda (x : A). b \) of type \( \Pi_{x : A} B(x) \).

For \( f : \Pi_{x : A} B(x) \) and \( a : A \) then \( f(a) : B[a/x] \) and the computation rule is

\[
(\lambda (x : A). b)(a) \equiv b[a/x]
\]

When \( B \) does not depend on the objects of \( A \), the product type is the function type \( A \to B \):

\[
\Pi_{x : A} B(x) \equiv A \to B.
\]

The propositional reading of \( f : \Pi_{x : A} B(x) \) is that \( f \) is a proof that all objects of type \( A \) satisfy the property \( B \). We will use this semantic throughout the paper as necessary. By the way, the elimination rules of Σ-types, co-product types, N-type, and W-types, establish that to prove that all objects of these types satisfy a property, you have to prove that their constructed objects satisfy the property, and for this, the rule introduces an induction operator fulfilling that task.

One useful property of Π types is Π-distribution over arrows. Let us suppose that for each \( x : A \) we have a function \( \varphi_x : P(x) \to Q(x) \). Then we can define the function

\[
\Delta : (\Pi_{x : A} P(x)) \to (\Pi_{x : A} Q(x))
\]

by \( \Delta(u)(x) \equiv \varphi_x(u(x)) \). This shows that if \( \Pi_{x : A} P(x) \lessdot \Pi_{x : A} Q(x) \lessdot \) then \( (\Pi_{x : A} P(x)) \to \Pi_{x : A} Q(x) \lessdot \). This property is known as Π-distribution over arrows and is frequently used in deductive chains as the following consequence link

\[
\Pi_{x : A} Q(x) \quad \leftarrow \quad \langle : \Delta ; \text{Definition of } \varphi_x \rangle \quad (1)
\]

Later, in the section [10] we will explain a method to find definitions of functions such as the one for \( \Delta \).

**Σ-types.** The dependent pair types or Σ-types, are the types whose inhabitants are dependent pairs.

Given \( A : \mathcal{U} \) and \( B : A \to \mathcal{U} \) we form \( \Sigma_{x : A} B(x) : \mathcal{U} \) and if \( a : A \) and \( b : B[a] \) then \( (a, b) : \Sigma_{x : A} B(x) \).

In order to prove a property \( C : \sum_{x : A} B(x) \to \mathcal{U} \) for all objects of the Σ-type, i.e., to inhabit \( \prod_{p : \sum_{x : A} B(x)} C(p) \), we must prove the property for its constructed objects, i.e., to inhabit \( \prod_{x : A} \prod_{y : B(x)} C((x, y)) \) For this there is a function \( \sigma(C) \) carrying a proof \( g \) of this latter expression to the proof \( \sigma(C)(g) \) of the former
expression. Therefore, the elimination rule is given by the following consequence link
\[ \prod_{p: \sum_{x:A} B(x)} C(p) \]
\[ \leftarrow \langle \sigma_C \rangle \]
\[ \prod_{x:A} \prod_{y:B(x)} C((x, y)) \]

The computation rule states a definition of the function \( \sigma_C \):
\[ \sigma_C(g)((a, b)) \equiv g(a)(b). \]

For the case when \( C \) is a constant family, we have that the induction operator link reduces to
\[ (\sum_{x:A} B(x)) \rightarrow C \]
\[ \leftarrow \langle \sigma_C \rangle \]
\[ \prod_{x:A} (B(x) \rightarrow C) \]

With the induction operator we can also define functions on \( \Sigma \)-types. For instance, projection functions \( \text{pr}_1 \) and \( \text{pr}_2 \) are defined by
\[ \text{pr}_1 : \equiv \sigma_A(g) \text{ and } \text{pr}_2 : \equiv \sigma_B \circ \text{pr}_1(h), \]
where \( g : \equiv \lambda(x : A). \lambda(y : B(x)). x \), and \( h : \equiv \lambda(x : A). \lambda(y : B(x)). y \). When \( B \) does not depend on the objects of \( A \), the \( \Sigma \)-type is the type \( A \times B \), the cartesian product type of \( A \) and \( B \):
\[ \sum_{x:A} B(x) \quad \equiv \\ A \times B. \]

**Coproduct types.** The coproduct corresponds to the disjoint union of sets in Set Theory.

Given \( A: \mathcal{U} \) and \( B: \mathcal{U} \) we form \( A + B: \mathcal{U} \) and if \( a:A \) and \( b:B \) then \( \text{inl}(a): A + B \) and \( \text{inr}(b): A + B \).

In order to prove a property \( C: A + B \rightarrow \mathcal{U} \) for all objects of the coproduct type, i.e., to inhabit \( \prod_{p:A+B} C(p) \), we must prove the property for its constructed objects, i.e., to inhabit \( \prod_{x:A} C(\text{inl}(x)) \times \prod_{y:B} C(\text{inr}(y)) \). For this there is a function \( \kappa_C \) carrying a proof \( g \) of the latter type to the proof \( \kappa_C(g) \) of the former one. Therefore, the elimination rule is given by the following consequence link
\[ \prod_{p:A+B} C(p) \]
\[ \leftarrow \langle \kappa_C \rangle \]
\[ \prod_{x:A} C(\text{inl}(x)) \times \prod_{y:B} C(\text{inr}(y)) \]

The computation rule states a definition of the function \( \kappa_C \):
\[ \kappa_C(g)(\text{inl}(a)) :\equiv (\text{pr}_1(g))(a) \quad \text{and} \quad \kappa_C(g)(\text{inr}(b)) :\equiv (\text{pr}_2(g))(b) \]
Empty type. It is presented as $\emptyset$. This type has no objects and its elimination rule is given by the function

$$o_C : \prod_{x: \emptyset} C(x),$$

which states that all the objects of $\emptyset$ satisfy any property $C : \emptyset \rightarrow \mathcal{U}$, and there is no computation rule.

Unit type. It is presented as $1$. This type has just one object, its constructor is $* : 1$, and its elimination rule is given by the following link:

$$\prod_{x : 1} C(x) \leftarrow \langle : \mu_C \rangle C(*)$$

which states that in order to prove a property $C : 1 \rightarrow \mathcal{U}$ it is enough to inhabit $C(*)$. Its computation rule is $\mu_C(u)(x) : \equiv u$.

The type of natural numbers is presented as $\mathbb{N}$ and its constructors are $0 : \mathbb{N}$ and $s : \mathbb{N} \rightarrow \mathbb{N}$.

In order to prove a property $C : \mathbb{N} \rightarrow \mathcal{U}$ for all objects of $\mathbb{N}$, i.e., to inhabit $\prod_{p : \mathbb{N}} C(p)$, we must prove the property for its constructed objects, i.e., to inhabit $C(0) \times \left( \prod_{p : \mathbb{N}} C(p) \rightarrow C(s(p)) \right)$. For this, there is a function $\nu_C$ carrying a proof $g$ of the latter type to the proof $\nu_C(g)$ of the former one. Therefore, the elimination rule is given by the following consequence link

$$\prod_{p : \mathbb{N}} C(p) \leftarrow \langle : \nu_C \rangle C(0) \times \prod_{p : \mathbb{N}} C(p) \rightarrow C(s(p))$$

The computation rule states a definition of the function $\nu_C$:

$$\nu_C(g)(0) \equiv (pr_1 g)(0) \text{ and } \nu_C(g)(s(p))) \equiv (pr_2 g)(p, \nu_C(g)(p)).$$

Identity type. Given any pair of objects $a$ and $b$ of a type $P : \mathcal{U}$, there is a type $(a =_P b) : \mathcal{U}$, called identity type. There is only one constructor:

$$\text{refl} : \prod_{x : P} (x =_P x)$$

that states de identification of an object with itself. The objects of $x = y$ are called paths from $x$ to $y$.

In order to prove a property $C : \prod_{x,y : P, x = y} x = y \rightarrow \mathcal{U}$ for all objects of the identity type, i.e., to inhabit $\prod_{x,y} \prod_{p : x = y} C(p)$, we must prove the property for
its constructed objects, i.e., to inhabit $\prod_{x,p} C(\text{refl}_x)$. For this there is a function $\iota_C$ carrying a proof $g$ of the latter type to the proof $\iota_C(g)$ of the former one. Therefore, the elimination rule is given by the following consequence link

$\prod_{x,y:p:x=y} C(x, y, p)$

$\leftarrow \{ \iota_C \}
\prod_{x:p} C(x, x, \text{refl}_x)$

The computation rule states the definition of the function $\iota_C$:

$\iota_C(g)(x, x, \text{refl}_x) : \equiv g(x)$.

**Remark.** Induction operators depend on a type family; however, the corresponding computation rules do not. Recall that computation rules for $\sigma$, $\kappa$, $\iota$ and $\mu$, for example, are respectively: $\sigma(u)((x, y)) : \equiv u(x)(y)$, $\kappa(u, v)(\text{inl}(x)) : \equiv u(x)$, $\kappa(u, v)(\text{inr}(y)) : \equiv v(y)$, $\iota(u)(x, x, \text{refl}_x) : \equiv u(x)$, and $\mu(u)(\ast) : \equiv u$. These computations are independent of the family type to which they apply. From now on, we will not mention the type families to which they apply.

With the identity induction operator, one can characterize the inhabitants of cartesian product types and coproduct types, this allows us to present the first examples of deductive chains. For the case of the cartesian product type, if $A$ and $B$ are types, then

$\prod_{u:A \times B} u = (\text{pr}_1(u), \text{pr}_2(u)) <:\ (2)$

In fact,

$\prod_{u:A \times B} u = (\text{pr}_1(u), \text{pr}_2(u))$

$\leftarrow \{ \sigma \}
\prod_{x:A \times B} (x, y) = (\text{pr}_1((x, y)), \text{pr}_2((x, y)))$

$\equiv \{ \text{Definition of pr}_1 \text{ and pr}_2 \}
\prod_{x:A \times B} (x, y) = (x, y)$

$\wedge \{ h(x)(y) : \equiv \text{refl}_{(x, y)} \}
\hat{h}$.

And, for the case of coproduct type, if $A$ and $B$ are types, then

$\prod_{p:A+B} \sum_{x:A}(p = \text{inl}(x)) + \sum_{y:B}(p = \text{inr}(y)) <:\ (2)$
In fact,
\[
\prod_{p:A+B} \sum_{x:A} (p = \text{inl}(x)) + \sum_{y:B} p = \text{inr}(y)
\]
\[
\leftarrow \langle \kappa \rangle
\prod_{a:A} \sum_{x:A} (\text{inl}(a) = \text{inl}(x)) + \sum_{y:B} \text{inl}(a) = \text{inr}(y)
\times \prod_{b:B} \sum_{x:A} (\text{inl}(b) = \text{inl}(x)) + \sum_{y:B} \text{inr}(b) = \text{inr}(y)
\leftarrow \langle \varphi ; \varphi(u, v) \equiv (\text{inl} \circ u, \text{inr} \circ v) \rangle
\prod_{a:A} (\sum_{x:A} \text{inl}(a) = \text{inl}(x)) \times \prod_{b:B} (\sum_{y:B} \text{inr}(b) = \text{inr}(y)
\leftarrow \langle h \equiv (\lambda a.(a, \text{refl}_{\text{inl}(a)}), \lambda b.(b, \text{refl}_{\text{inr}(b)})) \rangle
\]

\section{Equivalence of types}

Now, we introduce the notion of equivalence of types, but first, we need the one of homotopic functions. Details of this topic may be found in [14].

Let \( f \) and \( g \) be two dependent functions inhabiting \( \prod_{x:A} P(x) \). We will say that \( f \) and \( g \) are homotopic if the type \( f \sim g \) defined by
\[
f \sim g \equiv \prod_{x:A} (f(x) = g(x))
\]
is inhabited. Two types \( A \) and \( B \) are equivalent if there is a function \( f : A \to B \) such that the type \( \text{isequiv}(f) \) defined by
\[
\text{isequiv}(f) \equiv (\sum_{g:B \to A} (f \circ g \sim \text{id}_B) \times (\sum_{h:B \to A} h \circ f \sim \text{id}_A)
\]
is inhabited. Therefore, \( A \) and \( B \) are equivalent if the type \( A \simeq B \) defined by \( \sum_{f:A \to B} \text{isequiv}(f) \) is inhabited. However, in order to prove equivalence in this paper, we will not use the type \( \text{isequiv}(f) \), but the type \( \text{qinv}(f) \), which is a simpler equivalent version (see [14], 2.4 p. 76) and is defined by
\[
\text{qinv}(f) \equiv \sum_{g:B \to A} ((f \circ g \sim \text{id}_B) \times (g \circ f \sim \text{id}_A))
\]
This means that in order to show that types \( A \) and \( B \) are equivalent we must exhibit a 4-tuple
\[
f \equiv (f, f', \alpha, \alpha')
\]
where
\[
f : A \to B, \quad f' : B \to A, \quad \alpha : f \circ f' \sim \text{id}_B, \quad \text{and} \quad \alpha' : f' \circ f \sim \text{id}_A.
\]
For instance, let us show that given types $A$ and $B$,
\[
A + B \simeq B + A <: \tag{3}
\]
In fact, let $f : A + B \to B + A$ and $f' : B + A \to A + B$ be defined by $f(\lnl(a)) \equiv \lnr(a)$, $f(\lnr(b)) \equiv \lnl(b)$, $f'(\lnl(b)) \equiv \lnr(b)$ and $f'(\lnr(a)) \equiv \lnl(a)$. Then, the following deductive chain shows that $f \circ f' \sim \text{id}_{B + A}$ is inhabited:
\[
f \circ f' \sim \text{id}_{B + A}
\equiv \text{ (Definition of $\sim$)}
\prod_{p : B + A} f(f'(p)) = p
\leftarrow \text{ (Definition of $f$ and $f'$)}
\prod_{b : B} (f(f'(\lnl(b))) = \lnl(b)) \times \prod_{a : A} (f(f'(\lnr(a))) = \lnr(a))
\equiv \text{ (Definition of $f$ and $f'$)}
\prod_{b : B} (\lnl(b) = \lnl(b)) \times \prod_{a : A} (\lnr(a) = \lnr(a))
\uparrow \text{ (} u \equiv \lambda b.\text{refl}_{\lnl(b)} ; v \equiv \lambda a.\text{refl}_{\lnr(a)} \text{)}
(u, v)
\]
We prove $f' \circ f \sim \text{id}_{A + B} <:$ in the same way.

We present three equivalences characterizing the identification of objects of certain types: pairs, functions, and natural numbers.

**Identification of pairs.** Let $A, B$ be types. Then for all $u$ and $v$ inhabitants of $A \times B$ we have that
\[
u = v \simeq (\text{pr}_1(u) = \text{pr}_1(v)) \times (\text{pr}_2(u) = \text{pr}_2(v)) <:
\]
*Proof.* First of all, we define $P_1(u, v) \equiv \text{pr}_1(u) = \text{pr}_1(v)$ and $P_2(u, v) \equiv \text{pr}_2(u) = \text{pr}_2(v)$. And now, we define $f : u = v \to P_1(u, v) \times P_2(u, v)$, by means of the following deductive chain:
\[
f : u = v \to P_1(u, v) \times P_2(u, v)
\leftarrow \text{ (} \bullet_{\text{t}_1} \text{)}
\prod_{u, v : A \times B} P_1(u, v) \times P_2(u, v)
\uparrow \text{ (} h \equiv \lambda u.\text{refl}_{\text{pr}_1(u)}, \text{refl}_{\text{pr}_2(u)} \text{)}
\]
In order to define a function \( f' : P_1(u, v) \times P_2(u, v) \rightarrow u = v \), let us consider the following deductive chain:

\[
\prod_{u,v,A \times B} P_1(u, v) \times P_2(u, v) \rightarrow u = v
\]

\[
\Leftarrow \langle \text{Definition of } \sigma, \sigma'(u)((a, c), (b, d), (p, q)) \equiv w(a)(b)(c)(d)(p)(q) \rangle
\]

\[
\prod_{a,b,c,d:B \mid p = a = b = c = d} \prod (a, c) = (b, d)
\]

\[
\Leftarrow \langle \text{Definition of } \tau_2 \rangle \langle \tau_2(z)(a, a, c, c, \text{refl}_a, \text{refl}_c) \equiv z(a)(c) \rangle
\]

\[
\prod_{a:A \times B} (a, c) = (a, c)
\]

\[
\wedge \langle k(a, c) \equiv \text{refl}_{(a, c)} \rangle
\]

Therefore, we can put \( f' : (\sigma \circ \tau_2)(k)(u, v) \).

Now, let us show that \( \prod_{u,v:A \times B} f \circ f' \sim \text{id} < \):

\[
\prod_{u,v:A \times B} \prod_{g:P_1(u, v) \times P_2(u, v)} f(f'(g)) = g
\]

\[
\equiv \langle \text{Definition of } f \text{ and } f' \rangle
\]

\[
\prod_{u,v:A \times B} \prod_{g:P_1(u, v) \times P_2(u, v)} (\tau_1(h)(u, v))(\sigma \circ \tau_2)(k)(u, v))(p, q) = (p, q)
\]

\[
\Leftarrow \langle \text{Definition of } \sigma, \tau_2, \text{ and } k \rangle
\]

\[
\prod_{a:A \times B} (\tau_1(h)((a, c), (a, c)))(\sigma \circ \tau_2)(k)((a, c), (a, c))(\text{refl}_a, \text{refl}_c) = (\text{refl}_a, \text{refl}_c)
\]

\[
\equiv \langle \text{Definition of } \sigma, \tau_2, \text{ and } k \rangle
\]

\[
\prod_{a:A \times B} (\text{refl}_a, \text{refl}_c) = (\text{refl}_a, \text{refl}_c)
\]

\[
\wedge \langle \text{j } \equiv \lambda a. \lambda c. \text{refl}_{(\text{refl}_a, \text{refl}_c)} \rangle
\]

The proof of \( \prod_{u,v:A \times B} f' \circ f \sim \text{id} < \) is done in the same way.

As a particular case, we have that if \( a, c : A \), and \( b, d : B \), then

\[
(a, b) = (c, d) \simeq a = c \times b = d <:
\]

(4)

**Identification of functions.** Let \( A \) and \( B \) be two types, and \( f \) and \( g \) objects of \( A \rightarrow B \). Then

\[
f = g \simeq f \sim g <:
\]

(5)

The inhabitation cannot be proved with the theory introduced till now but introduced as an axiom in \[14\] as function extensionality.
Identification of natural numbers. If one introduces the type family

\[ \text{code} : \mathbb{N} \to \mathbb{N} \to \mathcal{U} \]

defined by

\[ \text{code}(0, 0) \equiv 1, \quad \text{code}(s(n), 0) \equiv 0, \quad \text{code}(0, s(n)) \equiv 0, \quad \text{and} \]

\[ \text{code}(s(m), s(n)) \equiv \text{code}(m, n) \]

then, theorem 2.13.1 in [14] states that, for all \( m, n : \mathbb{N} \), we have that

\[ m = n \simeq \text{code}(m, n) \prec \]

(6)

Its proof introduces the functions \( \text{encode} : \prod_{m,n:\mathbb{N}} m = n \to \text{code}(m, n) \) and \( \text{decode} : \prod_{m,n:\mathbb{N}} \text{code}(m, n) \to m = n \), and shows that the functions \( \text{encode}(m, n) \) and \( \text{decode}(m, n) \) are \( \eta \)-inverses of each other.

From here on, we introduce a new kind of link joining equivalent types in our deductive chains. Namely, homotopic type-equivalence links which we write as follows

\[ A \simeq \langle \text{Justification of the equivalence} \rangle B. \]

Most of the time, these links will be used in a deductive chain fulfilling the same purpose with which we use the consequence links.

In next sections, we will explore several properties related with equivalence.

7 Leibniz properties of type equivalence

By Leibniz properties, we refer to the replacement of equivalents by equivalents (or congruence) property of, in this case, homotopic type-equivalence.

7.1 Leibniz principles.

These principles refer to the fact that equality is preserved respectively, by function application and type dependency (through, equivalence).

Let \( A, B : \mathcal{U} \), \( f : A \to B \) and \( P : A \to \mathcal{U} \). Then

\[ \prod_{x,y:A} x = y \to f(x) = f(y) \prec \quad \text{and} \quad \prod_{x,y:A} x = y \to P(x) \simeq P(y) \prec \]

In fact,

\[ \prod_{x,y:A} \prod_{P : x = y} f(x) = f(y) \]

\[ \simeq \langle : \iota \rangle \]

\[ \prod_{x:A} f(x) = f(x) \]

\[ ^{\uparrow} \langle h(x) : \equiv \text{refl}_{f(x)} \rangle \]

\[ h \]
One defines $\text{ap}_f(x,y,p) := \iota(h)(x,y,p)$, and by definition of $\iota$, we get $\text{ap}_f(x,x,\text{refl}_x) := \iota(h)(x,x,\text{refl}_x) \equiv \text{refl}_f(x)$.

On the other hand, $\Pi_{x,y : A} P(x) \simeq P(y)$

$\simeq \langle \iota \rangle$

$\Pi_{x : A} P(x) \simeq P(x)$

$\uparrow \langle k(x) \equiv \text{id}_{P(x)} \rangle$

One defines $\text{tr}^P(x,y,p) := \iota(k)(x,y,p)$ and by definition of $\iota$, we get $\text{tr}^P(x,x,\text{refl}_x) := \iota(k)(x,x,\text{refl}_x) \equiv \text{id}_{P(x)}$.

7.2 Leibniz inference rules.

Leibniz inference rules generally express the fact that type equivalence is preserved by the replacement, in any given type expression, of any of its subexpressions by an equivalent one. We derive inference rules for $\Pi$ types, $\Sigma$ types and coproduct types endowing HoTT, by this means, with a calculational style of proof.

Let $A, B, C : U$ and $P, Q : A \rightarrow U$. Then

1. $\Pi_{x : A} P(x) \simeq Q(x) \quad \Pi_{x : A} P(x) \simeq \Pi_{x : A} Q(x) \quad \Pi_{x : A} P(x) \simeq Q(x)$

2. $f : A \simeq B \quad f : A \simeq \Pi_{y : B} P(f(y)) \quad f : A \simeq B$

3. $A \simeq B \quad +E1 \quad A \simeq B \quad +E2$

Proof of $\Pi E1$. Suppose that $\Phi : \Pi_{x : A} P(x) \simeq Q(x)$, with $\Phi(x) \equiv (\phi_x, \phi'_x, \alpha, \alpha')$, $\alpha : \phi_x \circ \phi'_x \simeq \text{id}_{Q(x)}$ and $\alpha' : \phi'_x \circ \phi_x \simeq \text{id}_{P(x)}$. Let

$\psi : \Pi_{x : A} P(x) \rightarrow \Pi_{x : A} Q(x)$

be defined by $\psi(f)(x) := \phi_x(f(x))$ and let

$\psi' : \Pi_{x : A} Q(x) \rightarrow \Pi_{x : A} P(x)$

---

$^4$This object is called transport$^5$ in the HoTT book [14]

$^5$ψ is precisely the function $\Delta$ of $\Pi$-distibution over arrows, see [14]
be defined by \( \psi'(g)(x) := \phi'_x(g(x)) \). Observe that

\[
\psi(\psi'(g))(x) \equiv \phi_x(\psi'(g)(x)) \equiv \phi_x(\phi'_x(g(x))) \equiv (\phi_x \circ \phi'_x)(g(x))
\]

(7)

Then, in order to prove \( \psi \circ \psi' \sim \text{id} \), it is enough to prove \( (\psi \circ \psi')(g) = g \) for all \( g : \prod_{x : A} Q(x) \). In fact,

\[
(\psi \circ \psi')(g) = g \\
\approx \prod_{x : A} (\psi \circ \psi')(g(x)) = g(x) \\
\equiv \prod_{x : A} (\phi_x \circ \phi'_x)(g(x)) = g(x)
\]

\( \psi : \sum_{x : A} P(x) \to \sum_{x : A} Q(x) \),

be defined by \( \psi(p) := (\text{pr}_1(p), \phi_{\text{pr}_2(p)}(\text{pr}_2(p))) \) and let

\[
\psi' : \sum_{x : A} Q(x) \to \sum_{x : A} P(x)
\]

be defined by \( \psi'(q) := (\text{pr}_1(q), \phi'_{\text{pr}_1(q)}(\text{pr}_2(q))) \). Observe that

\[
\psi(\psi'((x,y))) \equiv \psi((x, \phi'_x(y))) \equiv (x, \phi_x(\phi'_x(y))) \equiv (x, (\phi_x \circ \phi'_x)(y))
\]

(8)

Then,

\[
\psi \circ \psi' \sim \text{id} \\
\equiv \langle \text{Definition of } \sim \rangle \\
\frac{\prod_{q : \sum_{x : A} Q(x)} (\psi \circ \psi')(q) = q}{\langle \text{\sigma} \rangle} \\
\leftarrow \langle \text{See above calculations} \rangle \\
\prod_{x : A, y : Q(x)} \psi(\psi'((x,y))) = (x,y)
\]

\[
\equiv \langle \text{See above computations} \rangle \\
\prod_{x : A, y : Q(x)} (x, (\phi_x \circ \phi'_x)(y)) = (x,y)
\]

\( \approx \langle (a,b) = (c,d) \equiv a = c \times b = d <:\langle \text{\PiEQ1} \rangle \rangle \\
\prod_{x : A, y : Q(x)} x = x \times (\phi \circ \phi')(y) = y \\
\wedge \langle h(x,y) := (\text{refl}_z, \alpha(y)) ; \alpha : \phi \circ \phi' \sim \text{id} \rangle
\]

h

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We prove $\psi' \circ \psi \sim \text{id}$; similarly.

Proof of $\Pi\text{eq2}$. Suppose that $f : A \simeq B$. Let

$$\psi : \prod_{x:A} P(x) \to \prod_{y:B} P(f'(y))$$

be defined by $\psi(u)(y) :\equiv u(f'(y))$, and let

$$\psi' : \prod_{y:B} P(f'(y)) \to \prod_{x:A} P(x)$$

be defined by $\psi'(v)(x) :\equiv v(f(x))$. Let us see that $\psi'$ is a quasi-inverse of $\psi$. On one hand, we have

$$\psi \circ \psi' \sim \text{id}$$

$$\equiv \langle \text{Definition of } \sim \rangle \prod_{v:B} \psi(\psi'(v)) = v$$

$$\equiv \langle \text{Definition of } \psi \text{ and } \psi' \rangle \prod_{v:B} (v \circ f \circ f' = v)$$

$$\simeq \langle \text{Function extensionality } ; \Pi\text{eq1} \rangle \prod_{v:B} (v \circ f \circ f' \sim v)$$

$$\equiv \langle \text{Definition of } \sim \rangle \prod_{v:B} \prod_{y:B} v(f(f'(y))) = v(y)$$

$$\leftrightarrow \langle : \Delta ; \varphi(v,y) :\equiv \text{ap}_v(f(f'(y)),y), \text{ see (1)} \rangle \prod_{v:B} \prod_{y:B} f(f'(y)) = y$$

$$\leftrightarrow \langle : \lambda z. (\lambda v. z) \rangle \prod_{y:B} f(f'(y)) = y$$

$$\equiv \langle \text{Definition of } \sim \rangle f \circ f' \sim \text{id}_B$$

$$\alpha$$

On the other hand, we can show, exactly in the same way, that

$$h' \circ h \sim \text{id}_{\prod_{x:A} P(x)}$$

Application of $\Pi$-translation rule (to prove isSet($\mathbb{N}$) $\leftarrow$).

We can use the translation rule to prove isSet($\mathbb{N}$) $\leftarrow$. In fact, let $\Phi : m = n \to \mathbb{N}$.
code\( (m, n) \) be defined by \( \Phi \equiv \text{encode}(m, n) \) and let \( \Psi : \text{code}(m, n) \to m = n \) be defined by \( \Psi \equiv \text{decode}(m, n) \). Then,

\[
\text{isSet}(\mathbb{N}) \equiv \prod_{m,n : \mathbb{N}} \prod_{p,q : m = n} p = q
\]

\[
\cong \prod_{m,n : \mathbb{N}} \prod_{s,t : \text{code}(m, n)} \Psi(s) = \Psi(t)
\]

\[\overset{\wedge}{\text{See definition of } h \text{ below}}\]

\[h\]

where \( h \) is defined by

\[
h(m, n, s, t) = \begin{cases} 
\mu_1(\mu_2(\text{refl}_{\Psi(s)})) & \text{if } \text{code}(m, n) = 1 \\
\sigma_C(s)(t) & \text{if } \text{code}(m, n) = 0 
\end{cases}
\]

with \( C \equiv \prod_{t : 0} \Psi(s) = \Psi(t) \). The definition of \( h \) is justified by

\[
\prod_{s,t : 1} \Psi(s) = \Psi(t)
\]

\[\leftarrow \langle \mu_1 \rangle \]

\[
\prod_{t : 1} \Psi(s) = \Psi(t)
\]

\[\leftarrow \langle \mu_2 \rangle \]

\[
\Psi(s) = \Psi(t)
\]

\[\overset{\wedge}{\langle u : \equiv \text{refl}_{\Psi(s)} \rangle}\]

**Proof of \( \Sigma\text{eq2} \).** Suppose that \( f : A \simeq B \). Let

\[
\psi : \sum_{x : A} P(x) \to \sum_{y : B} P(f'(y))
\]

defined by \( \psi(u) \equiv (f(\text{pr}_1(u)), \text{pr}_2(u)) \) and let

\[
\psi' : \sum_{y : B} P(f'(y)) \to \sum_{x : A} P(x)
\]

defined by \( \psi'(v) \equiv (f'(\text{pr}_1(v)), \text{pr}_2(v)) \). Observe that

\[
\psi(\psi(v)) \equiv \psi((f'(\text{pr}_1(v)), \text{pr}_2(v)) \equiv ((f \circ f')(\text{pr}_1(v)), \text{pr}_2(v)) \quad (9)
\]

Then we have that

\[
\psi \circ \psi' \sim \text{id} \equiv \langle \text{Definition of } \sim \rangle
\]

\[\leftarrow \langle \sigma \rangle \]

\[
\prod_{y : B} \prod_{x : P(f'(y))} \psi(\psi'(y, z)) = (y, z)
\]
\[ \equiv \langle \text{See above calculations (9)} \rangle \]
\[ \prod_{y:B} \prod_{z:P(f(y))} ((f \circ f')(y), z) = (y, z) \]
\[ \simeq \langle (a, b) = (c, d) \simeq (a = c) \times (b = d) \rangle ; \text{\Pi EQ1} \]
\[ \prod_{y:B} \prod_{z:P(f(y))} ((f \circ f')(y) = y) \times (z = z) \]
\[ \uparrow \langle h(y, z) := (\alpha(y), \text{refl}_z) \rangle \]

The proof of \( \psi' \circ \psi \simeq \text{id}_X \)
\[ \sum_{x:A} P(x) \simeq \] is similar.

We can use \( \text{\Sigma EQ1, \Sigma EQ2 and transitivity of equivalence to derive the following inference rule which we will be using later:} \]
\[ f : A \simeq B \quad g : C \simeq D \]
\[ \prod_{x:A} P(x) \times \prod_{x:C} P(x) \simeq \prod_{x:B} P(x) \times \prod_{x:D} P(x) \quad \text{EQ} \]
\[ (10) \]

**Proof of +EQ1.** Suppose that \( f : A \simeq B \). Let \( \psi : A + C \to B + C \) be defined by \( \psi \equiv \kappa(\operatorname{inl} \circ f, \operatorname{inr} \circ \text{id}_C) \), and let \( \psi' : B + C \to A + C \) be defined by \( \psi' \equiv \kappa(\operatorname{inl} \circ f', \operatorname{inr} \circ \text{id}_C) \).

Let us see that \( \psi' \) is a quasi-inverse of \( \psi \). Observe that, by definition of \( \Psi \) and \( \Psi' \), we have

\[
\begin{align*}
\psi(\psi'(\operatorname{inl}(x))) & \equiv \psi'\kappa(\operatorname{inl} \circ f', \operatorname{inr} \circ \text{id}_C)(\operatorname{inl}(x)) & \equiv \psi(\operatorname{inl}(y)) \\
\psi(\operatorname{inl}(f'(x))) & \equiv \kappa(\operatorname{inl} \circ f, \operatorname{inr} \circ \text{id}_C)(\operatorname{inl}(f'(x))) & \equiv \operatorname{inr}(y)
\end{align*}

(11)

Then we have
\[
\begin{align*}
\psi \circ \psi' & \approx \text{id} \\
\prod_{p:B+C} \psi(\psi'(p)) & = p \\
\leftrightarrow & \langle \kappa \rangle \\
\prod_{x:B} (\psi(\psi'(\operatorname{inl}(x)))) & = \operatorname{inl}(x) \times \prod_{y:C} \psi(\psi'(\operatorname{inr}(y))) = \operatorname{inr}(y) \\
\equiv & \langle \text{Definition of } \psi \text{ and } \psi' \rangle \\
\prod_{x:B} (\operatorname{inl}(f'(x))) & = \operatorname{inl}(x) \times \prod_{y:C} \operatorname{inr}(y) = \operatorname{inr}(y) \\
\leftrightarrow & \langle k : k(u, v) \equiv (\lambda x. \text{ap}_{\text{inl}}(u(x)), \lambda x. \text{ap}_{\text{inl}}(v(x))) \} \\
\prod_{x:B} (f'(x)) & = x \times \prod_{y:C} y = y \\
\uparrow & \langle h : \equiv (\alpha, \text{refl}) \rangle \\
h
\end{align*}
\]

We can prove \( h' \circ h \approx \text{id}_{\prod_{x:A} P(x)} \simeq \) similarly.
Proof of +EQ2.

\[ C + A \]
\[ \simeq \quad (\text{Commutativity of } + \text{ (3)}) \]
\[ A + C \]
\[ \simeq \quad (+\text{EQ1}) \]
\[ B + C \]
\[ \simeq \quad (\text{Commutativity of } + \text{ (3)}) \]
\[ C + B \]

8 Induction operators as equivalences

In order to be able to restate HoTT giving equality and equivalence a preeminent role, it is convenient (and possible) to show that the inductive operators for the equality type, the Σ-type and the coproduct are actually, equivalences. We now proceed to show that this is actually so.

8.1 Identity type induction operator

We will prove that for all \( P : A \to U \), \( \iota \) is an equivalence, and then, 

\[ \prod_{x : A} (\prod_{p : x = y} P(x, y, p)) \simeq \prod_{x : A} P(x, x, \text{refl}_x) \Rightarrow: \]

This equivalence is inspired by the Equality rule in section 2.

Recall that \( \iota : (\prod_{x : A} P(x, x, \text{refl}_x)) \to \prod_{x, y : A} \prod_{p : x = y} P(x, y, p). \)

Now, let us define

\[ k : \prod_{x, y : A} (\prod_{p : x = y} P(x, y, p)) \to \prod_{x : A} P(x, x, \text{refl}_x) \]

by

\[ k(v)(x) : \equiv v(x, x, \text{refl}_x). \]

Let us prove that \( k \circ \iota \sim \text{id} \) and that \( \iota \circ k \sim \text{id} \). First, observe that for all \( u : \prod_{x : A} P(x, x, \text{refl}_x) \), by definition of \( k \) and \( \iota \),

\[ k(\iota(u))(x) \equiv \iota(u)(x, x, \text{refl}_x) \equiv u(x), \quad (12) \]

and for all \( v : \prod_{x, y : A} \prod_{p : x = y} P(x, y, p) \),

\[ \iota(k(v))(x, x, \text{refl}_x) \equiv k(v)(x) \equiv v(x, x, \text{refl}_x). \quad (13) \]

Then, in one hand, because of (12), we have that \( k \circ \iota \sim \text{id} \). On the other, for each \( v : \prod_{x, y : A} \prod_{p : x = y} P(x, y, p) \), let us show that \( \iota(k(v)) = v \lessgtr: \)

\[ \iota(k(v)) = v \]
\[ \simeq \quad (\text{Function extensionality (5)}) \]
\[ \prod_{x, y : A} \prod_{p : x = y} \iota(k(v))(x, y, p) = v(x, y, p) \]

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\[ \begin{align*}
\left\langle \varepsilon(x) \right\rangle \\
\prod_{x:A} \varepsilon(k(v))(x, x, \text{refl}_x) &= v(x, x, \text{refl}_x) \\
\equiv \quad \text{(See computation (13) above)} \\
\prod_{x:A} v(x, x, \text{refl}_x) &= v(x, x, \text{refl}_x) \\
\uparrow \quad \{ \ u(x) = \text{refl}_{u(x, x, \text{refl}_x)} \} \\
\end{align*} \]

Therefore, the equivalence is proven.

8.2 Identity type based-path induction operator

Let us suppose that \( a : A \) and that \( D : \prod_{x:A} \prod_{p:a=x} U \). Based path induction states the existence of a function \( \varepsilon' \) presented by the following consequence link

\[ \begin{align*}
\prod_{x:A} \prod_{p:a=x} D(x, p) &
\leftarrow \left\langle : \varepsilon'; \varepsilon'(u)(a, \text{refl}_a) : \equiv z \right\rangle \\
D(a, \text{refl}_a) &
\end{align*} \]

We have also that \( \varepsilon'_p \), the based path induction operator, is an equivalence, and then

\[ \prod_{x:A} \prod_{p:a=x} P(x, p) \simeq P(a, \text{refl}_a) \left\langle : \right\rangle \]

this equivalence corresponds to One-point rule in section 2.

Let us prove that the functions

\[ \begin{align*}
\prod_{x:A} \prod_{p:a=x} P(x, p) &
\leftarrow \left\langle : \varepsilon'; \varepsilon'(u)(a, \text{refl}_a) : \equiv u \right\rangle \\
P(a, \text{refl}_a) &
\end{align*} \]

and

\[ \begin{align*}
P(a, \text{refl}_a) &
\leftarrow \left\langle : k; k(v) : \equiv v(a, \text{refl}_a) \right\rangle \\
\prod_{x:A} \prod_{p:a=x} P(x, p) &
\end{align*} \]

are quasi-inverses. In fact,

\[ k(\varepsilon'(u)) \equiv \varepsilon'(u)(a, \text{refl}_a) \equiv u, \]

which shows that \( k \circ \varepsilon' \sim \text{id} \), and

\[ \varepsilon'(k(v))(a, \text{refl}_a) \equiv k(v)(x) \equiv v(a, \text{refl}_a). \quad (14) \]

And so, to prove \( \varepsilon' \circ k \sim \text{id} \), it is enough to perform the following calculation for all \( v : \prod_{x:A} \prod_{p:a=x} P(x, p) \),

\[ \begin{align*}
\varepsilon'(k(v)) &= v \\
&\quad \left\langle \left\langle \text{Function extensionality (5)} \right\rangle \right\rangle \\
\prod_{x:A} \prod_{p:a=x} \varepsilon'(k(v))(x, p) &= v(x, p)
\end{align*} \]
\[\begin{align*}
\&\left\langle \cdot \varepsilon' \right\rangle \\
\varepsilon'(k(v))(a, \text{refl}_a) &= v(a, \text{refl}_a) \\
\equiv \left\langle \text{See (14), above} \right\rangle \\
v(a, \text{refl}_a) &= v(a, \text{refl}_a) \\
\wedge \left\langle \text{Definition of refl} \right\rangle \\
\text{refl}_v(a, \text{refl}_a)
\end{align*}\]

Therefore, \(\prod_{x:A} \prod_{p:a=y} P(x,p) \simeq P(a, \text{refl}_a)\) \(\vdash\):

### 8.3 \(\Sigma\)-type induction operator

Now, we prove that for all \(P : A \rightarrow \mathcal{U}\), that \(\sigma\), the \(\Sigma\)-type induction operator, is an equivalence. And so,

\[
\prod_{x:A} \prod_{y:B(x)} P((x,y)) \simeq \prod_{g: \sum_{x:A} B(x)} P(g) \quad \quad (15)
\]

For the case of \(P\) being a non-dependent type, the intuitionistic logical theorem corresponding to this equivalence is

\[
(\forall x:T \cdot B \cdot P) \equiv (\exists x:T \cdot B) \Rightarrow P
\]

where \(x\) does not occur free in \(P\).

This motivate us to call the equivalence \(\Sigma\)-consequent rule.

Recall that

\[
\sigma : \prod_{x:A} \prod_{y:B(x)} P((x,y)) \rightarrow \prod_{g: \sum_{x:A} B(x)} P(g)
\]

and \(\sigma(u)((x,y)) \equiv u(x)(y)\). Let

\[
\Phi : \prod_{g: \sum_{x:A} B(x)} P(g) \rightarrow \prod_{x:A} \prod_{y:B(x)} P((x,y))
\]

be defined by \(\Phi(v)(x)(y) \equiv v((x,y))\). Composing \(\sigma\) with \(\Phi\) we get

\[
\Phi(\sigma(u))(x)(y) \equiv \sigma(u)((x,y)) \equiv u(x)(y).
\]

Then \(\Phi \circ \sigma\) is homotopic to the identity function. Conversely, let \(v\) be an inhabitant of \(\prod_{g: \sum_{x:A} B(x)} P(g)\), then

\[
\sigma(\Phi(v)) \equiv v
\]

\[
\simeq \left\langle \text{Function extensionality (5)} \right\rangle
\]

\[
\prod_{g: \sum_{x:A} B(x)} \sigma(\Phi(v))(g) = v(g)
\]

\[
\left\langle \cdot \sigma \right\rangle
\]

\[
\prod_{x:A} \prod_{y:B(x)} \sigma(\Phi(v))(x,y) = v((x,y))
\]

\[
\equiv \left\langle \sigma(\Phi(v))(x,y) \equiv \Phi(v)(x)(y) \equiv v((x,y)) \right\rangle
\]

\[
\prod_{x:A} \prod_{y:B(x)} v((x,y)) = v((x,y))
\]

\[
\wedge \left\langle h \equiv \lambda x.\lambda y.\text{refl}_v(x,y) \right\rangle
\]

\[
h
\]

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So, $\sigma \circ \Phi$ is homotopic to the identity function.

### 8.4 Coproduct induction operator

For all $A, B : \mathcal{U}$ and $P : A + B \to \mathcal{U}$ we have that

$$( \prod_{x : A + B} P(x) ) \simeq ( \prod_{x : A} P(\text{inl}(x))) \times ( \prod_{y : B} P(\text{inr}(x)))$$

This equivalence correspond to the $\forall$-range split rule in section 2.

**Proof.** We have the induction operator $\kappa$:

$$\prod_{x : A + B} P(x)$$

$$\leftarrow \langle : \kappa ; \kappa(u,v)(\text{inl}(x)) \equiv u(x) ; \kappa(u,v)(\text{inr}(x)) \equiv v(x) \rangle$$

$$( \prod_{x : A} P(\text{inl}(x)) ) \times ( \prod_{y : B} P(\text{inr}(x)) )$$

and let us define

$$\Psi : ( \prod_{x : A + B} P(x) ) \to ( \prod_{x : A} P(\text{inl}(x)) ) \times ( \prod_{y : B} P(\text{inr}(y)) )$$

by $\Psi(g) : \equiv (g \circ \text{inl}, g \circ \text{inr})$. Let us see that $\Psi$ is a quasi-inverse of $\kappa$. We will show that, the type $\kappa \circ \Psi \sim \text{id}$, which by definition is equivalent to

$$\prod_{x : A + B} \kappa(\Psi(g))(x) = g,$$

is inhabited. Let $g$ be an object of type $\prod_{x : A + B} P(x)$, then:

$$\kappa(\Psi(g)) = g$$

$$\equiv \langle \text{Definition of } \Psi \rangle$$

$$\kappa(g \circ \text{inl}, g \circ \text{inr}) = g$$

$$\simeq \langle \text{Function extensionality } \rangle$$

$$\kappa(g \circ \text{inl}, g \circ \text{inr}) \sim g$$

$$\equiv \langle \text{Definition of } \sim \rangle$$

$$\prod_{x : A + B} \kappa(g \circ \text{inl}, g \circ \text{inr})(z) = g(z)$$

$$\leftarrow \langle : \kappa \rangle$$

$$\prod_{x : A + B} \kappa(g \circ \text{inl}, g \circ \text{inr})(\text{inl}(x)) = g(\text{inl}(x))$$

$$\times \prod_{y : B} \kappa(g \circ \text{inl}, g \circ \text{inr})(\text{inr}(y)) = g(\text{inr}(y))$$

$$\equiv \langle \text{Definition of } \kappa \rangle$$

$$\prod_{x : A} ((g \circ \text{inl})(x) = (g \circ \text{inl})(x)) \times \prod_{y : B} (g \circ \text{inr})(y) = (g \circ \text{inr})(y)$$

$$\wedge \langle \text{h\text{\textunderscore}} \text{\textit{refl}}_g(\text{inl}(x)), \lambda x.\text{\textit{refl}}_g(\text{inr}(y)) \rangle \rangle$$

$$h(g)$$

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And now, we will show that $\Psi \circ \kappa \sim \text{id}$: In other words, that

$$
\prod_{x : A} P(\text{inl}(x)) \times \prod_{y : B} P(\text{inr}(y))
$$

Let $u$ be an object of type $\prod_{x : A} P(\text{inl}(x)) \times \prod_{y : B} P(\text{inr}(y))$, $p : u = (\text{pr}_1(u), \text{pr}_2(u))$ and $Q$ the type family defined by $Q(u) : \equiv (\kappa(u) \circ \text{inl}, \kappa(u) \circ \text{inr}) = u$, and so, by the second Leibniz principle,

$$
\text{tr}^Q(u, (\text{pr}_1(u), \text{pr}_2(u)), p) : Q(u) \simeq Q((\text{pr}_1(u), \text{pr}_2(u)))
$$

Then:

$$
\Psi(\kappa(u)) = u
\equiv (\text{Definition of } \Psi)
(\kappa(u) \circ \text{inl}, \kappa(u) \circ \text{inr}) = u
\simeq (\text{tr}^Q(u, (\text{pr}_1(u), \text{pr}_2(u)), p))
(\kappa(\text{pr}_1(u), \text{pr}_2(u)) \circ \text{inl} = \text{pr}_1(u)) \times (\kappa(\text{pr}_1(u), \text{pr}_2(u)) \circ \text{inr} = \text{pr}_2(u))
\equiv (\text{Definition of } \kappa)
(\text{pr}_1(u) = \text{pr}_1(u)) \times (\text{pr}_2(u) = \text{pr}_2(u))
\uparrow (h : \equiv \text{refl}_{\text{pr}_1(u)} ; k : \equiv \text{refl}_{\text{pr}_2(u)})
(h, k)
$$

As a matter of fact, the induction operators corresponding to $W$ type, $O$ type and $\Pi$ type could be similarly proved to be equivalences.

9 Operational properties of $\Pi$ and $\Sigma$ types

The following properties of $\Pi$ and $\Sigma$ types are induced by corresponding operational properties, enumerated in section 2 and associated respectively, to the universal and existential quantifiers of intuitionistic first order logic. This correspondence is not exact. Due to the higher-order nature of type theory, the $\Pi$ and $\Sigma$ type properties are more general than the logical properties that induce them. We would need a well established higher-order version of intuitionistic logic to be able to think of an extension of the Curry-Howard isomorphism.

One point rules. These rules correspond in first order logic when you are actually quantifying a property over exactly one element, and therefore, we have an equivalence with the property applied to just this element. For the case of HoTT, they are slightly more general.

$$
\prod_{x : A} \prod_{p : a = x} P(x, p) \simeq P(a, \text{refl}_a) <: 
$$

and

$$
\sum_{x : A} (\sum_{p : a = x} P(x, p) \simeq P(a, \text{refl}_a) < : 
$$

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We proved the Π-one-point rule in subsection 8.2. We now prove the Σ-one-point rule.

Given \( A : \mathcal{U} \), a : A and \( P : \prod_{x:A} \prod_{p:x=a} \mathcal{U} \), let us construct

\[
\Phi : \sum_{x:A} \left( \prod_{p:x=a} P(x,p) \right) \rightarrow P(a, \text{refl}_a).
\]

This can be done by means of the following deductive chain:

\[
\begin{align*}
\sum_{x:A} \left( \prod_{p:x=a} P(x,p) \right) & \quad \sim \quad \langle \sigma, \Sigma\text{-consequent rule} \rangle \\
\sum_{x:A} \sum_{p:x=a} P(x,p) & \quad \sim \quad \langle \Delta; \varphi_x : \equiv \sigma_x, \Pi EQ 1 \rangle \\
\prod_{x:A} \prod_{p:x=a} P(x,p) & \quad \sim \quad \langle \iota', \Pi\text{-one-point rule} \rangle \\
\prod_{x:P(a, \text{refl}_a)} P(x, \text{refl}_a) & \quad \sim \quad \langle \iota', \Pi\text{-one-point rule} \rangle \\
\prod_{x:P(a, \text{refl}_a)} P(a, \text{refl}_a) & \quad \equiv \quad \text{id}_{P(a, \text{refl}_a)}(t)
\end{align*}
\]

In the chain above, \( \sigma_x \) is the induction operator for \( \sum_{p:x=a} P(x,p) \) evaluated at the constant type family \( C(x,y) : = P(x, \text{refl}_a) \).

Now, let \( \Psi : P(a, \text{refl}_a) \rightarrow \sum_{x:A} \sum_{p:x=a} P(x,p) \) be defined by

\[
\Psi(u) : \equiv (a, (\text{refl}_a, u)).
\]

Let us verify that \( \Phi \circ \Psi \sim \text{id} \) and that \( \Psi \circ \Phi \sim \text{id} \). First of all observe that, making the compositions in the above chain, we get

\[
\Phi : \equiv \sigma\left( \Delta\left( \iota'\left( \text{id}_{P(a, \text{refl}_a)} \right) \right) \right).
\]

On one hand we have,

\[
\begin{align*}
\Phi(\Psi(t)) & \equiv \sigma\left( \Delta\left( \iota'\left( \text{id}_{P(a, \text{refl}_a)} \right) \right) \right) \left( a, \text{refl}_a, t \right) \\
\equiv & \Delta\left( \iota'\left( \text{id}_{P(a, \text{refl}_a)} \right) \right) \left( \text{id}_{P(a, \text{refl}_a)}(t) \right)
\end{align*}
\]

and, on the other hand,

\[
\begin{align*}
\Psi \circ \Phi & \sim \text{id} \\
\equiv & \langle \text{Definition of } \sim \rangle \\
\sum_{x:A} \sum_{p:x=a} P(x,p) & \quad \sim \quad \langle \sigma, \Sigma\text{-consequent rule} \rangle \\
(\sum_{x:A} \sum_{p:x=a} P(x,p)) & \quad \sim \quad \langle \sigma, \Sigma\text{-consequent rule} \rangle \\
(\sum_{x:A} \sum_{p:x=a} P(x,p)) & \quad \equiv \quad \text{id}_{P(a, \text{refl}_a)}(t) \equiv t
\end{align*}
\]
Equality rules. These equivalences correspond, in first order logic, to the case when we are quantifying over two variables that happen to be equal, then one of those quantified variables may be made equal to the other, and be, in this way, eliminated. 

\[ \prod_{x,y:A} P(x,y,p) \simeq \prod_{x:A} P(x,x,\text{refl}_x) \]

and

\[ \sum_{x,y:A} P(x,y,p) \simeq \sum_{x:A} P(x,x,\text{refl}_x) \]

\(\Pi\)-equality rule was proved in subsection 8.1. The proof of \(\Sigma\)-equality rule follows analogous steps to those of the \(\Sigma\)-one-point rule. We omit it.

Range split rules. The range split rule is a property of operationals in general. In the case of logical quantifications, it allows separating them into two quantifiers of the same kind of the original one: universal or existential. These operational parts are joined by conjunctions for the first kind, and by disjunctions for the second. Their ranges correspond to disjoint components of the range of the original quantification. In the case of HoTT, this splitting is possible when the range of a \(\Pi\)-type or a \(\Sigma\)-type corresponds to a coproduct type. For the case of a \(\Pi\)-type, its parts are joined by a cartesian product. In the case of a \(\Sigma\)-type they are joined by a coproduct operator. Namely,

\[ \prod_{x:P+Q} R(x) \simeq (\prod_{x:P} R(\text{inl}(x))) \times (\prod_{x:Q} R(\text{inr}(x))) \]

and

\[ \sum_{x:P+Q} R(x) \simeq (\sum_{x:P} R(\text{inl}(x))) + (\sum_{x:Q} R(\text{inr}(x))) \]

The \(\Pi\)-range split rule is related to the coproduct induction operator and was proved in subsection 3.4. We now prove \(\Sigma\)-range split rule. In order to get a function

\[ \Phi: \left( \sum_{x:P+Q} R(x) \right) \to \left( \sum_{y:P} R(\text{inl}(y)) \right) + \sum_{z:Q} R(\text{inr}(z)) \]

let us consider the following deductive chain:

\[ \left( \sum_{x:P+Q} R(x) \right) \to \left( \sum_{y:P} R(\text{inl}(y)) \right) + \sum_{z:Q} R(\text{inr}(z)) \]

\[ \simeq \langle \sigma, \Sigma\text{-consequent rule} \rangle \]

\[ \prod_{x:P+Q} (R(x) \to \left( \sum_{y:P} R(\text{inl}(y)) \right) + \sum_{z:Q} R(\text{inr}(z))) \]
\[ \cong \quad \langle \kappa, \text{(II-range split rule) } \rangle \\
(\prod_{y : P} R(\text{inl}(y))) \rightarrow (\sum_{z : Q} R(\text{inr}(z))) \\
\times (\prod_{y : P} R(\text{inr}(y))) \rightarrow (\sum_{z : Q} R(\text{inr}(z))) \]

\[ \triangleright \quad \langle \phi_0(u)(a) \equiv \text{inl}((u, a)); \quad \phi_1(v)(b) \equiv \text{inr}((v, b)) \rangle \\
(\phi_0, \phi_1) \]

Then we can put \( \Phi : \equiv \sigma(\phi_0, \phi_1) \)

Now, in order to get a function

\[ \Psi : \sum_{y : P} R(\text{inl}(y)) + \sum_{z : Q} R(\text{inr}(z)) \rightarrow \sum_{x : P + Q} R(x) \]

let us consider the following deductive chain:

\[ \langle \sum_{y : P} R(\text{inl}(y)) + (\sum_{z : Q} R(\text{inr}(z)) \rightarrow \sum_{x : P + Q} R(x) \rangle \\
\cong \quad \langle \kappa, \text{(II-range split rule) } \rangle \\
((\sum_{y : P} R(\text{inl}(y))) \rightarrow \sum_{z : P + Q} R(x)) \times ((\sum_{z : Q} R(\text{inr}(z))) \rightarrow \sum_{x : P + Q} R(x)) \]

\[ \cong \quad \langle \sigma_1 \times \sigma_2, \text{ EQ } \square \rangle \\
((\prod_{y : P} R(\text{inl}(y))) \rightarrow \sum_{x : P + Q} R(x)) \times ((\prod_{z : Q} R(\text{inr}(z))) \rightarrow \sum_{x : P + Q} R(x)) \]

\[ \triangleright \quad \langle \psi_0(y)(a) \equiv (\text{inl}(y), a); \quad \psi_1(z)(b) \equiv (\text{inr}(z), b) \rangle \\
(\psi_0, \psi_1) \]

Then we may define \( \Psi : \equiv \kappa(\sigma_1 \times \sigma_2(\psi_0, \psi_1)) : \equiv \kappa(\sigma_1(\psi_0), \sigma_2(\psi_1)) \)

Observe that

\[ \Phi(\Psi(\text{inl}(f_1, f_2))) \equiv \Phi(\text{inl}(f_1), f_2) \]
\[ \Phi(\kappa(\sigma_1(\psi_0), \sigma_2(\psi_1))(\text{inl}(f_1, f_2))) \equiv \kappa(\phi_0(\phi_1))(\text{inl}(f_1))(f_2) \]
\[ \Phi(\sigma_1(\psi_0)(f_1, f_2)) \equiv \phi_0(f_1)(f_2) \]
\[ \Phi(\sigma_2(\psi_1)(f_1, f_2)) \equiv \text{inl}(f_1, f_2) \]

In the same way we can prove that \( \Phi(\Psi(\text{inr}(g_1, g_2))) \equiv \text{inr}(g_1, g_2) \)

Then

\[ \prod_{p : \sum_{y : P} R(\text{inl}(y)) + \sum_{z : Q} R(\text{inr}(z))} \Phi(\Psi(p)) = p \]

\[ \cong \quad \langle \kappa, \text{(II-range split rule) } \rangle \\
\prod_{y : P} \Phi(\text{inl}(f)) = \text{inl}(f) \]
\[ \times \prod_{z : Q} \Phi(\text{inr}(g)) = \text{inr}(g) \]

\[ \cong \quad \langle \sigma_1 \times \sigma_2, \text{ EQ } \square \rangle \\
\prod_{f_1 : P} \Phi(\text{inl}(f_1, f_2)) = \text{inl}(f_1, f_2) \]
\[ \times \prod_{g_1 : Q} \Phi(\text{inr}(g_1, g_2)) = \text{inr}(g_1, g_2) \]
Term split rules. In logic, universal quantifications of conjunctions split (through an equivalence) into universal quantifications of each conjunct joined by conjunctions too. Dually, existential quantifications split into existential quantifications of each disjunct joined by disjunctions. In the case of HoTT, Π-types mapping into cartesian products split into Π-types for each factor joined by cartesian products. Dually, for Σ-types, we joined by disjunctions. In the case of HoTT, Π-types mapping into cartesian products Dually, existential quantifications split into existential quantifications of each disjunct joined by conjunctions too.

\[ \equiv \langle \text{Above computations} \rangle \]
\[ \prod_{f_1 : P \times R(\text{inl}(f_1))} \prod_{f_2 : R(\text{inl}(f_1))} \text{inl}(f_1, f_2) = \text{inl}(f_1, f_2) \]
\[ \times \prod_{g_1 : P \times R_2(\text{inr}(g_1))} \prod_{g_2 : R(\text{inr}(g_1))} \text{inr}(g_1, g_2) = \text{inr}(g_1, g_2) \]
\[ \uparrow \langle u(f_1, f_2) \equiv \text{refl}_{\text{inl}(f_1, f_2)} ; u(g_1, g_2) \equiv \text{refl}_{\text{inr}(g_1, g_2)} \rangle \]
\[ (u, v) \]

In the other direction, observe that

\[ \Psi(\Phi(\text{inl}(w), u_2)) \equiv \Psi(\text{inl}(w, u_2)) \]
\[ \equiv \Psi(\sigma_1(\text{inl}(w), u_2)) \equiv \sigma_1(\text{inl}(w), u_2) \]
\[ \equiv \Psi(\text{inl}(w)(u_2)) \equiv \text{inl}(w)(u_2) \]

In the same way we can prove that \( \Psi(\Phi(\text{inr}(z), u_2)) \equiv \text{inr}(z), u_2 \). Then

\[ \prod_{u : \sum_{x : P \times Q} R(x)} \Psi(\Phi(u)) = u \]
\[ \cong \langle \sigma, \Sigma\text{-consequent rule} \rangle \]
\[ \prod_{u_1 : P \times Q} \prod_{u_2 : R(u_1)} \Psi(\Phi(u_1, u_2)) = (u_1, u_2) \]
\[ \cong \langle \kappa, \Pi\text{-range split rule} \rangle \]
\[ \prod_{w : P \times Q} \prod_{u_2 : R(\text{inl}(w))} \Psi(\Phi(\text{inl}(w), u_2)) = (\text{inl}(w), u_2) \]
\[ \times \prod_{z : Q} \prod_{u_2 : R(\text{inr}(z))} \Psi(\Phi(\text{inr}(z), u_2)) = (\text{inr}(z), u_2) \]
\[ \equiv \langle \text{Above computations} \rangle \]
\[ \prod_{w : P \times Q} \prod_{u_2 : R(\text{inl}(w))} (\text{inl}(w), u_2) = (\text{inl}(w), u_2) \]
\[ \times \prod_{z : Q} \prod_{u_2 : R(\text{inr}(z))} (\text{inr}(z), u_2) = (\text{inr}(z), u_2) \]
\[ \uparrow \langle h : (\lambda w . \lambda u_2 . \text{refl}_{\text{inl}(w), u_2}), \lambda z . \lambda u_2 . \text{refl}_{\text{inr}(z), u_2} \rangle \}

h

Term split rules. In logic, universal quantifications of conjunctions split (through an equivalence) into universal quantifications of each conjunct joined by conjunctions too. Dually, existential quantifications split into existential quantifications of each disjunct joined by disjunctions. In the case of HoTT, Π-types mapping into cartesian products split into Π-types for each factor joined by cartesian products. Dually, for Σ-types, we have an analogous situation replacing cross products by coproducts. Namely,

\[ \prod_{x : A} (P(x) \times Q(x)) \simeq (\prod_{x : A} P(x)) \times (\prod_{x : A} Q(x)) \]
and

\[ \sum_{x : A} (P(x) + Q(x)) \simeq (\sum_{x : A} P(x)) + (\sum_{x : A} Q(x)) \]

To prove the Π-term split rule, let \( \Phi : (\prod_{x : A} P(x)) \times (\prod_{y : A} Q(y)) \rightarrow \prod_{x : A} P(x) \times Q(x) \) be defined by \( \Phi(u)(x) \equiv ((\text{pr}_1 u)(x), (\text{pr}_2 u)(x)) \), and also, let \( \Psi : \prod_{x : A} (P(x) \times Q(x)) \rightarrow \)
be defined by $\Psi(g) \equiv (\text{pr}_1 \circ g, \text{pr}_2 \circ g)$. Let us see that $\Psi$ is a quasi-inverse of $\Phi$:

$$
\Psi \circ \Phi \sim \text{id} \prod_x A P(x) \times \prod_y A Q(y)
$$

≡ \langle Definition of $\sim$ \rangle

$$
\prod_u \prod_x A P(x) \times \prod_y A Q(y)
$$

≡ \langle Definition of $\Psi$ \rangle

$$
\prod_u \prod_x A P(x) \times \prod_y A Q(y)
$$

≡ \langle Definition of $\Phi$ \rangle

$$
\prod_u \prod_x A P(x) \times \prod_y A Q(y)
$$

∧ \langle Uniqueness principle of pairs \rangle

Now let us show that $\Phi \circ \Psi \sim \text{id} <$:

$$
\Phi \circ \Psi \sim \text{id} \prod_x A P(x) \times Q(x)
$$

≡ \langle Definition of $\sim$ \rangle

$$
\prod_g \prod_x A P(x) \times Q(x)
$$

≡ \langle Definition of $\Psi$ \rangle

$$
\prod_g \prod_x A P(x) \times Q(x)
$$

≡ \langle Definition of $\Phi$ \rangle

$$
\prod_g \prod_x A P(x) \times Q(x)
$$

∧ \langle Uniqueness principle of pairs \rangle

And now, we prove the $\Sigma$-term split rule:

In order to get a function

$$
\Phi : \sum_{x : A} (P(x) + Q(x)) \rightarrow (\sum_{x : A} P(x)) + (\sum_{x : A} Q(x))
$$

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Then we may define $\Psi : Σ$-consequent rule

\[
\sum_{x:A} (P(x) + Q(x)) \to \left( \sum_{x:A} P(x) \right) + \left( \sum_{x:A} Q(x) \right)
\]

\[
\cong \quad \langle \Delta ; \varphi_x := \kappa_x, \text{HEQ1} \rangle
\]

\[
\prod_{x:A} \left( (P(x) \to (\sum_{x:A} P(x)) + (\sum_{x:A} Q(x))) \times (Q(x) \to (\sum_{x:A} P(x)) + (\sum_{x:A} Q(x))) \right)
\]

\[
\cong \quad \langle \eta \equiv \eta(u,v) := \lambda x. (u(x), v(x)), \text{II-term split rule} \rangle
\]

\[
\prod_{x:A} (P(x) \to \sum_{x:A} P(x)) \times \prod_{x:A} (Q(x) \to \sum_{x:A} P(x) + \sum_{x:A} Q(x))
\]

\[
\cong \quad \langle φ_1 := \lambda x. \lambda y. \text{inl}(x,y); φ_2 := \lambda x. \lambda y. \text{inr}(x,y) \rangle
\]

\[
(φ_1, φ_2)
\]

In the chain above, $\kappa_x$ is the induction operator for $P(x) + Q(x)$ evaluated at the constant type family $D : (\sum_{x:A} P(x)) + (\sum_{x:A} Q(x))$. Then, we may define $Ψ : Σ(Δ(η(φ_1, φ_2)))$.

In order to get a function $Ψ : \sum_{x:A} P(x) + \sum_{x:A} Q(x) \to \sum_{x:A} P(x) + Q(x)$

let us consider the following deductive chain:

\[
(\sum_{x:A} P(x)) + (\sum_{x:A} Q(x)) \to \sum_{x:A} (P(x) + Q(x))
\]

\[
\cong \quad \langle \sigma_1 \times \sigma_2, \text{EQ} \times \text{Σ-consequent rule} \rangle
\]

\[
(\sum_{x:A} P(x)) \to \sum_{x:A} P(x) \times (\sum_{x:A} Q(x)) \to \sum_{x:A} (P(x) + Q(x))
\]

\[
\cong \quad \langle \psi_1 := \lambda x. (\lambda y. (x, \text{inl}(y))); \psi_2 := \lambda x. (\lambda z. (x, \text{inr}(z))) \rangle
\]

\[
(ψ_1, ψ_2)
\]

Then we may define $Ψ : Σ(σ_1 \times σ_2(ψ_1, ψ_2))$

Observe that

\[
Φ(Ψ(\text{inl}(a_1, a_2))) \equiv σ(Δ(η(φ_1, φ_2))(a_1, \text{inl}(a_2))
\]

\[
Φ(σ_1(ψ_1), σ_2(ψ_2))\text{inl}(a_1, a_2)) \equiv Δ(η(φ_1, φ_2))(a_1)(\text{inl}(a_2))
\]

\[
Φ(σ_1(ψ_1)(a_1, a_2)) \equiv κ_{a_1}(φ_1(a_1), φ_2(a_1))(\text{inl}(a_2))
\]

\[
Φ(ψ_1(a_1), a_2)) \equiv φ_1(a_1)(a_2) \equiv \text{inl}(a_1, a_2).
\]

\[
Φ(ψ_1, \text{inl}(a_2))
\]

In the same way, $Φ(Ψ(\text{inr}(b_1, b_2))) \equiv \text{inr}(b_1, b_2)$. Then

\[
\prod_{a:Σ_{x:A} P(x)} \prod_{b:Σ_{x:A} Q(x)} Φ(Ψ(p)) = p
\]

\[
\cong \quad \langle \kappa_x, \text{II-range split rule} \rangle
\]

\[
\prod_{a:Σ_{x:A} P(x)} Φ(Ψ(\text{inl}(a))) \equiv \text{inl}(a) \times \prod_{b:Σ_{x:A} Q(x)} Φ(Ψ(\text{inr}(b))) \equiv \text{inr}(b)
\]

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\begin{align*}
\simeq \quad & \langle \cdot \sigma_1 \times \sigma_2, \text{EQ}_{\text{inl}} \rangle \\
& \prod_{a_1 : A \times P_{\langle a_1 \rangle}} \Phi(\Psi(\text{inl}(a_1, a_2))) = \text{inl}(a_1, a_2) \\
& \times \prod_{b_1 : A \times Q_{\langle b_1 \rangle}} \Phi(\Psi(\text{inr}(b_1, b_2))) = \text{inr}(b_1, b_2) \\
\equiv \quad & \langle \text{Above computations} \rangle \\
& \prod_{a_1 : A \times P_{\langle a_1 \rangle}} \text{inl}(a_1, a_2) = \text{inl}(a_1, a_2) \\
& \times \prod_{b_1 : A \times Q_{\langle b_1 \rangle}} \text{inr}(b_1, b_2) = \text{inr}(b_1, b_2) \\
\uparrow \quad & \langle u := \lambda a_1, \lambda a_2. \text{refl}_{\text{inl}(a_1, a_2)} ; v := \lambda b_1, \lambda b_2. \text{refl}_{\text{inr}(b_1, b_2)} \rangle \\
(u, v) \\
\text{In the other direction,} \\
& \prod_{p : \sum_{x : A} P_{\langle x \rangle} + Q_{\langle x \rangle}} \Psi(\Phi(p)) = p \\
\simeq \quad & \langle \cdot \sigma, \Sigma\text{-consequent rule} \rangle \\
& \prod_{x : A \times P_{\langle x \rangle} + Q_{\langle x \rangle}} \Psi(\Phi(x, y)) = (x, y) \\
\equiv \quad & \langle \text{Definition of } \Phi \rangle \\
& \prod_{x : A \times P_{\langle x \rangle} + Q_{\langle x \rangle}} \Psi(\kappa(\phi_1(x), \phi_2(x))(y)) = (x, y) \\
\simeq \quad & \langle \cdot \Delta ; \varphi_2 := \kappa_2 ; \text{IEQ1} \rangle \\
& \prod_{x : A \times P_{\langle x \rangle} + Q_{\langle x \rangle}} \left( \prod_{w : P_{\langle x \rangle}} \Psi(\kappa(\phi_1(x), \phi_2(x))(\text{inl}(w))) = (x, \text{inl}(w)) \right) \\
& \times \prod_{z : Q_{\langle x \rangle}} \Psi(\kappa(\phi_1(x), \phi_2(x))(\text{inr}(z))) = (x, \text{inr}(z)) \\
\equiv \quad & \langle \text{Definition of } \kappa \rangle \\
& \prod_{x : A \times P_{\langle x \rangle} + Q_{\langle x \rangle}} \left( \prod_{w : P_{\langle x \rangle}} \Psi(\phi_1(x)(w)) = (x, \text{inl}(w)) \right) \\
& \times \prod_{z : Q_{\langle x \rangle}} \Psi(\phi_2(x)(z)) = (x, \text{inr}(z)) \rangle \\
\equiv \quad & \langle \text{Definition of } \phi_1 \text{ and } \phi_2 \rangle \\
& \prod_{x : A \times P_{\langle x \rangle} + Q_{\langle x \rangle}} \left( \prod_{w : P_{\langle x \rangle}} \Psi(\text{inl}(x, w)) = (x, \text{inl}(w)) \right) \\
& \times \prod_{z : Q_{\langle x \rangle}} \Psi(\text{inr}(x, z)) = (x, \text{inr}(z)) \rangle \\
\equiv \quad & \langle \text{Definition of } \Psi \rangle \\
& \prod_{x : A \times P_{\langle x \rangle} + Q_{\langle x \rangle}} \left( \prod_{w : P_{\langle x \rangle}} (x, \text{inl}(w)) = (x, \text{inl}(w)) \right) \\
& \times \prod_{z : Q_{\langle x \rangle}} (x, \text{inr}(z)) = (x, \text{inr}(z)) \rangle \\
\uparrow \quad & \langle u := \lambda x. (\lambda w. \text{refl}_{\langle x, \text{inl}(w) \rangle}, \lambda z. \text{refl}_{\langle x, \text{inr}(z) \rangle}) \rangle \\
(u, v) \\
\end{align*}

**Translation rules.** These rules correspond to the derived inference rules \(\Pi\text{Eq2} \) and \(\Sigma\text{Eq2} \) which were proved in subsection 7.2.

**Congruence rules.** These rules correspond to the derived inference rules \(\Pi\text{Eq1} \) and \(\Sigma\text{Eq1} \) stated and proved in subsection 7.2.
Antecedent rules. These rules correspond to equivalences in first order logic that allow introducing the antecedent of an implication into the term of a logical operational when the quantified variables do not occur free in this antecedent. For HoTT, we only have an equivalence for the case of Π-types. For Σ-types we have an equivalence only if the antecedent is a mere proposition. Namely,

\[(P \rightarrow \prod_{x : A} Q(x)) \simeq \prod_{x : A} (P \rightarrow Q(x)) \simeq \]

and

\[\sum_{x : A} (P \rightarrow Q(x)) \rightarrow (P \rightarrow \sum_{x : A} Q(x)) \simeq \]

If \(P \simeq 1 \simeq \)

The proof of Π-antecedent rule appears in section 10. We prove Σ-antecedent rule. Let us consider the following deductive chain.

\[\sum_{x : A} (P \rightarrow Q(x)) \rightarrow (P \rightarrow \sum_{x : A} Q(x)) \simeq \]

\[\langle \sigma, \Sigma\text{-consequent rule} \rangle \]

\[\prod_{x : A} ((P \rightarrow Q(x)) \rightarrow (P \rightarrow \sum_{x : A} Q(x))) \simeq \]

\[\langle h, (h(x)(u)(y) : (x,u(y)) \rangle \]

\[h \]

This proves the first part. Now, If \(P \simeq 1 \simeq \)

be defined by

\[\psi(u) : (\text{pr}_1(u(u)), \text{pr}_2u).\]

10 Inhabiting arrows

One of the tasks in homotopy type theory is to determine a formula for a function from type \(A\) to a type \(B\). We found that in several cases the structures of types \(A\) and \(B\) determine a natural matching of their objects defining a function from \(A\) to \(B\). We will call such a mapping a canonical function. An attempt to systematize this task is to precise the way in which we can get out of type \(A\) through its eliminators and the way in which we can get in type \(B\) through its constructors. To do so, we define the exit door and the entry door of a type. Of course, there will be types \(A\) and \(B\) for which there is no canonical function. The entry door of a type is a \(\lambda\)-expression that represents a constructed object of the type, i.e., an object of the type obtained from its constructors. The exit door of a type is a \(\lambda\)-expression that represents an eliminated object of the type, i.e., an object of the type constructed from the elimination of a generic object. For instance, the entry door of the type \(\sum_{x : A} C(x)\) is the \(\lambda\)-expression

\[(u_1 : A, u_2 : C(u_1))\]
because a constructed object of the type is a dependent pair of objects \( u_1 \) of type \( A \) and \( u_2 \) of type \( C(u_1) \). Then, we write

\[
\sum_{x : A} C(x)
\]

\[
\uparrow \quad \langle \text{entry door} \rangle
\]

\[
( u_1 : A, u_2 : C(u_1) )
\]

The exit door of this type is the \( \lambda \)-expression

\[
( \text{pr}_1(u) : A, \text{pr}_2(u) : C(\text{pr}_1(u)) )
\]

because it is the dependent pair constructed from the elimination of a generic object \( u \) of type \( \sum_{x : A} C(x) \) through their projections. We write

\[
( \text{pr}_1(u) : A, \text{pr}_2(u) : C(\text{pr}_1(u)) )
\]

\[
\downarrow \quad \langle \text{exit door} \rangle
\]

\[
\sum_{x : A} C(x).
\]

The doors of a type can be used to determine a formula for a canonical function from a type to another, by matching the exit door of the source type with the entry door of the destination type. For instance, let us determine a function from \( \sum_{x : A} C(x) \) to itself. This means that we have to determine an object \( \Phi \) in the following link

\[
\sum_{x : A} C(x)
\]

\[
\leftarrow \quad \langle : \Phi \rangle
\]

\[
\sum_{x : A} C(x),
\]

i.e. we have to match the exit door \(( \text{pr}_1(u) : A, \text{pr}_2(u) : C(\text{pr}_1(u)) )\) and the entry door \((\Phi(u)_1 : A, \Phi(u)_2 : C(\Phi(u)_1))\) of the type \( \sum_{x : A} C(x) \), task that we represent with the following matching diagram

\[
\sum_{x : A} C(x)
\]

\[
\uparrow \quad \langle \text{entry door} \rangle
\]

\[
(\Phi(u)_1 : A, \Phi(u)_2 : C(\Phi(u)_1))
\]

\[
\leftarrow \quad \langle \text{Looked for definition} \rangle
\]

\[
(\text{pr}_1(u) : A, \text{pr}_2(u) : C(\text{pr}_1(u)))
\]

\[
\downarrow \quad \langle \text{exit door} \rangle
\]

\[
\sum_{x : A} C(x),
\]

where \( \leftarrow \) means that some sort of symbolic matching between two expressions must be discovered. By matching the doors we get

\[
\Phi(u) \equiv (\text{pr}_1(u), \text{pr}_2(u)).
\]

Observe that the canonical function in this case is not the identity function.
Let us determine the canonical function $\Phi$ from $\prod_{x:A} B(x)$ to itself. The corresponding matching diagram is

\[
\begin{array}{c}
\prod_{x:A} B(x) \\
\uparrow \text{(entry door)} \\
\lambda(x:A).\langle \Phi(f):B \rangle \\
\leftarrow \langle ? \rangle \\
\lambda(x:A).\langle f(x):B(x) \rangle \\
\downarrow \text{(exit door)} \\
\prod_{x:A} B(x).
\end{array}
\]

Therefore, by matching, we get

\[
\Phi(f)(x) :\equiv f(x).
\]

which, by uniqueness, is the identity function.

We now present some examples illustrating this technique.

**Π-distribution over arrows.** As we mention in section 5, we show how to obtain the canonical function $\Phi :\equiv \lambda u.\Phi(u)$ of the type

\[
\prod_{x:A} (P(x) \to Q(x)) \to (\prod_{x:A} P(x) \to \prod_{x:A} Q(x)).
\]

For that, the corresponding entrance and exit doors are made to coincide

\[
\begin{array}{c}
\prod_{x:A} P(x) \to \prod_{x:A} Q(x) \\
\uparrow \text{(entry door)} \\
\lambda(z: \prod_{x:A} P(x)).\lambda(x:A).\Phi(u)(z)(x) \\
\leftarrow \langle ? \rangle \\
\lambda(x:A).\lambda(y:P(x)).u(x)(y) \\
\downarrow \text{(exit door)} \\
\prod_{x:A} (P(x) \to Q(x))
\end{array}
\]

obtaining

\[
\Phi(u)(z)(x) :\equiv u(x)(z(x)).
\]

**Π-antecedent rule.** In order to prove that

\[
(P \to \prod_{x:A} Q(x)) \simeq \prod_{x:A} (P \to Q(x)) <:
\]

we have to determine a 4-tuple $(\Phi, \Phi', \alpha, \alpha')$ inhabiting the equivalence type. Consider
the following entry-exit door arguments:

\[
P \to \prod_{x:A} Q(x)
\]

\[
\xrightarrow{\text{entry door}} \lambda(y:P).\lambda(x:A). \Phi(u)(y)(x) : Q(x)
\]

\[
\xleftarrow{\text{exit door}} \lambda(x:A). \lambda(y:P). (u(x)(y) : Q(x))
\]

and

\[
\prod_{x:A} (P \to Q(x))
\]

\[
\xrightarrow{\text{entry door}} \lambda(x:A). \lambda(y:P). \Phi'(v)(x)(y) : Q(x)
\]

\[
\xleftarrow{\text{exit door}} \lambda(y:P). \lambda(x:A). (v(y)(x) : Q(x))
\]

Observe that, by definition of \(\Phi\) and \(\Phi'\),

\[
\Phi'(\Phi(u))(y)(x) \equiv \Phi(u)(y)(x) \equiv u(x)(y)
\]

and

\[
\Phi(\Phi'(v))(y)(x) \equiv \Phi'(v)(x)(y) \equiv v(y)(x).
\]

This shows that \(\Phi'\) and \(\Phi\) are each other inverses, and then, that \(\Phi' \circ \Phi \sim \text{id} <:\) and \(\Phi \circ \Phi' \sim \text{id} <:\).

11 Conclusions

Our goal of rephrasing HoTT giving homotopic equality and equivalence a preeminent role was achieved. Based on the algebraic identities and inference rules, expressing equalities and homotopic equivalences, provided by this restatement, we were able, using appropriate syntax and notations, to perform legible formal proofs.

We hope to have helped demythify the wide belief that formal proofs are messy and very long to be readable and performable, in a practical way, by humans. We also wonder if a higher order version of intuitionistic logic could be developed in order to extend the Curry-Howard isomorphism to cover HoTT.

Finally, we expect that our research will motivate exploring the proof theory associated to calculational methods of proof. We also think that it would be worthwhile to develop proof assistants and verifiers to support the automation of these methods.
References

[1] E. Acosta, B. Aldana, J. Bohórquez, and C. Rocha. Axiomatic set theory à la Dijkstra and Scholten. In A. Solano and H. Ordoñez, editors, Advances in Computing, pages 775–791, Cham, 2017. Springer International Publishing.

[2] R. Backhouse. Program Construction: Calculating Implementations from Specifications. John Wiley and Sons, Inc., 2003.

[3] H. Barendregt and E. Barendsen. Autarkic computations in formal proofs. J. Automated Reasoning, 28(3):321–336, 2002.

[4] J. Bohórquez and C. Rocha. Towards the effective use of formal logic in the teaching of discrete math. 6th International Conference on Information Technology Based Higher Education and Training, ITHET., 2005.

[5] J. A. Bohórquez. Intuitionistic logic according to Dijkstra’s calculus of equational deduction. Notre Dame J. Form. Log., 49(4):361–384, 2008.

[6] J. A. Bohorquez. Calculational solutions to combinatorial problems. 10th Computing Colombian Conference (10CCC), 2015.

[7] E. W. Dijkstra. How computing science created a new mathematical style. EWD 1073 in The writings of Edsger W. Dijkstra, 2000. URL http://www.cs.utexas.edu/users/EWD, 1994.

[8] E. W. Dijkstra and C. S. Scholten. Predicate Calculus and Program Semantics. Springer Verlag, 1990.

[9] W. H. J. Feijen and A. J. M. van Gasteren. On a method of multiprogramming. Springer-Verlag New York, Inc., New York, NY, USA, 1999.

[10] D. Gries. Foundations for calculational logic. In M. Broy and B. Schieder, editors, Mathematical Methods in Program Development, pages 83–126, Berlin, Heidelberg, 1997. Springer Berlin Heidelberg.

[11] D. Gries and F. B. Schneider. A Logical Approach to Discrete Math. Texts and Monographs in Computer Science. Springer Verlag, 1993.

[12] J. Misra. A Discipline of Multiprogramming: Programming Theory for Distributed Applications. Monographs in Computer Science. Springer-Verlag, New York, 2001.

[13] M. H. Sørensen and P. Urzyczyn. Lectures on the Curry-Howard Isomorphism. Elsevier, San Diego, CA, 2006.

[14] T. Univalent Foundations Program. Homotopy Type Theory: Univalent Foundations of Mathematics URL https://homotopytypetheory.org/book. Institute for Advanced Study, 2013.