LINEAR TROPICALIZATIONS

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Abstract. Let $X$ be a closed algebraic subset of $\mathbb{A}^n(K)$ where $K$ is an algebraically closed field complete with respect to a nontrivial non-Archimedean valuation. We show that there is a surjective continuous map from the Berkovich space of $X$ to an inverse limit of a certain family of embeddings of $X$ called linear tropicalizations of $X$. This map is shown to be injective on a subset of the Berkovich space which in particular contains all seminorms arising from closed points of $X$. We demonstrate some applications of this result to transversal intersections. In particular we prove that there exists a tropical line arrangement which is realizable by a complex line arrangement but not realizable by any real line arrangement.

1. Introduction

Suppose that $K$ is an algebraically closed field complete with respect to a nontrivial non-Archimedean valuation $\nu : K \to \mathbb{R} \cup \{-\infty\}$. The tropicalization of a closed algebraic subset $X$ of $K^n$ is the image $\nu(X)$ where $\nu$ stands for the coordinate-wise valuation map into $(\mathbb{R} \cup \{-\infty\})^n$. Studying tropicalizations of varieties proved to be useful for answering several questions about classical algebraic geometry. In order to see various aspects of this connection one can consult [8, 9, 10, 11].

The tropicalization construction is extrinsic which means that it depends on the particular embedding of $X$ in $K^n$ and not just on $X$ as an abstract variety. In [13], Payne considers all embeddings of $X$ into affine spaces (and later into toric varieties) and forms an inverse system of the resulting tropicalizations in the category of topological spaces (also see [6]). Then the inverse limit of all such tropicalizations can be regarded as an intrinsic tropicalization of $X$. It is proven in the same paper that this intrinsic tropicalization is homeomorphic to $X^{an}$, which is the analytification of the variety $X$ in the sense of Berkovich.

In this paper we consider a smaller subset of embeddings whose components are linear combinations of the coordinate functions of the original embedding. One advantage of working with such embeddings is that they preserve degree. The composition of a linear embedding with the valuation will be called a linear tropicalization. One of the main results of this paper is given in Theorem 3.1 which states that there is a surjective map from $X^{an}$ to the inverse limit of all linear tropicalizations which is also injective.

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on a certain subset of \( X^{\text{an}} \). All seminorms arising from the closed points of \( X \) belong to this subset.

The last section of the paper is devoted to applications of Theorem 3.1 to transversal intersections. In particular, Theorem 4.2 states that if \( X \) and \( Y \) are two varieties in \( \mathbb{A}^n \) intersecting at a set of \( m \) reduced points, then there exists a linear tropicalization \( \text{Trop}_i \) such that \( \text{Trop}_i(X) \cap \text{Trop}_i(Y) = \text{Trop}_i(X \cap Y) \). Determining tropicalizations of a variety having this property has been investigated by several authors, see [12, 14, 15]. Remark 4.3 explains that the tropical Bezout’s theorem for curves can be used to give a proof of the classical Bezout’s theorem. Corollary 4.4 shows that any line arrangement in \( \mathbb{CP}^2 \) admits a linear tropicalization such that along with the linearity and incidence relations, the transversality of the intersections are preserved. In Corollary 4.5, we show that there exists a tropical line arrangement that can be realized by a complex line arrangement but not by any real line arrangement.

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2. Linear Tropicalizations

Assume that \( K \) is an algebraically closed field complete with respect to a non-Archimedean valuation \( \nu \). Let us equip \( T = \mathbb{R} \cup \{-\infty\} \) with the topology extending the standard one on \( \mathbb{R} \) such that open neighborhoods of \( -\infty \) are semi-infinite open intervals and extend \( \nu \) to a map from \( K \) to \( T \) by setting \( \nu(0) = -\infty \). We will identify \( \mathbb{A}^n \) with \( K^n \) and by abuse of notation set \( \nu : \mathbb{A}^n \to T^n \) to be the map given by coordinate-wise valuation. Let \( X \) be a closed algebraic subset of \( \mathbb{A}^n \) and \( z_1, \ldots, z_n \) coordinate functions on \( \mathbb{A}^n \).

Let \( X^{\text{an}} \) denote the Berkovich analytic space of \( X \). Then \( X^{\text{an}} \), as a set, contains the multiplicative seminorms on the ring of regular functions \( K[X] \) of \( X \) compatible with \( \nu \) (a multiplicative seminorm \( [\cdot]_x \) on \( K[X] \) is compatible with \( \nu \) if \( [a]_x = \exp(\nu(a)) \) for any \( a \in K \)). For each closed point \( x \in X \) one has a multiplicative seminorm \( [f]_x = |f(x)|_x \). Denote the set of all such seminorms coming from closed points of \( X \) by \( X^{\text{cl}} \). One has \( X^{\text{cl}} \subsetneq X^{\text{an}} \). Indeed one major reason for studying \( X^{\text{an}} \) is that \( X^{\text{cl}} \) is totally disconnected with respect to weak topology whereas \( X^{\text{an}} \) is not and retains important attributes of \( X \) [2, 3, 17].

Let \( N \) be a positive integer. Pick an \( N \)-tuple \( f_1, \ldots, f_N \) of elements of \( K[z_1, \ldots, z_n] \). Using this \( N \)-tuple one can define a morphism \( i : X \to \mathbb{A}^N \) by setting \( i(x) = (f_1(x), \ldots, f_N(x)) \). For each such morphism, \( \nu \circ i : X \to T^N \) gives a tropicalization of \( X \) denoted by \( \text{Trop}_i(X) \). A single tropicalization evidently depends on the particular choice of the morphism, hence is extrinsic. In order to obtain an intrinsic tropicalization of \( X \), one can consider the family of “all” possible morphisms as follows:

First, suppose that \( \varphi : \mathbb{A}^N \to \mathbb{A}^M \) is a morphism such that \( \varphi \) restricts to a homomorphism of algebraic groups \( (K^*)^N \to (K^*)^M \). Important examples
are projections to any subset of coordinates of $\mathbb{A}^N$. Say $i_N : X \rightarrow \mathbb{A}^N$ and $i_M : X \rightarrow \mathbb{A}^M$ are two morphisms as above. We say that $\varphi$ is equivariant with respect to $(i_N, i_M)$ if $\varphi \circ i_N = i_M$. It is straightforward to see that if $\varphi$ is equivariant then it induces a linear map $\text{Trop}(\varphi) : \mathbb{T}^N \rightarrow \mathbb{T}^M$ restricting to $\text{Trop}_{i_N}(X) \rightarrow \text{Trop}_{i_M}(X)$. If $\varphi$ is a projection to a subset of coordinates of $\mathbb{A}^N$, then $\text{Trop}(\varphi)$ is the projection to the same subset of coordinates of $\mathbb{T}^N$.

Now consider the family $\mathcal{F}$ of all $i$ as above such that $i : X \rightarrow \mathbb{A}^N$ is an isomorphism onto its image, namely the family of all embeddings among such morphisms. Given two embeddings $i_N : X \rightarrow \mathbb{A}^N$ and $i_M : X \rightarrow \mathbb{A}^M$, set $i_N \times i_M : X \rightarrow \mathbb{A}^{N+M}$ to be the map given by $(i_N \times i_M)(x) = (i_N(x), i_M(x))$. Then $i_N \times i_M$ also belongs to $\mathcal{F}$. Furthermore, if $\varphi_1 : \mathbb{A}^{N+M} \rightarrow \mathbb{A}^N$ and $\varphi_2 : \mathbb{A}^{N+M} \rightarrow \mathbb{A}^M$ are projections to the first $N$ and last $M$ coordinates respectively, then $\varphi_1$ is equivariant with respect to $(i_N \times i_M, i_N)$ and $\varphi_2$ is equivariant with respect to $(i_N \times i_M, i_M)$.

Let us now consider the partial order defined on $\mathcal{F}$ such that $i$ dominates $j$ if there exists an equivariant morphism $\varphi$ with respect to $(i, j)$. Since any two embeddings in $\mathcal{F}$ are dominated by an element of $\mathcal{F}$, one can consider the inverse limit of tropicalizations of $X$ with respect to this partial order. Set

$$\text{Trop}(X) = \lim_{i \in \mathcal{F}} \text{Trop}_i(X)$$

where the inverse limit is in the category of topological spaces. Then $\text{Trop}(X)$ can be viewed as an intrinsic tropicalization of $X$. In [13], Payne proved that $\text{Trop}(X)$ is homeomorphic to the Berkovich space $X^{an}$ of $X$. More specifically, given $f_1, \ldots, f_N \in K[z_1, \ldots, z_n]$, let $\pi_i : X^{an} \rightarrow \mathbb{T}^N$ be the continuous map given by $x \mapsto (\log|f_1|_x, \ldots, \log|f_N|_x)$. Set

$$\pi(x) = \lim_{i \in \mathcal{F}} \pi_i(x).$$

Then, Payne’s theorem asserts that $\pi : X^{an} \rightarrow \text{Trop}(X)$ is a homeomorphism.

We would like to consider an inverse limit over a smaller subset of isomorphisms, namely the ones preserving degree.

**Definition 2.1.** Suppose that $f_1, \ldots, f_N$ are linear combinations of $z_1, \ldots, z_n$ and 1 in $K[z_1, \ldots, z_n]$. We say that the morphism $i : X \rightarrow \mathbb{A}^N$ given by $i(x) = (f_1(x), \ldots, f_N(x))$ is linear.

Let $\mathcal{G}$ be the set of all linear embeddings $i : X \rightarrow \mathbb{A}^N$ for an arbitrary value of $N$. We will say that $\text{Trop}_i(X)$ is a linear tropicalization of $X$. If $i_N$ and $i_M$ are two linear embeddings then it is clear that $i_N \times i_M$ is a linear embedding. Put a partial order on $\mathcal{G}$ such that $i$ dominates $j$ if there exists an equivariant morphism $\varphi$ with respect to $(i, j)$ which is a composition of a linear embedding and a projection. Then we can talk about the inverse limit of tropicalizations of $X$ over the family $\mathcal{G}$. We shall define an auxiliary subset of $X^{an}$ that will appear in the main theorem of the next section.
Definition 2.2. Let $X^l$ be the subset of $X^\text{an}$ consisting of multiplicative seminorms $[\cdot]_x$ such that for every $i$ there exists a linear polynomial $h = a_1z_1 + \ldots + a_{i-1}z_{i-1} + z_i + b$ and $[h]_x = 0$.

It is clear that $X^{cl} \subset X^l$ since for any closed point $x = (x_1, \ldots, x_n)$ one has $[z_i - x_i]_x = 0$.

Remark 2.3. In general $X^l \not\subset X^\text{an}$. For instance, type 2 points of the Berkovich space of $\mathbb{P}^1$ are norms, hence their kernel is zero [2, Section 1.2]. It is unclear to the authors whether $X^{cl} = X^l$ for an arbitrary variety $X$ or not.

3. Inverse Limit of All Linear Tropicalizations

Theorem 3.1. Let $X$ be a closed algebraic subset of $\mathbb{A}^n$.
Let $\pi = \lim_{\longleftarrow i \in G} \pi_i : X^\text{an} \to \lim_{\longleftarrow i \in G} \text{Trop}_i(X)$. Then
(i) $\pi$ is surjective.
(ii) $\pi$ restricted to $X^l$ is injective.

Proof. (i) Since the map $\lim_{\longleftarrow i \in F} \pi_i$ is surjective by Theorem 1.1 in [13] and $G$ is a sub-poset of $F$, we deduce that $\lim_{\longleftarrow i \in G} \pi_i$ is surjective.
(ii) The remainder of this section will be devoted to the proof of the injectivity of $\pi$ restricted to $X^l$. We need some preliminary results.
Let $K$ be an algebraically closed field complete with respect to a nontrivial nonarchimedean valuation $\nu$ and $|a| = \text{exp}(\nu(a))$. Define completions of the polynomial algebra $K[x_1, \ldots, x_n]$ with respect to two different norms. Let us use multi-index notation throughout: $\underline{x} = (x_1, x_2, \ldots, x_n), \underline{r} = (r_1, \ldots, r_n) \in (\mathbb{R}^+)^n, i = (i_1, \ldots, i_n) \in \mathbb{N}^n, |i| = i_1 + \ldots + i_n$ and $\underline{x}^i = x_1^{i_1}x_2^{i_2} \ldots x_n^{i_n}$ etc. Suppose that $f = \sum_{i \in \mathbb{N}^n} a_i \underline{x}^i$ with $a_i \in K$. For a fixed $\underline{x}$, set
\[ ||f|| = \sum_{i \in \mathbb{N}^n} |a_i|\underline{x}^i \]
Denote the completion of the polynomial ring with respect to the norm $||\cdot||$ by $K<\underline{x}^{-1}>$. Next, define
\[ |f| = \sup_{i \in \mathbb{N}^n} |a_i|\underline{x}^i \]
and denote the completion of the polynomial ring with respect to the norm $|\cdot|$ by $T_n = K\{\underline{x}^{-1}\}$. Both $K<\underline{x}^{-1}>$ and $K\{\underline{x}^{-1}\}$ are Banach algebras. Since in any Cauchy sequence of polynomials in $K\{\underline{x}^{-1}\}$ norms of differences must tend to 0, we see that $|a_i|\underline{x}^i \to 0$ as $|i| \to \infty$. Therefore $T_n = K\{\underline{x}^{-1}\}$ coincides with the Tate algebra [4, Section 5.1.1] and the supremum norm for any given $f$ is attained by at least by one monomial of $f$. 
Let $\mathcal{A}$ be a Banach algebra over $K$. Let $\mathcal{M}(\mathcal{A})$ denote the spectrum of $\mathcal{A}$, namely the set of all bounded multiplicative seminorms on $\mathcal{A}$ equipped with the weakest topology for which evaluation maps on members of $\mathcal{A}$ is continuous [3, Section 1.2]. Suppose that $A^n$ denotes the affine $n$-space over $K$. Then by [3, Remark 1.5.2], the affine space is a union of polydiscs:

$$(A^n)^{an} = \bigcup_{r} \mathcal{M}(K < r^{-1}x>).$$

**Lemma 3.2.** The Banach algebra $K < r^{-1}x>$ in the equality above can be replaced by $K\{r^{-1}x\}$. Namely,

$$(A^n)^{an} = \bigcup_{r} \mathcal{M}(K\{r^{-1}x\}).$$

**Proof.** Let $\mathcal{A} = K < r^{-1}x>$ and $\rho$ be the spectral seminorm on $\mathcal{A}$ defined by

$$\rho(f) = \lim_{N \to \infty} \sqrt[N]{||f^N||}.$$ 

Then $\rho$ is a bounded, power multiplicative seminorm on $\mathcal{A}$ characterized by being maximal with respect to these properties [17, Section 2.1.2]. The completion of $\mathcal{A}$ with respect to $\rho$ will be denoted by $\mathcal{A}^u$. Since completion gets rid of quasi-nilpotent elements, this operation does not change the maximal spectrum, hence $\mathcal{M}(\mathcal{A}) \cong \mathcal{M}(\mathcal{A}^u)$ (see [3, p.16]). Therefore it will suffice to show that $\mathcal{A}^u = K\{r^{-1}x\}$. First, let us show that $||$ is power multiplicative. Let $I \subset \mathbb{N}^n$ be the set of all indices $i$ such that $|f| = |a_i|^j$. Then $I$ is not empty since $|a_i|^j \to 0$, and the convex hull of $I$ in $\mathbb{R}^n$ has at least one vertex corresponding to some $j \in I$. Then in $f^N$, the norm of the coefficient of $x^j$ is precisely $|a_j|^N$. Hence $|f^N| = |a_j|^N x^j = |f|^N$. Thus $||$ is power multiplicative and consequently $|f| \leq \rho(f)$ for all $f$.

It suffices to prove the converse inequality for polynomials. Say $f = \sum_{|i| \leq d} a_i x^i$. Then $|a_i| \leq r^{-j}|f|$ for all $|i| \leq d$. Therefore, $||f|| \leq (d+1)^n|f|$ and consequently $||f^N|| \leq (Nd+1)^n|f|^N$. It follows that

$$\rho(f) \leq \lim_{N \to \infty} \sqrt[(Nd+1)^n]{|f|^N} = |f|.$$ 

Therefore the second inequality is justified. Hence $\rho(f) = |f|$ and the claim follows. $\square$

Let us now resume the proof of injectivity of $\lim \pi_i$. Suppose that $y, y' \in X^l$ and $\lim_{I \in G} \pi_i(y) = \lim_{I \in G} \pi_i(y')$. Notice that $K[X] = K[x_1, \ldots, x_n]/I(X)$ where $I(X)$ is the ideal of $X$. Let $f = a_1 x_1 + \ldots + a_n x_n + b$ be an affine linear function with $a_i, b \in K$. We will first show that $[f]_y = [f]_{y'}$ for any such $f$. Choose any generating set $f_1, \ldots, f_m$ for $K[X]$ with $f_1 = f$ and $f_j$ affine.
linear for all $j$. Then the induced embedding $i$ of $X$ is linear, hence belongs to $G$. Then $\pi_i(y) = \pi_i(y')$ by hypothesis. By looking at the first coordinates we see that $[f]_y = [f]_{y'}$.

It remains to show that $[g]_y = [g]_{y'}$ for any $g$ in the coordinate ring $K[X]$. We have $X^{an} \subset (\mathbb{A}^n)^{an}$ and by Lemma 3.2, $(\mathbb{A}^n)^{an} = \bigcup_r M(K\{r^{-1} z\})$. So, we may assume that there exists an $r$ such that $y, y' \in M(K\{r^{-1} z\})$. Thus it suffices to prove the claim for $y, y'$ belonging to this polydisc and $g \in T_n = K\{r^{-1} z\}$, assuming that it holds for all affine linear polynomials. The power series $g = \sum_{i \in \mathbb{N}^n} a_i x^i$ is said to be $x_n$-distinguished of degree $s$ if

$$g = \sum_{j=0}^{\infty} g_j(x_1, \ldots, x_{n-1}) x_n^j$$

with $g_j \in T_{n-1}$, $g_s$ a unit, $|g| = |g_s|$ and $|g_s| > |g_j|$ for all $j > s$ [4, Section 5.2.1]. It is clear that we can assume $g$ is $x_n$-distinguished for some degree $s$ if necessary by applying an affine linear transformation to coordinates, therefore we will assume so.

**Theorem 3.3.** (Weierstrass Preparation Theorem, [4, Theorem 5.2.2]) Let $g \in T_n$ be $x_n$-distinguished of degree $s$. Then $g$ can be expressed uniquely as

$$g = vp$$

where $v$ is an invertible power series in $T_n$ and $p = x_n^d + \sum_{i=0}^{d-1} a_i x_n^i \in T_{n-1}[x_n]$.

The following Lemma is a generalization of the argument in [2, Section 1.2]:

**Lemma 3.4.** Suppose that $v$ is an invertible power series in $T_n$. Then there exists $c \in K$ such that $v = cu$, where $u$ is an invertible power series in $T_n$ satisfying $[u]_y = 1$ for all $y \in M(T_n)$.

**Proof.** Say $v = \sum_{i \in \mathbb{N}^n} b_i x^i$ where $b_i \in K$, $\lim_{|i| \to \infty} |b_i| = 0$ and $v(0, \ldots, 0) \neq 0$. If we set $c = \max_{i \in \mathbb{N}^n} |b_i|$ then $v = c(\sum_{i \in \mathbb{N}^n} a_i x^i)$ with $|a_i| \leq 1$ for all $i$ and $|a_i| = 1$ for at least one $i$. Let $u = \sum_{i \in \mathbb{N}^n} a_i x^i$. Since $u$ is an invertible power series, it has a multiplicative inverse, say $u^{-1}$. Since all seminorms in $M(T_n)$ are bounded, we have $[u]_y \leq |u|$ and $|[u^{-1}]_y \leq |u^{-1}|$. By definition, $|u| = \max_{i \in \mathbb{N}^n} |a_i| = 1$ and $|[u^{-1}]_y[|u^{-1}]_y = [u \cdot u^{-1}]_y = 1$. The analogous equality also holds for the norm, $|u||u^{-1}| = |u \cdot u^{-1}| = 1$. Therefore $[u]_y = 1$ for all $y \in M(T_n)$. \hfill \Box

**Remark 3.5.** Note that the constant $c$ in the lemma above is independent of the seminorm $y \in M(T_n)$.

We will now proceed by induction on $n$. First let $g \in T_1$. By Theorem 3.3 and Lemma 3.4 $g = cu p$ where $[u]_y = [u]_{y'} = 1$, $p$ is a monic polynomial
of degree \(d\) in \(x_1\), and \(c\) is a constant. Since \(K\) is algebraically closed by assumption, we can write \(p = \prod_{j=1}^{d}(x_1 - b_j)\). Each factor \(x_1 - b_j\) is affine linear, therefore \([x_1 - b_j]_y = [x_1 - b_j]_{y'}\). It follows that \([g]_y = [g]_{y'}\).

Suppose now that the seminorms \([\cdot]_y\) and \([\cdot]_{y'}\) agree on all power series with respect to \(n - 1\) variables or less and say \(g \in T_n\). By Theorem 3.3 and Lemma 3.4 \(g = \cup\) where \(p = x_n^d + \sum_{i=0}^{d-1} a_i x_n^i\), \(u\) an invertible power series such that \([u]_y = [u]_{y'} = 1\) and \(a_i \in T_{n-1}\). We want to show that \([p]_y = [p]_{y'}\). Since \(y \in X^1\), there exists a linear polynomial \(h = x_n - k(x_1, \ldots, x_{n-1})\) such that \([h]_y = 0\). But \(y\) and \(y'\) agree on linear polynomials, so \([h]_{y'} = 0\) as well.

Now, since \([h^d]_y = [h^d]_{y'} = 0\),
\[
[p - h^d]_y = [p]_y \quad \text{and} \quad [p - h^d]_{y'} = [p]_{y'}
\]
But in \(p - h^d\) the variable \(x_n\) appears to \(d - 1\)'st power or less. Inductively, one can find \(b_i \in T_{n-1}\) such that the series
\[
p - h^d - b_{d-1}h^{d-1} - \ldots - b_1 h
\]
does not contain \(x_n\). So we strip off all the powers of \(x_n\) from \(p\). But we still have
\[
[p - h^d - b_{d-1}h^{d-1} - \ldots - b_1 h]_y = [p]_y
\]
and
\[
[p - h^d - b_{d-1}h^{d-1} - \ldots - b_1 h]_{y'} = [p]_{y'}
\]
since \([b_i h^i]_y = [b_i h^i]_{y'} = 0\). By the induction hypothesis saying that \([\cdot]_y\) and \([\cdot]_{y'}\) agree on \(T_{n-1}\), we must have \([p]_y = [p]_{y'}\). This finishes the proof of Theorem 3.1.

\[\square\]

**Remark 3.6.** It is unclear to the authors whether \(\pi\) is injective on all of \(X^{\text{an}}\) or not.

### 4. Transversal Intersections

**Lemma 4.1.** Say \(X, Y \subset \mathbb{A}^n\) are closed algebraic subsets intersecting transversally. Then

(a) \((X \cap Y)^{\text{an}} = X^{\text{an}} \cap Y^{\text{an}}\).

(b) \((X \cap Y)^{\text{cl}} = X^{\text{cl}} \cap Y^{\text{cl}}\).

(c) \((X \cap Y)^{l} = X^{l} \cap Y^{l} = X^{\text{an}} \cap Y^{l} = X^{l} \cap Y^{\text{an}}\).

**Proof.** Say \(I(X)\) and \(I(Y)\) are ideals of \(X\) and \(Y\) respectively, so that \(K[X] = K[z_1, \ldots, z_n]/I(X)\) and \(K[Y] = K[z_1, \ldots, z_n]/I(Y)\). Since \(X\) and \(Y\) are intersecting transversally, \(\text{rad}(I(X) + I(Y)) = I(X) + I(Y)\). Therefore \(K[X \cap Y] = K[z_1, \ldots, z_n]/(I(X) + I(Y))\).

(a) Suppose that \([\cdot]_x \in (X \cap Y)^{\text{an}}\). Then \([\cdot]_x\) is a multiplicative seminorm on \(K[z_1, \ldots, z_n]\) vanishing on \(I(X) + I(Y)\). Therefore \([\cdot]_x\) vanishes both on
\(I(X)\) and \(I(Y)\). This implies that \([\ ]_x\) is a multiplicative seminorm on \(K[X]\) and \(K[Y]\), hence \((X \cap Y)^{an} \subset X^{an} \cap Y^{an}\).

Conversely, say \([\ ]_x \in X^{an} \cap Y^{an}\). Then \([\ ]_x\) vanishes both on \(I(X)\) and \(I(Y)\). But the kernel of a seminorm is an ideal, therefore \([\ ]_x\) vanishes on \(I(X) + I(Y)\). So \(X^{an} \cap Y^{an} \subset (X \cap Y)^{an}\) and we get the desired equality.

(b) This is clear.

(c) Say \([\ ]_x \in X^l \cap Y^{an}\). Then \([\ ]_x \in X^{an} \cap Y^{an}\) by part (a). Since \([\ ]_x \in X^l\), for every \(i\) there exists \(h = a_1 z_1 + \ldots + a_{i-1} z_{i-1} + z_i + b\) such that \([h]_x = 0\). Therefore \([\ ]_x \in (X \cap Y)^l\). Therefore \(X^l \cap Y^{an} \subset (X \cap Y)^l\).

Similarly, \(X^{an} \cap Y^l \subset (X \cap Y)^l\). The inclusions \(X^l \cap Y^l \subset X^{an} \cap Y^{an}\) and \(X^l \cap Y^l \subset X^l \cap Y^{an}\) are clear.

Conversely, suppose now that \([\ ]_x \in (X \cap Y)^l\). Then \([\ ]_x \in (X \cap Y)^{an} = X^{an} \cap Y^{an}\) by part (a). Again, for every \(i\) there exists \(h = a_1 z_1 + \ldots + a_{i-1} z_{i-1} + z_i + b\) such that \([h]_x = 0\). Therefore \([\ ]_x \in X^l\) and \([\ ]_x \in Y^l\). Hence \((X \cap Y)^l \subset X^l \cap Y^l\). The equalities in the statement follow. \(\square\)

If \(X\) consists of a single reduced point, then \(K[X] = K\), therefore \(X^{an}\) consists of a single multiplicative seminorm, namely the one coming from the valuation on \(K\). Similarly, if \(X\) consists in \(m\) distinct reduced points, then \(X^{an}\) has \(m\) elements and \(X^l = X^l = X^{an}\) in this case.

**Theorem 4.2.** Suppose that \(X, Y \subset \mathbb{A}^n\) where \(X\) and \(Y\) intersect at a set of \(m\) reduced points. Then there exists a linear tropicalization \(\text{Trop}_i\) of \(X \cup Y\) such that \(\text{Trop}_i(X) \cap \text{Trop}_i(Y)\) consists of \(m\) distinct points and \(\text{Trop}_i(X) \cap \text{Trop}_i(Y) = \text{Trop}_i(X \cap Y)\).

**Proof.** Since \(X \cap Y\) consists of \(m\) distinct reduced points, \((X \cap Y)^{an} = (X \cap Y)^l\) is a set of \(m\) points. By Theorem 3.1 parts (a) and (b), \(\pi = \lim_{i \in I} \pi_i\) : \((X \cap Y)^{an} \to \lim_{i \in I} \text{Trop}_i(X \cap Y)\) is both surjective and injective, therefore there exists a linear tropicalization \(\text{Trop}_i\) of \(X \cap Y\) that has exactly \(m\) points. Note that for every \(j\) dominating \(i\) in the inverse system, \(\text{Trop}_j(X \cap Y)\) will also have \(m\) points.

The inclusion \(\text{Trop}(X \cap Y) \subset \text{Trop}(X) \cap \text{Trop}(Y)\) holds regardless of which tropicalization is chosen. Conversely, for the tropicalization chosen above, assume for a moment that for every \(j\) dominating \(i\) in the inverse system, \(\text{Trop}_j(X) \cap \text{Trop}_j(Y)\) contains a point \(p_j\) different from the \(m\) points in \(\text{Trop}_j(X \cap Y)\). If none of such sequences \(\{p_j\}\) survives to the inverse limit then \(\lim_{i \in I} \text{Trop}_i(X) \cap \text{Trop}_j(Y)\) contains \(m\) points, therefore for some \(j\), \(\text{Trop}_j(X) \cap \text{Trop}_j(Y)\) contains \(m\) points. So assume that \(\{p_j\}\) survives to the inverse limit. Then, \(\lim_{i \in I} p_j = [\ ]_x \in X^{an} \cap Y^{an}\). But then \(X^{an} \cap Y^{an} = (X \cap Y)^{an} = (X \cap Y)^l\) and \((X \cap Y)^l\) contains an \((m + 1)\)st point. This contradiction finishes the proof. \(\square\)

**Remark 4.3.** Suppose that \(X\) and \(Y\) are plane curves intersecting transversally at \(m\) points in \(\mathbb{C}P^2\) (or in a projective plane over any algebraically closed
field). We may visualize $X$ and $Y$ in $\mathbb{P}^2(K)$ where $K$ is the completion of the field of Puiseux series over $\mathbb{C}$. Applying Theorem 4.2 we see that there exists a linear tropicalization $Trop_i$ of the projective plane such that $Trop_i(X)$ and $Trop_i(Y)$ intersect at $m$ points. This can be used to produce a proof of (the classical) Bezout’s theorem for plane curves using the tropical Bezout’s theorem [7, 16]. This might be an indication of a promising path for using tropical geometry in order to obtain results in classical intersection theory.

**Corollary 4.4.** Let $X$ be a line arrangement in $\mathbb{C}\mathbb{P}^2$ (or over any algebraically closed field). Then there exists a linear tropicalization $Trop_i$ of $X$ such that all tropical lines in this tropicalization intersect transversally.

**Proof.** As in the previous remark, visualize the line arrangement in the projective plane over the completion of the field of Puiseux series. Then the result immediately follows from Theorem 4.2. $\square$

Recall that a tropical prevariety, namely a closed set with respect to the tropical semifield operations is said to be realizable (or a tropical variety) if it is the image $\nu$ of an algebraic variety over a non-Archimedean valued field $(K, \nu)$ [16].

**Corollary 4.5.** There exists a planar tropical line arrangement which is realizable by a complex line arrangement but not realizable by any real line arrangement.

**Proof.** The Hessian configuration in $\mathbb{C}\mathbb{P}^2$, namely the complex $(4,3)$-net, has a tropicalization $L$ that commutes with the intersection of the lines, by Corollary 4.4. However, the abstract $(4,3)$-net does not have any real embeddings. Therefore $L$ is not realizable by any real line arrangement. (A detailed study of the Hessian configuration can be found in [1] and $k$-nets in [18]. The nonexistence of real 4-nets is proved in Lemma 2.4 in [5].) $\square$

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