Multiplicity results for a fractional Schrödinger equation with sign-changing potentials

Sofiane Khoutir\textsuperscript{a,b} \ *

\textsuperscript{a}School of Mathematics and Statistics, Central South University
Changsha, 410083 Hunan, P.R.China
\textsuperscript{b}Faculty of Mathematics, Laboratory AMNEDP, USTHB, Algiers 16111, Algeria

Abstract: This paper is devoted to study a class of nonlinear fractional Schrödinger equations:
\[ (-\Delta)^s u + V(x)u = f(x, u), \text{ in } \mathbb{R}^N, \]
where \( s \in (0, 1), \) \( N > 2s, \) \((-\Delta)^s\) stands for the fractional Laplacian. The main purpose of this paper is to study the existence of infinitely many small solutions for the aforementioned equation. By using the variational methods and the genus properties in critical point theory, we establish the existence of at least one nontrivial solution as well as infinitely many small solutions for the above equation with a general potential \( V(x) \) which is allowed to be sign-changing and the nonlinearity \( f(x, u) \) is locally sublinear with respect to \( u. \)

Keywords: Fractional Schrödinger equation; sublinear; sign-changing potential.

2010 Mathematics Subject Classification. 35J20; 35J60.

1 Introduction and Main results

Consider the following fractional Schrödinger equations:
\[ (-\Delta)^s u + V(x)u = f(x, u), \text{ in } \mathbb{R}^N, \]  \hspace{1cm} (1.1)
where \( s \in (0, 1), \) \( N > 2s, \) \((-\Delta)^s\) stands for the fractional Laplacian.

The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. It was discovered by Laskin \([1, 2] \) as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths, where the Feynman path integral leads...
to the classical Schrödinger equation, and the path integral over Lévy trajectories leads to the fractional Schrödinger equation.

The equation (1.1) with $\alpha = 1$ is the nonlinear Schrödinger equation

$$ - \Delta u + V(x)u = f(x, u), \quad \text{in } \mathbb{R}^N, \quad (1.2) $$

which has been broadly studied in the last decade. Moreover, a lot of interesting studies by variational methods can be found in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] for the nonlinear Schrödinger equation with various growth conditions on the nonlinearity.

Recently, Shi and Chen [13] obtained the existence and multiplicity of nontrivial solutions for problem (1.1). By using Morse theory in combination with local linking arguments, they first proved the existence of at least two nontrivial solutions for the equation (1.1). Then, they obtained the existence of at least $k$ distinct pairs of solutions via Clark’s theorem, when the nonlinearity $f(x, u)$ is sublinear at infinity and the potential function satisfies the following assumption:

$$(V'_1) \quad V \in C(\mathbb{R}^N, \mathbb{R}) \text{ satisfies } \inf_{x \in \mathbb{R}^N} V(x) \geq a_1 > 0, \text{ where } a_1 \text{ is a constant.}$$

Zhang et al. [14] proved the existence of infinitely many radial and non-radial solutions for problem (1.1) by means of the Symmetric Mountain Pass Theorem, when $V(x)$ is radial and satisfies $(V'_1)$ and $f$ satisfies some general superlinear assumptions at infinity. Khoutir and Chen [15] obtained a sequence of high energy solutions for problem (1.1) by using the Symmetric Mountain Pass Theorem, when $f$ satisfied a superlinear growth conditions and the potential function $V$ satisfied $(V'_1)$ and the following assumption

$$(V'_2) \quad \text{for each } M > 0, \text{ meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty, \text{ where meas\{\} denotes the Lebesgue measure in } \mathbb{R}^N.$$  

Teng [16] established the existence of infinitely many high or small energy solutions for problem (1.1) via the variant Fountain theorems, when $V(x)$ satisfied $(V'_1)$ and a weaker condition than $(V'_2)$, that is,

$$(V_2) \quad \text{There exists } d_0 > 0 \text{ such that for any } M > 0,$$

$$\text{meas}\{x \in \mathbb{R}^N : |x - y| \leq d_0, V(x) \leq M\} < +\infty,$$

moreover, the nonlinear term $f(x, u)$ is assumed to be asymptotically linear or superquadratic growth.

In [17] Ge improved the conclusions of Teng [16]. When the potential $V(x)$ satisfied only the condition $(V'_1)$ and the nonlinear term $f$ satisfied some more relaxed superlinear assumptions, Ge proved the existence of infinitely many solutions of problem (1.1) by the aid of the variant Fountain Theorem.
In [18] Du and Tian considered the problem (1.1). Firstly, the authors studied the case when \( f(x, u) \) is sublinear at infinity with respect to \( u \) and \( V \) satisfied \((V_1')\), so in this case they obtained the existence of infinitely many small energy solutions via Dual Fountain Theorem. Then, the authors studied the existence of infinitely many high energy solution by using the Fountain Theorem, when \( f \) is superlinear at infinity and the potential \( V \) satisfied \((V_1)\) \( V \in C(\mathbb{R}^N, \mathbb{R}) \) satisfies \( \inf_{x \in \mathbb{R}^N} V(x) \geq -\infty \), and \((V_2)\). Finally, they proved the existence of infinitely many small energy solutions when the nonlinear term is a combination of sublinear and critical terms, and \( V \) satisfied \((V_1') - (V_2')\). Note that the results of Du and Tian extend and sharply improve the results of Teng [16].

For more interesting results on the existence and the multiplicity of solutions of problem (1.1), we refer the readers to [19, 20, 21, 22, 23, 24, 25, 26, 27] and the references therein.

Inspired by the above facts, In the present paper we prove the existence of nontrivial solution and infinitely many small solutions for problem (1.1) under appropriate assumptions on the nonlinear term \( f(x, u) \) which is only locally defined for \(|u|\) small, using the variational methods in combination with genus theory. Furthermore, the potential \( V \) is allowed to be sign-changing. Recent results from the literature are extended and improved. In order to state the main results of this paper, we make the following assumptions on \( f \):

\((F_1)\) There exists a constant \( \delta_1 > 0 \) such that \( f \in C(\mathbb{R}^N \times [-\delta_1, \delta_1], \mathbb{R}) \), and there exist a constant \( r \in (1, 2) \) and a positive function \( \xi \in L^{2/r}(\mathbb{R}^N) \) such that
\[
|f(x, u)| \leq r\xi(x)|u|^{r-1}, \quad |u| \leq \delta_1, \quad \forall x \in \mathbb{R}^N.
\]

\((F_2)\) There exists \( \delta_2 > 0 \) such that
\[
f(x, u) \geq M|u|, \quad |u| \leq \delta_2, \quad \forall x \in \mathbb{R}^N, \quad \forall M > 0.
\]

\((F_3)\) There exists a constant \( \delta_3 > 0 \) such that \( f(x, -u) = -f(x, u) \) for all \(|u| \leq \delta_3 \) and all \( x \in \mathbb{R}^N \).

Before stating our main results, we introduce the following notations. As usual, for \( 1 \leq p < +\infty \), we let
\[
\|u\|_p := \left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{1/p}, \quad u \in L^p(\mathbb{R}^N),
\]
and
\[
\|u\|_\infty = \text{ess} \sup_{x \in \mathbb{R}^N} |u(x)|, \quad u \in L^\infty(\mathbb{R}^N).
\]

Let \( C_0^\infty (\mathbb{R}^N) \) be the collection of smooth functions with compact support and \( S(\mathbb{R}^N) \) the Schwartz space of rapidly decreasing \( C^\infty \) functions in \( \mathbb{R}^N \). We recall that the Fourier transform \( \mathcal{F}\phi \) (or simply \( \hat{\phi} \)) is defined for any \( \phi \in S(\mathbb{R}^N) \) as
\[
\mathcal{F}\phi(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\xi} \phi(x) \, dx.
\]
Moreover, by Plancherel’s theorem we have \( \| \phi \|_2 = \| \hat{\phi} \|_2, \forall \phi \in \mathcal{S}(\mathbb{R}^N) \). The fractional Laplacian \((-\Delta)^s\) with \( s \in (0, 1) \) of a function \( \phi \in \mathcal{S}(\mathbb{R}^N) \) is defined by

\[
\mathcal{F}((-\Delta)^s \phi)(\xi) = |\xi|^{2s} \mathcal{F} \phi(\xi), \quad \forall s \in (0, 1).
\]

If \( \phi \) is sufficiently smooth, according to [29], the fractional Laplacian \((-\Delta)^s\) can be viewed as a pseudo-differential operator defined by

\[
(-\Delta)^s \phi(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x - y|^{N+2s}} dy,
\]

where \( P.V. \) is the principal value and \( C_{N,s} > 0 \) is a normalization constant. Consider the fractional Sobolev space

\[
H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{N+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}
\]

with the inner product and the norm

\[
\langle u, v \rangle_{H^s} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} u(x)v(x)dx,
\]

\[
\|u\|_{H^s}^2 = \langle u, u \rangle_{H^s} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} |u(x)|^2 dx,
\]

where the norm

\[
[u]_{H^s}^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy
\]

is the so called Gagliardo semi-norm of \( u \). The space \( H^s(\mathbb{R}^N) \) can also be described by means of the Fourier transform. Indeed, it is defined by

\[
H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\xi|^{2s} |\hat{u}(\xi)|^2 + |\hat{u}(\xi)|^2) d\xi < \infty \right\},
\]

and the norm is defined by

\[
\|u\|_{H^s} = \left( \int_{\mathbb{R}^N} (|\xi|^{2s} |\hat{u}(\xi)|^2 + |\hat{u}(\xi)|^2) d\xi \right)^{\frac{1}{2}}.
\]

In [29], the authors show that for \( u \in \mathcal{S}(\mathbb{R}^N) \), one has

\[
2C_{N,s}^{-1} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = 2C_{N,s}^{-1} \|(-\Delta)^{\frac{s}{2}} u\|_2^2 = [u]_{H^s}^2.
\]

Therefore, the norms on \( H^s(\mathbb{R}^N) \) defined below,

\[
u \mapsto \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} |u(x)|^2 dx \right)^{\frac{1}{2}},
\]

\[
u \mapsto \left( \int_{\mathbb{R}^N} (|\xi|^{2s} |\hat{u}(\xi)|^2 + |\hat{u}(\xi)|^2) d\xi \right)^{\frac{1}{2}},
\]

\[
u \mapsto \left( \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \int_{\mathbb{R}^N} |u(x)|^2 dx \right)^{\frac{1}{2}}.
\]
are all equivalent.

Denote \( V^{\pm} = \max\{\pm V(x), 0\} \) and
\[
S := \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{H^s}^2}{\|u\|_{L^2}^2}.
\] (1.3)

\( S \) is the best constant in the Sobolev embedding \( H^s(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N) \). Concerning the potential \( V \) we suppose that:

1. \( V \) is sing-changing, \( \lim_{|x| \to \infty} V(x) = V_{\infty} > 0 \), \( V(x) \leq V_{\infty} \) in \( \mathbb{R}^N \) and \( V^- \in L^\infty_{L^{2^*_s}}(\mathbb{R}^N) \) with \( \|V^-\|_{L^{N/2s}} < S \).

Now, we are ready to state the main result of this paper as follows.

**Theorem 1.1.** Assume that conditions (V), (F1) and (F2) hold. Then the problem (1.1) possesses at least one nontrivial solution.

**Theorem 1.2.** Assume that conditions (V), (F1) and (F3) hold. Then the problem (1.1) possesses infinitely many nontrivial solutions \( \{u_k\} \) satisfying
\[
\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} \, dxdy + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u_k^2 \, dx - \int_{\mathbb{R}^N} F(x, u_k) \, dx \leq 0
\]
and \( u_k \to 0 \) as \( k \to \infty \).

**Remark 1.3.**

1. Unlike [13, 18, 19], the nonlinear term \( f \) does not satisfy any growth condition and any control at infinity, and we just require \( f(x, u) \) locally odd with respect to \( u \) when we prove the existence of infinitely many small solutions. Moreover, there are functions \( f(x, u) \) satisfying (F1) − (F3), for example, let
\[
F(x, u) = \begin{cases} 
  a(x)|u|^r, & 0 < |u| \leq 1, \\
  0, & u = 0,
\end{cases}
\]
where \( r \in (1, 2) \) and \( a(x) \in L^{\frac{2}{2-r}}(\mathbb{R}^N, \mathbb{R}^+) \).

2. Under the condition (V) it is clear that \( V(x) \) is sign-changing and not satisfied any coerciveness condition, so the main difficulties of our problem is the lack of compactness of the Sobolev embedding. Noting that in [13, 18, 19] the authors studied problem (1.1) with a potential function \( V(x) \) which is strictly positive, so our results extend and improve the aforementioned works.
(3) To the best of our knowledge, conditions \((V)\) was introduced by Furtado et al. in [3]. Furthermore, it is not difficult to find a function \(V : \mathbb{R}^N \rightarrow \mathbb{R}\) satisfying the condition \((V)\), for example let

\[
V(x) = \begin{cases} 
\frac{|x|^2}{1+|x|^2}, & |x| > 1, \\
-\frac{\varepsilon}{|x|^{\alpha}}, & |x| \leq 1,
\end{cases}
\]

where \(\varepsilon > 0\) is small and \(0 < \alpha < 2s\), which is similar with the example appeared in [3] with a slight modification.

The paper is organized as follows. In Section 2, we prove some lemmas, which are crucial to prove our main results. Section 3 is devoted to the proof of Theorem 1.1 and Theorem 1.2.

2 Variational framework and technical lemmas

In the sequel, \(C, C_i > 0\) denote various positive constants which may change from line to line. Let \(\Omega \subset \mathbb{R}^N\), then, for \(1 \leq p \leq +\infty\), we denote by \(\|\cdot\|_{p,\Omega}\) the usual norm in \(L^p(\Omega)\).

Let \(\Omega \subset \mathbb{R}^N\) be a smooth bounded domain, then, we define a closed subspace

\[
X(\Omega) := \{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.
\]

Therefore, the fractional Sobolev inequality implies that \(X(\Omega)\) is a Hilbert space with inner product

\[
\langle u, v \rangle_X = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy,
\]

which induces a norm \(\|\cdot\|_X = [\cdot]_s\).

Let

\[
H = H(\mathbb{R}^N) := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V^+(x) u^2 \, dx < +\infty \right\},
\]

Obviously, \(H\) is a Hilbert space equipped with the inner product

\[
\langle u, v \rangle := \langle u, v \rangle_H = \langle u, v \rangle_X + \int_{\mathbb{R}^N} V^+(x) uv \, dx,
\]

and the norm \(\|u\| = \langle u, u \rangle^\frac{1}{2}_H\), furthermore, similar to [3, Lemma 2.1], it is easy to see that the norm \(\|\cdot\|\) is equivalent to the usual norm of \(H^s(\mathbb{R}^N)\). Let \(\Omega \subset \mathbb{R}^N\), we also define the following space

\[
H(\Omega) := \left\{ u \in X(\Omega) : \int_{\Omega} V^+(x) u^2 \, dx < +\infty \right\},
\]

with inner product and norm

\[
\langle u, v \rangle_{\Omega} := \langle u, v \rangle_{H(\Omega)} = \langle u, v \rangle_X + \int_{\Omega} V^+(x) uv \, dx, \quad \|u\|_{\Omega} = \langle u, u \rangle^\frac{1}{2}_{H(\Omega)}.
\]

Then, \(H(\Omega)\) is a Hilbert space.
Under the condition \((V)\) the embeddings \(H(\Omega) \hookrightarrow L^p(\Omega)\) is continuous for \(p \in [2, 2^*], \Omega \subseteq \mathbb{R}^N\), that is, there exist constants \(\mu_p > 0\) such that
\[
\|u\|_{p, \Omega} \leq \mu_p \|u\|_{\Omega}, \quad \forall u \in H(\Omega), \; p \in [2, 2^*],
\]
where \(2^* = \frac{2N}{N-2s}\) is the critical Sobolev exponent. Moreover, from [13, Lemma 2.1], we know that under the assumption \((V)\), the embedding \(H \hookrightarrow L^p_{loc}(\mathbb{R}^N)\) is compact for \(2 \leq p < 2^*\).

For the fractional Schrödinger equation \((1.1)\), the associated energy functional is defined on \(H\) as follows
\[
I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} F(x,u) dx.
\]
Let \(0 < l \leq \frac{1}{2} \min\{\delta_1, \delta_2, \delta_3\}\). We define an even function \(\eta \in C^1(\mathbb{R}, \mathbb{R})\) such that \(0 \leq \eta(t) \leq 1\),
\[
\eta(t) = \begin{cases} 1 & \text{for } |t| \leq l; \\ 0 & \text{for } |t| \geq 2l; \end{cases}
\]
and \(\eta\) is decreasing in \([l, 2l]\).

Let
\[
f_\eta(x,u) := \eta(u)f(x,u), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.
\]
Consider the cut-off functional \(I_\eta\):
\[
I_\eta(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} F_\eta(x,u) dx,
\]
where \(F_\eta(x,u) = \int_0^u \eta(s)f(x,s)ds\). Then, the critical points of \(I_\eta\) are weak solutions of the following equation
\[
(-\Delta)^s u + V(x)u = f_\eta(x,u), \quad x \in \mathbb{R}^N.
\]
Furthermore, \(u \in H\) satisfies \(\|u\|_{\infty, \mathbb{R}^N} \leq l\), is a critical point of the functional \(I_\eta\), then \(u\) is a weak solution of \((2.5)\).

**Lemma 2.1.** Suppose that \((V)\) and \((F_1)\) hold. Then, the functional \(I_\eta\) is well define and of class \(C^1(H, \mathbb{R})\) with
\[
\langle I_\eta'(u), v \rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} V(x)uv dx - \int_{\mathbb{R}^N} f_\eta(x,u)v dx,
\]
for all \(v \in H\). Moreover, the critical points of \(I_\eta\) in \(H\) are solutions of problem \((1.1)\).

**Proof.** By \((F_1)\) and \((2.3)\), we have
\[
|f_\eta(x,u)| \leq r \xi(x) |u|^{r-1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},
\]
and
which yields
\[
|F_\eta(x, u)| = |F_\eta(x, u) - F_\eta(x, 0)| \\
\leq \int_0^1 |f_\eta(x, tu)||u|dt \\
\leq \xi(x)|u|^r, \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R},
\]
where \( r \in (1, 2) \) and \( \xi \in L^{2-r}(\mathbb{R}^N) \). Then we obtain
\[
\int_{\mathbb{R}^N} F_\eta(x, u)dx \leq \int_{\mathbb{R}^N} \xi(x)|u|^r dx \\
\leq \left( \int_{\mathbb{R}^N} |\xi(x)|^{\frac{2}{2-r}} dx \right)^{\frac{2-r}{2}} \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{r}{2}} \quad (2.8)
\]
(2.8)

On the other hand, since \( V^- \in L^\infty(\mathbb{R}^N) \), by (1.3) and the Hölder inequality we have
\[
\int_{\mathbb{R}^N} V^-(x) u^2 dx \leq \|V^-\|_{\frac{N}{2}} \|u\|_{2_N}^2 \leq S^{-1} \|V^-\| \|u\|_{2_N} \leq S^{-1} \|V^-\| \|u\|^2,
\]
for all \( u \in H \). Hence, \( I_\eta \) is well defined on \( H \). Next, we prove that (2.6) holds. According to (2.4), it suffices to show that
\[
\Psi \in C^1(H, \mathbb{R}), \quad \langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} f_\eta(x, u)v dx, \quad \forall u, v \in H,
\]
where \( \Psi(u) = \int_{\mathbb{R}^N} F_\eta(x, u)dx \).

For any function \( \theta : \mathbb{R}^N \to (0, 1) \) and \( t \in (0, 1) \), by (F_1), (1.3) and the Hölder inequality, one has
\[
\int_{\mathbb{R}^N} \max_{t \in (0, 1)} |f_\eta(x, u + t\theta(x)v)| v dx \\
\leq \int_{\|u + t\theta(x)v\| \leq 2t} r\xi(x)|u + t\theta(x)v|^{r-1}|v| dx \\
\leq \int_{\|u + t\theta(x)v\| \leq 2t} r\xi(x) \left( |u|^{r-1} + |v|^{r-1} \right) |v| dx \quad (2.10)
\]
(2.10)
Then by (2.10) and the Lebesgue’s Dominated Convergence Theorem, we have
\[
\langle \Psi'(u), v \rangle = \lim_{t \to 0^+} \frac{\Psi(u + tv) - \Psi(u)}{t} = \lim_{t \to 0^+} \int_{\mathbb{R}^N} \frac{F_\eta(x, u + tv) - F_\eta(x, u)}{t} \, dx \\
= \lim_{t \to 0^+} \int_{\mathbb{R}^N} f_\eta(x, u + t\theta(x)v) v(x) \, dx \\
= \int_{\mathbb{R}^N} f_\eta(x, u) v \, dx,
\]
which implies that (2.6) holds. Moreover, by a standard argument, it is easy to show that the critical points of $I_\eta$ are solutions of problem (1.1). It remains to show that $\Psi'$ is continuous. Let $\{u_n\} \subset H$ be a sequence such that $u_n \to u \in H$, therefore $u_n \to u$ in $L^2(\mathbb{R}^N)$ and
\[
\lim_{n \to \infty} u_n(x) = u(x), \quad \text{a.e. } x \in \mathbb{R}^N. \tag{2.12}
\]
We claim that
\[
f_\eta(x, u_n) \to f_\eta(x, u) \text{ strongly in } L^2(\mathbb{R}^N). \tag{2.13}
\]
Arguing by contradiction, there exist $\varepsilon_0 > 0$ and a subsequence $\{u_{n_k}\}$ such that
\[
\int_{\mathbb{R}^N} |f_\eta(x, u_{n_k}) - f_\eta(x, u)|^2 \, dx \geq \varepsilon_0, \quad \forall k \in \mathbb{N}. \tag{2.14}
\]
Since $u_n \to u$ in $L^2(\mathbb{R}^N)$, passing to a subsequence if necessary, one can assume that
\[
\sum_{k=1}^{\infty} |u_{n_k} - u|_{2, \mathbb{R}^N}^2 < +\infty.
\]
Therefore, $g(x) := [\sum_{k=1}^{\infty} |u_{n_k} - u|^2]^{1/2} \in L^2(\mathbb{R}^N)$. On the other hand, by (2.7) and Hölder’s inequality we have
\[
|f_\eta(x, u_{n_k}) - f_\eta(x, u)|^2 \leq 2|f_\eta(x, u_{n_k})|^2 + 2|f_\eta(x, u)|^2 \\
\leq 4r^2|\xi(x)|^2 \left( |u_{n_k}(x)|^{2(r-1)} + |u(x)|^{2(r-1)} \right) \\
\leq C_0|\xi(x)|^2 \left( |g(x)|^{2(r-1)} + |u(x)|^{2(r-1)} \right) \\
:= w(x), \quad \forall k \in \mathbb{N}, \, x \in \mathbb{R}^N \tag{2.15}
\]
and
\[
\int_{\mathbb{R}^N} w(x) \, dx = C_0 \int_{\mathbb{R}^N} |\xi(x)|^2 \left( |g(x)|^{2(r-1)} + |u(x)|^{2(r-1)} \right) \, dx \\
\leq C_0\|\xi\|_{2(r-1)}^2 \left( \|g\|_{2(r-1)}^2 + \|u\|_{2(r-1)}^2 \right) < +\infty. \tag{2.16}
\]
Combining (2.12), (2.15), (2.16) with Lebesgue’s Dominated Convergent Theorem we conclude
\[
\lim_{k \to \infty} \int_{\mathbb{R}^N} |f_\eta(x, u_{n_k}) - f_\eta(x, u)|^2 \, dx = 0,
\]
which contradicts (2.14). Thus, (2.13) holds.

It follows from (2.11), (2.13) and Hölder’s inequality that

\[
\left| \langle \Psi'(u_n) - \Psi'(u), v \rangle \right| \leq \int_{\mathbb{R}^N} |f_\eta(x, u_n) - f_\eta(x, u)||v|dx
\]
\[
\leq \left( \int_{\mathbb{R}^N} |f_\eta(x, u_n) - f_\eta(x, u)|^2 dx \right)^{1/2} \|v\|_2
\]
\[
\leq \mu_2 \left( \int_{\mathbb{R}^N} |f_\eta(x, u_n) - f_\eta(x, u)|^2 dx \right)^{1/2} \|v\| \to 0,
\]
as \(n \to \infty\), which shows that \(\Psi'\) is continuous. The proof is completed. \(\square\)

Recall that a sequence \(\{u_n\} \subset E\) is said to be a Palais-Smale sequence at the level \(c \in \mathbb{R}\) ((PS)\(_c\) sequence for short) if \(I(u_n) \to c\) and \(I'(u_n) \to 0\), \(I\) is said to satisfy the Palais-Smale condition at the level \(c\) ((PS)\(_c\) condition for short) if any (PS)\(_c\)-sequence has a convergent subsequence.

Lemma 2.2. [28] Let \(E\) be a Banach space and \(I \in C^1(E, \mathbb{R})\) satisfy the (PS) condition. If \(I\) is bounded from below, then \(c = \inf_E I\) is a critical value of \(I\).

In order to find the multiplicity of nontrivial critical points of \(I\), we will use the genus properties, so we recall the following definitions and results (see [30]).

Let \(E\) be a Banach space and \(I \in C^1(E, \mathbb{R})\). We set

\[
\Gamma = \{ A \subset E - 0 : A \text{ is closed in } E \text{ and symmetric with respect to } 0 \}.
\]

Definition 2.3. For \(A \in \Gamma\), we say genus of \(A\) is \(k\) denoted by \(\gamma(A) = k\) if there is an odd map \(\Psi \in C(A, \mathbb{R}^N \setminus 0)\) and \(k\) is the smallest integer with this property.

For any \(k \in \mathbb{N}\), we set

\[
\Gamma_k = \{ A \in \Gamma : \gamma(A) \geq k \}.
\]

Then, we have the following lemma from [30].

Lemma 2.4. Let \(E\) be an infinite dimensional Banach space and \(I \in C^1(E, \mathbb{R})\) satisfy (\(A_1\)) and (\(A_2\)) below:

\((A_1)\) \(I\) is even, bounded from below, \(I(0) = 0\) and \(I\) satisfies the (PS) condition.

\((A_2)\) For each \(k \in \mathbb{N}\), there exists an \(A_k \in \Gamma_k\) such that \(\sup_{u \in A_k} I(u) < 0\).

Then \(I\) admits a sequence of critical points \(u_k\) such that \(I(u_k) \leq 0\), \(u_k \neq 0\) and \(\lim_{k \to \infty} u_k = 0\).
3 Proof of the main results

Lemma 3.1. Under the assumptions \((V)\) and \((F_1)\), the functional \(I_\eta\) is bounded from below and satisfies the (PS) condition.

Proof. By \((V), (2.4), (2.8)\) and \((2.9)\) we have

\[
I_\eta(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} V^-(x) u^2 \, dx - \int_{\mathbb{R}^N} F_\eta(x,u) \, dx \\
\geq \frac{1}{2} \|u\|^2 - \frac{1}{2} S^{-1} \|V^-\|_{\frac{N}{2s}} \|u\|^2 - \mu^* \|\xi(x)\|_{\frac{2}{2-r}} \|u\|^r \\
= \frac{1}{2} \left(1 - S^{-1} \|V^-\|_{\frac{N}{2s}}\right) \|u\|^2 - \mu^* \|\xi(x)\|_{\frac{2}{2-r}} \|u\|^r.
\]

Then by (3.1) we conclude that \(I_\eta\) is bounded from below since \(r \in (1, 2)\).

Next, we prove that \(I_\eta\) satisfies the (PS) condition. Let \(\{u_n\} \subset H \) be any (PS) sequence of \(I_\eta\), i.e., \(\{I_\eta(u_n)\}\) is bounded and \(I'_\eta(u_n) \to 0\) in \(H^*\).

From (3.1) we have

\[
C_1 \geq I_\eta(u_n) \geq \frac{1}{2} \left(1 - S^{-1} \|V^-\|_{\frac{N}{2s}}\right) \|u_n\|^2 - \mu^* \|\xi(x)\|_{\frac{2}{2-r}} \|u_n\|^r.
\]

This implies that \(\{u_n\}\) is bounded in \(H\) since \(r \in (1, 2)\) and \(C_1\) is independent of \(n\), that is, there exists a constant \(C_2 > 0\) which is independent of \(n\) such that

\[
\|u_n\| \leq C_2, \quad \forall n \in \mathbb{N}.
\]

Therefore, up to a subsequence, there exists \(u \in H\) such that \(u_n \rightharpoonup u\) in \(H\) and

\[
u_n \to u \text{ in } L^p_{\text{loc}}(\mathbb{R}^N), \quad p \in [2, 2^*_s)\]

By \((F_1)\), for any given \(\varepsilon > 0\), we can choose \(R > 0\) such that

\[
\left(\int_{|x| > R} |\xi(x)|^{\frac{2}{2-r}} \, dx\right)^{\frac{2-r}{2}} < \varepsilon.
\]

In the other hand, from (3.3) we get

\[
\lim_{n \to \infty} \int_{|x| \leq R} |u_n - u|^2 \, dx = 0,
\]

which implies that there exists \(n_0 \in \mathbb{N}\) such that

\[
\int_{|x| \leq R} |u_n - u|^2 \, dx \leq \varepsilon^2, \quad \text{for all } n \geq n_0.
\]
Therefore, for any \( n \geq n_0 \), it follows from \((F_1), (3.2), (3.5)\) and Hölder’s inequality that
\[
\int_{B_R} |f_\eta(x, u_n) - f_\eta(x, u)| |u_n - u| \, dx
\]
\[
\leq \left( \int_{B_R} |f_\eta(x, u_n) - f_\eta(x, u)|^2 \, dx \right)^{1/2} \, \|u_n - u\|_{2, B_R}
\]
\[
\leq \varepsilon \left[ \int_{B_R} 2 \left( |f_\eta(x, u_n)| + |f_\eta(x, u)|^2 \right) \, dx \right]^{1/2}
\]
\[
\leq 2\varepsilon \left[ r^2 \int_{B_R} |\xi(x)|^2 \left( |u_n(x)|^{2(r-1)} + |u(x)|^{2(r-1)} \right) \, dx \right]^{1/2}
\]
\[
\leq 2\varepsilon \left[ r^2 \|\xi\|_2^{2(r-1), B_R} \left( \|u_n\|_2^{2(r-1), B_R} + \|u\|_2^{2(r-1), B_R} \right) \right]^{1/2}
\]
\[
\leq \varepsilon C_3,
\]
where \( B_R := \{ x \in \mathbb{R}^N : |x| < R \} \). Let \( \Omega_R := \mathbb{R}^N \setminus B_R \), then combining \((F_1), (3.2), (3.4)\) with the Hölder inequality, one has
\[
\int_{\Omega_R} |f_\eta(x, u_n) - f_\eta(x, u)| |u_n - u| \, dx
\]
\[
\leq r \int_{\Omega_R} |\xi(x)| \left( |u_n(x)|^{(r-1)} + |u(x)|^{(r-1)} \right) \left( |u_n| + |u(x)| \right) \, dx
\]
\[
\leq 2r \int_{\Omega_R} |\xi(x)| \left( |u_n|^{r} + |u(x)|^{r} \right) \, dx
\]
\[
\leq 2r \left( \int_{\Omega_R} |\xi(x)|^{2(r+1)\gamma} \, dx \right)^{\frac{1}{2(r+1)\gamma}} \left( \|u_n\|_{2, \Omega_R}^{2} + \|u\|_{2, \Omega_R}^{2} \right)
\]
\[
\leq \varepsilon C_4,
\]
this together with \((3.6)\) implies that
\[
\int_{\mathbb{R}^N} |f_\eta(x, u_n) - f_\eta(x, u)| |u_n - u| \, dx \rightarrow 0,
\]
as \( n \rightarrow \infty \). Since \( I_\eta'(u_n) \rightarrow 0 \) in \( H^* \), it follows from \((V), (2.6)\) and \((2.9)\) that
\[
o_n(1) = \langle I_\eta'(u_n) - I_\eta'(u), u_n - u \rangle
\]
\[
= \|u_n - u\|^2 - \int_{\mathbb{R}^N} V^-(x) |u_n - u|^2 \, dx - \int_{\mathbb{R}^N} [f_\eta(x, u_n) - f_\eta(x, u)] (u_n - u) \, dx
\]
\[
\geq \left( 1 - S^{-1} \|V^-\|_{N/2} \right) \|u_n - u\|^2 - \int_{\mathbb{R}^N} [f_\eta(x, u_n) - f_\eta(x, u)] (u_n - u) \, dx,
\]
and then
\[
\|u_n - u\|^2 \leq o_n(1) + \frac{1}{1 - S^{-1} \|V^-\|_{N/2}} \int_{\mathbb{R}^N} [f_\eta(x, u_n) - f_\eta(x, u)] (u_n - u) \, dx.
\]
Consequently, by (3.8) and (3.9) we conclude that

\[ u_n \to u, \text{ strongly in } H \text{ as } n \to \infty. \]

Thus, the proof is completed. \(\square\)

**Proof of Theorem 1.1.** By Lemma 3.1, \(I_\eta\) is bounded from below and satisfies the (PS) condition. Then Lemma 2.2 implies that \(c = \inf_H I_\eta(u)\) is a critical value of \(I_\eta\), that is there exists a critical point \(u^* \in H\) such that \(I_\eta(u^*) = c\). Next, we show that \(u^* \neq 0\). Let \(u_0 \in H(\Omega) \setminus \{0\}\) with \(|u_0|_{\infty, \Omega} \leq 1\), where \(\Omega = \{x \in \mathbb{R}^N; |u_0(x)| \leq 1\}\), then by (F2) we have that there exists a small \(\delta_2 > 0\) such that

\[ F_\eta(x, u) \geq M|u|^2, \quad |u| \leq \delta_2, \forall x \in \mathbb{R}^N, \forall M > 0. \]  \(3.10\)

For \(0 < t < l\), it follows from (2.4) and (3.10) that

\[
I_\eta(tu_0) = \frac{1}{2}t^2\|u_0\|^2 - \frac{t^2}{2} \int_{\mathbb{R}^N} V^-(x)u_0^2dx - \int_{\mathbb{R}^N} F_\eta(x, tu_0)dx \\
\leq \frac{1}{2}t^2\|u_0\|^2 - t^2M \int_{\Omega} |u_0|^2dx. \]  \(3.11\)

By choosing large \(M > 0\) such that \(2M \int_{\Omega} |u_0|^2dx - \|u_0\|^2 > 0\), it follows from (3.11) that \(I_\eta(tu_0) < 0\) for \(t > 0\) small enough. Hence \(I_\eta(u^*) = c < 0\), which implies that \(u^*\) is a nontrivial critical point of \(I_\eta\) with \(|u^*|_{\infty, \Omega} \leq l\), and so \(u^*(x) = tu_0(x)\) is a critical point of \(I\), thus, \(u^*\) is a nontrivial solution of problem (1.1). The proof is complete. \(\square\)

**Lemma 3.2.** Assume that (V) and (F2) hold. Then, for any \(n \in \mathbb{N}\), there exists a closed symmetric subset \(A_n \subset H\) such that the genus \(\gamma(A_n) \geq n\) and \(\sup_{u \in A_n} I_\eta(u) < 0\).

**Proof.** Let \(H_n\) be any \(n\)-dimensional subspace of \(H\). Since all norms are equivalent in a finite dimensional space, there is a constant \(\beta = \beta(H_n)\) such that

\[ \|u\| \leq \beta|u|_2, \]  \(3.12\)

for all \(u \in H_n\).

**Claim.** There exists a constant \(\tau > 0\) such that

\[ \frac{1}{2} \int_{\mathbb{R}^N} |u|^2dx \geq \int_{|u|>\tau} |u|^2dx \]  \(3.13\)

for all \(u \in H_n\) with \(\|u\| \leq \tau\). In fact, if (3.13) is false, there exists a sequence \(\{u_k\} \in H_n\) such that \(u_k \to 0\) in \(H\) and

\[ \frac{1}{2} \int_{\mathbb{R}^N} |u_k|^2dx < \int_{|u_k|>\tau} |u_k|^2dx, \quad k \in \mathbb{N}. \]
Let \( w_k := \frac{u_k}{\|u_k\|_2} \). Then, we obtain

\[
\frac{1}{2} < \int_{|u_k| > l} |w_k|^2 dx, \quad k \in \mathbb{N}.
\] (3.14)

On the other hand, we can assume that \( w_k \to w \) in \( H \) since \( H_n \) is finite dimensional. Hence \( w_k \to w \) in \( L^2(\mathbb{R}^N) \). Moreover, it can be deduced from \( u_k \to 0 \) in \( H \) that

\[\text{meas}\{x \in \mathbb{R}^N : |u_k| > l\} \to 0, \quad k \to \infty.\]

Therefore,

\[
\int_{|u_k| > l} |w_k|^2 dx \leq 2 \int_{|u_k| > l} |w_k - w|^2 dx + 2 \int_{|u_k| > l} |w|^2 dx \to 0, \quad k \to \infty,
\]

which contradicts (3.14). Thus, (3.13) holds.

By (\( F_2 \)) we have

\[f_\eta(x, u) \geq 4\beta^2 u, \quad |u| \leq 2l, \quad \forall x \in \mathbb{R}^N.\]

This inequality implies that

\[F_\eta(x, u) = F(x, u) \geq 2\beta^2 u^2, \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}, \quad |u| \leq l.\] (3.15)

Therefore, it follows from (2.4), (3.13) and (3.15) that

\[
I_\eta(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} V^{-}(x)u^2 dx - \int_{\mathbb{R}^N} F_\eta(x, u) dx
\]

\[
\leq \frac{1}{2} \|u\|^2 - \int_{|u| \leq l} F_\eta(x, |u|) dx
\]

\[
\leq \frac{1}{2} \|u\|^2 - 2\beta^2 \int_{|u| \leq l} |u|^2 dx
\]

\[
\leq \frac{1}{2} \|u\|^2 - 2\beta^2 \left( \int_{\mathbb{R}^N} |u|^2 dx - \int_{|u| > l} |u|^2 dx \right)
\]

\[
\leq -\frac{1}{2} \|u\|^2,
\]

for all \( u \in H_n \) with \( \|u\| \leq \min\{\tau, 1\} \).

Let \( 0 < \rho \leq \min\{\tau, 1\} \) and \( A_n = \{u \in H_n : \|u\| = \rho\} \). We conclude that \( \gamma(A_n) \geq n \) and \( \sup_{u \in A_n} I_\eta(u) \leq -\frac{1}{2} \|u\|^2 < 0 \). The proof is completed. \( \square \)

**Lemma 3.3.** If \( u \in H \cap L^\infty(\mathbb{R}^N) \) is a critical point of \( I_\eta \), then \( \|u\|_{\infty, \mathbb{R}^N} \leq 3l. \)

**Proof.** Borrowing an idea from Lemma 3.3 in [31] we give the proof as follows.

If \( u \in H \cap L^\infty(\mathbb{R}^N) \) is a critical point of \( I_\eta \), then we have

\[
\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} V^+(x)uvdx - \int_{\mathbb{R}^N} V^-(x)uvdx = \int_{\mathbb{R}^N} f_\eta(x, u)vdx,
\] (3.16)
for any \( v \in H \cap L^\infty(\mathbb{R}^N) \).

Define a function \( \chi \in C^1(\mathbb{R}, \mathbb{R}) \) such that \( \chi(-s) = \chi(s) \), \( 0 \leq \chi(s) \leq 1 \), \( \chi(s) = 0 \) for \( |s| \leq 2l \), \( \chi(s) = 1 \) for \( |s| \geq 3l \) and \( \chi \) is increasing in \([2l, 3l]\). Taking \( v = \chi(u)u \) as a test function in (3.16), by (2.9) we obtain

\[
\int_{\mathbb{R}^N} \frac{(u(x) - u(y))(u(x)\chi(u) - u(y)\chi(u))}{|x - y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} V^+(x)u^2\chi(u)dx = \int_{\mathbb{R}^N} V^-(x)u^2\chi(u)dx \\
\leq S^{-1}\|V^-\|_{\frac{N}{2s}}\|u\chi(u)\|^2,
\]

which implies that

\[
\int_{\{|u(x)\geq 3l\} \times \{|u(y)\geq 3l\}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy + \int_{|u|\geq 3l} V^+(x)u^2dx \leq 0.
\]

Hence \( \text{meas}\{x \in \mathbb{R}^N : |u(x)| \geq 3l\} = 0 \). Thus, the proof is completed. \( \square \)

Proof of Theorem 1.2. By (F1) and (F3), we get that \( I_\eta \) is even and \( I_\eta(0) = 0 \). On the other hand, by Lemmas 3.1 and 3.2 all the conditions of Lemma 2.4 are satisfied, which implies that \( I_\eta \) has a sequence of critical points \( \{u_n\} \) converging to 0. By Lemma 3.3, we may get that there exist \( n_0 \in \mathbb{N} \) such that \( \|u_n\|_\infty \leq l \) for \( n \geq n_0 \). Then we get infinitely many small solutions of (1.1). This completes the proof. \( \square \)

References

[1] N. Laskin, Fractional quantum mechanics and Lévy path integrals, Phys. Lett. A 268(4) (2000) 298-305.

[2] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E 66(5) (2002) 056108.

[3] M. F. Furtado, L. A. Maia, and E. S. Medeiros, Positive and nodal solutions for a nonlinear Schrödinger equation with indefinite potential, Adv. Nonlinear Stud. 8(2) (2008) 353-373.

[4] X. H. Tang, New conditions on nonlinearity for a periodic Schrödinger equation having zero as spectrum, J. Math. Anal. Appl. 413(1) (2014) 392-410.

[5] X. H. Tang, New super-quadratic conditions on ground state solutions for superlinear Schrödinger equation, Adv. Nonlinear Stud. 14(2014) 349-361.

[6] D. Qin, X. H. Tang, Asymptotically linear Schrödinger equation with zero on the boundary of the spectrum, Electron. J. Differential Equations 2015(213) (2015) 1-15.

[7] D. Lv, X. Yang, Nonradial solutions for semilinear Schrödinger equations with sign-changing potential, Electron. J. Qual. Theo. Differential Equations, 2015(16) (2015) 1-12.
[8] Y. H. Cheng, T. F. Wu, Multiplicity and concentration of positive solutions for semilinear elliptic equations with steep potential, Commun. Pure Appl. Anal. 15(6) (2016) 2457-2473.

[9] X. H. Tang, S. Chen, Weak potential conditions for Schrödinger equations with critical nonlinearities. J. Aust. Math. Soc. 100(2) (2016) 272-288.

[10] J. Liu, J. F. Liao, and C. L. Tang, A positive ground state solution for a class of asymptotically periodic Schrödinger equations, Comput. Math. Appl. 71(4) (2016) 965-976.

[11] H. Liu, H. Chen, and X. Yang, Least energy sign-changing solutions for nonlinear Schrödinger equations with indefinite-sign and vanishing potential, Appl. Math. Lett. 53 (2016) 100-106.

[12] M. F. Furtado, R. Marchi, Existence of solutions to asymptotically periodic Schrödinger equations, Electron. J. Differential Equations 2017(15) (2017) 1-7.

[13] H. Shi, H. Chen, Multiple solutions for fractional Schrödinger equations, Electron. J. Differential Equations 2015(25) (2015) 1-11.

[14] W. Zhang, X. Tang, J. Zhang, Infinitely many radial and non-radial solutions for a fractional Schrödinger equation, Comput. Math. Appl. 71(3) (2016) 737-747.

[15] S. Khoutir, H. Chen, Existence of infinitely many high energy solutions for a fractional Schrödinger equation in $\mathbb{R}^N$, Appl. Math. Lett. 61 (2016) 156-162.

[16] K. Teng, Multiple solutions for a class of fractional Schrödinger equations in $\mathbb{R}^N$, Nonlinear Anal. Real World Applications 21 (2015) 76-86.

[17] B. Ge, Multiple solutions of nonlinear Schrödinger equation with fractional Laplacian, Nonlinear Anal. Real World Applications 30 (2016) 236-247.

[18] M. Du, L. Tian, Infinitely many solutions of the nonlinear fractional Schrödinger equation, Discrete Contin. Dyn. Syst. 21(10) (2016) 3407-3428.

[19] C. Chen, Infinitely many solutions for fractional Schrödinger equations in $\mathbb{R}^N$, Electron. J. Differential Equations 2016(88) (2016) 1-15.

[20] Z. Wang, H.-S. Zhou, Radial sign-changing solution for fractional Schrödinger equation, Discrete Contin. Dyn. Syst. 36(1) (2016) 499-508.

[21] K. Perera, M. Squassina, Y. Yang, Critical fractional p-Laplacian problems with possibly vanishing potentials, J. Math. Anal. Appl. 433(2) (2016) 818-831.

[22] J. M. DÓ, O. H. Miyagaki, M. Squassina, Critical and subcritical fractional problems with vanishing potentials, Commun. Contemp. Math. 18 (2015) 1-20.
[23] B. Barrios, E. Colorado, R. Servadei, F. Soria, A critical fractional equation with concave-convex power nonlinearities, Ann. Inst. H. Poincaré (C) Non-Linear Analysis 32(4) (2015) 875-900.

[24] P. Li, Y. Shang, Infinitely many solutions for fractional Schrödinger equations with perturbation via variational methods, Open Math. 15 (2017) 578-586.

[25] Y. Pu, J. Liu, and C. L. Tang, Existence of weak solutions for a class of fractional Schrödinger equations with periodic potential, Comput. Math. Appl. 73(3) (2017) 465-482.

[26] Q. Wang, Multiple positive solutions of fractional elliptic equations involving concave and convexe nonlinearities in $\mathbb{R}^N$, Commun. Pure Appl. Anal. 15(5) (2016) 1671-1688.

[27] Z. Gao, X. H. Tang, W. Zhang, Multiplicity and Concentration of Solutions for Fractional Schrödinger equation, Taiwan. J. Math. 21(1) (2017) 187-210.

[28] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, In: CBMS Reg. Conf. Ser. in Math., Vol. 65, Amer. Math. Soc. Providence, RI, 1986.

[29] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136(5) (2012) 521-573.

[30] R. Kajikiya, A Critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, J. Funct. Anal. 225 (2005) 352-370.

[31] K. Wu, X. Wu, Infinitely many small energy solutions for a modified Kirchhoff-type equation in $\mathbb{R}^N$, Comput. Math. Appl. 70 (2015) 592-602.