Recurrence in resonant transmission of one-dimensional array of delta potentials

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Abstract

The resonant transmission of a moving particle which interacts with an one-dimensional array of $N$ δ-function potentials is investigated. A suitable transfer matrix formulation is used to obtain the particle transmission. We give the parameters for perfect tunnelling and the transcendental equation for the quasi-bound state energies for $N = 2, 3$ and $4$. Conditions for perfect tunnelling and resonant transmission are discussed for arrays with arbitrary $N$. A model to explain how the tunnelling energy filter works in these systems is proposed here.
1 Introduction

The property of resonant transmission also known as resonant tunnelling, is an interesting topic from the point of view of both the physical understanding and the practical applications of potential barrier arrays. It has been extensively studied in different fields of physics involving wave propagation\[1, 2\]. The resonant transmission is present in mechanical waves analyzed in acoustic\[3, 4\], in electromagnetic waves studied in optics\[5, 6, 7\], and in quantum mechanics\[8, 9, 10, 11, 12\]. From Esaki seminal papers\[13, 14, 15\], electronic perfect tunnelling in a double barrier system has been explained in terms of the quasi-bound states (QBS) between the barriers\[16\]. This understanding allowed the fabrication of resonant tunnelling diodes (RTD), which present negative differential resistance and good performance in fast processes\[15, 17\]. Electronic resonance is also present with three\[14\] and more potential barriers. The specific conditions to obtain resonance depends on recurrences in the transmission functions, and on the shape of the barrier array, which can be divided into subsets of potentials or cells\[1, 10\].

In this work we consider $\delta$-function shaped potential barriers. Such a potential array has been useful to model several solid state systems like magnetic impurities and short range interactions\[18, 19, 20, 21\]. It has been used to study the conduction properties of crystals through the Kronig-Penney model\[22\] and Anderson localization in a disordered impurity array\[23, 24, 25, 26\]. In a solid state quantum information scenario, $\delta$-function potential barriers are used to depict the instantaneous interaction between a flying spin and a fixed magnetic impurity\[27, 28, 29, 30, 31, 32\], to implement teleportation\[33\] and quantum memory\[34\].
Here we study the conditions under which an array of $N$ $\delta$-function potential barriers, gives resonant transmission with an incident particle. The situation depicted here can be implemented, in a solid state scenario with a ballistic electron moving on a carbon nanotube$^{35,36}$, a heterostructure or a quantum Hall edge states$^{37,38}$, where short range potentials, impurities or quantum dots are located.

This paper is organized as follows. In order to clarify the results and discussion, in the section 2 we give the transfer matrix method and its representation which was given recently$^{39}$. In section 3 we present the conditions for resonant transmission and perfect tunnelling for arrays with $N = 2, 3$ and 4. Here we also calculate the transcendental equations to find the QBS energies of these arrays. In section 4, we generalize the results for arrays with arbitrary $N$ in terms of sub-arrays or cells. The intrinsic QBS energy concept helps us to explain the working of a tunnelling energy filtering in such arrays. Section 5 gives the conclusions.

2 Transfer matrix

We consider an $N$ $\delta$-function potential barriers on a one-dimensional quantum wire along the $x$-axis. A moving particle with energy $\epsilon$, is incident from the left end of the wire, as is shown in Figure 1. The Hamiltonian of the system is

$$\hat{H} = \frac{p^2}{2m} + \sum_{n=1}^{N} J_n \delta(x - x_n),$$

(1)

where $p$ and $m$ are the momentum and mass of the particle, respectively. $J_n$ is the strength and $x_n$ is position of the $n$-th potential barrier. The particle wavefunction $\psi_n(x)$ in the region $x_n < x < x_{n+1}$, is taken to be
Figure 1: (Color online) Scheme of one dimensional quantum wire with $N$ $\delta$-function potential barriers of strength $J_n$ located at $x = x_n$, $n = 1, 2, \ldots, N$. A moving particle (with wave number $k$) incident from the left, is scattered off the potential array. The transmitted (or reflected) particle is indicated by right (left) pointed arrows.

$$\psi_n(x) = A_n e^{ikx} + B_n e^{-ikx}, \quad (2)$$

where the wave number is $k = \sqrt{2m\epsilon / \hbar^2}$, and $A_n$ and $B_n$ are the probability amplitudes for incoming ($k$) and outgoing ($-k$) parts of the wavefunction, respectively. To relate the coefficients ($A_{n-1}$, $B_{n-1}$, $A_n$ and $B_n$) of the wavefunction on both sides of the $n$-th potential (see Figure 1), we use the transfer matrix $M_n$

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 - i\lambda_n & -i\lambda_n E_n \\ i\lambda_n E_n & 2 + i\lambda_n \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix} = M_n \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}, \quad n = 1, 2, \ldots, N. \quad (3)$$

in terms of the dimensionless strength parameter $\lambda_n = 2mJ_n / \hbar^2$ and $E_n = \exp(2ikx_n)$.

Another representation for the transfer matrix $M_n$ is

$$M_n = I - \frac{i\lambda_n}{2} L_n \quad (4)$$
where
\[ L_n(E_n) = \begin{pmatrix} 1 & E_n^* \\ -E_n & -1 \end{pmatrix}. \] \hspace{1cm} (5)

This representation is useful to find the transfer matrix for \( N \) potentials, defined as
\[ \mathcal{M}(N) \equiv M_N M_{N-1} \ldots M_2 M_1. \] \( \mathcal{M}(N) \) will relate the incident with the transmitted wavefunction, that is
\[ \begin{pmatrix} A_N \\ B_N \end{pmatrix} \equiv \mathcal{M}(N) \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}, \] \hspace{1cm} (6)

where \( \mathcal{M}(N) \) can be expressed as
\[ \mathcal{M}(N) = I - \frac{i}{2} \sum_{n_1=1}^{N} \lambda_{n_1} L_{n_1} + \left( -\frac{i}{2} \right)^2 \sum_{n_1,n_2=1}^{N} (\lambda_{n_1} L_{n_1})(\lambda_{n_2} L_{n_2}) + \ldots + \left( -\frac{i}{2} \right)^m \sum_{n_1,n_2\ldots,n_m=1}^{N} (\lambda_{n_1} L_{n_1})(\lambda_{n_2} L_{n_2})\ldots(\lambda_{n_m} L_{n_m}) + \ldots + \left( -\frac{i}{2} \right)^N (\lambda_{n_1} L_{n_1})(\lambda_{n_2} L_{n_2})\ldots(\lambda_{n_N} L_{n_N}). \] \hspace{1cm} (7)

This representation of \( \mathcal{M}(N) \) is very useful because of the interesting property of the \( L \)-matrices, namely
\[ L_n L_m + L_m L_n = (2 - E_n E_m^* - E_n^* E_m) I = 4 \sin^2(\phi_{nm}) I, \] \hspace{1cm} (8)

where \( \phi_{nm} = k(x_m - x_n) \). An obvious and useful consequence is that \( L_n^2 = 0 \). Eq. (8) is very useful in simplifying the multiple products of \( L \)'s in the expression for the general transfer matrix in Eq. (7) above.
In this way, for no particle incident from the right of the \( N \)-th potential, that is \( B_N = 0 \), the probability of transmission is

\[
T = \frac{1}{|\langle \mathcal{M}(N) \rangle_{22}|^2},
\]  

(9)

since \( \text{det} \mathcal{M}(N) = 1 \).

We now present results for resonant transmission and perfect tunnelling \((T = 1)\) for some specific arrays using the representation of \( \mathcal{M}(N) \) in Eq. (7).

3 Results for \( N = 2, 3 \) and 4

In our previous work\cite{39}, some specific regular arrays were considered. Here we note that \( E_n = \exp(2i k x_n) \) is the square of the wave function at \( x = x_n \) for particle travelling to the right. If the distance between two potentials, at \( x_n \) and \( x_m \) is such that \( k(x_n - x_m) = \pi \alpha_{nm} \) where \( \alpha_{nm} \) is an integer then \( E_n = E_m \), consequently \( L_n = L_m \).

If all the interpotential distances \( k(x_n - x_m), (n, m = 1, 2,...) \) are integer multiples of \( \pi \) then \( E_1 = E_2 = \ldots = E_N \), implying \( L_n = L \) for \( n = 1, 2, \ldots, N \). Since \( L^2 = 0 \), the transfer matrix in Eq. (7) reduces to

\[
\mathcal{M}(N) = I - \frac{i}{2} L \sum_{n=1}^{N} \lambda_n.
\]  

(10)

This means that the \( N \) potential array effectively acts like a single potential of strength \( \Lambda = \sum_{n=1}^{N} \lambda_n \). This behaviour is due to resonance between the particle wavefunction and the array geometry. Note that the distance between any two potential locations need not be equal. The only requirement is that all \( k(x_n - x_m) \) be integer multiple of

\footnote{This will also hold for all wave number \( k' = \beta' k \) (\( \beta \) an integer) since \( k'(x_n - x_m) = \pi \beta \alpha \).}
Clearly, a single potential can give $T = 1$ only if it has zero strength. In the above case this means $\sum_{n=1}^{N} \lambda_n = 0$.

### 3.1 Two-δ-function potential array

For $N = 2$ system Eq. (7) gives

$$\mathcal{M}(2) = I - \frac{i\lambda_2}{2} L_2 - \frac{i\lambda_1}{2} L_1 - \frac{\lambda_2\lambda_1}{4} L_2 L_1.$$  \hspace{1cm} (11)

If $E_2 = E_1$, then $L_2 = L_1$ and this reduces to an effective $N = 1$ case. Thus for a genuine $N = 2$ array we need $L_2 L_1$, to be non-zero. From Eq. (11) one obtains

$$(\mathcal{M}(2))_{22} = \frac{1}{4} z_2 z_1 + \frac{1}{4} \lambda_2\lambda_1 E_2 E_1^*,$$  \hspace{1cm} (12)

where $z_n \equiv (2 + i\lambda_n)$, $n = 1, 2, \ldots$. Thus, the transmission probability depends on three complex numbers namely $E_2 E_1^*$, $z_1$ and $z_2$. The last two are contained in $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. The condition for perfect tunnelling ($T = 1$), can be expressed as

$$E_2 E_1^* = \pm \frac{z_1 z_2}{|z_1 z_2|}.$$  \hspace{1cm} (13)

From this result, it is easy to see that if the two δ-function potential have the same strength ($\lambda_1 = \lambda_2 = \lambda$), then for $E_2 E_1 = -z/z^*$ one obtains $T = 1$ for any $\lambda$ (for graphical representation and other details see [39]). This can also be expressed as

$$\tan(2k(x_1 - x_2)) = \frac{4\lambda^2}{4 + \lambda^2},$$  \hspace{1cm} (14)

where the wave number is $k = \sqrt{2m\epsilon/\hbar}$, and the parameter $\lambda = 2mJ_n/k\hbar^2 = \sqrt{2mJ^2/\epsilon/\hbar}$. As the perfect tunnelling is present only when the incident particle energy is equal to
a QBS energy  \[16\], Eq. (14) is the transcendental equation for the QBS energies of an equal strength two-\(\delta\)-function potential array.

### 3.2 Three-\(\delta\)-function potential array

For a \(N = 3\) system Eq. (7) gives

\[
\mathcal{M}(3) = I - \frac{i}{2} \sum_{n_1=1}^{3} \lambda_{n_1} L_{n_1} - \frac{1}{4} \sum_{n_1,n_2=1}^{3} \lambda_{n_1} L_{n_1} \lambda_{n_2} L_{n_2} + \frac{i}{8} \lambda_3 \lambda_2 \lambda_1 L_3 L_2 L_1. \tag{15}
\]

Using the properties of the \(L\)-matrices in Eq. (8), one can easily read off the following special cases.

i) \(E_1 = E_2 = E_3\), that is \(L_1 = L_2 = L_3\) one has effectively a single \(\delta\)-function potential barrier of strength \((\lambda_1 + \lambda_2 + \lambda_3)\).

ii) For \(E_1 = E_2\) that is \(L_1 = L_2\) or \(E_2 = E_3\), that is \(L_2 = L_3\), one has an effective \(N = 2\) \(\delta\)-function potential array with strength \((\lambda_1 + \lambda_2)\) and \(\lambda_3\) or \(\lambda_1\) and \((\lambda_2 + \lambda_3)\).

This is so because for these cases \(L_3 L_2 L_1 = 0\) in Eq. (15).

iii) For a genuine \(N = 3\) array one needs the \(L_3 L_2 L_1\) term to be non-zero. In general, after a little algebra, Eq. (15) gives

\[
(\mathcal{M}(3))_{22} = \frac{1}{8} [z_1 z_2 z_3 + \lambda_1 \lambda_2 z_3 E_2 E_1^* + \lambda_1 \lambda_3 z_2 E_3 E_1^* + \lambda_2 \lambda_3 z_1 E_3 E_2^*]. \tag{16}
\]

Here \(z_n = (2 + i\lambda_n) = 2(M_n)_{22}\) for \(n = 1, 2\) and \(3\), while \(E_n = \exp(2i\phi_n)\) with \(\phi_n = kx_n\).

The first term in Eq. (16) represents the effect of single potential transfer matrices while the rest contain two independent relative phases from the wavefunction at the
three sites of the delta function potential. For perfect tunnelling ($T = 1$) these effects have to compensate each other. In other words, for $T = 1$ there will be specific relations between the strength parameters $\lambda_i$ (equivalently the phase of $z_i$) and the phase factors $E_i$ coming from the wavefunction.

For further analysis we set $E_1 = 1$. This is just a choice of the origin ($x_1 = 0$). Even so, Eq. (16) depends on five complex variables. Guided by the simplest non-trivial case of $N = 2$, we take the phases of $z_i$ which contain the strength parameters, to be equal. That is, we take $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. This reduces Eq. (16) to

$$\langle M(3) \rangle_{22} = \frac{1}{8}(z^3 + \lambda^2 z E_2 + \lambda^2 z^* E_3 + \lambda^2 z E_3 E_2^*).$$

(17)

The particle transmission ($T = 1/|\langle M(3) \rangle_{22}|^2$) is plotted in Figure 2 a), as a function of $kx_2$ and $kx_3$, with $\lambda_1 = \lambda_2 = \lambda_3 = 1$. We can see the points of perfect tunnelling, which follows a $\pi$ periodicity in both $kx_2$ and $kx_3$. In addition, notice that the perfect tunnelling points fulfil the relation $kx_3 = 2kx_2$. This behaviour can be explained if we consider the QBS energies in the three-potential array. Suppose that between every two contiguous $\delta$-function barriers, there is intrinsic QBS energy which depends on the geometry of the two barriers. Then, consider an asymmetrical array, like the one depicted in Figure 2 b). It has different values for the intrinsic QBS energies ($\epsilon_{1-2} \neq \epsilon_{2-3}$) and no perfect transmission is expected. By contrast, the symmetrical array described in Figure 2 c) presents the same geometry for every two contiguous barriers and consequently the intrinsic QBS energies are equal ($\epsilon_{1-2} = \epsilon_{1-2} = \epsilon_{qbs}$). In this case $\epsilon_{qbs}$ will be the QBS energy of the three-$\delta$-function barrier array. The necessary symmetry dictates that $x_3 = 2x_2$ or, given the periodicity of the wavefunction,
Figure 2: (Color online) Particle transmission for a genuine $N = 3$ δ-function potential array. a) Particle transmission $T$ as a function of $kx_2$ and $kx_3$, with $\lambda_1 = \lambda_2 = \lambda_3 = 1$, the white dotted line indicate the relation $kx_3 = 2kx_2$ and the black dots the perfect tunnelling. b) Scheme of an asymmetrical three δ-function potential array, where the intrinsic QBS energies are different ($\epsilon_{1-2} \neq \epsilon_{2-3}$) and $T < 1$. c) Scheme of a symmetrical three δ-function potential array, where the intrinsic QBS energies are equal ($\epsilon_{1-2} = \epsilon_{2-3}$) and we have perfect transmission.

\[
x_3 = 2x_2 + 2\pi/k.
\]

The last argument is incorporated in Eq. (17) by putting that $E_3 = E_2^2$, so that

\[
(\mathcal{M}(3))_{22} = \frac{1}{8}(z^3 + 2\lambda^2 z E_2 + \lambda^2 z^* E_2^3).
\]

Imposing the perfect tunnelling condition, that is $T = 1/(|\mathcal{M}(3)|^2 = 1$, gives

\[
\cos(2kx_2) = \frac{2 + \lambda^2 + 4\lambda \sin(2kx_2)}{\lambda^2 - 4}.
\]

This is the transcendental equation for the QBS energies of a three-equal δ-function potential array. In the specific case when $\lambda = 1$, Eq. (19) results in $kx_2 = \pi/2$ and
\[ kx_2 = \cos^{-1}(-4/5) = 2.498, \text{ which are shown in Figure 2 a).} \]

### 3.3 Four-\(\delta\)-function potential array

For a \(N = 4\) system, Eq. (7) reduces to

\[
\mathcal{M}(4) = I - i \sum_{n_1=1}^{4} L_{n_1} + \frac{1}{4} \sum_{n_1,n_2=1}^{4} \lambda_{n_1} L_{n_1} \lambda_{n_2} L_{n_2}
\]

\[
+ \frac{i}{8} \sum_{n_1,n_2,n_3=1}^{4} \lambda_{n_1} L_{n_1} \lambda_{n_2} L_{n_2} \lambda_{n_3} L_{n_3} + \frac{1}{16} \lambda_{n_1} L_{n_1} \lambda_{n_2} L_{n_2} \lambda_{n_3} L_{n_3} \lambda_{n_4} L_{n_4}. \tag{20}
\]

Note that \(\mathcal{M}(4)\) is a product of \(\mathcal{M}(2)\) for \(L_1\) and \(L_2\), and \(\mathcal{M}(2)\) for \(L_3\) and \(L_4\).

Depending on specific relations between the four phases \(E_i\) (equivalently \(L_i\)), Eq. (20) will represent arrays with \(N = 1, 2\) and 3. For a genuine \(N = 4\) array we need the quadrilinear term \(Q = L_1 L_2 L_3 L_4 \neq 0\). Using the properties of the \(L\)-matrices one can read off the following special cases:

i) The effect of resonance reduces the array to an effective \(N = 1\) system, if all \(E_i\) (or \(L_i\)) \(i = 1, 2, 3\) and 4 are equal.

ii) The array is reduced to an effective \(N = 2\) system. These can arise in 3-ways, namely a) \(L_1 = L_2 = L_3, L_4\), b) \(L_1, L_2 = L_3 = L_4\) and c) \(L_1 = L_2, L_3 = L_4\).

iii) The array is reduced to a three-\(\delta\)-function system. For an effective \(N = 3\) case, one of the trilinear products; \(\tau_1 \equiv L_4 L_3 L_1, \tau_2 \equiv L_4 L_3 L_2, \tau_3 \equiv L_3 L_2 L_1, \tau_4 \equiv L_4 L_2 L_1\) should be non-zero with the quadrilinear term \(Q = 0\). Then, two adjacent \(L\)-matrices should be equal and consequently the effective strength parameter would be the sum of the original parameters (\(\lambda_i\)'s). In this situation we distinguish the following different cases:
(a) $L_1 = L_2$, so $T_1 = T_2 = L_4 L_3 L_1$, with $T_3 = T_4 = 0$. The effective strength parameters at the three \(\delta\)-function potentials are \((\lambda_1 + \lambda_2), \lambda_3\) and \(\lambda_4\).

(b) $L_1 = L_2 = L_4$, so $T_1 = T_2 = L_1 L_3 L_1$. The effective strength parameters are as in case (a).

(c) $L_2 = L_3$, so $T_1 = T_4 = L_4 L_2 L_1$, with $T_2 = T_3 = 0$. The effective strength parameters are $\lambda_1, (\lambda_2 + \lambda_3)$ and $\lambda_4$.

(d) $L_4 = L_1$, gives $T_2 = T_4 = L_1 L_3 L_1$ gives a particular case of case (c) and it is the same as case (b) with the replacement $L_4 \rightarrow L_2$ and $\lambda_3 \rightarrow \lambda_2$.

(e) $L_4 = L_3$, gives $T_3 = T_4 = L_3 L_2 L_1$ with $T_1 = T_2 = 0$. The effective strength parameters are $\lambda_1, \lambda_2$ and $(\lambda_3 + \lambda_4)$.

(f) The choice $L_3 = L_1$ gives $T_3 = T_4 = L_1 L_2 L_1$, a particular case of case (e).

However, this case is mathematically equivalent to case (d) since the $\lambda_i$’s, the potential strength parameters are not fixed.

iv) A genuine four-\(\delta\) potential with $Q = L_1 L_2 L_3 L_1 \neq 0$. From Eq. (20) one obtains

\[
(M(4))_{22} = \frac{1}{16} (z_1 z_2 z_3 z_4 + \lambda_1 \lambda_2 z_3 z_4 E_2 E_1^* + \lambda_1 \lambda_3 z_2^* z_4 E_3 E_1^* + \lambda_1 \lambda_4 z_2^* z_3^* E_4 E_1^* + \lambda_2 \lambda_3 z_1 z_4 E_3 E_2^* + \lambda_2 \lambda_4 z_1 z_3^* E_4 E_2^* + \lambda_3 \lambda_4 z_1 z_2 E_4 E_3^* + \lambda_1 \lambda_2 \lambda_3 \lambda_4 E_4 E_2 E_3^* E_1^* ).
\] (21)

To find the conditions for perfect tunnelling in a genuine $N = 4$ case, we choose the origin at $x_1 = 0 \ (E_1 = 1)$, and we suppose that all \(\delta\)-potential have the same strength $(\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda)$. We expect the existence of intrinsic QBS energies between every two contiguous potential, which have to be all equal to allow perfect tunnelling. As these energies depend on the potential separation, we infer that all separations are
equal, meaning that $E_4 = E_2^3$ and $E_3 = E_2^2$. With this considerations Eq. (21) reduces to

$$(\mathcal{M}(4))_{22} = \frac{1}{16}(z^4 + \lambda^2 E_2(3z^2 + 2|z|^2 E_2 + z^*^2 E_2^2 + \lambda^2 E_2)). \quad (22)$$

Imposing the perfect tunnelling condition $T = 1/|(\mathcal{M}(3))_{22}|^2 = 1$, we obtain

$$4 \cos(2kx_2) + 6\lambda^2 \sin(kx_2) = \lambda \tan(kx_2)(2 - 12 \cos^2(kx_2) - \lambda^2 \sin^2(kx_2)), \quad (23)$$

which is the transcendental relation to find the QBS energies of a four-equal strength $\delta$-function potential array.

It should be noted that a special situation is present in the $N = 4$ potential array; when the first two and the last two potentials have perfect tunnelling separately. In this case, the four-$\delta$-function array can be seen as a system of two pairs of sub-arrays or cells with perfect tunnelling conditions. As we will discuss in the next section, we expect perfect tunnelling if the QBS energies of these two potential cells are the same.

### 4 Results for arbitrary $N$

As we seen in the cases of specific arrays, it is always possible to use the resonance of the wavefunction with the geometry of the potential array to effectively reduce the number of potentials, and the complexity of the problem. This reduction is allowed only when the resonance is present in a row of adjacent potential.

Similar to that seen in the previous section, to obtain perfect tunnelling in a $N$ equal-strength $\delta$-function potential array, we can locate them in a series with the same distance one after the other. This assures us the matching of all the intrinsic QBS energies which
will be the QBS energies of the whole array. Another way to obtain perfect tunnelling in an \( N \, \delta \)-function potential array is to separate it in cells of potentials, with perfect tunnelling conditions by themselves, as we in Figure 3 a). We expect to have perfect tunnelling when the incident particle energy is equal to one QBS energy in every cell. The advantage of this method is that in principle, the two cells in Figure 3 a) can be located at any distance one from each other with some resonance in the non perfect tunnelling energies.

Figure 3: (Color online) Schemes of perfect tunnelling in a cells configuration. a) Two perfect tunnelling cells can be located one after the other, at any separation. b) Two equal cells of three \( \delta \)-function potentials are located one after the other. It is expected perfect tunnelling in the QBS energies of the cells. c) Two different cells (three and two \( \delta \)-function potentials) are located in series. As they have different QBS energies, we expect to have perfect tunnelling only when two of these energies match.

For example, if we locate two or more identical cells, we can expect perfect tunnelling
at the incident energies which match all the QBS energies of a single cell (see Figure 3 b)). By contrast, if the cells are different, such as three and two $\delta$-function potentials as is shown in Figure 3 c), we only have the perfect tunnelling conditions when incident energy is equal to one QBS energy in every cell. Otherwise we can expect maximal values in the transmittivity but not perfect tunnelling.

Figure 4: (Color online) Perfect tunnelling selectivity in GaAs quantum wire. a) Particle transmittivity of two different two-$\delta$-function potential arrays as a function of incident energy. In blue line the transmittivity of two equal $\delta$-function potential barriers with $J = 2$ eVÅ, separated by 100 nm. In red line the transmittivity of two equal $\delta$-function potential barriers with $J = 2$ eVÅ, separated by 29 nm. Matching perfect tunnelling is found at 4.56 meV. b) Particle transmittivity of a four-$\delta$-function potential array as a function of incident energy. The four potential is arranged in two cells of two-$\delta$-function potentials depicted in a) separated by 150 nm. Perfect tunnelling is found at 4.56 meV.

The difference in QBS energies of each cell can be used as an energy selector or filter; showing perfect tunnelling only at one defined value of incident energy, and preventing the transmission in other energy values. To illustrate our proposal we consider a GaAs quantum wire (where the effective mass of electron is 0.067 times the electron mass)
and locate two $\delta$-function potential barriers with equal strength ($J = 2$ eVÅ). The transmitivity as a function of incident energy in this case is shown in Figure 4 a), in blue line when the potentials are located 100 nm one to each other. The red line shows the transmitivity when the potentials are separated by 29 nm. As can be seen, both cases share a QBS energy at 4.56 meV. Now, we take these two cases and put them together in series separated by 150 nm, to form a four-$\delta$-function potential array. In the Figure 4 b) we show the transmitivity as a function of the incident energy in this four-$\delta$-function potential array. Note that in this configuration the perfect tunnelling energy at 4.56 meV is preserved, while the other maximum values disappear. In this example the two cells behave like potential barriers for incident energies different than QBS energies. Considering the cells as potential barriers, the arbitrary separation between the two cells (in the example 150 nm) can create resonances for different incident energies, but never perfect tunnelling, unless the two cells are equal, as it was shown in [39].

5 Conclusions

Conditions for resonant transmission are considered, in detail, for arrays with $N = 1$, 2, 3 and 4. These results were applied to potential arrays with arbitrary $N$, using a simple representation for the transmission matrix. The resonant behaviour is also the cause of the perfect tunnelling present in these systems. In this context, we calculate the transcendental relations for QBS energies in specific arrays, using the concept of intrinsic QBS energies. For arrays with arbitrary $N$, we propose the separation of the array into subsets or cells, whose QBS energies are related with the QBS energies of the whole system. Using the relation between intrinsic QBS energies we showed how a
energy filter works.

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