New $G_2$ holonomy metrics, D6 branes with inherent $U(1) \times U(1)$ isometry and $\gamma$-deformations

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Abstract

It is found the most general local form of the 11-dimensional supergravity backgrounds which, by reduction along one isometry, give rise to IIA supergravity solutions with a RR field and a non trivial dilaton, and for which the condition $F^{(1,1)} = 0$ holds. This condition is stronger than the usual condition $F^{ab} J_{ab} = 0$, required by supersymmetry. It is shown that these D6 wrapped backgrounds arise from the direct sum of the flat Minkowski metric with certain $G_2$ holonomy metrics admitting an $U(1)$ action, with a local form found by Apostolov and Salamon. Indeed, the strong supersymmetry condition is equivalent to the statement that there is a new isometry on the $G_2$ manifold, which commutes with the old one; therefore these metrics are inherently toric. An example that is asymptotically Calabi-Yau is presented. There are found another $G_2$ metrics which give rise to half-flat $SU(3)$ structures. All this examples possess an $U(1) \times U(1) \times U(1)$ isometry subgroup. Supergravity solutions without fluxes corresponding to these $G_2$ metrics are constructed. The presence of a $T^3$ subgroup of isometries permits to apply the $\gamma$-deformation technique in order to generate new supergravity solutions with fluxes.

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1. Introduction

The AdS/CFT correspondence is a powerful tool in order to study strongly coupled regimes in gauge theories [1]. It relates field theories with gravitational theories satisfying particular boundary conditions. The original statement of the correspondence was that $\mathcal{N} = 4$ super-Yang Mills theory is dual to type IIB strings in $AdS^5 \times S^5$. This pioneer conjecture was developed further in [2] and even it was generalized to non conformal field theories [3].

In the particular case of $\mathcal{N} = 4$ super-Yang Mills there exist a three parameter deformations of its superpotential that preserves $\mathcal{N} = 1$ supersymmetry, these deformations are called $\beta$ deformations [10]. The original superpotential of the theory is transformed in the following way

$$Tr(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \rightarrow hTr(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) + h'\text{Tr}(\Phi_1^2 + \Phi_2^2 + \Phi_3^2),$$

being $h, h', \beta$ complex parameters, satisfying one condition by conformal invariance. One can select $h' = 0$. Besides the $U(1)_R$ symmetry, there is a $U(1) \times U(1)$ global symmetry generated by

$$U(1)_1: \ (\Phi_1, \Phi_2, \Phi_3) \rightarrow (\Phi_1, e^{i\varphi_1}\Phi_2, e^{-i\varphi_1}\Phi_3),$$

$$U(1)_2: \ (\Phi_1, \Phi_2, \Phi_3) \rightarrow (e^{-i\varphi_2}\Phi_1, e^{i\varphi_2}\Phi_2, \Phi_3),$$

which leaves the superpotential and the supercharges invariant. Therefore there is a two dimensional manifold of $\mathcal{N} = 1$ CFT with a torus symmetry. The physics contained in the deformed model is periodic in the variable $\beta$, which is parameterized as

$$\beta = \gamma - \tau s \sigma$$

where $\gamma$ and $\sigma$ are real variables with period one. The variable $\beta$ can be considered living on a torus with complex structure $\tau_s$, where $\tau_s$ is related to gauge coupling and theta parameter of the field theory [5]. The theory has an $SL(2, Z)$ duality group, in which $\beta$ transforms as a modular form

$$\tau_s \rightarrow \frac{a\tau_s + b}{c\tau_s + d}, \quad \beta \rightarrow \frac{\beta}{c\tau_s + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, Z).$$

If $\beta$ is chosen to be real then $\sigma = 0$. We will focus mainly in the case $\sigma = 0$, the corresponding deformations are called $\gamma$ deformations.

An interesting problem is to know how these deformations acts on the gravity dual, which has an $U(1) \times U(1)$ subgroup which is realized as an isometry. The answer is that the $\gamma$-deformation of $\mathcal{N} = 4$ super Yang-Mills induce in the gravity dual the simple transformation

$$\tau \equiv B + i\sqrt{g} \rightarrow \tau \rightarrow \tau' = \frac{\tau}{1 + \gamma \tau},$$

where $\sqrt{g}$ is the volume of the two torus [6]. The transformation (1.5) is in fact, a well known solution generating technique [4]. It is well known that when a closed string theory is compactified on a two torus the resulting eight dimensional theory has an exact $SL(2, Z) \times SL(2, Z)$ symmetry, which acts on the complex structure of the torus and on certain parameter

$$\tau = B_{12} + i\sqrt{g},$$
where $\sqrt{g}$ is the volume of the two torus in string metric. At supergravity level there is an enlarged $SL(2, R) \times SL(2, R)$ symmetry, which is not a symmetry of the full string theory. These $SL(2, R)$ symmetries can be used as solution generating transformation, at supergravity level.

As the reader can check, the transformations (1.5) are not the full $SL(2, R)$ transformations. Indeed (1.5) is the subgroup of $SL(2, R)$ for which $\tau \to 0$ implies that $\tau' \to 0$. Transformations with these properties are the only possible ones mapping a ten dimensional geometry which is non singular to a new one also without singularities. The reason is that the only points where a singularity can be introduced by performing an $SL(2, R)$ transformation is where the two torus shrinks to zero size. This shrink happens when $\tau' \to 0$ but for $\gamma$ transformations, this implies that $\tau \to 0$. Therefore, if the original metric was non-singular, then the deformed metric is also non singular [6]. The transformation (1.5) is the result of doing a T-duality on one circle, a change of coordinates, followed by another T-duality. This is another reason for which it can be interpreted as a solution generating technique [4].

In the present work new $G_2$ holonomy metrics will be constructed and the deformation procedure will be used to generate new supergravity backgrounds. The local form of such $G_2$ metrics is [8]

$$g_7 = \frac{(d\chi + A_2)^2}{\theta^2} + \theta [ u \, d\theta^2 + \frac{(dv + A_1)^2}{u} + g_4(\theta) ]. \tag{1.7}$$

All the quantities appearing in (1.7) are independent on the coordinates $v$ and $\chi$, and the vector fields $\partial_\chi$ and $\partial_v$ are Killing and commuting. Therefore by construction these metrics are toric. The metric $g_4(\theta)$ is a 4-metric at each level of constant $\theta$-surfaces. If $g_4(\theta)$ admit more isometries, then the full isometry group of (1.7) can be enlarged sometimes. In certain situations examples with $T^3$ or larger isometry groups can be generated. The undefined quantities appearing in the expression (1.7) are not arbitrary, the condition of holonomy to be included in $G_2$ gives an system of evolution equations relating all them [8].

There are different reasons for taking these 7-metrics as starting point. The general form (1.7) encodes known examples as particular subcases. For instance, the homothetic $G_2$ metrics, arising from an $SO(5)$ invariant $G_2$ metric by contraction of the isometry group, with local form

$$g_7 = \frac{(dv - xdz + ydc)^2}{\theta^2} + \frac{(d\chi - ydz - xdc)^2}{\theta^2} + \theta^4 \, d\theta^2 + \theta^2 \, ( dx^2 + dy^2 + dz^2 + dc^2 ),$$

found in [11], is an special subcase of (1.7). More general examples are also included. There are known some explicit examples of $G_2$ holonomy metrics [12]-[27], and one of the purpose of the present work is to enlarge the list.

There is another reason to consider the metrics (1.7). As is known, the problem of finding the conditions for preserving supersymmetry in type IIA string theory in the presence of a RR vector field and a nontrivial dilaton can be derived from reduction of the supersymmetry conditions of eleven-dimensional supergravity along certain isometry. If a further constraint $F^{(1,1)} = 0$ is imposed, which is stronger than the usual condition $F_{ab} J_{ab} = 0$, required by supersymmetry, then the ten-dimensional string frame metric in the presence of the D6-branes contains an internal 6-space that is Kahler. As it was shown in [7] such kind of geometries are characterized by certain holomorphic monopole equation. By use of the results given in [8] we will prove that indeed the metrics (1.7) parameterize the solution of the holomorphic monopole equation and the constraint $F^{(1,1)} = 0$. It will be shown that both conditions implies
that a second isometry is inherently defined on both the 7-dimensional $G_2$ manifold and the 6-dimensional Kahler manifold. Therefore the $G_2$ metrics characterized by these conditions are inherently toric, and in some cases they admit a larger group of isometries like $U(1)^3$ or others. One can immediately construct backgrounds that are suitable to apply the $\gamma$ solution generating technique. We are focused in the gravity part of the conjectured duality, mainly because we do not know yet the quantum field theory dual of our backgrounds.

In the first part of the present work it will be shown the reason for which these are the metrics solving the condition $F^{(1,1)} = 0$ and the holomorphic monopole equation. Explicit examples will be found, some of them are known and other are new. In the second part the $\gamma$-deformed backgrounds corresponding to these $G_2$ holonomy metrics are presented.

2. Toric D6 brane backgrounds with strong supersymmetry conditions

2.1 Wrapped IIA solutions with Kahler internal geometry

It is an standard fact that if the holonomy of a seven manifold $Y$ is included in $G_2$, then from the very beginning there exist at least one globally defined covariantly constant spinor $\eta$ over $Y$, that is, an spinor satisfying $D\eta = 0$ over $Y$, being $D$ standard covariant derivative in the representation of the field. This fact implies that $Y$ is Ricci-flat. It also implies that the curvature two-form $R_{ab}$ of $Y$ is octonion self-dual, that is

$$R_{ab} = \frac{c_{abcd}}{2} R_{cd},$$

being $c_{abcd}$ the dual octonion multiplication constants. The self-duality of the curvature implies the existence of a frame for which the spin connection $\omega_{ab}$ is also self-dual. All these conditions are equivalent to the existence of a $G_2$ invariant closed and co-closed three form $\Phi$ [12].

Along the present section, a generic solution of the eleven dimensional supergravity in which the fermions and the four form $F$ are zero is considered. Such solution is of the form

$$g_{11} = g_{(3,1)} + g_7. \quad (2.8)$$

The metric $g_7$ is defined on a seven manifold $Y$, and we assume that is non compact, with holonomy included in $G_2$ and that there is at least one $U(1)$ isometry. Without losing generality, the Killing vector field can be written as $\partial_\chi$, being $\chi$ certain coordinate. Then the full supergravity background takes the usual IIA form

$$g_{11} = e^{-2\alpha\phi} g_{10} + e^{2\beta\phi} (d\chi + A)^2 = g_{(3,1)} + g_7. \quad (2.9)$$

The metric $g_{10}$ is the physical metric IIA in ten dimensions obtained by reduction along the $U(1)$ isometry; the parameters $\alpha$ and $\beta$ determine the frame, with values $(\alpha, \beta) = (1/3, 2/3)$ for the string frame. The $G_2$ metric can be decomposed as

$$g_7 = e^{-2\alpha\phi} g_6 + e^{2\beta\phi} (d\chi + A)^2, \quad (2.10)$$

and the expression for $g_{10}$ is

$$g_{10} = e^{2\alpha\phi} g_{(3,1)} + g_6. \quad (2.11)$$
The metric $g_6$ is a six dimensional metric defined over a manifold $N$. The pair $(N, g_6)$ the Riemannian quotient of $(Y, g_7)$ by the $U(1)$ isometry, so that $N$ is a 6-dimensional manifold formed from the orbits of the Killing vector field $V = \partial \chi$. The IIA reduction gives the background

$$g_{10} = e^{2\alpha \phi} g_{(3,1)} + g_6, \quad F = dA$$

being $F$ the RR two form. The coordinate $\phi$ is interpreted as the dilaton field. As is was shown in [7], the supersymmetry condition $D\eta = 0$ implies the following system for the six dimensional manifold

$$F_{ab}J_{ab} = 0, \quad d(e^{-2\beta \phi}) = -* (e^{-\alpha \phi} \psi_3 \wedge F).$$

The second (2.13) is known as an holomorphic monopole equation in the terminology of the reference [7]. The following part is intended to find an explicit form of the $G_2$ holonomy metric (2.10) (and an explicit form of the background $g_{10}$) under the assumption that $g_6$ is Kahler. The Kahler condition implies that $F^{(1,1)} = 0$ for the six dimensional manifold, which is a requirement stronger that the first (2.13).

### 2.1.1 The IIA reduction and holomorphic monopole equations

If the internal manifold has holonomy in $G_2$, then the four dimensional theory obtained by dimensional reduction over the $G_2$ background will have $\mathcal{N} = 1$ supersymmetry. As is sketched above, the $G_2$ holonomy condition implies the existence of a closed and co-closed form $\Phi$ defined over our seven manifold $Y$. It always exist a seven-bein $e^a$ for which the metric $g_7$ is written in diagonal form $g = \delta_{ab} e^a \otimes e^b$, and for which the three form $\Phi$ is expressed as

$$\Phi = \frac{1}{3!} c_{abc} e^a \wedge e^b \wedge e^c,$$

being $c_{abc}$ the octonion multiplication constants. The fact that (2.14) is $G_2$ invariant is a consequence that $G_2$ is the automorphism group of the octonions. The holonomy of $Y$ will be included in $G_2$ if and only if [12]

$$d\Phi = d* \Phi = 0.$$

We will suppose that the Killing vector $V$ not only preserve $g_7$ but also $\Phi$. It means that the three form $\Phi$ is decomposed as

$$\Phi = e^{-3\alpha \phi} \psi_3 + e^{-2\alpha \phi} J \wedge e^z,$$

being $e^z = d\chi + A$ the co-tangent vector associated to the $\chi$ direction and $\psi_3$ certain 3-form, and the factors associated to the dilaton $\phi$ in (2.16) have been introduced by convenience. Then the two form $J = i_V \Phi$ satisfy

$$i_V J = 0,$$

where $i_V$ denote the contraction with the vector field $V$. A two form satisfying (2.17) is called horizontal and as a consequence of horizontality and that $V$, by assumption, preserve the $G_2$ structure, it is obtained that

$$dJ = d(i_V \Phi) = L_V \Phi = 0,$$

$$L_V J = d(i_V J) = 0.$$
The first of conditions (2.18) implies that $\mathcal{J}$ is closed, and the second that $\mathcal{J}$ it is preserved by $V$. It also hold the decomposition

$$
\ast \Phi = e^{-\alpha \phi} \psi_3' \wedge e^z + e^{-2\beta \phi} \mathcal{J} \wedge \mathcal{J},
$$

(2.19)

where the 3-form $\psi_3'$ is defined by

$$
e^{-\alpha \phi} \psi_3' = i_V(\ast \Phi),
$$

(2.20)

The form $e^{-\alpha \phi} \psi_3'$ is also closed and preserved by $V$, the proof has similarities with the one for $\mathcal{J}$. The antisymmetric tensor $J$ defined through the relation $g_6(J, \cdot, \cdot) = \mathcal{J}(\cdot, \cdot)$ is an almost complex structure, that is, it satisfies $J \cdot J = -I$. From (2.18) it is seen that the vector $V$ preserve $\mathcal{J}$, such isometries are called holomorphic. But a Killing vector preserving $\mathcal{J}$ always preserve $J$, such isometries are known as hamiltonian isometries. Therefore $V$ is a Killing, hamiltonian and holomorphic vector field.

Due to the fact that $\Phi \wedge \ast \Phi \sim \omega_7$, the three forms $\psi_3$ and $\psi_3'$ together with $\mathcal{J}$ should satisfy the following compatibility conditions

$$
\psi_3 \wedge \psi_3' = \frac{2}{3} \mathcal{J} \wedge \mathcal{J} \wedge \mathcal{J} = 4\omega_6,
$$

(2.21)

$$
\psi_3 \wedge \mathcal{J} = \psi_3' \wedge \mathcal{J} = 0,
$$

and therefore they conform an $SU(3)$ structure. Here $\omega_6$ and $\omega_7$ are the volume forms on $Y$ and $N$ respectively. The theory of $G_2$ structures gives the relations [9]

$$
\psi_3' (\cdot, \cdot, \cdot) = -\psi_3' (J \cdot, \cdot), \quad \psi_3 (\cdot, \cdot, \cdot) = \psi_3' (J \cdot, \cdot, \cdot) \quad (2.22)
$$

From the first (2.22) it is deduced that $\psi_3'$ has type $(0, 3) + (3, 0)$ with respect to $J$, and from the second it follows that the complex three form $\psi = \psi_3 + i\psi_3'$ is of $(0, 3)$ type with respect to $J$.

In [7]-[11] there were worked out the consequences of the $G_2$ conditions (2.15) for a manifold with a Killing vector $V$ preserving the $G_2$ structure, as in our case. It was found that one can reduce to six dimensions these equations and divide the resulting equations into pieces containing or not $e^z$. The system that is finally obtained is the following

$$
d(e^{-3\alpha \phi} \psi_3) + e^{(\beta - 2\alpha) \phi} \mathcal{J} \wedge F = 0, \quad d(e^{(\beta - 2\alpha) \phi} \mathcal{J}) = 0, \quad d(e^{-4\alpha \phi} (\ast \mathcal{J})) - e^{(\beta - 3\alpha) \phi} (\ast \psi_3) \wedge F = 0, \quad d(e^{(\beta - 3\alpha) \phi} (\ast \psi_3)) = 0.
$$

(2.23)

The Hodge operation $\ast$ is referred the physical metric $g_6$ and we have defined the "strength" 2-form $F = dA$.

Using the relation $\beta = 2\alpha$ of the ten-dimensional string frame it is obtained from the second equation (2.23) that $d\mathcal{J} = 0$. Therefore $J$, $\mathcal{J}$ and $g_6$ constitute an almost Kahler structure. As is well known, this structure will be Kahler if and only if the complex structure $J$ is integrable, that is, if its Niejenhuis tensor

$$
N^J(X, Y) = [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY]
$$

vanish identically. Let us assume that the Kahler condition hold and derive its consequences. If this is so, then the manifold is complex and it always exist a system of complex coordinates $(z^a, \bar{z}^\alpha)$ for which the tensors $\mathcal{J}$, $e^{-\alpha \phi} \psi_3$ and $e^{-\alpha \phi} \psi_3'$ take the simple form

$$
\mathcal{J} = \frac{i}{2} dz^a \wedge d\bar{z}^\alpha,
$$
\[ e^{-\alpha \phi} \psi_3 = \frac{1}{2} \left( \frac{1}{3!} \epsilon_{abc} dz^a \wedge dz^b \wedge dz^c + \frac{1}{3!} \epsilon_{abcd} dz^a \wedge dz^b \wedge dz^c \right), \quad (2.24) \]

\[ *e^{-\alpha \phi} \psi_3 = \frac{i}{2} \left( \frac{1}{3!} \epsilon_{abc} dz^a \wedge dz^b \wedge dz^c - \frac{1}{3!} \epsilon_{abcd} dz^a \wedge dz^b \wedge dz^c \right), \]

being \( dz^a \) and \( d\bar{z}^a \) frames on the holomorphic and anti-holomorphic cotangent bundle, respectively. The integrability condition implies that these forms are all closed. In this system of coordinates it is also simpler to see that the (2, 2) part of the first equation (2.23) implies that \( F^{(1,1)} \wedge J = 0 \), from where we obtain the well known result that a Kahler reduction implies that \[ F^{(1,1)} = 0, \quad (2.25) \]
a condition stronger than the usual one \( F^{ab} J_{ab} = 0 \), required by supersymmetry. We will to (2.25) as a strong supersymmetry preserving condition. If this condition is not satisfied, then the internal manifold \( N \) will be an \( SU(3) \) torsion manifold in general [9].

The (3, 1) and (1, 3) components of the first equation (2.23), together with the fourth equation, give the generalized monopole equations

\[ d(e^{-2\beta \phi}) = -* (e^{-\alpha \phi} \psi_3 \wedge F), \quad (2.26) \]

which in the basis defined by \( dz^a \) and \( d\bar{z}^a \) takes the form

\[ \partial(e^{-2\beta \phi}) = -* (e^{-\alpha \phi} \psi_3^{(3,0)} \wedge F^{(2,0)}). \quad (2.27) \]

The third equation (2.23) can be rewritten in the form

\[ d(*J) - 4\alpha d\phi \wedge *J - F \wedge *\psi_3 = 0, \quad (2.28) \]

but \( *J = \frac{1}{2} J \wedge J \) and this, together with the closure of \( J \) implies that \( d(*J) = 0 \). The sum of the last two terms of (2.28) results again into another form of the monopole equations (2.26). Equation (2.23) also implies that the internal six manifold admits a gauge covariantly constant spinor playing the role of supersymmetry generator [7]. The system (2.23) also characterize the sub-manifold \( M \) which the \( N \) supersymmetric D6-branes wrap. Being a magnetic source of charge \( N \) for the two form \( F \), one has \( dF = N \delta_M \), where \( \delta_M \) is the Poincare dual three-form of the cycle \( M \). One can deduce from (2.23) that

\[ J \wedge \delta_M = 0, \quad e^{-\alpha \phi} (*\psi_3) \wedge \delta_M = 0, \quad (2.29) \]

and the first of these equations implies that \( M \) is a Lagrangian cycle in the internal Kahler manifold.

A new isometry from the strong supersymmetry condition

The next task is to work out in detail the consequences of all the equations presented above. It is convenient, in order to simplify the following formulas and in order to compare them with those appearing in the mathematical literature, to define a new field (or a new coordinate)

\[ \theta = e^{-2\alpha \phi} = e^{-\beta \phi}, \quad \phi = -\frac{1}{2\alpha} \log(\theta), \quad (2.30) \]

which is entirely defined in terms of the dilaton. The monopole system (2.26) can be rewritten as

\[ d(*J) - 2 \log(\theta) \wedge *J - F \wedge *\psi_3 = 0. \quad (2.31) \]
Equation (2.23) is also reexpressed as
\[ \mathcal{J} \wedge dA + \theta^{1/2}d\theta \wedge \psi_3 = 0. \tag{2.32} \]
Now comes the crucial point. Let us define a vector field \( U \) by
\[ i_U \mathcal{J} = -d\theta. \tag{2.33} \]
From (2.33) it is concluded that
\[ \mathcal{L}_U \mathcal{J} = i_U d\mathcal{J} + d(i_U \mathcal{J}) = d(i_U \mathcal{J}) = -d(d\theta) = 0, \]
being \( i_U \) the contraction with the field \( U \). Therefore \( \mathcal{L}_U \mathcal{J} = 0 \) and \( U \) is an holomorphic vector field. We will prove now the crucial formula
\[ dA = -\theta^{1/2}i_U \psi_3, \tag{2.34} \]
which holds as a direct consequence of the (1,1) part of (2.32). A good consistency check of (2.32) comes from realizing that combining (2.33) and (2.34) with (2.32) gives
\[ \mathcal{J} \wedge dA + \theta^{1/2}d\theta \wedge \psi_3 = -\mathcal{J} \wedge \theta^{1/2}i_U \psi_3 - \theta^{1/2}i_U \mathcal{J} \wedge \psi_3 = -\theta^{1/2}i_U (\mathcal{J} \wedge \psi_3) = 0, \]
due to the second \( SU(3) \) condition (2.21). Therefore (2.34) implies (2.32). We need to prove the converse, that is, that (2.32) implies (2.34). Let us suppose that \( U \) is unitary (otherwise we will multiply it by its norm). An \( SU(3) \) basis \((e_i, J e_i)\) can be defined, in which \( e_1 = U \) and \( Je_1 = JU \). Then the complex basis \((z_i, \bar{z}_i)\) with \( z_i = e_i - iJe_i \) can be constructed. In particular \( z_1 = U - iJU, \bar{z}_1 = z_1^* \) and \( U = z_1 + \bar{z}_1 \). The dual basis is denoted here with upper indices as \((z^i, \bar{z}^i)\). The expression of \( \mathcal{J} \) and \( \theta^{1/2} \psi_3 \) in this basis is
\[ \mathcal{J} = z^1 \wedge \bar{z}^1 + z^2 \wedge \bar{z}^2 + z^3 \wedge \bar{z}^3, \quad \theta^{1/2} \psi_3 = -\Im(z^1 \wedge z^2 \wedge z^3), \tag{2.35} \]
where \( \Im \) denotes the imaginary part. The relation (2.33) implies that
\[ d\theta = -i_U \mathcal{J} = -i_{(z_1 + \bar{z}_1)} \mathcal{J} = -i_{z_1} \mathcal{J} - i_{\bar{z}_1} \mathcal{J}, \]
and from the last expression together with the first (2.35) it is found that
\[ d\theta(z_j) = -\mathcal{J}(z_1, z_j), \quad d\theta(z_j) = -\mathcal{J}(\bar{z}_1, z_j), \]
and this implies that
\[ d\theta(z_2) = d\theta(z_2) = d\theta(z_3) = d\theta(z_3) = 0, \]
so that, evaluating the equality (2.32) at the vectors \((z_j, z_j, z_i, z_k), (z_j, \bar{z}_j, z_i, \bar{z}_k), (z_j, z_j, \bar{z}_i, z_k)\) and \((z_j, \bar{z}_j, z_i, \bar{z}_k)\) and using the second (2.35) gives
\[ dA(z_i, z_k) = -\theta^{-1/2}\delta_{1j}\psi_3(z_j, z_i, z_k), \quad dA(\bar{z}_i, \bar{z}_k) = -\theta^{-1/2}\delta_{1j}\psi_3(\bar{z}_j, \bar{z}_i, \bar{z}_k) \tag{2.36} \]
\[ dA(\bar{z}_i, z_k) = -\theta^{-1/2}\delta_{1j}\psi_3(z_j, z_i, z_k) = 0, \quad dA(z_i, \bar{z}_k) = -\theta^{-1/2}\delta_{1j}\psi_3(z_j, z_i, \bar{z}_k) = 0. \]
The last two expressions shows that \( F = dA \) is of type \((2,0) + (0,2)\), which is in agreement with \( F^{(1,1)} = 0 \). We also have that the components
\[ \psi_3(z_j, z_i, z_k) = \psi_3(z_j, z_i, z_k) = \psi_3(\bar{z}_j, \bar{z}_i, z_k) = \psi_3(z_j, z_i, \bar{z}_k) = 0, \]
and so they can be conveniently added to (2.36) to give

\[
\begin{align*}
    dA(z_i, z_k) &= -\theta^{-1/2}\delta_{ij}\psi_3(z_j, z_i) - \theta^{-1/2}\delta_{1j}\psi_3(\bar{z}_j, z_i, z_k), \\
    dA(\bar{z}_i, \bar{z}_k) &= -\theta^{-1/2}\delta_{ij}\psi_3(\bar{z}_j, \bar{z}_i) - \theta^{-1/2}\delta_{1j}\psi_3(z_j, \bar{z}_i, \bar{z}_k),
\end{align*}
\]

and also

\[
\begin{align*}
    dA(\bar{z}_i, z_k) &= -\theta^{-1/2}\delta_{ij}\psi_3(\bar{z}_j, \bar{z}_i, z_k) - \theta^{-1/2}\delta_{1j}\psi_3(z_j, \bar{z}_i, \bar{z}_k) = 0, \\
    dA(z_i, \bar{z}_k) &= -\theta^{-1/2}\delta_{ij}\psi_3(z_j, \bar{z}_i, \bar{z}_k) - \theta^{-1/2}\delta_{1j}\psi_3(\bar{z}_j, \bar{z}_i, \bar{z}_k) = 0,
\end{align*}
\]

from where it clearly follows (2.34), which is the formula that we wanted to show.

From the identity

\[
    d(dA) = d(-\theta^{1/2}i_U\psi_3) = d(-i_U\psi_3) = \mathcal{L}_U\psi_3 = \theta^{1/2}\mathcal{L}_U\psi_3 = 0,
\]

it follows that \(U\) also preserve \(\psi_3\). Therefore \(U\) is an isometry of \(\mathcal{J}\) and \(\psi_3\). From the results of [29] it is known that a vector field preserving \(\mathcal{J}\) and \(\psi_3\) also preserve \(J\). Therefore \(U\) is not only holomorphic, but also hamiltonian. But an holomorphic and hamiltonian vector field is always Killing, that is, it preserve the metric \(g_6\). The \(SU(3)\) structure \((J, \mathcal{J}, \psi_3)\) is independent of the coordinate \(\chi\), then \(U\) can be selected independent of \(\chi\) and therefore it commutes with \(V = \partial_\chi\). This means that the isometry group is at least \(T^2 = U(1) \times U(1)\). It also seen from (2.33) that the variable \(\theta\) is the momentum map corresponding to \(U\).

In conclusion, the monopole system (2.26) is the \((3,1)\) part of the (2.32), and (2.26) is satisfied if and only if (2.34) is satisfied. This fact provides the link between the formalism of the references [8] and [7]. Therefore we have the following statement:

**Corollary** If a ten-dimensional string frame metric in the presence of the D6-branes is obtained by reduction of an eleven dimensional background with a \(G_2\) holonomy internal manifold which possess a Killing vector that preserve the \(G_2\) structure and is a warped product over a Kahler 6-dimensional metric, then the original \(G_2\) metric is necessarily toric, i.e, it has at least \(T^2 = U(1) \times U(1)\) isometry group.

**The torsion classes of the \(SU(3)\) structures**

From the previous discussion is clear that if a \(G_2\) holonomy metric admits a Kahler reduction along an \(U(1)\) isometry \(V\) implies the presence of a new holomorphic isometry \(U\) (which is therefore hamiltonian) such that \([U, V] = 0\), together with conditions (2.34) and \(F_{(1,1)} = 0\). The \(G_2\) structure is in this case

\[
    g_7 = \theta g_6 + \frac{(d\chi + A)^2}{\theta^2},
\]

\[
    \Phi = \theta^{3/2}\psi_3 + \mathcal{J} \wedge e^z, \tag{2.38}
\]

\[
    *\Phi = \theta^{1/2}\psi_3' \wedge e^z + \frac{1}{2}\theta^2 \mathcal{J} \wedge \mathcal{J}. \tag{2.39}
\]

The next task is to work out the consequences of the strong supersymmetry condition \(F_{(1,1)} = 0\). For this purpose, it is convenient to find the corresponding torsion classes for our \(SU(3)\) structure. In general, the five torsion classes of a given \(SU(3)\) structure are defined by [9]

\[
    d\mathcal{J} = \frac{3i}{4}(W_1 \psi^* - \overline{W_1} \psi) + W_4 \wedge \mathcal{J} + W_3, \tag{2.40}
\]
\[ d\psi = W_1 J \land J + W_2 J \land W_5^* \land \psi, \quad (2.41) \]

together with the conditions
\[ J \land W_3 = J \land J \land W_2 = \psi \land W_3 = 0. \]

It is seen from the closure of \( \bar{J} \) that \( W_1 = W_4 = W_3 = 0 \). The integrability condition implies that \( W_2 = 0 \), which is our assumption. Therefore the only non vanishing class is \( W_5 \) and it is defined by
\[ d\psi = W_5^* \land \psi. \quad (2.42) \]

It follows that \( W_5 \) is different from zero although \( W_4 = 0 \). This is, in principle, a situation different than those considered in [30]-[31], which arise in heterotic supersymmetric compactifications with fluxes and condensates. In order to define \( W_5 \) it is useful to consider the scaled three form
\[ \Psi = \Psi_3 + i\Psi_3' = \theta^{1/2}\psi. \quad (2.43) \]

As is seen below (2.20), \( \Psi_3' \) is preserved by \( V \) and \( d\Psi_3' = 0 \). The Kahler assumption implies that \( d\Psi_3 = 0 \). This also means that the transformation \( \psi \rightarrow \theta^{1/2}\psi \) takes the class \( W_5 \) to zero, and therefore this class is defined by means of a gradient. After some calculation it is finally obtained that
\[ d\psi_3 = -\frac{3}{2}d\log \theta \land \psi_3', \quad d\psi_3' = \frac{3}{2}d^c \log \theta \land \psi_3 \quad (2.44) \]

being \( d^c = Jd \) defined over the six manifold \( N \) [8]. Condition (2.44) is equivalent to \( F^{(1,1)} = 0 \) and should be supplied to (2.34) and to the requirement that \( U \) is an isometry, in order to have a Kahler reduction.

2.1.2 The \( G_2 \) toric metric and the logarithmic dilaton "evolution"

The presence of the new isometry \( U \) of the \( G_2 \) space \( Y \) allows to make a further reduction to a four dimensional \( M \) possessing a complex sympletic structure [8]. The isometry \( U \) is holomorphic and hamiltonian. Therefore there exists a coordinate system for which the metric \( g_6 \) and the Kahler form \( J \) takes the form
\[ g_6 = u \ d\theta^2 + \frac{(dv + A')^2}{u} + g_4(\theta), \quad (2.45) \]
\[ J = \bar{J}_1(\theta) + d\theta \land (dv + A'), \quad (2.46) \]

being the new Killing vector \( U = \partial_v \), and \( A' \) certain 1-form. Therefore the Kahler manifold \( N \) is locally the product \( N = R_{\theta} \times R_v \times M \) being \( M \) certain four dimensional manifold. The metric \( g_4(\theta) \) is a metric on \( M \) at each level of constant sets of the coordinate \( \theta \).

It is natural to consider the two forms \( \bar{J}_2 = i_U \Psi_3 \) and \( \bar{J}_3 = i_U \Psi_3' \), being \( \Psi_3 \) and \( \Psi_3' \) defined in (2.43). By use of these definitions and that \( \Psi_3 \) and \( \Psi_3' \) are closed, it follows that \( i_U \bar{J}_2 = i_U \bar{J}_3 = 0 \) and that \( d\bar{J}_2 = d\bar{J}_3 = 0 \). The compatibility conditions (2.21) implies that
\[ \bar{J}_2 \land \bar{J}_2 = \bar{J}_3 \land \bar{J}_3, \quad \bar{J}_2 \land \bar{J}_3 = 0. \quad (2.47) \]

These forms are the real and imaginary part of complex two form

\[ \Omega = i\varepsilon \Psi, \]
respectively, being $\Xi$ the holomorphic vector field $\Xi = U - iJU$. It also holds that $i_{\Xi}\Omega = 0$ and that $d\Omega = 0$. The form $\Omega = J_2 + iJ_3$ is known as a complex sympletic form on $M$. The complex structure $J$ on the Kahler 6-manifold descends to a complex structure $J_1$ on $M$ which is obtained from the relation

$$J_2(\cdot, \cdot) = J_3(J_1\cdot, \cdot),$$

(2.48)
or by the relation $g_4(J_1\cdot, \cdot) = \widetilde{J}_1$. Moreover the equation (2.34) implies that

$$dA = -\widetilde{J}_2.$$

(2.49)
The $U$-invariance of the three forms $\psi_3$ and $\psi'_3$ implies that their general form is

$$\psi_3 = \theta^{-1/2}[\ J_2 \wedge (dv + A') + uJ_3 \wedge d\theta ],$$

(2.50)
and the Kahler condition (2.44) is automatically satisfied for (2.50) and (2.51). Also the first (2.21) descends to the relation

$$2\theta\widetilde{J}_1(\theta) \wedge \widetilde{J}_1(\theta) = uJ_2 \wedge J_2 = uJ_3 \wedge J_3 = u\Omega \wedge \overline{\Omega},$$

(2.52)
and that the calibration three form $\Phi$ (2.38) is expressed as

$$\Phi = \widetilde{J}_1(\theta) \wedge (d\chi + A) + d\theta \wedge (dv + A') \wedge (d\chi + A)$$

$$+ \theta [\ J_2 \wedge (dv + A') + u\widetilde{J}_1(\theta) \wedge d\theta ].$$

(2.53)
Then from $d\Phi = d*\Phi = 0$ we obtain the additional equations [8]

$$\widetilde{J}_1'' = -dMd_c du, \quad dA' = (d^c du) \wedge d\theta + \widetilde{J}_1',$$

(2.55)
being $d_M$ defined on $M$ and $d^c_M = Jd_M$. In conclusion, and by denoting now $A = A_2$ and $A' = A_1$, it is concluded that the general form of the $G_2$ metrics is [8]

$$g_7 = \frac{(d\chi + A_2)^2}{\theta^2} + \theta [\ u \ d\theta^2 + \frac{(dv + A_1)^2}{u} + g_4(\theta) ],$$

(2.54)
being the quantities appearing in (2.54) defined by the evolution equations

$$\widetilde{J}_1'' = -dMd_c du,$$

(2.55)
$$2\theta\widetilde{J}_1(\theta) \wedge \widetilde{J}_1(\theta) = u\Omega \wedge \overline{\Omega},$$

(2.56)
and the forms $A_1$ and $A_2$ are defined on $M \times R_\theta$ and $M$ respectively by the equations

$$dA_1 = (d^c du) \wedge d\theta + \widetilde{J}_1', \quad dA_2 = -\widetilde{J}_2.$$

(2.57)
The symbol $'$ denote partial derivation with respect to the parameter $\theta$.

We will refer to the metrics (2.54) as the Apostolov-Salamon metrics [8]. It is seen from (2.30) that the dilaton has a logarithmic behaviour with respect to the ”time” $\theta^1$. The Kaluza-Klein anzatz (2.9) can be rewritten as

$$g_{11} = \theta g_{10} + \theta^{-2}(d\chi + A_2)^2 = g_{(3,1)} + g_7.$$

(2.58)

\footnote{It is more natural to denote the parameter $\theta$ as $t$. We did not use this notation in order that the reader do not get confused with the time coordinate appearing in the Minkowski metric $g_4$.}
Our manifolds reduce in the Type IIA language to a collection of wrapped D6-branes and the reduced 10-dimensional metric tensor is

\[ g_{10} = g_6 + \theta^{-1}g_{(3,1)} = u \, d\theta^2 + \frac{(dv + A_1)^2}{u} + g_4(\theta) + \theta^{-1}g_{(3,1)}, \quad \phi = -\frac{1}{2\alpha} \log(\theta), \]  

and being \( A_2 \) the potential for the 2 RR form \( F = dA_2 \). In all the examples that we can construct by use of the results of this subsection, the calibration form \( \Phi \) given by (2.53) is not \( L^2 \)-normalizable, that is, the norm

\[ ||\Phi|| = \int_Y \Phi \wedge *\Phi, \]  

badly diverges. Therefore the scalar mode could be only a real parameter after compactification to four dimensions.

3. Explicit toric \( G_2 \) holonomy metrics

In the present section certain solutions of the evolution equations (2.55) and (2.56) are presented, together with their respective Apostolov-Salamon metrics. Some of them are known but others are new. Examples of \( G_2 \) holonomy manifolds for which for large values of the evolution parameter \( \theta \) tends to a Ricci-flat metric with \( SU(3) \) holonomy are presented. These are asymptotically Calabi-Yau metrics. Other examples for which this property is not evident are also constructed. An interesting feature is that a large class of \( G_2 \) metrics can be constructed, based on an hyperkahler 4-manifold. There exist a well known family of \( G_2 \) metrics, the Bryant-Salamon metrics, that are based on a quaternion Kähler metric. The metrics of this section are constructed with an hyperkahler base and they have holonomy exactly \( G_2 \), even if the hyperkahler base is flat.

3.1 Asymptotically Calabi-Yau \( G_2 \) metrics

In searching particular solutions of the evolution equations (2.55) and (2.56) it is important to remark that \( \tilde{J}_1(\theta) \), \( \tilde{J}_2 \) and \( \tilde{J}_3 \) do not constitute an hyperkahler structure in general. But a particular set of solutions of the evolution equations can be found by assuming that the four manifold \( M \) admits an hyperkahler metric \( g_h \) independent on \( \theta \). The closed hyperkahler triplet is also independent of \( \theta \), it will be denoted as \( \tilde{J}_i \) in order to do not confuse with \( J_i(\theta) \). Then equation (2.55) is trivially satisfied by an anzatz of the form

\[ \tilde{J}_1(\theta) = \mathcal{J}_1 - \frac{1}{2}d_M d'_M G, \quad G'' = u, \]  

being \( G \) a function of \( \theta \) and of the coordinates of \( M \). It is convenient to introduce the operator \( M(G) \) defined through the relation

\[ (\tilde{J}_1 - \frac{1}{2}d_M d'_M G)^2 = M(G) \mathcal{J}_1 \wedge \mathcal{J}_1. \]  

This operator exists because \( \mathcal{J}_1 \wedge \mathcal{J}_1 \) is equal to the volume form on \( M \), and the square of any two form \( \omega \) on \( M \) is proportional to the volume form, that is

\[ \omega \wedge \omega = A(\omega) \mathcal{J}_1 \wedge \mathcal{J}_1, \]
being $A(\omega)$ a function over $M$. In particular by selecting $\omega = \mathcal{J}_1 - \frac{1}{2} d_M d_M^c G$ the last relation gives (3.62). The equation (2.56) can be expressed as

$$2 \theta M(G) = G'' ,$$

(3.63)

where, as before, the symbol ' denotes partial derivation with respect to the evolution parameter $\theta$. The one form $A_1$ is given in this case by

$$A_1 = - \frac{1}{2} d_M G'.$$

(3.64)

Equations of the form (3.63) have been investigated in the literature [40]. We see that the left side of (3.62) is explicitly

$$J_1 \wedge J_1 - \frac{1}{2} (d_M d_M^c G) \wedge J_1 - \frac{1}{2} (d_M d_M^c G) + \frac{1}{4} (d_M d_M^c G) \wedge (d_M d_M^c G).$$

(3.65)

From the last expression together with (3.62) it follows that the operator $M(G)$ is not linear in general. An obvious simplification is obtained when the last term in (3.65) can be deleted, i.e., when

$$(d_M d_M^c G) \wedge (d_M d_M^c G) = 0.$$ 

(3.66)

Nevertheless, once a solution of (3.63) is found by deleting the last term, it should be checked that such solution is consistent with (3.66).

As a ground to the earth, we will consider first the simplest hyperkahler 4-manifold, namely, $R^4$ with its flat metric $g_4 = dx^2 + dy^2 + dz^2 + d\varsigma^2$ and with the hyperkahler triplet

$$\mathcal{J}_1 = d\varsigma \wedge dy - dz \wedge dx, \quad \mathcal{J}_2 = d\varsigma \wedge dx - dy \wedge dz, \quad \mathcal{J}_3 = d\varsigma \wedge dz - dx \wedge dy,$$

(3.67)

which is automatically closed. Let us consider the simplification (3.66), then $M(G)$ reduce to the laplacian operator in flat space. If a functional dependence of the form $G = G(\theta, x, y)$ is selected, then (3.63) reduces simply to

$$G'' + \theta (\partial_{xx} G + \partial_{yy} G) = 2 \theta.$$ 

(3.68)

The separable solutions in the variable $\theta$ are of the form

$$G = \frac{1}{3} \theta^3 + V(x, y) K(\theta).$$

By introducing $G = G(\theta, x, y)$ into (3.68) it follows that $K(\theta)$ and $V(x, y)$ are solutions of the equations

$$K''(\theta) = p \theta K(\theta), \quad \partial_{xx} V + \partial_{yy} V + p V = 0,$$

(3.69)

being $p$ a parameter. By defining the $\tilde{\theta} = \theta / p^{1/3}$ the first of the equations (3.69) reduce to the Airy equation. The second is reduced to find eigenfunctions of the two dimensional Laplace operator, which is a well known problem in electrostatics. For $p > 0$ periodical solutions are obtained and for $p < 0$ there will appear exponential solutions. This solution is consistent with (3.66).

A simple example is given by the eigenfunction $V = q \sin(p x)$, being $q$ a constant. A solution of the Airy equation is given by

$$K = A i(\tilde{\theta}) = \frac{1}{3} \tilde{\theta}^{1/2} (J_{1/3}(\tau) + J_{-1/3}(\tau)), \quad \tau = i \frac{2 \theta^{3/2}}{3 p^{1/2}}.$$
Then the function $G$ is
\[
G = \frac{1}{3} \theta^3 + q \sin(p x) \text{Ai}(\frac{\theta}{p^{1/3}}).
\]

From (3.64) and both equation (3.61) it is obtained
\[
A_1 = -p q \text{Ai}(\bar{\theta})' \cos(p x) dy, \quad u = \theta(1 + p q \text{Ai}(\bar{\theta}) \sin(p x)). \quad (3.70)
\]

By defining the new function $H(\theta, x, y) = (1 + p q \text{Ai}(\bar{\theta}) \sin(p x))$ it is obtained the following $G_2$ holonomy metric [8]
\[
g_7 = \left( \frac{d\chi - x dz + y d\zeta}{\theta^2} \right)^2 + \left( \frac{d\nu - p q \text{Ai}(\bar{\theta})' \cos(p x) dy}{H} \right)^2 + \theta \left( H dx^2 + H dy^2 + dz^2 + d\zeta^2 \right) + \theta^2 d\theta^2 d\nu^2. \quad (3.71)
\]

The metric (3.71) has two parameters $p$ and $q$ with $p > 0$ and three commuting Killing vector fields $\partial_\chi, \partial_\nu$, and $\partial_\zeta$. The Airy function goes to zero for $\theta \to \infty$ values and therefore $H$ goes to 1 for large $\theta$. The asymptotic form of the metric is
\[
g_7 = \frac{(d\chi - x dz + y d\zeta)^2}{\theta^2} + \theta \left( dx^2 + dy^2 + dz^2 + d\zeta^2 \right) + \theta^2 d\theta^2 + d\nu^2, \quad (3.72)
\]
and it is seen that the dependence on $p$ and $q$ have disappeared. It is immediately seen that (3.71) is asymptotically of the form
\[
g_7 = d\nu^2 + g_6,
\]
being $g_6$ independent of the coordinate $\nu$. Therefore the holonomy has been reduced from $G_2$ to $SU(3)$, that is, the metric (3.71) has asymptotically $SU(3)$ holonomy. It means that the six dimensional part
\[
g_6 = \frac{(d\chi - x dz + y d\zeta)^2}{\theta^2} + \theta \left( dx^2 + dy^2 + dz^2 + d\zeta^2 \right) + \theta^2 d\theta^2,
\]
is Calabi-Yau. We do not know if there exist for (3.71) a coordinate system for which (3.71) is asymptotically conical. Therefore we ignore if (3.71) is reliable in order to obtain chiral matter after compactification. We only can say that (3.71) is asymptotically Calabi-Yau. In the limit $\theta \to \infty$ we have the scale invariance
\[
x \to \lambda^{3/2} x, \quad y \to \lambda^{3/2} y, \quad z \to \lambda^{3/2} z, \quad \zeta \to \lambda^{3/2} \zeta, \quad \chi \to \lambda^3 \chi, \quad \nu \to \lambda^4 \nu, \quad \theta \to \lambda \theta, \quad (3.73)
\]
which is generated by the homothetic Killing vector
\[
D = \frac{3}{2} x \partial_x + \frac{3}{2} y \partial_y + \frac{3}{2} z \partial_z + \frac{3}{2} \zeta \partial_\zeta + \theta \partial_\theta + 3 \chi \partial_\chi + 4 \nu \partial_\nu. \quad (3.74)
\]

Instead the full $G_2$ holonomy metric (3.71) is not invariant under (3.74), even by a redefinition of the values of $p$ and $q$.

As it has been seen in a previous section, after reduction along the coordinate $\chi$, the dilation has a logarithmic behaviour with respect to the coordinate $\theta$. An interesting question is which
dependence is obtained by making the IIA reduction along the coordinate \( v \). The Kaluza-Klein anzatz (2.9) can be rewritten as

\[
g_{11} = H^{1/2}g_{10} + H^{-1}(dv - p qAi(\bar{\theta})' \cos(px)dy)^2. \tag{3.75}
\]

The reduced IIA background given by

\[
g_{10} = \left(\frac{d\chi - xdz + yd\varsigma}{H^{1/2} \theta^2}\right)^2 + \theta \left( H^{1/2}dx^2 + H^{1/2}dy^2 + H^{-1/2}dz^2 + H^{-1/2}d\varsigma^2 \right) + \theta^2 H^{1/2} d\theta^2 + H^{-1/2}g_{(3,1)},
\]

\[
\phi = -\frac{1}{4\alpha} \log(H),
\]

and being \( A = -p qAi(\bar{\theta})' \cos(px)dy \) the potential for the 2 RR form \( F = dA \). The explicit form of \( F \) is

\[
F = p qAi(\bar{\theta})' \sin(px)dy \wedge dx - p q\bar{\theta}Ai(\bar{\theta}) \cos(px)dy \wedge d\bar{\theta}
\]

We see that for large \( \theta \) the function \( H \) goes to one and the dilaton decreases to zero. The new six dimensional metric will not satisfy the condition \( F^{(1,1)} = 0 \) but \( F^{ab}J_{ab} = 0 \). Therefore this new metric is not Kahler.

### 3.2 Two parameter \( G_2 \) metrics with \( U(1) \times U(1) \times U(1) \) isometry

There are more \( G_2 \) holonomy metrics that can be constructed by starting with an hyperkahler 4-manifold and by using the formalism of section 2.1.3. Solutions of the system (2.56), (2.57) and (2.55) will be found by assuming the additional condition \( d^c_Mu = 0 \). Then the first equation (2.57) is solved by

\[
\tilde{J}_1 = (r + s\theta) \bar{J}_1,
\]

the evolution equation (2.55) is trivially satisfied and (2.56) is an algebraic equation for \( u \) with solution

\[
u = \theta \left( r + s\theta \right)^2.
\]

The resulting \( G_2 \) metric is explicitly

\[
g_r = \frac{(dv + A_1)^2}{(r + s\theta)^2} + \frac{(d\chi + A_2)^2}{\theta^2} + \theta^2 (r + s\theta)^2 d\theta^2 + \theta (r + s\theta) g_h, \tag{3.76}
\]

being \( A_1 = A_1^i dx^i \) and \( A_2 = A_2^i dx^i \) one forms defined on the four manifold \( M \) by the equations

\[
dA_1 = s \bar{J}_1, \quad dA_2 = -\bar{J}_2. \tag{3.77}
\]

The integrability condition for (3.77) is satisfied because the triplet \( \bar{J}_i \) of an hyperkahler manifold is always closed.

There exist some cases in which a given hyperkahler metric admit more than one \( G_2 \) extension. There exist four dimensional metrics admitting an almost Kahler structure \((I, \bar{J}_0)\) compatible with the opposite orientation induced by the triplet \( \bar{J}_i \). A simple example is given by the flat \( R^4 \) metric, and hyperkahler examples admitting non integrable almost Kahler structure are known in the literature [39]. For all these cases a new solution of the evolution equations is obtained with corresponding \( G_2 \) metric

\[
\tilde{J}_1 = (p + q\theta) \bar{J}_0 + (r + s\theta) \bar{J}_1, \quad u = \theta \left[ (r + s\theta)^2 - (p + q\theta)^2 \right].
\]
metric. Nevertheless, as it will be shown below, an isometry preserving $\chi$ by a redefinition of the coordinates
\[ A \]
Because a total differential term, i.e.,
\[ A \]
tri-holomorphic. If an hyperkahler metric possesses an isometry that is not tri-holomorphic, it should preserve at least
\[ \varsigma \]
which it takes generically the Gibbons-Hawking form [34]. But if a larger isometry group including $T^2$ as a subgroup is desired, then the hyperkahler basis should have at least one Killing vector. If no isometries is lifted to a $G_2$ holonomy one by (3.76), the isometry group will be exactly $T^2$. An example of an hyperkahler metric without isometries is the well known Atiyah-Hitchin metric [32].

The local form (3.79) is general, and therefore in four dimensions an isometry that preserve two of the three Kahler forms of an hyperkahler metric is necessarily tri-holomorphic. For any hyperkahler metric possessing a tri-holomorphic isometry there exist a coordinate system for which it takes generically the Gibbons-Hawking form [34]
\[ g = V^{-1}(d\varsigma + A^3)^2 + V dx \cdot dx, \]
\[ (3.82) \]

\[ g_\tau = \frac{(dv + A^1)^2}{(r + s\theta)^2 - (p + q\theta)^2} + \frac{(d\chi + A^2)^2}{\theta^2} + \theta^2 \left[ (r + s\theta)^2 - (p + q\theta)^2 \right] d\theta^2 \]
\[ + \theta \left[ (r + p) + (q + s)\theta \right] g_h. \]

The one forms $A_i$ are defined in this case as
\[ dA^1 = s J_1 + p J_0, \quad dA^2 = -J_2. \]

These equations are well defined due to the closure of $J_i$. Only two of the four parameters $(p, q, r, s)$ are effective, because two of them can be fixed by a rescale of $\Phi$ and the Killing vector $V$.

If an hyperkahler metric with no isometries is lifted to a $G_2$ holonomy one by (3.76), the isometry group will be exactly $T^2$. An example of an hyperkahler metric without isometries is the well known Atiyah-Hitchin metric [32]. But if a larger isometry group including $T^2$ as a subgroup is desired, then the hyperkahler basis should have at least one Killing vector. If the new isometry is not tri-holomorphic, i.e., it do not preserve the sympletic forms $\varsigma$, it is seen that the action of this group could merely add a Killing vector of (3.79). The metric (3.79) is hyperkahler with respect to the $\varsigma$ dependent
\[ \tau \]
hyperkahler triplet
\[ \varsigma \]
\[ J_1 = e^\nu u_z dx \wedge dy + dz \wedge [d\varsigma + (u_x dy - u_y dx)], \]
\[ \varsigma \]
\[ \left( \begin{array}{c} J_2 \\ J_3 \end{array} \right) = e^{u/2} \left( \begin{array}{cc} \cos(\frac{\varsigma}{2}) & \sin(\frac{\varsigma}{2}) \\ \sin(\frac{\varsigma}{2}) & -\cos(\frac{\varsigma}{2}) \end{array} \right) \left( \begin{array}{c} J_2 \\ J_3 \end{array} \right). \]

In the last expression there have been defined the two forms
\[ \tilde{J}_2 = -u_z dz \wedge dy + [d\varsigma + u_y dx] \wedge dy, \quad \tilde{J}_3 = u_z dz \wedge dx + [d\varsigma + u_x dy] \wedge dx, \]
which should not be confused with the hyperkahler forms $J_i$ in (3.81). From (3.81) it is clear that $\partial_\varsigma$ preserve $J_1$, but the other two Kahler forms are $\varsigma$ dependent. It means that in the non-tri holomorphic case it is impossible to preserve two of the three $J_i$ without preserving the third. The local form (3.79) is general, and therefore in four dimensions an isometry that preserve two of the closed Kahler forms of an hyperkahler metric is necessarily tri-holomorphic. For any hyperkahler metric possessing a tri-holomorphic isometry there exist a coordinate system for which it takes generically the Gibbons-Hawking form [34]
\[ g = V^{-1}(d\varsigma + A^3)^2 + V dx \cdot dx, \]
\[ (3.82) \]
with a one form $A^3$ and a function $V$ satisfying the following linear system of equations
\[ \nabla V = \nabla \times A^3 \quad \iff \quad dV = *dA^3, \] (3.83)
where the operation $*$ is refereed to the three dimensional flat part. These metrics are hyperkahler with respect to the hyperkahler triplet
\[ \mathcal{J}_1 = (d\zeta + A^3) \wedge dx + Vdy \wedge dz, \]
\[ \mathcal{J}_2 = (d\zeta + A^3) \wedge dy + Vdz \wedge dx, \]
\[ \mathcal{J}_3 = (d\zeta + A^3) \wedge dz + Vdx \wedge dy, \] (3.84)
which is clearly $\zeta$ independent. Only for the tri-holomorphic case one can expect to obtain $\zeta$ independent 1-forms $A^1$ and $A^2$, up to a total differential term that can be managed by a redefinition of the coordinates $\chi$ and $\upsilon$. The isometry group will be enlarged to $T^3$. In general an arbitrary isometry group $G$ on $M$ will be enlarged to the total $G_2$ space if is tri-holomorphic. The total isometry group will be $T^2 \times G$. We will construct examples with larger isometry group in the next subsection.

### 3.3 The flat hyperkahler case

As before we consider $R^4$ with its flat metric $g_4 = dx^2 + dy^2 + dz^2 + d\zeta^2$ and with the closed hyperkahler triplet
\[ \overline{J}_1 = d\zeta \wedge dy - dz \wedge dx, \quad \overline{J}_2 = d\zeta \wedge dx - dy \wedge dz, \quad \overline{J}_3 = d\zeta \wedge dz - dx \wedge dy. \]

This innocent looking case is indeed rather rich and instructive. In principle for any hyperkahler metric there exist three different $G_2$ holonomy metrics that can be constructed by use of the results of section 2.1.3. This is due to the fact that in order to integrate equations (3.77) we should select a pair of elements of the triplet $\overline{J}_i$ as $\overline{J}_1$ and $\overline{J}_2$, and there are three possible choices. But in the flat case this selection corresponds to a permutation of coordinates and this ambiguity can be ignored. Equations (3.77) are easily integrated for (3.67) up to a total differential term and, from (3.76), it is directly obtained a $G_2$ metric. The result is the following
\[ A_1 = -s(xdz + yd\zeta), \]
\[ A_2 = -ydz - xd\zeta. \]

\[ g_7 = \frac{(dv - s(xdz + yd\zeta))^2}{(r + s\theta)^2} + \frac{(d\chi - ydz - xd\zeta)^2}{\theta^2} + \theta^2 (r + s\theta)^2 d\theta^2 \]
\[ + \theta (r + s\theta)(dx^2 + dy^2 + dz^2 + d\zeta^2). \] (3.85)

If the values $r = 0, s = 1$ for the parameters are selected the metric tensor (3.85) reduce to
\[ g_7 = \frac{(dv - xdz + yd\zeta)^2}{\theta^2} + \frac{(d\chi - ydz - xd\zeta)^2}{\theta^2} + \theta^4 d\theta^2 + \theta^2 (dx^2 + dy^2 + dz^2 + d\zeta^2). \] (3.86)

The metrics (3.86) have been already obtained in the physical literature [11]. They have been constructed by use of oxidation methods in 11 dimensional supergravity, by starting with a domain wall solution in five dimensions of the form
\[ g_5 = H^{4/3}(dx^2 + dy^2 + dz^2 + d\zeta^2) + H^{16/3}d\theta^2, \]
being \( a = 1, \ldots, 4 \). By use of oxidation rules one obtains a 11 dimensional background \( g_{11} = g_{(3,1)} + g_7 \) with a 7-dimensional metric of the form

\[
g_7 = \frac{(dv - xdz + yds)^2}{H^2} + \frac{(d\chi - ydz - xds)^2}{H^2} + H^4 \, d\theta^2 + H^2 \left( dx^2 + dy^2 + dz^2 + d\zeta^2 \right). \tag{3.87}
\]

The last metric reduce to (3.86) by selecting \( H = \theta \). It can be shown that (3.87) arise from an \( SO(5) \) invariant \( G_2 \) metric by contraction on the isometry group \([11]\). The isometry corresponding to (3.87) and (3.85) is the same \( SU(2) \) group that acts linearly on the coordinates \( (x, y, z, \zeta) \) on \( M \) and which preserve simultaneously the two forms \( \mathbf{J}_1 \) and \( \mathbf{J}_2 \). The nilpotent 6-dimensional algebra is the complexification of the 3-dimensional ur-Heisenberg algebra. The \( SU(2) \) action on \( M \) adds to \( A_1 \) and \( A_2 \) a total differential term that can be absorbed by a redefinition of the coordinates \( \chi \) and \( v \). For instance, by general translation of the form

\[
x \rightarrow x + \alpha_1, \quad y \rightarrow y + \alpha_2, \quad z \rightarrow z + \alpha_3, \quad \zeta \rightarrow \zeta + \alpha_4, \quad \tag{3.88}
\]

preserves \( \mathbf{J}_1 \) and \( \mathbf{J}_2 \) but not the one forms \( A_1 \) and \( A_2 \). Nevertheless the effect of (3.88) can be compensated by a transformation of the form

\[
v \rightarrow v + \alpha_5 + \alpha_1 z - \alpha_2 \zeta, \quad \chi \rightarrow \chi + \alpha_6 + \alpha_2 z + \alpha_1 \zeta. \tag{3.89}
\]

More general \( SU(2) \) transformations on \( M \) preserving \( \mathbf{J}_1 \) and \( \mathbf{J}_2 \) will also be absorbed by a redefinition of the coordinates \( \chi \) and \( v \). For the metric (3.86) we also have the scale invariance

\[
x \rightarrow \lambda x, \quad y \rightarrow \lambda y, \quad z \rightarrow \lambda z, \quad \zeta \rightarrow \lambda \zeta, \quad \chi \rightarrow \lambda^4 \chi, \quad v \rightarrow \lambda^4 v, \quad \theta \rightarrow \lambda^2 \theta,
\]

which is generated by the homothetic Killing vector

\[
D = 2x \partial_x + 2y \partial_y + 2z \partial_z + 2\zeta \partial_{\zeta} + \theta \partial_{\theta} + 4\chi \partial_{\chi} + 4v \partial_v. \tag{3.90}
\]

It can be shown by explicit calculation of the curvature tensor that (3.87) is irreducible and has holonomy exactly \( G_2 \), even in the subcase given in (3.86).

But more metrics can be found starting with the flat metric in \( R^4 \). This metric is also hyperkahler with respect to the opposite orientation triplet

\[
\mathbf{J}_1 = d\zeta \wedge dy + dz \wedge dx, \quad \mathbf{J}_2 = d\zeta \wedge dx + dy \wedge dz, \quad \mathbf{J}_3 = d\zeta \wedge dz + dx \wedge dy, \tag{3.91}
\]

and the three one forms

\[
A_1' = xdz + yd\zeta, \quad A_2' = zdy + xdz, \quad A_3' = ydx + zd\zeta, \tag{3.92}
\]

satisfy \( dA'_i = \mathbf{J}_i \). By denoting \( \mathbf{J}_0 = \mathbf{J}_1 \), then \( g_6 \) and \( \mathbf{J}_0 \) are a Kahler structure compatible with the opposite orientation defined by \( \mathbf{J}_i \). Therefore from (3.78) we obtain the new \( G_2 \) metric

\[
g_7 = \frac{[dv - (s - q)x dz + (s + q)y d\zeta]^2}{(r + s\theta)^2 - (p + q\theta)^2} + \frac{(d\chi - ydz - xds)^2}{\theta^2} + \theta \left( r + p + (q + s)\theta \right) \left( \theta \left( r - p + (s - q)\theta \right) d\theta^2 + dx^2 + dy^2 + dz^2 + d\zeta^2 \right). \tag{3.93}
\]
For the value \( s = 1 \) and all the remaining parameters equal to zero, the last metric reduce to (3.86). The metric (3.93) is not defined for the values \( p = r \) and \( s = q \). If \( s = q \) but \( p \neq q \), then the expression of the (3.93) is simplified as

\[
g_f = \frac{(dv + 2syd\zeta)^2}{f(\theta)} + \frac{(d\chi - ydz - xd\zeta)^2}{\theta^2} + \theta^2 f(\theta)d\theta^2 \\
+ \frac{\theta f(\theta)}{(r - p)}(dx^2 + dy^2 + dz^2 + d\zeta^2). \tag{3.94}
\]

being \( f(\theta) \) now a linear function of \( \theta \) defined by \( f(\theta) = (r - p)(2s\theta + r + p) \). It can be immediately checked that for \( s = -q \) it is obtained again the metric (3.94) up to a redefinition of the parameters and coordinates. The metric (3.93) is invariant under the linear action on the coordinates on the manifold preserving the two forms \( s\mathcal{J}_1 + p\mathcal{J}_0 \) and \( \mathcal{J}_2 \). If we instead choose \( \mathcal{J}_2 \) or \( \mathcal{J}_3 \) as \( \mathcal{J}_0 \), then we will find two more \( G_2 \) metrics and a similar analysis of the isometries and parameters could be done. We will not write the explicit expressions, but the method to find them is completely analogous to those described above. In all the cases presented here, there is a \( U(1) \times U(1) \times U(1) \) subgroup of isometries generated by shifts on \( \chi, v \) and \( \zeta \).

### 3.4 The stringy cosmic string case

After this warm up, more complicated examples can be analyzed. Let us consider hyperkahler metrics with two commuting Killing vectors, one of which is tri-holomorphic and other that is not. The presence of a tri-holomorphic Killing vector implies that the existence of a coordinate system for which such metrics will take the Gibbons-Hawking form (3.82). It is convenient to write the flat 3-dimensional part in cylindrical coordinates \((\eta, \rho, \varphi)\), the result is

\[
g_f = dx^2 + dy^2 + dz^2 = d\eta^2 + d\rho^2 + \rho^2 d\varphi^2.
\]

Then the subclass of the metrics (3.82) for which \( \varphi \) is a Killing vector take in cylindrical coordinates the following form

\[
g_h = V(d\eta^2 + d\rho^2 + \rho^2 d\varphi^2) + V^{-1}(d\zeta + Ad\phi)^2. \tag{3.95}
\]

The functions \( V \) and \( A \) for (3.95) satisfy the linear system the linear system

\[
\rho V_\eta = A_\rho, \quad \rho V_\rho = -A_\eta, \tag{3.96}
\]

which in particular implies that \( V \) satisfy the Ward integrability condition

\[
(\rho V_\eta)_\eta + (\rho V_\rho)_\rho = 0. \tag{3.97}
\]

The Killing vector \( \partial_\zeta \) and \( \partial_\phi \) are commuting and therefore there is an \( T^2 \) action over the 4-manifold. Nevertheless the former isometry is not tri-holomorphic and therefore will not be an isometry of the \( G_2 \) extension.

A simple example of (3.95) occurs when \( V \) is independent on the coordinate \( \eta \). The Ward equation gives \( V_\eta = \log(\rho) \) and the distance element is

\[
g_h = \log(\rho)(d\eta^2 + d\rho^2 + \rho^2 d\varphi^2) + \frac{1}{\log(\rho)}(d\zeta - \eta d\phi)^2. \tag{3.98}
\]
The metric (3.98) appears in many contexts in physics as, for instance, in the study of stringy cosmic strings [36]. It is the asymptotic form of the ALG gravitational instantons found in [35]. Also it describe the single matter hypermultiplet target metric for type IIA superstrings compactified on a Calabi-Yau threefold when supergravity and D-instanton effects are absent [55]. Our strategy will be, as before, to express the hyperkahler triplet (3.84) corresponding to the metric (3.98) as the differential of certain one forms $A_i$, and to construct the corresponding $G_2$ metrics by use of the formulas (3.77) and (3.76). After certain calculation it is found that the triplet can be expressed in cylindrical coordinates as $\hat{J}_i = dA_i$ being $A_i$ defined by

$$
A_1 = \rho \cos(\phi) d\zeta - \rho \eta \cos(\phi) d\phi + \sin(\phi)(\rho \log(\rho) - \rho) d\eta,
$$

$$
A_2 = \rho \sin(\phi) d\zeta + \rho \eta \sin(\phi) d\phi + \cos(\phi)(\rho \log(\rho) - \rho) d\eta,
$$

$$
A_3 = \eta d\zeta + (\eta^2 + \rho \log(\rho) - \rho) d\phi.
$$

With the help of the last expressions and (3.76) and (3.77) we can construct three, in principle different, $G_2$ holonomy metrics, depending on which pair of 2-forms $\hat{J}_i$ will play the role of $J_1$ and $J_2$. For instance, by selecting $\hat{J}_1$ and $\hat{J}_3$ as $J_1$ and $J_2$, and by using (3.99), (3.76) and (3.77) it is obtained the following $G_2$ metric

$$
g_7 = \left[ \frac{d\nu + \rho \cos(\phi) d\zeta - \rho \eta \cos(\phi) d\phi + \sin(\phi)(\rho \log(\rho) - \rho) d\eta}{(r + s\theta)^2} \right]^2 + \theta^2 (r + s\theta)^2 d\theta^2
$$

$$
+ \theta (r + s\theta) \left[ \log(\rho)(d\eta^2 + d\rho^2 + \rho^2 d\phi^2) + \frac{1}{\log(\rho)} (d\zeta - \eta d\phi)^2 \right]
$$

$$
+ \left[ \frac{d\chi - s \eta d\zeta - s(\eta^2 + \rho \log(\rho) - \rho) d\phi}{\theta^2} \right]^2.
$$

We will not write the expressions of the other two $G_2$ metrics, but the procedure for constructing them is completely analogous. The Killing vector fields $\partial_\eta$ and $\partial_\phi$ are not tri-holomorphic and therefore are not Killing vectors of the full $G_2$ metric.

More examples can be constructed by finding solutions that depends on the coordinate $\eta$. For instance, we can consider solutions of the form

$$
V = a \log(\rho) + b \eta + c \log\left( \frac{\eta + \sqrt{\rho^2 + \eta^2}}{\rho} \right),
$$

which give rise to Taub-Nut solutions, and also solutions like

$$
V = a \log(\rho) + \frac{1}{2} (b + c/\epsilon) \log\left( \frac{\eta - \epsilon + \sqrt{\rho^2 + (\eta - \epsilon)^2}}{\rho} \right) + \frac{1}{2} (b - c/\epsilon) \log\left( \frac{\eta + \epsilon + \sqrt{\rho^2 + (\eta + \epsilon)^2}}{\rho} \right),
$$

with $\epsilon^2 = \pm 1$. The case corresponding to minus sign corresponds to the potential for the axially symmetric circle of charge, while the other case corresponds to two sources on the axis of symmetry. This are known as Eguchi-Hanson I and II, the second is complete but the first is not. Finally one can consider multi-center Gibbons-Hawking metrics as well, which in the case of $N$ coincident centers is related to a Wu-Yang solution. To find the $G_2$ metrics corresponding to all this cases is straightforward, but the expressions are rather cumbersome and we will not write them explicitly.
3.5 Half-flat associated metrics

It is not difficult to see that there exist a coordinate system for which the metrics (3.78) takes the form

\[ g_\tau = d\tau^2 + g_6(\tau), \]  

(3.103)

being \( g_6(\tau) \) a six dimensional metric depending on \( \tau \) as an evolution parameter. In fact, by introducing the new variable \( \tau \) defined by

\[ \theta^2 \left[ (r + s\theta)^2 - (p + q\theta)^2 \right] d\theta^2 = d\tau^2, \]  

(3.104)

it is seen that (3.78) takes the desired form. Therefore these \( G_2 \) holonomy metrics are a wrapped product \( Y = I_{\tau} \times N' \) being \( I \) a real interval. The coordinate \( \tau \) is just a function of \( \theta \) and is given by

\[ \tau = \int \theta \left[ (r + s\theta)^2 - (p + q\theta)^2 \right]^{1/2} d\theta. \]  

(3.105)

The point is that \( g_6(\tau) \) is a half-flat metric on any hypersurface \( Y_{\tau} \) for which \( \tau \) has constant value. Indeed the \( G_2 \) structure can be decomposed as

\[ \Phi = \hat{J} \wedge d\tau + \hat{\psi}_3, \]  

(3.106)

\[ *\Phi = \hat{\psi}'_3 \wedge d\tau + \frac{1}{2} \hat{J} \wedge \hat{J}, \]  

(3.107)

where we have defined

\[ \hat{J} = z^{1/2} J_3 + z^{-1/2} (dv + A_1) \wedge (d\chi + A_2), \]  

(3.108)

\[ \hat{\psi}_3 = z^{-1/2} J_1 \wedge (d\chi + A_2) + \theta J_2 \wedge (dv + A_1), \]  

(3.109)

\[ \hat{\psi}'_3 = \theta z^{-1/2} J_2 \wedge (d\chi + A_2) - \theta^2 z^{1/2} J_1 \wedge (dv + A_1), \]  

(3.110)

and \( z = \theta^2 \left[ (r + s\theta)^2 - (p + q\theta)^2 \right] \). Then the \( G_2 \) holonomy conditions \( d\Phi = d*\Phi = 0 \) are

\[ d\Phi = d\hat{\psi}_3 + (d\hat{J} - \frac{\partial \hat{\psi}_3}{\partial \tau}) \wedge d\tau = 0, \]

\[ d*\Phi = \hat{J} \wedge d\hat{J} + (d\hat{\psi}_3 + \hat{J} \wedge \frac{\partial \hat{J}}{\partial \tau}) \wedge d\tau = 0. \]

The last equations are satisfied if and only if

\[ d\hat{\psi}_3 = \hat{J} \wedge d\hat{J} = 0 \]  

(3.111)

for every fixed value of \( \tau \), and

\[ \frac{\partial \hat{\psi}_3}{\partial \tau} = d\hat{J}, \quad \hat{J} \wedge \frac{\partial \hat{J}}{\partial \tau} = -d\hat{\psi}_3. \]  

(3.112)

The flow equations (3.112) were considered by Hitchin and are related to certain Hamiltonian system [28], [8]. Equations (3.111) implies that for every constant value \( \tau \) the metric \( g_6 \) together with \( \hat{J}, \hat{\psi}_3 \) and \( \hat{\psi}'_3 \) form a half-flat or half-integrable structure [9]. It means that the only non vanishing torsion classes defined in (2.40) and (2.41) are \( W_1, W_2 \) and \( W_3 \). A more precise...
description is as follows. Let us consider the decomposition \( W_1 = W_1^+ + W_1^- \) and \( W_2 = W_2^+ + W_2^- \) given by

\[
d\hat{\psi}_3 \wedge \mathcal{J} = \hat{\psi}_3 \wedge d\mathcal{J} = W_1^+ \mathcal{J} \wedge \mathcal{J} \wedge \mathcal{J},
\]

\[
d\hat{\psi}_3' \wedge \mathcal{J} = \hat{\psi}_3' \wedge d\mathcal{J} = W_1^- \mathcal{J} \wedge \mathcal{J} \wedge \mathcal{J},
\]

\[
(d\hat{\psi}_3)^{(2,2)} = W_1^+ \mathcal{J} \wedge \mathcal{J} + W_2^+ \mathcal{J} \wedge \mathcal{J},
\]

\[
(d\hat{\psi}_3')^{(2,2)} = W_1^- \mathcal{J} \wedge \mathcal{J} + W_2^- \mathcal{J} \wedge \mathcal{J}.
\]

Then for a half-flat manifold \( W_1^+ = W_2^+ = W_4 = W_5 = 0 \) and therefore the intrinsic torsion take values in \( W_1^- \oplus W_2^- \oplus W_3 \). It is direct to construct the half-flat metrics corresponding to the \( G_2 \) holonomy metrics presented of this section. For instance for (3.85) and (3.114) we obtain

\[
g_6 = \frac{(dv - s(xdz + yd\xi))^2}{(r + sa)^2} + \frac{(d\chi - ydz - xd\xi)^2}{a^2}
\]

\[
+ a (r + sa)(dx^2 + dy^2 + dz^2 + d\xi^2),
\]

and

\[
g_6 = \frac{[dv + p\cos(\phi)d\xi - p\eta\cos(\phi)d\phi + \sin(\phi)(p\log(\rho) - \rho)d\psi]^2}{(r + sa)^2}
\]

\[
+ a b[\log(\rho)(dn^2 + dp^2 + p^2d\phi^2) + \frac{1}{\log(\rho)}(d\xi - \eta d\phi)^2]
\]

\[
+ \frac{[d\chi - s\eta d\xi - s(\eta^2 + \rho \log(\rho) - \rho)d\phi]^2}{a^2},
\]

respectively. Here \( a \) was a function of \( \tau \) in the original \( G_2 \) metric, but because we are considering constant \( \tau \) it plays the role as a parameter in the six dimensional metric \( g_6 \).

## 4. D6 brane backgrounds and their \( \gamma \)-deformations

It is convenient to describe in more detail the \( SL(2, R) \) solution generating technique sketched in the introduction and developed in [6]. One usually starts with a solution of the eleven dimensional supergravity with \( U(1) \times U(1) \times U(1) \) isometry. Such solution can be written in the generic form

\[
g_{11} = \Delta^{1/3} M_{ab} D\alpha_a D\alpha_b + \Delta^{-1/6} \tilde{g}_{\mu \nu} dx^\mu dx^\nu,
\]

\[
C_3 = C D\alpha_1 \wedge D\alpha_2 \wedge D\alpha_3 + C_{1(ab)} \wedge D\alpha_a \wedge D\alpha_b + C_{2(a)} D\alpha_a + C_{(3)},
\]

with the indices \( a,b = 1,2,3 \) are associated to three isometries \( \alpha_1 = \upsilon, \alpha_2 = \chi \) and \( \alpha_3 = \zeta \) and the Greek indices \( \mu, \nu \) run over the remaining eight dimensional coordinates. Here \( D\alpha_i = d\alpha_i + A_i \), being \( A_i \) a triplet of one forms defined over the remaining eight dimensional manifold. In the present framework there is a manifest \( SL(3, R) \) symmetry which acts on the coordinates \( \alpha_i \) as

\[
\left( \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \end{array} \right) = (\Lambda^T)^{-1} \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \end{array} \right),
\]

and over the field tensors as \( M \) and \( A_i \) as

\[
M' = \Lambda M \Lambda^T \quad \left( \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ \end{array} \right) = (\Lambda^T)^{-1} \left( \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ \end{array} \right).$

(4.117)
The full isometry group of 11-dimensional supergravity compactified on a three torus is $SL(3, R) \times SL(2, R)$. The $SL(3, R)$ group leaves the background (4.115) unaltered. Following [6] we will deform these $T^3$ invariant backgrounds by an element of $SL(2, R)$. The strategy for deforming (4.115) is to use an $SL(2, R)$ transformation described as follows [38]. Let us define the complex parameter $\tau = C + i\Delta^{1/2}$. Under the $SL(2, R)$ action $\tau \rightarrow a\tau + b/c\tau + d$; $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$. (4.118)

The eight dimensional metric $g_{\mu\nu}$ and the tensor $C_2$ are instead invariant. The tensor $C_{(1)}^{\alpha\beta\gamma}$ and $A^\alpha_{\mu}$ form a doublet in similar way that the RR and NSNS two form fields do in IIB supergravity, their transformation law is given by

$$B^\alpha = \begin{pmatrix} 2A^\alpha_{\mu} \\ -e^{abc}C_{(1)bc} \end{pmatrix}, \quad B^\alpha \rightarrow \Lambda^{-T}B^\alpha$$

(4.119)

The field strenght $C_3$ also form a doublet with its magnetic dual with consequent transformation law

$$H = \begin{pmatrix} \Delta^{-1/2} F_4 \\ F_4 + C_{(0)}F_4 \end{pmatrix}, \quad H \rightarrow \Lambda^{-T}H,$$

(4.120)

being the Hodge operation taken with respect to the eight dimensional metric $g$. As we discussed in the introduction, this transformation deform the original metric (4.115) and the deformed metric will be regular only with elements of the form [6]

$$\Lambda = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \in SL(2, R),$$

(4.121)

which constitute a subgroup called $\gamma$-transformations. We will be concerned with such transformation in the following.

If the fields $C, C_1$ and $C_2$ are zero, it follows that $A^i$ and $\tilde{g}_{\mu\nu}$ are unchanged by a $\gamma$-transformation and $C_1$ and $C_2$ remains zero. The deformation then give the new fields

$$\Delta' = G^2\Delta, \quad C' = -\gamma G\Delta, \quad G = \frac{1}{1 + \gamma^2\Delta}.$$  

(4.122)

By inspection of the transformation rule (4.120) it follows that

$$F_4' = F_4 - \gamma\Delta^{-1/2} *_8 F_4 - \gamma d(G\Delta D\alpha_1 \wedge D\alpha_2 \wedge D\alpha_3).$$

(4.123)

The $\gamma$-deformed eleven dimensional metric results [38]

$$g_{11} = G^{-1/3}(G\Delta^{1/3}M_{ab}D\alpha_a D\alpha_b + \Delta^{-1/6}\tilde{g}_{\mu\nu}dx^\mu dx^\nu).$$

(4.124)

Note that if the initial four form $F_4$ was zero, then from the last term in (4.123) a non trivial flux is obtained.

The gamma-deformation procedure can be applied to a generic solution of the eleven dimensional supergravity over a manifold of the form $M_{3,1} \times Y$, being $M_{3,1}$ the flat Minkowski
four manifold and $Y$ a manifold with holonomy in $G_2$ and $T^3$ isometry. In particular it can be applied for the $G_2$ metrics presented along this work. The general solution is of the form

$$g_{11} = g_{(3,1)} + g_7.$$  

(4.125)

being $g_{(3,1)}$ the flat Minkowski metric in four dimensions. The form $C_3$ vanish identically and the only non trivial field is the graviton. The local form of the metric will be

$$g_7 = \frac{(d\chi + A_2)^2}{\theta^2} + \theta \left[ u \, d\theta^2 + \left( \frac{d\upsilon + A_1}{u} + \frac{(d\varsigma + A_3)^2}{W} \right) + g_3(\theta),$$  

(4.126)

being $W$ a new function, and being all the quantities appearing in (4.126) independent on the coordinates $\chi$, $\upsilon$ and $\varsigma$. The metric $g_3(\theta)$ is a three dimensional metric at every constant $\theta$ level surface. The reader can check that all the $G_2$ holonomy metrics presented along this work can be expressed as (4.126). The components of the 7-metric are

$$g_{\upsilon\upsilon} = \frac{\theta}{u}, \quad g_{\chi\chi} = \frac{1}{\theta^2}, \quad g_{\varsigma\varsigma} = \frac{\theta}{W}, \quad g_{\theta\theta} = \theta u,$$

$$g_{\upsilon\varsigma} = \frac{\theta A_1^1}{u}, \quad g_{\upsilon x^i} = \frac{\theta A_1^{x_i}}{u},$$

$$g_{\chi\varsigma} = \frac{A_2^2}{\theta^2}, \quad g_{\chi x^i} = \frac{A_2^{x_i}}{\theta^2}, \quad g_{\varsigma x^i} = \frac{\theta A_3^{x_i}}{W},$$

$$g_{x^i x^j} = g_{3 x^i x^j} + \frac{\theta A_1^{x_i} A_1^{x_j}}{u} + \frac{A_2^{x_i} A_2^{x_j}}{\theta^2} + \frac{\theta A_3^{x_i} A_3^{x_j}}{W},$$

$$g_{x^i x^i} = g_{3 x^i x^i} + \frac{\theta (A_1^{x_i})^2}{u} + \frac{(A_2^{x_i})^2}{\theta^2} + \frac{\theta (A_3^{x_i})^2}{W},$$

(4.127)

and the remaining components are zero.

The task to write the background (4.125) in the $SL(3, \mathbb{R})$ manifest form (4.115) presents no difficulties for the metrics (4.127). By defining the coordinates $\alpha_1 = \upsilon$, $\alpha_2 = \chi$ and $\alpha_3 = \varsigma$ the metric (4.125) can be expressed as

$$g_{11} = g_{(3,1)} + \Omega_{ab} (d\alpha_a + A^a)(d\alpha_b + A^b) + h,$$

where the matrix $\Omega_{ab}$ and the eight dimensional metric $h$ are defined by

$$\Omega_{ab} = \left[ \begin{array}{ccc} \frac{\theta}{u} & 0 & 0 \\ 0 & \theta^{-2} & 0 \\ 0 & 0 & \frac{\theta}{W} \end{array} \right], \quad \det(\Omega) = \frac{1}{u \, W^3},$$

$$h = g_3(\theta) + \theta \, u \, d\theta^2 + g_{(3,1)}.$$  

Therefore the unit determinant matrix $M_{ab}$ and the scalar $\Delta$ are given by

$$M_{ab} = \frac{\Omega_{ab}}{\det(\Omega)}, \quad \Delta = \det(\Omega)^3 = \frac{1}{u^3 \, W^3}.$$  

Introducing the covariant derivative $D\alpha_i = d\alpha_i + A^i$ gives, after making the identification $\tilde{g} = \Delta^{1/6} h$, the desired $SL(3, \mathbb{R})$ form (4.115). The deformation technique gives the following deformation

$$\Delta' = G^2 \Delta, \quad C' = -\gamma G \Delta, \quad G = \frac{1}{1 + \gamma^2 \Delta}.$$  

(4.128)
\[ F_4' = -\gamma d(G\Delta D\alpha_1 \wedge D\alpha_2 \wedge D\alpha_3). \]

and the new metric tensor
\[ g_{11} = G^{-1/3}(G^{1/3}M_{ab}D\alpha_a D\alpha_b + \Delta^{-1/6}\tilde{g}_{\mu\nu}dx^\mu dx^\nu) \]  \hspace{1cm} (4.129)

The new solution include a flux term \( F_4' \) that was absent in the starting background (4.115).

Also the presence of an isometry in the 11-dimensional background allows to find an IIA background by reduction along the Killing vector. If after this reduction a new isometry is preserved, then T-duality rules can be used in order to construct IIB backgrounds. In principle there are six possible reductions that can be done, depending on which pair of isometries is choose to make a reduction along a circle and a T-duality afterwards. In order to perform the IIA reduction the \( T^3 \) part of the metric should be decomposed as
\[ M_{ab}D\alpha_a D\alpha_b = e^{-2\phi/3}h_{mm}D\alpha_m D\alpha_n + e^{4\phi/3}(D\alpha_3 + N_mD\alpha_m)^2, \]  \hspace{1cm} (4.130)

with the indices \( m, n = 1, 2 \). The field \( \phi \) will be related to the dilaton of the reduced theory. By reducing along the isometry and, after that, making a T-duality along one of the remaining isometries, say \( \alpha_1 \), an IIB supergravity solution will be obtained. The final result can be found elsewhere, for instance in \( SL(3,R) \) form in [6] . In the case corresponding to (4.125) and (4.126), the \( T^3 \) metric can be decomposed as in (4.130) easily, and it is seen that the quantities \( N_m \) are zero and the term \( e^{4\phi/3} \) associated to the dilaton will be one of diagonal elements of \( \Omega_{ab} \), associated to our choice of the pair of isometries. The resulting background is
\[ g_{IIB} = \frac{1}{h_{11}} \left[ \frac{1}{\sqrt{\Delta}}(D\alpha^1)^2 + \sqrt{\Delta}(D\alpha^2)^2 \right] + e^{2\phi/3}\tilde{g}_{\mu\nu}dx^\mu dx^\nu, \]
\[ B = \frac{h_{12}}{h_{11}}D\alpha^1 \wedge D\alpha^2, \quad C^{(2)} = -D\alpha^1 \wedge A^3_\mu dx^\mu, \quad C^{(4)} = 0 \]  \hspace{1cm} (4.131)

It is worthy to recall that, in general, a IIB solution contains more tensor fields than those appearing in (4.132). This tensors have well defined transformation properties under the \( SL(3,R) \) action. But these fields are zero in our case. Instead under the \( SL(2,R) \) group that generates new solutions, we have a complex parameter \( C + i\sqrt{\Delta} \) transforming as a \( \tau \) parameter. There is a four form \( F_4 \) defined in terms of certain field \( C_{\mu\nu\alpha} \) appearing in the general expression \( C^{(4)} \). This form transforms into its magnetic dual in eight dimensions as in (4.120). Although this field is zero in our case, the \( SL(R,2) \) transformation induce a non trivial \( F_4 \) term. We have checked that this IIB deformed background is indeed the background obtained by reduction of the deformed 11-supergravity solution given in (4.129), the final result is
\[ g_{IIB} = \frac{1}{h_{11}} \left[ \frac{1}{\sqrt{\Delta'}}(D\alpha^1 - CD\alpha_2)^2 + \sqrt{\Delta'}(D\alpha^2)^2 \right] + e^{2\phi/3}\tilde{g}_{\mu\nu}dx^\mu dx^\nu, \]
\[ B = \frac{h_{12}}{h_{11}}D\alpha^1 \wedge D\alpha^2, \quad C^{(2)} = -D\alpha^1 \wedge A^3_\mu dx^\mu, \quad C^{(4)} = 0 \]  \hspace{1cm} (4.132)
\[ e^{2\phi} = \frac{e^{2\phi}}{h_{11}}, \quad C^{(0)} = 0. \]
being \( C' \) and \( \Delta' \) defined in (4.128).

It is important to recall that a D-brane on the original background that is invariant under both \( U(1) \) symmetries, will be left invariant under the action of (1.5). Therefore the new generated background will contain also a D-brane. It has been conjectured [6] that if the original brane gave rise to a certain open string field theory, then the open string field theory on the brane living on the new background is given by changing the start product

\[
f \star_{\gamma} g \rightarrow e^{i\pi\gamma(Q_f^1Q_g^2-Q_f^2Q_g^1)} f \star_0 g \tag{4.133}
\]

where \( \star_0 \) is the original star product and \( Q_f^i,g \) are the \( U(1) \) charges of the fields \( f \) and \( g \). If one consider branes sitting at the origin, the transformation (4.133) does not lead to a noncommutative field theory at low energies, because the \( U(1) \) directions are global symmetries of the field theory. The effect of this transformation for the field theory living on a brane is just to introduce certain phases in the lagrangian according to the rule in (4.133). If we know the gravity dual of the field theory living on a D-brane in the original background, then the gravity dual of the deformed field theory corresponding to the D-brane on the new background will be obtained by performing the \( SL(2, R) \) transformation on the original solution.

5. Discussion

Along the present work, new and old examples of toric metrics with holonomy \( G_2 \) has been presented. The direct sum of such metrics with the flat four dimensional Minkowski one are the most general solutions of the eleven dimensional supergravity which give rise to IIA backgrounds satisfying the strong supersymmetry condition (or Kahler condition) \( F^{(1,1)} = 0 \) of reference [7]. The equivalence between the formalisms of [7] and the Apostolov-Salamon one [8] has been explained in detail. In some sense, the statement that Apostolov-Salamon metrics [8] ”solve” the conditions of [7] (that is, the holomorphic monopole equation and the strong supersymmetry condition) could be a little misleading, because the general solution of Apostolov-Salamon evolution equations is not known. We just passed from one formalism into another, and presented some simple examples. Nevertheless, the formalism of [8] has the advantage that the presence of the toric isometry group is immediately seen. Another interesting feature of this formalism is that, as explained in section 3.2, any 4-dimensional hyperkahler metric can be extended to one with holonomy in \( G_2 \) by means of a linear system. Surprisingly, if the trivial flat hyperkahler metric is used in this construction, the resulting metric has irreducible curvature tensor and holonomy exactly \( G_2 \). These examples are all related to half-flat six dimensional structures, by means of Hitchin equations. An asymptotically Calabi-Yau \( G_2 \) metric related to the flat hyperkahler metric has also been presented, but the equations corresponding to other hyperkahler basis is non linear in general and more difficult to solve.

Conditions in order to have \( T^3 \) instead of \( T^2 \) have been worked out. One possible way to construct a \( G_2 \) holonomy metric with \( T^3 \) isometry is to use an hyperkahler basis with a \textit{triholomorphic} isometry. These spaces correspond to 11-dimensional supergravity solutions for which the \( \gamma \) deformation technique can be applied. This was done to our examples and new supergravity solutions with four dimensional fluxes turned on were found. The deformation technique was used as a solution generating technique only, because we do not know the gravity duals of our backgrounds.

It is not clear for us whether or not it exist a coordinate system for which these \( G_2 \) holonomy metrics are asymptotically conical. Therefore we ignore if these metrics are suitable for
obtaining chiral matter after compactification to four dimensions. But even if this is not so, there are many potential applications. To analyze supersymmetry breaking mechanisms by the presence of non zero flux of M-theory compactified on our manifolds [70], to find the membrane dynamics on our manifolds, to find the conserved quantities associated with these isometries and to investigate $N = 1$ dual theories [44]-[45] could be some of them. Moreover, these $G_2$ manifolds give rise to a dual theory in $3 + 1$ dimensions with minimal supersymmetry and extra KK modes. The techniques in [6] could be useful in order to determine which of these modes are relevant and which are not [46].

There exist also applications related to the construction of a topological string theory in seven dimensions [47] and to the study of domain wall solutions [48]-[49]. The half-flat backgrounds presented here are of relevance to type IIB and heterotic string compactifications [50]-[51]. Another interesting task could be to see if it is possible to lift these metrics to $Spin(7)$ ones as by use of the methods of [52]. Further applications can be found in [54]-[80]. We will return to some of these points in a future investigation.

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Note added When this work was finished, there appeared the references [81]-[83]. Perhaps the results presented in along our work have applications related to these references, and also to reference [53].

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