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COMPATIBILITY OF LOCAL AND GLOBAL LANGLANDS CORRESPONDENCES

RICHARD TAYLOR AND TERUYOSHI YOSHIDA

Abstract. We prove the compatibility of local and global Langlands correspondences for $GL_n$, which was proved up to semisimplification in [HT]. More precisely, for the $n$-dimensional $l$-adic representation $R_l(\Pi)$ of the Galois group of a CM-field $L$ attached to a conjugate self-dual regular algebraic cuspidal automorphic representation $\Pi$, which is square integrable at some finite place, we show that Frobenius semisimplification of the restriction of $R_l(\Pi)$ to the decomposition group of a prime $v$ of $L$ not dividing $l$ corresponds to $\Pi_v$ by the local Langlands correspondence.

Introduction

This paper is a continuation of [HT]. Let $L$ be an (imaginary) CM field and let $\Pi$ be a regular algebraic cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ which is conjugate self-dual ($\Pi \circ c \cong \Pi^\vee$) and square integrable at some finite place. In [HT] it is explained how to attach to $\Pi$ and an arbitrary rational prime $l$ (and an isomorphism $\mathcal{O}_{l}^c \cong \mathbb{C}$) a continuous semisimple representation $R_l(\Pi) : \text{Gal}(\mathbb{L}_{ac}/L) \rightarrow GL_n(\mathbb{Q}_{l}^c)$ which is characterised as follows. For every finite place $v$ of $L$ not dividing $l$,

$$iR_l(\Pi)|_{W_{L_v}} = \text{rec}(\Pi_v^\vee | \det |^{\frac{1-n}{2}})^{ss},$$

where rec denotes the local Langlands correspondence and ss denotes the semisimplification (see [HT] for details). In that book it is also shown that $\Pi_v$ is tempered for all finite places $v$.

In this paper we strengthen this result to completely identify $R_l(\Pi)|_{I_v}$ for $v \not| l$. In particular, we prove the following theorem.

**Theorem A.** If $v \not| l$ then the Frobenius semisimplification of $R_l(\Pi)|_{W_{L_v}}$ is the $l$-adic representation attached to $i^{-1}\text{rec}(\Pi_v^\vee | \det |^{\frac{1-n}{2}})$.

As $R_l(\Pi)$ is semisimple and $\text{rec}(\Pi_v^\vee | \det |^{\frac{1-n}{2}})$ is indecomposable if $\Pi_v$ is square integrable, we obtain the following corollary.

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Corollary B. If $\Pi_v$ is square integrable at a finite place $v \nmid l$, the representation $R_l(\Pi)$ is irreducible.

Using base change it is easy to reduce to the case that $\Pi_v$ has an Iwahori fixed vector. We descend $\Pi$ to an automorphic representation $\pi$ of a unitary group $G$ which locally at $v$ looks like $GL_n$ and at infinity looks like $U(n-1,1) \times U(n,0)^{(L:Q)/2-1}$. Then we realise $R_l(\Pi)$ in the cohomology of a Shimura variety $X$ associated to $G$ with Iwahori level structure at $v$. More precisely, for some $l$-adic sheaf $\mathcal{L}$, the $\pi^p$-isotypic component of $H^{n-1}(X, \mathcal{L})$ is, up to semisimplification and some twist, $R_l(\Pi)^a$ (for some $a \in \mathbb{Z}_{>0}$). We show that $X$ has semistable reduction and use the results of [HT] to calculate the cohomology of the (smooth, projective) strata of the reduction of $X$ above $p$ as a virtual $G(\mathbb{A}^\infty \mathbb{F}) \times \mathbb{F}^\mathbb{Z}$-module (where $F$ denotes Frobenius). This description and the temperedness of $\Pi_v$ shows that the $\pi^p$-isotypic component of the cohomology of any strata is concentrated in the middle dimension. This implies that the $\pi^p$-isotypic component of the Rapoport-Zink weight spectral sequence degenerates at $E_1$, which allows us to calculate the action of inertia at $v$ on $H^{n-1}(X, \mathcal{L})$.

In the special case that $\Pi_v$ is a twist of a Steinberg representation and $\Pi_\infty$ has trivial infinitesimal character, the above theorem presumably follows from the results of Ito [I].

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1. THE MAIN THEOREM

We write $F^{ac}$ for an algebraic closure of a field $F$. Let $l$ be a rational prime and fix an isomorphism $\iota : \mathbb{Q}^{ac}_l \cong \mathbb{C}$.

Suppose that $p \neq l$ is another rational prime. Let $K/\mathbb{Q}_p$ be a finite extension. We will let $\mathcal{O}_K$ denote the ring of integers of $K$, $v_K$ the unique maximal ideal of $\mathcal{O}_K$, $v_K$ the canonical valuation $K^\times \to \mathbb{Z}$, $k(v_K)$ the residue field $\mathcal{O}_K/v_K$ and $|.|_K$ the absolute value normalised by $|x|_K = (\#k(v_K))^{-v_K(x)}$. We will let $\text{Frob}_{v_K}$ denote the geometric Frobenius element of $\text{Gal}(k(v_K)^{ac}/k(v_K))$. We will let $I_{v_K}$ denote the kernel of the natural surjection $\text{Gal}(K^{ac}/K) \to \text{Gal}(k(v_K)^{ac}/k(v_K))$. We will let $W_K$ denote the preimage under $\text{Gal}(K^{ac}/K) \to \text{Gal}(k(v_K)^{ac}/k(v_K))$ of $\text{Frob}_{v_K}$ endowed with a topology by decreeing that $I_K$ with its usual topology is an open subgroup of $W_K$. Local class field theory provides a canonical isomorphism $\text{Art}_K : K^\times \cong W_K^{ab}$, which takes uniformisers to lifts of $\text{Frob}_{v_K}$.

Let $\Omega$ be an algebraically closed field of characteristic 0 and of the same cardinality as $\mathbb{C}$. (Thus in fact $\Omega \cong \mathbb{C}$.) By a Weil-Deligne representation of $W_K$ over $\Omega$ we mean a finite dimensional $\Omega$-vector space $V$ together with a homomorphism $r : W_K \to GL(V)$ with open kernel and an element $N \in \text{End}(V)$ which satisfies

$$r(\sigma)Nr(\sigma)^{-1} = |\text{Art}_K^{-1}(\sigma)||_K N.$$
We sometimes denote a Weil-Deligne representation by \((V, r, N)\) or simply \((r, N)\).

We call \((V, r, N)\) Frobenius semisimple if \(r\) is semisimple. If \((V, r, N)\) is any Weil-Deligne representation we define its Frobenius semisimplification \((V, r, N)^{F-ss} = (V, r^{ss}, N)\) as follows. Choose a lift \(\phi\) of \(\text{Frob}_{v_K}\) to \(W_K\). Let \(r(\phi) = su = us\) where \(s \in GL(V)\) is semisimple and \(u \in GL(V)\) is unipotent. For \(n \in \mathbb{Z}\) and \(\sigma \in I_K\) set \(r^{ss}(\phi^n \sigma) = s^n r(\sigma)\). This is independent of the choices, and gives a Frobenius semisimple Weil-Deligne representation.

One of the main results of [HT] is that, given a choice of \((\# k(v_K))^{1/2} \in \Omega\), there is a bijection \(\text{rec}\) (the local Langlands correspondence) from isomorphism classes of irreducible smooth representations of \(GL_n(K)\) over \(\Omega\) to isomorphism classes of \(n\)-dimensional Frobenius semisimple Weil-Deligne representations of \(W_K\), and that this bijection is natural in a number of respects. (See [HT] for details.)

We will call a Weil-Deligne representation of \(W_K\) over \(\mathbb{Q}_{ac}\) bounded if for some (and hence all) \(\sigma \in W_K - I_K\) all the eigenvalues of \(r(\sigma)\) are \(l\)-adic units. There is an equivalence of categories between bounded Weil-Deligne representations of \(W_K\) over \(\mathbb{Q}_{ac}\) and continuous representations of \(\text{Gal}(\mathbb{Q}_{ac}/K)\) on finite dimensional \(\mathbb{Q}_{ac}\)-vector spaces as follows. Fix a lift \(\phi \in W_K\) of \(\text{Frob}_{v_K}\) and a continuous homomorphism \(t : I_K \rightarrow \mathbb{Z}_l\). Send a Weil-Deligne representation \((V, r, N)\) to \((V, \rho)\), where \(\rho\) is the unique continuous representation of \(\text{Gal}(\mathbb{Q}_{ac}/K)\) on \(V\) such that

\[
\rho(\phi^n \sigma) = r(\phi^n \sigma) \exp(t(\sigma)N)
\]

for all \(n \in \mathbb{Z}\) and \(\sigma \in I_K\). Up to natural isomorphism this functor is independent of the choices of \(t\) and \(\phi\). We will write \(\text{WD}(V, \rho)\) for the Weil-Deligne representation corresponding to a continuous representation \((V, \rho)\). If \(\text{WD}(V, \rho) = (V, r, N)\), then have \(\rho|_{W_K}^{ss} \cong r^{ss}\). (See [T], §4 and [D], §8 for details.)

Now suppose that \(L\) is a finite, imaginary CM extension of \(\mathbb{Q}\). Let \(c \in \text{Aut}(L)\) denote complex conjugation. Suppose that \(\Pi\) is a cuspidal automorphic representation of \(GL_n(\mathbb{A}_L)\) such that

- \(\Pi \circ c \cong \Pi^\vee\);
- \(\Pi_{\infty}\) has the same infinitesimal character as some algebraic representation over \(\mathbb{C}\) of the restriction of scalars from \(L\) to \(\mathbb{Q}\) of \(GL_n\);
- and for some finite place \(x\) of \(L\) the representation \(\Pi_x\) is square integrable.

(In this paper ‘square integrable’ (resp. ‘tempered’) will mean the twist by a character of a pre-unitary representation which is square integrable (resp. tempered).) In [HT] (see theorem C in the introduction of [HT]) it is shown that there is a unique continuous semisimple representation

\[
R_l(\Pi) : \text{Gal}(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}_{ac})
\]
such that for each finite place $v \nmid l$ of $L$

$$\text{rec}(\Pi'_v | \det |^{\frac{1-n}{2}}) = (iR_l(\Pi)|_{W_{L_v}}, N)$$

for some $N$. Moreover it is shown that $\Pi_v$ is tempered for all finite places $v$ of $L$, which completely determines the $N$ (see lemma 1.3 below). If $n = 1$ both these assertions are true without the assumption that $\Pi \circ c \cong \Pi'$. 

The main theorem of this paper identifies the $N$ of $\text{WD}(R_l(\Pi)|_{\text{Gal}(L_{ac}^e/L_v)})$ with the above $N$. More precisely we prove the following.

**Theorem 1.1.** Keep the above notation and assumptions. Then for each finite place $v \nmid l$ of $L$ there is an isomorphism

$$i\text{WD}(R_l(\Pi)|_{\text{Gal}(L_{ac}^e/L_v)})^{F-ss} \cong \text{rec}(\Pi'_v | \det |^{\frac{1-n}{2}})$$

of Weil-Deligne representations over $\mathbb{C}$.

As $R_l(\Pi)$ is semisimple and $\text{rec}(\Pi'_v | \det |^{\frac{1-n}{2}})$ is indecomposable if $\Pi_v$ is square integrable, we have the following corollary.

**Corollary 1.2.** If $\Pi_v$ is square integrable for a finite place $v \nmid l$, then the representation $R_l(\Pi)$ is irreducible.
$i << 0$, such that the $i$-th graded piece is strictly pure of weight $i$. If $(V, r, N)$ is mixed then there is a unique choice of filtration $\text{Fil}_i^W$, and $N(\text{Fil}_i^W V) \subset \text{Fil}_{i-2}^W V$. Finally we will call $(V, r, N)$ pure of weight $k$ if it is mixed with all weights in $k + \mathbb{Z}$ and if for all $i \in \mathbb{Z}_{>0}$
\[ N^i : \text{gr}_{k+i}^W V \sim \text{gr}_{k-1}^W V. \]

If $W$ is strictly pure of weight $k$, then $\text{Sp}_s(W)$ is pure of weight $k - (s - 1)$ for any $s \in \mathbb{Z}_{>1}$.

(1) $(V, r, N)$ is pure if and only if $(V, r, N)^{F-ss}$ is.

(2) If $L/K$ is a finite extension, then $(V, r, N)$ is pure if and only if $(V, r, N)|_{W_L}$ is pure.

(3) An irreducible smooth representation $\pi$ of $GL_n(K)$ has $\sigma \pi$ tempered for all $\sigma : \Omega \hookrightarrow \mathbb{C}$ if and only if $\text{rec}(\pi)$ is pure of some weight.

(4) Given $(V, r)$ with $r$ semisimple, there is, up to equivalence, at most one choice of $N$ which makes $(V, r, N)$ pure.

(5) If $(V, r, N)$ is a Frobenius semisimple Weil-Deligne representation which is pure of weight $k$ and if $W \subset V$ is a Weil-Deligne subrepresentation, then the following are equivalent:
   - (a) $\bigwedge^{\dim W} W$ is pure of weight $k \dim W$,
   - (b) $W$ is pure of weight $k$,
   - (c) $W$ is a direct summand of $V$.

Proof: The first two parts are straightforward (using the fact that the filtration $\text{Fil}_i^W$ is unique). For the third part recall that an irreducible smooth representation $\text{Sp}_{s_1}(\pi_1) \oplus \cdots \oplus \text{Sp}_{s_t}(\pi_t)$ (see section I.3 of [HT]) is tempered if and only if the absolute values of the central characters of the $\text{Sp}_{s_i}(\pi_i)$ are all equal.

Suppose that $(V, r, N)$ is Frobenius semisimple and pure of weight $k$. As a $W_K$-module we can write uniquely $V = \bigoplus_{i \in \mathbb{Z}} V_i$ where $(V_i, r, 0)$ is strictly pure of weight $k + i$. For $i \in \mathbb{Z}_{>0}$ let $V(i)$ denote the kernel of $N^{i+1} : V_i \to V_{i-2}$. Then $N : V_{i+2} \hookrightarrow V_i$ and $V_i = NV_{i+2} \oplus V(i)$. Thus
\[ V = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^i N^j V(i), \]
and for $0 \leq j \leq i$ the map $N^j : V(i) \to V_{i-2j}$ is injective. Also note that as a virtual $W_K$-module $[V(i)] = [V_i] - [V_{i+2} \otimes \text{Art}_K^{-1}]$. Thus if $r$ is semisimple then $(V, r)$ determines $(V, r, N)$ up to isomorphism. This establishes the fourth part.

Now consider the fifth part. If $W$ is a direct summand it is certainly pure of the same weight $k$ and $\bigwedge^{\dim W} W$ is then pure of weight $k \dim W$. Conversely if $W$ is pure of weight $k$ then

$$W = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^i N^j W(i),$$

where $W(i) = W \cap V(i)$. As a $W_K$-module we can decompose $V(i) = W(i) \oplus U(i)$. Setting

$$U = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^i N^j U(i),$$

we see that $V = W \oplus U$ as Weil-Deligne representations. Now suppose only that $\bigwedge^{\dim W} W$ is pure of weight $k \dim W$. Write

$$W \cong \bigoplus_j \text{Sp}_{s_j}(X_j)$$

where each $X_j$ is strictly pure of some weight $k + k_j + (s_j - 1)$. Then, looking at highest exterior powers, we see that $\sum j k_j = 0$. On the other hand as $V$ is pure we see that $k_j \leq 0$ for all $j$. We conclude that $k_j = 0$ for all $j$ and hence that $W$ is pure of weight $k$.

The final part follows from the fifth part by a simple inductive argument. □

Now let $L$ denote a number field. Write $| |_L$ for

$$\prod_x | |_{L_x} : \mathbb{A}^x_L / L^x \to \mathbb{R}_{>0}^\times,$$

and write $\text{Art}_L$ for

$$\prod_x \text{Art}_{L_x} : \mathbb{A}^x_L / L^x \to \text{Gal}(L^{ac}/L)^{\text{ab}}.$$

We will call a continuous representation

$$R : \text{Gal}(L^{ac}/L) \to GL_n(\mathbb{Q}_l^{ac})$$

pure of weight $k$ if for all but finitely many finite places $x$ of $L$ the representation $R$ is unramified at $x$ and every eigenvalue $\alpha$ of $R(\text{Frob}_x)$ is a Weil $(\#k(x))^k$-number. Note that if $n = 1$ then $R$ is pure of weight $k$ if and only if for all $\iota : \mathbb{Q}_l^{ac} \to \mathbb{C}$ we have $|\iota R \circ \text{Art}_L|^2 = | |_L^{-k}$. In particular if $n = 1$ and $R$ is pure then $R|_{W_{L_x}}$ is strictly pure for all finite places $x$ of $L$.

We have the following lemma.
Lemma 1.4. Suppose that \( M/L \) is a finite extension of number fields. Suppose also that
\[ R : \text{Gal}(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}^{ac}) \]
is a continuous semisimple representation which is pure of weight \( k \). Suppose that
\[ S : \text{Gal}(M^{ac}/M) \longrightarrow GL_n(\mathbb{Q}^{ac}) \]
is another continuous representation with \( S^{ss} \cong R^a_{|\text{Gal}(M^{ac}/M)} \) for some \( a \in \mathbb{Z}_{>0} \). Suppose finally that \( w \) is a place of \( M \) above a finite place \( v \) of \( L \). If \( \text{WD}(S|_{\text{Gal}(M^{ac}/M_w)}) \) is pure of weight \( k \), then \( \text{WD}(R|_{\text{Gal}(L^{ac}/L_v)}) \) is also pure of weight \( k \).

Proof: Write
\[ R|_{\text{Gal}(M^{ac}/M)} = \bigoplus_i R_i \]
where each \( R_i \) is irreducible. Then \( R_i \) is pure of weight \( k \dim R_i \) and so that the top exterior power \( \bigwedge^{\dim R_i} \text{WD}(R_i|_{\text{Gal}(M^{ac}/M_w)}) \) is also pure of weight \( k \dim R_i \). Lemma 1.3(6) tells us that
\[ \text{WD}(S|_{\text{Gal}(M^{ac}/M_w)})^{F-ss} \cong \left( \bigoplus_i \text{WD}(R_i|_{\text{Gal}(M^{ac}/M_w)})^{F-ss} \right)^a \cong (\text{WD}(R|_{\text{Gal}(M^{ac}/M_w)})^{F-ss})^a, \]
and that \( \text{WD}(R|_{\text{Gal}(M^{ac}/M_w)})^{F-ss} \) is pure of weight \( w \). Applying lemma 1.3(1) and (2), we see that \( \text{WD}(R|_{\text{Gal}(L^{ac}/L_v)}) \) is also pure. \( \square \)

2. Shimura varieties

In this section we study the geometry of integral models of Shimura varieties of the type considered in [HT], but with Iwahori level. It may be viewed as a generalisation of the work of Deligne-Rapoport [DR] in the case of modular curves.

In this section,

- let \( E \) be an imaginary quadratic field, \( F^+ \) a totally real field and set \( F = EF^+ \);
- let \( p \) be a rational prime which splits \( p = uu^c \) in \( E \);
- and let \( w = w_1, ..., w_r \) be the primes of \( F \) above \( u \);
- and let \( B \) be a division algebra with centre \( F \) such that
  - \( \dim_F B = n^2 \),
  - \( B^{op} \cong B \otimes_{F,c} F \),
  - at every place \( x \) of \( F \) either \( B_x \) is split or a division algebra,
  - if \( n \) is even then the number of finite places of \( F^+ \) above which \( B \) is ramified is congruent to \( 1 + \frac{n}{2}[F^+:\mathbb{Q}] \) modulo 2.
Pick a positive involution $\ast$ on $B$ with $\ast|_F = c$. Let $V = B$ as a $B \otimes_F B^{\text{op}}$-module. For $\beta \in B^{\ast = -1}$ define a pairing
\[(\ ,
\ ) : V \times V \longrightarrow \mathbb{Q}
(x_1, x_2) \longmapsto \text{tr}_{F/\mathbb{Q}} \text{tr}_{B/F}(x_1\beta x_2^\ast).
\] Also define an involution $\#$ on $B$ by $x^\# = \beta x^\ast \beta^{-1}$ and a reductive group $G/\mathbb{Q}$ by setting, for any $\mathbb{Q}$-algebra $R$, the group $G(R)$ equal to the set of
\[(\lambda, g) \in R^\times \times (B^{\text{op}} \otimes_\mathbb{Q} R)^\times
\] such that
\[gg^\# = \lambda.
\] Let $\nu : G \to \mathbb{G}_m$ denote the multiplier character sending $(\lambda, g)$ to $\lambda$. Note that if $x$ is a rational prime which splits $x = yy^c$ in $E$ then
\[G(\mathbb{Q}_x) \sim \cong (B^y)^{\times} \times \mathbb{Q}_x^{\times}
(\lambda, g) \longmapsto (g_y, \lambda).
\] We can and will assume that
\begin{itemize}
  \item if $x$ is a rational prime which does not split in $E$ the $G \times \mathbb{Q}_x$ is quasisplit;
  \item the pairing $(\ , \ )$ on $V \otimes_\mathbb{Q} \mathbb{R}$ has invariants $(1, n-1)$ at one embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and invariants $(0, n)$ at all other embeddings $F^+ \hookrightarrow \mathbb{R}$. (See section I.7 of [HT] for details.)
\end{itemize}

Let $U$ be an open compact subgroup of $G(\mathbb{A}^\infty)$. Define a functor $\mathfrak{X}_U$ from the category of pairs $(S, s)$, where $S$ is a connected locally noetherian $F$-scheme and $s$ is a geometric point of $S$, to the category of sets, sending $(S, s)$ to the set of isogeny classes of four-tuples $(A, \lambda, i, \eta)$ where
\begin{itemize}
  \item $A/S$ is an abelian scheme of dimension $[F^+ : \mathbb{Q}]n^2$;
  \item $i : B \hookrightarrow \text{End}(A) \otimes_\mathbb{Z} \mathbb{Q}$ such that $\text{Lie} A \otimes_{(E \otimes \mathbb{Q} \mathcal{O}_S), 1 \otimes 1} \mathcal{O}_S$ is locally free over $\mathcal{O}_S$ of rank $n$ and the two actions of $F^+$ coincide;
  \item $\lambda : A \to A^\vee$ is a polarisation such that for all $b \in B$ we have $\lambda \circ i(b) = i(b^\ast) \circ \lambda$;
  \item $\eta$ is a $\pi_1(S, s)$-invariant $U$-orbit of isomorphisms of $B \otimes_\mathbb{Q} \mathbb{A}^\infty$-modules $\eta : V \otimes_\mathbb{Q} \mathbb{A}^\infty \to VA_s$ which take the standard pairing $(\ , \ )$ on $V$ to a $(\mathbb{A}^\infty)^{\times}$-multiple of the $\lambda$-Weil pairing on $VA_s$.
\end{itemize}
Here $VA_s = \left(\lim_{\to N} A[N](k(s))\right) \otimes_\mathbb{Z} \mathbb{Q}$ is the adelic Tate module. For the precise notion of isogeny class see section III.1 of [HT]. If $s$ and $s'$ are both geometric points of a connected locally noetherian $F$-scheme $S$ then $\mathfrak{X}_U(S, s)$ and $\mathfrak{X}_U(S, s')$ are in canonical bijection. thus we may think of $\mathfrak{X}_U$ as a functor from connected locally noetherian $F$-schemes to sets. We
may further extend it to a functor from all locally noetherian $F$-schemes to sets by setting

$$X_U\left(\prod_i S_i\right) = \prod_i X_U(S_i).$$

If $U$ is sufficiently small (i.e. for some finite place $x$ of $\mathbb{Q}$ the projection of $U$ to $G(\mathbb{Q}_x)$ contains no element of finite order except 1) then $X_U$ is represented by a smooth projective variety $X_U/F$ of dimension $n-1$. The inverse system of the $X_U$ for varying $U$ has a natural action of $G(\mathbb{A}^\infty)$.

Choose a maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ of $B$ with $\mathcal{O}_B^* = \mathcal{O}_B$. Also fix an isomorphism $\mathcal{O}_{B,w}^\text{op} \cong M_n(\mathcal{O}_{F,w})$, and let $\varepsilon \in B_w$ denote the element corresponding to the diagonal matrix $(1,0,0,...,0) \in M_n(\mathcal{O}_{F,w})$. We decompose $G(\mathbb{A}^\infty)$ as

$$G(\mathbb{A}^\infty) = G(\mathbb{A}^\infty_p) \times \left(\prod_{i=2}^r (B_{w_i}^\text{op})^\times\right) \times GL_n(F_w) \times \mathbb{Q}_p^\times.$$

Let $\varpi$ denote a uniformiser for $\mathcal{O}_{F,w}$. For $m = (m_2, ..., m_r) \in \mathbb{Z}_{\geq 0}^{r-1}$, set

$$U_p^w(m) = \prod_{i=2}^r \ker((\mathcal{O}_{B,w_i}^\text{op})^\times \to (\mathcal{O}_{B,w_i}^\text{op}/w_i^{m_i})^\times) \subset \prod_{i=2}^r (B_{w_i}^\text{op})^\times.$$

Let $B_n$ denote the Borel subgroup of $GL_n$ consisting of upper triangular matrices and let $N_n$ denote its unipotent radical. Let $Iw_{n,w}$ denote the subgroup of $GL_n(\mathcal{O}_{F,w})$ consisting of matrices which reduce modulo $w$ to $B_n(k(w))$. We will consider the following open subgroups of $G(\mathbb{Q}_p)$:

$$\text{Ma}(m) = U_p^w(m) \times GL_n(\mathcal{O}_{F,w}) \times \mathbb{Z}_p^\times,$$

$$\text{Iw}(m) = U_p^w(m) \times Iw_{n,w} \times \mathbb{Z}_p^\times.$$

Let $U^p$ be an open compact subgroup of $G(\mathbb{A}^\infty_p)$. Write $U_0$ (resp. $U$) for $U^p \times \text{Ma}(m)$ (resp. $U^p \times \text{Iw}(m)$).

We recall that in section III.4 [HT] integral model of $X_{U_0}$ over $\mathcal{O}_{F,w}$ is defined. It represents the functor $X_{U_0}$ from locally noetherian $\mathcal{O}_{F,w}$-schemes to sets. As above, $X_{U_0}$ is initially defined on the category of connected locally noetherian $\mathcal{O}_{F,w}$ schemes with a geometric point to sets. It sends $(S, s)$ to the set of prime-to-$p$ isogeny classes of $(r + 3)$-tuples $(A, \lambda, i, \tau^n, \alpha_i)$, where

- $A/S$ is an abelian scheme of dimension $[F^+: \mathbb{Q}]n^2$;
- $i : \mathcal{O}_B \twoheadrightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that $\text{Lie} A \otimes (\mathcal{O}_{F,w} \otimes_{\mathbb{Z}_p} \mathcal{O}_S), 1 \otimes 1 \mathcal{O}_S$ is locally free of rank $n$ and the two actions of $\mathcal{O}_F$ coincide;
- $\lambda : A \to A^\vee$ is a prime-to-$p$ polarisation such that for all $b \in \mathcal{O}_B$ we have $\lambda \circ i(b) = i(b^s)^\vee \circ \lambda$;
Lemma 2.1. If \( \eta : V \otimes _{\mathbb{Q}} \mathbb{A}_{\mathbb{R}}^{\infty \cdot p} \to V^p A_s \) which take the standard pairing \(( \ , \ )\) on \( V \) to a \( (\mathbb{A}_{\mathbb{R}}^{\infty \cdot p})^* \)-multiple of the \( \lambda \)-Weil pairing on \( V^p A_s \);

for \( 2 \leq i \leq r \), \( \alpha_i : (w_i^{-m_i}O_{B,w_i}/O_{B,w_i})_S \sim A[w_i^{m_i}] \) is an isomorphism of \( S \)-schemes with \( \mathcal{O}_B \)-actions;

Then \( X_{U_0} \) is smooth and projective over \( \mathcal{O}_{F,w} \) ([HT], page 109). As \( U^p \) varies, the inverse system of the \( X_{U_0} \)'s has an action of \( G(\mathbb{A}_{\mathbb{R}}^{\infty \cdot p}) \).

Given an \( (r + 3) \)-tuple as above we will write \( G_A \) for \( \varepsilon A[w^\infty] \) a Barsotti-Tate \( \mathcal{O}_{F,w} \)-module. Over a base in which \( p \) is nilpotent it is one dimensional. If \( A \) denotes the universal abelian scheme over \( X_{U_0} \), we will write \( \mathcal{G} \) for \( \mathcal{G}_A \). This \( \mathcal{G} \) is compatible, i.e. the two actions of \( \mathcal{O}_{F,w} \) on \( \text{Lie} G \) coincide (see [HT]).

Write \( \overline{X}_{U_0} \) for the special fibre \( X_{U_0} \times_{\text{Spec} \mathcal{O}_{F,w}} \text{Spec} k(w) \). For \( 0 \leq h \leq n - 1 \), we let \( \overline{X}_{U_0}^{[h]} \) denote the reduced closed subscheme of \( \overline{X}_{U_0} \) whose closed geometric points \( s \) are those for which the maximal etale quotient of \( \mathcal{G}_s \) has \( \mathcal{O}_{F,w} \)-height at most \( h \), and let

\[
\overline{X}_{U_0}^{[h]} = \overline{X}_{U_0}^{[h]} - \overline{X}_{U_0}^{[h-1]}
\]

(while we set \( \overline{X}_{U_0}^{[-1]} = \emptyset \)). Then \( \overline{X}_{U_0}^{[h]} \) is smooth of pure dimension \( h \) (corollary III.4.4 of [HT]), and on it there is a short exact sequence

\[
(0) \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^{et} \longrightarrow (0)
\]

where \( \mathcal{G}^0 \) is a formal Barsotti-Tate \( \mathcal{O}_{F,w} \)-module and \( \mathcal{G}^{et} \) is an etale Barsotti-Tate \( \mathcal{O}_{F,w} \)-module with \( \mathcal{O}_{F,w} \)-height \( h \).

**Lemma 2.1.** If \( 0 \leq h \leq n - 1 \) then the Zariski closure of \( \overline{X}_{U_0}^{(h)} \) contains \( \overline{X}_{U_0}^{(0)} \).

**Proof:** This is ‘well known’, but for lack of a reference we give a proof. Let \( x \) be a closed geometric point of \( \overline{X}_{U_0}^{(0)} \). By lemma II.4.1 of [HT] the formal completion of \( X_{U_0} \times_{\text{Spec} k(w)^{ac}} \) at \( x \) is isomorphic to the equicharacteristic universal deformation ring of \( \mathcal{G}_x \). According to the proof of proposition 4.2 of [Dr] this is \( \text{Spf} k(w)^{ac}[[T_1, \ldots, T_{n-1}]] \) and we can choose the \( T_i \) and a formal parameter \( S \) on the universal deformation of \( \mathcal{G}_x \) such that

\[
[w(w)](S) \equiv w w S + \sum_{i=1}^{n-1} T_i S^{#k(w)^{ac} + S^{#k(w)^{ac} + 1}} \pmod{S^{#k(w)^{ac} + 1}}.
\]

Thus we get a morphism

\[
\text{Spec} k(w)^{ac}[[T_1, \ldots, T_{n-1}]] \longrightarrow \overline{X}_{U_0}
\]

lying over \( x : k(w)^{ac} \to \overline{X}_{U_0} \), such that, if \( k \) denotes the algebraic closure of the field of fractions of \( k(w)^{ac}[[T_1, \ldots, T_{n-1}]]/(T_1, \ldots, T_{n-1}) \), then the induced map

\[
\text{Spec} k \longrightarrow \overline{X}_{U_0}
\]
factors through $X_{U_0}^{(h)}$. Thus $x$ is in the closure of $X_{U_0}^{(h)}$, and the lemma follows. □

Now we define the functor $\mathfrak{X}_U$. Again we initially define it as a functor from the category of connected locally noetherian schemes with a geometric point to sets, but then (as above) we extend it to a functor from locally noetherian schemes to sets. The functor $\mathfrak{X}_U$ will send $(S, s)$ to the set of prime-to-$p$ isogeny classes of $(r + 4)$-tuples $(A, \lambda, i, \mathfrak{P}, \mathcal{C}, \alpha_i)$, where $(A, \lambda, i, \mathfrak{P}, \alpha_i)$ is as in the definition of $gX_{U_0}$ and $\mathcal{C}$ is a chain of isogenies of compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules, each of degree $\# k(w)$ and with composite equal to the canonical map $\mathcal{G} \to \mathcal{G}/\mathcal{G}[w]$. There is a natural transformation of functors $\mathfrak{X}_U \to \mathfrak{X}_{U_0}$.

Lemma 2.2. The functor $\mathfrak{X}_U$ is represented by a scheme $X_U$ which is finite over $X_{U_0}$. The scheme $X_U$ has some irreducible components of dimension $n$.

Proof: By denoting the kernel of $\mathcal{G}_0 \to \mathcal{G}_j$ by $\mathcal{K}_j \subset \mathcal{G}[w]$, we can view the above chain as a flag

$$0 = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = \mathcal{G}[w]$$

of closed finite flat subgroup schemes with $\mathcal{O}_{F,w}$-action, with each $\mathcal{K}_j/\mathcal{K}_{j-1}$ having order $\# k(w)$. Let $\mathcal{H}$ denote the sheaf of Hopf algebras over $X_{U_0}$ defining $\mathcal{G}[w]$. Then $\mathfrak{X}_U$ is represented by a closed subscheme $X_U$ of the Grassmanian of chains of locally free direct summands of $\mathcal{H}$. (The closed conditions require that the subsheaves are sheaves of ideals defining a flag of closed subgroup schemes with the desired properties.) Thus $X_U$ is projective over $\mathcal{O}_{F,w}$. At each closed geometric point $s$ of $X_{U_0}$ the number of possible $\mathcal{O}_{F,w}$-submodules of $\mathcal{G}[w]_s \cong \mathcal{G}[w]_s^0 \times \mathcal{G}[w]_s^{et}$ is finite, so $X_U$ is finite over $X_{U_0}$. To see that $X_U$ has some components of dimension $n$ it suffices to note that on the generic fibre the map to $X_{U_0}$ is finite etale. □

We say an isogeny $\mathcal{G} \to \mathcal{G}'$ of one-dimensional compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules over a scheme $S$ of characteristic $p$ has connected kernel if it induces the zero map on Lie $\mathcal{G}$. We will denote the relative Frobenius map by $F : \mathcal{G} \to \mathcal{G}^{(p)}$ and let $f = [k(w) : \mathbb{F}_p]$, and then $F^f : \mathcal{G} \to \mathcal{G}^{(# k(w))}$ is an isogeny of compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules of degree $\# k(w)$ and has connected kernel.

We have the following rigidity lemma.

Lemma 2.3. Let $W$ denote the ring of integers of the completion of the maximal unramified extension of $F_w$. Suppose that $R$ is an Artinian local $W$-algebra with residue field $k(w)^{ac}$. Suppose also that

$$\mathcal{C} : \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_n = \mathcal{G}_0$$

is a chain of isogenies of degree $\# k(w)$ of one-dimensional compatible formal Barsotti-Tate $\mathcal{O}_{F,w}$-modules of $\mathcal{O}_{F,w}$-height $n$ with composite equal to multiplication by $\varpi_w$. If every isogeny $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel (for $i = 1, \ldots, n$) then $R$ is a $k(w)^{ac}$-algebra and $\mathcal{C}$
is the pull-back of a chain of Barsotti-Tate \( \mathcal{O}_{F,w} \)-modules over \( k(w)^{ac} \), with all the isogenies isomorphic to \( F^f \).

\textbf{Proof}: As the composite of the \( n \) isogenies induces multiplication by \( \varpi_w \) on the tangent space, \( \varpi_w = 0 \) in \( R \), i.e. \( R \) is a \( k(w)^{ac} \)-algebra. Choose a parameter \( T_i \) for \( G_i \) over \( R \). With respect to these choices, let \( f_i(T_i) \in R[[T_i]] \) represent \( G_{i-1} \to G_i \). We can write \( f_i(T_i) = g_i(T_i^{p^{h_i}}) \) with \( h_i \in \mathbb{Z}_{>0} \) and \( g'_i(0) \neq 0 \). (See [F], chapter I, §3, Theorem 2.) As \( G_{i-1} \to G_i \) has connected kernel, \( f'_i(0) = 0 \) and \( h_i > 0 \). As \( f_i \) commutes with the action \([r]\) for all \( r \in \mathcal{O}_{F,w} \), we have \( \pi g^{h_i}_i = \pi \) for all \( \pi \in k(w) \), hence \( h_i \) is a multiple of \( f = [k(w) : \mathbb{F}_p] \). Reducing modulo the maximal ideal of \( R \) we see that \( h_i \leq f \) and so in fact \( h_i = f \) and \( g'_i(0) \in R^\times \). Thus \( G_i \cong G_0^{(\#k(w)^{ac})} \) in such a way that the isogeny \( G_0 \to G_i \) is identified with \( F^{f_i} \). In particular \( G_0 \cong G_0^{(\#k(w)^{ac})} \) and hence \( G_0 \cong G_0^{(\#k(w)^{ac})} \) for any \( m \in \mathbb{Z}_{\geq 0} \). As \( R \) is Artinian some power of the absolute Frobenius on \( R \) factors through \( k(w)^{ac} \). Thus \( G_0 \) is a pull-back from \( k(w)^{ac} \) and the lemma follows. \( \square \)

Now let \( Y_{U,i} \) denote the closed subscheme of \( \overline{X}_U = X_U \times_{\text{Spec} \mathcal{O}_{F,w}} \text{Spec} k(w) \) over which \( G_{i-1} \to G_i \) has connected kernel.

\textbf{Proposition 2.4.} (1) \( X_U \) has pure dimension \( n \) and semistable reduction over \( \mathcal{O}_{F,w} \), that is, for all closed points \( x_0 \) of the special fibre \( X_U \times_{\text{Spec} \mathcal{O}_{F,w}} \text{Spec} k(w) \), there exists an etale morphism \( V \to X_U \) with \( x \in \text{Im} V \) and an etale \( \mathcal{O}_{F,w} \)-morphism:

\[ V \to \text{Spec} \mathcal{O}_{F,w}[T_1, \ldots, T_n]/(T_1 \cdots T_m - \varpi_w) \]

for some \( 1 \leq m \leq n \), where \( \varpi_w \) is a uniformizer of \( \mathcal{O}_{F,w} \).

(2) \( X_U \) is regular and the natural map \( X_U \to X_{U_0} \) is finite and flat.

(3) Each \( Y_{U,i} \) is smooth over \( \text{Spec} k(w) \) of pure dimension \( n - 1 \), \( \overline{X}_U = \bigcup_{i=1}^n Y_{U,i} \) and, for \( i \neq j \) the schemes \( Y_{U,i} \) and \( Y_{U,j} \) share no common connected component. In particular, \( X_U \) has strictly semistable reduction.

\textbf{Proof}: In this proof we will make repeated use of the following version of Deligne’s homogeneity principle ([DR]). Write \( W \) for the ring of integers of the completion of the maximal unramified extension of \( F_w \). In what follows, if \( s \) is a closed geometric point of an \( \mathcal{O}_{F,w} \)-scheme \( X \) locally of finite type, then we write \( \mathcal{O}_{X,s}^{\mathcal{X}} \) for the completion of the strict Henselisation of \( X \) at \( s \), i.e.\( \mathcal{O}_{X,s}^{\mathcal{X}} \times_{\text{Spec} \mathcal{W},s} \). Let \( P \) be a property of complete noetherian local \( W \)-algebras such that if \( X \) is an \( \mathcal{O}_{F,w} \)-scheme locally of finite type then the set of closed geometric points \( s \) of \( X \) for which \( \mathcal{O}_{X,s}^{\mathcal{X}} \) has property \( P \) is Zariski open. Also let \( X \to X_{U_0} \) be a finite morphism with the following properties

(i) If \( s \) is a closed geometric point of \( \overline{X}_{U_0}^{(h)} \) then, up to isomorphism, \( \mathcal{O}_{X,s}^{\mathcal{X}} \) does not depend on \( s \) (but only on \( h \)).

(ii) There is a unique geometric point of \( X \) above any geometric point of \( \overline{X}_{U_0}^{(0)} \).
If $\mathcal{O}_{X,s}^\wedge$ has property $P$ for every geometric point of $X$ over $\overline{X}_{U_0}^{(0)}$, then $\mathcal{O}_{X,s}^\wedge$ has property $P$ for every closed geometric point of $X$. Indeed, if we let $Z$ denote the closed subset of $X$ where $P$ does not hold, then its image in $X_{U_0}$ is closed and is either empty or contains some $\overline{X}_{U_0}^{(h)}$. In the latter case, by lemma 2.1, it also contains $\overline{X}_{U_0}^{(0)}$, which is impossible. Thus $Z$ must be empty.

Note that both $X = X_U$ and $X = Y_{U,i}$ satisfy the above condition (ii) for the homogeneity principle, by letting $R = k(w)^{ac}$ in lemma 2.3.

(1): The dimension of $\mathcal{O}_{X_U,s}^\wedge$ as $s$ runs over geometric points of $X_U$ above $\overline{X}_{U_0}^{(0)}$ is constant, say $m$. Applying the homogeneity principle to $X = X_U$ with $P$ being ‘dimension $m’$, we see that $X_U$ has pure dimension $m$. By lemma 2.2 we must have $m = n$ and $X_U$ has pure dimension $n$.

Now we will apply the above homogeneity principle to $X = X_U$ taking $P$ to be ‘isomorphic to $W[[T_1, ..., T_n]]/(T_1 \cdots T_m - \varpi_w)$ for some $m \leq n'$. By a standard argument (see e.g. the proof of proposition 4.10 of [Y]) the set of points with this property is open and if all closed geometric points of $X_U$ have this property $P$ then $X_U$ is semistable of pure dimension $n$.

Let $s$ be a geometric point of $X_U$ over a point of $\overline{X}_{U_0}^{(0)}$. Choose a basis $e_i$ of Lie $\mathcal{G}_i$ over $\mathcal{O}_{X_U,s}^\wedge$ such that $e_n$ maps to $e_0$ under the isomorphism $\mathcal{G}_n = \mathcal{G}_0/\mathcal{G}_0[w] \cong \mathcal{G}_0$, which is impossible. Thus

$$X_1 \cdots X_n = \varpi_w.$$ 

Moreover it follows from lemma 2.3 that $\mathcal{O}_{X_U,s}^\wedge/(X_1, ..., X_n) = k(w)^{ac}$. (Because, by lemma III.4.1 of [HT], $\mathcal{O}_{X_U,s}^\wedge$ is the universal deformation space of $\mathcal{G}_s$. Hence by lemma 2.3, $\mathcal{O}_{X_U,s}^\wedge$ is the universal deformation space for the chain

$$\mathcal{G}_s \xrightarrow{F^f} \mathcal{G}_s^{\#(w)} \xrightarrow{F^f} \cdots \xrightarrow{F^f} \mathcal{G}_s^{\#(w)^n} \cong \mathcal{G}_s/\mathcal{G}_s[\varpi_w].$$

Thus we get a surjection

$$W[[X_1, ..., X_n]]/(X_1 \cdots X_n - \varpi_w) \twoheadrightarrow \mathcal{O}_{X_U,s}^\wedge$$

and as $\mathcal{O}_{X_U,s}^\wedge$ has dimension $n$ this map must be an isomorphism.

(2): We see at once that $X_U$ is regular. Then [AK] V, 3.6 tells us that $X_U \rightarrow X_{U_0}$ is flat.

(3): We apply the homogeneity principle to $X = Y_{U,i}$ taking $P$ to be ‘formally smooth of dimension $n - 1’$. If $s$ is a geometric point of $Y_{U,i}$ above $\overline{X}_{U_0}^{(0)}$ then we see that $\mathcal{O}_{Y_{U,i,s}}^\wedge$ is cut out in $\mathcal{O}_{X_U,s}^\wedge \cong W[[X_1, ..., X_n]]/(X_1 \cdots X_n - \varpi_w)$ by the single equation $X_i = 0$. (We are using the parameters $X_1$ defined above.) Thus

$$\mathcal{O}_{Y_{U,i,s}}^\wedge \cong k(w)^{ac}[[X_1, ..., X_i-1, X_{i+1}, ..., X_n]]$$
is formally smooth of dimension \( n - 1 \). We deduce that \( Y_{U,i} \) is smooth of pure dimension \( n - 1 \).

As our \( G/X_U \) is one-dimensional, over a closed point, at least one of the isogenies \( G_{i-1} \rightarrow G_i \) must have connected kernel, which shows that \( X_U = \bigcup_i Y_{U,i} \). Suppose \( Y_{U,i} \) and \( Y_{U,j} \) share a connected component \( Y \) for some \( i \neq j \). Then \( Y \) would be finite flat over \( X_{U_0} \) and so the image of \( Y \) would meet \( X_{U_0}^{(n-1)} \). This is impossible as above a closed point of \( X_{U_0}^{(n-1)} \) one isogeny among the chain can have connected kernel. Thus, for \( i \neq j \) the closed subschemes \( Y_{U,i} \) and \( Y_{U,j} \) have no connected component in common. \( \square \)

By the strict semistability, if we write, for \( S \subset \{1, ..., n\} \),

\[
Y_{U,S} = \bigcap_{i \in S} Y_{U,i}, \quad Y_{U,S}^0 = Y_{U,S} - \bigcup_{T \supset S} Y_{U,T}
\]

then \( Y_{U,S} \) is smooth over \( \text{Spec } k(w) \) of pure dimension \( n - \#S \) and \( Y_{U,S}^0 \) are disjoint for different \( S \). With respect to the finite flat map \( X_U \rightarrow X_{U_0} \), the inverse image of \( X_{U_0}^{(h)} \) is exactly the locus where at least \( n - \#S \) of the isogenies have connected kernel, i.e. \( \bigcup_{\#S \geq n-h} Y_{U,S} \).

Hence the inverse image of \( X_{U_0}^{(h)} \) is equal to \( \bigcup_{\#S = n-h} Y_{U,S}^0 \). Also note that the inverse system of \( Y_{U,S}^0 \) for varying \( U^p \) is stable by the action of \( G(\mathbb{A}^{\infty,p}) \).

Next we will relate the open strata \( Y_{U,S}^0 \) to the Igusa varieties of the first kind defined in [HT]. For \( 0 \leq h \leq n - 1 \) and \( m \in \mathbb{Z}_{\geq 0}^r \), we write \( I_{U,1,m} \) for the Igusa varieties of the first kind defined on page 121 of [HT]. We also define an \textit{Iwahori-Igusa variety of the first kind} \( I_{U}^{(h)}/X_{U_0}^{(h)} \) as the moduli space of chains of isogenies

\[
G^{et} = G_{0} \rightarrow G_{1} \rightarrow \cdots \rightarrow G_{h} = G^{et}/G^{et}[w]
\]

of etale Barsotti-Tate \( \mathcal{O}_{F,w} \)-modules, each isogeny having degree \( \#k(w) \) and with composite equal to the natural map \( G^{et} \rightarrow G^{et}/G^{et}[w] \). Then \( I_{U}^{(h)} \) is finite etale over \( X_{U_0}^{(h)} \), and as the Igusa variety \( I_{U,1,m}^{(h)} \) classifies the isomorphisms

\[
\alpha_{1}^{et} : (w^{-1}\mathcal{O}_{F,w}/\mathcal{O}_{F,w})^{\mathbb{A}_{(h)}_{U_0}^{(h)}} \rightarrow G^{et}[w],
\]

the natural map

\[
I_{U,1,m}^{(h)} \rightarrow I_{U}^{(h)}
\]

is finite etale and Galois with Galois group \( B_h(k(w)) \). Hence we can identify \( I_{U}^{(h)} \) with \( I_{U,1,m}^{(h)}/B_h(k(w)) \). Note that the system \( I_{U}^{(h)} \) for varying \( U^p \) naturally inherits the action of \( G(\mathbb{A}^{\infty,p}) \).

**Lemma 2.5.** For \( S \subset \{1, ..., n\} \) with \( \#S = n - h \), there exists a finite map of \( X_{U_0}^{(h)} \)-schemes

\[
\varphi : Y_{U,S}^0 \rightarrow I_{U}^{(h)}
\]
which is bijective on the geometric points.

Proof: The map is defined in a natural way from the chain of isogenies $\mathcal{C}$ by passing to the etale quotient $\mathcal{G}^{et}$, and it is finite as $Y_{U,S}^{0}$ (resp. $I_{U}^{(h)}$) is finite (resp. finite etale) over $\overline{X}_{U_{0}}$. Let $s$ be a closed geometric point of $I_{U}^{(h)}$ with a chain of isogenies

$$G_{s}^{et} = G_{0}^{et} \to \cdots \to G_{n}^{et} = G_{s}^{et} / G_{s}^{et}[w].$$

For $1 \leq i \leq n$ let $j(i)$ denote the number of elements of $S$ which are less than or equal to $i$. Set $G_{i} = (G_{i})^{et} \times (\mathbb{A}^{\infty})^{0}$. If $i \not\in S$, define an isogeny $G_{i-1} \to G_{i}$ to be the identity times the given isogeny $G_{i-1}^{et} \to G_{i}^{et}$. If $i \in S$, define an isogeny $G_{i-1} \to G_{i}$ to be $F^{j}$ times the identity. Then

$$G_{0} \to \cdots \to G_{n}$$

defines the unique geometric point of $Y_{U,S}^{0}$ above $s$. □

Now recall from [HT] III.2, that for an irreducible algebraic representation $\xi$ of $G$ over $\mathbb{Q}^{ac}$, one can associate a lisse $\mathbb{Q}^{ac}$-sheaf $\mathcal{L}_{\xi}/X_{U}$ for every $U$ such that $X_{U}$ is defined, and the action of $G(\mathbb{A}^{\infty})$ extends to $\mathcal{L}_{\xi}$. The sheaf $\mathcal{L}_{\xi}$ is extended to the integral models and Igusa varieties, and on $I_{U,n}^{(h)}$, $I_{U}^{(h)}$ and $Y_{U,S}^{0}$ they are the pull back of $\mathcal{L}_{\xi}$ on $\overline{X}_{U_{0}}^{(h)}$.

Corollary 2.6. For every $i \in \mathbb{Z}_{\geq 0}$, we have isomorphisms

$$H_{c}^{i}(Y_{U,S}^{0} \times k(w)^{ac}, \mathcal{L}_{\xi}) \sim H_{c}^{i}(I_{U}^{(h)} \times k(w)^{ac}, \mathcal{L}_{\xi}) \sim H_{c}^{i}(I_{U,n}^{(h)} \times k(w)^{ac}, \mathcal{L}_{\xi})$$

that are compatible with the actions of $G(\mathbb{A}^{\infty})$ when we vary $U^{p}$.

Proof: By lemma 2.5, for any lisse $\mathbb{Q}^{ac}$-sheaf $\mathcal{F}$ on $I_{U}^{(h)}$, we have $\mathcal{F} \cong \varphi^{*} \varphi^{*} \mathcal{F}$ by looking at the stalks at all geometric points. As $\varphi$ is finite the first isomorphism follows. The second isomorphism follows easily as $I_{U,n}^{(h)} \to I_{U}^{(h)}$ is finite etale and Galois with Galois group $B_{k}(k(w))$. □

In the next section, we will be interested in the $G(\mathbb{A}^{\infty} \times \text{Frob}_{w}^{\mathbb{Z}})$-modules

$$H^{i}(Y_{w(m),S}, \mathcal{L}_{\xi}) = \lim_{U^{p}} H^{i}(Y_{U,S} \times k(w)^{ac}, \mathcal{L}_{\xi}).$$

Here we relate the alternating sum of these modules to the cohomology of Igusa varieties. We will define the elements of Groth $(G(\mathbb{A}^{\infty} \times \text{Frob}_{w}^{\mathbb{Z}}))$ (we write Groth $(G)$ for the
Grothendieck group of admissible $G$-modules) as follows:

$$[H(Y_{Iw(m)}, S, \mathcal{L}_\xi)] = \sum_i (-1)^{n-\#S-h_i} H^i(Y_{Iw(m)}, S, \mathcal{L}_\xi),$$

$$[H_c(Y^0_{Iw(m)}, S, \mathcal{L}_\xi)] = \sum_i (-1)^{n-\#S-h_i} \lim_{U^p} H^i_c(Y^0_{U,S} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi),$$

$$[H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)] = \sum_i (-1)^{h_i-\#S} \lim_{U^p} H^i_c(I^{(h)}_{U} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi).$$

Then, because $Y_U, S = \bigcup_{T \supset S} Y^0_{U,T}$ for each $U = U^p \times Iw(m)$, we have equalities

$$[H(Y_{Iw(m)}, S, \mathcal{L}_\xi)] = \sum_{T \supset S} (-1)^{(n-\#S)-(n-\#T)} [H_c(Y^0_{Iw(m)}, T, \mathcal{L}_\xi)]$$

$$= \sum_{T \supset S} (-1)^{(n-\#S)-(n-\#T)} [H_c(I^{(n-\#T)}_{Iw(m)}, \mathcal{L}_\xi)].$$

As there are $\binom{n-\#S}{h}$ subsets $T$ with $\#T = n - h$ and $T \supset S$, we conclude:

**Lemma 2.7.** We have an equality

$$[H(Y_{Iw(m)}, S, \mathcal{L}_\xi)] = \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} [H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)]$$

in the Grothendieck group of admissible $G(\mathbb{A}_{\infty,p}) \times \text{Frob}_{Iw}$-modules over $\mathbb{Q}_{l}^{ac}$.

### 3. Proof of the main theorem

We now return to the situation in theorem 1.1. Recall that $L$ is an imaginary CM field and that II is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ such that

- $\Pi \circ c \cong \Pi^\vee$;
- $\Pi_\infty$ has the same infinitesimal character as some algebraic representation over $\mathbb{C}$ of the restriction of scalars from $L$ to $\mathbb{Q}$ of $GL_n$;
- and for some finite place $x$ of $L$ the representation $\Pi_x$ is square integrable.

Recall also that $v$ is a place of $L$ above a rational prime $p$, that $l \not= p$ is a second rational prime and that $\iota : \mathbb{Q}_{l}^{ac} \rightarrow \mathbb{C}$. Recall finally that $R_l(\Pi)$ is the $l$-adic representation associated to $\Pi$.

Choose a quadratic CM extension $L'/L$ in which $v$ and $x$ split. Choose places $v' \not= x'$ of $L'$ above $v$ and $x$ respectively. Also choose an imaginary quadratic field $E$ and a totally real field $F^+$ such that
• \([F^+ : \mathbb{Q}]\) is even;
• \(F = EF^+\) is soluble and Galois over \(L'\);
• \(p\) splits as \(uv^*\) in \(E\);
• there is a place \(w\) of \(F\) above \(u\) and \(v'\) such that \(\Pi_{F,w}\) has an Iwahori fixed vector;
• \(x\) lies above a rational prime which splits in \(E\) and \(x'\) splits in \(F\).

Denote by \(\Pi_F\) the base change of \(\Pi\) to \(GL_n(\mathbb{A}_F)\). Note that the component of \(\Pi_F\) at a place above \(x'\) is square integrable and hence \(\Pi_F\) is cuspidal.

Choose a division algebra \(B\) with centre \(F\) as in the previous section and satisfying

• \(B_x\) is split for all places \(x \neq z, z^c\) of \(F\).

Also choose \(\alpha, \beta\) and \(G\) as in the previous section. Then it follows from theorem VI.2.9 and lemma VI.2.10 of [HT] that we can find

• a character \(\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times\),
• an irreducible algebraic representation \(\xi\) of \(G\) over \(\mathbb{Q}_E^{ac}\),
• and an automorphic representation \(\pi\) of \(G(\mathbb{A})\),

such that

• \(\pi_\infty\) is cohomological for \(\iota \xi\),
• \(\psi\) is unramified above \(p\),
• \(\psi^c|_{E^\times}\) is the inverse of the restriction of \(\iota \xi\) to \(E^\times \subset G(\mathbb{R})\),
• \(\psi^c/\psi\) is the restriction of the central character of \(\Pi_F\) to \(\mathbb{A}_E^\times\),
• and if \(x\) is a rational prime which splits \(yf^c\) in \(E\) then \(\pi_x = (\bigotimes_{z|y} JL^{-1}(\Pi_z)) \otimes \psi_y\) as a representation of \((B_y^{op})^\times \times \mathbb{Q}_E^\times \cong (\bigotimes_{z|y}(B_y^{op})^\times) \times \mathbb{Q}_E^\times\).

Here \(JL\) denotes the identity if \(B_z\) is split and denotes the Jacquet-Langlands correspondence if \(B_z\) is a division algebra. (See section I.3 of [HT].)

We will call two irreducible admissible representations \(\pi'\) and \(\pi''\) of \(G(\mathbb{A}_\infty)\) nearly equivalent if \(\pi'_x \cong \pi''_x\) for all but finitely many rational primes \(x\). If \(M\) is an admissible \(G(\mathbb{A}_\infty)\)-module and \(\pi'\) is an irreducible admissible representation of \(G(\mathbb{A}_\infty)\) then we define the \(\pi'\)-near isotypic component \(M[\pi']\) of \(M\) to be the largest \(G(\mathbb{A}_\infty)\)-submodule of \(M\) all whose irreducible subquotients are nearly equivalent to \(\pi'\). Then

\[
M = \bigoplus M[\pi']
\]

as \(\pi'\) runs over near equivalence classes of irreducible admissible \(G(\mathbb{A}_\infty)\)-modules. (This follows from the following fact. Suppose that \(A\) is a (commutative) polynomial algebra over \(\mathbb{C}\) in countably many variables, and that \(M\) is an \(A\)-module which is finitely generated over...
We can write
\[ M = \bigoplus \mathbb{M}_m, \]
where \( m \) runs over maximal ideals of \( A \) with residue field \( \mathbb{C}. \)

We consider the Shimura varieties \( X_U/F \) for open compact subgroups \( U \) of \( G(\mathbb{A}^\infty) \) as in the last section. Then
\[ H^i(X, \mathcal{L}_\xi) = \lim_{\to} H^i(X_U \times F F^{ac}, \mathcal{L}_\xi) \]
is a semisimple, admissible \( G(\mathbb{A}^\infty) \)-module with a commuting continuous action of the Galois group \( \text{Gal}(\mathbb{F}^{ac}/\mathbb{F}). \) (For details see III.2 of [HT].)

The following lemma follows from [HT], particularly corollary VI.2.3, corollary VI.2.7 and theorem VII.1.7.

**Lemma 3.1.** Keep the notation and assumptions above. (In particular we are assuming that \( \pi \) arises from a cuspidal automorphic representation \( \Pi \) of \( GL_n(\mathbb{A}_F). \))

1. If \( i \neq n-1 \) then \( H^i(X, \mathcal{L}_\xi)[\pi] = (0). \)
2. As \( G(\mathbb{A}^\infty) \times \text{Gal}(\mathbb{F}^{ac}/\mathbb{F}) \)-modules,
\[ H^{n-1}(X, \mathcal{L}_\xi)[\pi] = \bigoplus_{\pi'} \pi' \otimes R'_t(\Pi)^m(\pi') \otimes R_t(\psi), \]
where \( \pi' \) runs over irreducible admissible representations of \( G(\mathbb{A}^\infty) \) nearly equivalent to \( \pi \) and where \( m(\pi') \in \mathbb{Z}_{\geq 0} \), and \( R_t(\Pi) = R'_t(\Pi)^{ss}. \)
3. \( m(\pi) > 0. \)
4. If \( m(\pi') > 0 \) then \( \pi'_p \cong \pi_p. \)

If \( \pi' \) is an irreducible admissible representation of \( G(\mathbb{A}^\infty) \) we can decompose it as \( (\pi')^p \otimes \left( \prod_{i=2}^{n-1} \pi_i \right) \otimes \pi_0^p \otimes \pi_0^p, \) corresponding to the decomposition (1). If \( \pi'' \) is an irreducible admissible representation of \( G(\mathbb{A}^{\infty,p}) \) and \( N \) is an admissible \( G(\mathbb{A}^{\infty,p}) \)-module then we can define the \( \pi'' \)-near isotypic component of \( N \) in the same manner as we did for \( G(\mathbb{A}^\infty) \)-modules. If \( M \) is an admissible \( G(\mathbb{A}^{\infty}) \)-module and \( \pi' \) is an irreducible admissible representation of \( G(\mathbb{A}^{\infty}) \) then
\[ M^{Iw(m)}[(\pi')^p] = M[\pi^p]^{Iw(m)}. \]

We will write
\[ H^i(X_{Iw(m)}, \mathcal{L}_\xi) = \lim_{\to} H^i(X_U \times F^{ac}, \mathcal{L}_\xi) \cong H^i(X, \mathcal{L}_\xi)^{Iw(m)}. \]

It is a semisimple admissible \( G(\mathbb{A}^{\infty,p}) \)-module with a commuting continuous action of \( \text{Gal}(\mathbb{F}^{ac}/\mathbb{F}). \)

**Theorem 3.2.** Keep the above notation and assumptions. (In particular we are assuming that \( \pi \) arises from a cuspidal automorphic representation \( \Pi \) of \( GL_n(\mathbb{A}_F). \)) Let \( U^p \) be a
sufficiently small open compact subgroup of \( G(\mathbb{A}^{\infty,p}) \). Then
\[
WD(H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi]\mid_{U^p})
\]
is pure.

**Proof:** As \( X_U = X_{U^p \times Iw(m)} \) is strictly semistable by proposition 2.4, we can use the Rapoport-Zink weight spectral sequence [RZ] to compute \( H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi) \). For \( X_U \), it reads

\[
E_1^{i,j}(U) = \bigoplus_{t \geq \max(0,-i)} \bigoplus_{\#S = i+2t+1} H^{j-2t}(Y_{U,S} \times_k k(w)^{ac}, \mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_U \times_F F_w^{ac}, \mathcal{L}_\xi).
\]

Passing to the limit with respect to \( U^p \), it gives rise to the following spectral sequence of admissible \( (\mathbb{A}^{\infty,p}) \times \text{Frob}_w \)-modules

\[
E_1^{i,j}(Iw(m)) = \bigoplus_{t \geq \max(0,-i)} \bigoplus_{\#S = i+2t+1} H^{j-2t}(Y_{Iw(m),S}, \mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_{Iw(m)}, \mathcal{L}_\xi).
\]

Hence we get a spectral sequence of \( \text{Frob}_w \)-modules

\[
E_1^{i,j}(Iw(m))_{\mid [\pi]^{U^p}} \Rightarrow H^{i+j}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi]\mid_{U^p}.
\]

The sheaf \( \mathcal{L}_\xi \) is pure, say of weight \( w_\xi \). Thus the action of \( \text{Frob}_w \) on \( E_1^{i,j} \) is pure of weight \( w_\xi + j \) by the Weil conjectures. The theory of weight spectral sequence ([RZ]) defines an operator

\[
N : E_1^{i,j}(Iw(m))_{\mid [\pi]^{U^p}}(1) \rightarrow E_1^{i+2j-2}(Iw(m))_{\mid [\pi]^{U^p}},
\]

which induces the \( N \) for \( WD(H^{i+j}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi]\mid_{U^p}) \) and has the property that

\[
N^i : E_1^{i,j+i}(Iw(m))_{\mid [\pi]^{U^p}}(i) \sim \rightarrow E_1^{i,j-i}(Iw(m))_{\mid [\pi]^{U^p}}
\]

for all \( i \). If the spectral sequence (2) degenerates at \( E_1 \), then it follows that the Weil-Deligne representation \( WD(H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi]\mid_{U^p}) \) is pure of weight \( w_\xi + (n-1) \). Thus it suffices to show that

\[
E_1^{i,j}(Iw(m))_{\mid [\pi]^{U^p}} = (0)
\]

if \( i + j \neq n - 1 \), i.e. that

\[
H^j(Y_{Iw(m),S}, \mathcal{L}_\xi)[\pi]\mid_{U^p} = (0)
\]

if \( j \neq n - \#S \).

We first recall some notation from [HT]. For \( h = 0, \ldots, n-1 \) let \( P_h \) denote the maximal parabolic in \( GL_n \) consisting of matrices \( g \in GL_n \) with \( g_{ij} = 0 \) for \( i > n - h \) and \( j \leq n - h \). Also let \( N_h \) denote the unipotent radical of \( P_h \), let \( P_h^{op} \) denote the opposite parabolic and let \( N_h^{op} \) denote the unipotent radical of \( P_h^{op} \). Let \( D_{F_{w,n-h}} \) denote the division algebra with centre \( F_w \) and Hasse invariant \( 1/(n-h) \). If \( \pi' \) is a square integrable representation of \( GL_{n-h}(F_w) \), let \( \varphi_{\pi'} \) denote a pseudo-coefficient for \( \pi' \) as in section I.3 of [HT]. (Note that this depends on the choice of a Haar measure, but in the formulae below this choice will always be cancelled by the choice of an associated Haar measure on \( D_{F_{w,n-h}}^{\times} \). See [HT] for details.)
If we introduce the limit of cohomology groups of Igusa varieties for varying level structure at \( p \) as in (see p.136 of [HT]):

\[
[H_c(I^{(h)}, \mathcal{L}_\xi)] = \sum_i (-1)^{h-i} \lim_{U^p, m} H^i_c(I^{(h)}_{U^p, m} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi),
\]

then the second isomorphism of corollary 2.6 and theorem V.5.4 of [HT] tell us that

\[
n[H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)] = n[H_c(I^{(h)}, \mathcal{L}_\xi)]_{U^p(m) \times Iw, w} = \sum_i (-1)^{n-1-i} \text{Red}^{(h)} [H^i(X, \mathcal{L}_\xi)_{U^p(m)}]
\]

in Groth \((G(\mathbb{A}^\infty, p) \times \text{Frob}_w^\mathbb{Z})\), where

\[
\text{Red}^{(h)} : \text{Groth} (GL_n(F_w) \times \mathbb{Q}_p^\times) \longrightarrow \text{Groth} (\text{Frob}_w^\mathbb{Z})
\]

is the composite of the normalised Jacquet functor

\[
J_{\mathcal{N}^\mathbb{Z}} : \text{Groth} (GL_n(F_w) \times \mathbb{Q}_p^\times) \longrightarrow \text{Groth} (GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^\times)
\]

with the functor

\[
\text{Groth} (GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^\times) \longrightarrow \text{Groth} (\text{Frob}_w^\mathbb{Z})
\]

which sends \([\alpha \otimes \beta \otimes \gamma] \) to

\[
\sum_{\phi} \text{vol}(D^\infty_{F_w, n-h}/F_w^\times)^{-1} \text{tr} (\varphi_{\text{Sp}_{n-h}(\phi)})(\dim \beta_{Iw, w}) \left[ \text{rec}(\phi^{-1} | \frac{1+n}{\omega^2} (\gamma \mathbb{Z}_p \circ N_{F_w/F_n})^{-1}) \right],
\]

where the sum is over characters \( \phi \) of \( F_w^\times/O_{F_w}^\times \). (We just took the \( Iw, w \)-invariant part of the \( \text{Red}_1^{(h)} \), which is defined on p.182 of [HT]. Note that \( \text{Frob}_w \) acts on \( H_c(I^{(h)}, \mathcal{L}_\xi) \) as

\[
(1, p^{-[k(w):\mathbb{F}_p]}, -1, 1, 1) \in G(\mathbb{A}^\infty, p) \times (\mathbb{Q}_p^\times/O_{F_w}^\times) \times \mathbb{Z} \times GL_h(F_w) \times (\prod_{i=2}^r (B_{w,i}^{op})^\times),
\]

where we have identified \( D^\infty_{F_w, n-h}/O_{D_{F_w, n-h}}^\times \) with \( \mathbb{Z} \) via \( w(\det) \).

In particular, by lemma 3.1(1), we have an equality in Groth \((\text{Frob}_w^\mathbb{Z})\):

\[
n[H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)|_{\mathbb{F}_p}^{U^p}] = \text{Red}^{(h)} [H^{n-1}(X, \mathcal{L}_\xi)_{U^p(m)}|_{\mathbb{F}_p}^{U^p}].
\]

Moreover \( H^{n-1}(X, \mathcal{L}_\xi)_{U^p(m)}|_{\mathbb{F}_p}^{U^p} \) is \( \pi_w \otimes \pi_{p,0} \)-isotypic as a \( GL_n(F_w) \times \mathbb{Q}_p^\times \)-module by lemma 3.1(4). As \( \pi_w = \Pi_{F_w} \) has an Iwahori fixed vector and \( \pi_{p,0} = \psi_u \) is unramified,

\[
(\dim \Pi_{F_w, w}^{Iw, w}) [H^{n-1}(X, \mathcal{L}_\xi)_{U^p(m)}|_{\mathbb{F}_p}^{U^p}] = (\dim H^{n-1}(X, \mathcal{L}_\xi)_{Iw(m)}|_{\mathbb{F}_p}^{U^p}) [\Pi_{F,w} \otimes \psi_u],
\]

and

\[
n(\dim \Pi_{F_w, w}^{Iw, w}) [H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)|_{\mathbb{F}_p}^{U^p}] = (\dim H^{n-1}(X, \mathcal{L}_\xi)_{Iw(m)}|_{\mathbb{F}_p}^{U^p}) \text{Red}^{(h)} [\Pi_{F,w} \otimes \psi_u].
\]
Combining this with lemma 2.7, we get
\[ n(\dim \Pi_{F,w}^{Iw,n,w}) [H(Y_{Iw(m),S},L_\xi)\{\pi^p\}_{UP}] \]
\[ = (\dim H^{n-1}(X,L_\xi)^{Iw(m)}\{\pi^p\}_{UP}) \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} \text{Red}^h[\Pi_{F,w} \otimes \psi_u]. \]

As \( \Pi_{F,w} \) is tempered, it is a full normalised induction of the form
\[ n \text{-Ind}_{P(F_w)}^{GL_n(F_w)}(\text{Sp}_{s_1}(\pi_1) \otimes \cdots \otimes \text{Sp}_{s_t}(\pi_t)), \]
where \( \pi_i \) is an irreducible cuspidal representation of \( GL_n \) with Levi component \( GL_{s_1,g_1} \times \cdots \times GL_{s_t,g_t} \). As \( \Pi_{F,w} \) has an Iwahori fixed vector, we must have \( g_i = 1 \) and \( \pi_i \) unramified for all \( i \). Note that, for this type of representation (full induced from square integrables \( \text{Sp}_{s_i}(\pi_i) \) with \( \pi_i \) an unramified character of \( F_w^\times \)),
\[ \dim (\text{n-Ind}_{P(F_w)}^{GL_n(F_w)}(\text{Sp}_{s_1}(\pi_1) \otimes \cdots \otimes \text{Sp}_{s_t}(\pi_t)))^{Iw,n,w} \]
\[ = \#P(k(w))\text{GL}_n(k(w))/B_n(k(w)) = \frac{n!}{\prod_j s_j!}. \]

We can compute \( \text{Red}^h[\Pi_{F,w} \otimes \psi_u] \) using lemma I.3.9 of [HT] (but note the typo there — “positive integers \( h_1,\ldots,h_t \)” should read “non-negative integers \( h_1,\ldots,h_t \)” ). Putting \( V_i = \text{rec}(\pi_i^{-1}| \psi_u \circ N_{F,w/E_0}^{-1}) \), we see that
\[ \text{Red}^h[\Pi_{F,w} \otimes \psi_u] = \sum_i \dim (\text{n-Ind}_{P'(F_w)}^{GL_k(F_w)}(\text{Sp}_{s_i+h-n}(\pi_i) \otimes_{j \neq i} \text{Sp}_{s_j}(\pi_j)))^{Iw,k,w}[V_i] \]
\[ = \sum_i \frac{h!}{(s_i + h - n)! \prod_{j \neq i} s_j!} [V_i] \]
where the sum runs only over those \( i \) for which \( s_i \geq n - h \), and \( P' \subset GL_h \) is a parabolic subgroup. Thus
\[ n \frac{n!}{\prod_j s_j!} [H(Y_{Iw(m),S},L_\xi)\{\pi^p\}_{UP}] \]
\[ = D \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} \sum_{i: s_i \geq n-h} \frac{h!}{(s_i + h - n)! \prod_{j \neq i} s_j!} [V_i] \]
\[ = D \sum_{i=1}^{t} \frac{(n - \#S)!}{(s_i - \#S)! \prod_{j \neq i} s_j!} \sum_{h=n-s_i}^{n-\#S} (-1)^{n-\#S-h} \binom{s_i - \#S}{h + s_i - n} [V_i] \]
\[ = D \sum_{s_i = \#S}^{n-\#S} \frac{(n - \#S)!}{\prod_{j \neq i} s_j!} [V_i], \]
where \( D = \dim H^{n-1}(X,L_\xi)^{Iw(m)}\{\pi^p\}_{UP} \), and so
\[ n \binom{n}{\#S} [H(Y_{Iw(m),S},L_\xi)\{\pi^p\}_{UP}] = (\dim H^{n-1}(X,L_\xi)^{Iw(m)}\{\pi^p\}_{UP}) \sum_{s_i = \#S}^{n} [V_i]. \]
As $\Pi_{F,w}$ is tempered, $\text{rec}(\Pi_{F,w}^\vee \otimes (\psi_u \circ N_{F_w/E_u})) \det \frac{1-n}{2}$ is pure of weight $w_\xi + (n - 1)$. Hence

$$V_i = \text{rec}\left(\pi_i^{-1} | \frac{1-n}{w} (\psi_u \circ N_{F_w/E_u})^{-1} | \frac{n-n-\#S}{w-2}\right)$$

is strictly pure of weight $w_\xi + (n - \#S)$. The Weil conjectures then tell us that

$$H^j(Y_{1w(m)}, S, L_{\xi} \pi^p U_p) = (0)$$

for $j \neq n - \#S$. The theorem follows. □

We can now conclude the proof of theorem 1.1. Choose $k$ so that $|\chi_{\Pi}| = |\chi_{\Pi}|_L^{k+n-\#S}$ where $\chi_{\Pi}$ is the central character of $\Pi$. Set

$$V = H^{n-1}(X_{1w(m)}, L_{\xi} \pi^p U_p) \otimes R(\psi)^{-1},$$

a continuous representation of $\text{Gal}(F^{ac}/F)$. We know that

1. $V_{ss} \cong R(\Pi)_{\text{Gal}(F^{ac}/F)}^a$ for some $a \in \mathbb{Z}_{>0}$,
2. $V$ is pure of weight $k$ (proposition III.2.1 of [HT] and a computation of the determinant),
3. $\text{WD}(V_{\text{Gal}(F^{ac}/F)})$ is pure of weight $k$ (use theorem 3.2 and a computation of the determinant).

Thus lemma 1.4 tells us that $\text{WD}(R(\Pi)_{\text{Gal}(L^{ac}/L_v)})^{F-ss}$ is pure. On the other hand, as $\Pi_v$ is tempered (corollary VII.1.11 of [HT]), $\text{rec}(\Pi_v^\vee \det \frac{1-n}{2})$ is pure by lemma 1.3(3). As the representation of the Weil group in $\text{rec}(\Pi_v^\vee \det \frac{1-n}{2})$ and $\text{WD}(R(\Pi)_{\text{Gal}(L^{ac}/L_v)})^{F-ss}$ are equivalent, we deduce from lemma 1.3(4) that

$$\pi_{\text{WD}}(R(\Pi)_{\text{Gal}(L^{ac}/L_v)})^{F-ss} \cong \text{rec}(\Pi_v^\vee \det \frac{1-n}{2}),$$

as desired.

References

[AK] A. Altman, S. Kleiman, Introduction to Grothendieck Duality Theory, Lecture Notes in Math. 146, Springer-Verlag, 1970.

[D] P. Deligne, Les Constantes des Equations Fonctionnelles des Fonctions L, in: Modular Functions of One Variable II (Springer LNM 349, 1973), pp.501-597.

[DR] P. Deligne, M. Rapoport, Schémas de modules de courbes elliptiques, in: Modular Functions of One Variable II (Springer LNM 349, 1973), pp.143-316.

[Dr] V. Drinfeld, Elliptic modules, Math. USSR Sbornik 23-4 (1974), 561-592.

[F] A. Fröhlich, Formal Groups, Springer LNM 74, 1968.

[HT] M. Harris, R. Taylor, The Geometry and Cohomology of Some Simple Shimura Varieties, Ann. of Math. Studies 151, Princeton Univ. Press, Princeton-Oxford, 2001.

[I] T. Ito, Weight-monodromy conjecture for p-adically uniformized varieties, math-NT/0301201, to appear in Inventiones Mathematicae.

[KM] N. Katz, B. Mazur, Arithmetic Moduli of Elliptic Curves, Ann. of Math. Studies 108, Princeton Univ. Press, Princeton, 1985.
[RZ] M. Rapoport, T. Zink, Über die lokale Zetafunktion von Shimuravarietäten, Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik, Invent. Math. 68 (1982), no. 1, 21-101.

[T] J. Tate, Number Theoretic Background, in: A. Borel and W. Casselman, ed., Automorphic Forms, Representations and L-functions, Proc. Symp. in Pure Math. 33-2, AMS, 1979.

[Y] T. Yoshida, On non-abelian Lubin-Tate theory via vanishing cycles, math-NT/0404375, to appear in Ann. de l’Institut Fourier.

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