Curvature line parametrization from circle patterns

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Abstract

We study local and global approximations of smooth nets of curvature
lines and smooth conjugate nets by respective discrete nets (circular nets and
planar quadrilateral nets) with edges of order $\epsilon$. It is shown that choosing
the points of discrete nets on the smooth surface one can obtain the order $\epsilon^2$
approximation globally. Also a simple geometric construction for approximate
determination of principal directions of smooth surfaces is given.

1 Introduction

Discrete conjugate nets, defined as mappings $\mathbb{Z}^2 \rightarrow \mathbb{R}^3$ with the condition that
each elementary quadrangle is flat, and discrete nets of curvature lines, defined as
discrete conjugate nets with additional property of circularity of the four vertices of
every elementary quadrangle, play an important role in the contemporary discrete
differential geometry (see e.g. [2] for a review) and have important applications to
computer-aided geometric design [12].

This paper is devoted to a careful study of the order of approximation of discrete
nets to a given smooth conjugate net or the net of curvature lines on a smooth surface
that can be achieved. We also present a remarkably simple approximate construction

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of principal directions on a given smooth surface using the standard primitive of discrete differential geometry: an elementary circular quadrangle inscribed into the surface. Roughly speaking, our results show that the previously known upper bounds (3) can be made one order better for conjugate nets and discrete nets of curvature lines. Moreover, one can impose one additional geometric requirement: all vertices of approximating discrete nets should lie on the original smooth surface.

The paper is organized as follows. In Section 2 we present a method of construction of principal directions in a neighborhood of a non-umbilic point on a smooth surface using only the four points of intersection of the surface with an infinitesimal circle. Our method is in fact based on a third-order approximation of the given smooth surface and gives an $\epsilon^2$-approximation of principal directions if the infinitesimal circle has the radius $\epsilon$. Different methods for approximation of principal directions have been proposed recently, see for example [8]. Re-meshing techniques based on determination of principal directions are widely used in contemporary computer graphics ([1]).

Section 3 is devoted to the study of the order of approximation of a single elementary quad of a discrete conjugate net (resp. discrete circular net) to the respective infinitesimal quadrangle of a the smooth net of conjugate lines (resp. curvature lines) on the original smooth surface. The results of this Section are used in Section 4 where we prove the main global results on the order of approximation of respective discrete nets to a given smooth conjugate net or the net of curvature lines on a smooth surface.

We use the following terminology and notations: $d(A, S)$ is the distance from a point $A$ to another point $S$ (or a line, plane, etc.). A point $A$ is said to be $\epsilon^k$-close to $S$ if $d(A, S) = O(\epsilon^k)$, that is $d(A, S) \leq C\epsilon^k$ for $\epsilon \to 0$. A is said to lie at $\epsilon^k$-distance from $S$, if $d(A, S) \sim C\epsilon^k$ for some $C \neq 0$, we will denote this hereafter as $d(A, S) \simeq \epsilon^k$. The same terminology and notations will be used in characterization of the asymptotic behavior for other (numeric) quantities depending on $\epsilon$.

2 Principal curvature directions from circles

In this Section we show that one can use the elementary quadrangle of a discrete circular net inscribed into a smooth surface (see Section 3.2 for more details) for a rather precise construction of the principal directions on a smooth surface.

- **Euclidean construction.** Take three points $A, B, C$ on a smooth surface $f : \Omega \rightarrow \mathbb{R}^3$ without umbilic points, such that $d(A, B) \simeq \epsilon$, $d(B, C) \simeq \epsilon$ and the angle $\angle ABC$ is not $\epsilon$-close to 0 or $2\pi$. Then the circle $\omega$ passing through $A, B, C$ has radius of order $O(\epsilon)$ as well; from elementary topological considerations one concludes that $\omega$ has at least one more point of intersection with $f(\Omega)$. Denote it (or one of them) as $D$. Take the straight lines $(AC), (BD)$ and the point of their intersection $Z = (AC) \cap (BD)$ (see Fig. 1). The bisectors $b_1$ and $b_2$ of the angles formed at $Z$ by $(AC)$ and $(BD)$ are obviously perpendicular.
In fact $b_1$ and $b_2$ approximate the directions of smooth curvature lines on the surface: as we prove below, the directions of $b_1$ and $b_2$ are $\epsilon$-close to the principal directions at any point $M \in f(\Omega)$ which is $\epsilon$-close to $A, B, C, D$. In fact one can find the principal directions with only $O(\epsilon^2)$-error at some specific point $Z_f$ of $f(\Omega)$ which “lies over $Z$”. Note that the order of the points $A, B, C, D$ on the circle $\omega$ is important for finding the principal direction with the error of order $\epsilon^2$.

Namely, let $Z_f$ be the intersection of $f(\Omega)$ with the straight line $l$ passing through $Z$ and perpendicular to the plane $\pi_\epsilon = (ABC)$. Take the orthogonal (parallel to $l$) projections $p_1$ and $p_2$ of the bisectors $b_1$ and $b_2$ onto the tangent plane $\pi_f$ of $f(\Omega)$ at $Z_f$.

**Theorem 1** For a general smooth surface $f(\Omega)$ in a neighborhood of a non-umbilic point the Euclidean construction given above produces the directions $p_1, p_2$ which are $\epsilon^2$-close to the exact principal directions at $Z_f$.

In order to prove this Theorem we establish at first some simple but remarkable facts about planar quadrics and Dupin cyclides (Lemma 2 and Theorem 3) and give another (Möbius-invariant) construction of the principal directions (Fig. 2). In fact, the intersection curve $\delta = f(\Omega) \cap \pi_\epsilon$ is $\epsilon^2$-close to a quadric — the Dupin indicatrix of $f(\Omega)$ at $P$ (see the Appendix); this explains in particular why we may assume that the number of intersection points of $\omega$ and $f(\Omega)$ is exactly four. If we take then an arbitrary circle $\omega$ in a plane, intersecting a quadric in points $A, B, C, D$ (cf. Fig. 1 for the elliptic case; the order of the points $A, B, C, D$ on the circle is not necessarily consecutive for Lemma 2 below to hold), then the directions of the angles formed by the straight lines $(AC)$ and $(BD)$ are parallel to axes of the quadric:

Figure 1: Construction of the axes directions of a quadric: $A, B, C, D$ are the four points of intersection with a circle, $b_1$ and $b_2$ are the bisectors of the angles between $(AC)$ and $(BD)$. The bisectors $b_1$ and $b_2$ are parallel to the axes of the quadric
Lemma 2 For any plane quadric with four points of intersection with a circle the bisectors $b_1$ and $b_2$ (shown on Fig. 1 for the elliptic case) are parallel to the axes of the quadric.

Proof. For simplicity we give the proof only for the case of a non-degenerate central quadric $Q(x, y) = (K_1 x^2 + K_2 y^2) - 2\epsilon^2 = 0$; the other cases can be proved along the same lines. The coordinates of the intersection points $A$, $B$, $C$, $D$ satisfy the system

\[
\begin{align*}
Q(x, y) &= (K_1 x^2 + K_2 y^2) - 2\epsilon^2 = 0, \\
\omega(x, y) &= (x - x_e)^2 + (y - y_e)^2 - r^2 = 0. 
\end{align*}
\]  

They also lie on the one-parametric sheaf of quadrics $Q_\lambda$: $\{Q(x, y) + \lambda \omega(x, y) = 0\}$. Obviously the directions of the axes of all $Q_\lambda$ are the same. On the other hand, for some values of $\lambda$ this quadric is degenerate. These values are the three roots $\lambda_i$ of the equation

\[
\begin{vmatrix}
K_1 + \lambda & 0 & -2\lambda x_e \\
0 & K_2 + \lambda & -2\lambda y_e \\
-2\lambda x_e & -2\lambda y_e & 2\epsilon^2 - \lambda r^2
\end{vmatrix} = 0. 
\]  

Two of them correspond to the degenerate quadrics composed of the pairs of straight lines $\{(AB), (CD)\}$ and $\{(AD), (BC)\}$ shown on Fig. 1. The third degenerate quadric is the pair of straight lines $\{(AC), (BD)\}$, shown on Fig. 1.

Since the bisectors of these pairs of lines define exactly the axes of the respective degenerate quadric $Q_\lambda$, we have proved that their directions are the same as for $Q_0 = \{Q(x, y) = 0\}$. □

Note that we have not used the ordering of the points $A$, $B$, $C$, $D$ on the circle $\omega$ or on the quadric, so the statement of Lemma 2 holds true also for the bisectors of the angles formed by the lines $(AB)$ and $(CD)$, as well as $(AD)$ and $(BC)$. On the other hand in order to obtain the approximation of order $\epsilon^2$ below we fix the ordering as shown on Fig. 1.

Next we need to reformulate the Euclidean construction given above in terms of Möbius geometry:

- Möbius-invariant construction. We start with a smooth surface $f : \Omega \rightarrow \mathbb{R}^3$. Take a sphere $S$ tangent to $f(\Omega)$ at some point $P$, such that for its radius $R$ and the principal curvatures $K_1$, $K_2$ ($K_1 < K_2$) of $f(\Omega)$ at $P$ the inequalities $K_1 < 1/R < K_2$ hold (with non-infinitesimal differences $|K_1 - 1/R|$ and $|1/R - K_2|$). In this case $S$ intersects an $\epsilon$-patch $\Omega_\epsilon$ on $f(\Omega)$ along two smooth lines $\gamma_1$, $\gamma_2$ (Fig. 2): $\Omega_f \cap S = \gamma_1 \cup \gamma_2$, the angle between $\gamma_1$ and $\gamma_2$ is non-zero and non-infinitesimal (this follows from the classical lemma of Morse, cf. for example [10]). Take any circle $\omega \subset S$ of radius $\simeq \epsilon$ with $P$ inside such that the distances between $P$ and the four points of intersection of $\omega$ and $\Omega$: $\omega \cap \gamma_1 = A \cup C$, $\omega \cap \gamma_2 = B \cup D$ are of order $O(\epsilon)$ as well. Construct two auxiliary circles $\omega_1$, $\omega_2$ on $S$ defined by the triples of points $\{A, P, C\}$ and $\{B, P, D\}$ respectively (not shown on Fig. 2). $\omega_i$ are very close to $\gamma_i$. Then
the bisectors $b_1$, $b_2$ of the angles formed by $\omega_1$, $\omega_2$ at $P$ obviously lie in the tangent plane to $f(\Omega)$ at $P$. As we show below, $b_1$ and $b_2$ are $\epsilon^2$-close to the principal directions of $f(\Omega)$ at $P$.

![Diagram of the construction]

Figure 2: Approximate Möbius-invariant construction of the principal directions. A sphere $S$ is tangent to $f(\Omega)$ at $P$, it intersects $f(\Omega)$ along two smooth lines $\gamma_1$, $\gamma_2$. A circle $\omega \subset S$ of radius $\simeq \epsilon$ with $P$ inside gives four points: $\omega \cap \gamma_1 = A \cup C$, $\omega \cap \gamma_2 = B \cup D$, the points $A$, $B$, $C$, $D$ should lie on $\epsilon$-distance from $P$. The bisectors $b_1$, $b_2$ of the angles formed by the circles $\omega_1$, $\omega_2$ (not shown) passing through the triples of points $\{A, P, C\}$ and $\{B, P, D\}$ are $\epsilon^2$-close to the principal directions of $f(\Omega)$ at $P$.

First we prove that for Dupin cyclides (standard Möbius primitives, playing the role of the osculating paraboloids of the classical Euclidean differential geometry, see [6, 9]) $b_i$ are exactly the principal directions:

Theorem 3 For a smooth non-umbilic point $P$ on a Dupin cyclide $D$, the bisectors $b_1$ and $b_2$ of the Möbius-invariant construction give the principal directions of $D$ at $P$.

Proof. Since our construction of $\omega_i$ is Möbius-invariant, we may transform $D$ into one of the normalized Dupin cyclides: a torus, a circular cone or a circular cylinder. We give below a detailed proof for the case of a torus; the other cases can be easily proved along the same lines.

Making, if necessary, another Möbius transformation, we can reduce the configuration to the following: the torus is given by the equation

$$ (R^2 + x^2 + y^2 + z^2 - r^2)^2 = 4R^2(x^2 + y^2), $$  

(3)

the point $P$ has the coordinates $(R+r, 0, 0)$ and the sphere $S$ is given by the equation

$$ (x + \rho - (R + r))^2 + y^2 + z^2 = \rho^2. $$  

(4)
After a Möbius inversion \((x, y, z) \rightarrow (x_1, y_1, z_1)\) with the center \(P\):

\[
\begin{align*}
  x_1 &= \frac{x - R - r}{(x - (R + r))^2 + y^2 + z^2}, \\
  y_1 &= \frac{y}{(x - (R + r))^2 + y^2 + z^2}, \\
  z_1 &= \frac{z}{(x - (R + r))^2 + y^2 + z^2},
\end{align*}
\]

(5)

the sphere \(S\) will be transformed into a plane \(x_1 = \text{const} = \mu\). We prove now that the intersection of the image of the torus with any plane \(x_1 = \mu\) is a quadric of the form \(ay_1^2 + bz_1^2 = \kappa\). In fact, from (5) we have:

\[
\left( (x - (R + r))^2 + y^2 + z^2 \right) = (x - R - r)/\mu
\]

(6)

so

\[
z^2 = (x - R - r)/\mu - \left( (x - (R + r))^2 + y^2 \right).
\]

(7)

(5) may be now reshaped to

\[
\left( (x - (R + r))/\mu + 2(R + r)x - 2rR \right)^2 = 4R^2(x^2 + y^2),
\]

so one can express \(y^2\) in terms of \(x, R, r, \mu\). Substituting (6), (7) and the found expression for \(y^2\) into \(ay_1^2 + bz_1^2 = (ay_1^2 + bz_1^2)/\left( (x - (R + r))^2 + y^2 + z^2 \right)\) one gets a rational expression containing only \(x\), the constants \(R, r, \mu\) and \(a, b\). A straightforward calculation shows that for \(a = r(1 + 2\mu(R + r)), b = (1 + 2r\mu)(r + R)\) this expression is a constant \(\kappa = -R(1 + 2\mu R)(1 + 2r\mu)(1 + 2\mu r + 2\mu R)/4\).

Since the images of the circles \(\omega_1, \omega_2\) defined above are now straight lines passing through the points of intersection of the image of the circle \(\omega\) and the found quadric on the plane \(x_1 = \mu\), the statement of this Theorem is now equivalent to the statement of Lemma 2. \(\square\)

**Theorem 4** For an arbitrary smooth surface \(f(\Omega)\) in a neighborhood of a non-umbilic point \(P\) the Möbius-invariant construction produces the directions \(b_1, b_2\) which are \(\epsilon^2\)-close to the exact principal directions at \(P\).

**Proof.** First we perform a Möbius transformation that maps the sphere \(S\) and the circle \(\omega\) into themselves and brings \(P\) into the (Euclidean) center of \(\omega\) on \(S\), i.e. into the point \(P' \in S\) such that the distance from \(P'\) to all the points of \(\omega\) are equal. One can easily check that our requirement \(d(P, A) \simeq \epsilon, \ldots, d(P, D) \simeq \epsilon\) guarantees that after the transformation these distances will remain of order \(\simeq \epsilon\). Inside the circle \(\omega\) on \(S\) and in its \(\epsilon\)-neighborhood in \(\mathbb{R}^3\) where \(f(\Omega)\) lies, the Jacobian of this transformation is limited so the distances of order \(O(\epsilon^p), p \geq 2\) will be transformed
into distances of the same order. Since the angles do not change, $\epsilon^2$-close directions at $P$ remain $\epsilon^2$-close at $P'$ and vice versa. Introduce now a Cartesian coordinate system such that $P'$ is its origin, the coordinate plane $(xy)$ is tangent to $S$ (so to $f(\Omega)$ as well) and the $x, y$-axes are the principal directions of $f(\Omega)$ at $P'$. The plane where $\omega$ lies will be given by the equation $z = c_1\epsilon^2$, the circle $\omega$ will be defined as $x^2 + y^2 = c_2^2\epsilon^2$ and $f(\Omega)$ will be defined in a neighborhood of $P'$ by its Taylor expansion

$$z = (K_1x^2 + K_2y^2)/2 + (a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3) + o(\epsilon^3).$$

(8)

So the intersection points $P_1 = A$, $P_2 = B$, $P_3 = C$, $P_4 = D$ of $\omega$ and $f(\Omega)$ will have respectively the coordinates $x_i = \epsilon x_{i0} + \epsilon^2 x_{i1} + o(\epsilon^2)$ with $(x_{i0}, y_{i0})$ being the solutions of

$$\begin{align*}
\begin{cases}
x^2 + y^2 = c_2^2, \\
(K_1x^2 + K_2y^2) = 2c_1,
\end{cases}
\end{align*}$$

(9)

so we see that

$$(x_{10}, y_{10}) = (x_{20}, -y_{20}) = (-x_{30}, -y_{30}) = (-x_{40}, y_{40}).$$

(10)

The next terms $(x_{11}, y_{11})$ are found after substitution of $(x_i, y_i)$ into

$$\begin{align*}
\begin{cases}
x^2 + y^2 = c_2^2, \\
(K_1x^2 + K_2y^2)/2 + \epsilon(a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3) = c_1,
\end{cases}
\end{align*}$$

(11)

cancellation of the second-order terms using (9) and retaining only third-order terms:

$$\begin{align*}
\begin{cases}
2x_{i0}x_{i1} + 2y_{i0}y_{i1} = 0, \\
K_1x_{i0}x_{i1} + K_2y_{i0}y_{i1} + (a_{30}x_{i0}^3 + a_{21}x_{i0}^2y_{i0} + a_{12}x_{i0}y_{i0}^2 + a_{03}y_{i0}^3) = 0.
\end{cases}
\end{align*}$$

(12)

From (12) and (10) we easily conclude that

$$(x_{11}, y_{11}) = (x_{31}, y_{31}), (x_{21}, y_{21}) = (x_{41}, y_{41}),$$

(13)

so the $O(\epsilon^2)$-shifted lines $l_{11}, l_{21}$ defined by the pairs of points $\{(\epsilon x_{10} + \epsilon^2 x_{11}, \epsilon y_{10} + \epsilon^2 y_{11}), (\epsilon x_{30} + \epsilon^2 x_{31}, \epsilon y_{30} + \epsilon^2 y_{31})\}$ and $\{(\epsilon x_{20} + \epsilon^2 x_{21}, \epsilon y_{20} + \epsilon^2 y_{21}), (\epsilon x_{40} + \epsilon^2 x_{41}, \epsilon y_{40} + \epsilon^2 y_{41})\}$ are parallel to the lines $l_{10}, l_{20}$ defined by the pairs $\{(\epsilon x_{10}, \epsilon y_{10}), (\epsilon x_{30}, \epsilon y_{30})\}$ and $\{(\epsilon x_{20}, \epsilon y_{20}), (\epsilon x_{40}, \epsilon y_{40})\}$.

Thus the bisector directions between $l_{11}, l_{21}$ in the plane are the same as for the unperturbed lines $l_{10}, l_{20}$; so they are obviously parallel to the coordinate axes — the principal directions of $f(\Omega)$ at $P'$ which is $\epsilon^2$-close to both $Z_0 = l_{10} \cap l_{20}$ and $Z_1 = l_{11} \cap l_{21}$.

Now we have two pairs of circles in $\mathbb{R}^3$: $\omega_{10}$, $\omega_{20}$ passing through the triples of points $\{(\epsilon x_{10}, \epsilon y_{10}), P', (\epsilon x_{30}, \epsilon y_{30})\}$ and $\{(\epsilon x_{20}, \epsilon y_{20}), P', (\epsilon x_{40}, \epsilon y_{40})\}$ and the pair $\omega_{11}, \omega_{21}$ passing through $\{(\epsilon x_{10} + \epsilon^2 x_{11}, \epsilon y_{10} + \epsilon^2 y_{11}), P', (\epsilon x_{30} + \epsilon^2 x_{31}, \epsilon y_{30} + \epsilon^2 y_{31})\}$ and $\{(\epsilon x_{20} + \epsilon^2 x_{21}, \epsilon y_{20} + \epsilon^2 y_{21}), P', (\epsilon x_{40} + \epsilon^2 x_{41}, \epsilon y_{40} + \epsilon^2 y_{41})\}$. One easily concludes
from (13) that $\omega_{10}, \omega_{11}$ are tangent at $P'$, as well as $\omega_{20}, \omega_{21}$. Thus their bisectors are the same and parallel to the axes; the next $O(\varepsilon^3)$-terms in approximations for the intersection points $A, B, C, D$ will give a $O(\varepsilon^2)$-change in the directions of $b_1, b_2$. □

**Remark.** From the proof of the previous Theorem we see that in fact the Möbius-invariant construction gives the same (with $O(\varepsilon^2)$-error) principal directions if we will substitute instead of the original surface $f(\Omega)$ its tangent paraboloid at $P$ (after the Möbius transformation described in the beginning of the proof). Also it should be noted that the $O(\varepsilon^2)$-approximation was achieved taking into account approximating paraboloid of *third order*; the results obtained in this Section are therefore *third order results* despite the seemingly second-order construction based on an infinitesimal circle.

**Proof of Theorem 1.** Now we are in a position to prove the statement about $O(\varepsilon^2)$-approximation of principal directions at $Z_f$ in our Euclidean construction. For this we first perform an auxiliary construction in $\mathbb{R}^3$ (Fig. 3):

![Diagram](image.png)

Figure 3: Auxiliary construction

...
at \( Z_f \) are \( \epsilon^2 \)-close. (Here we use the same terminology of “\( \epsilon^p \)-closeness” for pairs of lines or planes, this means that the respective angles are \( O(\epsilon^p) \).) This shows that our sphere \( \Sigma \) is “very close” to the sphere \( S \) (which should be tangent to \( f(\Omega) \) at \( Z_f = P \)) used in the Möbius-invariant construction. We also see that our Euclidean construction is nearly reduced to the Möbius-invariant construction, with the sphere \( S \) substituted by \( \Sigma \): the orthogonal projections (along the line \( l = (ZZ_f) \perp \pi_{ABCD} \)) of \( b_1, b_2 \) onto \( \pi_\Sigma \) are nothing but the tangents to the circles \( \beta_1, \beta_2 \) which are \( \epsilon^2 \)-close to the circles which exactly bisect the angles between the circles \( \omega_1, \omega_2 \) at \( Z_f \). This follows from the fact, that the angle between \( \pi_{ABCD} \) and \( \pi_f \) is \( O(\epsilon) \): as one can easily check by a direct computation, in this situation the (non-infinitesimal) angle between any two lines on one of the planes and the angle between their orthogonal projections on the other plane are \( \epsilon^2 \)-close.

Let us now estimate the angles between the tangents to the circles \( \beta_i \) (lying in \( \pi_\Sigma \)) and the principal directions of \( f(\Omega) \) at \( Z_f \) (lying in \( \pi_f \), which is \( \epsilon^2 \)-close to \( \pi_\Sigma \)). For this we use the same technology as in the proof of Theorem 4: first we perform a Möbius transformation leaving \( \Sigma \) and \( \omega \) invariant and bringing the point \( Z_f \) to the center \( O \) of \( \omega \) on \( \Sigma \), then introduce a Cartesian coordinate system with \( O \) as its origin and \( \pi_\Sigma \) being its \((xy)\)-plane. Now the equation (8) of the surface \( f(\Omega) \) will be modified by small linear terms:

\[
z = (K_1 x^2 + K_2 y^2)/2 + (a_{30} x^3 + a_{21} x^2 y + a_{12} x y^2 + a_{03} y^3) + \sigma_1 x + \sigma_2 y + o(\epsilon^3), \tag{14}
\]

with \( \sigma_i = O(\epsilon^2) \). The first terms \( x_{i0}, y_{i0} \) in the Taylor expansions of the coordinates of the points \( A, B, C, D \) will be found from the same system (9) so we have the relations (10). The next terms \( x_{i1}, y_{i1} \) are found from a modified system of the form (12):

\[
\begin{align*}
2x_{i0}x_{i1} + 2y_{i0}y_{i1} &= 0, \\
K_1 x_{i0}x_{i1} + K_2 y_{i0}y_{i1} + (a_{30} x_{i0}^3 + a_{21} x_{i0}^2 y_{i1} + a_{12} x_{i0} y_{i1}^2 + a_{03} y_{i1}^3) + \frac{\alpha_1^2}{4} x_{i0} + \frac{\alpha_2^2}{4} y_{i1} &= 0.
\end{align*}
\]

We see that the main conclusion (13) still holds; from this point we just follow the guidelines of the proof of Theorem 4 and show that the bisectors of the angles between the circles \( \omega_1, \omega_2 \) coincide (up to a \( O(\epsilon^2) \)-error) with the directions of the \( x \)- and \( y \)-axes. On the other hand since we know that \( \sigma_i = O(\epsilon^2) \), using the standard differential-geometric formulas for calculation of the coefficients of the second fundamental form and the principal directions we conclude that the principal directions of \( f(\Omega) \) given by the equations (14) are \( \epsilon^2 \)-close to the \( x \)- and \( y \)-axes at \( Z_f \) as well (as the directions of the straight lines in \( \mathbb{R}^3 \)). \( \Box \)

**Remark.** From the proof of this Theorem we see that in fact one can assume (with \( O(\epsilon^2) \)-error) in the Euclidean construction that the tangent plane at \( Z_f \) is the readily constructible plane \( \pi_\Sigma \).
3 Local results

3.1 Smooth and discrete conjugate quads

Suppose that a smooth surface \( f(\Omega) \) parametrized locally by some curvilinear net of conjugate lines is given: \( f: \Omega \rightarrow \mathbb{R}^3, \Omega \subset \mathbb{R}^2 = \{(u,v)\} \). Take some initial point \( A = f(u_0, v_0) \) and points \( B = f(u_0 + \epsilon, v_0) \), \( C = f(u_0, v_0 + \epsilon) \) at \( \epsilon \)-distance on the two conjugate lines of the net on \( f(\Omega) \) and let \( D = f(u_0 + \epsilon, v_0 + \epsilon) \) be the fourth point of the curvilinear quad on \( f(\Omega) \).

Using the conjugacy condition

\[
f_{uv} = a(u,v)f_u + b(u,v)f_v
\]

(15)

and its derivatives, one can easily estimate the distance from this fourth point to the plane \( \pi_{ABC} \) defined by \( A, B, C \); we formulate this as our next Theorem.

We remind that smooth conjugate nets are supposed to be non-degenerate, that is, the directions of the curvilinear coordinate lines of such a net are non-asymptotic, so the angle between the tangent vectors \( f_u \) and \( f_v \) is everywhere different from zero.

Theorem 5 For an arbitrary smooth non-degenerate conjugate net \( f(\Omega) \), \( d(D, \pi_{ABC}) = O(\epsilon^4) \).

Proof. Using the standard Taylor expansions one gets:

\[
\overrightarrow{AB} = \epsilon f_u + \frac{\epsilon^2}{2} f_{uu} + \frac{\epsilon^3}{3!} f_{uuu} + o(\epsilon^3),
\]

\[
\overrightarrow{AC} = \epsilon f_v + \frac{\epsilon^2}{2} f_{vv} + \frac{\epsilon^3}{3!} f_{vvv} + o(\epsilon^3),
\]

\[
\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} + \epsilon^2 f_{uv} + \frac{\epsilon^3}{3} (f_{uu} + f_{vv}) + o(\epsilon^3),
\]

(16)

where all derivatives are taken at the point \((u_0, v_0)\).

Taking into account (15) and its derivatives

\[
f_{uv} = a_u f_u + b_u f_v + a f_{uu} + b a f_u + b f_v,
\]

\[
f_{uv} = a_v f_u + b_v f_v + a f_u + b_v f_v + b f_{vv},
\]

we can compute the triple product \( (\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} \) which gives the volume of the parallelepiped spanned by the vectors \( \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD} \):

\[
(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = \epsilon^2 (\overrightarrow{AB} \times \overrightarrow{AC}) \cdot f_{uv} + \frac{\epsilon^3}{2} (\overrightarrow{AB} \times \overrightarrow{AC}) \cdot (f_{uu} + f_{vv}) + o(\epsilon^5) =
\]

\[
\frac{\epsilon^5}{2} \left[ (f_u \times f_v) \cdot (f_{uu} + f_{vv}) + (f_u \times f_v) \cdot f_{uv} + (f_u \times f_v) \cdot f_u \right] + o(\epsilon^5) =
\]

\[
\frac{\epsilon^5}{2} \left[ a(f_u \times f_v) \cdot f_{uu} + b(f_u \times f_v) \cdot f_{vv} + a(f_u \times f_v) \cdot f_u + b(f_u \times f_v) \cdot f_v \right] + o(\epsilon^5) = o(\epsilon^5).
\]
Since the area of the base of this parallelepiped $|\overrightarrow{AB} \times \overrightarrow{AC}| \simeq \epsilon^2$, we get that its height $d = o(\epsilon^3)$, so at least we may state that $d = O(\epsilon^4)$. In fact, using Taylor expansions in (15) up to order $O(\epsilon^4)$ one can obtain after a lengthy computation that

\[
(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = \frac{\epsilon^6}{12} \left[(a_u - ab)(f_uu \times f_u) \cdot f_v - (b_v - ab)(f_vv \times f_u) \cdot f_v\right] + o(\epsilon^6) \tag{17}
\]

which proves that for a generic conjugate net we have $d \simeq \epsilon^4$. \(\square\)

In fact we can even choose $M$ (the fourth point of an elementary planar quad $ABCM$) on the plane $\pi_{ABC}$ sufficiently close to $D$ and lying on the given smooth surface $f(\Omega)$:

**Theorem 6** For a given smooth non-degenerate conjugate net $f : \Omega \rightarrow \mathbb{R}$ and the plane $\pi_{ABC}$ constructed as above one can choose a point $M \in \pi_{ABC} \cap f(\Omega)$ such that $d(M, D) = O(\epsilon^3)$.

**Proof.** As we prove in the Appendix, the intersection of the plane $\pi_{ABC}$ and the surface $f(\Omega)$ in the $\epsilon$-neighborhood of the points $A, B, C, D$ for sufficiently small $\epsilon$ is a curve $I_{ABC}$ close to a quadric – the Dupin indicatrix of $f(\Omega)$, and the angle between $\pi_{ABC}$ and $f(\Omega)$ is $\simeq \epsilon$ in all points of $I_{ABC}$. According to Theorem 5 $d(D, \pi_{ABC}) = O(\epsilon^4)$. From this we can easily see that the distance between $D$ and the closest to $D$ point $M$ on $I_{ABC}$ should be $O(\epsilon^4/\epsilon) = O(\epsilon^3)$. \(\square\)

### 3.2 Curvilinear quads of curvature lines and plane circular quads

For a given smooth surface $f(\Omega)$ parametrized by curvature lines one can prove a similar result, if we take the circle $\omega_{ABC}$ passing through the points $A, B, C$ defined as in Section 3.1.

**Theorem 7** For arbitrary smooth net of curvature lines in a neighborhood of a non-umbilic point $A$ on $f(\Omega)$, the distance between the fourth point $M$ of intersection of the circle $\omega_{ABC}$ with $f(\Omega)$ and the point $D$ has order 3: $d(D, M) = O(\epsilon^3)$.

In order to prove this theorem we need to establish a few auxiliary Lemmas.

**Lemma 8** For arbitrary smooth net of curvature lines in a neighborhood of a non-umbilic point $A$ on $f(\Omega)$, $d(D, \omega_{ABC}) = O(\epsilon^3)$.

**Proof.** Following an approach proposed in [4], we introduce for each of the points $A, B, C, D$ purely imaginary quaternion $A, B, C, D$. The point $A$ will be chosen as the origin, i.e. $A = 0$. The basic quaternions $I, J, K$ are chosen to be tangent to the curvature line directions $f_u, f_v$ and the normal vector to the surface in the initial point $A$ respectively. In view of future applications in the theory of triply
orthogonal coordinate systems we include the given surface \( f(\Omega) \) into such a system \( f(u, v, w) \), \( f(\Omega) \) being one of the coordinate surfaces \( f(u, v, w = w_0) \). This is always possible (see e.g. [7]): one can take for example the one-parametric family of surfaces parallel to \( f(\Omega) \) and the other two one-parametric families of developable surfaces defined by the curvature lines of \( f(\Omega) \) and the normals to \( f(\Omega) \).

Introducing for this 3-orthogonal system the Lamé coefficients \( H_i(u, v, w) = |\partial f|, \partial_1 = \partial/\partial u, \partial_2 = \partial/\partial v, \partial_3 = \partial/\partial w, \) normalized vectors \( \vec{V}_i = \partial_i f / H_i \) and the rotation coefficients \( \beta_{ik}(u) = \partial_i H_k / H_i, i \neq k \), \( \beta_{ii}(u) = 0 \), we have the following relations ([7]):

\[
\begin{align*}
\partial_i H_k &= \beta_{ik} H_i, \\
\partial_i \vec{V}_k &= \beta_{ki} \vec{V}_i, \\
\partial_i \vec{V}_i &= -\sum_{s \neq i} \beta_{si} \vec{V}_s, \\
\partial_j \beta_{ik} &= \beta_{ij} \beta_{jk}, \quad i \neq j \neq k, \\
\partial_i \beta_{ik} + \partial_k \beta_{ii} + \sum_{s \neq i, k} \beta_{si} \beta_{sk} &= 0,
\end{align*}
\]

(18)

where \( i, j, k \in \{1, 2, 3\}, i \neq j \neq k \). In the initial point \( A \) we have \( \vec{V}_1(u_0, v_0) = \vec{I}, \vec{V}_2(u_0, v_0) = \vec{J}, \vec{V}_3(u_0, v_0) = \vec{K} \). The Taylor expansions for the other points are now easily obtained after differentiation of \( f_u = H_1 \vec{V}_1, f_v = H_2 \vec{V}_2 \) using (18):

\[
\begin{align*}
\vec{B} &= \epsilon H_1 \vec{I} + \epsilon^2 \left( \partial_1 H_1 \vec{I} - \beta_{21} H_1 \vec{J} - \beta_{31} H_1 \vec{K} \right) + o(\epsilon^2), \\
\vec{C} &= \epsilon H_2 \vec{I} + \epsilon^2 \left( -\beta_{12} H_2 \vec{J} + \partial_2 H_2 \vec{J} - \beta_{32} H_2 \vec{K} \right) + o(\epsilon^2), \\
\vec{D} &= \vec{B} + \vec{C} + \epsilon^2 \left( \beta_{21} H_2 \vec{I} - \beta_{12} H_1 \vec{J} \right) + o(\epsilon^2).
\end{align*}
\]

Calculating now the quaternionic cross-ratio ([7]) \( \vec{Q} = (\vec{A} - \vec{B})(\vec{B} - \vec{C})^{-1}(\vec{C} - \vec{D})(\vec{D} - \vec{A})^{-1} \) and taking its imaginary part, one can see that \( \text{Im} \vec{Q} = o(\epsilon) \). According to [7] we conclude that \( \rho = o(\epsilon^2) \) for the distance \( \rho \) from the point \( D \) to the circle \( \omega_{ABC} \) of size \( \simeq \epsilon \) defined by the points \( A, B, C \). \( \square \)

In the following we fix an \( \epsilon \)-patch \( f(\Omega_\epsilon) \) of size \( \simeq \epsilon \) on the surface \( f(\Omega) \), such that all the points \( A, B, C, D \) belong to it. All subsequent constructions will be applied only to this \( \epsilon \)-patch \( f(\Omega_\epsilon) \).

From Lemma [16] (proved in the Appendix) we know that the intersection curve \( I_{ABC} = f(\Omega) \cap \pi_{ABC} \) is \( \epsilon^2 \)-close to the Dupin indicatrix of \( f(\Omega) \) at some point \( M \). In fact we can take the indicatrix at the following point \( P \) instead: choose the “center point” \( P \) on \( f(\Omega) \) such that \( P \) is the intersection of \( f(\Omega) \) and the normal to the plane \( \pi_{ABC} \) passing through the center of the circle \( \omega_{ABC} \). In the Cartesian coordinate system (similar to the system used in the proof of Lemma [16]), where the osculating paraboloid at \( P \) is \( \mathcal{P} = \{ z = (K_1 x^2 + K_2 y^2)/2 \} \), the points \( A, B, C, D \) will have
coordinates
\[
\begin{align*}
A & \left( -\frac{\epsilon}{2} + o(\epsilon), -\frac{\epsilon}{2} + o(\epsilon), \frac{K_1 + K_2}{8} \epsilon^2 + o(\epsilon^2) \right), \\
B & \left( \frac{\epsilon}{2} + o(\epsilon), -\frac{\epsilon}{2} + o(\epsilon), \frac{K_1 + K_2}{8} \epsilon^2 + o(\epsilon^2) \right), \\
C & \left( -\frac{\epsilon}{2} + o(\epsilon), \frac{\epsilon}{2} + o(\epsilon), \frac{K_1 + K_2}{8} \epsilon^2 + o(\epsilon^2) \right), \\
D & \left( \frac{\epsilon}{2} + o(\epsilon), \frac{\epsilon}{2} + o(\epsilon), \frac{K_1 + K_2}{8} \epsilon^2 + o(\epsilon^2) \right),
\end{align*}
\] (20)

so for the angle \( \theta \) between the plane \( \pi_{ABC} \) and the tangent plane \( \pi_P \) to \( f(\Omega) \) at \( P \) we have \( \theta = O(\epsilon^2) \). Moreover, the following is true:

**Lemma 9** The centers of the Dupin indicatrix \( Q_P = \mathcal{P} \cap \pi_{ABC} \) and \( \omega_{ABC} \) are \( \epsilon^2 \)-close. The angles of intersection of \( \omega_{ABC} \) and \( I_{ABC} \) are \( \approx 0 = 1 \).

**Proof.** From [20] we easily deduce the first statement. The radius of \( \omega_{ABC} \) is \( \epsilon \sqrt{2} + o(\epsilon) \), while the axes of \( Q_P \) are \( \frac{K_1 + K_2}{K_1} + o(\epsilon) \) and \( \frac{K_1 + K_2}{K_2} + o(\epsilon) \). Since we assume \( |K_2 - K_1| \geq \epsilon \) on \( \Omega \) we see that the intersection angles of \( Q_P \) and \( \omega_{ABC} \), so also of \( \omega_{ABC} \) and \( I_{ABC} \), are \( \approx 0 = 1 \). \( \square \)

Now using the same technique as in the proof of Lemma [16] we deduce that the Dupin indicatrix at \( P \) and \( I_{ABC} \) are \( \epsilon^2 \)-close.

**Proof of Theorem 7** We know (Theorem 5) that \( D \) is \( \epsilon \)-close to \( \pi_{ABC} \). Using the fact that the angle of intersection of \( \pi_{ABC} \) and \( f(\Omega) \) is \( \approx \epsilon \) (so \( \geq \epsilon^2 \)) we see that \( D \) is \( \epsilon^3 \)-close to \( I_{ABC} = \pi_{ABC} \cap f(\Omega) \). On the other hand (Lemma 8) \( D \) is \( \epsilon^3 \)-close to the circle \( \omega_{ABC} \). Since the angles of intersection of \( \omega_{ABC} \) and \( I_{ABC} \) are \( \approx 0 = 1 \), we see that \( D \) and \( M = I_{ABC} \cap \omega_{ABC} \) are \( \epsilon^3 \)-close as well. \( \square \)

4 Global results

4.1 Conjugate nets

Using the results of Section 3.1 one can try to construct inductively an approximating discrete conjugate net \( f^\epsilon \) for sufficiently small \( \epsilon > 0 \). At the first glance the following simplest strategy may be applied: starting from the initial point \( A \equiv f_{00} = f(u_0, v_0) \) on a finite piece of a smooth surface \( f : \Omega \to \mathbb{R}^3 \) parametrized by conjugate coordinate lines, fix two series of points \( f_{i,0} = f(u_0 + i\epsilon, v_0), f_{0,i} = f(u_0, v_0 + i\epsilon) \) at \( \epsilon \)-distances on two curvilinear coordinate lines on \( f(\Omega) \) passing through \( A \). We can construct the next point \( f_{11} \) as the orthogonal projection of \( f_{11} = f(u_0 + \epsilon, v_0 + \epsilon) \) onto the plane passing through \( f_{00}, f_{10}, f_{01} \); then \( f_{21} \) as the orthogonal projection of \( f_{21} = f(u_0 + 2\epsilon, v_0 + \epsilon) \) onto the plane passing through \( f_{10}, f_{20} \) and \( f_{11} \), etc. This inductive process is shown on Figure 4 with initial points \( f_{ij} = f(u_0 + i\epsilon, v_0 + j\epsilon) \) on the surface \( f(\Omega) \) with smooth coordinate lines (shown as dash-lines). The approximating discrete conjugate net consists of the points \( f_i^\epsilon, f_i^{\epsilon_0} \equiv f_{i0}, f_{0i}^{\epsilon_0} \equiv f_{0i} \).

This “geometric analogue” of the analytic approximation results of [3, 14] should give us, if the results of [3, 14] were directly applicable, an estimate \( d(f_{ij}, f_{ij}^\epsilon) \leq C(i+ \)
Paradoxically, a better strategy consists in choosing the points \( f_{ij}^\epsilon \) of the approximating discrete net on the given smooth surface \( f \), although Theorem 6 suggests that one has worse local error \( \simeq \epsilon^4 \) than the local error \( \simeq \epsilon^3 \) of Theorem 5. The construction we propose below is applicable to arbitrary simple pieces \( f(\Omega) \) without parabolic points, i.e. points where along with the diagonal coefficient \((f_{uv} \times f_u) \cdot f_v\) of the second fundamental form (cf. (14)) at least one of the other coefficients \((f_{uu} \times f_u) \cdot f_v\), \((f_{vv} \times f_u) \cdot f_v\) also vanishes.

First we fix the vertices \( f_{i0}^\epsilon = f_{i0} = f(u_0 + i\epsilon, v_0) \) and all even “columns” \( f_{2i,j}^\epsilon = f(u_0 + 2i\epsilon, v_0 + j\epsilon) \) in the new approximating discrete conjugate net (marked on Figure 5 as \( \otimes \)-points).

Then we proceed as follows. Two planes defined by the triples of points \( \{f_{00}, f_{10}, f_{01}\}, \{f_{10}, f_{20}, f_{21}\} \) intersect along a straight line passing through \( f_{10}^\epsilon = f_{10} \). As we show below in Lemma 10 this straight line intersects \( f(\Omega) \) in another point \( f_{11}^\epsilon \) which is close to \( f_{11} \), \( d(f_{11}^\epsilon, f_{11}) = O(\epsilon^3) \), whereas the distance between this new point \( f_{11}^\epsilon \) and the line \( v = v_0 + \epsilon \) of the smooth conjugate net is of order \( O(\epsilon^4) \). Other
points \( f^0_{i1}, f^0_{11}, \ldots \) of the first row are constructed in the same way and obviously have the same error estimates. The second row \( f^2_{2i+1,2}, i = 0, 1, \ldots \) of the vertices of the approximating discrete net is constructed analogously using the fixed points \( f^2_{2k,2}, k = 0, 1, \ldots \) of the second row on \( f(\Omega) \) and the constructed complete set of the points \( f^2_{i1} \) of the first row (cf. Figure 5).

As Lemma 11 below shows, \( f^2_{ij} \) are now \( \epsilon^5 \)-close to the smooth line \( v = v_0 + 2\epsilon \) whereas the distance between \( f^2_{2i+1,2} \) and \( f^2_{2i+1,2} \) along this line (the difference of their \( u \)-coordinates) is doubled: \( \delta u(f^2_{12}) = \delta u(f^1_{12}) + \mu(u_0, v_0 + \epsilon)\epsilon^3 + o(\epsilon^3) = 2\mu(u_0, v_0)\epsilon^3 + o(\epsilon^3), \delta v(f^2_{12}) = -\delta v(f^1_{12}) + \nu(u_0, v_0 + \epsilon)\epsilon^4 + o(\epsilon^4) = o(\epsilon^4). \)

Here and afterwards we use the following notations: \( \delta v(f^2_{ij}) \equiv v(f^2_{ij}) - v(f_{ij}), \delta u(f^2_{ij}) \equiv u(f^2_{ij}) - u(f_{ij}), \) where \( u(P) \) and \( v(P) \) denote the respective curvilinear coordinates \( u, v \) for any point \( P \) on the surface \( f(\Omega) \).

For the third row we will have \( \delta v(f^3_{13}) = 3\mu(u_0, v_0)\epsilon^3 + o(\epsilon^3), \delta u(f^3_{13}) = \nu(u_0, v_0)\epsilon^4 + o(\epsilon^4). \)

This linear ”\( u \)-drift” of the points \( f^1_{1,k} \) in contrast to ”\( v \)-oscillation” of the same points is easily explained: the position of the points \( f^0_{i0} = f_{i0} = f(u_0 + i\epsilon, v_0) \) on the initial conjugate line \( v = v_0 \) may be changed using a reparametrization \( u \mapsto \bar{u} = f(u) \) of the first conjugate coordinate \( u \); thus the \( u \)-coordinates of the points \( f^0_{i1} \) of the first row may be considered as some new choice of \( u \)-parametrization; all subsequent rows simply shift accordingly.

**Lemma 10** For two adjacent infinitesimal curvilinear coordinate quadrangles \( ABDC \) and \( BDFE \) of \( \epsilon \)-size (see Fig. 4) on a smooth conjugate net \( f : \Omega \to \mathbb{R}^3 \) without parabolic points one can find in a \( \epsilon^3 \)-neighborhood of \( D \) a unique point \( M \) on \( f(\Omega) \) such that the quadrangles \( ABMC \) and \( BMFE \) are planar. The curvilinear coordinates of \( M \) are \( u(M) = u_0 + \epsilon + \delta u_M, v(M) = v_0 + \epsilon + \delta v_M, \) with \( \delta u_M \sim \mu(u_0, v_0)\epsilon^3, \) \( \delta v_M \sim \nu(u_0, v_0)\epsilon^4. \)
Proof. We will follow the guidelines of the proof of Theorem 5. Using the Taylor expansions up to order $O(\epsilon^4)$ for the Cartesian coordinates of $A(u_0, v_0)$, $B(u_0 + \epsilon, v_0)$, $C(u_0, v_0 + \epsilon)$, $D(u_0 + \epsilon, v_0 + \epsilon)$, $E(u_0 + 2\epsilon, v_0)$, $F(u_0 + 2\epsilon, v_0 + \epsilon)$ and some point $D_1(u_0 + \epsilon + \delta u_1, v_0 + \epsilon + \delta v_1)$ with some $\delta u_1 = O(\epsilon^3)$, $\delta v_1 = O(\epsilon^3)$ and taking into consideration (15), we compute the triple products $W_1 = ((\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}_1) = e^6 \gamma(u_0, v_0) + e^3 \left( \delta u_1 (f_{uu} \times f_u) \cdot f_v + \delta v_1 (f_{vv} \times f_u) \cdot f_v \right) + o(\epsilon^6)$, $W_2 = ((\overrightarrow{BE} \times \overrightarrow{BD}) \cdot \overrightarrow{F}) = e^6 \gamma(u_0, v_0) + e^3 \left( \delta u_1 (f_{uu} \times f_u) \cdot f_v - \delta v_1 (f_{vv} \times f_u) \cdot f_v \right) + o(\epsilon^6)$ where $\gamma(u, v)$ is a combination of the coefficients $f(u, v), \beta(u, v)$ of (15), their derivatives and the triple products $(f_{uu} \times f_u) \cdot f_v, (f_{vv} \times f_u) \cdot f_v$. So if both of the latter triple products do not vanish, one can choose unique $\delta u_1 = \tilde{\mu}(u_0, v_0)\epsilon^3$, $\delta v_1 = 0$ such that $W_1 = O(\epsilon^7)$, $W_2 = O(\epsilon^7)$.

This shows (as in the proof of Theorem 6) that $D_1$ lies in $\epsilon^4$-neighborhood of the curves $I_{ABC}, I_{BEF}$ of intersection of $f(\Omega)$ with the planes $(ABC), (BEF)$. Since the angle of intersection of $I_{ABC}, I_{BEF}$ is $\approx 1$, we conclude that the point $M$ of their intersection, close to $D_1$, is in fact $\epsilon^3$-close to $D_1$.

Using Taylor expansions for the same points up to order $O(\epsilon^5)$, one can prove that for a generic surface actually $\delta u_M \sim \mu(u_0, v_0)\epsilon^3$, $\delta v_M \sim \nu(u_0, v_0)\epsilon^4$ with $\mu \neq 0$, $\nu \neq 0$. □

Figure 6: The adjacent infinitesimal smooth quads are defined by the points $A, B, C, D, E$ and $F$ (schematically shown as rectangles) and the planar quads $ABMC, BMFE$.

If one repeats the same calculation using instead of $B$ a point $B_1(u_0 + \epsilon + \delta u(B_1), v_0 + \delta v(B_1))$ with some $\delta u(B_1) = O(\epsilon^2), \delta v(B_1) = O(\epsilon^3)$, one will get from the conditions $W_1 = O(\epsilon^7), W_2 = O(\epsilon^7)$ that $\delta u_1 = \delta u(B_1) + \mu(u_0, v_0)\epsilon^3 + o(\epsilon^3)$, $\delta v_1 = -\nabla v(B_1) + \nu(u_0, v_0)\epsilon^4 + \lambda(u_0, v_0)\delta v(B_1)\epsilon + \gamma(u_0, v_0)\delta u(B_1)\epsilon^2 + o(\epsilon^4)$ where $\mu(u_0, v_0), \nu(u_0, v_0), \lambda(u_0, v_0), \gamma(u_0, v_0)$ are some algebraic combinations of the coefficients $f(u, v), \beta(u, v)$ of (15), their derivatives and the triple products $(f_{uu} \times f_u) \cdot f_v,$
(\text{f}_{uv} \times \text{f}_u) \cdot \text{f}_v$. Hence the following estimates hold for the second point $M$ of intersection of $I_{AB, C}$ and $I_{B, E F}$:

**Lemma 11** For two adjacent infinitesimal curvilinear coordinate quadrangles $ABDC$ and $BDFE$ of $\epsilon$-size on a smooth conjugate net $f : \Omega \rightarrow \mathbb{R}^3$ without parabolic points and some point $B_1(u_0 + \epsilon + \delta u(B_1), v_0 + \delta v(B_1))$, $\delta u(B_1) = O(\epsilon^2)$, $\delta v(B_1) = O(\epsilon^3)$, one can find in a $\epsilon^2$-neighborhood of $D$ a unique point $M$ on $f(\Omega)$ such that the quadrangles $AB_1MC$ and $B_1MFE$ are planar. The curvilinear coordinates of $M$ are $u(M) = u_0 + \epsilon + \delta u_M; v(M) = v_0 + \epsilon + \delta v_M$, with $\delta u_M = \delta u(B_1) + \mu(u_0, v_0)\epsilon^3 + o(\epsilon^3)$, $\delta v_M = -\delta v(B_1) + \nu(u_0, v_0)\epsilon^4 + \lambda(u_0, v_0)\delta v(B_1)\epsilon + \gamma(u_0, v_0)\delta u(B_1)\epsilon^2 + o(\epsilon^4)$.

**Remark.** As before, one can explain the “$\epsilon^2$-tolerance” along the $u$-lines simply by removing this $u$-shift using a reparametrization of the curvilinear coordinate $u$. Note that we need this reparametrization only locally, for a given pair of elementary infinitesimal quadrangles, in order to establish this estimate for them, and not globally, for the complete discrete conjugate net.

For global estimates we need the following simple result ([3]):

**Lemma 12** (Discrete Grönwall estimate) Assume that a nonnegative function $\Delta : \mathbb{N} \rightarrow \mathbb{R}$ satisfies

$$\Delta(n+1) \leq (1 + \epsilon K)\Delta(n) + \kappa$$

with nonnegative constants $K, \kappa$. Then

$$\Delta(n) \leq (\Delta(0) + n\kappa) \exp(Kn).$$

**Theorem 13** For a smooth conjugate net without parabolic points $f : \Omega \rightarrow \mathbb{R}^3$ and sufficiently small $\epsilon > 0$, there exists a discrete conjugate net $f^\epsilon$ with all its points $f^\epsilon_{ij}$ on $f(\Omega)$, such that $d(f_{ij}, f^\epsilon_{ij}) = O(\epsilon^2)$.

**Proof.** From Lemma [1] we immediately see that for each column $k = 2i + 1$ the function $\Delta(n) = |\delta u(f^\epsilon_{2i+1,j})|$ satisfies the estimate (21) with $K = 0$, $\kappa = 2\epsilon^3 \max_{(u,v) \in \Omega} |\mu(u, v)|$ for sufficiently small $\epsilon$, so for all $n$ we have (22). Since the number of steps in each column is $\approx 1/\epsilon$ we obtain the global estimate $|\delta u(f^\epsilon_{2i+1,j})| \leq C\epsilon^2$.

Now we set $\Delta(n) = |\delta v(f^\epsilon_{2i+1,j})|$ for each column $k = 2i + 1$. Taking the estimates of Lemma [1] and the already obtained global estimate for $|\delta u(f^\epsilon_{2i+1,j})|$ we get (21) with $K = \max_{(u,v) \in \Omega} |\lambda(u, v)|$, $\kappa = 2\epsilon^4 \max_{(u,v) \in \Omega} (|\mu(u, v)| + |\gamma(u, v)|)$, so (22) gives the global estimate $|\delta v(f^\epsilon_{2i+1,j})| \leq C\epsilon^3$. □

**Remark.** As we have observed above, the $u$-shift of the points $f^\epsilon_{2i+1,j}$ has a much more interesting oscillating behavior; in particular one may conjecture that $|\delta(f^\epsilon_{2i+1,j})| = O(\epsilon^4)$ globally.
4.2 Curvature lines and circular nets

We start the construction of a circular discrete approximation of a given finite piece \( f(\Omega) \) of a smooth surface without umbilic points parametrized by curvature lines fixing points \( f_i \) on two initial curvature lines passing through a point \( f_0 = f(u_0, v_0) \) (cf. Figure 7).

\[
\begin{align*}
\delta u & = \delta u_A + \delta u_B - \delta u_C + \epsilon (\delta v_B - \delta v_A) \mu_1(u_0, v_0) \\
& \quad + \epsilon (\delta u_C - \delta u_A) \mu_2(u_0, v_0) + \mu_3(u_0, v_0) \epsilon^3 + \mu_4(u_0, v_0) \epsilon^4 + o(\epsilon^4), \\
\delta v & = \delta v_A + \delta v_B - \delta v_C + \epsilon (\delta u_B - \delta u_A) \mu_1(u_0, v_0) \\
& \quad + \epsilon (\delta u_C - \delta u_A) \mu_2(u_0, v_0) + \mu_3(u_0, v_0) \epsilon^3 + \mu_4(u_0, v_0) \epsilon^4 + o(\epsilon^4),
\end{align*}
\]

Figure 7: A circular net \( f_{ij}^\epsilon \) with its vertices on \( f(\Omega) \), which are \( \epsilon^2 \)-close to the vertices of the smooth net of conjugate lines \( f_{ij} \)

The first circle is defined by the triple \( \{f_{00}, f_{10}, f_{01}\} \), its fourth point of intersection with the surface will be chosen as the next constructed point \( f_{11}^\epsilon \) of the discrete circular net. On the next step we can define two circles by the triples of points \( \{f_{01}, f_{02}, f_{11}^\epsilon\} \) and \( \{f_{10}, f_{20}, f_{11}^\epsilon\} \), the fourth points of intersection of these circles with the surface will give us \( f_{12}^\epsilon \) and \( f_{21}^\epsilon \). The behavior of the new points on the third step is of crucial importance. Their error estimates can be obtained from the following generalization of Theorem 7.

**Lemma 14** Let the points \( A, B \) and \( C \) on \( f(\Omega) \) be chosen, such that \( u(A) = u_0 + \delta u_A, v(A) = v_0 + \delta v_A, u(B) = u_0 + \epsilon + \delta u_B, v(B) = v_0 + \delta v_B, u(C) = u_0 + \delta u_C, \) \( v(C) = v_0 + \epsilon + \delta v_C \), with all \( \delta u, \delta v \) of order \( = O(\epsilon^3) \). Then the curvilinear coordinates of the fourth point \( M \) of intersection of the circle \( \omega_{ABC} \) with \( f(\Omega) \) are: \( u(M) = u(f_{11}^\epsilon) + \delta u_M, v(M) = v(f_{11}^\epsilon) + \delta v_M \), with

\[
\begin{align*}
\delta u_M &= \delta u_A + \delta u_B - \delta u_C + \epsilon (\delta v_B - \delta v_A) \mu_1(u_0, v_0) \\
& \quad + \epsilon (\delta u_C - \delta u_A) \mu_2(u_0, v_0) + \mu_3(u_0, v_0) \epsilon^3 + \mu_4(u_0, v_0) \epsilon^4 + o(\epsilon^4), \\
\delta v_M &= \delta v_A + \delta v_B - \delta v_C + \epsilon (\delta u_B - \delta u_A) \mu_1(u_0, v_0) \\
& \quad + \epsilon (\delta u_C - \delta u_A) \mu_2(u_0, v_0) + \mu_3(u_0, v_0) \epsilon^3 + \mu_4(u_0, v_0) \epsilon^4 + o(\epsilon^4),
\end{align*}
\]
Proof. We will use the same quaternionic cross-ratio \( Q = (A - B)(B - C)^{-1}(C - D)(D - A)^{-1} \). The obvious changes are necessary in the Taylor expansions \([19]\), where we shall take into consideration the \( \delta u \)- and \( \delta v \)-shifts of the points; the expansions themselves should include terms up to order \( O(\epsilon^4) \). For simplicity we will assume \( \delta u_A = 0, \delta v_A = 0 \), the general case requires only a shift by \( \delta u_A, \delta v_A \) in the net of curvature lines on \( f(\Omega) \).

Fixing for the moment the points \( A, B, C \) and introducing \( D \in f(\Omega), u(D) = u_0 + \epsilon + \delta u_D, v(D) = v_0 + \epsilon + \delta v_D \) with some \( \delta u_D = O(\epsilon^3), \delta v_D = O(\epsilon^3) \), one can compute the imaginary part of \( Q \):

\[
Im(Q) = I(\rho_1(u_0, v_0)e^3 + \theta H_2(\delta v_D - \delta v_C + \delta v_B) + o(\epsilon^3)) + J(\rho_2(u_0, v_0)e^3 + \theta H_1(-\delta u_D - \delta u_C + \delta u_B) + o(\epsilon^3)) + K(\rho_3e^2 + \epsilon^{-1}H_1H_2(H_1^2(\delta u_B - \delta u_D - \delta u_C) + H_2^2(\delta v_D + \delta v_C - \delta v_B)) + o(\epsilon^2)),
\]

where \( \theta = \frac{1}{2}(H_1^2H_2^2 - H_1H_2^3) = \frac{1}{2}H_1^2H_2^2(K_2 - K_1)/2 \), \( K_i \) being the principal curvatures at the point \( A \). Since we assume \( K_1 \neq K_2 \), choosing

\[
\delta u_D = \delta u_B - \delta u_C + \epsilon^3 \mu_3(u_0, v_0) + o(\epsilon^3), \quad \delta v_D = -\delta v_B + \delta v_C + \epsilon^3 \nu_3(u_0, v_0) + o(\epsilon^3),
\]

one will achieve \( Im(Q) = o(\epsilon^3)I + o(\epsilon^3)J + o(\epsilon^3)K \). As a lengthier computation shows, adding appropriate fourth-order corrections in Taylor expansions, given in \([23]\) (set there \( \delta u_A = 0, \delta v_A = 0 \)), we get \( Im(Q) = o(\epsilon^4)I + o(\epsilon^4)J + o(\epsilon^3)K \). This means that the point \( D \) with \( \delta u_D, \delta v_D \) given in \([23]\) is \( \epsilon^6 \)-close to the plane \( (ABC) \) (this can be also checked using the triple product \( \overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{AD} \) and \( \epsilon^5 \)-close to the circle \( \omega_{ABC} \) passing through \( A, B, C \). Using the that the angle between \( f(\Omega) \) and the plane \( (ABC) \) is \( \sim \epsilon \) at \( D \), we conclude that for the fourth point \( M \) of intersection of \( \omega_{ABC} \) and \( f(\Omega) \) we can keep the same expansions \([23]\). □

Remark. Linear behavior of \( \delta u_D \) and \( \delta v_D \) in \([23]\) is valid precisely in orders \( O(\epsilon^3) \) and \( O(\epsilon^4) \), corrections of order \( O(\epsilon^5) \) are nonlinear w.r.t. \( \delta u \)- and \( \delta v \)-shifts of the points \( A, B \) and \( C \).

Using \([23]\) we calculate \( \delta u(f_{12}) = \delta u(f_{01}) + \delta u(f_{11}) \delta u(f_{02}) + \mu_3(u_0, v_0 + 0) e^3 + o(\epsilon^3) = 2\mu_3(u_0, v_0) e^3 + o(\epsilon^3), \quad \delta v(f_{12}) = \delta v(f_{01}) + \delta v(f_{11}) + \delta v(f_{02}) + \epsilon(\delta v(f_{01}) + \delta v(f_{02}) - \delta u_1(u_0, v_0) + \epsilon + \epsilon(\delta u(f_{02}) - \delta u(f_{01})) + \epsilon) + \epsilon + o(\epsilon^4) = \epsilon(\delta u_1(u_0, v_0) + \epsilon + \epsilon) + \epsilon + o(\epsilon^4) = \epsilon(\delta u_1(u_0, v_0) + \epsilon + \epsilon) + \epsilon + o(\epsilon^4) = \epsilon(\delta u_1(u_0, v_0) + \epsilon + \epsilon) + \epsilon + o(\epsilon^4) = \epsilon(\delta u_1(u_0, v_0) + \epsilon + \epsilon) + \epsilon + o(\epsilon^4) = \epsilon(\delta u_1(u_0, v_0) + \epsilon + \epsilon) + \epsilon + o(\epsilon^4) = \epsilon(\delta u_1(u_0, v_0) + \epsilon + \epsilon) + \epsilon + o(\epsilon^4) = \epsilon(\delta u_1(u_0, v_0) + \epsilon + \epsilon) + \epsilon + o(\epsilon^4) = \epsilon(\delta u_1(u_0, v_0) + \epsilon + \epsilon) + \epsilon + o(\epsilon^4).
\]

As we see now, for points \( f_{12} \) (respectively \( f_{21} \)) only the shifts along the respective curvature lines is of order \( O(\epsilon^3) \) (only for surfaces of special type this shift may degenerate to 0), while the perpendicular shift is of order \( O(\epsilon^4) \). For \( f_{12} \) we now obtain a remarkable result: both its shifts are of order \( O(\epsilon^4) \), compared to the error estimates of order \( O(\epsilon^3) \) for \( f_{11}: \delta u(f_{22}) = 2\epsilon^4(\delta u\mu_3(u_0, v_0) + \mu_2(u_0, v_0)\mu_3(u_0, v_0)) + o(\epsilon^4), \quad \delta v(f_{22}) = 2\epsilon^4(\delta v\mu_3(u_0, v_0) + \mu_1(u_0, v_0)\nu_3(u_0, v_0)) + o(\epsilon^4) \).
This observation suggests us to partition the complete discrete lattice \( f^e_{ij} \) into 3 sublattices:

a) the even sublattice of points \( f^e_{2i,2j} \),
b) the odd sublattice of points \( f^e_{2i+1,2j+1} \),
c) the intermediate sublattice of points \( f^e_{2i+1,2j}, f^e_{2i,2j+1} \).

Infinitesimally, in a \((N\varepsilon)\)-neighborhood of the initial point \( f^e_{00} \) (with \( N \ll 1/\varepsilon \)), the even sublattice has both \( \delta u_D \) - and \( \delta v_D \)-shifts of order \( O(\varepsilon^4) \), for the odd sublattice they have order \( O(\varepsilon^3) \), for the intermediate one only the shifts along the respective curvature lines are of order \( O(\varepsilon^3) \), the perpendicular shifts being of order \( O(\varepsilon^4) \) again.

Accumulation of errors for these sublattices is also different: linear accumulation of \( O(\varepsilon^3) \)-errors for the odd sublattice, quadratic for the even sublattice and mixed for the intermediate sublattice: linear accumulation of \( O(\varepsilon^3) \)-shifts along the respective curvature lines and quadratic in perpendicular direction: \( \delta u(f^e_{2i,2j}) \approx i \cdot j \cdot \varepsilon^4 \), \( \delta v(f^e_{2i+1,2j+1}) \approx (i + j) \cdot \varepsilon^3 \), \( \delta u(f^e_{2i+1,2j+1}) \approx i \cdot j \cdot \varepsilon^4 \), \( \delta v(f^e_{2i,2j+1}) \approx (i + j) \cdot \varepsilon^3 \).

These estimates are easily obtained using (23) (see below the proof of Theorem 14), for example the first one follows from:

\[
\begin{align*}
\delta u(f^e_{2(i+1),2(j+1)}) &= \delta u(f^e_{2i+1,2j+1}) + \delta u(f^e_{2i+2,2j+1}) - \delta u(f^e_{2i,2j}) + 2\varepsilon^4(\partial_u\mu_3 + \mu_2\mu_3) - 2\varepsilon\mu_2(\delta u(f^e_{2i+1,2j+1}) - \delta u(f^e_{2i,2j})) + o(\varepsilon^4). \\
\text{(24)}
\end{align*}
\]

Note that the estimates of Lemma 14 do not require the initial three points \( A, B, C \) to lie near the grid points \( f_{ij} \) of the original smooth net and the distances \( d(A, B), d(A, C) \) need not to be equal, both just have to be of order \( O(\varepsilon) \).

**Theorem 15** For a smooth surface without umbilic points parametrized by curvature lines \( f : \Omega \rightarrow \mathbb{R}^3 \), \( \Omega = \{(i, v)|u^2 + v^2 < 1\} \) and sufficiently small \( \varepsilon > 0 \), there exists a discrete circular net \( f^e \) with all its points \( f^e_{ij} \) on \( f(\Omega) \), such that \( d(f_{ij}, f^e_{ij}) = O(\varepsilon^2) \).

**Proof.** We will give the details for the global estimate \( \delta u(f^e_{2i,2j}) \leq C \cdot i \cdot j \cdot \varepsilon^4 \), the other are proved in the same way.

First define a function \( S(k, n) = |\delta u(f^e_{2k,2(n+1)}) - \delta u(f^e_{2k,2n})| \). From (24) we see that

\[
S(k, n + 1) \leq S(k, n) + 2\varepsilon|\mu_2|S(k, n) + 2\varepsilon^4|\partial_u\mu_3 + \mu_2\mu_3| + o(\varepsilon^4)
\]

and using Lemma 12 we immediately obtain

\[
S(k, n) \leq C\varepsilon^4 k.
\]

Now for the functions \( \Delta_k(n) = |\delta u(f^e_{2k,2n})| \) taken separately for each column of the even sublattice one gets from (24) that

\[
\Delta_k(n + 1) \leq \Delta_k(n) + (1 + 2\varepsilon|\mu_2|)S(k, n) + 2\varepsilon^4K \leq \Delta_k(n) + \varepsilon^4(2Ck + K)
\]

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with $K = 2 \max_{(u,v) \in \Omega} |\partial_u \mu_3 + \mu_2 \mu_3|$. So (22) gives us the global estimate $\Delta_k(n) \leq \tilde{C} \cdot k \cdot n \cdot \epsilon^4$.

In [3] one can find a similar result with the same order $\epsilon^2$ of approximation but without the requirement $f_{ij}^e \in f(\Omega)$.

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**Appendix**

**Lemma 16** Let an $\epsilon$-neighborhood $f : \Omega_\epsilon \to \mathbb{R}^3$ of a point $M$ on a smooth surface be given, and for the principal curvatures one has $K_1^2 + K_2^2 > 0$ at $M$. Then the intersection $I_{ABC}$ of a plane $\pi_{ABC}$ lying at $\epsilon^2$-distance from the tangent plane $\pi_M$ to $f(\Omega_\epsilon)$ at $M$ is a curve lying in $\epsilon^2$-neighborhood of a quadric — the corresponding Dupin indicatrix of the surface $f(\Omega)$ in the central point $M$ of the $\epsilon$-patch.

**Proof.** Taking the appropriate Cartesian coordinate system with the origin $M$ we can approximate the chosen $\epsilon$-patch by the osculating paraboloid $P = \{z = (K_1 x^2 + K_2 y^2)/2\}$ with error terms of order $o(\epsilon^2)$. We obtain the quadric $Q_M = P \cap \pi_{ABC}$ called the Dupin indicatrix of $f(\Omega)$ at $M$. As one can easily check, the angle between $\pi_M$ and any of the tangent planes, taken at a point of $f(\Omega_\epsilon)$, $\epsilon^2$-close to $Q_M$, is $\simeq \epsilon$: the normals $\vec{n} = f_u \times f_v$ to $f(\Omega_\epsilon)$ at such points are $\epsilon^2$-close to the normals $\vec{n}_1$ of the osculating paraboloid at the corresponding points with the same $(x, y)$-coordinates. For the latter one has $\vec{n}_1(x, y) = -K_1 x \vec{i} + K_2 y \vec{j} + \vec{k}$ and for $x \simeq \epsilon$ and/or $y \simeq \epsilon$, the angle between $\vec{n}_1(x, y)$ and $\vec{n}_1(0, 0) = \vec{k}$ is $\simeq \epsilon$.

Now using the fact that $f(\Omega_\epsilon)$ is $\epsilon^3$-close to the osculating paraboloid and standard estimates for the values of implicit functions and their derivatives we conclude that $I_{ABC}$ and $Q_M$ are $\epsilon^2$-close and the tangent directions at their $\epsilon^2$-close points are also $\epsilon^2$-close. \(\square\)

**References**

[1] P. Alliez, D. Cohen-Steiner, O. Devillers, B. Lévy,, m. Desbrun, *Anisotropic polygonal remeshing*. ACM Transactions on Graphics. v. 22 , Iss. 3 (2003) p. 485–493.

[2] A.I. Bobenko, Yu.B. Suris, *Discrete differential geometry. Consistency as integrability*, Preliminary version of a book (2005). Preprint arXiv:math.DG/0504358.
[3] A.I. Bobenko, D. Matthes, Yu.B. Suris, *Discrete and smooth orthogonal systems: $C^\infty$-approximation*, Internat. Math. Research Notices 45 (2003), 2415–2459. Also preprint arXiv:math.DG/0303333.

[4] A.I. Bobenko, U. Pinkall. *Discrete isothermic surfaces*. J. reine andew. Math. 1996 v. 475, p. 187–208.

[5] A.I. Bobenko, Yu.B. Suris, *On organizing principles of Discrete Differential Geometry. Geometry of spheres*. preprint arXiv:math.DG/0608291, 2006.

[6] G. Darboux. *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*. T.I–IV. 3rd edition. Paris: Gauthier-Villars, 1914–1927.

[7] G. Darboux, *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes*, Paris (1910).

[8] J. Goldfeather, V. Interrante, *A novel cubic-order algorithm for approximating principal direction vectors*. ACM Transactions on Graphics. v. 23, Iss. 1 (2004) p. 45–63.

[9] U. Hertrich-Jeromin, *Introduction to Mobius differential geometry*. Cambridge University Press, 2003. xii+413 pp.

[10] J. Milnor, *Morse theory*. Princeton, NJ: Princeton Univ. Pr., 1970. - VIII, 153 p. (Annals of mathematics studies; v. 51)

[11] D. Matthes. *Discrete surfaces and coordinate systems: approximation theorems and computation*. PhD TU-Berlin, 2003, http://edocs.tu-berlin.de/diss/2003/matthes_daniel.htm

[12] H. Pottmann, Y. Liu, J. Wallner, A. Bobenko, W. Wang, *Geometry of Multi-layer Freeform Structures for Architecture*, ACM Trans. Graphics 26(3) (2007), SIGGRAPH 2007.