The structure of sandpile groups of outerplanar graphs

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Abstract
We compute the sandpile groups of families of planar graphs having a common weak dual by evaluating the critical ideals of the weak dual at the lengths of the cycles bounding the interior faces. This methodology allows us to compute the algebraic structure of the sandpile groups of outerplanar graphs, and can be used to compute the sandpile groups of many other planar graph families.

Keywords: sandpile group, outerplanar graphs, critical ideals, tree.

1 Introduction

Given a graph $G$ without loops, the $(u, v)$-entry of Laplacian matrix $L(G)$ of $G$ without loops is defined as

$$L(G)_{u,v} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m(u,v) & \text{otherwise}, \end{cases}$$

where $m(u,v)$ is the number of edges between the vertices $u$ and $v$. By considering an $m \times n$ matrix $M$ with integer entries as a linear map $M : \mathbb{Z}^n \to \mathbb{Z}^m$, the cokernel $\text{coker}(M)$ of $M$ is the quotient module $\mathbb{Z}^m/\text{Im} M$. The torsion part of the cokernel of the Laplacian matrix $L(G)$ of a graph $G$ is known as the sandpile group $K(G)$ of $G$. The sandpile group is especially interesting for connected graphs, since the order $|K(G)|$ of the sandpile group of $G$ is equal to the number $\tau(G)$ of spanning trees of the graph $G$. The sandpile group has been studied intensively over the last 30 years on several contexts: the group of
components [22], critical group [9, 11], the Picard group [6, 9], the Jacobian group [6, 9], the sandpile group [4, 13], chip-firing game [9, 23], or Laplacian unimodular equivalence [24]. The books [16, 20] are excellent references on the theory of chip-firing and its connections to other combinatorial objects like hyperplane arrangements, parking functions, dominoes, etc.

Let $M$ and $N$ be two $n \times n$ matrices with integer entries. We say that $M$ and $N$ are equivalent, denoted by $N \sim M$, if there exist $P, Q \in GL_n(\mathbb{Z})$ such that $N = PMQ$. Given a square integer matrix $M$, its Smith normal form (SNF) is the unique equivalent diagonal matrix $\text{diag}(d_1, d_2, \ldots, d_n)$ whose non-zero entries are non-negative and $d_i$ divides $d_{i+1}$. The influence of the SNF in combinatorics can be found in [20]. The diagonal elements of the SNF are known as invariant factors. The computation of the SNF of a matrix is a standard technique to determine the structure of cokernel. This is because if $N \sim M$, then $\text{coker}(M) = \mathbb{Z}^n/\text{Im}M \cong \mathbb{Z}^n/\text{Im}N = \text{coker}(N)$. Therefore, as the fundamental theorem of finitely generated Abelian groups states, the cokernel of $M$ can be described as: $\text{coker}(M) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n} \oplus \mathbb{Z}^{n-r}$, where $r$ is the rank of $M$. Let $f_1(M)$ denote the number of invariant factors of $\text{SNF}(M)$ equal to 1. The minimal number $\phi(M)$ of generators of the torsion part of the cokernel of $M$ equals the number of positive invariant factors of $\text{SNF}(M)$, thus $\phi(M) = \text{rank}(M) - f_1(M)$.

Let $f_1(G)$ and $\phi(G)$ denote the number of invariant factors of the sandpile group of $G$ equal to 1 and the minimal number of generators of the sandpile group, respectively. Several researchers have addressed the question of how often the sandpile group of a connected graph is cyclic, that is, how often $f_1(G)$ is equal to $n - 2$ or $n - 1$? In [22] and [28] D. Lorenzini and D. Wagner, based on numerical data, suggest we could expect to find a substantial proportion of graphs having a cyclic sandpile group. Based on this, D. Wagner conjectured [28] that almost every connected simple graph has a cyclic sandpile group. A recent study [30] concluded that the probability that the sandpile group of a random graph is cyclic is asymptotically at most

$$\zeta(3)^{-1}\zeta(5)^{-1}\zeta(7)^{-1}\zeta(9)^{-1}\zeta(11)^{-1}\cdots \approx 0.7935212,$$

where $\zeta$ is the Riemann zeta function; differing from Wagner’s conjecture. Besides, [11] that for any given connected simple graph, there is an homeomorphic graph with cyclic sandpile group. We say that two graphs $G_1$ and $G_2$ are in the same homeomorphism class if there exists a graph $G$ that is a subdivision of both $G_1$ and $G_2$.

When the graph is connected, it is convenient to compute the cokernel of a reduced Laplacian matrix since its rank is $n - 1$. The reduced Laplacian matrix $L_k(G)$ for a connected graph $G$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the row and column $k$ from $L(G)$. There are $n$ different reduced Laplacian matrices and $\text{K(G)} = \text{coker}(L_k(G))$ and $|\text{K(G)}| = \text{det}(L_k(G)) = \tau(G)$ for any $k \in [n] := \{1, \ldots, n\}$, see details in [9].

An useful way to compute the non-zero invariant factors of $M$ is the following.
Lemma 1. For $k \in [\text{rank}(M)]$, let $\Delta_k(M)$ be the gcd of the $k$-minors of $M$, and $\Delta_0(M) = 1$. Then the $k$-th invariant factor $d_k(M)$ of $M$ equals

$$
\frac{\Delta_k(M)}{\Delta_{k-1}(M)}.
$$

This relation inspired H. Corrales and C. Valencia to introduce in [14] the definition of the critical ideals of a graph, which are determinantal ideals generalizing the sandpile group. Let $A(G)$ be the adjacency matrix of the graph $G$ with $n$ vertices. Let $A_X(G) = \text{diag}(x_1, \ldots, x_n) - A(G)$, where $X$ is the set $\{x_1, \ldots, x_n\}$ of $n$ indeterminates associated with the vertices of $G$. For $k \in [n]$, the $k$-th critical ideal $I_k(G)$ of $G$ is the ideal in $\mathbb{Z}[X]$ generated by the $k$-minors of the matrix $A_X(G)$. For example, the evaluation of the $k$-th critical ideal of $G$ at $X = \text{deg}(G)$ will be an ideal in $\mathbb{Z}$ generated by $\Delta_k(L(G))$.

We will use $G^*$ to denote the dual of a plane graph $G$, and the weak dual, denoted by $G_*$, is constructed the same way as the dual graph, but without placing the vertex associated with the outer face. It is known [8, 13, 27] that the sandpile group of a planar graph is isomorphic to the sandpile group of its dual. Since the dual of any plane graph is connected [10], then $K(G) \cong \text{coker}(L_k(G^*))$ and $\tau(G) = \det(L_k(G^*))$.

Despite this, the structure of the sandpile groups of the outerplanar graphs have been largely unknown. Recently, the structure of the sandpile group of some subfamilies of the outerplanar graphs were established, see for example [7, 12, 21].

In [25], C. Phifer gave a nice interpretation of this relation by introducing the cycle-intersection matrix of a plane graph as follows. Given a plane graph $G$ with $s$ interior faces $F_1, \ldots, F_s$, let $c(F_i)$ denote the length of the cycle which bounds interior face $F_i$. We define the cycle-intersection matrix, $C(G) = (c_{ij})$, to be a symmetric matrix of size $s \times s$, where $c_{ii} = c(F_i)$, and $c_{ij}$ is the negative of the number of common edges in the cycles bounding the interior faces $F_i$ and $F_j$, for $i \neq j$. This matrix differs from the fundamental circuits intersection matrix used in [11]. Note that $C(G)$ is the reduced Laplacian of $G^*$, where the column and row associated with the outer face are removed from $L(G^*)$. Therefore we have the following.

Lemma 2. Let $G$ a plane graph with $G_*$ its weak dual. Then $K(G) \cong \text{coker}(C(G))$ and $\tau(G) = \det(C(G))$.

In Section 2, we explore this relation under the lenses of the critical ideals of graphs. This gives us a methodology to compute the sandpile groups of the family $\mathcal{F}$ of plane graphs having a common weak dual by evaluating the critical ideals of the weak dual at the lengths of the cycles bounding the interior faces of any plane graph in $\mathcal{F}$. In Section 3, we use this methodology and the property that the weak dual of outerplane graphs are trees to compute the sandpile groups of outerplanar graphs, which was suggested by Chen and Mohar in [12].
2 Sandpile group of planar graphs

In this section we will introduce a procedure that can be applied to compute the sandpile groups of families of planar graphs having a common weak dual graph in terms of the critical ideals of the weak dual of a plane graph and the lengths of the cycles bounding the interior faces of a plane embedding.

The basic properties about critical ideals and determinatal ideals of graphs can be found in [1,14], and in [2] can be found other applications of the critical ideals not considered there. By convention $I_k(G) = \langle 1 \rangle$ if $k < 1$, and $I_k(G) = \langle 0 \rangle$ if $k > n$. An ideal is called trivial if it is equal to $\langle 1 \rangle$. The algebraic co-rank of $G$, denoted by $\gamma(G)$, is the number of critical ideals of $G$ equal to $\langle 1 \rangle$. It is known that if $i \leq j$, then $I_j(G) \subseteq I_i(G)$. Moreover, if $H$ is an induced subgraph of $G$, then $I_i(H) \subseteq I_i(G)$ for all $i \leq |V(H)|$ and therefore $\gamma(H) \leq \gamma(G)$.

The classic relation between critical ideals and the invariant factors of the sandpile groups of families of planar graphs having a common weak dual is given by the following results. First, we recall Lemma 3 applied to Laplacian matrix and reduced Laplacian matrix implies the next result that states the relations between critical ideals and the sandpile groups of some families of graphs.

**Lemma 3.** Let $G$ be a graph with $n$ vertices and $c \in \mathbb{Z}^n$. Let $M = cI_n - A(G)$. Then, the $k$-th critical ideal $I_k(G)$ evaluated at $X = c$ is generated by $\Delta_k(M)$, that is, the gcd of the $k$-minors of the matrix $M$. And the $k$-th invariant factor of the SNF of $M$ is $d_k(M) = \frac{\Delta_k(M)}{\Delta_{k-1}(M)}$.

Lemma 3 applied to Laplacian matrix and reduced Laplacian matrix implies that the critical ideals and the invariant factors of the sandpile groups are related.

**Proposition 4.** Let $G$ be a graph with vertex set $\{v_1, \ldots, v_n\}$. Then,

1. if $\deg(G) = (\deg_G(v_1), \ldots, \deg_G(v_n))$, then the $k$-th critical ideal of $G$ evaluated at $X = \deg(G)$ is generated by $\Delta_k(L(G))$, and $\gamma(G) \leq f_1(G)$,

2. let $H$ be the graph constructed from $G$ by adding a new vertex $v_{n+1}$, and let $m \in \mathbb{N}^n$, where $m_i$ is the number of edges between $v_{n+1}$ and $v_i$, then the $k$-th critical ideal of $G$ evaluated at $X = \deg(G) + m$ is generated by $\Delta_k(L_{n+1}(H))$, and $\gamma(G) \leq f_1(H)$.

**Proof.** It follows from Lemma 3 note that in case (1) the evaluation of $AX(G)$ at $X = \deg(G)$ equals $L(G)$. Moreover, note that $\Delta_j(L(G)) = 1$ for all $1 \leq j \leq \gamma(G)$, therefore the first $\gamma(G)$ invariant factors are 1. In case (2) the evaluation of $AX(G)$ at $X = \deg(G) + m$ equals $L_{n+1}(H)$ and similarly to case (1) we have that $f_1(H) \geq \gamma(G)$.

The next example will illustrate how the critical ideals can be used to compute the sandpile group of the family of graphs obtained from a graph $G$ by adding a new vertex $v$ with an arbitrary number of edges between $v$ and the vertices of $G$. 


Example 5. Let $H$ be the plane graph shown in Figure 1. Let $C_8$ be the cycle with 8 vertices obtained from $H$ by removing vertex $v_9$ and the edges incident to it. The algebraic co-rank of $C_8$ is 6, and for the next critical ideal we will give their Gröbner bases since we need a simple basis that describe the ideal. The Gröbner basis of the 7-th critical ideal of $G'$ is generated by the following 3 polynomials:

\[p_1 = x_1 + x_3x_4x_5x_6x_7 - x_3x_4x_5 - x_3x_4x_7 - x_3x_4x_7 + x_3 - x_5x_6x_7 + x_5 + x_7,\]
\[p_2 = x_2 + x_2x_3x_6x_7x_8 - x_4x_5x_6 - x_4x_5x_8 - x_4 - x_6x_7x_8 + x_6 + x_8,\]
\[p_3 = x_2x_4x_5x_6x_7x_8 - x_3x_4x_5x_6 - x_3x_4x_5x_8 - x_3x_4x_7x_8 + \]
\[x_3x_4 - x_3x_6x_7x_8 + x_3x_6 + x_3x_8 - x_5x_6x_7x_8 + x_5x_6 + x_5x_8 + x_7x_8.\]

The 8-th critical ideal of $C_8$ is generated by the determinant of $A_X(C_8)$:

\[
x_1x_2x_3x_4x_5x_6x_7x_8 - x_1x_2x_3x_4x_5x_6 - x_1x_2x_3x_4x_7x_8 - x_1x_2x_3x_4x_7x_8 \\
+x_1x_2x_3x_4 - x_1x_2x_3x_4x_7x_8 + x_1x_2x_3x_6 + x_1x_2x_3x_8 - x_1x_2x_3x_6x_7x_8 \\
+x_1x_2x_5x_6 + x_1x_2x_5x_8 + x_1x_2x_7x_8 - x_1x_2 - x_1x_4x_5x_6x_7x_8 + x_1x_4x_5x_6 \\
+x_1x_4x_5x_8 + x_1x_4x_7x_8 - x_1x_4 + x_1x_6x_7x_8 - x_1x_6 - x_1x_8 - x_2x_3x_4x_5x_6x_7 + \\
+2x_2x_3x_4x_5 + 2x_2x_3x_4x_7 + x_2x_3x_6x_7 - 2x_2x_3 + 2x_2x_5x_6x_7 - 2x_2x_7 - \\
-3x_4x_5x_6x_7x_8 + 3x_4x_5x_6 + 3x_4x_5x_8 + x_4x_5x_7x_8 - x_3x_4 + x_3x_6x_7x_8 - x_3x_6 \\
-3x_3x_6 + x_4x_5x_6x_7 - x_4x_5 - x_4x_7 + x_5x_6x_7x_8 - x_5x_6 - x_5x_8 - x_6x_7 - x_7x_8.
\]

In particular, by evaluating the polynomials $p_1, p_2, p_3$ and $\det(A_X(C_8))$ at $X = \deg(C_8) + (0, 1, 0, 1, 0, 1, 0, 1, 0, 1)$, we obtain $\Delta_{7}(L_9(H)) = \gcd(32, 48, 72) = 8$, and $\Delta_{8}(L_9(H)) = 192$. From which follows that the sandpile group $K(H)$ is isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_{24}$.

The Gröbner basis for the critical ideals of the complete graphs, the cycles and the paths were computed in [14]. In [15], it was given a description of a set of generators of the $k$-th-critical ideal of any tree as a function of a set of special
2-matchings. The generators of the critical ideals of other graph families have been computed in [3,5,11].

A new relation is explored next based on the cycle-intersection matrix $C(H)$ of a plane graph $H$.

**Theorem 6.** Let $G$ be a graph with vertex set $\{v_1, \ldots, v_n\}$. If $G$ is the weak dual of the plane graph $H$ and $c \in \mathbb{N}^n$ is such that $c_i$ is the length of the cycle bounding the $i$-th finite face, then the $k$-th critical ideal of $G$ evaluated at $x = c$ is generated by $\Delta_k(C(H))$. And $f_1(C(H)) \geq \gamma(G)$.

**Proof.** We have that $H = H^\ast$. Let us assume that $v_{n+1} \in H^\ast$ is the vertex that corresponds to the outer face of $H$. Then $C(H)$ is the reduced Laplacian matrix $L_{n+1}(H^\ast)$. Now, set $c = \deg(G) + m$, where $m_i$ is the number of edges between the vertex associated with the $i$-th interior face and the outer face. Thus the result follows by applying Proposition[4,2].

Let $G$ be a plane graph. Therefore by Lemma[2] and Theorem[3] the sandpile group of any plane graph $H$ having $G$ as weak dual can be obtained from the critical ideals of $G$ by evaluating the indeterminates $X = (x_1, \ldots, x_n)$ at the lengths $c = (c_1, \ldots, c_n)$ of the cycles bounding the interior faces of $H$. Also $\det(A_X(G))|_{X=c} = \tau(H)$. Let us illustrate this with the following example.

**Example 7.** Let $G$ be the graph described in the right-hand side in Figure[4]. Then

$$A_Y(G) = \begin{bmatrix} y_1 & -1 & 0 & -1 \\ -1 & y_2 & -1 & 0 \\ 0 & -1 & y_3 & -1 \\ -1 & 0 & -1 & y_4 \end{bmatrix}. $$

Since there are 2-minors in $A_Y(G)$ equal to $\pm 1$, then $\gamma(G) \geq 2$, the equality follows since the third critical ideal if $G$ is non-trivial. The Gröbner basis of $I_3(G)$ is

$$\langle y_1 + y_3, y_2 + y_4, y_3y_4 \rangle$$

Moreover, $I_4(G) = \langle \det(A_Y(G)) \rangle = \langle y_1y_2y_3y_4 - y_1y_2 + y_1y_4 - y_2y_3 - y_3y_4 \rangle$. These critical ideals can be used to compute the sandpile groups of any plane graph $H$ whose weak dual is isomorphic to $G$. Thus, we only need to evaluate the lengths of the cycles bounding the interior faces of $H$. Note that the length of the interior faces of $H$ is at least 2 and at least one of the interior faces has length at least 3. One of such cases is when all interior faces of $H$ have the same length, say $t$. Hence having $\Delta_3(C(H)) = \gcd(2t, t^2)$ and $\Delta_4(C(H)) = |t^4 - 4t^2|$. It is not difficult to see that $\Delta_3(C(H))$ is equal to $t$ whenever $t$ is odd and it is equal to $2t$ whenever $t$ is even. Therefore, if the interior faces of $H$ have the same length $t$, the sandpile group $K(H)$ of $H$ is isomorphic to $\mathbb{Z}_{\gcd(2t,t^2)} \oplus \mathbb{Z}_{|t^4 - 4t^2|}$ and $\tau(H) = |t^4 - 4t^2|$. Since $t \geq 3$, then the sandpile group of $H$ is not cyclic.
3 Sandpile groups of outerplanar graphs

We call a graph outerplanar if it has a planar embedding with the outer face containing all the vertices. An outerplanar graph equipped with such embedding is known as outerplane graph.

Lemma 8. [17] A graph $G$ is outerplanar if and only if it has a weak dual $G^*$ which is a forest.

One advantage of the outerplane graphs is that when the outerplanar has been embedded in the plane with all all the vertices lying on the outer face, then the weak dual is the union of the weak duals of the blocks of $G$.

Next result implies that we should focus in computing sandpile groups of biconnected outerplanar graphs.

Lemma 9. [29] Let $G$ be a graph with $b$ non-trivial blocks $B_1, \ldots, B_b$. Then $K(G) \cong K(B_1) \oplus \cdots \oplus K(B_b)$.

The following result is an specialization of Lemma 8.

Corollary 10. A graph $G$ is biconnected outerplane if and only if its weak dual $G^*$ is a tree.

Now we will give a description of the generators of the critical ideals of any tree $T$, which were obtained in [15] in terms of the 2-matchings of the graph $T^l$, where $T^l$ is the graph obtained from $T$ by adding a loop at each vertex of $T$.

Recall that a 2-matching is a set of edges $\mathcal{M} \subseteq E(G)$ such that every vertex of $G$ isincident to at most two edges in $\mathcal{M}$ and note that a loop counts as two incidences for its respective vertex. The set of 2-matchings of $T^l$ with $k$ edges is denoted by $2M(T^l, k)$. Given a 2-matching $\mathcal{M}$ of $T^l$, the loops $\ell(\mathcal{M})$ of $\mathcal{M}$ is the edge set $\mathcal{M} \cap \{uu : u \in V(G)\}$. A 2-matching $\mathcal{M}$ of $T^l$ is minimal if there does not exist a 2-matching $\mathcal{M}'$ of $T^l$ such that $\ell(\mathcal{M}') \subsetneq \ell(\mathcal{M})$ and $|\mathcal{M}'| = |\mathcal{M}|$. The set of minimal 2-matchings of $T^l$ will be denoted by $2M^*(T^l)$, and the set of minimal 2-matchings of $T^l$ with $k$ edges will be denoted by $2M^*_k(T^l)$. Let $d_X(\mathcal{M})$ denote $\det(A_X(T)[V(\ell(\mathcal{M}))])$, that is, the determinant of the submatrix of $A_X(T)$ formed by selecting the columns and rows associated with the loops of $\mathcal{M}$.

Lemma 11. [15] Theorem 3.7] Let $T$ be a tree with $n$ vertices. Then

$$I_k(T) = \langle \{d_X(\mathcal{M}) : \mathcal{M} \in 2M^*_k(T^l)\} \rangle,$$

for $k \in [n]$.

It follows directly from Theorem 6 and Lemma 11 that the sandpile groups of outerplanar graphs are determined in terms of the length of the cycles bounding the interior faces of their outerplane embeddings and the 2-matching of the weak dual with loops.
Theorem 12. Let $G$ be a biconnected outerplane graph whose weak dual is the tree $T$ with $n$ vertices, and let $c = (c_1, \ldots, c_n)$ be the vector of the lengths of the cycles bounding the finite faces $F_1, \ldots, F_n$. Let

$$\Delta_k = \gcd \left( \{ d_X(M) : X = c, M \in 2M^*_k(T^l) \} \right),$$

for $k \in [n]$. Then $K(G) \cong \mathbb{Z}_{\Delta_1} \oplus \mathbb{Z}_{\Delta_2} \oplus \cdots \mathbb{Z}_{\Delta_{n-1}}$ and $\tau(G) = \Delta_n$.

Let us illustrate the utility of Theorem 12 in the following

![Figure 2: An outerplane graph $G$ with 6 interior faces and its weak dual $T$.](image)

Example 13. Let $G$ be the outerplane graph in figure 2 then $G_\ast = T$. We will use Theorem 12 to compute the sandpile group of $K(G)$. We need to compute $2M^*_k(T^l)$ for $1 \leq k \leq 6$. First, note that if $T^l$ has minimal 2-matching of size $k$ without loops, then $I_k(T^l) = \langle 1 \rangle$. It is easy to see that this is the case for $k \leq 4$ and then $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$. On the other hand, for $k = 5$,

$$2M^*_5(T^l) = \left\{ \begin{array}{ll} \{(11), (22), (33), (45), (46)\}, & \{(13), (23), (44), (55), (66)\} \end{array} \right\}.$$

Therefore, by Lemma 11

$$I_5(T^l) = \langle x_1x_2x_3 - x_1 - x_2, x_4x_5x_6 - x_5 - x_6, x_1x_5, x_1x_6, x_2x_5, x_2x_6 \rangle.$$

Moreover, the 6-th critical ideal of $T$ is generated by $\det(A_X(T))$;

$$x_1x_2x_3x_4x_6x_5 - x_1x_2x_3x_5 - x_1x_2x_3x_6 - x_1x_2x_6x_5 - x_1x_4x_6x_5 - x_2x_4x_6x_5 + x_1x_5 + x_2x_5 + x_1x_6 + x_2x_6$$

Now, since $c = (3, 3, 4, 5, 3, 3)$ and by Theorem 12 $\Delta_5 = \gcd(30, 39, 9) = 3$, $\Delta_6 = 1089$ and thus $K(G) = \mathbb{Z}_3 \oplus \mathbb{Z}_{363}$. Note that we can easily compute the
sandpile group of any graph with $T$ as its weak dual, using the corresponding cycle-lengths. For instance, some allowed edge contractions or vertex splittings of $G$ as in Figure 3. Let $c_1 = (3, 3, 3, 3, 3, 3)$ and $c_2 = (3, 3, 5, 6, 3, 3)$ be the vectors of lengths of the cycles bounding the interior faces of $G_1$ and $G_2$ respectively. Then

$$K(G_1) = \mathbb{Z}_{\gcd(39, 48, 9)} \oplus \mathbb{Z}_{\frac{1791}{\gcd(39, 48, 9)}} = \mathbb{Z}_3 \oplus \mathbb{Z}_{597}$$

and

$$K(G_2) = \mathbb{Z}_{\gcd(21, 9)} \oplus \mathbb{Z}_{\frac{360}{\gcd(21, 9)}} = \mathbb{Z}_3 \oplus \mathbb{Z}_{120}.$$

Observation 14. Note that if $G$ is an outerplane graph with weak dual $G^*$. Then any subdivision of $G$ has the same weak dual. Therefore the problem of computing sandpile groups of outerplane graphs can be reduced (up to evaluations) to computing the sandpile group of a representative graph of each homeomorphism class.

In particular, if $G$ is a biconnected outerplane graph whose weak dual is the tree $T$, then $f_1(C(G)) \geq \gamma(T)$. The 2-matching number $\nu_2(G)$ is the maximum cardinality (number of edges) of a 2-matching of $G$. In [15] it was proved that $\gamma(T) = \nu_2(T)$ for any tree $T$. In [2] it was shown that $\nu_2(T) = n - \delta(T)$ for any tree $T$ on $n$ vertices, where the parameter $\delta(T)$ is defined as the maximum of $p - q$ such that by deleting $q$ vertices from $T$ the remaining graph becomes $p$ paths. In [19], it was found a linear-time algorithm for finding $\delta(T)$. From which in [2], it was concluded that there is a polynomial time algorithm to compute the algebraic co-rank for trees. Also, in [2], it was proved that for any tree $T$, $\gamma(T)$ coincides with the minimum rank $\text{mr}(T)$ of $T$ and with $\text{mz}(T) := |V(T)| - Z(T)$, where $Z(T)$ denote the zero-forcing number of $T$.

In the following the sandpile groups of some outerplanar graphs are further simplified.
3.1 Outerplane graphs whose weak dual is a path

Let us consider the outerplane graphs whose common weak dual is a path. Let \((k_1, \ldots, k_n)\) be a sequence of integers with each \(k_i \geq 2\). Define the graph \(PC_0\) to be a path of one edge, and for each \(1 \leq i \leq n\), define the graph \(PC_i\) by starting with graph \(PC_{i-1}\) and adding a path of \(k_i - 1\) edges between any two consecutive vertices of the path added at the previous step. The resulting graph \(PC_n\) will consist of a stack of polygons with \(k_1, \ldots, k_n\) sides. We call \(PC_n\) a polygon chain. Polygon chains are the outerplanar graphs having the path as a weak dual.

It is not difficult to see that \(\gamma(G) = n - 1\) if \(G\) is a path with \(n\) vertices. The opposite is also true, see [15, Corollary 3.9]. From which follows that the sandpile group of a polygon chain is cyclic. The last critical ideal \(I_n(P_n)\) of the path \(P_n\) with \(n\) vertices is generated by the determinant of \(AX(P_n)\). Next relations follow directly from the determinant of \(AX(P_n)\). These were already noticed in [7, 12, 21].

Lemma 15. Let \(P_n\) be the path with \(n\) vertices and let \(X = \{x_1, \ldots, x_n\}\) a set of indeterminates associated with the vertices of \(P_n\). Then

\[
\det(AX(P_n)) = x_n \det(AX(P_{n-1})) - \det(AX(P_{n-2}))
\]

and \(\tau(PC_n) = k_n \tau(PC_{n-1}) - \tau(PC_{n-2})\).

In [14], an explicit computation of the determinant of \(AX(P_n)\) was obtained in terms of the matchings.

Lemma 16. [14, Corollary 4.5] Let \(P_n\) be the path with \(n\) vertices. Then

\[
\det(AX(P_n)) = \sum_{\mu \in M(P_n)} (-1)^{|\mu|} \prod_{v \notin V(\mu)} x_v, \text{ where } M(P_n) \text{ is the set of matchings of } P_n.
\]

Next result follows directly from previous lemma and Theorem 6.

Theorem 17. Let \(PC_n\) be a polygon chain whose stack of polygons have \(k_1, \ldots, k_n\) sides. Then the sandpile group \(K(PC_n)\) of \(PC_n\) is cyclic of order

\[
\tau(PC_n) = \sum_{\mu \in M(P_n)} (-1)^{|\mu|} \prod_{v \notin V(\mu)} k_v,
\]

where \(M(P_n)\) is the set of matchings of \(P_n\).

3.2 Outerplane graphs whose weak dual is a starlike tree

We denote by \(S(n_1, \ldots, n_l)\) a starlike tree in which removing the central vertex leaves disjoint paths \(P_{n_1}, \ldots, P_{n_l}\) in which exactly one endpoint of each path is a leaf on \(S(n_1, \ldots, n_l)\).

Let \(C_l = v_1 e_1 v_2 e_2 \cdots v_l e_l v_1\) be a cycle of length \(l\), and \(PC_{n_1}, \ldots, PC_{n_l}\) be \(l\) polygon chains. A polygon flower \(F = F(C_l; PC_{n_1}, \ldots, PC_{n_l})\) is constructed by identifying \(e_i\) with an edge \(e_i'\) that belongs to an end-polygon of \(PC_{n_i}\) which
is not contained in another polygon of \(PC_{n_i}\), for \(i \in [l]\). The weak dual of an outerplane embedding of polygon flowers are starlike trees.

In [12], the sandpile groups the polygon flowers were obtained in terms of the spanning tree numbers of the polygon chains.

**Lemma 18.** [12, Theorem 4.3] Let \(F = F(C_l; PC_{n_1}, \ldots, PC_{n_l})\) be a polygon flower. For \(j \in [l-2]\), \(\Delta_j = \gcd(\tau(PC_{n_{i_1}}) \cdots \tau(PC_{n_{i_j}}) : 1 \leq i_1 < \cdots < i_j \leq l)\). Then

\[
K(F) = \mathbb{Z}_{\Delta_1} \oplus \mathbb{Z}_{\frac{\Delta_2}{\Delta_1}} \oplus \cdots \oplus \mathbb{Z}_{\frac{\Delta_{t-2}}{\Delta_{t-3}}} \oplus \mathbb{Z}_{\tau(F)}. 
\]

By using Lemma 18 and Theorem 17, we can obtain an equivalent result stated in terms of matchings of the path and the length of the polygons.

**Theorem 19.** Let \(F = F(C_l; PC_{n_1}, \ldots, PC_{n_l})\) be a polygon flower, where \(k_{n_1}^1, \ldots, k_{n_1}^l\) are the sizes of the polygons of \(PC_{n_i}\). Let

\[
\omega(n_i, k_{n_i}^1, \ldots, k_{n_i}^l) = \sum_{\mu \in M(P_{ni})} (-1)^{|\mu|} \prod_{v \in V(\mu)} k_{i_v}^l.
\]

For \(j \in [l-2]\), \(\Delta_j = \gcd(\omega(n_{i_1}, k_{n_1}^{i_1}, \ldots, k_{n_{i_1}}^{i_1}) \cdots \omega(n_{i_j}, k_{n_j}^{i_j}, \ldots, k_{n_{i_j}}^{i_j}) : 1 \leq i_1 < \cdots < i_j \leq k)\). Then

\[
K(F) = \mathbb{Z}_{\Delta_1} \oplus \mathbb{Z}_{\frac{\Delta_2}{\Delta_1}} \oplus \cdots \oplus \mathbb{Z}_{\frac{\Delta_{t-2}}{\Delta_{t-3}}} \oplus \mathbb{Z}_{\tau(F)}. 
\]

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