COHOMOLOGY JUMP LOCI OF QUASI-PROJECTIVE VARIETIES

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Abstract. We prove that the cohomology jump loci in the space of rank one local systems over a smooth quasi-projective variety are finite unions of torsion translates of subtori. The main ingredients are a recent result of Dimca-Papadima, some techniques introduced by Simpson, together with properties of the moduli space of logarithmic connections constructed by Nitsure and Simpson.

1. Introduction

Let $X$ be a connected, finite-type CW-complex. Define

$$M_B(X) = \text{Hom}(\pi_1(X), \mathbb{C}^*)$$

to be the variety of $\mathbb{C}^*$ representations of $\pi_1(X)$. Then $M_B(X)$ is a direct product of $(\mathbb{C}^*)^{b_1(X)}$ and a finite abelian group. For each point $\rho \in M_B(X)$, there exists a unique rank one local system $L_\rho$, whose monodromy representation is $\rho$. The cohomology jump loci of $X$ are the natural strata

$$\Sigma^i_k(X) = \{ \rho \in M_B(X) \mid \dim \mathbb{C}H^i(X, L_\rho) \geq k \}.$$ 

$\Sigma^i_k(X)$ is a Zariski closed subset of $M_B(X)$. A celebrated result of Simpson says that if $X$ is a smooth projective variety defined over $\mathbb{C}$, then $\Sigma^i_k(X)$ is a union of torsion translates of subtori of $M_B(X)$.

In this paper, we generalize Simpson’s result to quasi-projective varieties.

Theorem 1.1. Suppose $U$ is a smooth quasi-projective variety defined over $\mathbb{C}$. Then $\Sigma^i_k(U)$ is a finite union of torsion translates of subtori of $M_B(U)$.

When $U$ is compact, the theorem is proved in [GL1, GL2, A1, S2], with the strongest form appearing in the latter. When $b_1(U) = 0$, Arapura [A2] showed that $\Sigma^i_k(U)$ are union of translates of subtori. The case of unitary rank one local systems on $U$ has been considered in [B] and [L]. Libgober [L] also proved the same theorem for $U = \mathcal{X} - \mathcal{D}$ where $\mathcal{X}$ is a germ of a smooth analytic space, and $\mathcal{D}$ is a divisor of $\mathcal{X}$. Dimca and Papadima were able to prove the following:

Theorem 1.2. [DP, Theorem C] Under the same assumption as Theorem [11] every irreducible component of $\Sigma^i_k(U)$ containing $1 \in M_B(U)$ is a subtorus.

The proof of this result reduces to the study of the infinitesimal deformations with cohomology constraints of the trivial local system. These are governed in general by infinite-dimensional models. In [DP] it is shown that, in this case, the finite-dimensional Gysin model due to Morgan provides the necessary linear algebra description for the infinitesimal deformations.

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The result of Dimca and Papadima serves as a key ingredient of our theorem. In Section 2, we will show that each irreducible component of $\Sigma_k^i(U)$ contains a torsion point. Then, in Section 3, we will see that, thanks to Theorem 1.2 having a torsion point on an irreducible component of $\Sigma_k^i(U)$ forces this component to be a translate of subtorus.

There are two other proofs of Simpson’s theorem: one via positive characteristic methods [PR], and one via D-modules [Sc1, Sc2]. However, in this paper we follow the original approach of Simpson. There are no analogous results for higher rank local systems even in the projective case.

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2. **Torsion points on the cohomology jump loci**

Let $X$ be a smooth complex projective variety, and let $D = \sum_{\lambda=1}^n D_\lambda$ be a simple normal crossing divisor on $X$ with irreducible components $D_\lambda$. Let $U = X - D$. Thanks to Hironaka’s theorem on resolution of singularities, every smooth quasi-projective variety $U$ can be realized in this way. The goal of this section is to prove the following:

**Theorem 2.1.** Each irreducible component of $\Sigma_k^i(U)$ contains a torsion point.

First, we want to reduce to the case when $X$ and each $D_\lambda$ are defined over $\bar{\mathbb{Q}}$. This can be done using a technique which we have learnt from the proof of [S2 Theorem 4.1]. We reproduce it here.

We can assume $X$ and each $D_\lambda$ to be defined over a subring $O$ of $\mathbb{C}$, which is finitely generated over $\mathbb{Q}$. Denote the embedding of $O$ to $\mathbb{C}$ by $\sigma: O \to \mathbb{C}$. Each ring homomorphism $O \to \mathbb{C}$ corresponds to a point in $\text{Spec}(O)(\mathbb{C})$. Denote by $X^0$ and $D_\lambda^0$ the schemes over $\text{Spec}(O)$ which give rise to $X$ and $D_\lambda$ respectively after tensoring with $\mathbb{C}$, that is $X = X^0 \times_{\text{Spec}(O)} \text{Spec}(\mathbb{C})$ and $D_\lambda = D_\lambda^0 \times_{\text{Spec}(O)} \text{Spec}(\mathbb{C})$. By possibly replacing $O$ by $O[1/h]$ for some $h \in O$, we can assume $X^0$ and every $D_\lambda^0$ are smooth over $\text{Spec}(O)$, and all the intersections of $D_\lambda^0$’s are transverse. Since each connected component of $\text{Spec}(O)(\mathbb{C})$ contains a $\bar{\mathbb{Q}}$ point, there exists a point $P \in \text{Spec}(O)(\bar{\mathbb{Q}})$, and a continuous path from $\sigma \in \text{Spec}(O)(\mathbb{C})$ to $P$ in $\text{Spec}(O)(\mathbb{C})^{\text{top}}$. Then, according to Thom’s First Isotopy Lemma [Di Ch. 1, Theorem 3.5], $X^0(\bar{\mathbb{Q}})$ together with its strata given by the $D_\lambda^0(\mathbb{C})$, is a topologically locally trivial fibration in the stratified sense over $\text{Spec}(O)(\mathbb{C})^{\text{top}}$. In particular, letting $X'$ and $D_\lambda'$ be the corresponding fibers over $P$, transporting along the path gives an isomorphism $(X-D)^{\text{top}} \cong (X'-D')^{\text{top}}$. Recall that $\text{M}_\text{B}(U)$ and $\Sigma_k^i(U)$ depend only on the topology of $U$. Hence replacing $U = X - D$ by $U' = X' - D'$, we may assume that $X$ and each $D_\lambda$ are defined over $\mathbb{Q}$.

Next, we introduce the other side of the story, namely the logarithmic flat bundles on $(X, D)$. A logarithmic flat bundle on $(X, D)$ consists of a vector bundle $E$ on $X$, and a logarithmic connection $\nabla : E \to E \otimes \Omega_X^1(\log D)$, satisfying the integrability condition $\nabla^2 = 0$. Given a logarithmic flat bundle $(E, \nabla)$, the flat sections of $E$ on $U$ (by which we will always mean on $U^{\text{top}}$) form a local system. And conversely, given any local system $L$ on $U$ (by which, as in the introduction, we will always mean a local system on $U^{\text{top}}$), it is always obtained from some logarithmic flat bundle $(E, \nabla)$. However, different logarithmic flat bundles may give the same local system.
This correspondence between local systems on $U$ and logarithmic flat bundles on $(X, D)$ is very well understood (e.g. [D], [S1], [M]).

For a vector bundle $E$ on $X$, the structure of a logarithmic flat bundle $(E, \nabla)$ on $(X, D)$ is the same as a $\mathcal{D}_X(\log D)$-module structure on $E$, where $\mathcal{D}_X(\log D)$ is the sheaf of logarithmic differentials.

Nitsure [N] and Simpson [S3] constructed coarse moduli spaces, which are separated quasi-projective schemes, for Jordan-equivalence classes of semistable $\Lambda$-modules. This correspondence between local systems on $(X, D)$ and logarithmic flat bundles on $(X, D)$ is very well understood (e.g. [D], [S1], [M]).

On the other hand, the embedding $\mathcal{M}_{\text{DR}}(X) \hookrightarrow \mathcal{M}_{\text{DR}}(X/D)$ has infinitely many connected components. $\mathcal{M}_{\text{DR}}(X)$, $\mathcal{M}_{\text{DR}}(X/D)$, $\mathcal{M}_{B}(X)$ and $\mathcal{M}_{B}(U)$ are all algebraic groups, except $\mathcal{M}_{\text{DR}}(X/D)$ may not be of finite type.

The following diagram plays an essential role in our proof.

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\mathbb{Z}^n & \mathcal{M}_{\text{DR}}(X) & \to \mathcal{M}_{\text{DR}}(X/D) \\
\downarrow{\text{RH}} & \downarrow{\text{RH}} & \downarrow{\text{exp}} \\
0 & \to & \mathcal{M}_{B}(X) \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{M}_{B}(U) \\
\downarrow{\text{ev}} & & \downarrow \\
0 & \to & (\mathbb{C}^*)^n
\end{array}
$$

Let us first explain how the arrows are defined. Since every $\mathcal{D}_X$-module is naturally a $\mathcal{D}_X(\log D)$-module, there is a natural embedding $\mathcal{M}_{\text{DR}}(X) \hookrightarrow \mathcal{M}_{\text{DR}}(X/D)$. On the other hand, the embedding $U \hookrightarrow X$ induces a surjective map on the fundamental group $\pi_1(U) \to \pi_1(X)$. Composing this map with the representations, we have $\mathcal{M}_{B}(X) \hookrightarrow \mathcal{M}_{B}(U)$. For every rank one logarithmic flat bundle $(E, \nabla)$, taking the residue along each $D_\lambda$ is the map $\text{res}$. In other words, $\text{res}((E, \nabla)) = \{\text{res}_{D_\lambda}(\nabla)\}_{1 \leq \lambda \leq n}$. Around each $D_\lambda$, we can take a small loop $\gamma_\lambda$. The map $\text{ev}$ is the evaluation at the loops $\gamma_\lambda$. More precisely $\text{ev}(\rho) = \{\rho(\gamma_\lambda)\}_{1 \leq \lambda \leq n}$.

For the horizontal arrows, $\text{RH} : \mathcal{M}_{\text{DR}}(X) \to \mathcal{M}_{B}(X)$ is taking the monodromy representations for flat bundles. Since every logarithmic flat bundle on $(X, D)$ restricts to a flat bundle on $U$, taking the monodromy representation on $U$ is $\text{RH} : \mathcal{M}_{\text{DR}}(X/D) \to \mathcal{M}_{B}(U)$. The map $\text{exp} : \mathbb{C}^n \to (\mathbb{C}^*)^n$ is component-wise defined to be multiplying by $2\pi \sqrt{-1}$, then taking exponential. On $\mathcal{M}_{\text{DR}}(X/D)$, there are some special elements. Let $(\mathcal{O}_X, d)$ be the trivial rank one logarithmic flat bundle.
on \((X,D)\). Notice that \(\mathcal{O}_X(-D_\lambda)\) is preserved under \(d\), that is, there is an induced map \(d: \mathcal{O}_X(-D_\lambda) \to \mathcal{O}_X(-D_\lambda) \otimes \Omega_X^1(\log D)\). Therefore, \((\mathcal{O}_X(-D_\lambda), d)\) is also a logarithmic flat bundle on \((X,D)\). The map \(\mathbb{Z}^n \to \mathbb{M}_{\text{DR}}(X/D)\) is defined by \(\{m_\lambda\}_{1 \leq \lambda \leq n} \mapsto \bigotimes_{1 \leq \lambda \leq n}(\mathcal{O}_X(-D_\lambda), d)^{\otimes m_\lambda}\). The map \(\mathbb{Z}^n \to \mathbb{C}^n\) is the natural inclusion map.

Notice that all the maps are group homomorphisms, all the rows and columns are exact. The first map \(\text{RH}\) is an analytic isomorphism, since \(\mathbb{M}_{\text{DR}}(X)\) and \(\mathbb{M}_{\text{B}}(X)\) analytically represent the same functor \([\mathbb{S}]\). Similarly, the quotient \(\mathbb{M}_{\text{DR}}(X/D)/\mathbb{Z}^n\) and \(\mathbb{M}_{\text{B}}(U)\) represent the same functor from the category of analytic spaces to the category of sets. Therefore, by Yoneda’s lemma, \(\text{RH}: \mathbb{M}_{\text{DR}}(X/D) \to \mathbb{M}_{\text{B}}(U)\) is an analytic covering map with transformation group \(\mathbb{Z}^n\). The map \(\exp\) is obviously an analytic covering map.

According to the discussion following Theorem 2.1, we can assume \(X\) and each \(D_\lambda\) to be defined over \(\overline{\mathbb{Q}}\) without loss of generality. Then \(\mathbb{M}_{\text{DR}}(X/D)\) and \(\mathbb{M}_{\text{DR}}(X)\) are also defined over \(\overline{\mathbb{Q}}\). The representation varieties \(\mathbb{M}_{\text{B}}(U)\) and \(\mathbb{M}_{\text{B}}(X)\) are always defined over \(\overline{\mathbb{Q}}\). Therefore, all the horizontal arrows in the above diagram are maps defined over \(\overline{\mathbb{Q}}\). From now on, we should think of \(\mathbb{C}^n\) and \((\mathbb{C}^*)^n\) as varieties defined over \(\overline{\mathbb{Q}}\), or in other words, as \(\mathbb{A}^n = \mathbb{A}^n_{\overline{\mathbb{Q}}} \times \overline{\mathbb{Q}}\mathbb{C}\) and \((\mathbb{G}_{m,C})^n = (\mathbb{G}_{m,\overline{\mathbb{Q}}})^n \times \overline{\mathbb{Q}}\mathbb{C}\), respectively.

**Lemma 2.2.** Suppose \(Z \subset \mathbb{C}^n\) is a non-empty Zariski constructible subset defined over \(\overline{\mathbb{Q}}\). Suppose \(\exp(Z) \subset (\mathbb{C}^*)^n\) is also a a Zariski constructible subset defined over \(\overline{\mathbb{Q}}\). Then \(\exp(Z)\) contains a torsion point.

**Proof.** When \(n = 1\), this follows from the Gelfond-Schneider theorem, which says if \(\alpha\) and \(e^{2\pi i \sqrt{-1}}\) are both algebraic numbers, then \(\alpha \in \overline{\mathbb{Q}}\).

We use induction on \(n\). Suppose the lemma is true for \(\mathbb{C}^{n-1}\). Let \(p_1: \mathbb{C}^n \to \mathbb{C}^{n-1}\) and \(p_2: (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-1}\) be the projections to the first \(n-1\) factors. Then \(p_1(Z) \subset \mathbb{C}^{n-1}\) and \(p_2(\exp(Z)) \subset (\mathbb{C}^*)^{n-1}\) are both defined over \(\overline{\mathbb{Q}}\). Since \(\exp(p_1(Z)) = p_2(\exp(Z))\), by induction hypothesis, \(\exp(p_1(Z))\) contains a torsion point \(\tau\) in \((\mathbb{C}^*)^{n-1}\).

Let \(M = p_2^{-1}(\tau)\). Then \(\exp^{-1}(M)\) is a disjoint union of infinitely many copies of \(\mathbb{C}\). Choose one copy of those, which intersects with \(Z\). Denote this copy by \(N\). Since \(\tau\) is a torsion point in \((\mathbb{C}^*)^{n-1}\), \(N\) is defined by equations with \(\overline{\mathbb{Q}}\) coefficients. Consider the following diagram,

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* \\
\downarrow{\exp} & & \downarrow{\exp} \\
\mathbb{C} & \xleftarrow{q_2} & \mathbb{C}^* \\
\end{array}
\]

where \(q_1\) and \(q_2\) are projections to the last coordinates respectively. Then \(q_1\) and \(q_2\) are isomorphisms defined over \(\overline{\mathbb{Q}}\). If \(M \subset \exp(Z)\), then every torsion point in \(\mathbb{C}^*\) via \(q_2^{-1}\) gives a torsion point in \(\exp(Z)\). If \(M \not\subset \exp(Z)\), then \(M \cap \exp(Z)\) contains finitely many points. Hence, \(N \cap Z\) also contains finitely many points. In this case, let \(\sigma\) be any point in \(N \cap Z\), \(q_1(\sigma) \in \mathbb{C}\) is defined over \(\overline{\mathbb{Q}}\). On the other hand, \(\exp(\sigma)\) is a point in \(M \cap \exp(Z)\), and hence defined over \(\overline{\mathbb{Q}}\). Thus, \(q_2(\exp(\sigma)) = \exp(q_1(\sigma)) \in (\mathbb{C}^*)^{n-1}\) is defined over \(\overline{\mathbb{Q}}\). Now, the Gelfond-Schneider theorem implies that \(p_2(\exp(\sigma))\) is torsion in \(\mathbb{C}^*\). Since \(q_2(\exp(\sigma))\) is torsion in \(\mathbb{C}^*\) and \(p_2(\exp(\sigma)) = \tau\) is torsion in \((\mathbb{C}^*)^{n-1}\), \(\exp(\sigma) \in \exp(Z)\) is a torsion point in \((\mathbb{C}^*)^n\). \(\square\)
Remark 2.3. In fact, Jiu-Kang Yu has pointed out to us that, using Hilbert’s irreducibility theorem one, can prove that if $Z$ and $\exp(Z)$ are closed irreducible subvarieties defined over $\overline{Q}$, then $\exp(Z)$ is a torsion translate of subtorus. We give the proof in the appendix.

Remember that we assume that $X$ and each $D_\lambda$ are defined over $\overline{Q}$.

Lemma 2.4. Let $T$ be an irreducible component of $\Sigma^i_k(U)$. Then there exists an irreducible subvariety $S$ of $M_{\text{DR}}(X/D)$ defined over $\overline{Q}$ such that $RH(S) = T$.

Proof. For any $\rho \in M_{\overline{B}}(U)$, $RH^{-1}(\rho)$ contains all the possible extensions of $L_\rho$ to a logarithmic flat bundle over $(X, D)$. Suppose $(E, \nabla) \in RH^{-1}(L)$, and suppose $\nabla$ does not have any residue being equal to a positive integer, that is, $res((E, \nabla))$ does not have any positive integer in its coordinates. Then by a theorem of Deligne [D, II, 6.10], the hypercohomology of the algebraic de Rham complex

$$
E \otimes \Omega^*_X(log D) = [E \xrightarrow{\nabla} E \otimes \Omega^*_X(log D) \xrightarrow{\nabla} E \otimes \Omega^*_X(log D) \xrightarrow{\nabla} \cdots]
$$

computes the cohomology of the local system $L$, i.e., $H^i(X, E \otimes \Omega^*_X(log D)) \cong H^i(U, L_\rho)$.

Define the bad locus $BL \subset M_{\text{DR}}(X/D)$ to be the locus where one of the residues of $\nabla$ is a positive integer. Then $BL$ is the preimage of infinitely many hyperplanes in $\mathbb{C}^n$ via $res$. Define

$$
\Sigma^i_k(X/D) = \{(E, \nabla) \in M_{\text{DR}}(X/D) \mid \dim H^i(X, E \otimes \Omega^*_X(log D)) \geq k\}.
$$

Given any point $\rho_0$ in $\Sigma^i_k(U)$, one can always find an extension $(E_0, \nabla_0) \in M_{\text{DR}}(X/D)$ of $L_{\rho_0}$, which is not in $BL$, e.g., the Deligne extension. Then $RH(\Sigma^i_k(X/D) - BL) = \Sigma^i_k(U)$.

Now, given $T \subset \Sigma^i_k(U)$ as an irreducible component, take any point $\rho_0$ in $T$. Since $RH$ is analytically a covering map, there is a unique irreducible component $S$ of $RH^{-1}(T)$ containing the Deligne extension $(E_0, \nabla_0)$ of $L_{\rho_0}$. Since $S \nsubseteq BL$ and since $RH$ is analytically a covering map, we have $RH(S) = T$. By semicontinuity theorem, $\Sigma^i_k(X/D) \subset M_{\text{DR}}(X/D)$ is closed and defined over $\overline{Q}$. Since $S$ is an irreducible component of $\Sigma^i_k(X/D)$, $S$ is closed and defined over $\overline{Q}$. \hfill $\square$

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $T$ be an irreducible component of $\Sigma^i_k(U)$. By [DP, Lemma 9.2], $\Sigma^i_k(U)$ is defined by some Fitting ideal coming from the CW-complex structure of $U$. Thus, $\Sigma^i_k(U) \subset M_{\text{DR}}(U)$ is defined over $\mathbb{Q}$. Hence, as an irreducible component of $\Sigma^i_k(U)$, $T$ is defined over $\overline{Q}$. According to Lemma 2.4, there exists $S \subset M_{\text{DR}}(X/D)$ defined over $\overline{Q}$ such that $RH(S) = T$. Then $res(S) \subset \mathbb{C}^n$ and $ev(T) \subset (\mathbb{C}^*)^n$ are defined over $\overline{Q}$, and moreover, $\exp(res(S)) = ev(T)$. According to Lemma 2.2, $ev(T)$ contains a torsion point $\tau$.

Since $\tau \in (\mathbb{C}^*)^n$ is torsion, we can take $l \in \mathbb{Z}_+$ such that $\tau^l = 1 \in (\mathbb{C}^*)^n$. Then the image of the $l$-power map $\cdot^l : ev^{-1}(\tau) \rightarrow M_{\text{B}}(U)$ is equal to $M_{\text{B}}(X)$. Choose $\eta \in ev^{-1}(\tau)$, such that $\eta^l = 1$. Every $\xi \in RH^{-1}(\eta)$ is a $\overline{Q}$ point in $M_{\text{DR}}(X/D)$. In fact, since $\eta^l = 1$ in $M_{\text{B}}(U)$, $\xi^l$ is in the image of $\mathbb{Z}^n \rightarrow M_{\text{DR}}(X/D)$. Recall that the image of $\{m_\lambda\}_{1 \leq \lambda \leq n}$ under $\mathbb{Z}^n \rightarrow M_{\text{DR}}(X/D)$ is $\otimes_{1 \leq \lambda \leq n}(O_X(-D_\lambda), d)^{\otimes m_\lambda}$, which is clearly a $\overline{Q}$ point in $M_{\text{DR}}(X/D)$. Therefore, $\xi$, as an $l$-th root of a $\overline{Q}$ point, has to be a $\overline{Q}$ point.
Moreover, since \( RH(S) = T \),

\[
T \cap ev^{-1}(\tau) = RH(S) \cap ev^{-1}(\tau) = \left( RH(S \cap (RH \circ ev)^{-1}(\tau)) \right) = \bigcup_{\xi \in RH^{-1}(\eta)} RH\left(S \cap (\xi \cdot M_{\text{DR}}(X))\right).
\]

Each \( RH(S \cap (\xi \cdot M_{\text{DR}}(X))) \) is closed in \( M_B(U) \), and \( T \cap ev^{-1}(\tau) \) is a noetherian topological space. Hence, for some \( \xi_0 \in RH^{-1}(\eta) \), \( RH(S \cap (\xi_0 \cdot M_{\text{DR}}(X))) \) contains an irreducible component of \( T \cap ev^{-1}(\tau) \). Since \( RH(\xi_0) = \eta \), \( RH((\xi_0^{-1} \cdot S) \cap M_{\text{DR}}(X)) \) contains an irreducible component of \( \eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \). Recall that \( \eta \in ev^{-1}(\tau) \). Thus, \( \eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subseteq M_B(X) \).

Now, \( RH \) maps an irreducible component \( W \) of \( (\xi_0^{-1} \cdot S) \cap M_{\text{DR}}(X) \) to an irreducible component \( RH(W) \) of \( \eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subseteq M_B(X) \). Both of these irreducible components are defined over \( \overline{\mathbb{Q}} \). Indeed, since \( \xi_0 \) and \( S \) are defined over \( \mathbb{Q} \) in \( M_{\text{DR}}(X/D) \), and \( \eta \), \( T \), \( ev^{-1}(\tau) \) are defined over \( \overline{\mathbb{Q}} \) in \( M_B(U) \), \( (\xi_0^{-1} \cdot S) \cap M_{\text{DR}}(X) \) and \( \eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subseteq M_B(X) \) are defined over \( \overline{\mathbb{Q}} \) in \( M_{\text{DR}}(X/D) \) and \( M_B(U) \), respectively. Hence, the same is true for their irreducible components. Thus, we can apply [S2, Theorem 3.3] which says that this irreducible component \( RH(W) \) of \( \eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subseteq M_B(X) \) is a torsion translate of a subtorus. In particular, \( \eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subseteq M_B(X) \) contains a torsion point. Since \( \eta \) is also a torsion point, \( T \) must contain a torsion point. \( \square \)

3. Finite abelian covers

First, we consider a more general situation. Let \( U \) be a connected, finite-type CW-complex, and let \( M_B(U) = \text{Hom}(\pi_1(U), \mathbb{C}^*) \) be the moduli space of rank one local systems on \( U \), which is naturally an algebraic group. Suppose \( \tau \in M_B(U) \) is a torsion point. Denote the universal cover of \( U \) by \( U \), and let \( H \) be the kernel of \( \tau : \pi_1(U) \to \mathbb{C}^* \). Then \( H \) acts on \( U \) and we denote the quotient \( U/H \) by \( V \). Now, \( \langle \tau \rangle \), the subgroup of \( M_B(U) \) generated by \( \tau \), acts on \( V \), and the quotient \( V/\langle \tau \rangle = U \).

Denote this quotient by \( f : V \to U \). Composing with \( f_* : \pi_1(V) \to \pi_1(U) \), \( f \) induces a homomorphism of algebraic groups \( f^* : M_B(U) \to M_B(V) \). Under this construction, we immediately have \( f^*(\tau) = 1 \in M_B(V) \) is the identity element, i.e., \( f^*(\tau) \) maps every element in \( \pi_1(V) \) to 1. The main result of this section is the following:

**Proposition 3.1.** Fixing \( i \), suppose that for every \( k \in \mathbb{Z}_+ \), each irreducible component of \( \Sigma_k^i(V) \) containing 1 is a subtorus. Then for every \( k \in \mathbb{Z}_+ \), each irreducible component of \( \Sigma_k^i(U) \) containing \( \tau \) is a translate of subtorus.

**Proof.** Denote the order of \( \tau \) in \( M_B(X) \) by \( r \). For any local system \( L \) on \( U \),

\[
f_* f^*(L) \cong \bigoplus_{j=0}^{r-1} L \otimes_{\mathbb{C}} L_{\tau}^{\otimes j}.
\]
According to the projection formula, \( H^i(V, f^*(L)) \cong H^i(U, f_* f^*(L)) \). Therefore,

\[
\dim H^i(V, f^*(L)) = \sum_{j=0}^{r-1} \dim H^i(U, L_\tau \otimes L^\otimes_j).
\]

Let \( T \) be an irreducible component of \( \Sigma^i_k(U) \) containing \( \tau \), and let \( \rho \) be a general point in \( T \). Define \( \beta_j = \dim H^i(U, L_\rho \otimes L^\otimes_j) \), for \( 0 \leq j \leq r-1 \), and \( \beta = \sum_{0 \leq j \leq r-1} \beta_j \).

It is possible that \( T \subset \Sigma^i_{k+1}(U) \), and in this case, \( \beta > k \).

**Claim.** \( f^*(T) \) is an irreducible component of \( \Sigma^i_\beta(V) \).

**Proof of Claim.** By the definition of \( \rho \) and \( \beta \), it is clear that \( f^*(T) \subset \Sigma^i_\beta(V) \). Let \( S \) be the irreducible component of \( \Sigma^i_\beta(V) \) containing \( f^*(T) \). We want to show that \( S = f^*(T) \). Let \( \tilde{S} \) be a connected component of \( (f^*)^{-1}(S) \) containing \( T \). Since \( f^* \) is a covering map, \( \tilde{S} \) is irreducible and is a covering space of \( S \).

Suppose \( T \subsetneq \tilde{S} \). Take a general point \( \rho' \) in \( \tilde{S} \). Since \( \tilde{S} \) is irreducible, and since \( T \) is an irreducible component of \( \Sigma^i_k(U) \), we can assume \( \rho' \notin \Sigma^i_k(U) \). Therefore, \( \dim H^i(U, L_{\rho'}) < \dim H^i(U, L_\rho) \). Since \( \rho' \) is more general than \( \rho \), \( \dim H^i(U, L_{\rho'} \otimes L^\otimes_j) \leq \dim H^i(U, L_\rho \otimes L^\otimes_j) \), for every \( 1 \leq j \leq r-1 \). Thus,

\[
\sum_{j=0}^{r-1} \dim H^i(U, L_{\rho'} \otimes L^\otimes_j) < \sum_{j=0}^{r-1} \dim H^i(U, L_\rho \otimes L^\otimes_j).
\]

Now, equality (1) implies that \( \dim H^i(V, f^*(L_{\rho'})) < \beta \), and hence \( f^*(\rho') \) is not contained in \( \Sigma^i_\beta(V) \). This is a contradiction to the definition of \( \rho' \) and \( \tilde{S} \). So we have proved \( T = \tilde{S} \). Therefore, \( f^*(T) = S \) is an irreducible component of \( \Sigma^i_\beta(V) \). \( \square \)

Since \( \tau \in T \), \( f^*(T) \) contains 1. By the assumption of the theorem, \( f^*(T) \) is a subtorus in \( M_\beta(V) \). Since \( f^* \) is a covering map, obviously \( T \) must be a translate of subtorus. We finished the proof of the proposition. \( \square \)

Theorem 1.1 is a direct consequence of Theorem 2.1, Proposition 3.1, and Theorem 1.2.

4. APPENDIX

We prove the following strengthening of Lemma 2.2 pointed out to us by Jiu-Kang Yu.

**Lemma 4.1.** Suppose \( S \subset \mathbb{C}^n \) is a Zariski closed subset defined over \( \bar{\mathbb{Q}} \). Suppose \( T \subset (\mathbb{C}^*)^n \) is also a Zariski closed subset defined over \( \bar{\mathbb{Q}} \) such that \( \dim S = \dim T \) and \( \exp(S) \subset T \). Then \( T \) is a torsion translate of a subtorus.

**Proof.** First we prove the lemma for the case \( \text{codim}(S) = 1 \). Denote the projections to the first \( n-1 \) coordinates by \( p_1 : \mathbb{C}^n \to \mathbb{C}^{n-1} \) and \( p_2 : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-1} \). After a change of bases, we can assume that \( \dim(p_1(S)) = \dim(p_2(T)) = n-1 \).

Let \( \rho \in \mathbb{Q}^{n-1} \subset \mathbb{C}^{n-1} \) be a point with rational coordinates. Denote \( p_1^{-1}(\rho) \) and \( p_2^{-1}(\exp(\rho)) \) by \( A_\rho \) and \( B_\rho \), respectively. Since \( \dim(p_2(T)) = n-1 \), for a general \( \rho \), \( B_\rho \cap T \) consists of finitely many points. Since \( \exp(S) \subset T \) and \( \exp(A_\rho) = B_\rho \), we have

\[
\exp(A_\rho \cap S) \subset B_\rho \cap T.
\]
The projection to the last coordinate defines an isomorphism $A_\rho \cong \mathbb{C}$. Similarly we have $B_\rho \cong \mathbb{C}^*$. Under these isomorphisms, $A_\rho \cap S$ and $B_\rho \cap T$ are both defined over $\mathbb{Q}$. This means any point in $A_\rho \cap S$ is a $\mathbb{Q}$ point, and its image under the exponential is also a $\mathbb{Q}$ point. Now, according to Gelfond-Schneider theorem, the points in $A_\rho \cap S$ must be rational points.

We have shown that for a general $\rho \in \mathbb{C}^{n-1}$, $A_\rho \cap S$ consists of only points with rational coordinates. Suppose $S$ is defined by a polynomial $f(x_1, \ldots, x_n) = 0$ with coefficients in $\overline{\mathbb{Q}}$. Since $S$ is irreducible, $f$ is irreducible over $\mathbb{Q}$. Let $\bar{f}$ be the irreducible polynomial defined over $\mathbb{Q}$ that has $f$ as a factor over $\mathbb{Q}$. Now, for a general $\rho \in \mathbb{Q}^{n-1}$, the intersection of the zero locus of $\bar{f}$ and $A_\rho$ must contain at least one point with rational coordinates. This means that by plugging in a general $(n-1)$-tuple of rational numbers into the first $n-1$ variables, $\bar{f}(x_1, \ldots, x_n) = 0$ has at least one solution $x_n \in \mathbb{Q}$. However, by Hilbert irreducibility theorem, after plugging in such a general $(n-1)$-tuple of rational numbers, $\bar{f}$ is irreducible over $\mathbb{Q}$ as a polynomial in $x_n$. Therefore, $\bar{f}$ must be of degree one in $x_n$. Since the coordinates can be chosen generically, $\bar{f}$ itself is of degree one. Now, it is obvious that $S$ is a translate of a linear subspace defined over $\mathbb{Q}$, and $T$ is a translate of a subtorus by a torsion point.

Next, we use induction on the codimension of $S$. Suppose $\text{codim}(S) \geq 2$. We define the projections $p_1 : \mathbb{C}^n \to \mathbb{C}^{n-1}$ and $p_2 : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-1}$ as before. After a change of bases, we can assume that $\dim(p_1(S)) = \dim(S)$. Then, $\dim(p_2(T)) = \dim(T) = \dim(S)$.

Let $S' = \overline{p_1(S)}$ and $T' = \overline{p_2(T)}$ be the closures in the usual Euclidean topology. Since $p_1(S)$ and $p_2(T)$ are Zariski constructible sets, both the Zariski topology and the usual topology define the same closure. Hence $S'$ and $T'$ are defined over $\overline{\mathbb{Q}}$. Since the exponential map is continuous in the usual topology,

\[ \exp(p_1(S)) \subset \exp(p_1(S)). \]

Since $\exp(S) \subset T$ and $\exp(p_1(S)) = p_2(\exp(S))$, we have

\[ \exp(S') = \exp(p_1(S)) \subset \exp(p_1(S)) = p_2(\exp(S)) \subset p_2(T) = T'. \]

Using the induction hypothesis on the pair $S' \subset \mathbb{C}^{n-1}$ and $T' \subset (\mathbb{C}^*)^{n-1}$, we conclude that $T'$ is a torsion translate of a subtorus. Now, by choosing a torsion point of $T'$ as origin, we can identify $T'$ as $(\mathbb{C}^*)^{\dim(T)}$. Taking the connected component of $\exp^{-1}(p_2^{-1}(T'))$ containing $S$ and choosing a compatible origin on this connected component, the problem is reduced to a codimension one case, which is already solved.

\[ \square \]

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