The post-Newtonian mean anomaly advance as further post-Keplerian parameter in pulsar binary systems.

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Abstract

The post-Newtonian gravitoelectric secular rate of the mean anomaly $\dot{M}$ is worked out for a two-body system in the framework of the General Theory of Relativity. The possibility of using such an effect, which is different from the well known decrease of the orbital period due to gravitational wave emission, as a further post-Keplerian parameter in binary systems including one pulsar is examined. The resulting effect is almost three times larger than the periastron advance $\dot{\omega}$. E.g., for the recently discovered double pulsar system PSR J0737-3039 A+B it would amount to $-47.79$ deg yr$^{-1}$. This implies that it could be extracted from the linear part of a quadratic fit of the orbital phase because the uncertainties both in the linear drift due to the mean motion and in the quadratic shift due to the gravitational wave are smaller. The availability of such additional post-Keplerian parameter would be helpful in further constraining the General Theory of Relativity, especially for such systems in which some of the other post-Keplerian parameters can be measured with limited accuracy. Moreover, also certain pulsar-white dwarf binary systems, characterized by circular orbits like PSR B1855+09 and a limited number of measured post-Keplerian parameters, could be used for constraining competing theories of gravity.

1 The post-Newtonian rate of the mean anomaly

According to the Einsteinian General Theory of Relativity (GTR), the post-Newtonian gravitoelectric two-body acceleration of order $O(c^{-2})$ (1PN) is, in the post-Newtonian centre of mass frame (see [3] and, e.g., [12] and references therein)

$$a_{GE} = \frac{Gm}{c^2 r^3} \left\{ \frac{Gm}{r} (4 + 2\nu) - (1 + 3\nu)v^2 + \frac{3\nu}{2r^2} (r \cdot v)^2 \right\} r + (r \cdot v)(4 - 2\nu)v,$$

(1)
where \( r \) and \( v \) are the relative position and velocity vectors, respectively, \( G \) is the Newtonian constant of gravitation, \( c \) is the speed of light, \( m_1 \) and \( m_2 \) are the rest masses of the two bodies, \( m \equiv m_1 + m_2 \) and \( v \equiv m_1 m_2 / m^2 < 1 \).

The orbital phase can be characterized by the mean anomaly \( M \) defined as

\[
M = n(t - T_0),
\]

where the unperturbed mean motion \( n \) is defined as

\[
n = \frac{2\pi}{P_b}.
\]

\( P_b \) is the anomalistic period, i.e. the time elapsed between two consecutive pericentre crossings, which is

\[
2\pi\sqrt{\frac{a^3}{Gm}}
\]

for an unperturbed Keplerian ellipse, and \( T_0 \) is the date of a chosen pericentre passage.

The variation of the mean anomaly can be written, in general, as

\[
\frac{dM}{dt} = n - 2\pi \left( \frac{\dot{P}_b}{P_b^2} \right) (t - T_0) - \frac{2\pi}{P_b} \left( \frac{dT_0}{dt} \right).
\]

The second term of the right-hand side of (4) accounts for any possible variation of the anomalistic period. The third term, induced by any small perturbing acceleration with respect to the Newtonian monopole, whether relativistic or not, is the change of the time of the pericentre passage, which we will define as

\[
\frac{d\xi}{dt} = -\frac{2\pi}{P_b} \left( \frac{dT_0}{dt} \right).
\]

It can be calculated with the aid of the Gauss perturbative equation [13]

\[
\frac{d\xi}{dt} = -\frac{2\pi}{naR} \left( \frac{r}{a} \right) - \sqrt{1 - e^2} \left( \frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt} \right),
\]

where \( R \) is the radial component of the disturbing acceleration, \( a, e, i, \Omega \) and \( \omega \) are the semimajor axis, the eccentricity, the inclination, the longitude of the ascending node and the argument of pericentre, respectively, of the orbit. In order to obtain the secular effects, we must evaluate the right-hand-side of (6) on the unperturbed Keplerian ellipse and, then, average the result over one orbital revolution. In regard to the physical interpretation of \( d\xi/dt \), the apsidal line can be crossed at a different time with respect to

\footnote{For a different approach based on a modified form of the Lagrange planetary equations see [2].}
the unperturbed Keplerian case either because there is an additional radial acceleration which accelerates/decelerates the moving particle or because the pericentre is no more fixed in space. The first case is accounted for by the first term of the right-hand side of (6), while the second term of the right-hand side of (6) explains the second case. Indeed, for, e.g., an inward radial acceleration which increases the gravitational strength the particle moves more rapidly and the pericentre, assumed fixed in space, is reached in advance with respect to the unperturbed case: \( \frac{dT_0}{dt} < 0 \). If the angle \( \omega + \Omega \cos i \), which defines an angular variable around the axis of the orbital angular momentum, increases, the passage at pericentre will occur later with respect to the unperturbed case: \( \frac{dT_0}{dt} > 0 \).

We will now consider \( \frac{d}{dt} \) as perturbing acceleration. Let us start with the first term of the right-hand-side of (6). By defining

\[
\begin{align*}
A & \equiv \frac{(Gm)^2}{c^2}(4 + 2\nu), \\
B & \equiv -\frac{Gm}{c^2}(1 + 3\nu), \\
C & \equiv \frac{Gm}{c^2}(4 - \frac{\nu}{2}),
\end{align*}
\]

it is possible to obtain from (6)

\[
R_{GE} = \frac{A}{r^3} + B \left( \frac{v^2}{r^2} \right) + C \left( \frac{\dot{r}}{r^2} \right).
\]

Now the term \(-2Rr/na^2\), with \( R \) given by (8), must be evaluated on the unperturbed Keplerian ellipse characterized by

\[
\begin{align*}
r & = \frac{a(1-e^2)}{1+e \cos f}, \\
\dot{r} & = \frac{nae \sin f}{\sqrt{1-e^2}}, \\
v^2 & = \frac{n^2a^2}{(1-e^2)}(1 + e^2 + 2e \cos f)
\end{align*}
\]

where \( f \) is the true anomaly, and averaged over one orbital period by means of

\[
\frac{dt}{P_b} = \frac{r^2df}{2\pi a^2\sqrt{1-e^2}}.
\]

Thus,

\[
- \left( \frac{2}{na} R_{GE} \right) \left( \frac{dt}{P_b} \right) = -\frac{1}{na^4\pi \sqrt{1-e^2}} \left[ A + Brv^2 + Cr(\dot{r})^2 \right] df.
\]
In the expansion of $r$ in (11) the terms of order $\mathcal{O}(e^4)$ are retained. The final result is
\[
\left\langle -\frac{2}{na}R_{GE}\frac{r}{a} \right\rangle_{P_b} = \frac{nGm}{c^2a\sqrt{1 - e^2}}H(e; \nu),
\]  
(12)

with
\[
H \simeq 2(4 + 2\nu) + (1 + 3\nu)\left(2 + e^2 + \frac{e^4}{4} + \frac{e^6}{8}\right) - \left(4 - \frac{\nu}{2}\right)\left(e^2 + \frac{e^4}{4} + \frac{e^6}{8}\right).
\]  
(13)

The post-Newtonian gravitoelectric secular rate of pericentre is independent of $\nu$ and is given by the well known formula
\[
\left.\frac{d\omega}{dt}\right|_{GE} = \frac{3nGm}{c^2a(1 - e^2)},
\]  
(14)

while there are no secular effects on the node.

The final expression for the post-Newtonian secular rate of the mean anomaly is obtained by combining (12)-(14) and by considering that, for a two-body system, it is customarily to write
\[
\frac{nGm}{c^2} = \left(\frac{P_b}{2\pi}\right)^{-5/3}(T_\odot M)^{2/3},
\]  
(15)

where $M = m/M_\odot$ is the sum of the masses in units of solar mass and $T_\odot = GM_\odot/c^3 = 4.925490947 \times 10^{-6}$ s. It is
\[
\left.\frac{d\xi}{dt}\right|_{GE} = -9\left(\frac{P_b}{2\pi}\right)^{-5/3}(T_\odot M)^{2/3}(1 - e^2)^{-1/2}F(e; \nu)
\]  
(16)

with
\[
F = \left[\left(1 + \frac{e^2}{3} + \frac{e^4}{12} + \frac{e^6}{24} + \ldots\right) - \frac{2}{9}\nu\left(1 + \frac{7}{4}e^2 + \frac{7}{16}e^4 + \frac{7}{32}e^6 + \ldots\right)\right].
\]  
(17)

Note that (16) is negative because (17) is always positive; thus the crossing of the apsidal line occurs at a later time with respect to the Kepler-Newton case.

Note that, for $\nu \to 0$, i.e. $m_1 \ll m_2$, (16) does not vanish and, for small eccentricities, it becomes
\[
\left.\frac{d\xi}{dt}\right|_{GE} \simeq -\frac{9nGm_2}{c^2a\sqrt{1 - e^2}}\left(1 + \frac{e^2}{3}\right),
\]  
(18)
which could be used for planetary motion in the Solar System [7]. E.g., for Mercury it yields a secular effect of almost -130 arcsec cy\(^{-1}\). It is important to note that the validity of the present calculations has also been numerically checked by integrating over 200 years the Jet Propulsion Laboratory (JPL) equations of motion of all the planets of the Solar System with and without the gravitoelectric 1/c\(^2\) terms in the dynamical force models [5] in order to single out just the post-Newtonian gravitoelectric effects. They fully agree with (18) (E.M. Standish, private communication, 2004). Another analytical calculation of the post-Newtonian general relativistic gravitoelectric secular rate of the mean anomaly was performed [14] in the framework of the Lagrangian perturbative scheme for a central body of mass \(M\)-test particle system. The author of [14] starts from the space-time line element of the Schwarzschild metric written in terms of the Schwarzschild radial coordinate \(r'\). Instead, (1) and the equations of motion adopted in the practical planetary data reduction at, e.g. JPL, are written in terms of the standard isotropic radial coordinate \(r\) related to the Schwarzschild coordinate by \(r' = r(1 + GM/2c^2r)^2\). As a consequence, the obtained exact expression

\[
\frac{d\xi}{dt}\bigg|_{\text{GE}}^{\text{(Rubincam)}} = \frac{3nGM}{c^2a\sqrt{1 - e^2}},
\]

contrary to the pericentre case, agrees neither with (18) nor with the JPL numerical integrations yielding, e.g., a secular advance of +42 arcsec cy\(^{-1}\) for Mercury. For a better understanding of such comparisons, let us note that both the numerical analysis by Standish and of the author of [14] are based on the \(\dot{P}_b = 0\) case; \(n\) gets canceled by construction in the Standish calculation, while in [14] the numbers are put just into (19), which is the focus of that work.

### 1.1 Testing gravitational theories with binary pulsars

In general, in the pulsar’s timing data reduction process\(^2\) five Keplerian orbital parameters and a certain number of post-Keplerian parameters are determined with great accuracy in a phenomenological way, independently of any gravitational theory [19, 16]. The Keplerian parameters are the projected semimajor axis \(x = a\sin i/c\), where \(i\) is the angle between the plane of the sky, which is normal to the line of sight and is assumed as reference plane, and the pulsar’s orbital plane, the eccentricity \(e\), the orbital period \(P_b\), the time of periastron passage \(T_0\) and the argument of periastron \(\omega_0\) at

\(^2\)For all general aspects of the binary pulsar systems see [19, 16] and references therein.
the reference time $T_0$. The most commonly used post-Keplerian parameters are the periastron secular advance $\dot{\omega}$, the combined time dilation and gravitational redshift due to the pulsar’s orbit $\gamma$, the variation of the anomalistic period $\dot{P}_b$, the range $r$ and the shape $s$ of the Shapiro delay. These post-Keplerian parameters are included in the timing models \cite{19,16} of the so-called Roemer, Einstein and Shapiro $\Delta_R, \Delta_E, \Delta_S$ delays\footnote{For the complete expression of the timing models including, e.g., also the delays occurring in the Solar System due to the solar gravity see \cite{19,16}.} occurring in the binary pulsar system\footnote{The aberration parameters $\delta_r$ and $\delta_\theta$ are not, in general, separately measurable.}.

\[
\begin{align*}
\Delta_R &= x \sin \omega [\cos E - e(1 + \delta_r)] + x \cos \omega \sin E \sqrt{1 - e^2(1 + \delta_\theta)^2}, \\
\Delta_E &= \gamma \sin E, \\
\Delta_S &= -2r \ln \{1 - e \cos E - s[\sin \omega (\cos E - e) + \sqrt{1 - e^2 \cos \omega \sin E}]\},
\end{align*}
\]

where $E$ is the eccentric anomaly defined as $E - e \sin E = M$. $\cos E$ and $\sin E$ appearing in (20) can be expressed in terms of $M$ by means of the following elliptic expansions \cite{17}

\[
\begin{align*}
\cos E &= -\frac{e}{2} + \sum_{j=1}^{\infty} \frac{2}{j^2} \frac{\gamma}{\pi} [J_j(e)] \cos(jM), \\
\sin E &= \frac{2}{e} \sum_{j=1}^{\infty} \frac{J_j(e)}{j} \sin(jM),
\end{align*}
\]

where $J_j(y)$ are the Bessel functions defined as

\[
\pi J_j(y) = \int_0^\pi \cos(j\theta - y \sin \theta) d\theta.
\]
\[
\begin{align*}
\dot{\omega} &= 3 \left( \frac{P_{\text{b}}}{2\pi} \right)^{-5/3} (T_\odot M)^{2/3} (1 - e^2)^{-1}, \\
\gamma &= e \left( \frac{P_{\text{b}}}{2\pi} \right)^{1/3} T_\odot^{2/3} M^{-4/3} m_c (m_p + 2m_c), \\
\dot{P}_b &= -\frac{192\pi}{5} T_\odot^{5/3} \left( \frac{P_{\text{b}}}{2\pi} \right)^{-5/3} \frac{\left( 1 + \frac{7}{24} e^2 + \frac{37}{96} e^4 \right) m_p m_c}{(1 - e^2)^{7/4}} m_c, \\
r &= T_\odot m_c, \\
s &= x T_\odot^{-1/3} \left( \frac{P_{\text{b}}}{2\pi} \right)^{-2/3} \frac{M^{2/3}}{m_c}
\end{align*}
\]  

(23)

It is important to note that the relativistic expression of \( \dot{P}_b \) in (23), should not be confused with \( \dot{\xi}_{\text{GE}} \) of (16). Indeed, it refers to the shrinking of the orbit due to gravitational wave emission which vanishes in the limit \( \nu \to 0 \), contrary to (16) which expresses a different, independent phenomenon. The measurement of two post-Keplerian orbital parameters allows to determine \( m_p \) and \( m_c \), assumed the validity of a given theory of gravity\(^5\). Such values can, then, be inserted in the analytical expressions of the remaining post-Keplerian parameters. If the so obtained values are equal to the measured ones, or the curves for the \( 2 + N \), with \( N \geq 1 \), measured post-Keplerian parameters in the \( m_p - m_c \) plane all intersect in a well determined \((m_p, m_c)\) point, the theory of gravity adopted is consistent. So, in order to use the pulsar binary systems as valuable tools for testing GTR the measurement of at least three post-Keplerian parameters is required. The number of post-Keplerian parameters which can effectively be determined depends on the characteristics of the particular binary system under consideration. For the pulsar-neutron star PSR B1913+16 system\(^6\) the three post-Keplerian parameters \( \dot{\omega}, \gamma \) and \( \dot{P}_b \) were measured with great accuracy. For the pulsar-neutron star PSR B1534+12 system\(^7\) the post-Keplerian parameters reliably measured are \( \dot{\omega}, \gamma, r \) and \( s \). For the recently discovered pulsar-pulsar PSR J0737-3039 A+B\(^8\) system the same four post-Keplerian parameters as for PSR B1534+12 are available plus \( \dot{P}_b \) and a further constraint on \( m_p/m_c \) coming from the measurement of both the projected semimajor axes. On the contrary, in the pulsar-white dwarf binary systems, which are the majority of the binary systems with one pulsar and present almost circular orbits, it

\(^5\)This would still not be a test of the GTR because the masses must be the same for all the theories of gravity, of course.
is often impossible to measure $\dot{\omega}$ and $\gamma$. Up to now, only $r$ and $s$ have been measured, with a certain accuracy, in the PSR B1855+09 system [8], so that it is impossible to use its data for testing the GTR as previously outlined.

1.2 The secular decrease of the mean anomaly and the binary pulsars

Let us investigate the magnitude of the mean anomaly precession in some systems including one or two pulsars.

For PSR B1913+16 we have [18] $m_p = 1.4414M_\odot$, $m_c = 1.3867M_\odot$, $e = 0.6171338$, $P_b = 0.322997448930$ d. Then, $\nu = 0.2499064$, $F = 0.04459537192$ and $\dot{\xi}\big|_{GE} = -10.422159$ deg yr$^{-1}$. For PSR J0737-3039 A we have [10] $m_p = 1.337M_\odot$, $m_c = 1.250M_\odot$, $e = 0.087779$, $P_b = 0.102251561$ d, so that $\nu = 0.249721953643$, $F = 0.946329857430$. Thus, $\dot{\xi}\big|_{GE} = -47.79$ deg yr$^{-1}$.

This implies that the ratio of the post-Newtonian gravitoelectric secular rate of the mean anomaly to the mean motion amounts to $\sim 10^{-5}$. Let us see if such post-Newtonian shift is detectable from quadratic fits of the orbital phases of the form $\mathcal{M} = a_0 + b_0 t + c_0 t^2$. For PSR B1913+16 the quadratic advance due to the gravitational wave emission over thirty years amounts to $(\dot{P}_b = -2.4184 \times 10^{-12})$

$$\Delta \mathcal{M} = -\pi \frac{\dot{P}_b}{P_b^2} (t - T_0)^2 = 0.5 \text{ deg}, \quad (24)$$

with an uncertainty $\delta(\Delta \mathcal{M})$ fixed to 0.0002 deg by $\delta \dot{P}_b = 0.0009 \times 10^{-12}$.

The linear shift due to [16] amounts to

$$\Delta \mathcal{M} = \dot{\xi}\big|_{GE} (t - T_0) = -312.6647 \text{ deg} \quad (25)$$

over the same time interval. The uncertainty in $n$ amounts to $1 \times 10^{-9}$ deg yr$^{-1}$ due to $\delta \dot{P}_b = 4 \times 10^{-12}$ d. For PSR J0737-3039 A the gravitational wave emission over three years $(\dot{P}_b = -1.20 \times 10^{-12})$ induces a quadratic shift of 0.008 deg, with an uncertainty $\delta(\Delta \mathcal{M})$ fixed to 0.0005 deg by $\delta \dot{P}_b = 0.08 \times 10^{-12}$. The linear shift due to [16] amounts to -143.3700 deg over the same time interval. The uncertainty in $n$ amounts to $7 \times 10^{-7}$ deg yr$^{-1}$ due

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\[^6\] The sum of the masses and the semimajor axis entering $n$ amounts to timing data processing independently of $\mathcal{M}$ itself, e.g. from the periastron rate and the projected barycentric semimajor axis.
to $\delta P_b = 2 \times 10^{-10}$ d. Thus, it should be possible to extract $\dot{\xi}|_{GE}$ from the measured coefficient $b_0$; both the corrupting bias due to the uncertainties in the quadratic signature and the errors in $n$ would be negligible.

Measuring $\dot{\xi}|_{GE}$ as a further post-Keplerian parameter would be very useful in those scenarios in which some of the traditional post-Keplerian parameters are known with a modest precision or, for some reasons, cannot be considered entirely reliable\footnote{The measured value of the derivative of the orbital period $\dot{P}_b$ is aliased by several external contributions which often limit the precision of the tests of competing theories of gravity based on this post-Keplerian parameter\cite{15}.}. E.g., in the double pulsar system PSR J0737-3039 A+B the parameters $r$ and $\gamma$ are measured with a relatively low accuracy\cite{11}. Moreover, there are also pulsar binary systems in which only the periastron rate has been measured\cite{9}: in this case the knowledge of another post-Keplerian parameter would allow to determine the masses of the system, although it would not be possible to constraint alternative theories of gravity.

2 Conclusions

In this paper we have analytically derived for a two-body system in eccentric orbits the secular variation $\dot{\xi}|_{GE}$ yielding the post-Newtonian general relativistic gravitoelectric part of the precession of the mean anomaly not due to the variation of the orbital period. In the limit of small eccentricities and taking the mass of one of the two bodies negligible, our results have been compared to the outcome of a numerical integration of the post-Newtonian general relativistic gravitoelectric equations of motion of the planets of the Solar System performed by JPL: the agreement between the analytical and numerical calculation is complete. Subsequently, we have investigated the possibility of applying the obtained results to the binary systems in which one pulsar is present. In particular, it has been shown that the variation of the orbital period $\dot{P}_b|_{gw}$ due to gravitational wave emission and the effect derived by us are different ones. Indeed, the post-Newtonian gravitoelectric precession of the mean anomaly, which is always negative, is related to the secular increase of the time of pericentre passage and occurs even if the orbital period does not change in time. A quadratic fit of the orbital phase of the pulsar would allow to measure $\dot{\xi}|_{GE}$ because the biases due to the errors in the quadratic shift due to $\dot{P}_b$ and in the linear shift due of the mean mo-
tion $n$ are smaller. The use of $\dot{\xi}_{GE}$ as a further post-Keplerian parameter would allow to improve and enhance the tests of post-Newtonian gravity especially for those systems in which only few post-Keplerian parameters can be reliably measured.

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