On Secure Network Coding with Nonuniform or Restricted Wiretap Sets

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Abstract

The secrecy capacity of a network, for a given collection of permissible wiretap sets, is the maximum rate of communication such that observing links in any permissible wiretap set reveals no information about the message. This paper considers secure network coding with nonuniform or restricted wiretap sets, for example, networks with unequal link capacities where a wiretapper can wiretap any subset of \( k \) links, or networks where only a subset of links can be wiretapped. Existing results show that for the case of uniform wiretap sets (networks with equal capacity links/packets where any \( k \) can be wiretapped), the secrecy capacity is given by the cut-set bound, and can be achieved by injecting \( k \) random keys at the source which are decoded at the sink along with the message. This is the case whether or not the communicating users have information about the choice of wiretap set. In contrast, we show that for the nonuniform case, the cut-set bound is not achievable in general when the wiretap set is unknown, whereas it is achievable when the wiretap set is made known. We give achievable strategies where random keys are canceled at intermediate non-sink nodes, or injected at intermediate non-source nodes. Finally, we show that determining the secrecy capacity is a NP-hard problem.

Index Terms

Secrecy capacity, network coding, information-theoretic security, cut-set bound, network interdiction, NP-hard.

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I. Introduction

Information-theoretically secure communication uses coding to ensure that an adversary that wiretaps a subset of network links obtains no information about the secure message. The secrecy capacity of a network, for a given collection of permissible wiretap sets, is defined as the maximum rate of communication such that any one of the permissible wiretap sets reveals no information about the message. In general, the choice of wiretap set is unknown to the communicating users, though we also discuss the case of known wiretap set where the encoding and decoding functions are allowed to depend on the choice of wiretap set, in which case the secrecy capacity is the maximum rate achievable under the worst case wiretap set.

A theoretical basis for information-theoretic security was given in the seminal paper by Wyner [1] using Shannon’s notion of perfect secrecy [2], where a coset coding scheme based on a linear maximum distance separable code was used to achieve security for a wiretap channel. More recently, information-theoretic security has been studied in networks with general topologies. The secure network coding problem, where a wiretapper observes an unknown set of links, was introduced by Cai and Yeung [3]. They proposed a coding strategy, which we refer to as the global key strategy, in which the source injects random key symbols that are decoded at the sink along with the message. They showed achievability of this strategy in the nonuniform case where a wiretapper can observe one of an arbitrary given collection of wiretap link sets, and optimality of this strategy for multicast in the uniform case where each link has equal capacity and a wiretapper can observe up to $k$ links. For the uniform case, various constructions of secure linear network codes have been proposed in e.g. [4], [5]. Other related work on secure network communication includes weakly secure codes [6] and wireless erasure networks [7].

In this paper, we consider secure communication over wireline networks in the nonuniform case. In the case of throughput optimization without security requirements, the assumption that all links have unit capacity is made without loss of generality, since links of larger capacity can be modeled as multiple unit capacity links in parallel. However, in the secure communication problem, such an assumption cannot be made without loss of generality. Indeed, we show in this paper that there are significant differences between the uniform and nonuniform cases. For the case of uniform wiretap sets, the multicast secrecy capacity is given by the cut set bound, whether or not the choice of wiretap set is known, and is achieved by the global key strategy [3]. In contrast, the nonuniform case is more complicated, even for a single source and sink. We
show that the secrecy capacity is not the same in general when the location of the wiretapped
links is known or unknown. We give new achievable strategies where random keys are canceled
at intermediate non-sink nodes or injected at intermediate non-source nodes, and show that these
strategies can outperform the global key strategy. Finally, we show that determining the secrecy
capacity is an NP-hard problem.

II. NETWORK MODEL AND PROBLEM FORMULATION

In this paper we focus on acyclic graphs for simplicity; we expect that our results can be
generalized to cyclic networks using the approach in [8], [9] of working over fields of rational
functions in an indeterminate delay variable.

We model a wireline network by a directed acyclic graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set
and $\mathcal{E}$ is the directed link set. Each link $(i, j) \in \mathcal{E}$ is a noise-free bit-pipe with a given capacity
$c_{i,j}$. We denote the set of incoming links $(w, v)$ of a node $v$ by $\mathcal{I}(v)$ and the set of outgoing
links $(v, w)$ of $v$ by $\mathcal{O}(v)$.

A source node $s \in \mathcal{V}$ observes a random source process $X_s$ taking values from a discrete
alphabet $\mathcal{X}_s$. A sink node $d \in \mathcal{V}$ wishes to reconstruct $X_s$ with probability of error going to zero
with the coding blocklength.

An eavesdropper can wiretap a set $\mathcal{A}$ of links chosen from a known collection $\mathcal{W}$ of possible
wiretap sets. Without loss of generality we can restrict our attention to maximal wiretap sets,
i.e. no set in $\mathcal{W}$ is a subset of another. The choice of wiretap set $\mathcal{A}$ is unknown to the
communicating nodes, except where otherwise specified in this paper. In the case of known
wiretap set, the wiretapper can choose an arbitrary wiretap set $\mathcal{A}$ in $\mathcal{W}$ which is then revealed
to the communicating nodes.

A block code of blocklength $n$ is defined by a mapping

$$f_e^{(n)}: \mathcal{X}_s^n \rightarrow \{1, \ldots, 2^{n c_e}\}, e \in \mathcal{O}(s)$$

from $\mathcal{X}_s^n$ to the vector transmitted on each outgoing link $e$ of the source $s$, a mapping

$$f_e^{(n)}: \prod_{d \in \mathcal{I}(v)} \{1, \ldots, 2^{n c_d}\} \rightarrow \{1, \ldots, 2^{n c_e}\}, e \in \mathcal{O}(v)$$

from the vectors received by a non-source node $v$ to the vectors transmitted on each outgoing
link $e$ of $v$, and a mapping

$$g_d^{(n)}: \prod_{d \in \mathcal{I}(d)} \{1, \ldots, 2^{n c_d}\} \rightarrow \mathcal{X}_s^n$$
from the vectors received by the sink $d$ to the decoded output. Node mappings are applied in topological order; each node receives input vectors from all its incoming links before applying the mappings corresponding to its outgoing links.

The secrecy capacity is defined as the highest possible source-sink communication rate for which there exists a sequence of block codes such that the probability of decoding error at the sink goes to zero and, for any choice of $A \in \mathcal{W}$, the message communicated is information theoretically secret, i.e. has zero mutual information with the wiretapper’s observations.

In Section III we give a cut set bound and achievable strategies for this general problem. In Sections IV and V, we show that the cut set bound is unachievable and that finding the secrecy capacity is NP hard, even for the following special cases:

1) Scenario 1 is a wireline network with equal link capacities, where the wiretapper can wiretap an unknown subset of $k$ links from a known collection of vulnerable network links.

2) Scenario 2 is a wireline network with unequal link capacities, where the wiretapper can wiretap an unknown subset of $k$ links from the entire network.

It is convenient to show these results for Scenario 1 first, and then show the corresponding results for Scenario 2, by converting the Scenario 1 networks considered into corresponding Scenario 2 networks for which the same result holds.

Although, for the sake of simplicity, we focus on single-source single-sink networks, the cut-set bound and strategy 2 in Section III can be easily extended to multicast networks, whereas the results discussed in Sections IV and V directly apply to both multicast and non-multicast cases since the single-source single-sink case represents a special case for both.

III. CUT-SET BOUND AND ACHIEVABLE STRATEGIES

In this section, we consider the general wireline problem with unequal link capacities where the eavesdropper can wiretap an unknown set $A$ of links chosen from a known collection $\mathcal{W}$ of possible wiretap sets. We state a cut-set upper bound on capacity, and give two new achievable strategies and examples in which they outperform the existing global key strategy. Using the combined intuition from these examples, we show in Section IV that the cut-set bound is unachievable in general. One of the achievable strategies is used in Section V to show that finding the secrecy capacity in general is NP-hard.
A. Cut-Set Bound

Let $S^c$ denote the set complement of a set $S$. A cut for $x, y \in V$ is a partition of $V$ into two sets $V_x$ and $V_x^c$ such that $x \in V_x$ and $y \in V_x^c$. For the $x - y$ cut given by $V_x$, the cut-set $[V_x, V_x^c]$ is the set of links going from $V_x$ to $V_x^c$, i.e.,

$$[V_x, V_x^c] = \{(u, v) | (u, v) \in E, u \in V_x, v \in V_x^c\}. \quad (1)$$

**Theorem 1:** Consider a network of point-to-point links, where link $(i, j)$ has capacity $c_{i,j}$. The secrecy capacity is upper bounded by

$$\min_{\{V_s: V_s \text{ is an } s-d \text{ cut}\}} \min_{A \in \mathcal{W}} \sum_{(i,j) \in [V_s, V_s^c] \cap A^c} c_{i,j}. \quad (2)$$

This bound applies whether or not the communicating nodes have knowledge of the chosen wiretap set $A$.

**Proof:** Consider any source-sink cut $V_s$ and any wiretap set $A \in \mathcal{W}$. Denote by $X$ the transmitted signals from nodes in $V_s$ over links in $[V_s, V_s^c]$ and denote by $Y$ and $Z$ the observed signals from links in $[V_s, V_s^c]$ and in $[V_s, V_s^c] \cap A$ respectively. We consider block coding with block length $n$ and secret message rate $R_s$. By the perfect secrecy requirement $H(M|Z^n) = H(M)$ we have

$$nR_s \leq H(M|Z^n)$$

$$\leq H(M|Z^n) - H(M|Y^n) + n\epsilon_n$$

$$= H(M|Z^n) - H(M|Y^n, Z^n) + n\epsilon_n$$

$$= I(M; Y^n|Z^n) + n\epsilon_n \quad (a)$$

$$\leq I(X^n; Y^n|Z^n) + n\epsilon_n$$

$$= H(X^n|Z^n) - H(X^n|Y^n, Z^n) + n\epsilon_n$$

$$\leq H(X^n|Z^n) + n\epsilon_n$$

$$\leq n \sum_{(i,j) \in [V_s, V_s^c] \cap A^c} c_{i,j} + n\epsilon_n, \quad (b)$$

where $\epsilon_n \to 0$ as $n \to +\infty$ and

(a) is due to Fano’s inequality;

(b) is due to the data processing inequality and the fact that $M \to X^n \to Y^n \to Z^n$ forms a Markov chain;
If the choice of wiretap set $\mathcal{A}$ is known to the communicating nodes, the cut-set bound in this case is also (2), which is achievable using a network code that does not send any flow on links in $\mathcal{A}$. In contrast, we show in Section [IV] that the cut-set bound is not achievable in general when the wiretap set $\mathcal{A}$ is unknown.

B. Achievable Strategies for Unknown Wiretap Set

In the case of unit link capacities, the secrecy capacity can be achieved using global keys generated at the source and decoded at the sink [3]. The source transmits $R_s$ secret information symbols and $R_w$ random key symbols, where $R_s + R_w$ is equal to the min-cut of the network. This scheme does not achieve capacity in general networks with unequal link capacities. Intuitively, this is because the total rate of random keys is limited by the min cut from the source to the sink, whereas more random keys may be required to fully utilize large capacity cuts with large capacity links.

Capacity can be improved by using a combination of local and global random keys. A local key is injected at a non-source node and/or canceled at a non-sink node. However, it is complicated to optimize over all possible combinations of nodes at which keys are injected and canceled. Thus, we propose the following more tractable constructions, which we will use to develop further results in subsequent sections.

**Strategy 1: Random Keys Injected by Source and Possibly Canceled at Intermediate Nodes**

Our first construction achieves secrecy with random keys injected only at the source. The source carries out pre-coding so that random keys are canceled at intermediate nodes and the sink receives the intended message without interference from the random keys. As such, it applies in the single-source, single-sink case, and is useful in networks where the incoming capacity of the sink is too small to accommodate the message plus all the keys needed in the network. An example is given in Fig. [I] where each link has unit capacity, the number of wiretapped links is $k=2$, and only the first layer of the three layer network is allowed to be wiretapped. The secret message rate $R_s=3$ is achievable by using the strategy in Fig. [I] where the operation is on a finite field $GF(5)$. In Fig. [I] $a,b,c$ are secret messages and $f,g$ are keys. The message on the $i$-th link in the first layer is denoted as $x_i$, $i=1,2,3,4,5$. The key $f$ is canceled at the second layer and the key cancelation scheme is labeled on the last layer links. It is easy to see that $H(x_i,x_j|a,b,c)=2$, $\forall i \neq j$ which means perfect secrecy is achieved. At the same time,
the sink $d$ can decode $a, b, c$ and the key $g$. When key cancelation is not applied, let $R_s$ and $R_w$ be the secrecy rate and the random key rate at the source, respectively. Let $z$ be the total rate of transmission on the first layer. To achieve secrecy, we must have $R_w \geq \frac{2}{5}z$, where the cut-set condition on the first layer requires $R_s + R_w \leq z$. Since the sink needs to decode both message and random key symbols from the source, the cut-set condition on the last layer requires $R_s + R_w \leq 4$. Combining these we obtain $R_s \leq \max_z \min(4 - \frac{2}{5}z, \frac{3}{5}z) = \frac{12}{5}$, which is strictly less than 3.

To formally develop the Strategy 1 construction, we will use the following result:

**Claim 1** ([10, Corollary 19.21]): Given an acyclic network, there exists, for a sufficiently large finite field, a linear network code in which the dimension of the received subspace at each non-source node $t$ is $\min(\omega, \maxflow(t))$, where $\omega$ is the dimension of the message subspace.

Let $R_s$ denote the secret message rate and $z_{i,j}$ the transmission rate on each network link $(i,j) \in \mathcal{E}$, whose values we will discuss how to choose below. Consider the graph $\mathcal{G}$ with the capacity of each link $(i,j) \in \mathcal{E}$ set as $z_{i,j} \leq c_{i,j}$. As illustrated in Fig. 2, augment the graph as follows:

- Connect each subset of links $\mathcal{A} \in \mathcal{W}$ to a virtual node $t^\mathcal{A}$: more precisely, for each directed
Fig. 2. Illustration of Strategy 1, an achievable construction where random keys are injected by the source and possibly canceled at intermediate nodes. In this figure, \( k = 2 \) and only the 5 links in the first layer can be wiretapped.

link \((i,j) \in \mathcal{E}\) in the network, create a node \(v_{i,j}\) and replace \((i,j)\) by two links \((i, v_{i,j})\) and \((v_{i,j}, j)\) of capacity \(z_{i,j}\), and for each \((i,j) \in \mathcal{A}\) create a link \((v_{i,j}, t^A)\) of capacity \(v_{i,j}\). Let \(R_{s \rightarrow A}\) be the max flow/min cut capacity between \(s\) and \(t^A\).

- Add a virtual sink node \(d'\) and join the actual sink \(d\) to \(d'\) by a link \((d, d')\) of capacity of \(R_s\).
- Connect both \(t^A\) and the virtual sink \(d'\) to a virtual sink \(d^A\) by adding a link \((t^A, d^A)\) of capacity \(R_{s \rightarrow A}\) and a link \((d', d^A)\) of capacity \(R_s\), respectively.

The source sends a secret message \(v = [v_1, \ldots, v_{R_s}]^T\) along with \(R_w\) random key symbols \(w = [w_1, \ldots, w_{R_w}]^T\). The values of \(R_s\), \(R_w\), and \(z_{i,j}\) are chosen such that each virtual sink \(d^A\) can decode \(R_s + R_{s \rightarrow A}\) linear combinations of message and random key symbols, and the sink \(d\) can decode the \(R_s\) message symbols. Specifically, if for each \(\mathcal{A}\) the rate \(R_s + R_{s \rightarrow A}\) equals the min-cut capacity between the source and the virtual sink \(d^A\) and \(R_{s \rightarrow A} \leq R_w\), by using Claim [1] there exists a network code such that each \(d^A\) receives \(R_s + R_{s \rightarrow A}\) linearly independent combinations.

\[1\] We assume that \(R_s\) and \(R_{s \rightarrow A}\) are integers, which can be approximated arbitrarily closely by scaling the capacity of all links by the same factor.
of \( v \) and \( w \) when the finite field size is sufficiently large \((q > \binom{|E|}{k})\). Let the signals received at a particular virtual sink \( d^B \) be denoted as \( M_{B}[v^T, w^T]^T \), where \( M_{B} \) is an \((R_s + R_s \rightarrow B) \times (R_s + R_w)\) received coding matrix with full row rank. We can add \( R_w - R_s \rightarrow B \) rows to \( M_{B} \) to get a full rank \((R_s + R_w) \times (R_s + R_w)\) square matrix \( \tilde{M}_{B} \). We then precode the secret message and keys using \( \tilde{M}_{B}^{-1} \), i.e., the source transmits \( \tilde{M}_{B}^{-1}[v^T, w^T]^T \), so that link \((d', d^B)\) carries \( v \).

**Claim 2:** Strategy 1 allows the sink to decode the message \( v \) and achieves perfect secrecy.

**Proof:** Since \((d, d')\) is the only incoming link of \((d', d^B)\), and both links have capacity \( R_s \) which is equal to the rate of the message \( v \), link \((d, d')\) carries exactly \( v \). This implies that sink \( d \) receives \( v \). Furthermore, for any virtual sink \( d^A \), the received coding matrix with precoding is \( M_{A}\tilde{M}_{B}^{-1} \), which is a full row rank matrix. As \( M_{A}\tilde{M}_{B}^{-1} \) is a full row rank matrix, the coding vectors of the received signals from the set \( A \) of wiretapping links span a rank \( R_s \rightarrow A \) subspace that is linearly independent of the set of coding vectors corresponding to message \( v \) received on \((d', d^A)\). Therefore, the signals received on \( A \) are independent of the message \( v \), and perfect secrecy is achieved.

Note that applying \( \tilde{M}_{B}^{-1} \) causes the random keys injected by the source to be either canceled at intermediate nodes or decoded by the sink.

It remains to optimize over values of \( R_s \), \( R_w \) and \( z_{i,j} \) such that for each \( A \) the rate \( R_s + R_s \rightarrow A \) equals the min-cut capacity between \( s \) and \( d^A \) and \( R_s \rightarrow A \leq R_w \). Since computing \( R_s \rightarrow A \) (the min-cut capacity between \( s \) and \( t^A \)) involves a separate max flow computation, to simplify the optimization, we can constrain \( R_s \rightarrow A \) to be equal to some upper bound \( U_A \) on \( R_s \rightarrow A \), and thereby obtain an achievable secrecy rate using Strategy 1. For instance, we can take \( U_A \) to be \( \Sigma_{(i,j) \in A} z_{i,j} \), or alternatively take \( U_A \) to be the min-cut capacity between \( s \) and \( t^A \) on the graph with the original link capacities \( c_{i,j} \). We can write a linear program (LP) for this key cancelation strategy as follows:

\[
\begin{align*}
\max & \quad R_s \\
\text{subject to} & \quad \sum_{(i,j) \in E} f^A_{i,j} - \sum_{(i,j) \in E} f^A_{j,i} = \begin{cases} 
R_s + U_A, & \text{if } i = s, \\
-R_s - U_A, & \text{if } i = d^A, \\
0, & \text{otherwise},
\end{cases} \\
& \quad \forall A \in \mathcal{W}, \\
& \quad f^A_{i,j} \leq z_{i,j} \leq c_{i,j}, \forall (i,j) \in E,
\end{align*}
\]

where \( f^A_{i,j} \) represents the rate of flow on link \((i,j)\) intended for the virtual sink \( d^A \). The conditions
on conservation of flow $f_{i,j}^A$ ensure that the min cut between the source and $d^A$ is at least $R_s + U_A$. Since the only incoming links of $d^A$ are $(t^A, d^A)$ of capacity $R_{s \rightarrow A}$ and $(d', d^A)$ of capacity $R_s$, this implies that $R_{s \rightarrow A}$ equals the upper bound $U_A$. Thus, the optimal value of (4) gives an achievable secrecy rate.

**Strategy 2: Random Keys Injected by Source and/or Intermediate Nodes and Decoded at Sink**

In strategy 2, any node in the network can inject random keys. The sink is required to decode both the secret message and the random keys from all nodes, i.e. keys are not canceled within the network, while the random key rates must be sufficient to “fill” each wiretap set (in a sense that is made precise below). Although for simplicity of notation the algorithm description below is for the single-source, single-sink case, this strategy applies directly to multiple-source multicast case. If random keys are injected only at the source, the strategy reduces to the global key strategy in [3]. Note that under the assumption that only the source knows the message and different nodes do not have common randomness, here we cannot apply the key cancelation and precoding idea from Strategy 1, since after applying the precoding matrix each node may potentially be required to transmit a mixture of the source message and other nodes’ random keys.

Let $R_{w,v}$ be the random key injection rate at node $v$. As before, $R_s$ denotes the secret message rate at the source and $z_{i,j}$ the transmission rate on link $(i,j)$. We will address the choice of these rates below. Consider the graph $G$ with the capacity of each link $(i,j) \in \mathcal{E}$ set as $z_{i,j}$. Connect each subset of links $A \in \mathcal{W}$ to a virtual node $d^A$: more precisely, for each directed link $(i,j) \in \mathcal{E}$ in the network, create a node $v_{i,j}$ and replace $(i,j)$ by two links $(i, v_{i,j})$ and $(v_{i,j}, j)$ of capacity $z_{i,j}$, and for each $(i,j) \in A$ create a link $(v_{i,j}, d^A)$ of capacity $v_{i,j}$. Intuitively, we want the max flow/min cut capacity from the message and random key sources to $d^A$ to be equal to that in the absence of the message. Similarly to strategy 1, we simplify the optimization by constraining this max flow/min cut capacity to be equal to an upper bound, $\sum_{(i,j) \in A} z_{i,j}$. Specifically, we have
the following LP:

\[
\begin{align*}
\text{max } & R_s \\
\text{subject to } & \sum_j f_{i,j}^A - \sum_j f_{j,i}^A = \begin{cases} 
- \sum_{(i',j') \in A} z_{i',j'}, & \text{if } i = d^A, \\
R_{w,i}, & \text{otherwise,}
\end{cases} \forall A \in \mathcal{W}, \\
& \sum_j f_{i,j}^d - \sum_j f_{j,i}^d = \begin{cases} 
-R_s + \sum_{v \in V, v \neq d} R_{w,v}, & \text{if } i = d, \\
R_s + R_{w,s}, & \text{if } i = s, \\
R_{w,i}, & \text{otherwise,}
\end{cases} \\
& f_{i,j}^A \leq z_{i,j}, \quad f_{i,j}^d \leq z_{i,j}, \quad z_{i,j} \leq c_{i,j}, \forall (i,j) \in \mathcal{E},
\end{align*}
\]

(5)

where the first set of equations corresponds to the requirement that the network accommodates a flow \( f^A \) of size \( \sum_{(i,j) \in A} z_{i,j} \) from the random key sources to \( d^A \), the second set of equations corresponds to the requirement that the network accommodates a flow \( f^d \), of size equal to the sum of the message and random key rates, from the message and random key sources to the sink \( d \), and the third set of inequalities corresponds to the link capacity constraints.

**Claim 3:** Strategy 2 allows the sink to decode the message \( v \), and achieves perfect secrecy.

**Proof:** As illustrated in the example of Fig. 3, consider an augmented network with

- a virtual source node \( u_s \) connected to the source node \( s \) by a directed link \((u_s, s)\) of capacity \( R_s \), and connected to each virtual sink \( d^A \) by a directed link \((u_s, d^A)\) of capacity \( R_s \), and
- a virtual node \( u_k \) connected to each node \( v \) by a directed link \((v, u_k)\) of capacity \( R_{w,v} \), and connected to each virtual sink \( d^A \) by a directed link \((u_k, d^A)\) of capacity \( \sum_{v} R_{w,v} - \sum_{(i,j) \in A} z_{i,j} \).

The source information enters the network at the virtual source node \( u_s \) and is transmitted to each virtual sink \( d^A \). Consider a multi-source multicast problem on this network, where the actual sink node and the virtual sinks \( d^A \) each demand the source message and all the random keys. By the first constraint of the LP, the max flow from the random key sources to \( d^A \) in the original network equals \( \sum_{(i,j) \in A} z_{i,j} \); together with the additional capacity in the augmented network \( \sum_{v} R_{w,v} - \sum_{(i,j) \in A} z_{i,j} \) from the random key sources and \( R_s \) from \( u_s \), the max flow from the message and random key sources to each virtual sink \( d^A \) is sufficient to ensure that the multicast problem is feasible. A capacity-achieving code for this multicast problem in the transformed graph corresponds to a code for the original secrecy problem, since the information received by each virtual sink \( d^A \) from the set \( A \) of original network links must be independent of information received from the additional links, which includes the entire source message. 

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Fig. 3. An example of the augmented network construction for the proof of correctness of strategy 2, where $s,a,b,d$ are nodes of the original graph, and only one of the two links $(s,a)$ and $(s,b)$ can be wiretapped.

Fig. 4. Example of the usefulness of Strategy 2.
An example where this strategy is useful is given in Fig. 4, which is obtained by interchanging the source and the sink as well as reversing all the links in Fig. 2. At most three links in the last layer can be wiretapped. By injecting one local key at node $j_2$ and two global keys at the source, Strategy 2 can achieve secrecy rate 2. On the other hand, if random keys are only injected at the source, the secrecy rate is at most $\frac{8}{5}$. Let $R_s$ and $R_w$ be the secrecy rate and the random key rate at the source, respectively. Let $z$ be the total rate of transmission on the last layer. To achieve secrecy, we must have $R_w \geq \frac{3}{5} z$, where the min-cut condition on the last layer requires $R_s + R_w \leq z$. Since the source injects all the random keys, the min-cut condition on the first layer requires $R_s + R_w \leq 4$. Combining these we obtain $R_s \leq \frac{8}{5}$, which is strictly less than 2.

From the examples, we see that the types of scenarios in which Strategy 1 and Strategy 2 are useful seem to be complementary. In general, these two strategies can be combined to obtain a higher secrecy rate. We use these strategies conceptually in the following sections to develop theoretical results. However, for numerical computation of achievable rates in scenarios 1 and 2, we note that the number of possible wiretapping sets, and thus the size of the LPs, are exponential in the size $k$ of each wiretap set, so they are useful for small $k$.

IV. Unachievability of Cut Set Bound

In the case of unrestricted wiretapping sets and unit link capacities, the secrecy capacity is equal to the cut-set bound [3]. In this section we show that the cut-set bound (2) is not achievable in general, by considering the example in Fig. 5 where the set of wiretappable links is restricted (Scenario 1). We give an explicit proof that the cut set bound is not achievable for the case when the wiretap set is unknown. We also use the program Information Theoretic Inequalities Prover (Xitip) [12] to show that the secrecy capacity is bounded away from the cut set bound. We then convert the example into one with unequal link capacities (Scenario 2), and show the unachievability of the cut set bound for this case also.

A. Restricted Wiretap Set (Scenario 1)

Consider the example in Fig. 5 where all links have unit capacity and any three of the five middle layer links can be wiretapped. Let the middle layer links be 1-5 (from top to bottom) and the last layer links be 6-8 (from top to bottom). Let the signal carried by link $i$ be called signal $S_i$. Let the source information be denoted $X$. The cut-set bound, or the secrecy capacity with known wiretap set, is 2.
Fig. 5. An example to show that the secrecy rate without knowledge of wiretapping set is smaller than that with such knowledge. The wiretapper can wiretap any three of the five links in the middle layer.

To provide intuition for the case when the wiretap set is unknown, we first show that secrecy rate 2 cannot be achieved by using scalar linear coding. Then, the argument is converted to an information theoretic proof that secrecy rate 2 cannot be achieved by using any possible coding scheme.

Suppose secrecy rate 2 is achievable with a scalar linear network code. First note that the source cannot inject more than unit amount of random key, otherwise the first layer cannot carry two units of source data. Let the random key injected by the source be denoted $K$. For the case when the source injects a unit amount of random key, we first have the following observations. Signal 6 must be a function of signal 1, otherwise if the adversary sees the signals 2-4 then he knows signals 6-7. Also, signal 8 must be a function of signal 5, otherwise if the adversary sees signals 1, 2 and 4, then he knows signals 7-8. Similarly we can show that signal 8 must be a function of signal 1, and signal 7 must be a function of signal 2. We consider the following two cases.

Case 1: signal 5 is a linear combination of signals present at the source node. To achieve the full key rank condition on links 1, 2 and 5, node a must put two independent local keys $k_1$.
and $k_2$ on links 1 and 2 respectively. Link 7, whose other input is independent of $k_2$, is then a function of $k_2$. Similarly, Link 8 is a function of $k_1$. This means that the last layer has two independent local keys on it.

Case 2: signal 5 is a linear combination of signals present at the source node as well as a local key $k$ injected by node c.

Case 2a: $k$ is also present in signal 1. Then $k$ is present in signal 6, and is independent of the key present in signal 7.

Case 2b: $k$ is not present in signal 1. Then $k$ is present in signal 8, and is independent of the key present in signal 7.

In all three cases 1, 2a, and 2b, there is a pair of last layer links which are functions of two independent random keys, leaving capacity for only one unit of secret message. Thus, we conclude that the secrecy rate without knowledge of the wiretapping set by using only linear network coding is less than two.

We now extend the above argument to any coding scheme which leads to the following theorem.

Theorem 2: For the wireline network in Fig. 5 a secrecy rate of 2 is not achievable with any possible coding scheme, if any three out of the five links (1-5) in the middle layer are wiretapped and the location of those links is unknown.

Proof: See Appendix.

We can also show that the secrecy rate is bounded away from 2 by using the framework for linear information inequalities [13]. Let $X$ be the message sent from the source and $Z_i$, $i=1,2,3$ be the signals on the links adjacent to the source. We want to check whether $H(X) \leq \omega$
is implied by

\( (1) \quad H(Z_i) \leq 1, H(S_j) \leq 1, i = 1, 2, 3, j = 1, \ldots, 8, \)

\( (2) \quad H(X|S_6, S_7, S_8) = 0, \)

\( (3) \quad I(X; Z_1, Z_2, Z_3, S_4, S_5, S_7, S_8; S_6|S_1, S_2, S_3) = 0, \)

\( (4) \quad I(X; Z_1, Z_2, Z_3, S_1, S_3, S_5, S_6, S_8; S_7|S_2, S_4) = 0, \)

\( (5) \quad I(X; Z_1, Z_2, Z_3, S_2, S_3, S_6, S_7, S_8|S_1, S_4, S_5) = 0, \)

\( (6) \quad I(X; S_1, S_2, S_3) = 0, I(X; S_1, S_2, S_4) = 0, \)

\( (7) \quad I(X; S_1, S_2, S_5) = 0, I(X; S_1, S_3, S_4) = 0, \)

\( (8) \quad I(X; S_1, S_3, S_5) = 0, I(X; S_1, S_4, S_5) = 0, \)

\( (9) \quad I(X; S_2, S_3, S_4) = 0, I(X; S_2, S_3, S_5) = 0, \)

\( (10) \quad I(X; S_2, S_4, S_5) = 0, I(X; S_3, S_4, S_5) = 0, \)

\( (11) \quad I(S_1; Z_2|Z_1, Z_3) = 0, I(S_2; Z_2, Z_3|Z_1) = 0, \)

\( (12) \quad I(S_3; Z_3|Z_1, Z_2) = 0, I(S_4; Z_1, Z_3|Z_2) = 0, \)

\( (13) \quad I(S_5; Z_1, Z_2|Z_3) = 0, I(S_1; S_4|Z_1, Z_2, Z_3) = 0, \)

\( (14) \quad I(S_2; S_4, S_5|Z_1, Z_2, Z_3) = 0, I(S_3; S_5|Z_1, Z_2, Z_3) = 0, \)

\( (15) \quad I(S_4; S_1, S_2, S_5|Z_1, Z_2, Z_3) = 0, I(S_5; S_2, S_3, S_1|Z_1, Z_2, Z_3) = 0, \)

\( (16) \quad I(S_1, S_2, S_3, S_4, S_5; X|Z_1, Z_2, Z_3) = 0, \)

where the first inequality is the capacity constraint, the second constraint shows that the sink can decode \( X \), constraints (3) to (5) mean that the signals in the last layer are independent of other signals given the incoming signals from the middle layer, constraints (6) to (10) represent the secrecy constraints when any three links in the middle layer are wiretapped, and constraints (11) to (16) represent the conditional independence between the signals in the first layer and those in the middle layer. In particular, (16) shows that \( X \to (Z_1, Z_2, Z_3) \to (S_1, \ldots, S_5) \) forms a Markov chain. Note that constraints (3) to (5) and (11) to (16) implicitly allow some randomness to be injected at the corresponding nodes. We use the Xitip program [12], which relies on the framework in [13], to show that \( H(X) \leq 5/3 \) is implied by the set of equalities (6). Therefore, 5/3 is an upper bound on the secrecy rate when the location of wiretapper is unknown, which is less than the secrecy rate 2 achievable when such information is known. Therefore, there is
a strict gap between the secrecy capacity and the cut set bound.

On the other hand, the secrecy rate for the wireline network in Fig. 5 is at least 1 which is shown by the example in Fig. 6 where a finite field $GF(7)$ is used. In this example, a combination of strategies 1 and 2 is used, where keys are injected inside the network and are also canceled at intermediate nodes.

**B. Unequal Link Capacities (Scenario 2)**

We have restricted the wiretapped links to be in the middle layer in Fig. 5. We next show that the unachievability of the cut-set bound also holds for the secure network coding problem with unequal link capacities (Scenario 2). We convert the example of Fig. 5 by partitioning each non-middle layer link into $\frac{1}{\epsilon}$ parallel small links each of which has capacity $\epsilon$. Any three links can be wiretapped in the transformed graph. We prove the unachievability of the cut-set bound in the transformed network.

First, we show a lower bound on the min-cut between the source and the sink in the transformed network.
network when three links are deleted. Note that deleting any $k'$ ($k' \leq 3$) non-middle layer links reduces the min-cut by at most $k'\epsilon$. When $k'=0$, the min-cut is 2. When $k'=1$ or at most two middle layer links are deleted, the min-cut is at least 2 after deleting these middle layer links, and the min-cut is at least $2 - k'\epsilon \geq 2 - \epsilon$ after further deleting the $k'=1$ non-middle layer link. When $k'=2$ or at most one middle layer link is deleted, the min-cut between the source and the sink is 3 after deleting this middle layer link, and the min-cut is at least $3 - k'\epsilon \geq 3 - 3\epsilon$ after further deleting the $k'$ non-middle layer links. Therefore, the cut-set bound is at least $\min(2 - \epsilon, 3 - 3\epsilon)$.

For the case where the location of the wiretap links is unknown, we prove the unachievability of the cut-set bound in the transformed network. First, consider the transformed network with the restriction that the wiretapper can only wiretap any 3 links in the middle layer. The optimal solution is exactly the same as for the original network of the previous subsection, and achieves secrecy rate at most $5/3$. Now, consider the transformed network without the restriction on wiretapping set, i.e., the wiretapper can wiretap any 3 links in the entire network. As wiretapping only the middle layer links is a subset of all possible strategies that the wiretapper can have, the secrecy rate in the transformed network is less than or equal to that in the former case, which is strictly smaller than the cut-set bound for $\epsilon$ strictly smaller than $\frac{1}{4}$. Therefore, the cut-set bound is still unachievable when the wiretap links are unrestricted in the transformed graph.

V. NP-HARDNESS

We show in the following that determining the secrecy capacity is NP-hard by reduction from the clique problem, which determines whether a graph contains a clique of at least a given size $r$.

When the choice of the wiretap set is made known to the communicating nodes, the secrecy capacity is given by the cut-set bound, from Theorem 1, and is achieved by not transmitting on the wiretapped links. Finding the cut-set bound involves determining the worst case wiretap set. This is equivalent to the network interdiction problem [14], which is to minimize the maximum flow of the network when a given number of links in the network are removed. It is shown in [14] that the network interdiction problem is NP-hard. Therefore, determining the secrecy capacity for the case where the location of the wiretap links is known is NP-hard.

2A clique in a graph is a set of size $r$ of pairwise adjacent vertices, or in other words, an induced subgraph which is a complete graph.
To show that determining the secrecy capacity for the case where the location of the wiretap links is unknown is NP-hard, we use the construction in \[14\] showing that for any clique problem on a given graph \(H\), there exists a corresponding network \(G^H\) whose secrecy capacity is \(r\) when the location of the wiretap links is known if and only if \(H\) contains a clique of size \(r\). We then show that for all such networks \(G^H\), the secrecy capacity for the case when the location of the wiretap links is unknown is equal to that for the case when this information is known, which shows that there is a one-to-one correspondence between the clique problem and the secrecy capacity problem.

We briefly describe the approach in \[14\] in the following. Given an undirected graph \(H=(V_h,E_h)\), we will define a capacitated directed network \(G^H\) such that there exists a set of links \(\hat{A}'\) in \(G^H\) containing less than or equal to \(|E_h|-(\binom{r}{2})\) links such that \(G^H - \hat{A}'\) has a maximum flow of \(r\) if and only if \(H\) contains a clique of size \(r\). For a given undirected graph \(H=(V_h,E_h)\) without parallel links and self loops, we create a capacitated, directed graph \(G^H=(N,A)\) as follows: For each link \(e\in E_h\) create a node \(i_e\) in a node set \(N_1\) and for each vertex \(v\in V_h\) create a node \(j_v\) in a node set \(N_2\). In addition, create source node \(s\) and destination node \(d\). For each link \(e\in E_h\), direct a link in \(G^H\) from \(s\) to \(i_e\) with capacity 2 and call this set of links \(A_1\). For each link \(e=(u,v)\in E_h\), direct two links in \(G^H\) from \(i_e\) to \(j_v\) and \(j_u\) with capacity 1, respectively and call this set of

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Fig. 7. Example of NP-hardness proof for the case with knowledge of the wiretapping set.
For each vertex \( v \in V_h \), direct a link with capacity 1 from \( j_v \) to \( d \). Let this be the set of links \( A_3 \). This completes the construction of \( G^H=(N,A)=(\{s\} \cup \{d\} \cup N_1 \cup N_2, A_1 \cup A_2 \cup A_3) \).

In Fig. 7 we give an example of the graph transformation, where \( H=(\{1,2,3,4\}, \{a,b,c,x,y\}) \).

We use the following result from [14]:

**Lemma 1** ([14, Lemma 2]): Let \( G^H \) be constructed from \( H \) as above. Then, there exists a set of links \( A_1 \subseteq A \) with \( |A_1| = |E_h| - \binom{r}{2} \) such that the maximum flow from \( s \) to \( d \) in \( G^H - A_1 \) is \( r \) if and only if \( H \) contains a clique of size \( r \).

After obtaining \( G^H \), we generate \( \hat{G}^H \) by replacing each link \( (i_v, j_v) \) with \( |E_h| \) parallel links each with capacity \( 1/|E_h| \) and call this link set \( \hat{A}_2 \). We carry out the same procedure for links \( (j_v, d) \) and call this link set \( \hat{A}_3 \). Then \( \hat{G}^H=(N,A)=(\{s\} \cup \{d\} \cup N_1 \cup N_2, A_1 \cup \hat{A}_2 \cup \hat{A}_3) \). For the case when the location of wiretap links is known, it is shown in [14] that the worst case wiretapping set \( \hat{A}' \) must be a subset of \( A_1 \). By using Lemma [1], this case is NP-hard.

Now, we consider the secrecy capacity when \( k = |E_h| - \binom{r}{2} \) and the wiretapping set is unknown. From Lemma [1] the condition that \( H \) contains a clique of size \( r \) is equivalent to the condition that the max-flow to the sink in \( G^H \) after removing any \( k \) links from \( A_1 \) is at least \( r \). We now show that the latter condition is equivalent to the condition that the secrecy capacity of \( G^H \) when the wiretapper accesses any unknown subset of \( k \) links from \( A_1 \) (Scenario 1) is at least \( r \).

For each subset \( A' \) of \( k \) links from \( A_1 \), we create nodes \( t^{A_1} \) and \( d^{A_1} \) with their corresponding incident links as described in Strategy 1. As the wiretapped links each have capacity 2 and are connected to the source directly, the min-cut between the source and each virtual sink \( d^{A_1} \) is at least \( 2k+r \). Then, by using Strategy 1 the secrecy rate \( r \) is achievable.

Finally, we show that the same condition is also equivalent to the condition that the secrecy capacity of \( \hat{G}^H \) when any \( k \) links are wiretapped (Scenario 2) is at least \( r \). Since each second layer link has a single first layer link as its only input, wiretapping a second layer link yields no more information to the wiretapper than wiretapping a first layer link. When some links in the third layer are wiretapped, let the wiretapping set be \( \hat{A}'=\hat{A}_1 \cup \hat{A}_2 \) where \( |\hat{A}_2| \geq 1 \) and \( |\hat{A}_1| \leq k-1 \). Thus \( A_1 - \hat{A}_1 \) contains at least \( \binom{k}{2} + 1 \) links. We create nodes \( t^{\hat{A}'} \) and \( d^{\hat{A}'} \) with their corresponding incident links as described in Strategy 1. Since removing links in \( A_1 \) corresponds to removing links in \( H \), after removing links in \( H \) corresponding to \( \hat{A}_1 \), \( H \) contains a subgraph \( \hat{H}_1 \) containing \( \binom{k}{2} \) links plus at least one link \( e=(u,v) \).

Case 1: \( \hat{H}_1 \) is a clique of size \( r \). In this case, the number of vertices with degree greater than 0 in \( \hat{H}_1 \cup e \) is \( r+2 \).
Case 2: $H_1$ is not a clique. $H_1$ contains at least $r+1$ vertices with degree greater than 0. According to [14, Lemma 1], the max-flow in $G^H$ is equal to the number of vertices in $H$ with degree greater than 0. In both cases, the max-flow of $G^H$ after removing links in $\hat{A}_1'$ is at least $r+1$. Let $\hat{R}_{s\rightarrow\hat{A}_3'}$ be the max-flow capacity from the source to $\hat{A}_3'$ in $\hat{G}^H - \hat{A}_1'$.

We can use a variant of the Ford-Fulkerson (augmenting paths) algorithm, e.g., [15], as follows to construct a max-flow subgraph $D$ from $s$ to $\hat{A}_3'$ in $\hat{G}^H - \hat{A}_1'$ satisfying the property that after removing $D$ from $\hat{G}^H - \hat{A}_1'$, the min-cut between $s$ and $d$ is at least $r+1 - \hat{R}_{s\rightarrow\hat{A}_3'} \geq r+1 - (|\hat{A}_3'|)/|E_h|$.

\[
\begin{align*}
    r+1 - \hat{R}_{s\rightarrow\hat{A}_3'} & \geq r+1 - |\hat{A}_3'|/|E_h| \\
    & \geq r+1 - (|E_h|-1)/|E_h| \\
    & > r,
\end{align*}
\]

where we have used $|\hat{A}_3'| \leq |E_h|-1$. Considering the network $\hat{G}^H - \hat{A}_1'$ with all link directions reversed, we construct augmenting paths via depth first search from $d$ to $s$, starting first by constructing augmenting paths via links in $\hat{A}_3'$, until we obtain a set of paths corresponding to a max flow of capacity $\hat{R}_{s\rightarrow\hat{A}_3'}$ between $s$ and $\hat{A}_3'$. We add further augmenting paths until we obtain a max flow (of capacity at least $r+1$) between $s$ and $d$, which may cause some of the paths traversing links in $\hat{A}_3'$ to be redefined but without changing their total capacity. The subgraph $D$ consists of the final set of paths traversing links in $\hat{A}_3'$. Thus, the paths remaining after removing $D$ have a total capacity lower bounded by (7).

Therefore, the min-cut between the source and $d\hat{A}_1'$ in $\hat{G}^H - \hat{A}_1' - D$ is at least $r$, and the min-cut between the source and $d\hat{A}_1'$ in $\hat{G}^H$ is at least $r + R_{s\rightarrow\hat{A}_1'} + \hat{R}_{s\rightarrow\hat{A}_3'} = r + R_{s\rightarrow\hat{A}}$. By using Strategy 1, a secure rate of $r$ is achievable when $\hat{A}_1'$ is wiretapped. Thus, the secrecy rate for the case when the location of the wiretap links is unknown is equal to that for the case when such information is known with an unrestricted wiretapping set. We have thus proved the following theorem.

**Theorem 3:** For a single-source single-sink network consisting of point-to-point links and an unknown wiretapping set, computing the secrecy capacity is NP-hard.

### VI. Conclusion

In this paper, we addressed the secrecy capacity of wireline networks where different links have different capacities. In particular, it was shown that the secrecy capacity is not the same in general when the location of the wiretapped links is known or unknown; in the former case the...
capacity is given by a cut-set bound, which is unachievable in general in the latter case. Further, we proposed achievable strategies where random keys are canceled at intermediate non-sink nodes, or injected at intermediate non-source nodes. Finally, we showed that determining the secrecy capacity is an NP-hard problem.

**Appendix: Proof of Theorem 2**

We prove Theorem 2 by contradiction. Suppose that a secrecy rate of 2 is achievable for the network in Fig. 5. As before, let $X$ and $K$ denote respectively the secret message and random key injected by the source node, and $S_i$ the signal on link $i$. Then each triple of links in the middle layer has zero mutual information with the source data, and each pair of links in the middle layer has joint conditional entropy 2 given the other three links.

Since the message $X$ is decodable from information on the last layer, we have $I(S_0, S_7, S_8; X) = 2$. Since $I(S_1, S_2, S_3; X) = 0$, by the data processing inequality $I(S_0; X) = 0$, therefore, $I(S_7, S_8; X | S_6) = 2$ and $H(S_7|S_6) = I(S_7; X|S_6) = 1$. Then, $H(S_7; X, S_6) = H(S_7|S_6) = I(S_7; X|S_6) = 0$. This implies that $S_7$ does not depend on random keys injected by node $S_f$ or head $(4)$ which would be independent of $X, S_6$. Similarly, $I(S_8; X) = 0$, implying $H(S_7|S_8) = I(S_7; X|S_8) = 1$ and $H(S_7; X, S_8) = 0$. Thus, $S_7$ does not depend on random keys injected by node head $(2)$ which would be independent of $X$ and $S_6$. In a similar manner, we can show that $S_6$ and $S_8$ also do not depend on any random keys injected after the middle layer. Also, since $H(S_7, S_8|S_6) = I(S_7, S_8; X | S_6) = 2$ and $H(S_6) = H(S_6|S_8) = 1$, therefore $H(S_6, S_7, S_8) = 3$. Let $S_A$ denote the adversary’s observations. By the secrecy requirement, $H(S_6, S_7, S_8|S_A) = 2$, which implies $I(S_6, S_7, S_8; S_A) = H(S_6, S_7, S_8) - H(S_6, S_7, S_8|S_A) = 1$.

Then, the mutual information $I(S_6; S_2, S_3) = 0$, otherwise, if the adversary sees signals 2-4 his mutual information with signals 6-7 is greater than 1. The mutual information $I(S_8; S_1, S_4) = 0$, otherwise if the adversary sees signals 1, 2, 4 his mutual information with signals 7-8 is greater than 1. The mutual information $I(S_8; S_4, S_5) = 0$, otherwise if the adversary sees signals 2, 4, 5 his mutual information with signals 7-8 is greater than 1. The mutual information $I(S_7; S_4, S_5) = 0$, otherwise if the adversary sees signals 1, 4, 5, his mutual information with signals 7-8 is greater than 1.

Case 1: signal 5 is a function of only signals present at the source node, i.e., $H(S_6|X, K) = 0$. By the zero mutual information condition for links 1, 2 and 5, $H(S_1, S_2, S_5|X) = 3$, so

$$H(S_1, S_2, S_5|X, K) = H(S_1, S_2|X, K, S_5) = 2.$$  (8)
Since $S_4$ is conditionally independent of $S_1, S_2$ given $X$ and $K$, we have $H(S_1, S_2|X,K,S_4, S_5) = 2$, $I(S_1, S_2; X, K, S_4, S_5) = 0$ and $I(S_1, S_2; X, K|S_4, S_5) = 0$. Now
\[
I(S_1, S_2, S_7, S_8; X, K|S_4, S_5) = I(S_7, S_8; X, K|S_4, S_5) + I(S_1, S_2; X, K|S_7, S_8, S_4, S_5)
\]
\[
= I(S_7, S_8; X, K|S_4, S_5) + I(S_7, S_8; X, K|S_1, S_2, S_4, S_5).
\]
Since $S_7, S_8$ is conditionally independent of $X, K$ given $S_1, S_2, S_4, S_5$, we have
\[
I(S_7, S_8; X, K|S_1, S_2, S_4, S_5) = 0.
\]

Then by the non-negativity of conditional mutual information,
\[
I(S_7, S_8; X, K|S_4, S_5) \leq I(S_1, S_2; X, K|S_4, S_5) = 0.
\]

Next, note that $S_1$ and $S_2$ are conditionally independent given $S_4$ and $S_5$, since $H(S_1|S_4, S_5) = H(S_2|S_1, S_4, S_5) = 1$. Therefore $S_7$ and $S_8$ are conditionally independent given $S_4$ and $S_5$, i.e. $I(S_7, S_8|S_4, S_5) = 0$. Since $H(S_7|S_4, S_5) = H(S_7) - I(S_7; S_4, S_5) = 1$, it follows that $H(S_7|S_8, S_4, S_5) = 1$. Then we have
\[
I(S_7, S_8; S_4, S_5) = I(S_8; S_4, S_5) + I(S_7; S_4, S_5|S_8)
\]
\[
= I(S_8; S_4, S_5) + H(S_7|S_8) - H(S_7|S_4, S_5, S_8) = 0 + 1 - 1 = 0.
\]

So, $I(S_7, S_8; X, K, S_4, S_5) = I(S_7, S_8; X, K|S_4, S_5) + I(S_7, S_8; S_4, S_5) = 0$, and therefore $H(S_7, S_8|X) \geq H(S_7, S_8|X, K, S_4, S_5) = 2$, which contradicts the requirement that there is at most 1 unit of random key on the last layer.

Case 2: signal 5 is not a function only of signals present at the source

Case 2a: signal 1 has nonzero mutual information with some random key injected at node $c$. Then $H(S_1|X, K, S_2, S_3, S_4) > 0$. For brevity, let $A = (S_2, S_3)$ and $Y = (X, K, S_4)$. Since $I(S_6; A) = 0$ and $H(S_6|S_1, A) = 0$, we have $H(A) + H(S_6) = H(A, S_6) \leq H(A, S_1) = H(A) + H(A|S_1)$. Since $H(S_6) = H(S_1)$, we have $H(A) = H(A|S_1)$ and so $H(S_1|A) = H(S_1)$. Then from $H(S_1, A|S_6) = H(S_1|A, S_6) + H(S_6|A) = H(S_6|A, S_1) + H(S_1|A)$, we have $H(S_1|A, S_6) = 0$. Since $H(S_1|A, Y, S_6) \leq H(S_1|A, S_6) = 0$ and $H(S_6|A, Y, S_1) \leq H(S_6|A, S_1) = 0$, from
\[
I(S_1; S_6|Y, A) = H(S_1|A, Y) - H(S_1|A, Y, S_6) = H(S_6|A, Y) - H(S_6|A, Y, S_1) > 0
\]
we have $H(S_6|A, Y) = H(S_1|A, Y) > 0$. Then since $H(S_7|S_2, S_4) = 0$, we have $H(S_6|S_7, X) > 0$. Also, since $H(S_7|X) = 1$, we have $H(S_6, S_7|X) > 1$.
Case 2b: signal 1 has zero mutual information with any random key injected at node c. Then $H(S_5|X,K,S_1,S_2,S_4) > 0$. Similar reasoning as for case 2a applies with $A = (S_1,S_4)$, $Y = (X,K,S_2)$, $S_5$ in place of $S_1$, and $S_8$ in place of $S_6$.

From Cases 1, 2a, and 2b, we conclude that the secrecy rate without knowledge of the wiretapping set by using any nonlinear or linear coding strategy is smaller than two obtained for the case where such knowledge is present at the source.

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