Research Article

Design of Nonbinary LDPC Cycle Codes with Large Girth from Circulants and Finite Fields

Hengzhou Xu,1,2 Huaan Li,1 Jixun Gao,3 Guixiang Zhang,1 Hai Zhu,1 and Xiao-Dong Zhang1

1School of Network Engineering, Zhoukou Normal University, Zhoukou 466000, China
2School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China
3School of Computer, Henan University of Engineering, Zhengzhou 451191, China

Correspondence should be addressed to Hengzhou Xu; hzxu@zknu.edu.cn

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In this paper, we study a class of nonbinary LDPC (NBLDPC) codes whose parity-check matrices have column weight 2, called NBLDPC cycle codes. We propose a design framework of \((2, \rho)\)-regular binary quasi-cyclic (QC) LDPC codes and then construct NBLDPC cycle codes of large girth based on circulants and finite fields by randomly choosing the nonzero field elements in their parity-check matrices. For enlarging the girth values, our approach is twofold. First, we give an exhaustive search of circulants with column/row weight \(\rho\) and design a masking matrix with good cycle distribution based on the edge-node relation in undirected graphs. Second, according to the designed masking matrix, we construct the exponent matrix based on finite fields. The iterative decoding performances of the constructed codes on the additive white Gaussian noise (AWGN) channel are finally provided.

1. Introduction

Nonbinary low-density parity-check (NBLDPC) codes based on modulo arithmetics were first discovered by Gallager in 1960s [1] and redefined over finite fields GF\((q)\) by Davey and MacKay in 1998 [2]. Similar to binary LDPC codes, NBLDPC codes also have the ability of approaching capacity when decoded with the iterative algorithms. Moreover, NBLDPC codes have much better performance than binary LDPC codes for the short and moderate code lengths. As much more low-complexity decoding algorithms were proposed [3–8], NBLDPC codes provide a promising coding scheme for 6G communications [9].

As shown in [10], NBLDPC codes over larger finite fields will have much better performance for a constant code length. However, when the finite field size is sufficiently large, the performance improvement is little. Moreover, when the finite field size is equal or greater than 64, the column weights of the parity-check matrices of good NBLDPC codes tend to 2. Since NBLDPC cycle codes perform well over various channels [11–13], it is worth studying NBLDPC codes over large finite fields whose parity-check matrices have column weight 2, referred to as NBLDPC cycle codes. As an important cycle codes, \((2, \rho)\)-regular NBLDPC codes also perform well under iterative decoding; lots of methods for constructing such codes were proposed [14–17]. Among these works on the construction of NBLDPC codes, the codes can be mainly classified into two categories: the first one is constructed by means of computer search under the algorithms satisfying certain rules, and the other one is constructed based on combinatorial designs, graph theory, matrix theory, and finite fields [18]. Simulation results show that they all have good performance. For a given code rate and length, it is of great interest to study which one of them has the best error performance.

Cycle structure plays an important role in binary/non-binary LDPC codes. Research results show that NBLDPC codes with large girth have good iterative performance [19]. In general, NBLDPC codes with large girth have large
Hamming minimum distance, and it can be ensured that NBLDPC codes have good performance in the waterfall and error-floor region. Hence, it is interesting to construct LDPC cycle codes with large girth.

In this paper, we focus on the construction of \((2, \rho)\)-regular quasi-cyclic LDPC (QC-LDPC) codes with large girth. We first proposed the construction framework of \((2, \rho)\)-regular QC-LDPC codes based on the edge-node relation in undirected graphs and transfer the construction of \((2, \rho)\)-regular QC-LDPC codes into two main parts, i.e., circulants and exponent matrices. In the first part, we find circulants with good cycle distribution by performing an exhaustive search. In order to prune the search space of circulants, isomorphism theory of circulants is proposed. For the second part, we directly employ finite fields to construct exponent matrices of QC-LDPC codes. Here, the employed finite fields are divided two types, i.e., prime fields and finite fields of characteristic 2. Finally, numerical results to show the good performance of our proposed codes are provided.

The rest of this paper is organized as follows. Section 2 introduces the definitions of LDPC codes and their associated Tanner graphs. Section 3 presents the design framework of \((2, \rho)\)-regular QC-LDPC codes. Design of NBLDPC cycle codes with large girth is proposed in Section 4, and numerical results are also provided in this section. Finally, Section 5 concludes this paper.

2. Preliminaries

2.1. LDPC Codes. A binary \((\gamma, \rho)\)-regular LDPC code is generated by the null space of an \(m \times n\) sparse parity-check matrix \(H\) over GF \((2)\), and the matrix \(H\) has the following properties: (1) each column has \(\gamma\) 1's; (2) each row has \(\rho\) 1's; (3) \(\gamma \leq m\) and \(\rho \leq n\). If the sparse matrix \(H\) is over GF \((q)\) for \(q\) being a prime power, then LDPC codes generated by such \(H\) are called nonbinary codes or \(q\)-ary codes. Binary LDPC codes are referred to as quasi-cyclic (QC) [20], if their parity-check matrices \(H\) have the following form

\[
H = \begin{bmatrix}
I(p_{1,1}) & I(p_{1,2}) & I(p_{1,3}) & \cdots & I(p_{1,\rho}) \\
I(p_{2,1}) & I(p_{2,2}) & I(p_{2,3}) & \cdots & I(p_{2,\rho}) \\
I(p_{3,1}) & I(p_{3,2}) & I(p_{3,3}) & \cdots & I(p_{3,\rho}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
I(p_{\gamma,1}) & I(p_{\gamma,2}) & I(p_{\gamma,3}) & \cdots & I(p_{\gamma,\rho})
\end{bmatrix}_{\gamma \times \rho}
\]

where for \(1 \leq i \leq \gamma, 1 \leq j \leq \rho, -1 \leq p_{ij} \leq Q - 1\), \(I(p_{ij})\) is a \(Q \times Q\) circulant permutation matrix (CPM) formed by cyclically shifting each row of a \(Q \times Q\) identity matrix \(I\) to the right (or left) by \(p_{ij} \mod Q\) positions, and \(I(-1)\) is a zero matrix of size \(Q \times Q\). Obviously, \(I(0)\) is an identity matrix of size \(Q \times Q\). For example, if \(Q = 4\), then

\[
I(0) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
I(1) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
I(2) = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

\[
I(3) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

We can easily see that the positions of 1's of \(H\) in (1) are uniquely determined by the following matrix \(P\), called exponent matrix (or permutation shift matrix),

\[
P = \begin{bmatrix}
p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,\rho} \\
p_{2,1} & p_{2,2} & p_{2,3} & \cdots & p_{2,\rho} \\
p_{3,1} & p_{3,2} & p_{3,3} & \cdots & p_{3,\rho} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{\gamma,1} & p_{\gamma,2} & p_{\gamma,3} & \cdots & p_{\gamma,\rho}
\end{bmatrix}
\]

It is not hard to see that the correspondence between \(P\) and \(H\) is one-to-one. It is noticeable that the parameter \(Q\) is called expansion factor (or lifting degree) [21]. By replacing 1's in a CPM \(I(p_{ij})\) of \(H\) in (1) with the same nonzero field element in finite field GF \((q)\), the resulting code is nonbinary QC-LDPC codes [22].

2.2. Tanner Graph. Apart from the matrix representation, an LDPC code can be also described in a simple and intuitive way, i.e., a graphical model called Tanner graph [23]. In fact, the Tanner graph of an LDPC code with the parity-check matrix \(H = [h_{ij}]\) is a bipartite graph in which the two classes of nodes are variable nodes (representing the code-bit nodes) and check nodes (representing the constraint nodes),
respectively. An edge in a Tanner graph connects the check node $s$ to the variable node $t$ if and only if the row-$s$ and column-$t$ element $h_{st}$ in $H$ is nonzero. A cycle in a Tanner graph is a sequence of the connected check nodes and variable nodes which start and end at the same node in the graph and contain no other nodes more than once. The cycle length is simply the number of the contained edges (or nodes), and the length of the shortest cycle is referred to as girth of the Tanner graph (or an LDPC code). As an example, Figure 1 shows the Tanner graph of $H_b$ (or $H_{ab}$) and an associate cycle of length 6 (6-cycle for short).

It is well-known that the iterative decoding algorithm converges to the optimal solution provided that the Tanner graph of an LDPC code is free of cycles [24]. In other words, short cycles, especially, the cycles of length 4, affect the decoding performance when decoded with the iterative algorithms based on belief propagation. In fact, there exist algorithms based on belief propagation. In fact, there exist many construction methods and techniques to avoid short cycles or obtain LDPC codes with large girth, many construction methods and techniques are proposed [25–33].

3. Design Framework of $(2, \rho)$-Regular Binary QC-LDPC Codes

3.1. Edge-Node Relation in Undirected Graphs. Let $G = (V, E)$ be an undirected graph, where $V$ is a set of nodes and $E$ is some subset of the pairs (called edges) $\{(a, b) : a, b \in V, a \neq b\}$. A cycle of $G = (V, E)$ has distinct nodes (or edges), and an edge in a cycle has two distinct nodes. If we treat the nodes and edges of $G = (V, E)$ as the check nodes and variable nodes, respectively, then a bipartite graph $G_B$ can be obtained. Consider a cycle of length $k$ (denoted by $k$-cycle for short) in $G = (V, E)$. This $k$-cycle will be turned into a $2k$-cycle in the above bipartite graph $G_B$. In other words, the girth of $G_B$ is double that of $G = (V, E)$. Based on this process, we can construct bipartite graphs (or Tanner graphs) with large girth from an undirected graph. In order to make it clearly, we give an example.

Consider the following $4 \times 4$ matrix

\[
B_{4\times4} = \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1
\end{bmatrix}
\]

(4)

Figure 1: Tanner graph of $H_b$ (or $H_{ab}$): (a) $H_b$ over GF (2); (b) $H_{ab}$ over GF (4); (c) Tanner graph.

Figure 2: Tanner graph of $B_{4\times4}$.

It is easy to plot the Tanner graph of $B_{4\times4}$, given in Figure 2. By treating the nodes and edges of the Tanner graph in Figure 2 as the check nodes and variable nodes, respectively, we can construct a new bipartite graph, given in Figure 3. We can see from Figures 2 and 3 that a 4-cycle in the Tanner graph of $B_{4\times4}$ becomes a 8-cycle in the bipartite graph.

3.2. Construction Framework of $(2, \rho)$-Regular Binary QC-LDPC Codes. In this subsection, we will present the framework for constructing $(2, \rho)$-regular binary QC-LDPC codes by using the edge-node relation in an undirected graph in Section 3.1. In order to design $(2, \rho)$-regular codes, the node degree of $G = (V, E)$ should be $\rho$. Furthermore, to guarantee $(2, \rho)$-regular codes are quasi-cyclic, the incidence matrix of $G = (V, E)$ should possess quasi-cyclic structure. In conclusion, the incidence matrix of $G = (V, E)$ is $(\rho, \rho)$-regular and quasi-cyclic. Hence, in order to obtain $(2, \rho)$-regular binary QC-LDPC codes with large girth, we need to design a $(\rho, \rho)$-regular quasi-cyclic matrix with large girth. For convenience, this $(\rho, \rho)$-regular quasi-cyclic matrix is called base matrix. Next, we will give the construction framework.

First, we design a $(\rho, \rho)$-regular base matrix $B$ of size $L \times L$. By employing the edge-node relation in Section 3.1, we can transfer the Tanner graph of $B$ into a new bipartite graph, and the incidence matrix $B_M$ of such a bipartite graph is obtained. It is obvious that $B_M$ is a $(2, \rho)$-regular quasi-cyclic matrix of size $2L \times \rho L$. Second, we construct an exponent matrix $P$ of size $2L \times \rho L$, and the corresponding expansion factor is $Q$. Third, we use $B_M$ to mask the exponent matrix $P$, and a $2L \times \rho L$ array $H_M$ of $Q \times Q$ CPMs is constructed. The null space of $H_M$ gives a $(2, \rho)$-regular binary QC-LDPC code of length $\rho LQ$. 
4. Design of Nonbinary LDPC Cycle Codes with Large Girth

In order to construct \((2, \rho)\)-regular binary QC-LDPC codes with large girth, we only design a base matrix and a corresponding exponent matrix based on the construction framework in Section 3.2. By replacing the nonzero element in the parity-check matrices of binary QC-LDPC cycle codes with the nonzero field elements, nonbinary LDPC cycle codes can be obtained. In this paper, we do not optimize the nonzero field elements and adopt the optimized row elements in [34]. Next, we will provide the construction of the base matrices and exponent matrices.

4.1. Exhaustive Search of Circulants Based on Isomorphism Theory

In this paper, we employ the circulant as the base matrix. It can be seen from the construction framework in Section 3.2 that the size of the base matrix is not too large since the code lengths of NBLDPC codes are short or moderate. In the following, we will give the design of the circulants.

A circulant is a square matrix whose \(i\)-th row is generated by cyclically shifting the first row to the right (or left) by \((i-1)\) positions. Hence, the first row of a circulant is referred to as the generator of the circulant. For a circulant of size \(L \times L\), each row (or column) is a rightward (or downward) cyclic-shift of its above (or left) row (or column), and the first row (or column) is the rightward (or downward) cyclic-shift of the last row (or column). Therefore, the rows and columns of a circulant have the same weight. It is clear that the row (or column) weight is associated with the row weight of the generator.

Consider a circulant \(C\) of size \(L \times L\), and its generator is \(G = (g_1, g_2, \cdots, g_L)\) where \(g_i \in \{0, 1\}\) for \(1 \leq i \leq L\). Let \(\rho\) be the number of the nonzero components of \(G\). Hence, \(C\) is \((\rho, \rho)\)-regular. We select the nonzero components from \(g_1, g_2, \cdots, g_L\), and record their subscripts in a set \(S\), called location set in this paper. Then, the location set \(S\) has \(\rho\) elements. Without loss of generality, the location set \(S\) is denoted by

\[
S = \{s_1, s_2, \cdots, s_\rho\},
\]

where \(1 \leq s_1 < s_j \leq L\) for \(1 \leq i < j \leq \rho\). It is obvious that the generator \(G\) and the location set \(S\) have a one-to-one correspondence. Based on the isomorphism theory of LDPC codes (or their parity-check matrices) in [16, 35, 36], we can directly give the isomorphism theory of the circulants as follows.

**Theorem 1.** Let \(S_1 = \{s_1, 1, s_2, 2, \cdots, s_\rho, \rho\}\) and \(S_2 = \{s_2, 1, s_3, 2, \cdots, s_\rho, \rho\}\) be two location sets of the circulants \(C_1\) and \(C_2\) of size \(L \times L\), respectively. Then, \(C_1\) is isomorphic to \(C_2\), denoted by \(C_1 \cong C_2\), if \(S_2\) is derived from \(S_1\) with either of the following two methods.

1. For \(r \in \{0, 1, \cdots, L-1\}\), the elements of \(S_2\) are derived from those of \(S_1\) by adding a constant \(r\) to the elements of \(S_1\) modulo \(L\), i.e., \(s_{2j} = s_{1j} + r \mod L\) for \(1 \leq i \leq \rho\).
2. Suppose that \(r\) and \(L\) are coprime. The elements of \(S_2\) are derived from those of \(S_1\) using \(s_{2j} = r \cdot s_{1j} \mod L\) for \(1 \leq i \leq \rho\).

Note that in the calculation process, if the element in \(S_1\) and \(S_2\) equals 0, it actually equals \(L\). Moreover, in the case (2) of Theorem 1, the number of \(r\) can be determined by a well-known function, called Euler’s phi function, i.e.,

\[
\phi(L) = L \prod_{r=L}^{1} \left( 1 - \frac{1}{r} \right).
\]

If \(C_1 \cong C_2\), we say \(S_1\) is isomorphic to \(S_2\), denoted by \(S_1 \cong S_2\).

In general, the size of the employed circulants in this paper is not large. Hence, we can make an exhaustive search of the circulants by using the computer. The search space of the location sets of the \(L \times L\) circulants with row/column weight \(\rho\) is

\[
\binom{L}{\rho} = \frac{L!}{(L-\rho)! \cdot \rho!}.
\]

Based on the case (1) in Theorem 1, we can see that all the location sets of the \(L \times L\) circulants have the following isomorphic form: \(S = \{s_1 = 1, s_2, \cdots, s_\rho\}\), where \(2 \leq s_i < s_j \leq L\) for \(2 \leq i < j \leq \rho\). Hence, the search space of such location sets \(S\) is

\[
\binom{L-1}{\rho-1} = \frac{(L-1)!}{(L-\rho)! \cdot (\rho-1)!}.
\]

That is, an exhaustive search of such location sets (or circulants) is feasible. Here, we do not provide the specific exhaustive search algorithm. Combined with the cycle-counting algorithms [37–39], the optimal \(L \times L\) circulants with row/column weight \(\rho\) can be found. In this paper, the optimal ones are such circulants whose Tanner graphs have fewer short cycles and larger girths. In order to facilitate the readers, some optimal circulants are presented in Table 1.

4.2. Review of Finite Fields Based QC-LDPC Codes and Their Exponent Matrices

In this subsection, we will review the
finite field based method for constructing QC-LDPC codes, and provide two classes of exponent matrices of these QC-LDPC codes [18].

4.2.1. A General Construction of QC-LDPC Codes Based on Finite Fields of Characteristic 2. Let $GF(q)$ be a finite field with $q = 2^t$ with $t \geq 2$, and let $\alpha$ be a primitive element of $GF(q)$. For each nonzero element $\alpha^i$ with $0 \leq i \leq q - 2$, we define the location vector $\nu(\alpha^i)$ as a $(q - 1)$-tuple over $GF(2)$,

$$\nu(\alpha^i) = (v_1, v_2, \ldots, v_{q-1}),$$

whose components correspond to the nonzero elements $\alpha_0, \alpha_1, \ldots, \alpha_{q-2}$ of $GF(q)$, where the $i$-th component $v_i$ is set to 1 and all the other $(q - 2)$ components are 0. Hence, based on the nonzero element $\alpha^i$ of $GF(q)$, we can uniquely form a $(q - 1) \times (q - 1)$ square matrix $M(\alpha^i)$ whose $j$-th row is obtained by cyclically shifting every component of the $(q - 1)$-ary location vector $\nu(\alpha^i)(j - 1)$ places to the right (or left) for $0 \leq i \leq q - 2$, $1 \leq j \leq q - 1$. The resulting square matrix $M(\alpha^i)$ is a CPM, and it is also referred to as the $(q - 1)$-fold matrix dispersion (or expansion) of the nonzero field element $\alpha^i$ over $GF(2)$ [18].

\begin{table}
\caption{Some optimal $L \times L$ circulants with row/column weight $\rho$ and location set $S$.}
\begin{tabular}{cccccccc}
$L$ & $\rho$ & Location set $S$ & $L$ & $\rho$ & Location set $S$ & $L$ & $\rho$ & Location set $S$ \\
4 & 4 & [0, 1, 2, 3] & 5 & 5 & [0, 1, 2, 3, 4] & 6 & 6 & [0, 1, 2, 3, 4, 5] \\
5 & 4 & [0, 1, 2, 3] & 6 & 5 & [0, 1, 2, 3, 4] & 7 & 6 & [0, 1, 2, 3, 4, 5] \\
6 & 4 & [0, 1, 2, 3] & 7 & 5 & [0, 1, 2, 3, 4] & 8 & 6 & [0, 1, 2, 3, 4, 5] \\
7 & 4 & [0, 1, 2, 4] & 8 & 5 & [0, 1, 2, 3, 5] & 9 & 6 & [0, 1, 2, 3, 4, 6] \\
8 & 4 & [0, 1, 2, 4] & 9 & 5 & [0, 1, 2, 3, 5] & 10 & 6 & [0, 1, 2, 3, 4, 6] \\
9 & 4 & [0, 1, 2, 4] & 10 & 5 & [0, 1, 2, 3, 6] & 11 & 6 & [0, 1, 2, 3, 4, 7] \\
10 & 4 & [0, 1, 2, 5] & 11 & 5 & [0, 1, 2, 4, 7] & 12 & 6 & [0, 1, 2, 3, 4, 8] \\
11 & 4 & [0, 1, 2, 5] & 12 & 5 & [0, 1, 2, 4, 7] & 13 & 6 & [0, 1, 2, 3, 5, 9] \\
12 & 4 & [0, 1, 3, 7] & 13 & 5 & [0, 1, 2, 4, 7] & 14 & 6 & [0, 1, 2, 3, 5, 9] \\
13 & 4 & [0, 1, 3, 9] & 14 & 5 & [0, 1, 2, 4, 7] & 15 & 6 & [0, 1, 2, 3, 6, 10] \\
14 & 4 & [0, 1, 3, 9] & 15 & 5 & [0, 1, 2, 4, 7] & 16 & 6 & [0, 1, 2, 3, 6, 10] \\
15 & 4 & [0, 1, 3, 7] & 16 & 5 & [0, 1, 2, 5, 8] & 17 & 6 & [0, 1, 2, 3, 6, 10] \\
16 & 4 & [0, 1, 3, 7] & 17 & 5 & [0, 1, 2, 4, 12] & 18 & 6 & [0, 1, 2, 4, 8, 13] \\
17 & 4 & [0, 1, 3, 7] & 18 & 5 & [0, 1, 2, 5, 11] & 19 & 6 & [0, 1, 2, 4, 8, 13] \\
18 & 4 & [0, 1, 3, 7] & 19 & 5 & [0, 1, 2, 6, 9] & 20 & 6 & [0, 1, 2, 4, 7, 12] \\
19 & 4 & [0, 1, 3, 8] & 20 & 5 & [0, 1, 2, 5, 14] & 21 & 6 & [0, 1, 2, 4, 7, 12] \\
20 & 4 & [0, 1, 3, 14] & 21 & 5 & [0, 1, 4, 14, 16] & 22 & 6 & [0, 1, 2, 4, 7, 13] \\
21 & 4 & [0, 1, 3, 9] & 22 & 5 & [0, 1, 3, 14] & 23 & 6 & [0, 1, 2, 4, 7, 15] \\
22 & 4 & [0, 1, 3, 9] & 23 & 5 & [0, 1, 3, 8, 14] & 24 & 6 & [0, 1, 2, 4, 12, 19] \\
23 & 4 & [0, 1, 3, 10] & 24 & 5 & [0, 1, 3, 11, 20] & 25 & 6 & [0, 1, 2, 4, 9, 15] \\
24 & 4 & [0, 1, 3, 10] & 25 & 5 & [0, 1, 3, 15, 21] & 26 & 6 & [0, 1, 2, 4, 7, 13] \\
25 & 4 & [0, 1, 3, 10] & 26 & 5 & [0, 1, 3, 17, 12] & 27 & 6 & [0, 1, 2, 4, 7, 17] \\
26 & 4 & [0, 1, 3, 11] & 27 & 5 & [0, 1, 3, 7, 18] & 28 & 6 & [0, 1, 4, 15, 20, 22] \\
27 & 4 & [0, 1, 4, 10] & 28 & 5 & [0, 1, 3, 13, 24] & 29 & 6 & [0, 1, 2, 4, 18, 23] \\
28 & 4 & [0, 1, 3, 12] & 29 & 5 & [0, 1, 3, 17, 19] & 30 & 6 & [0, 1, 2, 5, 14, 24] \\
\end{tabular}
\end{table}

\begin{align}
\alpha \rho^{-1} \gamma \rho^{-1} & = \begin{bmatrix}
\alpha_{p_1} & \alpha_{p_2} & \ldots & \alpha_{p_r}
\end{bmatrix} \\
\rho^{-1} & = \begin{bmatrix}
\alpha_{p_1} & \alpha_{p_2} & \ldots & \alpha_{p_r}
\end{bmatrix} \\
\gamma \rho^{-1} & = \begin{bmatrix}
\alpha_{p_1} & \alpha_{p_2} & \ldots & \alpha_{p_r}
\end{bmatrix}
\end{align}

whose rows satisfy the following two constraints: (1) for $1 \leq i < \gamma$, $0 \leq k, l \leq q - 2$ and $k \neq l$, $\alpha^k p_i$ and $\alpha^l p_j$ have at most one position where both of them have the same element of $GF(q)$ (i.e., they differ in at least $(\rho - 1)$ positions); (2) for $1 \leq i < j < \gamma$, $i \neq j$, and $0 \leq k, l \leq q - 2, \alpha^k p_i$ and $\alpha^l p_j$ differ in at least $(\rho - 1)$ positions. These constraints are called $\alpha$-multiplied row constraints in [18]. By replacing each of its entry $p_{ij}$ with the binary matrix $M(p_{ij})$, we obtain a $\gamma \times \rho$ array $H_{\gamma \rho}$ of $(q - 1) \times (q - 1)$ CPMs. Then, the null space of $H_{\gamma \rho}$ gives a binary $(\gamma, \rho)$-regular QC-LDPC code. Moreover, the $\alpha$-multiplied row constraints ensure the Tanner graph of this code free of cycle of length $4$. Hence, the constructed QC-LDPC codes have girth at least 6.
| Row index | Nonzero field elements | Row index | Nonzero field elements | Row index | Nonzero field elements |
|-----------|------------------------|-----------|------------------------|-----------|------------------------|
| 1         | 37 9 22 0              | 2         | 7 0 18 44              | 3         | 0 37 9 19              |
| 4         | 0 37 19 9              | 5         | 7 18 44 0              | 6         | 7 18 0 44              |
| 7         | 7 18 44 0              | 8         | 0 18 44 7              | 9         | 19 9 0 37              |
| 10        | 0 18 44 7              | 11        | 0 9 37 22              | 12        | 18 44 7 0              |
| 13        | 9 19 37 0              | 14        | 18 0 7 44              | 15        | 22 37 0 9              |
| 16        | 0 44 7 18              | 17        | 9 19 0 37              | 18        | 37 22 0 9              |
| 19        | 44 18 7 0              | 20        | 9 37 22 0              | 21        | 19 37 9 0              |
| 22        | 0 22 37 9              | 23        | 44 7 0 18              | 24        | 0 7 44 18              |
| 25        | 9 37 0 22              | 26        | 19 9 37 0              | 27        | 22 37 0 9              |
| 28        | 9 22 37 0              | 29        | 9 37 22 0              | 30        | 37 9 22 0              |
| 31        | 37 9 22 0              | 32        | 22 9 37 0              | 33        | 7 18 44 0              |
| 34        | 37 0 9 22              | 35        | 0 37 22 9              | 36        | 7 44 0 18              |
| 37        | 0 44 7 18              | 38        | 9 0 37 19              | 39        | 37 9 22 0              |
| 40        | 0 9 19 37              | 41        | 44 7 18 0              | 42        | 37 19 9 0              |
| 43        | 9 37 0 19              | 44        | 37 9 22 0              | 45        | 0 9 19 37              |
| 46        | 19 9 37 0              | 47        | 0 19 37 9              | 48        | 48 18 0 44              |
| 49        | 0 19 9 37              | 50        | 22 37 9 0              | 51        | 37 0 9 22              |
| 52        | 18 7 0 44              | 53        | 0 9 37 22              | 54        | 44 7 0 18              |
| 55        | 7 0 44 18              | 56        | 9 0 37 22              | 57        | 9 19 37 0              |
| 58        | 22 9 37 0              | 59        | 9 19 37 0              | 60        | 0 44 18 7              |
| 61        | 18 7 0 44              | 62        | 9 37 0 19              | 63        | 22 0 9 37              |
| 64        | 7 0 18 44              | 65        | 18 44 0 7              | 66        | 19 37 9 0              |
| 67        | 9 37 22 0              | 68        | 18 7 0 44              | 69        | 9 22 37 0              |
| 70        | 37 22 0 9              | 71        | 18 7 44 0              | 72        | 7 0 18 44              |
| 73        | 0 22 37 9              | 74        | 7 0 44 18              | 75        | 9 0 19 37              |
| 76        | 37 0 9 19              | 77        | 9 37 0 22              | 78        | 0 37 9 22              |
| 79        | 37 9 0 22              | 70        | 9 0 37 19              | 81        | 7 44 18 0              |
| 82        | 0 44 18 7              | 83        | 9 37 22 0              | 84        | 44 0 7 18              |
| 85        | 9 22 37 0              | 86        | 9 19 37 0              | 87        | 22 9 0 37              |
| 88        | 0 18 7 44              | 89        | 9 37 22 0              | 90        | 18 0 7 44              |
| 91        | 37 22 9 0              | 92        | 0 22 37 9              | 93        | 37 0 9 22              |
| 94        | 9 22 37 0              | 95        | 44 0 7 18              | 96        | 0 9 19 37              |
| 97        | 7 44 18 0              | 98        | 9 19 37 0              | 99        | 7 44 18 0              |
| 100       | 37 22 0 9              | 101       | 0 22 9 37              | 102       | 22 0 37 9              |
| 103       | 44 0 7 18              | 104       | 22 9 0 37              | 105       | 22 0 37 9              |
| 106       | 44 7 18 0              | 107       | 37 19 0 9              | 108       | 0 9 37 22              |
| 109       | 9 37 19 0              | 110       | 44 18 0 7              | 111       | 37 22 9 0              |
| 112       | 7 0 18 44              | 113       | 0 18 44 7              | 114       | 37 22 0 9              |
| 115       | 0 9 37 19              | 116       | 7 18 0 44              | 117       | 9 0 22 37              |
| 118       | 0 37 19 9              | 119       | 0 9 22 37              | 120       | 19 37 9 0              |
| 121       | 37 0 22 9              | 122       | 37 0 9 22              | 123       | 0 7 44 18              |
| 124       | 19 37 0 9              | 125       | 7 44 18 0              | 126       | 7 44 0 18              |
| 127       | 9 37 22 0              | 128       | 44 18 0 7              | 129       | 7 18 0 44              |
| 130       | 0 37 22 9              | 131       | 7 0 44 18              | 132       | 0 37 9 22              |
| 133       | 0 44 18 7              | 134       | 44 18 7 0              | 135       | 9 37 0 22              |
| 136       | 9 37 19 0              | 137       | 0 9 37 22              | 138       | 18 44 7 0              |
| 142       | 0 18 7 44              | 143       | 44 18 0 7              | 144       | 9 37 0 19              |
| 145       | 0 18 7 44              | 146       | 18 0 7 44              | 147       | 19 9 37 0              |
According to the definition in Section 2, we can see that the above $\gamma \times p$ matrix $P$ is the exponent matrix, and the associate expansion factor is $(q - 1)$. A framework for constructing such matrix $P$ based on two arbitrary subsets of a finite field was proposed in [40]. Let $\eta$ be a nonzero element in $\text{GF}(q)$ and $\alpha$ be a primitive element. For $1 \leq \gamma, \rho \leq q$, let $T_1 = \{\alpha^i, \alpha^{i+\rho}, \ldots, \alpha^\rho\}$ and $T_2 = \{\alpha^i, \alpha^{i+1}, \ldots, \alpha^{i+p}\}$ be two arbitrary subsets of elements in $\text{GF}(q)$ with $i_1$ and $j_1$ in the set $\{-\infty, 0, 1, 2, \ldots, q-2\}$, $i_1 < j_1 < \cdots < j_p$, and $j_1 < j_2 < \cdots < j_p$.

The $\gamma \times \rho$ matrix $P$ can be formed by

$$P(\eta) = [\eta \alpha^i + \alpha^j]_{1 \leq i \leq \gamma, 1 \leq j \leq \rho}.$$

(11)

Under this framework, some well-known constructions of QC-LDPC codes based on finite fields and combinatorial designs are special cases [14, 41, 42]. For example, when $T_1 \subseteq T_2 = \text{GF}(q)$, $P(\eta)$ is a Latin square over $\text{GF}(q)$ [43].

4.2.2. Construction of QC-LDPC Codes Based on Prime Fields. Let $p$ be a prime. Consider a $p \times p$ matrix

$$P = \begin{bmatrix}
p_{1,1} & p_{1,2} & \cdots & p_{1,p} \\
p_{2,1} & p_{2,2} & \cdots & p_{2,p} \\
\vdots & \vdots & \ddots & \vdots \\
p_{p,1} & p_{p,2} & \cdots & p_{p,p}
\end{bmatrix},$$

(12)

where $p_{ij} = (i - 1) \cdot (j - 1) \mod p$ for $1 \leq i \leq p$ and $1 \leq j \leq p$.

We select a $\gamma \times \rho$ submatrix from $P$ and replace its elements $p_{ij}$ with the CPMs $I(p_{ij})$ for $1 \leq s \leq \gamma$ and $1 \leq t \leq \rho$. The

| Row index | Nonzero field elements | Row index | Nonzero field elements | Row index | Nonzero field elements |
|-----------|------------------------|-----------|------------------------|-----------|------------------------|
| 148       | 44 18 0 7              | 149       | 37 0 19 9              | 150       | 19 0 9 37             |
| 151       | 0 37 22 9              | 152       | 7 18 44 0              | 153       | 44 0 7 18             |
| 154       | 0 37 9 22              | 155       | 18 0 7 44              | 156       | 44 7 0 18             |
| 157       | 0 37 9 19              | 158       | 22 37 9 0              | 159       | 7 18 44 0             |
| 160       | 0 37 9 19              | 161       | 0 18 7 44              | 162       | 0 37 9 22             |
| 163       | 7 18 44 0              | 164       | 37 22 9 0              | 165       | 0 9 37 22             |
| 166       | 0 37 19 9              | 167       | 18 44 7 0              | 168       | 9 0 22 37             |
| 169       | 0 37 9 19              | 170       | 44 0 18 7              | 171       | 9 0 19 37             |
| 172       | 9 0 37 19              | 173       | 9 37 19 0              | 174       | 44 7 18 0             |
| 175       | 19 0 37 9              | 176       | 0 22 9 37              | 177       | 9 0 37 19             |
| 178       | 0 18 7 44              | 179       | 0 18 44 7              | 180       | 37 0 9 22             |
| 181       | 9 37 0 22              | 182       | 0 37 22 9              | 183       | 37 0 22 9             |
| 184       | 0 9 22 37              | 185       | 9 19 37 0              | 186       | 44 7 18 0             |
| 187       | 19 9 0 37              | 188       | 37 0 9 22              | 189       | 18 44 7 0             |
| 190       | 22 37 9 0              | 191       | 37 19 0 9              | 192       | 37 9 19 0             |
| 193       | 0 37 19 9              | 194       | 22 0 37 9              | 195       | 7 0 18 44             |
| 196       | 37 19 0 9              | 197       | 9 37 22 0              | 198       | 22 9 0 37             |
| 199       | 44 18 0 7              | 200       | 22 37 9 0              | 201       | 7 18 0 44             |
| 202       | 0 19 9 37              | 203       | 0 44 18 7              | 204       | 9 0 19 37             |
| 205       | 7 18 0 44              | 206       | 37 19 0 9              | 207       | 7 44 18 0             |
| 208       | 44 0 18 7              | 209       | 19 37 0 9              | 210       | 44 18 0 7             |
| 211       | 19 37 0 9              | 212       | 18 0 7 44              | 213       | 37 9 22 0             |
| 214       | 0 9 37 22              | 215       | 22 37 0 9              | 216       | 19 37 9 0             |
| 217       | 7 44 18 0              | 218       | 44 0 7 18              | 219       | 44 0 7 18             |
| 220       | 37 0 19 9              | 221       | 22 9 0 37              | 222       | 44 18 7 0             |
| 223       | 19 9 37 0              | 224       | 18 0 44 7              | 225       | 37 22 9 0             |
| 226       | 22 9 37 0              | 227       | 19 37 9 0              | 228       | 18 44 7 0             |
| 229       | 9 19 37 0              | 230       | 18 44 7 0              | 231       | 18 7 0 44             |
| 232       | 44 7 18 0              | 233       | 44 18 0 7              | 234       | 37 22 9 0             |
| 235       | 22 9 37 0              | 236       | 22 9 0 37              | 237       | 44 0 18 7             |
| 238       | 0 37 22 9              | 239       | 37 22 0 9              | 240       | 37 19 0 9             |
| 241       | 0 18 7 44              | 242       | 19 37 0 9              | 243       | 22 9 0 37             |
| 244       | 44 7 18 0              | 245       | 9 37 19 0              | 246       | 37 22 0 9             |
| 247       | 44 7 18 0              | 248       | 19 9 0 37              |           |                       |
following $\gamma \times \rho$ array $H(P_{\gamma \times \rho})$ of $p \times p$ CPMs over GF (2) is obtained.

$$H(P_{\gamma \times \rho}) = [I(P_{i,j})]_{1 \leq i \leq \gamma, 1 \leq j \leq \rho}$$  \hspace{1cm} (13)

Actually, the null space of $H(P_{\gamma \times \rho})$ gives a $(\gamma, \rho)$-regular QC-LDPC code of length $p\rho$ with girth at least 6. Notice that the exponent matrix of this code is $P_{\gamma \times \rho}$ and the expansion factor is $p$. How to select good $\gamma$ rows and $\rho$ columns of CPMs from $P$ in (12) can be found in [44, 45].

4.3. Nonbinary LDPC Cycle Codes and Numerical Results.

Combined with Sections 3.2, 4.1, and 4.2, we can easily construct a binary QC-LDPC cycle code. Based on the replacement of the nonzero elements in finite fields, nonbinary LDPC cycle codes can be constructed. In order to show the advantages of our proposed construction methods, we next provide some examples.

Table 3: The nonzero field elements in the corresponding CPMs of the parity-check matrices $H_{M,256}(P_{8 \times 16})$ in Example 2.

| Row index | Nonzero field elements |
|-----------|------------------------|
| 1         | 0                      |
| 2         | 173                    |
| 3         | 40                     |
| 4         | 40                     |
| 5         | 40                     |
| 6         | 172                    |
| 7         | 8                      |
| 8         | 40                     |

Example 1. Consider the all-one matrix (or circulant) of size $4 \times 4$. It is clear that there is only one such circulant for $\rho = 4$ and $L = 4$. The following $8 \times 16$ matrix $B_{8 \times 16}$ can be obtained based on the construction framework in Section 3.2.
Based on the prime field GF (31), we can construct an 8 × 16 array \( H(P_{8 \times 16}) \) of 31 × 31 CPMs in the form of (13) by choosing the 8 rows and 16 columns from the exponent matrix \( P \) in equation (12). The indices of the chosen 8 rows and 16 columns from \( P \) are \([3, 4, 5, 10, 15, 17, 24, 28]\) and \([1, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 19, 26, 28, 29, 30]\), respectively. The row and column selection method is based on the proposed method in [45]. By employing \( B_{8 \times 16} \) to mask \( H(P_{8 \times 16}) \), we can obtain an array \( H_M(P_{8 \times 16}) \) of 31 × 31 CPMs. By replacing the 1’s of each row in \( H_M(P_{8 \times 16}) \) with nonzero elements of GF (64) in the corresponding rows of Table 2, a (2,4)-regular matrix \( H_{M,M}(P_{8 \times 16}) \) over GF (64) is obtained. Note that the numbers in Table 2 are the power numbers of \( \alpha \), where \( \alpha \) is a primitive element of GF (64) created by using the primitive polynomial \( p(x) = 1 + x + x^6 \). The null space of \( H_{M,M}(P_{8 \times 16}) \) gives a (2,4)-regular (496,248) LDPC cycle code over GF (64). The girth of this code is 16, and the number of 16-cycles is 775. For comparison, we reconstruct the irregular (504,252) QC-LDPC code over GF (64) based on finite field GF (64) and the masking matrix \( B_{mask}(4,8) \) in [15]. The average column and row weights in the parity-check matrix of (504,252) QC-LDPC code over GF (64) are 2.5 and 5, respectively. Moreover, we also design a comparable (2,4)-regular (496,248) LDPC code over GF (64) based on the progressive edge-growth (PEG) algorithm in [46], called PEG-LDPC code. Note that the nonzero elements of the parity-check matrix of the (2,4)-regular (496,248) PEG-LDPC code over GF (64) are randomly chosen. When decoded with the fast-Fourier-transform (FFT) based Q-ary sum-product algorithm (QSPA) with 50 iterations, the bit/word error rates (BERs/WERs) of these three nonbinary LDPC codes are shown in Figure 4. In the simulations, the transmissions on the BPSK modulated AWGN channel are assumed. We can see that at the BER of 2 × 10^{-6}, the proposed nonbinary (496,248) LDPC cycle code outperforms the (504,252) QC-LDPC code over GF (64) and the (2,4)-regular (496,248) PEG-LDPC code over GF (64) about 0.45 dB and 0.1 dB, respectively. In the above example, we can see that the constructed nonbinary LDPC cycle code is not quasi-cyclic because of the randomly chosen nonzero field elements in its parity-check matrix. Hence, we next show the performances of the proposed nonbinary QC-LDPC cycle codes.

Example 2. Consider the prime field GF (19), we can easily construct an 8 × 16 matrix \( P_{8 \times 16} \) over GF (19) based on the exponent matrix \( P \) in equation (12) and the construction framework in Section 3.2. The indices of the chosen 8 rows and 16 columns from \( P \) are \([1, 2, 4, 5, 7, 8, 14, 15]\) and \([1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]\), respectively. The row and column selection method is also based on the proposed method in [45]. By dispersing each entry of \( P_{8 \times 16} \) into the corresponding CPMs of size 19 × 19, an 8 × 16 array \( H(P_{8 \times 16}) \) of 19 × 19 CPMs is obtained. By employing \( B_{8 \times 16} \) in equation (14) to mask the array \( H(P_{8 \times 16}) \), we can construct an array \( H_M(P_{8 \times 16}) \) of 19 × 19 CPMs. By replacing the 1’s of each CPM in \( H_M(P_{8 \times 16}) \) with the same nonzero element of GF (256) in the corresponding row of Table 3, a (2,4)-regular matrix \( H_{M,M}(P_{8 \times 16}) \) over GF (256) is obtained. It is noticeable that the nonzero elements in Table 3 are the power numbers of \( \alpha \), where \( \alpha \) is a primitive element of GF (256) created by using the primitive polynomial \( p(x) = 1 + x^2 + x^3 + x^4 + x^6 \). The null space of \( H_{M,M}(P_{8 \times 16}) \) gives a (2,4)-regular (304,152) LDPC cycle code over GF (256). The girth of this code is 16, and the number of 16-cycles is 969. Since the nonzero field elements of a
CPM in $H_{M,256}(P_{8x162})$ are the same, and the resulting (304,152) LDPC cycle code over GF (256) is quasi-cyclic.

For comparison, we construct the (3,6)-regular (2432, 1216) LDPC code over GF (2) based on the progressive edge-growth (PEG) algorithm in [46]. The error performances of these two LDPC codes are shown in Figure 5. In the simulations, the employed decoding algorithms of the constructed (304,152) QC-LDPC cycle code over GF (256) and the (2432,1216) LDPC code over GF (2) are the QSPA (50 iterations) and the sum-product algorithm (SPA) with 50 iterations, respectively. The transmissions on the BPSK modulated AWGN channel are assumed. We can see that the constructed (304,152) QC-LDPC cycle code over GF (256) can outperform the (2432,1216) LDPC code over GF (2) about 0.75 dB at the BER of $10^{-6}$.

5. Conclusion

This paper proposed a design framework of binary QC-LDPC cycle codes and constructed nonbinary LDPC (NBLDPC) cycle codes based on circulants and finite fields. The presented construction method consists of three parts. First, the masking matrices are designed based on circulants and the point-line relation in graph theory. Second, the exponent matrices of binary QC-LDPC cycle codes are constructed from finite fields and the designed masking matrices. Third, by replacing 1’s in the parity-check matrices of binary QC-LDPC cycle codes with the nonzero field elements, NBLDPC cycle codes are obtained. Numerical results show that the constructed NBLDPC cycle codes have good iterative decoding performance.

Data Availability

The data used to support the findings of this study is available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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