Recent positivity constraints for
spin observables and parton distributions

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Abstract
Spin observables allow a deeper understanding of the nature of the underlying
dynamics and positivity reduces substantially their allowed domains. We
will present some new positivity constraints for spin observables and their
implications for parton distributions. We will also make some comparisons
with recent data.

1 Introduction
Spin observables for any particle reaction, contain some unique information
which is very useful to check the validity of theoretical assumptions. Positiv-
ity reduces substantially the allowed domain for spin observables and can be
used to determine these domains for exclusive and inclusive spin-dependent
reactions. We emphasize the relevance of positivity in spin physics, which
puts non-trivial model independent constraints on spin observables. If one,
two or several observables are measured, the constraints can help to decide
which new observable will provide the best improvement of knowledge. Dif-
f erent methods can be used to establish these constraints and they have been
presented together with many interesting cases in a recent review article [1].
Here we will present some new positivity constraints for spin observables and
their implications for parton distributions with some comparisons with re-
cent data.

1Invited talk presented at the 14th Workshop on Elastic and Diffractive Scattering
(EDS Blois Workshop), December 15-21, 2011, Qui Nhon, Vietnam
2 The quark transversity distribution $\delta q(x, Q^2)$

The quark transversity distribution was first mentioned by Ralston and Soper in 1979, in $pp \rightarrow \mu^+\mu^-X$, with transversely polarized protons, but forgotten until 1990, where it was realized that it completes the description of the quark distribution in a nucleon as a density matrix $Q(x, Q^2) = q(x, Q^2)I \otimes I + \Delta q(x, Q^2)\sigma_3 \otimes \sigma_3 + \delta q(x, Q^2)(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+).$

(1)

So in addition to $q(x, Q^2)$ and $\Delta q(x, Q^2)$, the helicity distribution, there is a new distribution function $\delta q(x, Q^2)$, (also denoted $\Delta_Tq(x, Q^2)$), chiral odd and leading twist, which decouples from Deep Inelastic Scattering. By requiring the positivity of the above density matrix, one finds that $\delta q(x, Q^2)$ must satisfy the following positivity bound $[2]$

$$q(x, Q^2) + \Delta q(x, Q^2) \geq 2|\delta q(x, Q^2)|,$$

which survives up to next-to leading order corrections.

We show in Fig. 1 (Top), that the region restricted by Eq. (2) is only half of the entire square. This theoretical result is not new, but recently it became very interesting to test, with the first extraction of $\delta q$. The transversity distributions for $u$ and $d$ quarks were determined indirectly from a global analysis $[3]$ of the experimental data on azimuthal asymmetries in seminclusive deep inelastic scattering (SIDIS), from the HERMES and COMPASS Collaborations, and in $e^+e^- \rightarrow h_1h_2X$ processes, from the Belle Collaboration. The results at $Q^2 = 2.4\text{GeV}^2$ are shown in Fig. 1 (Bottom) and one notices that in both cases $|\Delta_Tq| \leq |\Delta q|$. It is clear that the uncertainty on this determination is still rather large. The positivity bound is well respected for the $u$ quark, but saturation, or perhaps violation, seems to occur for the $d$ quark in the high $x$ region and this is partly related to the fact that $\Delta d$ is large and negative in this region. A special effort should be made on the experimental side to clarify this very important issue.

3 General positivity bounds in particle inclusive production

To start, we consider the inclusive reaction of the type $A(\text{spin } 1/2)+B(\text{spin } 1/2) \rightarrow C+X$, where both initial spin 1/2 particles can be in any possible directions
Figure 1: (Top): Allowed domain corresponding to the constraint Eq. (2). (Bottom): Transversity distributions for $u$ and $d$ quarks (solid curves inside uncertainties bands), positivity bounds (highest and lowest lines) and helicity distributions (dashed curves) (Taken from Ref. (3)).
and no polarization is observed in the final state. The spin-dependent corresponding cross section \( \sigma(P_a, P_b) = \text{Tr}(M\rho) \), can be defined through the \( 4 \times 4 \) cross section matrix \( M \) and the spin density matrix \( \rho \), where \( P_a, P_b \) are the spin unit vectors of \( A \) and \( B \), \( \rho = \rho_a \otimes \rho_b \) is the spin density matrix with \( \rho_a = (I_2 + P_a \cdot \vec{\sigma}_a)/2 \), and similar for \( \rho_b \). Here \( I_2 \) is the \( 2 \times 2 \) unit matrix, and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) stands for the \( 2 \times 2 \) Pauli matrices. For the parity-conserving process, \( M \) could be parametrized in the following way

\[
M = \sigma_0[I_4 + A_{aN}\sigma_{ay} \otimes I_2 + A_{bN}I_2 \otimes \sigma_{by} + A_{NN}\sigma_{ay} \otimes \sigma_{by} + A_{LL}\sigma_{az} \otimes \sigma_{bz} + A_{SS}\sigma_{ax} \otimes \sigma_{bx} + A_{LS}\sigma_{az} \otimes \sigma_{bx} + A_{SL}\sigma_{ax} \otimes \sigma_{bz}].
\]

Here \( I_4 \) is the \( 4 \times 4 \) unit matrix and \( \sigma_0 \) stands for the spin-averaged cross section. In other words, for a parity-conserving process, there are eight independent spin-dependent observables: the unpolarized cross section \( \sigma_0 \), two single transverse spin asymmetries \( A_{aN} \) and \( A_{bN} \), and five double spin asymmetries \( A_{NN}, A_{LL}, A_{SS}, A_{LS}, \) and \( A_{SL} \). Here the subscript \( L, N, S \) represents the unit vectors along the spin directions of initial particles \( A \) and \( B \). Specifically in the center-of-mass system of \( A \) and \( B \), \( L, N, S \) are along the incident momentum, along the normal to the scattering plane which contains \( A, B \) and \( C \), and along \( N \times L \), respectively.

The crucial point is that \( M \) is a Hermitian and positive matrix and in order to derive the positivity conditions. In the transverse basis where \( \sigma_y \) is diagonal, we have found that the diagonal matrix elements \( M_{ii} \) are given by

\[
M_{11} = (1 + A_{NN}) + (A_{aN} + A_{bN}), \quad M_{22} = (1 - A_{NN}) + (A_{aN} - A_{bN}),
\]

\[
M_{33} = (1 - A_{NN}) - (A_{aN} - A_{bN}), \quad M_{44} = (1 + A_{NN}) - (A_{aN} + A_{bN}).
\]

Since one of the necessary conditions for a Hermitian matrix to be positive definite is that all the diagonal matrix elements has to be positive \( M_{ii} \geq 0 \), we thus derive \( 1 \pm A_{NN} \geq |A_{aN} \pm A_{bN}| \), which is valid in full generality, for both parity-conserving and parity-violating processes.

Back to the case for \( p^+ + p^+ \to C + X \) where the initial particles are identical, we have \( A_{aN}(y) = -A_{bN}(-y) \). Using this relation, one obtains,

\[
1 \pm A_{NN}(y) \geq |A_{N}(y) \mp A_{N}(-y)|.
\]

This is an interesting result which, can be used, in principle, with available data on \( A_N \) for \( \pi^\pm, K^\pm, \pi^0, \eta \) production, to put some non trivial
Figure 2: (Top): The dependence of $A_N$ on $x_F$ is plotted for $2\gamma$ production in the $\pi^0$ and $\eta$ mass regions (Taken from Ref. [5]). (Bottom): $A_N$ for the reaction $pp \rightarrow nX$ versus the neutron transverse momentum $q_T$, measured at various energies. The asterisks are the result of the calculation (Taken from Ref. [6]).
contraints on $A_{NN}(y)$. For illustration we display in Fig. 2 some recent unexpected and large results on single spin asymmetries. In the case of neutron production, the asymmetry corresponds to very forward neutrons and the big effect observed is due to an interference of $\pi - a_1$ Regge exchanges [6]. Conversely, for prompt photon and jet production, theoretically $A_{NN}$ is expected to be small, except in the high $p_T$ region, because since there is no transversity for the gluons, its sensitivity to $\delta q$ is only in the high $p_T$ region, dominated by $qq \rightarrow qq$. By making use of the positivity bound on transversity, it was possible to estimate $A_{NN}$ [7] and it was observed that $|A_{NN}| << |A_{LL}|$. It is important to verify this theoretical expectation.

4 Positivity bounds for Sivers functions

Now let us study the implication of $1 \pm A_{NN}(y) \geq |A_N(y) \pm A_N(-y)|$, in the parity-violating process $p^\uparrow + p^\uparrow \rightarrow W^\pm + X$. We denote by $y$ and $q_\perp$ the rapidity and transverse momentum of the $W$ boson. Since we will assume $A_N \approx 0$, for $y = 0$, the bound reduces to $1 \geq 2|A_N(y = 0)|$, to be compared with the usual trivial bound $1 \geq |A_N(y = 0)|$.

The TMD quark distribution in a transversely polarized hadron of spin $\vec{S}$, can be expanded as

$$f_{q/h}(x, k_\perp) \equiv f_{q/h}(x, k_\perp) + \frac{1}{2} \Delta_N f_{q/h^\uparrow}(x, k_\perp) \vec{S} \cdot (\hat{p} \times \hat{k}_\perp), \quad (6)$$

where $\hat{p}$ and $\hat{k}_\perp$ are the unit vectors of $\vec{p}$ and $k_\perp$, respectively. $f_{q/h}(x, k_\perp)$ is the spin-averaged TMD distribution, and $\Delta_N f_{q/h^\uparrow}(x, k_\perp)$ is the Sivers function. There is a trivial positivity bound for the Sivers functions which reads $|\Delta_N f_{q/h^\uparrow}(x, k_\perp)| \leq 2f_{q/h}(x, k_\perp)$. $A_N$ is expressed by an integral in terms of the Sivers functions, and it is usually assumed in the phenomenological studies that the $x$ and $k_\perp$ dependence of the TMD distributions can be further factorized as follows

$$f_{q/h}(x, k_\perp) = f_q(x)g(k_\perp), \quad (7)$$
$$\Delta_N f_{q/h^\uparrow}(x, k_\perp) = \Delta^N f_{q/h^\uparrow}(x) h(k_\perp)g(k_\perp).$$

For the $k_\perp$ dependence, a Gaussian ansatz is usually introduced, which has the form

$$g(k_\perp) = \frac{1}{\pi \langle k^2_\perp \rangle} e^{-k^2_\perp / \langle k^2_\perp \rangle}, \quad (8)$$
\[ h(k_\perp) = \sqrt{2} e^{\frac{k_\perp}{M_1}} e^{-\frac{k_\perp^2}{M_1^2}}. \]

This choice satisfies \( h(k_\perp) \leq 1 \). Thus the previous positivity bound implies \( |\Delta^N f_{q/h^+(x)}| \leq 2f_q(x) \). Then one can carry out the integration analytically in \( A_N \) to obtain

\[ A_N(y = 0) = H(q_\perp) \frac{\sum_{ab} |V_{ab}|^2 \Delta^N f_{a/p^+}(x) f_b(x)}{\sum_{ab} |V_{ab}|^2 f_a(x) f_b(x)}, \quad (9) \]

where \( x = M_W/\sqrt{s} \) for \( y = 0 \) and \( H(q_\perp) \) is given by

\[ H(q_\perp) = \vec{S}_\perp \cdot (\hat{p} \times q_\perp) \frac{\sqrt{2} e^{\frac{k_\perp^2}{M_1}}}{\langle k_\perp^2 \rangle + \langle k_\perp^2 \rangle} e^{-\left[ \frac{\langle k_\perp^2 \rangle - \langle k_\perp^2 \rangle}{\langle k_\perp^2 \rangle + \langle k_\perp^2 \rangle} \right] \frac{q_\perp^2}{2\langle k_\perp^2 \rangle}}, \quad (10) \]

with \( \langle k_\perp^2 \rangle = M_1^2 \langle k_\perp^2 \rangle/[M_1^2 + \langle k_\perp^2 \rangle] \).

The \( q_\perp \)-dependent function \( H(q_\perp) \) reaches its maximum \( H(q_\perp)_{\text{max}} \) when \( q_\perp^2 = \langle k_\perp^2 \rangle (\langle k_\perp^2 \rangle + \langle k_\perp^2 \rangle) / (\langle k_\perp^2 \rangle - \langle k_\perp^2 \rangle) \), with \( H(q_\perp)_{\text{max}} \) given by

\[ H(q_\perp)_{\text{max}} = \frac{\langle k_\perp^2 \rangle^2}{\langle k_\perp^2 \rangle + \langle k_\perp^2 \rangle} \left[ \frac{2\langle k_\perp^2 \rangle}{M_1^2} \langle k_\perp^2 \rangle - \langle k_\perp^2 \rangle \right]^{\frac{1}{2}}. \quad (11) \]

Using the fact that \( 1 \geq \frac{1}{2} |A_N(y = 0)| \) for any \( q_\perp \) and \( \sqrt{s} \), we thus derive a new bound for the Sivers functions

\[ \frac{\left| \sum_{ab} |V_{ab}|^2 \Delta^N f_{a/p^+}(x) f_b(x) \right|}{\sum_{ab} |V_{ab}|^2 f_a(x) f_b(x)} \leq \frac{1/2}{H(q_\perp)_{\text{max}}}. \quad (12) \]

For \( W^+ \), it can be simplified as \( \left| \frac{\Delta^N u(x)}{u(x)} + \frac{\Delta^N d(x)}{d(x)} \right| \leq \frac{1}{H(q_\perp)_{\text{max}}} \). For \( W^- \), one obtains the following constraint \( \left| \frac{\Delta^N d(x)}{d(x)} + \frac{\Delta^N u(x)}{u(x)} \right| \leq \frac{1}{H(q_\perp)_{\text{max}}} \). \( \langle k_\perp^2 \rangle = 0.25 \text{ GeV}^2 \) and \( M_1^2 = 0.34^{+0.30}_{-0.16} \text{ GeV}^2 \), one gets \( 2.6 < 1/H(q_\perp)_{\text{max}} < 4.75 \), to be compared with the trivial bound, which gives the number 4 on the r.h.s. Although it is not a spectacular result, these new bounds are useful for consistency checks and we note that this can be also applied to the Sivers gluon function accessible in direct photon production [9].

5 Positivity bounds involving parity-violating asymmetries

In the helicity basis it is easy to obtain the explicit form of \( M \) and now from \( M_{ii} \geq 0 \), we have \( 1 \pm A_{LL} > |A_{aL} \pm A_{bL}| \). It is important to note that for
Figure 3: Longitudinal asymmetries are plotted as a function of rapidity $y$ of the $W^-$ (Top Left), $W^+$ (Top Right), $Z^0$ (Bottom), produced in $pp$ collisions at $\sqrt{s} = 500\text{GeV}$. The solid curves are the single longitudinal spin asymmetry $A_L$, the dashed curves are the double longitudinal spin asymmetry $A_{LL}$ and the dotted curves are the combination of $1 + A_{LL}(y) - \left| A_L(y) + A_L(-y) \right|$ (Taken from Ref. [4]).
These bounds should be tested in RHIC experiments for $W^\pm$ or $Z^0$ production in longitudinal $pp$ collisions, $\vec{p}\vec{p} \rightarrow W^\pm/Z^0 + X$. In perturbative QCD formalism, at leading-order and restricting to only up and down quarks, one has the following simple expressions for the single helicity asymmetries

$$A_{LL}^W(y) = \frac{-\Delta u(x_a) \overline{d}(x_b) + \Delta \overline{d}(x_a) u(x_b)}{u(x_a)d(x_b) + d(x_a)u(x_b)},$$

$$A_{L}^W(y) = \frac{-\Delta d(x_a) \overline{u}(x_b) + \Delta \overline{u}(x_a) d(x_b)}{d(x_a)\overline{u}(x_b) + \overline{u}(x_a)d(x_b)},$$

$$A_{L}^{Z^0}(y) = \frac{\sum_q (2v_q a_q) [-\Delta q(x_a) \overline{q}(x_b) + \Delta \overline{q}(x_a) q(x_b)]}{\sum_q (v_q^2 + a_q^2) [q(x_a)\overline{q}(x_b) + \overline{q}(x_a)q(x_b)]},$$

and for the double helicity asymmetries

$$A_{LL}^{W^+}(y) = \frac{-\Delta u(x_a) \Delta \overline{d}(x_b) + \Delta \overline{d}(x_a) \Delta u(x_b)}{u(x_a)d(x_b) + d(x_a)u(x_b)},$$

$$A_{LL}^{W^-}(y) = \frac{-\Delta d(x_a) \Delta \overline{u}(x_b) + \Delta \overline{u}(x_a) \Delta d(x_b)}{d(x_a)\overline{u}(x_b) + \overline{u}(x_a)d(x_b)},$$

$$A_{LL}^{Z^0}(y) = \frac{-\sum_q (v_q^2 + a_q^2) [\Delta q(x_a) \Delta \overline{q}(x_b) + \Delta \overline{q}(x_a) \Delta q(x_b)]}{\sum_q (v_q^2 + a_q^2) [q(x_a)\overline{q}(x_b) + \overline{q}(x_a)q(x_b)]},$$

where $\Delta q(x)$ and $q(x)$ are the helicity distribution and unpolarized parton distribution function, respectively. $v_q$ and $a_q$ are the vector and axial couplings of the $Z^0$ boson to the quark. $x_{a,b}$ are the parton momentum fractions given by $x_a = m_Q/\sqrt{s} e^y$, $x_b = m_Q/\sqrt{s} e^{-y}$, with $m_Q$ and $y$, the mass and rapidity of the $W$ (or $Z$) boson and $\sqrt{s}$ the center-of-mass energy.

One notices that $A_L$ and $A_{LL}$ involve only quark helicity distributions. To estimate these asymmetries numerically, we choose the unpolarized and polarized quark distributions functions based on a statistical approach [10]. The results of the calculations are displayed in Fig. 3 and show that positivity is satisfied. It is not always the case for other sets of parton distributions [4].

6 Concluding remarks

Although many interesting cases were not presented here, for lack of time, we have seen on these recent results that positivity provides very useful non-
trivial bounds on spin observables and parton distributions. One should not forget this important tool to check the consistency of model builders and of forthcoming data, in the exciting field of spin physics.

Acknowledgments

I am grateful to Prof. Chung-I Tan for organizing such an interesting scientific program. My warmest congratulations go to Prof. Tran Thanh Van for his wonderful project, in Quy Nhon, of the International Center for Interdisciplinary Science and Education (ICISE) in Vietnam, which will become soon a reality.

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