The Schlesinger System and the Riemann-Hilbert Problem

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Abstract

We generalize some classical results for the Schlesinger system of partial differential equations and give the explicit form of its solution, associated with rational matrix functions in general position.
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The Schlesinger system first appeared in L. Schlesinger’s work [Sch12] as a completely integrable non-linear Pfaffian system, governing the isomonodromic deformations in the class of non-resonant Fuchsian systems. It is closely related with the Riemann-Hilbert monodromy problem, which requires to find a Fuchsian system with prescribed monodromy. This problem was included by D. Hilbert in his list of problems [Hil00] as the 21st problem and solved in the non-resonant case by D. Hilbert himself in [Hil05], I. Plemelj in [Ple08] and G. D. Birkhoff in [Bir13a]. These solutions are based on passing from the Riemann-Hilbert monodromy problem to the Riemann-Hilbert boundary problem, which requires to find the factorization of a matrix function on a contour into the product of two factors, holomorphic and invertible, respectively, inside and outside the contour. In [Miw81] T. Miwa has employed a new method, based on the holonomic quantum fields theory and Clifford operators, to construct the isomonodromic deformation of a non-resonant Fuchsian system. Assuming the appropriate restrictions on the initial values, he has proved that all the singularities, except the fixed ones, of a solution of the Schlesinger system are poles.

The main goal of this Thesis is to investigate the Schlesinger system without such restrictions. Omitting the assumption of non-resonance, we encounter several difficulties. First of all, the class of isomonodromic deformations of a resonant Fuchsian system can be very rich. An example of this phenomenon, concerning rational matrix functions in general position, was described by V. E. Katsnelson in [Kat97], [Kat01]. Although the Schlesinger system preserves monodromy in general, it is unclear \textit{a priori}, how to distinguish the isomonodromic deformation, corresponding to its solution. On the
other hand, A. A. Bolibruch has demonstrated (see [AB94]) that in general the Riemann-Hilbert monodromy problem may have no solution at all. We cannot overcome these difficulties by adapting T. Miwa’s construction because of its heavy reliance on the assumption of non-resonance. Instead, we achieve our goal in the following way.

Given a linear differential system with rational coefficients (not necessarily Fuchsian), we can factorize its fundamental solution in a neighborhood of a singular point into the product of two factors. One of them is holomorphic and invertible in a neighborhood of the singular point, and we call it the non-singular factor. The other is a ”translation” of a matrix function, holomorphic and invertible in the Riemann surface of the logarithm. We call this matrix function the principal factor. The existence of such local factorization was established by G. D. Birkhoff (see [Bir09]), using his method of solution of the Riemann-Hilbert boundary problem. Furthermore, we consider isomonodromic\(^1\) deformations, which preserve also all the principal factors. We call such deformations isoprincipal. The isoprincipal deformation of a given linear differential system is uniquely determined, and we derive a completely integrable non-linear Pfaffian system, which governs the isoprincipal deformations. We call it the generalized Schlesinger system. In the case of Fuchsian systems this is just the classical Schlesinger system, which explains the terminology. The Cauchy problem for the generalized Schlesinger system can be reduced to the Riemann-Hilbert boundary problem, depending on a parameter. We solve this problem by D. Hilbert’s method, based on Fredholm theory of linear integral equations, and obtain a generalization of Miwa’s theorem for the generalized Schlesinger system.

The presentation of our results is organized as follows.

- Chapter 2 is dedicated to G. D. Birkhoff’s and D. Hilbert’s methods for the Riemann-Hilbert boundary problem. We present these classical results in some detail, since they are going to play the crucial role in our derivations.

- In Chapter 3 we define the generalized Schlesinger system and the isoprincipal deformation and formulate two of the main results of this Thesis. These are Theorem 3 which states that the isoprincipal deformations are governed by the generalized Schlesinger system, and The-

\(^1\)The monodromy being understood in the strictly ”Fuchsian” sense, without taking into account the Stokes phenomena.
orem 4, which states that all the singularities, except the fixed ones, of a solution of the generalized Schlesinger system are poles.

• In Chapter 4 we apply our results, obtained in Chapter 3, to the case of Fuchsian systems. In particular, we show that Schlesinger theorem (Theorem 5 here), formulated for the non-resonant Fuchsian systems, is a special case of our Theorem 3. We also consider V. E. Katsnelson’s example, concerning rational matrix functions in general position, and formulate the third main result of this Thesis - Theorem 8. It gives the explicit form of solution of the Schlesinger system in this case. This result was obtained in joint work with V. E. Katsnelson.

We would also like to mention, in addition to the literature cited above, the following textbooks:

• on general theory of linear differential systems with rational coefficients: [Hil76], [Gol58], [CL55], [Sib90];

• on the Schlesinger system: [IKSY91];

• on the Riemann-Hilbert boundary problem: [Hil12], [Gak66], [AF97].
Chapter 2

The Riemann-Hilbert Boundary Problem

2.1 A Theorem of G. D. Birkhoff

Let $\gamma$ be a closed analytic Jordan curve in the complex plane $\mathbb{C}$, let $D$ be the simply connected domain in $\mathbb{C}$, bounded by $\gamma$:

$$\gamma = \partial D,$$

(2.1)

and let $A$ be an open neighborhood of $\gamma$. We denote by $D^+, D^-$ the domains

$$D^+ = D \cup A,$$

(2.2)

$$D^- = \{\mathbb{C} \setminus D\} \cup A,$$

(2.3)

so that

$$D^+ \cup D^- = \mathbb{C},$$

(2.4)

$$D^+ \cap D^- = A.$$  

(2.5)

We also denote by $\mathcal{O}(A)$ the algebra of all functions, holomorphic\(^1\) in $A$, by $\mathcal{O}^+(A)$ the group of all functions, holomorphic and nowhere vanishing in $A$, by $gl(m, \mathcal{O}(A))$ the algebra of all $m \times m$ matrix functions, holomorphic in $A$, and by $GL(m, \mathcal{O}(A))$ the group of all $m \times m$ matrix functions, holomorphic and invertible in $A$.

---

\(^1\)By "holomorphic" we mean "analytic univalued".
**Definition 1** Let $F(x) \in GL(m, \mathcal{O}(A))$. The Riemann-Hilbert boundary problem (for $F(x)$ on $\gamma$) is to find matrix functions

\begin{align*}
X^+(x) & \in GL(m, \mathcal{O}(D^+)), \\
X^-(x) & \in GL(m, \mathcal{O}(D^-)),
\end{align*}

such that

$$X^-(x)F(x) = X^+(x), \quad x \in A. \quad (2.8)$$

**Remark 1** We would like to note here that

1. $A$ can always be replaced in (2.6) - (2.8) by a smaller neighborhood of $\gamma$, because $X^+(x)$ and $X^-(x)$ admit the analytic continuation wherever $Q$ does;

2. $X^-(x)$, in general, need not be holomorphic or invertible at $\infty$;

3. $X^+(x)$ and $X^-(x)$ are determined up to multiplication by an arbitrary entire invertible matrix function from the left.

Let us start dealing with the Riemann-Hilbert boundary problem by recalling the following classical result, which follows immediately from Cauchy and Liouville theorems:

**Lemma 1** Let $Z(x) \in gl(m, \mathcal{O}(A))$. Then there exist unique

\begin{align*}
Z^+(x) & \in gl(m, \mathcal{O}(D^+)), \\
Z^-(x) & \in gl(m, \mathcal{O}(D^- \cup \{\infty\})),
\end{align*}

such that

$$Z(x) = Z^+(x) - Z^-(x), \quad x \in A \quad (2.11)$$

and

$$Z^-(\infty) = 0. \quad (2.12)$$

These matrix functions $Z^+(x), Z^-(x)$ are given by

$$Z^+(x) = \begin{cases} 
\int_{\gamma} Z(y) \frac{dy}{y - x} 2\pi i, & x \in D, \\
Z(x) + \int_{\gamma} \frac{Z(y) - Z(x)}{y - x} \frac{dy}{2\pi i}, & x \in A,
\end{cases} \quad (2.13)$$
\[
Z^-(x) = \begin{cases} 
\oint_{\gamma} \frac{Z(y)}{y-x} \frac{dy}{2\pi i}, & x \in \mathbb{C} \setminus \overline{D}, \\
\oint_{\gamma} \frac{Z(y) - Z(x)}{y-x} \frac{dy}{2\pi i}, & x \in \mathcal{A}.
\end{cases}
\tag{2.14}
\]

Let us observe that Lemma 11 provides a solution to the Riemann-Hilbert boundary problem in the case \(m = 1\), i.e. when \(F(x) \in \mathcal{O}^*(\mathcal{A})\). Indeed, let \(\mu \in \mathbb{Z}\) be the index of \(F(x)\):

\[
\mu = \oint_{\gamma} \frac{dF}{dx} F^{-1}(x) \frac{dx}{2\pi i}.
\tag{2.15}
\]

Then, having fixed a point \(x_0 \in \mathcal{D} \setminus \mathcal{A}\), we can define the univalued function

\[
Z(x) = \log(F(x)(x-x_0)^{-\mu}) \in \mathcal{O}(\mathcal{A}).
\tag{2.16}
\]

In view of Lemma 11 we can let functions \(Z^+(x), Z^-(x)\) satisfy relations (2.9) - (2.12) for \(m = 1\) and define

\[
X^+(x) = \exp(Z^+(x)),
\tag{2.17}
\]
\[
X^-(x) = (x-x_0)^{-\mu} \exp(Z^-(x)).
\tag{2.18}
\]

Then we can conclude that relations (2.7) - (2.8) hold true. Thus the Riemann-Hilbert boundary problem in this case has solution \(X^+(x), X^-(x)\), such that \(X^-(x)\) has at most a pole at \(\infty\). Although this method is unsuitable for \(m > 1\), it turns out that the same conclusion is also true in the general case. This result is due to G. D. Birkhoff (see [Bir13a]):

**Theorem 1 (Birkhoff)** Let \(F(x) \in \text{GL}(m, \mathcal{O}(\mathcal{A}))\). Then there exist \(X^+(x), X^-(x)\), such that relations (2.7) - (2.8) are satisfied and \(X^-(x)\) has at most a pole at \(\infty\).

**Proof:** We shall present a sketch of the proof, which can be found in [Bir13a]. For simplicity, we assume

\[
\mathcal{D} = \{x : |x| < 1\}.
\tag{2.19}
\]

In the general case, the reasonings below can be slightly modified, using the Riemann conformal map theorem.
The main idea is to consider the following system of boundary problems:

\[
\begin{align*}
Z^+_1(x) - Z^-_1(x) &= x^\nu Z^+_2(x)F(x), \\
Z^+_2(x) - Z^-_2(x) &= I - x^{-\nu} Z^-_1(x)F^{-1}(x), \\
Z^-_1(\infty) &= Z^-_2(\infty) = 0,
\end{align*}
\]  

(2.20)

where \(\nu\) is a positive integer. We shall show that, for \(\nu\) sufficiently large, system (2.20) is solvable by successive approximations.

Indeed, let us look for solution in the form

\[
Z^\varsigma_j(x) = \sum_{k=0}^{\infty} Z^\varsigma_{j,k}(x), \quad j = 1, 2, \varsigma = +, -
\]

(2.21)

where \(Z^\varsigma_{j,k}(x)\) are recursively defined by

\[
\begin{align*}
Z^+_{1,k+1}(x) - Z^-_{1,k+1}(x) &= x^\nu Z^+_{2,k}(x)F(x), \\
Z^+_{2,k+1}(x) - Z^-_{2,k+1}(x) &= -x^{-\nu} Z^-_{1,k}(x)F^{-1}(x), \\
Z^-_{1,k}(\infty) &= Z^-_{2,k}(\infty) = 0
\end{align*}
\]

(2.22)

with the initial conditions

\[
Z^+_{1,0}(x) \equiv 0, Z^-_{1,0}(x) \equiv 0, Z^+_{2,0}(x) \equiv I, Z^-_{2,0}(x) \equiv 0.
\]

(2.23)

Furthermore, let \(\epsilon \in (0, 1)\) be such that

\[
\{x : \epsilon \leq |x| \leq \frac{1}{\epsilon}\} \subset A,
\]

(2.24)

and for \(Z \in \text{gl}(m, \mathcal{O}(A))\) let us denote

\[
\|Z\| = \max_{1 \leq \alpha, \beta \leq m, 1 \leq |x| \leq \frac{1}{\epsilon}} |Z(x)_{\alpha\beta}|.
\]

(2.25)
Now for $\epsilon < |x| < \frac{1}{\epsilon}$

$$Z_{1,k+1}^{-}(x) = \oint_{|y|=1} \frac{y^{\nu} Z_{2,k}^{+}(y) F(y) - x^{\nu} Z_{2,k}^{+}(x) F(x)}{y - x} \frac{dy}{2\pi i} = \oint_{|y|=\epsilon} \frac{y^{\nu} Z_{2,k}^{+}(y) F(y)}{y - x} \frac{dy}{2\pi i} = \oint_{|y|=\epsilon} \frac{y^{\nu} Z_{2,k}^{+}(y) F(y) - F(x)}{y - x} \frac{dy}{2\pi i}. \quad (2.26)$$

Analogously,

$$Z_{2,k+1}^{+}(x) = -\oint_{|y|=1} \frac{y^{-\nu} Z_{1,k}^{-}(y) F^{-1}(y) - F^{-1}(x)}{y - x} \frac{dy}{2\pi i}. \quad (2.27)$$

Hence we obtain

$$\|Z_{1,k+1}^{-}\| \leq \epsilon^{\nu} a \|Z_{2,k}^{+}\|, \quad (2.28)$$

$$\|Z_{2,k+1}^{+}\| \leq \epsilon^{\nu} a \|Z_{1,k}^{-}\|, \quad (2.29)$$

where

$$a = \max\left\{ \max_{1 \leq \alpha, \beta \leq m} \left| \frac{F(x)_{\alpha \beta} - F(y)_{\alpha \beta}}{x - y} \right|, \max_{1 \leq \alpha, \beta \leq m} \left| \frac{F^{-1}(x)_{\alpha \beta} - F^{-1}(y)_{\alpha \beta}}{x - y} \right| \right\}, \quad (2.30)$$

and uniform convergence of series (2.21) for $\nu$ sufficiently large follows immediately.

Now we can define

$$X_{0}^{+}(x) = Z_{1}^{+}(x), \quad (2.31)$$

$$X_{0}^{-}(x) = x^{\nu}(I + Z_{2}^{-}(x)), \quad (2.32)$$

and observe that

$$X_{0}^{+}(x) \in gl(m, \mathcal{O}(\mathcal{D}^{+})), \quad (2.33)$$

$$X_{0}^{-}(x) \in gl(m, \mathcal{O}(\mathcal{D}^{-})), \quad (2.34)$$

$$X_{0}^{-}(x) F(x) = X_{0}^{+}(x), \quad x \in \mathcal{A}, \quad (2.35)$$

and $X_{0}^{-}(x)$ has at most a pole at $\infty$. Also, since $Z_{2}^{-}(\infty) = 0$, $|X_{0}^{-}(x)|$ does not identically vanish, and, therefore, neither does $|X_{0}^{+}(x)|$. The isolated zeroes of $|X_{0}^{+}(x)|, |X_{0}^{-}(x)|$ can now be successively eliminated by the following
procedure. Let $x_0 \in \mathcal{D}$ be a zero of $|X_0^+(x)|$, then there exists $T \in GL(m, \mathbb{C})$, such that the first row of $TX_0^+(x)$ vanishes at $x_0$ (if $x_0 \in \mathcal{A}$ then the same is true for $TX_0^-(x)$). Let us consider
\begin{align*}
X_1^+(x) &= E_{x_0}(x)TX_0^+(x), \tag{2.36} \\
X_1^-(x) &= E_{x_0}(x)TX_0^-(x), \tag{2.37}
\end{align*}
where $E_{x_0}(x) = \text{diag}(\frac{1}{x-x_0}, 1, \ldots, 1)$. Then
\begin{align*}
X_1^+(x) &\in gl(m, \mathcal{O}(\mathcal{D}^+)), \tag{2.38} \\
X_1^-(x) &\in gl(m, \mathcal{O}(\mathcal{D}^-)), \tag{2.39} \\
X_1^-(x)F(x) &= X_1^+(x), \quad x \in \mathcal{A}, \tag{2.40}
\end{align*}
and the multiplicity of zero of $|X_1^+(x)|$ at $x_0$ is less by 1. Repeating this procedure, we finally obtain $X^+(x), X^-(x)$, satisfying (2.6) - (2.8), where $X^-(x)$ has at most a pole at $\infty$.

One can also consider the Riemann-Hilbert boundary problem when $\mathcal{D}$ is a finite disjoint union of simply connected domains, rather than a single domain. Let us denote for the moment
\begin{equation}
\mathcal{D} = \bigcup_{j=1}^n \mathcal{D}_j, \tag{2.41}
\end{equation}
where for $j = 1, \ldots, n \mathcal{D}_j$ is a bounded simply connected domain in $\mathbb{C}$, such that
\begin{equation}
\overline{\mathcal{D}}_j \cap \overline{\mathcal{D}}_{j'} = \emptyset, \quad j' \neq j''. \tag{2.42}
\end{equation}
Then $\gamma = \partial \mathcal{D} = \bigcup \gamma_j$, where $\gamma_j = \partial \mathcal{D}_j$, and we can assume that neighborhood $\mathcal{A}$ of $\gamma$ is a disjoint union of neighborhoods $\mathcal{A}_j$ of $\gamma_j$:
\begin{align*}
\mathcal{A} &= \bigcup_{j=1}^n \mathcal{A}_j, \tag{2.43} \\
\mathcal{A}_{j'} \cap \mathcal{A}_{j''} &= \emptyset, \quad j' \neq j''. \tag{2.44}
\end{align*}
Solution of the Riemann-Hilbert boundary problem in this case can be reduced to the previously considered one. Indeed, as we have already shown, for $j = 1, \ldots, n$ one can successively find
\begin{align*}
X_j^+(x) &\in GL(m, \mathcal{O}(\mathcal{D}_j \cup \mathcal{A}_j)), \tag{2.45} \\
X_j^-(x) &\in GL(m, \mathcal{O}((\mathbb{C} \setminus \mathcal{D}_j) \cup \mathcal{A}_j)), \tag{2.46}
\end{align*}
such that
\[ X_j^-(x) \left( X_{j-1}^- \cdots X_1^- F(x) \right) = X_j^+(x), \quad x \in A_j. \] (2.47)

Define
\[ X^+(x) = X_n^-(x) \cdots X_{j+1}^- X_j^+(x), \quad x \in D_j \cup A_j, \] (2.48)
\[ X^-(x) = X_n^-(x) \cdots X_1^- (x), \quad x \in D^-, \] (2.49)

then \( X^+, X^- \) give a solution to the Riemann-Hilbert boundary problem for \( F(x) \). If each \( X_j^-(x) \) has at most a pole at \( \infty \) then so does \( X^-(x) \), i.e. the conclusion of Theorem 1 holds true also in this case.

We would also like to mention another approach to the Riemann-Hilbert boundary problem, concerning holomorphic vector bundles. In view of (2.4), (2.5) matrix function \( F(x) \in GL(m, \mathcal{O}(A)) \) can be used to define a holomorphic vector bundle over \( \mathbb{C} \). The existence theorem of H. Grauert (see [GR79]) states that any holomorphic vector bundle over an open Riemann surface (in particular, over \( \mathbb{C} \)) is holomorphically isomorphic to the trivial one. This means that there exist
\[ X^+(x) \in GL(m, \mathcal{O}(D^+)), \] (2.50)
\[ X^-(x) \in GL(m, \mathcal{O}(D^-)), \] (2.51)
such that
\[ X^-(x) F(x) = X^+(x), \quad x \in A. \] (2.52)
Thus the Riemann-Hilbert boundary problem is always solvable. Furthermore, we can use \( F(x) \) to construct a holomorphic vector bundle over \( \mathbb{C} \). It was proved by A. Grothendieck in [Gro57] that any holomorphic vector bundle over the Riemann sphere is holomorphically isomorphic to a direct sum of line bundles. Recalling our solution of the Riemann-Hilbert boundary problem in the case \( m = 1 \) (the case of a line bundle), we can conclude that there exist
\[ X^+(x) \in GL(m, \mathcal{O}(D^+)), \] (2.53)
\[ X^-(x) \in GL(m, \mathcal{O}(D^- \cup \{\infty\})), \] (2.54)
and \( \mu_1, \ldots, \mu_m \in \mathbb{Z} \), such that
\[ X^-(x) F(x) = D(x) X^+(x), \quad x \in A, \] (2.55)
where
\[ D(x) = \text{diag}((x - x_0)^{\mu_1}, \ldots, (x - x_0)^{\mu_m}), \quad (2.56) \]
\[ x_0 \in D \setminus A. \]
This is, of course, a stronger result than Theorem 1. A similar result was obtained by G. D. Birkhoff in [Bir13b]. It states that there exist
\[ X^+(x) \in GL(m, \mathcal{O}(D^+)), \quad (2.57) \]
\[ X^-(x) \in GL(m, \mathcal{O}(D^- \cup \{\infty\})), \quad (2.58) \]
and \( \mu_1, \ldots, \mu_m \in \mathbb{Z} \), such that
\[ X^-(x)F(x) = X^+(x)D(x), \quad x \in A, \quad (2.59) \]
where
\[ D(x) = \text{diag}((x - x_0)^{\mu_1}, \ldots, (x - x_0)^{\mu_m}), \quad (2.60) \]
\[ x_0 \in D \setminus A. \] It should be noted, however, that in general factors \( X^+, X^-, D \) are different in (2.55) and (2.59).

2.2 Regular Solutions, Depending on a Parameter

Let us investigate, when the Riemann-Hilbert boundary problem has solution \( X^+(x), X^-(x) \), such that \( X^-(x) \) is holomorphic and invertible also at \( \infty \). We shall call such a solution regular. First of all, we observe that if \( X^+(x), X^-(x) \) give a regular solution to the Riemann-Hilbert boundary problem for \( F(x) \), then the determinants of \( X^+(x), X^-(x) \) give a regular solution to the Riemann-Hilbert boundary problem for the determinant of \( F(x) \):
\[ |X^+(x)| \in \mathcal{O}^*(D^+), \quad (2.61) \]
\[ |X^-(x)| \in \mathcal{O}^*(D^- \cup \{\infty\}), \quad (2.62) \]
\[ |X^-(x)||F(x)| = |X^+(x)|, \quad x \in A. \quad (2.63) \]
Hence a necessary condition for the existence of a regular solution is
\[ \mu = \oint_{\gamma} \frac{d|F(x)|}{dx}|F^{-1}(x)| \frac{dx}{2\pi i} = 0. \quad (2.64) \]
However, for $m > 1$ condition (2.64) is not sufficient. For example, if $x_0 \in \mathcal{D}$ and
\[
F(x) = \text{diag}(x - x_0, \frac{1}{x - x_0}) \in \text{GL}(2, \mathcal{O}(\mathbb{C} \setminus \{x_0\})),
\]
then one can check that the corresponding Riemann-Hilbert boundary problem has no regular solution. In order to investigate the problem further, we shall employ a method, based on the Fredholm theory of linear integral equations. It originated with D. Hilbert (see [Hil05], [Hil12]) and was employed, in the same way as below, by T. Miwa in [Miw81].

**Lemma 2** Let $F(x) \in \text{GL}(m, \mathcal{O}(\mathcal{A}))$. Assume that $X^+(x), X^-(x)$ give the regular solution to the Riemann-Hilbert boundary problem for $F(x)$, normalized by
\[
X^-(\infty) = I.
\]
Then $X^-(x)$ satisfies in $\mathcal{A}$ the linear integral equation
\[
X^-(x) - \oint_{\gamma} X^-(y)K(y, x)\frac{dy}{2\pi i} = I
\]
with kernel $K(y, x) \in \text{gl}(m, \mathcal{O}(\mathcal{A} \times \mathcal{A}))$ given by
\[
K(y, x) = \frac{F(y)F^{-1}(x) - I}{y - x}.
\]

**Proof:** Using again the properties of the Cauchy integral, we obtain for $x \in \mathcal{A}$
\[
\oint_{\gamma} X^-(y)F(y) - X^-(x)F(x)\frac{dy}{y - x} \frac{dy}{2\pi i} = \oint_{\gamma} X^+(y) - X^+(x)\frac{dy}{y - x} \frac{dy}{2\pi i} = 0. \tag{2.69}
\]
Hence
\[
\oint_{\gamma} X^-(y)\frac{F(y) - F(x)}{y - x} \frac{dy}{2\pi i} = \left(\oint_{\gamma} X^-(y) - X^-(x)\frac{dy}{y - x} \frac{dy}{2\pi i}\right)F(x). \tag{2.70}
\]
Since
\[
\oint_{\gamma} X^-(y) - X^-(x)\frac{dy}{y - x} \frac{dy}{2\pi i} = X^-(x) - I, \tag{2.71}
\]
we obtain equation (2.67) with kernel $K$, given by (2.68). \[\square\]
Let us investigate equation (2.67). According to Fredholm theory, we should consider Fredholm determinant $f(\lambda) \in \mathcal{O}(\mathbb{C})$, defined by

$$f(\lambda) = \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{l!} \sum_{\alpha_1, \ldots, \alpha_l = 1}^{m} \oint_{\gamma} \cdots \oint_{\gamma} K \left( x_1, \ldots, x_l \left| \alpha_1, \ldots, \alpha_l \right. \right) \frac{dx_1}{2\pi i} \cdots \frac{dx_l}{2\pi i},$$

(2.72)

where

$$K \left( x_1, \ldots, x_l \left| \alpha_1, \ldots, \alpha_l \right. \right) = \left| \left( K(y_j, y_k)_{\alpha_j \beta_k} \right)_{j,k=1}^{l} \right|. \quad (2.73)$$

If $f(1) \neq 0$, then equation (2.67) has unique solution $X^{-}(x) \in gl(m, \mathcal{O}(A))$, given by

$$X^{-}(x) = I + \frac{1}{f(1)} \oint_{\gamma} \hat{K}(y, x, 1) \frac{dy}{2\pi i}, \quad x \in A,$$

(2.75)

where $\hat{K}(y, x, \lambda) \in gl(m, \mathcal{O}(A \times A \times \mathbb{C}))$ is of the form

$$\hat{K}(y, x; \lambda)_{\alpha, \beta} = \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{l!} \sum_{\alpha_1, \ldots, \alpha_l = 1}^{m} \oint_{\gamma} \cdots \oint_{\gamma} K \left( y_1, \ldots, y_l \left| \alpha_1, \ldots, \alpha_l \right. \right) \frac{dy_1}{2\pi i} \cdots \frac{dy_l}{2\pi i}. \quad (2.76)$$

Now we shall show that conditions (2.64) and (2.74) are sufficient in order for the Riemann-Hilbert boundary problem to have a regular solution.

**Lemma 3** Let $F(x) \in GL(m, \mathcal{O}(A))$. Let $K(y, x) \in gl(m, \mathcal{O}(A \times A))$ be given by (2.68) and let $f(\lambda) \in \mathcal{O}(\mathbb{C})$ be given by (2.72), (2.73). Assume that conditions (2.64), (2.74) are satisfied. Then unique solution $X^{-}(x) \in gl(m, \mathcal{O}(A))$ of equation (2.67), given by (2.73), (2.76), can be analytically continued in $D^{-} \cup \{\infty\}$, is invertible there and satisfies at $\infty$ relation (2.67). Also, matrix function $X^{+}(x) \in gl(m, A)$, defined by

$$X^{+}(x) = X^{-}(x)F(x), \quad x \in A,$$

(2.77)

can be analytically continued in $D^{+}$ and is invertible there.
Proof: Since
\[ \text{trace}(K(x, x)) = \text{trace}(\frac{dF(x)}{dx}F^{-1}(x)) = \frac{d|F(x)|}{dx}|F^{-1}(x)|, \quad (2.78) \]
relation (2.64) implies
\[ \oint_{\gamma} \text{trace}(K(y, y)) \frac{dy}{2\pi i} = 0. \quad (2.79) \]
On the other hand, it follows from (2.68) that for \( j = 0, 1, \ldots \)
\[ \oint_{\gamma} \cdots \oint_{\gamma} \text{trace}(K(x_1, x_2)K(x_2, x_3) \cdots K(x_{2j+1}, x_1)) \frac{dx_1}{2\pi i} \cdots \frac{dx_{2j+1}}{2\pi i} = \]
\[ \oint_{\gamma} \text{trace}(K(x, x)) \frac{dx}{2\pi i}. \quad (2.80) \]
In view of (2.73) we can conclude that for \( j = 0, 1, \ldots \)
\[ \oint_{\gamma} \cdots \oint_{\gamma} K \left( x_1, \ldots, x_{2j+1} \mid \alpha_1, \ldots, \alpha_{2j+1} \right) \frac{dx_1}{2\pi i} \cdots \frac{dx_{2j+1}}{2\pi i} = 0. \quad (2.81) \]
Therefore, \( f(\lambda) \) is an even function:
\[ f(\lambda) = f(-\lambda). \quad (2.82) \]
Thus relation (2.71) implies
\[ f(-1) \neq 0. \quad (2.83) \]
Let us define matrix function \( Y(x) \) by
\[ Y(x) = X^{-}(x) - I + \oint_{\gamma} \frac{X^{-}(y) - X^{-}(x)}{y - x} \frac{dy}{2\pi i}. \quad (2.84) \]
Then \( Y(x) \) satisfies the linear integral system
\[ Y(x) + \oint_{\gamma} Y(y)K(y, x) \frac{dy}{2\pi i} = 0, \quad x \in \mathcal{A}. \quad (2.85) \]
In view of (2.83) we conclude that \( Y = 0 \), i.e.
\[ X^{-}(x) = I - \oint_{\gamma} \frac{X^{-}(y) - X^{-}(x)}{y - x} \frac{dy}{2\pi i}, \quad x \in \mathcal{A}. \quad (2.86) \]
It follows that we can analytically continue $X^-(x)$ in $\mathbb{C} \setminus \overline{\mathcal{D}}$ with
\[
X^-(x) = I - \oint_{\gamma} \frac{X^-(y) \, dy}{y - x} \, 2\pi i, \quad x \in \mathbb{C} \setminus \overline{\mathcal{D}}
\]
(2.87)
and obtain
\[
X^-(x) \in \text{gl}(m, \mathcal{O}(\mathcal{D}^- \cup \{\infty\})),
\]
(2.88)
\[
X^-(\infty) = I.
\]
(2.89)
Having defined $X^+(x) \in \text{gl}(m, \mathcal{A})$ by (2.77), we can conclude from (2.67) and (2.88) that
\[
\oint_{\gamma} \frac{X^+(y) - X^+(x) \, dy}{y - x} \, 2\pi i = 0, \quad x \in \mathcal{A}.
\]
(2.90)
Hence we can analytically continue $X^+(x)$ in $\mathcal{D}$ with
\[
X^+(x) = \oint_{\gamma} \frac{X^+(y) \, dy}{y - x} \, 2\pi i, \quad x \in \mathcal{D}
\]
(2.91)
and obtain
\[
X^+(x) \in \text{gl}(m, \mathcal{O}(\mathcal{D}^+)).
\]
(2.92)
It remains to establish the invertibility of $X^+, X^-$. Relation (2.64) implies that the Riemann-Hilbert boundary problem for $|F(x)|$ has regular solution $g^+(x), g^-(x)$, normalized by $g^-(\infty) = 1$. Then, according to Liouville theorem,
\[
|X^+(x)| = g^+(x), \quad x \in \mathcal{D}^+,
\]
(2.93)
\[
|X^-(x)| = g^-(x), \quad x \in \mathcal{D}^- \cup \{\infty\}.
\]
(2.94)
Hence $X^+(x), X^-(x)$ are invertible in $\mathcal{D}^+, \mathcal{D}^- \cup \{\infty\}$, respectively. ■

**Remark 2** In the discussion above we have not made clear, whether $\mathcal{D}$ is a single simply connected domain or a disjoint (with closures) union of such. In fact, it does not matter, as long as it is understood that
\[
\oint_{\cup \gamma_j} = \sum_j \oint_{\gamma_j}.
\]
(2.95)
Now let us consider the Riemann-Hilbert boundary problem, depending on a parameter. Let $U \subset \mathbb{C}^n$ be a polydisk and let $F(x, t) \in GL(m, \mathcal{O}(A \times U))$. We consider the Riemann-Hilbert boundary problem for each $t \in U$ and assume that for a fixed $t^0 \in U$ there exists a regular solution, $X^+(x, t^0), X^-(x, t^0)$. In fact, since we can always consider
\[ F'(x, t) = X^-(x, t^0)F(x, t)(X^+(x, t))^{-1}, \] (2.96)
there is no loss of generality in the assumption
\[ F(x, t^0) \equiv I. \] (2.97)
Let us observe that function
\[ \mu(t) = \oint_\gamma \frac{\partial[F(y, t)]}{\partial y} |F^{-1}(y, t)| \frac{dy}{2\pi i} \] (2.98)
is integer-valued and continuous, hence constant:
\[ \mu(t) \equiv \mu(t^0) = 0. \] (2.99)
Thus condition (2.64) is satisfied for every $t$. Furthermore, kernel $K$ is holomorphic with respect to $t$ and, because of the absolute convergence of the Fredholm series, the Fredholm determinant and the resolvent kernel are holomorphic with respect to $t$:
\[ K(y, x, t) \in gl(m, \mathcal{O}(A \times A \times U)), \] (2.100)
\[ f(\lambda, t) \in \mathcal{O}(\mathbb{C} \times U), \] (2.101)
\[ \hat{K}(y, x, \lambda, t) \in gl(m, \mathcal{O}(A \times A \times \mathbb{C} \times U)). \] (2.102)
Furthermore, assumption (2.97) implies that
\[ K(y, x, t^0) \equiv 0, \] (2.103)
\[ f(\lambda, t^0) \equiv 1. \] (2.104)
Therefore, there exists neighborhood $V$ of $t^0$, where $f(1, t)$ does not vanish. It follows, in view of Lemma 3, that for $t \in V$ the Riemann-Hilbert boundary problem for $F(x, t)$ has regular solution $X^+, X^-$, normalized by
\[ X^-(\infty, t) \equiv I \] (2.105)
and holomorphic with respect to $t$. Furthermore, we observe that
\[
f(1,t)X^+(x,t) \in \text{gl}(m, \mathcal{O}(\mathcal{D}^+ \times \mathcal{U})), \tag{2.106}
\]
\[
f(1,t)X^-(x,t) \in \text{gl}(m, \mathcal{O}((\mathcal{D}^- \cup \{\infty\}) \times \mathcal{U})). \tag{2.107}
\]
Since (2.99) implies
\[
|X^+(x,t)| \in \mathcal{O}^*(\mathcal{D}^+ \times \mathcal{U}), \tag{2.108}
\]
\[
|X^-(x,t)| \in \mathcal{O}^*((\mathcal{D}^- \cup \{\infty\}) \times \mathcal{U}), \tag{2.109}
\]
we can also observe that
\[
(f(1,t))^{m-1}(X^+(x,t))^{-1} \in \text{gl}(m, \mathcal{O}(\mathcal{D}^+ \times \mathcal{U})), \tag{2.110}
\]
\[
(f(1,t))^{m-1}(X^-(x,t))^{-1} \in \text{gl}(m, \mathcal{O}((\mathcal{D}^- \cup \{\infty\}) \times \mathcal{U})). \tag{2.111}
\]
Thus we obtain

**Lemma 4** Let $t^0 \in \mathcal{U}$ and assume that matrix function $F(x,t) \in \text{GL}(m, \mathcal{O}(\mathcal{A} \times \mathcal{U}))$ satisfies
\[
F(x,t^0) \equiv I. \tag{2.112}
\]
Then there exist neighborhood $\mathcal{V}$ of $t^0$ and matrix functions
\[
X^+(x,t) \in \text{GL}(m, \mathcal{O}(\mathcal{D}^+ \times \mathcal{V})), \tag{2.113}
\]
\[
X^-(x,t) \in \text{GL}(m, \mathcal{O}((\mathcal{D}^- \cup \{\infty\}) \times \mathcal{V})), \tag{2.114}
\]
such that
\[
X^-(x,t)F(x,t) = X^+(x,t), \quad x \in \mathcal{A}, t \in \mathcal{V}, \tag{2.115}
\]
\[
X^-(\infty, t) \equiv I. \tag{2.116}
\]
Moreover, matrix functions $X^+(x,t), (X^+(x,t))^{-1}, X^-(x,t), (X^-(x,t))^{-1}$ are meromorphic with respect to $t$ in $\mathcal{U}$.

We would like to mention that Lemma 4 can also be proved in another way. The existence of neighborhood $\mathcal{V}$ of $t^0$ and matrix functions $X^+(x,t), X^-(x,t)$, satisfying (2.113) - (2.116) follows immediately from H. Grauert’s theorems on semi-continuity and on direct images of coherent sheaves under proper analytic maps, proved in [Gra60]. Alternatively, it can be verified by utilizing Taylor expansions with respect to parameter $t$, recurrence relations for Taylor coefficients and Cauchy majorization principle.
Furthermore, let $t^1 \in \partial \mathcal{V}$ be a fixed value of the parameter, for which the Riemann-Hilbert boundary problem has no regular solutions. Then we can use either A. Grothendieck’s or G. D. Birkhoff’s improvement of Theorem [1] mentioned in the previous section of this Chapter. It follows from these results that matrix functions $X^+(x, t), (X^+(x, t))^{-1}, X^-(x, t), (X^-(x, t))^{-1}$ are meromorphic with respect to $t$ in a neighborhood of $t^1$. Hence we can conclude that matrix functions $X^+(x, t), (X^+(x, t))^{-1}, X^-(x, t), (X^-(x, t))^{-1}$ are meromorphic with respect to $t$ in $\mathcal{U}$. 
Chapter 3

The Generalized Schlesinger System

3.1 Basic Notions

We consider the linear differential system with rational coefficients

\[
\frac{dY}{dx} = \left( \sum_{j=1}^{n} \sum_{k=0}^{p_j} Q_{j,k} \frac{1}{(x-t_j)^{k+1}} \right) Y,
\]

(3.1)

where \( Q_{j,k} \in gl(m, \mathbb{C}) \), and \( t_1, \ldots, t_n \) are fixed mutually distinct points in \( \mathbb{C} \). To simplify our presentation, we assume\(^1\) that system (3.1) is regular at \( x = \infty \), i.e., that

\[
\sum_{j=1}^{n} Q_{j,0} = 0.
\]

(3.2)

According to Cauchy theorem, in a neighborhood of \( x = \infty \) there exists unique holomorphic fundamental solution \( Y \) of system (3.1), satisfying the initial condition

\[
Y(\infty) = I.
\]

(3.3)

Moreover, \( Y(x) \) admits the analytic continuation along any path in

\[
\mathcal{R}_t = \mathbb{C} \setminus \{t_j\}_{j=1}^{n}.
\]

(3.4)

\(^1\)There is no loss of generality in this assumption. One can use a Möbius transformation to ensure that it is satisfied, although this will increase \( n \) by 1.
staying invertible. Since for \( n > 1 \), \( \mathcal{R}_t \) is not simply connected, the result of the analytic continuation depends, in general, on the homotopy class of the path along which it is performed. Therefore, we should consider the universal covering

\[
\psi_t : \tilde{\mathcal{R}}_t \mapsto \mathcal{R}_t.
\]  

(3.5)

We denote a point of \( \tilde{\mathcal{R}}_t \) in \( \psi^{-1}(x) \) by \( \tilde{x} \) and the group of deck transformations, acting on \( \tilde{\mathcal{R}}_t \), by \( \Delta_t \). We recall that \( \tilde{\mathcal{R}}_t \) is a simply connected Riemann surface with the complex differential structure provided by \( \psi_t \).

If one distinguishes a point in \( \psi_t^{-1}(\infty) \) then \( \tilde{x} \) can be interpreted as a homotopy class of paths in \( \mathcal{R}_t \), starting at \( \infty \) and ending at \( x \). This interpretation allows us to identify \( \Delta_t \) with fundamental group \( \pi_1(\mathcal{R}_t, \infty) \), the action being the usual multiplication of homotopy classes. Namely, for \( \sigma \in \Delta_t \) and \( \tilde{x} \in \tilde{\mathcal{R}}_t \) we interpret \( \sigma \tilde{x} \in \psi_t^{-1}(x) \) as the homotopy class of the path, obtained by traversing first a loop in class \( \sigma \), beginning and ending at \( \infty \), and then a representative of class \( \tilde{x} \) from \( \infty \) to \( x \). Thus we can define

\[
Y(\tilde{x}) \in GL(m, \mathcal{O}(\tilde{\mathcal{R}}_t)),
\]  

(3.6)

satisfying system (3.1) in \( \tilde{\mathcal{R}}_t \), and initial condition (3.3), where \( \infty \) now denotes the distinguished point. Let \( \sigma \in \Delta_t \), then \( Y \circ \sigma(\tilde{x}) \) is another fundamental solution of system (3.1) in \( \tilde{\mathcal{R}}_t \), determined by the initial condition

\[
Y \circ \sigma(\infty) = Y(\sigma \infty).
\]  

(3.7)

Hence we can consider mapping

\[
\Phi : \Delta_t \mapsto GL(m, \mathbb{C}),
\]  

(3.8)

defined by

\[
\Phi(\sigma) = Y^{-1}(\sigma \infty) \in GL(m, \mathbb{C})
\]  

(3.9)

and satisfying

\[
(Y \circ \sigma(\tilde{x}))\Phi(\sigma) = Y(\tilde{x}).
\]  

(3.10)

Note that for any \( \sigma', \sigma'' \in \Delta_t \),

\[
Y \circ (\sigma' \sigma'') = (Y \Phi(\sigma')^{-1}) \circ \sigma'' = Y \Phi(\sigma'') \Phi(\sigma')^{-1},
\]  

(3.11)
\[ \Phi(\sigma') \Phi(\sigma'') = \Phi(\sigma' \sigma''), \quad (3.12) \]

and
\[ \Phi(1) = I. \quad (3.13) \]

We conclude that \( \Phi \) is a linear representation of group \( \Delta_t \).

**Definition 2** We shall call representation \( \Phi \) the monodromy representation of \( Y \).

Now we would like to introduce a certain canonical factorization of \( Y(\tilde{x}) \) for \( x \) in a simply connected open neighborhood of singular point \( t_j \). In order to do this, we need to introduce some notations first.

For \( j = 1, \ldots, n \) let \( L_j \) be a simply connected open neighborhood of \( t_j \), such that \( L_j \cap t_j'' = \emptyset \), \( j' \neq j'' \).

Denote
\[ L_{t_j} = L_j \setminus t_j \subset \mathbb{R}_t. \quad (3.15) \]

Also, let \( x_j \in L_{t_j}, \tilde{x}_j \in \tilde{R}_t, \) and denote by \( \tilde{L}_{t_j} \) the connected component\(^2\) of \( \psi_t^{-1}(L_{t_j}) \), containing \( \tilde{x}_j \). Then \( \tilde{x} \in \tilde{L}_{t_j} \) if, and only if, the homotopy class \( \tilde{x}_j^{-1} \tilde{x} \) of a path in \( \mathbb{R}_t \), starting at \( x_j \) and ending at \( x \), has a representative, which lies entirely in \( L_{t_j} \). Moreover, such a representative is unique up to homotopy in \( L_{t_j} \). It follows that \( \tilde{L}_{t_j} \) is simply connected and that
\[ \psi_t \mid_{\tilde{L}_{t_j}} : \tilde{L}_{t_j} \mapsto L_{t_j} \quad (3.16) \]

is the universal covering over \( L_{t_j} \). The group of deck transformations of covering (3.10), which we denote by \( \Delta_{t_j} \), is a cyclic subgroup of \( \Delta_t \). Its generator \( \sigma_j \) is defined in \( \pi_1(\mathbb{R}_t, \infty) \) by
\[ \sigma_j = \tilde{x}_j \gamma_j \tilde{x}_j^{-1}, \quad (3.17) \]

where \( \gamma_j \in \pi_1(L_{t_j}, x_j) \) is the homotopy class of a simple loop \( \gamma_j \), making one positive circuit of \( t_j \). Fundamental group \( \pi_1(L_{t_j}, x_j) \) can be identified with \( \pi_1(\mathbb{R}_{t_j}, x_j) \), where
\[ \mathbb{R}_{t_j} = \mathbb{C} \setminus t_j. \quad (3.18) \]

\(^2\)For \( n > 2 \) set \( \psi_t^{-1}(L_{t_j}) \) is disconnected.
Denote by
\[ \psi_{t_j} : \tilde{R}_{t_j} \rightarrow R_{t_j} \] (3.19)
the universal covering over \( R_{t_j} \), then \( \tilde{L}_{t_j} \) can be identified with \( \psi_{t_j}^{-1}(L_{t_j}) \subset \tilde{R}_{t_j} \), and \( \Delta_{t_j} \) - with the group of deck transformations of covering (3.19). Furthermore, we can assume that for \( j = 1, \ldots, n \) homotopy class \( \tilde{x}_j \) has a representative, which is simple and has no points of intersection with \( \gamma_j \), except \( x_j \). Then \( \sigma_1, \ldots, \sigma_n \), defined by (3.17), generate group \( \pi_1(R_t, \infty) \). Hence subgroups \( \Delta_{t_1}, \ldots, \Delta_{t_n} \) generate group \( \Delta_t \), and the monodromy generators \( \Phi(\sigma_1), \ldots, \Phi(\sigma_n) \) completely determine the monodromy representation of \( Y \). Let \( \mathbb{C}^* \) denote the Riemann surface of the logarithm, i.e. the universal covering surface over \( \mathbb{C}^* = \mathbb{C} \setminus 0 \). Considering \( \tilde{R}_{t_j} \), with the projection
\[ \psi_{t_j} - t_j : \tilde{R}_{t_j} \rightarrow \mathbb{C}^* \] (3.20)
as a covering surface over \( \mathbb{C}^* \), we can observe that there exists an isomorphism of covering spaces, mapping \( \tilde{R}_{t_j} \) onto \( \tilde{\mathbb{C}}^* \). Such an isomorphism is not unique, but we shall fix for the moment some arbitrary one and denote it by
\[ \tilde{x} \mapsto \tilde{x} - t_j. \] (3.21)
This allows us to consider
\[ \log(\tilde{x} - t_j) \in O(\tilde{R}_{t_j}). \] (3.22)
Let \( \sigma_0 \) be the generator of the group of deck transformations, acting on \( \tilde{\mathbb{C}}^* \) and corresponding to one positive circuit of 0. Then
\[ \log \sigma_0 \tilde{x} = \log \tilde{x} + 2\pi i, \quad \tilde{x} \in \tilde{\mathbb{C}}^* \] (3.23)
and
\[ \sigma_0(\tilde{x} - t_j) = \sigma_j \tilde{x} - t_j, \quad \tilde{x} \in \tilde{R}_{t_j}. \] (3.24)
Thus we have
\[ \log(\sigma_j \tilde{x} - t_j) = \log(\tilde{x} - t_j) + 2\pi i. \] (3.25)

With these notations we can formulate the following result, due to G. D. Birkhoff (see [Bir09]):

\[ \text{These are not free generators; taking more care in choice of } \tilde{x}_j \text{, we can always obtain, for example, } \sigma_1 \cdots \sigma_n = 1. \]
**Theorem 2 (Birkhoff)** Let \( Y(\tilde{x}) \in GL(m, \mathcal{O}(\tilde{R}_t)) \) be the fundamental solution of system (3.1) with initial condition (3.3). Then for \( j = 1, \ldots, n \) \( Y(\tilde{x}) \) admits in \( \mathcal{L}_{t_j} \) the factorization

\[
Y(\tilde{x}) = H_j(x)M_j(\tilde{x} - t_j),
\]

where

\[
H_j(x) \in GL(m, \mathcal{O}(L_j)), \quad (3.27)
\]

\[
M_j(\tilde{x}) \in GL(m, \mathcal{O}(\tilde{C}^*)).(3.28)
\]

**Proof:** Let us fix \( j \). Since monodromy generator \( \Phi(\sigma_j) \) is non-degenerate, we can choose some value of the matricial logarithm \( \log \Phi(\sigma_j) \) and consider

\[
X_j(\tilde{x}) = \tilde{x}^{\frac{1}{2\pi i} \log \Phi(\sigma_j)} = \exp(\frac{1}{2\pi i} \log \Phi(\sigma_j) \log \tilde{x}) \in GL(m, \mathcal{O}(\tilde{C}^*)).
\]

In view of (3.25), \( X_j(\tilde{x} - t_j) \in GL(m, \mathcal{O}(\tilde{R}_{t_j})) \) satisfies

\[
X_j(\sigma_j \tilde{x} - t_j) = \Phi(\sigma_j)X_j(\tilde{x} - t_j). \quad (3.30)
\]

Hence \( Y(\tilde{x})X_j(\tilde{x} - t_j) \) is actually univalued in \( \mathcal{L}_{t_j} \), and we can define

\[
F(x) = (Y(\tilde{x})X_j(\tilde{x} - t_j))^{-1} \in GL(m, \mathcal{O}(\mathcal{L}_{t_j})). \quad (3.31)
\]

Now we can consider the Riemann-Hilbert boundary problem for \( F(x) \), which we have discussed in Chapter 2. By Theorem 11 there exist

\[
X^+(x) \in GL(m, \mathcal{O}(L_j)), \quad (3.32)
\]

\[
X^-(x) \in GL(m, \mathcal{O}(R_{t_j})), \quad (3.33)
\]

such that

\[
X^-(x)F(x) = X^+(x), \quad x \in \mathcal{L}_{t_j}. \quad (3.34)
\]

It suffices to define

\[
H_j(x) = (X^+)^{-1}(x) \in GL(m, \mathcal{O}(L_j)), \quad (3.35)
\]

\[
M_j(\tilde{x}) = X^-(x + t_j)X_j^{-1}(\tilde{x}) \in GL(m, \mathcal{O}(\tilde{C}^*)) \quad (3.36)
\]

in order to complete the proof.
Definition 3 We shall call $H_j(x)$ and $M_j(\tilde{x})$, respectively, the non-singular and principal factors of $Y$ at $t_j$.

Remark 3 1. Of course, factors $H_j$ and $M_j$ are defined up to the transformation

$$
\begin{align*}
H_j'(x) &= H_j(x)E_j(x-t_j), \\
M_j'(\tilde{x}) &= E_j^{-1}(x)M_j(\tilde{x}),
\end{align*}
$$

for arbitrary $E_j(x) \in GL(m, O(\mathbb{C}))$.

2. Although our choice of $\tilde{L}_{t_j}$ and $\tilde{x} - t_j$ is somewhat ambiguous, one should keep in mind that any other connected component of $\psi_t^{-1}(L_{t_j})$ is of the form $\sigma \tilde{L}_{t_j}$ for $\sigma \in \Delta_t$, and any other isomorphism, mapping $\tilde{R}_{t_j}$ onto $\tilde{C}^*$, is of the form $\tilde{x} \mapsto \sigma_j^l \tilde{x} - t_j$ for $l \in \mathbb{Z}$. According to (3.10), relation (3.26) implies

$$
M_j(\sigma_0(\tilde{x} - t_j)) = M_j(\tilde{x} - t_j) (\Phi(\sigma_j))^{-1}. \quad (3.38)
$$

Hence we can fix non-singular factor $H_j(x)$ independently of these choices and define for arbitrary $\sigma \in \Delta_t$ the corresponding principal factor by

$$
M_j(\sigma \tilde{x} - t_j) = M_j(\tilde{x} - t_j) (\Phi(\sigma_j))^{-1}, \quad (3.39)
$$

so that

$$
H_j(x)M_j(\sigma \tilde{x} - t_j) = Y(\sigma \tilde{x}) = Y(\tilde{x})\Phi(\sigma^{-1}). \quad (3.40)
$$

holds true.

We describe some basic properties of the non-singular and principal factors in the following

Lemma 5 1. Let $Y(\tilde{x}) \in GL(m, O(\tilde{R}_t))$ be the fundamental solution of system (3.1) with initial condition (3.3). Let $H_j, M_j$ be the non-singular and principal factors of $Y$ at $t_j$. Then there exists $P_j(x) \in gl(m, O(\mathbb{C}))$, such that $H_j$ and $M_j$ satisfy the systems

$$
\frac{dH_j}{dx} = \left( \sum_{j' = 1}^{n} \sum_{k=0}^{p_j} \frac{Q_{j',k}}{(x-t_{j'})^{k+1}} \right) H_j(x) - \\
H_j(x) \left( \sum_{k=0}^{p_j} \frac{J_{j,k}}{(x-t_j)^{k+1}} + P_j(x-t_j) \right), \quad x \in L_{t_j}
$$

(3.41)
and
\[
\frac{dM_j}{dx} = \left( \sum_{k=0}^{p_j} \frac{J_{j,k}}{x^{k+1}} + P_j(x) \right) M_j(\bar{x}),
\]
where \( J_{j,k} \in \text{gl}(m, \mathbb{C}) \) is given by
\[
J_{j,k} = \sum_{k'=0}^{p_j-k} \oint_{\gamma_j} H_j^{-1}(x) Q_{j,k+k'} H_j(x) \frac{dx}{(x-t_j)^{k'+1}}.
\]

2. Let \( Y(\bar{x}) \in GL(m, \mathcal{O}(\tilde{\mathcal{R}}_i)) \) be the fundamental solution of system (3.1) with initial condition (3.3). Let \( J_{j,k} \in \text{gl}(m, \mathbb{C}), P_j(x) \in \text{gl}(m, \mathcal{O}(\mathbb{C})) \) be such that system (3.41) has solution \( H_j(x) \in GL(m, \mathcal{O}(\mathcal{L}_j)) \). Then \( H_j \) is the non-singular factor of \( Y \) at \( t_j \), and corresponding principal factor \( M_j \) is a fundamental solution of system (3.42).

3. Assume that \( Y(\bar{x}) \in GL(m, \mathcal{O}(\tilde{\mathcal{R}}_i)) \) satisfies (3.3) and, for \( j = 1, \ldots, n \) admits in \( \mathcal{L}_j \) factorization (3.26), where \( H_j(x) \in GL(m, \mathcal{O}(\mathcal{L}_j)) \) and \( M_j(\bar{x}) \) is a fundamental solution of system (3.42) with some \( J_{j,k} \in \text{gl}(m, \mathbb{C}), P_j(x) \in \text{gl}(m, \mathcal{O}(\mathbb{C})) \). Then \( Y \) is a fundamental solution of system (3.1), where \( Q_{j,k} \in \text{gl}(m, \mathbb{C}) \) are given by
\[
Q_{j,k} = \sum_{k'=0}^{p_j-k} \oint_{\gamma_j} H_j(x) J_{j,k+k'} H_j^{-1}(x) \frac{dx}{(x-t_j)^{k'+1}}.
\]

**Proof:** The proof is done by straightforward computation. We shall prove statement 1, and the rest can be done absolutely analogously.

Let \( Y(\bar{x}) \in GL(m, \mathcal{O}(\tilde{\mathcal{R}}_i)) \) be the fundamental solution of system (3.1) with initial condition (3.3) and let \( H_j, M_j \) be the non-singular and principal factors of \( Y \) at \( t_j \). Then it follows from (3.38) that the logarithmic derivative of \( M_j(\bar{x}) \),
\[
\frac{dM_j}{dx} M_j^{-1}(\bar{x}),
\]
is univalued in \( C^* \). In view of (3.41) and (3.20), it has at \( x = 0 \) a pole of order \( p_j + 1 \). Thus \( M_j(\bar{x}) \) is a fundamental solution of system (3.42) where \( J_{j,k} \in \text{gl}(m, \mathbb{C}), P_j(x) \in \text{gl}(m, \mathcal{O}(\mathbb{C})) \). Substituting (3.42) into (3.41), (3.26), we conclude that \( H_j(x) \) satisfies system (3.41) and that \( J_{j,k} \) are given by (3.43).
Remark 4 1. If we retrace our proof of Theorem 4 and use there Theorem 1 to demand from $X^{-}$ to have at most a pole at $\infty$, then we can conclude that also the logarithmic derivative of $M_j$ has at most a pole at $\infty$. (Actually, this stronger formulation of Theorem 4 appears in G. D. Birkhoff’s work [Bir09].) Hence $P_j(x)$ in system (3.43) can always be chosen to be a polynomial.

2. We would like to note here that one should be able, in principle, to recover $Q_{j,k}$ from principal factors $M_j$ alone. Indeed, it follows from Liouville theorem that if for $j = 1, \ldots, n$ $M_j \in GL(m, O(\tilde{\mathbb{C}}^*))$ satisfy monodromy relations (3.38) then there exists at most one $Y \in GL(m, O(\tilde{\mathbb{R}}_t))$, normalized by (3.3), which admits in $\tilde{L}_t$ factorizations (3.26) with some $H_j(x) \in GL(m, O(\tilde{L}_j))$.

3.2 The Generalized Schlesinger System

Let us consider an analytic family of linear systems of type (3.1), parameterized by position of the singular points. Let us denote

$$S = \mathbb{C}^n \setminus \bigcup_{j' \neq j''} \{t = (t_1, \ldots, t_n) : t_{j'} = t_{j''}\}$$

and let $\mathcal{U} = U_1 \times \cdots \times U_n \subset \mathbb{C}^n$ be a polydisk, such that $\overline{\mathcal{U}} \subset S$. Assume that $Q_{j,k}(t) \in gl(m, O(\mathcal{U}))$ satisfy

$$\sum_{j=1}^n Q_{j,0}(t) \equiv 0$$

and consider for each fixed $t = (t_1, \ldots, t_n) \in \mathcal{U}$ the system

$$\frac{dY}{dx} = \sum_{j=1}^n \sum_{k=0}^{p_j} \frac{Q_{j,k}(t)}{(x - t_j)^{k+1}} Y$$

with the initial condition

$$Y(\infty, t) \equiv I.$$  

We encounter here a slight difficulty, because for different $t$ we have to consider $\tilde{x}$ on different surfaces $\tilde{\mathcal{R}}_t$. However, it can be overcome if we consider

$$\mathcal{T}_j = \{(x, t) \in \mathbb{C} \times \mathcal{U} : x = t_j\},$$

$$\mathcal{R} = \mathbb{C} \times \mathcal{U} \setminus \bigcup_j \mathcal{T}_j,$$
and the universal covering
\[ \psi : \tilde{\mathcal{R}} \mapsto \mathcal{R}, \tag{3.52} \]
with the group of deck transformations \( \Delta \). Our reasoning here is analogous to what we did in the previous section, so we shall be as brief as possible.

First of all, for a fixed \( t \in \mathcal{U} \) let us denote
\[ (\mathcal{R}_t, t) = \{(x, t) : x \in \mathcal{R}_t\} \subset \mathcal{R}, \tag{3.53} \]
\[ (\tilde{\mathcal{R}}_t, t) = \psi^{-1}(\mathcal{R}_t, t). \tag{3.54} \]
Then set \((\tilde{\mathcal{R}}_t, t)\) is simply connected, hence isomorphic to \( \tilde{\mathcal{R}}_t \), and we can, indeed, interpret a point of \( \psi^{-1}(x, t) \) as a pair \((\tilde{x}, t)\), where \( \tilde{x} \in \tilde{\mathcal{R}}_t \). Analogously, for a fixed \( x \in \mathcal{C} \setminus \bigcup_{j=1}^{n} \mathcal{U}_j \) we denote
\[ (x, \mathcal{U}) = \{(x, t) : t \in \mathcal{U}\} \subset \mathcal{R} \tag{3.55} \]
and observe that the set \( \psi^{-1}(x, \mathcal{U}) \) is disconnected. Let us choose a connected component, and denote it by \((\tilde{x}, \mathcal{U})\). Then \((\tilde{x}, \mathcal{U})\) is simply connected (and, therefore, isomorphic to \( \mathcal{U} \)), hence for any \( t \in \mathcal{U} \) the intersection \((\tilde{x}, \mathcal{U}) \cap (\tilde{\mathcal{R}}_t, t)\) consists of a single point \((\tilde{x}, t)\). Thus we can identify \( \tilde{x} \in \mathcal{R}_{t'} \) and \( \tilde{x} \in \mathcal{R}_{t''} \), such that \((\tilde{x}, t')\) and \((\tilde{x}, t'')\) belong to the same connected component \((\tilde{x}, \mathcal{U})\) of \( \psi^{-1}(\{(x, t) : t \in \mathcal{U}\}) \). In particular, if we assume that initial condition (3.49) is taking place at some such connected component of \( \psi^{-1}(\infty, t) \) then, because of analytic dependence of solution of a linear system on the coefficients, the family of systems (3.48) determines \( Y(\tilde{x}, t) \in GL(m, \mathcal{O}(\tilde{\mathcal{R}})) \), and, instead of (3.48), we can actually write
\[ \frac{\partial Y}{\partial x} = \sum_{j=1}^{n} \sum_{k=0}^{p_j} \frac{Q_{j,k}(t)}{(x - t_j)^{k+1}} Y. \tag{3.56} \]
Also, since group of deck transformations \( \Delta \) acts on each surface \((\tilde{\mathcal{R}}_t, t)\), it can be identified with the corresponding group of deck transformations \( \Delta_t \), with the understanding
\[ \sigma(\tilde{x}, t) = (\sigma \tilde{x}, t), \tag{3.57} \]
and thus the monodromy representations of \( Y \) form an analytic family, that is, for \( \sigma \in \Delta \) we can consider the monodromy matrix function
\[ \Phi(\sigma)(t) = (Y \circ \sigma(\tilde{x}, t))^{-1} Y(\tilde{x}, t) = Y^{-1}(\sigma \infty, t) \in GL(m, \mathcal{O}(\mathcal{U})). \tag{3.58} \]

\[4\] We assume, of course, \( n > 1 \).
Definition 4 We shall say that system (3.56) gives an isomonodromic deformation if the monodromy representation of $Y$ is independent of $t$, i.e. for each $\sigma \in \Delta$ the monodromy matrix function $\Phi(\sigma)$ is constant in $U$.

Let us consider now a polydisk $L = L_1 \times \ldots \times L_n \subset S$, such that $\overline{U} \subset L$, and for $j = 1, \ldots, n$ let us denote

$$
\mathcal{L}_{T_j} = L_j \times U \setminus T_j, \quad (3.59)
$$

$$
\mathcal{R}_{T_j} = \mathbb{C} \times U \setminus T_j. \quad (3.60)
$$

As before, we can choose a connected component $\tilde{\mathcal{L}}_{T_j}$ of $\psi^{-1}(L_j)$ in such a way that the corresponding cyclic subgroups $\Delta_{T_j}$ of $\Delta$, acting on $\tilde{\mathcal{L}}_{T_j}$, generate $\Delta$, and fix an isomorphism

$$(\tilde{x}, t) \mapsto (\tilde{x} - t_j, t), \quad (3.61)$$

which maps the universal covering space $\tilde{\mathcal{R}}_{T_j}$ over $\mathcal{R}_{T_j}$ onto $\tilde{\mathbb{C}}^* \times U$.

Definition 5 We shall say that system (3.56) gives the isoprincipal deformation, if $Y$ has the principal factors, independent of $t$, i.e. for $j = 1, \ldots, n$ $Y$ admits in $\tilde{\mathcal{L}}_{T_j}$ the factorization

$$
Y(\tilde{x}, t) = H_j(x, t)M_j(\tilde{x} - t_j), \quad (3.62)
$$

where

$$
H_j(x, t) \in GL(m, \mathcal{O}(L_j \times U)), \quad (3.63)
$$

$$
M_j(\tilde{x}) \in GL(m, \mathcal{O}(\tilde{\mathbb{C}}^*)). \quad (3.64)
$$

Remark 5 In view of relation (3.38) in Remark 3, the isoprincipal deformation belongs to the class of isomonodromic deformations.

Now we shall try to develop some simple criteria of whether given system (3.56) gives the isoprincipal deformation.

Lemma 6 System (3.56) gives the isoprincipal deformation if, and only if, $Y$ satisfies the linear Pfaffian system

$$
dY = \sum_{j=1}^{n} \sum_{k=0}^{p_j} Q_{j,k}(t) \frac{dx - dt_j}{(x - t_j)^{k+1}} Y. \quad (3.65)
$$
Proof: Let us assume first that system (3.56) gives the isoprincipal deformation. Since $Y(\bar{x},t)$ satisfies, by definition, system (3.56), we only have to prove that
\[
\frac{\partial Y}{\partial t_j} Y^{-1} = - \sum_{k=0}^{p_j} \frac{Q_{j,k}(t)}{(x - t_j)^{k+1}}. \tag{3.66}\]

Let us observe that, since the isoprincipal deformation is isomonodromic, for any $\sigma \in \Delta$
\[
\left( \frac{\partial Y}{\partial t_j} Y^{-1} \right) \circ \sigma = \frac{\partial Y}{\partial t_j} Y^{-1} - Y \left( \Phi(\sigma^{-1}) \frac{\partial \Phi(\sigma)}{\partial t_j} \right) Y^{-1} = \frac{\partial Y}{\partial t_j} Y^{-1}, \tag{3.67}\]

hence,
\[
\frac{\partial Y}{\partial t_j} Y^{-1} \in gl(m, \mathcal{O}(\mathcal{R})). \tag{3.68}\]

Taking into account factorization (3.62) of $Y$ for $j' \neq j$, we can observe that in $\mathcal{L}_{T_j'}$
\[
\frac{\partial Y}{\partial t_j} Y^{-1} = \frac{\partial H_{j'}}{\partial t_j} H^{-1}_{j'} \in gl(m, \mathcal{O}(\mathcal{L}_{j'} \times \mathcal{U})), \tag{3.69}\]

hence,
\[
\frac{\partial Y}{\partial t_j} Y^{-1} \in gl(m, \mathcal{O}(\mathcal{R}_{T_j})). \tag{3.70}\]

Analogously, since
\[
\frac{\partial (M_j(\bar{x} - t_j))}{\partial t_j} = - \frac{\partial (M_j(\bar{x} - t_j))}{\partial x}, \tag{3.71}\]

we can observe that in $\mathcal{L}_{T_j}$
\[
\left( \frac{\partial Y}{\partial t_j} + \frac{\partial Y}{\partial x} \right) Y^{-1} = \left( \frac{\partial H_j}{\partial t_j} + \frac{\partial H_j}{\partial x} \right) H^{-1} \in gl(m, \mathcal{O}(\mathcal{L}_j \times \mathcal{U})), \tag{3.72}\]

hence
\[
\frac{\partial Y}{\partial t_j} Y^{-1} + \sum_{k=0}^{p_j} \frac{Q_{j,k}(t)}{(x - t_j)^{k+1}} \in gl(m, \mathcal{O}(\overline{\mathcal{C}} \times \mathcal{U})). \tag{3.73}\]

But initial condition (3.49) implies
\[
\left. dY \right|_{\bar{x}=\infty} = 0, \quad (3.74)\]
and, applying Liouville theorem, we can conclude

\[
\frac{\partial Y}{\partial t_j} Y^{-1} + \sum_{k=0}^{p_j} \frac{Q_{j,k}(t)}{(x - t_j)^{k+1}} \equiv 0. 
\]  

(3.75)

Thus, indeed, (3.66) holds true and \( Y \) satisfies system (3.65).

Conversely, if \( Y \) satisfies system (3.65), then let us choose a simply connected neighborhood \( \mathcal{V} \) of 0 in \( \mathbb{C} \), such that

\[
\forall t \in \mathcal{U}, \forall j \quad (3.76)
\]

and consider for any fixed \( \tilde{v} \in \tilde{\mathcal{V}}^* \) the surface \( \tilde{x} = \tilde{v} + t_j \), on which \( Y \) satisfies

\[
dY = \sum_{j' \neq j} \sum_{k=0}^{p_{j'}} Q_{j',k} \frac{dt_{j'} - dt_{j'}}{(v + t_j - t_{j'})^{k+1}} Y. 
\]  

(3.77)

Hence, the linear Pfaffian system

\[
dH_j = \sum_{j' \neq j} \sum_{k=0}^{p_{j'}} Q_{j',k} \frac{dt_{j'} - dt_{j'}}{(v + t_j - t_{j'})^{k+1}} H_j, 
\]  

(3.78)

where \( v \in \mathcal{V} \) is a parameter, is compatible\(^5\) and thus, according to Frobenius theorem, completely integrable. Let us fix \( t^0 \in \mathcal{U} \) and let \( H_j(x, t^0) \in GL(m, \mathcal{O}(\mathcal{L}_j)) \) and \( M_j(\tilde{x}) \in GL(m, \mathcal{O}(\tilde{\mathcal{C}}^*)) \) be the non-singular and principal factors of \( Y(\tilde{x}, t^0) \). Because of the linearity of system (3.78), it has a solution \( H_j(v, t) \in GL(m, \mathcal{O}(\mathcal{V} \times \mathcal{U})) \), satisfying

\[
H_j(v, t^0) = H_j^0(v + t^0_j). 
\]  

(3.79)

The product \( H_j^{-1}(v, t)Y(\tilde{v} + t_j, t) \in GL(m, \mathcal{O}(\tilde{\mathcal{V}}^* \times \mathcal{U})) \) is independent of \( t \), hence

\[
H_j^{-1}(v, t)Y(\tilde{v} + t_j, t) = M_j(\tilde{v}) 
\]  

(3.80)

and \( H_j(v, t) \) can be analytically continued into the simply connected domain \( \{(v, t) : t \in \mathcal{U}, v \in \mathcal{L}_j - t_j \} \subset \mathbb{C}^* \times \mathcal{U} \). Returning to \( \tilde{x} = \tilde{v} + t_j \), we obtain that \( H_j(x, t) \in GL(m, \mathcal{O}(\mathcal{L}_j \times \mathcal{U})) \) and

\[
Y(\tilde{x}, t) = H_j(x, t)M_j(\tilde{x} - t_j), \quad (\tilde{x}, t) \in \tilde{\mathcal{L}}_j, 
\]

(3.81)

\(^5\)Compatibility of a Pfaffian system means that the one-form on the right-hand side is closed.
\(\text{i.e. system } (3.56) \text{ gives the isoprincipal deformation, and this completes the proof. } \)

Let us observe that, since system (3.65) implies (3.56) and (3.74), Lemma 6 actually means that system (3.56) gives the isoprincipal deformation if, and only if, system (3.65) is integrable, i.e. compatible. The compatibility condition is

\[
\sum_{j=1}^{n} \sum_{k=0}^{p_j} \frac{dQ_{j,k} \wedge d(x - t_j)}{(x - t_j)^{k+1}} = \sum_{j,j'=1}^{n} \sum_{k=0}^{p_j} \sum_{k'=0}^{p_{j'}} \frac{Q_{j,k}Q_{j',k'}d(x - t_j) \wedge d(x - t_{j'})}{(x - t_j)^{k+1}(x - t_{j'})^{k'+1}}.
\]

(3.82)

Since \(Q_{j,k}\) are independent of \(x\), both sides of (3.82) are rational with respect to \(x\). Comparing on both sides the principal parts of Laurent series at each pole, we obtain the following non-linear Pfaffian system:

\[
dQ_{j,k} = \sum_{l=0}^{p_j-k} \sum_{j' \neq j} \sum_{k'=0}^{p_{j'}} (-1)^{(l+k')} [Q_{j',k'},Q_{j,k+l}] \frac{d(t_j - t_{j'})}{(t_j - t_{j'})^{k'+l+1}},
\]

(3.83)

\(j = 1, \ldots, n; \ k = 1, \ldots, p_j.\)

**Definition 6** We shall call system (3.83) the generalized Schlesinger system.

Thus we obtain the first main result of this Thesis:

**Theorem 3** System (3.56) gives the isoprincipal deformation if, and only if, \(Q_{j,k}(t)\) satisfy the generalized Schlesinger system.

**Remark 6** 1. Condition (3.47) is compatible with the generalized Schlesinger system. Indeed, (3.83) implies

\[
\sum_{j=1}^{n} dQ_{j,0} = - \sum_{j=1}^{n} dQ_{j,0} = 0,
\]

(3.84)

i.e. \(\sum_{j=1}^{n} Q_{j,0}(t)\) is a first integral of the generalized Schlesinger system.

2. One can check by straightforward computation that the generalized Schlesinger system is compatible, and thus completely integrable. Since in the next section we are going to prove independently a stronger result (Lemma 7), we do not give the details here.
3.3 The Painlevé Property

Let us consider the Cauchy problem for the generalized Schlesinger system. Let us fix point \( t^0 \in \mathcal{U} \) and, for \( j = 1, \ldots, n, \ k = 0, \ldots, p_j \), matrices \( Q_{j,k}^0 \in gl(m, \mathbb{C}) \) and consider system (3.83) in \( \mathcal{U} \) with the initial condition

\[
Q_{j,k}(t^0) = Q_{j,k}^0, \quad j = 1, \ldots, n, \ k = 0, \ldots, p_j.
\]  

To begin with, let us assume that \( Q_{j,k}^0 \) satisfy

\[
\sum_{j=1}^n Q_{j,0}^0 = 0.
\]  

Let \( \mathcal{L} \) be a polydisk, such that

\[
\overline{\mathcal{U}} \subset \mathcal{L} \subset \mathcal{S},
\]  

and for \( j = 1, \ldots, n \) let \( H_j(x, t^0) \in GL(m, \mathcal{O}(\mathcal{L}_j)), \ M_j(\bar{x}) \in GL(m, \mathcal{O}(\bar{\mathcal{C}}^*)) \) be the non-singular and principal factors of \( Y(\bar{x}, t^0) \in GL(m, \mathcal{O}(\bar{\mathcal{R}}_{t^0})) \), which satisfies the system

\[
\frac{dY}{dx} = \sum_{j=1}^n \sum_{k=0}^{p_j} \frac{Q_{j,k}^0}{(x - t^0_j)^{k+1}} Y,
\]  

with the initial condition

\[
Y(\infty, t^0) \equiv I.
\]  

By Lemma \[ M_j(\bar{x}) \] satisfies in \( \bar{\mathcal{C}}^* \) linear system (3.42), with \( J_{j,k} \in gl(m, \mathbb{C}) \) are given by

\[
J_{j,k} = \sum_{k'=0}^{p_j-k} \oint_{\gamma_j} \frac{H_j^{-1}(x, t^0)Q_{j,k+k'}^0H_j(x, t^0)}{(x - t^0_j)^{k'+1}} \frac{dx}{2\pi i}
\]  

and \( \gamma_j \) is a homotopically non-trivial simple contour in \( \mathcal{L}_j \setminus \overline{\mathcal{U}}_j \). Let us observe that the ratio \( M_j(\bar{x} - t^0_j)M_j^{-1}(\bar{x} - t_j) \) is a univalued matrix function in \( \{\mathcal{L}_j \setminus \overline{\mathcal{U}}_j\} \times \mathcal{U} \) and define

\[
F(x, t) \in GL(m, \mathcal{O}(\bigcup_{j=1}^n {\mathcal{L}_j \setminus \overline{\mathcal{U}}_j} \times \mathcal{U}))
\]
by

\[ F(x, t) = H_j(x, t^0)M_j(x - t_j^0) \quad M_j^{-1}(x - t_j)H_j^{-1}(x, t^0), \]

\[ x \in \mathcal{L}_j \setminus \overline{\mathcal{U}_j}, t \in \mathcal{U}_j, j = 1, \ldots, n. \]

(3.92)

Since

\[ F(x, t^0) \equiv I, \]

(3.93)

it follows from Lemma 4 in Chapter 2 that there exist a neighborhood \( \mathcal{V} \) of \( t_0 \) and matrix functions

\[ X^+(x, t) \in \text{GL}(m, \mathcal{O}(\bigcup_j \mathcal{L}_j \times \mathcal{V})), \]

\[ X^-(x, t) \in \text{GL}(m, \mathcal{O}(\{\mathcal{C} \setminus \bigcup_j \overline{\mathcal{U}_j}\} \times \mathcal{V})), \]

(3.94)

(3.95)

such that

\[ X^-(x, t)F(x, t) = X^+(x, t), \quad (x, t) \in \bigcup_j (\mathcal{L}_j \setminus \overline{\mathcal{U}_j}) \times \mathcal{V} \]

(3.96)

and

\[ X^-(\infty, t) \equiv I. \]

(3.97)

Also, \( X^+(x, t) \) and \( (X^+)^{-1}(x, t) \) are meromorphic with respect to \( t \) in \( \mathcal{U} \). Now we can formulate the following

**Lemma 7** Solution \( Q_{j,k}(t) \) of system (3.83) with initial condition (3.85), satisfying (3.86), is given in \( \mathcal{U} \) by

\[ Q_{j,k}(t) = \sum_{k' = 0}^{p_j - k} \oint_{\gamma_j} X^+(x, t)H_j(x, t^0)J_{j,k+k'}H_j^{-1}(x, t^0)(X^+)^{-1}(x, t) \frac{dx}{(x - t_j)^{k'+1}} \]  

(3.98)

In particular, \( Q_{j,k}(t) \) are holomorphic in \( \mathcal{V} \) and meromorphic in \( \mathcal{U} \).

**Proof:** Since (3.93) implies

\[ X^+(x, t^0) \equiv I, \]

(3.99)
we can define
\[ H_j(x,t) \in GL(m, \mathcal{O}(\mathcal{L}_j \times \mathcal{V})) \] (3.100)
by
\[ H_j(x,t) = X^+(x,t)H_j(x,t^0). \] (3.101)
Then (3.96) implies that there exists
\[ Y(\bar{x},t) \in GL(m, \mathcal{O}(\mathcal{R} \cap \psi^{-1}(\bar{\mathcal{C}} \times \mathcal{V}) \setminus \bigcup_j T_j))), \] (3.102)
which admits representation (3.62) in \( \tilde{\mathcal{L}}_j \cap \psi^{-1}(\{\mathcal{L}_j \times \mathcal{V}\} \setminus T_j) \) for \( j = 1, \ldots, n \) and the representation
\[ Y(\bar{x},t) = X^-(x,t)Y(\bar{x},t^0) \] (3.103)
in \( \psi^{-1}(\bar{\mathcal{C}} \setminus \bigcup_j \mathcal{U}_j) \times \mathcal{V} \). In particular, (3.97) means that \( Y(\infty,t) \equiv I \) for \( t \) in \( \mathcal{V} \). According to Lemma 5, \( Y(\bar{x},t) \) satisfies system (3.36), where \( Q_{j,k}(t) \in gl(m, \mathcal{O}(\mathcal{V})) \) are given by (3.37) and satisfy (3.85). By our construction, this system gives the isoprincipal deformation in \( \mathcal{V} \). Hence, by Theorem 3.83, \( Q_{j,k}(t) \) satisfy system (3.83) in \( \mathcal{V} \). Finally, since \( X^+(x,t) \) and \( (X^+)^{-1}(x,t) \) are meromorphic with respect to \( t \) in \( \mathcal{U} \), we conclude that \( Q_{j,k}(t) \), defined by (3.98), give in \( \mathcal{U} \) meromorphic solution to system (3.83) with initial condition (3.85). \( \square \)

We would like to note here that there is no loss of generality in assumption (3.86). Indeed, we can always fix \( a \in \mathbb{C} \setminus \bigcup_j \mathcal{L}_j \) and define map \( \mu : \mathcal{U} \mapsto \mathbb{C}^{n+1} \) by
\[ \mu(t_1, \ldots, t_n) = (\frac{1}{t_1 - a}, \ldots, \frac{1}{t_n - a}, 0). \] (3.104)
Furthermore, for \( j = 1, \ldots, n + 1, k = 0, \ldots, p_j \), \( p_{n+1} = 0 \), we can consider solution \( Q'_{j,k}(\tau) \) of the generalized Schlesinger system in \( n + 1 \) variables \( \tau = (\tau_1, \ldots, \tau_{n+1}) \) with the initial condition
\[ Q'_{j,k}(\mu(t^0)) = Q^0_{j,k}, \quad j = 1, \ldots, n, \] (3.105)
\[ Q'_{n+1,0}(\mu(t^0)) = -\sum_{j=1}^n Q^0_{j,0}, \]
so that
\[ \sum_{j=1}^{n+1} Q'_{j,0}(\mu(t^0)) = 0. \] (3.106)
According to Lemma 7, \( Q'_{n+1,0}(\tau) \) is of the form

\[
Q'_{n+1,0}(\tau) = X(\tau)Q'_{n+1,0}(\mu(t^0))X^{-1}(\tau),
\]

where meromorphic matrix function \( X(\tau) \) satisfies

\[
X(\mu(t^0)) = I
\]

(3.108)

and, in view of Lemma 6,

\[
\frac{\partial X}{\partial \tau_j} X^{-1}(\tau) = -\sum_{k=0}^{p_j} \frac{Q'_{j,k}(\tau)}{(\tau_{n+1} - \tau_j)^{k+1}}, \quad j = 1, \ldots, n.
\]

(3.109)

Now it can be checked by straightforward computation that \( Q_{j,k}(t) \), defined by

\[
Q_{j,k}(t) = (X^{-1}Q'_{j,k}X) \circ \mu(t), \quad j = 1, \ldots, n; k = 0, \ldots, p_j
\]

satisfy system (3.83) with initial condition (3.85).

Thus solution of system (3.83) with arbitrary initial values at \( t^0 \) is meromorphic in \( U \). Since our choice of \( U \) in the first place was limited only by the assumption \( U \subset S \), this solution can be continued into universal covering space \( \tilde{S} \) over \( S \), and we obtain the second main result of this Thesis:

**Theorem 4** Any solution of the generalized Schlesinger system (3.83) is meromorphic in \( \tilde{S} \).

Theorem 4 implies, in particular, that the generalized Schlesinger system enjoys the Painlevé property, that is, it has no movable critical points.
Chapter 4

Fuchsian Systems

4.1 Non-Resonant Fuchsian Systems

Definition 7 A linear differential system of the form

\[
\frac{dY}{dx} = \sum_{j=1}^{n} \frac{Q_j}{x-t_j} Y,
\]

(4.1)

where \( Q_j \in \text{gl}(m, \mathbb{C}) \), is called a Fuchsian system.

System (4.1) is a special case of system (3.1) with

\( p_1 = \ldots = p_n = 0 \), \( Q_{j,0} = Q_j \).

(4.2)

As before, we assume that \( x = \infty \) is a regular point, i.e.

\[
\sum_{j=1}^{n} Q_j = 0,
\]

(4.3)

and set the initial condition

\( Y(\infty) = I \).

(4.4)

Then we can apply the theory developed in the previous Chapter to the Fuchsian systems.

To begin with, we consider the following

Definition 8 Fuchsian system (4.1) is called non-resonant if for \( j = 1, \ldots, n \) no two eigenvalues of \( Q_j \in \text{gl}(m, \mathbb{C}) \) differ by an integer.
It is well-known\(^1\) that for the non-resonant Fuchsian systems most of the information about the monodromy representation and the local factorization of the fundamental solution can be obtained very easily:

**Lemma 8** Let system (4.1) be non-resonant. Then for \( j = 1, \ldots, n \) \( Y \) has at \( t_j \) the principal factor of the form

\[
M_j(\tilde{x}) = \tilde{x}^J K_j,
\]

where \( J \) is the Jordan form of \( Q_j \), and \( K_j \in GL(m, \mathbb{C}) \). In particular, the corresponding monodromy generator \( \Phi(\sigma_j) \) is given by

\[
\Phi(\sigma_j) = K_j^{-1} e^{-2\pi i J} K_j.
\]

**Proof:** We give a sketch of the proof, adapted from [CL55]. Let us fix \( j \). Considering, if necessary, the transformation

\[
Y' = TY,
\]

where

\[
TQ_j T^{-1} = J_j,
\]

we can assume that

\[
Q_j = J_j.
\]

In view of Lemma [5], it suffices to show that the system

\[
\frac{dH_j}{dx} = \left[ J_j, H_j(x) \right] + \sum_{j' \neq j} \frac{Q_j'}{x - t_j'} H_j(x)
\]

has solution \( H_j(x) \), holomorphic in a neighborhood of \( t_j \) and normalized by

\[
H_j(t_j) = I.
\]

Considering the Taylor expansions

\[
H_j(x) = I + \sum_{k=1}^{\infty} H_{j,k}(x - t_j)^k,
\]

\[
\sum_{j' \neq j} \frac{Q_j'}{x - t_j'} = \sum_{k=0}^{\infty} R_{j,k}(x - t_j)^k,
\]

\(^1\)See any textbook on Fuchsian systems, such as [CL55].
we obtain the recurrence relation for $H_{j,k}$:

$$(k + 1)H_{j,k+1} - [J_j, H_{j,k+1}] = \sum_{k'=0}^{k} R_{j,k'}H_{j,k-k'}.$$  

(4.14)

The left-hand side of (4.14) can be considered as an expression for a linear operator, acting on $gl(m, \mathbb{C})$. The non-resonance condition guarantees that this operator is invertible, i.e. the recurrence relation (4.14) is solvable. The convergence of the obtained Taylor series for $H_j(x)$ can be established by the Cauchy majorization method. This completes the proof. ■

Substituting (4.2) into (3.83), we observe that in the case of Fuchsian systems the generalized Schlesinger system takes the form

$$dQ_j = \sum_{j' \neq j} [Q_{j'}, Q_j] \frac{dt_j - dt'_j}{t_j - t'_j}, \quad j = 1, \ldots, n.$$  

(4.15)

System (4.15) was introduced by L. Schlesinger in [Sch12] and is known as the Schlesinger system. This explains our term “generalized” for system (3.83). In order to consider the implications of Lemma 8 for the Schlesinger system, let us recall the notation

$$\mathcal{S} = \mathbb{C}^n \setminus \bigcup_{j' \neq j''} \{t = (t_1, \ldots, t_n) : t_{j'} = t_{j''}\},$$  

(4.16)

fix point $t^0 \in \mathcal{S}$, matrices $Q_j^0 \in gl(m, \mathbb{C})$, such that

$$\sum_{j=1}^{n} Q_j^0 = 0,$$  

(4.17)

and assume that the Fuchsian system

$$\frac{\partial Y}{\partial x} = \sum_{j=1}^{n} \frac{Q_j^0}{x - t_j^0} Y$$  

(4.18)

is non-resonant. Now we consider an isomonodromic deformation of system (4.18). Let polydisk $\mathcal{U}$ and matrix functions $Q_1(t), \ldots, Q_n(t) \in$
$gl(m, \mathcal{O}(\mathcal{U}))$ be such that
\begin{align}
t^0 &\in \mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{S}, \\
\sum_{j=1}^{n} Q_j(t) &\equiv 0, \\
Q_j(t^0) &\equiv Q_j^0, 
\end{align}
and system
\begin{equation}
\frac{\partial Y}{\partial x} = \sum_{j=1}^{n} \frac{Q_j(t)}{x - t_j} Y,
\end{equation}
with initial condition
\begin{equation}
Y(\infty, t) \equiv I
\end{equation}
gives an isomonodromic deformation in $\mathcal{U}$. Then, by continuity, system \((4.22)\) is non-resonant for each fixed $t$ in a neighborhood of $t^0$. Since for $j = 1, \ldots, n$ the monodromy generator $\Phi(\sigma_j)$ is independent of $t$, Lemma \ref{lemma} implies that $Q_j(t)$ has Jordan form $J_j$, independent of $t$ in the whole of $\mathcal{U}$. Since matrix $K_j$ in expression \((4.16)\) is determined by matrices $\Phi(\sigma_j)$ and $J_j$ up to multiplication from the left by a non-degenerate matrix, commuting with $J_j$, we can observe in view of Remark \ref{remark} that the principal factors of $Y(\tilde{x}, t)$ can be chosen to be independent of $t$. Hence, system \((4.22)\) actually gives the isoprincipal deformation in $\mathcal{U}$. On the other hand, we know (see Remark \ref{remark}) that the isoprincipal deformation is isomonodromic. Thus Theorem \ref{theorem} implies the following result, originally proved by L. Schlesinger in \cite{Sch12}:

**Theorem 5 (Schlesinger)** Assume that Fuchsian system \((4.18)\) is non-resonant. Then $Q_j(t)$ satisfy system \((4.15)\) with initial condition \((4.21)\) if, and only if, system \((4.22)\) gives the isomonodromic deformation of Fuchsian system \((4.18)\).

We would like to note that without the assumption of non-resonance the conclusion of Theorem \ref{theorem} fails in one direction. There exist isomonodromic deformations of resonant Fuchsian systems, which are not governed by the Schlesinger system. In the subsequent sections of this Chapter we shall present V. E. Katsnelson’s example, concerning rational matrix functions in general position, illustrating this phenomenon.

Theorem \ref{theorem} on the other hand, is applicable in the resonant case as well, i.e. in order to solve the Cauchy problem for the Schlesinger system one
has to construct the isoprincipal deformation. We shall see that in the case of rational matrix functions in general position this approach allows to solve the Cauchy problem for the Schlesinger system explicitly. In this context we would like to mention a generalization of Lemma 8 due to A. H. M. Levelt (see [Lev61]). It states that in general fundamental solution $Y(x)$ of Fuchsian system (4.1) has at $t_j$ the principal factor of the form

$$M_j(\tilde{x}) = x^{D_j} \tilde{x}^{E_j} K_j,$$

where $D_j$ is a diagonal matrix with integer entries, $E_j$ is an upper-triangular matrix, whose spectrum coincides modulo $\mathbb{Z}$ with the spectrum of $Q_j$ and consists of eigenvalues, pairwise distinct modulo $\mathbb{Z}$, and $K_j \in GL(m, \mathbb{C})$.

We would also like to note that Theorem 4 implies that any solution of the Schlesinger system (4.15) is meromorphic in the universal cover over $\mathcal{S}$. This result was originally proved by T. Miwa in [Miw81] under the assumption of non-resonance, which now can be removed.

### 4.2 Rational Matrix Functions in General Position

Now our goal is to deal with the class of Fuchsian systems, whose fundamental solutions are rational matrix functions in general position. General theory of such systems was discussed in depth in V. E. Katsnelson’s works [Kat97], [Kat01]. We follow these sources in the presentation below.

**Definition 9** Rational square matrix function $Y(x)$ is said to be a rational matrix function in general position, if:

1. $|Y(x)|$ does not vanish identically;
2. the polar set of $Y(x)$ and the polar set of $Y^{-1}(x)$ do not intersect;
3. all the poles of $Y(x)$ and $Y^{-1}(x)$ are simple;
4. all the residues of $Y(x)$ and $Y^{-1}(x)$ are matrices of rank one;
5. both $Y(x)$ and $Y^{-1}(x)$ are holomorphic at $\infty$. 

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Let \( Y(x) \) be an \( m \times m \) rational matrix function in general position. Without loss of generality, we assume

\[
Y(\infty) = I. \tag{4.25}
\]

Let us order somehow the poles of \( Y(x) \) (respectively, the poles of \( Y^{-1}(x) \)) and denote them by \( t_1, \ldots, t_s \) (respectively, by \( t_{s+1}, \ldots, t_{s+s'} \)). Since, by definition, all the poles of \( Y(x) \) and \( Y^{-1}(x) \) are simple and all the residues of \( Y(x) \) and \( Y^{-1}(x) \) have rank one, it follows from (4.25) that

\[
|Y(x)| = \frac{\prod_{j=1}^{s'}(x - t_{s+j})}{\prod_{j=1}^{s}(x - t_j)}, \tag{4.26}
\]

and, in particular, that \( s = s' \). Thus the polar sets of \( Y(x) \) and \( Y^{-1}(x) \) have the same cardinality. We shall call the ordered sets

\[
\mathcal{P} = \{t_1, \ldots, t_s\}, \tag{4.27}
\]

\[
\mathcal{Z} = \{t_{s+1}, \ldots, t_{2s}\}, \tag{4.28}
\]

respectively, the pole and zero sets of \( Y(x) \). We shall also associate with these sets the diagonal matrices

\[
A_P = \text{diag}(t_1, \ldots, t_s), \tag{4.29}
\]

\[
A_Z = \text{diag}(t_{s+1}, \ldots, t_{2s}), \tag{4.30}
\]

which we shall call, respectively, the pole and zero matrices of \( Y(x) \). Since for \( j = 1, \ldots, s \),

\[
\text{rank}(\text{Res}(Y; x = t_j)) = 1, \tag{4.31}
\]

there exist \( m \times 1 \) matrix \( c_j \) and \( 1 \times m \) matrix \( b_j \), such that

\[
\text{Res}(Y; x = t_j) = c_j b_j. \tag{4.32}
\]

We shall call matrices \( c_1, \ldots, c_s \) (respectively, \( b_1, \ldots, b_s \)) the left (respectively, right) pole semi-residues of \( Y(x) \). Analogously, we shall call \( m \times 1 \) matrices \( c_{s+1}, \ldots, c_{2s} \) and \( 1 \times m \) matrices \( b_{s+1}, \ldots, b_{2s} \), such that for \( j = s + 1, \ldots, 2s \),

\[
\text{Res}(Y^{-1}; x = t_j) = c_j b_j, \tag{4.33}
\]

the left and right zero semi-residues of \( Y(x) \). From the left semi-residues of \( Y(x) \) we construct two \( m \times s \) matrices

\[
C_P = (c_1 \cdots c_s), \quad C_Z = (c_{s+1} \cdots c_{2s}), \tag{4.34}
\]
which are called, respectively, the left pole and zero semi-residual matrices of $Y(x)$. From the right semi-residues of $Y(x)$ we construct two $s \times m$ matrices

$$B_P = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix}, \quad B_Z = \begin{pmatrix} b_{s+1} \\ \vdots \\ b_{2s} \end{pmatrix},$$

which are called, respectively, the right pole and zero semi-residual matrices of $Y(x)$.

**Remark 7** Note that semi-residual matrices $C_P, B_P, C_Z, B_Z$ of $Y(x)$ are defined up to the transformation

$$\begin{align*}
C'_P &= C_P E_P, \\
B'_P &= E_P^{-1} B_P, \\
C'_Z &= C_Z E_Z, \\
B'_Z &= E_Z^{-1} B_Z
\end{align*}$$

for arbitrary diagonal $E_P, E_Z \in GL(s, \mathbb{C})$.

In terms of the semi-residual matrices we can write down the following realizations of $Y(x)$ and $Y^{-1}(x)$:

$$\begin{align*}
Y(x) &= I + C_P (x I - A_P)^{-1} B_P, \\
Y^{-1}(x) &= I + C_Z (x I - A_Z)^{-1} B_Z.
\end{align*}$$

In order to explain, what kind of relations the identity

$$Y(x)Y^{-1}(x) = Y^{-1}(x)Y(x) = I$$

imposes on the semi-residual matrices of $Y(x)$, we formulate the following version of a result, proved by I. Gohberg, M. A. Kaashoek, L. Lerer and L. Rodman in [GKLR84]:

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Theorem 6 (Gohberg, Kaashoek, Lerer, Rodman) 1. Let $Y(x)$ be a rational matrix function in general position, normalized at $\infty$ by (4.25), with pole and zero matrices $A_P, A_Z$ and left pole, left zero, right pole, right zero semi-residual matrices $C_P, C_Z, B_P, B_Z$, respectively. Let $S_{PZ}$ be the solution of the matricial equation

$$A_P S_{PZ} - S_{PZ} A_Z = B_P C_Z$$

(4.40)

and let $S_{ZP}$ be the solution of the matricial equation

$$A_Z S_{ZP} - S_{ZP} A_P = B_Z C_P.$$  

(4.41)

Then matrices $S_{PZ}$ and $S_{ZP}$ are mutually inverse:

$$S_{PZ} = S_{ZP}^{-1},$$

(4.42)

and the following relations hold true:

$$C_P S_{PZ} = C_Z,$$  \hspace{1cm} (4.43)

$$S_{PZ} B_Z = -B_P.$$  \hspace{1cm} (4.44)

2. Let $P, Z$ be two disjoint finite subsets of $\mathbb{C}$ of the same cardinality $s$ and let $A_P, A_Z$ be the associated diagonal matrices. Let $C_Z, B_P$ (respectively, $C_P, B_Z$) be an $m \times s$ matrix and an $s \times m$ matrix. Assume that $s \times s$ matrix solution $S_{PZ}$ (respectively, $S_{ZP}$) of equation (4.40) (respectively, (4.41)) is invertible. Then there exists unique $m \times m$ rational matrix function in general position $Y(x)$, normalized at $\infty$ by (4.25), with pole and zero matrices $A_P, A_Z$ and left zero, right pole (respectively, left pole, right zero) semi-residual matrices $C_Z, B_P$ (respectively, $C_P, B_Z$).

Proof:

1. Substituting expressions (4.37), (4.38) and (4.40), rewritten in the form

$$B_P C_Z = S_{PZ} (xI - A_Z) - (xI - A_P) S_{PZ},$$

(4.45)

into the identity $Y(x)Y^{-1}(x) = I$, we obtain

$$C_P (xI - A_P)^{-1} (B_P + S_{PZ} B_Z) = (C_Z - C_P S_{PZ}) (xI - A_P)^{-1} B_P.$$  \hspace{1cm} (4.46)
Since the spectra of $A_P$ and $A_Z$ are disjoint, this means that relations (4.43) and (4.44) hold true. Analogously, we can derive from equation (4.41) and the identity $Y^{-1}(x)Y(x) = I$ the following relations:

$$C_Z S_{ZP} = C_P, \quad (4.47)$$
$$S_{ZP} B_P = -B_Z. \quad (4.48)$$

It follows from (4.40) and (4.47) that

$$A_P S_{PZ} S_{ZP} - S_{PZ} A_Z S_{ZP} = B_P C_P. \quad (4.49)$$

On the other hand, it follows from (4.41) and (4.44) that

$$S_{PZ} A_Z S_{ZP} - S_{PZ} S_{ZP} A_P = -B_P C_P. \quad (4.50)$$

Thus

$$[A_P, S_{PZ} S_{ZP}] = 0, \quad (4.51)$$

and, since all eigenvalues of $A_P$ are of multiplicity 1, matrix $S_{PZ} S_{ZP}$ is diagonal. Since (4.43) and (4.47) imply

$$C_P S_{PZ} S_{ZP} = C_P, \quad (4.52)$$

and since, by definition, left pole semi-residual matrix $C_P$ has no zero columns, we conclude that

$$S_{PZ} S_{ZP} = I. \quad (4.53)$$

2. The proof is absolutely analogous for both versions, so we assume that matrices $C_Z, B_P$ are given and that matrix $S_{PZ}$, satisfying equation (4.40), is invertible. If rational matrix function in general position $Y(x)$, normalized at $\infty$ by (4.25), with pole and zero matrices $A_P, A_Z$ and left zero, right pole semi-residual matrices $C_Z, B_P$ exists, then, in view of part 1) of the Theorem, it admits realizations (4.37), (4.38) with left pole, right zero semi-residual matrices $C_P, B_Z$ given by

$$C_P = C_Z S_{PZ}^{-1}, \quad (4.54)$$
$$B_Z = -S_{PZ}^{-1} B_P. \quad (4.55)$$

Hence it is uniquely determined. In order to prove the existence of such matrix function $Y(x)$, it suffices to verify that expressions (4.37), (4.38) agree with each other, i.e. that the identity $Y(x)Y^{-1}(x) = I$ holds true. This can be done by straightforward computation, utilizing (4.40) in the same way as above.
Remark 8  

1. In principle, the existence and uniqueness of solution for equations (4.40), (4.41) follows from the disjointedness of the spectra of matrices \(A_P\) and \(A_Z\). However, since these matrices are diagonal, we can give the explicit solutions of these equations, with notations (4.29), (4.30), (4.34), (4.35):

\[
S_{PZ} = \left( \frac{b_\alpha c_{s+\beta}}{t_\alpha - t_{s+\beta}} \right)_\alpha,\beta = 1^s, \tag{4.56}
\]

\[
S_{ZP} = \left( \frac{b_{s+\alpha}c_\beta}{t_{s+\alpha} - t_\beta} \right)_\alpha,\beta = 1^s. \tag{4.57}
\]

If \(Y(x)\) is a rational matrix function in general position, we shall call matrices \(S_{PZ}\) and \(S_{ZP}\), constructed in (4.56), (4.57) from the semi-residues, the poles and the zeroes of \(Y(x)\), the pole-zero and zero-pole core matrices of \(Y(x)\).

2. Actually, the result proved in [GKLRS4] is more general than Theorem 6. It concerns invertible rational matrix functions, not necessarily in general position. Such matrix functions also have realizations of form (4.37), (4.38). However, in general, \(s \times s\) matrices \(A_P, A_Z\) in these expressions need not be diagonal and their spectra need not be disjoint. If realization (4.37) is minimal, i.e. dimension \(s\) is minimal possible, then the pair \((A_P, B_P)\) is called a pole pair of \(Y(x)\). If realization (4.38) is minimal, then the pair \((C_Z, A_Z)\) is called a zero pair of \(Y(x)\). Under the assumption that the pairs \((C_Z, A_Z)\) and \((A_P, B_P)\) are minimal, i.e. that

\[
\bigcap_{j=0}^{s-1} \ker(C_Z A_Z^j) = 0, \tag{4.58}
\]

\[
\sum_{j=0}^{s-1} \text{Im}(A_P^j) B_P = \mathbb{C}^s, \tag{4.59}
\]

Gohberg-Kaashoek-Lerer-Rodman theorem asserts that \((C_Z, A_Z)\) and \((A_P, B_P)\) are a zero pair and a pole pair of some rational matrix function \(Y(x)\) if, and only if, equation (4.40) has an invertible solution.
There is one-to-one correspondence between such rational matrix functions and invertible solutions of equation (4.40), given by (4.43), (4.44).

From Theorem 6 we can derive several results, illustrating the connection between rational matrix functions in general position and Fuchsian systems. For brevity we shall use the following notation: $I_j$ is the diagonal matrix

$$I_j = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0), \quad (4.60)$$

whose only non-zero element is in the $j$-th position on the diagonal.

**Lemma 9** Let $Y(x)$ be a rational matrix function in general position, normalized at $\infty$ by (4.25), with pole and zero matrices $A_P, A_Z$ of form (4.29), (4.30), and left pole, right zero semi-residual matrices $C_P, B_Z$. Let $S_{ZP}$ be the zero-pole core matrix of $Y(x)$. Then $Y(x)$ is a fundamental solution of the Fuchsian system

$$\frac{dY}{dx} = \sum_{j=1}^{2s} \frac{Q_j}{x-t_j} Y, \quad (4.61)$$

where for $j = 1, \ldots, 2s$ $Q_j$ is given by

$$Q_j = \begin{cases} C_P I_j S_{ZP}^{-1} (t_j I - A_Z)^{-1} B_Z, & j = 1, \ldots, s, \\ C_P (t_j I - A_P)^{-1} S_{ZP}^{-1} I_{j-s} B_Z, & j = s + 1, \ldots, 2s. \end{cases} \quad (4.62)$$

**Proof:** In the manner completely analogous to the proof of Theorem 6 we can derive from (4.37), (4.38) and (4.40) the following expression for the logarithmic derivative of $Y(x)$:

$$Q(x) = \frac{dy}{dx} Y^{-1}(x) \quad (4.63)$$

$$= C_P (xI - A_P)^{-1} S_{ZP} (xI - A_Z)^{-1} B_Z.$$

In order to obtain (4.62), it remains only to substitute (4.42) into (4.63) and to compute the residue of $Q(x)$ at $t_j$. $\blacksquare$

In view of Lemma 9 it makes sense to consider the principal factors of a rational matrix function in general position.
Lemma 10 Let $\mathcal{P}, \mathcal{Z}$ be two disjoint finite subsets of $\mathbb{C}$ of the same cardinality $s$, with notation $(4.27), (4.28)$, and let $Q_j \in gl(m, \mathbb{C})$ for $j = 1, \ldots, 2s$. Assume that
\[ \sum_{j=1}^{2s} Q_j = 0 \quad (4.64) \]
and let $Y(x)$ be the fundamental solution of Fuchsian system $(4.61)$ with initial condition $(4.25)$. Then $Y(x)$ is a rational matrix function in general position with pole and zero sets $\mathcal{P}, \mathcal{Z}$ if, and only if, $Y(x)$ has the principal factors of the form
\[ M_j(x) = \begin{cases} 
  x^{-l_1} K_j, & j = 1, \ldots, s, \\
  x^{l_1} K_j, & j = s + 1, \ldots, 2s, 
\end{cases} \quad (4.65) \]
where for $j = 1, \ldots, 2s$ $K_j \in Gl(m, \mathbb{C})$. In this case for $j = 1, \ldots, s$ the first row of matrix $K_j$ is the right pole semi-residue of $Y(x)$ at $t_j$ and for $j = s + 1, \ldots, 2s$ the first column of matrix $K_j^{-1}$ is the left zero semi-residue of $Y(x)$ at $t_j$.

**Proof:** Let us assume first that $Y(x)$ is a rational matrix function in general position with pole and zero sets $\mathcal{P}, \mathcal{Z}$. Let $b_1, \ldots, b_{2s}, c_1, \ldots, c_{2s}$ be the right pole, right zero, left pole, left zero semi-residues of $Y(x)$ and let us denote by $i_j$ the $j$-th vector-column of the standard basis. Now for $j = 1, \ldots, 2s$ let $K_j \in Gl(m, \mathbb{C})$ satisfy
\[ i^*_j K_j = b_j, \quad j = 1, \ldots, s, \quad (4.66) \]
\[ K_j^{-1} i_1 = c_j, \quad j = s + 1, \ldots, 2s. \quad (4.67) \]
We are going to prove that for $j = 1, \ldots, 2s$ matrix function $M_j(x)$, defined by (4.65) is the principal factor of $Y(x)$ at $t_j$. By definition of the principal factors, we have to show for $j = 1, \ldots, 2s$ that matrix function
\[ H_j(x) = Y(x)M_j^{-1}(x - t_j). \quad (4.68) \]
is holomorphic and invertible in a neighborhood of $t_j$. Both $H_j$ and $H_j^{-1}$ have at most a simple pole at $t_j$, so we only have to compute the appropriate residues. For $j = 1, \ldots, s$ we have
\[ \text{Res}(H_j; x = t_j) = c_j b_j K_j^{-1}(I - I_1) = c_j i^*_1(I - I_1) = 0, \quad (4.69) \]
and also, since
\[ \text{Res}(YY^{-1}; x = t_j) = c_j b_j Y^{-1}(t_j) = 0, \quad (4.70) \]
we obtain
\[ \text{Res}(H_j^{-1}; x = t_j) = I_1 K_j Y^{-1}(t_j) = i_1 b_j Y^{-1}(t_j) = 0. \quad (4.71) \]
For \( j = s + 1, \ldots, 2s \) the computations are absolutely analogous.

Conversely, let us assume that for \( j = 1, \ldots, 2s \) \( Y(x) \) admits in a neighborhood of \( t_j \) the factorization
\[ Y(x) = H_j(x) M_j(x - t_j), \quad (4.72) \]
where matrix function \( H_j(x) \) is holomorphic and invertible in a neighborhood of \( t_j \) and matrix function \( M_j(x) \) is given by (4.65). Then \( Y(x) \) is an invertible rational matrix function, \( P \) is the polar set of \( Y(x) \) and \( Z \) is the polar set of \( Y^{-1}(x) \). Moreover,
\[ \begin{align*}
\text{Res}(Y; x = t_j) &= c_j b_j, & j = 1, \ldots, s, \\
\text{Res}(Y^{-1}; x = t_j) &= c_j h_j, & j = s + 1, \ldots, 2s, 
\end{align*} \quad (4.73)(4.74) \]
where \( c_1, \ldots, c_{2s} \) and \( b_1, \ldots, b_{2s} \) are, respectively, \( m \times 1 \) and \( 1 \times m \) matrices, given by
\[ 
\begin{align*}
c_j &= \begin{cases} 
H_j(t_j)i_1, & j = 1, \ldots, s, \\
K_j^{-1}i_1, & j = s + 1, \ldots, 2s, 
\end{cases} \quad (4.75) \\
b_j &= \begin{cases} 
i_1^*K_j, & j = 1, \ldots, s, \\
i_1^*H_j^{-1}(t_j), & j = s + 1, \ldots, 2s. 
\end{cases} \quad (4.76)
\end{align*} \]

Hence, \( Y(x) \) is a rational matrix function in general position and \( b_1, \ldots, b_s, c_{s+1}, \ldots, c_{2s} \) are its right pole and left zero semi-residues.

Using Lemma 10, we can give the complete description of the class of Fuchsian systems, whose fundamental solutions are rational matrix functions in general position, in terms of the coefficients. The appropriate result, formulated below, was obtained by V. E. Katsnelson in [Kat97], [Kat01].
Theorem 7 (Katsnelson) Let $t_1, \ldots, t_{2s}$ be $2s$ distinct points in $\mathbb{C}$ and let $Q_1, \ldots, Q_{2s} \in \text{gl}(m, \mathbb{C})$. Let $Y(x)$ be a fundamental solution of the Fuchsian system

$$\frac{dY}{dx} = \sum_{j=1}^{2s} \frac{Q_j}{x-t_j}Y.$$ (4.77)

Then $Y(x)$ is a rational matrix function in general position with the pole and zero sets

$$\mathcal{P} = \{t_1, \ldots, t_s\},$$ (4.78)

$$\mathcal{Z} = \{t_{s+1}, \ldots, t_{2s}\}$$ (4.79)

if, and only if, $Q_1, \ldots, Q_{2s}$ satisfy the following relations:

$$\sum_{j=1}^{2s} Q_j = 0,$$ (4.80)

$$\text{rank}(Q_j) = 1, \quad j = 1, \ldots, 2s,$$ (4.81)

$$Q_j^2 = \begin{cases} -Q_j, & j = 1, \ldots, s, \\ Q_j, & j = s+1, \ldots, 2s, \end{cases}$$ (4.82)

$$Q_j R_j Q_j = \begin{cases} -R_j Q_j, & j = 1, \ldots, s, \\ Q_j R_j, & j = s+1, \ldots, 2s, \end{cases}$$ (4.83)

where $R_1, \ldots, R_{2s} \in \text{gl}(m, \mathbb{C})$ are given by

$$R_j = \sum_{j' \neq j} \frac{Q_{j'}}{t_j - t_{j'}},$$ (4.84)

Proof: First of all, condition (4.80) means that system (4.77) is regular at $x = \infty$. By definition, this condition is necessary in order for $Y$ to be a rational matrix function in general position. Lemma 10 implies that conditions (4.81), (4.82), which mean that for $j = 1, \ldots, 2s$ $Q_j$ has the Jordan form

$$J_j = \begin{cases} -I_1, & j = 1, \ldots, s, \\ I_1, & j = s+1, \ldots, 2s, \end{cases}$$ (4.85)
are necessary as well. Thus we assume that conditions $(4.80)$ - $(4.82)$ are satisfied. In view of Lemma 5 in Chapter 3 and Lemma 10, we have to show that for $j = 1, \ldots, 2s$ the linear system

$$
\frac{dH_j}{dx} = \frac{Q_j H_j(x) - H_j(x) J_j}{x - t_j} + \sum_{j' \neq j} \frac{Q_{j'}}{x - t_{j'}} H_j(x)
$$

(4.86)

has solution $H_j(x)$, holomorphic and invertible in a neighborhood of $t_j$, if, and only if, conditions $(4.83)$, $(4.84)$ are satisfied. This can be done in the manner similar to the proof of Lemma 8. Let us fix $j$, such that $1 \leq j \leq s$ (in the case $s + 1 \leq j \leq 2s$ the proof is absolutely analogous). Then we may assume, without loss of generality, that

$$Q_j = J_j = -I_1
$$

(4.87)

and consider the system

$$
\frac{dH_j}{dx} = \frac{[H_j(x), I_1]}{x - t_j} + \sum_{j' \neq j} \frac{Q_{j'}}{x - t_{j'}} H_j(x)
$$

(4.88)

with initial condition

$$H_j(t_j) = I.
$$

(4.89)

Furthermore, we substitute into (4.88) the Taylor expansions

$$H_j(x) = I + \sum_{k=1}^\infty H_{j,k}(x - t_j)^k,
$$

(4.90)

$$\sum_{j' \neq j} \frac{Q_{j'}}{x - t_{j'}} = R_j + \sum_{k=1}^\infty R_{j,k}(x - t_j)^k,
$$

(4.91)

where $R_j$ is given by (4.84), and obtain the recurrence relation for $H_{j,k}$:

$$(k + 1) H_{j,k+1} - [H_{j,k+1}, I_1] = R_j H_{j,k} + \sum_{k'=1}^k R_{j,k'} H_{j,k-k'}.
$$

(4.92)

Let us consider the linear operator

$$\Psi_k : gl(m, \mathbb{C}) 	o gl(m, \mathbb{C}),
$$

(4.93)
depending on parameter $k \in \mathbb{Z}$ and defined by

$$\Psi_k(X) = (k + 1)X - [X, I_1].$$  \hspace{1cm} (4.94)

We observe that $\Psi_k$ is invertible for $k > 0$ and that $X \in \text{Im}(\Psi_0)$ if, and only if,

$$XI_1 = I_1XI_1.$$ \hspace{1cm} (4.95)

Hence, recurrence relation (4.92) is solvable if, and only if, $R_j$ satisfies

$$R_jI_1 = I_1R_jI_1,$$ \hspace{1cm} (4.96)

which is precisely the form taken by condition (4.83) under assumption (4.87).

It remains to observe that if recurrence relation (4.92) is solvable, then the convergence of series (4.90) and the invertibility of $H_j(x)$ in a neighborhood of $t_j$ follow from the estimate

$$\|\Psi_k^{-1}\| = O(\frac{1}{k}), \hspace{1cm} k \to +\infty,$$ \hspace{1cm} (4.97)

on the norm of operator $\Psi_k^{-1}$ and from initial condition (4.89), respectively.

\[\blacksquare\]

**Remark 9** Note that if conditions (4.80) - (4.84) for system (4.77) are satisfied then it is possible to obtain the explicit form of fundamental solution $Y(x)$, satisfying the initial condition (4.25), directly from $Q_1, \ldots, Q_{2s}$. Indeed, in view of (4.81), for $j = 1, \ldots, 2s$ $Q_j$ admits the factorization

$$Q_j = \begin{cases} c_jq_j, & j = 1, \ldots, s, \\ q_jb_j, & j = s + 1, \ldots, 2s, \end{cases}$$ \hspace{1cm} (4.98)

where $c_1, \ldots, c_s, q_{s+1}, \ldots, q_{2s}$ and $q_1, \ldots, q_s, b_{s+1}, \ldots, b_{2s}$ are, respectively, $m \times 1$ and $1 \times m$ matrices. Furthermore, Lemma \[17\] implies that matrices $c_1, \ldots, c_s; b_{s+1}, \ldots, b_{2s}$ are the left pole and right zero semi-residues of $Y$. Hence, using (4.97), we can construct from these semi-residues zero-pole core matrix $S_{ZP}$ of $Y(x)$ and determine, according to Theorem \[6\], the rest of the semi-residues of $Y(x)$.
4.3 The Schlesinger System: Rational Solutions

Let us consider an isomonodromic deformation of a Fuchsian system, whose fundamental solutions are rational matrix functions in general position. Let set $S$ be defined by

$$S = \mathbb{C}^{2s} \setminus \bigcup_{j' \neq j''} \{ t = (t_1, \ldots, t_{2s}) : t_{j'} = t_{j''} \}. \quad (4.99)$$

Let us fix $t^0 = (t_1^0, \ldots, t_{2s}^0) \in S$ and $Q_1^0, \ldots, Q_{2s}^0 \in gl(m, \mathbb{C})$, such that

$$\sum_{j=1}^{2s} Q_j^0 = 0, \quad (4.100)$$

$$\text{rank}(Q_j^0) = 1, \quad j = 1, \ldots, 2s, \quad (4.101)$$

$$(Q_j^0)^2 = \begin{cases} -Q_j^0, & j = 1, \ldots, s, \\ Q_j^0, & j = s + 1, \ldots, 2s, \end{cases} \quad (4.102)$$

$$Q_j^0 R_j^0 Q_j^0 = \begin{cases} -R_j^0 Q_j^0, & j = 1, \ldots, s, \\ Q_j^0 R_j^0, & j = s + 1, \ldots, 2s, \end{cases} \quad (4.103)$$

with $R_j^0$ given by

$$R_j^0 = \sum_{j' \neq j} \frac{Q_{j'}^0}{t_{j'}^0 - t_j^0}. \quad (4.104)$$

Let us consider solution $Y(x, t^0)$ of the Fuchsian system

$$\frac{dY}{dx} = \sum_{j=1}^{2s} \frac{Q_j^0}{x - t_j^0} Y, \quad (4.105)$$

with initial condition

$$Y(\infty, t^0) = I. \quad (4.106)$$
According to Theorem 7, \( Y(x, t^0) \) is a rational matrix function in general position with the pole and zero sets

\[
\mathcal{P}^0 = \{t_1^0, \ldots, t_s^0\}, \quad (4.107)
\]

\[
\mathcal{Z}^0 = \{t_{s+1}^0, \ldots, t_{2s}^0\}. \quad (4.108)
\]

In view of Remark 9, let us fix the factorization

\[
Q_j^0 = \begin{cases}
    c_j^0 q_j^0, & j = 1, \ldots, s, \\
    q_j^0 b_j^0, & j = s + 1, \ldots, 2s,
\end{cases} \quad (4.109)
\]

where \( c_1^0, \ldots, c_s^0, q_{s+1}^0, \ldots, q_{2s}^0 \) and \( q_1^0, \ldots, q_s^0, b_{s+1}^0, \ldots, b_{2s}^0 \) are, respectively, \( m \times 1 \) and \( 1 \times m \) matrices, and construct the left pole and right zero semi-residual matrices of \( Y(x, t^0) \):

\[
C_P^0 = \begin{pmatrix} c_1^0 \cdots c_s^0 \end{pmatrix}, \quad (4.110)
\]

\[
B_Z^0 = \begin{pmatrix} b_{s+1}^0 \\ \vdots \\ b_{2s}^0 \end{pmatrix}. \quad (4.111)
\]

Furthermore, let \( U \) be a neighborhood of \( t^0 \) in \( S \) and let \( C_P(t) \) and \( B_Z(t) \) be, respectively, an \( m \times s \) and an \( s \times m \) matrix functions, holomorphic in \( U \) and satisfying

\[
C_P(t^0) = C_P^0, \quad (4.112)
\]

\[
B_Z(t^0) = B_Z^0. \quad (4.113)
\]

Let matrix function \( S_{ZP}(t) \in \text{gl}(s, \mathcal{O}(U)) \) satisfy for each \( t \) the equation

\[
A_Z(t) S_{ZP}(t) - S_{ZP}(t) A_P(t) = B_Z(t) C_P(t), \quad (4.114)
\]

where

\[
A_P(t) = \text{diag}(t_1, \ldots, t_s), \quad (4.115)
\]

\[
A_Z(t) = \text{diag}(t_{s+1}, \ldots, t_{2s}). \quad (4.116)
\]

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Then $S_{ZP}(t^0)$ is the zero-pole core matrix of $Y(x,t^0)$. Hence, $|S_{ZP}(t^0)| \neq 0$ and we can assume that $S_{ZP}(t) \in GL(s,U)$. According to Theorem 3, we can consider for each fixed $t$ in $U$ rational matrix function in general position $Y(x,t)$, normalized by
\[ Y(\infty,t) \equiv I, \tag{4.117} \]
with the pole and zero sets
\[ P = \{t_1, \ldots, t_s\}, \tag{4.118} \]
\[ Z = \{t_{s+1}, \ldots, t_{2s}\} \tag{4.119} \]
and left pole and right zero semi-residual matrices $C_P(t)$, $B_Z(t)$. By Lemma 9, $Y(x,t)$ satisfies the system
\[ \frac{\partial Y}{\partial x} = \sum_{j=1}^{2s} \frac{Q_j(t)}{x-t_j}Y, \tag{4.120} \]
where $Q_1(t), \ldots, Q_{2s}(t) \in gl(m, \mathcal{O}(U))$ are given by
\[
Q_j(t) = \begin{cases} 
C_P(t)I_jS_{ZP}^{-1}(t)(t_jI - A_Z(t))^{-1}B_Z(t), & j = 1, \ldots, s, \\
C_P(t)(t_jI - A_P(t))^{-1}S_{ZP}^{-1}(t)I_{j-s}B_Z(t), & j = s + 1, \ldots, 2s.
\end{cases} \tag{4.121}
\]
In particular,
\[ Q_j(t^0) = Q_j^0, \quad j = 1, \ldots, 2s. \tag{4.122} \]
Since the monodromy representation of a rational matrix function in general position is trivial, we conclude that system (4.120), thus constructed, gives in $U$ an isomonodromic deformation of Fuchsian system (4.18). However, this deformation need not be governed by the Schlesinger system. Indeed, solution of the latter must be completely determined at each $t$ by initial condition (4.122), whereas in the construction above our choice of $C_P(t), B_Z(t)$ is restricted by (4.112), (4.113) only at $t = t^0$. Thus the conclusion of Theorem 5 fails here. This does not, however, contradict Theorem 5 because (4.102) means that the non-resonance condition is violated.

The question remains, how to find the solution of the Schlesinger system with initial condition (4.122), satisfying (4.100) - (4.104). According to Theorem 3, it is necessary and sufficient to find the isoprincipal deformation of
Fuchsian system (4.105), constructed from initial condition (4.122). In view of Lemma 10, we can achieve this by modifying the construction above so that the right pole and left zero semi-residual matrices of \( Y(x, t) \) are independent of \( t \). Thus, employing Theorem 6, we can prove the third main result of this Thesis, obtained in joint work with V. E. Katsnelson:

**Theorem 8** Let \( t^0 = (t_1^0, \ldots, t_{2s}^0) \in S \) and \( Q_1^0, \ldots, Q_{2s}^0 \in gl(m, \mathbb{C}) \) satisfy the relations

\[
\begin{align*}
\sum_{j=1}^{2s} Q_j^0 &= 0, \quad (4.123) \\
\text{rank}(Q_j^0) &= 1, \quad j = 1, \ldots, 2s, \quad (4.124) \\
(Q_j^0)^2 &= \begin{cases} 
-Q_j^0, & j = 1, \ldots, s, \\
Q_j^0, & j = s + 1, \ldots, 2s,
\end{cases} \quad (4.125) \\
Q_j^0 R_j^0 Q_j^0 &= \begin{cases} 
-R_j^0 Q_j^0, & j = 1, \ldots, s, \\
Q_j^0 R_j^0, & j = s + 1, \ldots, 2s,
\end{cases} \quad (4.126)
\end{align*}
\]

with \( R_j^0 \) given by

\[
R_j^0 = \sum_{j' \neq j} \frac{Q_j^0}{t_j^0 - t_{j'}^0}. \quad (4.127)
\]

Then solution \( Q_1(t), \ldots, Q_{2s}(t) \) of the Schlesinger system

\[
dQ_j = \sum_{j' \neq j} [Q_j^0, Q_j^0] \frac{dt_j - dt_{j'}}{t_j - t_{j'}}, \quad j = 1, \ldots, 2s \quad (4.128)
\]

with the initial condition

\[
Q_j(t^0) = Q_j^0, \quad j = 1, \ldots, 2s \quad (4.129)
\]

can be constructed in the following way.

1. For \( j = 1, \ldots, 2s \) factorize matrix \( Q_j^0 \) (of rank 1) in the form

\[
Q_j^0 = \begin{cases} 
c_j^0 q_j^0, & j = 1, \ldots, s, \\
q_j^0 t_j^0, & j = s + 1, \ldots, 2s,
\end{cases} \quad (4.130)
\]
where \( c^0_1, \ldots, c^0_s, q^0_1, \ldots, q^0_{2s} \) and \( q^0_0, \ldots, q^0_s, b^0_{s+1}, \ldots, b^0_{2s} \) are, respectively, \( m \times 1 \) and \( 1 \times m \) matrices. Form \( m \times s \) matrix

\[
C^0_P = (c^0_1 \cdots c^0_s)
\]

and \( s \times m \) matrix

\[
B^0_Z = \begin{pmatrix}
b^0_{s+1} \\
\vdots \\
b^0_{2s}
\end{pmatrix}.
\]

2. Define \( s \times s \) matrix \( S^0_{ZP} \) by the equation

\[
A^0_Z S^0_{ZP} - S^0_{ZP} A^0_P = B^0_Z C^0_P,
\]

where

\[
A^0_P = \text{diag}(t^0_1, \ldots, t^0_s),
\]

\[
A^0_Z = \text{diag}(t^0_{s+1}, \ldots, t^0_{2s}).
\]

That is, set

\[
S^0_{ZP} = \left( \frac{b^0_{s+\alpha} c^0_{\beta}}{t^0_{s+\alpha} - t^0_{\beta}} \right)^s_{\alpha,\beta=1}.
\]

3. Determine the matrix \( S^0_{ZP} = (S^0_{ZP})^{-1} \) and define \( s \times m \) matrix \( B^0_P \) and \( m \times s \) matrix \( C^0_Z \) by

\[
B^0_P = -S^0_{ZP} B^0_Z,
\]

\[
C^0_Z = C^0_P S^0_{ZP}.
\]

4. Define rational \( s \times s \) matrix function \( S_{PZ}(t) \) by the equation

\[
A_P(t) S_{PZ}(t) - S_{PZ}(t) A_Z(t) = B^0_P C^0_Z,
\]

where

\[
A_P(t) = \text{diag}(t_1, \ldots, t_s),
\]

\[
A_Z(t) = \text{diag}(t_{s+1}, \ldots, t_{2s}).
\]
That is, set

$$S_{PZ}(t) = \left( \frac{b^0_\alpha c^0_{s+\beta}}{t_\alpha - t_{s+\beta}} \right)_{\alpha,\beta=1}^s,$$  \hspace{1cm} (4.142)

where $b^0_\alpha$ and $c^0_{s+\beta}$ are the $\alpha$-th row of $B^0_\alpha$ and the $\beta$-th column of $C^0_Z$, respectively.

5. Determine rational $s \times s$ matrix function $S_{ZP}(t) = S^{-1}_{PZ}(t)$ and define rational $m \times s$ and $s \times m$ matrix functions $C_P(t)$ and $B_Z(t)$, respectively, by

$$B_Z(t) = -S_{ZP}(t)B^0_\alpha,$$  \hspace{1cm} (4.143)

$$C_P(t) = C^0_Z S_{ZP}(t).$$  \hspace{1cm} (4.144)

6. Set

$$Q_j(t) = \begin{cases} 
C_P(t) I_j S_{ZP}(t) (t_j I - A_Z(t))^{-1} B_Z(t), & j = 1, \ldots, s, \\
C_P(t) (t_j I - A_P(t))^{-1} S_{ZP}(t) I_j S_{ZP}(t), & j = s + 1, \ldots, 2s,
\end{cases}$$

where for $k = 1, \ldots, s$ $I_k$ is the diagonal $s \times s$ matrix

$$I_k = \left( \delta^k_\alpha \delta^k_\beta \right)_{\alpha,\beta=1}^s = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0),$$  \hspace{1cm} (4.145)

whose only non-zero element is in the $k$-th position on the diagonal.

Remark 10 \hspace{0.5cm} 1. In order to allow steps 3. and 5. of the construction in Theorem 3, matrix $S^0_{ZP}$ must be invertible and the determinant of rational matrix function $S_{PZ}(t)$ must not vanish identically. We can see, in view of Theorem 4, that this is indeed the case, because, by our construction and by Theorem 7, $S^0_{ZP}$ is the zero-pole core matrix of a rational matrix function in general position and

$$S_{PZ}(t^0) = (S^0_{ZP})^{-1}.$$  \hspace{1cm} (4.147)
2. There is a certain ambiguity in the factorization of $Q_j^0$ at step 1. of the construction in Theorem 8. It corresponds to the ambiguity in the definition of the semi-residual matrices, described in Remark 7. However, this ambiguity disappears in the definition of $Q_j(t)$ at step 6.

3. Note that solution $Q_1(t), \ldots, Q_{2s}(t)$ of the Schlesinger system, constructed in Theorem 8, is rational.
Bibliography

[AB94] D. V. Anosov and A. A. Bolibruch. The Riemann-Hilbert Problem, volume 22 of Aspects of Mathematics: E. Vieweg-Verlag, Braunschweig · Wiesbaden, 1994.

[AF97] Mark J. Ablowitz and Athanassios S. Fokas. Complex Variables: Introduction and Applications. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge · New York · Melbourne, 1997.

[Bir09] G. D. Birkhoff. Singular points of ordinary linear differential equations. Trans. Amer. Math. Soc., 10, 1909.

[Bir13a] G. D. Birkhoff. The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations. In Proc. Amer. Acad. Arts and Sci., volume 49, 1913.

[Bir13b] G. D. Birkhoff. A theorem on matrices of analytic functions. Math. Ann., 74, 1913.

[CL55] E. A. Coddington and N. Levinson. Theory of Ordinary Differential Equations. McGraw-Hill, New York · Toronto · London, 1955.

[Gak66] F. D. Gakhov. Boundary Value Problems. Pergamon Press, New York, 1966.

[GKLR84] I. Gohberg, M. A. Kaashoek, L. Lerer, and L. Rodman. Minimal divisors of rational matrix functions with prescribed zero and pole structure. In H. Dym and I. Gohberg, editors, Topics in Operator Theory, Systems and Networks, volume 12 of
Operator Theory: Advances and Applications. Birkhäuser-Verlag, Basel · Berlin · Boston, 1984.

[Gol58] W. W. Golubew. Vorlesungen über Differentialgleischungen im Komplexen. Deutcher Verlag der Wiss, Berlin, 1958.

[GR79] H. Grauert and R. Remmert. Theory of Stein Spaces. Springer-Verlag, New York · Heidelberg · Berlin, 1979.

[Gra60] H. Grauert. Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Structuren. Publ. Math. IHES, 5, 1960.

[Gro57] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. Amer. J. Math., 79, 1957.

[Hil00] D. Hilbert. Mathematische Probleme. Nachr. Ges. Wiss, 1900.

[Hil05] D. Hilbert. Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen (Dritte Mitt.). Nachr. Ges. Wiss, 1905.

[Hil12] D. Hilbert. Grundzüge der Integralgleichungen. Drittes Abschnitt, Leipzig · Berlin, 1912.

[Hil76] E. Hille. Ordinary Differential Equations in the Complex Plane. Wiley-Interscience, New York, 1976.

[IKSY91] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida. From Gauß to Painlevé. A Modern Theory of Special Functions, volume 16 of Aspects of Mathematics: E. Vieweg, Braunschweig, 1991.

[Kat97] V. E. Katsnelson. Fuchsian differential systems related to rational matrix functions in general position and the joint system realization. In Proceedings of the Ashkelon Workshop on Complex Function Theory (1996), volume 11 of Israel Math. Conf. Proc. Bar-Ilan Univ., 1997.

[Kat01] V. E. Katsnelson. Right and left joint system representation of a rational matrix function in general position (system representation theory for dummies). In D. Alpay and V. Vinnikov, editors, Operator Theory, System Theory and Related Topics (The Moshe
Livšic Anniversary Volume), Operator Theory: Advances and Applications. Birkhäuser, Basel, 2001.

[Lev61] A. H. M. Levelt. Hypergeometric functions. In Nederl. Acad. Wetensch. Proc., volume 64, 1961.

[Miw81] T. Miwa. Painlevé property of monodromy preserving deformation equations and the analyticity of $\tau$ functions. Publ. RIMS, Kyoto Univ., 17, 1981.

[Ple08] I. Plemelj. Riemannsche Funktionenscharen mit gegebener Monodromiegruppe. Monatschefte für Math. u. Phys., 19, 1908.

[Sch12] L. Schlesinger. Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten. Journ. für die Reine und Angew. Math., 141, 1912.

[Sib90] Y. Sibuya. Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation, volume 82 of Translations of Mathematical Monographs. Amer. Math. Soc., 1990.