A Model Analyzing Life Cycle of Periodical Cicadas

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Abstract

This paper focuses on a mathematical model interpreting the prime number life cycle of periodical cicadas, *Magicicada spp.* Changed the viewpoint to predators rather than the prey, this model fits reality very well by utilizing some principles and assumption. With the definition of the predator income, natural selection from predators seems to be the main reason for such a long life cycle. Consequent solution of this model is exactly the fact of real nature.

1 Introduction

Periodical cicadas (*Magicicada spp.*) distribute in North America. Their nymphae feed on juice of young plant root systems, living underground for 17 or 13 years. So they are also called 17 years cicadas or 13 years cicadas. Smith’s research revealed some species of spiders are nature enemies of periodical cicadas.

As to the long prime number life cycle of North America cicadas, there has been dispute among academicians. May [1] and Murray [2] believed it was only coincidence and without much biological meanings. Japanese scientist Yoshimura [3] argued such long period was to prevent inbreeding. Lloyd and Dybas [4] suggested some parasites caused this periodical life cycle.

Other scientists, however, hold the belief that prime number life cycle could avoid killing from predators. Gould [5] argued if life cycle of cicadas was 12 years, it would possibly encounter predators whose life cycles are 2, 3, 4, 6 or 12 years; but while the cicadas have prime number life cycle such as 7 or 11 years, only those predators who have exactly the same life cycle could devour them. That is to say prime number life cycle tends to minimize the loss.

This model is terse and clear. But it is lack of material interpretation of long period cycle and cannot explain differences between 11 years and 13 years life cycle: why no 11 year cicadas has been found in North America. Consequently, a more detailed model with biological meanings is required. Hoppensteadt and Keller [6] set up a differential equation but their results seemed not very prefect. The calculated life cycle may vary with different parameter values. Goles group [7] [8] also presented some papers but their research were based on cicadas whose model setting and discussion are comparatively difficult. So a predator-centered model is achieved and solves the fascinating long life cycle problem.

2 Establishing the Model

Webb [9] in his research assumed that the life cycle of the ancestor of periodical cicadas varied between 2 to 18 years, while that of the predator varied between 2 to 5 years. To get more precise conclusion, we assume the life cycle of predators is between 2 and 9 years.

According to the opinion of Hoppensteadt and Keller [6], if the life cycle of predator equals that of the prey (cicada), the predator obtains its maximum satisfactory: its chance of survival and reproduction. We define its income of the predator is "1".

Denote the life cycle of predator and prey as *N* and *n* respectively.

If *n* < *N*, during the entire life of the predator, only in one year it has the opportunity to catch the prey. In this year, we say the predator synchronize with the prey and we define the income of predator as $\frac{1}{N}$ for this condition.
If \( n > N \) and \( n \) is a multiple of \( N \), \((n = kN, k = 1, 2, \ldots)\), the \((k-1)\)-th generation offspring of the predator has the opportunity to catch the prey and the \((k-1)\)-th generation offspring takes \( \frac{1}{2^{n/N-1}} \) genetic material of the predator. In this case, we define the income of predator as

\[
\frac{1}{N^{2^{[n/N]-1}}}
\]

where the symbol \([ ]\) means floor.

From the assumptions above, the income of the predator is function of both the predator and the prey life cycles. Denoting the income as \( f \), we obtain

\[
f(n, N) = \begin{cases} 
1 & n = N \\
\frac{1}{N} & n < N \\
\frac{1}{2^{n/N-1}} & n > N \text{ and } n \neq kN, k = 1, 2, \ldots
\end{cases}
\]

(1)

3 Solution of this model

According to Eq. 1, we can calculate out the income of predator in different permutation of life cycle of the prey and the predator. The result is listed in Table 1.

| Life Cycle of Predators(unit:year) | Total |
|---|---|
| prey’s | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1.0000 | 0.5000 | 0.3333 | 0.2500 | 0.2000 | 0.1667 | 0.1429 | 0.1250 | 0.1111 | 2.8290 |
| 2 | 0.5000 | 1.0000 | 0.3333 | 0.2500 | 0.2000 | 0.1667 | 0.1429 | 0.1250 | 0.1111 | 2.8290 |
| 3 | 0.2500 | 0.5000 | 1.0000 | 0.2500 | 0.2000 | 0.1667 | 0.1429 | 0.1250 | 0.1111 | 2.7456 |
| 4 | 0.1250 | 0.5000 | 0.3333 | 1.0000 | 0.2000 | 0.1667 | 0.1429 | 0.1250 | 0.1111 | 2.7040 |
| 5 | 0.0625 | 0.2500 | 0.3333 | 0.2500 | 1.0000 | 0.1667 | 0.1429 | 0.1250 | 0.1111 | 2.4415 |
| 6 | 0.0313 | 0.2500 | 0.5000 | 0.2500 | 0.2000 | 1.0000 | 0.1429 | 0.1250 | 0.1111 | 2.6102 |
| 7 | 0.0156 | 0.1250 | 0.1667 | 0.2500 | 0.2000 | 0.1667 | 1.0000 | 0.1250 | 0.1111 | 2.1601 |
| 8 | 0.0078 | 0.1250 | 0.1667 | 0.5000 | 0.2000 | 0.1667 | 0.1429 | 1.0000 | 0.1111 | 2.4201 |
| 9 | 0.0039 | 0.0625 | 0.2500 | 0.1250 | 0.2000 | 0.1667 | 0.1429 | 0.1250 | 1.0000 | 2.0759 |
| 10 | 0.0020 | 0.0625 | 0.0833 | 0.1250 | 0.5000 | 0.1667 | 0.1429 | 0.1250 | 0.1111 | 1.3184 |
| 11 | 0.0010 | 0.0313 | 0.0833 | 0.1250 | 0.1000 | 0.1667 | 0.1429 | 0.1250 | 0.1111 | 0.8862 |
| 12 | 0.0005 | 0.0313 | 0.1250 | 0.2500 | 0.1000 | 0.5000 | 0.1429 | 0.1250 | 0.1111 | 1.3857 |
| 13 | 0.0002 | 0.0156 | 0.0417 | 0.0625 | 0.1000 | 0.0833 | 0.1429 | 0.1250 | 0.1111 | 0.6823 |
| 14 | 0.0001 | 0.0156 | 0.0417 | 0.0625 | 0.1000 | 0.0833 | 0.5000 | 0.1250 | 0.1111 | 1.0394 |
| 15 | 0.0000 | 0.0078 | 0.0625 | 0.0625 | 0.2500 | 0.0833 | 0.0714 | 0.1250 | 0.1111 | 0.7737 |
| 16 | 0.0000 | 0.0078 | 0.0208 | 0.1250 | 0.0500 | 0.0833 | 0.0714 | 0.0625 | 0.1111 | 0.9695 |
| 17 | 0.0000 | 0.0039 | 0.0208 | 0.0313 | 0.0500 | 0.0833 | 0.0714 | 0.1250 | 0.1111 | 0.4344 |
| 18 | 0.0000 | 0.0039 | 0.0313 | 0.0313 | 0.0500 | 0.2500 | 0.0714 | 0.0625 | 0.5000 | 1.0003 |
| Total | 2.0000 | 3.4922 | 3.9271 | 4.0000 | 3.9000 | 3.8333 | 3.5000 | 3.3750 | 3.2778 |

Assuming the life cycle of ancestor of the prey and the predator is random distributed in 1 to 18 and 1 to 9 respectively, we can calculate out income of the predator in different life cycle choice of the prey and the predator by adding value of \( f(n, N) \) in rows and in columns respectively. The result is listed in the column and row labeled "Total".

Data in "Total" column indicates the income of the predator the different life cycle choice of the prey whereas data in "Total" row reflects the income of the predator vs. its life cycle. They are represented in dash line with round marker and solid line with cross marker in Figure 1 respectively.

According to Figure 1 the prey choose 13 and 17 years as its life cycle can minimize the income of predator while the predator choose 4 years as its life cycle can maximize its income.
4 Accordance with Reality

From Table 1 and Figure 1, the 13 and 17 years life cycle could reduce the income of predator significantly, which accords with the fact. Moreover, the troublesome "11" year period problem is solved that 11 years life cycle is not so efficient in reducing income of predators.

Solid line in Figure 1 indicates 4 years should be the best choice for predators. Vail [10] pointed out one nature enemy of periodical cicadas Mud dauber has life cycle of 4 years which accords with this model well.

5 Conclusion

Unlike traditional researches, this paper is a predator-focused model to interpret the phenomena of prime number life cycle of periodical cicadas in north America. The model utilizes some basic principles and assumptions. Defined the income of predator, this simple fits the reality very well. Both the fact from Magicicada spp. and its predator supports our conclusion.

References

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**Appendix**

The MATLAB code for calculation and plotting.

```matlab
% Matlab code for calculating.Cicada V.2 Last Modified at Oct.9,2005
% Programmed by Sheng Bao <shengbao@ieee.org> \protect\href{http://grandlab.cer.n...% --This code is distributed under the terms of GNU GPL (General Public License)
% --You can modify it and redistribute it freely but you must keep is open source.

for n=1:18 % prey
    for N=1:9 % predator
        if n==N
            f(n,N)=1;
        elseif n<N
            f(n,N)=1/N;
        elseif (n>N)&&(rem(n,N)==0)
            f(n,N)=2^(1-(n/N));
        else
            f(n,N)=1/(N*2^(fix(n/(N)-1)));
        end
    end
end

for i=1:18
    m1(i)=sum(f(i,:));
end

for i=1:9
    m2(i)=sum(f(:,i));
end

box on plot(m1,'--ko','MarkerEdgeColor','k','MarkerFaceColor',[1 1 1],'MarkerSize',6)
hold on
axis([1 18 0 4.1])
set(gca,'XTick',1:1:18)
set(gca,'XTickLabel',{'1','2','3','4','5','6','7','8',...'
    '9','10','11','12','13','14','15','16','17','18'})
plot(m2,'-kx','LineWidth',1,'MarkerEdgeColor','k',...'
    'MarkerFaceColor',[1 1 1],'MarkerSize',8)
legend('income vs. life cycle of the prey','income vs. life cycle of the predator',3)
xlabel('Year of life cycle')
ylabel('income of predator')
title('')
```