Experimental test of Hardy’s paradox on a five-qubit quantum computer

Soumya Das and Goutam Paul
Cryptology and Security Research Unit, R. C. Bose Centre for Cryptology and Security, Indian Statistical Institute, Kolkata 700 108, India.
Email: soumya06.das@gmail.com, goutam.paul@isical.ac.in

We test Hardy’s paradox of non-locality experimentally on the IBM five-qubit quantum computer for the first time. The quantum circuit is constructed on superconducting qubits corresponding to the original Hardy’s test of non-locality. Results confirmed the theory that any non-maximally entangled state of two qubits violates Hardy’s Equations, whereas any maximally entangled state and product state of two qubits do not exhibit Hardy’s non-locality. We also point out the difficulties associated with the practical implementation of any Hardy’s paradox based quantum protocol and propose three performance measures for any two qubits of any quantum computer.

I. INTRODUCTION

In 1935, the Einstein-Podolsky-Rosen (EPR) Paradox raised the question about the completeness of the quantum theory [1] and claimed that nature should be described by any local-realistic theory. In 1964, the non-local characteristics of quantum theory was demonstrated by Bell’s Theorem [2]. Since then, a significant number of experiments have been conducted favoring the correctness of Bell’s Theorem [3–17]. For this reason, Bell type inequalities are used to differentiate between quantum physics and classical physics. A comprehensive review of Bell’s Theorem including theoretical and experimental aspects can be found in [18].

In [19], the authors have demonstrated non-locality without using inequalities for three and four qubits. In 1992, through a thought experiment, Hardy constructed the test of local-realism without using inequalities for two qubits. This is called Hardy’s test [20, 21]. It is known as the “Best version of Bell’s Theorem” as indicated by Mermin [22]. This test provides a direct contradiction between the predictions of a quantum theory and Local Hidden Variable (LHV) theory.

Several experiments have been performed to demonstrate Hardy’s paradox using polarization, energy-time and orbital angular momentum of photons, entangled qubits, classical light and two-level quantum states [23–32]. The applications of Hardy’s paradox includes Device Independent Randomness [33], Device Independent Quantum Key Distribution [34] and Quantum Byzantine Agreement [35].

In the case of superconducting qubits, Clauser-Horne-Shimony-Holt (CHSH) inequality and Greenberger-Horne-Zeilinger (GHZ) test are already performed in IBM five-qubit quantum computer [36]. IBM has given access to its quantum computer that uses superconducting qubits in the cloud and this opens a new door for testing of quantum phenomena for researchers.

In [37], the author has implemented some protocols in quantum error correction, quantum arithmetic, quantum graph theory and fault-tolerant quantum computation in IBM five-qubit quantum computer. In [38], the authors have tested the theoretical predictions of entropic uncertainty relation with quantum side information (EURQSI) in IBM five-qubit quantum computer. Compressed quantum computation [39], Leggett-Garg test [40], Quantum cheque [41] are also recently performed in IBM five-qubit quantum computer.

Though Mermin inequalities have been tested experimentally using photons and ion traps [42, 43], subsequently the authors of [44] have tested three, four and five qubits Mermin polynomials in IBM five-qubit quantum computer. We have already discussed the works related to the experimental verification of Hardy’s non-locality. However, none of them used any real quantum computer using superconducting qubits. This motivates us to test the Hardy’s paradox for two qubits in IBM five-qubit quantum computer.

II. HARDY’S TEST OF NON-LOCALITY

Hardy’s test of non-locality for two qubits involves two distant parties (may be space-like separated), Alice and Bob. A physical system consisting of two subsystems is shared between them. Alice and Bob can freely measure and observe the measurement results of their own subsystems. Let us consider, Alice can perform the test of measurement on her own subsystem by choosing freely one of the two \{+1, −1\}-valued random variables \(A_1\) and \(A_2\). Similarly, Bob can also choose freely one of the two \{+1, −1\}-valued random variables \(B_1\) and \(B_2\) for measuring the subsystem in his possession.

Hardy’s test of non-locality starts with the following set of joint probability equations.

\[
P(+1, +1|A_1, B_1) = 0, \quad P(+1, −1|A_2, B_1) = 0, \quad P(−1, +1|A_1, B_2) = 0, \quad P(+1, +1|A_2, B_2) = \begin{cases} 0 & \text{for LHV theory,} \\ q & \text{for non-locality,} \end{cases}
\]

where \(q > 0\). Here \(P(x, y|A, B)\) denotes the joint probability of obtaining outcomes \(x\) and \(y\) given that \(A\) and \(B\) were the experimental choices made. If an experiment
is designed in such a way that Equations 1, 2 and 3 are satisfied, then for any LHV theory, the right hand side of Equation 4 becomes zero. But if this value is found to be greater than zero for some value of \( q \), then non-locality is established. The set of Equations (1)-(4) are called Hardy’s Equations.

The maximum value of \( q \) is found to be \( q_{\text{max}} = \frac{\sqrt{5} - 1}{2} \approx 0.61803 \) for two qubits \([21, 45]\). For two qubits system, every Maximally Entangled State (MES) or every Product State (PS) does not obey Hardy’s non-locality, but every Non-Maximally Entangled State (NMES) exhibits Hardy’s non-locality \([46]\). This is the specialty of Hardy’s Equations that for ideal case, only a single event can discard all LHV theories. The motivation of this work is to validate this statement for any practical experiment. As every MES of three or higher qubits exhibits Hardy’s non-locality, we have restricted our discussion for two qubits only.

For any practical experimental set up, it is quite obvious that the joint probabilities described in Equations (1)-(4) may not be zero due to errors caused by any external environment or internal device or both. So, Equations (1)-(4) can be written by some error parameter \( \epsilon \) as follows \([47]\):

\[
P(+1, +1|A_1, B_1) = \epsilon_1, \quad P(+1, -1|A_2, B_1) = \epsilon_2, \quad P(-1, +1|A_1, B_2) = \epsilon_3, \quad P(+1, +1|A_2, B_2) = \begin{cases} \epsilon_4 + q = \epsilon_5, & \text{for LHV theory,} \\ \epsilon_4, & \text{for non-locality,} \end{cases}
\]

where \( q \) is as described in Equation 1. For every MES and every PS of two qubits, the right hand side of Equation 4 is \( \epsilon_4 \). But for every NMES, it is \( \epsilon_4 + q = \epsilon_5 \). Therefore, if in an experiment, there is a clear distinction between the range of values of \( \epsilon_4 \) for every MES and PS and the values of \( \epsilon_5 \) for every NMES, then all LHV theories are discarded and non-locality is established.

### III. CIRCUITS FOR HARDY’S EQUATIONS

We have performed a series of experiments to validate Hardy’s non-locality for two qubits in IBM five-qubit quantum computer (ibmqx4 chip) \([36]\). It uses a particular physical type of qubit called a superconducting transmon qubit made from superconducting materials such as niobium and aluminum, patterned on a silicon substrate. During all the experiments, the fridge temperature was maintained at 0.021K. Any experiment in IBM quantum computer can be performed for 1 shot, 1024 shots, 4096 shots or 8192 shots in every run.

In current IBM ibmqx4 chip topology, for using multi-qubit gates like CNOT, there is a restriction, i.e., not all pair of qubits can be used for circuit implementation. List of possible combinations is given in details in IBM website \([36]\) and also discussed in Section IV. It should be noted that all the qubits are subjected to different types of errors given in IBM website \([36]\). Initially, we have implemented our circuit by choosing any possible pair of qubits and then validated the results for the rest of the possible combinations of qubits.

#### Table I: MES and PS based on different values of \( \theta \) and \( \phi \) in between 0 to 90 degrees.

| \( \theta \) | \( \phi \) | state \( |\psi\rangle \) |
|---|---|---|
| 0 | any value | PS \( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \) |
| any value | 0 | PS \( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (\cos \theta |0\rangle + \sin \theta |1\rangle) \) |
| 90 | any value | PS \( \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle) \otimes |1\rangle \) |
| 45 | 90 | MES \( \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \) |

In \([48]\), a mesoscopic circuit consists of two coupled electronic Mach-Zehnder (MZ) interferometers has been proposed for Hardy’s test which is similar to the original Hardy’s thought experiment \([20]\). This circuit is being implemented in IBM quantum computer. As described in \([48]\), three important parameters of this experiment namely beam splitters \( U_B(\theta) = (\cos \theta - i \sin \theta) \), phase shifter \( U_P(\phi) = (1 0 \ e^{i\phi}) \) and the coupling \( U_C(\phi) = \begin{pmatrix} 1 & 1 \\ 1 & e^{i\phi} \end{pmatrix} \) which can be decomposed as

\[
U_B(\theta) = U_3(2\theta, 0, 0), \\
U_P(\phi) = U_1(\lambda), \\
U_C(\phi) = M_3 \cdot \text{CNOT} \cdot M_2 \cdot \text{CNOT} \cdot M_1,
\]

where

\[
U_1(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix}, \\
U_3(\theta, \lambda, \phi) = \begin{pmatrix} \cos \frac{\theta}{2} & -e^{-i\lambda} \sin \frac{\theta}{2} \\ e^{-i\lambda} \sin \frac{\theta}{2} & e^{(i\phi + i\lambda)} \cos \frac{\theta}{2} \end{pmatrix}, \\
M_1 = \text{Id} \otimes U_1(-\lambda), \\
M_2 = U_1(\lambda) \otimes U_1(-\lambda), \\
M_3 = \text{Id} \otimes U_1(2\lambda),
\]

CNOT is controlled-NOT gate and \( \text{Id} \) is the identity gate. Here \( U_1(\lambda), U_3(\theta, \lambda, \phi), \text{CNOT} \) and \( \text{Id} \) are available as standard gates provided by IBM quantum computer \([36]\). The coupling \( U_C(\phi) \) may also be decomposed by standard IBM gates to others ways such that the total number of gates are reduced and that is left as a future work. The entangled state \( |\psi\rangle = V_0(V_1 \otimes V_2)(00) \) is ex-
Equation (4) is found when \( \cot \chi \) is given in Table I. The variation of values of \( \theta \) that is for pressed as \( P_{\theta} = \frac{\cos \theta}{\sqrt{2}}(\langle 00 \rangle + \langle 10 \rangle) + \frac{\sin \theta}{\sqrt{2}}(\langle 01 \rangle + e^{i2\phi}\langle 11 \rangle) \), (10)

where \( V_0 = U_C(\phi) \), \( V_1 = U_B(\frac{\pi}{4}) \) and \( V_2 = U_B(\theta) \). The state \( |\psi\rangle \) is expressed by IBM gates as discussed in Equation (9) which is a function of two parameters \( \theta \) and \( \phi \). The measurements for Alice and Bob are described as follows.

\[
\begin{align*}
  a_1 &= U_B(\frac{\pi}{4}) U_3(\frac{\pi}{2}, 0, 0), \\
  b_1 &= U_B(0) U_3(0, 0, 0), \\
  a_2 &= U_P(2\phi) U_B(\frac{\pi}{4}, 0) U_P(-2\phi) \\
  &= U_1(2\lambda U_3(\frac{\pi}{2}, 0, 0) U_1(-2\lambda), \\
  b_2 &= U_P(\phi) U_B(\chi) U_P(-\phi) \\
  &= U_1(\lambda U_3(2\chi, 0, 0) U_1(-\lambda),
\end{align*}
\]

where \( \cot \chi = \tan \theta \cos \phi \). The maximum value of \( q \) in Equation (4) is found when

\[
\cos(2\theta) = \cos(2\phi) = 2 - \sqrt{5},
\]

that is for \( \theta = \phi = 51.827 \) degrees approximately. The values of \( \theta \) and \( \phi \) for which \( |\psi\rangle \) is found to be MES and PS is given in Table I. The variation of \( \theta \) and \( \phi \) are carried out in between 0 to 90 degrees, but in general similar analysis can be done for any values of \( \theta \) and \( \phi \).

The experimental circuits for Hardy’s paradox in IBM five-qubit quantum computer for Equations (5),(6),(7) and (8) are given in Figure 1, 2, 3 and 4 respectively. The measurements are done in \( \sigma_z \) basis. The theoretical value of Equation (4) in this experimental set up is

\[
P_{\theta} = \frac{1}{2} \cos \theta \cos \chi \left( 1 - e^{-2i\phi} \right)^2.
\]

For \( \phi = 90 \) and \( \theta \neq \{0, 45, 90\} \) degrees, from Equation (10), we get \( |\psi\rangle \) as NMES. This means, if we perform Hardy’s test, a non-zero value of \( q \) in Equation (4) has to be found. But when \( \phi = 90 \) degree, we get \( \chi = 90 \) degree which means the right hand side of Equation (13) is zero. So, in this experimental set up, Hardy’s test fails for all NMES for the values of \( \phi = 90 \) and \( \theta \neq \{0, 45, 90\} \) degrees.

IV. EXPERIMENTAL RESULTS AND DISCUSSION

The values of \( q \) for all MES and all PS are 0, and for NMES it can be found from Equation (13) by varying \( \theta \) and \( \phi \) from 0 to 360 degrees. The results are depicted in Figure 5. In this Figure, we can see that \( q_{\text{max}} \) is achieved for \( \theta = \phi = 51.827 \) degrees and also for other values of \( \theta \) and \( \phi \) such that Equation (12) is satisfied.

Each of the experiments given in Figure 1, 2, 3, and 4 is run ten times with 8192 shots per run, and so total 81920 shots for different values of \( \theta \) and \( \phi \), whereas the authors of [37, 41] have run their experiments only for 8192 shots. The statistical error can be estimated by the expression \( \sqrt{P(1-P)/8192n} \), where \( n \) is the number of runs and \( P \) is the probability of a given experimental result. As we have limited access to the IBM quantum computer, we have taken \( n = 10 \) and the experiments were run for MES, PS and NMES for some values of \( \theta \) and \( \phi \) in between 0 to 90 degrees as shown in Table I. Initially, we have chosen \( Q_3 \) qubit for Alice (control qubit for CNOT) and \( Q_4 \) for Bob (target qubit for CNOT)
our experiments as discussed below.

A. Experiments for Hardy’s non-locality by choosing qubit $Q_1$ for Alice and qubit $Q_4$ for Bob

1. Experimental validation of the circuit for Hardy’s test

We have performed the experiments for Equations (5), (6) and (7) (Figure 1, 2 and 3 respectively) for the values of $\theta$ and $\phi$ in degrees given in Table II. The average values of $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ in each of the experiments are found to be less than 0.1, implying that Equations (5), (6) and (7) are satisfied. These results indicate that this experimental set up is now valid for Hardy’s test. As discussed in Section II, for Hardy’s test, only experiment for Equation (5) (Figure 3) has to be performed to establish non-locality; that’s why the details of the results for $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ are not presented. The experimental results of Equation (8) are given in Table II.

2. Test of non-locality when $q = q_{\max}$

From Table II, we can see that for the MES when $\theta = 45$ and $\phi = 90$ degrees, the value of $\epsilon_5 = 0.0807 \pm 0.0037$ which is equal to the estimated $\epsilon_4$ as $q = 0$. In the same way, for the Product States as shown in the Table II, we get the average values of the estimated $\epsilon_4$ to be less than 0.03, when again $q = 0$. As stated earlier, theoretically when $\theta = \phi = 51.827$ degrees, we get $q = q_{\max}$ for NMES. So, to test non-locality for $q = q_{\max}$, we have to check whether the average value of experimental $\epsilon_5$ is greater than the maximum value of the estimated $\epsilon_4$ for all MES and PS as shown in Table II.

From the experiment, the average value of $\epsilon_5$ is found to be 0.1281 $\pm$ 0.0039 which is greater than the maximum value of the estimated $\epsilon_4$ for all MES and PS as given in Table II. For NMES, where $\epsilon_5$ is found to be 0.1281 $\pm$ 0.0039 which is greater than the maximum value of the estimated $\epsilon_4$ for all MES and PS, the non-locality is established by Hardy’s test as we get a clear difference between the values of the estimated $\epsilon_4$ for all MES and PS and $\epsilon_5$ for NMES, where $q_{\max}$ is achieved.

3. Test of non-locality when $q < q_{\max}$

Now we want to check whether the same conclusion for non-locality can be drawn when $q < q_{\max}$ as we get above for NMES. When $\theta = \phi = 45$ degrees, we get $q = 0.0833$ and experiment gives $\epsilon_5 = 0.1041 \pm 0.0044$ which is greater than the maximum of the estimated $\epsilon_4$ for all MES and PS as shown in Table II. Similar kind of result is obtained when $\theta = \phi = 55$ degrees. Clearly these results support non-locality.

But when $q = 0.0433$, with $\theta = 30$, $\phi = 60$, we get $\epsilon_5 = 0.0832 \pm 0.0052$ which is in the same range of the maximum value of the estimated $\epsilon_4$ for all MES and PS. Also, for the same value of $q$, but with $\theta = 60$, $\phi = 30$, we get $\epsilon_5 = 0.0553 \pm 0.0028$. From these results, non-locality cannot be guaranteed because we are not sure that these
values of $\epsilon_5$ are due to non-locality or due to errors in the experiment.

When the value of $q$ is decreased further, i.e., $q = 0.00088$, the value of $\epsilon_5$ comes to be less that of the estimated $\epsilon_4$ for all MES and PS as shown in the Table I which supports the LHV theory.

So, we can observe that when $q$ is larger than the maximum value of the estimated $\epsilon_4$ for all MES and PS, the distinction between $\epsilon_5$ for NMES and the maximum value of the estimated $\epsilon_4$ for all MES and PS can be done which supports non-locality. But when the value of $q$ is less than the maximum value of the estimated $\epsilon_4$ for all MES and PS, it is hard to distinguish between $\epsilon_5$ for NMES and the maximum value of the estimated $\epsilon_4$ for all MES and PS.

4. Consistency check: whether for at $q = q_{\text{max}}$, we get $\epsilon_5 = \epsilon_{\text{max}}$

To check the consistency of the results of Table I we have conducted another set of experiments. If we take $\theta = \phi$ and vary it from 0 to 90 degrees, a bell shaped curve can be found with peak at $\theta = \phi = 51.827$ degrees for $q$ as shown in Figure 5. For limited control of the IBM quantum computer, we have plotted the values of experimental $\epsilon_5$, theoretical $q$ and estimated $\epsilon_4$ for $\theta = \phi$ varying from 0 to 90 degrees with an increment of 5 degree, i.e., $\theta = \phi = 5n$, where $n = \{0, 1, \ldots, 18\}$ (when $n = 18$ we get $\theta = \phi = 90$, then the value of $\chi$ is undefined. So we take $\theta = \phi = 89.99$ for this case).

From Figure 6, it can be seen that the maximum value of $\epsilon_5$ is shifted towards right when $\theta = \phi = 65$ degrees, where it should be at $\theta = \phi = 51.827$ degrees. To get more accurate result of $\epsilon_{\text{max}}$, we have repeated this experiment from 55 degree to 75 degree with an increment of one degree. Result shows that $\epsilon_{\text{max}}$ occurs when $\theta = \phi = 62$ degrees (not shown in Figure 6). This can be repeated again if more precise value of where $\epsilon_{\text{max}}$ occurs is needed. The curve for estimated $\epsilon_4$ is also not stable, where it should be a straight line. From these cases it can be concluded that although by IBM quantum computer non-locality can be established, but the errors induced by the computer need to be more stable.

5. Summary

In summery, for the parameters ($\theta$ and $\phi$) that lead to $q = 0$, our experimental outcome gives us estimated $\epsilon_4$ for all MES and PS. For the parameters ($\theta$ and $\phi$) that lead to $q > 0$, our experimental outcome gives us $\epsilon_5$ for NMES. When the value of $\epsilon_5$ for NMES is greater than maximum value of the estimated $\epsilon_4$ for all MES and PS, the non-locality is established. But when the value of $\epsilon_5$ is in the same range or less the maximum value of estimated $\epsilon_4$ for all MES and PS, non-locality cannot be guaranteed. Our experimental results exactly show this nature.

B. Check for other possible combinations of multi-qubit gate

1. Check for non-locality

There are currently six combinations of multi-qubit gate implementations available in IBM quantum computer (ibmqx4). [36]. For the multi-qubit CNOT gate that we have used, the possible control qubit and target qubit pairs other than $(Q_3, Q_4)$, are summarized in Table III. We have calculated the values of $\epsilon_5$ for $\theta = \phi = \{45, 51.827, 55\}$ degrees with all combinations of pair of qubits. It can be seen that the value of $\epsilon_5$
is minimum for the pair \((Q_2, Q_0)\) and maximum for the pair \((Q_2, Q_4)\) when \(\theta = \phi = 51.827\) degrees. All the experiments described in earlier have been done using all combination of these pair of qubits described in Table III and similar kind of results for non-locality are obtained as discussed in Table II.

2. Consistency check

We want to see whether \(\epsilon_{5\text{max}}\) occurs when \(q = q_{\text{max}}\) for other possible two qubit pairs or whether there is any shift of \(\epsilon_{5\text{max}}\) as shown in Figure 6 for the pair \((Q_4, Q_1)\). From Table III it can be seen that for the pair \((Q_2, Q_1)\), the average value of \(\epsilon_5\) for \(\theta = \phi = 51.827\) degrees is less than the average value for \(\theta = \phi = 45\) degrees and greater than the average value for \(\theta = \phi = 55\) degrees. To verify this result, we have done a similar experiment for the pair \((Q_2, Q_1)\) as we did for the pair \((Q_3, Q_4)\) which is shown Figure 6. Results are shown in Figure 7 which indicates that there is a shift of the value of \(\epsilon_{5\text{max}}\) to the left for the pair \((Q_2, Q_1)\) when \(\theta = \phi = 40\) degrees.

For the rest of the pairs, we have found that \(\epsilon_{5\text{max}}\) is shifted to the right as the average value of \(\epsilon_5\) for \(\theta = \phi = 51.827\) degrees is greater than the value for \(\theta = \phi = 45\) degrees and less than the average value for \(\theta = \phi = 55\) degrees as shown in Table III. So, we can conclude that the maximum value of \(\epsilon_5\) didn’t occur at \(q = q_{\text{max}}\), rather it is shifted towards right or left.

3. Summary

For some of the protocols like Quantum Byzantine Agreement (QBA) [34], it is necessary to check that Hardy’s state is actually prepared or not. For that, a specific value of experimental parameter say \(\rho\), we have to verify that the value of \(q = q_{\text{max}}\) is achieved or not. While implementing it practically, let the experimental value of \(q_{\text{max}}\) with addition of errors be \(Q_{\text{max}}\) (like \(\epsilon_5\) in Equation 6). Now if the errors in the experiment are not stable, so it is expected that this experimental value of \(Q_{\text{max}}\) will lie in an interval \(\{\rho - \delta, \rho + \delta\}\). So, to perform any quantum protocols based on Hardy’s test in any quantum computer, for checking of Hardy’s state, instead of choosing any specific value of the experimental parameters \(\rho\), where \(Q_{\text{max}}\) should occur, an interval of parameters values \(\{\rho - \delta, \rho + \delta\}\) can be chosen, where it is guaranteed to get \(Q_{\text{max}}\). In our experimental set up, \(\delta\) is found to be 12 degrees when \(\rho = \theta = \phi\).

C. Check whether reducing the number of gate reduces the error in the circuit

To verify that whether reducing the number of gates in the circuit reduces the error or not, we have performed another series of experiments for \((Q_3, Q_4)\) pair of qubits.

For \(\theta = \phi = 0\), we get a PS, i.e., \(|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle\). This state can be created easily by using a Hadamard gate \(H\) in the Alice’s qubit and \(U_1\) and \(U_3\) used for measurement in Figure 4, becomes identity (Id) gate. So, the number of gate is reduced significantly. For the modified circuit as shown in Figure 8(a), the value of \(\epsilon_5\) is 0.0084 \pm 0.0014 which is less than the value what we have found previously as shown in Table II i.e., 0.0193 \pm 0.0014.

Also, for \(\theta = 90, \phi = 0\), we get a PS \(|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle\). For this state, a Hadamard gate on Alice’s qubit and a bit flip gate \(X\) on Bob’s qubit is required as shown in Figure 8(b). Experimental results show that \(\epsilon_5 = 0.0079 \pm 0.001\) which is again less that what we have found earlier from Table II i.e., 0.0209 \pm 0.0015.

These experiments were repeated for the rest of the pair of qubits and similar kind of results are found. So, we can conclude that number of gate is a major parameter of error in any quantum circuit of IBM quantum computer.

D. Proposed performance measures of quantum computer for two qubits

From the above experimental results, we can propose three performance measures of any quantum computer. First, it can be seen that, when \(q = q_{\text{max}}\), value of \(\epsilon_4\) is greater than the maximum value of the estimated \(\epsilon_4\) for all MES and PS which supports non-locality. When \(q < q_{\text{max}}\), but it is greater than the maximum value of
the estimated \( \epsilon_4 \), non-locality is still valid. But when the value of \( q \) is less than the maximum value of the estimated \( \epsilon_4 \), non-locality cannot be guaranteed. So, minimum value of \( q \) up which non-locality is established can be the performance measure of a quantum computer. Lesser the value of \( q \) which supports non-locality, better the performance of the quantum computer.

Second, theoretically although we get \( q_{\text{max}} \) at \( \theta = \phi = 51.827 \) degree, but during experiment, due to unstable errors, the value of \( \epsilon_{5\text{max}} \) may be shifted to any nearby value. In our case it shifted to the left for the pair \((Q_2, Q_1)\) and to the right for the rest of the pairs of qubits. For our experiment when \( \theta = \phi \), the amount of shift is 12 degree. So, the amount of shift can be considered as a performance measure of a quantum computer. The smaller the shift, the better is the performance.

Third, we can see from Figure 6 and 7 that the graph for estimated \( \epsilon_4 \) is not stable, where it should be a straight line. So, the amount of fluctuations can be the indicator of the stability of the quantum computer. The less the graph of the estimated \( \epsilon_4 \) fluctuates, the more stable a quantum computer is.

In [14], the authors have also demonstrated non-locality in the case of Mermin polynomials for three, four and five qubits. They have concluded that the fidelity of quantum computer decreases when the number of qubits are increased from three. However, they do not mention anything for two qubits. From the experimental test of Hardy’s paradox, we have proposed three performance measures for two qubits of any quantum computer as discussed above. So, using these three procedures, the fidelity of any two qubits of any quantum computer can be measured.

During the experiments, IBM quantum computer was undergoing maintenance for nearly one month when some of the experiments were done. To present all the results in same time-line, we have repeated all the previous experiments. When we compare the data of one month earlier experiments, we have found a significant change of values, similar to what was reported in [14]. For the pair \((Q_3, Q_4)\) when \( \theta = \phi = 51.827 \) degrees, one month earlier, we got the value of \( \epsilon_5 = 0.16254 \pm 0.0078 \) which is greater than the value we get one month later as indicated in Table IV. Similar trend is noticed for the rest of the data.

V. CONCLUSION AND FUTURE WORK

We have performed an experimental verification of Hardy’s paradox of non-locality for the first time in IBM five-qubit quantum computer as proposed in [48] for two qubits.

Our first motivation was to check non-locality using Hardy’s test for two qubits in IBM five-qubit quantum computer choosing any two qubits from the five qubits. The experimental results favour non-locality when \( q = q_{\text{max}} \). Next, we have performed Hardy’s test when \( q < q_{\text{max}} \). Experimental results show that when the value of \( \epsilon_5 \) is greater than the maximum value of the estimated \( \epsilon_4 \), non-locality is established. But when the value of \( \epsilon_5 \) is in the range or less than the maximum value of the estimated \( \epsilon_4 \), non-locality can’t be guaranteed because of the errors in the experiments.

As there are errors in the system, it was necessary to check whether the experimental maximum Hardy’s probability, i.e., \( \epsilon_{5\text{max}} \) really occurs when \( q = q_{\text{max}} \) or not. Results show that for this test, \( \epsilon_{5\text{max}} \) didn’t occur at \( q = q_{\text{max}} \), rather we get a right shift of \( \epsilon_{5\text{max}} \) for the \((Q_3, Q_4)\) pair of qubits. Next, we wanted to verify whether for all possible combinations of two qubits, we get a right shift of \( \epsilon_{5\text{max}} \) or not. Results show that for the \((Q_2, Q_0)\) pair of qubits, the \( \epsilon_{5\text{max}} \) value is shifted to the left and for the rest it is shifted to the right. So, it is expected that the \( \epsilon_{5\text{max}} \) value will occur within an interval of \( \{ \rho - \delta, \rho + \delta \} \), where \( \rho \) is the value of experimental parameters achieving \( q_{\text{max}} \) and value of \( \delta \) depends on experimental errors. For our experiments, \( \delta \) is found to be 12 degrees when \( \theta = \phi \).

We have also performed experiments to show how decreasing the number of gates in the circuits decreases the errors in the circuit for all possible pairs of qubits.

Based on these results, we have proposed three performance measures of any quantum computer for two qubits. First, the minimum value of \( q \) up to which non-locality is established. Second, the amount of shift needed to get the experimental maximum value of Hardy’s probability, i.e., \( \epsilon_{5\text{max}} \). Third, the amount of fluctuations present in the values of the estimated \( \epsilon_4 \).

From the theoretical analysis of the Hardy’s experimental set up, we have found that this test fails for all NMES, where the value of \( \phi = 90 \) and \( \theta \neq \{0, 45, 90\} \) degrees. As future work, we plan to devise a new test for Hardy’s paradox for two qubits, so that it does not fail for any NMES.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[2] J. S. Bell, Physics 1, 195 (1964).
[3] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).
[4] W. Tittel, J. Brendel, B. Gisin, T. Herzog, H. Zbinden, and N. Gisin, Phys. Rev. A 57, 3229 (1998).
[5] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and
