On the cohomology of a Galois entwining

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1. Introduction

1.1. An extension of algebras $A/B$ is said to be Hopf-Galois over a Hopf algebra $H$ if $A$ is an $H$-comodule algebra, $B = A^{co H}$ is the subalgebra of coinvariants, and a certain Galois condition is satisfied, cf. [5]. Such an extension can be viewed as a non-commutative principal bundle with structure group $H$. It turns out, though, that to develop a satisfactory theory of principal bundles in the non-commutative setting the notion of Hopf-Galois extensions is too restrictive; it does not apply, for example, to all the quantum spheres of Podleś [6].

One considers then a more general situation in which the rôle of the structure group is played by a coalgebra. T. Brzeziński and P. Hajac [2] have proposed a corresponding notion of coalgebra Galois extensions, tightly related to that of entwining structures introduced in [3]. In this context, one can fit the Podleś spheres in coalgebra Galois extensions $SU_q(2)/S^2_{q,s}$ for appropriate coactions of coalgebras on $SU_q(2)$.

1.2. In a later paper [1], Brzeziński introduced two cohomology theories for an entwining structure, and, in particular, for coalgebra $C$-Galois extensions $A/B$: the entwined cohomology $H^*_{\psi}(A, -)$ of $A$ with values in $A$-bimodules, and a $C$-equivariant version. He computed the entwined cohomology of $A$ when the algebra $B$ of coinvariants in $A$ is the ground field $k$, and noted it is essentially trivial.

1.3. The purpose of the present note is to record the extension of Brzeziński’s computation of cohomology to the general case of a flat coalgebra Galois extension $A/B$. We show below that in that situation $H^v_{\psi}(A, -)$ coincides with the Hochschild cohomology $HH^*(B, -)$ of the subalgebra $B$ of coinvariants.

We plan to study the equivariant cohomology of the entwined structure corresponding to a coalgebra Galois extension in a future paper.

2. Coalgebra Galois extensions and the theorem

2.4. Fix a field $k$. All spaces and (co)algebras considered below are $k$-vector spaces and $k$-(co)algebras, and all unadorned tensor products are taken over $k$. Most of our statements can be extended to the slightly more general situation in

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which $k$ is simply a ring, provided one adds appropriate projectivity or flatness hypotheses.

2.5. An entwining structure is a triple $(A, C, \psi)$ consisting of an algebra $A$, a coalgebra $C$ and a map $\psi : C \otimes A \to A \otimes C$—which we write à la Sweedler, with implicit sums over greek indices, as in $\psi(c \otimes a) = a_\alpha \otimes c^\alpha$—satisfying the following compatibility conditions:

\[(aa')_\alpha \otimes c^\alpha = a_\alpha a'_\beta \otimes c^{\alpha \beta}, \quad 1_\alpha \otimes c^\alpha = 1 \otimes c,\]
\[a_\alpha \otimes c_1^\alpha \otimes c_2^{\alpha \beta} = a_\beta a_\alpha \otimes c_1^\alpha \otimes c_2^{\beta}, \quad a_\alpha \varepsilon(c^\alpha) = a \varepsilon(c).\]

2.6. Entwining structures arise naturally in the following situation. Let $C$ be a coalgebra. Let $A$ be an algebra which is a right $C$-comodule, and let

\[B = \{b \in A : (ba)_0 \otimes (ba)_1 = ba_0 \otimes a_1 \text{ for all } a \in A\}.\]

This is a subalgebra. There is a linear map $\beta : A \otimes_B A \to A \otimes C$ such that $\beta(a \otimes a') = a_0 a'_1 \otimes 1$, which is evidently left $A$-linear and right $C$-colinear; when $\beta$ is bijective, we say that $A/B$ is a $C$-Galois extension. If this is the case, we let $\gamma : C \to A \otimes_B A$ be the unique map such that $\beta \circ \gamma = \eta \otimes 1$, which we shall write $\gamma(c) = l(c) \otimes r(c)$ with an implicit summation over an implicit index. Then there is a canonical entwining structure $(A, C, \psi)$ associated to the extension $A/B$ in which the map $\psi$ is given by $\psi(c \otimes a) = \beta(\gamma(c)a) = l(c)(r(c)a)_0 \otimes (r(c)a)_1$.

2.7. Let $(A, C, \psi)$ be an entwining structure. We shall always consider the space $A \otimes C$ to be endowed with the structure of an $A$-bimodule with left and right actions given by

\[
\lambda \mapsto a \otimes c = \lambda a \otimes c, \quad \quad a \otimes c \leftarrow \rho = a \mapsto \psi(c \otimes \rho)
\]

for each $a, \lambda, \rho \in A, c \in C$.

2.8. We note that when $A/B$ is a $C$-Galois extension, the Galois map $\beta : A \otimes_B A \to A \otimes C$ is a map of $A$-bimodules. Indeed, we have

\[
\beta(a \otimes a') \leftarrow b = a_0 a'_1 \otimes 1 \leftarrow b = a_0 a'_1 l(a'_1)(r(a'_1)b)_0 \otimes (r(a'_1)b)_1
= a(a'_1b)_0 \otimes (a'_1b)_1 = \beta(a \otimes a' \leftarrow b),
\]

where the third equality follows from the fact, stated as property (iii) in the proof of theorem 2.7 in [3], that $a_0 l(a_1) \otimes r(a_1) = 1 \otimes a$ for all $a \in A$.

2.9. In [1], Brzeziński considers the complex $\text{Bar}^\psi_A = (A \otimes C) \otimes_A \text{Bar}^\psi_A$; here $\text{Bar}^\psi_A$ is the usual Hochschild resolution of $A$ as an $A$-bimodule. Since of course $A$ is flat as a left $A$-module, this complex is acyclic over $A \otimes C$, and since its components are clearly free as $A$-bimodules, we have in fact a projective resolution of $A \otimes C$ as an $A$-bimodule.

2.10. For each $A$-bimodule $M$, [1] defines the cohomology of the entwining structure $(A, C, \psi)$ with values in $M$ to be the graded space $H^\bullet_C(A, M)$ obtained by taking the homology of the cochain complex $\text{Hom}_A(\text{Bar}^\psi_A, M)$. In view of the observation made in 2.9 we have at once that $H^\bullet_C(A, M) = \text{Ext}^\bullet_A(A \otimes C, M)$.

Observe that with this identification in mind, proposition 2.3 in [1], stating that $A \otimes C$ is a projective $A$-bimodule iff $H^1_C(A, -)$ vanishes identically, becomes immediate.
2.11. Proposition 2.6 in [1] and the comments after its proof hint that when \( A/B \) is a \( C \)-Galois extension, the cohomology of the corresponding entwining structure \((A, C, \psi)\) is related to the Hochschild cohomology of \( B \). In that paper the case where \( B = k \) is considered; we have, more generally,

2.12. Theorem. Let \( C \) be a coalgebra. Let \( A/B \) be a \( C \)-Galois extension, and let \((A, C, \psi)\) be the corresponding entwining structure. Then we have \( H^0_\psi(A, -) \cong H^0(B, -) \) as functors of \( A \)-bimodules. In fact, if \( A \) is flat as a (left or right) \( B \)-module,

\[
H^\bullet_\psi(A, -) \cong H^\bullet(B, -)
\]

as \( \partial \)-functors on the category of \( A \)-bimodules. In that paper the case where \( B = k \) is considered; we have, more generally,

Proof. Because \( \beta : A \otimes_B A \to A \otimes C \) is an isomorphism of \( A \)-bimodules,

\[
H^\bullet_\psi(A, -) \cong \text{Ext}^\bullet_{A^e}(A \otimes C, -) \cong \text{Ext}^\bullet_{A^e}(A \otimes_B A, -)
\]

naturally on \( A \)-bimodules. On the other hand, the change-of-rings spectral sequence XVI.5.2, constructed in [1], when specialised to the morphism \( B^e \to A^e \), has \( E^{p,q}_2 \cong \text{Ext}^p_{A^e}(\text{Tor}^{B^e}_p(A^e, B), -) \) and converges to \( \text{Ext}^\bullet_{B^e}(B, -) \); note that we know from corollary IX.4.4, loc. cit., that \( \text{Tor}^{B^e}_p(A^e, B) \cong \text{Tor}^B_p(A, A) \). Now, since this spectral sequence lives on the first quadrant, [1] and convergence immediately imply that \( H^0_\psi(A, -) \cong H^0(B, -) \). When \( A \) is flat as a \( B \)-module, the spectral sequence degenerates at once, and this, together with [1], gives an isomorphism \( \text{Ext}^\bullet_{A^e}(A \otimes_B A, -) \cong \text{Ext}^\bullet_{B^e}(B, -) \).

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