MULTIPLE POSITIVE SOLUTIONS OF SATURABLE NONLINEAR SCHröDINGER EQUATIONS WITH INTENSITY FUNCTIONS

TAI-CHIA LIN

Department of Mathematics
National Taiwan University
Taipei 10617, Taiwan

and
Mathematics Division
National Center for Theoretical Sciences
Taipei 10617, Taiwan

TSUNG-FANG WU*

Department of Applied Mathematics
National University of Kaohsiung
Kaohsiung 811, Taiwan

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Abstract. In this paper, we study saturable nonlinear Schrödinger equations with nonzero intensity function which makes the nonlinearity become not superlinear near zero. Using the Nehari manifold and the Lusternik-Schnirelman category, we prove the existence of multiple positive solutions for saturable nonlinear Schrödinger equations with nonzero intensity function which satisfies suitable conditions. The ideas contained here might be useful to obtain multiple positive solutions of the other non-homogeneous nonlinear elliptic equations.

1. Introduction. Here we study multiple positive solutions of a saturable nonlinear Schrödinger equation denoted as

\[
\begin{cases}
-\Delta u + \lambda u = \frac{\Gamma I(x) + u^2}{1 + I(x) + u^2} u & \text{in } \mathbb{R}^N, \\
\lim_{|x| \to \infty} u(x) = 0,
\end{cases}
\]

(E)

for \( 0 < \lambda < \Gamma \) and \( N \geq 1 \), where solution \( u = u(x) > 0 \) for \( x \in \mathbb{R}^N \) and function \( I \in C(\mathbb{R}^N, \mathbb{R}) \) satisfies \( \lim_{|x| \to \infty} I(x) = 0 \) and \( I(x) > -1 \) for \( x \in \mathbb{R}^N \).
(E) may come from the solitary wave solutions of nonlinear Schrödinger (NLS) equations with saturable nonlinearity (called saturable NLS equations) as follows:

\[ -i \frac{\partial \phi}{\partial z} = \Delta \phi + \Gamma \frac{I(x)}{1 + I(x)} + |\phi|^2 \phi, \quad \text{for} \quad x \in \mathbb{R}^N, \quad z > 0, \quad (1) \]

where \( i \) is the imaginary unit, \( \Gamma \) is a coupling constant and \( \phi = \phi(x, z) : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{C} \). Setting \( \phi(x, z) = e^{i \lambda z} u(x) \), we may transform equation (1) into equation (E).

Physically, \( \phi = \phi(x, z) \) is the electromagnetic field along the propagation coordinate \( z \) in a photorefractive material equipped with the photonic lattice described by an intensity (distribution) function \( I = I(x) \) (cf. \([9, 10, 12, 18, 27]\)). Besides, \( x = (x_1, \cdots, x_N) \in \mathbb{R}^N \) are the transverse coordinates, \( \Delta = \sum_{j=1}^{N} \partial^2_{x_j} \) is the transverse Laplacian, and \( \Gamma > 0 \) is the beam coupling constant.

To get positive solutions of equation (E), we may consider the following energy minimization problem.

\[ e_T = \text{Minimize} \ H[u] \text{ over } u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} u^2 = 1, \quad (2) \]

where \( H[u] = \int_{\mathbb{R}^N} |\nabla u|^2 - \Gamma \left[ u^2 - \ln \left( 1 + \frac{u^2}{1 + I(x)} \right) \right] \). This approach was taken in \([20]\), not in this one. Note that the Euler-Lagrange equation of problem (2) is equation (E) and \( \lambda > 0 \) is an eigenvalue which comes from the Lagrange multiplier of the \( L^2 \)-normalization condition \( \int_{\mathbb{R}^N} u^2 = 1 \). When function \( I \) becomes zero, spatial dimension \( N = 2 \) and \( \Gamma > T_1 > 0 \), the existence of minimizer of problem (2) can be proved by the energy estimate method (cf. \([20]\)), where \( T_1 \) is a positive constant. Moreover, Lin et al. \([21]\) get the estimate of the eigenvalue \( \lambda \) and the minimum (ground state) energy \( e_T \) by developing a virial theorem. Very recently, as function \( I \) becomes nonzero, Lin et al. \([22]\) use a convexity argument to prove the existence of minimizer of problem (2) with orbital stability of equation (1) if \( \Gamma > 0 \) sufficiently large and function \( I \) satisfies

\[ I \in C(\mathbb{R}^N, \mathbb{R}), \quad \lim_{|x| \to \infty} I(x) = 0 \quad \text{and} \quad I(x) > -1 \quad \text{for} \quad x \in \mathbb{R}^N. \quad (3) \]

On the other hand, under the assumption \( I \neq 0 \), equation (E) can be regarded as a perturbation problem of the nonlinear Schrödinger equation as follows:

\[
\begin{align*}
-\Delta u + \lambda u &= \Gamma \frac{u^2}{1 + u^2} \quad \text{in} \ \mathbb{R}^N, \\
\lim_{|x| \to \infty} u &= 0.
\end{align*}
\]

(4)

It is well known that the assumptions on the nonlinearity like \( g(s) = s^3 / (1 + s^2) \), asymptotically linear at infinity, can be found in Berestycki and Lions \([4]\). The existence of the radial ground state positive solution for equation (4) was given by Stuart and Zhou \([30]\) and the uniqueness is guaranteed by Serrini and Tang \([29]\). Furthermore, a more general situation has been studied by various authors who considered the following problem:

\[
\begin{align*}
-\Delta u + \lambda u &= f(x, u) u \quad \text{in} \ \mathbb{R}^N, \\
\lim_{|x| \to \infty} u &= 0,
\end{align*}
\]

(5)

where the nonlinear term \( f(x, s) \) is asymptotically linear in \( s \) at infinity, that is, \( f(x, s) \sim q(x) \in L^\infty(\mathbb{R}^N) \) as \( |s| \to +\infty \). We refer readers to \([6, 7, 8, 14, 15, 16, 17, 19, 25, 26, 30, 31]\) and the references therein. In most of these papers, \( f(x, s) \) is generally assumed to be superlinear in \( s \) near zero, that is, \( f(x, s) \sim 0 \) as \( s \to 0 \).
In addition, either \( q(x) \) is a constant or \( \lim_{|x| \to +\infty} q(x) \) exists, or \( f(x, s) \equiv f(s) \) is required. For example, by using Mountain Pass Theorem suggested by Ambrosetti-Rabinowitz \([3]\), the existence of positive solution for equation (5) was obtained in \([6]\). Applying the sub-supersolution method, the existence of positive solution for equation (5) was also discussed in \([8]\) when \( f(x, u) \sim p(x) \in L^\infty(\mathbb{R}^N) \) and \( q(x) \in L^\infty(\mathbb{R}^N) \), respectively, as \( u \) approaches the origin and infinity, where both \( p(x) \) and \( q(x) \) are required to vanish at infinity. In fact, there are a few papers concerned with the existence of infinitely many solutions for the asymptotically linear case when \( f \) is periodic in \( x \); e.g. see \([7]\). However, it is remarkable that only the existence of positive solution is considered on this problem while the multiplicity of positive solutions is not concerned so far, because of limitation of research methods.

In view of this, the main purpose of this paper is to present a new approach to search for multiple positive solutions of asymptotically linear Schrödinger equations. Here we require that the nonlinear term in the equation is non-homogeneous and non-autonomous, leading to complicate the problem. By showing that all functions on an open subset of \( H^1(\mathbb{R}^N) \setminus \{0\} \) can be projected on the Nehari manifold \( N \) and minimizing the energy functional on an appropriate subset of \( N \), we prove the existence and multiplicity of positive solutions for equation (\( E \)), with the aid of the shape of the graph of \( I(x) \) and Lusternik Schnirelman category.

Here we consider the following nonlinear Schrödinger equation:

\[
\begin{cases}
-\Delta u + \lambda u = \Gamma \frac{I_\mu(x) + u^2}{1 + I_\mu(x) + u^2} u \quad \text{in} \ \mathbb{R}^N, \\
\lim_{|x| \to \infty} u = 0,
\end{cases}
\]

where the parameter \( \mu \geq 0 \). We assume that \( I_\mu(x) = \mu I^+(x) - I^-(x) \), where the functions \( I^\pm = \max \{ \pm I, 0 \} \) satisfy the following conditions:

\( D1 \) \( I \in C(\mathbb{R}^N) \) and there is positive number \( r < 2 \) such that

\[
0 \leq I^-(x) \leq c_0 \quad \text{for some } c_0 < 1 \quad \text{and for all } x \in \mathbb{R}^N
\]

and

\[
I^+(x) \geq d_0 \exp(-r|x|) \quad \text{for some } d_0 > 0 \quad \text{and for all } x \in \mathbb{R}^N;
\]

\( D2 \) \( \text{supp} I^- \) is a bounded set with positive measure;

\( D3 \) \( I^+(x) \to 0 \) as \( |x| \to \infty \).

Then we summarize our main result as follows.

**Theorem 1.1.** Suppose that \( 0 < \lambda < \Gamma \) and the function \( I(x) \) satisfies (\( D1 \))–(\( D3 \)). Let \( \mu_*(N) = \frac{\lambda}{(\Gamma - \lambda)} I^+ \). Then we have the following results.

(i) for every \( \mu \in (0, \mu_*(N)) \), equation (\( E_\mu \)) has a least energy positive solution;

(ii) there exists a positive number \( \mu_0 \leq \mu_*(N) \) such that for every \( \mu \in (0, \mu_0) \), equation (\( E_\mu \)) has at least two positive solutions.

**Remark 1.** Suppose that function \( I \in C(\mathbb{R}^N, \mathbb{R}) \) satisfies \( \lim_{|x| \to \infty} I(x) = 0 \) and \( I(x) > -1 \) for \( x \in \mathbb{R}^N \). Then this is easy to see that equation (\( E \)) does not admit any nontrivial solution, for all \( \lambda > \Gamma \). Indeed, if \( \lambda > \Gamma \) and \( u_0 \) is a nontrivial solution, then

\[
\begin{align*}
0 &= \|u_0\|^2_{H^1} - \Gamma \int_{\mathbb{R}^N} \frac{I + u_0^2}{1 + I + u_0} u_0^2 dx \\
&= \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + (\lambda - \Gamma) \int_{\mathbb{R}^N} u_0^2 dx + \int_{\mathbb{R}^N} \frac{\Gamma u_0^2}{1 + I + u_0^2} dx
\end{align*}
\]
> 0,
which is a contradiction.

For the proof of Theorem 1.1, we have to face two challenges because of condition \( \lim_{s \to 0} f (\cdot, s) \equiv 0 \). The first one is how to prove that the Nehari manifold is a natural constraint. The second one is to show that all functions on an open subset of \( H^1 (\mathbb{R}^N) \setminus \{0\} \) can be projected on the Nehari manifold, since the uniqueness of critical point of fibering map associated with the energy functional is not sure. In order to overcome these difficulties, we establish some accurate inequalities using two power-law nonlinearities to estimate the saturable nonlinearity with nonzero intensity function (see Lemma 2.2, 2.3), which can help us to verify that the Nehari manifold is a natural constraint with nice properties (see Lemma 2.4-2.7). Furthermore, we provide a new estimation method to prove that the center mass function is non-zero on an appropriate sublevel set of the Nehari manifold. It is worth emphasizing that the ideas contained here might be useful to obtain multiple positive solutions of the other non-homogeneous nonlinear elliptic equations.

The rest of this paper is organized as follows. After giving some notations and preliminaries in Section 2, we establish some energy estimates in Section 3. In Section 4, we prove the existence of least energy positive solutions of equation \( (E_\mu) \). In Section 5, the existence of two positive solutions is obtained for \( \mu \) sufficiently small.

2. Notations and preliminaries. Let us define the energy functional \( J_\mu \) in \( H^1 (\mathbb{R}^N) \) associated with equation \( (E_\mu) \) by

\[
J_\mu (u) = \frac{1}{2} \| u \|_{H^1}^2 - \frac{\Gamma}{2} \int_{\mathbb{R}^N} u^2 - \ln \left( 1 + \frac{u^2}{1 + I_\mu} \right) \, dx,
\]

where \( \| u \|_{H^1} = \left( \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 \, dx \right)^{1/2} \) is the standard norm in \( H^1 (\mathbb{R}^N) \). It is well known that \( J_\mu \in C^1 (H^1 (\mathbb{R}^N), \mathbb{R}) \) and the solutions of equation \( (E_\mu) \) are the critical points of the energy functional \( J_\mu \) in \( H^1 (\mathbb{R}^N) \).

Next, we define the Palais–Smale (or simply (PS))–sequences, (PS)–values, and (PS)–conditions in \( H^1 (\mathbb{R}^N) \) for \( J_\mu \) as follows.

Definition 2.1. (i) For \( \beta \in \mathbb{R} \), a sequence \( \{ u_n \} \) is a (PS)\( _{\beta} \)–sequence in \( H^1 (\mathbb{R}^N) \) for \( J_\mu \) if \( J_\mu (u_n) = \beta + o(1) \) and \( J_\mu' (u_n) = o(1) \) strongly in \( H^{-1} (\mathbb{R}^N) \) as \( n \to \infty \);
(ii) \( J_\mu \) satisfies the (PS)\( _{\beta} \)–condition in \( H^1 (\mathbb{R}^N) \) if every (PS)\( _{\beta} \)–sequence in \( H^1 (\mathbb{R}^N) \) for \( J_\mu \) contains a convergent subsequence.

We establish the following estimates on the nonlinearity.

Lemma 2.2. Suppose that \( 0 < \lambda < \Gamma \). Then for each \( \mu > 0 \) and \( 2 < p \leq \min \{4, 2^*\} (2^* = \infty, \text{ if } N = 1, 2; 2^* = \frac{2N}{N-2}, \text{ if } N \geq 3) \), there exists

\[
A_p = \begin{cases} 
1/2, & \text{if } p = 4; \\
\frac{p(2-p)}{2p}, & \text{if } 2 < p \leq \min \{4, 2^*\} \text{ and } p \neq 4
\end{cases}
\]

such that

\[
s^2 - \ln \left( 1 + \frac{s^2}{1 + I_\mu (x)} \right) \leq \frac{I_\mu (x)}{1 + I_\mu (x)} s^2 + \frac{A_p}{(1 + I_\mu (x))^{p/2}} s^p \quad \text{for all } s \geq 0.
\]
Proof. Let \( f(s) = c(x) s^p - b(x) s^2 + \log \left(1 + \frac{s^2}{a(x)}\right) \) for \( 2 < p \leq \min\{4, 2^*\} \). Clearly, \( f(0) = 0 \) and a direct calculation shows that

\[
f'(s) &= pc(x) s^{p-1} - 2b(x) s + \frac{2s}{a(x) + s^2} \\
&= s \left( pc(x) s^{p-2} + \frac{2}{a(x) + s^2} - 2b(x) \right) \\
&= s \left( pc(x) s^{p-2} (a(x) + s^2) + 2b(x) \right) \\
&= s \left( pc(x) s^{p-2} + pc(x) s^p \right) \\
&= s \left( pc(x) s^p + pa(x) c(x) s^{p-2} - 2b(x) s^2 + 2(1 - a(x) b(x)) \right).
\]

We take \( a(x) = 1 + I_\mu(x) \), \( b(x) = \frac{1}{1+I_\mu(x)} \) and \( c(x) = \frac{A_p}{(1+I_\mu(x))^{p/2}} \). Then

\[
1 - a(x) b(x) = 0
\]

and

\[
pc(x) s^p + pa(x) c(x) s^{p-2} - 2b(x) s^2 + 2(1 - a(x) b(x))
\]

\[
= \frac{A_p}{(1+I_\mu(x))^{p/2}} g(s),
\]

where \( g(s) = \frac{pA_p}{1+I_\mu(x)} s^2 - 2 \left( \frac{pA_p}{1+I_\mu(x)} \right)^{p-2} s^{4-p} + pA_p \). Since

\[
g(s) \geq g(s_0) = g(s_0) = 0 \text{ for all } s \geq 0,
\]

where \( s_0 = \left( \frac{4-p}{pA_p} \right)^{1/(p-2)} (1+I_\mu(x))^{1/2} \), we have

\[
pc(x) s^p + pa(x) c(x) s^{p-2} - 2b(x) s^2 + 2(1 - a(x) b(x)) \geq 0
\]

for all \( s \geq 0 \). This implies that \( f'(s) \geq 0 \) and

\[
s^2 - \log \left(1 + \frac{s^2}{1+I_\mu(x)}\right) \leq \frac{I_\mu(x)}{1+I_\mu(x)} s^2 + \frac{A_p}{(1+I_\mu(x))^{p/2}} s^p \text{ for all } s \geq 0.
\]

This completes the proof. \(\square\)

**Lemma 2.3.** Suppose that \( 0 < \lambda < \Gamma \). Then for each \( \mu > 0 \) and \( 2 < p \leq \min\{4, 2^*\} \) there exists

\[
B_p = \begin{cases} 
1, & \text{if } p = 4; \\
\frac{32 \sqrt{3}(p-2)^{5/2}}{p(\sqrt{p+1} - \sqrt{p-2})^3}, & \text{if } 2 < p \leq \min\{4, 2^*\} \text{ and } p \neq 4 \\
\end{cases}
\]

such that

\[
\frac{I_\mu + s^2}{1+I_\mu + s^2} \leq \frac{I_\mu}{1+I_\mu} s^2 + \frac{B_p}{(1+I_\mu)^{p/2}} s^p \text{ for all } s \geq 0.
\]

**Proof.** The proof is almost the same as Lemma 2.2, and we omit it here. \(\square\)
Remark 2. From the proof of Proposition 2.2 in Costa-Tehrani [6], one can see that the uniqueness of critical points of fibering map \( h_u(t) = J_\mu (tu) \) is not obvious. However, by the inequalities obtained in Lemma 2.3, we can prove that the uniqueness of critical points of \( h_u(t) \) satisfying same condition as in Proposition 2.2 [6].

As the energy functional \( J_\mu \) is not bounded from below on \( H^1 (\mathbb{R}^N) \), it is useful to consider the functional on the Nehari manifold
\[
N_\mu = \{ u \in H^1 (\mathbb{R}^N) \setminus \{0\} \mid \langle J'_\mu (u) , u \rangle = 0 \} .
\]
Thus, \( u \in N_\mu \) if and only if
\[
\| u \|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} \frac{I_\mu + u^2}{1 + I_\mu + u^2} u^2 dx = \| u \|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} \left[ u^2 - \frac{u^2}{1 + I_\mu + u^2} \right] dx = 0
\]
and
\[
\| u \|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} u^2 dx = -\Gamma \int_{\mathbb{R}^N} \frac{u^2}{1 + I_\mu + u^2} dx < 0
\]
Define
\[
\psi_\mu (u) = \| u \|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} \left( \frac{I_\mu + u^2}{1 + I_\mu + u^2} \right) u^2 dx .
\]
Then for \( u \in N_\mu \),
\[
\langle \psi'_\mu (u) , u \rangle = 2 \| u \|_{H^1}^2 - 2 \Gamma \int_{\mathbb{R}^N} \left[ 1 - \frac{1 + I_\mu}{(1 + I_\mu + u^2)^2} \right] u^2 dx
\]
\[
= 2 \Gamma \left[ \int_{\mathbb{R}^N} \left( \frac{1 + I_\mu}{1 + I_\mu + u^2} - 1 \right) \frac{1}{1 + I_\mu + u^2} u^2 dx \right] < 0 .
\]
Then we have the following results.

Lemma 2.4. Suppose that \( 0 < \lambda < \Gamma \) and \( 0 < \mu < \mu_\ast (N) \). Then we have the following results.
(i) The set \( \{ u \equiv 0 \} \) is an isolated point of \( J^{-1}_\mu (0) \).
(ii) \( N_\mu \) is a closed set.
(iii) \( N_\mu \) is a \( C^1 \) manifold.
(iv) There exists \( \sigma_\mu > 0 \) such that \( \| u \| > \sigma_\mu \) for all \( u \in N_\mu \).

Proof. (i) We divide the proof into the following two cases:
(i - a) For \( N = 1, 2, 3 \) and \( 0 < \mu < \mu_\ast (N) \). By Lemma 2.2 and the Sobolev inequality, we take \( p = 4 \),
\[
J_\mu (u) \geq \frac{1}{2} \| u \|_{H^1}^2 - \frac{\Gamma}{2} \int_{\mathbb{R}^N} \frac{I_\mu}{1 + I_\mu} u^2 + \frac{1}{2(1 + I_\mu)^2} u^4 dx
\]
\[
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \left( \lambda - \frac{\mu |I_\mu|_{\infty}}{1 + \mu |I_\mu|_{\infty}} \right) u^2 dx - \frac{\Gamma}{4(1 - c_0)^2} \int_{\mathbb{R}^N} u^4 dx
\]
\[
\geq \min \left\{ \frac{1}{2} - 1 - \frac{\mu \Gamma |I_\mu|_{\infty}}{\lambda (1 + \mu |I_\mu|_{\infty})} \right\} \| u \|_{H^1}^2 - \frac{\Gamma}{4(1 - c_0)^2} \int_{\mathbb{R}^N} u^4 dx
\]
\[
\geq \min \left\{ \frac{1}{2} - 1 - \frac{\mu \Gamma |I_\mu|_{\infty}}{\lambda (1 + \mu |I_\mu|_{\infty})} \right\} \| u \|_{H^1}^2 - \frac{\Gamma |u|_{\infty}^2}{4(1 - c_0)^2} \int_{\mathbb{R}^N} u^2 dx
\]
\begin{align*}
\geq \|u\|^2_{H^1} & \left( \min \left\{ \frac{1}{2}, 1 - \frac{\mu \Gamma |I^+|_{\infty}}{\lambda (1 + \mu |I^+|_{\infty})} \right\} - \frac{\Gamma S^2_2}{4 (1 - c_0)^2} \right) \int_{\mathbb{R}^N} u^2 \, dx. \\
\text{Thus, it is easy to obtain that } J_\mu > 0 & \text{ if } 0 < \|u\|_{H^1} < D \text{ for some } D > 0. \\
(i - b) & \text{ For } N \geq 4 \text{ and } 0 < \mu < \mu_* (N). \text{ By Lemma 2.2, we take } p = 2^*, \text{ Since } \\
J_\mu (u) & \geq \frac{1}{2} \|u\|^2_{H^1} - \frac{\Gamma}{2} \int_{\mathbb{R}^N} \frac{I_{\mu} - u^2}{1 + I_{\mu}} \, dx + \frac{A_{2^*}}{(1 + I_{\mu})^{N/(N-2)}(x)} |u|^{2^*} \, dx \\
& \geq \frac{1}{2} \int_{\mathbb{R}^N} \|\nabla u\|^2 + \lambda \left( 1 - \frac{\mu \Gamma |I^+|_{\infty}}{\lambda (1 + \mu |I^+|_{\infty})} \right) u^2 \, dx \\
& \quad - \frac{\Gamma A_{2^*}}{2 (1 - c_0)^{N/(N-2)}} \int_{\mathbb{R}^N} |u|^{2^*} \, dx \\
& \geq \frac{1}{2} \left( 1 - \frac{\mu \Gamma |I^+|_{\infty}}{\lambda (1 + \mu |I^+|_{\infty})} \right) \|u\|^2_{H^1} - \frac{\Gamma A_{2^*} \cdot S^{2^*/2}}{2 (1 - c_0)^{N/(N-2)}} \|u\|^2_{H^1}. \\
\text{Thus, it is easy to obtain that } J_\mu > 0 & \text{ if } 0 < \|u\|_{H^1} < D \text{ for some } D > 0. \\
(ii) & \text{ } \psi_\mu (u) \text{ is a } C^1 \text{ functional, thus } \mathcal{N}_\mu \cup \{0\} = \psi_\mu^{-1}(0) \text{ is a closed subset. Moreover, } \\
\{u \equiv 0\} & \text{ is an isolated point in } \psi_\mu^{-1}(0) \text{ and the claim in (ii) follows.} \\
(iii) & \text{ By (9), } \\
\langle \psi_\mu' (u), u \rangle & < 0 \text{ for all } u \in \mathcal{N}_\mu. \\
\text{This shows that } \mathcal{N}_\mu & \text{ is a } C^1 \text{ manifold.} \\
(iv) & \text{ The proofs of all cases } N \geq 1 \text{ are similar. So, we only prove the cases } N \geq 4. \\
\text{For } u \in \mathcal{N}_\mu, \text{ it follows that } \\
\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 \, dx & = \Gamma \int_{\mathbb{R}^N} \frac{I_{\mu} + u^2}{1 + I_{\mu}} \, dx, \\
\text{then by Lemma 2.3 and the Sobolev inequality, for } 0 < \mu < \mu_* (N) \\
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \lambda \int_{\mathbb{R}^N} u^2 \, dx & \leq \frac{\mu \Gamma |I^+|_{\infty}}{1 + \mu |I^+|_{\infty}} \int_{\mathbb{R}^N} u^2 \, dx + \frac{\Gamma B_{2^*} \cdot S^{2^*/2}}{(1 - c_0)^{N/(N-2)}} \|\nabla u\|^2_{L^2}. \\
\text{This implies that } \\
\left( 1 - \frac{\mu \Gamma |I^+|_{\infty}}{\lambda (1 + \mu |I^+|_{\infty})} \right) \|u\|^2_{H^1} & \leq \frac{\Gamma B_{2^*} \cdot S^{2^*/2}}{(1 - c_0)^{N/(N-2)}} \|u\|^2_{H^1}. \\
\text{Thus,} \\
\|u\|_{H^1} & \geq \left( \frac{(1 - c_0)^N}{\Gamma(N-2)/4 B_{2^*}^{(N-2)/4} S^{N/2}} \right) \left( 1 - \frac{\mu \Gamma |I^+|_{\infty}}{\lambda (1 + \mu |I^+|_{\infty})} \right)^{(N-2)/4} = \sigma > 0. \\
\text{This completed the proof.} \quad \square
\textbf{Lemma 2.5.} \text{ Suppose that } 0 < \lambda < \Gamma. \text{ Then } \mathcal{N}_\mu \text{ is a natural constraint of equation } (E_\mu).
Proof. Let \( u \) be a critical point of the functional \( J_\mu \) restricted to the manifold \( N_\mu \).

By the theorem of Lagrange multipliers, there exists a \( \zeta \in \mathbb{R} \) such that
\[
J'_\mu(u) + \zeta \psi'_\mu(u) = 0.
\]

That is,
\[
0 = \langle J'_\mu(u) + \zeta \psi'_\mu(u), u \rangle = \zeta \langle \psi'_\mu(u), u \rangle.
\]

From (9), we have \( \zeta = 0 \), gives \( J'_\mu(u) = 0 \) and so \( u \) is a critical point of \( J_\mu \).

**Lemma 2.6.** Suppose that \( 0 < \lambda < \Gamma \) and \( 0 < \mu < \mu_* (N) \). Then for each \( u \in H^1 (\mathbb{R}^N) \setminus \{ 0 \} \), we have the following.

(i) If \( \| u \|^2_{H^1} - \Gamma \int_{\mathbb{R}^N} u^2 \, dx \geq 0 \), then \( J_\mu (tu) > 0 \) for all \( t > 0 \).

(ii) If \( \| u \|^2_{H^1} - \Gamma \int_{\mathbb{R}^N} u^2 \, dx < 0 \), then there is a unique \( 0 < t_0 (u) \leq t_\mu (u) \) such that \( t_\mu (u) u \in N_\mu \). Furthermore,
\[
J_\mu (t_\mu (u) u) = \sup_{t \geq 0} J_\mu (tu).
\]

(iii) \( t_\mu (u) \) is a continuous function for \( u \in H^1 (\mathbb{R}^N) \) with \( \| u \|^2_{H^1} - \Gamma \int_{\mathbb{R}^N} u^2 \, dx < 0 \).

(iv) \( t_\mu (u) = \frac{\| u \|^2_{H^1}}{\mu} \), \( t_\mu (\frac{u}{\| u \|^2_{H^1}}) \).

(v) \( N_\mu = \{ u \in H^1 (\mathbb{R}^N) \mid \| u \|^2_{H^1} - \Gamma \int_{\mathbb{R}^N} u^2 \, dx < 0 \} \).

**Proof.** The proofs of all cases \( N \geq 1 \) are similar. So, we only prove the cases \( N \geq 4 \).

Fix \( u \in H^1 (\mathbb{R}^N) \setminus \{ 0 \} \). Let
\[
h_u (t) = J_\mu (tu) = \frac{t^2}{2} \| u \|^2_{H^1} - \frac{\Gamma}{2} \int_{\mathbb{R}^N} (tu)^2 - \ln \left( 1 + \frac{t^2 u^2}{1 + t^2 u^2} \right) \, dx.
\]

Then
\[
th'_u (t) = \langle J'_\mu (tu) , tu \rangle = \| tu \|^2_{H^1} - \Gamma \int_{\mathbb{R}^N} t^2 u^2 - \frac{t^2 u^2}{1 + t^2 u^2} \, dx
\]
\[
= \| tu \|^2_{H^1} - \Gamma \int_{\mathbb{R}^N} \left( \frac{I_\mu + t^2 u^2}{1 + t^2 u^2} \right) t^2 u^2 \, dx
\]
\[
\geq \| tu \|^2_{H^1} - \Gamma \int_{\mathbb{R}^N} \left( \frac{I_\mu}{1 + t^2 u^2} \right) (tu)^2 + \frac{B_{2^*}}{(1 + t^2 u^2)^{(N-2)/N}} |tu|^{2^*} \, dx.
\]

and the arguments of (10) show that \( h_u (t) > 0 \) and \( h'_u (t) > 0 \) for \( t > 0 \) small.

**Claim (I):** \( \lim_{t \to \infty} \frac{1}{t^2} \int_{\mathbb{R}^N} \ln \left( 1 + \frac{t^2 u^2}{1 + t^2 u^2} \right) \, dx = 0. \)

**Claim (II):** \( \lim_{t \to \infty} \int_{\mathbb{R}^N} \frac{u^2}{1 + t^2 u^2} \, dx = 0. \)

The proofs of these claims are similar and the second one is simpler. So we only prove Claim (I) (see also [30]). We have
\[
\int_{\mathbb{R}^N} \ln \left( 1 + \frac{t^2 u^2}{1 + t^2 u^2} \right) \, dx = \int_{\mathbb{R}^N} \left( \int_0^{t^2 u^2} \frac{2s}{1 + s^2} \, ds \right) \, dx
\]
\[
= \int_{\mathbb{R}^N} \left( t^2 u^2 (x) \int_0^{1} \frac{2\tau}{1 + I_\mu + \tau^2 t^2 u^2} \, d\tau \right) \, dx,
\]
so that Lebesgue’s dominated convergence theorem yields
\[
\lim_{t \to \infty} \frac{1}{t^2} \int_{\mathbb{R}^N} \ln \left( 1 + \frac{t^2 u^2}{1 + t^2 u^2} \right) \, dx = \lim_{t \to \infty} \int_{\mathbb{R}^N} \left( u^2 (x) \int_0^{1} \frac{2\tau}{1 + I_\mu + \tau^2 t^2 u^2} \, d\tau \right) \, dx.
\]
Since $\frac{I_\mu + s^2}{1 + t_\mu + s^2}$ as a increasing function for $s \geq 0$, (13) shows that $h_u'(t)/t$ is a decreasing function of $t$ and Claim (I) implies

$$
\lim_{t \to \infty} \frac{h_u'(t)}{t} = \|u\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} u^2 \, dx.
$$

Similarly, we obtain from (12) and Claim (II) that

$$
\lim_{t \to \infty} \frac{h_u(t)}{t^2} = \frac{1}{2} \left( \|u\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} u^2 \, dx \right).
$$

(i) If $\|u\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} u^2 \, dx \geq 0$, then $h_u(t) = J_\mu(tu) > 0$ for all $t > 0$;

(ii) If $\|u\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} u^2 \, dx < 0$, then there is a unique $t_\mu > 0$ with $t_\mu = t_\mu(u)$, such that $h_u(t) > 0$ for $t < t_\mu$, $h_u(t) = 0$ for $t = t_\mu$, and $h_u(t) < 0$ for $t > t_\mu$. In particular, we have

$$
t_\mu h_u(t_\mu) = \langle J_\mu'(t_\mu u), t_\mu u \rangle = \|t_\mu u\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} \left( \frac{I_\mu + t^2_\mu u^2}{1 + I_\mu + t^2_\mu u^2} \right) t^2_\mu u^2 \, dx = 0. \quad (14)
$$

Moreover,

$$
\max_{0 < t < \infty} J_\mu(tu) = J_\mu(t_\mu u) \quad \text{and} \quad \lim_{t \to \infty} J_\mu(tu) = -\infty. \quad (15)
$$

Next, we show that $\tilde{t}(u) \leq t_\mu(u)$. First, we consider the function $m_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$
m_u(t) = \|tu\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} \left[ \frac{I_\mu}{1 + I_\mu} (tu)^2 + \frac{B_2^*}{(1 + I_\mu)^{N/(N-2)}} |tu|^2 \right] \, dx \quad \text{for } t \geq 0.
$$

Clearly, $m_u(0) = m_u(\tilde{t}(u)) = 0$ and $\lim_{t \to \infty} m_u(t) = -\infty$, where

$$
\tilde{t}(u) = \left( \frac{\|u\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} \frac{I_\mu u^2}{1 + I_\mu} \, dx}{B_2^* \int_{\mathbb{R}^N} \frac{|u|^2}{(1 + I_\mu)^{N/(N-2)}} \, dx} \right)^{(N-2)/4} > 0. \quad (16)
$$

Moreover,

$$
m_u'(t) = 2t \|u\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} \left[ \frac{2I_\mu t}{1 + I_\mu} u^2 + \frac{2NB_2^* t^{(N+2)/(N-2)}}{(N-2)(1 + I_\mu)^{N/(N-2)}} |u|^2 \right] \, dx,
$$

which implies that $m_u$ is strictly increasing on $(0, \tilde{t}(u))$ and is strictly decreasing on $(\tilde{t}(u), \infty)$, where

$$
\tilde{t}(u) = \left[ \frac{2 \left( \|u\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} \frac{I_\mu u^2}{1 + I_\mu} \, dx \right)^{(N-2)/4}}{\Gamma \int_{\mathbb{R}^N} \frac{2NB_2^* |u|^2}{(N-2)(1 + I_\mu)^{N/(N-2)}} \, dx} \right]^{(N-2)/4}.
$$

Since

$$
\psi(u) = \psi(u) = \|tu\|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} \left( \frac{I_\mu + t^2 u^2}{1 + I_\mu + t^2 u^2} \right) t^2 u^2 \, dx
$$

and

$$
\psi_\mu(t_\mu(u)u) = 0
$$
Thus, by the uniqueness of \( t_\mu (u) \) and \( m_u (\widehat{t}(u)) = 0 \), we can conclude that \( \widehat{t} (u) \leq t_\mu (u) \).

(iii) By the uniqueness of \( t_\mu (u) \) and the extrema property of \( t_\mu (u) \), we have \( t_\mu (u) \) is a continuous function for \( u \in H^1 (\mathbb{R}^N) \) with \( \| u \|_{H^1}^2 - \Gamma \int u^2 \, dx < 0 \).

(iv) Let \( v = \frac{u}{\| u \|_{H^1}} \). Then by parts (i) and (ii), there is a unique \( t_\mu (v) > 0 \) such that \( t_\mu (v) v \in N_\mu \) or \( t_\mu (\frac{u}{\| u \|_{H^1}}) \frac{u}{\| u \|_{H^1}} \in N_\mu \). Thus, by the uniqueness of \( t_\mu (v) \), we can conclude that \( t_\mu (u) = \frac{1}{\| u \|_{H^1}} t_\mu (\frac{u}{\| u \|_{H^1}}) \).

(v) For \( u \in N_\mu \). By parts (i) – (iii), \( t_\mu (\frac{u}{\| u \|_{H^1}}) \frac{u}{\| u \|_{H^1}} \in N_\mu \). Moreover, by (8),

\[
\| u \|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} u^2 \, dx < 0
\]

Since \( u \in N_\mu \), we have \( t_\mu (\frac{u}{\| u \|_{H^1}}) \frac{1}{\| u \|_{H^1}} = 1 \), which implies that

\[
N_\mu \subset \left\{ u \in H^1 (\mathbb{R}^N) \mid \| u \|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} u^2 \, dx < 0 \text{ and } \frac{1}{\| u \|_{H^1}} t_\mu (\frac{u}{\| u \|_{H^1}}) = 1 \right\}.
\]

Conversely, let \( u \in H^1 (\mathbb{R}^N) \) with \( \| u \|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} u^2 \, dx < 0 \) such that

\[
\frac{1}{\| u \|_{H^1}} t_\mu (\frac{u}{\| u \|_{H^1}}) = 1.
\]

Then by part (iii),

\[
t_\mu (\frac{u}{\| u \|_{H^1}}) \frac{u}{\| u \|_{H^1}} \in N_\mu.
\]

Thus,

\[
N_\mu = \left\{ u \in H^1 (\mathbb{R}^N) \mid \| u \|_{H^1}^2 - \Gamma \int_{\mathbb{R}^N} u^2 \, dx < 0 \text{ and } \frac{1}{\| u \|_{H^1}} t_\mu (\frac{u}{\| u \|_{H^1}}) = 1 \right\}.
\]

This completes the proof.

Furthermore, we have the following results.

**Lemma 2.7.** Suppose that \( 0 < \mu < \mu_+ (N) \). Then we have the following results.

(i) The energy functional \( J_\mu \) is bounded below on \( N_\mu \). Furthermore, there exists \( \tilde{c}_0 > 0 \) such that \( J_\mu (u) \geq \tilde{c}_0 \) for all \( u \in N_\mu \).

(ii) Let \( \{ u_n \} \subset N_\mu \) be a (PS)\( \beta \)-sequence in \( H^1 (\mathbb{R}^N) \) for \( J_\mu \). Then \( \{ u_n \} \) is a bounded sequence.

**Proof.** (i) The proofs of all cases \( N \geq 1 \) are similar. So, we only prove the cases \( N \geq 4 \). By Lemma 2.6 (v), if \( u \in N_\mu \), then

\[
J_\mu (u) = \sup_{t > 0} J_\mu (tu).
\]

Moreover, by Lemma 2.2 and the Sobolev inequality,

\[
\sup_{t > 0} J_\mu (tu)
\]
Let the following results.

We consider the energy functional \( J \) associated to equation (\( E^\infty \)),

\[
J^\infty (u) = \frac{1}{2} \| u \|_{H^1}^2 - \frac{\Gamma}{2} \int_{\mathbb{R}^N} [u^2 - \ln (1 + u^2)] \, dx.
\]

Consider the minimizing problem:

\[
\inf_{u \in \mathbb{N}^\infty} J^\infty (u) = \alpha^\infty,
\]

where

\[
\mathbb{N}^\infty = \{ u \in H^1 (\mathbb{R}^N) \setminus \{0\} \mid \langle (J^\infty)' (u) , u \rangle = 0 \}.
\]

It is known that equation (\( E^\infty \)) has a unique positive radial solution \( w (x) \) such that \( J^\infty (w) = \alpha^\infty \) and \( w (0) = \max_{x \in \mathbb{R}^N} w (x) \) (see \([13, 29]\)). Then we have the following results.

**Proposition 1.** Let \( \{ u_n \} \) be a \((PS)_c\)-sequence in \( H^1 (\mathbb{R}^N) \) for \( J_c \). Then there exist a subsequence \( \{ u_{n_j} \} \), \( m \in \mathbb{N} \), sequences \( \{ x_{n_j}^{l} \} \) \( j = 1, \ldots, m \) in \( \mathbb{R}^N \), and functions \( v_{c} \in H^1 (\mathbb{R}^N) \), \( 0 \neq w^l \in H^1 (\mathbb{R}^N) \), \( j = 1, \ldots, m \) such that

\[\begin{align*}
&(i) \ | x_{n_j}^{l} | \to \infty \text{ and } | x_{n_j}^{l} - x_{n_j}^{j} | \to \infty \text{ as } n \to \infty, \text{ for } 1 \leq i \neq j \leq m; \\
&(ii) \ -\Delta v_{c} + \lambda v_{c} = \frac{I_{c} (x) + v_{c}^2}{1 + I_{c} (x) + v_{c}^2} v_{c} \text{ in } \mathbb{R}^N;
\end{align*}\]
(iii) 
\[-\Delta u^i + \lambda u^i = \frac{1}{2} (w_i)^2 \quad \text{in} \, \mathbb{R}^N;\]
(iv) \( u_n = v_\mu + \sum_{i=1}^{m} w_i (\cdot - x_i) + o(1) \) strongly in \( H^1 (\mathbb{R}^N) \);
(v) \( J_\mu(u_n) = J_\mu(v_\mu) + \sum_{i=1}^{m} J_\infty(w_i) + o(1). \)

In addition, if \( u_n \geq 0 \), then \( v_\mu \geq 0 \) and \( w_i \geq 0 \) for each \( 1 \leq i \leq m \).

**Proof.** Similar to the argument in Lions [23, 24].

For \( \mu \geq 0 \), we define
\[ \alpha_\mu = \inf_{u \in \mathcal{N}_\mu} J_\mu(u). \]

Then, by Proposition 1, we have the following compactness result.

**Corollary 1.** Suppose that \( \{ u_n \} \) is a \((PS)_\beta\)-sequence in \( H^1 (\mathbb{R}^N) \) for \( J_\mu \) with \( 0 < \beta < \alpha_\infty + \min \{ \alpha_\mu, \alpha_\infty \} \) and \( \beta \neq \alpha_\infty \). Then there exists a subsequence \( \{ u_{n_k} \} \) and a non-zero \( u_\mu \) in \( H^1 (\mathbb{R}^N) \) such that \( u_n \to u_\mu \) strongly in \( H^1 (\mathbb{R}^N) \) and \( J_\mu(u_\mu) = \beta \).

3. **Existence of positive solutions.** Let \( w(x) \) be a positive radial solution of equation \((E_\infty)\) such that \( J_\infty(w) = \alpha_\infty \). Then by Gidas, Ni and Nirenberg [13], for any \( \varepsilon > 0 \), there exist positive numbers \( A_\varepsilon \) and \( B_0 \) such that
\[ A_\varepsilon \exp(-((1 + \varepsilon) |x|)) \leq w(x) \leq B_0 \exp(-|x|) \quad \text{for all} \ x \in \mathbb{R}^N. \]

Let \( e \in \mathbb{S}^{N-1} = \{ x \in \mathbb{R}^N \mid |x| = 1 \} \). Define
\[ w_{e,l}(x) = w(x - le) \quad \text{for} \ l \geq 0 \quad \text{and} \ e \in \mathbb{S}^{N-1}. \]

Clearly, \( w_{e,l} \) is also least energy positive solutions of equation \((E_\infty)\) for all \( l \geq 0 \).

**Proposition 2.** For each \( 0 < \mu < \mu_\ast (N) \), there exists \( \hat{l}_1 = \hat{l}_1(\mu) > 0 \) such that for all \( l \geq \hat{l}_1 \),
\[ \sup_{l \geq 0} J_\mu(tw_{e,l}) < \alpha_\infty \quad \text{for all} \ e \in \mathbb{S}^{N-1}. \]

Furthermore, there is a unique \( t_\mu(w_{e,l}) > 0 \) such that \( t_\mu(w_{e,l})w_{e,l} \in \mathcal{N}_\mu \).

**Proof.** We have
\[ J_\mu(tw_{e,l}) = \frac{t^2}{2} \| w_{e,l} \|_{H^1}^2 - \frac{\Gamma}{2} \int_{\mathbb{R}^N} t^2 w_{e,l}^2 - \ln \left( 1 + \frac{t^2 w_{e,l}^2}{1 + l_\mu} \right) dx, \]
this implies that \( J_\mu(tw_{e,l}) \to -\infty \) as \( t \to \infty \) for all \( e \in \mathbb{S}^{N-1} \). Thus, there exists \( t_1 > 0 \) such that for every \( t \geq 0 \),
\[ J_\mu(tw_{e,l}) < \alpha_\infty \quad \text{for all} \ t \geq t_1 \quad \text{and for all} \ e \in \mathbb{S}^{N-1}. \]

Moreover, \( J_\mu(0) = 0 < \alpha_\infty, J_\mu \in C^2 (H^1 (\mathbb{R}^N), \mathbb{R}) \) and \( \| w_{e,l} \|_{H^1}^2 = \| w \|_U^2 \) for all \( l \geq 0 \), this implies that there exists \( t_2 > 0 \) such that for every \( l \geq 0 \),
\[ J_\mu(tw_{e,l}) < \alpha_\infty \quad \text{for all} \ 0 \leq t \leq t_2 \quad \text{and for all} \ e \in \mathbb{S}^{N-1}. \]

By (20) and (21), we only need to show that there exists \( \hat{l}_1 > 0 \) such that, for any \( l > \hat{l}_1 \),
\[ \sup_{t_2 \leq t \leq t_1} J_\mu(tw_{e,l}) < \alpha_\infty \quad \text{for all} \ e \in \mathbb{S}^{N-1}. \]
Moreover, by Brown and Zhang [5] and Willem [32], we know that
\[ J^\infty(tw_{e,l}) = \frac{t^2}{2} \|w_{e,l}\|_{H^1}^2 - \frac{\Gamma}{2} \int_{\mathbb{R}^N} t^2 w_{e,l}^2 - \ln \left( 1 + t^2 w_{e,l}^2 \right) dx \leq \alpha^\infty \text{ for all } t > 0. \] (22)

Thus, by (19) and (22),
\[ J_\mu(tw_{e,l}) \leq \alpha^\infty + \frac{\Gamma}{2} \int_{\mathbb{R}^N} \ln \left( 1 + \frac{t^2 w_{e,l}^2}{1 + I_\mu} \right) dx - \frac{\Gamma}{2} \int_{\mathbb{R}^N} \ln \left( 1 + t^2 w_{e,l}^2 \right) dx \text{ for all } t > 0. \] (23)

Since
\[
\begin{align*}
\int_{\mathbb{R}^N} \left[ \ln \left( 1 + \frac{t^2 w_{e,l}^2}{1 + I_\mu} \right) - \ln \left( 1 + t^2 w_{e,l}^2 \right) \right] dx
&= 2 \int_{\mathbb{R}^N} \int_0^{tw_{e,l}} -I_\mu s \left( \frac{1}{1 + I_\mu + s^2} \right) ds dx \\
&\leq 2 \int_{\mathbb{R}^N} \int_0^{tw_{e,l}} \left( 1 - \frac{1}{1 - I_\mu - (tw_{e,l})^2} \right) ds dx - \int_{\Omega^c} \int_0^{tw_{e,l}} \frac{2\mu I^+ s}{(1 + I^+ + s^2)^2} ds dx \\
&= \int_{\Omega} \frac{I^-}{1 - I^-} - \frac{I^-}{1 - I^- + (tw_{e,l})^2} dx - \int_{\Omega^c} \frac{\mu I^+}{1 + \mu I^+} - \frac{\mu I^+}{1 + \mu I^+ + (tw_{e,l})^2} dx \\
&= - \frac{\mu I^+}{1 + I^+} dx + \int_{\Omega^c} \frac{\mu I^+}{1 + \mu I^+ + (tw_{e,l})^2} dx \\
&= \int_{\mathbb{R}^N} \left( 1 + I_\mu + tw_{e,l}^2 \right) \left( 1 + I_\mu \right) dx \\
&= t^2 \left[ \int_{\mathbb{R}^N} \frac{I^- w_{e,l}^2}{1 + I_\mu + tw_{e,l}^2} \left( 1 + I_\mu \right) dx \\
&\quad - \mu \int_{\mathbb{R}^N} \frac{I^+ w_{e,l}^2}{1 + I_\mu + tw_{e,l}^2} \left( 1 + I_\mu \right) dx \right],
\end{align*}
\]

where \( \Omega = \text{supp} I^- \). We set
\[ C_0 = \min_{x \in B^N(0,1)} w_0^2(x) > 0, \]
and
\[ \widehat{C}_0 = \max_{x \in \mathbb{R}^N} w_0^2(x) > 0 \]

where \( B^N(0,1) = \{ x \in \mathbb{R}^N \mid |x| < 1 \} \). Then, by the condition (D2),
\[
\int_{\mathbb{R}^N} \frac{I^+ w_{e,l}^2}{1 + \mu I^+ + tw_{e,l}^2} \left( 1 + \mu I^+ \right) dx \geq \int_{|x| \geq R_0} \frac{I^+ w_{e,l}^2}{1 + \mu I^+ + tw_{e,l}^2} dx \\
\geq \int_{|x + le| \geq R_0} \frac{I^+ (x + le) w_0^2(x)}{1 + \mu |I^+| + l^2 \widehat{C}_0} dx.
\]
For all \( l \geq 2 \max \{1, R_0\} \). Moreover, by (17) and the conditions (D1) and (D2),
\[
\int_{\mathbb{R}^N} \frac{I \cdot w_{e,l}^2}{(1 - I + t^2 w_{e,l}^2) (1 - I^{-})} \leq \frac{\alpha_0}{(1 - c_0)^2} \int_{\Omega} \exp (-2 |x - le|) \, dx \\
\leq \frac{\tau_0}{(1 - c_0)^2} \exp (-2l)
\]
Since \( r_+ < 2 \) and \( t_2 \leq t \leq t_3 \), we can find \( \tilde{l}_3 > 2 \max \{1, R_0\} \) such that, for any \( l > \tilde{l}_1 \),
\[
\int_{\mathbb{R}^N} \frac{I \cdot w_{e,l}^2}{(1 - I + t^2 w_{e,l}^2) (1 - I^{-})} < \mu_0 \int_{\mathbb{R}^N} \frac{I \cdot w_{e,l}^2}{(1 + \mu I + t^2 w_{e,l}^2) (1 + \mu I^+)} \, dx
\]
for all \( e \in \mathcal{S} \) and for all \( t \in [t_2, t_3] \). Thus, by (20) – (23) and (24), we obtain that
\[
\sup_{t \geq 0} J_{\mu}(tw_{e,l}) < \alpha^\infty \text{ for all } e \in S^{N-1}.
\]
Moreover, by Lemma 2.6, there is a unique \( t_{\mu}(w_{e,l}) > 0 \) such that \( t_{\mu}(w_{e,l}) w_{e,l} \in N_\mu \). This completes the proof.

First, we establish the existence of least energy positive solutions of equation \((E_\mu)\).

**Theorem 3.1.** For each \( 0 < \mu < \mu_\ast(N) \), equation \((E_\mu)\) has a least energy positive solution \( u_\mu \) such that
\[
J_{\mu} (u_\mu) = \inf_{u \in N_\mu} J_{\mu} (u) < \alpha^\infty.
\]

**Proof.** By analogy with the proof of Ni and Takagi [28], one can show that by the Ekeland variational principle (see [11]), there exists a minimizing sequence \( \{u_n\} \subset N_\mu \) such that
\[
J_{\mu} (u_n) = \alpha_\mu + o(1) \text{ and } J_{\mu}' (u_n) = o(1) \text{ in } H^{-1} (\mathbb{R}^N).
\]
Since \( \inf_{u \in N_\mu} J_{\mu} (u) < \alpha^\infty \) from Proposition 2 (ii), by Lemma 2.7 and Corollary 1 there exists a subsequence \( \{u_n\} \) and \( u_\mu \in N_\mu \), a nonzero solution of equation \((E_\mu)\), such that
\[
u_n \to u_\mu \text{ strongly in } H^1 (\mathbb{R}^N) \text{ and } J_{\mu} (u_\mu) = \inf_{u \in N_\mu} J_{\mu} (u).
\]
Since \( J_{\mu} (u_\mu) = J_{\mu} (|u_\mu|) \) and \( |u_\mu| \in N_\mu \), by Lemma 2.5 and the maximum principle, we obtain \( u_\mu > 0 \) in \( \mathbb{R}^N \). This completes the proof.

**Theorem 3.2.** Suppose that \( \mu = 0 \). Then we have
\[
\alpha_0 = \inf_{u \in N_0} J_0 (u) = \inf_{u \in N^\infty} J^\infty (u) = \alpha^\infty,
\]
where \( \alpha_0 = \alpha_\mu \) with \( \mu = 0 \). Furthermore, equation \((E_\mu)\) does not admit any least energy solutions.
4. Existence of two positive solutions.

Proof. Let \( w_{e,l} \) be as in (18). Then, by \( \|w_{e,l}\|^2_{H^1} - \int_{\mathbb{R}^N} w_{e,l}^2 dx < 0 \) and Lemma 2.6 (ii), there is a unique \( t_\mu (w_{e,l}) > 0 \) such that \( t_0 (w_{e,l}) w_{e,l} \in \mathbf{N}_0 \) for all \( e \in \mathbb{S}^{N-1} \) and for \( \mu = 0 \) that is

\[
\|t_0 (w_{e,l}) w_{e,l}\|_{H^1}^2 = \int_{\mathbb{R}^N} \left( \frac{I_0 + t_0^2 (w_{e,l}) w_{e,l}^2}{1 + I_0 + t_0^2 (w_{e,l}) w_{e,l}^2} \right) t_0^2 (w_{e,l}) w_{e,l}^2 dx
\]

Since

\[
\int_{\mathbb{R}^N} \left( \frac{I_0 + t_0^2 (w_{e,l}) w_{e,l}^2}{1 + I_0 + t_0^2 (w_{e,l}) w_{e,l}^2} \right) t_0^2 (w_{e,l}) w_{e,l}^2 dx \to \int_{\mathbb{R}^N} \left( \frac{t_0^4 (w_0) w_0^2}{1 + t_0^2 (w_0) w_0^2} \right) w_0^2 dx \text{ as } l \to \infty,
\]

and

\[
\|w_0\|^2_{H^1} = \|w_{e,l}\|^2_{H^1} = \Gamma \int_{\mathbb{R}^N} \left( \frac{w_{e,l}^2}{1 + w_{e,l}^2} \right) w_{e,l}^2 dx = \Gamma \int_{\mathbb{R}^N} \left( \frac{w_0^2}{1 + w_0^2} \right) w_0^2 dx,
\]

for all \( l \geq 0 \) and for all \( e \in \mathbb{S}^{N-1} \), we have \( t_0 (w_{e,l}) \to 1 \) as \( l \to \infty \). Thus,

\[
\lim_{l \to \infty} J_0 (t_0 (w_{e,l}) w_{e,l}) = \lim_{l \to \infty} J^\infty (w_0) = \alpha^\infty \text{ for all } e \in \mathbb{S}^{N-1}.
\]

Then

\[
\alpha_0 = \inf_{u \in \mathbf{N}_0} J_0 (u) \leq \inf_{u \in \mathbb{N}^\infty} J^\infty (u) = \alpha^\infty.
\]

Let \( u \in \mathbf{N}_0 \). Then, by Lemma 2.6, \( J_0 (u) = \sup_{t \geq 0} J_0 (tu) \). Moreover, there is a unique \( t^\infty > 0 \) such that \( t^\infty u \in \mathbb{N}^\infty \). Thus,

\[
J_0 (u) \geq J_0 (t^\infty u) \geq J^\infty (t^\infty u) \geq \alpha^\infty
\]

and so \( \alpha_0 \geq \alpha^\infty \). Therefore,

\[
\alpha_0 = \inf_{u \in \mathbf{N}_0} J_0 (u) = \inf_{u \in \mathbb{N}^\infty} J^\infty (u) = \alpha^\infty.
\]

Next, we will show that for \( \mu = 0 \), equation \( (E_\mu) \) does not admit any positive solution \( u_0 \) such that \( J_0 (u_0) = \alpha_0 \). Suppose the contrary. Then we can assume that there exists \( u_0 \in \mathbf{N}_0 \) such that \( J_0 (u_0) = \alpha_0 \). Then, by Lemma 2.6 (i), \( J_0 (u_0) = \sup_{t \geq 0} J_0 (tu_0) \). Moreover, there is a unique \( t^\infty (u_0) > 0 \) such that \( t^\infty (u_0) u_0 \in \mathbb{N}^\infty \). Thus,

\[
\alpha^\infty = \inf_{u \in \mathbf{N}_0} J_0 (u) = J_0 (u_0) \geq J_0 (t^\infty (u_0) u_0) > J^\infty (t^\infty (u_0) u_0) \geq \alpha^\infty,
\]

which is a contradiction. This completes the proof.

4. Existence of two positive solutions. By Theorem 3.2, for \( \mu = 0 \), equation \( (E_\mu) \) does not admit any solution \( u_0 \) such that \( J_0 (u_0) = \inf_{u \in \mathbf{N}_0} J_0 (u) \) and

\[
\alpha_0 = \inf_{u \in \mathbf{N}_0} J_0 (u) = \inf_{u \in \mathbb{N}^\infty} J^\infty (u) = \alpha^\infty.
\]

Furthermore, we have the following result.

**Lemma 4.1.** Suppose that \( \{u_n\} \) is a minimizing sequence for \( J_0 \) in \( \mathbf{N}_0 \). Then

\[
\int_{\mathbb{R}^N} \left[ \ln \left( 1 + \frac{u_n^2}{1 - I} \right) - \ln \left( 1 + u_n^2 \right) \right] dx = o (1).
\]

Furthermore, there exists a subsequence \( \{u_n\} \) which is \((PS)_{\alpha^\infty}\)–sequence for \( J^\infty \) in \( H^1 (\mathbb{R}^N) \).
Proof. For each $n$, by $\|u_n\|_2^2 - \Gamma \int_{\mathbb{R}^N} u_n^2 \, dx < 0$ and Lemma 2.6 (ii), there is a unique $t_n^\infty > 0$ such that $t_n^\infty u_n \in \mathbb{N}^\infty$, that is
\[
(t_n^\infty)^2 \|u_n\|_{L^1}^2 = \Gamma \int_{\mathbb{R}^N} \frac{(t_n^\infty u_n)^2}{1 + (t_n^\infty u_n)^2} (t_n^\infty u_n)^2 \, dx
\]
Then
\[
J_0(u_n) \geq J_0(t_n^\infty u_n) = J^\infty(t_n^\infty u_n) + \frac{\Gamma}{2} \left[ \int_{\mathbb{R}^N} \ln \left( 1 + \frac{(t_n^\infty u_n)^2}{1 - I^-} \right) \, dx - \int_{\mathbb{R}^N} \ln \left( 1 + \frac{(t_n^\infty u_n)^2}{1 - I^-} \right) \, dx \right]
\geq \alpha^\infty + \frac{\Gamma}{2} \left[ \int_{\mathbb{R}^N} \ln \left( 1 + \frac{(t_n^\infty u_n)^2}{1 - I^-} \right) \, dx - \int_{\mathbb{R}^N} \ln \left( 1 + \frac{(t_n^\infty u_n)^2}{1 - I^-} \right) \, dx \right].
\]
Since $J_0(u_n) = \alpha^\infty + o(1)$ from Theorem 3.2, we have
\[
o(1) = \int_{\mathbb{R}^N} \ln \left( 1 + \frac{(t_n^\infty u_n)^2}{1 - I^-} \right) \, dx
= 2 \int_{\mathbb{R}^N} \int_0^{t_n^\infty u_n} \frac{I^- s}{1 - I^- + s^2} \, ds \, dx
\geq 2 \int_{\mathbb{R}^N} \int_0^{t_n^\infty u_n} \frac{I^- s}{1 + s^2} \, ds \, dx = \int_{\mathbb{R}^N} \frac{I^- u_n^2}{1 + (t_n^\infty u_n)^2} \, dx
= \frac{(t_n^\infty)^2}{1 - I^-} \int_0^{t_n^\infty u_n} I^- u_n^2 \, dx
\]
We will show that there exists $C_0 > 0$ such that $t_n^\infty > C_0$ for all $n$. Suppose the contrary. Then we may assume $t_n \to 0$ as $n \to \infty$. Since $J_0(u_n) = \alpha^\infty + o(1)$, by Lemma 2.7, we have $\|u_n\|$ is uniformly bounded and so $\|t_n u_n\|_{L^1} \to 0$ or $J^\infty(t_n u_n) \to 0$, and this contradicts the fact that $J^\infty(t_n u_n) \geq \alpha^\infty > 0$. Thus,
\[
0 \leq \int_{\Omega} \frac{I^- u_n^2}{1 - I^- + u_n^2} \, dx \leq \int_{\Omega} \frac{I^- u_n^2}{(t_n^\infty)^2 + u_n^2} \, dx = o(1),
\]
this implies that $u_n \to 0$ a.e. in $\Omega$.

Since
\[
0 \leq \int_{\mathbb{R}^N} \ln \left( 1 + \frac{u_n^2}{1 - I^-} \right) - \ln \left( 1 + u_n^2 \right) \, dx
= 2 \int_0^{u_n} \frac{I^- s}{1 - I^- + s^2} \, ds \, dx
\leq 2 \int_0^{u_n} \frac{I^- s}{1 - I^- + s^2} \, ds \, dx
= \int_0^{u_n} \frac{I^-}{1 - I^-} - \frac{I^- u_n^2}{1 - I^- + u_n^2} \, dx,
\]
by the Lebesgue dominated convergence theorem,
\[
\int_0^{\infty} \frac{I^-}{1 - I^-} - \frac{I^- u_n^2}{1 - I^- + u_n^2} \, dx \to 0 \text{ as } n \to \infty.
\]
where \(N\)\(\frac{u_n^2}{1 - I^n}\), which implies that
\[
\|u_n\|^2_{H^1} - \Gamma \int_{\mathbb{R}^N} \left( \frac{u_n^2}{1 + u_n^2} \right) u_n^2 dx = o(1)
\]
and
\[
J^\infty(u_n) = \alpha^\infty + o(1).
\]
Then by the Ekeland variational principle (see [11]), there exists a subsequence \(\{u_{n_k}\}\) such that \(J^\infty(u_n)\) is \((PS)\)-sequence for \(J^\infty\) in \(H^1(\mathbb{R}^N)\).

We need the following result.

**Lemma 4.2.** Suppose that \(\mu = 0\). Then there exists \(\xi_0 > 0\) such that if \(u \in \mathbb{N}_0\) and \(J_0(u) \leq \alpha^\infty + \xi_0\), then
\[
\int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla u|^2 + \lambda u^2 \right) dx \neq 0,
\]
where \(\mathbb{N}_0 = \mathbb{N}_\mu\) and \(J_0 = J_\mu\) with \(\mu = 0\).

**Proof.** Suppose the contrary. Then there exists a sequence \(\{u_n\} \subset \mathbb{N}_0\) such that \(J_0(u) = \alpha^\infty + o(1)\) and
\[
\int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla u_n|^2 + u_n^2 \right) dx = 0.
\]
Moreover, by Lemma 4.1, there exists a subsequence \(\{u_{n_k}\}\) which is \((PS)\)-sequence in \(H^1(\mathbb{R}^N)\) for \(J^\infty\). By the concentration–compactness principle (see Lions [23, 24]) and the fact that \(\alpha^\infty > 0\), there exist a subsequence \(\{u_{n_k}\}\), a sequence \(\{x_{n_k}\} \subset \mathbb{R}^N\), and a positive solution \(w \in H^1(\mathbb{R}^N)\) of equation (\(E^\infty\)) such that
\[
||u_{n_k}(x) - w(x - x_{n_k})||_{H^1} \to 0 \text{ as } n \to \infty.
\]  
(25)

Now we will show that \(|x_{n_k}| \to \infty\) as \(n \to \infty\). Suppose the contrary. Then we may assume that \(\{x_{n_k}\}\) is bounded and \(x_{n_k} \to x_0\) for some \(x_0 \in \mathbb{R}^N\). Thus, by (25),
\[
\int_{\mathbb{R}^N} \frac{I^-}{1 - I^-} - \frac{I^-}{1 - I^- + u_{n_k}^2} dx = \int_{\mathbb{R}^N} \frac{I^-}{1 - I^-} - \frac{I^-}{1 - I^- + w^2(x - x_{n_k})} dx + o(1)
\]
\[
= \int_{\mathbb{R}^N} \frac{I^-}{1 - I^-} - \frac{I^-}{1 - I^- + w^2(x - x_0)} dx + o(1),
\]
which contradicts the result of Lemma 4.1:
\[
\int_{\mathbb{R}^N} \frac{I^-}{1 - I^-} - \frac{I^-}{1 - I^- + u_{n_k}^2} dx = o(1).
\]
Hence we may assume \(x_{n_k}/|x_{n_k}| \to e_0\) as \(n \to \infty\), where \(e_0 \in \mathbb{S}^{N-1}\). Then, by the Lebesgue dominated convergence theorem, we have
\[
0 = \int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla u_{n_k}|^2 + u_{n_k}^2 \right) dx = \int_{\mathbb{R}^N} \frac{x + x_{n_k}}{|x + x_{n_k}|} \left( |\nabla w|^2 + w^2 \right) dx + o(1)
\]
\[
= e_0 \|w\|^2_{H^1} + o(1),
\]
which is a contradiction. This completes the proof. \(\square\)
For $\mu > 0$ and $u \in N_\mu$, by Lemma 2.6, there is a unique $t_0 (u) > 0$ such that $t_0 (u) u \in N_0$ where $N_0 = N_\mu$ with $\mu = 0$. Moreover, by the proof of Proposition 2, there exist positive numbers $t_\mu (w_{e,t})$ and $\hat{t}_1$ such that $t_\mu (w_{e,t}) w_{e,t} \in N_\mu$ and

$$ J_\mu (t_\mu (w_{e,t}) w_{e,t}) < \alpha^\infty $$

for all $l > \hat{t}_1$.

Then we have the following result.

**Lemma 4.3.** There exists a positive number $\mu_0 \leq \mu_* (N)$ such that for every $\mu \in (0, \mu_0)$, we have

$$ \int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla u|^2 + u^2 \right) dx \neq 0 $$

for all $u \in N_\mu$ with $J_\mu (u) < \alpha^\infty$.

**Proof.** Let $u \in N_\mu$ with $J_\mu (u) < \alpha^\infty$. Then, by Lemma 2.6, there exists $t_0 (u) > 0$ such that $t_0 (u) u \in N_0$. Moreover,

$$ J_\mu (u) = \sup_{t \geq 0} J_\mu (tu) \geq J_\mu (t_0 (u) u) $$

$$ = J_0 (t_0 (u) u) - \frac{\Gamma}{2} \int_{\mathbb{R}^N} \ln \left( 1 + \frac{t_0^2 (u) u^2}{1 + I_0} \right) - \ln \left( 1 + \frac{t_0^2 (u) u^2}{1 + I_\mu} \right) dx. $$

Since

$$ \int_{\mathbb{R}^N} \ln \left( 1 + \frac{t_0^2 (u) u^2}{1 + I_0} \right) - \ln \left( 1 + \frac{t_0^2 (u) u^2}{1 + I_\mu} \right) dx $$

$$ = 2 \int_{\mathbb{R}^N} \int_0^{t_0 (u) u} \frac{s}{1 + I_0 + s^2} - \frac{s}{1 + I_\mu + s^2} ds dx $$

$$ = 2 \mu \int_{\mathbb{R}^N} \int_0^{t_0 (u) u} \frac{s I^+}{(1 - I_0 + s^2) (1 + I_\mu + s^2)} ds dx $$

$$ \leq 2 \mu \int_{\mathbb{R}^N} \int_0^{t_0 (u) u} \frac{s I^+}{(1 - I_0 + s^2)^2} ds dx $$

$$ = \mu \int_{\mathbb{R}^N} \left( \frac{I^+}{1 - I^-} - \frac{I^+}{1 - I^- + (t_0 (u) u)^2} \right) dx, $$

We have that

$$ J_0 (t_0 (u) u) \leq J_\mu (u) + \frac{\Gamma}{2} \int_{\mathbb{R}^N} \ln \left( 1 + \frac{t_0^2 (u) u^2}{1 + I_0} \right) - \ln \left( 1 + \frac{t_0^2 (u) u^2}{1 + I_\mu} \right) dx $$

$$ \leq J_\mu (u) + \frac{\mu \Gamma}{2} \int_{\mathbb{R}^N} \left( \frac{I^+}{1 - I^-} - \frac{I^+}{1 - I^- + (t_0 (u) u)^2} \right) dx $$

$$ \leq J_\mu (u) + \frac{\mu \Gamma}{2} \int_{\mathbb{R}^N} \frac{I^+}{1 - I^-} dx. \quad (26) $$

Let $\xi_0 > 0$ be as in Lemma 4.2. Then there exists a positive number $\mu_0 \leq \mu_* (N)$ such that for $\mu \in (0, \mu_0)$,

$$ J_0 (t_0 (u) u) < \alpha^\infty + \xi_0. \quad (27) $$

Since $t_0 (u) u \in N_0$ and $t_0 (u) > 0$, by Lemma 4.2 and (27)

$$ \int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla (t_0 (u) u)|^2 + (t_0 (u) u)^2 \right) dx \neq 0, $$
which implies that there exists a positive number $\mu_0$ such that for every $\mu \in (0, \mu_0)$,

$$\int_{\mathbb{R}^N} \frac{x}{|x|} \left( |\nabla u|^2 + u^2 \right) \, dx \neq 0$$

for all $u \in N_\mu$ with $J_\mu (u) < \alpha^\infty$.

In the following, we use an idea of Adachi and Tanaka [1]. For $c \in \mathbb{R}^+$, we define $[J_\mu \leq c] = \{ u \in N_\mu | u \geq 0, J_\mu (u) \leq c \}$.

We then try to show that for a sufficiently small $\sigma > 0$,

$$\text{cat} \left( [J_\mu \leq \alpha^\infty - \sigma] \right) \geq 2.$$  \hfill (28)

To prove (28), we need some preliminaries. Recall the definition of the Lusternik-Schnirelman category.

**Definition 4.4.** (i) For a topological space $X$, we say that a non-empty, closed subset $Y \subset X$ is contractible to a point in $X$ if and only if there exists a continuous mapping $\xi : [0, 1] \times Y \rightarrow X$ such that, for some $x_0 \in X$ $\xi (0, x) = x$ for all $x \in Y$ and $\xi (1, x) = x_0$ for all $x \in Y$.

(ii) We define

$$\text{cat} (X) = \min \{ k \in \mathbb{N} | \text{there exist closed subsets } Y_1, \ldots, Y_k \subset X \text{ such that } Y_j \text{ is contractible to a point in } X \text{ for all } j \text{ and } \bigcup_{j=1}^{k} Y_j = X \}.$$  

When there do not exist finitely many closed subsets $Y_1, \ldots, Y_k \subset X$ such that $Y_j$ is contractible to a point in $X$ for all $j$, and $\bigcup_{j=1}^{k} Y_j = X$, we say that $\text{cat} (X) = \infty$.

We need the following two lemmas.

**Lemma 4.5.** Suppose that $X$ is a Hilbert manifold and $F \in C^1 (X, \mathbb{R})$. Assume that there exist $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$ such that

(i) $F (x)$ satisfies the Palais–Smale condition for energy levels $c \leq c_0$;
(ii) $\text{cat} \left( \{ x \in X | F (x) \leq c_0 \} \right) \geq k$.

Then $F (x)$ has at least $k$ critical points in $\{ x \in X ; F (x) \leq c_0 \}$.

**Proof.** See Ambrosetti [2, Theorem 2.3].

We have the following results.

**Lemma 4.6.** Let $X$ be a topological space. Suppose that there are two continuous maps $\Phi : S^{N-1} \rightarrow X$, $\Psi : X \rightarrow S^{N-1}$ such that $\Psi \circ \Phi$ is homotopic to the identity map of $S^{N-1}$, that is, there exists a continuous map $\zeta : [0, 1] \times S^{N-1} \rightarrow S^{N-1}$ such that $\zeta (0, x) = (\Psi \circ \Phi) (x)$ for each $x \in S^{N-1}$,

$$\zeta (1, x) = x \text{ for each } x \in S^{N-1}.$$  

Then

$$\text{cat} (X) \geq 2.$$
Proof. See Adachi and Tanaka [1, Lemma 2.5].

For \( l > \tilde{l}_1 \), we may define a map \( \Phi_{\mu,l} : \mathbb{S}^{N-1} \to H^1(\mathbb{R}^N) \) by

\[
\Phi_{\mu,l}(e)(x) = t_\mu(w(x - le))w_0(x - le) \quad \text{for } e \in \mathbb{S}^{N-1},
\]

where \( t_\mu(w(x - le))w_0(x - le) \) is as in the proof of Proposition 2. Then we have the following result.

**Lemma 4.7.** There exists a sequence \( \{\sigma_l\} \subset \mathbb{R}^+ \) with \( \sigma_l \to 0 \) as \( l \to \infty \) such that

\[
\Phi_{\mu,l}(\mathbb{S}^{(N-1)-1}) \subset [J_\mu \leq \alpha^\infty - \sigma_l].
\]

**Proof.** By Proposition 2, for each \( l > \tilde{l}_1 \) we have

\[
t_\mu(w_0(x - le))w_0(x - le) \in N_\mu
\]

and

\[
\sup_{l > \tilde{l}_1} J_\mu(t_\mu(w_0(x - le)))w_0(x - le) < \alpha^\infty \quad \text{for all } e \in \mathbb{S}^{N-1}.
\]

Since \( \Phi_{\mu,l}(\mathbb{S}^{N-1}) \) is compact,

\[
J_\mu(t_\mu(w_0(x - le)))w_0(x - le) \leq \alpha^\infty - \sigma_l,
\]

so the conclusion follows.

From Lemma 4.3, we define \( \Psi_\mu : [J_\mu < \alpha^\infty] \to \mathbb{S}^{N-1} \) by

\[
\Psi_\mu(u) = \frac{\int_{\mathbb{R}^N} \frac{2}{l^2} \left( |\nabla u|^2 + \lambda u^2 \right) dx}{\int_{\mathbb{R}^N} \frac{2}{l^2} \left( |\nabla u|^2 + \lambda u^2 \right) dx}.
\]

Then we have the following results.

**Lemma 4.8.** Let \( \mu_0 > 0 \) be as in Lemma 4.3. Then for each \( \mu \in (0, \mu_0) \) there exists \( \tilde{l}_0 \geq \tilde{l}_1 \) such that for \( l > \tilde{l}_0 \), the map

\[
\Psi_\mu \circ \Phi_{\mu,l} : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}
\]

is homotopic to the identity.

**Proof.** Let \( \Sigma = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \int_{\mathbb{R}^N} \frac{2}{l^2} \left( |\nabla u|^2 + u^2 \right) dx \neq 0 \right\} \). We define

\[
\overline{\Psi}_\mu : \Sigma \to \mathbb{S}^{N-1}
\]

by

\[
\overline{\Psi}_\mu(u) = \frac{\int_{\mathbb{R}^N} \frac{2}{l^2} \left( |\nabla u|^2 + \lambda u^2 \right) dx}{\int_{\mathbb{R}^N} \frac{2}{l^2} \left( |\nabla u|^2 + \lambda u^2 \right) dx},
\]

an extension of \( \Psi_\mu \). Since \( w_0(x - le) \) for all \( e \in \mathbb{S}^{N-1} \) and for \( l \) sufficiently large, we let \( \gamma : [s_1, s_2] \to \mathbb{S}^{N-1} \) be a regular geodesic between \( \overline{\Psi}_\mu(w_0(x - le)) \) and \( \overline{\Psi}_\mu(\Phi_{\mu,l}(e)) \) such that \( \gamma(s_1) = \overline{\Psi}_\mu(w_0(x - le)), \gamma(s_2) = \overline{\Psi}_\mu(\Phi_{\mu,l}(e)) \). By an argument similar to that in Lemma 4.2, there exists a positive number \( \tilde{l}_0 \geq \tilde{l}_1 \) such that, for \( l > \tilde{l}_0 \),

\[
w_0\left(x - \frac{le}{2(1 - \theta)}\right) \in \Sigma \quad \text{for all } e \in \mathbb{S}^{N-1} \quad \text{and } \theta \in [1/2, 1).
\]

We define

\[
\zeta_t(\theta, e) : [0, 1] \times \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}
\]
Conclusion.

5. 

By

\[
\zeta_t (\theta, e) = \begin{cases} 
\gamma (2\theta (s_1 - s_2) + s_2) & \text{for } \theta \in [0, 1/2); \\
\Psi_\mu \left( w \left( x - \frac{le}{2(1-\theta)} \right) \right) & \text{for } \theta \in [1/2, 1); \\
e & \text{for } \theta = 1.
\end{cases}
\]

Then \( \zeta_t (0, e) = \Psi_\mu (\Phi_{\mu, t} (e)) = \Psi_\mu (\Phi_{\mu, t} (e)) \) and \( \zeta_t (1, e) = e \). First, we claim that \( \lim_{\theta \rightarrow 1^-} \zeta_t (\theta, e) = e \) and \( \lim_{\theta \rightarrow \frac{1}{2}^-} \zeta_t (\theta, e) = \Psi_\mu (w_0 (x - le)) \).

(a) \( \lim_{\theta \rightarrow 1^-} \zeta_t (\theta, e) = e \) : since

\[
\int_{\mathbb{R}^N} \frac{x}{|x|} \left( \left\| \nabla \left( w_0 \left( x - \frac{le}{2(1-\theta)} \right) \right) \right\|^2 + \left( w_0 \left( x - \frac{le}{2(1-\theta)} \right) \right)^2 \right) dx
\]

\[
= \int_{\mathbb{R}^N} \frac{x + \frac{le}{2(1-\theta)}}{\left( x + \frac{le}{2(1-\theta)} \right)} \left( \nabla w_0 (x)^2 + w_0^2 (x) \right) dx
\]

\[
= e \|w_0\|_{\mu, 1}^2 + o(1) \text{ as } \theta \rightarrow 1^-,
\]

it follows that \( \lim_{\theta \rightarrow 1^-} \zeta_t (\theta, e) = e \).

(b) \( \lim_{\theta \rightarrow \frac{1}{2}^-} \zeta_t (\theta, e) = \Psi_\mu (w_0 (x - le)) \) : since \( \Psi_\mu \in C \left( \Sigma, S^{N-1} \right) \), we obtain

\[
\lim_{\theta \rightarrow \frac{1}{2}^-} \zeta_t (\theta, e) = \Psi_\mu (w_0 (x - le)).
\]

Thus, \( \zeta_t (\theta, e) \in C \left( [0, 1] \times S^{N-1}, S^{N-1} \right) \) and

\[
\zeta_t (0, e) = \Psi_\mu (\Phi_{\mu, t} (e)) \text{ for all } e \in S^{N-1},
\]

\[
\zeta_t (1, e) = e \text{ for all } e \in S^{N-1},
\]

provided \( l > \hat{l}_0 \). This completes the proof. \( \square \)

**Theorem 4.9.** For each \( \mu \in (0, \mu_0) \), the functional \( J_\mu \) has at least two critical points in \([J_\mu < \alpha^\infty]\). In particular, equation \((E_\mu)\) has two positive solutions \( u_0^{(i)} \) and \( u_0^{(2)} \) such that \( u_0^{(i)} \in \mathbb{N}_\mu \) for \( i = 1, 2 \).

**Proof.** Applying Lemmas 4.6, 4.8, we have for \( \mu \in (0, \mu_0) \),

\[
\text{cat} \left( [J_\mu \leq \alpha^\infty - \sigma_1] \right) \geq 2.
\]

By Proposition 1 and Lemma 4.5, \( J_\mu (u) \) has at least two critical points in \([J_\mu < \alpha^\infty]\). This implies that equation \((E_\mu)\) has two positive solutions \( u_0^{(1)} \) and \( u_0^{(2)} \) such that \( u_0^{(i)} \in \mathbb{N}_\mu \) for \( i = 1, 2 \). \( \square \)

We are now ready to prove Theorem 1.1: Theorem 1.1 can be obtained directly from Theorems 3.1, 4.9.

5. **Conclusion.** To obtain multiple positive solutions of saturable nonlinear Schrödinger equations with nonzero intensity function, we use two power-law nonlinearities to estimate the saturable nonlinearity with nonzero intensity function and derive some accurate inequalities (see Lemmas 2.2, 2.3) which can be used to verify that the Nehari manifold is a natural constraint with nice properties (see Lemmas 2.4-2.7). Furthermore, we provide a new estimation method to prove that the center mass function is non-zero on an appropriate sublevel set of the Nehari manifold.
Our ideas here might be generalized to the other non-homogeneous nonlinear elliptic equations.

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E-mail address: tclin@math.ntu.edu.tw
E-mail address: tfwu@nuk.edu.tw