1 Introduction

Let $C$ be a $\mathbb{Q}$-linear, pseudoabelian and symmetric monoidal category with a product $\otimes$. Let $n$ be a natural number and let $\Sigma_n$ be the symmetric group of permutations of $n$ elements. For any $X \in C$ one can define its wedge $X^{(n)}$ and symmetric $X^{[n]}$ powers as images of the idempotents in $\text{End}_C(X^{\otimes n})$ corresponding to the ”vertical” and ”horizontal” irreducible representations of $\Sigma_n$ over $\mathbb{Q}$. These powers generalize usual wedge and symmetric powers of vector spaces over a field of characteristic zero. Then $X$ is called to be evenly (oddly) finite dimensional if $X^{(n)}$ (correspondingly $X^{[n]}$) is a zero object for some natural $n$. In general, $X$ is called to be finite dimensional if $X \cong X_+ \oplus X_-$ where $X_+$ is evenly and $X_-$ is oddly finite dimensional.

The theory of finite dimensional Chow motives was introduced by S.-I. Kimura in [Kim98], and then considered in [GP02] and [GP03]. The abstract
theory was developed independently by O'Sullivan, see [AKO02]. It turns out that finite dimensionality is closely connected with some important problems in algebraic geometry. In particular, if X is a smooth projective complex surface without non-trivial globally holomorphic 2-forms, then Bloch's conjecture on Albanese kernel ² for X holds if and only if the motive $M(X)$ of the surface X is finite dimensional, see [GP03]. For surfaces of general type ³ it holds iff $M(X)$ is evenly finite dimensional, i.e. $M(X)^{(n)} = 0$ for some n, loc. cit.

Finite dimensional objects have good properties with respect to their tensor products and quotients. The motive of a smooth projective curve over a field is finite dimensional, see [Kim98], Prop. 5.10, 6.9 and Th. 4.2. It follows that motives of finite quotients of products of curves and abelian varieties are finite dimensional. Moreover, the motives of Fermat hypersurfaces are finite dimensional and motivic finite dimensionality is a birational invariant for surfaces, see [GP02]. Now we'd like to ask about further properties of finite dimensional objects and apply them to the geometry of varieties. With this in mind we consider Voevodsky's triangulated category $\text{DM}_\mathbb{Q}$ of motives over a field $k$ with coefficients in $\mathbb{Q}$, see [Voe00], [SV98] and [MVW] for its construction. Here we now can use motives of Zariski open and even non-smooth varieties, if the ground field $k$ admits a resolution of singularities, see [Voe00] and [SV98]. For example, let $X$ be a smooth proper scheme of finite type over $k$, $Z$ a closed subscheme in $X$, and let $U = X - Z$. Then we have the distinguished triangle

$$M(Z) \rightarrow M(X) \rightarrow M^c(U) \rightarrow M(Z)[1]$$

in $\text{DM}_\mathbb{Q}$, where $M^c$ is a motive with compact support, see [Voe00]. If $X$ is a surface, then $Z$ is just a union of curves and points on $X$. Assume that the motive of a curve (not necessary smooth or irreducible) is finite dimensional. Then $\text{dim}(M(Z)) < \infty$. In view of that and of birational invariance of motivic finite dimensionality for surfaces, [GP02], one may ask: is it true that $M(X)$ is finite dimensional if and only if $M(U)$ is finite dimensional? Or, more generally⁵: is the full subcategory generated by finite dimensional objects a thick triangulated subcategory in $\text{DM}_\mathbb{Q}$?⁶

This paper is based on two general ideas. The first one is due to Uwe Jannsen and consists of the expectation of a nice filtration on wedge (correspondingly, symmetric) powers of an object $Y$, inserted into a dist. triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$. It should be determined by corresponding powers of objects $X$ and $Y$, similarly to the filtration for a short exact sequence of locally

---

¹see also more general concept of a Schur functor in [De] ²see [Bl80], [Jann94] and [Voi93] for its formulation and motivations ³the unknown and hard part of Bloch’s conjecture ⁴in the sense of a pseudoabelian category ⁵the question stated by Claudio Pedrini ⁶as it was pointed out to me by Bruno Kahn and Chuck Weibel, the answer is negative in the general setting of a pseudoabelian $\mathbb{Q}$-linear tensor and triangulated category
free sheaves of modules on a manifold, see [Hart77], p. 127. Without further assumptions, however, there arise problems to show the required compatibilities in diagrams of distinguished triangles related to the above filtration. The second idea is then to use a triangulated category $T$, which is the homotopy category of an underlying pointed model and monoidal category $C$, with the monoidal structure on $T$ induced by the monoidal structure on $C$. Note that $T$ is a simplicial homotopy category, see [Ho99], so that we naturally assume that the shift functor $\Sigma$ in $T$ is a suspension $\Sigma X = X \wedge S^1$ by the simplicial circle $S^1$. In so structured category $T$ it is easy to control powers of vertices in distinguished triangles using cofiber sequences in the underlying category $C$. This second idea takes its roots in the paper [May01]. Our main result then is:

**Theorem 1** Let $T$ be a pseudoabelian, $\mathbb{Q}$-linear, monoidal and triangulated category, which is, at the same time, the homotopy category of a pointed model and monoidal category $C$. Assume, furthermore, that the monoidal structure on $T$ is induced by the monoidal structure on $C$ and that the shift functor in $T$ is a simplicial suspension $\Sigma X = X \wedge S^1$. Then, for any distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

in $T$, if $X$ and $Z$ are evenly (oddly) finite dimensional, it follows that $Y$ is also evenly (oddly) finite dimensional. Equivalently: if $X$ is evenly (oddly) finite dimensional and $Y$ is oddly (evenly) finite dimensional, it follows that $Z$ is oddly (evenly) finite dimensional.

The important example of the above category $T$ is $\mathbb{Q}$-localized Voevodsky’s motivic stable homotopy category, which we denote by $\text{SH}_\mathbb{Q}$, see [Voe98], [VW99], [Mor1] and [Mor1]. The underlying category $C$ is then the category of motivic symmetric spectra constructed by Jardine in [Jar00]. The recent result due to Morel [Mor3] asserts that, if $\text{char}(k) = 0$ and $-1$ is a sum of squares in $k$, then there exists an exact and monoidal equivalence between $\mathbb{Q}$-localized categories $\text{SH}_\mathbb{Q}$ and $\text{DM}_\mathbb{Q}$ over the ground field $k$. Applying this and Theorem 1 to schemes of dimension one we obtain the following generalization of Kimura’s result:

**Theorem 2** Let $k$ be a field of characteristic zero and let $X$ be an integral scheme of dimension one, separated and of finite type over $k$. Then its motive $M(X)$, considered in Voevodsky’s category $\text{DM}_\mathbb{Q}$, is finite dimensional.

---

7(i) for short of notation, a monoidal category is symmetric and closed monoidal; (ii) sometimes we will use the word ”tensor” instead of the word ”monoidal”

8in other words, the localization functor $C \rightarrow T = Ho(C)$ is monoidal
After the publishing of the first version of this paper I was informed that the same result as in Theorem 2 has been independently obtained by Carlo Mazza⁹.

The paper is organized as follows. For the convenience of the reader, in the second section we recall definitions and basic results on finite dimensional objects, known results on triangulated categories, which are homotopy categories of model monoidal categories, contained in [Hø99] and [May01], and outline that the motivic stable homotopy category is an example of such a category. In the third section we develop a homotopy technique to deal with finite dimensionality of vertices in dist. triangles and show the existence of the above filtration on $Y^{(n)}$ with graded pieces $Z^{(p)} \wedge X^{(q)}$ where $p + q = n$ (and the same for symmetric powers), and then prove Theorem 1. In the last section we prove Theorem 2.

Acknowledgements. The author wish to thank to Uwe Jannsen for many useful conversations and suggestions on the subject of the paper, and to Jens Hornbostel for consultations on stable homotopy categories. Also we thank Luca Barbieri-Viale, Bruno Kahn, Fabien Morel, Ivan Panin, Claudio Pedrini and Charles Weibel for useful discussions of the theme. The work was done while the author enjoyed the hospitality of the University of Regensburg.

2 Preliminary results

2.1 Basics on finite dimensional objects

Let $C$ be a monoidal $\mathbb{Q}$-linear and pseudoabelian category with a product $\otimes : C \times C \to C$ and a unite object $S$. For any natural number $n$ and any object $X$ in $C$ by $X^{(n)}$ we denote the $n$-fold product $X^{\otimes n}$ in $C$ and set $X^{(0)} = S$. If $f : X \to Y$ is a morphism in $C$ then let $f^{(n)} : X^{(n)} \to X^{(n)}$ be the $n$-fold tensor product of $f$.

Let $n$ be a natural number and let $\Sigma_n$ be the group of permutations of a finite set consisting of $n$ elements. Let also $A = \mathbb{Q}\Sigma_n$ be the group algebra (over $\mathbb{Q}$) of the group $\Sigma_n$. A classical result asserts that the set of all irreducible representations of $\Sigma_n$ over $\mathbb{Q}$ is in one-to-one correspondence with the set $P_n$ of all partitions $\lambda$ of the integer $n$, and that there exists a finite collection $\{e_\lambda\}$ of pairwise orthogonal idempotents in $A$, such that $\sum_{\lambda \in P_n} e_\lambda = 1$, and each $e_\lambda$ induces the corresponding irreducible representation of $\Sigma_n$ up to an isomorphism.

For any natural $n$ and for any $X \in Ob(C)$ let $\Gamma : A \to End_C(X^{(n)})$ be a natural homomorphism sending any $\sigma \in \Sigma_n$ into its ”graph”, i.e. the endomorphism of $M^{(n)}$ permuting factors according to $\sigma$ and the commutativity and associativity axioms from the definition of a monoidal category (see, for

⁹as far as I know, his proof involves some relevant filtration due to C. Weibel, and Schur functors due to P. Deligne.
example, [DM1982] or [Ho99]). For any $\lambda \in P_n$, let $d_\lambda$ be the graph of the idempotent $e_\lambda$. Since $\sum_{\lambda \in P_n} e_\lambda = 1$ in $A$, it follows that $\sum_{\lambda \in P_n} d_\lambda = 1$ in $\text{End}_C(X^{(n)})$. The category $C$ being pseudoabelian, it follows that $X^{(n)}$ is a direct sum of the images $\text{im}(d_\lambda)$ of the idempotents $d_\lambda$.

Let $n$ be a natural number. Let $d^+_n$ be the projector $d_{(\lambda)}$ when $\lambda$ is the partition $(1, \ldots, 1)$ of $n$, and let $d^-_n$ be the projector $d_{(\lambda)}$ when $\lambda$ is the partition $(n)$ of $n$. In other words,

$$d^+_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \Gamma_\sigma$$

and

$$d^-_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \Gamma_\sigma$$

Then let

$$X^{[n]} = \text{im}(d^+_n)$$

and

$$X^{(n)} = \text{im}(d^-_n) .$$

Note that, if $X$ is a vector space over a field of characteristic zero, then $X^{[n]}$ is a usual wedge and $X^{(n)}$ is a usual symmetric powers of $X$, see [FH91], §B.2.

Then we say that $X$ is evenly (oddly) finite dimensional if $X^{[n]} = 0$ (correspondingly $X^{(n)} = 0$) for some natural number $n$. In general, $X$ is said to be finite dimensional if it can be decomposed into a direct sum $X = X_+ \oplus X_-$, such that $X^{[m]}_+ = 0$ and $X^{(n)}_- = 0$.

The dimension $\text{dim}(X)$ of an evenly (oddly) finite dimensional object $X$ is, by definition, the smallest natural number $n$, such that $X^{[n]} \neq 0$ (correspondingly, $X^{(n)} \neq 0$). For example, the motive $\mathbb{Q}(d)[2d]$ is evenly one dimensional in $\text{DM}_Q$ for any $d \in \mathbb{Z}$. In the mixed case: $\text{dim}(X) = \text{dim}(X_+) + \text{dim}(X_-)$. It’s clear that $\text{dim}(X \oplus Y) = \text{dim}(X) + \text{dim}(Y)$.

The following result involves just properties of pseudoabelian monoidal categories and the theory of representations of symmetric groups (see [Kim98]):

**Proposition 3** The tensor product of two finite dimensional objects in $C$ is finite dimensional and a subobject of a finite dimensional object is also finite dimensional. Moreover, if $X$ and $Y$ are two purely finite dimensional objects with the same parity, then $X \otimes Y$ is evenly finite dimensional, and if $X$ and $Y$ have different parity, then $X \otimes Y$ is oddly finite dimensional.

**Proof.** See [Kim98], Prop. 5.10 and Cor. 5.11  

\footnote{in the sense of S.-I. Kimura} \footnote{in the sense of a pseudoabelian category}  

5
**Remark 4** Geometrically it means that the motive of a fibered product of two varieties with finite dimensional motives is finite dimensional, and, if \( X \to Y \) is a finite cover of varieties, then \( \dim(M(X)) < \infty \) implies \( \dim(M(Y)) < \infty \). In particular, any Fermat hypersurface is motivically finite dimensional and motivic finite dimensionality is a birational invariant for surfaces, see [GP02], Th. 2.8.

**Proposition 5** Let \( X \) be a finite dimensional object in \( C \) and let \( X \cong X_+ \oplus X_- \cong Y_+ \oplus Y_- \) be two decompositions of \( X \) into its even and odd parts. Then it follows that \( X_+ \cong Y_+ \) and \( X_- \cong Y_- \).

**Proof.** [Kim98], Prop. 6.3

**Theorem 6** The motive of a smooth projective curve \( X \) over a field is finite dimensional. If \( M(X) = \mathbb{Q} \oplus M^1(X) \oplus \mathbb{Q}(1)[2] \) is the decomposition\(^{12}\) of \( M(X) \) given by a \( k \)-rational point on \( X \), then \( M^1(X) \) is oddly finite dimensional of dimension \( 2g \), where \( g \) is the genus of \( X \).

**Proof.** See [She74] and [Kim98]. The proof essentially involves the fact that a big enough symmetric power of a smooth projective curve is a projective bundle over its Jacobian variety by Riemann-Roch theorem.

**Remark 7** It follows that any abelian variety and all motives, which can be reconstructed using motives of curves via their fibered products and finite covers are finite dimensional.

**Remark 8** Motivic finite dimensionality controls ”phantom motives”: if \( M \) is a Chow-motive\(^{13}\) and \( \dim(M) < \infty \), then \( H^*(M) = 0 \) implies \( M = 0 \), see [Kim98], Propositions 7.2 and 7.5, as well as Ex. 9.2.4 in [AKO02].

### 2.2 Homotopy category of a pointed model monoidal category

The basic tool in the below proof of Theorem II is a monoidal triangulated category which can be represented as the homotopy category of a pointed model and monoidal category. Here we recall basics on such categories following [Ho99].

Let \( C \) be a pointed model and monoidal category with a monoidal product \( \wedge : C \times C \to C \) and the unite object \( S \). The coproduct of two objects \( X \) and \( Y \) in \( C \) will be denoted by \( X \vee Y \). Let \( f : X \to Y \) and \( f' : X' \to Y' \) be two

\(^{12}\)see [Mu90] or [Sch]

\(^{13}\)see [Man68], [Jann92], [Mu90] and [Sch] for the definition of Chow-motives
maps in \( C \). Consider the coproduct \( X \wedge Y' \coprod_{X \times X'} Y \wedge X' \), that is colimit of the diagram

\[
\begin{array}{ccc}
X \wedge X' & \xrightarrow{f \wedge 1} & Y \wedge X' \\
\downarrow 1 \wedge f' & & \downarrow \\
X \wedge Y' & & 
\end{array}
\]

Then so called pushout smash product of \( f \) and \( f' \) is a unique map

\[
f \Box f' : X \wedge Y' \coprod_{X \times X'} Y \wedge X' \rightarrow Y \wedge Y'
\]
determined by the above colimit. The connection between model and monoidal structures can be then expressed by the following two axioms, see [Ho99], 4.2:

- If \( f \) and \( f' \) are cofibrations then \( f \Box f' \) is also a cofibration. If, in addition, one of two maps \( f \) and \( f' \) is a weak equivalence, then so is \( f \Box f' \).

- If \( q : QS \rightarrow S \) is a cofibrant replacement for the unite object \( S \), then the maps \( q \wedge 1 : QS \wedge X \rightarrow S \wedge X \) and \( 1 \wedge q : X \wedge QS \rightarrow X \wedge S \) are weak equivalences for all cofibrant \( X \).

Now we need some series of sophisticated definitions. Let \( M \) be a category and let \( R \) be a monoidal category with a product \( \otimes \) and the unite object \( S \). Then \( M \) is called to be a (right) \( R \)-module if we have a functor \( \otimes : M \times R \rightarrow M \) and two natural coherent isomorphisms \((M \otimes K) \otimes L \cong M \otimes (K \otimes L)\) and \( M \otimes S \cong M \) for any \( M \in \text{Ob}(M) \), \( K, L \in \text{Ob}(R) \). An \( R \)-module is closed if there is the corresponding adjunction, see [McL71]. If \( A \) and \( B \) are two monoidal categories and \( F : A \rightarrow B \) is a monoidal functor, then we say that \( B \) is an algebra over \( A \), see [Ho99], p. 104.

Let \( C \) be a monoidal model category. A model category \( D \) is called to be a \( C \)-model category if (i) it is a right \( C \)-module, (ii) the action \( D \times C \rightarrow D \) is a Quillen bifunctor\(^{14}\) and (iii) for an cofibrant \( X \in \text{Ob}(D) \) and for cofibrant replacement \( q : QS \rightarrow S \) the map \( 1 \otimes q : X \otimes QS \rightarrow X \otimes S \) is a weak equivalence, see [Ho99], p. 114.

Let now \( \triangle \) be the category of ordered finite sets and order preserving maps between them. Let \( S \) be the category of sets and let \( SS = \triangle^{op}S \) be the category of simplicial sets. We always consider \( SS \) with a standard model structure on it. Correspondingly, one has a pointed model category of simplicial sets \( SS_\ast \). Then an \( SS \)-model category is called a simplicial model category\(^{15}\). A pointed

\[14\text{see [Ho99], p. 107} \]
\[15\text{see an equivalent definition in Ch. II, §2 and 3 of [Jar99]} \]
simplicial model category can be considered as a $\text{SS}_*$-model category, see Prop. 4.2.19 in [Ho99].

The homotopy category $\text{Ho}(C)$ of any simplicial model category $C$ (pointed simplicial model category $C$) is a closed $\text{SS}$-module ($\text{SS}_*$-module). Moreover, this fact can be gereralized to an arbitrary situation: the homotopy category $\text{Ho}(C)$ of a model category $C$ (pointed model category $C$) is a closed $\text{SS}$-module ($\text{SS}_*$-module), see [Ho99], Ch.5. If the category $C$ is monoidal (pointed monoidal), then the monoidal structure on $C$ induces a monoidal structure on $\text{Ho}(C)$, see [Ho99], Th.4.3.2, and, moreover, $\text{Ho}(C)$ is an algebra over $\text{Ho}(\text{SS})$ (over $\text{Ho}(\text{SS}_*)$), loc. cit., Ch.5.

Now we’d like to recall (following [Ho99]) a triangulated structure on $\text{Ho}(C)$, compatible with the monoidal structure, so that $\text{Ho}(C)$ is a monoidal (=tensor) triangulated category in the sense of Appendix 8A in [MVW]. Let $C$ be a pointed model and monoidal category. Since it is pointed, it’s better to denote its product by $\wedge$. The category $\text{Ho}(C)$ is an algebra over $\text{Ho}(\text{SS}_*)$. Then one can consider smash products $X \wedge K$ of any object $X \in \text{Ob}(\text{Ho}(C)) = \text{Ob}(C)$ with any pointed simplicial set $K$, in particular, with the simplicial interval $I$ and simplicial circle $S^1$. Therefore, for any $X$ we have a natural notion of its cone $CX = X \wedge I$ and suspension $\Sigma X = X \wedge S^1$, as well as mapping cone $Cf$, say, for any cofibration $f : X \to Y$ between cofibrant objects defined as a colimit of the diagram

$$
\begin{array}{ccc}
X & \rightarrow & CX \\
\downarrow & \ & \downarrow \\
Y & & 
\end{array}
$$

Assume now that $\Sigma$ is a Quillen equivalence with an adjoint loop functor $\Omega$. Then $\text{Ho}(C)$ is naturally a triangulated category with an endofunctor $\Sigma$, see [Ho99], 6.5, 6.6, 7.1, and the triangulated structure is compatible with the monoidal structure in the usual sense, see [MVW], A8. To be more precise, $\text{Ho}(C)$ is a triangulated category in Hovey’s sense, i.e. it is a pre-triangulated category (see [Ho99], 6.5) and the suspension functor $\Sigma$ is an autoequivalence on $\text{Ho}(C)$. It can be shown that any triangulated category in the Hovey’s sense is a classically triangulated category, see [Ho99], Prop. 7.1.6.

The triangulated category $T = \text{Ho}(C)$ has at least two important advantages: strong compatibility of monoidal and triangulated structures in $T$, see [May01], and the possibility to describe distinguished triangles in $T$ in terms of cofiber sequences in $C$, see [Ho99] and [May01], pp. 18-19. If $f : X \to Y$ is a map in the category $T$, then, using cofibre replacement in $C$, one can assume that $f$ is a cofibration between cofibrant objects $X$ and $Y$. Let $Z = Y/X$ be
a quotient object, that is colimit of the diagram

\[
\begin{array}{c}
X \\ f \\
\downarrow \\
* \\
\end{array} \quad \begin{array}{c} \longrightarrow \\ \quad \longrightarrow \end{array} \quad \begin{array}{c}
Y \\
\downarrow \\
\end{array}
\]

where \(*\) is an initial-terminal object in the pointed category \(C\). Then \(\Sigma X\), being a cogroup object, coacts on \(Z\), see \([Ho99]\), Th. 6.2.1. In particular, one can define a standard boundary map \(\partial : Z \to \Sigma X\) as a composition of the coaction \(Z \to Z \amalg \Sigma X\) with the evident map \(Z \amalg \Sigma X \to \Sigma X\). Then we have a cofiber distinguished triangle

\[
X \xrightarrow{f} Y \to Z \xrightarrow{\partial} \Sigma X
\]

where \(Z\) is a quotient object \(Y/X\). And, in fact, any distinguished triangle in \(\mathcal{T} = Ho(C)\) is isomorphic to a cofiber distinguished triangle of the above type, see \([Ho99]\), 6.2-7.1, as well as \([May01]\), Section 5.

**Lemma 9** Let \(f : X \to Y\) and \(f' : X' \to Y'\) be two cofibrations of cofibrant objects in \(C\) with cofibers \(Z\) and \(Z'\) correspondingly. Let \(a : X \to X'\) and \(b : Y \to Y'\) be maps, such that \(bf = f'a\), and let \(c : Z \to Z'\) be the induced map on cofibers. Then \(c\) is equivariant in \(\mathcal{T}\) with respect to the cogroup homomorphism \(\Sigma a\), so that we have the corresponding map of distinguished triangles in \(\mathcal{T}\):

\[
\begin{array}{c}
X \\ a \\
\downarrow \\
X' \\
\end{array} \quad \begin{array}{c} \longrightarrow \\ \quad \longrightarrow \end{array} \quad \begin{array}{c}
Y \\ b \\
\downarrow \\
Y' \\
\end{array} \quad \begin{array}{c} \longrightarrow \\ \quad \longrightarrow \end{array} \quad \begin{array}{c}
Z \\ c \\
\downarrow \\
Z' \\
\end{array} \quad \begin{array}{c} \longrightarrow \\ \quad \longrightarrow \end{array} \quad \begin{array}{c}
\Sigma X \\
\Sigma a \\
\downarrow \\
\Sigma X' \\
\end{array}
\]

*Proof.* See \([Ho99]\), Prop. 6.2.5. \(\Box\)

### 2.3 Motivic stable homotopy category

Let’s consider now an important particular case of the above abstract situation. Let \(k\) be a field and let \(\text{Sm}\) be the category of all smooth schemes, separated and of finite type over \(k\). Let \(\text{Ps}(\text{Sm})\) be the category of presheaves of sets on \(\text{Sm}\). Let, further, \(\text{Spc} = \Delta^{op}\text{Ps}(\text{Sm})\) be the category of simplicial presheaves\(\footnote{further called simply smooth schemes}\)

\(\footnote{\text{the homotopy categories of simplicial sheaves and presheaves are canonically isomorphic via the forgetful functor, see \([Jar00]\), Th. 1.2 (2), p. 453, and we prefer to follow notation of \([Jar00]\).}}^{17}\)
of sets on $\text{Sm}$ and let also $\text{Spc}_*$ be the corresponding pointed category with an evident notion of a terminal-initial object $\ast$. Simplicial presheaves on $\text{Sm}$ play the role of spaces in algebraic topology, whence the notation $\text{Spc}$ and $\text{Spc}_*$, see [Voe98]. The model structure on $\text{Spc}$ (and, therefore, on $\text{Spc}_*$) depends on the Nisnevich topology on the category $\text{Sm}$, see [Jar87], [Jar00], [Voe98], [Mor1] and [Mor2]. The composition of the Yoneda embedding with the functor from presheaves into simplicial presheaves allows to identify a smooth scheme with the corresponding simplicial presheaf, see [Voe98], [Mor1] and [Mor2]. Then, since $\text{Spc}$ is cocomplete, one can consider colimits of spaces, for example, quotiens, contractions, glueings, etc. in $\text{Spc}$. In particular, let

$$T = \mathbb{A}^1/ (\mathbb{A}^1 - 0)$$

be the quotient of $\mathbb{A}^1$ by $\mathbb{A}^1 - 0$, i.e. the colimit of the diagram

$$\begin{array}{ccc}
\mathbb{A}^1 - 0 & \longrightarrow & \mathbb{A}^1 \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & \mathbb{A}^1 \\
\end{array}$$

In the homotopy category $\text{Ho}(\text{Spc}_*)$ one has

$$T \cong \mathbb{P}^1 \cong S^1 \wedge (\mathbb{A}^1 - 0)$$

where $S^1$ is a simplicial circle represented by the corresponding constant (and pointed) simplicial presheaf coming from simplicial sets.

Now a $T$-spectrum $X$ (or a motivic spectrum) is a sequence of objects $X_n \in \text{Spc}_*$ and bonding maps $T \wedge X^n \rightarrow X^{n+1}$ for each $n$. A map of spectra $f : X \rightarrow Y$ consists of maps $f^n : X^n \rightarrow Y^n$ commuting with bonding maps. A motivic symmetric spectrum $X$ is a motivic spectrum $X$ with an extra (left) action of the symmetric group $\Sigma_n$ on each $X^n$ and with $\Sigma_m \times \Sigma_p$-equivariant compositions of bonding maps $T^m \wedge X^n \rightarrow X^{m+n}$, where $T^m = T^{(m)}$. A map of motivic symmetric spectra should be, of course, also equivariant for the symmetric group action. All of these can be found in [Jar00], Section 4, and should be compared with the theory in [HSS].

Let $\text{Spc}_T^\Sigma$ be the category of motivic symmetric spectra. In [Jar00], Th. 4.15, Jardine constructed a model structure on $\text{Spc}_T^\Sigma$ and Th. 4.30, loc. cit., asserts that the resulting homotopy category $\text{Ho}(\text{Spc}_T^\Sigma)$ is the desired motivic stable homotopy category $\text{SH}$ on the Nisnevich site, see [Voe98], [Mor1], [Mor2].

Since the category $\text{Spc}_T^\Sigma$ is a pointed monoidal model category, it follows a monoidal structure on $\text{SH}$, such that the corresponding localization functor $\text{Spc}_T^\Sigma \rightarrow \text{SH}$ is monoidal, see Th. 4.3.2 in [Ho99]. Moreover, applying the above general method, briefly described in Section 2.2 (see [Ho99], Sections 6.1 - 7.1 for details), one can construct the structure of a triangulated category
on $\mathbf{SH} = Ho(\text{Spc}^\Sigma_{\mathcal{F}})$ with the shift functor being induced by the suspension $\Sigma X = X \wedge S^1$. So, the category $\mathbf{SH}$ is a typical example of a category in the assumptions in Theorem 1.

It’s important to connect now the motivic stable homotopy category $\mathbf{SH}$ with the triangulated category $\mathbf{DM}$ of motives over the ground field $k$, built in [Voe00]. One can construct another monoidal and triangulated category $\mathbf{DM}$, see [Mor2], Sections 4.3 - 5.2, of, so called, $\mathbb{P}^1$-motivic unbounded complexes $^18$ of abelian sheaves on the Nisnevich site. Taking free presheaves of abelian groups, generated by presheaves of sets, and associating normalized chain complexes to free generated simplicial presheaves, one can construct a monoidal triangulated functor

$$C_* : \mathbf{SH} \to \mathbf{DM},$$

which is a monoidal triangulated equivalence after the localization by $\mathbb{Q}$, see [Mor2], p. 32. Moreover, there is a canonical triangulated and monoidal functor $\Phi : \mathbf{DM} \to \mathbf{DM}$, which is not an equivalence in general. The recent result due to Fabien Morel asserts that, if $\text{char}(k) = 0$ and $-1$ is a sum of squares in $k$, then $\Phi$ induces an equivalence of categories after the localization by $\mathbb{Q}$:

**Theorem 10** Let $k$ be a field, such that $\text{char}(k) = 0$ and $-1$ is a sum of squares in $k$. Then

$$\Phi C_* : \mathbf{SH}_\mathbb{Q} \to \mathbf{DM}_\mathbb{Q}$$

is a monoidal and triangulated equivalence of categories.

*Proof. [Mor2], [Mor3] + [Mor4]*

**Remark 11** If $F : T \to T'$ is an additive and monoidal equivalence of $\mathbb{Q}$-linear monoidal categories, then $X \in \text{Ob}(T)$ is (evenly, oddly) finite dimensional in $T$ iff $FX \in \text{Ob}(T')$ is (evenly, oddly) finite dimensional in $T'$. Thus we see now that Theorem 10 implies the same additivity for finite dimensional objects in distinguished triangles in the category $\mathbf{DM}_\mathbb{Q}$, constructed over an arbitrary field $k$ of characteristic zero.$^19$

**Remark 12** Let $X$ be a smooth projective complex surface with $p_g = 0$. Let $M(X)$ be its motive in $\mathbf{DM}_\mathbb{Q}$ and let $E^\Sigma(X)$ be its symmetric spectrum in $\mathbf{SH}_\mathbb{Q}$. It is known, see [GP03], Theorem 27, that Bloch’s conjecture holds for $X$ if and only if $M(X)$ is finite dimensional in $\mathbf{DM}_\mathbb{Q}$. Applying Theorem 10 we have that Bloch’s conjecture holds for $X$ if and only if $E^\Sigma(X)$ is finite dimensional in $\mathbf{SH}_\mathbb{Q}$. This gives two interesting sides of the matter: a motivic stable homotopy view on the Bloch conjecture and, at the same time, a new application of the motivic stable homotopy category to a very concrete geometrical problem.

---

$^18$or $\mathbb{P}^1$-stabilized category of unbounded complexes

$^19$the condition $-1$ to be a sum of squares is not important since we consider motives with coefficients in $\mathbb{Q}$
3 Finite dimensional objects in dist. triangles

3.1 Cofiber sequences and combinatorics of powers

Let $T$ be a $\mathbb{Q}$-linear, monoidal and triangulated category reinforced by a pointed closed monoidal and model category $C$ in the sense of Section 2.2, so that $T = Ho(C)$. The monoidal product in $C$ we denote by $\wedge$ and the coproduct by $\vee$. The monoidal product in $T$ will be denoted by $\otimes$ and the direct sum by the symbol $\oplus$. The canonical (localization) functor $C \to T$ is monoidal, i.e. it carries an object $X \wedge Y$ in $C$ into the object $X \otimes Y$ in $T$. The endofunctor $X \mapsto \Sigma X = X \wedge S^1$ is a suspension by a simplicial circle $S^1$. Let’s also recall that ”monoidal” always means ”closed and symmetric monoidal”.

In particular, for any fibrant $X \in Ob(C)$ both functors $- \wedge X$ and $X \wedge -$ preserve colimits in $C$.

Let $X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z \overset{h}{\longrightarrow} \Sigma X$ be a distinguished triangle in $T$. Without loss of generality, applying cofibrant replacement, we may assume that both $X$ and $Y$ are cofibrant and the above dist. triangle is a cofibration triangle. In particular, $Z = Y/X$.

For any natural number $m$ a set of the cardinal $m$ will be called $m$-set and a 0-set is, by definition, an empty set. Fix a natural number $m$. For any integer $0 \leq i \leq m$ let $S_i$ be a set of $i$-subsets in the set $\{1, \ldots, m\}$. For any $S \in S_i$ let

$$(Y, X)_S$$

be a smash-product $A_1 \wedge \ldots \wedge A_m$ in $C$, such that, for each $j \in \{1, \ldots, m\}$, the object $A_j$ coincides with $X$ if $j \in S$, and with $Y$ otherwise. Surely, there are $C^i_m$ objects of the type $(Y, X)_S$. For any $S \in S_i$ let

$$t_S : X^{(m)} \to (Y, X)_S$$

be an evident morphism induced by the morphisms $1_X$ and $f$. Let

$$(Y, X, m - i, i)$$

be a colimit in $C$ of all morphisms $t_S$. In particular, $(Y, X, m, 0) = Y^{(m)}$ and $(X, Y, 0, m) = X^{(m)}$. Let also

$$t_{m,i} : X^{(m)} \to (Y, X, m - i, i)$$

be the corresponding canonical map.

Since smashings with cofibrants preserve colimits, for any $i \in \{1, \ldots, m\}$ there exists a canonical morphism

$$w_{m,i} : (Y, X, m - i, i) \longrightarrow (Y, X, m - i + 1, i - 1)$$
in C induced by the morphism \( f : X \to Y \) applied on different factors. In particular, \( w_{2,1} \) is the pushout square \( f \Box f \). Let

\[
v_{m,i} = w_{m,1} \circ \ldots \circ w_{m,i},
\]

so that \( v_{m,i} \) is a map

\[
v_{m,i} : (Y, X, m - i, i) \to Y^{(m)}.
\]

Then \( v_{m,1} \) can be considered as a multi-pushout product of \( f \) by \( f \). Clearly, \( v_{m,m} \) coincides with the morphism \( f^{(m)} : X^{(m)} \to Y^{(m)} \). More generally, we have two commutative diagrams

For any \( S \in S_i \) let \((Z, X)_{S} \) be a smash product \( B_1 \wedge \ldots \wedge B_m \) where, for each \( j \in \{1, \ldots, m\} \), \( B_j = X \) if \( j \in S \) and \( B_j = Z \) otherwise. Let also \([Z, X]_{S} \) be \((Z, X)_{S} \), but considered in \( T \), that is to say \([Z, X]_{S} \) is a tensor product \( B_1 \otimes \ldots \otimes B_m \) where, for each \( j \in \{1, \ldots, m\} \), \( B_j = X \) if \( j \in S \) and \( B_j = Z \) otherwise. There are \( C_i^m \) objects of type \([X, Z]_{S} \). Let \((Z, X, m - i, i) \) be a coproduct \( \vee_{S \in S_i} (Z, X)_{S} \) in the category \( C \). And let \([Z, X, m - i, i] \) be a direct sum \( \oplus_{S \in S_i} [Z, X]_{S} \) in the triangulated category \( T \), that is the object \((Z, X, m - i, i) \) viewing as an object in \( T \).

**Proposition 13** The morphism \( w_{m,i} \) is a cofibration for any \( i \). Moreover, the corresponding quotient object \((X, Y, m - i + 1, i - 1)/(X, Y, m - i, i) \) is canonically isomorphic in \( T \) to the object \([Z, X, m - i + 1, i - 1] \), so that we have a cofibration distinguished triangle

\[
(Y, X, m - i, i) \xrightarrow{w_{m,i}} (Y, X, m - i + 1, i - 1) \to [Z, X, m - i + 1, i - 1] \to \Sigma(Y, X, m - i, i)
\]

in the category \( T \).
Proof. Since \( C \) is a closed monoidal model category, it follows that, for any cofibrant object \( D \) in \( C \), both functors \( − \wedge D : C \to C \) and \( D \wedge − : C \to C \) are Quillen functors, see \([Ho99], 4.2\). In particular, they preserve cofibrations. By our assumption, \( X \) and \( Y \) are cofibrant, so that \( f(m) : X(m) \to X(m) \) is a cofibration. More generally: if \( T \) is a \( j \)-subset in an \( i \)-set \( S \in S_i \), then the corresponding map \((Y, X, m, j) \to (Y, X, m, i)\) is a cofibration. It follows that any map \( t_{m,i} \) is also a cofibration. Since \( f(m) \) is a cofibration and \( f(m) = v_{(m,i)} \circ t_{(m,i)} \), we have that \( v_{(m,i)} \) is a cofibration by 2-of-3 lemma. By the pushout product axiom for monoidal model category, the map \( w_{m,1} = v_{m,1} = f \square \ldots \square f \) is a cofibration. This is a basis for an induction. Assume that all \( w_{m,1}, \ldots, w_{m,i-1} \) are cofibrations. Then their composition \( v_{m,i-1} \) is a cofibration. The map \( v_{(m,i)} \) being a cofibration it follows that \( w_{(m,i)} \) is a cofibration, again by 2-of-3 lemma.

To show the second assertion of the lemma let’s recall that \( C \) is a pointed category, so that, for any two objects \( A \) and \( B \) in \( C \), their coproduct \( A \vee B \) is a colimit of the diagram

\[
\begin{array}{c}
\ast \\
\downarrow \\
A
\end{array} \quad \begin{array}{c}
\downarrow \\
B
\end{array}
\]

If \( A \to B \) is a cofibration, then the quotient object \( A/B \) is a colimit of the diagram

\[
\begin{array}{c}
A \\
\downarrow \\
\ast
\end{array} \quad \begin{array}{c}
\downarrow \\
B
\end{array}
\]

and it may be thought as a contraction of the subobject \( A \) in \( B \) into the fixed point \( \ast \to B \) in \( B \). Then it is easy to see that the quotient object \((X, Y, m - i + 1, i - 1)/(X, Y, m - i, i)\) is exactly the coproduct \((Z, X, m - i, i)\) in \( C \) (for the proof one has just to draw colimit diagrams carefully). Therefore, as an object it \( T \), it is equal to the direct sum \([Z, X, m - i, i]\), so that we obtain the desired cofibration exact triangle from the statement of the lemma. \( \square \)

For example, let \( m = 2 \). Then

\[
(Y, X, 1, 1) = (Y \wedge X) \coprod_{X \wedge X} (X \wedge Y)
\]
and we have the following commutative diagram

\[
\begin{array}{ccc}
(Z \otimes X) & \oplus & (X \otimes Z) \\
\downarrow & & \downarrow \\
(Y \wedge X) \coprod_{X \wedge X}(X \wedge Y) & \rightarrow & Y \otimes Y \\
\downarrow & & \downarrow \\
X \otimes X & \rightarrow & Z \otimes Z
\end{array}
\]

where the vertical row and the horizontal column both are cofibration dist. triangles from Prop. 13.

Let \( m = 3 \). Then we have three cofiber distinguished triangles coming from Proposition 13. The dist. triangle "of the first iteration" is

\[
\begin{array}{ccc}
Z \otimes Z \otimes Z & \\
\downarrow & \\
Y \otimes Y \otimes Y & \\
\downarrow & \\
(X \wedge Y \wedge Y) \coprod_{X \wedge X \wedge X}(Y \wedge Y \wedge X) \coprod_{X \wedge X \wedge X}(Y \wedge X \wedge Y) & \\
\end{array}
\]

Here the map \( w_{3,1} \) is a fiber (in the sense of triangulated categories) of the map \( g^{(3)} \). Or, in a homotopical language: \( Z^{(3)} \) is a quotient of \( Y^{(3)} \) by a subobject \((Y, X, 2, 1)\). The dist. triangle of the second iteration looks like

\[
\begin{array}{ccc}
(X \otimes Z \otimes Z) & \oplus & (Z \otimes Z \otimes X) \oplus (Z \otimes X \otimes Z) \\
\downarrow & & \\
(X \wedge Y \wedge Y) \coprod_{X \wedge X \wedge X}(Y \wedge Y \wedge X) \coprod_{X \wedge X \wedge X}(Y \wedge X \wedge Y) & \\
\downarrow & & \\
(X \wedge Y \wedge X) \coprod_{X \wedge X \wedge X}(Y \wedge X \wedge X) \coprod_{X \wedge X \wedge X}(X \wedge X \wedge Y) & \\
\end{array}
\]
and the third one is

\[(X \otimes Z \otimes X) \oplus (Z \otimes X \otimes X) \oplus (X \otimes X \otimes Z)\]

\[(X \wedge Y \wedge X) \coprod_{X \wedge X \wedge X} (Y \wedge X \wedge X) \coprod_{X \wedge X \wedge X} (X \wedge X \wedge Y)\]

\[X \otimes X \otimes X\]

### 3.2 \(2^m\)-diagram

Here we show how to build commutative diagrams illustrating the above iterations. Let \(K_m\) be \(m\)-dimensional cube in \(\mathbb{R}^m\) and let \(V\) be a set of vertices of \(K_m\). Fix a vertex \(v_0 \in V\) a "place" the object \(Y^{(m)}\) on it. Let \(V_1\) be the set of all vertices of \(K_m\) adjoint with \(v_0\). Place objects \((Y, X)_S\) in vertices from \(V_1\) when \(S\) runs \(S_1\) and orient edges connecting vertices from \(V_1\) with \(v_0\) so that they start in \(V_1\) and has \(v_0\) as a target. Such oriented edges from \(V_1\) to \(v_0\) can be considered as cofibrations from the above objects \((Y, X)_S\), \(S \in S_1\), into \(Y^{(m)}\) induced by the cofibration \(f : X \to Y\). Let \(V_0 = \{v_0\}\) and let \(V_2\) be all vertices from \(V \setminus (V_1 \cup V_0)\) adjoint with vertices from \(V_1\). Place objects \((Y, X)_S\), \(S \in S_2\) in vertices from \(V_2\) and orient edges connecting vertices from \(V_2\) with vertices from \(V_1\) so that they start in \(V_2\) and has vertices from \(V_1\) as their targets. Again, these oriented edges are the cofibrations from the objects \((Y, X)_S\), \(S \in S_2\), into the objects \((Y, X)_S\), \(S \in S_1\), induced by the cofibration \(f\). And so on. On the final step we obtain that in the vertex \(v_{2^m-1}\) opposite to \(v_0\) we place the object \(X^{(m)}\) and consider the edges from \(v_{2^m-1}\) to the vertices from \(V_{m-1}\) as the cofibrations from \(X^{(m)}\) into the objects \((Y, X)_S\) where \(S\) runs \(S_{m-1}\).

As a result we obtain the commutative \(2^m\)-diagram associated with the cofibration \(f : X \to Y\) and the natural number \(m\). We will denote it by the same symbol \(K_m\). For any \(i \in \{0, 1, \ldots, m\}\) the objects \((Y, X)_S\), \(S \in S_i\), can be identified with vertices from \(V_i\). We will say that the objects from \(V_i\) are objects of the \(i\)-th iteration. In this terminology, any object \((Y, X, m - i, i)\) is a colimit of all compositions of arrows in \(K_m\) starting in \(X^{(m)}\) and with a target in the objects of the \(i\)-th iteration.
For example, $2^2$-diagram is

$$
\begin{array}{c}
X \wedge X \xrightarrow{1 \wedge f} X \wedge Y \\
\downarrow f \wedge 1 \quad \quad \quad \downarrow f \wedge 1 \\
Y \wedge X \xrightarrow{1 \wedge f} Y \wedge Y
\end{array}
$$

Here the objects $X \wedge Y$ and $Y \wedge X$ are objects of the first iteration, that is they are vertices from $V_1$.

$2^3$-diagram looks as follows:

$$
\begin{array}{c}
X \wedge X \wedge X \xrightarrow{f \wedge 1 \wedge 1} Y \wedge X \wedge X \\
\downarrow f \wedge 1 \wedge 1 \quad \quad \quad \downarrow f \wedge 1 \wedge 1 \\
X \wedge X \wedge Y \xrightarrow{f \wedge 1 \wedge 1} Y \wedge X \wedge Y \\
\downarrow f \wedge 1 \wedge 1 \quad \quad \quad \downarrow f \wedge 1 \wedge 1 \\
X \wedge Y \wedge X \xrightarrow{f \wedge 1 \wedge 1} Y \wedge Y \wedge X \\
\downarrow f \wedge 1 \wedge 1 \quad \quad \quad \downarrow f \wedge 1 \wedge 1 \\
X \wedge Y \wedge Y \xrightarrow{f \wedge 1 \wedge 1} Y \wedge Y \wedge Y
\end{array}
$$

Here the objects $X \wedge Y \wedge Y$, $Y \wedge X \wedge Y$ and $Y \wedge Y \wedge X$ are objects of the first iteration and the objects $X \wedge Y \wedge X$, $X \wedge X \wedge Y$ and $Y \wedge X \wedge X$ are objects of the second iteration.

### 3.3 Mixed idempotents and their images

Now let $\Sigma_n$ be a symmetric group of permutations of an $n$-set and let $\sigma$ be any permutation from $\Sigma_n$. Let $i \in \{0, 1, \ldots, m\}$ and let $S \in S_i$. Let $A_S = \{a_1, \ldots, a_m\}$ be an ordered binary $m$-set, such that, for any $j \in \{1, \ldots, m\}$, $a_j = 0$ if $j$-th factor in the product $(Y, X)_S$ coincides with $X$, and $a_j = 1$ otherwise, i.e. when its coincides with $Y$. Then the permutation $\sigma$ induces a
uniquely defined permutation $\sigma_S$ on $A_S$. We will say that the product $(Y, X)_S$ is of an inner type with respect to the permutation $\sigma$ if $\sigma_S(a_j) = a_j$ for any $j \in \{1, \ldots, m\}$, and we will say that $(Y, X)_S$ is of an outer type with respect to $\sigma$ if there exists $j \in \{1, \ldots, m\}$, such that $\sigma_S(a_j) \neq a_j$.

If $(Y, X)_S$ is of the inner type with respect to $\sigma$, then let

$$\Gamma_{\sigma, S} : (Y, X)_S \longrightarrow (Y, X)_S$$

be a uniquely defined automorphism induced by $\sigma$ using commutativity and associativity axioms in the monoidal category $C$. We may say again that $\Gamma_{\sigma, S}$ is a graph of the permutation $\sigma$. If $(Y, X)_S$ is of the outer type with respect to $\sigma$, then we still can define a graph of $\sigma$ as a uniquely defined isomorphism

$$\Gamma_{\sigma, S} : (Y, X)_S \longrightarrow (Y, X)_{\sigma(S)} ,$$

again induced by $\sigma$ using commutativity and associativity in $C$.

Let

$$\Gamma'_{\sigma, i} : \coprod_{S \in S_i} (Y, X)_S \longrightarrow \coprod_{S \in S_i} (Y, X)_S$$

a morphism on coproducts induced by all maps $\Gamma_{\sigma, S}$ when $S$ runs $S_i$, but $\sigma$ is fixed. Then, for any $i \in \{1, \ldots, m\}$, we have the commutative diagram

$$\begin{array}{ccc}
\coprod_{S \in S_i} (Y, X)_S & \longrightarrow & \coprod_{S \in S_{i-1}} (Y, X)_S \\
\downarrow & & \downarrow \\
\coprod_{S \in S_i} (Y, X)_S & \longrightarrow & \coprod_{S \in S_{i-1}} (Y, X)_S \\
\end{array}$$

where horizontal arrows are induced by all possible maps from the objects of $V_i$ into the objects of $V_{i-1}$. In particular, we have the commutative diagram

$$\begin{array}{ccc}
X^{(m)} & \longrightarrow & \coprod_{S \in S_i} (Y, X)_S \\
\downarrow & & \downarrow \\
\coprod_{S \in S_i} (Y, X)_S & \longrightarrow & \coprod_{S \in S_{i-1}} (Y, X)_S \\
\end{array}$$

Therefore, $\Gamma'_{\sigma, i}$ induces an endomorphism

$$\Gamma_{\sigma, i} : (Y, X, m - i, i) \longrightarrow (Y, X, m - i, i)$$
on the level of colimits and, moreover, we have the commutative diagram

\[
\begin{array}{ccc}
(Y, X, m - i, i) & \xrightarrow{w_{m,i}} & (Y, X, m - i + 1, i - 1) \\
\downarrow \Gamma_{\sigma,i} & & \downarrow \Gamma_{\sigma,i-1} \\
(Y, X, m - i, i) & \xrightarrow{w_{m,i}} & (Y, X, m - i + 1, i - 1)
\end{array}
\]

for each \( i \in \{1, \ldots, m\} \).

In the same fashion, any product \((Z, X)_S\) can be of inner or of outer type with respect to a permutation \( \sigma \in \Sigma_m \). If \((Z, X)_S\) is of the inner type, then let

\[
\Xi_{\sigma,S} : (Z, X)_S \longrightarrow (Z, X)_S
\]

be a unique automorphism induced by \( \sigma \), and if \((Y, X)_S\) is of the outer type, then we can define its graph as a uniquely defined isomorphism

\[
\Xi_{\sigma,S} : (Z, X)_S \longrightarrow (Z, X)_{\sigma(S)},
\]

induced by \( \sigma \) and the commutativity and associativity in \( C \). Let

\[
\Xi'_{\sigma,i} : \forall S \in S, (Z, X)_S \longrightarrow \forall S \in S, (Z, X)_S
\]

a morphism on coproducts induced by all maps \( \Xi_{\sigma,S} \) when \( S \in S_i \) and let

\[
\Xi_{\sigma,i} : [Z, X, m - i, i] \longrightarrow [Z, X, m - i, i]
\]
be the corresponding morphism in the category \( T \). Then, for any \( i \in \{1, \ldots, m\} \) we have the morphism of cofibered sequences

\[
\begin{array}{ccc}
(Y, X, m - i, i) & \xrightarrow{w_{m,i}} & (Y, X, m - i + 1, i - 1) \\
\downarrow \Gamma_{\sigma,i} & & \downarrow \Gamma_{\sigma,i-1} \\
(Y, X, m - i, i) & \xrightarrow{w_{m,i}} & (Y, X, m - i + 1, i - 1)
\end{array}
\]

\[
\begin{array}{ccc}
& [Z, X, m - i + 1, i - 1] & [Z, X, m - i + 1, i - 1] \\
& \downarrow \Xi_{\sigma,i-1} & \downarrow \Xi_{\sigma,i-1} \\
& [Z, X, m - i + 1, i - 1] & [Z, X, m - i + 1, i - 1]
\end{array}
\]

Applying Lemma 9 we obtain the corresponding morphisms of distinguished triangles

\[
\begin{array}{ccc}
(Y, X, m - i, i) & \xrightarrow{w_{m,i}} & (Y, X, m - i + 1, i - 1) \\
\downarrow \Gamma_{\sigma,i} & & \downarrow \Gamma_{\sigma,i-1} \\
(Y, X, m - i, i) & \xrightarrow{w_{m,i}} & (Y, X, m - i + 1, i - 1)
\end{array}
\]

\[
\begin{array}{ccc}
& [Z, X, m - i + 1, i - 1] & \Sigma(Y, X, m - i, i) \\
& \downarrow \Xi_{\sigma,i-1} & \downarrow \Sigma \Gamma_{\sigma,i} \\
& [Z, X, m - i + 1, i - 1] & \Sigma(Y, X, m - i, i)
\end{array}
\]

19
Let
\[ d^+_{m,i} = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \text{sgn}(\sigma) \Gamma_{\sigma,i}, \]
\[ d^-_{m,i} = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \Gamma_{\sigma,i}, \]
\[ e^+_{m,i} = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \text{sgn}(\sigma) \Xi_{\sigma,i}, \]
and
\[ e^-_{m,i} = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \Xi_{\sigma,i}, \]
for any \( i \in \{0, 1, \ldots, m\} \). It is not hard to see that all of these maps are idempotents in \( T \). Note that \( d^\pm_{m,0} = d^\pm_m \) for \( Y \), and \( d^\pm_{m,m} = d^\pm_m \) for \( X \), where \( d^\pm_m \) are the idempotents defined in Section 2.1. Similarly, \( e^\pm_{m,0} = d^\pm_m \) for \( Z \), and \( e^\pm_{m,m} = d^\pm_m \) for \( X \). Therefore we will say that \( d^\pm_m \) are pure idempotents and that \( d^\pm_{m,i} \) and \( e^\pm_{m,i} \) are mixed idempotents.

Now, summing vertical maps in the last morphism of distinguished triangles, we obtain two morphisms of distinguished triangles

\[ (Y, X, m - i, i) \overset{w_{m,i}}{\longrightarrow} (Y, X, m + 1, i - 1) \overset{\Sigma(Y, X, m - i, i)}{\longrightarrow} [Z, X, m + 1, i - 1] \overset{\Sigma d^+_m}{\longrightarrow} \]
\[ (Y, X, m - i, i) \overset{w_{m,i}}{\longrightarrow} (Y, X, m + 1, i - 1) \overset{\Sigma d^-_m}{\longrightarrow} \]

and

\[ (Y, X, m - i, i) \overset{w_{m,i}}{\longrightarrow} (Y, X, m + 1, i - 1) \overset{\Sigma(Y, X, m - i, i)}{\longrightarrow} [Z, X, m + 1, i - 1] \overset{\Sigma d^-_m}{\longrightarrow} \]
\[ (Y, X, m - i, i) \overset{w_{m,i}}{\longrightarrow} (Y, X, m + 1, i - 1) \overset{\Sigma d^+_m}{\longrightarrow} \]

To extract images of these idempotents of distinguished triangles we need some trivial, but useful lemmas.

Let \( X \) be a category and let \( f : X \to X \) be an idempotent in \( X \), i.e. \( f^2 = f \). The typical example: if \( g : X \to Y \) and \( h : Y \to X \) are two morphisms in \( X \), such that \( gh = 1_Y \), then \( f = hg \) is an idempotent. In such a case \( f \) is called to
be a splitting idempotent. If the idempotent $f$ has two splittings $f = hg = st$. Then there exists a unique isomorphism $d : Y \cong Z$, such that the diagram commutes. Indeed, define $d$ as $th$ and set $c = gs$. Then $dc = thgs = tf s = tsts = 1_Z$ and $cd = gsth = gsth = ghgh = 1_Y$. Furthermore, $dg = thg = tf = tst = t$ and $sd = sth = fh = hgh = h$. If there exists another such $d'$, then $sd = h = sd'$, whence $t sd = t sd'$. Then $d = d'$ because of $ts = 1_Z$. So, we may speak about the image $im(f)$ of a splitting idempotent $f$ which is defined up to a canonical isomorphism.

In fact, this is a corollary of more general

**Lemma 14** Let $f : X \to X$ and $g : Y \to Y$ be splitting idempotents in $X$ and let $t : X \to Y$ be a morphism from $f$ to $g$, i.e. $gt = tf$. Then there exists a unique morphism $q : im(f) \to im(g)$, such that the diagram commutes.

**Proof.** To see the uniqueness assume that $q$ exists. Then $qa = ct$. Since $ab = 1_X$, it follows that $q$ is uniquely determined as $q = ctb$. Defining $q$ as $ctb$ we get: $qa = c t b a = c t f = c g t = c d c t = c t$ and $dq = d c t b = g t b = tf b = tb a b = tb$.

One can say that an image of idempotents is a functor on the evident category of splitting idempotents over the category $X$.

**Lemma 15** Let

\[
\begin{array}{c}
X & \xrightarrow{t} & Y & \xrightarrow{s} & Z & \xrightarrow{u} & \Sigma X \\
\downarrow{f} & & \downarrow{g} & & \downarrow{u} & & \Sigma f \\
X & \xrightarrow{t} & Y & \xrightarrow{s} & Z & \xrightarrow{u} & \Sigma X
\end{array}
\]
be a commutative diagram, where $f$ and $g$ are splitting idempotents and both rows are the same distinguished triangle. Then there exists a splitting idempotent $h : Z \to Z$ (not necessary unique), such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{t} & Y & \xrightarrow{s} & Z & \xrightarrow{u} & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
im(f) & \xrightarrow{im(t)} & \text{im}(g) & \xrightarrow{im(s)} & \text{im}(h) & \xrightarrow{im(u)} & \Sigma(\text{im}(f)) \\
X & \xrightarrow{t} & Y & \xrightarrow{s} & Z & \xrightarrow{u} & \Sigma X \\
\end{array}
\]

is commutative, where all vertical compositions are splittings of the corresponding idempotents. Moreover, the middle triangle is distinguished.

**Proof.** Indeed, let $C$ be a cone of the morphism $\text{im}(t) : \text{im}(f) \to \text{im}(g)$ and let $a$ and $b$ be two morphisms on cones included in the commutative diagram

\[
\begin{array}{ccc}
im(f) & \xrightarrow{im(t)} & \text{im}(g) & \xrightarrow{a} & C & \xrightarrow{b} & \Sigma(\text{im}(f)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{t} & Y & \xrightarrow{s} & Z & \xrightarrow{u} & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
im(f) & \xrightarrow{im(t)} & \text{im}(g) & \xrightarrow{s} & C & \xrightarrow{\text{im}(u)} & \Sigma(\text{im}(f)) \\
\end{array}
\]

Since the compositions $\text{im}(f) \to X \to \text{im}(f), \text{im}(g) \to Y \to \text{im}(g)$ are the identity, $c = ba$ is an isomorphism. Let $d = c^{-1}b$. Then we have the
where all three vertical compositions are identical morphisms. Then \( h = ad \) and \( C = \text{im}(h) \).

The triangle

\[
\text{im}(f) \to \text{im}(g) \to \text{im}(h) \to \Sigma \text{im}(f)
\]

is a candidate triangle in \( T \), see [Nec01], i.e. all three compositions in it are trivial. This is so because, if we consider the commutative diagram

\[
\begin{array}{cccccc}
X & \to & Y & \to & Z & \to \Sigma X \\
| & \downarrow t & | & \downarrow s & | & \downarrow u \\
\text{im}(f) & \to & \text{im}(g) & \to & \text{im}(h) & \to \Sigma \text{im}(f) \\
\end{array}
\]

then, say, \( \text{im}(s) \circ \text{im}(t) \circ f' = h' \circ s \circ t = 0 \), whence \( \text{im}(s) \circ \text{im}(t) = 0 \) because \( f' \) has a section.

By symmetry the same holds for the triangle

\[
\text{im}((1_X - f) \to \text{im}(1_Y - g) \to \text{im}(1_Z - h) \to \Sigma \text{im}(1_X - f))
\]

At the same time, evidently, the dist. triangle \( X \to Y \to Z \to \Sigma X \) is a direct sum of the two above candidate triangles. Therefore, both candidate triangles are distinguished, see [Nec01].

**Proposition 16** There are commutative diagrams
where

\[ I_{m,i}^{\pm} = \text{im}(d_{m,i}^{\pm}) \]

and

\[ J_{m,i}^{\pm} = \text{im}(e_{m,i}^{\pm}) \]

are images of the mixed idempotents and the maps \( w_{m,i}^+ \) and \( w_{m,i}^- \) are induced on images of idempotents by the map \( w_{m,i} \). All rows in these commutative diagrams are distinguished triangles and all columns are splittings of the corresponding idempotents \( d_{m,i}^{\pm} \) and \( e_{m,i}^{\pm} \).

**Proof.** Apply Lemma 15 to the above maps of distinguished triangles induced by the mixed idempotents \( d_{m,i}^{\pm} \) and \( e_{m,i}^{\pm} \).

Now we want to compute the images \( J_{m,i}^{\pm} \) of idempotents \( e_{m,i}^{\pm} \). We again need an abstract lemma.
Lemma 17  Let $X$ be a $\mathbb{Q}$-linear category with splitting idempotents, and let

$$
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow{a} & & \downarrow{b} \\
A & \xleftarrow{d} & B
\end{array}
$$

be a diagram in $X$, commutative up to a scalar $\alpha$, i.e. $aa = dbu$ for some $\alpha \in \mathbb{Q}$. Assume, furthermore, that $b^2 = b$ and $ud = \alpha$ (to be more precise, $ud = \alpha 1_B$). Then $a^2 = a$, $ua = bu$, $ad = db$ and the map $im(u) : im(a) \to im(b)$ (which exists because $ua = bu$, see Lemma 14) is an isomorphism.

Proof. Indeed, since $aa = dbu$ and $ud = \alpha$, it follows that $aa = \alpha bu$, whence $ua = bu$. Similarly, multiplying $aa = dbu$ on $d$ from the right hand side, we have that $aad = dbud$. Since $ud = \alpha$, we get $aad = \alpha db$, whence $ad = db$.

Further, $a^2 = \alpha^{-2}(dbu)(dbu) = \alpha^{-2}dbu(ud)bu = \alpha^{-2}dbabu = \alpha^{-1}dbbu$. Since $b^2 = b$ by assumptions, we get $a^2 = \alpha^{-1}dbu = a$.

Now let’s consider the commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{d} & A \\
\downarrow{\pi_B} & & \downarrow{\pi_A} \\
I_B & \xrightarrow{I_d} & I_A \\
\downarrow{\iota_B} & & \downarrow{\iota_A} \\
B & \xleftarrow{d} & A
\end{array}
$$

where the columns are splittings of the idempotents $a$ and $b$, $I_A = im(a)$, $I_B = im(b)$, etc. Easy chasing on this diagram shows that $I_d I_u \pi_B = bud = \alpha t_B \pi_B$, whence $I_B (\alpha^{-1}I_u I_d) \pi_B = \alpha t_B \pi_B$. Since $t_B$ is a left inverse for $\pi_B$, it follows that $\alpha^{-1}I_u I_d = 1_B$. Further, $I_d I_u = (\pi_A t_B)(\pi_B \iota_A) = \pi_A (t_B \pi_B) \iota_A = \pi_A dbu \iota_A = \pi_A (aa) \iota_A = \alpha \pi_A \iota_A = \alpha \pi_A \iota_A = \alpha$, whence $\alpha^{-1}I_d I_u = 1_I_A$. \qed

Proposition 18

$$
J^+_{m,i} \cong Z^{[m-i]} \otimes X^{[i]}$$

$$
J^-_{m,i} \cong Z^{[m-i]} \otimes X^{[i]}
$$

25
Proof. Let $\Sigma_m$ be a symmetric group of permutations of an $m$-set $\{1, \ldots, m\}$. Let $i$ be an integer, such that $1 \leq i \leq m$. The group $\Sigma_i$ may be considered as a subgroup in $\Sigma_m$ consisting of permutations which preserve the set $\{1, \ldots, i\}$, and $\Sigma_{m-i}$ as a subgroup in $\Sigma_m$ of permutations preserving the set $\{i+1, \ldots, m\}$. Then $\Sigma_{m-i} \times \Sigma_i$ is a subgroup in $\Sigma_m$ consisting of the product of permutations acting on the sets $\{1, \ldots, i\}$ and $\{i+1, \ldots, m\}$ in an inner fashion.

The number of elements in $S_i$ is equal to the number $C_{m}^r = \frac{m!}{(m-i)!i!}$, so that it coincides with the number of left cosets in $\Sigma_m$ modulo the subgroup $\Sigma_{m-i} \times \Sigma_i$. For any $S \in S_i$ fix an isomorphism

$$u_S : [Z, X]_S \cong Z^{(m-i)} \otimes X^{(i)}$$

fixing, in fact, a representative $\varsigma_S$ of the corresponding right coset in $\Sigma_m$ modulo the subgroup $\Sigma_{m-i} \times \Sigma_i$. Let

$$u : [Z, X, m-i, i] \rightarrow Z^{(m-i)} \otimes X^{(i)}$$

be a sum of isomorphisms $u_S$, $S \in S_i$, and let

$$d : Z^{(m-i)} \otimes X^{(i)} \rightarrow [Z, X, m-i, i]$$

be a morphism, such that its composition with any projection

$$p_S : [Z, X, m-i, i] \rightarrow [Z, X]_S$$

coincides with the corresponding inverse isomorphism $u_S^{-1}$. Let’s consider the diagram

$$
\begin{array}{ccc}
 [Z, X, m-i, i] & \xrightarrow{u} & Z^{(m-i)} \otimes X^{(i)} \\
 & \downarrow{e_{m,i}^\pm} & \downarrow{d_{m-i}^\pm \otimes d_i^\pm} \\
 [Z, X, m-i, i] & \xrightarrow{d} & Z^{(m-i)} \otimes X^{(i)} \\
\end{array}
$$

Since $\{\varsigma_S\}_{S \in S_i}$ is a set of representatives of the right cosets in $\Sigma_m$ modulo the subgroup $\Sigma_{m-i} \times \Sigma_i$, $\{\varsigma_S^{-1}\}_{S \in S_i}$ is a collection of representatives of the right cosets $\Sigma_m$ modulo $\Sigma_{m-i} \times \Sigma_i$. Then

$$\Sigma_m = \Sigma_m \varsigma_S = \cup_{T \in S_i} \varsigma_T^{-1}(\Sigma_{m-i} \times \Sigma_i) \varsigma_S,$$

so that

$$(m-i)!l! (p_T \circ d \circ (d_{m-i}^\pm \otimes d_i^\pm) \circ u_S) = m! (p_T \circ e_{m,i}^\pm \circ u_S),$$

or, equivalently,

$$p_T \circ d \circ (d_{m-i}^\pm \otimes d_i^\pm) \circ u_S = \frac{m!}{(m-i)!l!} p_T \circ e_{m,i}^\pm \circ u_S,$$

26
for any $S$ and $T$ from $\mathcal{S}_i$. This shows that the above diagram is commutative modulo the scalar $\frac{m!}{(m-i)!i!}$, i.e. 
\[
\frac{m!}{(m-i)!i!} \cdot e_{m,i}^\pm = d \circ (d_{m-i}^\pm \otimes d_i^\pm) \circ u.
\]
Moreover, the composition $ud$, evidently, coincides with the multiplication by $\frac{m!}{(m-i)!i!}$. Now it remains just to apply Lemma 17 and observe that 
\[
im(d_{m-i}^\pm \otimes d_i^\pm) = \nim(d_{m-i}^\pm) \otimes \nim(d_i^\pm).
\]

3.4 Triangle filtrations: the proof of Theorem 1

Let’s recall that the following approach to Theorem 1 was suggested by Uwe Jannsen: build a filtration on wedge (symmetric) powers of vertices in a distinguished triangle, similar to the filtration for powers in a short exact sequence of locally free sheaves of modules on a manifold. Now it remains just to claim that the middle rows in the diagrams from Proposition 16 provide, in fact, the desired filtration.

Let $T$ be an arbitrary triangulated category and let $X$ be an object in $T$. The following definition appears in [Ka99], p. 152 - 153. A finite (decreasing) filtration on $X$ is a sequence of objects $A_{-1}, A_0, A_1, A_2, \ldots$ and of distinguished triangles 
\[\mathcal{F}^i X : A_i \xrightarrow{a_i} A_{i-1} \rightarrow \text{Gr}^i_{\mathcal{F}} X \rightarrow \Sigma A_i,\]
such that 
\[\mathcal{F}^0 X : X \rightarrow X \rightarrow 0 \rightarrow \Sigma X,\]
i.e. $A_{-1} = X$, $A_0 = X$ and $a_0 = 1_X$, and 
\[\mathcal{F}^{m+1} X : 0 \rightarrow A_m \rightarrow A_{m-1} \rightarrow 0,\]
for some natural number $m$, i.e. $A_{m+1} = 0$ and $\text{Gr}^m_{\mathcal{F}} X = A_m$. The last condition may be interpreted also like $\mathcal{F}^{m+1} = 0$. Note that if graded pieces $\text{Gr}^i_{\mathcal{F}} X$ are equal to zero for all $i$, then it follows that $X$ is also trivial.

Proposition 19 Let $T = \text{Ho}(\mathcal{C})$ be a triangulated category, reinforced by an underlying pointed model and monoidal category $\mathcal{C}$ in the sense of Section 2.2, and let 
\[X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X\]
be a distinguished triangle in $T$. Then, for any natural $m$, there exist a finite decreasing filtration $\mathcal{F}^* Y^{(m)}$ on $Y^{(m)}$ and a finite decreasing filtration $\mathcal{F}^* Y^{(m)}$ on $Y^{(m)}$, such that 
\[\text{Gr}^i_{\mathcal{F}} Y^{(m)} \cong Z^{(m-i)} \otimes X^{(i)}\]
and
\[ Gr_Y^i Y^{(m)} \cong Z^{(m-i)} \otimes X^{(i)} \]
for any \( i \), and, in both cases, \( F^{m+1} = 0 \).

**Proof.** Recall that, without loss of generality, applying cofibrant replacement, we may assume that \( f \) is a cofibration and both \( X \) and \( Y \) are cofibrant, so that the above dist. triangle is a cofibration triangle. Then we define a filtration on \( Y^{(m)} \) using distinguished triangles of images of mixed idempotents from Section 3.3. In other words, let
\[
F^0 Y^{(m)} : Y^{(m)} \xrightarrow{1} Y^{(m)} \longrightarrow 0 \longrightarrow \Sigma Y^{(m)},
\]
and let
\[
F^i Y^{(m)} : I_{m,i}^+ \xrightarrow{w_{m,i}^+} I_{m,i-1}^+ \longrightarrow J_{m,i-1}^+ \longrightarrow \Sigma I_{m,i}^+
\]
for all \( i \in \{1, \ldots, m\} \). Also, let
\[
F^{m+1} Y^{(m)} : 0 \longrightarrow I_{m,m}^+ \xrightarrow{1} I_{m,m}^+ \longrightarrow 0
\]
be the last term of the filtration. Then it is easy to see that the first term of this filtration is
\[
F^1 Y^{(m)} : I_{m,1}^+ \xrightarrow{w_{m,1}^+} Y^{(m)} \xrightarrow{g^{(m)}} Z^{(m)} \longrightarrow \Sigma I_{m,1}^+
\]
and the last one is the triangle
\[
F^{m+1} Y^{(m)} : 0 \longrightarrow X^{(m)} \xrightarrow{1} X^{(m)} \longrightarrow 0
\]
By Prop. 18, the graded pieces can be computed by the formula
\[ Gr_Y^i Y^{(m)} = J_{m,i}^+ \cong Z^{(m-i)} \otimes X^{(i)} \]
for any \( i \).

Similarly, we define a filtration on \( Y^{(m)} \):
\[
F^0 Y^{(m)} : Y^{(m)} \xrightarrow{1} Y^{(m)} \longrightarrow 0 \longrightarrow \Sigma Y^{(m)},
\]
\[
F^i Y^{(m)} : I_{m,i}^- \xrightarrow{w_{m,i}^-} I_{m,i-1}^- \longrightarrow J_{m,i-1}^- \longrightarrow \Sigma I_{m,i}^-
\]
for all \( i \in \{1, \ldots, m\} \), and set
\[
F^{m+1} Y^{(m)} : 0 \longrightarrow I_{m,m}^- \xrightarrow{1} I_{m,m}^- \longrightarrow 0
\]
as a last term of the filtration. Then
\[
F^1 Y^{(m)} : I_{m,1}^- \xrightarrow{w_{m,1}^-} Y^{(m)} \xrightarrow{g^{(m)}} Z^{(m)} \longrightarrow \Sigma I_{m,1}^-
\]
and

\[ F^{m+1}E^{[m]} : 0 \to X^{[m]} \xrightarrow{1} X^{[m]} \to 0 \]

The graded pieces:

\[ Gr^i_F E^{[m]} = J_{m,i}^{-} \cong Z^{(m-i)} \otimes X^{[i]} \]

for any \( i \).

Now we can finish the proof of Theorem 1. Assume that \( X \) and \( Z \) in the triangle from Theorem 1 are evenly finite dimensional. It means that \( X^{[t]} = 0 \) and \( Z^{[t]} = 0 \) for some natural \( t \). Then, for a big enough \( m \), all graded pieces \( Gr^i_F E^{[m]} \cong Z^{(m-i)} \otimes X^{[i]} \) of the even filtration are equal to zero. Therefore, \( Y^{[m]} = 0 \). And similarly in the odd case.

As to the the second part of Theorem 1, it is equivalent to the first one. To see this we need just to observe that the shift functor \( \Sigma \) in any \( \mathbb{Q} \)-linear monoidal triangulated category \( T \) carries evenly (oddly) finite dimensional objects into oddly (evenly) finite dimensional objects. This is due to the axioms coding the compatibility of the monoidal and the triangulated structures in \( T \), see A8 in [MVW].

\section{Finite dimensionality of motives of curves over a field}

In this section we prove Theorem 2. The word ”scheme” means a separated scheme of finite type over a field \( k \) and the word ”curve” means a an integral one-dimensional scheme over \( k \). We will also assume that \( char(k) = 0 \).

\subsection{Scalar extension and a splitting lemma}

If \( k \) is algebraically closed, then \( X \) can be considered as a Zariski open subset in a projective curve \( Y \), see [Hart77], p. 105, and [Nag62]. Let \( p : W \to Y \) be a resolution of singularities of \( Y \) and let \( U = p^{-1}(X) \), so that we have a commutative square

\[
\begin{array}{ccc}
W & \xrightarrow{p} & Y \\
\uparrow & & \uparrow \\
U & \xrightarrow{p} & X
\end{array}
\]  

(1)

Let also \( Z = Y - X \) and \( V = W - U \) be complements of Zariski open subsets \( Y \) and \( U \) in the projective curves \( X \) and \( W \) respectively.

29
If $k$ is not algebraically closed, then we may consider the square (1) first over an algebraic closure of $k$ and then take a finite extension $L/k$, such that all varieties and maps in (1), as well as $Z$ and $V$, are defined over $L$, and $L$ contains $\sqrt{-1}$. But, since we work with motives with coefficients in $\mathbb{Q}$, one can use transfer arguments to show that finite dimensionality of the motive $M(X_L)$ in $\text{DM}(L)_{\mathbb{Q}}$ implies finite dimensionality of $M(X)$ in $\text{DM}(k)_{\mathbb{Q}}$. It means that, proving Theorem 2, we may assume, without loss of generality, that all the data in the square (1) is rational over $k$.

We will also need the following useful

**Lemma 20** Let $\mathcal{T}$ be a triangulated category with the shift functor $\Sigma$. Assume that we have a distinguished triangle

$$
A \oplus B \xrightarrow{(a \ b \ c \ d)} A \oplus C \rightarrow D \rightarrow \Sigma(A \oplus B)
$$

where $a$ is an automorphism of the object $A$. Then this triangle is isomorphic to the direct sum of triangles

$$
A \xrightarrow{1} A \rightarrow 0 \rightarrow \Sigma A
$$

and

$$
B \xrightarrow{t} C \rightarrow D \rightarrow \Sigma B
$$

where $t = d - ca^{-1}b$.

**Proof.** This is just a reformulation of Lemma 1.2.4 in [Nee01].

4.2 **The proof of Theorem 2**

We will consider three cases separately: (a) when a curve is projective, but not necessary smooth; (b) when it is not projective, but smooth, and (c) when it is not projective and, probably, not smooth. Then, to prove Theorem 2 we will have just to join all together.

**Proposition 21** Assume that $X$ is projective. Then $M(X)$ is finite dimensional and, moreover,

$$
M(X) \cong \mathbb{Q} \oplus G \oplus \mathbb{Q}(1)[2],
$$

where $G$ is an oddly finite dimensional object in $\text{DM}_{\mathbb{Q}}$.

**Proof.** If $X$ is projective and smooth, then the proposition holds by Th. 6. Assume that $X$ is singular. For simplicity, we will consider the case when $X$ has only one singular point (the other case can be proved by the same
methods, but with more cumbrous formulas). Then $p : U \to X$ contracts points $\{u_1, \ldots, u_n\}$ onto a singular point in $X$. Let

$$Q^\oplus n \to Q \oplus M(U) \to M(X) \to Q^\oplus n[1]$$  \hspace{1cm} (2)

be a blow up distinguished triangle corresponding to the map $p$, see [SV98], Th. 5.2. Note that $Q^\oplus n$ is just a motive of the finite set $\{u_1, \ldots, u_n\}$.

For any $i$ the composition

$$Q \to Q^\oplus n \to Q \oplus M(U) \to Q ,$$

corresponding to the point $u_i$, is an isomorphism. By Lemma 20, the triangle (2) is isomorphic to a direct sum of two distinguished triangles

$$Q^\oplus n-1 \xrightarrow{t} M(U) \to M(X) \to Q[1]^\oplus n-1$$  \hspace{1cm} (3)

and

$$Q \to Q \to 0 \to Q[1] ,$$

In other words, we split the isomorphism $Q \xrightarrow{\sim} Q$ corresponding to a point from $\{u_1, \ldots, u_n\}$, say, to $u_1$.

Since $U$ is smooth projective, we have a decomposition

$$M(U) = Q \oplus M^1(U) \oplus Q(1)[2]$$

induced by some $k$-rational point on $U$. Then we rewrite (3) as follows:

$$Q^\oplus n-1 \xrightarrow{t} Q \oplus M^1(U) \oplus Q(1)[2] \to M(X) \to Q[1]^\oplus n-1$$  \hspace{1cm} (4)

For any $i$, let $\nu_i : Spec(k) \to U$ be the map corresponding to the point $u_i$. Let also $\gamma : U \to Spec(k)$ be the structure map for $U$. If $i > 1$, then the composition

$$Q \to Q^\oplus n-1 \xrightarrow{t} M(U) ,$$

corresponding to the point $u_i$, coincides with the difference $M(\nu_i) - M(\nu_1)$ (here we use the general expression for the morphism $t$ given by Lemma 20).

The projection

$$M(U) = Q \oplus M^1(U) \oplus Q(1)[2] \to Q$$

is, in fact, the morphism $M(\gamma) : M(U) \to M(Spec(k))$. Therefore, for any $u_i, i > 1$, the composition

$$Q \to Q^\oplus n-1 \xrightarrow{t} M(U) \to Q$$

is equal to the difference $M(\gamma \nu_i) - M(\gamma \nu_1)$, which is equal to zero. And, in addition, any map from $Q$ to $Q(1)[2]$ is zero. This shows that the triangle (4) is a direct sum of two distinguished triangles

$$Q^\oplus n-1 \xrightarrow{t} M^1(U) \to G \to Q[1]^\oplus n-1$$  \hspace{1cm} (5)
and
\[0 \rightarrow \mathbb{Q} \oplus \mathbb{Q}(1)[2] \rightarrow \mathbb{Q} \oplus \mathbb{Q}(1)[2] \rightarrow 0.\]

In particular,
\[M(X) = \mathbb{Q} \oplus G \oplus \mathbb{Q}(1)[2].\]

Now recall that \(M^1(U)\) is oddly finite dimensional by Th. Then \(G\) is oddly finite dimensional by Theorem + Theorem

\[\text{Proposition 22} \quad \text{Assume that } U \text{ is not projective, i.e. } V \neq \emptyset. \text{ Then } M(U) \text{ is finite dimensional and, for any decomposition}\]
\[M(U) = \mathbb{Q} \oplus H,\]

the motive \(H\) is oddly finite dimensional.

\[\text{Proof. Since } V \neq \emptyset, \text{ we have the canonical distinguished triangle}\]
\[M(V) \rightarrow M(W) \rightarrow M^c(U) \rightarrow M(V)[1] \quad (6)\]
in \(\text{DM}_\mathbb{Q}\), where the motive \(M(V)\) is just a direct sum \(\mathbb{Q}^\oplus n\) of \(n\) copies of the unit motive \(\mathbb{Q}\). Let
\[M(W) = \mathbb{Q} \oplus M^1(W) \oplus \mathbb{Q}(1)[2]\]
be a decomposition of the motive of the smooth projective curve \(W\) by means of a point from \(V\), say \(v \in V\). Then we rewrite the triangle \((6)\) as follows:
\[\mathbb{Q}^\oplus n \rightarrow \mathbb{Q} \oplus M^1(W) \oplus \mathbb{Q}(1)[2] \rightarrow M^c(U) \rightarrow \mathbb{Q}[1]^\oplus n \quad (7)\]

The composition
\[\mathbb{Q} \rightarrow \mathbb{Q}^\oplus n \rightarrow \mathbb{Q} \oplus M^1(W) \oplus \mathbb{Q}(1)[2] \rightarrow \mathbb{Q},\]
corresponding to the point \(v\), is an isomorphism. Applying Lemma we have that \((7)\) is a direct sum of the distinguished triangles
\[\mathbb{Q}^\oplus n-1 \rightarrow M^1(W) \oplus \mathbb{Q}(1)[2] \rightarrow M^c(U) \rightarrow \mathbb{Q}[1]^\oplus n-1 \quad (8)\]
and
\[\mathbb{Q} \rightarrow \mathbb{Q} \rightarrow 0 \rightarrow \mathbb{Q}[1]\]
Since any map from \(\mathbb{Q}\) into \(\mathbb{Q}(1)[2]\) is zero, the triangle \((8)\) is a direct sum of the triangles
\[\mathbb{Q}^\oplus n-1 \rightarrow M^1(W) \rightarrow N \rightarrow \mathbb{Q}[1]^\oplus n-1 \quad (9)\]
and
\[0 \rightarrow \mathbb{Q}(1)[2] \rightarrow \mathbb{Q}(1)[2] \rightarrow 0,\]

32
so that
\[ M^c(U) \cong N \oplus \mathbb{Q}(1)[2]. \]

Note that \( M^1(W) \) is oddly finite dimensional by Th. 6. Applying Theorem 10 jointly with Theorem 11 to the triangle (9) we claim that the motive \( N \) is oddly finite dimensional.

Furthermore, since \( U \) is a smooth scheme of pure dimension one,
\[ M(U) \cong N^\ast(1)[2] \oplus \mathbb{Q} \]

by [Voe00], Th. 4.3.7 (3), where \( N^\ast \) is a motive dual to \( N \). The dualization is a tensor endofunctor of \( DM_{\mathbb{Q}} \), whence \( N^\ast \) is oddly finite dimensional because \( N \) is so. The motive \( N^\ast(1)[2] \) is oddly finite dimensional as a product of motives with different parities, see Th. 8.

Assume now that we have a splitting \( M(U) = H \oplus Q \). Since \( M(U) \) is finite dimensional as a sum of finite dimensional motives \( N^\ast(1)[2] \) and \( Q \), one can use Proposition 5 to show that \( H \cong N^\ast(1)[2] \).

**Proposition 23** Assume that \( X \) is not projective. Then \( M(X) \) is finite dimensional, and, moreover,
\[ M(X) = \mathbb{Q} \oplus D, \]

where \( D \) is oddly finite dimensional in \( DM_{\mathbb{Q}} \).

**Proof.** If \( X \) is smooth then \( U = X \) and the proposition follows from Proposition 22. Assume that \( X \) is singular. Again, for simplicity, we will prove the proposition in the case when \( X \) has only one singular point. Due to Proposition 22 the present proof is almost the same like the proof of Proposition 21.

We consider a blow up distinguished triangle
\[ \mathbb{Q}^{\oplus n} \rightarrow \mathbb{Q} \oplus M(U) \rightarrow M(X) \rightarrow \mathbb{Q}^{\oplus n}[1] \]

(10)
associated with the contraction \( p \) of points \( \{u_1, \ldots, u_n\} \) onto a singular point of \( X \). By Lemma 20 the triangle (10) splits into two distinguished triangles
\[ \mathbb{Q}^{\oplus n-1} \rightarrow M(U) \rightarrow M(X) \rightarrow \mathbb{Q}[1]^{\oplus n-1} \]

(11)
and
\[ \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow 0 \rightarrow \mathbb{Q}[1], \]

where the last one corresponds to, for example, the point \( u_1 \). Let
\[ M(U) = \mathbb{Q} \oplus \tilde{M}(U) \]

20 recall that DM is rigid.
be a splitting induced by a $k$-rational point on $U$, say, by any point $u_i$. Then, as above, for any $x_i, i > 1$, the composition
\[ \mathbb{Q} \to \mathbb{Q}^{\oplus n - 1} \xrightarrow{t} M(U) = \mathbb{Q} \oplus \tilde{M}(U) \to \mathbb{Q} \]
is equal to zero, whence the triangle (11) is a direct sum of two distinguished triangles
\[ \mathbb{Q}^{\oplus n - 1} \xrightarrow{t} \tilde{M}(U) \to D \to \mathbb{Q}[1]^{\oplus n - 1} \quad (12) \]
and
\[ 0 \to \mathbb{Q} \to \mathbb{Q} \to 0 . \]
In particular,
\[ M(X) = \mathbb{Q} \oplus D . \]
Note that $\tilde{M}(U)$ is oddly finite dimensional by Prop. 22. Then, again, applying Theorem 1 together with Theorem 10 to the triangle (12) we have that $D$ is oddly finite dimensional because $\tilde{M}(U)$ is so.

**4.3 A remark on surfaces**

Let $X$ be a smooth projective surface with $p_g = q = 0$ and let $b_2$ be its second Betti number in the sense of some Weil cohomology theory. Assume that the motive $M(X)$ is finite dimensional. Then it is evenly finite dimensional, and, in fact, it is isomorphic to the direct sum of $\mathbb{Q}$, $b_2$ copies of the Lefschetz motive $\mathbb{Q}(1)[2]$ and its tensor square $\mathbb{Q}(2)[4]$, see [GP02]. Let $U$ be a Zariski open subset in $X$. Theorem 1 and the methods from Section 4.2 allow then to prove that $M(U)$ is also finite dimensional. In particular, if Bloch’s conjecture holds for a smooth projective complex surface $X$ of general type with $p_g = 0$, i.e. $M(X)$ is finite dimensional, see [GP03], then $M(U)$ is finite dimensional for any Zariski open subset $U$ in $X$.

**References**

[AKO02] Y. Andre, B. Kahn and P. O’Sullivan *Nilpotence, radicaux et structure monoidales.* (French) Rend. Sem. Mat. Univ. Padova 108 (2002), 107 - 291.

[Bh80] S. Bloch. *Lectures on algebraic cycles.* Duke Univ. Math. Series IV, 1980.

[De] P. Deligne. *Catégories tensorielles.* Preprint, [www.math.ias.edu/~phares/deligne/preprints.html](http://www.math.ias.edu/~phares/deligne/preprints.html)

\[^{21}\text{where } p_g \text{ is a geometrical genus and } q \text{ is an irregularity of } X\]
[DM1982] P. Deligne and J. Milne *Tannakian categories*. In Hodge Cycles and Shimura Varieties, Lecture Notes in Math. 900, Springer-Verlag, 1982, 101 - 208

[GP02] V. Guletskii, C. Pedrini. *The Chow motive of the Godeaux surface*. In Algebraic Geometry, a volume in memory of Paolo Francia, M.C. Beltrametti, F. Catanese, C. Ciliberto, A. Lanteri and C. Pedrini, editors. Walter de Gruyter, Berlin New York, 2002, 179 - 195

[GP03] V. Guletskii, C. Pedrini. *Finite dimensional motives and the Conjectures of Murre and Beilinson*. Preprint 2003, [www.math.uiuc.edu/K-theory/0617/](http://www.math.uiuc.edu/K-theory/0617/)

[FH91] W. Fulton, J. Harris. *Representation theory, a first course*. Grad. Texts in Math, Springer-Verlag, 1991.

[Hart77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics, Springer-Verlag, 1977.

[Ho99] M. Hovey. *Model Categories*. Math. Surveys and Monographs. Vol. 63. AMS. 1999.

[HSS] M. Hovey, B. Shipley, J. Smith. *Symmetric spectra*. J. Amer. Math. Soc. 13 (2000), no. 1, 149–208.

[Jann92] U. Jannsen. *Motives, numerical equivalence and semi-simplicity*. Inventiones Math. Vol. 107 (1992), pp. 447 - 452.

[Jann94] U. Jannsen, *Motivic Sheaves and Filtrations on Chow Groups*, In ”Motives”, Proc. Symposia in Pure Math. Vol. 55, Part 1 (1994), 245-302.

[Jar00] J.F. Jardine. *Motivic symmetric spectra*. Documenta Mathematica, 2000.

[Jar99] J.F. Jardine. *Simplicial homotopy theory*. Birkhäuser, 1999.

[Jar87] J.F. Jardine. *Simplicial presheaves*. J. Pure Applied Algebra, 47 (1987).

[Ka99] B. Kahn. *Geometrically cellular varieties*. In ”Algebraic K-theory”, Proc. Symposia in Pure Math. Vol. 67 (1999), 149 - 174.

[Kim98] S.-I. Kimura. *Chow groups can be finite dimensional, in some sense*. Preprint 1998, to appear in the Journal of Algebraic Geometry.

[McL71] S. MacLane. *Categories for the working mathematicians*. Grad. Texts in Math. Springer-Verlag, 1971.

[Man68] Yu. I. Manin. *Correspondences, motives and monoidal transformations*. Math. USSR Sb. 6 (1968) 439 - 470.

[May01] J. P. May. *Additivity of traces in triangulated categories*. Adv. Math. 163 (2001), no. 1, 34 - 73.
[Mor1] F. Morel. *An introduction to \(A^1\)-homotopy theory*. Preprint, July 2002. [www.math.jussieu.fr/~morel/](http://www.math.jussieu.fr/~morel/)

[Mor2] F. Morel. *On the motivic \(\pi_0\) of the sphere spectrum*. Preprint, May 2003 [www.math.jussieu.fr/~morel/](http://www.math.jussieu.fr/~morel/)

[Mor3] F. Morel. *On stable \(A^1\)-homotopy groups I, II*. Preprint, in preparation.

[Mor4] F. Morel. Private communication, May 2003.

[Mu90] J. P. Murre. *On the motive of an algebraic surface*. J. für die reine und angew. Math. Bd. 409 (1990), S. 190-204.

[Nag62] M. Nagata. *Imbedding of an abstract variety in a complete variety*. J. Math. Kyoto Univ. 2 (1962), 1 - 10.

[Nee01] A. Neeman. *Triangulated Categories*. Annals of Math. Studies, vol.148. Princeton University Press (2001).

[She74] A. M. Shermenev. *The motive of an abelian variety*. Funct. Anal. 8 (1974), 47 - 53.

[Sch] A. J. Scholl. *Classical motives*. In "Motives", Proc. Symposia in Pure Math. Vol.55, Part 1 (1994), pp.163-187.

[SV98] A. Suslin, V. Voevodsky. *Bloch-Kato conjecture and motivic cohomology with finite coefficients*. The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117 - 189, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

[Voe98] V. Voevodsky. *\(A^1\)-homotopy theory*. Doc. Math. (1998), ICM 1998, Berlin, 417 - 442.

[Voe00] V. Voevodsky. *Triangulated categories of motives over a field*. In: V.Voevodsky, A. Suslin and E. Friedlander. Cycles, Transfers and Motivic Cohomology Theories. Annals of Math. Studies, 143. P.U.P. Princeton, N.J., U.S.A.

[VW99] V. Voevodsky (notes by C. Weibel). *Voevodsky’s Seattle lectures: \(K\)-theory and motivic cohomology*. In "Algebraic \(K\)-theory", Proc. Symposia in Pure Math. Vol. 67 (1999), 283 - 303.

[MVV] C. Mazza, V. Voevodsky, C. Weibel. *Lectures on motivic cohomology*. [www.math.uiuc.edu/K-theory/0486/](http://www.math.uiuc.edu/K-theory/0486/)

[Voi93] C. Voisin. *Trascendental methods in the study of algebraic cycles*. In Algebraic Cycles and Hodge Theory, Lecture Notes in Math. 1594, Springer-Verlag, 1993, 153 - 222

guletskii@im.bas-net.by

INSTITUTE OF MATHEMATICS, SURGANOVA 11, 220072 MINSK, BELARUS