The class of second order quasilinear equations: models, solutions and background of classification

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Abstract. The paper is concerned with the unsteady solutions to the model of mutually penetrating continua and quasilinear hyperbolic modification of the Burgers equation (QHMB). The studies were focused on the peculiar solutions of models in question. On the base of these models and their solutions, the ideas of second order quasilinear models classification were developed.

Keyword: hyperbolic equations, attractors, multivaluedness

Introduction

Nonlinear models for phenomena and systems are the cornerstones in modern physics. The examples of these models are well known, namely the Burgers, KdV, Liouville, nonlinear Shredinger equations and etc. The derivation of analytical solutions for these equations is a challenge for scientists. Therefore, the numerical treatments of such models are developed intensively.

Among the models mentioned above it is worth accentuating the second order in time nonlinear hyperbolic differential equations [1, 2] having a broad applications recently. Note that the partial solutions for these equations had been derived [3].

Due to the significance of considered equation and their solutions, this paper deals with the classification of these equations, definition of general expression of quasilinear models. Models’ solutions obtained via the numerical modelling are analyzed in detail. We also discuss the possible ways of investigations, in particular, combining the concept of dynamical systems (attractors) and artificial intelligence methods (neural networks). The aspects related to the multivaluedness of solutions including symmetries are considered as well.

1 Wave regimes in media with oscillating inclusions

To begin with, let us note that the class of second order quasilinear models is not empty and covers many different models originated from the physics and biology. In particular,
consider the model of mutually penetrating continua which uses for the description of physical processes in complex media [5, 6, 12]. It turned out that this model possesses the specific wave solutions. Consider these solutions in more detail.

The model we are going to deal with has the following form

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} - m\rho \frac{\partial^2 w}{\partial t^2}, \quad \frac{\partial^2 w}{\partial t^2} + \Phi (w - u) = 0,$$

(1)

where \(\rho\) is medium’s density, \(u\) and \(w\) are the displacements of carrying medium and oscillator from the rest state, \(m\rho\) is the density of oscillating continuum. We also use the cubic constitutive equation for the carrying medium

$$\sigma = e_1 u_x e_3 w^3,$$

where \(e_1, e_3\) are the elastic moduli, and relation for applied force \(\Phi(x) = \omega^2 x + \delta x^3\), where \(\omega\) denotes the natural frequency of oscillator, whereas the parameter \(\delta\) appears due to accounting for the cubic term in the expansion of restoring force in a power series.

The traveling wave solutions of model (1) have the following form

$$u = U(s), \quad w = W(s), \quad s = x - Dt,$$

(2)

where the parameter \(D\) is a constant velocity of the wave front.

Inserting (2) into model (1), it is easy to see that the functions \(U\) and \(W\) satisfy the dynamical system

$$D^2 U' = \rho^{-1} \sigma(U') - mD^2 W', \quad W'' + \Omega^2 (W - U) + \delta D^{-2} (W - U)^3 = 0,$$

where \(\Omega = \omega D^{-1}\).

This system can be written in the form

$$W' = \alpha_1 R + \alpha_3 R^3, \quad U' = R, \quad (\alpha_1 + 3\alpha_3 R^2) R' + \Omega^2 (W - U) + \delta D^{-2} (W - U)^3 = 0,$$

(3)

where \(\alpha_1 = \frac{e_1 - D^2 \rho}{m\rho D^2}, \quad \alpha_3 = \frac{e_3}{m\rho D^2} > 0\). Through the report we fix

$$e_1 = \rho = 1, \quad e_3 = 0.5, \quad m = 0.6, \quad \omega = 0.9$$

in numerical treatments.

At first, consider system (3) at \(\delta = 0\). The detail description of phase plane of dynamical system had been done in the paper [9], we thus summarize the main results only.

At \(\alpha_1 < 1\) the phase plane contains three fixed points, whereas at \(\alpha > 1\) the only one fixed point (center) remains. For \(\alpha_1 < 0\), when \(D = 1.2\) is fixed for definiteness, a typical phase portrait is depicted in the Fig.1a. In this case, all fixed points are centers surrounded by periodic orbits. There are separatrices separated the regions with periodic and unbounded trajectories and two lines corresponding to discontinuity of system.

When \(0 < \alpha < 1\), at \(D = 0.9\) for instance, in the phase portrait (Fig.1b) one can distinguish the homoclinic trajectories that go through the origin. The homoclinic loop can be written in the explicit form

$$s - s_0 = \frac{3}{2\Omega} \arcsin \left( \frac{4\alpha_3 R^2 - 3 + 4\alpha_1}{\sqrt{9 - 8\alpha_1}} \right) - \frac{1}{2\Omega} \sqrt{\frac{\alpha_1}{1 - \alpha_1}} \times$$

$$\ln \left( \frac{1}{R^2} + \frac{3\alpha_3 - 4\alpha_1\alpha_3}{4(\alpha_1 - \alpha_1^2)} + \sqrt{\left\{ \frac{1}{R^2} + \frac{3\alpha_3 - 4\alpha_1\alpha_3}{4(\alpha_1 - \alpha_1^2)} \right\}^2 - \frac{\alpha_3^2 (9 - 8\alpha_1)}{16(\alpha_1 - \alpha_1^2)^2}} \right) R^2,$$

(4)
Figure 1: The phase portraits for system (3) at (a): \( \alpha_1 < 0 \) \((D = 1.2)\) and (b): \( 0 < \alpha_1 < 1 \) \((D = 0.9)\).

Solution (4) corresponds to the solitary wave solution with infinite support.

When \( \alpha_1 \) tends to zero, the angles between separatrices of saddle point \( O \) are growing. As a result, at \( \alpha_1 = 0 \) we observe the transformation of solitary wave into the compacton, i.e., solutions with finite support [10, 11]. These orbits are described by the following expressions

\[
U_s' = \begin{cases} 
\sqrt{3} \sin \frac{\Omega_s}{3}, & \text{if } \frac{\Omega_s}{3} \in [0; \pi] \\
0, & \text{if } \frac{\Omega_s}{3} \notin [0; \pi] 
\end{cases}
\]

and

\[
U = \begin{cases} 
0, & \text{if } \frac{\Omega_s}{3} \in [-\infty; 0], \\
\frac{3}{\Omega} \sqrt{\frac{3}{2\alpha_3}} (1 - \cos \frac{\Omega_s}{3}), & \text{if } \frac{\Omega_s}{3} \in (0; \pi], \\
\frac{6}{\Omega} \sqrt{\frac{3}{2\alpha_3}}, & \text{if } \frac{\Omega_s}{3} \in (\pi; \infty).
\end{cases}
\]

As above, there are the periodic orbits enclosed in the homoclinic loops and periodic trajectories \( l \) lying beyond the homoclinic contour.

1.1 Phase diagrams in the model with cubic nonlinearity in the equation of motion for oscillating inclusions

If \( \delta \neq 0 \), then system (3) does not reduce to the dynamical system in the plane \((R; R')\). But the first integral for (3) can still be derived in the form

\[
I = \frac{\mu_1}{2} (W - U)^4 + \mu_2 (W - U)^2 + f(R),
\]

where \( \mu_1 = \delta D^{-2} \), \( \mu_2 = \omega^2 D^{-2} \), \( f(R) = \alpha_3 R^6 + \frac{4\alpha_1 - 3}{2} \alpha_3 R^4 + (\alpha_1^2 - \alpha_1) R^2 \). Since \( dI/ds = 0 \) on the trajectories of system (3), then \( I \equiv \text{const} \).

It is easy to see that expression (5) can be used for splitting system (3). Indeed, solving (5) with respect \( W - U \) we obtain

\[
W - U = \pm \sqrt{\frac{-\mu_2 \pm \sqrt{\mu_2^2 - 2\mu_1 f(R) + 2\mu_1 I}}{\mu_1}}.
\]

This allows us to separate the last equation of (3) from other ones. Unfortunately, the resulting equation cannot be integrated for general case, therefore, let us consider its phase plane structure, which is equivalent to the structure of level curves for function (6).
Figure 2: Left: Position of level curves $I(\delta) = 0$ at different values of $\delta$. Curve 1 is plotted at $\delta = 0$, curve 2 at $\delta_0$, curve 3 at $-2.7 < \delta_0$, curve 4 at $\delta = -1.5 > \delta_0$. Right: Homoclinic trajectories from the left diagram corresponding to $\delta = -2.7 < \delta_0$ (upper panel) and $\delta = -1.5 > \delta_0$ (lower panel).

Now consider the position and the form of homoclinic trajectories when the parameter $\delta$ is varied. Starting from the loop at $\delta = 0$ which coincides with the orbits of Fig. 1b, we see that increasing $\delta$ causes the attenuation of loop’s size along vertical axis. Actually, the level curve consists of the closed curve (homoclinic loops) and unbounded trajectories. If $\delta$ decreases, loop’s size grows, but at $\delta_0$ the additional heterocycle connecting four new saddle points appears. The bifurcational value $\delta_0$ can be derived via analyzing the function (6). Namely, $\delta_0$ corresponds to the moment when different branches of (6) are tangent. This happens when $\mu_2^2 - 2\mu_1 f(R) = 0$. Thus, the condition of contact for two branches leads us to a cubic equation with respect to $R^2$ with zero discriminant. Then $\delta_0 = -\frac{\alpha_2^2 \omega^4}{D^2 (\alpha_1 - 1)^2}$ or $\delta_0 = \frac{27 \alpha_3 \omega_1^4}{D^2 (\alpha_1 - 1)^2}$. The last value of $\delta_0$ is not interesting because four branches of (6) degenerate into two ones forming homoclinic loops.

Considering the first value $\delta_0$, we put $D = 0.9$ and derive $\delta_0 = -2.24$. For $\delta > \delta_0$ we have homoclinic loops placed along horizontal axis accompanied by appearing the unbounded curves in the upper and lower parts of diagram. When $\delta < \delta_0$, the homoclinic loops are placed in the vertical quarters of the phase plane, whereas the unbounded orbits appear at the left and at the right sides of diagram (Fig. 2a). Note that the profiles of the resulting solitary waves are different (Fig. 2b), namely, at $\delta < \delta_0$ the $U$ profile looks like a bell-shape curve, but at $\delta > \delta_0$ it is a kink-like regime.

The homoclinic orbits divide the phase plane into parts filled by closed curves corresponding to the periodic regimes. If we choose $\delta = -1.5 > \delta_0$, we get the typical phase portrait of system (3) plotted in the Fig. 3a. In the portrait two pairs of nontrivial fixed points $A_\pm(\pm Q; 0)$ and $B_\pm(0; \pm \omega/\sqrt{-\delta})$ can be distinguished. Inserting the coordinates of these points into relation (5), we obtain the values of $I_1 = -\frac{(\alpha_1 - 1)^2}{2\alpha_2}$ and $I_2 = \frac{\omega^4}{D^2 |\delta|}$ which allows us to state the conditions of periodic regimes existence. For fixed $\delta = -1.5$, $I_1 = -0.18$ and $I_2 = 0.27$. We thus get that periodic regimes exist if $I_1 < I < I_2$ only.

Now let us choose $\delta = -2.7 < \delta_0$. In this case the homoclinic loop going through the origin is placed along vertical axis and the phase portrait looks like Fig. 3b. As above, we have $I_1 = -0.18$, but $I_2 = 0.15$. 


Figure 3: Phase portraits at $\delta = -1.5(a)$ and $\delta = -2.7(b)$.

1.2 Wave dynamics of model (1)

To model the wave dynamics, we used the three level finite-difference numerical scheme for model (1).

Figure 4: The propagation of solitary waves at $\delta = 0$ starting from the two-arch initial profile.

Solitary waves. To consider the evolution of solitary waves, let us construct the initial data $v_i, q_i, G_i, F_i$ for numerical simulation on the base of homoclinic contour. To do this, we integrate dynamical system (3) with initial data $R(0) = 10^{-8}, Z(0) = 0, s \in [0; L]$ and choose the right homoclinic loop in the phase portrait (Fig. 1b). Then the profiles of $W(s)$, $U(s)$, and $R(s)$ can be derived. Joining the proper arrays, we can build the profile in the form of arch:

$$v = U(ih) \cup U(L - ih), q = u(x + \tau D) = [U(ih) + \tau DR(ih)] \cup [U(L - ih) + \tau DR(L - ih)].$$
The arrays $G$ and $F$ are formed in similar manner. Combining two arches and continuing the steady solutions at the ends of graph, we get more complicated profile. We apply the fixed boundary conditions, i.e. $u(x = 0, t) = v_1$, $u(x = Kh, t) = v_K$, where $K$ is the length of an array.

Starting from the two-arch initial data, we see (Fig. 4) that solitary waves move to each other, vanish during approaching, and appear with negative amplitude and shift of phases. After collision in the zones between waves some ripples are revealed. Secondary collisions of waves are watched also. Note that the simulation of compacton solutions displays similar properties.

![Figure 5: Collision of solitary waves (a) and evolution of periodic wave (b) at $\delta = -0.2$.](image)

Propagation of solitary waves at $\delta \neq 0$ depends on the sign of $\delta$. Numerical simulations show that the collision of waves at $\delta > 0$ is similar to the collision at $\delta = 0$. Behavior of waves after interaction does not change essentially when $\delta < 0$ and close to zero (Fig. 5a). But if $\delta$ is not small, after collision the amplitude of solution is increasing in the place of soliton’s intersection and after a while the solution is destroyed [7].

This suggests that we encounter the unstable interaction of solitary waves or the numerical scheme we used possesses spurious solutions. But if we take half spatial step and increase the scheme parameter $r$ up to 0.8, the scenario of solitary waves collision is not changed qualitatively. Therefore, the assertion on the unstable nature of collision is more preferable.

**Periodic waves.** To simulate the evolution of periodic waves, we take the initial profile corresponding to a periodic orbit (red curve in Fig. 5b) surrounding the homoclinic loop at the phase portrait of Fig. 3a at $\delta = -0.2$ and $D = 0.9$. Due to the periodicity of the problem, the boundary conditions for numerical scheme should be modified [7]. The resulting profile derived after time interval $1200\tau$ has passed is depicted in Fig. 5b with dark green curve. It is obvious that this wave is shifted a distance $1200\tau D$ to the right in a self-similar manner.
2 Unsteady solutions to QHMB

Let us consider the evolution of localized perturbation within the framework of QHMB:

\[
\frac{\tau}{\partial t^2} \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \beta \phi(u) \frac{\partial u}{\partial x} = \mu k(u) \frac{\partial^2 u}{\partial x^2} + \nu \psi(u) \left( \frac{\partial u}{\partial x} \right)^2 + \theta f(u),
\]

(7)

where \( \tau, \alpha, \beta, \mu, \nu, \theta \) are constants.

As an initial data for equation (7) we chose the profile \( u(x, 0) = \exp \left( \frac{x + a}{b} \right) + \exp \left( \frac{x - a}{b} \right) \) (fig. 6a), where \( a = 5, b = 4.5 \). The evolution of this two-hump perturbation is shown in figs. 6b,c,d.

![Graphs of u over time](image)

Figure 6: The initial profile for model (7) and its evolution.

From the analysis of these figures it follows that the number of oscillations increases during evolution. The oscillations have non-regular character but remain localized in spatial domain. Thus, one can call them the “pre-turbulent oscillations” or, due to the localization, the “turbulons”. Let us also note that the appearance of this type solutions was discovered in other numerical simulations of 2D and 3D problems [1].
3 The class of quasilinear equations and their formal description

The presented examples of modelling the carrying processes manifest the variety of solutions of models considered. Analyzing these models in details, one can be convinced that they belong to the second order quasilinear equations, i.e., the relation is a linear with respect to higher derivatives whereas it is nonlinear with respect to lower derivatives, unknown functions, and independent variables. It is obvious that this class of equations is very broad and needs to systematize.

Let us start from the defining the general expression of quasilinear second order models, namely

$$\tau r(u) \frac{\partial^2 u}{\partial t^2} + \alpha s(u) \frac{\partial u}{\partial t} + \beta \varphi(u) \frac{\partial u}{\partial x} = \mu k(u) \frac{\partial^2 u}{\partial x^2} + \nu \psi(u) \left( \frac{\partial u}{\partial x} \right)^2 + \gamma h(u) \frac{\partial^2 u}{\partial x \partial t} +$$

$$+ \xi b(u) \left( \frac{\partial u}{\partial t} \right)^2 + \theta f(u) + \chi I(u, x, t, \pm \Delta t, \pm \Delta x; ...)$$

where $I$ is the source (integral form addmited), $\tau, \alpha, \beta, \mu, \nu, \theta$ are constants, $r, s, k, \psi, h, b, f$ are the specified functions.

Note that, depending on the coefficients and functions, class (8) covers the classical linear equations (elliptic, parabolic and hyperbolic) and well-known nonlinear models (sin-Gordon, Burgers, Liouville, Hopf, hyperbolic modification for the Burgers equation, equations with blow-up solutions).

For future handling the variety of models and their solutions, we need a convenient method for their identification. We propose the following descriptor for such objects:

$$EQ(\tau, \alpha, \beta, \mu, \nu, \gamma, \xi, \theta, \chi;$$

{NONLINEAR_FUNCTION : $r, s, k, \varphi, \psi, h, b, f, I$};

{TITLE_OF_EQUATIONS}; {TYPES_OF_SOLUTIONS})

For instance, the descriptor for QHMB can be chosen in the form

$$EQ(1, 1, 1, 1, 0, 0, 1, 1, 0;$$

{NONLINEAR_FUNCTION : $k, \varphi, \psi, f$};

{QHMB}; {compactons; blow-up; oscillations}).

When the system of equations is considered, the coefficients and functions in (8) should be assumed as matrices. In particular, rewrite system (1) as follows

$$\frac{\partial^2 u}{\partial t^2} = \rho^{-1} \left( e_1 + 3e_3 \left( \frac{\partial u}{\partial x} \right)^2 \right) \frac{\partial^2 u}{\partial x^2} + m \left( \omega^2 (w - u) + \delta (w - u)^3 \right) ,$$

$$\frac{\partial^2 w}{\partial t^2} = \omega^2 (w - u) - \delta (w - u)^3.$$
Then its descriptor is

\[
EQ\left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0, \begin{pmatrix} \rho^{-1} & 0 \\ 0 & 1 \end{pmatrix}, 0, 0, \begin{pmatrix} m & 0 \\ 0 & -1 \end{pmatrix} \right), \hat{0};
\]

\{NONLINEAR\_FUNCTION : 

\[
k = \left( e_1 + 3e_3 \left( \frac{\partial w}{\partial x} \right)^2 \right), 
\]

\[
f = \left( \omega^2(w - u) + \delta(w - u)^3 \right) \}
\]

\{EXAMPLE\_ONE}; \{solitary\_waves; compactones\}.

Remark that the construction of descriptor helps us not only to classify the equations but at the statement of problems. In particular, the Hopf equation can be considered as a limit case of hyperbolic Burgers equation.

Let us also outline the types of solutions of such models. For generality, it is worth to mention that certain equations from class \(^8\) possess interesting solutions, namely, autowave solutions to the Burgers equation, singular solutions to the Liouville equation, blow-up solutions to parabolic and hyperbolic equations with nonlinear sources, solitons for the sin-Gordon equation. Among the solutions to the hyperbolic modification of the Burgers equation we encounter the packets of oscillations, compactons, autowaves.

Another type of solutions is related to the multivaluedness. This means that the solutions have several values in given spatial point at fixed moment of time \(^4\). In particular, the Hopf equation admits the solutions when their profiles “overturn” and becomes multivalued. Similar behavior of wave solutions is observed in the Vakhnenko equation which admits the loop solutions. Studies of such solutions cause the introduction of new class of solutions named the “foldons”. Foldons are the multivalued autowaves. It is important that both the Hopf equation and Vakhnenko equation belong to class \(^8\) and their multivalued solutions have analytical expressions.

Up today we have hardly any methods to treat multivalued models. In this case the group analysis methods seem to be useful. To apply group methods, the multivalued differential equations should be considered as multivalued surfaces (geometric objects). The promising approach of multivalued model studies is concerned with the asymptotic transition from the singlevalued to multivalued models. We encounter this when transform the hyperbolic Burgers equation to the Hopf equation. Another way to treat the multivaluedness is the expanding the inverse scattering transform.

It is worth to mention the development of novel approach dealing with the specific type of inverse problems. The examples considered in the paper are concerned with analyzing the solutions and their dependence on the parameters when the equations subjected to the initial and boundary data are available. Traditionally, such problems have been solved via the trial and error methods to find the self-similar solutions. But the inverse statement of the problem is possible. Indeed, we can take the functions \(\tilde{u}(x, t)\) in such a way that they satisfy equation \(^8\), when the components in \(^8\) are chosen correctly. In general case we would like to have the method when the set of functions \(\tilde{u}_1(x, t), \tilde{u}_2(x, t), \ldots, \tilde{u}_n(x, t)\) satisfies equation \(^8\) subjected to proper set of initial conditions. This problem is similar to approaches in the theory of neural networks. The initial and boundary conditions are regarded as inputs of neural networks, whereas the functions \(\tilde{u}_1(x, t), \tilde{u}_2(x, t), \ldots, \tilde{u}_n(x, t)\) considered as potential solutions are identified as outputs of neural networks. It is obvious that for arbitrary equation with fixed initial conditions the solutions \(u_1(x, t), u_2(x, t), \ldots, u_n(x, t)\)
differ from the “desired” set of function \( \tilde{u}_1(x,t), \tilde{u}_2(x,t), \ldots, \tilde{u}_n(x,t) \). Then one can find the values of equation’s coefficients providing the small deviation \( \Delta = \| \tilde{u}_i(x,t) - u_i(x,t) \| \).

To realize this procedure, the iteration process of approaching to the equation’s coefficients can be applied. This process is an analog of studying process in neural networks. We can use multivalued neural networks in these problems as well.

4 Conclusion

Summarizing, we presented the specific solutions occurred in quasilinear models for the carrying processes in nature. The results obtained were encouraged us to develop the systematic approach to the studies of nonlinear dynamical models within the framework of quasilinear second order equations and their solutions. We outlined new ways to analyze quasi-linear models including the multivalued cases.

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