Probabilistic Analysis of Tournament Organization Systems

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Abstract—In this paper a criteria of comparison of different tournament organization systems in sport-
ing contests is offered; the criteria uses the probability of winning the fairly strongest player. Two prob-
abilistic models have been analyzed. Calculating formulas for estimating the probability and probabil-
ity density of score points gained by one or another player were obtained. Some really used tournament
systems were analyzed with the stochastic modeling method. The available results also provide an
order of objects presenting to experts while organizing the examination by paired comparison. An ana-
lytical estimation of probability of tournament results (or pared comparison) was obtained. In many
cases it allows to avoid a time-consuming procedure of sorting out possible variants.

Keywords: tournament organization, probability, paired comparisons

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INTRODUCTION

The emergence and development of probability theory is largely due to the need for game analysis [1].
Under some conditions imposed on a random element of the game methods of probability theory allow
predicting the average values characterizing game’s results with multiple repetitions. Extensive literature
is devoted to probabilistic methods of assessing the solution of combinatorial problems [2–13]. The solu-
tion of such a problem which is defined as an extreme problem, the criterion of which is defined on a set
of possible systems for processing the results of pair comparisons is discussed below.

The goal of many types of games is the identification  of the relative strength of players. At the same
time, the organization of tournaments can be different for different competitions, different sports, etc. In
some cases, the tournament is held on a round-robin system, in others – on a Cup system or on a “Swiss”
one. In some cases, players are pre-divided into groups with the subsequent holding of the championship
between the winners of the groups, etc. At the same time, the results of paired comparisons are the basis
for the conclusion about the winner in any organization of the tournament [14–19].

This raises the questions:
—How to evaluate the tournament organization?
—What should be the minimum number of games to reveal the strongest player with a given probability
at a given number of players?
—What is the probability that as a result of the tournament, the order of the occupied places by the
players coincides with their actual “force”?
—How does the score rule affect this probability?

Let us say the number of players and the total number of games are specified. Since the goal of a tour-
nament is to identify the strongest player, then intuitively clear that a tournament must be organized so
that the number of games between close in force players should be greater, but the number of games in
which a result can be predicted with a probability which is very close to one should be less.

To analyze a tournament scheme, let us use a probabilistic model of the result of a game of two players,
and the probability of any outcome of a game should depend on their “force.” This indicator can be
entered in different ways: as a probability of a player is in any possible states, as an average value of some
random rate, the comparison of which for two players determines the result of a game, etc.
The attempt to provide analytical and numerical analysis of various tournament systems is made below. At the same time, we initially use the most simple probability model of a player and a rule of scoring points. Then we will generalize the consideration for a model that is closer to reality.

The structure of the tournament is the better when a probability that with a given total number of games \( N \) the player with the greatest “force” will be the winner is higher. Here by the “winner” we mean a player who scores at least as many points as any other player in the tournament.

1. DISCRETE DISTRIBUTION OF PLAYER STATUS AND PROBABILITY DENSITY OF THE NUMBER OF SCORED POINTS

Let the state \( \xi_j \) of \( j \)th player is a random discrete value which takes a value of one with probability \( s_j \) and zero with probability \( 1 - s_j \). The value \( s_j \) will be called the force of \( j \)th player. Player states are independent of each other. Let us mark the total number of players as \( M \).

A tournament is a series of paired comparisons (games), in which states of players are compared. If a state of \( j \)th player is more than a state of \( k \)th one, he gets two points, and his opponent gets zero. If their states are the same then everyone gets one point. The players are ordered in such a way so \( s_1 > s_2 > ... s_M \). The result of the game which means the number of points \( r \) scored by each participant is a random value that is characterized by the probability density \( P(r) \) defined on the set of values \( r = 0, 1, 2 \) points.

The probabilistic nature of the results of each game leads to random errors. A tournament can be considered as a filter that separates the useful signal (a priori distribution of forces) from the interference. This filter is as better as the final placement of players closer to the a priori arrangement of their forces.

Let us order the players by the value \( s_j \) so that \( s_1 > s_2 > s_3 > ... \). The result of a tournament will be called ideal if the places occupied by the players (the number of points scored) are such that \( r_1 \geq r_2 \geq r_3 \), i.e. if the order of places determined by the number of scored points and playing system corresponds to the distribution of forces. The result of a tournament will be called correct if the player having the force \( s_j \) won first place or shared it with other players, i.e. \( r_j \geq r_i \), \( j = 2, ..., M \).

Since the number of points scored by a player at the end of a tournament is random, the ideal or correct results can be expected with some probability \( P(A(M, N)) \) and \( P(B(M, N)) \), where \( N \) — is the total number of games played in the tournament. Now let us focus on determining the probability that the tournament is correct.

We will assume that the system of drawing \( A \) is better than the system \( B \) if

\[
P^A_p(M, N) > P^B_p(M, N). \tag{1}
\]

In this work, a comparison of the playing systems — round-robin, round-robin with preliminary division into groups and elimination, a cup with elimination after each game, etc. — was made.

1.1. The Density of the Number of Points Scored by Results of a Tournament

**Single game.** The state of \( j \)th player is a random value, its expected value is

\[
E_{s_j} = s_j, \tag{2}
\]

and dispersion is

\[
D_{s_j} = s_j(1 - s_j). \tag{3}
\]

Expected value of the number of points scored by \( j \)th player in a game with \( k \)th player is

\[
E_{r_{jk}} = 2P_{jk}(2) + 1P_{jk}(1) + 0P_{jk}(0),
\]

where \( P_{jk}(2) \) is a probability of winning of \( j \)th player, \( P_{jk}(0) \) is a probability of losing of \( j \)th player, \( P_{jk}(1) \) is a probability of draw. The number of points of \( k \)th player \( r_{jk} \) (1) in one game of \( j \)th and \( k \)th players takes the values 0, 1, 2 with the probabilities
Thus, the average number of points scored by \( j \)th player in the game with \( k \)th one and the dispersion \( \sigma_{jk} \) are equal to:

\[
E_{jk} = 1 + s_j - s_k, \quad D_{jk} = s_j(1 - s_j) + s_k(1 - s_k).
\]

For the number of points \( r_j < 0 \) and \( r_j > 2 \) \( P^j(r_j) = 0 \). The sum of the points scored by both players in each game is 2.

When calculating the number of points scored in a tournament by \( j \)th player, it should be taken into account that the player does not play with himself, and therefore he gets no point. The easiest way to put it is to set that

\[
P^j_0(0) = 1, \quad P^j_1(1) = P^j_1(2) = 0.
\]

From now on, we assume that the results of the games are independent of each other. Then the number of points in two games is \( \nu = r_{jk}(2) \), it takes the values from 0 to 4, the density of this value is equal to the convolution

\[
P^2_{jk}(\nu) = \sum_{\mu=0}^{2} P^1_{jk}(\mu)P^1_{jk}(\nu - \mu), \quad \nu = 0, 1, 2, 3, 4.
\]

So, for \( \nu = 0 \)

\[
P^2_{jk}(0) = (P^1_{jk}(0))^2,
\]

for \( \nu = 1 \)

\[
P^2_{jk}(1) = P^1_{jk}(0)P^1_{jk}(1) + P^1_{jk}(1)P^1_{jk}(0) = 2P^1_{jk}(0)P^1_{jk}(1),
\]

and etc.

According to (6) for \( n \) games we have the following recursive equation:

\[
P^n_{jk}(\nu) = \sum_{\mu=0}^{2} P^{n-1}_{jk}(\nu - \mu)P^1_{jk}(\mu), \quad \nu = 0, \ldots, 2n.
\]

As \( n \) is more, then the distribution of the total number of points is closer to the normal discrete distribution law defined on a set of natural numbers.

Since the convolution operation

\[
z(\nu) = x(\nu) * y(\nu) = \sum_{t} x(t)y(\nu - t) = \sum_{t} y(t)x(\nu - t)
\]

will be used repeatedly below, let us recall some of its properties [20, 21]:

(1) The convolution is transposition, i.e. the result does not depend on the order in which the convolution functions are placed under the sign of the sum (see (9)).

(2) The domain \( z \) is a combination of the domains \( x \) and \( y \). For example, if one of the functions is defined for \( \nu = 0, 1, 2 \) and the other for \( \nu = 0, 1, 2, 3 \), then the function \( z_{\nu} \) has the domain \( \nu = 0, 1, \ldots, 5 \).

(3) Since each of the convolution densities is not negative and the sum of its values for all points of the domain is equal to one, then the result of the convolution also corresponds to the same conditions.

(4) The expected value and dispersion of a sum of independent random variables are equal to the sum of expected values and dispersions of each of them. It allows us to find the expected value and dispersion of the distribution which is a result of the convolution.

\subsection*{1.2. Round-Robin Tournament}

Let \( M \) be the number of players. The total number of games is \( N = 0.5M(M - 1) \), and the total number of points scored by all players is \( R = M(M - 1) \). Number of games played by each player is \( n = M - 1 \).
Since the order of the games does not affect the number of scored points, then let us assume that each \( j \)th player plays sequentially with the first, second, etc. up to \( M \)th player.

Let us denote as \( r_j \) the number of points received by \( j \)th player in all \((M-1)\) games of the tournament. The number of points in each meeting is a discrete random value that has a density (4). The density of the number of points after \( m \) games is related to the density of the number of points after \( m-1 \) game by a ratio similar to (8)

\[
P_j^m(r_j) = \sum_{\mu=0}^{2} P_j^{m-1}(r_j - \mu)P_j^1(\mu) = P_j^{m-1} \ast P_j^1, \quad r_j = 0, \ldots, 2m, \tag{10}
\]

where \( \ast \) — convolution operation sign. It is considered that in the last game \( j \)th player plays with \( M \)th one.

In accordance with (4)

\[
P_j^1(0) = s_j(1 - s_j), \]
\[
P_j^1(1) = 1 - (s_j + s_j - 2s_j), \]
\[
P_j^1(2) = s_j(1 - s_j). \tag{11}
\]

The random value \( r_j \) takes integer values in the range from zero to \( 2(M-1) \) and its density is identified recursively with initial conditions (11) by the formula (10):

\[
P_j(r_j) = \sum_{\mu=r_j-2}^{r_j} P_j^{M-2}(\mu)P_j^1(r_j - \mu) = \sum_{\mu=0}^{2} P_j^{M-2}(r_j - \mu)P_j^1(\mu). \tag{12}
\]

The summation limits are determined by the fact that the argument in the function \( P_j^1 \) takes values from zero to two.

The average number of points \( \overline{r}_j \) scored in a single-round tournament by \( j \)th player is equal to the sum

\[
\overline{r}_j = (M - 1)(1 + s_j) - \sum_{k=1,k \neq j}^{M} s_k,
\]
\[
D_{r_j} = (M - 1)s_j(1 - s_j) + \sum_{k=1,k \neq j}^{M} s_k(1 - s_k). \tag{13}
\]

**Calculation of probability of tournament correctness.** Knowing \( P(r_j) \) for all players, we need to find the probability that one player (for certainty the first player) scored no less points than \( j \)th one. The domain of probability densities satisfies the conditions

\[
0 \leq r_i \leq 2(M-1), \quad i = 1, \ldots, M, \tag{14}
\]
\[
\sum_{i=1}^{M} r_i = R = M(M - 1).
\]

Let the first player scored the maximal number of points \( 2(M - 1) \), then, with a probability of one, he scored more than any \( j \)th player. A tournament is knowingly correct if at the end of the tournament \( r_1 \geq 2(M-1) - 1 = 2M - 3 \). Otherwise, when calculating the probability that the first player will score \( r_1 \) points, and at the same time \( j \)th player will score \( r_j \geq r_1 \), we need to take into account the results of the personal game separately since these events are not independent.

Let us denote by \( P_{1j-}(r) \) the density of probability that in all games of the first player, except for his game with the \( j \)th player, the total number of points he scored is \( r \). This density for \( 0 \leq r \leq 2(M - 2) \) is as follows:

\[
\tilde{P}_{1j-}(r) = P_{12}(r) \ast \ldots \ast P_{1(j-1)}(r) \ast P_{1(j+1)}(r) \ast \ldots \ast P_{1M}(r), \tag{15}
\]

where \( P_{ik}(r) \) is the density of the probability of scoring \( r \) points by \( i \)th player in a game with \( k \)th player. The symbol \( \ast \) is used for the convolution operation:
Similarly, there is \( \bar{p}_{ji}(r) \) — a density of points scored by the player in all games with except the game of this player with the first one:

\[
\bar{p}_{ji}(r) = P_{j3}(r) \ast P_{j4}(r) \ast P_{j5}(r) \ast \ldots \ast P_{jM}(r).
\]  

Let us consider 3 possible outcomes of the game between the 1st and \( j \)th players.

(A) Player 1 lost to player \( j \). In this case, the probability \( P_{i}(r) \) that the first player scored \( r \) points for the whole tournament is equal to

\[
P_{i}(r) = \bar{P}_{1j-}(r).
\]

(B) The result of the game between the 1st and \( j \)th players was a draw.

\[
P_{i}(r) = \bar{R}_{1j-}(r - 1).
\]

(C) The player 1 beat the player \( j \):

\[
P_{i}(r) = \bar{P}_{1j-}(r - 2).
\]

Let us find the probability of \( P(\eta > r_j) \) that the number of points \( \eta \) scored by the 1st player will be no less than the number of points \( r_j \) scored by the player \( j \). We will write expressions for this probability depending on the results of the personal game of the 1st and \( j \)th players:

(A)

\[
P_{a}(\eta \geq r_j) = \sum_{r_j = 2}^{2(M-2)} \left( \bar{P}_{1j-}(r_j) \sum_{r_j = 2}^{r_j} \bar{p}_{ji}(r_j - 2) \right),
\]

(B)

\[
P_{b}(\eta \geq r_j) = \sum_{r_j = 1}^{2(M-2)+1} \left( \bar{P}_{1j-}(r_j - 1) \sum_{r_j = 1}^{r_j} \bar{p}_{ji}(r_j - 1) \right),
\]

(C)

\[
P_{c}(\eta \geq r_j) = \sum_{r_j = 0}^{2(M-2)+2} \left( \bar{P}_{1j-}(r_j - 2) \sum_{r_j = 0}^{r_j} \bar{p}_{ji}(r_j) \right).
\]

The probability \( P(\eta \geq r_j) \) of the fact that the first player will not be lower than \( j \)th player in the final results table, is calculated as the weighted average of the recorded values taking into account the probabilities of personal game results:

\[
P(\eta \geq r_j) = P_{i}(0)P_{a}(\eta \geq r_j) + P_{i}(1)P_{b}(\eta \geq r_j) + P_{i}(2)P_{c}(\eta \geq r_j),
\]

where \( P_{i}(r) \) is the probability density of the fact that the 1st player will score \( r \) points in the game with \( j \)th player.

However, multiplying the probabilities that the first player gaining more points than \( j \)th one, where \( j = 2, 3, \ldots, M \), can not be used for calculating the probability that the tournament is correct since each of these probabilities depends on whether the first player scored more points than \( k \)th. It is true that: *The tournament is correct if for any possible number of \( \eta_j \) points scored by the first player, none of the other players scored more points.* The minimum number of points that the first player must score in order to share the first place with a probability greater than zero is \( M - 1 \). In this case, all the games will end in a draw and he will share the first place with the rest of the participants. If, as mentioned above, he gets \( 2M - 3 \) points, then he will obviously be the first or will share the first place with one of the remaining participants of the tournament. So in this case the probability that the tournament is correct is equal to one.

It is necessary to note that the total number of points scored by all participants in the tournament is fixed and equal to \( M(M - 1) \). This means that if the first player scored \( \eta_j \) points, then the remaining players will score \( M(M - 1) - \eta_j \) points. Let us denote by \( P_{\sigma} \) the probability that all players in total will score some
It is equal to the convolution of the densities of the number of points scored by the second, third, ..., \( M \)th players based on tournament results:

\[
P_\sigma(r_\sigma) = P_2(r_2) \ast P_3(r_3) \ast \ldots \ast P_M(r_M).
\]

The probability of tournament correctness in general case can be expressed by formula

\[
P_p = \sum_{r_\sigma=M-1}^{2(M-1)} P_2(r_2) \sum_{r_{j2} \in V, j=2,...,M} P_3(r_3) \ldots P_M(r_M).
\]  

(19)

In this case, the set \( V \) of acceptable values of arguments \( r_j, j = 2, 3, \ldots, M \) is defined by the constraints:

\[
\sum_{j=2}^{M} r_j = M(M-1) - r_\sigma, \quad r_j \leq r_\sigma \forall j.
\]

The practical use of formula (19) requires a cumbersome enumeration and at \( M > 10 \) becomes too expensive in the calculation time. Below are formulas that allow you to get an estimate of the probability of the tournament is correct with a much lower labor intensity.

Since \( r_\sigma = M(M-1) - r_1 \), then the probability \( P_\sigma \) is determined for a fixed number of players for each value \( r_1 \). Therefore, with regard to such a replacement, we will write \( P_\sigma(r_1) \). To evaluate the probability of tournament correctness, we get an expression

\[
P_p(M, N) = \sum_{r_\sigma=M-1}^{2M-3} P_2(r_2) \prod_{i=2}^{M} \left[ 1 - \sum_{r_{i+1}=r_{i+1}}^{2M-3} P_i(r_i) \right] P_2(r_2).
\]  

(20)

Here random events “no player except the first one will score more points than \( r_1 \)” and “all players, except the first one, will get in total \( r_\sigma \) points” are assumed to be independent. Therefore, the evaluation of the correctness of the tournament thus found is higher than the value found by the formula (19), however, the calculation does not require an enumeration.

To assess the accuracy of the last formula, numerical experiments were carried out for various variants of the initial distribution of forces \( s_1, s_2, \ldots, s_M \). Table 1 shows the results of these calculations for various a priori distribution of forces. It shows the probability \( P_p \) obtained by the formula (19), assessment of the probability \( P_p(M, N) \) found by the formula (20), and the error \( \Delta, \% \) obtained from the formula

\[
\Delta, \% = \frac{P_p(M, N) - P_p}{P_p} \times 100\%.
\]

These calculations have shown that the assessment (20) has an error about 4%, which allows to use it to compare different tournament systems.
2. PLAYING WITH PRELIMINARY SPLIT INTO GROUPS AND KNOCK-OUT

Let us evaluate the structure of the tournament by the probability of its correctness \( P_p \). Let the number of participants \( M \) be divided into 4 and the teams are divided into subgroups by the rule: the first subgroup consists of odd numbers, and the second — of even numbers. The playing is held in subgroups on a round-robin system. Half of the teams that took the last places in the subgroups are cut off, the remaining teams play the championship on a round-robin system. Total number of games

\[
N_{ng} = \frac{3}{4} M \left( \frac{M}{2} - 1 \right).
\]

(21)

The number of games \( N_{ng} \) is less than for a round-robin system. Let us compare the value of \( P_p \) for two playing systems at \( M = 4 \), when the system with selection becomes a Cup system, and the number of games in a round-robin system \( N = 6 \) exceeds twice the number of games in a Cup system.

For a round-robin system, \( P_p \) can be found by the formulas obtained in the previous section. For the system with the selection, the subgroups include teams with the forces \( s_1, s_2, s_3, s_4 \). To win in the draw, the player with the force \( s_1 \) must win in the subgroup, and then win in the final. Since these events are independent, the probability of victory is equal to

\[
P_p = P_{i1} P_{f} = s_1 (1 - s_3) P_{i2}.
\]

(22)

In turn, the probability of winning in the final game is equal to

\[
P_{if} = P_{i2} \cdot P_{24} + P_{i4} \cdot P_{42} = s_2 (1 - s_2) s_3 (1 - s_4) + s_1 (1 - s_3) s_4 (1 - s_2) = s_1 (1 - s_2) (1 - s_4) (s_2 + s_4).
\]

(23)

After the substitution of this expression in (22), we get

\[
P_p = s_1^2 (1 - s_1) (1 - s_3) (1 - s_4) (s_2 + s_4).
\]

(24)

Since the number of games in a round-robin system is twice as large than with a Cup system, you can conduct not one, but two games between players, and consider the one who scored more points in two games as the winner. Then the probability of the first player will win in his subgroup (the probability of winning more than two points) is determined by the formula (7)

\[
\bar{p}_{i1} = s_1 (1 - s_3) [1 + 2(1 - s_1 - s_3 + 2 s_3 s_4)].
\]

(25)

The probability of reaching the final of the second and fourth players is equal, respectively

\[
\bar{p}_{24} = s_2 (1 - s_4) [1 + 2(1 - s_2 - s_4 + 2 s_3 s_4)],
\]

\[
\bar{p}_{42} = 1 - \bar{p}_{24}.
\]

(26)

The probability of winning the first player in the game is equal to

\[
\bar{p}_p = \bar{p}_{i1} \left( \bar{p}_{24} \cdot \bar{p}_{i2} + \bar{p}_{42} \cdot \bar{p}_{i4} \right)
\]

\[
= \bar{p}_{i1} \left\{ \bar{p}_{24} \cdot s_1 (1 - s_2) [1 + 2(1 - s_1 - s_2 + 2 s_3 s_4)] + \bar{p}_{42} \cdot s_1 (1 - s_4) [1 + 2(1 - s_1 - s_4 + 2 s_3 s_4)] \right\}.
\]

(27)

Here the line corresponds to the probability of winning in two games.

Numerical calculations, which were performed according to the formula (27) and with an equal distribution of the players’ forces, showed that with such an organization the probability of the tournament’s correctness is equal to \( \bar{p}_p = 0.645 \). At the same time, the probability of a correct tournament in a round-robin system, which was found using the formula (20) is equal \( P_p = 0.670 \).

3. PLAYER’S STATE IS A RANDOM VALUE WITH A CONTINUOUS DISTRIBUTION

The case where the force of each \( j \)th player \( s_j \) is the probability that his state is equal to one is considered above. This circumstance facilitated the calculation of paired comparisons results. Let us consider what will change if the density of state \( \xi_j \) is continuous with expected value \( \bar{s}_j \) and dispersion \( D_j = \sigma_j^2 \). The player \( j \) wins the player \( k \) and gets two points when in the pair comparison \( \xi_j > \xi_k \). Since the densities are continuous, then \( \xi_j = \xi_k \) with any small probability, and thus a tie is excluded. The formulas for calculating the density of the amount of points scored in the tournament remain correct because they use only the results of paired comparisons.
The probability of winning of the player in a single game with the (see (4)) is equal to

\[ P_{jk}(2) = \int_{0}^{\infty} f_{j}(\xi)F_{k}(\xi)d\xi, \quad P_{jk}(0) = 1 - P_{jk}(2) = P_{kj}(2), \]  

(28)

where

\[ F_{k}(\xi) = \int_{0}^{\xi} f_{k}(\xi)d\xi, \]  

(29)

Fig. 1. The density of the states of the th and th players (a) and the number of scored points by the th player on the result of their game (b) for \( i < j \).

Fig. 2. The density of the number of points scored by the th player in two games with the th player when \( i < j \).

Fig. 3. Densities of states of the th and th players for the normal law of distribution when \( s_i < s_j \) and various \( D_i \) and \( D_j \).

The probability of winning of the th player in a single game with the (see (4)) is equal to

\[ P_{jk}(2) = \int_{0}^{\infty} f_{j}(\xi)F_{k}(\xi)d\xi, \quad P_{jk}(0) = 1 - P_{jk}(2) = P_{kj}(2), \]  

(28)

where

\[ F_{k}(\xi) = \int_{0}^{\xi} f_{k}(\xi)d\xi, \]  

(29)
These expressions allow analyzing the outcome of tournaments and find the probability of their “correctness” when the number of games is fixed.

If $f_j(\xi_j)$ is the normal law of distribution

$$f_j(\xi_j) = \frac{1}{\sqrt{2\pi D_j}} e^{-\frac{(\xi_j - \eta_j)^2}{2D_j}},$$

as $f_k(\xi_k)$, then the probability of winning $j$th player from $k$th one (losing $j$th player from $k$th) is equal to

$$P_{jk}(2) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi D_j}} e^{-\frac{(\xi_j - \eta_j)^2}{2D_j}} \left[ \Phi\left(\frac{\xi_j - \eta_k}{\sigma_k \sqrt{2}}\right) - \Phi\left(\frac{-\eta_k}{\sigma_k \sqrt{2}}\right) \right] d\xi_j,$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} \exp{-t^2} dt$$

is the Laplace function that is defined in tables [21].

In the case of the uniform law of distribution, its value is zero for each player outside of the segment $\eta_k - \delta_k$, $\eta_k + \delta_k$ and is equal to $\frac{1}{2\delta_k}$ within that segment.
Let us compare several systems, in which the total number of games is the same. In addition to the round-robin system, three more playing systems were considered.

**System 1.** All players are divided into 2 subgroups that are identical in average strength. The round-robin tournament is held in each subgroup. Then half of the players of each subgroup that took the last places form a new subgroup, in which another round-robin tournament is held and its winner is revealed. This winner and the remaining best players from each subgroup conduct the fourth round tournament. By the total amount of points scored in this tournament, 2 players who took 1 and 2 places are revealed. There are several games between them so the total number of games played is equal to the number of games in a round-robin system, so it is \( M(M - 1) \). The winner is the player who scored the maximum number of points in these games.

**System 2.** All players are divided into 2 identical subgroups, in each subgroup a round-robin tournament is held. One player who took the last place in each subgroup is removed from the tournament. The remaining players conduct a round-robin tournament, after which the two best players hold a series of final games, the number of which is determined from the condition that the total number of games should be equal to the number of games in a round-robin system.

**System 3.** All players are divided into 2 identical subgroups, in each of them a round-robin tournament is held. Half of the players of each subgroup who took the last places are removed from the tournament. The remaining players hold another two round-robin tournaments, then the two best players hold a series of final meetings, the number of which is such a way so the total number of games is equal to the number of games in the round-robin system.

In addition, two variants of the initial division of all players into subgroups were considered. In the first case (method A), the split was conducted by taking into account the force of the players \( s_i \); All players were ordered by their force. After that the first player in the list went to subgroup 1, the second – to subgroup 2, the third – to 1, the fourth – to 2, and so on. The second method (B) simulates the widely used division into subgroups in sports competitions by drawing of lots, so the players are divided into the groups randomly.

The distribution of the player’s forces was made using two approaches: An equal distribution of forces, when the difference between \( s_j \) is the same, and a random distribution of forces. In the case of the random distribution two different approaches were used: “Random 1” – when the forces of the strongest players are close to each other, and “Random 2” – when the force of the strongest player is significantly greater than the force of the second player.

The following results were obtained by stochastic modeling. For the given values of players’ force \( s_1, s_2, \ldots, s_M \), several iterations were performed. On each iteration a table of the results of all games was build and the player who took the first place was determined. The random value \( \zeta_j \) takes the value 1 if the event (in this case the strongest player took the first place) occurred, or 0 otherwise. The estimation of probability \( P^* \) of the correctness of a tournament was defined as the estimation of the expected value \( E\zeta \) of a random value \( \zeta \).

The number of iterations \( K \) was determined from the conditions that with the given probability \( Q \) the deviation of the estimation \( P^* \) from the actual probability of tournament correctness was less than the given accuracy \( \varepsilon \).

\[
P(\{P^* - \rho\} < \varepsilon) = Q.
\]

For these conditions, \( K \) is defined as follows [22]:

\[
K \leq \left( \frac{\sigma_\zeta^2}{\varepsilon} \left[ \Phi^{-1}(Q/2) \right]^2 \right)^{-2},
\]

where \( \sigma_\zeta \) is an estimation of the standard deviation of a random value \( \zeta \), \( \Phi^{-1} \) – inverse Laplace function.
In order to find the first estimation of the standard deviation $\sigma_{\zeta}$, at first, there was a certain number $k_0$ of values $\zeta_i$, after which the estimation for $\sigma_{\zeta}$ and the value of $K$ were calculated. These values were further refined after each iteration.

The results of the comparative experiments are provided in Table 2. It shows probabilities that the strongest player will win first place. In the experiments:

—number of players $M = 8$;
—total number of games in the tournament $N = 56$;
—level of trust $Q = 0.999$;
—accuracy $\epsilon = 0.001$;
—initial number of iterations $k_0 = 1000000$.

Analyzing the results of the experiments, the following conclusions can be drawn:

1. The selection of a playing system can have a significant impact on the effectiveness of a tournament.
2. The third system of organizing tournaments for all variants of a priori distribution of forces is the best among the systems considered.

5. CONCLUSIONS

A criterion that makes it possible to assess the way in which tournaments are organized with a given number of participants and a limited number of games was offered. Expressions for analytical estimation of the quality of tournament organization were received. Statistical modeling for calculating the criterion of the correctness of one or another system of the tournament organization was carried out.

All the obtained recommendations, but in other terms, relate to the task of determining the best product by means of an expertise in which an expert chooses the best one in a series of paired comparisons.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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