The quasi-two-body decays $B_{(s)} \to (D_{(s)}, \bar{D}_{(s)}) \rho \to (D_{(s)}, \bar{D}_{(s)}) \pi \pi$ in the perturbative QCD factorization approach

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(Dated: September 18, 2017)

In this paper, we studied the $B_{(s)} \to (D_{(s)}, \bar{D}_{(s)}) \rho \to (D_{(s)}, \bar{D}_{(s)}) \pi \pi$ decays by employing a framework for the quasi-two-body decays in the perturbative QCD (PQCD) factorization approach. We use the two-pion distribution amplitudes $\Phi_{\pi \pi}$, which contains both resonant and nonresonant contributions from the pion pair, to describe the final state interactions (FSIs) between the pions in the resonant region. We found that (a) for all considered decays, the PQCD predictions for their branching ratios based on the quasi-two-body and the two-body framework agree well with each other due to $B(\rho \to \pi \pi) \approx 100\%$; For $B^+ \to D^0 \rho^+ \to D^0 \pi^+ \pi^0$ and other four considered decay modes, the PQCD predictions do agree well with the measured values within errors; (b) the great difference between the PQCD predictions for $B(B \to D \rho \to D \pi \pi)$ and $B(B \to D \rho \to D \pi \pi)$ can be understood by the strong CKM suppression factor $R_{CKM} \approx 3 \times 10^{-4}$; (c) for the $B_s \to D \rho \to D \pi \pi$ and $B_s \to D \rho \to D \pi \pi$ decays, however, the PQCD predictions of $R_{s1} \approx 0.13$ and $R_{s2} \approx 0.14$ do agree very well with the moderate CKM suppression factor $R^{s}_{CKM} \approx 0.14$; and (d) the PQCD predictions for the ratios $R_{D \rho}$ and the strong phase difference $\cos \delta_{D \rho}$ of the three $B \to D \rho$ decay modes agree well with the LHCb measurements within one standard deviation.

PACS numbers: 13.25.Hw, 12.38.Bx, 14.40.Nd

I. INTRODUCTION

The hadronic two-body and three-body $B$ meson decays provide rich information for studying the heavy flavor physics in and beyond the Standard Model (SM), but the three-body decays are clearly more complicated than the two-body cases due to the involvement of the resonant and nonresonant contributions, as well as the possible FSIs. In experiments, the BaBar [1–5], Belle [6–11] and LHCb Collaborations [12–17] have reported their measurements for the branching ratios and CP violations of some hadronic three-body $B/B_s$ meson decay modes. The large localized
CP asymmetries in a number of such decay channels, specifically, have raised great interests in theoretical studies [18–48].

It is fair to say that the nonresonant contributions in the three-body $B$-meson decays are quite difficult to calculate, since we can not separate the nonresonant contributions from the resonant ones clearly and have no good methods to estimate the non-resonant contributions reliably [45]. In the so called “quasi-two-body” approximation, the two-body scattering and all possible interactions between the two involved particles are included but the interactions between the third particle (usually referred to as bachelor) and the pair of mesons are ignored.

In a recent work [23], the authors studied the quasi-two-body decay $B \rightarrow K \rho \rightarrow K \pi \pi$ by employing the PQCD factorization approach based on $k_T$ factorization theorem. The resonant and nonresonant contributions between two final pions are parameterized into the time-like pion form factors involved in the $P$-wave two-pion distribution amplitudes $\phi^{(2)}_{\pi\pi}$, the PQCD predictions for the branching ratios and the CP-violating asymmetries as presented in Ref. [23] are in good agreement with currently available experimental measurements. By analyzing the distribution of the branching ratios and direct CP asymmetries in the pion-pair invariant mass $w$, they also found that the main portion of the branching ratios lies in the region around the pole mass of the $\rho$ resonance as expected. For $B^+ \rightarrow K^+ \rho^0 \rightarrow K^+ \pi^+ \pi^-$ decay, for example, its differential decay rate $dB/dw$ exhibits peak at $w = m_\rho$, the central value of its branching ratio is $B = 2.46 \times 10^{-6}$ in the range of $w = [m_\rho - \Gamma_\rho, m_\rho + \Gamma_\rho]$, which is around 72% of the total decay rate $B = 3.42 \times 10^{-6}$ [23]. In Ref. [24], we extend this work [23] to the cases $B \rightarrow P \rho \rightarrow P \pi \pi$, where the $P$ standing for kaon and other light pseudoscalar mesons $(\pi, \eta, \eta')$ as well. For all $B \rightarrow P \rho \rightarrow P \pi \pi$ decays studied in Refs. [23, 24], the PQCD predictions for their branching ratios of those quasi-two-body modes in the three-body and the two-body frameworks are well consistent with each other. This fact is generally expected since $B(\rho \rightarrow \pi \pi) \approx 100\%$, and it does suggest that the PQCD approach is a consistent theory for exclusive hadronic $B$ meson decays [23, 24].

Besides the above mentioned $B$ meson charmless decays, the two-body hadronic charmed decays $B \rightarrow DM$ ($M$ denotes the light pseudoscalar and vector mesons) have also been studied by many authors based on rather different theoretical approaches [49–63]. Since such charm hadronic $B$ decay modes involve the tree operators $O_{1,2}$ only, there are much less theoretical uncertainties from the relevant QCD dynamics.

In the PQCD factorization approach, the factorization for $B \rightarrow DM$ decays was approved at the leading order of $m_D/m_B$ expansion [53, 54]. Many two-body charmed decays $B_{(s)} \rightarrow D_{(s)}^{(*)}(P, V, T)$ have been studied in Refs. [55–63]. In Refs. [60, 61], specifically, the authors analyzed $B_{(s)} \rightarrow D_{(s)}^{(*)}(P, V)$ decays. By using the data of six $B \rightarrow DP$ channels available at 2008, they firstly made a selection for the expression of $D/D_s$ meson wave functions by $\chi^2$ fit, presented the PQCD predictions for the considered charmed $B$ decays and found that most of them agreed very well with experiments. Some predictions for $B_s$ decays [60], such as $B(\bar{B}_s^0 \rightarrow D_s^+ K^-) \approx 1.70 \times 10^{-4}$ and $B(\bar{B}_s^0 \rightarrow D_s^{*+} \pi^-) \approx 18.9 \times 10^{-4}$, are confirmed by later experimental measurements [64, 65].

In this paper, we will extend previous studies as presented in Refs. [23, 24] to the cases of the quasi-two-body charmed decays $B_{(s)} \rightarrow (D_{(s)}, \bar{D}_{(s)}) \rho \rightarrow (D_{(s)}, \bar{D}_{(s)}) \pi \pi$ by employing the PQCD factorization approach, and to examine if the PQCD approach are applicable to the cases involving a charmed meson as one of the three final state mesons. Since only the tree diagrams contribute to the considered decay processes, there is no direct CP asymmetry for these decays in

\[ B(\bar{B}_s^0 \rightarrow D_s^+ \pi^-) = (24_{-6}^{+7}) \times 10^{-4}. \]

\[ B(\bar{B}_s^0 \rightarrow D_s^{*+} \pi^-) = (24_{-6}^{+7}) \times 10^{-4}. \]
the standard model. We consider the decays $B(s) \to \bar{D}(s)\rho \to \bar{D}(s)\pi\pi$, where $B(s) = (B^+, B^0, B_s^0)$, $\bar{D}(s) = (D^0, D^-, D_s^-)$ and $\rho = (\rho^+, \rho^-, \rho^0)$.

This paper is organized as follows. In Sec. II, we give a brief introduction for the theoretical framework, calculate and present the decay amplitudes. The numerical values, some discussions from the different framework of the two-body decay and the quasi-two-body decay. In the light-cone coordinates, the $B$ meson momentum $p_B$, the $\rho$ meson momentum $p$, and the $D$ meson momentum $p_3$ are chosen as

$$p_B = \frac{m_B}{\sqrt{2}}(1, 1, 0_T), \quad p = \frac{m_B}{\sqrt{2}}(1 - r^2, \eta, 0_T), \quad p_3 = \frac{m_B}{\sqrt{2}}(r^2, 1 - \eta, 0_T),$$

where $O_{1,2}(\mu)$ are the tree operators, $C_{1,2}(\mu)$ are the Wilson coefficients, $q = (d, s)$ and $V_{ij}$ are the CKM matrix elements. The typical Feynman diagrams for the decays $B(s) \to \bar{D}(s)\rho \to \bar{D}(s)\pi\pi$ and $B(s) \to \bar{D}(s)\rho \to \bar{D}(s)\pi\pi$ are shown in the Fig. 1 and 2, respectively.

FIG. 1. Typical Feynman diagrams for the decays $B(s) \to \bar{D}(s)\rho \to \bar{D}(s)\pi\pi$, where $B(s) = (B^+, B^0, B_s^0)$, $\bar{D}(s) = (D^0, D^-, D_s^-)$ and $\rho = (\rho^+, \rho^-, \rho^0)$.

II. THE THEORETICAL FRAMEWORK

For $B(s) \to \bar{D}(s)\rho \to \bar{D}(s)\pi\pi$ decays and the CKM-suppressed $B(s) \to D(s)\rho \to D(s)\pi\pi$ decays, the effective Hamiltonian are of the form

$$\mathcal{H}_{eff} = \begin{cases} g_{\rho\bar{D}\pi\pi} V_{d\bar{s}} V_{cq} \left[ C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu) \right], & \text{for } B(s) \to \bar{D}(s)\rho \to \bar{D}(s)\pi\pi \text{ decays,} \\ g_{\rho\bar{D}\pi\pi} V_{sb\bar{u}} V_{cq} \left[ C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu) \right], & \text{for } B(s) \to D(s)\rho \to D(s)\pi\pi \text{ decays,} \end{cases} \quad (1)$$

where $O_{1,2}(\mu)$ are the tree operators, $C_{1,2}(\mu)$ are the Wilson coefficients, $q = (d, s)$ and $V_{ij}$ are the CKM matrix elements. The typical Feynman diagrams for the decays $B(s) \to \bar{D}(s)\rho \to \bar{D}(s)\pi\pi$ and $B(s) \to \bar{D}(s)\rho \to \bar{D}(s)\pi\pi$ are shown in the Fig. 1 and 2, respectively.
where \( m_B \) denotes the \( B \) meson mass, the variable \( \eta \) is defined as \( \eta = w^2/[(1 - r^2)m_B^2] \) with the mass ratio \( r = m_D/m_B \) and the invariant mass squared \( w^2 = p^2 = m^2(\pi\pi) \) of the pion pair. The momentum of the light quark in the \( B \) meson, \( \rho \) and \( D \) meson are denoted as \( k_B, k \) and \( k_3 \) respectively

\[
k_B = (0, x_B \frac{m_B}{\sqrt{2}}, k_{BT}), \quad k = (z \frac{(1 - r^2)m_B}{\sqrt{2}}, 0, k_t), \quad k_3 = (0, x_3 \frac{(1 - \eta)m_B}{\sqrt{2}}, k_{3T}),
\]

where the momentum fraction \( x_B, z \) and \( x_3 \) run between zero and unity. The momentum of pion pair \( p_1, p_2 \) are expressed as

\[
p_1^+ = \zeta p^+, \quad p_2^+ = (1 - \zeta)p^+, \quad p_1^- = (1 - \zeta)p^-, \quad p_2^- = \zeta p^-,
\]

in which \( \zeta \) is the momentum fraction for one of the pion pair and \( p = p_1 + p_2 \). For the decays involving \( B_s \) or \( D_s \) mesons, one can get the relevant definitions or expressions from those as given in Eqs. (2-4) by simple replacement \( m_B \to m_{B_s} \) and \( m_D \to m_{D_s} \). All discussions for the \( B \) meson decays are applicable for the cases of \( B_s \) decays, unless explained specifically.

The total decay amplitude \( \mathcal{A} \) for the quasi-two-body decays \( B \to D\rho \to D\pi\pi \) in PQCD approach can be written conceptually as the convolution \([18, 19, 23]\)

\[
\mathcal{A} = \Phi_B \otimes H \otimes \Phi_D \otimes \Phi_{\pi\pi}^{I=1},
\]

where the hard kernel \( H \) describes the dynamics of the strong and electroweak interactions in the decays, the functions \( \Phi_B, \Phi_D \) and \( \Phi_{\pi\pi} \) are wave functions for the \( B \) meson, the final state \( D/\bar{D} \) meson and the pair of pions, which absorb the non-perturbative dynamics in the process.

The wave function of \( B \) meson can be written as the one for example in Ref. [66]

\[
\Phi_B = \frac{i}{\sqrt{2N_c}}(\bar{p}_B + m_B)\gamma_5\phi_B(\mathbf{k}_1),
\]

where the \( B \)-meson distribution amplitude \( \phi_B \) is of the form

\[
\phi_B(x, b) = N_B x^2(1-x)^2 \exp \left[-\frac{M_B^2 x^2}{2\omega_B^2} - \frac{1}{2}(\omega_B b)^2 \right],
\]

where the normalization factor \( N_B \) is determined through the relation \( \int_0^1 dx \phi_B(x, 0) = f_B/(2\sqrt{6}) \), \( \omega_B \) is a free parameter and one usually take \( \omega_B = 0.40 \pm 0.04 \) GeV and \( \omega_{B_s} = 0.50 \pm 0.05 \) GeV in the numerical calculations \([18, 19, 23]\) for the case of \( B \) and \( B_s \) decays respectively.

**FIG. 2.** Typical Feynman diagrams for the CKM-suppressed decays \( B(s) \to D(s)\rho \to D(s)\pi\pi \), where \( D(s) = (D^0, D^+, D_s^+) \).
For $D$ meson, in the heavy quark limit, the two-parton LCDAs can be written as [53, 54, 60–63]

$$
\langle D(p_3)|q_\alpha(z)c_\beta(0)|0\rangle = \frac{i}{\sqrt{2N_c}} \int_0^1 dx e^{ixp_3z} [\gamma_5(p_3 + m_D)\phi_D(x, b)]_{\alpha\beta},
$$

(8)

where

$$
\phi_D(x, b) = \frac{1}{2\sqrt{2N_c}} f_D 6x(1-x) [1 + C_D(1-2x)] \exp \left[ -\frac{\omega^2 b^2}{2} \right],
$$

(9)

with $C_D = 0.5 \pm 0.1, \omega = 0.1$ GeV and $f_D = 204.6$ MeV for $D$ meson, and $C_{D_s} = 0.4 \pm 0.1, \omega = 0.2$ GeV and $f_{D_s} = 257.5$ MeV for $D_s$ meson. In the above models, $x$ is the momentum fraction of the light quark in $D$ ($D_s$) meson.

For the quasi-two-body decays, the dynamics associated with the pair of the pion mesons are factorized into two-meson distribution amplitudes for the following two reasons [18, 19, 67, 68]:

1) It is not practical to make a direct evaluation for the hard $b$-quark decay kernels containing two virtual gluons at leading order due to the enormous number of diagrams, while the contribution from such kinematic region is in fact not important due to the power-suppression;

2) The dominant contribution comes from the region where the involved two energetic mesons are almost collimating to each other and having an invariant mass below $O(\Lambda m_B)$ ($\Lambda = m_B - m_b$).

The longitudinal distribution amplitude of $\rho$ meson is defined as the two-pion distribution amplitudes [23]:

$$
\Phi^{I=1}_\pi = \frac{1}{\sqrt{2N_c}} \left[ \phi_0(z, \zeta, w^2) + w\phi_1(z, \zeta, w^2) + \frac{\phi_1^0 - \phi_1^1}{w(2\zeta - 1)} \phi_1(z, \zeta, w^2) \right],
$$

(10)

where

$$
\phi_0(z, \zeta, w^2) = \frac{3F_\pi(w^2)}{2\sqrt{2N_c}} z(1-z) \left[ 1 + a_2^0 C_2^{3/2}(t) \right] P_1(2\zeta - 1),
$$

$$
\phi_1(z, \zeta, w^2) = \frac{3F_\pi(w^2)}{2\sqrt{2N_c}} (1-2z) \left[ 1 + a_2^2(1 - 10z + 10z^2) \right] P_1(2\zeta - 1),
$$

$$
\phi_1^I(z, \zeta, w^2) = \frac{3F_\pi(w^2)}{2\sqrt{2N_c}} (1-2z)^2 \left[ 1 + a_2^2 C_2^{3/2}(t) \right] P_1(2\zeta - 1),
$$

(11)

with the Gegenbauer polynomial $C_2^{3/2}(t) = \frac{5}{2}(5t^2 - 1), t = 2z - 1$, the Legendre polynomial $P_1(2\zeta - 1) = 2\zeta - 1$. We choose the Gegenbauer coefficient $a_2^0 = 0.25 \pm 0.10, a_2^2 = -0.60 \pm 0.20$ and $a_2^1 = 0.75 \pm 0.25$ determined in Ref. [23].

The strong interactions between the $\rho$ meson and the pion pair with the inclusion of the elastic rescattering among the two pions are factorized into $F_\pi$. The form factor $F_\pi$ for $\rho$ meson is chosen as [69]

$$
F_\pi(w^2) = \frac{1}{1 + \sum c_i} \left\{ BW^G_\rho(w^2, m_\rho, \Gamma_\rho) \frac{1 + c_\omega BW^G_{\omega}(w^2, m_\omega, \Gamma_\omega)}{1 + c_\omega} + \sum c_i BW^G_i(w^2, m_i, \Gamma_i) \right\},
$$

(12)
with

$$BW^{KS}_\omega(s, m, \Gamma) = \frac{m^2}{m^2 - s - i m \Gamma},$$

$$BW^{GS}_{\rho,i}(s, m, \Gamma) = \frac{m^2 \left[ 1 + d(m) \Gamma/m \right]}{m^2 - s + f(s, m, \Gamma) - i m \Gamma(s, m, \Gamma)},$$

(13)

here $BW^{KS}_\omega(s, m, \Gamma)$ are the ordinary Breit-Wigner (BW) function [70] for $\omega$ meson, while $BW^{GS}_{\rho,i}$ are the functions for the $\rho$ and its excited states $i = \left( \rho' \left( 1450 \right), \rho'' \left( 1700 \right), \rho''' \left( 2254 \right) \right)$ as described by the Gounaris-Sakurai (GS) model [71]. The explicit expressions of the functions $\Gamma(s, m, \Gamma), d(m)$ and $f(s, m, \Gamma)$ in Eq. (13) can be found in Eqs. (29-31) of Ref. [69], other relevant parameters such as $c_\omega$ and $c_i$ in Eq. (12-13) can also be found in the Appendix of Ref. [69]. Here, we single out the part of $\rho$ meson component. The equivalence between the framework with the $\rho$ meson propagator and the present one with the two-pion distribution amplitudes leads to the relations [23]:

$$F_\pi^\rho(w^2) \approx \frac{g_{\rho\pi\pi} w f_\rho}{D_\rho(w^2)}, \quad F_{s,i}^\rho(w^2) \approx \frac{g_{\rho\pi\pi} w f_{\rho}^T}{D_\rho(w^2)},$$

(14)

where $g_{\rho\pi\pi}$ describes the strength of the $\rho \to \pi\pi$ transition, $D_\rho$ represents the denominator of the BW function for the $\rho$ resonance and $f_\rho(f_{\rho}^T)$ is associated with the normalization of the twist-2 (twist-3) $\rho$ meson distribution amplitudes ($f_\rho = 0.216 \text{ GeV}$ [72, 73] and $f_{\rho}^T = 0.184 \text{ GeV}$ [74] numerically).

After the integration for $\zeta$, the differential decay rate is written as

$$\frac{d\mathcal{B}}{dw^2} = \frac{\tau_B |\vec{p}_\pi| |\vec{p}_D|}{96\pi^3 m_B^3} |A|^2,$$

(15)

where $\tau_B$ is the mean lifetime of $B$ meson, and $|\vec{p}_\pi|$ and $|\vec{p}_D|$ denote the magnitudes of the $\pi$ and $D$ momenta in the center-of-mass frame of the pion pair,

$$|\vec{p}_\pi| = \frac{1}{2} \sqrt{w^2 - 4m_\pi^2},$$

$$|\vec{p}_D| = \frac{1}{2} \sqrt{[(m_B^2 - m_D^2)^2 - 2(m_B^2 + m_D^2)w^2 + w^4]/w^2}. (16)$$

For the considered $B(s) \to \bar{D}(s) \rho \to \bar{D}(s) \pi\pi$ decays, the analytic formula for the corresponding decay amplitudes are of the following form:

$$\mathcal{A}(B^+ \to \bar{D}^0 \rho^+ (\rho^+ \to \pi^+ \pi^0)) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [a_1 F_{e\rho}^{LL} + C_2 M_{e\rho}^{LL} + a_2 F_{eD}^{LL} + C_1 M_{eD}^{LL}],$$

(17)

$$\mathcal{A}(B^0 \to D^- \rho^+ (\rho^+ \to \pi^+ \pi^0)) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [a_1 F_{a\rho}^{LL} + C_2 M_{a\rho}^{LL} + a_2 F_{aD}^{LL} + C_1 M_{aD}^{LL}],$$

(18)

$$\mathcal{A}(B^0 \to \bar{D}^0 \rho^0 (\rho^0 \to \pi^+ \pi^-)) = \frac{G_F}{2} V_{cb}^* V_{ud} [a_1 (-F_{e\rho}^{LL} + F_{a\rho}^{LL}) + C_2 (-M_{e\rho}^{LL} + M_{a\rho}^{LL})],$$

(19)

$$\mathcal{A}(B_s^0 \to D^- \rho^+ (\rho^+ \to \pi^+ \pi^0)) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} [a_1 F_{a\rho}^{LL} + C_2 M_{a\rho}^{LL}],$$

(20)

$$\mathcal{A}(B_s^0 \to \bar{D}^0 \rho^0 (\rho^0 \to \pi^+ \pi^-)) = \frac{G_F}{2} V_{cb}^* V_{us} [a_1 F_{a\rho}^{LL} + C_2 M_{a\rho}^{LL}],$$

(21)

$$\mathcal{A}(B_s^0 \to D_s^- \rho^+ (\rho^+ \to \pi^+ \pi^0)) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [a_2 F_{eD}^{LL} + C_1 M_{eD}^{LL}].$$

(22)
where the Wilson coefficients \( a_1 = C_1 + C_2 / 3 \) and \( a_2 = C_2 + C_1 / 3 \), the individual amplitude \( F^{LLL}_\rho \), \( M^{LLL}_\rho \), \( F^{LL}_\rho \), \( M^{LL}_\rho \), \( F^{LL}_\rho \) and \( M^{LL}_\rho \) denote the amplitudes from different sub-diagrams in Fig. 1.

\[
F^{LLL}_\rho = -8\pi C_F m_B^4 f_B \int_0^1 dx_B dz \int_0^{1/A} b_B db_B db \phi_B(x_B, b_B) \times \left\{ \left[ r^2 ((1 - \eta)(1 + z) + (1 - (1 - \eta)r^2)z) - (1 - \eta)(1 + z) \phi_0(z) - \sqrt{\eta(1 - r^2)} \right] x - [(1 - \eta)(1 - 2z(1 - r^2))(\phi_s(z) + \phi_t(z)) + r^2 (\phi_s(z) - \phi_t(z))] E_a(t_a) h_a(x_B, z, b, B) \times S_t(z) - [(1 - r^2)[r^2(x_B - \eta) - (1 - \eta) \eta] \phi_0(z) + 2\sqrt{\eta(1 - r^2)}(1 - \eta)(1 - r^2) - r^2(x_B - \eta)] \phi_s(z) E_c(t_b) h_b(x_B, z, b, b) S_t(|x_B - \eta|) \right\} , \tag{23}
\]

\[
M^{LLL}_\rho = 32\pi C_F m_B^4 / \sqrt{6} \int_0^1 dx_B dz dx_3 \int_0^{1/A} b_B db_B db_3 db \phi_B(x_B, b_B) \phi_D(x_3, b_3) \times \left\{ \left[ ((1 - r^2)(r^2 - \eta) - (1 - \eta)) (x_B - \eta z - (1 - \eta)(1 - x_3)) + r(\eta r - r_\rho (\eta + r_\rho) \times (r^2 - \eta) \phi_0(z) + \sqrt{\eta(1 - r^2)}(r^2(1 - \eta) x_3 + x_B)(\phi_s(z) + \phi_t(z)) - (1 - \eta) \times (1 - r^2) z (\phi_s(z) - \phi_t(z)) - 2r((1 - \eta)r - 2r_\rho) \phi_s(z)] E_a(t_a) h_e(x_B, z, x_3, b, b_3) \times [(1 - \eta - r^2(1 - 2\eta))(1 - r^2)z + (1 - \eta)x_3 - x_B \phi_0(z) + \sqrt{\eta(1 - r^2)} \times [r^2(x_B - (1 - \eta)x_3)(\phi_s(z) - \phi_t(z)) - (1 - \eta)(1 - r^2)z(\phi_s(z) + \phi_t(z))] \times E_n(t_d) h_d(x_B, x_3, b, b_3, b_3), \right\} , \tag{24}
\]

\[
F^{LL}_\rho = -8\pi C_F m_B^4 f_B \int_0^1 dx_B dz \int_0^{1/A} b_3 db_3 db \phi_D(x_3, b_3) \times \left\{ \left[ ([r^2((1 - r^2)(1 - 2rr_\rho) - \eta(1 - 2r^2 + 2rr_\rho) - (1 - \eta)(1 - r^2)z)] \phi_0(z) + \sqrt{\eta(1 - r^2)}(r_\rho(1 - \eta) \phi_s(z) + \phi_t(z)) + r(2(1 - r^2)z + rr_\rho)(\phi_s(z) - \phi_t(z)) \times -4r \phi_s(z)] E_a(t_a) h_e(x_B, z, x_3, b, b_3) S_t(z) + \left[ ((r^2 - 1)((1 - \eta - r^2) \eta(y(1 - \eta)^2 x_3)] \phi_0(z) + 2r \sqrt{\eta(1 - r^2)}(1 + \eta)(1 + x_3) - 2\eta x_3 - r^2) \phi_s(z)] E_a(t_f) h_f(x_B, x_3, b, b) \times S_t(|\eta x_3 - 1 - x_3|) \right\} , \tag{25}
\]

\[
M^{LL}_\rho = 32\pi C_F m_B^4 / \sqrt{6} \int_0^1 dx_B dz dx_3 \int_0^{1/A} b_B db_B db_3 db \phi_B(x_B, b_B) \phi_D(x_3, b_3) \times \left\{ \left[ r^2((1 - r^2) + \eta(1 - \eta) + (1 - \eta - r^2(1 - \eta) + (1 - \eta - r^2(r^2 - \eta))) (x_3(\eta - 1) - x_B + \eta(z - 2))] \phi_0(z) + r \sqrt{\eta(1 - r^2)}(x_B - (1 - \eta)(1 - x_3})(\phi_s(z) - \phi_t(z)) - (1 - z)^2 + z(\phi_s(z) + \phi_t(z)) + 4\phi_s(z)] E_n(t_g) h_g(x_B, x_3, b, b_3, b) + (r^2 - \eta - 1) (r^2((1 - \eta)(1 - x_3) + x_B - \eta) + (1 - \eta)(1 - z)] \phi_0(z) + r \sqrt{\eta(1 - r^2)}(1 - \eta)(1 - x_3) + x_B) \phi_s(z) + \phi_t(z) \times + (1 - \eta)(1 - z)] \phi_0(z) + r \sqrt{\eta(1 - r^2)}(1 - \eta)(1 - x_3) + x_B) \phi_s(z) + \phi_t(z) \times + (1 - \eta)(1 - z)] \phi_0(z) + 2\phi_s(z)] E_n(t_h) h_h(x_B, x_3, b, b_3, b_3) \right\} , \tag{26}
\]

\[
F^{LL}_\rho = -8\pi C_F m_B^4 F^\rho_{\pi}(w^2) \int_0^1 dx_B dz \int_0^{1/A} b_B db_B db_3 db \phi_B(x_B, b_B) \phi_D(x_3, b_3) \times \left\{ \left[ (1 + r)(r^2 + (1 - \eta)(1 - x_3(1 - r)) + (1 - \eta - 2r)] E_e(t_m) h_m(x_B, x_3, b, b_3) \times S_t(x_3) + [(1 - \eta - r^2)((1 - 2r)(1 + r_\rho) - \eta) + \eta x_B(1 - \eta - 2r)] \right\} E_e(t_n) h_n(x_B, x_3, b_3, b_3) S_t(x_B) \right\} , \tag{27}
\]
\[ M_{cD}^{LL} = 32\pi C_F m_B^4 / \sqrt{6} \int_0^1 dx_B dz dx_3 \int_0^{1/\Lambda} db dB db dB \phi_B(x_B, b_B) \phi_D(x_3, b_3) \phi_0(z) \]
\[ \times \left\{ \left[ (x_3 + z - 2)(\eta r(1 - \eta r) + r^2(1 - r)(1 + r - \eta r)) + (1 - x_B - z)(1 - \eta) \right] \times (1 + \eta - r^2) + (1 - \eta)(1 + r - \eta) \left[ E_n(t_o) h_o(x_B, z, x_3, b_B, b) \right. \right. \]
\[ \left. \left. - [(1 - r)((1 - \eta)(1 + r) - \eta r)((1 - r^2)z - x_B) + (1 - \eta)x_3(1 - \eta + r(2\eta + r - 1) - 1)) \right] E_n(t_p) h_p(x_B, z, x_3, b_B, b) \right\}. \] (28)

The hard function \( h_i \) with \( i = (a, b, c, d, e, f, g, h, m, n, o, p) \) are obtained from the Fourier transformation of the hard kernels. The explicit expressions of \( h_i \) and the hard scales \( t_i \) will be given in Appendix. The six decay amplitudes (\( F_{ep}^{LL}, \cdots, M_{ep}^{LL} \)) as given in Eqs. (23-28) are different from those as given in Eqs. (31-33,38-40) of Ref. [60]: the terms proportional to \( r^2, r\eta \) or higher order factors are all kept here but neglected in Ref. [60], although the resulted changes in the PQCD predictions for branching ratios are not large in magnitude.

The evolution factors \( E_e(t), E_o(t) \) and \( E_n(t) \) in above equations are written as the form

\[ E_e(t) = \alpha_s(t) \exp[-S_B(t) - S_\rho(t)], \]
\[ E_o(t) = \alpha_s(t) \exp[-S_D(t) - S_\rho(t)], \]
\[ E_n(t) = \alpha_s(t) \exp[-S_B(t) - S_\rho(t) - S_D(t)], \] (29)

where the Sudakov exponents are defined as

\[ S_B = S(x_B \frac{m_B}{\sqrt{2}}, b_B) + \frac{5}{3} \int_{1/b_B}^t \frac{d\bar{u}}{\bar{u}} \gamma_q(\alpha_s(\bar{u})), \] (30)
\[ S_\rho = S(z(1 - r^2) \frac{m_B}{\sqrt{2}}, b) + S((1 - z)(1 - r^2) \frac{m_B}{\sqrt{2}}, b) + 2 \int_{1/b}^t \frac{d\bar{u}}{\bar{u}} \gamma_q(\alpha_s(\bar{u})), \] (31)
\[ S_D = S(x_3(1 - \eta) \frac{m_B}{\sqrt{2}}, b_3) + 2 \int_{1/b_3}^t \frac{d\bar{u}}{\bar{u}} \gamma_q(\alpha_s(\bar{u})), \] (32)

with the quark anomalous dimension \( \gamma_q = -\alpha_s/\pi \). The explicit expressions of the functions \((S(x_B m_B / \sqrt{2}, b_B), \cdots)\) can be found for example in Appendix A of Ref. [75]. The threshold resummation factor \( S_t(x) \) in Eqs. (23,25) is of the form [75]:

\[ S_t(x) = \frac{2^{1+2c} \Gamma(3/2 + c)}{\sqrt{\pi} \Gamma(1+c)} (3 - x) c \]. (33)

We here choose \( c = 0.3 \) in numerical calculations.

For the eight CKM suppressed \( B_{(s)} \rightarrow D_{(s)} \rho \rightarrow D_{(s)} \pi \pi \) decays, on the other hand, we can find the same set of analytic formula for the decay amplitudes and relevant functions by following the same procedure as for \( B_{(s)} \rightarrow D_{(s)} \rho \rightarrow D_{(s)} \pi \pi \) decays. The explicit expressions of all relevant decay amplitudes and functions will be given in Appendix.

### III. NUMERICAL RESULTS

Besides the quantities specified in previous sections, the following input parameters (the masses, decay constants and QCD scale are in units of GeV) will be used in the numerical
calculations \[64\]:

\[
\Lambda_{M^2S}^{(f=4)} = 0.25, \quad m_B = 5.280, \quad m_{B_s} = 5.367, \quad m_{D^0} = 1.870, \quad m_{D^0/\rho^0} = 1.865, \\
m_{D^0_\pm} = 1.968, \quad m_{\rho} = 0.775, \quad m_{\pi^\pm} = 0.140, \quad m_{\pi^0} = 0.135, \quad m_b = 4.8, \quad m_c = 1.27, \\
f_B = 0.19, \quad f_{B_s} = 0.236, \quad \tau_{\rho^0} = 1.520 \text{ ps}, \quad \tau_{B^+} = 1.638 \text{ ps}, \quad \tau_{B_s} = 1.510 \text{ ps}. \quad (34)
\]

For the Wolfenstein parameters \((A, \lambda, \bar{\rho}, \bar{\eta})\) we use the following values: \(A = 0.811 \pm 0.026, \quad \lambda = 0.22506 \pm 0.00050, \quad \bar{\rho} = 0.124^{+0.019}_{-0.016}, \quad \bar{\eta} = 0.356 \pm 0.011\).

We calculate the branching ratios of \(B(s) \to (D(s), \bar{D}(s))\rho \to (D(s), \bar{D}(s))\pi\pi\) decays in the quasi-two-body and the two-body framework in the PQCD factorization approach by using the same set of the Gegenbauer moments. Taking \(B^+ \to \bar{D}^0\rho^+ \to \bar{D}^0\pi^+\pi^0\) as one example, we find the PQCD prediction for its branching ratio in the quasi-two-body framework:

\[
\mathcal{B}(B^+ \to \bar{D}^0\rho^+ \to \bar{D}^0\pi^+\pi^0) = \left[115^{+58}_{-36}(\omega_B)\times\frac{7}{8}(a_\rho) \pm 8(C_D)\right] \times 10^{-4},
\]

where the first error comes from the uncertainties of the input parameters \(\omega_B = 0.40 \pm 0.04\) or \(\omega_{B_s} = 0.50 \pm 0.05\); the second one is induced by the uncertainties of the Gegenbauer moments: \(a_\rho^2 = -0.60 \pm 0.20, a_\rho^2 = 0.25 \pm 0.10\) and \(a_\rho^2 = 0.75 \pm 0.25\); the third one is due to \(C_D = 0.5 \pm 0.1\) or \(C_{D_s} = 0.4 \pm 0.1\).

**TABLE I.** The PQCD predictions for the branching ratios (in units of \(10^{-4}\)) of \(B(s) \to \bar{D}(s)\rho \to \bar{D}(s)\pi\pi\) decays in the quasi-two-body (second column) and the two-body (third column) framework. We also list those currently available measured values \[64, 65\] of the two-body cases and the central values of the theoretical predictions as given in Ref. [60] and Ref. [52].

| Decays | Quasi-two-body | Two-body | Data \[64, 65\] | Two-body[60] | FAT [52] |
|--------|----------------|----------|----------------|-------------|----------|
| \(\mathcal{B}(B^+ \to D^0\rho^+ \to D^0\pi^+\pi^0)\) | \(115^{+58}_{-36}\) | \(116^{+56}_{-37}\) | \(134 \pm 18\) | \(111\) | \(105\) |
| \(\mathcal{B}(B^0 \to D^-\rho^+ \to D^-\pi^+\pi^0)\) | \(82.3^{+49.2}_{-29.0}\) | \(88.2^{+49.7}_{-30.7}\) | \(79 \pm 13\) | \(67.0\) | \(65.3\) |
| \(\mathcal{B}(B^0 \to \bar{D}^0\rho^0 \to \bar{D}^0\pi^+\pi^-)\) | \(1.30^{+1.24}_{-0.90}\) | \(1.23^{+0.90}_{-0.64}\) | \(2.9 \pm 1.1\) | \(1.90\) | \(2.60\) |
| \(\mathcal{B}(B^0_\rho \to D^0\rho^0 \to D^0\pi^+\pi^-)\) | \(0.026^{+0.010}_{-0.006}\) | \(0.022^{+0.006}_{-0.005}\) | \(-\) | \(0.042\) | \(0.010\) |
| \(\mathcal{B}(B^0_\rho \to D^-\rho^+ \to D^-\pi^+\pi^0)\) | \(0.051^{+0.022}_{-0.014}\) | \(0.044^{+0.012}_{-0.011}\) | \(-\) | \(0.079\) | \(0.019\) |
| \(\mathcal{B}(B^0_\rho \to D_s^0\rho^+ \to D_s^0\pi^+\pi^0)\) | \(77.2^{+40.2}_{-25.6}\) | \(79.5^{+30.6}_{-26.3}\) | \(85 \pm 21\) | \(47.0\) | \(78.6\) |

The numerical results for all fourteen considered decay modes are listed in Table I and II, where the individual errors have been added in quadrature. As a comparison, we list the PQCD predictions for the branching ratios in both the quasi-two-body framework and the ordinary two-body framework \(^2\). The central values of the theoretical predictions obtained by employing the factorization-assisted topological-amplitude (FAT) approach [52] also be listed in the last column of the two Tables. For some decay modes considered here, the currently available experimental measurements of the two-body cases \(B(s) \to (\bar{D}(s), D(s))\rho\) as given in PDG 2016 [64] or HFAG 2016 [65] are included in Table I and II as well.

Since only the tree diagrams contribute to the considered processes here, there is no direct CP asymmetry for these considered decays. From the calculation and numerical results as listed in Table I and II, we have the following observations:

\(^2\) We calculated the PQCD predictions in the ordinary two-body framework as listed in the third column of Table I and II. These numerical results are obtained by using the formulae as given in Refs. [60, 61] but with the updated Gegenbauer moments and other input parameters, and they agree well with those as given in Refs. [60, 61] (fifth column).
TABLE II. The PQCD predictions for the branching ratios of the CKM suppressed $B_s \to D_s \rho \to D_s \pi\pi$ decays in the quasi-two-body (second column) and the two-body (third column) framework. We also list those currently available measured values \cite{s43,s40} of the two-body cases and the central values of the theoretical predictions as given in Ref. \cite{s48} and Ref. \cite{s47}.

| Decays                        | Quasi-two-body | Two-body | Data \cite{s43,s40} | Two-body \cite{s48} | FAT \cite{s47} |
|-------------------------------|----------------|----------|---------------------|---------------------|---------------|
| $B(B^+ \to D^0 \rho^+ \to D^0 \pi^+ \pi^0)(10^{-7})$ | $0.50^{+0.22}_{-0.14}$ | $0.53^{+0.26}_{-0.14}$ | –                   | 0.93                | 4.80          |
| $B(B^0 \to D^+ \rho^- \to D^+ \pi^- \pi^0)(10^{-7})$ | $7.63^{+5.92}_{-3.08}$ | $9.45^{+6.48}_{-4.89}$ | –                   | 12.7                | 9.40          |
| $B(B^0 \to D^0 \rho^0 \to D^0 \pi^+ \pi^-)(10^{-7})$ | $0.13^{+0.09}_{-0.08}$ | $0.13^{+0.10}_{-0.05}$ | –                   | 0.34                | 1.20          |
| $B(B^+ \to D^+ \rho^0 \to D^+ \pi^+ \pi^-)(10^{-7})$ | $5.33^{+3.60}_{-2.65}$ | $5.99^{+3.93}_{-2.91}$ | –                   | 7.50                | 3.30          |
| $B(B_s^0 \to D^0 \rho^0 \to D^0 \pi^+ \pi^-)(10^{-7})$ | $3.41^{+1.03}_{-0.75}$ | $3.13^{+0.98}_{-0.64}$ | –                   | 1.90                | 1.30          |
| $B(B_s^0 \to D^+ \rho^- \to D^+ \pi^- \pi^0)(10^{-7})$ | $6.88^{+1.98}_{-1.58}$ | $6.30^{+1.96}_{-1.29}$ | –                   | 3.70                | 2.50          |
| $B(B^+ \to D_s^+ \rho^0 \to D_s^+ \pi^+ \pi^-)(10^{-5})$ | $1.52^{+0.11}_{-0.82}$ | $1.82^{+1.19}_{-0.91}$ | $< 30$              | 1.94                | 1.68          |
| $B(B^0 \to D_s^+ \rho^- \to D_s^+ \pi^- \pi^0)(10^{-5})$ | $2.82^{+2.04}_{-1.53}$ | $3.37^{+2.19}_{-1.63}$ | $1.1 \pm 0.9$       | 3.59                | 3.12          |

FIG. 3. The PQCD predictions for the $w$-dependence of the differential branching ratios for the $B^+ \to D^0 \rho^+ \to D^0 \pi^+ \pi^0$ (red curve) and $B^0 \to D^- \rho^+ \to D^- \pi^+ \pi^0$ (blue curve).

(1) Although the PQCD formalism are rather different for the case of the quasi-two-body and the two-body decay analysis, the PQCD predictions for the branching ratios of all considered decays obtained in both frameworks do agree very well with each other, as generally expected. The reason indeed very simple: $B(\rho \to \pi\pi) \approx 100\%$. Consequently, there exist a simple relation between the decay rate of the same kinds of decays evaluated in the
quasi-two-body and the ordinary two-body framework:

\[ \mathcal{B}(B_s \to \bar{D}_s \rho \to \bar{D}_s \pi \pi) = \mathcal{B}(B_s \to \bar{D}_s \rho) \cdot \mathcal{B}(\rho \to \pi \pi) \approx \mathcal{B}(B_s \to \bar{B}_s \rho), \]

\[ \mathcal{B}(B_s \to D_s \rho \to D_s \pi \pi) = \mathcal{B}(B_s \to D_s \rho) \cdot \mathcal{B}(\rho \to \pi \pi) \approx \mathcal{B}(B_s \to D_s \rho). \] (36)

For \( B^+ \to \bar{D}^0 \rho^+ \to \bar{D}^0 \pi^+ \pi^0 \) and other four considered decay modes, furthermore, the PQCD predictions do agree well with those currently available experimental measurements [64, 65] within errors. We can take above two ”good behavior” as a new indication for the reliability of the PQCD factorization approach and its applicability for the charmed two-body and/or quasi-two-body hadronic decays of \( B \) and \( B_s \) mesons.

(2) For the four CKM suppressed \( B \to D \rho \to D \pi \pi \) decays as listed in the first four lines of Table II, the PQCD predictions for their branching ratios are much smaller than those for the three \( B \to D \rho \to D \pi \pi \) decays as given in Table I, say by about 3–5 orders. The major reason is the strong CKM suppression factor:

\[ R_{\text{CKM}} = \left| \frac{V_{ub}^* V_{cd}}{V_{cb}^* V_{ud}} \right|^2 \approx \lambda^4 (\rho^2 + \eta^2) \approx 3 \times 10^{-4}, \] (37)

which can be seen easily from the decay amplitudes as given in Eqs. (17-19) and Eqs. (A1-A4). Taking the corresponding pairs of \( B \to \bar{D} \rho \) and \( B \to D \rho \) decays into account, for example, the ratios of their branching ratios are of the form

\[ R_1 = \frac{\mathcal{B}(B^+ \to D^0 \rho^+ \to D^0 \pi^+ \pi^0)}{\mathcal{B}(B^+ \to D^0 \rho^+ \to D^0 \pi^+ \pi^0)} \approx 0.44 \times 10^{-5}, \]

\[ R_2 = \frac{\mathcal{B}(B^0 \to D^+ \rho^- \to D^+ \pi^- \pi^0)}{\mathcal{B}(B^0 \to D^+ \rho^- \to D^+ \pi^- \pi^0)} \approx 0.93 \times 10^{-4}, \]

\[ R_3 = \frac{\mathcal{B}(B^0 \to D^0 \rho^0 \to D^0 \pi^+ \pi^-)}{\mathcal{B}(B^0 \to D^0 \rho^0 \to D^0 \pi^+ \pi^-)} \approx 0.94 \times 10^{-4}. \] (38)

(3) For the \( B_s \to D \rho \to D \pi \pi \) decays and \( B_s \to \bar{D} \rho \to \bar{D} \pi \pi \) decays, there still exist the CKM suppression but it is now much moderate in size than the previous cases:

\[ R_{\text{CKM}}^s = \left| \frac{V_{ub}^* V_{cs}}{V_{cb}^* V_{us}} \right|^2 \approx (\rho^2 + \eta^2) \approx 0.14. \] (39)

We can again define the ratios of the branching ratios for the corresponding pairs of \( B_s \) decays in the following form:

\[ R_{s1} = \frac{\mathcal{B}(B_s^0 \to D^0 \rho^0 \to D^0 \pi^+ \pi^-)}{\mathcal{B}(B_s^0 \to D^0 \rho^0 \to D^0 \pi^+ \pi^-)} \approx 0.13, \]

\[ R_{s2} = \frac{\mathcal{B}(B_s^0 \to D^+ \rho^- \to D^+ \pi^- \pi^0)}{\mathcal{B}(B_s^0 \to D^- \rho^+ \to D^- \pi^+ \pi^0)} \approx 0.14. \] (40)

The PQCD predictions for both \( R_{s1} \) and \( R_{s2} \) indeed agree very well with \( R_{\text{CKM}}^s \).
(4) In Fig. 3, we show the $w$-dependence of the differential decay rate $d\mathcal{B}/dw$ for the first two decay modes listed in Table I. One can see directly that the main contribution to the decay rates lies in the region around the pole mass $m_\rho = 775$ MeV of the $\rho$ resonance. Taking the decay $B^+ \rightarrow \bar{D}^0 \rho^+ \rightarrow \bar{D}^0 \pi^+ \pi^0$ as an example, the central values (in units of $10^{-4}$) of its branching ratios after making the integration over different ranges of $w$ are of the following form:

$$B(B^+ \rightarrow \bar{D}^0 \rho^+ \rightarrow \bar{D}^0 \pi^+ \pi^0) = \begin{cases} 
89, & \text{for } w = [m_\rho - \Gamma_\rho, m_\rho + \Gamma_\rho], \\
109, & \text{for } w = [m_\rho - 3\Gamma_\rho, m_\rho + 3\Gamma_\rho], \\
115, & \text{for } 2m_\pi \leq w \leq m_B - m_D.
\end{cases}$$

(41)

This is an indication that the quasi-two-body framework is a very good approximation for the charmed $B/B_s$ three-body decays considered in this paper.

(5) The color-allowed emission diagrams Fig.1(a), 1(b), 2(a) and 2(b) are generally dominant for the considered decays, but the color-suppressed nonfactorizable emission diagrams and annihilation diagrams can also provide considerable contributions to those decays with the $D$ or $D_s$ meson as one of the final state mesons.

Because of the isospin symmetry [17], there is a relation between the decay amplitudes $A_{1/2}$ and $A_{3/2}$ of the charmed decays $B^+ \rightarrow \bar{D}^0 \rho^+$, $B^0 \rightarrow D^- \rho^+$ and $B^0 \rightarrow \bar{D}^0 \rho^0$ considered here:

$$A(\bar{D}^0 \rho^+) = A(D^- \rho^+) + \sqrt{2}A(\bar{D}^0 \rho^0).$$

(42)

Based on such isospin symmetry, one can further define the amplitude ratio $R_{D\rho}$ and the strong phase difference $\delta_{D\rho}$ between the amplitudes $A_{1/2}$ and $A_{3/2}$ in the following forms [17]

$$R_{D\rho} = \sqrt{\frac{1}{2}} \frac{\tau_{B^+}}{\tau_{B^0}} \times \frac{3(B(D^- \rho^+) + B(\bar{D}^0 \rho^0))}{B(\bar{D}^0 \rho^+)} - 1,$$

(43)

$$\cos \delta_{D\rho} = \frac{1}{4R_{D\rho}} \times \left[ \frac{\tau_{B^+}}{\tau_{B^0}} \cdot \frac{3(B(D^- \rho^+) - 2B(\bar{D}^0 \rho^0))}{B(\bar{D}^0 \rho^+)} + 1 \right].$$

(44)

We found $R_{D\rho} = 0.82$ (0.86) and $\cos \delta_{D\rho} = 0.983$ (0.981) in the quasi-two-body (two-body) PQCD framework, while the measured values from LHCb Collaboration [17] are of the form:

$$R_{D\rho} = 0.69 \pm 0.15, \quad \cos \delta_{D\rho} = 0.984_{-0.048}^{+0.113}, \quad \text{for Isobar model},$$

(45)

$$R_{D\rho} = 0.69 \pm 0.15, \quad \cos \delta_{D\rho} = 0.987_{-0.048}^{+0.114}, \quad \text{for K-matrix model}.$$  

(46)

It is easy to see that our PQCD predictions for both $R_{D\rho}$ and $\cos \delta_{D\rho}$ agree well with the measured values within one standard deviation.

IV. SUMMARY

In this paper, we studied the contributions from the $P$-wave resonant states $\rho$ to the $B(s) \rightarrow (D(s), \bar{D}(s)) \rho \rightarrow (D(s), \bar{D}(s)) \pi \pi$ decays. We considered fourteen decay modes, calculated the branching ratios by employing the quasi-two-body and the two-body framework respectively in the PQCD factorization approach. The two-pion distribution amplitude $\Phi_{\pi \pi}^{1/2}$ was introduced, the time-like form factor $F_\rho$ was employed to describe the strong interactions between the resonance $\rho$ and the pion pair including two meson final state interactions.

From the analytical and numerical calculations, we found the following points:
(1) For all considered decays, the PQCD predictions based on the quasi-two-body and the two-body framework agree well with each other, as generally expected. For $B^+ \rightarrow D^0 \rho^+ \rightarrow \bar{D}^0 \pi^+ \pi^0$ and other four considered decay modes, the PQCD predictions do agree well with the measured values within errors. It is a new indication for the reliability of the PQCD factorization approach and its applicability for the charmed hadronic decays of $B$ and $B_s$ mesons.

(2) The great difference between the PQCD predictions for the branching ratios of $B \rightarrow \bar{D}\rho \rightarrow \bar{D}\pi\pi$ decays and the $B \rightarrow D\rho \rightarrow D\pi\pi$ decays can be explained by a strong CKM suppression factor $R_{CKM} \approx \lambda^4(\bar{\rho}^2 + \bar{\eta}^2) \approx 3 \times 10^{-4}$.

(3) For the $B_s \rightarrow D\rho$ and $B_s \rightarrow \bar{D}\rho$ decays, however, the CKM suppression factor is moderate in size: $R_{CKM}^s \approx (\bar{\rho}^2 + \bar{\eta}^2) \approx 0.14$, which agrees very well with the PQCD prediction $R_{s1} \approx 0.13$ and $R_{s2} \approx 0.14$.

(4) The PQCD predictions for the ratios $R_{D\rho}$ and the strong phase difference $\cos \delta_{D\rho}$ defined based on the isospin symmetry between the three $B \rightarrow \bar{D}\rho$ decay modes agree well with the LHCb measurements.

**ACKNOWLEDGMENTS**

Many thanks to Hsiang-nan Li, Cai-Dian Lü and Xin Liu for valuable discussions. This work was supported by the National Natural Science Foundation of China under the No. 11235005 and 11547038.

**Appendix A: Some decay amplitudes and relevant functions**

The analytic formula for the $B_{(s)} \rightarrow D_{(s)}\rho \rightarrow D_{(s)}\pi\pi$ decay amplitudes are of the following form:

$$A(B^+ \rightarrow D^0 \rho^+ (\rho^+ \rightarrow \pi^+ \pi^0)) = \frac{G_F}{\sqrt{2}}V_{ub}V_{cd}[a_1 F^L_{e\rho} + C_2 M^L_{e\rho} + a_2 F^L_{a\rho} + C_1 M^L_{a\rho}], \quad (A1)$$

$$A(B^+ \rightarrow D^+ \rho^0 (\rho^0 \rightarrow \pi^+ \pi^-)) = \frac{G_F}{2}V_{ub}V_{cd}[a_2 F^L_{e\rho} - F^L_{a\rho}) + C_1 (M^L_{e\rho} - M^L_{a\rho})], \quad (A2)$$

$$A(B^0 \rightarrow D^0 \rho^0 (\rho^0 \rightarrow \pi^+ \pi^-)) = \frac{G_F}{2}V_{ub}V_{cd}[a_1 (-F^L_{e\rho} + F^L_{a\rho}) + C_2 (-M^L_{e\rho} + M^L_{a\rho})], \quad (A3)$$

$$A(B^0 \rightarrow D^+ \rho^- (\rho^- \rightarrow \pi^- \pi^0)) = \frac{G_F}{\sqrt{2}}V_{ub}V_{cd}[a_2 F^L_{e\rho} + C_1 M^L_{e\rho} + a_1 F^L_{a\rho} + C_2 M^L_{a\rho}], \quad (A4)$$

$$A(B^+ \rightarrow D^0 \rho^0 (\rho^0 \rightarrow \pi^+ \pi^-)) = \frac{G_F}{2}V_{ub}V_{cd}[a_1 F^L_{e\rho} + C_1 M^L_{e\rho}], \quad (A5)$$

$$A(B^0 \rightarrow D^0 \rho^- (\rho^- \rightarrow \pi^- \pi^0)) = \frac{G_F}{\sqrt{2}}V_{ub}V_{cd}[a_2 F^L_{e\rho} + C_1 M^L_{e\rho}], \quad (A6)$$

$$A(B^0 \rightarrow D^0 \rho^0 (\rho^0 \rightarrow \pi^+ \pi^-)) = \frac{G_F}{2}V_{ub}V_{cs}[a_1 F^L_{a\rho} + C_2 M^L_{a\rho}], \quad (A7)$$

$$A(B^0 \rightarrow D^+ \rho^- (\rho^- \rightarrow \pi^- \pi^0)) = \frac{G_F}{\sqrt{2}}V_{ub}V_{cs}[a_1 F^L_{a\rho} + C_2 M^L_{a\rho}], \quad (A8)$$
with the functions $F_{ep}^{LL}$, $M_{ep}^{LL}$, $F_{aD}^{LL}$ and $M_{aD}^{LL}$ denote the amplitudes as illustrated by Fig. 2.

\[
F_{ep}^{LL} = -8\pi C_F m_4^4 f_D \int_0^1 dx_B dz \int_0^{1/A} b_B db_B db dB \phi_B(x_B, b_B) \\
\times \left\{ [r^2((1-2\eta)(1+z) + (1-(1-\eta)r^2)z) - (1-\eta)(1+z)]\phi_0(z) - \sqrt{\eta(1-r^2)} \\
\times [(1-\eta)(1-2z(1-r^2))(\phi_\sigma(z) + \phi_\pi(z)) + r^2(\phi_s(z) - \phi_i(z)))] E_n(t_a) h_a(x_B, z, b, b_B) \\
\times S_i(z) - [(1-r^2)[r^2(x_B - \eta) - (1-\eta)\eta]\phi_0(z) + 2\sqrt{\eta(1-r^2)}[(1-\eta)(1-r^2) \\
-r^2(x_B - \eta)]\phi_s(z) \right\} E_c(t_b) h_b(x_B, z, b, b_B) S_i([x_B - \eta]) \right\}, \tag{A9}
\]

\[
M_{ep}^{LL} = -32\pi C_F m_4^4 / \sqrt{6} \int_0^1 dx_B dF dz x_3 \int_0^{1/A} b_B db_B db_B db_B \phi_B(x_B, b_B) \phi_D(x_3, b_3) \\
\times \left\{ [(1-\eta + r^2)(1-r^2)((\eta - 1)x_3 + x_B - \eta z)]\phi_0(z) + \sqrt{\eta(1-r^2)}r^2((1-\eta)x_3 \\
-x_B)(\phi_i(z) + \phi_\pi(z)) + (1-\eta)(1-r^2)z(\phi_s(z) - \phi_i(z))] \right\} E_n(t_c) h_c(x_B, z, x_3, b_B, b_3) \\
\times [rr_e(1+\eta) - r^2(2(1-\eta)z - 1) + rr_e - 2\eta^2(1-x_3) + \eta(3(x_B - z) + 2(1-x_3)) \\
x_B + x_3 - 2z - (1\eta)((1-\eta)(1-x_3) - x_B + z)]\phi_0(z) - \sqrt{\eta(1-r^2)}[(1-1) \\
x(1-r^2)z(\phi_s(z) + \phi_\pi(z)) + r^2((1-\eta)x_3 + x_B)(\phi_s(z) - \phi_\pi(z)) + 2(2rr_e - (1-\eta)r^2) \\
\times \phi_s(z)] E_n(t_d) h_d(x_B, z, x_3, b_B, b_3) \right\} \tag{A10}
\]

\[
F_{aD}^{LL} = -8\pi C_F m_4^4 f_D \int_0^1 dx_3 dz \int_0^{1/A} b_B db_B db_B \phi_D(x_3, b_3) \\
\times \left\{ [(1-r^2)[\eta(1-\eta + r^2) + (1-\eta)x_3]\phi_0(z) + 2r\sqrt{\eta(1-r^2)}[(1+\eta + (1-\eta)x_3 \\
-r^2]\phi_s(z)] \right\} E_a(t_y) h_y(z, x_3, x_3, b_3) \right\} S_i(x_3) - [(1-\eta)(r^2(z - 1) + r^2(1+\eta - 2z) + z) \\
+2rr_c(r^2 - 1 - \eta)]\phi_0(z) + \sqrt{\eta(1-r^2)}r(2z + 2r^2(1-z) - rr_c)(\phi_s(z) + \phi_\pi(z)) \\
+(1-\eta)(2r - rr_c)(\phi_s(z) - \phi_i(z)) \right\} E_n(t_y) h_y(z, x_3, b, b_3) \right\} S_i(x_3) \right\} \tag{A11}
\]

\[
M_{aD}^{LL} = 32\pi C_F m_4^4 / \sqrt{6} \int_0^1 dx_B dF dz x_3 \int_0^{1/A} b_B db_B db_B \phi_B(x_B, b_B) \phi_D(x_3, b_3) \\
\times \left\{ [r^2((1-\eta)(x_3 - z) - 1) - x_3(1-\eta^2) + (1-\eta)x_B - (\eta^2 + \eta - 2)z + 1) \\
-(1-\eta)((1+\eta)(x_B + z) - \eta)]\phi_0(z) - r\sqrt{\eta(1-r^2)}[(z(1-r^2) + x_B)(\phi_s(z) + \phi_\pi(z)) \\
+(1-\eta)x_3(\phi_s(z) - \phi_\pi(z)) + 2\phi_s(z)] \right\} E_n(t_v) h_v(x_B, z, x_3, b, b_B) + [(1-\eta + r^2) \\
\times [(1-r^2)((1-\eta)x_3 + \eta z) - x_B\eta\phi_0(z) + r\sqrt{\eta(1-r^2)}(1-\eta)x_3(\phi_s(z) + \phi_\pi(z)) \\
+((1-r^2)z - x_B)(\phi_\pi(z) - \phi_\pi(z))] \right\} E_n(t_w) h_w(x_B, z, x_3, b, b_B) \right\} \tag{A12}
\]

We show here the explicit expressions of the hard functions $h_i(i = a, b, c, d, e, f, g, h, m, n, o, p, x, y, v, w)$, coming from the Fourier transform of hard kernel:

\[
h_i(x_1, x_2(x_3), b_1, b_2) = h_1(\beta, b_2) \times h_2(\alpha, b_1, b_2),
\]

\[
h_1(\beta, b_2) = \begin{cases} 
K_0(\sqrt{\beta}b_2), & \beta > 0 \\
K_0(i\sqrt{-\beta}b_2), & \beta < 0
\end{cases}
\]

\[
h_2(\alpha, b_1, b_2) = \begin{cases} 
\theta(b_2 - b_1)I_0(\sqrt{\alpha}b_1)K_0(\sqrt{\alpha}b_2) + (b_1 \leftrightarrow b_2), & \alpha > 0 \\
\theta(b_2 - b_1)I_0(\sqrt{-\alpha}b_1)K_0(i\sqrt{-\alpha}b_2) + (b_1 \leftrightarrow b_2), & \alpha < 0
\end{cases} \tag{A13}
\]
where $K_0, I_0$ are modified Bessel function with $K_0(ix) = \frac{2}{\pi}(-N_0(x)+iJ_0(x))$ and $J_0$ is the Bessel function, $\alpha$ and $\beta$ are the factor $i_1, i_2(\hat{i} = a, b, c, d, e, f, g, \hat{h}, m, n, o, p, x, y, v, w)$ as defined in the following paragraph.

The hard scale $t_1$ is chosen as the maximum of the virtuality of the internal momentum transition in the hard amplitudes. For $B(s) \to D(s)\rho \to D(s)\pi\pi$ decays, we have

\[
\begin{align*}
t_a &= \max\{m_B\sqrt{|a_1|}, m_B\sqrt{|a_2|}, 1/b, 1/b_B\}, & \quad t_b &= \max\{m_B\sqrt{|b_1|}, m_B\sqrt{|b_2|}, 1/b_B, 1/b\}; \\
t_c &= \max\{m_B\sqrt{|c_1|}, m_B\sqrt{|c_2|}, 1/b_B, 1/b_3\}, & \quad t_d &= \max\{m_B\sqrt{|d_1|}, m_B\sqrt{|d_2|}, 1/b_B, 1/b_3\}; \\
t_e &= \max\{m_B\sqrt{|e_1|}, m_B\sqrt{|e_2|}, 1/b, 1/b_3\}, & \quad t_f &= \max\{m_B\sqrt{|f_1|}, m_B\sqrt{|f_2|}, 1/b_3, 1/b\}; \\
t_g &= \max\{m_B\sqrt{|g_1|}, m_B\sqrt{|g_2|}, 1/b, 1/b_B\}, & \quad t_h &= \max\{m_B\sqrt{|h_1|}, m_B\sqrt{|h_2|}, 1/b, 1/b_B\}; \\
t_m &= \max\{m_B\sqrt{|m_1|}, m_B\sqrt{|m_2|}, 1/b_3, 1/b_B\}, & \quad t_n &= \max\{m_B\sqrt{|n_1|}, m_B\sqrt{|n_2|}, 1/b_3, 1/b_B\}; \\
t_o &= \max\{m_B\sqrt{|o_1|}, m_B\sqrt{|o_2|}, 1/b_B, 1/b\}, & \quad t_p &= \max\{m_B\sqrt{|p_1|}, m_B\sqrt{|p_2|}, 1/b_B, 1/b\}.
\end{align*}
\]

(A14)

with the factors

\[
\begin{align*}
a_1 &= (1-r^2)z, & \quad a_2 &= (1-r^2)x_Bz; \\
b_1 &= (1-r^2)(x_B - \eta), & \quad b_2 &= a_2; \\
c_1 &= a_2, & \quad c_2 &= r_c^2 - [(1 - z)r^2 + z][(1 - \eta)(1 - x_3) - x_B]; \\
d_1 &= a_2, & \quad d_2 &= (1 - r^2)z[x_B - (1 - \eta)x_3]; \\
e_1 &= r_c^2 - [1 - z(1 - r^2)], & \quad e_2 &= (1 - r^2)(1 - z)[(\eta - 1)x_3 - \eta]; \\
f_1 &= (1 - r^2)[(\eta - 1)x_3 - \eta], & \quad f_2 &= e_2; \\
g_1 &= e_2, & \quad g_2 &= 1 - [(1 - z)r^2 + z][(1 - \eta)(1 - x_3) - x_B]; \\
h_1 &= e_2, & \quad h_2 &= (1 - r^2)(1 - z)[(\eta - 1)x_3 - \eta + x_B]; \\
m_1 &= (1 - \eta)x_3, & \quad m_2 &= (1 - \eta)x_3x_B; \\
n_1 &= r_c^2 - (r^2 - x_B)(1 - \eta), & \quad n_2 &= m_2; \\
o_1 &= m_2, & \quad o_2 &= [(\eta - 1)x_3 - \eta][(1 - z)(1 - r^2) - x_B]; \\
p_1 &= m_2, & \quad p_2 &= (1 - \eta)x_3[x_B - (1 - r^2)z].
\end{align*}
\]

(A15)

For $B(s) \to D(s)\rho \to D(s)\pi\pi$ decays, similarly, we have

\[
\begin{align*}
t_a &= \max\{m_B\sqrt{|a_1|}, m_B\sqrt{|a_2|}, 1/b, 1/b_B\}, & \quad t_b &= \max\{m_B\sqrt{|b_1|}, m_B\sqrt{|b_2|}, 1/b_B, 1/b\}; \\
t_c &= \max\{m_B\sqrt{|c_1|}, m_B\sqrt{|c_2|}, 1/b_B, 1/b_3\}, & \quad t_d &= \max\{m_B\sqrt{|d_1|}, m_B\sqrt{|d_2|}, 1/b_B, 1/b_3\}; \\
t_e &= \max\{m_B\sqrt{|e_1|}, m_B\sqrt{|e_2|}, 1/b_3, 1/b\}, & \quad t_f &= \max\{m_B\sqrt{|f_1|}, m_B\sqrt{|f_2|}, 1/b_3, 1/b\}; \\
t_g &= \max\{m_B\sqrt{|g_1|}, m_B\sqrt{|g_2|}, 1/b, 1/b_B\}, & \quad t_h &= \max\{m_B\sqrt{|h_1|}, m_B\sqrt{|h_2|}, 1/b, 1/b_B\}; \\
t_m &= \max\{m_B\sqrt{|m_1|}, m_B\sqrt{|m_2|}, 1/b_3, 1/b_B\}, & \quad t_n &= \max\{m_B\sqrt{|n_1|}, m_B\sqrt{|n_2|}, 1/b_3, 1/b_B\}; \\
t_o &= \max\{m_B\sqrt{|o_1|}, m_B\sqrt{|o_2|}, 1/b_B, 1/b\}, & \quad t_p &= \max\{m_B\sqrt{|p_1|}, m_B\sqrt{|p_2|}, 1/b_B, 1/b\}.
\end{align*}
\]

(A16)
with the factors

\[ a_1 = (1 - r^2)z, \quad a_2 = (1 - r^2)x_B z; \]
\[ b_1 = (1 - r^2)(x_B - \eta), \quad b_2 = a_2; \]
\[ c_1 = a_2, \quad c_2 = (1 - r^2)(x_B - (1 - \eta)x_3); \]
\[ d_1 = a_2, \quad d_2 = r^2 - [(1 - z)r^2 + z][(1 - \eta)(1 - x_3) - x_B]; \]
\[ x_1 = (1 - r^2)[(\eta - 1)x_3 - \eta], \quad x_2 = (1 - \eta)(r^2 - 1)x_3 z; \]
\[ y_1 = r^2 - (1 - \eta)[z + r^2(1 - z)], \quad y_2 = x_2; \]
\[ v_1 = x_2, \quad v_2 = [1 - (1 - \eta)x_3][[(1 - r^2)z + x_B] + (1 - \eta)x_3; \]
\[ w_1 = x_2, \quad w_2 = (1 - \eta)x_3[x_B - (1 - r^2)z]. \]

(A17)
[33] H.Y. Cheng and C.K. Chua, Phys. Rev. D 89, 074025 (2014).
[34] H.Y. Cheng, C.K. Chua, and Z.Q. Zhang, Phys. Rev. D 94, 094015 (2016).
[35] B. Bhattacharya, M. Gronau and J. L. Rosner, Phys. Lett. B 726, 337 (2013).
[36] M. Gronau, Phys. Lett. B 727, 136 (2013).
[37] C. Wang, Z.H. Zhang, Z.Y. Wang and X.H. Guo, Eur. Phys. J. C 75, 536 (2015).
[38] L. Lesfiaik and P. Żenczykowski, Phys. Lett. B 737, 201 (2014).
[39] Y. Li, Phys. Rev. D 89, 094007 (2014).
[40] D. Xu, G.N. Li and X.G. He, Phys. Lett. B 728, 579 (2014).
[41] X.G. He, G.N. Li and D. Xu, Phys. Rev. D 91, 014029 (2015).
[42] Z.H. Zhang, X.H. Guo and Y.D. Yang, Phys. Rev. D 87, 076007 (2013).
[43] I. Bediaga, T. Frederico, and O. Lourenc¸o, Phys. Rev. D 89, 094013 (2014).
[44] J.H. Alvarenga Nogueira et al., Phys. Rev. D 92, 054010 (2015).
[45] S. Kr¨ ankl, T. Mannel and J. Virto, Nucl. Phys. B 899, 247-264 (2015).
[46] N.R.-L. Lorier, M. Imbeault and D. London, Phys. Rev. D 84, 034040 (2011).
[47] M. Imbeault, N.R.-L. Lorier and D. London, Phys. Rev. D 84, 034041 (2011).
[48] N.R.-L. Lorier and D. London, Phys. Rev. D 85, 016010 (2012).
[49] M. Beneke, G. Buchalla, M. Neubert and C.T. Sachrajda, Nucl. Phys. B 591, 313 (2000).
[50] C.W. Bauer, D. Pirjol and I.W. Stewart, Phys. Rev. Lett. 87, 201806 (2001).
[51] C.W. Chiang and E. Senaha, Phys. Rev. D 75, 074021 (2007).
[52] S.H. Zhou, Y.B. Wei, Q. Qin, Y. Li, F.S. Yu and C.D. Lü, Phys. Rev. D 92, 094016 (2015).
[53] T. Kurimoto, H.n. Li and A.I. Sanda, Phys. Rev. D 67, 054028 (2003).
[54] Y.Y. Keum, T. Kurimoto, H.n. Li, C.D. Lü and A.I. Sanda, Phys. Rev. D 69, 094018 (2004).
[55] H.n. Li, Phys. Rev. D 52, 3958 (1995).
[56] C.Y. Wu, T.W. Yeh and H.n. Li, Phys. Rev. D 53, 4982 (1996).
[57] C.D. Lü, Eur. Phys. J. C 24, 121 (2002).
[58] C.D. Lü, Phys. Rev. D 68, 097502 (2003).
[59] Y. Li and C.D. Lü, High Energy Phys. & Nucl. Phys. 27, 1062 (2003).
[60] R.H. Li, C.D. Lü and Z. Hao, Phys. Rev. D 78, 014018 (2008).
[61] Z. Hao, R.H. Li, X.X. Wang and C.D. Lü, J. Phys. G 37, 015002 (2010).
[62] Z.T. Zou, X. Yu and C.D. Lü, Phys. Rev. D 86, 094001 (2012).
[63] Z.T. Zou, R. Zhou and C.D. Lü, Chin. Phys. C 37, 013103 (2013).
[64] C. Patrignani et al., (Particle Data Group), Chin. Phys. C 40, 100001 (2016)and 2017 update.
[65] Y. Amhis et al., (Heavy Flavor Averaging Group), arXiv:1612.07233v1 [hep-ex].
[66] Y.Y. Keum, H.n. Li and A.I. Sanda, Phys. Rev. D 63, 054008 (2001).
[67] M. Diehl, T. Gousset, B. Pire, and O. Teryaev, Phys. Rev. Lett. 81, 1782 (1998).
[68] M. V. Polyakov, Nucl. Phys. B 555, 231 (1999).
[69] J. P. Lees et al. [BaBar Collaboration], Phys. Rev. D 86, 032013 (2012).
[70] G. Breit and E. Wigner, Phys. Rev. 49, 519 (1936).
[71] G. J. Gounaris and J.J. Sakurai, Phys. Rev. Lett. 21, 244 (1968).
[72] P. Ball, G.W. Jones and R. Zwicky, Phys. Rev. D 75, 054004 (2007).
[73] A. Bharucha, D.M. Straub and R. Zwicky, JHEP 1608, 098 (2016).
[74] K. Jansen, C. McNeile, C. Michael and C. Urbach, Phys. Rev. D 80, 054510 (2009).
[75] W.F. Wang and Z.J. Xiao, Phys. Rev. D 86, 114025 (2012).