The virtual-casing principle and Helmholtz’s theorem

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Abstract

The virtual-casing principle is used in plasma physics to convert a Biot–Savart integration over a current distribution into a surface integral over a surface that encloses the current. In many circumstances, use of virtual casing can significantly speed up the computation of magnetic fields. In this paper, a virtual-casing principle is derived for a general vector field with arbitrary divergence and curl. This form of the virtual-casing principle is thus applicable to both magnetostatic fields and electrostatic fields. The result is then related to Helmholtz’s theorem.

Keywords: virtual casing, Helmholtz’s theorem, vector fields

1. Introduction

Shafranov and Zakharov originally formulated the virtual-casing principle [1] for an axisymmetric current-carrying plasma. Assuming that the magnetic field in the plasma region was known, they imagined surrounding the plasma with a virtual superconducting surface that was everywhere tangent to the magnetic field. Since the magnetic field in the interior of the superconductor (exterior to the current carrying region) would be zero, there must be surface currents on the virtual surface of the superconductor. Writing the zero magnetic field in the virtual superconducting region as a Biot–Savart integral over the currents in the interior volume and on the surface, one obtains

$$0 = \frac{\mu_0}{4\pi} \int_V J(r') \times \frac{r - r'}{|r - r'|^3} dV' + \frac{\mu_0}{4\pi} \oint_{\partial V} K(r') \times \frac{r - r'}{|r - r'|^3} d\sigma' \quad r \not\in V \tag{1}$$

where $K$ represents the surface current density. Using Ampère’s law and a standard Stokesian path argument, the surface current density can be expressed as $\mu_0 K = \hat{n} \times (-B)$, where $B$ is the magnetic field evaluated just inside the virtual surface and $\hat{n}$ is the outward normal unit vector on the virtual surface. Thus the volume Biot–Savart integral can be replaced with a surface integration

$$\frac{\mu_0}{4\pi} \int_V J(r') \times \frac{r - r'}{|r - r'|^3} dV' = \frac{1}{4\pi} \oint_{\partial V} (\hat{n} \times B) \times \frac{r - r'}{|r - r'|^3} d\sigma' \quad r \not\in V \tag{2}$$

The form of the virtual-casing principle is especially helpful for plasma equilibrium computations [2]. Such calculations often result in obtaining the total magnetic field in the plasma region, leaving the magnetic field outside the plasma region undetermined. The magnetic field due to currents in coils or external structures is readily computable in all regions from the Biot–Savart law. The remaining computation of the magnetic field due to currents in the plasma region is precisely what the virtual-casing principle addresses. The reduction of the Biot–Savart integration by one dimension—from (3) to (2) in (non)axisymmetric situations—accomplished by the virtual-casing principle can significantly speed up computation of the magnetic field. But perhaps as important as the computational savings is the clear exemplification of the common result that what happens within a volume can often be fully described by quantities on the surface of the volume.

One recent example of this integration-by-parts principle is the work of Ludwig, Rodrigues and Bizarro [3] on tokamak...
equilibria with strong current density reversals, where a virtual superconducting shell was used to separate the issue of plasma equilibrium from the related issue of determining currents in external coils consistent with the plasma equilibrium.

The virtual-casing principle has been used for many plasma physics computations, some of which are follow-on computations to equilibrium results. For example, virtual casing has been used in the design of tokamak control coils [4, 5], computation of plasma inductances [6], theory of perturbed equilibria [7], magnetic field line tracing [8], magnetic diagnostic response calculation [9–11] and magnetic diagnostic design [12].

The original formulation of the virtual-casing principle was only applicable to the case where the magnetic field was tangential to the surface of the volume in question. Two papers [13, 14] have addressed the case when the magnetic field has a component normal to the virtual surface. In both papers, the correct result was stated, but without careful justification. The discussion of the extension of the virtual-casing principle invoked in one paper [13] a surface magnetic charge density, and in the other paper [14] a surface magnetic dipole moment density.

In this paper I generalize the virtual-casing principle to a general vector field. There is no restriction to either divergence-free (magnetostatic) or curl-free (electrostatic) fields. The derivation does not rely on a virtual superconducting surface, and the field components on the surface are not restricted. The rest of the paper is structured as follows. The restricted case will be explicitly stated for the important special cases of magnetostatic and electrostatic fields.

2. Derivation of the virtual-casing principle

Consider a general vector field \( \mathbf{b} \) with curl \( \mathbf{c} \) and divergence \( d \):

\[
\nabla \times \mathbf{b} = \mathbf{c} \quad \nabla \cdot \mathbf{b} = d. \tag{3}
\]

The special cases of a magnetostatic field can be obtained by the choices

\[
\mathbf{b} \rightarrow \mathbf{B} \quad \mathbf{c} \rightarrow \mu_0 \mathbf{J} \quad d \rightarrow 0 \tag{4}
\]

and that of an electrostatic field by the choices

\[
\mathbf{b} \rightarrow \mathbf{E} \quad \mathbf{c} \rightarrow 0 \quad d \rightarrow \frac{\rho}{\varepsilon_0}. \tag{5}
\]

2.1. Field definition

Define the related field \( \mathbf{b}_V \) arising from combined Coulomb and Biot–Savart integration of the field sources over a volume \( V \)

\[
\mathbf{b}_V(r) \equiv \frac{1}{4\pi} \int_V \mathbf{a}(\nabla' \cdot \mathbf{b}) \mathbf{r}' - \mathbf{a} \times (\nabla' \times \mathbf{b}) \, d^3r' \tag{6}
\]

Writing the sources explicitly in terms of the original field \( \mathbf{b} \) and using the shorthand \( \mathbf{a} \equiv (r - r')/|r - r'|^3 \) we can write

\[
b_V \equiv \frac{1}{4\pi} \int_V \mathbf{a}(\nabla' \cdot \mathbf{b}) - \mathbf{a} \times (\nabla' \times \mathbf{b}) \, d^3r' \tag{7}
\]

where \( \nabla' \) is the del operator on primed coordinates. The goal is now to move the derivatives from the field \( \mathbf{b} \) to the field \( \mathbf{a} \). To proceed further we will need a vector identity.

2.2. Vector identity

The product rule for the divergence of a symmetric dyadic is

\[
\nabla \cdot (a \cdot ba + ba \cdot a) = a(\nabla \cdot b) + (b \cdot \nabla)a + a(\nabla \cdot b) + b(\nabla \cdot a) \tag{8}
\]

and combining this with the product rule for the gradient of a dot product, we have

\[
\nabla \cdot (ab + ba) = \nabla \cdot (ab) + \nabla \cdot (ba) + b(\nabla \cdot a) - a \times (\nabla \times b) - b \times (\nabla \times a) \tag{9}
\]

for arbitrary vector fields \( \mathbf{a} \) and \( \mathbf{b} \). Rearranging terms gives

\[
a(\nabla \cdot b) - a \times (\nabla \times b) = \nabla \cdot (ab + ba) - \nabla (a \cdot b) - b(\nabla \cdot a) + b \times (\nabla \times a). \tag{10}
\]

2.3. Derivation

Using the vector identity equation (10) in the definition of \( b_V \), we obtain

\[
b_V = \frac{1}{4\pi} \int_V \nabla' \cdot (ab + ba) - \nabla'(a \cdot b) - b(\nabla' \cdot a) + b \times (\nabla' \times a) \, d^3r'. \tag{11}
\]

Using the fact that \( (\nabla' \times a) = 0 \) and corollaries of the divergence theorem, we obtain

\[
b_V = \frac{1}{4\pi} \int_{\partial V} (\nabla' \cdot a)b + (\nabla' \cdot b)a - \nabla'(a \cdot b) \, d^3r' - \frac{1}{4\pi} \int_V b(\nabla' \cdot a) \, d^3r' \tag{12}
\]

where \( \hat{n}' \) is the unit outward normal to the volume \( V \). To evaluate the remaining volume integral, we will use the relation \( \nabla' \cdot a = -4\pi \delta(r - r') \). The volume integral evaluates to different functional forms depending on whether the field point \( r \) is inside, on the boundary of, or outside the integration volume. Performing the delta-function integration and using the BAC-CAB rule to combine the first and third terms of the surface integral, we obtain the main result of this paper:

\[
b_V(r) \equiv \frac{1}{4\pi} \int_V a(\nabla' \cdot b) - a \times (\nabla' \times b) \, d^3r' = \frac{1}{4\pi} \int_{\partial V} a(\nabla' \cdot b) - a \times (\nabla' \times b) \, d^3r' \tag{13}
\]

\[
\begin{cases}
\mathbf{b} & r \in V \\
\mathbf{b}/2 & r \in \partial V \\
0 & r \notin V, \partial V
\end{cases}
\]
where \( a \equiv (r - r')/|r - r'|^3 \), and we have repeated the definition of \( b_V \).

### 3. Discussion

A careful reader may be concerned about the existence and interpretation of the integrals in this paper which contain singularities. The best interpretation is in terms of the language of generalized functions or distributions [15]. In particular the Biot–Savart and Coulomb integrals that we start with in the definition of the field \( b_V \) are convolutions of distributions. In concrete terms, integrals containing \( a \) should be interpreted as a limiting process as \( \epsilon \to 0 \) of the well-behaved \( a_\epsilon \equiv \nabla \left( \frac{1}{\sqrt{|r - r'|^2 + \epsilon^2}} \right) \). The classic text of Jackson [16] contains an example of of this limiting process, as well as an example of dealing with singularities by excision of a small region surrounding the singularity.

Equation (13), the primary result of this paper, is closely related to Helmholtz’s theorem. Helmholtz’s theorem, discussed in many text books [17–19], states that an arbitrary vector field defined on all space which decays sufficiently rapidly with distance may be written as a sum of a gradient and a curl

\[
b = -\nabla U + \nabla \times W \tag{14}
\]

and the functions \( U \) and \( W \) are given by Coulomb and Biot–Savart all-space integrations over the divergence and curl:

\[
U = \frac{1}{4\pi} \int \frac{\nabla' \cdot b}{|r - r'|} \, d^3r' \quad \quad W = \frac{1}{4\pi} \int \frac{\nabla' \times b}{|r - r'|} \, d^3r'. \tag{15}
\]

To obtain Helmholtz’s theorem, start from (13), and hold the position \( r \) fixed. Let the volume of integration \( V \) expand to infinity. With the assumption that the field \( b \) decays sufficiently rapidly with distance, and since the \( a \) term decreases like \( r^{-2} \), the surface integral becomes negligible. With the position \( r \) inside the volume of integration, (13) becomes

\[
b_V(r) = b(r) = \frac{1}{4\pi} \int a(\nabla' \cdot b) - a \times (\nabla' \times b) \, d^3r'. \tag{16}
\]

Since

\[
-\nabla U = -\nabla \frac{1}{4\pi} \int \frac{\nabla' \cdot b}{|r - r'|} \, d^3r' = \frac{1}{4\pi} \int \frac{(r - r')}{|r - r'|^3} (\nabla' \cdot b) \, d^3r'
\]

and

\[
\nabla \times W = \nabla \times \frac{1}{4\pi} \int \frac{\nabla' \times b}{|r - r'|} \, d^3r' = -\frac{1}{4\pi} \int \frac{(r - r')}{|r - r'|^3} \times (\nabla' \times b) \, d^3r'. \tag{17}
\]

we recover Helmholtz’s theorem. The derivation of the virtual-casing principle in the previous section is then the first part of a different approach to the proof of Helmholtz’s theorem.

In this paper the virtual-casing principle has been derived for an arbitrary vector field. For reference, it is helpful to write the result for the two special cases of magnetostatic fields

\[
b \to B \quad \nabla \cdot B = 0 \quad \nabla \times B = \mu_0 J
\]

\[
B_V(r) = \frac{\mu_0}{4\pi} \int_V J(r') \times \frac{(r - r')}{|r - r'|^3} \, d^3r'
\]

\[
= \frac{1}{4\pi} \oint_{\partial V} a(\hat{n} \cdot B) - a \times (\hat{n} \times B) \, d\vec{r}'
\]

\[
\begin{cases}
B(r) & r \in V \\
0 & r \notin V, \partial V
\end{cases}
\]

and electrostatic fields

\[
b \to E \quad \nabla \cdot E = 0 \quad \nabla \times E = 0
\]

\[
E_V(r) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(r') \frac{(r - r')}{|r - r'|^3} \, d^3r'
\]

\[
\begin{cases}
\Phi(r) & r \in V \\
\Phi(r)/2 & r \in \partial V \\
0 & r \notin V, \partial V
\end{cases}
\]

The electrostatic (curl-free) result (20) is consistent with Green’s theorem applied to an electrostatic potential [16]:

\[
\begin{aligned}
\Phi(r) & \quad r \in V \\
\Phi(r)/2 & \quad r \in \partial V \\
0 & \quad r \notin V, \partial V
\end{aligned}
\]

as we will now show. Taking the gradient of equation (21), using \( E = -\nabla \Phi \) and rearranging gives

\[
E_V(r) = \frac{1}{4\pi\varepsilon_0} \oint_{\partial V} a(\hat{n} \cdot E) + \Phi(r')(\hat{n} \cdot \nabla) \nabla \left( \frac{1}{|r - r'|} \right) \, d\vec{r}'
\]

\[
= \oint_{\partial V} E(r) \, d\vec{r}'. \tag{22}
\]

Note that taking the gradient of the left hand side of equation (21) gives a singular result for the component normal to the surface when \( r \in \partial V \), due the discontinuity across the surface \( \partial V \). Because of this, equation (22) is not valid for \( r \in \partial V \). Comparison with equation (20) show that the two are consistent in the regions \( r \in V \) and \( r \notin V, \partial V \) if

\[
\int_{\partial V} -a \times (\hat{n} \times E) \, d\vec{r}' = \int_{\partial V} \Phi(r')(\hat{n} \cdot \nabla) \nabla \left( \frac{1}{|r - r'|} \right) \, d\vec{r}'. \tag{23}
\]

With an application of Stokes’s theorem to a closed (boundaryless) surface with the vector field \( h \times (\phi \nabla g) \), where \( h \) is a uniform vector field, and \( \phi \) and \( g \) are scalar fields, one can show that
\[ \oint_{\partial V} \hat{n}(\nabla \cdot \nabla g) + \hat{n} \phi \nabla^2 g - \nabla \phi(\hat{n} \cdot \nabla g) - \phi(\hat{n} \cdot \nabla) \nabla g \, d^2 \mathbf{r}' = 0. \]

(24)

Using (24) with \( \phi = \Phi \) and \( g = |\mathbf{r} - \mathbf{r}'|^{-1} \), recognizing \( \mathbf{a} = \nabla |\mathbf{r} - \mathbf{r}'|^{-1} \) and a use of the BAC-CAB identity yields (23). The term \( \oint_{\partial V} \hat{n} \phi \nabla^2 g \, d^2 \mathbf{r}' \) was dropped from consideration because it is zero in the regions of interest, \( \mathbf{r} \in V \) and \( \mathbf{r} \notin V, \partial V \).

Note that when \( \mathbf{r} \in \partial V \), the surface integral of the volume delta function in this term gives a singular component in the surface normal direction, consistent with the singularity (mentioned above) in the gradient of the left hand side of equation (21). Thus we see that the electrostatic virtual-casing result is indeed consistent with Green’s theorem applied to an electrostatic field.

The usefulness of the virtual-casing principle (13) hinges on the assumption that the original field \( \mathbf{b} \) is known. This is the appropriate assumption when dealing with solutions to the MHD equilibrium equations. In other contexts, for computation of the Biot–Savart integrals, the current density \( \mathbf{J} \) is assumed known. Researchers in these areas [20, 21] have derived similar but less general integration by parts formulae, sometimes invoking fictitious magnetization densities.

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