A FULLY NONCOMMUTATIVE ANALOG OF THE PAINLEVÉ IV EQUATION AND A STRUCTURE OF ITS SOLUTIONS

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Dedicated to the 70th birthday of V.V. Sokolov

Abstract. We study a fully noncommutative generalisation of the commutative fourth Painlevé equation that possesses solutions in terms of an infinite Toda system over an associative unital division ring equipped by a derivation.

Introduction

The main purpose of the present paper is to illustrate the “pure noncommutative” approach to the theory of integrable systems by applying it to the noncommutative Painlevé equations, more precisely to the PIV equation and the related τ-functions. This approach was first introduced in the paper [RR10] and can be regarded as a universal approach to the quantization or deformation of the well-known classical objects. In the cited paper the authors constructed an integrable fully noncommutative analog of the second Painlevé equation and gave an explicit description of its solutions in terms of Hankel quasideterminants. They also generalized results related to the solutions of the fully noncommutative Toda chain, suggested in papers [EGR97], [GR92]. Different versions of the noncommutative Toda equations can also be found in [Mik81], [Kri81]. We will apply these results to construct a fully noncommutative version of the fourth Painlevé equation that admits solutions in terms of the infinite Toda system. The subject of our study is motivated by papers [JKM04], [JKM06], where the authors have explained the nature of the determinant solutions for the commutative PII and PIV equations.

Consider an associative unital division ring $R$ over a field $F$. Let $D : R \to R$ be a derivation of $R$, i.e. an $F$-linear map that satisfies the Leibniz rule; for any $f \in R$ we put $Df = f'$. Below we will often refer to the elements of the ring $R$ as to functions. We will fix an element $t \in R$ such that $t' = 1$ (so we assume that the differential equation $f' = 1$ in $R$ has solutions) and for any scalar parameter $\alpha \in F$ we have $\alpha' = 0$. In the paper [RR10] the authors constructed solutions of the infinite Toda system in terms of quasideterminants of Hankel matrices over $R$ (see Theorem 2.1 in [RR10]). This system contains two parts, “positive” and “negative”, and can be written as

\[
\begin{align*}
(n')^n &= \theta_{n+1}^{-1} - \theta_n^{-1}, \\
(n)^m &= \eta_m^{-1} - \eta_{m+1}^{-1},
\end{align*}
\]

Here $\theta_1 = \eta_0^{-1} = \kappa_1$ and $\theta_0 = \eta_{-1}^{-1} = \kappa_{-1}$ for some generic initial functions $\kappa_{-1}$ and $\kappa_1$. It turns out that if we impose some conditions on the functions $\kappa_{-1}$ and $\kappa_1$, then one can use the solutions of the noncommutative Toda equations to construct the solutions of the fully noncommutative Painlevé II equation of the form

\[
\text{PII}[u; \beta] \quad u'' = 2u^3 - 2tu - 2ut + 4 \left( \beta + \frac{1}{2} \right),
\]

where $u, t \in R$, $t' = 1$, and $\beta$ is a scalar parameter, i.e. $\beta \in F$ (in particular $\beta' = 0$). Note that the r.h.s. of the PII[$u; \beta$] equation is written in symmetric or anticommutator form; this form is useful to construct some generalizations of the commutative PII equation. Equation PII[$u; \beta$] is a generalization of the matrix Painlevé II equation, obtained in [BS98], [AS21], and of the quantum Painlevé II equation, suggested in [NGR+08], since in contrast with these two examples, there are no additional assumptions for the algebra $R$ in it.

In this paper we go further in this direction and suggest a fully noncommutative version of the commutative Painlevé IV equation

\[
y'' = \frac{1}{2} y^{-1} (y')^2 + \frac{3}{2} y^3 - 2ty^2 + \left( \frac{1}{2} t^2 - (\alpha_1 - \alpha_0) \right) y - \frac{1}{2} \alpha_2 y^{-1},
\]
where $y = y(t)$ and the scalar parameters $\alpha_0, \alpha_1, \alpha_2$ are such that $\alpha_0 + \alpha_1 + \alpha_2 = 1$. The solutions of this equation are expressed in terms of the solutions of noncommutative Toda equations under a certain ansatz for the functions $\kappa_{-1}$ and $\kappa_1$. This analog reads as

$$
\begin{align*}
    f_0' &= f_0 f_1 - f_2 f_0 + \alpha_0, \\
    f_1' &= f_1 f_2 - f_0 f_1 + \alpha_1, \\
    f_2' &= f_2 f_0 - f_1 f_2 + \alpha_2,
\end{align*}
$$

where again $\alpha_0 + \alpha_1 + \alpha_2 = 1$; this system can be regarded as a fully noncommutative generalization of the PIV symmetric system (in the commutative case system (1) defines 3-periodic solutions of the dressing chain [VS93]). This system admits the same Bäcklund transformations as in the commutative case (see Table 1) and has an isomonodromic Lax representation, presented in Section 3.2.2, that is equivalent to the Noumi-Yamada pair for the commutative PIV symmetric form [NY00]. To construct solutions of (1) in terms of the noncommutative Toda equations we use noncommutative analogs of the translation operators (for more details see Section 3). Therefore, the existence of Bäcklund transformations, compositions of which form an affine Weyl group of type $A_2^{(1)}$, is significant for determining solutions of (1). Unlike the PII($u; \beta$) equation, system (1) is not written in the symmetric form. It means that a noncommutative analog of the PIV equation cannot be obtained by using the Weyl ordering.

We also remark that the order of the system can be reduced by the first integral

$$I = f_0 + f_1 + f_2 - t,$$

but in noncommutative situation the resulting system cannot be written as an ODE of the second order, since to do this we have to invert an operator of the form $f g + g f$.

Observe that unlike in the papers [Nag04] and [NGR+08], where quantum versions of the Painlevé equations were considered, we do not impose any additional conditions on $R$. So, system (1) is a generalization of the quantum version of the PIV equation to a “pure noncommutative” case.

On the other hand, with the help of the matrix generalization of the Painlevé-Kovalevskaya test, authors of the papers [BS21a], [BS21b] have derived three non-equivalent matrix Painlevé IV systems, labeled $P_4^i$, $i = 0, 1, 2$. The $P_4^0$ system with scalar parameters is equivalent to (1) with central variable $t$. There is also a fully noncommutative version of the $P_4^0$ system:

$$
\begin{align*}
    u' &= -u^2 + uv + vu + (k - 2) \bar{z}u - k \bar{z}v + \gamma_1, \\
    v' &= -v^2 + vu + uv + k \bar{z}v - (k - 2) v\bar{z} + \gamma_2,
\end{align*}
$$

where $\bar{z}' = 1$ and $k \in \mathbb{C}$; this system admits an isomonodromic Lax pair [BS21b]. It turns out that solutions of this system can be expressed via solutions of the semi-infinite noncommutative Toda equations when $k = 0$ or $k = 2$. The noncommutative version (1) allows us to express its solutions in both directions.

Up to the authors’ knowledge, particular cases of the other two systems from the list in [BS21a], $P_4^1$ and $P_4^2$, do not have Bäcklund transformations that define an affine Weyl group of type $A_2^{(1)}$. Therefore they cannot be solved by the noncommutative Toda equations. The same remark holds for noncommutative systems of the PIV type, obtained in the paper [CdII14] by using the Riemann-Hilbert problem.

Organization of the paper. In Section 1, we give a short overview of the known facts about Toda equations and their solutions in the commutative (Section 1.1) and noncommutative cases (Section 1.2). The overview is based on the papers [KMN+99] and [RR10]. In Section 2 we recall the Hirota bilinear form of the commutative PIV equation, that is derived by Bäcklund transformations of the commutative PIV symmetric form. We note that, unlike the paper [JKM06], we present a direct proof of the fact that solutions of the Painlevé IV equation can be constructed by solutions of the Toda equations under some assumptions for the general initial functions $\kappa_{-1}$ and $\kappa_1$ (see Theorem 2.3 in Section 2.3). In Section 3 we give the generalizations of the results from Section 2 to a “pure noncommutative” case. Namely, we prove that solutions of the system (1) can be expressed via solutions of the infinite noncommutative Toda system (see Theorem 3.2 in Section 3.1). In Sections 3.2.1 and 3.2.2 we discuss the “Hamiltonians” and isomonodromic properties of the system (1). We move some proofs to the appendices (see Appendix A) to make the main text more straightforward and readable.

But they do possess a Bäcklund transformation (see [Adl21]). We are grateful to V. E. Adler, who drew our attention to this fact.
1. Toda equations and their solutions

1.1. Solutions of the commutative Toda chain. In this section, we review the results from [KMN+99].

The Toda equations can be viewed as the recurrence relation for a sequence of functions \( \{\tau_n\}_{n\in\mathbb{Z}}, \tau_n = \tau_n(t) \):

\[
\tau_{n+1} = \tau_n'' - (\tau_n')^2,
\]

with general initial conditions \( \tau_0 \) and \( \tau_1 \). Equation (2) is called the bilinear form of the Toda chain. It is convenient to introduce a sequence of functions \( \{\kappa_n\}_{n\in\mathbb{Z}} \) with \( \kappa_0 = 1 \) by applying the gauge transformation to \( \tau_n \) given by

\[
(\ln \tau_n)'' = (\ln \kappa_n)'' + \kappa_1 \kappa_{n-1}. 
\]

Then the Toda equations (2) become

\[
\kappa_{n+1}\kappa_{n-1} = \kappa_n'\kappa_n - (\kappa_n')^2 + \kappa_1\kappa_n^2, \quad \kappa_0 = 1.
\]

It turns out that \( \kappa_n \) are uniquely determined for the given \( \kappa_1 \) and \( \kappa_1 \); it is also known that \( \kappa_n \) are expressible in a determinant form (see Theorem 2.1 in [KMN+99]):

**Theorem 1.1.** Let \( \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \) be two sequences of functions \( a_n = a_n(t), b_n = b_n(t) \) defined recursively as

\[
a_n = a_{n-1}' + \kappa_1 \sum_{i+j=n-2} a_i a_j, \quad a_0 = \kappa_1,
\]

\[
b_n = b_{n-1}' + \kappa_1 \sum_{i+j=n-2} b_i b_j, \quad b_0 = \kappa_1. 
\]

Let \( \kappa_n \) for \( n \in \mathbb{Z} \) be an \( |n| \times |n| \) Hankel determinant:

\[
\kappa_n = \begin{vmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \ldots & a_{2n-2} \end{vmatrix}, \quad n > 0,
\]

\[
\begin{cases} 
1, & n = 0, \\
\begin{vmatrix} b_0 & b_1 & \ldots & b_{n-1} \\ b_1 & b_2 & \ldots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & b_{n-2} & \ldots & b_{2n-2} \end{vmatrix}, & n < 0.
\end{cases}
\]

Then \( \kappa_n \) satisfies the bilinear Toda equations (3) for general initial conditions \( \kappa_1 \) and \( \kappa_1 \).

**Proof.** Let us denote the determinant of an \( n \times n \) matrix \( X \) from the list above by \( D \) and let \( D \left( \begin{smallmatrix} i_1 & i_2 & \ldots & i_k \\ j_1 & j_2 & \ldots & j_k \end{smallmatrix} \right) \) be the determinant of the matrix obtained from \( X \) by removing \( i_1, \ldots, i_k \) rows and \( j_1, \ldots, j_k \) columns. Then the statement of Theorem 1.1 follows from the relations

\[
D \equiv \kappa_{n+1}, \quad D \begin{pmatrix} n+1 \\ n \end{pmatrix} = \kappa_n, \quad D \begin{pmatrix} n+1 \\ n \end{pmatrix} = \kappa_{n-1},
\]

\[
D \begin{pmatrix} n+1 \\ n \end{pmatrix} = D \begin{pmatrix} n+1 \\ n \end{pmatrix} = \kappa_n', \quad D \begin{pmatrix} n+1 \\ n \end{pmatrix} = \kappa_n + \kappa_{n-1}\kappa_1\kappa_n,
\]
and the Jacobi identity
\[
D \left( \frac{n}{n} \right) \cdot D \left( \frac{n+1}{n+1} \right) - D \left( \frac{n}{n+1} \right) \cdot D \left( \frac{n+1}{n} \right) = D \cdot D \left( \frac{n}{n} \frac{n}{n+1} \right).
\]

Remark 1.1. Let us put \( \theta_n = \kappa_n \kappa_{n-1}^{-1} \) for \( n \geq 0 \), in particular \( \theta_0 = \kappa_0^{-1} \) and \( \theta_1 = \kappa_1 \). Then the Toda equations (3) can be rewritten as
\[
(\ln \theta_n)'' = \theta_{n+1} \theta_n^{-1} - \theta_1 \theta_n^{-1},
\]
Similarly, we can put \( \eta_m = \kappa_m \kappa_{m+1}^{-1} \) for \( m \leq 0 \) to get
\[
(\ln \eta_m)'' = \eta_{m+1} \eta_m^{-1} - \eta_{m-1} \eta_m^{-1}.
\]
Note that \( \eta_0 = \kappa_1^{-1} = \theta_1^{-1} \) and \( \eta_1 = \kappa_1^{-1} = \theta_0^{-1} \). In the commutative case, there is no difference between “positive” and “negative” parts of the Toda equations, but in the noncommutative setting the difference is significant.

Remark 1.2. The substitution \( \theta_n = e^{u_n} \) brings the Toda chain (4) to the classical form:
\[
u_n'' = e^{u_{n+1}} - u_n - e^{u_n} - u_{n-1}.
\]

1.2. Solutions of noncommutative Toda equations. In the current section we give an overview of Section 2 from the paper [RR10].

1.2.1. Quasideterminants and almost Hankel matrices. Consider a matrix algebra over an associative unital ring \( R \) and its element \( X = (x_{ij}) \), where \( 1 \leq i, j \leq n \). We denote by \( X^{ij} \) the \((n-1) \times (n-1)\) matrix obtained from \( X \) by removing the \( i \)-th row and the \( j \)-th column. Let \( r_i \) and \( c_j \) be the \( i \)-th row and the \( j \)-th column of the matrix \( X \):
\[
r_i = (x_{i1}, x_{i2}, \ldots, x_{ij}, \ldots, x_{im})^T, \quad c_j = (x_{1j}, x_{2j}, \ldots, x_{ij}, \ldots, x_{nj})^T.
\]
According to [GR91], the quasideterminant \(|X|_{ij}\) is defined as
\[
\bullet \text{ for } n = 1, |X|_{11} = x_{11};
\bullet \text{ for } n > 1 \text{ under the condition that } X^{ij} \text{ is an invertible matrix, } |X|_{ij} = x_{ij} - r_i (X^{ij})^{-1} c_j.
\]

Example 1.1.
\[
\bullet \text{ Consider a } 2 \times 2 \text{ matrix } A \text{ with generic entries } a_{ij}, \ i, j = 1, 2. \text{ Then it has four quasideterminants:}
\]
\[
|A|_{11} = a_{11} - a_{12} a_{22}^{-1} a_{21}, \quad |A|_{12} = a_{12} - a_{11} a_{22}^{-1} a_{21},
\]
\[
|A|_{21} = a_{21} - a_{22} a_{12}^{-1} a_{11}, \quad |A|_{22} = a_{22} - a_{21} a_{11}^{-1} a_{12}.
\]
\[
\bullet \text{ In the case of } 3 \times 3 \text{ matrix } A \text{ with generic entries } a_{ij}, \ i, j = 1, 2, 3, \text{ there are nine quasideterminants. }
\]
\[
|A|_{11} = a_{11} - a_{12} \left( a_{22} - a_{23} a_{33}^{-1} a_{32} \right)^{-1} a_{21} - a_{12} \left( a_{32} - a_{33} a_{23}^{-1} a_{22} \right)^{-1} a_{31}
\]
\[
- a_{13} \left( a_{23} - a_{22} a_{33}^{-1} a_{32} \right)^{-1} a_{21} - a_{13} \left( a_{33} - a_{32} a_{22}^{-1} a_{23} \right)^{-1} a_{31}.
\]

Note that in the case of commutative ring \( R \), the quasideterminant is given simply by
\[
|X|_{ij} = (-1)^{i+j} \frac{\det X}{\det X^{ij}}, \quad \text{for any } i, j = 1, 2, \ldots, n.
\]

Following the paper [RR10], we define almost Hankel matrices \( H_n(i, j) = (x_{st}) \), where \( s, t = 0, 1, \ldots, n \) and \( i, j \geq 0 \), for a sequence \( x_0, x_1, \ldots \) of the elements of \( R \) in the following way:
\[
x_{st} = x_{s+t}, \quad x_{nt} = x_{i+t}, \quad x_{sn} = x_{s+j}, \quad x_{nn} = x_{i+j},
\]
where \( s, t < n \). In particular, \( H_n(n, n) \) is a Hankel matrix.

Let us denote by \( h_n(i, j) \) the quasideterminant \(|H_n(i, j)|_{nn}\). If at least one of the conditions \( i < n, j < n \) holds, we have \( h_n(i, j) = 0 \).
1.2.2. Noncommutative Toda equations. Consider an associative unital division ring $R$ over a field $F$ equipped with a derivation $D : R \rightarrow R$ such that

1) $D(\alpha) = 0$ for any $\alpha \in F$;
2) there exists an element $t \in R$ such that $D(t) = 1$.

Let us use the following notation $D(u) = u'$, $D^2(u) = u''$, \ldots, and recall that $D(v^{-1}) = -v^{-1}v'v^{-1}$ for any invertible element $v \in R$. We assume that all objects related to the noncommutative case belong to the ring $R$. Elements of the ring $R$ we will call functions.

In the paper [RR10] the following noncommutative analogs of the Toda equations (4) and (5) were introduced:

(6) \[(\theta'_n \theta^{-1}_n)' = \theta_{n+1} \theta^{-1}_{n+1} - \theta_n \theta^{-1}_n, \quad n \geq 0,\]
(7) \[(\eta^{-1}_m \eta'_m)' = \eta_{m+1} \eta^{-1}_{m+1} - \eta_m \eta^{-1}_m, \quad m \leq 0,\]

where $\theta_1 = \kappa_1 = \eta_0^{-1}$ and $\theta_0 = \kappa^{-1}_1 = \eta^{-1}_1$ with general initial conditions $\kappa_{-1}$ and $\kappa_1$. Further in the text, we will refer to the expressions $\theta' \theta^{-1}$ and $\theta^{-1} \theta'$ as the left and right noncommutative logarithmic derivatives of $\theta$, respectively.

There is a noncommutative generalization of Theorem 1.1, obtained in [RR10].

**Theorem 1.2.** Let $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ be two sequences of elements in $R$ defined recursively as

(8) \[a_n = a'_{n-1} + \sum_{i+j = \kappa_{-2}} a_i \kappa_j a_j, \quad a_0 = \kappa_1,\]
(9) \[b_n = b'_{n-1} + \sum_{i+j = \kappa_{-2}} b_i \kappa_j b_j, \quad b_0 = \kappa_{-1}.\]

Let $A_{p+1} = [A_p]_{pp}$ and $\eta_{q-1} = [B_{q-1}]_{qq}$, where $A_n = (a_{i+j})$ and $B_n = (b_{i+j})$, $i, j = 0, 1, \ldots, n$, are Hankel matrices.

Then $\theta_n$, $n \geq 0$ and $\eta_m$, $m \leq 0$ satisfy the systems (6) and (7) respectively for general initial conditions $\kappa_{-1}$ and $\kappa_1$.

**Remark 1.3.** The proof of Theorem 1.2 is based on the properties of almost Hankel matrices and the noncommutative generalization of the Sylvester identity [GR91], [GR92].

**Example 1.2.** Let us show that under the conditions of Theorem 1.2 the left logarithmic derivative $\theta'_1 \theta^{-1}_1$ satisfies the noncommutative Toda chain (6):

$$ (\theta'_1 \theta^{-1}_1)' = \theta_2 \theta^{-1}_1 - \theta_1 \theta^{-1}_0, $$

where

$$ \theta_0 = b_0^{-1}, \quad \theta_1 = a_0, \quad \theta_2 = a_2 - a_1 a_0^{-1} a_1, $$

and the functions $a_n$ are defined by recurrence relations (8). Indeed,

$$ (\theta'_1 \theta^{-1}_1)' = (a'_0 a_0^{-1})' = (a'_0 a_{0-1}) = a'_1 a_{0-1} - (a'_0 a_{0-1})^2 = (a_2 - a_0 b_0 a_0) a_0^{-1} - (a_1 a_{0-1})^2 $$

$$ = (a_2 - a_1 a_0^{-1} a_1) a_0^{-1} - a_0 b_0 = \theta_2 \theta^{-1}_1 - \theta_1 \theta^{-1}_0. $$

Similar computations show that the right logarithmic derivative $\eta^{-1}_{-1} \eta'_{-1}$, of Hankel quasideterminant $\eta_{-1}$ (where the entries $b_n$ of $\eta_{-m}$ are defined by (9)), is a solution of the noncommutative Toda chain (7):

$$ (\eta^{-1}_{-1} \eta'_{-1})' = (b_0^{-1} b'_0)' = (b_0^{-1} b_1)' = -(b_0^{-1} b'_0)^{-2} + b_0^{-1} b_1' = b_0^{-1} (b_2 - b_0 a_0 b_0) - (b_0^{-1} b_1)^{-2} $$

$$ = b_0^{-1} (b_2 - b_1 b_0^{-1} b_1) - a_0 b_0 = \eta_{-2}^{-1} \eta_{-2} - \eta_{-1}^{-1} \eta_{-1}. $$

2. Commutative Painlevé IV equation

In this section we consider the Painlevé IV equation written in the form

$$ y'' = \frac{1}{2} y^{-1} (y')^2 + \frac{3}{2} y^2 - 2 t y^2 + (\frac{1}{2} t^2 - (\alpha_1 - \alpha_0)) y - \frac{1}{2} \alpha_2^2 y^{-1}, $$

where $y = y(t)$ and $\alpha_0 + \alpha_1 + \alpha_2 = 1$, and describe its Bäcklund transformations and some of its representations in the Hirota bilinear form (see Sections 2.1, 2.2). We will follow [Oka81], [Nou04] and refer the reader to these texts for details.
In Section 2.3, we prove that solutions of the Painlevé IV equation are expressible in terms of solutions of the Toda equations under some assumptions for the general initial functions $\kappa_{-1}$ and $\kappa_1$ (see Theorem 2.3 in Section 2.3).

2.1. Symmetric form and Bäcklund transformations. The symmetric form of the PIV equation (10) is given by

\[ \begin{aligned}
    f'_0 &= f_0 f_1 - f_0 f_2 + \alpha_0, \\
    f'_1 &= f_1 f_2 - f_0 f_1 + \alpha_1, \\
    f'_2 &= f_0 f_2 - f_1 f_2 + \alpha_2,
\end{aligned} \tag{11} \]

where $f_i = f_i(t)$ and $\alpha_0 + \alpha_1 + \alpha_2 = 1$. This system has the first integral $I = f_0 + f_1 + f_2 - t$ and is equivalent to PIV (10) for $y = f_2$ by eliminating $f_0$ and $f_1$. We define $\tau$-functions $\tau_i$, $i = 0, 1, 2$ by the formulas

\[ h_0 = \tau_0^{-1}, \quad h_1 = \tau_1^{-1}, \quad h_2 = \tau_2^{-1}, \]

where $h_i$, $i = 0, 1, 2$, are Hamiltonians:

\[ h_0 = f_0 f_1 f_2 + \frac{\alpha_1 - \alpha_2}{3} f_0 + \frac{\alpha_1 + 2 \alpha_2}{3} f_1 - \frac{2 \alpha_1 + \alpha_2}{3} f_2, \]

\[ h_1 = f_0 f_1 f_2 - \frac{2 \alpha_2 + \alpha_0}{3} f_0 + \frac{\alpha_2 - \alpha_0}{3} f_1 + \frac{\alpha_2 + 2 \alpha_0}{3} f_2, \]

\[ h_2 = f_0 f_1 f_2 + \frac{\alpha_0 + 2 \alpha_1}{3} f_0 - \frac{2 \alpha_0 + \alpha_1}{3} f_1 + \frac{\alpha_0 - \alpha_1}{3} f_2. \]

The symmetric form (11) is invariant under the following Bäcklund transformations defined by Table 1:

| $s_0$ | $-\alpha_0$ | $\alpha_1 + \alpha_0$ | $\alpha_2 + \alpha_0$ | $f_0$ | $f_1$ | $f_2$ |
|------|-------------|----------------|----------------|------|------|------|
| $s_1$ | $\alpha_0 + \alpha_1$ | $-\alpha_1$ | $\alpha_2 + \alpha_1$ | $f_0 - \frac{\alpha_1 f_2}{f_1}$ | $f_1$ | $f_2 + \frac{\alpha_1 f_1}{f_2}$ |
| $s_2$ | $\alpha_0 + \alpha_2$ | $\alpha_1 + \alpha_2$ | $-\alpha_2$ | $f_0 + \frac{\alpha_2 f_2}{f_1}$ | $f_1$ | $f_2$ |
| $\pi$ | $\alpha_1$ | $\alpha_2$ | $\alpha_0$ | $f_1$ | $f_2$ | $f_0$ |

Table 1. Bäcklund transformations of the symmetric form (11)

Bäcklund transformations form an extended affine Weyl group of type $A_2^{(1)}$:

\[ \tilde{W} \left( A_2^{(1)} \right) = \langle s_0, s_1, s_2, \pi \rangle, \]

where the generators $s_i$, $i = 0, 1, 2$ and $\pi$ commute with the derivation and satisfy the following fundamental relations

\[ s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad \pi^3 = 1, \quad \pi s_i = s_i \pi, \quad i \in \mathbb{Z}_3. \]

Using the relations between the $f$-variables and the $\tau$-functions

\[ f_0 = (\ln \tau_2^{-1} + \frac{1}{3} t), \quad f_1 = (\ln \tau_2^{-1} + \frac{1}{3} t), \quad f_2 = (\ln \tau_1^{-1} + \frac{1}{3} t), \]

one can show that $\tau$-functions satisfy different bilinear differential equations of the Hirota type. Here are some of them:

\[ (D_t^2 + \frac{1}{3} D_t - \frac{2}{3} t^2 + \frac{1}{3} (\alpha_0 - \alpha_1)) \tau_0 \cdot \tau_1 = 0, \]

\[ (D_t^2 + \frac{1}{3} D_t - \frac{2}{3} t^2 + \frac{1}{3} (\alpha_1 - \alpha_2)) \tau_1 \cdot \tau_2 = 0, \]

\[ (D_t^2 + \frac{1}{3} D_t - \frac{2}{3} t^2 + \frac{1}{3} (\alpha_2 - \alpha_0)) \tau_2 \cdot \tau_0 = 0, \]

and

\[ (\frac{1}{3} D_t^2 - \frac{1}{3} (\alpha_1 - \alpha_2)) \tau_0 \cdot \tau_0 = s_1(\tau_1) s_2(\tau_2), \]

\[ (\frac{1}{3} D_t^2 - \frac{1}{3} (\alpha_1 - \alpha_2)) \tau_1 \cdot \tau_1 = s_2(\tau_2) s_0(\tau_0), \]

\[ (\frac{1}{3} D_t^2 - \frac{1}{3} (\alpha_1 - \alpha_2)) \tau_2 \cdot \tau_2 = s_0(\tau_0) s_1(\tau_1), \]
where $D^n_t$ is the Hirota operator:

$$D^n_t f \cdot g = \left( \frac{d}{dt} - \frac{d}{ds} \right)^n f(t) g(s) \bigg|_{s=t}.$$

2.2. Translations and $\tau$-functions on the lattice. By composing Bäcklund transformations, we define the translation operators $T_i$, $i = 1, 2, 3$:

$$T_1 = \pi s_2 s_1, \quad T_2 = s_1 \pi s_2, \quad T_3 = s_2 s_1 \pi.$$

From this definition, it follows that $T_1 T_2 T_3 = 1$ and

$$T_1^l(\alpha_0) = \alpha_0 + l, \quad T_1^l(\alpha_1) = \alpha_1 - l, \quad T_1^l(\alpha_2) = \alpha_2;$$
$$T_2^m(\alpha_0) = \alpha_0, \quad T_2^m(\alpha_1) = \alpha_1 + m, \quad T_2^m(\alpha_2) = \alpha_2 - m;$$
$$T_3^n(\alpha_0) = \alpha_0 - n, \quad T_3^n(\alpha_1) = \alpha_1, \quad T_3^n(\alpha_2) = \alpha_2 + n.$$

Thus, we define the $\tau$-functions on the lattice by

$$\tau_{l,m,n} = T_1^l T_2^m T_3^n(\tau_0).$$

In particular,

$$\tau_{0,0,0} = \tau_0, \quad \tau_{1,0,0} = \tau_1, \quad \tau_{1,1,0} = \tau_2.$$

Since $T_1 T_2 T_3 = 1$, for any $k \in \mathbb{Z}$ we have $\tau_{l+k,m+k,n+k} = \tau_{l,m,n}$. Applying $T_1^l T_2^m T_3^n$ on the bilinear equations (13) – (14), we obtain the following equations:

$$\left( D^2_t + \frac{1}{3} t D_t - \frac{2}{9} t^2 + \frac{1}{3} (\alpha_0 - \alpha_1 + 2l - m - n) \right) \tau_{l,m,n} \cdot \tau_{l+1,m,n} = 0,$$

and

$$\left( \frac{3}{2} D^2_t - \frac{1}{3} (\alpha_1 - \alpha_2 + 2l - m) \right) \tau_{l,m,n} \cdot \tau_{l+1,m,n} = \tau_{l,m+1,n} \tau_{l,m-1,n}$$

Following the notation in Section 1.1, we introduce the variables $\kappa_{l,m,n}$ by

$$\kappa_{l,m,n} = e^{-\frac{1}{4} ((l+m+n)^2 \tau_{l,m,n} \tau_{0,0,0}^{-1}}.$$

Here $\kappa_{l,m,n}$ depend on three indices corresponding to the three operators $T_1$, $T_2$, and $T_3$ (unlike in the paper [JKM06]): shift of the $i$-th index corresponds to the action of the operator $T_i$, $i = 1, 2, 3$. Then the bilinear equations (15) – (16) become

$$\left( D^2_t - t D_t + 2 \kappa_{l-1,0,0} \kappa_{l,0,0} + (\alpha_0 - \alpha_1 + 2l + m + n) \right) \kappa_{l,m,n} \cdot \kappa_{l+1,m,n} = 0,$$

$$\left( D^2_t - t D_t + 2 \kappa_{0,-1,0} \kappa_{0,1,0} + (\alpha_1 - \alpha_2 + l + 2m + n + 2) \right) \kappa_{l,1,m,n} \cdot \kappa_{l+1,m+1,n+1} = 0,$$

and

$$\left( \frac{1}{2} D^2_t - \kappa_{l-1,0,0} \kappa_{l,0,0} + (m + n) \right) \kappa_{l,m,n} \cdot \kappa_{l,m,n} = \kappa_{l+1,m,n} \kappa_{l-1,m,n},$$
$$\left( \frac{1}{2} D^2_t + \kappa_{0,-1,0} \kappa_{0,1,0} + (l + n) \right) \kappa_{l,m,n} \cdot \kappa_{l,m,n} = \kappa_{l+1,m+1,n} \kappa_{l,m-1,n},$$
$$\left( \frac{1}{2} D^2_t + \kappa_{0,0,-1} \kappa_{0,0,1} + (l + m) \right) \kappa_{l,m,n} \cdot \kappa_{l,m,n} = \kappa_{l+1,m,n+1} \kappa_{l,m,n-1}.$$
2.3. **Hankel determinant formula.** In this section we consider a sequence of the $\kappa$-functions $\kappa_{n,0,0} = \kappa_n$, $n \in \mathbb{Z}$, only in the $T_1$-direction (similar results can be formulated in the directions of the operators $T_2$ and $T_3$). This sequence can be regarded as being generated by the Toda equations (18):

\[
(\frac{1}{2}D_t^2 + \kappa_{-1} \kappa_1) \kappa_n \cdot \kappa_n = \kappa_{n-1} \kappa_{n+1}, \quad \kappa_0 = 1,
\]

where $\kappa_{-1}$ and $\kappa_1$ are general initial conditions. As we have mentioned in Section 1.1, solutions of this equation are given by the Hankel determinants (see Theorem 1.1). For the sake of completeness, we are going to show that solutions of the PIV equation

\[
y'' = \frac{1}{2}y^{-1}(y')^2 + \frac{3}{2}y^3 - 2ty_n^2 + \left(\frac{1}{2}t^2 + \alpha_0 - \alpha_1 + 2n\right)y_n - \frac{1}{2}\alpha_2 y_n^{-1},
\]

where

\[
y_n = (\ln \kappa_{n+1} \kappa_{n}^{-1})' + t,
\]

are also given by the Hankel determinants, if we impose some conditions for $\kappa_{-1}$ and $\kappa_1$.

**Proposition 2.1.** Let $\kappa_{-1}$ and $\kappa_1$ satisfy the following relations

\[
\kappa''_{-1} - t\kappa'_{-1} + 2\kappa^2_{-1} \kappa_1 + (\alpha_0 - \alpha_1 - 2) \kappa_{-1} = 0,
\]

\[
\kappa''_1 + t\kappa'_1 + 2\kappa_{-1} \kappa_1^2 + (\alpha_0 - \alpha_1) \kappa_1 = 0.
\]

Let $z_0 = \kappa_{-1} \kappa_1 - (\alpha_1 + \alpha_2)$. Then

(a) $y_0 = \kappa'_{1} \kappa_{1}^{-1} + t$ is a solution of PIV[y_0;0] (20), if

\[
z_0' = y_0^{-1}z_0' + (\alpha_2 - y_0^2)y_0^{-1}z_0 - (\alpha_1 + \alpha_2)y_0;
\]

(b) $y_{-1} = -\kappa_{-1} \kappa_{1}^{-1} + t$ is a solution of PIV[y_{-1};-1] (20), if

\[
z_0' = y_{-1}^{-1}z_0' + (\alpha_2 - y_{-1}^2)y_{-1}^{-1}z_0 - (\alpha_1 + \alpha_2)y_{-1}.
\]

**Proof.** First of all, we notice that the conditions for $\kappa_{-1}$ and $\kappa_1$ follow from the bilinear equation (17) for $(l,m,n) = (-1,0,0)$ and $(l,m,n) = (0,0,0)$.

- **Case (a).** Consider the case of $y_0$. Using the definition of $y_0$, we take the derivative w.r.t. $t$:

\[
y_0' = (\kappa_1' \kappa_1^{-1} + t) = \kappa''_{1} \kappa_{1}^{-1} - (\kappa_1' \kappa_{1}^{-1})^2 + 1.
\]

Then we use the condition for $\kappa_1$ to reduce the order of the relation:

\[
y_0'' = \kappa''_{1} \kappa_{1}^{-1} - (\kappa_1' \kappa_{1}^{-1})^2 + 1 = -t\kappa_1' \kappa_{1}^{-1} - 2\kappa_{-1} \kappa_1 - (\kappa_1' \kappa_{1}^{-1})^2 + (2\alpha_1 + \alpha_2).
\]

Substituting $\kappa_1' \kappa_{1}^{-1} = y_0 - t$ and $\kappa_{-1} \kappa_1 = z_0 + (\alpha_1 + \alpha_2)$, we arrive at the equation

\[
y_0'' = -2z_0 - y_0^2 + ty_0 - \alpha_2,
\]

that, with the assumption (23) for $z_0$, gives the following system

\[
\begin{cases}
-\kappa_0' &= y_0^{-1}z_0' + (\alpha_2 - y_0^2)y_0^{-1}z_0 - (\alpha_1 + \alpha_2)y_0, \\
-\kappa_0 &= 2z_0 + y_0^2 - ty_0 + \alpha_2,
\end{cases}
\]

which is equivalent to the PIV[y_0;0] equation (20) with $n = 0$:

\[
y''_0 = \frac{1}{2}y_0^{-1}(y')^2 + \frac{3}{2}y_0^3 - 2ty_0^2 + \left(\frac{1}{2}t^2 + \alpha_0 - \alpha_1\right)y_0 - \frac{1}{2}\alpha_2 y_0^{-1}.
\]

- **Case (b).** The case of $y_{-1}$ can be regarded in a similar way. The resulting system is given by

\[
\begin{cases}
\kappa_{-1}' &= y_{-1}^{-1}z_0' + (\alpha_2 - y_{-1}^2)y_{-1}^{-1}z_0 - (\alpha_1 + \alpha_2)y_{-1}, \\
y'_{-1} &= 2z_0 + y_{-1}^2 - ty_{-1} + \alpha_2,
\end{cases}
\]

where the first equation is just condition (24). Eliminating $z$ from this system, we arrive at the PIV[y_{-1};-1] equation (20) with $n = -1$:

\[
y''_{-1} = \frac{1}{2}y_{-1}^{-1}(y'_{-1})^2 + \frac{3}{2}y_{-1}^3 - 2ty_{-1}^2 + \left(\frac{1}{2}t^2 + \alpha_0 - \alpha_1 - 2\right)y_{-1} - \frac{1}{2}\alpha_2 y_{-1}^{-1}.
\]
Remark 2.1. If we substitute $y_{-1}$ and $z_{0}$ into (24) or $y_{0}$ and $z_{0}$ into (23), we get the following additional condition for $\kappa_{-1}$, $\kappa_{1}$:

\[
\kappa'_{-1} \kappa_{1}' + t \left( \kappa_{-1}^2 \kappa_{1}^2 - \kappa_{-1}' \kappa_{1}' \right) + \kappa_{-1}' \kappa_{1}'^2 - \left( t^2 - \alpha_0 + \alpha_1 + 1 \right) \kappa_{-1} \kappa_{1} = \alpha_1 (\alpha_0 - 1).
\]

So, as in the case of the PII equation (see Proposition 2.2 in [JKM04]), the initial functions $\kappa_{-1}$ and $\kappa_{1}$ should satisfy three conditions: (21), (22), and (25).

Condition (25) can be derived without the use of the auxiliary function $z_{0}$. If we require that $y_{0} = \kappa_{1}' \kappa_{1}^{-1} + t$ (resp. $y_{-1} = - \kappa_{1}' \kappa_{1}^{-1} + t$) is a solution of the PIV[$y_{0}; 0$] (resp. PIV[$y_{-1}; -1$]) equation, where $\kappa_{-1}$ and $\kappa_{1}$ satisfy (21) – (22), then $\kappa_{-1}$, $\kappa_{1}$ must satisfy condition (25). The converse statement is also true, but it is more convenient to prove it in other notation and using Bäcklund transformations.

A generalization of Proposition 2.1 to the case of arbitrary $n \in \mathbb{Z}$ is given in Theorem 2.1:

**Theorem 2.1.** Let $\kappa_{n}$, $n \in \mathbb{Z}$, satisfy the Toda equations (19) and the bilinear equation

\[
(D_t^2 - tD_t + 2\kappa_{-1} \kappa_{1} + (\alpha_0 - \alpha_1 + 2n)) \kappa_{n} \cdot \kappa_{n+1} = 0.
\]

Let $y_{n} = (\ln \kappa_{n+1} \kappa_{n}^{-1})' + t$ and the function $z_{n}$ be defined as

\[
z_{n} = \kappa_{n-1} \kappa_{n}^{-2} \kappa_{n+1} - (\alpha_1 + \alpha_2 - n).
\]

Then

(a) for $n \geq 0$, $y_{n} = y_{n}(t)$ is a solution of the PIV[$y_{n}; n$] equation (20), if

\[
-z_{n}' = y_{n-1}^{-1} z_{n} + (\alpha_2 - y_{n-1}^2) y_{n-1}^{-1} z_{n} - (\alpha_1 + \alpha_2 - n) y_{n};
\]

(b) for $n \leq 0$, $y_{n-1} = y_{n-1}(t)$ is a solution of the PIV[$y_{n-1}; n - 1$] equation (20), if

\[
z_{n}' = y_{n-1}^{-1} z_{n} + (\alpha_2 - y_{n-1}^2) y_{n-1}^{-1} z_{n} - (\alpha_1 + \alpha_2 - n) y_{n-1}.
\]

**Proof.** The proof is given by a straightforward computation, in which we will use the bilinear equation (26):

\[
\kappa_{n}'' \kappa_{n}^{-1} - 2 \kappa_{n}' \kappa_{n}^{-1} \kappa_{n+1} \kappa_{n+1}^{-1} + \kappa_{n+1}' \kappa_{n+1}^{-1} - t \left( \kappa_{n}' \kappa_{n+1}^{-1} \kappa_{n+1}' - \kappa_{n} \kappa_{n+1}^{-1} \kappa_{n+1}' \right) + 2 \kappa_{-1} \kappa_{1} + (\alpha_0 - \alpha_1 + 2n) = 0,
\]

which is just the equation (17) for the set $(n, 0, 0)$, and the Toda equations (19):

\[
\kappa_{n+1}' \kappa_{n+1}^{-1} - \left( \kappa_{n}' \kappa_{n}^{-1} \right)^2 + \kappa_{-1} \kappa_{1} = \kappa_{n+1}' \kappa_{n+1}^{-1} \kappa_{n+1}^{-2}.
\]

Note that case (a) and case (b) coincide with statements in cases (a) and (b) from Proposition 2.1 for $n = 0$.

- **Case (a).** Take the derivative of $y_{n} = \kappa_{n+1}' \kappa_{n+1}^{-1} - \kappa_{n}' \kappa_{n}^{-1} + t$ w.r.t. $t$:

\[
y_{n}' = \left( \kappa_{n+1}' \kappa_{n+1}^{-1} - \kappa_{n}' \kappa_{n}^{-1} + t \right)' = \kappa_{n+1}'' \kappa_{n+1}^{-1} - \left( \kappa_{n+1}' \kappa_{n+1}^{-1} \right)^2 - \left( \kappa_{n}' \kappa_{n}^{-1} \right)^2 + 1.
\]

From the bilinear equation it follows that

\[
kappa_{n+1}'' \kappa_{n+1}^{-1} = - \kappa_{n+1}' \kappa_{n+1}^{-1} + 2 \kappa_{n}' \kappa_{n}^{-1} \kappa_{n+1}' \kappa_{n+1}^{-1} + t \left( \kappa_{n}' \kappa_{n+1}^{-1} - \kappa_{n+1}' \kappa_{n+1}^{-1} \right) + 2 \kappa_{-1} \kappa_{1} + (\alpha_0 - \alpha_1 + 2n).
\]

Thus,

\[
y_{n}' = -2 \kappa_{n+1}'' \kappa_{n}^{-1} - 2 \kappa_{-1} \kappa_{1} + 2 \kappa_{n}' \kappa_{n}^{-1} \kappa_{n+1}'' \kappa_{n+1}^{-1} + t \left( \kappa_{n}' \kappa_{n+1}^{-1} - \kappa_{n+1}' \kappa_{n+1}^{-1} \right) - \left( \kappa_{n}' \kappa_{n+1}^{-1} \right)^2 + \left( \kappa_{n}' \kappa_{n+1}^{-1} \right)^2 + (2 \alpha_1 + \alpha_2 - 2n),
\]

where we have used the condition $\alpha_0 + \alpha_1 + \alpha_2 = 1$. Since $\kappa_{n}$ satisfy the Toda equations, we replace $\kappa_{n+1}'' \kappa_{n}^{-1} + \kappa_{-1} \kappa_{1}$ by $\kappa_{n+1}'' \kappa_{n+1}^{-1} + \kappa_{n+1}' \kappa_{n+1}^{-1}$. The result can be written as

\[
y_{n}' = -2 \kappa_{n+1}'' \kappa_{n}^{-1} - 2 \kappa_{n}' \kappa_{n}^{-1} \kappa_{n+1}'' \kappa_{n+1}^{-1} - t \left( \kappa_{n+1}' \kappa_{n+1}^{-1} \kappa_{n+1}'' \kappa_{n+1}^{-1} - \kappa_{n+1}' \kappa_{n+1}^{-1} \kappa_{n+1}'' \kappa_{n+1}^{-1} \right) - \left( \kappa_{n}' \kappa_{n+1}^{-1} \kappa_{n+1}'' \kappa_{n+1}^{-1} \right)^2 + (2 \alpha_1 + \alpha_2 - 2n).
\]

Recall that from the definitions of $y_{n}$ and $z_{n}$, it follows that

\[
(\ln \kappa_{n+1} \kappa_{n}^{-1})' = y_{n} - t
\]

and $\kappa_{n+1}^{-2} \kappa_{n+1} = z_{n} + (\alpha_1 + \alpha_2 - n)$. Therefore, we obtain

\[
y_{n}' = -2 \left( z_{n} + (\alpha_1 + \alpha_2 - n) \right) - t \left( y_{n} - t \right) - (y_{n} - t)^2 + (2 \alpha_1 + \alpha_2 - 2n) = -2z_{n} + y_{n}^2 + ty_{n} - \alpha_2.
\]

This equation with condition (27) give the system

\[
\begin{cases}
-z_{n}' = y_{n}' z_{n} + (\alpha_2 - y_{n}^2) y_{n}^{-1} z_{n} - (\alpha_1 + \alpha_2 - n) y_{n}, \\
y_{n}' = 2z_{n} + y_{n}^2 - ty_{n} + \alpha_2,
\end{cases}
\]

that is equivalent to the PIV[$y_{n}; n$] equation (20):

\[
y_{n}' = \frac{1}{2} y_{n}' (y_{n}' + 2 \alpha_2 - 2n) + \frac{1}{2} y_{n}^2 - 2ty_{n} + (\frac{1}{2} t^2 + \alpha_0 - \alpha_1 + 2n) y_{n} - \frac{1}{2} \alpha_2 y_{n}^{-1}.
\]
• Case (b). Similarly, for the case (b), we have the following chain of identities:

\[ y''_{n-1} = \kappa'_n \kappa_n^{-1} - (\kappa'_n \kappa_n^{-1})^2 - \kappa''_{n-1} \kappa_n^{-1} + (\kappa'_n \kappa_n^{-1})^2 + 1 \]

\[ = 2\kappa'_n \kappa_n^{-1} + 2\kappa_1 \kappa_n^{-1} - (\kappa'_n \kappa_n^{-1})^2 - 2\kappa''_n \kappa_n^{-1} + (\kappa'_n \kappa_n^{-1})^2 - t\left((\kappa'_n \kappa_n^{-1} - \kappa_n^{-1}) + (-2\alpha_1 - \alpha_2 + 2n)\right) \]

\[ = 2\kappa_{n-1} \kappa_2 \kappa_{n+1} + (\kappa'_n \kappa_n^{-1} - \kappa''_n \kappa_n^{-1})^2 + \left((\kappa'_n \kappa_n^{-1} - \kappa''_n \kappa_n^{-1}) + (-2\alpha_1 - \alpha_2 + 2n)\right). \]

Replacing \((\ln \kappa \kappa_n^{-1})'\) by \(y_{n-1} - t\) and \(\kappa_{n-1} \kappa_2 \kappa_{n+1}\) by \(z_n + (\alpha_1 + \alpha_2 - n)\) and taking into account the condition \((28)\), one can obtain the system

\[
\begin{cases}
  z_n' = y_{n-1}^{-1} z_n + (\alpha_2 - y_n^{-1}) y_{n-1}^{-1} z_n + (\alpha_1 + \alpha_2 - n) y_{n-1}, \\
  y_{n-1}' = 2z_n + y_{n-1} - ty_{n-1} + \alpha_2,
\end{cases}
\]

which is also equivalent to the \(\text{PIV}[y_{n-1}^*; n - 1]\) equation \((20)\):

\[ y_{n-1}'' = \frac{1}{2} y_{n-1}' \left(y_{n-1}^{-1}\right)^2 + \frac{3}{2} y_{n-1}' - 2t y_{n-1}' + \left(\frac{1}{2} t^2 + \alpha_0 - \alpha_1 + 2(n - 1)\right) y_{n-1} - \frac{1}{2} \alpha_0^2 y_{n-1}^{-1}. \]

\[ \square \]

For our next goal, it is convenient to introduce variables \(\theta_n = \kappa_n \kappa_{n+1}^{-1}, n \geq 0\), and \(\eta_m = \kappa_m \kappa_{m+1}^{-1}, m \leq 0\). Then bilinear equation \((26)\) can be rewritten as

\[ \theta''_{n+1} + t \theta'_{n+1} + 2\theta^2_{n+1} \theta_n^{-1} + (\alpha_0 - \alpha_1 + 2n) \theta_{n+1} = 0, \]

\[ \eta''_{m-1} - t \eta'_{m-1} + 2\eta^2_{m-1} \eta_m^{-1} + (\alpha_0 - \alpha_1 + 2(m - 1)) \eta_{m-1} = 0. \]

For instance, in the case of \(\theta_{n+1}\), we have the following identities:

\[ \kappa_n^{-2} \left(D^2_t - t D_t + 2\kappa_1 \kappa_1 + (\alpha_0 - \alpha_1 + 2n)\right) \kappa_n \cdot \kappa_{n+1} = 0, \]

\[ \theta''_{n+1} + 2\theta_{n+1} \kappa_n^{-2} \left(\kappa'_n \kappa_n^{-1} - (\kappa'_n \kappa_n^{-1})^2\right) + t\theta'_{n+1} + 2\kappa_1 \kappa_1 \theta_{n+1} + (\alpha_0 - \alpha_1 + 2n) \theta_{n+1} = 0, \]

\[ \theta''_{n+1} + 2\theta_{n+1} \kappa_n^{-2} \left((\kappa_n \kappa_{n+1}^{-1} - \kappa_1 \kappa_1)^2\right) + t\theta'_{n+1} + 2\kappa_1 \kappa_1 \theta_{n+1} + (\alpha_0 - \alpha_1 + 2n) \theta_{n+1} = 0, \]

\[ \theta''_{n+1} + 2\theta^2_{n+1} \theta_n^{-1} + t\theta'_{n+1} + (\alpha_0 - \alpha_1 + 2n) \theta_{n+1} = 0. \]

where we used the Toda chain \((19)\) and the definition of \(\theta_{n+1}\).

Remark 2.2. The resulting conditions, \((29)\) and \((30)\), are consequences of the bilinear equation \((26)\) and the Toda equations \((19)\), but they implicitly require the Toda chain to hold.

Theorem 2.1 can be reformulated in the following way.

**Theorem 2.2.** Let \(\kappa_n, n \in \mathbb{Z}\), be solution of the Toda chain \((19)\). Assume that the functions \(\theta_n = \kappa_n \kappa_{n+1}^{-1}, n \geq 0\), and \(\eta_m = \kappa_m \kappa_{m+1}^{-1}, m \leq 0\), satisfy the following equations

\[ \theta''_{n+1} + t \theta'_{n+1} + 2\theta^2_{n+1} \theta_n^{-1} + (\alpha_0 - \alpha_1 + 2n) \theta_{n+1} = 0, \]

\[ \eta''_{m-1} - t \eta'_{m-1} + 2\eta^2_{m-1} \eta_m^{-1} + (\alpha_0 - \alpha_1 + 2(m - 1)) \eta_{m-1} = 0. \]

Under these conditions

(a) if the function

\[ z_n = \theta_{n+1}^{-1} \theta_{n+1} - (\alpha_1 + \alpha_2 - n) \]

satisfies the equation

\[ z_n' = y_n^{-1} z_n^2 + (\alpha_2 - y_n^{-1}) y_n^{-1} z_n - (\alpha_1 + \alpha_2 - n) y_n, \]

then \( y_n = (\ln \theta_{n+1})' + t \) is a solution of the \(\text{PIV}[y_n; n]\) equation;
Let \( \text{Theorem 2.3.} \) conditions so that coincide with those obtained by substitution of \( (36) \).

**Proof.** Using the definitions of \( y_n, z_n \) and conditions for \( \theta_{n+1} \) and \( \eta_{m-1} \), we can derive systems that are equivalent to the corresponding PIV equations (see Appendix A.1).

**Remark 2.3.** Note that for \( n = m = 0 \), the equations \((29)\) and \((30)\) take the form
\[
(33) \quad \theta_n'' + t \theta_n' + 2 \theta_n \theta_0' + (\alpha_0 - \alpha_1) \theta_0 = 0, \\
(34) \quad \eta_n'' - t \eta_n' + 2 \eta_n \eta_0' + (\alpha_0 - \alpha_1) \eta_0 = 0,
\]
where \( \theta_1 = \kappa_1, \eta_{-1} = \kappa_{-1} \) and \( \theta_0 = \eta_{-1}^{-1}, \eta_0 = \theta_1^{-1} \). One can see that these equations coincide with \((22)\) and \((21)\) from Proposition 2.1, respectively. To obtain from these conditions the equations \((29)\) and \((30)\), one can apply the \( T^k_1 \)-operator to \((33)\) and \((34)\), since for any \( k \in \mathbb{Z} \)
\[
T^k_1(\alpha_0) = \alpha_0 - k, \quad T^k_1(\alpha_1) = \alpha_1 + k,
\]
and
\[
T^k_1(\theta_0) = \theta_k, \quad T^k_1(\eta_0) = \eta_k.
\]
The latter formulas are proved using the definition of \( \tau_n \), namely \( \tau_n = T^n_1(\tau_0) \), and mathematical induction.

**Remark 2.4.** The additional condition \((25)\) for \( \kappa_{-1}, \kappa_1 \) is rewritten in terms of the variables \( \theta_0, \theta_1 \) and \( \eta_0, \eta_{-1} \) as
\[
\theta_n'' + t (\theta_n' + \theta_0 \theta_n') - \theta_n^2 + (t^2 - \alpha_0 + \alpha_1 + 1) \theta_n \theta_0 = \alpha_1 (1 - \alpha_0) \theta_0^2,
\]
\[
\eta_n'' - t (\eta_n' + \eta_0 \eta_n') - \eta_n^2 + (t^2 - \alpha_0 + \alpha_1 + 1) \eta_n \eta_0 = \alpha_1 (1 - \alpha_0) \eta_0^2.
\]
Applying the \( T^k_1 \)-operator to these equalities, we obtain conditions
\[
(35) \quad \theta_n'' + t (\theta_n' + \theta_n' \theta_n') - \theta_n^2 + (t^2 - \alpha_0 + \alpha_1 + 2n + 1) \theta_n \theta_n' = (\alpha_1 - n)(1 - \alpha_0 - n) \theta_n^2, \quad n \geq 0,
\]
\[
(36) \quad \eta_n'' - t (\eta_n' + \eta_n' \eta_n') - \eta_n^2 + (t^2 - \alpha_0 + \alpha_1 + 2m + 1) \eta_n \eta_n' = (\alpha_1 - m)(1 - \alpha_0 - m) \eta_n^2, \quad m \leq 0,
\]
that coincide with those obtained by substitution of \( y_n, z_n \) and \( y_{m-1}, z_m \) into \((31)\) and \((32)\), respectively.

The following proposition connects solutions of the Toda chain \((2)\) constructed from the Hankel matrix representation and those obtained by Bäcklund transformations.

**Proposition 2.2.** Solutions defined by Bäcklund transformations and by Hankel matrices of the Toda equations are equivalent.

**Proof.** Note that solutions of the Toda equations are uniquely determined by the initial conditions \( \kappa_{-1} \) and \( \kappa_1 \). Thus the statement of this proposition follows from the fact that one can choose the same initial conditions so that \( \alpha_0 = \kappa_1 \) and \( \alpha_0 = \kappa_{-1} \).

The main result of this section is the following

**Theorem 2.3.** Let \( \kappa_n, n \in \mathbb{Z} \), be a function generated by the Toda chain \((19)\),
\[
(4\frac{1}{2}D_t^2 + \kappa_{-1} \kappa_1) \kappa_n \cdot \kappa_n = \kappa_{n-1} \kappa_{n+1}, \quad \kappa_0 = 1,
\]
and the initial conditions \( \kappa_{-1}, \kappa_1 \) satisfy the equations
\[
\kappa_{-1}' + t \kappa_{-1} + 2 \kappa_{-1}^2 \kappa_1 + (\alpha_0 - \alpha_1 - 2) \kappa_{-1} = 0, \quad \kappa_1' + t \kappa_1 + 2 \kappa_{-1} \kappa_1^2 + (\alpha_0 - \alpha_1) \kappa_1 = 0,
\]
\[
\kappa_{-1}' + t (\kappa_{-1}' \kappa_1 - \kappa_{-1} \kappa_1') + \kappa_{-1} \kappa_1^2 - (t^2 - \alpha_0 + \alpha_1 + 1) \kappa_{-1} \kappa_1 = \alpha_1 (\alpha_0 - 1).
\]
Then the function \( y_n = (\ln \kappa_{n+1} \kappa_n')' + t \) is a solution of the PIV[\( y_n; n \)] equation:
\[
y_n'' = \frac{1}{2} y_n^{-1} (y_n')^2 + \frac{1}{2} y_n^3 - 2 \frac{t}{y_n} - (\frac{t^2}{2} + \alpha_0 - \alpha_1 + 2n) y_n - \frac{1}{4} \alpha_2 y_n^{-1},
\]
where $\alpha_0 + \alpha_1 + \alpha_2 = 1$.

Proof. As we have already shown, this theorem can be proved by introducing new notation $\theta_n$, $n \geq 0$, and $\eta_m$, $m \leq 0$, and applying the $T^n_{\Pi}$-operator to the conditions for $\kappa_{-1}$, $\kappa_1$. It allows us to derive conditions for $\theta_n$ and $\eta_m$ that lead to the statement of the theorem.

Since in the noncommutative case it is sometimes impossible to reduce a system of ODEs to a single ODE, we prefer to use the auxiliary function $z_n$ and a condition for it instead of (35) or (36).

Now we are going to reformulate Theorem 2.2 for the PIV[$f_{i,n};n$] symmetric system

$$
\begin{aligned}
&f_{0,n} = f_{0,n} f_{1,n} - f_{2,n} f_{0,n} + (\alpha_0 + n), \\
&f_{1,n} = f_{1,n} f_{2,n} - f_{0,n} f_{1,n} + (\alpha_1 - n), \\
&f_{2,n} = f_{2,n} f_{0,n} - f_{1,n} f_{2,n} + \alpha_2,
\end{aligned}
$$

that can be obtained from (11) by applying the $T^n_{\Pi}$-operator to it, by setting

$$
T^n_{\Pi}(f_0) = f_{0,n}, \quad T^n_{\Pi}(f_1) = f_{1,n}, \quad T^n_{\Pi}(f_2) = f_{2,n}.
$$

**Theorem 2.4.** Let $\theta_n$, $n \geq 0$, and $\eta_m$, $m \leq 0$, be chosen as in Theorem 2.2. Then

(a) the functions

$$
\begin{aligned}
f_{0,n} = -f_{1,n} - f_{2,n} + t, \quad \tilde{f}_{1,n} = \theta_{n+1} \theta^{-1}_n - (\alpha_1 - n) = f_{1,n} f_{2,n}, \quad \tilde{f}_{2,n} = (\ln \theta_n + 1)' + t,
\end{aligned}
$$

are solutions of the PIV[$f_{i,n};n$] symmetric form (37), if $f_{1,n}$ satisfies the equation

$$
f'_{1,n} = f_{1,n}^2 + 2 f_{1,n} f_{2,n} - t f_{1,n} + (\alpha_1 - n);$$

(b) the functions

$$
\begin{aligned}
f_{1,m-1} = -f_{0,m-1} - f_{2,m-1} + t, \\
\tilde{f}_{0,m-1} = \eta_{m-1} \eta^{-1}_m + (\alpha_0 + m - 1) = f_{0,m-1} f_{2,m-1}, \quad \tilde{f}_{2,m-1} = -(\ln \eta_{m-1})' + t,
\end{aligned}
$$

are solutions of the PIV[$f_{i,m-1}; m-1$] symmetric form (37), if $f_{0,m-1}$ satisfies the equation

$$
\begin{aligned}
f'_{0,m-1} = -f_{0,m-1}^2 - 2 f_{0,m-1} f_{2,m-1} + t f_{0,m-1} + (\alpha_0 + m - 1).
\end{aligned}
$$

Proof. We repeat the reasoning of the proof of Theorem 2.2.

- **Case (a).** By a straightforward computation, one can get the equation

$$
f_{2,n} = -f_{2,n}^2 - 2 \tilde{f}_{1,n} + t f_{2,n} + \alpha_2 = -f_{2,n}^2 - 2 f_{1,n} f_{2,n} + t f_{2,n} + \alpha_2.
$$

Taking it together with (38), we obtain the system

$$
\begin{aligned}
f'_{1,n} &= f_{1,n}^2 + 2 f_{1,n} f_{2,n} - t f_{1,n} + (\alpha_1 - n), \\
|f'_{2,n} &= -f_{2,n}^2 - 2 f_{1,n} f_{2,n} + t f_{2,n} + \alpha_2.
\end{aligned}
$$

Using the definition of $f_{0,n}$, it can be rewritten as

$$
\begin{aligned}
f'_{1,n} &= f_{1,n} f_{2,n} - f_{0,n} f_{1,n} + (\alpha_1 - n), \\
|f'_{2,n} &= f_{2,n} f_{0,n} - f_{1,n} f_{2,n} + \alpha_2,
\end{aligned}
$$

where

$$
\begin{aligned}
f'_{0,n} &= (-f_{1,n} - f_{2,n} + t)' = -f'_{1,n} - f'_{2,n} + 1 \\
&= -(f_{1,n} f_{2,n} - f_{0,n} f_{1,n} + (\alpha_1 - n)) - (f_{2,n} f_{0,n} - f_{1,n} f_{2,n} + \alpha_2) + (\alpha_0 + \alpha_1 + \alpha_2) \\
&= f_{0,n} f_{1,n} - f_{2,n} f_{0,n} + (\alpha_0 + n).
\end{aligned}
$$

So, we arrive at the PIV[$f_{i,m};n$] symmetric form (37).

- **Case (b).** In a similar way we obtain the equation

$$
f'_{2,m-1} = f_{2,m-1}^2 + 2 f_{0,m-1} - t f_{2,m-1} + \alpha_2 = f_{2,m-1}^2 + 2 f_{0,m-1} f_{2,m-1} - t f_{2,m-1} + \alpha_2,
$$

that, with the assumption (39) and the definition of $f_{1,m-1}$, gives the PIV[$f_{i,m-1}; m-1$] symmetric form (37).
3. Noncommutative version of the Painlevé IV system

A noncommutative version of the PIV symmetric form (11), that has the first integral
\[ I = f_0 + f_1 + f_2 - t, \]
can be written as
\[
\begin{align*}
f_0' &= a_0 f_0 f_1 + (1 - a_0) f_1 f_0 - a_2 f_2 f_0 - (1 - a_2) f_0 f_2 + a_0, \\
f_1' &= a_1 f_1 f_2 + (1 - a_1) f_2 f_1 - a_0 f_0 f_1 - (1 - a_0) f_1 f_0 + a_1, \\
f_2' &= a_2 f_2 f_0 + (1 - a_2) f_0 f_2 - a_1 f_1 f_2 - (1 - a_1) f_2 f_1 + a_2,
\end{align*}
\]
where \(a_0 + a_1 + a_2 = 1\) and \(a_i\) are arbitrary parameters.

**Remark 3.1.** We remark that when \(a_0 + a_1 + a_2 = 0\), this system can be regarded as a fully noncommutative analog of the Lotka-Volterra system.

**Remark 3.2.** The non-abelian generalizations for the Volterra lattices with central time and their Painlevé type reductions were studied in the paper [Adl21]. In particular, the authors have derived two non-equivalent Painlevé IV type systems that admit a Bäcklund transformation different from those given in Table 1.

It turns out that this system admits the same Bäcklund transformations as in the commutative case (see Table 1) with nonzero parameters \(a_i\), if \(a_0 = a_1 = a_2 = a\). Then it takes the form
\[
\begin{align*}
f_0' &= a f_0 f_1 + (1 - a) f_1 f_0 - a f_2 f_0 - (1 - a) f_0 f_2 + a_0, \\
f_1' &= a f_1 f_2 + (1 - a) f_2 f_1 - a f_0 f_1 - (1 - a) f_1 f_0 + a_1, \\
f_2' &= a f_2 f_0 + (1 - a) f_0 f_2 - a f_1 f_2 - (1 - a) f_2 f_1 + a_2.
\end{align*}
\]

**Remark 3.3.** The involution \(xy \mapsto yx\) preserves the form of system (40) changing the parameter \(a\) to \(1 - a\).

### 3.1. Solutions of the noncommutative PIV system

In Section 2.3, we have established that the commutative PIV equation possesses solutions expressible in terms of the Hankel determinant, if we impose some restrictions on the initial conditions \(\kappa_{-1}\) and \(\kappa_1\). In the current section, we are going to generalize this result to a fully noncommutative version of the PIV symmetric system of the form (40). Since as we have remarked earlier, in noncommutative case it is sometimes impossible to reduce a system of ODEs to a single ODE (see, for instance, system (40)), we will generalize Theorem 2.4.

Consider the functions \(\theta_n, n \geq 0\), and \(\eta_m, m \leq 0\) that are defined by the noncommutative Toda equations (6) and (7), respectively. Note that \(\theta_0 = \eta_{-1}^{-1}\) and \(\eta_0 = \theta_1^{-1}\).

**Proposition 3.1.** Let \(\theta_0, \theta_1\) and \(\eta_0, \eta_{-1}\) satisfy the conditions
\[
\begin{align*}
\theta_n'' + t \theta_n' + 2 \theta_1 \theta_0^{-1} \theta_1 + (\alpha_0 - \alpha_1) \theta_1 &= 0, \\
\eta_{-1}' - \eta_{-1} + 2 \eta_{-1} \eta_0^{-1} \eta_{-1} + (\alpha_0 - \alpha_1 - 2) \eta_{-1} &= 0,
\end{align*}
\]
where \(\theta_0 = \eta_{-1}^{-1}\) and \(\eta_0 = \theta_1^{-1}\).

Then
(a) the functions
\[
\begin{align*}
f_0 &= -f_1 - f_2 + t, \\
\tilde{f}_0 &= \theta_1 \theta_0^{-1} - \alpha_1 = \frac{1}{2} f_1 f_2 + \frac{1}{2} f_2 f_1, \\
f_2 &= \theta_1 \theta_1^{-1} + t,
\end{align*}
\]
are solutions of the PIV symmetric form (40) with \(a = 1\), if \(f_1\) satisfies the equation
\[
f_1' = f_1' + f_1 f_2 + f_2 f_1 - t f_1 + \alpha_1;
\]
(b) the functions
\[
\begin{align*}
f_1 &= -f_0 - f_2 + t, \\
\tilde{f}_0 &= \eta_0^{-1} \eta_{-1} + (\alpha_0 - 1) = \frac{1}{2} f_0 f_2 + \frac{1}{2} f_2 f_0, \\
f_2 &= -\eta_{-1}^{-1} \eta_{-1} + t,
\end{align*}
\]
are solutions of the PIV symmetric form (40) with \(\tilde{a}_0 = \alpha_0 - 1, \tilde{\alpha}_1 = \alpha_1 + 1, \) and \(a = 1\), if \(f_0\) satisfies the equation
\[
f_0' = -f_0^2 - f_0 f_2 - f_2 f_0 + \alpha_0 t + \alpha_0.
\]
Proof. The proof is given by a straightforward computation, using definitions of \( f_i \) and conditions (41) – (42).

- **Case (a).** The derivative of \( f_2 \) can be written in the following way:
  \[
  f'_2 = \theta'_1 \theta_1^{-1} - (\theta'_1 \theta_1^{-1})^2 + 1
  = -(\theta'_1 + 2 \theta_1 \theta_0^{-1} \theta_1 + (a_0 - a_1) \theta_1) \theta_1^{-1} - (\theta'_1 \theta_1^{-1})^2 + (a_0 + a_1 + a_2)
  = -t \theta'_1 \theta_1^{-1} - (\theta'_1 \theta_1^{-1})^2 - 2 \theta_1 \theta_0^{-1} + (2a_1 + a_2)
  = -t (f_2 - t) - (f_2 - t)^2 - 2 \left( \tilde{f}_1 + a_1 \right) + (2a_1 + a_2)
  = -f_2^2 - 2 \tilde{f}_1 + f_2 t + a_2 = -f_2^2 - f_2 f_1 - f_1 f_2 + f_2 t + a_2.
  \]

Taking it together with the condition (43), we obtain the system
  \[
  \begin{align*}
  f'_1 &= f_1^2 + f_1 f_2 + f_2 f_1 - t f_1 + a_1, \\
  f'_2 &= -f_2^2 - f_2 f_1 - f_1 f_2 + f_2 t + a_2.
  \end{align*}
  \]

Since \( f_0 = -f_1 - f_2 + t \), the system takes the form
  \[
  \begin{align*}
  f'_1 &= f_1 f_2 - (f_1 - f_2 + t) f_1 + a_1, \\
  f'_2 &= f_2 (f_2 - f_1 + t) - f_1 f_2 + a_2.
  \end{align*}
  \]

Finally, adding to this system the following equation for \( f_0 \):
  \[
  f'_0 = (-f_1 - f_2 + t)' = -f'_1 - f'_2 + 1
  = -(f_1 f_2 - f_0 f_1 + a_1) - (f_2 f_0 - f_1 f_2 + a_1) + (a_0 + a_1 + a_2)
  = f_0 f_1 - f_2 f_0 + a_0.
  \]

we arrive at the PIV symmetric form (40) with \( a = 1 \).

- **Case (b).** Similarly, for \( f_2' \) we have:
  \[
  f'_2 = (\eta^{-2}_1 \eta^{-1}_2 - \eta^{-1}_1 \eta^{-2}_2 + 1
  = (\eta^{-2}_1 \eta^{-1}_2) + \eta^{-1}_2 (-\eta^{-1}_1 t + 2 \eta^{-1}_0 \eta^{-1}_2 + (a_0 - a_1 - 2) \eta^{-1}_0) + (a_0 + a_1 + a_2)
  = (\eta^{-2}_1 \eta^{-1}_2) + \eta^{-1}_0 (-\eta^{-1}_1 t + 2 \eta^{-1}_0 \eta^{-1}_2 + (2a_0 + a_2 - 2)
  = (-f_2 + t)^2 - (-f_2 + t) t + 2 \left( f_0 - (a_0 - 1) \right) + (2a_0 + a_2 - 2)
  = f_2^2 + 2 f_0 - f_2 t + a_2 = f_2^2 + f_2 f_0 + f_0 f_2 - t f_2 + a_2.
  \]

The equation and condition (44) give the system
  \[
  \begin{align*}
  f'_0 &= -f_0^2 - f_0 f_2 - f_2 f_0 + f_0 t + (a_0 - 1), \\
  f'_2 &= f_2^2 + f_2 f_0 + f_0 f_2 - t f_2 + a_2,
  \end{align*}
  \]

that, by the definition of \( f_1' \), can be rewritten as
  \[
  \begin{align*}
  f'_0 &= f_0 f_1 - f_2 f_0 + (a_0 - 1), \\
  f'_2 &= f_2 f_0 - f_1 f_2 + a_2.
  \end{align*}
  \]

Supplementing it by the equation for \( f_1' \):
  \[
  f'_1 = (-f_0 - f_2 + t)' = -f'_0 - f'_2 + 1
  = -(f_0 f_1 - f_2 f_0 + (a_0 - 1)) - (f_2 f_0 - f_1 f_2 + a_2) + (a_0 + a_1 + a_2)
  = f_1 f_2 - f_0 f_1 + (a_1 + 1),
  \]

we get the PIV symmetric form (40) with \( \alpha_0 = a_0 = 1, \alpha_1 = a_1 + 1, \) and \( a = 1 \). \( \square \)

Remark 3.4. Conditions (41) – (42) are noncommutative analogs of conditions (33) – (34).
Proposition 3.1 means that there is only one system (up to the $T$-involution given in Remark 3.3) of the form (40):

$$\begin{align*}
    f'_0 &= f_0 f_1 - f_2 f_0 + \alpha_0, \\
    f'_1 &= f_1 f_2 - f_0 f_1 + \alpha_1, \\
    f'_2 &= f_2 f_0 - f_1 f_2 + \alpha_2,
\end{align*}$$

(45)

that has solutions in the quasideterminant Hankel form. This system is a generalization of the quantum PIV symmetric form defined in the paper [NGR+08] and the matrix $P_4^0$ system from [BS21b] to the fully noncommutative case. In the paper [BS21b], the authors have suggested a noncommutative version of the PIV system with the noncommutative independent variable. This system admits solutions in the quasideterminant form only in the "positive" or "negative" directions for $k = 2$ or $k = 0$. The noncommutative version (45) of the PIV symmetric form allows us to write solutions in both directions.

Now we are going to define solutions in the "positive" and "negative" directions. Since system (45) has the same Weyl group as in the commutative case, we are able to define the $T_1$-operator. Applying it to (45), we obtain the noncommutative PIV[$f_{i,n}; n$] symmetric form

$$\begin{align*}
    f'_{0,n} &= f_{0,n} f_{1,n} - f_{2,n} f_{0,n} + (\alpha_0 + n), \\
    f'_{1,n} &= f_{1,n} f_{2,n} - f_{0,n} f_{1,n} + (\alpha_1 - n), \\
    f'_{2,n} &= f_{2,n} f_{0,n} - f_{1,n} f_{2,n} + \alpha_2,
\end{align*}$$

(46)

where $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and $T^n_1(f_i) = f_{i,n}$. Note that the system has the following first integral

$$I = f_{0,n} + f_{1,n} + f_{2,n} - t.$$

**Theorem 3.1.** Let the functions $\theta_n$, $n \geq 0$ and $\eta_m$, $m \leq 0$ satisfy the noncommutative Toda equations (6) – (7) and the following equations

$$\begin{align*}
    \theta''_{n+1} + t \theta'_{n+1} + 2 \theta_{n+1} \theta_n^{-1} \theta_{n+1} + (\alpha_0 - \alpha_1 + 2n) \theta_{n+1} &= 0, \\
    \eta_{m-1}' - \eta_{m-1} t + 2 \eta_{m-1} \eta_{m-1}' + (\alpha_0 - \alpha_1 + 2(m-1)) \eta_{m-1} &= 0.
\end{align*}$$

(47) (48)

Then

(a) the functions

$$f_{0,n} = -f_{1,n} - f_{2,n} + t, \quad \hat{f}_{1,n} = \theta_{n+1} \theta_n^{-1} - (\alpha_1 - n) = \frac{1}{2} f_{1,n} f_{2,n} + \frac{1}{2} f_{2,n} f_{1,n}, \quad f_{2,n} = \theta_{n+1}' \theta_n^{-1} + t,$$

are solutions of the PIV[$f_{i,n}; n$] symmetric form (46), if $f_{1,n}$ satisfies the equation

$$f'_{1,n} = f^2_{1,n} + f_{1,n} f_{2,n} + f_{2,n} f_{1,n} - t f_{1,n} + (\alpha_1 - n);$$

(49)

(b) the functions

$$f_{1,m-1} = -f_{0,m-1} - f_{2,m-1} + t,$$

$$\hat{f}_{0,m-1} = \eta_m^{-1} \eta_{m-1} + (\alpha_0 + m - 1) = \frac{1}{2} f_{0,m-1} f_{2,m-1} + \frac{1}{2} f_{2,m-1} f_{0,m-1}, \quad f_{2,m-1} = -\eta_{m-1}' \eta_{m-1}' + t,$$

are solutions of the PIV[$f_{i,m}; m$] symmetric form (46), if $f_{0,m-1}$ satisfies the equation

$$f'_{0,m-1} = -f^2_{0,m-1} - f_{0,m-1} f_{2,m-1} + f_{2,m-1} f_{0,m-1} + t (\alpha_0 + m - 1).$$

**Proof.** Using the conditions (47) – (48) and definitions of $f_{i,n}$, one is able to derive the corresponding PIV symmetric form (46) by the same computations as in Proposition 3.1. For more details see Appendix A.2. □

**Remark 3.5.** Conditions (47) and (48) are derived from (41) and (42), respectively, by acting the $T^n_1$-operator on them.

The next proposition follows from the fact that in the noncommutative case Bäcklund transformations preserve the Hankel property just like in the commutative case (see Proposition 2.2).

**Proposition 3.2.** Solutions defined by Bäcklund transformations and by Hankel matrices of the noncommutative Toda equations are equivalent.

**Proof.** Follows from the same arguments as in Proposition 2.2. □

Summarising results of this section, we formulate the following
Theorem 3.2. Let the functions \( \theta_n, n \geq 0 \) and \( \eta_m, m \leq 0 \) be defined by the noncommutative Toda chains (6) - (7) and initial conditions \( \theta_0, \theta_1 \) and \( \eta_0, \eta_1 \) satisfy the equations
\[
\begin{align*}
\theta''_n + t \theta'_n + 2 \theta_1 \theta_0^{-1} \theta_1 + (\alpha_0 - \alpha_1) \theta_1 &= 0, \\
\eta''_{-1} - \eta'_{-1} t + 2 \eta_{-1} \eta_0^{-1} \eta_{-1} + (\alpha_0 - \alpha_1 - 2) \eta_{-1} &= 0,
\end{align*}
\]
where \( \theta_0 = \eta_1^{-1} \) and \( \eta_0 = \theta_1^{-1} \).
Then
(a) the functions
\[
\begin{align*}
f_{0,n} &= -f_{1,n} - f_{2,n} + t, \\
f'_{1,n} &= \theta_{n+1} \theta_n^{-1} - (\alpha_1 - n) = \frac{1}{2} f_{1,n} f_{2,n} + \frac{1}{2} f_{1,n} f_{1,n}, \\
f_{2,n} &= \theta_{n+1} \theta_n^{-1} + t,
\end{align*}
\]
are solutions of the PIV\([f_i,n; n]\) symmetric form (46), if \( f_{1,n} \) satisfies the equation
\[
f'_{1,n} = f^2_{1,n} + f_{1,n} f_{2,n} + f_{2,n} f_{1,n} - t f_{1,n} + (\alpha_1 - n);
\]
(b) the functions
\[
\begin{align*}
f_{1,m-1} &= -f_{0,m-1} - f_{2,m-1} + t, \\
f'_{0,m-1} &= \eta_{m-1}^{-1} \eta_{m-1} + (\alpha_0 + m - 1) = \frac{1}{2} f_{0,m-1} f_{2,m-1} + \frac{1}{2} f_{2,m-1} f_{0,m-1}, \\
f_{2,m-1} &= -\eta_{m-1}^{-1} \eta_{m-1} + t,
\end{align*}
\]
are solutions of the PIV\([f_i,m-1; m-1]\) symmetric form (46), if \( f_{0,m-1} \) satisfies the equation
\[
f'_{0,m-1} = -f^2_{0,m-1} - f_{0,m-1} f_{2,m-1} - f_{2,m-1} f_{0,m-1} + f_{0,m-1} + t (\alpha_0 + m - 1).
\]

3.2. "Hamiltonian" structure and the Lax representation.

3.2.1. "Hamiltonian" structure. The Hamiltonian for the commutative PIV symmetric form is given by (12):
\[
H(f_0, f_1, f_2) = f_0 f_1 f_2 + \frac{1}{2} (\alpha_1 - \alpha_2) f_0 + \frac{1}{2} (\alpha_1 + 2 \alpha_2) f_1 - \frac{1}{3} (2 \alpha_1 + \alpha_2) f_2.
\]
Using a skew-symmetric matrix \( U \),
\[
U = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = ||u_{ij}||, \quad i, j = 0, 1, 2,
\]
one can define a Poisson bracket as
\[
\{ f_i, f_j \} = u_{i,j}.
\]
It turns out that the commutative PIV symmetric form is represented in terms of the linear Poisson bracket (51) with Hamiltonian (50) in the following form
\[
\begin{align*}
f'_0 &= \{ H, f_0 \} + 1, \\
f'_1 &= \{ H, f_1 \}, \\
f'_2 &= \{ H, f_2 \},
\end{align*}
\]
and is Hamiltonian iff \( \alpha_0 + \alpha_1 + \alpha_2 = 0 \). On the other hand, if we introduce the canonical variables
\[
q := f_1, \quad p := f_2, \quad t := f_0 + f_1 + f_2,
\]
then the Hamiltonian takes the form
\[
H(q, p, t) = -p^2 q - pq^2 + pq t - \alpha_1 p + \alpha_2 q + \frac{1}{3} (\alpha_1 - \alpha_2) t
\]
and gives the system
\[
\begin{align*}
q' &= \{ H, q \} = -\partial_p H = q^2 + 2pq - qt + \alpha_1, \\
p' &= \{ H, p \} = \partial_q H = -p^2 - 2pq + pt + \alpha_2.
\end{align*}
\]
In the noncommutative case we have a similar structures.

Proposition 3.3. Let us introduce the following "canonical" variables
\[
q := f_1, \quad p := f_2, \quad t := f_0 + f_1 + f_2,
\]
where functions $f_i$, $i = 0, 1, 2$ satisfy the fully noncommutative PIV symmetric form. In terms of $p$ and $q$ the function
$$H(f_0, f_1, f_2) = a_0 f_0 f_1 f_2 + (2 - a_0 - a_1) f_1 f_2 f_0 + a_1 f_2 f_0 f_1$$
$$- (1 - a_1) f_1 f_2 f_0 + (1 - a_0 - a_1) f_0 f_2 f_1 - (1 - a_0) f_2 f_1 f_0$$
$$+ \frac{1}{3}(\alpha - \alpha_2) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2$$
obtains the form
$$H(q, p, t) = -(1 - a_0)p^2 q + (1 - 2a_0)pqp - (1 - a_0)qp^2 + (1 - a_0 - a_1)q^2 p - (3 - 2a_0 - a_2)qp + (1 - a_0 - a_1)pq^2$$
$$+ (1 - a_0 - a_1)tpq + a_1 ptq - (1 - a_0)ptq + (2 - a_0 - a_1)qpt - (1 - a_1)qtp + a_0 tqp$$
$$\quad - \alpha_1 p + \alpha_2 q + \frac{1}{3}(\alpha_1 - \alpha_2) t$$
and the PIV system is equivalent to the following "Hamiltonian" system
$$\begin{align*}
q' &= -\partial_p H = q^2 + pq + pt - q + \alpha_1, \\
p' &= \partial_q H = -p^2 - pq - pt + \alpha_2.
\end{align*}$$

Proof. By a straightforward computation. □

Remark 3.6. When $a_0 = a_1 = 1$, the "Hamiltonian" reads
$$H = f_0 f_1 f_2 + f_2 f_0 f_1 - f_0 f_2 f_1 + \frac{1}{3}(\alpha_1 - \alpha_2) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2$$
$$= -pq - q^2 p + qqp - pq^2 - tpq + ptq + tqp - \alpha_1 p + \alpha_2 q + \frac{1}{3}(\alpha_1 - \alpha_2) t$$
and is equivalent to a matrix Hamiltonian found in [Kaw15].

Remark 3.7. Since the “Poisson brackets” are defined by the matrix $U$, one can consider their naive “quantization”, which is the replacement of the Poisson brackets by commutators of the generators $f_i$. This can always be done thanks to Theorem 1.2 from the paper [FL98]. A “quantized Hamiltonian” takes the form
$$H = f_2 f_0 f_1 + \frac{1}{3}(\alpha_1 - \alpha_2 + 3\lambda) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2,$$
where \{f_i, f_j\} = \lambda^{-1} [f_i, f_j], \lambda \in F.

3.2. Lax representation. In this section we will discuss different isomonodromic Lax pairs for the noncommutative PIV$[f_i; n]$ symmetric form (46).

Proposition 3.4. For any $n \in \mathbb{Z}$, system (46) can be written as the linear system
$$\begin{align*}
\begin{cases}
\partial_\lambda \Psi_n(\lambda, t) = A_n(\lambda, t) \Psi_n(\lambda, t), \\
\partial_t \Psi_n(\lambda, t) = B_n(\lambda, t) \Psi_n(\lambda, t),
\end{cases}
\end{align*}$$
where matrices $A_n(\lambda, t)$ and $B_n(\lambda, t)$ have the following $\lambda$-dependence
$$A_n(\lambda, t) = A_0 + A_{-1} \lambda^{-1}, \quad B_n(\lambda, t) = B_1 \lambda + B_0,$$
with matrices $A_0$, $A_{-1}$, $B_1$, and $B_0$ given by
$$A_0 = \begin{pmatrix} 0 & 1 & f_{0,n} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{-1} = \begin{pmatrix} \beta_0 & 0 & 0 \\ f_{1,n} & \beta_1 & 0 \\ 1 & f_{2,n} & \beta_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} -f_{2,n} & 0 & 0 \\ 1 & -f_{0,n} & 0 \\ 0 & 1 & -f_{1,n} \end{pmatrix}.$$

Here scalar parameters $\beta_0$, $\beta_1$, $\beta_2$ are related to the $\alpha$’s parameters as
$$\alpha_0 = 1 + \beta_2 - \beta_0 - n, \quad \alpha_1 = \beta_0 - \beta_1 + n, \quad \alpha_2 = \beta_1 - \beta_2.$$

Proof. The compatibility condition
$$\partial_t A_n - \partial_\lambda B_n = [B_n, A_n]$$
leads to the system
$$\begin{align*}
f_{0,n}' &= f_{0,n} f_{1,n} - f_{2,n} f_{0,n} + (1 + \beta_2 - \beta_0), \\
f_{1,n}' &= f_{1,n} f_{2,n} - f_{0,n} f_{1,n} + (\beta_0 - \beta_1), \\
f_{2,n}' &= f_{2,n} f_{0,n} - f_{1,n} f_{2,n} + (\beta_1 - \beta_2)
\end{align*}$$
that is exactly the PIV$[f_i; n]$ symmetric form (46), since relations (53) hold. □
Remark 3.8. The pair is constructed by using the method of non-abelianization of the well-known scalar pairs, suggested in the paper [BS21b].

Note that in the commutative case the Lax pair $A_0(\lambda, t)$, $B_0(\lambda, t)$ is equivalent to the Noumi-Yamada pair [NY00] for the PIV symmetric form (11). The pair is reduced to the Jimbo-Miwa pair [JM81] that has the following dependence on the spectral parameter $\mu$

$$A_0(\mu, t) = A_1 \mu + A_0 + A_{-1} \mu^{-1},$$
$$B_0(\mu, t) = B_1 \mu + B_0,$$
where $A_1$, $A_0$, $A_{-1}$, $B_1$, $B_0$ are $2 \times 2$ matrices and $A_1$, $B_1$ are constant diagonal matrices with different eigenvalues. The reduction is given by a generalized Laplace transform and some constraints on the parameters $\beta_i$ [JKT07]. One can establish the same fact in the noncommutative case.

Proposition 3.5. The Noumi-Yamada pair (52) can be reduced to a pair of the Jimbo-Miwa type.

Proof. We will repeat the proof of this fact in the commutative case, presented in the paper [JKT07].

- **Laplace transform.** First we formally consider the following Laplace transform

$$\Psi_n(\lambda, t) = \int_C e^{\lambda \mu} \Phi_n(\mu, t) \, d\mu,$$
where we assume that one can choose a contour $C$ such that the corresponding terms arising from integration by parts will cancel out.

Remark 3.9. Strict conditions of the existence such a transformation will be explored in forthcoming articles.

This transformation turns the linear system

$$\begin{aligned}
\partial_\lambda \Psi_n(\lambda, t) &= (A_0 + A_{-1} \lambda^{-1}) \Psi_n(\lambda, t), \\
\partial_t \Psi_n(\lambda, t) &= (B_1 \lambda + B_0) \Psi_n(\lambda, t)
\end{aligned}$$

into the following linear system

$$\begin{aligned}
\partial_\mu \Phi_n(\mu, t) &= (A_0 - \mu I)^{-1} (A_{-1} + I) \Phi_n(\mu, t) = A_n(\mu, t) \Phi_n(\mu, t), \\
\partial_t \Phi_n(\mu, t) &= -B_1 \partial_\mu \Phi_n(\mu, t) + B_0 \Phi_n(\mu, t) = B_n(\mu, t) \Phi_n(\mu, t),
\end{aligned}$$

where matrices $A_n$ and $B_n$ have the form

$$A_n(\mu, t) = A_{-1} \mu^{-1} + A_{-2} \mu^{-2} + A_{-3} \mu^{-3},$$
$$B_n(\mu, t) = B_0 + B_{-1} \mu^{-1}.$$
Making a gauge transformation by the matrix \( g = \begin{pmatrix} 1 & f_{2,n} \\ 0 & 1 \end{pmatrix} \), the pair is written in the Jimbo-Miwa form:

\[
\tilde{A}_n(\mu, t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu + \begin{pmatrix} f_{0,n} + f_{1,n} + f_{2,n} & -f_{1,n}f_{2,n} + (\beta_1 + 1) \\ 1 & 0 \end{pmatrix}
+ \begin{pmatrix} f_{2,n}f_{1,n} + (\beta_0 + 1) & -f_{2,n}f_{1,n}f_{2,n} - (\beta_0 - \beta_1)f_2 \\ f_{1,n} & -f_{1,n}f_{2,n} + (\beta_1 + 1) \end{pmatrix} \mu^{-1},
\]

(54)

\[
\tilde{B}_n(\mu, t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu + \begin{pmatrix} 0 & -f_{1,n}f_{2,n} + (\beta_1 + 1) \\ 1 & -f_{0,n} - f_{2,n} \end{pmatrix}.
\]

The compatibility condition

\[
\partial_t \tilde{A}_n - \partial_\mu \tilde{B}_n = \left[ \tilde{B}_n, \tilde{A}_n \right]
\]
leads to the system that is equivalent to the noncommutative PIV\([f_{i,n}; n]\) symmetric form (46).

**Remark 3.10.** If we use the condition \( f_{0,n} + f_{1,n} + f_{2,n} = t \) with an appropriate choice of index \( n \) in the Jimbo-Miwa pair (54), we obtain pairs for systems (55) and (56). In particular, the substitution of \( f_{0,n} = -f_{1,n} - f_{2,n} + t \) into (54) leads to a pair that is a particular case of a slightly general pair written in Conclusion of the paper [BS21b].

**Conclusion**

Using Bäcklund transformations and the symmetric form of the commutative PIV equation, we construct its fully noncommutative version that possesses solutions in terms of the infinite Toda system. Our approach can be useful for constructing such analogs for other Painlevé equations. Their connection to the solutions of the noncommutative Toda system can be applied to describing Darboux-Bäcklund transformations of their solutions.

Our analog leads to a fully noncommutative hierarchy of the PIV systems and to fully noncommutative analogs of the systems of type \( A_l^{(1)} \), \( l = 2, 3, \ldots, N \). One of such infinite sequence of the PIV systems is suggested in [GP21] in the matrix case. The second type of the hierarchy that contains the commutative PIV symmetric form was introduced in [VS93] as a dressing chain and was investigated in the paper [NY00] with the help of the affine Weyl group symmetries.

In the commutative case, the generating functions for the entries of the Hankel determinants are connected with the asymptotic solution at infinity of the isomonodromic problem for the PII and PIV equations [JKM04], [JKM06]. It would be interesting to study this problem in a fully noncommutative case.

**Appendix A.**

**A.1. Proof of Theorem 2.2.**

- **Case (a).** Take the derivative of \( y_n \) w.r.t. \( t \) and use condition (29):

\[
y'_n = (\theta_{n+1}^{-1} + t)' = \theta_{n+1}'\theta_{n+1}^{-2} - (\theta_{n+1}'\theta_{n+1}^{-1})^2 + 1
= -((\theta_{n+1}' + 2\theta_{n+1}^{-1}\theta_{n+1}^{-1} + (\alpha_0 - \alpha_1 + 2n)\theta_{n+1}^{-1})\theta_{n+1}^{-1} - (\theta_{n+1}'\theta_{n+1}^{-1})^2 + 1
= -\theta_{n+1}'\theta_{n+1}^{-1} - 2\theta_{n+1}^{-1}\theta_{n+1}^{-1} - (\alpha_0 - \alpha_1 + 2n) - (\theta_{n+1}'\theta_{n+1}^{-1})^2 + 1.
\]

Replace \( \theta_{n+1}'\theta_{n+1}^{-1} \) by \( y_n - t \) and \( \theta_{n+1}'\theta_{n+1}^{-1} \) by \( z_n + (\alpha_1 + \alpha_2 - n) \) and use \( \alpha_0 + \alpha_1 + \alpha_2 = 1 \):

\[
y'_n = -t(y_n - t) - 2(z_n + (\alpha_1 + \alpha_2 - n)) - (\alpha_0 - \alpha_1 + 2n) - (y_n - t)^2 + 1
= -y_n^2 - 2z_n + ty_n - \alpha_2.
\]

Hence, we obtain the system

\[
\begin{cases}
-\varphi'_n = e^{-2} + (\alpha_2 - y_n^2)z_n - (\alpha_1 + \alpha_2 - n)y_n, \\
y'_n = e^2 + 2z_n - ty_n + \alpha_2,
\end{cases}
\]

that is equivalent to the PIV\([y_n; n]\) equation (20).
• **Case (b).** There is the same chain of identities:

\[ \begin{align*}
    y''_{m-1} &= (-\eta'_{m-1} \eta_{m-1} + t)'' = -\eta''_{m-1} \eta_{m-1} + (\eta'_{m-1} \eta^{-1}_{m-1})^2 + 1 \\
    &= (-t \eta'_{m-1} + 2 \eta'_{m-1} \eta_{m-1} + (\alpha_0 - \alpha_1 + 2(m - 1)) \eta_{m-1}) \eta^{-1}_{m-1} + (\eta'_{m-1} \eta^{-1}_{m-1})^2 + 1 \\
    &= -t \eta''_{m-1} \eta_{m-1} + 2 \eta_{m-1} \eta_{m-1} + (\eta'_{m-1} \eta^{-1}_{m-1})^2 + (\alpha_0 - \alpha_1 + 2m - 1).
\end{align*} \]

Replace \( \eta'_{m-1} \eta^{-1}_{m-1} \) by \( -y_{m-1} + t \) and \( \eta_{m-1} \eta^{-1}_{m-1} \) by \( z_m + (\alpha_1 + \alpha_2 - m) \) and use \( \alpha_0 + \alpha_1 + \alpha_2 = 1 \):

\[ \begin{align*}
    y''_{m-1} &= -t(-y_{m-1} + t) + 2(z_m + (\alpha_1 + \alpha_2 - m)) + (-y_{m-1} + t) + (-2\alpha_1 - \alpha_2 + 2m) \\
    &= y''_{m-1} + 2z_m - ty_{m-1} + \alpha_2.
\end{align*} \]

So, the resulting system

\[ \begin{align*}
    z'_m &= y_m z_m + (\alpha_2 - y_{m-1}) \eta_{m-1} z_m - (\alpha_1 + \alpha_2 - m) y_{m-1}, \\
    y'_{m-1} &= 2z_m - ty_{m-1} + \alpha_2,
\end{align*} \]

reduces to the PIV \([y_{m-1}; m] \) equation (20) for \( y_{m-1} = y_{m-1}(t) \).

### A.2. Proof of Theorem 3.1.

• **Case (a).** The derivative of \( f_{2,n} \):

\[ f'_{2,n} = \theta'_n(\theta^{-1}_{n-1} - \eta_{n-1} - \eta_{n-1})^2 + 1 \]

\[ = -(\theta'_n + 2\theta_n \theta_{n-1} + (\alpha_0 - \alpha_1 + 2\eta_1) \theta^{-1}_{n-1} - (\theta'_n \theta^{-1}_{n-1})^2 + (\alpha_0 + \alpha_1 + \alpha_2) \\
= -t \theta''_n - \theta'_n \theta_{n-1} - \eta_{n-1} \theta_{n-1} + (\alpha_0 - \alpha_1 + 2\eta_1) \theta^{-1}_{n-1} + (\alpha_0 + \alpha_1 + \alpha_2) \\
= -t(f_{2,n} - t) - (f_{2,n} - t)^2 - 2(\hat{f}_{2,n} + (\alpha_1 - n)) + (2\alpha_1 + \alpha_2 - 2n) \\
= -f_{2,n}^2 - 2f_{2,n} t + \alpha_2 = -f_{2,n}^2 - f_{2,n} f_{1,n} - f_{1,n} f_{2,n} + f_{2,n} t + \alpha_2.
\]

This equation together with (49) give the system

\[ (55) \]

\[ \begin{align*}
    f'_{1,n} &= f_{2,n} + f_{1,n} f_{2,n} + f_{1,n} f_{1,n} - t f_{1,n} + (\alpha_1 - n), \\
    f'_{2,n} &= -f_{2,n} - f_{2,n} f_{1,n} - f_{1,n} f_{2,n} + f_{2,n} t + \alpha_2.
\end{align*} \]

Since \( f_{0,n} = -f_{1,n} - f_{2,n} + t \), it can be rewritten as

\[ \begin{align*}
    f'_{1,n} &= f_{1,n} f_{2,n} - f_{0,n} f_{1,n} + (\alpha_1 - n), \\
    f'_{2,n} &= f_{2,n} f_{0,n} - f_{1,n} f_{2,n} + \alpha_2.
\end{align*} \]

Supplementing the system with the following equation for \( f_{0,n} \):

\[ f''_{0,n} = (-f_{1,n} - f_{2,n} + t)'' = -f'_1, n - f'_2, n + 1 \]

\[ = -(f_{1,n} f_{2,n} - f_{0,n} f_{1,n} + (\alpha_1 - n)) - (f_{2,n} f_{0,n} - f_{1,n} f_{2,n} + \alpha_2) + (\alpha_0 + \alpha_1 + \alpha_2) \\
= f_{0,n} f_{1,n} - f_{2,n} f_{0,n} + (\alpha_0 + n),
\]

we get the PIV \([f_{1,n}; n] \) symmetric form (46).

• **Case (b).** Similarly, we have the following chain of identities:

\[ f'_{2,m-1} = (\eta_{m-1} \eta_{m-1} - \eta^{-1}_{m-1} \eta^{-1}_{m-1})^2 + 1 \]

\[ = (\eta_{m-1} \eta_{m-1})^2 + \eta_{m-1} \eta_{m-1} + (-\eta_{m-1} t + 2 \eta_{m-1} \eta_{m-1} \eta_{m-1} + (\alpha_0 - \alpha_1 + 2\eta_1) \eta_{m-1}) + (\eta_{m-1} \eta_{m-1})^2 + 1 \\
= (-f_{2,m-1} - t)^2 - (-f_{2,m-1} + t + 2(\hat{f}_{0,m-1} - (\alpha_0 + (m - 1))) + (2\alpha_0 + \alpha_2 + 2(m - 1)) \\
= f_{2,m-1}^2 + 2\hat{f}_{0,m-1} - t f_{2,m-1} + \alpha_2 = f_{2,m-1}^2 + f_{2,m-1} f_{0,m-1} + f_{0,m-1} f_{2,m-1} - t f_{2,m-1} + \alpha_2.
\]

So we arrive at the system

\[ (56) \]

\[ \begin{align*}
    f'_{0,m-1} &= -f_{0,m-1}^2 - f_{0,m-1} f_{2,m-1} - f_{2,m-1} f_{0,m-1} + f_{0,m-1} t + (\alpha_0 + (m - 1)), \\
    f'_{2,m-1} &= f_{2,m-1}^2 + f_{2,m-1} f_{0,m-1} + f_{0,m-1} f_{2,m-1} - t f_{2,m-1} + \alpha_2.
\end{align*} \]
that by the definition of \( f_{1,m-1} \), can be represented as

\[
\begin{align*}
    f'_{0,m-1} &= f_{0,m-1}f_{1,m-1} - f_{2,m-1}f_{0,m-1} + (\alpha_0 + (m-1)), \\
    f'_{2,m-1} &= f_{2,m-1}f_{0,m-1} - f_{1,m-1}f_{2,m-1} + \alpha_2.
\end{align*}
\]

Taking it with the equation for \( f'_{1,m-1} \),

\[
f'_{1,m-1} = (-f_{0,m-1} - f_{2,m-1} + t)' = -f'_{0,m-1} - f'_{2,m-1} + 1
\]

\[
= -(f_{0,m-1}f_{1,m-1} - f_{2,m-1}f_{0,m-1} + (\alpha_0 + (m-1)))
\]

\[
- (f_{2,m-1}f_{0,m-1} - f_{1,m-1}f_{2,m-1} + \alpha_2) + (\alpha_0 + \alpha_1 + \alpha_2)
\]

\[
= f_{1,m-1}f_{2,m-1} - f_{0,m-1}f_{1,m-1} + (\alpha_1 - (m-1)),
\]

one can get the PIV\([f_{i,m-1}; m-1]\) symmetric form (46).

**Appendix B.**

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