On The Emergence of a New Prime Number
And Omega Sequences

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Abstract
This paper highlights the emergence of the Omega sequence in number theory and its connection with the emergence of a new prime number, and also highlights its theoretical applications for Lucas-Lehmer primality test, and Euclid-Euler theory for even perfect numbers. We also show that Omega sequences unify and give new representations for Mersenne numbers, Fermat numbers, Lucas numbers, Fibonacci numbers, Chebyshev sequence, Dickson sequence, and others.

1 Summary for the main results
For a natural number $n$, we define $\delta(n) = n \pmod{2}$. For an arbitrary real number $x$, $\lfloor x^2 \rfloor$ is the highest integer less than or equal $x^2$. Due to the work of Euclid and Euler, it is well-known that an even integer $n$ is perfect if and only if $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime. A prime of the form $2^p - 1$ is called a Mersenne prime. Here is a summary of some results.

Theorem 1. (Lucas-Lehmer-Moustafa) For any given prime $p \geq 5$, $n := 2^{p-1}$, we associate the double-indexed polynomial sequences $A_r(k), B_r(k)$, which are defined by

\[
A_r(k) = (p - r - k) A_r(k - 1) + 4 (p - 2r) A_{r+1}(k - 1), \quad A_r(0) = 1 \quad \text{for all } r,
\]

\[
B_r(k) = -2 (n - r - k) B_r(k - 1) - 2 (n - 2r - 1) B_{r+1}(k - 1), \quad B_r(0) = 1 \quad \text{for all } r.
\]

Then both of the ratios

\[
\frac{A_0(\lfloor \frac{p}{2} \rfloor)}{(p-1)(p-2)\cdots(p-\lfloor \frac{p}{2} \rfloor)}, \quad \frac{B_0(\lfloor \frac{n}{2} \rfloor)}{(n-1)(n-2)\cdots(n-\lfloor \frac{n}{2} \rfloor)}
\]

are integers. Moreover the number $2^p - 1$ is prime if and only if

\[
\frac{A_0(\lfloor \frac{p}{2} \rfloor)}{(p-1)(p-2)\cdots(p-\lfloor \frac{p}{2} \rfloor)} \mid \frac{B_0(\lfloor \frac{n}{2} \rfloor)}{(n-1)(n-2)\cdots(n-\lfloor \frac{n}{2} \rfloor)}.
\]
Theorem 2. (New representation of Mersenne numbers) For any given odd natural number \( p \), the number \( 2^p - 1 \) can be represented by
\[
2^p - 1 = \frac{A_0(\lfloor \frac{p}{2} \rfloor)}{(p-1)(p-2) \cdots (p-\lfloor \frac{p}{2} \rfloor)},
\]
where the double-indexed polynomial sequence \( A_r(k) \) is defined by the recurrence relation
\[
A_r(k) = (p - r - k) A_r(k - 1) + 4 (p - 2r) A_{r+1}(k - 1), \quad A_r(0) = 1 \quad \text{for all } r.
\]

Theorem 3. For any given natural number \( n \), we associate the double-indexed polynomial sequence \( U_r(k) \), which is defined by
\[
U_r(k) = (n - r - k) U_r(k - 1) - 2 (n - 2r - \delta(n - 1)) U_{r+1}(k - 1),
\]
\[
U_r(0) = 1 \quad \text{for all } r.
\]

Then
\[
U_0(\lfloor \frac{n}{2} \rfloor) \mod (n-1)(n-2) \cdots (n-\lfloor \frac{n}{2} \rfloor) = \begin{cases} 
+2 & n \equiv 0 \pmod{6} \\
+1 & n \equiv 1 \pmod{6} \\
-1 & n \equiv 2 \pmod{6} \\
-2 & n \equiv 3 \pmod{6}
\end{cases}.
\]

Theorem 4. For any given natural number \( n \), we associate the double-indexed polynomial sequence \( V_r(k) \), which is defined by
\[
V_r(k) = +2 (n - r - k) V_r(k - 1) - 2 (n - 2r - \delta(n - 1)) V_{r+1}(k - 1),
\]
\[
V_r(0) = 1 \quad \text{for all } r.
\]

Then
\[
V_0(\lfloor \frac{n}{2} \rfloor) \mod (n-1)(n-2) \cdots (n-\lfloor \frac{n}{2} \rfloor) = \begin{cases} 
+2 & n \equiv 0 \pmod{8} \\
+1 & n \equiv 1, 7 \pmod{8} \\
0 & n \equiv 2 \pmod{8} \\
-1 & n \equiv 3 \pmod{8} \\
-2 & n \equiv 4, 6 \pmod{8}
\end{cases}.
\]

Theorem 5. For any given natural number \( n \), we associate the double-indexed polynomial sequence \( W_r(k) \), which is defined by
\[
W_r(k) = 3 (n - r - k) W_r(k - 1) - 2 (n - 2r - \delta(n - 1)) W_{r+1}(k - 1),
\]
\[
W_r(0) = 1 \quad \text{for all } r.
\]

Then
\[
W_0(\lfloor \frac{n}{2} \rfloor) \mod (n-1)(n-2) \cdots (n-\lfloor \frac{n}{2} \rfloor) = \begin{cases} 
+2 & n \equiv 0 \pmod{12} \\
+1 & n \equiv 1, 11 \pmod{12} \\
0 & n \equiv 3 \pmod{12} \\
-1 & n \equiv 4, 10 \pmod{12} \\
-2 & n \equiv 6 \pmod{12}
\end{cases}.
\]
Theorem 6. For any given natural number $n$, we associate the double-indexed polynomial sequence $T_r(k)$, which is defined by

$$T_r(k) = 4(n - r - k)T_r(k - 1) - 2(n - 2r - \delta(n - 1))T_{r+1}(k - 1),$$

$$T_r(0) = 1 \quad \text{for all } r. \quad (12)$$

Then

$$\frac{T_0(\lfloor \frac{n}{2} \rfloor)}{(n - 1)(n - 2) \cdots (n - \lfloor \frac{n}{2} \rfloor)} = 2^{\delta(n-1)}. \quad (13)$$

Theorem 7. (New representation for Lucas sequence) For any given natural number $n$, we associate the double-indexed polynomial sequence $H_r(k)$, which is defined by

$$H_r(k) = (n - r - k)H_r(k - 1) + 2(n - 2r - \delta(n - 1))H_{r+1}(k - 1),$$

$$H_r(0) = 1 \quad \text{for all } r. \quad (14)$$

Then

$$L(n) = \frac{H_0(\lfloor \frac{n}{2} \rfloor)}{(n - 1)(n - 2) \cdots (n - \lfloor \frac{n}{2} \rfloor)}, \quad (15)$$

where $L(n)$ is Lucas sequence defined by $L(m+1) = L(m) + L(m-1)$, $L(1) = 1$, $L(0) = 2$.

Theorem 8. (New representation for Fermat numbers) For any given natural number $n$, we associate the double-indexed polynomial sequence $F_r(k)$, which is defined by

$$F_r(k) = (2^n - r - k)F_r(k - 1) + 4(2^n - 2r - 1)F_{r+1}(k - 1),$$

$$F_r(0) = 1 \quad \text{for all } r. \quad (16)$$

Then the Fermat number $F_n = 2^{2^n} + 1$ can be represented by

$$F_n = \frac{F_0(2^{n-1})}{(2^{n-1})(2^{n-2}) \cdots (2^n - 1)}. \quad (17)$$

Theorem 9. (New representation for Fibonacci-Lucas oscillating sequence) For any given natural number $n$, we associate the double-indexed polynomial sequence $G_r(k)$, which is defined by

$$G_r(k) = 5(n - r - k)G_r(k - 1) - 2(n - 2r - \delta(n - 1))G_{r+1}(k - 1),$$

$$G_r(0) = 1 \quad \text{for all } r. \quad (18)$$

Then

$$\frac{G_0(\lfloor \frac{n}{2} \rfloor)}{(n - 1)(n - 2) \cdots (n - \lfloor \frac{n}{2} \rfloor)} = \begin{cases} F(n) & \text{if } n \text{ odd} \\ L(n) & \text{if } n \text{ even} \end{cases}, \quad (19)$$

where $F(n)$ is Fibonacci sequence defined by $F(m+1) = F(m) + F(m-1)$, $F(1) = 1$, $F(0) = 0$, and $L(n)$ is Lucas sequence defined by $L(m+1) = L(m) + L(m-1)$, $L(1) = 1$, $L(0) = 2$. 

3
2 The Omega sequence

**Definition 10.** (The Omega sequence associated with \( n \) and a point \( (\zeta, \xi) \))

For any given natural number \( n \), and a point \( (\zeta, \xi) \), \( (\zeta, \xi) \neq (0, 0) \), we associate the double-indexed sequence \( \Omega_r(k|\zeta, \xi|n) \), where \( 0 \leq r + k \leq \left\lfloor \frac{n}{2} \right\rfloor \), such that

\[
\begin{align*}
\Omega_r(k|\zeta, \xi|n) &= (2\zeta - \xi) (n - r - k) \Omega_r(k - 1|\zeta, \xi|n) \\
&\quad - 2 \zeta (n - 2r - \delta(n - 1)) \Omega_{r+1}(k - 1|\zeta, \xi|n), \\
\Omega_r(0|\zeta, \xi|n) &= 1 \quad \text{for all } r.
\end{align*}
\]

In Section 3, we give detailed example to compute the Omega sequence associated with some point.

**Notation 11.** For a given point \( (\zeta, \xi) \), we put

\[\Omega_r(k|\zeta, \xi|n) = \Omega_r(k|n).\]

For a given \( n \), and point \( (\zeta, \xi) \), we put

\[\Omega_r(k|\zeta, \xi|n) = \Omega_r(k).\]

3 An illustrative example of the calculations of \( \Omega \)

**The Omega sequence associated with \((1, -2)\)**

We know that Omega sequence associated with \( n \) and \((1, -2)\) is given by the recurrence relation

\[
\Omega_r(k|1, -2|n) = 4 \ (n - r - k) \ \Omega_r(k - 1|1, -2|n) \\
- 2 \ (n - 2r - \delta(n - 1)) \ \Omega_{r+1}(k - 1|1, -2|n),
\]

\[\Omega_r(0|1, -2|n) = 1 \quad \text{for all } r.\]  \hspace{1cm} (21)

Solving (21), one by one, we get

\[
\Omega_r(1|1, -2|n) = 4 \ (n - r - 1) \ \Omega_r(0|1, -2|n) - 2 \ (n - 2r - \delta(n - 1)) \ \Omega_{r+1}(0|1, -2|n) \\
= 4 \ (n - r - 1) \ (1 - 2 \ (n - 2r - \delta(n - 1)) \ (1) \\
= 2 \ (n + \delta(n - 1) - 2). 
\]

\[\Omega_r(2|1, -2|n) = 4 \ (n - r - 2) \ \Omega_r(1|1, -2|n) - 2 \ (n - 2r - \delta(n - 1)) \ \Omega_{r+1}(1|1, -2|n) \\
= 2^2 \ (n + \delta(n - 1) - 2) \ (n + \delta(n - 1) - 4). \]

Continue the process, we get

\[
\Omega_r(3|1, -2|n) = 4 \ (n - r - 3) \ \Omega_r(2|1, -2|n) - 2 \ (n - 2r - \delta(n - 1)) \ \Omega_{r+1}(2|1, -2|n) \\
= 2^3 \ (n + \delta(n - 1) - 2) \ (n + \delta(n - 1) - 4) \ (n + \delta(n - 1) - 6). \]

Then again we compute

\[
\Omega_r(3|1, -2|n) = 4 \ (n - r - 3) \ \Omega_r(2|1, -2|n) - 2 \ (n - 2r - \delta(n - 1)) \ \Omega_{r+1}(2|1, -2|n) \\
= 2^3 \ (n + \delta(n - 1) - 2) \ (n + \delta(n - 1) - 4) \ (n + \delta(n - 1) - 6). \]

Finally, we obtain the following result
Theorem 12. The Omega sequence associated with \( n \) and the point \((1, -2)\) is given by

\[
\Omega_r(k|1, -2|n) = 2^k \prod_{\lambda=1}^{k} (n + \delta(n - 1) - 2\lambda).
\]  

(25)

Put \( r = 0, k = \left\lfloor \frac{n}{2} \right\rfloor \) in (25), we immediately get

Theorem 13.

\[
\Omega_0\left(\left\lfloor \frac{n}{2} \right\rfloor |1, -2|n\right) = 2^{\left\lfloor \frac{n}{2} \right\rfloor} \prod_{\lambda=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (n + \delta(n - 1) - 2\lambda).
\]  

(26)

Noting that

\[
n + \delta(n - 1) - 2 \left\lfloor \frac{n}{2} \right\rfloor = n + \delta(n - 1) - 2 \frac{n - \delta(n)}{2} = \delta(n - 1) + \delta(n) = 1.
\]  

(27)

Therefore we get the following explicit formula.

Theorem 14.

\[
\Omega_0\left(\left\lfloor \frac{n}{2} \right\rfloor |1, -2|n\right) = 2^{\left\lfloor \frac{n}{2} \right\rfloor} (n + \delta(n - 1) - 2) (n + \delta(n - 1) - 4) \ldots (1).
\]  

(28)

4 Definition and properties of \( \Psi(a, b, n) \)

In [1], the author of the current paper first introduced the following definition

Definition 15. For any given variables \( a, b \), \((a, b) \neq (0, 0)\), and for any natural number \( n \), we define the sequence \( \Psi(a, b, n) = \Psi(n) \), by the following recurrence relation

\[
\Psi(0) = 2, \Psi(1) = 1, \Psi(n + 1) = (2a - b)\delta(n)\Psi(n) - a\Psi(n - 1).
\]  

(29)

4.1 Computing \( \Psi(a, b, n) \)

The polynomials \( \Psi(a, b, n) \) enjoy natural arithmetical and also differential properties and unify many well-known polynomials. One of the methods to compute \( \Psi \)-sequence, see [1], for details, is the following explicit formula

\[
\Psi(a, b, n) = \frac{(2a - b)^{\left\lfloor \frac{n}{2} \right\rfloor}}{2^n} \left\{ \left( 1 + \sqrt{\frac{b + 2a}{b - 2a}} \right)^n + \left( 1 - \sqrt{\frac{b + 2a}{b - 2a}} \right)^n \right\}.
\]  

(30)

In [1], using the formal derivation, we proved the following identity

\[
x^n + y^n = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^i \frac{n}{n-i} \binom{n-i}{i} (xy)^i (x+y)^{n-2i}.
\]  

(31)

From (30), (31) we get the proof of the following
Theorem 16. For any natural number \( n \), the following formula is true

\[
\Psi(a, b, n) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-i}{n-i} \binom{n-i}{i} (-a)^{i}(2a-b)^{\left\lfloor \frac{n}{2} \right\rfloor-i}.
\]  

(32)

5 Arithmetic differential properties

Theorem 17. For any natural number \( n \), and any real numbers \( a, b, \alpha, \beta \), \( \beta a - \alpha b \neq 0 \), there exist unique polynomials in \( a, b, \alpha, \beta \) with integer coefficients, that we call \( \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ r \end{array} \right) \), that depend only on \( a, b, \alpha, \beta, n, \) and \( r \), and satisfy the following polynomial identity

\[
(\beta a - \alpha b)^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^n + y^n}{(x+y)^{\text{g}(n)}} = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ r \end{array} \right) (ax^2 + \alpha y^2)^{\left\lfloor \frac{n}{2} \right\rfloor-r}(bx^2 + by + ay^2)^r.
\]

Moreover

\[
\Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ 0 \end{array} \right) = \Psi(a, b, n),
\]

(34)

and

\[
\Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ \left\lfloor \frac{n}{2} \right\rfloor \end{array} \right) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi(\alpha, \beta, n).
\]

(35)

Proof. We first prove the existence and uniqueness of the integer coefficients of (33). Multiplying (31) by \( (\beta a - \alpha b)^{\left\lfloor \frac{n}{2} \right\rfloor} \), we get integer coefficients for expansion (33) after substituting

\[
(\beta a - \alpha b)(x+y)^2 = (2a-b)(\alpha x^2 + \beta xy + \alpha y^2) + (\beta - 2\alpha)(ax^2 + bxy + ay^2),
\]

(36)

\[
(\beta a - \alpha b)xy = a(\alpha x^2 + \beta xy + \alpha y^2) + (-\alpha)(ax^2 + bxy + ay^2).
\]

(37)

The uniqueness of the coefficients come from the fact that \( \alpha x^2 + \beta xy + \alpha y^2 \) and \( ax^2 + bxy + ay^2 \) are algebraically independent for \( \beta a - \alpha b \neq 0 \). Put \( x = x_0 = -b + \sqrt{b^2 - 4a^2} \), \( y = y_0 = 2a \), then \( ax_0^2 + bx_0y_0 + ay_0^2 = 0 \). It follows that

\[
\Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ 0 \end{array} \right) = (2a-b)^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{2^n}{\left\lfloor \frac{b+2a}{b-2a} \right\rfloor} \left\{ \left( 1 + \sqrt{\frac{b+2a}{b-2a}} \right)^n + \left( 1 - \sqrt{\frac{b+2a}{b-2a}} \right)^n \right\}.
\]

(38)

From (38) and (30) we get (34). Now put \( x = x_1 = -\beta + \sqrt{\beta^2 - 4\alpha^2} \), \( y = y_1 = 2\alpha \), then \( \alpha x_1^2 + \beta x_1 y_1 + \alpha y_1^2 = 0 \). Hence

\[
\Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ \left\lfloor \frac{n}{2} \right\rfloor \end{array} \right) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(2\alpha - b)^{\left\lfloor \frac{n}{2} \right\rfloor}}{2^n} \left\{ \left( 1 + \sqrt{\frac{\beta + 2\alpha}{\beta - 2\alpha}} \right)^n + \left( 1 - \sqrt{\frac{\beta + 2\alpha}{\beta - 2\alpha}} \right)^n \right\}.
\]

(39)

From (39) and (30) we get (35). This completes the proof.
Theorem 18. For $\beta a - \alpha b \neq 0$, the polynomials $\Psi_r(n) := \Psi \left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \right) \frac{n^r}{r!}$ satisfy

$$
\left( \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} \right) \Psi_r(n) = -(r+1)\Psi_{r+1}(n),
$$

$$
\left( a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} \right) \Psi_r(n) = -\left( \lfloor \frac{n}{2} \rfloor - r + 1 \right) \Psi_{r-1}(n).
$$

(40)

Proof. We differentiate (33) with respect to the particular differential operator

$$
\left( \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} \right).
$$

Noting that

$$
\left( \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} \right)(\beta a - \alpha b)^{\lfloor \frac{n}{2} \rfloor - r} (x^n + y^n) = 0,
$$

(41)

we get

$$
0 = \left( \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} \right) \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_r(n) (ax^2 + bxy + ay^2)^{\lfloor \frac{n}{2} \rfloor - r} (ax^2 + bxy + ay^2)^r
$$

$$
= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (ax^2 + bxy + ay^2)^{\lfloor \frac{n}{2} \rfloor - r} \left( \frac{\partial}{\partial a} + \frac{\partial}{\partial b} \right) \Psi_r(n) + (r+1)\Psi_{r+1}(n)
$$

$$
+ \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_r(n) (ax^2 + bxy + ay^2)^{\lfloor \frac{n}{2} \rfloor - r} \left( \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} \right) (ax^2 + bxy + ay^2)^r
$$

$$
+ \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_r(n) (ax^2 + bxy + ay^2)^r \left( \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} \right) (ax^2 + bxy + ay^2)^{\lfloor \frac{n}{2} \rfloor - r}.
$$

(42)

Consequently, from (42), we obtain the following desirable polynomial expansion

$$
0 = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \left( \left( \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} \right) \Psi_r(n) + (r+1)\Psi_{r+1}(n) \right) (ax^2 + bxy + ay^2)^{\lfloor \frac{n}{2} \rfloor - r} (ax^2 + bxy + ay^2)^r.
$$

(43)

As $\beta a - \alpha b \neq 0$, the polynomials $(ax^2 + bxy + ay^2)$ and $(ax^2 + bxy + ay^2)$ are algebraically independent which means that all of the coefficients of (43) must vanish. This means that

$$
\left( \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} \right) \Psi_r(n) + (r+1)\Psi_{r+1}(n) = 0 \quad \text{for all } r.
$$

(44)

Similarly, we differentiate (33) with respect to the particular differential operator

$$
\left( a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} \right).
$$
We get
\[ (a \frac{\partial}{\partial \alpha} + b \frac{\partial}{\partial \beta}) \Psi_r(n) + (\lfloor \frac{n}{2} \rfloor - r + 1) \Psi_{r-1}(n) = 0 \quad \text{for all } r. \] (45)

This completes the proof. \( \square \)

This result immediately gives the following desirable theorem

**Theorem 19.** For \( \beta a - \alpha b \neq 0 \), the polynomials \( \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ r \end{array} \right) \) satisfy
\[ \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ r \end{array} \right) = \frac{(-1)^r}{r!} \left( \alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} \right)^r \Psi(a, b, n), \] (46)

and
\[ \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ r \end{array} \right) = \frac{(-1)^r}{(\lfloor \frac{n}{2} \rfloor - r)!} \left( \alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} \right)^{\lfloor \frac{n}{2} \rfloor - r} \Psi(\alpha, \beta, n). \] (47)

6 An illustrative example of the calculations of \( \Psi \)

It is desirable to clarify how we apply the methods of Theorem (19) to compute the polynomial coefficients \( \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ n \\ r \end{array} \right) \). We choose to compute
\[ \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ 4 \\ r \end{array} \right) \quad \text{for } r = 0, 1, 2 = \lfloor 4 \rfloor. \]

Then, from Theorem (17), we get
\[ \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ 4 \\ 0 \end{array} \right) = \Psi(a, b, 4) = -2a^2 + b^2. \]

Hence, from Theorem (19), we get
\[ \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ 4 \\ 1 \end{array} \right) = \frac{-1}{1} (\alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta})(-2a^2 + b^2) = 4a\alpha - 2b\beta; \]
\[ \Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ 4 \\ 2 \end{array} \right) = \frac{-1}{2} (\alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta})(4a\alpha - 2b\beta) = -2a^2 + b^2. \]

Then from Theorem (17) we immediately obtain the following polynomial identity
\[ (\beta a - \alpha b)^2(x^4 + y^4) = (-2a^2 + b^2)(ax^2 + \beta xy + ay^2)^2 \]
\[ + (4a\alpha - 2b\beta)(\alpha x^2 + \beta xy + \alpha y^2)(ax^2 + bxy + ay^2) \]
\[ + (-2a^2 + b^2)(ax^2 + bxy + ay^2)^2. \] (48)
Generally, it is desirable to search for values for the parameters $a, b, \alpha, \beta$ that make the middle term, $\Psi \left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \right)$ get vanished. Therefore, we put $\left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \right) = \left( \begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right)$. Then the middle coefficient, $4a\alpha - 2b\beta$, of (48), is vanished, and we obtain the following special case for a well-known identity in the history of number theory that is used extensively in the study of equal sums of like powers and in discovering new formulas for Fibonacci numbers

$$x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2.$$  \hspace{0.5cm} (49)

Volume 2, [2], attributes this special case to C. B. Haldeman (1905), although Proth (1878) used it in passing.

7 The fundamental theorem of $\Psi-$ sequence

Now, put $r = \left\lfloor \frac{n}{2} \right\rfloor$ in equation (46) of Theorem (19), together with Theorem (17), we get the following immediate consequence

**Theorem 20.** (The fundamental theorem of $\Psi-$sequence)

For any numbers $a, b, \alpha, \beta$, $\beta a - \alpha b \neq 0$, and any natural number $n$, we have

$$\frac{1}{(\left\lfloor \frac{n}{2} \right\rfloor)!} \left( \frac{\alpha \partial}{\partial a} + \frac{\beta \partial}{\partial b} \right)^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi(a, b, n) = \Psi(\alpha, \beta, n).$$  \hspace{0.5cm} (50)

Also, it is useful to deduce the following relations. Replace each $\alpha$ and $\beta$ by $\lambda \alpha$ and $\lambda \beta$ respectively in Theorem (17), we get the following polynomial identity for any $\lambda$

$$(\lambda \beta a - \lambda \alpha b)^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^n + y^n}{(x + y)^{\delta(n)}} = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{cc} a & b \\ \lambda \alpha & \lambda \beta \end{array} \right) \left( \right) \left( \lambda \alpha x^2 + \lambda \beta xy + \lambda \alpha y^2 \right)^{\left\lfloor \frac{n}{2} \right\rfloor} - r (ax^2 + bxy + ay^2)^r.$$  \hspace{0.5cm} (51)

Then

$$(\beta a - \alpha b)^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^n + y^n}{(x + y)^{\delta(n)}} = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \lambda^{-r} \Psi \left( \begin{array}{cc} a & b \\ \lambda \alpha & \lambda \beta \end{array} \right) \left( \right) \left( \alpha x^2 + \beta xy + \alpha y^2 \right)^{\left\lfloor \frac{n}{2} \right\rfloor} - r (ax^2 + bxy + ay^2)^r.$$  \hspace{0.5cm} (51)

Comparing (51) with (17), we obtain

$$\lambda^{-r} \Psi \left( \begin{array}{cc} a & b \\ \lambda \alpha & \lambda \beta \end{array} \right) \left( \right) = \Psi \left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \right) \left( \right).$$

Similarly, we can prove the following useful relations.
Theorem 21. For any numbers \(a, b, \alpha, \beta, \beta a - \alpha b \neq 0, \lambda, r, n\), we get

\[
\Psi \left( \begin{array}{c|c|c} a & b & n \\ \lambda & \alpha & r \\ \beta & \beta & \lambda \end{array} \right) = \lambda^r \Psi \left( \begin{array}{c|c|c} a & b & n \\ \alpha & \alpha & r \\ \beta & \lambda & \beta \end{array} \right),
\]

and

\[
\Psi \left( \begin{array}{c|c|c} a & b & n \\ \alpha & \alpha & r \\ \beta & \beta & \lambda \end{array} \right) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor - r} \Psi \left( \begin{array}{c|c|c} \alpha & \beta & n \\ \alpha & \alpha & \left\lfloor \frac{n}{2} \right\rfloor - r \\ a & b & \beta \end{array} \right).
\]

(52)

8 The \(\Psi\)-representation for the \(\Psi\)-sequence

We are ready to prove the following theorem.

Theorem 22. For any \(a, b, \alpha, \beta, \theta, n\), \(\beta a - \alpha b \neq 0\), the following identities are true

\[
\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{c|c|c} a & b & n \\ \alpha & \alpha & r \\ \beta & \beta & \lambda \end{array} \right) \theta^r = \Psi(a - \alpha \theta, b - \beta \theta, n).
\]

(53)

Proof. Define \(q_1 := \alpha x^2 + \beta xy + \alpha y^2\) and \(q_2 := \alpha x^2 + bxy + ay^2\) and

\[\Lambda_\theta := \theta q_1 - q_2 = (\alpha \theta - a)x^2 + (\beta \theta - b)xy + (\alpha \theta - a)y^2.\]

From Theorem (17), we know that

\[
(\beta a - \alpha b)^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^n + y^n}{(x + y)^{\theta(n)}} = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{c|c|c} a & b & n \\ \alpha & \alpha & r \\ \beta & \beta & \lambda \end{array} \right) (q_1)^{\left\lfloor \frac{n}{2} \right\rfloor - r} (q_2)^r.
\]

As \(q_2 \equiv \theta q_1 \ (\text{mod} \ \Lambda_\theta)\), we get

\[
(\beta a - \alpha b)^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^n + y^n}{(x + y)^{\theta(n)}} \equiv q_1^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{c|c|c} a & b & n \\ \alpha & \alpha & r \\ \beta & \beta & \lambda \end{array} \right) \theta^r \ (\text{mod} \ \Lambda_\theta).
\]

(54)

Replace each of \(a, b\) by \(\alpha \theta - a, \beta \theta - b\) respectively, in Theorem (17), we obtain

\[
(\beta [\alpha \theta - a] - \alpha [\beta \theta - b])^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^n + y^n}{(x + y)^{\theta(n)}} \equiv \Psi(\alpha \theta - a, \beta \theta - b, n)q_1^{\left\lfloor \frac{n}{2} \right\rfloor} \ (\text{mod} \ \Lambda_\theta).
\]

(55)

As \(\beta [\alpha \theta - a] - \alpha [\beta \theta - b] = -(\beta a - \alpha b)\), and noting from Theorem (21) that

\[
(\beta a - \alpha b)^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi(\alpha \theta - a, \beta \theta - b, n) = \Psi(a - \alpha \theta, b - \beta \theta, n),
\]

(56)
we immediately get the following congruence
\[(\beta a - \alpha b) \left\lfloor \frac{n}{2} \right\rfloor \frac{x^n + y^n}{(x+y)^{\delta(n)}} \equiv \Psi(a - \alpha \theta, b - \beta \theta, n)q_1^{\left\lfloor \frac{n}{2} \right\rfloor} \pmod{\Lambda_\theta}.\] (56)

Now, subtracting (54) and (56), we obtain
\[0 \equiv \left( \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \middle| n \atop r \right) \right) \theta^r - \Psi(a - \alpha \theta, b - \beta \theta, n)q_1^{\left\lfloor \frac{n}{2} \right\rfloor} \pmod{\Lambda_\theta}.\] (57)

As the congruence (57) is true for any \(x, y\), and as \((\beta \theta - b)\alpha - (\alpha \theta - a)\beta = \beta a - \alpha b \neq 0\), then the binary quadratic forms \(\Lambda_\theta\) and \(q_1\) are algebraically independent. This immediately leads to
\[0 = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \middle| n \atop r \right) \theta^r - \Psi(a - \alpha \theta, b - \beta \theta, n).\] (58)

Hence we obtained the proof of (53). This completes the proof of Theorem (22).

\[\square\]

9 Specialization and lifting

The following desirable generalization is important

**Theorem 23.** For any \(a, b, \alpha, \beta, \eta, \xi, n, \beta a - \alpha b \neq 0\), the following identities are true
\[\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \middle| n \atop r \right) \xi^{\left\lfloor \frac{n}{2} \right\rfloor - \eta^r} = \Psi(a \xi^\eta - \alpha \xi, b \xi^\eta - \beta \eta, n).\] (59)

**Proof.** Without loss of generality, let \(\xi \neq 0\). We obtain the proof by replacing each \(\theta\) in equation (53) of Theorem (22) by \(\frac{\eta}{\xi}\), and multiplying each side by \(\xi^{\left\lfloor \frac{n}{2} \right\rfloor}\), and noting from Theorem (21) that
\[\xi^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi(a - \alpha \xi, b - \beta \xi, n) = \Psi(a \xi - \alpha \eta, b \xi - \beta \eta, n).\]

\[\square\]

Replacing \(\theta\) by \(\pm 1\) in (53), we obtain the following desirable special cases

**Theorem 24.** For any \(a, b, \alpha, \beta, n, \beta a - \alpha b \neq 0\) the following identities are true
\[\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \middle| n \atop r \right) = \Psi(a - \alpha, b - \beta, n),\] (60)
\[\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi \left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \middle| n \atop r \right) (-1)^r = \Psi(a + \alpha, b + \beta, n).\]
9.1 More generalizations

Now we can generalize Theorem (22) by applying the following specific differential map
\[
\left(-\frac{\partial}{\partial \theta}\right)^k
\]
on equation (53), and noting that
\[
\left(-\frac{\partial}{\partial \theta}\right)^k \Psi(a - \alpha \theta, b - \beta \theta, n) = (-1)^k(k!\Psi\left(\begin{array}{c}a - \alpha \theta \\ \alpha \\
\end{array} \begin{array}{c}b - \beta \theta \\ \beta \end{array} \bigg| n \right) \bigg| k \right).
\]
Hence we immediately obtain the following desirable generalization

**Theorem 25.** For any \(n, k, a, b, \alpha, \beta, \theta, n\), \(\beta a - \alpha b \neq 0\), the following identity is true
\[
\sum_{r=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{r}{k} \Psi\left(\begin{array}{c}a \\ \alpha \\
b \\ \beta \end{array} \bigg| n \right) \theta^{r-k} = \Psi\left(\begin{array}{c}a - \alpha \theta \\ \alpha \\
b - \beta \theta \\ \beta \end{array} \bigg| n \right) . \quad (61)
\]

Again, without loss of generality, let \(\xi \neq 0\). By replacing each \(\theta\) in (61) by \(\frac{\xi}{\alpha}\), and multiplying each side by \(\xi^{\left\lceil \frac{n}{2} \right\rceil - k}\), and noting the properties of \(\Psi\) of Theorem (21) that
\[
\xi^{\left\lceil \frac{n}{2} \right\rceil - k} \Psi\left(\begin{array}{c}a - \alpha \frac{\xi}{\alpha} \\ \alpha \\
b - \beta \frac{\xi}{\beta} \\ \beta \end{array} \bigg| n \right) = \Psi\left(\begin{array}{c}a\xi - \alpha \eta \\ \alpha \\
b\xi - \beta \eta \\ \beta \end{array} \bigg| n \right) ,
\]
we obtain the following generalization for Theorem (25)

**Theorem 26.** For any \(n, k, a, b, \alpha, \beta, \theta, n\), \(\beta a - \alpha b \neq 0\), the following identity is true
\[
\sum_{r=k}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{r}{k} \Psi\left(\begin{array}{c}a \\ \alpha \\
b \\ \beta \end{array} \bigg| n \right) \xi^{\left\lceil \frac{n}{2} \right\rceil - r} \eta^{r-k} = \Psi\left(\begin{array}{c}a\xi - \alpha \eta \\ \alpha \\
b\xi - \beta \eta \\ \beta \end{array} \bigg| n \right) . \quad (62)
\]

10 The \(\Psi\)-representation for sums of powers

Now, put \(\xi = \alpha x^2 + \beta xy + \alpha y^2\) and \(\eta = ax^2 + bxy + ay^2\) in equation (59) of Theorem (23), we obtain
\[
\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi\left(\begin{array}{c}a \\ \alpha \\
b \\ \beta \end{array} \bigg| n \right) (\alpha x^2 + \beta xy + \alpha y^2)^{\left\lceil \frac{n}{2} \right\rceil - r} (ax^2 + bxy + ay^2)^r = (\beta a - \alpha b)^{\left\lfloor \frac{n}{2} \right\rceil} \Psi(xy, -x^2 - y^2, n) . \quad (63)
\]
Now, from (17), we get
\[
\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rceil} \Psi\left(\begin{array}{c}a \\ \alpha \\
b \\ \beta \end{array} \bigg| n \right) (\alpha x^2 + \beta xy + \alpha y^2)^{\left\lceil \frac{n}{2} \right\rceil - r} (ax^2 + bxy + ay^2)^r = (\beta a - \alpha b)^{\left\lfloor \frac{n}{2} \right\rceil} \frac{x^n + y^n}{(x + y)^{\delta(n)}} . \quad (64)
\]
From (63), (64), we get the following desirable $\Psi$–representation for the sums of powers

**Theorem 27. (The $\Psi$–representation for sums of powers)** For any natural number $n$, the $\Psi$–polynomial satisfy the following identity

$$
\Psi(xy, -x^2 - y^2, n) = \frac{x^n + y^n}{(x + y)^{\delta(n)}}.
$$

**11 The representation of $\Psi$ in terms of $\Omega$ sequence**

**Theorem 28.** For any $n, k, a, b, \alpha, \beta, \theta, n, \beta a - \alpha b \neq 0$, the following expansion is true

$$
\Psi\left(\begin{array}{ccc}
\alpha & b \\
\beta & n
\end{array}\right) = \frac{\left[\frac{n}{2}\right] - k}{\lambda r(k|\alpha, \beta|n)a^r(2a - b)|\frac{n}{r}|^{-k - r}}
$$

where the numbers $\lambda_r(k|\alpha, \beta|n)$ are integers and divisible by $k!$ and satisfy the double-indexed recurrence relation

$$
\begin{cases}
\lambda_r(k|\alpha, \beta|n) = (2\alpha - \beta) \left(\left[\frac{n}{2}\right] - k - r + 1\right) \lambda_r(k - 1|\alpha, \beta|n) \\
\lambda_r(0|\alpha, \beta|n) = (-1)^r \frac{n!}{n - r} (n - r) \end{cases}
$$

*Proof.* From Theorem(19), we know that

$$
\Psi\left(\begin{array}{ccc}
\alpha & b \\
\beta & n
\end{array}\right) = \frac{(-1)^k}{k!} \left(\alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b}\right)^k \Psi(a, b, n).
$$

Then from equation (32), it follows

$$
\Psi\left(\begin{array}{ccc}
\alpha & b \\
\beta & n
\end{array}\right) = \frac{(-1)^k}{k!} \left(\alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b}\right)^k \left[\frac{n}{2}\right] \sum_{r=0}^{\left[\frac{n}{r}\right]} \frac{1}{n - r} \binom{n - r}{r} (-a)^r (2a - b)|\frac{n}{r}|^{-r}
$$

$$
= \frac{(-1)^k}{k!} \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r \frac{n!}{n - r} \binom{n - r}{r} \left(\alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b}\right)^k a^r(2a - b)|\frac{n}{r}|^{-r}.
$$

From (68), and for $0 \leq k + r \leq \left[\frac{n}{2}\right]$, there exist integers $\lambda_r(k|\alpha, \beta|n)$, which are divisible by $k!$, and depend only on the numbers $r, k, \alpha, \beta, n$ (and independent on $a, b$), such that

$$
\Psi\left(\begin{array}{ccc}
\alpha & b \\
\beta & n
\end{array}\right) = \frac{(-1)^k}{k!} \lambda_r(k|\alpha, \beta|n)a^r(2a - b)|\frac{n}{r}|^{-k - r}.
$$
This proves (66). Now, to study the coefficients $\lambda_r(k|\alpha, \beta|n)$, we need to compute the recurrence relation that arise up easily once we notice, from Theorem (18), that the following differential property of $\Psi$ for any non-negative integer $k$

$$
\left(\alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b}\right) \Psi \left(\begin{array}{c} a \\ \alpha \\ \beta \end{array} \bigg| n \bigg| k \right) = -(k + 1) \Psi \left(\begin{array}{c} a \\ \alpha \\ \beta \end{array} \bigg| n \bigg| k + 1 \right),
$$

(70)

Then from (69), (70), we get

$$
\left(\alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b}\right) \left(\frac{-1}{k} \right)^k \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \lambda_r(k|\alpha, \beta|n)a^r(2a - b)\left[\frac{n}{2}\right]^{-k-r}
$$

$$
= -(k + 1)\frac{(-1)^{(k+1)}}{(k+1)!} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \lambda_r(k+1|\alpha, \beta|n)a^r(2a - b)\left[\frac{n}{2}\right]^{-k-1-r}.
$$

(71)

Simplifying, and noting that the coefficients $\lambda_r(k|\alpha, \beta|n)$ are independent on $a, b$, we get

$$
\left(\alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b}\right) \lambda_r(k|\alpha, \beta|n) = 0.
$$

Therefore

$$
\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \lambda_r(k|\alpha, \beta|n)\left(\alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b}\right) a^r(2a - b)\left[\frac{n}{2}\right]^{-k-r}
$$

$$
= \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \lambda_r(k+1|\alpha, \beta|n)a^r(2a - b)\left[\frac{n}{2}\right]^{-k-1-r}.
$$

(72)

Hence

$$
\sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \alpha (r + 1) \lambda_{r+1}(k|\alpha, \beta|n) a^r(2a - b)\left[\frac{n}{2}\right]^{-k-r-1}
$$

$$
+ \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (2\alpha - \beta)\left(\left\lfloor \frac{n}{2} \right\rfloor - k - r\right) \lambda_r(k|\alpha, \beta|n) a^r(2a - b)\left[\frac{n}{2}\right]^{-k-r-1}
$$

(73)

$$
= \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \lambda_r(k+1|\alpha, \beta|n)a^r(2a - b)\left[\frac{n}{2}\right]^{-k-r-1}.
$$

From (73), comparing the coefficients, and noting that $a, 2a-b$ are algebraically independent, we immediately get
\[ \alpha \left( r + 1 \right) \lambda_{r+1}(k|\alpha, \beta|n) + (2\alpha - \beta) \left( \left\lfloor \frac{n}{2} \right\rfloor - k - r \right) \lambda_r(k|\alpha, \beta|n) = \lambda_r(k+1|\alpha, \beta|n). \]  

(74)

Now the initial value for \( \lambda_r(k|\alpha, \beta|n) \), that corresponds to \( k = 0 \), is given by

\[
\Psi \left( \begin{array}{c} a \\ \alpha \\ b \\ \beta \\ 0 \end{array} \bigg| n \right) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor - k} \lambda_r(0|\alpha, \beta|n) a^r (2a - b)^{\left\lfloor \frac{n}{2} \right\rfloor - r}.
\]

(75)

Then, from (34), we get

\[
\Psi(a, b, n) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor - k} \lambda_r(0|\alpha, \beta|n) a^r (2a - b)^{\left\lfloor \frac{n}{2} \right\rfloor - r}.
\]

(76)

Consequently, from (32), we immediately get

\[
\lambda_r(0|\alpha, \beta|n) = (-1)^r \frac{n}{n-r} \binom{n-r}{r},
\]

(77)

which completes the proof.

**Theorem 29.** For any \( n, r, k, \alpha, \beta, n \), the following relation is true

\[
\lambda_r(k|\alpha, \beta|n) = (-1)^r \frac{n (n-r-k-1)! \left( \left\lfloor \frac{n}{2} \right\rfloor - r \right)!}{(n-2r)! \quad r! \quad \left( \left\lfloor \frac{n}{2} \right\rfloor - r - k \right)!} \Omega_r(k|\alpha, \beta|n).
\]

(78)

**Proof.** To prove (78), define \( \bar{\Omega}_r(k|\alpha, \beta|n) \) as following

\[
\lambda_r(k|\alpha, \beta|n) = (-1)^r \frac{n (n-r-k-1)! \left( \left\lfloor \frac{n}{2} \right\rfloor - r \right)!}{(n-2r)! \quad r! \quad \left( \left\lfloor \frac{n}{2} \right\rfloor - r - k \right)!} \bar{\Omega}_r(k|\alpha, \beta|n).
\]

(79)

We need to prove that

\[
\bar{\Omega}_r(k|\alpha, \beta|n) = \Omega_r(k|\alpha, \beta|n),
\]

(80)

as following. From (79), (67), we get

\[
(-1)^r \frac{n (n-r-k-1)! \left( \left\lfloor \frac{n}{2} \right\rfloor - r \right)!}{(n-2r)! \quad r! \quad \left( \left\lfloor \frac{n}{2} \right\rfloor - r - k \right)!} \bar{\Omega}_r(k|\alpha, \beta|n)
\]

\[
= (2\alpha - \beta) \left( \left\lfloor \frac{n}{2} \right\rfloor - k - r + 1 \right)(-1)^r \frac{n (n-r-k)! \left( \left\lfloor \frac{n}{2} \right\rfloor - r \right)!}{(n-2r)! \quad r! \quad \left( \left\lfloor \frac{n}{2} \right\rfloor - r - k + 1 \right)!} \bar{\Omega}_r(k-1|\alpha, \beta|n)
\]

\[
+ \alpha (r+1) (-1)^{r+1} \frac{n (n-r-k-1)! \left( \left\lfloor \frac{n}{2} \right\rfloor - r - 1 \right)!}{(n-2r-2)! \quad (r+1)! \quad \left( \left\lfloor \frac{n}{2} \right\rfloor - r - k \right)!} \Omega_{r+1}(k-1|\alpha, \beta|n).
\]

(81)
Simplifying again, we get

\[
\bar{\Omega}_r(k|\alpha, \beta | n) \\
= (2\alpha - \beta) (n - r - k) \bar{\Omega}_r(k - 1|\alpha, \beta | n) \\
- \alpha \frac{(n - 2r) (n - 2r - 1)}{\left(\left\lfloor \frac{n}{2} \right\rfloor - r \right)!} \bar{\Omega}_{r+1}(k - 1|\alpha, \beta | n). \tag{82}
\]

If \(\delta(n) = 0\) then \(\delta(n - 1) = 1\), and if \(\delta(n) = 1\) then \(\delta(n - 1) = 0\). Therefore, for either case, we get the following

\[
\frac{(n - 2r) (n - 2r - 1)}{\left(\left\lfloor \frac{n}{2} \right\rfloor - r \right)!} = \frac{(n - 2r - \delta(n)) (n - 2r - \delta(n - 1))}{\frac{n - \delta(n)}{2} - r}
\]

\[
= 2 \frac{(n - 2r - \delta(n)) (n - 2r - \delta(n - 1))}{n - 2r - \delta(n)}
\]

\[
= 2 (n - 2r - \delta(n - 1)). \tag{83}
\]

Hence, from (82), (83), we get

\[
\bar{\Omega}_r(k|\alpha, \beta | n) \\
= (2\alpha - \beta) (n - r - k) \bar{\Omega}_r(k - 1|\alpha, \beta | n) \\
- 2 \alpha (n - 2r - \delta(n - 1)) \bar{\Omega}_{r+1}(k - 1|\alpha, \beta | n). \tag{84}
\]

Now, it remains to compute the initial value

\[
\bar{\Omega}_r(0|\alpha, \beta | n),
\]

as following. Put \(k = 0\) in (79), and noting (67), we get

\[
(-1)^r \frac{n}{n - r} \binom{n - r}{r} = (-1)^r \frac{n (n - r - 1)!}{(n - 2r)!} \frac{\left(\left\lfloor \frac{n}{2} \right\rfloor - r \right)!}{\left(\left\lfloor \frac{n}{2} \right\rfloor - r \right)!} \bar{\Omega}_r(0|\alpha, \beta | n). \tag{85}
\]

Consequently, for any \(r\), we get

\[
1 = \bar{\Omega}_r(0|\alpha, \beta | n). \tag{86}
\]

From (86), (84), and from definition (10) of Omega sequence, we immediately conclude

\[
\bar{\Omega}_r(k|\alpha, \beta | n) = \Omega_r(k|\alpha, \beta | n). \tag{87}
\]

This completes the proof of Theorem (29).

Therefore, from Theorem (28) and Theorem (29), we get the following representation for

\[
\Psi \left( \begin{array}{c|c|c}
\alpha & b & n \\
\hline
\alpha & \beta & k
\end{array} \right)
\]

in terms of \(\Omega\)-sequence, which is quite desirable.
Theorem 30. (The representation of $\Psi$–sequence in terms of $\Omega$–sequence)
For any numbers $a, b, \alpha, \beta, n$, $\beta a - \alpha b \neq 0$, we get the following expansion

$$
\Psi \left( \begin{array}{c} a \\ \alpha \\
\beta \\
\end{array} \mid \begin{array}{c} b \\ n \\
\end{array} \right) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor - k} (-1)^{r+k} \frac{(n-r-k-1)! n}{(n-2r)! r!} \left( \begin{array}{c} \frac{n}{2} \\ k \\
\end{array} - r \right) \Omega_r(k \mid \alpha, \beta \mid n) \quad (88)
$$

where the coefficients

$$
(-1)^{r+k} \frac{(n-r-k-1)! n}{(n-2r)! r!} \left( \begin{array}{c} \frac{n}{2} \\ k \\
\end{array} - r \right) \Omega_r(k \mid \alpha, \beta \mid n) \quad (89)
$$

are integers.

Now, we get the following desirable theorem.

Theorem 31. (The first fundamental theorem of $\Omega$–sequence)
For any numbers $\alpha, \beta, n$, $(\alpha, \beta) \neq (0, 0)$, the ratio

$$
\frac{\Omega_0 \left( \left\lfloor \frac{n}{2} \right\rfloor \mid \alpha, \beta \mid n \right)}{(n-1)(n-2) \cdots (n-\left\lfloor \frac{n}{2} \right\rfloor)} \quad (90)
$$

is integer. Moreover this ratio gives $\Psi(\alpha, \beta, n)$. Namely

$$
\Psi(\alpha, \beta, n) = \frac{\Omega_0 \left( \left\lfloor \frac{n}{2} \right\rfloor \mid \alpha, \beta \mid n \right)}{(n-1)(n-2) \cdots (n-\left\lfloor \frac{n}{2} \right\rfloor)}. \quad (91)
$$

Proof. To deduce the formula (91) for

$$
\Omega_0 \left( \left\lfloor \frac{n}{2} \right\rfloor \mid \alpha, \beta \mid n \right),
$$

we put $k = \left\lfloor \frac{n}{2} \right\rfloor$ in Theorem (30) as following

$$
\Psi \left( \begin{array}{c} a \\ \alpha \\
\beta \\
\end{array} \mid \begin{array}{c} b \\ \left\lfloor \frac{n}{2} \right\rfloor \\
\end{array} \right) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor - k} (-1)^{r+k} \frac{(n-r-\left\lfloor \frac{n}{2} \right\rfloor - 1)! n}{(n-2r)! r!} \left( \begin{array}{c} \frac{n}{2} \\ k \\
\end{array} - r \right) \Omega_r \left( \left\lfloor \frac{n}{2} \right\rfloor \mid \alpha, \beta \mid n \right) \quad (92)
$$

$$
\quad = (\alpha) \frac{\Omega_0 \left( \left\lfloor \frac{n}{2} \right\rfloor \mid \alpha, \beta \mid n \right)}{(n-1)(n-2) \cdots (n-\left\lfloor \frac{n}{2} \right\rfloor)}. \quad (92)
$$
From Theorem (17), it follows that

\[ \Psi \left( \begin{array}{cc} a & b \\ \alpha & \beta \end{array} \right) \equiv \frac{n}{2^k} \right) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi(\alpha, \beta, n). \] (93)

From (92) and (93), we immediately get the proof.

Now, from Theorems (31) and (27), we get the following theorem

**Theorem 32. (The \(\Omega\)–representation for sums of powers)**

\[ \frac{x^n + y^n}{(x + y)^\delta(n)} = \Omega_0 \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \frac{xy, -x^2 - y^2 | n}{(n - 1)(n - 2) \cdots (n - \left\lfloor \frac{n}{2} \right\rfloor)} \] (94)

Before exploring the second fundamental theorem of Omega sequence, we need to define the Omega space at level \(n\) as following.

**12 The Omega space at level \(n\)**

**Definition 33. (The Omega space at level \(n\))**

For any given natural number \(n\), we define the Omega space, \(\omega(n)\), at level \(n\) as following

\[ \omega(n) := \{(a, b) \mid \Psi(a, b, n) \neq 0\}. \] (95)

Here are some specific examples for points belonging to Omega space at level \(n\).

- If \(n \equiv \pm 1 \mod 8\) ⇒ \(\Psi(1, 0, n) = 1 \neq 0 \Rightarrow (1, 0) \in \omega(n)\)
- If \(n \equiv \pm 2 \mod 12\) ⇒ \(\Psi(1, -1, n) = 1 \neq 0 \Rightarrow (1, -1) \in \omega(n)\)
- If \(n \equiv \pm 3 \mod 16\) ⇒ \(\Psi(1, \sqrt{2}, n) = -1 - \sqrt{2} \neq 0 \Rightarrow (1, \sqrt{2}) \in \omega(n)\)
- If \(n \equiv \pm 4 \mod 20\) ⇒ \(\Psi(1, \varphi - 1, n) = -\varphi \neq 0 \Rightarrow (1, \varphi - 1) \in \omega(n)\)
- If \(n \equiv \pm 5 \mod 24\) ⇒ \(\Psi(1, \sqrt{3}, n) = 2 + \sqrt{3} \neq 0 \Rightarrow (1, \sqrt{3}) \in \omega(n)\) (96)

where \(\varphi\) is the Golden ratio.

Here are some specific examples for points not belonging to Omega space at level \(n\).

- If \(n \equiv \pm 2 \mod 8\) ⇒ \(\Psi(1, 0, n) = 0 \Rightarrow (1, 0) \notin \omega(n)\)
- If \(n \equiv \pm 3 \mod 12\) ⇒ \(\Psi(1, -1, n) = 0 \Rightarrow (1, -1) \notin \omega(n)\)
- If \(n \equiv \pm 4 \mod 16\) ⇒ \(\Psi(1, \sqrt{2}, n) = 0 \Rightarrow (1, \sqrt{2}) \notin \omega(n)\) (97)
- If \(n \equiv \pm 5 \mod 20\) ⇒ \(\Psi(1, \varphi - 1, n) = 0 \Rightarrow (1, \varphi - 1) \notin \omega(n)\)
- If \(n \equiv \pm 6 \mod 24\) ⇒ \(\Psi(1, \sqrt{3}, n) = 0 \Rightarrow (1, \sqrt{3}) \notin \omega(n)\)

These examples suggest one to define the Kernel of Omega space for further research developments.
Definition 34. (The Kernel of the Omega space at level $n$) For any given natural number $n$, we define the Kernel of the Omega space, $\text{Ker}_\omega(n)$, at level $n$ as following

$$\text{Ker}_\omega(n) := \{(a, b) \mid \Psi(a, b, n) = 0\}. \quad (98)$$

From (65), we immediately get

$$\Psi(xy, -x^2 - y^2, n) \neq 0$$

for any integers $x, y$ where $x \neq -y$. Hence we get

Theorem 35. For any natural number $n$, the space $\omega(n)$ is infinite and include all the integer points $(xy, -x^2 - y^2)$ where $x, y$ any integers such that $x \neq -y$.

For any natural number $n > 1$, we should observe from formula (91) that

$$\Psi(\alpha, \beta, n) \neq 0 \iff \Omega_0(\left\lfloor \frac{n}{2} \right\rfloor \mid \alpha, \beta \mid n) \neq 0.$$

Hence, the following ratio

$$\frac{\Omega_0(\left\lfloor \frac{n}{2} \right\rfloor \mid \alpha, \beta \mid n)}{\Psi(\alpha, \beta, n)} \quad (99)$$

is well-defined for any point $(\alpha, \beta) \in \omega(n)$. Consequently, for any point $(\alpha, \beta) \in \omega(n)$, the product

$$(n - 1)(n - 2) \cdots (n - \left\lfloor \frac{n}{2} \right\rfloor)$$

can be represented as following

$$(n - 1)(n - 2) \cdots (n - \left\lfloor \frac{n}{2} \right\rfloor) = \frac{\Omega_0(\left\lfloor \frac{n}{2} \right\rfloor \mid \alpha, \beta \mid n)}{\Psi(\alpha, \beta, n)} \quad (100)$$

Substitute $n$ by $2n$ in (100) we get the following theorem

Theorem 36. (The first fundamental theorem of $\Omega$-sequence) (version 2)
For any natural number $n$, and any point $(\alpha, \beta) \in \omega(2n)$, we get

$$n(n + 1) \cdots (2n - 1) = \frac{\Omega_0(n \mid \alpha, \beta \mid 2n)}{\Psi(\alpha, \beta, 2n)}. \quad (101)$$

The following unexpected theorem proves that any new prime must be a factor to Omega sequence.

Notation 37. Everywhere below, $p_k$ denotes the $k$-th prime, $k > 1$. 

19
13 The emergence of a new prime number

In 1850, Chebyshev proved Bertrand postulate (1845) which states that for any integer \( m > 3 \) there always exists at least one prime number \( p \) with \( m < p < 2m - 2 \). The postulate is also called the Bertrand–Chebyshev theorem. For a proof of the theorem and for some additional information, for example, see [36]. Therefore, from Bertrand–Chebyshev theorem, one get

\[
p_{k+1} \mid (p_k + 1)(p_k + 2) \ldots (2p_k - 3).
\]  

(102)

Hence

\[
p_{k+1} \mid p_k(p_k + 1)(p_k + 2) \ldots (2p_k - 1).
\]  

(103)

From (101), put \( n = p_k \), we get

\[
p_k(p_k + 1) \ldots (2p_k - 1) = \frac{\Omega_0(p_k \mid \alpha, \beta \mid 2p_k)}{\Psi(\alpha, \beta, 2p_k)}.
\]  

(104)

where \((\alpha, \beta) \in \omega(p_k)\). Hence from (103), and (104), we get the following desirable result

**Theorem 38.** (The second fundamental theorem of \( \Omega \)-sequence)

For any point \((\alpha, \beta) \in \omega(2p_k)\), the ratio

\[
\frac{\Omega_0(p_k \mid \alpha, \beta \mid 2p_k)}{\Psi(\alpha, \beta, 2p_k)}
\]  

is integer. And

\[
p_{k+1} \mid \frac{\Omega_0(p_k \mid \alpha, \beta \mid 2p_k)}{\Psi(\alpha, \beta, 2p_k)}.
\]  

(105)

Moreover, for any finite set \( \varpi \subset \omega(2p_k) \), and any finite set \( I \) of integers we get

\[
p_{k+1} \mid \sum_{(\alpha, \beta) \in \varpi} \lambda \frac{\Omega_0(p_k \mid \alpha, \beta \mid 2p_k)}{\Psi(\alpha, \beta, 2p_k)}.
\]  

(106)

We should observe the following generalization for Theorem (38)

**Theorem 39.** For any point \((\alpha, \beta) \in \omega(2p_k)\), the ratio

\[
\frac{\Omega_0(p_k \mid \alpha, \beta \mid 2p_k)}{p_k(2p_k - 1)(2p_k - 2)\Psi(\alpha, \beta, 2p_k)}
\]  

is integer. Moreover

\[
p_{k+1} \mid \frac{\Omega_0(p_k \mid \alpha, \beta \mid 2p_k)}{p_k(2p_k - 1)(2p_k - 2)\Psi(\alpha, \beta, 2p_k)}.
\]  

(107)

We should also observe the following specialization for Theorem (38)

**Theorem 40.** For any point \((\alpha, \beta)\), we get the following property

\[
p_{k+1} \mid \Omega_0(p_k \mid \alpha, \beta \mid 2p_k).
\]  

(108)
14 Some useful special cases

14.1 $\Psi(1, 1, n)$

For $\alpha = 1, \beta = 1$ we get the following formula for the $\Psi$-sequence

$$\Psi(1, 1, n) = \begin{cases} +2 & n \equiv \pm 0 \pmod{6} \\ +1 & n \equiv \pm 1 \pmod{6} \\ -1 & n \equiv \pm 2 \pmod{6} \\ -2 & n \equiv \pm 3 \pmod{6} \end{cases} \quad (111)$$

Therefore, we get

**Theorem 41.**

$$\Omega_0\left(\left\lfloor \frac{n}{2} \right\rfloor \mid 1, 1 \mid n\right) \equiv \begin{cases} +2 & n \equiv \pm 0 \pmod{6} \\ +1 & n \equiv \pm 1 \pmod{6} \\ -1 & n \equiv \pm 2 \pmod{6} \\ -2 & n \equiv \pm 3 \pmod{6} \end{cases} \quad (112)$$

14.2 $\Psi(1, 0, n)$

For $\alpha = 1, \beta = 0$ we get the following formula for the $\Psi$-sequence

$$\Psi(1, 0, n) = \begin{cases} +2 & n \equiv \pm 0 \pmod{8} \\ +1 & n \equiv \pm 1 \pmod{8} \\ 0 & n \equiv \pm 2 \pmod{8} \\ -1 & n \equiv \pm 3 \pmod{8} \\ -2 & n \equiv \pm 4 \pmod{8} \end{cases} \quad (113)$$

Therefore, we get

**Theorem 42.**

$$\Omega_0\left(\left\lfloor \frac{n}{2} \right\rfloor \mid 1, 0 \mid n\right) \equiv \begin{cases} +2 & n \equiv \pm 0 \pmod{8} \\ +1 & n \equiv \pm 1 \pmod{8} \\ 0 & n \equiv \pm 2 \pmod{8} \\ -1 & n \equiv \pm 3 \pmod{8} \\ -2 & n \equiv \pm 4 \pmod{8} \end{cases} \quad (114)$$
14.3 \( \Psi(1, -1, n) \)

For \( a = 1, b = -1 \) we get the following formula for the \( \Psi \)-sequence

\[
\Psi(1, -1, n) = \begin{cases} 
+2 & n \equiv \pm 0 \pmod{12} \\
+1 & n \equiv \pm 1, \pm 2 \pmod{12} \\
0 & n \equiv \pm 3 \pmod{12} \\
-1 & n \equiv \pm 4, \pm 5 \pmod{12} \\
-2 & n \equiv \pm 6 \pmod{12} 
\end{cases}
\] (115)

Therefore, we get

Theorem 43.

\[
\Omega_0(\left\lfloor \frac{n}{2} \right\rfloor | 1, -1 | n) = \begin{cases} 
+2 & n \equiv \pm 0 \pmod{12} \\
+1 & n \equiv \pm 1, \pm 2 \pmod{12} \\
0 & n \equiv \pm 3 \pmod{12} \\
-1 & n \equiv \pm 4, \pm 5 \pmod{12} \\
-2 & n \equiv \pm 6 \pmod{12} 
\end{cases}
\] (116)

14.4 Combination of Fibonacci and Lucas sequences

For any natural number \( n \), we get, from (29), the following relation

\[
\Psi(1, \sqrt{5}, n) = \begin{cases} 
L\left( \frac{n}{2} \right) & n \equiv 0 \pmod{4} \\
L\left( \frac{n+1}{2} \right) + F\left( \frac{n-1}{2} \right) \sqrt{5} & n \equiv 1 \pmod{4} \\
-F\left( \frac{n}{2} \right) \sqrt{5} & n \equiv 2 \pmod{4} \\
-L\left( \frac{n-1}{2} \right) - F\left( \frac{n+1}{2} \right) \sqrt{5} & n \equiv 3 \pmod{4} 
\end{cases}
\] (117)

Therefore, from Theorem (31), we immediately get the proof of the following theorem.

Theorem 44. (Representation for a combination of Fibonacci and Lucas sequences)

\[
\Omega_0(\left\lfloor \frac{n}{2} \right\rfloor | 1, \sqrt{5} | n) = \begin{cases} 
L\left( \frac{n}{2} \right) & n \equiv 0 \pmod{4} \\
L\left( \frac{n+1}{2} \right) + F\left( \frac{n-1}{2} \right) \sqrt{5} & n \equiv 1 \pmod{4} \\
-F\left( \frac{n}{2} \right) \sqrt{5} & n \equiv 2 \pmod{4} \\
-L\left( \frac{n-1}{2} \right) - F\left( \frac{n+1}{2} \right) \sqrt{5} & n \equiv 3 \pmod{4} 
\end{cases}
\] (118)

14.5 The Omega sequence associated with \( (1, 2) \)

From (32)

\[
\Psi(1, 2, n) = \frac{n}{n - \left\lfloor \frac{n}{2} \right\rfloor} \left( n - \left\lfloor \frac{n}{2} \right\rfloor \right) (-1)^{\left\lfloor \frac{n}{2} \right\rfloor}.
\]
Consequently
\[ \Psi(1, 2, n) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} 2^{\delta(n-1)} n^{\delta(n)}. \]

Therefore, from Theorem (31), we immediately obtain the following explicit formula

**Theorem 45.**
\[ \Omega_0 \left( \left\lfloor \frac{n}{2} \right\rfloor |1, 2|n \right) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} 2^{\delta(n-1)} n^{\delta(n)} (n - 1)(n - 2) \cdots (n - \left\lfloor \frac{n}{2} \right\rfloor). \] (119)

However, if we desire to compute all the terms of Omega sequence that is associated with \( n \) and the point \((1, 2)\), we need to compute the terms of Omega sequence, one by one, as following:

\[ \Omega_r (k|1, 2|n) = (-2) (n - 2r - \delta(n - 1)) \Omega_{r+1}(k-1|1, 2|n), \]
\[ \Omega_r (0|1, 2|n) = 1 \quad \text{for all } r. \] (120)

Solving (120), we immediately get

**Theorem 46.** The Omega sequence associated with \( n \) and \((1, 2)\) is given by
\[ \Omega_r (k|1, 2|n) = (-2)^k \prod_{\lambda=0}^{k-1} (n - \delta(n + 1) - 2r - 2\lambda). \] (121)

**14.6 The Omega sequence associated with \((0, -1)\)**

Noting that \( \Psi(0, -1, n) = 1 \), for any natural number \( n \), we can deduce the following result

**Theorem 47.** The Omega sequence associated with \( n \) and \((0, -1)\) is given by
\[ \Omega_r (k|0, -1|n) = \prod_{\lambda=1}^{k} (n - r - \lambda). \] (122)

Moreover
\[ \Omega_0 \left( \left\lfloor \frac{n}{2} \right\rfloor |0, -1|n \right) = (n - 1)(n - 2) \cdots (n - \left\lfloor \frac{n}{2} \right\rfloor). \] (123)

**15 New combinatorial identity**

From (29), it follows that
\[ \Psi(1, -2, n) = 2^{\delta(n+1)}. \] (124)

From (124), (28) and (91), we immediately get the following identity

**Theorem 48.** (New combinatorial identity)
\[ (n - 1)(n - 2) \cdots \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right) = 2^{\left\lfloor \frac{n}{2} \right\rfloor - \delta(n+1)} (n + \delta(n - 1) - 2)(n + \delta(n - 1) - 4)(n + \delta(n - 1) - 6) \cdots (3)(1). \] (125)
Remark
Put \( r = 0, k = \left\lfloor \frac{n}{2} \right\rfloor \) in (121), and compare the result with (119), we also get identity (125).

16 The product of the first odd primes

Observe that
\[
\prod_{i=2}^{k} p_i \mid (2p_k - 1) (2p_k - 3) (2p_k - 5) \cdots (3) (1). \quad (126)
\]

Put \( n = 2p_k \) in (125), and from (126), we deduce
\[
\prod_{i=2}^{k} p_i \mid (2p_k - 1) (2p_k - 2) \cdots (2p_k - \left\lfloor \frac{2p_k}{2} \right\rfloor ). \quad (127)
\]

From Bertrand–Chebyshev theorem, it follows that
\[
p_{k+1} \mid (2p_k - 1) (2p_k - 2) \cdots (2p_k - \left\lfloor \frac{2p_k}{2} \right\rfloor ). \quad (128)
\]

From (127), and (128), we obtain
\[
\prod_{i=2}^{k+1} p_i \mid (2p_k - 1) (2p_k - 2) \cdots (2p_k - \left\lfloor \frac{2p_k}{2} \right\rfloor ). \quad (129)
\]

Now, from (104), we get the following desirable generalization.

**Theorem 49.** *(The third fundamental theorem of \( \Omega \)-sequence)*

For any point \((\alpha, \beta) \in \omega(2p_k)\), the ratio
\[
\frac{\Omega_0(p_k \mid \alpha, \beta \mid 2p_k)}{\Psi(\alpha, \beta, 2p_k)} \quad (130)
\]
is integer. And
\[
\prod_{i=2}^{k+1} p_i \mid \frac{\Omega_0(p_k \mid \alpha, \beta \mid 2p_k)}{\Psi(\alpha, \beta, 2p_k)}. \quad (131)
\]

Furthermore, for any finite set \( \mathcal{W} \subset \omega(2p_k) \), and any finite set \( I \) of integers, we get
\[
\prod_{i=2}^{k+1} p_i \mid \sum_{(\alpha, \beta) \in \mathcal{W}} \lambda \frac{\Omega_0(p_k \mid \alpha, \beta \mid 2p_k)}{\Psi(\alpha, \beta, 2p_k)}. \quad (132)
\]
17 Representation for Chebyshev polynomial sequence

The Chebyshev polynomials first appeared in his paper [17]. The Chebyshev polynomial sequence of the first kind, $T_n(x)$, is defined by

\[
\begin{align*}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_{n+1}(x) &= 2x T_n(x) - T_{n-1}(x).
\end{align*}
\]

Chebyshev polynomials are important in approximation theory, polynomial approximation, rational approximation, integration, integral equations and in the development of spectral methods for the solution of ordinary and partial differential equations and numerical analysis and in some quadrature rules based on these polynomials such as Gauss-Chebyshev rule that appears in the theory of numerical integration (see for example [22], [24]). Of all polynomials with leading coefficient unity, it is a well-known property of the Chebyshev polynomials that they possess the smallest absolute upper bound when the argument is allowed to vary between their limits of orthogonality and this property suggests the use of Chebyshev polynomials as a means of interpolation. Also, it is well-known that the Nobel Prize-winning physicist Enrico Fermi is the creator of the world’s first nuclear reactor, the Chicago Pile-1, and his work led to the discovery of nuclear fission, the basis of nuclear power and the atom bomb,[28] and [29], and [30]. One of the common approaches to the approximation of Fermi-Dirac integrals is the use of Chebyshev rational approximations, [25],[26],[27], [31],[32],[33], to the Fermi-Dirac integrals defined by

\[
F_s(x) = \frac{1}{\Gamma(s + 1)} \int_0^\infty \frac{t^s}{e^{t-x} + 1} \, dt.
\]

Another common approach to the approximation of Riemann Zeta Function is the use of Chebyshev rational approximations, [34], [35], where the Riemann Zeta Function, or Euler-Riemann Zeta Function, $\zeta(s)$, is a function of a complex variable $s$ that analytically continues the sum of the Dirichlet series

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

for when the real part of $s$ is greater than 1.

Also, the periodicity of Chebyshev polynomials over finite fields and its tremendous applications on the security of cryptosystems based on Chebyshev polynomials had been studied recently (see for example [18], [19], [20], [21]). The integer coefficients of Chebyshev polynomial are given explicitly by the following formula (see for example [22], [23]):

\[
T_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \frac{n!}{n-i} \binom{n-i}{i} (2)^{n-2i-1} x^{n-2i}.
\]
From (32), and (133), we can deduce the following formula

\[ T_n(x) = \frac{x^{\delta(n)}}{2^{\delta(n-1)}} \Psi(1, 2 - 4x^2, n). \] (134)

Therefore, from Theorem (31), we immediately get

**Theorem 50.** (Representation for Chebyshev polynomial sequence)

\[ T_n(x) = \frac{x^{\delta(n)}}{2^{\delta(n-1)}} \Omega_0\left( \left\lfloor \frac{n}{2} \right\rfloor | 1, 2 - 4x^2 | n \right). \] (135)

### 17.1 Representation for Dickson polynomial sequence

The Dickson polynomial, \( D_n(x, \alpha) \), of the first kind of degree \( n \) with parameter \( \alpha \) is defined by

\[
\begin{align*}
D_0(x, \alpha) &= 2 \\
D_1(x, \alpha) &= x \\
D_{n+1}(x, \alpha) &= 2x \, D_n(x, \alpha) - D_{n-1}(x, \alpha).
\end{align*}
\]

Modern cryptography is heavily based on mathematical theory and computer science practice. Properties of polynomials over finite fields play a vital role not only in mathematics, but also useful in many other applications like error correcting codes, pseudo random sequences used in code-division multiple access (CDMA) systems. CDMA technology was initially used in World War II military operations to thwart enemy attempts to access radio communication signals. As the name suggests, permutation polynomials permute the elements of a ring or field over which they are defined. Permutation Polynomials are the roots of public key methods like RSA Cryptosystem and Dickson cryptographic schemes and they are very important in the development of cryptographic schemes. Recently, permutations of finite fields have become of considerable interest in the construction of cryptographic systems for the secure transmission of data, see [16]. Also, permutation polynomials have been an active topic of study in recent years due to their important applications in cryptography, coding theory, combinatorial designs theory. Also the encryption polynomials \( x^k \) of the RSA-scheme are replaced by another class of polynomials, namely by the so-called Dickson-polynomials. Cryptographers call this Cryptosystem the Dickson-scheme. Also, Fried [5] proved that any integral polynomial that is a permutation polynomial for infinitely many prime fields is a composition of Dickson polynomials and linear polynomials (with rational coefficients). Permutation polynomials were studied first by Hermite [6] and later by Dickson [2] and [7]. Dickson polynomials form an important class of permutation polynomials and have been extensively investigated in recent years under different contexts. See for instance [8],[9], [10],[11],[12], [13], [14], and [15], where the work on Dickson polynomials, and its developments are presented.
For integer $n > 0$ and $\alpha$ in a commutative ring $R$ with identity. The Dickson polynomials (of the first kind) over $R$ are given by

$$D_n(x, \alpha) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (-\alpha)^i x^{n-2i}$$

(136)

From (32), and (136) we can deduce the following formula

$$D_n(x, \alpha) = x^{\delta(n) \Omega_0(\lfloor \frac{n}{2} \rfloor | \alpha, 2\alpha - x^2, n)}.$$  

(137)

Therefore, from Theorem (31), we immediately get the proof of the following theorem.

**Theorem 51. (Representation for Dickson polynomial sequence)**

$$D_n(x, \alpha) = x^{\delta(n) \Omega_0(\lfloor \frac{n}{2} \rfloor | \alpha, 2\alpha - x^2, n)}.$$  

(138)

### 18 Mersenne primes and even perfect numbers

Mersenne numbers $2^p - 1$ with prime $p$ form the sequence

$$3, 7, 31, 127, 2047, 8191, 131071, 524287, 8388607, 536870911, \ldots$$

(sequence A001348 in [4]). For $2^p - 1$ to be prime, it is necessary that $p$ itself be prime.

#### 18.1 Primality test for Mersenne primes

**Theorem 52. (Lucas-Lehmer-Moustafa) Given prime $p \geq 5$. The number $2^p - 1$ is prime if and only if**

$$2n - 1 \mid \Psi(1, 4, n),$$

(139)

where $n := 2^{p-1}$.

**Proof.** Given prime $p \geq 5$, let $n := 2^{p-1}$. From Lucas-Lehmer test, [3], we have

$$2^p - 1 \text{ is prime} \iff 2^p - 1 \mid (1 + \sqrt{3})^n + (1 - \sqrt{3})^n.$$  

As $n$ even, $\delta(n) = 0$, and from Theorem (27), we get the following equivalent statement:

$$2^p - 1 \text{ is prime} \iff 2^p - 1 \mid \Psi(x_0 y_0, -x_0^2 - y_0^2, n),$$

where $x_0 = 1 + \sqrt{3}$, $y_0 = 1 - \sqrt{3}$. As $(x_0 y_0, -x_0^2 - y_0^2) = (-2, -8)$, and from Theorem (21), and noting $(2^p - 1, 2) = 1$, we get the following equivalent statements:

$$2^p - 1 \text{ is prime} \iff 2^p - 1 \mid \Psi(-2, -8, n)$$

$$\iff 2^p - 1 \mid (-2)^{\frac{p}{2}} \Psi(1, 4, n)$$

$$\iff 2^p - 1 \mid \Psi(1, 4, n).$$

$\Box$

27
From Theorem(52) and Theorem(31), we immediately get the following result

**Theorem 53.** (Lucas-Lehmer-Moustafa) Given prime \( p \geq 5 \). The number \( 2^p - 1 \) is prime if and only if

\[
2n - 1 \mid \frac{\Omega_0\left(\left\lfloor \frac{n}{2} \right\rfloor \mid 1, 4 \mid n\right)}{(n-1)(n-2) \cdots (n - \left\lfloor \frac{n}{2} \right\rfloor)},
\]

where \( n := 2^{p-1} \).

Therefore, from Theorem (53) and from Euclid-Euler theorem for even perfect numbers, we get the following result

**Theorem 54.** (Euclid-Euler-Lucas-Lehmer-Moustafa) A number \( N \) is even perfect number if and only if \( N = 2^p - 1 \) for some prime \( p \), and

\[
2n - 1 \mid \frac{\Omega_0\left(\left\lfloor \frac{n}{2} \right\rfloor \mid 1, 4 \mid n\right)}{(n-1)(n-2) \cdots (n - \left\lfloor \frac{n}{2} \right\rfloor)},
\]

where \( n := 2^{p-1} \).

From Theorem(31), and the fact that

\[
\Psi(-2, -5, p) = 2^p - 1,
\]

we immediately get the following desirable representation for Mersenne numbers.

**Theorem 55.** (Representation of Mersenne numbers) For any given odd natural number \( p \), the number \( 2^p - 1 \) can be represented by

\[
2^p - 1 = \frac{\Omega_0\left(\left\lfloor \frac{p}{2} \right\rfloor \mid 1, 4 \mid p\right)}{(p-1)(p-2) \cdots (p - \left\lfloor \frac{p}{2} \right\rfloor)},
\]

where the double-indexed polynomial sequence \( \Omega_r(k) \) is associated with the point \((-2, -5)\), \( 0 \leq r + k \leq \left\lfloor \frac{p}{2} \right\rfloor \), and defined by the recurrence relation

\[
\Omega_r(k) = (p - r - k) \Omega_r(k - 1) + 4 (p - 2r) \Omega_{r+1}(k - 1),
\]

\( \Omega_r(0) = 1 \) for all \( r \).

Hence from Theorem (53) and Theorem (55), we get the following

**Theorem 56.** (Lucas-Lehmer-Moustafa) For any given prime \( p \geq 5 \), \( n := 2^{p-1} \), the number \( 2^p - 1 \) is prime if and only if

\[
\frac{\Omega_0\left(\left\lfloor \frac{n}{2} \right\rfloor \mid 1, 4 \mid n\right)}{(n-1)(n-2) \cdots (n - \left\lfloor \frac{n}{2} \right\rfloor)} \mid \frac{\Omega_0\left(\left\lfloor \frac{n}{2} \right\rfloor \mid 1, 4 \mid n\right)}{(p-1)(p-2) \cdots (p - \left\lfloor \frac{p}{2} \right\rfloor)}.
\]
Now, from Theorem (47), we get
\[(n - 1)(n - 2) \cdots (n - \left\lfloor \frac{n}{2} \right\rfloor) = \Omega_0\left( \left\lfloor \frac{n}{2} \right\rfloor 0, -1 \right| n), \]
\[(p - 1)(p - 2) \cdots (p - \left\lfloor \frac{p}{2} \right\rfloor) = \Omega_0\left( \left\lfloor \frac{p}{2} \right\rfloor 0, -1 \right| p). \]

Consequently, from Theorem (56), and from (146), we get

**Theorem 57. (Lucas-Lehmer-Moustafa)** For any given prime \(p \geq 5, n := 2^{p-1}\). The number \(2^p - 1\) is prime if and only if
\[\Omega_0\left( \left\lfloor \frac{n}{2} \right\rfloor 0, -1 \right| n) \cdot \Omega_0\left( \left\lfloor \frac{p}{2} \right\rfloor 0, -1 \right| p) \cdot \ldots \cdot \Omega_0\left( \left\lfloor \frac{n}{2} \right\rfloor 0, -1 \right| n) \cdot \Omega_0\left( \left\lfloor \frac{p}{2} \right\rfloor 0, -1 \right| p). \]  

**18.2 Mersenne composite numbers**

The number \(2^p - 1\) is called Mersenne composite number if \(p\) is prime but \(2^p - 1\) is not prime.

**Theorem 58.** Given prime \(p, n := 2^{p-1}\). If
\[2n - 1 \mid \Psi(1, 4, n, \pm 1), \]
then \(2^p - 1\) is a Mersenne composite number.

**Proof.** Let \(2^p - 1 \mid \Psi(1, 4, n, \pm 1)\). Now, suppose the contrary, and let \(2^p - 1\) be prime. Then from Theorem (52), and from the recurrence relation (29), we get \(2^p - 1 \mid \Psi(1, 4, 1) = 1\). Contradiction.

\(\square\)

**19 Clarifications on the theorems of the summary**

Now need to provide clarifications for how the theorems in the summary arose up easily, one by one, as special cases of the main theorems of the this paper.

- **Clarifications on Theorem (1):**
  It is clear that the sequences \(A_r(k)\) and \(B_r(k)\), which are defined in (1), satisfy:
  \[A_r(k) = \Omega_r(k| - 2, -5 | p), \]
  \[B_r(k) = \Omega_r(k| 1, 4 | n). \]
  Hence from Theorem (56) and Theorem (31) we get the proof of Theorem (1).

- **Clarifications on Theorem (2):**
  Again, it is clear that
  \[A_r(k) = \Omega_r(k| - 2, -5 | p). \]
  Hence from Theorem (55) and Theorem (31) we get the proof of Theorem (2).
• Clarifications on Theorem (3):
Again, it is clear that
\[ U_r(k) = \Omega_r(k|1, 1|n). \]  \hfill (151)
Hence from Theorem (41) we get the proof of Theorem (3).

• Clarifications on Theorem (4):
We should notice that
\[ V_r(k) = \Omega_r(k|1, 0|n). \]  \hfill (152)
Hence from Theorem (42) we get the proof of Theorem (4).

• Clarifications for Theorem (5):
We should notice that
\[ W_r(k) = \Omega_r(k|1, -1|n). \]  \hfill (153)
Hence from Theorem (43) we get the proof of Theorem (5).

• Clarifications on Theorem (6):
We should notice that
\[ T_r(k) = \Omega_r(k|1, -2|n). \]  \hfill (154)
Hence from (124) and Theorem (31) we get the proof of Theorem (6).

• Clarifications on Theorem (7):
We should notice that
\[ H_r(k) = \Omega_r(k|1, -3|n). \]  \hfill (155)
It is straightforward to see
\[ \Psi(-1, -3, n) = L(n). \]
Hence from Theorem (31) we get the proof of Theorem (7).

• Clarifications on Theorem (8):
We should notice that
\[ F_r(k) = \Omega_r(k|-2, 2^n). \]  \hfill (156)
Noting that
\[ \Psi(-2, -5, 2^n) = F_n, \]
hence, from Theorem (31), we get the proof of Theorem (8).

• Clarifications on Theorem (9):
We should notice that
\[ G_r(k) = \Omega_r(k|1, -3|n). \]  \hfill (157)
Noting that
\[ \Psi(1, -3, n) = \begin{cases} F(n) & \text{if } n \text{ odd} \\ L(n) & \text{if } n \text{ even} \end{cases}, \]
hence, from Theorem (31), we get the proof of Theorem (9).
19.1 Fibonacci-Lucas oscillating sequence

It is nice to study the sequence \( G(n) \)

\[
G(n) = \begin{cases} 
F(n) & \text{if } n \text{ odd} \\
L(n) & \text{if } n \text{ even}
\end{cases}
\]

and I call it Fibonacci-Lucas oscillating sequence. Theorem (9) should motivate researchers for further new investigations towards the arithmetic of this sequence. The sequence \( G(n) \) corresponds to the sequence \( \text{A005247} \) in [4]. This sequence alternates Lucas \( \text{A000032} \) and Fibonacci \( \text{A000045} \) sequences for even and odd \( n \).

20 Further research investigations

20.1 Fibonacci numbers and \( p_k \)

With some extra work one can prove the following result:

For any given natural number \( n \), if we associate the double-indexed polynomial sequence \( \Lambda_r(k) \) which is defined by

\[
\Lambda_r(k) = (n - r - k) \Lambda_r(k - 1) + 2 (n - 1 - 2r - \delta(n)) \Lambda_{r+1}(k - 1),
\]

\[
\Lambda_r(0) = 1 \quad \text{for all } r,
\]

then

\[
F(n) = \frac{\Lambda_0(\lfloor \frac{n-1}{2} \rfloor)}{(n - 1)(n - 2) \cdots (n - \lfloor \frac{n-1}{2} \rfloor)},
\]

where \( F(n) \) is Fibonacci sequence. Moreover

\[
p_{k+1} \mid \frac{\Lambda_0(p_k - 1)}{F(2p_k)},
\]

20.2 Conjecture

While my studies for the connections between Omega sequences, Omega spaces, the natural relationships between Omega sequences and the prime \( p_{k+1} \), and my studies how the primes naturally arise up, I feel strongly compelled to propose the following conjecture:

**Conjecture 59.** (The Omega conjecture for the prime numbers) Based on the properties of Omega sequences, Omega spaces, and the Kernel of Omega space, we can find algorithm to determine the prime \( p_{k+1} \) based only on the knowledge of \( p_k, p_{k-1}, p_{k-2}, \ldots \) and on the knowledge of some points in Omega space at level \( n \) for some natural number \( n \), where \( n \) is independent on \( p_{k+1} \).
Of course, my remark only begins the story, and I have told only of those formulas that show various natural deep connections of Omega sequences with the nature of prime numbers, Mersenne primes, even perfect numbers, and unify many well-known sequences in number theory.

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