On the Limiting Shape of Random Young Tableaux Associated to Inhomogeneous Words

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Abstract

The limiting shape of the random Young tableaux associated to the inhomogeneous word problem is identified as a multidimensional Brownian functional. This functional is thus identical in law to the spectrum of a certain matrix ensemble. The Poissonized word problem is also studied, and the asymptotic behavior of the shape analyzed.

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1 Introduction

One of the most remarkable achievements in modern random matrix theory is the identification by Tracy and Widom [28] of the distribution which now bears their names. The Tracy-Widom distribution, denoted by $F_{TW}$, gives the fluctuations of, the properly centered and normalized, largest eigenvalue of a matrix taken from the Gaussian Unitary Ensemble (GUE). Since then, the fluctuations of some apparently disconnected models have also been

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shown to be governed by the same limiting law. For example, this is the case of the length of the longest increasing subsequence of a random permutation (Baik, Deift and Johansson \cite{1}) as well as of a last-passage time percolation problem (Johansson \cite{20}, \cite{21}). We refer the reader to \cite{22} for a survey of such topics. Along this path, connections between the eigenvalues of some random matrices and longest increasing subsequence problems will be further studied below.

Let \( X_1, X_2, ..., X_N \) be a finite sequence of random variables taking values in an ordered set. The length of the longest (weakly) increasing subsequence of \( X_1, X_2, ..., X_N \), denoted by \( LI_N \), is the maximal \( k \leq N \) such that there exists an increasing sequence of integers \( 1 \leq i_1 < i_2 < \cdots < i_k \leq N \) with \( X_{i_1} \leq X_{i_2} \leq \cdots \leq X_{i_k} \), i.e.,

\[
LI_N = \max \{ k : \exists 1 \leq i_1 < i_2 < \cdots < i_k \leq N, \text{ with } X_{i_1} \leq X_{i_2} \leq \cdots \leq X_{i_k} \}.
\]

When \( X_1, X_2, ..., X_N \) is a random permutation of \( 1, 2, ..., N \), Baik, Deift and Johansson \cite{1} showed that,

\[
\frac{LI_N - 2\sqrt{N}}{N^{1/6}} \Rightarrow FTW,
\]

where \( \Rightarrow \) indicates convergence in distribution. When each \( X_i \) takes values independently and uniformly in an \( M \)-letter ordered alphabet, through a careful analysis of the exponential generating function of \( LI_N \), Tracy and Widom \cite{30} gave the limiting distribution of \( LI_N \) (properly centered and normalized) as that of the largest eigenvalue of a matrix drawn from the \( M \times M \) traceless GUE. This was extended to the non-uniform setting by Its, Tracy and Widom (\cite{18}, \cite{19}).

A very efficient method to study longest increasing subsequences is through Young tableaux (\cite{12}, \cite{27}). Recall that a Young diagram of size \( n \) is a collection of \( n \) boxes arranged in left-justified rows, with a weakly decreasing number of boxes from row to row. A (semi-standard) Young tableau is a Young diagram, with a filling of a positive integer in each box, in such a way that the integers are weakly increasing along the rows and strictly increasing down the columns. A standard Young tableau of size \( n \) is a Young tableau in which the fillings are the integers from 1 to \( n \). The shape of a Young tableau is the vector \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \), where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \) and for each \( i \), \( \lambda_i \) is the number of boxes in the \( i \)th row while \( k \) is the total number of rows of the tableau (and so \( \lambda_1 + \cdots + \lambda_k = n \)).
Let \( \{1, 2, ..., M\} \) be an \( M \)-letter alphabet. A word of length \( N \) is a mapping \( W \) from \( \{1, 2, ..., N\} \) to \( \{1, 2, ..., M\} \), and let \([M]^N\) denotes the set of words of length \( N \) with letters taken from the alphabet \( \{1, 2, ..., M\} \). A word is a permutation if \( M = N \), and \( W \) is onto. The Robinson-Schensted correspondence is a bijection between the set of words \([M]^N\) and the set of pairs of Young tableaux \( \{(P, Q)\} \), where \( P \) is a semi-standard Young tableau with entries from \( \{1, 2, ..., M\} \), \( Q \) is a standard Young tableau with entries from \( \{1, 2, ..., N\} \). Moreover \( P \) and \( Q \) share the same shape which is a partition of \( N \). From now on, we do not distinguish between shape and partition. If the word is a permutation, then \( P \) is also a standard Young tableau. A word \( W \) in \([M]^N\) can be represented uniquely as an \( M \times N \) matrix \( X_W \) with entries
\[
(X_W)_{i,j} = 1_{W(j)=i}.
\]

The Robinson-Schensted correspondence actually gives a one to one correspondence between the set of pairs of Young tableaux and the set of matrices whose entries are either 0 or 1 and with exactly a unique 1 in each column. This was generalized by Knuth to the set of \( M \times N \) matrices with nonnegative integer entries. Let \( \mathcal{M}(M, N) \) be the set of \( M \times N \) matrices with nonnegative integer entries. Let \( \mathcal{P}(P, Q) \) be the set of pairs of semi-standard Young tableaux \( (P, Q) \) sharing the same shape and whose size is the sum of all the entries, where \( P \) has elements in \( \{1, ..., M\} \) and \( Q \) has elements in \( \{1, ..., N\} \). The Robinson-Schensted-Knuth (RSK) correspondence is a one to one mapping between \( \mathcal{M}(M, N) \) and \( \mathcal{P}(P, Q) \). If the matrix corresponds to a word in \([M]^N\), \( Q \) is a standard tableau.

Johansson [21], using orthogonal polynomial methods, proved that the limiting shape of the Young tableaux, associated to the iid uniform \( m \)-letter framework through the RSK correspondence, is given by the joint distribution of the eigenvalues of the traceless \( M \times M \) GUE. Since \( LI_N \) is equal to the length of the first row of the associated Young tableaux, the corresponding asymptotic results follow. The permutation case is also obtained by Johansson [21] and, independently, by Okounkov [25] as well as Borodin, Okounkov and Olshanski [7]. Its, Tracy and Widom [18, 19] further analyzed the independent non uniform framework. They showed that the corresponding limiting distribution of \( LI_N \) can be written in terms of the distribution function of the eigenvalues of the direct sum of mutually independent GUEs with an overall trace constraint. More recently, in [16] and [17], via simple probabilistic tools, the limiting law of the longest increasing subsequence for finite and countable alphabets is given as a Brownian functional. We investigate
below the connections between random matrix models and longest increasing subsequence problems, for finite alphabets, through these Brownian functionals. Our tools rely on basic probabilistic and combinatorial arguments.

Let us now describe the contributions of the present paper. In Section 2, we list some simple properties of a matrix ensemble, which we call generalized traceless GUE; the proofs of these simple facts are relegated to the Appendix. We relate various properties of the GUE to this generalized one. In particular, it is shown that their respective spectrum differ by a multivariate degenerate Gaussian vector. In Section 3, we obtain the limiting length of the longest increasing subsequence of an inhomogeneous random word as q Brownian functional. This Brownian functional has thus the same law as the largest eigenvalue of the block of the $M \times M$ generalized traceless GUE, corresponding to the most probable letters. In fact, a more general result with a multivariate Brownian functional representation for the limiting shape of the associated Young tableaux is actually obtained. Finally, the corresponding Poissonized measure is studied in Section 4.

2 Generalized Traceless GUE

In this section, we list some elementary properties of the generalized traceless GUE and defer the proofs to the Appendix.

Recall that an element of the $n \times n$ GUE is an $n \times n$ Hermitian random matrix $G = (G_{i,j})$ where $i,j \leq n$, whose entries are such that: $G_{i,i} \sim N(0,1)$, for $1 \leq i \leq n$, $\text{Re}(G_{i,j}) \sim N(0,1/2)$ and $\text{Im}(G_{i,j}) \sim N(0,1/2)$, for $1 \leq i < j \leq n$, and $G_{i,j}$, $\text{Re}(G_{i,j})$, $\text{Im}(G_{i,j})$ are mutually independent for $1 \leq i \leq j \leq n$.

Now, for $M \geq 1$, $K = 1, \ldots, M$ and $d_1, \ldots, d_K$ such that $\sum_{k=1}^K d_k = M$, let $G_M(d_1, \ldots, d_K)$ be the set of random matrices $X$ which are direct sums of mutually independent elements of the $d_k \times d_k$ GUE, $k = 1, \ldots, K$ (i.e., $X$ is an $M \times M$ block diagonal matrix whose $K$ blocks are mutually independent elements of the $d_k \times d_k$ GUE, $k = 1, \ldots, K$). Let $p_1, \ldots, p_M > 0$, $\sum_{j=1}^M p_j = 1$, be such that the multiplicities of the $K$ distinct probabilities $p^{(1)}, \ldots, p^{(K)}$ are respectively $d_1, \ldots, d_K$, i.e., let $m_1 = 0$ and for $k = 2, \ldots, K$, let $m_k = \sum_{j=1}^{k-1} d_j$, and so $p_{m_k+1} = \cdots = p_{m_k+d_k} = p^{(k)}$, $k = 1, \ldots, K$. The generalized $M \times M$ traceless GUE associated to the probabilities $p_1, \ldots, p_M$ is the set, denoted by
\( G^0 (p_1, ..., p_M) \), of \( M \times M \) matrices \( X^0 \), of the form

\[
X_{i,j}^0 = \begin{cases} 
X_{i,i} - \sqrt{p_i} \sum_{l=1}^{M} \sqrt{p_l} X_{l,l}, & \text{if } i = j; \\
X_{i,j}, & \text{if } i \neq j,
\end{cases}
\] (2.1)

where \( X \in G_M (d_1, ..., d_K) \). Clearly, from (2.1), \( \sum_{i=1}^{M} \sqrt{p_i} X_{i,i}^0 = 0 \). Note also that the case \( K = 1 \) (for which \( d_1 = M \)), recovers the traceless GUE, whose elements are of the form \( X - tr(X) I_M / M \), with \( X \) an element of the GUE and \( I_M \) the \( M \times M \) identity matrix.

Here is an equivalent way of defining the generalized traceless GUE: let \( X^{(k)} \) be the \( M \times M \) diagonal matrix such that

\[
X_{i,i}^{(k)} = \begin{cases} 
\sqrt{p^{(k)}} \sum_{l=1}^{M} \sqrt{p_l} X_{l,l}, & \text{if } m_k < i \leq m_k + d_k; \\
0, & \text{otherwise},
\end{cases}
\] (2.2)

and let \( X \in G_M (d_1, ..., d_K) \). Then, \( X^0 := X - \sum_{k=1}^{K} X^{(k)} \in G^0 (p_1, ..., p_M) \).

Equivalently, there is an "ensemble" description of \( G^0 (p_1, ..., p_M) \).

**Proposition 2.1** \( X^0 \in G^0 (p_1, ..., p_M) \) if and only if \( X^0 \) is distributed according to the probability distribution

\[
P \left( dX^0 \right) = C \gamma \left( dX_{1,1}^0, ..., dX_{M,M}^0 \right) \prod_{k=1}^{K} \left( e^{- \sum_{m_k < i \leq m_k + d_k} |X_{i,j}^0|^2} \right) \prod_{m_k < i < j \leq m_k + d_k} dRe \left( X_{i,j}^0 \right) dIm \left( X_{i,j}^0 \right),
\] (2.3)

on the space of \( M \times M \) Hermitian matrices, which are direct sum of \( d_k \times d_k \) Hermitian matrices, \( k = 1, ..., K \), \( \sum_{k=1}^{K} d_k = M \), and where \( m_1 = 0, m_k = \sum_{j=1}^{k-1} d_j, k = 2, ..., K \). Above, \( C = \pi^{- \sum_{k=1}^{K} d_k (d_k - 1) / 2} \) and \( \gamma \left( dX_{1,1}^0, ..., dX_{M,M}^0 \right) \) is the distribution of an \( M \)-dimensional centered (degenerate) multivariate Gaussian law with covariance matrix

\[
\Sigma_0 = \begin{pmatrix}
1 - p_1 & -\sqrt{p_1 p_2} & \cdots & -\sqrt{p_1 p_M} \\
-\sqrt{p_2 p_1} & 1 - p_2 & \cdots & -\sqrt{p_2 p_M} \\
\vdots & \ddots & \ddots & \vdots \\
-\sqrt{p_M p_1} & \cdots & -\sqrt{p_M p_{M-1}} & 1 - p_M
\end{pmatrix}.
\]
We provide next a relation in law between the spectrum of $X$ and of $X^0$.

**Proposition 2.2** Let $X \in G_M(d_1, \ldots, d_K)$, and let $X^0 \in G^0(p_1, \ldots, p_M)$. Let $\xi^1, \ldots, \xi^M$ be the eigenvalues of $X$, where for each $k = 1, \ldots, K$, $\xi^m_{k+1}, \ldots, \xi^{m_k+d_k}$ are the eigenvalues of the $k$th diagonal block (an element of the $d_k \times d_k$ GUE). Then, the eigenvalues of $X^0$ are given by:

$$
\xi^i_0 = \xi^i - \sqrt{p_i} \sum_{l=1}^{d_k} \sqrt{p_l} \xi^l = \xi^i - \sqrt{p_i} \sum_{l=1}^{M} \sqrt{p_l} \xi^l, \quad i = 1, \ldots, M.
$$

Let $\xi^1_{GUE,M}, \xi^2_{GUE,M}, \ldots, \xi^M_{GUE,M}$ be the eigenvalues of an element of the $M \times M$ GUE. It is well known that the empirical distribution of the eigenvalues $(\xi^i_{GUE,M}/\sqrt{M})_{1 \leq i \leq M}$ converges almost surely to the semicircle law $\nu$ with density $\sqrt{4-x^2}/2\pi$, $-2 \leq x \leq 2$. Equivalently, the semicircle law is also the almost sure limit of the empirical spectral measure for the $k$th block of the generalized traceless GUE, provided $d_k \to \infty$, $k = 1, \ldots, K$. This is, for example, the case of the uniform alphabet, where $K = 1$, $d_1 = M$ and $p^{(1)} = 1/M$.

**Proposition 2.3** Let $\xi^1_0, \xi^2_0, \ldots, \xi^M_0$ be the eigenvalues of an element of the $M \times M$ generalized traceless GUE, such that $\xi^m_{0k+1}, \ldots, \xi^{m_k+d_k}$ are the eigenvalues of the $k$th diagonal block, for each $k = 1, \ldots, K$. For any $k = 1, \ldots, K$, the empirical distribution of the eigenvalues $(\xi^i_0/\sqrt{d_k})_{m_k < i \leq m_k+d_k}$ converges almost surely to the semicircle law $\nu$ with density $\sqrt{4-x^2}/2\pi$, $-2 \leq x \leq 2$, whenever $d_k \to \infty$.

Now for $p_1, \ldots, p_M$ considered, so far, i.e., such that the multiplicities of the $K$ distinct probabilities $p^{(1)}, \ldots, p^{(K)}$ are respectively $d_1, \ldots, d_K$ and $p_{m_k+1} = \cdots = p_{m_k+d_k} = p^{(k)}$, $k = 1, \ldots, K$, let

$$
\mathcal{L}^{p_1, \ldots, p_M} := \left\{ x = (x_1, \ldots, x_M) \in \mathbb{R}^M : x_{m_k+1} \geq \cdots \geq x_{m_k+d_k}, \quad k = 1, \ldots, K ; \right. \\
\left. \sum_{j=1}^{M} \sqrt{p_j} x_j = 0 \right\}.
$$

(2.4)
In other words, $L_{p_1,\ldots,p_M}$ is a subset of the hyperplane $\sum_{j=1}^{M} \sqrt{p_j} x_j = 0$, where within each block of size $d_k$, $k = 1,\ldots,K$, the coordinates $x_{m_k+1},\ldots,x_{m_k+d_k}$ are ordered. For any $s_1,\ldots,s_M \in \mathbb{R}$, let also

$$L_{p_1,\ldots,p_M}(s_1,\ldots,s_M) := L_{p_1,\ldots,p_M} \cap \left\{(x_1,\ldots,x_M) \in \mathbb{R}^M : x_i \leq s_i, \ i = 1,\ldots,M \right\}. \quad (2.5)$$

The distribution function of the eigenvalues, written in non-increasing order within each $d_k \times d_k$ GUE, of an element of $G^0(p_1,\ldots,p_M)$ is given now.

**Proposition 2.4** The joint distribution function of the eigenvalues, written in non-increasing order within each $d_k \times d_k$ GUE, of an element of $G^0(p_1,\ldots,p_M)$ is given, for any $s_1,\ldots,s_M \in \mathbb{R}$, by

$$\mathbb{P}\left(\xi_0^1 \leq s_1, \xi_0^2 \leq s_2,\ldots,\xi_0^M \leq s_M\right) = \int_{L_{p_1,\ldots,p_M}(s_1,\ldots,s_M)} f(x) dx_1 \cdot \cdots \cdot dx_{M-1}, \quad (2.6)$$

where for $x = (x_1,\ldots,x_M) \in \mathbb{R}^M$,

$$f(x) := c_M \prod_{k=1}^{K} \Delta_k(x)^2 e^{-\sum_{i=1}^{M} x_i^2/2} 1_{L_{p_1,\ldots,p_M}}(x), \quad (2.7)$$

with $c_M = (2\pi)^{-(M-1)/2} \prod_{k=1}^{K} (0!! \cdots (d_k - 1))^{-1}$ and where $\Delta_k(x)$ is the Vandermonde determinant associated to those $x_i$ for which $p_i = p^{(k)}$, i.e.,

$$\Delta_k(x) = \prod_{m_k \leq i < j \leq m_k + d_k} (x_i - x_j).$$

**Remark 2.5** When the eigenvalues are not ordered within each $d_k \times d_k$ GUE, the identity (2.6) remains valid, multiplying $c_M$, above, by $\prod_{k=1}^{K} (d_k!)^{-1}$, and also by omitting the ordering constraints $x_{m_k+1} \geq \cdots \geq x_{m_k+d_k}$, $k = 1,\ldots,K$, in the definition of $L_{p_1,\ldots,p_M}$.

The forthcoming proposition gives a relation in law between the spectra of elements of $G_M(d_1,\ldots,d_K)$ and of $G^0(p_1,\ldots,p_M)$.

**Proposition 2.6** For any $M \geq 2$, let $X \in G_M(d_1,\ldots,d_K)$ and let $X^0 \in G^0(p_1,\ldots,p_M)$. Let $\xi_1,\ldots,\xi_M$ be the eigenvalues of $X$, and let $\xi_0^1,\ldots,\xi_0^M$ be the eigenvalues of $X^0$ as given in Proposition 2.2. Then

$$\left(\xi_1,\ldots,\xi_M\right) \overset{d}{=} \left(\xi_0^1,\ldots,\xi_0^M\right) + (Z_1,\ldots,Z_M),$$

where $Z_1,\ldots,Z_M$ are independent centred Gaussian random variables with variance $\frac{d_k}{2}$.
where \((Z_1, \ldots, Z_M)\) is a centered (degenerate) multivariate Gaussian vector with covariance matrix \((\sqrt{p_ip_j})_{1 \leq i, j \leq M}\). Moreover, \((\xi_0^1, \ldots, \xi_0^M)\) and \((Z_1, \ldots, Z_M)\) are independent.

The asymptotic behavior of the maximal eigenvalues, within each block, of \(X^0 \in G^0(p_1, \ldots, p_M)\) is well known and well understood (see also Theorem 5.2 and Theorem 5.4 of the Appendix for elementary arguments leading to the result below).

**Corollary 2.7** For \(k = 1, \ldots, K\), let \(\max_{m_k \leq i \leq m_k + d_k} \xi_0^i\) be the largest eigenvalue of the \(d_k \times d_k\) block of \(X^0 \in G^0(p_1, \ldots, p_M)\), then

\[
\lim_{d_k \to \infty} \max_{m_k \leq i \leq m_k + d_k} \frac{\xi_0^i}{\sqrt{d_k}} = 2,
\]

with probability one, or in the mean.

### 3 Random Young Tableaux and Inhomogeneous Words

Throughout the rest of this paper, let \(\mathcal{A}_M = \{\alpha_1, \ldots, \alpha_M\}, \alpha_1 < \cdots < \alpha_M\), be an \(M\)-letter ordered alphabet and let \(W = X_1X_2 \cdots X_N\) be a random word, where \(X_1, X_2, \ldots, X_N\) are iid random variables with \(\mathbb{P}(X_1 = \alpha_j) = p_j\), where \(p_j > 0\), and \(\sum_{j=1}^M p_j = 1\). Let \(\tau\) be a permutation of \(\{1, \ldots, M\}\) corresponding to a non-increasing ordering of \(p_1, p_2, \ldots, p_M\), i.e., \(p_{\tau(1)} \geq \cdots \geq p_{\tau(M)}\). Assume also there are \(K = 1, \ldots, M\), distinct probabilities in \(\{p_1, p_2, \ldots, p_M\}\), and reorder them as \(p^{(1)} > \cdots > p^{(K)}\), in such a way that the multiplicity of each \(p^{(k)}\) is \(d_k, k = 1, \ldots, K\). In our notation, \(K = 1\) corresponds to the uniform alphabet case, where \(d_1 = M\). Let \(m_1 = 0\) and for any \(k = 2, \ldots, K\), let \(m_k = \sum_{j=1}^{k-1} d_j\) and so the multiplicity of each \(p_{\tau(j)}\) is \(d_k\) if \(m_k < \tau(j) \leq m_k + d_k\), \(j = 1, \ldots, M\). Finally, let \(X_W\) be as in (1.1) the matrix corresponding to such a random word \(W\) of length \(N\).

Its, Tracy and Widom (18, 19) have obtained the limiting law of the length of the longest increasing subsequence of such a random word.
To recall their result, let \((\xi_1, ..., \xi_M)\) be the eigenvalues of an element of 
\(\mathcal{G}^0(p_{\tau(1)}, ..., p_{\tau(M)})\), written in such a way that 
\((\xi_1, ..., \xi_M) = (\xi_1^1, ..., \xi_1^{d_1}, ..., \xi_K^1, ..., \xi_K^{d_K})\), i.e., \(\xi_1^k, ..., \xi_K^k\) are the eigenvalues of the \(k\)th block, \(k = 1, ..., K\). Then (see [12]), the limiting law of the length of the longest increasing subsequence, properly centered and normalized, is the law of 
\(\max_{1 \leq i \leq d_k} \xi_i^k\).

A representation of this limiting law, as a Brownian functional is given in [16]. A multidimensional Brownian functional representation of the whole tableaux associated to a Markovian random word is further given in [17].

Below, we recover the convergence of the whole tableau, in the iid nonuniform case via a different set of techniques which is related to the work of Baryshnikov [4], Gravner, Tracy and Widom [13], Doumerc [10] and [17].

Let 
\[
(\hat{B}^1(t), \hat{B}^2(t), ..., \hat{B}^M(t))
\]
be the \(M\)-dimensional Brownian motion having covariance matrix

\[
\Sigma_t := t \begin{pmatrix}
  p_{\tau(1)} (1 - p_{\tau(1)}) & -p_{\tau(1)} p_{\tau(2)} & \cdots & -p_{\tau(1)} p_{\tau(M)} \\
  -p_{\tau(2)} p_{\tau(1)} & p_{\tau(2)} (1 - p_{\tau(2)}) & \cdots & -p_{\tau(2)} p_{\tau(M)} \\
  \vdots & \vdots & \ddots & \vdots \\
  -p_{\tau(M)} p_{\tau(1)} & -p_{\tau(M)} p_{\tau(2)} & \cdots & p_{\tau(M)} (1 - p_{\tau(M)})
\end{pmatrix}.
\]

(3.1)

For each \(l = 1, ..., M\), there is a unique \(1 \leq k \leq K\) such that \(p_{\tau(l)} = p^{(k)}\), and let

\[
\hat{L}_M^l = \sum_{j=1}^{m_k} \hat{B}^{(j)}(1) + \sup_{J(l-m_k,d_k)} \sum_{j=m_k+1}^{m_k+d_k} \sum_{i=1}^{k-m_k} (\hat{B}^{(j)}(t_{j-i+1}^l) - \hat{B}^{(j)}(t_{j-i}^l)).
\]

(3.2)

where the set \(J(l-m_k,d_k)\) consists of all the subdivisions \((t_j^l)\) of \([0,1]\), \(1 \leq j \leq l - m_k\), \(0 \leq i \leq d_k\), of the form:

\[
t_j^l \in [0,1]; \ t_j^{i+1} \leq t_j^i \leq t_j^i \text{ for } j \leq 0; \ t_j^i = 1 \text{ for } j \geq M - k + 1.
\]

(3.3)

With these preliminaries, we have:

**Theorem 3.1** Let \(\lambda(RSK(X_W)) = (\lambda_1, ..., \lambda_M)\) be the common shape of the Young tableaux associated to \(W\) through the RSK correspondence. Then, as
$N \to \infty$,
\[
\left( \frac{\lambda_1 - N p_{r(1)}}{\sqrt{N p_{r(1)}}}, \ldots, \frac{\lambda_M - N p_{r(M)}}{\sqrt{N p_{r(M)}}} \right) \Rightarrow \left( \hat{L}_M^1, \hat{L}_M^2 - \hat{L}_M^1, \ldots, \hat{L}_M^M - \hat{L}_M^{M-1} \right).
\]

\textbf{Proof.} Let $(e_j)_{j=1,\ldots,M}$ be the canonical basis of $\mathbb{R}^M$, and let $V = (V_1, \ldots, V_M)$ be the random vector such that
\[
\mathbb{P}(V = e_j) = p_j, \quad j = 1, \ldots, M.
\]

Clearly, for each $1 \leq p \leq M$,
\[
\mathbb{E}(V_j) = p_j, \quad \text{Var}(V_j) = p_j(1 - p_j),
\]
and for $j_1 \neq j_2$, $\text{Cov}(V_{j_1}, V_{j_2}) = -p_{j_1}p_{j_2}$. Hence the covariance matrix of $V$ is
\[
\Sigma = \begin{pmatrix}
    p_j (1 - p_j) & -p_1 p_2 & \cdots & -p_1 p_M \\
    -p_2 p_1 & p_2 (1 - p_2) & \cdots & -p_2 p_M \\
    \vdots & \vdots & \ddots & \vdots \\
    -p_M p_1 & -p_M p_2 & \cdots & p_M (1 - p_M)
\end{pmatrix}.
\]

Let $V_1, V_2, \ldots, V_N$ be independent copies of $V$, where $V_i = (V_{i,1}, V_{i,2}, \ldots, V_{i,M})$, $i = 1, \ldots, N$. Then $X_W$ has the same law as the matrix formed by all the $V_{i,j}$ on the lattice $\{1, \ldots, N\} \times \{1, \ldots, M\}$.

It is a well known combinatorial fact (see Lemma 1 of Section 3.2 in [12]) that, for all $1 \leq l \leq M$,
\[
\lambda_1 + \cdots + \lambda_l = G^l(M, N) := \sup \left\{ \sum_{(i,j) \in \pi_1 \cup \cdots \cup \pi_l} V_{i,j} : \pi_1, \ldots, \pi_l \in \mathcal{P}(M, N), \right. \\
\left. \text{and } \pi_1, \ldots, \pi_l \text{ are all disjoint} \right\},
\]

where $\mathcal{P}(M, N)$ is the set of all paths $\pi$ taking only unit steps up or to the right in the rectangle $\{1, \ldots, N\} \times \{1, \ldots, M\}$ and where, by disjoint, it is meant that any two paths do not share a common point $(i, j)$ in the rectangle $\{1, \ldots, N\} \times \{1, \ldots, M\}$. We prove next that, for any $k = 1, \ldots, M$,
\[
\frac{G^l(M, N) - N s_l}{\sqrt{N}} \xrightarrow{N \to \infty} \hat{L}_M^l,
\]

(3.7)
where \( s_l = \sum_{j=1}^{l} p_r(j) \). For \( l = 1 \),

\[
G^1(M, N) = \max \left\{ \sum_{(i,j) \in \pi} V_{i,j} : \pi \in \mathcal{P}(M, N) \right\} .
\]  

(3.8)

Moreover, each path \( \pi \) is uniquely determined by the weakly increasing sequence of its \( M - 1 \) jumps, namely \( 0 = t_0 \leq t_1 \leq \cdots \leq t_{M-1} \leq 1 \), such that \( \pi \) is horizontal on \( [[t_{j-1} N], [t_j N]] \times \{j\} \) and vertical on \( \{t_j N\} \times [j, j+1] \). Hence

\[
G^1(M, N) = \sup_{0=0 \leq t_0 \leq t_1 \leq \cdots \leq t_{M-1} \leq 1} \sum_{j=1}^{M-1} \sum_{i=0}^{t_j - 1} V_{i,j}.
\]

(3.9)

Let \( p_{\text{max}} = \max_{1 \leq j \leq M} p_j \), \( J(M) = \{ j : p_j = p_{\text{max}} \} \subset \{1, \ldots, M\} \) and so \( d_1 = \text{card}(J(M)) \) (the \( \alpha_j \), where \( j \in J(M) \), correspond to the most probable letters). As shown in [17, Section 5], the distribution of \( G^1(M, N) \) is very close, for large \( N \), to that of a very similar expression which involves only those \( V_{i,j} \) for which \( j \in J(M) \). To recall this result, if

\[
\hat{G}^1(M, N) = \sup_{0=0 \leq t_0 \leq t_1 \leq \cdots \leq t_{M-1} \leq 1} \sum_{j=1}^{M-1} \sum_{i=0}^{t_j - 1} V_{i,j} - (t_j - t_{j-1}) N p_{\text{max}}
\]

then, as \( N \to \infty \),

\[
\frac{G^1(M, N)}{\sqrt{N}} - \frac{\hat{G}^1(M, N)}{\sqrt{N}} \overset{p}{\to} 0,
\]

(3.10)

i.e., as \( N \to \infty \), the distribution of the maximum (over all the northeast paths) in (3.8) is approximately the distribution of the maximum over the northeast paths going eastbound only along the rows corresponding to the most probable letters. Now,

\[
\frac{\hat{G}^1(M, N) - N p_{\text{max}}}{\sqrt{N}} = \sup_{0=0 \leq t_0 \leq t_1 \leq \cdots \leq t_{M-1} \leq t_{M-1} \leq 1} \sum_{j=1}^{M-1} \sum_{i=[t_{j-1} N]}^{t_j - 1} V_{i,j} - (t_j - t_{j-1}) N p_{\text{max}}
\]

(3.10)

We next claim that, as \( N \to \infty \), for any \( t > 0 \),

\[
\left( \frac{\sum_{i=1}^{[t N]} V_{i,j} - t N p_j}{\sqrt{N}} \right)_{1 \leq j \leq M} \Rightarrow \left( \hat{B}^j(t) \right)_{1 \leq j \leq M},
\]

(3.10)
where \( \tilde{B}^j(t) \) is an \( M \)-dimensional Brownian motion with covariance matrix \( t \Sigma \). Indeed, for any \( t > 0 \), since \( V_1, V_2, \ldots \) are independent, each with mean vector \( p = (p_1, \ldots, p_M) \), and covariance matrix \( \Sigma \),

\[
\frac{\sum_{i=1}^{\lfloor tN \rfloor} (V_i - tNp)}{\sqrt{N}} \rightarrow \left( \tilde{B}^j(t) \right)_{1 \leq j \leq M} ,
\]

by the central limit theorem for iid random vectors and Slutsky’s lemma. Next, for any \( t > s > 0 \),

\[
\left( \sum_{i=1}^{\lfloor (t-s)N \rfloor} (V_i - (t-s)Np), \sum_{i=1}^{\lfloor sN \rfloor} (V_i - sNp) \right) \rightarrow \left( \left( \tilde{B}^j(t-s) \right)_{1 \leq j \leq M}, \left( \tilde{B}^j(s) \right)_{1 \leq j \leq M} \right). \tag{3.11}
\]

The continuous mapping theorem and Slutsky’s lemma allow us to conclude that

\[
\left( \sum_{i=1}^{\lfloor tN \rfloor} (V_i - tNp), \sum_{i=1}^{\lfloor sN \rfloor} (V_i - sNp) \right) \rightarrow \left( \left( \tilde{B}^j(t) \right)_{1 \leq j \leq M}, \left( \tilde{B}^j(s) \right)_{1 \leq j \leq M} \right). \tag{3.12}
\]

The convergence for the time points \( t_1 > t_2 > \cdots > t_n > 0 \) can be treated in a similar fashion. Thus the finite dimensional distribution converges to that of \( \left( \tilde{B}^j(t) \right)_{1 \leq j \leq M} \). Since tightness in \( (C([0, 1]^M), d^M(\cdot, \cdot)) \) is as in the proof of Donsker’s invariance principle (e.g., see [5]), we are just left with identifying the covariance structure of the limiting Brownian motion \( \left( \tilde{B}^j(t) \right)_{1 \leq j \leq M} \). For each \( N \),

\[
\mathbb{E} \left( \left| \sum_{i=1}^{\lfloor tN \rfloor} (V_{i,j} - tNp_j) \right|^2 \right) = \frac{N}{N} \mathbb{E} \left( \left| V_{i,j} - p_j \right|^2 \right) \leq t \mathbb{E} \left( \left| V_{i,j} - p_j \right|^2 \right). \tag{3.13}
\]
Therefore, \( \sup_{N \geq 1} \mathbb{E} \left( \left| \sum_{i=1}^{\lfloor tN \rfloor} (V_{i,j} - \lfloor tN \rfloor p_j) / \sqrt{N} \right|^2 \right) < \infty \). As \( N \to \infty \), for each \( 1 \leq j \leq M \),

\[
Var \left( \hat{B}^j(t) \right) = \lim_{N \to \infty} Var \left( \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j}}{\sqrt{N}} \right) = \lim_{N \to \infty} \frac{\lfloor tN \rfloor}{N} Var (V_{1,j}) = tp_j (1 - p_j).
\]

Moreover, for any \( j_1 \neq j_2 \), by the continuous mapping theorem,

\[
\left( \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j_1} - \lfloor tN \rfloor p_{j_1}}{\sqrt{N}} \right) \left( \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j_2} - \lfloor tN \rfloor p_{j_2}}{\sqrt{N}} \right) \xrightarrow{N \to \infty} \hat{B}^{j_1}(t) \hat{B}^{j_2}(t).
\]

Since

\[
\sup_{N \geq 1} \left( \mathbb{E} \left( \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j_1} - \lfloor tN \rfloor p_{j_1}}{\sqrt{N}} \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j_2} - \lfloor tN \rfloor p_{j_2}}{\sqrt{N}} \right) \right)^2
\]

\[
\leq \sup_{N \geq 1} \left( \mathbb{E} \left( \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j_1} - \lfloor tN \rfloor p_{j_1}}{\sqrt{N}} \right)^2 \right) \sup_{N \geq 1} \left( \mathbb{E} \left( \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j_2} - \lfloor tN \rfloor p_{j_2}}{\sqrt{N}} \right)^2 \right)
\]

\[
< \infty,
\]

therefore,

\[
Cov \left( \hat{B}^{j_1}(t), \hat{B}^{j_2}(t) \right) = \lim_{N \to \infty} Cov \left( \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j_1}}{\sqrt{N}}, \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j_2}}{\sqrt{N}} \right)
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{\lfloor tN \rfloor} Cov (V_{1,j_1}, V_{1,j_2})
\]

\[
= t Cov (V_{1,j_1}, V_{1,j_2}).
\]

Hence the \( M \)-dimensional Brownian motion \( \left( \hat{B}^{j}(t) \right)_{1 \leq j \leq M} \) has covariance matrix \( t\Sigma \) with \( \Sigma \) given in (3.5). In particular, as \( N \to \infty \), for any \( t > 0 \),

\[
\left( \sum_{i=1}^{\lfloor tN \rfloor} \frac{V_{i,j} - tN p_{\max}}{\sqrt{N}} \right)_{1 \leq j \leq M, j \in J(M)} \Rightarrow \left( \hat{B}^{j}(t) \right)_{1 \leq j \leq M, j \in J(M)}.
\]
It is also straightforward to see that the covariance matrix of \( \hat{B}_j(t) \) is the \( d_1 \times d_1 \) matrix

\[
\begin{pmatrix}
p_{\text{max}} (1 - p_{\text{max}}) & -p_{\text{max}}^2 & \cdots & -p_{\text{max}}^2 \\
-p_{\text{max}}^2 & p_{\text{max}} (1 - p_{\text{max}}) & \cdots & -p_{\text{max}}^2 \\
\vdots & \vdots & \ddots & \vdots \\
-p_{\text{max}}^2 & -p_{\text{max}}^2 & \cdots & p_{\text{max}} (1 - p_{\text{max}})
\end{pmatrix}.
\]

By the continuous mapping theorem,

\[
\frac{\hat{G}^1(M, N) - Np_{\text{max}}}{\sqrt{N}} \overset{N \to \infty}{\Rightarrow} \sup_{J(1, d_1)} \sum_{j=1}^{d_1} \left( \hat{B}^{\tau(j)}(t_j) - \hat{B}^{\tau(j)}(t_{j-1}) \right),
\]

and the right hand side of (3.17) is exactly \( \hat{L}_M^1 \), then (3.9), leads to

\[
\frac{G^1(M, N) - Np_{\text{max}}}{\sqrt{N}} \overset{N \to \infty}{\Rightarrow} \hat{L}_M^1.
\]

Now, for \( l \geq 2 \), \( G^l(M, N) \) is the maximum, of the sums of the \( V_{i,j} \), over \( l \) disjoint paths. Still by the argument in [17], \( \frac{\hat{G}^l(M, N) - G^l(M, N)}{\sqrt{N}} \overset{p}{\Rightarrow} 0 \), as \( N \to \infty \), where \( \hat{G}^l(M, N) \) is the maximal sums of the \( V_{i,j} \) over \( l \) disjoint paths we now describe. Let \( 1 \leq k \leq K \) be the unique integer such that \( p_{\tau(l)} = p_{\text{max}}^{(k)} \). Denote by \( \alpha_{j(1)}, .., \alpha_{j(m_k)} \) the letters corresponding to the \( m_k \) probabilities that are strictly larger than \( p_{\tau(l)} \). For each \( 1 \leq s \leq m_k \), the horizontal path from \((1, j(s))\) to \((N, j(s))\) is included, and thus so are these \( m_k \) paths. The remaining \( l - m_k \) disjoint paths only go eastbound along the rows corresponding to the \( d_k \) letters having probability \( p_{\tau(l)} \). The set of these \( l - m_k \) paths is in a one to one correspondence with the set of subdivisions of \([0, 1]\) given in (3.3). Therefore

\[
\hat{G}^l(M, N) = \sum_{j=1}^{m_k} \sum_{i=1}^{N} V_{i,\tau(j)}
\]

\[
+ \sup_{J(l-m_k,d_k)} \sum_{j=m_k+1}^{m_k+d_k} \sum_{i=1}^{k-m_k} \sum_{[t_{j-1},N]} \left( V_{[t_{j-1},N],\tau(j)} - V_{[t_{j-1},N],\tau(j)} \right)
\]

\[
(3.19)
\]
Now,

\[ \frac{G^l(M, N) - Ns_l}{\sqrt{N}} = \sum_{j=1}^{m_k} \frac{\sum_{i=1}^{N} V_{i,\tau(j)} - Np_{\tau(j)}}{\sqrt{N}} + \sup_{j(l-m_k,d_k)} \sum_{j=m_k+1}^{m_k+d_k} \sum_{i=1}^{k-m_k} \sum_{[t'_{j-i+1}N]} \frac{(V_{[t'_{j-i+1}N],\tau(j)} - V_{[t'_{j-i+1}N],\tau(j)}) - Np^{(k)}}{\sqrt{N}}. \]

(3.20)

Since the column vectors \( V'_1, V'_2, ..., V'_N \) are iid, again, as \( N \to \infty \), for any \( t > 0 \),

\[ \left( \sum_{i=1}^{[tN]} V_{i,\tau(j)} - tNp_{\tau(j)} \right)_{1 \leq j \leq M} \Rightarrow \left( \hat{B}^j(t) \right)_{1 \leq j \leq M}, \]

where \( \left( \hat{B}^j(t) \right)_{1 \leq j \leq M} \) is an \( M \)-dimensional Brownian motion with covariance matrix given in (3.1). Hence, (3.20) and standard arguments give

\[ \frac{G^l(M, N) - Ns_l}{\sqrt{N}} \xrightarrow{N \to \infty} \hat{L}_M^l. \]

Finally, by the Cramér-Wold theorem, as \( N \to \infty \),

\[ \left( \frac{\lambda_1 - Ns_1}{\sqrt{N}}, \frac{\lambda_2 - Ns_2}{\sqrt{N}}, ..., \frac{\lambda_M - Ns_M}{\sqrt{N}} \right) \xrightarrow{N \to \infty} \left( \hat{L}_M^1, \hat{L}_M^2, ..., \hat{L}_M^M \right), \]

(3.21)

therefore, as \( N \to \infty \), by the continuous mapping theorem,

\[ \left( \frac{\lambda_1 - Np_{\tau(1)}}{\sqrt{N}}, \frac{\lambda_2 - Np_{\tau(2)}}{\sqrt{N}}, ..., \frac{\lambda_M - Np_{\tau(M)}}{\sqrt{N}} \right) = \left( \frac{G^1 - Ns_1}{\sqrt{N}}, \frac{G^2 - Ns_2}{\sqrt{N}} - (G^1 - Ns_1), ..., \frac{(G^M - Ns_M) - (G^{M-1} - Ns_{M-1})}{\sqrt{N}} \right) \xrightarrow{N \to \infty} \left( \hat{L}_M^1, \hat{L}_M^2 - \hat{L}_M^1, ..., \hat{L}_M^M - \hat{L}_M^{M-1} \right). \]

(3.22)
The proof is now complete. □

Remark 3.2  
(i) Let \((\xi_0^1, \ldots, \xi_0^M)\) be the vector of eigenvalues of an element of \(G^0(p_{\tau(1)}, \ldots, p_{\tau(M)})\), written in such a way that \(\xi_0^{m_k+1} \geq \cdots \geq \xi_0^{m_k+d_k}\) for \(k = 1, \ldots, K\). Its, Tracy and Widom [18] proved that the limiting density of \(\left(\frac{\lambda_1 - N p_{\tau(1)}}{\sqrt{N p_{\tau(1)}}}, \ldots, \frac{\lambda_M - N p_{\tau(M)}}{\sqrt{N p_{\tau(M)}}}\right)\), as \(N \to \infty\), is the joint density of the eigenvalues of an element of \(G^0(p_{\tau(1)}, \ldots, p_{\tau(M)})\), given by (2.7). By a simple Riemann integral approximation argument, it follows that

\[
\left(\frac{\lambda_1 - N p_{\tau(1)}}{\sqrt{N p_{\tau(1)}}}, \ldots, \frac{\lambda_M - N p_{\tau(M)}}{\sqrt{N p_{\tau(M)}}}\right) \Rightarrow (\xi_0^1, \ldots, \xi_0^M).
\]

Thus, from Theorem 3.1

\[
\left(\hat{L}_M^1, \hat{L}_M^2 - \hat{L}_M^1, \ldots, \hat{L}_M^M - \hat{L}_M^{M-1}\right) \overset{d}{=} (\xi_0^1, \ldots, \xi_0^M). \quad (3.23)
\]

4 The Poissonized Word Problem

"Poissonization" is another useful tool in dealing with length asymptotics for longest increasing subsequence problems. It was introduced by Hammersley in [14] in order to show the existence of \(\lim_{N \to \infty} \mathbb{E}(LI_N)/\sqrt{N}\), for a random permutation of \(\{1, 2, \ldots, N\}\). Since then, this technique has been widely used, and we intend, below, to use it in connection with the inhomogeneous word problem.

Johansson [21] studied the Poissonized measure on the set of shapes of Young tableaux associated to the homogeneous random word, while Its Tracy and Widom [19] also studied the Poissonization of \(LI_N\) for inhomogeneous random words. They showed that the Poissonized distribution of the length of the longest increasing subsequence, as a function of \(p_1, \ldots, p_M\), can be identified as the solution of a certain integrable system of nonlinear PDEs. Below, we show that the Poissonized distribution of the shape of the whole Young tableaux associated to an inhomogeneous random word converges to the spectrum of the corresponding direct sum of GUEs. Next, using this result, together with "de-Poissonization", we obtain the asymptotic behavior of the shape of the tableaux.
Let $W = X_1X_2 \cdots X_N$ be a random word of length $N$, with each letter independently drawn from $\mathcal{A}_M = \{\alpha_1 < \ldots < \alpha_M\}$ with $\mathbb{P}_M (X_i = \alpha_j) = p_j$, $i = 1, \ldots, N$, where $p_j > 0$ and $\sum_{j=1}^{M} p_j = 1$, i.e., the random word is distributed according to $\mathbb{P}_{W;M,N} = \mathbb{P}_M \times \cdots \times \mathbb{P}_M$ on the set of words $[M]^N$. Using the terminology of [21], with $N = \{0, 1, 2, \ldots\}$, let $P_M^{(N)} := \left\{ \lambda = (\lambda_1, \ldots, \lambda_M) \in \mathbb{N}^M : \lambda_1 \geq \cdots \geq \lambda_M, \sum_{i=1}^{M} \lambda_i = N \right\}$, denote the set of partitions of $N$, of length at most $M$. The RSK correspondence defines a bijection from $[M]^N$ to the set of pairs of Young tableaux $(P, Q)$ of common shape $\lambda \in P_M^{(N)}$, where $P$ is semi-standard with elements in $\{1, \ldots, M\}$ and $Q$ is standard with elements in $\{1, \ldots, N\}$.

For any $W \in [M]^N$, let $S(W)$ be the common shape of the Young tableaux associated to $W$ by the RSK correspondence. Then $S$ is a mapping from $[M]^N$ to $P_M^{(N)}$, which, moreover, is a surjection. The image (or push-forward) of $\mathbb{P}_{W,M,N}$ by $S$ is the measure $\mathbb{P}_{M,N}$ given, for any $\lambda_0 \in P_M^{(N)}$, by

$$\mathbb{P}_{M,N} (\lambda_0) := \mathbb{P}_{W,M,N} (\lambda (\text{RSK}(X_W)) = \lambda_0).$$

Next, let

$$P_M := \left\{ \lambda = (\lambda_1, \ldots, \lambda_M) \in \mathbb{N}^M : \lambda_1 \geq \cdots \geq \lambda_M \right\},$$

be the set of partitions, of elements of $\mathbb{N}$, of length at most $M$. The set $P_M$ consists of the shapes of the Young tableaux associated to the random words of any finite length made up from the $M$ letter alphabet $\mathcal{A}_M$.

For $\alpha > 0$, the Poissonized measure of $\mathbb{P}_{M,N}$ on the set $P_M$ is then defined as

$$\mathbb{P}_M^\alpha (\lambda_0) := e^{-\alpha} \sum_{N=0}^{\infty} \mathbb{P}_{M,N} (\lambda_0) \frac{\alpha^N}{N!}. \quad (4.1)$$

The Poissonized measure $\mathbb{P}_M^\alpha$ coincides with the distribution of the shape of the Young tableaux associated to a random word, taking its values in the alphabet $\mathcal{A}_M$ and whose length is a Poisson random variable with mean $\alpha$. Such a random word is called Poissonized, and $LI_\alpha$ denote the length of its longest increasing subsequence.

The Charlier ensemble is closely related to the Poissonized word problem. It is used by Johansson [21] to investigate the asymptotics of $LI_N$ for finite
uniform alphabets. For the non-uniform alphabets we consider, let us define the generalized Charlier ensemble to be:

\[ P_{\alpha}^{\text{Ch},M}(\lambda^0) = \prod_{1 \leq i < j \leq M} (\lambda_i^0 - \lambda_j^0 + j - i) \prod_{j=1}^{M} \frac{1}{(\lambda_j^0 + M - j)!} s_{\lambda^0}(p)e^{-\alpha} \prod_{i=1}^{M} \alpha \lambda_i^0, \]

for all \( \lambda^0 = (\lambda_0^1, \lambda_0^2, ..., \lambda_0^M) \in \mathcal{P}_M \), and where \( s_{\lambda^0}(p) \) is the Schur function of shape \( \lambda^0 \) in the variable \( p = (p_{\tau(1)}, ..., p_{\tau(M)}) \) which we describe next. Let \( A_1, ..., A_K \) be the decomposition of \( \{1, ..., M\} \) such that \( p_{\tau(i)} = p_{\tau(j)} = p^{(k)} \) if and only if \( i, j \in A_k \), for some \( 1 \leq k \leq K \). Clearly, \( d_k = \text{card}(A_k) \).

Then,

\[ s_{\lambda^0}(p) = \frac{\sum_{\sigma \in S_M} (-1)^\sigma \prod_{k=1}^{K} \prod_{i \in A_k} \left( p_{M-\sigma(i)-m_k+d_k+\tau(i)} h_{\sigma(i)}^{m_k+d_k-\tau(i)} \right)}{\prod_{k=1}^{K} (0! \cdot \cdots \cdot (d_k - 1)! \prod_{k<l} (p^{(k)} - p^{(l)})^{d_k d_l})}, \]

where \( S_M \) is the set of all the permutations of \( \{1, ..., M\} \) and where \( h_i = \lambda_i^0 + M - i \) for \( i = 1, ..., M \).

The next theorem gives, for inhomogeneous random words, both \( P_{W,M,N}(\lambda_0) \) and the distribution of \( LI_\alpha \). The first statement is due to Its, Tracy and Widom ([18], [19]), while the second follows directly from the fact that the length of the longest increasing subsequence is equal to the length of the first row of the corresponding Young tableaux.

**Theorem 4.1** (i) On \( [M]^N \), the image (or push-forward) of \( P_{W,M,N} \) by the mapping \( S : [M]^N \rightarrow \mathcal{P}^{(N)}_M \) is, for any \( \lambda^0 = (\lambda_0^1, \lambda_0^2, ..., \lambda_0^M) \in \mathcal{P}^{(N)}_M \), given by

\[ P_{M,N}(\lambda^0) = s_{\lambda^0}(p)f^{\lambda^0}. \]

Above, \( f^{\lambda^0} \) is the number of Young tableaux of shape \( \lambda^0 \) with elements in \( \{1, ..., N\} \):

\[ f^{\lambda^0} = N! \prod_{1 \leq i < j \leq M} (\lambda_i^0 - \lambda_j^0 + j - i) \prod_{j=1}^{M} \frac{1}{(\lambda_j^0 + M - j)!}, \]

and \( s_{\lambda^0}(p) \) is the Schur function of shape \( \lambda^0 \) in the variable \( p = (p_{\tau(1)}, ..., p_{\tau(M)}) \) given in (4.3), with \( \tau \) a permutation of \( \{1, ..., M\} \) corresponding to a non-increasing ordering of \( p_1, p_2, ..., p_M \).
(ii) The Poissonization of $\mathbb{P}_{M,N}$ is the generalized Charlier ensemble $\mathbb{P}_{Ch,M}^\alpha$ defined in (4.3). In particular, for the Poissonized word problem,

$$\mathbb{P}_{W,M}^\alpha (LI_\alpha \leq t) := e^{-\alpha} \sum_{N=0}^{\infty} \mathbb{P}_{M,N} (\lambda_1 \leq t) \frac{\alpha^N}{N!} = \mathbb{P}_{Ch,M}^\alpha (\lambda_1 \leq t). \quad (4.5)$$

For uniform alphabet, Johansson [21] obtained the convergence, as $\alpha \to \infty$, of the Poissonized measure on $\mathbb{P}_M$ to the joint law of the ordered eigenvalues of the GUE. Next, following his lead and techniques, we generalize this result to the nonuniform case, where the convergence is towards the joint law of the eigenvalues $(\xi_1, ..., \xi_M)$, ordered within each block, of an element of $\mathcal{G}_M (d_1, ..., d_K)$. The density of $(\xi_1, ..., \xi_M)$ is, for any $x \in \mathbb{R}^M$, given by

$$f_{\xi_1, ..., \xi_M} (x) = \frac{1}{\sqrt{2\pi c_M}} \prod_{k=1}^{K} \Delta_k (x)^2 e^{-\sum_{i=1}^{M} x_i^2 / 2}, \quad (4.6)$$

where $c_M = (2\pi)^{-(M-1)/2} \prod_{k=1}^{K} (0!1! \cdots (d_k - 1)!)^{-1}$, and where

$$\Delta_k (x) = \prod_{m_k \leq i < j \leq m_k + d_k} (x_i - x_j).$$

**Theorem 4.2** Let $\lambda (RSK(X_W)) = (\lambda_1, ..., \lambda_M)$ be the common shape of the Young tableau associated to $W$ through the RSK correspondence. Let $(\xi_1, ..., \xi_M)$ be the eigenvalues of an element of $\mathcal{G}_M (d_1, ..., d_K)$, written in such a way that $\xi_{m_k+1} \geq \cdots \geq \xi_{m_k+d_k}$ for $k = 1, ..., K$, and let $f_{\xi_1, ..., \xi_M}$ be its density given by (4.6). Then, for any continuous function $g$ on $\mathbb{R}^M$,

$$\lim_{\alpha \to \infty} \mathbb{E}_M^\alpha \left( g \left( \frac{\lambda_1 - \alpha P_{r(1)}}{\sqrt{\alpha P_{r(1)}}}, ..., \frac{\lambda_M - \alpha P_{r(M)}}{\sqrt{\alpha P_{r(M)}}} \right) \right) = \int_{\mathbb{R}^M} g(x) f_{\xi_1, ..., \xi_M} (x) dx. \quad (4.7)$$

**Proof.** By Theorem 4.1, for any partition $\lambda^0 = (\lambda_1^0, \lambda_2^0, ..., \lambda_M^0)$ of $N \in \mathbb{N}$,

$$\mathbb{P}_{M,N} (\lambda (RSK(X_W)) = \lambda^0) = s^\lambda (p) f^{\lambda^0},$$

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where
\[ f^{\lambda_0} = N! \prod_{1 \leq i < j \leq M} (\lambda_i^0 - \lambda_j^0 + j - i) \prod_{j=1}^{M} \frac{1}{(\lambda_j^0 + M - j)!}, \]
and where \( s_{\lambda^0}(p) \) is the Schur function of shape \( \lambda^0 \) in the variable \( p = (p_{\tau(1)}, ..., p_{\tau(M)}) \) as given in (4.3). Hence the Poissonized measure is
\[ P_{M}^{\alpha} (\lambda^0) = e^{-\alpha} \sum_{N=0}^{\infty} N! \prod_{1 \leq i < j \leq M} (\lambda_i^0 - \lambda_j^0 + j - i) \prod_{j=1}^{M} \frac{1}{(\lambda_j^0 + M - j)!} s_{\lambda^0}(p)\alpha^N / N!. \]

Next, for \( i = 1, ..., M \), let
\[ x_i = \frac{\lambda_i^0 - \alpha p_{\tau(i)}}{\sqrt{\alpha p_{\tau(i)}}}, \]
then, as \( \alpha \to \infty \),
\[ \prod_{j=1}^{M} \frac{1}{(\lambda_j^0 + M - j)!} \sim (2\pi)^{-M/2} \frac{e^\alpha}{\alpha^N} \alpha^{-M(M-1)/2} \left( \prod_{i=1}^{M} p_{\tau(i)}^M \right) e^{-\sum_{i=1}^{M} x_i^2 / 2}, \]

and
\[ \prod_{1 \leq i < j \leq M} (\lambda_i^0 - \lambda_j^0 + j - i) \sim \alpha^{M(M-1)/2 - \sum_{k=1}^{K} d_k(d_k-1)/4} \prod_{k=1}^{K} \left( (p(k))^{d_k(d_k-1)/4} \Delta_k(x) \right) \prod_{k<l} (p(k) - p(l))^{d_kd_l}. \]

Together with
\[ \sum_{\sigma \in S_M} (-1)^\sigma \prod_{k=1}^{K} \prod_{i \in A_k} \left( p_{M-\sigma(i)-m_k-d_k+\tau(i)} \right) \]
\[ \sim \prod_{i=1}^{M} p_{\tau(i)}^{M-\tau(i)} \prod_{k=1}^{K} \left( (p(k))^{d_k(d_k-1)/4} \Delta_k(x) \right) \]

(4.10)
the limiting density of \( \left( \left( \lambda_1 - \alpha p_1 \right) / \sqrt{\alpha p_1}, \ldots, \left( \lambda_M - \alpha p_M \right) / \sqrt{\alpha p_M} \right) \), as \( \alpha \to \infty \), is

\[
\sqrt{2\pi c_M} \prod_{k=1}^{K} \Delta_k(x)^2 e^{-\sum_{i=1}^{M} x_i^2/2}, \quad x = (x_1, \ldots, x_M) \in \mathbb{R}^M,
\]

which is just the joint density of the eigenvalues, ordered within each block, of an element of \( G_M(d_1, \ldots, d_K) \). The statement then follows from a Riemann sums approximation argument as in [21]. \( \square \)

The next result is concerned with "de-Poissonization", and again is the nonuniform version (with a similar proof) of a result of Johansson.

**Proposition 4.3** Let \( \alpha_N = N + 3\sqrt{N \ln N} \) and \( \beta_N = N - 3\sqrt{N \ln N} \). Then there is a constant \( C \) such that, for sufficiently large \( N \), and for any \( 0 \leq n_i \leq N, i = 1, \ldots, M \),

\[
\mathbb{P}_M^{\alpha_N} (\lambda_1 \leq n_1, \ldots, \lambda_M \leq n_M) - \frac{C}{N^2} \leq \mathbb{P}_{M,N} (\lambda_1 \leq n_1, \ldots, \lambda_M \leq n_M) \leq \mathbb{P}_M^{\beta_N} (\lambda_1 \leq n_1, \ldots, \lambda_M \leq n_M) + \frac{C}{N^2}.
\]

(4.11)

**Proof.** The proof is analogous to the proof of the corresponding uniform alphabet result, given in [21] (see also Lemma 4.7 in [7]). It suffices to prove that \( \mathbb{P}_{M,N} (\lambda_1 \leq n_1, \ldots, \lambda_M \leq n_M) \) is decreasing in \( N \). Denote a random word in \( [M]^N \) by \( W^{(N)} = X_1 X_2 \cdots X_N \) and let \( [M]^{N+1} (j) \) be the set of the random words \( W^{(N+1)} = X_1 X_2 \cdots X_{N+1} \) such that \( X_{N+1} = j, j = 1, \ldots, M \). Each word \( W^{(N+1)} \) in \( [M]^{N+1} (j) \) is mapped into a word \( F_j \left( W^{(N+1)} \right) \in [M]^N \), by deleting the last letter \( X_{N+1} \). Clearly, \( F_j \) is a bijection from \( [M]^{N+1} (j) \) to \( [M]^N \). Moreover adding the letter \( X_{N+1} \) can only increase the length of a row. Therefore, for all \( i = 1, \ldots, M \),

\[
\lambda_i (F_j \left( W^{(N+1)} \right)) \leq \lambda_i (W^{(N+1)}).
\]

Now, let \( g \left( W^{(N)} \right) = 1 \), if \( \lambda_i (W^{(N)}) \leq n_i \), for all \( i = 1, \ldots, M \), and let \( g \left( W^{(N)} \right) = 0 \), otherwise. We have

\[
g \left( F_j \left( W^{(N+1)} \right) \right) \geq g \left( W^{(N+1)} \right).
\]
Hence,

\[
\mathbb{P}_{M,N+1} (\lambda_1 \leq n_1, \ldots, \lambda_M \leq n_M) \\
= \sum_{W^{(N+1)} \in [M]^{N+1}} g \left( W^{(N+1)} \right) \mathbb{P} \left( W^{(N+1)} \right) \\
= \sum_{j=1}^{M} \left( p_k \sum_{W^{(N+1)} \in [M]^{N+1} (j)} g \left( W^{(N+1)} \right) \mathbb{P} \left( W^{(N+1)} \right) \right) \\
\leq \sum_{j=1}^{M} \left( p_k \sum_{W^{(N+1)} \in [M]^{N+1} (j)} g \left( F_j \left( W^{(N+1)} \right) \right) \mathbb{P} \left( W^{(N+1)} \right) \right) \\
= \sum_{W^{(N)} \in [M]^N} g \left( W^{(N)} \right) \mathbb{P} \left( W^{(N)} \right) \sum_{j=1}^{M} p_j \\
= \mathbb{P}_{M,N} (\lambda_1 \leq n_1, \ldots, \lambda_M \leq n_M). \tag{4.12}
\]

This completes the proof. \(\square\)

We are now ready to obtain asymptotics for the shape of the Young tableaux associated to a random word \(W \in [M]^N\), when \(M\) and \(N\) go to infinity. Before stating our result, let us recall the well known, large \(M\), asymptotic behavior of the spectrum of the \(M \times M\) GUE (\cite{28}, \cite{29}, \cite{21}).

**Theorem 4.4 (Tracy-Widom)** Let \(\xi_{GUE,M}^j\) be the \(j\)th largest eigenvalue of an element of the \(M \times M\) GUE. For each \(r \geq 1\), there is a distribution function \(F_r\) on \(\mathbb{R}^r\), such that, for all \((t_1, \ldots, t_r) \in \mathbb{R}^r\),

\[
\lim_{M \to \infty} \mathbb{P}_{GUE,M} \left( \xi_{GUE,M}^j \leq 2\sqrt{M} + t_j/M^{1/6}, j = 1, \ldots, r \right) = F_r(t_1, \ldots, t_r).
\]

**Remark 4.5** The multivariate distribution function \(F_r\) originates in \cite{28} and \cite{23}, another expression for it is also given in \cite{21} (see (3.48) there). For each \(r = 1, 2, \ldots\), the first marginal of \(F_r\) is the Tracy-Widom distribution \(F_{TW}\).

Once more, our next theorem is already present, for uniform alphabets, in Johansson \cite{21}.
Theorem 4.6 For each \( r \geq 1 \), let \( F_r(t_1, ..., t_r) \) on \( \mathbb{R}^r \) be the distribution function obtained in Theorem 4.4. Assume that \( d_1 \to +\infty \), as \( M \to +\infty \). Then, for all \( (t_1, ..., t_r) \in \mathbb{R}^r \),
\[
\lim_{M \to \infty} \lim_{\alpha \to \infty} \mathbb{P}_M^{\alpha} \left( \lambda_j \leq \alpha p_{\text{max}} + 2 \sqrt{d_1 \alpha p_{\text{max}}} + t_j d_1^{-1/6} \sqrt{\alpha p_{\text{max}}}, j = 1, ..., r \right) = F_r(t_1, ..., t_r), \tag{4.13}
\]
and,
\[
\lim_{d_1 \to \infty} \lim_{N \to \infty} \mathbb{P}_{M,N} \left( \lambda_j \leq N p_{\text{max}} + 2 \sqrt{d_1 N p_{\text{max}}} + t_j d_1^{-1/6} \sqrt{N p_{\text{max}}}, j = 1, ..., r \right) = F_r(t_1, ..., t_r). \tag{4.14}
\]

In particular, for any \( t \in \mathbb{R} \),
\[
\lim_{d_1 \to \infty} \lim_{N \to \infty} \mathbb{P}_{W,M,N} \left( \frac{L_{1N} \leq N p_{\text{max}} + 2 \sqrt{d_1 N p_{\text{max}}} + t d_1^{-1/6} \sqrt{N p_{\text{max}}}}{N} \right) = F_{TW}(t). \tag{4.15}
\]

Proof. By Theorem 4.2, for each \( r \geq 1 \), and for all \( (s_1, ..., s_r) \in \mathbb{R}^r \),
\[
\lim_{\alpha \to \infty} \mathbb{P}_{W,M}^{\alpha} \left( \frac{\lambda_j - \alpha p_{\text{max}}}{\sqrt{\alpha p_{\text{max}}}} \leq s_j, j = 1, ..., r \right) = \mathbb{P}_{GUE,d_1} \left( \xi_j \leq s_j, j = 1, ..., r \right), \tag{4.16}
\]
where \( \xi_j \) is the \( j \)-th largest eigenvalue of the \( d_1 \times d_1 \) GUE. Hence, for any \( (t_1, ..., t_r) \in \mathbb{R}^r \),
\[
\lim_{\alpha \to \infty} \mathbb{P}_{M}^{\alpha} \left( \lambda_j \leq \alpha p_{\text{max}} + 2 \sqrt{d_1 \alpha p_{\text{max}}} + t_j d_1^{-1/6} \sqrt{\alpha p_{\text{max}}}, j = 1, ..., r \right) = \lim_{\alpha \to \infty} \mathbb{P}_{M}^{\alpha} \left( \frac{\lambda_j - \alpha p_{\text{max}}}{\sqrt{\alpha p_{\text{max}}}} \leq 2 \sqrt{d_1} + t_j d_1^{-1/6}, j = 1, ..., r \right) = \mathbb{P} \left( \xi_j \leq 2 \sqrt{d_1} + t_j d_1^{-1/6}, j = 1, ..., r \right). \tag{4.17}
\]
As \( d_1 \to \infty \), Theorem 4.4 gives the first conclusion, proving (4.13). Next, by Proposition 4.3, with \( \alpha_N = N + 3 \sqrt{N \ln N} \) and \( \beta_N = N - 3 \sqrt{N \ln N} \), there is a constant \( C \) such that, for sufficiently large \( N \), and for any \( 0 \leq s_j \leq N \), \( j = 1, ..., r \),
\[
\mathbb{P}_M^{\alpha_N} \left( \lambda_j \leq s_j, j = 1, ..., r \right) - \frac{C}{N^2} \leq \mathbb{P}_{M,N} \left( \lambda_j \leq s_j, j = 1, ..., r \right) \leq \mathbb{P}_M^{\beta_N} \left( \lambda_j \leq s_j, j = 1, ..., r \right) + \frac{C}{N^2}. \tag{4.18}
\]
Next, $N = (1 - \varepsilon_\alpha) \alpha N$, with $\varepsilon_\alpha = 3\sqrt{N \log N} / (N - 3\sqrt{N \log N})$, whereas $N = (1 + \varepsilon_\beta) \beta N$ with $\varepsilon_\beta = 3\sqrt{N \log N} / (N + 3\sqrt{N \log N})$. Since $\varepsilon_\alpha, \varepsilon_\beta \to 0$, as $N \to \infty$, it follows from (4.18), by setting $s_j = p_{\text{max}} N + 2\sqrt{d_1 p_{\text{max}} N + t_j d_1^{1/6} \sqrt{p_{\text{max}} N}}$, that

$$
\lim_{N \to \infty} \mathbb{P}_{M, \alpha}^{p_{\text{max}} N} \left( \lambda_j \leq \alpha N p_{\text{max}} + 2\sqrt{d_1 \alpha N p_{\text{max}} + t_j d_1^{1/6} \alpha N p_{\text{max}}}, j = 1, \ldots, r \right) \\
\leq \lim_{N \to \infty} \mathbb{P}_{M, \beta}^{p_{\text{max}} N} \left( \lambda_j \leq \beta N p_{\text{max}} + 2\sqrt{d_1 \beta N p_{\text{max}} + t_j d_1^{1/6} \beta N p_{\text{max}}}, j = 1, \ldots, r \right) \\
\leq \lim_{N \to \infty} \mathbb{P}_{M}^{\beta N} \left( \lambda_j \leq \beta N p_{\text{max}} + 2\sqrt{d_1 \beta N p_{\text{max}} + t_j d_1^{1/6} \beta N p_{\text{max}}}, j = 1, \ldots, r \right).
$$

(4.19)

Now, (4.17) holds true with $\alpha$ replaced by $\alpha N$ or $\beta N$. Finally, (4.14) follows from (4.19) by letting $d_1 \to \infty$. □

**Remark 4.7**  
(i) The convergence results in Theorem 4.6 are obtained by taking successive limits, i.e., first in $N$ and then in $M$. For uniform finite alphabets, in which case $d_1 = M$, Johansson [21] had previously obtained the convergence, towards $F_1$, for the length of the longest increasing subsequence, via a careful analysis of corresponding Kernel and methods of orthogonal polynomials. His results, which are for simultaneous limits, require $(\ln N)^{3/2}/M \to 0$ and $\sqrt{N}/M \to \infty$. Also in the uniform case, under the assumption $M = o\left(N^{3/7}(\ln N)^{-6/7}\right)$, the convergence result (4.14) is obtained in [8] for simultaneous limits, via Gaussian approximation and a method originating in Baik and Suidan [3] and Bodineau and Martin [6]. Non-uniform results are also given in [8].

(ii) In the permutation case, Baik, Deift and Rains [2] and Widom [32] proved the convergence of the moments of the rows of the associated Young tableaux. In the finite alphabet case, this convergence also holds. First, the convergence of the moments for the eigenvalues of the GUE towards the moments of Tracy-Widom distribution is well known. Then, since the eigenvalues of the traceless GUE and of the corresponding GUE only differ by a centered multivariate Gaussian vector, the convergence of the moments for the eigenvalues of the traceless GUE also holds. We are thus only left with proving the convergence of the moments of $((\lambda_1 - N p_{\tau(1)}) / \sqrt{N p_{\tau(1)}}, \ldots, (\lambda_M - N p_{\tau(M)}) / \sqrt{N p_{\tau(M)}})$, towards the eigenvalues of the traceless GUE, as $N \to \infty$. This, for example, can be proved through a concentration argument, as illustrated
in Section 2.3 of [22], which we describe next. For each \( i = 1, \ldots, M \), let \( \lambda_i(W) \) denote the length of the \( i \)-th row of the Young tableaux associated to the random word \( W = X_1 \cdots X_N \). Then, on the product space \( \{1, \ldots, M\}^N \), \( \lambda_i \) is a Lipschitz function in that \( |\lambda_i(W) - \lambda_i(Y)| \leq 1 \), as long as \( W = X_1 \cdots X_N \) and \( Y = Y_1 \cdots Y_N \) differ only by one coordinate. Thus, by a standard concentration inequality, for any \( t > 0 \),

\[
P(|\lambda_i - E\lambda_i| \geq t) \leq 2e^{-\frac{t^2}{2}}. \quad (4.20)
\]

Therefore, for any \( q > 0 \),

\[
E\left[ \left| \frac{\lambda_i - Np_{r(i)}}{\sqrt{Np_{r(i)}}} \right|^q \right]
\leq \int_0^\infty P\left(|\lambda_i - E\lambda_i| + |E\lambda_i - Np_{r(i)}| \geq \sqrt{Np_{r(i)}} t^{1/q} \right) dt
\leq \frac{|E\lambda_i - Np_{r(i)}|^q}{\sqrt{Np_{r(i)}}} + 2 \int_0^\infty e^{-\frac{p_{r(i)} t^2}{4}} dt \quad (4.21)
\]

Now, for any \( q_1, \ldots, q_M \geq 0 \),

\[
E\left[ \left| \frac{\lambda_i - Np_{r(i)}}{\sqrt{Np_{r(i)}}} \right|^{q_i} \cdots \left| \frac{\lambda_M - Np_{r(M)}}{\sqrt{Np_{r(M)}}} \right|^{q_M} \right]
\leq \sum_{i=1}^M \int_0^\infty P\left(|\lambda_i - Np_{r(i)}|^M \leq \frac{1}{\sqrt{Np_{r(i)}}} |E\lambda_i - Np_{r(i)}|^M q_i t \right) dt
\leq \sum_{i=1}^M \left\{ \left| \frac{E\lambda_i - Np_{r(i)}}{\sqrt{Np_{r(i)}}} \right|^{M q_i} + 2 \int_0^\infty e^{-\frac{p_{r(i)} t^2}{2}} dt \right\} \quad (4.22)
\]

If \( \sup_{N \geq 1} |E\lambda_i - Np_{r(i)}|/\sqrt{Np_{r(i)}} < +\infty \), which is the case if all the probabilities \( p_1, \ldots, p_M \) are distinct (see [19]), it will follow that

\[
\sup_{N \geq 1} E\left[ \left| \frac{\lambda_1 - Np_{r(1)}}{\sqrt{Np_{r(1)}}} \right|^{q_1} \cdots \left| \frac{\lambda_M - Np_{r(M)}}{\sqrt{Np_{r(M)}}} \right|^{q_M} \right] < \infty,
\]

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which, when combined with weak convergence, gives the convergence of the moments of the rows of the tableaux.

5 Appendix

We provide here the proofs of the various properties of the generalized traceless GUE listed in Section 2.

Proof of Proposition 2.1. Let $X \in G_M (d_1, \ldots, d_K)$, and let $X^0$ be given as in (2.1). Then,

$$(X^0_{1,1}, X^0_{2,2}, \ldots, X^0_{M,M})' = \Sigma_0 \left(X_{1,1}, X_{2,2}, \ldots, X_{M,M}\right)' .$$

Since $(X_{1,1}, \ldots, X_{M,M}) \sim N(0, I_M)$, and $\Sigma_0 \Sigma_0 = \Sigma_0$, it follows that

$$(X^0_{1,1}, \ldots, X^0_{M,M}) \sim N(0, \Sigma_0).$$

Now, the off diagonal entries of $X$ are independent of the random variables $X_{1,1} - \sqrt{p_1} \sum_{l=1}^M \sqrt{p_l} X_{1,l}, \ldots, X_{M,M} - \sqrt{p_M} \sum_{l=1}^M \sqrt{p_l} X_{l,l}$, and thus the distribution of $X^0$ is given by (2.3). On the other hand, suppose the matrix $X^0$ is distributed according to the probability distribution (2.3). Clearly, the diagonal entries $X^0_{1,1}, \ldots, X^0_{M,M}$ are independent of the off diagonal ones. Moreover,

$$\mathbb{E} \left[ \left( \sum_{i=1}^M \sqrt{p_i} X^0_{i,i} \right)^2 \right] = \mathbb{E} \left[ \sum_{i,j=1}^M \sqrt{p_i} \sqrt{p_j} X^0_{i,i} X^0_{j,j} \right]$$

$$= (\sqrt{p_1}, \ldots, \sqrt{p_M}) \Sigma^0 (\sqrt{p_1}, \ldots, \sqrt{p_M})'$$

$$= \sum_{i=1}^M p_i (1 - p_i) - \sum_{i=1}^M \sum_{j \neq i} p_i p_j$$

$$= \sum_{i=1}^M p_i - \sum_{i=1}^M p_i$$

$$= 0. \quad (5.1)$$

Therefore, $\sum_{i=1}^M \sqrt{p_i} X^0_{i,i} = 0$. Since $(X^0_{1,1}, \ldots, X^0_{M,M}) \sim N(0, \Sigma^0)$ and $\Sigma^0 \Sigma^0 = \Sigma^0$, there exists a vector $(Z_1, \ldots, Z_M) \sim N(0, I_M)$ such that

$$(X^0_{1,1}, \ldots, X^0_{M,M})' = \Sigma^0 (Z_1, \ldots, Z_M)' ,$$
and, moreover, the vector \((Z_1, ..., Z_M)\) can also be chosen to be independent of the off diagonal entries of \(X^0\). Let \(X\) be the matrix \(X^0\) with the diagonal entries \(X_{1,1}^0, ..., X_{M,M}^0\) replaced by \(Z_1, ..., Z_M\). Then \(X \in G_M (d_1, ..., d_K)\) and \(X^0\) is given as in (2.1).

**Proof of Proposition 2.2.** It is clear that \(\xi_i^0 = \xi_i - \sqrt{p_i} \sum_{l=1}^{M} \sqrt{p_l} X_{l,l}\), \(i = 1, ..., M\), are the eigenvalues of \(X^0\). Next, to prove that \(\sum_{l=1}^{M} \sqrt{p_l} X_{l,l} = \sum_{l=1}^{M} \sqrt{p_l} \xi_l\), let \(X_p\) be the \(M \times M\) matrix obtained by multiplying the \(k\)th diagonal block of \(X\) by \(\sqrt{p^{(k)}}\). For each \(i = 1, ..., M\), there exists a unique \(1 \leq k \leq K\), such that \(m_k < i \leq m_k + d_k\), and \(\xi_i\) is an eigenvalue of the \(k\)th diagonal block of \(X\). Moreover, \(p_i = p^{(k)}\), thus \(\sqrt{p_l} \xi_l\) is an eigenvalue of the \(k\)th diagonal block of \(X_p\), which is an eigenvalue of \(X_p\) as well. Then, \(\sqrt{p_1} \xi_1, ..., \sqrt{p_M} \xi_M\) are the eigenvalues of \(X_p\).

**Proof of Proposition 2.3.** By Proposition 2.2, there exists an \(X \in G_M (d_1, ..., d_K)\), whose eigenvalues \(\xi_1, \cdots, \xi_M\) satisfy,

\[
\xi_0^i = \xi_i - \sqrt{p_i} \sum_{l=1}^{M} \sqrt{p_l} X_{l,l} = \xi_i - \sqrt{p_i} \sum_{l=1}^{M} \sqrt{p_l} \xi_l, \quad i = 1, ..., M.
\]

If \(d_k \to \infty\) for some \(k = 1, ..., K\), by Wigner’s theorem [24], the spectral measure of \((\xi_i/\sqrt{d_k})_{m_k < i \leq m_k + d_k}\) converges weakly to the semicircle law \(\nu\) almost surely, i.e., for any bounded continuous function \(f : \mathbb{R} \to \mathbb{R}\),

\[
\frac{1}{d_k} \sum_{i=m_k+1}^{m_k+d_k} f \left( \frac{\xi_i}{\sqrt{d_k}} \right) \to \int f d\nu,
\]

almost surely. Now, \(\sum_{l=1}^{M} \sqrt{p_l} X_{l,l} \sim N(0, 1)\), and \(p^{(k)}/d_k \to 0\) as \(d_k \to \infty\), hence,

\[
\frac{\sqrt{p^{(k)}} \sum_{l=1}^{M} \sqrt{p_l} X_{l,l}}{\sqrt{d_k}} \xrightarrow{a.s.} 0.
\]
Next, for any bounded Lipschitz function $f$, almost surely,

$$
\left| \frac{1}{d_k} \sum_{i=m_k+1}^{m_k+d_k} f \left( \frac{\xi_0^i}{\sqrt{d_k}} \right) - \int f \, d\nu \right| \\
\leq \left| \frac{1}{d_k} \sum_{i=m_k+1}^{m_k+d_k} f \left( \frac{\xi_0^i}{\sqrt{d_k}} \right) - \frac{1}{d_k} \sum_{i=m_k+1}^{m_k+d_k} f \left( \frac{\xi^i}{\sqrt{d_k}} \right) \right| + \left| \frac{1}{d_k} \sum_{i=m_k+1}^{m_k+d_k} f \left( \frac{\xi^i}{\sqrt{d_k}} \right) - \int f \, d\nu \right|.
$$

(5.2)

Now,

$$
\left| \frac{1}{d_k} \sum_{i=m_k+1}^{m_k+d_k} f \left( \frac{\xi_0^i}{\sqrt{d_k}} \right) - \frac{1}{d_k} \sum_{i=m_k+1}^{m_k+d_k} f \left( \frac{\xi^i}{\sqrt{d_k}} \right) \right| \leq \|f\|_{Lip} \left| \sum_{l=1}^{M} \frac{\sqrt{p^l}}{\sqrt{d_k}} X_{l,l} \right| \\
\rightarrow 0,
$$

(5.3)

and the proposition is proved, since the bounded Lipschitz functions form a determining class for weak convergence ([11, Section 9.3]).

**Proof of Proposition 2.4.** From Proposition 2.2,

$$
\xi_0^i = \xi^i - \sqrt{p^i} \sum_{l=1}^{M} \sqrt{p^l} \xi^l, \quad i = 1, \ldots, M,
$$

where $\xi^1, \ldots, \xi^M$ are the eigenvalues of an element of $G_M(d_1, \ldots, d_K)$, and where $\xi^{m_k+1} \geq \cdots \geq \xi^{m_k+d_k}$ are the eigenvalues of the $k$th diagonal block (an element of the $d_k \times d_k$ GUE), for each $k = 1, \ldots, K$. Clearly, $\sum_{l=1}^{M} \sqrt{p^l} \xi^l = 0$. Let us now compute the joint density of $\xi_0^1, \ldots, \xi_0^{M-1}$. Recall that the joint density of $(\xi^1, \ldots, \xi^M)$ is, for any $x \in \mathbb{R}^M$, given by

$$
\frac{1}{\sqrt{2\pi}} c_M K \prod_{k=1}^{K} \Delta_k(x)^2 e^{-\sum_{i=1}^{M} x_i^2 / 2},
$$

(5.4)

where $c_M = (2\pi)^{-(M-1)/2} \prod_{k=1}^{K} (0! \cdots (d_k - 1)!)^{-1}$, and where

$$
\Delta_k(x) = \prod_{m_k \leq i < j \leq m_k+d_k} (x_i - x_j),
$$

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with again \( m_1 = 0 \) and \( m_k = \sum_{j=1}^{k-1} d_j \), for \( k = 2, \ldots, K \). Consider the change of variables from \( (\xi^1, \ldots, \xi^{M-1}, \xi^M) \) to \( (\xi^1_0, \ldots, \xi^{M-1}_0, Y) \), where

\[
Y = \sqrt{p_M} \sum_{l=1}^{M} \sqrt{p_l} \xi^l.
\]

Then,

\[
\frac{1}{\det(J)} = \det \begin{pmatrix} 1 - p_1 & -\sqrt{p_1 \sqrt{p_2}} & \cdots & -\sqrt{p_1 \sqrt{p_M}} \\
\vdots & \ddots & \ddots & \vdots \\
-\sqrt{p_{M-1} \sqrt{p_1}} & \cdots & 1 - p_{M-1} & -\sqrt{p_{M-1} \sqrt{p_M}} \\
\sqrt{p_1} & \cdots & \sqrt{p_{M-1}} & \sqrt{p_M} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
\sqrt{p_1} & \cdots & \sqrt{p_{M-1}} & \sqrt{p_M} \end{pmatrix} = \sqrt{p_M},
\]

(5.5)

where \( J \) is the Jacobian of this transformation. Thus, the joint density of \( (\xi^1_0, \ldots, \xi^{M-1}_0, Y) \) is given by:

\[
f_{\xi^1_0, \ldots, \xi^{M-1}_0, Y}(x^1_0, \ldots, x^{M-1}_0, y) = \frac{1}{\sqrt{2\pi p_M}} c_M e^{-\frac{y^2}{2p_M} \sum_{j=1}^{M-1} x^2_j} \prod_{1 \leq i < j \leq M} (x^0_i - x^0_j)^2 1_{E_{p_1 \cdots p_M}}(x^0_1, \ldots, x^0_{M-1}, y) = \frac{1}{\sqrt{2\pi p_M}} e^{-\frac{y^2}{2p_M} f(x^0)},
\]

(5.6)

where \( x^0 = (x^0_1, \ldots, x^0_M) \) and \( x^0_M = -\sum_{j=1}^{M-1} x^0_j \). Integrating \( y \) from \(-\infty\) to \( \infty \), shows that the joint density of \( (\xi^1_0, \ldots, \xi^{M-1}_0) \) is \( f(x^0) \).

**Proof of Proposition 2.6.** For any \( s_1, \ldots, s_M \in \mathbb{R} \),

\[
\mathbb{P}\left( \xi^1 \leq s_1, \ldots, \xi^M \leq s_M \right) = \frac{c_M}{\sqrt{2\pi}} \int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_M} \prod_{k=1}^{K} \Delta_k(x)^2 e^{-\sum_{j=1}^{M} x^2_j/2} dx_1 \cdots dx_M,
\]

(5.7)
where \( c_M = (2\pi)^{-(M-1)/2} \prod_{k=1}^K (0! \cdots (d_k - 1))^{-1} \). Consider the change of variables
\[
y = \sum_{j=1}^M \sqrt{p_j} x_j, \quad x_j = x_j^0 + \sqrt{p_j} y, \quad j = 1, \ldots, M.
\]
(5.8)

Clearly, \( \sum_{j=1}^M \sqrt{p_j} x_j^0 = 0 \). With \( L_{(s_1, \ldots, s_M)} \) as in (2.5), we have
\[
P(\xi_1^0 \leq s_1, \ldots, \xi_M^0 \leq s_M)
= \frac{c_M}{\sqrt{2\pi}} \int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_M} \prod_{k=1}^K \left( \Delta_k(x)^2 e^{-\sum_{l=m_k}^{m_k+d_k} x_l^2/2} \right) dx_1 \cdots dx_M
= \frac{c_M}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy \int_{\hat{L}_{(s_1-\sqrt{p_1} y, \ldots, s_M-\sqrt{p_M} y)}} \prod_{k=1}^K \left( \Delta_k(x)^2 e^{-\sum_{l=m_k}^{m_k+d_k} x_l^2/2} \right) e^{-y^2/2} dx
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} P(\xi_0^1 < s_1 - \sqrt{p_1} y, \cdots, \xi_0^M < s_M - \sqrt{p_M} y) dy
= \mathbb{E} \left[ \mathbb{P}(\xi_0^1 < s_1 - \sqrt{p_1} Y, \cdots, \xi_0^M < s_M - \sqrt{p_M} Y | Y) \right],
\]
(5.9)

where \( dx^0 \) is the Lebesgue measure on \( \{ x = (x_1, \ldots, x_M) \in \mathbb{R}^M : \sum_{l=1}^M \sqrt{p_l} x_l = 0 \} \). The right hand side of (5.9) is the distribution function of the sum of the mutually independent random vectors \( (\xi_0^1, \cdots, \xi_0^M) \) and \( (Z_1, \cdots, Z_M) \), where \( (Z_1, \cdots, Z_M) \overset{d}{=} (\sqrt{p_1}, \ldots, \sqrt{p_M}) Z \) with \( Z \sim N(0, 1) \). □

Let \( \xi_{\text{max}}^\text{GU,E,M} \) be the maximal eigenvalue of an element of the \( M \times M \) GUE. Below, we give a simple proof of the fact that \( \xi_{\text{max}}^\text{GU,E,M} / \sqrt{M} \to 2 \) almost surely. This proof is based on a "tridiagonalization" technique originating in Trotter [31] (see also Silverstein [26] where similar ideas are used). Our first result is the well known Householder representation of Hermitian matrices.

**Lemma 5.1** Let \( G = (G_{i,j})_{1 \leq i,j \leq M} \) be a matrix from the GUE. Then, there exists a unitary matrix \( U \), such that
\[
U GU^* = \begin{pmatrix}
A_{1,1} & \chi_{M-1} & 0 & \cdots & 0 \\
\chi_{M-1} & A_{2,2} & \chi_{M-2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \chi_2 & A_{M-1,M-1} & \chi_1 \\
0 & \cdots & 0 & \chi_1 & A_{M,M} \\
\end{pmatrix},
\]
(5.10)
where $A_{1,1}, ..., A_{M,M}$ are independent $N(0,1)$ random variables, and for each $1 \leq k \leq M - 1$, $\chi_{M-k}$ has a chi distribution, with $M - k$ degrees of freedom. Moreover, for each $k = 1, ..., M - 1$, $A_{k,k}$ is independent of $\chi_{M-k}, ..., \chi_1$.

**Theorem 5.2** As $M \to \infty$,

$$\frac{\xi_{\max}^{\text{GUE},M}}{\sqrt{M}} \to 2, \quad \text{almost surely.}$$

**Proof.** By Lemma 5.1, there exists a unitary matrix $U$, such that

$$T := UGU^* = \begin{pmatrix} A_{1,1} & \chi_{M-1} & \cdots & 0 \\ \chi_{M-1} & A_{2,2} & \chi_{M-2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \chi_2 & A_{M-1,M-1} & \chi_1 \\ 0 & \cdots & 0 & \chi_1 & A_{M,M} \end{pmatrix}, \quad (5.11)$$

where $A_{1,1}, ..., A_{k,k}$ are independent $N(0,1)$ random variable, and for each $1 \leq k \leq M - 1$, $\chi_k$ has a chi distribution with $k$ degrees of freedom. Clearly $G$ and $T$ share the same eigenvalues.

By the Geršgorin circle theorem (see [15]), for any eigenvalue $\xi_i$ of $G$, letting also $\chi_0 = \chi_M = 0$,

$$\xi_i \in \bigcup_{k=1, ..., M} [A_{k,k} - \chi_{M-k+1} - \chi_{M-k}, A_{k,k} + \chi_{M-k+1} + \chi_{M-k}].$$

Hence

$$\frac{\xi_{\max}^{\text{GUE},M}}{\sqrt{M}} \leq \max_{k=1, ..., M} \left( \frac{A_{k,k}}{\sqrt{M}} + \frac{\chi_{M-k+1}}{\sqrt{M}} + \frac{\chi_{M-k}}{\sqrt{M}} \right). \quad (5.12)$$

For each $k = 1, ..., M$, $A_{k,k} \sim N(0,1)$, thus, for any fixed $\varepsilon > 0$,

$$\mathbb{P}\left( \max_{k=1, ..., M} \frac{A_{k,k}}{\sqrt{M}} > \varepsilon \right) \leq \sum_{k=1}^M \mathbb{P}\left( \frac{|A_{k,k}|}{\sqrt{M}} > \varepsilon \right) \leq \frac{\sqrt{2M}}{\varepsilon \sqrt{\pi}} e^{-\varepsilon^2 M/2}. \quad (5.13)$$

Therefore, $\sum_{M=1}^\infty \mathbb{P}\left( \max_{k=1, ..., M} \frac{A_{k,k}}{\sqrt{M}} > \varepsilon \right) < \infty$, and thus $\max_{k=1, ..., M} A_{k,k}/\sqrt{M} \xrightarrow{a.s.} 0$. Next, for any fixed $\varepsilon > 0,$
\[ P \left( \max_{k=1, \ldots, M} \frac{\chi^2_{M-k+1} - 1}{M} > \varepsilon \right) \]

\[ = P \left( \max_{k=1, \ldots, M} \chi^2_k < M(1 - \varepsilon) \right) + P \left( \max_{k=1, \ldots, M} \chi^2_k > M(1 + \varepsilon) \right) \]

\[ \leq P \left( \chi^2_M < M(1 - \varepsilon) \right) + MP \left( \chi^2_M > M(1 + \varepsilon) \right). \tag{5.14} \]

Now, for any \( y > 0 \),

\[ \int_y^\infty u^{\frac{M}{2} - 1} e^{-\frac{u}{2}} \, du \]

\[ = 2e^{-\frac{y}{2}} y^{\frac{M}{2}} \left( \sum_{k=1}^{[M/2]} \prod_{l=1}^{k} \frac{M - 2l}{y} \right) + 2(M - 2)! \int_{\sqrt{y}}^\infty x^{M \mod(2)} e^{-\frac{x^2}{2}} \, dx, \tag{5.15} \]

where \( M \geq 2 \). For \( y = M(1 + \varepsilon) \),

\[ P \left( \chi^2_M > M(1 + \varepsilon) \right) = \frac{1}{\Gamma \left( \frac{M}{2} \right) 2^{\frac{M}{2}}} \int_{M(1+\varepsilon)}^\infty u^{\frac{M}{2} - 1} e^{-\frac{u}{2}} \, du \]

\[ \leq \frac{Me^{-\frac{M(1+\varepsilon)}{2}} (M(1+\varepsilon))^{\frac{M}{2}}}{\Gamma \left( \frac{M}{2} \right)} + \frac{2e^{-\frac{M(1+\varepsilon)}{2}} (M - 2)!!}{\Gamma \left( \frac{M}{2} \right)}. \tag{5.16} \]

But, as \( M \to \infty \),

\[ \frac{Me^{-\frac{M(1+\varepsilon)}{2}} (M(1+\varepsilon))^{\frac{M}{2}}}{\Gamma \left( \frac{M}{2} \right) 2^{\frac{M}{2}}} \sim \sqrt{M} e^{-\frac{M\varepsilon}{2}} \frac{1}{\sqrt{\pi}} \]

and

\[ (M - 2)!! \sim \Gamma \left( \frac{M}{2} \right) 2^{\frac{M}{2}}, \]

thus \( \sum_{M=1}^\infty M P \left( \chi^2_M > M(1 + \varepsilon) \right) < \infty \). Next, since \( f(u) = u^{\frac{M}{2} - \varepsilon} e^{-\frac{u}{2}} \) is increasing for \( u \leq M - 2\varepsilon \), we have for \( M \geq 2 \):

\[ P \left( \chi^2_M < M(1 - \varepsilon) \right) \leq \frac{(M(1 - \varepsilon))^{\frac{M}{2}} e^{-\frac{M - M\varepsilon}{2}}}{\varepsilon \Gamma \left( \frac{M}{2} \right) 2^{\frac{M}{2}}}, \tag{5.17} \]
and by Stirling’s Formula, $\sum_{M=2}^{\infty} \mathbb{P}(\chi_M^2 < M(1-\varepsilon)) < \infty$. Therefore,

$$\sum_{M=1}^{\infty} \mathbb{P}\left( \left| \max_{k=1,\ldots,M} \frac{\chi_{M-k+1}^2}{M} - 1 \right| > \varepsilon \right) < \infty, \quad \max_{k=1,\ldots,M} \frac{\chi_{M-k+1}^2}{M} \overset{a.s.}{\to} 1,$$

and almost surely,

$$\limsup_{M \to \infty} \frac{\xi_{\text{GUE},M}^{\max}}{\sqrt{M}} \leq 2. \quad (5.18)$$

Next, since the empirical distribution of the eigenvalues $\left( \xi_{\text{GUE},M}/\sqrt{M} \right)_{1 \leq i \leq M}$ converges almost surely to the semicircle law $\nu$ with density $\sqrt{4-x^2}/2\pi$. We claim that, for any $\varepsilon > 0$,

$$\mathbb{P}\left( \liminf_{M \to \infty} \frac{\xi_{\text{GUE},M}^{\max}}{\sqrt{M}} > 2 - \varepsilon \right) = 1, \quad (5.19)$$

If not, i.e., if

$$\mathbb{P}\left( \liminf_{M \to \infty} \frac{\xi_{\text{GUE},M}^{\max}}{\sqrt{M}} > 2 - \varepsilon \right) < 1,$$

then,

$$\mathbb{P}\left( \liminf_{M \to \infty} \frac{\xi_{\text{GUE},M}^{\max}}{\sqrt{M}} \leq 2 - \varepsilon \right) > 0,$$

and thus,

$$\mathbb{P}\left( \liminf_{M \to \infty} \frac{\xi_{\text{GUE},M}^{i}}{\sqrt{M}} \leq 2 - \varepsilon, i = 1, \ldots, M \right) > 0.$$

Let the event $A_\varepsilon := \left\{ \liminf_{M \to \infty} \frac{\xi_{\text{GUE},M}^{i}}{\sqrt{M}} \leq 2 - \varepsilon, i = 1, \ldots, M \right\}$ and consider the bounded continuous function

$$f_\varepsilon(x) = \begin{cases} 
1, & x < 2 - \varepsilon; \\
\frac{2-x}{\varepsilon}, & 2 - \varepsilon \leq x \leq 2; \\
0, & x > 2. 
\end{cases} \quad (5.20)$$

Then, on $A_\varepsilon$, $\sum_{i=1}^{M} f_\varepsilon \left( \xi_{\text{GUE},M}/\sqrt{M} \right) / M = 1$ and $\int f_\varepsilon d\nu < 1$, and so

$$\mathbb{P}\left( \lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^{M} f_\varepsilon \left( \xi_{\text{GUE},M}/\sqrt{M} \right) \neq \int f_\varepsilon d\nu \right) \geq \mathbb{P}(A_\varepsilon) > 0,$$
which is clearly a contradiction. Letting $\varepsilon \to 0$ in (5.19) yields,

$$\liminf_{M \to \infty} \frac{s_{\text{max}}^{\text{GUE},M}}{\sqrt{M}} \geq 2. \quad \text{a.s.}$$  \hspace{1cm} (5.21)

Therefore, combining (5.18) and (5.21), $s_{\text{max}}^{\text{GUE},M}/\sqrt{M} \to 2$ almost surely. □

To prove our next convergence result, we first need a simple lemma.

**Lemma 5.3** For each $k = 1, \ldots, M$, let $\chi_k^2$ be a chi-square random variable with $k$ degrees of freedom. Then,

$$\lim_{M \to \infty} \mathbb{E} \left( \frac{\max_{k=1,\ldots,M} \chi_k^2}{M} \right) = 1. \quad (5.22)$$

**Proof.** First,

$$\mathbb{E} \left( \max_{k=1,\ldots,M} \chi_k^2 \right) \geq \mathbb{E} (\chi_M^2) = M.$$ 

Next, by the concavity of the logarithm, for any $0 < t < 1/2$,

$$t \mathbb{E} \left( \frac{\max_{k=1,\ldots,M} \chi_k^2}{M} \right) \leq \frac{1}{M} \ln \left( \sum_{k=1}^{M} \mathbb{E} e^{t \chi_k^2} \right)$$

$$\leq \frac{1}{M} \ln \left( M \frac{1}{(1-2t)^{M/2}} \right)$$

$$= \frac{\ln M}{M} - \frac{1}{2} \ln (1-2t). \quad (5.23)$$

Hence,

$$t \limsup_{M \to \infty} \mathbb{E} \left( \frac{\max_{k=1,\ldots,M} \chi_k^2}{M} \right) \leq -\frac{1}{2} \ln (1-2t),$$

and letting $t \to 0$,

$$\limsup_{M \to \infty} \mathbb{E} \left( \frac{\max_{k=1,\ldots,M} \chi_k^2}{M} \right) \leq \lim_{t \to 0} -\frac{\ln (1-2t)}{2t} = 1.$$
\[
\left( \text{Since } -\ln(1-2t) \leq 2t + 4t^2, \text{ for } 0 \leq t \leq 1/3, \text{ taking } t = \sqrt{\ln M/2M} \text{ in } (5.23) \right), \quad \text{will give } \mathbb{E} \left( \max_{k=1,\ldots,M} \frac{\chi^2_k}{M} \right) \leq 1 + 2\sqrt{2 \ln M/M}, \text{ for } M > 10. \] □

Again, in the uniform finite alphabet case, where \( p_1 = \cdots = p_M = 1/M \), we have \( K = 1, \ d_1 = M. \) For \( k = 1, \ldots, M, \) and to keep up with the notation of [17], denote by \( \tilde{H}_M^k \) the particular version of \( \hat{L}_M^k, \) as in (3.2). Let \( (\tilde{B}^1(t), \tilde{B}^2(t), \ldots, \tilde{B}^M(t)) \) be the \( M \)-dimensional Brownian motion having covariance matrix
\[
t \begin{pmatrix}
1 & \rho & \cdots & \rho \\
\rho & 1 & \cdots & \rho \\
\vdots & \vdots & \ddots & \vdots \\
\rho & \rho & \cdots & 1
\end{pmatrix},
\]
with \( \rho = -1/(M - 1). \) Then, for \( k = 1, \ldots, M \) (see also [17], [10]),
\[
\tilde{H}_M^k = \sqrt{\frac{M-1}{M}} \sup \sum_{i=1}^M \sum_{p=1}^k (\tilde{B}^i(t^p_i) - \tilde{B}^i(t^p_{i-1})),
\]
where the sup is taken over all the subdivisions \( (t^p_i) \) of \([0,1]\). As a corollary to Theorem 3.1 (see also [16]), for each \( M \geq 2, \)
\[
\left( \tilde{H}_M^1, \tilde{H}_M^2 - \tilde{H}_M^1, \ldots, \tilde{H}_M^M - \tilde{H}_M^{M-1} \right) \overset{d}{=} \left( \xi^{1.0}_{GU,EU, M}, \xi^{2.0}_{GU,EU, M}, \ldots, \xi^{M.0}_{GU,EU, M} \right). \quad (5.25)
\]
Moreover, the following convergence, in \( L^1, \) holds.

**Theorem 5.4** Let \( \xi^{\max.0}_{GU, EU, M} \) be the maximal eigenvalue of an element of the \( M \times M \) traceless GUE. As \( M \to \infty, \)
\[
\frac{\xi^{\max.0}_{GU, EU, M}}{\sqrt{M}} \to 2, \quad \text{in } L^1.
\]
Equivalently,
\[
\frac{\xi^{\max.0}_{GU, EU, M}}{\sqrt{M}} \to 2, \quad \text{in } L^1.
\]
Equivalently,
\[
\frac{\tilde{H}_M^1}{\sqrt{M}} \to 2, \quad \text{in } L^1.
\]
Proof. Note that when \( p_1 = \cdots = p_M = 1/M \), \( \mathcal{L}_{(s_1, \ldots, s_M)} \), given by (2.4) is the empty set when \( s_1 < 0 \). Hence \( \xi_{\text{GUE},M}^{\text{max},0} \) is nonnegative (this is actually clear from the traceless requirement). By Theorem 3.1, \( \tilde{H}_M^1 \) and \( \xi_{\text{GUE},M}^{\text{max},0} \) are equal in distribution, and so it suffices to prove that, as \( M \to \infty \),

\[
\mathbb{E} \left( \frac{\xi_{\text{GUE},M}^{\text{max},0}}{\sqrt{M}} \right) \to 2. \tag{5.26}
\]

Next, by Proposition 2.6, \( \mathbb{E} \left( \xi_{\text{GUE},M}^{\text{max},0} \right) = \mathbb{E} \left( \xi_{\text{GUE},M}^{\text{max}} \right) \). Moreover, taking expectations on both sides of (5.12) gives:

\[
\mathbb{E} \left( \xi_{\text{GUE},M}^{\text{max}} \right) \leq \mathbb{E} \left( \max_{k=1,\ldots,M} A_{k,k} \right) + \mathbb{E} \left( \max_{k=1,\ldots,M} \chi_{M-k+1} \right) + \mathbb{E} \left( \max_{k=1,\ldots,M} \chi_{M-k} \right).
\]

It is well known (see [23]) that,

\[
\mathbb{E} \left( \max_{k=1,\ldots,M} A_{k,k} \right) \leq \sqrt{2 \ln M},
\]

while, by Lemma 5.3,

\[
\limsup_{M \to \infty} \mathbb{E} \left( \max_{k=1,\ldots,M} \frac{\chi_k}{\sqrt{M}} \right) = 1,
\]

leading to

\[
\limsup_{M \to \infty} \mathbb{E} \left( \frac{\xi_{\text{GUE},M}^{\text{max},0}}{\sqrt{M}} \right) \leq 2.
\]

Now, \( \xi_{\text{GUE},M}^{\text{max},0} \) is nonnegative and by Theorem 5.2, \( \xi_{\text{GUE},M}^{\text{max},0} / \sqrt{M} \to 2 \), almost surely. Thus, by Fatou’s Lemma,

\[
\liminf_{M \to \infty} \mathbb{E} \left( \frac{\xi_{\text{GUE},M}^{\text{max},0}}{\sqrt{M}} \right) \geq \mathbb{E} \left( \liminf_{M \to \infty} \frac{\xi_{\text{GUE},M}^{\text{max},0}}{\sqrt{M}} \right) = 2,
\]

and so, \( \lim_{M \to \infty} \mathbb{E} \left( \frac{\xi_{\text{GUE},M}^{\text{max},0}}{\sqrt{M}} \right) = 2 \). Using once more the fact that \( \xi_{\text{GUE},M}^{\text{max},0} \) is nonnegative, we conclude that \( \lim_{M \to \infty} \mathbb{E} \left| \frac{\xi_{\text{GUE},M}^{\text{max},0}}{\sqrt{M}} - 2 \right| = 0 \), and by the weak law of large number, \( \lim_{M \to \infty} \mathbb{E} \left| \frac{\xi_{\text{GUE},M}^{\text{max},0}}{\sqrt{M}} - 2 \right| = 0 \). \qed
Remark 5.5  A small and elementary tightening of the arguments of Davidson and Szarek [9] (see also [22]) will also provide an alternative proof of Theorem 5.4.

Proof of Corollary 2.7.  By Proposition 2.2

\[
\max_{m_k \leq i \leq m_k + d_k} \xi_0^i = \max_{m_k \leq i \leq m_k + d_k} \xi^i - \sqrt{p(k)} \sum_{l=1}^{M} \sqrt{p_l} X_{l,l}.
\]

Since \( \max_{m_k \leq i \leq m_k + d_k} \xi^i \) is the maximal eigenvalue of an element of the \( d_k \times d_k \) GUE, with probability one or in the mean, \( \lim_{d_k \to \infty} \max_{m_k \leq i \leq m_k + d_k} \xi^i / \sqrt{d_k} = 2 \).

Moreover, \( \sum_{l=1}^{M} \sqrt{p_l} X_{l,l} \) is a centered Gaussian random variable with variance \( \text{Var} \left( \sum_{l=1}^{M} \sqrt{p_l} X_{l,l} \right) = \sum_{l=1}^{M} p_l = 1 \). Hence, with probability one or in the mean, \( \lim_{d_k \to \infty} \sqrt{p(k)} \sum_{l=1}^{M} \sqrt{p_l} X_{l,l} / \sqrt{d_k} = 0. \)

\[ \square \]

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