The $q$-tetrahedron algebra and its finite dimensional irreducible modules

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Abstract

Recently B. Hartwig and the second author found a presentation for the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To obtain this presentation they defined an algebra $\mathfrak{R}$ by generators and relations, and displayed an isomorphism from $\mathfrak{R}$ to the three-point $\mathfrak{sl}_2$ loop algebra. We introduce a quantum analog of $\mathfrak{R}$ which we call $\mathfrak{R}_q$. We define $\mathfrak{R}_q$ via generators and relations. We show how $\mathfrak{R}_q$ is related to the quantum group $U_q(\mathfrak{sl}_2)$, the $U_q(\widehat{\mathfrak{sl}}_2)$ loop algebra, and the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$. We describe the finite dimensional irreducible $\mathfrak{R}_q$-modules under the assumption that $q$ is not a root of 1, and the underlying field is algebraically closed.

Keywords. Quantum group, quantum affine algebra, loop algebra, Onsager algebra, tridiagonal pair.

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1 Introduction

In [15] B. Hartwig and the second author gave a presentation of the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To obtain this presentation they defined a Lie algebra $\mathfrak{R}$ by generators and relations, and displayed an isomorphism from $\mathfrak{R}$ to the three-point $\mathfrak{sl}_2$ loop algebra. $\mathfrak{R}$ has essentially six generators, and it is natural to identify these with the six edges of a tetrahedron. For each face of the tetrahedron the three surrounding edges form a basis for a subalgebra of $\mathfrak{R}$ that is isomorphic to $\mathfrak{sl}_2$ [15, Corollary 12.4]. Any five of the six edges of the tetrahedron generate a subalgebra of $\mathfrak{R}$ that is isomorphic to the $\mathfrak{sl}_2$ loop algebra [15, Corollary 12.6]. Each pair of opposite edges generate a subalgebra of $\mathfrak{R}$ that is isomorphic to the Onsager algebra [15, Corollary 12.5]. Let us call these Onsager subalgebras. Then $\mathfrak{R}$ is the direct sum of its three Onsager subalgebras [15, Theorem 11.6].

In [15] Elduque found an attractive decomposition of $\mathfrak{R}$ into a direct sum of three abelian subalgebras, and he showed how these subalgebras are related to the Onsager subalgebras. In [30] Pascasio and the second author give an action of $\mathfrak{R}$ on the standard module of the Hamming association scheme. In [8] Bremner obtained the universal central extension of

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the three-point \( \mathfrak{sl}_2 \) loop algebra. By modifying the defining relations for \( \mathfrak{A} \), Benkart and the second author obtained a presentation for this extension by generators and relations \([7]\). In \([17]\) Hartwig obtained the irreducible finite dimensional \( \mathfrak{A} \)-modules over an algebraically closed field with characteristic 0.

In this paper we introduce a quantum analog of \( \mathfrak{A} \) which we call \( \mathfrak{A}_q \). We define \( \mathfrak{A}_q \) using generators and relations. We show how \( \mathfrak{A}_q \) is related to \( \mathfrak{A} \) in roughly the same way that \( \mathfrak{A} \) is related to the \( \mathfrak{sl}_2 \) loop algebra. In \([20]\) we considered an algebra \( \mathfrak{A}_q \) on two generators subject to the cubic \( q \)-Serre relations. \( \mathfrak{A}_q \) is often called the positive part of \( U_q(\mathfrak{sl}_2) \). We showed how \( \mathfrak{A}_q \) is related to \( \mathfrak{A} \) in roughly the same way that \( \mathfrak{A} \) is related to the Onsager algebra. We describe the finite dimensional irreducible \( \mathfrak{A}_q \)-modules under the assumption that \( q \) is not a root of 1, and the underlying field is algebraically closed. As part of our description we relate these \( \mathfrak{A}_q \)-modules to the following kind of \( \mathfrak{A}_q \)-module. A finite dimensional irreducible \( \mathfrak{A}_q \)-module \( V \) is called NonNil whenever neither of the two \( \mathfrak{A}_q \) generators is nilpotent on \( V \). We described these \( \mathfrak{A}_q \)-modules in \([20]\). Associated with such a module there is an ordered pair of nonzero scalars called its type. Also, for each finite dimensional irreducible \( \mathfrak{A}_q \)-module there is a scalar in the set \( \{1, -1\} \) called its type. The main result of this paper is an explicit bijection between the following two sets:

(i) the isomorphism classes of finite dimensional irreducible \( \mathfrak{A}_q \)-modules of type 1;

(ii) the isomorphism classes of NonNil finite dimensional irreducible \( \mathfrak{A}_q \)-modules of type \((1, 1)\).

All of the original results in this paper are about \( \mathfrak{A}_q \), although we will initially discuss \( \mathfrak{A} \) in order to motivate things. The paper is organized as follows. In Section 2 we define \( \mathfrak{A} \) and discuss how it is related to \( \mathfrak{sl}_2 \). In Section 3 we recall how \( \mathfrak{A} \) is related to the \( \mathfrak{sl}_2 \) loop algebra. In Section 4 we recall how \( \mathfrak{A} \) is related to the Onsager algebra. In Section 5 we discuss the finite dimensional irreducible \( \mathfrak{A}_q \)-modules. In Section 6 we introduce the algebra \( \mathfrak{A}_q \). In Section 7 we discuss how \( \mathfrak{A}_q \) is related to \( U_q(\mathfrak{sl}_2) \). In Section 8 we discuss how \( \mathfrak{A}_q \) is related to the \( U_q(\mathfrak{sl}_2) \) loop algebra. In Section 9 we discuss how \( \mathfrak{A}_q \) is related to the algebra \( \mathfrak{A}_q \). In Sections 10–18 we describe the finite dimensional irreducible \( \mathfrak{A}_q \)-modules. In Section 19 we mention some open problems.

Throughout the paper \( \mathbb{K} \) denotes a field.

## 2 The tetrahedron algebra

In this section we recall the tetrahedron algebra and discuss how it is related to \( \mathfrak{sl}_2 \).

Until further notice assume the field \( \mathbb{K} \) has characteristic 0.

**Definition 2.1** \([18]\) Let \( \mathfrak{A} \) denote the Lie algebra over \( \mathbb{K} \) that has generators

\[
\{x_{ij} \mid i, j \in \mathbb{I}, i \neq j\} \quad \mathbb{I} = \{0, 1, 2, 3\}
\]

and the following relations:
(i) For distinct $i,j \in \mathbb{I}$,
\[ x_{ij} + x_{ji} = 0. \]

(ii) For mutually distinct $h, i, j \in \mathbb{I}$,
\[ [x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}. \]

(iii) For mutually distinct $h, i, j, k \in \mathbb{I}$,
\[ [x_{hi}, [x_{hi}, [x_{hi}, x_{jk}]]] = 4[x_{hi}, x_{jk}]. \]

We call $\mathfrak{T}$ the tetrahedron algebra.

We comment on how $\mathfrak{T}$ is related to $\mathfrak{sl}_2$. Recall that $\mathfrak{sl}_2$ is the Lie algebra over $\mathbb{K}$ with a basis $h, e^\pm$ and Lie bracket
\[ [h, e^\pm] = \pm 2e^\pm, \quad [e^+, e^-] = h. \]

Define
\[ x = h, \quad y = 2e^+ - h, \quad z = -2e^- - h. \]

Then $x, y, z$ is a basis for $\mathfrak{sl}_2$ and
\[ [x, y] = 2x + 2y, \quad [y, z] = 2y + 2z, \quad [z, x] = 2z + 2x. \]  
(2)

We call $x, y, z$ the equitable basis for $\mathfrak{sl}_2$.

**Proposition 2.2** [18, Corollary 12.1] Let $h, i, j$ denote mutually distinct elements of $\mathbb{I}$. Then there exists an injection of Lie algebras from $\mathfrak{sl}_2$ to $\mathfrak{T}$ that sends
\[ x \to x_{hi}, \quad y \to x_{ij}, \quad z \to x_{jh}. \]

3 The $\mathfrak{sl}_2$ loop algebra

In this section we discuss how $\mathfrak{T}$ is related to the $\mathfrak{sl}_2$ loop algebra. We begin with a definition.

**Definition 3.1** [22, p. 100] The loop algebra $L(\mathfrak{sl}_2)$ is the Lie algebra over $\mathbb{K}$ that has generators $h_i, e^\pm_i, i \in \{0,1\}$ and the following relations:
\begin{align*}
    h_0 + h_1 &= 0, \\
    [h_i, e^\pm_i] &= \pm 2e^\pm_i, \\
    [h_i, e^\pm_j] &= \mp 2e^\pm_j, \quad i \neq j, \\
    [e^+_i, e^-_i] &= h_i, \\
    [e^+_0, e^-_1] &= 0, \\
    [e^+_i, [e^+_i, [e^+_i, e^-_j]]] &= 0, \quad i \neq j.
\end{align*}
The following presentation of $L(sl_2)$ will be useful.

Lemma 3.2 [18, Lemma 5.3] $L(sl_2)$ is isomorphic to the Lie algebra over $\mathbb{K}$ that has generators $x_i, y_i, z_i, i \in \{0, 1\}$ and the following relations:

\[
\begin{align*}
    x_0 + x_1 & = 0, \\
    [x_i, y_i] & = 2x_i + 2y_i, \\
    [y_i, z_i] & = 2y_i + 2z_i, \\
    [z_i, x_i] & = 2z_i + 2x_i, \\
    [z_i, y_j] & = 2z_i + 2y_j, \quad i \neq j, \\
    [y_i, [y_i, [y_i, y_j]]] & = 4[y_i, y_j], \quad i \neq j, \\
    [z_i, [z_i, [z_i, z_j]]] & = 4[z_i, z_j], \quad i \neq j.
\end{align*}
\]

An isomorphism with the presentation in Lemma 3.1 is given by

\[
\begin{align*}
x_i & \rightarrow h_i, \\
y_i & \rightarrow 2e^+_i - h_i, \\
z_i & \rightarrow -2e^-_i - h_i.
\end{align*}
\]

The inverse of this isomorphism is given by

\[
\begin{align*}
h_i & \rightarrow x_i, \\
e^+_i & \rightarrow (x_i + y_i)/2, \\
e^-_i & \rightarrow -(z_i + x_i)/2.
\end{align*}
\]

Proof: One routinely checks that each map is a homomorphism of Lie algebras and that the maps are inverses. It follows that each map is an isomorphism of Lie algebras. \hfill \Box

Proposition 3.3 [18, Corollary 12.3] Let $h, i, j, k$ denote mutually distinct elements of $\mathbb{I}$. Then there exists an injection of Lie algebras from $L(sl_2)$ to $\mathit{G}$ that sends

\[
\begin{align*}
x_1 & \rightarrow x_{hi}, \\
y_1 & \rightarrow x_{ij}, \\
z_1 & \rightarrow x_{jh}, \\
x_0 & \rightarrow x_{ih}, \\
y_0 & \rightarrow x_{hk}, \\
z_0 & \rightarrow x_{ki}.
\end{align*}
\]

4 The Onsager algebra

In this section we discuss how $\mathit{G}$ is related to the Onsager algebra.

Definition 4.1 [29, 31] The Onsager algebra $\mathcal{O}$ is the Lie algebra over $\mathbb{K}$ that has generators $x, y$ and relations

\[
\begin{align*}
    [x, [x, [x, y]]] & = 4[x, y], \quad (3) \\
    [y, [y, [y, x]]] & = 4[y, x]. \quad (4)
\end{align*}
\]

Definition 4.2 Referring to Definition 4.1 we call $x, y$ the standard generators for $\mathcal{O}$.

We refer the reader to [10], [11], [12], [31], [32] for a mathematical treatment of the Onsager algebra. For connections to solvable lattice models see [1], [2], [3], [4], [5], [6], [13], [14], [16], [24], [25], [26], [27], [34].
Proposition 4.3 [18, Corollary 12.2] Let \( h, i, j, k \) denote mutually distinct elements of \( I \). Then there exists an injection of Lie algebras from \( O \) to \( \mathfrak{X} \) that sends
\[
    x \mapsto x_{hi}, \quad y \mapsto x_{jk}.
\]

We have a remark. Let \( \Omega \) (resp. \( \Omega' \) (resp. \( \Omega'' \)) denote the subalgebra of \( \mathfrak{X} \) generated by \( x_{01} \) and \( x_{23} \) (resp. \( x_{02} \) and \( x_{31} \)) (resp. \( x_{03} \) and \( x_{12} \)). By Proposition 4.3 each of \( \Omega, \Omega', \Omega'' \) is isomorphic to \( O \). By [18, Theorem 11.6] we have
\[
    \mathfrak{X} = \Omega + \Omega' + \Omega'' \quad \text{(direct sum)}.
\]

5 The finite dimensional irreducible \( \mathfrak{X} \)-modules

Throughout this section assume that the field \( K \) is algebraically closed with characteristic 0.

In [17] Hartwig classifies the finite dimensional irreducible \( \mathfrak{X} \)-modules. He does this by relating them to the finite dimensional irreducible \( O \)-modules which were previously classified by Davies [12]; see also [10], [32]. We will state Hartwig’s main results after a few comments.

Let \( V \) denote a finite dimensional irreducible \( O \)-module. Define a scalar \( \alpha \) (resp. \( \alpha^* \)) to be \( (\dim(V) - 1) \) times the trace of \( x \) (resp. \( y \)) on \( V \), where \( x, y \) are the standard generators for \( O \). We call the ordered pair \( (\alpha, \alpha^*) \) the type of \( V \). Replacing \( x \) and \( y \) by \( x - \alpha I \) and \( y - \alpha^* I \) respectively, the type becomes \( (0,0) \).

**Theorem 5.1** (Hartwig [17, Theorem 1.2]) Let \( V \) denote a finite dimensional irreducible \( \mathfrak{X} \)-module. Then there exists a unique \( O \)-module structure on \( V \) such that the standard generators \( x \) and \( y \) act as \( x_{01} \) and \( x_{23} \) respectively. This \( O \)-module structure is irreducible of type \( (0,0) \).

**Theorem 5.2** (Hartwig [17, Theorem 1.3]) Let \( V \) denote a finite dimensional irreducible \( O \)-module of type \( (0,0) \). Then there exists a unique \( \mathfrak{X} \)-module structure on \( V \) such that the standard generators \( x \) and \( y \) act as \( x_{01} \) and \( x_{23} \) respectively. This \( \mathfrak{X} \)-module structure is irreducible.

**Remark 5.3** (Hartwig [17, Remark 1.4]) Combining Theorem 5.1 and Theorem 5.2 we obtain a bijection between the following two sets:

(i) the isomorphism classes of finite dimensional irreducible \( \mathfrak{X} \)-modules.

(ii) the isomorphism classes of finite dimensional irreducible \( O \)-modules of type \( (0,0) \).

6 The \( q \)-tetrahedron algebra

We are now ready to introduce the \( q \)-tetrahedron algebra. Until further notice assume that the field \( K \) is arbitrary. We fix a nonzero scalar \( q \in K \) such that \( q^2 \neq 1 \) and define
\[
    [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n = 0, 1, 2, \ldots
\]

We let \( \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} \) denote the cyclic group of order 4.
Definition 6.1 Let $\mathfrak{X}_q$ denote the unital associative $K$-algebra that has generators
\[ \{x_{ij} \mid i, j \in \mathbb{Z}_4, j - i = 1 \text{ or } j - i = 2\} \]
and the following relations:

(i) For $i, j \in \mathbb{Z}_4$ such that $j - i = 2$,
\[ x_{ij} x_{ji} = 1. \]

(ii) For $h, i, j \in \mathbb{Z}_4$ such that the pair $(i - h, j - i)$ is one of $(1, 1), (1, 2), (2, 1)$,
\[ \frac{qx_{hi} x_{ij} - q^{-1} x_{ij} x_{hi}}{q - q^{-1}} = 1. \]

(iii) For $h, i, j, k \in \mathbb{Z}_4$ such that $i - h = j - i = k - j = 1$,
\[ x_{3h} x_{jk} - [3] q x_{hi} x_{jk} x_{hi} + [3] q x_{hi} x_{jk} x_{hi}^2 - x_{jk} x_{hi}^3 = 0. \]

We call $\mathfrak{X}_q$ the \emph{$q$-tetrahedron algebra}.

Note 6.2 The equations (6) are the cubic $q$-Serre relations.

We make some observations.

Lemma 6.3 There exists a $K$-algebra automorphism $\rho$ of $\mathfrak{X}_q$ that sends each generator $x_{ij}$ to $x_{i+1, j+1}$. Moreover $\rho^4 = 1$.

Lemma 6.4 There exists a $K$-algebra antiautomorphism $\omega$ of $\mathfrak{X}_q$ that sends
\[ x_{01} \rightarrow x_{01}, \quad x_{12} \rightarrow x_{30}, \quad x_{23} \rightarrow x_{23}, \quad x_{30} \rightarrow x_{12}, \]
\[ x_{02} \rightarrow x_{31}, \quad x_{13} \rightarrow x_{20}, \quad x_{20} \rightarrow x_{13}, \quad x_{31} \rightarrow x_{02}. \]
Moreover $\omega^2 = 1$.

Lemma 6.5 There exists a $K$-algebra automorphism of $\mathfrak{X}_q$ that sends each generator $x_{ij}$ to $-x_{ij}$.

7 The algebra $U_q(\mathfrak{sl}_2)$

In this section we discuss how the algebra $\mathfrak{X}_q$ is related to $U_q(\mathfrak{sl}_2)$. We start with a definition.

Definition 7.1 Let $U_q(\mathfrak{sl}_2)$ denote the unital associative $K$-algebra with generators $K^{\pm 1}, e^\pm$ and the following relations:
\[ KK^{-1} = K^{-1} K = 1, \]
\[ Ke^\pm K^{-1} = q^{\pm 2} e^\pm, \]
\[ [e^+, e^-] = \frac{K - K^{-1}}{q - q^{-1}}. \]
The following presentation of $U_q(\mathfrak{sl}_2)$ will be useful.

**Lemma 7.2** [21, Theorem 2.1] The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators $x^{\pm 1}, y, z$ and the following relations:

\[
\begin{align*}
xx^{-1} &= x^{-1}x = 1, \\
nxy - qy^{-1}yx &= 1, \\
nyz - qz^{-1}zy &= 1, \\
nzx - qz^{-1}xz &= 1.
\end{align*}
\]

An isomorphism with the presentation in Definition 7.1 is given by:

\[
x^{\pm 1} \rightarrow K^{\pm 1}, \\
y \rightarrow K^{-1} + e^{-}, \\
z \rightarrow K^{-1} - K^{-1}e^+q(q^{-1})^2.
\]

The inverse of this isomorphism is given by:

\[
\begin{align*}
K^{\pm 1} &\rightarrow x^{\pm 1}, \\
e^{-} &\rightarrow y - x^{-1}, \\
e^{+} &\rightarrow (1 - xz)q^{-1}(q - q^{-1})^{-2}.
\end{align*}
\]

**Proof:** One readily checks that each map is a homomorphism of $\mathbb{K}$-algebras and that the maps are inverses. It follows that each map is an isomorphism of $\mathbb{K}$-algebras. \qed

**Definition 7.3** [21] Referring to Lemma 7.2 We call $x^{\pm 1}, y, z$ the equitable generators for $U_q(\mathfrak{sl}_2)$.

**Proposition 7.4** For $i \in \mathbb{Z}_4$ there exists a $\mathbb{K}$-algebra homomorphism from $U_q(\mathfrak{sl}_2)$ to $\mathfrak{g}_q$ that sends

\[
x \rightarrow x_{i,i+2}, \quad x^{-1} \rightarrow x_{i+2,i}, \quad y \rightarrow x_{i+2,i+3}, \quad z \rightarrow x_{i+3,i}.
\]

**Proof:** Compare the defining relations for $U_q(\mathfrak{sl}_2)$ given in Lemma 7.2 with the relations in Definition 6.1(i),(ii). \qed

**Conjecture 7.5** The map in Proposition 7.4 is an injection.
8  The $U_q(\mathfrak{sl}_2)$ loop algebra

In this section we consider how $\boxslash q$ is related to the $U_q(\mathfrak{sl}_2)$ loop algebra. We start with a definition.

**Definition 8.1** [9, p. 266] Let $U_q(L(\mathfrak{sl}_2))$ denote the unital associative $\mathbb{K}$-algebra with generators $K_i, e_i^\pm, i \in \{0, 1\}$ and the following relations:

$$
K_0K_1 = K_1K_0 = 1,
K_i e_i^\pm K_i^{-1} = q^{\pm 2} e_i^\pm,
K_i e_j^\pm K_i^{-1} = q^{\mp 2} e_j^\pm, \quad i \neq j,
[e_i^+, e_i^-] = K_i - K_i^{-1} / (q - q^{-1}),
[e_0^+, e_1^-] = 0,

(e_i^\pm)^3 e_j^\pm - [3]_q (e_i^\pm)^2 e_j^\pm e_i^\mp + [3]_q e_i^\pm e_j^\mp (e_i^\pm)^2 - e_j^\mp (e_i^\pm)^3 = 0, \quad i \neq j.
$$

We call $U_q(L(\mathfrak{sl}_2))$ the $U_q(\mathfrak{sl}_2)$ loop algebra.

The following presentation of $U_q(L(\mathfrak{sl}_2))$ will be useful.

**Theorem 8.2** ([19, Theorem 2.1], [33]) The loop algebra $U_q(L(\mathfrak{sl}_2))$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators $x_i, y_i, z_i, i \in \{0, 1\}$ and the following relations:

$$
x_0 x_1 = x_1 x_0 = 1,
q x_i y_i - q^{-1} y_i x_i = 1,
q y_i z_i - q^{-1} z_i y_i = 1,
q z_i x_i - q^{-1} x_i z_i = 1,
q z_i y_j - q^{-1} y_j z_i = 1, \quad i \neq j,

y_i^2 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0, \quad i \neq j,
z_i^3 z_j - [3]_q z_i^3 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0, \quad i \neq j.
$$

An isomorphism with the presentation in Definition 8.1 is given by:

$$
x_i \rightarrow K_i,
y_i \rightarrow K_i^{-1} + e_i^-,
z_i \rightarrow K_i^{-1} - K_i^{-1} e_i^+ q(q - q^{-1})^2.
$$
The inverse of this isomorphism is given by:

\[ K_i \to x_i, \]
\[ e_i^- \to y_i - x_i^{-1}, \]
\[ e_i^+ \to (1 - x_i z_i) q^{-1}(q - q^{-1})^{-2}. \]

**Proof:** One readily checks that each map is a homomorphism of \( \mathbb{K}\)-algebras and that the maps are inverses. It follows that each map is an isomorphism of \( \mathbb{K}\)-algebras. \(\square\)

**Proposition 8.3** For \( i \in \mathbb{Z}_4 \) there exists a \( \mathbb{K}\)-algebra homomorphism from \( U_q(L(\mathfrak{sl}_2)) \) to \( \Xi_q \) that sends

\[ x_1 \to x_{i,i+2}, \quad y_1 \to x_{i+2,i+3}, \quad z_1 \to x_{i+3,i}, \]
\[ x_0 \to x_{i+2,i}, \quad y_0 \to x_{i,i+1}, \quad z_0 \to x_{i+1,i+2}. \]

**Proof:** Compare the defining relations for \( U_q(L(\mathfrak{sl}_2)) \) given in Theorem 8.2 with the relations in Definition 6.1. \(\square\)

**Conjecture 8.4** The map in Proposition 8.3 is an injection.

## 9 The algebra \( A_q \)

In this section we discuss how \( \Xi_q \) is related to the algebra \( A_q \).

**Definition 9.1** Let \( A_q \) denote the unital associative \( \mathbb{K}\)-algebra defined by generators \( x, y \) and relations

\[ x^3 y - [3]_q x^2 y x + [3]_q x y x^2 - y x^3 = 0, \quad (7) \]
\[ y^3 x - [3]_q y^2 x y + [3]_q y x y^2 - x y^3 = 0. \quad (8) \]

**Definition 9.2** Referring to Definition 9.1 we call \( x, y \) the **standard generators** for \( A_q \).

**Note 9.3** [28, Corollary 3.2.6] The algebra \( A_q \) is often called the **positive part** of \( U_q(\hat{\mathfrak{sl}}_2) \).

**Proposition 9.4** For \( i \in \mathbb{Z}_4 \) there exists a homomorphism of \( \mathbb{K}\)-algebras from \( A_q \) to \( \Xi_q \) that sends the standard generators \( x, y \) to \( x_{i,i+1}, x_{i+2,i+3} \) respectively.

**Proof:** Compare the relations (7), (8) with the relations (6). \(\square\)

**Conjecture 9.5** The map in Proposition 9.4 is an injection.
The finite dimensional irreducible $\boxtimes_q$-modules

For the rest of this paper assume that $q$ is not a root of 1, and the field $\mathbb{K}$ is algebraically closed.

Our next goal is to describe the finite dimensional irreducible $\boxtimes_q$-modules. As part of this description we relate these modules to a type of $A_q$-module that we discussed in [20]. In order to define this type of $A_q$-module we recall a concept. Let $V$ denote a finite-dimensional vector space over $\mathbb{K}$. A linear transformation $A : V \to V$ is said to be nilpotent whenever there exists a positive integer $n$ such that $A^n = 0$.

**Definition 10.1** [20, Definition 1.3] Let $V$ denote a finite-dimensional $A_q$-module. We say this module is NonNil whenever the standard generators $x, y$ are not nilpotent on $V$.

**Note 10.2** In [20, Theorems 1.6, 1.7] we classified up to isomorphism the NonNil finite-dimensional irreducible $A_q$-modules assuming the field $\mathbb{K}$ has characteristic 0 in addition to being algebraically closed.

We now describe how the finite dimensional irreducible $\boxtimes_q$-modules are related to the NonNil finite dimensional irreducible $A_q$-modules. We begin with a few comments. Let $V$ denote a NonNil finite-dimensional irreducible $A_q$-module. By [20, Corollary 2.8] the standard generators $x, y$ are semisimple on $V$. Moreover there exist an integer $d \geq 0$ and nonzero scalars $\alpha, \alpha^* \in \mathbb{K}$ such that the set of distinct eigenvalues of $x$ (resp. $y$) on $V$ is $\{\alpha q^{d-2n} \mid 0 \leq n \leq d\}$ (resp. $\{\alpha^* q^{d-2n} \mid 0 \leq n \leq d\}$). We call the ordered pair $(\alpha, \alpha^*)$ the type of $V$. Replacing $x, y$ by $x/\alpha, y/\alpha^*$ the type becomes $(1, 1)$. Now let $V$ denote a finite dimensional irreducible $\boxtimes_q$-module. As we will see, there exist an integer $d \geq 0$ and a scalar $\varepsilon \in \{1, -1\}$ such that for each generator $x_{ij}$ the action on $V$ is semisimple with eigenvalues $\{\varepsilon q^{d-2n} \mid 0 \leq n \leq d\}$. We call $\varepsilon$ the type of $V$. Replacing each generator $x_{ij}$ by $\varepsilon x_{ij}$ the type becomes 1. The main results of the present paper are contained in the following two theorems and subsequent remark.

**Theorem 10.3** Let $V$ denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1. Then there exists a unique $A_q$-module structure on $V$ such that the standard generators $x$ and $y$ act as $x_{01}$ and $x_{23}$ respectively. This $A_q$-module is NonNil irreducible of type $(1, 1)$.

**Theorem 10.4** Let $V$ denote a NonNil finite-dimensional irreducible $A_q$-module of type $(1, 1)$. Then there exists a unique $\boxtimes_q$-module structure on $V$ such that the standard generators $x$ and $y$ act as $x_{01}$ and $x_{23}$ respectively. This $\boxtimes_q$-module structure is irreducible and type 1.

**Remark 10.5** Combining Theorem 10.3 and Theorem 10.4 we obtain a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1;

(ii) the isomorphism classes of NonNil finite-dimensional irreducible $A_q$-modules of type $(1, 1)$.

We prove Theorems 10.3 and 10.4 in Sections 17 and 18, respectively. In Sections 11–16 we will obtain some results used in these proofs.
11 Some linear algebra

In this section we obtain some linear algebraic results that will help us describe the finite dimensional irreducible $\mathbb{F}_q$-modules.

We will use the following concepts. Let $V$ denote a finite-dimensional vector space over $\mathbb{K}$ and let $A : V \to V$ denote a linear transformation. For $\theta \in \mathbb{K}$ we define

$$V_A(\theta) = \{v \in V \mid Av = \theta v\}.$$

Observe that $\theta$ is an eigenvalue of $A$ if and only if $V_A(\theta) \neq 0$, and in this case $V_A(\theta)$ is the corresponding eigenspace. The sum $\sum_{\theta \in \mathbb{K}} V_A(\theta)$ is direct. Moreover this sum is equal to $V$ if and only if $A$ is semisimple.

**Lemma 11.1** Let $V$ denote a finite-dimensional vector space over $\mathbb{K}$. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{K}$ the following are equivalent:

(i) The expression $A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3$ vanishes on $V_A(\theta)$.

(ii) $BV_A(\theta) \subseteq V_A(q^2\theta) + V_A(\theta) + V_A(q^{-2}\theta)$.

**Proof:** For $v \in V_A(\theta)$ we have

$$\begin{align*}
(A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3)v &= (A^3 - \theta[3]_q A^2 + \theta^2[3]_q A - \theta^3 I)Bv \\
 &= (A - q^2\theta I)(A - \theta I)(A - q^{-2}\theta I)Bv,
\end{align*}$$

where $I : V \to V$ is the identity map. The scalars $q^2\theta, \theta, q^{-2}\theta$ are mutually distinct since $\theta \neq 0$ and since $q$ is not a root of 1. The result follows. \qed

**Lemma 11.2** Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{K}$ the following are equivalent:

(i) The expression $qAB - q^{-1}BA - (q - q^{-1})I$ vanishes on $V_A(\theta)$.

(ii) $(B - \theta^{-1} I)V_A(\theta) \subseteq V_A(q^{-2}\theta)$.

**Proof:** For $v \in V_A(\theta)$ we have

$$(qAB - q^{-1}BA - (q - q^{-1})I)v = q(A - q^{-2}\theta I)(B - \theta^{-1} I)v$$

and the result follows. \qed

For later use we give a second version of Lemma 11.2.

**Lemma 11.3** Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{K}$ the following are equivalent:
(i) The expression $qAB - q^{-1}BA - (q - q^{-1})I$ vanishes on $V_B(\theta)$.

(ii) $(A - \theta^{-1}I)V_B(\theta) \subseteq V_B(q^2\theta)$.

Proof: In Lemma 11.2 replace $(A, B, q)$ by $(B, A, q^{-1})$. □

**Lemma 11.4**

Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A : V \to V$ and $B : V \to V$ denote linear transformations such that

$$
\frac{qAB - q^{-1}BA}{q - q^{-1}} = I.
$$

Then for all nonzero $\theta \in \mathbb{K}$,

$$
\sum_{n=0}^{\infty} V_A(q^{-2n}\theta) = \sum_{n=0}^{\infty} V_B(q^{2n}\theta^{-1}).
$$

(9)

Proof: Let $S_A$ (resp. $S_B$) denote the sum on the left (resp. right) in (9). We show $S_A = S_B$. Since $V$ has finite dimension there exists an integer $t \geq 0$ such that $V_A(q^{-2n}\theta) = 0$ and $V_B(q^{2n}\theta^{-1}) = 0$ for $n > t$. So $S_A = \sum_{n=0}^{t} V_A(q^{-2n}\theta)$ and $S_B = \sum_{n=0}^{t} V_B(q^{2n}\theta^{-1})$. By construction $S_B$ is the set of vectors in $V$ on which the product

$$
\prod_{n=0}^{t} (B - q^{2n}\theta^{-1}I)
$$

(10)

is zero. Using Lemma 11.2 we find (10) is zero on $S_A$ so $S_A \subseteq S_B$. By construction $S_A$ is the set of vectors in $V$ on which the product

$$
\prod_{n=0}^{t} (A - q^{-2n}\theta I)
$$

(11)

is zero. Using Lemma 11.3 we find (11) is zero on $S_B$ so $S_B \subseteq S_A$. We conclude $S_A = S_B$ and the result follows. □

**Lemma 11.5**

With the notation and assumptions of Lemma 11.4 for all nonzero $\theta \in \mathbb{K}$ we have

$$
\dim V_A(\theta) = \dim V_B(\theta^{-1}).
$$

(12)

Proof: By Lemma 11.4 (with $\theta$ replaced by $q^{-2}\theta$),

$$
\sum_{n=1}^{\infty} V_A(q^{-2n}\theta) = \sum_{n=1}^{\infty} V_B(q^{2n}\theta^{-1}).
$$

(13)

Let $S$ denote the sum on either side of (13). Comparing (9) and (13) we find

$$
V_A(\theta) + S = V_B(\theta^{-1}) + S
$$

(14)

and the sum on either side of (14) is direct. The result follows. □
12 The type of a finite dimensional irreducible $\mathfrak{S}_q$-module

Let $V$ denote a finite dimensional irreducible $\mathfrak{S}_q$-module. In this section we show that each generator $x_{ij}$ of $\mathfrak{S}_q$ is semisimple on $V$. We also find the eigenvalues. Using these eigenvalues we associate with $V$ a parameter called the type.

Before proceeding we refine our notation.

**Definition 12.1** Let $V$ denote a finite dimensional irreducible $\mathfrak{S}_q$-module. For each generator $x_{ij}$ of $\mathfrak{S}_q$ and for each $\theta \in \mathbb{K}$ we write $V_{ij}(\theta) = \{v \in V \mid x_{ij}v = \theta v\}$.

**Lemma 12.2** Let $V$ denote a finite dimensional irreducible $\mathfrak{S}_q$-module and choose a generator $x_{ij}$ of $\mathfrak{S}_q$. Then for all nonzero $\theta \in \mathbb{K}$ the spaces $V_{ij}(\theta)$ and $V_{ij}(\theta^{-1})$ have the same dimension. This dimension is independent of our choice of generator.

**Proof:** By Definition 6.1(ii) we have

\[ x_{02} \to x_{23} \to x_{31} \to x_{12} \to x_{20} \to x_{01} \to x_{13} \to x_{30} \quad (15) \]

where $r \to s$ means

\[ \frac{qrs - q^{-1}sr}{q - q^{-1}} = 1. \]

Applying Lemma 11.5 to each arrow in (15) we find

\[ \dim V_{02}(\theta) = \dim V_{23}(\theta^{-1}) = \dim V_{31}(\theta) = \cdots = \dim V_{30}(\theta^{-1}) \quad (16) \]

and

\[ \dim V_{02}(\theta^{-1}) = \dim V_{23}(\theta) = \dim V_{31}(\theta^{-1}) = \cdots = \dim V_{30}(\theta). \quad (17) \]

Also $V_{02}(\theta) = V_{20}(\theta^{-1})$ since $x_{02}, x_{20}$ are inverses; therefore

\[ \dim V_{02}(\theta) = \dim V_{20}(\theta^{-1}). \quad (18) \]

Combining (16)–(18) we obtain the result. \qed

**Theorem 12.3** Let $V$ denote a finite dimensional irreducible $\mathfrak{S}_q$-module. Then the following hold:

(i) Each generator $x_{ij}$ of $\mathfrak{S}_q$ is semisimple on $V$.

(ii) There exist an integer $d \geq 0$ and a scalar $\varepsilon \in \{1, -1\}$ such that for each generator $x_{ij}$ the set of distinct eigenvalues of $x_{ij}$ on $V$ is $\{\varepsilon q^{d-2n} \mid 0 \leq n \leq d\}$. 

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Proof: Since \( \mathbb{K} \) is algebraically closed and since \( V \) has finite positive dimension, there exists a scalar \( \eta \in \mathbb{K} \) such that \( V_{02}(\eta) \neq 0 \). Observe \( \eta \neq 0 \) since \( x_{02} \) is invertible. By Lemma 12.2 we find \( V_{01}(\eta) \neq 0 \). Since \( \eta \neq 0 \) and since \( q \) is not a root of 1, the scalars \( \eta, q^2\eta, q^4\eta, \ldots \) are mutually distinct, so they cannot all be eigenvalues for \( x_{01} \) on \( V \). Therefore there exists a nonzero \( \theta \in \mathbb{K} \) such that \( V_{01}(\theta) \neq 0 \) and \( V_{01}(q^2\theta) = 0 \). Similarly there exists an integer \( d \geq 0 \) such that \( V_{01}(q^{-2n}\theta) \) is nonzero for \( 0 \leq n \leq d \) and zero for \( n = d + 1 \). We show that

\[
V_{01}(\theta) + V_{01}(q^{-2}\theta) + \cdots + V_{01}(q^{-2d}\theta)
\]

(19)
is equal to \( V \). By Lemma 11.2 and since \( V_{01}(q^{-2d-2}\theta) = 0 \) the space (19) is invariant under each of \( x_{12}, x_{13} \). The space (19) is invariant under \( x_{31} \) since \( x_{31} \) is the inverse of \( x_{13} \). By Lemma 11.3 and since \( V_{01}(q^2\theta) = 0 \) the space (19) is invariant under each of \( x_{20}, x_{30} \). The space (19) is invariant under \( x_{02} \) since \( x_{02} \) is the inverse of \( x_{20} \). By Lemma 11.1 and since each of \( V_{01}(q^2\theta), V_{01}(q^{-2d-2}\theta) \) is zero, the space (19) is invariant under \( x_{23} \). We have now shown that (19) is invariant under each generator \( x_{ij} \) of \( \mathbb{K}_q \), so (19) is a \( \mathbb{K}_q \)-submodule of \( V \). Each term in (19) is nonzero and there is at least one term so (19) is nonzero. By these comments and since the \( \mathbb{K}_q \)-module \( V \) is irreducible we find (19) is equal to \( V \). This shows that the action of \( x_{01} \) on \( V \) is semisimple with eigenvalues \( \Delta := \{ q^{-2n}\theta \mid 0 \leq n \leq d \} \). By Lemma 12.2 \( \Delta \) contains the multiplicative inverse of each of its elements; therefore \( q^{-2d}\theta = \theta^{-1} \) so \( \theta^2 = q^{2d} \). Consequently there exists a scalar \( \varepsilon \in \{1, -1\} \) such that \( \theta = \varepsilon q^d \), and we get \( \Delta = \{ \varepsilon q^{d-2n} \mid 0 \leq n \leq d \} \). So far we have shown that the action of \( x_{01} \) on \( V \) is semisimple with eigenvalues \( \{ \varepsilon q^{d-2n} \mid 0 \leq n \leq d \} \). Applying Lemma 12.2 we find that for each generator \( x_{ij} \) the action on \( V \) is semisimple with eigenvalues \( \{ \varepsilon q^{d-2n} \mid 0 \leq n \leq d \} \). □

Definition 12.4 Referring to Theorem 12.3 we call \( d \) the diameter of \( V \). We call \( \varepsilon \) the type of \( V \).

Note 12.5 Let \( V \) denote a finite dimensional irreducible \( \mathbb{K}_q \)-module of type \( \varepsilon \). Applying the automorphism from Lemma 6.5 the type becomes \( -\varepsilon \).

Note 12.6 In view of Note 12.5 as we proceed we will focus on the finite dimensional irreducible \( \mathbb{K}_q \)-modules of type 1.

13 The shape of a finite dimensional irreducible \( \mathbb{K}_q \)-module

Let \( V \) denote a finite dimensional irreducible \( \mathbb{K}_q \)-module of type 1. In this section we associate with \( V \) a finite sequence of positive integers called the shape of \( V \).

We will use the following notation. Let \( V \) denote a vector space over \( \mathbb{K} \) with finite positive dimension. Let \( (s_0, s_1, \ldots, s_d) \) denote a finite sequence consisting of positive integers whose sum is the dimension of \( V \). By a decomposition of \( V \) of shape \( (s_0, s_1, \ldots, s_d) \) we mean a sequence \( V_0, V_1, \ldots, V_d \) of subspaces of \( V \) such that \( V_n \) has dimension \( s_n \) for \( 0 \leq n \leq d \) and

\[
V = V_0 + V_1 + \cdots + V_d
\]

(direct sum).
We call $d$ the diameter of the decomposition. For $0 \leq n \leq d$ we call $V_n$ the $n$th component of the decomposition. For notational convenience we define $V_{-1} = 0$, $V_{d+1} = 0$. By the inversion of $V_0, V_1, \ldots, V_d$ we mean the decomposition $V_d, V_{d-1}, \ldots, V_0$.

**Definition 13.1** Let $V$ denote a finite dimensional irreducible $\mathbb{F}_q$-module of type 1 and diameter $d$. For each generator $x_{ij}$ of $\mathbb{F}_q$ we define a decomposition of $V$ which we call $[i, j]$. The decomposition $[i, j]$ has diameter $d$. For $0 \leq n \leq d$ the $n$th component of $[i, j]$ is the eigenspace of $x_{ij}$ on $V$ associated with the eigenvalue $q^{d-2n}$.

**Note 13.2** With reference to Definition 13.1 for $i \in \mathbb{Z}_d$ the decomposition $[i, i+2]$ is the inversion of $[i+2, i]$.

**Proposition 13.3** Let $V$ denote a finite dimensional irreducible $\mathbb{F}_q$-module of type 1 and diameter $d$. Choose a generator $x_{ij}$ of $\mathbb{F}_q$ and consider the corresponding decomposition $[i, j]$ of $V$ from Definition 13.1. Then the shape of this decomposition is independent of the choice of generator. Denoting the shape by $(\rho_0, \rho_1, \ldots, \rho_d)$ we have $\rho_n = \rho_{d-n}$ for $0 \leq n \leq d$.

**Proof:** Immediate from Lemma 12.2 \hfill \square

**Definition 13.4** Let $V$ denote a finite dimensional irreducible $\mathbb{F}_q$-module of type 1 and diameter $d$. By the shape of $V$ we mean the sequence $(\rho_0, \rho_1, \ldots, \rho_d)$ from Proposition 13.3.

## 14 Finite dimensional irreducible $\mathbb{F}_q$-modules; the $x_{ij}$ action

Let $V$ denote a finite dimensional irreducible $\mathbb{F}_q$-module of type 1. In this section we describe how each generator $x_{ij}$ of $\mathbb{F}_q$ acts on the eigenspaces of the other generators. We will treat separately the cases $j - i = 1$ and $j - i = 2$.

**Theorem 14.1** Let $V$ denote a finite dimensional irreducible $\mathbb{F}_q$-module of type 1 and diameter $d$. Let $V_0, V_1, \ldots, V_d$ denote a decomposition of $V$ from Definition 13.1. Then for $i \in \mathbb{Z}_d$ and for $0 \leq n \leq d$ the action of $x_{i,i+1}$ on $V_n$ is given as follows.

| decomposition $[i, i+1]$ | action of $x_{i,i+1}$ on $V_n$ |
|--------------------------|----------------------------------|
| $[i, i+1]$               | $(x_{i,i+1} - q^{d-2n}I)V_n = 0$ |
| $[i+1, i+2]$             | $(x_{i,i+1} - q^{2n-d}I)V_n \subseteq V_{n-1}$ |
| $[i+2, i+3]$             | $x_{i,i+1}V_n \subseteq V_{n-1} + V_n + V_{n+1}$ |
| $[i+3, i]$               | $(x_{i,i+1} - q^{2n-d}I)V_n \subseteq V_{n+1}$ |
| $[i, i+2]$               | $(x_{i,i+1} - q^{d-2n}I)V_n \subseteq V_{n+1}$ |
| $[i+1, i+3]$             | $(x_{i,i+1} - q^{d-2n}I)V_n \subseteq V_{n-1}$ |

**Proof:** We consider each of the six rows of the table.

- $[i, i+1]$: By Definition 13.1 $V_n$ is the eigenspace for $x_{i,i+1}$ associated with the eigenvalue $q^{d-2n}$.
- $[i+1, i+2]$: Apply Lemma 11.3 (with $A = x_{i,i+1}$ and $B = x_{i+1,i+2}$).
\[ [i+2, i+3] \]: Apply Lemma \textbf{11.1} (with \(A = x_{i+2,i+3}\) and \(B = x_{i,i+1}\)).
\[ [i+3, i] \]: Apply Lemma \textbf{11.2} (with \(A = x_{i+3,i}\) and \(B = x_{i,i+1}\)).
\[ [i, i+2] \]: Apply Lemma \textbf{11.2} (with \(A = x_{i+2,i}\) and \(B = x_{i,i+1}\)).
\[ [i+1, i+3] \]: Apply Lemma \textbf{11.3} (with \(A = x_{i,i+1}\) and \(B = x_{i+1,i+3}\)).

\[ \square \]

**Theorem 14.2** Let \(V\) denote a finite dimensional irreducible \(\mathbb{F}_q\)-module of type 1 and diameter \(d\). Let \(V_0, V_1, \ldots, V_d\) denote a decomposition of \(V\) from Definition \textbf{13.3}. Then for \(i \in \mathbb{Z}_4\) and for \(0 \leq n \leq d\) the action of \(x_{i,i+2}\) on \(V_n\) is given as follows.

| Decomposition | Action of \(x_{i,i+2}\) on \(V_n\) |
|---------------|----------------------------------|
| \([i, i+1]\)  | \((x_{i,i+2} - q^{d-2n}I)V_n \subseteq V_0 + \cdots + V_{n-1}\) |
| \([i+1, i+2]\) | \((x_{i,i+2} - q^{d-2n}I)V_n \subseteq V_{n+1} + \cdots + V_d\) |
| \([i+2, i+3]\) | \((x_{i,i+2} - q^{2n-d}I)V_n \subseteq V_{n-1}\) |
| \([i, i+2]\)  | \((x_{i,i+2} - q^{2n-d}I)V_n \subseteq V_{n+1}\) |
| \([i+1, i+3]\) | \((x_{i,i+2} - q^{d-2n}I)V_n = 0\) |
| \([i+2, i+3]\) | \(x_{i,i+2}V_n \subseteq V_{n-1} + \cdots + V_d\) |

**Proof:** We consider each of the six rows of the table.

\[ [i, i+1]: \] By Lemma \textbf{11.3} (with \(A = x_{i+2,i}\) and \(B = x_{i,i+1}\)) we find
\[
(x_{i,i+2} - q^{2r-d}I)V_r \subseteq V_{r-1} \quad (0 \leq r \leq d).
\]

The result follows from this and since \(x_{i,i+2}\) is the inverse of \(x_{i+2,i}\).

\[ [i+1, i+2]: \] By Lemma \textbf{11.2} (with \(A = x_{i+1,i+2}\) and \(B = x_{i+2,i}\)) we find
\[
(x_{i,i+2} - q^{2r-d}I)V_r \subseteq V_{r+1} \quad (0 \leq r \leq d).
\]

The result follows from this and since \(x_{i,i+2}\) is the inverse of \(x_{i+2,i}\).

\[ [i+2, i+3]: \] Apply Lemma \textbf{11.3} (with \(A = x_{i,i+2}\) and \(B = x_{i+2,i+3}\)).

\[ [i+3, i]: \] Apply Lemma \textbf{11.2} (with \(A = x_{i+3,i}\) and \(B = x_{i,i+2}\)).

\[ [i, i+2]: \] Recall that \(V_n\) is the eigenspace of \(x_{i,i+2}\) associated with the eigenvalue \(q^{d-2n}\).

\[ [i+1, i+3]: \] Let \(U_0, U_1, \ldots, U_d\) denote the decomposition \([i+2, i+3]\). From row \([i+2, i+3]\) in the table of this theorem we find
\[
x_{i,i+2}V_r \subseteq U_{r-1} + U_r \quad (0 \leq r \leq d).
\]

By Lemma \textbf{11.4} (with \(A = x_{i+2,i+3}\) and \(B = x_{i+3,i+1}\)),
\[
V_r + \cdots + V_d = U_r + \cdots + U_d \quad (0 \leq r \leq d).
\]

We may now argue
\[
x_{i,i+2}V_n \subseteq x_{i,i+2}(V_n + \cdots + V_d)
\]
\[
= x_{i,i+2}(U_n + \cdots + U_d) \quad \text{(by (21))}
\]
\[
\subseteq U_{n-1} + \cdots + U_d \quad \text{(by (20))}
\]
\[
= V_{n-1} + \cdots + V_d \quad \text{(by (21)).}
\]

\[ \square \]
15 Flags

In this section we recall the notion of a flag.

Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $(s_0, s_1, \ldots, s_d)$ denote a sequence of positive integers whose sum is the dimension of $V$. By a flag on $V$ of shape $(s_0, s_1, \ldots, s_d)$ we mean a nested sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ of subspaces of $V$ such that the dimension of $U_n$ is $s_0 + \cdots + s_n$ for $0 \leq n \leq d$. We call $U_n$ the $n$th component of the flag. We call $d$ the diameter of the flag. We observe $U_d = V$.

The following construction yields a flag on $V$. Let $V_0, V_1, \ldots, V_d$ denote a decomposition of $V$ of shape $(s_0, s_1, \ldots, s_d)$. Define

$$U_n = V_0 + V_1 + \cdots + V_n \quad (0 \leq n \leq d).$$

Then the sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ is a flag on $V$ of shape $(s_0, s_1, \ldots, s_d)$. We say this flag is induced by the decomposition $V_0, V_1, \ldots, V_d$.

We now recall what it means for two flags to be opposite. Suppose we are given two flags on $V$ with the same diameter: $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ and $U'_0 \subseteq U'_1 \subseteq \cdots \subseteq U'_d$. We say these flags are opposite whenever there exists a decomposition $V_0, V_1, \ldots, V_d$ of $V$ such that

$$U_n = V_0 + V_1 + \cdots + V_n, \quad U'_n = V_d + V_{d-1} + \cdots + V_{d-n}$$

for $0 \leq n \leq d$. In this case

$$U_i \cap U'_j = 0 \quad \text{if} \quad i + j < d \quad (0 \leq i, j \leq d) \quad (22)$$

and

$$V_n = U_n \cap U'_{d-n} \quad (0 \leq n \leq d). \quad (23)$$

In particular the decomposition $V_0, V_1, \ldots, V_d$ is uniquely determined by the given flags.

16 Flags on finite dimensional irreducible $\boxtimes_q$-modules

We return our attention to the finite dimensional irreducible $\boxtimes_q$-modules.

**Theorem 16.1** Let $V$ denote a finite dimensional irreducible $\boxtimes_q$-module of type 1 and diameter $d$. Then there exists a collection of flags on $V$, denoted $[h], h \in \mathbb{Z}_4$, that have the following property: for each generator $x_{ij}$ of $\boxtimes_q$ the decomposition $[i, j]$ of $V$ induces $[i]$ and the inversion of $[i, j]$ induces $[j]$.

**Proof:** For all $h \in \mathbb{Z}_4$ let $[h]$ denote the flag on $V$ induced by the inversion of $[h-1, h]$. By Lemma 11.4 (with $A = x_{h-1, h}$ and $B = x_{h, h+1}$) the flag on $V$ induced by $[h, h+1]$ is equal to $[h]$. By Lemma 11.4 (with $A = x_{h-1, h}$ and $B = x_{h, h+2}$) the flag on $V$ induced by $[h, h+2]$ is equal to $[h]$. The result follows. □
Lemma 16.2 Let $V$ denote a finite dimensional irreducible $\mathfrak{g}_q$-module of type 1. Then for $i \in \mathbb{Z}_4$ the shape of the flag $[i]$ coincides with the shape of $V$.

Proof: Let $(\rho_0, \rho_1, \ldots, \rho_d)$ denote the shape of $V$. By Proposition 13.3 the decomposition $[i, i + 1]$ has shape $(\rho_0, \rho_1, \ldots, \rho_d)$. This decomposition induces the flag $[i]$ so the flag $[i]$ has shape $(\rho_0, \rho_1, \ldots, \rho_d)$. \hfill \Box

Theorem 16.3 Let $V$ denote a finite dimensional irreducible $\mathfrak{g}_q$-module of type 1. Then the flags $[i], i \in \mathbb{Z}_4$ on $V$ from Theorem 16.1 are mutually opposite.

Proof: We invoke Theorem 16.1. For $i \in \mathbb{Z}_4$ the flags $[i], [i + 1]$ are opposite since the decomposition $[i, i + 1]$ induces $[i]$ and the inversion of this decomposition induces $[i + 1]$. The flags $[i], [i + 2]$ are opposite since the decomposition $[i, i + 2]$ induces $[i]$ and the inversion of this decomposition induces $[i + 2]$. The result follows. \hfill \Box

Theorem 16.4 Let $V$ denote a finite dimensional irreducible $\mathfrak{g}_q$-module of type 1 and diameter $d$. Pick a generator $x_{ij}$ of $\mathfrak{g}_q$ and consider the corresponding decomposition $[i, j]$ of $V$ from Definition 13.1. For $0 \leq n \leq d$ the $n$th component of $[i, j]$ is the intersection of the following two sets:

(i) component $n$ of the flag $[i]$;

(ii) component $d - n$ of the flag $[j]$.

Proof: Combine Theorem 16.1 and line (23). \hfill \Box

17 From $\mathfrak{g}_q$-modules to $A_q$-modules

In this section we give the proof of Theorem 10.3. Our proof is based on the following proposition.

Proposition 17.1 Let $V$ denote a finite dimensional irreducible $\mathfrak{g}_q$-module of type 1. Let $W$ denote a nonzero subspace of $V$ such that $x_{01}W \subseteq W$ and $x_{23}W \subseteq W$. Then $W = V$.

Proof: Without loss we may assume $W$ is irreducible as a module for $x_{01}, x_{23}$. Let $V_0, V_1, \ldots, V_d$ denote the decomposition $[0, 1]$ and let $V'_0, V'_1, \ldots, V'_d$ denote the decomposition $[2, 3]$. Recall that $x_{01}$ (resp. $x_{23}$) is semisimple on $V$ with eigenspaces $V_0, V_1, \ldots, V_d$ (resp. $V'_0, V'_1, \ldots, V'_d$). By this and since $W$ is invariant under each of $x_{01}, x_{23}$ we find

$$W = \sum_{n=0}^{d} W \cap V_n, \quad W = \sum_{n=0}^{d} W \cap V'_n. \quad (24)$$
Define

\[ m = \min \{ n \mid 0 \leq n \leq d, \ W \cap V_n \neq \emptyset \}. \tag{25} \]

We claim

\[ m = \min \{ n \mid 0 \leq n \leq d, \ W \cap V_n' \neq \emptyset \}, \tag{26} \]
\[ m = \min \{ n \mid 0 \leq n \leq d, \ W \cap V_{d-n} \neq \emptyset \}, \tag{27} \]
\[ m = \min \{ n \mid 0 \leq n \leq d, \ W \cap V_{d-n}' \neq \emptyset \}. \tag{28} \]

To prove the claim we let \( m', m'', m''' \) denote the integers on the right-hand side of (25)–(28) respectively. We will show \( m, m', m'', m''' \) coincide by showing \( m \leq m' \leq m'' \leq m''' \leq m \).

Suppose \( m > m' \). By (25) and the equation on the left in (24), the space \( W \) is contained in component \( d - m \) of the flag \([1]\). By construction \( W \) has nonzero intersection with component \( m' \) of the flag \([2]\). Since \( m > m' \) the component \( d - m \) of \([1]\) has zero intersection with component \( m' \) of \([2]\), for a contradiction. Therefore \( m \leq m' \). Next suppose \( m' > m'' \). By the definition of \( m' \) and the equation on the right in (24), the space \( W \) is contained in component \( d - m' \) of the flag \([3]\). By construction \( W \) has nonzero intersection with component \( m'' \) of the flag \([1]\). Since \( m' > m'' \) the component \( d - m' \) of \([3]\) has zero intersection with component \( m'' \) of \([1]\), for a contradiction. Therefore \( m' \leq m'' \). Next suppose \( m'' > m''' \). By the definition of \( m'' \) and the equation on the left in (24), the space \( W \) is contained in component \( d - m'' \) of the flag \([0]\). By construction \( W \) has nonzero intersection with component \( m''' \) of the flag \([2]\). Since \( m'' > m''' \) the component \( d - m'' \) of \([0]\) has zero intersection with component \( m''' \) of \([2]\), for a contradiction. Therefore \( m'' \leq m''' \). Now suppose \( m''' > m \). By the definition of \( m''' \) and the equation on the right in (24), the space \( W \) is contained in component \( d - m''' \) of the flag \([2]\). By (25) \( W \) has nonzero intersection with component \( m \) of the flag \([0]\). Since \( m''' > m \) the component \( d - m''' \) of \([2]\) has zero intersection with component \( m \) of \([0]\), for a contradiction. Therefore \( m''' \leq m \). We have now shown \( m \leq m' \leq m'' \leq m''' \leq m \).

Therefore \( m, m', m'', m''' \) coincide and the claim is proved. The claim implies that for all \( i \in \mathbb{Z}_4 \) the component \( d - m \) of the flag \([i]\) contains \( W \), and component \( m \) of \([i]\) has nonzero intersection with \( W \). We can now easily show \( W = V \). Since \( V \) is irreducible as a \( \mathbb{Z}_q \)-module it suffices to show that \( W \) is invariant under \( \mathbb{Z}_q \). By construction \( W \) is invariant under each of \( x_{01}, x_{23} \). We now let \( x_{rs} \) denote one of \( x_{12}, x_{13}, x_{20}, x_{30} \) and show \( x_{rs}W \subseteq W \). Let \( W' \) denote the span of the set of vectors in \( W \) that are eigenvectors for \( x_{rs} \). By construction \( W' \subseteq W \) and \( x_{rs}W' \subseteq W' \). We show \( W' = W \). To this end we show that \( W' \) is nonzero and invariant under each of \( x_{01}, x_{23} \). We now show \( W' \neq 0 \). By the comment after the preliminary claim, \( W \) has nonzero intersection with component \( m \) of the flag \([r]\) and \( W \) is contained in component \( d - m \) of the flag \([s]\). By Theorem 16.4 the intersection of component \( m \) of \([r]\) and component \( d - m \) of \([s]\) is equal to component \( m \) of the decomposition \([r, s]\), which is an eigenspace for \( x_{rs} \). The intersection of \( W \) with this eigenspace is nonzero and contained in \( W' \), so \( W' \neq 0 \). We now show \( x_{01}W' \subseteq W' \). To this end we pick \( v \in W' \) and show \( x_{01}v \in W' \). Without loss we may assume that \( v \) is an eigenvector for \( x_{rs} \); let \( \theta \) denote the corresponding eigenvalue. Then \( \theta \neq 0 \) by Theorem 12.3. Recall \( v \in W' \) and \( W' \subseteq W \) so \( v \in W \). The space \( W \) is \( x_{01} \)-invariant so \( x_{01}v \in W \). By these comments \( (x_{01} - \theta^{-1}I)v \in W \). By Lemma 11.2 or Lemma 14.3 the vector \( (x_{01} - \theta^{-1}I)v \) is contained in an eigenspace of \( x_{rs} \), so \( (x_{01} - \theta^{-1}I)v \in W' \). By this and since \( v \in W' \) we have \( x_{01}v \in W' \). We have now shown
We now define two decompositions on \( i, j \), \( i, i \).\( \tag{12.3} \)

Note that \( (x_{23} - \eta^{-1}I)u \in W' \). By Lemma 16.2 or Lemma 16.3 the vector \((x_{23} - \eta^{-1}I)u \) is contained in an eigenspace of \( x_{rs} \), so \((x_{23} - \eta^{-1}I)u \in W' \). By this and since \( u \in W' \) we have \( x_{23}u \in W' \). We have now shown \( x_{23}W' \subseteq W' \) as desired. So far we have shown that \( W' \) is nonzero and invariant under each of \( x_{01}, x_{23} \). Now \( W' = W \) by the irreducibility of \( W \), so \( x_{rs}W \subseteq W \). It follows that \( W \) is \( \mathbb{H}_q \)-invariant. The space \( V \) is irreducible as a \( \mathbb{H}_q \)-module so \( W = V \). \( \square \)

It is now a simple matter to prove Theorem 10.3.

**Proof of Theorem 10.3.** The generators \( x_{01}, x_{23} \) satisfy the cubic \( q \)-Serre relations \( (\varnothing) \); by this and \( (\mathcal{A}), (\mathcal{B}) \) there exists an \( \mathcal{M}_q \)-module structure on \( V \) such that the standard generators \( x, y \) act as \( x_{01} \) and \( x_{23} \) respectively. This module structure is unique since \( x, y \) generate \( \mathcal{A}_q \). This module structure is irreducible by Proposition 17.1. By Theorem 12.3 and the construction, for each of \( x, y \) the action on \( V \) is semisimple with eigenvalues \( \{q^{d-2n} \mid 0 \leq n \leq d \} \). Therefore the \( \mathcal{M}_q \)-module structure is NonNil and type \((1,1)\). \( \square \)

18  **From \( \mathcal{A}_q \)-modules to \( \mathbb{H}_q \)-modules**

In this section we give the proof of Theorem 10.4.

**Proof of Theorem 10.4.** By 20 Corollary 2.8] and since the \( \mathcal{A}_q \)-module \( V \) is NonNil, the standard generators \( x, y \) are semisimple on \( V \). Since \( V \) has type \((1,1)\) there exists an integer \( d \geq 0 \) such that for each of \( x, y \) the set of distinct eigenvalues on \( V \) is \( \{q^{d-2n} \mid 0 \leq n \leq d \} \). We now define two decompositions on \( V \), denoted \([0,1]\) and \([2,3]\), where we view \( 0,1,2,3 \) as the elements of \( \mathbb{Z}_4 \). Each of these decompositions has diameter \( d \). For \( 0 \leq n \leq d \) the \( n \)th component of \([0,1]\) (resp. \([2,3]\)) is the eigenspace for \( x \) (resp. \( y \)) on \( V \) associated with the eigenvalue \( q^{d-2n} \). We now define some flags on \( V \), denoted \([i] \), \( i \in \mathbb{Z}_4 \). The flag \([0] \) is induced by \([0,1]\) and the flag \([1] \) is induced by the inversion of \([0,1]\). The flag \([2] \) is induced by \([2,3]\) and the flag \([3] \) is induced by the inversion of \([2,3]\). By construction the flags \([0],[1]\) are opposite and the flags \([2],[3]\) are opposite. By 19 Lemma 4.3 the flags \([i], i \in \mathbb{Z}_4 \) are mutually opposite. Now for \( i, j \in \mathbb{Z}_4 \) such that \( j - i = 1 \) or \( j - i = 2 \), there exists a decomposition \([i,j]\) of \( V \) such that \([i,j] \) induces \([i] \) and the inversion of \([i,j] \) induces \([j] \). We note that \([i,i+2] \) is the inversion of \([i+2,i] \) for \( i \in \mathbb{Z}_4 \). For \( i,j \in \mathbb{Z}_4 \) such that \( j - i = 1 \) or \( j - i = 2 \), let \( x_{ij} : V \rightarrow V \) denote the linear transformation such that for \( 0 \leq n \leq d \) the \( n \)th component of \([i,j]\) is the eigenspace for \( x_{ij} \) associated with the eigenvalue \( q^{d-2n} \). We observe that the standard generators \( x, y \) act on \( V \) as \( x_{01} \) and \( x_{23} \) respectively. We now show that the above transformations \( x_{ij} \) satisfy the defining relations for \( \mathbb{H}_q \). The \( x_{ij} \) satisfy Definition 6.1(i) by the construction. The \( x_{ij} \) satisfy Definition 6.1(ii) by 19 Theorems 7.1,10.1,10.2. The \( x_{ij} \) satisfy Definition 6.1(iii) by 19 Theorem 12.1. We have now shown that the transformations \( x_{ij} \) satisfy the defining relations for \( \mathbb{H}_q \). Therefore they induce a \( \mathbb{H}_q \)-module structure on \( V \). So far we have shown that there exists a \( \mathbb{H}_q \)-module structure on
V such that the standard generators $x$ and $y$ act as $x_{01}$ and $x_{23}$ respectively. Next we show that this $\mathbb{A}_q$-module structure is unique. Suppose we are given any $\mathbb{A}_q$-module structure on $V$ such that the standard generators $x$ and $y$ act as $x_{01}$ and $x_{23}$ respectively. This $\mathbb{A}_q$-module structure is irreducible by construction and since the $A_q$-module $V$ is irreducible. This $\mathbb{A}_q$-module structure is type 1 and diameter $d$, since the action of $x_{01}$ on $V$ has eigenvalues \(\{q^{d-2n} \mid 0 \leq n \leq d\}\). For each generator $x_{ij}$ of $\mathbb{A}_q$ the action on $V$ is determined by the decomposition $[i, j]$. By Theorem 16.4 the decomposition $[i, j]$ is determined by the flags $[i]$ and $[j]$. Therefore our $\mathbb{A}_q$-module structure on $V$ is determined by the flags $[h], h \in \mathbb{Z}_4$. By construction the flags $[0]$ and $[1]$ are determined by the decomposition $[0, 1]$ and hence by the action of $x$ on $V$. Similarly the flags $[2]$ and $[3]$ are determined by the decomposition $[2, 3]$ and hence by the action of $y$ on $V$. Therefore the given $\mathbb{A}_q$-module structure on $V$ is determined by the action of $x$ and $y$ on $V$, so this $\mathbb{A}_q$-module structure is unique. We have now shown that there exists a unique $\mathbb{A}_q$-module structure on $V$ such that the standard generators $x$ and $y$ act as $x_{01}$ and $x_{23}$ respectively. We mentioned earlier that this $\mathbb{A}_q$-module structure is irreducible and has type 1. 

\[\square\]

19 Suggestions for further research

In this section we give some suggestions for further research.

Problem 19.1 Find a basis for the $K$-vector space $\mathbb{A}_q$.

Problem 19.2 Find all the Hopf-algebra structures on $\mathbb{A}_q$.

Problem 19.3 Find the automorphism group of the $K$-algebra $\mathbb{A}_q$.

Problem 19.4 Find all the 2-sided ideals of the $K$-algebra $\mathbb{A}_q$.

Conjecture 19.5 Let $V$ denote a finite dimensional irreducible $\mathbb{A}_q$-module of type 1. Then there exists a nondegenerate symmetric bilinear form $\langle , \rangle$ on $V$ such that

\[
\langle r.u, v \rangle = \langle u, \omega(r).v \rangle \quad r \in \mathbb{A}_q, \ u, v \in V,
\]

where $\omega$ is the antiautomorphism from Lemma 6.4.

Conjecture 19.6 Let $\Omega$ (resp. $\Omega'$) (resp. $\Omega''$) denote the subalgebra of $\mathbb{A}_q$ generated by $x_{01}$ and $x_{23}$ (resp. by $x_{12}$ and $x_{30}$) (resp. by $x_{02}, x_{20}, x_{13}, x_{31}$). Then the map

\[
\Omega \otimes \Omega' \otimes \Omega'' \rightarrow \mathbb{A}_q
\]

\[
u \otimes v \otimes w \rightarrow uvw
\]

is an isomorphism of $K$-vector spaces. By $\otimes$ we mean $\otimes_K$. 

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Problem 19.7 For \( i \in \mathbb{Z}_4 \) let \( \mathfrak{g}^{(i)}_q \) denote the set of elements \( r \in \mathfrak{g}_q \) that have the following property: for all finite dimensional irreducible \( \mathfrak{g}_q \)-modules \( V \) of type 1, \( r \) leaves invariant each component of \([i]\), where \([i]\) is the flag on \( V \) from Theorem [16.1]. We observe that \( \mathfrak{g}^{(i)}_q \) is a subalgebra of \( \mathfrak{g}_q \) that contains each of the generators \( x_{i,i+1}, x_{i,i+2}, x_{i+1,i}, x_{i-1,i} \). Find a generating set for the \( \mathbb{K} \)-algebra \( \mathfrak{g}^{(i)}_q \). Find a basis for the \( \mathbb{K} \)-vector space \( \mathfrak{g}^{(i)}_q \). For each subset \( S \subseteq \mathbb{Z}_4 \) describe the subalgebra \( \cap_{i \in S} \mathfrak{g}^{(i)}_q \).

Problem 19.8 Recall that for each element \( r \in \mathfrak{g}_q \) the centralizer \( C(r) \) is the subalgebra of \( \mathfrak{g}_q \) consisting of the elements in \( \mathfrak{g}_q \) that commute with \( r \). For each generator \( x_{ij} \) of \( \mathfrak{g}_q \) describe \( C(x_{ij}) \). Find a generating set for the \( \mathbb{K} \)-algebra \( C(x_{ij}) \). Find a basis for the \( \mathbb{K} \)-vector space \( C(x_{ij}) \).

Problem 19.9 Recall that an element \( r \in \mathfrak{g}_q \) is central whenever it commutes with each element of \( \mathfrak{g}_q \). The center of \( \mathfrak{g}_q \) is the subalgebra of \( \mathfrak{g}_q \) consisting of the central elements. Find the center of \( \mathfrak{g}_q \).

Before we state our last problem we have two remarks.

Remark 19.10 Let \( V \) denote a finite dimensional irreducible \( \mathfrak{g}_q \)-module. Then the dual vector space \( V^* \) has a \( \mathfrak{g}_q \)-module structure such that
\[
(r,f)(v) = f(\omega(r).v) \quad r \in \mathfrak{g}_q, \quad f \in V^*, \quad v \in V,
\]
where \( \omega \) is the antiautomorphism of \( \mathfrak{g}_q \) from Lemma 6.4.

Remark 19.11 Let \( V \) denote a finite dimensional irreducible \( \mathfrak{g}_q \)-module. For each automorphism \( \sigma \) of \( \mathfrak{g}_q \) there exists a \( \mathfrak{g}_q \)-module structure on \( V \), called \( V \) twisted via \( \sigma \), that has the following property: for all \( r \in \mathfrak{g}_q \) and for all \( v \in V \) the vector \( r.v \) computed in \( V \) twisted via \( \sigma \) coincides with \( \sigma(r).v \) computed in the original \( \mathfrak{g}_q \)-module \( V \).

Problem 19.12 Let \( V \) denote a finite dimensional irreducible \( \mathfrak{g}_q \)-module of type 1, and let the automorphism \( \rho \) of \( \mathfrak{g}_q \) be as in Lemma 6.3. Using Remark 19.10 and Remark 19.11 we obtain eight \( \mathfrak{g}_q \)-module structures on \( V \); these are \( V \) twisted via \( \rho^n \) for \( 0 \leq n \leq 3 \) and \( V^* \) twisted via \( \rho^n \) for \( 0 \leq n \leq 3 \). How are these eight \( \mathfrak{g}_q \)-module structures related up to isomorphism?

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