Unusual pseudo-Hermiticity in two-sided deformation of Heisenberg algebra

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Abstract

The recently introduced by us two- and three-parameter \((p,q)\)- and \((p,q,\mu)\)-deformed extensions of the Heisenberg algebra were explored under the condition of their connectedness with the respective (nonstandard) deformed quantum oscillator algebras. In this paper we show that such connectedness dictates certain \(\eta(N)\)-pseudo-Hemitian conjugation (with \(\eta(N)\) depending on the particle number operator \(N\)) between the creation and annihilation operators. Likewise, proper \(\eta(N)\)-pseudo-Hemiticity characterizes position and momentum operators, while the involved Hamiltonian is Hermitian. Different possible cases are analyzed, and some interesting features stemming from the use of such \(\eta(N)\)-based conjugation are emphasized.

Keywords: deformed Heisenberg algebra; position and momentum operators; deformed oscillator algebra; \(\eta(N)\)-pseudo-Hermitian conjugation; \(\eta(N)\)-pseudo-Hermiticity.

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1 Introduction

Non-Hermitian modifications of quantum mechanics [1-12] which lead nevertheless to real spectra of operators, attract great interest. A significant part of such investigations in recent years has been crystalized into an important branch encompassing works on pseudo-Hermitian [5] representation in quantum mechanics, see the comprehensive review [10] which gives a plenty of references and discusses main ideas and results. This approach have its impact on a variety of applications, ranging from nuclear and quantum field theory to nonlinear optics and biophysics [10].

On the other hand, significant attention is devoted to generalized versions [12-20] of Heisenberg algebra (HA), obtained through appropriate extension of its basic relation $[X, P] = i\hbar$. That implies respective modifications of the uncertainty relation (see e.g. [15, 16, 21-24]). In our recent paper [20], the so-called two-sided or three-parameter ($p, q, \mu$)-deformation of HA has been introduced and studied. Its particular $p = 1, \mu = 0$ case yields simple modified HA with $q$-commutator involved which was studied in [17], where the explicit relation with certain non-standard $q$-deformed oscillator algebra was found (for some recent applications of deformed oscillators or deformed bosons see e.g. [25-32]). In the more general ”left-handed” + ”right-handed” generalization [20], the analogous relation with deformed oscillator has been derived in the case of the $p, q$-deformation of HA (involving $p, q$-commutator) as well as for the case of three-parameter deformed HA, where the related non-standard deformed oscillator algebra (DOA) was obtained in a somewhat restricted situation. In conjunction with the mentioned relation, an important property was deduced that the deformation parameter $\mu$ and also the parameters $p$ and $q$ explicitly depend on the particle number operator $N$.

In all three mentioned cases, i.e. $q$-, $(p, q)$- and $(\mu, p, q)$-deformations of HA, the formulas relating the position and momentum operators $X, P$ and creation/destruction operators $a^+, a^-$ are not those of usual harmonic oscillator, see e.g. [33], but involve $N$-dependent coefficients. This fact, rooted in the required realizability of particular deformed HA through respective DOA, is of basic importance for us as it dictates principal distinction from the usual (Hermitian) conjugation rules of the operators involved. However, such rules were not considered in [20].

The goal of the present paper is to examine those aspects of 2- and 3-parameter deformed HA which concern modified rules for conjugation of the engaged operators. In our treatment, most important aspect is the encountered unusual $\eta(N)$-pseudo-Hermitian conjugation and the related $\eta(N)$-pseudo-Hermiticity of $X$ and/or $P$. The unusual feature consists in the inevitable (within our approach) dependence of $\eta$-function(s) on the particle number operator $N$, and this differs from the common pseudo-Hermiticity studied e.g. in [8, 7, 10, 12].

About the plan of our paper. Sections 2-4 give a sketch of those deformed versions of HA which serve as playground. The reasons of why, instead of usual Hermitian conjugation and usual Hermiticity, there inevitably emerges the concept of $\eta(N)$-pseudo-Hermitian conjugation, along with $\eta(N)$-pseudo-Hermiticity, are described in Sec. 5. Therein, we also study the cases with partial $\eta(N)$-pseudo-Hermiticity (when one of the operators, $X$ or $P$, remains Hermitian). In Sec. 6 we show that both $X$ and $P$ should obey $\eta(N)$-pseudo-Hermiticity in the case when $a^+$ and $a^-$ are usual
Hermitian conjugates of each other. General situation when $\eta(N)$-pseudo-Hermitian conjugation concerns all the four operators is treated in Sec. 7. Next section deals with commutation properties of $X$, $P$ with the particle number operator $N$, while the Hamiltonian (in terms of $X$, $P$) and its Hermiticity are the subject of Sec. 9. The paper ends with concluding remarks.

2 Extended Heisenberg algebra with Hamiltonian in the r.h.s.

The Heisenberg algebra (HA), based on the well-known relation of commutation

$$\left[ X, P \right] = i\hbar \ ,$$

during last decades serves as starting point for diverse modifications or generalizations. Rather general and one of most natural extensions of (1) involves in its r.h.s. some function $f(\mathcal{H})$ of the Hamiltonian $\mathcal{H}$, that is

$$\left[ X, P \right] = i\hbar f(\mathcal{H}) \ .$$

Different versions of this modification of HA were studied e.g. in [13, 14, 19], exploiting either $f(P^2)$ or $f(P^2, X^2)$, or exp ($\kappa P^2$). With constant term only (i.e. zeroth order in $\mathcal{H}$), relation (2) reduces to the customary HA (1).

An important special case of Eq. (2), namely the algebra based on the relation

$$\left[ X, P \right] = i\hbar (1 + \mu \mathcal{H}) \ , \quad \mu \in \mathbb{R},$$

was explored in [13] with the impact on quantum mechanics at the extreme conditions of high energy physics and quark physics. This line of research was developed in many other papers such as [12, 14–16, 19, 20] and others.

3 A $p, q$-deformation of the Heisenberg algebra

Another way of deforming the HA affects left-hand side of basic relation, yielding the two-parameter or $p, q$-deformation

$$pXP - qPX = i\hbar \ ,$$

introduced and studied in Ref. 12. Note that its special case $p = 1$ was earlier analyzed by Chung and Klimyk in [17].

It is really important that $X$ and $P$ when related with the creation, annihilation, and particle number operators are expressed as [20]

$$X = \frac{1}{\sqrt{2}} \left[ Q^{2N}a^+ + Q^N a^- \right], \quad P = \frac{i}{\sqrt{2}} \left[ Q^N a^+ - Q^{2N} a^- \right], \quad Q \equiv q/p.$$

The inverse relations (to be used below) readily follow, so that

$$a^- = d_{N,Q} (Q^{-N} X + i P), \quad a^+ = d_{N,Q} (X - i Q^{-N} P), \quad d_{N,Q} \equiv \sqrt{2} (1 + Q^{2N})^{-1}.$$
Obviously, the restriction \( Q = 1 \) implies \( d_{\mathcal{N},1} = \frac{1}{\sqrt{2}} \) and brings us back to the usual relations (see e.g. \cite{33}) between \( a^+, a^- \) and \( X, P \).

**Skew-Hermiticity of basic relation (1)**

Here we examine consistency of the basic relation (1) from the viewpoint of conjugation: since r.h.s. of (1) is skew-Hermitian, the same property should be valid for the l.h.s. To this end, consider the cases of real and complex \( p, q \) separately.

**(A) Let \( p, q \in \mathbb{R} \).**

Assume that \( X^\dagger = X \) and \( P^\dagger = P \). Then, skew-hermiticity of the l.h.s. of (1) does hold only if \( p = q \). This case however is not interesting for us as it reduces to the non-deformed one for the operators \( \tilde{\mathcal{X}} \) and \( \tilde{\mathcal{P}} \) such that \( \tilde{\mathcal{X}} = \sqrt{q} X \) and \( \tilde{\mathcal{P}} = \sqrt{q} P \).

Now let \( P^\dagger = P \), but \( X^\dagger = \kappa X \neq X \) with constant \( \kappa \). By demanding skew-hermiticity of the l.h.s. of (1) we deduce:

\[
(kp - q)PX + (p - \kappa q)XP = 0.
\]

For \( XP \neq 0 \), it must be that either (i) \( q = \kappa p \) and \( p = \kappa q \) which implies \( \kappa^2 = 1 \) i.e. \( \kappa = \pm 1 \) (the first option is trivial and the second one is unphysical), or (ii) there should be \( XP = \omega PX \) where \( \omega = \frac{kp - q}{\kappa p - \kappa q} \). Using (1) we infer that \( PX = i\hbar I/(\omega p - q) \) which means \( P \) is proportional to inverse of \( X \), which is also rather exotic.

Same conclusion is drawn if \( X^\dagger = X \) and \( P^\dagger = \kappa P \), or if \( X^\dagger = \kappa X \) and \( P^\dagger = \kappa' P \).

More general case involves \( P^\dagger = P \) and \( X^\dagger = \tilde{\eta}X\tilde{\eta}^{-1} \) (i.e. the operator \( X \) is pseudo-Hermitian), and the case with both \( P^\dagger = \eta'P(\eta')^{-1} \) and \( X^\dagger = \tilde{\eta}X\tilde{\eta}^{-1} \) (the two operators are pseudo-Hermitian). This will be considered below, see Sec. 7.

**(B) Let \( p, q \in \mathbb{C} \).**

With same assumption that \( X^\dagger = X \) and \( P^\dagger = P \), we infer:

\[
(\bar{p} - q)PX + (p - \bar{q})XP = 0.
\]

For \( XP \neq 0 \), we have that either

(i) \( p = \bar{q} = re^{-i\theta} \) and thus \( e^{-i\theta} XP - e^{i\theta} PX = i\hbar \frac{\partial}{\partial \tau} \), or

(ii) \( p \neq \bar{q} \) and then \( PX = \frac{\bar{q} - p}{\bar{p} - q}XP \), or equivalently \([P, X]_\tilde{Q} = 0 \) where \([A, B]_s \equiv AB - sBA \) and \( \tilde{Q} \equiv \frac{\bar{q} - p}{\bar{p} - q} = -\frac{\bar{q} - p}{(p - q)^2} = -e^{2i\arg(p - q)} \).

The found restrictions concern parameters \( p, q \). In general (and more realistic in presence of deformation) case we will deal with pseudo-Hermitian \( X \) and/or \( P \).

## 4 Two-sided (three-parameter) deformed Heisenberg algebra

The two-sided, 3-parameter deformed extension of HA recently introduced in \cite{20} combines different modifications of the HA (1) and reads

\[
qXP - pPX = i\hbar(1 + \mu \mathcal{H}) .
\]
It is linked with a deformed boson algebra such that the relations

\[ H(N)a^- a^+ - G(N)a^+ a^- = 1, \quad a^- a^+ - a^+ a^- = \phi(N + 1) - \phi(N) \tag{8} \]

are valid where \( \phi(N) \) is the structure function of deformation (DSF), see e.g. \[34, 35\]. This SF was derived \[20\] using the functions \( H(n) \) and \( G(n) \) for two important cases:

(i) when \( \mu = 0 \) is set in (7), the proper structure function in (8) is

\[ \phi(N) = \tilde{\Phi}(n) = \frac{2p^{-1}Q^{-n}}{(1 + Q^{2n-2})(1 + Q^{2n})} \left( 1 + \frac{Q^n - Q^{-n+1}}{Q - 1} \right) = \frac{2q^{-n}p^{5n-3}}{(q^{2n-2} + p^{2n-2})(q^{2n} + p^{2n})} \left( 1 + \frac{2n-1}{(qp)^{n-1}} \right) \tag{9} \]

(here \([m]_{q,p} \equiv \frac{q^m - p^m}{q - p} \) denotes the \( q, p \)-number corresponding to a number \( m \)), and

(ii) for \( H(N) = G(N) \) at \( p \neq q \) we obtain (denote \( Q = p/q \)):

\[ \tilde{\Phi}(n) = \frac{4Q^2}{p(1 + Q^2)(1 + Q^3)} = \frac{4}{p(1 + Q)} \left( \frac{1 - Q^{2-2n}}{1 - Q^2} + \sum_{j=1}^{n-1} \frac{1 + Q^5}{Q^2(1 + Q) + Q^{2j}(1 + Q^5)} \right). \]

The structure function \( \phi(N) \) relates \( a^+ a^- \) and \( a^- a^+ \) with \( N \) according to formulas

\[ a^+ a^- = \phi(N), \quad a^- a^+ = \phi(N + 1), \]

and determines the corresponding action formulas for \( a^+, a^- \) in deformed analog of Fock space (see e.g. \[35\]) so that

\[ a^\pm |n\rangle = \sqrt{\phi(n + \frac{1 \pm 1}{2})} |n \pm 1\rangle, \quad |n\rangle = (\phi(n)!)^{-\frac{1}{2}} (a^+)^n |0\rangle, \quad a^- |0\rangle = 0, \]

where \( \phi(n)! = \phi(n) \phi(n-1) \ldots \phi(2) \phi(1) \).

Formula (9) gives the SF of nonstandard two-parameter deformed quantum oscillator. Nonstandard means it is nonsymmetric under \( q \leftrightarrow p \) because of the factor \( q^{-n}p^{5n-3} \) in the numerator. Thus it obviously differs from the well-known \( q, p \)-oscillator \[36\] whose structure function \( \varphi_{q,p}(n) = [n]_{q,p} \) is \( (q \leftrightarrow p) \)-symmetric.

Let us note that formulas (5)-(6) and the conclusions in the preceding Section about skew-Hermiticity extend to the two-sided deformation of HA, see Eq. (7), under the condition that \( \mu \) is real and the Hamiltonian \( \mathcal{H} \) is Hermitian (the Hermiticity of \( \mathcal{H} \) is discussed in Sec. 9 below).

## 5 An \( \eta(N) \)-pseudo-Hermitian conjugation of \( a^+ \) and \( a^- \)

In this Section, two distinct cases will be considered.

**Case A.** Assume, at Hermitian \( N \), the Hermiticity for the momentum operator

\[ P^\dagger = P, \tag{10} \]
and then infer conjugation rules for \( a^\pm \). From Eq. (11), using Eq. (5) we have
\[
P^\dagger = \frac{-i}{\sqrt{2}} \left[ (a^+)\dagger Q^N - (a^-)\dagger Q^{2N} \right] = \frac{i}{\sqrt{2}} \left[ Q^N a^+ - Q^{2N} a^- \right] = \frac{i}{\sqrt{2}} \left[ a^+ Q^{N+1} - a^- Q^{2N-2} \right]
\]
that yields
\[
(a^+)\dagger Q^N = a^- Q^{2N-2}, \quad (a^-)\dagger Q^{2N} = a^+ Q^{N+1}.
\]
From this we infer the rules:
\[
(a^+)\dagger = \eta(N) a^- , \quad (a^-)\dagger = a^+ \eta^{-1}(N) , \quad \eta(N) \equiv Q^{N-1} . \quad (11)
\]
We call this new kind of conjugation \( \eta(N) \)-\textit{pseudo-Hermitian conjugation}: it generalizes (from \( \eta \) to \( \eta = \eta(N) \)) the known \( \eta \)-\textit{pseudo-Hermitian conjugation} [10][12].

Thus, \( a^+ \) and \( a^- \) are mutual \( \eta(N) \)-\textit{pseudo-Hermitian} conjugates of each other.

It is worth noting that instead of (11), we can adopt the equivalent definition of \( \eta(N) \)-\textit{pseudo-Hermitian conjugation}, namely
\[
(a^+)\dagger = a^- \eta(N) , \quad (a^-)\dagger = \eta^{-1}(N)a^+ , \quad \eta(N) \equiv Q^{N-2} . \quad (12)
\]
Obviously, when \( Q \to 1 \) (i.e. at \( p = q \)), the both versions of \( \eta(N) \)-\textit{pseudo-Hermitian} conjugation, (11) and (12), go over into the usual Hermitian (mutual) conjugation of \( a^+ \) and \( a^- \). We stress that this concerns the \( p, q \) differing from unity, and also when each of these is equal to 1.

**Remark 1.** With account of (11), we have usual Hermiticity for the bilinears,
\[
(a^+ a^-)\dagger = (a^-)\dagger (a^+)\dagger = a^+ \eta^{-1}(N)\eta(N)a^- = a^+ a^- , \quad (13)
\]
\[
(a^- a^+)\dagger = (a^+)\dagger (a^-)\dagger = \eta(N)a^- a^+ \eta^{-1}(N) = a^- a^+ , \quad (14)
\]
where at the last step in (14) the permutation rule
\[
F(N)a^\pm = a^\mp F(N \pm 1) , \quad (15)
\]
stemming from the relation \( [N, a^\pm] = \pm a^\pm \) and valid for general function \( F(N) \), has been utilized. The same is true if one takes (12).

**Remark 2.** Instead of (11) (resp. (12)) we of course could take the standard, as for operators, shape of mutual conjugation, i.e. \( (a^+)\dagger = \zeta(N)a^- \zeta^{-1}(N) \) and \( (a^-)\dagger = \zeta(N)a^+ \zeta^{-1}(N) \). However in view of (15), after redefinition \( \zeta(N-1)\zeta^{-1}(N) \to \eta(N) \) (resp. \( \zeta(N)\zeta^{-1}(N+1) \to \eta(N) \)), that would reduce to (11) (resp. (12)).

**Pseudo-Hermiticity of the position operator** \( X \)

Recall that \( \eta(N) \)-\textit{pseudo-Hermitian conjugation} (11) of \( a^+ \) and \( a^- \) has been inferred in view of the requirement (10). Besides, the very dependence of \( \eta \) on the operator \( N \) is dictated by formulas (5). On the other hand, the property (11) causes non-Hermiticity of the operator \( X \). That is, we have to modify conjugation rule for the operator \( X \). So let us find the modified rule of self-conjugation for the position operator in the assumed form \( X^\dagger = \bar{\eta}^{-1}(N)X\bar{\eta}(N) \). Using (11) we have
\[
X^\dagger = \frac{1}{\sqrt{2}} \left( (a^+)\dagger Q^{2N} + (a^-)\dagger Q^N \right) = \frac{Q}{\sqrt{2}} \left( a^+ + Q^N a^- \right),
\]
and it can be easily verified that

\[ X^\dagger = \tilde{\eta}^{-1}(N) X \tilde{\eta}(N) \quad \text{with} \quad \tilde{\eta}(N) = Q^{N^2}. \]  
(16)

The same does follow if we take the rule of \( \eta(N) \)-conjugation in the form (12).

Thus, for the conjugation properties of the momentum and position operators we have usual Hermiticity of \( P \) jointly with \( \tilde{\eta}(N) \)-pseudo-Hermiticity of \( X \), i.e.

\[ P^\dagger = P \quad \text{and} \quad X^\dagger = Q^{-N^2} X Q^{N^2}. \]  
(17)

That is certainly linked with the rule of \( \eta(N) \)-pseudo-Hermitian mutual conjugation for \( a^+ \) and \( a^- \) given by (11) or (12).

Similar analysis can be carried out if one exchanges the roles of \( X \) and \( P \).

Case B. This time let us require that

\[ X^\dagger = X. \]  
(18)

Then we are led to the conjugation rule

\[ (a^+)^\dagger = \tilde{\eta}(N)a^-, \quad (a^-)^\dagger = a^+\tilde{\eta}^{-1}(N), \quad \tilde{\eta}(N) = Q^{-N-2}. \]  
(19)

As a consequence we arrive at \( \tilde{\eta}(N) \)-pseudo-Hermiticity of \( P \), i.e.

\[ P^\dagger = \tilde{\eta}^{-1}(N) P \tilde{\eta}(N) \quad \text{with} \quad \tilde{\eta}(N) = Q^{-N^2}. \]  
(20)

Thus, for the (self)conjugation rules for momentum/position operators in this case we have usual Hermiticity of \( X \) jointly with \( \tilde{\eta}(N) \)-pseudo-Hermiticity of \( P \):

\[ X^\dagger = X \quad \text{and} \quad P^\dagger = Q^{N^2} P Q^{-N^2}, \]  
(21)

the both linked with \( \tilde{\eta}(N) \)-pseudo-Hermitian conjugation of \( a^+ \) and \( a^- \) in (19).

It is interesting to compare the coordinated triple of conjugation rules (11) and (17), with the respective coordinated triple of conjugation rules (19) and (21).

6 The case when \( a^+ \) and \( a^- \) are usual conjugates of each other

Let us require for \( a^+ \) and \( a^- \) the customary conjugation property: \( (a^\pm)^\dagger = a^\mp \). Then it is easy to see that both \( X^\dagger \neq X \) and \( P^\dagger \neq P \). Therefore we consider these operators as \( \eta(N) \)-pseudo-Hermitian ones and impose

\[ X^\dagger = \eta^{-1}_X(N) X \eta_X(N), \quad P^\dagger = \eta^{-1}_P(N) P \eta_P(N). \]  
(22)

To find \( \eta_X(N) \) and \( \eta_P^{-1}(N) \) explicitly, we use the formulas (5) for \( X \) and \( P \). Then, by a simple algebra we deduce the following recurrence relations:

\[ \eta_X(N + 1) = \eta_X(N) Q^{N+2}, \quad \eta_P(N + 1) = \eta_P(N) Q^{-N+1}. \]

Solving them we find respectively

\[ \eta_X(N) = Q^{\frac{1}{2}N(N+3)} \eta_X(0), \quad \eta_P(N) = Q^{\frac{1}{2}N(-N+3)} \eta_P(0). \]

Obviously, the convenient choice is to set \( \eta_X(0) = \eta_P(0) = 1. \)
7 On the $\eta(N)$-pseudohermitian conjugation of $a^{\pm}$, $X$ and $P$

To consider most general situation when the rules of pseudo-Hermitian conjugation concern both the pair $a^+, a^-$ and the operators $X, P$, we impose the relations

\begin{align*}
(a^+)^\dagger &= \eta_a(N) a^- , & (a^-)^\dagger &= a^+ \eta_a^{-1}(N) , \\
X^\dagger &= \eta_X^{-1}(N) X \eta_X(N) , & P^\dagger &= \eta_P^{-1}(N) P \eta_P(N) ,
\end{align*}

where all the three eta’s are different.

We wish to find relations governing the eta’s. For this, we take conjugate $X^\dagger$ of $X$ in (5), then use (23) and compare with $X^\dagger$ in (24). That results in the equations

\begin{align*}
Q^{2N+2} \eta_a(N) &= Q^N \frac{\eta_X(N+1)}{\eta_X(N)} , & \frac{Q^{N-1}}{\eta_a(N-1)} &= Q^{2N} \frac{\eta_X(N-1)}{\eta_X(N)} ,
\end{align*}

or equivalently in the equations

\begin{align*}
\eta_a(N) &= Q^{-N-2} \frac{\eta_X(N+1)}{\eta_X(N)} , & \frac{1}{\eta_a(N-1)} &= Q^{N+1} \frac{\eta_X(N-1)}{\eta_X(N)} .
\end{align*}

The latter two are not independent, being inverse of each other (shift $N \to N+1$).

Likewise, taking conjugate of $P$ in (5), then using (23) and comparing with $P^\dagger$ in (24), we obtain the equations

\begin{align*}
Q^{N+1} \eta_a(N) &= Q^{2N} \frac{\eta_P(N+1)}{\eta_P(N)} , & \frac{Q^{2N-2}}{\eta_a(N-1)} &= Q^N \frac{\eta_P(N-1)}{\eta_P(N)} ,
\end{align*}

or equivalently the equations

\begin{align*}
\eta_a(N) &= Q^{-N-1} \frac{\eta_P(N+1)}{\eta_P(N)} , & \frac{1}{\eta_a(N-1)} &= Q^{-N+2} \frac{\eta_P(N-1)}{\eta_P(N)} .
\end{align*}

Again the latter two are not independent, but inverse of each other.

At last, from (25) and (26) by excluding $\eta_a$ we infer the relation connecting $\eta_X(N)$ with $\eta_P(N)$, namely

\begin{align*}
\frac{\eta_X(N+1)}{\eta_X(N)} &= Q^{2N+1} \eta_P(N+1) \eta_P(N) .
\end{align*}

Thus, for finding $\eta_a(N)$, $\eta_X(N)$ and $\eta_P(N)$ we have three relations: that is Eq.(27) and, say, the first ones in (25), (26), so that any two of the three are independent.

Now let us examine different possible situations.

(i) It follows from (27) that $\eta_X(N) \neq \text{const} \cdot \eta_P(N)$ for any $Q \neq 1$.

(ii) If $\eta_X(N)$ is known (or chosen), then $\eta_a(N)$ follows explicitly, see (25), and for $\eta_P(N)$ we have recursion relation which can be easily solved.

(iii) Likewise, if $\eta_P(N)$ is known (or chosen), then $\eta_a(N)$ follows explicitly, see (26), and for $\eta_X(N)$ we have recursion relation which can be easily solved.
(iv) If \( \eta_a(N) \) is fixed (chosen), then we have two similar, though not identical, recursion relations for \( \eta_X(N) \) and \( \eta_P(N) \) to be solved.

It is worth to consider some particular cases:

(a) Put \( \eta_a(N) = Q^{-N-2} \) in (25). Then \( \eta_X(N) = \text{const} \) and thus \( X \) is Hermitian: \( X^\dagger = X \). For \( \eta_P(N) \), from recurrence relation (26) we then find \( \eta_P(N) = Q^{-N^2} \).

(b) Put \( \eta_a(N) = Q^{N-1} \) in (26). Then \( \eta_P(N) = \text{const} \) and thus \( P \) is Hermitian: \( P^\dagger = P \). For \( \eta_X(N) \), from recurrence relation (25) we then find \( \eta_X(N) = Q^{N^2} \).

(c) Put \( \eta_a(N) = 1 \) that implies \((a^\pm)^\dagger = a^\mp\) (see also Sec. 6). Then from the respective recursion relations we find \( \eta_X(N) = Q^{\frac{1}{2}N(N+3)} \) and \( \eta_P(N) = Q^{\frac{1}{2}N(-N+3)} \).

(d) Let \( \eta_a(N) = \text{const} \neq 1 \), for instance \( \eta_a = Q^\alpha \) with real \( \alpha \). Then for \( a^\pm \) we have standard pseudo-Hermitian conjugation of the shape \((a^\pm)^\dagger = Q^{\pm\alpha}a^\mp\). The remaining \( \eta_X(N) \) and \( \eta_P(N) \) are found from the relevant recurrence relations, and the result is \( \eta_X(N) = Q^{\frac{1}{2}N(N+3\pm 2\alpha)} \) and \( \eta_P(N) = Q^{\frac{1}{2}N(-N+3\pm 2\alpha)} \).

8 Commutation of \( X \) and \( P \) with the particle number operator \( N \)

For what follows we need the relations of permutation of the particle number operator \( N \) with the position or momentum operators,

\[
[N, X] = \frac{1}{\sqrt{2}}(Q^{2N}a^+ - Q^N a^-) = X - 2Q^N a^- ,
\]

\[
[N, P] = \frac{i}{\sqrt{2}}(Q^N a^+ + Q^{2N} a^-) = P + 2iQ^{2N} a^- ,
\]

from which we find

\[
q^{\pm N}[N, X] \equiv i[N, P] = -iP \pm q^{\pm N} X
\]

and, denoting \( q^{\pm N} X \equiv X_{N,a}^{(\pm)} \), infer

\[
[N, X_{N,a}^{(\pm)} \equiv iP] = \pm(X_{N,a}^{(\pm)} \equiv iP) \iff N(X_{N,a}^{(\pm)} \equiv iP) = (X_{N,a}^{(\pm)} \equiv iP)(N \pm 1). \tag{28}
\]

It is also possible to infer an interesting relations (containing \( a^- \) explicitly), e.g.

\[
N X = X(N+1) - \sqrt{2}q^N a^- ,
\]

\[
N^2 X = X(N+1)^2 - 2q^N a^- (2N) ,
\]

\[
N^3 X = X(N+1)^3 - \sqrt{2}q^N a^- (3N^2 + 1) ,
\]

\[
N^4 X = X(N+1)^4 - \sqrt{2}q^N a^- (4N^3 + 4N) ,
\]

and so on. It is easily seen that these particular cases generalize to

\[
N^k X = X(N+1)^k - \sqrt{2}q^N a^- A_k(N)
\]

where \( A_k(N) \) obeys the recurrence formula

\[
A_{k+1}(N) = 2NA_k(N) - (N-1)(N+1)A_{k-1}(N)
\]
solved by
\[ A_k(N) = \sum_{r=0}^{k-1} (N+1)^{k-1-r}(N-1)^r = \frac{(N+1)^k - (N-1)^k}{2}. \] (29)

Equivalently,
\[ N^kX = X(N+1)^k - \sqrt{2}q^N A_k(N+1)a^- \]. (30)

Using the latter, we arrive at the desired relation involving general function \( \mathcal{F}(N) \):
\[ \mathcal{F}(N)X = X\mathcal{F}(N+1) - [\mathcal{F}(N+2) - \mathcal{F}(N)]\hat{a}^- \], \( \hat{a}^- \equiv \frac{1}{\sqrt{2}}Q^N a^- \). (31)

Likewise, for the pair \( N \) and \( P \) we obtain (compare with (30))
\[ N^kP = P(N+1)^k + i\sqrt{2}q^{2N} A_k(N+1)a^- \] (32)
with the same \( A_k(N) \) as in (29) above. Again, from the latter formula we find for general function \( \mathcal{F}(N) \) the relation
\[ \mathcal{F}(N)P = P\mathcal{F}(N+1) + [\mathcal{F}(N+2) - \mathcal{F}(N)]\hat{a}^- \], \( \hat{a}^- \equiv \frac{i}{\sqrt{2}}Q^{2N} a^- \). (33)

Note that for particular \( \mathcal{F}(N) = Q^N \) or \( Q^{-N} \), the above formulas take simpler form:
\[ Q^{\pm N}X = XQ^{\pm(N+1)} \pm (1 - Q^2)Q^{\pm(N+1)-1}a^- \],
\[ Q^{\pm N}P = PQ^{\pm(N+1)} \mp (1 - Q^2)Q^{\pm(N+1)-1}a^- \].

We see that under the replacement \( N \to N \pm 1 \) the entities \( \hat{a}^- \) and \( \hat{a}^- \) in these formulas do not change. Remark also that \( a^-\hat{a}^- = Q\hat{a}^-a^- \), \( a^-\hat{a}^- = Q^2\hat{a}^-a^- \). Using the above results (31) and (33) we deduce the following relation of permutation
\[ \mathcal{F}(N)(X_{N,q}^{(\pm)} \mp iP) = (X_{N,q}^{(\pm)} \mp iP) \mathcal{F}(N \pm 1) \] (34)
for an operator function \( \mathcal{F}(N) \) (possessing expansion into formal series). This is nothing but generalization of Eq. (28).

Remark 3. Let us stress that the obtained relations of commutation between \( X, \)
\( P \) and (a function of) \( N \), see (31), (33) and (34), are of importance just for the chosen (in ref. [20] and herein) line of research based on the link: deformed Heisenberg algebra \( \Leftrightarrow \) deformed oscillator algebra. That will be used in our subsequent work.

9 Hamiltonian in terms of the position and momentum operators

We consider first the particular case \( \mu = 0 \) of the algebra (7). Let us use the Hamiltonian taken in the conventional form [35]
\[ \mathcal{H} = \frac{1}{2}(aa^+ + a^+a) = \frac{1}{2} \left( \Phi(N + 1) + \Phi(N) \right) \] (35)
which yields the energy spectrum $E(n) = \frac{1}{2}(\hat{\Phi}(n+1) + \hat{\Phi}(n))$ in the Fock-like basis. With account of eq. (35) and recalling that $d_N = d_{N,Q} \equiv \sqrt{2}(1 + Q^{2N})^{-1}$, we find the Hamiltonian in terms of the position and momentum operators, namely

$$
\mathcal{H} = \frac{1}{2}d_N Q^{-N} \{ (d_{N+1} + Qd_{N-1})(X^2 + Q^{-1}P^2) +
+ i (Q^N d_{N+1} - Q^{1-N} d_{N-1})PX + i(Q^N d_{N-1} - Q^{1-N} d_{N+1})XP \},
$$

which is somewhat reminiscent of the Swanson model [37].

With the use of (35) this Hamiltonian takes the form

$$
\mathcal{H} = \frac{1}{2}d_N Q^{-N} \{ (d_{N+1} + Qd_{N-1})[X^2 + Q^{-1}P^2 + iQ^{-1}(Q^N - Q^{-N})XP +
+ (1/q)(Q^N d_{N+1} - Q^{1-N} d_{N-1})] \}
$$

(37)

with $PX$ term now absent. Note that at $p \rightarrow 1$ the results obtained here for the $p,q$-deformed HA reduce to those of the one-parameter case (since (35) reduces to $\phi$ comes to the structure function $X$ and $P$ to the usual harmonic oscillator, whose spacing in the (linear) energy spectrum gets $\frac{1}{q}$-scaled. Concerning $X$, $P$ vs. $a^+$, $a^-$ see the phrase just after Eq. (35).

**Hermiticity of the Hamiltonian**

We take again the Hamiltonian in the form $\mathcal{H} = \frac{1}{2}(a^-a^+ + a^+a^-)$, see (35). Recall that the creation and annihilation operators are in general not Hermitian conjugates of each other but instead satisfy the rules of generalized $\eta(N)$-pseudo-Hermitian conjugation, see eq. (11) or eq. (23). However, in view of (13) and (14) this form of Hamiltonian guarantees that it is Hermitian. The same is true for (36) and (37) as these are related with (35) through identical transformation.

The Hamiltonian $\mathcal{H}$, with account of the equality

$$
\frac{p}{2}Q^{2N+1}(1 + Q^{2N+2})a^-a^+ - \frac{p}{2}Q^2N(1 + Q^{2N-2})a^+a^- = 1,
$$

see (14) and our earlier paper [20], can be presented as

$$
\mathcal{H} = \frac{1}{p} \frac{Q^{-2N-1}}{1 + Q^{2N+2}} + \frac{1}{2} \left( 1 + Q^{-1} \frac{1 + Q^{2N-2}}{1 + Q^{2N+2}} \right) a^+a^-.
$$

(38)

This is still Hermitian, in view of Hermiticity of (an operator function of) $N$ and the property (13) of $a^+a^-$. At $p = q$, we have $a^-a^+ - a^+a^- = q^{-1}$ and $\mathcal{H} = \frac{1}{2q} + a^+a^-$. When $q = 1$, the usual harmonic oscillator with $\mathcal{H} = \mathcal{H}_0 = \frac{1}{2} + a^+a^-$ is recovered.

**Remark 4.** The versions of Hamiltonian $\mathcal{H}$ in (36), (37) and (38) are equivalent to the initial one (35) and thus also Hermitian. On the other hand, the form of Hamiltonian $H = \frac{1}{2}(X^2 + P^2)$ (the standard one for harmonic oscillator) is not plausible, being neither Hermitian nor pseudo-Hermitian in the deformed case, i.e. when $Q \neq 1$. We can however suggest natural and simple modification of $H$ given in terms of $\eta_X$-pseudo-Hermitian operator $X$ and $\eta_P$-pseudo-Hermitian operator $P$:

$$
\hat{\mathcal{H}} = \frac{1}{2} \left( (\eta_X)^{-\frac{1}{2}} X^2 (\eta_X)^{\frac{1}{2}} + (\eta_P)^{-\frac{1}{2}} P^2 (\eta_P)^{\frac{1}{2}} \right).
$$

(39)
With the particular $\eta_X$ and $\eta_P$ linked with $\eta_a = 1$, see case (c) at the end of Sec. 7, we have

$$\tilde{\mathcal{H}} = \frac{1}{2}\left(Q^{-\frac{1}{4}N(N+3)}X^2 Q^{\frac{1}{4}N(N+3)} + Q^{\frac{1}{4}N(N-3)}P^2 Q^{-\frac{1}{4}N(N-3)}\right). \tag{40}$$

One can easily check Hermiticity of (39) and (40). Note that if $Q \to 1$, then $\tilde{\mathcal{H}} \to \mathcal{H} = \frac{1}{2}(X^2 + P^2)$.

**Remark 5.** Returning to the skew-hermiticity of Eq. (41) especially its l.h.s., as discussed in the last part of Sec. 3, we may state the following: since the Hamiltonian is Hermitian, and $\mu \in \mathbb{R}$, all the conclusions made at the end of Sec. 3 extend completely to the (skew-Hermiticity of) *three-parameter deformation* of the Heisenberg algebra, with its $p, q, \mu$-deformed basic relation Eq. (7).

**Discussion**

In this paper, for the 2- and 3-parameter extensions [20] of the Heisenberg algebra, assuming either usual or generalized (with $\eta_a(N)$ involved) conjugation properties of $a^-$ and $a^+$, we studied the special non-Hermiticity of $X, P$, realized explicitly as $\eta_X(N)$-pseudo-Hermiticity of $X$ and/or $\eta_P(N)$-pseudo-Hermiticity of $P$. In other words, our main results concern precise and fully-coordinated conjugation properties of the four involved operators, with crucial $N$-dependence of the eta-functions $\eta_a$ and $\eta_X, \eta_P$. Such dependence is rooted in the link: deformed Heisenberg algebra $\leftrightarrow$ deformed oscillator algebra, established in Ref. [20] (just this link produces formula (31) that was essentially used in our analysis).

It is worth noting that all the metric operators $\eta_a(N), \eta_X(N)$ and $\eta_P(N)$ are Hermitian as respective functions of the (Hermitian) particle number operator. We also stress that the Hamiltonian $\mathcal{H}$ in our treatment is Hermitian, as it is formed from the bilinears $a^+a^-$ and $a^-a^+$. Note that these bilinears are Hermitian although individual $a^+$ and $a^-$ may be not (mutual) Hermitian conjugates, but the $\eta_a(N)$-pseudo-Hermitian conjugates of each other. Also, we have proposed the new Hamiltonian $\tilde{\mathcal{H}}$, see Eqs. (39)-(40), which is yet another *Hermitian deformation* of well-known Hamiltonian $H = \frac{1}{2}(X^2 + P^2)$ of harmonic oscillator.

In a forthcoming work we intend to explore the spectra (eigenvalues, eigenfunctions) of the position and momentum operators along with the Hamiltonian (40), within the coordinate realization. The energy spectrum of the Hamiltonian $H = \frac{1}{2}\{a^+, a^-\}$ is known: in the deformed Fock-like basis it is given explicitly (see Eq. (35)) through respective structure function such as $\tilde{\Phi}(n)$ in Sec. 4, and thus real.

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