Computing Least and Greatest Fixed Points in Absorptive Semirings

Matthias Naaf

November 5, RAMiCS 2021
Q: Minimal cost of an infinite path?

\[ X_a = 1 + X_b \]
\[ X_b = \min(1 + X_a, 20 + X_c) \]
\[ X_c = 0 + X_c \]

\[ X_a = \infty \]
\[ X_b = 20 \]
\[ X_c = 0 \]
Q: Minimal cost of an infinite path?

$$X_a = 1 + X_a$$
$$X_b = \min(1 + X_a, 20 + X_c)$$
$$X_c = 0 + X_c$$

polynomial equation system

$$= (\mathbb{R}_{\geq 0}^\infty, \min, +, \infty, 0)$$

$$X_a = \infty$$
$$X_b = 20$$
$$X_c = 0$$
greatest sol.
Motivation: Semiring Provenance for Logics

Semiring Provenance

▶ Unify provenance analyses for databases
▶ Generalize to logics: Semiring semantics for FO, LFP, ... 

Semiring Semantics

▶ Idea: Replace Boolean model by semiring annotation:

\[
G \models E_{aa} \land E_{ab} \quad \leadsto \quad \pi \left[ E_{aa} \land E_{ab} \right] = 1 + 20
\]
Motivation: Semiring Provenance for Logics

Fixed-Point Logic

- $\varphi(v) = \lfloor \text{gfp } R \; x. \; (\exists y \; Exy \land Ry) \rfloor(v)$
  - minimal cost of an infinite path from $v$ (in $\mathcal{F}$)

- $\varphi_{\text{win}}(v)$: winning region in Büchi games
  - modify the game so that Player 0 wins (polynomial semiring)

How to evaluate LFP-formulae?

- least/greatest solutions of PES (in absorptive semirings)

\[
\pi[\varphi_{\text{win}}(v)] = a + \overline{c}
\]
Motivation: Semiring Provenance for Logics

Fixed-Point Logic

- $\varphi(v) = \left[ \text{gfp } R \ x. \ (\exists y \ E x y \land R y) \right](v)$
  - minimal cost of an infinite path from $v$ (in)

- $\varphi_{\text{win}}(v)$: winning region in Büchi games
  - modify the game so that Player 0 wins (polynomial semiring)

![Diagram]

- $\pi(\varphi_{\text{win}}(v)) = a + \overline{c}$

How to evaluate LFP-formulae?

- least/greatest solutions of PES (in absorptive semirings)
Fixed-Point Iteration?

\[ F: \begin{pmatrix} X_a \\ X_b \\ X_c \end{pmatrix} \mapsto \begin{pmatrix} 1 + X_a \\ \min(1 + X_a, 20 + X_c) \\ 0 + X_c \end{pmatrix} \]

\[
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \mapsto \cdots \mapsto \begin{pmatrix} 20 \\ 0 \end{pmatrix}
\]

M. Naaf (RWTH Aachen)

Computing Least and Greatest Fixed Points in Absorptive Semirings
Fixed-Point Iteration?

\[
F: \begin{pmatrix} X_a \\ X_b \\ X_c \end{pmatrix} \mapsto \begin{pmatrix} 1 + X_a \\ \min(1 + X_a, 20 + X_c) \\ 0 + X_c \end{pmatrix}
\]

\[
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \mapsto \cdots \mapsto \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 21 \\ 20 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 22 \\ 20 \\ 0 \end{pmatrix} \mapsto \cdots \mapsto \begin{pmatrix} \infty \\ 20 \\ 0 \end{pmatrix}
\]
Faster Computation

Main Result

Let \((K, +, \cdot, 0, 1)\) be an absorptive, fully-continuous semiring. Given a PES with \(n\) variables over \(K\), we can compute:

\[
\begin{align*}
\text{lfp}(F) &= F^n(0). \\
\text{gfp}(F) &= F^n \left( (F^n(1))^\infty \right).
\end{align*}
\]

We only need a polynomial number of semiring operations.
Chapter I

Absorptive Semirings
Semirings with Orders

Commutative Semiring

\((K, +, \cdot, 0, 1)\) such that \((K, +, 0)\) and \((K, \cdot, 1)\) are commutative monoids, \(\cdot\) distributes over \(+\), \(0 \neq 1\) and \(0 \cdot a = 0\).

A semiring is naturally ordered if

\[ a \leq b \iff \exists c. a + c = b \]

defines a partial order.

Examples: Boolean semiring, \(\mathbb{R}_{\geq 0}\), \(\mathbb{N}[X]\)
Absorptive Semirings

Absorption

A semiring is absorptive if \( a + ab = a \) for all \( a, b \).

Some facts

- Absorptive semirings are idempotent and naturally ordered
- Equivalent definitions:

\[
 a + ab = a \quad \iff \quad \top = 1 \quad \iff \quad ab \leq a
\]
Absorptive Semirings

Absorption

A semiring is absorptive if $a + ab = a$ for all $a, b$.

Some facts

▶ Absorptive semirings are idempotent and naturally ordered

▶ Equivalent definitions:

\[ a + ab = a \iff \top = 1 \iff ab \leq a \]

Remember: Absorption = decreasing multiplication
An absorptive semiring is $K$ is **fully continuous** if $\leq$ is a complete lattice satisfying the continuity property:

$$\bigsqcup (a \circ C) = a \circ \bigsqcup C \quad \text{and} \quad \bigcap (a \circ C) = a \circ \bigcap C$$

for all non-empty chains $C \subseteq K$ and all $a \in K$, $\circ \in \{+, \cdot\}$.

For $a \in K$ we define $a^\infty := \bigcap_{n \in \mathbb{N}} a^n$. 
Absorptive Semirings with Fixed Points

Examples

- **Boolean semiring** \((\{0, 1\}, \lor, \land, 0, 1)\)

- **Łukasiewicz semiring** \(([0, 1], \max, \star, 0, 1)\)
  with \(a \star b = \max(0, a + b - 1)\)

- **Any distributive lattice or min-max semiring**

\[
a^\infty = a
\]

\[
a^\infty = \begin{cases} 0, & a = 0 \\ \infty, & \text{else} \end{cases}
\]

\[
a^\infty = \begin{cases} 1, & a = 1 \\ 0, & \text{else} \end{cases}
\]

Problem: \(\mathbb{N}\) and \(\mathbb{N}[X]\) not absorptive!
Modify $\mathbb{N}[X]$ by

- dropping coefficients,

$$2x^2y + xy^2 + 5x^2 + 3z^{10}$$
Absorptive Polynomials

Modify \( \mathbb{N}[X] \) by

- dropping coefficients,
- absorption among monomials (by comparing exponents),

\[ 2x^2y + xy^2 + 5x^2 + 3z^{10} \]
Absorptive Polynomials

Modify $\mathbb{N}[X]$ by

- dropping coefficients,
- absorption among monomials (by comparing exponents),
- allowing $\infty$ as exponent.

\[
2x^2y + xy^2 + 5x^2 + 3z^\infty
\]

Absorptive polynomials $S^\infty[X]$ are

- always finite (Dickson’s lemma),
- the most general absorptive, fully-continuous semiring.
Chapter II

Proof Sketch
Proof Overview: Least Solution

Main Result

Let \((K, +, \cdot, 0, 1)\) be an absorptive, fully-continuous semiring. Given a PES with \(n\) variables over \(K\), we can compute:

- \(\text{lfp}(F) = F^n(0)\).
- \(\text{gfp}(F) = F^n((F^n(1))^\infty)\).

Remark: lfp follows from [Esparza, Kiefer, Luttenberger, ICALP’08]
Proof Overview: Least Solution

**Main Result**

Let \((K, +, \cdot, 0, 1)\) be an absorptive, fully-continuous semiring. Given a PES with \(n\) variables over \(K\), we can compute:

- \(\text{lfp}(F) = F^n(0)\).
- \(\text{gfp}(F) = F^n\left((F^n(1))^{\infty}\right)\).

Remark: \(\text{lfp}\) follows from [Esparza, Kiefer, Luttenberger, ICALP’08]

Newton’s method for \(\text{lfp}(F)\) converges in \(n\) steps in idempotent semirings

Newton’s method = fixed-point iteration
Proof Overview: Greatest Solution

Main Result

Let \((K, +, \cdot, 0, 1)\) be an absorptive, fully-continuous semiring. Given a PES with \(n\) variables over \(K\), we can compute:

- \(\text{lfp}(F) = F^n(0)\).
- \(\text{gfp}(F) = F^n\left((F^n(1))^\infty\right)\).

Proof:

1. Express \(\text{gfp}(F)\) using derivation trees
2. Apply absorption to derivation trees
Derivation Trees

\[ X = aXY + b \]
\[ Y = cZ^2 \]
\[ Z = dZ + e \]

Yield: \( b \)

Yield: \( abce^2 \)

Yield: \( a \cdot b \cdot c \cdot d^\infty \)
Derivation Trees

\[ X = aXY + b \]
\[ Y = cZ^2 \]
\[ Z = dZ + e \]

\[
\begin{align*}
X_a & \rightarrow X_b \quad \text{yield: } b \\
& \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow
Y_c & \rightarrow Z_e \\
& \quad \quad \quad \downarrow \\
Z_e & \quad \quad \quad \Downarrow
\end{align*}
\]

\[ X_a \rightarrow X_b \quad \text{yield: } abce^2 \]

\[
\begin{align*}
\text{lfp} &= \sum \{(\text{yield})(\text{finite}) \}
\text{gfp} &= \sum \{(\text{yield})(\text{finite, infinite}) \}
\end{align*}
\]

Yield: \(a \cdot b \cdot c \cdot d^\infty\)
Observation: Prefixes of \( \downarrow \) correspond to iteration steps.

\[
\sum \{ \text{yield}(\cdot) \mid \text{finite/infinite} \} = \mathbf{F}(1)
\]
**Observation:** Prefixes of $\mathcal{T}$ correspond to iteration steps.

\[ F(1) + F(1) = F^2(1) \]
**Observation:** Prefixes of \( \uparrow \) correspond to iteration steps.

\[
F(1) + F^2(1) + F^3(1) + \cdots
\]
Observation: Prefixes of \( \text{★} \) correspond to iteration steps.

\[
\prod_{n \in \mathbb{N}} \sum \left\{ \text{yield(★)} \mid \text{finite/infinite} \right\} = \text{gfp}(F)
\]
Main Result

Let \((K, +, \cdot, 0, 1)\) be an absorptive, fully-continuous semiring. Given a PES with \(n\) variables over \(K\), we can compute:

- \(\text{lfp}(F) = F^n(0)\).
- \(\text{gfp}(F) = F^n \left( (F^n(1))^{\infty} \right)\).

Proof:

1. Express \(\text{gfp}(F)\) using derivation trees
2. Apply absorption to derivation trees
Absorption on Derivation Trees

If each coefficient occurs more often in $\bullet$ than in $\circ$, then $\text{yield}(\bullet)$ is absorbed by $\text{yield}(\circ)$.
Absorption on Derivation Trees

If each coefficient occurs more often in \( \mathbb{E} \) than in \( \mathbb{F} \), then \( \text{yield}(\mathbb{E}) \) is absorbed by \( \text{yield}(\mathbb{F}) \).

complicated tree \( \mathbb{E} \) \quad \leq \quad \text{ultimately periodic} \quad \leq \quad \text{nice tree} \( \mathbb{F} \)
If each coefficient occurs more often in $\bullet$ than in $\bigcirc$, then yield($\bullet$) is absorbed by yield($\bigcirc$).
Absorption on Derivation Trees

If each coefficient occurs more often in \( \bullet \) than in \( \mathcal{F} \), then \( \text{yield}(\bullet) \) is absorbed by \( \text{yield}(\mathcal{F}) \).

complicated tree \( \bullet \)

ultimately periodic

nice tree \( \mathcal{F} \)
Computing Nice Trees

\[ \text{gfp}(F') = \sum \left\{ \text{yield}(\text{nice tree}) \mid \text{nice tree} \right\} = \ldots \]
Computing Nice Trees

\[ \text{gfp}(F') = \sum \{ \text{yield}(\bullet) \mid \text{nice } \bullet \} = \ldots \]
Computing Nice Trees

\[ \text{gfp}(F') = \sum \left\{ \text{yield}(\text{nice tree}) \mid \text{nice tree} \right\} = \ldots \]
Computing Nice Trees

$$\text{gfp}(F') = \sum \left\{ \text{yield}(\text{nice tree}) \mid \text{nice tree} \right\} = F^n(F^n(1)^\infty)$$
Result

- Greatest solutions of PES in absorptive, fully-continuous semirings ...
- ... are computable in a polynomial number of semiring operations

Alternative: Symbolic approach for $S_\infty [X]$

- Solve first equation for $X$, substitute and solve recursively

Future Work: Compute nested fixed points in absorptive semirings