ČECH COHOMOLOGY AND THE CAPITULATION KERNEL

CRIŞTIAN D. GONZÁLEZ-AVILÉS

Abstract. We discuss the capitulation kernel associated to a degree \( n \) covering using Čech cohomology and the Kummer sequence. The main result is a five-term exact sequence that relates the capitulation kernel to the Čech cohomology of the \( n \)-th roots of unity and a certain subquotient of the group of units modulo \( n \)-th powers.

1. Introduction

Let \( F \) be a global field, let \( \Sigma \) be a nonempty finite set of primes of \( F \) containing the archimedean primes and let \( \mathcal{O}_{F, \Sigma} \) and \( C_{F, \Sigma} \) denote, respectively, the ring of \( \Sigma \)-integers and the \( \Sigma \)-ideal class group of \( F \). Now let \( K/F \) be a finite Galois extension with Galois group \( \Delta \) of order \( n \geq 2 \) and let \( \mathcal{O}_{K, \Sigma} \) and \( C_{K, \Sigma} \) denote, respectively, the ring of \( \Sigma_K \)-integers and the \( \Sigma_K \)-ideal class group of \( K \), where \( \Sigma_K \) denotes the set of primes of \( K \) lying over the primes in \( \Sigma \). The extension of ideals from \( F \) to \( K \) defines a map
\[
j_{K/F, \Sigma} : C_{F, \Sigma} \to C_{K, \Sigma}
\]
which is often called the capitulation map, after Arnold Scholz. An ideal class in \( C_{F, \Sigma} \) is said to capitulate in \( K \) if it lies in \( \text{Ker} j_{K/F, \Sigma} \), which is often called the capitulation kernel associated to the pair \(( K/F, \Sigma) \). It was shown in [5, Theorem 2.3] that \( \text{Ker} j_{K/F, \Sigma} \) is naturally isomorphic to the kernel of the canonical localization map in \( \Delta \)-cohomology
\[
H^1(\Delta, \mathcal{O}_{K, \Sigma}^*) \to \prod_{v \notin \Sigma} H^1(\Delta_{w_v}, \mathcal{O}_{w_v}^*),
\]
where, for each prime \( v \notin \Sigma \), \( w_v \) denotes a fixed prime of \( K \) lying over \( v \) and \( \Delta_{w_v} \) is the corresponding decomposition subgroup of \( \Delta \). Unfortunately,
the $n$-torsion abelian group $H^1(\Delta, \mathcal{O}_{K, \Sigma}^*)$ (and therefore also $\ker j_{K/F, \Sigma}$) is difficult to compute and can vary widely. In this paper we obtain additional information on $\ker j_{K/F, \Sigma}$ that should be helpful in determining its structure, at least in some particular cases of interest. To explain our results, let $S = \text{Spec} \mathcal{O}_{F, \Sigma}, S' = \text{Spec} \mathcal{O}_{K, \Sigma}$ and note that, via the well-known isomorphisms $C_{F, \Sigma} \simeq \text{Pic } S \simeq H^1(S_{et}, \mathbb{G}_m)$ (and similarly for $S'$), the map $j_{K/F, \Sigma}$ can be identified with the canonical restriction map in étale cohomology $j^{(1)}: H^1(S_{et}, \mathbb{G}_m) \to H^1(S'_{et}, \mathbb{G}_m)$. Now, it is well-known [13, III, proof of Proposition 4.6, p. 123] that $\ker j^{(1)}$ is canonically isomorphic to the first Čech cohomology group $\check{H}^1(S'/S, \mathbb{G}_m)$ associated to the fppf covering $S'/S$. Thus there exists a canonical isomorphism of finite $n$-torsion abelian groups $\ker j_{K/F, \Sigma} \simeq \check{H}^1(S'/S, \mathbb{G}_m)$. This observation motivated the central idea of this paper, which is that the tools afforded by the Čech cohomology theory and the cohomology sequences arising from the Kummer sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$ can be combined to yield new information on $\ker j_{K/F, \Sigma}$.

Set

(1.2) \[ \Psi(K/F, \Sigma) = \{ u \in \mathcal{O}_{K, \Sigma}^*: u \otimes u^{-1} \in ((\mathcal{O}_{K, \Sigma} \otimes \mathcal{O}_{F, \Sigma})^*)^n \}. \]

Then the following holds.

**Theorem 1.1.** (=Proposition 4.1) There exists a canonical exact sequence of finite $n$-torsion abelian groups

\[
0 \to \mathcal{O}_{F, \Sigma}^*/(\mathcal{O}_{F, \Sigma}^*)^n \to \check{H}^1(\mathcal{O}_{K, \Sigma}/\mathcal{O}_{F, \Sigma}, \mu_n) \to \ker j_{K/F, \Sigma} \to \Psi(K/F, \Sigma)/\mathcal{O}_{F, \Sigma}^*(\mathcal{O}_{K, \Sigma}^*)^n \to \check{H}^2(\mathcal{O}_{K, \Sigma}/\mathcal{O}_{F, \Sigma}, \mu_n),
\]

where $\Psi(K/F, \Sigma)$ is the subgroup of $\mathcal{O}_{K, \Sigma}^*$ given by (1.2).

The following corollary of the proposition is part of Corollary 4.2.

**Corollary 1.2.** Assume that $\Sigma$ contains all primes of $F$ that ramify in $K$. Assume, furthermore, that $n$ is odd and $\mu_n(K) = \{1\}$. Then there exists a canonical isomorphism of finite $n$-torsion abelian groups

\[
\ker j_{K/F, \Sigma} \sim \Psi(K/F, \Sigma)/\mathcal{O}_{F, \Sigma}^*(\mathcal{O}_{K, \Sigma}^*)^n.
\]

In particular, at most $[\mathcal{O}_{K, \Sigma}^*: \mathcal{O}_{F, \Sigma}^*(\mathcal{O}_{K, \Sigma}^*)^n] \Sigma$-ideal classes of $F$ capitulate in $K$. 

The preceding discussion, which is a summary of the contents of the last section of the paper (section 4), is a particular case of the more general developments of section 3. Our main theorem, namely Theorem 3.4, applies to admissible pairs \((G, f)\) (see Definition 3.1), of which \((\mathbb{G}_m, S'/S)\) is an example. See Examples 3.2 for additional examples of admissible pairs. Section 2 consists of various preliminaries.

2. Preliminaries

2.1. Generalities. The category of abelian groups will be denoted by \(\textbf{Ab}\). If \(n \geq 1\) is an integer and \(A\) is an object of an abelian category \(\mathcal{A}\), \(A_n\) (respectively, \(A/n\)) will denote the kernel (respectively, cokernel) of the multiplication by \(n\) morphism on \(A\). If \(\psi: A \to B\) is a morphism in \(\mathcal{A}\), \(\psi_n: A_n \to B_n\) and \(\psi/n: A/n \to B/n\) will denote the morphisms in \(\mathcal{A}\) induced by \(\psi\). Note that

\[
\text{(2.1)} \quad \text{Ker}(\psi_n) = (\text{Ker} \psi)_n.
\]

However, \(\text{Ker}(\psi/n) \neq (\text{Ker} \psi)/n\) (in general).

All schemes appearing in this paper are tacitly assumed to be non-empty.

If \(S\) is a scheme and \(\tau\) denotes either the étale (ét) or fppf (fl) topology on \(S\), \(S_\tau\) will denote the small \(\tau\) site over \(S\). Thus \(S_\tau\) is the category of \(S\)-schemes that are étale (respectively, flat) and locally of finite presentation over \(S\) equipped with the étale (respectively, fppf) topology. If \(f: S' \to S\) is a faithfully flat morphism locally of finite presentation, then \(f\) is an fppf covering of \(S\). We will write \(S_\tau\) for the (abelian) category of sheaves of abelian groups on \(S_\tau\). If \(G\) is a commutative \(S\)-group scheme, then the presheaf represented by \(G\) is an object of \(S_\tau\). In particular, if \(f: S' \to S\) is as above, the induced map \(G(f): G(S) \to G(S')\) is an injection that will be regarded as an inclusion. If \(n \geq 1\), the object \(G_n\) of \(S_\tau\) is represented by the \(S\)-group scheme \(G \times_{n, G, \varepsilon} S\), where \(n_G: G \to G\) is the \(n\)-th power morphism on \(G\) and \(\varepsilon: S \to G\) is the unit section of \(G\).

Let \(\mathcal{F}_S: S_\tau^\sim \to \textbf{Ab}\) be the \(S\)-section functor on \(S_\tau^\sim\). For every object \(\mathcal{F}\) of \(S_\tau^\sim\) and every integer \(r \geq 0\), let \(H^r(S_\tau, \mathcal{F}) = R^r\mathcal{F}_S(\mathcal{F})\) be the corresponding \(r\)-th \(\tau\) cohomology group of \(\mathcal{F}\). If \(G\) is a commutative \(S\)-group scheme (regarded as an object of \(S_\tau^\sim\)) and \(S' \to S\) is a morphism of schemes, we will write \(H^r(S'_\tau, G)\) for \(H^r(S'_\tau, G_{S'})\), where \(G_{S'} = G \times_S S'\). The abelian group \(H^1(S_\tau, G)\) will be identified with the group of isomorphism classes of right (sheaf) \(G\)-torsors over \(S\) relative to the \(\tau\) topology on \(S\). If \(G = \mathbb{G}_m, S\) (respectively, \(G = \mu_n, S\) for
some integer \( n \geq 2 \), the groups \( H^r(S_\tau, G) \) will be denoted by \( H^r(S_\tau, \mathbb{G}_m) \) (respectively, \( H^r(S_\tau, \mu_n) \)). If \( G \) is smooth over \( S \), the groups \( H^r(S_{\text{fl}}, G) \) and \( H^r(S_{\text{ét}}, G) \) will be identified via [9, Theorem 11.7(1), p. 180]. If \( f : S' \to S \) is a morphism of schemes, the \( r \)-th restriction morphism associated to \((G, f)\) is the morphism of abelian groups

\[
(2.2) \quad j^{(r)} = R^r f_* : H^r(S_\tau, G) \to H^r(S'_\tau, G).
\]

When reference to \( G \) and \( \tau \) (respectively, \( G, S'/S \) and \( \tau \)) is necessary, \( j^{(r)} \) will be denoted by \( j^{(r)}_{G, \tau} \) (respectively, \( j^{(r)}_{G, S'/S, \tau} \)). If \( G \) is smooth over \( S \), the maps \( j^{(r)}_{G, \text{fl}} \) and \( j^{(r)}_{G, \text{ét}} \) will be identified and denoted by \( j^{(r)}_G \), i.e.,

\[
(2.3) \quad j^{(r)}_G = j^{(r)}_{G, \text{fl}} = j^{(r)}_{G, \text{ét}} \quad \text{ (if } G \text{ is smooth over } S). \]

Note that \( j^{(0)} \) is the inclusion \( G(f) : G(S) \to G(S') \). The map

\[
(2.4) \quad j^{(1)} : H^1(S_\tau, G) \to H^1(S'_\tau, G)
\]

sends the class in \( H^1(S_\tau, G) \) of a right (sheaf) \( G \)-torsor \( X \) over \( S \) (relative to the \( \tau \) topology on \( S \)) to the class in \( H^1(S'_\tau, G) \) of the right (sheaf) \( G_{S'} \)-torsor \( X_{S'} \) over \( S' \). A class \([X] \in H^1(S_\tau, G)\) lies in \( \text{Ker} j^{(1)} \) if, and only if, \( X(S') \neq \emptyset \). See [13, III, comments after Proposition 4.1, p. 120]. After the classical case \( G = \mathbb{G}_m, \mathcal{O} \), where \( \mathcal{O} \) is the ring of integers of a number field, we call \( \text{Ker} j^{(1)}_G \) the capitulation kernel associated to the pair \((G, f)\).

For every scheme \( X \), the group of global units on \( X \) is the abelian group

\[
(2.5) \quad U(X) = \Gamma(X, \mathcal{O}_X)^* = \mathbb{G}_m(X).
\]

We will identify \( \text{Pic} X \) and \( H^1(X_{\text{ét}}, \mathbb{G}_m) \) via [13, Theorem 4.9, p. 124]. A morphism of schemes \( f : S' \to S \) induces a morphism of abelian groups

\[
(2.6) \quad \text{Pic } f : \text{Pic } S \to \text{Pic } S', [E] \mapsto [E \times_S S'],
\]

where \([E]\) denotes the isomorphism class of the right (sheaf) \( \mathbb{G}_m,S \)-torsor \( E \) and \([E \times_S S']\) denotes the isomorphism class of the right (sheaf) \( \mathbb{G}_m,S' \)-torsor \( E \times_S S' \).
2.2. Čech cohomology. If $n \geq 1$ is an integer and $X$ is an $S$-scheme, $X^n$ will denote the $S$-scheme defined recursively by $X^1 = X$ and $X^n = X \times_S X^{n-1}$, where $n \geq 2$. By [12, §10.2, p. 97], $\{X^n\}_{n \geq 1}$ is a simplicial $S$-scheme with face maps

$\partial^i_n : X^{n+1} \to X^n, (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{n+1}),$

where $1 \leq i \leq n + 1$, i.e., $\partial^i_n$ is the canonical projection that omits the $i$-th factor. When $n = 1$, we will use the following (standard) alternative notation for the maps (2.7):

$\partial^1_1 = p_2 \quad \text{and} \quad \partial^2_1 = p_1,$

where $p_r : X^2 \to X$ ($r = 1, 2$) is the projection onto the $r$-th factor.

Now assume that $S$ is locally noetherian, let $\mathcal{F}$ be an abelian presheaf on $S_{fl}$ and let $X \to S$ be a finite and faithfully flat morphism of schemes. Then we may consider the Čech cohomology groups of $\mathcal{F}$ relative to the fppf covering $X \to S$, i.e., the cohomology groups $\check{H}^i(X/S, \mathcal{F})$ of the complex of abelian groups $\{\mathcal{F}(X^n), \partial^n\}_{n \geq 1}$, where

$$\partial^n = \sum_{j=1}^{n+1} (-1)^{j+1} \mathcal{F}(\partial^j_n) : \mathcal{F}(X^n) \to \mathcal{F}(X^{n+1}).$$

If $\mathcal{F} = \mathbb{G}_{m,S}$ (respectively, $\mathcal{F} = \mu_{n,S}$), the groups $\check{H}^i(X/S, \mathcal{F})$ will be denoted by $\check{H}^i(X/S, \mathbb{G}_m)$ (respectively, $\check{H}^i(X/S, \mu_n)$).

In terms of the standard notation (2.3),

$$\check{H}^0(X/S, \mathcal{F}) = \text{Ker}[\mathcal{F}(p_2) - \mathcal{F}(p_1) : \mathcal{F}(X) \to \mathcal{F}(X^2)].$$

Note that, since $f \circ p_2 = f \circ p_1$, the map $\mathcal{F}(f) : \mathcal{F}(S) \to \mathcal{F}(X)$ factors through $\check{H}^0(X/S, \mathcal{F})$, i.e.,

$$\mathcal{F}(f) : \mathcal{F}(S) \xrightarrow{f^\natural} \check{H}^0(X/S, \mathcal{F}) \hookrightarrow \mathcal{F}(X),$$

where the second map is the inclusion.

The category of abelian presheaves on $S_{fl}$ is abelian. Thus, given any integer $n \geq 1$, we may consider the presheaves $\mathcal{F}_n$ and $\mathcal{F}/n$ with values in the category of $n$-torsion abelian groups. Explicitly, $\mathcal{F}_n$ and $\mathcal{F}/n$ are defined as follows. If
$T \to S$ is an object of $S_n$ and $h: T \to T'$ is a morphism in $S_n$, then

(2.11) \[ \mathcal{F}_n(T) = \mathcal{F}(T)_n \quad \text{and} \quad \mathcal{F}_n(h) = \mathcal{F}(h)_n, \]

(2.12) \[ (\mathcal{F}/n)(T) = \mathcal{F}(T)/n \quad \text{and} \quad (\mathcal{F}/n)(h) = \mathcal{F}(h)/n. \]

It follows from (2.1), (2.9) and (2.10) that the following diagram exists and is commutative

(2.13) \[
\begin{array}{ccc}
\mathcal{F}_n(S) & \xrightarrow{\mathcal{F}_n(f)'} & \mathcal{H}^0(X/S, \mathcal{F}_n) \\
\| & & \| \\
\mathcal{F}(S) & \xrightarrow{\mathcal{F}(f)'} & \mathcal{H}^0(X/S, \mathcal{F})
\end{array}
\]

Thus

(2.14) \[ \mathcal{F}_n(f)' = \mathcal{F}(f)'_n. \]

Note that, by the comment made right after (2.1), no analogous statements can be expected to hold for the presheaf $\mathcal{F}/n$.

Next recall that, if $f: X \to S$ is endowed with a (right) action of a finite group $\Delta$, then $f$ is called a \textit{Galois covering with Galois group $\Delta$} if the canonical map

(2.15) \[
\prod_{\delta \in \Delta} X \to X \times_X X, (x, \delta) \mapsto (x, x\delta),
\]

is an isomorphism of $S$-schemes. If $f$ is a Galois covering with Galois group $\Delta$ and $\mathcal{F}$ transforms finite sums of schemes into direct products of abelian groups (e.g., is representable \cite[p. 231, line 7]{ref}), then there exists a canonical isomorphism of abelian groups

(2.16) \[
\mathcal{H}^i(X/S, \mathcal{F}) \simeq H^i(\Delta, \mathcal{F}(X))
\]

for every $i \geq 0$, where $\mathcal{F}(X)$ is a left $\Delta$-module via the given right action of $\Delta$ on $X$ over $S$. See \cite[III, Example 2.6, p. 99]{ref}.

2.3. \textbf{Restriction and corestriction maps.} Let $f: S' \to S$ be a morphism of schemes and let $X'$ be an $S'$-scheme. The \textit{Weil restriction of $X'$ along $f$} is the contravariant functor $(\text{Sch}/S) \to (\text{Sets}), T \mapsto \text{Hom}_{S'}(T \times_S S', X')$. The latter functor is representable if there exist an $S$-scheme $R_{S'/S}(X')$ and a morphism of $S'$-schemes $\theta_{X', S'/S}: R_{S'/S}(X') \to X'$ such that the map

(2.16) \[
\text{Hom}_S(T, R_{S'/S}(X')) \to \text{Hom}_{S'}(T \times_S S', X'), \quad g \mapsto \theta_{X', S'/S} \circ g_{S'},
\]
is a bijection (functorially in $T$). See [11 §7.6] and [3 Appendix A.5] for basic information on the Weil restriction functor. We will write
\[(2.17)\]

\[j_{X, S'/S} : X \to R_{S'/S}(X_{S'})\]

for the canonical adjunction $S$-morphism, i.e., the $S$-morphism that corresponds to the identity morphism of $X_{S'}$ under the bijection (2.16).

Now assume that $S$ is locally noetherian and $f : S' \to S$ is finite, faithfully flat and of constant rank $n \geq 1$ (in particular, $f$ is an fppf covering of $S$). Then $S'$ is also locally noetherian and $j_{S, S'/S}$ is an isomorphism that will be regarded as an identification, i.e., we will write
\[(2.18)\]

\[R_{S'/S}(S') = S.\]

See [3, Proposition A.5.7, p. 510]. Now let $G$ be a commutative and quasi-projective $S$-group scheme. Then $j_{G, S'/S} : G \to R_{S'/S}(G_{S'})$ (2.17) is a closed immersion of commutative and quasi-projective $S$-group schemes. Further, if $G$ is smooth over $S$, then $R_{S'/S}(G_{S'})$ is smooth over $S$. See [11 §7.6, Theorem 4, p. 194], [7, II, Corollary 4.5.4] and [3 Propositions A.5.2(4), A.5.7 and A.5.8, pp. p. 506-513].

For $r \geq 0$ and $\tau = \text{ét or fl}$, let $e^{(r)}_\tau : H^r(S_\tau, R_{S'/S}(G_{S'})) \to H^r(S'_\tau, G)$ be the $r$-th edge morphism\(^{1}\) induced by the Cartan-Leray spectral sequence associated to the pair $(G, f)$ relative to the $\tau$ topology:
\[(2.19)\]

\[H^r(S_\tau, R^sf_*(G_{S'})) \Rightarrow H^{r+s}(S'_\tau, G).\]

Note that, by [16 Theorem 6.4.2(ii), p. 128], the maps $e^{(r)}_\text{ét}$ are isomorphisms for every $r \geq 0$. However, the maps $e^{(r)}_\text{fl}$ are not isomorphisms in general [8 XXIV, Remarks 8.5]. We also note that, via the bijection (2.16), $e^{(0)}_\tau$ is the identity map:
\[(2.20)\]

\[H^0(S_\tau, R_{S'/S}(G_{S'})) = R_{S'/S}(G_{S'})(S) = G_{S'}(S') = G(S') = H^0(S'_\tau, G).\]

The $r$-th restriction morphism associated to $(G, f)$ (2.2) factors as
\[(2.21)\]

\[j^{(r)} : H^r(S_\tau, G) \xrightarrow{H^r(S_\tau, j)} H^r(S_\tau, R_{S'/S}(G_{S'})) \xrightarrow{e^{(r)}_\tau} H^r(S'_\tau, G),\]

\(^{1}\) As noted by Brian Conrad, the restriction in [3 Appendix A.5] to an affine base $S$ can be removed since all assertions in [loc.cit.] are local on $S$. Further, the noetherian hypotheses in [loc.cit.] are satisfied if $S$ and $S'$ are locally noetherian.

\(^{2}\) This is an instance of the first edge morphism in [16 Proposition 2.3.1, p. 14].
where \( j = j_{G,s'/s} \).

Now, for every object \( T \to S \) of \( S_{fl} \), set

\[
\mathcal{H}^r(G)(T) = H^r(T_{fl}, G).
\]

Further, if \( h: T^* \to T \) is a morphism in \( S_{fl} \), set

\[
H^r(G)(T^*) = H^r(T_{fl}, G) \to H^r(T^*_{fl}, G).
\]

Then (2.22) and (2.23) define an abelian presheaf \( \mathcal{H}^r(G) \) on \( S_{fl} \) such that

\[
\mathcal{H}^0(G) = G.
\]

By (2.10), \( j_{G, fl} = \mathcal{H}^r(G)(f) \) factors as

\[
H^r(S_{fl}, G) \overset{f^r_G}{\to} \tilde{H}^0(S'/S, \mathcal{H}^r(G)) \to H^r(S'_{fl}, G),
\]

where

\[
f^r_G = \mathcal{H}^r(G)(f)^r
\]

is the map in (2.10) associated to the presheaf \( \mathcal{F} = \mathcal{H}^r(G) \). Thus

\[
\text{Ker} j_{G, fl} = \text{Ker}[f^r_G: H^r(S_{fl}, G) \to \tilde{H}^0(S'/S, \mathcal{H}^r(G))].
\]

The maps \( f^r_G \) are the second edge morphisms in [16, Proposition 2.3.1, p. 14] associated to the spectral sequence for Čech cohomology [16, Theorem 3.4.4(i), p. 58]:

\[
\tilde{H}^*(S'/S, \mathcal{H}^r(G)) \Rightarrow H^{r+s}(S_{fl}, G).
\]

By [13, p. 309, line 8] and (2.27) for \( r = 2 \), the preceding spectral sequence yields an exact sequence of abelian groups

\[
0 \to \tilde{H}^1(S'/S, G) \overset{e^1_G}{\to} H^1(S_{fl}, G) \overset{f^1_G}{\to} \tilde{H}^0(S'/S, \mathcal{H}^1(G)) \overset{d^0_1}{\to} \tilde{H}^2(S'/S, G) \to \text{Ker} j_{G, fl} \overset{f^2_G}{\to} \tilde{H}^1(S'/S, \mathcal{H}^1(G)) \to \tilde{H}^3(S'/S, G),
\]

where \( e^1_G \) is an instance of the first edge morphism in [16, Proposition 2.3.1, p. 14] and \( d^0_1 \) is the first transgression map [15, p. 49] induced by the spectral
sequence (2.28). In particular, the exactness of the above sequence together with (2.27) for $r = 1$ implies the following well-known fact:

**Proposition 2.1.** There exists a canonical isomorphism of abelian groups

$$\text{Ker} j^{(1)}_G \cong \check{H}^1(S'/S, G).$$

Now let

$$N_{G, S'/S} : R_{S'/S}(G_{S'}) \to G$$

be the norm (or trace) morphism defined in [10, XVII, 6.3.13.1 & 6.3.14(a)]. By [10, XVII, Proposition 6.3.15(iv)], the composition

$$N_{G, S'/S} \circ j_{G, S'/S} = n_G : G \xrightarrow{j_{G, S'/S}} R_{S'/S}(G_{S'}) \xrightarrow{N_{G, S'/S}} G$$

is the $n$-th power morphism on $G$. The $r$-th corestriction map associated to $G$ and $f : S' \to S$, denoted by

$$N^{(r)} : H^r(S'_{\text{ét}}, G) \to H^r(S_{\text{ét}}, G),$$

is the composition

$$H^r(S'_{\text{ét}}, G) \xrightarrow{(e_{\text{ét}}^r)^{-1}} H^r(S_{\text{ét}}, R_{S'/S}(G_{S'})) \xrightarrow{H^r(S_{\text{ét}}, N)} H^r(S_{\text{ét}}, G),$$

where $N = N_{G, S'/S}$.

Now, by the factorization (2.21), $N^{(r)} \circ j_{G, S'/S}^{(r)} = H^r(S_{\text{ét}}, N) \circ H^r(S_{\text{ét}}, j)$. Thus, by (2.31), the composition

$$H^r(S_{\text{ét}}, G) \xrightarrow{j_{G, S'_{\text{ét}}}^{(r)}} H^r(S'_{\text{ét}}, G) \xrightarrow{N^{(r)}} H^r(S_{\text{ét}}, G)$$

is the multiplication by $n$ map on $H^r(S_{\text{ét}}, G)$. Consequently, Ker $j_{G, S'_{\text{ét}}}^{(r)} \subseteq H^r(S_{\text{ét}}, G)_n$, whence (2.3) and (2.27) yield the following statement.

**Lemma 2.2.** If $G$ is smooth over $S$, then

$$\text{Ker} j^{(r)}_G = \text{Ker} [H^r(S_{\text{ét}}, G)_n \xrightarrow{(j^{(r)}_G)^n} \check{H}^0(S'/S, \mathcal{H}^r(G))_n],$$

where $j^{(r)}_G = \mathcal{H}^r(G)(f')$ is the map in (2.25).

---

3 The standard proof of this fact can be found, for example, in [13 III, proof of Proposition 4.6, p. 123].
3. The capitulation kernel

In this section we establish the main theorem of the paper (Theorem 3.4) which applies to the following types of pairs \((G, f)\):

**Definition 3.1.** The pair \((G, f)\) is called **admissible** if

(i) \(S\) is locally noetherian,

(ii) \(f: S' \to S\) is finite, faithfully flat and of constant rank \(n \geq 2\),

(iii) \(G\) is smooth, commutative, quasi-projective, of finite presentation over \(S\) and its fibers are connected, and

(iv) for every point \(s \in S\) such that \(\text{char}\ k(s)\) divides \(n\), \(G_s\) is a semiabelian variety over \(k(s)\).

**Examples 3.2.** The following are examples of admissible pairs:

(a) \(f: S' \to S\) is a finite and faithfully flat morphism of connected noetherian schemes and \(G\) is a semiabelian \(S\)-scheme. For example, \(G = \mathbb{G}_{m,S}\).

(b) \(f: S' \to S\) is a finite and faithfully flat morphism of rank \(n \geq 2\) between connected Dedekind schemes, \(F\) is the function field of \(S\), \(G_F\) is a smooth, commutative and connected \(F\)-group scheme of finite type, \(G_F\) has a Néron \(S\)-model \(\mathcal{N}\) with semiabelian reduction at all \(s \in S\) such that \(\text{char}\ k(s)\) divides \(n\) and \(G = \mathcal{N}^0\) is the identity component of \(\mathcal{N}\). That \(G\) satisfies condition (iii) of Definition 3.1 follows from [1, §6.4, Theorem 1, p. 153, and §10.1, p. 290, line 6].

**Proposition 3.3.** Let \(n \geq 1\) be an integer, let \(S\) be a locally noetherian scheme and let \(G\) be a smooth and commutative \(S\)-group scheme with connected fibers. Assume that, for every point \(s \in S\) such that \(\text{char}\ k(s)\) divides \(n\), \(G_s\) is a semiabelian \(k(s)\)-variety. Then \(n_G: G \to G\) is faithfully flat and locally of finite presentation.

**Proof.** By [6, (6.2.1.2) and Proposition 6.2.3(v), p. 298], \(n_G\) is locally of finite presentation. Now, by [7 IV\textsubscript{3}, Corollary 11.3.11], to establish the flatness and surjectivity of \(n_G\) we may assume that \(S = \text{Spec} \ F\), where \(F\) is a field. If \(n\) is prime to \(\text{char}\ F\), then \(n_G\) is étale and therefore flat [8 VII\textsubscript{A}, §8.4, Proposition]. Thus, since \(G\) is connected, \(n_G\) is surjective by [8 VI\textsubscript{B}, Proposition 3.11 and its proof]. Assume now that \(\text{char}\ F\) divides \(n\) and let \(\bar{F}\) be an algebraic closure of \(F\). Since the \(n\)-th power morphism \(\bar{F}^* \to \bar{F}^*\) is surjective, \(n_G(\bar{F})\) is surjective if \(G\) is an \(F\)-torus. If \(G\) is an abelian variety over \(F\), then \(n_G(\bar{F})\) is also
surjective by \cite[p. 62]{14}. Now let $G$ be a semiabelian variety over $F$, i.e., an extension $1 \to T \to G \to A \to 1$, where $T$ is an $F$-torus and $A$ is an abelian variety over $F$. Then the following exact and commutative diagram of abelian groups shows that the map $n_G(\bar{F}) : G(\bar{F}) \to G(\bar{F})$ is surjective as well:

\[
\begin{array}{ccccccccc}
0 & \to & T(\bar{F}) & \to & G(\bar{F}) & \to & A(\bar{F}) & \to & 0 \\
\downarrow n_T(\bar{F}) & & \downarrow n_G(\bar{F}) & & & & \downarrow n_A(\bar{F}) & & \\
0 & \to & T(\bar{F}) & \to & G(\bar{F}) & \to & A(\bar{F}) & \to & 0.
\end{array}
\]

We conclude from \cite[§3, Corollary 6.10, p. 96]{4} that $n_G$ is surjective. Thus, since $G$ is smooth, $n_G$ is flat by \cite[VI B, Proposition 3.11]{8}. □

Let $f : S' \to S$ be a morphism of schemes and let $G$ be an $S$-group scheme such that the pair $(G, f)$ is admissible (see Definition 3.1). Let $n \geq 2$ be the rank of $f$. By Proposition 3.3

\[(3.1)\quad 0 \to G_n \to G \overset{n}{\to} G \to 0\]

is an exact sequence in $S_{\tilde{\mathbb{A}}}$. For every object $T \to S$ of $S_{\tilde{\mathbb{A}}}$ and every integer $r \geq 1$, (3.1) induces an exact sequence of abelian groups

\[(3.2)\quad 0 \to H^{r-1}(T_{\acute{e}t}, G) / n \to H^r(T_{\tilde{\mathbb{A}}}, G_n) \to H^r(T_{\acute{e}t}, G) \to 0,
\]

where the left-hand nontrivial map above is induced by the connecting morphism $H^{r-1}(T_{\tilde{\mathbb{A}}}, G) \simeq H^{r-1}(T_{\acute{e}t}, G) \to H^r(T_{\tilde{\mathbb{A}}}, G_n)$ in fppf cohomology induced by the sequence (3.1). Thus there exists a canonical exact sequence of abelian presheaves on $S_{\tilde{\mathbb{A}}}$

\[(3.3)\quad 0 \to \mathcal{H}^{r-1}(G) / n \to \mathcal{H}^r(G_n) \to \mathcal{H}^r(G) \to 0,
\]

where the left-hand term is the presheaf (2.12) associated to $\mathcal{H}^{r-1}(G)$, the middle term is the presheaf (2.22) associated to $G_n$, and the right-hand term is the presheaf (2.11) associated to $\mathcal{H}^r(G)$. By (2.10), (2.14) and (2.26), the maps $(\mathcal{H}^{r-1}(G)/n)(f)$ and $\mathcal{H}^r(G)(f)$ factor as

\[(3.4)\quad H^{r-1}(S_{\acute{e}t}, G) / n \overset{f_{\acute{e}t}^{(r-1)}}{\to} \check{H}^0(S'/S, \mathcal{H}^{r-1}(G)/n) \hookrightarrow H^{r-1}(S'_{\acute{e}t}, G) / n
\]

and

\[(3.5)\quad H^r(S_{\acute{e}t}, G) \overset{f_{\acute{e}t}^r}{\to} \check{H}^0(S'/S, \mathcal{H}^r(G))_n \hookrightarrow H^r(S'_{\acute{e}t}, G)_n,
\]
respectively, where \( f^{(r-1)}_n = (\mathcal{H}^{r-1}(G)/n)(f)^{\mathbb{Z}}. \)

Now set \( S'' = S' \times_S S' \) and, for \( i = 1 \) or \( 2 \), let \( p_i: S'' \to S' \) be the projection onto the \( i \)-th factor. We will write \( p_i^* = G(p_i): G(S') \to G(S'') \) for the morphism of abelian groups induced by \( p_i \). We now define
\[
(3.6) \quad \Psi(G, f) = \{ x \in G(S') : p_1^*(x)p_2^*(x)^{-1} \in G(S'') \}.
\]
When \( G = \mathbb{G}_{m, S} \), \( (3.6) \) will be written
\[
(3.7) \quad \Psi(U, f) = \{ x \in U(S') : p_1^*(x)p_2^*(x)^{-1} \in U(S'') \},
\]
where \( U \) is the global units functor \( (2.3) \).

Since \( f \circ p_1 = f \circ p_2 \), the image of \( G(f): G(S) \to G(S') \) is contained in \( \Psi(G, f) \). Thus \( (3.6) \) is a subgroup of \( G(S') \) containing \( G(S)G(S')^n \). By \( (2.9), (2.12) \) and \( (2.24) \), we have \( \check{H}^0(S'/S, \mathcal{H}^0(G)/n) = \Psi(G, f)/G(S')^n \) and \( f^{(0)}_n: G(S)/n \to \check{H}^0(S'/S, \mathcal{H}^0(G)/n) \) is the map \( G(S)/n \to \Psi(G, f)/G(S')^n \) induced by \( G(f): G(S) \to G(S') \). Thus there exist canonical isomorphisms of abelian groups
\[
(3.8) \quad \text{Ker} f^{(0)}_n \simeq \frac{G(S) \cap G(S')^n}{G(S)^n}
\]
and
\[
(3.9) \quad \text{Coker} f^{(0)}_n \simeq \frac{\Psi(G, f)}{G(S)G(S')^n},
\]
where the intersection takes place inside \( G(S') \).

We now consider the exact and commutative diagram of abelian groups
\[
(3.10) \quad \begin{array}{c}
0 \rightarrow \check{H}^{r-1}(S_{\text{et}}, G)/n \rightarrow \check{H}^r(S_{\text{et}}, G_n) \rightarrow \check{H}^r(S_{\text{et}}, G) \\
\left| f^{(r-1)}_n \right| \quad \left| f^{(r)}_G \right| \quad \left( f^{(r)}_G \right)_n \\
0 \rightarrow \check{H}^0(S'/S, \mathcal{H}^{r-1}(G)/n) \rightarrow \check{H}^0(S'/S, \mathcal{H}^r(G_n)) \rightarrow \check{H}^0(S'/S, \mathcal{H}^r(G))
\end{array}
\]
where the top row is the sequence \( (3.2) \) for \( T = S \), the bottom row is the beginning of the Čech cohomology sequence induced by \( (3.3) \) \( (13) \) p. 97, line –4 \) and the vertical maps are given, respectively, by \( (3.4), (2.25) \) (for \( G_n \) and \( (3.5) \)) via the equality \( \check{H}^0(S'/S, \mathcal{H}^r(G)_n) = \check{H}^0(S'/S, \mathcal{H}^r(G))_n \) \( (2.13) \). Using

\[4\] Note that, in general, \( f^{(r-1)}_n \neq f^{(r-1)}_G / n \), by the comment following \( (2.14) \).
ČECH COHOMOLOGY AND CAPITULATION

(2.27) for \( G_n \) and Lemma 2.24, the above diagram yields an exact sequence of abelian groups

\[
0 \to \text{Ker} f^{(r-1)}_{/n} \to \text{Ker} j^{(r)}_{G_n,fl} \to \text{Ker} j^{(r)}_G \to \text{Coker} f^{(r-1)}_{/n} \to \text{Coker} f^{(r)}_{G_n}.
\]

We now observe that (2.29) (applied to \( G_n \)) induces an injection \( \text{Coker} f^{(1)}_G \xrightarrow{\cong} \check{\text{H}}^2(S'/S, G_n) \). Thus Proposition 2.1, (3.8), (3.9) and the exactness of (3.11) for \( r = 1 \) yield the main result of this paper:

**Theorem 3.4.** Assume that the pair \((G, f)\) is admissible (see Definition 3.1) and let \( n \geq 2 \) be the rank of \( f \). Then there exists a canonical exact sequence of \( n\)-torsion abelian groups

\[
0 \to G(S) \cap G(S')^n / G(S)^n \to \check{\text{H}}^1(S'/S, G_n) \to \text{Ker} j^{(1)}_G \to \Psi(G, f) / G(S) G(S')^n \to \check{\text{H}}^2(S'/S, G_n),
\]

where \( j^{(1)}_G : H^1(S_{\text{ét}}, G) \to H^1(S'_{\text{ét}}, G) \) is the first restriction map associated to \((G, f)\) (2.4) and \( \Psi(G, f) \) is the subgroup (3.0) of \( G(S') \). If, in addition, \( f \) is a Galois covering with Galois group \( \Delta \) and the groups \( \check{\text{H}}^i(S'/S, G_n) \) above are replaced with \( H^i(\Delta, G(S')_n) \) via (2.15), where \( i = 1 \) and \( 2 \), then the resulting sequence is also exact.

**Remark 3.5.** The nontrivial maps in the sequence of the theorem can be described as follows. The map \( G(S) \cap G(S')^n / G(S)^n \to \check{\text{H}}^1(S'/S, G_n) \) is the composition of the inverse of the isomorphism (3.8) and the top horizontal arrow in the following commutative square

\[
\begin{array}{ccc}
\text{Ker} f^{(0)}_{/n} & \xrightarrow{\text{inj}} & \check{\text{H}}^1(S'/S, G_n) \\
\downarrow & & \downarrow e_{G_n} \\
G(S)/n & \xrightarrow{\epsilon_{G_n}} & H^1(S_{\text{fl}}, G_n),
\end{array}
\]

where \( e_{G_n} \) is the edge morphism in the exact sequence (2.29) associated to \( G_n \) and the bottom horizontal arrow is the left-hand nontrivial map in (3.2) for \( r = 1 \) and \( T = S \). The map \( \check{\text{H}}^1(S'/S, G_n) \to \text{Ker} j^{(1)}_G \) is the composition of the map \( \check{\text{H}}^1(S'/S, G_n) \to \check{\text{H}}^1(S'/S, G) \) induced by the immersion \( G_n \xrightarrow{\epsilon} G \) and the isomorphism \( \check{\text{H}}^1(S'/S, G) \cong \text{Ker} f^{(1)}_G = \text{Ker} j^{(1)}_G \) induced by the edge morphism \( e_G \) in (2.29) (note that the latter map is the inverse of the isomorphism
of Proposition 2.1. The map $\text{Ker} j_G^{(1)} \to \Psi(G, f)/G(S')^n$ is the composition of the connecting morphism $\text{Ker} j_G^{(1)} = \text{Ker} (f_G^{(1)})_n \to \text{Coker} f/n$ in the snake-lemma exact sequence arising from diagram (3.10) and the isomorphism (3.9). Finally, $\Psi(G, f)/G(S')^n \to \check{H}^2(S'/S, G_n)$ is the composition of the connecting morphism $\text{Ker} j_G^{(1)} = \text{Ker} (f_G^{(1)})_n \to \text{Coker} f_n$ induced by $H^0(S'/S, \phi)$, where $\phi: \mathcal{H}^0(G)/n \to \mathcal{H}^1(G_n)$ is the left-hand nontrivial map in the exact sequence (3.3) for $r = 1$, and the injection $\text{Coker} f_G^{(1)} \hookrightarrow \check{H}^2(S'/S, G_n)$ induced by the transgression map $d_{a_2^{0,1}}$ appearing in the exact sequence (2.29).

Via the identifications $H^1(S'_\text{ét}, \mathbb{G}_m) = \text{Pic} S$ and $H^1(S'_\text{ét}, \mathbb{G}_m) = \text{Pic} S'$, the first restriction map $j_{G_m, S}^{(1)}: H^1(S'_\text{ét}, \mathbb{G}_m) \to H^1(S'_\text{ét}, \mathbb{G}_m)$ (2.4) associated to the pair $(\mathbb{G}_m, S, f)$ can be identified with the canonical map $\text{Pic} f: \text{Pic} S \to \text{Pic} S'$ (2.6). Further, by Example 3.2(a), $(\mathbb{G}_m, S, f)$ is an admissible pair. Thus the theorem immediately yields

**Corollary 3.6.** Let $f: S' \to S$ be a finite and faithfully flat morphism of rank $n \geq 2$ between connected noetherian schemes. Then there exists a canonical exact sequence of $n$-torsion abelian groups

$$1 \to U(S) \cap U(S')^n/U(S)^n \to \check{H}^1(S'/S, \mu_n) \to \text{Ker} \text{Pic} f \to \Psi(U, f)/U(S)U(S')^n \to \check{H}^2(S'/S, \mu_n),$$

where $U$ is the global units functor (2.5), $\text{Pic} f$ is the canonical map (2.6) and $\Psi(U, f) \subset U(S')$ is the group (3.7).

### 4. Capitulation of ideal classes

In this final section we specialize Corollary 3.6 to the case $S = \text{Spec} \mathcal{O}_{F, \Sigma}$. The notation is as in the Introduction.

As noted in the Introduction, the capitulation map (1.1) can be identified with the canonical map $\text{Pic} f$ (2.3).

Now observe that $S'' = S' \times_S S' = \text{Spec} (\mathcal{O}_{K, \Sigma} \otimes_{\mathcal{O}_{F, \Sigma}} \mathcal{O}_{K, \Sigma})$ and the canonical projection morphisms $p_i: S'' \to S'$ for $i = 1$ and 2 are induced, respectively, by the morphisms of rings $\mathcal{O}_{K, \Sigma} \to \mathcal{O}_{K, \Sigma} \otimes_{\mathcal{O}_{F, \Sigma}} \mathcal{O}_{K, \Sigma}$ given by $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$, where $x \in \mathcal{O}_{K, \Sigma}$. Thus, if $G = \mathbb{G}_m, S$, then the morphisms of abelian groups $p_i^* = G(p_i): G(S') \to G(S'')$ induced by $p_i$ are the maps...
\( \mathcal{O}_{K,\Sigma}^{*} \to (\mathcal{O}_{K,\Sigma} \otimes_{\mathcal{O}_{F,\Sigma}} \mathcal{O}_{K,\Sigma})^{*} \) defined by \( u \mapsto u \otimes 1 \) and \( u \mapsto 1 \otimes u \), where \( u \in \mathcal{O}_{K,\Sigma}^{*} \). Consequently, the group \( \Psi(U, f) \) (3.7) can be identified with

\[
(4.1) \quad \Psi(K/F, \Sigma) = \{ u \in \mathcal{O}_{K,\Sigma}^{*} : u \otimes u^{-1} \in ((\mathcal{O}_{K,\Sigma} \otimes_{\mathcal{O}_{F,\Sigma}} \mathcal{O}_{K,\Sigma})^{*})^{n} \}.
\]

Thus Corollary 3.6 yields the following statement.

**Proposition 4.1.** There exists a canonical exact sequence of finite \( n \)-torsion abelian groups

\[
0 \to \mathcal{O}_{F,\Sigma}^{*} \cap (\mathcal{O}_{K,\Sigma}^{*})^{n} / (\mathcal{O}_{F,\Sigma}^{*})^{n} \to H^{1}(\mathcal{O}_{K,\Sigma}/\mathcal{O}_{F,\Sigma}, \mu_{n}) \to \text{Ker} j_{K/F,\Sigma}
\]

\[
\to \Psi(K/F, \Sigma)/\mathcal{O}_{F,\Sigma}^{*}(\mathcal{O}_{K,\Sigma}^{*})^{n} \to \tilde{H}^{2}(\mathcal{O}_{K,\Sigma}/\mathcal{O}_{F,\Sigma}, \mu_{n}),
\]

where \( j_{K/F,\Sigma} \) is the capitulation map (1.1) and \( \Psi(K/F, \Sigma) \) is the subgroup of \( \mathcal{O}_{K,\Sigma}^{*} \) given by (4.1).

If \( \Sigma \) contains all primes of \( F \) that ramify in \( K \), then \( f : \text{Spec} \mathcal{O}_{K,\Sigma} \to \text{Spec} \mathcal{O}_{F,\Sigma} \) is a Galois covering with Galois group \( \Delta \). Thus the following statement is immediate from the proposition using (2.15).

**Corollary 4.2.** Let \( K/F \) be a finite Galois extension of global fields with Galois group \( \Delta \) of order \( n \) and let \( \Sigma \) be a nonempty finite set of primes of \( F \) containing all archimedean primes and all primes that ramify in \( K \). Then there exists a canonical exact sequence of finite \( n \)-torsion abelian groups

\[
0 \to \mathcal{O}_{F,\Sigma}^{*} \cap (\mathcal{O}_{K,\Sigma}^{*})^{n} / (\mathcal{O}_{F,\Sigma}^{*})^{n} \to H^{1}(\mathcal{O}_{K,\Sigma}/\mathcal{O}_{F,\Sigma}, \mu_{n}) \to \text{Ker} j_{K/F,\Sigma}
\]

\[
\to \Psi(K/F, \Sigma)/\mathcal{O}_{F,\Sigma}^{*}(\mathcal{O}_{K,\Sigma}^{*})^{n} \to \tilde{H}^{2}(\mathcal{O}_{K,\Sigma}/\mathcal{O}_{F,\Sigma}, \mu_{n}),
\]

Consequently, if \( n \) is odd and \( \mu_{n}(K) = \{1\} \), then there exists a canonical isomorphism of finite \( n \)-torsion abelian groups

\[
\text{Ker} j_{K/F,\Sigma} \cong \Psi(K/F, \Sigma)/\mathcal{O}_{F,\Sigma}^{*}(\mathcal{O}_{K,\Sigma}^{*})^{n}.
\]

In particular, at most \( [\mathcal{O}_{K,\Sigma}^{*} : \mathcal{O}_{F,\Sigma}^{*}(\mathcal{O}_{K,\Sigma}^{*})^{n}] \) \( \Sigma \)-ideal classes of \( F \) capitate in \( K \).

**Remark 4.3.** The group \( \Psi(K/F, \Sigma) \) (111) seems difficult to compute in general since the structure of \( (\mathcal{O}_{K,\Sigma} \otimes_{\mathcal{O}_{F,\Sigma}} \mathcal{O}_{K,\Sigma})^{*} \) is rather mysterious. Perhaps a more sensible approach would be to investigate whether or not \( \Psi(K/F, \Sigma) \) satisfies a Hasse principle, i.e., whether the condition that defines \( \Psi(K/F, \Sigma) \), namely \( u \otimes u^{-1} \in ((\mathcal{O}_{K,\Sigma} \otimes_{\mathcal{O}_{F,\Sigma}} \mathcal{O}_{K,\Sigma})^{*})^{n} \), is determined locally. In general, if \( B \) is an \( A \)-algebra, where \( A \) and \( B \) are (commutative and unital) rings,
then there exists a canonical map $B^* \times B^* \to (B \otimes_A B)^*$, $(b_1, b_2) \mapsto b_1 \otimes b_2$, which is not surjective in general. When $A$ is a field and $B$ and $C$ are finitely generated $A$-algebras, the structure of $(B \otimes_A C)^*$ has been discussed by Jaffe [11]. However, no general structure theorem for $(B \otimes_A B)^*$ seems to be known. Of course, it may still be possible to determine the subgroup $\Psi(K/F, \Sigma)$ of $O_{K, \Sigma}^*$ explicitly in some particular cases of interest and then apply Corollary 4.2 to obtain nontrivial information on the capitulation kernel, e.g., when $K/F$ is a (possibly ramified) quadratic extension of number fields. For the quadratic unramified case (when $F$ is a totally real quadratic number field), see [2].

References

[1] Bosch, S., Lütkebohmert, W. and Raynaud, M. Néron Models. Springer Verlag, Berlin 1989. 2.3, 2.3, 3.2
[2] Benjamin, E., Sanborn, F. and Snyder, C. Capitulation in unramified quadratic extensions of real quadratic number fields. Glasgow Math. J. 36 (1994), no. 3, 385–392. 1.3
[3] Conrad, B., Gabber, O. and Prasad, G.: Pseudo-reductive groups. Second Ed. New Math. Monographs 26, Cambridge U. Press, 2015. 2.3, 2.3, 1
[4] Demazure, M. and Gabriel, P.: Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. Masson & Cie, Éditeur, Paris, 1970. (with an Appendix by M. Hazewinkel: Corps de classes local). ISBN 7204-2034-2 3
[5] González-Avilés, C.D. Capitulation, ambiguous classes and the cohomology of the units. J. reine angew. Math. 613 (2007), 75–97. 1
[6] Grothendieck, A. and Dieudonné, J.: Éléments de géométrie algébrique I. Le langage des schémas. Grund. der Math. Wiss. 166 (1971). 2.2, 3
[7] Grothendieck, A. and Dieudonné, J.: Éléments de géométrie algébrique. Publ. Math. IHES 8 (= EGA II) (1961), 11 III1 (1961), 20 (= EGA IV1) (1964), 24 (= EGA IV2) (1965), 32 (= EGA IV4) (1967). 2.3, 3
[8] Demazure, M. and Grothendieck, A. (Eds.): Schémas en groupes. Séminaire de Géométrie Algébrique du Bois Marie 1962-64 (SGA 3). Augmented and corrected 2008-2011 re-edition of the original by P.Gille and P.Polo. Available at http://www.math.jussieu.fr/~polo/SGA3 [2.3, 3] Reviewed at http://www.jmilne.org/math/xnotes/SGA3r.pdf 2.3, 3
[9] Grothendieck, A: Le groupe de Brauer I-III. In: Dix Exposés sur la cohomologie des schémas, North Holland, Amsterdam, 1968, 46–188. 2.1
[10] Grothendieck, A. et al.: Théorie des Topos et Cohomologie Étale des Schémas. Séminaire de Géométrie Algébrique du Bois Marie 1963-64 (SGA 4). Lect. Notes in Math. 270, 305 and 370, Springer-Verlag 1973. 2.3
[11] Jaffe, D.B.: Functorial structure of units in a tensor product. Trans. Amer. Math. Soc. 348 (1996), no. 11, 4339–4353. 4.3
[12] May, P.J.: The geometry of iterated loop spaces. Lecture Notes in Math. 271, Springer-Verlag, 1975.

[13] Milne, J.S.: *Étale cohomology*. Princeton University Press, Princeton, 1980.

[14] Mumford, D.: *Abelian varieties*. Reprint of the Second (1974) Edition. Oxford Univ. Press, 1985.

[15] Shatz, S.: *Profinite groups, arithmetic, and geometry*, Annals of Math. Studies 67. Princeton Univ. Press 1972.

[16] Tamme, G.: Introduction to Étale Cohomology. Translated from the German by Manfred Kolster. Universitext. Springer-Verlag, Berlin, 1994.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LA SERENA, LA SERENA, CHILE

E-mail address: cgonzalez@userena.cl