Aspects of the Supersymmetry Algebra

in Four Dimensional Euclidean Space

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1 Abstract

The simplest supersymmetry (SUSY) algebra in four dimensional Euclidean space ($4dE$) has been shown to closely resemble the $N = 2$ SUSY algebra in four dimensional Minkowski space ($4dM$). The structure of the former algebra is examined in greater detail in this paper. We first present its Clifford algebra structure. This algebra shows that the momentum Casimir invariant of physical states has an upper bound which is fixed by the central charges. Secondly, we use reduction of the $N = 1$ SUSY algebra in six dimensional Minkowski space ($6dM$) to $4dE$; this reproduces our SUSY algebra in $4dE$. Moreover, this same reduction of supersymmetric Yang-Mills theory (SSYM) in $6dM$ reproduces Zumino’s SSYM in $4dE$. We demonstrate how this dimensional reduction can be used to introduce additional generators into the SUSY algebra in $4dE$.

2 Introduction

The nature of SUSY in $4dE$ is surprisingly different from that of SUSY in $4dM$ due to the fact that spinors in these two spaces have distinct structures. The fundamental reason for this difference is that in the decomposition of $SO(4)$ into $SU(2) \times SU(2)$, the generators of the two $SU(2)$ groups are not Hermitian conjugates of each other and this has the consequence that one cannot define Majorana spinors in $4dE$. A detailed analysis of spinors in $4dE$ and the simplest attendant SUSY algebra is presented in [1]. There it was noted that this symmetry algebra more closely resembles that of $N = 2$ SUSY in $4dM$ rather than $N = 1$
SUSY in 4dM. There are two distinct SUSY generators, their hermitian conjugates and two central charges in this algebra in 4dE; however, unlike N = 2 SUSY in 4dM, no SU(2) internal symmetry exists between the distinct SUSY generators.

In this paper, we analyze further the algebra found in [1] in 4dE. Initially, by choosing a suitable linear combinator of SUSY generators, the Clifford algebra structure of this SUSY algebra is made explicit. There is an immediate consequence of this algebra: the requirement that the anticommutator of an operator and its Hermitian conjugate be positive definite places an upper bound on magnitude of the eigenvalue associated with the Casimir $P^2$ (where $P^\mu$ is the four-momentum) that is dictated by the central charges of the algebra. Furthermore, these central charges all have to be negative definite. We note that this scenario is very different to what happens in 4dM, where the central charges can be consistently set to zero. In 4dE the central charges must be non-zero.

The second approach to analyzing the structure of our algebra is to perform a dimensional reduction of the N = 1 SUSY algebra from 6dM to 4dE. This is motivated by a similar reduction from 6dM to 4dM done in [2,3,4]; in these references the N = 1 SSYM model in 6dM is used to derive the N = 2 SSYM model in 4dM. We actually reproduce the 4dE supersymmetry algebra presented previously [1]. Surprisingly, by using this procedure, we are able to extend the SUSY algebra in 4dE by considering generators initially associated with rotation operators in 6dE that involve those two dimensions eliminated by dimensional reduction. We find also that by using this same reduction in conjunction not with the SUSY algebra, but with the SSYM theory itself in 6dE, the SSYM model of Zumino [5] in 4dE is
automatically generated. Furthermore, we speculate on the likelihood of generating a SUSY algebra in $4dE$ with an internal symmetry by applying dimensional reduction to the $N = 1$ SUSY algebra in $10dM$.

### 3 The Clifford Algebra

The simplest SUSY algebra in $4dE$ was found in [1] to be

\[
\{ g, g^\dagger \} = 0 \quad (1a)
\]

\[
\{ g, g^\dagger \} = \gamma^\mu P^\mu + Z_+ + Z_- \gamma_5 \quad (1b)
\]

\[
[P^\mu, g] = 0 \quad (1c)
\]

\[
[M^\mu\nu, g] = -\frac{1}{2} \Sigma^\mu\nu g \quad (1d)
\]

\[
[M^\mu\nu, P^\lambda] = i \left( \delta^\nu^\lambda P^\mu - \delta^\mu^\lambda P^\nu \right) \quad (1e)
\]

\[
[M^\mu\nu, M^{\rho\sigma}] = i \left( \delta^{\nu^\rho} M^{\mu^\sigma} - \delta^{\mu^\rho} M^{\nu^\sigma} + \delta^{\mu^\sigma} M^{\nu^\rho} - \delta^{\nu^\sigma} M^{\mu^\rho} \right) . \quad (1f)
\]

The notation used is explained in reference [1]. We note here only that the charge conjugate $g^c$ of the Dirac spinor generator $g$ cannot be consistently set equal to $g$ and that $g$ cannot decompose into a linear combination of two such (self-conjugate) Majorana spinors; this accounts for the difference between (1b) and the analogous equation in $N = 2$ SUSY in $4dM$.

When using two dimensional notation,

\[
g = \begin{pmatrix} Q_a \\ R^\dagger \end{pmatrix} \quad g^c = \begin{pmatrix} -\tilde{Q}_a \\ \overline{R}^\dagger \end{pmatrix} \quad (2a)
\]
\[ g^+ = (\overline{Q}^a, -\overline{R}_a) \quad g^{c+} = (Q^a, R_a) \] (2b)  

the algebraic expressions in (1) which involve \( g \) become  

\[ \{Q_a, R_b\} = 0 = \{\overline{Q}^j, \overline{R}^i\} \] (3a)  

\[ \{Q_a, \overline{R}_b\} = i\sigma^{\mu}_{ab} P^\mu = \{\overline{Q}_a, \overline{R}_b\} \] (3b)  

\[ \{Q_a, \overline{Q}_b\} = \epsilon_{ab} Z_{Q\overline{Q}} \] (3c)  

\[ \{\overline{R}^a, \overline{R}^b\} = \epsilon^{\bar{a}\bar{b}} Z_{R\overline{R}} \] (3d)  

where \( Z_{Q\overline{Q}} = Z_+ + Z_- \), \( Z_{R\overline{R}} = Z_+ - Z_- \).

If now we make the identifications  

\[ (\alpha_a, \beta_a) = (Q_a, -R_a^a) \quad (Q = 1, 2) \] (4a)  

\[ (\alpha_a^+, \beta_a^+) = (\overline{Q}^a, \overline{R}_a) \quad (a = 1, 2) \] (4b)  

then in the frame oriented so that \( P^\mu = (0, 0, 0, P) \), (3) becomes  

\[ \{\alpha_a, \beta_b\} = 0 = \{\alpha_a^+, \beta_b^+\} \] (5a)  

\[ \{\alpha_a, \beta_b^+\} = i\delta_{ab} P = -\{\alpha_a^+, \beta_b\} \] (5b)  

\[ \{\alpha_a, \alpha_b^+\} = -\delta_{ab} Z_{Q\overline{Q}} \] (5c)  

\[ \{\beta_a, \beta_b^+\} = -\delta_{ab} Z_{R\overline{R}}. \] (5d)  

Since the left side of (5c) and (5d) are non-negative, we conclude that  

\[ Z_{Q\overline{Q}} \leq 0 \] (6a)
We consider first the case where \( Z_{QQ} \) and \( Z_{RR} \) are both non-zero. In this instance, we first make a rescaling

\[
\alpha_a \to \alpha_a \sqrt{-Z_{QQ}} \quad (7a)
\]

\[
\beta_a \to \beta_a \sqrt{-Z_{RR}} \quad (7b)
\]

and then let

\[
A_a = \frac{\alpha_a^+ - i\beta_a^+}{\sqrt{2}} \quad A_a^+ = \frac{\alpha_a + i\beta_a}{\sqrt{2}} \quad (8a)
\]

\[
B_a = \frac{\beta_a + i\alpha_a}{\sqrt{2}} \quad B_a^+ = \frac{\beta_a^+ - i\alpha_a^+}{\sqrt{2}} \quad (8b)
\]

The anticommutation relations of (5) then become

\[
\{ A_a, A_b^+ \} = \delta_{ab}(1 + \delta \mathbb{P}) \quad (9a)
\]

\[
\{ B_a, B_b^+ \} = \delta_{ab}(1 - \delta \mathbb{P}) \quad (9b)
\]

(where \( \delta \equiv (Z_{QQ}Z_{RR})^{-1/2} \)) and all other anticommutators involving \( A_a \) and \( B_a \) are zero.

For eq. (9) to be consistent, we see that we must have \((1 \pm \delta P) \geq 0\) so that

\[
\mathbb{P} \leq (Z_{QQ}Z_{RR})^{+1/2}; \quad (10)
\]

this places a bound on the magnitude of the eigenvalue associated with the Casimir \( \mathbb{P}^2 \).

If now we were to have \( Z_{QQ} = 0 = Z_{RR} \) then it is apparent from (5) that all states have zero norm; this case we will discard as uninteresting.

The last situation involves having one, but not both, of the central charges \( Z_{QQ} \) and \( Z_{RR} \) equal to zero. Without loss of generality, let us consider the case \( Z_{RR} = 0 \). It is easily
established that one can then choose suitable linear combinations of $\alpha_i$, $\beta_i$, $\alpha^+_i$ and $\beta^+_i$ so that
\[
\{A_i, A^+_j\} = \delta_{ij} \left( \frac{1 + \sqrt{1 + 4a^2}}{2} \right) \quad (11a)
\]
\[
\{B_i, B^+_j\} = \delta_{ij} \left( \frac{1 - \sqrt{1 + 4a^2}}{2} \right) \quad (11b)
\]
(where $a \equiv \mathcal{P}/\sqrt{-Z_{QQ}}$) and all other anticommutators are zero. For $a^2 \neq 0$, the right side of (11b) is negative. Since the left side of (11b) is positive definite, we therefore have an inconsistency, allowing to exclude the possibility of having one of the central charges equal to zero and the other non-zero.

From the analysis of this section, we see that the SUSY algebra of eq. (1) is equivalent to a Clifford algebra; the structure of this Clifford algebra imposes the restriction that the central charges $Z_{RR}$ and $Z_{QQ}$ be negative definite and that the magnitude of the eigenvalue of the Casimir operator $\mathcal{P}^2$ be always less than or equal to $Z_{QQ}Z_{RR}$. The saturation of this bound (so that $1 - \delta\mathcal{P}^2 = 0$ in (9b)) eliminate half of the states present in the case where the bound is not saturated.

It is interesting to compare this situation to what occurs in $N = 2$ SUSY in $4dM$. In this latter case [6], the Clifford algebra has a structure identical to that of eq. (9); however the roles of the eigenvalue of $\mathcal{P}^2$ and the central charges are reversed; one finds that the central charges can be zero and that there is a lower bound on the eigenvalue of $\mathcal{P}^2$ which depends on the central charge.

We now turn to the analysis of the properties of our algebra using dimensional reduction of the SUSY algebra in $6dM$ to $4dE$. 

7
4 Dimensional Reduction

Most often the SUSY algebra in $4dM$ is presented using either two-component notation or four-component Majorana spinors; however, one can also employ four-component Weyl spinors to this end. The six-dimensional analogue of this appears in [4]:

\[ \{Q, Q\} = 0 \] (12a)

\[ \{Q, \overline{Q}\} = \frac{1}{2} (1 + \Gamma_7) \Gamma^a P_a \] (12b)

\[ [M^{ab}, P^c] = i \left( g^{bc} P^a - g^{ac} P^b \right) \] (12c)

\[ [M^{ab}, M^{cd}] = i \left( g^{bc} M^{ad} - g^{ac} M^{bd} + g^{ad} M^{bc} - g^{bd} M^{ac} \right) \] (12d)

\[ [M^{ab}, Q] = -\frac{1}{2} \Sigma^{ab} Q. \] (12e)

Here, we have taken $Q$ to be an eight component Dirac spinor with $\overline{Q} = Q^+ \Gamma^0$.

The Dirac matrices are taken to be

\[
\begin{align*}
\Gamma^0 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \Gamma^i &= \begin{pmatrix} 0 & i \gamma^i \\ i \gamma^i & 0 \end{pmatrix} & (i = 1, 2, 3) \\
\Gamma^5 &= \begin{pmatrix} 0 & i \gamma_5 \\ i \gamma_5 & 0 \end{pmatrix} & \Gamma^6 &= \begin{pmatrix} 0 & i \gamma^0 \\ i \gamma^0 & 0 \end{pmatrix} \\
\Sigma^{ab} &= \frac{i}{2}[\Gamma^a, \Gamma^b] \\
\Gamma_7 &= \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^5 \Gamma^6 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]

where $\{\gamma^\mu, \gamma^\nu\} = 2 \delta^{\mu\nu}$ ($\mu, \nu = 1 \ldots 4$) and $\gamma_5^+ = \gamma_5$. 

8
In order to reduce the algebra of (12) to that of (1), we make the identification

\[ Q = \begin{pmatrix} 0 \\ g \end{pmatrix} \]  

where \( g \) is a four component spinor, and define,

\[ IP_0 = Z_+ \]  \hspace{1cm} (15a)

\[ IP_5 = Z_- \]  \hspace{1cm} (15b)

\[ (IP_1, IP_2, IP_3, IP_6) = P^\mu \]  \hspace{1cm} (15c)

\[ M^{ij} = -M^{ij} \quad (i, j = 1, 2, 3) \]  \hspace{1cm} (15d)

\[ M^{i4} = -M^{i6}. \]  \hspace{1cm} (15e)

It is easily verified that with these identifications, the algebra of (12) reduces to that of (1).

The role of the rotation operators in (12) merits consideration. We can define

\[ J^\mu = M^{\mu0} \]  \hspace{1cm} (16a)

\[ K^\mu = M^{\mu5} \]  \hspace{1cm} (16b)

\[ L = M^{05} \]  \hspace{1cm} (16c)

and obtain from (12) the non-zero commutators (which are consistent with those of (1), in the sense that all Jacobi identities are satisfied):

\[ [L, g] = \frac{i}{2} \gamma_5 g \]  \hspace{1cm} (17a)
\[
[L, Z_\pm] = iZ_\pm. \tag{17b}
\]
\[
[J^\mu, g] = -\frac{i}{2}\gamma^\mu g \tag{17c}
\]
\[
[K^\mu, g] = \frac{i}{2}\gamma^\mu\gamma_5 g \tag{17d}
\]
\[
[J^\mu, P^\nu] = -i\delta^{\mu\nu} Z_+ \tag{17e}
\]
\[
[K^\mu, P^\nu] = i\delta^{\mu\nu} Z_- \tag{17f}
\]
\[
[J^\mu, Z_+] = -iP^\mu \tag{17g}
\]
\[
[K^\mu, Z_-] = -iP^\mu \tag{17h}
\]
\[
[J^\mu, L] = iK^\mu \tag{17i}
\]
\[
[K^\mu, L] = iJ^\mu. \tag{17j}
\]
\[
[J^\mu, K^\nu] = i\delta^{\mu\nu} L \tag{17k}
\]
\[
[J^\mu, J^\nu] = iM^{\mu\nu} = -[K^\mu, K^\nu] \tag{17l}
\]

Together, (1) and (17) constitute an algebra which is an extension of the algebra presented in [1] by itself. It is evident that the \(N = 2\) SUSY algebra in 4\(dM\) can be extended in a similar fashion by dimensionally reducing the 6\(dM\) algebra of (12), as outlined in ref. [4].

It is also possible to perform a dimensional reduction of the \(N = 1\) SUSY algebra in 10\(dM\) to 4\(dM\) to obtain the \(N = 4\) SUSY algebra in 4\(dM\). It is likely that an extended SUSY algebra in 4\(dE\) can be generated by a similar dimensional reduction.

Dimensional reduction was primarily used in ref. [2-3] to generate a SSYM model with extended SUSY in 4\(dM\). It is interesting to note at this point that the SUSY model of
Zumino [5] can similarly be generated. One starts with the action in 6dM

\[ I = \int d^6x \left[ -\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \bar{\lambda} \Gamma^a \nabla_a \lambda \right] \]  

(18)

with

\[ \lambda = \frac{i}{2} (1 - \Gamma_7) \lambda \]

\[ \nabla_a = \partial_a \lambda + i [A_a, \lambda]. \]

To dimensionally reduce this action to 4dE, one employs the representation of the \( \Gamma^a \) given in (13), sets \( \partial_0 = \partial_5 = 0 \) and makes the identifications

\[
\begin{align*}
\lambda &= \begin{pmatrix} \psi \\ 0 \end{pmatrix} \\
\bar{\lambda} &= (0, -i \psi^+) \\
A_0 &= A \\
A_5 &= B
\end{align*}
\]

in order to produce the Zumino model. (One could presumably deduce a SSYM with extended supersymmetry in 4dE by a dimensional reduction of the 10dM version of the model of eq. (18).) We should keep in mind that the momentum of “physical” states in the Zumino model are bounded by the inequality of (10).

5 Discussion

In this paper we have considered two aspects of the SUSY algebra in 4dE introduced in [1]. First of all, the algebra has been written as a Clifford algebra. This has demonstrated
that the algebra is closely linked to that of $N = 2$ SUSY algebra in $4dM$, the principle
difference being the inversion of the roles of momentum and central change so that the
eigenvalue of the Casimir operator $P^2$ faces an upper bound determined by the central
charges. Secondly, we have generated both the SUSY algebra in $4dE$ and the Zumino model
of ref. [5] by dimensional reduction from $6dM$. This procedure has shown how additional
Bosonic operators $J^\mu$, $K^\mu$ and $L$ can be introduced, as in (17), thereby extending the algebra
of (1). This is unexpected in view of the theorems in ref. [7]. We also note that dimensional
reduction was also used in ref. [8] to study spinors in $4dE$.

6 Acknowledgement

The authors would like to thank each other’s home institutions for hospitality while this
work was being done. Financial support was provided by NSERC and Forbairt.

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