APPROXIMATION OF SOLUTIONS TO PARABOLIC LAMÉ TYPE OPERATORS IN CYLINDER DOMAINS AND CARLEMAN’S FORMULAS FOR THEM

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Abstract. Let \( s \in \mathbb{N}, T_1, T_2 \in \mathbb{R}, T_1 < T_2, \) and let \( \Omega, \omega \) be bounded domains in \( \mathbb{R}^n, n \geq 1 \) such that \( \omega \subset \Omega \) and the complement \( \Omega \setminus \omega \) have no non-empty compact components in \( \Omega \). We investigate the problem of approximation of solutions to parabolic Lamé type system from the Lebesgue class \( L^2(\omega \times (T_1, T_2)) \) in a cylinder domain \( \omega \times (T_1, T_2) \subset \mathbb{R}^{n+1} \) by more regular solutions in a bigger domain \( \Omega \times (T_1, T_2) \). As an application of the obtained approximation theorems we construct Carleman’s formulas for recovering solutions to these parabolic operators from the Sobolev class \( H^{2s,s}(\Omega \times (T_1, T_2)) \) via values the solutions on a part of the lateral surface of the cylinder and the corresponding them stress tensors.

Introduction

It was known from the middle of the XX-th century that the approximation theorems for solutions of homogeneous elliptic equations (and systems) are closely related to ill-posed problems for the corresponding elliptic operators, see, for instance, [1], [2] for the Laplace equation or [3] ch. 5–8, 10] for general elliptic operators with the uniqueness condition in small. Actually, the key role in the development of the approximation theory has the approach by C. Runge [4] for the uniform approximation of holomorphic functions on compact sets; see also [5] for (non-necessarily elliptic) operators with constant coefficients and [6] ch. 4, 5] for elliptic operators with sufficiently smooth coefficients). However, more delicate approximation theorems in various function spaces, where behaviour of the elements are controlled up to the boundary of the considered sets, appeared to be more important for applications, see, for instance, the pioneer papers by A.G. Vitushkin [7] and V.P. Havin [8] for analytic functions or the monograph [9] ch. 5–8] for Sobolev solutions to systems of differential equations with surjective/injective symbols.

Taking in account general suggestions by M.M. Lavrent’ev [9], S. Bergman [10], I.F. Krasichkov [11], a way to find systems with the double orthogonality property was indicated in the paper by L.A. Aizenberg and A.M. Kytmanov [12]. Combined with the approximation theorems and the integral representation method, this approach by [12] has lead to the construction of Carleman’s formulas for exact and approximate solutions to the Cauchy problem for holomorphic functions. This scheme was successfully adopted for the investigation of the Cauchy problem for a wide class of elliptic equations, see [3] ch. 10, 12], [13, 14, 15], or elliptic complexes, see [16].

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In the last decades, the area of the application of the ill-posed Cauchy type problems has expanded due to the theory of parabolic equations, see, for instance, L.A. Aizenberg in order to study the ill-posed Cauchy problem for the parabolic Lamé type operator $\mathcal{L}$ in $\mathbb{R}^{n+1},$

\begin{equation}
\mathcal{L} = \partial_t - \mu \Delta - (\mu + \lambda) \nabla \cdot \mathbf{A} \partial_j + A_0,
\end{equation}

where $\Delta$ is the Laplace operator in $\mathbb{R}^n$, $\nabla$ is the gradient operator, $\nabla \cdot$ is the divergence operator, $\mu$ and $\lambda$ are the Lamé constants, $\mu > 0$, $\lambda + \mu \geq 0$, and $A_j$ are $(n \times n)$-matrices over $\mathbb{R}$; actually, equations with the operator $\mathcal{L}$ can be considered as one of the linearisation of the evolution Navier-Stokes Equations, see, for instance, ch. 2, §1.

More precisely, according to [17], the ill-posed Cauchy problem for the operator $\mathcal{L}$ in a cylinder domain in $\mathbb{R}^{n+1}$ with the data on its lateral surface can be reduced to the problem of the extension of solutions to the operator $\mathcal{L}$ from a lesser cylinder domain to a bigger one. The last problem could be solved with the use of the systems with the double orthogonality property when considered in suitable Hilbert spaces. However, in [17] a solvability criterion for the Cauchy problem was obtained in the Hölder spaces and the solutions were constructed as formal power series, only.

In the present paper we obtain the solvability criterion in the anisotropic Sobolev spaces in terms systems with the double orthogonality property and, using them, we construct Carleman’s formulas for the exact and the approximate solution to the Cauchy problem cf. [19] for the heat operator. Since Aizenberg’s method relies on the completeness of the used doubly orthogonal system, we also prove the theorem on the approximation from the Lebesgue class $L^2((\omega \times (T_1, T_2)))$ in the cylinder domain $\omega \times (T_1, T_2) \subset \mathbb{R}^{n+1}$ by more regular solutions in a bigger domain $\Omega \times (T_1, T_2)$, cf. [20] for the heat operator.

1. Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with the coordinates $x = (x_1, \ldots, x_n)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain (open connected set). As usual, denote by $\overline{\Omega}$ the closure of $\Omega$, and by $\partial \Omega$ its boundary. We assume that $\partial \Omega$ is piece-wise smooth hypersurface. We denote also by $\Omega_{T_1, T_2}$ a bounded open cylinder $\Omega \times (T_1, T_2)$ in $\mathbb{R}^{n+1}$ with $T_1 < T_2$; for $\Omega \times (0, T)$ we write simply $\Omega_T$.

We consider functions over $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. As usual, for $s \in \mathbb{Z}_+$ we denote by $C^s(\Omega)$ the space of all $s$ times continuously differentiable functions on $\Omega$ and by $C^{s,\gamma}(\Omega)$ we denote the corresponding Hölder space with the power $\gamma \in (0, 1)$. Next, for $\gamma \in [0, 1)$ and relatively open set $S \subset \partial \Omega$, we denote by $C^{s,\gamma}(\Omega \cup S)$ the set of functions from $C^{s,\gamma}(\Omega)$ such that all their derivatives up to order $s$ extend continuously on $\Omega \cup S$ (the last space will be considered as a Banach space if $S = \partial \Omega$). Let $L^2(\Omega)$ be the Lebesgue space over $\Omega$ with the standard inner product $(u, v)_{L^2(\Omega)}$, and $H^s(\Omega)$, $s \in \mathbb{N}$, be the Sobolev space with the standard inner product $(u, v)_{H^s(\Omega)}$. Investigating spaces of solutions to the heat equation, we need the anisotropic Sobolev spaces $H^{2s, s}(\Omega_{T_1, T_2})$, $s \in \mathbb{Z}_+$, with the standard inner product and the anisotropic Hölder spaces $C^{2s, \gamma, \eta}(\Omega \cup S)_{T_1, T_2}$ (the last space will be considered as a Banach space if $S = \partial \Omega$). see, for instance, ch. 2, [21], [22].
Finally, for $k \in \mathbb{Z}_+$, we denote by $H^{k,2s,s}(\Omega_{I_1,T_2})$ the set of all functions $u \in H^{2s,s}(\Omega_{I_1,T_2})$ such that $\partial^\beta u \in H^{2s,s}(\Omega_{I_1,T_2})$ for all $|\beta| \leq k$. Similarly, one introduces that space $C^{k,2s,\gamma,s,\tau}(\Omega \cup S)_{I_1,T_2}$.

It is convenient to denote by $C^{\sigma,\gamma}(\Omega \cup S)$ the space of vector fields ($n$-vector functions) with components of the class $C^{\sigma,\gamma}(\Omega \cup S)$ and, similarly, for the spaces $L^2(\Omega)$, $H^s(\Omega)$, $H^{2s,s}(\Omega_{I_1,T_2})$, etc.

We will also use the so-called Bocher spaces of functions depending on $(x,t)$ from the strip $\mathbb{R}^n \times [T_1,T_2]$. Namely, for a Banach space $B$ (for example, the space of functions on a subdomain of $\mathbb{R}^n$) and $p \geq 1$, we denote by $L^p(I,B)$ the Banach space of all the measurable mappings $u : [T_1,T_2] \to B$ with the finite norm

$$\|u\|_{L^p(I,B)} := \|u(\cdot,t)\|_B \|_{L^p([T_1,T_2])},$$

see, for instance, [24], ch. § 1.2, [25], ch. III, § 1.

The space $C([T_1,T_2],B)$ is introduced with the use of the same scheme; this is the Banach space of all the continuous mappings $u : [T_1,T_2] \to B$ with the finite norm

$$\|u\|_{C([T_1,T_2],B)} := \sup_{t \in [T_1,T_2]} \|u(\cdot,t)\|_B.$$

Obviously, the steady differential Lamé type operator

$$L = -\mu \Delta - (\mu + \lambda) \nabla \text{div} + \sum_{m=1}^n A_m \partial_m + A_0$$

is strongly elliptic, if $\mu > 0$ and $\lambda + \mu \geq 0$. Hence the operator $\mathcal{L}$, given by formula (0.1.1), is strongly uniformly parabolic, see, for instance, [26], [27]; besides, as the coefficients of the operator are constant, it is well known, it admits a fundamental solution $\Phi(x,y,t,\tau)$ of the convolution type, i.e. there is a kernel $\Phi(x,t)$ that can be constructed with the use of the Fourier transform and such that $\Phi(x,y,t,\tau) = \Phi(x-y,t-\tau)$, see [26], Ch. 2, § 1.2]. In the simplest case where $A_m = 0$ for all $m = 0,1,\ldots,n$, the components $\Phi_{i,j}(x,t)$ of the kernel $\Phi(x,t)$ of the fundamental solution may be written as

$$\Phi_{i,j}(x,t) = \varphi(x,\mu t)\delta_{i,j} + \int_{\mu t}^{(2\mu+\lambda)t} \partial^2 \varphi(x,\zeta) \frac{\partial^2 \varphi(x,\zeta)}{\partial x_i \partial x_j} \, d\zeta, \quad 1 \leq i,j \leq n,$$

where $\delta_{i,j}$ is the Kronecker symbol and

$$\varphi(x,t) = \begin{cases} 
\frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x|^2}{2t^2}} & \text{if } t > 0, \\
0 & \text{if } t \leq 0,
\end{cases}$$

is the fundamental solution of the heat operator $\partial_t - \Delta$, see [26]. Formulas for the components of the fundamental solutions $\varphi(x,t)$ and $\Phi(x,t)$ imply immediately that they are smooth outside the diagonal $\{(x,t) = 0\}$ and real analytic with respect to the space variables. In particular, this means that the parabolic operator $\mathcal{L}$ is hypoelliptic. Besides, the fundamental solution allows to construct a useful integral Green formula for the operator $\mathcal{L}$. With this purpose, fix a $(n \times n)$-matrix differential first order operator $B_1$ with the principal symbol non-degenerate on the normal vectors $\nu(x) = (\nu_1(x),\ldots,\nu_n(x))$ to the surface $\partial \Omega$ at all the points $x \in \partial \Omega$. Denote by $\{C_0,C_1\}$ the Dirichlet pair associated with the Dirichlet pair $\{I_n,B_1\}$ via first Green formula for the operator $L$, i.e. $(n \times n)$-matrix differential
operators $C_j$ of order $j$, with the principal symbol non-degenerate on the normal vectors $\nu(x)$ and such that
\[
\int_{\partial \Omega} ((C_1v)^*u + (C_0v)^*B_1u) \, ds = (Lu, v)_{L^2(\Omega)} - (u, L^*v)_{L^2(\Omega)}
\]
for all $u, v \in C^\infty(\Omega)$, where $I_n$ is the unit $(n \times n)$-matrix, and $L^*$ is the formal adjoint operator for $L$. As the operator $B_1$ one usually takes the boundary stress tensor $\sigma$ with the components
\[
\sigma_{i,j} = \mu \delta_{i,j} \sum_{k=1}^n \nu_k \frac{\partial}{\partial x_k} + \mu \nu_j \frac{\partial}{\partial x_i} + \lambda \nu_i \frac{\partial}{\partial x_j}.
\]
Then, in the simplest case where $A = 0$, we obtain $C_0 = I_n$, $C_1 = \sigma$. Next, for vector functions $f \in L^2(\Omega_{T_1, T_2})$, $v \in L^2([T_1, T_2]), H^{1/2}(\partial \Omega))$, $w \in L^2([T_1, T_2]), H^{3/2}(\partial \Omega))$, $h \in H^{1/2}(\Omega)$, we consider parabolic potentials:
\[
I_{\Omega, T_1}(h)(x, t) = \int_{\Omega} \Phi(x - y, t)h(y, T_1) \, dy,
\]
\[
G_{\Omega, T_1}(f)(x, t) = \int_{T_1}^t \int_{\Omega} \Phi(x - y, t - \tau)f(y, \tau) \, dy \, d\tau,
\]
\[
V_{\partial \Omega, T_1}(v)(x, t) = \int_{T_1}^t \int_{\partial \Omega} (C_{0,y}^*\Phi^*(x - y, t - \tau))^*(x - y, t - \tau)v(y, \tau) \, dy \, d\tau,
\]
\[
W_{\partial \Omega, T_1}(w)(x, t) = -\int_{T_1}^t \int_{\partial \Omega} (C_{1,y}^*\Phi^*(x - y, t - \tau))^*w(y, \tau) \, dy \, d\tau.
\]
(see, for instance, [28] ch. 1, §3, ch. 5, §2]. By the definition, these are (improper) integrals, depending on the parameter $(x, t)$.

**Lemma 1.1.** Let $\partial \Omega \subset C^2$. For any $T_1 < T_2$ and all $u \in H^{2,1}(\Omega_{T_1, T_2})$ the following formula holds:
\[
\begin{cases}
  u(x, t) \text{ in } \Omega_{T_1, T_2} \\
  0 \text{ outside } \Omega_{T_1, T_2}
\end{cases}
= I_{\Omega, T_1}(u) + G_{\Omega, T_1}(Lu) + V_{\partial \Omega, T_1}(B_1u) + W_{\partial \Omega, T_1}(u).
\]

**Proof.** See, [29] ch. 6, §12] (and also [6] theorem 2.4.8] for more general operators, admitting fundamental solutions/parametresces).

Now, let $S_L(\Omega_{T_1, T_2})$ be the set of all the generalized $n$-vector functions on $\Omega_{T_1, T_2}$, satisfying (homogeneous) Lamé type equation
\[
Lu = 0 \text{ in } \Omega_{T_1, T_2}
\]
in the sense of distributions. First of all we note that the hypoellipticity of the operator $L$ means that all the solutions to equation (1.2) are infinitely differentiable on their domain, i.e.
\[
S_L(\Omega_{T_1, T_2}) \subset C^\infty(\Omega_{T_1, T_2}).
\]
As it is known, this is a closed subspace in the space $C^\infty(\Omega_{T_1, T_2})$ with the standard Fréchet topology (inducing the uniform convergence together with all the derivatives on compact subsets of $\Omega_{T_1, T_2}$).
Also, we need the space $S_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}})$, defined as the union of the spaces
\[ \cup_{G \supset \overline{\Omega_{T_1,T_2}}} S_{\mathcal{L}}(G), \]
where the union is with respect to all the domains $G \subset \mathbb{R}^{n+1}$, containing the closure of the domain $\Omega_{T_1,T_2}$.

Let $\mathbf{H}^{k,2s,s}_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}}) = \mathbf{H}^{k,2s,s}(\overline{\Omega_{T_1,T_2}}) \cap S_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}})$, $s \in \mathbb{Z}_+$, $k \in \mathbb{Z}_+$. As it is known, this is a closed subspace of the Sobolev space $\mathbf{H}^{k,2s,s}(\overline{\Omega_{T_1,T_2}})$. Similarly, $C^\infty_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}}) = C^\infty(\overline{\Omega_{T_1,T_2}}) \cap S_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}})$ is a closed subspace, consisting of solutions to equation (1.2), in the space $C^\infty(\overline{\Omega_{T_1,T_2}})$. It follows from the hypoellipticity of the operator $\mathcal{L}$ that the following (continuous) embeddings
\[ S_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}}) \subset C^\infty_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}}) \subset \mathbf{H}^{k,2s,s}_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}}) \]
are fulfilled for all $k, s \in \mathbb{Z}_+$.

2. An approximation theorem

In this section we discuss an approximation theorem for solutions to the operator $\mathcal{L}$. Actually, it is quite similar to the approximation theorems for elliptic operators mentioned in the introduction. Also, they are well known for the heat equation, see, for instance, [30] for the spaces with uniform convergence on compact sets or [20] for the Sobolev type spaces.

Theorem 2.1 (P.Yu. Vilkov, A.A. Shlapunov). Let $\omega \subset \Omega \subset \mathbb{R}^n$, $\partial \omega \in C^2$, $\partial \Omega \in C^1$. If the complement $\Omega \setminus \omega$ has no compact components in $\Omega$ then $S_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}})$ is everywhere dense in $L^2_{\mathcal{L}}(\omega_{T_1,T_2})$. Conversely, if $A = 0$, then the absence of the compact components of $\Omega \setminus \omega$ in $\Omega$ is also necessary for the density of $S_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}})$ in $L^2_{\mathcal{L}}(\omega_{T_1,T_2})$.

Proof. Sufficiency. Clearly, the set $S_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}})$ is everywhere dense in $L^2_{\mathcal{L}}(\omega_{T_1,T_2})$ if and only if the following relations
\[ (u, w)_{L^2(\omega_{T_1,T_2})} = 0 \]
for all $w \in S_{\mathcal{L}}(\overline{\Omega_{T_1,T_2}})$, means precisely for the field $u \in L^2_{\mathcal{L}}(\omega_{T_1,T_2})$ that $u \equiv 0$ in $\omega_{T_1,T_2}$.

Assume that the complement $\Omega \setminus \omega$ has no (non-empty connected) compact components in $\Omega$. In order to prove the sufficiency of the statement we will use the fact that the operator $\mathcal{L}$ admits the bilateral fundamental solution of the convolution type. By the definition,
\[ \mathcal{L}_{x,t} \Phi(x - y, t - \tau) = I_n \delta(x - y, t - \tau), \]
where $\delta(x,t)$ is the Dirac functional supported at the point $(x,t)$. Besides, the convolution type provides the normality property for the fundamental solution, i.e.
\[ \mathcal{L}^*_{y,\tau} \Phi^*(x - y, t - \tau) = I_n \delta(x - y, t - \tau). \]
We note that the normality property is not self-evident, though it can be provided under some not very restrictive assumptions, see [20, Property 2.2].

Let for the field $u \in L^2_{\mathcal{L}}(\omega_{T_1,T_2})$ relation (2.1) is fulfilled. Consider an auxiliary vector field
\[ v(y, \tau) = \int_{\omega_{T_1,T_2}} \Phi^*(x - y, t - \tau) u(x, t) dx dt. \]
According to (2.2) we have \( \mathcal{L}_x \Phi(x - y, t - \tau) = 0 \), if \((x, t) \neq (y, \tau)\). That is why \( \mathcal{L}_x \Phi(x - y, t - \tau) = 0 \) in \( \Omega_{T_1, T_2} \) for each fixed pair \((y, \tau) \notin \Omega_{T_1, T_2}\). Now, using the hypoellipticity of the operator \( \mathcal{L} \), we conclude that \( \Phi(x - y, t - \tau) \in S_{C}(\Omega_{T_1, T_2}) \) with respect to variables \((x, t) \in \Omega_{T_1, T_2}\) for each fixed pair \((y, \tau) \notin \Omega_{T_1, T_2}\). In particular, relations (2.5) imply

\[
(2.6) \quad v(y, \tau) = 0 \text{ in } \mathbb{R}^{n+1} \backslash \Omega_{T_1, T_2}.
\]

On the other hand, (2.3) yields

\[
(2.7) \quad L^* v = \chi_{\omega_{T_1, T_2}} u \text{ in } \mathbb{R}^{n+1},
\]

where \( \chi_{\omega_{T_1, T_2}} \) is the characteristic function of the domain \( \omega_{T_1, T_2} \). Obviously,

\[
L^*_{y, \tau} v(y, \tau) = 0 \text{ in } \mathbb{R}^{n+1} \backslash \omega_{T_1, T_2},
\]

and then, by the discussed above properties of the fundamental solution, the vector field \( v \) is real analytic with respect to the space variables of \( \mathbb{R}^{n+1} \backslash \omega_{T_1, T_2} \).

Since \( \omega \) has a smooth boundary, each component of \( \mathbb{R}^n \backslash \omega \) is itself a smooth domain and the similar conclusion is valid for the domain \( \Omega \). However the complement \( \mathbb{R}^n \backslash \omega \) has no compact components in \( \Omega \) and hence each component of \( \mathbb{R}^n \backslash \omega \) intersects with \( \mathbb{R}^n \backslash \Omega \) by a non-empty open set. Thus, (2.5) and the uniqueness theorem for the real analytic functions imply that

\[
(2.8) \quad v(y, \tau) = 0 \text{ in } \mathbb{R}^{n+1} \backslash \omega_{T_1, T_2}.
\]

In addition, (2.6), (2.7) mean that the field \( v \) is a solution to the Cauchy problem

\[
\begin{cases}
L^* v = \chi_{\omega_{T_1, T_2}} u \text{ in } \mathbb{R}^n \times (-T_2 - 1, 1 - T_1), \\
v(y, 1 - T_1) = 0 \text{ on } \mathbb{R}^n.
\end{cases}
\]

Taking in account the natural relation between parabolic and backward-parabolic operators and using arguments from [22] ch. 2, §5 theorem 3], we may conclude that \( v \in H^{2,1}(\mathbb{R}^n \times (-T_2 - 1, 1 - T_1)) \) and the solution is unique in this class. The regularity of this unique solution to the Cauchy problem can be expressed in term of the Bochner classes, too. Namely, \( v \in C([-T_2 - 1, 1 - T_1], H^1(\mathbb{R}^n)) \cap L^2([-T_2 - 1, 1 - T_1], H^2(\mathbb{R}^n)) \), see, for instance, [25] ch. 3, §1], where similar linear problems for Stokes’ equations are considered. In particular, the vector field \( v \) belongs to the space

\[
(2.9) \quad C([-T_1 - 1, T_2 + 1], H^1(\mathbb{R}^n)) \cap L^2([-T_1 - 1, T_2 + 1], H^2(\mathbb{R}^n)) \cap H^{2,1}(\mathbb{R}^n \times (T_1 - 1, T_2 + 1)).
\]

**Lemma 2.2.** Any vector field of type (2.1), satisfying (2.7), can be approximated by the fields from \( C^\infty(\omega_{T_1, T_2}) \) in the topology of the Hilbert space \( H^{2,1}(\omega_{T_1, T_2}) \).

**Proof.** Let \( \partial_v = \sum_{j=1}^n \nu_j(x) \partial_{x_j} \) be the derivative with respect to the exterior unit normal \( \nu(x) = (\nu_1(x), ..., \nu_n(x)) \) to the surface \( \partial \Omega \) at the point \( x \). If \( \partial \omega \) is a \( C^2 \)-smooth surface then, as the field \( v \) belongs to the space (2.5), we see that there are traces

\[
v|_{\partial(\omega_{T_1, T_2})} \in H^{1/2}(\partial(\omega_{T_1, T_2})),
\]

\[
v|_{(\partial \omega_{T_1, T_2})} \in L^2([T_1, T_2], H^{1/2}(\partial \omega)),
\]

\[
\partial_v v|_{(\partial \omega_{T_1, T_2})} \in L^2([T_1, T_2], H^{1/2}(\partial \omega)),
\]

cf. [31] ch. 3, §7, property 7]. Moreover, by (2.7), we have

\[
v = 0 \text{ on } \partial(\omega_{T_1, T_2}), \quad \partial_v v = 0 \text{ on } (\partial \omega)_{T_1, T_2}.
\]
Hence $v$ belongs to both the space $L^3$ and the space
\[(2.9) \quad C([T_1, T_2], H_0^1(\omega)) \cap L^2([T_1, T_2], H_0^2(\omega)) \cap H^2,1(\omega T_1, T_2),\]
where $H_0^1(\omega)$ is the closure in $H^1(\omega)$ of the space $C_0^\infty(\omega)$ consisting of infinitely smooth fields with compact supports in $\omega$. Now the statement of the lemma easily follows with the use of the standard regularisation, see, for instance, [31, ch. 3, §7,] for the isotropic Sobolev spaces.

□

Next, using lemma 2.2 and fixing a sequence $\{v_k\} \subset C_0^\infty(\omega T_1, T_2)$ converging to the field $v$ in $H^1,1(\omega T_1, T_2)$, we see that
\[\|u\|_{L^2(\omega T_1, T_2)} = \lim_{k \to +\infty} (u, L^*v_k)_{L^2(\omega T_1, T_2)} = 0,\]

because $L^*u = 0$ in $\omega T_1, T_2$ in the sense of distributions. Thus, $u \equiv 0$ in $\omega T_1, T_2$, that was to be proved.

\textbf{Necessity.} We use the arguments similar to that in [30] for the case of the uniform approximation of the solutions to the heat equation, cf. also [20] for the approximation in the Lebesgue space. Let the complement $\Omega \setminus \omega$ has at least one compact component. As we noted before, since the domains $\Omega, \omega$ has smooth boundaries, this component is the closure of a non-empty domain $\omega^{(0)}$. Moreover, the set $\omega \cup \omega^{(0)}$ is a domain with smooth boundary in $\mathbb{R}^n$.

Fix a point $(x_0, t_0) \in \omega^{(0)} \times (T_1, T_2)$.

\textbf{Lemma 2.3.} Let $A = 0$. Then for each $\delta > 0$ there is such a vector function $v_0 \in \mathcal{S}^2(\mathbb{R}^{n+1})$ that $v_0(x_0, t_0) \neq 0$ and $v_0(x, t) = 0$ for all $t$, $|t - t_0| \geq \delta$.

\textbf{Proof.} For the heat operator $(\partial_t - a\Delta)$ such a function $\hat{v}_0$ was constructed by A.N. Tikhonov [32]. Take a function $\hat{v}_0$, depending on $t$ and $x_1$, only, and related to $a = 2\mu + \lambda$, then the vector field $v_0$, with the first component $\hat{v}_0$ and all other components being zero, fits our requirements.

Next, there is an infinitely times differentiable function $\phi$ supported in $\omega$ such that $\phi(x_0) \equiv 1$ in a neighbourhood $U$ of the compact $\omega^{(0)}$. Then the vector field $v_1(x, t) = \phi(x)v_0(x, -t)$ is infinitely smooth in $\mathbb{R}^{n+1}$, supported in $\omega \times [t_0 - \delta, t_0 + \delta]$ and, moreover,
\[(2.10) \quad L^*v_1 = 0 \text{ in } U \times (T_1, T_2),\]

Denote by $\Pi_0$ the orthogonal projection from $L^2(\omega T_1, T_2)$ onto $L_2^2(\omega T_1, T_2)$.

Using properties of the projection $\Pi_0$, the function $v_1$ and the fundamental solution $\Phi$, we obtain for all $(y, \tau) \notin \partial \omega \times (T_1, T_2)$:
\[(2.11) \quad \int_{T_1}^{T_2} \int_{\omega \cup \omega^{(0)}} \Phi^*(x - y, t - \tau)(\Pi_0 L^*v_1)(x, y)dx \, dt = v_1(y, \tau).\]

Therefore the function $\Pi_0 L^*v_1 \in L_2^2(\omega T_1, T_2)$ is not $L^2(\omega T_1, T_2)$-orthogonal to the columns $\Phi_j(x - x_0, t - t_0) \in L_2^2(\omega T_1, T_2)$ of the matrix $\Phi(x - x_0, t - t_0)$, but it is
$L^2(\omega_{T_1,T_2})$-orthogonal to the vector fields $\Phi_j(x - y, t - \tau) \in L^2(\omega_{T_1,T_2})$ for any vectors $(y, \tau) \notin (\omega \cup \omega^{(0)}) \times (T_1, T_2)$.

If the function $u$ belongs to $S_L(\Omega_{T_1,T_2})$ then it belongs to $H^{2,1}_L(\Omega_{T_1,T_2})$ for some numbers $T'_1 < T_1 < T_2 < T'_2$ and a bounded domain $\Omega' \supseteq \Omega$. Now the Green formula yields

$$\begin{align*}
u(x,t) \text{ in } \Omega_{T'_1,T'_2}, \\
0 \text{ outside } \Omega_{T'_1,T'_2}
\end{align*}$$

According to (2.10) $L^* v_1 = 0$ in $\omega^{(0)} \times (T_1, T_2)$, and then Fubini theorem and formulas (2.11) for $(y, \tau) \in (\partial \Omega \times [T_1, T_2]) \cup \{(\Omega \times \{T'_1\})$ imply that

$$(\Pi_0 L^* v_1,u)_{L^2_2(\omega_{T_1,T_2})} = (L^* v_1, I_{\Omega',T'_1}(u) + V_{\partial \Omega',T'_1}(B_1 u) + W_{\partial \Omega',T'_1}(u))_{L^2_2(\omega_{T_1,T_2})} =$$

$$\int_{\Omega'} u^* (y) \left( \int_{T_1}^{T_2} \Phi^*(x - y, t - T'_1) (L^* v_1(x,y)) dx dt \right) dy +$$

$$\int_{T'_1}^{t} \int_{\partial \Omega'} (B_1 u)^* (y) C_{0,y} \left( \int_{T_1}^{T_2} \Phi^*(x - y, t - \tau) (L^* v_1(x,y)) dx dt \right) dy +$$

$$\int_{T'_1}^{t} \int_{\partial \Omega'} u^* (y) C_{1,y} \left( \int_{T_1}^{T_2} \Phi^*(x - y, t - \tau) (L^* v_1(x,y)) dx dt \right) dy = 0.$$
It was proved in [17] that Problem [3.1] has no more than one solution (if relative interior of $\Gamma$ is not empty), it is densely solvable (if the relative interior of $\partial \Omega \setminus \Gamma$ on $\partial \Omega$ is not empty) and it is ill-posed in the sense of Hadamard and Also a solvability criterion in Hölder space was obtained in [17] for Problem [3.1] but, unfortunately, there were no Hilbert space methods involved for investigation of solvability conditions and the construction of Carleman’s formulas for the problem.

We showed how to do the last two steps for a similar problem related to the heat equation in paper [19]; in this section we apply these ideas to the Lamé type operator. For this purpose we need to increase essentially the smoothness of the surface $\partial \Omega$ and the boundary data $u_1, u_2$. More precisely, we need the following lemma.

**Lemma 3.2.** Let $\gamma \in (0, 1), \partial \Omega \in C^{3+\gamma}$ and let $\Gamma$ be relatively open non-empty connected set on $\partial \Omega$ with boundary $\partial \Gamma \in C^{2+\gamma}$. If $u_1 \in C^{2, 1, \gamma/2}(\Gamma_T)$, $u_2 \in C^{2, 1, \gamma/2}(\Gamma_T)$ then there exist vector fields $\tilde{u}_j \in C^{2, 1, \gamma/2}(\partial \Omega_T)$ and $\tilde{u} \in C^{2, 1, \gamma/2}(\Omega_T)$ such that $\tilde{u}_j = u_j$ on $\Gamma_T$, $j = 1, 2$, and $\tilde{u} = \tilde{u}_1$ on $(\partial \Omega)_T$, $B_1 u = \tilde{u}_2$ on $(\partial \Omega)_T$.

**Proof.** It is quite similar to [19] lemma 4]: the only difference is that instead of the Dirichlet pair $(1, \partial_\nu)$ on $\partial \Omega$ and the bi-Laplacian $\Delta^2$ one has to consider the Dirichlet pair $(1, B_1)$, and the strongly elliptic operator $L^2$. \hfill $\square$

Under the assumptions of lemma [3.2] we set

$$
\tilde{F} = G_{0,0}(f) + V_{\partial \Omega,0}(\tilde{u}_2) + W_{\partial \Omega,0}(\tilde{u}_1) + I_{\Omega,0}(\tilde{u}).
$$

Assuming that $\Gamma$ is a relatively open connected set on $\partial \Omega$, we find an open set $\Omega^+ \subset \mathbb{R}^n$ such that the set $D = \Omega \cup \Gamma \cup \Omega^+$ is a domain with piece-wise smooth boundary; it is convenient to denote $\Omega^- = \Omega$. Then for a vector function $v$ on $D_T$ we denote by $v^+$ its restriction to $\Omega^+_T$ and, similarly, we denote by $v^-$ its restriction to $\Omega^-_T$. It is natural to denote the boundary values (traces) $v^\pm$ on $\Gamma_T$, if defined, by $v^\pm_{|\Gamma_T}$.

**Theorem 3.3** (Kurilenko I.A.). Let $\partial \Omega \in C^{3+\gamma}$, and let $\Gamma$ be a relatively open connected set on $\partial \Omega$ with the boundary $\partial \Gamma$ of the class $C^{2+\gamma}$, such that $\partial \Omega \setminus \Gamma$ have non-empty interior on $\partial \Omega$. If $f \in C^{0, 0, \gamma/2}(\Omega_T)$, $w_1 \in C^{2, 1, \gamma/2}(\Gamma_T)$, $u_2 \in C^{2, 1, \gamma/2}(\Gamma_T)$ then problem [3.1] is solvable in the space $C^{2, 1, \gamma/2}(\Omega_T) \cap C^{1, 0, \gamma/2}(\Omega_T \cup \Gamma_T) \cap H^{2, 1}(\Omega_T)$ if and only if there exists a vector field $\tilde{F} \in C^\infty(D_T) \cap H^{2, 1}(D_T) \cap S_{\lambda}(D_T)$, satisfying $\tilde{F} = \tilde{F}$ in $\Omega^+_T$.

**Proof.** We slightly modify the arguments from [17] theorem 5], where a solvability criterion for problem [3.1] was obtained in terms of the potential

$$
F(x, t) = G_{0,0}(f) + V_{\Gamma,0}(u_2) + W_{\Gamma,0}(u_1).
$$

Namely, it was proved in [17] theorem 5] that problem is solvable [3.1] if and only if there is a vector field $F \in C^{2, 1}(D_T) \cap S_{\lambda}(D_T)$ satisfying $F^+ = \tilde{F}^+$ in $\Omega^+_T$ and, moreover, if the (unique) solution $u$ exists then it is given by the following formula

$$
u_x = \chi_{\Omega^+_T} u(x, t) (x, t) \in \Omega_T,
$$

and the (unique) extension $F$ of the potential $F$, if exists, can be expressed via $u$ as

$$F(x, t) = F(x, t) - \chi_{\Omega^+_T} u(x, t) (x, t) \in D_T,$$
where \( \chi_{\Omega_T} \) is the characteristic function of the domain \( \Omega_T \). Unfortunately, the potential \( \mathcal{F} \) is not regular enough near the surface \( \partial \Gamma \). However, the potential \( \tilde{\mathcal{F}} \), defined with the use of lemma 3.2 helps to improve the situation.

More precisely, by Green formula (1.1), we obtain \( \tilde{\mathcal{F}} = G_{\Omega,0}(f - \mathcal{L}\tilde{u}) + \chi_{\Omega_T}\tilde{u} \). Since under the assumptions of the theorem, \( f, \mathcal{L}\tilde{u} \in C^{0,0,\gamma,\gamma/2}(\Omega_T) \), then the results [21] ch. 4, §§11-14, [28] ch. 1, §3 imply that
\[
G_{\Omega,0}(f - \mathcal{L}\tilde{u}) \in C^{2,1,\gamma,\gamma/2}(\Omega_T) \cap C^{1,0,\gamma,\gamma/2}(D_T),
\]
i.e. \( \tilde{\mathcal{F}} \in C^{2,1,\gamma,\gamma/2}(\Omega_T) \). On the other hand,
\[
\tilde{\mathcal{F}} - \mathcal{F} = V_{\Omega T \setminus \Gamma,0}(\tilde{u}_2) + W_{\Omega T \setminus \Gamma,0}(\tilde{u}_1) + I_{\Omega,0}(\tilde{u}).
\]

This means that the vector field \( \tilde{\mathcal{F}} - \mathcal{F} \) satisfies equation \( \mathcal{L}(\tilde{\mathcal{F}} - \mathcal{F}) = 0 \) in \( D_T \), and therefore the potential \( \mathcal{F} \) extends from \( \Omega_T^+ \) onto \( D_T \) as a solution to the operator \( \mathcal{L} \) if and only if the potential \( \tilde{\mathcal{F}} \) extends from \( \Omega_T^+ \) onto \( D_T \) as a solution to the operator \( \mathcal{L} \).

If problem 3.4 is solvable in the space \( C^{2,1,\gamma,\gamma/2}(\Omega_T) \cap C^{1,0,\gamma,\gamma/2}(\Omega_T \cup \Gamma_T) \cap H^{2,1}(\Omega_T) \) then formulas (3.3), (3.5) and (3.7) imply
\[
\tilde{\mathcal{F}} = \tilde{\mathcal{F}} - \chi_{\Omega_T}u \in H^{2,1}(\Omega_T^+) \text{ and } \mathcal{L}\tilde{\mathcal{F}} = 0 \text{ in } D_T.
\]

As \( \tilde{\mathcal{F}} \in H^{2,1}(\Omega_T^+) \cap C^\infty(D_T) \) (this follows, for instance, from [31] ch. VI, §1, theorem 1)), we conclude that \( \tilde{\mathcal{F}} \in H^{2,1}(D_T) \cap S_C(D_T) \).

If there is a vector field \( \tilde{\mathcal{F}} \in H^{2,1}(D_T) \cap S_C(D_T) \), coinciding with the potential \( \tilde{\mathcal{F}} \) in \( \Omega_T^+ \), then the potential \( \mathcal{F} \) extends from \( \Omega_T^+ \) onto \( D_T \) as a solution to the operator \( \mathcal{L} \), i.e. problem 3.4 is solvable.

Moreover, it follows from formulas (3.4) and (3.7) that in \( D_T \) we have
\[
U = \mathcal{F} - \mathcal{F} = \tilde{\mathcal{F}} - \tilde{\mathcal{F}} \in H^{2,1}(\Omega_T^+),
\]
and then \( U^- \) is the unique solution to problem 3.4 in the space \( C^{2,1,\gamma,\gamma/2}(\Omega_T) \cap C^{1,0,\gamma,\gamma/2}(\Omega_T \cup \Gamma_T) \cap H^{2,1}(\Omega_T) \) by [17] theorem 5.

On the other hand, theorem 2.4 allows to prove the existence of a basis with the double orthogonality property in the spaces of solutions to the operator \( \mathcal{L} \) that we will use in order to construct formulas for the precise and approximate solutions to problem 3.4.

**Corollary 3.4** (Kurilenko I.A.). Let \( s \in \mathbb{N} \), \( k \in \mathbb{Z}_+ \), \( \partial \Omega \subset C^1 \) and let \( \omega \) be a relatively compact subdomain in \( \Omega \subset \mathbb{R}^n \) such that \( \partial \omega \subset C^2 \) and the complement \( \Omega \setminus \omega \) have no compact components in \( \Omega \). Then there is an orthonormal basis \( \{b_\nu\} \) is the space \( H^{k,2s,\gamma}(\omega_{T_1,T_2}) \) such that its restriction \( \{b_\nu|_{\omega_{T_1,T_2}}\} \) to \( \omega_{T_1,T_2} \) is on orthonormal basis in the space \( L^2(x_{\omega_{T_1,T_2}}) \).

**Proof.** By the definition, the space \( H^{k,2s,\gamma}(\omega_{T_1,T_2}) \) is embedded continuously into the space \( L^2(x_{\omega_{T_1,T_2}}) \). We denote by \( R_{\Omega,\omega} \) the natural embedding operator
\[
R_{\Omega,\omega} : H^{k,2s,\gamma}(\omega_{T_1,T_2}) \rightarrow L^2(x_{\omega_{T_1,T_2}}).
\]
The analyticity of solutions to the operator \( \mathcal{L} \) with respect to the space variables implies that the operator \( R_{\Omega,\omega} \) is injective. Besides, it follows from theorem 2.4 that the range of the operator \( R_{\Omega,\omega} \) is everywhere dense in the space \( L^2(x_{\omega_{T_1,T_2}}) \).

By Fubini theorem, anisotropic Sobolev space \( H^{2,1}(\omega_{T_1,T_2}) \) is embedded continuously to the Bochner space \( \mathcal{B}(\{T_1,T_2, H^2(\Omega), L^2(\Omega)\}) \), consisting of mappings
v : [T_1, T_2] → H^2(Ω) such that ∂_tv ∈ L^2(Ω), see [24] ch. 1, §5. By Rellich-Kondrashov theorem the embedding H^2(Ω) → L^2(Ω) is compact. Using famous compact embedding theorem for the Bochner type spaces o (see, for example, [24, ch. 1, §5, theorem 5.1]), we conclude that the space \( E((T_1, T_2, H^2(Ω), L^2(Ω)) \) is embedded compactly into \( L^2(T_1, T_2, Ω) = L^2(Ω_{T_1, T_2}). \) Thus, the space \( H^{2,1}_L(Ω_{T_1, T_2}) \) is embedded compactly into \( L^2(Ω_{T_1, T_2}), \) and the into \( L^2(ω_{T_1, T_2}). \) Therefore the space \( H^{2,1}_L(Ω_{T_1, T_2}) \) is embedded to \( L^2(ω_{T_1, T_2}) \) compactly, too, i.e. the operator \( R_{Ω,ω} \) is compact.

Finally, [14 example 1.9] implies that the complete system of eigen-vectors of the compact self-adjoint operator \( R^*_{Ω,ω}R_{Ω,ω} : H^{k,2,s}_L(Ω_{T_1, T_2}) → H^{k,2,s}_L(Ω_{T_1, T_2}) \) is the basis looked for; here \( R^*_{Ω,ω} \) is the adjoint operator for \( R_{Ω,ω} \) in the sense of the Hilbert space theory.

Since in theorem 3.3 we have \( ∂Ω ∈ C^{3+γ} \) and \( ∂Γ ∈ C^{2+γ} \), then there is a bounded domain \( Ω^+ ⊂ R^n \), such that the domain \( D = Ω^+ \cup Γ \cup Ω \) has the boundary of the class \( C^1 \). Fix a relatively compact subdomain \( ω \in Ω^+ \) with \( ∂ω \in C^2 \) and a basis \( \{b_ν\} \) with the double orthogonality property granted by corollary 3.4 for the pair \( R^*_{Ω,ω} \) and \( R_{Ω,ω} \), and a basis \( \{b_ν\} \) in the space \( L^2_{2}(ω_{T}) \).

Let also \( c_ν(\tilde{F}) \) be the Fourier coefficients of the potential \( \tilde{F} \) with respect to the basis \( \{b_ν\} \) in the space \( L^2_{2}(ω_{T}) \):

\[
(3.9) \quad c_ν(\tilde{F}) = \left( \int_{ω^+} \tilde{F}^*(z, \tilde{τ})b_ν(z, \tilde{τ})dzd\tilde{τ} \right)/\|b_ν\|^2_{L^2_{2}(ω_{T})}.
\]

**Corollary 3.5** (Kurilenko I.A.). *Under assumptions of theorem 3.3, if \( ∂D \subset C^{1, 1}/ζ \) and \( ∂Ω, ∂Γ ∈ C^2 \), then problem (3.7) is solvable in the space \( C^{1, 1/2}(Ω_T)∩C^{1, 1/2}(Ω_T∪Γ) \) and H^{2,1}_L(Ω_T) if and only if the series \( ∑_{ν=1}^{∞} |c_ν(\tilde{F})|^2 \) converges. Besides, if exists, the solution to the problem is given by the following formula:

\[
(3.10) \quad u(x, t) = \lim_{N→+∞} \left( \int_{∂Ω} c_N^{(ω)}(x, y, t, τ)f(y, τ)dydτ + \int_{(∂Ω)_x} c_N^{(ω)}(x, y, t, τ)\tilde{u}_2(y, τ)ds(y)dτ + \int_{(∂Ω)_x} (B_1(y)c_N^{(ω)})^*(x, y, t, τ)\tilde{u}_1(y, τ)ds(y)dτ \right),
\]

Proof. As we already noted, \( H^{2,1}_L(Ω_T) ⊂ C^{∞}(Ω_T). \) Moreover, as the solutions to the operator \( L \) are real analytic with respect to the space variables, then the solvability conditions from theorem 3.3 are equivalent to the following: \( R_{D,ω}F = \tilde{F}. \) Thus the first statement of the corollary follows immediately from 3.3 and [14 example 1.9]. On the other hand, according to [14 example 1.9], if \( \tilde{F} ∈ H^{2,1}_L(D_T) \) is the extension of the vector field \( \tilde{F} \) from \( ω_{T} \) onto \( D_T \) then

\[
\tilde{F} = ∑_{ν=0}^{∞} c_ν(\tilde{F})b_ν(x, t), \quad (x, t) ∈ D_T.
\]
Hence (3.8) yields

\[
 u(x, t) = \lim_{N \to +\infty} \left( \tilde{F}(x, t) - \sum_{\nu=0}^{N} c_\nu(\tilde{F})b_\nu(x, t) \right), \quad (x, t) \in \Omega_T.
\]

Note that if \( y \in \Omega \) and \( x \in \omega \), then \( x \neq y \) and the components of the kernel \( \Phi(x - y, t - \tau) \) are integrable over \( \Omega_T \times \Omega_T \). Hence we may use integral formula (3.2) for \( \tilde{F} \) and Fubini theorem to change the order the integration in (3.9). Thus, (3.11) implies that formula (3.10) is true. \( \square \)

**Remark 3.1.** Note that the approximation theorem 2.1 and the results of this paper related to the ill-posed Cauchy problem 3.1 may be easily adopted to the backward parabolic operator \( L^* \), if we will use instead of Green formula 1.1 its backward parabolic analogue, including an integral on final data instead of the integral on the initial ones.

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