On discretization of the Euler top

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Abstract
Application of the intersection theory to construction of \( n \)-point finite-difference equations associated with classical integrable systems is discussed. As an example, we present a few new discretizations of motion of the Euler top sharing the integrals of motion with the continuous time system and the Poisson bracket up to the integer scaling factor.

1 Introduction
This paper deals with discretization equations of motion for one of the basic integrable systems, the three-dimensional Euler top, which describes motion of a free rigid body with a fixed point

\[
\begin{align*}
A\dot{p} + (C - B)qr &= 0, \\
B\dot{q} + (A - C)rp &= 0, \\
C\dot{r} + (B - A)pq &= 0.
\end{align*}
\]

(1.1)

Here \( \Omega = (p,q,r) \) is the vector of angular velocity in the coordinate system attached firmly to the body; axes of this system coincide with the principal axes of inertia, and numbers \( A,B,C > 0 \) are the corresponding principal moments of inertia.

In the world coordinate system angular momentum vector

\[ M = (Ap, Bq, Cr) = (M_1, M_2, M_3) \]

is fixed and its length in body coordinate system

\[ K = (M, M) = A^2p^2 + B^2q^2 + C^2r^2, \]

(1.2)

is a first integral of equations (1.1). Another first integral is a kinetic energy for the body

\[ T = \frac{1}{2} (\Omega, M) = \frac{1}{2} (Ap^2 + Bq^2 + Cr^2) = \frac{1}{2} \left( \frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C} \right). \]

(1.3)

The first discretization of the free \( n \)-dimensional top was constructed by Moser and Veselov [17, 29] by refactorization of matrix polynomials. This discretization is represented by an isospectral transformation

\[ L_{k+1} = A_k L_k A_k^{-1} \]

(1.4)

which does not explicitly involve a time step. Here \( L_k = M_k + \lambda J^2, \ A_k = \Omega_k + \lambda J, \) \( \lambda \) is a spectral parameter and \( M, \Omega, J \) are matrices associated with angular momentum, angular velocity and tensor of inertia, respectively. Detailed exposition of this integrable discretization may be found in the textbook [19].

In [3] authors discuss the following finite-difference equation

\[ \tilde{M}_i - M_i = \gamma \alpha_i (\tilde{M}_i + M_j)(\tilde{M}_k + M_k), \]

where \( \gamma \) is a function on phase space and \( (ijk) \) stands for any cyclic permutation of (123). This implicit map \( M \to \tilde{M} \) preserves first integrals of the continuous problem and Poisson structure only for the special choice of function \( \gamma \).
In [12] explicit discrete map
\[ \dot{M}_i - M_i = \delta_i (M_j M_k + M_j \dot{M}_k), \]
where \( \delta_i \) are some parameters, was constructed by applying the Hirota method. This map does not preserve first integrals of the equations (1.1), but has all the integrability attributes, i.e. there are two compatible invariant Poisson structures; two independent integrals of motion which are in involution with respect to any of the invariant Poisson brackets; a Lax representation; explicit solutions in terms of elliptic functions and so on, see [18].

In [6] discretization of the classical Euler top
\[ \dot{M}_i - M_i = \phi_i (M_1, M_2, M_3), \]
where \( \phi_i \) are some special functions was obtained by using the Poinsot model of motion. In this model conditions \( T = E \) (1.3) and \( K = k \) (1.2) define the inertia ellipsoid and it’s confocal ellipsoid, respectively. Path of the angular velocity vector in three dimensional space (polhode) is the intersection of these two ellipsoids, a closed quartic curve on the inertia ellipsoid. Point \( P \) on a polhode corresponds to some partial solution \( \Omega(t) \) of (1.1) at fixed time \( t \). According to [6] two points \( P \) and \( \tilde{P} \) on the polhode are related by transformations (1.5), which can be obtained using the Chasles theorem on tangent lines to a geodesic on a quadric.

Instead of the Chasles theorem on contact curves, we propose to use the intersection theory. Indeed, for a given integrable system we can consider solutions of equations of motion at \( t = t_1, t_2, \ldots, \) which form a set of points \( P_1, P_2, \ldots \) on a common level surface \( X \) of integrals of motion \( H_1, H_2, \ldots \). Finite-difference equation
\[ \mathcal{Y}(P_k - t, \ldots, P_k \ldots P_{k+m}; k) = 0, \quad k \in \mathbb{Z} \]
or \( \ell + m + 1 \)-point map [11] can be considered as a correspondence between the intersection points of surface \( X \) with an auxiliary curve \( Y \). If \( X \) is the algebraic surface, we can study standard configurations of intersection points and the corresponding finite-difference equations [10] in the framework of the classical intersection theory.

Because points \( P_1, P_2, \ldots \) are solutions of equations of motion with fixed values of the first integrals we may suppose that \( \ell + m + 1 \)-point maps (1.6) preserve the integrals of motion defining surface \( X \) and the Poisson brackets defining equations of motion. For the Euler top two point maps \( \ell + m = 1 \) sharing the integrals of motion with the continuous time system and the Poisson bracket are well-known, see [3] [6] [17] [18] [29] and references within. The main aim of this article is to discuss construction of finite-difference equations in the framework of the classical intersection theory and present a few finite-difference equations for the Euler top at \( \ell + m > 1 \) and two-point maps sharing original first integrals up to the integer scaling factor.

Intersection theory is at the heart of algebraic geometry and its history is rich and fascinating. To construct finite-difference equation for the Euler case, when generic level curves of \( T \) and \( K \) are elliptic curves, it will be enough to use Abel’s results and Clebsch’s geometric interpretation of these results based on algebraic theory of curves which can be found in the classical text books [2] [9]. Of course, this construction could also be described using modern mathematical language of intersection theory [5] [8] [10] [14].

2 The Euler top
Equations (1.1) in terms of angular momenta look like
\[
\begin{align*}
\dot{M}_1 &= \left( \frac{1}{B} - \frac{1}{C} \right) M_2 M_3 = (\alpha_2 - \alpha_3)M_2 M_3, \\
\dot{M}_2 &= \left( \frac{1}{C} - \frac{1}{A} \right) M_3 M_1 = (\alpha_3 - \alpha_1)M_1 M_3, \\
\dot{M}_3 &= \left( \frac{1}{A} - \frac{1}{B} \right) M_1 M_2 = (\alpha_1 - \alpha_2)M_1 M_2,
\end{align*}
\]
where \( \alpha_1 = A^{-1}, \quad \alpha_2 = B^{-1}, \quad \alpha_3 = C^{-1}. \)

These equations are Hamiltonian with respect to Poisson bracket

\[
\{ M_1, M_2 \} = M_3, \quad \{ M_2, M_3 \} = M_1, \quad \{ M_3, M_1 \} = M_2. \tag{2.8}
\]

and Hamilton function \( H = 2T. \) The bracket \( \{ 2.8 \} \) is degenerate and its Casimir polynomial coincides with the first integral \( K = (M, M) \) \( \{ 1.2 \}. \)

Let us express \( M_{1,2} \) via third component of momenta \( M_3 \) and values of the first integrals \( 2T = h \) and \( K = k: \)

\[
M_1 = \sqrt{\frac{(\alpha_3 - \alpha_2)M_3^2 + \alpha_2 k - h}{\alpha_2 - \alpha_1}}, \quad M_2 = \sqrt{\frac{(\alpha_1 - \alpha_3)M_3^2 - \alpha_1 k + h}{\alpha_2 - \alpha_1}}. \tag{2.9}
\]

Substituting these expressions into the third equation in \( 2.7 \) and taking \( M_3 = x \) one gets standard quadrature

\[
\int_{M_3} \frac{dx}{y} = 2t, \quad y^2 = \left( (\alpha_1 - \alpha_3)x^2 - \alpha_1 k + h \right) \left( (\alpha_3 - \alpha_2)x^2 + \alpha_2 k - h \right) \tag{2.10}
\]

on elliptic curve \( X: \)

\[
y^2 = f(x), \quad f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0. \tag{2.11}
\]

with coefficients

\[
a_4 = (\alpha_1 - \alpha_3)(\alpha_3 - \alpha_2), \quad a_3 = 0,
\]

\[
a_2 = (2\alpha_3 - \alpha_1 - \alpha_2)h + (2\alpha_1 \alpha_2 - \alpha_1 \alpha_3 - \alpha_2 \alpha_3)k, \quad a_1 = 0, \tag{2.12}
\]

\[
a_0 = (\alpha_1 k - h)(h - \alpha_2 k).
\]

Solution \( M_3(t) \) of the equation \( 2.10 \) and, therefore, solutions of the original equations \( 2.7 \) are expressed via Jacobi elliptic functions \( 13, \) see also \( 9, \) p. 101-103. This solution \( M_3(t) \) of \( 2.10 \) at \( t = t_i \) is point \( P_i \) on curve \( X \) with abscissa \( x_i = M_3(t_i) \) and ordinate \( y_i = M_3(t_i). \)

Let \( X \) be a smooth nonsingular algebraic curve on a projective plane over field \( k. \) Prime divisors are points on \( X \) denoted \( P_i = (x_i, y_i), \) and \( P_{\infty}, \) which is a point at infinity. Divisor

\[
D = \sum m_i P_i, \quad m_i \in \mathbb{Z}
\]

is a formal sum of prime divisors, and the degree of divisor \( D \) is a sum \( \deg D = \sum m_i \) of multiplicities of points in support of the divisor. Group of divisors \( \text{Div} X \) is an additive Abelian group under the formal addition rule

\[
\sum m_i P_i + \sum n_i P_i = \sum (m_i + n_i) P_i.
\]

To define a linear equivalence relation on divisors we can use the rational functions on \( X. \) Function \( f \) is a quotient of two polynomials; they are each zero only on a finite closed subset of codimension one in \( X, \) which is therefore the union of finitely many prime divisors. The difference of these two subsets define a principal divisor \( \text{div} f \) associated with function \( f. \) The subgroup of \( \text{Div} X \) consisting of the principal divisors is denoted by \( \text{Prin} X. \) So, two divisors \( D, D' \in \text{Div} X \) are linearly equivalent

\[
D \approx D'
\]

if their difference \( D - D' \) is principal divisor

\[
D - D' = \text{div}(g) \equiv 0 \mod \text{Prin} X,
\]
i.e. divisor of rational function $g$ on $X$. The Picard group of $X$ is quotient group

$$\text{Pic} X = \frac{\text{Div} X}{\text{Prin} X} = \frac{\text{Divisors defined over } k}{\text{Divisors of functions defined over } k}.$$ 

If $X$ is a “projective variety”, then the group $\text{Pic} X$ underlies a natural $k$-scheme (Picard scheme), which is a disjoint union of quasi-projective schemes, and the operations of multiplying and inverting are given by $k$-maps [14]. Then we can extend the theory of schemes to stacks and so on [5].

We can consider the finite-difference equation (1.6), associated with quadrature (2.10), as a corresponding between prime divisors on $X$

$$\mathcal{Y}(P_{k-1}, \ldots, P_{k+m}; k) = 0 \mod \text{Prin} X.$$ 

If we restrict ourselves by linear correspondences

$$\mathcal{Y}(P_{k-1}, \ldots, P_k, \ldots, P_{k+m}; k) = \sum_{i=k-\ell}^{k+m} m_i P_i = 0 \mod \text{Prin} X,$$

these equations can be identified with an intersection divisor of $X$ with some auxiliary curve $Y$

$$\text{div}(X \cdot Y) = 0.$$ 

It allows us to construct finite-difference equations (1.6) using well-known group operations, operations on schemes and stacks [5, 8, 14].

### 2.1 Examples of intersection divisors

Let us consider intersection of plane curve $X$ (2.11) with a parabola

$$Y : \quad y = \mathcal{P}(x), \quad \mathcal{P}(x) = b_2 x^2 + b_1 x + b_0$$

and the corresponding intersection divisor $\text{div}(X \cdot Y)$ of degree four, see [2], p.113 or [9], p.166. Following Abel we substitute $y = \mathcal{P}(x)$ into (2.11) and obtain the so-called Abel polynomial

$$\psi(x) = \mathcal{P}(x)^2 - f(x).$$

Divisor of this polynomial on $X$ coincides with $\text{div}(X \cdot Y)$, i.e. roots of this polynomial are abscissas of intersection points $P_1, P_2, P_3$ and $P_4$ forming support of the intersection divisor $\text{div}(X \cdot Y)$.

At $b_2 = \sqrt{a_4}$ one of the intersection points is $P_\infty$. In this case $\psi(x)$ is equal to

$$\psi(x) = (2b_1 b_2 - a_3)x^3 + (2b_0 b_2 + b_1^2 - a_2)x^2 + (2b_0 b_1 - a_1)x + b_0^2 - a_0$$

$$= (2b_1 b_2 - a_3)(x - x_1)(x - x_2)(x - x_3).$$

Equating coefficients of $\psi$ one gets relation between abscissas of the remaining points $P_1, P_2$ and $P_3$ in support of the intersection divisor

$$x_1 + x_2 + x_3 = -\frac{2b_0 b_2 + b_1^2 - a_2}{2b_1 b_2 - a_3}. \quad (2.13)$$

If $P_1 \neq P_2$, we can define parabola $Y$ using the Lagrange interpolation by any pair of points $(P_1, P_2)$, $(P_1, P_3)$ or $(P_2, P_3)$. For instance, taking the following pair of points $(P_1, P_2)$ one gets

$$\mathcal{P}(x) = b_2 x^2 + b_1 x + b_0 = \sqrt{a_4}(x - x_1)(x - x_2) + \frac{(x - x_2)y_1}{x_1 - x_2} + \frac{(x - x_1)y_2}{x_2 - x_1},$$

which allows us to determine $b_2, b_1, b_0$ as functions on $x_{1,2}$ and $y_{1,2}$. Substituting coefficients of $\mathcal{P}(x)$ into the equation (2.13) we obtain an explicit expression for abscissa $x_3$ as a function on coordinates $x_{1,2}$ and $y_{1,2}$

$$x_3 = -x_1 - x_2 + \phi(x_1, y_1, x_2, y_2), \quad \phi = -\frac{2b_0 b_2 + b_1^2 - a_2}{2b_1 b_2 - a_3} \quad (2.14)$$

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Then we can calculate ordinate $y_3 = P(x_3)$ of the third intersection point $P_3$.

If we have a double intersection point, for instance $P_1 = P_3$, then

$$x_2 = -2x_1 + \phi(x_1, y_1), \quad \phi = -\frac{2b_0b_2 + b_1^2 - a_2}{2b_1b_2 - a_3},$$

(2.15)

where function $\phi(x_1, y_1)$ is defined by $P(x)$ due to the Hermite interpolation

$$P(x) = b_2x^2 + b_1x + b_0 = \sqrt{a_4}(x - x_1)^2 + \frac{(x - x_1)(4a_4x_1^3 + 3a_3x_1^2 + 2a_2x_1 + a_1)}{2y_1} + y_1.$$

In modern terms, we consider two partitions of the intersection divisor

$div(X \cdot Y) = (P_1 + P_2) + P_3 + P_\infty$, \quad and \quad $div(X \cdot Y) = (2P_1) + P_2 + P_\infty$.

Here we use brackets (.) in order to visually separate a part of the intersection divisor which will be used for polynomial interpolation of auxiliary curve $Y$. Because

$$div(X \cdot Y) = 0,$$

these partitions can be rewritten as addition and doubling of prime divisors

$$-P_3 = P_1 + P_2, \quad -P_2 = 2P_1,$$

where inversion is $(x, y) \rightarrow (x, -y)$, see Figure 1.

![Figure 1: Intersection of curve $X$ with parabola $Y: y = \sqrt{a_4}x^2 + b_1x + b_0$](image)

**Figure 1**: Intersection of curve $X$ with parabola $Y: y = \sqrt{a_4}x^2 + b_1x + b_0$

At $b_2 \neq \sqrt{a_4}$ support of the intersection divisor consists of four points $P_1 \neq P_\infty$ up to multiplicity.

Let us consider the following partitions of this divisor

$div(X \cdot Y) = (P_1 + P_2 + P_3) + P_4$, \quad $div(X \cdot Y) = (2P_1 + P_2) + P_3$, \quad $div(X \cdot Y) = (3P_1) + P_2$,

(see Figure 2).

In the first case parabola $Y$ is defined by the Lagrange interpolation using three ordinary points $P_1, P_2$ and $P_3$. In the second and third cases parabola $Y$ is defined by the Hermite interpolation using either double and ordinary points $2P_1, P_2$ or one triple point $3P_1$, respectively.

In the first case abscissa of the fourth intersection point is

$$x_4 = -x_1 - x_2 - x_3 + \varphi(x_1, x_2, x_3, y_1, y_2, y_3), \quad \varphi = -\frac{a_3 - 2b_1b_2}{a_4 - b_2^2},$$

(2.16)

where function $\varphi$ is defined using coefficients of quadratic polynomial $P(x) = b_2x^2 + b_1x + b_0$

$$P(x) = \frac{(x - x_2)(x - x_3)y_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{(x - x_1)(x - x_3)y_2}{(x_2 - x_1)(x_2 - x_3)} + \frac{(x - x_1)(x - x_2)y_3}{(x_3 - x_1)(x_3 - x_2)}.$$

(2.17)
In the second case expression for the abscissa looks like

\[ x_3 = -2x_1 - x_2 + \varphi(x_1, x_2, y_1, y_2), \quad \varphi = -\frac{a_3 - 2b_1b_2}{a_4 - b_2^2}. \]

Here function \( \varphi \) is defined via coefficients of the same polynomial \( P(x) = b_2x^2 + b_1x + b_0 \) and Hermite interpolation formulae

\[ P(x) = \frac{(x - x_1)^2y_2 - (x - 2x_1 + x_2)(x - x_2)y_1}{(x_1 - x_2)^2} + \frac{(x - x_1)(x - x_2)(4a_4x_1^3 + 3a_3x_1^2 + 2a_2x_1 + a_1)}{2y_1(x_1 - x_2)}. \]

In the third case, when we consider tripling the prime divisor on \( X \)

\[ (x_2, y_2) = 3(x_1, y_1), \]

second abscissa is equal to

\[ x_2 = -3x_1 + \varphi(x_1, y_1), \quad \varphi = -\frac{a_3 - 2b_1b_2}{a_4 - b_2^2}, \quad (2.18) \]

where function \( \varphi \) is defined via coefficients of the polynomial

\[ P(x) = b_2x^2 + b_1x + b_0 = \frac{-(x - x_1)^2(4a_4x_1^3 + 3a_3x_1^2 + 2a_2x_1 + a_1)^2}{8y_1^2} \]

\[ + \frac{(x - x_1)(x(6a_4x_1^2 + 3a_3x_1 + a_2) - 2a_4x_1^3 + a_2x_1 + a_1)}{2y_1} + y_1. \quad (2.19) \]

In the generic cases, using intersection divisors of plane curve \( X \) with auxiliary curves

\[ Y : \quad y = b_Nx^N + b_{N-1}x^{N-1} + \cdots + b_0, \quad N = 1, 2, 3, \ldots \]

we can describe multiplication of the prime divisor on integer \( \mathcal{P}_1 = n\mathcal{P}_2 \), which is a key ingredient of the modern elliptic curve cryptography, and other configurations of the prime divisors.

### 2.2 Examples of finite-difference equations

Let us identify curve \( X \) with one of the common level curves of \( K \) and \( T \) and consider intersection divisor

\[ \text{div}(X \cdot Y) = (P_1 + P_2) + P_3 + P_\infty \]
of this curve with parabola $Y$, see Figure 1a. Substituting

$$
\begin{align*}
  x_1 &= M_3, \\
  y_1 &= (\alpha_2 - \alpha_1)M_1M_2, \\
  x_2 &= \tilde{M}_3, \\
  y_2 &= (\alpha_2 - \alpha_1)\tilde{M}_1\tilde{M}_2, \\
  x_3 &= \tilde{M}_3, \\
  y_3 &= (\alpha_2 - \alpha_1)\tilde{M}_1\tilde{M}_2,
\end{align*}
$$

where $M_j, \tilde{M}_j$ and $\tilde{M}_j$ are solutions of equations (2.7) at $t = t_1, t_2, t_3$, in (2.13) one gets a 3-point finite-difference equation

$$
M_3 + \tilde{M}_3 + \tilde{M}_3 = \frac{\sqrt{a_4} \left( M_3^2 - \tilde{M}_3^2 \right) + (\alpha_1 - \alpha_2) \left( M_1M_2 - \tilde{M}_1\tilde{M}_2 \right)}{2\sqrt{a_4} \left( M_3 - \tilde{M}_3 \right)} \tag{2.20}
$$

whereas expression $y_3 = -\mathcal{P}(x_3)$ gives rise to another finite-difference equation

$$(M_3 - \tilde{M}_3)(M_3 - \tilde{M}_3)(\tilde{M}_3 - \tilde{M}_3) = (a_2-a_1)((\tilde{M}_1\tilde{M}_2-\tilde{M}_1\tilde{M}_2)\tilde{M}_3 + (\tilde{M}_1\tilde{M}_2-\tilde{M}_1\tilde{M}_2)\tilde{M}_3 + (\tilde{M}_1\tilde{M}_2-\tilde{M}_1\tilde{M}_2)\tilde{M}_3),$$

which is compatible with (2.20). Here coefficients $a_1$ and $a_2$ of $X$ are given by (2.12). These equations explicitly define non-invertible multivalued map

$$
\begin{pmatrix}
  M_1, \tilde{M}_1 \\
  M_2, \tilde{M}_2 \\
  M_3, \tilde{M}_3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  \tilde{M}_1 \\
  \tilde{M}_2 \\
  \tilde{M}_3
\end{pmatrix},
$$

which preserves the first integrals and the Poisson bracket. To calculate Poisson brackets between $\tilde{M}_i$ we have to take into account that $h = 2T$ has nontrivial Poisson brackets with $M_i$ and $\tilde{M}_i$, see discussion in [3, 6].

If we put

$$
\begin{align*}
  x_1 &= M_3^{(k)} , \\
  y_1 &= (\alpha_2 - \alpha_1)M_1^{(k)}M_2^{(k)}, \\
  x_2 &= \lambda_k , \\
  y_2 &= \pm\sqrt{f(\lambda_k)} , \quad \lambda_k \in \mathbb{C}, \\
  x_3 &= M_3^{(k+1)} , \\
  y_3 &= (\alpha_2 - \alpha_1)M_1^{(k+1)}M_2^{(k+1)},
\end{align*}
$$

where $f(x)$ is a function on the phase space defined by (2.10, 2.12), into the same expressions (2.13) and $y_3 = -\mathcal{P}(x_3)$, one gets a recurrence chain of 2-point invertible maps

$$
\cdots \rightarrow \begin{pmatrix}
  M_1^{(k-1)} \\
  M_2^{(k-1)} \\
  M_3^{(k-1)}
\end{pmatrix}
\rightarrow \begin{pmatrix}
  M_1^{(k)} \\
  M_2^{(k)} \\
  M_3^{(k)}
\end{pmatrix}
\rightarrow \begin{pmatrix}
  M_1^{(k+1)} \\
  M_2^{(k+1)} \\
  M_3^{(k+1)}
\end{pmatrix}
\rightarrow \cdots
$$

depending on parameters of discretization $\lambda_1, \lambda_2, \ldots$. Similar to the Moser-Veselov correspondence [1.3] and the Fedorov discretization [1.5] these maps do not explicitly involve a time step $t_j - t_{j+1}$. Straightforward calculations allow us to prove that these 2-point maps preserve first integrals $K, T$ [1.2, 1.3] and the Poisson bracket (2.8).

Doubling of prime divisor

$$-P_2 = 2P_1$$

at

$$
\begin{align*}
  x_1 &= M_3, \\
  y_1 &= (\alpha_2 - \alpha_1)M_1M_2, \\
  x_2 &= \tilde{M}_3, \\
  y_2 &= (\alpha_2 - \alpha_1)\tilde{M}_1\tilde{M}_2
\end{align*}
$$
The proof is a straightforward calculation.

Proposition 1  Two point map \( M_i \rightarrow \tilde{M}_i \) [2.21] preserves the form of first integrals \( K, T \) [1.8, 1.9] and doubles the Poisson bracket [2.8], i.e.

\[
\{ M_i, M_j \} = \varepsilon_{ijk} M_k, \quad \Rightarrow \quad \{ \tilde{M}_i, \tilde{M}_j \} = 2\varepsilon_{ijk} \tilde{M}_k.
\]

Map \( M_j \rightarrow L_j = \tilde{M}_j/2 \) preserves the original Poisson bracket [2.8]

\[
\{ M_i, M_j \} = \varepsilon_{ijk} M_k, \quad \Rightarrow \quad \{ L_i, L_j \} = \varepsilon_{ijk} L_k
\]

and multiplies the first integrals on the scaling factor due to \( e \) homogeneity of integrals of motion

\[
H = \alpha_1 M_1^2 + \alpha_2 M_2^2 + \alpha_3 M_3^2 = 4 \left( \alpha_1 L_1^2 + \alpha_2 L_2^2 + \alpha_3 L_3^2 \right),
\]

\[
K = M_1^2 + M_2^2 + M_3^2 = 4 \left( L_1^2 + L_2^2 + L_3^2 \right).
\]

The proof is a straightforward calculation.

Similarly we can obtain more complicated finite-difference equations using intersection divisors with four points, see Figure 2. For instance, at

\[
x_1 = M_3, \quad y_1 = (\alpha_2 - \alpha_1) M_1 + \alpha_1 M_3, \quad x_2 = \tilde{M}_3, \quad y_2 = (\alpha_2 - \alpha_1) \tilde{M}_1 + \tilde{M}_3,
\]

\[
x_3 = \tilde{M}_3, \quad y_3 = (\alpha_2 - \alpha_1) \tilde{M}_3 + \tilde{M}_1, \quad x_4 = \tilde{M}_3, \quad y_4 = (\alpha_2 - \alpha_1) \tilde{M}_1 + \tilde{M}_3,
\]

equations [2.16], [2.17] give rise to the following finite-difference equation

\[
M_3 + \tilde{M}_3 + \tilde{M}_3 + \tilde{M}_3 = \varphi(M, \tilde{M}, \tilde{M}),
\]  

where function \( \varphi(M, \tilde{M}, \tilde{M}) = \varphi_1/\varphi_2 \) is given by

\[
\varphi_1 = 2(\alpha_1 - \alpha_2)^2 \left( M_1 M_2 (M_3 - \tilde{M}_3) + \tilde{M}_1 M_2 (M_3 - \tilde{M}_3) + \tilde{M}_1 \tilde{M}_2 (\tilde{M}_3 - M_3) \right)
\]

\[
\times \left( M_1 M_2 (M_3^2 - \tilde{M}_3^2) + \tilde{M}_1 M_2 (M_3^2 - \tilde{M}_3^2) + \tilde{M}_1 \tilde{M}_2 (\tilde{M}_3^2 - M_3^2) \right),
\]

and

\[
\varphi_2 = (\alpha_1 - \alpha_2)^2 \left( M_1 M_2 (M_3 - \tilde{M}_3) + \tilde{M}_1 M_2 (M_3 - \tilde{M}_3) + \tilde{M}_1 \tilde{M}_2 (\tilde{M}_3 - M_3) \right)
\]

\[
+ (\alpha_2 - \alpha_3)(\alpha_1 - \alpha_3)(M_3 - \tilde{M}_3)(\tilde{M}_3 - \tilde{M}_3)^2. \]

Straightforward calculations allow us to check that this 4-point non invertible map

\[
\begin{pmatrix}
M_1, \tilde{M}_1, \tilde{M}_1 \\
M_2, \tilde{M}_2, \tilde{M}_2 \\
M_3, \tilde{M}_3, \tilde{M}_3
\end{pmatrix} \rightarrow \begin{pmatrix}
\tilde{M}_1 \\
\tilde{M}_2 \\
\tilde{M}_3
\end{pmatrix}
\]
preserves first integrals $K, T$ and the Poisson bracket (2.8).

Substituting parameter $\lambda_k$ instead of variable $\tilde{M}_i$ into (2.22) one gets a 3-point map depending on parameter of discretization $\lambda_k$

$$\begin{pmatrix} M_1, \tilde{M}_1 \\ M_2, \tilde{M}_2 \\ M_3, \tilde{M}_3 \end{pmatrix} \underset{\lambda_k}{\rightarrow} \begin{pmatrix} \tilde{M}_1 \\ \tilde{M}_2 \\ \tilde{M}_3 \end{pmatrix},$$

If we take $x_2 = \lambda_1 k$ and $x_3 = \lambda_2 k$ instead of $x_2 = \tilde{M}_3$ and $x_3 = \tilde{M}_3$, we obtain a recurrence chain of 2-point invertible maps

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} \underset{\lambda_1k, \lambda_2k}{\rightarrow} \begin{pmatrix} \tilde{M}_1 \\ \tilde{M}_2 \\ \tilde{M}_3 \end{pmatrix},$$

depending on two parameters of discretization. These maps are algebraic multivalued mappings containing the following functions on phase space

$$\mu_{ik} = \pm \sqrt{a_4 \lambda_{ik}^4 + a_2 \lambda_{ik}^2 + a_0}, \quad \lambda_{ik} \in \mathbb{C},$$

where $a_4, a_2$ and $a_0$ are given by (2.12).

Let us come back to construction of single-valued maps and substitute

$$x_1 = M_3, \quad y_1 = (\alpha_2 - \alpha_1)M_1M_2, \quad x_2 = \tilde{M}_3, \quad y_2 = (\alpha_2 - \alpha_1)\tilde{M}_1\tilde{M}_2$$

into the expressions (2.18) and (2.19) for tripling a point on elliptic curve

$$-P_2 = 3P_1,$$

see Figure 2c. As a result one gets the following map $M_i \rightarrow \tilde{M}_i$:

$$\tilde{M}_1 = 3M_1 - \frac{4M_3^3}{D} \left(M_1^2(\beta_1 M_2^2 - \beta_3 M_3^2)^2 - \beta_2 M_2^2 M_3^2(\beta_1 M_1^2 + \beta_3 M_3^2)\right),$$

$$\tilde{M}_2 = -3M_2 + \frac{4M_3^3}{D} \left(M_2^2(\beta_1 M_2^2 - \beta_3 M_3^2)^2 - \beta_3 M_1^2 M_3^2(\beta_1 M_1^2 + \beta_2 M_2^2)\right),$$

$$\tilde{M}_3 = -3M_3 + \frac{4M_3^3}{D} \left(M_3^2(\beta_2 M_2^2 - \beta_3 M_3^2)^2 - \beta_1 M_1^2 M_2^2(\beta_3 M_1^2 + \beta_2 M_2^2)\right),$$

where

$$D = M_3^4(\beta_3 M_2^2 - \beta_2 M_3^2)^2 - 2\beta_1 M_1^2 M_2^2 M_3^2(\beta_3 M_1^2 + \beta_2 M_2^2) + \beta_1^2 M_1^4 M_2^4.$$

**Proposition 2** Two point map $M_i \rightarrow \tilde{M}_i$ preserves first integrals $K, T$ and changes the Poisson bracket (2.8) by the following rule

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \Rightarrow \quad \{\tilde{M}_i, \tilde{M}_j\} = 3\varepsilon_{ijk} \tilde{M}_k.$$ 

**Map** $M_j \rightarrow L_j = \tilde{M}_j/3$ preserves the original Poisson bracket (2.8)

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \Rightarrow \quad \{L_i, L_j\} = \varepsilon_{ijk} L_k$$

and the first integrals up to the integer scaling factor

$$H = \alpha_1 M_1^2 + \alpha_2 M_2^2 + \alpha_3 M_3^2 = 9(\alpha_1 L_1^2 + \alpha_2 L_2^2 + \alpha_3 L_3^2),$$

$$K = M_1^2 + M_2^2 + M_3^2 = 9(L_1^2 + L_2^2 + L_3^2).$$
The proof is a straightforward calculation.

So, equations (2.21) and (2.23) give rise to a recurrence chain of 2-point mappings

\[
\cdots \rightarrow N \left( \begin{array}{c}
M_1^{(k-1)} \\
M_2^{(k-1)} \\
M_3^{(k-1)}
\end{array} \right) \rightarrow N \left( \begin{array}{c}
M_1^{(k)} \\
M_2^{(k)} \\
M_3^{(k)}
\end{array} \right) \rightarrow N \left( \begin{array}{c}
M_1^{(k+1)} \\
M_2^{(k+1)} \\
M_3^{(k+1)}
\end{array} \right) \rightarrow \cdots
\]

with \( N = 2, 3 \), which can be considered as a counterpart of geometric progression. In contrast with 2-point algebraic maps from the Theorem 3.3 in [6] these maps change original Poisson bracket and, therefore, they are different from the known 2-point maps preserving original bracket.

Because points on plane curve \( X \) (2.11) are solutions of equation (2.10) at fixed time and, therefore, \( M_i^{(k)} = M_i(t_k) \) are proportional to the Jacobi elliptic functions \( sn(u_k), cn(u_k), dn(u_k) \), with \( u_k = \text{const} \cdot t_k \), this recurrence chain represents a set of additional theorems for elliptic functions.

3 Conclusion

In [3, 6, 17, 19, 29] various Lax matrices for the Euler top, their refactorizations and the underlying addition theorems for the elliptic functions were used to obtain integrable 2-point finite-difference equations associated with motion of the Euler top. Nowadays, refactorization in Poisson-Lie groups is viewed as one of the most universal mechanisms of integrability for integrable 2-point maps. In this note, we come back to the Abel construction of addition theorems in order to study \( n \)-point finite-difference equations sharing integrals of motion and Poisson bracket up to the integer scaling factor.

If a common level surface \( X \) of integrals of motion can be represented as a symmetric product of an algebraic curve (for instance, the spectral curve of the Lax matrix), we can apply both of these constructions to:

- construction of integrable discrete maps [11, 16, 17, 19, 29, 30];
- study of relations between different integrable systems [22, 23, 24];
- construction of new integrable systems [25, 26, 27, 28];
- integration by quadratures.

Let us explain the last item by an example of the Steklov-Lyapunov system. The corresponding Kirchhoff equations of motion were solved explicitly by Kötter [15] using separation of variables and reduction of equations of motion to quadratures which have the form of the Abel-Jacobi map associated with a genus two hyperelliptic curve. The main problem is that original phase variables are expressed in terms of separating variables in a complicated way and, therefore, we cannot use Kötter’s results for analysis of motion in practice, see discussion in [4, 7, 20].

In [21] we introduce other variables of separation associated with a simple polynomial bi-Hamiltonian structure for the Steklov-Lyapunov system. In this case original phase variables are easily expressed via variables of separation, which allows us to integrate by quadratures the nontrivial integrable generalization of the Steklov–Lyapunov system discovered by Rubanovsky. Using modern computer algebra systems we can prove that canonical transformation relating Kötter’s variables and "new" variables of separation is the doubling divisor operation on genus two hyperelliptic curves discussed in [28]. Consequently, sometimes we can apply divisor arithmetic to simplification of integration by quadratures.

Another reason to conduct these calculations is related to consideration of finite-difference equations (1.6) relating points on the common level surface \( X \) of first integrals, which can not be realized as a product of the plane algebraic curves. In this generic case when we do not know variables of separation or the Lax matrices, we can continue to study various configurations of points on algebraic surface \( X \) in the framework of the standard intersection theory [2, 3, 8, 10, 14].

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