AUTOMORPHISMS OF POINTLESS SURFACES
CONSTANTIN SHRAMOV AND VADIM VOLOGODSKY

Abstract. For a geometrically rational surface $X$ over a perfect field of characteristic different from 2 that contains all roots of 1, we show that either $X$ is birational to a product of a projective line and a conic, or the group of birational automorphisms of $X$ has bounded finite subgroups. As an auxiliary result, we show boundedness of finite subgroups in anisotropic linear algebraic groups of bounded rank and number of connected components. Also, we provide applications to Jordan property for groups of birational automorphisms.

CONTENTS

1. Introduction 1
2. Quotients 5
3. Tori 6
4. Linear algebraic groups 9
5. Varieties with zero irregularity 11
6. Severi–Brauer varieties 14
7. Conic bundles 20
8. Proof of the main theorem 24
9. Quadrics 28
10. Del Pezzo surfaces 30
11. Jordan property 32
References 33

1. Introduction

Groups of birational automorphisms of some varieties are rather difficult to understand. However, in many examples the structure of its finite subgroups is much more accessible. In particular, it is often the case that a variety is either birationally “nice” itself, or there are strong restrictions on finite subgroups of its birational automorphism group. An amazing example of such a behavior is the following result.

Theorem 1.1 ([BZ17a, Corollary 4.11]). Let $\mathbb{K}$ be a field of characteristic zero that contains all roots of 1. Let $C$ be a conic over $\mathbb{K}$. Assume that $C$ is not $\mathbb{K}$-rational, i.e., that $C(\mathbb{K}) = \emptyset$. Then every non-trivial element of finite order in $\text{Aut}(C)$ has order 2, and every finite subgroup of $\text{Aut}(C)$ has order at most 4.

We say that a group $\Gamma$ has bounded finite subgroups if there exists a constant $B = B(\Gamma)$ such that for any finite subgroup $G \subset \Gamma$ one has $|G| \leq B$. If this is not the case, we say that $\Gamma$ has unbounded finite subgroups.

A partial two-dimensional analog of Theorem 1.1 can be stated as follows.
Theorem 1.2 ([PS16b, Theorem 1.6]). Let $\mathbb{K}$ be a field of characteristic zero that contains all roots of 1, and $X$ be a geometrically rational surface over $\mathbb{K}$. Assume that $X$ is not $\mathbb{K}$-rational but has a $\mathbb{K}$-point. Then the group of birational automorphisms of $S$ has bounded finite subgroups.

The goal of this paper is to prove the following assertion that can be interpreted as a two-dimensional generalization of Theorem 1.1 and that applies both to surfaces with $\mathbb{K}$-points and without them, and holds over all perfect fields of characteristic different from 2. We will say that a field $\mathbb{K}$ contains all roots of 1, if it contains all roots of 1 available in its algebraic closure, that is, those of degrees coprime to the characteristic of $\mathbb{K}$.

Theorem 1.3. Let $\mathbb{K}$ be a perfect field of characteristic different from 2 that contains all roots of 1. Let $X$ be a geometrically rational surface over $\mathbb{K}$. Then the group of birational automorphisms of $X$ has unbounded finite subgroups if and only if $X$ is $\mathbb{K}$-birational to $\mathbb{P}^1 \times C$, where $C$ is a conic.

Using Theorem 1.3 in the case when the surface $X$ has a $\mathbb{K}$-point, one proves a generalization of Theorem 1.2 over perfect fields of characteristic different from 2; see Remark 8.2 below for details. We point out that such a generalization does not hold in general over perfect fields of characteristic 2, see Example 8.6. As another application of Theorem 1.3 for surfaces without $\mathbb{K}$-points, we obtain the following result concerning groups of birational automorphisms of Severi–Brauer surfaces.

Corollary 1.4. Let $\mathbb{K}$ be a perfect field of characteristic different from 2 that contains all roots of 1. Let $X$ be a Severi–Brauer surface over $\mathbb{K}$ without $\mathbb{K}$-points. Then the group of birational automorphisms of $X$ has bounded finite subgroups.

The key step in the proof of Theorem 1.3 is the following property of linear algebraic groups that holds in arbitrary dimension. Recall that a linear algebraic group is said to be anisotropic if it does not contain a subgroup isomorphic to $\mathbb{G}_m$.

Theorem 1.5. Let $r$ and $n$ be positive integers. Then there exists a constant $L = L(r, n)$ with the following property. Let $\mathbb{K}$ be a field that contains all roots of 1, and let $\Gamma$ be an anisotropic linear algebraic group over $\mathbb{K}$ such that the number of connected components of $\Gamma$ is at most $r$ and the rank of $\Gamma$ is at most $n$. Let $G$ be a finite subgroup of $\Gamma(\mathbb{K})$. The following assertions hold.

(i) If $\Gamma$ is reductive and $\mathbb{K}$ is perfect, then $|G| \leq L$.

(ii) Suppose that $\Gamma$ is an arbitrary linear algebraic group. If $\text{char}\, \mathbb{K} > 0$, denote by $|G'|$ the largest factor of $|G|$ which is coprime to $\text{char}\, \mathbb{K}$; otherwise put $|G'| = |G|$. Then $|G'| \leq L$.

As an application of Theorem 1.5 we obtain the following result. Recall that the irregularity of a smooth projective variety $X$ is defined as $h^1(\mathcal{O}_X)$. Abusing the terminology, we will define the irregularity of singular projective varieties in the same way.

Theorem 1.6. Let $\mathbb{K}$ be a perfect field that contains all roots of 1. Let $S$ be a variety over $\mathbb{K}$, and let $\pi : \mathcal{X} \to S$ be a projective morphism with reduced and geometrically irreducible fibers over closed points of $S$, such that the irregularity of a fiber over every closed point of $S$ equals 0. Then there exists a constant $B = B(\mathcal{X})$ with the following property. Suppose that a fiber $\mathcal{X}_s$, $s \in S(\mathbb{K})$, is not birational to $\mathbb{P}^1 \times Y$ for any variety $Y$ over $\mathbb{K}$. Then any finite subgroup of $\text{Aut}(\mathcal{X}_s)$ has order at most $B$. 

2
Example 1.7. Let \( K \) be a field of characteristic zero. For every positive integer \( n \) and every \( \varepsilon > 0 \), Fano varieties of dimension \( n \) over \( K \) with \( \varepsilon \)-log canonical singularities are bounded by [Bir16, Theorem 1.1]. Moreover, one can check that some multiple of the relative anticanonical class provides a line bundle on the corresponding family that is ample over the base; this follows for instance from [Kol93, Theorem 1.1 and Lemma 1.2]. Therefore, Theorem 1.6 applies to such varieties provided that \( K \) contains all roots of 1.

A particular case of Theorem 1.3 for automorphism groups of certain flag varieties was considered in [G18].

One can apply Theorem 1.5 to study the automorphisms groups of Severi–Brauer varieties. For a Severi–Brauer variety \( X \) associated to a central simple algebra \( A \) over a perfect field \( K \) that contains all roots of 1, Theorem 1.5 implies that \( \text{Aut}(X) \) has bounded finite subgroups if and only if \( A \) is a division algebra; see Remark 6.7 for details. The following proposition (which we prove directly) amplifies this observation and partially extends it to the case of non-perfect base fields.

Proposition 1.8. Let \( K \) be a field that contains all roots of 1. Let \( X \) be a Severi–Brauer variety of dimension \( n - 1 \) over \( K \), and let \( A \) be the corresponding central simple algebra. Assume that the characteristic \( \text{char} \ K \) of \( K \) does not divide \( n \). The following assertions hold.

(i) The group \( \text{Aut}(X) \) has bounded finite subgroups if and only if \( A \) is a division algebra; in particular, if \( n \) is a prime number, then \( \text{Aut}(X) \) has bounded finite subgroups if and only if \( X(K) = \emptyset \), i.e., \( X \) is not isomorphic to \( \mathbb{P}^{n-1} \).

(ii) Suppose that \( A \) is a division algebra. Let \( g \in \text{Aut}(X) \) be an element of finite order, and \( G \subset \text{Aut}(X) \) be a finite subgroup. Then \( g^n = 1 \) and \( |G| \leq n^{2(n-1)} \).

(iii) Suppose that \( \text{char} \ K = 0 \), \( n = 3 \), and \( X(K) = \emptyset \). Let \( G \subset \text{Aut}(X) \) be a finite subgroup. Then \( |G| \leq 27 \).

In particular, if \( K \) is a perfect field that contains all roots of 1, then Proposition 1.8 applies to Severi–Brauer varieties over \( K \); indeed, in this case \( \text{char} \ K \) cannot divide the dimension of a division algebra over \( K \), see Remark 6.6 below. In the case of an arbitrary field whose characteristic divides the dimension of the division algebra, the structure of finite subgroups of the corresponding automorphism group is still rather simple.

Proposition 1.9. Let \( K \) be a field of characteristic \( p > 0 \) that contains all roots of 1. Let \( A \) be a division algebra over \( K \) of dimension \( n^2 \), and let \( X \) be a corresponding Severi–Brauer variety. Write \( n = n'p^m \) for some non-negative integers \( m \) and \( n' \) such that \( n' \) is coprime to \( p \). Then every finite subgroup \( G \subset \text{Aut}(X) \) is a semi-direct product of its normal subgroup \( H \) whose order is coprime to \( p \) and is less than or equal to \( n'^{2(n-1)} \), and an abelian \( p \)-group of exponent less than or equal to \( p^m \).

As another application of Theorem 1.5, one can prove that the automorphism group of a smooth quadric \( Q \) over a field \( K \) of characteristic different from 2 that contains all roots of 1 has bounded finite subgroups if and only if \( Q(K) = \emptyset \). In the case when \( K \) has zero characteristic, this result was proved earlier in [BZ17a, §4]. We find explicit bounds for orders of finite automorphism groups of quadrics over appropriate fields, thus generalizing the results of [BZ17a, §4] and making them more precise.
Proposition 1.10. Let $K$ be a field that contains all roots of 1. Assume that $\text{char } K \neq 2$ or $K$ is perfect. Let $n \geq 3$ be an integer, and let $Q \subset \mathbb{P}^{n-1}$ be a smooth quadric hypersurface over $K$. The following assertions hold.

(i) The group $\text{Aut}(Q)$ has bounded finite subgroups if and only if $Q(K) = \emptyset$.

(ii) If $n$ is odd and $Q(K) = \emptyset$, then every non-trivial element of finite order in the group $\text{Aut}(Q)$ has order 2, and every finite subgroup of $\text{Aut}(Q)$ has order at most $2^{n-1}$.

(iii) If $n$ is even and $Q(K) = \emptyset$, then every non-trivial element of finite order in the group $\text{Aut}(Q)$ has order 2 or 4, and every finite subgroup of $\text{Aut}(Q)$ has order at most $8^{n-1}$.

(iv) If $n = 4$ and $Q(K) = \emptyset$, then every finite subgroup of $\text{Aut}(Q)$ has order at most 64.

In particular, we see from Proposition 1.8(ii) or Proposition 1.10(ii) that Theorem 1.1 actually holds over any perfect field, and also over any field of characteristic different from 2 (we will use this in the proof of our Theorem 1.3). Note that over a non-perfect field of characteristic 2 this is not the case: for every smooth conic over any infinite field of characteristic 2 its automorphisms group has unbounded finite subgroups; see Lemma 6.12(ii) below for details. Note also that one cannot drop the assumption about roots of 1 in Theorem 1.1, as well as in Propositions 1.8 and 1.10. Indeed, the conic over the field of real numbers defined by the equation $x^2 + y^2 + z^2 = 0$ has automorphisms of arbitrary finite order.

Next, we provide explicit bounds for orders of finite subgroups in the automorphism groups of smooth del Pezzo surfaces.

Proposition 1.11. Let $K$ be a perfect field that contains all roots of 1. Let $X$ be a smooth del Pezzo surface over $K$ such that $X$ is not birational to $\mathbb{P}^1 \times C$, where $C$ is a conic. Suppose that either $K^2_X \geq 6$, or $K$ has characteristic 0. Then every finite subgroup of $\text{Aut}(X)$ has order at most 432.

We remind the reader that there are well-known bounds for orders of automorphism groups of del Pezzo surfaces $X$ with $K^2_X \leq 5$, see Remark 10.4 below.

Similarly to [BZ17a], where Theorem 1.1 was applied to study groups of birational self-maps of conic bundles over non-uniruled varieties, we will apply Theorem 1.3 to higher dimensional varieties whose maximal rationally connected fibration has relative dimension 2, see Proposition 11.6 below.

The plan of the paper is as follows. In §2 we collect some facts about quotients of varieties by the groups $\mathbb{G}_m$ and $\mathbb{G}_a$. In §3 we prove auxiliary assertions about elements of finite order in algebraic tori. In §4 we study finite subgroups of linear algebraic groups and prove Theorem 1.5. In §5 we discuss automorphism groups of varieties with zero irregularity and prove Theorem 1.6. In §6 we describe automorphism groups of Severi–Brauer varieties and prove Propositions 1.8 and 1.9. In §7 we study groups acting on conic bundles. In §8 we prove Theorem 1.3 and Corollary 1.4. We also provide a counterexample to Theorem 1.3 over a perfect field of characteristic 2. In §9 we prove Proposition 1.10. In §10 we study groups acting on del Pezzo surfaces and prove Proposition 1.11. In §11 we derive some consequences of Theorem 1.3 for birational automorphism groups of higher dimensional varieties.

Throughout the paper by $\overline{K}$ we denote an algebraic closure of a field $K$, and by $K_{\text{sep}}$ we denote a separable closure of $K$ (recall that $K_{\text{sep}} = \overline{K}$ provided that $K$ is perfect; we prefer
to use the notation $\overline{K}$ in this case). Given a variety $X$ or a morphism $\phi$ defined over $K$, for an arbitrary field extension $K \supset K$ we denote by $X_K$ and $\phi_K$ the corresponding scalar extensions to $K$, and by $X(K)$ we denote the set of $K$-points of $X$. Abusing notation a bit, we write $P^n$ for a projective space over a field $K$, and similarly write $G_m$ and $G_a$ for the multiplicative and additive groups, respectively.

We are grateful to I. Cheltsov, J.-L. Colliot-Thélène, A. Duncan, S. Gorchinsky, A. Kuznetsov, Ch. Liedtke, V. Popov, Yu. Prokhorov, D. Timashev, and A. Trepalin for useful discussions. Constantin Shramov was partially supported by the Russian Academic Excellence Project “5-100” and by Young Russian Mathematics award. Vadim Vologodsky was partially supported by the Laboratory of Mirror Symmetry NRU HSE, RF government grant, ag. № 14.641.31.0001.

2. Quotients

In this section we collect several (well-known) facts about quotients of varieties by the groups $G_m$ and $G_a$.

**Lemma 2.1.** Let $K$ be an arbitrary field. Let $G$ be an algebraic group over $K$ isomorphic either to $G_m$ or to $G_a$. Let $X$ be an irreducible variety over $K$ with a faithful action of $G$. Then $X$ is birational to $\mathbb{P}^1 \times Y$ for some variety $Y$.

**Proof.** The argument is contained in the proof of [BB73, Theorem 2], but we provide it here for the reader’s convenience. It follows from [Ros56, Theorem 2] that there exists a (dense) open subset $U$ of $X$ such that the action of $G$ on $U$ is free and there is a variety $Y = U/G$ and a morphism $\pi: U \to Y$ such that the fibers of $\pi$ are exactly $G$-orbits contained in $U$. By [Ros56, Theorem 10], the morphism $\pi$ has a section (see also [Ros67] for a different approach to a proof of this statement). This implies that $U \cong G \times Y$, and the assertion follows.

**Corollary 2.2.** Let $X$ be an irreducible variety over a perfect field with a faithful action of a connected linear algebraic group $\Gamma$. Suppose that $\Gamma$ is not reductive. Then $X$ is birational to $\mathbb{P}^1 \times Y$ for some variety $Y$.

**Proof.** Since $\Gamma$ is not reductive, its unipotent radical (that is, the largest normal unipotent connected closed subgroup of $\Gamma$) is non-trivial. Hence, it contains a subgroup isomorphic to $G_a$; this follows from the Lie–Kolchin theorem (see e.g. [Bor91, Corollary 10.5]). Now the assertion follows from Lemma 2.1.

**Lemma 2.3.** Let $T$ be a torus of dimension $n$ over a field $K$, and $S$ be its subtorus of dimension $n - 1$. Let $U$ be a $T$-torsor, and let $Y$ be the (geometric) quotient of $U$ by $S$. Suppose that the variety $Y$ is isomorphic to $G_m$. Then $U$ is birational to $G_m \times Z$ for some variety $Z$.

**Proof.** Note that $Y$ has a natural structure of a torsor under the torus $T/S$. Since $Y$ has a point, it is isomorphic to $T/S$. Therefore, $T/S$ is isomorphic to $G_m$ as a variety, and thus also as an algebraic group. We conclude that $T$ contains a subtorus $\tilde{R} \cong G_m$; this can be seen from the Galois action on the lattice $\tilde{T} \cong \text{Hom}(G_m, T_{\overline{K}_{\text{sep}}})$, see [Bor91, §8.12]. The rest follows from Lemma 2.1.

Note that in the notation of Lemma 2.3 and its proof the natural rational map

$$U \dashrightarrow G_m \times Z \to G_m$$
may disagree with the projection from $U$ to $Y \cong \mathbb{G}_m$.

**Example 2.4.** Put $\hat{T} = \mathbb{Z}^2$ and $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Let the non-trivial element of $\Gamma$ act on $\hat{T}$ by interchanging two vectors of its basis, so that $\Gamma$ becomes a subgroup of $\text{GL}_2(\mathbb{Z})$. Suppose that $T$ is a two-dimensional torus over a field $\mathbb{K}$ such that

$$\hat{T} \cong \text{Hom}(\mathbb{G}_m, T_{\text{sep}})$$

as a $\text{Gal}(\mathbb{K}_{\text{sep}}/\mathbb{K})$-module, where the image of the group $\text{Gal}(\mathbb{K}_{\text{sep}}/\mathbb{K})$ in $\text{Aut}(\hat{T}) \cong \text{GL}_2(\mathbb{Z})$ is the group $\Gamma$. In the notation of [Vos65] the torus $T$ has type $2c$. There exists a basis $\{v_+, v_-\}$ of $\hat{T} \otimes \mathbb{Q}$ such that $v_+$ is invariant and $v_-$ is anti-invariant with respect to $\Gamma$. Thus the torus $T$ contains a subtorus $R$ isomorphic to $\mathbb{G}_m$, and also a one-dimensional subtorus $S$ not isomorphic to $\mathbb{G}_m$ corresponding to the vectors $v_+$ and $v_-$, respectively (see [Bor91, §8.12]). One has $T/S \cong \mathbb{G}_m$, so that $T$ is birational to $\mathbb{G}_m \times S$ by Lemma 2.3.

Moreover, one has $T \setminus R \cong \mathbb{G}_m \times (S \setminus \{0\})$.

Note however that $R \subset T$ is not a section of the quotient morphism $\pi: T \to \mathbb{G}_m$, but the restriction of $\pi$ to $R$ is a double cover.

A straightforward application of Lemma 2.3 gives the following.

**Corollary 2.5.** Let $T$ be a torus of dimension $n$ over a field $\mathbb{K}$. Let $U$ be a $T$-torsor, and let $\pi: U \to \mathbb{G}_m$ be a morphism of varieties. Suppose that there is a subtorus $\bar{S}$ in $T_{\bar{\mathbb{K}}}$ such that $\pi_{\bar{\mathbb{K}}}$ is the (geometric) quotient of $U_{\bar{\mathbb{K}}}$ by $\bar{S}$. Then $U$ is birational to $\mathbb{P}^1 \times Y$ for some variety $Y$.

**Proof.** The morphism $\pi$ defines a subtorus $S \subset T$ such that $\bar{S} = S_{\bar{\mathbb{K}}}$. Moreover, $\pi$ is the (geometric) quotient of $U$ by $S$. Now everything follows from Lemma 2.3. \(\square\)

### 3. Tori

In this section we will study elements of finite order in algebraic tori.

The following is a famous theorem of Minkowski, see [Min1887], or [Ser07, Theorem 5] together with [Ser07, §4.3].

**Theorem 3.1.** For any positive integer $n$, the group $\text{GL}_n(\mathbb{Z})$ has bounded finite subgroups.

Theorem 3.1 tells us that there is a constant $\Upsilon(n) < \infty$ equal to the maximal order of a finite subgroup in $\text{GL}_n(\mathbb{Z})$.

**Remark 3.2.** One can deduce Theorem 3.1 from the following assertion: for every integer $m > 2$, the kernel of the reduction homomorphism $\text{GL}_n(\mathbb{Z}) \to \text{GL}_n(\mathbb{Z}/m\mathbb{Z})$ is torsion free. In particular, one has $\Upsilon(n) \leq |\text{GL}_n(\mathbb{Z}/3\mathbb{Z})|$. However, in general this bound is not sharp. However, for small $n$ one can find the precise value of the constant $\Upsilon(n)$. For instance, we have $\Upsilon(1) = 2$, $\Upsilon(2) = 12$, and $\Upsilon(3) = 48$, see e.g. [Ser07, §1.1] or [Tah71, §1].

The following is the main technical result of this section.

**Lemma 3.3.** Let $n$ and $d > \Upsilon(n)$ be positive integers. Let $\mathbb{K}$ be a field such that the characteristic of $\mathbb{K}$ does not divide $d$, and $\mathbb{K}$ contains a primitive $d$-th root of 1. Let $T$ be an $n$-dimensional algebraic torus over $\mathbb{K}$ such that $T(\mathbb{K})$ contains a point of order $d$. Then $T$ contains a subtorus isomorphic to $\mathbb{G}_m$. 

6
Proof. Let \( \tilde{T} = \text{Hom}(\mathbb{G}_m, T_{\text{sep}}) \) be the lattice of cocharacters of \( T \). Recall (see [Bor91 §8.12]) that the functor \( T \mapsto \tilde{T} \) induces an equivalence between the category of algebraic tori over \( K \) and the category of free abelian groups of finite rank equipped with an action of the Galois group \( \text{Gal}(K_{\text{sep}}/K) \) such that the image of the homomorphism \( \text{Gal}(K_{\text{sep}}/K) \to \text{Aut}(\tilde{T}) \) is finite. Denote this image by \( \Gamma \).

The group of \( d \)-torsion elements of \( T(K_{\text{sep}}) \) is isomorphic, as a Galois module, to \( \tilde{T} \otimes \mu_d \), where \( \mu_d \) is the group of \( d \)-th roots of unity in \( K_{\text{sep}} \). Since \( K \) contains a primitive \( d \)-th root of 1, the Galois module \( \mu_d \) is the trivial module \( \mathbb{Z}/d\mathbb{Z} \), so that the Galois module \( \tilde{T} \otimes \mu_d \) is isomorphic to \( \tilde{T}/d\tilde{T} \). Hence, a point \( x \in T(K) \) of order \( d \) can be viewed as a \( \text{Gal}(K_{\text{sep}}/K) \)-invariant element \( \bar{v} \in \tilde{T}/d\tilde{T} \) of order \( d \) (so that \( m\bar{v} \neq 0 \) for \( m < d \)). Let \( v \in \tilde{T} \) be any preimage of \( \bar{v} \) under the projection \( \tilde{T} \to \tilde{T}/d\tilde{T} \), and let

\[
    w = \sum_{v \in \Gamma} \gamma(v).
\]

Since \( \bar{v} \) is \( \text{Gal}(K_{\text{sep}}/K) \)-invariant, the image of \( w \) in \( \tilde{T}/d\tilde{T} \) is equal to \( |\Gamma|\bar{v} \). By assumption, the latter is not equal to 0. Hence \( w \neq 0 \). On the other hand, it is clear that \( w \) is a \( \text{Gal}(K_{\text{sep}}/K) \)-invariant element of \( \tilde{T} \). By the above mentioned equivalence of categories, \( w \) gives rise to a nonzero homomorphism \( \mathbb{G}_m \to T \) whose image is the required subtorus. \( \square \)

Remark 3.4. J.-L. Colliot-Thélène pointed out to us that the proof of Lemma 3.3 can be reformulated in the following way. The short exact sequence of \( \Gamma \)-modules

\[
    0 \longrightarrow \tilde{T} \xrightarrow{d} \tilde{T} \longrightarrow \tilde{T}/d\tilde{T} \longrightarrow 0
\]

gives rise to the long exact sequence of cohomology groups

\[
    \ldots \longrightarrow H^0(\Gamma, \tilde{T}) \longrightarrow H^0(\Gamma, \tilde{T}/d\tilde{T}) \longrightarrow H^1(\Gamma, \tilde{T}) \longrightarrow \ldots
\]

If \( H^0(\Gamma, \tilde{T}) = 0 \) then the second map in (3.5) is injective. On the other hand, the group \( H^1(\Gamma, \tilde{T}) \) is annihilated by \( |\Gamma| \) (see e.g. [CF67 Proposition IV.6.3]). It follows that the group \( H^0(\Gamma, \tilde{T}/d\tilde{T}) \) of \( d \)-torsion points of \( T(K) \) is also annihilated by \( |\Gamma| \).

Lemma 3.3 immediately implies the following result.

Corollary 3.6. Let \( K \) be a field that contains all roots of 1, and let \( T \) be an anisotropic \( n \)-dimensional torus over \( K \). Then every element of finite order in \( T(K) \) has order at most \( \Upsilon(n) \), and every finite subgroup in \( T(K) \) has order at most \( \Upsilon(n)^n \). Moreover, every element of finite order and every finite subgroup of \( T(K) \) has order coprime to the characteristic of \( K \).

Proof. Note that if \( \text{char} K = p \) is positive, then \( T \) does not contain elements of order \( p \), because there are no such elements even in \( T_{\text{sep}} \cong \mathbb{G}_m^n \). Thus, if there is an element of finite order \( d \) in \( T(K) \), then \( d \) is coprime to \( p \), so that \( d \leq \Upsilon(n) \) by Lemma 3.3. It remains to notice that every finite subgroup of \( T(K) \) is an abelian group generated by at most \( n \) elements, and thus has order at most \( \Upsilon(n)^n \). \( \square \)

Remark 3.2 and Corollary 3.6 immediately imply the following (well-known) assertion.

Lemma 3.7. Let \( K \) be a field that contains all roots of 1, and let \( T \) be a one-dimensional torus over \( K \) that is different from \( \mathbb{G}_m \). Then every non-trivial finite subgroup of \( T(K) \) has order 2.
Lemma 3.7 implies the following assertion that is in some sense analogous to Theorem 1.1.

**Corollary 3.8** (cf. [BZ17b, Lemma 3.3]). Let $\mathbb{K}$ be a field that contains all roots of 1. Let $O_1, O_2 \in \mathbb{P}^1(\mathbb{K}_{\text{sep}})$ be two distinct points that form a $\text{Gal}(\mathbb{K}_{\text{sep}}/\mathbb{K})$-orbit (so that $\{O_1, O_2\}$ is a closed point of $\mathbb{P}^1$ whose residue field $\mathbb{K}_{O_1, O_2}$ is a separable quadratic extension of $\mathbb{K}$). Consider the affine curve

$$U = \mathbb{P}^1 \setminus \{O_1, O_2\}$$

defined over $\mathbb{K}$. Let $G$ be a finite group acting by automorphisms of $U$. Then $|G| \leq 4$.

**Proof.** Let $T$ be the one-dimensional torus that corresponds to the quadratic field extension $\mathbb{K} \subset \mathbb{K}_{O_1, O_2}$. In other words, one can describe $T$ as the group of all automorphisms $\phi$ of $\mathbb{P}^1$ such that $\phi|_{\mathbb{K}_{\text{sep}}}$ preserves both points $O_1$ and $O_2$. The curve $U$ is a torsor over some torus $T$ over $\mathbb{K}$ different from $\mathbb{G}_m$. Moreover, since $\mathbb{P}^1$ has $\mathbb{K}$-points, so has $U$. This means that $U$ and $T$ are isomorphic as varieties over $\mathbb{K}$. Therefore, we have $\text{Aut}(U) \cong T \rtimes \text{Aut}(T)$, so that the assertion follows from Lemma 3.7. \qed

Now we will provide more precise bounds for the possible finite orders of elements and subgroups of two-dimensional tori that will be used in §10. Note that there is an explicit classification of two-dimensional tori (see [Vos65]), but we will not use it. Our approach here is similar to the proof of Lemma 3.3 but instead of the bounds on the orders of finite subgroups of $\text{GL}_n(\mathbb{Z})$ we use the bounds on the orders of elements of $\text{GL}_2(\mathbb{Z})$ together with the following easy observation that is specific to rank 2.

**Lemma 3.9.** Put $\Lambda = \mathbb{Z}^2$, and let $\Gamma \subset \text{GL}(\Lambda)$ be a finite subgroup. Suppose that every element of $\Gamma$ has a non-zero invariant vector in $\Lambda$. Then $\Gamma$ has a non-zero invariant vector in $\Lambda$.

**Proof.** Let $\gamma$ be an element of $\Gamma$. Since $\gamma \in \text{GL}(\Lambda) \cong \text{GL}_2(\mathbb{Z})$, its determinant is $\pm 1$. Since one of the eigen-values of $\gamma$ equals 1, the other eigen-value equals $\pm 1$. Since $\gamma$ is of finite order, this implies that $\gamma^2 = 1$. Hence the group $\Gamma$ is abelian, and thus there are two non-proportional vectors $v_1$ and $v_2$ in $\Lambda$ that generate one-dimensional $\Gamma$-invariant sublattices. If there exist two non-trivial elements $\gamma_1$ and $\gamma_2$ in $\Gamma$ such that $\gamma_i v_i = -v_i$, then it is easy to see that there is an element $\gamma'$ in $\Gamma$ such that $\gamma' v_i = -v_i$ for both $i = 1, 2$. This means that $\gamma'$ acts on $\Lambda$ by multiplication by $-1$, which is impossible because $\gamma'$ must have a non-zero invariant vector by assumption. Therefore, one of the vectors $v_1$ and $v_2$ is invariant with respect to the whole group $\Gamma$. \qed

**Lemma 3.10.** Put $\Lambda = \mathbb{Z}^2$, and let $\Gamma \subset \text{GL}(\Lambda)$ be a finite subgroup. Let $d > 6$ be an integer. Suppose that there is a $\Gamma$-invariant vector $v$ in $\Lambda \pmod{d}$ such that $mv \neq 0$ for $m < d$. Then there is a non-zero $\Gamma$-invariant vector in $\Lambda$.

**Proof.** Let $\gamma$ be an element of the group $\Gamma$, and $r$ be the order of $\gamma$. It is easy to check that $r \in \{1, 2, 3, 4, 6\}$, see for instance [Tah71], §1.

Suppose that $\gamma$ has no non-zero invariant vectors in $\Lambda$. Then the matrix $\gamma - 1$ is invertible in $\text{GL}_2(\mathbb{Q})$. Since we know that $\gamma^r - 1 = 0$, this implies that the matrix

$$\delta = \gamma^{r-1} + \ldots + \gamma + 1 \in \text{Mat}_2(\mathbb{Z}) \subset \text{Mat}_2(\mathbb{Q})$$

is the zero matrix, where $\text{Mat}_2$ denotes the ring of $2 \times 2$-matrices. Thus

$$\delta_d = \delta \pmod{d}$$
is a zero matrix as well. Write
\[ 0 = \delta_d v = (\gamma^{r-1} + \ldots + \gamma + 1) v \pmod{d} = rv, \]
which is a contradiction since \( r \leq 6 < d \).

Therefore, we see that every element of \( \Gamma \) has an invariant vector in \( \Lambda \). Now it remains to apply Lemma 3.9. \( \square \)

**Lemma 3.11.** Let \( K \) be a field that contains all roots of 1, and let \( T \) be an anisotropic two-dimensional torus over \( K \). Then every element of finite order in \( T(\bar{K}) \) has order at most 6, and every finite subgroup of \( T(\bar{K}) \) has order at most 36.

*Proof.* As in the proof of Lemma 3.9, we put \( \hat{T} = \text{Hom}(G_m, T_{\bar{K}}) \) and let \( \Gamma \) be the image of the Galois group \( \text{Gal}(\bar{K}/K) \) in \( \text{Aut}(\hat{T}) \). Then \( \Gamma \) is a finite subgroup of \( \text{Aut}(\hat{T}) \cong \text{GL}_2(\mathbb{Z}) \).

Suppose that \( T(\bar{K}) \) has an element of finite order \( d > 1 \). We note that \( d \) is coprime to the characteristic of the field \( K \). As in the proof of Lemma 3.9, we see that \( \hat{T}/d\hat{T} \) has a \( \Gamma \)-invariant vector \( v \) such that \( mv \neq 0 \) for \( m < d \). By Lemma 3.10, if \( d > 6 \), then there is a non-zero \( \Gamma \)-invariant vector in \( \hat{T}^v \). Hence there is an embedding \( G_m \hookrightarrow \hat{T} \). Since the latter is not the case by assumption, we conclude that \( d \leq 6 \). As in the proof of Corollary 3.6, we observe that every finite subgroup of \( T \) is an abelian group generated by at most two elements, and thus has order at most 36. \( \square \)

**Remark 3.12.** The proofs of Lemmas 3.10 and 3.11 also imply that an anisotropic two-dimensional torus over \( K \) does not have elements of order 5.

### 4. Linear algebraic groups

In this section we study finite subgroups in linear algebraic groups and prove Theorem 1.5.

Recall that a linear algebraic group \( \Gamma \) over a field \( K \) is a smooth closed subgroup scheme of \( \text{GL}_N \) over \( K \). In particular, the group \( \Gamma(\mathbb{K}) \) of its \( \mathbb{K} \)-points has a faithful finite-dimensional representation in a \( \mathbb{K} \)-vector space. We refer the reader to [Bor91] and [Spr98] for the basics of the theory of linear algebraic groups.

Similarly to the case of an algebraically closed field, many properties of linear algebraic groups are determined by their maximal tori. Note that in general a linear algebraic group \( \Gamma \) over a non-algebraically closed field \( K \) may contain non-isomorphic maximal tori, but their dimension still equals the dimension of maximal tori in \( \Gamma_{\mathbb{K}} \), see [Spr98, Theorem 13.3.6(i)] and [Spr98, Remark 13.3.7]; this dimension is called the *rank* of \( \Gamma \).

Recall that an element \( \gamma \in \Gamma(\mathbb{K}) \) is called semi-simple if its image in \( \text{GL}_N(\mathbb{K}) \) is diagonalizable over an algebraic closure \( \bar{K} \) of \( K \). The notion of a semi-simple element is intrinsic, that is, it does not depend on the choice of \( N \) and an embedding \( \Gamma \hookrightarrow \text{GL}_N(\mathbb{K}) \), see [Spr98, §2.4]. The main tool that will allow us to apply the results of §3 is the following theorem.

**Theorem 4.1** (see [Spr98, Corollary 13.3.8(i)]). Let \( \Gamma \) be a connected linear algebraic group over a field \( \mathbb{K} \), and let \( \gamma \in \Gamma(\mathbb{K}) \) be a semi-simple element. Then there exists a torus \( T \subset \Gamma \) such that \( \gamma \) is contained in \( T(\mathbb{K}) \).

**Corollary 4.2.** Let \( \Gamma \) be a connected linear algebraic group over a field \( \mathbb{K} \), and let \( \gamma \in \Gamma(\mathbb{K}) \) be a finite order element whose order is coprime to the characteristic of \( \mathbb{K} \). Then there exists a torus \( T \subset \Gamma \) such that \( \gamma \) is contained in \( T(\mathbb{K}) \).
For anisotropic reductive groups over perfect fields one has a stronger result.

**Theorem 4.3** ([BT71, Corollary 3.8]). Let $\mathbb{K}$ be a perfect field, and $\Gamma$ be a connected anisotropic reductive linear algebraic group over $\mathbb{K}$. Then, for every element $\gamma \in \Gamma(\mathbb{K})$, there exists a torus $T \subset \Gamma$ such that $\gamma$ is contained in $T(\mathbb{K})$.

**Corollary 4.4.** Under the assumptions of Theorem 4.3 the order of every finite order element of $\Gamma(\mathbb{K})$ is coprime to the characteristic of $\mathbb{K}$.

**Proof.** By Theorem 4.3 it suffices to prove the assertion in the case when $\Gamma$ is torus, in which case it is given by Corollary 3.6. □

Note that over fields of positive characteristic non-reductive algebraic groups may have unbounded finite subgroups. For instance, the $p$-torsion subgroup of $G_a$ over an infinite field of characteristic $p$ is an infinite dimensional vector space over the field $\mathbb{F}_p$ of $p$ elements. However, this example is in certain sense the only source of such unboundedness for unipotent groups.

**Lemma 4.5.** Let $\mathbb{K}$ be a field, and let $\Gamma$ be a unipotent group over $\mathbb{K}$. Then $\Gamma(\mathbb{K})$ does not contain elements of finite order coprime to the characteristic of $\mathbb{K}$.

**Proof.** Without loss of generality we may assume that $\mathbb{K}$ is algebraically closed. In this case the Lemma follows from a similar assertion for $G_a$ together with Lie–Kolchin theorem, which implies that any unipotent group over an algebraically closed field can be obtained as a consecutive extensions of groups isomorphic to $G_a$. □

We will need the following auxiliary fact about orders of finite groups with given exponents.

**Theorem 4.6** (see [HP76, Theorem 1]). Let $n$ and $d$ be positive integers, and let $\mathbb{K}$ be a field. Let $G \subset \text{GL}_n(\mathbb{K})$ be a finite subgroup. If $\text{char } \mathbb{K} > 0$, denote by $|G|^\prime$ the largest factor of $|G|$ which is coprime to $\text{char } \mathbb{K}$; otherwise put $|G|^\prime = |G|$. Suppose that for every $g \in G$, whose order is coprime to the characteristic of $\mathbb{K}$, one has $g^d = 1$. Then $|G|^\prime \leq d^n$.

Now we prove Theorem 1.5.

**Proof of Theorem 1.5**. Clearly, we may assume that $\Gamma$ is connected. Note that assertion (i) follows from assertion (ii) and Corollary 4.4.

Let $\Gamma$ be an arbitrary linear algebraic group of rank $n$, and let $\gamma \in \Gamma(\mathbb{K})$ be an element of finite order coprime to $\text{char } \mathbb{K}$. Then, by Corollary 4.2 the element $\gamma$ is contained in some subtorus of $\Gamma$. Thus, it follows from Corollary 3.6 that the order of $\gamma$ is bounded by some constant that depends only on $n$. In other words, there exists a constant $d(n)$, such that for every connected linear algebraic group of rank $n$ and every element $\gamma \in \Gamma(\mathbb{K})$ of order coprime to $\text{char } \mathbb{K}$, one has $\gamma^{d(n)} = 1$.

Let $\Gamma_{\bar{\mathbb{K}}}$ be the algebraic group over $\bar{\mathbb{K}}$ obtained from $\Gamma$ by the base change, and let $\Delta_{\bar{\mathbb{K}}}$ be the unipotent radical of $\Gamma_{\bar{\mathbb{K}}}$. (Note that unless $\mathbb{K}$ is perfect the group $\Delta_{\bar{\mathbb{K}}}$ need not be defined over $\mathbb{K}$.) Then $\Gamma_{\bar{\mathbb{K}}}/\Delta_{\bar{\mathbb{K}}}$ is a reductive linear algebraic group over $\bar{\mathbb{K}}$. Moreover, the group $\Gamma_{\bar{\mathbb{K}}}/\Delta_{\bar{\mathbb{K}}}$ has the same rank as $\Gamma_{\bar{\mathbb{K}}}$ (which is equal to the rank of $\Gamma$). Indeed, the rank of a unipotent algebraic group is zero. Hence, the rank of $\Gamma_{\bar{\mathbb{K}}}/\Delta_{\bar{\mathbb{K}}}$ is greater than or equal to the rank of $\Gamma_{\bar{\mathbb{K}}}$. On the other hand, by [Bor91, Theorem 10.6(4)], every extension of a torus by a unipotent group admits a section, which means that the rank of $\Gamma_{\bar{\mathbb{K}}}$ is greater
than or equal to the rank of the quotient $\Gamma_{\bar{K}}/\Delta_{\bar{K}}$. By a theorem of Chevalley (see \cite[Theorem 9.6.2]{Spr98}) there are only finitely many isomorphism classes of connected reductive groups of given rank over an algebraically closed field. Every such group is linear, that is, it admits a faithful finite-dimensional representation. Applying this to $\Gamma_{\bar{K}}/\Delta_{\bar{K}}$ we infer that there exists a constant $N(n)$, which depends only on $n$, such that, for every connected linear algebraic group $\Gamma$ of rank $n$, the group $(\Gamma_{\bar{K}}/\Delta_{\bar{K}})(\bar{K})$ admits a faithful representation in an $N(n)$-dimensional vector space over $\bar{K}$:

$$(\Gamma_{\bar{K}}/\Delta_{\bar{K}})(\bar{K}) \hookrightarrow \text{GL}_{N(n)}(\bar{K}).$$

Composing this embedding with the projection $\Gamma(\bar{K}) \to (\Gamma_{\bar{K}}/\Delta_{\bar{K}})(\bar{K})$ we construct a homomorphism

$$\phi: \Gamma(\bar{K}) \to \text{GL}_{N(n)}(\bar{K}),$$

whose kernel is contained in $\Delta_{\bar{K}}(\bar{K})$. By Lemma 4.3, every element of finite order in $\Delta_{\bar{K}}(\bar{K})$ has order divisible by $\text{char}\, \bar{K}$. This means that the image $\phi(G)$ of a finite subgroup $G \subset \Gamma(\bar{K})$ in $\text{GL}_{N(n)}(\bar{K})$ has order divisible by the largest factor $|G|$ of $|G|$ coprime to $\text{char}\, \bar{K}$; in particular, if $\text{char}\, \bar{K} = 0$, then $G$ projects isomorphically to $(\Gamma_{\bar{K}}/\Delta_{\bar{K}})(\bar{K})$. Therefore, assertion (ii) is implied by Theorem 4.6 applied to $\phi(G)$. $\square$

5. Varieties with zero irregularity

In this section we study automorphism groups of varieties with zero irregularity and prove Theorem 1.6. In general, such groups can be very far from linear algebraic groups. For instance, there exists a surface $Z$ that is a blow up of a $K3$ surface such that $\text{Aut}(Z)$ is a discrete infinitely generated group, see \cite{DO17}. However, we will see below that on the level of finite subgroups the difference is not so large.

We start with a few general auxiliary facts. Given a scheme $S$ and a locally free sheaf $E$ on $S$, we denote by $\text{PGL}(E)$ the group scheme over $S$ of projective linear automorphisms of $E$, that is, the group scheme $\text{Aut}(\text{Proj}(S\cdot E))$ of automorphisms (over $S$) of the relative projective space. Thus, over an open subscheme $U \subset S$, over which $E$ is free of rank $n$, the group scheme $\text{PGL}(E)$ is isomorphic to $\text{PGL}_n \times U$. The following observation is well known.

**Lemma 5.1.** Let $S$ be a locally Noetherian scheme, and let $\pi: X \to S$ be a flat projective morphism with geometrically connected fibers. Let $L$ be a line bundle over $X$ that is very ample relative to $S$. Assume that $R^m\pi_*L = 0$, for every $m > 0$. Then the coherent sheaf $E = \pi_*L$ is locally free and the functor that carries a $S$-scheme $T$ to the group of automorphisms

$$\phi: X \times_S T \to X \times_S T$$

over $T$, such that Zariski locally over $T$ the line bundle $\phi^*\text{pr}_X^*L$ is isomorphic to $\text{pr}_X^*L$, is representable by a closed group subscheme $\text{Aut}(X, L)$ of $\text{PGL}(E)$.

**Proof.** The coherent sheaf $E$ is locally free by the base change theorem (see e.g. \cite[§5]{Mum70}) and the vanishing of the higher direct images $R^m\pi_*L$. It also follows from the same theorem that, for every scheme $T$ over $S$ with a structure morphism $g: T \to S$, we have

$$g^*E \sim \text{pr}_T^*\text{pr}_X^*L.$$
Let $i: \mathcal{X} \hookrightarrow \text{Proj}(S^*E)$ be the embedding defined by $\mathcal{L}$. We claim that the functor carrying an $S$-scheme $\mathcal{T}$ to the subgroup $\mathcal{G}(T) \subset \text{PGL}(E)(T)$ of automorphisms of $\text{Proj}(S^*E) \times_S \mathcal{T}$ preserving the closed subscheme

$$\mathcal{X} \times_S \mathcal{T} \xrightarrow{i \times \text{Id}} \text{Proj}(S^*E) \times_S \mathcal{T}$$

is represented by a closed subgroup scheme $\mathcal{G}$ of $\text{PGL}(E)$. Indeed, the image of $i$ is given by a sheaf $\mathcal{I}_*$ of graded ideals in $S^*E$. For sufficiently large $m$, the $\mathcal{O}_S$-submodule $I^m \subset S^mE$ is locally a direct summand. Thus, $I^m$ determines an $S$-point of the relative Grassmannian associated to $S^mE$. The group $\mathcal{G}(T)$ consists of the elements of $\text{PGL}(E)(T)$ which preserve $I_m \subset S^mE$ for all sufficiently large $m$. Thus, our subgroup $\mathcal{G} \subset \text{PGL}(E)$ can be defined as the stabilizer of a point in the (infinite) product of relative Grassmann schemes $\text{Gr}(S^mE)$ under the action of $\text{PGL}(E)$. Alternatively, $\mathcal{G}$ can be described as the stabilizer in $\text{PGL}(E)$ of the point of the Hilbert scheme $\text{Hilb} (\text{Proj}(S^*E))$ determined by the embedding $i: \mathcal{X} \hookrightarrow \text{Proj}(S^*E)$.

We claim that $\mathcal{G}$ represents the functor defined in the assertion of the lemma, that is, one has $\text{Aut}(\mathcal{X}, \mathcal{L}) \cong \mathcal{G}$. Let $\mathcal{T}$ and $\phi$ be as in the assertion of the lemma, and let $g: \mathcal{T} \rightarrow S$ be the structure morphism. Choose, Zariski locally on $\mathcal{T}$, an isomorphism

$$\alpha: \phi^* \text{pr}_X^* \mathcal{L} \cong \text{pr}_X^* \mathcal{L}.$$ 

Then $\phi$ and $\alpha$ induce an automorphism of the $\mathcal{O}_\mathcal{T}$-module $g^* \mathcal{E}$ and, thus, a $\mathcal{T}$-valued point of $\text{Aut}(\mathcal{X}, \mathcal{L})$. Since the fibers of $\pi$ are geometrically connected, using the Theorem on Formal Functions (see [EGA III-1, Theorem 4.1.5]) we conclude that

$$(\pi \times \text{Id})_* \mathcal{O}_{\mathcal{X} \times_S \mathcal{T}} = \mathcal{O}_\mathcal{T}.$$ 

Thus, $\alpha$ is well defined locally on $\mathcal{T}$ up to multiplication by an invertible function. Hence, the $\mathcal{T}$-valued point of $\text{Aut}(\mathcal{X}, \mathcal{L})$ does not depend on the choice of $\alpha$ and is defined on the whole $\mathcal{T}$. We leave it to the reader to check that the constructed morphism of functors is an isomorphism.

We will need the following boundedness result on the Picard groups.

**Lemma 5.2.** Let $S$ be a Noetherian scheme, and let $\pi: \mathcal{X} \rightarrow S$ be a flat projective morphism with geometrically reduced and geometrically irreducible fibers. Suppose that for every point $s \in S$ the irregularity of the fiber $\mathcal{X}_s$ is zero. Then the minimal number of generators of $\text{Pic}(\mathcal{X}_s)$ and the order of the torsion subgroup in $\text{Pic}(\mathcal{X}_s)$ are both bounded by a constant that depends only on $\mathcal{X}$.

**Proof.** Let $\bar{s}$ be a geometric point of $S$ lying over $s$ (in particular, if the residue field of $s$ is algebraically closed, then $\bar{s} = s$). Note that the pullback morphism $\text{Pic}(\mathcal{X}_s) \rightarrow \text{Pic}(\mathcal{X}_s)$ is injective (see for instance [Stack, Lemma 32.30.3]). Next, by a result of Grothendieck [Kle75, Theorem 9.4.8], the group $\text{Pic}(\mathcal{X}_s)$ can be identified with the group of closed points of the Picard group scheme $\text{Pic}(\mathcal{X}_s)$. By [Kle75, Theorem 9.5.11], the Lie algebra of $\text{Pic}(\mathcal{X}_s)$ is isomorphic to $H^1(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s})$. The latter space is zero by our assumption. Therefore, the homomorphism from the Picard group of the fiber $\mathcal{X}_s$ to the group of connected components of the Picard scheme of the corresponding geometric fiber $\mathcal{X}_s$ is injective. Thus, our result follows from the boundness of the latter [Kle77, Theorem 5.1].

In a particular case when $S$ is a point, Lemma 5.2 gives a well-known assertion that the Picard group of a variety with zero irregularity is finitely generated.
Lemma 5.3 (cf. the proof of [MZ15, Lemma 2.5]). Let $X$ be a geometrically reduced and geometrically irreducible projective variety over an arbitrary field. Suppose that the irregularity of $X$ equals 0. Let $\Gamma_{\text{Pic}}$ be the kernel of the action of $\text{Aut}(X)$ on the Picard group $\text{Pic}(X)$. Then there is a constant $\Xi$ that depends only on the (minimal) number of generators of $\text{Pic}(X)$ and the order of the torsion subgroup in $\text{Pic}(X)$, such that for every finite subgroup $G \subset \text{Aut}(X)$ the intersection $G \cap \Gamma_{\text{Pic}}$ has index at most $\Xi$ in $G$.

Proof. In our case $\text{Pic}(X)$ is a finitely generated abelian group by Lemma 5.2. Thus, Theorem 3.1 implies that the automorphism group of $\text{Pic}(X)$ has bounded finite subgroups. This means that the orders of all finite subgroups in the quotient group

$$\text{Aut}(X)/\Gamma_{\text{Pic}} \hookrightarrow \text{Aut}(\text{Pic}(X))$$

are also bounded by some constant $\Xi$ that depends only on $\text{Pic}(X)$. □

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. Since $\mathcal{S}$ is Noetherian, arguing by induction it suffices to prove the theorem for the restriction of $\pi$ to a non-empty open subscheme of $\mathcal{S}$. Thus, by generic flatness (see [Nit05, Theorem 5.12]), we may assume that $\pi$ is flat. Choose a line bundle $\mathcal{L}$ on $\mathcal{X}$ that is very ample relative to $\mathcal{S}$. Replacing if necessary $\mathcal{L}$ by its power $\mathcal{L}^N$, we may assume that the higher direct images $R^j \pi_* \mathcal{L}$ vanish for all $j > 0$. Then, by Lemma 5.1, the coherent sheaf $\mathcal{E} = \pi_* \mathcal{L}$ is free of some rank $m$ over $\mathcal{O}_S$. Let

$$\text{Aut}(\mathcal{X}, \mathcal{L}) \subset \text{PGL}(\mathcal{E}) \cong \text{PGL}_m \times \mathcal{S}$$

be the closed subgroup scheme from Lemma 5.1. By generic flatness we may assume that $\text{Aut}(\mathcal{X}, \mathcal{L})$ is flat over $\mathcal{S}$. By [Stack, Lemma 36.25.7] the number of connected components of the fibers of the morphism $\text{Aut}(\mathcal{X}, \mathcal{L}) \to \mathcal{S}$ is bounded. In fact, it follows from [Stack, Lemma 36.25.7] that after shrinking $\mathcal{S}$ we have a short exact sequence

$$(5.4) \quad 1 \to \text{Aut}^\circ(\mathcal{X}, \mathcal{L}) \to \text{Aut}(\mathcal{X}, \mathcal{L}) \to C \to 1,$$

where $\text{Aut}^\circ(\mathcal{X}, \mathcal{L})$ is a flat group scheme with connected fibers and $C$ is a finite étale group scheme over $\mathcal{S}$.

For $s \in \mathcal{S}(\mathbb{K})$, let $\Gamma_{\text{Pic}}(\mathcal{X}_s)$ be the kernel of the action of $\text{Aut}(\mathcal{X}_s)$ on $\text{Pic}(\mathcal{X}_s)$. Let

$$\mathcal{G}_s = \text{Aut}(\mathcal{X}_s, \mathcal{L}_s).$$

Then $\Gamma_{\text{Pic}}(\mathcal{X}_s)$ is a subgroup of $\mathcal{G}_s(\mathbb{K})$. In particular, by Lemmas 5.2 and 5.3 there exists a constant $\Theta$ such that for every finite subgroup $G \subset \text{Aut}(\mathcal{X}_s)$, the index of the intersection $G \cap \mathcal{G}_s(\mathbb{K})$ in $G$ is at most $\Theta$. Thus, to prove the theorem it is enough to bound the orders of finite subgroups in $\mathcal{G}_s(\mathbb{K})$. Keeping in mind the exact sequence (5.4), we see that it is actually enough to bound the orders of finite subgroups of the group $\mathcal{G}_s^\circ(\mathbb{K})$, where

$$\mathcal{G}_s^\circ = \text{Aut}^\circ(\mathcal{X}_s, \mathcal{L}_s)$$

is the connected component of identity in $\mathcal{G}_s$. Moreover, by Lemma 2.1 we may assume that $\mathcal{G}_s^\circ$ is anisotropic, and by Corollary 2.2 we may assume that $\mathcal{G}_s^\circ$ is reductive. Now we win by Theorem 1.5(i). □
6. SEVERI–BRAUER VARIETIES

In this section we describe automorphism groups of Severi–Brauer varieties and prove Propositions 1.8 and 1.9. We refer the reader to [Ar82] for the definition and basic facts concerning Severi–Brauer varieties.

Let \( A \) be a central simple algebra of dimension \( n^2 \) over an arbitrary field \( K \), and \( X \) be the corresponding Severi–Brauer variety of dimension \( n - 1 \). As usual, for a field extension \( K \supset K \) we denote by \( A^*(K) \) the multiplicative group of invertible elements in \( A \otimes_K K \). In particular, one has \( A^*(K_{sep}) = GL_n(K_{sep}) \). If \( A \) is a division algebra, then \( A^*(K) = A - \{0\} \).

The following fact is well known to experts (cf. Theorem E on page 266 of [Ch44], or [Ar82, §1.6.1]), but for the reader’s convenience we provide a proof.

**Lemma 6.1.** One has \( \text{Aut}(X) \cong A^*(K)/K^* \).

**Proof.** Recall that the scheme \( X \) represents the functor that takes a scheme \( S \) over \( K \) to the set of right ideals \( I \) in the sheaf of algebras \( A \otimes_K \mathcal{O}_S \) which are locally free of rank \( n \) as \( \mathcal{O}_S \)-modules and are locally direct summands, that is, \( I \oplus J = A \otimes_K \mathcal{O}_S \) for some ideal \( J \). The action of the group \( A^*(K) \) on \( A \) by conjugation induces an action of \( A^*(K) \) on the above functor and thus, by Yoneda Lemma, on \( X \). Obviously, the action of the central subgroup \( K^* \subset A^*(K) \) is trivial on \( A \) and on \( X \). This gives a homomorphism of groups

\[
\xi_K : A^*(K)/K^* \to \text{Aut}(X).
\]

Recall that \( X_{K_{sep}} \cong \mathbb{P}^{n-1}_{K_{sep}} \) is the Severi–Brauer variety associated with the split central simple algebra \( A \otimes_K K_{sep} \) over \( K_{sep} \). Let \( \Gamma = \text{Gal}(K_{sep}/K) \). We have a commutative diagram

\[
\begin{array}{ccc}
A^*(K)/K^* & \xrightarrow{\xi_K} & \text{Aut}(X) \\
\downarrow & & \downarrow \\
(A^*(K_{sep})/K_{sep}^*)^\Gamma & \xrightarrow{\xi_{K_{sep}}^\Gamma} & (\text{Aut}(X_{K_{sep}}))^\Gamma
\end{array}
\]

Here, for a group \( C \) with an action of \( \Gamma \), we write \( C^\Gamma \) for the subgroup of \( \Gamma \)-invariant elements. The homomorphism

\[
\xi_{K_{sep}} : A^*(K_{sep})/K_{sep}^* \to \text{Aut}(X_{K_{sep}}) \cong \text{PGL}_n(K_{sep})
\]

is an isomorphism. Thus, the lower horizontal arrow in the diagram is also an isomorphism. In addition, the vertical arrows are injections. Hence, to complete the proof it suffices to show that the morphism \( A^*(K)/K^* \to (A^*(K_{sep})/K_{sep}^*)^\Gamma \) is surjective. Indeed, the exact sequence of groups with \( \Gamma \)-action

\[
1 \to K_{sep}^* \to A^*(K_{sep}) \to A^*(K_{sep})/K_{sep}^* \to 1
\]

gives rise to the exact sequence of Galois cohomology groups

\[
1 \to K^* \to A^*(K) \to (A^*(K_{sep})/K_{sep}^*)^\Gamma \to H^1(\Gamma, K^*).
\]

The latter cohomology group vanishes by Hilbert’s Theorem 90, and the assertion of the lemma follows. \( \square \)
Remark 6.2. The above argument can be restated as follows: let \( \mathcal{A}^* \) be the algebraic group whose \( S \)-points are invertible elements in the algebra \( A \otimes \mathcal{O}_S \). There is a natural embedding \( \mathbb{G}_m \to \mathcal{A}^* \) and the quotient group scheme is identified with the group scheme \( \text{Aut}(Y) \). Hilbert's Theorem 90 implies that the group of \( K \)-points of the quotient \( \mathcal{A}^*/\mathbb{G}_m \) is \( \mathcal{A}(\mathbb{K})/\mathbb{K}^* \).

**Lemma 6.3.** Let \( K \) be a field that contains all roots of 1. Let \( A \) be a central simple algebra of dimension \( n^2 \) over a field \( K \), and let \( x \) be an element of \( \mathcal{A}(K) \) whose image in \( \mathcal{A}^*(K)/\mathbb{K}^* \) has finite order. Then the minimal polynomial \( f(y) \in K[y] \) of \( x \) has the form
\[
f(y) = \prod_i (y^r - a_i),
\]
for some positive integer \( r \) that divides \( n \) and some \( a_i \in K \). In particular, if \( A \) is a division algebra, then the order of the image of \( x \) in \( \mathcal{A}(K)/\mathbb{K}^* \) divides \( n \).

**Proof.** We know that for some positive integer \( m \), the element \( a = x^m \) is contained in \( K^* \). We claim that there exists a positive integer \( r \) dividing \( m \) such that every irreducible factor of the polynomial \( y^m - a \in K[y] \) has the form \( y^r - b \) for some \( b \in K \). To show this, let \( \alpha \) be a root of \( y^m - a \) in \( K \), and let \( r \) be the minimal positive integer such that \( b = \alpha^r \) is contained in \( K \). Clearly, \( r \) divides \( m \). We need to check that the polynomial \( y^r - b \) is irreducible in \( K[y] \). Consider the extension \( F = K(\alpha) \supset K \). Since \( K \) contains all roots of 1, the extension \( F \) is the splitting field of the polynomial \( y^r - b \).

If the characteristic of \( K \) does not divide \( r \), then \( F \) is separable over \( K \). We have an injective (Kummer) homomorphism \( \text{Gal}(F/K) \hookrightarrow \mu_r \) to the cyclic group of \( r \)-th roots of 1 which sends \( g \in \text{Gal}(F/K) \) to \( \frac{2\alpha^g}{\alpha} \). If \( \mu_{r'} \subseteq \mu_r \) is its image, then \( \alpha^{r'} \) is invariant under the action of the Galois group and, thus, belongs to \( K \). Therefore, one has \( r = r' = [F : K] \).

This implies that \( y^r - b \) is irreducible in \( K[y] \).

If \( K \) has positive characteristic \( p \) which divides \( r \), write \( r = p^k m \) with \( m \) coprime to \( p \). The subfield \( K(\alpha^{p^k}) \subset F \) is the splitting field of the separable polynomial \( y^m - b \), which, by the previous argument, is irreducible over \( K \). On the other hand, \( K(\alpha^{p^k}) \subset F \) is the splitting field of the polynomial
\[
y^{p^k} - b = (y - \alpha^m)^{p^k}
\]
which is also irreducible over \( K \). Since the extension \( K \subset K(\alpha^{p^k}) \) is separable and the extension \( K \subset K(\alpha^m) \) is purely inseparable, the degree of their composite is equal to the product
\[
[K(\alpha^{p^k}) : K] \cdot [K(\alpha^m) : K] = r.
\]
Hence, the degree of \( F \) over \( K \) is at least \( r \), and therefore \( y^r - b \) is irreducible in \( K[y] \).

Since the minimal polynomial \( f(y) \) of \( x \) divides \( y^m - a \), each of its irreducible factors also has the form \( y^r - b \) for some integer \( r \) which divides \( m \) and \( b \in K \) such that \( b^{\frac{m}{p^k}} = a \).

We need to show that \( r \) divides \( n \). Consider the subalgebra
\[
R = K[x] \subset A.
\]
By definition of the minimal polynomial, the algebra \( R \) is isomorphic to the quotient \( K[y]/(f(y)) \). If \( y^r - b \) is an irreducible factor of \( f(y) \) then, using the assumption that \( K \) contains all roots of 1 and the Chinese Remainder theorem, there exist an integer \( e \) and a unital monomorphism
\[
K[y]/((y^r - b)^e) \hookrightarrow R.
\]
If the characteristic of $\mathbb{K}$ does not divide $r$, then by Hensel's lemma the projection
$$\mathbb{K}[y]/((y^r - b)^n) \to \mathbb{K}[y]/(y^r - b)$$
splits, and thus $A$ contains a subfield isomorphic to $\mathbb{K}[y]/(y^r - b)$. On the other hand, the degree $[F: \mathbb{K}]$ of any subfield $F \subset A$ divides $n$. Indeed, the central simple algebra $A \otimes_{\mathbb{K}} F$ over $F$ acts on the $F$-vector space $A$. Hence, $[F: \mathbb{K}] = \dim_F A$ is divisible by $n$. Therefore, we see that $r$ divides $n$.

If $\text{char } \mathbb{K} = p$ divides $r$, write $r = pkm$ with $m$ coprime to $p$. Then $R$ contains a subfield isomorphic to $\mathbb{K}[z]/(z^m - b)$. Therefore, $m$ divides $n$. Next, the algebra $A \otimes_{\mathbb{K}} \mathbb{K}_{\text{sep}}$ splits and thus has a representation of dimension $n$ over $\mathbb{K}_{\text{sep}}$. On the other hand, $R \otimes_{\mathbb{K}} \mathbb{K}_{\text{sep}}$ contains a subalgebra of the form $\mathbb{K}_{\text{sep}}[u]/((u^{pk} - b)^n)$. Since the polynomial $u^{pk} - b$ remains irreducible over $\mathbb{K}_{\text{sep}}$, the dimension of any representation of $R \otimes_{\mathbb{K}} \mathbb{K}_{\text{sep}}$ in a $\mathbb{K}_{\text{sep}}$-vector space is divisible by $p^k$. In particular, $n$ is divisible by $p^k$.

If $A$ is a division algebra the minimal polynomial of any of its elements is irreducible and the last assertion of the lemma follows. \hfill $\square$

**Remark 6.4.** T. Bandman and Yu. Zarhin proved in [BZ17b, Theorem 3.4] a special case of Lemma 6.3 for $A = \text{Mat}_n(\mathbb{K})$ and $\text{char } \mathbb{K} = 0$.

Theorem 4.6 implies the following result about orders of finite groups with given exponents.

**Lemma 6.5.** Let $n$ and $d$ be positive integers, and let $\mathbb{K}$ be a field such that $\text{char } \mathbb{K}$ does not divide $d$. If $\text{char } \mathbb{K} > 0$, denote by $n'$ the largest factor of $n$ which is coprime to $\text{char } \mathbb{K}$; otherwise put $n' = n$. Let $G \subset \text{PGL}_n(\mathbb{K})$ be a finite subgroup. Suppose that for every $g \in G$ one has $g^d = 1$. Then
$$|G| \leq (n'd)^{n-1}.$$  
Moreover, if $n = d = 3$ and $\text{char } \mathbb{K} = 0$, then
$$|G| \leq 27.$$  

**Proof.** We can assume that the field $\mathbb{K}$ is algebraically (or separably) closed. Let $\tilde{G} \subset \text{SL}_n(\mathbb{K})$ be the preimage of $G$ with respect to the natural projection $\phi: \text{SL}_n(\mathbb{K}) \to \text{PGL}_n(\mathbb{K})$. The kernel of $\phi$ is a cyclic group of order $n'$ that consists of scalar matrices. Thus, for every $g \in \tilde{G}$, one has $g^{n'd} = 1$ and $|\tilde{G}| = n'|G|$. Let $\mu_{n'd} \subset \text{GL}_n(\mathbb{K})$ be the subgroup of scalar matrices whose order divides $n'd$. Consider the subgroup $\tilde{G} \mu_{n'd} \subset \text{GL}_n(\mathbb{K})$ generated by $G$ and $\mu_{n'd}$. The order of any of its elements still divides $n'd$. Thus, by Theorem 4.6 we have
$$|\tilde{G} \mu_{n'd}| \leq (n'd)^n.$$  

On the other hand, we also have
$$|\tilde{G} \mu_{n'd}| = n'd|G|,$$
and the first assertion of the lemma follows.

If $n = d = 3$, then $|G| = 3^r$ for some $r$, so that the second assertion of the lemma follows from the classification of finite subgroups of $\text{PGL}_3(\mathbb{K})$ over a field of characteristic zero, see [Bli17, Chapter V]. \hfill $\square$

Now we are ready to prove Proposition 1.8.
Proof of Proposition 1.8. Suppose that $A$ is a division algebra. Let $g \in \text{Aut}(X)$ be an element of finite order. We claim that $g^n = 1$. Indeed, by Lemma 6.1 the element $g$ corresponds to some invertible element $x$ of $A$ (defined up to $K^*$). Since $A$ is a division algebra, the minimal polynomial of $x$ must be irreducible over $K$. By Lemma 6.3 the minimal polynomial of $x$ has the form $y^r - a$ for some $a \in K^*$ and some $r$ which divides $n$.

Furthermore, one has

$$G \subset \text{Aut}(X) \subset \text{Aut}(X_{K_{\text{sep}}}) \cong \text{PGL}_m(K_{\text{sep}}).$$

Therefore, by Lemma 6.5 we have $|G| \leq n^{2(n-1)}$ in general, and also $|G| \leq 27$ in the case when char $K = 0$ and $n = 3$. Moreover, every central simple algebra of dimension $n^2$ is a division algebra provided that $n$ is a prime number. This proves assertions (ii) and (iii).

Now suppose that $A$ is not a division algebra. Then

$$A \cong D \otimes_K \text{Mat}_m(K)$$

for some $2 \leq m \leq n$, where $\text{Mat}_m(K)$ denotes the algebra of $m \times m$-matrices. Thus $A$ contains $\text{Mat}_m(K)$ as a subalgebra. Since the field $K$ contains roots of 1 of arbitrarily large degree, we see from Lemma 6.1 that the group $\text{Aut}(X)$ contains elements of arbitrarily large finite order. This completes the proof of assertion (i).

□

Remark 6.6. Let $K$ be a perfect field of positive characteristic $p$, and let $A$ be a division algebra of dimension $n^2$ over $K$. Then $p$ does not divide $n$. Indeed, the Frobenius morphism $\text{Fr}: K^* \to K^*$ is an isomorphism and, hence, the Brauer group

$$\text{Br}(K) \cong H^2(\text{Gal}(\overline{K}/K), K^*)$$

of $K$ has no $p$-torsion elements. Therefore, our claim follows from the fact that, over any field, the dimension of a central division algebra and the order of its class in the Brauer group have the same prime factors (see for instance [Lieb08, Lemma 2.1.1.3]).

Remark 6.7. Let $A$ be a central simple algebra over a field $K$. Denote by $A^*$ the algebraic group whose $S$-points are invertible elements in the algebra $A \otimes_K O_S$. We have a natural embedding $G_m \hookrightarrow A^*$ induced by the homomorphism $O_S^* \hookrightarrow (A \otimes_K O_S)^*$. The quotient group scheme $A^*/G_m$ is anisotropic if and only if $A$ is a division algebra (see, for instance, [Bor91, §23.1]). In particular, if $K$ is perfect, then Proposition 1.8(i) follows from Theorem 1.5 applied to the reductive group $A^*/G_m$.

The restriction on the characteristic of $K$ in Proposition 1.8 is essential for validity of the statement.

Example 6.8. Let $F$ be a field of characteristic $p > 0$, and let $K = F(x,y)$ be the field of rational functions in two variables, so that $K$ is a non-perfect field of characteristic $p$. Let $A$ be an algebra over $K$ with generators $u$ and $v$ and relations

$$v^p = x, \quad u^p = y, \quad vu - uv = 1.$$

Then $A$ is a central division algebra of dimension $p^2$ over $K$. This is a special case of the Azumaya property of the ring of differential operators in characteristic $p$ (see [BMR08, Theorem 2.2.3]) but can be also checked directly. The group $A^*(K)/K^*$ contains $F(v)^*/F(v^p)^*$ as a subgroup. The latter is an infinite dimensional vector space over the field $F_p$ of $p$ elements. In particular, for $p = 2$ this construction provides a conic $C$ over a non-perfect field of characteristic $2$ such that $C$ is acted on by elementary $2$-groups of arbitrary order.
Now we prove Proposition 1.9.

**Proof of Proposition 1.9.** By Lemma 6.1 we have \( \text{Aut}(\mathcal{X}) \cong A^*(\mathbb{K})/\mathbb{K}^* \). Recall the reduced norm homomorphism:

\[
\text{Norm}: A^*(\mathbb{K}) \to \mathbb{K}^*.
\]

One has \( \text{Norm}(cx) = c^n \text{Norm}(x) \) for every \( c \in \mathbb{K}^* \) and \( x \in A^*(\mathbb{K}) \). Hence, \( \text{Norm} \) induces a homomorphism

\[
A^*(\mathbb{K})/\mathbb{K}^* \to \mathbb{K}^*/(\mathbb{K}^*)^n,
\]

where \((\mathbb{K}^*)^n \subset \mathbb{K}^*\) is the subgroups of \( n \)-th powers. Composing homomorphism (6.9) with the projection \( \mathbb{K}^*/(\mathbb{K}^*)^n \to \mathbb{K}^*/(\mathbb{K}^*)^{p^m} \), we get

\[
A^*(\mathbb{K})/\mathbb{K}^* \to \mathbb{K}^*/(\mathbb{K}^*)^{p^m},
\]

Let \( \Gamma(A) \) be the kernel of homomorphism (6.10). We claim that the order of every element of \( \Gamma(A) \) of finite order divides \( n' \). Indeed, if \( n = n' \), this follows from Proposition 1.8(ii). Hence, we may assume that \( m > 0 \). By Lemma 6.3 the order of every element of \( \Gamma(A) \) of finite order divides \( n \). Thus, it suffices to check that \( \Gamma(A) \) has no elements of order \( p \). Assuming the contrary, let \( g \in \Gamma(A) \) be an element of order \( p \), and let \( x \) be its preimage in \( A^*(\mathbb{K}) \). Then \( x^p = a \) for some \( a \in \mathbb{K}^* \). Since \( a \) does not belong to \((\mathbb{K}^*)^p\), the image of \( a \) in \( \mathbb{K}^*/(\mathbb{K}^*)^{p^m} \) has order \( p^m \). On the other hand, we have \( \text{Norm}(x) = a^{p^m} \). Thus, \( \text{Norm}(x) \) is not equal to \( 1 \) in \( \mathbb{K}^*/(\mathbb{K}^*)^{p^m} \), that is, \( g \) does not belong to \( \Gamma(A) \), which gives a contradiction.

It follows that for every finite subgroup \( H \subset \Gamma(A) \), one has \( |H| \leq n'^{2(n-1)} \). The proof repeats verbatim the proof of the corresponding estimate from Proposition 1.8(ii), using the above multiplicative bound on the orders of finite order elements of \( \Gamma(A) \).

Now let \( G \subset A^*(\mathbb{K})/\mathbb{K}^* \) be a finite subgroup. Set \( H = G \cap \Gamma(A) \). Then \( H \) is normal and the quotient \( G/H \) is a subgroup of \( \mathbb{K}^*/(\mathbb{K}^*)^{p^m} \). Moreover, a \( p \)-Sylow subgroup of \( G \) must project isomorphically to \( G/H \). Thus, \( G \) is isomorphic to a semi-direct product of \( H \) and \( G/H \).

We conclude this sections with a few further remarks on division algebras over a field of characteristic \( p \).

**Lemma 6.11.** Let \( A \) be a division algebra of dimension \( n^2 \) over a field \( \mathbb{K} \) of finite characteristic \( p \) that contains all roots of \( 1 \). Then the group \( A^*(\mathbb{K})/\mathbb{K}^* \) has bounded finite subgroups if and only if every element of \( v \in A \) is separable over \( \mathbb{K} \), that is, the field extension \( \mathbb{K}(v) \supset \mathbb{K} \) is separable.

**Proof.** If \( v \in A \) is not separable, then there exists a subfield \( \mathbb{K} \subset L \subset \mathbb{K}(v) \) which is a purely inseparable extension of \( \mathbb{K} \) of degree \( p \). Then every non-trivial element of the group \( L^*/\mathbb{K}^* \) has order \( p \). Since \( |L^*/\mathbb{K}^*| = \infty \), we conclude that the group \( L^*/\mathbb{K}^* \subset A^*(\mathbb{K})/\mathbb{K}^* \) has unbounded finite subgroups.

Conversely, suppose that every element of \( A \) is separable over \( \mathbb{K} \). Denote by \( n' \) the largest factor of \( n \) which is coprime to \( \text{char} \mathbb{K} \). Then by Lemma 6.3 for every element \( v \in A^*(\mathbb{K}) \) whose image in \( A^*(\mathbb{K})/\mathbb{K}^* \) has finite order, one has \( v^{n'} \in \mathbb{K}^* \) (otherwise \( v^{n'} \) would be inseparable over \( \mathbb{K} \)). Hence, the assertion follows from Lemma 6.5. \( \square \)

There are examples of division algebras of dimension \( p^4 \) over a field \( \mathbb{K} \) of characteristic \( p \) such that all their elements are separable over \( \mathbb{K} \). On the other hand, we do not know if it is true that every division algebra \( A \) of dimension \( p^2 \) over a field \( \mathbb{K} \) of characteristic \( p \) contains
an element $v \in A$ which is inseparable over $\mathbb{K}$. We refer the reader to [ABGV11, §1] for a further discussion of this problem and references therein.

The following result shows that the division algebra in Example 6.8 provides a universal example of a division algebra of dimension $p^2$ over a field of characteristic $p$ which contains an inseparable element.

**Lemma 6.12** (cf. “Albert’s cyclicity criterion”, [Alb61, Theorem XI.4.4]). Let $A$ be a division algebra of dimension $p^2$ over a field $\mathbb{K}$ of finite characteristic $p$.

(i) The following conditions are equivalent.

1. $A$ contains an element $v \in A$ which is inseparable over $\mathbb{K}$.
2. $A$ is generated by two elements $v, u \in A$ such that $v^p, u^p \in \mathbb{K}$ and $vu - uv = 1$.
3. $A$ contains a subfield $L \subset A$ that is a Galois extension of $\mathbb{K}$ of degree $p$.

(ii) If $p = 2$ or $p = 3$, then every division algebra $A$ of dimension $p^2$ over a field $\mathbb{K}$ of characteristic $p$ contains an element $v \in A$ which is inseparable over $\mathbb{K}$. In particular, for every Severi–Brauer variety $X$ of dimension $p - 1$ over an infinite field $\mathbb{K}$ of characteristic $p$, the group $\text{Aut}(X)$ has unbounded finite subgroups.

**Proof.** The equivalence of conditions (1) and (3) is due to Albert [Alb61].

Let us show that condition (1) implies condition (2). Suppose that an element $v \in A$ is inseparable over $\mathbb{K}$. Consider the linear map $\text{ad} v : A \to A$, which takes an element $w \in A$ to the commutator $vw - vw$. Since $v^p \in \mathbb{K}$, we see that $(\text{ad} v)^p = \text{ad} v^p = 0$. On the other hand, the kernel of $\text{ad} v$ has dimension $p$ over $\mathbb{K}$. Hence, the image of $\text{ad} v$ is equal to the kernel of $(\text{ad} v)^{p-1}$. In particular, the element 1 is in the image of $\text{ad} v$. Thus, there exists an element $u \in A$ such that $vu - uv = 1$. Let

$$y^p + a_{p-1}y^{p-1} + \ldots + a_0 \in \mathbb{K}[y]$$

be the minimal polynomial for $u$. Applying the operator $\text{ad} v$ to the equality

$$w^p + a_{p-1}w^{p-1} + \ldots + a_0 = 0,$$

we compute that

$$(p - 1)a_{p-1}w^{p-1} + \ldots + 2a_2u + a_1 = 0,$$

which is impossible unless $a_{p-1} = \ldots = a_1 = 0$. Thus, we have $w^p \in \mathbb{K}$.

To prove that condition (2) implies condition (3), assume that $v, u \in A$ are as in (2). Then one can easily check (or deduce from [BMR08, Lemma 1.3.1]) that

$$(uv)^p - uv = w^pv^p \in \mathbb{K}.$$ 

Thus, the subfield $\mathbb{K}(uv) \subset A$ is a Galois extension of $\mathbb{K}$.

Finally, assume that a subfield $L \subset A$ is a Galois extension of $K$. Its Galois group is a cyclic group of order $p$. Let $\sigma$ be its generator. By the Skolem–Noether theorem the action of $\sigma$ on $L$ extends to an inner automorphism of $A$ given by an element $v \in A$, that is, $\sigma(v) = \sigma(v)u \in L$. Then, since the normalizer of $L$ in $A$ is $L$ itself, we have that $v^p \in L$. Since $L$ is separable over $K$, the element $v^p$ is, in fact, contained in $\mathbb{K}$. This completes the proof of assertion (i).

To prove assertion (ii), recall that every division algebra of dimension $p^2$ contains a subfield $L \subset A$ of degree $p$ over $\mathbb{K}$, which is separable over $\mathbb{K}$. If $p = 2$, then any such subfield is a Galois extension of $\mathbb{K}$. For $p = 3$, existence of a subfield $L \subset A$ that is a Galois extension $\mathbb{K}$ of degree $p$ was proved in [Wed21]. The last assertion follows from Lemmas 6.1 and 6.11.\qed
7. Conic bundles

In this section we study groups acting on conic bundles. Some of the minor auxiliary assertions below will be valid over arbitrary fields, but in spite of this we will always assume that the base field is perfect, to make sure that Galois-invariant objects are defined over the base field and thus to make the arguments more transparent.

By a conic bundle we will mean a smooth projective geometrically irreducible surface $X$ with a morphism $\phi: X \to C$ to a smooth curve $C$ such that everything is defined over some field $K$ and every fiber of $\phi_K$ is isomorphic to a (possibly reducible) conic in $\mathbb{P}^2$; note that the fibers are reduced due to the assumption that $S$ is smooth. By the discriminant locus of $\phi$ we mean the complement to the maximal open subset of $C$ over which the morphism $\phi$ is smooth; this set is defined over $K$. We say that $\phi$ is relatively minimal if $\text{rk Pic}(X) - \text{rk Pic}(S) = 1$.

Let $\phi: X \to C$ be a conic bundle, and $G$ be a group acting by birational automorphisms of $X$ such that $\phi$ is $G$-equivariant. Then there is an exact sequence of groups
\begin{equation}
1 \to G_\phi \to G \to G_C \to 1,
\end{equation}
where the action of $G_\phi$ is fiberwise with respect to $\phi$, and $G_C \subset \text{Aut}(C)$. The group $G_\phi$ can be also considered as a subgroup of $\text{Aut}(X_\eta)$, where $X_\eta$ is a schematic general fiber of $\phi$.

Remark 7.2 (cf. [Pro17, Corollary 3.3.6]). In the above notation, let $\Omega \subset C$ be the discriminant locus of $\phi$, and suppose that $\phi$ is relatively minimal. Suppose also that the base field has characteristic different from 2. Then the group $G_C$ preserves $\Omega$. This is obvious in the case when $G$ acts by biregular transformations of $X$, and follows from explicit computations of the action of $G$ on the quadratic form that defines the schematic general fiber of $\phi$ in the general case. We refer the reader to [Pro17, Lemma 3.3.5] for details.

The following assertion is well known to experts (see [Isk96, §1.1]), but we provide its proof for the reader’s convenience.

Lemma 7.3. Let $K$ be a perfect field. Let $C$ be a conic, and $\phi: X \to C$ be a conic bundle over $K$. The following assertions hold.

(i) If $X_K$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then $X \cong C \times C'$ for some conic $C'$, and $\phi$ is the projection to the first factor.

(ii) If $\phi$ is smooth and does not have sections, then $X \cong C \times C'$ for some conic $C'$, and $\phi$ is the projection to the first factor.

Proof. Suppose that $X_K \cong \mathbb{P}^1 \times \mathbb{P}^1$. Since $X_K$ has exactly two extremal contractions, and one of them is defined over $K$ by assumption, the other one is defined over $K$ as well, which implies assertion (i).

Suppose that $\phi$ is smooth, so that every fiber of $\phi_K$ is irreducible. Then $X_K$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, hence it is isomorphic to one of the surfaces
\[ \mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n)), \quad n \geq 0, \]
over $K$. If $n \geq 1$, then $\phi_K$ has a unique section $D$ with negative self-intersection, which is thus defined over $K$ and provides a section of $\phi$. Thus, we have $n = 0$. We conclude that $X_K \cong \mathbb{P}^1 \times \mathbb{P}^1$, so that $X \cong C \times C'$ for some conic $C'$ by assertion (i). This proves assertion (ii).
Remark 7.4. Let $\phi: X \to C$ be a conic bundle over a perfect field $K$ such that $\phi$ has a unique geometrically reducible fiber. Then this fiber is Galois-invariant, and its unique singular point is Galois-invariant as well. Thus, $X$ has a $K$-point in this case.

For Severi–Brauer curves, Proposition 1.8 provides the following generalization of Theorem 1.1.

Corollary 7.5. Let $K$ be either a perfect field, or a field of characteristic different from 2. Suppose that $K$ contains all roots of 1. Let $C$ be a conic over $K$ with $C(K) = \emptyset$. Then every non-trivial element of finite order in $\text{Aut}(C)$ has order 2, and every finite subgroup of $\text{Aut}(C)$ has order at most 4.

Proof. If $K$ is perfect, then $\text{char } K \neq 2$ by Remark 6.6. Thus, the assertion follows from Proposition 1.8(ii).

Lemma 7.6. Let $K$ be a perfect field that contains all roots of 1. Let $C$ be a conic over $K$ with $C(K) = \emptyset$, and $\phi: X \to C$ be a conic bundle over $K$. Suppose that $\phi$ has no sections. Let $G$ be a finite group acting by birational automorphisms of $X$ such that $\phi$ is $G$-equivariant. Then $|G| \leq 16$.

Proof. Since $K$ is perfect, one has $\text{char } K \neq 2$ by Remark 6.6.

We use the exact sequence (7.1). The group $G_\phi$ is a subgroup of $\text{Aut}(X_\eta)$, where $X_\eta$ is a schematic general fiber of $\phi$. Since $\phi$ has no sections, the conic $X_\eta$ has no points over the function field $K(C)$. Therefore, one has $|G_\phi| \leq 4$ by Corollary 7.5. (Note that the field $K(C)$ is not perfect if the characteristic of $K$ is positive, but the characteristic of $K(C)$ is still different from 2 by assumption!) Similarly, since the conic $C$ has no $K$-points, we also have $|G_C| \leq 4$ by Corollary 7.5. Hence $|G| = |G_\phi| \cdot |G_C| \leq 16$.

Remark 7.7. Let $C$ be a conic over a field $K$. Then the diagonal in $X = C \times C$ provides a section of the projection $\phi: X \to C$ on the first factor. This means that $X$ is birational to $C \times \mathbb{P}^1$. So the group of birational automorphisms of $X$ has unbounded finite subgroups provided that the field $K$ contains all (or at least an infinite number of) roots of 1, and the assertion of Lemma 7.6 fails in this case. Note that $X$ can be embedded as a quadric into $\mathbb{P}^3$, see [Lied17, Proposition 5.2(1)]. We will see below in Corollary 7.8 that if $C(K) = \emptyset$, then the group $\text{Aut}(X)$ still has bounded finite subgroups provided that $K$ is a perfect field that contains all roots of 1.

Corollary 7.8 immediately implies the following.

Corollary 7.8. Let $K$ be a perfect field that contains all roots of 1. Let $C$ and $C'$ be conics over $K$ without $K$-points, and put $X = C \times C'$. Then every finite subgroup of $\text{Aut}(X)$ has order at most 32.

Proof. Let $G$ be a finite subgroup of $\text{Aut}(X)$. Then $G$ acts on the set of extremal rays (of the cone of effective 1-cycles) of $X$. In particular, $G$ has a subgroup $\bar{G}$ of index at most 2 that preserves both extremal contractions from $X$, i.e., the projections from $X$ to both factors. We conclude that $\bar{G}$ is a subgroup of $\text{Aut}(C) \times \text{Aut}(C')$. Thus the assertion follows from Corollary 7.6.

It appears that in certain cases conic bundles over $\mathbb{P}^1$ behave similarly to those over conics without points, and the following assertion analogous to Lemma 7.6 holds.
Lemma 7.9. Let \( K \) be a perfect field of characteristic different from 2 that contains all roots of 1. Let \( \phi: X \to \mathbb{P}^1 \) be a relatively minimal conic bundle over \( K \) without sections. Suppose that \( \phi_K \) has exactly two reducible fibers over some points \( O_1, O_2 \in \mathbb{P}^1_K \), and that the points \( O_1 \) and \( O_2 \) are \( \text{Gal}(\overline{K}/K) \)-conjugate. Let \( G \) be a finite group acting by birational automorphisms of \( X \) such that \( \phi \) is \( G \)-equivariant. Then \( |G| \leq 16 \).

Proof. We use the exact sequence (7.1) in the same way as in the proof of Lemma 7.6. The group \( G_\phi \) acts by automorphisms of a conic over the function field \( K(\mathbb{P}^1) \) that has no \( K(\mathbb{P}^1) \)-points, and thus \( |G_\phi| \leq 4 \) by Corollary 7.5. On the other hand, by Remark 7.2 the group \( G_C \) is contained in the automorphism group of the affine curve

\[ U = \mathbb{P}^1 \setminus \{O_1, O_2\}. \]

By Corollary 3.8 this gives \( |G_C| \leq 4 \), so that \( |G| = |G_\phi| \cdot |G_C| \leq 16 \). \( \Box \)

The following result is also well-known (see \cite{Isk96, §4.7} or \cite{Pro17, Proposition 5.2}), but we provide its proof for completeness.

Lemma 7.10. Let \( K \) be a perfect field. Let \( C \) be a conic, and \( \phi: X \to C \) be a conic bundle over \( K \) such that \( \phi_K \) has exactly two reducible fibers. Suppose that \( \phi \) has no sections. Then \( X \) is a del Pezzo surface with \( K_X^2 = 6 \).

Proof. Contracting one irreducible component of each reducible fiber on \( X_\phi \) we obtain a conic bundle \( \phi': X' \to \mathbb{P}^1 \) over \( K \) that has no reducible fibers. Thus

\[ X' \cong \mathbb{F}_n = \mathbb{P}^1(\mathcal{O} \oplus \mathcal{O}(n)) \]

for some \( n \geq 0 \), and \( X_\phi \) is obtained from \( X' \) by blowing up two points \( P_1 \) and \( P_2 \) not contained in one fiber of \( \phi' \).

Suppose that \( n \geq 1 \). Denote by \( D \) the unique section of \( \phi' \) with negative self-intersection. Let \( \overline{D} \) be its proper transform on \( X_\phi \), so that \( \overline{D} \) is a section of \( \phi_\phi \). Note that any section of \( \phi' \) except \( D \) has self-intersection at least \( n \). If \( n \geq 2 \), or if \( n = 1 \) and at least one of the points \( P_1 \) and \( P_2 \) is contained in \( D \), then \( \overline{D} \) is the only section of \( \phi_\phi \) with negative self-intersection; hence it is Galois-invariant, which gives a contradiction. Thus we see that \( n = 1 \) and none of the points \( P_1 \) and \( P_2 \) is contained in \( D \), which means that \( X_\phi \), and thus also \( X \), is a del Pezzo surface.

Now suppose that \( n = 0 \). If \( P_1 \) and \( P_2 \) are contained in one section \( D \) of \( \phi_\phi \) with self-intersection 0, then the proper transform of \( D \) on \( X_\phi \) is the only section of \( \phi_\phi \) with negative self-intersection, which gives a contradiction. Thus we conclude that \( P_1 \) and \( P_2 \) are contained in different sections of \( \phi_\phi \) with self-intersection 0, which implies that \( X_\phi \), and thus also \( X \), is a del Pezzo surface.

The equality \( K_X^2 = 6 \) follows from Noether’s formula. \( \Box \)

Corollary 7.11. Let \( K \) be a perfect field of characteristic different from 2 that contains all roots of 1. Let \( C \) be a conic, and \( \phi: X \to C \) be a relatively minimal conic bundle over \( K \) such that \( \phi_K \) has exactly two reducible fibers. Suppose that \( \phi \) has no sections. Let \( G \) be a finite group acting by birational automorphisms of \( X \) such that \( \phi \) is \( G \)-equivariant. Then \( |G| \leq 16 \).

Proof. If \( C(K) = \emptyset \), then the assertion follows from Lemma 7.6. Therefore, we will assume that \( C \cong \mathbb{P}^1 \). Moreover, if the images of the reducible fibers of \( \phi_K \) are \( \text{Gal}(\overline{K}/K) \)-conjugate, the assertion follows from Lemma 7.9. Therefore, we will assume that both
points in the discriminant locus \( \Omega \) of \( \phi \) are defined over \( K \). We claim that under such assumptions \( X \) is birational to \( \mathbb{P}^1 \times C' \) for some conic \( C' \), so that this case does not arise.

We know from Lemma 7.10 that \( X \) is a del Pezzo surface with \( K_X^2 = 6 \). Let \( Z \subset X \) be the union of all \((-1)\)-curves on \( X \) (defined over \( \overline{K} \)). Then \( Z \) is defined over \( K \). Put \( U = X \setminus Z \). The open subset \( U \) is a torsor under some two-dimensional torus \( T \), which can be seen from a construction of \( X_{\overline{K}} \) as a blow up of three points on \( \mathbb{P}^2 \).

The restriction of \( \phi \) to \( U \) provides a morphism \( \pi: U \to \mathbb{P}^1 \setminus \Omega \sim = \mathbb{G}_m \).

One can check that there is a one-dimensional subtorus \( \overline{S} \) in \( T_{\overline{K}} \) such that \( \pi_{\overline{K}} \) is the geometric quotient of \( U_{\overline{K}} \) by \( \overline{S} \); this follows for instance from a description of \( X_{\overline{K}} \) as a toric variety. Now Corollary 2.5 implies that \( U \), and thus also \( X \), is birational to \( \mathbb{P}^1 \times C' \) for some conic \( C' \), which is a contradiction. \( \square \)

**Remark 7.12.** Relative minimality in Corollary 7.11 can be deduced from the remaining assumptions.

While conic bundles that have few reducible fibers over \( K \) and have no \( K \)-points may possess non-trivial birational properties, those with \( K \)-points are easy to understand due to the following general result.

**Theorem 7.13** (see [Isk96, §4]). Let \( K \) be a perfect field, and let \( X \) be a smooth projective geometrically rational surface over \( K \). Suppose that \( X(K) \neq \emptyset \). The following assertions hold.

(i) If \( K_X^2 \geq 5 \), then \( X \) is rational.

(ii) If \( X \) has a structure of a relatively minimal conic bundle and \( K_X^2 \leq 4 \), then \( X \) is not rational.

**Corollary 7.14.** Let \( K \) be a perfect field of characteristic different from 2 that contains all roots of 1. Let \( C \) be a conic, and \( \phi: X \to C \) be a conic bundle over \( K \) such that \( \phi_{\overline{K}} \) has at most two reducible fibers. Suppose that \( \phi \) has no sections. Let \( G \) be a finite subgroup acting by birational automorphisms of \( X \) so that \( \phi \) is \( G \)-equivariant. Then \( |G| \leq 16 \).

**Proof.** The surface \( X \) has no \( K \)-points, since otherwise it is rational by Theorem 7.13(i). Also, our assumptions imply that the morphism \( \phi \) has no sections.

Let \( m \) be the number of reducible fibers of \( \phi_{\overline{K}} \). If \( m = 2 \), then the assertion follows from Corollary 7.11. By Remark 7.3 we have \( m \neq 1 \). Thus we can assume that \( m = 0 \), so that by Lemma 7.3(ii) one has \( X \cong C \times C' \) for some conic \( C' \), and \( \phi \) is the projection to the first factor. In this case the conic \( C \) has no \( K \)-points, and the assertion follows from Lemma 7.6. \( \square \)

**Lemma 7.15.** Let \( K \) be a perfect field of characteristic different from 2 that contains all roots of 1. Let \( C \) be a conic, and \( \phi: X \to C \) be a conic bundle over \( K \). Let \( G \subset \text{Aut}(X) \) be a finite group such that \( \phi \) is \( G \)-equivariant. Suppose that \( \phi \) has no sections. Suppose also that there is a (closed) finite subset \( \Omega \subset C \) such that \( |\Omega_{\overline{K}}| = m \geq 3 \), and \( \phi^{-1}(\Omega) \) is \( G \)-invariant. Then

\[ |G| \leq 4m! \]

**Proof.** There is an exact sequence of groups (7.1), where

\[ G_C \subset \text{Aut}(C) \subset \text{Aut}(C_{\overline{K}}) \cong \text{PGL}_2(\mathbb{K}). \]
The set $\Omega$ is $G_\phi$-invariant, so that the estimate $|G_\phi| \leq m!$ holds.

On the other hand, the group $G_\phi$ is a subgroup of $\text{Aut}(X_\eta)$, where $X_\eta$ is a schematic general fiber of $\phi$. Since $\phi$ has no sections, the conic $X_\eta$ has no points over the function field $K(C)$. Therefore, one has $|G_\phi| \leq 4$ by Corollary 7.5.

Remark 7.16. If in the assumptions of Lemma 7.15 we also suppose that the conic bundle $\phi$ is relatively minimal, and $\Omega$ is taken to be the discriminant locus of $\phi$, then the assertion of the lemma holds for any finite group $G$ acting by birational automorphisms of $X$ so that $\phi$ is $G$-equivariant. This follows from Remark 7.2.

We conclude this section by recalling that relatively minimal conic bundles with many geometrically reducible fibers are known to have few birational models.

Theorem 7.17 (see [Isk96, Theorem 1.6(iii)]). Let $K$ be a perfect field. Let $X$ and $X'$ be smooth projective geometrically rational surfaces over $K$, and $X \to B$ and $X' \to B'$ be relatively minimal conic bundles. Suppose that $X$ and $X'$ are birational, and $K_X^2 \leq 0$. Then $K_X^2 = K_{X'}^2$.

8. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 1.3 and Corollary 1.4 and provide a counterexample to Theorem 1.3 over a perfect field of characteristic 2. We start by observing that Theorem 1.6 applied to del Pezzo surfaces implies the following result (which is a particular case of Example 1.7, cf. Proposition 1.11).

Corollary 8.1. Let $K$ be a perfect field that contains all roots of 1. Then there exists a constant $\delta$ with the following property. Let $X$ be a smooth del Pezzo surface over $K$ such that $X$ is not birational to $\mathbb{P}^1 \times C$, where $C$ is a conic. Then every finite subgroup of $\text{Aut}(X)$ has order at most $\delta$.

Now we are ready to prove our main result.

Proof of Theorem 1.3. The proof is similar to that of Theorem 1.2; see [PS16b §2]. If $X$ is birational to $\mathbb{P}^1 \times C$, then the assertion is clear. Thus we will assume that $X$ is not birational to such a surface.

Let $G$ be a finite group acting by birational automorphisms of $X$. Replace $X$ with its $G$-minimal model $Y$. Then $Y$ is either a $G$-del Pezzo surface (not necessarily minimal if we drop the action of $G$ and consider only the action of the Galois group), or a $G$-conic bundle (again not necessarily relatively minimal if we drop the action of $G$), see [Isk80]. In the former case the order of $G$ is bounded by some universal constant due to Corollary 8.1. So we can assume that there is a structure $\phi: Y \to C$ of a $G$-conic bundle over some conic $C$.

Let $\Delta \subset C$ be the discriminant locus of $\phi$. Every fiber $\bar{F}_i$ of $\phi_\bar{K}$ over a point of $\Delta_\bar{K}$ has the form $\bar{F}_i^{(1)} \cup \bar{F}_i^{(2)}$, where $\bar{F}_i^{(1)}$ and $\bar{F}_i^{(2)}$ are $(-1)$-curves meeting transversally at one point. Relabelling the fibers if necessary, we may assume that $\bar{F}_1, \ldots, \bar{F}_m$ are fibers whose irreducible components are Gal($\bar{K}/K$)-conjugate, and $\bar{F}_{m+1}, \ldots, \bar{F}_n$ are ones whose irreducible components are not Gal($\bar{K}/K$)-conjugate. Let $\Omega$ be the subset of $C_\bar{K}$ that consists of the images of the fibers $\bar{F}_1, \ldots, \bar{F}_m$. Then $\Omega$ is a closed subset of $C$ defined over $K$, and $\phi^{-1}(\Omega)$ is $G$-invariant. Moreover, if we discard the action of $G$, then $Y$ becomes birational to a relatively minimal conic bundle $\phi': Y' \to C$ such that $\phi'$ has $m$ geometrically reducible fibers exactly over the points of $\Omega$ (so that $\Omega$ is the discriminant
locus of \( \phi' \)). Note that the group \( G \) acts by birational automorphisms of \( Y' \) so that \( \phi' \) is \( G \)-equivariant. Therefore, applying Corollary 7.14 to the conic bundle \( \phi' \) we deduce the assertion of the theorem in the case \( m \leq 2 \). Therefore, we may assume that \( m \geq 3 \). Now applying Lemma 7.15 to the conic bundle \( \phi \) (or applying Remark 7.16 to the conic bundle \( \phi' \)), we see that the estimate \( |G| \leq 4m! \) holds.

We conclude that the only way how the group of birational automorphisms of \( X \) can have unbounded finite subgroups is that there is a sequence of finite groups \( G_j \) acting by birational automorphisms of \( X \), such that \( G_j \) have unbounded order, the action of \( G_j \) is regularized on a conic bundle \( \phi_j \): \( Y_j \to C_j \), the morphism \( (\phi_j)_R \) has \( m_j \) reducible fibers whose irreducible components are Galois-conjugate, and \( m_j \) is unbounded. However, Theorem 7.17 implies that for \( m_j \geq 8 \) the number \( m_j \) is a birational invariant of \( Y_j \), and thus also of \( X \), which shows that this situation is impossible. \( \square \)

**Remark 8.2.** If a smooth projective variety \( X \) over a field \( \mathbb{K} \) is birational to \( \mathbb{P}^1 \times Y \) for some smooth projective variety \( Y \), and \( X \) has no \( \mathbb{K} \)-points, then \( \mathbb{P}^1 \times Y \), and thus also \( Y \), has no \( \mathbb{K} \)-points as well. Indeed, having \( \mathbb{K} \)-points is a birationally invariant property for smooth proper varieties by the theorem of Lang and Nishimura, see for instance [Ko96, Proposition IV.6.2]. In particular, if \( \mathbb{K} \) is a perfect field of characteristic different from 2 that contains all roots of 1, and \( X \) is a smooth geometrically rational surface over \( \mathbb{K} \) that has a \( \mathbb{K} \)-point, then by Theorem 1.3 the surface \( X \) is birational to a surface \( Y \cong \mathbb{P}^1 \times C \) for some conic \( C \), and the surface \( Y \) has a \( \mathbb{K} \)-point. The latter implies that \( C \) has a \( \mathbb{K} \)-point as well, so that \( C \cong \mathbb{P}^1 \) and \( X \) is rational. Thus, Theorem 1.3 implies a generalization of Theorem 1.2 for the fields \( \mathbb{K} \) with the above properties.

Now we prove Corollary 7.14.

**Proof of Corollary 7.14** Suppose that the group of birational automorphisms of \( X \) has unbounded finite subgroups. Then by Theorem 1.3 the surface \( X \) is birational to a surface \( Y \cong \mathbb{P}^1 \times C \) for some conic \( C \). Since \( \mathbb{P}^1 \) has a \( \mathbb{K} \)-point, and \( C \) has a point defined over a quadratic extension of \( \mathbb{K} \), we conclude that \( Y \) has a point defined over a quadratic extension of \( \mathbb{K} \) as well. Therefore, the same holds for \( X \) by the theorem of Lang and Nishimura. However, a Severi–Brauer surface with a point defined over a quadratic extension is isomorphic to \( \mathbb{P}^2 \), so we obtain a contradiction. \( \square \)

Note that there is a description of birational automorphism groups of Severi–Brauer surfaces via their generators and relations, see [IT91, §3]. We do not know whether it can be used to prove Corollary 7.14.

Below, we will give an example showing that Theorem 1.3 fails over perfect fields of characteristic 2. We start with the following general construction.

**Lemma 8.3.** Let \( \mathbb{K} \) be a perfect field of characteristic \( p > 0 \), let \( \Delta \subset \mathbb{P}^1(\mathbb{K}) \) be a finite subset consisting of more than one point, and let \( Y = \mathbb{P}^1 \setminus \Delta \). Assume that \( \mathbb{K} \) admits a Galois extension of degree \( p \) and either \( p \neq 2 \) or the cardinality of \( \Delta \) is even. Then the following assertions hold.

(i) There exists an element \( \alpha \in \text{Br}(Y) \) of order \( p \) in the Brauer group of \( Y \) which is not contained in the image of the restriction map \( \text{Br}(\mathbb{P}^1 \setminus \Delta') \to \text{Br}(Y) \), for any proper subset \( \Delta' \subset \Delta \).

(ii) For any \( \alpha \) as above, there exists an Azumaya algebra \( A \) over \( Y \) of rank \( p^2 \) whose class in the Brauer group of \( \text{Br}(Y) \) is equal to \( \alpha \). If \( X \) is the corresponding
Severi–Brauer scheme over $Y$, then the structure morphism $\pi: X \to Y$ cannot be extended to a smooth proper morphism $\pi': X' \to Y'$, for any strictly larger open subscheme $Y' \subset \mathbb{P}^1$.

Proof. We compute $\text{Br}(Y) \cong H^2(Y_{\text{ét}}, \mathcal{O}^*_Y)$ using the Leray spectral sequence that converges to $H^*(Y_{\text{ét}}, \mathcal{O}^*_Y)$ and whose second page is given by $H^i(\text{Gal}(\overline{K}/K), H^j(Y_{\overline{\text{K}}, \text{ét}}, \mathcal{O}^*_Y))$. By Tsen’s theorem, we know that

$$H^2(Y_{\overline{\text{K}}, \text{ét}}, \mathcal{O}^*_Y) \cong \text{Br}(Y_{\overline{\text{K}}}) = 0.$$ 

In addition, one has $H^1(Y_{\overline{\text{K}}, \text{ét}}, \mathcal{O}^*_Y) \cong \text{Pic}(Y_{\overline{\text{K}}}) = 0$. Hence, we conclude that

$$\text{Br}(Y) \cong H^1(\text{Gal}(\overline{K}/K), \mathcal{O}^*(Y_{\overline{\text{K}}})).$$

We have an exact sequence of Galois modules

$$(8.4) \quad 0 \to K^* \to \mathcal{O}^*(Y_{\overline{\text{K}}}) \to Z[\Delta] \to Z \to 0,$$ 

where $Z[\Delta]$ is the group of divisors on $\Delta$ and the last morphism is the degree map. The sequence $(8.4)$ splits as a sequence of Galois modules, that is, it is a direct sum of sequences of Galois modules of length 2. Since $(8.4)$ splits, it remains exact (and split) after passing to Galois cohomology:

$$0 \to \text{Br}(K) \to \text{Br}(Y) \xrightarrow{\text{Res}} H^2(\text{Gal}(\overline{K}/K), Z[\Delta]) \to H^2(\text{Gal}(\overline{K}/K), Z) \to 0.$$ 

Since $H^i(\text{Gal}(\overline{K}/K), Q) = 0$ for every $i > 0$, we have

$$H^2(\text{Gal}(\overline{K}/K), Z) \cong \text{Hom}(\text{Gal}(\overline{K}/K), Q/Z).$$

Thus, the above exact sequence takes the form

$$0 \to \text{Br}(K) \to \text{Br}(Y) \xrightarrow{\text{Res}} \bigoplus_{\Delta} \text{Hom}(\text{Gal}(\overline{K}/K), Q/Z) \xrightarrow{\Sigma} \text{Hom}(\text{Gal}(\overline{K}/K), Q/Z) \to 0.$$ 

Tensoring this exact sequence $(8.4)$ by $Z/pZ$ and using the vanishing of the $p$-torsion part of $\text{Br}(K)$ (see Remark $6.6$), we find the following short exact sequence

$$0 \to \text{Br}(Y)_p \xrightarrow{\text{Res}} \bigoplus_{\Delta} \text{Hom}(\text{Gal}(\overline{K}/K), Z/pZ) \xrightarrow{\Sigma} \text{Hom}(\text{Gal}(\overline{K}/K), Z/pZ) \to 0,$$ 

where we write $\text{Br}(Y)_p$ for the group of $p$-torsion elements in $\text{Br}(Y)$. Since $K$ has a Galois extension of degree $p$, the group $\text{Hom}(\text{Gal}(\overline{K}/K), Z/pZ)$ is nontrivial (and, thus, it contains a cyclic subgroup of order $p$). Under our assumptions on $p$ and the cardinality of $\Delta$ we can find an element of

$$\ker \Sigma \subset \bigoplus_{\Delta} \text{Hom}(\text{Gal}(\overline{K}/K), Z/pZ)$$

such that each of its components in $\text{Hom}(\text{Gal}(\overline{K}/K), Z/pZ)$ is not equal to zero. Let $\alpha$ be the corresponding element of $\text{Br}(Y)_p$. The compatibility of all our constructions with the restriction map $\text{Br}(\mathbb{P}^1 \setminus \Delta') \to \text{Br}(Y)$ shows that $\alpha$ is not contained in the image of such map. This proves assertion (i).

To prove assertion (ii), we recall from [OV07, Proposition 4.2] that for every smooth affine scheme $Z$ over a perfect field $K$ of characteristic $p > 0$ there is a surjective homomorphism

$$\Psi: \Omega^1(Z) \to \text{Br}(Z)_p,$$
which takes a differential form $\eta = xdy$ to the class of the Azumaya algebra $A_2$ generated over $O(Z)$ by elements $v$ and $u$ subject to the relations $v^p = x$, $u^p = y$ and $vu - uv = 1$. This Azumaya algebra has rank $p^2$. Applying this to $Z = Y$ and observing that every differential form on $Y$ can be written as $xdy$ for some regular functions $x$ and $y$ on $Y$ (which holds since $Y$ is one-dimensional and $\Delta \neq \emptyset$) we conclude that every $p$-torsion class in $\text{Br}(Y)$ is represented by an Azumaya algebra of rank $p^2$. This proves the existence of an Azumaya algebra $A$ as stated in assertion (ii). Let $\pi: \hat{X} \to Y$ be the corresponding Severi–Brauer scheme.

Suppose that $\pi': X' \to Y'$ as in assertion (ii) exists. We may assume that $Y' \neq \mathbb{P}^1$. Consider the base change $\pi'_Y: X'_Y \to Y'_Y$. Using Tsen's theorem and the triviality of vector bundles over $Y'_K$, we conclude that $X'_K$, as a scheme over $Y'_K$, is the product $\mathbb{P}^1 \times Y'_K$. But then $X'_K$ is also isomorphic to the product $\mathbb{P}^1 \times Y'_K$. It implies that $\pi': X' \to Y'$ is a Severi–Brauer scheme over $Y'$ whose class in the Brauer group extends $\alpha$. A contradiction with assertion (i) completes the proof of assertion (ii). $\square$

**Remark 8.5.** The Artin–Schreier short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to K \xrightarrow{\phi} K \to 0,$$

where the homomorphism $\phi$ is defined by $\phi(c) = c - c^p$, gives rise to an isomorphism

$$\text{Hom}(\text{Gal}(K/K), \mathbb{Z}/p\mathbb{Z}) \cong \text{coker} \phi.$$

One can verify that the composition

$$\Omega^1(Y) \xrightarrow{\Psi} \text{Br}(Y)_p \xrightarrow{\text{Res}} \bigoplus_{\Delta} \text{Hom}(\text{Gal}(K/K), \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{\Delta} \text{coker} \phi$$

carries a differential form $\eta$ to its residues (modulo the image of $\phi$) at all points of $\Delta$. This can be used to construct a class $\alpha$ and an Azumaya algebra $A$ satisfying the requirements of Lemma 8.3 explicitly. Namely, take an invertible function $f$ on $Y$ whose order at every point of $\Delta$ is not equal to 0 modulo $p$, and an element $t \in K$, which is not in the image of the Artin–Schreier homomorphism. Then the class $\Psi(t^2_f) \in \text{Br}(Y)_p$ satisfies the requirements of assertion (i), and the Azumaya algebra $A_{t^2_f}$ satisfies the requirements of assertion (ii).

Finally, we present a counterexample to Theorem 1.3 over a perfect field of characteristic 2.

**Example 8.6.** Let $k$ be an algebraically closed field of characteristic 2, and let

$$K = k(t^{2^{-\infty}}),$$

that is, $K$ is obtained from $k$ by adjoining all roots of $t$ of degrees $2^r$, for all positive integers $r$. Then $K$ is a perfect field of characteristic 2 and it has a Galois extension of degree 2. Pick a subset $\Delta \subset \mathbb{P}^1(K)$ of 4 points. Set $Y = \mathbb{P}^1 \setminus \Delta$, and let $X \to Y$ be the smooth conic family from Lemma 8.3(ii). Consider a relatively minimal conic bundle $\phi: \hat{X} \to \mathbb{P}^1$ such that $\hat{X}$ is a smooth compactification of $X$. Then $\phi$ has degenerate fibers exactly over the points of $\Delta$. This means that $K_{\hat{X}}^2 = 4$. Also, we know that $\hat{X}$ has a $K$-point, because every fiber of $\phi$ has a $K$-point (see Remark 6.6). Hence, the surface $\hat{X}$ is not rational by Theorem 7.13(ii). On the other hand, the group of birational selfmaps of $\hat{X}$ contains an automorphism group of the schematic general fiber $\hat{X}_v$ of $\phi$, which is a
conic over a non-perfect field $\mathbb{K}(\mathbb{P}^1)$ of characteristic 2. The latter automorphism group has unbounded finite subgroups by Lemma 6.12. We conclude that the generalization of Theorem 1.2 (and thus also Theorem 1.3, cf. Remark 8.2) fails over perfect fields of characteristic 2.

9. Quadrics

In this section we study automorphism groups of quadrics and prove Proposition 1.10.

Let $V$ be a finite-dimensional vector space over a field $\mathbb{K}$, and let $q$ be a non-degenerate quadratic form on $V$. Recall that a quadratic form $q$ is said to be non-degenerate if the associated symmetric bilinear form

$$B_q : V \times V \to \mathbb{K}, \quad B_q(v, w) = q(v + w) - q(v) - q(w),$$

is non-degenerate. If $\text{char} \mathbb{K} = 2$ the form $B_q$ is also alternating. Hence, in this case the dimension of $V$ must be even. We refer to the pair $(V, q)$ as a quadratic module over $\mathbb{K}$.

Denote by $O(V, q)$ the corresponding orthogonal (linear algebraic) group.

The following result is well known (see, for instance, [Bor91, §§22.4, 22.6]).

**Lemma 9.1.** The group $O(V, q)$ is reductive and it is anisotropic if and only if $q$ does not represent 0.

We will also need the following structural result on quadratic forms over a perfect field of characteristic 2 due to Arf ([Arf41]).

**Lemma 9.2.** Let $V$ be a finite-dimensional vector space over a perfect field $\mathbb{K}$ of characteristic 2, and let $q$ be a non-degenerate quadratic form on $V$ (so that in particular $\dim V = 2k$ is even). Then, for some coordinates $x_1, \ldots, x_{2k}$ on $V$, the quadratic form $q$ is given by

$$(9.3) \qquad q_a(x_1, \ldots, x_{2k}) = x_1^2 + x_1 x_2 + a x_2^2 + x_3 x_4 + \ldots + x_{2k-1} x_{2k},$$

where $a$ is an element of $\mathbb{K}$. Moreover, two quadratic forms $q_a(x_1, \ldots, x_{2k})$ and $q_{a'}(x_1, \ldots, x_{2k})$ are equivalent if and only if $a$ and $a'$ have the same image in the cokernel of Artin–Schreier homomorphism

$$\mathbb{K} \to \mathbb{K}, \quad c \mapsto c^2 - c,$$

which is the Arf invariant of the quadratic form. In particular, if $\dim V > 2$ then every non-degenerate quadratic form on $V$ represents 0.

**Lemma 9.4** (cf. [Gar13, Lemma 2.1]). Let $\mathbb{K}$ be a field of characteristic $p > 2$, and let $(V, q)$ be a non-degenerate quadratic module over $\mathbb{K}$. Assume that $q$ does not represent 0. Then the group of $\mathbb{K}$-points of $O(V, q)$ has no elements of order $p$.

**Proof.** Assuming the contrary, let $g \in O(V, q)(\mathbb{K})$ be an element of order $p$. Viewing $g$ as a linear endomorphism of $V$, we have

$$g^p - 1 = (g - 1)^p = 0.$$

Applying the Jordan Normal Form theorem to $g - 1$, we can find linearly independent vectors $v_1, v_2 \in V$ such that $g(v_1) = v_1$ and $g(v_2) = v_1 + v_2$. Thus, we have

$$B_q(v_1, v_2) = B_q(g(v_1), g(v_2)) = B_q(v_1, v_1) + B_q(v_1, v_2).$$

Hence, we obtain $2q(v_1) = B_q(v_1, v_1) = 0$, that is, $q$ represents 0. \qed
Lemma 9.6. (cf. [BZ17a, Corollary 4.4]) Let \( K \) be a field that contains all roots of 1. Assume that \( \text{char } K \neq 2 \) or \( K \) is perfect. Suppose that \( q \) does not represent 0. Then every non-trivial element of finite order in \( O(V,q)(K) \) has order 2, and every finite subgroup of \( O(V,q)(K) \) is abelian of order less or equal to \( 2^{\dim V} \).

Proof. By Lemma 9.2 if \( K \) is perfect and \( \text{char } K = 2 \), then we must have \( \dim V = 2 \). In this case, by Lemma 9.1 the group \( O(V,q)(K) \) is isomorphic to the product of an anisotropic torus of rank 1 and the finite group \( \mathbb{Z}/2\mathbb{Z} \). Therefore, the assertion of the lemma follows from Lemma 3.6.

Now, assume that \( \text{char } K \neq 2 \). Let \( g \) be an element of finite order. By Lemma 9.1 the order of \( g \) is coprime to the characteristic of \( K \). Since \( K \) contains all roots of unity, it follows that every such element \( g \in O(V,q)(K) \) viewed as a linear endomorphism of \( V \) is diagonalizable in an appropriate basis for \( V \). Moreover, since \( q \) does not represent 0, the diagonal entries of the matrix of \( g \) in this basis must be equal to \( \pm 1 \). Hence \( g^2 = 1 \). It follows that every finite subgroup \( G \subset O(V,q)(K) \) is abelian, and \( G \) is conjugate to a subgroup of the group of diagonal matrices in \( GL(V) \). Hence, the order of \( G \) is at most \( 2^{\dim V} \). \( \square \)

Remark 9.5. If the field \( K \) is perfect, Lemma 9.4 can be deduced from Corollary 4.4 and Lemma 9.1.

If \( K \) is a perfect field that contains all roots of 1, then it follows from Theorem 1.5 and Lemma 9.1 that the orthogonal group \( O(V,q) \) has bounded finite subgroups if and only if \( q \) does not represent 0. In fact, we have a stronger result.

Lemma 9.6 (cf. [BZ17a, Corollary 4.4]). Let \( K \) be a field that contains all roots of 1. Assume that \( \text{char } K \neq 2 \) or \( K \) is perfect. Suppose that \( q \) does not represent 0. Then every non-trivial element of finite order in \( O(V,q)(K) \) has order 2, and every finite subgroup of \( O(V,q)(K) \) is abelian of order less or equal to \( 2^{\dim V} \).

Proof. By Lemma 9.2 if \( K \) is perfect and \( \text{char } K = 2 \), then we must have \( \dim V = 2 \). In this case, by Lemma 9.1 the group \( O(V,q)(K) \) is isomorphic to the product of an anisotropic torus of rank 1 and the finite group \( \mathbb{Z}/2\mathbb{Z} \). Therefore, the assertion of the lemma follows from Lemma 3.6.

Now, assume that \( \text{char } K \neq 2 \). Let \( g \in O(V,q)(K) \) be an element of finite order. By Lemma 9.1 the order of \( g \) is coprime to the characteristic of \( K \). Since \( K \) contains all roots of unity, it follows that every such element \( g \in O(V,q)(K) \) viewed as a linear endomorphism of \( V \) is diagonalizable in an appropriate basis for \( V \). Moreover, since \( q \) does not represent 0, the diagonal entries of the matrix of \( g \) in this basis must be equal to \( \pm 1 \). Hence \( g^2 = 1 \). It follows that every finite subgroup \( G \subset O(V,q)(K) \) is abelian, and \( G \) is conjugate to a subgroup of the group of diagonal matrices in \( GL(V) \). Hence, the order of \( G \) is at most \( 2^{\dim V} \). \( \square \)

Now we are ready to prove Proposition 1.10.

Proof of Proposition 1.10. Let \( V \) be an \( n \)-dimensional vector space such that \( \mathbb{P}^{n-1} \) is identified with the projectivization \( \mathbb{P}(V) \), and let \( q \) be a quadratic form corresponding to the quadric \( Q \).

First, assume that \( K \) is a perfect field of characteristic 2. If \( n \) is even, then the quadratic form is non-degenerate; indeed, otherwise its kernel \( T \) would be at least two-dimensional, so that the singular locus of \( Q \), which is \( Q \cap \mathbb{P}(T) \), would be non-empty. Thus, by Lemma 9.2 one has \( Q(K) \neq \emptyset \). Moreover, writing \( q \) in the form \( \langle x, x \rangle \), we see that \( \text{Aut}(Q) \) contains a subgroup isomorphic to \( K^* \). Hence, \( \text{Aut}(Q) \) has unbounded finite subgroups. If \( n \) is odd, the symmetric bilinear form \( B_q : V \times V \to K \) associated to \( q \) has a one-dimensional kernel. In this case \( q \) can be written as

\[
q(x_1, \ldots, x_n) = x_1^2 + r(x_2, \ldots, x_n)
\]

for some coordinates \( x_1, \ldots, x_n \) on \( V \) and some non-degenerate quadratic form \( r \) in \( n-1 \) variables. Applying Lemma 9.2 to \( r \) we see that \( Q(K) \neq \emptyset \) and \( \text{Aut}(Q) \) has unbounded finite subgroups. In fact, in this case \( \text{Aut}(Q) \) is isomorphic to the group of linear transformations of the quotient \( \tilde{V} \) of \( V \) by \( T \) which preserve the induced bilinear form \( B_q \) on \( \tilde{V} \), i.e., to the symplectic group \( \text{Sp}(\tilde{V}, B_q)(K) \); see [Bor91 §22.6]. We see that in the case when \( \text{char } K = 2 \), assertion (i) holds, while the assumptions of assertions (ii), (iii), and (iv) do not hold.

From now on we assume that \( \text{char } K \neq 2 \). Then the group \( \text{Aut}(Q) \) is isomorphic to the group of \( K \)-points of the quotient \( O(V,q)/\mu_2 \), where \( \mu_2 \subset O(V,q) \) is the central subgroup of order 2. (More geometrically, \( \text{Aut}(Q) \) can be identified with the group of automorphisms
of the projective space $\mathbb{P}(V)$ that preserve $q$ up to a scalar multiple.) Thus, $\text{Aut}(Q)$ fits into an exact sequence
\[(9.7)\quad 0 \to \mu_2 \to O(V,q)(\mathbb{K}) \to \text{Aut}(Q) \to \mathbb{K}^*/(\mathbb{K}^*)^2.\]
By Lemma\[9.4\] the group $O(V,q)/\mu_2$ is anisotropic if and only if $Q(\mathbb{K}) = \emptyset$. If $Q(\mathbb{K}) = \emptyset$, then Lemma\[9.4\] together with exact sequence \[(9.7)\] shows that the order of any element of $\text{Aut}(Q)$ of finite order is coprime to $\text{char} \, \mathbb{K}$. Thus, assertions (i) follows from Theorem\[1.3\] applied to the reductive group $O(V,q)/\mu_2$.

If $n$ is odd, then the embedding $\mu_2 \hookrightarrow O(V,q)$ splits, so that $\text{Aut}(Q)$ is isomorphic to the subgroup $\text{SO}(V,q)(\mathbb{K}) \subset O(V,q)(\mathbb{K})$ of the orthogonal group that consists of matrices whose determinant is equal to 1. Thus, assertion (ii) follows from Lemma\[9.6\].

Suppose that $n$ is even. Then, using exact sequence \[(9.7)\] and Lemma\[9.6\] we infer that every non-trivial element of $O(V,q)(\mathbb{K})$ of finite order has order 2. Hence, every non-trivial element of $\text{Aut}(Q)$ of finite order has order 2 or 4. Let $G \subset \text{Aut}(Q)$ be a finite subgroup. Consider the embedding
\[\text{Aut}(Q) \cong (O(V,q)/\mu_2)(\mathbb{K}) \hookrightarrow (O(V,q)/\mu_2)(\mathbb{K}) \cong O(V,q)(\mathbb{K})/\{\pm 1\},\]
and let $\tilde{G}$ be the preimage of $G$ in $O(V,q)(\mathbb{K})$. The order of every element of $\tilde{G}$ divides 8, and the same is true for the subgroup $\tilde{G}_{\mu_8} \subset \text{GL}(V)(\mathbb{K})$ generated by $\tilde{G}$ and scalar matrices whose orders divide 8. Thus, by Theorem\[1.6\] we have $|\tilde{G}_{\mu_8}| \leq 8^n$. On the other hand, we know that $|\tilde{G}_{\mu_8}| = 8|G|$. This proves assertion (iii).

Finally, to get the stronger bound in the case $n = 4$, observe that the group of $\bar{\mathbb{K}}$-points of $O(V,q)/\mu_2$, that is the group of automorphisms of $Q$ over $\bar{\mathbb{K}}$, is isomorphic to the semi-direct product $(\text{PGL}_2(\mathbb{K}) \times \text{PGL}_2(\mathbb{K})) \rtimes \mathbb{Z}/2\mathbb{Z}$, where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\text{PGL}_2(\mathbb{K}) \times \text{PGL}_2(\mathbb{K})$ by a permutation of the factors. We claim that every finite subgroup of the latter group whose non-trivial elements are of order 2 or 4 has order at most 64. Indeed, let $G$ be such a subgroup. Since $\text{char} \mathbb{K} \neq 2$, the order of the intersection
\[H = G \cap (\text{PGL}_2(\mathbb{K}) \times \text{PGL}_2(\mathbb{K}))\]
divides 64 by Lemma\[6.3\]. If $|H| < 64$ we are done. Otherwise, $H$ is the product $H_1 \times H_2$, for some subgroups $H_i \subset \text{PGL}_2(\mathbb{K})$ of order 8 each of which contains an element of order 4. But then $G$ must coincide with $H$, because otherwise $G$ would contain an element of the form
\[(g, 1, \sigma) \in (\text{PGL}_2(\mathbb{K}) \times \text{PGL}_2(\mathbb{K})) \rtimes \mathbb{Z}/2\mathbb{Z},\]
where $g$ is an element of order 4 and $\sigma$ is the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$. But such element has order 8, and assertion (iv) follows. \hfill $\Box$

10. Del Pezzo surfaces

In this section we study groups acting on del Pezzo surfaces and prove Proposition\[1.11\].

Recall that a del Pezzo surface is a (normal) projective geometrically irreducible surface with an ample anticanonical divisor. By the degree of a smooth del Pezzo surface $X$ we always mean its anticanonical degree $K_X^2$.

**Lemma 10.1.** Let $\mathbb{K}$ be a perfect field that contains all roots of 1. Let $X$ be a smooth del Pezzo surface of degree 6 over $\mathbb{K}$. Suppose that $X$ is not birational to $\mathbb{P}^1 \times C$, where $C$ is a conic. Then every finite subgroup of $\text{Aut}(X)$ has order at most 432.
Proof. One has
\[ \text{Aut}(X_{\bar{K}}) \cong (\bar{K}^*)^2 \times D_{12}, \]
where \( D_{12} \) is the dihedral group of order 12, cf. [Dol12, Theorem 8.4.2]. Thus the group \( \text{Aut}(X) \) has a subgroup \( T \) of index at most 12 such that \( T \) is a two-dimensional torus (cf. the proof of Corollary 7.14).

If \( T \) contains a subtorus isomorphic to \( \mathbb{G}_m \), then \( X \) is birational to \( \mathbb{P}^1 \times C \) for some conic \( C \) by Lemma 2.11, which is not the case by assumption. Thus we see that \( T \) is anisotropic, so that every finite subgroup of \( T \) has order at most 36 by Lemma 3.11. Therefore, every finite subgroup of \( \text{Aut}(X) \) has order at most 12 \( \cdot \) 36 \( = \) 432. \( \square \)

Note that the assumptions of Lemma 10.1 imply that \( X(\bar{K}) = \emptyset \), since otherwise \( X \) is rational by Theorem 7.13(i).

Now we proceed to del Pezzo surfaces of degree 8. We will say that a del Pezzo surface \( X \) of degree 8 over a field \( K \) is of product type, if \( X \cong \mathbb{P}^1 \times \mathbb{P}^1 \).

**Remark 10.2.** Let \( X \) be a smooth del Pezzo surface of degree \( 7 \leq K_X^2 \leq 8 \) over a perfect field \( K \). If either \( K_X^2 = 7 \), or \( K_X^2 = 8 \) and \( X_{\bar{K}} \) is a blow up of \( \mathbb{P}^2 \) at a point, then \( X \) contains a Galois-invariant \((-1)\)-curve, which in turn contains a \( K \)-point; in this case \( X \) is rational by Theorem 7.13(i). Therefore, if \( X \) is not rational, then \( K_X^2 = 8 \) and \( X_{\bar{K}} \cong \mathbb{P}^1 \times \mathbb{P}^1 \), so that \( X \) is of product type.

**Lemma 10.3.** Let \( K \) be a perfect field that contains all roots of 1. Let \( X \) be a del Pezzo surface of degree \( K_X^2 = 8 \) of product type over \( K \). Suppose that \( X \) is not birational to \( \mathbb{P}^1 \times C \), where \( C \) is a conic. Then every finite subgroup of \( \text{Aut}(X) \) has order at most 64.

**Proof.** We may assume that \( X(\bar{K}) = \emptyset \), since otherwise \( X \) is rational by Theorem 7.13(i). It follows from [Lied17, Proposition 5.2] that either \( X \cong C \times C' \), where \( C \) and \( C' \) are conics without \( K \)-points, or \( X \) is a quadric in \( \mathbb{P}^3 \). In the former case the assertion follows from Corollary 7.8. In the latter case the assertion follows from Proposition 10.1(iv). \( \square \)

Now we are ready to prove Proposition 10.1.

**Proof of Proposition 10.1.** Put \( d = K_X^2 \). One has \( 1 \leq d \leq 9 \). Let \( G \) be a finite subgroup in \( \text{Aut}(X) \).

If \( d = 9 \), then \( X \) is a Severi–Brauer surface without \( K \)-points, so that by Proposition 10.3(ii) we have \( |G| \leq 81 \). If \( 7 \leq d \leq 8 \), then it follows from Remark 10.2 that \( d = 8 \) and \( X \) is of product type, so that \( |G| \leq 64 \) by Lemma 10.3. If \( d = 6 \), then \( |G| \leq 432 \) by Lemma 10.1. \( \square \)

**Remark 10.4.** Let \( X \) be a del Pezzo surface of degree \( d \leq 5 \) over a field \( K \). Then the group \( \text{Aut}(X) \) is finite. Moreover, in the case when \( \text{char} \ K = 0 \), one has \( \text{Aut}(X) \) is algebraically closed, and use an explicit classification of automorphism groups of del Pezzos surfaces of low degree, see [Dol12 §8.5.4], [Dol12 §8.6.4], [Dol12 §8.7.3], [Dol12 §8.8.4], and [Dol12 §9.5.3].

If \( K \) is an arbitrary field, the known upper bound is much higher. Namely, one can check that \( |\text{Aut}(X)| \) is bounded by the order of the Weyl group W(E8), that is, by the number 696 729 600, cf. [Man67 Theorem 4.5] and [Dol12 Corollary 8.2.40]. This bound is not sharp. However, over appropriate fields there exist del Pezzo surfaces of low degree with automorphism groups of rather large order. For instance, the automorphism group of the Fermat cubic surface over an algebraically closed field of characteristic 2 has order 25 920, see [DD17, Table 1].
11. Jordan property

In this section we apply Theorem 1.3 to study groups of birational automorphisms of higher-dimensional varieties. The group-theoretic property we will be interested in here is described as follows.

Definition 11.1 (see [Pop16, Definition 1]). A group $\Gamma$ is called Jordan if there is a constant $J$ such that for any finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most $J$.

The first result concerning Jordan property (and motivating the modern terminology) is an old theorem by C. Jordan that asserts this property for a general linear group over a field of characteristic zero (see e.g. [CR62, Theorem 36.13]).

J.-P. Serre noticed that Jordan property sometimes holds for groups of birational automorphisms.

Theorem 11.2 ([Ser09, Theorem 5.3]). The group of birational automorphisms of $\mathbb{P}^2$ over a field of characteristic zero is Jordan.

In [PS16a, Theorem 1.8], Yu. Prokhorov and C. Shramov generalized Theorem 11.2 to the case of rationally connected varieties of arbitrary dimension (actually, their results were initially obtained modulo boundedness of terminal Fano varieties, which was recently proved by C. Birkar in [Bir16, Theorem 1.1]). Jordan property was also proved in many other cases for groups of birational automorphisms and for automorphism groups, see for instance [PS14, MZ15, BZ17a, BZ17b, PST17, Mun14, Mun16], and references therein. However, there are varieties whose groups of birational automorphisms are not Jordan.

Theorem 11.3 ([Zar14]). Let $A$ be a positive-dimensional abelian variety. Then the group of birational automorphisms of $A \times \mathbb{P}^1$ is not Jordan.

Recall that to any variety $X$ one can associate the maximal rationally connected fibration, which is a canonically defined rational map with rationally connected fibers and non-uniruled base (see [Kol96, §IV.5], [CHS03, Corollary 1.4]). The maximal rationally connected fibration is equivariant with respect to the whole group of birational automorphisms of $X$. Keeping in mind Theorem 11.3, it seems natural to try to understand the groups of birational automorphisms for varieties with maximal rationally connected fibration of small relative dimension; note that the case when the relative dimension is 0 (that is, the maximal rationally connected fibration is birational or, equivalently, the variety is not uniruled) is settled by [PS14, Theorem 1.8(ii)].

Theorem 11.4 ([BZ17a, Theorem 1.6]). Let $X$ be an irreducible variety over a field of characteristic zero, and $\phi: X \dashrightarrow Y$ be its maximal rationally connected fibration. Suppose that the relative dimension of $\phi$ equals 1, and that $\phi$ has no rational sections. Then the group of birational automorphisms of $X$ is Jordan.

Theorem 11.5 ([PS16b, Lemma 4.6]). Let $X$ be an irreducible variety over a field of characteristic zero, and $\phi: X \dashrightarrow Y$ be its maximal rationally connected fibration. Suppose that the relative dimension of $\phi$ equals 2, and that $\phi$ has a rational section. Suppose also that $X$ is not birational to $Y \times \mathbb{P}^2$. Then the group of birational automorphisms of $X$ is Jordan.
Using Theorem 1.3 we obtain the following result that is somewhat similar to Theorems 11.4 and 11.5.

**Proposition 11.6.** Let $X$ be an irreducible variety over a field $k$ of characteristic zero, and $\phi: X \rightarrow Y$ be its maximal rationally connected fibration. Suppose that the relative dimension of $\phi$ equals 2, and that $\phi$ has no rational sections. Suppose also that $X$ is not birational to $Z \times \mathbb{P}^1$, where $Z$ is a conic bundle over $Y$. Then the group of birational automorphisms of $X$ is Jordan.

**Proof.** The proof is similar to those of Theorems 11.4 and 11.5, see [BZ17a, §5] and [PS16b, §4]. We may assume that the field $k$ is algebraically closed.

Let $S$ be the fiber of $\phi$ over the general schematic point of $Y$. Then $S$ is a geometrically rational surface defined over the field $K = k(Y)$, and $S$ has no $K$-points by assumption. Since $\phi$ is equivariant with respect to Bir$(X)$, we have an exact sequence

$$1 \rightarrow \text{Bir}(X)_\phi \rightarrow \text{Bir}(X) \rightarrow \text{Bir}(Y),$$

where the action of Bir$(X)_\phi$ is fiberwise with respect to $\phi$.

Our assumptions imply that $S$ is not birational to $\mathbb{P}^1 \times C$ over $K$, where $C$ is a conic. The group Bir$(X)_\phi$ is isomorphic to the group Bir$(S)$, and thus has bounded finite subgroups by Theorem 1.3. On the other hand, the variety $Y$ is not uniruled. Thus the group Bir$(Y)$ is Jordan by [PS14, Theorem 1.8(ii)]. Moreover, there is a constant $R$ such that every finite subgroup in Bir$(Y)$ is generated by at most $R$ elements, see e.g. [PS14, Remark 6.9]. Therefore, the group Bir$(X)$ is Jordan by a simple group-theoretic argument, see [PS14, Lemma 2.8]. □

Note that if the field $k$ is algebraically closed, the assumptions of Proposition 11.6 imply that the dimension of $Y$ is at least 2 (so that the dimension of $X$ is at least 4) due to the theorem of Graber, Harris, and Starr, see [GHS03]. It would be interesting to describe more explicitly the varieties with maximal rationally connected fibration of relative dimension 2 whose birational automorphism groups are not Jordan, similarly to what was done in [PS16b, Theorem 1.8]. Also, it would be interesting to try to generalize Theorem 1.3 to the case of threefolds, and to derive conclusions for birational automorphisms of varieties with maximal rationally connected fibration of relative dimension 3.

**References**

[Alb61] A. Albert. *Structure of algebras*. AMS Coll. Pub. 24, AMS, Providence, RI, 1961.

[Arf41] C. Arf. *Untersuchungen der quadratischen Formen in Korpern de Charakteristik 2*. Journal für die reine und angewandte Mathematik 183, 148–167 (1941).

[Ar82] M. Artin. *Brauer–Severi varieties*. In: van Oystaeyen F.M.J., Verschoren A.H.M.J. (eds). Brauer Groups in Ring Theory and Algebraic Geometry. Lecture Notes in Math. 917, 194–210. Springer-Verlag, 1982.

[ABGV11] A. Auel, E. Brussel, S. Garibaldi, U. Vishne. *Open problems on central simple algebras*. Transform. Groups 16, no. 1, 219–264 (2011).

[BZ17a] T. Bandman, Yu. Zarhin. *Jordan groups, conic bundles and abelian varieties*. Alg. Geom. 4 (2), 229–246 (2017).

[BZ17b] T. Bandman, Yu. Zarhin. *Jordan properties of automorphism groups of certain open algebraic varieties*. [arXiv:1705.07523] (2017).

[BM08] R. Bezrukavnikov, I. Mirković, D. Rumynin. *Localization of modules for a semisimple Lie algebra in prime characteristic*. Ann. Math. 167, 945–991 (2008).

[BB73] A. Bialynicki-Birula. *On fixed point schemes of actions of multiplicative and additive groups*. Topology 12, 99–103 (1973).
[Bir16] C. Birkar. Singularities of linear systems and boundedness of Fano varieties. arXiv:1609.05543 (2016).

[Bli17] H. F. Blichfeldt. Finite collineation groups. The Univ. Chicago Press, Chicago, 1917.

[Bor91] A. Borel. Linear algebraic groups. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.

[BT71] A. Borel, J. Tits. Éléments unipotents et sous-groupes paraboliques de groupes réductifs. I. Invent. Math. 12, 95–104 (1971).

[CF67] J. Cassels and A. Fröhlich (eds). Algebraic Number Theory. Academic Press, 1967.

[Ch44] F. Châtelet. Variations sur un thème de H. Poincaré. Ann. Sci. École Norm. Sup. 61, 249–300 (1944).

[CR62] Ch. W. Curtis, I. Reiner. Representation theory of finite groups and associative algebras. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York–London, 1962.

[DO17] T.-C. Dinh, K. Oguiso. A surface with discrete and non-finitely generated automorphism group. arXiv:1710.07019 (2017).

[Dol12] I. Dolgachev. Classical algebraic geometry: a modern view. Cambridge University Press, Cambridge, 2012.

[DD17] I. Dolgachev, A. Duncan. Automorphisms of cubic surfaces in positive characteristic. arXiv:1712.01167 (2017).

[EGA III-1] A. Grothendieck, J. Dieudonné. Eléments de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents, Première partie. Publ. Math. Inst. Hautes Étud. Sci. 11, 1961.

[HP76] M. Herzog, Ch. E. Praeger. On the order of linear groups of fixed finite exponent. J. Algebra 43, 216–220 (1976).

[GA13] M. Garcia-Armas. Finite group actions on curves of genus zero. Journal of Algebra, 394, 173–181 (2013).

[GHS03] T. Graber, J. Harris, J. Starr. Families of rationally connected varieties. J. Am. Math. Soc. 16 (1), 57–67 (2003).

[G18] A. Guld. Boundedness properties of automorphism groups of forms of flag varieties. arXiv:1806.05400 (2018).

[Isk80] V. A. Iskovskikh. Minimal models of rational surfaces over arbitrary fields. Math. USSR-Izv., 14(1):17–39 (1980).

[Isk96] V. A. Iskovskikh. Factorization of birational mappings of rational surfaces from the point of view of Mori theory. Russian Math. Surveys, 51(4):585–652 (1996).

[IT91] V. A. Iskovskikh, S. L. Tregub. Birational automorphisms of rational surfaces. Math. USSR-Izv., 55(2):254–281 (1991).

[Kle71] S. Kleiman. Les théorèmes de finitude pour le foncteur de Picard. Exposé XIII in SGA6, “Théorie des intersections et théorème de Riemann–Roch”, Lecture Notes in Math., 225, 616–666 (1971).

[Kle05] S. Kleiman. The Picard scheme. Fundamental algebraic geometry, 235–321, Math. Surveys Monogr., 123, AMS, 2005.

[Kol93] J. Kollár. Effective base point freeness. Math. Ann. 296, no. 4, 595–605 (1993).

[Kol96] J. Kollár. Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 32. Springer-Verlag, Berlin, 1996.

[Lieb08] M. Lieblich. Twisted sheaves and the period-index problem. Compositio Math. 144, 1, 1–31 (2008).

[Lied17] C. Liedtke. Morphisms to Brauer–Severi varieties, with applications to del Pezzo surfaces. Geometry over Nonclosed Fields, 157–196. Springer, 2017.

[MZ15] S. Meng, D.-Q. Zhang. Jordan property for non-linear algebraic groups and projective varieties. arXiv:1507.02230 (2015).

[Man67] Yu. I. Manin. Rational surfaces over perfect fields. II. Mat. Sb. (N.S.), 72 (114):161–192 (1967).

[Min1887] H. Minkowski. Zur Theorie der positiven quadratische Formen. J. Reine Angew. Math. 101, 196–202 (1887).
[Mum70] D. Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, 5. Oxford University Press, London, 1970.

[OV07] A. Ogus, V. Vologodsky. *Nonabelian Hodge theory in characteristic p*. Publ. Math. Inst. Hautes Études Sci. 106, 1–138 (2007).

[Mun14] I. Mundet i Riera. *Finite group actions on homology spheres and manifolds with nonzero Euler characteristic*. arXiv:1403.0383 (2014).

[Mun16] I. Mundet i Riera. *Finite subgroups of Ham and Symp*. Math. Ann. 370, No. 1–2, 331–380 (2018).

[Nit05] N. Nitsure. *Construction of Hilbert and Quot schemes*. Fundamental algebraic geometry, 105–137, Math. Surveys Monogr., 123, AMS, 2005.

[Pop16] V. L. Popov. *Finite subgroups of diffeomorphism groups*. Proc. Steklov Inst. Math. 289 (1), 221–226 (2016).

[PS14] Yu. Prokhorov, C. Shramov. *Jordan property for groups of birational selfmaps*. Compositio Math. 150 (12), 2054–2072 (2014).

[PS16a] Yu. Prokhorov, C. Shramov. *Jordan property for Cremona groups*. Amer. J. Math. 138 (2), 403–418 (2016).

[PS16b] Yu. Prokhorov, C. Shramov. *Finite groups of birational selfmaps of threefolds*. arXiv:1611.00789 (2016).

[PS17] Yu. Prokhorov, C. Shramov. *Automorphism groups of compact complex surfaces*. arXiv:1708.03566 (2017).

[Ros56] M. Rosenlicht. *Some basic theorems on algebraic groups*. Amer. J. Math. 78, 401–443 (1956).

[Ros67] M. Rosenlicht. *Another proof of a theorem on rational cross sections*. Pacific J. Math. 20, 129–133 (1967).

[Ser07] J.-P. Serre. *Bounds for the orders of the finite subgroups of G(k)*. Group Representation Theory, 405–450. EPFL Press, 2007.

[Ser09] J.-P. Serre. *A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field*. Mosc. Math. J. 9 (1), 193–208 (2009).

[Spr98] T. A. Springer. *Linear algebraic groups*. Second edition. Progress in Mathematics, 9. Birkhäuser Boston, 1998.

[Stack] *The stacks project*. Available electronically at https://stacks.math.columbia.edu/

[Tah71] K. Tahara. *On the finite subgroups of GL(3, Z)*. Nagoya Math. J. 41, 169–209 (1971).

[Vos65] V. E. Voskresenskii. *On two-dimensional algebraic tori*. Izv. Akad. Nauk SSSR Ser. Mat., 29:1, 239–244 (1965).

[Wed21] J. H. M. Wedderburn. *On division algebras*. Trans. Amer. Math. Soc. 22, 129-135 (1921).

[Zar14] Yu. G. Zarhin. *Theta groups and products of abelian and rational varieties*. Proc. Edinburgh Math. Soc. 57 (1), 299–304 (2014).

Constantin Shramov
Steklov Mathematical Institute of RAS, 8 Gubkina street, Moscow 119991, Russia.
National Research University Higher School of Economics, Laboratory of Algebraic Geometry, NRU HSE, 6 Usacheva str., Moscow, 117312, Russia.
costya.shramov@gmail.com

Vadim Vologodsky
National Research University Higher School of Economics, Laboratory of Mirror Symmetry, NRU HSE, 6 Usacheva str., Moscow, 117312, Russia.
vologod@gmail.com