SRB MEASURES FOR SOME DIFFEOMORPHISMS WITH DOMINATED SPLITTINGS AS ZERO NOISE LIMITS

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Abstract. In this paper, we provide a technical result on the existence of Gibbs cu-states for diffeomorphisms with dominated splittings. More precisely, for given $C^2$ diffeomorphism $f$ with dominated splitting $T\Lambda M = E \oplus F$ on an attractor $\Lambda$, by considering some suitable random perturbation of $f$, we show that for any zero noise limit of ergodic stationary measures, if it has positive integrable Lyapunov exponents along invariant sub-bundle $E$, then its ergodic components contain Gibbs cu-states associated to $E$. With this technique, we show the existence of SRB measures and physical measures for some systems exhibiting dominated splittings and weak hyperbolicity.

1. Introduction and main result. Our aim of this paper is to show the existence of SRB (or physical) measures for some systems beyond hyperbolicity. The existence is based on a criterion established to get Gibbs cu-states by techniques of adding random perturbations.

SRB measures were discovered by Sinai, Ruelle and Bowen in 1970s for uniformly hyperbolic systems [30, 13, 12, 29]. After that, the SRB theory rapidly became a central topic of dynamical systems. Beyond the uniform hyperbolicity, up to now, there is no systematic approach for the construction of SRB measures, it remains to be a challenging problem in the ergodic theory of dynamical systems (see [33, 26] for related conjectures). An invariant Borel probability is said to be an SRB measure if it has positive Lyapunov exponents almost everywhere and its disintegration along Pesin unstable manifolds are absolutely continuous w.r.t. Lebesgue measures. SRB measures can also characterized by the fact they have positive Lyapunov exponents and satisfy the Pesin entropy formula [19]. Sometimes, SRB measures are expected to be physically observable [16], known as physical measures, i.e., the sets of generic points have positive Lebesgue measure on the manifold (e.g. see [37] for a survey).

As for the construction of SRB (or physical) measures one should refer the Gibbs u-states and Gibbs cu-states. Gibbs u-states were introduced and studied by Pesin and Sinai [27] for partially hyperbolic diffeomorphisms admitting strong unstable sub-bundles, which requires the absolute continuity of conditional measures along strong unstable manifolds (see [10, Chapter 11] for more details). In [2], Alves,
Bonatti and Viana proposed the notion of Gibbs $cu$-states (see Definition 2.3) for diffeomorphisms with dominated splittings as a nonuniform notion of Gibbs $u$-states. It turns out that Gibbs $u$-states and Gibbs $cu$-states are crucial candidates of SRB measures and physical measures (e.g. [2, 11, 15, 5, 25]).

In this paper, we deal with the problem of the existence of SRB measures for some systems exhibiting dominated splitting. The key idea is to add small random noise to the deterministic dynamical system and prove that as noise levels tend to zero, the limit of the ergodic stationary measures, called the randomly ergodic limit (see Definition 3.6), has ergodic components to be Gibbs $cu$-states associated to some sub-bundle $E$ whenever this randomly ergodic limit appears some weak expansion along $E$.

To formulate the main result more precisely, let $f$ be a $C^2$ diffeomorphism on a compact Riemannian manifold $M$. Let $\Lambda$ be an attractor for $f$ with the trapping region $U$, i.e., $\Lambda$ is a compact $f$-invariant subset of $M$, and $U$ is the open neighborhood of $\Lambda$ such that $f(U) \subset U$ and $\Lambda = \cap_{n \geq 0} f^n(U)$. We say $\Lambda$ admits a dominated splitting, if there exists a $Df$-invariant splitting $T\Lambda M = E \oplus F$ and constant $0 < \tau < 1$ such that $\|Df|_E(x)\| \leq \tau \|Df^{-1}|_{E(f(x))}\|$ for any $x \in \Lambda$. In this case, we say $E$ dominates $F$, and denote it by $E \oplus_\tau F$.

**Theorem A.** Let $f$ be a $C^2$ diffeomorphism with an attractor $\Lambda$ admitting a dominated splitting $T\Lambda M = E \oplus F$. If $\mu$ is a randomly ergodic limit supported on $\Lambda$ such that

$$\int \log \|Df^{-1}|_E\|^{-1} d\mu > 0,$$

then there exist ergodic components of $\mu$ as Gibbs $cu$-states associated to $E$.

In the previous work of Cowieson-Young [15], they showed the existence of SRB measures as zero noise limits in partially hyperbolic attractors with one-dimensional centers. The underlying principle there is that if the zero noise limit (accumulation point of stationary measures as noise levels tend zero) has no mixed behavior and the system is random entropy expansive, then it satisfies the Pesin entropy formula by passing limit of the random Pesin entropy formula. With the similar token, but using asymptotical entropy expansiveness property instead, Liu-Lu [22] obtained SRB measures by showing Pesin entropy formula holds for every zero noise limit for some partially hyperbolic attractors. Note that without using the tool of random perturbations, [14] got the existence of invariant measures satisfying Pesin entropy formula for more general dynamical systems (see [38] also).

In contrast to their strategy, in the proof of Theorem A, we do not involve the discussion on Pesin entropy formula, and different to them, we consider special zero noise limits—randomly ergodic limits, and take advantage of this modification, as ergodic stationary measures inherit more information from their limit than stationary measures. Indeed, as a central step, we deduce some uniform behaviors with respect to these ergodic stationary measures, which guarantee that the randomly ergodic limit admits the absolute continuity property along a family of unstable manifolds tangent to $E$ (see Lemma A in §4).

Recently, [5, 25] showed the existence (and finiteness) of SRB/physical measures on mostly expanding diffeomorphisms. Recall that in both works, the authors make effort to show the existence of positive Lebesgue measure set of (weakly) non-uniformly expanding points, then in terms of the previous techniques from [3] or [2] to find SRB measures. In §6, as a byproduct of our result, we provide a proof of
the existence of SRB measures for systems considered in [25] (it works for [5] with simpler arguments).

The remainder of this paper is organized as follows. In §2, we recall the definitions of SRB measures and Gibbs cu-states. In §3, we study the dynamics under regular random perturbations. §4 is dedicated to prove Lemma A, a key lemma which shows that any randomly ergodic limit of Theorem A has the absolute continuity property along a family of local unstable disks. In §5, we complete the proof of Theorem A, by using a result which asserts the arising the Gibbs cu-states from ergodic components of randomly ergodic limit. In §6, as applications of Theorem A, we give some results on the existence of SRB measures and physical measures. Appendix A is devoted to showing the Lipschitz continuity of random local unstable disks.

2. SRB measures and Gibbs cu-states. Let $M$ be a compact Riemannian manifold, use $\text{Leb}$ represent the Lebesgue measure of $M$. Given a sub-manifold $\gamma \subset M$, denote by $\text{Leb}_\gamma$ the Lebesgue measure on $\gamma$ induced by the restriction of the Riemannian structure to $\gamma$. Let $d$ denote the distance in $M$, and $\rho$ the distance in the Grassmannian bundle of $TM$ generated by the Riemannian metric. Denote by $\text{Diff}^2(M)$ the space of $C^2$ diffeomorphisms on $M$.

Given diffeomorphism $f$ on $M$, let $E$ be a $Df$-invariant sub-bundle and $\mu$ an $f$-invariant measure, we define the integrable Lyapunov exponent along $E$ of $\mu$ by

$$\lambda_E(\mu) = \int \log \|Df^{-1}|_E\|^{-1} d\mu.$$ 

Given $f \in \text{Diff}^2(M)$, let $\Lambda$ be an attractor admitting the dominated splitting $T_\Lambda M = E \oplus F$. Since the distributions $E$ and $F$ are continuous, we may extend them continuously to some trapping region $U$ of $\Lambda$, denoted by $E$ and $F$ as well. Given $a > 0$, define the $E$-direction cone field $\mathcal{E}_a = \{\mathcal{E}_a(x) : x \in U\}$ of width $a$ by

$$\mathcal{E}_a(x) = \{v = \nu_E + \nu_F \in E(x) \oplus F(x) : \|v_F\| \leq a\|v_E\|\}$$

for every $x \in U$. We say a smooth embedded sub-manifold $D$ is tangent to $\mathcal{E}_a$ if $\dim D = \dim E$ and $T_x D \subset \mathcal{E}_a(x)$ for any $x \in D$.

Let $(X, \mathcal{A}, \mu)$ be a probability space, we say $\mathcal{P}$ is a measurable partition of $X$, when there exists a sequence of countable partitions $\{P_k : k \in \mathbb{N}\}$ of $X$ such that $\mathcal{P} = \bigvee_{k=0}^\infty P_k \pmod 0$. By Rokhlin’s disintegration theorem [28] (see [10, Appendix C.4] also), for measurable partition $\mathcal{P}$, there exists a unique family of conditional measures $\{\mu_P : P \in \mathcal{P}\}$ of $\mu$ w.r.t. $\mathcal{P}$ such that $\mu_P(P) = 1$ for $\mu$-almost every $P \in \mathcal{P}$, and for any measurable set $A$, $P \mapsto \mu_P(A)$ is measurable with $\mu(A) = \int \mu_P(A) d\hat{\mu}$, where $\hat{\mu}$ is the quotient measure of $\mu$ w.r.t. $\mathcal{P}$.

**Definition 2.1.** Let $\mu$ be a Borel probability and $\mathcal{P}$ a measurable partition formed by disjoint smooth sub-manifolds, whose union admits positive $\mu$ measure. We say $\mu$ has absolutely continuous conditional measures along $\mathcal{P}$, if the conditional measures $\{\mu_\gamma : \gamma \in \mathcal{P}\}$ of $\mu$ (restricted to $\cup_{\gamma \in \mathcal{P}} \gamma$) w.r.t. $\mathcal{P}$ are absolutely continuous w.r.t. corresponding Lebesgue measures on these sub-manifolds, i.e., $\mu_\gamma \ll \text{Leb}_\gamma$ for $\mu$-almost every $\gamma \in \mathcal{P}$.

Let $\mu$ be an $f$-invariant Borel probability that has positive Lyapunov exponents almost everywhere, then there exist a family of Pesin unstable manifolds $W^u(x)$ for $\mu$-almost every $x \in M$.
\(\mu\)-almost every \(x\) (see [9] for Pesin theory). A measurable partition \(\xi\) is said to be subordinate to \(W^u\) if for \(\mu\)-almost every \(x\),

- \(\xi(x) \subset W^u(x)\), where \(\xi(x)\) is the element of \(\xi\) containing \(x\),
- \(\xi(x)\) contains an open neighborhood of \(x\) in \(W^u(x)\).

In particular, for diffeomorphism \(f\) with dominated splitting of type \(E \oplus F\), if \(\mu\) is a Borel probability that has positive Lyapunov exponents along \(E\), then there exists a Pesin unstable manifold \(W^{E,u}(x)\) tangent to \(E\) at \(x\) with dimension \(\dim E\), for \(\mu\)-almost every \(x\). Thus, measurable partitions subordinate to \(W^{E,u}\) can be defined similarly.

Now we recall the definition of SRB measures, following [37, 16]:

**Definition 2.2.** Let \(f\) be a \(C^2\) diffeomorphism on \(M\), an \(f\)-invariant Borel probability \(\mu\) is an SRB measure if \(\mu\) has positive Lyapunov exponents almost everywhere, and it has absolutely continuous conditional measures along any measurable partitions subordinate to \(W^u\).

**Definition 2.3.** Assume that \(f\) is a \(C^2\) diffeomorphism with an attractor \(\Lambda\) admitting a dominated splitting \(T_\Lambda M = E \oplus F\).

An \(f\)-invariant Borel probability \(\mu\) supported on \(\Lambda\) is called a Gibbs cu-state (associated to \(E\)) if the Lyapunov exponents of \(\mu\) along \(E\) are positive and \(\mu\) has absolutely continuous conditional measures along any measurable partition subordinate to \(W^{E,u}\).

We remark that to check the absolute continuity property for SRB measures and Gibbs cu-states, it suffices to verify it for one (not all) measurable partition subordinate to Pesin unstable manifolds (see e.g. [21, Chapter IV: Remark 2.1]).

### 3. Random perturbations.

#### 3.1. Regular random perturbations.

In this work, we use the random perturbation model of iterations of random maps, the strategy is to consider random orbits generated by an independent and identically distributed random choice of map at each iteration (see [32], [4], [10, Appendix D.2]). Given \(f \in \text{Diff}^2(M)\), take a metric space \(\Omega\) and define the continuous map \(F\) from \(\Omega\) to \(\text{Diff}^2(M)\):

\[
F : \Omega \to \text{Diff}^2(M)
\]

\[
\omega \mapsto f_\omega
\]

with property \(F(\omega_f) = f\) for some \(\omega_f \in \Omega\).

For each \(x \in M\), define \(\ell_x : \Omega \to M\) such that \(\ell_x(\omega) = f_\omega(x)\) for every \(\omega \in \Omega\). Let us introduce a sequence of Borel probability measures \(\{\nu_\varepsilon\}_{\varepsilon > 0}\) on \(\Omega\) satisfying:

**(H1):** \(\{\text{supp}(\nu_\varepsilon)\}_{\varepsilon > 0}\) form a nested family of compact connected sets such that \(\text{supp}(\nu_\varepsilon) \to \{\omega_f\}\), when \(\varepsilon \to 0\).

**(H2):** For each \(\varepsilon > 0\), \((\ell_x)_*\nu_\varepsilon \ll \text{Leb}\) for each \(x \in M\), where \((\ell_x)_*\nu_\varepsilon(A) = \nu(\ell_x^{-1}A)\) for any measurable subset \(A\).

**Definition 3.1.** We refer to \(\mathcal{R}_f = \{F, (\nu_\varepsilon)_{\varepsilon > 0}\}\) with \(F\) having above settings and \(\{\nu_\varepsilon\}_{\varepsilon > 0}\) satisfying (H1) and (H2) as a regular random perturbation of \(f\). Given \(\varepsilon > 0\), \(\mathcal{R}_f^\varepsilon = \{F, \nu_\varepsilon\}\) is denoted to be the \(\varepsilon\)-regular random perturbation of \(f\) with \(\varepsilon\) as a noise level.
Remark. We will also use $R_f = \{ F, (\nu_n)_{n \in \mathbb{N}} \}$ to denote a regular random perturbation of $f$ if $(\nu_n)_{n \in \mathbb{N}}$ satisfies $\text{(H1)}$ and $\text{(H2)}$ in discrete sense: $\text{supp}(\nu_n) \to \{ \omega_f \}$ as $n \to +\infty$, and $(\ell_x)_* \nu_n \ll \text{Leb}$ for every $x \in M$ and $n \in \mathbb{N}$.

Let us mention that the regular random perturbations of $C^2$ diffeomorphisms are always exist, see [4, Example 5.2] for details. This kind of random perturbations were previously considered in [6, 7, 1] also.

For each $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) \in \Omega^\mathbb{Z}$, we use the presentation

$$f^N_\omega = \begin{cases} f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0} & n > 0 ; \\ id & n = 0 ; \\ f_{\omega_1}^{-1} \circ \cdots \circ f_{\omega_{-1}}^{-1} & n < 0. \end{cases}$$

As an important idea to investigate the dynamics under random perturbations, it is useful to introduce the two-sided skew-product map

$$G : \Omega^\mathbb{Z} \times M \to \Omega^\mathbb{Z} \times M \\
(\omega, x) \mapsto (\sigma(\omega), f_{\omega_0}(z)).$$

where $\sigma$ is the left shift operator. Let us define the projection maps

$$P : \Omega^\mathbb{Z} \times M \to \Omega^\mathbb{N} \times M \\
(\omega, x) \mapsto (\omega^+, x)$$

$$P_M : \Omega^\mathbb{Z} \times M \to M \\
(\omega, x) \mapsto x$$

where $\omega^+ = (\omega_0, \omega_1, \cdots) \in \Omega^\mathbb{N}$ for $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) \in \Omega^\mathbb{Z}$.

**Definition 3.2.** Given a Borel probability $\nu$ on $\Omega$, we say $\mu$ is a stationary measure for $\nu$ if

$$\mu(A) = \int \mu(f_{\omega}^{-1}A)d\nu(\omega)$$

for every Borel subset $A$.

For given regular random perturbation $R_f = \{ F, (\nu_\epsilon)_{\epsilon > 0} \}$, one says that $\mu$ is a stationary measure for $R_f$ if it is a stationary measure for some $\nu_\epsilon$. Furthermore, we say $\mu$ is a stationary measure for $R_f$ = $\{ F, \nu_\epsilon \}$ as long as it is a stationary measure for $\nu_\epsilon$. Note that when $\nu$ is reduced to $\delta_{\omega_f}$, then the stationary measures for $\nu$ are $f$-invariant measures. Moreover, we have the following result, see [4, Lemma 5.5] for a proof.

**Lemma 3.3.** Let $R_f = \{ F, (\nu_\epsilon)_{\epsilon > 0} \}$ be a random perturbation of $f$. If $\mu_\epsilon$ is a stationary measure for $\nu_\epsilon$ for any $\epsilon > 0$, then all the accumulation points of $\{ \mu_\epsilon \}_{\epsilon > 0}$ as $\epsilon \to 0$ are $f$-invariant measures.

We need the following observation:

**Lemma 3.4.** Let $R_f$ be a regular random perturbation of $f$, then any the stationary measure for $R_f$ is absolutely continuous with respect to Lebesgue measure.

**Proof.** Fix $R_f = \{ F, (\nu_\epsilon)_{\epsilon > 0} \}$ and a stationary measure $\mu_\epsilon$ for $\nu_\epsilon$ for some $\epsilon > 0$. Let $A$ be a Borel subset such that $\text{Leb}(A) = 0$, then the hypothesis (H2) of regular
random perturbation implies that \((\ell_x)_* \nu_\varepsilon(A) = 0\) for every \(x \in M\). Therefore, by recalling the definition of \(\ell_x, x \in M\), one gets

\[
\int \mu_\varepsilon(f_\omega^{-1}A) d\nu_\varepsilon(\omega) = \int \int \chi_{f_\omega^{-1}A}(x) \, d\mu_\varepsilon(x) \, d\nu_\varepsilon(\omega)
= \int \int \chi_A(f_\omega(x)) \, d\nu_\varepsilon(\omega) \, \mu_\varepsilon(x)
= \int \nu_\varepsilon(\{\omega : f_\omega(x) \in A\}) \, d\mu_\varepsilon(x)
= \int \nu_\varepsilon(\ell_x^{-1}(A)) \, d\mu_\varepsilon(x)
= \int (\ell_x)_* \nu_\varepsilon(A) \, d\mu_\varepsilon(x)
= 0.
\]

Then by definition of stationary measure, we have

\[
\mu(A) = \int \mu(f_\omega^{-1}A) d\nu(\omega) = 0,
\]

which completes the proof of absolute continuity.

For a stationary measure \(\mu\) for some Borel probability \(\nu\) on \(\Omega\), we say a Borel set \(A\) is random invariant if for \(\mu\)-a.e. \(x \in M\),

\[
x \in A \implies f_\omega(x) \in A \quad \text{for } \nu\text{-a.e. } \omega,
\]

\[
x \in M \setminus A \implies f_\omega(x) \in M \setminus A \quad \text{for } \nu\text{-a.e. } \omega.
\]

**Definition 3.5.** A stationary measure \(\mu\) is ergodic if every random invariant set has either \(\mu\) measure 0 or \(\mu\) measure 1.

By the ergodic decomposition theorem in random case (see e.g. [35, Appendix A.1],[4, Proposition 5.9]), every stationary measure can be written as a convex combination of ergodic stationary measures. Therefore, the existence of ergodic stationary measures is guaranteed. We introduce the following notion, as a special kind of zero noise limits.

**Definition 3.6.** Given \(f \in \text{Diff}^2(M)\) and an \(f\)-invariant measure \(\mu\), if there is a regular random perturbation \(\mathcal{F}_f = \{\mathcal{F}_\varepsilon, (\nu_\varepsilon)_{\varepsilon > 0}\}\) of \(f\) such that \(\lim_{n \to +\infty} \mu_{\varepsilon_n} = \mu\) for some subsequence \(\{\varepsilon_n\}\), where \(\mu_{\varepsilon_n}\) is an ergodic stationary measure for \(\nu_{\varepsilon_n}\) for every \(n \in \mathbb{N}\). Then \(\mu\) is said to be a randomly ergodic limit.

**Lemma 3.7.** For any stationary measure \(\mu\) for \(\nu\), there is a unique \(G\)-invariant measure \(\mu^G\) on \(\Omega^\mathbb{Z} \times M\) such that \(\mathbb{P}_* \mu^G = \nu^\mathbb{Z} \times \mu\). We have the following properties:

- \(\mu\) is an ergodic stationary measure if and only if \(\mu^G\) is ergodic for \(G\).
- Suppose that \(\mu_n\) is a stationary measure for \(\nu_n\) for any \(n \in \mathbb{N}\), if \(\mu_n \to \mu\), \(\nu_n \to \nu\) as \(n \to +\infty\), then \(\mu_n^G \to \mu^G\) as \(n \to +\infty\).

**Proof.** It follows form [21, Proposition 1.2] that for every stationary measure \(\mu\) for some probability \(\nu\) on \(\Omega\), there exists the unique \(G\)-invariant measure \(\mu^G\) such that \(\mathbb{P}_* \mu^G = \nu^\mathbb{Z} \times \mu\). Moreover, by [21, Proposition 1.3] we have that \(\mu\) is ergodic if and only if \(\mu^G\) is ergodic for \(G\).

Under the assumptions of the second item, by passing to the limit, one gets that \(\mu\) is a stationary measure for \(\nu\). Assume that \(\lim_{k \to +\infty} \mu_{\varepsilon_k} = \eta\) for some subsequence
\{n_k\}. Since each \( \mu_n^G \) is \( G \)-invariant, \( \eta \) is \( G \)-invariant. Moreover, by applying the continuity of \( \mathbb{P} \) one gets
\[
\mathbb{P}_* \eta = \mathbb{P}_* \lim_{k \to \infty} \mu_{n_k}^G = \lim_{k \to \infty} \mathbb{P}_* \mu_{n_k} = \lim_{k \to \infty} \nu_{n_k}^N \times \mu_{n_k} = \nu_N^N \times \mu.
\]
Since \( \mu^G \) is the only \( G \)-invariant measure such that \( \mathbb{P}_* \mu^G = \nu_N^N \times \mu \), we obtain \( \eta = \mu^G \) and thus \( \lim_{n \to \infty} \mu_n^G = \mu^G \).

As a consequence of Lemma 3.7, we have:

**Corollary 3.8.** Let \( \mu_n \) be a sequence of stationary measures for \( \nu_n \), if \( \nu_n \to \delta_{\omega_j} \) and \( \mu_n \to \mu \) as \( n \to +\infty \), then \( \lim_{n \to \infty} \mu_n^G = \delta_{\omega_j}^G \times \mu \). In particular, if \( \mu \) is a randomly ergodic limit, then \( \mu^G = \delta_{\omega_j}^G \times \mu \).

### 3.2. Dominated splittings under random perturbations

Let \( \Lambda \) be an attractor for a \( C^2 \) diffeomorphism \( f \) with trapping region \( U \), and there exists the dominated splitting \( T_\Lambda M = E \ominus F \). Fixing a regular random perturbation \( R_f = \{ \mathcal{F}, (\nu_\varepsilon)_{\varepsilon > 0} \} \).

Since \( f(U) \subset U \), we can choose \( \varepsilon_0 \) small enough such that: for every \( \varepsilon \leq \varepsilon_0 \), \( f_{\omega}(U) \subset U \) for any \( \omega \in N_{\varepsilon_0}^\varepsilon \) as well. For every \( \omega = (\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}) \in N_{\varepsilon_0}^\varepsilon \), \( \varepsilon \leq \varepsilon_0 \), we write \( \Lambda_{\omega} = \bigcap_{n \geq 1} U_{f_{\omega}^n - \omega} (U) \). By construction, \( \{ \Lambda_{\omega} \} \) is a family of compact subsets satisfying \( f_{\omega_0} \Lambda_{\omega} = \Lambda_{\sigma \omega} \) for each \( \omega \in N_{\varepsilon_0}^\varepsilon \). For each \( \varepsilon \leq \varepsilon_0 \), we define \( \Lambda_\varepsilon = \bigcup_{\omega \in N_{\varepsilon_0}^\varepsilon} \Lambda_{\omega} \) it is a \( G \)-invariant compact subset contained in \( N_{\varepsilon_0}^\varepsilon \times U \). By the choice of \( \varepsilon_0 \), for every \( \varepsilon \leq \varepsilon_0 \), any stationary measure for \( \nu_\varepsilon \) supported on \( U \) is concentrated on \( \mathbb{P}_M(\Lambda_\varepsilon) \) and the corresponding \( G \)-invariant measure \( \mu^G \) is concentrated on \( \Lambda_\varepsilon \).

By standard cones argument as in [15, §4.3] one has

**Lemma 3.9.** For each \( \varepsilon \leq \varepsilon_0 \), there are random sub-bundles \( E(\omega, x) \) and \( F(\omega, x) \) for every \( (\omega, x) \in \Lambda_\varepsilon \) corresponding to \( E \) and \( F \). Moreover, \( E(\omega, x) \) and \( F(\omega, x) \) depend continuously on \( (\omega, x) \).

Recall the notion of local unstable disk in deterministic dynamical systems. We represent \( W_d^E(x) \) as the local unstable disk of \( x \) with radius \( \delta \) around \( x \), which is tangent to \( E \) everywhere and \( (C, \lambda) \)-backward contracting for some \( C > 0 \) and \( \lambda \in (0,1) \), i.e.,
\[
d(f^{-n}(y), f^{-n}(z)) \leq C \lambda^n d(y, z)
\]
for every \( n \in \mathbb{N} \), whenever \( y, z \in W_d^E(x) \).

Similarly, using \( W_d^E(u)(\omega, x) \) to denote the random local unstable disk of \( (\omega, x) \) with radius \( \delta \) around \( x \), which is tangent to the random sub-bundle \( E(\omega, \cdot) \) everywhere and \( (C, \lambda) \)-backward contracting for some \( C > 0 \), \( \lambda \in (0,1) \), in the sense that
\[
d(f^{-n}(y), f^{-n}(z)) \leq C \lambda^n d(y, z)
\]
for every \( n \in \mathbb{N} \), whenever \( y, z \in W_d^E(u)(\omega, x) \). Note that we have \( W_d^E(u)(\omega, x) \subset \Lambda_{\omega} \) if \( (\omega, x) \in \Lambda_{\varepsilon_0} \).

Sometimes, one needs to look random unstable disks on the product space \( \Omega \times M \). For this, we use \( \{ \omega \} \times W_d^E(u)(\omega, x) \subset \{ \omega \} \times M \) to denote the endowed random local unstable disk associated to \( W_d^E(u)(\omega, x) \).

Now we state a result which asserts the uniform Lipschitz continuity of random local unstable disks, it is a simple generalization of its deterministic version (see e.g. [17, § 6.1]). (One can see a direct proof in Appendix A).
Lemma 3.10. Given $C > 0$ and $0 < \lambda < 1$ there exists constant $L_0 > 0$, such that for any $(C, \lambda)$-backward contracting random local unstable disk $\gamma = W_{\delta}^{E,u}(\omega, x)$ of $(\omega, x) \in \Lambda_{\varepsilon_{0}}$, we have

$$\rho(T_y \gamma, T_z \gamma) \leq L_0 d(x, y)$$

for any $y, z \in \gamma$.

We shall consider following typical (random) local unstable disks.

Definition 3.11. Given $\delta > 0, \lambda \in (0, 1)$, we say a random local unstable disk $W_{\delta}^{E,u}(\omega, x)$ of $(\omega, x)$ is $\lambda$-backward contracting if

- $W_{\delta}^{E,u}(\omega, x)$ is $(1, \lambda)$-backward contracting;
- $\|Df^{-n}|_{E(\omega, y)}\| \leq \lambda^n$ for every $n \in \mathbb{N}$ and $y \in W_{\delta}^{E,u}(\omega, x)$.

Similarly, a local unstable disk $W_{\delta}^{E,u}(x)$ is said to be $\lambda$-backward contracting whenever it is $(1, \lambda)$-backward contracting and $\|Df^{-n}|_{E(y)}\| \leq \lambda^n$ for every $n \in \mathbb{N}$ and $y \in W_{\delta}^{E,u}(x)$.

3.3. Random Gibbs cu-states. Throughout this subsection, let $f \in \text{Diff}^2(M)$ and fix a stationary measure $\mu$ of $\mathcal{F}_{\varepsilon} = \{F, \nu_\varepsilon\}$ for some small $\varepsilon > 0$ and assume $\mu^G$ is the $G$-invariant measure associated to $\mu$ given by Lemma 3.7.

According to the hypotheses (H1), $\mathcal{N}_{\varepsilon} = \text{supp}(\nu_\varepsilon)$ is compact, then use the continuity of $\mathcal{F}$, we have that $\mathcal{F}(\mathcal{N}_{\varepsilon})$ is a compact subset of $\text{Diff}^2(M)$ around $f$, which implies that

$$\int (\log^+ |f_\omega|_{C^1} + \log^- |f_\omega^{-1}|_{C^1}) d\nu_\varepsilon(\omega) < +\infty,$$

$$\int \log^+ |f_\omega^{-1}|_{C^2} d\nu_\varepsilon(\omega) < +\infty,$$

where we use the notation $\log^+ a = \max\{\log a, 0\}$ and $\log^- a = \max\{-\log a, 0\}$.

In terms of (1), we have the following proposition [21, Chapter VI: Proposition 1.2]

Proposition 3.12. For $\mu^G$-a.e. $(\omega, x)$, there exists $r(x) \in \mathbb{N}$ and a decomposition

$$T_x M = \bigoplus_{i=1}^{r(x)} E_i(\omega, x)$$

such that there are numbers $\lambda_1(x) > \cdots > \lambda_{r(x)}(x)$ satisfying

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \|Df^n_\omega(x)v\| = \lambda_i(\omega, x) = \lambda_i(x)$$

for any non-zero vector $v \in E_i(\omega, x)$, and every $1 \leq i \leq r(x)$.

Definition 3.13. The numbers $\lambda_i(x), 1 \leq i \leq r(x)$ we introduced above are called the random Lyapunov exponents of $\mathcal{F}_{\varepsilon}$.

Since condition (2) holds, corresponding to the positive(resp. negative) random Lyapunov exponents, one may construct random unstable(resp. stable) manifolds analogous in deterministic dynamical systems. See [21, Chapter VI: Proposition 1.4] for details.
Proposition 3.14. For $\mu^G$-a.e. $(\omega, x)$, if $\lambda_1(x) > \cdots > \lambda_d(x)$ are the positive random Lyapunov exponents of $(\omega, x)$, then there exist random unstable manifolds $W^{u,i}(\omega, x), 1 \leq i \leq d$ defined by

$$W^{u,i}(\omega, x) = \left\{ y \in M : \limsup_{n \to +\infty} \frac{1}{n} \log d(f_{\omega}^{-n}(x), f_{\omega}^{-n}(y)) \leq -\lambda_i(x) \right\}.$$  

In addition, for every $1 \leq i \leq d$, $W^{u,i}(\omega, x)$ is a $C^1$ sub-manifold tangent to \( \bigoplus_{j=1}^d E(\omega, x) \) at $(\omega, x)$ with dimension equal to $\dim(\bigoplus_{j=1}^d E(\omega, x))$. Also, we have $W^{u,1}(\omega, x) \subset \cdots \subset W^{u,d}(\omega, x)$.

Now we assume that $\Lambda$ is an attractor of $f$ with dominated splitting $T_\Lambda M = E \oplus F$. Shrinking $\varepsilon$ if necessary, we assume $\varepsilon$ is less than $\varepsilon_0$ given in Lemma 3.9, it follows that for $\mu^G$-a.e. $(\omega, x)$ there exists an integer $1 \leq d(x) \leq r(x) - 1$ such that

$$E(\omega, x) = \bigoplus_{i=1}^{d(x)} E_i(\omega, x), \quad F(\omega, x) = \bigoplus_{i=d(x)+1}^{r(x)} E_i(\omega, x).$$

Moreover, if the Lyapunov exponents of $\mu^G$ along $E$ are positive, namely $\lambda_{d(x)}(x) > 0$ for $\mu^G$-a.e. $(\omega, x)$, then there exists a Pesin unstable manifold $W^{E,u}(\omega, x) := W^{v,d(x)}(\omega, x)$ tangent to $E$ for $\mu^G$-almost every $(\omega, x)$.

Analogous to the deterministic setting, we can introduce the measurable partition subordinate to Pesin unstable manifolds (in different levels), follows after [21, Chapter VI: E].

Definition 3.15. $\xi$ is called a measurable partition subordinate to $W^{u,1}$ w.r.t. $\mu^G$ if for $\mu^G$-a.e $(\omega, x)$, $\xi(\omega, x) \subset \{\omega\} \times M$, and $\xi(\omega, x) = \{y : (\omega, y) \in \xi(\omega, x) \} \subset W^{u,1}(\omega, x)$ contains a neighborhood of $x$ open in $W^{u,1}(\omega, x)$.

Similarly, one can define the measurable partitions subordinate to $W^{E,u}$. Now we can give the precise definition of random Gibbs cu-state as below.

Definition 3.16. We say $\mu^G$ is a Gibbs cu-state (associated to $E$), if the Lyapunov exponents of $\mu^G$ along $E$ are positive and $\mu^G$ has absolutely continuous conditional measures along $\xi$ for any measurable partition $\xi$ subordinate to $W^{E,u}$.

As a result of the regularity of the random perturbations, we have the following important fact.

Lemma 3.17. If the Lyapunov exponents of $\mu^G$ along $E$ are positive, then $\mu^G$ is the random Gibbs cu-states associated to $E$.

Proof. We sketch the proof of this lemma. By Lemma 3.4, $\mu$ is absolutely continuous w.r.t. Lebesgue measure. This together with Theorem 1.1 of [21, Chapter VI] imply that $\mu$ has absolutely continuous conditional measures along any measurable partition subordinate to $W^u$. Use the same manner of [20, Lemma 9.11] one can construct two measurable partitions $\xi \prec \xi^E$ (\( \xi^E \) is finer than $\xi$) such that $\xi$ and $\xi^E$ are subordinate to $W^u$ and $W^{E,u}$, respectively. Thus, the essential uniqueness of conditional measures [36, Exercise 5.21] implies that $\mu$ has absolutely continuous conditional measures along $\xi^E$, this shows that $\mu^G$ is a random Gibbs cu-state associated to $E$ by definition.

Now we discuss the densities of conditional measures of random Gibbs cu-states along measurable partitions consisted of random local unstable disks that are backward contracted by a uniform rate.
Lemma 3.18. Let $\mathcal{P}$ be a measurable partition formed by disjoint endowed random local unstable disks with uniform radius, and all these random local unstable disks are $(C, \sigma)$-backward contracting for some $C > 0$ and $\sigma \in (0, 1)$. Then there exists $K > 0$ such that for any random Gibbs cu-state admitting positive measure on the union of the elements from $\mathcal{P}$, the densities of the conditional measures of $\mu$ along $\mathcal{P}$ are between $1/K$ and $K$.

Proof. Rewrite

$$\mathcal{P} = \{\{\omega\} \times \gamma : \text{each } \gamma \text{ is } (C, \sigma)-\text{backward contracting}\}.$$ 

Let $P$ be the union of the element of $\mathcal{P}$. Let $\alpha, \beta$ be the constants satisfying that for each $\{\omega\} \times \gamma \in \mathcal{P}$,

- $d(y, z) \leq \alpha$ for every $y, z \in \gamma$;
- $1/\beta \leq \text{Leb}(\gamma) \leq \beta$.

Since $f_\omega$ are $C^2$ diffeomorphisms nearby $f$ with a fixed radius, thus there exists a uniform bound on the Lipschitz constant of $Df_\omega$. Combining this fact with Lemma 3.10 yields that for every $y, z$,

$$|\log J^E(\omega, x) - \log J^E(\omega, y)| \leq L_1 d(x, y)$$

for some constant $L_1 > 0$ depending only on $f$. Therefore, for any random local unstable disk $\gamma$ which is $(C, \sigma)$-backward contracting, for every $y, z \in \gamma$ and any $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} \left| \log \frac{J^E(G^{-j}(\omega, y))}{J^E(G^{-j}(\omega, z))} \right| = \sum_{j=1}^{n} \left| \log J^E(G^{-j}(\omega, y)) - J^E(G^{-j}(\omega, z)) \right| \leq L_1 \sum_{j=1}^{n} d(f^{-j}_\omega(y), f^{-j}_\omega(z))$$

$$\leq L_1 \sum_{j=1}^{n} C\sigma^j d(y, z)$$

$$\leq \frac{L_1 C\sigma \alpha}{1 - \sigma} := \log M.$$

Consequently, we obtain

$$\frac{1}{M} \leq \prod_{j=1}^{\infty} \frac{J^E(G^{-j}(\omega, y))}{J^E(G^{-j}(\omega, z))} \leq M.$$  (3)

Now let us take $\mu$ as a random Gibbs cu-state such that $\mu(P) > 0$. Let $\rho$ be the density of the conditional measures of $\mu$ along $\mathcal{P}$ w.r.t. its Lebesgue measures. Moreover, for $\mu$-almost every $\{\omega\} \times \gamma \in \mathcal{P}$, the density $\rho$ has following formula (see [21, Chapter VI: Proposition 2.2 and Corollary 8.1] or [31, Proposition 2.1]):

$$\frac{\rho(\omega, y)}{\rho(\omega, z)} = \prod_{j=1}^{\infty} \frac{J^E(G^{-j}(\omega, z))}{J^E(G^{-j}(\omega, y))}$$

for $\mu(\omega) \times \gamma$-almost every $(\omega, y)$ and $(\omega, z)$ in $\{\omega\} \times \gamma$. Moreover, we have

$$\rho(\omega, z) = \frac{\kappa(\omega, y), (\omega, z))}{\int_{\gamma} \kappa((\omega, y), (\omega, z)) d\text{Leb}_\gamma(z)}.$$  (4)
where
\[ \kappa((\omega, y), (\omega, z)) = \prod_{j=1}^{\infty} \frac{J_u(G^{-j}(\omega, y))}{J_u(G^{-j}(\omega, z))}. \]

By the choices of \( \beta \) and the estimate (3), the formula (4) yields
\[ \frac{1}{\beta M^2} \leq \rho(\omega, z) \leq \beta M^2. \]
Thus, it suffices to take \( L = \beta M^2 \) to complete the proof. \( \square \)

4. **Absolutely continuous conditional measures.** As the main technical step for proving Theorem A, we shall make effort to show the following result in this section.

**Lemma A.** Let \( f \) be a \( C^2 \) diffeomorphism with an attractor \( \Lambda \) admitting a dominated splitting \( T_\Lambda M = E \oplus F \).

If \( \mu \) is a randomly ergodic limit satisfying \( \lambda_E(\mu) > 0 \), then there exists \( \delta > 0 \), \( 0 < \sigma < 1 \) and a measurable partition \( \mathcal{F} \) with following properties:

- the union of elements from \( \mathcal{F} \) has positive \( \mu \)-measure;
- each \( \gamma \in \mathcal{F} \) is a \( \sigma \)-backward contracted local unstable disk with radius \( \delta \);
- \( \mu \) has absolutely continuous conditional measures along \( \mathcal{F} \).

We will complete the proof of Lemma A in §4.4, here we briefly outline the strategy for proving Lemma A. To begin with, let us fix ergodic stationary measures \( \mu_n \) converges to \( \mu \), consider \( \mu^G_n \) and \( \mu^G \) as the corresponding \( G \)-invariant measures of \( \mu_n \) and \( \mu \) introduced in Lemma 3.7.

Firstly, we show that there exists a “Pesin block” \( \mathcal{H}^G_{\lambda_0} \) for some \( \lambda_0 \in (0, 1) \) in \( \Omega^Z \times M \) possessing positive \( \mu^G \) measure for all sufficiently large \( n \), which consists of points admitting random local unstable disks tangent to \( E \) of uniform size and backward contraction rate. This is done in §4.3.

Secondly, we get a measurable partition \( \mathcal{F}^G \) formed by random local unstable disks generated by points from \( \mathcal{H}^G_{\lambda_0} \), which is induced by a foliated chart established in lift space \( \Omega^Z \times M \), this is a consequence of Proposition 4.6.

Finally, by the construction of \( \mathcal{F}^G \) one knows that the densities of disintegrations of \( \mu^G_n \) along \( \mathcal{F}^G \) are uniformly bounded from above and from below by applying Lemma 3.18, then passing to the limit as noise level tends zero, we get the desired absolute continuity of \( \mu \) along a measurable partition \( \mathcal{F} \), where \( \mathcal{F} \) is a family of local unstable disks arising from \( \mathcal{F}^G \).

**Settings of this section.** Throughout this section, let \( \Lambda \) be an attractor with dominated splitting \( T_\Lambda M = E \oplus F \). Let us take \( \varepsilon_0 \) as in Lemma 3.9 and thus, one can fix constant \( a > 0 \) such that \( \rho(E(\omega, x), E(x)) \leq a \) for every \( (\omega, x) \in \Lambda_{\varepsilon_0} \). Moreover, we assume that all the noise levels considered in this section are bounded from above by \( \varepsilon_0 \).

4.1. **Unstable disks at backward contracting points.** Given \( \lambda \in (0, 1) \), consider the following “Pesin block”:
\[ \mathcal{H}^G_\lambda = \left\{ (\omega, x) : \prod_{i=0}^{n-1} \| Df^{\sigma^{-1}}_{\omega} | E(G^{-i}(\omega, x)) \| \leq \lambda^n, \quad \forall n \in \mathbb{N} \right\}. \]
Analogously, we define
\[ \mathcal{H}_\lambda = \left\{ x : \prod_{i=0}^{n-1} \| Df^{-1}\|_{E(f^{-1}x)} \leq \lambda^n, \forall n \in \mathbb{N} \right\}. \]

**Definition 4.1.** Given \( \delta > 0, \lambda \in (0,1) \), we say \((\omega, x)\) is \(\lambda\)-backward contracting if \((\omega, x) \in \mathcal{H}_\lambda^\mathcal{G}\). Similarly, we also say a point \(x\) is \(\lambda\)-backward contracting if it is contained in \(\mathcal{H}_\lambda\).

The lemmas below stress that any backward contracting points inherits the (random) local unstable disk, it is a folklore result which can be deduced from the Plaque family theorem [18], see [8, Theorem 4.7] for a direct proof.

**Lemma 4.2.** Given \( \lambda \in (0,1) \), there exist \( \sigma \in (\lambda, 1) \) and \( \delta = \delta(\lambda) > 0 \), such that if \((\omega, x) \in \mathcal{H}_\lambda^\mathcal{G}\), then there exists a random local unstable disk \(W^{E,u}_\delta(\omega, x)\) of \((\omega, x)\) that is \(\sigma\)-backward contracting.

We have deterministic version of Lemma 4.2 as follows:

**Lemma 4.3.** Given \( \lambda \in (0,1) \), there exist \( \sigma \in (\lambda, 1) \) and \( \delta = \delta(\lambda) > 0 \), such that if \(x \in \mathcal{H}_\lambda\), then there exists a local unstable disk \(W^{E,u}_\delta(x)\) of \(x\) which is \(\sigma\)-backward contracting.

### 4.2. A foliated chart of random unstable disks

Denote by \(D^k\) the \(k\)-dimensional compact unit ball of \(\mathbb{R}^k\).

**Definition 4.4.** Let \(\mathcal{K}\) be a compact metric space. A foliated chart associated to \(\mathcal{K}\) is a homeomorphism \(\Phi : \mathcal{K} \times D^k \rightarrow \mathcal{B}\) such that
- \(\Phi_p = \Phi|\{p\} \times D^k\) is a diffeomorphism for each \(p \in \mathcal{K}\).
- \(\Phi_p\) maps \(D^k\) to disjoint (endowed random) local unstable disks with dimension \(k\).

We will also say the above \(\Phi\) is a foliated chart of (endowed random) local unstable disks. It follows from Definition 4.4 that any foliated chart induces a measurable partition formed by disjoint (endowed random) local unstable disks. This suggests that one can also describe the absolute continuity via foliated charts.

More precisely, let \(\mu\) be a Borel probability, one gets that \(\mu\) has absolutely continuous conditional measures along the measurable partition induced by a foliated chart \(\Phi\), if and only if the pullback \(\nu := \Phi^* \mu = \mu \circ \Phi\) has the absolutely continuous conditional measures along the “vertical lines” \(\{\{x\} \times D^k : x \in \mathcal{K}\}\), which means that there exists a measurable function \(\rho : \mathcal{K} \times D^k \rightarrow [0, \infty)\) such that
\[ \nu(A) = \int_A \rho(x, y) d\mathrm{Leb}_{\mathbb{R}^k}(y) d\hat{\nu}(x) \]
for every measurable subset \(A\) of \(\mathcal{K} \times D^k\), where \(\hat{\nu}\) is the quotient measure of \(\nu\) w.r.t. “vertical lines” defined by \(\hat{\nu}(\xi) = \nu(\xi \times D^k)\) for measurable \(\xi \subset \mathcal{K}\).

We will use the following argument (see e.g. [34, Proposition 7.3]).

**Lemma 4.5.** Let \(\mathcal{K}\) be a compact metric space and \(\nu\) a measure on product space \(\mathcal{K} \times D^k\), if there exists \(C > 0\) such that for every open subset \(\xi \subset \mathcal{K}\) satisfying \(\hat{\nu}(\partial \xi) = 0\) and open subset \(\eta \subset D^k\) one has
\[ \nu(\xi \times \eta) \leq C \cdot \mathrm{Leb}_{\mathbb{R}^k}(\eta) \hat{\nu}(\xi), \]
then \(\nu\) has absolutely continuous conditional measures along the vertical lines with density bounded from above by \(C\).
Proposition 4.6. Let $f$ be a $C^2$ diffeomorphism with an attractor $\Lambda$ admitting
dominated splitting $T\lambda M = E \oplus F$. Given $0 < \lambda_0 < 1$, assume that $(\mu_n)_{n \in \mathbb{N}}$
is a sequence of stationary measures for $\nu_n$ so that $\liminf_{n \to \infty} \mu_n^G |H_{\lambda_0}^G > 0$, and
$\lim_{n \to +\infty} \mu_n = \mu$ for some stationary measure $\mu$.

Then there exist $0 < \delta < \delta_u$, and a foliated chart $\Phi : K \times \mathbb{R}^m \rightarrow B$ such that

- $\mu^G(\partial(B)) = 0$, $\mu^G(B) > 0$ and $\mu^G_n(B) > 0$ for all sufficiently large $n$.
- each $\Phi(\omega, x)(\mathbb{R}^m) = \{\omega\} \times \gamma_{\delta}(\omega, x)$, where $\gamma_{\delta}(\omega, x) \subset W_{\delta_n}(\omega, y)$ is a random
local unstable disk with radius $\delta$ around $x$, for some $(\omega, y) \in H_{\lambda_0}^G$.

Proof. Suppose that $N_0 := \text{supp}\nu_{\epsilon_0}$, then we have $\text{supp}(\nu_n) \subset N_0$ for any $n \in \mathbb{N}$
by our setting on random perturbations. By Lemma 4.2, we take $\delta_u$ as the
radius of random local unstable disks of points in $H_{\lambda_0}^G$. Since $\mu_n \to \mu$, we have
$\mu_n^G \to \mu^G$ by Lemma 3.7, together with the compactness of $H_{\lambda_0}^G$ and assumption
$\liminf_{n \to \infty} \mu_n^G |H_{\lambda_0}^G := \alpha_0 > 0$ yield that $\mu^G |H_{\lambda_0}^G \geq \alpha_0$. Choose any $(\omega_0, x_0)$ in
the supported of $\mu^G |H_{\lambda_0}^G$, hence for any $r \ll \delta_u$, one has $\mu^G(N_0 \times B(x_0, r)) > 0$, and
thus $\mu^G(N_0 \times B(x_0, r)) > 0$. Rewrite $C_r = N_0 \times B(x_0, r)$, it is compact.
Up to reducing $r$, one can take a smooth compact disk $\Delta^F$ such that for every
$x \in B(x_0, r_0)$, if $N_x$ is a smooth disk tangent to $E_u$ with radius $\delta_u$ around $x$, then
$N_x$ transverse to $\Delta^F$ at a single point. This implies that for every $(\omega, x) \in H_{\lambda_0}^G \cap C_r$,
the random local unstable disk $W_{\delta_n}^{E,u}(\omega, x)$ transverse to $\Delta^F$ at a point, use $L$
to denote the family of these random local unstable manifolds. More precisely, we take
$L = \{\gamma(\omega, x) : \gamma(\omega, x) = W_{\delta_n}^{E,u}(\omega, x), \text{ for some } (\omega, x) \in H_{\lambda_0}^G \cap C_r\}$.

Then, consider the following subset of $N_0^G \times \Delta^F$,
$K = \{(\omega, x) \in N_0^G \times \Delta^F : x = \gamma(\omega, y) \cap \Delta^F, \text{ for some } \gamma(\omega, y) \in L\}$.

Lemma 4.7. $K$ is a compact subset of $N_0^G \times \Delta^F$.

Proof. Let $(\omega_m, x_m) \in K$ be a sequence of points which converges to a point $(\omega, x) \in N_0^G \times M$ as $n \rightarrow +\infty$. One knows that $x$ is contained in $\Delta^F$, as $\Delta^F$ is compact, thus it suffices to show that there is $\gamma(\omega, y) \in L$ containing $x$. By construction of $K$, for each $(\omega_m, x_m) \in K$, there exists a point $y_m$ so that $x_m \in \gamma(\omega_m, y_m) \in L$.
Considering subsequences if necessary, we assume that $(\omega_m, y_m)$ converges to a point $(\omega, y)$, using Ascoli-Arzelà theorem, there is a smooth disk $\gamma$ containing $y$ such that $\gamma(\omega_m, y_m)$ converges to $\gamma$ in $C^1$-topology. This also implies that $\gamma$ is tangent to $\mathcal{E}(\omega, \cdot)$ everywhere. Since $H_{\lambda_0}^G \cap C_r$ is compact, we have $(\omega, y) \in H_{\lambda_0}^G \cap C_r$, and so $\gamma \in L$. Now we deduce that $x \in \gamma$, indeed, for any $\varepsilon > 0$, there exists $m$ large enough such that $d(x, x_m) < \varepsilon/2$, and some point $x_\varepsilon \in \gamma$ satisfying $d(x, x_\varepsilon) < \varepsilon/2$, since $\gamma(\omega_m, y_m)$ converges to $\gamma$. Consequently, $d(x, x_\varepsilon) < \varepsilon$, by compactness of $\gamma$ we obtain $x \in \gamma$.

It follows from the construction of $K$ that there exists $\delta \leq \delta_u$ and a family of
random unstable disks $\mathcal{F}_\delta = \{\gamma_{\delta}(\omega, x) : (\omega, x) \in K\}$ such that for each $(\omega, x) \in K$,
- there exists $\gamma(\omega, y) \in L$ such that $\gamma_{\delta}(\omega, x) \subset \gamma(\omega, x)$ and has radius $\delta$ with
center $x$;
- the intersection of $\gamma_{\delta}(\omega, x)$ with $C_r$ is contained in the interior of $\gamma_{\delta}(\omega, x)$.

Furthermore, as the random local unstable disks of $\mathcal{F}_\delta$ depend continuously on the
points of $K$, there is a continuous map $\phi : K \rightarrow \text{Emb}^1(\mathbb{R}^m \times M)$ such that
\( \phi(\omega, x)(\mathbb{D}^E) = \{\omega\} \times \gamma^E_\delta(\omega, x) \). As a result, one can define the map

\[
\Phi : K \times \mathbb{D}^E \rightarrow \Omega^E \times M
\]

\[
(\omega, x, y) \mapsto \phi(\omega, x)(y).
\]

Let \( B = \Phi(K \times \mathbb{D}^E) \), then \( \Phi : K \times \mathbb{D}^E \rightarrow B \) is a foliated chart. Since \( B \) consisted of all the endowed disks from \( \mathcal{F}_\delta \), thus \( \mu^G(B) \geq \mu^G(H^G_{\lambda_0} \cap C_r) > 0 \). Reducing \( r, \delta \) if necessary, one can assume \( \mu^G(\partial(B)) = 0 \). Then \( \lim_{n \to \infty} \mu^G_n(B) = \mu^G(B) > 0 \), which ends the proof of Proposition 4.6.

4.3. Existence of backward contracting points. The goal of this subsection is to show the existence of enough backward contracting points in measure-theoretic sense, under the assumption that the randomly ergodic limit has positive integration Lyapunov exponents along \( E \). More precisely, we have

**Proposition 4.8.** Let \( f \) be a \( C^2 \) diffeomorphism with an attractor \( \Lambda \) admitting a dominated splitting \( T\Lambda M = E \oplus F \). If \( \mu \) is a randomly ergodic limit such that \( \lambda_E(\mu) > 0 \), then there exist \( \lambda_0, \alpha_0 \in (0, 1) \) with following properties:

- \( \mu^G(H^G_{\lambda_0}) \geq \alpha_0 \) and \( \mu(H_{\lambda_0}) \geq \alpha_0 \).
- If \( \{\mu_n\}_{n \in \mathbb{N}} \) is the sequence of ergodic stationary measures such that \( \lim_{n \to \infty} \mu_n = \mu \), then \( \liminf_{n \to \infty} \mu^G_n(H^G_{\lambda_0}) \geq \alpha_0 \).

Recall the well known Pliss Lemma, one can see a proof in [2, Lemma 3.1].

**Lemma 4.9.** Given constants \( p_0 \geq p_1 > p_2 > 0 \), there exists \( \rho = \rho(p_0, p_1, p_2) > 0 \) such that for any integer \( N \in \mathbb{N} \), and real numbers \( a_1, \cdots, a_N \), if

\[
\frac{1}{N} \sum_{i=1}^{N} a_i \geq p_1, \ a_i \leq p_0 \text{ for every } 1 \leq i \leq N.
\]

Then there is an integer \( \ell > \rho N \) and a sequence \( 1 \leq n_1 \leq \cdots \leq n_\ell \leq N \) such that

\[
\frac{1}{n_j - n} \sum_{i=n+1}^{n_j} a_i \geq p_2 \text{ for any } 0 \leq n < n_j, \text{ and } 1 \leq j \leq \ell.
\]

We need the following lemma, see a proof in [24, Lemma 2.1].

**Lemma 4.10.** Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of real numbers, for \( N \in \mathbb{N} \), if

\[
\sum_{i=1}^{n} a_i \leq 0, \ \forall \ n \geq N.
\]

Then there exists \( 1 \leq k \leq N \) such that

\[
\sum_{i=1}^{n} a_{i+k-1} \leq 0, \ \forall \ n \in \mathbb{N}.
\]

**Definition 4.11.** Given subset \( L \subset \mathbb{N} \) we define the density of \( L \) as

\[
\limsup_{n \to +\infty} \frac{\#([1, n] \cap L)}{n}.
\]

By Pliss Lemma (Lemma 4.9) and Lemma 4.10, one can get the following result.
Lemma 4.12. Given constants $p_0 \geq \max\{0, p_1\} \geq p_1 > p_2$, there exists $\rho = \rho(p_0, p_1, p_2) > 0$ such that for real numbers $a_1, a_2, \cdots, a_n, \cdots$, if
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} a_i \leq p_2, \ |a_i| \leq p_0,
\]
then there is a subset $K \subset \mathbb{N}$ whose density is larger than $\rho$ such that for any $k \in K$, we have
\[
\sum_{i=0}^{n-1} a_{i+k} \leq np_1, \ \forall n \in \mathbb{N}.
\]

Proof. In what follows, we shall show that there exists a sequence of integers $\{\ell_m\}_{m \in \mathbb{N}}$ such that $\ell_0 = 1$, $\ell_m \to \infty$, and $I_m \subset [\ell_{m-1}, \ell_m]$ such that every $k \in I_m, m \in \mathbb{N}$,
\[
\sum_{i=0}^{n-1} a_{i+k} \leq np_1, \ \forall n \in \mathbb{N}.
\]
Moreover, they admit the following property:
\[
\#(I_1 \cup \cdots I_m) / \ell_m \geq \rho
\]
for some $\rho$ depending only on $p_0, p_1, p_2$. Therefore, by taking $K = \bigcup_{m \in \mathbb{N}} I_m$, we then have
\[
\limsup_{n \to \infty} \frac{1}{n} \#([1, n] \cap K)) n \geq \rho.
\]
to end the proof.

Let us fix $p = (p_1 + p_2) / 2$. By assumption, for any $j \in \mathbb{N}$ we have
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} a_{i+j} \leq p_2.
\]
Thus, there exists $N(j) \in \mathbb{N}$ such that
\[
\frac{1}{n} \sum_{i=0}^{n-1} a_{i+j} \leq p, \ \forall n \geq N(j).
\]
By applying Lemma 4.10, we know that there exists $j \leq \ell(j) \leq N(j) + j$ so that
\[
\frac{1}{n} \sum_{i=0}^{n-1} a_{i+\ell(j)} \leq p, \ \forall n \geq 1.
\]
Take $j_1$ large enough such that $\ell_1 := \ell(j_1) \geq j_1 \geq N(1)$, then by (7) we get
\[
\frac{1}{\ell_1} \sum_{i=1}^{\ell_1} a_{i} \leq p.
\]
By applying Lemma 4.9, there exists $\rho = \rho(p_0, p_1, p_2) \in (0, 1)$ so that, there is a sequence of integers $I_1 = \{k_1, \cdots, k_{\ell_1(1)}\} \subset \{1, \cdots, \ell_1\}$ satisfying $j(1) \geq \rho \ell_1$, and for each $k \in I_1$, one has
\[
\sum_{i=0}^{m-1} a_{i+k} \leq mp_1, \ \forall 1 \leq m \leq \ell_1 - k + 1.
\]
It follows from (8) and (10) that for \( m > \ell_1 - k + 1 \) we have
\[
\sum_{i=0}^{m-1} a_{i+k} \leq \sum_{i=0}^{\ell_1-k} a_{i+k} + \sum_{i=0}^{m-k-\ell_1-1} a_{\ell_1+i} \\
\leq (\ell_1-k)p_1 + (m+k-\ell_1-1)p \\
\leq mp_1.
\]

This together with (10) ensure that each \( k \in I_1 \) has property (5).

Now we take \( j_2 = \ell_1 \), and then choose \( \ell_2 = \ell(j_2) \) with properties (8) and (9) by letting \( j = j_2 \) and replacing \( \ell_1 \) with \( \ell_2 \), respectively. Again, by applying Lemma 4.9 for \( \ell_2 \), one can find a sequence of integers \( I_2 \subset [\ell_1, \ell_2] \) such that

- each \( k \in I_2 \) exhibits the property (10) for \( \ell_2 \), i.e.,
  \[
  \sum_{i=0}^{m-1} a_{k+i} \leq mp_1, \ \forall 1 \leq m \leq \ell_2 - k + 1.
  \]

- \( \#\{I_1 \cup I_2\} \geq \rho \ell_2 \).

Here we use the fact that each \( k \leq \ell_1 \) satisfies (10) for \( \ell_2 \) whenever it satisfies (10) for \( \ell_1 \). Combining (11) with (8) for \( j_2 \), one can conclude that each \( k \) in \( I_2 \) satisfies the property (5), as in \( \ell_1 \) case.

By taking \( j_k = \ell_{k-1} \) and \( \ell_k = \ell(j_k) \), \( k \geq 2 \) we can select a sequence of integers \( \ell_m, m \in \mathbb{N} \) and \( I_m \subset [\ell_{m-1}, \ell_m] \) such that each \( k \in \cup_{m \in \mathbb{N}} I_m \) satisfies property (5), with density estimate (6) for \( \rho(p_0, p_1, p_2) \).

Now we can give the proof of Proposition 4.8 by using Lemma 4.12.

**Proof of Proposition 4.8.** One observes that the first item can be deduced from the second item by taking into the account the compactness of \( H_{\mathcal{G}_0}^G \). In fact, if we take ergodic stationary measures \( \mu_n \) converges to \( \mu \), then \( \mu_n^G \) converges to \( \mu^G = \delta_{\omega_f} \times \mu \) by applying Corollary 3.8, and thus
\[
\mu^G(H_{\mathcal{G}_0}^G) \geq \limsup_{n \to \infty} \mu^G_n(H_{\mathcal{G}_0}^G) \geq \liminf_{n \to \infty} \mu^G_n(H_{\mathcal{G}_0}^G) \geq \alpha_0.
\]

Moreover, we have the following relation
\[
\mu^G(H_{\mathcal{G}_0}^G) = \int \int \chi_{H_{\mathcal{G}_0}^G}(\omega, x) d\delta_{\omega_f}^G(\omega) d\mu(x) \\
= \int \chi_{H_{\mathcal{G}_0}^G}(\omega_f, x) d\mu(x) \\
= \int \chi_{H_{\mathcal{G}_0}}(x) d\mu(x) \\
= \mu(H_{\mathcal{G}_0}).
\]

Therefore, \( \mu(H_{\mathcal{G}_0}) \geq \alpha_0 \). Now it suffices to give the proof of the second item.

Put
\[
\alpha = \int \log \|Df^{-1}|E(x)\|^{-1} d\mu.
\]
By convergence \( \mu_n^G \to \delta_{\omega_j} \times \mu \) and the continuity of \((\omega, x) \mapsto \|Df_{\omega}^{-1}\|_{E(\omega, x)}\|\), one obtains
\[
\lim_{n \to \infty} \int \log \|Df_{\omega}^{-1}\|_{E(\omega, x)}\| d\mu_n^G = \int \log \|Df_{\omega}^{-1}\|_{E(\omega, x)}\| d(\delta_{\omega_j} \times \mu) = \int \log \|Df^{-1}\|_{E(x)}\| d\mu = -\alpha.
\]

Therefore, there exists \( n_0 \in \mathbb{N} \) such that for any \( m \geq n_0 \),
\[
\int \log \|Df_{\omega}^{-1}\|_{E(\omega, x)}\| d\mu_m^G \leq -\alpha/2.
\]

For every \( m \geq n_0 \), as \( \mu_m^G \) is ergodic, by Birkhoff ergodic theorem, for \( \mu_m \)-almost every \((\omega, x)\), we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log \|Df_{\omega}^{-1}\|_{E(G^{-i}(\omega, x))} = \int \log \|Df_{\omega}^{-1}\|_{E(\omega, x)}\| d\mu_m^G \leq -\alpha/2.
\]

For any fixed \((\omega, x)\) satisfying above equality, setting
\[ a_i = \log \|Df_{\omega}^{-1}\|_{E(G^{-i}(\omega, x))}, \forall i \in \mathbb{N} \]
and
\[ p_0 = \max_{(\omega, x) \in \Lambda_0} \left| \log \|Df_{\omega}^{-1}\|_{E(\omega, x)}\| \right|, \quad p_1 = -\alpha/4, \quad p_2 = -\alpha/2. \]

By Lemma 4.12, there is \( \rho = \rho(p_0, p_1, p_2) \) independent of \( m \) and a subset \( K \subset \mathbb{N} \) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \#([1, n] \cap K) \geq \rho. \tag{12}
\]

and for each \( k \in K \), one has
\[
\sum_{i=0}^{n-1} \log \|Df_{\omega}^{-1}\|_{E(G^{-k-i}(\omega, x))} \leq -\frac{\alpha}{4} n, \quad \forall n \in \mathbb{N}. \tag{13}
\]

Take \( \lambda_0 = e^{-\frac{\alpha}{4}} \), property (13) is equivalent to
\[
\prod_{i=0}^{n-1} \|Df_{\omega}^{-1}\|_{E(G^{-k-i}(\omega, x))} \leq \lambda_0^n, \quad \forall n \in \mathbb{N}, \quad \forall k \in K. \tag{14}
\]

It follows from the definition of \( \mathcal{H}_0^G \) that \( G^{-k}(\omega, x) \) belongs to \( \mathcal{H}_0^G \) as long as \( k \in K \). Therefore, use the ergodicity of \( \mu_m^G \) again, one gets
\[
\mu_m^G(\mathcal{H}_0^G) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{\mathcal{H}_0^G}(G^{-i}(\omega, x)) = \lim_{n \to \infty} \frac{1}{n} \# \{ i : 1 \leq i \leq n, \ G^{-i}(\omega, x) \in \mathcal{H}_0^G \} = \limsup_{n \to \infty} \frac{1}{n} \#([1, n] \cap K)
\]

By applying the density estimate (12) and taking \( \alpha_0 = \rho \), we get \( \mu_m^G(\mathcal{H}_0^G) \geq \alpha_0 \) to complete the proof of the second item. \( \square \)
4.4. Proof of Lemma A. In light of above preparations, we can complete the proof of Lemma A.

Proof of Lemma A. Under the assumption of Lemma A, assume that \( \{\mu_n\} \) is a sequence of ergodic stationary measures converges to \( \mu \) in weak* topology. By Proposition 4.8, there exist \( \lambda_0, \alpha_0 \in (0, 1) \) such that \( \liminf_{n \to \infty} \mu_n^G(H_{\lambda_0}^0) \geq \alpha_0 \), recall that \( \{\mu_n^G\} \) are \( G \)-invariant measures defined by Lemma 3.7. By applying Proposition 4.6, one can built a foliated chart \( \Phi : K \times \mathbb{D}^{\dim E} \to B \) of random local unstable disks generated by points from \( H_{\lambda_0}^0 \) such that \( \mu^G(B) > 0 \) and there exists \( N \in \mathbb{N} \) satisfying \( \mu_n^G(B) > 0 \) for any \( n \geq N \). More precisely, the construction of \( B \) suggests that there exist \( \delta > 0, \sigma \in (0, 1) \) and a family of random local unstable disks \( F_\delta = \{\gamma_\delta^0(\omega, x) : (\omega, x) \in K\} \) with following properties:

- \( \gamma_\delta^0(\omega, x) \) is a random local unstable disk of radius \( \delta \) around \( x \);
- each element of \( F_\delta \) is \( \sigma \)-backward contracting;
- \( B \) is the union of the endowed disks from \( F_\delta \), i.e.,
  \[
  B = \bigcup_{\gamma_\delta(\omega, x) \in F_\delta} \{\omega\} \times \gamma_\delta^0(\omega, x);
  \]

- \( \mu^G(\partial(B)) = 0 \).

Define \( F^G = \{\{\omega\} \times \gamma_\delta^0(\omega, x) : \gamma_\delta^0(\omega, x) \in F_\delta\} \). The assumption \( \lambda_E(\mu) > 0 \) implies that the Lyapunov exponents of \( \mu_n^G \) along \( E \) are positive for all \( n \geq N \), up to increasing \( N \). By Lemma 3.17, \( \{\mu_n^G\} \) are random Gibbs \( cu \)-state associated to \( E \). Therefore, according to Lemma 3.18, the densities of the conditional measures of \( \mu_n^G \) along \( F^G \) are bounded from above by a constant \( C \) independent of \( n \). This implies that when pulling back to \( K \times \mathbb{D}^{\dim E} \) by foliated chart \( \Phi \), and denote by \( \nu_n = \mu_n^G \circ \Phi \) for every \( n \geq N \), there is a constant \( C_\Phi \) such that for every measurable subset \( \xi \subset K \) and \( \zeta \subset \mathbb{D}^{\dim E} \) we have

\[
\nu_n(\xi \times \zeta) \leq C_\Phi \cdot \text{Leb}_{\mathbb{D}^{\dim E}}(\zeta) \nu_n(\xi) \quad \text{for every } n \geq N.
\]  

(15)

Define \( \nu = \mu^G \circ \Phi \), then the convergence \( \mu_n^G \to \mu^G \) together with \( \mu^G(\partial(B)) = 0 \) gives \( \nu_n \to \nu \) and \( \hat{\nu}_n \to \hat{\nu} \) as \( n \to \infty \). Take any open subset \( \zeta \subset \mathbb{D}^{\dim E} \) and open subset \( \xi \subset K \) with \( \hat{\nu}(\partial\xi) = 0 \). Applying (15), one gets

\[
\nu(\xi \times \zeta) \leq \liminf_{n \to \infty} \nu_n(\xi \times \zeta) \\
\leq C_\Phi \text{Leb}_{\mathbb{D}^{\dim E}}(\zeta) \liminf_{n \to \infty} \hat{\nu}_n(\xi) \\
= C_\Phi \text{Leb}_{\mathbb{D}^{\dim E}}(\zeta) \hat{\nu}(\xi)
\]

Hence, when applying Lemma 4.5 we get that \( \mu^G \) has absolutely continuous conditional measures along \( F^G \). Due to \( \mu^G(B) > 0 \) and \( \mu^G = \delta_{\omega_0} \times \mu \), when denoting \( \mathcal{F} = \{\gamma_\delta^0(\omega, x) \in F_\delta : \omega = \omega_0\} \), then \( \mu \) has positive measure on the union of disks of \( \mathcal{F} \), and moreover \( \mu \) has absolutely continuous conditional measures along \( \mathcal{F} \). Now we complete the proof of Lemma A.

5. Gibbs \( cu \)-states as ergodic components: Proof of Theorem A. Given a point \( x \in M \), denote by \( \mu_x \) probability measure given by the time average along the orbit of \( x \):

\[
\mu_x = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}.
\]
By ergodic decomposition theorem [23, Chapter II.6], there exists a subset \( \Sigma \) of \( M \) that has full measure for every \( f \)-invariant measure, such that \( \mu_x \) is well-defined and ergodic for every \( x \in \Sigma \). Moreover, every \( \varphi \in L^1(M, \nu) \) is \( \mu_x \) integrable for \( \nu \)-almost every \( x \in \Sigma \) and
\[
\int \left( \int \varphi \, d\mu_x \right) \, d\nu = \int \varphi \, d\nu.
\]
In fact, for such \( \varphi \) the integrable \( \int \varphi \, d\mu_x \) coincides with the time average of \( \varphi \) along the orbit of \( x \) for \( \nu \)-almost everywhere \( x \).

Let \( \mathcal{L} \) be the set of Lyapunov regular points, where we say \( x \) is Lyapunov regular if its Lyapunov exponent of every non-zero vector is well-defined (both forward and backward). A point \( x \in M \) is Birkhoff regular if for any continuous function \( \varphi : M \to \mathbb{R} \), both the forward and backward time averages
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \quad \text{and} \quad \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^{-i}(x))
\]
exist and they coincide. Denote by \( \mathcal{R} \) the set of Birkhoff regular points of \( f \).

It is well known that both \( \mathcal{R} \) and \( \mathcal{L} \) are \( f \)-invariant set having full measure for any \( f \)-invariant probability measure.

By using the backward contraction property of local unstable disks, we have the next result directly.

**Lemma 5.1.** If \( \gamma \) is a local unstable disk, then \( \mu_x = \mu_y \) for any \( x, y \in \gamma \cap \Sigma \cap \mathcal{R} \).

In this section, we will make effort to show the following result.

**Proposition 5.2.** Given \( 0 < \sigma < 1 \) and an \( f \)-invariant probability measure \( \mu \). Assume that \( \mathcal{F} \) is a measurable partition formed by local unstable disks satisfying:

- each local unstable disk of \( \mathcal{F} \) is \( \sigma \)-backward contracting;
- the union \( L \) of the elements of \( \mathcal{F} \) has positive \( \mu \) measure;
- \( \mu \) has absolutely continuous conditional measures along \( \mathcal{F} \).

Then for \( \mu \)-almost every \( x \in L \), \( \mu_x \) is a Gibbs cu-state.

Now we can conclude the proof of Theorem A by applying Proposition 5.2 together with Lemma A.

**Proof of Theorem A.** Under the settings of Theorem A, by Lemma A one knows that \( \mu \) has absolutely continuous conditional measures along a measurable partition \( \mathcal{F} \) formed by local unstable disks, all of which are \( \sigma \)-backward contracting for some \( \sigma \in (0,1) \). Consequently, by applying Proposition 5.2, there exist ergodic components of \( \mu \) to be Gibbs cu-states.

For proving Proposition 5.2, we need the next result which is essentially contained in [23, Lemma 6.3]. Here we give a proof for completeness.

**Lemma 5.3.** Assume that \( \mu \) is an \( f \)-invariant measure. For a Borel set \( B \), if \( \mu(B) > 0 \), then \( \mu_x(B) > 0 \) for \( \mu \)-almost every \( x \in B \).

**Proof.** Take \( A = B \cap \{ x : \mu_x(B) = 0 \} \), it suffices to show \( \mu(A) = 0 \). By contradiction, we assume \( \mu(A) > 0 \). If \( \mu(A \cap \{ x : \mu_x(A) > 0 \}) = 0 \), it follows from the definition that
\[
\{ x : \mu_x(A) > 0 \} = \bigcup_{n \geq 0} f^{-n}(A \cap \{ x : \mu_x(A) > 0 \}).
\]
and thus \( \mu(\{x : \mu_x(A) > 0\}) = 0 \). However, ergodic decomposition theorem tell us that
\[
\mu(A) = \int \mu_x(A) d\mu = \int_{\{x : \mu_x(A) > 0\}} \mu_x(A) d\mu = 0,
\]
which is a contradiction. Hence, we must have \( \mu(A \cap \{x : \mu_x(A) > 0\}) > 0 \). As a consequence,
\[
\mu(A \cap \{x : \mu_x(B) > 0\}) \geq \mu(A \cap \{x : \mu_x(A) > 0\}) > 0,
\]
it is contradict to the fact that \( A \cap \{x : \mu_x(B) > 0\} \) is an empty set by construction. Therefore, we get \( \mu(A) = 0 \).

Recall the following abstract Lemma, see [2, Lemma 6.2].

**Lemma 5.4.** Let \( M \) be a measurable space with a finite measure \( \eta \). Let \( \mathcal{P} \) be a measurable partition of \( M \) and \( \{\eta_P\}_{P \in \mathcal{P}} \) be the conditional measure of \( \eta \) w.r.t. \( \mathcal{P} \).

Assume that \( \{\eta_x\}_{x \in M} \) is a family of finite measures on \( M \) with the following properties:

1. \( x \mapsto \eta_x(A) \) is measurable for every measurable subset \( A \subset M \) and it is constant on each element of \( \mathcal{P} \).
2. there exists an integrable function \( k : M \to [0, \infty) \) such that
\[
\eta(A) = \int k(x) \eta_x(A) d\eta(x)
\]
for any measurable subset \( A \subset M \).
3. \( \{z : \eta_z = \eta_x\} \) is a measurable set with full \( \eta_x \)-measure, for every \( x \in M \).

Then for \( \eta \)-almost every \( x \in M \) and \( \tilde{\eta}_x \)-almost every \( P \), for the conditional measures \( \{\eta_x,P, P \in \mathcal{P}\} \) of \( \eta_x \) w.r.t. \( \mathcal{P} \), we have
\[
\tilde{\eta}_P = \eta_x,P,
\]
where \( \tilde{\eta}_x \) denotes the quotient measure induced by \( \eta_x \) on \( M \).

**Proof of Proposition 5.2.** Since \( \mu(L) > 0 \), by Lemma 5.3, we have that \( \mu_x(L) > 0 \) for \( \mu \)-almost every \( x \in L \).

For every measurable subset \( A \) of \( L \), by the ergodic decomposition theorem,
\[
\mu(A) = \int \mu_x(A) d\mu(x),
\]
where
\[
\mu_x(A) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(f^i(x)).
\]
This suggests that \( \mu_x(A) > 0 \) only if \( x \) has some iterate in \( A \subset L \). By applying Poincaré's recurrence theorem, one can choose \( r(x) \) to be the smallest positive integer such that \( f^{-r(x)}(x) \in L \) for almost every \( x \in L \). The construction of \( r(x) \) implies the following fact:

**Claim.** \( r(x) \) is integrable on \( L \), and
\[
\mu(A) = \int_L r(x) \mu_x(A) d\mu(x). \tag{16}
\]
Now we show that $\mu_x$ has positive Lyapunov exponents along $E$ for $\mu$-almost every $x \in L_\infty$. Lemma 5.3 implies that $\mu_x(L) > 0$ for $\mu$-almost every $x \in L$. Since $\mu_x$-almost every point of $L$ is Lyapunov regular, each $\gamma \in F$ is $\sigma$-backward contracting, we obtain that for almost every $y \in L$

$$\lim_{n \to +\infty} \frac{1}{n} \log (Df^n|_{E(y)})^2 = \lim_{n \to +\infty} \frac{1}{n} \log \|Df^{-n}|_{E(y)}\| \geq -\log \sigma > 0.$$  

Therefore, for $\mu$-almost every $x \in L$, $\mu_x$-almost every point has positive Lyapunov exponents along $E$. According to the ergodic decomposition theorem, with Lemma 5.1 and expression (16), one can apply Lemma 5.4 by letting

$$\mathcal{M} = L, \quad \eta = \mu|_{L}, \quad \mathcal{P} = L, \quad \eta_x = \mu_x|_{L}, \quad k(x) = r(x), \quad \forall x \in L.$$  

Consequently, for $\mu|_{L}$-almost every point $x$, the conditional measures of $\mu_x|_{L}$ along the disks of $F$ coincides almost everywhere with the conditional measures of $\mu|_{L}$ along the same family of disks. Since the conditional measures of $\mu|_{L}$ are absolutely continuous w.r.t. Lebesgue measures on these unstable disks by assumption, we get that $\mu_x|_{L}$ has absolutely continuous conditional measures. This together with the ergodicity imply that $\mu_x$ is a Gibbs $cu$-state for $\mu$-almost every $x \in L$.  

6. Consequences of Theorem A. As we mentioned before, Gibbs $cu$-states are crucial candidates of SRB measures and physical measures. As applications of Theorem A, in this section we establish the existence of SRB measures or physical measures for systems with dominated splittings.

The following result may be seen as a variation of Theorem A.

**Theorem 6.1.** Let $f$ be a $C^2$ diffeomorphism with an attractor $\Lambda$ admitting a dominated splitting $T_\Lambda M = E \oplus F$. If $\mu$ is a randomly ergodic limit supported on $\Lambda$ such that all the Lyapunov exponents along $E$ are positive for $\mu$-almost everywhere, then there exist ergodic Gibbs $cu$-states associated to $E$.  

**Proof.** Let us take $\mu$ as the randomly ergodic limit such it has positive Lyapunov exponents along $E$. More precisely, for $\mu$-almost every $x$ we have

$$\lim_{n \to +\infty} \frac{1}{n} \log \|Df^{-n}|_{E(x)}\|^{-1} > 0.$$  

Integrating over $\Lambda$ we obtain

$$\lim_{n \to +\infty} \int \frac{1}{n} \log \|Df^{-n}|_{E(x)}\|^{-1} d\mu > 0.$$  

In particular, there exists $N \in \mathbb{N}$ such that

$$\int \log \|Df^{-N}|_{E(x)}\|^{-1} d\mu > 0.$$  

By applying Theorem A, we know that there exists an ergodic component $\nu$ of $\mu$ (for $f^N$) such that $\nu$ is a Gibbs $cu$-state associated to $E$ for $f^N$. Taking $\nu_0 = \frac{1}{N} \sum_{i=0}^{N-1} f^i \nu$, one can check that $\nu_0$ is a Gibbs $cu$-state associated to $E$ for $f$ by construction.  

**Theorem 6.2.** Let $f$ be a $C^2$ diffeomorphism with an attractor $\Lambda$ admitting a dominated splitting $T_\Lambda M = E \oplus_{\sigma} F$. If $\mu$ is a randomly ergodic limit supported on $\Lambda$ such that

\[ \text{Here } \|m\| \text{ is the mini-norm defined by } m(A) = \inf_{\|\nu\| = 1} \|A\nu\| = \|A^{-1}\|^{-1} \text{ for linear isomorphism } A. \]
\[ \int \log \| Df^{-1} \|^{-1} \, d\mu > 0; \]
\[ \lim_{n \to +\infty} \frac{1}{n} \log \| Df^n \|_{F(x)} \leq 0 \text{ for } \mu\text{-almost every } x. \]

Then there exist ergodic components of \( \mu \) as SRB measures.

**Proof.** On the one hand, it follows from Theorem A that there exist ergodic components of \( \mu \) to be Gibbs \( cu \)-states. On the other hand, since \( \mu \) has non-positive Lyapunov exponents along \( F \), the ergodic decomposition theorem implies that almost every ergodic component of \( \mu \) has non-positive Lyapunov exponents along \( F \). Altogether one gets the desired result that there are ergodic components of \( \mu \) to be SRB measures. \( \Box \)

By adjusting the condition on \( F \)-direction one can deduce the following corollary.

**Corollary 6.3.** Under the assumption of Theorem 6.2, if we have
\[ \lim_{n \to +\infty} \frac{1}{n} \log \| Df^n \|_{F(x)} < 0 \]
for \( \mu\)-almost every \( x \), then there exist physical measures in the ergodic components of \( \mu \).

**Proof.** It can be deduced by Theorem 6.2 and absolute continuity of Pesin stable foliation. One can see [37] for more details. \( \Box \)

By applying Theorem 6.2, we can prove the following result. Note that it has been proved by Cowieson-Young in [15, Theorem C] with a different approach.

**Theorem 6.4.** Let \( f \) be a \( C^2 \) diffeomorphism with an attractor \( \Lambda \) admitting partially hyperbolic splitting \( T_{\Lambda}M = E^u \oplus \left\langle \right\rangle E^c \oplus \left\langle \right\rangle E^s \), if \( \dim E^c = 1 \) then there exist some SRB measures as the ergodic components of any randomly ergodic limit.

**Proof.** Consider \( \mu \) as a randomly ergodic limit, since \( \dim E^c = 1 \), there exist following two possibilities: either \( \lambda_{E^c}(\mu) \leq 0 \) or \( \lambda_{E^c}(\mu) > 0 \). In the first case, since every zero noise limit of regular random perturbation of \( f \) is a Gibbs \( u \)-state (e.g. see [15, Proposition 5]), in particular, the randomly ergodic limit \( \mu \) is a Gibbs \( u \)-state, which implies that almost every ergodic component of \( \mu \) is a Gibbs \( u \)-state [10, Lemma 11.13]. Since \( \lambda_{E^c}(\mu) \leq 0 \), by using ergodic decomposition theorem one gets that there are ergodic components of \( \mu \) such that they admit non-positive Lyapunov exponents along \( E^c \), and thus they are SRB measures. For the later case, one can apply Theorem 6.2 by taking \( E = E^u \oplus E^c \) and \( F = E^s \) to get SRB measures from ergodic components of \( \mu \). \( \Box \)

Now we provide a new proof of the existence of SRB/physical measures for systems considered in [25]. Recall that we say \( f \) is mostly expanding(resp. contracting) along an invariant sub-bundle \( E \) if all the Lyapunov exponents of every Gibbs \( u \)-state along \( E \) are positive(resp. negative).

**Theorem 6.5.** Let \( f \) be a \( C^2 \) diffeomorphism with an attractor \( \Lambda \) admitting partially hyperbolic splitting \( T_{\Lambda}M = E^u \oplus \left\langle \right\rangle E^{cu} \oplus \left\langle \right\rangle E^{cs} \) such that \( f \) is mostly expanding along \( E^{cu} \) and mostly contracting along \( E^{cs} \), then there exist physical measures as the ergodic components of any randomly ergodic limit.
Proof. By using the compactness of the set of Gibbs $u$-states, we know that there exists $N \in \mathbb{N}$ such that
\[
\int \log \|Df^{-N}|_{E^u}\|^{-1}\nu > 0
\] (17)
for any Gibbs $u$-state $\nu$ of $f^N$, see [25, Proposition 3.3] for details. By adding the regular random perturbation of $f^N$, if we take $\mu$ as a randomly ergodic limit, then it is a Gibbs $u$-state, and thus $\mu$ satisfies (17). Then applying Theorem A we conclude that there is an ergodic component $\eta$ of $\mu$ to be a Gibbs $cu$-state for $f^N$. Since the assumption that $f$ is mostly contracting along $E^c$, implies that $f^N$ is mostly contracting along $E^c$ [25, Lemma 2.4], we get that $\eta$ has negative Lyapunov exponents along $E^c$. Hence, the absolute continuity of Pesin stable foliation ensures that $\eta$ is a physical measure for $f^N$. Taking $\xi = \frac{1}{N} \sum_{i=0}^{N-1} f^i \eta$, one can check that $\xi$ is a physical measure for $f$ immediately.

We emphasize that the partially hyperbolic splittings considered in [25](also for [5]) occurs only on $M$, can not generalized to attractors due to the technique they used there. Moreover, after showing the existence of physical measures, by more accurate arguments(see the proof of [25, Theorem 5.1]) one can obtain the finiteness of these measures, i.e., there exist at most finitely many ergodic Gibbs $cu$-states for $f$, which are physical measures.

Appendix A. Proof of Lemma 3.10.

A sketched proof. Since $T_\Lambda M = E \oplus F$ is dominated, there exists constant $\tau \in (0,1)$ such that
\[
\|Df^n|_{F(\omega,x)}\| \cdot \|(Df^n|_{E(\omega,x)})^{-1}\| \leq \tau
\] (18)
for every $(\omega,x) \in \Lambda_{\epsilon_0}$. As a consequence of (18), for any two subspaces different to random $F$ sub-bundle, the distance of their forward images under $Df^n_\omega$ are contracted exponentially with rate $\tau$. Now we take smooth distribution $\hat{E}$ close to the random sub-bundle $E$, as smoothness implies Lipschitz, one can choose constants $\hat{C} > 0$ such that $\rho(\hat{E}_x, \hat{E}_y) \leq \hat{C}d(x,y)$ for any $x,y \in P_M(\Lambda_{\epsilon_0})$. Therefore, we have the following estimations:
\[
\rho \left( Df^n_{\sigma^{-n}} \hat{E}(G^{-n}(\omega,y)), Df^n_{\sigma^{-n}} \hat{E}(G^{-n}(\omega,z)) \right)
\leq \tau^n \rho \left( \hat{E}(G^{-n}(\omega,y)), \hat{E}(G^{-n}(\omega,z)) \right)
\leq \tau^n \hat{C}d \left( f^{-n}(y), f^{-n}(z) \right)
\leq \tau^n \hat{C}C\lambda^n d(y,z)
\leq \hat{C}C(\tau\lambda)^n d(y,z)
\leq \hat{C}Cd(y,z).
\]
Moreover, by property (18) one knows that $Df^n_{\sigma^{-n}} \hat{E}(G^{-n}(\omega,y))$ converges to $E(\omega,y)$, and $Df^n_{\sigma^{-n}} \hat{E}(G^{-n}(\omega,z))$ converges to $E(\omega,z)$. Hence, by taking $n \to +\infty$ to the left hand of the above estimation we get
\[
\rho(T^\gamma \gamma, T^\gamma \gamma) = \rho(E(\omega,y), E(\omega,z)) \leq C\hat{C}d(x,y).
\]
Thus, we complete the proof of this lemma as long as we put $L_0 = C\hat{C}$. 
\[\square\]
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