Anomalous transport on a corrugated ratchet potential

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Abstract. We present an extensive study the motion of particles in a rocking ratchet potential corrugated by quenched correlated disorder. The spatial disorder slows down the collective motion of particles. Anomalous transport in its both forms, subdiffusion and superdiffusion has been observed at long times. We provide a qualitative explanation for the origin of these anomalies. These behaviors are a direct consequence of the interplay between the ratchet potential roughness and the temperature of the system. In the same way as Khoury et al [1], we find a relation between the mean velocity of the particles and the different kinds of diffusion at long time.

1. Introduction
Disorder on periodic potentials, even in small amounts, can lead to important changes in transport and diffusive behavior. Among the more impressive effects are anomalous diffusion and the several orders of magnitude increase in the diffusion coefficient in correlated disordered tilted periodic potentials [1, 2]. These anomalous behaviors were observed experimentally in the motion on corrugated optical vortices [3].

Although anomalies in diffusion have been studied mainly in weakly disordered tilted periodic potentials disorder may also appear in ratchet potentials. Disorder on ratchets arises in systems with structural bottlenecks [4], antisymmetric dc voltage drop along amorphous indium oxide wires exposed to an ac bias source [5] and zero-energy states emerge in the form of Dirac points in asymmetric potentials in heterosubstrate-induced graphene superlattices [6].

Usually CTRW models or fractional diffusion equations are used to study anomalous diffusion [7–9]. Nevertheless, we choose to solve the Langevin equation to obtain anomalous diffusion behavior [1, 2, 10, 11].

Diffusion enhancement and superdiffusion has been obtained recently for a weakly disordered ratchet potential, in the absence of thermal noise [10] where trapping of particles in the disordered potential causes the coexistence of locked and running states producing both anomalous superdiffusive behavior and several orders of magnitude growth of the diffusion coefficient.

A recent work on a rocking ratchet potential with spatially correlated weak disorder was focused on the particle trapping probability density, namely the probability density of existence of locked states, and its opposite behavior, the existence of running states [11]. The study was
done without thermal noise, therefore superdiffusive motion of the particles have only transient nature. Nevertheless it may be detectable experimentally because this dynamical behavior could last much longer than the characteristic time scale of the system.

The barriers and wells of the disorder potential seem to hinder particle motion, and this leads to superdiffusion because some particles are trapped on the wells while others overcome the barriers running by the ratchet motion. When thermal noise is considered, more particles will be able to overcome the barriers and the diffusive motion will change. To study the effect of correlated spatial disorder and thermal noise on an overdamped rocking ratchet we perform simulations of particle motion on the disordered ratchet potential under different external force intensities and different temperatures.

The presence of particle motion in the steady state caused by the interplay between thermal noise and spatial disorder leads to different kind of diffusion, that is, normal, subdiffusion and superdiffusion depending on the temperature. The mean velocity of the particles decreases with time or is at most constant. The asymptotic decrease of the mean velocity is related to the kind of diffusion.

The outline of this paper is as follows. We provide a detailed description of model and its associated dynamical equations, in section 2. A brief analysis of the magnitudes of the quenched disorder is presented in the same section. The different kind of diffusion characterized by the mean-square displacement and their relations to the mean velocity of the particles is discussed in section 3. Finally, section 4 consists of concluding remarks.

2. Model and simulation method

We consider the overdamped motion of identical non-interacting Brownian particles moving in a rocking ratchet potential with a quenched disorder following the Langevin equation

$$\gamma \dot{x} = -dU(x)/dx + F_0 \sin(\omega t) + \xi(t).$$  \hspace{1cm} (1)

Here $x$ is the position of the particle, $t$ denotes time, $\gamma$ is the dissipation parameter, $\omega$ and $F_0$ are the frequency and amplitude of the applied sinusoidal force, and $\xi(t)$ is Gaussian thermal noise at temperature $T$. The correlation function of the noise obeys the fluctuation-dissipation relation $\langle \xi(t)\xi(t') \rangle = 2k_B T \delta(t - t')$. The potential, $U(x)$ consists of two parts: the archetypal ratchet potential of double-sine [11]

$$V_p(x) = -V_0 \left[ \sin \left( \frac{2\pi x}{\lambda_p} \right) + \frac{1}{4} \sin \left( \frac{4\pi x}{\lambda_p} \right) \right]$$  \hspace{1cm} (2)

and a Gaussian spatially random contribution $V_r(x)$ with correlation function

$$g_r(x) = \langle V_r(x)V_r(0) \rangle = g_0 \exp \left( -\frac{2\pi^2 x^2}{l_r^2} \right),$$  \hspace{1cm} (3)

that reflects quenched correlated disorder on the ratchet potential.

The ratchet potential $V_p(x)$ and the quenched spatial disorder $V_r(x)$ relatively contribute to the total potential $U(x)$ through the parameter $\sigma = [0, 1]$ according to

$$U(x) = (1 - \sigma)V_p(x) + \sigma V_r(x)$$  \hspace{1cm} (4)

In addition, in order that the total potential amplitude is of order $V_0$ independently of $\sigma$, we choose correlations of both ratchet and random potentials to be the same at $x = 0$. The correlation function for the ratchet potential is

$$g_p(x) = V_0^2/2 \left[ \cos \left( \frac{2\pi x}{\lambda_p} \right) + \frac{1}{16} \cos \left( \frac{4\pi x}{\lambda_p} \right) \right]$$  \hspace{1cm} (5)
Thus, from eqs. (3) and (5) and matching \( g_r(0) = g_p(0) \), the correlation random function amplitude is \( g_0 = 17V_0^2/32 \).

The equation of motion (1) can be reduced into a dimensionless form in terms of the rescaled spatial and temporal quantities \( z = 2\pi/\lambda_p x \) and \( \tau = (2\pi)^2 V_0/\gamma \lambda_p^2 t \) as [1]

\[
\dot{z} = -(1 - \sigma)f_p(z) - (\sigma/\lambda)f_r(z/\lambda) + F_0 \sin(\Omega \tau) + \eta(\tau),
\]

where \( f_p \) and \( f_r \) are the dimensionless forces arising from the ratchet \( V_p \) and random \( V_r \) potentials, respectively, \( \Omega \) and \( F_0 \) are the frequency and amplitude of the applied dimensionless sinusoidal force, and \( \eta(\tau) \) is the dimensionless noise. The dimensionless parameters

\[
\lambda = \frac{l_r}{\lambda_p}, \quad F_0 = F_0 \frac{\lambda_p}{2\pi V_0}, \quad \text{and} \quad \tilde{T} = \frac{k_BT}{V_0}
\]

combined with \( \sigma = [0, 1] \) and \( \Omega = \omega \frac{\gamma \lambda_p^2}{(2\pi)^2 V_0} \) define the motion of the particles subjects to equation (6).

Note that, in equation (6), a decrease of \( \lambda \) even for fixed \( \sigma \) leads to an increase in the relative contribution of the random force. On the other hand, according to equation (3) in its dimensionless form, \( \lambda \) is a measure of the ratchet potential roughness, smaller values of \( \lambda \) lead to a greater number and height of the barriers. Throughout this work we set \( \sigma = 0.025 \) and \( \lambda = 0.095 \), associated to a rough total potential. In figure 1(a), an example of contribution of disorder to the total potential is shown, set for one realization of the total potential in one spatial period \( \lambda_p \). The heights and locations of the barriers are random. Adding this disorder (part (a) of figure 1) to the contribution of the ratchet potential (part (b) of the same figure) results a disorder level perceptible in the scale of the total potential (part (c)). Note also that the number of barriers is indeed significant in one period \( \lambda_p \).

In addition, \( \Omega \) and \( \lambda_p \) are also set to 0.1 and 2\pi, respectively. Variations in \( \Omega \) do not lead to any additional phenomenology, while the specific choice of \( \lambda_p \) is only important as a reference value.

We have carried out numerical simulations of the equation (6) over a large number \( (10^3-10^4) \) of particle trajectories, each one in a different random potential. The method to generate the random force values is described in [11] in more detail. In order to characterize the transport, we measured the mean-square displacement \( MSD \) and the mean velocity \( v \) of the particles, which are given by

\[
MSD = \langle (z(\tau) - \langle z(\tau) \rangle)^2 \rangle, \quad v(\tau) = \frac{\langle z(\tau) \rangle}{\tau},
\]

where \( \langle \ldots \rangle \) indicates the average taken over the many trajectories.
3. Results and Discussion
The particle motion given by equation (6) strongly depends on the levels of thermal noise and spatial disorder at the system. In the absence of thermal noise ($T = 0$), the particles can travel long distances when the amplitude of the external force is greater than the amplitude of the ratchet force. In this case, quenched spatial disorder, whatever its magnitude, delays the particle movement until stopping. Due to this delay, transient superdiffusive motion over long distances compared to the spatial period of the ratchet potential $\lambda_p$ is obtained as previously studied in detail [11].

Although their dynamical behavior could last much longer than the characteristic time scale of the system, and therefore may be measurable, trapping processes without thermal noise have only transient nature, and a more realistic scenario is obtained when thermal noise is considered. This behavior is shown in figure 2, in which $MSD$ versus $\tau$ is plotted for $\lambda = 0.095$, $F_0 = 1.47$ and $T = 0$, $10^{-4}$, $3 \times 10^{-4}$, $10^{-3}$, $3 \times 10^{-3}$ and $5 \times 10^{-3}$. The curve in solid black line corresponds to $T = 0$, in which, after a superdiffusive transient all the particles are stopped due to the barriers imposed by quenched disorder. When thermal noise is considered, the particles may overcome the disorder barriers and move along the landscape. Thus, as $T$ increases, a variety of motions are obtained. The temporal behavior of MSD scales with time as $MSD(\tau) \sim \tau^\beta$, where the exponent $\beta$ characterizes the different regimes observed: $\beta < 1$ corresponds to subdiffusion, $\beta > 1$ to superdiffusion and $\beta = 1$ to normal diffusion. For large enough time, the motion tends to be subdiffusive at low temperatures (see zoom of the MSD curves in blue and green color of figure 2, in which their slopes tend to values less than 1), superdiffusive at intermediate temperatures (see zoom of the curves in red and magenta color of the same figure, in which their slopes tend to values greater than 1) and brownian at high temperatures (curve in red color with slope approximately equal to 1).

![Figure 2](image_url)

Figure 2. A log-log plot of the mean-square displacement versus the rescaled time, corresponding to different choices of the dimensionless temperature $T$ as indicated and for fixed external force amplitude $F_0 = 1.47$ and correlation length of quenched noise $\lambda = 0.095$. Dashed lines with slopes equal to 1 (normal diffusion) are shown for comparison purposes only. On the right, zoomed box of the figure to show the long time behaviors.

The spatial disorder slows down the collective motion of the particles and the mean velocity decreases over time (subtransport) or at most, it can be constant. This is clearly observed in figure 3, where we show the mean velocity as a function of $\tau$, for the same values of thermal noise as those corresponding to the previous figure. The mean velocity tends to decrease as $v \sim \tau^\alpha$, with $\alpha < 0$ and for $\tau \gg 1$. Following the reasoning in [1] applied to anomalous diffusion on systems in steady states, we establish bounds for the exponent $\alpha$ delimiting the different kinds of diffusion,
\[ \begin{align*} 
\alpha = 0 & \quad \rightarrow \text{normal or superdiffusion} \\
0 > \alpha > -1/2 & \quad \rightarrow \text{superdiffusion} \\
-1/2 > \alpha > -1 & \quad \rightarrow \text{subdiffusion} 
\end{align*} \] (9)

The dashed lines in figure 3 show the different bonds for the diffusive subtransport of the particles. The data for each temperature agree with the behaviors shown in figure 2, for large enough \( \tau \).

![Figure 3. A log-log plot of the mean velocity versus the rescaled time, for the same parameters as those corresponding to figure 2. Dashed lines with slopes equals to \(-1/2\) and \(-1\) delimit the different kinds of diffusion, in the steady state (see equation (9)). On the right, zoomed box of the figure to show the long time behaviors.](image)

In the previous section, we pointed out that increasing \( \lambda \) diminishes the effects of the disorder. When the correlation length is much greater than a single period of the ratchet potential, that is, the randomness is very smooth, the steady state subdiffusion is not reached. This has been previously studied in [1] for particles driven by a constant external force over a landscape consisting of a symmetric periodic potential with a quenched disorder identical to the Eq. (3). Finally, if the external force amplitude is changed to a value less than the amplitude of the ratchet, due to the quenched disorder, a transient subdiffusive is obtained and then the particles achieve constant mean velocity with normal diffusion. In essence, this is a regular ratchet movement in which the particles motion is slowed down by quenched disorder.

4. Conclusions

In this paper, we have studied the motion of particles on a ratchet potential with both, correlated weak disorder and thermal noise for different temperatures. In a previous work [11] where correlated weak disorder without thermal noise was considered on a ratchet potential transient superdiffusive particle motion appears until the particles get trapped. When thermal noise is present, particles may overcome the spatial disorder and a variety of different kinds of diffusion are obtained according to the temperature of the system. For large enough times, we found, for high temperatures, normal diffusion, for intermediate temperatures, superdiffusion and for low temperatures, subdiffusion. In all cases, the mean velocity of the particles decreases with time or is at most constant, due to the interplay between the thermal noise and spatial disorder. The asymptotic decrease of the mean velocity is a power function of time related to the kind of diffusion. Thus, we found bounds for the asymptotic behaviors of the mean velocities that agree with the different diffusion regimes of the particles. The asymptotic behaviors of the mean velocities with exponents between -1 and -1/2 correspond to subdiffusive motion, between -1/2
and 0 to superdiffusive motion and constant mean velocities can be superdiffusion or normal diffusion. These movements are obtained after a superdiffusive transient due to a high quenched disorder and for an amplitude of the external force higher than the amplitude of the ratchet force. In contrast, a regular ratchet movement is found for an amplitude of the external force less than the amplitude of the ratchet force after a subdiffusive transient. In this case, a net motion is obtained for high enough temperatures that combined with the external force let the particle overcome both the ratchet potential and the quenched disorder. In summary, a set of dramatic anomalous behaviors as diverse as subtransport, subdiffusion, and superdiffusion appear when the periodic ratchet potential is modified with a small amount of correlated week disorder and thermal noise.

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