The subgroup normalizer problem for integral group rings of nilpotent groups

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For a group $G$ and a subgroup $H$ of $G$ this article deals with the normalizer of $H$ in the units of a group ring $RG$. We prove that if $H$ is cyclic it is only normalized by the ‘obvious’ units. Moreover we prove similar results for subgroups of nilpotent groups. This article presents work of the author’s PhD thesis [Bäc12].

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1 Introduction

Let throughout $G$ denote a (possibly infinite) group. Let $R$ be a commutative ring with identity element 1. By $RG$ we denote the group ring of $G$ over $R$ and by $U(RG)$ its group of units. $RG$ is equipped with a natural augmentation map $\varepsilon: RG \to R$, sending an element (expressed as a linear combination with respect to the basis $G$) to the sum of its coefficients. Restricting this map to the group of units, we obtain as kernel $V(RG) = \text{Ker } \varepsilon|_{U(RG)}$, the group of units of augmentation 1, also called normalized units. Evidently, $U(RG) = U(R) \cdot V(RG)$, here $U(R)$ denotes the units of the ring $R$.

The group $U(RG)$ acts by conjugation on itself. Clearly $G \leq U(RG)$ is normalized by all elements of $G \cdot Z(U(RG))$, where $Z$ denotes the center. The question if equality holds was raised by Jackowski andMarciniak [JM87, 3.7. Question] and is also a research problem in the fundamental book of Sehgal [Seh93, Problem 43]:

$$N_{U(RG)}(G) = G \cdot Z(U(RG)) \quad \quad \text{NP}(G, R)$$

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It turned out that answering it in the negative was a crucial step for constructing a counter example for the isomorphism problem, the long-standing problem if a group is determined by its group ring over the integers. This was done by Hertweck in his PhD thesis [Her98, Theorem A, Theorem B] and published in the Annals of Mathematics in [Her01, Theorem A, Theorem B]. Although this example is known, there are important classes of groups for which $\text{NP}(G, \mathbb{Z})$ holds: for finite groups with normal Sylow 2-subgroups (Jackowski, Marciniak, [JM87]), finite metabelian groups with abelian Sylow 2-subgroups (Marciniak, Roggenkamp [MR01]), finite Blackburn Groups and groups having an abelian subgroup of index 2 (Li, Parmenter, Sehgal [LPS99]). Also for Frobenius groups (Petit Lobão, Polcino Milies [PLPM02]), finite quasi-nilpotent and finite 2-constrained groups with $G/\text{O}_2(G)$ having no chief factor of order 2 (Hertweck, Kimmerle [HK02]). Furthermore for locally nilpotent groups (Jespers, Juriaans, de Miranda, Rogerio [JJdMR02]), groups with every finite normal subgroup having a normal Sylow 2-subgroup (Hertweck [Her04]) and for arbitrary Blackburn groups (Hertweck, Jespers [HJ09]) $\text{NP}(G, \mathbb{Z})$ has been verified. (In some results more general coefficient rings than the integers are allowed.)

Considering subgroups $H \leq G$ of the group basis $G$, the statement corresponding to $\text{NP}(G, R)$ is

$$\text{NU}(RG)(H) = N_G(H) \cdot C_{U(RG)}(H).$$

We say that $H \leq G$ together with the coefficient ring $R$ has the subgroup normalizer property, if this equality holds true. It is interesting to investigate for which groups $\text{NP}(H \leq G, R)$ holds and possibly find counterexamples with $G$ having smaller order than in the previous mentioned example of Hertweck (there $|G| = 2^{25} \cdot 97^2$) to obtain a better insight into the structural properties that cause such phenomena.

A group $G$ together with a commutative ring $R$ has the subgroup normalizer property if $\text{NP}(H \leq G, R)$ holds for all subgroups $H$ of $G$, i.e. if

$$\forall H \leq G: \text{NU}(RG)(H) = N_G(H) \cdot C_{U(RG)}(H).$$

The article is organized as follows: we restate the problem in terms of automorphisms (section 2). In the next section we prove that $\text{NP}(H \leq G, R)$ holds whenever $H$ is cyclic. In section 4 we first prove a generalization of the famous ‘Coleman lemma’, which is subsequently used to show that $\text{SNP}(G, R)$ holds for locally-nilpotent groups. We also establish that $\text{SNP}(G, R)$ holds for finitely-generated torsion-free nilpotent groups $G$. It is shown that $\text{NP}(H \leq G, R)$ holds for torsion subgroups $H$ of finitely-generated nilpotent groups $G$.

The notation is mainly standard. Let $x$ and $y$ be elements of a group. By $x^y = y^{-1}xy$ we denote the conjugate of $x$ by $y$ and by $x^G$ the set of all $G$-conjugates of $x$. Further, $[x, y] = x^{-1}y^{-1}xy$ is the commutator of $x$ and $y$. For subsets $X$ and $Y$ of a group $[X, Y]$ denotes the subgroup generated by all commutators $[x, y]$ for $x \in X$ and $y \in Y$. 

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(if one set is a singleton we omit the set braces). For the order of an element $x$ we write $o(x)$. Moreover, $\text{Aut}(G)$ denotes the group of group automorphisms of the group $G$ and $\text{Inn}(G)$ its subgroup of automorphisms induced by conjugation $\text{conj}(x) = (g \mapsto g^x = x^{-1}gx)$ by an element $x \in G$.

For an element $u = \sum_{g \in G} u_g g \in RG$ define by $\text{supp}(u) = \{ g \in G \mid u_g \neq 0 \}$ the support of $u$. For a group $G$, a $G$-adapted ring is an integral domain of characteristic zero in which a rational prime $p$ is not invertible, whenever there exists an element of order $p$ in $G$. Note that homomorphisms are written from the right (with exception of the augmentation of the group ring). We always assume that a ring has an identity element. $\text{U}(R)$ denotes the group of units of a ring $R$.

## 2 Automorphisms

The question $\text{NP}(H \leq G, R)$ can be restated using automorphisms.

**Definition 1.** Let $G$ be a group, $H \leq G$ and $R$ be a commutative ring. Set

$\text{Aut}_G(H) = \{ \varphi \in \text{Aut}(H) \mid \varphi = \text{conj}(g) \text{ for some } g \in N_G(H) \}$

and

$\text{Aut}_{RG}(H) = \{ \varphi \in \text{Aut}(H) \mid \varphi = \text{conj}(u) \text{ for some } u \in N_{\text{U}(RG)}(H) \}$,

the groups of automorphisms of $H$ induced by elements of $G$ and $\text{U}(RG)$, respectively.

Clearly, for $H \leq G$ we have $\text{Inn}(H) \leq \text{Aut}_G(H) \leq \text{Aut}_{RG}(H) \leq \text{Aut}(H)$.

**Lemma 2.** For a group $G$ with subgroup $H$ and a commutative ring $R$ the following statements are equivalent

1. $\text{NP}(H \leq G, R)$ holds
2. $\text{Aut}_{RG}(H) = \text{Aut}_{G}(H)$
3. for every $u \in N_{\text{U}(RG)}(H)$ there exists a $g \in N_G(H)$ such that $[ug, H] = 1$. $\square$

## 3 Cyclic subgroups

We prove that $\text{NP}(H \leq G, R)$ holds for all rings $R$, provided $H$ is cyclic with a similar calculation that occurred in the proof of [PLPM02, Theorem 3.1] and the simplified proof of [Her04, 17.3 Theorem] suggested by I. B. S. Passi. To this end we need the notation of an additive commutator: For $x, y \in RG$ set $[x, y]_L = xy - yx$, the additive
commutator of $x$ and $y$. Let $[RG, RG]_L$ be the $R$-submodule of $RG$ generated by all $[x, y]_L$, for $x, y \in RG$. The map $[-, -]_L : RG \times RG \to RG : (x, y) \mapsto [x, y]_L$ is $R$-bilinear. Let $\text{ccl}(G)$ denote the collection of all conjugacy classes of $G$. For $C \in \text{ccl}(G)$ define $\epsilon_C : RG \to R : \sum_{g \in G} u_g g \mapsto \sum_{g \in C} u_g$, the partial augmentation map with respect to the conjugacy class $C$. Direct calculations show that

$$[RG, RG]_L = \{ u \in RG \mid \forall C \in \text{ccl}(G) : \epsilon_C(u) = 0 \}.$$

**Lemma 3.** Let $H \leq G$, $R$ a commutative ring, $\sigma \in \text{Aut}_{RG}(H)$. Then $x\sigma \in x^G$ for every $x \in H$.

**Proof.** There is an element $u \in N_U(RG)(H)$ such that $\sigma = \text{conj}(u)$. We have

$$x\sigma - x = u^{-1} xu - x = [u^{-1} x, u]_L \in [RG, RG]_L,$$

and consequently $\epsilon_C(x\sigma) = \epsilon_C(x)$ for all $C \in \text{ccl}(G)$. Together with $x\sigma \in H \leq G$ this implies $x\sigma \in x^G$. $\square$

**Proposition 4.** Let $H \leq G$. If $H$ is cyclic, then $\text{NP}(H \leq G, R)$ holds for a commutative rings $R$.

**Proof.** Let $H = \langle x \rangle$ and $\sigma \in \text{Aut}_{RG}(H)$, then $x\sigma \in x^G$ by the previous Lemma 3 and hence there is a $g \in G$ such that $x\sigma = x\text{conj}(g)$, so $\sigma = \text{conj}(g)|_H \in \text{Aut}_G(H)$. $\square$

### 4 Nilpotent groups

We first prove a generalization of a well known result on the normalizer of $p$-subgroups, the so-called Coleman lemma (cf. [Col64]). We adapt the line of proof in [Her04, 19.4 Lemma] to boil it down to a finite group problem.

**Lemma 5** (Coleman lemma, relative version). Let $H \leq G$ and $R$ a commutative ring and $p$ a rational prime such that $p \notin U(R)$. Let $u \in N_U(RG)(H)$. Then there exists $P \leq H$ with $|H : P| < \infty$, $p \nmid |H : P|$ and $x \in \text{supp}(u) \cap N_G(P)$ such that $x^{-1} u \in C_U(RG)(P)$.

**Proof.** Let $H \leq G$ and $u = \sum_{g \in G} u_g g \in N_U(RG)(H)$. For every $h \in H$ we have

$$\sum_{g \in G} u_g g = \sum_{g \in G} u_g h^{-1} gh^u,$$

and hence

$$g \in \text{supp}(u) \iff \forall h \in H : h^{-1} gh^u \in \text{supp}(u).$$
In particular, we obtain the following (right) action of the group $H$ on the support of $u$:

$$\text{supp}(u) \times H \to \text{supp}(u) \quad (x, h) \mapsto h^{-1}xh^u.$$  

The coefficients $u_g$ of $u$ are constant on the orbits of this action by (1). Let $K = \{ h \in H \mid \forall x \in \text{supp}(u): h^{-1}xh^u = x \} \leq H$ be the kernel of the action. Then $H/K$ is isomorphic to a subgroup of the finite group $\text{Sym}(\text{supp}(u))$. Let $K \leq P \leq H$, such that $P/K$ is a Sylow $p$-subgroup of $H/K$. The induced action of the $p$-group $P/K$ on $\text{supp}(u)$ must have a fixed point $x \in \text{supp}(u)$, as $\varepsilon(u) \in U(R)$ and $p \not\in U(R)$ by assumption on $R$. But this implies $x^{-1}u \in C_{U(RG_1)}(P).$

The subgroup normalizer property behaves well with respect to direct products: Let $G_1$ and $G_2$ be groups and $j \in \{1, 2\}$. The natural projections $\pi_j: G_1 \times G_2 \to G_j$ give rise to ring homomorphisms $R[G_1 \times G_2] \to RG_j$ by the universal property of the group ring, which can be restricted to homomorphisms of the unit groups, $\Pi_j: U(R[G_1 \times G_2]) \to U(RG_j)$ (the box brackets are included for better readability). With the obvious inclusion maps we have the following diagram

$$
\begin{array}{cccc}
G_1 & \xleftarrow{\pi_1} & G_1 \times G_2 & \xrightarrow{\pi_2} & G_2 \\
U(RG_1) & \xleftarrow{\Pi_1} & U(R[G_1 \times G_2]) & \xrightarrow{\Pi_2} & U(RG_2)
\end{array}
$$

**Lemma 6.** Let $G = G_1 \times G_2$, $R$ a commutative ring, and $H \leq G$. Assume that $\text{NP}(H \pi_1 \leq G \pi_1, R)$ and $\text{NP}(H \pi_2 \leq G \pi_2, R)$ hold, then also $\text{NP}(H \leq G, R)$ holds.

**Proof.** Let $u \in N_{U(RG_1)}(H)$. Set $H_j = H \pi_j \leq G_j$ and $u_j = u \Pi_j$. For every $x_1 \in H_1$ there exists $x_2 \in H_2$, such that $x_1x_2 \in H$. Now

$$
x_1^{u_1} = (x_1x_2)^{u \Pi_1} = (x_1x_2)^{u} \pi_1 \in H_1,
$$

hence $u_1 \in N_{U(RG_1)}(H_1)$. Analogously, we see that $u_2 \in N_{U(RG_2)}(H_2)$, so we get from the assumption elements $g_j \in N_{G_j}(H_j)$ and $z_j \in C_{U(RG_j)}(H_j)$ such that $u_j = g_jz_j$. 

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To construct the corresponding units in the group ring of $G$, set $w = u_1^{-1}u_2^{-1}u$. This unit is centralizing $H$: To see this, take any $x \in H$, then
\[
x^w = ((x\pi_1)(x\pi_2))^{(u^{-1}\pi_1)(u^{-1}\pi_2)} = ((x\pi_1)(u^{-1}\pi_1)(x\pi_2)(u^{-1}\pi_2))^{u} = (x^{u^{-1}}\pi_1(x^{u^{-1}}\pi_2))^{u} = x.
\]
A similar calculation shows that the element $g$ defined as $g = g_1g_2$ acts on $H$ by conjugation like $u$ does, in particular $g \in N_G(H)$. Obviously, we have $z_j \in C_{U(RG)}(H)$. Hence
\[
u = u_1u_2u_1^{-1}u_2^{-1}u = (g_1g_2)(z_1z_2w) \in N_G(H) \cdot C_{U(RG)}(H).
\]

**Proposition 7.** If $G = G_1 \times G_2$ and $R$ is a commutative ring, then
\[
\text{SNP}(G, R) \text{ holds } \iff \text{SNP}(G_1, R) \text{ and SNP}(G_2, R) \text{ hold}.
\]

**Proof.** 
"$\Rightarrow$": Let $H_1 \leq G_1$ and $u_1 \in N_{U(RG_1)}(H_1)$. Set $H = H_1 \times 1 \leq G$. Using the inclusion map, $u_1 \in N_{U(RG)}(H)$. By assumption there are $g \in N_G(H)$ and $z \in C_{U(RG)}(H)$ such that $u_1 = g z$. Clearly $g_1 = g\pi_1 \in N_{G_1}(H_1)$ and $z_1 = z\pi_1 \in C_{U(RG_1)}(H_1)$. Hence we get $u_1 = u_1\pi_1 = (gz)\pi_1 = g_1z_1 \in N_{G_1}(H_1) \cdot C_{U(RG_1)}(H_1)$. In the same way we can verify the normalizer property for the second factor.

"$\Leftarrow$": Follows from Lemma 6.

A group is called **locally nilpotent**, if every finite subset is contained in a nilpotent subgroup or, equivalently, every finitely generated subgroup is nilpotent.

**Theorem 8.** Let $G$ be a locally nilpotent torsion group. Then SNP($G, R$) holds for $G$-adapted rings $R$. In particular SNP($G, R$) holds for finite nilpotent groups $G$ and $G$-adapted rings $R$.

**Proof.** By [Rob96, 12.1.1] we have that $G = \prod_{p \in P} G_p$, where $P$ runs through all primes, $G_p$ denotes the unique maximal $p$-subgroup of $G$, and $Dr$ stands for the restricted direct product (i.e. the subgroup of the direct product containing those elements for which all but finitely many coordinates are equal to the identity element). Let $H \leq G$ and $u \in N_{U(RG)}(H)$. Define the set $P$ of rational primes 'occurring' in the support of $u$:
\[
P = \{ p \in \mathbb{N} \mid p \text{ a prime and } \exists g \in \text{ supp}(u): p \mid o(g) \}.
\]
Now $G$ can be decomposed as $G = X \times Y$, where $X = \prod_{p \notin P} G_p$ and $Y = Dr_{p \in P} G_p$. As $[X, Y] = 1$ and supp($u$) $\subseteq X$ it follows that $[u, Y] = 1$. Let $\kappa : RG \to RX$ denote the natural projection. By Lemma 6 it is enough to show that $u\kappa$ acts on $H\kappa$ like an element of $G\kappa$. But this follows by induction on the finite number of primes in $P$, using Lemma 5 and Lemma 6.
Next, we aim for a generalization of this result for nilpotent groups which are not necessarily torsion.

**Lemma 9.** Let $G$ be a group, $K \leq G$ and $C \leq Z(G)$ a central subgroup and $R$ a commutative ring. Assume that $NP(KC \leq G, R)$ holds, then so does $NP(K \leq G, R)$.

**Proof.** Let $u \in N_{U(RG)}(K)$. As $C \leq Z(G)$ we have

$$u \in N_{U(RG)}(K) \subseteq N_{U(RG)}(KC) = N_G(KC) \cdot C_{U(RG)}(KC).$$

Hence there is a factorization $u = gz$ with $g \in N_G(KC)$, $z \in C_{U(RG)}(KC)$. For every $k \in K$ we get $K \ni k^u = k^g = (k^g)^2 = k^g$, so in fact $g \in N_G(K)$. Furthermore, $C_{U(RG)}(KC) = C_{U(RG)}(K)$, and hence $u = gz \in N_G(K) \cdot C_{U(RG)}(K)$. $\square$

For the next lemma we need some additional notation: Let $1 \to K \to E \to Q \to 1$ be a short exact sequence of groups, where $K$ is an abelian group. For an automorphism $\varphi$ of $E$ fixing $K$ as set denote by $\varphi^Q$ the automorphism induced on the quotient group $Q$. Set

$$\text{Aut}(Q, K) = \{ \varphi \in \text{Aut}(E) \mid \varphi|_K = \text{id}_K \text{ and } \varphi^Q = \text{id}_Q \},$$

the group of automorphisms of $E$ restricting to the identity on $K$ and inducing the identity on the quotient group $Q$. The inner automorphisms of $E$ contained therein form the normal subgroup $\text{Inn}(Q, K) = \text{Aut}(Q, K) \cap \text{Inn}(E)$. Note that after having fixed a $Q$-module structure on $K$ the group $\text{Aut}(Q, K)$ does not depend on the concrete extension (cf. [Rot07, Corollary 9.17]). We need the following result and state it for the convenience of the reader.

**Proposition 10 ([Rot07, Corollary 9.20]).** For every group $Q$ and every $Q$-module $K$, $H^1(Q, K) \simeq \text{Aut}(Q, K)/\text{Inn}(Q, K)$, where $H^1$ denotes the first cohomology group.

**Lemma 11.** Let $1 \to C \to G \to \overline{G} \to 1$ be a central extension, where $C$ is torsion-free and let $R$ be a commutative ring. Assume that $L \leq G$ with $C \leq L$ such that $|L : C| < \infty$. If $NP(L/C \leq G/C, R)$ holds, then $NP(L \leq G, R)$ holds as well.

**Proof.** Let $u \in N_{U(RG)}(L)$. Denote by bar the ring homomorphism $RG \to R\overline{G}$ induced by the natural homomorphism $G \to \overline{G}$. From $\overline{u} \in N_{U(R\overline{G})}(\overline{L})$ and the assumption we get that there is a $g \in G$ with $\text{conj}(\overline{u})|_{\overline{G}} = \text{conj}(\overline{g})|_{\overline{G}}$. We have $g \in C \cdot N_G(L) = N_G(L)$. Let $\sigma = \text{conj}(ug^{-1}) \in \text{Aut}_{RG}(L)$. As $\sigma$ induces the identity on $L/C$ and $C \leq Z(G)$ we have the following commuting diagram with the obvious horizontal maps

$$\begin{array}{ccc}
C & \longrightarrow & L \\
\downarrow{id} & & \downarrow{id} \\
C & \longrightarrow & L
\end{array}$$

$$\begin{array}{ccc}
\overline{C} & \longrightarrow & \overline{L} \\
\downarrow{\overline{\sigma}} & & \downarrow{\overline{id}} \\
\overline{C} & \longrightarrow & \overline{L}
\end{array}$$

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So σ ∈ Aut(Δ, C). Now 1 → C ↪ L → T → 1 is a central extension of a finite group by a torsion-free group, hence \( H^1(Δ, C) = \text{Hom}(Δ, C) = 1 \). Hence Proposition 10 implies that σ = \( \text{conj}(t) \in \text{Inn}(Δ, C) \) for some \( t \in L \), so that \( \text{conj}(u)|_{L} = \text{conj}((t)j)|_{L} \in \text{Aut}_{G}(L) \).

**Proposition 12.** Let G be finitely generated nilpotent and \( H \leq G \) a torsion subgroup. Then NP(\( H \leq G, R \)) holds for G-adapted rings R.

**Proof.** First define a special central series of G in the following way. By [Rob96, 5.2.22 (ii)], a finitely generated nilpotent group X is infinite if and only if there is an element of infinite order in the center of X. Let \( U_0 = 1 \). Assume by induction that \( U_j \leq G \) is already defined and \( G/U_j \) is finitely-generated nilpotent. If this quotient is infinite then there is an element \( y_{j+1} \in Z(G/U_j) \) of infinite order, the pre-image \( U_{j+1} \) of \( \langle y_{j+1} \rangle \) in G; if \( G/U_j \) is finite, define \( U_{j+1} = U_j \). In any case the resulting quotient group is finitely generated nilpotent and consequently has the desired property for the induction process. This yields a chain

\[
1 = U_0 \leq U_1 \leq U_2 \leq U_3 \leq \ldots
\]

of normal subgroups of G. Define \( U = \bigcup_{j} U_j \). This is finitely generated as a subgroup of a finitely generated nilpotent group [Rob96, 3.1.6], hence the above chain is eventually constant, i.e. there is a minimal \( n \in \mathbb{N} \) such that \( U_n = U_{n+k} \) for all \( k \geq 0 \). But then \( G/U_n \) is finite nilpotent.

For \( 0 \leq j \leq n \) define \( G_j = G/U_j \). The natural map \( G_j \to G_{j+1} \) is injective on torsion elements, as the kernel is torsion-free. Hence the torsion subgroup \( \text{Tor}(G) \) of G is isomorphic to a subgroup of the finite group \( G_n \) and hence finite. Thus, also \( H \) is finite.

Set \( C_j = U_{j+1}/U_j \leq Z(G_j) \). Additionally, let \( K_j = U_j H/U_j \leq G_j \), a finite subgroup, and \( L_j = U_{j+1} H/U_j \leq G_j \). Note that \( L_j = K_j C_j \). For every \( 0 \leq j \leq n-1 \) we get a short exact sequence

\[
1 \to C_j \to G_j \to G_{j+1} \to 1.
\]

NP(\( K_n \leq G_n, R \)) holds by Theorem 8, as \( G_n \) is a finite nilpotent group. Using Lemma 11 and Lemma 9 while proceeding inductively along the sequence

\[
G = G_0 \to G_1 \to \ldots \to G_{n-1} \to G_n
\]

shows that NP(\( K_0 \leq G_0, R \)) holds.

We also have the following lifting lemma:

**Lemma 13.** Let G be a group, \( H \leq G \), R a commutative ring, and \( u \in \text{N}_{U(RG)}(H) \). Assume there is a normal subgroup \( N \) of G with \( N \leq C_G(H) \). Set \( \overline{\sigma} = G/N \) and denote by bar the reduction homomorphism, \( \overline{\sigma} : RG \to R\overline{G} \). Assume there is \( \overline{\pi} \in \text{supp}(\overline{\pi}) \) with \( \text{conj}(\overline{\pi})|_{\overline{T}} = \text{conj}(\overline{\pi})|_{\overline{T}} \). If \( N \) is torsion-free, then there is \( g \in \text{supp}(u) \) with \( \text{conj}(u)|_{H} = \text{conj}(g)|_{H} \).

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Proof. We have \( \overline{\pi} \in N_{U(RG)}(\overline{H}) \), so by assumption there is a \( g \in \text{supp}(u) \) such that \( \text{conj}(\overline{\pi})|_{\overline{H}} = \text{conj}(\overline{\pi})|_{\overline{H}} \) (we may take any \( g \in H \cap \text{supp}(u) \neq \emptyset \)). Hence for all \( h \in H \) we have \( h^{-1}gh^{u} = \overline{g} \), this means \( h^{-1}gh^{u} = z_{h}g \), for some \( z_{h} \in N \).

Using \( z_{h} \in N \leq C_{G}(H) \) for all \( h \in H \) we have for \( h_{1}, h_{2} \in H \)

\[
\begin{align*}
    z_{h_{1}h_{2}}g &= (h_{1}h_{2})^{-1}g(h_{1}h_{2})^{u} = h_{2}^{-1}(h_{1}^{-1}gh_{1}^{u})h_{2}^{u} = h_{2}^{-1}z_{h_{1}}gh_{2}^{u} \\
    &= z_{h_{2}}^{-1}gh_{2}^{u} = z_{h_{1}h_{2}}g,
\end{align*}
\]

and hence \( z: H \to N: h \mapsto z_{h} \) is a homomorphism. Now (2) in the proof of Lemma 5 (page 4) shows that \( z_{h}g \in \text{supp}(u) \) for all \( h \in H \). As \( N \) is torsion-free, the finiteness of the support implies \( z_{h} = 1 \) for all \( h \in H \). By (3) we obtain \( \text{conj}(u)|_{H} = \text{conj}(g)|_{H} \).

Corollary 14. If \( G \) is nilpotent of class 2 and the center of \( G \) is torsion-free, then SNP\((G, R)\) holds for every commutative ring \( R \).

Proof. Set \( N = Z(G) \) in Lemma 13 and note that the assumption implies \( G' \leq Z(G) \), and hence that \( G/Z(G) \) is abelian.

Proposition 15. Let \( G \) be finitely-generated torsion-free nilpotent, then SNP\((G, R)\) holds for all commutative rings \( R \).

Proof. Let \( H \leq G \) and \( u \in N_{U(RG)}(H) \). Let \( 1 = U_{0} < U_{1} < \ldots < U_{n+1} = G \) be a central series of \( G \), such that the factors \( C_{j} = U_{j+1}/U_{j} \) are all infinite cyclic groups (such a series exists by \([\text{Rob}96, \text{5.2.20}]\)). Define \( G_{j} = G/U_{j} \) and note that there is a natural exact sequence \( 1 \to C_{j} \to G_{j} \to G_{j+1} \to 1 \) for every \( j \in \{1, \ldots, n - 1\} \). The natural projection maps make the following diagram commute

\[
\begin{array}{cccccc}
  & & RG & & & \\
  & & \downarrow \text{id} & & & \\
  & RG_{0} & \longrightarrow & RG_{1} & \longrightarrow & \cdots & \longrightarrow RG_{n-1} & \longrightarrow RG_{n}
\end{array}
\]

Denote by \( H_{j} \) the image of \( H \) in \( G_{j} \) and by \( u_{j} \) the image of \( u \) in \( U(RG_{j}) \). Obviously every element of the support of \( u_{n} \) acts on \( H_{n} \) like the element \( u_{n} \), so fix any such support element \( g_{n} \). Using induction together with Lemma 13 yields an element \( g_{j} \in \text{supp}(u_{j}) \) acting like \( u_{j} \) on \( H_{j} \) for every \( 0 \leq j \leq n \). Consequently there is an element \( g = g_{0} \in \text{supp}(u) \) such that \( \text{conj}(u)|_{H} = \text{conj}(g)|_{H} \).

Remark 16. The last proof gives a direct argument for the proposition. The result can also be deduced by combining the theorems \([\text{MR}77, \text{Theorem 2.2.4}]\) and \([\text{Seh}93, \text{Lemma (45.3)}]\).
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