THE ELLIPTIC HALL ALGEBRA AND THE K-THEORY OF THE HILBERT SCHEME OF $\mathbb{A}^2$

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Abstract. In this paper we compute the convolution algebra in the equivariant $K$-theory of the Hilbert scheme of $\mathbb{A}^2$. We show that it is isomorphic to the elliptic Hall algebra, and hence to the spherical DAHA of $GL_\infty$. We explain this coincidence via the geometric Langlands correspondence for elliptic curves, by relating it also to the convolution algebra in the equivariant $K$-theory of the commuting variety. We also obtain a few other related results (action of the elliptic Hall algebra on the $K$-theory of the moduli space of framed torsion free sheaves over $\mathbb{P}^2$, virtual fundamental classes, shuffle algebras,...).

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0. Introduction and notation

0.1. Let $\text{Hilb}_n$ denote the Hilbert scheme of $n$ points in $\mathbb{C}^2$. The dimension of the homology groups $H^*(\text{Hilb}_n, \mathbb{Q})$, which were first determined by Ellingsrød and Strömme using a cell decomposition [ES], are given by

$$
\sum_{n \geq 0} \sum_{i=0}^{2n} \dim H^i(\text{Hilb}_n, \mathbb{Q})p^i q^n = \prod_{k \geq 1} \frac{1}{1 - p^{2k}q^k}.
$$

In a groundbreaking work [N] (see also [G2]), Nakajima obtained a much more precise understanding of the space

$$
\mathbf{V} = \bigoplus_n H^*(\text{Hilb}_n, \mathbb{Q})
$$

by geometrically realizing it as the Fock space representation of a Heisenberg algebra

$$
\mathbb{H} = \mathbb{C}\langle p_{\pm 1}, p_{\pm 2}, \ldots \rangle / [(p_i, p_j) = i \delta_{i+j,0}].
$$
More precisely, for \( k \in \mathbb{Z} \) and \( n, n+k \geq 0 \) let \( Z_{n+k,n} \subset \text{Hilb}_{n+k} \times \text{Hilb}_n \) stand for the nested Hilbert scheme and let \([Z_{n+k,n}] \in H^*(\text{Hilb}_{n+k} \times \text{Hilb}_n)\) be its fundamental class. The element \([Z_k] = \prod_n [Z_{n+k,n}]\) acts on \( V\) by convolution. Nakajima’s theorem may be stated as follows.

**Theorem (Nakajima)**. The following hold:

(a) the operators \([Z_k] : k \in \mathbb{Z}\) generate a Heisenberg algebra \(\mathbb{H} \subset \text{End}(V)\),

(b) as an \(\mathbb{H}\)-module, \(V\) is isomorphic to the Fock space representation, i.e., there exists a (unique) intertwining operator \(V \sim \mathbb{C}[p_1, p_2, \ldots]\) in which the action of the operators \([Z_k]\) is given by

\[
1 + \sum_{k \geq 1} [Z_k] z^k = \exp \left( - \sum_{n \geq 1} \frac{P_n}{n} z^n \right),
\]

\[
1 + \sum_{k \geq 1} [Z_{-k}] z^k = \exp \left( - \sum_{n \geq 1} \frac{1}{n} \frac{\partial}{\partial p_n} z^n \right).
\]

Note that this result extends to the Hilbert scheme of an arbitrary smooth quasiprojective surface. One has to replace \(\mathbb{H}\) by a Heisenberg-Clifford algebra modeled over the homology lattice \(H_n(S, \mathbb{Z})\). See also \([LQW], [LL], [LS], [V]\), etc, for other works in this direction.

In the well-documented analogy between the Hilbert-Chow resolution \(\text{Hilb}_n \rightarrow S^n \mathbb{C}^2\) and the Springer resolution \(T^*B \rightarrow \mathcal{N}\) of the nilpotent cone of a simple Lie algebra \(\mathfrak{g}\), Nakajima’s construction corresponds to the Springer representation of the Weyl group \(W\) of \(\mathfrak{g}\) in the homology of the Springer fibers.

One aim of this paper is to generalize Nakajima’s work for equivariant \(K\)-theory. The torus \(T = (\mathbb{C}^*)^2\) acts on \(\mathbb{C}^2\) and on \(\text{Hilb}_n\) for any \(n \geq 1\). Let \(K^T(\text{Hilb}_n)\) denote the (algebraic) equivariant \(K\)-theory group and set

\[L_R = \bigoplus_n K^T(\text{Hilb}_n),\]

a \(R = R_T = \mathbb{C}[q^{\pm 1}, t^{\pm 1}]\)-module. We are interested in a natural \(K\)-theoretic analog of the convolution algebra considered by Nakajima. It is not a priori clear how to define such a natural convolution algebra because the nested Hilbert schemes \(Z_{n+k,n}\) are in general singular when \(k \not\in \{-1, 0, 1\}\) and the classes \(\mathcal{O}_{Z_{n+k,n}}\) are typically badly behaved.

We get around this difficulty by considering only the smooth nested Hilbert schemes \(Z_{n+1,n}, Z_{n,n}, Z_{n,n+1}\) and their respective tautological bundles \(\tau_{n+1,n}, \tau_{n,n}, \tau_{n,n+1}\). Namely, we let \(K = \mathbb{C}(q^{1/2}, t^{1/2})\) and let

\[H_K \subset \bigoplus_k \prod_n K^T(\text{Hilb}_{n+k} \times \text{Hilb}_n) \otimes_R K\]

be the subalgebra generated by the elements

\[f_{-1,l} = \prod_n \tau^l_{n,n+1}, \quad f_{1,l} = \prod_n \tau^l_{n+1,n}, \quad l \in \mathbb{Z}\]

and the Adams operations

\[f_{0,l} = \prod_n \Psi^l(\tau_{n,n}), \quad l \in \mathbb{Z}.
\]

Our first main result identifies \(H_K\) and its action on \(L_K = L_R \otimes_R K\) (see Theorem 3.1).

**Theorem 1.** The following hold:

(a) there is an isomorphism \(\Omega : E_c \simeq H_K\), where \(E_c\) is a certain one-dimensional central extension of the spherical Double Affine Hecke Algebra \(SH_\infty\) of type \(GL_\infty\),

(b) as an \(E_c\)-module, \(L_K\) is isomorphic to the standard representation on the space of symmetric polynomials \(\Lambda_K = K[x_1, x_2, \ldots]^{S_\infty}\).
The algebras $\mathcal{E}_c$ and $\mathbf{SH}_\infty$ were introduced in [BS], [SV1]. The standard representation on $\mathbf{L_K}$ (which we define in Section 4.7) involves operators of multiplication by symmetric polynomials, differential operators as well as Macdonald’s difference operators. It may be viewed, in a certain sense, as a stable limit as $n \to \infty$ of the polynomial representations of the spherical DAHAs of finite type $GL_n$. The group $SL(2, \mathbb{Z})$ acts by automorphisms on $\mathbf{SH}_\infty$ (but not on $\mathcal{E}_c$). It would be interesting to find a geometric interpretation for this (almost) symmetry from the point of view of the Hilbert scheme.

The occurrence of Macdonald operators and polynomials comes as no surprise here: these are ubiquitous in Haiman’s work on the Hilbert scheme, see [H]. In fact the intertwining operator $\mathbf{L_K} \simeq \mathbf{L_K}$ in Theorem 1 coincides with Haiman’s identification of $\mathbf{L_K}$ with the space of symmetric polynomials.

While we were finishing this paper B. Feigin and A. Tsymbaliuk sent us a copy of their preprint [FT] where the equivariant $K$-theory of the Hilbert schemes is studied by a different approach based on shuffle algebras, see Section 9.

What about operators associated to higher rank nested Hilbert schemes? We do not know whether the classes $[O_{Z_r}] = \prod_n [O_{Z_{n+k,n}}]$ belong to $\mathbf{H_K}$ or not (for $k \notin \{-1, 0, 1\}$). However, the virtual classes $\Lambda_{\mathbf{W}_k} = \prod_n \Lambda_{\mathbf{W}_{n+k,n}}$ introduced by Carlsson and Okounkov in [CO] do indeed belong to $\mathbf{H_K}$ and seem to be much better behaved than the $O_{Z_r}$. In Theorem 2 and Corollary 5.2 we prove the following.

**Theorem 2.** The following hold:

(a) for any $k \in \mathbb{Z}$ the virtual class $\Lambda_{\mathbf{W}_k}$ belongs to $\mathbf{H_K}$.

(b) under the isomorphism $\mathbf{L_K} \simeq \mathbf{L_K}$ the action of the operators $\Lambda_{\mathbf{W}_k}$ is given by

\[
1 + \sum_{n \geq 1} \tau_n^* \otimes \Lambda_{\mathbf{W}_k} z^n = \exp \left( - \sum_{n \geq 1} (-1)^n \frac{1 - t^n q^n}{1 - q^n} \frac{p_n}{n} z^n \right),
\]

(0.3)

\[
1 + \sum_{n \geq 1} \Lambda_{qt\mathbf{W}_k} z^n = \exp \left( - \sum_{n \geq 1} \frac{1 - t^n q^n}{1 - t^n} \frac{\partial}{\partial p_n} z^n \frac{1}{n} \right).
\]

(0.4)

So the elements $\tau_n^* \otimes \Lambda_{\mathbf{W}_k}$, $\Lambda_{qt\mathbf{W}_k}$, $n \geq 0$, generate a Heisenberg subalgebra of $\mathbf{H_K}$.

It would be very interesting to generalize the above Theorem 1 and Theorem 2 to the case of the Hilbert scheme of an arbitrary smooth quasiprojective surface $S$. This would presumably involve a “global” version of $\mathbf{SH}_\infty$ living over the base $K(S)$. The approach used in the present paper, based on localization to $T$-fixed points, seems, unfortunately, restricted to toric surfaces.

The Hilbert scheme $\text{Hilb}_n$ may be interpreted as the moduli space of framed rank one torsion free-sheaves over $\mathbb{P}^2(\mathbb{C})$ of chern class $c_2 = n$. For $r \geq 1$, let $M_{r,n}$ be the moduli space of framed torsion free-sheaves over $\mathbb{P}^2(\mathbb{C})$ of chern class $c_2 = n$ and rank $r$. One may define “nested schemes” $M_{r,n+k,n}$ and tautological bundles over them, and one may consider the convolution algebra $H_{K(r)}^*$. The Theorem 1 has an analogue in this setting (see Theorem 8.1).

**0.2.** Our original motivation for studying the convolution algebra $H_{K(r)}$ stems not so much from a desire to understand the geometry of $\text{Hilb}_n$ as from its interpretation in the framework of the geometric Langlands program for elliptic curves, which we now describe. Recall that Beilinson and Drinfeld’s geometric Langlands program, for $GL_r$, predicts the existence, for any fixed smooth projective curve $C$ over $\mathbb{C}$, of an equivalence of derived categories

\[
\text{Coh}(\text{Loc}_C) \overset{\mathcal{D}}{\simeq} \text{D-mod}(\text{Bun}_r C),
\]

(0.5)

see, e.g., [F]. Here $\text{Loc}_C$ is the moduli stack parametrizing local systems over $C$ of rank $r$ and $\text{Bun}_r C$ is the moduli stack of vector bundles of rank $r$ over $C$. This correspondence should intertwine the natural action of $\text{Rep GL}_r$ on $\text{Coh}(\text{Loc}_C)$ by tensor product and the action of
$Rep \ GL_r$ on $D-mod(Bun_r C)$ by means of Hecke operators. The image under this correspondence of the skyscraper sheaves at points of $Loc, C$ corresponding to irreducible local systems has been determined in [FGV]. The case which is relevant to us is rather orthogonal. Namely, we are interested in coherent sheaves supported in the formal neighborhood of the trivial local system $C_{\sigma}$ over $C$.

Let us specialize from now on $C$ to be an elliptic curve. Then

$$\text{Loc}_C = \{ \rho : \pi_1(C) \to GL_r \}/GL_r = \{(A, B) \in GL_r^2 : AB = BA\}/GL_r$$

and the formal neighborhood of the trivial local system is the formal neighborhood of $(0, 0)$ in

$$\text{Loc}_C = \{(a, b) \in gl_r^2 : [a, b] = 0\}/GL_r = C_{gl_r}/GL_r$$

where $C_r \subset gl_r \times gl_r$ is the commuting variety. Observe that this formal neighborhood is independent of the choice of $C$. Moreover, the torus $T = (\mathbb{C}^*)^2$ naturally acts on $\text{Loc}_C$ by $(z_1, z_2) \cdot (a, b) = (z_1 a, z_2 b)$.

The relation with the Hilbert schemes is given by the following simple observation. We have

$$Z_{n+r, n} = \{(I, J) ; I \subset J \subset \mathbb{C}[x, y] \text{ ideals}, \text{ codim } I = n + r, \text{ codim } J = n\}.$$ 

Hence if $(I, J) \in Z_{n+r, n}$ then $(x_i; y_j, y_j/1)$ define a point in $C_r/GL_r$. One may try to use this map $Z_{n+r, n} \to C_r/GL_r$ to lift classes from $C_{R} = K^{GL_r \times T}(C_r)$ to $K^{T}(Z_{n+r, n})$. In this spirit we define a convolution algebra structure on the direct sum

$$C_R = \bigoplus_{r \geq 0} C_r = \bigoplus_{r \geq 0} K^{T}(\text{Loc}_r C)$$

as well as an action of $C_{R} = C_{R} \otimes R K$ on $L_{K}$. Let $H_{K}^2 \subset H_{K}$ be the subalgebra generated by the classes $f_{I, l}$ for $l \in \mathbb{Z}$, and let $C_{K} \subset C_{R}$ be the subalgebra generated by $C_{R}$.

**Theorem 3.** There is a natural isomorphism of algebras

$$\rho : C_{K}/\text{torsion} \sim H_{K}^2.$$ 

Moreover, $\rho$ is compatible with the actions of $C_{K}$ and $H_{K}^2$ on $L_{K}$.

We conjecture that $C_{K}$ is actually torsion free (see Theorem 7.13 and Conjecture 7.13).

On the other hand, it is proved in [SV1] that the positive part $E^\geq \simeq SH^\geq_{\mathbb{C}}$ is isomorphic to the (universal, spherical) elliptic Hall algebra. In other words, for any elliptic curve $C_{r}$ defined over some finite field $k = \mathbb{F}_l$ with Frobenius eigenvalues $\{\sigma, \overline{\sigma}\}$ the algebra $E^\geq$ specializes at $t = \sigma, q = \overline{\sigma}$ to the spherical Hall algebra $H_{C_k}^{\text{ph}}$ of $C_k$ (more precisely, to its vector bundle part $H_{C_k}^{\text{ph, vec}}$, see [BS]). By definition $H_{C_k}^{\text{ph, vec}}$ is a convolution algebra of functions on the moduli stacks $\bigcup_r Bun_r C_k$. It is shown in [SI] that $H_{C_k}^{\text{ph, vec}}$ is the Grothendieck group $\bigoplus$ of a certain category $Eis_{C_k}$ of semisimple perverse sheaves over $\bigcup_r Bun_r C_k$, the so-called Eisenstein sheaves generated by the constant sheaves over the Picard stacks $\text{Pic}^d_{C_k}$ for $d \in \mathbb{Z}$. There is a similar category $Eis_{C}$ for our complex elliptic curve $C$. We expect that $Eis_{C}$ carries a natural $\mathbb{Z}^2$-grading and that the associated graded Grothendieck group $\bigoplus K_0^g(Eis_{C})$ is isomorphic to $E^\geq$ after a suitable extension of scalars.

Combining Theorem 1 and Theorem 3, we may draw the diagram

$$\begin{array}{ccc}
\bigoplus_r K_0(\text{Coh}^T(\text{Loc}_r C)) & \xrightarrow{\rho} & \bigoplus_r K_0(\text{Perv}(\text{Bun} r C)) \\
\bigcup_r K_0(\text{Loc}_r C) & \xrightarrow{\Omega} & E^\geq = K_0^g(Eis_{C})
\end{array}$$

1For simplicity we ignore in this introduction the (important) questions concerning completions of Hall algebras, see [BS], [SI].

2Again, we ignore problems of completions.
The vertical arrows are embeddings. The last equality is conjectural. The faithful actions of $\mathcal{C}_K$ and $\mathcal{E}^\gg$ on $\mathcal{L}_K$ identify these two algebras. They are both naturally $\mathbb{N}$-graded by the rank $r$. For each $r$ the action of $R_{GL_r \times T}$ on $\mathcal{C}_K$ coincides with the action of $R_{GL_r \times T}$ on $\mathcal{E}^\gg[r]$ by means of Hecke operators, see Proposition 7.10. We summarize this in the following corollary, which is the second main result of this paper.

**Corollary.** There is an algebra isomorphism $\mathcal{C}_K/torsion \simeq \mathcal{E}^\gg$ intertwining, on the graded components of degree $r$, the natural actions of $R_{GL_r \times T}$ by tensor product and Hecke operators respectively.

This corollary may be seen as a version, at the level of Grothendieck groups, of a geometric Langlands correspondence [1.5] for local systems living in the formal neighborhood of the trivial local system (or, equivalently, for the category of Eisenstein sheaves generated by constant sheaves over $\text{Pic}_C$). We point, however, two notable differences with (0.5). First, we have not just an equality of Grothendieck groups $\mathcal{C}_K \simeq \mathcal{E}^\gg[r]$ for each $r$, but an equality of convolution algebras. This convolution product only exists because we are dealing with the groups $GL_r$. Secondly, we have considered $T$-equivariant coherent sheaves over $\text{Loc}_r\mathcal{C}$. There is no corresponding $\mathbb{Z}^2$ grading on the categories $\mathcal{E}isc_C$ in our picture. See [52], Lecture 5, for more in that direction.

0.3. The plan of the paper is as follows. Sections 1 and 2 serve as reminders on the elliptic Hall algebra $\mathcal{E}$ and the Cherednik algebra $\mathcal{SH}_K$ and on the Hilbert schemes respectively. The construction of the standard representation of $\mathcal{SH}_K\mathcal{E}$ and $\mathcal{E}^\gg$ is given in Section 1.4. In Section 3 we consider the convolution algebra $\mathcal{H}_K$ in the equivariant $K$-theory of Hilbert schemes and state our first main Theorem (see Theorem 1). Section 4 is devoted to the proof of that result. In Section 5 we explain how to relate the virtual classes $\Lambda(\mathcal{V}_n)$ with $\mathcal{H}_K$. Section 6 contains some preparatory results pertaining to the Hecke action on the Hall algebra $\mathcal{E}$. The convolution algebras $\mathcal{C}_K$, $\mathcal{C}_K$ in the equivariant $K$-theory of the commuting variety are defined and studied in Section 7, where $\mathcal{C}_K$ is compared with the positive part $\mathcal{H}_K^\gg$ (see the above Theorem 3 and Corollary). Section 8 deals with the case of moduli spaces of framed torsion sheaves of rank $r > 1$ over $\mathbb{P}^2(\mathbb{C})$. Finally, Sections 9 and 10 contain some remarks on some natural Heisenberg subalgebras of the convolution algebra $\mathcal{H}_K^\gg$ as well as a description of $\mathcal{H}_K^\gg$ as a shuffle algebra, and a comparison between the present paper and the recent work [FT]. In an attempt at making this paper more readable, several technical points have been postponed to the Appendix. The most important notations have been gathered together in an index at the very end of the paper.

0.4. We will use the standard notation for partitions. If $\lambda = (\lambda_1 \geq \lambda_2 \ldots)$ is a partition then $\lambda'$ is its conjugate partition, $l(\lambda)$ is its length and $|\lambda|$ is the number of boxes in the Young diagram associated with $\lambda$. We’ll identify $\lambda$ and its Young diagram, i.e., we write $\lambda = \{(i,j) \in \mathbb{N}^2; 1 \leq j \leq \lambda_i\}$. If $s \in \lambda$ is a box then we write $s = (i,j)$. Let

$$a(s) = a_\lambda(s) = \lambda_i - j, \quad l(s) = l_\lambda(s) = \lambda'_j - i,$$

the arm length and the leg length of $s$. We also set

$$i(s) = i_\lambda(s) = i, \quad j(s) = j_\lambda(s) = j.$$

These of course do not depend on $\lambda$. Occasionally, we will write

$$x(s) = i(s) - 1, \quad y(s) = j(s) - 1.$$

So we have the following picture for $\lambda = (10, 9^3, 6, 3^2)$
Denote the set of partitions by $\Pi$. For $s$ a box of a partition $\lambda$ we let $C_s$ and $R_s$ be the column and row of $s$.

Given a field $K$ let $\Lambda_K = K[x_1, x_2, \ldots]^S$ be the $K$-vector space of symmetric polynomials and $\Lambda_K^\infty = K[x_1, x_2, \ldots, x_n]^S_n$ be the space of symmetric polynomials in $n$ variables.

Let $E$ be a complex vector space of dimension $n$. Let us denote by $R_{GL(E)}$ the complexified representation ring of $GL(E)$. There is a canonical ring isomorphism $R_{GL(E)} \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^S_n$ determined by $\Lambda_E \mapsto e_l(x_1, \ldots, x_n)$ for $1 \leq l \leq n$. Here $E$ is the tautological representation of $GL(E)$. Under this isomorphism, the Adams operations $\Psi_l(E)$ are mapped to the power sum functions $p_l(x_1, \ldots, x_n)$ for any $l \in \mathbb{Z}^*$. If $v$ is any variable, we put as usual $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}$, $[n]_v! = [2]_v \cdots [n]_v$.

1. The elliptic Hall algebra, the spherical DAHA and the polynomial representation

We first recall some result and definition from [BS] and [SV1], to which the reader is referred for details. Set $K = \mathbb{C}(\sigma^{1/2}, \bar{\sigma}^{1/2})$, where $\sigma, \bar{\sigma}$ are formal variables.

1.1. We begin with some recollection on the elliptic Hall algebra $E^+$ and its Drinfeld double $\hat{E}$. We set $Z = \mathbb{Z}^2$, $Z^* = \mathbb{Z}\setminus\{(0,0)\}$, $Z^+ = \{(i, j) \in Z; i > 0 \text{ or } i = 0, j > 0\}$, $Z^- = -Z^+$.

For a future use we set also

$Z^k = \{(i, j) \in Z; i > 0\}$, $Z^k = \{(i, j) \in Z; i = k\}$, $Z^k = \{(i, j) \in Z; i > 0\}$,

for each $k \in \mathbb{Z}$. For any $x = (i, j) \in Z^*$ let $d(x)$ denote the greatest common divisor of $i, j$.

We’ll also use the following polynomials

$\alpha_n = \alpha_n(\sigma, \bar{\sigma}) = (1 - (\sigma \bar{\sigma})^{-n})(1 - \sigma^n)(1 - \bar{\sigma}^n)/n$.

Note that $\alpha_n = \alpha_{-n}$.

We set $\epsilon_x = 1$ if $x \in Z^+$ and $\epsilon_x = -1$ if $x \in Z^-$. For a pair of non-collinear elements $x, y \in Z^*$ we set $\epsilon_{x,y} = \text{sgn}(|\text{det}(x,y)|)$, an element in $\{\pm 1\}$. Finally let $\Delta_{x,y}$ be the triangle in $Z$ with vertices $(0,0), x, x+y$. 

![Figure 1. Notations for partitions.](image-url)
**Definition.** Let $\hat{E}$ be the $K$-algebra generated by elements $u_x, \kappa_x, x \in \mathbb{Z}^*$, modulo the following set of relations

(a) $\kappa_x$ is central for all $x$, and $\kappa_{x+y} = \kappa_x \kappa_y$,
(b) if $x, y$ belong to the same line in $\mathbb{Z}$ then
$$[u_y, u_x] = \frac{\kappa_x - \kappa_x^{-1}}{\alpha(d(x))} \text{ if } x = -y, \quad [u_y, u_x] = 0 \text{ else},$$
(c) if $x, y \in \mathbb{Z}^*$ are such that $d(x) = 1$ and that $\Delta_{x,y}$ has no interior lattice point then
$$[u_y, u_x] = \epsilon_{x,y} \kappa_{\alpha(x,y)} \frac{\theta_{x+y}}{\alpha},$$
where
$$\alpha(x, y) = \begin{cases} \epsilon_x (\epsilon_x x + \epsilon_y y - \epsilon_{x+y}(x + y))/2 & \text{if } \epsilon_{x,y} = 1, \\ \epsilon_y (\epsilon_x x + \epsilon_y y - \epsilon_{x+y}(x + y))/2 & \text{if } \epsilon_{x,y} = -1, \end{cases}$$
and where the elements $\theta_z, z \in \mathbb{Z}^*$, are given by
$$\sum_i \theta_{i x_0} s^i = \exp \left( \sum_{r \geq 1} \alpha_r u_{r x_0} s^r \right),$$
for any $x_0 \in \mathbb{Z}^*$ such that $d(x_0) = 1$.

For a future use, note that $\theta_z = \alpha_1 u_z$ if $d(z) = 1$. Observe also that the $K$-algebra $\hat{E}$ is $\mathbb{Z}$-graded and has a natural $SL(2, \mathbb{Z})$-symmetry. We write $K$ for the central subalgebra generated by $\{\kappa_x; x \in \mathbb{Z}\}$. We'll abbreviate $c_1 = \kappa_{0,1}$ and $c_2 = \kappa_{1,0}$. We have $K = K[c_1^{\pm 1}, c_2^{\pm 1}]$. We'll view $\hat{E}$ as a $K$-algebra.

Let $\hat{E}^\pm$ be the $K$-subalgebra of $\hat{E}$ generated by $\{u_x; x \in \mathbb{Z}^\pm\}$. It is shown in [BS], Section 5, that relations (b), (c) restricted to $\mathbb{Z}^\pm$ give a presentation of $\hat{E}^\pm$, and that there is a biangular decomposition

\begin{equation}
\hat{E} \simeq \hat{E}^+ \otimes \hat{E}^-.
\end{equation}

For our convenience we put $u_{0,0} = 1$. It is also proved in loc. cit. that the $K$-algebra $\hat{E}$ is generated by the elements $u_x$ with $x \in \mathbb{Z}^1 \cup \mathbb{Z}^0 \cup \mathbb{Z}^{-1}$. It will be helpful to consider a slight refinement of the isomorphism (1.1). Let $\hat{E}^+, \hat{E}^\prec, \hat{E}^\succ$ be the subalgebras of $\hat{E}$ generated by the elements $u_x$ with $x \in \mathbb{Z}^+, \mathbb{Z}^-, \mathbb{Z}^0$ respectively. From (1.1) and the defining relations of $\hat{E}$ one deduces the triangular decomposition

\begin{equation}
\hat{E} \simeq \hat{E}^+ \otimes \hat{E}^0 \otimes \hat{E}^\prec.
\end{equation}

The algebras $\hat{E}^+, \hat{E}^\prec$ are isomorphic. By [BS], Corollary 5.2, the algebras $\hat{E}^+, \hat{E}^\prec$ contains all the elements $u_x$ for $x \in \mathbb{Z}^+, \mathbb{Z}^\prec$ respectively. Let $\hat{E}^\succ$ be the subalgebra generated by $\hat{E}^+, \hat{E}^0$. We define $\hat{E}^\prec$ in the same way. We have

$$\hat{E}^+ \subset \hat{E}^\succ \subset \hat{E}^\succ \subset \hat{E}^\prec.$$

For any $\omega = (\omega_1, \omega_2) \in (\mathbb{C}^*)^2$, write $E_\omega$ for the specialization of $\hat{E}$ at $c_1 = \omega_1, c_2 = \omega_2$. The notations $E_\omega^+, E_\omega^\prec, E_\omega^\succ$, etc, are clear. We will simply write $E$ for $E_{(1,1)}$. We'll be mostly interested in the algebra $E_\omega$, where

$$c = (1, q^{1/2} t^{1/2}).$$

When $\sigma, \bar{\sigma}$ are specialized to the Frobenius eigenvalues of an elliptic curve $X$ defined over a finite field, $E^+$ is equal to the spherical Hall algebra of $X$. It is endowed with a natural coproduct and $\hat{E}$ is isomorphic to its Drinfeld double.
1.2. The following result will be useful.

**Proposition 1.1.** The algebra $\hat{E}$ is isomorphic to the algebra generated by $\hat{E}^<, \hat{E}^0, \hat{E}^>$ modulo the following relations

\begin{align}
[u_{0,d}, u_{1,l}] &= u_{1,l+d} \quad (\text{if } d > 0), \\
[u_{0,d}, u_{1,l}] &= -c_1^d u_{1,l+d} \quad (\text{if } d < 0),
\end{align}

and

\begin{align}
[u_{-1,k}, u_{0,d}] &= c_1^{-d} u_{1,k+d} \quad (\text{if } d > 0), \\
[u_{-1,k}, u_{0,d}] &= -u_{1,k+d} \quad (\text{if } d < 0),
\end{align}

and finally

\begin{align}
[u_{-1,k}, u_{1,l}] &= c_2 c_1^{-k} \frac{\theta_{0,k+l}}{\alpha_1} \quad (\text{if } k + l > 0), \\
[u_{-1,k}, u_{1,l}] &= -c_2^{-1} c_1^{l} \frac{\theta_{0,k+l}}{\alpha_1} \quad (\text{if } k + l < 0).
\end{align}

*Proof.* Let $\hat{E}$ momentarily denote the algebra generated by $\hat{E}^>, \hat{E}^0, \hat{E}^<$ modulo the relations (1.6). We have a natural surjective morphism $\phi : \hat{E} \to \hat{E}$ which is the identity on the subspace $\hat{E}^> \hat{E}^0 \hat{E}^<$ of $\hat{E}$. Hence it suffices to prove that $\hat{E}^> \hat{E}^0 \hat{E}^< = \hat{E}$. The algebras $\hat{E}^>, \hat{E}^<$ are generated by the elements $u_x$ with $x \in \mathbb{Z}^1$, $\mathbb{Z}^{-1}$ respectively. Hence we have to prove that any monomial of $\hat{E}$ in the elements $u_{1,l}, u_{0,n}, u_{-1,k}$ may be straightened to similar monomials in which the $u_{1,l}, u_{0,n}$ and $u_{-1,k}$ appear in that order. It is easy to see that relations (1.6) precisely enable one to perform such straightening. \hfill $\Box$

1.3. We briefly recall here the relation between $E$ and the spherical double affine Hecke algebras. See [SV1], Section 2, for details.\footnote{The conventions used in [SV1] differ slightly from the more standard ones of [C2].} Let $\hat{H}_n$ be the double affine Hecke algebra (DAHA for short) of type $GL_n$. It is a $C(q^{1/2}, t^{1/2})$-algebra generated by elements $X_i^{\pm 1}, Y_i^{\pm 1}$ for $i = 1, \ldots, n$ and $T_j$ for $j = 1, \ldots, n - 1$ subject to some relation which we won’t write here. Let $S$ be the complete idempotent in the finite Hecke algebra $H_n \subset \hat{H}_n$ generated by the $T_1, T_2, \ldots, T_{n-1}$. It is characterized by the property that $T_j S = S T_j = t^{-1/2} S$ for all $j$. The spherical DAHA is the subalgebra $\hat{S}_n = S \hat{H}_n S$ of $\hat{H}_n$. For $l > 0$ we set

\begin{align}
P_{0,l}^n &= S \sum_i Y_i^l S, & P_{0,-l}^n &= q^l S \sum_i Y_i^{-1} S, & P_{l,0}^n &= q^l S \sum_i X_i^l S, & P_{-l,0}^n &= S \sum_i X_i^{-1} S.
\end{align}

These elements generate $\hat{S}_n$. More general elements $P_x^n$ for $x \in \mathbb{Z}^*$ are defined in [SV1] via the $SL(2, \mathbb{Z})$ action on $\hat{S}_n$. We put also $P_{0,0}^0 = 1$.

Let $\hat{S}_n^+$ be the subalgebra of $\hat{S}_n$ generated by the elements $P_x^n$ for $x \in \mathbb{Z}^+$. It is shown in [SV1], Proposition 4.1, that the assignment $P_x^n \mapsto P_x^m$ extends to a surjective algebra homomorphism $\hat{S}_n^+ \to \hat{S}_m^+$ for $n > m$. This allows one to consider the stable limit $\hat{S}_\infty^+ \subset \varprojlim \hat{S}_n^+$ of the algebras $\hat{S}_n^+$, along with its set of generators $\{P_x^\infty; x \in \mathbb{Z}^+\}$.

We identify $K$ and $C(q^{1/2}, t^{1/2})$ by means of

\begin{align}
\sigma \mapsto q^{-1}, & \quad \bar{\sigma} \mapsto t^{-1}.
\end{align}
Theorem 1.2 ([SVI]). For any \( n \) the assignment
\[
\mathbf{u}_x \mapsto \frac{1}{q^{d(x)} - 1} P^n_x, \quad x \in \mathbb{Z}^n
\]
extends to a surjective algebra homomorphism
\[
\Phi_n : \mathcal{E} \to \mathcal{S} \tilde{H}_n .
\]
The restrictions \( \Phi^+_n \) of \( \Phi_n \) to \( \mathcal{E}^+ \) form a projective system of maps and give rise to an algebra isomorphism
\[
\Phi^+_\infty : \mathcal{E}^+ \cong \mathcal{S} \tilde{H}_\infty^+ , \quad \mathbf{u}_x \mapsto \frac{1}{q^{d(x)} - 1} P^\infty_x .
\]

Note that it is not clear a priori how to make sense of a stable limit \( \mathcal{S} \tilde{H}_\infty \). For this reason, we may take \( \mathcal{E} \) as a definition of \( \mathcal{S} \tilde{H}_\infty \).

1.4. We now define a faithful representation of \( \mathcal{E}^+ \). First, recall that there is a faithful representation \( \varphi_n \) of \( \mathcal{H}_n \) on the space \( K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) defined by
\[
\varphi_n(X_i) = x_i , \quad \varphi_n(T_i) = t^{1/2} s_i + \frac{t^{-1/2} - t^{1/2}}{x_i / x_{i+1} - 1} (s_i - 1) , \quad \varphi_n(Y_i) = \varphi_n(T_1) \cdots \varphi_n(T_{i-1}) \omega \varphi_n(T_1^{-1}) \cdots \varphi_n(T_{i-1}^{-1}) .
\]
Here \( s_i \) stands for the transposition \( x_i \leftrightarrow x_{i+1} \), \( \partial_i \) is the linear operator defined by \( \partial_i \cdot p(x_j) = p(q^{i-1} x_j) \) and \( \omega = s_{n-1} \cdots s_1 \partial_1 \). See [C2] or [SVI], Section 4.

Let \( \mathcal{S} \tilde{H}_n^+, \mathcal{S} \tilde{H}_n^0, \mathcal{S} \tilde{H}_n^\geq \subset \mathcal{S} \tilde{H}_n \) be the subalgebras generated by the elements \( P^n_x \) with \( x \in \mathbb{Z}^+ , \mathbb{Z}^0 , \mathbb{Z}^\geq \) respectively. Note that
\[
\mathcal{S} \tilde{H}_n^+ \subset \mathcal{S} \tilde{H}_n^0 \subset \mathcal{S} \tilde{H}_n^\geq .
\]
The representation \( \varphi_n \) restricts to a representation of \( \mathcal{S} \tilde{H}_n^\geq \) on the space of symmetric polynomials \( A^n_K \). In this representation the element \( P^n_{0,1} \) acts as Macdonald’s difference operator
\[
\Delta^n_i = \sum_{j \neq i}^n \left( \prod_{j \neq i} \frac{t^{-1/2} x_i - t^{1/2} x_j}{x_i - x_j} \right) \partial_i .
\]
It has as eigenvectors the Macdonald polynomials \( P^n_\lambda(q,t^{-1}) \) where \( \lambda \) runs among all partitions with at most \( n \) parts. The \( \Delta^n_i \)-eigenvalue of \( P^n_\lambda(q,t^{-1}) \) is given by the formula
\[
\beta^1_{\lambda,n} = t^{n-1} \sum_{i=1}^n q^{\lambda_i} t^{(i-1)} ,
\]
see [M], Section VI.3. For \( l > 1 \) the element \( P^n_{0,l} \) acts as the linear operator \( \Delta^n_l \) whose eigenvalues on the Macdonald polynomials read
\[
\beta^l_{\lambda,n} = t^{-i} \sum_{i=1}^n q^{\lambda_i} t^{(i-1)} ,
\]
see [M], Section VI.4, (4.15). Similar results also hold for the elements \( P^n_{0,-l} \) with \( l \geq 1 \). Namely, the operator \( \Delta^n_{-l} = \varphi_n(P^n_{0,-l}) \) is diagonalizable in the basis of Macdonald polynomials with eigenvalues given by
\[
\beta^{-l}_{\lambda,n} = q^l t^{i-1} \sum_{i=1}^n q^{-\lambda_i} t^{-(i-1)} .
\]
There is a unique algebra homomorphism \( \rho_\lambda : A^n_K \to A^n_K^{-1} \) given by \( x_i \mapsto x_i \) for \( i \leq n - 1 \) and \( x_n \mapsto 0 \). It takes \( P^n_\lambda(q,t^{-1}) \) to \( P^{-1}_\lambda(q,t^{-1}) \) if \( l(\lambda) \leq n - 1 \) and to zero if \( l(\lambda) = n \). It is easy
to see from the formulas (1.7), (1.8) that the map \( \rho_n \) is not compatible with the representations \( \varphi_n \), i.e., we have

\[ \varphi_{n-1} \circ \rho_n \neq \rho_n \circ \varphi_n. \]

To remedy this, we need to renormalize the action \( \varphi_n \) slightly. Put

\[ \Delta_l^n = t^{\Delta_l^n - [n]_{l,1}/2}. \]

One checks using (1.7), (1.8) that we have \( \rho_n \circ \Delta_l^n = \Delta_l^{n-1} \circ \rho_n \) and that for each \( l \geq 1 \) the eigenvalue of \( \Delta_l^n \) on \( P^n_X(q,t^{-1}) \) is equal to

\[ \beta_{\lambda,n} = \sum_{i=1}^{n} (q^{l \lambda_i} - 1)t^{l(i-1)}, \quad \beta_{\lambda,n}^{-1} = q^{l} \sum_{i=1}^{n} (q^{-l \lambda_i} - 1)t^{-l(i-1)}. \]

We denote by \( \Delta_l^n \) the stable limit of the operators \( \Delta_l^n \).

By [BS], Corollary 5.2, and Theorem 1.2 the algebra \( \tilde{S}_H^n \) is generated by the elements \( P^n_X \) with \( x \in \mathbb{Z}^1 \) and we have the following relation in \( \tilde{S}_H^n \)

\[ [P^n_{0,l}, P^n_{1,j}] = \epsilon_l(q^l - 1)P^n_{1,j+l}. \]

Here \( \epsilon_l = 1 \) if \( l \geq 1 \) and \( \epsilon_l = -1 \) if \( l \leq -1 \). Let \( \tilde{S}_H^n \times \tilde{S}_H^n \) be the algebra generated by \( \tilde{S}_H^n \) and \( \tilde{S}_H^n \) modulo the relations (1.10).

**Lemma 1.3.** The \( K \)-algebra \( \tilde{S}_H^n \) is isomorphic to \( \tilde{S}_H^n \times \tilde{S}_H^n \).

**Proof.** The multiplications in \( \tilde{S}_H^n \times \tilde{S}_H^n \) and \( \tilde{S}_H^n \) yield right \( \tilde{S}_H^n \)-module homomorphisms

\[ \tilde{S}_H^n \otimes_K \tilde{S}_H^n \to \tilde{S}_H^n \times \tilde{S}_H^n \to \tilde{S}_H^n. \]

The first map is surjective by (1.10). The second one is also surjective by Theorem 1.2 and (1.2). We must prove that it is injective. It is enough to prove that the surjective map

\[ m : \tilde{S}_H^n \otimes_K \tilde{S}_H^n \to \tilde{S}_H^n \]

given by the multiplication in \( \tilde{S}_H^n \) is injective.

There is a unique \( \mathbb{Z} \)-grading on the algebra \( \tilde{S}_H^n \) such that the element \( P^n_X \) has the degree \( x \). For each integer \( i \geq 0 \) let \( \tilde{S}_H^n_i \subset \tilde{S}_H^n \) be subspace spanned by the homogeneous elements whose degree belongs to \( \mathbb{Z}^i \). It is a right \( \tilde{S}_H^n_i \)-submodule of \( \tilde{S}_H^n \) such that \( \tilde{S}_H^n = \bigoplus_{i \geq 0} \tilde{S}_H^n_i \). The map \( m \) is a surjective right graded \( \tilde{S}_H^n_i \)-module homomorphism. Thus it restricts to a surjective right \( \tilde{S}_H^n_i \)-module homomorphism

\[ m^i : \tilde{S}_H^n_i \otimes_K \tilde{S}_H^n_i \to \tilde{S}_H^n_i, \quad \tilde{S}_H^n_i = \tilde{S}_H^n_i \cap \tilde{S}_H^n_i. \]

Let \( A \) be the localization of the ring \( \mathbb{C}[t, t^{-1}] \) with respect to the multiplicative set generated by \( [n]_{l,1/2} \). Let \( \tilde{H}_{n,A} \subset \tilde{H}_n \) be the \( A \)-subalgebra generated by the elements \( X_i, Y_i^\pm 1, T_j \), and let \( \tilde{S}_H^n_{n,A}, \tilde{S}_H^n_{0,A} \) be the \( A \)-subalgebras generated by the elements \( P^n_{x} \) with \( x \in \mathbb{Z}^i, \mathbb{Z}^0 \) respectively. We put \( \tilde{S}^n_{H,n,A} = \tilde{S}^n_{H,n,A} \). We define also \( \tilde{S}^n_{H,n,A}, \tilde{S}^n_{H,n,A} \) in the obvious way. The map \( m^i \) restricts to an \( A \)-linear map

\[ m^i_A : \tilde{S}^n_{H,n,A} \otimes_A \tilde{S}^n_{H,n,A} \to \tilde{S}^n_{H,n,A}. \]

We’ll abbreviate \( \mathbb{C} = A/(q - 1, t - 1) \). We have the following

- \( \tilde{S}^n_{H,n,A} \) is a finitely generated \( A \)-module such that \( \tilde{S}^n_{H,n,A} \otimes_A K = \tilde{S}^n_{H,n,A} \).
- \( \tilde{S}^n_{H,n,A} \) is a free \( \tilde{S}^n_{H,n,A} \)-module, \( \tilde{S}^n_{H,n,A} \) is a direct summand of finite rank, and \( \tilde{S}^n_{H,n,A} \otimes_A K = \tilde{S}^n_{H,n,A} \).
- \( m_A^i \otimes \text{id}_\mathbb{C} \) is an isomorphism.
The first claim is obvious. It implies that
\[ \dim_K(\tilde{\mathbf{SH}}^0_n) \leq \dim_C(\tilde{\mathbf{SH}}^0_{n,A} \otimes_A \mathbb{C}). \]

The two other yield the reverse inequality. Indeed, the third claim implies that \( \tilde{\mathbf{SH}}^0_{n,A} \otimes_A \mathbb{C} \) is a free \( \tilde{\mathbf{SH}}^0_{n,A} \otimes_A \mathbb{C} \)-module of rank \( \dim_C(\tilde{\mathbf{SH}}^0_{n,A} \otimes_A \mathbb{C}) \). Thus the second claim implies that \( \tilde{\mathbf{SH}}^0_n \) is a free \( \tilde{\mathbf{SH}}^0_n \)-module of rank \( \dim_C(\tilde{\mathbf{SH}}^0_{n,A} \otimes_A \mathbb{C}) \). Hence the surjectivity of \( m^i \) implies that
\[ \dim_K(\tilde{\mathbf{SH}}^0_n) \geq \dim_C(\tilde{\mathbf{SH}}^0_{n,A} \otimes_A \mathbb{C}). \]

Therefore \( m^i \) is a surjective homomorphism of projective \( \tilde{\mathbf{SH}}^0_n \)-modules of the same (finite) rank. Hence it is invertible.

The proof of the third claim is immediate. It is also easy to check that \( \tilde{\mathbf{SH}}^0_{n,A} \) is a free \( \tilde{\mathbf{SH}}^0_{n,A} \)-module, because it is a direct summand of \( \tilde{\mathbf{H}}^0_{n,A} \), the later is free over \( \tilde{\mathbf{SH}}^0_{n,A} \) by the PBW-theorem and the Steinberg-Pittie theorem, and \( \tilde{\mathbf{SH}}^0_{n,A} \) is a polynomial ring. Therefore we are reduced to check that \( \tilde{\mathbf{SH}}^0_n = \tilde{\mathbf{SH}}^0_n S \). This is proved as in [SV], Proposition 2.6. □

By Lemma 1.3 above we may construct a family of automorphisms of \( \tilde{\mathbf{SH}}^0_n \) by the assignment
\[ P^n_{0,l} \mapsto P^n_{0,l} + x_l \text{ if } l \in \mathbb{Z}^+, \quad P^n_x \mapsto P^n_x \text{ if } x \in \mathbb{Z}^+. \]

Here \( x_l \in \mathbb{K}, \ l \in \mathbb{Z}^+ \), are arbitrary. The algebra \( \tilde{\mathbf{SH}}^0_n \) is \( \mathbb{Z} \)-graded. Thus we can also construct automorphisms by the assignment
\[ P^n_{i,j} \mapsto y_1y_2P^n_{i,j} \]
for all \( i, j \). Here \( y_1, y_2 \in \mathbb{K} \) are arbitrary. Therefore there is an automorphism \( \theta \) of \( \tilde{\mathbf{SH}}^0_n \) such that
\[ \theta(P^n_{0,l}) = l^{i/2}(P^n_{0,l} - [n]_{i/2}) \text{ if } l \in \mathbb{Z}^+, \quad \theta(P^n_{i,j}) = t^{-j}P^n_{i,j} \text{ if } (i, j) \in \mathbb{Z}^+. \]

Set \( \tilde{\varphi}_n = \varphi_n \circ \theta \), a representation of \( \tilde{\mathbf{SH}}^0_n \). The discussion above implies that
\[ \tilde{\varphi}_n^{-1}(P^n_{0,1}) \circ \rho_n = \rho_n \circ \tilde{\varphi}_n(P^n_{0,1}), \quad \tilde{\varphi}_{n-1}(P^n_{1,0}) \circ \rho_n = p_1 \circ \tilde{\varphi}_n(P^n_{1,0}). \]

Since the algebra \( \tilde{\mathbf{SH}}^0_n \) is generated by the elements \( P^n_{0,l}, P^n_{1,0} \), we have \( \tilde{\varphi}_n^{-1} \circ \rho_n = \rho_n \circ \tilde{\varphi}_n \). Thus the representations \( \tilde{\varphi}_n \) yield a representation \( \tilde{\varphi}_\infty \) of \( \tilde{\mathbf{SH}}^0_\infty \) on the \( \mathbb{K} \)-vector space \( \Lambda_\mathbb{K} \). This representation is faithful because each of the \( \tilde{\varphi}_n \) is faithful. Further, it may be characterized as in the following proposition. Let \( P_\lambda(q,t^{-1}) \in \Lambda_\mathbb{K} \) be the Macdonald polynomial associated with the partition \( \lambda \).

**Proposition 1.4.** There is a unique representation \( \tilde{\varphi}_\infty \) of \( \tilde{\mathbf{SH}}^0_\infty \) on \( \Lambda_\mathbb{K} \) such that \( \tilde{\varphi}_\infty(P^n_{0,1}) = p_1 \) and such that for any partition \( \lambda \) and any integer \( l \geq 1 \) we have
\[ \tilde{\varphi}_\infty(P^n_{0,l}) \cdot P_\lambda(q,t^{-1}) = \left( \sum_i (q^{l\lambda_i} - 1)t^{l(i-1)} \right) P_\lambda(q,t^{-1}) \]
and
\[ \tilde{\varphi}_\infty(P^n_{0,1}) \cdot P_\lambda(q,t^{-1}) = q^l \left( \sum_i (q^{-l\lambda_i} - 1)t^{l(i-1)} \right) P_\lambda(q,t^{-1}). \]

This representation is faithful.
Corollary 1.5. There is a unique representation \( \tilde{\varphi} \) of \( E^\ge \) on \( \mathbb{A}_K \) such that \( \tilde{\varphi}(u_{1,0}) = p_1/(q-1) \) and such that for any partition \( \lambda \) and any integer \( l \geq 1 \) we have

\[
\tilde{\varphi}(u_{0,t}) \cdot P_\lambda(q,t^{-1}) = \left( \sum_i \frac{q^{\lambda_i} - 1}{q^i - 1} t^{i(l-1)} \right) P_\lambda(q,t^{-1})
\]

and

\[
\tilde{\varphi}(u_{0,-t}) \cdot P_\lambda(q,t^{-1}) = -\left( \sum_i \frac{q^{-\lambda_i} - 1}{q^{-i} - 1} t^{-i(l-1)} \right) P_\lambda(q,t^{-1}).
\]

This representation is faithful.

We call \( \tilde{\varphi} \) the polynomial, or standard representation of \( E^\ge \). In Section 4.7 we will extend this representation to the whole algebra \( \mathcal{E} \).

2. Hilbert schemes of \( \mathbb{A}^2 \)

This section contains some standard fact on Hilbert schemes. Most of this may be found in [ES], [G1].

2.1. Throughout the paper, our ground field will be \( \mathbb{C} \). Let \( \text{Hilb}_n \) denote the Hilbert scheme parametrizing length \( n \) subschemes of \( \mathbb{A}^2 \). By Fogarty’s theorem it is a smooth irreducible variety of dimension \( 2n \). Associating to a closed point of \( \text{Hilb}_n \) its ideal sheaf yields a bijection

\[
\text{Hilb}_n(\mathbb{C}) = \{ I \subset \mathbb{C}[x,y]; I \text{ is an ideal of codimension } n \}.
\]

We will denote by \( S = \mathbb{C}[x,y] \) the ring of regular functions on \( \mathbb{A}^2 \). The tangent space \( T_I \text{Hilb}_n \) at a closed point \( I \in \text{Hilb}_n(\mathbb{C}) \) is canonically isomorphic to the vector space \( \text{Hom}_S(I,S/I) \).

2.2. The torus \( T = \mathbb{C}^* \times \mathbb{C}^* \) acts on \( \mathbb{A}^2 \) via \((z_1, z_2) \cdot (x,y) = (z_1 x, z_2 y)\). There is an induced action on \( S \) given by \((z_1, z_2) \cdot P(x,y) = P(z_1^{-1} x, z_2^{-1} y)\) and one on \( \text{Hilb}_n \) such that

\[
(z_1, z_2) \cdot I = \{ P(z_1^{-1} x, z_2^{-1} y); P(x,y) \in I \}, \quad \forall I \in \text{Hilb}_n(\mathbb{C}).
\]

This action has a finite number of isolated fixed points, indexed by the set of partitions of the integer \( n \). To such a partition \( \lambda \vdash n \) corresponds the fixed point \( I_\lambda \) where

\[
I_\lambda = \bigoplus_{s \not\in \lambda} \mathbb{C} x^{i(s)-1} y^{j(s)-1}.
\]

When \( I = I_\lambda \) is a \( T \)-fixed point, there is an induced \( T \)-action on \( T_I \text{Hilb}_n \).

In order to describe this action, we fix a few notations concerning \( T \). Let \( R = R_T \) denote the complexified representation ring of \( T \). We have \( R = \mathbb{C}[q^{\pm 1}, t^{\pm 1}] \) where

\[
q : T \to \mathbb{C}^*, (z_1, z_2) \mapsto z_1^{-1}, \quad t : T \to \mathbb{C}^*, (z_1, z_2) \mapsto z_2^{-1}.
\]

For \( V \) a \( T \)-module let \([V]\) be its class in \( R \). We abbreviate \( T_\lambda = [T_I \text{Hilb}_n] \). It is given by

\[
T_\lambda = \sum_{s \in \lambda} (t^{i(s)} q^{-a(s)-1} + t^{-l(s)-1} q^{a(s)}).
\]
2.3. Let $\Theta_n \subset \text{Hilb}_n \times \mathbb{A}^2$ be the universal family and let $p : \text{Hilb}_n \times \mathbb{A}^2 \to \text{Hilb}_n$ be the projection. The tautological bundle of $\text{Hilb}_n$ is the locally free sheaf $\pi_\ast \mathcal{O}_{\Theta_n}$. The fiber of $\pi_\ast$ at a point $I \in \text{Hilb}_n(\mathbb{C})$ is $S/I$. The character of the $T$-action on its fiber at the fixed point $I_\lambda$ is
\begin{equation}
\tau_\lambda = [\tau_n|_{I_\lambda}] = \sum_{s \in \lambda} t^{j(s)-1} q^{i(s)-1}.
\end{equation}

2.4. Let $k \geq 0$. The nested Hilbert scheme $Z_{n,n+k}$ is the reduced closed subscheme of $\text{Hilb}_{n+k}$ parametrizing pairs of ideals $(I,J)$ where $J \subset I$. One defines the nested Hilbert scheme $\text{Hilb}_{n+k,n}$ in a similar fashion. Of course $Z_{n,n+k}$ is simply the diagonal of $\text{Hilb}_{n+k}$, The schemes $Z_{n,n+k}$ are singular in general for $k \geq 2$, but they are smooth if $k = 0$ or $k = 1$, see [CI]. The tangent space at a point $(I,J) \in Z_{n,n+k}$ is the kernel of the natural map
\begin{equation}
\psi : \text{Hom}_S(I,S/I) \oplus \text{Hom}_S(J,S/J) \to \text{Hom}_S(J,S/I).
\end{equation}

When $k = 1$ the map $\psi$ is surjective.

The diagonal $T$-action on $\text{Hilb}_n \times \text{Hilb}_{n+k}$ preserves $Z_{n,n+k}$. The fixed points contained in $Z_{n,n+k}$ are those pairs $I_{\mu,\lambda} = (I_{\mu}, I_\lambda)$ for which $\mu \subset \lambda$. When $k = 1$ this may be used to give a cell decomposition of $\text{Hilb}_{n+k+1}$, but we won’t need this here. We abbreviate $T_{\mu,\lambda} = [T_{I_{\mu,\lambda}} \text{Hilb}_{n+k+1}]$. We may use (2.3) to write a formula for $T_{\mu,\lambda}$
\begin{equation}
T_{\mu,\lambda} = [\text{Hom}_S(I_{\mu},S/I_{\mu})] + [\text{Hom}_S(I_{\lambda},S/I_{\lambda})] - [\text{Hom}_S(I_{\lambda},S/I_{\mu})] = T_{\mu} + T_{\lambda} - N_{\mu,\lambda},
\end{equation}
where the character of the fiber of the normal bundle $N_{\mu,\lambda} = [N_{I_{\mu,\lambda},\text{Hilb}_{n+k+1}]$ is equal to
\begin{equation}
N_{\mu,\lambda} = \sum_{s \in \mu} \left(t^{l_{\mu}(s)} q^{-a_{\lambda}(s)-1} + t^{-l_{\lambda}(s)-1} q^{a_{\mu}(s)}\right).
\end{equation}

Of course, similar formulas hold for the nested Hilbert scheme $Z_{n+1,n}$.

2.5. Let $\pi_1, \pi_2$ be the natural projections of $\text{Hilb}_n \times \text{Hilb}_{n+1}$ to $\text{Hilb}_n$ and $\text{Hilb}_{n+1}$ respectively. Over $Z_{n,n+1}$ there is a natural surjective map $\phi : \pi_2^\ast (\tau_{n+1}) \to \pi_1^\ast (\tau_n)$. Over the point $(I,J) \in Z_{n,n+1}$ it specializes to the map $S/J \to S/I$. The kernel sheaf $\text{Ker}(\phi)$ is a line bundle, which we call the tautological bundle of $Z_{n,n+1}$ and which we denote by $\tau_{n,n+1}$. Over a $T$-fixed point $I_{\mu,\lambda}$ its character is
\begin{equation}
\tau_{\mu,\lambda} = [\tau_{n+1}|_{I_{\mu,\lambda}}] = t^{j(s)-1} q^{i(s)-1}
\end{equation}
where $s = \lambda \setminus \mu$ is the unique box of $\lambda$ not contained in $\mu$. Write
\begin{equation}
\tau_{l,n,n+1} = (\tau_{n,n+1})^{\otimes l}, \quad \tau_{-l,n,n+1} = (\tau_{n,n+1}^\ast)^{\otimes l}, \quad l > 0.
\end{equation}

2.6. Let $\pi_1, \pi_2$ be the natural projections of $\text{Hilb}_n \times \text{Hilb}_n$ to $\text{Hilb}_n$. Over $Z_{n,n}$ we have the vector bundle $\tau_{n,n} = \pi_2^\ast (\tau_n) = \pi_1^\ast (\tau_n)$. We call it the tautological bundle of $\text{Hilb}_{n,n}$. Over a $T$-fixed point $I_{\lambda,\lambda}$ its character is $\tau_{\lambda,\lambda} = \tau_{\lambda}$.

3. CONVOLUTION ALGEBRAS IN K-theory

3.1. Let $G$ be a complex linear algebraic group. By a $G$-variety we’ll mean a quasi-projective complex variety with an action of the algebraic group $G$. Let $K^G(X)$ be the complexified Grothendieck group of the category of $G$-equivariant coherent sheaves on $X$. It is a $R_G$-module, where $R_G$ is the representation ring of $G$. We will usually denote the class of a $G$-equivariant sheaf $\mathcal{F}$ by the symbol $[\mathcal{F}]$. If $X' \subset X$ is a $G$-stable closed subvariety let $[X'] \in K^G(X)$ stand
for the class of the structure sheaf of \( X' \). If \( X' = \{ x \} \), a closed point of \( X \), we abbreviate \([x] = [\{x\}]\).

Let \( X_1, X_2, X_3 \) be smooth quasi-projective \( G \)-varieties admitting proper maps to a (possibly singular) quasi-projective \( G \)-variety \( Y \). Let \( p_{12}, p_{13}, p_{23} \) be the projections from the triple fiber product \( X_1 \times_Y X_2 \times_Y X_3 \) along the factor not named. There is a natural map
\[
\ast : K^G(X_1 \times_Y X_2) \otimes K^G(X_2 \times_Y X_3) \to K^G(X_1 \times_Y X_3), \quad (w, z) \mapsto R_{p_{13}^*} \left( p_{12}^*(w) \otimes L p_{23}^*(z) \right).
\]

If \( X_1 = X_2 = X_3 = X \) this map endows \( K^G(X \times_Y X) \) with the structure of an associative \( R_G \)-algebra with unit given by \([\Delta_X] \) where \( \Delta_X \subseteq X \times_Y X \) is the diagonal. Now assume in addition that \( Y = \{ pt \} \). The map \( \ast \) with \( X_1 = X_2 = X \) and \( X_3 = \{ pt \} \) endows \( K^G(X) \) with the structure of a \( K^G(X \times X) \)-module.

### 3.2. We apply this formalism to \( X \) equal to the union of Hilbert schemes \( \text{Hilb} = \bigsqcup_{n \geq 0} \text{Hilb}_n \), and \( G = T, \ Y = \{ pt \} \). This is not directly possible since \( \text{Hilb}_n \) is not proper if \( n > 0 \). We may get around this difficulty however by first localizing to the \( T \)-fixed loci, which consists (for each \( n \)) of a finite number of points. For any \( R \)-module \( M = M_R \) we write \( M_K = M_R \otimes_R K \). Recall that \( K \) is identified with \( \mathbb{C}(q^{1/2}, t^{1/2}) \) by (1.1). The direct image provides us with an isomorphism
\[
(3.1) \quad i_* : K^T(\text{Hilb}_T^n) \otimes_R K = \bigoplus_{\lambda \vdash n} K[I_\lambda] \xrightarrow{\sim} K^T(\text{Hilb}_n) \otimes_R K
\]
where \( i : \text{Hilb}_T^n \to \text{Hilb}_n \) is the embedding. We have also
\[
K^T(\text{Hilb}_n \times \text{Hilb}_m) \otimes_R K = \bigoplus_{\lambda \vdash n, \mu \vdash m} K[I_{\lambda, \mu}].
\]

By (3.1) each element in \( K^T(\text{Hilb}_n) \) or \( K^T(\text{Hilb}_n \times \text{Hilb}_m) \) is a linear combination of classes of coherent sheaves with proper support. This allows us to define convolution operations
\[
\ast : K^T(\text{Hilb}_n) \otimes_K K^T(\text{Hilb}_m) \to K^T(\text{Hilb}_n \times \text{Hilb}_m)_K
\]
and
\[
\ast : K^T(\text{Hilb}_n \times \text{Hilb}_m)_K \otimes_K K^T(\text{Hilb}_m)_K \to K^T(\text{Hilb}_n)_K.
\]
We consider the following associative \( K \)-algebra
\[
(3.2) \quad E_K = \bigoplus_{n \in \mathbb{Z}} \prod_{k \geq n} K^T(\text{Hilb}_{n+k} \times \text{Hilb}_n)_K
\]
where the product ranges over all integers \( n \) for which \( n \geq 0 \) and \( n + k \geq 0 \). It acts on the \( K \)-vector space
\[
L_K = \bigoplus_{n \geq 0} K^T(\text{Hilb}_n)_K.
\]
We will denote this representation as \( \psi \). The integer \( k \) in (3.2) yields a \( \mathbb{Z} \)-grading on the \( K \)-algebra \( E_K \). There is also an obvious \( \mathbb{Z} \)-grading on the \( K \)-vector space \( L_K \) such that the action of \( E_K \) on \( L_K \) is compatible with these gradings. The representation \( \psi \) is faithful.

### 3.3. The \( K \)-vector space \( L_K \) is spanned by the elements \([I_\lambda] \), \( \lambda \in \Pi \), and \( E_K \) is its endomorphism ring. Following [H] we may identify \( L_K \) with \( A_K \) via the map
\[
L_K \xrightarrow{\sim} A_K, \quad [I_\lambda] \mapsto \tilde{H}_\lambda(q, t).
\]
Here \( \tilde{H}_\lambda(q, t) \) is the cocharge Macdonald polynomial. It is a renormalization of Macdonald’s original polynomials \( P_\lambda(q, t) \). The precise relation is as follows. Let \( \gamma_t : A_K \to A_K \) be the unique \( K \)-algebra homomorphism such that \( \gamma_t(p_r) = (1 - t^r)p_r \). Then
\[
(3.4) \quad t^{-\lambda(\lambda)} c_\lambda(q, t^{-1}) \cdot P_\lambda(q, t^{-1}) = \gamma_t(\tilde{H}_\lambda(q, t))
\]

Here \( \tilde{H}_\lambda(q, t) \) is the cocharge Macdonald polynomial. It is a renormalization of Macdonald’s original polynomials \( P_\lambda(q, t) \). The precise relation is as follows. Let \( \gamma_t : A_K \to A_K \) be the unique \( K \)-algebra homomorphism such that \( \gamma_t(p_r) = (1 - t^{r^2})p_r \). Then
\[
(3.4) \quad t^{-\lambda(\lambda)} c_\lambda(q, t^{-1}) \cdot P_\lambda(q, t^{-1}) = \gamma_t(\tilde{H}_\lambda(q, t))
\]
where
\[ n(\lambda) = \sum_i (i-1)\lambda_i, \quad c_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{\alpha(s)}t^{l(s)+1}). \]

Under the identification\footnote{33} of elements of \( E_K \), elements appearing as \( q, t \)-operators on symmetric functions. For any partitions \( \mu, \lambda \) and any operator \( z \) on \( L_K \) we denote by \( \langle \mu, z \cdot \lambda \rangle \) the coefficient of \( H_\mu(q, t) \) in \( z(\hat{H}_\lambda(q, t)) \).

3.4. We would ideally like to understand the action on \( L_K \) of the classes of the nested Hilbert schemes \( Z_{n, n+k} \) and of their tautological bundles, and abstractly describe the algebra which these generate. It is not clear, however, how to deal with the classes of \( Z_{n, n+k} \) when \( |k| > 1 \) since these varieties are singular. Consider the \( K \)-subalgebra \( H_K \subset E_K \) generated by
\[ f_{-1,l} = \prod_n \tau_{n,n+1}^l, \quad f_{1,l} = \prod_n \tau_{n+1,n}^l, \quad l \in \mathbb{Z}, \]
\[ e_{0,l} = \prod_n \Lambda^l \tau_{n,n}, \quad e_{0,-l} = \prod_n \Lambda^l \tau^*_n, \quad l \in \mathbb{Z}_{>0}. \]

It acts on the \( K \)-vector space \( L_K \cong A_K \) in the obvious way. Note that here, to unburden the notation, we have abbreviated
\[ \Lambda^{\pm l} \tau_{n,n} = [\Lambda^{\pm l}, \tau_{n,n}], \quad \tau_{n+1,n} = [\tau_{n+1,n}], \quad \tau_{n,n+1} = [\tau_{n,n+1}]. \]

Here the brackets denote the classes in the Grothendieck groups \( \text{Hilb}_n \), \( \text{Hilb}_{n+1} \times \text{Hilb}_n \) and \( \text{Hilb}_n \times \text{Hilb}_{n+1} \) respectively. For a future use, we define another class of elements \( f_{0,l} \in E_K \) for \( l \in \mathbb{Z}^* \) through the relations
\[ \sum_{l \geq 1} (-1)^l f_{0,\pm l} s^{l-1} = -\frac{d}{ds} \log(E_\pm(s)), \quad E_\pm(s) = 1 + \sum_{k \geq 1} (-1)^k e_{0,\pm k} s^k. \]

So \( f_{0,l} \) is obtained from the classes of the tautological bundles \( \tau_{n,n} \) by the Adams operations
\[ f_{0,l} = \prod_n \Psi_l(\tau_{n,n}). \]

We can now state our first result. Recall the algebra \( \mathcal{E}_c \) associated with the central charge \( c = (1, q^{1/2} t^{1/2}) \).

For \( l \in \mathbb{Z}, n \in \mathbb{N} \) we consider the following elements of \( \mathcal{E}_c \)
\[ h_{1,l} = t^{1/2} f_{1,l-1}, \quad h_{-1,k} = -q^{1/2} f_{-1,k}, \]
\[ h_{0,n} = f_{0,n} - \frac{1}{(1 - q^n)(1 - t^n)}, \]
\[ h_{0,\pm n} = f_{0,\pm n} + \frac{1}{(1 - q^{-n})(1 - t^{-n})}. \]

The following Theorem is proved in the next section.

**Theorem 3.1.** There is an isomorphism of \( K \)-algebras
\[ \Omega : \mathcal{E}_c \cong H_K, \quad u_{i,l} \mapsto h_{i,l}, \quad \forall i = -1, 0, 1, \ l \in \mathbb{Z}. \]

4. Construction of the isomorphism

Our proof of Theorem 3.1 is based on the characterization of \( \hat{E} \) given in Proposition 1.4 and on the faithful polynomial representation \( \varphi \) of \( \mathcal{E}^\vee \) given in Section 1.4.
4.1. We begin by computing the action of the class $[I_{\lambda,\mu}]$ on $K^T(\text{Hilb}_n)$. For any $T$-equivariant vector bundle $\mathcal{V}$ on a smooth $T$-variety $X$ we set

$$\Lambda[\mathcal{V}] = \sum_{i \geq 0} (-1)^i [\Lambda^i(\mathcal{V})],$$

where $\Lambda^i$ is the usual wedge power. This operation descends to a well-defined morphism

$$\Lambda : K^T(X) \to K^T(X)$$

which satisfies $\Lambda(x + y) = \Lambda(x) \cdot \Lambda(y)$. Recall that if $\mathcal{V}$ is a $T$-equivariant vector bundle on $X$ and $x \in X$ a $T$-fixed point then $[\mathcal{V}|_x] \in R$ is the character of the fiber of $\mathcal{V}$ over $x$, as a $T$-module. Again, this descends to the Grothendieck group, yielding the pullback morphism $K^T(X) \to K^T(\{x\}) = R$. The following lemma is well-known.

**Lemma 4.1.** (a) For partitions $\lambda \vdash n$ and $\mu, \nu \vdash m$ we have

$$[I_{\lambda,\mu}] \star [I_{\nu}] = \delta_{\mu,\nu} \cdot \Lambda(T^*_\nu) \cdot [I_\lambda].$$

(b) For any $T$-equivariant vector bundle $\mathcal{V}$ on $\text{Hilb}_n$ we have in $K^T(\text{Hilb}_n)$

$$[\mathcal{V}] = \sum_\lambda \Lambda(T^*_\lambda)^{-1} \cdot [\mathcal{V}|_\lambda] \cdot [I_\lambda].$$

For any partitions $\lambda, \mu$ we abbreviate $\Lambda_\lambda = \Lambda(T^*_\lambda)$ and $\Lambda_{\lambda,\mu} = \Lambda(T^*_\lambda) \boxtimes \Lambda(T^*_\mu)$. As a corollary of the above lemma, we get the following

**Corollary 4.2.** For any $r \in \mathbb{Z}$ it holds

$$[\tau^r_{n,n+1}]= \sum_{\mu \subseteq \lambda} \tau^r_{\mu,\lambda} \cdot \Lambda(N^*_\mu,\lambda) \cdot \Lambda^{-1}_{\mu,\lambda} \cdot [I_{\mu,\lambda}],$$

$$[\tau^r_{n+1,n}]= \sum_{\mu \subseteq \lambda} \tau^r_{\lambda,\mu} \cdot \Lambda(N^*_\lambda,\mu) \cdot \Lambda^{-1}_{\lambda,\mu} \cdot [I_{\lambda,\mu}],$$

$$[\tau^r_{n,n}]= \sum_{\lambda \vdash n} \tau^r_{\lambda,\lambda} \cdot \Lambda^{-1}_{\lambda,\lambda} \cdot [I_{\lambda,\lambda}],$$

where the sums range over all pairs of partitions $\lambda \subseteq \mu$ such that $|\mu| = n$, $|\lambda| = n + 1$.

4.2. We now write explicit formulas for the action of the elements $f_{\pm,d}, f_{0,d}$ of $H_K$ on the $K$-vector space $\Lambda_K$. By (2.7) we have

$$\Lambda_\lambda = \prod_{s \in \lambda} (1 - t^{-l(s)}q^{a(s)+1})(1 - t^{l(s)+1}q^{-a(s)}).$$

For $\mu \subseteq \lambda$, by (2.8) we have

$$\Lambda(N^*_\mu,\lambda) = \prod_{s \in \mu} (1 - t^{-l_s(s)}q^{a_s(s)+1})(1 - t^{l_s(s)+1}q^{-a_s(s)}) = \Lambda(N^*_\mu,\lambda).$$

Abbreviate $x(s) = i(s) - 1$ and $y(s) = j(s) - 1$. We have also

$$\tau^r_{\mu,\lambda} = \tau^r_{\lambda,\mu} = t^{r \cdot y(\lambda \mu)} q^{r \cdot x(\lambda \mu)}, \quad \tau^r_{\mu,\mu} = (\sum_{s \in \mu} t^{y(s)}q^{x(s)})^r.$$
Lemma 4.3. (a) For any \( r \in \mathbb{Z} \) and any partition \( \nu \) we have
\[
f_{1,r} \cdot \tilde{H}_\nu(q,t) = \frac{1}{(1-q)(1-t)} \sum_{\mu \supset \nu} q^{(r+1)\chi(\mu\setminus\nu)} t^{(r+1)\rho(\mu\setminus\nu)} L_{\nu,\mu}(q,t) \tilde{H}_\mu(q,t),
\]
where the sum ranges over all \( \mu \supset \nu \) with \(|\mu| = |\nu| + 1\), and
\[
L_{\nu,\mu}(q,t) = \prod_{s \in C_{\mu\setminus\nu}} \frac{t_{\nu}(s) - q_{\nu}(s)+1}{t_{\nu}(s) - q_{\nu}(s)} \prod_{s \in B_{\nu\setminus\mu}} \frac{t_{\nu}(s) + 1 - q_{\nu}(s)}{t_{\nu}(s) - q_{\nu}(s)}.
\]
(b) For any \( r \in \mathbb{Z} \) and any partition \( \nu \) we have
\[
f_{-1,r} \cdot \tilde{H}_\nu(q,t) = \sum_{\lambda \subset \nu} q^{\chi(\nu\setminus\lambda)} t^{\rho(\nu\setminus\lambda)} L_{\nu,\lambda}(q,t) \tilde{H}_\lambda(q,t),
\]
where the sum ranges over all \( \lambda \subset \nu \) with \(|\lambda| = |\nu| - 1\), and
\[
L_{\nu,\lambda}(q,t) = \prod_{s \in C_{\nu\setminus\lambda}} \frac{t_{\nu}(s) + 1 - q_{\nu}(s)}{t_{\nu}(s) - q_{\nu}(s)} \prod_{s \in B_{\nu\setminus\lambda}} \frac{t_{\nu}(s) - q_{\nu}(s)}{t_{\nu}(s) - q_{\nu}(s)}.
\]

In particular we have the following formulas, see [GT], Theorem 1.4.

Corollary 4.4. The following relations holds in \( \text{End}_K(A_K) \)
\[
f_{1,-1} = \frac{1}{(1-q)(1-t)} p_1, \quad f_{-1,0} = \frac{\partial}{\partial p_1}.
\]

The action of \( f_{0,r} \) is given by the following formula, compare [M], I. (2.10').

Lemma 4.5. For any \( r \in \mathbb{Z}^* \) and any partition \( \nu \) we have
\[
f_{0,r} \cdot \tilde{H}_\nu(q,t) = \left( \sum_{s \in \nu} q^{\chi(\nu)} t^{\rho(\nu)} \right) \tilde{H}_\nu(q,t).
\]

4.3. Next, we check that the generators \( f_{1,r}, f_{0,l} \) of \( H_K \) satisfy relations \([1.3], [1.4]\).

Proposition 4.6. For any \( l, k \in \mathbb{Z}^* \) we have \([f_{0,l}, f_{0,k}] = 0\).

Proof. This is obvious since the convolution product of two classes supported on the diagonal in \( \text{Hilb} \times \text{Hilb} \) is their tensor product. \(\square\)

Proposition 4.7. For any \( l, k \in \mathbb{Z}^* \) we have \([f_{0,l}, f_{1,k}] = \pm f_{1,l+k}\).

Proof. For any partition \( \lambda \) we set \( B^l_\lambda(q,t) = \sum_{s \in \lambda} q^{\chi(s)} t^{\rho(s)} \). Then we have
\[
f_{0,l} f_{1,k} \cdot \tilde{H}_\nu(q,t) = \frac{1}{(1-q)(1-t)} \sum_{\mu \supset \nu} B^l_\mu(q,t) q^{(r+1)\chi(\mu\setminus\nu)} t^{(r+1)\rho(\mu\setminus\nu)} L_{\nu,\mu}(q,t) \tilde{H}_\mu(q,t),
\]
while
\[
f_{1,k} f_{0,l} \cdot \tilde{H}_\nu(q,t) = \frac{1}{(1-q)(1-t)} \sum_{\mu \supset \nu} B^l_\mu(q,t) q^{(r+1)\chi(\mu\setminus\nu)} t^{(r+1)\rho(\mu\setminus\nu)} L_{\nu,\mu}(q,t) \tilde{H}_\mu(q,t).
\]
Substituting the relation
\[
B^l_\mu(q,t) - B^l_\nu(q,t) = q^{\chi(s)} t^{\rho(s)}
\]
for \( s = \mu \setminus \nu \) we obtain \([f_{0,l}, f_{1,k}] = f_{1,l+k}\). The second part of the proposition is identical. \(\square\)
4.4. Using Corollary 4.4 and Lemma 4.5 we may now compare the representation

$$\psi: H_K \to \text{End}_K(L_K)$$

defining Section 3.2 with the representation

$$\tilde{\varphi}: E^\gg \to \text{End}_K(A_K)$$

doing Corollary 1.5, under the isomorphism $L_K \cong A_K$ in (3.3). Recall the plethystic substitution $\gamma_t \in \text{End}_K(A_K)$ of Section 3.3. By (3.1) we have

$$\gamma_t(\tilde{H}_\lambda(q,t)) = u_\lambda(q,t)P_\lambda(q,t^{-1})$$

for some scalar $u_\lambda(q,t)$. Using Corollary 1.5 and Lemma 4.5 we get, for each $n \geq 1$

$$\tilde{\varphi}(u_{0,n}) \circ \gamma_t(\tilde{H}_\lambda(q,t)) = \left( \sum_{s \in \lambda} q^{nx(s)t^{ny(s)}} u_\lambda(q,t)P_\lambda(q,t^{-1}) \right)$$

$$= \left( \sum_{s \in \lambda} q^{nx(s)t^{ny(s)}} \gamma_t(\tilde{H}_\lambda(q,t)) \right)$$

$$= \gamma_t \circ (\psi(f_{0,n}))(\tilde{H}_\lambda(q,t))$$

and

$$\tilde{\varphi}(u_{0,-n}) \circ \gamma_t(\tilde{H}_\lambda(q,t)) = \left( -\sum_{s \in \lambda} q^{-nx(s)t^{-ny(s)}} u_\lambda(q,t)P_\lambda(q,t^{-1}) \right)$$

$$= \left( -\sum_{s \in \lambda} q^{-nx(s)t^{-ny(s)}} \gamma_t(\tilde{H}_\lambda(q,t)) \right)$$

$$= -\gamma_t \circ (\psi(f_{0,-n}))(\tilde{H}_\lambda(q,t)).$$

In addition, we have

$$\tilde{\varphi}(u_{1,0}) \circ \gamma_t = \frac{p_1}{(q-1)} \circ \gamma_t = \gamma_t \circ \frac{p_1}{(q-1)(1-t)} = -\gamma_t \circ \psi(f_{1,-1}).$$

Now, let $H_K^{\gg} \subset H_K$ be the subalgebra generated by $f_{1,-1}$ and the elements $f_{0,n}$ for $n \in \mathbb{Z}^*$. By Proposition 1.7 the elements $f_{1,n}$ belong to $H_K^{\gg}$ for all $n \in \mathbb{Z}$. Because the representations $\psi$, $\tilde{\varphi}$ are both faithful and because the $K$-algebras $H_K^{\gg}$ and $E^\gg$ are respectively generated by $f_{1,-1}$, $f_{0,l}$, $l \in \mathbb{Z}^*$ and $u_{1,0}$, $u_{0,l}$, $l \in \mathbb{Z}^*$ we deduce from the above formulas that the assignment

$$u_{1,0} \mapsto -f_{1,-1}, \quad u_{0,n} \mapsto f_{0,n}, \quad u_{0,-n} \mapsto -f_{0,-n}, \quad n \geq 1$$

extends to an isomorphism $E^\gg \to H_K^{\gg}$. Twisting by automorphisms (1.11), (1.12) we see that the assignment

$$u_{1,0} \mapsto t^{1/2}f_{1,-1}, \quad u_{0,n} \mapsto f_{0,n} - \frac{1}{(1-q^n)(1-t^n)}, \quad u_{0,-n} \mapsto -f_{0,-n} + \frac{1}{(1-q^{-n})(1-t^{-n})}$$

also extends to an isomorphism of algebras $E^\gg \to H_K^{\gg}$. In other words, restricting the map $\Omega$ in Theorem 3.1 we get an isomorphism $E^\gg \to H_K^{\gg}$. In the same way we prove that $\Omega$ restricts also to an isomorphism $E^\ll \to H_K^{\ll}$.

4.5. We have just proved that $\Omega$ restricts to $K$-algebra isomorphisms $E^\gg \to H_K^{\gg}$ and $E^\ll \to H_K^{\ll}$. By Proposition 1.1 these two morphisms extend to the whole of $E_c$ if and only if relation (1.5) holds with the appropriate specialization of the center. The proof of this fact, which requires somewhat involved computations, is given in Appendix A.1.
4.6. Let $H_K^∞$, $H_K^0$, $H_K^< \subset H_K$ be the subalgebras generated by the elements $f_x$ with $x \in Z^1, Z^0, Z^{-1}$ respectively. By Sections 4.4 and 4.5 there is a well-defined surjective algebra homomorphism $\Omega : E_e \rightarrow H_K$ which restricts to isomorphisms $E_e^> \rightarrow H_K^>$, $E_e^0 \rightarrow H_K^0$, and $E_e^< \rightarrow H_K^<$. Recall the triangular decomposition $E_e \simeq E_e^> \otimes E_e^0 \otimes E_e^<$. Thus to prove that the map $\Omega$ is injective it is enough to prove the following result.

**Proposition 4.8.** The algebra $H_K$ has a triangular decomposition, i.e., the multiplication map induces an isomorphism

$m : H_K^> \otimes H_K^0 \otimes H_K^< \rightarrow H_K$.

**Proof.** Since $\Omega$ is surjective and $E_e$ has a triangular decomposition, the multiplication map

$m : H_K^> \otimes H_K^0 \otimes H_K^< \rightarrow H_K$ is onto. We will now prove that it is also injective. The basic idea is to mimick the construction of a coproduct on $H_K$.

Let us argue by contradiction and let $x = \sum P_i \otimes R_i \otimes Q_i$ be a nonzero homogeneous element in $\text{Ker}(m)$. We may assume that the $R_i = R_i(f_{i,\pm 1}, f_{i,\pm 2}, \ldots)$ are linearly independent polynomials in the variables $f_{i,l}$, $l \in Z$, and that $P_i$ and $Q_i$ are nonzero. Multiplying by an element of $H_K^>$ or $H_K^<$ if necessary, we may also assume that $\deg(x) = 0$. By definition, we have

\[
(4.2) \quad \sum_i P_i \circ R_i \circ Q_i \cdot \tilde{H}_\lambda(q,t) = 0
\]
for all partitions $\lambda$. We will apply (4.2) to a certain (asymptotic) kind of partition. Given partitions $\lambda_1, \lambda_2, \ldots, \lambda_k$, and given an integer $n \gg |\lambda_1|, \ldots, |\lambda_k|$ we let $(\lambda_1, \ldots, \lambda_k)_n$ stand for the following partition.

![Figure 2. An asymptotic partition $(\lambda_1, \ldots, \lambda_k)_n$](image)

Note that $(\lambda_1, \ldots, \lambda_k)_n$ is well-defined as soon as $n > \sup_l (l(\lambda_i), l(l_i'))$. Put

$r = \sup_i (\deg(P_i)) = \sup_i (-\deg(Q_i)).$

Recall that for partitions $\nu, \gamma$ and $z$ an operator on $L_K$ we denote by $\langle \nu, z \cdot \gamma \rangle$ the coefficient of $H_{\nu}(q,t)$ in $z(H_{\gamma}(q,t))$. For $n \gg 0$ we consider the coefficients

\[
\langle \langle \lambda^#, \gamma, \mu^# \rangle_n, P_i R_i Q_i \cdot (\lambda, \gamma, \mu)_n \rangle, \quad \lambda^# \subset \lambda, \quad \mu \subset \mu^#, \quad |\lambda \setminus \lambda^#| = |\mu^# \setminus \mu| = r.
\]

Since the $Q_i$ are annihilation operators while the $P_i$ are creation operators, and because $r$ is maximal, the only way to obtain $\langle \lambda^#, \gamma, \mu^# \rangle_n$ from $(\lambda, \gamma, \mu)_n$ is to use all of $Q_i$ to reduce $\lambda$ to

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4The existence of this coproduct is a consequence of the identification $H_K \simeq E$. 

\(\lambda^\#\) and to use all of \(P_t\) to increase \(\mu\) to \(\mu^\#\). Therefore we have
\[
(4.3) \quad \langle (\lambda^\#, \gamma, \mu^\#)_{(\cdot)} , P_t R_t Q_t \cdot (\lambda, \gamma, \mu)_{(\cdot)} \rangle =
\]
\[
= \langle (\lambda^\#, \gamma, \mu^\#)_{(\cdot)} , P_t \cdot (\lambda^\#, \gamma, \mu^\#)_{(\cdot)} \rangle \langle (\lambda^\#, \gamma, \mu)_{(\cdot)} , R_t \cdot (\lambda^\#, \gamma, \mu)_{(\cdot)} \rangle \langle (\lambda^\#, \gamma, \mu)_{(\cdot)} , Q_t \cdot (\lambda, \gamma, \mu)_{(\cdot)} \rangle.
\]
Note that (4.3) is equal to zero unless \(\text{deg}(P_t) = - \text{deg}(Q_t) = r\).

Next, we define automorphisms \(\tau \in \text{Aut}(H_K^\gamma)\), \(\rho \in \text{Aut}(H_K^\gamma)\) by
\[
\tau(f_{-1,k}) = t^k f_{-1,k}, \quad \rho(f_{t, i}) = q^i f_{t, i}, \quad \forall k, l \in \mathbb{Z}.
\]

The existence of \(\tau, \rho\) is a consequence of the isomorphisms
\[
H_K^\gamma \simeq \mathcal{E}^+, \quad H_K^\gamma \simeq \mathcal{E}^-.
\]

**Lemma 4.9.** There are constants \(c = c(\lambda, \lambda^\#, \gamma, \mu, \mu^\#)\) and \(d = d(\lambda^\#, \gamma, \mu, \mu^\#, \mu)\) such that
\[
(4.4) \quad \langle (\lambda, \gamma, \mu)_{(\cdot)} , Q \cdot (\lambda, \gamma, \mu)_{(\cdot)} \rangle = c(\lambda^\#, \gamma, \mu^\#) \cdot \lambda^2\gamma,
\]
\[
(4.5) \quad \langle (\lambda^\#, \gamma, \mu^\#)_{(\cdot)} , P \cdot (\lambda^\#, \gamma, \mu^\#)_{(\cdot)} \rangle = d(\mu^\#, \mu^2\gamma) \cdot \mu.
\]

**Proof.** We prove the first statement only, the second one is identical. If \(Q = f_{-1,k_1} \cdots f_{-1,k_i}\) then
\[
\langle (\lambda^\#, \gamma, \mu)_{(\cdot)} , Q \cdot (\lambda, \gamma, \mu)_{(\cdot)} \rangle = \sum \left( t^{\sum_{j=1}^i j} q^{\sum_{j=1}^i j} \cdot \prod_{j=1}^r L_{(\lambda_j, \gamma, \mu)_{n}, (\lambda_{j+1}, \gamma, \mu)_{n}}(q_t, t) \right).
\]

Here we have
- the sum runs over all sequences \(\lambda = \lambda_1 \supseteq \lambda_2 \supseteq \cdots \supseteq \lambda_{r+1} = \lambda^\#\) and \(s_i = \lambda_i \setminus \lambda_{i+1}\),
- for partitions \(\alpha \supset \beta\) with \(|\alpha| = |\beta| + 1\) we have
\[
L_{\alpha, \beta}(q, t) = \prod_{s \in C_{\alpha \setminus \beta}} \frac{t^{l_{\alpha}(s)+1} - q^{a_{\alpha}(s)}}{t^{l_{\alpha}(s)} - q^{a_{\alpha}(s)}} \cdot \prod_{s \in R_{\alpha \setminus \beta}} \frac{t^{l_{\alpha}(s)} - q^{a_{\alpha}(s)+1}}{t^{l_{\alpha}(s)} - q^{a_{\alpha}(s)}}.
\]

For \(s_j\) a box in \(\lambda_j\) we have \(x(s_j) = x(s_j)\) and \(y(s_j) = y(s_j) = 2\gamma\), where \(x\) and \(y\) denote the coordinate values when we place the origin at the bottom left corner of \(\lambda\), i.e., at the point \((0, 2n)\), as opposed to the coordinate values when the origin is at the bottom left corner of \((\lambda, \gamma, \mu)\).

Similarly we have, for the row and columns of a box \(s_j\) in \(\lambda_j\)
\[
R(s_j) = R_{\lambda_j}(s_j),
\]
\[
C(s_j) = C_{\lambda_j}(s_j) \cup C'(s_j),
\]
where \(C'(s_j) = \{(x(s_j), 0), \ldots, (x(s_j), 2n-1)\}\). Finally, observe that the armlength \(a(u)\) or the leglength \(l(u)\) of a box \(u \in \lambda_j\) are the same whether we consider \(u\) as belonging to \(\lambda_j\) or to \((\lambda_j, \gamma, \mu)\) as the same whether we consider \(u\) as belonging to \(\lambda_j\) or to \((\lambda_j, \gamma, \mu)\).

From the above formulae we deduce that
\[
t^{\sum_{j=1}^i j} q^{\sum_{j=1}^i j} \cdot \prod_{j=1}^r L_{(\lambda_j, \gamma, \mu)_{n}, (\lambda_{j+1}, \gamma, \mu)_{n}}(q_t, t) =
\]
\[
= t^{\sum_{j=1}^i j} q^{\sum_{j=1}^i j} \cdot \prod_{j=1}^r L_{\lambda_j, \lambda_{j+1}}(q_t, t) \cdot t^{2n} \sum_{j=1}^r \prod_{u \in C'(s_j)} \frac{t^{l_{\lambda_j}(u)+1} - q^{a_{\lambda_j}(u)}}{t^{l_{\lambda_j}(u)} - q^{a_{\lambda_j}(u)}},
\]
where we have set \(\sigma_j = (\lambda_j, \gamma, \mu)\). It remains to note that
\[
\prod_{j=1}^r \prod_{u \in C'(s_j)} \frac{t^{l_{\lambda_j}(u)+1} - q^{a_{\lambda_j}(u)}}{t^{l_{\lambda_j}(u)} - q^{a_{\lambda_j}(u)}} = \prod_{s \in \lambda \setminus \lambda^\# \setminus \lambda^\#} \prod_{u \in C'(s)} \frac{t^{y(u)+1} - q^{a(u)}}{t^{y(u)} - q^{a(u)}}.
\]
where we have set \( \sigma = (\lambda, \gamma, \mu)_n \), is independent of the choice of the chain of subdiagrams \((\lambda_j)_j\), and that
\[
\sum \left( \sum_{j=1}^r \Delta_{\lambda_j, \lambda_{j+1}}(q, t) \right) = (\lambda^#, Q \cdot \lambda).
\]
The lemma is proved. \(\square\)

Using (4.3) together with the above lemma, the linear relation (4.2) may be rescaled to
\[
\sum_i \langle \mu^#, \rho^{2n} P_i \rangle \langle (\lambda^#, \gamma, \mu)_n, R_i \cdot (\lambda^#, \gamma, \mu)_n \rangle \langle \lambda^#, \rho^{2n} Q_i \rangle \cdot \lambda = 0
\]
for all \( \lambda, \lambda^#, \gamma, \mu, \mu^# \) as above, and all \( n \gg 0 \). Let us choose some \( \mu, \mu^# \) and \( \lambda, \lambda^# \) such that \( \langle \mu^#, \rho^{2n} P_i \rangle \neq 0 \) and \( \langle \lambda^#, \rho^{2n} Q_i \rangle \cdot \lambda \neq 0 \) for at least one value of \( i \). Let us fix \( n \gg 0 \) and let us vary \( \gamma \). Recall that \( R_i \) is a polynomial in the operators \( f_{0,1} \), and observe that
\[
\langle (\lambda^#, \gamma, \mu)_n, f_{0,1} \cdot (\lambda^#, \gamma, \mu)_n \rangle =
= t^{2nl} \langle \lambda^#, f_{0,1} \cdot \lambda^# \rangle + q^{nlt} \langle \gamma, f_{0,1} \cdot \gamma \rangle + q^{2nl} \langle \mu, f_{0,1} \cdot \mu \rangle + \langle (\emptyset, \emptyset, \emptyset)_n, f_{0,1} \cdot (\emptyset, \emptyset, \emptyset)_n \rangle.
\]
Setting
\[
R_i' = \langle \mu^#, \rho^{2n} P_i \rangle \langle \lambda^#, \rho^{2n} Q_i \rangle R_i(t^{\pm nq^{l-n}} f_{0,1, \pm 2 l} + \alpha_{\pm 1}, t^{\pm 2nq^{\pm 2n}} f_{0,1, \pm 2} + \alpha_{\pm 2}, \ldots)
\]
where
\[
\alpha_l = \langle (\emptyset, \emptyset, \emptyset)_n, f_{0,1} \cdot (\emptyset, \emptyset, \emptyset)_n \rangle + t^{2nl} \langle \lambda^#, f_{0,1} \cdot \lambda^# \rangle + q^{2nl} \langle \mu, f_{0,1} \cdot \mu \rangle, \quad l \in \mathbb{Z}^*,
\]
we may rewrite (4.6) as
\[
\sum_i \langle \gamma, R_i' \cdot \gamma \rangle = 0.
\]
Since this holds for all \( \gamma \) with \( l(\gamma), l(\gamma') < n \), taking \( n \) large enough we deduce that \( \sum_i R_i' = 0 \). Remember that the \( R_i \)s were chosen to be linearly independent; but then the \( R_i' \)s are also linearly independent and we arrive at a contradiction. This finishes the proof of Proposition 4.8. Theorem 3.1 follows. \(\square\)

**4.7.** Theorem 3.1 allows us to extend the polynomial representation \( \tilde{\mathcal{E}} \) of \( E_c \) defined in Section 1.4 to the whole algebra \( E_c \). We simply use the isomorphism \( \Omega \) to transport the representation of \( H_K \) on \( A_K \) to \( E_c \). Recall that \( \sigma = q^{-1} \) and \( \bar{\sigma} = t^{-1} \). Recall also the operators \( \Delta_{\pm 1}^\infty \) on \( A_K \) defined in Section 1.4 for each \( l \geq 1 \). We set
\[
\Delta_{\pm l}^\infty = \Delta_{\pm 1}^\infty/(q^l - 1).
\]
The eigenvalue of \( \Delta_{\pm 1}^\infty \) on \( P_{l}(q, t^{-1}) \) is equal to \( \pm B_{\lambda}^{\pm 1}(q, t) \) by (1.9).

**Proposition 4.10.** There is a unique faithful representation \( \varphi \) of \( E_c \) on \( A_K \) such that
\[
\varphi(u_{l,0}) = \frac{t^{l/2}}{1 - q^l} p_l,
\]
\[
\varphi(u_{-l,0}) = -\frac{q^{l/2}}{1 - t^l} \frac{\partial}{\partial q^l},
\]
\[
\varphi(u_{0,l}) = \Delta_l^\infty - \frac{1}{(1 - q^l)(1 - t^l)},
\]
\[
\varphi(u_{0,-l}) = \Delta_{-l}^\infty + \frac{1}{(1 - q^{-l})(1 - t^{-l})}.
\]
Since the $K$-algebra $E_\varphi$ is generated by $u_{0, \pm 1}$, $u_{\pm 1, 0}$ it is enough to specify the action of these elements to determine the representation $\varphi$. Note that there are no finite rank analogues of the representation $\varphi$ because the ring $\mathcal{SH}$ acts only on the space of symmetric Laurent polynomials $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathbb{G}_m}$.

5. Virtual classes and their action on $K^T(\text{Hilb})$

We have defined the algebra $H_K$ as the subalgebra of $E_K$ generated by the classes of the tautological bundles on the smooth nested Hilbert schemes $Z_{n,m}$, i.e., when $|n-m| \leq 1$. The aim of this section is to show that $H_K$ contains the virtual classes of the more general (singular) nested Hilbert schemes $Z_{n,m}$ as well as the tautological bundles over them. We also explicitly describe the action of these virtual classes in the natural representation $\Lambda_K$.

5.1. Consider the virtual vector bundle $\mathcal{V}$ over $\text{Hilb} \times \text{Hilb}$ with fiber

\[ \mathcal{V}|_{(I,J)} = \chi(\mathcal{O}) - \chi(I, J). \]

Here $I, J$ are closed points of $\text{Hilb}$ which are viewed as ideal sheaves on $\mathcal{H}^2$ and

\[ \chi(\mathcal{F}, \mathcal{G}) = \sum_{i=0}^{2} (-1)^i \text{Ext}^i(\mathcal{F}, \mathcal{G}) \]

for any coherent sheaves $\mathcal{F}, \mathcal{G}$ on $\mathcal{H}^2$. We abbreviate $V_{\lambda, \mu} = [\mathcal{V}|_{I_{\lambda, \mu}}]$, an element of $R$.

Lemma 5.1. For any partitions $\lambda, \mu$ such that $|\lambda| = n$, $|\mu| = m$ the following hold

(a) $V_{\lambda, \mu} = \sum_{s \in \mu} t_{l(s)+1} q^{-a_s(s)} + \sum_{s \in \lambda} t^{-l(s)} q^{a_s(s)+1} = qtV^*_{\lambda, \mu}$,

(b) $T_{\lambda} = V_{\lambda, \lambda}$ if $n = m$,

(c) $N_{\lambda, \mu}^\lambda = V_{\lambda, \mu} - qt$ if $n = m + 1$ and $\lambda \supset \mu$,

(d) $N_{\lambda, \mu}^\mu = qtV^*_{\lambda, \mu} - qt$ if $n = m - 1$ and $\lambda \subset \mu$,

(e) $\Lambda(V_{\lambda, \mu}) = 0$ unless $\mu \subset \lambda$ and $\Lambda(qtV^*_{\lambda, \mu}) = 0$ unless $\lambda \subset \mu$.

Proof. Part (a) is proved in [CO]. Part (b) is obvious. Now, assume that $n = m + 1$ and $\lambda \supset \mu$. Let $s = \lambda \setminus \mu$. From (2.3) and (a) we get the following formula $V_{\lambda, \mu} = N_{\lambda, \mu}^\lambda + t_{l(s)+1} q^{-a_s(s)} = N_{\lambda, \mu}^\mu + qt$. This yields (c). Part (d) follows from (a) and (c). To prove the first statement in part (e) suffices to notice that if $\lambda \not\supset \mu$ then there exists a box $s \in \lambda$ with $l_s(s) = 0$, $a_s(s) = -1$ or a box $s \in \mu$ with $l_s(s) = -1$, $a_s(s) = 0$ (such a box is located at the intersection of the right boundaries of the Young diagrams of $\lambda$ and $\mu$). The second statement of (e) is obtained by duality.

5.2. The associative $R$-algebra

\[ E_R = \bigoplus_{n \in \mathbb{Z}} K^T(\text{Hilb}_{n+k} \times \text{Hilb}_n) \]

where the product ranges over all integers $n$ for which $n \geq 0$ and $n+k \geq 0$, acts on the $R$-module

\[ L_R = \bigoplus_{n \geq 0} K^T(\text{Hilb}_n). \]

Abbreviate $\mathcal{V} = [\mathcal{V}]$, a class in $E_R$. Let $\mathcal{V}_{n,m}$ be the restriction of $\mathcal{V}$ to $K^T(\text{Hilb}_n \times \text{Hilb}_m)$ and consider the elements of $E_R$

\[ \mathcal{V}_k = \prod_{n} \mathcal{V}_{k+n,n}, \quad \mathcal{V}^*_{-k} = \prod_{n} \mathcal{V}^*_{n+n,k}, \quad k > 0. \]

By part (e) of the above Lemma the classes $\Lambda(\mathcal{V}_k)$, $\Lambda(qt\mathcal{V}^*_{-k})$ are supported on the union of nested Hilbert schemes $\bigsqcup_{n,m} Z_{n,m}$. The class $\Lambda(\mathcal{V}_{n,m})$ for $n > m$ or $\Lambda(qt\mathcal{V}^*_{n,m})$ for $n < m$ is called the virtual fundamental class of $Z_{n,m}$, see [CO].
There is a natural embedding $E_R \subset E_K$. Recall also the subalgebra $H_K \subset E_K$ introduced in Section 3.4. As we will show in this Section,

$$\Lambda(V_k) \in H_K, \quad \forall k > 0.$$ 

More precisely, let us consider the generating series

$$\Lambda^+(\mathcal{V})(z) = 1 + \sum_{k \geq 1} \Lambda(V_k)z^k \in E_R[[z]],$$

$$\Lambda^-(\mathcal{V})(z) = 1 + \sum_{k \geq 1} \Lambda(q\mathcal{V}_k)z^{-k} \in E_R[[z^{-1}]].$$

Let us set

$$q_{l,0} = \begin{cases} t^{-l/2}\Omega(u_{l,0}) & \text{if } l > 0, \\ q^{l/2}\Omega(u_{l,0}) & \text{if } l < 0, \end{cases}$$

where $\Omega : E_c \xrightarrow{\sim} H_K$ is the isomorphism given in Theorem 3.1 and the elements $u_{r,d}$ are the standard generators of $E_c$.

**Theorem 5.2.** We have

\begin{equation}
\Lambda^+(\mathcal{V})(z) = \exp \left( - \sum_{n \geq 1} (-1)^n(1 - t^nq^n) a_{n,0} \frac{z^n}{n} \right),
\end{equation}

\begin{equation}
\Lambda^-(\mathcal{V})(z) = \exp \left( - \sum_{n \geq 1} (1 - t^nq^n) a_{-n,0} \frac{z^n}{n} \right).
\end{equation}

**Proof.** We will deal with (5.3) only. The proof of (5.4) goes along similar lines. In degree one, (5.3) reads

$$\Lambda(V_{n+1,n}) = (1 - tq)\Omega_{Z_{n+1,n}} = (1 - tq)\Lambda(N_{Z_{n+1,n}})$$

which holds by virtue of Lemma 5.1 (c). We will prove (5.3) by showing that both sides are solution to a certain recurrence equation. Recall the elements

$$f_{0,1} = \prod_n \tau_{n,n}, \quad f_{1,1} = \prod_n \tau_{n+1,n}$$

of $H_K$. For a series $A(z) \in 1 + E_K[[z]]$ we consider the functional equation

\begin{equation}
[f_{0,1}, A(z)] = z \left( A(z)f_{1,1} - tqf_{1,1}A(z) \right)
\end{equation}

A solution $A(z) = 1 + \sum_i x_i z^i$ of (5.5) is uniquely determined by its first Fourier coefficient $x_1$. Thus (5.3) will be a consequence of the following two lemmas.

**Lemma 5.3.** The series $\Lambda^+(\mathcal{V})(z)$ satisfies (5.3).

**Proof.** See Appendix A.2

**Lemma 5.4.** The series $A(z) = \exp \left( - \sum_{n \geq 1} (-1)^n(1 - t^nq^n) a_{n,0} \frac{z^n}{n} \right)$ satisfies (5.3).

**Proof.** This is again a direct computation. Set $B(z) = \sum_n (-1)^n(t^nq^n - 1)a_{n,0} \frac{z^n}{n}$ so that $A(z) = \exp(B(z))$. Introduce the elements $q_{l,1} = t^{-l/2}\Omega(u_{l,f+1})$. We have

$$[f_{0,1}, A(z)] = \sum_{k=1}^{\infty} \frac{1}{k!} [f_{0,1}, B(z)^k] = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} B(z)^j [f_{0,1}, B(z)] B(z)^{k-1-j}.$$

Using the relation $[f_{0,1}, a_{n,0}] = a_{n,1}$ we get

$$[f_{0,1}, B(z)] = \sum_{n \geq 1} (-1)^n(t^nq^n - 1)a_{n,1} \frac{z^n}{n}.$$
and more generally if we set
\[B^{(s)}(z) = \sum_{l_1, \ldots, l_s \geq 1} (-1)^{i_1 + \cdots + i_s} (t^{l_1} q^{l_1} - 1) \cdots (t^{l_s} q^{l_s} - 1) a_{l_1 + \cdots + l_s, 1} \frac{z^{l_1 + \cdots + l_s}}{l_1 \cdots l_s}\]
then
\[[B^{(s)}(z), B(z)] = B^{(s+1)}(z).\]

We deduce that
\[(5.6)\]
\[\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \left( k - 1 - j \right) B^{(k-j)}(z) B^{(k-j-1)}(z) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{u=0}^{k} \left( k \right) B^{(u)}(z) B^{(k-u)}(z).\]

In a similar fashion, starting from the relation \[[f_{0,1}, A(z)] = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{l=0}^{k} \left( k \right) B^{(l)}(z) B^{(k-l)}(z)\]
(5.7)
we get
\[z[f_{1,1}, A(z)] = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{u=0}^{k} \left( k \right) B^{(u)}(z) B^{(k-u)}(z).\]

where
\[\tilde{B}^{(s)}(z) = \sum_{l_1, \ldots, l_s \geq 1} (-1)^{i_1 + \cdots + i_s} (t^{l_1} q^{l_1} - 1) \cdots (t^{l_s} q^{l_s} - 1) a_{l_1 + \cdots + l_s + 1, 1} \frac{z^{l_1 + \cdots + l_s + 1}}{l_1 \cdots l_s}.\]

From (5.6) and (5.7) we see that (5.5) reduces to the identity
\[\sum_{s} \frac{1}{s!} \sum_{l_1 + \cdots + l_s = n} \frac{(t^{l_1} q^{l_1} - 1) \cdots (t^{l_s} q^{l_s} - 1)}{l_1 \cdots l_s} = tq \sum_{s} \frac{1}{s!} \sum_{l_1 + \cdots + l_s + 1 = n-1} \frac{(t^{l_1} q^{l_1} - 1) \cdots (t^{l_s} q^{l_s} - 1)}{l_1 \cdots l_s}\]
for all \(n\). This identity is in turn a corollary of the following formula, whose proof is left to the reader
\[\sum_{s} \frac{1}{s!} \sum_{l_1 + \cdots + l_s = n} \frac{(t^{l_1} q^{l_1} - 1) \cdots (t^{l_s} q^{l_s} - 1)}{l_1 \cdots l_s} = t^{n-1} q^{n-1} (tq - 1).\]

Lemma 5.4 and Theorem 5.2 are proved.

5.3. Once Theorem 5.2 is established, it is a simple task to describe the action of the virtual classes \(\Lambda V_n\) and \(\Lambda V_{-n}\) on \(L_K\). The following corollary of Theorem 5.2 may be viewed as a K-theoretic analog of Nakajima’s formulas in Borel-Moore homology (see (0.1), (0.2)).

Corollary 5.5. As operators in \(L_K \simeq \Lambda_K\), we have
\[1 + \sum_{n \geq 1} \tau_n^* \otimes \Lambda(V_n) z^n = \exp \left( - \sum_{n \geq 1} (-1)^n \frac{1 - t^n q^n}{1 - q^n} p_n \frac{z^n}{n} \right),\]
\[1 + \sum_{n \geq 1} \Lambda(q t V_{-n}) z^n = \exp \left( - \sum_{n \geq 1} \frac{1 - t^n q^n}{1 - t^n} \frac{\partial}{\partial p_n} \frac{z^n}{n} \right).\]

Here we have set
\[\tau_n^* \otimes \Lambda(V_n) = \prod_k \tau_{n+k,k}^* \otimes \Lambda(V_{n+k,k}).\]

As mentioned to us by A. Oblomkov, a very similar result appears in [MO].
6. Hecke operators

This section is devoted to the action of the Hecke operators on the (positive part of the) elliptic Hall algebra $\mathcal{E}^>$. The definition of the action of $\mathcal{E}^0$ on $\mathcal{E}^>$ is given in Section 6.1. In Proposition 6.2 this action is expressed in terms of the tensor product by the tautological bundle over Hilb$_{k}$. In Proposition 6.3 we prove that this action equips $\mathcal{E}^>$ with the structure of a tensor product of sheaves. We extend this notation to any symmetric function bundles. Recall the algebra isomorphism $\Omega : \mathcal{E}^0 \rightarrow \mathcal{E}^>$. We define the map $\bullet : \mathcal{E}^0 \otimes \mathcal{E}^> \rightarrow \mathcal{E}^>$, $u \bullet v = \omega(u)v$, where $\omega : \mathcal{E}^> \rightarrow \mathcal{E}^>$ is the projection along $\mathcal{E}^> \otimes J$. It is clear that $(uv') \bullet v = u \bullet (u' \bullet v)$ so that $\bullet$ equips $\mathcal{E}^>$ with the structure of a $\mathcal{E}^0$-module. This action restricts to the graded pieces $\mathcal{E}^>[(r,d)] = \bigoplus_{d \in \mathbb{Z}} \mathcal{E}^>[(r,d)]$, $r \geq 1$.

From [SV1], Theorem 6.3, we have $u_{0,l} \bullet v = \text{sgn}(l) [u_{0,l}, v]$ for any $l \in \mathbb{Z}$ and $v \in \mathcal{E}^>$. For any $r \geq 1$ we define also a projection $\pi_r : \mathcal{E}^0 \rightarrow K[z_1^{\pm 1}, \ldots, z_r^{\pm 1}]^{E_r}$, $u_{0,l} \mapsto p_l(z_1, \ldots, z_r)$ where $p_l(z_1, \ldots, z_r) = \sum_i z_i^l$ is the power sum function.

**Remark 6.1.** When $\mathcal{E}^>$ is interpreted as the Hall algebra of an elliptic curve $\bullet$ is identified with the action of Hecke operators, see [SV1], Section 6. Indeed $\mathcal{E}^0 \cap \mathcal{E}^+ = \mathcal{E}^0$. Indeed $\mathcal{E}^0 \cap \mathcal{E}^+$ is the Hall algebra of the category of torsion sheaves, $\mathcal{E}^>$ is the Hall algebra of vector bundles, and the map $\omega : \mathcal{E}^> \rightarrow \mathcal{E}^>$ is the restriction map from the Hall algebra of all coherent sheaves to that of vector bundles.

6.2. Fix an integer $r \geq 1$. The nested Hilbert scheme $Z_{r+k,k} \subset \text{Hilb}_{r+k} \times \text{Hilb}_k$ carries a tautological bundle $\tau_{r+k,k}$ whose fiber over a point $(I,J)$ is equal to $J/I$. The character of $\tau_{r+k,k}$ over the $T$-fixed point $I_{\lambda,\mu}$ is given by the following expression:

$$(6.1) \quad [\tau_{r+k,k}|_{I_{\lambda,\mu}}] = \sum_{s \in \lambda \setminus \mu} t^j(s)^{-1} q^i(s)^{-1}.$$ 

Let us put $\tau_r = \prod_k \tau_{r+k,k} \in \prod_k K^T(\text{Hilb}_{r+k} \times \text{Hilb}_r)$. We introduce one final piece of notation: for any $l \in \mathbb{Z}^*$ let $p_l(\tau_r) = \Psi_l(\tau_r) = \prod_k \Psi_l(\tau_{r+k,k})$ denote the $l$th Adams operation. If $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a partition then we set $p_\lambda(\tau_r) = p_{\lambda_1}(\tau_r) \otimes \cdots \otimes p_{\lambda_s}(\tau_r)$, the tensor product of sheaves. We extend this notation to any symmetric function $\theta = \sum a_\lambda p_\lambda$ by linearity. Note that because $\tau_r$ is of rank $r$ we have $c_l(\tau_r) = \lambda^l \tau_r = 0$ for $l > r$ and therefore $\theta(\tau_r)$ makes sense for $\theta \in K[z_1^{\pm 1}, \ldots, z_r^{\pm 1}]^{E_r}$. In this notation, we have $1(\tau_r) = \prod_k O_{Z_{r+k,k}}$.

The following proposition connects the action of the Hecke operators with the tautological bundles. Recall the algebra isomorphism $\Omega : \mathcal{E} \rightarrow H_K$ given in Section 4.6.
Proposition 6.2. For any \( v \in \mathcal{E}^r \) and any \( u \in \mathcal{E}^0 \) we have
\[
\Omega(u \bullet v) = \pi_r(u)(\tau_r) \otimes \Omega(v).
\]

Proof. Fix \( v \in \mathcal{E}^r \). It is enough to prove (6.2) for a system of generators of \( \mathcal{E}^0 \), such as \( \{u_{0,l} : l \in \mathbb{Z}^+\} \). We have, by definition
\[
\Omega(u_{0,l} \bullet v) = \text{sgn}(l) \Omega([u_{0,l}, v]) = \text{sgn}(l) [\Omega(u_{0,l}), \Omega(v)] = [f_{0,l}, \Omega(v)].
\]

In the convolution diagram
\[
\text{Hilb}_{r+k} \leftarrow \text{Hilb}_{r+k} \times \text{Hilb}_k \xrightarrow{\varphi_2} \text{Hilb}_k
\]
there is, over \( Z_{r+k,k} \), a short exact sequence
\[
0 \rightarrow \tau_{r+k,k} \rightarrow q^1 \tau_{r+k} \rightarrow q^2 \tau_k \rightarrow 0
\]
and hence also a short exact sequence
\[
0 \rightarrow \Psi_1(\tau_{r+k,k}) \rightarrow q^1 \Psi_1(\tau_{r+k}) \rightarrow q^2 \Psi_1(\tau_k) \rightarrow 0
\]
Equation (6.2) for \( u = u_{0,l} \) follows from (6.4).

6.3. Put \( J_r = \text{Ker}(\pi_r) \subset \mathcal{E}^0 \). By Proposition 6.2 above and because \( \Omega : \mathcal{E} \rightarrow \text{H}_K \) is an isomorphism, the Hecke action of \( \mathcal{E}^0 \) on \( \mathcal{E}^r \) factors through \( \pi_r \) and yields an action
\[
\bullet : K[z_1^{\pm 1}, \ldots, z_r^{\pm 1}] \otimes \mathcal{E}^r \rightarrow \mathcal{E}^r.
\]
Observe in particular that the element \( z_1 \cdots z_r \) acts on \( \mathcal{E}^r \) by an automorphism which we denote by \( \xi_r \in \text{Aut}(\mathcal{E}^r) \).

Proposition 6.3. Under the above action, \( \mathcal{E}^r \) is a torsion-free \( K[z_1^{\pm 1}, \ldots, z_r^{\pm 1}] \xi_r \)-module.

Proof. We will use some results from [BS], to which we refer for definitions. The algebra \( \mathcal{E}^+ \subset \mathcal{E}^0 \) is equipped with a (topological) comultiplication
\[
\Delta : \mathcal{E}^+ \rightarrow \mathcal{E}^+ \hat{\otimes} \mathcal{E}^+
\]
which satisfies
\[
\Delta(u_{0,l}) = u_{0,l} \otimes 1 + 1 \otimes u_{0,l},
\]
\[
\Delta(u_{1,l}) = u_{1,l} \otimes 1 + \sum_{d \geq 0} \theta_{0,d} \otimes u_{1,l-d}.
\]
Here \( \hat{\otimes} \) is the completed tensor product defined in [BS]. When we need to specify graded components, we write \( \Delta_{r,r'} : \mathcal{E}^+[r + r'] \rightarrow \mathcal{E}^+[r] \hat{\otimes} \mathcal{E}^+[r'] \). We begin with a couple of easy lemmas.

Lemma 6.4. For any \( r \geq 1 \), the iterated coproduct map \( \Delta_{1,\ldots,1} : \mathcal{E}^+[r] \rightarrow \mathcal{E}^+[1] \hat{\otimes} \cdots \otimes \mathcal{E}^+[1] \) is injective. Moreover, the map \( \omega^{\otimes r} \circ \Delta_{1,\ldots,1} : \mathcal{E}^+[r] \rightarrow \mathcal{E}^+[1] \otimes \cdots \otimes \mathcal{E}^+[1] \) is injective.

Proof. Let \( (\ , \ ) \) be Green’s scalar product on \( \mathcal{E}^+ \). It is nondegenerate and satisfies the Hopf property, i.e. \( (x y, z) = (x \otimes y, \Delta(z)) \) for any \( x, y, z \in \mathcal{E}^+_K \). The first statement of the Lemma follows from the fact that \( \bigoplus_{r \geq 1} \mathcal{E}^+[r] \) is generated by elements of \( \mathcal{E}^+[1] \). In a similar vein, the second statement follows from the fact that \( \bigoplus_{r \geq 1} \mathcal{E}^+[r] \) is generated by elements of \( \mathcal{E}^+[1] \). □
Lemma 6.5. For any \( u \in K[u_{0,1}, u_{0,2}, \ldots] \) and any \( v \in E^> \) we have \( \omega \otimes \omega(\Delta(u \cdot v)) = \Delta(u) \cdot [\omega \otimes \omega(\Delta(v))] \).

Proof. Using the formula \( (u_1 u_2) \cdot v = u_1 \cdot (u_2 \cdot v) \) we see that it is enough to prove the Lemma for \( u = u_{0,l} \). In that case, we have

\[
\begin{align*}
\omega \otimes \omega(\Delta(u_{0,l} \cdot v)) &= \omega \otimes \omega(\Delta([u_{0,l}, v])) \\
&= \omega \otimes \omega([u_{0,l} \otimes 1 + 1 \otimes u_{0,l}, \Delta(v)]) \\
&= (u_{0,l} \otimes 1 + 1 \otimes u_{0,l}) \cdot [\omega \otimes \omega(\Delta(v))] \\
&= \Delta(u_{0,l}) \cdot [\omega \otimes \omega(\Delta(v))].
\end{align*}
\]

\( \square \)

We may now proceed with the proof of Proposition 6.3. Let \( u \in E^0, v \in E^> [r], v \neq 0 \), and suppose that \( u \cdot v = 0 \). Our aim is to show that \( u \in J_r \). Twisting by the automorphism \( \xi_r \) if necessary, we may assume that \( u \in K[u_{0,1}, u_{0,2}, \ldots] \). Let us consider the subspace

\[
N_r = \sum_{s=1}^r (E^0)^{\otimes (s-1)} \otimes J_1 \otimes (E^0)^{\otimes (r-s)} \subset (E^0)^{\otimes r}.
\]

We claim that it suffices to prove that \( \Delta^{(r)}(u) \in N_r \). Indeed, using the bialgebra isomorphism

\[
\pi : K[u_{0,1}, u_{0,2}, \ldots] \cong K[z_1, z_2, \ldots]^{\mathbb{S}_\infty}, \quad u_{0,l} \mapsto p_l
\]

we have \( J_r \cap K[u_{0,1}, u_{0,2}, \ldots] = \pi^{-1}K[e_{r+1}, e_{r+2}, \ldots] \) and standard symmetric functions arguments show that \( (\Delta^{(r)})^{-1}(\pi(N_r)) = K[e_{r+1}, e_{r+2}, \ldots] \).

Now, by Lemma 6.3 we have

\[
0 = \Delta_{1,\ldots,1}(u \cdot v) = \Delta^{(r)} \cdot \omega^{\otimes r}(\Delta_{1,\ldots,1}(v)).
\]

By Lemma 6.4 we have \( \omega^{\otimes r}(\Delta_{1,\ldots,1}(v)) \neq 0 \), and we may write

\[
(6.8) \quad \omega^{\otimes r}(\Delta_{1,\ldots,1}(v)) = \sum \sum a_d u_{1,1} \otimes \cdots \otimes u_{1,d}, \quad a_d \neq 0.
\]

The above sum is in general infinite. However, it follows from [BS], Proposition 2.1, that \( d_r \) is bounded from above, for \( d_r \) fixed \( d_r-1 \) is bounded above, and more generally, for fixed \( d_r, d_r-1, \ldots, d_{r+1} \) the possible values of \( d_{r-i} \) are bounded above. In particular, there exists a unique element \( \overline{d} = (d_1, \ldots, d_r) \) appearing in \( (6.8) \) which is maximal for the (right) lexicographic order. Similarly, modulo \( N_r \) we may write

\[
(6.9) \quad \Delta^{(r)}(u) = \sum b_L u_{0,1}^1 \otimes \cdots \otimes u_{0,1}^l, \quad (b_L \neq 0).
\]

This sum is finite and may be empty. If \( \Delta^{(r)}(u) \notin N_r \) then there exists a unique \( \overline{l} = (l_1, \ldots, l_r) \) which is maximal for the (right) lexicographic order. But then, considering the \( \mathbb{Z} \)-graded component

\[
\Delta_{(l_1, l_1+1),\ldots,(l_r, l_r+1)}(u \cdot v) = a_{\overline{d}} b_{\overline{l}} (u_{1,1} \otimes \cdots \otimes u_{1,l_1}) \cdot (u_{1,d_1} \otimes \cdots \otimes u_{1,d_r})
\]

we reach a contradiction. Thus \( \Delta^{(r)}(u) \in N_r \) and \( u \in J_r \). We are done. \( \square \)
6.4. By “transport de structure” and using Theorem 5.1 we get a Hecke action 

\[ \bullet : H_K^0 \otimes H_K^0 \rightarrow H_K^0 \]

which is determined by

\[ (u_1 u_2) \bullet v = u_1 \bullet (u_2 \bullet v), \quad f_{0,l} \bullet v = [f_{0,l}, v], \quad \forall l \in \mathbb{Z}. \]

This action is torsion free in the same sense as Proposition 6.3. Note that we may also use formulas (6.10) to extend this to a Hecke action

\[ \bullet : H_K^0 \otimes \text{End}_K(L_K) \rightarrow \text{End}_K(L_K). \]

Using this notation we have

\[ h \bullet \psi(u) = \psi(h \bullet u) \]

for any \( h \in H_K^0 \) and \( u \in H_K^\geq \).

7. Cohomology of the commuting variety

The aim of this section is to introduce a ring structure on the equivariant Grothendieck group of the commuting variety and to compare this ring with the positive part of the elliptic Hall algebra. The ring \( C_H \) is given in Proposition 7.5. Then we prove in Proposition 7.9 that it acts on the \( R \)-module \( L_R \) equal to the \( K \)-theory of the Hilbert scheme. Next, in Proposition 7.10 we compare the Hecke action on the algebra \( H_K^\geq \), which is the positive part of the elliptic Hall algebra by Theorem 3.1 with the natural action of the representation ring on the equivariant \( K \)-theory of the commuting variety. Finally, in Theorem 7.14, we compare \( C_K \) with \( H_K^\geq \).

7.1. First let us recall a few general facts. Let \( G \) be a complex linear algebraic group. By a variety we’ll always mean a quasi-projective complex variety. We call \( G \)-variety a variety with a rational action of \( G \).

Let \( P \subset G \) a parabolic subgroup and \( H \subset P \) a Levi subgroup. Fix a \( H \)-variety \( Y \). The group \( P \) acts on \( Y \) through the obvious group homomorphism \( P \rightarrow H \). Let \( X = G \times_Y Y \) be the induced \( G \)-variety.

Now assume that \( Y \) is smooth. Given a smooth subscheme \( O \subset Y \) let \( T_O^* Y \subset T^* Y \) be the conormal bundle to \( O \). It is well-known that the induced \( H \)-action on \( T^* Y \) is Hamiltonian and that the zero set of the moment map is the closed \( H \)-subvariety

\[ T_H^* Y = \bigcup_O T_O^* Y \subset T^* Y, \]

where \( O \) runs over the set of \( G \)-orbits. The following lemma is left to the reader.

**Lemma 7.1.** We have \( T^* X = T^*_{G \times p}(G \times Y)/P \) and \( T^*_{G \times p}X = G \times_Y T_H^* Y \). The induction yields a canonical isomorphism \( K^H(T_H^* Y) = K^G(T_G^* X) \).

We’ll call fibration a smooth morphism which is locally trivial in the Zariski topology. Let \( X' \) be a smooth \( G \)-variety and \( V \) be a smooth \( H \)-variety. Assume that we are given \( H \)-equivariant homomorphisms \( p : V \rightarrow Y \) and \( q : V \rightarrow X' \) which are a fibration and a closed embedding respectively. Set \( W = G \times_Y V \) and consider the following maps

\[ g : W \rightarrow X', \quad (g, v) \mod P \mapsto gq(v), \]
\[ f : W \rightarrow X, \quad (g, v) \mod P \mapsto (g, p(v)) \mod P. \]

The following properties are immediate.

**Lemma 7.2.** The map \( f \) is a \( G \)-equivariant fibration, the map \( g \) is a \( G \)-equivariant proper morphism, and the map \( (f, g) \) is a closed embedding \( W \subset X \times X' \). The varieties \( V, W, X, X' \) are smooth.
We'll identify $W$ with its image in $X \times X'$. Let $Z = T_W^*(X \times X')$ the conormal bundle. It is again a smooth $G$-variety. The obvious projections yield $G$-equivariant maps

$$\phi : Z \to T^*X, \quad \psi : Z \to T^*X'.$$

Consider the $G$-variety

$$Z_G = Z \cap (T^*_G X \times T^*_G X').$$

Recall that a morphism of varieties $S \to T$ is called regular if it is the composition of a regular immersion $S \subset S'$, i.e., an immersion which is locally defined by a regular sequence, and of a smooth map $S' \to T$. Note that a regular map has a finite tor-dimension and that a morphism $S \to T$ is regular whenever $S$ and $T$ are smooth.

**Lemma 7.3.** (a) The map $\psi$ is proper and regular, the map $\phi$ is regular.

(b) We have $\phi^{-1}(T^*_G X) = Z_G$ and $\psi(Z_G) \subset T^*_G X'$.

**Proof.** To prove that $\psi$ is a proper morphism we may assume that the fibration $f$ is indeed the obvious projection

$$f : W = X \times U \to X,$$

where $U$ is a smooth $G$-variety. Since the map $g$ is proper, it is enough to check that for all $w = (x, u) \in W$ we have

$$\{ \xi \in T^*_w(X \times X') : \xi(T_w W) = \xi(T_{x'} X') = 0 \} = \{0\}.$$ 

This is obvious because we have

$$T_w W + T_u X' = T_x X + T_u U + T_u X' = T_x X + T_u X'.$$

The maps $\phi$, $\psi$ are regular because $Z$ is smooth. For instance $\phi$ is the composition of the projection $T^*(X \times X') \to T^*X$, which is smooth, and of the obvious inclusion $Z \subset T^*(X \times X')$, which is regular. Claim (a) is proved.

Now let us concentrate on (b). The second claim is obvious. Let us prove the first one. Since the set $W \subset X \times X'$ is preserved by the diagonal action of $G$ we have

$$T_{x'}(Gx') \subset T_w W + T_x(Gx), \quad \forall w = (x, x') \in W.$$

Thus we have also

$$Z \cap (T^*_G X \times T^*_G X') \subset T_G^* X \times T_G^* X'.$$

□

To avoid confusions we may abbreviate

$$\phi_G = \phi|_{Z_G} : Z_G \to T_G^* X, \quad \psi_G = \psi|_{Z_G} : Z_G \to T_G^* X'.$$

**7.2.** Recall that for any $G$-variety $M$ and any closed $G$-stable subvariety $N \subset M$ the direct image by the obvious inclusion $N \to M$ identifies $K^G(N)$ with the complexified Grothendieck group $K^G(M)$ on $N$ of the category of $G$-equivariant coherent sheaves on $M$ supported on $N$. Since the map $\psi$ is a proper morphism the derived direct image yields a map

$$R\psi_* : K^G(Z) \to K^G(T^*X').$$

By Lemma 7.3(b) we have also a map

$$R\psi_* : K^G(Z_G) = K^G(Z \text{ on } Z_G) \to K^G(T^*X' \text{ on } T_G^* X') = K^G(T_G^* X').$$

Since the map $\phi$ has a finite tor-dimension the derived pull-back yields a map

$$L\phi^* : K^G(T^*X) \to K^G(Z).$$

By definition $L\phi^*$ is the composition of the pull-back by the projection $T^*X \times T^*X' \to T^*X$ and the derived pull-back by the regular immersion $Z \subset T^*X \times T^*X'$. By Lemma 7.3(b) we have also a map

$$L\phi^* : K^G(T_G^* X) = K^G(T^* X \text{ on } T_G^* X) \to K^G(Z \text{ on } Z_G) = K^G(Z_G).$$
Composing $R\psi_*$ and $L\phi^*$ we get a map
\[ R\psi_* \circ L\phi^* : K^G(T^*_G X) \rightarrow K^G(T^*_G X'). \]
By Lemma 7.1 the induction yields also an isomorphism
\[ K^H(T^*_H Y) = K^G(T^*_G X). \]
Composing it by $R\psi_* \circ L\phi^*$ we obtain a map
\[ (7.1) \quad K^H(T^*_H Y) \rightarrow K^G(T^*_G X'). \]

7.3. Now, we apply the general construction recalled above to the particular case of the commuting variety. First, let us fix some notation. Let $E$ be a finite dimensional $\mathbb{C}$-vector space. We'll frequently use the following isomorphisms without mentioning them explicitly. Proof.

7.4. Next we fix a subspace $E_1 \subset E$ and we set $E_2 = E/E_1$. We may write $G_i = G_{E_i}$, $g_i = g_{E_i}$, $C_i = C_{E_i}$, etc, for $i = 1, 2$. Set
\[ H = G_1 \times G_2, \quad P = \{ g \in G; g(E_1) = E_1 \}. \]
Let $g, p$ and $h$ be the corresponding Lie algebras. Put
\[ Y = h, \quad X' = g, \quad V = p. \]
The $G$-action on $X'$ and the $H$-action on $Y$ are the adjoint ones. Put
\[ C_g = C, \quad C_h = C_1 \times C_2, \quad C_p = p^2 \cap C. \]
For each $a \in p$ let $a_h \in h$ be the graded linear map associated with $a$. We apply the general construction in Section 7.2 with the map $\phi : V \rightarrow Y$, $a \mapsto a_h$ and the obvious inclusion $q : V \rightarrow X'$. By the canonical isomorphisms $g^* = g, h^* = h$ we'll always mean the isomorphisms given by the trace.

Lemma 7.4. (a) We have $X = G \times P h$ and $W = G \times P p$. The maps $W \rightarrow X$, $W \rightarrow X'$ are given by $(g, a) \mod P \mapsto (g, a_h) \mod P$ and $(g, a) \mod P \mapsto gag^{-1}$.

(b) The canonical isomorphisms $g^* = g, (g \times h)^* = g \times h$ yield isomorphisms of $G$-varieties
\[ T^*X' = g^2, \quad T^*X = G \times p \{(c, a, b) \in p \times h \times h; c = [a, b] \}, \quad Z = G \times p p^2. \]
For each $a, b \in p$ we have
\[ \phi((g, a, b) \mod P) = (g, [a, h], a_h, b_h) \mod P, \quad \psi((g, a, b) \mod P) = (gag^{-1}, gb g^{-1}). \]

(c) We have canonical isomorphisms of $G$-varieties
\[ T^*_G X = G \times P C_h, \quad T^*_G X' = C, \quad Z_G = G \times p C_p. \]
The maps $\phi_G, \psi_G$ are the obvious ones.

Proof. We'll frequently use the following isomorphisms without mentioning them explicitly
\[ G \times P (h \times g) \rightarrow X \times X', \quad (g, a, b) \mod P \mapsto ((g, a) \mod P, gb g^{-1}), \]
\[ G \times g^* \rightarrow T^*G, \quad (g, f) \mapsto gf. \]
By Lemma 7.1 we have
\[ T^*(X \times X') = T^*_P(G \times h \times g)/P \]
\[ = G \times P \{(f, a) \in (g \times h \times g)^* \times (h \times g); f(-b, [\delta b, a]) = 0, \forall b \in p \}. \]
Let $\delta$ be the linear map
\[ \delta : p \to h \times g, \quad a \mapsto (a_b, a). \]

We have
\[ Z = T^*_p(X \times X'), \quad W = G \times_p \delta p, \quad T^*W = T^*_p(G \times p)/P. \]

Let $\delta^P_\perp \subset (h \times g)^*$ be the orthogonal of $\delta p$. We have also
\[ T^*(X \times X') |_{W} = G \times_p \{(f, a) \in (g \times h \times g)^* \times p; f(-b, \delta[b, a]) = 0, \forall b \in p\}, \]
\[ T^*W = G \times_p \{(f, a) \in (g \times p)^* \times p; f(-b, [b, a]) = 0, \forall b \in p\}, \]
\[ Z = G \times_p (\delta^P_\perp \times p). \]

Here the inclusion $Z \subset T^*(X \times X')$ is given by the inclusion $\delta : p \to h \times g$ and the inclusion
\[ \delta^P_\perp = \{0\} \times \delta^P_\perp \subset \{0\} \times (h \times g)^* \subset (g \times h \times g)^*. \]

The canonical isomorphism $(h \times g)^* \to h \times g$ identifies $\delta^P_\perp$ with $\delta^P \simeq p$, where
\[ \delta' : p \to h \times g, \quad a \mapsto (-a_b, a). \]

This yields an isomorphism
\[ Z = G \times_p p^2. \]

By Lemma 7.1 we have
\[ T^*X = T^*_p(G \times h)/P \]
\[ = G \times_p \{(f, a) \in (g \times h)^* \times h; f(-b, [b, a]) = 0, \forall b \in p\}, \]
\[ T^*_G X = G \times_p T^*_p h, \]
\[ = G \times_p C_p, \]
\[ = G \times_p \{(f, a) \in (g \times h)^* \times h; f(-b, [b, a]) = f(c, 0) = 0, \forall b \in p, c \in g\}. \]

Here the inclusion $h^* \times h \subset (g \times h)^* \times h$ is given by the map
\[ h^* = \{0\} \times h^* \subset (g \times h)^*. \]

The map $\phi$ is the composition of the chain of maps
\[ Z \subset T^*(X \times X') \to T^*X. \]

Fix $a, b \in p$. Consider the element $\xi = (g, a, b) \mod P$ of $Z$. We may identify $a$ with $\delta' a$, which can be regarded as an element in
\[(h \times g)^* = \{0\} \times (h \times g)^* \subset (g \times h \times g)^* = g^* \times h^* \times g^*, \]
and $b$ with $\delta b$, which is an element of $h \times g$. So $\xi$ can be viewed as an element in $T^*(X \times X')$, see above. Set $\delta' a = (0, f_b, f)$. A short computation yields
\[ \phi(\xi) = (g, -[b, f], f_b, b) \mod P \]
where the bracket is the coadjoint action. Now, observe that the canonical map identifies $f_b, f$ with $-a_b, a$ respectively. This yields the formula for $\phi$ in part (b). Finally Lemma 7.3(b) yields
\[ Z_G = \phi^{-1}(T^*_G X). \]

Therefore we have $Z_G = G \times_p C_p$. The other claims are left to the reader. \qed

Next, we set $C_{E, R} = K^{G \times T}(C_E)$. A vector space isomorphism $E \simeq E'$ yields a $R$-module isomorphism $C_{E, R} \simeq C_{E', R}$. Let
\[ C_R = \lim_{\to E} C_{E, R}, \]
where the limit runs over the groupoid formed by all finite dimensional vector spaces with their isomorphisms. There is a $G \times T$-action on $T^*X$ and $T^*X'$ is given by
\[ (z_1, z_2) \cdot (g, c, a, b) \mod P = (g, z_1 z_2 c, z_1 a, z_2 b) \mod P, \]
\[ (z_1, z_2) \cdot (g, a, b) \mod P = (g, z_1 a, z_2 b) \mod P. \]
We define as in (6.1) a $R$-linear homomorphism
\[ K^{H \times T}(C_{0}) \to K^{G \times T}(C) = C_{E,R}. \]
We'll abbreviate $C_i = C_{E,R}$ for $i = 1, 2$. By the Kunneth formula, see [CG], Chapter 5.6, it can be viewed as a map
\[ (7.3) \quad C_1 \otimes_R C_2 \to C_{E,R}. \]

**Proposition 7.5.** The map (7.3) equips $C_R$ with the structure of an associative unital $R$-algebra.

*Proof.* Fix a flag $E_1 \subset E_2 \subset E$. First, we define the following varieties
- $X_1$ is the set of tuples $(F_1, F_2, a)$ where $F_1 \subset F_2 \subset E$ is a flag such that $F_1 \simeq E_1$, $F_2 \simeq E_2$ and $a$ is an endomorphism of the graded vector space $F_1 \oplus (F_2/F_1) \oplus (E/F_2)$,
- $X_2$ is the set of pairs $(F_1, a)$ where $F_1 \subset E$ is a vector subspace isomorphic to $E_1$ and $a$ is an endomorphism of the graded vector space $F_1 \oplus (E/F_1)$,
- $X_3 = \mathfrak{g}$.

Next, we define the following ones
- $W_1$ is the set of pairs $(F_1, a)$ where $F_1 \subset E$ is a vector subspace isomorphic to $E_1$ and $a \in \mathfrak{g}$ preserves $F_1$,
- $W_2$ is the set of tuples $(F_1, F_2, a)$ where $F_1 \subset F_2 \subset E$ is a flag such that $F_1 \simeq E_1$, $F_2 \simeq E_2$ and $a \in \mathfrak{g}$ preserves $F_1$ and $F_2$,
- $W_3$ is the set of tuples $(F_1, F_2, a)$ where $F_1 \subset F_2 \subset E$ is a flag such that $F_1 \simeq E_1$, $F_2 \simeq E_2$ and $a$ is an endomorphism of the graded vector space $F_1 \oplus (E/F_1)$ which preserves the subspace $\{0\} \oplus (F_2/F_1)$.

There are obvious inclusions $W_1 \subset X_3 \times X_2$, $W_2 \subset X_3 \times X_1$ and $W_3 \subset X_2 \times X_1$. These inclusions factor through an isomorphism
\[ W_2 = W_1 \times_{X_2} W_3. \]

Now, we consider the smooth varieties
\[ Z_1 = T_{W_1}(X_3 \times X_2), \quad Z_2 = T_{W_2}(X_3 \times X_1), \quad Z_3 = T_{W_3}(X_2 \times X_1). \]

The intersection of $(W_1 \times X_1) \cap (X_3 \times W_2)$ is transverse in $X_3 \times X_2 \times X_1$. Thus, by [CG], Theorem 2.7.26, the obvious projection $T(X_3 \times X_2 \times X_1) \to T^*(X_3 \times X_1)$ factors to an isomorphism
\[ Z_1 \times_{T^*X_2} Z_3 \cong Z_2. \]

Therefore, we have the following diagram with a Cartesian square
\[ (7.4) \quad \begin{array}{ccc}
T^*X_3 & \xrightarrow{\psi_1} & Z_1 & \xrightarrow{\phi_1} & T^*X_2 \\
\downarrow{\psi_2} & & \downarrow{\alpha} & & \downarrow{\psi_3} \\
Z_2 & \xrightarrow{\beta} & Z_3 & \xrightarrow{\phi_2} & T^*X_1.
\end{array} \]

The maps are all regular (all the varieties are smooth), and $\psi_1, \psi_3$ are proper. Put $\psi_2 = \psi_1 \circ \alpha$ and $\phi_2 = \phi_3 \circ \beta$. Then $\psi_2$ is also proper. Therefore we can define the following maps
\[ (7.5) \quad \begin{align*}
I_1 &= R(\psi_1) \circ L\phi_1^* : K^{G \times T}(T^*X_2 \text{ on } T^*_G X_2) \to K^{G \times T}(T^*X_3 \text{ on } T^*_G X_3), \\
I_2 &= R(\psi_2) \circ L\phi_2^* : K^{G \times T}(T^*X_1 \text{ on } T^*_G X_1) \to K^{G \times T}(T^*X_2 \text{ on } T^*_G X_2), \\
I_3 &= R(\psi_3) \circ L\phi_3^* : K^{G \times T}(T^*X_1 \text{ on } T^*_G X_1) \to K^{G \times T}(T^*X_3 \text{ on } T^*_G X_3).
\end{align*} \]
Note that $Z_1$, $Z_2$, $Z_3$, $T^*X_2$ are smooth with $\dim Z_1 + \dim Z_3 = \dim Z_2 + \dim T^*X_2$, that $\psi_3$ is proper and that $\alpha \times \beta$ is a closed embedding $Z_2 \subset Z_1 \times Z_3$. Therefore, by base change we have $I_2 = I_1 \circ I_3$, see Proposition[4.1]. Now, set

$$P_1 = \{g \in G; g(E_1) = E_1\}, \quad P = \{g \in P_1; g(E_2) = E_2\}.$$  

The Lie algebras of $P_1$, $P$ and their Levi factors are denoted by $p_1$, $p$, $\mathfrak{h}$, $\mathfrak{h}$. Lemma 7.4 yields

$$T_G^*X_3 = C, \quad T_G^*X_2 = G \times P_1 \cdot C_h, \quad T_G^*X_1 = G \times P \cdot C_h.$$  

Thus $I_1$, $I_3$ yield maps

$$I_1 : K^{G_{E_1} \times T}(C_1) \otimes_R K^{G_{E_1} / E_1 \times T}(C_{E_1 / E_1}) \to K^{G \times T}(C),$$

$$I_3 : K^{H \times T}(C_h) \to K^{G_{E_1} \times E_1 \times T}(C_1) \otimes_R K^{G_{E_1} / E_1 \times T}(C_{E_1 / E_1}).$$

The equality $I_2 = I_1 \circ I_3$ implies that $\mathcal{C}_R$ is associative in the same way as in [7.2], Lemma 3.4.  

**Remark 7.6.** Although $Z$ is smooth the variety $Z_G$ is not locally a complete intersection in general. This explains why we used $L\phi^*$ rather than $L(\phi_G)^*$ (which may not be well defined) in the definition of the map $[7.1]$. Note also that the map $\phi$ is not flat in general. This explains why we used the derived functor $L\phi^*$.

### 7.5.  
Next, we modify slightly the construction in Section 7.3 in order to get a $\mathcal{C}_R$-module. First, let us recall the relation between the commuting variety and the Hilbert scheme. Set

$$N = N_E = g^2 \times E^* \times E, \quad M = M_E = \{(a, b, \varphi, v) \in N; [a, b] + v \circ \varphi = 0\}.$$  

Here $v \circ \varphi$ is regarded as an element of $g$. Let $N^* \subset N$ be the set of tuples $(a, b, v, \varphi)$ such that the elements $a^1 b^2 \cdots (v)$ span $E$ as $(i_1, i_2, \ldots)$ runs over the set of all tuples of integers $\geq 0$. Put $M^* = M \cap N^*$. The group $G \times T$ acts on $M^*$ as

$$(g, z_1, z_2) \cdot (a, b, \varphi, v) = (z_1 g a g^{-1}, z_2 g b g^{-1}, \varphi g^{-1}, z_1 z_2 g v).$$

We set $M_{E, R} = K^{G \times T}(M^*)$. A vector space isomorphism $E \simeq E'$ yields an $R$-module isomorphisms $M_{E, R} \simeq M_{E', R}$. Let

$$M_R = \lim_{\rightarrow E} M_{E, R},$$

where the limit runs over the groupoid formed by all finite dimensional vector spaces with their isomorphisms. If $n = \dim E$ there is a $G$-torsor

$$M^* \to \text{Hilb}_n, \quad (a, b, v, \varphi) \mapsto \{p(x, y) \in \mathbb{C}[x, y]; p(a, b)v = 0\}.$$  

Recall the $R$-module $L_R$ from [5.2]. The following is now obvious.

**Lemma 7.7.** If $\dim E = n$ there are canonical $R$-module isomorphisms $M_{E, R} = K^T(\text{Hilb}_n)$ and $M_R = L_R$.

### 7.6.  
Now fix a subspace $E_1 \subset E$ and we set $E_2 = E/E_1$. Let $\pi : E \to E_2$ be the obvious projection. Let $H$, $P$, $\mathfrak{h}$ and $\mathfrak{p}$ be as in Section 7.4. We set

$$X' = g \times E, \quad Y = \mathfrak{h} \times E_2, \quad V = \mathfrak{p} \times E, \quad X = G \times P \cdot Y, \quad W = G \times P \cdot V.$$  

The $G$-action on $E$ is the obvious one, the $P$ action on $E_2$ is the composition of the obvious map $P \to H$ and the $H$-action on $E_2$. The $G$-action on $X'$, the $H$-action on $Y$ and the $P$-action on $V$ are the diagonal ones. We’ll also write

$$N_g = N, \quad N_{\mathfrak{h}} = \mathfrak{h}^2 \times E_2 \times E_2 = (\mathfrak{g}_1)^2 \times N_2, \quad N_{\mathfrak{p}} = \mathfrak{p}^2 \times E_2^* \times E,$$

$$M_g = M, \quad M_{\mathfrak{h}} = C_1 \times M_2, \quad M_{\mathfrak{p}} = N_{\mathfrak{p}} \cap M.$$
Here we have identified $E_2^*$ with a subspace of $E^*$ via the transpose of $\pi$. We’ll apply the general construction in Section 7.2 with the map $p : V \to Y$, $(a, v) \mapsto (a_b, \pi(v))$ and the obvious inclusion $q : V \to X'$. The following is proved as Lemma 7.4.

**Lemma 7.8.** (a) We have $X = G \times P (b \times E_2)$ and $W = G \times P (p \times E)$. The maps $W \to X$, $W \to X'$ are given by $(g, a, v) \mapsto (g, a_b, \pi(v))$ and $(g, a, v) \mapsto (gag^{-1}, gv)$. (b) We have canonical isomorphisms of $G$-varieties

\[ T^*X' = N, \quad T^*X = G \times P ((c, a, b, \varphi, v) \in p \times N_b; c_b = [a, b] + v \circ \varphi), \quad Z = G \times P N_p. \]

For all $(a, b, \varphi, v) \in N_p$ we have

\[ \phi((g, a, b, \varphi, v) \mod P) = (g, [a, b] + v \circ \varphi, a_b, b_h, \varphi, \pi(v)) \mod P, \]

\[ \psi((g, a, b, \varphi, v) \mod P) = (gag^{-1}, gb^{-1}, (\varphi \circ \pi)g^{-1}, gv). \]

(c) We have canonical isomorphisms of $G$-varieties

\[ T^*_G X = G \times P M_b, \quad T^*_G X' = M, \quad Z_G = G \times P M_p. \]

The maps $\phi_G, \psi_G$ are the obvious ones.

We can now prove the following.

**Proposition 7.9.** There is a representation $\rho : C_R \to \text{End}_R(M_R)$.

**Proof.** We’ll abbreviate $M_i = M_{E_i, R}$ for $i = 1, 2$. First we define a $R$-linear homomorphism

\[ C_1 \otimes_R M_2 \to M_{E_i, R}. \]

We set $N^*_p = N_p \cap N^*, M^*_p = M_p \cap M^*$ and

\[ Z^* = Z \cap \psi^{-1}(N^*) = G \times P N^*_p, \quad Z^*_G = Z_G \cap Z^* = G \times P M^*_p. \]

Note that $N^*_p, M^*_p, Z^*, Z^*_G$ are open in $N_p, M_p, Z, Z_G$. The map $\psi$ restricts to proper morphisms

\[ \psi_* : Z^* \to N^*_p, \quad \psi_{G,*} : Z^*_G \to M^*_p. \]

Taking the derived direct image we get a $R$-linear map

\[ R(\psi)_* : K^{G \times T}(Z^*_G) = K^{G \times T}(Z^* \subseteq Z_G) \to K^{G \times T}(N^* \subseteq M^*). \]

Next, we set $N^*_p = (g_1)^2 \times N^*_2, M^*_p = C_1 \times M^*_2$ and

\[ T^*X^* = T^* X \cap (G \times P (p \times N^*_p)), \quad T^*_G X^* = T^*_G X \cap T^* X^* = G \times P M^*_p. \]

We have $\phi(Z^*) \subseteq T^* X^*$ by Lemma 7.8(b). Hence the restriction of $\phi$ yields morphisms

\[ \phi_* : Z^* \to T^* X^*, \quad \phi_{G,*} : Z^*_G \to T^*_G X^*. \]

The map $\phi_*$ has a finite tor-dimension because $\phi$ is regular. Hence the derived pull-back $L\phi_*^*$ is well-defined, and it yields a $R$-linear homomorphism

\[ L\phi_*^* : K^{G \times T}(T^*_G X^*) = K^{G \times T}(T^* X^* \subseteq T^*_G X^*) \to K^{G \times T}(Z^* \subseteq Z^*_G). \]

Finally, by induction we have a canonical isomorphism of $R$-modules

\[ C_1 \otimes_R M_2 = K^{G \times T}(T^*_G X^*). \]

We define the map (7.7) to be the composition of (6.10), (7.9) and (7.8).

Now we must prove that the map (7.7) defines a $C_R$-action on $M_R$. The proof is the same as the proof of Proposition 7.5. More precisely, fix a flag $E_1 \subset E_2 \subset E$ and define $P_1, P, H_1, H$ and their Lie algebras as in loc. cit. We define the following varieties

- $X_1$ is the set of tuples $(F_1, F_2, a, v)$ where $F_1 \subset F_2 \subset E$ is a flag such that $F_1 \cong E_1$, $F_2 \cong E_2$ and $a$ is an endomorphism of the graded vector space $F_1 \oplus (F_2/F_1) \oplus (E/F_2)$, and $v$ is an element of $E/F_2$,
- $X_2$ is the set of pairs $(F_1, a, v)$ where $F_1 \subset E$ is a vector subspace isomorphic to $E_1$, $a$ is an endomorphism of the graded vector space $F_1 \oplus (E/F_1)$, and $v$ is an element of $E/F_1$. The following is proved as Lemma 7.4.

- $X_1$ is the set of tuples $(F_1, F_2, a, v)$ where $F_1 \subset F_2 \subset E$ is a flag such that $F_1 \cong E_1$, $F_2 \cong E_2$ and $a$ is an endomorphism of the graded vector space $F_1 \oplus (F_2/F_1) \oplus (E/F_2)$, and $v$ is an element of $E/F_2$,
- $X_2$ is the set of pairs $(F_1, a, v)$ where $F_1 \subset E$ is a vector subspace isomorphic to $E_1$, $a$ is an endomorphism of the graded vector space $F_1 \oplus (E/F_1)$, and $v$ is an element of $E/F_1$. The following is proved as Lemma 7.4.
• $X_3 = g \times E$.

Next, we define the following ones

• $W_1$ is the set of tuples $(F_1, a, v)$ where $F_1 \subset E$ is a vector subspace isomorphic to $E_1$, $a \in g$ preserves $F_1$, and $v \in E$,

• $W_2$ is the set of tuples $(F_1, F_2, a, v)$ where $F_1 \subset F_2 \subset E$ is a flag such that $F_1 \simeq E_1$, $F_2 \simeq E_2$, $a \in g$ preserves $F_1$ and $F_2$, and $v \in E$,

• $W_3$ is the set of tuples $(F_1, F_2, a)$ where $F_1 \subset F_2 \subset E$ is a flag such that $F_1 \simeq E_1$, $F_2 \simeq E_2$, $a$ is an endomorphism of the graded vector space $F_1 \oplus (E/F_1)$ which preserves the subspace $\{0\} \oplus (F_2/F_1)$, and $v \in E/F_1$.

We have canonical inclusions $W_1 \subset X_3 \times X_2$, $W_3 \subset W_2 \times X_2$ and $W_2 \subset X_3 \times X_1$. Set

$$Z_1 = T_{W_1}(X_3 \times X_2), \quad Z_2 = T_{W_2}(X_3 \times X_1), \quad Z_3 = T_{W_3}(X_2 \times X_1).$$

We have the subspace $\{0\}$ by base change we get $I_2 = I_1 \circ I_3$. Further, by Lemma 7.8 we have

$$I_1 : K^G \times T(C_1) \otimes_R K^{G/E_1} \times T(M_{E/E_1}) \to K^G \times T(M),$$

$$I_3 : K^H \times T(M_b) \to K^G \times T(C_1) \otimes_R K^{G_2} \times T(M_{E/E_1}),$$

where $M_b = C_1 \times C_{E_2/E_1} \times M_{E/E_2}$. This implies that the ring $C_R$ acts on the Abelian group $R \to K^G \times T(M_E)$.

Finally we must prove that $C_R$ acts also on $M_R$. By Lemma 7.8 we have

$$T^*X_3 = N, \quad T^*X_2 \subset G \times P_1 (p_1 \times N_{b_1}), \quad T^*X_1 \subset G \times P (p \times N_b),$$

where $N_{b_1} = (g_1)^2 \times N_{E/E_1}$ and $N_b = (g_1)^2 \times (g_{E_2/E_1})^2 \times N_{E/E_2}$. We set

$$T^*X_3^s = N^s, \quad T^*X_2^s = T^*X_2 \cap (G \times P_1 (p_1 \times N_{b_1})), \quad T^*X_1^s = T^*X_1 \cap (G \times P (p \times N_b^s)).$$

Set also

$$Z_3^s = \psi_3^{-1}(T^*X_3^s), \quad Z_2^s = \psi_2^{-1}(T^*X_2^s), \quad Z_1^s = \psi_1^{-1}(T^*X_1^s).$$

We still have a commutative diagram with a Cartesian square

$$\begin{array}{ccc}
T^*X_3^s & \xrightarrow{\psi_3} & Z_1^s \\
\downarrow{\alpha} & & \downarrow{\beta} \\
Z_2^s & \xrightarrow{\psi_2} & Z_3^s \\
\downarrow{\phi_2} & & \downarrow{\phi_3} \\
T^*X_1^s & & 
\end{array}$$

We can check that $Z_2^s = Z_3^s \times T^*X_3^s$. Then the rest of the proof is as above. \qed

7.7. We keep the same notation as in the previous section. Note that we have $M_R = L_R$ by Lemma 7.7. We’ll denote as usual by $C_K$, $M_K$, etc, the extensions of scalars from $R$ to $K$. Our next goal is to compare the representation $\rho$ with the faithful representation

$$\psi : H_K \to \text{End}_K(L_K)$$

given in Section 3.4. For each finite dimensional vector space $E$ we have a $R$-submodule $C_{E,R} \subset C_R$ which depends only on the dimension of $E$.

Recall that $E^0 \simeq H_K^0$ is a free polynomial algebra in the tautological classes $f_{0,l} = \prod_n \Psi_l(\tau_{n,n})$, $l \in \mathbb{Z}$. We define, for each $E$, a projection map

$$(7.11) \quad \pi_E : H_K^0 \to R_{G \times T} \otimes_R K, \quad f_{0,l} \mapsto \Psi_l(E)$$
where we also write $E$ for the standard $G$-module. The collection of maps $\pi_E$ endows $C_K$ with the structure of an $H^0_K$-module:

$$H^0_K \otimes_K C_{E,K} \to C_{E,K}, \quad (h, u) \mapsto h u = \pi_E(h)u,$$

where the multiplication in the rhs is the tensor product of a class in equivariant K-theory by a representation. In Section 6.4 we have also defined a Hecke action

$$\bullet : H^0_K \otimes_K \text{End}_K(L_K) \to \text{End}_K(L_K).$$

**Proposition 7.10.** The map $\rho : C_K \to \text{End}_K(L_K)$ intertwines the $H^0_K$-module structure on $C_K$ with the Hecke action on $\text{End}_K(L_K)$, i.e., for $h \in H^0_K$ and $u \in C_K$, we have $\rho(hu) = h \bullet \rho(u)$.

**Proof.** Fix a flag $E_1 \subset E$, $E_2 = E/E_1$ with $\dim E_1 = n$, $\dim E_2 = k$. By (6.11) we must check that

$$\rho(\Psi(E_1)u)(x) = \Psi(\tau_{n+k}) \otimes \rho(u)(x) - \rho(u)(\Psi(\tau_k) \otimes x),$$

for each

$$u \in C_1 = K^{G_1 \times T}(C_1), \quad x \in M_2 = K^{G_2 \times T}(M_2) = K^T(\text{Hilb}_k).$$

Here the tensor product is the tensor product of coherent sheaves over $\text{Hilb}_{n+k}$ and $\text{Hilb}_k$ respectively. Recall the diagram

$$T^*_G X^* \xrightarrow{\phi_{G^*}} Z^*_G \xrightarrow{\psi_{G^*}} (T^*_G X')^*$$

$$\xrightarrow{g} \quad \xrightarrow{g}$$

$$T^*_G X^* \xrightarrow{\psi_{G^*}} (T^*_G X')^*,$$

where $T^*_G X^* = G \times_P (C_1 \times M_2^*)$, $Z^* = G \times_P N^*$ and $(T^*_G X')^* = M^*$. Recall also the induction

$$\text{Ind} : C_1 \otimes_R M_2 \to K^{G \times T}(T^*_G X^*).$$

We have

$$\rho(u)(x) = R(\psi_s)_* L\phi_s^* g_*(I_{u,x}), \quad I_{u,x} = \text{Ind}(u \otimes x).$$

Therefore we have

$$\rho(u)(\Psi(E_1)u)(x) = R(\psi_s)_* L\phi_s^* g_*(\theta_{\Psi(E_2)}(I_{u,x})),$$

where $\theta_{\Psi(E_2)} \in K^{G \times T}(T^*_G X^*)$ is the class induced from the class of the representation $\Psi(E_2)$ in $R^{P \times T}$. Note that the $G_2$-module $E_2$ is regarded as a $P$-module via the obvious map $P \to H = G_1 \times G_2$, and that it is equipped with the trivial $T$-action. Thus, the projection formula yields

$$\rho(u)(\Psi(E_1)u)(x) = R(\psi_s)_* (\theta_{\Psi(E_2)} \otimes L\phi_s^* g_*(I_{u,x})),

$$

where $\theta_{\Psi(E_2)} \in K^{G \times T}(Z^*)$ is induced from the class of $\Psi(E_2)$ in $R^{P \times T}$. For a similar reason we have also

$$\rho(\Psi(E_1)u)(x) = R(\psi_s)_* (\theta_{\Psi(E_2)} \otimes L\phi_s^* g_*(I_{u,x})),

$$

Finally, we have

$$\Psi(E_1)u)(x) = R(\psi_s)_* (\theta_{\Psi(E_2)} \otimes L\phi_s^* g_*(I_{u,x})),

$$

where $\Psi(E)$ is identified with its class in $R^{G \times T}$. The formula (7.12) is a direct consequence of (7.13), (7.14), (7.15). It is enough to observe that (7.15) is equivalent to the equality

$$\Psi(E_1)u)(x) = R(\psi_s)_* (\Psi(E) L\phi_s^* g_*(I_{u,x})),

$$

and that $\Psi(E) = \theta_{\Psi(E_1)} + \theta_{\Psi(E_1)}$ in $K^{G \times T}(Z^*)$.

$\square$
If $E$ is one-dimensional then

$$C_R^1 = C_{E,R} = R_{C^\ast \times T}[C], \quad R_{C^\ast \times T} = R[z,z^{-1}].$$

Recall that the symbol $[C]$ denotes the class of $\mathcal{O}_C$ in $C_{E,R} = K^{G \times T}(C)$ and that $R = R_T$.

We can now compare the representation $\psi : H_K \to \text{End}_K(L_K)$ from Section 3.2 with the representation $\rho : C_K \to \text{End}_K(L_K)$ from Proposition 7.10.

**Proposition 7.11.** If $E$ is one-dimensional then we have $\rho(z^l[C]) = \psi(f_1 l)$ for each $l \in \mathbb{Z}$.

*Proof.* Let us change the notation. Set $E_2 = E/E_1$ where $E_1 \subset E$ is a line and $\dim E = n + 1$. Let $x_1 = z^l[C_1]$ and $x_2 \in K^T(\text{Hilb}_n)$. We view $x_1, x_2$ as elements of

$$C_R^1 = K^{G_1 \times T}(C_1), \quad M_2 = K^{G_2 \times T}(M_2).$$

We have

$$\tau^l_{n+1,n} \ast x_2 \in K^T(\text{Hilb}_{n+1}) = M_{E,R}.$$ 

We must check that the image $x_3 \in M_{E,R}$ of $x_1 \otimes x_2$ by the map (7.7) is equal to $\tau^l_{n+1,n} \ast x_2$. In Section 7.6 we have defined a map $\phi_s : Z_s \to T^s X^s$ which restricts to the map

$$\phi_{G,s} : Z_G^s = G \times_p M_p^s \to T_G^s X^s = G \times_p (C_1 \times M_2^s).$$

Consider the Cartesian square of smooth connected varieties

$$\begin{array}{ccc}
T_G^s X^s & \xrightarrow{\phi_{G,s}^s} & Z_G^s \\
\downarrow g & & \downarrow g' \\
T^s X^s & \xrightarrow{\phi_s} & Z^s.
\end{array}$$

The vertical maps are the canonical closed embeddings. The induction yields an isomorphism

$$\text{Ind} : C_1 \otimes R M_2 \to K^{G \times T}(T_G^s X^s)$$

such that $x_3 = \text{R}(\psi_s)(y_3)$, where

$$y_3 = L\phi^s_{G,s} \text{Ind}(x_1 \otimes x_2) \in K^{G \times T}(Z_G^s).$$

By Lemma 7.8(c) the variety $Z_G^s$ is the set of tuples $(a,b,v,\varphi) \in M^s$ such that $a, b \in p$ and $\varphi(E_1) = 0$. It is well-known that the stability condition implies that $\varphi = 0$. Thus $Z_G^s$ is a $G$-torsor over $\text{Hilb}_{n+1,n}$. Hence we have

$$\dim Z_G^s = \dim G + \dim \text{Hilb}_{n+1,n} = n^2 + 4n + 3,$n^2 + 4n + 3,$

$$\dim Z^s = \dim G - \dim P + \dim N_p = 2n^2 + 5n + 3,$

$$\dim T_G^s X^s = \dim G - \dim P + \dim M_p^s = n^2 + 3n + 2,$$

$$\dim T^s X^s = 2\dim X = 2n^2 + 4n + 2.$$ 

Thus Proposition [C4] yields $L\phi^s_{G,s} \circ g_s = g'_s \circ L\phi^s_{G,s}$. Therefore we have

$$x_3 = \text{R}(\psi_{G,s}) L\phi^s_{G,s} \text{Ind}(x_1 \otimes x_2).$$

Thus we are reduced to observe that the canonical isomorphism

$$K^{G \times T}(Z_G^s) = K^T(Z_{n+1,n})$$

takes the class $L\phi^s_{G,s} \text{Ind}(x_1 \otimes x_2)$ to $\tau^l_{n+1,n} \otimes \pi_2^s x_2$. Recall that $\tau^l_{n+1,n}$ is the $l$-th power of the tautological bundle over $\text{Hilb}_{n+1,n}$. Thus the claim follows from the definition of $x_1$ and the formula for $\phi_{G,s}$ recalled above.

$\square$
7.8. The $R$-algebra $C_R$ is naturally $\mathbb{N}$-graded, with the piece $C^n_R \subset C_R$ of degree $n$ equal to the limit

$$C^n_R = \lim_{\to E} C_{E,R},$$

over the groupoid formed by all $n$-dimensional vector spaces with their isomorphisms. Consider the $R$-subalgebra $C_R \subset C_R$ generated by $C^n_R$. We'll abbreviate

$$C^n_R = C^n_R \cap C_R, \quad C_K = C_R \otimes_R K.$$

We have defined in Section 4.6 a $K$-subalgebra $H^n_K \subset H_K$. Proposition 7.11 bears the following consequence.

**Corollary 7.12.** For $n = 1$ the assignment $z^l[C] \mapsto f_{i,l}$, $l \in \mathbb{Z}$, defines uniquely a surjective $K$-algebra homomorphism $\Xi : \bar{C}_K \to H^n_K$ such that $\rho = \psi \circ \Xi$.

7.9. Our next goal is to prove that the map $\Xi$ is indeed an isomorphism. For each vector space $E$ of dimension $n$ the direct image by the inclusion $C \subset g \times g$ yields a $R_{G \times T}$-module homomorphism

$$C^n_R \to K^{G \times T}(g \times g).$$

We conjecture that (7.16) is an injective map. By the Thomason concentration theorem this conjecture is equivalent to the following one.

**Conjecture 7.13.** The $R_{G \times T}$-module $C^n_R$ is torsion-free.

The image of (7.16) is the quotient of $C^n_R$ by its torsion $R_{G \times T}$-submodule. Let $\bar{C}^n_R$ be the image of $C^n_R$ by (7.16) and let $C_R = \bigoplus_{n \geq 0} \bar{C}^n_R$.

**Theorem 7.14.** (a) The map (7.16) factors to a surjective algebra homomorphism $C_R \to \bar{C}_R$.

(b) The map $\Xi$ factors to a $K$-algebra isomorphism $\Xi : \bar{C}_K \to H^n_K$.

**Proof.** (a) The map (7.16) yields a surjective $R$-linear homomorphism $C_R \to \bar{C}_R$. We must check that the multiplication on $\bar{C}_R$ descends to $C_R$. This is obvious.

(b) Set $n = \dim E$. In Section 5.2 we have defined a class $\Lambda(V_n) \in H^n_K$. Since the map $\Xi : C_K \to H^n_K$ is surjective we may fix an element $\nu_n \in C_K$ which maps to $\Lambda(V_n)$. By the Thomason concentration theorem, the direct image by the inclusion $\{0\} \subset g^2$ yields an isomorphism

$$\text{Frac}(R_{G \times T}) \to C^n_R \otimes_{R_{G \times T}} \text{Frac}(R_{G \times T}).$$

Hence, for each $x \in C^n_K$ there are non-zero elements $\alpha, \beta \in R_{G \times T}$ such that

$$\alpha x = \beta \nu_n.$$

Now, assume that $\rho(x) = 0$. Then we have using Proposition 7.11 and 6.12

$$\psi(\beta \cdot \Lambda(V_n)) = \beta \cdot \psi(\Lambda(V_n)) = \beta \cdot \rho(\nu_n) = \rho(\beta \nu_n) = \rho(\alpha x) = \alpha \cdot \rho(x) = 0.$$

Thus $\beta \cdot \Lambda(V_n) = 0$ because $\psi$ is faithful and $\beta = 0$ by Proposition 6.3. Hence $x$ is a torsion element of $C^n_K$.

Combining the above result and Theorem 3.1 we obtain the following corollary.

**Corollary 7.15.** There is a graded algebra isomorphism $\Gamma : \bar{C}_K \to \mathcal{E}$ such that $\Gamma(\theta x) = \theta \cdot \Gamma(x)$ for any $x \in \bar{C}_K$ of degree $r$ and $\theta \in R_{GL_{r \times T}}.$
7.10. The associativity of the multiplication, proved in Proposition 7.5, yields a \( R \)-linear map
\[
(C^1_R)^\otimes n \to C^0_R.
\]
The lhs is identified with \( R[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_n^{\pm 1}] \), see Section 7.7. For \( \ell = (l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n \) we set
\[
z^\ell = z_1^{l_1} z_2^{l_2} \cdots z_n^{l_n} \in (C^1_R)^\otimes n.
\]
Let us fix a \( n \)-dimensional vector space \( E \). We have a canonical isomorphism
\[
C^0_R = K^{G \times T}(C).
\]
Let \( B \subset G \) be a Borel subgroup and \( H \subset B \) be a maximal torus. Let \( b, h \) be the Lie algebras of \( B, H \) and \( n \subset b \) be the nilpotent radical. Let \( \theta_\ell \) be the character of \( H \) associated with \( \ell \). It yields a 1-dimensional representation of \( B \). Set \( Z = G \times_B b^2 \). For each \( r \geq 0 \) we have the \( G \times T \)-equivariant vector bundle over \( Z \)
\[
\Lambda^r_Z(\ell) = G \times_B (b^2 \times \theta_\ell \otimes \Lambda^r n^*).
\]
Consider the complex
\[
\Lambda_Z(\ell) = \left\{ \cdots \to q^{-2t} t^{-2} \Lambda_Z^2(\ell) \to q^{-1} t^{-1} \Lambda_Z^1(\ell) \to \Lambda_Z^0(\ell) \right\},
\]
where the differential is given by the following map
\[
b^2 \times \Lambda^{r+1} n^* \to b^2 \times \Lambda^r n^*, \quad (a, b, \omega) \mapsto \iota_{(a, b)} \omega.
\]
Here \( \iota \) denotes the contraction. The cohomology sheaves of this complex are supported on the subset \( Z_G = G \times_B C_b \). Consider the proper map
\[
\psi : Z \to g^2, \quad (g, a, b) \mod B \mapsto (g a g^{-1}, g b g^{-1}).
\]
We have \( \psi(Z_G) \subset C \). Thus the class \( \Lambda(\ell) = [\psi_\ast \Lambda_Z(\ell)] \) belongs to \( C^0_R \). The \( R \)-module \( C^0_R \) is described by the following proposition. Since we’ll not use it, the proof is left to the reader.

**Proposition 7.16.** The following holds:

(a) for each \( \ell \in \mathbb{Z}^n \) the image of \( z^\ell \) by the map \( (7.17) \) is equal to \( \Lambda(\ell) \),

(b) the \( R \)-module \( C^0_R \) is spanned by the elements \( \Lambda(\ell) \) with \( \ell \in \mathbb{Z}^n \).

8. Higher rank

8.1. Fix integers \( r > 0, n \geq 0 \). Let \( M_{r, n} \) be the moduli space of framed torsion-free sheaves on \( \mathbb{P}^2 \) with rank \( r \) and second Chern class \( n \) (over \( \mathbb{C} \)). More precisely, closed points of \( M_{r, n} \) are isomorphism classes of pairs \( (E, \Phi) \) where \( E \) is a torsion-free sheaf which is locally free in a neighborhood of \( \ell_\infty \) and \( \Phi : E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^r \) is a framing at infinity. Here \( \ell_\infty = \{ [x : y : 0] \in \mathbb{P}^2 \} \) is the line at infinity. Recall that \( M_{r, n} \) is a smooth variety of dimension \( 2rn \) which admits the following alternative description. Let \( E \) be a \( n \)-dimensional vector space. As above we’ll abbreviate \( G = G_E, g = g_E \). There is an isomorphism of algebraic variety \( M_{r, n} = M_{r, E}^r / G \) where
\[
M_{r, E}^r = \{(a, b, \varphi, v) \in M_{r, E}; (a, b, \varphi, v) \text{ is stable}\},
\]
\[
M_{r, E} = \{(a, b, \varphi, v) \in N_{r, E}; [a, b] + v \circ \varphi = 0\},
\]
\[
N_{r, E} = g^2 \times \text{Hom}(E, C^r) \times \text{Hom}(C^r, E).
\]
The \( G \)-action is given by \( g(a, b, \varphi, v) = (g a g^{-1}, g b g^{-1}, \varphi g^{-1}, g v) \) and \( (a, b, \varphi, v) \) is stable iff there is no proper subspace \( E_1 \subset E \) which is preserved by \( a, b \) and contains \( v(C^r) \).
8.2. Consider the tori $T^* = (\mathbb{C}^*)^r$ and $T^2 = (\mathbb{C}^*)^2$. Let $R = R_T$ be the complexified Grothendieck ring of $T = T^* \times T^2$. We have $R = \mathbb{C}[q^{\pm 1}, t^{\pm 1}, e_1^{\pm 1}, \ldots, e_r^{\pm 1}]$ where $q: T \to \mathbb{C}^*(h, z_1, z_2) \mapsto z_1^{-1}$, $t: T \to \mathbb{C}^*(h, z_1, z_2) \mapsto z_2^{-1}$, $e_\alpha: T \to \mathbb{C}^*(h, z_1, z_2) \mapsto h_\alpha^{-1}$, and $h = (h_1, h_2, \ldots, h_r)$, $\alpha = 1, 2, \ldots, r$. In Section 8, we make the following change of notation:

$$K = \mathbb{C}(q^{1/2}, t^{1/2}, e_1^{1/2}, \ldots, e_r^{1/2})$$

and we extend the scalars of all the algebras $E, C, \ldots$ defined in the previous section to $K$.

We equip $M_{r,n}$ and $M^{*}_{r,E}$ with the $T$-action given by

$$(h, z_1, z_2) \cdot (a, b, v, \varphi) = (z_1 a, z_2 b, v h^{-1}, z_1 z_2 h \varphi).$$

In other words, given a pair $(\mathcal{E}, \Phi)$ as above, the element $h$ acts on the framing at infinity in the obvious way while $(z_1, z_2)$ acts on $\mathbb{P}^2$ by the formula

$$(z_1, z_2) \cdot [x : y : w] = [z_1 x : z_2 y : w].$$

This action has a finite number of isolated fixed points, indexed by the set of $r$-tuples of partitions with total weight $n$. To such a tuple $\lambda = (\lambda(1), \lambda(2), \ldots, \lambda(r))$ corresponds a fixed point $I_\lambda$ such that the class $T_\lambda = [T_{I_\lambda} M_{r,n}]$ in $R$ is given by

$$T_\lambda = \sum_{\alpha, \beta = 1}^r e_\beta^{-1} e_\alpha \sum_{\alpha \in \lambda(\alpha)} t^{\lambda(\alpha)(s)} q^{-a_{\lambda(\alpha)}(s)} - 1 + \sum_{\alpha \in \lambda(\beta)} t^{-l_{\lambda(\alpha)}(s)} q^{\lambda(\beta)(s)},$$

see [NY], Theorem 2.11.

8.3. The tautological bundle of $M_{r,n}$ is the $T$-equivariant locally free sheaf $\tau_n$ given by

$$\tau_n = M^{*}_{r,E} \times_G E.$$  

The character of the $T$-action on its fiber at the fixed point $I_\lambda$ is

$$\tau_\lambda = [\tau_n|_{I_\lambda}] = \sum_{\alpha} \sum_{\alpha \in \lambda(\alpha)} e_\alpha^{-1} t^{\lambda(\alpha)(s)} - 1 q^{\lambda(\alpha)(s)} - 1,$$

see [NY], Theorem 2.11 and [VV], Lemma 6. The classes of the tangent bundles and the tautological bundles are related by the following equation. For each $\lambda$ we have

$$T_\lambda = -(1 - q^{-1})(1 - t^{-1}) \tau_\lambda \otimes \tau_\lambda^* + \tau_\lambda \otimes W^* + q^{-1} t^{-1} \tau_\lambda^* \otimes W$$

where $W = e_1^{-1} + \cdots + e_r^{-1}$ is the class of the tautological representation of the torus $T^r$.

8.4. We can now define the Hecke correspondence. It is the variety

$$M_{r,n,n+1} = Z^{r,E}_{r,n+1},$$

where $Z^{r,E}_{r,n+1}$ is the variety of all tuples $(a, b, \varphi, v, E_1)$ where $(a, b, \varphi, v) \in M^{*}_{r,E}$ and $E_1 \subset E$ is a line preserved by $a, b$ such that $\varphi(E_1) = 0$. The variety $Z^{r,E}_{r,n+1}$ is a $G$-torsor over $M_{r,n,n+1}$, a smooth variety of dimension $2rn + r + 1$. Further the assignments

$$(a, b, \varphi, v) \mapsto (\bar{a}, \bar{b}, \bar{\varphi}, \bar{v}), (a, b, \varphi, v)$$

yield a closed immersion $M_{r,n,n+1} \subset M_{r,n} \times M_{r,n+1}$. Here $\bar{a}, \bar{b} \in g_{E_2}$, $\bar{\varphi} \in \text{Hom}(E_2, \mathbb{C}^*)$ are the induced linear maps and $\bar{v} = \pi \circ v$. As before we have set $E_2 = E/E_1$. The fixed points contained in $M_{r,n,n+1}$ are those pairs $I_{\mu,\lambda} = (I_\mu, I_\lambda)$ for which $\mu \subset \lambda$, i.e., we have $\mu(\alpha) \subset \lambda(\alpha)$ for all $\alpha$, and $\mu, \lambda$ have total weight $n, n + 1$ respectively. The class in $R$ of the fiber of the normal bundle to $M_{r,n,n+1}$ at the point $I_{\mu,\lambda}$ has the following expression

$$N_{\mu,\lambda} = -(1 - q^{-1})(1 - t^{-1}) \tau_{\mu} \otimes \tau_{\mu}^* + \tau_{\mu} \otimes W^* + q^{-1} t^{-1} \tau_{\mu}^* \otimes W - q^{-1} t^{-1}.$$  

See Appendix B for details.
Finally we introduce some elements $\phi:\tau_{n,n+1}\to\tau_1^{*}(\tau_n)$. The kernel sheaf $\text{Ker}(\phi)$ is a line bundle called the tautological bundle of $M_{r,n+1}$ which we denote by $\tau_{n,n+1}$. Over a $T$-fixed point $I_{\mu,\lambda}$ its character is

$$\tau_{\mu,\lambda} = [\tau_{n,n+1}|_{I_{\mu,\lambda}}] = e^{-1/p_\lambda}(s-1)q^{\lambda(s)-1},$$

where $s = \lambda^{(\alpha)}\backslash\mu^{(\alpha)}$ is the unique box of $\lambda$ not contained in $\mu$. We define the Hecke correspondence $M_{r,n+1,n}$ and the tautological bundle $\tau_{n+1,n}$ over it in the obvious way.

8.6. We’ll abbreviate $M_r = \bigsqcup_{n\geq 0} M_{r,n}$. Now, we apply the formalism in Section 3.1 to $X = M_r$, $G = T$ and $Y = \{pt\}$. Note that $M_{r,n}$ is not proper but has a finite number of fixed points. Hence the direct image provides us with an isomorphism

$$i_* : K^T(M_{r,n}) \otimes_R K \cong \bigoplus_{\lambda} K[\lambda] \longrightarrow K^T(M_{r,n}) \otimes_R K$$

where $i : M_{r,n}^T \to M_{r,n}$ is the embedding. We have also

$$K^T(M_{r,n} \times M_{r,m}) \otimes_R K = \bigoplus_{\lambda,\mu} K[I_{\lambda,\mu}].$$

This allows us to define convolution operations

$$\ast : K^T(M_{r,n} \times M_{r,m})_K \otimes K^T(M_{r,m} \times M_{r,k})_K \to K^T(M_{r,n} \times M_{r,k})_K,$$

$$\ast : K^T(M_{r,n} \times M_{r,m})_K \otimes K^T(M_{r,m})_K \to K^T(M_{r,n})_K.$$

Therefore, the associative $K$-algebra

$$E_K = \bigoplus_{k \in \mathbb{Z}} \prod_{n} K^T(M_{r,n+k} \times M_{r,n})_K,$$

where the product ranges over all integers $n \geq 0$ with $n + k \geq 0$, acts on the $K$-vector space

$$L_K = \bigoplus_{n \geq 0} K^T(M_{r,n})_K.$$

The integer $k$ yields a $\mathbb{Z}$-grading on $E_K$. There is an obvious $\mathbb{Z}$-grading on the module $L_K$.

8.7. We’ll write

$$\tau^l_{n,n+1} = (\tau_{n,n+1})^{\otimes l}, \quad \tau^{-l}_{n,n+1} = (\tau^*_{n,n+1})^{\otimes l}, \quad l \in \mathbb{Z}_{>0},$$

$$f_{-l} = \prod_n \tau^l_{n,n+1}, \quad f_l = \prod_n \tau^{-l}_{n+1,n}, \quad l \in \mathbb{Z},$$

$$e_{0,l} = \prod_n \Lambda^l \tau_{n,n}, \quad e_{0,-l} = \prod_n \Lambda^{-l} \tau^*_{n,n}, \quad l \in \mathbb{Z}_{>0}.$$

Once again, we define the elements $f_{0,l} \in E_K$ for $l \in \mathbb{Z}^+$ through the relations

$$\sum_{l \geq 1} (-1)^l f_{0,\pm l}s^{l-1} = -\frac{d}{ds} \log(E_{\pm}(s)), \quad E_{\pm}(s) = 1 + \sum_{k \geq 1} (-1)^k e_{0,\pm k}s^k.$$

So $f_{0,l}$ is obtained from the classes of the tautological bundles $\tau_{n,n}$ by the Adams operations

$$f_{0,l} = \prod_n \Psi_l(\tau_{n,n}).$$

Finally we introduce some elements $h_{i,l}$, $i = -1, 0, 1, l \in \mathbb{Z}$ defined as

$$h_{1,l} = \kappa^{-1/2}q^{1/2}f_{1,l-r}, \quad h_{-1,k} = (-1)^r \kappa^{-1/2}q^{1-r/2}f_{-1,k},$$

$$h_{0,n} = f_{0,n} - \frac{1}{(1 - q^n)(1 - \ell^n)}.$$
where $\kappa = e_1 \cdots e_r$ and we consider the $K$-subalgebra $H_K \subset E_K$ generated by $h_{\pm1,l}$, $h_{0,l}$, $l \in \mathbb{Z}$. The $K$-subalgebras $H_{K}^2, H_{K}^0, H_{K}^*$ are defined as in Section 4.6. We have a canonical faithful representation $\psi : H_K \to \text{End}_K(L_K)$. Compare Section 3.2. Consider the central charge $c^* = (1,q^{r/2}r^2/2)$.

**Theorem 8.1.** There is a $K$-algebra homomorphism $E_{c^*} \to H_K$ such that $u_{i,l} \mapsto h_{i,l}$ for all $i = -1,0,1$ and $l \in \mathbb{Z}$.

### 8.8. The proof is the same as the proof of Theorem 3.1. Let us just recall the main arguments.

(a) First we construct a $K$-algebra homomorphism $\Xi : \mathcal{C}_K \to H_K^*$ as in Proposition 7.9. Composing $\Xi$ with the isomorphism $E^* \simeq \mathcal{C}_K$ in Theorem 3.1 and Theorem 7.14 we get a $K$-algebra homomorphism $E^* \to H_K$.

To define $\Xi$ we first construct a representation of $C_K$ on $L_K$. To do that we let $E_1, E_2, \pi, H, P, h$ and $p$ be as in Sections 7.4, 7.6, and we set

\[
X' = g \times \text{Hom}(C', E), \quad Y = h \times \text{Hom}(C', E_2), \quad V = p \times \text{Hom}(C', E),
\]

\[
X = G \times_p Y, \quad W = G \times_p V.
\]

The $G$-action on $X'$, the $H$-action on $Y$ and the $P$-action on $V$ are the obvious ones. Set also

\[
N_h = h^2 \times \text{Hom}(E_2, C') \times \text{Hom}(C', E), \quad N_p = p^2 \times \text{Hom}(E_2, C') \times \text{Hom}(C', E),
\]

\[
N_g = N_{r,E}, \quad M_g = M_{r,E}, \quad M_h = C_{E_1 \times M_{r,E}}, \quad M_p = N_p \cap M_p.
\]

Then we apply the general construction in Section 7.2 with the map $p : V \to Y, (a, v) \mapsto (a_h, \pi \circ v)$ and the obvious inclusion $q : V \to X'$. We have the following lemma.

**Lemma 8.2.** (a) We have canonical isomorphisms of $G$-varieties

\[
T^*X' = N_g, \quad T^*X = G \times_p \{(c,a,b,\varphi,v) \in p \times N_h; c_b = [a,b] + v \circ \varphi\}, \quad Z = G \times_p N_p.
\]

For all $(a,b,\varphi,v) \in N_p$ we have

\[
\phi((g,a,b,\varphi,v) \text{ mod } P) = (g, [a,b] + v \circ \varphi, a_h, b_h, \varphi \circ \pi \circ v) \text{ mod } P,
\]

\[
\psi((g,a,b,\varphi,v) \text{ mod } P) = (g a g^{-1}, g b g^{-1}, (\varphi \circ \pi) g^{-1}, g v).
\]

(b) We have canonical isomorphisms of $G$-varieties

\[
T_{G}^*X = G \times_p M_g, \quad T_{G}^*X' = M_g, \quad Z_G = G \times_p M_p.
\]

The maps $\phi_G, \psi_G$ are the obvious ones.

Note that $K^T(M_{r,n}) = K^{G \times T}(M_{g}^*).$ Thus the proof of Proposition 7.9 and the formulas above yield the representation $\rho : \mathcal{C}_K \to \text{End}_K(L_K)$ we need. Next, the proof of Proposition 7.11 implies that if $E$ is one-dimensional then we have $\rho(z^l[C]) = \psi(f_{l,1})$ for each $l \in \mathbb{Z}$. Since the representation $\psi$ is faithful this yields a $K$-algebra homomorphism $\Xi : \mathcal{C}_K \to H_K^*$ which factors to $\mathcal{C}_K \to H_K^*$. Details are left to the reader.

(b) In the same way we define a $K$-algebra homomorphism $E^* \to H_K^*$. (c) Finally one checks that the class $[f_{1,l}, f_{-1,l}]$ is supported on the diagonal of $M_{r,n} \times M_{r,n}$ as in [VV]. Lemma 9, and this contribution is computed as in [VV], Section 4.5.

### 9. Heisenberg subalgebras

This short section contains a few remarks concerning a natural family of Heisenberg subalgebras in $H_K$ and their action on $L_K = \bigoplus_n K^T(\text{Hilb}_n)_K$. 
9.1. Heisenberg subalgebras. Fix $\mu \in \mathbb{Q} \cup \{\infty\}$. Write $\mu = d/r$ with $r \geq 0$, $d$ and $r$ coprime (and $d = 1$ if $r = 0$). Let us set for simplicity $u^\mu_l = u_{r,l}^\mu$ for any $l \in \mathbb{Z}$. Let $\hat{\mathcal{E}}^\mu$ be the subalgebra of $\hat{\mathcal{E}}$ generated by $K$ and the elements $\{u^\mu_{-1}, u^\mu_{-2}, \ldots\}$. This algebra is isomorphic to a (quantum) Heisenberg algebra. The defining relations are

$$[K, u^\mu_l] = 0, \quad [u^\mu_l, u^\mu_n] = \delta_{l,-n} \frac{(c^\mu_l c^\mu_n)^n - (c^\mu_l c^\mu_n)^{-n}}{\alpha_n}.$$ 

As before we have set $\alpha_n = (1 - (c \sigma)^{-n})(1 - \sigma^{-n})$. We define subalgebras $\mathcal{E}^\mu_c$ in the obvious way. Two cases are of special interest

i) $\mu = \infty$. The subalgebra $\mathcal{E}^\infty_c$ acts on $L_K$ by Macdonald operators (see Section 1.4.). It also gives rise, through the adjoint action, to the action of the Hecke operators on $\mathcal{E}_c$ (see Section 6). In this case the central charge vanishes and $\mathcal{E}^\infty_c$ is a polynomial algebra,

ii) $\mu = 0$. The subalgebra $\mathcal{E}^0_c$ acts on $L_K$ by means of the correspondences induced by the virtual classes $\tau^*_n \otimes \Lambda(V_n)$ and $\Lambda(qt^*V_n^*)$ for $n \geq 1$ (see Section 5).

Note that (when the central charge is set to zero) the various Heisenberg subalgebras $\hat{\mathcal{E}}^\mu$ are interchanged by the $SL(2, \mathbb{Z})$-action. When viewed as the Hall algebra of an elliptic curve, $\hat{\mathcal{E}}^\mu$ consists of the functions supported on the set of semistable coherent sheaves of slope $\mu$ (see [BS], Section 5).

9.2. Casimir operators. There is a natural nondegenerate pairing on $\hat{\mathcal{E}}^\infty$, coming from its realization as a Hall algebra (see [BS], Section 2 and Section 4.5). It is given by

$$\langle u^\mu_l, u^\nu_n \rangle = \delta_{l,-n} \frac{1}{\alpha_l}.$$ 

More generally we have

$$\langle u^\mu_\lambda, u^\nu_\nu \rangle = \delta_{\lambda,\nu} \prod_i m_i(\lambda)! \prod_i \frac{1}{\alpha_i m_i(\lambda)}$$ 

where $\lambda, \nu$ are partitions, $\lambda = (\lambda_1, \lambda_2, \ldots) = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \ldots)$ and $u^\mu_\lambda = \prod_i u^\mu_{\lambda_i}$. We introduce the canonical Casimir operator

$$C^\mu = \sum_\lambda \frac{1}{\langle u^\mu_\lambda, u^\mu_\lambda \rangle} u^\mu_\lambda u^\mu_\lambda^{-\lambda}$$ 

where the sum ranges over all partitions and where we have set $u^\mu_\lambda = \prod_i u^\mu_{\lambda_i}$. Although the sum defining it is infinite, the operator $C^\mu$ acts on $L_K$ since $u^\mu_\lambda$ acts by zero on $K^T(\text{Hilb}_n)K$ as soon as $|\lambda| > n$. We expect that these Casimir operators are relevant to the study of the monodromy of the so-called quantum differential equation arising in GW/DT theory of the Hilbert scheme of points in the plane (see [OP]).

Remarks 9.1. i) There is a factorization

$$C^\mu = \left( \sum_n \frac{1}{n!} \alpha_1^n(u^\mu_1)^n(u^\mu_{-1})^n \right) \left( \sum_n \frac{1}{n!} \alpha_2^n(u^\mu_2)^n(u^\mu_{-2})^n \right) \cdots.$$ 

The factors in the expression of $C^\mu$ above are not the exponential of any natural expressions because the algebra $\mathcal{E}^\mu_c$ is not commutative. The Heisenberg algebras $\hat{\mathcal{E}}^\mu$ have a Hopf algebra structure coming from their Hall algebra realization. The primitive elements are exactly the $u^\mu_l$ with $l \in \mathbb{Z}$. This leads to another variant of the Casimir operator, given by

$$C'^\mu = \exp \left( \alpha_1 u^\mu_1 u^\mu_{-1} \right) \exp \left( \alpha_2 u^\mu_2 u^\mu_{-2} \right) \cdots = \exp \left( \sum_l \alpha_l u^\mu_l u^\mu_{-l} \right).$$ 

For instance, using Section 4.7, we get $C'^0 = \exp \left( \sum_{l \geq 1} ((tq)^{l/2} - (tq)^{-l/2}) \frac{1}{l \alpha_l} \right)$. Note that $C^\mu$ and $C'^\mu$ coincide if the central charge vanishes, i.e., when $\mu = \infty$. 
ii) The group $SL(2, \mathbb{Z})$ does not act on $E_c$ (because of the central charge), but its unipotent subgroup $\left( \begin{array}{cc} 1 & Z \\ 0 & 1 \end{array} \right) \simeq \mathbb{Z}$ does, via automorphisms $\rho_n : u_{r,d} \mapsto u_{r,d+nr}$. In particular, $\rho_n(C^\mu) = C^{\mu+\nu}$ for any $\mu \in \mathbb{Q}$. Furthermore, the representation $L_K$ and its twist $L_K^{\rho_1}$ are isomorphic, with intertwining operator

$$\nabla : L_K \rightarrow L_N, \quad [I_{\lambda}] \mapsto t^{\mu(\lambda)}q^{\nu(\lambda')}[I_{\lambda}],$$

As a consequence, the Casimir operators on $L_K$ satisfy

$$C^{\mu+1} = \nabla C^\mu \nabla^{-1}$$

for any $\mu \in \mathbb{Q}$. Note that the operator $\nabla$ restricts, for any given $n$, to the action on the K-theory of the tensor product by the relative ample line bundle $O(1)$ on Hilb$_n$ (see [H]).

10. The shuffle algebra

In this last section we provide an alternative algebraic description of the algebras $H^\otimes_K$ or $E^\otimes$. This presentation involves a certain (noncommutative) shuffle product on the algebra of symmetric functions $A_K$. It appeared independently in [FT]. We also discuss and compare the results of loc. cit. with ours.

10.1. One may use the faithful action of $E^\otimes$ on the representation $L_K$ together with the virtual classes $V_l$, $l \geq 1$, to give a new presentation of the $E^\otimes$ as some kind of $q,t$-shuffle algebra. The precise formalism, which we now briefly recall, was introduced by B. Feigin and A. Odesskii in [FO]. Let $g(z) \in \mathbb{C}(z)$ be any rational function. For $r \geq 1$ we put $g(z_1, \ldots, z_r) = \prod_{1 \leq j \leq r} g(z_j)$. Let us denote by

$$Sy^m_r : \mathbb{C}(z_1, \ldots, z_r) \rightarrow \mathbb{C}(z_1, \ldots, z_r)^{\otimes r}, \quad P(z_1, \ldots, z_r) \mapsto \sum_{\sigma \in S_r} P(z_{\sigma(1)}, \ldots, z_{\sigma(r)})$$

the standard symmetrization operator and let us consider the weighted symmetrization

$$\Psi_r : \mathbb{C}[z_1^{\pm1}, \ldots, z_r^{\pm1}] \rightarrow \mathbb{C}(z_1, \ldots, z_r)^{\otimes r}, \quad P(z_1, \ldots, z_r) \mapsto \Psi_r(g(z_1, \ldots, z_r)P(z_1, \ldots, z_r)).$$

We denote by $S_r$ the image of $\Psi_r$. There is a unique map $m_{r,r'} : S_r \otimes S_{r'} \rightarrow S_{r+r'}$ which makes the following diagram commute

$$\begin{array}{ccc}
\mathbb{C}[z_1^{\pm1}, \ldots, z_r^{\pm1}] \otimes \mathbb{C}[z_1^{\pm1}, \ldots, z_{r'}^{\pm1}] & \xrightarrow{\Psi_r \otimes \Psi_{r'}} & S_r \otimes S_{r'} \\
i_{r,r'} \sim & & m_{r,r'} \downarrow \\
\mathbb{C}[z_1^{\pm1}, \ldots, z_{r+r'}^{\pm1}] & \xrightarrow{\Psi_{r+r'}} & S_{r+r'}
\end{array}$$

where $i_{r,r'}(P(z_1, \ldots, z_r) \otimes Q(z_{r+1}, \ldots, z_{r+r'})) = P(z_1, \ldots, z_r)Q(z_{r+1}, \ldots, z_{r+r'})$. It is easy to check that the maps $m_{r,r'}$ endow the space $S = \mathbb{C}1 \oplus \bigoplus_{r \geq 1} S_r$ with the structure of an associative algebra. The product in $S$ may be explicitly written as the shuffle operation

$$h(z_1, \ldots, z_r) \cdot f(z_1, \ldots, z_{r'}) = \frac{1}{r!r'!} Sy^m_{r+r'} \left( \prod_{1 \leq i \leq r} g(z_i) \cdot h(z_1, \ldots, z_r)f(z_{r+1}, \ldots, z_{r+r'}) \right).$$

Note that by construction the algebra $S$ is generated by the subspace $S_1$. We may replace the ground field $\mathbb{C}$ by $K$ and define $S$ for any $g(z) \in K(z)$.

From now on we fix

$$g(z) = \frac{(1 - tz)(1 - qz)}{(1 - z)(1 - tqz)}.$$
Recall that the action of $\mathcal{E}^0$ on $\mathcal{E}^>$ by Hecke operators factors yields, for each $r$, an action
\[ \bullet : K[z_1^{\pm 1}, \ldots, z_r^{\pm 1}]^{S_r} \otimes \mathcal{E}^> [r] \to \mathcal{E}^> [r]. \]

**Theorem 10.1.** The assignment $u_{1,1} \mapsto z_1^l$ for $l \in \mathbb{Z}^*$ extends to a graded algebra isomorphism $\Upsilon : \mathcal{E}^> \cong S$. Moreover, for any $P(z_1, \ldots, z_r) \in K[z_1^{\pm 1}, \ldots, z_r^{\pm 1}]^{S_r}$ and any $u \in \mathcal{E}^> [r]$ we have
\[ \Upsilon(P(z_1, \ldots, z_r) \bullet u) = P(z_1, \ldots, z_r) \Upsilon(u). \]

As mentioned above, the above theorem may be proved by considering the action of $\mathcal{E}^>$ on $L_K$ and expanding it in terms of the virtual classes. However, since a more general theorem (giving a shuffle presentation for the Hall algebra of any smooth projective curve) appears in [SV2] we omit the proof here.

**10.2.** Consider the generating series
\[ E^+(z) = \sum_{p \in \mathbb{Z}} u_{1,p} z^p, \]
Kapranov showed in [K], Theorem 3.3, that the following relation holds:
\[ \zeta(z_1/z_2) E^+(z_1) E^+(z_2) = \zeta(z_2/z_1) E^+(z_2) E^+(z_1), \quad \zeta(z) = \frac{(1 - \sigma z)(1 - \bar{\sigma} z)}{(1 - z)(1 - \sigma \bar{\sigma} z)} = g(z^{-1}). \]
Equations (10.3) is the so-called functional equation for Eisenstein series. Let $\mathcal{E}^>$ be the associative algebra generated by some elements $u_{1,p}, \ p \in \mathbb{Z}$ subject to (10.3). There is a surjective map $\pi^> : \mathcal{E}^> \to \mathcal{E}^>$. It is known however, that this map is NOT an isomorphism. Its kernel is presumably described by some higher rank analogs of (10.3). One can complete the algebra $\mathcal{E}^>$ by adding new generators
\[ E^-(z) = \sum_{p \in \mathbb{Z}} u_{-1,p} z^p, \quad \psi^\pm(z) = \exp \left( \sum_{p \in \mathbb{Z}} u_{0,\pm p}/\alpha_r \right), \]
and certain new relations, see, e.g., [FT]. One gets in this way the so-called Ding-Iohara algebra $\mathcal{E}$ associated to $\zeta(z)$. There is a surjective (but not injective) algebra map $\pi : \mathcal{E} \to \mathcal{E}_c$.

**10.3.** In [FT] the authors define the algebras $\mathcal{E}$, $S$ and construct two representations
\[ \rho : \mathcal{E} \to \text{End}_K(L_K), \quad \rho' : S \to \text{End}_K(L_K). \]
They prove that $\rho$, $\rho'$ are compatible in the sense that we have $\rho|_{\mathcal{E}^>} = \rho' \circ \Upsilon \circ \pi|_{\mathcal{E}^>}$. The representation $\rho'$ coincides, via $\Upsilon$, with our representation $\varphi : \mathcal{E}_c \to \text{End}_K(L_K)$ in Proposition 4.10, i.e., we have $\rho' = \varphi \circ \Upsilon^{-1}$ over $S$.

Our results imply also that the representation $\rho$ factors through our $\varphi$, i.e., we have $\rho = \phi \circ \pi$, that $\varphi$ is faithful, that $\mathcal{E}_c$ is related to the Double Affine Hecke algebra, that $\varphi$ can be expressed in terms of Macdonald’s difference operator, and that $\mathcal{E}^>$ is related to the $K$-theory of the commuting variety. Note also that the subalgebra of $\mathcal{E}^>$ generated by the virtual fundamental classes is probably the same as the commutative subalgebra of $S$ studied in [FT] (we have not checked this).
APPENDIXES

A.1. Proof of relation \(1.5\). We begin with the following lemma.

Lemma A.1. For any \(k, l \in \mathbb{Z}\) the class \([f_{1,l}, f_{-1,k}]\) is supported on the diagonal of \(\text{Hilb} \times \text{Hilb}\).

Proof. Let \(\lambda\) be any partition and \(s\) be an addable box of \(\lambda\). Let \(\mu\) be the partition such that \(\mu \setminus \lambda = s\). Let \(s'\) be a removable box of \(\mu\) different from \(s\). It is also a removable box of \(\lambda\). Define partitions \(\nu\) and \(\sigma\) by \(\mu \setminus \nu = s'\) and \(\lambda \setminus \sigma = s'\). We have

\[
\langle \nu, f_{1,l} f_{-1,k} : \lambda \rangle = \langle \nu, f_{1,l} : \sigma \rangle \langle \sigma, f_{-1,k} : \lambda \rangle,
\]

\[
\langle \nu, f_{-1,k} f_{1,l} : \lambda \rangle = \langle \nu, f_{-1,k} : \mu \rangle \langle \mu, f_{1,l} : \lambda \rangle.
\]

Lemma 4.3 yields

\[
\langle \sigma, f_{-1,k} : \lambda \rangle \langle \nu, f_{-1,k} : \mu \rangle^{-1} = L_{\sigma,\nu}(q,t)L_{\lambda,\mu}(q,t)^{-1}
\]

(A.1)

where \(u\) is the (unique) intersection point of \(C_s \cup R_s\) and \(C_t \cup R_t\). Similarly,

\[
\langle \nu, f_{1,l} : \sigma \rangle \langle \mu, f_{1,l} : \lambda \rangle^{-1} = L_{\lambda,\sigma}(q,t)L_{\mu,\nu}(q,t)^{-1}
\]

(A.2)

Hence \(\langle \nu, f_{1,l} f_{-1,k} : \lambda \rangle = \langle \nu, f_{-1,k} f_{1,l} : \lambda \rangle\) for all \(\nu \neq \lambda\). We are done. \(\square\)

The precise computation of the quantities \(\langle \lambda, [f_{1,l}, f_{-1,k}] : \lambda \rangle\) is more involved. For this it is convenient to use the following changes of variables, see \(\text{GT}\), Section 2. Let \(\lambda\) be our fixed partition. We label the removable boxes of \(\lambda\) by \(A_1, A_2, \ldots, A_r\) from left to right, and we set \(\alpha_k = y(A_k), \beta_k = x(A_k)\). We introduce new variables \(x_1, \ldots, x_r, u_0, \ldots, u_r\) by

\[
x_k = t^{\alpha_k} q^{\beta_k}, \quad u_l = t^{\alpha_{l+1}} q^{\beta_l}
\]

where by convention \(\beta_0 = \alpha_{r+1} = 0\). We also let \(C_0, \ldots, C_r\) stand for the addable boxes of \(\lambda\), so that \(y(C_l) = \alpha_{l+1} + 1, x(C_l) = \beta_l + 1\). Here is an example with \(\lambda = (10, 9^3, 6, 3^2)\)

![Diagram of Garsia and Tesler’s change of variables](image)

**Figure 3.** Garsia and Tesler’s change of variables.

Lemma A.2 \([\text{GT}]\). The following formulas hold

(a) for any \(m \in \mathbb{Z}\) we have

\[
\sum_i x_i^m - \sum_j y_j^m = (1 - t^{-m})(1 - q^{-m})B_m^\nu(q,t) - t^{-m} q^{-m},
\]

(A.3)
(b) if \( \mu \subset \lambda \) is such that \( \lambda \backslash \mu = A_i \) then we have
\[
(A.4) \quad L_{\lambda, \mu}(q, t) = \frac{t q x_i^{-1}}{(1 - t)(1 - q)} \prod_{j=0}^{r} (u_j - x_i) \prod_{j=1}^{r} (x_j - x_i)^{-1},
\]

(c) if \( \mu \supset \lambda \) is such that \( \mu \backslash \lambda = C_i \) then we have
\[
(A.5) \quad L_{\lambda, \mu}(q, t) = \frac{u_i^{-1}}{q t} \prod_{j=0}^{r} (u_i - x_j) \prod_{j \neq i}^{r} (u_i - u_j)^{-1}.
\]

Using the above \([A.4]\) and \([A.5]\) we obtain, after a little arithmetic
\[
(A.6) \quad \langle \mu, f_{1, l} \cdot \lambda \rangle = \frac{1}{(1 - q)(1 - t)}(qt)^l u_l^i \cdot \prod_{j=0}^{r} (u_j - x_i) \prod_{j \neq i}^{r} (x_j - x_i)^{-1} \text{ if } \mu \supset \lambda \text{ and } \mu \backslash \lambda = C_i,
\]
\[
(A.7) \quad \langle \nu, f_{-1, k} \cdot \lambda \rangle = \frac{t q}{(1 - q)(1 - t)} x_i^{k-1} \cdot \prod_{j=0}^{r} (u_j - x_i) \prod_{j \neq i}^{r} (x_j - x_i)^{-1} \text{ if } \nu \subset \lambda \text{ and } \lambda \backslash \nu = A_i,
\]
\[
(A.8) \quad \langle \lambda, f_{1, l} f_{-1, k} \cdot \lambda \rangle = \frac{1}{(1 - q)(1 - t)} \sum_{i=0}^{r} \frac{(qt)^l - (1 - t^n q^{-n})}{q t u_i^i - x_i} \prod_{j=1}^{r} \frac{u_i - x_j}{t q u_i - x_j} \prod_{j \neq i}^{r} \frac{t q u_i - u_j}{u_i - u_j}.
\]

In particular, the commutator \([f_{-1, k}, f_{1, l}]\), being supported on the diagonal, only depends on \(k + l\). We put
\[
\gamma_m = [f_{-1, k}, f_{1, l - 1}]
\]
for any \((k, l)\) such that \(k + l = m\). We now establish a formula for the generating series of \(\gamma_m\).

**Proposition A.3.** We have
\[
(1 - q)(1 - t) \sum_{m \geq 0} \gamma_m s^m = \frac{1}{(1 - q t)} \left[ 1 - q t \exp \left( \sum_{n \geq 1} (1 - t^n q^n) \left( (1 - t^{-n})(1 - q^{-n}) f_{0, n} - t^{-n} q^{-n} \right) \frac{s^n}{n} \right) \right]
\]
and
\[
(1 - q)(1 - t) \sum_{m \geq 0} \gamma_{-m} s^m = \frac{1}{(1 - q t)} \left[ -q t + \exp \left( \sum_{n \geq 1} (1 - t^{-n} q^{-n}) \left( (1 - t^n)(1 - q^n) f_{0, -n} - t^n q^n \right) \frac{s^n}{n} \right) \right].
\]

**Proof.** We sketch the proof of the first equality, the second is similar. Using the change of variables introduced above, \([A.3]\) and \([A.8]\), we are reduced to showing the following identity
\[
(A.9) \quad \sum_{i=0}^{r} \frac{1}{1 - t q u_i s} T_i - \sum_{i=1}^{r} \frac{t q}{1 - x_i s} S_i = \frac{1}{1 - t q} - \frac{t q}{1 - t q} \prod_{i=1}^{r} \frac{1 - t q x_i s}{1 - x_i s} \prod_{i=0}^{r} \frac{1 - u_i s}{1 - t q u_i s}
\]
where we have set
\[ T_i = \prod_{j=1}^{r} \frac{u_i - x_j}{tq u_i - x_j} \prod_{j=0}^{r} \frac{tq u_i - u_j}{u_i - u_j}, \quad S_i = \prod_{j=0}^{r} \frac{u_j - x_i}{tq u_j - x_i} \prod_{j \neq i}^{r} \frac{tq x_j - x_i}{x_j - x_i}. \]

The proof of (A.9) is standard. Namely, by direct computations we check that the l.h.s and the r.h.s. have the same poles and residues. Since both are rational functions of degree zero, they differ by at most a constant. To compute this constant we set \( s = 0 \). This reduces (A.9) to
\[ \sum_{i=0}^{r} \prod_{j=1}^{r} \frac{u_i - x_j}{tq u_i - x_j} \prod_{j=0}^{r} \frac{tq u_i - u_j}{u_i - u_j} - tq \sum_{i=0}^{r} \prod_{j=0}^{r} \frac{u_j - x_i}{tq u_j - x_i} \prod_{j \neq i}^{r} \frac{tq x_j - x_i}{x_j - x_i} = 1 \]
which is itself a corollary of the following formula: for any variables \( \xi_1, \ldots, \xi_n, p \) we have
\[ \sum_{i=1}^{n} \left( \prod_{j \neq i} p \xi_i - \xi_j \right) = 1 + p + \ldots + p^{n-1}. \]
We leave the details to the reader. \( \Box \)

It is straightforward to check (3.5) from the above proposition and (3.5, 3.7).

A.2. Proof of Lemma 5.3 We have to show that for any \( n \geq 1 \) we have
\[ [f_{0,1}, \Lambda E_n] = \Lambda E_{n-1} f_{1,1} - tq f_{1,1} \Lambda E_{n-1}. \]
This means that for every pair of partitions \( \lambda, \mu \) with \( \mu \subset \lambda \) and \( |\lambda| - |\mu| = n \) it holds
\[ [f_{0,1}, \Lambda E_n]|_{\lambda, \mu} = (\Lambda E_{n-1} f_{1,1})|_{\lambda, \mu} - tq (f_{1,1} \Lambda E_{n-1})|_{\lambda, \mu}. \]
By means of Lemma 5.1 (a) and Lemma 4.5 we have
\[ [f_{0,1}, \Lambda E_n]|_{\lambda, \mu} = \left( \sum_{s \in \lambda \setminus \mu} q^x(s) t^y(s) \right) \Lambda E_{\lambda, \mu} \]
while
\[ (\Lambda E_{n-1} f_{1,1})|_{\lambda, \mu} = \sum_{s} q^x(s) t^y(s) \cdot \Lambda E_{\lambda, \mu+s} \cdot \Lambda N^*_{\mu+s, \mu} \cdot \Lambda_{\mu+s}^{-1} \]
where the sum ranges over all adable boxes of \( \mu \) which lie in \( \lambda \), and
\[ (f_{1,1} \Lambda E_{n-1})|_{\lambda, \mu} = \sum_{s} q^{-x(s)} t^{-y(s)} \cdot \Lambda N^*_{\lambda \setminus s} \cdot \Lambda E_{\lambda \setminus s, \mu} \cdot \Lambda_{\lambda \setminus s}^{-1} \]
where the sum now ranges over all removable boxes of \( \lambda \) which do not lie in \( \mu \). Quantities (A.12) and (A.13) may be nicely expressed via the change of variables introduced by Garsia and Tesler, as in Appendix A. Namely, let \( x_1, \ldots, x_r, u_0, \ldots, u_r \) be the variables associated to \( \lambda \) and likewise let \( x'_1, \ldots, x'_r, u'_0, \ldots, u'_r \) be those associated to \( \mu \). Then we have
\[ \frac{(\Lambda E_{n-1} f_{1,1})|_{\lambda, \mu}}{\Lambda_{\lambda, \mu}} = \frac{1}{(1-q)(1-t)} \sum_{i=1}^{r} \left\{ \prod_{j=0}^{r} (u_j - x_i) \prod_{j \neq i}^{r} (x_j - x_i)^{-1} \cdot \prod_{j=1}^{p} (x'_j - x_i) \prod_{j=0}^{p} (u'_j - x_i)^{-1} \right\} \]
and
\[ \frac{(f_{1,1} \Lambda E_{n-1})|_{\lambda, \mu}}{\Lambda_{\lambda, \mu}} = \frac{qt}{(1-q)(1-t)} \sum_{i=0}^{p} \left\{ \prod_{j=0}^{r} (u'_i - u_j) \prod_{j=1}^{r} (u'_i - x_j)^{-1} \cdot \prod_{j=0}^{p} (u'_i - x'_j) \prod_{j \neq i}^{p} (u'_i - u'_j)^{-1} \right\}. \]
Note that by Lemma A.2 (a),
\[(1 - t^{-1})(1 - q^{-1}) \sum_{s \in \lambda, \mu} q^{\gamma(s)} y^{(s)} = \left( \sum_{i=1}^{r} x_i - \sum_{i=0}^{r} u_i \right) - \left( \sum_{i=1}^{p} x'_i - \sum_{i=0}^{p} u'_i \right)\]
so that (A.11) reduces to the following

**Claim.** The following identity holds (in the field of rational functions in the variables involved)
\[(A.14)\]
\[
\left( \sum_{i=1}^{r} x_i - \sum_{i=0}^{r} u_i \right) - \left( \sum_{i=1}^{p} x'_i - \sum_{i=0}^{p} u'_i \right) =
\sum_{i=0}^{r} \left\{ \prod_{j=0}^{r} (u_j - x_i) \cdot \prod_{j=1}^{r} (x_j - x_i)^{-1} \cdot \prod_{j=1}^{r} (x'_j - x_i) \cdot \prod_{j=0}^{r} (u'_j - x_i)^{-1} \right\}
- \sum_{i=0}^{p} \left\{ \prod_{j=1}^{p} (u'_j - u_j) \cdot \prod_{j=1}^{r} (u'_j - x_j)^{-1} \cdot \prod_{j=0}^{p} (u_j - x_j) \cdot \prod_{j=0}^{p} (u'_j - u'_j)^{-1} \right\}.
\]

**Proof of claim.** Let us denote by \(A_{r,s} = A_{r,s}(x_i, x'_i, u_i, u'_i)\) the r.h.s of (A.14). This is a rational function of degree one, whose poles are of order at most one and are located along the hyperplanes \(x_i = x_j, u'_j = u'_j\) and \(u'_i = x_j\) for \(i \neq j\). A simple calculation shows that the residues of \(A_{r,s}\) vanish along all these hyperplanes, hence \(A_{r,s}\) is in fact a polynomial in the variables \(x_i, x'_i, u_i, u'_i\) which we may write as
\[A_{r,s} = \sum_{i} \alpha_i x_i + \sum_{i} \alpha'_i x'_i + \sum_{i} \beta_i u_i + \sum_{i} \beta'_i u'_i + \gamma\]
for some scalars \(\alpha_i, \alpha'_i, \beta_i, \beta'_i\) and \(\gamma\). By considering the leading coefficients of \(A_{r,s}\) in \(x_i\) and \(u'_i\), one sees that \(\alpha_i = \beta'_i = -1\) for all \(i\). Next, observe that the specialization of \(A_{r,s}\) at \(u_r = x_r\) is equal to \(A_{r-1,s}\) and that similarly the specialization of \(A_{r,s}\) at \(x'_r = u'_r\) is equal to \(A_{r,s-1}\). It follows that \(\beta_i = -\alpha_i = 1\) and \(\alpha'_i = -\beta'_i = 1\), and that \(\gamma = A_{0,0} = 0\). The claim, and with it Lemma 5.3, are proved.

**Appendix B. Hecke correspondences in higher rank**

Fix a pair of vector space \(E_1, E_2\) of dimension \(n, n+1\) respectively. We define a \(T\)-equivariant complex of vector bundles over \(M_{r,n} \times M_{r,n+1}\) by
\[
q\text{Hom}(E_1, E_2) \oplus t\text{Hom}(E_1, E_2)
\]
(B.1) \[
\text{Hom}(E_1, E_2) \xrightarrow{\sigma} \oplus \xrightarrow{\tau} q\text{Hom}(E_1, E_2) \oplus qt,
\]
where the maps \(\sigma, \tau\) are defined by
\[
\sigma(\xi) = \begin{pmatrix}
\xi a_1 - a_2 & \xi b_1 - b_2 \\
\xi c_1 - c_2 & -\xi v_1
\end{pmatrix}, \quad \tau \begin{pmatrix}
c \\
d \\
p \\
v
\end{pmatrix} = ([a, d] + [c, b] + V \varphi_1 + v_2 \Phi) \oplus (\text{tr}_{E_1}(v_1 \Phi) \oplus \text{tr}_{E_2}(V \varphi_2)),
\]
with
\[
[a, d] = a_2 d - da_1, \quad [c, b] = cb_1 - b_2 c.
\]
It is well-known that the unique non-zero cohomology group is \(\mathcal{V} = \text{Ker}(\tau)/\text{Im}(\sigma)\) and that the latter is a \(T\)-equivariant vector bundle of rank \(r(2n + 1) - 1\) which admits a section which vanishes precisely on the Hecke correspondence \(M_{r,n,n+1}\). Therefore, for each \(\mu \subset \lambda\) we have
\[
N_{\mu, \lambda} = [\mathcal{V}^* |_{\mu, \lambda}].
\]
Appendix C. Complements on base change

Consider the following Cartesian square of smooth connected varieties

\[
\begin{array}{ccc}
X' & \xleftarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xleftarrow{f} & Y \\
\end{array}
\]

Observe that the maps \(f, f', g, g'\) are regular. So the derived pull-back in K-theory are well-defined. Let \(x, y, x', y'\) be the dimensions of \(X, Y, X', Y'\). The aim of this appendix is to prove the following.

**Proposition C.1.** Assume that \(g\) is proper, that \(f' \times g'\) is a closed embedding \(Y' \to X' \times Y\) and that \(x + y = x' + y\). Then \(g'\) is proper and we have \(L f^* \circ R g_* = R g'_* \circ L (f')^*\), an equality of group homomorphisms \(K(X') \to K(Y)\).

**Proof.** First, observe that \(g'\) is proper because the base-change of a proper map is again proper. Now we prove the proposition in two steps.

(a) Assume that \(g, g'\) are closed (regular) embeddings. The hypothesis on the dimensions implies that \(\text{Tor}_k^X(\mathcal{O}_{X'}, \mathcal{O}_Y) = 0\) for each \(k > 0\), see [T], Lemma 3.2. Therefore \(X'\) and \(Y\) are tor-independent over \(X\), see [SGA6], Definition 1.5.

(b) Now, assume that \(g, g'\) are any regular morphisms. Consider the following diagram with Cartesian squares

\[
\begin{array}{ccc}
X' & \xleftarrow{p_1} & X' \times Y \xleftarrow{f' \times g'} Y' \\
\downarrow{g} & & \downarrow{g \times 1} \\
X & \xleftarrow{p_1} & X \times Y \xleftarrow{f \times 1} Y.
\end{array}
\]

Here \(p_1\) means the projection along the second factor. Since \(p_1\) is a smooth morphism the varieties \(X'\) and \(X \times Y\) are tor-independent over \(X\). Since \(f \times 1\) and \(f' \times g'\) are closed (regular) embeddings the varieties \(X' \times Y\) and \(Y\) are tor-independent over \(X \times Y\) by part (a). Thus the varieties \(X'\) and \(Y\) are tor-independent over \(X\) by [SGA6], Lemma III.1.5.1. Thus [SGA6], Proposition IV.3.1.1, yields

\[L f^* \circ R g_* = R g'_* \circ L (f')^*\].

\(\square\)

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**Index of Notations**

### Varieties, Bundles

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| $R = R_T = \mathbb{C}[q^{\pm 1}, t^{\pm 1}]$ | 2.2 |
| $K = K_T = \mathbb{C}(q, t)$ | 3.2 |
| $\mathbb{K} = \mathbb{C}(q^{1/2}, \sigma^{1/2}) = \mathbb{C}(q^{1/2}, t^{1/2})$ | 1.1, 1.3 |
| $\sigma = q^{-1}, \sigma = t^{-1}$ | 1.1, 1.3 |

### Algebras

| Notation | Page |
|----------|------|
| $\varnothing, \varnothing^+, \varnothing^0, \varnothing^\ast, \varnothing^<, \varnothing^\varnothing, \varnothing^\varnothing\varnothing$ | 1.1 |
| $E, E^\pm, E^0, E^\ast, E^<, E^\varnothing, E^\varnothing\varnothing$ | 1.1 |
| $\mathcal{H}_n, \mathcal{SH}_n, \mathcal{SH}_n^\ast, \mathcal{SH}_n^\varnothing$ | 1.3 |
| $\mathcal{SH}_K, \mathcal{H}_K, \mathcal{H}_K^\ast, \mathcal{H}_K^\varnothing$ | 1.4 |
| $\mathcal{H}_K^\ast, \mathcal{H}_K^\varnothing, \mathcal{H}_K^0$ | 3.4 |
| $\mathcal{H}_K, \mathcal{H}_K^\varnothing, \mathcal{H}_K^0$ | 4.6 |
| $\mathcal{H}_K, \mathcal{H}_K^\varnothing, \mathcal{H}_K^0$ | 4.4 |
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| $S = S_{\mathcal{R}, \mathcal{C}}$ | 9.1 |

### Other

| Notation | Page |
|----------|------|
| $\pi_r : \mathcal{E}^0 \to K[z_1^{\pm 1}, ..., z_r^{\pm 1}]^\varnothing$ | 6.1 |
| $\pi^r : \mathcal{E}^0 \to R_{T \times \text{GL}(E)} \otimes R K$ | 7.7 |

### Spaces

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| $\Lambda_K$ | 0.4 |
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| $M_K \simeq L_K$ | 7.5 |

### Maps, Representations

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| $\varnothing_\ast : \mathcal{SH}_\mathcal{R} \to \text{End}(\Lambda_K)$ | 1.4 |
| $\varnothing : \mathcal{E}^\varnothing \to \text{End}(\Lambda_K)$ | 1.4 |
| $\varnothing : \mathcal{E}_c \to \text{End}(\Lambda_K)$ | 4.7 |
| $\psi : \mathcal{H}_K \to \text{End}(\Lambda_K)$ | 3.2 |
| $\rho : \mathcal{C}_K \to \text{End}(M_K) = \text{End}(\Lambda_K)$ | 7.6 |
| $\Omega : \mathcal{E}_c \varnothing \simeq \mathcal{H}_K$ | 3.4 |
| $\Phi : \mathcal{E}^+ \varnothing \simeq \mathcal{SH}_K$ | 1.3 |
| $\Xi : \mathcal{C}_K \varnothing \simeq \mathcal{H}_K$ | 7.9 |
| $\Gamma : \mathcal{C}_K \varnothing \simeq \mathcal{E}^\varnothing$ | 7.9 |
| $\Upsilon : \mathcal{E}^\varnothing \simeq \mathcal{S}$ | 9.1 |
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