Continuous frames in tensor product Hilbert spaces, localization operators and density operators

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Abstract
Continuous frames and tensor products are important topics in theoretical physics. This paper combines those concepts. We derive fundamental properties of continuous frames for tensor product of Hilbert spaces. This includes, for example, the consistency property, i.e. preservation of the frame property under the tensor product, and the description of the canonical dual tensors by those on the Hilbert space level. We show the full characterization of all dual systems for a given continuous frame, a result interesting by itself, and apply this to dual tensor frames. Furthermore, we discuss the existence on non-simple tensor product (dual) frames. Continuous frame multipliers and their Schatten class properties are considered in the context of tensor products. In particular, we give sufficient conditions for obtaining partial trace multipliers of the same form, which is illustrated with examples related to short-time Fourier transform and wavelet localization operators. As an application, we offer an interpretation of a class of tensor product continuous frame multipliers as density operators for bipartite quantum states, and show how their structure can be restricted to the corresponding partial traces.

Keywords: continuous frame, dual frame, tensor product, bilinear continuous frame multiplier, localization operator, density operator

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1. Introduction

Continuous frames extend the concept of frames when the indices are related to some measurable space, see [1, 4, 27, 35]. Apart from expected similarities, this extension pointed out various differences between the ‘discrete’ and ‘continuous’ theories. For example, continuous frames need not be norm bounded, and they may describe the states of quantum systems in a neighborhood of a point in phase space $\mathbb{R}^{2d}$, which is a more realistic situation than the corresponding discrete case related to some lattice in $\mathbb{R}^{2d}$, cf [29].

On the other hand the tensor product of Hilbert spaces is a very important topic in mathematics [45] and theoretical physics [15]. Here we combine those two approaches. We introduce the notion of continuous frames (and Bessel mappings) for tensor products of Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with respect to a (tensor product) measure space $(X, \mu)$. When the measure $\mu$ is chosen to be the counting measure, the main properties of tensor products of (discrete) frames considered in [14, 28, 36, 46] are recovered.

We show the expected consistence property, i.e. that the continuous frame/Bessel mapping condition is preserved by the tensor product, theorem 3.3. To tackle the issue of representing vectors in tensor product Hilbert spaces, different systems can be used for analysis and synthesis, which gives rise to the notion of dual pairs of continuous frames. We study the corresponding operators, and give a representation of canonical dual frames for the tensor product continuous frames. In addition, we briefly discuss the existence of non-simple tensor product (dual) frames. For that result, a full characterization of all dual continuous frames is needed. We prove the generalization of the well-known result for (discrete) frames [16, lemma 6.3.6] to the continuous frame setting, solving an open question. We use the powerful technique of reproducing kernel Hilbert spaces (RKHS) in these investigations, see theorem 4.5, and derive the corresponding property for tensor product continuous frames.

Let us recall that tensor product Hilbert spaces are important in many different contexts. For example, as noted in [10], ‘the theory of tensor products is at the heart of kernel theorems for operators’. In fact, tensor product of two Hilbert spaces can be introduced in terms of Hilbert–Schmidt operators which, in turn, can be identified with their kernels. In this paper we focus our attention to other aspects of tensor products, and the approach based on kernel theorems will be given in a separate contribution.

For example, in section 5 we study the tensor product continuous frame multipliers and their compactness properties, thus extending results from [9] to the tensor product setting. In addition, we recall the partial trace theorem which is an important tool related to applications of our results to quantum systems. As an illustration, in section 6 we consider particular examples of continuous frame multipliers in the form of familiar localization operators in the context of the short-time Fourier transform (STFT) and wavelet multipliers. Localization operators are used in the context of quantization [12], in signal analysis [22], or as an approximation of pseudodifferential operators, cf [21] and the references given there. We recover some well-known results, but also point out some Schatten class results related to the wavelet and mixed type multipliers that so far seems to remain unconsidered.

Specific instances of our general theory, which is one of the main motivations for our study could be related to the states of quantum systems. More precisely, we propose the interpretation of a family of trace class operators as density operators also called density matrices for composite (bipartite) quantum systems. Recently, de Gosson in [29, 30] considered Toeplitz density operators by using the approach which is closely related to the STFT multipliers of section 6. The main feature of operators considered in section 7 is that their partial traces (or reduced density operators) are operators of the same form. Thus we propose the study of
bilinear localization operators which, in principle, could be used to describe the state of subsystem in a prescribed region of the phase space. This is analogous to the use of localization operators in extracting an information about a signal in a specific region of time–frequency plane.

In our opinion the results from sections 6 and 7, open the perspective of using mathematical tools developed in sections 3 and 5 in the future study of bipartite quantum systems and their subsystems. For example, theorem 7.4 provides a description of the separable state of a composite system, and a partial affirmative answer to the question of de Gosson [29, section 5] which can be roughly rephrased as follows: can the structure of a density operator be appropriately restricted to its partial traces?

2. Preliminaries

For the reader’s convenience in this section we collect some basic facts from operator theory and tensor products of Hilbert spaces which will be used in the sequel. We refer to [17, 24, 39] for details.

2.1. Operator theory

By \( \mathcal{H} \) we denote a complex Hilbert space with the inner product \( \langle x, y \rangle \) (linear in the first and conjugate linear in the second coordinate) and norm \( \|x\| = \sqrt{\langle x, x \rangle} \), \( x, y \in \mathcal{H} \). In the sequel we consider separable Hilbert spaces. A map \( \Psi : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) is a sesquilinear form if it is linear in the first variable and conjugate-linear in the second. The sesquilinear form is bounded if there exists a constant \( C > 0 \) such that \( |\Psi(x, y)| \leq C \cdot \|x\|\|y\|, x, y \in \mathcal{H} \). The smallest, optimal, such constant is called the bound of \( \Psi \) denoted by \( \|\Psi\| \). There is a unique operator \( O \) on \( \mathcal{H} \) such that

\[
\Psi(x, y) = \langle O(x), y \rangle \quad x, y \in \mathcal{H},
\]

and \( \|O\| = \|\Psi\| \).

A bounded operator \( T : \mathcal{H} \to \mathcal{H} \) is positive (respectively non-negative), if \( \langle Tx, x \rangle > 0 \) for all \( x \neq 0 \) (respectively \( \langle Tx, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \)).

A linear operator \( T \) from the Banach space \( X \) into the Banach space \( Y \) is compact if the image of the closed unit ball in \( X \) is a relatively compact subset of \( Y \), or, equivalently, if the image of any bounded sequence contains a convergent subsequence. If \( T \) is a compact operator on Hilbert space \( \mathcal{H} \) and if \( T^* \) is the adjoint of \( T \) (i.e. \( \langle Tx, y \rangle = \langle x, T^*y \rangle, \forall x, y \in \mathcal{H} \) then the eigenvalues of the unique non-negative and compact operator \( S \) such that \( S^2 = T^*T \) are called the singular values of \( T \). An operator \( T \) belongs to the Schatten class \( S_p(\mathcal{H}) \), \( 1 \leq p < \infty \), if the sequence of its singular values \( (s_n) \) belongs to \( l^p \). In particular, \( S_1(\mathcal{H}) \) consists of the trace class operators, and \( S_2(\mathcal{H}) \) is the class of Hilbert–Schmidt operators (see also below), and \( S_p(\mathcal{H}) \subseteq S_q(\mathcal{H}) \), when \( 1 \leq p \leq q \leq \infty \), where \( S_\infty(\mathcal{H}) = B(\mathcal{H}) \) denotes the set of all bounded linear operators on \( \mathcal{H} \).

Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be separable Hilbert spaces. The set \( B(\mathcal{H}_2, \mathcal{H}_1) \) of all bounded linear operators from \( \mathcal{H}_2 \) to \( \mathcal{H}_1 \) is a Banach space with the usual operator norm \( \|T\| = \sup_{\|x\|=1} \|Tx\| \), and \( GL(\mathcal{H}_2, \mathcal{H}_1) \) denotes the set of all bounded linear operators from \( \mathcal{H}_2 \) to \( \mathcal{H}_1 \) with bounded inverse. If \( \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H} \), we write \( B(\mathcal{H}) \) and \( GL(\mathcal{H}) \) for short.

If \( T \in B(\mathcal{H}_2, \mathcal{H}_1) \) and

\[
\|T\|_{HS}^2 := \sum_{n=1}^{\infty} \|Te_n\|_{\mathcal{H}_1}^2 < \infty
\]
for some orthonormal basis (ONB) \((e_n)\) in \(\mathcal{H}_2\), then \(T\) is called a Hilbert–Schmidt (HS) operator from \(\mathcal{H}_2\) to \(\mathcal{H}_1\). We denote the class of Hilbert–Schmidt operators by \(\mathcal{HS}(\mathcal{H}_2, \mathcal{H}_1)\). If \(\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}\), then \(\mathcal{HS}(\mathcal{H}, \mathcal{H}) = S_2(\mathcal{H})\). \(\mathcal{HS}(\mathcal{H}_2, \mathcal{H}_1)\) is a Hilbert space (of compact operators) with the inner product

\[
\langle S, T \rangle_{\mathcal{HS}} = \sum_{n=1}^{\infty} \langle S e_n, T e_n \rangle_{\mathcal{H}_1}.
\]

If \(x \in \mathcal{H}_1\) and \(y \in \mathcal{H}_2\), then their tensor product \(x \otimes y : \mathcal{H}_2 \rightarrow \mathcal{H}_1\) is defined by

\[
(x \otimes y)h = \langle h, y \rangle x, \quad h \in \mathcal{H}_2,
\]

belonging to \(\mathcal{HS}(\mathcal{H}_2, \mathcal{H}_1)\).

For \(P \in B(\mathcal{H}_2)\) and \(Q \in B(\mathcal{H}_1)\) we define the tensor product of operators \(Q \otimes P : B(\mathcal{H}_2, \mathcal{H}_1) \rightarrow B(\mathcal{H}_2, \mathcal{H}_1)\) by \((Q \otimes P)T = Q \cdot T \circ P\). It is invertible if and only if \(P\) and \(Q\) are invertible, and \((Q \otimes P)^{-1} = Q^{-1} \otimes P^{-1}\).

### 2.2. Tensor product of Hilbert spaces

Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be separable Hilbert spaces. Equipping the algebraic tensor product with the (extension of) the inner product

\[
\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\otimes} = \langle x_1, x_2 \rangle_{\mathcal{H}_1} \langle y_1, y_2 \rangle_{\mathcal{H}_2}, \quad x_1, x_2 \in \mathcal{H}_1, \quad y_1, y_2 \in \mathcal{H}_2,
\]

makes it into a Hilbert space, denoted by \(\mathcal{H}_1 \otimes \mathcal{H}_2\), and \(\|\cdot\|_{\otimes} = \langle \cdot, \cdot \rangle_{\otimes}\).

The space \(\mathcal{H}_1 \otimes \mathcal{H}_2\) is unitary isomorphic to the class of Hilbert–Schmidt operators \(\mathcal{HS}(\mathcal{H}_2, \mathcal{H}_1)\). The unitary operator maps \((x_1 \otimes y_1)\) onto the operator given by (2.2) cf [33].

Let us collect basic properties of tensor products given in the following lemma.

**Lemma 2.1.** Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be separable Hilbert spaces and \(\mathcal{H}_1 \otimes \mathcal{H}_2\) their tensor product. Then we have:

(a) \(\|u \otimes v\| = \|u\| \|v\|\), \(u \in \mathcal{H}_1, v \in \mathcal{H}_2\).

(b) If \(S \in B(\mathcal{H}_1)\) and \(T \in B(\mathcal{H}_2)\), then \(\|S \otimes T\| = \|S\| \|T\|\), and

\[
(S \otimes T)(u \otimes v) = Su \otimes Tv, \quad u \in \mathcal{H}_1, \quad v \in \mathcal{H}_2.
\]

(c) \(\mathcal{H}_1 \otimes \mathcal{H}_2 = \text{span}\{u \otimes v, u \in \mathcal{H}_1, v \in \mathcal{H}_2\}\), i.e. \(\mathcal{H}_1 \otimes \mathcal{H}_2\) is the closure of the set of all finite linear combinations of elements of the form \(u \otimes v\), \(u \in \mathcal{H}_1, v \in \mathcal{H}_2\).

(d) The tensor product of two ONBs is an ONB in the tensor product space.

(e) (Schmidt decomposition) For every \(x \in \mathcal{H}_1 \otimes \mathcal{H}_2\) there are non-negative numbers \(c_n\) and ONB \(e_n \in \mathcal{H}_1\) and \(f_n \in \mathcal{H}_2\), such that

\[
x = \sum_{n=1}^{\infty} c_n (e_n \otimes f_n), \quad \|x\| = \sum_{n=1}^{\infty} c_n^2.
\]

**Proof.** The proof (a)–(d) is folklore, see e.g. [24, 26, 32]. For the proof of Schmidt decomposition (e) we refer to [13].
3. Frames in tensor products of Hilbert spaces

In this section we derive fundamental properties of continuous frames for tensor product of Hilbert spaces.

The usual definition of frames use discrete index sets [16], one can also give a definition using continuous ones [1, 2, 9]. As an introduction we transfer the basic definitions directly for the case used in this manuscript.

3.1. Continuous frames in tensor product Hilbert spaces

Definition 3.1. Let $H$ be the tensor product $H = H_1 \otimes H_2$ of separable complex Hilbert spaces, and $(X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2)$ be the product of measure spaces with $\sigma$-finite positive measures $\mu_1$, $\mu_2$. The mapping $F : X \rightarrow H$ is called a continuous frame for the tensor product Hilbert space $H$ with respect to $(X, \mu)$, if

(a) $F$ is weakly-measurable, i.e., for all $\vec{f} \in H$,

$$x = (x_1, x_2) \rightarrow \langle \vec{f}, F(x) \rangle$$

is a measurable function on $X$;

(b) There exist constants $A > 0$ and $B < \infty$ such that

$$A \| \vec{f} \|^2 \leq \int_X |\langle \vec{f}, F(x) \rangle|^2 \, d\mu(x) \leq B \| \vec{f} \|^2, \quad \forall \vec{f} \in H. \tag{3.1}$$

The constants $A$ and $B$ are called the lower and the upper continuous frame bound, respectively. If $A = B$, then $F$ is called a tight continuous frame, if $A = B = 1$ a Parseval frame.

The mapping $F$ is called the Bessel mapping if only the second inequality in (3.1) is considered. In this case, $B$ is called the Bessel constant or the Bessel bound$^3$.

To each continuous frame we define the frame related operators as follows.

Let $(X, \mu)$ and $H$ be as in definition 3.1, and let $L^2(X, \mu)$ be the space of square-integrable functions on $(X, \mu)$. The operator $T_F : L^2(X, \mu) \rightarrow H$ defined by

$$T_F \vec{\varphi} = \int_X \varphi(x) F(x) \, d\mu(x) = \int_{X_1} \int_{X_2} \varphi(x_1, x_2) F(x_1, x_2) \, d\mu_1(x_1) \, d\mu_2(x_2) \tag{3.2}$$

is called the synthesis operator, and the operator $T_F^* : H \rightarrow L^2(X, \mu)$, given by

$$(T_F^* \vec{f})(x) = \langle \vec{f}, F(x) \rangle, \quad x \in X \tag{3.3}$$

is called the analysis operator of $F$.

The continuous frame operator $S_F$ of $F$ is given by $S_F = T_F T_F^*$.

Remark 3.2. In discrete frame theory, it is of interest to consider Riesz bases. It does not make sense to address this question in the context of continuous frames, since all continuous Riesz bases are actually discrete, cf [4, 34, 43].

The first inequality in (3.1), shows that $F$ is complete, i.e.,

$$\text{span}\{F(x)\}_{x \in X} = H.$$

$^3$ Please be aware, that this concept of Bessel mappings does not coincide with Bessel functions.
where we have \( \overline{\text{span}} \{ F(x) \}_{x \in X} := \{ f \in H | \mu \left( \{ x \mid \langle f, F(x) \rangle \neq 0 \} \right) \neq 0 \} \). In contrast to discrete setting, in the continuous setting one has to be a bit more careful with this definition due to the null sets in \( X \), cf [11].

The next result shows that the continuous frame condition is preserved by the tensor product, generalizing the result for discrete frames.

**Theorem 3.3.** Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be separable Hilbert spaces, \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \), and let \( (X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2) \) be the product of measure spaces with \( \sigma \)-finite positive measures \( \mu_1, \mu_2 \). The mapping \( F = F_1 \otimes F_2 : X \to \mathcal{H} \) is a continuous frame for \( \mathcal{H} \) with respect to \( (X, \mu) \) if and only if \( F_1 \) is a continuous frame for \( \mathcal{H}_1 \) with respect to \( (X_1, \mu_1) \), and \( F_2 \) is a continuous frame for \( \mathcal{H}_2 \) with respect to \( (X_2, \mu_2) \).

Furthermore, if \( F = F_1 \otimes F_2 \) is a continuous frame for \( \mathcal{H} \) with frame bounds \( A \) and \( B \), then the continuous frame bounds for \( F_1 \) can be chosen as \( A_1 = A/C_{F_2} \) and \( B_1 = B/D_{F_2} \), where

\[
C_{F_2} = \inf_{\|f\|_{\mathcal{H}_2} = 1} \int_{X_2} |\langle g, F_2(x_2) \rangle|^2 \, d\mu_2(x_2), \tag{3.4}
\]

\[
D_{F_2} = \sup_{\|f\|_{\mathcal{H}_2} = 1} \int_{X_2} |\langle g, F_2(x_2) \rangle|^2 \, d\mu_2(x_2), \tag{3.5}
\]

and the continuous frame bounds for \( F_2 \) can be chosen as \( A_2 = A/C_{F_1} \) and \( B_2 = B/D_{F_1} \), where

\[
C_{F_1} = \inf_{\|f\|_{\mathcal{H}_1} = 1} \int_{X_1} |\langle f, F_1(x_1) \rangle|^2 \, d\mu_1(x_1), \tag{3.6}
\]

\[
D_{F_1} = \sup_{\|f\|_{\mathcal{H}_1} = 1} \int_{X_1} |\langle f, F_1(x_1) \rangle|^2 \, d\mu_1(x_1). \tag{3.7}
\]

Vice versa, if \( F_j \) is a continuous frame for \( \mathcal{H}_j \) with the frame bounds \( A_j \) and \( B_j \), \( j = 1, 2 \), then the frame bounds for \( F = F_1 \otimes F_2 \) can be chosen as \( A = A_1 A_2 \) and \( B = B_1 B_2 \).

**Proof.** Assume that \( F = F_1 \otimes F_2 \) is a continuous frame for \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) with respect to \( (X, \mu) \). Let \( f \in \mathcal{H}_1 \setminus \{0\} \), and fix \( g \in \mathcal{H}_2 \setminus \{0\} \). Then \( f \otimes g \in \mathcal{H} \), and

\[
T^*_F(f \otimes g) := \langle f \otimes g, F_1(x_1) \otimes F_2(x_2) \rangle = \langle f, F_1(x_1) \rangle \langle g, F_2(x_2) \rangle
\]

implies that (by Fubini’s theorem)

\[
\int_X |\langle f \otimes g, F_1(x_1) \otimes F_2(x_2) \rangle|^2 \, d\mu(x) = \int_{X_1} |\langle f, F_1(x_1) \rangle|^2 \, d\mu_1(x_1) \int_{X_2} |\langle g, F_2(x_2) \rangle|^2 \, d\mu_2(x_2).
\]

Now, (3.1) and

\[
\| f \otimes g \|_\circ = \| f \|_{\mathcal{H}_1} \| g \|_{\mathcal{H}_2}
\]

imply

\[
A \| f \otimes g \|_\circ^2 \leq \int_{X_1} |\langle f, F_1(x_1) \rangle|^2 \, d\mu_1(x_1) \int_{X_2} |\langle g, F_2(x_2) \rangle|^2 \, d\mu_2(x_2) \leq B \| f \otimes g \|_\circ^2,
\]

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so that
\[
\frac{A\|g\|^2_{\mathcal{H}_2}}{\int_{X_1}|(g, F_2(x_2))|^2 \, d\mu_2(x_2)} \|f\|^2_{\mathcal{H}_1} \leq \int_{X_1} |(f, F_1(x_1))|^2 \, d\mu_1(x_1) \\
\leq \frac{B\|g\|^2_{\mathcal{H}_2}}{\int_{X_1}|(g, F_2(x_2))|^2 \, d\mu_2(x_2)} \|f\|^2_{\mathcal{H}_1}.
\]

Notice that \(\int_{X_1}|(g, F_2(x_2))|^2 \, d\mu_2(x_2) \neq 0\) for all \(g \in \mathcal{H}_2 \setminus \{0\}\), and choose

\[
A_1 := \sup_{\|g\|_{\mathcal{H}_2} = 1} \frac{A}{\int_{X_1}|(g, F_2(x_2))|^2 \, d\mu_2(x_2)} = \frac{A}{C_{F_2}} > 0,
\]

\[
B_1 := \inf_{\|g\|_{\mathcal{H}_2} = 1} \frac{B}{\int_{X_1}|(g, F_2(x_2))|^2 \, d\mu_2(x_2)} = \frac{B}{D_{F_2}} < \infty,
\]

with \(C_{F_2}\) and \(D_{F_2}\) given by (3.4) and (3.5) respectively.

Thus we conclude that \(F_1\) is a continuous frame for \(\mathcal{H}_1\) with respect to \((X_1, \mu_1)\) with the continuous frame bounds \(A_1\) and \(B_1\).

By similar arguments we conclude that \(F_2\) is a continuous frame for \(\mathcal{H}_2\) with respect to \((X_2, \mu_2)\) with continuous frame bounds \(0 < A_2 = A/C_{F_1}\) and \(B_2 = B/D_{F_1} < \infty\), with \(C_{F_1}\) and \(D_{F_1}\) given by (3.6) and (3.7) respectively.

For the converse, by the assumptions it immediately follows that \(F = F_1 \otimes F_2\) is weakly measurable on \(\mathcal{H}\) with respect to \((X, \mu)\), so it remains to check (3.1).

Let \(f \otimes g\) be a simple tensor. Then
\[
\|T_{F_1 \otimes F_2}(f \otimes g)\|^2 = \int_{X_1} \int_{X_2} |(f \otimes g, F_1(x_1) \otimes F_2(x_2))|^2 \, d\mu_1(x_1) \, d\mu_2(x_2)
\]
\[
= \int_{X_1} |(f, F_1(x_1))|^2 \, d\mu_1(x_1) \int_{X_2} |(g, F_2(x_2))|^2 \, d\mu_2(x_2)
\]
\[
\leq B_1 B_2 \|f\|^2_{\mathcal{H}_1} \|g\|^2_{\mathcal{H}_2} = B_1 B_2 \|f \otimes g\|^2_{\mathcal{H}},
\]
and similarly
\[
\|T_{F_1 \otimes F_2}(f \otimes g)\|^2 \geq A_1 A_2 \|f \otimes g\|^2_{\mathcal{H}}.
\]

This is true for the span of \(f \otimes g\) which is dense in \(\mathcal{H}_1 \otimes \mathcal{H}_2\). By [41, proposition 2.5] it follows that \(F_1 \otimes F_2\) is a continuous frame with the frame bounds \(A = A_1 A_2\) and \(B = B_1 B_2\).

From the proof of theorem 3.3 we also have the following observation.

**Corollary 3.4.** Let the assumptions of theorem 3.3 hold. Then the mapping \(F = F_1 \otimes F_2 : X \rightarrow \mathcal{H}\) is a continuous bilinear Bessel mapping for \(\mathcal{H}\) with respect to \((X, \mu)\) if and only if \(F_1\) is a continuous Bessel mapping for \(\mathcal{H}_1\) with respect to \((X_1, \mu_1)\) and \(F_2\) is a continuous Bessel mapping for \(\mathcal{H}_2\) with respect to \((X_2, \mu_2)\).
3.2. Dual pairs of continuous frames

Next we discuss dual continuous frames. If \( F_j \) are continuous frames for \( \mathcal{H}_j, j = 1, 2 \), then we may consider dual frames \( G_j \) which fulfill

\[
\langle f, g \rangle = \int_{\mathcal{X}_j} \langle f, F_j(x_j) \rangle \langle G_j(x_j), g \rangle d\mu(x_j), \quad \forall f, g \in \mathcal{H}_j, \quad j = 1, 2,
\]

and \( G_j \) be a Bessel mapping from \( \mathcal{X} \) to \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). Hence \( SF \) is invertible, positive and \( 1/\mathcal{B} \leq S_F^{-1} \leq 1/A \).

By definition 3.5 and (3.8) it follows that for a given continuous frame \( F \) there always exists an associated dual pair, i.e. \((F, S_F^{-1}F)\) and \((S_F^{-1}F, F)\) are dual pairs. The frame \( S_F^{-1}F \) is called the canonical dual frame for \( F \), denoted by \( F(x) \).

To each continuous frame \( F \) one can associate a dual continuous frame which is introduced as follows.

**Definition 3.5.** Let \( F \) and \( G \) be continuous frames for \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) with respect to \((X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2)\). The frame \( G \) is a continuous dual frame of \( F \) if

\[
\tilde{f} = S_F^{-1}F(x)G(x) d\mu(x), \quad \forall \tilde{f} \in \mathcal{H},
\]

in the weak sense, i.e. if

\[
\langle \tilde{f}, \tilde{g} \rangle = \int_{\mathcal{X}} \langle \tilde{f}, F(x) \rangle \langle G(x), \tilde{g} \rangle d\mu(x), \quad \forall \tilde{f}, \tilde{g} \in \mathcal{H}.
\]

In this case the pair \((F, G)\) is called a dual pair of continuous frames.

By definition 3.5 and (3.8) it follows that for a given continuous frame \( F \) there always exists an associated dual pair, i.e. \((F, S_F^{-1}F)\) and \((S_F^{-1}F, F)\) are dual pairs. The frame \( S_F^{-1}F \) is called the canonical dual frame for \( F \), denoted by \( F(x) \).

In the next theorem we establish the tensor product version of the usual identification of continuous frame operator in terms of analysis and synthesis operators. We refer to [7, 41] when \( \mathcal{H} \) is a Hilbert space.

**Theorem 3.6.** Let \((X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2)\) be a tensor product measure space and let \( F \) be a Bessel mapping from \( X \) to \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). Then the synthesis operator \( T_F : L^2(X, \mu) \to \mathcal{H} \) given by (3.2) is a well defined, linear and bounded operator, and its adjoint operator \( T_F^* : \mathcal{H} \to L^2(X, \mu) \) is given by (3.3).
If $F = F_1 \otimes F_2$ is a continuous frame for $\mathcal{H}$ with respect to $(X, \mu)$, and $\tilde{f} = f_1 \otimes f_2 \in \mathcal{H}$, then the analysis operator can be represented by

$$(T_F^* \tilde{f})(x) = \langle f_1, F_1(x_1) \rangle \langle f_2, F_2(x_2) \rangle.$$ (3.10)

The continuous frame operator $S_F$ is given by $S_F = T_F T_F^*$, and

$$S_{F_1 \otimes F_2} = S_{F_1} \otimes S_{F_2}.$$ The canonical dual frame for $F$ is $G = S_F^{-1} F_1 \otimes S_F^{-1} F_2$.

**Proof.** The first part of the claim follows immediately from the definition of $T_F$ given by (3.2). Furthermore, if $F = F_1 \otimes F_2$ is a continuous bilinear frame for $\mathcal{H}$ with respect to $(X, \mu)$, then the representation (3.10) follows directly from (2.3).

It remains to show the second part of theorem 3.6. Let $f_j \in \mathcal{H}_j$, $j = 1, 2$. Then

$$T_F T_F^*(f_1 \otimes f_2) = \int_X \langle f_1, F_1(x_1) \rangle \langle f_2, F_2(x_2) \rangle F_1(x_1) \otimes F_2(x_2) d\mu(x)$$

$$= \int_{X_1} \langle f_1, F_1(x_1) \rangle F_1(x_1) d\mu_1(x_1) \otimes \int_{X_2} \langle f_2, F_2(x_2) \rangle F_2(x_2) d\mu_2(x_2)$$

$$= S_{F_1} f_1 \otimes S_{F_2} f_2 = \langle S_{F_1} \otimes S_{F_2} \rangle (f_1 \otimes f_2),$$

and

$$T_F^* T_F(f_1 \otimes f_2) = \int_X \langle f_1, f_1(x_1) \rangle \langle f_2, F_2(x_2) \rangle F_1(x_1) \otimes F_2(x_2) d\mu(x)$$

$$= \int_X \langle f_1 \otimes f_2, F_1(x_1) \otimes F_2(x_2) \rangle F_1(x_1) \otimes F_2(x_2) d\mu(x)$$

$$= S_{F_1 \otimes F_2} (f_1 \otimes f_2).$$

Therefore on simple tensors we have that $S_F = S_{F_1} \otimes S_{F_2}$. By lemma 2.1 (see also [41, proposition 2.5]) this is true on all of $\mathcal{H}$.

Moreover, $S_F$ is self-adjoint and we have

$$S_F^{-1} = (S_{F_1} \otimes S_{F_2})^{-1} = S_{F_1}^{-1} \otimes S_{F_2}^{-1} = S_{G_1} \otimes S_{G_2} = S_{G_1 \otimes G_2},$$

where $G_1$ and $G_2$ are canonical dual frames of $F_1$ and $F_2$ respectively.

Furthermore,

$$S_{F_1 \otimes F_2}^{-1}(F_1 \otimes F_2) = S_{F_1}^{-1} \otimes S_{F_2}^{-1}(F_1 \otimes F_2) = (S_{F_1}^{-1} F_1) \otimes (S_{F_2}^{-1} F_2) = G_1 \otimes G_2,$$

which proves the claim. □

Recall that in $L^2(X_1 \times X_2, \mu_1 \otimes \mu_2)$ a simple tensor $f \otimes g$ is just the product $f \otimes g(x) = f(x_1)g(x_2)$ which is commonly identified with an operator with the integral kernel $f \otimes g$. Thus, in (3.10) we may put

$$T_F^* = T_{F_1}^* \otimes T_{F_2}^*.$$

Obviously, for a pair of continuous frames $F$ and $G$ the condition (3.9) can be written as $T_G T_F^* = I$ (in the weak sense).
3.3. Non-simple frames

Let us digress a bit, and see if ‘everything is solved’ now considering theorem 3.3. In this subsection, we actually discuss the existence of non-simple tensor frames, therefore the result mentioned above does not cover the full tensor frame theory (since it concerns only simple tensors). Let us stress that by definition 3.1 it follows that not every frame in a tensor product Hilbert space has to be represented as a (sequence of) simple tensor(s). We will show that any continuous frame admits a non-simple dual frame.

Our first result shows that tensor Bessel sequences can be constructed with ranks different than 1.

**Lemma 3.7.** Let $\ell_k(\omega)$ and $g_k(\nu)$ be continuous Bessel mappings in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively with bounds $B_k$ and $B'_k$ such that $B := \sum_k B_k \cdot \sum_\nu B'_\nu < \infty$, then $F(\omega, \nu) = \sum_k \ell_k(\omega) \otimes g_k(\nu)$ is a Bessel mapping in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with the Bessel bound $B$.

**Proof.** Note that

$$\left| \langle \psi \otimes \phi, F(x_1, x_2) \rangle \right|^2 = \left| \left\langle \psi \otimes \phi, \sum_k \ell_k(x_1) \otimes g_k(x_2) \right\rangle \right|^2 = \left| \sum_k \langle \psi, \ell_k(x_1) \rangle \langle \phi, g_k(x_2) \rangle \right|^2.

Then we have

$$\int \left| \langle \psi \otimes \phi, F(x_1, x_2) \rangle \right|^2 d\mu(x_1, x_2) = \int \left| \sum_k \langle \psi, \ell_k(x_1) \rangle \langle \phi, g_k(x_2) \rangle \right|^2 d\mu(x_1, x_2)

\leq \sum_k \int \left| \langle \psi, \ell_k(x_1) \rangle \right|^2 d\mu(x_1) \sum_\nu \int \left| \langle \phi, g_\nu(x_2) \rangle \right|^2 d\mu(x_2)

\leq \sum_k B_k \|\psi\|^2 \sum_\nu B'_\nu \|\phi\|^2 \leq \left( \sum_k B_k \sum_\nu B'_\nu \right) \|\psi\|^2 \cdot \|\phi\|^2.

The result now follows by extension from simple tensors to all of $\mathcal{H}_1 \otimes \mathcal{H}_2$ (cf [41, proposition 2.5]).

A direct converse can never be true (consider e.g. any $f_k$ and $g_k = 0$).

So, we now know that non-simple Bessel sequence exist. If we now consider a fixed frame, do there dual frames exist, which are not necessarily of the form given by theorem 3.3 (see also [5]). More precisely, if $F_1 \otimes F_2$ is a frame for $\mathcal{H}$ with respect to $(X, \mu)$, then we examine the existence of its dual frame $G$ such that $G \neq G_1 \otimes G_2$ for any $G_1 \in \mathcal{H}_1, G_2 \in \mathcal{H}_2$. Let us shortly digress from the logical order of results and rather use the proof of this result as a motivation for the next section.

We first recall that a continuous frame $F$ is redundant if

$$R(F) := \dim(\text{ran} (T_F)) > 0.$$

It has been observed that $R(F)$ depends on the underlying measure space $(X, \mu)$. For example, if $(X, \mu)$ is non-atomic, then $R(F) = \infty$. We refer to [42] for details.

**Lemma 3.8.** Let $\dim(\mathcal{H}_1), \dim(\mathcal{H}_2) > 1$, and let $F_1 \otimes F_2$ be a redundant frame for $\mathcal{H}$. Then $F_1 \otimes F_2$ admits at least one non-simple tensor product dual.

**Proof.** The idea is to follow the steps of the proof of [46, theorem 2.3], and the case study examination given there. This proof uses the fact that for $T \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim(\text{ran}(T)) \leq 1$ if
and only if $T = f \otimes g$ for some $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ ([46, lemma 2.2]). Replacing sums by integrations the proof of [46, theorem 2.3] can be generalized in a straightforward way, if we can show the tensor product version of [16, theorem 6.3.7], i.e. a description of all dual tensor frames of a given tensor frame. This is corollary 4.7.

In order to make this proof complete we have to introduce the next section, which by itself answers an open question in frame theory.

4. Full classification of dual continuous frames

In this section we extend the well known classification of dual discrete frames [16, theorem 6.3.7] to the continuous frames setting. This is a new result in continuous frame theory, and we apply it to describe dual frames in the context of tensor products.

It turned out that the theory of RKHS provides convenient tools for the result in this section. The interplay between RKHS and frame theory is recently used in [43] in the study of stable theory, and we apply it to describedualframes in the context of tensor products.

Corollary 4.1. Let $\mathcal{H}$ and $\mathcal{G}$ be subspaces of a normed space $X$, where the closures are RKHS. Then $\mathcal{H} \cup \mathcal{G} = \overline{\mathcal{H} \cup \mathcal{G}}$ is a RKHS.

Proof. Let us denote by $k^H$ and $k^G$ the respective reproducing kernels for the closures. Then, for $f \in \overline{\mathcal{H}}$ we have that $|f(x)| \leq \|k^H||f||$, and $|f(x)| \leq \|k^G||f||$ for $f \in \overline{\mathcal{G}}$, so that

$$|f(x)| \leq \max \left\{ \|k^H||f||, \|k^G||f|| \right\} \quad \text{for} \quad f \in \overline{\mathcal{H} \cup \mathcal{G}}.$$ 

The following result [43, proposition 11], which relates frames with RKHS, is needed later:

Lemma 4.2. If $F$ satisfies the lower frame inequality, then $\text{ran} \left( T^*_F \right)$ is a RKHS. Moreover, for any subspace $\mathcal{H}_K$ of $L^2(X, \mu)$, the following are equivalent:

(a) $\mathcal{H}_K$ is a RKHS.

(b) There exists a continuous frame $F$ such that $\text{ran} \left( T^*_F \right) = \mathcal{H}_K$.

We are now ready to attack the main question in this section. We first prove the continuous counterpart of [16, lemma 6.3.5].

Lemma 4.3. Let $F(x)$ be a continuous frame for the Hilbert space $\mathcal{H}$. Let $e_k$ be an ONB for $\mathcal{H}$ and let $V : L^2(X, \mu) \rightarrow \mathcal{H}$ be a bounded left-inverse of $T_F$, such that $(\ker V)^\perp$ is a reproducing kernel subspace of $L^2(X, \mu)$. Then the dual frames of $F$ are precisely the functions $G(x) = \sum_{k \in \mathbb{K}} V^*(e_k)(x) e_k$.

Proof. The function $G(x)$ is well defined if $\sum_{k \in \mathbb{K}} |V^*(e_k)(x)|^2 < \infty$. By [43, proposition 6], $\sum_{k \in \mathbb{K}} |V^*(e_k)(x)|^2 < \infty$ if $V^*(e_k)$ is a discrete Bessel sequence in the RKHS $(\ker V)^\perp$. Now, from the proof of [16, proposition 5.3.1] it follows that this is true.


Next, following [43, proposition 21], we have

\[
\langle f, g \rangle_{\mathcal{H}} = \langle VT_F f, g \rangle_{\mathcal{H}} = \langle T_F f, V^* g \rangle_{L^2(X, \mu)} = \left( T_F f, \sum_{k \in K} \langle g, e_k \rangle V^* e_k \right)_{L^2(X, \mu)} = \left( T_F f, \left\{ g, \sum_{k \in K} V^*(e_k) \cdot e_k \right\} \right)_{L^2(X, \mu)} = \left( T_F f, T^*_G g \right)_{L^2(X, \mu)}.
\]

On the other hand, let \( G_0(x) \) be a frame and set \( V = T_{G_0} \). We have that \( V^\perp = \text{ran} \left( T_{G_0} \right) \), which is a reproducing kernel Hilbert space by lemma 4.2. Thus

\[
T_G^0(g)(x) = \langle g, G(x) \rangle_{\mathcal{H}} = \langle g, \sum_{k \in K} V^* e_k(x) e_k \rangle_{\mathcal{H}} = \sum_{k \in K} V^* e_k(x) \langle g, e_k \rangle_{\mathcal{H}} = \langle V^* g \rangle(x).
\]

Thus \( T_G^0 = T_{G_0}^* \) a.e. and so \( G(x) = G_0(x) \) a.e. \( \square \)

Note that the left-inverse \( V \) in lemma 4.3 can never be invertible. (Because then ker \( V^\perp = L^2(\mathbb{R}) \), which is not a RKHS).

The continuous version of [16, lemma 6.3.6] can now be given as follows.

**Lemma 4.4.** Let \( F(x) \) be a continuous frame for \( \mathcal{H} \). The bounded left-inverses of \( T_F \), i.e. \( VT_F = \text{id}_{\mathcal{H}} \), with ker \( (V)^\perp \) being a RKHS are precisely the operators of the form

\[
V = S_F^{-1}T_F + W \left( \text{id}_{\mathcal{H}} - T_F S_F^{-1}T_F \right),
\]

where \( W : L^2(X, \mu) \to \mathcal{H} \) is a bounded operator with ker \( (W)^\perp \) being a RKHS.

**Proof.** The proof of [16, lemma 6.3.6] can be used directly in the sense that all left-inverses \( V \) can be exactly represented by (4.1). It remains to prove the transfer of the RKHS property.

Consider the mapping \( W_0 : \ker(T_F) \to \mathcal{H} \), defined by \( W_0 := W_{\pi_{\ker(T_F)}} \). The operator \( W \) can then be written as \( W = W_0\pi_{\ker(T_F)} + W_{\pi_{\text{ran}(T_F^*)}} \).

By the assumptions we have that

\[
V = S_F^{-1}T_F + W_{\pi_{\ker(T_F)}} = S_F^{-1}T_F + W_0\pi_{\ker(T_F)}.
\]

and

\[
V^* = T_F^*S_F^{-1} + \pi_{\ker(T_F)}W^* = T_F^*S_F^{-1} + \pi_{\ker(T_F)}W_0^*.
\]

Clearly, ker \( (W_0) = \ker(W) \cap \ker(T_F) \). In particular, ker \( (W)^\perp \subseteq \ker(W_0)^\perp \) and ker \( (W_0)^\perp = \ker(W)^\perp \cup \ker(T_F)^\perp = \ker(W)^\perp \cup \text{ran} \left( T_F \right) \). This tells us that (by corollary 4.1) that ker \( (W)^\perp \) is a RKHS if and only if ker \( (W_0)^\perp \) is.

Additionally, by construction, ker \( (W_0) \subseteq \ker(V) \). Therefore, if we assume that ker \( (W)^\perp \) is a RKHS, then ker \( (W_0)^\perp \) is a RKHS, and so is ker \( (V)^\perp \). This shows the first direction.

For the opposite direction, assume that ker \( (V)^\perp \) is a RKHS. Let \( k_1^\perp \) be the kernel of \( \text{ran}(V^*) \) and \( k_2^\perp \) the one on \( \text{ran}(T_F^*) \). Then
\[(W_\alpha^* f)(x) = |(V^* f)(x) - (T_F^* S_F^{-1} f)(x)|\]
\[\leq |(V^* f)(x)| + |(T_F^* S_F^{-1} f)(x)|\]
\[\leq \|k_{V}^*\| \|V^* f\| + \|k_{F}^*\| \|T_F^* S_F^{-1} f\|\]
\[\leq \left(\|k_{V}^*\| \|V\| + \|k_{F}^*\| \frac{1}{A}\right) \|f\|,\]
where \(A\) is the lower frame bound of \(F(x)\). This shows the other direction. \(\square\)

Next we prove the continuous frame counterpart of [16, theorem 6.3.7].

**Theorem 4.5.** Let \(F\) be a continuous frame for \(\mathcal{H}\). The dual frames of \(F\) are precisely the functions

\[G(x) = S_F^{-1} F(x) + \Theta(x) - \int \langle S^{-1} F(x), F(y) \rangle \Theta(y) d\mu(y),\]

(4.2)

where \(\Theta\) is a Bessel mapping.

**Proof.** Equation (4.2) is equivalent to

\[G(x) = (T_F - T_\Theta) T_F^* S_F^{-1} F(x) + \Theta(x).\]

By the construction it is a Bessel mapping as a sum of Bessel mappings. Note that a bounded operator applied to a Bessel mapping again gives a Bessel mapping. Since \(T_\Theta T_F = \text{id}_H\), it is a dual frame.

On the other hand let \(G_\Theta\) be dual frame of \(F\). Then \(V = T_{G_\Theta}\) is a left inverse of \(T_F\), where \(\ker(V)\) is a RKHS. By lemma 4.4 it follows that

\[V = S_F^{-1} T_F + W(I - T_F^* S_F^{-1} T_F),\]

where \(W\) is a bounded operator with \(\ker(W)\) being a RKHS. By lemma 4.3 we have \(G(x) = \sum_{k \in K} \bar{W}(e_k)(x)e_k\). Therefore

\[G(x) = \sum_{k \in K} T_F^* S_F^{-1} (e_k)(x)e_k + \sum_{k \in K} \bar{W}(e_k)(x)e_k - \sum_{k \in K} T_F^* S_F^{-1} T_F \bar{W}(e_k)(x)e_k\]
\[= \sum_{k \in K} T_F^* S_F^{-1} (e_k)(x)e_k + (\text{id}_H - T_F^* S_F^{-1} T_F) \sum_{k \in K} \bar{W}(e_k)(x)e_k.\]

The sequence \(W^*(e_k)\) is a Bessel sequence in the RKHS \(\text{ran}(W^*) = \ker(W)\) and so \(\Theta(x)\) is well-defined. Furthermore

\[\langle f, \Theta(x) \rangle = \left(\int f \sum_{k \in K} \bar{W}(e_k)(x)e_k\right) = \sum_{k \in K} W^*(e_k)(x) \langle f, e_k \rangle = ev_x \left(\sum_{k \in K} W^*(e_k) \langle f, e_k \rangle\right)\]
\[= ev_x \left(W^* \sum_{k \in K} \langle f, e_k \rangle e_k\right) = ev_x \left(W^* f\right).\]

Therefore
\[
\int |\langle f, \Theta(x) \rangle|^2 \, d\mu(x) = \| W^* f \|_{L^2(X, \mu)} \leq \| W \|_0 \| f \|_{L^2(X, \mu)},
\]
and \( \Theta(x) \) is a continuous Bessel mapping.  

Like in the discrete setting [3] this can be reformulated as

**Corollary 4.6.** Let \( F \) be a continuous frame for \( \mathcal{H} \). The dual frames of \( F \) are precisely the functions

\[
G(x) = S_{F(x)}^{-1}F(x) + \Theta(x),
\]
where \( \Theta \) is a Bessel mapping with \( \text{ran} \, (T_{F(x)})^* \subseteq \ker(T_F) \).

Adapting theorem 4.5 to the tensor frame setting we reach the following:

**Corollary 4.7.** Let \( F_1 \otimes F_2 \) be a frame for \( \mathcal{H} \). Then the dual frames of \( F_1 \otimes F_2 \) are precisely the families of the form

\[
S_{F_1(x_1)}^{-1}F_1(x_1) \otimes S_{F_2(x_2)}^{-1}F_2(x_2) + W(x_1, x_2)
\]
\[
- \int_X \langle S_{F_1(x_1)}^{-1}F_1(x_1), F(y_1) \rangle \langle S_{F_2(x_2)}^{-1}F_2(x_2), F(y_2) \rangle W(y_1, y_2) \, d\mu(y),
\]
where \( W \) is a Bessel mapping in \( \mathcal{H} \).

Now let us come back to lemma 3.8: to show that there are non-simple dual tensor frames, one has to find a non-simple Bessel mapping \( W \) such that the result is also non-simple. This can be done as in the case study of the proof of [46, lemma 2.2].

## 5. Tensor product continuous frame multipliers

Gabor multipliers [23] led to the introduction of Bessel and frame multipliers for abstract Hilbert spaces. These operators are defined by a fixed multiplication pattern (the symbol) which is inserted between the analysis and synthesis operators [6–8]. This section is inspired by the continuous frame multipliers studied in [9]. We are interested in the tensor product setting as follows.

**Definition 5.1.** Let \( \mathcal{H} \) be the tensor product \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) of complex Hilbert spaces, and \( (X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2) \) be the product of measure spaces with \( \sigma \)-finite positive measures \( \mu_1, \mu_2 \). Also, let \( F \) and \( G \) be Bessel mappings for \( \mathcal{H} \) with respect to \( (X, \mu) \) and \( m : X \to \mathbb{C} \) be a measurable function. The operator \( M_{m,F,G} : \mathcal{H} \to \mathcal{H} \) weakly defined by

\[
\langle M_{m,F,G} \vec{f}, \vec{g} \rangle = \int_X m(x) \langle \vec{f}, F(x) \rangle \langle G(x), \vec{g} \rangle \, d\mu(x),
\]
for all \( \vec{f}, \vec{g} \in \mathcal{H} \), is called tensor product continuous Bessel multiplier of \( F \) and \( G \) with respect to the symbol \( m \). If, in addition, \( F \) and \( G \) are continuous frames, then \( M_{m,F,G} \) given by (5.1) is called tensor product continuous frame multiplier.

Equation (5.1) is equivalent to the weak formulation of

\[
M_{m,F,G} \vec{f} := \int_X m(x) \langle \vec{f}, F(x) \rangle G(x) \, d\mu(x). \tag{5.2}
\]

**Remark 5.2.** If \( m \equiv 1 \) and \( F \) and \( G \) are Bessel mappings for \( \mathcal{H} \) with respect to \( (X, \mu) \), then \( M_{1,F,G} \) given by (5.1) is a well-defined and bounded sesquilinear form on \( \mathcal{H} \), which could be called the cross-frame operator.
If, in addition, the corresponding operator given by (5.2) has a bounded inverse, then \((F, G)\) is a reproducing pair for \(\mathcal{H}\) in the sense of [43] (when the definition of reproducing pairs is suitably interpreted for tensor product of Hilbert spaces).

If \((F, G)\) is a dual pair of continuous frames (cf definition 3.5), then \(\mathbf{M}_{F,G}\) given by (5.2) is the identity operator (and vice-versa, as we have assumed the Bessel property).

If \(\mathbf{M}_{m,F,G}\) is given by (5.1), then it immediately follows that \((\mathbf{M}_{m,F,G})^* = \mathbf{M}_{m,F,G}\), cf [9, proposition 3.4].

**Lemma 5.3.** Let \(F\) and \(G\) be as in definition 5.1, with the Bessel bounds \(B_F\) and \(B_G\) respectively. If \(m \in L^\infty(X, \mu)\), then the continuous tensor product Bessel multiplier \(\mathbf{M}_{m,F,G}\) given by (5.1) is well defined and bounded with

\[
\|\mathbf{M}_{m,F,G}\| \leq \|m\|_\infty \sqrt{B_FB_G}.
\]

**Proof.** The proof is a modification of the proof of [9, lemma 3.3] for the case of tensor products, and is therefore omitted. □

Here and in what follows the norm in Lebesgue spaces \(L^p(X, \mu)\), \(1 \leq p \leq \infty\) is denoted by \(\|\cdot\|_p\). As usual, we shorten notation by setting \(\|\cdot\| = \|\cdot\|_2\).

If \(m(x) > 0\) a.e., then for any Bessel mapping \(F\) the multiplier \(\mathbf{M}_{m,F}\) is a positive operator, and if \(m(x) \geq \delta > 0\) almost everywhere for some positive constant \(\delta\) and \(\|m\|_\infty < \infty\) then \(\mathbf{M}_{m,F}\) is just the frame operator of \(\sqrt{mF}\) and so it is positive, self-adjoint and invertible, cf [9].

By using analysis and synthesis operators for \(F\) and \(G\), it is easy to see that

\[
\mathbf{M}_{m,F,G} = T_G \circ D_m \circ T_F,
\]
where \(D_m : L^2(X, \mu) \to L^2(X, \mu)\) is given by \((D_m \varphi)(x) = m(x) \varphi(x)\). If \(m \in L^\infty(X, \mu)\), then \(D_m\) is bounded and \(\|D_m\| = \|m\|_\infty\), [17].

If \(m \in L^\infty(\mathbb{R}^d, dx)\), then [9, proposition 3.6] implies that the multiplication operator \(D_m\) on \(L^2(\mathbb{R}^d, dx)\) (with \(dx\) denoting the Lebesgue measure) is compact if and only if \(m \equiv 0\). This constitutes an important difference between the discrete and the continuous case, see [7]. To prove sufficient conditions for compactness of tensor product continuous frame multipliers a different approach than in the discrete setting has to be taken. We closely follow the approach suggested in [9].

### 5.1. Compact multipliers

Recall, a mapping \(F\) is called norm bounded on \((X, \mu)\) if there exists a constant \(C > 0\) such that \(\|F(x)\| \leq C\) for almost every \(x \in X\). Furthermore, the support of measurable function \(m : X \to \mathbb{C}\) is of a finite measure if there exists a subset \(K \subseteq X\) with \(\mu(K) < \infty\) such that \(m(x) = 0\) for almost every \(x \in X\backslash K\).

We can formulate [9, theorem 3.7] in the tensor product setting:

**Theorem 5.4.** Let \(F\) and \(G\) be as in definition 5.1, and let either \(F\) or \(G\) be norm bounded. If \(m : X \to \mathbb{C}\) is a (essentially) bounded measurable function with support of finite measure, then \(\mathbf{M}_{m,F,G}\) given by (5.1) is a compact operator.

The conclusion of theorem 5.4 remains the same if, instead of having the support of finite measure, we assume that \(m : X \to \mathbb{C}\) vanishes at infinity, i.e. for every \(\varepsilon > 0\) there is a set of finite measure \(K = K(\varepsilon) \subseteq X\), \(\mu(K) < \infty\), such that \(m(x) \leq \varepsilon\) for almost every \(x \in X\backslash K\) (cf [9, corollary 3.8]).
If, in addition, we assume that both $F$ and $G$ are norm bounded, then we have the following trace class, and Schatten $p$-class result which is a reformulation of [9, theorems 3.10 and 3.11] to our setting:

**Theorem 5.5.** Let $F$ and $G$ be as in definition 5.1 which are norm bounded with norm bounds $L_F$ and $L_G$, respectively. Then the following is true:

(a) If $m \in L^1(X, \mu)$, then $M_{m,F,G}$ is a trace class operator with the trace norm estimate given by

$$\|M_{m,F,G}\|_{S_1} \leq \|m\|_{L_F}L_G.$$

(b) If $m \in L^p(X, \mu)$, $1 < p < \infty$, then $M_{m,F,G}$ belongs to the Schatten $p$-class $S_p(H)$, with norm estimate

$$\|M_{m,F,G}\|_{S_p} \leq \|m\|_p(L_FL_G)^{\frac{1}{p} - \frac{1}{p'}}.$$

We omit the proof since it follows by slight modifications of the proofs of [9, theorems 3.10 and 3.11].

Recall, if $A \in S_1(H)$, then its trace is defined to be

$$\text{Tr}_H(A) = \sum_n \langle Ae_n, e_n \rangle,$$

for any ONB in $H$. We have $|\text{Tr}_H(A)| \leq \|A\|_{S_1}$, with the equality if $A$ is a positive operator.

For tensor product Hilbert spaces $H = H_1 \otimes H_2$, the following partial trace theorem holds.

**Theorem 5.6.** Let $H$ be a tensor product Hilbert space $H = H_1 \otimes H_2$, and let $A \in S_1(H)$. Then there is a continuous and linear map

$$T : S_1(H) \rightarrow S_1(H_1)$$

such that the following properties hold:

$$T(A_1 \otimes A_2) = A_1 \text{Tr}_{H_2}(A_2), \quad \forall A_j \in S_1(H_j), \quad j = 1, 2;$$

$$\text{Tr}_{H_1}(T(A)) = \text{Tr}_H(A), \quad \forall A \in S_1(H).$$

Proof of theorem 5.6 is contained in the proof of [13, theorem 26.7], and therefore omitted.

If $T$ is the mapping given by (5.4), then $T(A)$ is called the partial trace of $A$ with respect to $H_1$. In a similar way we may define the partial trace of $A$ with respect to $H_2$.

In section 7 we will use the following simple consequence of definition 5.1 and theorem 5.6.

**Corollary 5.7.** Let $m_j$ be measurable functions on $X_j$, let $F_j$ and $G_j$ be continuous Bessel mappings (frames) for $H_j$, $j = 1, 2$, and let $m = m_1 \otimes m_2$, $F = F_1 \otimes F_2$, and $G = G_1 \otimes G_2$. If $M_{m,F,G} \in S_1(H_1 \otimes H_2)$, then its partial trace $T(M_{m,F,G})$ with respect to $H_1$ is a continuous Bessel (frame) multiplier given by

$$T(M_{m_1,F_1,G_1} \otimes M_{m_2,F_2,G_2}) = M_{m_1,F_1,G_1} \text{Tr}_{H_1}(M_{m_2,F_2,G_2}),$$

i.e. it is a trace class operator of ‘the same form’ as $M_{m,F,G}$.

Similar holds for the partial trace of $M_{m,F,G}$ with respect to $H_2$. 


6. Bilinear localization operators

In this section, we reveal bilinear localization operators as examples of tensor product continuous frame multipliers. In the case of short-time Fourier transform multipliers (STFT multipliers), the results from section 5 are in line with those of [20, 44], while their interpretation in the case of wavelet multipliers (Calderón–Toeplitz operators) and mixed STFT/wavelet multipliers seems to be new, although their ‘linear’ counterparts are well studied, see e.g. [9, section 3.4] for a brief survey. In addition, let us mention that the continuity properties of multipliers for the ridgelet transform given in [38] can be derived from the results of [9].

STFT multipliers, also known as time–frequency localization operators, are used in signal analysis as a mathematical tool to extract specific features of a signal from its phase space. The ridgelet transform given in [38] can be derived from the results of [9].

Bilinear localization operators

Let $T_x f(\cdot) := f(\cdot - x)$, $M_x f(\cdot) := e^{2\pi i x \cdot \cdot} f(\cdot)$, and $D_a f(\cdot) := |a|^{-d/2} f(\cdot/|a|)$, denote translation, modulation, and dilation operators, respectively, $x, \omega \in \mathbb{R}^d$, $a \in \mathbb{R} \setminus \{0\}$. These operators are unitary on $L^2(\mathbb{R}^d)$, and we use the notation

\[
\pi(x, \omega) = M_x T_x, \quad \text{for } (x, \omega) \in \mathbb{R}^{2d},
\]

\[
\pi_{ab}(b, a) = T_b D_a, \quad \text{for } (b, a) \in \mathbb{R}^d \times (\mathbb{R} \setminus \{0\}).
\]

Let $\hat{g}$ denote the Fourier transform of $g \in L^1(\mathbb{R}^d)$ given by $\hat{g}(\omega) = \int g(t) e^{-2\pi i t \omega} \, dt$. This definition extends to $L^2(\mathbb{R}^d)$ by density arguments. We say that $g \in L^2(\mathbb{R}^d)$ is an admissible wavelet if

\[
0 < C_g := \int_{\mathbb{R}^d} \frac{|\hat{g}(\omega)|^2}{|\omega|} \, d\omega < +\infty.
\]

Definition 6.1. Let $g \in L^2(\mathbb{R}^d) \setminus \{0\}$. The STFT of a function $f \in L^2(\mathbb{R}^d)$ with respect to the window function $g$ is given by

\[
V_g f(x, \omega) := \int_{\mathbb{R}^d} f(t) \hat{g}(t-x)e^{-2\pi i t \omega} \, dt = \langle f, M_x T_x g \rangle = \langle f, \pi(x, \omega) g \rangle,
\]

$(x, \omega) \in \mathbb{R}^{2d}$. If, in addition, (6.1) holds, i.e. $g$ is an admissible wavelet, then the (continuous) wavelet transform of $f \in L^2(\mathbb{R}^d)$ with respect $g$ is given by

\[
W_g(f)(b, a) := \int_{\mathbb{R}^d} f(t) \frac{1}{|a|} g(a^{-1}(t-b)) \, dt = \langle f, T_b D_a g \rangle = \langle f, \pi_{ab}(b, a) g \rangle, \quad b \in \mathbb{R}^d, \ a \in \mathbb{R} \setminus \{0\}.
\]

Definition 6.1 can be extended to various spaces of (generalized) functions, but we focus our attention here to $L^2(\mathbb{R}^d)$ to make the exposition of our main ideas more transparent.

By the orthogonality relation (see e.g. [31, theorem 3.2.1])

\[
\langle V_{g_1} f_1, V_{g_2} f_2 \rangle = \langle f_1, f_2 \rangle \langle \hat{g}_1, \hat{g}_2 \rangle, \quad f_1, f_2, \in L^2(\mathbb{R}^d), \ g_1, g_2 \in L^2(\mathbb{R}^d) \setminus \{0\},
\]

if $g_1 = g_2 = g$ it follows that $\pi(x, \omega) g$ is a continuous tight frame for $L^2(\mathbb{R}^d)$ with respect to $(\mathbb{R}^d \times \mathbb{R}^d, dx \, dw)$ and with bound $|\hat{g}|^2$, for any $g \in L^2(\mathbb{R}^d) \setminus \{0\}$. If $||g|| = 1$, then we have a continuous Parseval frame.
Likewise, for the wavelet transform the following orthogonality relation holds:

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_{f_1}(f_1)(b, a) W_{f_2}(f_2)(b, a) \frac{db \, da}{a^{d+1}} = C_{g_1, g_2} \langle f_1, f_2 \rangle, \quad f_1, f_2 \in L^2(\mathbb{R}^d),
\]

(6.3)

if \( g_1, g_2 \in L^2(\mathbb{R}^d) \) are such that for almost all \( \omega \in \mathbb{R}^d \) with \( |\omega| = 1 \),

\[
\int_{0}^{\infty} |\hat{g}_1(s\omega)||\hat{g}_2(s\omega)| \frac{ds}{s} < \infty,
\]

and the constant \( C_{g_1, g_2} \) given by

\[
C_{g_1, g_2} := \int_{0}^{\infty} \frac{\hat{g}_1(s\omega)\hat{g}_2(s\omega)}{s} \frac{ds}{s}
\]

is finite, non-zero, and independent on \( \omega \), cf [31, theorem 10.2].

If \( g \in L^2(\mathbb{R}^d) \) is an admissible and rotation invariant function, then the orthogonality relation holds for \( g = g_1 = g_2 \), and \( \pi_{ad}(b, a)g \) is a continuous tight frame for \( L^2(\mathbb{R}^d) \) with respect to \( (\mathbb{R}^d \times \mathbb{R}\{0\}, \frac{db \, da}{a^{d+1}}) \). The frame bound is \( 1/C_{g, g} \), and if \( g \) is suitably normed so that \( C_{g, g} = 1 \), then we have a continuous Parseval frame.

Related continuous frame multipliers, called STFFT and Calderon–Toeplitz multipliers, were discussed in [9]. Here we consider the tensor product space \( \mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \) instead.

If \( \vec{f}, \vec{\varphi} \in \mathcal{H} \), then

\[
V_\varphi \vec{f}(x, \omega) = \langle \vec{f}, \pi(x, \omega) \vec{\varphi} \rangle = \int_{\mathbb{R}^{2d}} \hat{f}(t) \pi(x, \omega) \overline{\varphi(t)} \, dt, \quad x, \omega \in \mathbb{R}^{2d},
\]

and if \( \vec{\varphi}(t) = \varphi_1 \otimes \varphi_2(t) = \varphi_1(t_1) \varphi_2(t_2) \), \( t = (t_1, t_2) \in \mathbb{R}^d \times \mathbb{R}^d \), then \( V_\varphi \) acts on a simple tensor \( f_1 \otimes f_2 \in \mathcal{H} \) as

\[
V_{\varphi_1 \otimes \varphi_2}(f_1 \otimes f_2)(x, \omega) = \int_{\mathbb{R}^{2d}} (f_1 \otimes f_2)(t) \pi(x, \omega) \varphi_1(t_1) \varphi_2(t_2) \, dt
\]

(6.4)

**Lemma 6.2.** Let \( \mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \), and \( \vec{\varphi} = \varphi_1 \otimes \varphi_2 \in \mathcal{H} \setminus \{0\} \). Then

\[
\pi(x, \omega) \vec{\varphi}(t) = \pi(x_1, \omega_1) \varphi_1(t_1) \pi(x_2, \omega_2) \varphi_2(t_2)
\]

is a continuous tight frame for the tensor product space \( \mathcal{H} \) with respect to \( (\mathbb{R}^d \times \mathbb{R}^d, dx \, dw) \).

**Proof.** From (6.2) and (6.4) it follows that the orthogonality relation holds for simple tensors. Now by [41, proposition 2.5] the orthogonality relation can be extended to \( \mathcal{H} \), and we conclude that

\[
\pi(x, \omega) \vec{\varphi}(t) = \pi(x_1, \omega_1) \varphi_1(t_1) \pi(x_2, \omega_2) \varphi_2(t_2), \quad x, \omega \in \mathbb{R}^{2d},
\]

with \( t = (t_1, t_2) \in \mathbb{R}^d \times \mathbb{R}^d \), is a continuous tight frame for \( \mathcal{H} \), i.e.

\[
\langle \vec{f}, \pi(x, \omega) \vec{\varphi} \rangle = \| \vec{f} \| \| \vec{\varphi} \|.
\]

If, in addition, \( \vec{\varphi} \in \mathcal{H} \) is chosen so that \( \| \vec{\varphi} \| = 1 \), then \( \pi(x, \omega) \vec{\varphi} \) is a Parseval frame. \( \square \)
Let $\tilde{\varphi} = \varphi_1 \otimes \varphi_2$, $\tilde{\phi} = \phi_1 \otimes \phi_2 \in \mathcal{H}$, $\|\tilde{\varphi}\| = \|\tilde{\phi}\| = 1$, and let $m : \mathbb{R}^{2d} \to \mathbb{C}$ be a measurable function. Then the tensor product continuous frame multipliers of the form $M_{m,\pi(x,\omega)}\tilde{\varphi}$ can be identified with bilinear localization operators considered in [18, 44] (see Remark 1.2 in [44]), i.e.

$$\langle M_{m,\pi(x,\omega)}\tilde{\phi}, \tilde{g} \rangle = \langle m \nu_{\varphi_1 \otimes \varphi_2}(f_1 \otimes f_2), \nu_{\phi_1 \otimes \phi_2}(g_1 \otimes g_2) \rangle,$$  \hspace{1cm} (6.5)

$f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. The function $m$ is commonly called the symbol of the operator $M_{m,\pi(x,\omega)}\tilde{\varphi}$.

Certain Schatten class properties of bilinear localization operators given by (6.5) can be deduced from their linear counterparts given in e.g. [18, 19]. In these investigations, localization operators are interpreted as Weyl pseudodifferential operators. We note that these results extend results from section 5 in the considered special case. However, we present here a simple alternative proof of related particular result for the linear case given in [18].

**Proposition 6.3.** Let $\mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$, and let $\varphi_1, \varphi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}^d) \backslash \{0\}$. If $m \in L^p(\mathbb{R}^{2d})$, $1 \leq p < \infty$, then $M_{m,\pi(x,\omega)}\tilde{\varphi}$ given by (6.5) belongs to Schatten class $S_p(\mathcal{H})$.

**Proof.** By lemma 6.2 it follows that $F = \pi(x,\omega)\tilde{\varphi}$ and $G = \pi(x,\omega)\tilde{\phi}$ are continuous tight frames for $\mathcal{H}$. Thus $M_{m,F,G}$ is a tensor product continuous frame multiplier, and by theorem 5.5 it follows that $M_{m,F,G} \in S_p(\mathcal{H})$. □

Next we discuss bilinear Calderón–Toeplitz operators. To that end we consider time-scale shifts, and the left Haar measure $\mu = db da / (2d+1)$.

Let $\varphi_1, \varphi_2$ be admissible rotation invariant wavelets, $\varphi_\varphi = \varphi_1 \otimes \varphi_2 \in \mathcal{H}$, and let $\tilde{\psi} \in \mathcal{H}$. Then the tensor product continuous wavelet transform is given by

$$W_{\varphi}(\tilde{\psi})(b, a) = \langle f, \pi_{aff}(b, a)\varphi \rangle, \quad b \in \mathbb{R}^{2d}, \quad a \in \mathbb{R}^2 \backslash \{0\}.$$  

It acts on a simple tensor $f_1 \otimes f_2 \in \mathcal{H}$ as

$$W_{\varphi}(f_1 \otimes f_2)(b, a) = W_{\varphi_1}(f_1) \otimes W_{\varphi_2}(f_2)(b_1, b_2, a_1, a_2)$$

$$= \langle f_1, \pi_{aff}(b_1, a_1)\varphi_1 \rangle \langle f_2, \pi_{aff}(b_2, a_2)\varphi_2 \rangle,$$ \hspace{1cm} (6.6)

where $b = (b_1, b_2) \in \mathbb{R}^{2d}$, $a = (a_1, a_2) \in \mathbb{R}^2 \backslash \{0\}$, and

$$\pi_{aff}(b,a)\varphi = \pi_{aff}(b_1, a_1)\varphi_1 \otimes \pi_{aff}(b_2, a_2)\varphi_2.$$  

**Lemma 6.4.** Let $\mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$, and $\varphi = \varphi_1 \otimes \varphi_2 \in \mathcal{H}$, where $\varphi_1$ and $\varphi_2$ are admissible rotation invariant wavelets. Then

$$\pi_{aff}(b,a)\varphi(t) = \tilde{\psi}(t), \quad b \in \mathbb{R}^{2d}, \quad a \in \mathbb{R}^2 \backslash \{0\},$$

is a continuous tight frame for the tensor product space $\mathcal{H}$ with respect to $(\mathbb{R}^{2d} \times \mathbb{R}^2 \backslash \{0\}, \frac{db da}{(2d+1)})$.

The proof is similar to the proof of lemma 6.2, and therefore omitted.
If \( m : \mathbb{R}^d \times \mathbb{R}^2 \setminus \{0\} \mapsto \mathbb{C} \) is a measurable function, then the tensor product continuous frame multipliers of the form
\[
M_m,\pi_{(a),\varphi} \pi_{(b),\psi}(f_1 \otimes f_2, g_1 \otimes g_2) = \langle mW_{f_1 \otimes \psi}(f_1 \otimes f_2), W_{g_1 \otimes \phi}(g_1 \otimes g_2) \rangle,
\]
equation (6.7)
f and \( g_1, g_2 \in L^2(\mathbb{R}^d) \), can be interpreted as a bilinear extension of (two)wavelet localization operators considered in [47]. More precisely, we have the following result, which seems to be new (see also [47, theorem 19.11]).

**Proposition 6.5.** Let \( \mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \), and let \( \varphi_1, \varphi_2, \phi_1, \) and \( \phi_2 \) be admissible rotation invariant wavelets such that \( \| \varphi_j \| = \| \phi_j \| = 1, j = 1, 2 \). If \( m \in L^p(\mathbb{R}^d \times \mathbb{R}^2 \setminus \{0\}) \), \( 1 \leq p < \infty \), then \( M_m,\pi_{(a),\varphi} \pi_{(b),\psi} \) given by (6.7) belongs to Schatten class \( S_p(\mathcal{H}) \).

**Proof.** By lemma 6.4 it follows that \( F = \pi_{(a),\varphi} \pi_{(b),\psi} \) and \( G = \pi_{(a),\varphi} \pi_{(b),\psi} \) are continuous tight frames for \( \mathcal{H} \). Thus \( M_{m,F,G} \) is a tensor product continuous frame multiplier, and by theorem 5.5 it follows that \( M_{m,F,G} \in S_p(\mathcal{H}) \).

Finally, we combine STFT and wavelet continuous tight frames and consider bilinear localization operators of the ‘mixed-form’.

Consider the measurable space \( (X, \mu) = (\mathbb{R}^d \times (\mathbb{R}^d \times \mathbb{R}^2 \setminus \{0\}), \mu) \) where \( \mu \) is the product of 2\(d\)-dimensional Lebesgue measure and the left Haar measure \( \frac{dx}{|x|^d} \). If \( \varphi \in L^2(\mathbb{R}^d) \setminus \{0\} \) and if \( \phi \in L^2(\mathbb{R}^d) \) is an admissible and rotation invariant wavelet, then we define the STFT-wavelet transform on \( \mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \) as follows
\[
(V_\varphi \otimes W_\phi)(f_1 \otimes f_2) = \langle f_1, \pi(x, \omega)\varphi \rangle \otimes \langle f_2, \pi_{(a),\psi} \rangle.
\]
equation (6.8)

By orthogonality relations (6.2) and (6.3), it follows that
\[
\langle (V_\varphi \otimes W_\phi)(f_1 \otimes f_2), (V_\varphi \otimes W_\phi)(g_1 \otimes g_2) \rangle_{L^2(X)} = \langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle \| \varphi \|^2 \| \phi \|^2.
\]

Thus we conclude that \( \pi(x, \omega)\varphi \pi_{(a),\psi} \) is continuous tight frame for the tensor product space \( H \) (cf lemmas 6.2 and 6.4).

If \( m : X \mapsto \mathbb{C} \) is a measurable function, then the related tensor product continuous frame multiplier is given by
\[
M_m,\pi(x, \omega)\varphi \pi_{(a),\psi} \pi(x, \omega)\varphi \pi_{(b),\psi}(f_1 \otimes f_2, g_1 \otimes g_2) = \int_X m(x)(V_\varphi \otimes W_\phi)(f_1 \otimes f_2)(x), (V_\varphi \otimes W_\phi)(g_1 \otimes g_2)(x)\,d\mu(x),
\]
equation (6.9)
for \( f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d) \).

In the same way as propositions 6.3 and 6.5 we obtain the following.

**Proposition 6.6.** Let \( \mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \), and \( (X, \mu) = (\mathbb{R}^d \times (\mathbb{R}^d \times \mathbb{R}^2 \setminus \{0\}), dx \, \frac{dx}{|x|^d}) \). Moreover, let \( \varphi \in L^2(\mathbb{R}^d) \setminus \{0\} \) and let \( \phi \in L^2(\mathbb{R}^d) \) be an admissible and rotation invariant wavelet, such that \( \| \varphi \| = \| \phi \| = 1 \). If \( m \in L^p(X) \), and \( F = \pi(x, \omega)\varphi \pi_{(a),\psi} \pi_{(b),\psi} \), \( 1 \leq p < \infty \), then \( M_{m,F,F} \) given by (6.9) belongs to Schatten class \( S_p(\mathcal{H}) \).

**Proof.** The result is a consequence of theorem 5.5 and the fact that \( F = \pi(x, \omega)\varphi \pi_{(a),\psi} \) is a continuous tight frame for \( \mathcal{H} \).

We refer to [10] where a general approach based on the coorbit space theory is used to obtain deep continuity results for related kernel type operators.
7. Localization operators as density operators of quantum systems

In this section we first briefly recall the notion of a density operator or a density matrix (as presented in e.g. [32, section 19]), and then identify specific tensor product continuous frame multipliers as density operators. This opens the possibility to use more general results from sections 3 and 5 in the study of quantum systems.

If \( \psi \) represents the wave function which describes the quantum system of e.g. two spinless ‘distinguishable’ particles moving in \( \mathbb{R}^3 \), then typically \( \psi = \psi(x, y) \in L^2(\mathbb{R}^6) \), where \( x \) is the position of the first particle, and \( y \) is the position of the second particle. In general, there does not seem to be a way to associate a vector \( \tilde{\psi} \in L^2(\mathbb{R}^3) \) which could sensibly describe the state of the first (or second) particle, see [13, 32]. To overcome this obstacle a more general notion of the ‘state’ of a quantum system is introduced by associating expectation value of an observable on \( L^2(\mathbb{R}^3) \) with respect to the wave function \( \psi \). This turned out to be the notion of density operator or density matrix, which is uniquely determined by a given family of expectation values. A density operator on the Hilbert space \( \mathcal{H} \) is simply a non-negative, self-adjoint operator \( \rho \in S_1(\mathcal{H}) \) such that \( \Tr_H(\rho) = 1 \).

A class of density operators, called Toeplitz operators is recently studied in [29, 30]. They correspond to quantum states obtained from a fixed function by position-momentum translations. This approach is closely related to the STFT multipliers, and we complement the investigations from [29] by considering the corresponding partial traces (reduced density operators).

By partial trace theorem (theorem 5.6), a density operator of a subsystem can be related to partial trace of the density operator for the whole system. This procedure may give a reasonable description of a subsystem of a bipartite system given by the tensor product Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). In particular, if \( \rho \in S_1(\mathcal{H}_1 \otimes \mathcal{H}_2) \) is of the form \( \rho = \rho_1 \otimes \rho_2 \), then the corresponding density operators for subsystems \( \mathcal{H}_j, j = 1, 2 \), given by partial trace theorem are exactly \( \rho_j, j = 1, 2 \), cf [32, theorem 19.13]. Then the state is said to be a separable state. The opposite direction, i.e. the existence of a pure state \( \rho \) such that given \( \rho_j, j = 1, 2 \), are its partial traces is considered in e.g. [37]. Recently, for given \( \rho_j, j = 1, 2 \), necessary and sufficient conditions for the existence of \( \rho \) with supp \( \rho \subset X \subseteq \mathcal{H} \) such that \( \rho_j, j = 1, 2 \), are its partial traces are given in [25]. These investigations lead to interesting insights related to different types of operator convergence. For example, the weak convergence is not preserved under the partial trace. We refer to [25] for details in that direction.

It is known that characteristic functions of a certain region in phase space give rise to trace class localization operators and may serve to extract time–frequency features of a signal when restricted to that region, see [22]. Thus, it seems plausible to identify convenient tensor product continuous frame multipliers as ‘localized versions’ of density operators of bipartite systems, and use their partial traces to study the features of a subsystem. Of course, to be appropriate candidate of a density operator, a multipliers has to satisfy certain conditions. For the convenience we call them admissible multipliers.

**Definition 7.1.** Let \( M_{m,F,G} \) be a tensor product continuous Bessel (frame) multiplier of \( F \) and \( G \) with respect to the symbol \( m \). Then, \( M_{m,F,G} \) is admissible if it is non-negative, self-adjoint trace class operator such that

\[
\Tr_H(M_{m,F,G}) = 1.
\]

Therefore, any admissible tensor product continuous Bessel (frame) multiplier is a density operator.
As noted in section 5, if $F$ is a continuous frame, $m(x) \geq \delta > 0$ and $\|m\|_{\infty} < \infty$, then $M_{m,F,F}$ is positive, self-adjoint and invertible. For a given $F$, the trace of $M_{m,F,F}$ depends on the symbol $m$, which can be designed in such a way to ensure that $M_{m,F,F}$ is in fact an admissible multiplier.

To illustrate this idea we consider particular case of STFT multipliers.

**Theorem 7.2.** Let there be given $\varphi, \phi \in L^2(\mathbb{R}^d)$ such that $\langle \varphi, \phi \rangle \neq 0$. If $m \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$, then

$$\text{Tr}(M_{m,\pi(x,\omega),\pi(x,\omega)}) = \langle \varphi, \phi \rangle \int_{\mathbb{R}^{2d}} m(x,\omega) \, dx \, d\omega, \quad x, \omega \in \mathbb{R}^d,$$

where $M_{m,\pi(x,\omega),\pi(x,\omega)}$ is weakly given by

$$\langle M_{m,\pi(x,\omega),\pi(x,\omega)} f, g \rangle = \langle mV_{\varphi} f, V_{\phi} g \rangle, \quad f, g \in L^2(\mathbb{R}^d).$$

**Proof.** By definition 5.1 and lemma 6.2 it follows that $M_{m,\pi(x,\omega),\pi(x,\omega)}$ is a tensor product continuous frame multiplier. Furthermore, since $m \in L^1(\mathbb{R}^{2d})$ by proposition 6.3 we have that $M_{m,\pi(x,\omega),\pi(x,\omega)}$ is a trace class operator.

The rest of the proof is similar to the proof of [47, theorem 16.1] which is formulated in terms of irreducible and square-integrable representations of locally compact Hausdorff groups. We give it here for the sake of completeness. Let $(e_n)_{n \in \mathbb{N}}$ be an ONB in $L^2(\mathbb{R}^d)$. Then, by Fubini’s theorem, Parseval’s equality, and since $\pi(x,\omega)$ acts unitary on $L^2(\mathbb{R}^d)$ we obtain

$$\text{Tr}(M_{m,\pi(x,\omega),\pi(x,\omega)}) = \sum_{n=1}^{\infty} \langle M_{m,\pi(x,\omega),\pi(x,\omega)} e_n, e_n \rangle$$

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}^{2d}} m(x,\omega) \langle e_n, \pi(x,\omega) \varphi \rangle \langle \pi(x,\omega) \phi, e_n \rangle \, dx \, d\omega$$

$$= \int_{\mathbb{R}^{2d}} m(x,\omega) \sum_{n=1}^{\infty} \langle e_n, \pi(x,\omega) \varphi \rangle \langle \pi(x,\omega) \phi, e_n \rangle \, dx \, d\omega$$

$$= \int_{\mathbb{R}^{2d}} m(x,\omega) \langle \pi(x,\omega) \varphi, \pi(x,\omega) \phi \rangle \, dx \, d\omega$$

$$= \langle \varphi, \phi \rangle \int_{\mathbb{R}^{2d}} m(x,\omega) \, dx \, d\omega,$$

and the proof is finished. \(\square\)

**Proposition 7.3.** Let there be given $\varphi_1, \phi_1 \in L^2(\mathbb{R}^d \setminus \{0\})$, and let $\vec{\varphi} = \varphi_1 \otimes \varphi_2$, $\vec{\phi} = \phi_1 \otimes \phi_2$. If $m \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$ is chosen so that

$$\int_{\mathbb{R}^{2d}} m(x,\omega) \, dx \, d\omega = \frac{1}{\langle \vec{\varphi}, \vec{\phi} \rangle},$$

then $\text{Tr}(M_{m,\pi(x,\omega),\pi(x,\omega)}) = 1$, where $M_{m,\pi(x,\omega),\pi(x,\omega)}$ is given by (6.5).

If, in addition $\vec{\varphi} = \vec{\phi}$, and $m > 0$, then $M_{m,\pi(x,\omega),\pi(x,\omega)}$ is an admissible multiplier.

**Proof.** To prove the first part, it is enough to consider the extension of theorem 7.2 to tensor product Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$.
The second part follows from the fact that $M_{m,\pi(x,\omega)}\varphi$, $\pi(x,\omega)\varphi$ is the frame operator of $\sqrt{m}\varphi$ and so it is positive, self-adjoint and invertible. Since by theorem 7.2 and (7.1)

$$\text{Tr}_H(M_{m,\pi(x,\omega)}\varphi,\pi(x,\omega)\varphi) = \langle m(x,\omega)dx \, d\omega, \frac{1}{\langle \varphi, \varphi \rangle} = 1,$$

it follows that $M_{m,\pi(x,\omega)}\varphi$, $\pi(x,\omega)\varphi$ is an admissible multiplier. □

By proposition 7.3 we have the following important conclusion, which can be interpreted as a description of a separable state of a bipartite quantum system. This also gives an affirmative partial answer to the question of de Gosson [29, section 5] related to the restriction of the structure of a density operator to its partial traces.

**Theorem 7.4.** Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$, $\varphi = \varphi_1 \otimes \varphi_2 \in L^2(\mathbb{R}^{2d}) \otimes L^2(\mathbb{R}^{2d}) \setminus \{0\}$, and let $m_j(x_j,\omega_j) \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$ be positive functions such that

$$\int_{\mathbb{R}^{2d}} m_j(x_j,\omega_j)dx_j \, d\omega_j = \frac{1}{\|\varphi_j\|}, \quad j = 1, 2. \tag{7.2}$$

Put $m(x,\omega) = m_1(x_1,\omega_1)m_2(x_2,\omega_2)$, and

$$F = \pi(x,\omega)\varphi = \pi(x_1,\omega_1)\varphi_1 \otimes \pi(x_2,\omega_2)\varphi_2.$$

Then the operator $M_{m,F,F}$ given by (6.5) is a density operator, and its partial trace $T(M_{m,F,F})$ with respect to $\mathcal{H}_j$ is the density operator $M_{m_j,\pi_j^{\varphi_j},\pi_j^{\varphi_j}}$, $j = 1, 2$.

**Proof.** By proposition 7.3 it follows that $M_{m,F,F}$ is an admissible multiplier, and therefore it is a density operator.

Next, by corollary 5.7 it follows that

$$T(M_{m,\pi(x,\omega)}\varphi,\pi(x,\omega)\varphi) = M_{m_1,\pi_1^{\varphi_1}} \text{Tr}(M_{m_2,\pi_2^{\varphi_2}}).$$

From the assumptions of the theorem it follows that $M_{m_1,\pi_1^{\varphi_1}}$ and $M_{m_2,\pi_2^{\varphi_2}}$ are both admissible multipliers, so that

$$T(M_{m,\pi(x,\omega)}\varphi,\pi(x,\omega)\varphi) = M_{m_1,\pi_1^{\varphi_1}},$$

and it is a density operator. Similarly for $M_{m_2,\pi_2^{\varphi_2}}$. □

In the same manner one can consider multipliers given by (6.7) and (6.9), and use propositions 6.5 and 6.6 to obtain another types of density operators for which theorem 7.4 holds as well. These considerations can be further used in the study of different aspects of bipartite quantum systems.

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Data availability statement

No new data were created or analysed in this study.

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