Quasi-conserved quantities in the perturbed XXX spin chain

Savvas Malikis,1,* Denis Kurlov,2 and Vladimir Gritsev3,2

1 Instituut-Lorentz, Universiteit Leiden, Leiden, The Netherlands
2 Russian Quantum Center, Skolkovo, Moscow 143025, Russia
3 Institute for Theoretical Physics Amsterdam, Universiteit van Amsterdam, Amsterdam, The Netherlands

We consider the spin-isotropic integrable Heisenberg spin chain weakly perturbed by a local translationally-invariant perturbation. Starting from the local integrals of motion of the unperturbed model, we correct them in order to define proper quasi-conserved integrals of motion (charges) for the perturbed model. To achieve this, we exploit the symmetries of the Hamiltonian and introduce a basis for the set of all possible corrections. It is believed that quasi-conserved quantities are responsible for the prethermalization phase at the intermediate time scales. We found that for a sufficiently local perturbation only the first few integrals of motion can be extended to the quasi-conserved charges, whereas higher-order integrals of motion die out.

I. INTRODUCTION

In classical mechanics the Kolmogorov-Arnold-Moser (KAM) theorem predicts that weakly perturbed integrable systems are stable for a sufficiently long time [1–5]. It is believed that it should be possible to extend the KAM theorem to the quantum case [6]. However, a formal derivation seems to be difficult, since in the quantum case even the very definition of integrability is subtle [7].

Naively, one would expect that nearly-integrable quantum systems will simply thermalize according to the Eigenstate Thermalization Hypothesis (ETH) [8, 9]. However, it is widely believed that such systems also exhibit a different, so-called pre-thermal, behaviour for intermediate time scales [10–12]. Different studies [13–15] suggest that the eventual thermalization occurs at much later times \( t_{\text{therm}} \sim \lambda^{-2} \), where \( \lambda \) is the strength of the perturbation.

It is well understood that the dynamics of integrable models is described by a generalization of the Generalized Gibbs Ensemble (GGE) [16–18], which should include quasi-local conserved charges in addition to the local ones [19–22]. On the other hand, it is believed that the very nature of a pre-thermal phase is described by some effective GGE [23]. Thus, it is natural to look for the corresponding quasi-conserved (perhaps quasi-local) integrals of motion. Additionally, based on the previous reasoning, we may assume that their commutator with the perturbed Hamiltonian should be proportional to \( \lambda^2 \). Since the exact conservation laws of the unperturbed system constrain the dynamics of an integrable system, one can expect that the dynamics of a perturbed system should be contained in the quasi-conserved integrals.

The conjecture above is also supported by the developments in the context of the slowest operators [24, 25]. Indeed, for an operator \( O \) that commutes with a Hamiltonian \( H \), the time evolution \( e^{iHt}Oe^{-iHt} \) is trivial. In terms of the quantum information language this means that the information encoded in \( O(0) \) does not spread. On the contrary, if \( [O(0), H] \neq 0 \), the typical timescale is inversely proportional to a degree of noncommutativity (as follows from the Baker-Campbell-Hausdoff formula). Thus, to slow down the spread of the quantum information it is desirable to (at least approximately) suppress (at least) the first order of the expansion.

The story of existence of quasi-conserved charges can be linked with an old problem in functional analysis, related to almost commuting matrices [26] and explicitly stated by Halmos in [27]. This long-standing question “when two almost commuting matrices are close to matrices that exactly commute” was solved by H. Lin, who proved that “almost commuting” Hermitian matrices are “nearly commuting” [28]. To be more precise, Lin showed that given \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( N \times N \) matrices \( A, B \) are Hermitian, with \( \| AB - BA \| < \delta \), and \( \| A \|, \| B \| \leq 1 \), then there exists commuting Hermitian \( N \times N \) matrices \( X, Y \) such that \( \| A - X \| + \| B - Y \| < \epsilon \). Here, \( \| \cdot \| \) is the usual operator norm. Importantly, \( \delta = \delta(\epsilon) \) does not depend on the dimension \( N \). Recently, Hastings [29] obtained an explicit estimate \( \epsilon(\delta) \sim \delta^{1/5} \), where the exponent may depend on the choice of the operator norm. Note that the question of existence of triples of almost commuting matrices has generically a negative answer [30]. This perhaps is connected to what we found in this paper: it is probably impossible to construct a set of higher-order quasi-commuting quasi-conserved charges.

Similar story about unitary matrices is more involved [31]. There, the existence of almost commuting unitary matrices have some topological obstructions given by the so-called Bott indices. There is an extensive mathematical literature on the subject, see, e.g., [32]. This may suggest that an analogue of the KAM theory for quantum systems can not be generically defined.

The aim of this paper is a construction of nontrivial quasi-conserved integrals of motion for perturbed integrable models. The measure of “non-triviality of these quasi-conserved charges \( Q_n \) is a nonlinear scaling of the norm \( \| [H, Q_n] \| \) with the perturbation strength. While our method (Section III) is quite general, here we focus...
on the integrable XXX spin chain perturbed by the next-nearest neighbour exchange (Section II). Results are discussed in Section IV and can be summarized as follows: one can construct (beyond the Hamiltonian) a few first nontrivial quasi-conserved charges, while it seems that all higher-order charges die out beyond a timescale which scales linearly with the perturbation strength. Details of our procedure are discussed in the Appendix.

II. THE MODEL AND THE BOOST OPERATOR

The model we are going to work with is the XXX spin-1/2 chain with a next-nearest neighbour interaction:

\[ H = H_0 + \lambda h = \sum_{j \in \Lambda} J (\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + \lambda \vec{\sigma}_j \cdot \vec{\sigma}_{j+2}) \].

(1)

Without any loss of generality, we can set \( J = 1 \). Here \( \sigma_j^a \) is a set of Pauli matrices at the \( j \)-th site of the lattice, which is defined in some domain \( \Lambda \) (the number of sites). In this work, we impose periodic boundary conditions and consider \( |\Lambda| \gg 1 \). The corresponding Hilbert space \( \mathcal{H} \) has dimensionality \( 2^{|\Lambda|} \). We assume a small perturbation \( |\lambda| \ll 1 \) from the integrable limit \( \lambda = 0 \).

For \( \lambda = 0 \), the spectrum and the eigenstates are given by the Bethe ansatz \([33]\). The XXX chain is a popular representative of a broader class of exactly solvable quantum systems \([34]\). While for \( \lambda \neq 0 \) the Bethe Ansatz techniques are not applicable, in the limit \( |\lambda| \rightarrow \infty \) the resulting system is essentially equivalent to two decoupled XXX spin chains and thus is again solvable.

The symmetries of the model will be important for our analysis. In particular, the Hamiltonian is translationally invariant and local. This implies that we can write it as a sum of some densities,

\[ H = \sum_{j \in \Lambda} h_{j,k}. \]

(2)

The subscript \( k \) indicates the support of the density \( h_{j,k} \), i.e., the number of consecutive sites where the density acts non-trivially (the XXX Hamiltonian has support \( k = 2 \), while in the presence of the perturbation \( k = 3 \)). Moreover, the total spin is conserved:

\[ [H, \sum_{j \in \Lambda} \sigma_j^a] = 0, \quad a = \{1, 2, 3\}. \]

(3)

This means that the system is \( SU(2) \) invariant. Any relevant operators can be written as a sum of tensor products of \( SU(2) \) invariant densities.

Integrability of \( H_0 \) implies the existence of many nontrivial integrals of motion, which are in an involution,

\[ [H, Q_n] = 0, \quad [Q_n, Q_m] = 0, \quad \forall n, m \in [1, 2, ..., \dim(\mathcal{H})]. \]

(4)

These integrals can be obtained using the quantum inverse scattering method. However, in practice this is too difficult as it requires computing derivatives of a product of many matrices. For certain models, a shorter pathway to conserved charges is via the boost operator. This operator acts as a ladder operator on the set of conserved charges and generates \( Q \)'s iteratively \([35–37]\),

\[ i[B, Q_n] = Q_{n+1}. \]

(5)

The procedure starts from the Hamiltonian (the convention is that \( Q_1 \equiv H_0 \), while \( Q_1 \) is the total magnetization). Consequently, the derivation of different \( Q_n \)'s boils down to the computation of a single commutator. The cost we pay for this is that we obtain only a subset of \( |\Lambda| \) integrals, namely the local ones. We also note that there are quasi-local integrals of motion for the XXX chain \([19]\). To our knowledge, their boost operator construction is not known. For the XXX chain, the boost operator is given by

\[ B = \sum_{j \in \Lambda} j\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} = \sum_{j \in \Lambda} j h_j^{(XXX)}. \]

(6)

Note that it is not translationally invariant but has all the other symmetries. Nevertheless, conserved charges generated by the boost operator \( B \) are translationally invariant and local. Another important property is that it generates quantities that have increasing support, i.e., \( Q_k \) has a support on \( k \) sites. The explicit form of the first integrals can be found in Ref. \([36]\) and will not be reproduced here.

III. METHOD

Considering an obvious idea to treat the unperturbed integrals of motion as the candidates for the perturbed Hamiltonian, we arrive at the trivial statement that \( ||Q_n, H|| \propto \lambda \). Scaling with the first power in perturbation strength implies that we cannot consider them as sufficiently good candidates (see discussion in the Introduction). Ideally, we would like the commutators of quasi-conserved charges with the perturbed Hamiltonian \((1)\) to be proportional to some higher power of \( \lambda \), thus giving a room for prethermalized states and some form of the quantum version of KAM theory.

Having as a starting point the charges created by the boost operator, we will add corrections to them such that the commutator of the new operator with the Hamiltonian would be as small as possible. In addition, we are going to use the symmetries of the problem as much as possible. Thus, quasi-conserved charges have to be some local, translationally invariant operators supporting the symmetries of the (perturbed) model. Regarding locality, we are going to weaken this condition, assuming that the new operator may have a greater support than the original one. This makes sense for two reasons: first, the perturbed Hamiltonian has a greater support than \( H_0 \), and second, quasi-local charges are important already in
the integrable limit. On a practical level, this means that we consider the following ansatz:

\[ Q_{n,M} = Q_n + \sum_{k,i} a_{(k+1,i)}^n O_{k+1}(i). \]  \hfill (7)

Here the symbol \( M \) is the maximal support of the densities of the quasi-conserved charges \( Q_{n,M} \). The symbol \( k \) is used for the support, as before, and we assume that \( 2 \leq k \leq M \). The quantities \( a_{(k+1,i)}^n \) are treated as variational parameters and will be fixed by minimizing the commutator norm. The subscript “\( k + 1 \)” is our convention in order for \( \sigma_a \sigma_{b+1} \) to have support \( k = 2 \). Apart from the support, we also need to label the sites where the density acts non-trivially. By the translational invariance

\[ O_{k+1}(i) = \sum_j o_j(i), \]  \hfill (8)

\[ i = \{j, j + l_2, j + l_3, ..., j + k\}, \quad l_2 < l_3 < ... < k. \]

We assume \( l_i \in \mathbb{N} \). Note that the choice \( l_2 < l_3 < ... < k \) is not the most general one. However, after using our variational method, we are convinced that this choice is efficient enough for both saving computational power and being consistent with the locality. Moreover, due to the \( SU(2) \) symmetry, we do not have to specify the directions of the spin operators at each site and only the specification of the site itself is enough. For more information we refer to the Appendix. Thus, the set of numbers \( a_{(k+1,i)}^n \) defines the operator content of the final quasi-local charge.

We fix these coefficients by minimizing the squared Frobenius norm of the following quantity:

\[ K_{n,M} = i[Q_{n,M}, H]. \]  \hfill (9)

This yields a scalar expression that depends on the set \{\( a_{(k+1,i)}^n \}\} and \( \lambda \). For practical reasons we consider the dimensionless parameter \( L[Q_{n,M}] \), defined as

\[ L^2[Q] \equiv 2^N N \frac{||[H,Q]||_F^2}{||H||_F^2 ||Q||_F^2}. \]  \hfill (10)

We then minimize it numerically for different values of \( \lambda \). The \( 2^N N \) factor is introduced to make the result independent of the system size. The Frobenius norm is chosen for computational convenience.

IV. RESULTS

Using the method described above, we correct the first four initially conserved quantities constructed using the boost operator. To have an estimate of the result, we fitted our data with the power law \( a \lambda^b \).

We observe that for the first quasi-conserved charge \( Q_3 \), the elimination of the first power of \( \lambda \) is exact, while

![FIG. 1. The commutativity measure for the four first quasi-conserved integrals for different maximal support. The fitting parameters are given for every \( M \).](image-url)
the remaining operator is proportional to $\lambda^2$. For the next charge, $Q_4$, the scaling is consistent with a power law $1 < b < 2$. Finally, for the next integrals (up to the sixth) we can optimise the power-law prefactor $a$ while the power $b = 1$ itself seems to be robust when fitting $\mathcal{L}[Q_n,M]$ as a function of $\lambda$.

Thus, we deduce that $Q_3$ does scale as $\lambda^2$ and therefore can be considered as a good candidate for a quasi-conserved charge. The next one, $Q_4$, is kind of marginal, while all the higher-order integrals scale linearly with a perturbation strength and thus die out in the time evolution.

On the other hand, the natural question is whether this method itself converges. In case we need a larger and larger support to have a better "commutativity" for the higher charges, one can imagine that the current results could be unreliable. To check whether this is the case, it is worth to see how the corrections are distributed for different support sizes $k$.

According to Fig. 2, the contribution of the corrections from support $k = 8$ is practically negligible for the first three integrals, and it is even smaller for the fourth one. For this reasons we exclude possible corrections coming from the operators defined on sufficiently large supports $k > 5$. In terms of dynamical time scales it means that contributions coming from higher conserved charges are negligible.

The next question we would like to address is how generic our results are in terms of the perturbation form variability. We have checked the following perturbation that preserves the overall symmetry of the model

$$h' = \sum_{j \in \Lambda} \vec{\sigma}_j \cdot \vec{\sigma}_{j+3}. \quad (11)$$

We concluded that it was not possible to find any quasi-conserved charges (in terms of the same criterion as before). We conjecture that the situation will be the same for any other less local perturbation. The argument for this is that we would need some operators whose commutators with the unperturbed Hamiltonian cancel the non-zero commutator of the original integrals with the perturbation of $H$ (e.g., the first order terms in the perturbation strength). This is quite unlikely when $H_0$ and $h$ have different support $k$.

V. DISCUSSION

Our study indicates that at the timescale where the prethermalization is supposed to work, only (at most) few conserved quantities are important. Influence of the higher order conserved charges seems to vanish in the long-time limit (compared to the inverse of the perturbation strength) dynamics. Thus, it is tempting to suggest that the prethermal phase is governed only by the first few conserved charges, those ones that are consistent with a timescale of the order of $\lambda^{-2}$, while the higher order charges give some later-time corrections.

FIG. 2. The distribution of the corrections for different support for the first four integrals.
The fact that we were able to tune sufficiently well only the first few integrals for the perturbed case is not a great obstacle for making claims about the stability of quantum integrability. Indeed, even in the unperturbed case, it was argued that only a few integrals, the most local ones, are required to describe approximately the full GGE [16–18]. This is known under the name of truncated GGE (tGGE), which gives reliable approximate results. Although the studies of the XXZ chain for $\Delta > 1$ [18] suggest that the formal convergence of the truncated GGE in the limit $\Delta \to 1$ requires many integrals, the corresponding Lagrange multipliers for higher charges are considerably smaller than for the lowest ones. It is then natural to assume that the truncation does make sense for the perturbed case as well.

Our study suggests different possible answers for the existence of the quantum-KAM theory. First of all, there is a possibility that the maximal support used in our computations is not large enough to capture the necessary corrections. It is also possible that the stability of integrability (and, consequently, the existence of the prethermalization plateaus) is not a generic property for perturbations of arbitrary nature. If this is the case, then the identification of quasi-conserved charges should be considered only as a sufficient (and not necessary) condition for the prethermalization.

One possible way out for not excluding the quantum version of the KAM theorem is an extension of the previous observations to include quasi-local quasi-conserved integrals of motion. It is known [19, 20] they play an important role in constructing the full GGE for the integrable case [21]. Therefore, including the quasi-local quasi-conserved integrals of motion could be as important for the perturbed case. However, this is a subject for further studies.

VI. ACKNOWLEDGEMENTS

We would like to thank Anatoli Polkovnikov for igniting this project and for the insightful discussions. This work is part of the DeltaITP consortium, a program of the Netherlands Organization for Scientific Research (NWO) that is funded by the Dutch Ministry of Education, Culture and Science (OCW). This work was supported by the Russian Science Foundation Grant No. 20-42-05002.

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VII. APPENDIX

Here we describe how to construct a basis of local translationally- and $SU(2)$-invariant spin-1/2 operator densities. It is easy to show that one can construct nontrivial quantities by contracting indices only with the use of $\delta_{a,b}$ and $\epsilon_{a,b,c}$ symbols [38]. Examples of these densities are:

$$\delta_{a_1a_2}\sigma_{j_1}^{a_1}\sigma_{j_2}^{a_2}, \quad (12)$$

$$\epsilon_{a_1a_2a_3}\sigma_{j_1}^{a_1}\sigma_{j_2}^{a_2}\sigma_{j_3}^{a_3}, \quad (13)$$

$$\epsilon_{a_1a_2b}\epsilon_{b}a_3a_4\sigma_{j_1}^{a_1}\sigma_{j_2}^{a_2}\sigma_{j_3}^{a_3}\sigma_{j_4}^{a_4}, \quad (14)$$

$$\epsilon_{a_1a_2b}\epsilon_{b}a_3a_4\epsilon_{c}a_5\sigma_{j_1}^{a_1}\sigma_{j_2}^{a_2}\sigma_{j_3}^{a_3}\sigma_{j_4}^{a_4}\sigma_{j_5}^{a_5}. \quad (15)$$

Here indices $j_i$'s represent the lattice sites, while $a_i$'s are the directions in the spin space at that site. The Einstein convention is used implicitly. Since all the densities should be $SU(2)$-symmetric the only degree of freedom is the range of a support where these operators act nontrivially.

It is therefore convenient to introduce the following conventions

$$b_2(j_1,j_2) = -\frac{1}{2}\delta_{a_1a_2}\sigma_{j_1}^{a_1}\sigma_{j_2}^{a_2} \quad (16)$$

$$b_3(j_1,j_2,j_3) = \frac{1}{2}\epsilon_{a_1a_2a_3}\sigma_{j_1}^{a_1}\sigma_{j_2}^{a_2}\sigma_{j_3}^{a_3} \quad (17)$$

$$b_4(j_1,j_2,j_3,j_4) = -\frac{1}{2}\epsilon_{a_1a_2b}\epsilon_{b}a_3a_4\sigma_{j_1}^{a_1}\sigma_{j_2}^{a_2}\sigma_{j_3}^{a_3}\sigma_{j_4}^{a_4} \quad (18)$$

$$b_5(j_1,j_2,j_3,j_4,j_5) = \frac{1}{2}\epsilon_{a_1a_2b}\epsilon_{b}a_3a_4\epsilon_{c}a_5\sigma_{j_1}^{a_1}\sigma_{j_2}^{a_2}\sigma_{j_3}^{a_3}\sigma_{j_4}^{a_4}\sigma_{j_5}^{a_5} \quad (19)$$

Thus we can relate $\alpha_I(i) \leftrightarrow b_n(i)$ from Eq. (8), given that $I = \{j, j+i, \ldots, j+l_n\}$ and $l_n = k$. Note that $b_n$ is a sum of $3 \times 2^{n-2}$ terms with $n$ spins involved. From the basic invariance properties one can see that $b_2(i_1,i_2) = b_2(i_2,i_1)$ and $b_3(i_1,i_2,i_3) = b_3(i_1,i_2,i_3)$. Here we used the round brackets for the totally symmetric expressions, while the square brackets for totally antisymmetric --- the same notations as in the tensor algebra. For $b_n$'s with $n > 3$ there is antisymmetry in the first and the last two arguments, but no global symmetry for all the arguments. Thus, the order of the arguments does matter. Nevertheless we prefer to order the arguments. With these conventions the general operator with a support, e.g., $k = 3$ is written as

$$I_k = k_{3,1}b_2(j,j+2) + k_{3,2}b_3(j,j+1,j+2). \quad (20)$$

In this expression $k_{3,i}$ are arbitrary scalars, which are the general counterparts for the variational parameters $a_{(k+1,i)}^{\alpha}$ used in the main text to construct the $Q_n, M$ properly. Similarly for the support $k = 4$, we have:

$$I_k = k_{4,1}b_2(j,j+3) + k_{4,2}b_3(j,j+1,j+3) + k_{4,3}b_4(j,j+2,j+3) + k_{4,4}b_5(j,j+1,j+2,j+3). \quad (21)$$

This is a generic construction for $M = 4$ (see the main text for notations). We can proceed similarly for higher maximal supports. Let us comment on the complexity of these expressions. Without the $SU(2)$ symmetry there

[38] Examples of these densities
would be $3 \times 4^{M-1}$ independent parameters. The symmetry reduces this number to $\sum_{n=2}^{M} 2^{n-2}$ parameters. Indeed, for a given support $k$, one could have different $b_n$’s for every $n \leq k$. There will be $\binom{k-1}{n-1}$ different $b_n$’s for a given $k$. Thus, for a given $k$, there will be in total $2^{k-2}$ terms for $I_k$.

We now proceed by showing how to generate all $b_n$’s recursively. Starting with $b_2(i_1, i_2)$ we can write $b_3(i_1, i_2, i_3)$ as

$$ b_3(i_1, i_2, i_3) = i[b_2(i_1, i_2), b_2(i_2, i_3)], \quad i_1 \neq i_2 \neq i_3. \quad (22) $$

Similarly, we can construct $b_4(i_1, i_2, i_3, i_4)$ in terms of $b_{2,3}$ as

$$ b_4(i_1, i_2, i_3, i_4) = i[b_2(i_1, i_2), b_3(i_2, i_3, i_4)], \quad i_1 \neq i_4. \quad (23) $$

This can be generalized to generate an arbitrary $b_n$:

$$ b_n(i_1, i_2, \ldots, i_n) = i[b_2(i_1, i_2), b_{n-1}(i_2, \ldots, i_n)], \quad i_1 \neq i_n. \quad (24) $$

In all the previous expressions, all the arguments are distinct. This is important because different arguments ensure commutativity of spin operators entering $b_n$. In the case of coinciding indices, the resulting expression will have terms that contain $m$ spins for $m < n$. The latter can be seen by recalling that $\sigma_a^b \sigma_b^c = \sigma_c^a \delta_{ab} + i \sigma_j^c \epsilon_{abc}$. Note that eventually every $b_n$ can be expressed in terms of $b_2$ and $b_3$ only. To understand this one should realize that all higher $b_n$’s are expressed in terms of the Levi-Civita symbols that have a common index. The use of a simple relation $\epsilon_{abc} \epsilon_{cde} = \delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd}$ proves the above statement.

Following the above observations it remains to establish some algebraic relations for $b_{2,3}$. We found that:

\[
\begin{align*}
    b_2(i, i) &= -\frac{3}{2} \quad (25) \\
    b_3(i_1, i_1, i_2) &= -2ib_2(i_1, i_2) \quad (26) \\
    b_3(i, i, i) &= 3i \quad (27) \\
    b_2(i_1, i_2)b_2(i_2, i_3) &= -\frac{1}{2}b_2(i_1, i_3) - \frac{i}{2}b_3(i_1, i_2, i_3) \quad (28) \\
    b_2(i_1, i_2)b_2(i_1, i_2) &= \frac{3}{4} + b_2(i_1, i_2) \quad (29) \\
    b_3(i_1, i_2, i_3)b_3(i_1, i_2, i_3) &= \frac{3}{2} + b_2(i_1, i_2) + b_2(i_1, i_3) + b_2(i_2, i_3) \quad (30) \\
    b_3(i_1, i_2, i_3)b_3(i_4, i_5, i_3) &= b_2(i_1, i_4)b_2(i_2, i_5) - b_2(i_1, i_5)b_2(i_2, i_4) + ib_2(i_3, i_5)b_3(i_1, i_2, i_4) - ib_2(i_4, i_3)b_3(i_1, i_2, i_5) \quad (31) \\
    b_3(i_1, i_2, i_3)b_3(i_1, i_2, i_4) &= -b_2(i_3, i_4) + \frac{i}{2}b_3(i_1, i_3, i_4) + \frac{i}{2}b_3(i_2, i_3, i_4) - b_2(i_1, i_3)b_2(i_2, i_3) - b_2(i_1, i_4)b_2(i_2, i_3) \quad (32) \\
    b_2(i_1, i_2)b_2(i_2, i_1, i_4) &= -\frac{1}{2}b_3(i_2, i_3, i_4) - ib_2(i_1, i_3)b_2(i_2, i_4) + ib_2(i_1, i_4)b_2(i_2, i_3) \quad (33) \\
    b_2(i_1, i_2)b_3(i_1, i_2, i_3) &= -ib_2(i_1, i_3) + ib_2(i_2, i_3) + \frac{1}{2}b_3(i_1, i_2, i_3) \quad (34) \\
    b_3(i_1, i_2, i_3)b_2(i_1, i_4) &= -\frac{1}{2}b_3(i_2, i_3, i_4) + ib_2(i_1, i_2)b_2(i_3, i_4) - ib_2(i_1, i_3)b_2(i_2, i_4) \quad (35) \\
    b_3(i_1, i_2, i_3)b_2(i_1, i_2) &= ib_2(i_1, i_3) - ib_2(i_2, i_3) + \frac{1}{2}b_3(i_1, i_2, i_3) \quad (36) \\
    b_3(i_1, i_2, i_3)b_2(i_1, i_2) &= -\frac{1}{2}b_3(i_2, i_3, i_4) + ib_2(i_1, i_2)b_2(i_3, i_4) - ib_2(i_1, i_3)b_2(i_2, i_4) \quad (37) \\
    b_3(i_1, i_2, i_3)b_2(i_1, i_2) &= ib_2(i_1, i_3) - ib_2(i_2, i_3) + \frac{1}{2}b_3(i_1, i_2, i_3). \quad (38) \\
\end{align*}
\]

Provided we know these relations and the symmetries of $b_2, b_3$ we know all the possible combinations therein and the higher-order $b_n$’s. We stress again that in the previous expressions $i_a$’s are all distinct. Moreover, a product of two $b_3$ can be reduced to terms that have three $b_2$. This can be shown by using the well-known identity

$$
\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix}
\delta_{il} & \delta_{im} & \delta_{in} \\
\delta_{jl} & \delta_{jm} & \delta_{jn} \\
\delta_{kl} & \delta_{km} & \delta_{kn}
\end{vmatrix}. \quad (39)
$$
Also, there is the following identity \[ [39] \]

\[
\begin{align*}
  & b_3(i_1, i_2, i_3) b_2(i_4, i_5) - b_3(i_2, i_3, i_4) b_2(i_1, i_5) + \\
  & b_3(i_3, i_4, i_1) b_2(i_2, i_5) - b_3(i_4, i_1, i_2) b_2(i_3, i_5) = 0. 
\end{align*}
\] (40)

This can be used to define some sort of a normal-ordering type prescription: the smallest argument in all expressions is contained in \( b_3 \).