ON THE GOOD FILTRATION DIMENSION OF WEYL MODULES FOR A LINEAR ALGEBRAIC GROUP

ALISON E. PARKER

Abstract. Let \( G \) be a linear algebraic group over an algebraically closed field of characteristic \( p \) whose corresponding root system is irreducible. In this paper we calculate the Weyl filtration dimension of the induced \( G \)-modules, \( \nabla(\lambda) \) and the simple \( G \)-modules \( L(\lambda) \), for \( \lambda \) a regular weight. We use this to calculate some Ext groups of the form \( \text{Ext}^i(\nabla(\lambda), \Delta(\mu)) \), \( \text{Ext}^i(L(\lambda), L(\mu)) \), and \( \text{Ext}^i(\nabla(\lambda), \nabla(\mu)) \), where \( \lambda, \mu \) are regular and \( \Delta(\mu) \) is the Weyl module of highest weight \( \mu \). We then deduce the projective dimensions and injective dimensions for \( L(\lambda), \nabla(\lambda) \) and \( \Delta(\lambda) \) for \( \lambda \) a regular weight in associated generalised Schur algebras. We also deduce the global dimension of the Schur algebras for \( GL_n, S(n, r) \), when \( p > n \) and for \( S(mp, p) \) with \( m \) an integer.

Introduction

In this paper we consider the notion of the Weyl filtration dimension and good filtration dimension of modules for a linear algebraic group. These concepts were first introduced by Friedlander and Parshall [15] and may be considered a variation of the notion of projective dimension and injective dimension respectively. (The precise definition is given in 2.2.) The Weyl filtration dimension of a module is always at most its projective dimension. In fact, it is often much less. In the situation of algebraic groups the Weyl and good filtration dimensions are always finite for a finite dimensional module (unlike the projective and injective dimensions which are usually infinite). Thus knowing these dimensions give us another tool for calculating the cohomology of an algebraic group. Indeed we use knowledge of these dimensions to calculate various Ext groups for \( G \).

We had previously calculated the good filtration dimension of the irreducible modules for \( S(n, r) \), the Schur algebra corresponding to \( GL_n(k) \) when \( n = 2 \) and \( n = 3 \) in [21]. We were then able to determine the global dimension of \( S(n, r) \). The proof in [21] relies heavily on the use of filtrations of the induced modules \( \nabla(\lambda) \), \( \lambda \) a dominant weight, by modules of the form \( \nabla(\mu) \otimes L(\nu) \).

In this paper we instead use the translation functors introduced by Jantzen to calculate properties of the induced modules and the Weyl modules (denoted \( \Delta(\lambda) \)) for an algebraic group. We first calculate the Weyl filtration dimension (abbreviated wfd), of the induced modules for regular weights (theorem 4.2). We then prove \( \text{Ext}^i(\nabla(\lambda), \Delta(\mu)) \cong k \) when \( i = \text{wfd}(\nabla(\lambda)) + \text{wfd}(\nabla(\mu)) \) and \( \lambda, \mu \) regular (theorem 4.3). We can then deduce that \( \text{Ext}^i(L(\lambda), L(\mu)) \cong k \) for \( i = \text{wfd}(L(\lambda)) + \text{wfd}(L(\mu)) \).

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wfd\((L(\mu))\) (corollary 4.5). These results then enable us to write down the injective and projective dimensions of \(L(\lambda), \nabla(\lambda)\) and \(\Delta(\lambda)\) for \(\lambda\) a regular weight in associated generalised Schur algebras (theorem 4.7).

We can deduce the value of the global dimension of \(S(n, r)\) when \(p > n\) and \(S(p, mp)\) with \(m \in \mathbb{N}\) (theorems 5.8 and 5.9). This gives us an alternative proof for \(S(2, r)\) (all \(p\)) and for \(S(3, r)\) with \(p \geq 5\). Some of this work also appears in the author’s PhD thesis [22], chapter 6.

In general the global dimension of \(S(n, r)\) is still not known. Previous values were calculated for \(r \leq n\) by Totaro [24] (for the classical case) and Donkin [12], section 4.8, (for the quantum case). The semi-simple Schur algebras (that is the Schur algebras with zero global dimension) have been determined in [13] for the classical case and [14], theorem (A), for the quantum case. Conjectured values for the remaining cases are presented in [22], section 6.5.

We conclude by showing that analogous results for the Dipper–Donkin quantum group hold and hence for the \(q\)-Schur algebra. The extent to which similar methods may be applied to category \(\mathcal{O}\) is also discussed.

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1. Preliminaries

We first review the basic concepts and most of the notation that we will be using. The reader is referred to [16] and [23] for further information. This material is also in [18] where it is presented in the form of group schemes.

Throughout this paper \(k\) will be an algebraically closed field of characteristic \(p\). Let \(G\) be a linear algebraic group which is connected and reductive. We fix a maximal torus \(T\) of \(G\) of dimension \(n\), the rank of \(G\). We also fix \(B\), a Borel subgroup of \(G\) with \(B \supseteq T\) and let \(W\) be the Weyl group of \(G\).

We will write \(\text{mod}(G)\) for the category of finite dimensional rational \(G\)-modules. Most \(G\)-modules considered in this paper will belong to this category. Let \(X(T) = X\) be the weight lattice for \(G\) and \(Y(T) = Y\) the dual weights. The natural pairing \(\langle -, - \rangle : X \times Y \to \mathbb{Z}\) is bilinear and induces an isomorphism \(Y \cong \text{Hom}_\mathbb{Z}(X, \mathbb{Z})\). We take \(R\) to be the roots of \(G\). For each \(\alpha \in R\) we take \(\check{\alpha} \in Y\) to be the coroot of \(\alpha\). Let \(R^+\) be the positive roots, chosen so that \(B\) is the negative Borel and let \(S\) be the set of simple roots. Set \(\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in X \otimes \mathbb{Z}\).

We have a partial order on \(X\) defined by \(\mu \leq \lambda \Leftrightarrow \lambda - \mu \in \mathbb{N}S\). A weight \(\lambda\) is dominant if \(\langle \lambda, \check{\alpha} \rangle \geq 0\) for all \(\alpha \in S\) and we let \(X^+\) be the set of dominant weights.

Take \(\lambda \in X^+\) and let \(k_\lambda\) be the one-dimensional module for \(B\) which has weight \(\lambda\). We define the induced module, \(\nabla(\lambda) = \text{Ind}_G^B(k_\lambda)\). This module has formal character given by Weyl’s character
formula and has simple socle $L(\lambda)$, the irreducible $G$-module of highest weight $\lambda$. Any finite dimensional, rational irreducible $G$-module is isomorphic to $L(\lambda)$ for a unique $\lambda \in X^+$. 

Since $G$ is split, connected and reductive we have an antiautomorphism, $\tau$, which acts as the identity on $T$ ([18], II, corollary 1.16). From this morphism we may define $^\circ$, a contravariant dual. It does not change a module’s character, hence it fixes the irreducible modules. We define the Weyl module, to be $\Delta(\lambda) = \nabla(\lambda)^{^\circ}$. Thus $\Delta(\lambda)$ has simple head $L(\lambda)$.

We return to considering the weight lattice $X$ for $G$. There are also the affine reflections $s_{\alpha,mp}$ for $\alpha$ a positive root and $m \in \mathbb{Z}$ which act on $X$ as $s_{\alpha,mp}(\lambda) = \lambda - (\langle \lambda, \alpha^{\vee} \rangle - mp)\alpha$. These generate the affine Weyl group $W_p$. We mostly use the dot action of $W_p$ on $X$ which is the usual action of $W_p$, with the origin shifted to $-\rho$. So we have $w \cdot \lambda = w(\lambda + \rho) - \rho$. If $F$ is an alcove for $W_p$ then its closure $\bar{F} \cap X$ is a fundamental domain for $W_p$, operating on $X$. The group $W_p$ permutes the alcoves simply transitively. We set $C = \{ \lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle \lambda + \rho, \alpha^{\vee} \rangle < p \ \forall \alpha \in R^+ \}$ and call $C$ the fundamental alcove. We also set $h = \max \{ \langle \rho, \beta^{\vee} \rangle + 1 \mid \beta \in R^+ \}$. When $R$ is irreducible then $h$ is the Coxeter number of $R$. In general, it is the maximum of all Coxeter numbers of the irreducible components of $R$. We have $C \cap X \neq \emptyset \iff \langle \rho, \beta^{\vee} \rangle < p \ \forall \beta \in R^+ \iff p \geq h$.

A facet $F$ is a wall if there exists a unique $\beta \in R^+$ with $\langle \lambda + \rho, \beta^{\vee} \rangle = mp$ for some $m \in \mathbb{Z}$ and for all $\lambda \in F$. Let $s_F = s_{\beta,mp}$. This is the unique reflection in $W_p$ which acts as the identity on $F$ and we call $s_F$ the reflection with respect to $F$.

Let $\text{Stab}_{W_p}(\lambda)$ be all the elements of $W_p$ which stabilise $\lambda \in X$. We take $\Sigma$ to be the set of all reflections $s_F$ where $F$ is a wall (for $W_p$) with $F \subset \bar{C}$. Thus the set $\Sigma$ consists of the reflections $s_{\alpha,0}$ with $\alpha \in S$ together with $s_{\beta,p}$ with $\beta$ the longest short root of each irreducible component of the root system $R$. Let $\Sigma^0(\mu)$ be the subset of $\Sigma$ where each element of $\Sigma^0(\mu)$ fixes $\mu$. The affine Weyl group $W_p$ is generated by $\Sigma$. These generators form a presentation for $W_p$ as a Coxeter group so we may define a length function $l(w)$ for $w \in W_p$ which is the length of a reduced expression for $w$ in terms of elements of $\Sigma$.

We say that $\lambda$ and $\mu$ are linked if they belong to the same $W_p$ orbit on $X$ (under the dot action). If two irreducible modules $L(\lambda)$ and $L(\mu)$ are in the same $G$ block then $\lambda$ and $\mu$ are linked.

The category of rational $G$-modules has enough injectives and so we may define $\text{Ext}^*_G(-,-)$ as usual by using injective resolutions (see [3], section 2.4 and 2.5). We will usually just write $\text{Ext}$ for $\text{Ext}_G$.

2. Quasi-Hereditary Algebras

In this section we prove some lemmas about module category $\text{mod}(A)$, for a quasi-hereditary algebra $A$ with poset $(\Lambda, \leq)$, standard modules $\Delta(\lambda)$ and costandard modules $\nabla(\lambda)$. We will later lift these results to $\text{mod}(G)$.

We say $X \in \text{mod}(A)$ has a good filtration if it has a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_i = X$ with quotients $X_j/X_{j-1}$ isomorphic to $\nabla(\mu_j)$ for some $\mu_j \in \Lambda$. The class of $A$-modules with good
filtration is denoted $\mathcal{F}(\nabla)$, and dually the class of modules filtered by $\Delta(\mu)$’s is denoted $\mathcal{F}(\Delta)$. We say that $X \in \mathcal{F}(\Delta)$ has a Weyl filtration. The multiplicity of $\nabla(\mu)$ in a filtration of $X \in \mathcal{F}(\nabla)$ is independent of the filtration chosen and is denoted by $(X : \nabla(\mu))$. The composition multiplicity of $L(\mu)$ in $X \in \mod(S)$ is denoted by $[X : L(\mu)]$. We point out that even when $\nabla(\mu) = L(\mu)$ then it is still not necessarily true that $(X : \nabla(\mu)) = [X : L(\mu)]$.

Some of the important properties of $\mathcal{F}(\nabla)$ and $\mathcal{F}(\Delta)$ are stated below.

**Proposition 2.1.**

(i) Let $X \in \mod(A)$ and $\lambda \in \Lambda$. If $\text{Ext}_A^1(X, \nabla(\lambda)) \neq 0$ then $X$ has a composition factor $L(\mu)$ with $\mu > \lambda$.

(ii) For $X \in \mathcal{F}(\Delta)$, $Y \in \mathcal{F}(\nabla)$ and $i > 0$, we have $\text{Ext}_A^i(X, Y) = 0$.

(iii) Suppose $\text{Ext}_A^1(\Delta(\mu), M) = 0$ for all $\mu \in \Lambda$ then $M \in \mathcal{F}(\nabla)$.

(iv) Let $X \in \mathcal{F}(\nabla)$ (resp. $X \in \mathcal{F}(\Delta)$) and $Y$ a direct summand of $X$ then $Y \in \mathcal{F}(\nabla)$ (resp. $Y \in \mathcal{F}(\Delta)$).

**Proof.** See [12], A2.2. □

Suppose $X \in \mod(A)$. We can resolve $X$ by modules $M_i \in \mathcal{F}(\nabla)$ as follows

$$0 \to X \to M_0 \to M_1 \to \ldots \to M_d \to 0.$$

Such a resolution a good resolution for $X$. Good resolutions exist for all $A$-modules as $A$ has enough injectives and an injective resolution is also a good resolution.

The following definition may be found in [15] where a proof of the equivalence of properties (i) and (ii) may be found (see [15], proposition 3.4).

**Definition 2.2.** Let $X \in \mod(A)$. We say $X$ has good filtration dimension $d$, denoted $gfd(X) = d$, if the following two equivalent conditions hold:

(i) $0 \to X \to M_0 \to M_1 \to \ldots \to M_d \to 0$ is a resolution for $X$ with $M_i \in \mathcal{F}(\nabla)$, of shortest possible length.

(ii) $\text{Ext}_A^i(\Delta(\lambda), X) = 0$ for all $i > d$ and all $\lambda \in \Lambda$, but there exists $\lambda \in \Lambda$ such that $\text{Ext}_A^d(\Delta(\lambda), X) \neq 0$.

Similarly we have the dual notion of the Weyl filtration dimension of $M$ which we will denote $wfd(M)$.

**Lemma 2.3.** Given $A$-modules $M$ and $N$, we have

$$\text{Ext}_A^i(N, M) = 0 \text{ for } i > wfd(N) + gfd(M).$$

**Proof.** See [21], lemma 2.2. □
Definition 2.4. Let \( g = \sup \{ \text{gfd}(X) \mid X \in \text{mod}(A) \} \). We say \( A \) has good filtration dimension \( g \) and denote this by \( \text{gfd}(A) = g \). Let \( w = \sup \{ \text{wfd}(X) \mid X \in \text{mod}(A) \} \). We say \( A \) has Weyl filtration dimension \( w \) and denote this by \( \text{wfd}(A) = w \).

Remark 2.5. In general \( \text{gfd}(A) \) is not the good filtration dimension of \( A \) when considered as its own left (or right) module. Similar remarks apply to \( \text{wfd}(A) \). We will only use \( \text{gfd}(A) \) and \( \text{wfd}(A) \) in the sense that they are defined above.

For a finite dimensional \( k \)-algebra \( A \), the \textit{injective dimension} of an \( A \)-module \( M \), is the length of a shortest possible injective resolution and is denoted by \( \text{inj}(M) \). Equivalently we have \( \text{inj}(M) = \sup \{ d \mid \text{Ext}_{A}^{d}(N, M) \neq 0 \text{ for } N \in \text{mod}(A) \} \). The \textit{global dimension} of \( A \) is the supremum of all the injective dimensions for \( A \)-modules, and is denoted by \( \text{glob}(A) \). This is equivalent to \( \text{glob}(A) = \sup \{ d \mid \text{Ext}_{A}^{d}(N, M) \neq 0 \text{ for some } N, M \in \text{mod}(A) \} \). We will also denote the \textit{projective dimension} of an \( A \)-module \( M \) by \( \text{proj}(M) \).

Corollary 2.6. The global dimension of \( A \) has an upper bound of \( \text{wfd}(A) + \text{gfd}(A) \).

Definition 2.7. We say a module \( T \) is a \textit{tilting module} if \( T \) has both a good filtration and a Weyl filtration. That is \( T \in \mathcal{F}(\nabla) \cap \mathcal{F}(\Delta) \).

For each \( \lambda \in \Lambda \) there is a unique indecomposable tilting module, \( T(\lambda) \), of highest weight \( \lambda \) with \( \{ T(\lambda) : L(\lambda) \} = 1 \). Every tilting module \( T \) can be written as a direct sum of indecomposable tilting modules \( T(\mu) \) with \( \mu \in \Lambda \) ([12], theorem A4.2).

Definition 2.8. Take \( \lambda \in \Lambda \). We take a chain \( \mu_{0} < \mu_{1} < \cdots < \mu_{l-1} < \mu_{l} = \lambda \) with \( l \) maximal and \( \mu_{i} \in \Lambda \). We define the length of \( \lambda \), \( l(\lambda) \) to be \( l \). We also define \( l(\Lambda) = \max \{ l(\lambda) \mid \lambda \in \Lambda \} \).

Lemma 2.9. We have \( \text{wfd}(\nabla(\lambda)) \leq l(\lambda) \).

Proof. If \( l(\lambda) = 0 \) then \( \lambda \) is minimal so \( \nabla(\lambda) = \Delta(\lambda) \) and \( \text{wfd}(\nabla(\lambda)) = 0 = l(\lambda) \).

Now suppose the lemma is true for \( \mu < \lambda \). We have a short exact sequence

\[ 0 \to N \to T(\lambda) \to \nabla(\lambda) \to 0 \]

where \( T(\lambda) \) is the indecomposable tilting module of highest weight \( \lambda \). Applying \( \text{Ext}_{A}^{i}(\nabla(\nu)) \) for \( \nu \in \Lambda \) gives us

\[ \text{Ext}_{A}^{i-1}(N, \nabla(\nu)) \to \text{Ext}_{A}^{i}(\nabla(\lambda), \nabla(\nu)) \to \text{Ext}_{A}^{i}(T(\lambda), \nabla(\nu)) \]

We now take \( i > l(\lambda) \). All the \( \nabla(\mu) \) appearing in a good filtration of \( N \) have \( \mu < \lambda \). Hence \( l(\mu) < l(\lambda) \) and so \( i-1 > l(\mu) \). We now use the induction hypothesis to get \( \text{Ext}_{A}^{i-1}(N, \nabla(\nu)) = 0 \).

We also have \( \text{Ext}_{A}^{i}(T(\lambda), \nabla(\mu)) = 0 \) as \( T(\lambda) \) is tilting and using proposition 2.1 (ii). Hence \( \text{Ext}_{A}^{i}(\nabla(\lambda), \nabla(\mu)) = 0 \) for \( i > l(\lambda) \) and \( \text{wfd}(\nabla(\lambda)) \leq l(\lambda) \).

We may similarly prove that \( \text{gfd}(\Delta(\lambda)) \leq l(\lambda) \).
Lemma 2.10.  

\[ \text{wfd } A = \max \{ \text{wfd}(\nabla(\lambda)) \mid \lambda \in \Lambda \}. \]

Proof. We certainly have  

\[ \text{wfd } A \geq \max \{ \text{wfd}(\nabla(\lambda)) \mid \lambda \in \Lambda \}. \]

Take \( \lambda \in \Lambda \) with \( \text{wfd}(L(\lambda)) = d = \text{wfd}(A) \). Let \( Q \) be the quotient \( \nabla(\lambda)/L(\lambda) \). Since \( \text{wfd}(L(\lambda)) \) was maximal we must have \( \text{wfd}(Q) \leq d \). Let \( \mu \in \Lambda \), the corresponding long exact sequence for the short exact sequence for \( Q \) gives us  

\[ \text{Ext}^d_A(\nabla(\lambda), \nabla(\mu)) \rightarrow \text{Ext}^d_A(L(\lambda), \nabla(\mu)) \rightarrow \text{Ext}^{d+1}_A(Q, \nabla(\mu)) \]

Now \( \text{Ext}^{d+1}_A(Q, \nabla(\mu)) = 0 \) for all \( \mu \in \Lambda \) by lemma 2.3. But there exists \( \mu \in \Lambda \) with \( \text{Ext}^d_A(L(\lambda), \nabla(\mu)) \neq 0 \). Hence \( \text{Ext}^d_A(\nabla(\lambda), \nabla(\mu)) \neq 0 \). Thus there exists \( \lambda \in \Lambda \) with \( \text{wfd}(\nabla(\lambda)) = d = \text{wfd}(A) \).  

\[ \square \]

Remark 2.11. We may replace the set of \( \nabla(\lambda) \) with any set of \( A \)-modules \( X \) with the property that for all \( \lambda \in \Lambda \) there exists \( X \in \mathcal{X} \) with \( L(\lambda) \) contained in the socle of \( X \). We can then repeat the argument above to get \( \text{wfd}(A) = \max \{ \text{wfd}(X) \mid X \in \mathcal{X} \} \).

Now suppose \( A \) is a quasi-hereditary algebra with contravariant duality preserving simples. That is there exists an involutory, contravariant functor \( \circ : \text{mod}(A) \rightarrow \text{mod}(A) \) such that, \( \Delta(\lambda) \circ \cong \nabla(\lambda) \) (and \( \text{Ext}^i_A(M, N) \cong \text{Ext}^i_A(N \circ, M \circ) \)). We will usually shorten this and say \( A \) has a simple preserving duality.

Remark 2.12. It is clear (given the equivalences in the definition for the good filtration dimension) that for \( A \) with simple preserving duality and \( M \) an \( A \)-module we have \( \text{wfd}(M) = \text{gfd}(M \circ) \). We will use this without further comment.

Thus lemma 2.9 gives an upper bound for \( \text{wfd}(A) \) of \( l(\Lambda) \). Corollary 2.6 gives, for \( A \) with simple preserving duality that  

\[ \text{glob}(A) \leq 2 \text{gfd}(A) = 2 \text{wfd}(A) \leq 2l(\Lambda). \]

We say a subset \( \Pi \) of a poset \( (\Lambda, \leq) \) (not necessarily finite) is saturated if for all \( \lambda \in \Pi \) then \( \mu \leq \lambda \) implies that \( \mu \in \Pi \).

Take \( G \) to be a split, connected reductive algebraic group with weight lattice \( X \). Suppose \( \Pi \) is a finite saturated subset of \( X^+ \) with respect to the dominance ordering. We may consider \( G \)-modules all whose composition factors have highest weights lying in \( \Pi \). These modules form a subcategory of \( \text{mod}(G) \) which is a highest weight category corresponding to a quasi-hereditary algebra which we denote \( S(\Pi) \), the generalised Schur algebra (see [9] for more information). We have a natural isomorphism  

\[ \text{Ext}^i_{S(\Pi)}(M, N) \cong \text{Ext}^i_G(M, N) \]
for $S(\Pi)$-modules $M$ and $N$ \cite{9}, 2.2d. The costandard and standard modules for $S(\Pi)$ are exactly the induced and Weyl modules for $G$ respectively. Thus as long as we restrict our attention to finite dimensional $G$-modules then we can lift the results from quasi-hereditary algebras to $G$.

Generally speaking, a finite dimensional $G$-module does not have a finite injective or projective resolution. It will have, however, have a finite good (and Weyl) resolution. Thus we can lift the definitions of good (and Weyl) filtration dimension to $\text{mod}(G)$.

If we take $G = \text{GL}_n(k)$ and $\Pi = \Lambda^+(n,r)$ then $S(\Pi)$ is isomorphic to $S(n,r)$, the usual Schur algebra. Thus Schur algebras are quasi-hereditary with poset $\Lambda^+(n,r)$ ordered by dominance.

3. Properties of Translation Functors

For any $G$-module $V$ and any $\mu \in \bar{C}$, set $\text{pr}_\mu V$ equal to the sum of submodules of $V$ such that all the composition factors have highest weight in $W_p \cdot \mu$. Then $\text{pr}_\mu V$ is the largest submodule of $V$ with this property. The following definition is due to Jantzen \cite{18}, II, 7.6.

**Definition 3.1.** Suppose $\lambda, \mu \in \bar{C}$. There is a unique $\nu_1 \in X^+ \cap W(\mu - \lambda)$. We define the translation functor $T^\mu_{\lambda}$ from $\lambda$ to $\mu$ via

$$T^\mu_{\lambda}V = \text{pr}_{\mu}(L(\nu_1) \otimes \text{pr}_{\lambda}V)$$

for any $G$-module $V$. It is a functor from $\text{mod}(G)$ to itself.

**Lemma 3.2.** Let $\lambda$ and $\mu \in \bar{C}$, then the functors $T^\mu_{\lambda}$ and $T^\lambda_{\mu}$ are adjoint to each other. For $M, N \in \text{mod}(G)$ we have $\text{Ext}^i(T^\mu_{\lambda}M, N) \cong \text{Ext}^i(M, T^\lambda_{\mu}N)$.

**Proof.** See \cite{18}, II, lemma 7.6 (b) and remark 7.6 (2). \hfill \Box

**Proposition 3.3.** Let $\mu, \lambda \in \bar{C}$ and $w \in W_p$ with $w \cdot \mu \in X^+$, then $T^\lambda_{\mu} \nabla(w \cdot \mu)$ has a good filtration. Moreover the factors are $\nabla(ww_1 \cdot \lambda)$ with $w_1 \in \text{Stab}_{W_p}(\mu)$ and $ww_1 \cdot \lambda \in X^+$. Each different $ww_1 \cdot \lambda$ occurs exactly once.

**Proof.** See \cite{18}, proposition 7.13. \hfill \Box

**Corollary 4.** Let $\lambda \in C$ and $\mu \in \bar{C}$. Suppose there is $s \in \Sigma$ with $\Sigma^0(\mu) = \{s\}$. Let $w \in W_p$ with $w \cdot \lambda \in X^+$ and $w \cdot \lambda < ws \cdot \lambda$. Then we have a short exact sequence

$$0 \to \nabla(w \cdot \lambda) \to T^\lambda_{\mu} \nabla(w \cdot \mu) \to \nabla(ws \cdot \lambda) \to 0.$$ 

**Proof.** See \cite{18}, lemma 7.19 (a). \hfill \Box

We would like to know when such a situation in the above corollary occurs. Firstly we need a $\lambda \in C$ and this happens when $p \geq h$. We also need a weight $\mu$ lying on the wall between $\lambda$ and $s \cdot \lambda$. This happens when the derived group of $G$ is simply connected and $p \geq h$. See \cite{18}, II, 6.3 (1), for details. We will henceforth assume that $p \geq h$ and that the derived group of $G$ is simply
connected. We will also assume that the root system $R$ of $G$ is irreducible, although we believe that theorem 4.2 is also true in the more general case.

We have another partial order on $X$ denoted $\uparrow$. If $\alpha$ is a positive root and $m \in \mathbb{Z}$ then we set

$$s_{\alpha,mp} \cdot \lambda \uparrow \lambda$$

if and only if $\langle \lambda + \rho, \alpha^\vee \rangle \geq mp$.

This then generates an order relation on $X$. So $\mu \uparrow \lambda$ if there are reflections $s_i \in W_p$ with

$$\mu = s_ms_{m-1} \cdots s_1 \cdot \lambda \uparrow s_{m-1} \cdots s_1 \cdot \lambda \uparrow \cdots \uparrow s_1 \cdot \lambda \uparrow \lambda.$$  

We define $l(\lambda)$ for $\lambda \in X^+$ to be the length of a maximal chain $\mu_0 \uparrow \mu_1 \uparrow \cdots \uparrow \mu_{l-1} \uparrow \mu_l = \lambda$ with $\mu_0 \in \mathcal{C}$, each $\mu_i \neq \mu_{i+1}$ and $\mu_i \in X$. We will also define $\bar{l}(\lambda)$ for $\lambda \in X^+$ to be the length of a maximal chain $\mu_0 \uparrow \mu_1 \uparrow \cdots \uparrow \mu_{l-1} \uparrow \mu_l = \lambda$ with all $\mu_i \in X^+$.

We define $d(\lambda)$ to be the number of hyperplanes separating $\lambda$ and a weight lying in $C$ (we do not count any hyperplanes that $\lambda$ may lie on). Take $n_\alpha$, $d_\alpha \in \mathbb{Z}$ with $\langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p + d_\alpha$ and $0 < d_\alpha \leq p$ for all $\alpha$ a positive root. If $\lambda$ is dominant then $d(\lambda) = \sum_{\alpha \geq 0} n_\alpha$.

**Lemma 3.5.** If $\lambda \in C$ and $w \in W_p$ with $w \cdot \lambda \in X^+$ then $\bar{l}(w \cdot \lambda) = l(w) = \bar{l}(w) = d(w \cdot \lambda)$

**Proof.** Since $w \cdot \lambda$ lies inside an alcove we have that $d(w \cdot \lambda) = l(w)$. (This is true as the alcoves in $X$ can be identified with chambers in the Coxeter complex associated to $W_p$.) It is clear that $l(w \cdot \lambda) \geq \bar{l}(w \cdot \lambda)$. We have using [18], proposition 6.8, that $\bar{l}(w \cdot \lambda) \geq d(w \cdot \lambda)$. Now take a maximal chain for $w \cdot \lambda$, $\mu_0 \uparrow \mu_1 \uparrow \cdots \uparrow \mu_l = w \cdot \lambda$ with $\mu_0 \in C$ and $\mu_i \in X$. We know that in this chain for $w \cdot \lambda$ we have $d(\mu_i) < d(\mu_{i+1})$, by applying [18], lemma 6.6. Thus $\bar{l}(w \cdot \lambda) \geq l(w \cdot \lambda)$. Hence we have the equalities as claimed. \hfill \Box

**Remark 3.6.** If $\lambda \in C$ then the $\uparrow$-ordering on $X^+ \cap W_p \cdot \lambda$ is equivalent to the Bruhat ordering on $W_p$. That is we have for $\lambda \in C$ and $w, v \in W_p$ with $w \cdot \lambda$ and $v \cdot \lambda \in X^+$ that

$$w \cdot \lambda \uparrow v \cdot \lambda$$

if and only if $w \leq v$.

This can be seen from the definition of the Bruhat order in [17], section 5.9, and using the previous lemma. See also [25], section 1.6.

We have that $[\nabla(\lambda) : L(\mu)] \neq 0$ implies $\mu \uparrow \lambda$ [1], corollary 3, (known as the strong linkage principle). Thus when we take $\Pi$, a finite saturated subset of $X^+$ with respect to the $\uparrow$ ordering, the corresponding algebra $S(\Pi)$ is quasi-hereditary, thus we may apply lemma 2.9 to deduce that $\text{wfd}(\nabla(\lambda)) \leq \bar{l}(\lambda)$.

4. The Weyl Filtration Dimension of the Induced Modules

**Lemma 4.1.** Suppose we have the situation of corollary 3.4. So we have $\lambda \in C$, $\mu \in \mathcal{C} \setminus C$, $w \cdot \lambda < ws \cdot \lambda$, $w \cdot \lambda \in X^+$ and $\Sigma^0(\mu) = \{s\}$. If $l(w) \geq 1$ then $\text{wfd}(\nabla(w \cdot \mu)) < l(w)$.
Theorem 4.2. Suppose the root system $R$ of $G$ is irreducible and $\lambda \in C$. Then

$$\text{wfd}(\nabla(w \cdot \lambda)) = l(w).$$

Proof. We proceed by induction on $l(w)$. If $l(w) = 0$ then $\nabla(\lambda) = \Delta(\lambda) = L(\lambda)$ so $\text{wfd}(\nabla(\lambda)) = 0$.

Now let $w = s \cdot s'$ with $s \cdot s' \in X^+$. Take $\mu$ to be a dominant weight on the wall separating $\lambda$ and $s \cdot s'$. Such a $\mu$ has $\text{wfd}(\nabla(\mu)) = 0$. Thus $T_\mu^*(\nabla(\mu))$ is a tilting module of highest weight $s \cdot s'$. The short exact sequence of corollary 3.4 is a Weyl resolution of $\nabla(\Delta(\lambda))$ and so $\text{wfd}(\nabla(s \cdot s')) \leq 1$.

But $\text{Ext}^1(\nabla(s \cdot s'), \nabla(\lambda)) = k$ by [18], II, proposition 7.21, and so $\text{wfd}(\nabla(s \cdot s')) = 1 = l(s)$.

Now suppose the theorem is true for all $w \in W_p$ with $l(w) \leq l$, $l \geq 1$. We will show the result holds for $ws$ with $s \in \Sigma$. We take $\mu \in \check{C} \setminus C$ with $\Sigma_0(\mu) = \{s\}$. We have for all $i, v \in W_p$ and $v \cdot \lambda \in X^+$

$$\text{Ext}^1(T_\mu^*(\nabla(w \cdot \mu)), \nabla(v \cdot \lambda)) \cong \text{Ext}^1(\nabla(w \cdot \mu), T_\mu^*(\nabla(v \cdot \lambda))) \cong \text{Ext}^1(\nabla(w \cdot \mu), \nabla(v \cdot \mu))$$

by lemma 3.2 and proposition 3.3. So we have

$$(1) \quad \text{wfd}(T_\mu^*(\nabla(w \cdot \mu))) = \text{wfd}(\nabla(w \cdot \mu)) < l(w)$$

by lemma 4.1.

Applying $\text{Ext}^*(\nabla(v \cdot \lambda))$ with $\nu \in X^+$ to the short exact sequence of corollary 3.4 gives us

$$\text{Ext}^i(T_\mu^*(\nabla(w \cdot \mu)), \nabla(v \cdot \lambda)) \to \text{Ext}^i(\nabla(w \cdot \lambda), \nabla(v))$$

$$\to \text{Ext}^{i+1}(\nabla(ws \cdot \lambda), \nabla(v)) \to \text{Ext}^{i+1}(T_\mu^*(\nabla(w \cdot \mu)), \nabla(v)).$$

Thus for $i \geq l(w)$ we have

$$\text{Ext}^i(\nabla(w \cdot \lambda), \nabla(v)) \cong \text{Ext}^{i+1}(\nabla(ws \cdot \lambda), \nabla(v))$$

using (1) and lemma 2.3. Hence $\text{wfd}(\nabla(ws \cdot \lambda)) = \text{wfd}(\nabla(w \cdot \lambda)) + 1 = l(w) + 1 = l(ws)$, as required. □

We may use the $\circ$-duality to get that $\text{gfd}(\Delta(w \cdot \lambda)) = l(w)$. The previous theorem and lemma 2.3 give us that for $v \in W_p$ with $v \cdot \lambda \in X^+$ we have $\text{Ext}^i(\nabla(w \cdot \lambda), \Delta(v \cdot \lambda)) = 0$ for $i > l(w) + l(v)$. The following corollary tells us that this bound is strict.
Theorem 4.3. Suppose $\lambda \in C$, and $w, v \in W_p$ with $w \cdot \lambda, v \cdot \lambda \in X^+$. Then

$$\operatorname{Ext}^{l(w) + l(v)}(\nabla(w \cdot \lambda), \Delta(v \cdot \lambda)) \cong k.$$ 

Proof. We proceed by induction on $l(w) + l(v)$. If $l(w) + l(v) = 0$ then $w \cdot \lambda = v \cdot \lambda = \lambda$ so $\operatorname{Hom}(\nabla(w \cdot \lambda), \Delta(v \cdot \lambda)) \cong k$.

If $l(w) + l(v) = 1$ then either $w \cdot \lambda = \lambda$ or $v \cdot \lambda = \lambda$. Without loss of generality (using the $^\circ$-duality), take $w \cdot \lambda \neq \lambda$. Thus $\Delta(v \cdot \lambda) = \nabla(\lambda)$. Also $l(w) = 1$ so $w = s \in \Sigma$. By [18], II, proposition 7.21 (c), we have $\operatorname{Ext}^1(\nabla(s \cdot \lambda), \nabla(\lambda)) \cong k$. Thus the corollary is true for $l(w) + l(v) = 1$.

Now take $l(w) = l(v) = 1$ Applying $\operatorname{Ext}^* (\nabla(s \cdot \lambda), -)$ to the $^\circ$-dual of the short exact sequence of corollary 3.4 gives us

$$\operatorname{Ext}^1(\nabla(s \cdot \lambda), T_\mu^\lambda(\Delta(\mu))) \to \operatorname{Ext}^1(\nabla(s \cdot \lambda), \Delta(\lambda)) \to \operatorname{Ext}^2(\nabla(s \cdot \lambda), \Delta(s \cdot \lambda)) \to 0.$$ 

The last zero follows by lemma 2.3. Also

$$\operatorname{Ext}^1(\nabla(s \cdot \lambda), T_\mu^\lambda(\Delta(\mu))) \cong \operatorname{Ext}^1(T_\lambda^\mu(\nabla(s \cdot \lambda)), \nabla(\mu)) \cong \operatorname{Ext}^1(\nabla(\mu), \nabla(\mu)) \cong 0.$$ 

Hence

$$\operatorname{Ext}^2(\nabla(s \cdot \lambda), \Delta(s \cdot \lambda)) \cong \operatorname{Ext}^1(\nabla(s \cdot \lambda), \Delta(\lambda)) \cong k.$$

Now suppose the corollary is true for all $w, v \in W_p$ with $w \cdot \lambda, v \cdot \lambda \in X^+$ and $l(w) + l(v) \leq m$, for some $m \geq 1$. We need to show the result holds for $l(w') + l(v') = m + 1$, $w', v' \in W_p$ and $w' \cdot \lambda, v' \cdot \lambda \in X^+$. Without loss of generality we may take $v' = v$ and $w' = ws$ with $s \in \Sigma$. We may also assume that $l(v')$ or $l(w')$ is at least 2 so that we can assume $w \neq 1$. (As we have already covered the case with $l(w) = l(v) = 1$.)

Apply $\operatorname{Ext}^* (-, \Delta(v \cdot \lambda))$ to the short exact sequence of corollary 3.4 to get

$$\operatorname{Ext}^m(T_\mu^\lambda(\nabla(w \cdot \mu)), \Delta(v \cdot \lambda)) \to \operatorname{Ext}^m(\nabla(w \cdot \lambda), \Delta(v \cdot \lambda))$$

$$\to \operatorname{Ext}^{m+1}(\nabla(ws \cdot \lambda), \Delta(v \cdot \lambda)) \to \operatorname{Ext}^{m+1}(T_\mu^\lambda(\nabla(w \cdot \mu)), \Delta(v \cdot \lambda)).$$

But $\operatorname{wd}(T_\mu^\lambda(\nabla(w \cdot \mu))) < l(w)$ by (1) (provided $w \neq 1$). Now we may apply lemma 2.3 to get that the first and last Ext groups above are zero. Thus the middle two groups are isomorphic. So by induction we have

$$\operatorname{Ext}^{l(w') + l(v')}(\nabla(w' \cdot \lambda), \Delta(v \cdot \lambda)) \cong \operatorname{Ext}^{l(w) + l(v)}(\nabla(w \cdot \lambda), \Delta(v \cdot \lambda)) \cong k. \quad \square$$

Corollary 4.4. For $\lambda, \mu \in X^+$ lying inside an alcove and in the same $W_p$-orbit we have

$$\operatorname{wd}(\nabla(\lambda)) = d(\lambda), \quad \operatorname{gfd}(\Delta(\mu)) = d(\mu) \quad \text{and} \quad \operatorname{Ext}^{d(\lambda) + d(\mu)}(\nabla(\lambda), \Delta(\mu)) \cong k.$$ 

Proof. We have that $\lambda = w \cdot \lambda_0$ and $\mu = v \cdot \lambda_0$ for some $\lambda_0 \in C$. Lemma 3.5, theorem 4.2 and the previous corollary then give us the result. \quad \square
Corollary 4.5. For \( \lambda, \mu \in X^+ \) lying inside an alcove and in the same \( W_p \)-orbit we have

\[
\wfd(L(\lambda)) = d(\lambda) \quad \text{and} \quad \Ext^{d(\lambda)+d(\mu)}(L(\lambda), L(\mu)) \cong k.
\]

Proof. Let \( Q \) be the quotient \( \nabla(\lambda)/L(\lambda) \). If \( L(\nu) \) is a composition factor of \( Q \) then \( \nu \uparrow \lambda \) and \( \nu \neq \lambda \). Thus \( l(\nu) < l(\lambda) \). Hence \( \wfd(Q) < l(\lambda) = d(\lambda) = l \). Now apply \( \Ext^*(\nabla(\nu)) \) to the short exact sequence

\[
0 \to L(\lambda) \to \nabla(\lambda) \to Q \to 0
\]

to get

\[
\cdots \to \Ext^i(Q, \nabla(\nu)) \to \Ext^i(\nabla(\lambda), \nabla(\nu)) \to \Ext^i(L(\lambda), \nabla(\nu)) \to 0
\]

where the last zero follows by lemma 2.3. We also have that \( \Ext^i(Q, \nabla(\nu)) = 0 \) by lemma 2.3. Thus \( \wfd(L(\lambda)) = d(\lambda) = l \) as required.

A similar argument yields that

\[
\Ext^{d(\lambda)+d(\mu)}(L(\lambda), L(\mu)) \cong \Ext^{d(\lambda)+d(\mu)}(\nabla(\lambda), \Delta(\mu)) \cong k. \quad \square
\]

The result of Ryom-Hansen's in the appendix, theorem 2.4, states that for \( \lambda \in C \) and \( w, v \in W_p \) with \( v \leq w \) and \( w \cdot \lambda, v \cdot \lambda \in X^+ \)

\[
\Ext^{l(w)-l(v)}(L(w \cdot \lambda), \nabla(v \cdot \lambda)) \cong k.
\]

We also know that if \( i > l(w) - l(v) \) then \( \Ext^i(L(w \cdot \lambda), \nabla(v \cdot \lambda)) \cong 0 \) by the appendix, lemma 2.1, (see also [18], proposition 6.20). So using this result and given remark 3.6 and lemma 3.5 we may now prove

Proposition 4.6. Let \( \lambda \in C, w, v \in W_p \) with \( w \cdot \lambda, v \cdot \lambda \in X^+ \) and \( v \cdot \lambda \uparrow w \cdot \lambda \) then

\[
\Ext^{l(w)-l(v)}(\nabla(w \cdot \lambda), \nabla(v \cdot \lambda)) \cong k.
\]

Proof. We may argue along similar lines to the proof of corollary 4.5. \quad \square

We now are in a position where we may deduce the projective and injective dimensions of several modules for the generalised Schur algebras. We define \( \Pi(\lambda) \) to be the (finite) saturated subset of \( X^+ \) with respect to the \( \uparrow \)-ordering whose highest weight is \( \lambda \).

Theorem 4.7. Suppose \( \lambda \in X^+ \) is regular (lies inside an alcove) then in \( \mod(S(\Pi(\lambda))) \) for \( \mu \in \Pi(\lambda) \) we have

\[
\inj(L(\mu)) = \proj(L(\mu)) = \proj(\nabla(\mu)) = \inj(\Delta(\mu)) = d(\mu) + d(\lambda)
\]

\[
\inj(\nabla(\mu)) = \proj(\Delta(\mu)) = d(\lambda) - d(\mu).
\]

In particular this gives us information for the blocks of the Schur algebra whose weights are regular.
5. The Global Dimension of $S(n, r)$ when $p > n$

We will now focus on the classical Schur algebra. So $G = \text{GL}_n(k)$, the root system of $\text{GL}_n$ is irreducible and its derived subgroup $\text{SL}_n$ is simply connected. We wish to determine the good filtration dimension and global dimension for $S(n, r)$ (that is for the whole Schur algebra, not just for the regular blocks). So we need to know what $d(\lambda)$ is for a partition and a condition for $\lambda$ to lie inside an alcove. The next two lemmas do this.

**Lemma 5.1.** Suppose $G = \text{GL}_n(k)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in X^+$. Then we have

$$d(\lambda) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left\lfloor \frac{\lambda_i - \lambda_j - i + j - 1}{p} \right\rfloor.$$

**Proof.** Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in X$ with a one in the $i$th position. The $e_i$ form the usual basis of $X$, so $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^n \lambda_i e_i$. We take $\omega_i = \sum_{j=1}^i e_j$. We can write $\lambda = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) \omega_i + \lambda_n \omega_n$. Thus for $\alpha = e_i - e_j \in R^+$, we have

$$\langle \lambda + \rho, \alpha^- \rangle = \lambda_i - \lambda_j + j - i.$$

The definition of $d(\lambda)$ then gives us the result. \qed

**Lemma 5.2.** A weight $\lambda \in X^+$ lies inside an alcove if there exist no integers $i$ and $j$ such that $\lambda_i - \lambda_j \equiv i - j \pmod{p}$.

**Proof.** A weight $\lambda \in X^+$ lies on a wall if there exists $\alpha \in R^+$ such that $\langle \lambda + \rho, \alpha^- \rangle = mp$ for some $m \in \mathbb{Z}$. So a weight $\lambda$ does not lie on a wall if for all $\alpha = e_i - e_j$ we have $\lambda_i - \lambda_j + j - i \neq 0 \pmod{p}$. \qed

We first calculate an upper bound for $\text{wfd}(S(n, r))$.

Let $E = L(1, 0, \ldots, 0)$ be the natural module for $\text{GL}_n$. We take $S^r E$ to be the $r$th symmetric power of $E$ and $\Lambda^r E$ to be the $r$th exterior power. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda^+(n, r)$ we take $S^\lambda E = S^{\lambda_1} E \otimes S^{\lambda_2} E \otimes \cdots \otimes S^{\lambda_n} E$ with $S^1 E = E$ and $S^0 E = k$.

**Lemma 5.3.**

$$\text{wfd}(S(n, r)) = \max\{\text{wfd}(S^\lambda E) \mid \lambda \in \Lambda^+(n, r)\}.$$

**Proof.** We have that $L(\lambda)$ embeds in $S^\lambda E$ by [12], section 2.1 (15)(i)(b). So the set $\mathcal{X} = \{S^\lambda E \mid \lambda \in \Lambda^+(n, r)\}$ satisfies the requirements of remark 2.11. \qed

For all $\lambda$ and $\mu \in X^+$ the module $\nabla(\lambda) \otimes \nabla(\mu)$ has a good filtration. A proof of this property, for type $A_n$, is given in [26]. It is proved for most other cases in [8]. The general proof is given in [20]. The $\nabla(\nu)$ which appear as quotients in this filtration are given by Brauer’s character formula [18], II, lemma 5.8. We can generalise this property to good and Weyl filtration dimensions as below.
Lemma 5.4. Let $X, Y$ be $G$-modules then we have
\[
\text{wfd}(X \otimes Y) \leq \text{wfd}(X) + \text{wfd}(Y).
\]

Proof. See [15], proposition 3.4 (c), where the corresponding result for good filtration dimensions is proved. \hfill \square

Lemma 5.5. We have the short exact sequence,
\[
0 \to \nabla(mp - j, 1^j, 0^{n-j-1}) \to S^{mp-j}E \otimes \wedge^jE \to \nabla(mp - j + 1, 1^{j-1}, 0^{n-j}) \to 0.
\]

Proof. Since $S^{mp-j}E \otimes \wedge^jE$ has a good filtration by the dual version of lemma 5.4, this follows using characters. \hfill \square

Proposition 5.6.
\[
\text{wfd}(S^rE) \leq (n-1) \left\lfloor \frac{r}{p} \right\rfloor.
\]

Proof. We first reduce to the case $S^{mp}E$. Write $r = r_0 + pm$. If $0 < r_0 < p$ then the multiplication $S^{r_0}E \otimes S^{rm}E \to S^rE$ splits [12], section 4.8, proposition (12), so that \text{wfd}(S^rE) \leq \text{wfd}(S^{r_0}E \otimes S^{rm}E) \leq \text{wfd}(S^{rm}E) + \text{wfd}(S^{r_0}E) = \text{wfd}(S^{rm}E)$ as $S^{r_0}(E) \in F(\Delta)$. So suppose $r = mp$. We prove this proposition by induction on $m$. The proposition is clearly true for $m = 0$

Let $\lambda \in \Lambda^+$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. We define
\[
\left\lfloor \frac{\lambda}{p} \right\rfloor = \sum_{i=1}^{n} \left\lfloor \frac{\lambda_i}{p} \right\rfloor.
\]

Corollary 5.7.
\[
\text{wfd}(S(n, r)) \leq (n-1) \left\lfloor \frac{r}{p} \right\rfloor.
\]

Proof. We have \text{wfd}(S^\lambda E) \leq \left\lfloor \frac{\lambda}{p} \right\rfloor$ using lemma 5.4 and proposition 5.6. The result now follows using lemma 5.3 and noting that $\left\lfloor \frac{\lambda}{p} \right\rfloor \leq \left\lfloor \frac{\lambda}{p} \right\rfloor$ for all $\lambda \in \Lambda^+$. \hfill \square
Theorem 5.8. If \( p > n \) then the Weyl (and the good) filtration dimension of the Schur algebra \( S(n, r) \) is
\[
\text{wfd}(S(n, r)) = (n - 1) \left\lfloor \frac{r}{p} \right\rfloor.
\]
The global dimension of \( S(n, r) \) is twice this value.

Proof. The previous corollary tells us that this value for \( \text{wfd}(S) \) is an upper bound for all \( p \). So for \( p > n \) we just need to give a weight in \( \Lambda^+(n, r) \) whose Weyl filtration dimension attains this bound. We write \( r = r_1 p + r_0 \) for \( r_1, r_0 \in \mathbb{N} \) and \( 0 \leq r_0 \leq p - 1 \).

Since \( p > n \) we can write \( r_0 = bn + a \) where \( a, b \in \mathbb{N} \) and \( 0 \leq a \leq n - 1 \). Consider the weight \( \mu = (r_1 p + 1, 1^{a-1}, 0^{n-a}) + b(1^n) \in \Lambda^+(n, r) \). If \( a = 0 \) then we take \( \mu = (r_1 p, 0^{n-1}) + b(1^n) \). The weight \( \mu \) lies inside an alcove by lemma 5.2. Also \( d(\mu) = (n - 1)r_1 \). Hence \( \text{wfd}(\nabla(\mu)) = (n - 1)r_1 \), and so the bound is attained.

Theorem 4.2 also tells us that there is a non-zero Ext group in degree \( 2(n - 1)r_1 \). Hence the global dimension of \( S(n, r) \) is twice the Weyl filtration dimension by corollary 2.6.

Theorem 5.9. Let \( m \in \mathbb{N} \) then the Weyl (and the good) filtration dimension of the Schur algebra \( S(p, mp) \) is
\[
\text{wfd}(S(p, mp)) = (p - 1)m.
\]
The global dimension of \( S(p, mp) \) is twice this value.

Proof. The weight \( (mp, 0, \ldots, 0) \in \Lambda^+ \) lies inside an alcove by lemma 5.2. The same argument as in the previous proof then gives us the result.

The values calculated for \( \text{wfd}(L(\lambda)) \) with \( \lambda \) inside an alcove for \( n = 2 \) and \( n = 3 \) agree with our previous results for \( \text{SL}_2 \) and \( \text{SL}_3 \) calculated in [21, sections 3 and 5] This gives a new proof for [21], theorem 3.7, (for all \( p \)) and [21], theorem 5.12, in the cases where \( p \geq 5 \) and \( p = 3 \) and \( 3 \mid r \).

It is still an open problem to determine what happens for weights which are not regular. Many of the results above give upper bounds but most of time these bounds are not sharp. Various conjectures are presented for the value of \( \text{wfd}(S(n, r)) \) in [22], section 6.5.

6. The quantum case

We now show that the arguments in sections 4 and 5 generalise to the quantum case. To do this we need the appropriate quantum versions of the results used. We will be using the Dipper-Donkin quantum group \( q-\text{GL}_n \) defined in [7]. Our field \( k \) remains algebraically closed but \( k \) may now also have zero as well as positive characteristic. Background information can be found in [12]. The cohomological theory of quantum groups and their \( q \)-Schur algebras appears in [11]. When \( q = 1 \) then the module category for \( q-\text{GL}_n \) is the same as for \( \text{GL}_n \). If \( q \) is not a root of unity then \( \text{mod}(q-\text{GL}_n) \) is semi-simple. We will consider the case where \( q \) is a primitive \( l \)th root of unity with \( l \geq 2 \).
All of the structures defined in section 1 have their quantum analogues, which are essentially the same. The most significant difference for us will be that $p$-alcoves and $p$-hyperplanes will be replaced by $l$-alcoves and $l$-hyperplanes. We need the quantum version of translation functors. These are defined in [2], section 8, together with the quantum version of proposition 3.3, [2], theorem 8.3.

All of our proofs in section 4 now carry through in the quantum case with $p$ replaced by $l$. So the statement of theorem 4.2 and 4.3 and their corollaries 4.4 and 4.5 are equally valid for the quantum case when $l \geq h$ (even if $k$ has characteristic 0). We also expect that the result in the appendix carries through in the quantum case so that we would also have the quantum version of proposition 4.6 and theorem 4.7.

We now consider the quantised Schur algebra, $S_q(n,r)$. This can be constructed in the same way as in the classical case. Take the saturated subset of dominant weights $\Pi = \Lambda^+(n,r)$, then the quasi-hereditary algebra $S(\Pi)$ is isomorphic to $S_q(n,r)$, the quantised Schur algebra. Moreover we have the same ordering – namely the $\uparrow$-ordering defined using the action of the affine Weyl group. The proofs in section 5 work equally well in the quantum case with $p$ replaced by $l$ where $q$ is an $l$th root of unity with $l \geq 2$. So we get an upper bound for $\mathrm{wfd}(S_q(n,r))$ of $(n-1)\lfloor \frac{r}{l} \rfloor$. Together with the quantum version of the results of section 4 we may now deduce the following theorem.

**Theorem 6.1.** If $q$ is a primitive $l$th root of unity with $l > n$ then the Weyl (and the good) filtration dimension of the quantised Schur algebra $S_q(n,r)$ is

$$\mathrm{wfd}(S_q(n,r)) = (n-1)\lfloor \frac{r}{l} \rfloor.$$

Suppose $l \geq 2$ and let $m \in \mathbb{N}$. Then we have

$$\mathrm{wfd}(S_q(l, ml)) = (l-1)m.$$

In both these case the global dimension of $S_q(n,r)$ and $S_q(l, ml)$ is twice its Weyl filtration dimension.

Again, this result is dependent only on $l$ and not on the characteristic of the field $k$.

7. **Category $\mathcal{O}$**

There are analogous situations in Category $\mathcal{O}$ defined by Bernstein, Gel'fand and Gel'fand, [5]. Category $\mathcal{O}$ is known to be a highest weight category (see [19], section 4.1 for a basic introduction) so we can apply the general theory of section 2. We use the setup of [6], although note that [6] uses the terminology ‘$p$-filtration’ for what we have defined to be a Weyl filtration. There $\mathfrak{g}$ is a complex, semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Weyl group $W$. We denote the longest element of $W$ by $w_0$. The standard modules for $\mathcal{O}$ are the well-known Verma modules, denoted $M(\lambda)$ for $\lambda \in \mathfrak{h}^*$. We also have that $[M(\mu) : L(\lambda)] \neq 0$ if and only if there are positive
roots $\gamma_1, \ldots, \gamma_m$ such that there is a chain of inequalities $\mu \geq s_{\gamma_1}(\mu) \geq \cdots \geq s_{\gamma_m} \cdots s_{\gamma_1}(\mu) = \lambda$ ([4]). We may use [6], proposition 3.7, theorem 3.8 and theorem 4.6 to deduce that $\text{gfd}(M(w \cdot \lambda)) = \text{gfd}(L(w \cdot \lambda)) = l(w_0) - l(w)$ and $\text{proj}(M(w \cdot \lambda)) = l(w)$ for $\lambda$ an integral weight inside the dominant Weyl chamber. We may deduce that $\text{proj}(L(w \cdot \lambda)) \leq 2l(w_0) - l(w)$. These last two statements are consistent with [5], statements 1 and 2. We also have translation functors and the analogue of proposition 3.3 and hence the corollary 3.4. Unfortunately the analogue of lemma 4.1 may no longer be true. Our argument does not work in this situation and indeed already fails for type $A_2$. However, in [5], remark in §7, it is stated that $\text{Ext}^{2l(w_0)}(L(\lambda), L(\lambda)) \cong \mathbb{C}$. So there is strong evidence to suggest that $\text{Ext}^{2l(w_0) - l(w) - l(v)}(L(w \cdot \lambda), L(v \cdot \lambda)) \cong \mathbb{C}$ for $v, w \in W$. The results of [5], §7, are already enough to deduce that the global dimension of $O$ is $2l(w_0)$.

References

[1] H. H. Andersen, The strong linkage principle, J. reine angew. Math. 315 (1980), 53–59.
[2] H. H. Andersen, P. Polo, and K. X. Wen, Representations of quantum algebras, Invent. Math. 104 (1991), no. 1, 1–59.
[3] D. J. Benson, Representations and Cohomology I, Cambridge Studies in Advanced Mathematics, no. 30, Cambridge University Press, 1995.
[4] I. N. Bernstein, I. M. Gel’fand, and S. I. Gel’fand, Structure of representations generated by highest weight, Funct. Anal. and Appl. 5 (1971), 1–8.
[5] ———, A category of $g$–modules, Funct. Anal. and Appl. 10 (1976), 87–92.
[6] K. J. Carlin, Extensions of Verma modules, Trans. Amer. Math. Soc. 294 (1986), no. 1, 29–43.
[7] R. Dipper and S. Donkin, Quantum $GL_n$, Proc. London Math. Soc. (3) 63 (1991), 165–211.
[8] S. Donkin, Rational Representations of Algebraic Groups: Tensor Products and Filtrations, Lecture Notes in Mathematics, vol. 1140, Springer–Verlag, Berlin/Heidelberg/New York, 1985.
[9] ———, On Schur algebras and related algebras I, J. Algebra 104 (1986), 310–328.
[10] ———, On tilting modules for algebraic groups, Math. Z. 212 (1993), 39–60.
[11] ———, Standard homological properties for quantum $GL_n$, J. Algebra 181 (1996), 235–266.
[12] ———, The $q$–Schur Algebra, London Math. Soc. Lecture Note Ser., vol. 253, Cambridge University Press, Cambridge, 1998.
[13] S. R. Doty and D. K. Nakano, Semi–simplicity of Schur algebras, Math. Proc. Cambridge Philos. Soc. 124 (1998), 15–20.
[14] K. Erdmann and D. K. Nakano, Representation type of $q$–Schur algebras, Trans. Amer. Math. Soc. 353 (2001), no. 12, 4729–4756.
[15] E. M. Friedlander and B. J. Parshall, Cohomology of Lie algebras and algebraic groups, Amer. J. Math. 108 (1986), 235–253.
[16] J. E. Humphreys, Linear Algebraic Groups, Graduate Texts in Mathematics, vol. 21, Springer–Verlag, Berlin/Heidelberg/New York, 1975.
[17] ———, Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics, no. 30, Cambridge University Press, 1990.
[18] J. C. Jantzen, Representations of Algebraic Groups, Pure Appl. Math., vol. 131, Academic Press, San Diego, 1987.
[19] M. Klucznik and S. König, Characteristic Tilting Modules over Quasi–hereditary Algebras, unpublished notes, 1999.

[20] O. Mathieu, Filtrations of $G$–modules, Ann. Sci. École Norm. Sup. (4) 23 (1990), no. 4, 625–644.

[21] A. E. Parker, The global dimension of Schur algebras for $GL_2$ and $GL_3$, J. Algebra 241 (2001), 340–378.

[22] ———, On the global dimension of Schur algebras and related algebras, Ph.D. thesis, University of London, 2001.

[23] T. A. Springer, Linear Algebraic Groups, Progress in Mathematics, vol. 9, Birkhäuser, Boston/Basel/Stuttgart, 1981.

[24] B. Totaro, Projective resolutions of representations of $GL(n)$, J. reine angew. Math. 482 (1997), 1–13.

[25] D. N. Verma, The rôle of affine Weyl groups, Lie Groups and their representations (I.M. Gel’fand, ed.), 1975, pp. 653–705.

[26] Jian-pan Wang, Sheaf cohomology of $G/B$ and tensor products of Weyl modules, J. Algebra 77 (1982), 162–185.

School of Mathematics and Statistics F07, University of Sydney, NSW 2006, Australia

E-mail address: alisonp@maths.usyd.edu.au