Electrical resistivity near Pomeranchuk instability
in two dimensions

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We analyze the DC charge transport in the quantum critical regime near a d-wave Pomeranchuk instability in two dimensions. The transport decay rate is linear in temperature everywhere on the Fermi surface except at cold spots on the Brillouin zone diagonal. For pure systems, this leads to a DC resistivity proportional to $T^{3/2}$ in the low-temperature limit. In the presence of impurities the residual impurity resistance at $T=0$ is approached linearly at low temperatures.

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Pomeranchuk instabilities leading to symmetry-breaking deformations of the Fermi surface in interacting electron systems have attracted much interest in the last few years. Interactions favoring a Pomeranchuk instability with d-wave symmetry have been found in the two most intensively studied single-band models for cuprate superconductors, that is, the two-dimensional t-J and Hubbard model. These models thus exhibit enhanced "nematic" correlations, as usually discussed in the context of fluctuating stripe order. Signatures for incipient nematic order with d-wave symmetry have been observed in various cuprate materials. In particular, nematic correlations close to a d-wave Pomeranchuk instability provide a natural explanation for the relatively strong in-plane anisotropy observed in the magnetic excitation spectrum of YBa$_2$Cu$_3$O$_y$. A spin dependent Pomeranchuk instability was recently invoked to explain a new phase observed in ultrapure crystals of the layered ruthenate metal Sr$_3$Ru$_2$O$_7$, and also to account for
a puzzling phase transition in URu$_2$Si$_2$ [10].

Critical fluctuations near a Pomeranchuk instability provide an interesting route to non-Fermi liquid behavior in two dimensions [11, 12]. The properties of single-particle excitations near a quantum critical point associated with a Pomeranchuk instability have been studied already in considerable detail [13, 14]. For a d-wave Pomeranchuk instability in an electron system on a square lattice the singular part of the electronic self-energy is proportional to $d_k^2$, where $d_k$ is a form factor with d-wave symmetry [12]. At the quantum critical point, the real and imaginary parts of the self-energy scale as $|\omega|^{2/3}$ with energy [11, 12]. This leads to a complete destruction of quasi-particles near the Fermi surface except for the ”cold spots” on the Brillouin zone diagonal, where the form factor $d_k$ vanishes. In the quantum critical regime at $T > 0$ the self-energy consists of a ”classical” and a ”quantum” part with very different dependences on $T$ and $\omega$. The classical part, which is due to classical fluctuations, dominates at $\omega = 0$ and yields a contribution proportional to $\sqrt{T}/\log T$ to the imaginary part of the self-energy on the Fermi surface [13].

In this letter we compute the temperature dependence of the DC resistivity in the quantum critical regime near a Pomeranchuk instability in two dimensions. We obtain a momentum dependent transport decay rate $\gamma_{tr}^k(T)$ which is linear in temperature for all momenta on the Fermi surface except at the cold spots on the Brillouin zone diagonal. Adding a conventional $T^2$-term to $\gamma_{tr}^k(T)$ we obtain an overall resistivity $\rho(T)$ proportional to $T^{3/2}$ at low temperatures. In the presence of impurities, the residual resistivity at zero temperature is approached linearly.

Our calculations are based on a phenomenological lattice model [12],

$$H = H_0 + \frac{1}{2V} \sum_{k,k',q} f_{kk'}(q) n_k(q) n_{k'}(-q),$$

where $H_0$ is a kinetic energy, $n_k(q) = \sum_\sigma c_{k-q/2,\sigma}^\dagger c_{k+q/2,\sigma}$, and $V$ is the volume of the system. Since the Pomeranchuk instability is driven by interactions with small momentum transfers (forward scattering), we choose a coupling function $f_{kk'}(q)$ which contributes
only for relatively small momenta $q$. This excludes other instabilities such as superconductivity or density waves. We consider an interaction of the form \[ f_{kk'}(q) = u(q) + g(q) d_k d_{k'} \] (2)

with $u(q) \geq 0$ and $g(q) < 0$, and a form factor $d_k$ with $d_{x^2-y^2}$ symmetry, such as $d_k = \cos k_x - \cos k_y$. The coupling functions $u(q)$ and $g(q)$ vanish if $|q|$ exceeds a certain small momentum cutoff $q_c$. This ansatz mimics the effective interaction in the forward scattering channel as obtained from renormalization group calculations \[3\] for the two-dimensional Hubbard model. The uniform term originates directly from the repulsion between electrons and suppresses the electronic compressibility of the system. The $d$-wave term drives the Pomeranchuk instability.

The Pomeranchuk instability can actually be preempted by a first order transition at low temperatures, where the Fermi surface symmetry changes abruptly before the fluctuations become truly critical \[16\]. However, for reasonable choices of hopping and interaction parameters the system is nevertheless characterized by strong fluctuations on the symmetric side of the transition \[15\]. The first order character of the transition is suppressed by the uniform repulsion $u$ in (2), and for a favorable but not unphysical choice of model parameters a genuine quantum critical point can be realized \[15\].

Near the Pomeranchuk instability, the electrons interact via a singular effective interaction of the form \[12, 13\]

\[ \Gamma_{kk'}(q, \nu) = \frac{g d_k d_{k'}}{(\xi_0/\xi)^2 + (\xi_0^2/|q|^2 - i\nu/(c|q|))}, \] (3)

where $q$ and $\nu$ is the momentum and energy transfer, respectively. The parameters $g = g(0)$, $\xi_0$ and $c$ can be treated as constants, whereas the correlation length $\xi$ depends sensitively on control parameters and temperature. In the quantum critical regime $\xi(T)$ is proportional to $(T|\log T|)^{-1/2}$.

The electron self-energy $\Sigma(k, \omega)$ has been computed previously \[12, 13, 14\] in random phase approximation (RPA) with the effective interaction (3). In the quantum critical
regime one finds \[13\]

\[
\text{Im}\Sigma(k_F, 0) = \frac{g d_{k_F}^2}{4v_{k_F} \xi_0^2} T \xi(T)
\]

for momenta \(k_F\) on the Fermi surface \((v_{k_F}\) is the Fermi velocity\). The corresponding approximation for the electrical resistivity involves the RPA self-energy and current vertex corrections due to particle-hole ladder diagrams, in close analogy to the Born approximation for impurity scattering \[17\]. We assume that the Pomeranchuk fluctuations thermalize sufficiently rapidly such that the effective interaction (3) is not modified by the electric current. This relaxation to equilibrium is not described by the model (1) and has to be provided by additional terms such as umklapp, impurity or phonon scattering \[18, 19\].

The DC conductivity (inverse resistivity) can be obtained from the retarded current-current correlation function \(\Pi\) as

\[
\sigma_{jj'} = -\lim_{\omega \to 0} \lim_{q \to 0} \frac{e^2}{\omega} \text{Im} \Pi_{jj'}(q, \omega)
\]

For cubic symmetry the conductivity tensor is diagonal. The current-current correlator can be expressed (exactly) in terms of single-particle Green functions \(G\) and current vertices. Performing an analytic continuation from Matsubara to real frequencies and taking the DC limit, one obtains

\[
\sigma_{jj'} = -\frac{e^2}{\pi} \int d\omega f'(\omega) \int \frac{d^2k}{(2\pi)^2} \Lambda^0_j(k) |G(k, \omega)|^2 \Lambda_j'(k, \omega),
\]

where \(f(\omega)\) is the Fermi function, \(\Lambda^0_j(k) = v_k = \nabla \epsilon_k\) the bare current vertex, and \(\Lambda(k, \omega)\) is the interacting current vertex in the mixed advanced-retarded DC limit, that is, \(\Lambda(k, \omega) = \Lambda(k, \omega + i0^+; k, \omega - i0^+)\). The product \(|G|^2\) can be expressed in terms of the single-particle spectral function \(A(k, \omega) = -\frac{1}{\pi} \text{Im} G(k, \omega)\) and the (retarded) self-energy as

\[
|G(k, \omega)|^2 = -\frac{\pi A(k, \omega)}{\text{Im} \Sigma(k, \omega)}. \tag{7}
\]

At low temperatures the derivative of the Fermi function \(f'(\omega)\) has a sharp peak of width \(T\) at \(\omega = 0\). Since all other factors under the integral in Eq. (6) have a broader \(\omega-\)
dependence, one can replace $f'(\omega)$ by $-\delta(\omega)$, such that the conductivity can be written as

$$\sigma_{jj'} = -e^2 \int \frac{d^2k}{(2\pi)^2} \Lambda_j^0(k) \frac{A(k,0)}{\text{Im}\Sigma(k,0)} \Lambda_{j'}(k,0).$$  

The interacting current vertex includes all particle-hole ladder vertex corrections. It is thus obtained from a linear integral equation, which can be written as

$$\Lambda(k,\omega) = \Lambda^0(k) + \int d\epsilon \int \frac{d^2q}{(2\pi)^2} \left[b(\epsilon) + f(\omega + \epsilon)\right] \times \text{Im}\Gamma_{kk}(q,\epsilon) \frac{A(k + q,\omega + \epsilon)}{\text{Im}\Sigma(k + q,\omega + \epsilon)} \Lambda(k + q,\omega + \epsilon).$$

(9)

after analytic continuation to the real frequency axis. Here $b(\epsilon)$ is the Bose function.

For $T \to 0$ the correlation length $\xi(T)$ diverges. Repeating the arguments used for the calculation of $\Sigma(k,\omega)$ in Ref. 13, one finds that the integration variable $\epsilon$ in Eq. (9) scales as $\xi^{-3}$ and can therefore be set to zero in the arguments of $A$, $\Sigma$, and $\Lambda$ on the right hand side of Eq. (9). Expanding the Bose function as $b(\epsilon) \sim T/\epsilon$, one can carry out the $\epsilon$-integration explicitly,

$$\int d\epsilon \frac{1}{\epsilon} \text{Im}\Gamma_{kk}(q,\epsilon) = \pi \Gamma_{kk}(q,0),$$

yielding a closed equation for the static current vertex $\Lambda(k) = \Lambda(k,0)$

$$\Lambda(k) = \Lambda^0(k) + T \int \frac{d^2q}{(2\pi)^2} \Gamma_{kk}(q,0) \frac{\pi A(k + q,0)}{\text{Im}\Sigma(k + q,0)} \Lambda(k + q).$$

(10)

The same result could have been obtained by considering only the classical fluctuations, that is, by including only the term $\Gamma_{kk}(q,i\epsilon_n)$ with Matsubara frequency $\epsilon_n = 0$ in the Matsubara sums for the current vertex corrections. At this point the equations for $\sigma_{jj'}$ and $\Lambda(k)$ are formally identical to those obtained from the Born approximation in disordered electron systems with a $k$-dependent long-ranged disorder correlator given by $\Gamma_{kk}(q,0)$.

Inserting the ansatz $\Lambda(k) = \lambda(k)v_k$ into the equation for the current vertex, one obtains the following equation for the function $\lambda(k)$:

$$\lambda(k) = 1 + T \int \frac{d^2q}{(2\pi)^2} \Gamma_{kk}(q,0) \frac{\pi A(k + q,0)}{\text{Im}\Sigma(k + q,0)} \frac{v_k \cdot v_{k+q}}{v_k^2} \lambda(k + q).$$

(11)

Since the conductivity is dominated by momenta near the Fermi surface, we now focus on the case $k = k_F$. For large $\xi$ the above integral is dominated by small momentum
transfers $q$ of order $\xi^{-1}$, due to the effective interaction $\Gamma_{k\bar{k}}(q,0)$. The spectral function is peaked for momenta on the Fermi surface, with a width determined by $\text{Im}\Sigma(k_F,0)$, which is proportional to $T\xi(T)$. The self-energy $\Sigma(k,0)$ varies on a momentum scale of order $\xi^{-1}$ for momentum shifts perpendicular to the Fermi surface \[13\]. The same can be expected for $\lambda(k)$, since the current vertex correction can be related to the shift of the self-energy in the presence of a field coupled to the current operator. Since $T\xi^2(T) \propto 1/\log T$ in the quantum critical regime, and since the tangential $q$-dependence of $\text{Im}\Sigma(k_F+q)$ and $\lambda(k_F+q)$ is negligible on the scale $\xi^{-1}$, we may neglect the $q$-dependence of $\text{Im}\Sigma(k_F+q)$ and $\lambda(k_F+q)$ in (11) altogether, which can then be solved explicitly, yielding

$$\lambda(k_F) = \left[1 - \frac{\pi T}{\text{Im}\Sigma(k_F,0)} \int \frac{d^2 q}{(2\pi)^2} \Gamma_{k_F,k_F}(q,0) A(k_F+q,0) \frac{v_{k_F} \cdot v_{k_F+q}}{v_{k_F}^2} \right]^{-1}. \quad (12)$$

Using

$$\text{Im}\Sigma(k_F,0) = \pi T \int \frac{d^2 q}{(2\pi)^2} \Gamma_{k_F,k_F}(q,0) A(k_F+q,0), \quad (13)$$

which is true within self-consistent RPA restricted to classical fluctuations \[13\], one can write $\lambda_{k_F}$ as

$$\lambda(k_F) = \frac{\gamma_{k_F}}{\gamma_{k_F}^\text{tr}}, \quad (14)$$

where $\gamma_{k_F} = -\text{Im}\Sigma(k_F,0)$ is the single-particle decay rate while

$$\gamma_{k_F}^\text{tr} = -\pi T \int \frac{d^2 q}{(2\pi)^2} \Gamma_{k_F,k_F}(q,0) A(k_F+q,0) \left(1 - \frac{v_{k_F} \cdot v_{k_F+q}}{v_{k_F}^2} \right). \quad (15)$$

is the scattering rate relevant for transport.

The momentum integral in the expression (8) for the conductivity is peaked at the Fermi surface. For $T \to 0$ with $\xi(T) \propto (T \log T)^{-1/2}$ one can replace $A(k,0)$ under the integral by $\delta(\epsilon_k - \mu)$, neglecting possible corrections of order $1/\log T$, such that the conductivity can be written as a Fermi surface integral. Inserting Eq. (14) for $\lambda(k_F)$, one obtains

$$\sigma = \frac{e^2}{8\pi^2} \int d\Omega_{k_F} \frac{v_{k_F}}{\gamma_{k_F}^\text{tr}} \quad (16)$$

for the diagonal part $\sigma = \sigma_{jj}$ of the conductivity tensor.
To compute $\gamma_{\mathbf{k}_F}^{\text{tr}}$, we parametrize the (small) momentum transfer $\mathbf{q}$ in Eq. (15) by radial and tangential components, $q_r$ and $q_t$, respectively. For $T \to 0$, we may again approximate $A(\mathbf{k}_F + \mathbf{q}, 0)$ by a $\delta$-function, $\delta(\epsilon_{\mathbf{k}_F + \mathbf{q}} - \mu)$. The dispersion relation can be expanded as $\epsilon_{\mathbf{k}_F + \mathbf{q}} - \mu = v_{\mathbf{k}_F} q_r + q_t^2/(2m_{\mathbf{k}_F}^t)$, where $(m_{\mathbf{k}_F}^t)^{-1} = \partial^2_{k_t} \epsilon_{\mathbf{k}}|_{\mathbf{k}_F}$. Since $\mathbf{k}_F + \mathbf{q}$ is confined to the Fermi surface, the momentum transfers $\mathbf{q}$ are predominantly tangential to the Fermi surface in $\mathbf{k}_F$, such that the term of order $q_t^2$ cannot be neglected compared to the term linear in $q_r$. After expanding also the kinematic factor $1 - v_{\mathbf{k}_F} \cdot v_{\mathbf{k}_F + \mathbf{q}}/v_{\mathbf{k}_F}^2$ to linear order in $q_r$ and quadratic order in $q_t$, the momentum integral in Eq. (15) can be performed analytically. For $\xi^{-1}(T) \ll m_{\mathbf{k}_F}^t v_{\mathbf{k}_F}$, one obtains

$$\gamma_{\mathbf{k}_F}^{\text{tr}} = \frac{|g|}{\pi \xi_0} m_{\mathbf{k}_F}^t \arctan \left( \frac{q_c}{2m_{\mathbf{k}_F}^t v_{\mathbf{k}_F}} \right) K_{\mathbf{k}_F} d_{\mathbf{k}_F}^2 T,$$

where $q_c$ is the momentum cutoff and

$$K_{\mathbf{k}_F} = \frac{1}{2v_{\mathbf{k}_F}^2} \left( \frac{v_{\mathbf{k}} \cdot \partial_{k_r} v_{\mathbf{k}}}{v_{\mathbf{k}} m_{\mathbf{k}_F}^t} - v_{\mathbf{k}} \cdot \partial^2_{k_t} v_{\mathbf{k}} \right) \bigg|_{k=k_F}. \tag{18}$$

The function $K_{\mathbf{k}_F}$ has units of inverse momentum squared. For a quadratic dispersion relation, $\epsilon_{\mathbf{k}} = k^2/(2m)$, one has $m_{\mathbf{k}}^t = m$ and $K_{\mathbf{k}_F} = 1/(2k_F^2)$. The scattering rate $\gamma_{\mathbf{k}_F}^{tr}$ is thus linear in $T$ at low temperatures. Note that the correlation length $\xi(T)$ does not appear in the asymptotic low temperature behavior of $\gamma_{\mathbf{k}_F}^{\text{tr}}$. This behavior in the quantum critical regime can be contrasted with the behavior in the Fermi liquid regime close to the quantum critical point. For the latter case $\gamma_{\mathbf{k}_F}^{\text{tr}}$ is proportional to $\xi^2 T^2 \log T$, where $\xi = \xi(T \to 0)$. \[19\]

Due to the d-wave form factor the prefactor of the $T$-linear behavior of $\gamma_{\mathbf{k}_F}^{\text{tr}}$ varies strongly along the Fermi surface and vanishes on the Brillouin zone diagonal. This is reminiscent of the cold spot scenario of transport in cuprates \[20\]. Inserting $\gamma_{\mathbf{k}_F}^{\text{tr}}$ from (17) in Eq. (16) for the conductivity one obtains a divergent Fermi surface integral, due to the zeros of $\gamma_{\mathbf{k}_F}^{\text{tr}}$ at the cold spots $\mathbf{k}_F^c$. In the absence of any other scattering mechanism, the conductivity would thus be infinite. However, other (than d-wave forward scattering) residual interactions will lead at least to the conventional Fermi liquid decay rate of order $T^2$ all over the Fermi surface, including the cold spots. Including a Fermi liquid term
of order $T^2$, the scattering rate has the form $\gamma_{k_F}^{\text{tr}}(T) = a_{k_F} T^2 + b_{k_F} d_{k_F}^2 T$, where the coefficients $a_{k_F}$ and $b_{k_F}$ are finite for all $k_F$. Inserting this ansatz into Eq. (16), one finds a resistivity

$$\rho(T) = \frac{2\pi}{e^2} \sqrt{a_{k_F} b_{k_F}} \frac{T^{3/2}}{v_{k_F}}$$

(19)

for low $T$. Taking the well-known logarithmic correction to the $T^2$-behavior of the scattering rate in two-dimensional Fermi liquids into account, one obtains $\rho(T) \propto T^{3/2} |\log T|^{1/2}$.

In the presence of impurities, the scattering rate has the form $\gamma_{k_F}^{\text{tr}}(T) = \gamma_{k_F}^{\text{imp}} + b_{k_F} d_{k_F}^2 T$, for temperatures low enough that the Fermi liquid term of order $T^2$ can be neglected compared to the impurity term. For $T \to 0$ one then obtains a finite residual resistivity determined exclusively by impurity scattering. For low finite temperatures the resistivity increases linearly with $T$ as long as $T \ll \gamma_{k_F}^{\text{imp}}/b_{k_F}$. For $T \gg \gamma_{k_F}^{\text{imp}}/b_{k_F}$ one obtains $\rho(T) \propto T^{1/2}$, with a prefactor proportional to $(\gamma_{k_F}^{\text{imp}} b_{k_F})^{1/2}/v_{k_F}$, provided that impurity scattering still dominates over the conventional Fermi liquid contribution to $\gamma_{k_F}^{\text{tr}}(T)$.

In summary, we have analyzed the DC charge transport in the quantum critical regime near a d-wave Pomeranchuk instability in two dimensions. It turned out that the relaxation of the electric current is dominated by classical fluctuations. The transport decay rate $\gamma_{k_F}^{\text{tr}}(T)$ is linear in temperature everywhere on the Fermi surface except at cold spots on the Brillouin zone diagonal. For pure systems, this leads to a DC resistivity proportional to $T^{3/2}$ in the low-temperature limit. It is tempting to associate this result with the unusual $T^{3/2}$-law observed for the resistivity in overdoped La$_{2-x}$Sr$_x$CuO$_4$ [21]. In the presence of impurities the residual impurity resistance at $T = 0$ is approached linearly at low temperatures.

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[1] I.J. Pomeranchuk, Sov. Phys. JETP 8, 361 (1958).
[2] H. Yamase and H. Kohno, J. Phys. Soc. Jpn. 69, 332 (2000); 69, 2151 (2000).
[3] C.J. Halboth and W. Metzner, Phys. Rev. Lett. 85, 5162 (2000).
[4] I. Grote, E. Körding, and F. Wegner, J. Low Temp. Phys. 126, 1385 (2002).
[5] S.A. Kivelson, E. Fradkin, and V.J. Emery, Nature 393, 550 (1998).
[6] S.A. Kivelson et al., Rev. Mod. Phys. 75, 1201 (2003).
[7] V. Hinkov et al., Nature 430, 650 (2004).
[8] H. Yamase and W. Metzner, Phys. Rev. B 73, 214517 (2006).
[9] S.A. Grigera et al., Science 306, 1154 (2004).
[10] C.M. Varma and L. Zhu, Phys. Rev. Lett. 96, 036405 (2006).
[11] V. Oganesyan, S.A. Kivelson, and E. Fradkin, Phys. Rev. B 64, 195109 (2001).
[12] W. Metzner, D. Rohe, and S. Andergassen, Phys. Rev. Lett. 91, 066402 (2003).
[13] L. Dell’Anna and W. Metzner, Phys. Rev. B 73, 045127 (2006).
[14] J. Rech, C. Pepin, and A.V. Chubukov, cond-mat/0605306.
[15] H. Yamase, V. Oganesyan, and W. Metzner, Phys. Rev. B 72, 035114 (2005).
[16] H.-Y. Kee, E.H. Kim, and C.-H. Chung, Phys. Rev. B 68, 245109 (2003); I. Khavkine et al., Phys. Rev. B 70, 155110 (2004).
[17] See, for example, G. Rickayzen, *Green’s Functions and Condensed Matter* (Academic Press, London, 1980).
[18] For a detailed discussion of this point for the case of a density wave instability, see S. Caprara et al., cond-mat/0610676.
[19] H. v. Löhneysen et al., cond-mat/0606317 to appear in Rev. Mod. Phys; P. Wölfle and A. Rosch, cond-mat/0609343.
[20] L.B. Ioffe, and A.J. Millis, Phys. Rev. B 58, 11631 (1998).
[21] H. Takagi et al., Phys. Rev. Lett. 69, 2975 (1992).