Abstract  Generators for the discrete U-duality groups of toroidally compactified
M-theory in \( d \geq 4 \) are presented and used to determine the \( d = 3 \)
U-duality group. This contribution summarizes the results of [1].

1. INTRODUCTION AND SUMMARY

The study of nonperturbative duality symmetries in the last years
has dramatically changed our understanding of string theory. U-duality,
introduced in [2], is one of these symmetries. It was conjectured to be
generated by the target-space duality of \( T^d \) (see [4] for a review) and the
modular group of the torus including the eleventh M-theory direction
[5]. This definition was adopted in the algebraic approach to U-duality
reviewed in [3].

Using instead the original conjecture of [2] that takes the hidden sym-
metries of low energy supergravity as a starting point, generators of dis-
crete U-duality in four dimensions may be determined directly. Groups
in higher dimensions can be found by embeddings, and a method to
determine the \( d = 3 \) U-duality is presented. Applied to a toy model
corresponding to a truncation of M-theory, the method is seen to give a
significantly different result than for the full theory.

2. U-DUALITY IN FOUR DIMENSIONS

Compactifying eleven dimensional supergravity, the low energy limit
of M-theory, on \( T^7 \) yields 70 scalars arising from the \( d = 11 \) 3-form
potential and the moduli of the torus. They may be joint in a field
\( Y^{(4)} \in E_{7(7)}/SU(8) \) [7], a representation matrix in the fundamental
56 representation. Furthermore, 28 \( U(1) \) gauge fields arise from the
$d = 11$ potential and metric. These fields and their magnetic duals can be arranged to 28 dimensional vectors $G_{\bar{\mu}\bar{\nu}}$ and $H_{\bar{\mu}\bar{\nu}}$, $\bar{\mu}, \bar{\nu} = 0 \ldots 3$.

The equations of motion of the theory are invariant under classical $E_{7(+7)}$ duality and local $SU(8)$ transformations, acting as

$$\mathcal{F}_{\bar{\mu}\bar{\nu}} \to \Lambda^{-1} \mathcal{F}_{\bar{\mu}\bar{\nu}}, \quad \mathcal{V}^{(4)} \to h \mathcal{V}^{(4)} \Lambda, \quad \Lambda \in E_{7(+7)}, \quad h \in SU(8)$$

where $\mathcal{F}^t = (G_{\bar{\mu}\bar{\nu}}, H_{\bar{\mu}\bar{\nu}})^t$. Defining charges $Z = \oint_\Sigma \mathcal{F}$, the DSZ quantization condition breaks $E_{7(+7)}$ to a discrete subgroup inducing integer shifts on the charge lattice. This group is the U-duality group and was proposed to extend to a nonperturbative quantum symmetry of M-theory.

To make U-duality transformations “manageable” in $d = 4$, the $56$ representation needs to be addressed. This can be done by an embedding into $\mathfrak{e}_{8(+8)}$ using Freudenthal’s realization of exceptional Lie algebras [8]. The $\mathfrak{e}_{8(+8)}$ generators are given by $E_{ij}$, $i, j = 1 \ldots 9$, corresponding to $\mathfrak{sl}_9$, and $E_{ijk}$, $E^*_{ijk}$. Their commutators are given in [1]. Defining the basis

$$S^t = \left(-E_{i\bar{j}9}^* + E_{i\bar{j}}^1, -E_{i\bar{j}9}^1, -E_{i\bar{j}}^9\right), \quad \mathcal{X}^t = \left(x^{i\bar{j}}, x^{i9}, y_{i\bar{j}}, y_{i9}\right)$$

where $i, j = 2 \ldots 8$, the adjoint action of the $\mathfrak{e}_{7(+7)}$ subalgebra on $\mathcal{X}S$ exactly spans the $56$ representation as defined in [7]. The basis (1.1) can therefore be used to study U-duality transformations on $\mathcal{F}$ and $\mathcal{V}^{(4)}$, as well as its subgroups T-duality and S-duality.

What are the generators of $E_{7(+7)}(\mathbb{Z})$? Using the fact that the $56$ representation of $E_{7(+7)}$ is the unique minimal representation of $E_{7(+7)}$, it may be proven using the Birkhoff decomposition of Lie groups that the subgroup of $E_{7(+7)}$ inducing integer shifts on the lattice defined by $S$ is generated by “fundamental unipotents”¹, that is, the action of the discrete subgroup is spanned by exponentiating the Chevalley generators for all positive and negative roots. From these, $T$ and $S$ generators may be built parallel to $SL(2, \mathbb{Z})$, the latter carrying a representation of the Weyl group modulo $\mathbb{Z}_2$. This together with the basis $S$ yields contact to the algebraic approach to M-theory, and it may be shown that the two approaches are equivalent.

Since the notion of the above generators is representation independent, the discrete U-duality groups of higher dimensional theories follow directly from truncating the Dynkin diagram. Their representations are minimal and can be read off from $S$.

¹See [9], where this group is defined in a more general context as homeomorphism of the group ring over $\mathbb{Z}$ to $\mathbb{Z}$.
2.1 U-DUALITY IN THREE DIMENSIONS

The $d = 3$ theory is known to have classical $E_{8(+8)}$ symmetry. As only scalars remain in the theory, the notion of electric charge seems ill defined and the meaning of a duality symmetry seems unclear. Therefore, in order to define U-duality in $d = 3$, a method proposed in [2] parallel to [6] may be extended to M-theory. By compactifying M-theory on the torus, we can choose eight different ways how to compactify first to four dimensions. This results in eight $E_{7(+7)}(\mathbb{Z})$'s acting differently on M-theory fields. By reducing the theory further to three dimensions, these groups are merged together to form the three dimensional duality group.

The reduction to $d = 3$ yields a scalar coset matrix of the form

$$V^{(3)} = V^{(4)} \exp\left(\frac{1}{2} \phi \sum_{i=1}^{8} h_i\right) \exp\left(Y \cdot S\right) \exp\left(f E^1\right). \quad (1.2)$$

$V^{(4)}$ is identical to $V^{(4)}$, but now in the 248 adjoint representation of $E_{8(+8)}$. $Y$ obeys $\partial_{\mu} \eta = \mathcal{G}_{\mu z}$, $\partial_{\mu} \bar{\eta} = \mathcal{H}_{\mu z}$, $Y = (\eta, \bar{\eta})$, $\mu, \nu = 0 \ldots 2$ and therefore carries the $d = 4$ charges, where $z$ is the compact fourth direction. $\varphi$ and $f$ are the $d = 3$ dilaton and dualized KK field strength respectively, and $h_i$ represents the $\mathfrak{sl}_9$ Cartan subalgebra.

A $\Lambda \in E_{7(+7)} \subset E_{8(+8)}$ transformation acts on $V^{(4)}$ and $Y$ exactly as discussed in the last section. It therefore represents the $d = 4$ U-duality in $d = 3$. Completing a circle around vortex solutions in $d = 3$ may be seen to correspond to an $E_{7(+7)}(\mathbb{Z})$ action on the fields, parallel to [6].

The different compactifications yield 8 different coset matrices in $d = 3$. Using explicitly the connection to M-theory fields, it can be seen that they are related by

$$V^{(3)}_{\#n} = (P_n S^1_n)^{-1} h_n \quad V^{(3)}_{\#1} \quad P_n S^1$$

where $h_n$ is a local transformation restoring upper triangular parameterization, $S^1_n$ corresponds to a Weyl reflection and $P_n$ corresponds to a charge conjugation in $d = 4$. Denoting the $d = 4$ U-duality generators of the $n$th compactification by $\Lambda$, the total $d = 3$ U-duality is given by joining all generators

$$P_n S^1_n \Lambda (P_n S^1_n)^{-1}$$

for all $n$. This can be seen to yield the whole $E_{8(+8)}(\mathbb{Z})$ defined by exponentiating all Chevalley generators. The intersection of two different $d = 4$ U-dualities is seen to be $E_{6(+6)}(\mathbb{Z})$ as expected. This determines U-duality in $d = 3$. 

Discrete U-duality
2.2 $G_{2(+2)}$ AS TOY MODEL

The described method to generate $d = 3$ U-duality can also be applied to five dimensional simple supergravity as toy model, which upon reduction to three dimensions exhibits a $G_{2(+2)}$ global symmetry [10]. It is known that this theory closely resembles $d = 11$ supergravity in many respects and corresponds to a direct truncation. The analogue of U-duality in $d = 4$ is $SL(2, \mathbb{Z})$, acting in the spin 3/2 representation.

Here, the joint U-duality $U(\mathbb{Z})$ in $d = 3$ is strictly smaller than $G_{2(+2)}(\mathbb{Z})$ defined by exponentiating the Chevalley generators for all roots. All generators of $U(\mathbb{Z})$ correspond to short roots of $G_{2(+2)}(\mathbb{Z})$. That the groups do not agree is therefore connected to the fact that $G_{2(+2)}$ is not simply laced. Since no string compactification described by this no-moduli supergravity at low energies is known, one cannot determine which group is the “correct” U-duality group until such a description has been found.

Acknowledgments

I wish to thank Shun’ya Mizoguchi for a stimulating and fruitful collaboration.

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