MASS GROWTH OF OBJECTS AND CATEGORICAL ENTROPY

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Abstract. In the pioneer work by Dimitrov-Haiden-Katzarkov-Kontsevich, they introduced various categorical analogies from classical theory of dynamical systems. In particular, they defined the entropy of an endofunctor on a triangulated category with a split generator. In the connection between categorical theory and classical theory, a stability condition on a triangulated category plays the role of a measured foliation so that one can measure the “volume” of objects, called the mass, via the stability condition. The aim of this paper is to establish fundamental properties of the growth rate of mass of objects under the mapping by the endofunctor and to clarify the relationship between the entropy and that. We also show that they coincide under a certain condition.

1. Introduction

In the pioneer work [DHKK14], Dimitrov-Haiden-Katzarkov-Kontsevich introduced various categorical analogies of classical theory from dynamical systems. In particular, they defined the entropy of an endofunctor on a triangulated category with a split generator. One of their motivations comes from the connection between theory of stability conditions on triangulated categories and Teichmüller theory of surfaces [GMN13, BS15]. In this connection, a stability condition on a triangulated category corresponds to a measured foliation (a quadratic differential) on a surface, and the mass of stable objects corresponds to the length of geodesics. Thus the mass of objects plays the role of “volume” in some sense. In the work [DHKK14], they also suggested that there is a connection between the growth rate of mass of objects under the mapping by an endofunctor and the entropy of that. In this paper, we establish fundamental properties of the mass growth and clarify the relationship between the entropy and that. We also show that they coincide under a certain condition. The result in this paper is motivated by the famous classical work “Volume growth and entropy” by Yomdin [Yom87] on classical dynamical systems.

1.1. Fundamental properties of mass growth. First we introduce the mass growth with respect to endofunctors. Let \( \mathcal{D} \) be a triangulated category and \( K(\mathcal{D}) \) be its Grothendieck group. A stability condition \( \sigma = (Z, \mathcal{P}) \) on \( \mathcal{D} \) [Bri07] is a pair of a linear map \( Z : K(\mathcal{D}) \to \mathbb{C} \) and a family of full subcategories \( \mathcal{P}(\phi) \subset \mathcal{D} \) for \( \phi \in \mathbb{R} \) satisfying some axioms (see Definition 2.8). A nonzero object in \( \mathcal{P}(\phi) \) is called a semistable object of phase \( \phi \). One of the axioms implies that any nonzero object \( E \in \mathcal{D} \) can be decomposed into semistable objects with decreasing phases, i.e. there
is a sequence of exact triangles, called a Harder-Narasimhan filtration,

\[ 0 = E_0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E_m = E \]

with \( A_i \in P(\phi_i) \) and \( \phi_1 > \phi_2 > \cdots > \phi_m \). Through the Harder-Narasimhan filtration, the mass of \( E \) with a parameter \( t \in \mathbb{R} \) (see Definition 3.1) is defined by

\[
m_{\sigma,t}(E) := \sum_{i=1}^{m} |Z(A_i)| e^{\phi_i t}.
\]

Thus a given stability condition defines the “volume” of objects in some sense. Actually in the connection between spaces of stability conditions and Teichmüller spaces, the mass of stable objects gives the length of corresponding geodesics [BS15, GMN13, HKK, Ike]. For an endofunctor \( F: \mathcal{D} \to \mathcal{D} \), we want to consider the growth rate of mass of objects under the mapping by \( F \). Therefore we introduce the following quantity. The mass growth with respect to \( F \) is the function \( h_{\sigma,t}(F): \mathbb{R} \to [-\infty, \infty] \) defined by

\[
h_{\sigma,t}(F) := \sup_{E \in \mathcal{D}} \left\{ \limsup_{n \to \infty} \frac{1}{n} \log(m_{\sigma,t}(F^n E)) \right\}.
\]

(As conventions, set \( m_{\sigma,t}(0) = 0 \) and \( \log 0 = -\infty \).) Fundamental properties of \( h_{\sigma,t}(F) \) are stated as the main result of this paper. We also recall the space of stability conditions to consider the behavior of \( h_{\sigma,t}(F) \) under the deformation of \( \sigma \).

In [Bri07], it was shown that the set of stability conditions \( \text{Stab}(\mathcal{D}) \) has a natural topology and in addition, \( \text{Stab}(\mathcal{D}) \) becomes a complex manifold.

Next we recall the entropy of endofunctors from [DHKK14]. Let \( \mathcal{D} \) be a triangulated category with a split-generator and \( F: \mathcal{D} \to \mathcal{D} \) be an endofunctor. In [DHKK14], they introduced the function \( h_t(F): \mathbb{R} \to [-\infty, \infty] \), called the entropy of \( F \) (see Definition 2.4), and showed various fundamental properties of \( h_t(F) \). In addition, they asked the relationship between the entropy \( h_t(F) \) and the mass growth \( h_{\sigma,t}(F) \) (see [DHKK14, Section 4.5]). Our result is the following.

**Theorem 1.1** (Theorem 3.5 and Proposition 3.10). Let \( \mathcal{D} \) be a triangulated category, \( F: \mathcal{D} \to \mathcal{D} \) be an endofunctor and \( \sigma \) be a stability condition on \( \mathcal{D} \). Assume that \( \mathcal{D} \) has a split-generator \( G \). Then the mass growth \( h_{\sigma,t}(F) \) satisfies the followings.

1. If a stability condition \( \tau \) lies in the same connected component as \( \sigma \) in the space of stability conditions \( \text{Stab}(\mathcal{D}) \), then

\[
h_{\sigma,t}(F) = h_{\tau,t}(F).
\]
(2) The mass growth of the generator $G$ determines $h_{\sigma,t}(F)$, i.e.

$$h_{\sigma,t}(F) = \limsup_{n \to \infty} \frac{1}{n} \log(m_{\sigma,t}(F^n G)).$$

(3) An inequality

$$h_{\sigma,t}(F) \leq h_t(F) < \infty$$

holds.

In the case $t = 0$, this result was stated in [DHKK14, Section 4.5] by using the triangle inequality for mass (see Proposition 3.3). However, the triangle inequality for mass is non-trivial even if $t = 0$ and actually the most technical part in this paper. Therefore we give a detailed proof of it with a parameter $t$ in Section 3.2.

1.2. Lower bound by the spectral radius. We consider the lower bound of the mass growth when $t = 0$. Since $F: \mathcal{D} \to \mathcal{D}$ preserves exact triangles, $F$ induces a linear map $[F]: K(\mathcal{D}) \to K(\mathcal{D})$. The spectral radius of $[F]$ is defined by

$$\rho([F]) := \max\{|\lambda| | \lambda \text{ is an eigenvalue of } [F]\}.$$

**Theorem 1.2** (Proposition 3.11). In the case $t = 0$, we have an inequality

$$\log \rho([F]) \leq h_{\sigma,0}(F) \leq h_0(F)$$

for any stability condition $\sigma \in \text{Stab}(\mathcal{D})$.

As known results, if $\mathcal{D}$ is saturated, then it was shown in [DHKK14, Theorem 2.9] that for a linear map $\text{HH}_*(F): \text{HH}_*(\mathcal{D}) \to \text{HH}_*(\mathcal{D})$ induced on the Hochschild homology of $\mathcal{D}$, the inequality $\log \rho(\text{HH}_*(F)) \leq h_0(F)$ holds under some condition for eigenvalues of $\text{HH}_*(F)$. They also conjectured that the inequality holds without that condition. Our result Theorem 1.2 holds without any conditions for $[F]$, however we use the existence of stability conditions on $\mathcal{D}$. For many examples in [DHKK14, Kik, KT], it was shown that the equality $\log \rho([F]) = h_0(F)$ holds. Kikuta-Takahashi gave a certain conjecture on the equality in [KT, Conjecture 5.3].

1.3. Equality between mass growth and entropy. The remaining important question is to ask when the equality $h_{\sigma,t}(F) = h_t(F)$ holds. In the following, we give a sufficient condition for the equality. For a stability condition $\sigma = (Z, \mathcal{P})$, we can associate an abelian category, called the heart of $\mathcal{P}$, as the extension-closed subcategory generated by objects in $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$. Denote it by $\mathcal{P}((0, 1])$. A stability condition $\sigma = (Z, \mathcal{P})$ is called algebraic if the heart $\mathcal{P}((0, 1])$ is a finite length abelian category with finitely many simple objects (see Definition 2.7 and Definition 2.11).
Theorem 1.3 (Theorem 3.14). Let $G \in \mathcal{D}$ be a split-generator and $F: \mathcal{D} \to \mathcal{D}$ be an endofunctor. If a connected component $\text{Stab}^0(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ contains an algebraic stability condition, then for any $\sigma \in \text{Stab}^0(\mathcal{D})$ we have

$$h_t(F) = h_{\sigma,t}(F) = \lim_{n \to \infty} \frac{1}{n} \log(m_{\sigma,t}(F^nG)).$$

Note that in the above theorem, a stability condition $\sigma$ is not necessarily an algebraic stability condition.

We see a typical example which satisfies the condition in Theorem 1.3 from Section 4.1. Let $A = \bigoplus_k A^k$ be a dg-algebra such that $H^0(A)$ is a finite dimensional algebra and $H^k(A) = 0$ for $k > 0$. Denote by $\mathcal{D}_{fd}(A)$ the derived category of dg-modules over $A$ with finite dimensional total cohomology, i.e. $\sum_k \dim H^k(M) < \infty$. Then there is a bounded t-structure whose heart is isomorphic to the abelian category of finite dimensional modules over $H^0(A)$. As a result, we can construct algebraic stability conditions on $\mathcal{D}_{fd}(A)$. Thus in the context of representation theory, Theorem 1.3 works well. As an application, we compute the entropy of spherical twists in Section 4.2.

On the other hand, for derived categories coming from algebraic geometry, we cannot find algebraic hearts in general. Only in special cases, for example in the case that the derived category has a full strong exceptional collection, the work by Bondal [Bon89] enables us to find algebraic hearts. It is an important problem to answer whether the equality $h_{\sigma,t}(F) = h_t(F)$ holds without the existence of algebraic stability conditions.

1.4. Categorical theory versus classical theory. We compare our result with the famous classical result “Volume growth and entropy” by Yomdin [Yom87]. Let $M$ be a compact smooth manifold and $f: M \to M$ be a smooth map. The map $f$ induces a linear map $f_*: H_*(M; \mathbb{R}) \to H_*(M; \mathbb{R})$ on the homology group $H_*(M; \mathbb{R})$. For the map $f$, we can define the topological entropy $h_{\text{top}}(f)$ [AKM65] and the inequality $\log \rho(f_*) \leq h_{\text{top}}(f)$ was conjectured in [Shu74]. In [Yom87], Yomdin introduced the volume growth $v(f)$ by using a Riemannian metric on $M$ and showed that

$$\log \rho(f_*) \leq v(f) \leq h_{\text{top}}(f).$$

Our result Theorem 1.2 looks like the categorical analogy of this classical result. On the other hand, the difference between categorical theory and classical theory is that the categorical entropy $h_t(F)$ and the mass growth $h_{\sigma,t}(F)$ have the parameter $t$ which measures the growth rate of degree shifts in a triangulated category. This point is an essentially new feature of categorical theory.

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Notations. We work over a field $\mathbb{K}$. All triangulated categories in this paper are $\mathbb{K}$-linear and their Grothendieck groups are free of finite rank, i.e. $K(D) \cong \mathbb{Z}^n$ for some $n$. An endofunctor $F : D \to D$ refers to an exact endofunctor, i.e. $F$ preserves all exact triangles and commutes with degree shifts. The natural logarithm is extended to $\log : [0, \infty) \to (-\infty, \infty)$ by setting $\log 0 := -\infty$.

2. Preliminaries

In this section, we prepare basic terminologies mainly from [DHKK14, Bri07].

2.1. Complexity and entropy. First we recall the notion of complexity and entropy from [DHKK14, Section 2].

Let $D$ be a triangulated category. A triangulated subcategory is called thick if it is closed under taking direct summands. For an object $E \in D$, we denote by $\langle E \rangle \subset D$ the smallest thick triangulated subcategory containing $E$. An object $G \in D$ is called a split-generator if $\langle G \rangle = D$. This implies that for any object $E \in D$, there is some object $E' \in D$ such that we have a sequence of exact triangles

$$0 = A_0 \to A_1 \to A_2 \to \ldots \to A_{k-1} \to A_k = E \oplus E'$$

with $n_i \in \mathbb{Z}$. We note that the object $E'$ and the above sequence are not unique.

**Definition 2.1** ([DHKK14, Definition 2.1]). Let $E_1$ and $E_2$ be objects in $D$. The complexity of $E_2$ relative to $E_1$ is the function $\delta_t(E_1, E_2) : \mathbb{R} \to [0, \infty)$ defined by

$$\delta_t(E_1, E_2) := \begin{cases} 0 & \text{if } E_2 \cong 0 \\ \inf \{ \sum_{i=1}^k e^{n_i} : \quad 0 \to A_1 \to \ldots \to A_{k-1} \to E_2 \oplus E' \to \ldots \to E_1[n_k] \} & \text{if } E_2 \in \langle E_1 \rangle \\ \infty & \text{if } E_2 \notin \langle E_1 \rangle. \end{cases}$$

By definition, we have an inequality $0 < \delta_t(G, E) < \infty$ for a split-generator $G \in D$ and a nonzero object $E \in D$. We recall fundamental inequalities for complexity.
Proposition 2.2 ([DHKK14], Proposition 2.3). For $E_1, E_2, E_3 \in \mathcal{D}$,

1. $\delta_t(E_1, E_3) \leq \delta_t(E_1, E_2)\delta_t(E_2, E_3)$,
2. $\delta_t(E_1, E_2 \oplus E_3) \leq \delta_t(E_1, E_2) + \delta_t(E_1, E_3)$,
3. $\delta_t(F(E_1), F(E_2)) \leq \delta_t(E_1, E_2)$ for an endofunctor $\mathcal{D} \to \mathcal{D}$.

Similar to [DHKK14, Proposition 2.3], it is easy to check the following.

Lemma 2.3. For objects $D, E_1, E_2, E_3 \in \mathcal{D}$, if there is an exact triangle $E_1 \to E_2 \to E_3 \to E_1[1]$, then

$$\delta_t(D, E_2) \leq \delta_t(D, E_1) + \delta_t(D, E_3).$$

Now we introduce the notion of the entropy of endofunctors. The entropy of an endofunctor $F$ measures the growth rate of complexity $\delta_t(G, F^nG)$ as $n \to \infty$.

Definition 2.4 ([DHKK14], Definition 2.5). Let $\mathcal{D}$ be a triangulated category with a split-generator $G$ and let $F: \mathcal{D} \to \mathcal{D}$ be an endofunctor. The entropy of $F$ is the function $h_t(F): \mathbb{R} \to [-\infty, \infty)$ defined by

$$h_t(F) := \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, F^nG).$$

By [DHKK14] Lemma 2.6, it follows that $h_t(F)$ is well-defined and $h_t(F) < \infty$.

2.2. Bounded t-structures and the associated cohomology.

Definition 2.5 ([BBDS82]). A t-structure on $\mathcal{D}$ is a full subcategory $\mathcal{F} \subset \mathcal{D}$ satisfying the following conditions:

(a) $\mathcal{F}[1] \subset \mathcal{F}$,
(b) define $\mathcal{F}^\perp := \{F \in \mathcal{D} | \text{Hom}(D, F) = 0 \text{ for all } D \in \mathcal{F}\}$, then for every object $E \in \mathcal{D}$ there is an exact triangle $D \to E \to F \to D[1]$ in $\mathcal{D}$ with $D \in \mathcal{F}$ and $F \in \mathcal{F}^\perp$.

In addition, the t-structure $\mathcal{F} \subset \mathcal{D}$ is said to be bounded if $\mathcal{F}$ satisfies the condition

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{F}^\perp[i] \cap \mathcal{F}[j].$$

For a t-structure $\mathcal{F} \subset \mathcal{D}$, we define the heart $\mathcal{H} \subset \mathcal{D}$ by

$$\mathcal{H} := \mathcal{F}^\perp[1] \cap \mathcal{F}.$$ 

It was proved in [BBDS82] that $\mathcal{H}$ becomes an abelian category. Bridgeland gave the characterization of the heart of a bounded t-structure as follows.

Lemma 2.6 ([Bri07], Lemma 3.2). Let $\mathcal{H} \subset \mathcal{D}$ be a full additive subcategory. Then $\mathcal{H}$ is the heart of a bounded t-structure if and only if the following conditions hold:

(a) if $k_1 > k_2 \in \mathbb{Z}$ and $A_i \in \mathcal{H}[k_i]$ ($i = 1, 2$), then $\text{Hom}_\mathcal{D}(A_1, A_2) = 0$, 

(b) for $0 \neq E \in \mathcal{D}$, there is a finite sequence of integers
\[ k_1 > k_2 > \cdots > k_m \]
and a sequence of exact triangles
\[
0 = E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{m-1} \rightarrow E_m = E
\]
with $A_i \in \mathcal{H}[k_i]$ for all $i$.

The above filtration in the condition (b) defines the $k$-th cohomology $H^k(E) \in \mathcal{H}$ of the object $E$ by
\[
H^k(E) := \begin{cases} 
A_i[-k_i] & \text{if } k = -k_i \\
0 & \text{otherwise}.
\end{cases}
\]
This cohomology becomes a cohomological functor from $\mathcal{D}$ to $\mathcal{H}$, i.e. if there is an exact triangle $D \rightarrow E \rightarrow F \rightarrow E[1]$, then we can obtain a long exact sequence
\[
\cdots \rightarrow H^{k-1}(F) \rightarrow H^k(D) \rightarrow H^k(E) \rightarrow H^k(F) \rightarrow H^{k+1}(D) \rightarrow \cdots
\]
in the abelian category $\mathcal{H}$. In the last of this section, we introduce the special class of bounded t-structures.

**Definition 2.7.** We say that the heart of a bounded t-structure is **algebraic** if it is a finite length abelian category with finitely many isomorphism classes of simple objects.

If $\mathcal{D}$ has an algebraic heart $\mathcal{H}$ with simple objects $S_1, \ldots, S_n$, then it is easy to see that the direct sum $G := \oplus_{i=1}^n S_i$ becomes a split-generator of $\mathcal{D}$.

2.3. **Bridgeland stability conditions.** In [Bri07], Bridgeland introduced the notion of a stability condition on a triangulated category as follows.

**Definition 2.8.** Let $\mathcal{D}$ be a triangulated category and $K(\mathcal{D})$ be its Grothendieck group. A **stability condition** $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$, called a **central charge**, and a family of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ satisfying the following conditions:

(a) if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,
(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
(c) if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$ ($i = 1, 2$), then $\text{Hom}_\mathcal{D}(A_1, A_2) = 0$,
(d) for $0 \neq E \in \mathcal{D}$, there is a finite sequence of real numbers
\[ \phi_1 > \phi_2 > \cdots > \phi_m \]
and a sequence of exact triangles

\[
0 = E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_{m-1} \longrightarrow E_m = E
\]

with \( A_i \in \mathcal{P}(\phi_i) \) for all \( i \).

We write \( \phi^+_\sigma(E) := \phi_1 \) and \( \phi^-_\sigma(E) := \phi_m \). Nonzero objects in \( \mathcal{P}(\phi) \) are called \( \sigma \)-semistable of phase \( \phi \) in \( \sigma \). The sequence of exact triangles in (d) is called a Harder-Narasimhan filtration of \( E \) with semistable factors \( A_1, \ldots, A_m \) of phases \( \phi_1 > \cdots > \phi_m \).

In addition to the above axioms, we always assume that our stability conditions have the support property in [KS]. Let \( \| \cdot \| \) be some norm on \( K(D) \otimes \mathbb{R} \). A stability condition \( \sigma = (Z, \mathcal{P}) \) satisfies the support property if there is a some constant \( C > 0 \) such that

\[
\frac{|Z(E)|}{\|E\|} > C
\]

for all \( \sigma \)-semistable objects \( E \in \mathcal{D} \).

For an interval \( I \subset \mathbb{R} \), we denote by \( \mathcal{P}(I) \) the extension-closed subcategory generated by objects in \( \mathcal{P}(\phi) \) for \( \phi \in I \), namely

\[
\mathcal{P}(I) := \{ E \in \mathcal{D} \mid \phi^+_\sigma(E) \in I \} \cup \{0\}.
\]

From a stability condition \( (Z, \mathcal{P}) \), we can construct a bonded t-structure \( \mathcal{F} := \mathcal{P}((0, \infty)) \) and its heart is given by \( \mathcal{H} = \mathcal{P}((0, 1]) \).

2.4. Algebraic stability conditions. In [Bri07], Bridgeland gave the alternative description of a stability condition on \( \mathcal{D} \) as a pair of a bounded t-structure and a central charge on its heart. By using this description, we construct algebraic stability conditions.

**Definition 2.9.** Let \( \mathcal{H} \) be an abelian category and let \( K(\mathcal{H}) \) be its Grothendieck group. A central charge on \( \mathcal{H} \) is a group homomorphism \( Z: K(\mathcal{H}) \rightarrow \mathbb{C} \) such that for any nonzero object \( 0 \neq E \in \mathcal{H} \), the complex number \( Z(E) \) lies in the semi-closed upper half-plane \( \mathbb{H} := \{ re^{i\pi\phi} \in \mathbb{C} \mid r \in \mathbb{R}_{>0}, \phi \in (0, 1] \} \).

For any nonzero object \( E \in \mathcal{H} \), define the phase of \( E \) by

\[
\phi(E) := \frac{1}{\pi} \arg Z(E) \in (0, 1].
\]

An object \( 0 \neq E \in \mathcal{H} \) is called \( Z \)-semistable if every subobject \( 0 \neq A \subset E \) satisfies \( \phi(A) \leq \phi(E) \). A Harder-Narasimhan filtration of \( 0 \neq E \in \mathcal{H} \) is the filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E
\]
whose extension factors $F_i := E_i/E_{i-1}$ are $Z$-semistable with decreasing phases

$$\phi(F_1) > \cdots > \phi(F_m).$$

A central charge $Z$ is said to have the \textit{Harder-Narasimhan property} if any nonzero object of $\mathcal{H}$ has a Harder-Narasimhan filtration. The following gives the another definition of a stability condition.

**Proposition 2.10** ([Bri07], Proposition 5.3). \textit{Giving a stability condition on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{H}$ of a bounded structure on $\mathcal{D}$ and a central charge on $\mathcal{H}$ with the Harder-Narasimhan property.}

In Proposition 2.10, the pair $(Z, \mathcal{H})$ is constructed from a stability condition $(Z, \mathcal{P})$ by setting $\mathcal{H} := \mathcal{P}((0, 1])$.

**Definition 2.11.** A stability condition $(Z, \mathcal{P})$ is called \textit{algebraic} if the corresponding heart $\mathcal{H} = \mathcal{P}((0, 1])$ is algebraic (for the definition of algebraic hearts, see Definition 2.7).

Algebraic stability conditions are constructed from algebraic hearts as follows. Let $\mathcal{H} \subset \mathcal{D}$ be an algebraic heart with simple objects $S_1, \ldots, S_n$. Then the Grothendieck group is given by $K(\mathcal{H}) \cong \oplus_{i=1}^n \mathbb{Z}[S_i]$. Take $(z_1, \ldots, z_n) \in \mathbb{H}^n$ and define the central charge $Z : K(\mathcal{H}) \to \mathbb{C}$ by the linear extension of $Z(S_i) := z_i$. Then $Z$ has the Harder-Narasimhan property by [Bri07] Proposition 2.4. Thus $(Z, \mathcal{H})$ becomes a stability condition on $\mathcal{D}$.

2.5. \textbf{Harder-Narasimhan polygons.} In this section, we discuss the Harder-Narasimhan polygon following [Bay]. This plays a key role to show the triangle inequality for mass in Section 3.2. The following is based on [Bay Section 3].

**Definition 2.12.** Let $\mathcal{H}$ be an abelian category and $Z$ be a central charge on it. For an object $E \in \mathcal{H}$, the \textit{Harder-Narasimhan polygon} $\text{HN}^Z(E)$ of $E$ is the convex hull of the subset $\{ Z(A) \in \mathbb{C} \mid A \subset E \} \subset \mathbb{C}$ in the complex plane.

It is clear from the definition that if $F \subset E$, then $\text{HN}^Z(F) \subset \text{HN}^Z(E)$. The Harder-Narasimhan polygon $\text{HN}^Z(E)$ is called \textit{polyhedral on the left} if it has finitely many extremal points $0 = z_0, z_1, \ldots, z_k = Z(E)$ such that $\text{HN}^Z(E)$ lies to the right of the path $z_0 z_1 \ldots z_k$. This implies that the intersection of $\text{HN}^Z(E)$ and the closed half-plane to the left of the line through 0 and $Z(E)$ becomes a polygon with vertices $z_0, z_1, \ldots, z_k$ (see Figure 1).

**Proposition 2.13** ([Bay], Proposition 3.3). \textit{The object $E$ has a Harder-Narasimhan filtration if and only if $\text{HN}^Z(E)$ is polyhedral on the left. In particular, if the Harder-Narasimhan filtration of $E$ is given by}

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots E_k = E,$$
then extremal points of $\text{HN}^Z(E)$ are given by $z_i = Z(E_i)$ for $i = 0, 1, \ldots, k$.

2.6. **Topology on the space of stability conditions.** In [Bri07], Bridgeland introduced a natural topology on the space of stability conditions and showed that this space becomes a complex manifold. In the following, we recall his construction.

Let $\text{Stab}(\mathcal{D})$ be the set of stability conditions on a triangulated category $\mathcal{D}$ with the support property. For stability conditions $\sigma = (Z, P)$ and $\tau = (W, Q)$ in $\text{Stab}(\mathcal{D})$, set

$$d(P, Q) := \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^{-}_\sigma(E) - \phi^{-}_\tau(E)|, |\phi^{+}_\sigma(E) - \phi^{+}_\tau(E)| \right\} \in [0, \infty].$$

and

$$\|Z - W\|_\sigma := \sup \left\{ \frac{|Z(E) - W(E)|}{|Z(E)|} \bigg| E \text{ is } \sigma\text{-semistable} \right\} \in [0, \infty].$$

Define a subset $B_\epsilon(\sigma) \subset \text{Stab}(\mathcal{D})$ by

$$B_\epsilon(\sigma) := \{ \tau = (W, Q) \in \text{Stab}(\mathcal{D}) \mid d(P, Q) < \epsilon, \|Z - W\|_\sigma < \sin(\pi \epsilon) \}$$

for $0 < \epsilon < \frac{1}{4}$.

In [Bri07, Section 6], it was shown that a family of subsets

$$\left\{ B_\epsilon(\sigma) \subset \text{Stab}(\mathcal{D}) \mid \sigma \in \text{Stab}(\mathcal{D}), 0 < \epsilon < \frac{1}{4} \right\}$$

becomes an open basis of a topology on $\text{Stab}(\mathcal{D})$. In [Bri07], Bridgeland showed a crucial theorem.

**Theorem 2.14** ([Bri07, Theorem 1.2]). *The projection map of central charges

$$\pi : \text{Stab}(\mathcal{D}) \rightarrow \text{Hom}_\mathbb{Z}(K(\mathcal{D}), \mathbb{C}), \ (Z, P) \mapsto Z$$

is a local isomorphism of topological spaces. In particular, $\pi$ induces a complex structure on $\text{Stab}(\mathcal{D})$.***
3. Mass growth of objects and categorical entropy

3.1. Mass with a parameter and complexity. In this section, we introduce the mass growth of objects and show fundamental properties of it.

Definition 3.1 ([DHKK14], Section 4.5). Take a stability condition \( \sigma = (Z, P) \) on \( \mathcal{D} \). Let \( E \in \mathcal{D} \) be a nonzero object with semistable factors \( A_1, \ldots, A_m \) of phases \( \phi_1 > \cdots > \phi_m \). The mass of \( E \) with a parameter \( t \in \mathbb{R} \) is the function \( m_{\sigma,t}(E): \mathbb{R} \to \mathbb{R}_{>0} \) defined by

\[
m_{\sigma,t}(E) := \sum_{i=1}^{m} |Z(A_i)| e^{\phi_i t}.
\]

When \( t = 0 \), \( m_{\sigma,0}(E) \) is called the mass of \( E \) and simply written as \( m_{\sigma}(E) := m_{\sigma,0}(E) \). As a convention, set \( m_{\sigma,t}(E) := 0 \) if \( E \cong 0 \).

In the following, if \( \sigma \) is clear in the context, we often drop it from the notation and write \( m_t(E) \). Similar to the growth rate of complexity of a generator with respect to endofunctors, we consider the growth rate of mass of objects.

Definition 3.2 ([DHKK14], Section 4.5). Let \( \sigma \) be a stability condition on \( \mathcal{D} \) and \( F: \mathcal{D} \to \mathcal{D} \) be an endofunctor. The mass growth with respect to \( F \) is the function \( h_{\sigma,t}(F): \mathbb{R} \to [-\infty, \infty] \) defined by

\[
h_{\sigma,t}(F) := \sup_{E \in \mathcal{D}} \left\{ \limsup_{n \to \infty} \frac{1}{n} \log(m_{\sigma,t}(F^n E)) \right\}.
\]

In the rest of this section, we study fundamental properties of \( h_{\sigma,t}(F) \). The triangle inequality for \( m_{\sigma,t} \) plays an important role.

Proposition 3.3. For objects \( D, E, F \in \mathcal{D} \), if there is an exact triangle \( D \to E \to F \to D[1] \), then

\[
m_{\sigma,t}(E) \leq m_{\sigma,t}(D) + m_{\sigma,t}(F).
\]

The proof of Proposition 3.3 is given in Section 3.2.

Proposition 3.4. Let \( \sigma \) be a stability condition on \( \mathcal{D} \). Then

\[
m_{\sigma,t}(E) \leq m_{\sigma,t}(D) \delta_t(D, E)
\]

for any objects \( D, E \in \mathcal{D} \).

Proof. It is sufficient to show the case \( E \in \langle D \rangle \). Then by the definition of complexity \( \delta_t(D, E) \), for any \( \epsilon > 0 \) there is a sequence of exact triangles

\[
0 \to A_1 \to A_2 \to \cdots \to A_{k-1} \to E \oplus E'
\]

with

\[
D[n_1] \to D[n_2] \to D[n_k]
\]
such that
\[
\sum_{i=1}^{k} e^{\epsilon n t} < \delta_t(D, E) + \epsilon.
\]

Note that \(m_{\sigma, t}\) satisfies \(m_{\sigma, t}(D[n]) = m_{\sigma, t}(D) \cdot e^{\epsilon n t}\) for \(D \in \mathcal{D}\) and \(n \in \mathbb{Z}\). By using the inequality in Proposition \([3.3]\) repeatedly, we have
\[
m_{\sigma, t}(E) \leq m_{\sigma, t}(E \oplus E') \leq \sum_{i=1}^{k} m_{\sigma, t}(D[n_i]) \leq m_{\sigma, t}(D) \cdot \left(\sum_{i=1}^{k} e^{\epsilon n i t}\right)
\]
\[
\leq m_{\sigma, t}(D)\delta_t(D, E) + \epsilon \cdot m_{\sigma, t}(D)
\]
for any \(\epsilon > 0\). This implies the result. \(\square\)

Now we show fundamental properties of the mass growth.

**Theorem 3.5.** Let \(F: \mathcal{D} \to \mathcal{D}\) be an endofunctor and \(\sigma\) be a stability condition on \(\mathcal{D}\). Assume that \(\mathcal{D}\) has a split-generator \(G \in \mathcal{D}\). Then the mass growth \(h_{\sigma, t}(F)\) satisfies the followings.

1. \(h_{\sigma, t}(F)\) is given by
   \[
   h_{\sigma, t}(F) = \limsup_{n \to \infty} \frac{1}{n} \log(m_{\sigma, t}(F^n G)).
   \]

2. We have an inequality
   \[
   h_{\sigma, t}(F) \leq h_t(F) < \infty
   \]
   where \(h_t(F)\) is the entropy of \(F\) (see Definition \([2.4]\)).

**Proof.** By Proposition \([2.2](3)\) and Proposition \([3.4]\) we have
\[
m_t(F^n E) \leq m_t(F^n G)\delta_t(F^n G, F^n E) \leq m_t(F^n G)\delta_t(G, E)
\]
for any object \(E \in \mathcal{D}\). Hence
\[
\limsup_{n \to \infty} \frac{1}{n} \log m_t(F^n E) \leq \limsup_{n \to \infty} \frac{1}{n} \log m_t(F^n G)
\]
and this inequality implies (1). Again by Proposition \([3.4]\) we have
\[
m_t(F^n G) \leq m_t(G)\delta_t(G, F^n G).
\]
Hence
\[
\limsup_{n \to \infty} \frac{1}{n} \log m_t(F^n G) \leq \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, F^n G)
\]
and this inequality implies (2). \(\square\)
3.2. Triangle inequality for mass with a parameter. We prove Proposition 3.3. Recall the notation \( \mathbb{H} = \{ re^{i\phi} \mid r > 0, \phi \in (0,1] \} \). For a complex number \( z \in \mathbb{H} \), define the function of \( t \in \mathbb{R} \) by

\[
g_t(z) := |z| e^{\phi(z)t}
\]

where \( \phi(z) \) is the phase of \( z \) given by \( \phi(z) := (1/\pi) \arg z \in (0,1] \). We start to show the triangle inequality for \( g_t(z) \).

**Lemma 3.6.** For \( z_1, z_2 \in \mathbb{H} \), an inequality

\[
g_t(z_1 + z_2) \leq g_t(z_1) + g_t(z_2)
\]

holds.

![Figure 2](image-url)

**Figure 2.** Triangle consisting of vertices 0, \( z_1 \), \( z_1 + z_2 \).

**Proof.** Set \( \phi_1 := \phi(z_1), \phi_2 := \phi(z_2) \) and \( \phi_3 := \phi(z_1 + z_2) \). If \( \phi_1 = \phi_2 \), the result is trivial. We consider the case \( \phi_1 > \phi_2 \). By applying the law of sine for the triangle consisting of vertices 0, \( z_1 \), \( z_1 + z_2 \) (see Figure 2), we obtain

\[
|z_1 + z_2| = d \sin(\pi a + \pi b), \quad |z_1| = d \sin \pi b, \quad |z_2| = d \sin \pi a
\]

where \( a = \phi_1 - \phi_3, b = \phi_3 - \phi_2 \) and \( d \) is the diameter of the excircle of the triangle. By inputting these parameters, the inequality \( g_t(z_1 + z_2) \leq g_t(z_1) + g_t(z_2) \) becomes

\[
\sin(\pi a + \pi b) \leq e^{at} \sin \pi b + e^{-bt} \sin \pi a
\]

where \( 0 < a, b < 1 \). Dividing by \( \sin \pi a \sin \pi b \) and applying the addition formula, we have

\[
\frac{e^{at} - \cos \pi a}{\sin \pi a} + \frac{e^{-bt} - \cos \pi b}{\sin \pi b} \geq 0.
\]

After setting \( c = -b \), the above inequality is equivalent to \( f(a) \geq f(c) \) for \(-1 < c < 0 < a < 1\) where

\[
f(x) = \frac{e^{xt} - \cos \pi x}{\sin \pi x}.
\]
It is easy to check that \( f(x) \) is increasing in the intervals \((-1,0)\) and \((0,1)\), and the limit of \( f(x) \) at the zero is given by \( \lim_{x \to \pm 0} f(x) = \frac{4}{\pi} \).

The triangle inequality for \( g_t(z) \) implies the following.

**Lemma 3.7.** Let \( z_1, \ldots, z_k \) and \( w_1, \ldots, w_l \) be complex numbers in \( \mathbb{H} \) with \( z_k = w_l \) and set \( z_0 = w_0 = 0 \). If they satisfy the following conditions (see the left of Figure 3):

(a) \( \phi(z_i - z_{i-1}) > \phi(z_{i+1} - z_i) \) and \( \phi(w_j - w_{j-1}) > \phi(w_{j+1} - w_j) \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, l \),

(b) the polygon \( w_0w_1w_2 \ldots w_lw_0 \) contains the polygon \( z_0z_1z_2 \ldots z_kz_0 \),

then

\[
\sum_{i=1}^{k} g_t(z_i - z_{i-1}) \leq \sum_{j=1}^{l} g_t(w_j - w_{j-1}).
\]

**Figure 3.** Polygons and a triangulation of the encircled domain.

**Proof.** By the condition \((b)\), there is a unique domain encircled by two paths \( z_0z_1z_2 \ldots z_k \) and \( w_0w_1w_2 \ldots w_l \). By the convexity condition \((a)\), we can triangulate this domain as in the right of Figure 3. Applying the triangle inequality for \( g_t(z) \) (Lemma 3.2) repeatedly, we obtain the result.

**Lemma 3.8.** Let \( \sigma = (Z, \mathcal{P}) \) be a stability condition and \( \mathcal{H} = \mathcal{P}((0,1]) \) be the associated heart. If there is a short exact sequence

\[
0 \to A \to B \to C \to 0
\]

in \( \mathcal{H} \) and \( C \in \mathcal{P}(1) \), then

\[
m_t(A) \leq m_t(B) + e^{-t}m_t(C).
\]

**Proof.** Let

\[
0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_l = A
\]

\[
0 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_{k-1} = B
\]
be Harder-Narasimhan filtrations of $A$ and $B$. Set $z_i := Z(A_i)$ for $i = 0, 1, \ldots, k$, $w_j := Z(B_j)$ for $j = 0, 1, \ldots, l - 1$ and $w_l := Z(B) - Z(C) = Z(A)$. Then by definition of the Harder-Narasimhan filtration, these complex numbers satisfy the condition $(a)$ in Lemma 3.7. Consider the Harder-Narasimhan polygons $HN^Z(A)$ and $HN^Z(B)$ (see Definition 2.12). By Proposition 2.13 complex numbers $z_0, z_1, \ldots, z_k$ and $w_0, w_1, \ldots, w_{l-1}$ are extremal points of $HN^Z(A)$ and $HN^Z(B)$ respectively. Thus the intersection of $HN^Z(A)$ and the left of the line through 0 and $z_k = Z(A)$ is the polygon $z_0 z_1 z_2 \ldots z_k z_0$ and the intersection of $HN^Z(B)$ and the left of the line through 0 and $w_l = Z(A)$ is the polygon $w_0 w_1 w_2 \ldots w_l w_0$. Since $HN^Z(A) \subset HN^Z(B)$, the polygon $w_0 w_1 w_2 \ldots w_l w_0$ contains the polygon $z_0 z_1 z_2 \ldots z_k z_0$ and this implies the condition $(b)$ in Lemma 3.7. Since

$$m_t(A) = \sum_{i=1}^{k} g_t(z_i - z_{i-1}), \quad m_t(B) = \sum_{j=1}^{l-1} g_t(w_j - w_{j-1}), \quad e^{-t} m_t(C) = g_t(w_l - w_{l-1}),$$

applying Lemma 3.7 we obtain the result. \hfill \Box

**Proof of Proposition 3.3.** Assume that there is a exact triangle $D \rightarrow E \rightarrow F \rightarrow D[1]$. From a Harder-Narasimhan filtration of $E$, we can construct the dual Harder-Narasimhan filtration

$$D = D_m \xrightarrow{A_m} D_{m-1} \xrightarrow{A_{m-1}} \cdots \xrightarrow{A_1} D_1 \xrightarrow{A_0} D_0 = 0$$

with $A_i \in \mathcal{P}(\phi_i)$ and $\phi_m > \phi_{m-1} > \cdots > \phi_1$. Applying the octahedra axiom for the above sequence together with the exact triangle $D \rightarrow E \rightarrow F \rightarrow D[1]$, we can construct a sequence of exact triangles

$$E = E_m \xrightarrow{A_m} E_{m-1} \xrightarrow{A_{m-1}} \cdots \xrightarrow{A_1} E_1 \xrightarrow{A_0} F.$$ 

Since $m_t(D) = \sum_{i=1}^{m} |Z(A_i)| e^{t\phi_i}$ and $A_i$ is semistable, the problem is reduced to the case that $D$ is semistable. Without loss of generality we can assume $D \in \mathcal{P}(1)$. By taking the cohomology associated with the heart $\mathcal{H} = \mathcal{P}((0,1])$ (see Section 2.2), we have a long exact sequence

$$0 \rightarrow H^{-1}(E) \rightarrow H^{-1}(F) \rightarrow H^0(D) \rightarrow H^0(E) \rightarrow H^0(F) \rightarrow 0$$

and isomorphisms $H^i(E) \cong H^i(F)$ for $i \neq -1, 0$ in $\mathcal{H}$. If $1 > \phi^+(H^0(E))$, then the map $f : H^0(D) \rightarrow H^0(E)$ is zero. Hence the long exact sequence splits into
Proposition 3.9. Let \( \sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{D}) \) be a stability condition on \( \mathcal{D} \). If \( \tau = (W, \mathcal{Q}) \in B_\epsilon(\sigma) \) with small enough \( \epsilon > 0 \), then there are functions \( C_1, C_2 : \mathbb{R} \to \mathbb{R}_{>0} \) such that
\[
C_1(t) \, m_{\tau,t}(E) < m_{\sigma,t}(E) < C_2(t) \, m_{\tau,t}(E)
\]
for all \( 0 \neq E \in \mathcal{D} \).

**Proof.** We use the argument similar to the proof of [Bri07, Proposition 8.1]. It is sufficient to show that for \( \tau = (W, \mathcal{Q}) \in B_\epsilon(\sigma) \) with small enough \( \epsilon > 0 \), there is some constants \( C > 1 \) and \( r > 0 \) such that
\[
m_{\tau,t}(E) < Ce^{rt}m_{\sigma,t}(E)
\]
for any nonzero object \( E \in \mathcal{D} \). We first consider the case \( \phi^+_\sigma(E) - \phi^-_\sigma(E) < \eta \) for \( 0 < \eta < \frac{1}{4} \). In this case, it was shown in the proof of [Bri07, Proposition 8.1] that there is a constant \( C(\epsilon, \eta) > 1 \) such that
\[
m_\tau(E) \leq C(\epsilon, \eta) \, m_\sigma(E)
\]
and \( C(\epsilon, \eta) \to 1 \) as \( \max\{\epsilon, \eta\} \to 0 \). Note that \( \phi^+_\sigma(E) - \phi^-_\sigma(E) < \eta \) implies \( \phi^+_\sigma(E) \in (\psi, \psi + \eta) \) for some \( \psi \in \mathbb{R} \). Since \( d(\mathcal{P}, \mathcal{Q}) < \epsilon \), we have \( \phi^+_\sigma(E) \in (\psi - \epsilon, \psi + \epsilon + \eta) \).

By definition of \( m_{\sigma,t}(E) \) and \( m_{\tau,t}(E) \), it follows that
\[
m_{\tau,t}(E) \leq m_\tau(E) \exp\left(\phi^+_\sigma(E)|t|\right), \quad m_\sigma(E) \exp\left(\phi^-_\sigma(E)|t|\right) \leq m_{\sigma,t}(E).
\]
Since $\psi < \phi_\sigma(E)$ and $\phi_\tau(E) < \psi + \epsilon + \eta$, we have an inequality

$$m_{\tau,t}(E) \leq C(\epsilon, \eta)e^{(\epsilon + \eta)|t|} m_{\sigma,t}(E).$$

Next we consider a general nonzero object $E$. Take a real number $\phi$ and a positive integer $n$. For $k \in \mathbb{Z}$, define intervals

$$I_k := [\phi + k\epsilon, \phi + (k + 1)n\epsilon), \quad J_k := [\phi + k\epsilon - \epsilon, \phi + (k + 1)n\epsilon + \epsilon)$$

and let $\alpha_k$ and $\beta_k$ be the truncation functors projecting into the subcategories $\mathcal{Q}(I_k)$ and $\mathcal{P}(J_k)$ respectively. Again by the argument in the proof of [Bri07, Proposition 8.1], for small enough $n\epsilon$, we have

$$m_{\tau,t}(E) = \sum_k m_{\tau,t}(\alpha_k(E)) \leq \sum_k m_{\tau,t}(\beta_k(E)) < C(n+2)\epsilon e^{(n+3)\epsilon|t|} \sum_k m_{\sigma,t}(\beta_k(E)).$$

On the other hand, we can choose $\phi$ so that

$$\sum_k m_{\sigma,t}(\beta_k(E)) \leq \frac{n+2}{n} m_{\sigma,t}(E).$$

By taking the limits $\epsilon \to 0$ and $n \to \infty$ in keeping with $n\epsilon \to 0$, the result follows.

□

From Proposition 3.9 we immediately have the following.

**Proposition 3.10.** Let $F: \mathcal{D} \to \mathcal{D}$ be an endofunctor, and $\sigma$ and $\tau$ be stability conditions on $\mathcal{D}$. If $\sigma$ and $\tau$ lie in the same connected component in $\text{Stab}(\mathcal{D})$, then

$$h_{\sigma,t}(F) = h_{\tau,t}(F).$$

**Proof.** Let $\sigma, \tau \in \text{Stab}(\mathcal{D})$ be stability conditions such that $\tau \in B_c(\sigma)$ for small enough $\epsilon > 0$. Then Proposition 3.9 implies $h_{\sigma,t}(F) = h_{\tau,t}(F)$. Thus $h_{\sigma,t}(F)$ is locally constant on the topological space $\text{Stab}(\mathcal{D})$.

□

3.4. **Lower bound of the mass growth by the spectral radius.** Let $F: \mathcal{D} \to \mathcal{D}$ be an endofunctor. Since $F$ preserves exact triangles in $\mathcal{D}$, $F$ induces a linear map

$$[F]: K(\mathcal{D}) \to K(\mathcal{D}).$$

The **spectral radius** of $[F]$ is defined by

$$\rho([F]) := \max\{|\lambda| | \lambda \text{ is an eigenvalue of } [F]\}.$$ 

**Proposition 3.11.** For any stability condition $\sigma \in \text{Stab}(\mathcal{D})$, we have an inequality

$$\log \rho([F]) \leq h_{\sigma,0}(F).$$

**Proof.** Set $K(\mathcal{D})_\mathbb{C} := K(\mathcal{D}) \otimes \mathbb{C}$. Let $A_1, \ldots, A_n \in \mathcal{D}$ be objects whose classes $[A_1], \ldots, [A_n]$ form a basis of $K(\mathcal{D})_\mathbb{C}$. Take an eigenvector

$$v = \sum_{i=1}^n a_i[A_i] \in K(\mathcal{D})_\mathbb{C} \quad (a_i \in \mathbb{C})$$
for the eigenvalue $\lambda \in \mathbb{C}$ of $[F]$ satisfying $|\lambda| = \rho([F])$. First we consider the case that a stability condition $\sigma = (Z, \mathcal{P})$ satisfies $Z(v) \neq 0$. Note that the mass satisfies $|Z(E)| \leq m_\sigma(E)$ and $m_\sigma(E \oplus E') = m_\sigma(E) + m_\sigma(E')$ for any objects $E, E' \in \mathcal{D}$. Then

$$|\lambda|^k |Z(v)| = |Z(\lambda^k v)| = |Z([F]^k v)| \leq \sum_{i=1}^n |a_i| \cdot |Z(F^k A_i)|$$

$$\leq \sum_{i=1}^n l_i m_\sigma(F^k A_i) = m_\sigma \left( F^k \left( \bigoplus_{i=1}^n A_i^{\otimes l_i} \right) \right)$$

where $l_1, \ldots, l_n$ are positive integers satisfying $|a_i| \leq l_i$. Since $|Z(v)| > 0$, we have

$$\log \rho([F]) = \lim_{k \to \infty} \frac{1}{k} \log(|\lambda|^k |Z(v)|) \leq \limsup_{k \to \infty} \frac{1}{k} \log(m_\sigma(F^k E)) \leq h_{\sigma,0}(F)$$

where $E = \bigoplus_{i=1}^n A_i^{\otimes l_i}$. Next consider the case $Z(v) = 0$. Then by Theorem 2.14 we can deform $\sigma = (Z, \mathcal{P})$ to $\sigma' = (Z', \mathcal{P}')$ so that $Z'(v) \neq 0$. Again we have $\log \rho([F]) \leq h_{\sigma',0}(F)$ and Proposition 3.10 implies $h_{\sigma,0}(F) = h_{\sigma',0}(F)$. □

3.5. Mass growth via algebraic stability conditions. If a triangulated category has an algebraic stability condition, then we can show that the mass growth coincides with the entropy. Let $\mathcal{H} \subset \mathcal{D}$ be an algebraic heart with simple objects $S_1, \ldots, S_n$. Then the Grothendieck group is given by

$$K(\mathcal{D}) \cong \bigoplus_{i=1}^n \mathbb{Z}[S_i].$$

The class of an object $E \in \mathcal{H}$ is written as $[E] = \sum_{i=1}^n d_i [S_i]$ with $d_i \in \mathbb{Z}_{\geq 0}$. We define the dimension of $E$ by $\dim E := \sum_{i=1}^n d_i \in \mathbb{Z}_{\geq 0}$. Then the dimension gives the upper bound of the complexity for objects in $\mathcal{H}$.

**Lemma 3.12.** Let $\mathcal{H} \subset \mathcal{D}$ be an algebraic heart with simple objects $S_1, \ldots, S_n$. Then for the split-generator $G := \bigoplus_{i=1}^n S_i$, we have an inequality

$$\delta_0(G, E) \leq \dim E.$$ 

**Proof.** Since $\mathcal{H}$ is a finite length abelian category, for any object $E \in \mathcal{H}$ there is a Jordan-Hölder filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l = E$$

of length $l = \dim E$ with $E_i / E_{i-1} \in \{S_1, \ldots, S_n\}$. As a result, we can construct a filtration

$$0 = E'_0 \subset E'_1 \subset E'_2 \subset \cdots \subset E'_l = E \oplus E'$$

of length $l = \dim E$ with $E'_i / E'_{i-1} = G$ and this implies the result. □
Following Section 2.4, we construct the special algebraic stability condition. For an algebraic heart $H \subset D$ with simple objects $S_1, \ldots, S_n$, define the central charge 

$$Z_0: K(D) \cong \bigoplus_{i=1}^{n} Z[S_i] \to \mathbb{C}$$

by $Z_0(S_i) := i$. Then the pair $\sigma_0 := (Z_0, H)$ becomes an algebraic stability condition. By definition, the mass of an object $E \in H$ is given by 

$$m_{\sigma_0,t}(E) = \dim E \cdot e^{\frac{1}{2}t}.$$ 

Together with Lemma 3.12, we obtain the following inequality.

**Proposition 3.13.** For the generator $G = \bigoplus_{i=1}^{n} S_i$ and the algebraic stability condition $\sigma_0 = (Z_0, H)$, we have an inequality

$$\delta_t(G, E) \leq e^{-\frac{1}{2}t} m_{\sigma_0,t}(E).$$

**Proof.** For an object $E \in D$, we denote by $H^k(E) \in H$ the cohomology associated with the heart $H$ (see Section 2.2). By using Lemma 2.3 and Lemma 3.12, we have

$$\delta_t(G, E) \leq \sum_k \delta_t(G, H^k(E)) e^{-kt} \leq \sum_k \dim H^k(E) e^{-kt}.$$ 

On the other hand, the definition of $m_{\sigma_0,t}(E)$ implies

$$m_{\sigma_0,t}(E) = \sum_k m_{\sigma_0,t}(H^k(E)) e^{-kt} = \sum_k \dim H^k(E) e^{\frac{1}{2}t} e^{-kt}.$$ 

Thus we obtain the result. \qed

We show the main result of this section.

**Theorem 3.14.** Let $G \in D$ be a split-generator and $F: D \to D$ be an endofunctor. If a connected component $\text{Stab}^\circ(D) \subset \text{Stab}(D)$ contains an algebraic stability condition, then for any $\sigma \in \text{Stab}^\circ(D)$ we have

$$h_t(F) = h_{\sigma,t}(F) = \lim_{n \to \infty} \frac{1}{n} \log(m_{\sigma,t}(F^n G)).$$

**Proof.** Let $H$ be an algebraic heart with simple objects $S_1, \ldots, S_n$ and set $G = \bigoplus_{i=1}^{n} S_i$. Consider the special algebraic stability condition $\sigma_0 = (Z_0, H)$ which is constructed in this section. By Proposition 3.10, it is sufficient to show that

$$h_{\sigma_0,t}(F) = h_t(F).$$

By [DHKK14] Lemma 2.6, the limit

$$h_t(F) = \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, F^n G)$$
converges. On the other hand, by Proposition 3.4 and Proposition 3.13, we have
\[ e^{\frac{1}{2}t} \delta_t(G, F^n G) \leq m_{\sigma_0, t}(F^n G) \leq m_{\sigma_0, t}(G) \delta_t(G, F^n G). \]

Hence the limit
\[ \lim_{n \to \infty} \frac{1}{n} \log(m_{\sigma_0, t}(F^n G)) \]
converges and coincides with \( h_t(F) \). \qed

4. Applications

4.1. Entropy on the derived categories of non-positive dg-algebras. In this section, we discuss the entropy of endofunctors on the derived categories of non-positive dg-algebras. In this case, we can describe the entropy as the growth rate of the Hilbert-Poincaré polynomial of a generator.

Let \( A = \oplus_{k \in \mathbb{Z}} A^k \) be a dg-algebra over \( \mathbb{K} \) satisfying the following conditions:

(a) \( H^k(A) = 0 \) for \( i > 0 \),
(b) \( H^0(A) \) is a finite dimensional algebra over \( \mathbb{K} \).

Let \( \mathcal{D}(A) \) be the derived category of dg-modules over \( A \) and \( \mathcal{D}_{fd}(A) \) be the full subcategory of \( \mathcal{D}(A) \) consisting of dg-modules with finite dimensional total cohomology, i.e.

\[ \mathcal{D}_{fd}(A) := \left\{ M \in \mathcal{D}(A) \middle| \sum_k \dim H^k(M) < \infty \right\}. \]

Define the full subcategory \( \mathcal{F} \subset \mathcal{D}_{fd}(A) \) by

\[ \mathcal{F} := \left\{ M \in \mathcal{D}_{fd}(A) \middle| H^k(M) = 0 \text{ for } k > 0 \right\}. \]

Then \( \mathcal{F} \) becomes a bounded t-structure on \( \mathcal{D}_{fd}(A) \). The heart \( \mathcal{H}_s \) of \( \mathcal{F} \) is called the standard heart. It is known that the 0-th cohomology functor \( H^0: \mathcal{D}_{fd}(A) \to \text{mod-} H^0(A) \) induces an equivalence of abelian categories:

\[ H^0: \mathcal{H}_s \cong \text{mod-} H^0(A) \]

where \( \text{mod-} H^0(A) \) is an abelian category of finite dimensional \( H^0(A) \)-modules. (For details, see [Ami09 Section 2].) Since \( H^0(A) \) is a finite dimensional algebra, \( \mathcal{H}_s \) becomes an algebraic heart. Thus we can construct an algebraic stability condition on \( \mathcal{D}_{fd}(A) \). Applying Theorem 3.14, we obtain the following.

**Proposition 4.1.** Let \( \text{Stab}^n(\mathcal{D}_{fd}(A)) \) be the connected component which contains stability conditions with the standard heart \( \mathcal{H}_s \). Then for any stability conditions \( \sigma \in \text{Stab}^n(\mathcal{D}_{fd}(A)) \) and an endofunctor \( F: \mathcal{D}_{fd}(A) \to \mathcal{D}_{fd}(A) \), we have

\[ h_t(F) = h_{\sigma, t}(F). \]
Next we describe $h_t(F)$ by using the Hillbert-Poincaré polynomial.

**Definition 4.2.** For a dg-module $M \in \mathcal{D}_{fd}(A)$, define the Hillbert-Poincaré polynomial of $M$ by

$$P_t(M) := \sum_{k \in \mathbb{Z}} \dim H^k(M) e^{-kt} \in \mathbb{Z}[e^t, e^{-t}].$$

As in Section 3.5, we construct the special stability condition $\sigma_0 = (Z_0, \mathcal{H}_s)$ by using the standard heart $\mathcal{H}_s$. Then by definition of $\sigma_0$, we have

$$m_{\sigma_0,t}(M) = e^{\frac{1}{2}t}P_t(M)$$

for any dg-module $M \in \mathcal{D}_{fd}(A)$. As a result, the entropy is described as follows.

**Proposition 4.3.** Let $F: \mathcal{D}_{fd}(A) \to \mathcal{D}_{fd}(A)$ be an endofunctor and $G \in \mathcal{D}_{fd}(A)$ be a split-generator. Then the entropy of $F$ is given by

$$h_t(F) = \lim_{n \to \infty} \frac{1}{n} \log P_t(F^nG).$$

### 4.2. Entropy of spherical twists.

In this section, we compute the entropy of Seidel-Thomas spherical twists on the derived categories of Calabi-Yau algebras associated with acyclic quivers. Let $Q$ be an acyclic quiver with vertices $\{1, \ldots, n\}$ and $\Gamma_N^Q$ be the Ginzburg $N$-Calabi-Yau dg-algebra associated with $Q$ for $N \geq 2$. (For the definition of $\Gamma_N^Q$, see [Gin, Section 4.2] or [Kel11, Section 6.2].) Set $\mathcal{D}_N^Q := \mathcal{D}_{fd}(\Gamma_N^Q)$. By [Kel11] Theorem 6.3, the category $\mathcal{D}_N^Q$ becomes a $N$-Calabi-Yau category, i.e. there is a natural isomorphism

$$\text{Hom}(E, F) \cong \text{Hom}(F, E[N])^*$$

for $E, F \in \mathcal{D}_N^Q$. (The notation $V^*$ is the dual space of a $\mathbb{K}$-linear space $V$.) In the Calabi-Yau category, we can consider a certain class of objects, called spherical objects. An object $S \in \mathcal{D}_N^Q$ is called $N$-spherical if

$$\text{Hom}(S, S[i]) = \begin{cases} \mathbb{K} & \text{if } i = 0, N \\ 0 & \text{otherwise}. \end{cases}$$

For a spherical object $S \in \mathcal{D}_N^Q$, Seidel-Thomas [ST01] defined an exact autoequivalence $\Phi_S \in \text{Aut}(\mathcal{D}_N^Q)$, called a spherical twist, by the exact triangle

$$\text{Hom}^*(S, E) \otimes S \to E \to \Phi_S(E)$$

for any object $E \in \mathcal{D}_N^Q$. The inverse functor $\Phi_S^{-1} \in \text{Aut}(\mathcal{D}_N^Q)$ is given by

$$\Phi_S^{-1}(E) \to E \to S \otimes \text{Hom}^*(E, S)^*.$$
standard algebraic heart $\mathcal{H}_s \subset \mathcal{D}^N_Q$ generated by simple $\Gamma_N Q$-modules $S_1, \ldots, S_n$ corresponding to vertices $\{1, \ldots, n\}$ of $Q$. In addition, these objects $S_1, \ldots, S_n$ become $N$-spherical by [Kel11, Lemma 4.4]. Thus we can define spherical twists $\Phi_{S_1}, \ldots, \Phi_{S_n} \in \text{Aut}(\mathcal{D}^N_Q)$. In the following, we compute the entropy of spherical twists $\Phi_{S_1}, \ldots, \Phi_{S_n}$ by using Proposition 4.3. For simplicity, write $\Phi_i := \Phi_{S_i}$.

**Lemma 4.4.** For a spherical twist $\Phi_i \in \text{Aut}(\mathcal{D}^N_Q)$ and a spherical object $S_j \in \mathcal{D}^N_Q$, the Hilbert-Poincaré polynomial of $\Phi_i^k S_j (k \geq 0)$ is given by

$$P_t(\Phi_i^k S_j) = \begin{cases} e^{k(1-N)t} & \text{if } i = j \\ 1 + q_{ij} \sum_{l=0}^{k-1} e^{l(1-N)t} & \text{if } q_{ij} > 0 \\ 1 + q_{ji} e^{(2-N)t} \sum_{l=0}^{k-1} e^{l(1-N)t} & \text{if } q_{ji} > 0 \\ 1 & \text{otherwise} \end{cases}$$

where $q_{ij}$ is the number of arrows from $i$ to $j$ in $Q$.

**Proof.** First we note that

$$\dim \text{Hom}(S_i, S_j[m]) = \begin{cases} 1 & \text{if } i = j \text{ and } m = 0, N \\ q_{ij} & \text{if } q_{ij} > 0 \text{ and } m = 1 \\ q_{ji} & \text{if } q_{ji} > 0 \text{ and } m = N - 1 \\ 0 & \text{otherwise} \end{cases}.$$ 

By the definition of spherical twists, it is easy to see that $\Phi_i^k S_i = S_i[k(1-N)]$ and hence $P_t(\Phi_i^k S_i) = e^{k(1-N)t}$. If $i \neq j$ and $q_{ij} = q_{ji} = 0$, then $\Phi_i^k S_j = S_j$ and hence $P_t(\Phi_i^k S_j) = 1$. Consider the case $q_{ij} > 0$. Since

$$\text{Hom}^\bullet(S_i, S_j) \otimes S_i = \bigoplus_{m \in \mathbb{Z}} \text{Hom}(S_i[m], S_j) \otimes S_i[m] \cong S_i^{\oplus q_{ij}}[-1],$$

we have an exact triangle

$$S_j \to \Phi_i S_j \to S_i^{\oplus q_{ij}} \to S_j[1].$$

Applying the spherical twist $\Phi_i$ for the above exact triangle repeatedly, we obtain a sequence of exact triangles

$$S_j \to \Phi_i S_j \to \Phi_i^2 S_j \to \ldots \to \Phi_i^{k-1} S_j \to \Phi_i^k S_j.$$ 

This implies the result in the case $q_{ij} > 0$ and the similar argument gives the result in the case $q_{ji} > 0$. \qed
Proposition 4.5. Let $Q$ be a connected acyclic quiver and assume that $Q$ is not a quiver with one vertex and no arrows. Then the entropy of spherical twists $\Phi_1, \ldots, \Phi_n$ is given by

$$h_t(\Phi_i) = \begin{cases} 0 & \text{if } t \geq 0 \\ (1 - N)t & \text{if } t < 0. \end{cases}$$

**Proof.** We use the generator $G = \oplus_{j=1}^n S_j$. Then $P_t(\Phi_i^k G) = \sum_{j=1}^n P_t(\Phi_i^k S_j)$. Recall from Proposition 4.4 that

$$P_t(\Phi_i^k S_j) = 1 + q_{ij} \sum_{l=0}^{k-1} e^{l(1-N)t} = 1 + q_{ij} \frac{1 - e^{k(1-N)t}}{1 - e^{(1-N)t}}$$

in the case $q_{ij} > 0$ and

$$P_t(\Phi_i^k S_j) = 1 + q_{ji} e^{(2-N)t} \sum_{l=0}^{k-1} e^{l(1-N)t} = 1 + q_{ij} e^{(2-N)t} \frac{1 - e^{k(1-N)t}}{1 - e^{(1-N)t}}$$

in the case $q_{ji} > 0$. First we consider the case $t > 0$. Then the above two terms converge to some positive real numbers as $k \to \infty$ since $(1-N)t < 0$. By the assumption of $Q$, the sum $\sum_{j=1}^n P_t(\Phi_i^k S_j)$ contains at least one of the above two. As a result, $\sum_{j=1}^n P_t(\Phi_i^k S_j)$ also converges to some positive real number as $k \to \infty$. Thus

$$h_t(\Phi_i) = \lim_{k \to \infty} \frac{1}{k} \log P_t(\Phi_i^k G) = 0$$

when $t > 0$. Next consider the case $t < 0$. Similarly we can show that $e^{-k(1-N)t} \sum_{j=1}^n P_t(\Phi_i^k S_j)$ converges to some positive real number as $k \to \infty$ since $-(1-N)t < 0$. Thus

$$h_t(\Phi_i) = \lim_{k \to \infty} \frac{1}{k} \log P_t(\Phi_i^k G) = \lim_{k \to \infty} \frac{1}{k} \log e^{k(1-N)t} e^{-k(1-N)t} P_t(\Phi_i^k G)$$

$$= (1 - N)t + \lim_{k \to \infty} \frac{1}{k} \log e^{-k(1-N)t} P_t(\Phi_i^k G) = (1 - N)t$$

when $t < 0$. Finally we can easily check that $h_t(\Phi_i) = 0$ when $t = 0$. $\square$

**Remark 4.6.** The subgroup of autoequivalences generated by spherical twists

$$\text{Sph}(\mathcal{D}^N_Q) := \langle \Phi_1, \ldots, \Phi_n \rangle \subset \text{Aut}(\mathcal{D}^N_Q)$$

is called the Seidel-Thomas braid group. Here we only computed the entropy of generators $\Phi_1, \ldots, \Phi_n$. It is important problem to compute the entropy $h_t(\Phi)$ for a general element $\Phi \in \text{Sph}(\mathcal{D}^N_Q)$. 
4.3. Lower bound of the entropy on the derived categories of surfaces.

Let \( X \) be a smooth projective variety over \( \mathbb{C} \) and denote by \( \mathbf{D}^b(X) \) the bounded derived category of coherent sheaves on \( X \). Define the Euler form \( \chi : \mathbf{K}(\mathbf{D}^b(X)) \times \mathbf{K}(\mathbf{D}^b(X)) \to \mathbb{Z} \) by

\[
\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \text{Hom}_{\mathbf{D}^b(X)}(E, F[i]).
\]

The numerical Grothendieck group \( N(X) \) is the quotient of \( \mathbf{K}(\mathbf{D}^b(X)) \) by the radical of the Euler form \( \chi \). Let \( \text{End}^{FM}(\mathbf{D}^b(X)) \) be the semi-group consisting of Fourier-Mukai type endofunctors. Since these endofunctors preserve the radical of \( \chi \), they induce linear maps on \( N(X) \), i.e. the semi-group homomorphism

\[
\text{End}^{FM}(\mathbf{D}^b(X)) \to \text{End}(N(X)), \quad F \mapsto [F]
\]

is well-defined (see [KT, Section 5.1]). A stability condition \( \sigma = (Z, P) \) is called numerical if \( Z : \mathbf{K}(\mathbf{D}^b(X)) \to \mathbb{C} \) factors through the numerical Grothendieck group \( N(X) \).

In [Bri08, AB13], a numerical stability condition on \( \mathbf{D}^b(X) \) was constructed when \( \dim_{\mathbb{C}} X = 2 \). Applying Theorem 3.5 and Proposition 3.11, we obtain the following lower bound of the entropy.

**Proposition 4.7.** Let \( X \) be a smooth projective surface over \( \mathbb{C} \) and \( F : \mathbf{D}^b(X) \to \mathbf{D}^b(X) \) be a Fourier-Mukai type endofunctor. Then

\[
\log \rho([F]) \leq h_0(F)
\]

where \( \rho([F]) \) is the spectral radius of the induced linear map \( [F] : N(X) \to N(X) \) and \( h_0(F) \) is the entropy of \( F \) at \( t = 0 \).

**Remark 4.8.** Let \( X \) be a smooth projective variety over \( \mathbb{C} \). In [KT], they conjectured that the equality \( \log \rho([F]) = h_0(F) \) holds for any autoequivalence \( F \in \text{Aut}(\mathbf{D}^b(X)) \). This conjecture was shown for a curve in [Kik] and for a variety with the ample canonical bundle or the ample anti-canonical bundle in [KT].

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