Galilean-Invariant (2 + 1)-Dimensional Models with a Chern-Simons-Like Term and $D = 2$ Noncommutative Geometry

Jerzy Lukierski*† Peter C. Stichel‡ Wojtek J. Zakrzewski*

Abstract
We consider a new $D = 2$ nonrelativistic classical mechanics model providing via the Noether theorem the $(2 + 1)$-Galilean symmetry algebra with two central charges: mass $m$ and the coupling constant $k$ of a Chern-Simons-like term. In this way we provide the dynamical interpretation of the second central charge of the $(2 + 1)$-dimensional Galilean algebra. We discuss also the interpretation of $k$ as describing the noncommutativity of $D = 2$ space coordinates. The model is quantized in two ways: using the Ostrogradski-Dirac formalism for higher order Lagrangians with constraints and the Faddeev-Jackiw method which describes constrained systems and produces nonstandard symplectic structures. We show that our model describes the superposition of a free motion in noncommutative $D = 2$ space as well as the “internal” oscillator modes. We add a suitably chosen class of velocity-dependent two-particle interactions, which is described by local potentials in $D = 2$ noncommutative space. We treat, in detail, the particular case of a harmonic oscillator and describe its quantization. It appears that the indefinite metric due to the third order time derivative term in the field equations, even in the presence of interactions, can be eliminated by the imposition of a subsidiary condition.

*Department of Mathematical Sciences, Science Laboratoires, University od Durham, South Road, Durham DH1 3LE, England
†On leave of absence from the Institute of Theoretical Physics, University of Wroclaw, pl. Maxa Borna 9, 50-204 Wroclaw, Poland
‡Faculty of Physics, University of Bielefeld, Universitätsstr.25, 33615 Bielefeld, Germany
1 Introduction

In a $N$-dimensional nonrelativistic classical mechanics the Galilean symmetry transformations $(i,j = 1, \ldots , N)$

$$
\begin{align*}
    x_i' &= R_{ij} x_j + v_i t + a_i, \\
    t' &= t + \tau
\end{align*}
$$

generated by the Galilei algebra $G_N$ leave the equation of motion invariant, but quite often the Lagrangian is changed under the transformations (1.1) by a total time derivative (see e.g [1,2]). Such a quasi-invariance of the Lagrangian leads to the appearance of a central extension $G_N \rightarrow \hat{G}_N$ of the Galilean algebra. Let us recall that $G_N$ is described by $\frac{1}{2} N(N-1)$ rotation generators $J_{ij} = -J_{ji}$ (generate $O(N)$ rotations $R^j$), $N$ momenta $P_i$ (generate space translations $a_i$), $N$ Galilean boosts $K_i$ (generate velocities $v_i$) and the energy operator $H$ (generates time shifts $\tau$). The best known central extension, occurring for any $N \geq 1$, describes the mass generator $M$ which modifies the commutativity of boosts and momenta as follows (see [3])

$$
[K_i, P_j] = i \delta_{ij} M.
$$

The relation (1.2) implies that for $M \neq 0$, by defining

$$
X_i = \frac{K_i}{M},
$$

one can embed the Heisenberg algebra

$$
[X_i, P_j] = i \delta_{ij}
$$

into the enveloping algebra of $\hat{G}_N$. This property of nonrelativistic systems has important consequences; in particular the no-interaction theorems valid for relativistic Poincaré-invariant two-particle systems (see [2,4,5]) are not true in the nonrelativistic case.

The simplest way of demonstrating the physical interpretation of the central charge $M$ involves considering a free nonrelativistic particle, with the Lagrangian $L_0 = \frac{1}{2} m \dot{x}_i^2$. Introducing the momenta $p_i = \frac{\partial L_0}{\partial \dot{x}_i} = m \dot{x}_i$ we find from the Noether theorem applied to the transformations (1.1) ($\vec{p} = (p_1 \ldots p_N)$) that

$$
\begin{align*}
    J_{ij} &= x_i p_j - p_i x_j, \\
    K_i &= m x_i, \\
    H &= \frac{\vec{p}^2}{2m}, \\
    P_i &= p_i, \\
    M &= m.
\end{align*}
$$
If we introduce the canonical commutation relations \((1.4)\) for \(X_i = x_i\) and \(P_j = p_i\) we can show that \((1.5)\) provides the one-particle realization of the Galilei algebra \(G_N\), centrally extended by the mass generator \(M = m \cdot 1\). Using the field equations \(\dot{p}_i = 0\) we obtain further that the generators \(J_{ij}, P_i\) and \(H\) are constant in time, and \(K_i\) depend on time in accordance with the Galilei algebra relation

\[
\dot{K}_i = i[H, K_i] = P_i. 
\] (1.6)

Let us add that it is the cohomological consideration (see e.g. [6]) which shows that in three space time dimensions \((N = 3)\) the mass generator \(M\) is the only central charge which can be added to the ten generators of the classical Galilei algebra \(G_3\). This conclusion is not valid for \(N = 1\) and \(N = 2\); for \(N = 1\) (one space, one time) we can introduce two central charges and for \(N = 2\) (two space, one time) we have the possibility of three central charges (the mass \(M\) and two additional central charges \(K, E\) — see [7]). In the latter case we have the following extended Galilei algebra \(\hat{G}_2\) \((J_{12} = J; i, j = 1, 2)\):

\[
\begin{align*}
[J, K_i] &= i\epsilon_{ij} K_j, & [J, P_i] &= i\epsilon_{ij} P_j, \\
[J, H] &= iE, & [K_i, P_j] &= i\delta_{ij} M, \\
[K_i, K_j] &= i\epsilon_{ij} K, & [K_i, H] &= iP_i, & [P_i, H] &= [P_1, P_2] = 0.
\end{align*}
\] (1.7)

Taking into consideration the mass dimensions of the generators \([K_i] = 0\), \([J] = 0\), \([P_i] = [H] = 1\) we see that the central generators have dimensions \([M] = [E] = 1\) and \([K] = 0\). In what follows we shall restrict ourselves to the case \(E = 0\) because, as shown by Levy-Leblond [7], if \(E \neq 0\) the algebra \((1.7)\) can not be integrated to the extended \(N = 2\) Galilei group \(\hat{E} \neq 0\). Indeed, assuming that \(E = e \cdot 1\), the finite \(O(2)\) rotations are generated by \(J\) as follows:

\[
e^{\theta J} H e^{-\theta J} = H + e\theta 1.
\] (1.8)

However, as \(\theta = 2\pi\) and \(\theta = 0\) should give identical results, one can integrate the Lie algebra \(\hat{G}_2\) to the corresponding Lie group only if \(e = 0\).

The representations of the Lie algebra \((1.7)\) with three central charges \(M, K, E\) as well as the projective representations of the corresponding \(N = 2\) Galilei group were studied by several authors (see e.g. [7-12]). Indeed, in accordance with the general scheme (see e.g. [3,6]) the appearance of central charges in a Lie algebra leads, on the group level, to the appearance of projective representations of the corresponding Lie group.
The main result of this paper involves finding a Lagrangian model which provides, via the Noether theorem, the $N = 2$ Galilean algebra $\hat{G}_2$ with two central charges $M = m \cdot 1$ and $K = 2k \cdot 1$. The interest of having such a model is threefold:

i) One obtains a clear physical interpretation of the second central charge of $\hat{G}_2$

ii) If we keep relation (1.3) the model possesses noncommutative space coordinates, i.e.\[ [X_i, X_j] = i \frac{k}{m^2} \epsilon_{ij} \] (1.9)

iii) It provides a good example of the Faddeev-Jackiw quantization method.

As we shall show, our model can be described either in terms of phase space variables with commuting space coordinates, or in terms of new phase space variables with noncommutative space coordinates given by relations (1.3). After considering free motion in the noncommutative space we shall introduce interactions in the classical $D = 2$ space generating a potential term which depends on noncommuting $D = 2$ space coordinates. Recently, there have been several proposals for deformations of space-time variables leading to their noncommutativity (see e.g. [15-18]) and also to deformed classical and quantum mechanics (see e.g. [19,20]). In our case we exploit the explicit relation between commuting and noncommuting position variables and we expect that our model can contribute also to a better understanding of dynamical models on other noncommutative spaces.

The plan of our presentation is as follows. In Sect. 2 we present our model in a Lagrangian as well as Hamiltonian formulation, write down the corresponding constraints, Dirac brackets and introduce the corresponding Ostrogradski-Dirac symplectic formalism. In the Hamiltonian formalism, following the scheme for higher order Lagrangians [21-24] one introduces besides the positions $x_i$ also two pairs of momenta $p_i, \tilde{p}_i$ as phase space variables. It should be mentioned that an equivalent formalism is also possible, with the canonical variables $(x_i, \dot{x}_i, p_i)$ and in Sect. 2 we shall show that the quantization with this choice of variables can be easily achieved using the geometrically motivated Faddeev-Jackiw method [13,14]. In Sect. 3, using the Ostrogradski phase space formalism and the Faddeev-Jackiw method we
discuss the Galilean symmetries: Noether charges and conservation laws. In Sect 4 we present the symplectic formalism for the choice of phase space with noncommuting space coordinates satisfying relations (1.9), and use this framework to consider the dynamics of the model. We arrive at the conclusion that the Hamiltonian of section 2 can be diagonalised and that it describes a free motion in the noncommutative phase space supplemented by the oscillator modes with negative sign of their energies. In sect. 5 we introduce the two particle $D = 2$ Galilean invariant dynamics and consider the class of velocity-dependent interactions, which imply the appearance of a potential term in the noncommutative $D = 2$ space. In particular, we study in detail a model with noncommutative harmonic forces, describing a harmonic oscillator in the $D = 2$ noncommutative space which was first introduced, in the Hamiltonian framework, in [10]. We find that although the parameter $k$ (see (1.9)) modifies the standard spectrum of the oscillator all its eigenvalues remain positive. In sect. 6 we discuss the problem of indefinite metric. We find that the modes carrying indefinite metric can always be removed by the imposition of a Gupta-Bleuler type of a subsidiary condition. The paper contains also an Appendix in which we show that our Lagrangian is the most general $D = 2$ Galilei-invariant Lagrangian linear in the second time derivative of the position variable.

2 A model with a Chern-Simons-like term

As is well known, in two dimensions, due to the existence of the Levi-Civita antisymmetric metric $\epsilon_{ij}$, one can introduce a free particle action with a Chern-Simons like term ($\lambda$ has dimension of mass/time):

$$L = \frac{m \dot{x}^2}{2} + \lambda \epsilon_{ij} x_i \dot{x}_j.$$  \hfill (2.1)

The second term can be interpreted as a coupling $\lambda A_j \dot{x}_j$ of a particular electromagnetic potential $A_j = \epsilon_{ij} x_i$ corresponding to constant magnetic field strength $F_{ij} = \partial_i A_j - \partial_j A_i = \lambda \epsilon_{ij}$. The Lagrangian (2.1) is neither invariant nor invariant up to a total derivative under the Galilean boost transformations; the symmetry algebra is described by the Hamiltonian $H$ and the $D = 2$ Euclidean inhomogeneous algebra $(J, P_1, P_2)$ centrally extended by
the central charge \( \Lambda = \lambda \cdot 1 \):

\[
[J, P_i] = i\epsilon_{ij} P_j, \quad [P_i, P_j] = 2i\epsilon_{ij}\Lambda .
\]  

(2.2)

In order to obtain a two-dimensional model which is quasi-invariant under \( D = 2 \) Galilei symmetry we modify the second term in (2.1) and consider \( (k \) has the physical dimension of mass \( \times \) time)

\[
L = \frac{m\dot{x}_i^2}{2} - k\epsilon_{ij}\ddot{x}_i\dot{x}_j .
\]  

(2.3)

It is interesting to observe that following the methods of [25] one can show that the Lagrangian (2.3) is the most general one which is quasi-invariant under the \( D = 2 \) Galilei transformations and which contains at most a linear dependence on the second derivative terms \( \ddot{x}_i \) (see the Appendix).

A) Quantization using the Ostrogradski-Dirac method

The Hamiltonian description of the Lagrangian (2.3) follows from the Ostrogradski formalism for higher order Lagrangians, supplemented by the Dirac bracket technique. Due to the presence of a second order derivative in the Lagrangian we have to introduce two momenta:

\[
p_i = \frac{\partial L}{\partial \dot{x}_i} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}_i} , \quad \tilde{p}_i = \frac{\partial L}{\partial \dddot{x}_i} .
\]  

(2.4a)

Hence in our case

\[
p_i = m\dot{x}_i - 2k\epsilon_{ij}\ddot{x}_j , \quad \tilde{p}_i = k\epsilon_{ij}\dot{x}_j .
\]  

(2.4b)

The Lagrange equation of motion

\[
\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}_i} = 0
\]  

(2.5)

takes in our case the form

\[
\dot{p}_i = m\ddot{x}_i - 2k\epsilon_{ij}\dddot{x}_j = 0 .
\]  

(2.6)
The Hamiltonian is given by

\[ H = \dot{x}_i p_i + \ddot{x}_i \tilde{p}_i - L = \frac{m \dot{x}_i^2}{2} - 2k \epsilon_{ij} \dot{x}_i \dot{x}_j. \]  

(2.7)

Because

\[ \frac{m \dot{x}_i^2}{2} = \frac{m}{2k^2} (\tilde{p}_j)^2 \]  

(2.8a)

\[ \epsilon_{ij} \dot{x}_i \dot{x}_j = -\frac{1}{2k^2} \tilde{p}_k \epsilon_{kli} p_l + \frac{m}{2k^3} (\tilde{p}_j)^2, \]  

(2.8b)

we obtain

\[ H = -\frac{m}{2k^2} (\tilde{p}_j)^2 + \frac{1}{k} \tilde{p}_k \epsilon_{kli} p_l. \]  

(2.9)

The Hamiltonian formalism for the Lagrangian (2.3) can be written in the eight-dimensional phase space \((x_i, \dot{x}_i, p_i, \tilde{p}_i)\) with two constraints

\[ \Phi_i = \dot{x}_i + \frac{1}{k} \epsilon_{ij} \tilde{p}_j = 0. \]  

(2.10)

These constraints lead to the replacement of the canonical Poisson brackets

\[
\begin{align*}
\{x_i, p_j\} &= \delta_{ij} , \\
\{x_i, \dot{x}_j\} &= \{p_i, \tilde{p}_j\} = 0 , \\
\{x_i, \tilde{p}_j\} &= \{\dot{x}_i, p_j\} = 0 ,
\end{align*}
\]

(2.11)

by the Dirac brackets

\[ \{X, Y\}_D = \{X, Y\} - \{X, \Phi_i\} \frac{k}{2} \epsilon_{ij} \{\Phi_j, Y\} , \]

(2.12)

where \(\frac{k}{2} \epsilon_{ij} = C_{ij}^{-1}\) and \(C_{ij} = \{\Phi_i, \Phi_j\}\). In particular, the fundamental Poisson bracket relations are replaced by the symplectic structure depending on the choice of six independent canonical variables. We have two possibilities:

i) The phase space with two sets of momenta \( y_A = (x_i, p_i, \tilde{p}_i) \).

Then from (2.12) we have

\[ \{y_A, y_B\}_D = \omega_{AB} , \]  

(2.13)
where
\[
\omega = \begin{pmatrix}
0 & 1_2 & 0 \\
-1_2 & 0 & 0 \\
0 & 0 & k/2 \epsilon
\end{pmatrix}.
\]
(2.14)

The Hamiltonian equations of motion
\[
y_A = \{y_A, H\}_D
\]
(2.15)
where \(H\) is given by (2.9), take the form:
\[
\dot{x}_i = \{x_i, H\}_D = -\frac{1}{k} \epsilon_{ij} \tilde{p}_j,
\]
(2.16a)
\[
\dot{p}_i = \{p_i, H\}_D = 0,
\]
(2.16b)
\[
\dot{\tilde{p}}_i = \{\tilde{p}_i, H\}_D = -\frac{m}{2k} \epsilon_{ij} \tilde{p}_j - \frac{1}{2} p_i.
\]
(2.16c)

Substituting the constraint equation (2.16a) into (2.16c) differentiating and using (2.16b) reproduces the equations (2.6).

To obtain the quantized form of the canonical commutation relations (2.13) as well as the Heisenberg equations of motion we perform the replacement
\[
\{y, y'\}_D \rightarrow \frac{1}{i\hbar} [\hat{y}, \hat{y}],
\]
(2.17)
where \(\hat{y}, \hat{y}'\) denote the quantized variables.

ii) The choice of independent variables \(\tilde{y}_A = (x_i, \dot{x}_i, p_i)\).

The symplectic structure is given by
\[
\{\tilde{y}_A, \tilde{y}_B\}_\tilde{D} = \tilde{\omega}_{AB},
\]
(2.18)
where
\[
\tilde{\omega} = \begin{pmatrix}
0 & 0 & 1_2 \\
0 & -\frac{1}{2k} \epsilon & 0 \\
-1_2 & 0 & 0
\end{pmatrix}.
\]
(2.19)

The Hamiltonian \(H\) reads using (2.10)
\[
H = \dot{x}_j p_j - \frac{m \dot{x}_j^2}{2}.
\]
(2.20)
The Hamiltonian equations are
\[
\ddot{x}_i = \{x_i, H\}_{\tilde{D}} = -\frac{m}{2k}\epsilon_{ij}\dot{x}_j + \frac{1}{2k}\epsilon_{ij}p_j, \quad (2.21a)
\]
\[
\dot{p}_i = \{p_i, H\}_{\tilde{D}} = 0, \quad (2.21b)
\]
and the relation \(\dot{x}_i = \{x_i, H\}_D\) has become an identity. One can easily see that the equations (2.16a)-(2.16c) and (2.21a)-(2.21b) supplemented by the constraints (2.10) are equivalent.

b) Quantization using the Faddeev-Jackiw method

It appears that Lagrangians with higher order derivatives can be also treated by the Faddeev-Jackiw method [13, 14]; particularly well-suited are the Lagrangians which are linear in the highest order derivatives. If we assume the Lagrangian of the form
\[
L(x_i, \dot{x}_i, \ldots, x_i^{(n)}) = L^{(0)}(x_i) + L^{(1)}_{i}(x_i, \ldots, x_i^{(n-1)}) (2.22)
\]
then by introducing \(n - 1\) momenta \((p_i, \ldots, p_{i;k}; k = 1, \ldots n - 1)\) as the Lagrange multipliers we can rewrite (2.22) as \((i, j = 1, \ldots d)\)
\[
L = L^{(0)}(x_j, y_{j;1}, \ldots, y_{j;n-1}) + L^{(1)}_{i}(x_j, y_{j;1}, \ldots, y_{j;n-1}) \cdot \dot{y}_{i;n-1}
\]
\[
+ p_i(x_j - y_{i;1}) + \sum_{k=1}^{n-2} p_{i;k}(\dot{y}_{i;k} - y_{i;k+1}) \quad (2.23)
\]
\[
= p_i\dot{x}_i + \sum_{k=1}^{n-2} p_{i;k}\dot{y}_{i;k} + L^{(1)}_{i}(x_j, y_{j;1}, \ldots, y_{j;n-1}) \cdot \dot{y}_{i;n-1} - H,
\]
where
\[
H = p_iy_{i;1} + \sum_{k=1}^{n-2} p_{i;k}y_{i;k+1} - L^{(0)}(x_j, y_{j;1}, \ldots, y_{j;n-1}). \quad (2.24)
\]

The relation (2.23) is written in the form presented in [13, 14]; in particular, the canonical one-form determining the equal-time commutator algebra is given by
\[
A_K(Y) dY_K = p_i dx_i + \sum_{k=1}^{n-2} p_{i;k} dy_{i;k} + L^{(1)}_{i}(x_j, y_{j;1}, \ldots, y_{j;n-1}) dy_{i;n-1}, \quad (2.25)
\]
where \( Y_K = (x_i, p_i, p_{i;1}, y_{i;1}, \ldots, p_{i;n-2}, y_{i;n-2}, y_{i;n-1}) \).

If we introduce the antisymmetric tensor

\[
f_{KL} = \frac{\partial A_K}{\partial Y_L} - \frac{\partial A_L}{\partial Y_K}.
\]

and assume that the matrix (2.26) is invertible we find that the basic Poisson brackets are given by

\[
\omega_{KL} = \{Y_K, Y_L\} = f_{KL}^{-1}.
\]

Our Lagrangian (2.3) fits very well into such a scheme, which is obtained from (2.23) by putting \( d = 2, n = 3, y_{i;1} \equiv y_i, Y_A = (x_1, x_2, p_1, p_2, y_1, y_2) \) and \((\vec{y} = (y_1, y_2))\)

\[
L^{(0)} = m\frac{\vec{y}^2}{2}, \quad L^{(1)} = -k\epsilon_{ij}y_j.
\]

We have

\[
f_{KL} = \begin{pmatrix} 0 & 1_2 & 0 \\ -1_2 & 0 & 0 \\ 0 & 0 & -2k\epsilon_{ij} \end{pmatrix}, \quad \text{(2.29)}
\]

i.e. from (2.27) we obtain the symplectic structure (2.19). It should be noted that the Faddeev-Jackiw method, presented in this subsection, provides the quantization of the model (2.3) in an easier way than the conventional Ostrogradski-Dirac approach.

### 3 Noether charges and the generalized \( D = 2 \) Galilean algebra

**a) Ostrogradski-Dirac method**

Let us consider a Lagrangian \( L(x_i, \dot{x}_i, \ddot{x}_i) \) which depends on the first and second time derivatives. The variation of the action \( S = \int dt L \) under the change \( x_i \rightarrow x_i + \delta x_i \) takes the form (see e.g [21,22])

\[
\delta S = \int dt \delta L = \int dt \left( \frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i + \frac{\partial L}{\partial \ddot{x}_i} \delta \ddot{x}_i \right) = \int dt \frac{d}{dt}(p_i \delta x_i + \ddot{p}_i \delta \dot{x}_i)
\]

(3.1)
If $\delta x_i = F_i^r(x_i, t) \delta \alpha_r$ describes the symmetry, i.e. $\delta L_{\alpha_r} = 0$, we obtain the following formulae for the generator

$$Q^r(t) = p_i(t) F_i^r(x_i(t), t) + \bar{p}_i(t) \frac{d}{dt} F_i^r(x_i(t), t)$$  \hspace{1cm} (3.2)

which is conserved

$$\frac{d}{dt} Q^r(t) = 0 \implies Q^r(t_1) = Q^r(t_2).$$  \hspace{1cm} (3.3)

If the Lagrangian $L$ is quasi-invariant under the symmetry transformation, i.e

$$\delta L_{\alpha_r} = \frac{d}{dt} (F^r \delta \alpha_r),$$  \hspace{1cm} (3.4)

the generators (3.2) are not conserved. However, as the nonconservation law takes the form

$$\frac{d}{dt} Q^r(t) = \frac{d}{dt} F^r,$$  \hspace{1cm} (3.5)

we can introduce modified generators $\tilde{Q}^r = Q^r - F^r$ which are conserved. The generators $\tilde{Q}^r$ correspond to the modified symmetry transformations with central charges.

Let us list below the generators of the $D = 2$ Galilei symmetry for the Lagrangian (2.3). We have

i) translations ($\delta x_i = \delta \alpha_i, \delta \dot{x}_i = 0; F_i^r = \delta^r_i, Q^i = P_i$)

$$P_i = p_i$$  \hspace{1cm} (3.6)

ii) rotations $O(2)$ ($\delta x_i = -\epsilon_{ij} x_j \delta \alpha, \delta \dot{x}_i = -\epsilon_{ij} \dot{x}_j \delta \alpha; F_i = -\epsilon_{ij} x_j, Q = J$)

$$J = x_i \epsilon_{ij} p_j - \bar{p}_i \epsilon_{ij} \dot{x}_j.$$  \hspace{1cm} (3.7)

Using the constraint equations (2.10) we find that

$$J = \epsilon_{ij} x_i p_j - \frac{1}{k} (\bar{p}_j)^2.$$  \hspace{1cm} (3.8)

iii) Galilean boosts $\delta x_i = v_i t, \delta \dot{x}_i = v_i; F_i^r = \delta^r_i \cdot t$. (we denote the nonconserved Noether boost charges by $B_i$)

From (3.2) we obtain

$$B_i = p_i \cdot t + \bar{p}_i.$$  \hspace{1cm} (3.9)
The Lagrangian (2.3) is quasi-invariant under Galilean boost transformations; in fact
\[ \delta L_{v_i} = \frac{d}{dt}(mx_i - k\epsilon_{ij}\dot{x}_j)v_j \] (3.10)
and the relation (3.5) takes the form
\[ \frac{d}{dt}B_i = \frac{d}{dt}(p_i \cdot t + \tilde{p}_i) = \frac{d}{dt}(mx_i - k\epsilon_{ij}\dot{x}_j) , \] (3.11)
or
\[ \tilde{B}_i = p_i \cdot t + \tilde{p}_i - mx_i + k\epsilon_{ij}\dot{x}_j . \] (3.12)

After inserting the constraint formula (2.10) we derive the following conserved generator
\[ \tilde{B}_i = p_i t - mx_i + 2\tilde{p}_i , \] (3.13)
Finally, we introduce the boost generators \( K_i \) by means of the formula\(^3\)
\[ \tilde{B}_i = p_i t - K_i \] (3.14)
or
\[ K_i = mx_i - 2\tilde{p}_i , \] (3.15)
in consistency with the relation (1.6)

Let us recall that the full \( D = 2 \) Galilean algebra is described by the generators (3.6), (3.8), (3.13)-(3.14) and the energy operator \( H \) is given by (2.9). If we use the Dirac brackets (2.13)-(2.19) it can be shown that we obtain the \( D = 2 \) Galilean algebra (1.7), with \( E = 0 \) and nonvanishing central charges \( M = m \cdot 1 \) and \( K = 2k \cdot 1 \).

**b) Faddeev-Jackiw method**

Let us write the Lagrangean (2.3) as a special case of formula (2.23)
\[ L = p_i\dot{x}_i - k\epsilon_{ij}y_i\dot{y}_j + \frac{m\tilde{y}^2}{2} - p_iy_i . \] (3.16)

The variations corresponding to translations, \( O(2) \) rotations and Galilei boosts take the form
\[ \delta x_i = \delta \alpha_i - \epsilon_{ij}x_j(\delta \alpha + \delta v_i t) , \]
\[ \delta p_i = -\epsilon_{ij}p_j(\delta \alpha + m\delta v_i) \]
\[ \delta y_i = -\epsilon_{ij}y_j(\delta \alpha + \delta v_i) . \] (3.17)
Then we obtain
\[ \delta L = \frac{d}{dt}(mx_i - k\epsilon_{ij}y_j) \delta v_i . \] (3.18)

Denoting the variations \( \delta Y_A = F_A^r(Y_A, t) \delta \alpha_r \) as \( \delta \alpha_i, \delta \alpha, \delta v_i \) we find
\[ Q^r(t) = P_A(t) F_A^r(Y_A, t) , \] (3.19)

where from \( (3.16) \) it follows that
\[ P_A = \frac{\partial L}{\partial \dot{Y}_A} = (p_i, 0, k\epsilon_{ij}y_j) . \] (3.20)

Taking into consideration \( (3.18) \) we see that
\[ P_i = p_i \]
\[ J = \epsilon_{ij}x_ip_j - ky_i^2 \] (3.21)
\[ K_i = p_it - mx_i + 2k\epsilon_{ij}y_j \]

Together with the formula for the Hamiltonian, which follows from \( (3.16) \)
\[ H = p_iy_i - \frac{m\dot{y}_i^2}{2} , \] (3.22)

and using \( (2.19) \) one can easily check that the formulae \( (3.21)-(3.22) \) lead to a realization of the \( D = 2 \) Galilei algebra \( (1.7) \) (with \( E=0 \))

4 Quantization of a free motion with non-commuting space coordinates and internal oscillator modes

For Galilean systems the position operator \( X_i \) can be expressed via formula \( (1.3) \) by the Galilean boost operators \( K_i \). Such position operators for \( D > 2 \) are commuting. In the \( D = 2 \) case in the presence of two central charges \( (m \neq 0, k \neq 0) \) if we keep the formula \( (1.3) \) we obtain noncommuting position variables. Using \( (3.15) \) we have
\[ X_i = x_i - \frac{2}{m} \bar{p}_i . \] (4.1)
If we put $P_i = p_j$ we obtain the standard canonical Poisson bracket
\[
\{X_i, P_j\}_D = \delta_{ij}.
\] (4.2)

In order to obtain $\{X_i, \tilde{P}_j\}_D = 0$ we should redefine the second pair of momenta as follows
\[
\tilde{P}_i = \frac{k}{m} p_i + \epsilon_{ij} \tilde{p}_j.
\] (4.3)

Introducing six phase space variables $Y_A = (X_i, p_i, \tilde{P}_i)$ we obtain the following symplectic structure
\[
\{Y_A, Y_B\}_D = \Omega_{AB},
\] (4.4)
where
\[
\Omega = \begin{pmatrix}
\frac{2k}{m^2} \epsilon & \mathbf{1}_2 & 0 \\
-\mathbf{1}_2 & 0 & 0 \\
0 & 0 & \frac{k}{2} \epsilon
\end{pmatrix}.
\] (4.5)

We see from (4.4)-(4.5) that the parameter $k$ introduces noncommutativity in the coordinate sector, in accordance with recent ideas of noncommutative geometry (see e.g. [13-15]). One can describe the model (2.3) in the Hamiltonian framework using variables $Y_A$. In particular, the symmetry generators calculated in Sect. 3 can be expressed as follows
\[
P_i = p_i, \quad K_i = m X_i, \quad (4.6a)
\]
\[
J = -\epsilon_{ij} \tilde{p}_i X_j + \frac{k}{m^2} \tilde{p}^2 - \frac{1}{k} \tilde{P}^2, \quad (4.6b)
\]
\[
H = \frac{\tilde{p}^2}{2m} - m \frac{k}{2k^2} \tilde{P}^2. \quad (4.6c)
\]

Then the dynamics of the model is described by the following set of equations, with $H$ given by (4.6c)
\[
\dot{Y}_A = \Omega_{AB} \frac{\partial H}{\partial Y_B}, \quad (4.7)
\]
or, more explicitly,
\[
\dot{X}_i = \frac{2k}{m^2} \epsilon_{ij} \frac{\partial H}{\partial X_j} + \frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \quad (4.8a)
\]
\[
\dot{p}_i = - \frac{\partial H}{\partial X_i} = 0, \quad (4.8b)
\]
\[
\dot{\tilde{P}}_i = \frac{k}{2} \epsilon_{ij} \frac{\partial H}{\partial \tilde{P}_j} = -m \frac{m}{2k} \epsilon_{ij} \tilde{p}_j. \quad (4.8c)
\]
We see from relations (4.8a,b) that $\bar{X}_i = 0$, i.e. that our model describes a free motion in the noncommutative two-dimensional space, supplemented by independent “internal” modes described by variables $\bar{P}_i$, which can be identified with a standard pair of canonical variables. Indeed, identifying $\bar{P}_1 = \sqrt{k/2} \bar{x}$, $\bar{P}_2 = \sqrt{k/2} \bar{p}$ and introducing oscillator variables

$$C = \frac{1}{\sqrt{2}}(\bar{x} + i \bar{p}), \quad C^* = \frac{1}{\sqrt{2}}(\bar{x} - i \bar{p})$$

(4.9)

we find from (4.3) that

$$\{C, C^*\}_D = -i$$

(4.10)

and

$$H = \frac{\bar{p}^2}{2m} - \frac{m}{4k}(CC^* + C^*C).$$

(4.11)

We see that our model describes a free motion in the noncommutative space $(X_1, X_2)$ supplemented by internal degrees of freedom described by oscillator modes with negative energies. Indeed, introducing the correspondence principle

$$\{A, B\}_D \rightarrow \frac{1}{i\hbar}[\hat{A}, \hat{B}]$$

(4.12)

we obtain from (4.12)

$$[C, C^\dagger] = \hbar$$

(4.13)

and the spectrum of the Hamiltonian (4.13) is given by

$$E_{\vec{p}, n} = \frac{\bar{p}^2}{2m} - \frac{h}{4k}(n + \frac{1}{2}).$$

(4.14)

We see that the energy spectrum (4.14) is not bounded from below due to the presence of the $C$-quanta. The physical space can be defined by means of the subsidiary condition

$$C |ph\rangle = 0$$

(4.15)

Next we shall introduce interactions which allow us to introduce, consistently, the subsidiary condition (4.15).
5 Local potentials in the $D = 2$ noncommutative space and the case of harmonic forces

In the previous section we have shown that our model, defined by the Lagrangian (2.3), can be decomposed into two decoupled sectors:

i) The external one described by variables $(X_i, P_i)$ with $P_i = p_i$ and $X_i$ describing noncommuting $D = 2$ space coordinates. It appears that our model (2.3) describes a free motion in the noncommutative space $X_i$.

ii) The internal sector which is described by auxiliary momenta $\tilde{P}_i$ which commute with the external variables. The states generated by the variables in the internal sector are eliminated by the subsidiary condition (4.15).

The model (2.3) describes the one-particle $D = 2$ Galilean dynamics and so is fixed uniquely by the Galilean invariance with $k \neq 0$ and $E = 0$ (see (1.7))\(^4\). In order to add to the free Lagrangian (2.3) a potential energy term, consistent with Galilean invariance, we have to consider the two-body particle dynamics. Denoting by $x_{i;1}, x_{i;2}$ ($i = 1, 2$) the positions of two point particles we consider the following Lagrangian\(^5\)

$$L_{1+2} = L_{0;1} + L_{0;2} - V(x_{i;1} - x_{i;2}, \dot{x}_{i;1} - \dot{x}_{i;1}), \quad (5.1)$$

where $V$ is a scalar with respect to the $D = 2$ space rotations $O(2)$ and is, by construction, invariant under translations and Galilean boosts. $L_{0;r}$ are given by (for simplicity we put $\tilde{m} = m_1 = m_2$ and $\tilde{k} = k_1 = k_2$)

$$L_{0;r} = \frac{\tilde{m}}{2} \ddot{x}_{i;r}^2 - \frac{\tilde{k}}{2} \epsilon_{ij} \dot{x}_{i;r} \dot{x}_{j;r}. \quad (5.2)$$

If we define the centre-of-mass (CM) and relative coordinates

$$x_i := x_{i;1} - x_{i;2}, \quad R_i := \frac{1}{2}(x_{i;1} + x_{i;2}) \quad (5.3)$$

we can rewrite (5.1) as

$$L_{1+2} = L_{CM} + L' \quad (5.4)$$
where \( M = 2\tilde{m}, \ K = 2\tilde{k} \; m = \frac{\tilde{m}}{2}, \ k = \frac{\tilde{k}}{2} \)

\[
L_{CM} = \frac{M}{2} \ddot{R}_i^2 - K\epsilon_{ij} \dot{R}_i \dot{R}_j , \quad (5.5a)
\]
\[
L' = \frac{m}{2} \ddot{x}_i^2 - k\epsilon_{ij} x_i \ddot{x}_j - V(x_i, \dot{x}_i) . \quad (5.5b)
\]

The global CM motion described by (5.5a) has exactly the structure of the one-particle dynamics discussed in sections 2-4. In the following we shall study the dynamics of the relative two-particle motion described by the Lagrangian (5.5b). We postulate that the Hamiltonian obtained from the Lagrangian (5.5b) should also split into the sum

\[
H = H^{(ext)}(P, X) + H^{(int)}(\tilde{P}) , \quad (5.6)
\]

where \( H^{(int)}(\tilde{P}) \) is the free internal oscillator Hamiltonian (see (4.9)). We shall consider here the interactions \( V(x_i, \dot{x}_i) \) which do not modify the choice of the internal Hamiltonian and add to the free external Hamiltonian an arbitrary potential \( U(X) \)

\[
H^{(ext)}(P, X) = \frac{P^2}{2m} + U(X) . \quad (5.7)
\]

For this purpose we will take

\[
V(x_i, \dot{x}_i) = U(x_i - \frac{2k}{m} \epsilon_{ij} \ddot{x}_j) . \quad (5.8)
\]

Using

\[
p_i = m \dot{x}_i - 2k\epsilon_{ij} \ddot{x}_j - \frac{2k}{m} \epsilon_{ij} \frac{\partial U}{\partial x_j} , \quad \tilde{p}_i = k\epsilon_{ij} \ddot{x}_j , \quad (5.9)
\]

and (1.1), (1.3), (5.9) we get

\[
H = \dot{x}_i p_i + \ddot{x}_i \tilde{p}_i - \mathcal{L} = \frac{p^2}{2m} - \frac{m}{2k^2} \tilde{P}^2 + U(X_i) . \quad (5.10)
\]

We see therefore that our particular velocity-dependent interaction (5.8) leads to local interactions involving noncommutative variables.
The property that quantum mechanical models on noncommutative spaces can be transformed into standard but complicated models on commuting spaces is known from the studies of quantum deformed models of quantum mechanics (see e.g. [17,18]). Our model provides one more example of such a construction. In the simplest case one can assume that the potential $U$ is quadratic and so consider the following form of the noncommutative oscillator Lagrangian

$$L_{osc} = \frac{m\dot{x}_i^2}{2} - k\epsilon_{ij}\dot{x}_i\dot{x}_j - \frac{m\omega^2}{2}(x_i - \frac{2k}{m}\epsilon_{ij}\dot{x}_j)^2 \tag{5.11}$$

which contains the Chern-Simons term. The equations (2.6) are now generalised to

$$\left(\delta_{ij} - \frac{2k}{m}\epsilon_{ij}\frac{d}{dt}\right)(m\ddot{x}_j - 2k\omega^2\epsilon_{jl}\dot{x}_l + m\omega^2x_j) = 0 \tag{5.12}$$

Introducing noncommutative coordinates (4.1) and the modified auxiliary momenta (4.3) we obtain (putting $P_i = p_i$) the expected special form of (5.7)

$$H^{(ext)}_{osc}(P,X) = \frac{\tilde{P}_2}{2m} + \frac{m\omega^2}{2}\tilde{X}^2. \tag{5.13}$$

Hamilton’s equations (4.7) then give us

$$\dot{X}_i = 2\frac{k\omega^2}{m}\epsilon_{ij}X_j + \frac{1}{m}P_i \tag{5.14}$$

$$\dot{P}_i = -m\omega^2X_i.$$  

Thus we obtain the following equation for our noncommutative $X_i$:

$$\ddot{X}_i - 2\frac{k\omega^2}{m}\epsilon_{ij}\dot{X}_j + m\omega^2X_i = 0. \tag{5.15}$$

We note that the velocity dependent term in (5.13) is due to the noncommutativity of space coordinates $X_i$ (see (1.9)). If we now introduce, in the standard way, the oscillator variables

$$A_i := \sqrt{\frac{m\omega}{2}}X_i + i\sqrt{\frac{1}{2m\omega}}P_i \tag{5.16a}$$

$$A_i^* := \sqrt{\frac{m\omega}{2}}X_i - i\sqrt{\frac{1}{2m\omega}}P_i \tag{5.16b}$$
we obtain
\[ H_{\text{osc}}^{(\text{ext})} = \frac{\omega}{2} (A_i A_i^* + A_i^* A_i). \] (5.17)

Calculating the Dirac brackets (see (4.5)) for the oscillator variables (5.16a) - (5.16b) and quantizing by the substitution \{ \ldots \} \rightarrow \frac{1}{\hbar} \{ \ldots \} \text{ we obtain}
\[
[A_i, A_j^\dagger] = \hbar \delta_{ij} + \frac{i \hbar \omega k}{m} \epsilon_{ij},
\]
\[
[A_i, A_j] = [A_i^\dagger, A_j^\dagger] = \frac{i \hbar \omega k}{m} \epsilon_{ij}.
\] (5.18)

The \( k \)-deformation of the harmonic oscillator is obtained therefore by the deformation of the Heisenberg commutation relations, describing modified equal time oscillator algebra (5.18). In principle one can quantize the \( k \)-deformed oscillator by introducing a \( k \)-deformed Fock space of the modified oscillator variables (5.18). Here we shall however solve the model by introducing the following commuting space coordinates
\[
\hat{X}_i = X_i + \frac{k}{m^2} \epsilon_{ij} P_j = x_i - \frac{2}{m} p_i + \frac{k}{m^2} \epsilon_{ij} p_j,
\] (5.19)
which satisfy two conditions:

i) the standard Poisson brackets are valid
\[
\{ \hat{X}_i, P_j \} = \delta_{ij}, \quad \{ \hat{X}_i, \hat{X}_j \} = 0.
\] (5.20)

ii) the Poissons bracket with internal symmetry variables vanish, i.e. \{ \hat{X}_i, \tilde{P}_j \} = 0.

In this way, following [10], we find that (5.13) gives us
\[
H_{\text{osc}}^{(\text{ext})} = \frac{p^2}{2m} + \frac{m \omega^2}{2} (\hat{X}_i - \frac{k}{m^2} \epsilon_{ij} P_j)^2 = \frac{p^2}{2\tilde{m}} + \frac{\tilde{m} \omega^2}{2} \hat{X}_i^2 - \frac{k \omega^2}{m} J^{(\text{ext})}
\] (5.21)

where
\[
\tilde{m} = m(1 + \omega^2 \frac{k^2}{m^2})^{-1},
\] (5.22a)
\[
\tilde{\omega}^2 = \omega^2 (1 + \frac{k^2 \omega^2}{m^2})
\] (5.22b)
and where, as is clear from (4.6b),

\[ J^{(ext)} = \epsilon_{ij}X_i P_j + \frac{k}{m^2}p^2 = \epsilon_{ij}\hat{X}_i P_j \]  

(5.23)

describes the \(O(2)\) angular momentum for the external dynamics.

The first part of the Hamiltonian (5.21) is the standard oscillator. If we introduce the standard quantized oscillator variables

\[
a_i := \sqrt{\frac{\tilde{m}\tilde{\omega}}{2\hbar}}\hat{X}_i + i\sqrt{\frac{1}{2\tilde{m}\tilde{\omega}\hbar}}P_i
\]

\[
a_i^\dagger := \sqrt{\frac{\tilde{m}\tilde{\omega}}{2\hbar}}\hat{X}_i - i\sqrt{\frac{1}{2\tilde{m}\tilde{\omega}\hbar}}P_i
\]

(5.24)

satisfying

\[ [a_i, a_j^\dagger] = \delta_{ij} \]  

(5.25)

we find that

\[ H_{osc}^{(ext)} = H_{osc}^{(0)} - \gamma J^{ext}, \quad \gamma = \frac{k\omega^2}{m}, \]  

(5.26)

where

\[
H_{osc}^{(0)} = \hbar\tilde{\omega}(a_i^\dagger a_i + 1) = \hbar\tilde{\omega}(N + \frac{1}{2})
\]

(5.27a)

\[ J^{ext} = i\hbar(a_i^\dagger a_1 - a_i a_1^\dagger a_2). \]

(5.27b)

As is well known, using the operators (5.25) and the \(2 \times 2\) Pauli matrices \(\sigma_r (r = 1, 2, 3)\), we can construct the following \(SU(2)\) Lie algebra generators

\[
J_r = \frac{1}{2}a_i^\dagger(\sigma_r)_{ij}a_j.
\]

(5.28)

The \(SU(2)\) Casimir \(J^2 = J_r J_r\) is related to the number operator (see (5.27a)) by

\[ J^2 = \frac{N}{2}(\frac{N}{2} + 1). \]

(5.29)

Furthermore

\[ J^{ext} = 2\hbar J_2. \]

(5.30)
Let us consider now the common eigenstates of $N$ and $J_2$ (see also [9])

\begin{align}
N|n; l> &= n|n; l> \quad n \in N \quad (5.31a) \\
J_2|n; l> &= \frac{l}{2}|n; l> \quad l \in Z. \quad (5.31b)
\end{align}

We see from (5.29), however, that in the oscillator representation the number $n$ plays the role of the half of the angular momentum eigenvalue. Following the standard procedure in quantum mechanics we see that we have the restriction

$$|l| \leq n. \quad (5.32)$$

From (5.26), (5.27a,b), (5.30) and (5.31a,b) we obtain

$$H_{osc}^{(ext)}|n; l> = E_{n,l}|n; l>, \quad (5.33)$$

where

$$E_{n,l} = \hbar \tilde{\omega}(n + 1) - \hbar \frac{k\omega^2}{m}l. \quad (5.34)$$

Using (5.32) we obtain the following lower bound on the energy spectrum

$$E_{n,l} \geq \hbar \tilde{\omega} + \hbar n(\tilde{\omega} - \frac{k\omega^2}{m}). \quad (5.35)$$

From (5.22b) we get

$$\tilde{\omega} - \frac{k\omega^2}{m} = \omega(\sqrt{1 + \frac{k^2\omega^2}{m^2} - \frac{k\omega}{m}}) > 0. \quad (5.36)$$

We see that the energy spectrum (5.35) is positive.

In order to describe states (5.31a-b) one should consider the oscillators $a_i$ as $SU(2)$ spinors and rotate them around the first axis by an angle $\frac{\pi}{4}$. Introducing the following unitary transformation

$$\tilde{a}_i = U_{ij}a_j, \quad (5.37)$$

where

$$U = [exp^{i\pi \sigma_1} 4]_{ij} = \frac{1}{\sqrt{2}}(1 + \imath \sigma_1) \quad (5.38)$$
we find that
\[ J_2 = \frac{1}{2}(\tilde{a}_1\tilde{a}_1 - \tilde{a}_2\tilde{a}_2). \]  
(5.39)

If we now introduce the states
\[ |n_1, n_2 > = \frac{1}{\sqrt{n_1}! \sqrt{n_2}!} (\tilde{a}_1^\dagger)^{n_1} (\tilde{a}_2^\dagger)^{n_2} |0 > \]  
(5.40)
where
\[ \tilde{a}_i |0 >= 0 \]  
(5.41)
we obtain the following formula for the eigenstates (5.33)
\[ |n; l >= \frac{1}{2} (n + l), \frac{1}{2} (n - l) >. \]  
(5.42)

We would like to make the following two additional remarks

i) The Lagrangian (5.5a) can also be discussed for nonharmonic potentials \( U(X_i) \). For example, one can assume that \( U(X_i) = \frac{\lambda}{4} (\vec{X}^2)^2 \). It is easy to see that such a model with noncommutative quartic interaction will lead, in the commuting space, to a Hamiltonian with the generalised kinetic term
\[ \frac{\vec{p}^2}{2m} \rightarrow \frac{\vec{p}^2}{2m} + \tilde{\lambda} (\vec{p}^2)^2, \quad \tilde{\lambda} = \frac{\lambda k^4}{m^8}. \]  
(5.43)
Such a model is currently under consideration.

ii) One can also ask if it is possible to generalize the free oscillator Hamiltonian \( H^{(\text{int})} \) (see (5.6)). It can be shown that such a generalization is possible only if we can introduce terms with second order time derivatives which are different from the Chern-Simons-like term in (2.3). It appears, however, that in such a case, the constraints (2.10) are not valid and the separation into “external” and “internal” degrees of freedom looses its meaning. Moreover, for the interacting “internal” degrees of freedom the negativemetric states cannot be consistently eliminated by the imposition of a subsidiary condition, and the Faddeev-Jackiw method cannot be applied.
6 Concluding Remarks

In sect. 2-4 we have presented a one-particle model with higher order derivatives, which provides a dynamical interpretation of the second central extension of $D = 2$ Galilei algebra. The model can be interpreted as describing a free motion in the $D = 2$ space with noncommuting coordinates and internal structure described by oscillator modes with negative energies. Further, in sect. 5, we have also considered a two-particle Galilean-invariant dynamics with relative motion described by a model with a potential depending on noncommuting coordinates. It appears that such models are obtained if our primary Lagrangian contains suitably chosen velocity-dependent interactions. In particular, we have fully discussed the case of a harmonic oscillator in noncommutative space. The modification due to the noncommutativity introduces a bilinear $SU(2)$-breaking term into the Hamiltonian of a two-dimensional oscillator.

It appears that the dynamics in the models considered in Sect. 5 can be separated into two independent sectors - describing external and internal dynamics. The external dynamics describes the quantum mechanics in the $D = 2$ space with noncommuting coordinates. The internal dynamics, in the presence of particularly chosen interaction in the external sector, is described by free oscillators (4.15).

From the general framework for higher order Lagrangians (see eg [21]) it can be deduced that in our model there exist states which, after quantization, are endowed with an indefinite metric. These states are generated by the internal oscillator variables and are eliminated from the physical spectrum by the imposition of the subsidiary condition (4.15). In the case of the interaction described by the potential (5.9) we obtain from the Lagrangian (5.5b) the following equations of motion

$$\hat{O}_{ij}(m\ddot{x}_j + \frac{\partial U}{\partial x_j}) = 0,$$

where

$$\hat{O}_{ij} = (\delta_{ij} - \frac{2k}{m} \epsilon_{ij} \frac{d}{dt}).$$

The subsidiary condition (4.15) eliminating the states with negative metric
can be expressed in the following way

\[ [\ddot{x}_i(t) + \frac{1}{m} \frac{\partial U}{\partial x_i}] P \phi = 0 \]  \hspace{1cm} (6.3)

i.e. in the physical sector we retain only the reduced dynamics in the external sector. The factorization of the Euler-Lagrange equations (6.3) shows that the operator \( \hat{O}_{ij} \) describing the modes which carry indefinite metric does not depend on the interaction term and the relation (6.3) is equivalent to the subsidiary condition (4.15).

In summary, one can treat the presented model as an explicit realization of a theory with higher derivatives, with interesting symmetry properties, constraints and several symplectic structures. The model also illustrates very well the Faddeev-Jackiw technique of quantization. In our view it is also important that the model provides an example of a Lagrangian dynamics which can be expressed equivalently in terms of commuting and noncommuting position variables. Moreover, although our model contains higher order derivatives, even in the interacting case, the ghost problem of higher order Lagrangian theories can be solved. The unphysical features (negative energies, negative metric states) which are linked to the introduction of a noncommutative structure are described by free modes and can be made harmless by the imposition of a suitable subsidiary condition eliminating negative metric states.

Finally we would like to add that since the appearance of recent models of strings with substructure described by the so-called 0-branes (see e.g. [28,29]) the (2+1) dimensional Galilei-invariant systems have become more important. Possible links with such applications should also be considered.

Acknowledgments.

One of the authors (JL) would like to thank the Department of Mathematical Sciences of the University of Durham for its warm hospitality and the EPSRC for financial support. He also acknowledges partial support from the KBN grant 2P30208706.

The authors would like to thank Roman Jackiw for pointing out that our model can be quantized in an easy way by the geometric method presented
in [13, 14]. The authors would also like to acknowledge Jose A. de Azcárraga and E.H. de Groot for helpful comments.

Appendix

In this appendix we prove the following

Theorem. The most general one-particle Lagrangian, which is at most linearly dependent on $\ddot{x}$, leading to the Euler-Lagrange equations of motion which are covariant with respect to the $D = 2$ Galilei group, is given, up to gauge transformations, by

$$L(x, \dot{x}, \ddot{x}, t) = \frac{m}{2} \dot{x}_i \dot{x}_i - k \epsilon_{ij} \dot{x}_i \ddot{x}_j$$  \hspace{1cm} (A.1)

with $m$ and $k$ constant.

Proof

• (i) Covariance of the equations of motion, with respect to the $D = 2$ Galilei group is equivalent to the statement that the Euler variation $f_i$ is independent of $t$, $x$ and $\dot{x}$ and transforms, under space rotations, as an $i^{th}$ component of a vector (see [23]), i.e.

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}_i} = f_i(\ddot{x}, \overline{x}, \overline{x}).$$  \hspace{1cm} (A.2)

• (ii) If we now suppose that $L$ is at most linearly dependent on $\ddot{x}$

$$L(x, \dot{x}, \ddot{x}, t) = L_1(x, \dot{x}, t) + L_2(x, \dot{x}, t) \ddot{x}_i$$  \hspace{1cm} (A.3)

we conclude from (A.2) that $f_i$ does not depend on $\overline{x}$ and is linearly dependent on $\ddot{x}$

$$f_i(\ddot{x}, \overline{x}) = f_{1i}(\ddot{x}) + f_{2ij}(\ddot{x}) \overline{x}_j$$  \hspace{1cm} (A.4)

with

$$f_{2ij}(\ddot{x}) = \frac{\partial L_{2i}}{\partial \dot{x}_j} - \frac{\partial L_{2j}}{\partial \dot{x}_i}.$$  \hspace{1cm} (A.5)

Because the r.h.s. of (A.5) is an antisymmetric tensor independent of $\ddot{x}$, we have

$$f_{2ij}(\ddot{x}) = 2k \epsilon_{ij}$$  \hspace{1cm} (A.6)
with \( K = \text{constant} \) (the factor 2 is a matter of convenience).

Putting (A.6) back into (A.5) we conclude that

\[
L_{2i}(x, \dot{x}, t) = L_{20i}(x, t) + L_{21ij}(x, t)\dot{x}_j + k\epsilon_{ij}\dot{x}_j
\]  \hspace{1cm} (A.7)

with \( L_{21ij} \) being a symmetric tensor.

Therefore we have as an intermediate result

\[
L(x, \dot{x}, \ddot{x}, t) = L_1(x, \dot{x}, t) + L_{20i}(x, t)\ddot{x}_i + k\epsilon_{ij}\ddot{x}_i \dot{x}_j + L_{21ij}(x, t)\dot{x}_i \dot{x}_j
\]  \hspace{1cm} (A.8)

• (iii) We now perform a gauge transformation

\[
L = \tilde{L} + \frac{d}{dt}\phi(x, \dot{x}, t)
\]  \hspace{1cm} (A.9)

with \( \phi := L_{20i}(x, y)\dot{x}_i + \frac{1}{2}L_{21ij}(x, t)\dot{x}_i \dot{x}_j \).

Then \( \tilde{L} \) reads

\[
\tilde{L}(x, \dot{x}, \ddot{x}, t) = L_0(x, \dot{x}, t) + k\epsilon_{ij}\ddot{x}_i \dot{x}_j
\]  \hspace{1cm} (A.10)

with

\[
L_0 := L_1(x, \dot{x}, t) - (\frac{d}{dt}L_{20i}(x, t))\dot{x}_i - \frac{1}{2}(\frac{d}{dt}L_{21ij}(x, t))\dot{x}_i \dot{x}_j.
\]  \hspace{1cm} (A.11)

Because the Euler-variation is invariant with respect to the gauge transformation (A.9) we find from (A.2) and (A.4) that

\[
\frac{\partial L_0(x, \dot{x}, t)}{\partial x_i} - \frac{d}{dt}\frac{\partial L_0(x, \dot{x}, t)}{\partial \dot{x}_i} = f_{1i}(\ddot{x}).
\]  \hspace{1cm} (A.12)

But (A.12) is a well known text-book problem (cp. Landau - Lifshitz, Vol.1) with the solution

\[
L_o(x, \dot{x}, t) = \frac{m}{2}\ddot{x}_i \dot{x}_i + \frac{d}{dt}\psi(x, t).
\]  \hspace{1cm} (A.13)

• (iv) Combining (A.8), (A.9) and A(13) we arrive at (A.1) up to a gauge transformation.
Footnotes

1. We consider here $\hbar = c = 1$, i.e. the mass dimensions of space and time coordinates are the same and equal to $-1$.

2. The noncommutativity (1.9) (though it describes the $D = 2$ case) resembles the four-dimensional noncommutative structure proposed in [17], where the space-time coordinates also commute to a number.

3. For convenience we have changed the overall sign of the modified boost generators.

4. Adding potential $V(x)$ to a one-particle dynamics leads to the broken translational invariance and we obtain $[P_i, H] = \frac{\partial}{\partial x_i} V(x) \neq 0$. In the $D = 3$ case the modification of the Galilei algebra obtained by requiring that the one-particle dynamics is described by a harmonic oscillator was first discussed by Sudbery [26].

5. Following [25] one can show that the $D = 2$ Lagrangian $L_{1+2}(x_{i;r}, \dot{x}_{i;r}, \ddot{x}_{i;r})$ $(i = 1, 2); (r = 1, 2)$ for the interacting identical point particles is the most general one which

   i) contains only linear acceleration-dependent terms
   
   ii) the potential $V$ depends only on coordinates $x_{i;r}$ and velocities $\dot{x}_{i;r}$
   
   iii) it leads to the Euler-Lagrange equations of motion which are form-invariant with respect to the $D = 2$ Galilei transformations (1.1)

6. This is the so-called Schwinger-Jordan representation (see [27]).
References

[1] J.M. Levy-Leblond, Comm. Math. Phys. 12, 64 (1969)

[2] N. Mukunda and E.C.G. Sudarshan, Classical Dynamics: A Modern Perspective, ed. Wiley, New York, 1974

[3] V. Bargmann, Ann. Math. 59, 1 (1954)

[4] D.G. Curie, T.F. Jordan and E.C.G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963)

[5] H. Leutwyler, Nuovo Cim. 37, 556 (1965)

[6] J.A. de Azcarraga and J.M. Izquierdo, Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics, Cambridge University Press, 1995

[7] J.M. Levy-Leblond, in “Group Theory and Applications”, ed. E. Loebel, Academic Press, New York 1971.

[8] V. Aldaya, J.A. de Azcarraga and R.W. Tucker, J. Geom. Phys. 3, 305 (1986)

[9] A. Ballesteros, M. Gadella and A. A. del Olmo, J. Math. Phys. 33, 3379 (1992)

[10] Y. Brihaye, C. Gonera, S. Giller and P. Kosinski, hep-th/9503046, unpublished

[11] S.K. Bose, J. Math. Phys. 36, 875 (1995) Comm. Math. Phys. 169, 385 (1995)

[12] D.R. Grigore, J. Math. Phys. 37, 460 (1996), Fortschr. d. Physik, 44, 63 (1996)

[13] L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60, 1962 (1988)

[14] R. Jackiw, in “Constraints Theory and Quantization Methods”, F. Colomo et al editors, World Scientific, Singapore, 1994, p.163.

[15] A. Kempf, J. Math. Phys. 35, 4483 (1994)
[16] J. Lukierski, H. Ruegg and W.J. Zakrzewski, Ann. Phys. 243, 90 (1995)
[17] S. Doplicher, K. Fredenhagen and J.E. Roberts, Comm. Math. Phys. 172, 187 (1995)
[18] P. Podles and S.L. Woronowicz, Comm. Math. Phys. 178, 61 (1996)
[19] J. Schwenk and J. Wess, Phys. Lett. 291B, 273 (1992)
[20] S.V. Shabanov, Journ. Phys. 26A, 2583 (1993)
[21] A. Barut and G.H. Mullen, Ann. Phys. 20, 203 (1962)
[22] D.M. Gitman and I.V. Tyutin, Quantization of Fields with Constraints, Springer Verlag 1990, Chapter 7
[23] J. Govaerts and M.S. Rashid, hep-th/9403009
[24] T. Nakamura and S. Hamamoto, Progr. Theor. Phys. 95, 469 (1996)
[25] J. Lopuszanski and P.C. Stichel, Fortschr. Phys. 45, 79 (1997)
[26] A. Sudbery, Nucl. Phys. B44, 520 (1972)
[27] J. Schwinger On Angular Momentum Report US.AEC.NYO-3071, (1951) unpublished
[28] O. Bergman and C.B. Thorn, Phys. Rev. D52, 5980 (1995)
[29] C. Thorn, “Substructure of Strings”, hep-th/9607204