Spontaneous breaking of the rotational symmetry
induced by monopoles in extra dimensions

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Abstract

We propose a field theoretical model that exhibits spontaneous breaking of the rotational symmetry. The model has a two-dimensional sphere as extra dimensions of the space-time and consists of a complex scalar field and a background gauge field. The Dirac monopole, which is invariant under the rotations of the sphere, is taken as the background field. We show that when the radius of the sphere is larger than a certain critical radius, the vacuum expectation value of the scalar field develops vortices, which pin down the rotational symmetry to lower symmetries. We evaluate the critical radius and calculate configurations of the vortices by the lowest approximation. The original model has a $U(1) \times SU(2)$ symmetry and it is broken to $U(1), U(1), D_3$ for each case of the monopole number $q = 1/2, 1, 3/2$, respectively, where $D_3$ is the symmetry group of a regular triangle. Moreover, we show that the vortex configurations are stable against higher corrections of the perturbative approximation.

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1 Introduction

Explorations of extra dimensions of the space-time have a rich history and is recently calling new interests of the physicists since, for example, existence of extra dimensions suggests a solution to the hierarchy problem [1]. Moreover, the latest developments in a brane world scenario [2] are clarifying that Nambu-Goldstone bosons appear as a result of spontaneous breaking of the translational symmetry in extra dimensions and that they play a role of a probe into the extra dimensions. Recently, some of the authors [3] constructed field theoretical models that exhibit spontaneous breaking of the translational symmetry in extra one dimension.

One of our motivations to study models of translational symmetry breaking is that they may function as a mechanism to break supersymmetry. Since the algebra of supersymmetric charges generates the algebra of translations, translational symmetry breaking involves supersymmetry breaking. Actually, some of the authors [4] have constructed models in which supersymmetry breaking is induced by spontaneous breaking of the translational symmetry. However, the previous studies are restricted to the models which have only one extra dimension, that is, $S^1$. To get a realistic model it is desirable to make models with higher extra dimensions, which have richer particle spectra.

In this Letter, we construct a model with two extra dimensions that are represented by $S^2$. This model consists of a complex scalar field and has the rotational symmetry described by the group $SU(2)$. As a seed that causes symmetry breaking, we put a background monopole field in $S^2$. We show that there is a critical radius of $S^2$: When the radius of $S^2$ exceeds the critical radius, the scalar field exhibits a vacuum expectation value that is not uniform over $S^2$, and therefore, the rotational symmetry is spontaneously broken. In the following, we will give the definition of our model and evaluate the critical radius. We will examine structure of vacua of this model for cases of a few monopole numbers, and show that the $SU(2)$ symmetry is broken to $U(1), U(1), D_3$ for each case of the monopole number $q = 1/2, 1, 3/2$, respectively. Here $D_3$ denotes the symmetry group of a regular triangle.

2 Model

First, we define a model which exhibits spontaneous breaking of the rotational symmetry. Our model is defined in a space-time $S^2 \times M^n$, where $S^2$ is a two-dimensional sphere of a radius $r$ and $M^n$ is an $n$-dimensional Minkowskian space. The spherical coordinate of $S^2$ is denoted by $(\theta, \phi)$, and the Cartesian coordinate of $M^n$ is denoted by $(x^0, x^1, \ldots, x^{n-1})$. The space $M^n$ is equipped with the metric $g_{\mu\nu} = \text{diag}(+1, -1, \ldots, -1)$. Our model consists of a complex scalar field $f$ over $S^2 \times M^n$ with a background gauge field $A$ over $S^2$. The gauge field $A$ is fixed to be the
Dirac-Wu-Yang monopole \cite{5}, which is defined as follows: Two open sets of $S^2$, $U_+ = \{(\theta, \phi) \mid \theta \neq \pi\}$ and $U_- = \{(\theta, \phi) \mid \theta \neq 0\}$, cover $S^2 = U_+ \cup U_-$. The monopole field $A$ is described by the pair of fields $(A_+, A_-)$ which consists of the components

$$A_\pm(\theta, \phi) = q(\pm 1 - \cos \theta)d\phi \quad \text{in } U_\pm,$$

respectively. The magnetic field is given by $B = dA_+ - dA_- = q \sin \theta d\theta \wedge d\phi$. Then the total magnetic flux is given by $\Phi = \int B = 4\pi q$. The scalar field $f$ is described by a pair of fields $(f_+, f_-)$, where $f_\pm$ is a complex field over $U_\pm$, respectively. The pairs of the fields, $(A_+, f_+)$ and $(A_-, f_-)$, are patched together by the gauge transformation

$$A_- = A_+ - 2q d\phi, \quad f_- = e^{-2iq\phi} f_+$$

in $U_+ \cap U_-$. To make the field $f_\pm$ single-valued, $2q$ must be an integer. The covariant derivative of the field $f$ is defined as $Df_\pm = df_\pm - iA_\pm f_\pm$. The action of our model is given by

$$S_A[f] = \int d^n x \, d\theta d\phi \, r^2 \sin \theta \left\{ -\frac{1}{r^2} \left| \frac{\partial f_\pm}{\partial \theta} \right|^2 - \frac{1}{r^2 \sin^2 \theta} \left| \frac{\partial f_\pm}{\partial \phi} - iq(\pm 1 - \cos \theta)f_\pm \right|^2 + g_{\mu\nu} \frac{\partial f_\pm^*}{\partial x^\mu} \frac{\partial f_\pm}{\partial x^\nu} + \mu^2 f_\pm^* f_\pm - \lambda (f_\pm^* f_\pm)^2 \right\}$$

with the real parameters $\mu^2, \lambda > 0$.

This model has a global symmetry $U(1) \times SU(2)$. The $U(1)$ symmetry is defined as a family of the transformations

$$(f_+, f_-) \to e^{it}(f_+, f_-)$$

with $t \in \mathbb{R}$. On the other hand, the elements of $SU(2)$ act on $S^2$ as rotation transformations. Of course, the explicit form of the background gauge field $A_\pm = q(\pm 1 - \cos \theta)d\phi$ is not invariant under arbitrary rotations. Actually, to leave the background field $A_\pm$ invariant, a rotation transformation is needed to be combined with a gauge transformation. Such a gauge transformation can be found by using the Wigner rotation, which is a well-known technique in the theory of induced representations \cite{6}. To describe the $SU(2)$ symmetry we need to provide some notation as follows: The usual spherical coordinate $(\theta, \phi)$ is assigned to a point $p \in S^2$. We define two maps $s_\pm : U_\pm \to SU(2)$ as

$$s_\pm(p) := e^{-i\sigma_3\phi/2} e^{-i\sigma_2\theta/2} e^{\pm i\sigma_3\phi/2}.$$ 

Let indices $\alpha$ and $\beta$ denote either $+$ or $-$. Suppose that a point $p \in U_\alpha$ is transformed to $g^{-1}p \in U_\beta$ by an element $g \in SU(2)$. Then the Wigner rotation is defined as

$$W_{\alpha\beta}(g; p) := s_\alpha(p)^{-1} \cdot g \cdot s_\beta(g^{-1}p).$$
It is easily verified that the value of the Wigner rotation has a form

$$W_{\alpha\beta}(g; p) = e^{-i\sigma_3 \omega/2}$$  \hspace{1cm} (2.7)

with a real number $\omega$, which defines a function $\omega_{\alpha\beta}(g; p)$. Using it we define the transformation of the fields by $g \in SU(2)$ as

$$A_\alpha(p) \rightarrow A'_\alpha(p) = A_\beta(g^{-1}p) + q \, d\omega_{\alpha\beta}(g;p), \hspace{1cm} (2.8)$$

$$f_\alpha(p) \rightarrow f'_\alpha(p) = e^{iq\omega_{\alpha\beta}(g;p)} f_\beta(g^{-1}p). \hspace{1cm} (2.9)$$

Under this transformation the monopole background field remains invariant as $A'_\pm = A_\pm$, and the action (2.3) is also left invariant.

Although calculation of the concrete value of the Wigner rotation (2.6) is cumbersome, to give a definite example we calculate it for $g = e^{-i\sigma_3 \gamma/2}$, which is a rotation around the $z$-axis. The point $p = (\theta, \phi) \in U_\pm$ is then transformed to $g^{-1}p = (\theta, \phi - \gamma) \in U_\pm$, and from the definitions (2.5) and (2.6) we get

$$W_{\pm\pm}(e^{-i\sigma_3 \gamma/2}; p) = e^{\mp i\sigma_3 \gamma/2}, \hspace{1cm} (2.10)$$

which implies $\omega_{\pm\pm}(g;p) = \pm \gamma = \text{constant in (2.7)}$. The transformations (2.8) and (2.9) now become

$$A_\pm(\theta, \phi) \rightarrow A'_{\pm}(\theta, \phi) = A_\pm(\theta, \phi - \gamma) \pm q \, d\gamma = A_\pm(\theta, \phi), \hspace{1cm} (2.11)$$

$$f_\pm(\theta, \phi) \rightarrow f'_{\pm}(\theta, \phi) = e^{\pm iq\gamma} f_\pm(\theta, \phi - \gamma). \hspace{1cm} (2.12)$$

Thus the invariance of $A_\pm$ is checked and the patching condition (2.2) is also satisfied by $(f'_+, f'_-)$. For later use, let us calculate the Wigner rotation (2.6) for the rotation around the $x$-axis with the angle $\pi$, which is given by $g = e^{-i\sigma_1 \pi/2} = -i\sigma_1$. The point $p = (\theta, \phi) \in U_\pm$ is then transformed to $g^{-1}p = (\pi - \theta, -\phi) \in U_\mp$. Then the corresponding Wigner rotation is

$$W_{\pm\mp}(e^{-i\sigma_1 \pi/2}; p) = -i\sigma_3 = e^{-i\sigma_3 \pi/2}, \hspace{1cm} (2.13)$$

and therefore we have $\omega_{\pm\mp}(g;p) = \pi$ in the place of (2.7). The transformations (2.8) and (2.9) now become

$$A_\pm(\theta, \phi) \rightarrow A'_{\pm}(\theta, \phi) = A_{\mp}(\pi - \theta, -\phi) = A_\pm(\theta, \phi), \hspace{1cm} (2.14)$$

$$f_{\pm}(\theta, \phi) \rightarrow f'_{\pm}(\theta, \phi) = e^{iq\pi} f_{\pm}(\pi - \theta, -\phi). \hspace{1cm} (2.15)$$

Thus the gauge field $A_\pm$ remains invariant again.

### 3 Rotational symmetry breaking

Assume that the monopole number $q$ is not zero and that the component fields $(f_+, f_-)$ are continuous functions. Then it is easily proved that if the field $f$ is rotationally
invariant, it must vanish identically as \( f \equiv 0 \) over \( S^2 \). The contraposition says that if \( f \) is not identically zero, \( f \) cannot be rotationally invariant over \( S^2 \). Actually, even if \( f \) takes nonzero values in some region, the value of \( f(\theta, \phi) \) must vanish generally at \( 2|q| \) points in \( S^2 \). Hence the zero points of \( f \) pin down the rotational symmetry. A zero point of \( f \) is called a vortex.

Here we describe only the outline of the proof of the above theorem. The statement that \( f \) is rotationally invariant means that both \( f_\pm \) remain invariant under the transformations (2.9) by \( SU(2) \). If \( f \) is rotationally invariant and takes nonzero value at some point in \( S^2 \), \( f_\pm \) should be nonvanishing everywhere because the group \( SU(2) \) acts on \( S^2 \) transitively. Suppose that \( f_+ \) does not vanish in the upper hemisphere, \( 0 \leq \theta \leq \pi/2 \). When the coordinate \( \phi \) runs over the range \( 0 \leq \phi \leq 2\pi \) with \( \theta \) fixed at \( \theta = \pi/2 \), the value of \( f_- = e^{-2iq\phi} f_+ \) runs around the zero in the complex plane \( 2q \) times. Thus the value of the continuous function \( f_- \) becomes zero as its value at least \( 2|q| \) times in the lower hemisphere, \( \pi/2 \leq \theta \leq \pi \). A sketch of the proof is then over.

The above theorem tells that if the scalar field exhibits a nonzero vacuum expectation value \( \langle f \rangle \), the rotational symmetry is necessarily broken. Now we would like to examine a condition for rotational symmetry breaking. In this paper we analyze the model only at the classical level. In other words, we will seek for a vacuum field configuration which minimizes the classical energy. Moreover, since the translational symmetry in \( M^n \) is kept unbroken, what we need to find is the vacuum field \( f(\theta, \phi) \) that minimizes the classical energy functional

\[
E = \int d\theta d\phi \ r^2 \sin \theta \left\{ \frac{1}{r^2} \frac{\partial f_\pm}{\partial \theta} \right\}^2 + \frac{1}{r^2 \sin^2 \theta} \left| \frac{\partial f_\pm}{\partial \phi} - iq(\pm 1 - \cos \theta) f_\pm \right|^2 - \mu^2 f_\pm^2 f_\mp^2 + \lambda (f_\pm f_\mp)^2 \right\}. \tag{3.1}
\]

Variation of the gradient energy with respect to \( f_\pm \) gives the Laplacian coupled with the monopole,

\[
- \Delta_q f_\pm := - \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial \phi} - iq(\pm 1 - \cos \theta) \right) \right]^2 f_\pm. \tag{3.2}
\]

The eigenvalue problem of the monopole Laplacian has been solved by Wu and Yang [3]; its eigenfunctions are expressed in terms of the matrix elements of unitary representations of \( SU(2) \) as

\[
f_\pm(\theta, \phi) = D_{mq}^j (\theta, \phi, \mp \phi) = \langle j, m|e^{-iJ_3 \phi} e^{-iJ_2 \theta} e^{\pm iJ_3 \phi} |j, q \rangle = e^{-i(m \mp q) \phi} d_{mq}^j (\theta) \tag{3.3}
\]

which has the eigenvalue

\[
\epsilon_j = j(j + 1) - q^2, \quad (j = |q|, |q| + 1, |q| + 2, \ldots). \tag{3.4}
\]

Each eigenvalue is degenerate with respect to the index \( m \), which has a range \( m = -j, -j + 1, \ldots, j - 1, j \). Since the lowest eigenvalue of the Laplacian is \( \epsilon_{|q|} = |q| \), the lower bound of the energy is given by

\[
E \geq \langle |q| - \mu^2 r^2 \rangle \int d\theta d\phi \sin \theta |f_\pm|^2 + \lambda r^2 \int d\theta d\phi \sin \theta |f_\pm|^4. \tag{3.5}
\]
If $|q| - \mu^2 r^2 > 0$, the inequality (3.3) is positive definite. Then the minimum of $E$ is realized only by the trivial vacuum $f \equiv 0$. On the contrary, if $|q| - \mu^2 r^2 < 0$, it is possible to find a field $f$ which has a negative energy. For example, take the eigenfunction (3.3) of the lowest eigenvalue, $f_{\pm}(\theta, \phi) = c e^{-i(m \mp q)\phi} d_{m,q}^{(\pm)}(\theta)$, where $c$ is a complex number. The energy of this field configuration can be written into the form

$$E = -a_2 |c|^2 + a_4 |c|^4$$ \hspace{1cm} (3.6)

with coefficients $a_2, a_4 > 0$. Thus, for $0 < |c|^2 < a_2/a_4$, this field realizes $E < 0$. Therefore, the minimum value of $E$ must be negative and it is realized by a nontrivial vacuum $f \neq 0$ for $|q| - \mu^2 r^2 < 0$. Thus we conclude that the rotational symmetry is spontaneously broken when the radius $r$ of $S^2$ is larger than the critical radius $r_q$, i.e.

$$r > r_q := \frac{\sqrt{|q|}}{\mu}. \hspace{1cm} (3.7)$$

We may rephrase this condition by borrowing the language of superconductivity. When the scalar field $f$ develops a nonzero vacuum expectation value in the Lagrangian (2.3), the Higgs boson with the mass $M^2 = 2 \mu^2$ would appear. The coherence length $\xi$ is defined as $\xi = M^{-1}$, which characterizes the radius of the core of a vortex. In terms of $\xi$, the condition for rotational symmetry breaking can be expressed as

$$r > \sqrt{2|q| \xi}. \hspace{1cm} (3.8)$$

### 4 Vacuum configurations

We shall now examine the concrete vacuum configuration of the scalar field. We will obtain approximate solutions of the scalar field by a variational method for three cases with the monopole number $q = 1/2, 1, 3/2$. Let us explain briefly our method of calculation. A generic field which satisfies the patching condition (2.2) can be expanded in a series of the eigenfunctions (3.3) as

$$f_{\pm}^q(\theta, \phi) = \sum_{m=-j}^{j} c_m \bar{D}_{m,q}^{(\pm)}(\theta, \phi, \mp \phi). \hspace{1cm} (4.1)$$

Here we use the lowest approximation for it; we restrict the above series to the leading terms

$$f_{\pm}^q(\theta, \phi) = \sum_{m=-|q|}^{|q|} c_m \bar{D}_{m,q}^{(\pm)}(\theta, \phi, \mp \phi) \hspace{1cm} (4.2)$$

and substitute it into the energy functional (3.1). Then the vacuum configuration is determined by the coefficients \{c_m\} that minimize the energy. In the next section this lowest approximation will be justified.
When the monopole has the smallest charge \( q = 1/2 \), in the lowest expansion (4.2) we put the coefficients as \( c_{1/2} = -ve^{i\gamma/2} \sin(\beta/2) \) and \( c_{-1/2} = ve^{-i\gamma/2} \cos(\beta/2) \) with the real parameters \((v, \beta, \gamma)\). Then we have

\[
f_{1/2}^\pm(\theta, \phi) = v \left[ -e^{-i(\phi-\gamma)/2} \sin(\beta/2) \cos(\theta/2) + e^{i(\phi-\gamma)/2} \sin(\theta/2) \right] e^{\pm i\phi/2}. \tag{4.3}
\]

Thus it can easily be seen that the value of \( f_{1/2}^\pm \) vanishes at the point \((\theta, \phi) = (\beta, \gamma)\).

Since the position of the zero point of \( f_{1/2}^\pm \) can be moved to the north pole of \( S^2 \) by using an \( SU(2) \) rotation, we can set \((\beta, \gamma) = (0, 0)\) without loss of generality. Then the field becomes simply

\[
f_{1/2}^\pm(\theta, \phi) = ve^{i(1\pm1)\phi/2} \sin(\theta/2), \tag{4.4}
\]

and thus a single vortex is located at the north pole of \( S^2 \). The energy (3.1) is then calculated as

\[
E = -2\pi \left( \mu^2 r^2 - \frac{1}{2} \right) v^2 + \frac{2}{3} \cdot 2\pi \lambda r^2 v^4. \tag{4.5}
\]

Hence the energy is minimized when

\[
v^2 = \begin{cases} 
0 & \text{for } r \leq (\sqrt{2} \mu)^{-1}, \\
\frac{3}{4\lambda r^2} \left( \mu^2 r^2 - \frac{1}{2} \right) & \text{for } r > (\sqrt{2} \mu)^{-1}.
\end{cases} \tag{4.6}
\]

This result is to be compared with the vacuum expectation value

\[
\langle f^0 \rangle^2 = \frac{1}{2\lambda} \mu^2 \tag{4.7}
\]

for the case of \( q = 0 \).

It can also be verified that the vacuum field configuration (4.4) is invariant under the combined transformation of the phase (2.4) with the rotation (2.12) of the angle \( \gamma = 2t \),

\[
(f_{1/2}^+, f_{1/2}^-)(\theta, \phi) \rightarrow e^{it}(e^{-it} f_{1/2}^+, e^{it} f_{1/2}^-)(\theta, \phi - 2t). \tag{4.8}
\]

Therefore, we conclude that when the monopole number is \( q = 1/2 \) and \( r > r_{1/2} \), the symmetry \( U(1) \times SU(2) \) is spontaneously broken to the subgroup \( U(1) \) which is defined by (4.8). Then three massless Nambu-Goldstone bosons appear due to the symmetry breaking. However, one of the three would be absorbed into the gauge boson via the Higgs mechanism if the gauge field has its own dynamical degrees of freedom.

Next let us consider the case of \( q = 1 \). Then a generic scalar field \( f \) has two vortices on \( S^2 \). The lowest expansion (4.2) now becomes

\[
f_{1}^\pm(\theta, \phi) = \frac{1}{2} \left[ c_{+1} e^{-i\phi} (1 + \cos \theta) + c_0 \sqrt{2} \sin \theta + c_{-1} e^{i\phi} (1 - \cos \theta) \right] e^{\pm i\phi}. \tag{4.9}
\]
By using the $U(1) \times SU(2)$ symmetry, which is defined in (2.4) and (2.9), it is always possible to bring the coefficients into the form
\[
(c_{+1}, c_0, c_{-1}) = \left( \frac{v}{\sqrt{2}} \sin \alpha, \frac{v}{\sqrt{2}} \cos \alpha, \frac{v}{\sqrt{2}} \sin \alpha \right)
\] (4.10)
with the real parameters $v$ and $\alpha$ ($0 \leq \alpha \leq \pi/4$). Then the field (4.9) becomes
\[
f_1^{\pm} (\theta, \phi) = \frac{1}{\sqrt{2}} v \left[ (\cos \alpha \sin \theta + \sin \alpha \cos \phi) - i \sin \alpha \cos \theta \sin \phi \right] e^{\pm i \phi}.\]
(4.11)
Thus two zero points of $f^1$ are located at $(\theta, \phi) = \left( \sin^{-1}(\tan \alpha), \pi \right)$. So, the relative displacement of two vortices is changed by the parameter $\alpha$.

When the trial function (4.11) is substituted, the total energy (3.1) is evaluated as
\[
E = \frac{4\pi}{3} (1 - \mu^2 r^2) v^2 + \frac{8\pi}{15} \left( 1 + \frac{1}{4} (1 - \cos 4\alpha) \right) \lambda r^2 v^4.
\] (4.12)
By changing $\alpha$, the minima of the potential is realized when $\cos 4\alpha = 1$, i.e. $\alpha = 0$. Then two vortices are located at opposite two points on $S^2$. On the other hand, the maxima of the potential is realized when $\cos 4\alpha = -1$, i.e. $\alpha = \pi/4$. Then two vortices coincide. Thus we observe that the vortices repel each other. The minimum of the energy is realized by $\alpha = 0$ with
\[
v^2 = \frac{5}{4\lambda} \left( 1 - \frac{1}{\mu^2 r^2} \right) \mu^2 \quad \text{for} \quad r > \frac{1}{\mu}.
\] (4.13)
The vacuum configuration (4.11) then becomes
\[
f_1^{\pm} (\theta, \phi) = \frac{1}{\sqrt{2}} v \sin \theta e^{\pm i \phi},
\] (4.14)
and has two vortices at the north and the south poles of $S^2$. This configuration (4.14) is invariant under the rotations around the $z$-axis (2.12). Thus, we conclude that when the monopole number is $q = 1$, the symmetry $U(1) \times SU(2)$ is spontaneously broken to $U(1)$ for $r > \mu^{-1}$. At this time also three Nambu-Goldstone bosons appear, although one of the three would disappear via the Higgs mechanism.

Finally, let us examine the case of $q = 3/2$. Then the number of vortices is three. By similar but tedious calculation we can determine the coefficients in the expansion (4.2). Here we show only the result: The minimum energy configuration is given, up to the $U(1) \times SU(2)$ symmetry, by
\[
f_{3/2}^{\pm} (\theta, \phi) = v \left[ e^{-i\phi/2} \cos(\theta/2) - e^{i\phi/2} \sin(\theta/2) \right]
\times \left[ e^{-i(\phi-2\pi/3)/2} \cos(\theta/2) - e^{i(\phi-2\pi/3)/2} \sin(\theta/2) \right]
\times \left[ e^{-i(\phi-4\pi/3)/2} \cos(\theta/2) - e^{i(\phi-4\pi/3)/2} \sin(\theta/2) \right] e^{\pm 3i\phi/2}
\] (4.15)
with
\[
v^2 = \frac{35}{44\lambda} \left( 1 - \frac{3}{2\mu^2 r^2} \right) \mu^2 \quad \text{for} \quad r > \sqrt{\frac{3}{2} \frac{1}{\mu}}.
\] (4.16)
We can read off the location of zero points from (4.15); they are located at $\phi = 0, 2\pi/3, 4\pi/3$ on $\theta = \pi/2$. Namely, the vortices are located at the vertices of the largest equilateral triangle on $S^2$.

Now we shall describe the symmetry of the vacuum. By the combination of the rotation around the $z$-axis (2.12) of the angle $\gamma = 2\pi/3$ with the $U(1)$ transformation (2.4) of the phase $t = \pi$, the scalar field is transformed accordingly as

$$(f_+^{3/2}, f_-^{3/2})(\theta, \phi) \rightarrow R(f_+^{3/2}, f_-^{3/2})(\theta, \phi - 2\pi/3).$$

(4.17)

This transformation is denoted by $R$ and it generates the cyclic group $Z_3$. Actually, the configuration (4.15) remains invariant under the operation of $R$. Moreover, under a combination of the $\pi$-rotation around the $x$-axis (2.13) with the $U(1)$ transformation (2.4) of the phase $t = -\pi/2$, the scalar field changes as

$$(f_+^{3/2}, f_-^{3/2})(\theta, \phi) \rightarrow T(e^{i\pi}(f_+^{3/2}, f_-^{3/2})(\pi - \theta, -\phi),$$

(4.18)

and this transformation $T$ generates another cyclic group $Z_2$. It can easily be verified that the configuration (4.15) remains invariant also under the operation of $T$. Notice that two operations $R$ and $T$ do not commute each other; they generate a nonabelian group $D_3$, which is called the dihedral group of the order three, i.e. the symmetry group of a regular triangle. We thus conclude that when the monopole number is $q = 3/2$, the symmetry $U(1) \times SU(2)$ is spontaneously broken to the discrete nonabelian group $D_3$ for $r > \sqrt{3/2} \mu^{-1}$. Therefore, the number of the Nambu-Goldstone bosons associated with the symmetry breaking is four, although one of the four would be absorbed into the gauge boson via the Higgs mechanism.

5 Stability of the vacuum

In the previous section we solved the problem to minimize the energy functional (3.1) by the variational method within the restricted function space (4.2), which is the lowest eigenspace of the monopole Laplacian (3.2). Here we would like to clarify validity of our analysis.

The first point to be clarified is that the critical radius (3.7) is exact in the context of the classical field theory. We did not recourse any approximation in the argument to decide the critical radius.

The second point to be noticed is accuracy of the concrete forms of the vacuum configurations, like (4.4), (4.14), (4.15). They are approximately calculated by the variational method with restricting the full function space (4.1) to the lowest eigenspace (4.2) of the monopole Laplacian. This restriction is a good approximation if the lowest eigenvalue $\epsilon_{|q|}$ makes the quadratic term in (3.1) negative but the next eigenvalue $\epsilon_{|q|+1}$ gives positive contribution to it. More explicitly, referring to the
eigenvalues (3.4), the necessary condition for the validity is
\[
\epsilon_{|q|} - \mu^2 r^2 < 0 < \epsilon_{|q|+1} - \mu^2 r^2, \quad \text{or} \quad \frac{\sqrt{|q|}}{\mu} < r < \frac{\sqrt{3|q| + 2}}{\mu}.
\] (5.1)

In other words, the restricted variational method is a good approximation when the radius of \(S^2\) is larger than just the critical radius but not too large.

The third point to be examined is the stability of the vacuum against perturbation by higher eigenvalue functions. The approximation is improved by including higher-order terms in the expansion (4.1). Now a question arises: If we include some of higher terms or all the terms in the expansion (4.1), does the better approximated or the precise vacuum have the same symmetry as the lowest-approximated vacuum has?

The answer is affirmative: When higher-order terms are included in the trial function (4.1), but if the radius of \(S^2\) is in the range (5.1), the vacuum calculated by the higher expansion has the same symmetry as the vacuum calculated by the lowest approximation has.

The above statement is proved as follows: First, notice that the space of the scalar fields \(f = (f_+, f_-)\) provides a unitary representation of \(U(1) \times SU(2)\) by the transformations (2.4) and (2.9). Let \(f^{(0)}\) denote the solution by the lowest approximation like (4.4), (4.14), (4.15). Assume that \(f\) is a solution by the higher-order approximation and define \(f^{(1)}\) as a correction to \(f^{(0)}\), i.e.
\[
f = f^{(0)} + f^{(1)}.
\] (5.2)

Let \(H\) be a subgroup of \(U(1) \times SU(2)\) that preserves \(f^{(0)}\) invariant. Then we decompose \(f^{(1)}\) into a component \((f^{(1)})_{\parallel}\) that is in the identity representation of \(H\), and its orthogonal complement \((f^{(1)})_{\perp}\) as
\[
f^{(1)} = (f^{(1)})_{\parallel} + (f^{(1)})_{\perp}.
\] (5.3)

When (5.2) is substituted, the energy functional (3.1) is symbolically written as
\[
E[f^{(0)} + f^{(1)}] = E[f^{(0)}] + \frac{\delta E}{\delta f}[f^{(0)}] \cdot f^{(1)} + \frac{1}{2} \frac{\delta^2 E}{\delta f^2}[f^{(0)}] (f^{(1)})^2 + \cdots.
\] (5.4)

The second term of the RHS is
\[
\frac{\delta E}{\delta f}[f^{(0)}] \cdot f^{(1)} = \int d\theta d\phi \sin \theta \left\{ f^{(0)*}(-\Delta_q - \mu^2 r^2 + 2\lambda r^2 |f^{(0)}|^2) f^{(1)} + f^{(1)*}(-\Delta_q - \mu^2 r^2 + 2\lambda r^2 |f^{(0)}|^2) f^{(0)} \right\}.
\] (5.5)

Since \(f^{(0)}\) is invariant under the actions of \(H\), \((-\Delta_q - \mu^2 r^2 + 2\lambda r^2 |f^{(0)}|^2) f^{(0)}\) is also invariant. Therefore, the term linear in the orthogonal component \((f^{(1)})_{\perp}\) vanishes as
\[
\frac{\delta E}{\delta f}[f^{(0)}] \cdot (f^{(1)})_{\perp} = 0.
\] (5.6)
Moreover, the third term of the RHS of (5.4) is
\[
\frac{1}{2} \frac{\delta^2 E}{\delta f^2} [f(0)] \cdot (f(1))^2 = \int d\theta d\phi \sin \theta \left[ (f(1))^* (-\Delta_q - \mu^2 r^2) f(1) \right.
\]
\[+ \lambda r^2 \left\{ 2 |f(0)|^2 |f(1)|^2 + (f(0)f(1)^* + f(0)^* f(1))^2 \right\} \] (5.7)

If the radius of \(S^2\) is in the range (5.1), the first term in the last line is positive definite for \((f^{(1)})_\perp\) since it is orthogonal to the space of the lowest-eigenvalue functions. It is clear that the second term is also positive definite. Thus we conclude that the quadratic form (5.7) is positive for any \((f^{(1)})_\perp\), and therefore, the lowest-approximated vacuum \(f^{(0)}\) is stable against symmetry-breaking perturbation by \((f^{(1)})_\perp\). Hence in the vacuum \(f\) the orthogonal component vanishes, i.e. \((f^{(1)})_\perp = 0\). This implies that the perturbed vacuum \(f = f^{(0)} + (f^{(1)})_\parallel\) remains invariant under the actions of \(H\). The proof is over.

6 Remarks

We would like to put a remark about a relation of our argument with the Coleman theorem [7], which forbids spontaneous breaking of continuous symmetries in two dimensions. Our model is built in higher dimensions than two; the space-time we concern is the direct product \(M^n \times S^2\) of the Minkowski space \(M^n\) with the extra \(S^2\). Thus the Coleman theorem is not applicable to our model.

Finally, we shall briefly mention directions for further development of our work. Models with higher-dimensional manifolds than \(S^2\) involve higher symmetries and richer matter contents, and such a generalization must be advantageous to construction of more realistic models. In particular, background gauge fields on higher-dimensional homogeneous spaces [8] provide a relevant basis for further development.

It is also strongly desirable to build a supersymmetric model, in which translational or rotational symmetry breaking induces supersymmetry breaking. It remains as an important question to seek for a dynamical origin of the background gauge field that triggers symmetry breaking. We expect that the Hosotani mechanism [9] or a similar mechanism functions as the origin of the background field, although it is not yet unveiled.

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