Obtaining the large angle MSW solution to the solar neutrino problem in models

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Abstract

The large mixing angle (LMA) MSW solution to the solar neutrino problem seems favored by the data at the moment over the small mixing angle (SMA) MSW solution and the vacuum (VAC) solution. In this paper the various main types of models of neutrino masses and mixings are studied from the point of view of how naturally they can give the LMA solution. Special attention is given to a very simple type of “lopsided” $SU(5)$ model.
1 Introduction

The main solutions to the solar neutrino problem are the SMA solution (small mixing angle MSW), the LMA solution (large mixing angle MSW), the LOW solution, and the VAC solution (vacuum oscillations). The experimental situation has been and remains very ambiguous. However, recent results seem somewhat to disfavor the SMA and VAC solutions. In fact the superKamiokande collaboration claims that they are excluded at the 95% confidence level [1].

On the other hand, a survey of the hundreds of models of neutrino masses and mixings published in the last three years shows that most of them yield the SMA or VAC solution, and even some that claim to obtain the LMA solution are only marginally consistent with the latest global analyses of the data. The purpose of this paper is to look at the main types of models of neutrino masses and mixing angles that have been proposed in the literature from the point of view of their ability to yield the LMA solution in a natural way. There are two aspects of this question that can be distinguished. First, one can ask whether a certain scheme or model can fit the LMA solar solution with some choice of model parameters that is not too badly fine tuned. Second, one can ask whether the model explains the LMA values of the neutrino masses and mixing angles. In order to say that a theoretical model really explains them, something close to the LMA best-fit values should emerge automatically when the parameters of the theoretical model take their most “natural values”. If a model parameter that is a priori of order one must be set to a value of ten or a tenth in order to fit the neutrino masses and mixings, one has accomodated them but not really explained them. What our survey will show is that of the great number of models that now exist in the literature, few can be said to provide an explanation (in this sense) of the LMA values of $\tan^2 \theta_{sol}$ and $\Delta m_{sol}^2$.

What are the values that are to be explained? A recent global analysis of Bahcall, Krastev and Smirnov [4] (published before the SNO results [3]) gives as the best fit values for the LMA solution $\Delta m_{sol}^2 = 4.2 \times 10^{-5} \text{eV}^2$, and $\tan^2 \theta_{sol} \simeq 0.26$. However, the allowed region is fairly broad. The 90% confidence-level allowed region given in [2] extends in $\Delta m_{sol}^2$ from about $2 \times 10^{-5} \text{eV}^2$ to $10^{-4} \text{eV}^2$, and in $\tan^2 \theta_{sol}$ from about 0.15 to 0.5; i.e. about a factor of two in either direction for both parameters. A significant aspect of the fits in [4] is that they nearly exclude exactly maximal mixing for the LMA solution. The 95% confidence level contour extends in $\tan^2 \theta_{sol}$ only up
to 0.55, and the 99% contour up to 0.7. A more recent analysis by Bahcall, Gonzalez-Garcia and Pena-Garay [4], which includes the SNO results, gives similar results for the LMA solution; their best-fit value of $\tan^2 \theta_{sol}$ being about 0.4. However, the allowed LMA regions of [4] extend somewhat higher in $\tan^2 \theta_{sol}$ than those obtained in [2]. For example, the fit of Fig. 9 of [4], in which sterile neutrinos are excluded \textit{a priori}, gives a 95% confidence-level region that extends up to about $\tan^2 \theta_{sol} = 0.8$, and a 99% confidence-level region that extends slightly past $\tan^2 \theta_{sol} = 1$.

As we shall see, the issue of how close to exact maximal mixing $\tan^2 \theta_{sol}$ is allowed to be is crucial for deciding whether several kinds of models can naturally give an acceptable LMA solution. In this paper we shall say that a model gives an LMA solution that is in comfortable agreement with the data if it predicts $\tan^2 \theta_{sol} \leq 0.8$. It is convenient to express the mass-squared splitting $\Delta m^2_{sol}$ as a fraction, which we shall call $r$, of the mass-squared splitting relevant to the atmospheric neutrino oscillations: $r \equiv \Delta m^2_{sol}/\Delta m^2_{atm}$. The best fit values give $r \simeq 1.4 \times 10^{-2}$.

The rest of the paper is organized as follows. In section 2, we shall look at the basic non-see-saw approaches to neutrino mass. We shall see that models based on such approaches have been constructed which can fit the LMA solution, but for the most part not comfortably, either because $r$ tends to come out too small or because the solar mixing angle tends to come out too close to maximality. In other words, most of the non-see-saw models that can fit the LMA solution do not really explain it in the sense that we have defined. In section 3, we look at see-saw models. Here too, most of the published models give the SMA or VAC solutions. However, we show that there are some reasonably simple “textures” that can reproduce nicely the LMA values of the neutrino masses and mixings. However, it remains unclear whether these simple textures can arise in simple models.

In section 4, we look at a well-known model that is particularly interesting for two reasons: (a) it is very simple in conception, and (b) it can explain at least the LMA value of the solar neutrino angle, although it does not explain the value of the neutrino mass splitting. It is an $SU(5)$ grand unified model with an abelian family symmetry. We shall analyze this model in some detail both analytically and numerically. We shall show how the predictions of this model can be studied statistically in a completely analytic way by assuming that the unknown parameters of the model have Gaussian distributions. This method should be easily applicable to many other kinds of models.

Section 5 is a brief summary.
2 Non-see-saw models

A. Non-see-saw models where $\theta_{atm}$ comes from $M_\nu$.

In non-see-saw models the mass matrix $M_\nu$ of the three light neutrinos is typically generated by new low-energy physics. It therefore has no relation, or only a very indirect relation, to the Dirac mass matrices of the charged leptons, the down quarks, and the up quarks (which matrices we denote $L$, $D$, and $U$ respectively). This has the great advantage of making it easy to explain why the atmospheric neutrino mixing angle is very large ($U_{\mu 3} \equiv \sin \theta_{atm} \approx 0.7$) while the corresponding quark mixing is so small ($V_{cb} \approx 0.04$). If the Dirac matrices are assumed to be hierarchical, then they would naturally give the small mixing angles seen in the quark sector. But if $M_\nu$ is unrelated to the Dirac mass matrices, it could easily have a very different form with large off-diagonal elements that gives large mixing angles. Non-see-saw models based on this idea are called Type I(1) in [5].

The tricky question for this type of model is to explain why $\Delta m^2_{sol} \ll \Delta m^2_{atm}$. If the large mixing $U_{\mu 3}$ comes from diagonalizing the 2-3 block of $M_\nu$, one would expect that $m_2$ and $m_3$, the second and third eigenvalues of $M_\nu$, would have similar magnitudes, in which case typically so would $\Delta m^2_{sol} = m_2^2 - m_1^2$ and $\Delta m^2_{atm} = m_3^2 - m_2^2$. The challenge then is to reconcile the hierarchy seen in the eigenvalues of $M_\nu$ with the large atmospheric mixing angle. To do this requires a special form of $M_\nu$. Two special forms have been found viable in constructing realistic models, one leads to a so-called “inverted hierarchy” $m_1 \approx m_2 \gg m_3$, and the other to the ordinary hierarchy $m_1 \ll m_2 \ll m_3$. We shall consider these in turn.

Inverted hierarchy models.

Inverted hierarchy models have the following special form for $M_\nu$:

$$M_\nu = \begin{pmatrix} m_{11} & cM & sM \\ cM & m_{22} & m_{23} \\ sM & m_{23} & m_{33} \end{pmatrix}.$$ (1)

Here $c \equiv \cos \theta$ and $s \equiv \sin \theta$, where $\theta \sim 1$, and $m_{ij} \ll M$. One can diagonalize this matrix in stages, the first step being to rotate by angle $\theta$ in the 2-3 plane, bringing the matrix to the form
\[ M'_\nu = \begin{pmatrix} m_{11} & M & 0 \\ M & m'_{22} & m'_{23} \\ 0 & m'_{23} & m'_{33} \end{pmatrix}. \] (2)

One sees immediately that \( m_1 \simeq m_2 \simeq M \gg m_3 \). The mass-squared splitting relevant to atmospheric oscillations is \( \Delta m^2_{atm} \simeq M^2 \), whereas the splitting relevant to solar oscillations is \( \Delta m^2_{sol} \simeq 2(m_{11} + m'_{22})M \), which is much smaller, as required by all the viable solar solutions. The atmospheric angle gets a contribution \( \theta \sim 1 \) from diagonalizing the 2-3 block of \( M_\nu \), so in the absence of some unlikely cancellation it will be large, as observed. But what of the solar angle? From the fact that the 1-2 block of Eq. (2) has a pseudo-Dirac form it is apparent that the solar mixing angle will be close to maximal. Consequently, inverted hierarchy models cannot give the SMA solar solution, but rather give “bimaximal” mixing.

The inverted hierarchy form of Eq. (1) can arise in several plausible ways. One example is the Zee type of model [6]. In the Zee model [7] there is a singly charged singlet scalar field \( h^+ \), which is allowed by the standard model quantum numbers to couple (antisymmetrically) to both a pair of lepton doublets \( (h^+L_iL_j) \) and a pair of Higgs doublets \( (h^+\Phi_a\Phi_b) \), assuming that more than one Higgs doublet exists. If both types of coupling are present, a conserved lepton number cannot be consistently assigned to \( h^+ \), and consequently \( \Delta L = 2 \) Majorana masses for the left-handed neutrinos arise at one-loop level. The resulting one-loop mass matrix has the form

\[ M_\nu = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}. \] (3)

For \( c \ll a \sim b \), this has the desired inverted hierarchy form.

The inverted hierarchy form can also arise in models with an approximately conserved \( L_e - L_\mu - L_\tau \) lepton number. If this quantum number is exactly conserved, then only the 12, 21, 13, and 31 elements of \( M_\nu \) can be nonvanishing. If there are small violations of \( L_e - L_\mu - L_\tau \) the form in Eq. (1) can result [8].

The question of present interest to us is whether the inverted hierarchy can give an acceptable LMA solution. To answer this one must look more closely at the solar neutrino mixing angle. This is given by \( \theta_{sol} = \theta^L_{12} - \theta^L_{12} \), where the two angles on the right-hand side are the contributions that come
from diagonalizing $M_\nu$ and $L$ respectively. From Eq. (2) it is easily found that $	an 2\theta^\nu_{12} \simeq 2M/(m'_{22} - m_{11})$, so that $\theta^\nu_{12} \simeq \pi/4 - (m'_{22} - m_{11})/4M$. We have already seen that $r \equiv \Delta m^2_{s\text{ol}}/\Delta m^2_{\text{atm}} \simeq 2(m'_{22} + m_{11})/M$. Requiring that this be of order $10^{-2}$ as needed for the LMA solution, and assuming that there are no accidental cancellations, one has that $\theta^\nu_{12} \simeq \pi/4 + O(10^{-3})$. If $\theta^L_{12}$ vanished, this would give $\tan^2 \theta_{s\text{ol}} = 1 + O(10^{-3})$, which is too close to maximal mixing to be in comfortable accord with the global fits. However, one expects that $\theta^L_{12} \sim \sqrt{m_e/m_\mu} \simeq 0.07$. This contribution can have any complex phase relative to the contribution from $M_\nu$, and can therefore increase or decrease $\tan^2 \theta_{s\text{ol}}$ from unity. If one assumes that $\theta^L_{12} = 0.07$ and has a relative minus sign to $\theta^\nu_{12}$, then $\tan^2 \theta_{s\text{ol}} \simeq 0.75$, which is consistent with the global LMA fits. However, one can see that the tendency of inverted hierarchy models is to give solar mixing that is closer to maximality than to the best-fit LMA value of $\tan^2 \theta_{s\text{ol}} \approx 0.4$. This is one reason why many of the published inverted hierarchy models claim a better fit to the VAC solution than to the LMA solution [9]. A significant reduction of the experimental upper limits on $\tan^2 \theta_{s\text{ol}}$ would make the inverted hierarchy idea much less plausible as an explanation of the LMA solution. For example, a value of $\tan^2 \theta_{s\text{ol}} = 0.5$, would imply in the inverted hierarchy context that $\tan \theta^L_{12} \simeq 0.17 \simeq 2.5\sqrt{m_e/m_\mu}$, which would require a very special form of the 1-2 block of $L$.

**Ordinary hierarchy models.**

The other possibility for non-see-saw models that gives a large atmospheric neutrino mixing angle coming from $M_\nu$ and a hierarchy in the mass-squared splittings is

$$M_\nu \simeq \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & s^2 M & s c M \\ m_{13} & s c M & c^2 M \end{pmatrix}. \quad (4)$$

Here, again, $c \equiv \cos \theta$ and $s \equiv \sin \theta$, where $\theta \sim 1$, and $m_{ij} \ll M$. As written, the 2-3 block of the matrix has vanishing determinant; however, it is assumed that there are small corrections to these elements, which we have not written.

As in the case of the inverted hierarchy models, one can diagonalize this in stages, starting with a rotation by angle $\theta$ in the 2-3 plane. The result of such a rotation is to bring the matrix to the form
\[ M'_\nu = \begin{pmatrix} m_{11} & m'_{12} & m'_{13} \\ m'_{12} & m'_{22} & 0 \\ m'_{13} & 0 & M \end{pmatrix} \]  

Because of the small corrections to the 2-3 block that were just mentioned, the 22 element in Eq. (5) does not vanish, but is small compared to \( M \). This matrix gives \( \Delta m^2_{sol} = O(m^2_{ij}) \) and \( \Delta m^2_{atm} \approx M^2 \). Thus the right hierarchy of splittings for any of the solutions can be achieved for the appropriate values of \( m_{ij}/M \). In contrast to the inverted hierarchy form, this form can give either small or large \( \theta_{sol} \), and in the large-angle case there is no preference for values of \( \theta_{sol} \) that are very close to maximal.

The form in Eq. (4) is clearly special in the sense that the 2-3 block is approximately of rank one. This would be unnatural unless some symmetry or mechanism guaranteed it. One possibility is that this form arises from a nonabelian flavor symmetry \( U \), however, this is difficult to achieve. Rather, almost all published models that achieve this form in a natural way use the idea of factorization. The idea of factorization is that the dominant contribution to the neutrino mass matrix has the form \( (M_\nu)_{ij} = f_i f_j \), which is obviously of rank one. If \( f_1 \ll f_2, f_3 \), this dominant term reproduces the large elements in Eq. (4). The condition that \( f_1 \) is small compared to \( f_2 \) and \( f_3 \) is necessary to satisfy the experimental constraint that \( U_{e3} \leq 0.15 \). One drawback of most models based on factorization is that they do not explain why \( f_1 \) is small.

A factorized form can arise in various ways in non-see-saw models. A much studied example is supersymmetry with terms in the superpotential that violate both lepton number and R-parity. Cubic terms of this type are \( \lambda_{ijk} L_i L_j E^+_k \) and \( \lambda'_{ijk} L_i Q_j D^c_k \). The latter leads to one-loop \( \Delta L = 2 \) neutrino mass diagrams, in which a neutrino converts into a virtual quark-squark pair. Assuming that the LR squark masses are proportional to the corresponding quark masses, this diagram gives \( (M_\nu)_{ij} \propto \lambda'_{klt} \lambda'_{ijk} m_{dk} m_{dl} \). Consequently, the b-quark/b-squark loop dominates, and gives a contribution that is proportional to \( \lambda'_{333} \lambda'_{333} m^2_b \), which obviously has a factorized form. This gives only the heaviest neutrino mass, \( m_3 \). The second largest neutrino mass comes from a similar diagram with both b and s quarks/squarks in the loop. Consequently, one has that \( r \cong (m_2/m_3)^2 \sim (m_s/m_b)^2 \sim 3 \times 10^{-4} \). This is much smaller than the value of \( 1.4 \times 10^{-2} \) preferred by experiment; however, there are several unknown parameters that come into this calculation, such as the couplings \( \lambda'_{ijk} \), so that nothing prevents the right LMA value of \( r \) from being
obtained [11]. However, the model does not really explain the magnitude of $r$.

We have only considered the effects of the cubic lepton-number-violating and R-parity-violating terms in the superpotential. There are also in general bilinear terms of the form $L_i H_u$. These have the effect of mixing leptons and Higgs fields, and so allow the sneutrino fields to acquire non-vanishing vacuum expectation values. That, in turn, through the sneutrino-neutrino-neutralino coupling gives a tree-level neutrino mass in which the neutralino plays the role of “right-handed neutrino”. It is easily seen that this tree-level mass has a factorized form and gives mass only to one neutrino, i.e. $m_3$. The other neutrino masses, $m_2$ and $m_1$, arise from the one-loop diagrams previously discussed. In consequence, in such models where both cubic and bilinear R-parity-violating terms contribute to $M_\nu$ one expects that $r \approx (m_2/m_3)^2 \sim (\text{loop/tree})^2 \ll 10^{-2}$. For this reason, most analyses of supersymmetric models in which the bilinear R-parity-violating terms contribute to $M_\nu$ conclude that there is much more parameter space for the VAC solution than for the LMA solution, i.e. the LMA solution requires special choices or tuning of parameters [12]. However, in [13] it is shown that under certain assumptions (specifically, that there are only bilinear R-parity-violating terms and that the SUSY-breaking terms are non-universal) the LMA solution can be achieved without fine tuning. Nevertheless, it seems, on the whole, that the SUSY models with R-parity breaking do not do well in explaining the LMA value of $\Delta m^2_{\text{sol}}$.

Another possibility for obtaining an approximately factorized form for $M_\nu$ that has been much studied in the literature is called “single right-handed neutrino dominance” (SRHND) [14]. As the name suggests, the idea here is that instead of there being three right-handed neutrinos, one in each family, as there are in typical grand unified theories or Pati-Salam models, there is just one right-handed neutrino, $N^c$, which can have mass terms $M_R N^c N^c + f_i (\nu_i N^c) \langle H \rangle$. Integrating out $N^c$ gives a rank-1 factorized contribution to $M_\nu$. If one assumes that $N^c$ couples with almost equal strength to the $\mu$ and $\tau$ neutrinos, and (for some reason not generally explained) only weakly to the electron neutrino, the large terms in Eq. (4) are reproduced.

One way to explain the smallness of the coupling of $N^c$ to the electron neutrino would be to impose a symmetry that distinguishes $\nu_e$ from the $\nu_\mu$ and $\nu_\tau$ (but does not distinguish the latter from each other). Such a symmetry would also tend to suppress mixing between the $\nu_e$ and the heavier neutrinos, and thus give the SMA solar solution, as in the model of [13].
Since the right-handed neutrino only gives mass to one neutrino, some other mechanism must be found to give mass to the other neutrinos. In [16] the lighter two neutrino masses arise from loop effects. In [17] they arise at tree level from integrating out other heavy states that have different flavor quantum numbers than $N^c$. In [18] they arise from operators of the form $\nu_i\nu_j H_u H_u /M_{Pl}$, which it is argued are generally there anyway, the idea being to avoid having to invent new beyond-the-standard-model physics to account for each kind of neutrino mass. In all these cases, $m_1$ and $m_2$ are much less than $m_3$, though the specific reason is different in each case: in [16] they are suppressed by loop factors, in [17] by small flavor-breaking parameters, and in [18] by $M_R /M_{Pl}$. That SRHND models tend to give a strong hierarchy in neutrino masses is what one would naturally expect. Since the mechanisms that generate the largest neutrino mass and the other neutrino masses are different, there is no reason a priori that they should yield masses of similar scale. Rather, it would be a coincidence calling for an explanation if they did. Because most SRHND models give $m_2 \ll m_3$ they yield the VAC solution or SMA solution to the solar neutrino problem rather than the LMA solution [16, 17, 18].

To obtain the LMA solution, one wants $m_3$ and $m_2$ to be only about a factor of ten in ratio. This suggests that they arise from the same basic mechanism. One possibility is that all three neutrino masses arise from integrating out right-handed neutrinos, but that one of those right-handed neutrinos is somewhat lighter than the others and so dominates to some extent, but not by a large factor. However, this would really be just a special case of the ordinary see-saw mechanism, which we will discuss in the next section. In fact, the structure given in Eq. (13) is really based on this idea, which was proposed in [19].

B. Non-see-saw models where $\theta_{atm}$ comes from $L$. 

There is another class of non-see-saw models in which the large atmospheric neutrino angle comes predominantly from the diagonalization of the charged lepton mass matrix $L$. (This class of models is called Type II(1) in [5].) This has the advantage that it becomes easy to reconcile the largeness of $\theta_{atm}$ with the smallness of $\Delta m^2_{sol}/\Delta m^2_{atm}$, since the former comes from $L$ while the latter comes from $M_\nu$. On the other hand another issue arises for this class of models, namely explaining why the CKM angles are small. Since the form of $L$ is such as to give a large mixing angle $\theta_{atm}$, one would naturally
expect that the Dirac mass matrices $D$ and $U$ of the quarks would be such as to give similarly large contributions to the CKM angles. The point is that in many kinds of models the Dirac mass matrices $L$, $D$, and $U$ are closely related to each other.

One possibility is that there are indeed large contributions to the CKM angles coming from $U$ and $D$, but that these nearly cancel. This possibility is realized in a much-studied class of models based on the idea of “flavor democracy” [20]. In flavor democracy models it is assumed that all the Dirac mass matrices have approximately the “democratic” form

\[
\begin{pmatrix}
  1 & 1 & 1 \\
  1 & 1 & 1 \\
  1 & 1 & 1
\end{pmatrix}
\]  

(6)

This form can be enforced by permutation symmetries among the three families. In the limit of exact flavor democracy, the matrices $U$ and $D$ are exactly of the same form, so that flavor mixing in the quark sector cancels out. On the other hand, it is assumed that the neutrino mass matrix $M_\nu$ has a very different form. In most papers it is assumed to be approximately proportional to the identity matrix, though in some papers it is only assumed to be nearly diagonal. As a result, for the leptonic mixing angles there is no cancellation such as makes the CKM angles small.

The MNS mixing matrix for the leptons is given by $U_{MNS} = U_L^\dagger U_\nu$, where $U_L$ and $U_\nu$ are the unitary matrices that diagonalize respectively $L^\dagger L$ and $M_\nu^\dagger M_\nu$. If $L$ has exactly the democratic form, then

\[
U_L^\dagger = \begin{pmatrix}
  1/\sqrt{2} & -1/\sqrt{2} & 0 \\
  1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\
  1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}
\end{pmatrix} \equiv U_{FD}.
\]  

(7)

If the mass matrix of the neutrinos is exactly diagonal, then $U_{MNS} = U_L^\dagger = U_{FD}$. This would give $\sin^2 2\theta_{atm} = 8/9$, which is consistent with the data, and $\tan^2 \theta_{sol} = 1$, i.e. exactly maximal mixing for solar neutrinos. However, the matrix $L$ clearly cannot have exactly the democratic form, as that is rank one and would give $m_e = m_\mu = 0$. There must therefore be small corrections to $L$ coming from the breaking of the permutation symmetries. These corrections not only generate masses for the electron and muon but also make the angle $\theta_{sol}$ deviate from maximality. For the simplest and most widely assumed form of these corrections to $L$, one can calculate the corrections to $\theta_{sol}$ in
terms of \( \sqrt{m_e/m_\mu} \). One finds, still assuming that \( M_\nu \) is diagonal, that

\[
\tan^2 \theta_{sol} = 1 - \frac{4}{3} \sqrt{m_e/m_\mu} \approx 0.84,
\]
or equivalently \( \sin^2 2\theta_{sol} = 0.993 \). This is too close to unity to be in comfortable agreement with the LMA global fits. Almost all published models based on flavor democracy have \( \tan^2 \theta_{sol} \approx 1 \), or else obtain smaller values by fine-tuning. However, Tanimoto, Watari, and Yanagida have a version in which there are small corrections to \( M_\nu \) that can reduce \( \tan^2 \theta_{sol} \) to the region preferred by the LMA fits \([21]\). While this shows that it is possible within the flavor democracy framework to construct LMA models that can fit the data, it does not appear that flavor democracy does a good job of explaining the LMA value of \( \tan^2 \theta_{sol} \). Flavor democracy is more naturally compatible with the VAC or LOW solutions.

We may summarize the situation by saying that most schemes that have been proposed based on non-see-saw mechanisms neither very comfortably fit nor really do much to explain the values of the neutrino parameters required for the LMA solution to the solar neutrino problem. The great majority of non-see-saw models in the literature more naturally give the SMA or VAC solutions. There are exceptions, which we have noted above. How close \( \tan^2 \theta_{sol} \) is to 1 is a crucial issue.

### 3 See-saw models

The see-saw mechanism is usually associated with grand unification. In \( SO(10) \) grand unified models, and in most other unified schemes except \( SU(5) \), the existence of one right-handed neutrino for each family is required to make up complete multiplets of the unified group. Moreover, the see-saw formula \( M_\nu = -N^T M_R^{-1} N \), where \( M_R \) is the Majorana mass matrix of the right-handed neutrinos and \( N \) is the Dirac mass matrix of the neutrinos, gives neutrino masses in the range required by experiment if the scale of \( M_R \) is near the grand unified scale. Thus both the existence of neutrino masses and their magnitude are elegantly accounted for by the related ideas of grand unification and the see-saw mechanism. In this section, we shall therefore assume that we are dealing with a grand unified model.

**A. See-saw models where \( \theta_{atm} \) comes from \( M_\nu \).**
In models based on $SO(10)$, there is generally a close relationship among the four Dirac mass matrices $N, U, D,$ and $L$. Indeed, in the minimal $SO(10)$ model (which is too simple to be realistic) $N = U \propto D = L$. The smallness of the CKM angles and the small interfamily mass ratios of the quarks can be explained by assuming that the matrices $U$ and $D$ are “hierarchical” in form. There are two simple kinds of hierarchical mass matrix that are frequently encountered in models

$$\begin{pmatrix}
(\epsilon')^2 & \epsilon' & \epsilon' \\
\epsilon & \epsilon^2 & \epsilon \\
\epsilon' & \epsilon & 1
\end{pmatrix}, \quad \begin{pmatrix}
\epsilon' & \epsilon' & \epsilon' \\
\epsilon' & \epsilon & \epsilon \\
\epsilon' & \epsilon & 1
\end{pmatrix},$$

(8)

where $\epsilon' \ll \epsilon \ll 1$. The entries in these matrices as written are to be understood as giving only the order in the small parameters of the entries. The first form in Eq. (8) has what may be called a “geometric hierarchy”, since an off-diagonal element is of the same order as the geometric mean of the corresponding diagonal elements. The second form in Eq. (8) has what may be called a “cascade hierarchy”, since the matrix is made up of successive tiers, a 1-by-1, a 2-by-2, and a 3-by-3, of ever smaller magnitude. Both forms in Eq. (8), if applied to the quark masses, give $V_{cb} \sim \epsilon, V_{us} \sim \epsilon'/\epsilon$, and $V_{ub} \sim \epsilon' \sim V_{us}V_{cb}$.

While there is as a rule a close relation among the four Dirac mass matrices in $SO(10)$, the Majorana mass matrix $M_R$ of the right-handed neutrinos can be quite different in form. For example, in minimal $SO(10)$ the Dirac mass matrices all come from the same term, $16,16,10_H$, whereas the matrix $M_R$ comes from different terms, either $16,16,126_H$ or $16,16,10_H,16_H$. A reasonable hypothesis is that the CKM angles are small because of the hierarchical nature of the Dirac matrices, while the largeness of $\theta_{\text{atm}}$ and possibly of $\theta_{\text{sol}}$ has to do with the very different form of $M_R$. Models based on this idea were classified in [5] as Type I(2).

A potential difficulty with this idea is that if the Dirac mass matrix of the neutrinos $N$ has a hierarchical form it tends, through the see-saw formula, to make $M_\nu$ also have a hierarchical form, indeed a more strongly hierarchical form. For example, suppose $N = \text{diag}(\epsilon', \epsilon, 1)$, and we parametrize $M_R^{-1}$ as $(M_R^{-1})_{ij} = a_{ij}$. Then

$$M_\nu = \begin{pmatrix}
(\epsilon')^2 a_{11} & \epsilon' a_{12} & \epsilon' a_{13} \\
\epsilon a_{12} & \epsilon^2 a_{22} & \epsilon a_{23} \\
\epsilon' a_{13} & \epsilon a_{23} & a_{33}
\end{pmatrix}.$$

(9)

12
If all the $a_{ij}$ are of the same order, then $\tan 2\theta_{23} \approx 2\epsilon a_{23}/(a_{33} - \epsilon^2 a_{22}) \sim \epsilon$. In order for $\theta_{atm}$ to come primarily from diagonalizing $M_\nu$, one must have $\theta_{23} \sim 1$. Clearly, this is only possible with some special form of $M_R$. One possibility is that $a_{23}/a_{33} \sim \epsilon^{-1}$. If this is true, then there is a hierarchy among the elements of $M_R$ that is related to the hierarchy among the elements of $N$. That is, the atmospheric neutrino mixing angle is of order unity because of a conspiracy between the Majorana and Dirac neutrino mass matrices. This would appear to be somewhat unnatural in a theory in which $M_R$ and $N$ have different origins, as is typically the case in unified models. On the other hand, this “Dirac-Majorana conspiracy” might not be unnatural in a model in which the same flavor symmetry, and the same small parameter characterizing the breaking of that symmetry, controlled the structure of both matrices. A good example of such “correlated hierarchies” is the model of [22].

A very important question is whether $\theta_{atm}$ can naturally be of order unity even if the Dirac matrices are hierarchical and the parameters in $M_R$ have no direct relationship to those of $N$. The answer is yes. In [23] and [15] interesting examples were found that satisfy these criteria. The specific forms given in those papers happen to lead to the SMA solar solution, but with some modifications they can also yield a satisfactory LMA solution, as we will now see.

**Example 1:** The following structure is closely related to that in [23]:

$$N = \begin{pmatrix} d\epsilon' & e\epsilon' & f\epsilon' \\ g\epsilon' & a\epsilon & b\epsilon \\ h\epsilon' & \epsilon\epsilon' & 1 \end{pmatrix} \ m_N, \quad M_R = \begin{pmatrix} 0 & 0 & A \\ 0 & 1 & 0 \\ A & 0 & 0 \end{pmatrix} \ m_R. \quad (10)$$

Here $a, ..., h$ are of order one, $\epsilon' \ll \epsilon \ll 1$, and $(\epsilon'/\epsilon)\epsilon^{-1} \ll A \ll \epsilon^{-1}$. Keeping only the significant terms, the resulting light neutrino mass matrix $M_\nu = -N^T M_R^{-1} N$ is

$$M_\nu \cong -\begin{pmatrix} O(\epsilon'^2) & O(\epsilon\epsilon') & d\epsilon'/A \\ O(\epsilon\epsilon') & a^2\epsilon^2 & ab\epsilon^2 + \epsilon\epsilon'/A \\ d\epsilon'/A & ab\epsilon^2 + \epsilon\epsilon'/A & b^2\epsilon^2 + 2\epsilon\epsilon'/A \end{pmatrix} \begin{pmatrix} m_N^2 \\ m_R^2 \end{pmatrix}. \quad (11)$$

A rotation in the 23 plane by angle $\theta \cong \tan^{-1}(a/b) \sim 1$ diagonalizes the 2-3 block and brings the matrix to the form
If one assumes that $e'/A \sim \epsilon^2/10$, then it is apparent that $\Delta m_{sol}^2/\Delta m_{atm}^2 \sim 10^{-2}$ as required for the LMA solution. It is to be observed that the 12 and 21 elements of this matrix are of the same order as the 22 element. This is just what is needed to get the right value of $\theta_{sol}$ for the LMA solution, i.e. a value that is of order unity, but not very close to maximal. For example, if the 12 element is exactly equal to the 22 element, then $\tan^2 \theta_{sol} \cong 0.39$, which is in excellent agreement with the LMA best-fit value given in [4].

An examination of this matrix reveals that in obtaining the LMA solution a crucial role is played by the “cascade hierarchy” form of $N$. In particular, it is important that $d$ be of the same order as $e$ and $f$, which would not be the case if $N$ had a “geometric hierarchy” form. It should also be noted that the largeness of the atmospheric angle is also traceable to the cascade hierarchy form of $N$, and specifically to the fact that $b$ is of the same order as $a$.

**Example 2:** The following structure is closely related to that given in [19]

$$N = \begin{pmatrix} de' & ce' & fe' \\ ge' & ae & be' \\ he' & ce & 1 \end{pmatrix} m_N, \quad M_R = \begin{pmatrix} B & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} m_R. \quad (13)$$

As in the last example, $a, \ldots, h$ are of order one, and $e' \ll \epsilon \ll 1$. If one also assumes that $e^2/A \gg \epsilon^2/B \gg 1$, then the light neutrino mass matrix $M_\nu = -N^TM_R^{-1}N$ takes the form (keeping only the important terms):

$$M_\nu \cong \begin{pmatrix} O(e^2/A) & gae'e'/A + de'e'^2/B & gbce'/A + df\epsilon^2/B \\ gae'/A + de'e'^2/B & a^2\epsilon^2/A + e^2\epsilon^2/B & ab^2/A + e\epsilon f^2/B \\ gbce'/A + df\epsilon^2/B & abe^2/A + e\epsilon f^2/B & b^2\epsilon^2/A + f^2\epsilon^2/B \end{pmatrix} \frac{m_N^2}{m_R}. \quad (12)$$

A rotation in the 23 plane by angle $\theta \cong \tan^{-1}(a/b) \sim 1$ diagonalizes the 2-3 block and brings the matrix to the form

$$M'_\nu \cong \begin{pmatrix} O(\epsilon^2/A) & d(\epsilon^2)/B & O(\epsilon e'/A) + O(\epsilon^2/B) \\ \frac{d(\epsilon^2)}{\sqrt{a^2+b^2}}\epsilon^2/B & O(\epsilon^2/A) & 0 \\ O(\epsilon e'/A) + O(\epsilon^2/B) & 0 & (a^2+b^2)\epsilon^2/A \end{pmatrix} \frac{m_N^2}{m_R}. \quad (14)$$
The same remarks apply as in the previous example. The largeness of both the atmospheric neutrino angle and the solar neutrino angle can be traced to the “cascade hierarchy” form of $N$. Because the 12, 21, and 22 element are of the same order, the solar angle is (as required for the LMA solution) of order one, but not very close to maximal. The right ratio of mass splittings for the LMA solution can be obtained if $e^2/B \sim 10^{-1} e^2/A$.

These two examples show that there are reasonable forms or “textures” for the mass matrices in the context of the see-saw mechanism that can quite naturally yield the LMA solution. However, actual detailed models based on these textures have not been constructed. It is also not clear how simple it is for the seemingly required “cascade hierarchy” form to arise in the framework of grand unified models. Finally, it should be noted that while some forms for $N$ and $M_R$ can be identified which would naturally give the LMA solution, most of the viable forms give the SMA or VAC solutions, and indeed the great majority of see-saw models published in the literature give the latter solutions rather than the LMA solution.

B. See-saw models where $\theta_{atm}$ comes largely from $L$.

As we saw in part B. of section 2, there are advantages to models in which the large atmospheric angle comes primarily from the charged lepton mass matrix $L$. In particular it becomes easy to reconcile the largeness of $\theta_{atm}$ with the smallness of $\Delta m^2_{sol}/\Delta m^2_{atm}$, since they come from different matrices: the former from $L$ and the latter from $M_\nu$. As we also saw, however, having a large angle arise from the diagonalization of $L$ raises the question of why a large CKM angle does not arise from the diagonalization of the quark mass matrices $D$ and $U$. The answer in the “flavor democracy” models, was that the CKM angles are small by a cancellation caused by an approximate symmetry. The possibility of a different and very elegant answer arises in the context of grand unified models, especially if an $SU(5)$ symmetry plays a role in the form of the fermion mass matrices. In $SU(5)$ the left-handed (right-handed) charged leptons are in the same multiplets as the (CP conjugates) of the right-handed (left-handed) down quarks. Since the large mixing angle $\theta_{atm}$ is a mixing of left-handed leptons, and specifically of left-handed charged leptons in the scenarios we are presently considering, it would generally be related by $SU(5)$ to a large mixing angle for the right-handed down quarks. But such a right-handed mixing angle has nothing to do with the observed CKM angles. On the other hand,
the small CKM angles are related by $SU(5)$ to small mixings of the right-handed leptons, which are irrelevant to neutrino oscillation phenomena.

These considerations lead naturally to the idea that the matrix $L$ is such as to give large left-handed mixings and small right-handed mixings, so that $\theta_{\text{atm}}$ can be large while $V_{cb}$ is small. In other words, $L$ must be highly asymmetric or “lopsided” to use the term suggested in [24]. (It should be noted that in $SU(5)$, $L$ is related to $D^T$, but not in general to $U$ or $N$. Thus while the lopsidedness of $L$ entails the lopsidedness of $D$, there is no reason to expect $N$ and $U$ to be lopsided, and in the examples we give below they are not.)

Many models have been proposed based on this idea of lopsided mass matrices [25, 26]. These are classified as Type II(2) in [3]. The great majority of these models give the SMA solution or the VAC solution to the solar neutrino problem. However, it is possible to obtain the LMA solution as well [26]. In a lopsided model in which the LMA, LOW, or VAC solution arises, the large atmospheric angle can come primarily from the matrix $L$ while the large solar angle can come from the matrix $M_\nu$. This actually has the virtue of simplicity, since the form of $M_\nu$ is less constrained than in models where it must give rise both to a large $\theta_{\text{atm}}$ and large $\theta_{\text{sol}}$.

An example of how the LMA solution might arise in a see-saw model with lopsided $L$ is provided by the following matrices:

$$N = \begin{pmatrix} d\epsilon' & e\epsilon' & f\epsilon' \\ g\epsilon' & a\epsilon & b\epsilon \\ h\epsilon' & c\epsilon & 1 \end{pmatrix} \ m_N, \quad M_R = \begin{pmatrix} 0 & A & 0 \\ A & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ m_R. \quad (14)$$

As before, it is assumed that $a, ..., h$ are of order one, and that $\epsilon' \ll \epsilon \ll 1$. Suppose the value of $A$ is such that $\epsilon^2, \epsilon' \ll \epsilon/\epsilon' \ll 1$. Then, keeping only the most important terms, the light neutrino mass matrix has the form:

$$M_\nu \cong - \begin{pmatrix} O(\epsilon'^2/A) & a\epsilon\epsilon'/A & b\epsilon\epsilon'/A \\ a\epsilon\epsilon'/A & 2a\epsilon\epsilon'/A & (af + be)\epsilon'/A + \epsilon \epsilon' \\ b\epsilon\epsilon'/A & (af + be)\epsilon'/A + c\epsilon \end{pmatrix} \frac{m_N^2}{m_R}. \quad (\text{16})$$

Here the 23 and 32 elements are small compared to the 33 element, leading to a small contribution to $\theta_{\text{atm}}$; but that is alright, since $\theta_{\text{atm}}$ is supposed to arise primarily from diagonalizing $L$ in this class of models. As in the previous examples, one sees that the 12, 21 and 22 elements are of the same order, giving a large, but not nearly maximal, value of $\theta_{\text{sol}}$ as required by the
LMA solution. To get the right ratio of neutrino mass-squared splittings one needs $\epsilon \epsilon'/A \sim 10^{-1}$.

4 An $SU(5)$ pattern

A particularly interesting kind of pattern can arise very simply in the context of $SU(5)$ with abelian flavor symmetry. Consider an $SU(5)$ model with a $U(1)$ flavor symmetry under which the quark and lepton multiplets have the following charge assignments: $10_1(2), 10_2(1), 10_3(0), \bar{5}_1(1), \bar{5}_2(0), \bar{5}_3(0)$. Let the breaking of the $U(1)$ flavor symmetry be done by a field $\chi$ having $U(1)$ charge $-1$, and an expectation value $\langle \chi \rangle / M_{\text{flavor}} = \epsilon \ll 1$. Then the mass matrices of the quarks and charged leptons will have the following forms.

$$D \sim \begin{pmatrix} \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} m_D, \quad U \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} m_U,$$

$$L \sim \begin{pmatrix} \epsilon^3 & \epsilon^2 & \epsilon^2 \\ \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \end{pmatrix} m_D.$$

(15)

Note that $L$ and $D$ have the lopsided form. It has been pointed out in several papers in the literature that these forms give a very good account of the mass ratios and mixing angles of the quarks and leptons [27].

One can see that the quantities $m_\mu/m_\tau \simeq 1/17$, $m_u/m_c \simeq 1/50$, and $V_{cb} \simeq 1/25$ all are of order $\epsilon$. Thus, $\epsilon$ is roughly of order $1/20$. A consistent value of $\epsilon$ is obtained from the fact that $m_c/m_t \simeq 1/400$, $m_u/m_c \simeq 1/200$, $m_e/m_\mu \simeq 1/200$, and $V_{ub} \simeq 1/300$ are all of order $\epsilon^2$. From the Cabibbo mixing and the ratio $m_d/m_s$, one would get the somewhat larger value $\epsilon \sim 1/5$.

The light neutrino mass matrix $M_\nu$ arises from the see-saw mechanism; so to know this matrix exactly it would be necessary to know $M_R$. However, to know merely the order in $\epsilon$ of the elements of $M_\nu$ it is not necessary to know the $U(1)$ family charges of the right-handed neutrinos at all, since the effective mass term in which $M_\nu$ appears involves only the left-handed lepton doublets, which are in the $\bar{5}_i$. Knowing the $U(1)$ charges of the $\bar{5}_i$ tells us that
$$M_\nu \sim \left( \begin{array}{ccc} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{array} \right) m_D^2/m_R. \tag{16}$$

From the forms of $L$ and $M_\nu$ it is obvious that the mixing $U_{\mu 3}$ of the second and third family neutrinos will get $O(1)$ contributions from both these matrices, thus explaining the largeness of the atmospheric neutrino mixing angle. Let us imagine now diagonalizing the 2-3 block of $M_\nu$ to get

$$M_\nu' \sim \left( \begin{array}{ccc} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & m_{2(0)} & 0 \\ \epsilon & 0 & 1 \end{array} \right) m_D^2/m_R. \tag{17}$$

The entry $m_{2(0)}$ would naturally be expected to be of order 1. However, for the ratio $r \equiv \Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}}$ to come out to be of order $10^{-2}$, as required by the LMA solution, $m_{2(0)}$ should rather be of order $1/10$. If we accept this rather mild fine-tuning, and assume that $m_{2(0)} \sim 1/10$, something interesting can be observed, namely that the 12 and 21 elements of $M_\nu'$ are of the same order as the 22 element, since $\epsilon \sim 1/20$. Recall that this is just what is needed for $\tan^2 \theta_{\text{sol}}$ to come out to be near the best-fit LMA value of about 0.3 or 0.4.

This model, then, naturally explains both the value of $\theta_{\text{atm}}$ and the LMA value of $\theta_{\text{sol}}$, provided that $r \equiv \Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}}$ is set to the LMA value.

Let us now imagine diagonalizing $M_\nu'$. The rotation needed to eliminate the 13 and 31 elements will give a contribution to $U_{e3}$ that is of order $\epsilon$, quite consistent with the present experimental limit of 0.15. This leaves the diagonalization of the 1-2 block. In doing this one may neglect the 11 element, since it is of order $\epsilon^2$. One then finds the simple relations (a) $\tan 2\theta_{\text{sol}} \sim 2\epsilon/m_{2(0)}$, and (b) $m_2 = m_{2(0)}/(1 - \tan^2 \theta_{\text{sol}})$. (Here we have ignored the $O(\epsilon)$ contribution to $\theta_{\text{sol}}$ coming from diagonalizing $L$, since we are interested in large values of $\theta_{\text{sol}}$.) From these relations one can infer roughly what region this model gives in the standard $\log(\tan^2 \theta_{\text{sol}}) - \log(\Delta m^2_{\text{sol}})$ plot. One sees from (a) that $\tan^2 \theta_{\text{sol}} \sim \epsilon^2 (m_{2(0)})^{-2}$, and from (b) that $\Delta m^2_{\text{sol}} \sim (m_{2(0)})^2$. In other words, in the standard plot the region corresponding to this model lies roughly along a line with slope $-1$ going through the LMA allowed region. We shall see shortly, both by a much more careful analytic calculation and by a Monte Carlo numerical calculation, that this conclusion is correct. The form of (b) tells us something else that is interesting. As the value of the
solar angle approaches maximality, i.e. \( \tan^2 \theta_{\text{sol}} \rightarrow 1 \), the denominator in (b) approaches zero. Therefore, to maintain a finite value of \( \Delta m_{\text{sol}}^2 \) the value of \( m_{2(0)} \) must be tuned to be extremely small. Thus, one expects the region of greatest probability in this model to fade away as \( \tan^2 \theta_{\text{sol}} \) approaches 1. This is confirmed by the analytic and Monte Carlo calculations, as we shall see.

We shall now study the predictions of this model in a statistical way, much in the spirit of [28]. Similar statistical analyses have been done in several recent papers [29], and our results are consistent insofar as they can be compared with theirs. However, our analysis differs in some respects. We do not treat \( \epsilon \) as a free parameter, and seek to find its optimal value for the various solar solutions. Rather, we fix \( \epsilon \) to the value that best reproduces the mass ratio \( m_\mu/m_\tau \) and then derive the full region of the \( \tan^2 \theta_{\text{sol}} - \Delta m_{\text{sol}}^2 \) plane which results. We also show that by treating the random variables as having a Gaussian distribution the statistical predictions of the model can be obtained analytically. We also carry out a numerical simulation using a non-Gaussian distribution similar to those used in previous analyses and show that it agrees remarkably well with the analytic results obtained using a Gaussian distribution.

To carry out the statistical analysis we parametrize the neutrino and charged lepton mass matrices as follows:

\[
M_\nu = \begin{pmatrix}
 f \epsilon^2 & d \epsilon / \sqrt{2} & e \epsilon / \sqrt{2} \\
 d \epsilon / \sqrt{2} & b & c / \sqrt{2} \\
 e \epsilon / \sqrt{2} & c / \sqrt{2} & a
\end{pmatrix} \frac{m_D^2}{m_R},
\]

and

\[
L = \begin{pmatrix}
 O(\epsilon^3) & O(\epsilon^2) & O(\epsilon^2) \\
 O(\epsilon^2) & D \epsilon & C \epsilon \\
 O(\epsilon) & B & A
\end{pmatrix} m_D.
\]

We will take the unknown order-one parameters \( a, b, c, d, e, f, A, B, C \), and \( D \), to be complex random variables whose real and imaginary parts have Gaussian distributions with standard deviation \( \sigma \). For example, if \( a = |a|e^{i\theta_a} \), then \( P(a) \, da = (2\pi\sigma^2)^{-1} \exp(-|a|^2/2\sigma^2)|a| \, d|a| \, d\theta_a \). It should be noted that we have put factors of \( 1/\sqrt{2} \) in the off-diagonal elements of \( M_\nu \). This is the appropriate normalization to use for a symmetric matrix.

What we want to calculate is the probability distribution \( P(r, t) \, dr \, dt \),
where \( r \equiv \Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}} \), as before, and \( t \equiv \tan^2 \theta_{\text{sol}} \), given that the order-one unknown parameters in the mass matrices have Gaussian distributions as described. We will first describe and give the results of an analytic calculation of \( P(r, t) \), and then present the results of a Monte Carlo numerical calculation of \( P(r, t) \).

A very important point in what follows is that if one does unitary changes of basis \( (\nu_L^2 \nu_L^3) \rightarrow V (\nu_L^2 \nu_L^3) \) or \( (\ell^-_L \ell^-_3) \rightarrow V (\ell^-_L \ell^-_3) \), the resulting parameters \( a', ..., e', A', ..., D' \), have exactly the same Gaussian distributions as the parameters in the original basis. (This would not be true without the factors of \( 1/\sqrt{2} \) in \( M_\nu \).) This is one fact that makes the analytic calculation tractable using Gaussian distributions. Moreover this basis independence is more consistent with the group-theoretical approach advocated in [28].

The first thing to do is diagonalize \( L \). For our purposes, we need only diagonalize the 2-3 block to find \( m_\mu/m_\tau \) and the contribution of \( L \) to \( \theta_{\text{atm}} \). This involves multiplying the 2-3 block of \( L \) from the right (which in our convention is a transformation on the left-handed leptons) by a unitary matrix

\[
U^{[23]}_\ell \cong \begin{pmatrix} A & B^* \\ -B & A^* \end{pmatrix} (|A|^2 + |B|^2)^{-1/2}.
\]

This eliminates the large element \( B \) in \( L \) and makes the 33 element become \( \sqrt{|A|^2 + |B|^2} \). The new 22 and 23 elements, which can be written \( D'\epsilon \) and \( C'\epsilon \) respectively, have the same Gaussian distribution as do \( A, B, C, \) and \( D \). Consequently, one has that

\[
(m_\mu)_{\text{rms}}/(m_\tau)_{\text{rms}} = \epsilon |D'|_{\text{rms}}/(\sqrt{|A|^2 + |B|^2})_{\text{rms}} = \epsilon/\sqrt{2}.
\]

Thus, the most reasonable value to choose for the small parameter, from the point of view of lepton physics, is \( \epsilon/\sqrt{2} = m_\mu/m_\tau \).

The first constraint that we shall impose is that the atmospheric neutrino mixing comes out to be very close to maximal, as found experimentally. This angle gets contributions from the diagonalizations of both \( L \) and \( M_\nu \). It would seem, then, that we must, in computing \( P(r, t) \), take into account the random variables in both mass matrices. However, a great simplification
occurs because of the use of Gaussian distributions and the resulting basis independence of the probability distributions. A little thought shows that one can compute $P(r, t)$ in a basis where the contribution to the atmospheric neutrino mixing coming from $L$ has some fixed value, and the result will not depend on that value. Thus the parameters in $L$ are irrelevant to $P(r, t)$. It is simplest in practice to choose the basis where the entire atmospheric neutrino mixing comes from $L$. A further simplification is achieved by neglecting the parameters $e$ and $f$. The parameter $e$ comes into calculating $U_{e3}$ (which is predicted to be of order $\epsilon$, and so consistent with the experimental bound $|U_{e3}| \leq 0.15$), but has a negligible effect on $\theta_{\text{sol}}, \theta_{\text{atm}}$, and the neutrino masses. The parameter $f$ is multiplied by $\epsilon^2$, and so is negligible also. Finally, one can choose a basis where the parameters $a$, $b$, and $d$ are real. That means that the only parameters that come into the calculation of $P(r, t)$ are $|a|$, $|b|$, $|c|$, $|d|$, and $\theta_c \equiv \arg c$. From now on, we shall drop the absolute value signs and denote $|a|$, for example, simply by $a$.

One begins, then, with a matrix

$$M_{\nu} = \begin{pmatrix} 0 & d\epsilon/\sqrt{2} & 0 \\ d\epsilon/\sqrt{2} & b & ce^{i\theta_c}/\sqrt{2} \\ 0 & ce^{i\theta_c}/\sqrt{2} & a \end{pmatrix} \frac{m_D^2}{m_R}, \quad (20)$$

and a probability distribution

$$P(a, b, c, \theta_c, d) = \frac{abcd}{2\pi \sigma^8} e^{-\frac{1}{2\sigma^2}(a^2 + b^2 + c^2 + d^2)}. \quad (21)$$

The first step is to diagonalize the 1-2 block of the matrix given in Eq. (20), which gives $\tan 2\theta_{\text{sol}} \equiv s = \sqrt{2}d\epsilon/b$. This allows the elimination of the random variable $d$ in favor of the measurable parameter $s$ or equivalently $t$. The 11 and 22 elements of the matrix then become $\frac{1}{2}(1 - \sqrt{1 + s^2})b$ and $\frac{1}{2}(1 + \sqrt{1 + s^2})b$ respectively. The latter quantity we shall denote as $b'$. The next step is to impose the atmospheric angle constraint. Since we are working in a basis where the contribution to this angle from $M_{\nu}$ is vanishingly small, the imposition of this constraint sets the parameter $c$ to zero. More precisely, if one requires that the (complex) contribution to the atmospheric angle from $M_{\nu}$ have magnitude bounded by some arbitrary small cutoff $\Delta_{\text{atm}} \ll 1$, the condition on $c$ becomes $(c/\sqrt{2})/(a - b') \leq \Delta_{\text{atm}}$. This means that the integration over $dc\, d\theta_c$ in the probability distribution can be done, yielding

$$\int c\, dc\, d\theta_c = \pi(\sqrt{2}(a - b')\Delta_{\text{atm}})^2. \quad \text{The only remaining variables are then } a, b,$$
The random variable \( s \) can be eliminated in favor of the measurable ratio \( r \) of mass-squared splittings using the relation

\[
r = \frac{b^2 \sqrt{1 + s^2}}{(a^2 - b^2)}.
\]

It is a very good approximation here to replace \( b' \) by \( b \), since for the whole region of interest either \( s \) or \( r \) is very small as we shall see. After all these steps, one is left with a probability distribution \( P(r, b, s) \). The final step is simply to integrate over the random variable \( b \). Since this integral is a Gaussian it is easily done. The final result is

\[
P(r, s) \, dr \, ds = N \frac{rs \, dr \, ds}{1 + s^2} \left[ \frac{1 + \frac{r}{\sqrt{1 + s^2}} - \frac{r}{\sqrt{1 + s^2}}}{1 + \frac{2r}{\sqrt{1 + s^2}} + \frac{rs^2}{2\epsilon^2 \sqrt{1 + s^2}}} \right]^4. \tag{22}
\]

where \( N \) is just a normalization constant. Changing variable from \( s \equiv \tan 2\theta_{\text{sol}} \) to \( t \equiv \tan^2 \theta_{\text{sol}} \) one finds that

\[
P(r, t) \, dr \, dt = N \frac{2r \, dr \, dt}{1 - t^2} \left[ \frac{1 + r \left( \frac{1 - t}{1 + t} \right) - r \left( \frac{1 - t}{1 + t} \right)}{1 + 2r \left( \frac{1 - t}{1 + t} \right) + \frac{2rt}{\epsilon^2 (1 - t^2)}} \right]^4. \tag{23}
\]

One can see the qualitative behavior of this function rather easily. The crucial term is the one containing \( \epsilon^2 \) in the denominator. This term forces the product \( rt \) to be of order \( \epsilon^2 \). This is consistent with what we argued above, namely that the region of greatest probability in this model has \( rt \sim \text{constant} \), i.e. a line of slope -1 in the \( \log(\tan^2 \theta_{\text{sol}}) - \log(\Delta m^2_{\text{sol}}) \) plane. Moreover, we see that as \( t \to 1 \), the product \( rt \) is forced to be less than or of order \( \epsilon^2 (1 - t^2) \to 0 \), so that the probability is suppressed for \( t \equiv 1 \).

In Figure [1], we give a contour plot of the probability function just computed analytically, and compare it to the results of a Monte Carlo calculation. For the Monte Carlo calculation we used the forms in Eqs. (18) and (19), with \( \epsilon/\sqrt{2} = m_\mu/m_\tau \), but assumed that the magnitudes of the complex random variables, instead of having a Gaussian distribution, had constant probability in the interval 0.5 to 2.0 and zero probability outside that interval. The phases of the complex variables were also treated as random numbers and were varied from 0 to 2\( \pi \). We then diagonalized randomly generated matrices to obtain the corresponding MNS mixing matrices \( U_{\text{MNS}} \) and neutrino
masses and analyzed the results by imposing the conditions \( \sin^2 2\theta_{\text{atm}} \geq 0.9 \) and \( |U_{e3}| \leq 0.15 \). These conditions reduced our initial set of 50,000 data points to 20,860 that were compatible with both the CHOOZ and the atmospheric neutrino experiments. The points that passed the cuts are given in Fig. 1.

One can see from the excellent agreement between the analytic and Monte Carlo results evident in Fig. 1 that the exact form used for the probability distributions of the random variables themselves makes little difference. This has also been found in other papers [28, 29]. One point that should be noted is that since Figure 1 is a log-log plot, the correct thing to plot and what has been plotted, is \( P(\log r, \log t) \sim P(r, t)rt \).

In Figure 2, we have taken the slice \( r = 1.4 \times 10^{-2} \), which comes from using the best-fit values from experiment, and plotted the resulting \( P(\log t) \) against a normalized, binned distribution. The binned distribution has been obtained by counting the number of data points in the strip \( \log r = \log(1.4 \times 10^{-2}) \pm 0.1 \), the width of one bin being 0.2, and the normalization has been carried out with respect to the maximum of \( P(\log t) \). Note that the most probable value for \( \tan^2 \theta_{\text{sol}} \) is about 0.1, with a very substantial part of the area under the curve being in the region \( [0.2, 0.8] \) preferred by the LMA solution global fits.

The one weakness of this model is that it does not explain why \( r \equiv \Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}} \approx 1.4 \times 10^{-2} \). From Figure 1 one sees that a value of \( 10^{-1} \) for this ratio is near the peak of the probability distribution \( P(r, t) \). However, the same figure shows that a value of \( 1.4 \times 10^{-2} \) is near the edge of the preferred region, and so requires a mild fine-tuning. However, once \( r \) is constrained to be the right value, the value of \( \tan^2 \theta_{\text{sol}} \) needed for the LMA solution emerges quite naturally, as can be seen from Fig. 2. The atmospheric mixing angle is, of course, also naturally explained. In Figure 1 the best fit values of the LMA, SMA, and LOW solutions are indicated by dots. One sees that this \( SU(5) \) lopsided model (with \( \epsilon/\sqrt{2} = m_\mu/m_\tau \)) naturally prefers the LMA solution over the others.

5 Conclusions

One can see from the foregoing that it is considerably easier to build satisfactory models of the VAC, LOW, or SMA type than of the LMA type. That is reflected in the models that have actually been constructed in the literature. One problem is that in many models which predict large solar mixing angle,
notably inverted hierarchy schemes and flavor democracy schemes, this angle tends to come out very close to maximal. They do not naturally explain why \( \tan^2 \theta_{sol} \) should come out in the range 0.2 to 0.8 preferred by the data. Other non-see-saw schemes, such as SUSY with R-parity breaking and single-right-handed-neutrino-dominance (SRHND) models, tend to predict a value of \( \Delta m_{sol}^2/\Delta m_{atm}^2 \) significantly less than that preferred by the LMA solution. While the LMA value of this ratio can be fit, it is not really explained.

The situation seems more promising for the see-saw approaches, although here also the great majority of published models give the small angle or vacuum solar solutions. We showed that certain fairly simple textures exist that would naturally reproduce the neutrino masses and mixings required by the LMA solution. Whether these textures can be implemented in simple models remains to be seen.

One of the few existing schemes that shows a natural preference for the LMA solution is the lopsided \( SU(5) \) model studied in Section 4. The value of \( \Delta m_{sol}^2/\Delta m_{atm}^2 \) requires a mild fine-tuning, but given that, both the atmospheric angle and the LMA value of the solar angle emerge quite naturally. We studied the predictions of this model in a statistical way, and found that by using Gaussian distributions the analysis could be carried out very simply and accurately by purely analytic means. We believe that the same methods should be applicable to many other models. The advantage of such statistical analyses is that they allow one to estimate in a somewhat objective and quantitative way how "fine tuned" models must be to reproduce the data.

References

[1] The Super-Kamiokande Collaboration, *Phys. Rev. Lett.* **86**, 5656 (2001).
[2] J.N. Bahcall, P.I. Krastev, and A.Yu. Smirnov, *JHEP* **0105**, 015 (2001).
[3] Q.R. Ahmad, et al., “Measurement of charged current interactions produced by \( ^8\)B solar neutrinos at the Sudbury Neutrino Observatory”, submitted to *Phys. Rev. Lett.*
[4] J.N. Bahcall, M.C. Gonzalez-Garcia, C. Pena-Garay, [hep-ph/0106258](https://arxiv.org/abs/hep-ph/0106258).
[5] S.M. Barr and I. Dorsner, *Nucl. Phys.* **B585**, 79 (2000).
[6] C. Jarlskog, M. Matsuda, S. Skadhauge, and M. Tanimoto, *Phys. Lett.* **B449**, 240 (1999); K. Cheung and O.C.W. Kong, *Phys. Rev.* **D61**, 113012 (2000); A.S. Joshipura and S.D. Rindani, *Phys. Lett.* **B464**, 239 (1999).

[7] A. Zee, *Phys. Lett.* **B93**, 389 (1980); *Phys. Lett.* **B161**, 141 (1985).

[8] A.S. Joshipura, hep-ph/9808261; A.S. Joshipura and S.D. Rindani, *Phys. Lett.* **B464**, 239 (1999); P. Frampton and S.L. Glashow, *Phys. Lett.* **B461**, 95 (1999); A.S. Joshipura and S.D. Rindani, *Eur. Phys. J.* **C14**, 85 (2000); R.N. Mohapatra, A. Perez-Lorenzana, C.A. deS. Pires, *Phys. Lett.* **B474**, 355 (2000); L. Lavoura, *Phys. Rev.* **D62**, 093011 (2000); L. Lavoura and W. Grimus, *JHEP* **0009**, 007 (2000); T. Kitabayashi and M. Yasue, *Phys. Lett.* **B508**, 85 (2001).

[9] P. Frampton and S.L. Glashow, *Phys. Lett.* **B461**, 95 (1999); A.S. Joshipura and S.D. Rindani, *Eur. Phys. J.* **C14**, 85 (2000); R.N. Mohapatra, A. Perez-Lorenzana, C.A. deS. Pires, *Phys. Lett.* **B474**, 355 (2000); L. Lavoura, *Phys. Rev.* **D62**, 093011 (2000); L. Lavoura and W. Grimus, *JHEP* **0009**, 007 (2000); R.N. Mohapatra, A. Perez-Lorenzana, C.A. deS. Pires, *Phys. Lett.* **B474**, 355 (2000); A.S. Joshipura and S.D. Rindani, *Eur. Phys. J.* **C14**, 85 (2000).

[10] R.N. Mohapatra and S. Nussinov, *Phys. Rev.* **D60**, 013002 (1999).

[11] M. Drees, S. Pakvasa, X. Tata, T. derVeldhuis, *Phys. Rev.* **D57**, 5335 (1998).

[12] E.J. Chun, S.K. Kang, C.W. Kim, and U.W. Lee, *Nucl. Phys.* **B544**, 89 (1999); K. Choi, E.J. Chun and K. Hwang, *Phys. Rev.* **D60**, 031301 (1999); D.E. Kaplan and A.E. Nelson, *JHEP* **0001**, 033 (2000); A.S. Joshipura and S.K. Vempati, *Phys. rev.* **D60**, 111303 (1999); E.J. Chun and S.K. Kang, *Phys. Rev.* **D61**, 075012 (2000);

[13] M. Hirsch, M.A. Diaz, W. Porod, J.C. Romao, J.W.F. Valle, *Phys. Rev.* **D62**, 113008 (2000).

[14] S.L. King, *Phys. Lett.* **B439**, 350 (1998); **B562**, 57 (1999).

[15] E. Ma and D.P. Roy, *Phys. Rev.* **D59**, 097702 (1999).
[16] S. Davidson and S.L. King, *Phys. Lett.* **B445**, 191 (1998).

[17] Q. Shafi and Tavartkiladze, *Phys. Lett.* **B451**, 129 (1999).

[18] A. de Gouvea and J.W.F. Valle, *Phys. Lett.* **B501**, 115 (2001).

[19] G. Altarelli, F. Feruglio, and I. Masina, *Phys. Lett.* **B472**, 382 (2000).

[20] H. Fritzsch and Z.Z. Xing, *Phys. Lett.* **B372**, 265 (1996); *Phys. Lett.* **B440**, 313 (1998); *Prog. Part. Nucl. Phys.* **45**, 1 (2000); M. Fukugita, M. Tanimoto and T. Yanagida, *Phys. Rev.* **D57**, 4429 (1998); *Phys. Rev.* **D59**, 113016 (1999); M. Tanimoto, *Phys. Rev.* **D59**, 017304 (1999); R. Mohapatra and S. Nussinov, *Phys. Lett.* **B441**, 299 (1998); S.K. Kang and C.S. Kim, *Phys. Rev.* **D59**, 091302 (1999).

[21] M. Tanimoto, T. Watari, and T. Yanagida, *Phys. Lett.* **B461**, 345 (1999).

[22] B. Stech, *Phys. Lett.* **B465**, 219 (1999).

[23] M. Jezabek and Y. Sumino, *Phys. Lett.* **B440**, 327 (1998).

[24] C.H. Albright, K.S. Babu and S.M. Barr, *Phys. Rev. Lett.* **81**, 1167 (1998).

[25] K.S. Babu and S.M. Barr, *Phys. Lett.* **B381**, 202 (1996); J. Sato and T. Yanagida, *Phys. Lett.* **B430**, 127 (1998); C.H. Albright and S.M. Barr, *Phys. Rev.* **D58**, 013002 (1998); C.H. Albright, K.S. Babu and S.M. Barr, *Phys. Rev. Lett.* **81**, 1167 (1998); N. Irges, S. Lavignac and P. Ramond, *Phys. Rev.* **D58**, 035003 (1998); J.K. Elwood, N. Irges and P. Ramond, *Phys. Rev. Lett.* **81**, 5064 (1998); Y. Nomura and T. Yanagida, *Phys. Rev.* **D59**, 017303 (1999); N. Haba, *Phys. Rev.* **D59**, 035011 (1999); G. Altarelli and F. Feruglio, *JHEP* **9811** 021 (1998); Z. Berezhiani and A. Rossi, *JHEP* **9903**, 002 (1999); K. Hagiwara and N. Okamura, *Nucl. Phys.* **B548**, 60 (1999); G. Altarelli and F. Feruglio, *Phys. Lett.* **B451**, 388 (1999); K.S. Babu, J. Pati and F. Wilczek, *Nucl. Phys.* **B566**, 39 (2000); C.H. Albright and S.M. Barr, *Phys. Lett.* **B452**, 287 (1999); M. Bando and T. Kugo, *Prog. Theor. Phys.* **101**, 1313 (1999); K. Izawa, K. Kurosawa, Y. Nomura and T. Yanagida, *Phys. Rev.* **D60**, 115016 (1999); P. Frampton and A. Rasin, *Phys. Lett.* **B478**, 424 (2000); R. Barbieri, G.L. Kane, L.J. Hall and G.G. Ross,
hep=ph/9901228; M. Bando, T. Kugo and K. Yoshioka, *Prog. Theor. Phys.* **104**, 211 (2000);

[26] Y. Nir and Y. Shadni, *JHEP* **9905**, 023 (1999); Y. Nomura and T. Sugimoto, *Phys. Rev.* **D61**, 093003 (2000); C.H. Albright and S.M. Barr, hep-ph/0104294.

[27] J. Sato and T. Yanagida, *Phys. Lett.* **B430**, 127 (1998); **B493**, 356 (2000); W. Buchmüller and T. Yanagida, *Phys. Lett.* **B445**, 399 (1999).

[28] N. Haba and H. Murayama, *Phys. Rev.* **D63**, 053010 (2001).

[29] J. Sato and T. Yanagida, *Phys. Lett.* **B493**, 356 (2000); F. Vissani, *Phys. Lett.* **B508**, 79 (2000); J. Sato and K. Tobe, *Phys. Rev.* **D63**, 116010 (2001).
Figure 1: Contour plot of the normalized probability distribution $P$ in
$\log(\tan^2 \theta_{\text{sol}}) - \log(\Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}})$ plane with the contour values 0.002, 0.02, 0.06, 0.1, and 0.14 superimposed on the numerically generated distribution of points. Large dots represent best-fit values for LMA, SMA, and LOW solar neutrino solutions, as indicated.
Figure 2: Normalized probability distribution $P$ (solid line) as a function of log($\tan^2 \theta_{\text{sol}}$) for best-fit LMA solution value $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2 = 1.4 \times 10^{-2}$ plotted against the normalized, binned distribution (dashed line).