The Completeness Problem for Modal Logic

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Abstract. We examine the completeness problem for Modal Logic: given a formula, is it complete? We discover that for most cases, completeness and validity have the same complexity – with certain exceptions for which there are, in general, no complete formulas. To prove upper bounds, we give a game which combines bisimulation games and tableaux, and which determines whether a formula is complete. The game then easily turns into an alternating algorithm.

Keywords: Modal Logic, Completeness, Computational Complexity, Bisimulation

1 Introduction

We examine the following problem: given a modal formula \( \phi \) (or, equivalently, a finite modal theory), is it complete? That is, can all modal formulas that are consistent with \( \phi \) be derived from \( \phi \)? Of course, this is a different problem for different modal logics. We call this the completeness problem for a modal logic and we examine its complexity. Our main results show that the completeness problem usually has the same complexity as provability.

Completeness is a very important notion in Logic. A logic is complete with respect to its semantics if all valid formulas are also provable; a theory in a logic is called complete if for every formula of the logic, either the formula or its negation are derivable from the theory. Together with soundness and consistency, it is one of the most important properties one usually considers for a logic or theory.

Modal Logic is a very well-known family of logics. Since it is used to formalize a variety of situations, it should be of great importance to know if a formula/finite theory formalizes exactly one situation (i.e. it is complete) or not. Given Modal Logic’s wide area of applications, we find it surprising that, to the best of our knowledge, the completeness problem for Modal Logic has not been studied as a computational problem so far. On the other hand, the complexity of satisfiability (and thus validity) for Modal Logic has been studied extensively [10, 8, 7].

We examine the completeness problem for the many well known modal logics, the extensions of \( K \) by the axioms Factivity, Consistency, Positive Introspection, and Negative Introspection – i.e. the ones between \( K \) and \( S5 \). We discover that the complexity of provability and completeness tend to be the same, i.e. if the logic
does not have Negative Introspection, then the completeness problem is \text{PSPACE}-complete; if the logic has Negative Introspection, then the problem is \text{coNP}-complete. There are exceptions: for certain logics (D and T), the completeness problem is trivial, as these logics have no finite complete theories.

Our motivation comes from [1], where Artemov raises the following issue. It is the usual practice in Game Theory (and Epistemic Game Theory) to reason about a game based on a model. On the other hand, it is often the case in an epistemic setting that the game specification is not complete, thus any conclusions reached by examining any single model are precarious. He proposes a syntactic, proof-centered approach, which is more robust and general and which is based on a syntactic formal description of the game specification. Artemov’s approach is more sound, but on the other hand, the model-based approach has been largely successful in Game Theory for a long time. If we can determine that the syntactic specification of a game is complete, then the syntactic and semantic approaches are equivalent and we can describe the game using one model.

Of course, the same can be said for other areas where Modal Logic is used as a logic for specification. Furthermore, if a formula is discovered to be complete, then by representing it using a model, we may be able to draw multiple conclusions efficiently – especially if the model happens to not be particularly large. On the other hand, if a formula is discovered to be incomplete, then, as a specification it may need to be refined.

A modal formula is complete if all its models are bisimilar to each other, therefore bisimulation is important as a notion for this paper. Bisimulation was introduced to Modal Logic by van Benthem in 1976 [14] and it is a central tool to study the definability of Modal Logic and to characterize it as a fragment of First-Order Logic.

For an overview of Modal Epistemic Logic, the reader can see [5, 12, 2]. For an introduction to the complexity of Modal Logic, the reader can see [7].

2 Background

Modal formulas are constructed from propositional variables, which we call \(p_1, p_2, \ldots\), the constants \(\bot, \top\), the usual operators \(\land, \lor, \neg\) of propositional logic, and the dual modal operators, \(\Box\) and \(\Diamond\). Specifically,

\[
\phi ::= \bot \mid \top \mid p \mid \neg \phi \mid \phi \land \phi \mid \phi \lor \phi \mid \Box \phi \mid \Diamond \phi.
\]

As usual, we may consider some of these operators to be constructed from the others.\(^1\) The language of Modal Logic is called \(L\). If \(P\) is a finite set of propositional variables, we define \(L(P) \subseteq L\) to be the set of formulas that use only variables from \(P\). If \(\phi\) is a modal formula, then \(P(\phi)\) is the set of propositional variables that appear in \(\phi\). Naturally, \(\phi \in L(P(\phi))\). If \(\phi \in L\), then \(\text{sub}(\phi)\) is the set of subformulas of \(\phi\).

\(^1\) Implication was not introduced, because we later need formulas in Negation Normal Form, which has no implications. When it appears, \(\phi \rightarrow \psi\) can be considered short for \(\neg \phi \lor \psi\).
Normal modal logics use a sufficient amount of propositional tautological schemes (we can take all tautologies, or a finite axiomatization of Propositional Logic) and the axiom

\[ K: \Box \phi \land \Box (\phi \rightarrow \psi) \rightarrow \Box \psi. \]

Furthermore, they use two rules: Modus Ponens and the Necessitation Rule:

\[ \phi \quad \Box \phi, \]

which claims that all provable formulas are provably necessary.

The logic which has exactly these axioms and rules is the smallest normal modal logic, K. We extend K with more axioms and thus introduce more modal logics. The axioms we consider are:

- \[ D: \neg \Box \bot \] (equivalently, \( \Box \phi \rightarrow \Diamond \phi \));
- \[ T: \Box \phi \rightarrow \phi; \]
- \[ 4: \Box \phi \rightarrow \Box \Box \phi; \]
- \[ 5: \neg \Box \phi \rightarrow \Box \neg \Box \phi. \]

We consider modal logics which are formed from a combination of these axioms. Of course, not all combinations make sense: axiom \( D \) (called the Consistency axiom) is a special case of \( T \) (the Factivity axiom). Axiom 4 is called Positive Introspection and 5 is called Negative Introspection. Given a logic \( \mathcal{L} \) and axiom \( a \), \( \mathcal{L} + a \) is the logic which has as axioms the axioms of \( \mathcal{L} \) with the addition of \( a \). Thus, modal logic D is \( K + D \), T is \( K + T \), K4 = \( K + 4 \), D4 = \( K + D + 4 = D + 4 \), S4 = \( K + T + 4 = T + 4 = K4 + T \), KD45 = \( D4 + 5 \), and S5 = \( S4 + 5 \).

A Kripke structure is a triple \( M = (W, R, V) \), where \( W \) is the set of states (or worlds), \( R \subseteq W \times W \) is an accessibility relation and \( V \) is a function which assigns to each state in \( W \) a set of propositional variables. If \( P \) is a set of propositional variables, then for every \( a \in W \), \( V_P(a) = V(a) \cap P \).

Truth in a Kripke model is defined in the following way:

- \( M, a \models \bot \) and \( M, a \models p \) iff \( p \in V(a) \);
- \( M, a \models \neg \phi \) iff \( M, a \not\models \phi \);
- \( M, a \models \phi \land \psi \) iff both \( M, a \models \phi \) and \( M, a \models \psi \);
- \( M, a \models \phi \lor \psi \) iff \( M, a \models \phi \) or \( M, a \models \psi \);
- \( M, a \models \Diamond \phi \) iff there is some \( b \in W \) such that \( aRb \) and \( M, b \models \phi \);
- \( M, a \models \Box \phi \) iff for all \( b \in W \) such that \( aRb \) it is the case that \( M, b \models \phi \).

If \( M, a \models \phi \), we say that \( \phi \) is true/satisfied in \( a \) of \( M \). To ease notation, when \( (s, t) \in R \) we usually write \( sRt \). \( (W, R) \) is called a frame. We call a Kripke model \( (W, R, V) \) (resp. frame \( (W, R) \)) finite if \( W \) is finite.

Each modal logic is associated with a class of frames. The accessibility relation for each frame \( (W, R) \) in a class associated to a logic must meet certain conditions, depending on the logic’s axioms: if the logic has axiom \( D \), then the accessibility relation must be serial (that is, for every state \( a \in W \) there must be some \( b \in W \) such that \( aRb \));
Let $T$, then the accessibility relation must be reflexive (for all $a \in W$, $aRa$); 4, then the accessibility relation must be transitive (if $aRbRc$, then $aRc$); 5, then the accessibility relation must be euclidean (if $aRb$ and $aRe$, then $bRe$).

We call a formula satisfiable for a modal logic $l$, if it is satisfied in a state of a model for $l$; we call a formula valid for a modal logic $l$, if it is satisfied in all states of all models for $l$. A formula is valid if and only if it is provable; it is satisfiable if and only if it is satisfied in a model with a finite amount of states. Therefore, we only consider models on finite frames.

### 2.1 Bisimulation

A very important notion in Modal Logic (and elsewhere) is that of bisimulation. Let $P$ be a (finite) set of propositional variables. For two Kripke models $M = (W, R, V)$ and $M' = (W', R', V')$ we say that $M$ is bisimilar (respectively, bisimilar modulo $P$) to $M'$ and write $M \sim M'$ (resp. $M \sim_P M'$) if there exists a non-empty relation $R \subseteq W \times W'$ such that:

- For all $(s, s') \in R$ we have $V(s) = V'(s')$ (resp. $V_P(s) = V'_P(s')$).
- For all $s \in W$, $s', t' \in W'$ such that $(s, s') \in R$ and $s'R' t'$ there exists $t \in W$ such that $(t, t') \in R$ and $sRt$.
- For all $s, t \in W$, $s' \in W'$ such that $(s, s') \in R$ and $sRt$ there exists $t' \in W'$ such that $(t, t') \in R$ and $s'R' t'$.

We call $R$ a bisimulation (resp. bisimulation modulo $P$) from $M$ to $M'$.

We call a pair $(M, a)$ a pointed model if $a$ is a state of $M$. We say that pointed models $(M, a), (M', a')$ are bisimilar (resp. bisimilar modulo $P$) if there is a bisimulation (resp. bisimulation modulo $P$) $R$ from $M$ to $M'$, such that $aRa'$. If $(M, a)$ is a pointed model, then $Th_P(M, a) = \{ \phi \in L(P) \mid M, a \models \phi \}$. We are interested in whether two pointed models satisfy the same formulas, therefore we say that two pointed models are equivalent and write $(M, a) \equiv_P (M', a')$ if and only if $Th_P(M, a) = Th_P(M', a')$.

The following simplification of the Hennessy-Milner Theorem gives a very useful characterization of pointed model equivalence.

#### Theorem 1 (Hennessy-Milner Theorem, [9]).

Let $(M, a), (M', a')$ be two finite pointed models. Then,

$$(M, a) \equiv_P (M', a') \iff (M, a) \sim_P (M', a')$$

The classical complexity results for Modal Logic are due to Ladner [10], who established PSPACE-completeness for the satisfiability of $K$, $T$, $D$, $K4$, $D4$, and $S4$ and NP-completeness for the satisfiability of $S5$. Halpern and Régol gave a more complete picture by characterizing the NP–PSPACE gap by the presence or absence of Negative Introspection [8], resulting in Theorem 2.
Theorem 2. Let $M$ be a modal logic from the following: $K$, $T$, $D$, $K4$, $D4$, $S4$. $M$-satisfiability is PSPACE-complete and $M + 5$-satisfiability is NP-complete. Equivalently, $M$-provability is PSPACE-complete and $M + 5$-provability is coNP-complete.

In the course of proving the NP upper bound for logics with Negative Introspection, Halpern and Régo give in [8] a construction which provides a small model for a satisfiable formula. By breaking down their construction, we can extract the following results.

Lemma 1 gives a normal form for models of logics with Negative Introspection.

Lemma 1. Let $(M, s)$ be a pointed $l + 5$-model for $\phi$. $(M, s)$ is bisimilar to some $(M', s')$, where $M' = (W, R, V)$, $W = \{s\} \cup W'$, $R = R_1 \cup R_2$, where $R_1 \subseteq \{s\} \times W'$ and if $W' \neq \emptyset$, then $R_2$ is an equivalence relation on $W'$, otherwise $R_2 = \emptyset$. If $l \in \{T, S4\}$, then $s' \in W'$.

Proof. Let $W$ be the set of states of $M$ reachable from $s$ and $R$ the restriction of the accessibility relation of $M$ on $W$. It is easy to see that $(M, s) \sim (M', s)$; let $W_R = \{w \in W \mid \exists w'R w\}$. Therefore $W = W_R \cup \{s\}$ and if $l \in \{T, S4\}$, then $s \in W_R$. Since $M$ is an $M + 5$-model, $R$ is euclidean (if $aRb, c$, then $b Rc$).

Therefore, the restriction of $R$ on $W_R$ is reflexive (for all $a \in W_R$, $aRa$). This in turn means $R$ is symmetric in $W_R$: if $a, b \in W_R$ and $aRb$, since $aRa$, we also have $bRa$. Finally, $R$ is transitive in $W_R$: if $aRbRc$ and $a, b, c \in W_R$, then $bRa$, so $a Rc$. Therefore $R$ is an equivalence relation when restricted on $W_R$. \qed

If we continue the construction from [10] and [8], we can filter the states of the model to end up with a small model for $\phi$. Using this construction, Halpern and Régo prove Corollary 1 [8], from which the NP upper bound for $l + 5$-satisfiability from Theorem 2 is a direct consequence:

Corollary 1 ([8]). $\phi$ is $l + 5$-satisfiable if and only if it is satisfied in an $l + 5$-model of at most $|\phi| + 1$ states.

Proof. We continue from the proof of Lemma 1. If $W_R = \emptyset$, then $M_\phi = M'$ and we are done. Otherwise, let $S_\phi$ be the set which contains every $\diamond \psi$ subformula of $\phi$ that is true in $s$ and every $\Box \psi$ subformula of $\phi$ that is not true in $s$; let $T_\phi$ be the set which contains every $\diamond \psi$ subformula of $\phi$ that is true in some state of $M'$ and every $\Box \psi$ subformula of $\phi$ that is not true in some state of $M'$. For every $\diamond \psi \in S_\phi$ we fix a state $a_\psi \in W_R$, such that $sRa_\psi$ and where $\psi$ is true; for every $\Box \psi \in S_\phi$ we fix a state $a_\psi \in W_R$, such that $sRa_\psi$ and where $\psi$ is true; for every $\diamond \psi \in T_\phi \setminus S_\phi$ we fix a state $a_\psi \in W_R$, where $\psi$ is true; finally, for every $\Box \psi \in T_\phi \setminus S_\phi$ we fix a state $a_\psi \in W_R$, where $\psi$ is true. Notice that $a_\psi \in W_R$ and thus it is accessible from every state in $W_R$. Let $M_\phi = (W_\phi, R_\phi, V_\phi)$, where

$$W_\phi = \{s\} \cup \{a_\psi \in W_R \mid \diamond \psi \in T_\phi \text{ or } \Box \psi \in T_\phi\},$$

$R_\phi$ the restriction of $R$ on $W_\phi$, and $V_\phi(a) = V(a)$ for all $a \in W_\phi$. 


It is not hard to confirm that $|W_\phi| \leq |\phi|$, since $|T_\phi| \leq |\phi| - 1$ (the subformulas of $\phi$ are at most $|\phi|$ in number and at least one of them is a propositional variable or $\bot$). Furthermore, $M_\phi, s \models \phi$. Specifically, for all $a \in W_\phi$ and $\psi \in \text{sub}(\phi)$, we prove by induction on $\psi$ that $M', a \models \psi$ if and only if $M_\phi, a \models \psi$. Propositional cases are easy. If $\psi = \Diamond \chi$ and $M', a \models \psi$, then there is some $a_\chi$, such that $M', a_\chi \models \chi$ and by the definition of $a_\chi$, $aR_\chi$, therefore by the inductive hypothesis, $M_\phi, a_\chi \models \chi$ and thus $M_\phi, a \models \psi$. If $\psi = \Diamond \chi$ and $M_\phi, a \models \psi$, then there is some $aR_\phi c \in W_\phi$, such that $M_\phi, c \models \chi$; by the inductive hypothesis, $M', c \models \chi$ and since $R_\phi$ is the restriction of $R$ on $W_\phi$, $aRc$, so $M', a \models \psi$. The cases where $\psi = \Box \chi$ are similar.

What remains is to demonstrate that the resulting model remains an $l + 5$-model. It is not hard to confirm that through this filtering, transitivity, euclide-ity, and reflexivity are preserved for the accessibility relation (since they are preserved by restrictions on subsets of binary relations). As for seriality, it is enough to run this construction on $\phi \land \Diamond \top$ if necessary, thus increasing the upper bound on the number of states from $|\phi|$ to $|\phi| + 1$. 

Since we are asking whether a formula is complete, instead of whether it is satisfiable, we can adapt the construction above to give us two small non-bisimilar models for $\phi$ when $\phi$ is incomplete. This gives us Corollary 2, which is a useful characterization of incomplete formulas.

**Corollary 2.** Formula $\phi$ is incomplete for $l + 5$ if and only if it has two non-bisimilar pointed models for $l + 5$ of at most $|\phi| + 2$ states and of the form described in Lemma 1.

**Proof.** If $\phi$ has two non-bisimilar pointed models for $l + 5$ of at most $|\phi| + 2$ states, then by Theorem 1, it is incomplete. On the other hand, if $\phi$ is incomplete, again by Theorem 1, $\phi$ has two non-bisimilar (pointed) models, $(M_1, s_1)$ and $(M_2, s_2)$. If we (separately) apply the construction from the proof of Lemma 1 on $M_1, M_2$, resulting with $M'_1 = (W_1, R_1, V_1)$ and $M'_2 = (W_2, R_2, V_2)$, since also $M'_1 \not\equiv_p M'_2$, one of these resulting models (w.l.o.g. $M'_1$) would have a state $x \in W_1$, such that $(s_1 R_1 x)$ and $V_1(x)$ would be different from $V_2(y)$ for all $y \in W_2$ (for which $s_2 R_2 y$). If through the filtering construction of the proof for Corollary 1, we additionally keep $x$ in the resulting model (we instead set $W_\phi = \{s, x\} \cup \{a_\psi \in W_R \mid \Diamond \psi \in T_\phi \text{ or } \Box \psi \in T_\phi\}$), we end up with two non-bisimilar models for $\phi$ of small size. Notice also that the filtering construction preserves the form described in Lemma 1.

In the following, when $P$ is evident from the context, we will often omit any reference to it and instead of bisimulation modulo $P$, we will call the relation simply bisimulation.

## 3 The Completeness Problem

We are interested in the completeness problem for Modal Logic. That is, we want to know for a finite theory $\Phi$ (equivalently, for a formula $\phi = \bigwedge \Phi$) if starting
from $\Phi$ we can prove or disprove all modal formulas. In other words, we want to know if $\Phi$ as a specification is complete.

Of course, no finite description can be really complete, as there is an infinite amount of propositional variables. Thus, we call a modal formula $\phi$ complete for logic $l$ if for every $\psi \in L(P(\phi))$, either $\phi \vdash_l \psi$ or $\phi \vdash_l \neg \psi$ – where $\phi \vdash_l \psi$ means that $\psi$ can be derived from $\phi$ and the axioms for $l$. Equivalently $\phi$ is complete for logic $l$ if for every $\psi \in L(P(\phi))$, either $\phi \models_l \psi$ or $\phi \models_l \neg \psi$ – where $\phi \models_l \psi$ means that every pointed model for $l$ which satisfies $\phi$ also satisfies $\psi$. Equivalently, a formula is complete for $l$ is any two models for $l$ which satisfy it are equivalent. Therefore, by Theorem 1, a formula is complete for $l$ if any two models for $l$ that satisfy it are bisimilar.

We consider the following problem for modal logic $l$: Given a modal formula, is it complete? We call this the completeness problem for $l$.

3.1 Is there any complete formula?

Naturally, the first question we need to answer about the completeness problem for $l$ is whether there are any satisfiable complete formulas for $l$ (and whether there are any incomplete formulas).\(^2\) If the answer is negative, then the problem is trivial. We examine this question depending on the logic and on whether $P$, the set of propositional variables we use is empty or not. If for some logic $l$ the problem in nontrivial, then we give a complete formula $\phi^l_P$, which depends on $P$ and uses exactly the propositional variables in $P$. Furthermore, we will see that sometimes when $P = \emptyset$, completeness becomes trivial for another reason: for some logics, when $P = \emptyset$, all formulas are complete.

For $K$ and $K_4$, the answer is simple: $\phi^K_P = \phi^{K_4}_P = \bigwedge P \land \Box \bot$ is such a complete formula, for whatever finite $P$ (we define $\bigwedge \emptyset$ to be $\top$).

**Lemma 2.** Formula $\bigwedge P \land \Box \bot$ is complete for $K$ and for $K_4$.

**Proof.** A model which satisfies $\phi^K_P$ is $\mathcal{M} = \{(a), \emptyset, V\}$, where $V(a) = P$. If there is another model $\mathcal{M}', a' \models \phi_0$, then there should be no accessible world from $a'$ in $\mathcal{M}'$ (since $\mathcal{M}', a' \models \Box \bot$); therefore, $R = \{(a, a')\}$ is a bisimulation. \(\Box\)

Note also that for all $P$, $\bigwedge P$ is an incomplete formula for $K$ and $K_4$. For this it is enough to see that $\{\{a\}, \emptyset, V\}$ and $\{\{a, b\}, \{(a, b)\}, V'\}$, where $V(a) = V'(a) = P$ both satisfy $\bigwedge P$ and they are non-bisimilar. Therefore, whether $P = \emptyset$ or not, completeness is nontrivial for $K$ and $K_4$.

When the logic $l$ has factivity or consistency, but not positive or negative introspection (i.e. $l$ is $D$ or $T$), whether a satisfiable formula is complete completely depends on $P$. In fact,

**Lemma 3.** Let $l$ be either $D$ or $T$. A satisfiable formula $\phi \in L$ is complete with respect to $l$ if and only if $P(\phi) = \emptyset$.

\(^2\) If a formula is not satisfiable, it is inconsistent, and therefore from it we can deduce all formulas.
Lemma 4. For every finite $P$, all formulas $\phi$, where $P(\phi) = \emptyset$ are trivially complete. When $P \neq \emptyset$, we have the opposite situation: there is no satisfiable formula that is complete. Let the modal depth$^3$ of $\phi$ be $d$ and let $\mathcal{M}, a \models \phi$, where $\mathcal{M} = (W, R, V)$. Let $x /\in W^*$. Let $a_0 = a,$ 

$$W' = \{a_0 \cdots a_k \in W^* \mid k \leq d \text{ and for all } 0 \leq i < k, a_iRa_{i+1} \} \cup \{x\},$$

$$R'_j = \{(a, ab) \in W'^2 \mid b \in W\} \cup \{(a, a) \in W'^2 \mid l = T\} \cup \{(a_0a_1 \cdots a_{d}, x) \in W'^2 \} \cup \{(x, x) \mid l = D\},$$

and for $j = 1$ or $2,$ $V'_j(ab) = V(b)$ if $|a| \leq d,$ $V'_j(a) = V(a),$ $V'_j(x) = \emptyset,$ and $V'_2(x) = P.$

Let $\mathcal{M}'_1 = (W', R', V'_1)$ and $\mathcal{M}'_2 = (W', R', V'_2)$. It is not hard to verify that $\mathcal{M}'_1, a \models \phi$ and $\mathcal{M}'_2, a \models \phi.$ If $(\mathcal{M}'_1, a) \sim (\mathcal{M}'_2, a)$ through bisimulation $\mathcal{R}$ from $\mathcal{M}'_1$ to $\mathcal{M}'_2$, then by the conditions of bisimulation, there must be a path of length $d + 2$ from $a$ to some $b \in W'$ s.t. $b\mathcal{R}x$; since from $b$ there is a path to $x$ and the paths from $x$ only end up to $x$, $x\mathcal{R}x$, which is a contradiction, since $V'_1(x) \neq V'_2(x)$.

Therefore, $\phi$ has two non-bisimilar models and it is not complete. \hfill \Box

For $D4$ and $S4$, $\phi^D_4 = \phi^S_4 = P \land \Box P$ is a complete formula:

**Lemma 4.** For every finite $P$, $\phi^D_4$ is complete for $D4$ and $S4$; all formulas in $L(\emptyset)$ are complete for $D4$ and $S4$.

**Proof.** Let $\mathcal{M}, a \models \phi$ and $\mathcal{M}', a' \models \phi$; let $\mathcal{R}$ be the relation that connects all states of $\mathcal{M}$ that are reachable from $a$ (including $a$) to all states of $\mathcal{M}'$ that are reachable from $a'$ (including $a'$); $\mathcal{R}$ is easily a bisimulation. Notice that if $P = \emptyset$, $\phi^D_4$ is a tautology, thus all formulas are complete. \hfill \Box

For logic $l = l' + 5$, let $\phi^l_5 = \land P \land \Box \land P$.

**Lemma 5.** $\phi^l_5$ is a complete formula for $l$.

**Proof.** The completeness of $\phi$ follows from Lemma 1. \hfill \Box

When $P = \emptyset$, there are two cases. If $l' \in \{D, D4, T, S4\}$, then $\phi^l_5$ is a tautology, therefore all formulas in $L(P)$ are complete for $l$; otherwise (if $l' \in \{K, K4\}$), by Lemma 1, an $l$-model would either satisfy $\phi^l_5$ or $\Box \bot$ – in the latter case, the model would be the singleton with an empty accessibility relation. Therefore, if $P = \emptyset$ the completeness problem for $l$ is not exactly trivial, but it is close: a formula is complete for $l$ if it is satisfied in at most one of the two possible (up to bisimulation) models.

**Corollary 3.** When $P = \emptyset$, the completeness problem for $K5$ and $K45$ is in $P$.

$^3$ The modal depth of a formula is the nesting depth of its modalities, defined: $md(\bot) = 0$; $md(\phi \land \psi) = \max\{md(\phi), md(\psi)\}$; and $md(\Box \phi) = md(\phi) + 1$. 
We say that a logic $l$ has a nontrivial completeness problem if for $P \neq \emptyset$, there are complete formulas for $l$. From the modal logics we examined, only $D$ and $T$ do not have nontrivial completeness problems. Table 1 comprehensively presents the observations of this section and of Section 4 regarding the completeness problem for Modal Logic.

| Modal Logic | $P = \emptyset$ | $P \neq \emptyset$ |
|-------------|-----------------|---------------------|
| K, K4       | PSPACE-complete | PSPACE-complete     |
| D, T        | trivial (all)   | trivial (none)      |
| D4, S4      | trivial (all)   | PSPACE-complete     |
| $l + 5$, $l \neq K, K4$ | trivial (all) | coNP-complete |
| K5, K45     | in $P$          | coNP-complete       |

Table 1. The Completeness Problem for Modal Logic.

4 The Complexity of Completeness

First we present a lower bound for the complexity of completeness for logics with nontrivial completeness problems: we show that completeness is at least as hard as provability.

**Theorem 3.** Let $l$ be a logic from the ones we presented in this paper, where $l$ has a nontrivial completeness problem and let $C$ be a complexity class. If $l$-provability is $C$-hard, then the completeness problem for $l$ is $C$-hard.

**Proof.** To prove the theorem we present a reduction from $l$-satisfiability to the completeness problem for $l$.

For each logic $l$ with nontrivial completeness and finite set of propositional variables $P$, we have provided a complete formula $\phi'_P$, satisfiable in a model of one or two states at most, which can be generated in at most time $O(|P|)$ (see Section 3). Let $(\mathcal{M}_l, a_l)$ be such a (small, pointed) model for $\phi'_P$. Notice that any pointed model which satisfies $\phi'_P$ is bisimilar to $(\mathcal{M}_l, a_l)$. Given a formula $\phi \in L(P)$, there are two cases: the first is that $\neg \phi \land \phi'_P$ is satisfiable (equivalently, $\mathcal{M}_l, a_l \models \phi$, which can be checked in linear time), in which case $\phi$ is provable if and only if $\top$ is complete (i.e. $\phi$ is not provable). The second case is that $\neg \phi \land \phi'_P$ is not satisfiable, in which case we demonstrate that $\phi$ is provable if and only if $\phi \rightarrow \phi'_P$ is complete. If $\phi$ is provable, then $\phi \rightarrow \phi'_P$ is equivalent to $\phi'_P$, which is complete. On the other hand, if $\phi \rightarrow \phi'_P$ is complete and $(\mathcal{M}, a)$ a pointed model, we show that $\mathcal{M}, a \models \phi$, implying that if $\phi \rightarrow \phi'_P$ is complete, then $\phi$ is provable. If $(\mathcal{M}, a) \sim_P (\mathcal{M}_l, a_l)$, then from our assumptions $\mathcal{M}, a \not\models \phi$, thus $\mathcal{M}, a \models \phi$. On the other hand, if $(\mathcal{M}, a) \not\sim_P (\mathcal{M}_l, a_l)$, since $(\mathcal{M}_l, a_l) \models \phi \rightarrow \phi'_P$ and $\phi \rightarrow \phi'_P$ is complete, $\mathcal{M}, a \not\models \phi \rightarrow \phi'_P$, therefore $\mathcal{M}, a \models \phi$. □

Naturally, Theorem 3 applies to more than the modal logics that we have defined in Section 2. For Propositional Logic, completeness amounts to the problem

\[4\] We consider hardness under polynomial-time reductions.
of determining whether a formula does not have two distinct satisfying assignments, therefore it is coNP-complete. Of course, for First-order Logic, completeness is undecidable, as satisfiability is undecidable.

For simplicity we assume in this section that formulas are given in Negation Normal Form (NNF): all negations are pushed to the level of propositions. If \( p \) is a propositional variable, \( p \) and \( \neg p \) are called literals. A set \( S \) of literals is complete for \( P \) if it either has both \( p \) and \( \neg p \) for some \( p \in P \), or if for every \( p \in P \), either \( p \in S \) or \( \neg p \in S \); \( S \) is consistent if for every propositional variable \( p \), at most one from \( p \) and \( \neg p \) can be found in \( S \). We also assume that when \( l \neq K, K4 \), a formula uses at least one propositional variable.

The easiest cases are the logics with Negative Introspection. Proposition 1 is a direct consequence of Theorem 3 and Corollary 2.

**Proposition 1.** Let \( l \) be a modal logic with Negative Introspection from Section 2. The completeness problem for \( l \) is coNP-complete.

**Proof.** For logics with Negative Introspection, by Theorem 2, satisfiability is NP-complete and provability coNP-complete. Therefore, by Theorem 3, completeness is coNP-hard. What remains is to show that completeness is in coNP.

To check for completeness, because of Corollary 2, we simply need to verify that a formula \( \phi \) does not have two non-bisimilar models of size at most \( |\phi| + 2 \). Given the normal form of Lemma 1, we can universally guess two models of size at most \( |\phi| + 2 \) and of the form presented by Lemma 1 and verify that these are bisimilar. In this case, due to their normal form, bisimilarity can be checked in polynomial time: let there be such \((M, a)\) and \((M', a')\), where \( M = (W \cup \{a\}, R, V) \), \( M' = (W' \cup \{a'\}, R', V') \), \( R, R' \) are equivalence relations when restricted on \( W, W' \) respectively, and all states in \( W, W' \) are reachable from \( a \) and \( a' \), respectively – that is, there are states \( b \in W, b' \in W' \) accessible from \( a \) and \( a' \), respectively.

Pointed models \((M, a)\) and \((M', a')\) are bisimilar (modulo \( P \)) if and only if:

- \( V_P(a) = V_P(a') \);
- for every \( b \in W \), there is some \( b' \in W' \) such that \( V_P(b) = V_{P'}(b') \);
- for every \( b' \in W' \), there is some \( b \in W \) such that \( V_P(b) = V_{P'}(b') \);
- for every \( b \in W \) for which \( aRb \), there is some \( b' \in W' \) such that \( a'Rb' \) and \( V_P(b) = V_{P'}(b') \); and
- for every \( b' \in W' \) such that \( a'Rb' \), there is some \( b \in W \) such that \( V_P(b) = V_{P'}(b') \) and \( aRb \).

If these conditions are met, we can define bisimulation \( \mathcal{R} \) such that \( a\mathcal{R}a' \) and for \( b \in W \) and \( b' \in W' \), \( b\mathcal{R}b' \) iff \( V_P(b) = V_{P'}(b') \); on the other hand, if there is a bisimulation, then it is not hard to see by the definition of bisimulation that these conditions hold – for both claims, notice that the conditions above, given the form of the models, correspond exactly to the conditions for bisimilarity. Given two models, it takes up to polynomial time to check these two conditions; therefore, completeness for logics with Negative Introspection is in coNP. \( \square \)
For the logics without Negative Introspection, by Theorem 2, satisfiability and provability are both \( \text{PSPACE} \)-complete. So, completeness is \( \text{PSPACE} \)-hard, if nontrivial. It remains to show that it is also in \( \text{PSPACE} \).

For this we present a family of games to decide completeness for a modal formula. These we call Model Constructing-Collapsing Games. They are essentially bisimulation games [13] combined with the tableaux by Fitting [6] and Massacci [11] for Modal Logic. For more on tableaux the reader can see [4]. These games can then be directly turned into an alternating polynomial time algorithm, thus establishing that the completeness problems for these logics is in \( \text{PSPACE} \).

For a state \( x \in W \) of a frame \((W,R)\), \( \text{Reach}(x) \subset W \) is the set of states reachable from \( x \); i.e. it is the smallest set such that \( x \in \text{Reach}(x) \) and if \( y \in \text{Reach}(x) \) and for some \( z \in W \ yRz \), then \( z \in \text{Reach}(x) \).

### 4.1 Model Constructing-Collapsing Game for logic \( l \) on formula \( \phi \) (assumed to be in NNF):

The game is played by two players: the Constructor and the Collapser. Intuitively, the Constructor tries to construct two models for a given formula and at the same time demonstrate that these models are not bisimilar. On the other hand the Collapser’s goal is to demonstrate that the models the Constructor provides are bisimilar (thus collapsing these to one model). We first give a few definitions which are needed to describe the game.

The game has two sides which we will call the left and right side (each corresponding to one of the constructed models). Both sides have a number of states, each of which is assigned a set of formulas. For each side there is always a parent state and possibly some children states.

Given a modal logic \( l \), a set \( s \) of formulas is called \( l \)-closed if the following conditions hold:

- a) if \( \phi_1 \land \phi_2 \in s \), then \( \phi_1, \phi_2 \in s \);
- b) if \( \phi_1 \lor \phi_2 \in s \), then \( \phi_1 \in s \) or \( \phi_2 \in s \);
- c) if \( \square \psi \in s \) and \( l \) has factivity, then \( \psi \in s \);
- d) for every \( p \in P \), either \( p \in s \) or \( \neg p \in s \).

A state is called \( l \)-closed if it is assigned with an \( l \)-closed set. A side is called \( l \)-complete if the following conditions hold:

- i) every state on that side is \( l \)-closed;
- ii) for every \( \Diamond \psi \) the parent state is assigned with, there must be a child state assigned with \( \psi \);
- iii) for every \( \square \psi \) the parent state is assigned with, all children states are assigned with \( \psi \);
- iv) if \( l \) has positive introspection, then for every \( \square \psi \) the parent state is assigned with, all children states are assigned with \( \square \psi \);
- v) if \( l \) has the consistency axiom, then there must be at least one child state.
If a state $a$ is assigned with a set of formulas $s$, then $th(a) = \bigwedge s$. A child state $c$ is called maximal if for every other child state $d$, $th(d) \not\models th(c)$.

Lemma 6 is fairly straightforward and needed for the coming proofs.

**Lemma 6.** If a side is $l$-complete, $a$ its parent state, and $C \neq \emptyset$ the set of its children states, then if $th(a)$ is consistent and for every $c \in C$, $th(c)$ is consistent, then

- if $l = K$, then
  $$th(a) \land \bigwedge_{c \in C} \diamond th(c) \land \Box \bigvee_{c \in C} th(c)$$
  is consistent;
- if $l$ has positive introspection, then
  $$th(a) \land \bigwedge_{c \in C} \diamond th(c)$$
  is consistent.

**Proof.** Let for $c \in C, M_c = (W_c, R_c, V_c)$ and $a_c \in W_c$, such that $M_c, a_c \models th(c)$. Then let $M = (W, R, V)$, where

- $W = \{a\} \cup \bigcup_{c \in C} W_c,$
- $R' = \{(a, a_c) \mid c \in C\} \cup \bigcup_{c \in C} R_c,$

$R$ is the transitive (if $l$ has positive introspection and not factivity) or reflexive and transitive (if $l$ has both axioms) closure of $R'$, or just $R'$ (if $l$ has neither axiom), and

$$V(a) = \{p \in P \mid p \text{ is assigned to } a\}$$

and for $b \in W_c$, $V(b) = V_c(b)$. Now, $M, a_c \models th(c)$, since $M_c$ is the result of restricting $M$ to only states reachable from $a_c$ and $M_c, a_c \models th(c)$. It is now not hard to conclude that

$$M, a \models th(a) \land \bigwedge_{c \in C} \diamond th(c) \land \Box \bigvee_{c \in C} th(c)$$

if $l = K$ or that

$$M, a \models th(a) \land \bigwedge_{c \in C} \diamond th(c)$$

otherwise – note that if $a$ is assigned with some formula $\psi$, then (by straightforward induction on $\psi$ and because of the closure properties of the $l$-complete side) $M, a \models \psi$. Therefore $M, a \models th(a)$.

**Lemma 7.** For a fixed side, let parent node $a$ be $l$-complete and a child state $c$ be maximal. If $th(a)$ is consistent and for every child node $d$, $th(d)$ is consistent, then if $th(a)$ is complete, so is $th(c)$.
Proof. If \( th(c) \) is not complete, then there is some \( \psi \) such that \( th(c) \not\vdash \psi \) and \( th(c) \not\vdash \neg \psi \). Therefore the side can be extended to a complete side where \( c \) is additionally assigned some \( \psi \) and to a complete side where \( c \) is additionally assigned some \( \neg \psi \) and every other child is also assigned with \( \neg th(c) \) (because \( c \) is maximal). By Lemma 6, \( th(a) \land \Diamond (th(c) \land \psi) \) is consistent and so is \( th(a) \land \neg \Diamond (th(c) \land \psi) \), which is a contradiction. \( \square \)

The game description: At the beginning of the game, each side has one state which has been assigned only with \( \phi \); these states are the current parent states for their respective sides. The Constructor starts with 2\(|\phi|\) points. If \( \phi \) is unsatisfiable, then the Collapser wins automatically.

The moves of the game occur in the following order:

1. The Constructor adds states to each side and assigns formulas to these states, so that each side is \( l \)-complete and every child state is maximal.\(^5\)
2. If a state is assigned with a set of formulas which is inconsistent, or if at both sides there are no children states, then the Collapser wins.
3. The Constructor picks a child state from one side.
4. The Collapser picks a child state from the other side; if there is no child to pick, then the Constructor wins.
5. Alternatively, if \( l \) has positive introspection, then let \( x \) be the child state that the Constructor picked from side \( A \) and let \( y \) be the parent state from the other side, \( B \); if \( th(y) \vdash \Diamond th(x) \), then a copy of \( x \) is made on side \( B \) and the Collapser can pick that copy.\(^6\)
6. If the picked states are assigned a different set of literals, then the Constructor wins.
7. If the Constructor has no more points, then the Collapser wins.
8. The Constructor pays one point to make the pair of chosen states into parent states in their respective sides and all other states are deleted; the game resumes from 1.

\(^5\) The maximality of children can be ensured by merging \( c \) and \( d \) whenever \( th(c) \vdash th(d) \).

\(^6\) Notice that the copy of \( x \) might not be maximal. This, however, will not affect the game.

Lemma 8. The Model Constructing-Collapsing Game for logic \( l \) on \( \phi \) ends after at most \( O(|\phi|) \) moves.

Proof. At the beginning the Constructor has 2\(|\phi|\) points, so each step from 1 to 8 may occur up to 2\(|\phi|\) times. \( \square \)

Since the game is finite, it is also determined.

Corollary 4. The Model Constructing-Collapsing Game for logic \( l \) is determined; that is, for every formula either the Constructor or the Collapser has a winning strategy.
Theorem 4. In the Model Constructing-Collapsing Game for logic $l$ on $|\phi|$, the Collapser has a winning strategy if and only if $\phi$ is complete.

Proof. We prove that the Constructor has a winning strategy if and only if $|\phi|$ can be satisfied by two non-bisimilar models. By Corollary 4 and Theorem 1, this is enough to prove the theorem.

First we assume that there are two non-bisimilar pointed models $(A, a)$ and $(B, b)$, such that $A, a \models \phi$ and $B, b \models \phi$, to prove that the Constructor has a winning strategy. We call these models the underlying models; the states of the underlying models are called model states to distinguish them from states of the game. The Constructor will maintain the following invariants:

- at every stage of the game, the parent state of the left side will correspond to a state $a'$ of $A$ and the parent state of the right side will correspond to a state $b'$ of $B$, such that $(A, a')$ is not bisimilar to $(B, b')$;
- the children states of each side will correspond to states that are accessible from the corresponding model state of the respective parent state of that side;
- finally, the formulas that are assigned to a state will always be formulas that are satisfied at the corresponding model state of that model.

We call any strategy of the Constructor that maintains these invariants a guided strategy based on pointed models $(A, a)$ and $(B, b)$. We first demonstrate that the Constructor can play in such a way as to maintain these invariants – thus there is always a guided strategy for the Constructor.

It is not hard to verify that the Constructor can maintain these invariants when playing move 1, since the states and assigned formulas the Constructor must provide can be copied directly from the underlying models. Because of the invariants, there is a state $c$ in $A$ (or in $B$, but without loss of generality we assume it is found in $A$), which is accessible from the corresponding state of the current parent state of the left side, such that $(A, c)$ is not bisimilar to any $(B, d)$, where $d$ accessible from the corresponding state of the current parent state of the right side; otherwise, the non-bisimilarity invariant would fail. The Constructor can make sure to create a new child state (say $c'$) at move 1 to correspond to $c$ (and is assigned a maximal set of literals and formulas which hold in $c$) and pick it at move 3.\footnote{Note that the Constructor needs to make at most $|\phi| + 1$ children (which is actually an overestimation) at move 1 to preserve a guided strategy.}

It is not hard to verify that under these invariants the Collapser cannot win as long as the Constructor still has points – only in move 2 the Collapser can win, but because of the underlying models and the invariants, all assigned sets of formulas are consistent and the Constructor can always create at least one new child state at move 1, from the observation above. Therefore we examine what happens when the Constructor has spent the last point. We distinguish two cases depending on whether $l$ has positive introspection.
If \( l \) does not have positive introspection, we further assume that the Constructor only assigns subformulas of \( \phi \) to the states – with the exception of literals – and that the assigned set of formulas to a state is minimal in that it is produced by a closure under the conditions for \( l \)-closed sets and \( l \)-complete sides. Under these assumptions, notice that every time the Constructor spends a point, the maximum modal depth of the formulas drops. The modal depth of \( \phi \) can be at most \(|\phi|\), so when the Constructor has no more points, all formulas in the game are propositional.\(^8\) Then, the Constructor can use move 1 and make one state for each side assigned with a different set of literals; the Collapser will have to make these two a pair, so the Constructor wins (and the game will not reach move 7 with no points).

For the cases where \( l \) has positive introspection, to help us prove that the Constructor has a winning strategy, we slightly modify the game in favor of the Collapser. We assume that the Constructor at move 1 assigns to a state all subformulas of \( \phi \) of the form \( \Diamond \psi \) which are true in the corresponding model state. We furthermore assume that for move 5, even if \( \text{th}(y) \not
ot\text{th}(x) \), as long as for the corresponding state \( g \) of \( y \), \( g \models \text{th}(x) \), then a copy of \( x \) is made and the Collapser can pick that copy. We prove that the Constructor can still win in this game which has further advantages for the Collapser.

First notice that the Constructor, given an infinite amount of initial points and non-bisimilar pointed models \((A, a)\) and \((B, b)\) of \( \phi \), can eventually win by following some guided strategy based on the pointed models: the Collapser cannot win through 2 or 7 and if the Collapser has a way to play so that the game never stops (i.e. the Constructor doesn’t win either) for all guided strategies of the Constructor, then \((A, a)\) and \((B, b)\) must be bisimilar, with the bisimulation relation defined by the Collapser’s choices of states.\(^9\) Move 5 can be interpreted as the Collapser claiming that there must be another accessible state, bisimilar to the one the Constructor has chosen, and which the Constructor has not revealed.

So, if \((A, a)\) and \((B, b)\) are non-bisimilar models of \( \phi \), then there is a winning guided strategy on \((A, a)\) and \((B, b)\) for the Constructor, such that every play in which the Constructor plays according to the strategy will end after using a finite number of moves (thus also points). Therefore, this guided strategy uses a maximum number of points (by König’s Lemma). Lets assume the Constructor is using the winning guided strategy that uses the least maximum number of points. If this number of points is more than \(2|\phi|\), then we can show how to find a winning guided strategy which uses even fewer points (thus reaching a contradiction; therefore, there is a winning guided strategy which uses at most \(2|\phi|\) points in any play).

In a play, \( B(x) \), the set of boxed formulas assigned to a state \( x \), can only increase from parent to child – that is, if \( r \) a parent state and \( c \) a child state of

\(^8\) Actually, this means that \(2|\phi|\) points are too many for the Constructor when \( l \) does not have positive introspection; \(|\phi| - 1\), or rather \(\text{md}(\phi)\) are enough.

\(^9\) The situation here is quite similar to the definition of bisimulation as a game, given in ex. [13].
the same side, \( B(r) \subseteq B(c) \; \) the number of formulas in such a \( B(x) \) can be at most \( n < |\phi| \). On the other hand, because of transitivity, if a formula of the form \( \diamond \psi \) (a diamond formula) is true at a corresponding state of a child \( c \), it will also hold at the corresponding state of the parent \( r \). Let \( D(x) \) be the set of diamond formulas assigned to a state \( x \). From the observation above, if \( r \) a parent state and \( c \) a child state of the same side, then \( D(c) \subseteq D(r) \). Furthermore, for all states \( x \), \( |D(x)| \leq |\phi| - n \). Therefore, the overall number of times the set \( A(x) = B(x) \cup D(x) \) changes from parent to child on one side is at most \( |\phi| \). So if the Constructor uses more than \( 2|\phi| \) points, there is a pair of parents \( r, r' \) and a pair of selected children \( c, c' \), so that the \( A(c) = A(r) \) and \( A(c') = A(r') \). In that case, we can modify the Constructor’s strategy so that they play (and win) on the parent-pair \( r, r' \) as they play on the children-pair \( c, c' \), effectively reducing the play’s maximum length by 1 – note that all moves of the (altered) game on \( r, r' \) can be played as on \( c, c' \) and vice-versa.

On the other hand, we prove that if the Constructor has a winning strategy, then there are two non-bisimilar pointed models for \( \phi \). In fact, by induction on the number of points remaining in the game, we prove that if the Constructor has a winning strategy at a certain point in the game, then there are two non-bisimilar pointed models for \( th(a) \) and \( th(b) \) respectively, where \( a \) and \( b \) the respective parent states for the two sides.

Claim: if the Constructor can force a win at move 4 or 6, then there are two non-bisimilar pointed models for \( th(a) \) and \( th(b) \) respectively. Since the Collapser was forced like this, because of Lemma 6 there is a formula \( \chi \) which is of the form \( \diamond \top, \Box \bot, \diamond s, \) or \( \Box \neg s \), where \( s \) a conjunction of literals, such that \( th(a) \land \chi \) and \( th(b) \land \neg \chi \) are both satisfiable. Therefore, there are two non-bisimilar pointed models for \( th(a) \) and \( th(b) \) respectively.

If there are 0 points remaining in the game, then since the Collapser does not win with move 7, it must be the case that the Constructor wins with move 4 or 6. From the claim above, the base case of the induction is complete.

If there are \( n + 1 \) points in the game, then there are three possible cases. Either the Constructor wins at move 4, or at move 6, or after spending one point and playing on a pair of children. If the Constructor wins at move 4 or 6, then as above, by the claim there are two non-bisimilar pointed models for \( th(a) \) and \( th(b) \) respectively. Therefore, we can assume that there is a child \( (w.l.o.g. \) of \( a ) \) \( x \) picked by the Constructor such that for all children \( y \) picked by the Collapser, there are two non-bisimilar pointed models for \( th(x) \) and \( th(y) \) respectively. If \( th(a) \) is incomplete, then we are done, because \( th(a) \) has two non-bisimilar models, thus one of them is non-bisimilar to a model of \( th(b) \); similarly for when \( th(b) \) is incomplete. Otherwise, by Lemma 7, \( th(x) \) is complete (since \( x \) is picked by the Constructor, \( x \) was introduced by the Constructor and not by move 5; therefore, \( x \) is maximal); therefore, it is satisfied in a unique up to bisimilarity model, \( M_x \), meaning that for every child \( y \) from the other side, there is a (pointed) model \( M_y \models th(y) \), such that \( M_x \not\models M_y \). This also means that

\[10\] Remember, \( t \) has positive introspection at the moment.
for every child $y$ from the other side, $M_y \not\models th(x)$, therefore $th(y) \land \neg th(x)$ is consistent and satisfied at $M_y$. By Lemma 6, $th(a) \land \Diamond th(x)$ and $th(b) \land \neg \Diamond th(x)$ are satisfiable (all $y$'s can be additionally assigned with a consistent closure – so that $y$ remains $l$-closed – of $\{th(y), \neg th(x)\}$), therefore there are two non-bisimilar pointed models for $th(a)$ and $th(b)$ respectively. □

Corollary 5. The completeness problem for $K$, $K4$, $D4$, and $S4$ is PSPACE-complete.

Proof. The Constructor-Collapser game can easily be turned into an alternating polynomial-time algorithm with an oracle from PSPACE (for when we need to determine the provability or satisfiability of formulas). The algorithm simply runs the game, trying to determine whether the Collapser wins, with the Collapser’s moves being simulated by existential nondeterministic choices and the choices of the Constructor by universal choices. Thus, the completeness problem for these logics is in $\text{AP}^{\text{PSPACE}} = \text{PSPACE}$. □

5 Variations and Further Considerations

In this section we make observations and take a look at variations on the completeness problem for Modal Logic.

5.1 Triviality

It is interesting to note that the completeness problem for $T$ and $D$ is trivial (while it is not for $K$, $S4$, or $D4$), in that there are no complete formulas for these logics. In other words, no finite specification can describe a unique model. Of course, this statement is not true for schemes, as with the addition of positive or negative introspection and some finite formula we can specify a specific model. The role of positive and negative introspection is important here, as they seem to extend the effect of boxed formulas to most of the model. On the other hand, what brings completeness to $K$ seems to be the ability to cut off the model with (satisfiable) formulas like $\Box \bot$.

5.2 Satisfiable and Complete Formulas

It may be more appropriate, depending on the case, to check whether a formula is satisfiable and complete. In this case, if the modal logic does not have Negative Introspection, we can simply alter the Constructor-Collapser games so that the Constructor wins if the formula is not satisfiable. Therefore, the problem remains in PSPACE; for PSPACE-completeness, notice that the products of the reduction for Theorem 3 are satisfiable formulas.

For logics with Negative Introspection (and plain Propositional Logic), the set of Satisfiable and Complete Formulas is US-complete, where US is the class of problems for which we demand unique solutions [3]: UniqueSAT is a complete problem for US and a special case of this problem.
5.3 Completeness with Respect to a Model

It would also be very natural to consider completeness of a formula over a satisfying model. That is, given a formula $\phi$ and model $(M, s)$, such that $M, s \models \phi$, is the formula complete? Now we are given one of $\phi$'s models, so it is a reasonable expectation that the problem just became easier. Note that in many cases, this problem may even be more natural than the original one, as we are now testing whether the formula completely describes the model.

This problem, though, has exactly the same complexity as plain completeness: we can easily reduce completeness with respect to a model to plain completeness by dropping the model from the input; on the other hand, the reduction from provability to completeness of Section 4 still works in this case, as it can be adjusted to additionally provide the satisfying model of the known complete formula ($\phi^*_p$).

5.4 Other Logics

Of course, this is not the end of Modal Logic and therefore neither of the completeness problem. Modal Logic is extensive and we can study the completeness problem for similar logics to the ones we examined here (for example $\mathcal{B}$, which has symmetric accessibility relations), or, perhaps even more appropriately, for logics such LTL or the $\mu$-calculus, which have a wide impact on applications as specification languages. Another direction of interest would be to consider schemes as part of the input – as we have seen, the Negative Introspection scheme together with $\phi^{S5}$ is complete for $T$, when no formula is.

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