Topological gauge invariant variables in QCD

D. Blaschke, V. N. Pervushin∗ and G. Röpke
Fachbereich Physik, Universität Rostock, D-18051 Rostock, Germany

Abstract

We suggest that proper variables for the description of non-Abelian theories are those gauge invariant ones which keep the invariance of the winding number functional with respect to topologically nontrivial (large) gauge transformations. We present a model for these variables using the zero mode of the Gauss law constraint and investigate their physical consequences for hadron spectrum and confinement on the level of the generating functional for two-color QCD.

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∗Permanent address: Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia
I. INTRODUCTION

The identification of physical degrees of freedom in non-Abelian gauge theories is a crucial point for understanding the physical phenomena hidden in their structure. According to Dirac [1], the principle of local gauge invariance has to be established not only on the level of the Lagrangian, but also on the level of the variables used in the formulation of the gauge theory, since we can observe only gauge-invariant quantities. Dirac has obtained the unconstrained form of QED in terms of gauge invariant variables for QED [1] as functionals of the initial gauge fields by explicitly resolving the Gauss law constraint. The resulting unconstrained formulation of QED coincides with the one obtained in the Coulomb gauge [2] with the physical phenomena of electrostatics, 'dressed' electrons, and two transverse photon degrees of freedom. The relativistic covariance of this formulation of QED has been proven by Zumino [3] on the level of the algebra of generators of the Poincare group.

The Dirac definition of the gauge-invariant variables can be treated also as a change of variables to construct the generating functional of the Green functions in any gauges including the Lorentz invariant ones. The invariance of the corresponding Green functions under a change of variables (which generates the Ward-Taylor-Slavnov identities [4]) is guaranteed by the Dirac factors in source terms, which restore the Coulomb gauge Feynman rules in any Lorentz invariant gauge. So, the Coulomb interaction and electrostatics are consequences of the identification of the physical degrees of freedom which correspond to an explicit solution of the Gauss law, but not primarily to the choice of the gauge. For example, if one would omit the Dirac factors in for the source terms in relativistically invariant Lorentz gauge formulations of QED, one would get the Wick-Cutkosky bound states formed by gauge propagators with light-cone singularities with a spectrum different from the observed one which corresponds to the instantaneous Coulomb interaction. Thus, the Dirac variables in QED are gauge-invariant, Lorentz covariant, and bear direct relation to measurable quantities.

The attempts to obtain the non-Abelian generalization of the Dirac gauge invariant variables [6–10] meet the difficulties of the nontrivial topological structure of gauge transformations which are classified as 'small' and 'large' ones in accordance with their homotopy group, e.g. \( \pi_3(SU(2)) = \mathbb{Z} \) [11,12] for the example of two-color QCD considered below. The quantization of the fermion sector of the theory leads to anomalies [13] and to the topological 'winding number' functional of the gauge fields which plays an important role in the description of physical phenomena connected with the \( U_A(1) \) anomaly [14–18]. It is, however, not invariant with respect to large gauge transformations and the problem arises: Is it possible to construct 'topological' gauge invariant variables which leave the winding number invariant under large gauge transformations?

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One of the authors (V.P.) thanks W. Kummer who pointed out that in Ref. [5] the difference between the Coulomb atom and the Wick-Cutkosky bound states in QED has been demonstrated.
In the present paper, we study the physical consequences of the nontrivial topology of non-Abelian gauge fields using topological Dirac variables. The resulting representation of the QCD generating functional is defined by the zero modes of the Gauss law constraint in the class of functions of large gauge transformations, which disappear at spatial infinity but have nonzero surface integrals.

The paper is organized as follows. Section 2 reviews the introduction of ordinary Dirac variables for non-Abelian theories using the example of the $SU_c(2)$ Yang-Mills theory. In Section 3, we give the definition of topological Dirac variables which are constructed by including a zero mode of the Gauss law constraint. We derive an equation for the topological invariance of non-perturbative fields in the class of functions corresponding to large gauge transformations. In Section 4, we show that the Wu-Yang monopole is a solution of this equation and use it to obtain the electric field of the $\theta$-vacuum, the instantaneous interaction for QCD, and equations for quasiparticle excitations. In Section 5, the generating functional for Green functions is constructed and a discussion of the problems of hadronization and confinement is given. The conclusions are presented in Section 6.

II. DIRAC VARIABLES IN NON-ABELIAN THEORIES

We consider two-color QCD, i.e. $SU_c(2)$ Yang-Mills theory coupled to fermionic fields (quarks), with the action functional

$$W[A, \Psi] = \int dt \int d^3x \left( -\frac{1}{4} G^2 + \bar{\psi} [i \gamma^\mu (\partial_\mu + \hat{A}_\mu) - m] \psi \right),$$

where $\hat{A} = g \frac{\tau^a A_a^a}{2i}$,

$$-\frac{1}{4} G^2 = \frac{1}{2} \sum_{i,a} (E^{a^2}_i - B^{a^2}_i),$$

and the conventional notations for the covariant derivative $D_i^{ab}(A) := \delta^{ab} \partial_i + g \epsilon^{abc} A^c_i$ as well as for the non-Abelian electric and magnetic fields

$$E_i^a = \partial_0 A_i^a - D_i^{ab}(A) A_0^b,$$

$$B_i^a = \epsilon_{ijk} \left( \partial_j A_k^a + \frac{g}{2} \epsilon^{abc} A_j^b A_k^c \right),$$

respectively, have been used.

The action (2.1) is invariant with respect to arbitrary gauge transformations $u(t, x)$

$$\psi^u := u(t, x) \psi ,$$

$$\hat{A}^u_i := u(t, x) \left( \hat{A}_i + \partial_i \right) u^{-1}(t, x).$$
In order to eliminate unphysical degrees of freedom, we introduce the non-Abelian generalization of the Dirac variables \( \psi \) according to \( \psi_D[\psi, A] := U[A] \psi \),

\[
\psi_D[\psi, A] := U[A] \psi ,
\]

\[
\hat{A}^D_\mu[A] := U[A] \left( \hat{A}_\mu + \partial_\mu \right) U^{-1}[A] ,
\]

where \( U[A] \) is a solution of the linear differential equation

\[
U[A] \left( \hat{a}_0[A] + \partial_0 \right) U^{-1}[A] = 0 ,
\]

and \( a_0[A] \) is a solution of the Gauss equation without the fermion source term

\[
[D^2(A)]^{bc} a_0^c = [D_k(A)]^{bc} \partial_0 A^k_c .
\]

By construction, the matrix \( U[A] \) has the transformation property

\[
U[A^u] = U[A] u^{-1} ,
\]

so that the Dirac variables \( A^D_i \) are gauge-invariant functionals of the initial gauge potentials

\[
\hat{A}^D[A^u] = \hat{A}^D[A] , \quad \psi^D[\psi^u, A^u] = \psi^D[\psi, A] ,
\]

which satisfy the identity

\[
D^a_{\mu} (A^D) \partial_0 (A^D)^b = 0 .
\]

The solution of the system of linear differential equations \( 2.9, 2.10 \) can be written in the form of the time ordered exponential

\[
U(t, \vec{x}; t_0) = v(\vec{x}) T \exp \left( \int_{t_0}^t dt \hat{a}_0(t, \vec{x}) \right) .
\]

The gauge-invariant Dirac variables \( A^D \) as solutions of the differential equation \( 2.9 \) are defined for the initial values

\[
U(t, \vec{x}) \mid_{t=t_0} = v(\vec{x}) .
\]

These values define the remaining group of stationary gauge transformations of the Dirac variables. The group of the stationary gauge transformations \( v(\vec{x}) \) in the three-dimensional coordinate space is topologically nontrivial and represents the group of three-dimensional paths lying on the three-dimensional space of the \( SU_c(2) \)-manifold with the homotopy group \( \pi_3(SU_c(2)) = Z \). The whole group of the stationary gauge transformations is split into topological classes marked by the degree of the map (i.e. the integer number \( n \)) which counts how many times a three-dimensional path turns around the \( SU(2) \)-manifold when the coordinate \( x_i \) runs over the space where it is defined. The
stationary transformations \( v^n(\vec{x}) \) with \( n = 0 \) are called the small ones; and those with \( n \neq 0 \)

\[
\hat{A}^{(n)} = v^{(n)}(\vec{x})(\hat{A} + \partial)v^{(n)}(\vec{x})^{-1},
\]

(2.16)

the large ones.

Quantization of the fermion sector of the theory leads to the well-known Adler-Bell-Jackiw anomaly \([13]\)

\[
W_{\text{anomal}}[A; j, \eta, \bar{\eta}] = C_\eta \int dt \bar{\eta} I_c \gamma_5 \eta \frac{d}{dt} X[A],
\]

(2.17)

where \( \eta, \bar{\eta} \) are fermion sources, \( C_\eta \) is a constant and \( I_c \) is the unit matrix in color space.

\[
X[A^D] = -\frac{1}{8\pi^2} \int d^3x \epsilon^{ijk} Tr \left[ \hat{A}^D_i \partial_j \hat{A}^D_k - \frac{2}{3} \hat{A}^D_i \hat{A}^D_j \hat{A}^D_k \right]
\]

(2.18)

is the topological winding number functional of the gauge fields. This functional plays an important role for the description of observable phenomena in hadronic physics connected with the violation of the axial \( U_A(1) \) symmetry and, in particular, with the occurrence of the \( \eta - \eta' \) mass difference \([14–18]\). The functional \( X[A] \) is not invariant with respect to large gauge transformations (2.16) \([11]\) (see defs. (3.33), (3.36) in \([12]\)):

\[
X[A^{(n)}] = X[A] + \mathcal{N}[b, n],
\]

(2.19)

where

\[
\mathcal{N}[b, n] = \frac{1}{8\pi^2} \int d^3x \epsilon^{ijk} Tr [\partial_i (\hat{b}_j L^n_k) - \frac{1}{3} (L^n_i L^n_j L^n_k)].
\]

(2.20)

\( L^n_k = v^n \partial_k (v^n)^{-1} \) is a pure gauge field and \( \hat{b}_i(\vec{x}) \) is an asymptotics of the field \( \hat{A}^D_i(t, \vec{x}) \) at the spatial infinity

\[
\lim_{\vec{x} \to \infty} A^D(t, \vec{x}) = b(x).
\]

(2.21)

Both the large transformations and 'asymptotic' field \( b(x) \) are given in the electrostatic-type class of the functions which disappear at spatial infinity but have nonvanishing surface integrals in Eq. (2.20).

In the context of the Dirac invariant description of the quantum theory the question appears: Is it possible to construct 'topological' non-Abelian variables as functionals of the Dirac ones \( A^T[A^D] \), so that the winding number (2.18) becomes invariant also with respect to the large gauge transformations, i.e.

\[
X[A^T[A^{D(n)}]] = X[A^T[A^D]]
\]

(2.22)

for \( n \neq 0 \)? The answer to this question is the subject of the present paper and will be given in the next Section.
III. TOPOLOGICALLY INvariant DIRAC VARIABLES

A. Zero mode of the Gauss constraint

Our idea is to construct such topological variables $A^T$ for which the winding number (2.22) converts into a zero mode of the Gauss constraint [6–8,20]. The latter, in terms of the Dirac variables (2.7), (2.8), has the form

$$\left( D^2(A) \right)^{ac} (A^c_0)^c = (j_0^D)^a,$$

where $j_0^a = \frac{1}{2} \bar{\psi} \tau^a \psi$ is the fermionic current. A general solution of this inhomogeneous equation is a sum of the solution $\Phi_c$ of the homogeneous equation

$$\left( D^2(A) \right)^{ac} \Phi_c = 0,$$

i.e. a zero mode of the Gauss law constraint, and a partial solution $\tilde{A}_0^D$ of the inhomogeneous one (3.1)

$$A_0^D = -\Phi + \tilde{A}_0^D.$$

The electric field strength

$$E^a_k = \partial_0 (A^D_0)^a + D^a_{kc} (A^D_0)(A^D_0)^c$$

can also be decomposed into a zero mode part and a perturbative part $E^c_k$,

$$E^c_k = D^c_{kb} (A^D_b) \Phi^b + \tilde{E}^c_k.$$

Then, the initial Yang-Mills action in (2.1) is a sum of global (G) and local (L) parts

$$W_{YM} = W_G + W_L.$$

The local part coincides with the gauge field sector of the initial action (2.1) with the fields $A_0$ replaced by $\tilde{A}_0$, and the global one is

$$W_G = \int dt \ (I_E + I_\Phi),$$

where

$$I_E = \int d^3x \ [\tilde{E}_i D_i(A) \Phi + j_0 \Phi],$$

$$I_\Phi = \frac{1}{2} \int d^3x \ (D_i(A) \Phi)^a (D_i(A) \Phi)^a.$$

To construct the generating functional in quantum theory [21], we need only the constraint-shell action, where the (global) zero-mode part converts into a sum of two surface integrals [8].
\begin{align}
I_E & \equiv \int d^3 x \, \partial_i (\vec{E}_i \Phi) = \oint d s_i (\vec{E}_i \Phi) \big|_{|\vec{x}| \to \infty} \quad (3.10) \\
I_\Phi & \equiv \frac{1}{4} \int d^3 x \, \Delta(\Phi^a)^2 . \quad (3.11)
\end{align}

Thus, we can restrict ourselves to the 'electrostatics' class of functions of topologically nontrivial gauge transformations, in the region of the spatial infinity. For simplicity, we suppose that the quasiparticle part disappears, $I_E = 0$, and the Dirac field $A^D$ at spatial infinity converts into a stationary one $(b_i(\vec{x})$, Eq. (2.21)). In this case, the zero mode field factorizes into

$$
\Phi^a(t, \vec{x}) \big|_{|\vec{x}| \to \infty} = \hat{N}_0(t) \Phi^0_0(\vec{x}) ,
$$

where $\Phi^0_0(\vec{x})$ satisfies the equation

$$
(D^2(b))^{ca} \Phi_0^a = 0 . \quad (3.13)
$$

$N_0$ is a zero mode with the dynamics of a free rotator defined by the Lagrangian (3.11)

$$
I_\Phi = \frac{M_N}{2} \hat{N}_0^2 ; \quad M_N = \frac{1}{2} \int \! d^3 x \, \Delta(\Phi_0^a)^2 . \quad (3.14)
$$

B. The topological Dirac variables

We define the topological variables

$$
\hat{A}^T[A] = \hat{U}[\Phi^D](\hat{A}^D + \partial)\hat{U}^{-1}[\Phi^D] , \quad (3.15)
$$

$$
\hat{U}[\Phi^D] = T \exp \left( \int_{t_0}^t \! dt \Phi^D(t, \vec{x}) \right) ,
$$

so that the topological functional (2.18) in terms of these variables depends only on the zero mode $N_0$:

$$
X[A^T] = X[A^D] + \mathcal{N}(b, N_0 - X[A^D]) = N_0 , \quad (3.17)
$$

where

$$
\mathcal{N}(b, N) = \frac{1}{8\pi^2} \int \! d^3 x \epsilon_{ijk} Tr [\partial_i (\hat{b}_j L^N_k) - \frac{1}{3} (L^N_i L^N_j L^N_k)] , \quad (3.18)
$$

and $L^N_i$ is the pure gauge field

$$
L^N_k = U(N) \partial_k (U(N))^{-1} , \quad (3.19)
$$

$$
U(N) = \lim_{|\vec{x}| \to \infty} \hat{U}[\Phi^D] = \exp[N(t) \Phi^0_0(\vec{x})] . \quad (3.20)
$$

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In order to fulfill the constraint \((3.17)\), it is sufficient to find the magnetostatic field \(\mathbf{b_i}(\mathbf{x})\) for which
\[
\mathcal{N}(b, N) = N. \tag{3.21}
\]
This self-consistency condition for the invariance of the topological functional \((3.18)\) is a necessary and sufficient condition for the topological invariance of the Dirac variables \((3.15), (3.16)\) constructed above.

Let us solve the constraint \((3.21)\) for the class of functions of topologically nontrivial gauge transformations \([11,12]\) with the boundary conditions \(f(0) = 0, \ f(\infty) = 1\) supposing that the corresponding zero mode solution \(\mathbf{\hat{\Phi_0}(\mathbf{x})}\) has the form
\[
\mathbf{\hat{\Phi_0}(\mathbf{x})} = -in^a\tau^a\pi f(r); \quad r = |\mathbf{x}|. \tag{3.22}
\]
In this case, the second part of \(\mathcal{N}(b, N)\) in Eq. \((3.18)\) is equal to \([12]\)
\[
(N - \frac{\sin(2\pi N)}{2\pi}). \tag{3.23}
\]
Then, the first part should be equal to
\[
\left(\frac{\sin(2\pi N)}{2\pi}\right),
\]
in order to fulfill the condition \((3.21)\) \(\mathcal{N}(b, N) = N.\) One can be show that this compensation is fulfilled for an asymptotic field \(b(\mathbf{x})\) which has the form of the Wu-Yang monopole \([22,23]\), i.e.
\[
b_i^a = \frac{1}{g}\epsilon_{iak}\frac{n_k(\Omega)}{r}, \quad n_k(\Omega) = \frac{x^l\Omega_{lk}}{r}, \tag{3.24}
\]
where \(\Omega^{lk}\) is an orthogonal matrix in color space: \(n_k(\Omega)n^k(\Omega) = 1.\)

The Wu-Yang monopole and similar solutions of the classical equations are also present in \(SU(3)\) theory, if we choose the minimal subgroup \(SU(2)\) of \(SU_c(3)\) (this means that the fundamental representation of \(SU_c(3)\) is an irreducible one of this subgroup).

For example, the role of matrices \(\tau_1, \tau_2, \tau_3\) of the minimal subgroup \(SU(2)\), in \(SU_c(3)\) theory, is played by \((\lambda_2, \lambda_5, \lambda_7)\). In this case,
\[
\Rightarrow b_i = g\frac{b_1^i\lambda^2 + b_2^i\lambda^5 + b_3^i\lambda^7}{2i}; \quad b_i^a = \frac{\epsilon^{aik}n^k}{gr}. \tag{3.25}
\]
Thus, we can reproduce the construction of the topological Dirac variables also in the \(SU_c(3)\) non-Abelian theory.
IV. PHYSICAL CONSEQUENCES

A. Non-perturbative asymptotic field

In order to illustrate physical consequences of the topological Dirac variables, we consider the Wu-Yang monopole more in detail.

The Wu-Yang monopole satisfies the classical equation everywhere except for the singularity at \( r = 0 \) with the corresponding magnetic field

\[
B^a_i(b) = \frac{n^a n^i}{gr^2} .
\]

Following Wu and Yang \[22\], we consider the whole finite space volume, excluding an \( \epsilon \)-region around the singular point. The size of this small region is chosen so that the vacuum energy of the monopole solution \[3.24\]

\[
\frac{1}{2} \int d^3 x (B^a_i(b))^2 = \frac{1}{2 \alpha_s} \int \frac{dr}{r^2} \sim \frac{1}{2 \alpha_s \epsilon} + O[\frac{1}{R}] , \quad \alpha_s = g_s^2 \frac{4 \pi}{g^2} \]

is removed by a finite counter-term in the Lagrangian,

\[
\bar{\mathcal{L}} = \mathcal{L} - \frac{\mu^4}{2 \alpha_s} .
\]

The choice \( \mu^4 = (\epsilon V)^{-1} \) leads to volume-independent results. The parameter \( \epsilon \) of the ultraviolet cutoff is proportional to \( V^{-1} \) and disappears in the infinite volume limit

\[
(\epsilon)^{-1} = V \mu^4 ,
\]

where the parameter \( \mu \) has the dimension of energy and determines the average density of the asymptotic field \( b \)

\[
\frac{1}{2V} \int d^3 x G^2(b) = \frac{\alpha_s G^2(b)}{2} = \mu^4 .
\]

B. \( \theta \)-vacuum

For the given regularization there is a solution of the zero mode equation \[3.2\] in the form of Eq. \[3.22\]

\[
\Phi^a = \frac{1}{g} n^a 2 \pi f(r) ; \quad f(r) = 1 - \frac{\epsilon}{r} ,
\]

with the function \( f(r) \) which fulfills the boundary conditions \( f(\epsilon) = 0 \) and \( f(\infty) = 1 \), in the considered region of space.
In this case, the functional \( (3.18) \) does not depend on the parameter of regularization and can be calculated exactly. As we have seen above, it coincides with the zero mode variable, \( \mathcal{N}(b, N) = N \).

The kinetic term \( I_\Phi \) is equal to

\[
I_\Phi = \frac{M_N}{2} \dot{N}_0^2 ; \tag{4.6}
\]

\[
M_N = \frac{1}{2} \int_V d^3x \, \triangle (\Phi_0) - \left( \frac{4\pi^2}{\alpha_s} \int dr \ \frac{d}{dr} (\nu^2 \frac{d}{dr} f) \right) = \frac{4\pi^2}{\alpha_s} (\mu^4 V)^{-1} ,
\]

where we took into account Eq. \( (4.3) \). The zero mode \( N_0 \) is given in the physical region \( 0 \leq N_0 \leq 1 \), where the endpoints \( N_0 = 0, N_0 = 1 \) are physically equivalent, so that the phase space of the physical variables has the topology of a cylinder \([11,7]\). The Lagrangian of the global motion \( (4.6) \) describes a free rotator with the momentum spectrum \([6]\)

\[
P_0 = \dot{N}_0 M_N = (2\pi k + \theta) , \quad k = 0, 1, 2, ...,
\]

which follows from the constraint on the wave function

\[\Psi(N_0 + 1) = \exp(i\theta)\Psi(N_0) .\]

The corresponding vacuum part of the electric field in \( (3.4) \) does not contain the regularization parameter and reads

\[
E^a_i = \dot{N}_0 (D_i(b)\Phi)^a = (k + \frac{\theta}{2\pi})\alpha_s B^a_i , \tag{4.7}
\]

where \( B^a_i \) is the vacuum magnetic field \( (1.1) \). The vacuum state with the minimal wave number \( k = 0 \) corresponds to a nonzero electric field

\[
(E^a_i)_{\text{min}} = \frac{\theta\alpha_s B^a_i}{2\pi} , \tag{4.8}
\]

i.e. a 'persistent field motion' around the 'cylinder'. We have obtained a field theoretical analogy of the Josephson effect: a circular current without sources. Coleman \([24]\) was the first who guessed an effect similar to \( (4.8) \) in \( QED_{1+1} \), but from a classical point of view. The quantum treatment of this effect was discussed in Refs. \([7,25]\).

This 'duality' of the vacuum in the Minkowski space \( (1.7) \) shows that there can be a value of the superfluid momentum \( P_0 = 2\pi/\alpha_s \) for which the vacuum Lagrangian vanishes

\[
\alpha_s = (k + \frac{\theta}{2\pi})^{-1} \rightarrow G^2 = 2(B^2 - E^2) = 0
\]

without any counterterm discussed before.
C. Non-Abelian generalization of the Coulomb potential

We can also calculate the instantaneous interaction which corresponds to the Green function of the gauge field when perturbations of the vacuum are neglected, see (3.5). In the presence of the Wu-Yang monopole the Green function satisfies the equation

\[(D^2)^{ab}(\vec{x})G^{bc}(\vec{x},\vec{y}) = \delta^{ac}\delta^3(\vec{x} - \vec{y}),\]

where

\[(D^2)^{ab}(\vec{x}) = \delta^{ab}\Delta - \frac{n^a n^b + \delta^{ab}}{r^2} + 2\left(\frac{n_a}{r}\partial_b - \frac{n_b}{r}\partial_a\right),\]

and \(n_a(x) = x_a/r; \ r = |\vec{x}|\). Let us decompose \(G^{ab}\) into a complete set of orthogonal vectors in color space

\[G^{ab}(\vec{x},\vec{y}) = [n^a(x)n^b(y)V_0(z) + \sum_{\alpha=1,2} e^a_{\alpha}(x)e^b_{\alpha}(y)V_1(z)]; \ (z = |\vec{x} - \vec{y}|).\]

Substituting the latter into the first equation, we get

\[\frac{d^2}{dz^2}V_n + \frac{2}{z}\frac{d}{dz}V_n - \frac{n}{z^2}V_n = 0.\]

The general solution for the last equation is

\[V_n(z) = d_n z^{l^n_1} + c_n z^{l^n_2}; \ (n = 0, 1),\]

where \(d_n, c_n\) are constants, and \(l^n_1, l^n_2\) can be found as roots of the equation \((l^n)^2 + l^n - n = 0; \ i.e.\)

\[l^n_1 = -\frac{1 + \sqrt{1 + 4n}}{2}; \ l^n_2 = \frac{-1 + \sqrt{1 + 4n}}{2}.\]

It is easy to see that for \(n = 0\) we get the Coulomb potential \(d_0 = -1/4\pi\), and for \(n = 1\) the 'golden section' potential with

\[l^1_1 = -\frac{1 + \sqrt{5}}{2} \approx -1.618; \ l^1_2 = \frac{-1 + \sqrt{5}}{2} \approx 0.618,\]

which can be used as the potential for the 'hadronization' of two-color QCD.

D. Quasiparticle excitations

The presence of the Wu-Yang monopole \(\hat{b}_i(\vec{x})\) does not change the qualitative character of the excitation spectrum of the perturbation theory, it only mixes the color and spin-orbital quantum numbers. To demonstrate this fact, let us consider the fundamental representation with the equation for a fermion in the Wu-Yang monopole

\[i\gamma_0\partial_0\psi + \gamma_j[i\partial_j\psi + \frac{1}{2r}\tau_a e^{ajl}n_l\psi] = 0.\]
In the two-component form of the fermion field, \((\psi_+, \psi_-)\), each component is a \(2 \times 2\) - matrix which can be decomposed into a scalar \(s_{\pm}\) and a vector \(v^j_{\pm}\),

\[
\psi^\alpha_{\pm} = s_{\pm} \delta^\alpha + v^j_{\pm} \tau_j^\alpha.
\] (4.10)

Finally, we obtain the following set of equations for the scalar \(s_{\pm}\) and vector \(v^j_{\pm}\) amplitudes

\[
(\mp q_0 + m)s_{\mp} \mp i(\partial_\alpha + \frac{n^\alpha}{r})v^\alpha_{\pm} = 0 ,
\] (4.11)

\[
(\mp q_0 + m)v^\alpha_{\mp} \mp i(\partial^\alpha - \frac{n^\alpha}{r})s_{\pm} - i\epsilon^{jab} \partial_j v^b_{\pm} = 0 ,
\] (4.12)

where \(q_0\) is an eigenvalue of the \(i\partial_0\) - operator. These equations are solved by decomposing the functions \((s, v)\) w.r.t. orbital momenta. As a result, we obtain a continuous spectrum of the conventional perturbation theory with a mixing of color and spin and the wave function asymptotics \(O(r^{-n})\), \(n > 1\).

V. GENERATING FUNCTIONAL

A. Problems of the covariant Coulomb gauge

Let us consider differences between the topological Dirac variables \(A^T, \psi^T\) and the conventional variables \([26,21]\) for covariant Coulomb gauge

\[
D_k(b)A_k = 0
\] (5.1)

on the level of the generating functional of the unitary perturbation theory treating the asymptotic field \(b(\vec{x})\) as a nonperturbative background. For comparison, it is worth to repeat the construction of the generating functional given in Ref. [21] for the case of an external field (here, the Wu-Yang monopole)

\[
Z_D[J, \eta, \bar{\eta}] = \int [dA_i][dE_i][d\psi][d\bar{\psi}]\Pi_0 \delta(D_k E_k)\delta(D_k A_k) \\
\exp \left\{ i \int d^4x \left[ E_k \cdot \dot{A}_k - \frac{1}{2}E_k^2 - \frac{1}{2}B_k^2 - \frac{1}{2}(D_k f)^2 - A_k \cdot J^c_k - \bar{\psi} \gamma^c - \bar{\eta} \gamma^c \psi \right] \right\} . \] (5.2)

This construction is based on an explicit solution of the Gauss law constraint obtained by decomposing the electric field components of the field strength tensor \(G_{0i}\) into transverse \(E_i\) and longitudinal \(G^{L}_{0i} = -D_i f\) parts,

\[
G_{0i} = E_i - D_i(b) f; \quad D_k(b)E_k = 0 .
\] (5.3)

Here the function \(f\) can be determined from the equation \([26,21]\)

\[
\left( (D^2)^{ab} + g\epsilon^{abc}A^d D^{ab} \right) f_b = \dot{J}^b_{\text{tot},0}; \quad \dot{J}^b_{\text{tot},0} = g\epsilon^{abc}A^a_i E^c_i + J^b_0 ,
\] (5.4)

where \(D \equiv D(b)\).
As we have seen above, the Wu-Yang monopole leads to the instantaneous potential of hadronization which can form mesonic bound states. In order to describe this bound state, we can use the bilocal representation of the generating functional in the meson channel with the external fields \((b + A)\) [27,28]

\[
Z_H[b, A, \eta, \bar{\eta}] = \int [dM_h] \exp \left\{ iW_{\text{eff}}(M_h) - i(G_{M_h}(b, A), \eta, \bar{\eta}) \right\},
\]  

(5.5)

where \(W_{\text{eff}}(M_h)\) is the effective action for the bilocal fields \(M_h\)

\[
W_{\text{eff}}(\mathcal{M}) = -\frac{1}{2} \left( \mathcal{M}, \mathbf{K}^{-1}(b, A) \mathcal{M} \right) - i \text{tr} \ln G_{M}(b, A).
\]  

(5.6)

with the non-Abelian Coulomb kernel \(\mathbf{K}(b, A)\) which depends on the transverse gluon fields \(A\) and the tr-symbol includes the trace over color indices.

This effective action contains the \(\eta_0\) -meson part (5.9)

\[
W_{\text{eff}}^{[0]}(\eta_0) = \int dt \left[ \frac{1}{2} \left( \eta_0^2 - m_0^2 \eta_0^2 \right) V + C_\eta(t) \frac{d}{dt} X[b + A] \right],
\]  

(5.7)

where

\[
C_\eta = \frac{N_f}{F_\eta} \sqrt{\frac{2}{3}}, \quad F_\eta \sim 1.1 \ F_\pi,
\]

\(F_\pi = 92.4\ \text{MeV}\) is the pion weak decay constant, \(N_f\) is the number of flavors, and \(X[A]\) is the topological winding number functional (2.18). On the level of conventional transverse variables [24,21], the problem appears, how to extract the dynamics of the collective variable \(X[A]\) from the QCD action. This calculation also meets the set of problems connected with

1. the Gribov ambiguity [23] due to the zero mode \(D^2(b) = 0\) in the perturbation theory,
2. the violation of gauge invariance due to the anomaly (5.7),
3. the violation of translational invariance due to the external field \(b(\vec{x})\),
4. missing Lorentz covariance.

Let us consider in the following how the introduction of topological Dirac variables helps to cure all these problems.

**B. Solution by topological Dirac variables**

First of all, to remove the Gribov ambiguity, we should extract zero modes of the Gauss law constraint to define the longitudinal function \(f\) and the non-Abelian instantaneous interactions on the class of functions without zero mode where the Faddev-Popov determinant is not equal to zero.
Then, we construct the topological Dirac variables
\[ A^T + (b^T) = U[\Phi][A + (b + \partial)]U[\Phi]^{-1}, \quad \psi^T = U[\Phi]\psi \] (5.8)
and announce that these variables correspond to observables. As a consequence, the topological functional \( X \) converts into the independent global variable \( N_0 \) and mixes with the \( \eta_0 \)-meson channel thus changing its mass.

The effective hadronic Lagrangian in the pseudoscalar, isoscalar (\( \eta \)-meson) channel in its rest frame has the form
\[ L_{\text{eff}}(N_0, \eta_0) = \frac{N_0^2 M_N}{2} + \frac{1}{2}(\dot{\eta}_0^2 - m_0^2 \eta_0^2) V + C_\eta \eta_0(t) \dot{N}_0. \] (5.9)

After diagonalization it reveals the mass shift of the \( \eta_0 \)-meson
\[ L_{\text{eff}}(\bar{N}_0, \eta_0) = \frac{\bar{N}_0^2 M_N}{2} + \frac{1}{2}(\dot{\eta}_0^2 - (m_0^2 + \Delta m^2) \eta_0^2) V, \] (5.10)
where
\[ \bar{N}_0 = \dot{N}_0 + C_\eta \eta_0(t)/M_N \] (5.11)
is a new topological variable, and
\[ \Delta m^2 = \frac{(C_\eta)^2}{M_N V} = \frac{\alpha_s \mu^4 N_f}{6\pi F_\pi} \] (5.12)
is the mass shift of the \( \eta_0 \)-meson which is in agreement with the experimental data when the Shifman-Vainshtein-Zakharov condensate value \( \langle \alpha_s G^2(b) \rangle = 2 \mu^4 \) is used.

Thus, we define both the gauge-invariant variables in terms of which the procedure of hadronization should be fulfilled and the effective infrared parameter of hadronization \( \mu \).

### C. Topological confinement

Another consequence of the topological gauge-fixing of sources of the observable fields is the nonperturbative zero-mode phase factor in front of any colored state, which depends on the topological variable \( N_0 \). In perturbation theory, instead of the colored states in the conventional gauge \( \langle 1|a^+|\psi^D(\vec{x}) \rangle \), we get the topologically ’dressed’ ones
\[ \langle 1|a^+|\psi^T(\vec{x}) \rangle = \exp[N_0 \hat{\Phi}_0(\vec{x})] \langle 1|a^+|\psi^D(\vec{x}) \rangle. \]

Fixing the topological momentum as a conserved quantum number requires the averaging of colored state amplitudes over the parameter of degeneration \( N_0 \) and over the Euler angles of the matrix \( \Omega \) in the color space.

This averaging can lead to confinement as a complete destructive interference phenomenon \([4,34]\). Thus, all colored local variables become ghosts. They occur in Feynman diagrams but not in the observable asymptotic states. The observable states are colorless bound states which depend on the translation-invariant relative coordinate \( (x - y) \).
D. Relativistic covariance

In QED, in terms of the Dirac variables, the Poincare symmetry is realized which is mixing with the gauge one \[3\]. This mixing was interpreted in 1930 by Heisenberg and Pauli \[2\] (with reference to the unpublished note by von Neumann) as the transition from the Coulomb gauge with respect to the time axis in the rest frame \(l_\mu^0 = (1, 0, 0, 0)\) to the Coulomb gauge with respect to the time axis in the moving frame.

\[
l_\mu = l_\mu^0 + \delta_L l_\mu^0 = (Ll_\mu^0)_\mu.
\]

The Coulomb interaction has the covariant form

\[
W_C = \int dxdy \frac{1}{2} j_l^T(x) V_C(z_\perp) j_l^T(y) \delta(l \cdot z), \tag{5.13}
\]

where

\[
 j_l^T = e \bar{\psi}^T \psi^T , \quad z_\perp = z_\mu - l_\mu(z \cdot l) , \quad z_\mu = (x - y)_\mu . \tag{5.14}
\]

This transformation law of the Lorentz covariance was reproduced in quantum theory by Zumino \[3\].

In the non-Abelian theory, in the moving Lorentz frame an observer sees the moving Wu-Yang monopole \[23\] with the corresponding dynamics. We should only manage to realize this transformation law on the level of operators.

VI. CONCLUSION

In this paper, we have presented a scheme for the introduction of topological gauge-invariant variables in QCD by generalizing the concept of Dirac variables from QED to non-Abelian gauge theories for the example of two-color QCD. We construct a model for these variables using a zero mode of the Gauss law constraint from the class of functions which disappear at spatial infinity but have nonzero surface integrals. The model is based on the condition that the winding number functional converts into the zero-mode. It is shown that the corresponding equation of invariance has a solution for asymptotically stationary fields in the form of the Wu-Yang monopole.

Following Wu and Yang \[22\], we consider this monopole except for the singularity at the origin. In the considered region of the space, the zero-mode solution of the Gauss law constraint forms the \(\theta\)-vacuum \[24, 25\] and leads to a finite constraint-shell action for definite values of the strong coupling constant.

To construct the generating functional of the quantum field theory, it requires only the constraint-shell action, where the (global) zero-mode part represents surface integrals defined in the region at spatial infinity. We suggest that in this region the gauge fields are stationary and the zero-mode solution is factorizing.

In the field of the Wu-Yang monopole the instantaneous quark-quark potential is a sum of the Coulomb potential and the golden-section one. The latter one can lead to
spontaneous chiral symmetry breaking and to mesonic bound states. The $\eta'$-meson mixes
with the zero-mode so that after diagonalization of this low-energy action a mass shift of
the $\eta'$-meson is obtained which resolves the $U_A(1)$ problem.

Color amplitudes contain additional phase factors which depend on the zero-mode. Averaging
the color amplitudes over the zero-mode parameters leads to the phenomenon of complete
destructive interference \[7\], so that the color amplitudes disappear.

According to Heisenberg, Pauli \[2\] and Zumino \[1\], the relativistic covariance is
established by a rotation of the timelike axis so that the Coulomb field moves together
with relativistic bound states. Recently, Faddeev and Niemi \[23\] constructed a similar
relativistically covariant form of effective Lagrangian using the Wu-Yang monopole.

In summary the present scheme for the introduction of topological gauge-invariant
variables is a promising tool for the investigation of the challenging properties of QCD
such as the meson spectrum, chiral symmetry breaking, quark and gluon confinement,
and the $U_A(1)$ anomaly. Detailed numerical analyses and the generalization to $SU_c(3)$ are
to be presented in a subsequent work.

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