Extending a system in the calculus of structures with a self-dual quantifier

Luca Roversi
Università di Torino — Dipartimento di Informatica

Abstract

We recall that SBV, a proof system developed under the methodology of deep inference, extends multiplicative linear logic with the self-dual non-commutative logical operator $\text{Seq}$. We introduce $\text{SBVQ}$ that extends SBV by adding the self-dual quantifier $\text{Sdq}$. The system $\text{SBVQ}$ is consistent because we prove that (the analogous of) cut elimination holds for it. Its new logical operator $\text{Sdq}$ operationally behaves as a binder, in a way that the interplay between $\text{Seq}$ and $\text{Sdq}$ can model $\beta$-reduction of linear $\lambda$-calculus inside the cut-free subsystem $\text{BVQ}$ of SBVQ. The long term aim is to keep developing a programme whose goal is to give pure logical accounts of computational primitives under the proof-search-as-computation analogy, by means of minimal, and incremental extensions of SBV.

1 Introduction.

This is a work in structural proof-theory. We extend SBV [5], the paradigmatic system of the deep inference methodology to design proof systems.

Deep inference (DI). One of the main aspects of DI is that logical systems can be designed as they were rewriting systems, namely, systems with rules that apply deeply inside terms, or, equivalently, in any suitable context. We must read “deep” as opposed to “shallow”. Rules of sequent and natural deduction systems are shallow because they build proofs whose form mimics the one of formulas. Thanks to the deep application of its rules, $\text{BV}$ substantially extends multiplicative linear logic (MLL) [3] with the non commutative binary operator $\text{Seq}$, whose logical properties are strictly connected to the expressiveness of $\text{BV}$ itself. Any limits we might put on the application depth of $\text{BV}$ rules would yield a strictly less expressive system [16] indeed. An extension of $\text{BV}$, by means of linear logic exponentials [6, 7, 8, 15] is NEL, whose provability is undecidable [13].

Contributions, and motivations. We introduce $\text{SBVQ}$. It is $\text{SBV}$ plus a quantifier that we identify as $\text{Sdq}$, which abbreviates “Self-dual quantifier”. The relevant feature of $\text{Sdq}$ is to bind variable names of SBVQ only. The consequence is twofold. First, we do not need to classify $\text{Sdq}$ as either an existential, or a universal quantifier. Indeed, binding variable names...
only, it never requires to distinguish if the quantification is over a variable which we can think of as an assumption or as a conclusion. Hence, a second consequence is that SBVQ naturally becomes self-dual. So, SBVQ can be viewed as a minimal extension of SBVQ by means of a logical operator whose instances identify regions of formulas where specific variable names can essentially change freely.

The work may be viewed as divided in two parts. The first is about proving that SBVQ is consistent. Namely, SBVQ enjoys Splitting (Section 3) which identifies the subset BVQ of SBVQ which plays the role of cut-free fragment.

The second part of the work gives to Sdq an operational semantics. Exploiting that Sdq is a binder, we show that its interplay with Seq makes proof-search inside BVQ complete w.r.t. the basic functional computation expressed by linear λ-calculus. We recall that functions linear λ-calculus represents use their arguments exactly once in the course of the evaluation. So, the set of functions it can express is quite limited, but large enough to let the decision about which is the normal form of two linear λ-terms a polynomial size complete problem [10]. Completeness amounts to first defining an embedding \( \langle \cdot \rangle \) from linear λ-terms to formulas of BVQ (Section 5). Then, completeness states that, for every linear λ-term \( M \), and every atom \( o \), which plays the role of an output-channel, if \( M \) reduces to \( N \), then there is a derivation \( \mathcal{D} \) of BVQ, that derives the conclusion \( \langle M \rangle_o \), from the assumption \( \langle N \rangle_o \). (Theorem 5.1) For example, let us recall a possible encoding of boolean values, and of boolean negation:

\[
\begin{align*}
\text{Not} & \equiv \lambda z.\lambda x.\lambda y.((z) y) x \\
\text{True} & \equiv \lambda w.\lambda z.(w) z \\
\text{False} & \equiv \lambda w.\lambda z.(z) w
\end{align*}
\]

Figure 1 shows (part of) a non trivial example of completeness. We have a derivation of BVQ whose conclusion encodes \( \langle \text{Not} \rangle \text{True} \), while the premise encodes its \( \beta \)-reduct \( \lambda x.\lambda y.((\text{True}) y) x \).

Finally, showing completeness means we keep developing a programme whose goal is to give pure logical accounts of computational primitives under the proof-search-as-computation analogy, by means of minimal extensions of SBV. This programme begins in [2]. It shows that Seq soundly, and completely models CCS, the restriction of Milner CCS [?] to a fragment that contains sequential, and parallel composition only.
Related works. This work directly relates to [11] and [12] as follows. First here we choose a better terminology. Current “self-dual quantifier” $Sdq$ were dubbed as “renaming” in both [11] and [12], putting too much emphasis about its operational meaning. Moreover, this work (i) cleans up the definition, and the properties of $Sdq$, (ii) generalizes the statements of some principal property, correcting non-crucial flows in their proofs, (iii) states and proves deduction and standardization properties, (iv) includes details of many proofs of the given statements, (v) simplifies the map $\langle \cdot \rangle$ from linear $\lambda$-terms to formulas of BVQ dropping any reference to explicit substitutions inside linear $\lambda$-terms, which was, instead, mandatory in [11, 12], (vi) among the conclusions (Section 6), anticipates that BVQ can be complete, and not only sound, w.r.t. a suitable extension of the above fragment CCS of Milner CCS.

Besides [2], further related works are [1, 4], and [17]. The former restates natural deduction of the negative fragment of intuitionistic logic into a DI system from which extracting an algebra of combinators where interpreting $\lambda$-terms.

Road map. Section 2 introduces the extension SBVQ (BVQ) of SBV (BV). Section 3 proves that SBVQ is consistent by extending the proof of (the analogous of) cut-elimination for SBV to SBVQ. Section 4 recalls linear $\lambda$-calculus, and defines the embedding of its terms to formulas of BVQ. Section 5 shows the completeness of BVQ w.r.t. linear $\lambda$-calculus, namely it shows that every computation in the latter corresponds to a proof-search in the former. Section 6 comments about the lack of a reasonable soundness of BVQ w.r.t. to $\lambda$-calculus, and points to future work.

2 Systems SBVQ and BVQ

We recall and clean-up the definitions of [11] [12].

Structures. Let $a, b, c, \ldots$ denote the elements of a countable set of positive propositional variables. Let $\overline{a}, \overline{b}, \overline{c}, \ldots$ denote the elements of a countable set of negative propositional variables. The set of names, which we range over by $l, m,$ and $n$, contains both positive, and negative propositional variables, and nothing else. Let $\circ$ be a constant, different from any name, which we call unit. The set of atoms contains both names and the unit, while the set
of *structures* identifies formulas of SBV. Structures belong to the language of the grammar in (1).

\[
R ::= \circ \mid 1 \mid \overline{R} \mid (R \otimes R) \mid (R \cdot R) \mid [R \otimes R] \mid [R]_n
\]

(1)

We use $K, P, R, T, U, V$ to range over structures. As in SBV, $\overline{R}$ is a Not structure, $(R \otimes T)$ is a CoPar structure, $(R \cdot T)$ is a Seq structure, and $[R \otimes T]$ is a Par structure. The $\text{Sdq}$ structure $[R]_a$ is new. It comes with the proviso that $a$ must be a positive atom. Namely, $[R]_T$ is not in the syntax. $\text{Sdq}$ induces notions of *free*, and *bound names*, defined in (2).

\[
\begin{align*}
[a] &= \text{fn}(a) \cup \text{fn}(\overline{a}) & \emptyset &= \text{bn}(a) \cup \text{bn}(\overline{a}) \\
a \in \text{fn}(R) &\iff a \in \text{fn}(R) & a \in \text{bn}(\overline{R}) &\iff a \in \text{bn}(\overline{R}) \\
a \in \text{fn}((R \otimes T)) &\iff a \in \text{fn}(R) \cup \text{fn}(T) & a \in \text{bn}(\overline{(R \otimes T)}) &\iff a \in \text{bn}(\overline{R}) \cup \text{bn}(T) \\
a \in \text{fn}((R \cdot T)) &\iff a \in \text{fn}(R) \cup \text{fn}(T) & a \in \text{bn}((R \cdot T)) &\iff a \in \text{bn}(R) \cup \text{bn}(T) \\
a \in \text{fn}([R \otimes T]) &\iff a \in \text{fn}(R) \cup \text{fn}(T) & a \in \text{bn}([R \otimes T]) &\iff a \in \text{bn}(R) \cup \text{bn}(T) \\
a \in \text{fn}([R]_b) &\iff a \neq b & a \in \text{fn}(R) &\iff a \neq b \text{ or } a \in \text{bn}(R)
\end{align*}
\]

(2)

Finally, (3) defines the substitution $R[^c_b]$ that replaces (i) the atom $a$ for the free occurrences of $b$, and (ii) the atom $\overline{a}$ for those ones of $\overline{b}$, in $R$.

\[
\begin{align*}
\circ[^c_b] &= \circ & c[^c_b] &= c \\
\overline{b}[^c_b] &= a & \overline{c}[^c_b] &= \overline{c} \\
\overline{[R]^c_b} &= [R]_b & (R \otimes T)[^c_b] &= (R[^c_b] \otimes T[^c_b]) \\
(R \cdot T)[^c_b] &= (R[^c_b] \cdot T[^c_b]) & [R \otimes T][^c_b] &= [R[^c_b] \otimes T[^c_b]] \\
\end{align*}
\]

(3)

Size of the structures. The size $|R|$ of $R$ is the number of occurrences of atoms in $R$ plus the number of occurrences of Sdq that effectively bind an atom.

Example 2.1 (Size of the structures) We have $|\overline{a} \otimes \overline{a}| = |\overline{[a \otimes \overline{a}]}_a| = 2$ for we do not count the occurrence of $[\cdot]$. Instead, we count it in $|[a \otimes \overline{a}]|_a$, getting $|[a \otimes \overline{a}]|_a = 3$.

(Structure) Contexts. We denote them by $S\{\}$. A context is a structure with a single hole $\{\}$ in it. If $S(R)$, then $R$ is a *substructure* of $S$. We shall tend to shorten $S([R \otimes U])$ as $S[R \otimes U]$ when $[R \otimes U]$ fills the hole $\{\}$ of $S\{\}$ exactly.

Congruence $\approx$ on structures. Structures are partitioned by the smallest congruence $\approx$ we obtain as reflexive, symmetric, transitive and contextual closure of the relation $\sim$ whose defining clauses are (4), through (20) here below.
The system \( \text{SBVQ} \). It contains the set of inference rules in (21) here below. Every rule has form \( \rho \vdash \frac{T}{R} \), name \( \rho \), premise \( T \), and conclusion \( R \).

\[
\begin{array}{l}
\begin{array}{c}
\text{Negation} \\
\neg \varphi \vdash \varphi' \\
\neg R \vdash R
\end{array} \\
\begin{array}{c}
\varphi' \vdash (R \odot \top) \\
\varphi \vdash (R \odot \top)
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{Unit} \\
\varphi \vdash (\varphi \odot R) \\
\top \vdash \top
\end{array}
\]

\[
\begin{array}{c}
\text{Symmetry} \\
(R \odot T) \vdash (T \odot R) \\
(R \odot T) \vdash (T \odot R)
\end{array}
\]

\[
\begin{array}{c}
\alpha\text{-rule} \\
(R \odot T) \vdash (\varphi \odot R) \\
(R \odot T) \vdash (\varphi \odot R)
\end{array}
\]

\[
\begin{array}{c}
\text{Associativity} \\
(R \odot T) \vdash (R \odot T) \\
(R \odot T) \vdash (R \odot T)
\end{array}
\]

\[
\begin{array}{c}
\text{Up and down fragments of SBVQ.} \\
\text{The set \{ai, s, q, u\} is the down fragment \text{BVQ} of SBVQ. The up fragment is \{ai, s, q, u\}. So s belongs to both. The rule ai plays the role of the cut rule of sequent calculus. The down rule for \text{SDq} restricts the following one (14):}
\end{array}
\]

\[
\forall a. (R \odot U) \vdash [R \odot U]_{\alpha}
\]

\[
\forall a. (R \odot U) \vdash [R \odot U]_{\alpha}
\]
to binding variable names only. Limiting Sdq to abstract variables implies that the difference between existentially, and universally quantified names disappears. The reason is that the cut-elimination will have no need to differentiate between the substitution of an existentially quantified variable for a universally quantified one, or vice versa. So, Sdq becomes self-dual.

Derivations vs. proofs. A derivation in SBVQ is either a structure or an instance of the above rules or a sequence of two derivations. Both $\mathcal{D}$, and $\mathcal{E}$ will range over derivations. The topmost structure in a derivation is its premise. The bottommost is its conclusion. The length $|\mathcal{D}|$ of a derivation $\mathcal{D}$ is the number of rule instances in $\mathcal{D}$. A derivation $\mathcal{D}$ of a structure $R$ in SBVQ from a structure $T$ in SBVQ, only using a subset $B \subseteq SBVQ$ is $\mathcal{D}|_B$. The equivalent space-saving form we shall tend to use is $\mathcal{D}: T \vdash_B R$. The derivation $\mathcal{D}|_B$ is a $\mathcal{D}$ proof whenever $T \approx \circ$. We denote it as $\mathcal{D}|_B$, or $\mathcal{D}|_B R$, or $\mathcal{D}: T \vdash B R$. Both $\mathcal{P}$, and $\mathcal{E}$ will range over proofs. In general, we shall drop $B$ when clear from the context. In a derivation, we write $T \vdash R_{1,\ldots,n_p}$ whenever we use the rules $\rho_1,\ldots,\rho_m$ to derive $R$ from $T$ with the help of $n_1,\ldots,n_p$ instances of $\{4,\ldots,11\}$. To avoid cluttering derivations, whenever possible, we shall tend to omit the use of negation axioms $\{4,\ldots,9\}$, associativity axioms $\{12\}, \{13\}, \{14\}$, and symmetry axioms $\{10\}, \{11\}$. This means we avoid writing all brackets, as in $[R \otimes [T \otimes U]]$, in favor of $[R \otimes T \otimes U]$, for example. Finally if, for example, $q > 1$ instances of some axiom $(n)$ of $\{4,\ldots,20\}$ occurs among $n_1,\ldots,n_p$, then we write $(n)^q$.

Admissible and derivable rules. A rule $\rho$ is admissible for the system SBVQ if $\rho \notin SBVQ$ and, for every derivation $\mathcal{D}$ such that $\mathcal{D}: T \vdash_{[\rho]} SBVQ R$, there is a derivation $\mathcal{D}'$ such that $\mathcal{D}' : T \vdash_{SBVQ} R$. A rule $\rho$ is derivable in $B \subseteq SBVQ$ if $\rho \notin B$ and, for every instance $\rho \vdash_{[\rho]} R$, there exists a derivation $\mathcal{D}$ in $B$ such that $\mathcal{D}: T \vdash_B R$.

The rules in (22) recall a core set of rules derivable in SBV, hence in SBVQ.

![Diagram of derivable rules](image)

General interaction down and up. In (22), general interaction up is $\uparrow$, derivable in the set $\{\alpha\uparrow, s, q\uparrow, u\uparrow\}$, reasoning by induction on $|R|$, and proceeding by cases on the form of $R$. We show the few steps of the proof, relative the case Sdq:

![Diagram of general interaction up](image)

6
Similar arguments apply to the cases relative to Not, CoPar, Seq, and Par. Symmetrically, *general interaction down* $\downarrow$ is derivable in $\{a|\downarrow, s, q, u\}$.

**General Seq-transitive up, and down rules.** In (26), $\downarrow$ is derivable by reasoning inductively on the size of $\{\ }$, and proceeding by cases on its structure, under the proviso (⋆) which says that $(\{a\cup fn(T)\} \cap bn(\{\ })) = \emptyset$. If $\{\ } \approx \{\ }$, then $\downarrow$ is:

\[
\begin{align*}
&\frac{}{\{R \supset T\}} \tag{5} \\
&\frac{\{R \supset \{a \supset \{\sigma \supset T\}\}\}}{\{R \supset \{a \supset \{\sigma \supset T\}\}\}} \tag{4} \\
&\frac{\{R \supset \{a \supset \{\sigma \supset T\}\}\}}{\{R \supset \{a \supset \{\sigma \supset T\}\}\}} \tag{3} \\
&\frac{\{R \supset \{a \supset \{\sigma \supset T\}\}\}}{\{R \supset \{a \supset \{\sigma \supset T\}\}\}} \tag{2} \\
&\frac{\{R \supset \{a \supset \{\sigma \supset T\}\}\}}{\{R \supset \{a \supset \{\sigma \supset T\}\}\}} \tag{1}
\end{align*}
\]

If $\{\ } \approx (\{S \{\} \supset U\})$, then:

\[
\begin{align*}
&\frac{\{S \{\} \supset U\}}{\{S \{\} \supset U\}} \tag{7} \\
&\frac{\{S \{\} \supset U\}}{\{S \{\} \supset U\}} \tag{6} \\
&\frac{\{S \{\} \supset U\}}{\{S \{\} \supset U\}} \tag{5} \\
&\frac{\{S \{\} \supset U\}}{\{S \{\} \supset U\}} \tag{4} \\
&\frac{\{S \{\} \supset U\}}{\{S \{\} \supset U\}} \tag{3} \\
&\frac{\{S \{\} \supset U\}}{\{S \{\} \supset U\}} \tag{2} \\
&\frac{\{S \{\} \supset U\}}{\{S \{\} \supset U\}} \tag{1}
\end{align*}
\]

If $\{\ } \approx [S \{\} \supset \{\}]$, then:

\[
\begin{align*}
&\frac{\{S \{\} \supset \{\}\}}{\{S \{\} \supset \{\}\}} \tag{7} \\
&\frac{\{S \{\} \supset \{\}\}}{\{S \{\} \supset \{\}\}} \tag{6} \\
&\frac{\{S \{\} \supset \{\}\}}{\{S \{\} \supset \{\}\}} \tag{5} \\
&\frac{\{S \{\} \supset \{\}\}}{\{S \{\} \supset \{\}\}} \tag{4} \\
&\frac{\{S \{\} \supset \{\}\}}{\{S \{\} \supset \{\}\}} \tag{3} \\
&\frac{\{S \{\} \supset \{\}\}}{\{S \{\} \supset \{\}\}} \tag{2} \\
&\frac{\{S \{\} \supset \{\}\}}{\{S \{\} \supset \{\}\}} \tag{1}
\end{align*}
\]

The case with $\{\ } \approx [S \{\} \supset \{\}]$ is simpler than the two here above.

**Mix rules.** In (22) both $\text{mixp}$, and $\text{pmix}$, show a hierarchy between connectives: Par is the lowermost, Seq lies in the middle, and CoPar on top (3). *Postfix mix rule* mixp is derivable in $\{q\}$.

Finally, some properties that formalize simple derivations we can always build inside BVQ. The first one says when two structures $R$, and $T$ of BVQ can be moved inside a context so that they get one aside the other.

**Proposition 2.2 (Context extrusion)** $S[R \supset T] \vdash_{\{q\}, \land, \land, a} [S[R] \supset T]$, for every $S, R, T$.

**Proof** By induction on $|S[\ ]|$, proceeding by cases on the form of $S[\ ]$. (Details in Appendix A).

The following statement highlights the scoping nature of $\text{Sdq}$. For proving it, it is enough to inspect the behavior of the rules in BVQ.

**Fact 2.3 (Sdq is a scoping operator)** Let $a, U$, and $V$ be given.

1. If $\mathcal{D} : V \vdash_{\text{BVQ}} [U]_a$, then there exist $R$, and $\mathcal{D}$ such that $\mathcal{D} : [R]_a \vdash_{\text{BVQ}} [U]_a$.
2. For every $R$, if $\mathcal{D} : [R]_a \vdash_{\text{BVQ}} [U]_a$, then $\mathcal{D}' : R \vdash_{\text{BVQ}} U$, for some $\mathcal{D}'$.

The last property says that no new variable can be introduced in the course of a derivation.

**Proposition 2.4 (BVQ is affine)** In every $\mathcal{D} : T \vdash_{\text{BVQ}} R$, we have $|R| \geq |T|$.

**Proof** By induction on $|\mathcal{D}|$, proceeding by cases on its last rule $\rho$. 

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3 Splitting for SBVQ

We recall, and clean the proof of Splitting for SBVQ in [11] [12]. Splitting can be viewed as a generalization of cut-elimination for sequent calculus-like systems. Proving Splitting of SBVQ amounts to proving that SBVQ, and BVQ are equivalent, namely that every up-rule is admissible in BVQ, or, equivalently, that we can eliminate every up-rule from any derivation of SBVQ. Since \( a↑ \) is an up-rule, and it plays the role of the cut rule, proving Splitting means proving also cut-elimination for SBVQ.

The first part of this section traces how Splitting, and some other properties it relies on, works to eliminate \( u↑ \). The second part, Subsection 3.1, is for technical eyes interested to the full formal details.

Let us see how Splitting eliminates an occurrence \((*)\) of \( u↑ \) from a proof \( P \) of SBVQ, so focusing on the case that differentiates the proof of Splitting for SBVQ from the one for SBV. Let:

\[
P \quad \Rightarrow \quad S[(R_a \otimes T_a)]_{\mu} \quad (*)
\]

be \( P \) with \((*)\) the instance of \( u↑ \) we want to eliminate. We are going to rewrite \( P \) to a proof of BVQ with the same conclusion as \( P \), but without \((*)\). The first step to get rid of \((*)\) is Splitting (Theorem 3.5). The instance of Splitting we need, up to some details we can ignore at this level, is:

\[
Q \quad \Rightarrow \quad \exists K, \vec{b} \text{ such that } \forall V, \text{ both } S[V] \quad \Rightarrow \quad S[(R_T)_{\mu}] \quad \text{and} \quad \exists K, \vec{b} \text{ such that } \forall V, \text{ both } S[V] \quad \Rightarrow \quad S[(R_T)_{\mu}]
\]

We remark that extracting \( K \), hidden inside \( P \), might require many instances of \( Sdq \) to emerge, as the outermost occurrence \([\cdot]_{\mu} \) in the premise of \( P \) shows. We can apply Splitting by taking \( P \) — beware, not \( P' \) — as \( Q \). Since \( V \) in \( P \) can be any, we choose \( V \cong [(R_T)_{\mu}] \), the conclusion of the instance of \( u↑ \) we want to eliminate. From such an instance of \( P \) we get:

\[
[(R_T)_{\mu}] \quad \Rightarrow \quad S[(R_T)_{\mu}]
\]

Now we extract from \( K \) the, usually called, killers of \( R \), and \( T \) inside \( (R_T)_{\mu} \). Namely, we apply the following instance of Shallow splitting (Proposition 3.2) to the above \( Q' \):

\[
Q' \quad \Rightarrow \quad \exists K_1, K_2, \vec{c} \text{ such that } \exists K, \vec{b} \text{ such that } \forall V, \text{ both } S[V] \quad \Rightarrow \quad S[(R_T)_{\mu}]
\]

which, once more, may let instances of \( Sdq \) to emerge. Composing \( Q', \delta', \delta_1, \) and \( \delta_2 \), we get...
the (⁺)-free proof we are looking for:

\[
\begin{align*}
&[a, b] \approx \circ \\
&[T \equiv K_2]_{a, b} \\
&[T \equiv K_1 \otimes T]_{a, b} \\
&[[R \equiv K_1] \otimes T \equiv K_2]_{a, b} \\
&[T \equiv T]_{a, b} \\
&[T \equiv K_2]_{a, b} \\
&[T \equiv K_1 \otimes T]_{a, b} \\
&[[R \equiv K_1] \otimes T \equiv K_2]_{a, b} \\
&[T \equiv T]_{a, b}
\end{align*}
\]

Proposition 3.1 (Provability of structures in BVQ) Let \( R, T \) be structures, and \( \rho \) be a name, and \( \mathcal{P}, \mathcal{P}_1, \) and \( \mathcal{P}_2 \) be proofs of BVQ.

1. \( \mathcal{P} : \vdash_{\text{BVQ}} (R \equiv T) \) iff \( \mathcal{P}_1 : \vdash_{\text{BVQ}} R \) and \( \mathcal{P}_2 : \vdash_{\text{BVQ}} T. \)
2. \( \mathcal{P} : \vdash_{\text{BVQ}} (R \otimes T) \) iff \( \mathcal{P}_1 : \vdash_{\text{BVQ}} R \) and \( \mathcal{P}_2 : \vdash_{\text{BVQ}} T. \)
3. \( \mathcal{P} : \vdash_{\text{BVQ}} [R]_{a} \) iff \( \mathcal{P'} : \vdash_{\text{BVQ}} R[a/b], \) for every variable \( b. \)

Proof. "If implication". The proofs of 1 and 2 given in 3 by induction on \( |\mathcal{P}| \) inside BV, extend to the cases when the last rule of \( \mathcal{P} \) is \( \mathcal{P}_1 \). Indeed, the redex of \( \mathcal{P}_1 \) can only be inside \( R \) or \( T \). Concerning 3 the assumption implies the existence of \( \mathcal{P'} : \vdash R[a\rightarrow b], \) namely of \( \mathcal{P'} : \vdash R[a]. \) So, we can “wrap” \( \mathcal{P'} \) with \( \mathcal{P}_1 \), exploiting 3, and apply every rule of \( \mathcal{P'} \) deep in the proof \( \mathcal{P} \) we are building.

"Only if implication". In all the three cases, the proof is by induction on \( |\mathcal{P}| \), proceeding by cases on its last rule \( \rho \). Concerning points 1 and 2 a redex can only be inside \( R \) or \( T \). So, the application of the inductive hypothesis is immediate. Instead, \( a \) may not belong to \( \text{fn}(R) \) in Point 3. If this is true, then 3 implies that every instance of \( \mathcal{P'} \) with \( b \) in place of \( a \) exists. The reason is that, by definition, the substitution 3 distributes over structures, preserving the scope of every instance of \( \text{Sdq} \). Otherwise, if \( a \in \text{fn}(R) \), then the redex of \( \rho \) can only be inside \( R \). So, we can conclude thanks to the inductive hypothesis.

Proposition 3.2 (Shallow Splitting in BVQ) Let \( R, T, \) and \( P \) be structures, and \( a \) be a name, and \( \mathcal{P} \) be a proof of BVQ.

1. If \( \mathcal{P} : \vdash_{\text{BVQ}} ([R \equiv T] \equiv P), \) then there are \( \mathcal{D} : [P_1 \equiv P_2] \vdash_{\text{BVQ}} P, \) and \( \mathcal{P}_1 : \vdash_{\text{BVQ}} [R \equiv P_1], \) and \( \mathcal{P}_2 : \vdash_{\text{BVQ}} [T \equiv P_2], \) for some \( P_1, \) and \( P_2. \)
2. If \( \mathcal{P} : \vdash_{\text{BVQ}} ([R \otimes T] \equiv P), \) then there are \( \mathcal{D} : [P_1 \equiv P_2] \vdash_{\text{BVQ}} P, \) and \( \mathcal{P}_1 : \vdash_{\text{BVQ}} [R \equiv P_1], \) and \( \mathcal{P}_2 : \vdash_{\text{BVQ}} [T \equiv P_2], \) for some \( P_1, \) and \( P_2. \)
3. Let \( \mathcal{P} : \vdash_{\text{BVQ}} [R \equiv P] \) with \( R \approx [l_1 \equiv \ldots \equiv l_m], \) such that \( i \neq j \) implies \( l_i \neq l_j, \) for every \( i, j \in \{1, \ldots, m\}, \) and \( m > 0. \) Then, for every structure \( R_0, \) and \( R_1, \) if \( R \approx [R_0 \equiv R_1], \) there exists \( \mathcal{D} : \vdash_{\text{BVQ}} [R_0 \equiv P]. \)
4. If $\mathcal{P} : \vdash \{[R] \ni P\}$, then there are $\mathcal{P} : \vdash [T] \ni P$, and $\mathcal{P}' : \vdash \mathbf{BVQ} [R \ni T]$, for some $T$.

**Proof** Following [5], both statements [1] and [2] must be proved simultaneously. We reason by induction on the lexicographic order of the pair $([V], \rho)$, where $V$ is one between $([R \ni T] \ni P)$ or $([R \ni T] \ni P)$, proceeding by cases on the last rule $\rho$ of $\mathcal{P}$.

Point [3] relies on points [1][2]. It holds by induction on $([R] \ni \mathcal{P})$, proceeding by cases on the last rule of $\mathcal{P}$. Point [4] relies on points [1][2]. It holds by induction on $([R] \ni \mathcal{P})$, proceeding by cases on the last rule of $\mathcal{P}$. (Details in Appendix B).

**Remark 3.3** The proviso “$i \neq j$ implies $l \neq i_j$, for every $i, j \in \{1, \ldots, m\}$” of Point (3) in Proposition 3.2 serves to let the killer of every $l$ be inside $[R_0 \ni P]$.

Proposition 3.4 here below says that $\mathcal{S} \{ \}$ supplies the “context” $U$, required for proving $R$, no matter which structure fills the hole of $\mathcal{S} \{ \}$.

**Proposition 3.4 (Context Reduction in BVQ)** Let $R$ be a structure, and $\mathcal{S} \{ \}$ be a context such that $\mathcal{P} : \vdash \mathbf{BVQ} \mathcal{S}[R]$. There are a structure $U$, and, possibly, some variables $b$ such that, for every $V$, if $\text{fn}(V) \cap \text{bn}(R) = \emptyset$, then both $\mathcal{P} : \vdash [V \ni U] \vdash \mathbf{BVQ} \mathcal{S}[V]$, and $\mathcal{P} : \vdash \mathbf{BVQ} [R \ni U]$.

**Proof** The proof is by induction on $[\mathcal{S} \{ ]$, proceeding by cases on the form of $\mathcal{S} \{ ]$. (Details in Appendix C).

**Theorem 3.5 (Splitting in BVQ)** Let $R$, and $T$, be structures, and $\mathcal{S} \{ ]$ be a context.

1. If $\mathcal{P} : \vdash \mathbf{BVQ} \mathcal{S}[R \ni T]$, then there are structures $K_1, K_2$, and, possibly, some variables $\bar{b}$ such that, for every $V$ with $\text{fn}(V) \cap \text{bn}(R \ni T) = \emptyset$, there are $\mathcal{P} : \vdash [V \ni (K_1 \ni K_2)] \vdash \mathbf{BVQ} \mathcal{S}[V]$, and $\mathcal{P}_1 : \vdash \mathbf{BVQ} [R \ni K_1]$, and $\mathcal{P}_2 : \vdash \mathbf{BVQ} [T \ni K_2]$.

2. If $\mathcal{P} : \vdash \mathbf{BVQ} \mathcal{S}[R \ni T]$, then there are structures $K_1, K_2$, and, possibly, some variables $\bar{b}$ such that, for every $V$ with $\text{fn}(V) \cap \text{bn}(R \ni T) = \emptyset$, there are $\mathcal{P} : \vdash [V \ni (K_1 \ni K_2)] \vdash \mathbf{BVQ} \mathcal{S}[V]$, and $\mathcal{P}_1 : \vdash \mathbf{BVQ} [R \ni K_1]$, and $\mathcal{P}_2 : \vdash \mathbf{BVQ} [T \ni K_2]$.

3. If $\mathcal{P} : \vdash \mathbf{BVQ} \mathcal{S}[R]$, then there are a structure $K$, and, possibly, some variables $\bar{b}$ such that, for every $V$ with $\text{fn}(V) \cap \text{bn}(R) = \emptyset$, there exist $\mathcal{P} : \vdash [V \ni K] \vdash \mathbf{BVQ} \mathcal{S}[V]$, and $\mathcal{P}' : \vdash \mathbf{BVQ} [R \ni K]$.

**Proof** We obtain the proof of the three statements by composing Context Reduction (Proposition 3.4), and Shallow Splitting (Proposition 3.2) in this order. (Details in Appendix D).

**Theorem 3.6 (Admissibility of the up fragment for BVQ)** The set $\{a\uparrow, q\uparrow, u\uparrow\}$ in SBVQ is admissible for BVQ.

**Proof** Use Splitting (Theorem 3.5), and Shallow Splitting (Proposition 3.2) (Details in Appendix E).

4 **Linear $\lambda$-calculus mapped to BVQ**

To show that SBVQ is not an extemporaneous logical operator we interpret it as binder that, together with Seq, models the renaming mechanism of linear $\beta$-reduction.
Linear $\lambda$-calculus. We recall that linear $\lambda$-calculus can be viewed as a pair (linear $\lambda$-terms, linear operational semantics). Let $\mathcal{V}$ be a countable set of variable names we range over by $x, y, w, z$. We call $\mathcal{V}$ the set of $\lambda$-variables. The set of linear $\lambda$-terms is $\Lambda = \bigcup_{X \in \mathcal{V}} \Lambda_X$ we range over by $M, N, P, Q$. For every $X \in \mathcal{V}$, the set $\Lambda_X$ contains the linear $\lambda$-terms whose free variables are in $X$, and which we define as follows: (i) $x \in \Lambda_{\{x\}}$; (ii) $\lambda X.M \in \Lambda_X$ if $M \in \Lambda_{X \cup \{x\}}$; (iii) $(M) N \in \Lambda_{X \cup Y}$ if $M \in \Lambda_X, N \in \Lambda_Y$, and $X \cap Y = \emptyset$; (iv) $M \{y/x\} \in \Lambda_{X \cup Y}$ if $M \in \Lambda_{X \cup \{y\}}, P \in \Lambda_Y$, and $X \cap Y = \emptyset$. The linear operational semantics that rewrites linear $\lambda$-terms is the relation $\Rightarrow \subseteq \Lambda \times \Lambda$ here below:

\[
\begin{align*}
\text{rn} & : M \Rightarrow M & \beta & : (\lambda x.M) N \Rightarrow M \{N/x\} & \text{ta} & : M \Rightarrow P \\
\text{tr} & : M \Rightarrow N & \alpha & : (M) P \Rightarrow (N) P & \text{tr} & : M \Rightarrow N
\end{align*}
\]

where $M \{N/x\}$ is the usual clash-free substitution, that replaces $N$ for the forcefully single occurrence of $x$ in $M$. We remark that (23) is the reflexive, contextual, and transitive closure of linear $\beta$-reduction we find in rule $\beta$. Finally, $|M \Rightarrow N|$ denotes the number of instances of rules in (23), used to derive some given $M \Rightarrow N$.

The map $\langle \cdot \rangle_o$. We define it here below, to map terms of $\Lambda$ into structures of BVQ.

\[
\begin{align*}
\langle x \rangle_o &= \langle x \ast \overline{\vartheta} \rangle \text{ with } \overline{\vartheta} \text{ fresh} & \text{(24a)} \\
\langle \lambda x.M \rangle_o &= \big[ \{M\} \big]_{\vartheta} & \text{(24b)} \\
\langle (M) N \rangle_o &= \big[ \{M\} \circ \{N\} \circ \langle \varphi \rangle \big]_{\vartheta} & \text{(24c)}
\end{align*}
\]

For every linear $\lambda$-term $M$, the structure $\langle M \rangle_o$ is such that (i) $o$ is a unique output channel, and (ii) every free variable of $M$ is used as positive atom name that plays the role of input channel. Clause (24a) associates the input channel $x$ to the fresh output channel $o$. Intuitively, $x$ shall be eventually forwarded to $o$, in accordance with terminology taken from [9]. Clause (24b) uses $\text{Seq}$ to abstract on the input channel $x$. This means to let $x$ ready to merge with any output channel of a linear $\lambda$-term that has to be substituted for $x$. Such a channel comes from the argument of an application, as translated by (24c). It wraps $\langle N \rangle_q$, abstracting on its output channel $q$ thanks to $\text{Seq}$. So, thanks to $\text{Seq}$, linear $\beta$-reduction, and its substitution mechanism, become an identification of channel names inside BVQ, as follows:

\[
\begin{align*}
\langle \{M\{y/x\}\} \rangle_o &= \big[ \{M\} \big]_{\vartheta} & \text{(25)} \\
\text{rn} & : \big[ \{M\} \big]_{\vartheta} \Rightarrow \big[ \{M\} \big]_{\vartheta} & \beta & : \big[ \{M\} \big]_{\vartheta} \Rightarrow \big[ \{M\} \big]_{\vartheta} \\
\text{sub} & : \big[ \{M\} \big]_{\vartheta} \Rightarrow \big[ \{M\} \big]_{\vartheta} & \text{tr} & : \big[ \{M\} \big]_{\vartheta} \Rightarrow \big[ \{M\} \big]_{\vartheta} \\
\text{tr} & : \big[ \{M\} \big]_{\vartheta} \Rightarrow \big[ \{M\} \big]_{\vartheta}
\end{align*}
\]

In (25) here above (i) (19) holds because we have that $\big[ \{N\} \big]_q \approx \big[ \{N\} \big]_q \approx \big[ \{N\} \big]_q$, holds thanks to the uniqueness of input, and output channels, and thanks to $\text{Seq}$ which never
5 Completeness of BVQ w.r.t. Linear $\lambda$-calculus

Completeness says that we can mimic every computation step of linear $\lambda$-calculus as proof-reconstruction inside BVQ.

Theorem 5.1 (Completeness of BVQ) For every $M$, and $N$, and $o$, if $M \Rightarrow N$, then $\mathcal{D} : \llbracket N \rrbracket_o \vdash_{\text{BVQ}} \llbracket M \rrbracket_o$.

The proof relies on some technical lemma that we detail out in the coming lines.

Lemma 5.2 (Output names are linear) For every $M$, and $o$, the output name $o$ of $\llbracket M \rrbracket_o$ occurs once.

Proof By induction on the definition of $\llbracket \cdot \rrbracket$, proceeding by cases on the form of $M$.

Lemma 5.3 (Substitution in BVQ) For every $M, N, o, p, x$, such that $x \in \text{fn}(\llbracket M \rrbracket_o)$, in BVQ, we can derive:

\[
\begin{align*}
\text{mt}_o: & \quad \frac{\llbracket M \rrbracket_o}{\llbracket M \rrbracket_o \bowtie (p \cdot o)} \\
\text{subst}: & \quad \frac{\llbracket M(\lambda \cdot \cdot) \rrbracket_o}{\llbracket M \rrbracket_o \bowtie \llbracket N \rrbracket_o}
\end{align*}
\]

Proof Concerning $\text{mt}_o$, we reason inductively on the size of $\llbracket \cdot \rrbracket$, proceeding by cases on $M$. (Details in Appendix F) Concerning $\text{subst}$, we reason inductively on the size of $\llbracket M \rrbracket_o \bowtie \llbracket N \rrbracket_o$, exploiting $\text{mt}_o$. (Details in Appendix F)

Lemma 5.4 (Linear $\beta$ reduction in BVQ) For every $M, N, o, x$, in BVQ, we can derive:

\[
\begin{align*}
\text{beta}: & \quad \frac{\llbracket M(\lambda \cdot \cdot) \rrbracket_o}{\llbracket (\lambda x M) N \rrbracket_o}
\end{align*}
\]

Proof The rule $\text{beta}$ is derived in \[23\] exploiting the definition of $\llbracket \cdot \rrbracket$, and Lemma 5.3.

Proof of Theorem 5.1 By induction on $\llbracket M \Rightarrow N \rrbracket$, proceeding by cases on the last rule in \[23\] used for proving $M \Rightarrow N$. If the last rule is $\text{beta}$, then Lemma 5.4 implies the thesis. Let the last rule be $\text{tra}$. The inductive hypothesis implies the existence of $\mathcal{D}_0$, and $\mathcal{D}_1$:

\[
\begin{align*}
\llbracket N \rrbracket_o \\
\mathcal{D}_0 \\
\llbracket P \rrbracket_o \\
\mathcal{D}_1 \\
\llbracket M \rrbracket_o
\end{align*}
\]

In all the remaining cases we proceed as here above, exploiting that BVQ is a DI system, so we can apply deeply, namely in any context, every of its rules.

Remark 5.5 As a corollary, under the same assumption as Theorem 5.1, we have $\vdash_{\text{BVQ}} \llbracket \llbracket M \rrbracket_o \bowtie \llbracket N \rrbracket_o \rrbracket$ because we can derive $\mathcal{D}_1$ in BVQ, and we can plug it on top of $\mathcal{D}$.
6 Conclusions and future work

On the computational interpretation side of proof-search inside BVQ, this work makes no reference to soundness of BVQ w.r.t. linear $\lambda$-calculus. Soundness is the reverse of completeness. For every $M, N$, and $o$, if $\not\models_{BVQ} o$ then $M \Rightarrow N$. A counter example to it is:

$\langle M \rangle_{o}$

$⟨(\langle x \rangle P) \vec{Q}⟩_{o} = \langle\langle \{\{M\}\}_{o} \circ \{\{P\}\}_{o} \circ (s \cdot \vec{T})\rangle_{o} \circ \{\{Q\}\}_{o} \circ (r \cdot \vec{T})\rangle_{o}$

where we would erroneously substitute (the mapping of) $Q$ for (the mapping of) $x$ in (the mapping of) $M$. We think essentially two ways exist to react to the lack of soundness of BVQ w.r.t. linear $\lambda$-calculus. The first is in [11][12] which proves a weak, and not so interesting form of soundness. The second way is replacing the target language linear $\lambda$-calculus, so moving towards the programme that [2] begins. It suggests that the natural computational paradigm w.r.t. which BVQ can be sound, is some extension of CCS, the fragment of Milner CCS with sequential and parallel composition only. This is coming work, indeed.

On the proof-theoretical side, whose concern is the minimal, and incremental extension of SBV, an example of which is SBVQ, we plan to keep investigating self-dual operators. By means of a self-dual operator, and in accordance with the proof-search-as-computation paradigm, we plan to model non deterministic choice. Candidate rules that model a self-dual operator, and in accordance with the proof-search-as-computation paradigm, we plan to model non deterministic choice. Candidate rules that model a self-dual operator, and in accordance with the proof-search-as-computation paradigm, we plan to model non deterministic choice. Candidate rules that model a self-dual operator, and in accordance with the proof-search-as-computation paradigm, we plan to model non deterministic choice.

| Candidate rules that model a self-dual non-deterministic choice are: |
| --- |
| $p_{\mathcal{R}} \frac{[(R \otimes T) \otimes (U \otimes T)]}{[(R \otimes U) \otimes T]}$ |
| $p_{\mathcal{R}} \frac{[(R \otimes U) \otimes (U \otimes T)]}{[(R \otimes T) \otimes (U \otimes T)]}$ |

We think they are interesting because they would internalize the non deterministic choice that we apply at the meta-level when searching for proofs, or derivations, inside SBVQ or SBV.

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A Proof of Context extrusion (Proposition 2.2, page 7)

By induction on $|S\{\}|$, proceeding by cases on the form of $S\{\}$. The base is with $S\{\} \equiv \{\}$. The statement holds simply because (i) $S[R \otimes T] \equiv [S[R] \otimes T] \equiv [R \otimes T]$, and (ii) $[R \otimes T]$ is a structure, so, by definition, a derivation.

As a first case, let $S\{\} \equiv \langle S'\{\} \cdot U \rangle$. Then:

\[
\langle S'[R \otimes T] \cdot U \rangle \equiv S[R \otimes T]
\]

\[
\frac{|S[R] \otimes T| \equiv |\langle S'[R] \cdot U \rangle \otimes T|}{|S[R] \otimes T| \equiv |\langle S'[R] \cdot U \rangle \otimes T|}
\]

where $D$ exists by inductive hypothesis which holds thanks to $|S'|(|) < |S\{\}|$. If, instead $S\{\} \equiv \langle S'\{\} \cdot U \rangle$, we can proceed as above, using $*$ in place of $\cdot$.

As a second case, let $S\{\} \equiv [S\{\}].$ Without loss of generality, thanks to (19), we can assume $a \notin \text{fn}(T)$. Then:

\[
[S'[R \otimes T]U] \equiv S[R \otimes T]
\]

\[
\frac{|S[R] \otimes T| \equiv |[S'[R] \cdot U] \otimes T|}{|S[R] \otimes T| \equiv |[S'[R] \cdot U] \otimes T|}
\]

where $D$ exists by inductive hypothesis which holds thanks to $|S'|(|) < |S\{\}|$.

B Proof of Shallow Splitting (Proposition 3.2, page 9)

Proof of Points 1 and 2 We prove the two statements simultaneously, by induction on the lexicographic order $(|U|, |\mathcal{D}|)$, where $U$ is one among $(R \cdot T) \otimes P$, and $(R \otimes T) \otimes P$, proceeding by cases on the last rule $\rho$ of $\mathcal{D}$.

As a first case for both points 1 and 2 we assume the redex of $\rho$ is inside one among $R,T$ or $P$. So, $\mathcal{D}$ is one between:

\[
\frac{|(R' \cdot T') \otimes P'|}{|R' \cdot T| \equiv P}| \quad \frac{|(R' \otimes T') \otimes P'|}{|R' \otimes T| \equiv P}
\]

where only one among $R', T', P'$ is the redct of $\rho$. We can conclude by applying the inductive hypothesis on $\mathcal{D}'$, or $\mathcal{D}''$, and $\rho$ in the obvious way.

As a second case of Point 1 let $\rho$ be $q_1$ with $|<(R' \cdot T') \otimes T' > \equiv [<(P' \cdot P') \otimes P'' >]|$ as its redex. So, $\mathcal{D}$ can be:

\[
\frac{|(R' \cdot P') \cdot [(R'' \cdot T') \otimes P'']|}{q_1 |(R' \cdot P') \cdot [(R'' \cdot T') \otimes P'']|}
\]

\[
\frac{|(R' \cdot T') \otimes P'| \equiv [P' \cdot P''] \otimes P''}{|Q| < |\mathcal{D}| \text{ the inductive hypothesis holds on } \mathcal{D}' \text{ which implies } \mathcal{D}' : \vdash (P_1 \cdot P_2) + P''', \text{ and } \mathcal{D}' : \vdash |(R' \cdot T') \otimes P'| \equiv P'}.\}
\]

Thanks to $|<(R' \cdot R'') \cdot T > \equiv [P' \cdot P'' \times P''']|$ and $|\mathcal{D}'| < |\mathcal{D}|$ the inductive hypothesis holds on $\mathcal{D}'$ which implies $\mathcal{D}' : \vdash (P_1 \cdot P_2) + P''$, and $\mathcal{D}' : \vdash |(R' \cdot T') \otimes P'| \equiv P'$.\}

Thanks to $|((R' \cdot T') \otimes P') |< |(R' \cdot P') \cdot [(R'' \cdot T') \otimes P'']| \equiv P''|$ the inductive hypothesis holds on $\mathcal{D}$ which implies $\mathcal{D}'' : \vdash (U_1 \cdot U_2) + [P'' \equiv P_2]$, and $\mathcal{D}'' : \vdash |(R'' \cdot U_1) \equiv [T \cdot U_2]|$.\}


The first derivation and the first proof of BVQ in the statement we have to prove are:

\[
\begin{align*}
&\langle \langle P' \otimes P_1 \rangle \cdot U_1 \cdot U_2 \rangle \\
&\langle \langle P' \otimes P_1 \rangle \cdot (U_1 \cdot U_2) \rangle \\
&\langle \langle P' \otimes P_1 \rangle \cdot [P'' \otimes P_3] \rangle \\
&\langle \langle P' \cdot P'' \rangle \otimes (P_1 \cdot P_2) \rangle \\
&\langle \langle P' \cdot P'' \rangle \otimes P' \rangle
\end{align*}
\]

\[
\begin{align*}
&\langle \langle P' \otimes P_1 \rangle \cdot \langle P'' \otimes P_1 \rangle \rangle \\
&\langle \langle P' \otimes P_1 \rangle \cdot \langle P'' \otimes P_1 \rangle \rangle \\
&\langle \langle P' \otimes P_1 \rangle \cdot [R \otimes P_1 \cdot \langle R' \otimes P_1 \rangle \rangle \\
&\langle \langle P' \otimes P_1 \rangle \cdot [R \otimes P_1 \cdot \langle R' \otimes P_1 \rangle \rangle \\
&\langle \langle P' \otimes P_1 \rangle \cdot [R \otimes P_1 \cdot \langle R' \otimes P_1 \rangle \rangle
\end{align*}
\]

The second proof of BVQ in the statement we have to prove is \(2\').

The situation with \(\rho \equiv q_1\) and \([\langle R \otimes \langle T' \cdot T'' \rangle \rangle \otimes \langle P' \cdot P'' \rangle \otimes P''']\) its redex is analogous to one one just developed.

As a third case of Point [1] let \(\rho\) be \(q_1\) with \([\langle R \otimes T \rangle \otimes \langle P' \cdot P'' \rangle \otimes [U' \otimes U'']\]) as its redex. So, \(\mathcal{P}\) can be:

\[
\begin{align*}
&\langle \langle P' \cdot [(R \cdot T) \otimes P'' \rangle \rangle \otimes [U' \otimes U''] \rangle \\
&\langle \langle (\cdot \cdot (R \cdot T) \otimes (P' \cdot P'' \rangle \rangle \otimes [U' \otimes U''] \rangle \\
&\langle \langle (R \cdot T) \otimes (P' \cdot P'' \rangle \rangle \otimes [U' \otimes U''] \rangle
\end{align*}
\]

Thanks to \([\langle (R \cdot T) \otimes [P' \cdot P'' \rangle \rangle \otimes [U' \otimes U''] \rangle) = \langle \langle P' \cdot [(R \cdot T) \otimes P'' \rangle \rangle \otimes [U' \otimes U''] \rangle\) and \(|\mathcal{P}'| < |\mathcal{P}|\) the inductive hypothesis holds on \(\mathcal{P}'\) yielding \(\mathcal{P}' : \langle P_1 \cdot P_2 \rangle \vdash \langle U' \otimes U'' \rangle\), and \(\mathcal{P}'' : \vdash \langle (R \cdot T) \otimes P''' \rangle \otimes P_2\).

Thanks to \([\langle (R \cdot T) \otimes P''' \rangle \otimes P_2 \rangle < \langle (R \cdot T) \otimes \langle P' \cdot P'' \rangle \rangle \otimes [U' \otimes U''] \rangle\) and \(|\mathcal{P}'| < |\mathcal{P}|\) the inductive hypothesis holds on \(\mathcal{P}'\) yielding \(\mathcal{P}' : \langle U_1 \cdot U_2 \rangle \vdash \langle P'' \otimes P_2 \rangle\), and \(\mathcal{P}'' : \vdash \langle R \otimes U_1 \rangle\), and \(\mathcal{P}''' : \vdash \langle T \cdot U_2 \rangle\).

Both \(\mathcal{P}''\) and \(\mathcal{P}'''\) are the two proofs of BVQ of the statement we have to prove. The derivation of BVQ is:

\[
\begin{align*}
&\langle (U_1 \cdot U_2) \\
&\langle (\cdot \cdot (U_1 \cdot U_2) \rangle \\
&\langle P' \otimes P_1 \cdot (U_1 \cdot U_2) \rangle \\
&\langle P' \otimes P_1 \cdot [P'' \otimes P_3] \rangle \\
&\langle P' \cdot P'' \rangle \otimes \langle P_1 \cdot P_2 \rangle \\
&\langle P' \cdot P'' \rangle \otimes P' \rangle
\end{align*}
\]

As a fourth case of Point [1] let \(\rho\) be \(s\) with \([\langle R \otimes T \rangle \otimes \langle P' \otimes P'' \rangle \otimes P''']\) as its redex. So, \(\mathcal{P}\) can be:

\[
\begin{align*}
&\langle \langle (R \cdot T) \otimes P' \rangle \otimes P''' \rangle \\
&\langle \langle (P' \otimes (R \cdot T) \rangle \otimes P'' \rangle \rangle \otimes P''' \rangle \\
&\langle \langle (P' \otimes P'' \rangle \otimes (R \cdot T) \rangle \rangle \otimes P''' \rangle \\
&\langle \langle (R \cdot T) \otimes [P' \otimes P'' \rangle \rangle \otimes P''' \rangle
\end{align*}
\]

Thanks to \([\langle (R \cdot T) \otimes [P' \otimes P'' \rangle \rangle \otimes P''' \rangle) = \langle \langle (R \cdot T) \otimes P' \rangle \rangle \otimes P''' \rangle\) and \(|\mathcal{P}'| < |\mathcal{P}|\), by the inductive hypothesis, Point [2] applies to \(\mathcal{P}'\). This means there exist \(\mathcal{P}' : \langle P_1 \otimes P_2 \rangle \vdash P'''\), and \(\mathcal{P}'' : \vdash \langle (R \cdot T) \otimes P' \rangle \otimes P_1\), and \(\mathcal{P}''' : \vdash \langle P' \otimes P_2 \rangle\).

Thanks to \([\langle (R \cdot T) \otimes P' \rangle \rangle < \langle \langle (R \cdot T) \otimes P' \rangle \rangle \otimes P'' \rangle\) the inductive hypothesis holds on \(\mathcal{P}'''\) which implies \(\mathcal{P}'' : \langle U_1 \cdot U_2 \rangle \vdash \langle P' \otimes P_1 \rangle\), and \(\mathcal{P}''' : \vdash \langle R \otimes U_1 \rangle\), and \(\mathcal{P}''' : \vdash \langle T \cdot U_2 \rangle\).
As a fifth case of Point 1 let \( \rho \) be \( \langle R \cdot T \rangle \) with \( \langle R \cdot T \rangle \approx P \) as its redex. This means \( P \approx [U]_\alpha \), for some \( U \) and \( \alpha \), that, without loss of generality, thanks to (19), we can assume such that \( a \in \text{fn}(U) \), and \( a \notin \text{fn}(R \cdot T) \). So, by (18), \( \langle R \cdot T \rangle \approx \langle R \cdot T \rangle \_\alpha \), the derivation is:

\[
\frac{[\langle R \cdot T \rangle \_\alpha]_\alpha}{[\langle R \cdot T \rangle \_\alpha]_\alpha} \]

As a fifth case of Point 1 let \( \rho \) be \( u \_1 \) with \( \langle \langle R \cdot T \rangle \_\alpha \_\beta \_\alpha \rangle \_\beta \) as its redex. This means \( P \approx [U]_\alpha \), for some \( U \) and \( \alpha \), that, without loss of generality, thanks to Fact 2.3.

We have exhausted the interesting cases relative to Point 1.

Recall that we prove Point 1 and Point 2 simultaneously, by induction on the lexicographic order \( |U|, |P| \), where \( U \) is one among \( \langle \langle R \cdot T \rangle \_\alpha \_\beta \_\alpha \rangle \_\beta \), and \( |R \cdot T| < |P| \). Proceeding by cases on the last rule \( \rho \) of \( \mathcal{P} \). Now we explore the cases relative to Point 2.

As a first case of Point 2 let \( \rho \) be \( q \_1 \) with \( \langle \langle R \cdot T \rangle \_\alpha \_\beta \_\alpha \rangle \_\beta \) as its redex. So, \( \mathcal{P} \) can be:

\[
\frac{[\langle \langle R \cdot T \rangle \_\alpha \_\beta \_\alpha \rangle \_\beta]_\alpha}{[\langle \langle R \cdot T \rangle \_\alpha \_\beta \_\alpha \rangle \_\beta]_\alpha} \]

Thanks to \( [\langle \langle R \cdot T \rangle \_\alpha \_\beta \_\alpha \rangle \_\beta]_\alpha \rangle \_\beta \) as its redex. So, \( \mathcal{P} \) can be:
Both in the statement we have to prove. The derivation of BVQ in the statement we have to prove is:

\[
\begin{align*}
\vdash & \\
\vdash & \frac{[[P \otimes U] \otimes U_1]}{[[P \otimes U_1] \cdot \circ \otimes U]} \\
\vdash & \frac{[[(P' \circ U_1) \cdot [P' \circ U_2]) \otimes U']}{[P' \circ U_1] \cdot [P' \circ U_2]} \\
\vdash & \frac{[P' \circ [P'' \circ (U_1 \cdot U_2)] \otimes U']}{[P' \circ P''] \circ [U' \otimes (U_1 \cdot U_2)]} \\
\vdash & \frac{[P' \circ P''] \circ [U' \otimes U'']}{}
\end{align*}
\]

As a second case of Point[2] let \( \rho \) be \( \underline{s} \) with \( (((R' \otimes R'') \otimes (T' \otimes T'')) \otimes [P' \otimes P'']) \) as its redex. So, \( \mathcal{P} \) can be:

\[
\begin{align*}
\vdash & \\
\vdash & \frac{[[(R' \otimes T') \otimes P'] \circ (R'' \otimes T'') \circ P'']}{[P' \circ P'']} \\
\vdash & \frac{[[(R' \otimes T') \otimes (R'' \otimes T'') \circ P'] \circ P'']}{[P' \circ P'']} \\
\vdash & \frac{[[(R' \otimes R'') \otimes (T' \otimes T'')] \circ [P' \circ P'']]}{}
\end{align*}
\]

Both \( [[(R' \otimes R'') \otimes (T' \otimes T'')] \circ [P' \otimes P'']] \) and \( |\mathcal{P}'| < |\mathcal{P}| \) imply that the inductive hypothesis applies to \( \mathcal{P}' \). There exist \( \mathcal{E} : [P_1 \circ P_2] \vdash P' \), and \( \mathcal{E}' : [P' \circ P''] \vdash [P' \circ P_1] \), and \( \mathcal{D} : [P'' \circ P_2] \vdash [R' \otimes U_1] \), and \( \mathcal{D}' : [R' \otimes U_2] \vdash [T' \otimes T''] \).

Thanks to \( [[(R'' \otimes T'') \otimes P_2]] \) and \( [[(R'' \otimes T'') \otimes P''] \) the inductive hypothesis holds on \( \mathcal{E}'' \) which implies \( \mathcal{E}'' : [U_1 \circ U_2] \vdash [P' \circ P_2] \), and \( \mathcal{D}_2' : [R' \otimes U_1] \vdash [T'' \otimes U_2] \).

The derivation and the two proofs of BVQ in the statement we have to prove are:

\[
\begin{align*}
\vdash & \\
\vdash & \frac{[[U_1 \circ U_2] \circ [U_1 \circ U_2]]}{[[U_1 \circ U_2] \circ [U_1 \circ U_2]]} \\
\vdash & \frac{[[U_1 \circ U_2] \circ [U_1 \circ U_2]]}{[[U_1 \circ U_2] \circ [U_1 \circ U_2]]} \\
\end{align*}
\]
As a third case of Point 3, let \( \rho \) be \( \$ \) with \( [(R \otimes T) \not\in (P' \otimes P'') \otimes [U' \otimes U'']] \) as its redex. So, \( \mathcal{P} \) can be:

\[
\mathcal{P} \vdash [(P' \otimes [R \otimes T] \not\in U') \otimes P'') \otimes U''] \\
\mathcal{P} \vdash [((P' \otimes P'') \otimes [R \otimes T] \not\in U') \otimes U''] \\
\mathcal{P} \vdash [(R \otimes T) \not\in (P' \otimes P'') \otimes [U' \otimes U'']] 
\]

Both \( [(P' \otimes [R \otimes T] \not\in U')] \otimes P'') \otimes U''] = \( [(P' \otimes P'') \otimes [U' \otimes U'']] \), and \( |\mathcal{P}'| < |\mathcal{P}| \) imply that the inductive hypothesis holds on \( \mathcal{P}' \). So, we have \( \mathcal{E} : [P_1 \otimes P_2] \vdash U', \) and \( \mathcal{E}'' : \vdash [P'' \otimes P_2], \) and \( \mathcal{D} : \vdash [(P' \otimes (R \otimes T) \not\in U') \otimes P_1] \).

Both \( [(P' \otimes (R \otimes T) \not\in U')] \otimes P_1] \not\in [(P' \otimes (R \otimes T) \not\in U') \otimes P'' \otimes U''], \) and \( |\mathcal{P}'| < |\mathcal{P}| \) imply that the inductive hypothesis holds on \( \mathcal{D} \). SO, we have \( \mathcal{E}'' : [U_1 \otimes U_2] \vdash [P' \otimes U' \otimes P_1], \) and \( \mathcal{D}'' : \vdash [T \otimes U_2] \).

Both \( \mathcal{D}' \), and \( \mathcal{D}'' \) are the two proofs of BVQ of the statement we have to prove. The derivation of BVQ is:

\[
[U_1 \otimes U_2] \\
\mathcal{E} : [P_1 \otimes P_2] \vdash U', \) and \( \mathcal{E}'' : \vdash [P'' \otimes P_2], \) and \( \mathcal{D} : \vdash [(P' \otimes (R \otimes T) \not\in U') \otimes P_1] \).

As a fourth case of Point 3, let \( \rho \) be \( \|$ with \( [(R \otimes T) \not\in P] \) as its redex. This means \( P \approx [U]_a \), for some \( U \) and \( a \), that, without loss of generality, thanks to \( [19] \), we can assume such that \( a \notin \text{fn}((R \otimes T)) \). So, by \( [13] \), \( (R \otimes T) \approx [(R \otimes T)]_a \), and \( \mathcal{P} \) is:

\[
\mathcal{P} \vdash [(R \otimes T) \not\in U]]_a \\
\mathcal{P} \vdash [[(R \otimes T)]_a \not\in [U]_a] 
\]

Point 3 of Proposition 3.1 applied on \( \mathcal{P}' \), implies:

\[
\mathcal{P}' \vdash [(R \otimes T) \not\in U] 
\]

Thanks to \( [(R \otimes T) \not\in U] \not\in [[(R \otimes T)]_a \not\in [U]_a] \) the inductive hypothesis holds on \( \mathcal{P}' \), which implies \( \mathcal{E} : [P_1 \otimes P_2] \vdash U', \) and \( \mathcal{D} : \vdash [T \otimes P_2], \) and \( \mathcal{D} : \vdash [T \otimes P_2]. \)

Both \( \mathcal{D}_1, \) and \( \mathcal{D}_2 \) are the two poofs of BVQ in the statement we have to prove. The derivation is \( [(P_1 \otimes P_2)]_a \vdash [U]_a, \) we obtain from \( \mathcal{E} \) thanks to Fact 2.3.

Proof of Point 3. It holds by induction on \( |\mathcal{P}| \), proceeding by cases on the last rule \( \rho \) of \( \mathcal{P} \).

As a first case let the redex of \( \rho \) be inside \( P \). So, \( \mathcal{P} \) is:

\[
\mathcal{P} \vdash [R \otimes P'] \\
\mathcal{P} \vdash [R \otimes P] 
\]
We can conclude by applying the inductive hypothesis on $\mathcal{P}'$, and $\rho$ in the obvious way.

As a second case, let $\rho$ be $q_1$ with $P \equiv [(P' \cdot P'') \equiv P''']$. Also, let $R_0$, and $R_1$ such that $R \equiv [R_0 \odot R_1]$. The proof $\mathcal{P}$ can be:

$$\begin{align*}
\text{Point } & \text{II applies to } \mathcal{P}' \text{. There are structures } P_1, P_2, \text{ such that } \\
& \mathcal{E}_0 : \langle P_1 \odot P_2 \rangle \vdash [R_0 \odot P'''] \text{, and } \mathcal{E}_0 : \vdash [[R_1 \odot P''] \odot P_1] \equiv [R_1 \odot [P' \odot P_1]] \text{, and } \mathcal{E}_0 \vdash [P'' \odot P_2] \text{.}
\end{align*}$$

We observe that $|R_1| < [|R_0 \odot R_1]|$. So, the inductive hypothesis holds on $\mathcal{E}_0$. It implies that, for every $R_0', R_1'$, if $R_1 \approx [R_0' \odot R_1']$, then $\mathcal{E}_0' \vdash [R_0' \odot [P'' \odot P_2]] \approx [P'' \odot P_2]$. In particular, it holds $\mathcal{E}_0' \vdash [R_0' \approx [P'' \odot P_2]]$ by taking $R_1 \approx R_1'$, and $\rho \approx R_0'$.

We can conclude as follows:

$$\begin{align*}
\text{As a second case let } \rho & \text{ be } q_1 \text{ with } P \equiv [(P' \odot P'') \equiv [R' \odot R'']] \text{. So, } \mathcal{P} \text{ can be:}
\end{align*}$$

$$\begin{align*}
\text{Point } & \text{II applies to } \mathcal{P}' \text{. There are structures } P_1, P_2 \text{ such that there exist } \\
& \mathcal{E}_0 : \langle P_1 \odot P_2 \rangle \vdash [R_0 \odot R'''], \text{ and } \mathcal{E}_0 : \vdash [P' \odot P_1] \text{, and } \\
& \mathcal{E}_0 \vdash [[R_0 \odot R_1] \odot [P' \odot P_2]] \approx [R_1 \odot [R_0 \odot [P'' \odot P_2]]].
\end{align*}$$

We observe that $|R_1| < [|R_0 \odot R_1]|$. So, the inductive hypothesis holds on $\mathcal{E}_0$. It implies that, for every $R_0', R_1'$, if $R_1 \approx [R_0' \odot R_1']$, then $\mathcal{E}_0' \vdash [R_0' \odot [R_0 \odot [P'' \odot P_2]]] \approx [R_0 \odot [P'' \odot P_2]]$. In particular, it holds $\mathcal{E}_0' \vdash [R_0' \approx [R_0 \odot [P'' \odot P_2]]] \approx [R_0 \odot [P'' \odot P_2]]$ by taking $R_1 \approx R_1'$, and $\rho \approx R_0'$.
We can conclude as follows:

\[
\begin{array}{c}
\frac{R_1}{\langle P' \equiv P_1 \rangle \cdot R_1} \\
\psi_0 \\
\langle P' \equiv P_1 \rangle \cdot R_1
\end{array}
\]

\[
\begin{array}{c}
\langle P' \equiv P_1 \rangle \cdot [R_0 \equiv [P'' \equiv P_3]] \\
\psi_1 \\
\langle P' \equiv P_1 \rangle \cdot [R_0 \equiv P''']
\end{array}
\]

\[
\begin{array}{c}
\langle P' \equiv P_1 \rangle \cdot [R_0 \equiv [P'' \equiv P_3]] \\
\psi_2 \\
\langle P' \equiv P_1 \rangle \cdot [R_0 \equiv P''']
\end{array}
\]

As a fourth case let \( \rho \) be \( S \) with \( P \equiv [P' \equiv P'''] \). Also, let \( R_0 \), and \( R_1 \) such that \( R \equiv [R_0 \equiv R_1] \). The proof \( \mathcal{P} \) can be:

\[
\begin{array}{c}
\mathcal{P} \\
\sigma \\
[[R_0 \equiv P'] \equiv R''']
\end{array}
\]

\[
\begin{array}{c}
[[R_0 \equiv P'] \equiv [R_0 \equiv P'''] \\
[[R_0 \equiv (P' \equiv P''')] \equiv [R_0 \equiv P'''] \\
[[R_0 \equiv R_1] \equiv [(P' \equiv P') \equiv P''']
\end{array}
\]

Point 4 applies to \( \mathcal{P}' \). There are structures \( P_1, P_2, \) such that there exist \( \rho_0 : [P_1 \equiv P_2] \vdash [R_0 \equiv P'''], \) and \( \mathcal{Q}_0 : \vdash [[R_0 \equiv P'] \equiv P_1] \), and \( \mathcal{Q}_1 : \vdash [P'' \equiv P_2] \).

We observe that \( |R_1| < |[R_0 \equiv R_1]| \). So, the inductive hypothesis holds on \( \mathcal{Q}_0 \). It implies that, for every \( R_0^1, R_1^1 \), if \( R_1 \equiv [R_0^1 \equiv R_1^1] \), then \( \rho_1 : [R_1^1] \vdash [R_0 \equiv [P' \equiv P_1]] \). In particular it holds \( \rho_1 : [R_1] \vdash [\circ \equiv [P' \equiv P_1]] \equiv [P' \equiv P_1] \) by taking \( R_1 \equiv R_1^1 \), and \( \circ \equiv R_0^1 \). We can conclude as follows:

\[
\begin{array}{c}
\frac{R_1}{\langle (P'' \equiv P_2) \equiv R_1 \rangle} \\
\psi_3 \\
\langle (P'' \equiv P_2) \equiv R_1 \rangle
\end{array}
\]

\[
\begin{array}{c}
\langle (P'' \equiv P_2) \equiv R_1 \rangle \\
\psi_4 \\
\langle (P'' \equiv P_2) \equiv R_1 \rangle
\end{array}
\]

\[
\begin{array}{c}
\langle (P'' \equiv P_2) \equiv [R_0 \equiv P'''] \\
\psi_5 \\
[R_0 \equiv [P'' \equiv P''']
\end{array}
\]

As a fifth case let \( \rho \) be \( u_1 \) with \( P \equiv [P'_1] \). The proof \( \mathcal{P} \) can be:

\[
\begin{array}{c}
\mathcal{P}' \\
\sigma_1 \\
[[R_0 \equiv R_1] \equiv [P'_1] \\
[[R_0 \equiv R_1] \equiv [P'_1] \\
[[R_0 \equiv R_1] \equiv [P'_1] \\
[[R_0 \equiv R_1] \equiv [P'_1] \\
[[R_0 \equiv R_1] \equiv [P'_1]
\end{array}
\]
because, thanks to \([13]\), we can always assume \(P'\) is such that \(a \notin \text{fn}(\lbrack R_0 \otimes R_1 \rbrack)\).

Point \([4]\) of Proposition \([5.1]\) applied on \(\mathcal{P}'\), implies:

\[
\mathcal{P}' \vdash \lbrack R_0 \otimes R_1 \rbrack \otimes P'
\]

We observe that \(|\mathcal{P}'| < |\mathcal{P}|\). So the inductive hypothesis holds on \(\mathcal{P}'\). It implies that, for every \(R_0, R_1\), if \(\lbrack R_0 \otimes R_1 \rbrack \approx \lbrack R_0' \otimes R_1' \rbrack\), there are \(E : \lbrack R_1' \rbrack \vdash \lbrack R_0' \otimes P' \rbrack\). In particular it holds \(E' : \lbrack R_0 \otimes P' \rbrack\) by taking \(R_1 \approx R_1'\), and \(R_0 \approx R_0'\). We can conclude as follows:

\[
\begin{array}{c}
\begin{array}{c}
\hline
\hline
R_1
\hline
\hline
\hline
\end{array}
\end{array}
\]

The topmost instance of \([13]\) is legal thanks to \(a \notin \text{fn}(\lbrack R_0 \otimes R_1 \rbrack)\).

**Proof of Point \([4]\).** The proof is by induction on \(|\mathcal{P}|\), proceeding by cases on the last rule \(\rho\) of \(\mathcal{P}\).

As a first case let the last rule of \(\mathcal{P}\) be \(\text{q}_{1}\) with \([\lbrack R_{1a} \otimes [(P' \cdot P'') \otimes [U' \otimes U'']]]\) as its redex. So, \(\mathcal{P}\) can be:

\[
\begin{array}{c}
\hline
\hline
\langle P' \cdot [\lbrack R_{1a} \otimes P''] \rbrack \otimes [U' \otimes U''] \rbrack \\
\hline
\hline
\end{array}
\]

Point \([1]\) applies to \(\mathcal{P}'\). There exist \(E : \langle P_1 \cdot P_2 \rangle \vdash [U' \otimes U'']\), and \(\mathcal{P}'' : \vdash [P' \otimes P_1]\), and \(\mathcal{Q} : \vdash [[R_{1a} \otimes P''] \otimes P_2]\). The inductive hypothesis holds on \(\mathcal{Q}\). Thanks to \(\langle [R_{1a} \otimes P''] \otimes P_2 \rangle \approx [\lbrack R_{1a} \otimes (P' \cdot P'') \otimes [U' \otimes U''] \rbrack\rangle\) we get \(E' : [U_{1a} \vdash [P'' \otimes P_2]\), and \(\mathcal{Q}'' : \vdash [R \otimes U]\). The proof of BVQ in the statement we have to prove is \(\mathcal{Q}''\). The derivation of BVQ in the statement we have to prove is:

\[
\begin{array}{c}
\hline
\hline
\langle P' \cdot [\lbrack R_{1a} \otimes P''] \rbrack \otimes [U' \otimes U''] \rbrack \\
\hline
\hline
\end{array}
\]

As a second case let the last rule of \(\mathcal{P}\) be \(\text{q}_{1}\) with \([\lbrack R_{1a} \otimes [(P' \cdot P'') \otimes P''']\] as its redex. So, \(\mathcal{P}\) can be:

\[
\begin{array}{c}
\hline
\hline
\langle P' \cdot [\lbrack R_{1a} \otimes P''] \rbrack \otimes P'' \rbrack \\
\hline
\hline
\end{array}
\]

Point \([1]\) applies to \(\mathcal{P}'\). There exist \(E : \langle P_1 \cdot P_2 \rangle \vdash P''\), and \(\mathcal{P}'' : \vdash [P' \otimes P_1]\), and \(\mathcal{Q} : \vdash [[R_{1a} \otimes P'' \rbrack \otimes P_2]\). Thanks to \(\langle [R_{1a} \otimes P'' \otimes P_2] \rangle \approx [\lbrack R_{1a} \otimes (P' \cdot P'') \otimes P''']\rangle\)
the inductive hypothesis holds on \( \mathcal{D} \) which implies \( \mathcal{D}' : \vdash [U]_a \vdash [P' \otimes P_2], \) and \( \mathcal{D} : \vdash [R \otimes U] \). The proof of BVQ in the statement we have to prove is \( \mathcal{D}' \). The derivation is:

\[
\begin{array}{c}
\vdash [U]_a \\
\vdash [P' \otimes P_2] \\
\vdash ([P \otimes P_1] \cdot [P' \otimes P_2]) \\
\vdash ([P' \cdot P'] \otimes (P_1 \cdot P_2)) \\
\vdash ([P' \cdot P'] \otimes P'')
\end{array}
\]

As a third case let the last rule of \( \mathcal{D} \) be \( s \) with \( [[R],_a \otimes [(P' \otimes P''') \otimes P''']] \) as its redex. So, \( \mathcal{D} \) can be:

\[
\begin{array}{c}
\vdash [((P' \otimes [R]_a) \otimes P'') \otimes P'''] \\
\vdash [[(P' \otimes P'') \otimes P''']] \\
\vdash [[[(P \otimes P') \otimes P_1]]] < [[[[P' \otimes P'] \otimes P_1]]]
\end{array}
\]

Point 2 applies to \( \mathcal{D}' \). There exist \( \mathcal{D}' : [P_1 \otimes P_2] \vdash P''' \), and \( \mathcal{D}'' : \vdash [((P_1 \otimes P_2) \cdot P') \otimes P_1] \), and \( \mathcal{D} : \vdash [P' \otimes P_2] \). Thanks to \( [[(P_1 \otimes P_2) \cdot P'] \otimes P_1] < [[[(P \otimes P') \otimes P''']]] \) the inductive hypothesis holds on \( \mathcal{D}' \) which implies \( \mathcal{D}' : [U]_a \vdash [P' \otimes P_1] \), and \( \mathcal{D} : \vdash [R \otimes U] \). The proof of BVQ in the statement we have to prove is \( \mathcal{D}' \). The derivation of BVQ in the statement we have to prove is:

\[
\begin{array}{c}
\vdash [U]_a \\
\vdash [P' \otimes P_1] \\
\vdash ([\circ \otimes P') \otimes P_1) \\
\vdash [([P' \otimes P_2] \otimes P') \otimes P_1] \\
\vdash [([P' \otimes P'] \otimes [P_1 \otimes P_2])] \\
\vdash ([P' \otimes P'] \otimes P''')
\end{array}
\]

As a fourth case let the last rule of \( \mathcal{D} \) be \( u_1 \) with \( [[R],_a \otimes P] \) as its redex. This means \( P \approx [U]_a \). So, \( \mathcal{D} \) is:

\[
\begin{array}{c}
\vdash [R \otimes U]_a \\
\vdash [R]_a \otimes [U]_a \\
\vdash [R]_a \otimes P
\end{array}
\]

Point 3 of Proposition 3.1, applied on \( \mathcal{D}' \), implies the existence of \( \mathcal{D}'' : \vdash [R \otimes U] \), which is the proof of BVQ in the statement we have to prove. The derivation is \( [U]_a \vdash [U]_a \).

**C Proof of Context Reduction (Proposition 3.4, page 10)**

The proof is by induction on \([S]_\{\} \) proceeding by cases on the form of \( S_\{\} \).

As a first case, let \( S_\{\} \approx \langle S'[R],_a \cdot P \rangle \). So, the assumption is \( \mathcal{D} : \vdash \langle S'[R],_a \cdot P \rangle \). Point 1 of Proposition 3.1 implies \( \mathcal{D}' : \vdash S'[R],_a \), and \( \mathcal{D}'' : \vdash P' \). Thanks to \( |S'[R]| < |S'[R],_a \cdot P| \) the inductive hypothesis holds on \( \mathcal{D}' \). There are \( U \), and \( \tilde{b} \) such that, for every \( V \) with \( \text{fn}(V) \cap \)}
bn(R) = 0, both $D : [[V \otimes U]]_g \vdash S'[V]$, and $\mathcal{P}''' : \vdash [R \otimes U]$. The proof $\mathcal{P}'''$ is the one we are looking for. To get the derivation we are looking for, we fix $V$ such that $fn(V) \cap bn(R) = 0$. This allows to use $D$ as follows:

\[
\begin{array}{c}
[[V \otimes U]]_g \\
\not\vdash \\
\vdash \\
S'[V] \\
(S'[V] \cap P) \\
\mathcal{P} \\
\end{array}
\]

As a second case, let $S\{\} \approx (S'[\}) \otimes P$. So, the assumption is $\mathcal{P} : \vdash (S'[R] \otimes P)$. Point 2 of Proposition 5.1 implies $\mathcal{P}' : \vdash S'[R]$, and $\mathcal{P}'' : \vdash P$. Thanks to $|S'[R]| < |(S'[R] \otimes P)|$ the inductive hypothesis holds on $\mathcal{P}'$. There are $U$, and $\bar{b}$ such that, for every $V$ with $fn(V) \cap bn(R) = 0$, both $D : [[V \otimes U]]_g \vdash S'[V]$, and $\mathcal{P}''': \vdash [R \otimes U]$. The proof $\mathcal{P}'''$ is the one we are looking for. To get the derivation we are looking for, we fix $V$ such that $fn(V) \cap bn(R) = 0$. This allows to use $D$ as follows:

\[
\begin{array}{c}
[[V \otimes U]]_g \\
\not\vdash \\
\vdash \\
S'[V] \\
(S'[V] \otimes P) \\
\mathcal{P} \\
\end{array}
\]

As a third case, let $S\{\} \approx [S'[\}]_b$ with $b \in fn(S\{\})$. Otherwise it would be meaningless assuming to have $S\{\}$ with such a form. So, the assumption is $\mathcal{P} : \vdash [S'[R]]_b$. Point 3 of Proposition 5.1 implies $\mathcal{P}' : \vdash S'[R]$. So, $|S'[R]| < |[S'[R]]_b|$ implies the inductive hypothesis holds on $\mathcal{P}'$. There are $U$, and $\bar{b}$ such that, for every $V$ with $fn(V) \cap bn(R) = 0$, both $D : [[V \otimes U]]_g \vdash S'[V]$, and $\mathcal{P}'' : \vdash [R \otimes U]$. The proof $\mathcal{P}''$ is the one we are looking for. To get the derivation we are looking for, we fix $V$ such that $fn(V) \cap bn(R) = 0$. This allows to use $D$ as follows:

\[
\begin{array}{c}
[[V \otimes U]]_g \\
\not\vdash \\
\vdash \\
[S'[V]]_b \\
\mathcal{P} \\
\end{array}
\]

As a fourth case, let $S\{\} \approx ([S'[\} \otimes P') \otimes P]$. The assumption is $\mathcal{P} : \vdash ([S'[R] \otimes P') \otimes P]$. Shallow splitting implies the existence of $P_1, P_2$ such that $D : (P_1 \otimes P_2) \vdash P$, and $\mathcal{P}_1 : \vdash [S'[R] \otimes P_1]$, and $\mathcal{P}_2 : \vdash [P' \otimes P]$. The relation $|[S'[R] \otimes P_1]| < |([S'[R] \otimes P')] \otimes P])$, which holds also thanks to $|P_1| < |P|$, implies the inductive hypothesis holds on $\mathcal{P}_1$. There are $U$, and $\bar{b}$ such that, for every $V$ with $fn(V) \cap bn(R) = 0$, both $D : [[V \otimes U]]_g \vdash [S'[V] \otimes P_1]$, and $\mathcal{P}'' : \vdash [R \otimes U]$. The proof $\mathcal{P}'''$ is the one we are looking for. To get the derivation we are looking for, we fix $V$ such that $fn(V) \cap bn(R) = 0$. This allows to use $D$ as follows:

\[
\begin{array}{c}
[[V \otimes U]]_g \\
\not\vdash \\
\vdash \\
[S'[V]]_b \\
\mathcal{P} \\
\end{array}
\]

As a fifth case, let $S\{\} \approx ([S'[R] \otimes P') \otimes P]$. The assumption is $\mathcal{P} : \vdash ([S'[R] \otimes P') \otimes P]$. Shallow splitting implies the existence of $P_1, P_2$ such that $D : [P_1 \otimes P_2] \vdash P$, and $\mathcal{P}_1 : \vdash \ldots
meaningless to assume splitting implies the existence of what we are looking for, we fix \( f_n(P) = 1 \).

We remark that (18) applies thanks to \( a \in \text{fn}(S') \).

We have the following derivation:

\[
\frac{[V \otimes U]_{(\ast \otimes [V] \otimes P_1)}}{[S [V] \otimes P_1)}
\]

As a sixth case, let \( S \{ \} \models [S'] \{ \} \otimes P \) with \( a \in \text{bn}(S') \). Otherwise, it would be meaningless to assume \( S \{ \} \). The assumption is \( \mathcal{D} : \vdash [S [R] \otimes P_1) \). Shallow splitting implies the existence of \( P' \) such that \( \mathcal{D} : [P']_a \vdash P, \) and \( \mathcal{D}' : [S [R] \otimes P'] \). The relation \( \|S' \{ R \} \otimes P'\| < \|S [R] \otimes P_1\| \), which holds also because \( a \in \text{fn}(S') \), implies that the inductive hypothesis on \( \mathcal{D}' \) is true. There are \( U, \) and \( b \) such that, for every \( V \) with \( \text{fn}(V) \cap \text{bn}(R) = \emptyset \), we have \( \mathcal{D} : [V \otimes U]_a \vdash [S [V] \otimes P_1) \), and \( \mathcal{D}' : \vdash [R \otimes U] \). The proof \( \mathcal{D}' \) is the one we are looking for. To get the derivation we are looking for, we fix \( V \) such that \( \text{fn}(V) \cap \text{bn}(R) = \emptyset \). This allows to use \( \mathcal{D} \) as follows:

\[
\frac{[[V \otimes U]]_a}{[V \otimes U]_{b_1...b_n,a}}
\]

As a seventh case, let \( S \{ \} \models [S'] \{ \} \otimes P_1 \) with \( a \in \text{bn}(P_1) \). Also, without loss of generality, can always choose \( a \) such that \( \notin \text{fn}(S') \). The assumption is \( \mathcal{D} : [S [R] \otimes \lfloor P \rfloor] \). Shallow splitting implies the existence of \( P' \) such that \( \mathcal{D} : [P']_a \vdash P_1, \) and \( \mathcal{D}' : \vdash [S [R] \otimes P'] \). The relation \( \|S' \{ R \} \otimes P'\| < \|S [R] \otimes P_1\| \), which holds also because \( a \in \text{bn}(P_1) \), implies that the inductive hypothesis on \( \mathcal{D}' \) is true. There are \( U, \) and \( b \) such that, for every \( V \) with \( \text{fn}(V) \cap \text{bn}(R) = \emptyset \), both \( \mathcal{D} : [V \otimes U]_{a} \vdash [S [V] \otimes P'], \) and \( \mathcal{D}' : \vdash [R \otimes U] \). The proof \( \mathcal{D}' \) is the one we are looking for. To get the derivation we are looking for, we fix \( V \) such that \( \text{fn}(V) \cap \text{bn}(R) = \emptyset \). This allows to use \( \mathcal{D} \) as follows:

\[
\frac{[[V \otimes U]]_a}{[V \otimes U]_{b_1...b_n,a}}
\]

We remark that (18) applies thanks to \( a \notin \text{fn}(S') \).
D Proof of Splitting (Theorem 3.5 page 10)

We obtain the proof of the three statements by composing Context Reduction (Proposition 3.4), and Shallow Splitting (Proposition 3.2) in this order. We develop the details of Points 1 and 3. The proof of Point 2 is analogous to the one of 1.

Point 1 Context Reduction (Proposition 3.4) applies to $\mathcal{P}$. So, there are $U$, and $b$ such that, for every $V$, with $\text{fn}(V) \cap \text{bn}(R \cdot T) = \emptyset$, there exist $\mathcal{D} : [[V \cdot U]]_e \vdash S[V]$, and $\mathcal{D} : \vdash [(R \cdot T) \cdot U]$. Shallow Splitting (Proposition 3.2) applies to $\mathcal{D}$. So, $\mathcal{D} : (K_1 \cdot K_2) \vdash U$, and $\mathcal{D}_1 : \vdash [R \cdot K_1]$, and $\mathcal{D}_2 : \vdash [T \cdot K_2]$, for some $K_1, K_2$. Both $\mathcal{D}_1$, and $\mathcal{D}_2$ are the two proofs we are looking for. The derivation is:

\[
\begin{align*}
[[V \cdot (K_1 \cdot K_2)]]_e \\
\vdash \\
[[V \cdot U]]_e \\
\vdash \\
S[V]
\end{align*}
\]

Point 3 Context Reduction (Proposition 3.4) applies to $\mathcal{P}$. So, there are $U$, and $b$ such that, for every $V$ with $\text{fn}(V) \cap \text{bn}(R \cdot a) = \emptyset$, there exist $\mathcal{D} : [[V \cdot U]]_e \vdash S[V]$, and $\mathcal{D} : \vdash [(R \cdot a) \cdot U]$. Shallow Splitting (Proposition 3.2) applies to $\mathcal{D}$. So, $\mathcal{D} : [K_a \cdot U]$, and $\mathcal{D}' : \vdash [R \cdot K]$, for some $K$. So, $\mathcal{D}'$ is the proof we are looking for. The derivation is:

\[
\begin{align*}
[[V \cdot K]]_a \\
\vdash \\
[[V \cdot [K_a \cdot U]]_e \\
\vdash \\
[[V \cdot U]]_e \\
\vdash \\
S[V]
\end{align*}
\]

The step (18) applies thanks to the assumption that $\text{fn}(V) \cap \text{bn}(R \cdot a) = \emptyset$, which implies $a \notin \text{fn}(V)$.

E Proof of Admissibility of the up fragment (Theorem 3.6 page 10)

As a first case we show that $a \uparrow$ is admissible for BVQ. So, we start by assuming:

\[
\begin{array}{c}
\mathcal{P} \\
\vdash \\
a \uparrow \\
S(a \otimes \overline{a}) \\
\vdash \\
S(\sigma)
\end{array}
\]

Point 2 of Splitting (Theorem 3.5) applies to $\mathcal{P}$, whose conclusion is $S(a \otimes \overline{a})$. There are $K_1, K_2$, and $b$ such that, for every $V$ with $\text{fn}(V) \cap \text{bn}(a \otimes \overline{a}) = \emptyset$, there exist $\mathcal{D} : [[V \cdot [K_1 \cdot K_2]]]_e \vdash S[V]$, and $\mathcal{D}_1 : \vdash [a \cdot K_1]$, and $\mathcal{D}_2 : \vdash [\overline{a} \cdot K_2]$. Shallow splitting (Proposition 3.2) on $\mathcal{D}_1$, and $\mathcal{D}_2$ implies $\mathcal{D}_1 : \overline{a} \vdash K_1$, and $\mathcal{D}_2 : a \vdash K_2$. To build the following proof with the same conclusion as $\mathcal{P}$, but without its bottommost instance of $a \uparrow$ it is enough to observe that among all the possible instances of $V$ there is $\sigma$, because
\[ \text{fn}(\varnothing) \cap \text{bn}((a \otimes \overline{a})) = \emptyset. \] So, we can prove:

\[
\begin{align*}
\varnothing \quad \text{13} \\
\alpha_1 \quad \text{13} \\
\upsilon_1 \quad \text{13} \\
\end{align*}
\]

where \( \mathcal{D}' \) is \( \mathcal{D} \) with \( V \) instantiated as \( \varnothing \).

As a second case we show that \( q^1 \) is admissible for BVQ. So, we start by assuming:

\[
\begin{align*}
\varnothing \quad \text{13} \\
\upsilon_1 \quad \text{13} \\
\end{align*}
\]

Point 1 of Splitting (Theorem 35) applies to \( \mathcal{D} \) — beware, not \( \mathcal{D}' \) —, whose conclusion is \( S ((R \cdot U) \otimes (T \cdot V)) \). There are \( K_1, K_2, \) and \( b \) such that, for every \( V' \) with \( \text{fn}(V') \cap \text{bn}(S ((R \cdot U) \otimes (U \cdot V))) = \emptyset \), there exist \( \mathcal{D} : [\varnothing \otimes (K_1 \cdot K_2)]_G \vdash S[V'] \), and \( \mathcal{D}_1 : \vdash [(R \cdot T) \otimes K_1] \), and \( \mathcal{D}_2 : \vdash [U \otimes K_2] \), and \( \mathcal{D}_3 : \vdash [V \otimes K_1] \). To build the following proof with the same conclusion as \( \mathcal{D} \), but without its bottommost instance of \( q^1 \), it is enough to observe that one of the possible instances of \( V' \) is \( (R \otimes T) \otimes (U \otimes V) \) because, thanks to (19), we can always assume \( \text{fn}((R \otimes T) \otimes (U \otimes V)) \cap \text{bn}((R \otimes T) \otimes (U \otimes V)) = \emptyset \):

\[
\begin{align*}
\varnothing \quad \text{13} \\
\upsilon_1 \quad \text{13} \\
\end{align*}
\]

where \( \mathcal{D}' \) is \( \mathcal{D} \) with \( V' \) instantiated as \( (R \otimes T) \otimes (U \otimes V) \).

As a third case we show that \( u^1 \) is admissible for BVQ. So, we start by assuming:

\[
\begin{align*}
\varnothing \quad \text{13} \\
\upsilon_1 \quad \text{13} \\
\end{align*}
\]
Point 3 of Splitting (Theorem 5.5) applies to $\mathcal{P}$ — beware, not $\mathcal{P}'$ —, whose conclusion is $S \{ (R \otimes T) \}_{\alpha}$. There is $K$ and $\delta$ such that, for every $V$ with $\text{fn}(V) \cap \text{bn}(S \{ (R \otimes T) \}_{\alpha}) = \emptyset$, there exist $\mathcal{P} : \{ [V \otimes K] \}_{\alpha, \beta} \vdash S[V]$, and $\mathcal{P}_1 : \vdash [ (R \otimes T) \otimes K ]$. Shallow splitting (Proposition 3.2) on $\mathcal{P}_1$ implies $\mathcal{S} : [K \otimes K_T] \vdash K$, and $\mathcal{P}_2 : \vdash [R \otimes K_R]$, and $\mathcal{P}_2 : \vdash [T \otimes K_T]$. To build the following proof with the same conclusion as $\mathcal{P}$, but without its bottommost instance of $\mathcal{U}$, it is enough to observe that one of the possible instances of $V$ is $S \{ (R \otimes T) \}_{\alpha}$ such that $\text{fn}(S \{ (R \otimes T) \}_{\alpha}) \cap \text{bn}(S \{ (R \otimes T) \}_{\alpha}) = \emptyset$:

$$
\begin{align*}
\text{Proof that } \text{mt}_{\downarrow} \text{ is derivable in BVQ (Lemma 5.3, page 12)}
\end{align*}
$$

We proceed by induction on the size $|\langle M \rangle_{\alpha}|$, of $\langle M \rangle_{\alpha}$, that occurs in the conclusion of $\text{mt}_{\downarrow}$, proceeding by cases on the form of $M$.

The first base case is $M \equiv x$.

$$
\begin{align*}
\text{by \{nal\}} \\
\text{by \{nal\}}
\end{align*}
$$

The second base case is $M \equiv (M') M''$.

$$
\begin{align*}
\text{by \{nal\}} \\
\text{by \{nal\}}
\end{align*}
$$

The unique inductive case is with $M \equiv \lambda y. M'$ that, without loss of generality, can have $y \neq x$.

$$
\begin{align*}
\text{by \{nal\}} \\
\text{by \{nal\}}
\end{align*}
$$

where $\text{mt}_{\downarrow}$ applies by induction because $|\langle M' \rangle_{\alpha}| < |\langle \lambda y. M' \rangle_{\alpha}|$.

$$
\begin{align*}
\text{Proof that } \text{subs} \text{ is derivable in BVQ (Lemma 5.3, page 12)}
\end{align*}
$$

We proceed by induction on the size $|\langle \{ M \} \otimes \langle N \rangle \rangle_{\alpha}|$ of $\langle \{ M \} \otimes \langle N \rangle \rangle_{\alpha}$, that occurs in the conclusion of $\text{subs}$, proceeding by cases on the form of $M$.

Let $M \equiv x$. We have three situations:
\[ N \equiv y. \]

\[
\begin{align*}
\forall u. \quad \{ x(y) \}_o & \equiv \{ y \}_o \equiv (y \cdot \overline{\text{t}}) \\
\{ (x)_o \otimes \{ y \}_o \} & \equiv \{ (x \cdot \overline{\text{t}}) \otimes (y \cdot \overline{\text{t}}) \}
\end{align*}
\]

\[ N \equiv (N') N''. \]

\[
\begin{align*}
\forall u. \quad \{ x((N')_o) \}_o & \equiv \{(N')_o \}_o \equiv [\{(N')_o \}_o \otimes [\{(N'')_o \}_q \otimes (p \cdot \overline{\text{t}})]_p \\
\{ (x)_o \otimes \{ (N')_o \}_o \} & \equiv \{ (x \cdot \overline{\text{t}}) \otimes \{ (N')_o \}_o \} \\
\{ (x)_o \} & \equiv \{ (N')_o \}_o \equiv \{ (N')_o \}_o \otimes \{ (N'')_o \}_q \otimes \{ (p \cdot \overline{\text{t}}) \}_p \\
\{ (x)_o \} & \equiv \{ (N')_o \}_o \equiv \{ (N')_o \}_o \otimes \{ (N'')_o \}_q \otimes \{ (p \cdot \overline{\text{t}}) \}_p
\end{align*}
\]

\[ N \equiv \lambda y. N' \] that, without loss of generality, can be \( y \neq x \).

\[
\begin{align*}
\forall u. \quad \{ x(\lambda y. N')_o \}_o & \equiv \{ \lambda y. N' \}_o \equiv [\{ (N')_o \}_o \otimes \{ (N'')_o \}_q \}
\{ (x \cdot \overline{\text{t}}) \otimes \{ (N')_o \}_o \} & \equiv \{ (x \cdot \overline{\text{t}}) \otimes \{ (N')_o \}_o \}
\end{align*}
\]

Let \( M \equiv \lambda y. M' \) that, without loss of generality, can always be such that \( y \neq x \).

\[
\begin{align*}
\forall u. \quad \{ x(\lambda y. M')_o \}_o & \equiv \{ \lambda y. M' \}_o \equiv [\{ (M')_o \}_o \otimes \{ (N')_o \}_q \}
\{ (x \cdot \overline{\text{t}}) \otimes \{ (M')_o \}_o \} & \equiv \{ (x \cdot \overline{\text{t}}) \otimes \{ (M')_o \}_o \}
\end{align*}
\]

where subst applies by induction because \( [\{ (M')_o \}_o \otimes \{ (N')_o \}_q] \) \( < \) \( [\{ (\lambda y. M')_o \}_o \otimes \{ (N')_o \}_q] \).

Let \( M \equiv (M') M'' \) with \( x \in \text{fv}(M') \).

\[
\begin{align*}
\forall u. \quad \{ (M'(N')_o) \}_o & \equiv \{ (M'N')_o \}_o \equiv [\{ (M')_o \}_o \otimes \{ (N')_o \}_q \}_p \}
\{ (x \cdot \overline{\text{t}}) \otimes \{ (M')_o \}_o \} & \equiv \{ (x \cdot \overline{\text{t}}) \otimes \{ (M')_o \}_o \}
\end{align*}
\]

where subst can be applied by induction because \( [\{ (M')_o \}_o \otimes \{ (N')_o \}_q] \) \( < \) \( [\{ (M') M'' \}_o \otimes \{ (N')_o \}_q] \).

Let \( M \equiv (M') M'' \) with \( x \in \text{fv}(M'') \).

\[
\begin{align*}
\forall u. \quad \{ (M' M'')_o \}_o & \equiv \{ (M'M'')_o \}_o \equiv [\{ (M')_o \}_p \otimes \{ (M'')_o \}_q \}_q \}
\{ (x \cdot \overline{\text{t}}) \otimes \{ (M')_o \}_p \} & \equiv \{ (x \cdot \overline{\text{t}}) \otimes \{ (M')_o \}_p \}
\end{align*}
\]

where subst applies by induction as \( [\{ (M'')_q \otimes \{ (N')_o \}_q] \) \( < \) \( [\{ (M') M'' \}_o \otimes \{ (N')_o \}_q] \).