Pancyclic zero divisor graph over the ring $\mathbb{Z}_n[i]$

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Abstract. Let $\Gamma(\mathbb{Z}_n[i])$ be the zero divisor graph over the ring $\mathbb{Z}_n[i]$. In this article, we study pancyclic properties of $\Gamma(\mathbb{Z}_n[i])$ and $L(\Gamma(\mathbb{Z}_n[i]))$ for different $n$. Also, we prove some results in which $L_p(\Gamma(\mathbb{Z}_n[i]))$ and $L_p(L(\Gamma(\mathbb{Z}_n[i])))$ to be pancyclic for different values of $n$.

1. INTRODUCTION

Let $R$ be a finite commutative ring with unity, $Z(R)$ the set of zero-divisors of $R$ and $Z^*(R) = Z(R) - \{0\}$. The zero-divisor graph of $R$, denoted by $\Gamma(R)$, is the graph in which the set of vertices $V(\Gamma(R))$ is $Z^*(R)$ and any two vertices $x, y \in V(\Gamma(R))$ are adjacent if and only if $xy = 0$.

It is known that the set of complex numbers forms a Euclidean domain under usual addition and multiplication of complex numbers where Euclidean norm is defined as $|a + ib| = a^2 + b^2$. The set of Gaussian integers $\mathbb{Z}[i]$ is a subset of $\mathbb{C}$ which is defined as $\mathbb{Z}[i] = \{\alpha = a + ib | a, b \in \mathbb{Z}\}$ and Gaussian norm $N(\alpha) = \alpha \overline{\alpha}$. It is obvious that a Gaussian integer is prime in $\mathbb{Z}[i]$ if its norm is prime in $\mathbb{Z}$. So the Gaussian prime can describe as follows:

1. $1 + i$ and $1 - i$ are Gaussian primes.
2. If $q$ is a prime integer such that $q \equiv 3 \pmod{4}$, then $q$ is a Gaussian prime.
3. If $p = a^2 + b^2$ is a prime for some integers $a$ and $b$ such that $p \equiv 1 \pmod{4}$, then $a + ib$, $a - ib$ are Gaussian primes.

Let $n$ be a positive integer and $\langle n \rangle$ be the principal ideal generated by $n$ in $\mathbb{Z}[i]$. Then $\mathbb{Z}[i]/\langle n \rangle \cong \mathbb{Z}_n[i]$ and if $n = \prod_{k=1}^{m} t_k^{n_k}$, then $\mathbb{Z}_n[i] \cong \prod_{k=1}^{m} \mathbb{Z}_{t_k^{n_k}}[i]$, for detail reader can see [5].

In 2008, Osba et al. [9] introduced the zero divisor graph for the ring of Gaussian integers modulo $n$, where they discussed several graph theoretic

2010 Mathematics Subject Classification. 13M99, 05C25, 05C76.

Key words and phrases. Pancyclic graph, Line graph, Zero divisor graph.
properties for $\Gamma(Z_n[i])$.
Through out the article, $p$ and $q$ represent the primes which are congruent to 1 modulo 4 and congruent to 3 modulo 4 respectively. For a connected graph $G$, the distance $d(u,v)$ is the shortest path between $u$ and $v$. A graph $G$ of order $n$ is said to be Hamiltonian if it contains a cycle of length $n$. The line graph of $G$, denoted by $L(G)$, is a graph whose vertices are the edges of $G$ and two vertices of $L(G)$ are adjacent whenever the corresponding edges of $G$ are adjacent. For basic definitions and results, we refer [2].
A graph $G$ of order $p \geq 3$ is said to be pancyclic if $G$ contains a cycle of length $n$ for every integer $n$ where $3 \leq n \leq p$. If a graph contains every cycle of even length $n$, where $4 \leq n \leq p$, then the graph is said to be bipancyclic.

2. When is $\Gamma(Z_n)$ and $\Gamma(Z_n[i])$ is Pancyclic.

In this section we discuss the cases in which the graph $\Gamma(Z_n)$ and $\Gamma(Z_n[i])$ are pancyclic.

**Theorem 2.1.** $\Gamma(Z_n[i])$ is not pancyclic for $n = 2^m, m > 1$.

**Proof.** For $n = 2^m$ where $m > 1$, Graph $\Gamma(Z_n[i])$ contains $2^{2m-2}$ pendent vertices. Since by Theorem 3 of [9], $|\Gamma(Z_{2^m}[i])| = 2^{2m-1} - 1$ but it can not contain a cycle of length $2^{2m-1} - 1$. Hence, $\Gamma(Z_{2^m}[i])$ is not a pancyclic.

**Theorem 2.2.** The graph $\Gamma(Z_{q^n}[i])$ is pancyclic if and only if $m = 2$.

**Proof.** Since $Z_q[i]$ is a field, therefore $\Gamma(Z_q[i])$ is an empty graph. For $m = 2$, $\Gamma(Z_{q^2}[i])$ is the complete graph $K_{q^2-1}$ and complete graph is always pancyclic. Now, for $m > 2$, from Theorem 8 of [10], $\Gamma(Z_{q^m}[i])$ is not Hamiltonian. Therefore, it does not contain a cycle of length $q^{2m-2} - 1$. Hence, $\Gamma(Z_{q^m}[i])$ is not pancyclic.

**Theorem 2.3.** The graph $\Gamma(Z_{q^m}[i])$ is bipancyclic if and only if $m = 1$.

**Proof.** $\Gamma(Z_{q^m}[i])$ is the complete bipartite graph $K_{p-1,p-1}$ with two sets of vertices $V_1 = \{ a + ib \}$ and $V_2 = \{ a - ib \}$. So it is a bipancyclic graph. Now, for $m > 1$, we know from Theorem 6 of [10] that $\Gamma(Z_{q^m}[i])$ is not a Hamiltonian graph. Therefore, it is not a pancyclic with $|\Gamma(Z_{q^m}[i])| = 2p^{2m-1} - p^{2m-2} - 1$, an even integer. Hence, it is not a bipancyclic graph.

It is known that for two primes $q_1$ and $q_2$ such that $q_j \equiv 3(mod 4)$ for $j = 1, 2$, $\Gamma(Z_{q_1q_2}[i]) \equiv \Gamma(Z_{q_1[i]} \times Z_{q_2[i]})$. Also, $\Gamma(Z_{q_1[i]})$ is a field, therefore $\Gamma(Z_{q_1q_2}[i]) \equiv K_{q_1^{2-1}q_2^{2-1}}$. Hence, $\Gamma(Z_{q_1q_2}[i])$ is a complete bipartite graph with $|\Gamma(Z_{q_1q_2}[i])| = q_1^2 + q_2^2 - 2$. Since there does not exist any cycle of length $q_1^2 + q_2^2 - 2$, thus, $\Gamma(Z_{q_1q_2}[i])$ is neither pancyclic nor bipancyclic.
Now, we present the well-known existence theorem (Theorem 6.3.4 of [2]) on Hamiltonian graph:

**Proposition 2.4.** If $G$ is a Hamiltonian, then for every nonempty proper subset $S$ of $V(G)$, $c(G - S) \leq |S|$.

**Lemma 2.5.** For $n = p_1p_2...p_n$ and $p_1 < p_2 < ... < p_n$ are distinct primes, $\Gamma(Z_n)$ is not Hamiltonian.

**Proof.** Suppose $n = p_1p_2...p_n$ where $p_i$'s are distinct primes. We know that the set of vertices of $\Gamma(Z_n)$ is all zero divisors of $Z_n$. Let $S = \{\alpha : \alpha \in \{(p_2p_3...p_n), 2(p_2p_3...p_n), ..., (p_1-1)(p_2p_3...p_n)\}\}$ and $H = \{p_1, 2p_1, ..., (p_2 - 1)p_1\}$, $p_1 < p_2$. Then $c(\Gamma(Z_n) - S) > |H| = p_2 - 1 > p_1 - 1 = |S|$. Follows from Proposition 2.4, $\Gamma(Z_n)$ is not a Hamiltonian.

From the Lemma 2.5, we can easily see that $\Gamma(Z_n)$ is not a pancyclic graph. Now, we will show that $\Gamma(Z_{p^m})$ is pancyclic if and only if $m = 2$.

**Lemma 2.6.** $\Gamma(Z_{p^m})$ is Hamiltonian if and only if $m = 2$ and $p$ is a prime.

**Proof.** Since for $m = 1$, $Z_p$ is a field so $\Gamma(Z_p)$ is a null graph. For $m = 2$, $\Gamma(Z_{p^2})$ is a complete graph of $p-1$ vertices, which is a Hamiltonian graph. Now, for $m > 2$, the vertex set in $\Gamma(Z_{p^m})$ is $(p-1)\{0\}$. Let $S = \{\alpha p^{m-1} : 1 \leq \alpha \leq p-1\}$ and $H = \{\alpha p \in Z_{p^m} : gcd(\alpha, p) = 1\}$. Here, elements of $H$ are only adjacent to elements of $S$. Then $c(\Gamma(Z_{p^m}) - S) > |H| > p - 1 = |S|$. Hence, $\Gamma(Z_{p^m})$ is not Hamiltonian.

**Theorem 2.7.** $\Gamma(Z_{p^m})$ is pancyclic if and only if $m = 2$.

**Theorem 2.8.** $\Gamma(Z_{p^s}^{-q})$ is not a Hamiltonian graph for all distinct prime $p$ and $q$ with $p < q$.

**Proof.** Let $S = \langle pq^2 \rangle$ and $H = \langle p \rangle \cap \langle pq \rangle$. Then $H \subseteq V(\Gamma(Z_{p^s}^{-q}))$. Now, $c(\Gamma(Z_{p^s}^{-q}) - S) > |H| = q(p - 1)(q - 1) > p - 1 = |S|$. So, it follows from Proposition 2.4, $\Gamma(Z_{p^s}^{-q})$ is not Hamiltonian.

**Theorem 2.9.** Let $R_1$ and $R_2$ be two rings and $R = R_1 \times R_2$. Then $\Gamma(R)$ is bipancyclic if and only if $R_1$ and $R_2$ are integral domains such that $|R_1| = m = |R_2|$.

**Proof.** Suppose $R = R_1 \times R_2$, where $R_1$ and $R_2$ are integral domains and $|R_1| = |R_2| = m$, then $\Gamma(R)$ is a complete bipartite graph with two vertex sets $A_1 = \{(x, 0) : x \in R_1\} \setminus \{0\}$ and $A_2 = \{(0, y) : y \in R_2\} \setminus \{0\}$. So $\Gamma(R)$ is a bipancyclic graph. Conversely, let $\Gamma(R)$ is bipancyclic. If possible, let $R_1$ be not an integral domain. Then there arises two cases:
(1) Let \( |Z(R_1)^*| = 2k \) and \( k \in \mathbb{Z}^* \). Then number of vertices in \( \Gamma(R) \) is always even. In order to prove \( \Gamma(R) \) is not Hamiltonian, consider the set \( S = \{ (x, 0) : x \in R_1 \} \) such that \( |S| = m - 1 \). Then \( c(\Gamma(R) - S) > m > m - 1 = |S| \). By proposition 2.4, \( \Gamma(R) \) is not Hamiltonian, i.e. it does not contain a cycle of even length of \( |\Gamma(R)| \). Hence, \( \Gamma(R) \) is not bipancyclic.

(2) Suppose \( |Z(R_1)^*| = 2k + 1 \) \( k \in \mathbb{Z}^* \). Here, \( |\Gamma(R)| = (2k + 3)(m - 1) \) and by proposition 2.4, it is clear that \( \Gamma(R) \) is not a Hamiltonian. If \( m \) is an odd, then \( \Gamma(R) \) is not bipancyclic and if \( m \) is even, then order of \( \Gamma(R) \) is odd. Remove one vertex from \( \Gamma(R) \) so that order of \( \Gamma(R) \) is even. Then by proposition 2.4, it is clear that \( \Gamma(R) \) is not Hamiltonian. Hence, \( \Gamma(R) \) is not bipancyclic.

Example 2.10. Take \( R_1 = \mathbb{Z}_4 \) and \( R_2 = \mathbb{Z}_5 \). Then \( R = \mathbb{Z}_3 \times \mathbb{Z}_5 \) and \( Z(R)^* = \{0, 1\} \). \( \Gamma(R) \) is a complete bipartite graph of order 6 i.e \( K_{2,4} \) but there is no cycle in \( \Gamma(R) \) of order 6.

3. When is \( L(\Gamma(\mathbb{Z}_n)) \) and \( L(\Gamma(\mathbb{Z}_n[i])) \) pancyclic?

For a commutative ring \( R \), it is clear that \( \Gamma(R) \) is connected by (Theorem 2.3 of [1]), so \( L(\Gamma(R)) \) is also connected. To characterise the graph \( L(\Gamma(\mathbb{Z}_n)) \) and \( L(\Gamma(\mathbb{Z}_n[i])) \) is pancyclic we use the following proposition.

Proposition 3.1. (Corollary 5 of [3]) Let \( G \) be a connected, almost bridgeless graph of order \( n \geq 4 \) such that \( \text{deg}(u) + \text{deg}(v) \geq \frac{(2n+1)}{3} \) for every edge \( uv \) of \( G \). Then \( L(G) \) is Hamiltonian. Moreover if \( G \not\cong C_4, C_5 \), then \( L(G) \) is pancyclic.

Corollary 3.2. (Corollary 8 of [1]) If \( G \) is a graph of diameter at most 2 with \( |V(G)| \geq 4 \), then \( L(G) \) is Hamiltonian.

Theorem 3.3. \( L(\Gamma(\mathbb{Z}_{pq})) \) is pancyclic graph for \( p < q \), where \( p \) and \( q \) are distinct primes.

Proof. Suppose \( p = 2 \) and \( q > 3 \), then \( \Gamma(\mathbb{Z}_{2q}) \) is a star graph and its line graph \( L(\Gamma(\mathbb{Z}_{2q})) \) is a complete graph of order \( q - 1 \). So \( L(\Gamma(\mathbb{Z}_{2q})) \) is pancyclic. Now, if \( p, q \) are distinct odd primes, then \( \Gamma(\mathbb{Z}_{pq}) \) is a complete bipartite graph \( K_{p-1, q-1} \) and \( \text{diam}(\Gamma(\mathbb{Z}_{pq})) \leq 2 \). Hence, by Corollary 3.2, \( L(\Gamma(\mathbb{Z}_{pq})) \) is a Hamiltonian graph. Now, for any \( uv \in E(\Gamma(\mathbb{Z}_{pq})) \), \( d(u) + d(v) = n > \frac{(2n+1)}{3} \) where \( n = |V(G)| \) and \( G \not\cong C_4, C_5 \). Thus, by Proposition 3.1, \( L(G) \) is pancyclic. □
THEOREM 3.4. \( L(\Gamma(Z_{p^{m}})) \) is a pancyclic for a prime \( p > 3 \) and \( m = 2, 3 \).

Proof. For \( m = 2 \), \( \Gamma(Z_{p^{2}}) \) is a complete graph of order \( p - 1 \) and complete graph is always bridgeless such that \( d(u) + d(v) = 2p - 4 > \frac{(2n+1)}{3} \), where \( n = |\Gamma(Z_{p^{2}})| \) and \( uv \in E(\Gamma(Z_{p^{2}})) \). Now, for \( m = 3 \), \( \Gamma(Z_{p^{3}}) \) is also bridgeless graph such that \( d(u) + d(v) \geq p^2 + p - 3 > \frac{(2n+1)}{3} \), where \( n = |\Gamma(Z_{p^{3}})| = p^2 - 1 \). Hence, for both cases, \( d(u) + d(v) \geq \frac{(2n+1)}{3} \) for every \( uv \in E(\Gamma(Z_{p^{m}})) \). Therefore, from proposition 3.1, \( L(G) \) is Hamiltonian and \( G \neq C_4, C_5 \). Thus, \( L(G) \) is a pancyclic graph.

\[ \square \]

DEFINITION 3.5. Let \( G \) be a graph of order \( p \geq 5 \). Then \( G \) is said to be an \( R \)-graph if there exist distinct \( r, s, t, u \in V(G) \) such that \( rs, st, tu, ur \in E(G) \) and for every \( v \in V(G - r - s - t - u) \), either \( rv \in E(G) \) or \( tv \in E(G) \).

LEMMA 3.6. [8] Let \( G \) be a graph of order \( p \geq 5 \). If \( G \) is an \( R \)-graph, then \( L(G) \) is pancyclic.

THEOREM 3.7. For integer \( m \geq 2 \), \( L(\Gamma(Z_{2^{m}}[i])) \) is pancyclic.

Proof. For \( m = 2 \), \( L(\Gamma(Z_{2^{2}}[i])) \) is a graph of order 7. Then it has induced complete subgraph of order 6 and a vertex of degree 2. Therefore, the graph contain all cycles of length \( k \), for \( 3 \leq k \leq 7 \). Hence, \( L(\Gamma(Z_{2^{2}}[i])) \) is pancyclic. Now, for \( m > 2 \), consider distinct four vertices \( r, s, t, u \) of \( V(\Gamma(Z_{2^{m}}[i])) \) where \( r = 2^{n-1} + i2^{n-1} \), \( s = 2^{n-1} \), \( t = 2 \), \( u = i2^{n-1} \) such that \( rs, st, tu, ur \) have an edge. Also, for every \( v \in V(\Gamma(Z_{2^{m}}[i])) \setminus \{r, s, t, u\} \) is adjacent to \( r \). Hence, the graph \( \Gamma(Z_{2^{m}}[i]) \) is an \( R \)-graph of order greater than 5. Thus, by Lemma 3.6, \( L(\Gamma(Z_{2^{m}}[i])) \) is pancyclic.

\[ \square \]

THEOREM 3.8. \( L(\Gamma(Z_{q^{m}}[i])) \) is pancyclic, for \( m \geq 2 \).

Proof. Since for the prime \( q \), \( Z_{q}[i] \) is a field, so its zero divisor graph is an empty graph. For \( m = 2 \), \( \Gamma(Z_{q^{2}}[i]) \) is a complete graph, therefore by Proposotion 3.1, \( L(\Gamma(Z_{q^{2}}[i])) \) is pancyclic. Now, for \( m > 2 \), we consider four distinct vertices \( r, s, t, u \) of \( V(\Gamma(Z_{q^{m}}[i])) \) such that \( r = q^{n-1} \), \( s = q^{\frac{1}{2}} \), \( t = iq^{\frac{1}{2}} \) and \( u = iq^{n-1} \). It is clear that \( rs, st, tu, ur \in E(\Gamma(Z_{q^{m}}[i])) \) and for every \( v \in V(\Gamma(Z_{q^{m}}[i])) \setminus \{r, s, t, u\} \) is adjacent to \( r \). Hence, the graph \( \Gamma(Z_{q^{m}}[i]) \) is an \( R \)-graph of order greater than 5. Thus, by Lemma 3.6, \( L(\Gamma(Z_{q^{m}}[i])) \) is pancyclic.

\[ \square \]

THEOREM 3.9. \( L(\Gamma(Z_{p^{m}}[i])) \) is pancyclic if and only if \( m = 1 \).

Proof. If \( m = 1 \), then \( \Gamma(Z_{p}[i]) \) is complete bipartite graph \( K_{p-1, p-1} \) and this implies it is a Hamiltonian. Now, taking \( uv \in E(\Gamma(Z_{p}[i])) \), \( d(u) + d(v) = p - 1 + p - 1 = 2p - 2 > \frac{(2n+1)}{3} \), where \( n = 2p - 2 \), then \( L(\Gamma(Z_{p}[i])) \) is Hamiltonian.
and also $G \not= C_4, C_5$, therefore by Proposition 3.1, $L(\Gamma(Z_p[i]))$ is pancyclic. If $m \neq 1$, then from (Corollary 4.4) of [7], $L(\Gamma(Z_{p^n}[i]))$ is not Hamiltonian and so $L(\Gamma(Z_{p^n}[i]))$ is not pancyclic.

\[
4. \text{ When is } L(\Gamma(Z_n[i])) \text{ pancyclic?}
\]

For $n = 2^m$, $\Gamma(Z_{2^m}[i])$ is not Hamiltonian because it contains isolated vertex $\langle (1 + i)^{2^m - 1}\rangle \setminus \{0\}$.

An important result given by G. H Fan in [6] will help to prove the Theorem 4.2.

\textbf{Theorem 4.1.} [6] \textit{Let } $G$ \textit{be a } 2-connected graph on $n > 3$ \textit{vertices and } $v, u$ \textit{be distinct vertices of } $G$. \textit{If}

\[d(v, u) = 2 \Rightarrow \max(d(v), d(u)) \geq n/2,\]

\textit{then } $G$ \textit{has a Hamiltonian cycle.}

\textbf{Theorem 4.2.} \textit{For } $m > 1$, $\Gamma(Z_{p^m}[i])$ \textit{is pancyclic graph.}

\textbf{Proof.} For $m = 1$, $\Gamma(Z_{p^m}[i]) \cong K_{p-1} \cup K_{p-1}$, which is disconnected. So $\Gamma(Z_p[i])$ is not a pancyclic. Now, for $m > 1$, $\Gamma(Z_{p^m}[i]) \cong \Gamma(Z_{p^n} \times Z_{p^n})$.

Here degree of each vertex is greater than $n/2$ where $n = |\Gamma(Z_{p^m}[i])|$. So $d(0, sp^{n-1}, 0)$, where $1 \leq s \leq p-1$. Since $d(0, sp^{n-1}, 0) = 3$, because $(0, sp^{n-1}) - (0, 1) - (1, p) - (sp^{n-1}, 0)$ or $(0, sp^{n-1}) - (p, 1) - (1, 0) - (sp^{n-1}, 0)$. Then for every $u, v \in \Gamma(Z_{p^m}[i])$, $d(u, v) = 2, \max(d(u), d(v)) \geq n/2$ hold. Also, this is a 2-connected, therefore by theorem 4.1, $\Gamma(Z_{p^m}[i])$ is Hamiltonian and edges of $\Gamma(Z_{p^m}[i])$ i.e $|E(G)| \geq n^2/4$. Hence, by Bondy [4], $\Gamma(Z_{p^m}[i])$ is either pancyclic or complete bipartite graph. Since $(0, 1) - (0, 2) - (-0, 3)$ in $\Gamma(Z_{p^m}[i])$, so it contains a cycle of length 3. Thus, $\Gamma(Z_{p^m}[i])$ is pancyclic graph.

For $\Gamma(Z_{p^m}[i])$, here every vertex $\langle q \rangle$ is adjacent to $\langle q^{n-1} \rangle$ in $\Gamma(Z_{p^m}[i])$, so $\langle q^{n-1} \rangle$ is an isolated vertex in $\Gamma(Z_{p^m}[i])$. Hence $\Gamma(Z_{p^m}[i])$ will never be pancyclic.

\textbf{5. When is } L(\Gamma(Z_n[i])) \textbf{ pancyclic?}

Since $V(\Gamma(Z_2[i])) = \{1 + i\}$, so its complement graph is $K_0$. $\Gamma(Z_{2^m}[i])$ is a complete graph and $L(\Gamma(Z_{2^m}[i]))$ is $K_0$. Now $\Gamma(Z_{2^m}[i])$, $m \geq 2$ is a disconnected graph with two component. One is isolated vertex $\{2^{m-1} + i2^{n-1}\}$ and other is connected subgraph called $H$. So, $L(\Gamma(Z_{2^m}[i])) \cong L(H)$. Similarly, $\Gamma(Z_{2^m}[i])$ is also disconnected graph in which connected component is called $H$. Then $L(\Gamma(Z_{2^m}[i])) \cong L(H)$. Now, we prove following theorem.

\textbf{Theorem 5.1.} If $n = 2^m$, $m \geq 2$, then $L(\Gamma(Z_n[i]))$ is pancyclic.
Proof. Since $\Gamma(Z_{2m}[i])$, has $2^{2m-2}$ vertices having degree 1. In $\Gamma(Z_{2m}[i])$, the connected component $H$ has complete subgraph of order $2^{2m-2}$ and other vertices of $H$ are adjacent to atleast $2^{2m-2}$ vertices. So, $\delta(H) \geq 2^{2m-2}$. Now taking for every $u, v \in H$, $\deg(u) + \deg(v) \geq 2^{2m-2} + 2^{2m-2} = 2^{2m-1} > \frac{2^{m+1}}{3}$ and $H \neq C_4, C_5$. Then by proposition 3.1, $L(H)$ is pancyclic and $L(\Gamma(Z_{2m}[i])) \cong L(H)$. Hence, $L(\Gamma(Z_{2m}[i]))$ is pancyclic.

□

Theorem 5.2. If $n = q^m$, $m \geq 3$, then $L(\Gamma(Z_n[i]))$ is pancyclic.

Proof. In $\Gamma(Z_q[i])$, the connected component is assumed to be $H$. Taking four distinct vertices $r, s, t, u$ of $V(H)$ where $r = q$, $s = q^2 + iq$, $t = q + iq^2$, $u = iq$ such that $rs, st, tu, ur$ have an edge in $E(H)$ and for every $v \in V(H) \setminus \{r, s, t, u\}$ is adjacent to $r$. So the graph $H$ is an $R$–graph. Thus by Lemma 3.7, $L(H)$ is pancyclic and hence $L(\Gamma(Z_{q^m}[i]))$ is pancyclic.

□

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