FINDING LINEAR PATTERNS OF COMPLEXITY ONE

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Abstract. We study the following generalization of Roth’s theorem for 3-term arithmetic progressions. For $s \geq 2$, define a nontrivial $s$-configuration to be a set of $s(s+1)/2$ integers consisting of $s$ distinct integers $x_1, \ldots, x_s$ as well as the averages $(x_i + x_j)/2$ ($1 \leq i < j \leq s$). Our main result states that if a set $A \subset [N]$ has density $\delta \gg (\log N)^{-c(s)}$ for some positive constant $c(s) > 0$ depending on $s$, then $A$ contains a nontrivial $s$-configuration. This improves on the previous bound of the form $\delta \gg (\log \log N)^{-c(s)}$ due to Dousse [3]. We also deduce, as a corollary, an improvement of a problem involving sum-free subsets.

1. Introduction

The celebrated Roth’s theorem [11] states that every dense subset of the integers contains a nontrivial 3-term arithmetic progression (3-AP).

Theorem (Roth). Let $0 < \delta < 1$ be a positive real. For a sufficiently large positive integer $N$, any subset $A \subset [N]$ with $|A| \geq \delta N$ must contain a nontrivial 3-term arithmetic progression.

Here we use $[N]$ to denote the set of positive integers up to $N$. In fact, Roth obtained the more precise bound $\delta \gg (\log \log N)^{-1}$. This has been subsequently improved by Szemerédi [17] and Heath-Brown [10] to $\delta \gg (\log N)^{-c}$ for some small positive constant $c$. Later Bourgain [1,2] improved this to $\delta \gg (\log N)^{-2/3+o(1)}$. Currently the best quantitative bound is due to Sanders [15]: $\delta \gg (\log N)^{-1+o(1)}$. On the other hand, a construction of Behrend shows that there exists a subset without nontrivial 3-APs with density about $\exp(-C\sqrt{\log N})$ for some constant $C > 0$.

Roth’s argument relies on a dichotomy which either guarantees the existence of a nontrivial 3-AP, or leads to a density increment in a shorter subprogression. See [19] for an exposition of Roth’s density increment argument as well as some of the later improvements.

We are interested in the following generalization of 3-APs. Fix a positive integer $s \geq 2$. An $s$-configuration is set of $s(s+1)/2$ integers

\begin{equation}
\{n_i + n_j + a : 1 \leq i \leq j \leq s\},
\end{equation}

where $n_1, \ldots, n_s, a \in \mathbb{Z}$. For example, when $s = 2$, the three integers $(2n_1 + a, n_1 + n_2 + a, 2n_2 + a)$ form a 3-AP. An $s$-configuration is said to be nontrivial if $n_1, \ldots, n_s$ are all distinct. A geometrical way to think about $s$-configurations is that they contain $s$ points on the real line and all midpoints between any two of these points.

It is natural to ask whether every dense subset of the integers contains a nontrivial $s$-configuration (for fixed $s$). This is proved by Dousse [3] using Roth’s density increment argument.
Theorem 1.1. Let $A \subset [N]$ with $|A| = \delta N$. Suppose that
\[ \delta \gg (\log \log N)^{-\frac{1}{s(s+1)}}. \]
Then $A$ contains a nontrivial $s$-configuration.

A crucial reason why Roth’s argument applies to $s$-configurations is that the set of linear forms in (1.1) has complexity 1 (regardless of the value of $s$), and thus controlled by Gowers $U^2$-norm. The Gowers uniformity norms are the main tools in Gowers’ proof [4, 5] of Szemerédi’s theorem. The notion of complexity of linear forms seems to originate from [9]; for the precise definition and related results see [18].

For comparison, the set of linear forms corresponding to $(k+1)$-APs has complexity $k-1$, and thus controlled by Gowers $U^k$-norm. While analyzing the $U^2$-norm is more or less equivalent to doing Fourier analysis, dealing with the $U^k$-norm when $k > 2$ becomes significantly more difficult. Indeed this is the main topic of the book [18]. Fortunately for us, to study $s$-configurations it is sufficient to consider the $U^2$-norm.

We now state our main result:

Theorem 1.2. Let $A \subset [N]$ with $|A| = \delta N$. Suppose that
\[ \delta \geq 100 \left( \frac{\log \log N}{\log N} \right)^{1/6s(s+1)}. \]
Then $A$ contains a nontrivial $s$-configuration.

The exponent $1/6s(s+1)$ can certainly be improved slightly by a more careful analysis. We shall not do so here. Our argument should work for other linear patterns of complexity 1 (and thus the title for this paper). However doing so in this generality involves some technical complications. We will thus focus solely on $s$-configurations.

The proof of Theorem 1.2 relies on Fourier analysis localized at certain approximate subgroups called Bohr sets. This technique is originated by Bourgain [1] in his work on finding 3-APs and developed further in [6, 14]. In order for the local Fourier analysis argument to work on $s$-configurations, we need a local analogue of Gowers $U^2$-norm; this is the main innovation of this paper.

It is worth mentioning that a weaker bound of the form $\delta \gg \exp(-c(s)\sqrt{\log \log N})$ can be obtained by following the argument of Green-Tao [8], where the problem of finding 4-APs is studied. The main idea behind their argument is in turn borrowed from Szemerédi and Heath-Brown’s work in the case of 3-APs. Our problem with $s$-configurations that are governed by Gowers $U^2$-norm is much simpler than the problem with 4-APs that are governed by Gowers $U^3$-norm. It is thus not surprising that we are able to get a better bound for $s$-configurations. In the ongoing work of Green-Tao they show that a bound of the same quality can be obtained in the case of 4-APs as well.

The improved bound in Theorem 1.2 combined with the argument of Sudakov, Szemerédi, and Vu [16], leads to an improvement in a problem involving sumfree sets.
Corollary 1.3. Any set $A \subset \mathbb{Z}$ of size $n$ contains a subset $B \subset A$ with size $|B| \geq (\log n)(\log \log n)^{1/2-o(1)}$ such that $B$ is sumfree with respect to $A$. In other words, $b + b' \notin A$ for any two distinct elements $b, b' \in B$.

Here $o(1)$ denotes a quantity that goes to zero as $n \to \infty$. For a more detailed discussion on this problem we refer the reader to [16], where the first superlogarithmic bound on $|B|$ is obtained. The bound $|B| \geq (\log n)(\log \log \log n)^c$ is obtained by Dousse [3] using Theorem 1.1. In exactly the same way, Theorem 1.2 immediately gives the bound $|B| \geq (\log n)(\log \log n)^c$ for some small positive constant $c > 0$. The improvement of the exponent $c$ is a consequence of using the Ruzsa embedding lemma instead of Freiman’s theorem to extract a dense subset of a long interval from a set with small doubling.

The rest of the paper is organized as follows. In Section 2 we collect basic properties of Bohr sets, a multi-dimensional analogue of arithmetic progressions. Theorem 1.2 is then proved in Sections 3 to 5. Section 3 develops the theory of a local analogue of Gowers $U^2$-norm; this captures the intuition of performing Fourier analysis on Bohr sets (approximate groups) instead of genuine groups. In Section 4 we establish a dichotomy which either guarantees the existence of a nontrivial $s$-configuration or leads to a large local $U^2$-norm. Finally in Section 5 we obtain a density increment from the large local $U^2$-norm and complete the proof of Theorem 1.2. The last section contains the proof of Corollary 1.3.

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2. Bohr Sets: Basic Properties

In this section we record some basic properties of Bohr sets, all of which can be found in standard texts in additive combinatorics such as [19].

Definition 2.1 (Bohr sets). Let $\theta = (\theta_1, \cdots, \theta_d) \in \mathbb{R}^d$, $0 < \epsilon < 1/2$, and $M \geq 1$. Define the Bohr set $\Lambda = \Lambda_{\theta, \epsilon, M}$ to be the set of all $n \in \mathbb{Z}$ with $|n| \leq M$ and $|n\theta_j| \leq \epsilon$ for each $1 \leq j \leq d$ (here $\|x\|$ denotes the distance from $x$ to its nearest integer). The positive integer $d$ is called the dimension of $\Lambda$. For any real number $c > 0$, we write $c\Lambda$ for the dilated Bohr set $c\Lambda = \Lambda_{\theta, \epsilon, cM}$.

For example, the interval $[-M, M]$ is the 1-dimensional Bohr set $\Lambda_{1,1/2,M}$. A simple heuristic argument shows that an integer $n \in [-M, M]$ lies in a $d$-dimensional Bohr set $\Lambda_{\theta, \epsilon, M}$ with probability $\epsilon^d$, and thus $\Lambda_{\theta, \epsilon, M}$ has expected size roughly $\epsilon^d M$. This is indeed correct as a lower bound.

Lemma 2.2 (Size bound). Let $\Lambda = \Lambda_{\theta, \epsilon, M}$ be a Bohr set of dimension $d$. Then $|\Lambda| \geq \epsilon^d M$.

Bohr sets are approximate groups in the sense that $|\Lambda + \Lambda| \approx 2^d|\Lambda|$. However, as $d$ gets large the doubling also gets large. To get around this unpleasant behavior, we will work with pairs of Bohr sets $(\Lambda, c\Lambda)$ for some small constant $c \in (0, 1)$. Heuristically we expect that $|\Lambda + c\Lambda| \approx |\Lambda|$. This leads to the following definition.
Definition 2.3 (Regular Bohr sets). A Bohr set $\Lambda = \Lambda_{\theta, \epsilon, M}$ of dimension $d$ is said to be regular if

$$1 - 100d|c| \leq \frac{|(1 + c)\Lambda|}{|\Lambda|} \leq 1 + 100d|c|$$

whenever $|c| \leq 1/100d$.

Not all Bohr sets are regular. For example, consider $\Lambda = \Lambda_{1/2, 0.499, M}$, and observe that while $\Lambda$ contains only even integers in $[-M, M]$, the dilated Bohr set $1.01\Lambda$ contains all integers in $[-M, M]$. On the other hand, it is not hard to see that $\Lambda_{1/2, \epsilon, M}$ is regular when $\epsilon$ is not too close to 0.5. In general, the following lemma shows that regular Bohr sets exist in abundance.

Lemma 2.4 (Finding regular Bohr sets). For any Bohr set $\Lambda$, there exists $\alpha \in [1/2, 1]$ such that $\alpha \Lambda$ is regular.

Suppose that $\Lambda$ is a regular Bohr set of dimension $d$, and $\Lambda' = c\Lambda$ for some small constant $0 < c < 1/100d$. As mentioned above, the approximate group structure of the pair $(\Lambda, \Lambda')$ is crucial in our argument. For example, let $f : \mathbb{Z} \to \mathbb{C}$ be a 1-bounded function (meaning that $|f| \leq 1$), and consider the average

$$\mathbb{E}_{n \in n' + \Lambda} f(n)$$

for some $n' \in \Lambda'$. The regularity of $\Lambda$ allows us to replace the range $n \in n' + \Lambda$ by $n \in \Lambda$, at the cost of a small error:

$$\mathbb{E}_{n \in n' + \Lambda} f(n) = \mathbb{E}_{n \in \Lambda} f(n) + O(cd).$$

We will use this type of estimate again and again without explicitly mentioning it.

3. The Local Gowers $U^2$-norm

In this section, we define a local Gowers $U^2$-norm and prove an inverse theorem for it. To motivate the definition, first recall the usual $U^2$-norm, defined for a 1-bounded function $f : G \to \mathbb{C}$ on a finite abelian group $G$:

$$\|f\|_{U^2}^4 = \mathbb{E}_{a, d_1, d_2 \in G} f(a) f(a + d_1) f(a + d_2) f(a + d_1 + d_2).$$

Geometrically $\|f\|_{U^2}$ counts parallelograms weighted by $f$, with the parameter $a$ representing one of the vertices of the parallelogram, and $d_1, d_2$ representing the two edges of it.

Definition 3.1 (Local Gowers $U^2$-norm). Let $\Lambda, \Lambda_1 = c_1\Lambda, \Lambda_2 = c_2\Lambda_1$ be regular Bohr sets of dimension $d$, where $0 < c_1, c_2 < 1/100d$. For a 1-bounded function $f : \mathbb{Z} \to \mathbb{C}$, its local Gowers $U^2$-norm with respect to $(\Lambda, \Lambda_1, \Lambda_2)$ is defined by

$$(3.1)\|f\|_{U^2(\Lambda, \Lambda_1, \Lambda_2)}^4 = \mathbb{E}_{a \in \Lambda} \mathbb{E}_{n_1, n'_1 \in \Lambda_1} \mathbb{E}_{n_2, n'_2 \in \Lambda_2} f(a + n_1 + n_2) f(a + n_1 + n'_2) f(a + n'_1 + n_2) f(a + n'_1 + n'_2).$$
Proof. Roughly speaking, the local Gowers $U^2$-norm counts those parallelograms with one side restricted to a narrower Bohr set $\Lambda_1$, and the other side restricted to an even narrower Bohr set $\Lambda_2$.

Clearly the local Gowers $U^2$-norm is always bounded by 1. We now turn to proving an inverse theorem, which says that if the local Gowers $U^2$-norm of a function $f$ is large, then $f$ must possess some structure.

**Theorem 3.2** (Inverse theorem for the $U^2$-norm, local version). Let $0 < \eta < 1$ be a parameter. Let $\Lambda, \Lambda_1 = c_1 \Lambda_1, \Lambda_2 = c_2 \Lambda_1$ be regular Bohr sets of dimension $d$, where $0 < c_1 \leq \eta^8/5000d$ and $0 < c_2 \leq \eta^2/400d$. If $\|f\|_{U^2(\Lambda, \Lambda_1, \Lambda_2)} \geq \eta$, then

$$ \mathbb{E}_{a \in \Lambda} \sup_{y \in \mathbb{R}} |\mathbb{E}_{n_2 \in \Lambda_2} f(a + n_2) e(n_2 y)|^2 \geq \frac{\eta^8}{40}. $$

When $\Lambda, \Lambda_1, \Lambda_2$ are replaced by a finite abelian group $G$, we recover the following inverse theorem for the usual Gowers $U^2$-norm: if $\|f\|_{U^2} \geq \eta$ for a 1-bounded function $f : G \to \mathbb{C}$, then $|\hat{f}(\gamma)| \gg \eta^4$ for some character $\gamma \in G$. This is slightly weaker than the usual inverse theorem which says that $|\hat{f}(\gamma)| \gg \eta^2$. An improvement in the exponent will lead to an improvement in the exponent in the final bound in Theorem 1.2 as well. This is, however, not our major concern.

**Proof.** By the definition (3.1), for at least $\eta^4 |\Lambda|/2$ values of $a \in \Lambda$ we have

$$ (3.2) \quad \mathbb{E}_{n_1, n'_1 \in \Lambda_1} \mathbb{E}_{n_2, n'_2 \in \Lambda_2} f(a + n_1 + n_2) \overline{f(a + n_1 + n'_2)} f(a + n'_1 + n_2) f(a + n'_1 + n'_2) \geq \frac{\eta^4}{2}. $$

Fix such an $a$ and we will work with condition (3.2). We may assume that $f$ is supported on the set $a + \Lambda_1 + \Lambda_2$, whose size is bounded by $|\Lambda_1| (1 + 100 dc_2)$. Consider

$$ (3.3) \quad \sum_{n_1, n'_1 \in \mathbb{Z}} \sum_{n_2, n'_2 \in \Lambda_2} f(a + n_1 + n_2) \overline{f(a + n_1 + n'_2)} f(a + n'_1 + n_2) f(a + n'_1 + n'_2). $$

The contributions from $n_1, n'_1 \in \Lambda_1$ are at least $\eta^4 |\Lambda_1|^2 |\Lambda_2|^2 / 2$ by (3.2). If $n_1, n'_1 \notin \Lambda_1$, then we must have $n_1, n'_1 \in \Lambda_1 + 2 \Lambda_2$ in order for the summand not to vanish. There are at most $(200 dc_2)^2 |\Lambda_1|^2$ such pairs $(n_1, n'_1)$. Hence (3.3) is at least

$$ [\eta^4/2 - (200 dc_2)^2 |\Lambda_1|^2 |\Lambda_2|^2 \geq \frac{\eta^4}{4} |\Lambda_1|^2 |\Lambda_2|^2], $$

provided that $c_2 \leq \eta^2/400d$.

On the other hand, we claim that (3.3) is equal to

$$ \int_0^1 \int_0^1 |\hat{\Lambda}_2(x)|^2 |\hat{f}(y)|^2 |\hat{f}(x + y)|^2 dxdy. $$

Indeed, the integral above can be written as

$$ \sum_{n_2, n'_2 \in \Lambda_2} \sum_{r, s, t, u \in \mathbb{Z}} f(r) f(s) f(t) f(u) \int_0^1 \int_0^1 e((n_2 - n'_2)x + (r - s)y + (t - u)(x + y)) dxdy. $$
By orthogonality the integral above vanishes unless \(u - t = n_2 - n'_2\) and \(r + t = s + u\). After making the change of variables \(n_1 = r - a - n_2\) and \(n'_1 = u - a - n_2\), it is evident that the expression above equals (3.3).

It then follows that
\[
\frac{\eta^4}{4}|\Lambda_1|^2|\Lambda_2|^2 \leq \int_0^1 \int_0^1 |\hat{\Lambda}_2(x)|^2|\hat{f}(y)|^2|\hat{f}(x + y)|^2dxdy
\leq \int_0^1 |\hat{f}(y)|^2 \sup_y \int_0^1 |\hat{\Lambda}_2(x)|^2|\hat{f}(x + y)|^2dx
\leq |\Lambda_1|(1 + 100dc_2) \sup_y \int_0^1 |\hat{\Lambda}_2(x)|^2|\hat{f}(x + y)|^2dx.
\]

Hence there exists \(y = y(a) \in \mathbb{R}\) such that
\[
(3.4) \quad \int_0^1 |\hat{\Lambda}_2(x)|^2|\hat{f}(x + y)|^2dx \geq \frac{\eta^4}{5}|\Lambda_1||\Lambda_2|^2.
\]

If we write \(f_y\) for the function \(f_y(a) = f(a)e(ay)\), then the left side above is
\[
(3.5) \quad \|\hat{\Lambda}_2 \cdot \hat{f}_y\|_2^2 = \|\hat{\Lambda}_2 \ast f_y\|_2^2 = \sum_{n \in \mathbb{Z}} |(\Lambda_2 * f_y)(n)|^2 = \sum_{n \in a + \Lambda_1 + 2\Lambda_2} |(\Lambda_2 * f_y)(n)|^2.
\]

Observe that
\[
|(\Lambda_2 * f_y)(n)| = \left| \sum_{n_2 \in \Lambda_2} f_y(n + n_2) \right| = \left| \sum_{n_2 \in \Lambda_2} f(n + n_2)e(n_2y) \right|.
\]

Combining this with (3.4) and (3.5) we get
\[
\mathbb{E}_{n \in a + \Lambda_1 + 2\Lambda_2} |\mathbb{E}_{n_2 \in \Lambda_2} f(n + n_2)e(n_2y)|^2 \geq \frac{\eta^4}{5} \cdot \frac{|\Lambda_1|}{|\Lambda_1 + 2\Lambda_2|} \geq \frac{\eta^4}{10}.
\]

Recall that the above is true for at least \(\eta^4|\Lambda|/2\) values of \(a \in \Lambda\). We have thus shown that
\[
(3.6) \quad \mathbb{E}_{a \in \Lambda} \mathbb{E}_{n \in a + \Lambda_1 + 2\Lambda_2} \sup_{y \in \mathbb{R}} |\mathbb{E}_{n_2 \in \Lambda_2} f(n + n_2)e(n_2y)|^2 \geq \frac{\eta^4}{20}.
\]

After changing the order of summation over \(a\) and \(n\) the left side above is at most
\[
\mathbb{E}_{n \in \Lambda} \sup_{y \in \mathbb{R}} |\mathbb{E}_{n_2 \in \Lambda_2} f(n + n_2)e(n_2y)|^2 + 300dc_1.
\]

The proof is completed by combining this with (3.6). □

Remark 3.3. In the proof above it is crucial that the variables \(n_2, n'_2\) are restricted to an even narrower Bohr set than the variables \(n_1, n'_1\). One might think that a more natural way to define the local Gowers \(U^2\)-norm is by restricting \(n_1, n_2, n'_1, n'_2\) to the same Bohr set. An inverse theorem for this definition is obtained by Green-Tao [7], but with a significantly more involved argument. We will be content with our slightly less orthodox definition (3.1).
4. A Local Generalized von-Neumann Theorem

Fix a positive integer $s \geq 2$. We will be looking for $s$-configurations $\{n_i + n_j + a : 1 \leq i < j \leq s\}$, where $a \in \Lambda$ and $n_i \in \Lambda$. Here $\Lambda, \Lambda_1, \cdots, \Lambda_s$ are all regular Bohr sets of dimension $d$, and $\Lambda_1 = c_1 \Lambda$, $\Lambda_i = c_i \Lambda_{i-1}$ for $2 \leq i \leq s$, where $0 < c_1, \cdots, c_s < 1/100d$. To this end, define the counting function

$$T_s(\mathcal{F}; \Lambda, \Lambda_1, \cdots, \Lambda_s) = \mathbb{E}_{a \in \Lambda} \mathbb{E}_{n_1 \in \Lambda_1} \cdots \mathbb{E}_{n_s\in \Lambda_s} \prod_{1 \leq i \leq j \leq s} f_{ij}(n_i + n_j + a),$$

where $\mathcal{F} = \{f_{ij}\}$ is a collection of 1-bounded functions.

**Proposition 4.1** (Generalized von-Neumann, local version). For any $1 \leq i < j \leq s$ we have $|T_s(\mathcal{F}; \Lambda_1, \cdots, \Lambda_s)| \leq \|f_{ij}\|U^2(\Lambda, \Lambda_1, \Lambda_j)$.

If one wants to control $T_s(\mathcal{F})$ by $f_{ii}$, it seems that a different definition of the local $U^2$-norm is needed (see Remark 3.3). Fortunately it is enough to have the result for $i \neq j$.

**Proof.** For notational convenience assume that $(i, j) = (1, 2)$; the other cases are treated in the same way. We write

$$T_s(\mathcal{F}) = \mathbb{E}_{a \in \Lambda} \mathbb{E}_{n_2 \in \Lambda_2} \cdots \mathbb{E}_{n_s\in \Lambda_s} \prod_{2 \leq i \leq j \leq s} f_{ij}(n_i + n_j + a) \mathbb{E}_{n_1 \in \Lambda_1} \prod_{j=1}^s f_{1j}(n_1 + n_j + a).$$

By Cauchy-Schwarz,

$$|T_s(\mathcal{F})|^2 \leq \mathbb{E}_{a \in \Lambda} \mathbb{E}_{n_2 \in \Lambda_2} \cdots \mathbb{E}_{n_s\in \Lambda_s} \left| \mathbb{E}_{n_1 \in \Lambda_1} \prod_{j=1}^s f_{1j}(n_1 + n_j + a) \right|^2.$$

Expanding the square and rearranging the order of summation we get

$$|T_s(\mathcal{F})|^2 \leq \mathbb{E}_{a \in \Lambda} \mathbb{E}_{n_1, n_1' \in \Lambda_1} \left| \mathbb{E}_{n_2 \in \Lambda_2} f_{12}(n_1 + n_2 + a) f_{12}(n_1' + n_2 + a) \right|^2.$$

Apply Cauchy-Schwarz again to get

$$|T_s(\mathcal{F})|^4 \leq \mathbb{E}_{a \in \Lambda} \mathbb{E}_{n_1, n_1' \in \Lambda_1} \left| \mathbb{E}_{n_2 \in \Lambda_2} f_{12}(n_1 + n_2 + a) f_{12}(n_1' + n_2 + a) \right|^2.$$

The right side above, after expanding out the square, is exactly $\|f\|U^2(\Lambda, \Lambda_1, \Lambda_2)^4$. \qed

For a Bohr set $\Lambda$, we write $2 \cdot \Lambda$ for the set $\{2n : n \in \Lambda\}$. Note that this is contained in, but usually much smaller than the dilated set $2\Lambda$.

**Corollary 4.2** (Dichotomy, local version). Let $s \geq 2$ be a positive integer. Let $\Lambda$ be a regular Bohr set of dimension $d$ and $\Lambda \subset \Lambda$ be a subset with $|\Lambda| = \delta|\Lambda|$. Let $0 < c_1, \cdots, c_s < 1/100d$ be real parameters with $c_1 \leq \delta^s/32000d^2$. Set $\Lambda_1 = c_1 \Lambda$, $\Lambda_i = c_i \Lambda_{i-1}$ for $2 \leq i \leq s$. Suppose that $\Lambda$ does not contain any nontrivial $s$-configurations. Then either one of the following statements holds:

1. $|\Lambda_s| \leq 32s^2\delta^{-(s+1)/2}$;
The density of $A$ on $a + 2 \cdot \Lambda_i$ is at least $(1 + 1/8 s^2)\delta$ for some $a \in \Lambda$ and $1 \leq i \leq s$ with $a + 2 \cdot \Lambda_i \subset \Lambda$.

(3) $\|1_A - \delta_1 A\|_{U^2(\Lambda,\Lambda_i,\Lambda_j)} \geq \delta^{(s+1)/2} / 32 s^2$ for some $1 \leq i < j \leq s$.

**Proof.** Assume that case (1) fails. Since $A$ does not contain any non-trivial $d$-configurations, we have

$$T_s(1_A) \leq \frac{s^2}{|\Lambda_1|} \leq \frac{1}{32} \delta^{(s+1)}.$$  

On the other hand, we have the decomposition into $L + 1 = \binom{s}{2} + 1$ components

$$T_s(1_A) = T_s(F_1) + \cdots + T_s(F_L) + T_s(\mathcal{G}),$$

such that each $F_k$ has $1_A - \delta 1_A$ as the $ij$-component function for some $i < j$, and $\mathcal{G} = (g_{ij})$ satisfies $g_{ij} = \delta 1_A$ for all $i < j$ and $g_{ii} = 1_A$. Using Proposition 4.1, we get

$$T_s(1_A) \geq T_s(\mathcal{G}) - s^2 \|1_A - \delta 1_A\|_{U^2(\Lambda,\Lambda_i,\Lambda_j)}$$

for some $1 \leq i < j \leq s$. To estimate $T_s(\mathcal{G})$, note that if $a \in (1 - 2c_1)\Lambda$ and $n_i, n_j \in \Lambda$, then $a + n_i + n_j \in \Lambda$. Hence by the regularity of $\Lambda$,

$$T_s(\mathcal{G}) \geq \delta^{(\frac{s}{2})} \left[ \mathbb{E}_{a \in \Lambda} \prod_{i=1}^{s} (\mathbb{E}_{n \in \Lambda_i} 1_A(2n + a)) - 200dc_1 \right].$$

Now write $\delta_i(a)$ for the density of $A$ on $a + 2 \cdot \Lambda_i$, and let $E_i$ be the set of $a \in (1 - 2c_1)\Lambda$ with $\delta_i(a) < (1 - 1/s)\delta$. Assume that case (2) of the conclusion fails, so that $\delta_i(a) \leq (1 + 1/8 s^2)\delta$ for all $a \in (1 - 2c_1)\Lambda$. Since

$$\mathbb{E}_{a \in \Lambda} \delta_i(a) = \mathbb{E}_{n \in \Lambda_i} \mathbb{E}_{a \in \Lambda} 1_A(2n + a) \geq \delta - 200dc_1 \geq (1 - 1/4s^2)\delta,$$

we have

$$(1 - 1/4s^2)\delta \leq \frac{|E_i|}{|\Lambda|} (1 - 1/s)\delta + \left(1 - \frac{|E_i|}{|\Lambda|}\right) (1 + 1/8s^2)\delta + 200dc_1.$$  

From this it follows that $|E_i| \leq |\Lambda|/2s$. Let $E$ be the union $E_1 \cup \cdots E_s$. Then $|E| \leq |\Lambda|/2$. Hence

$$T_s(\mathcal{G}) \geq \delta^{(\frac{s}{2})} \left[ \frac{1}{2} \mathbb{E}_{a \in \Lambda \setminus E} \delta_1(a) \cdots \delta_d(a) - 200dc_1 \right] \geq \delta^{(\frac{s}{2})} \left[ \frac{1}{8} \delta^s - 200dc_1 \right] \geq \frac{1}{16} \delta^{(s+1)/2}.$$  

Combining this with (4.1) and (4.2) we get case (iii) of the conclusion.

\[\square\]

Compared with the usual dichotomy in Roth’s argument, the second case above is new. However it is certainly harmless as it already implies a density increment.
5. Obtaining Density Increment in the Local Setting

The iterative procedure of attacking Theorem 1.2 is the following. Suppose that $A$ does not contain a nontrivial $s$-configuration. Then Corollary 1.2 implies that there is a large local Gowers $U^2$-norm. By Theorem 3.2 we thus have a large Fourier coefficient (in the average sense). Finally, this large Fourier coefficient leads to a density increment on a smaller Bohr set. This final step of the argument is recorded in the following lemma.

Lemma 5.1 (Large Fourier coefficient leads to density increment, local version). Let $0 < \eta < 1$ be a parameter. Let $\Lambda = \Lambda_{\eta, \varepsilon, M}$, $\Lambda_1 = c_1 \Lambda$ be regular Bohr sets of dimension $d$, where $c_1 \leq 2^{-15} \eta^3 d^{-1}$. Suppose that $E_{a \in \Lambda} f(a) = 0$. If

$$E_{a \in \Lambda} \sup_{y \in \mathbb{R}} |E_{n_1 \in \Lambda_1} f(a + n_1)e(n_1y)|^2 \geq \eta^2,$$

then either one of the following statements holds:

(1) $f$ has density increment on a translate of $\Lambda_1$: $E_{n_1 \in \Lambda_1} f(a + n_1) \geq \eta^3/128$ for some $a$ with $a + \Lambda_1 \subset \Lambda$;

(2) for any positive $c' \leq 2^{-13} \eta d^{-1}$, $f$ has density increment on some translate of a $(d+1)$-dimensional Bohr set $\Lambda' = \Lambda_{c', \varepsilon, M'}$ with $c' = c_1 c$ and $M' = c_1 M$:

$$E_{n' \in \Lambda'} f(a + n') \geq \eta/16$$

for some $a$ with $a + \Lambda' \subset \Lambda$.

Proof. For $a \in \Lambda$ write $\delta(a) = E_{n_1 \in \Lambda_1} f(a + n_1)$. Suppose that case (i) fails, so that $\delta(a) \leq \eta^3/128$ for each $a \in (1 - c_1) \Lambda$. Let $E$ be the set of $a \in (1 - c_1) \Lambda$ with $\delta(a) \leq -\eta/32$. Note that

$$E_{a \in \Lambda} \delta(a) = E_{a \in \Lambda} E_{n_1 \in \Lambda_1} f(a + n_1) \geq -200 c_1 d \geq -\eta^3/128$$

since $E_{a \in \Lambda} f(a) = 0$. This gives the upper bound $|E| \leq 3\eta^2 |\Lambda|/4$. It follows that there exists $a \in (1 - c_1) \Lambda \setminus E$ and $y \in \mathbb{R}$ such that

$$E_{n_1 \in \Lambda_1} f(a + n_1)e(n_1y) \geq \eta/2.$$

Fix such an $a$ and $y$. Define $\Lambda'$ by taking $\theta' = (\theta, y) \in \mathbb{R}^{d+1}$. Then for any $n' \in \Lambda'$,

$$|E_{n_1 \in \Lambda_1} f(a + n_1 + n')e((n_1 + n'y) - E_{n_1 \in \Lambda_1} f(a + n_1)e(n_1y)| \leq 200 c'd.$$

After averaging over $n'$ and changing the order of summation we get

$$|E_{n_1 \in \Lambda_1} E_{n' \in \Lambda'} f(a + n_1 + n')e((a + n_1 + n'y)) \geq E_{n_1 \in \Lambda_1} f(a + n_1)e(n_1y) - 200 c'd \geq \frac{\eta}{4}.$$

On the other hand, for $n' \in \Lambda'$ we have $|1 - e(n'y)| \leq 8\epsilon'$. Hence

$$|E_{n_1 \in \Lambda_1} E_{n' \in \Lambda'} f(a + n_1 + n')e((a + n_1 + n'y))| \leq E_{n_1 \in \Lambda_1} |E_{n' \in \Lambda'} f(a + n_1 + n')e(n'y)|$$

$$\leq E_{n_1 \in \Lambda_1} |E_{n' \in \Lambda'} f(a + n_1 + n')| + 8\epsilon'.$$

It then follows that

$$E_{n_1 \in \Lambda_1} |E_{n' \in \Lambda'} f(a + n_1 + n')| \geq \frac{\eta}{8}. $$
Note that
\[ \mathbb{E}_{n_1 \in \Lambda_1} \mathbb{E}_{n' \in \Lambda'} f(a + n_1 + n') \geq \delta(a) - 200c'd \geq -\frac{\eta}{32} - 200c'd \geq -\frac{\eta}{16} \]
since \( a \notin E \). Hence there exists \( n_1 \in \Lambda_1 \) such that
\[ \mathbb{E}_{n' \in \Lambda'} f(a + n_1 + n') \geq \frac{\eta}{16}. \]

\[ \square \]

**Proposition 5.2** (Iterative step for Theorem 1.2). Let \( \Lambda = \Lambda_{\theta, \epsilon, M} \) be a regular Bohr set of dimension \( d \) and let \( A \subset \Lambda \) be a subset with \(|A| = \delta|\Lambda| \). Set
\[ x_1 = 2^{-85}s^{-24}d^{-1}\delta^{6s(s+1)}, \quad x_2 = \cdots = x_s = 2^{-20}s^{-4}d^{-1}\delta^{s(s+1)}. \]
Then there exists \( c_i \in [x_1, 2x_i] \) such that the Bohr sets \( \Lambda_1 = c_1\Lambda \) and \( \Lambda_i = c_i\Lambda_{i-1} \) (\( 2 \leq i \leq s \)) are all regular. Moreover, either one of the following statements holds:

1. \( A \) contains a non-trivial \( s \)-configuration;
2. \(|\Lambda_1| \leq 32s^2\delta^{-\left(\frac{s+1}{2}\right)} \);
3. the density of \( A \) on \( a + 2 \cdot \Lambda_i \) is at least \((1 + 1/8s^2)\delta\) for some \( a \in \Lambda \) and \( 1 \leq i \leq s \);
4. the density of \( A \) on \( a + \Lambda_i \) is at least \( \delta + 2^{-54}s^{-16}\delta^{4s(s+1)} \) for some \( a \in \Lambda \) and \( 1 \leq i \leq s \);
5. there is a regular Bohr set \( \Lambda' = \Lambda_{\theta', \epsilon', M'} \) of dimension \( d + 1 \), where \( \epsilon' \geq c'c_1 \cdots c_s \epsilon \), \( M' \geq c'c_1 \cdots c_s M \) with
\[ c' \geq 2^{-37}s^{-8}d^{-1}\delta^{2s(s+1)} \]
such that the density of \( A \) on \( a + \Lambda' \) is at least \( \delta + 2^{-28}s^{-8}\delta^{2s(s+1)} \) for some \( a \).

**Proof.** Assume that case (1) fails. By Lemma 2.4 we may choose \( c_i \in [x_1, 2x_i] \) (\( 1 \leq i \leq s \)) such that \( \Lambda_1 = c_1\Lambda, \Lambda_2 = c_2\Lambda_1, \cdots \) are all regular Bohr sets. By Corollary 4.2 either we are in case (2) or (3), or
\[ \|1_A - \delta 1_{\Lambda}\|_{L^2(\Lambda_\Lambda, \Lambda_j)} \geq \delta^\left(\frac{s+1}{2}\right)/32s^2 \]
for some \( 1 \leq i < j \leq s \). By Theorem 3.2,
\[ \mathbb{E}_{a \in \Lambda} \sup_{y \in \mathbb{R}} |\mathbb{E}_{n_j \in \Lambda_j} f(a + n_j)e(n_j y)|^2 \geq 2^{-46}s^{-16}\delta^{4s(s+1)}. \]
Finally apply Proposition 5.1 to arrive at case (4) or (5). \[ \square \]

**Proof of Theorem 1.2.** Set
\[ d_0 = 1, \quad \epsilon_0 = 1, \quad M_0 = N, \quad \delta_0 = \delta/2. \]
Start with the \( d_0 \)-dimensional regular Bohr set \( \Lambda_0 = \Lambda_{1, \epsilon_0, M_0} = [-N, N] \). Let \( A_0 = A \) so that the density of \( A_0 \) in \( \Lambda_0 \) is \( \delta_0 \). We will apply Proposition 5.2 repeatedly to obtain sequences of Bohr sets \( \Lambda_0, \Lambda_1, \cdots, \Lambda_k \) and subsets \( A_1 \subset \Lambda_1, \cdots, A_k \subset \Lambda_k \) of densities \( \delta_1, \cdots, \delta_k \) satisfying the following properties:

1. (dimension bound) \( d_i \leq d_{i-1} + 1 \) for each \( i = 1, 2, \cdots, k \);
(2) (density increment) for each \(i = 1, 2, \ldots, k\) we have
\[
\delta_i \geq \delta_{i-1} + 2^{-54} s^{-16} \delta_{i-1}^{4s(s+1)};
\]
moreover if \(d_i = d_{i-1} + 1\) then
\[
\delta_i \geq \delta_{i-1} + 2^{-28} s^{-8} \delta_{i-1}^{2s(s+1)};
\]

(3) (size bound) \(\varepsilon_i \geq c\varepsilon_{i-1}\) and \(M_i \geq cM_{i-1}\) for each \(i = 1, 2, \ldots, k\), where
\[
c = s^{-100d_k^{-s}} \delta^{10s^4};
\]

(4) for each \(i = 1, 2, \ldots, k\), existence of nontrivial \(s\)-configurations in \(A_i\) implies existence of nontrivial \(s\)-configurations in \(A_{i-1}\);

(5) (end of iterations) either \(A_k\) contains a nontrivial \(s\)-configuration, or
\[
|c\Lambda_k| \leq 32s^2 \delta^{-(s+1)/2},
\]
where the constant \(c\) is defined in (5.3).

It remains to show that (5.4) does not occur. This will imply that \(A_k\) contains a nontrivial \(s\)-configuration, and thus so does the original set \(A\). By (5.1), (5.2), and (5.3), we have
\[
k \leq 2^{55} s^{16} \delta^{-4s(s+1)},
d_k \leq 2^{29} s^8 \delta^{-2s(s+1)},
\epsilon_k \geq c^k, M_k \geq c^k N.
\]

By Lemma 2.2 we have the lower bound
\[
|c\Lambda_k| \geq (c\epsilon_k)^d_k M_k \geq c^{(k+1)d_k} c^k N \geq N \exp(-2^{94} s^{27} \delta^{-6s(s+1)} \log(1/\delta)).
\]
The inequality
\[
N \exp(-2^{94} s^{27} \delta^{-6s(s+1)} \log(1/\delta)) > 32s^2 \delta^{-(s+1)/2}
\]
can be readily verified under the assumption on \(\delta\). This shows that (5.4) cannot happen. □

6. Proof of Corollary 1.3

The general strategy of proving Corollary 1.3 is the same as in [16] (see also [3]). It follows from the following proposition, just as Theorem 1.1 in [16] follows from Theorem 1.2 there.

**Proposition 6.1.** Let \(h\) be a sufficiently large positive integer. Let \(X, Y\) be two finite subsets of positive integers with
\[
h^{-29}|Y| \geq |X| \geq \exp(\exp(CH^2 \log h))
\]
for some sufficiently large constant \(C > 0\). Then \(Y\) contains a subset \(Z\) of size \(|Z| = h\) disjoint from \(X\) and is sumfree with respect to the union \(X \cup Y\). In other words, \(z_1 + z_2 \notin X \cup Y\) for distinct \(z_1, z_2 \in Z\).
Using the notation in [16], we could take $F(h) = \exp(\exp(CH^2 \log h))$, and thus Corollary 1.3 is true with the lower bound $|B| \geq g(n) \log n$, where $g(n) = cm/\log m$. Here $c$ is a small constant and $F(m) = n^{1/2}$. Thus $m$ can be taken to be $(\log \log n)^{1/2-o(1)}$. We now focus on proving Proposition 6.1.

Proposition 6.2. Let $h, X, Y$ be as in the statement of Theorem 6.1. Suppose that $Y$ does not contain any subset $Z$ of size $h$ which is sumfree with respect to $X \cup Y$. Then there is a subset $Y_3 \subset Y$ satisfying the following properties:

1. (large size) $|Y_3| \gg h^{-29}|Y|$;
2. (small doubling) $|Y_3 + Y_3| \ll h^{181}|Y_3|$;
3. the set $2 \cdot Y_3$ is disjoint from $X \cup Y$. In other words, $2y_3 \notin X \cup Y$ for any $y_3 \in Y_3$.

Proof. See Section 6 in [16].

To deduce Proposition 6.1 from Proposition 6.2, it suffices to find an $h$-configuration in $Y_3$. The small doubling property of $Y_3$ allows us to extract a subset of $Y_3$ lying densely inside an interval. This is achieved in [16] by an application of Freiman’s theorem. A more economical way of doing this is to use an embedding lemma due to Ruzsa [12, 13] (see also Lemma 5.26 in [19]).

Lemma 6.3 (Ruzsa’s embedding lemma). Let $A \subset \mathbb{Z}$ be a finite set with $|A - A| \leq K|A|$. Then there is a subset $A' \subset A$ with $|A'| \geq |A|/2$ such that $A'$ is Freiman isomorphic to a subset $A'' \subset \mathbb{Z}/NK$ with $N \ll K|A|$. In other words, there is a bijection $\phi : A' \to A''$ such that

$$a'_1 + a'_2 = a'_3 + a'_4 \iff \phi(a'_1) + \phi(a'_2) = \phi(a'_3) + \phi(a'_4)$$

whenever $a'_1, a'_2, a'_3, a'_4 \in A'$.

Proof of Proposition 6.1. As noted above, it suffices to show that $Y_3$ contains a nontrivial $h$-configuration. Since $|Y_3 + Y_3| \ll h^{181}|Y_3|$, we have $|Y_3 - Y_3| \ll h^{362}|Y_3|$ (this is a consequence of the Ruzsa triangle inequality; see Section 2.3 in [19]). By Ruzsa’s embedding lemma, there is a subset $Y_4 \subset Y_3$ with $|Y_4| \geq |Y_3|/2$ that is Freiman isomorphic to a subset $Y'_4 \subset \mathbb{Z}/NK$ with $N \ll h^{362}|Y_3|$.

Identify $\mathbb{Z}/NK$ with $[N]$ and view $Y'_4$ as a subset of the integers. To find a nontrivial $h$-configuration in $Y_3$, it suffices to find a nontrivial $h$-configuration in $Y'_4$. The density of $Y'_4$ in $[N]$ is

$$\delta = \frac{|Y'_4|}{N} \gg h^{-362}.$$ 

By Theorem 1.2 it thus suffices to show that

$$h^{-362} \gg (\log N)^{-1/20h^2}.$$ 

Simple algebra reveals that this is implied by the condition (6.1).
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