Adjustable propagating plasmons in $\alpha - T_3$ lattice-based armchair nanoribbons

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We have obtained and analyzed the electronic states, polarization function and the plasmon excitations for $\alpha - T_3$-based nanoribbons with armchair termination. The calculated plasmon dispersions strongly depend on the number of the atomic rows across the ribbon, and the presence of the energy gap between the valence and conduction bands which is also determined by the nanoribbon geometry. The bandgap was proven to have the strongest effect on both the plasmon dispersions and their Landau damping. We have also demonstrated that for a small electron doping the plasmon dispersions do not depend on the relative hopping parameter $\alpha$ of the considered $\alpha - T_3$ material in the long-wave limit and investigated the conditions when $\alpha$ becomes an important factor which strongly affects the plasmons. We believe that our new uncovered electronic and collective properties of nano-size $\alpha - T_3$ ribbons will find their applications in the field of modern electronics and nanodevices.

I. INTRODUCTION

Graphene plasmons, or the quantum collective oscillation of its free electrons has become one of the most important and quickly growing fields in connection with the unique electronic and optical properties of all recently discovered Dirac cone materials, graphene$^{11}$ and beyond. The plasmons are viewed a crucial instrument in optical manipulation, light sensing, nanoscale spectroscopy and other applications.$^{2}$ The plasmon-based optical devices demonstrated good efficiency in various frequencies, specifically in terahertz with a possibility to move them up to the visible light range by decreasing the size of a graphene sample to a nanoribbon (GNR).

One of the most crucial problems in connection with the graphene plasmonic applications is a precise estimate of plasmon frequencies or the energy range, regions of finite (or negligible) Landau damping where a plasmon is reduced into single particle excitations and its behavior in the long wave limit. After more than a decade of extensive studies,$^{3,4}$ modern many-body theory of low-dimensional materials has developed a huge arsenal of the tools of plasmon investigation, primarily based on calculating the dynamical polarization function,$^{11}$ which is also related to the static screening transport properties of the investigated material.

Graphene nanoribbons (GNR’s), or nano-size strips made from two-dimensional graphene, has become one of the hotspots in low-dimensional electronic because of their specific size and shape which are convenient and suggestive of their use in nanoelectronic devices. They also reveal a number of spectacular and technologically promising properties, as well as some unique physical phenomena which had not been observed in the corresponding bulk materials.$^{12}$ These includes exotic and non-trivial topological electronic states, (Majorana fermions,$^{13,14}$ spin–momentum correlation and locked transport channels, arrays of plasmonic nanoantennas$^{15}$ Specific electronic quantum phases could be created at the junctions of armchair nanoribbons.$^{16}$ Transport of charged carriers was studied in the networks of

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armchair nanoribbons and possibility of a reproducible field-effect transistor with higher field-effect mobilities was also demonstrated.\(^\text{12}\)

Plasmonics has become one of the central areas in low-dimensional device physics since the collective excitations could be confined inside the patterned ribbons which results in a distinct plasmon mode and strong enhancement of the external field.\(^\text{16-20}\) Specific types of plasmon excitations in NR’s have a number of important and sometimes unexpected applications in sensing and nano-imaging.\(^\text{23-24}\)

All the electronic properties of nanoribbons strongly depend on their size (atomically precise width of a ribbon) and specifically, on the type of the termination.\(^\text{29,30}\) Thus, the investigation of quantum finite-size effects and nonlocality on the plasmons, dielectric and optical response in nanoribbons and nanodisks demonstrated substantial plasmon broadening which is significantly larger for zigzag termination compared to the armchair case.\(^\text{33}\) A tunable band gap in the which is not generally present in graphene and could be only introduced in a limited range by applying an external field is essential for most of the semiconductor devices, directly depends on the width of an armchair nanoribbon and, therefore, could be set by creating and using an AGNR with a specific number of atomic cells across the ribbon.

It is important that efficient reliable and affordable techniques of the fabrication of nano-size ribbons with a given width have been developed using chemical vapor deposition, in addition to earlier existing atomically precise bottom-up fabrication based on the chemical or lithographic unzipping of carbon nanotubes which made it possible to consider the electronic properties of a nanoribbon with a fixed and precise atomic width.

Among all the newest and recently discovered two dimensional structures, \(\alpha - T_3\) model represents one of the most unusual and budding materials. Its atomic building represents a honeycomb lattice like we observed in graphene with an additional atom located at the center of each hexagon - a hub (or \(H\)) atom. The interaction strength and the resulting electron hopping integral between \(H\) and the remaining \(A\) and \(B\) rim atoms differ from such hopping coefficient between the nearest-neighbor rim atoms of hexagon. The relative hopping parameter \(\alpha = t_{\text{hub-rim}}/t_{\text{rim-rim}}\) could vary between 0 and 1. Its lowest value \(\alpha = 0\) corresponds to graphene with a completely detached set of hub atoms, and the opposite limit \(\alpha = 1\) is defined as a dice lattice. A general \(\alpha - T_3\) model is considered an interpolation between graphene and a dice lattice. A number of really existing at3 materials has been successfully fabricated.\(^\text{43}\)

Such an atomic structure results in a pseudospin-1 Dirac-Weyl Hamiltonian and the metallic (gapless) low-energy bandstructure which consists of a Dirac cone and an additional dispersionless flat band. The flat band makes the \(\alpha - T_3\) distinguished from any other Dirac materials and appears to be very stable and robust in the presence of external fields or a disorder. A energy bandgap could be also generated in a dice lattice similarly to how it was done in graphene. Recently, Dirac semifinals also demonstrates some interesting electronic phenomena close but not completely similar to \(\alpha - T_3\). The unique electron bandstructure of \(\alpha - T_3\) model leads to its unusual electronic, optical, collective, magnetic and topological properties which has been rigorously investigated over the last several years.

In spite of the fact that \(\alpha - T_3\) materials were discovered very recently, there have been a handful of crucial publications on the subject of the nanoribbons made from such materials. The group velocities and current distributions were also studied in such ribbons with both armchair and zigzag termination were investigated in Ref.\(^\text{63}\) in the presence of magnetic field. A comprehensive study in the electronic states in a dice lattice with \(\alpha = 1\) demonstrated new electronic states without a direct analogy in graphene nanoribbon for the case of zigzag edges.\(^\text{60}\) The corresponding electronic states in the presence of magnetic field was performed in Ref.\(^\text{71}\) for both armchair and zigzag terminations. A mean-field investigation of the strain effect revealed a transition antiferromagnetic to ferromagnetic with increasing \(\alpha\) in analogy with paramagnetic transition in a bulk \(\alpha - T_3\).\(^\text{72-74}\) A valley degree of freedom and lifting the valley degeneracy is also of the highest interest for both bulk and nanoribbons.\(^\text{66}\) Ferromagnetic ordering in dice ribbons was investigated in Ref.\(^\text{76}\)

Most of the existing theoretical papers on GNR plasmonics deal with a very specific case of semi-metallic gapless dispersions and low electron doping for which an approximated analytical derivation of the long-wave plasmon dispersions could be done, which are realistic and important but very limited because the Dirac Hamiltonian and \(kp\) approximation works well for the nanoribbons with sufficiently large width in which even for a moderated doping density several subbands become populated. A crucial case of a finite bandgap has never received sufficient attention and consideration even for earlier investigated graphene nanoribbons. In view of this, our primary focus for this paper is to investigate the plasmons, their dispersions and damping for the various non-trivial cases of nanoribbon widths which determine the gap in the single-particle dispersions, finite-level electron doping densities and, specifically, the dependence of the obtained plasmon excitations on the relative hopping parameter \(\alpha\).

The rest of this paper is organized in the following way. In Sec.\(^\text{11}\) we derive the phase-dependent electronic states and demonstrate that the energy dispersions are the same for graphene, dice lattice and all kinds of \(\alpha-T_3\) materials,
We limit our consideration to the low-energy states located in the vicinity of the valleys with transverse dispersion, as well as some crucial analytical expression for the long-wave limit are presented and analyzed in Sec. IV. Finally, Sec. V contains some concluding remarks, possible applications of our results and further research outlook.

The finite width and the edge termination of a ribbon establishes its all crucial electronic properties such as the quantization of the transverse electron momentum and possible existence of a bandgap in the electronic dispersions. Apart from these two bands, our numerically obtained plasmon dispersion, as well as some crucial analytical expression for the long-wave limit are presented and analyzed in Sec. IV. Finally, Sec. V contains some concluding remarks, possible applications of our results and further research outlook.

II. $\alpha$-DEPENDENT ELECTRONIC STATES IN FINITE-WIDTH NANORIBBONS

Our goal now is to calculate the wave function and the corresponding low-energy electron dispersions in a nanoribbon made from an $\alpha - T_3$ material described by a bulk pseudospin-1 Dirac-Weyl Hamiltonian

$$H^\tau - \phi_b (k) = \gamma_0 \begin{pmatrix} 0 & k_- \cos \phi & 0 \\ k_+ \cos \phi & 0 & k_- \sin \phi \\ 0 & k_+ \sin \phi & 0 \end{pmatrix} ,$$

where $k_\pm = \tau k_x \pm i k_y$ depends on the valley index $\tau$, i.e., is not the same for the two non-equivalent $K$ and $K'$ valleys. We limit our consideration to the low-energy states located in the vicinity of the valleys with $\delta K_x = (4 \pi)/(a_0) / (3 \sqrt{3})$, $\delta K_y = 0$. Relative hopping parameter $\alpha$ is related to the phase $\phi$ which is present in Eq. (1) (which is sometimes is also referred as Berry phase, even though the Berry phase in $\alpha - T_3$ has been obtained as $\pm \cos(2\phi)$) as $\alpha = \tan \phi$. Distance $a_0 = 0.142 \text{ nm}$ is the lattice constant (distance between the nearest identical atoms, such as $B$ and $B'$) and $a = a_0 / (2 \text{sin} 30^\circ) = a_0 / \sqrt{3}$ is the side of a hexagon in a lattice, as shown in Fig. 1.

In a bulk $\alpha - T_3$ model, the three solutions for the low-energy band structure $\varepsilon^{\tau - \phi}_{\gamma = \pm 1}(k) = \pm \gamma E_F$ correspond to the valence and conduction bands and are exactly similar to graphene. Apart from these two bands, Hamiltonian (1) allows for an additional solution to $\varepsilon^{\tau - \phi}_{\gamma = \pm 1}(k) = 0$ which represents a dispersionless or flat band. This general schematics with the division on the valence, conduction and the flat bands is also observed in nanoribbons. All the energy dispersions do not directly depend on the valley index $\tau$ in contrast to the phases of the corresponding electronic states.

The finite width and the edge termination of a ribbon establishes its all crucial electronic properties such as the quantization of the transverse electron momentum and possible existence of a bandgap in the electronic dispersions. The width of a ribbon $W_R$ is related to the number of the atomic rows $N_R$ across the ribbon as $W_R = a_0 / (2(N_R - 1) = a\sqrt{3}(N_R - 1)/2$, such as shown in Fig. 1. Here, $N_R$ is the total number of all atomic rows including all types (A, B and H) of the lattice atoms.

The case of a finite-width nanoribbon is distinguished primarily because there is no translational symmetry in the transverse ($x -$) directions so that $k_y \rightarrow -i \partial / \partial x$ and Hamiltonian (1) is modified as

\[ H^\tau - \phi_b (k) = \gamma_0 \begin{pmatrix} 0 & i k_0 \cos \phi & 0 \\ k_0 \cos \phi & 0 & i k_0 \sin \phi \\ 0 & i k_0 \sin \phi & 0 \end{pmatrix} , \]
\[ \mathbf{H}_r \phi(k) = \gamma_0 \begin{pmatrix} 0 & (\tau \partial / \partial x - i k_y) \cos \phi & 0 \\ (\tau \partial / \partial x + i k_y) \cos \phi & 0 & (\tau \partial / \partial x - i k_y) \sin \phi \\ 0 & (\tau \partial / \partial x + i k_y) \sin \phi & 0 \end{pmatrix}, \]  

which basically means that transverse momentum \( k_x \) cannot be introduced as a good quantum number.

The boundary conditions are obviously the same as in dice lattice \(^{20}\) and reflect the fact that all three sublattice wave functions, including the \( H \) atom, disappear at each boundary of our ribbon

\[ \varphi_\nu(x)|_{x=0} = \varphi_\nu(x)|_{x=0}, \]

\[ \varphi_\nu(x)|_{x=L} = \exp(i \delta K_x W_R) \varphi_\nu(x)|_{x=L}, \]

where \( \nu = A, B \) and \( H \). We see that boundary conditions \(^3\) contain the states for both \( K \) and \( K' \) valleys therefore, these electronic states are mixed similarly to graphene, \(^25\) and both of the valleys needs to be taken into our consideration.

Therefore, the complete wavefunctions are

\[ \Phi^\gamma_{\phi}(n|x,k_y) = \begin{cases} \Psi^\tau=1_{(K)\phi}(n|x,k_y) \\ \Psi^\tau=-1_{(K')\phi}(n|x,k_y) \end{cases}. \]  

Each of the two components in \(^4\) is related to a separate valley (\( \tau = \pm 1 \)) and is built as

\[ \Psi_{\gamma K \phi}(n|x,k_y) = \begin{cases} \varphi_A(x|\phi,n) \\ \varphi_B(x|\phi,n) \end{cases} e^{ik_y y} \]  

and

\[ \Psi_{\gamma K' \phi}(n|x,k_y) = \begin{cases} \varphi'_A(x|\phi,n) \\ \varphi'_B(x|\phi,n) \end{cases} e^{ik_y y}. \]

The unknown components of eigenstates \(^5\) and \(^6\) could be now obtained from Hamiltonian \(^2\). Specifically, the hub state \( \varphi_H(x) \) is given by \(^7\)

\[ \frac{1}{2} k_x^2 \varphi_H(x) = -\left[ \frac{\partial^2}{\partial x^2} - k_y \right] \varphi_H(x), \]

with the following general solution

\[ \varphi_H(x) = A e^{i \xi x} + B e^{-i \xi x}. \]  

While both terms in solution \(^5\) are relevant and could be a part of the sought wave function, we also keep in mind that the wave function corresponding to each valley has only one direction of the transverse momentum \( \xi \) (\( \sim e^{+i \xi x} \) for \( K \) and \( \sim e^{-i \xi x} \) for \( K' \)) and assume \( \varphi_H(x) = A e^{i \xi x} \) and \( \varphi_H(x) = A' e^{-i \xi x} \). The boundary conditions for \( \varphi_H^{(\nu)}(x) \) are as follows:

\[ A - B' = 0 \]

\[ A e^{i \xi W_R} - e^{i \delta K_x W_R} B' e^{-i \xi W_R} = 0. \]

System \(^9\) has non-trivial solutions
\[ e^{i\xi W_R} - \exp[i\delta K_x W_R] e^{-i\xi W_R} = 0 . \]  

Since \( W_R \) could be chosen arbitrarily, equation (10) is equivalent to

\[ \xi W_R = 2\pi N + \delta K_x W_R - \xi W_R, \]

where \( N = 0, \pm 1, \pm 2, \pm 3, \ldots \).

Condition (11) which determines the quantization of the transverse electron momentum \( \xi_N \) and the complete energies of our states, is equivalent to what was earlier found for both limiting cases of graphene (\( \alpha = 0 \)) \footnote{2} as a dice lattice (\( \alpha = 1 \)).

The obtained result in (12) also reflects the fact that the energy dispersions \( \varepsilon_n(k_y) \) are the same in graphene and \( \alpha - T_3 \) materials, except for the presence of a flat band. This is similar to what we earlier observed for the corresponding bulk materials.

From Eq. (13) we conclude that the transverse electron momentum is quantized as

\[ \xi_N = \frac{\pi N}{W_R} - \frac{4\pi}{3a_0} = \frac{2\pi}{3\sqrt{3}a} \left( \frac{N}{N_R + 1} - \frac{1}{3} \right) = \frac{2\pi}{3\sqrt{3}a} \frac{3N - N_R - 1}{N_R + 1}, \]  

The energy dispersions \( \varepsilon_{\pm n}(k_y) = \gamma_0 \xi^{\sigma=0}(k_y) \) are now given by

\[ \varepsilon_{\pm n}^{\sigma\pm 1}(k_y) = \pm \sqrt{k_y^2 + \xi_N(W_R)^2} = \pm \left\{ k_y^2 + \frac{\pi N}{W_R} - \frac{4\pi}{3\sqrt{3}a_0} \right\}^{1/2}. \]  

The obtained energy dispersions presented in Fig. 2 reflect a known fact that depending on the number \( N_R \) there might be either zero or finite bandgap. The gap between the valence (or conduction) and the flat band which is a half of the total bandgap is

\[ \Delta_0(N_R) = \frac{4\pi}{3\sqrt{3}a} \frac{\gamma_0}{N_R + 1} \min_N(3N - N_R - 1), \]  

where the minimal value of \( 3N - N_R - 1 \) for a given ribbon width \( N_R \) is achieved for the integer \( N^\Delta \) which is obviously zero if \( N_R + 1 \) is exactly divisible by 3. The obtained number \( N_0 = (N_R + 1)/3 \) specifies the lowest pair of metallic subbands which touch each other and the flat band at Dirac point. If this is not the case, the number of the two lowest subbands \( N^\Delta \) obtained as the least of the following two numbers:

\[ N^\Delta_{(1,2)} = \text{InP} \left( \frac{N_R + 1}{3} \right) \pm 1, \]  

where \( \text{InP}(x) \) means the integer part of a rational number \( x \). The dependence of the energy gap vs. the ribbon width \( N_R \) is presented in Fig. 2 (d). All the other (higher) subbands of the metallic dispersions are doubly degenerate, as we show in the remaining panels of Fig. 2. We also see that energy separation and the gap are large for a narrow ribbon, similarly to the case of a quantum well, which makes it difficult to populate more than one subband for an experimentally accessible electron doping density.

In all our calculations, the unit of momentum is chosen \( k_F^{(0)} = \pi/2(10^8 \text{ m}^{-1}) = 1.57 \cdot 10^8 \text{ m}^{-1} \), while the unit of length is its reciprocal \( L_0 = 1/k_F^{(0)} = 6.33 \text{ nm} \). As for the energy, its unit is \( E^{(0)} = \gamma_0 k_F^{(0)} = 93.15 \text{ meV} \) which corresponds to a plasmon frequency \( 1.42 \cdot 10^{14} \text{ Hz} \). The width of a ribbon is estimated as \( W_R = 6.27 \text{ nm} \) for \( N_R = 50 \) and \( 24.72 \text{ nm} \) for \( N_R = 200 \).

Once \( \varphi_H(x) \) is known, we can immediately find the remaining components of the wave functions from

\[ \varphi_{\alpha}^{(\sigma)}(x) = \frac{1}{k_x} \left( -i\tau \frac{\partial}{\partial x} - ik_y \right) \varphi_{H}^{(\sigma)}(x), \]

\[ \varphi_{C}^{(\sigma)}(x) = \frac{1}{k_x} \left( -i\tau \frac{\partial}{\partial x} + ik_y \right) \varphi_{H}^{(\sigma)}(x). \]
FIG. 2: (Color online) Low-energy subbands for an $\alpha − T_3$ armchair nanoribbons of different widths. Panels (a), (b) and (c) demonstrate the energy dispersions as a function of the longitudinal momentum $k_y$ for the ribbons with $N_R = 49$, 50 and 51 atomic rows, correspondingly. Plot (d) shows the energy bandgap $\Delta_0(N_R)$ between the conduction and flat bands in the nanoribbons with various number of atomic rows $N_R$.

The obtained expressions are as follows:

$$\Psi_{\sigma = \pm 1}^\tau(\phi | x, k_y) = \frac{1}{\sqrt{W_R}} \varphi_{\sigma = \pm 1}^\tau(n | x) e^{ik_y y},$$

where

$$\varphi_{\sigma = \pm 1}^\tau(n | x) = \frac{1}{\sqrt{2}} \begin{cases} \tau \cos \phi e^{-i\tau \Theta_n} \\ \sigma \sin \phi e^{i\tau \Theta_n} \end{cases} \exp[i\tau \xi_n x]$$

It is crucial to realize that only the electron/hole index $\sigma = \pm 1$ and not the subband number $N$ determines the $\pm$ sign of the energy dispersions $\epsilon_N(k_y)$.

Similarly to a bulk $\alpha − mcT_3$, a nanoribbon is expected to exhibit a zero-energy flat band because equation (7) allows for an additional solutions with $\epsilon(k) = 0$

$$\begin{bmatrix} \frac{\partial}{\partial x} - \tau k_y \\ \frac{\partial}{\partial x} + \tau k_y \end{bmatrix} \cos \phi \varphi'_A(x) + \begin{bmatrix} \frac{\partial}{\partial x} + \tau k_y \\ \frac{\partial}{\partial x} - \tau k_y \end{bmatrix} \sin \phi \varphi'_B(x) = 0$$

or, explicitly,

$$\begin{bmatrix} \frac{\partial}{\partial x} - k_y \\ \frac{\partial}{\partial x} + k_y \end{bmatrix} \cos \phi \varphi_A(x) + \begin{bmatrix} \frac{\partial}{\partial x} + k_y \\ \frac{\partial}{\partial x} - k_y \end{bmatrix} \sin \phi \varphi_B(x) = 0,$$

$$\begin{bmatrix} \frac{\partial}{\partial x} + k_y \\ \frac{\partial}{\partial x} - k_y \end{bmatrix} \cos \phi \varphi'_A(x) + \begin{bmatrix} \frac{\partial}{\partial x} - k_y \\ \frac{\partial}{\partial x} + k_y \end{bmatrix} \sin \phi \varphi'_B(x) = 0,$$
from which we see that the two equations for \( K \) and \( K' \) valley differ only by the sign of x-derivative \( \partial / \partial x \).

The two equations \([20]\) have an obvious solutions in the form of \( \varphi_\nu \sim e^{\pm i \xi x} \). However, we also keep in mind that the wave function corresponding to each valley has only one direction of the transverse momentum \( \xi \) (\( \sim e^{+i \xi x} \) for \( K \) and \( \sim e^{-i \xi x} \) for \( K' \)).

Therefore, we assume \( \varphi_A(x) = A e^{i \xi x} \), \( \varphi_B(x) = B e^{i \xi x} \), and \( \varphi'_A(x) = A' e^{i \xi x} \) and \( \varphi'_B(x) = B' e^{i \xi x} \), correspondingly. Eqs. \([20]\) could be now rewritten as

\[
\begin{align*}
\cos \phi (-i \xi + k_y) A + \sin \phi (-i \xi + k_y) B &= 0, \\
\cos \phi (+i \xi - k_y) A' + \sin \phi (+i \xi + k_y) B' &= 0,
\end{align*}
\]

where we took into account that Eqs. \([20]\) should be satisfied for all \( x \): \( 0 < x < W_R \). Another four equations come from the boundary conditions \([3]\):

\[
\begin{align*}
A - A' &= 0, \\
B - B' &= 0, \\
A e^{i \xi W_R} - e^{i \delta K_x W_R} A' e^{-i \xi W_R} &= 0, \\
B e^{i \xi W_R} - e^{i \delta K_x W_R} B' e^{-i \xi W_R} &= 0,
\end{align*}
\]

which obviously makes our system overdetermined. Eqs. \([21], [22] \) and \([24]\) are compatible and have a non-zero solution if

\[
- \sin^2 \phi \left\{ e^{i \xi W_R} - \exp[i \delta K_x W_R] e^{-i \xi W_R} \right\} (k_y^2 + \xi^2) = 0
\]

is satisfied. Since \( k_y \) could be chosen arbitrary, this is equivalent to the following condition

\[
\xi W_R = 2 \pi N + \delta K_x W_R - i \xi W_R,
\]

where \( N = 0, \pm 1, \pm 2, \pm 3, \ldots \)

or, equivalently,

\[
\xi_N = \frac{\pi N}{W_R} + \frac{1}{2} \delta K_x.
\]

Now, the wavefunctions for the flat band with \( \gamma = 0 \) could be immediately obtained as

\[
\Psi^{\tau,\phi}_{\gamma=0}(n \mid x, k_y) = \frac{1}{\sqrt{W_R}} \varphi^{\tau,\phi}_{\gamma=0}(n \mid x) e^{i k_y y}
\]

where

\[
\varphi^{\tau,\phi}_{\gamma=0}(n \mid x) = \begin{cases}
\sin \phi \left( \tau \xi_n - i k_y \right) & 0 \\
- \cos \phi \left( \tau \xi_n + i k_y \right) & \frac{e^{i \tau \xi_n x}}{k_x},
\end{cases}
\]

in which the \( K \) and \( K' \) valleys correspond to \( \tau = \pm 1 \). If we introduce the following notation \( \Theta_n = \Theta(\xi_n, k_y) \) is the angle associated with the quantized wave vector \( k_n = \{ \xi_n, k_y \} \) so that \( \Theta_n = \tan^{-1}(k_y/\xi_n) \), the result in Eq. \([30]\) could be simplified as

\[
\varphi^{\tau,\phi}_{\gamma=0}(n \mid x) = \begin{cases}
\tau \sin \phi e^{-i \tau \Theta_n} & 0 \\
- \tau \cos \phi e^{i \tau \Theta_n} & \exp[i \tau \xi_n x].
\end{cases}
\]

Finalizing the discussion of the obtained electronic states, we should say that the type, structure and dependence of the wave functions \([18]\) and \([31]\) are similar to those in a bulk material. However, the transverse momenta \( \xi_n \) and the quantum phase \( \Theta_n \) are now quantized (discrete), and the corresponding quantization rules \([28]\) account for the mixing of \( K \) and \( K' \) valleys.
III. POLARIZATION FUNCTION, PLASMON DISPERSIONS AND DAMPING

For an ideal graphene nanoribbon without edge defects, by using the standard many-body theory, the dielectric-function tensor within the random-phase approximation (RPA) can be generally written as

$$\epsilon_{\lambda,\rho}(q,\omega|E_F,\alpha) = \delta_{\lambda,\mu} \delta_{\rho,\nu} - V_{\lambda,\rho}(q) \Pi^{(0)}_{\mu,\nu}(q,\omega|E_F,\alpha),$$

where $q$ is the longitudinal transition wave number along a nanoribbon and $\omega$ is the angular frequency of a testing field. Each of $\lambda, \rho, \mu$ and $\nu$ is a composite index which includes subband number and the conduction, valence or flat band index, such as $\lambda = \{i,\sigma_i\}$ etc.

The plasmon modes of the system can be computed from the determinant of dielectric-function tensor in Eq. (32), i.e.,

$$\text{Det} \left[ \epsilon_{\lambda,\rho}(q,\omega|E_F,\alpha) \right] = \text{Det} \left[ \delta_{\lambda,\mu} \delta_{\rho,\nu} - V_{\lambda,\rho}(q) \Pi^{(0)}_{\mu,\nu}(q,\omega|E_F,\alpha) \right] = 0$$

which is in general could be related to a multi-dimensional matrix $\epsilon_{\lambda,\rho}(q,\omega|E_F,\alpha)$. The diagonal matrix elements in Eq. (32) are expected to give rise to the dispersion of individual plasmon modes, while the off-diagonal matrix elements in Eq. (32) describe the couplings between different plasmon modes.

The subband polarization function $\Pi^{(0)}_{\mu,\nu}(q,\omega|E_F,\alpha)$, defined as

$$\Pi^{(0)}_{\mu,\nu}(q,\omega|E_F,\alpha) = \frac{g_s}{2\pi} \int_{1st\ BZ} dk_y \left\{ f_0[\varepsilon_{\mu}(k_y)] - f_0[\varepsilon_{\nu}(k_y + q)] \right\} \Pi^{m,n}_{\sigma_i\leftrightarrow\sigma_i}(k_y, q|\alpha),$$
FIG. 4: (Color online) Single-particle excitation spectrum for nanoribbons with different widths. All panels (a)-(d) represent the density plots of the imaginary part of the non-interacting intra-subband polarization functions $\Pi^{(0)}_{n,n'}(q,\omega|E_F,\alpha)$ which depend on frequency $\omega$ and wave vector $q$ for $N_R = 20$, 21, 50 and 51, correspondingly. The energy bandstructure for $N_R = 20$ and $N_R = 50$ (plots (a) and (c)) is metallic, while the other two cases shown in plots (b) and (d) demonstrate finite energy bandgaps between the valence and conduction bands. Relative hopping parameter $\alpha$ was selected 0.32 (phase $\phi = \pi/10$) for all plots.

where the integral with respect to wave number $k_x$ is limited to the first Brillouin zone, $g_s = 2$ takes into account the spin degeneracy, $\delta \ll \omega$ is associated with a homogeneous diagonal-dephasing rate of electrons $\varepsilon_{\mu}(k_y)$ is the subband energy, $f_0(x) = \{1 + \exp[(x - u_0)/k_BT]\}^{-1}$ is the Fermi function for thermal-equilibrium electrons, $T$ is the temperature, and $u_0(T)$ is the chemical potential of electrons equal to the Fermi energy $E_F$ at $T = 0$ so that

$$f_0[\varepsilon_{\mu}(k_y)|E_F, T \to 0] = \left\{1 + \exp\left[\frac{\varepsilon_{\mu}(k_y) - E_F}{k_BT}\right]\right\}^{-1} \quad \rightarrow \delta_{\sigma\mu,-1} + \delta_{\sigma\mu,1} \Theta[E_F - \varepsilon_{\mu}(k_y)],$$

where $\Theta(x)$ is a Heaviside step function.

A crucial simplification comes from the fact that if the mirror symmetry between conduction and valence bands is maintained, as for ideal graphene nanoribbons, the orbital part of wave function becomes independent of band index. Consequently, the Coulomb interaction in Eq. (32) is simplified to

$$V^{j,m}_{j',m'}(q) = \frac{e^2}{2\pi\epsilon_0\epsilon_b} \int_0^1 du \int_0^1 du' \cos[\pi(j - m)u] \cos[\pi(j' - m')u'] K_0(|q_x|W|u - u'|)$$

is non-negligible only for $j - j' = 0$ and $m - m' = 0$, as demonstrated in Fig. 3 (a). The relevant matrix elements also do not depend on the band index $\sigma$ or phase $\phi$ and, therefore, must be essentially the same for all kinds of $\alpha - T_3$ materials including graphene. In the long-wave limit $q \to 0$, the Bessel function of the second kind $K_0(|q_x|W|u - u'|)$ diverges as $-\log(x)$ which looks qualitatively similar to $\simeq 1/q$ in two dimensions. The Coulomb matrix element combines two initial and final states with the same valley and sublattice indices.
FIG. 5: (Color online) Plasmon dispersions for nanoribbons with different number $N_R$ of atomic rows across. The density plots of the spectral loss function $S(q, \omega | E_F, \alpha) = -\text{Im} [1/(q, \omega | E_F, \alpha)]$ are shown for (a) $N_R = 20$, (b) $N_R = 21$, (c) $N_R = 50$ and (d) $N_R = 51$ as a function of frequency $\omega$ and wave vector $q$. Parameter $\alpha$ was chosen 0.32 (phase $\phi = \pi/10$) for all plots.

Our initial equation (32) for the dielectric function is now reduced to

$$\epsilon_{\lambda=\mu}^{\lambda=\mu}(q, \omega | E_F, \alpha) = \delta_{\sigma_\mu, \sigma_\nu} \delta_{m,n} - V_{0}^0(q) \Pi^{(0)}_{\nu,\nu}(q, \omega | E_F, \alpha),$$

which is a regular two-dimensional matrix $(3N \times 3N)$ where $N_{\text{max}}$ is the number of subbands which we take into consideration. Only intra-subband $(\mu = \nu)$ transition contribute into the polarization function $\Pi^{(0)}_{\mu,\mu}(q, \omega)$.

The obtained matrix in (37) could be even more simplified because the elements depend only on the row index, i.e, all the off-diagonal elements in the same row are identical. $\Pi^{(0)}_{\nu}(q, \omega)$

The determinant of a matrix with such composition

$$\text{Det} \left[ \delta_{\mu,\nu} - V_{0}^{0}(q) \Pi^{(0)}_{\nu,\nu}(q, \omega | E_F, \alpha) \right] =$$

$$\sum_{\mu_1, \mu_2, \ldots, \mu_{3N_{\text{max}}}} \text{L}_\epsilon(\mu_1, \mu_2, \ldots, 3N_{\text{max}}) \epsilon_\nu^{\mu}(q, \omega | E_F, \alpha)_{1,\mu_1} \cdots \epsilon_\nu^{\mu}(q, \omega | E_F, \alpha)_{3N_{\text{max}}, \mu_{3N_{\text{max}}}},$$

where $\text{L}_\epsilon(\mu_1, \mu_2, \ldots, 3N_{\text{max}})$ is the Levi-Civita tensor, is simplified to a single summation over the only remaining composite index $\nu = \{\sigma_\nu, n\}$

$$\text{Det} \left[ \epsilon_\nu^{\mu}(q, \omega | E_F, \alpha) \right] = 1 - V_{0}^{0}(q) \sum_{n=1}^{N_{\text{max}}} \Pi^{(0)}_{n,n}(q, \omega | E_F, \alpha),$$

where $\Pi^{(0)}_{n,n}(q, \omega)$ is the subband polarizability which already includes the summation over the valence, conductance and flat bands.
FIG. 6: (Color online) Polarization function $\Pi_{n,n}^{(0)}(q, \omega | E_F, \alpha)$, single-particle excitation spectrum and plasmon excitation dispersions for a wide metallic nanoribbon with $N_R = 200$ and $N_0 = (N_R + 1)/3 = 67$. Panel (a) shows a (unrealistic) situation when all the overlaps are equal to unity $O_{n,n}^\sigma (k_y, q) (\sigma_1, z = -1, 0, +1)$ and equally contribute to the polarization function $\Pi^{(0)}(q, \omega | E_F, \alpha)$. Plot (b) shows the contribution by the transitions from and to the flat band only by setting $O_{n,n}^\sigma (k_y, q) = 0$ for a dice lattice with $\phi = \pi/4$. Panels (c) and (d) display the imaginary and real part of $\Pi^{(0)}(q, \omega | E_F, \alpha)$ for a dice lattice when all the nine possible electron transitions ($\pm 1 \leftrightarrow \pm 1, 0 \leftrightarrow \pm 1, \pm 1 \leftrightarrow 0$ and $0 \leftrightarrow 0$) are taken into account. Plots (e) and (f) demonstrate the plasmon dispersions as the peaks of the spectral loss function $S(q, \omega | E_F, \alpha)$ for graphene with $\phi = 0$ and a dice lattice with $\phi = \pi/4$, correspondingly.

$$\Pi_{n,n}^{(0)}(q, \omega | E_F, \alpha) = \sum_{\sigma = \pm 1, 0} \Pi_{n,n}^{(0)}(q, \omega | E_F, \alpha).$$

This is especially straightforward to verify for the case of a smallest possible $2 \times 2$ matrix...
FIG. 7: (Color online) Precise adjustments (or fine-tuning) of the plasmon dispersions in $\alpha - T_1$-based nanoribbons with armchair edges. Panel (a) demonstrates the plasmons for different values of the electron doping densities $n = \ldots$ and ... which correspond to the Fermi energies $0.2 E^{(0)}$ and $2.0 E^{(0)}$. (b) shows the plasmon dispersions for graphene ($\alpha = 0$) and a dice lattice ($\alpha = 1$) for a wide nanoribbon with $N_R = 200$ atomic rows and $E_F = 1.0 E^{(0)} = 93.15 \text{meV}$. Inset (i1) shows the schematic representation of all the allowable electron transitions in a narrow ($N_R = 20$) metallic electron-doped $\alpha - T_1$ ribbon, and (i2) describes the single particle excitation spectrum for a wide metallic graphene nanoribbon with $N_R = 20$ and $\alpha = 0$.

\[
\text{Det} \begin{bmatrix}
-1 - V_0^0(q) \Pi_{1,1}^{(0)}(q, \omega | E_F, \alpha) & -V_0^0(q) \Pi_{2,1}^{(0)}(q, \omega | E_F, \alpha) \\
-V_0^0(q) \Pi_{1,1}^{(0)}(q, \omega | E_F, \alpha) & 1 - V_0^0(q) \Pi_{2,2}^{(0)}(q, \omega | E_F, \alpha)
\end{bmatrix} = \sum_{\mu = 1}^{3N_{\text{max}}} (-1)^{\mu + 3N_{\text{max}}} e_{3N_{\text{max}}}^0 M_{\mu,3N_{\text{max}}} =
\]

Relation (39) for a general $N \times N$ could be proven using the method of induction. We begin from the Laplace expansion of matrix (37) over its last row $\left\{ -V_0^0(q) \Pi_{3N_{\text{max}},3N_{\text{max}}}, -V_0^0(q) \Pi_{3N_{\text{max}},3N_{\text{max}}} \right\}$:

\[
\text{Det} \left[ e_{\nu}^\alpha(q, \omega | E_F, \alpha) \right] = \sum_{\mu = 1}^{3N_{\text{max}}} (-1)^{\mu + 3N_{\text{max}}} e_{3N_{\text{max}}}^0 M_{\mu,3N_{\text{max}}} =
\]

where the corresponding minor matrices are calculated as

\[
M_{\mu,3N_{\text{max}}} = -V_0^0(q) \Pi_{\mu,3N_{\text{max}}}^{(0)}(q, \omega | E_F, \alpha) \quad \text{for} \; \mu \neq 3N_{\text{max}}
\]

and

\[
M_{3N_{\text{max}},3N_{\text{max}}} = 1 - V_0^0(q) \sum_{\mu = 1}^{3N_{\text{max}} - 1} \Pi_{\mu,3N_{\text{max}}}^{(0)}(q, \omega | E_F, \alpha).
\]

The validity of Eq. (44) is assumed as the base of the induction. The complete summation in Eq. (42) amounts to expression (39). However, we must say that a complete proof of (39) falls off the scope of the present paper. Finally, the determinant $\text{Det}$ and the trace $\text{Tr}$ of matrix (39) are connected as

\[
\text{Det} \left[ e_{\nu}^\alpha(q, \omega | E_F, \alpha) \right] = \text{Tr} \left[ e_{\nu}^\alpha(q, \omega | E_F, \alpha) \right] - 3N_{\text{max}} + 1,
\]

which was also employed in Ref. [80] for graphene. In our calculation, the actual polarization function is obtained as

\[
\Pi^{(0)}(q, \omega | E_F, \alpha) = \sum_{n = N_0 - 10}^{N_0 + 10} \Pi_{n,n}^{(0)}(q, \omega | E_F, \alpha)
\]
around the lowest subband \( N_0 \).

Our next step is to calculate the wave function overlaps (prefactors) \( O_{n,n'}(k_y,q) \) which enter Eq. (34) for the polarization function. For a \( \alpha - \mathcal{T}_3 \) nanoribbon, it is calculated as

\[
O_{\pm 1,0\leftrightarrow\pm 1,0}(k_y,q) \equiv \left| \langle \Phi^{\gamma}_n \left| k_y \right| \Phi^{\gamma'}_{n'} \left| k_y + q \right| \rangle \right|^2
\]  

(47)

where the complete wave functions \( \Phi^{\gamma}_n \left| k_y \right| \) and \( \Phi^{\gamma'}_{n'} \left| k_y + q \right| \) given by Eq. (4).

The wave function overlaps \( O_{\pm 1,0\leftrightarrow\pm 1,0}(k_y,q) \) obtained from Eq. (47)

\[
O_{\pm 1\leftrightarrow1}(k_y,q) = \frac{1}{4} \left[ 1 \pm \cos[\Theta_{n,n'}(k_y,q)] \right]^2 + \frac{1}{4} \cos^2(2\phi) \sin^2[\Theta_{n,n'}(k_y,q)]
\]

(48)

and

\[
O_{\pm 1\leftrightarrow1}(k_y,q) = \frac{1}{2} \sin^2(2\phi) \sin^2[\Theta_{n,n'}(k_y,q)]
\]

(49)

are the same as we earlier obtained for the bulk \( \alpha - \mathcal{T}_3 \) since \( O_{\pm 1,0\leftrightarrow\pm 1,0}(k_y,q) \) do not depend on the valley index \( \tau \) and for an armchair nanoribbon wave function (4) is just a combination of the states from the valleys. Therefore, an overlap (47) is essentially an average between the two inequivalent valleys. At the same time, form factors (4) directly depend on the band indices so that the obtained result (48) and (49) are completely different for the various type of carrier transitions (from and to valence, flat and conduction bands).

For graphene with \( \alpha = 1 \), we easily recover \( O_{\pm 1,\pm 1\leftrightarrow1}(k_y,q) = 1/2(1 \pm \cos[\Theta_{n,n'}(k_y,q)]) \) obtained in Ref. [24]. For the opposite limiting case of a dice lattice, we obtain \( 1/4(1 \pm \cos[\Theta_{n,n'}(k_y,q)])^2 \).

We should emphasize that overlaps (48) and (49) are mainly responsible for all the crucial difference between the plasmons in nanoribbons and bulk \( \alpha - \mathcal{T}_3 \) materials. Angle \( \Theta_{n,n'}(k_y,q) \) between the wave vectors \( k_y \) and \( k_y + q \) is quantized due to the quantization of allowable values of the transverse momenta \( \xi_n(\gamma) \). The longitudinal momenta \( k_y \) and \( k_y + q \) are always directed along the \( y \)-axis only. Since the biggest contribution to the polarization function comes for several lowest subbands with the minimal values of \( \xi_n(\gamma) \), the calculated angles \( \Theta_{n,n'}(k_y,q) \) are often close to 0 or \( \pi \) so that large number of relevant overlaps are equal to either 1 or 0, especially in the long wave limit with a small transfer momentum \( q \). This is drastically different from the corresponding bulk material.

IV. RESULTS AND DISCUSSION

We begin our discussion with the case of an armchair \( \alpha - \mathcal{T}_3 \) nanoribbon with metallic (gapless) subbands in which in which the lowest and the next consequent subbands are far removed. [22][23][25] In general, the situation with the energy separation for the quantized subbands in a finite-width ribbon is qualitatively similar to a quantum well, i.e., such well-separated levels are found for a narrow ribbon.

The transverse momenta for the electrons at the Fermi surface are negligible and the angle \( \theta_n \) associated with wave vector \( \{\xi_n, k_y\} \) are

\[
\theta_n(k_y) = \begin{cases} 
-\pi/2 & \text{if } k_y < 0 \\
\pi/2 & \text{if } k_y > 0 
\end{cases}
\]

(50)

and for angle \( \Theta_{n,n'}(k_y,q) \) between the two different states \( \{\xi_n, k_y\} \) and \( \{\xi_n, k_y + q\} \) which enters the prefactors (48) and (49)

\[
\Theta_{n,n'}(k_y,q) = \begin{cases} 
-\pi & \text{if } k_y > 0 & k_y + q < 0 \\
0 & \text{if } k_y(k_y + q) > 0 \\
\pi & \text{if } k_y < 0 & k_y + q > 0
\end{cases}
\]

(51)
In all these cases, \( \sin[\Theta_{n,n'}(k_y, q)] = 0 \), \( \Theta_{0+1}^{n,n'}(k_y, q) = 0 \) and \( \Theta_{\pm 1+1}^{n,n'}(k_y, q) = 1/4(1 \pm \cos[\Theta_{n,n'}(k_y, q)])^2 \) is equal to either 0 or 1 according to

\[
\Theta_{\sigma+1}^{n,n'}(k_y, q) = \begin{cases} 
0 & \text{if } \sigma k_y(k_y + q) < 0, \\
1 & \text{if } \sigma k_y(k_y + q) > 0. 
\end{cases}
\]

(52)

The obtained polarization function does not depend on \( \phi \), and the whole situation is exactly similar to what was earlier found for graphene. \(^{27,80}\)

Therefore, the plasmons in the long-wave limit for \( \alpha - T_3 \) are given by

\[
\Omega_p(q) = \sqrt{-q \left( q + \frac{2E_F}{\gamma_0} \right) \ln(q W_R)}. 
\]

(53)

The obtained result implies that for the case described above the plasmon dispersions in the long-wave limit do not depend on \( \alpha \) and are exactly the same for all \( \alpha - T_3 \) lattices, including the limiting cases of a dice lattice and graphene.

For all the other cases, wave function overlaps \(^{[18]}\) and \(^{[19]}\) are finite and depend on \( q \). This occurs if angle \( \Theta_{n,n}(k_y, q) \) different from 0 and \( \pi \). This is achieved for a finite \( \xi_n \) disregarding of the ribbon width. For a wide ribbon, the situation is close to a bulk \( \alpha - T_3 \) since the increment in \( \xi_n \) between the subband levels is decreased. The effect of the transfer wave vector \( q \) which is normally considered limited to \( \sim k_F \) or much less than that in the long wave limit also depends on the magnitude of initial \( k_y \) value and its relation to \( \xi_n \), as we can see in Fig. 3.

The difference between graphene and other types of \( \alpha - T_3 \) lattices obviously originates from the transitions associated with the flat band. \(^{[3]}\) Since the form factors \(^{[19]}\) for such transitions are \( \sim \sin^2(2\phi) \sin^2(\Theta_{n,n}(k_y, q)) \), they are severely suppressed for all \( \alpha \equiv 0 \). i.e., for all materials close to graphene and for small transverse components \( \xi_n \) which is always the case for a narrow ribbon. Therefore, we conclude that the effect of parameter \( \alpha \) in our considered range of \( q \) is noticeable only for a sufficiently wide ribbon. If \( q \) is set large enough \( \sim 10k_F \) (see Fig. 5 (d)) overlaps \(^{[19]}\) become observable for any width of the ribbon.

The real and imaginary parts of the dynamical polarization function for armchair nanoribbons are presented in Fig. 4 and the corresponding plasmon dispersions - in Fig. 5. We see that energy bandgap plays a crucial role in shaping the plasmon and its damping for all widths of the nanoribbon. The single-particle excitation region is split into several areas corresponding to the different subbands in addition to the main split separating \(-1 \leftrightarrow 1\) and 2DEG with two parabolic edges.

The plasmon frequencies are decreased in the presence of the gap for all \( q \). This effect has some similarity to \( \sim (1 - \Delta_0/E_F^2) \) dependence in the bulk, but for a nanoribbon this reduction is less. For larger values of \( q \), the plasmon branch tends to the lower, intra-band section of the particle-hole modes and a special concave-convex shape of the plasmon dispersion curve is found (see Fig. 5 (b) and (d)).

For larger \( N_R \), the upper inter-band single particle excitation region (Inter-SPE) is observed within our range of the wave vector \( q \) and frequency \( \omega \). In the case of zero gap, the single-particle excitation regions are reduced into a small area along the main diagonal \( \omega = \gamma_0 q \) and the plasmon is not damped.

Fig. 6 shows the effect of relative hopping parameter \( \alpha \) for a wide nanoribbon with \( N_R = 200 \). The energy separation between the two closest subbands reduces to \( \sim 0.7E_F^0 \) which comparable with the regular doping density. As a result, several subbands become populated and the situation resembles a bulk material.

First, we examine a (unrealistic) situation when all the form factors are equal to unity. We clearly see the contributions from all possible types of transitions and a strong peak around the main diagonal \( h\omega = \gamma_0 q \). When the flat band contribution become important, a plasmons encounters additional Landau damping due to a particle-hole mode located at \( h\omega \geq E_F \) which appears due to the electron transition between the conduction and flat bands. The contribution from such transitions shown in Fig. 6 (b) is not uniform but is built of several discrete pieces due to each subband level. Its contribution to \( \text{Im} \left[ \Pi_{0,n}^{(0)}(q, \omega) \right] \) at the Fermi level is about 1/2 of its maximum value. It obviously increases at larger frequencies which is expected since more levels become relevant. Also, it is larger for larger values of wave vector \( q \) for the same frequency. Such a strong plasmon damping at relatively low \( q \) constitutes the most drastic difference between graphene and \( \alpha - T_3 \) for both bulk and the nanoribbons (see Figs. 6 (e) and (f)). The plasmon branch also demonstrates a special shape with a pinching to a single point \( \Omega_p = q = E_F \). The strength of this pinching is increased for a larger \( \alpha \) (see Fig. 6 (b)).

Importantly, we see that the plasmon frequency is much less sensitive to the electron doping (doping density or the chemical potential) than in the case of bulk graphene or a dice lattice, as shown at Fig. 7 (a). Indeed, the most crucial \( E_F \)-dependent term of the polarization function \(^{[34]}\) \( 1 \leftrightarrow 1 \) which monotonically depends on \( E_F \) and would disappear if \( E_F = 0 \) in fact becomes independent of \( E_F \) if \( E_F < q \) since this is always a transition from an occupied to a free state or zero. Also, there is no averaging by the direction which we needed to perform in bulk \( \alpha - T_3 \).
V. SUMMARY AND REMARKS

In this paper, we have calculated the dynamical polarization function, plasmon dispersions and Landau damping for the various types of α − T₃-based armchair nanoribbons. Specifically, we have considered the principally different cases of zero and finite energy gap between the valence and conduction bands which is regulated by the width or the number of atomic rows across the ribbon.

In the case of a finite gap, the inter- and intra-band portions of the particle-hole modes are split into two separate regions similarly to the bulk case. However, for a finite-width ribbon each mode is also divided into the distinguished discrete areas which correspond to the quantized energy levels of the single-electron dispersions. The location and intensity of these separate regions depends on the width of the ribbon. In particular, they could be very far removed for a narrow AGNR. The plasmon branch is located at the lower frequencies for the case of a finite gap but this effect is less noticeable compared to bulk graphene or a dice lattice.

The most substantial difference between the bulk α − T₃ and a ribbon stems from the wave function overlaps (prefactors) which depend on a single subband index n and most importantly become equal or close to zero for a number of relevant transitions in the case of a small ribbon width. Thus, we see that for NR ≤ 20 the obtained plasmon dispersion show almost no dependence on relative hopping parameter α, i.e., the situation is nearly the same for graphene and all types of α − T₃ materials including a dice lattice. However, more energy levels need to be taken into account in a wider ribbon since for a finite-level doping several subband could become populated. Apart from that, one cannot rely on the Dirac Hamiltonian approximation for a very narrow ribbon. The previously considered situation with a small doping and a large energy separation between the lowest metallic and its nearest neighbor subband is important but limited and cannot be applied for most realistic problems which motivated our present study.

We have also considered a wide ribbon with NR = 200 in which the 0 ↔ 1 transition associated with the flat band become substantial, and found a strong dependence of the obtained plasmons on α. First, the plasmon branch is damped at the Fermi level because of an additional piece of the particle hole modes. We also see a pinching of the plasmon dispersions around the Fermi level. Similar phenomena have been earlier observed in the bulk α − T₃ but in a ribbon these effects could be regulated from negligible to a very strong level by adjusting the ribbon width and the bandgap.

In conclusion, we have performed a comprehensive study of the plasmon excitations in α − T₃ nanoribbons and uncovered some very special types of dependence of the plasmon frequencies and the regions of finite Landau damping on the width of the ribbon, absence or presence of the energy bandgap, electron doping density and parameter α of an α − T₃ lattice. We are confident that these novel and earlier unseen plasmon dispersions will be widely employed in creating a new generation of nanoribbon-based electronic, optical and plasmonic devices.

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