EXPLICIT MOTIVIC MIXED ELLIPTIC CHABAUTY-KIM

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ABSTRACT. The main point of the paper is to take the explicit motivic Chabauty-Kim method developed in papers of Dan-Cohen–Wewers and Dan-Cohen and the author and make it work for non-rational curves. In particular, we calculate the abstract form of an element of the Chabauty-Kim ideal for \(\mathbb{Z}[1/\ell]\)-points on a punctured elliptic curve, and lay some groundwork for certain kinds of higher genus curves. For this purpose, we develop an “explicit Tannakian Chabauty-Kim method” using \(\mathbb{Q}_\ell\)-Tannakian categories of Galois representations in place of \(\mathbb{Q}\)-linear motives. In future work, we intend to use this method to explicitly apply the Chabauty-Kim method to a curve of positive genus in a situation where Quadratic Chabauty does not apply.

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1. Introduction

1.1. Extended Abstract. The main point of the paper is to take the explicit motivic Chabauty-Kim method developed in \cite{DCW16} and \cite{CDC20} and make it work for non-rational curves. In particular, we calculate the abstract form of an element of the Chabauty-Kim ideal for $\mathbb{Z}[1/\ell]$-points on a punctured elliptic curve (Theorem 1.4 and \S 11.4) and lay some groundwork for certain kinds of higher genus curves (\S 7.6). Some important themes include:

- Use of the Bloch-Kato conjectures and Poitou-Tate duality for explicitly bounding dimensions of Selmer groups (\S 2.3, \S 5, \S 7)
- Use of $\mathbb{Q}_\ell$-linear categories of Galois representations in place of $\mathbb{Q}$-linear motives (\S 4)
- Methods for dealing with primes of bad reduction via results of Betts–Dogra in [BD20] (\S 3.5, \S 12.2.1)
- A setup for Tannakian Selmer varieties (\S 8) and a description in terms of cocycles (Theorem A.4) generalizing \cite[Proposition 5.2]{DCW16}
- An $\mathfrak{p}$-adic period map for mixed elliptic motivic periods (\S 9.3)

1.2. The Problem of Effective Faltings. Let $X$ be a smooth proper curve of genus $g \geq 2$ over a number field $k$. The theorem of Faltings states that $X(k)$ is finite. A major open question is to find an algorithm for determining the finite set $X(k)$ given $X/k$.

More generally, the combination of the theorems of Faltings and Siegel imply that whenever $X$ is a smooth curve with negative (geometric) Euler characteristic, and $S$ is a finite set of places of $k$, we have $\mathcal{X}(\mathcal{O}_{k,S})$ finite, for an $\mathcal{O}_{k,S}$-model $\mathcal{X}$ of $X$. This formalism includes the case of rational points, as $\mathcal{X}(\mathcal{O}_{k,S}) = X(k)$ whenever $X$ is proper. We are thus interested in the general question of determining the set $\mathcal{X}(\mathcal{O}_{k,S})$ for $X/k$ and a finite set $S$ of places of $k$.

In practice, one may often conjecturally find the set $\mathcal{X}(\mathcal{O}_{k,S})$ by searching over points of bounded height. This produces a finite set of elements of $\mathcal{X}(\mathcal{O}_{k,S})$, and one hopes, after a diligent enough search that this is all of $\mathcal{X}(\mathcal{O}_{k,S})$. The challenge is in proving that one has found all of $\mathcal{X}(\mathcal{O}_{k,S})$.

1.3. The Chabauty-Skolem Method. Before Faltings’ proof in 1983, the primary method for proving finiteness of $\mathcal{X}(\mathcal{O}_{k,S})$ was via the method of Chabauty-Skolem (\cite{Cha41}). In the 1980’s, Chabauty’s method was upgraded to an effective method by Coleman (\cite{Col85}), using his theory known as “Coleman integration.” More specifically, using the generalized Jacobian $X \hookrightarrow J$, a basepoint $b \in \mathcal{X}(\mathcal{O}_{k,S})$ \footnote{We define an $\mathcal{O}_{k,S}$-model as a finite type, separated, faithfully flat scheme $\mathcal{X}$ over $\mathcal{O}_{k,S}$ with an isomorphism $\mathcal{X}_k \rightarrow X$.} and a finite place $p$ of $k$ not in $S$ with $k_p \cong \mathbb{Q}_p$.

\footnote{One might then prefer to replace $\mathcal{X}$ by $X$ and $\mathcal{O}_{k,S}$, but as hinted at in \S 3.1 and described in more detail in \S 3.3.2 it is best to set $S = \emptyset$ and thus consider $\mathcal{O}_k$-points in the proper case.}

\footnote{This is characterized by the fact that $J$ is a semi-abelian variety and that the closed embedding $X \hookrightarrow J$ is an isomorphism on first homology.}

\footnote{More generally, an integral tangential basepoint is allowed when $X$ is not proper.}
one constructs a diagram

\[ \mathcal{X}(\mathcal{O}_{k,S}) \longrightarrow \mathcal{X}(\mathcal{O}_p) \]

\[ \mathcal{J}(\mathcal{O}_{k,S}) \mathrel{\longrightarrow} \mathcal{J}(\mathcal{O}_p) \mathrel{\longrightarrow} \text{Lie } J_{\mathbb{Q}_p} \]

for an appropriate integral model \( \mathcal{J} \) of \( J \) when \( X \) is not proper.

By definition, \( \text{Lie } J_{k_p} \) is the tangent space to \( J_{k_p} \) at the identity. When \( J \) is proper, it is the linear dual of \( H^0(J_{k_p}, \Omega^1) \), and more generally, it is dual to the subspace \( H^0(J_{k_p}, \Omega^1)^J \) of translation invariant differential 1-forms. The map \( \int \) sends \( z \in \mathcal{X}(\mathcal{O}_p) \) to the functional sending \( \omega \in H^0(J_{k_p}, \Omega^1)^J \) to the Coleman integral

\[ \int_b \omega. \]

If \( r := \text{rank}_\mathbb{Q} J(k) < g \), then the image of \( J(k) \) in \( \text{Lie } J_{k_p} \) lies in a vector subspace of positive codimension. Therefore, the annihilator \( \mathcal{I}_J \) of \( \text{loc}(\mathcal{J}(\mathcal{O}_{k,S})) \) in \( (\text{Lie } J_{k_p})^\vee = H^0(J_{k_p}, \Omega^1)^J \) is nonzero, so its pullback \( \int^\#(\mathcal{I}_J) \) is a nonzero set of functions on \( \mathcal{X}(\mathcal{O}_p) \) that vanish on \( \mathcal{X}(\mathcal{O}_{k,S}) \).

By the theory of Coleman, each nonzero \( f \in \int^\#(\mathcal{I}_J) \) has finitely many zeroes, and the theory of Newton polygons allows one to \( p \)-adically estimate the set \( \mathcal{X}(\mathcal{O}_p)_f \) of common zeroes of all \( f \in \int^\#(\mathcal{I}_J) \). More details may be found in [MP12].

If \( r \geq g \), one is usually out of luck with Chabauty’s method. Moreover, even if \( r < g \), \( \mathcal{X}(\mathcal{O}_p)_f \setminus \mathcal{X}(\mathcal{O}_{k,S}) \) might be nonempty, and the fact that computations of zeroes are \( p \)-adic approximations means that one cannot then use \( \int^\#(\mathcal{I}_J) \) to determine \( \mathcal{X}(\mathcal{O}_p) \) (though it can be successful in conjunction with other methods; see [Poo02, §5.3]).

1.4. Non-Abelian Chabauty’s Method. The non-abelian Chabauty’s method of Minhyong Kim (Kim05, Kim09), also known as Chabauty-Kim, allows one to remove this restriction. For an integer \( n \) and basepoint \( b \in \mathcal{X}(\mathcal{O}_{k,S}) \), Kim constructs a diagram

\[ \mathcal{X}(\mathcal{O}_{k,S}) \longrightarrow \mathcal{X}(\mathcal{O}_p) \]

\[ \text{Sel}(\mathcal{X})_n \mathrel{\longrightarrow} \text{Sel}(\mathcal{X}/\mathcal{O}_p)_n \mathrel{\longrightarrow} U_n/F^0U_n \]

The set \( \text{Sel}(\mathcal{X})_n \) is the non-abelian cohomology set \( H^1_{f,S}(G_k; U_n)^\vee \) where \( f \) refers to the Selmer conditions of Bloch-Kato, and \( U_n \) is the \( n \)th descending central series quotient of the \( \mathbb{Q}_p \)-unipotent geometric fundamental group of \( X \) (based at \( b \)). More details may be found in [Cor20, §4].

\[^v\text{Often called the \textit{depth}, although this conflicts with the notion of \textit{depth} in the theory of multiple zeta values.}\]

\[^v\text{Technically, one needs to modify } H^1_{f,S}(G_k; U_n) \text{ by either expanding } S \text{ to include all places of bad reduction of } X \text{ or by taking a finite union of twists as described in \S 4.}\]
For \( n = 1 \), this is essentially the same as the diagram in classical Chabauty’s method. More precisely, \( \text{Sel}(\mathcal{X})_1 \) is the \( p \)-adic Selmer group of \( J \), and we have an embedding

\[
\kappa_J : J(\mathcal{O}_{k,S}) \otimes \mathbb{Q}_p \hookrightarrow \text{Sel}(\mathcal{X})_1
\]

that is conjecturally (by finiteness of \( \text{III}(J) \)) an isomorphism, and verifiably so in practice.

We define the \textit{Chabauty-Kim locus}:

\[
\mathcal{X}(\mathcal{O}_p)_n := \kappa_{p^{-1}}(\text{Im}(\text{loc}_n)) = \int_{1}^{-1} (\text{Im}(\text{log}_{\text{BK}} \circ \text{loc}_n)).
\]

The \textit{Chabauty-Kim ideal}

\[
\mathcal{I}_{CK,n}(\mathcal{X})
\]

of regular functions vanishing on the image of \( \text{loc}_n \) pulls back to a set \( \kappa_{p^{-1}}(\mathcal{I}_{CK,n}) \) of functions on \( \mathcal{X}(\mathcal{O}_p) \) vanishing on \( \mathcal{X}(\mathcal{O}_{k,S}) \) with \( \mathcal{X}(\mathcal{O}_p)_n \) as its set of common zeroes. As long as \( \kappa_J \) is an isomorphism, \( \mathcal{I}_{CK,1} \) is the ideal generated by \( \mathcal{I}_J \), and \( \mathcal{X}(\mathcal{O}_p)_1 = \mathcal{X}(\mathcal{O}_p)_J \). For \( n \geq 1 \),

\[
\mathcal{X}(\mathcal{O}_p)_{n+1} \subseteq \mathcal{X}(\mathcal{O}_p)_n.
\]

When

\[
\dim_{\mathbb{Q}_p} \text{Sel}(\mathcal{X})_n < \dim_{\mathbb{Q}_p} \text{Sel}(\mathcal{X}/\mathcal{O}_p)_n,
\]

the set \( \mathcal{X}(\mathcal{O}_p)_n \) is finite, a consequence of the fact that \( \kappa_p \) has Zariski dense image ([Kim09 Theorem 1]).

Kim shows ([Kim09 Theorem 2]) that this inequality holds for sufficiently large \( n \) if a part of the Bloch-Kato Conjecture (see Conjecture 2.2 below) holds.

The following appears as [BDCKW18, Conjecture 3.1] for \( S = \emptyset \) and as a remark about what one “might conjecture” in [BDCKW18 §8]:

**Conjecture 1.1** (Kim’s Conjecture). For \( k = \mathbb{Q} \), a regular minimal model \( \mathcal{X} \) and \( n \) sufficiently large, we have

\[
\mathcal{X}(\mathcal{O}_p)_n = \mathcal{X}(\mathcal{O}_{k,S}).
\]

**Remark 1.2.** It is mentioned in [BDCKW18 Remark 3.2] that there should be a suitable generalization to all number fields \( k \). One might expect such a conjecture to follow from the recent ideas of [Dog20], although no such conjecture is contained therein.

The intuition behind this conjecture is that a random \( p \)-adic analytic function should not vanish at a given point unless it has a very good reason to do so.

This conjecture implies that if we can compute \( \mathcal{X}(\mathcal{O}_p)_n \) up to some \( p \)-adic precision, then there is an effective version of Faltings’ Theorem. More precisely, if we have a subset \( F \) of \( \mathcal{X}(\mathcal{O}_{k,S}) \), then to check that \( F = \mathcal{X}(\mathcal{O}_{k,S}) \), we need only find some \( n \) for which \( |\mathcal{X}(\mathcal{O}_p)_n| = |F| \).

In particular, we need only the part of the theory of Newton polygons that determines the number of zeroes of a \( p \)-adic power series in a residue disc.

The set \( \mathcal{X}(\mathcal{O}_p)_n \) is defined as the common zero set of the pullbacks under \( \int \) of the set of functions vanishing on the image of \( \text{loc}_n \). Therefore, modulo Conjecture 1.1, effective Faltings over \( \mathbb{Q} \) is reduced to the problem of computing, up to some \( p \)-adic precision, the set of functions on \( U_n/F^0U_n \) that vanish on the image of \( \text{loc}_n \).

Let \( g \) denote the genus of the smooth projective closure \( \overline{X} \) of \( X \). Then \( \mathcal{X} \) is a \textit{regular minimal model} if it is the complement of a reduced horizontal divisor in the regular minimal model \( \overline{\mathcal{X}} \) of \( \overline{X} \) over \( \mathcal{O}_{k,S} \) (resp. in \( \mathbb{P}^1/\mathcal{O}_{k,S} \)) when \( g \geq 1 \) (resp. when \( g = 0 \)).
1.5. **Quadratic Chabauty.** The most successful method to-date for computing with non-abelian Chabauty is the Quadratic Chabauty method of Balakrishnan et al ([BBM16], [BD18a]). This method essentially computes part of \( \text{loc}_2 \) using the observation of Kim that a certain coordinate of \( \text{Sel}(X)_2 \) corresponds to the \( p \)-adic height pairing on \( J \), while a similar coordinate of \( \text{Sel}(X/\mathcal{O}_p)_2 \) corresponds to the local component at \( p \) of the \( p \)-adic height pairing. (More precisely, [BBM16] covers the case of integral points on affine curves using the Coleman-Gross \( p \)-adic height pairing, while [BD18a] covers the case of rational points on proper curves using a whole slew of pairings defined relative to a certain kind of divisor on \( X \times X \).)

In particular, let \( \rho := \text{rank}_\mathbb{Q} \text{NS}(J) \), with \( X \) proper. Then [BD18a, Lemma 3.2] shows that \( X(\mathcal{O}_p)_2 \) is finite whenever

\[
r < g + \rho - 1,
\]

and moreover, [BD18a, §8] gives a method to \( p \)-adically approximate a set containing \( X(\mathcal{O}_p)_2 \) in this case. In [BD18b], the authors relax the condition (partly dependent on Conjecture 2.2), but still restrict to the case \( n = 2 \).

1.6. **Explicit Motivic Non-Abelian Chabauty.** The only computed cases of Chabauty-Kim going beyond \( n = 2 \) are for \( S \)-integral points on \( X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \). More specifically, the case of \( S = \emptyset, \{2\} \) with \( n = 4 \) appeared in [DCW16], while \( S = \{3\}, n = 4 \) appeared in [CDC20]. Both cases were also done in [Bro17a].

To compute \( X(\mathcal{O}_p)_n \), one needs to understand \( \text{Sel}(X)_n \) and the map \( \text{loc}_n \) concretely. As mentioned above, (the set of \( \mathbb{Q}_p \)-points of) \( \text{Sel}(X)_n \) is (modulo technicalities described in §)

\[
H^1_{f,S}(G_k; U_n),
\]

the set of cohomology classes of \( G_k \) with coefficients in \( U_n \) that are crystalline at primes over \( p \) and unramified outside \( S \cup \{p\} \). Both the group \( G_k \) and the local conditions are hard to understand explicitly.

The key insight is that one need only understand the category of continuous \( p \)-adic representations of \( G_k \) that appear in \( U_n \) and its torsors. This category is Tannakian, and \( \text{Sel}(X)_n \) may be described as group cohomology of the Tannakian fundamental group with coefficients in \( U_n \).

In the case of \( X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \), the relevant category is the category

\[
\text{Rep}_{\mathbb{Q}_p}^{\text{MT}}(\mathcal{O}_{k,S})
\]

of mixed Tate geometric Galois representations with good reduction over \( \mathcal{O}_{k,S} \). Its Tannakian fundamental group

\[
\pi_1^{\text{MT}}(\mathcal{O}_{k,S})
\]

is isomorphic to an extension of \( \mathbb{G}_m \) by a pro-unipotent group. The pro-unipotent group may be determined by computing the Bloch-Kato Selmer groups \( H^1_f(G_k; \mathbb{Q}_p(n)) \) for each \( n \), and these are known ([Sou79]). In this way, the set

\[
H^1_{f,S}(G_k; U_n),
\]

becomes simply the group cohomology set

\[
H^1(\pi_1^{\text{MT}}(\mathcal{O}_{k,S}); U_n),
\]

with no further local conditions other than those encoded in the group \( \pi_1^{\text{MT}}(\mathcal{O}_{k,S}) \) itself.
Remark 1.3. In fact, $\text{Rep}_{\mathbb{Q}_p}^{\text{MT}}(\mathcal{O}_{k,S})$ has a $\mathbb{Q}$-form $\text{MT}(\mathcal{O}_{k,S}, \mathbb{Q})$, the category of mixed Tate motives over $\mathcal{O}_{k,S}$ with coefficients in $\mathbb{Q}$. This latter category was defined in [DG05] and first applied to Selmer varieties in [Had11] and [DCW16]. The group $\pi_1^{\text{MT}}(\mathcal{O}_{k,S})$ is $\pi_1(\text{MT}(\mathcal{O}_{k,S}, \mathbb{Q}))$, while the analogue of what we do in this paper is its tensorization with $\mathbb{Q}_p$.

As such a $\mathbb{Q}$-linear category is only conjectural in the mixed elliptic case (see [Pat13]), we work with categories of Galois representations. While it is in some ways nicer to work over $\mathbb{Q}$, it is not necessary, as the end result of the Chabauty-Kim method is still $p$-adic.

1.7. This Work: Explicit Tannakian Non-Abelian Chabauty. The main goal of this paper is to extend the methods of [DCW16] and [CDC20] (see also [DC20] and [DCC20]) beyond the mixed-Tate case. We develop some general foundations that we expect to apply to all curves, and we do some more explicit computations in the mixed elliptic case. This is the case in which the Jacobian $J$ of the smooth compactification $\overline{X}$ of $X$ is isogenous to a power of an elliptic curve.

We set up some explicit examples when $X = E' = E \setminus \{O\}$ is a punctured elliptic curve, and $\mathcal{O}_{k,S} = \mathbb{Z}[1/2]$. This is arguably the simplest non-rational example in which the Chabauty-Kim locus is infinite for $n = 2$. We plan to compute these examples when the necessary code for computing Coleman integrals is fully available.

More precisely, we replace $\text{Rep}_{\mathbb{Q}_p}^{\text{MT}}(\mathcal{O}_{k,S})$ with a category $\text{Rep}_{\mathbb{Q}_p}^{f,S}(G_k, E)$ of $S$-integral mixed elliptic Galois representations; i.e., iterated extensions of Galois representations appearing in tensor powers of the Tate module $h_1(E) := H^\text{et}_1(E_{\overline{\mathbb{F}}_p}; \mathbb{Q}_p)$ of $E$. The fundamental group $\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E) := \pi_1(\text{Rep}_{\mathbb{Q}_p}^{f,S}(G_k, E))$ of this Tannakian category is now an extension of $\text{GL}_2$ by a pro-unipotent group $U(\mathcal{O}_{k,S}, E)$, the latter of which is determined by Bloch-Kato Selmer groups $H^1_{f,S}(G_k; M_{a,b})$, where $M_{a,b} := \text{Sym}^a(h_1(E))(b)$, runs over all irreducible representations of $\text{GL}_2$; i.e., over $a, b \in \mathbb{Z}$ with $a \geq 0$.

We explain how to compute the dimensions of $H^1_{f,S}(G_k; M_{a,b})$ while assuming the Bloch-Kato conjectures when $k$ is $\mathbb{Q}$ or a quadratic imaginary field (§5). We show how to use this to explicitly bound the dimension of the level $n$ Selmer variety of a curve of mixed elliptic type (§7).

For an elliptic curve over $\mathbb{Q}$ with ordinary reduction, the necessary cases of the Bloch-Kato conjecture with $a = 1$ may be verified using [Kat04, Theorem 18.4] and computations of $p$-adic $L$-functions. More generally, other relevant cases should be accessible using [All16] (for $a = 2$) and work of Loeffler–Zerbes (for $a = 3$). We review this in §2.3.
We explain the basic setup of Selmer varieties in the Tannakian formalism in §8-9. Most notably, we prove an explicit description of cohomology sets, generalizing [DCW16, Proposition 5.2], in Theorem A.4. This is necessary in order to carry out explicit motivic Chabauty-Kim in general.

We give an example computation of an element of $\mathcal{I}_{\text{CK},n}(\mathcal{X})$ for $n = 3$ and $\mathcal{X}$, $\mathcal{O}_{\mathcal{X},S} = \mathbb{Z}[1/\ell]$, and $\mathcal{X}$ the punctured minimal Weierstrass model of an elliptic curve, using the Tannakian formalism (§11). From it, we deduce the following:

**Theorem 1.4** (Theorem 10.1). Let $\mathcal{E}$ be the minimal Weierstrass model of an elliptic curve $E$ over $\mathbb{Q}$ with $p$-Selmer rank 1, let $\alpha$ denote a choice of component of the Néron model of $E$ at each place of $\mathbb{Q}$, and let $S = \{\ell\}$ for some prime $\ell \neq p$. Then assuming Conjecture 2.2 for $h^1(E)$, there is a function of the form

$$c_1J_4 + c_2J_3 + c_3J_1J_2 + c_4J_1^3 + c_5J_1$$

vanishing on the subset $\mathcal{E}'(\mathbb{Z}[1/S])_\alpha$ of $\mathcal{E}'(\mathbb{Z}[1/S])$ reducing to $\alpha$ at each bad prime of $\mathcal{E}$, in which:

- The $c_i \in \mathbb{Q}_p$ arising as periods of elements of $\mathcal{O}(W)$ (c.f. §9), not all of which are zero, and
- The $J_i$ are explicit iterated integrals on $E'$ defined in §10.2.

**Remark 1.5.** As noted in §2.4, one can often computationally verify Conjecture 2.2 for $h^1(E)$ using $p$-adic $L$-functions. Therefore, for individual elliptic curves, one may apply Theorem 1.4 without relying on any conjectures.

**Remark 1.6.** While Theorem 1.4 does not formally assert that the function is nonzero, it would follow from the existence of a $\mathbb{Q}$-linear category of mixed elliptic motives and a $p$-adic Kontsevich-Zagier period conjecture (generalizing [Yam10, Conjecture 4]) for this category that the function is nonzero. Furthermore, one may in practice verify that the $c_i$ are not all zero by computing Coleman integrals.

We analyze some explicit examples, corresponding to the curves ‘102a1’ and ‘128a2’ (§12). We describe the appropriate integral models and list known $\mathbb{Z}[1/2]$-points. Most notably, we explain how to deal with primes of bad reduction in §12.2.1, based on the theory of [BD20] as described in §3.4.2.

1.7.1. **Paper Outline.** In §2 we review some background on Galois representations and Bloch-Kato Selmer groups. This includes the statement of the part of the Bloch-Kato Conjecture we need (Conjecture 2.2). We also introduce certain Grothendieck groups ($K_0$) of Galois representations to be used in §7.

In §3 we describe the different kinds of Selmer varieties we use, paying careful attention to integral models and local conditions at primes of bad reduction. We review the results of [BD20] as they apply to our current work.

In §4 we define $\text{Rep}_{\text{f}}^{S}(G_k, E)$ and related objects. In §5 we compute the dimension of $H^1_{f,S}(G_k; M_{a,b})$ in various cases. In §6 we compute the ranks of the corresponding local Selmer groups $H^1_p(G_p; M_{a,b})$, then summarize the differences between local and global ranks.

In §7 we explain a general procedure for determining the semisimplification of $U_n$ when it is mixed elliptic. We also carry this out up to $n = 5$ for $X = E'$ and up to $n = 3$ for
projective curves of genera 2 and 4 whose motive is mixed elliptic. We mention some explicit genus 2 curves to which our method should apply.

In §8 we define a Tannakian fundamental group $\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)$ and write Selmer varieties in terms of it. We also explain the relevance of an analogue of [DCW16 Proposition 5.2], proven in the appendix (Theorem A.4).

In §9 we explain how to understand the localization map $\text{loc}_{11}$ more explicitly. For this, we describe a $p$-adic period map and a universal cocycle evaluation map.

In §10 we specialize to $\Pi = U_3(E')$. We describe a basis for $\mathcal{O}(\Pi)$ in §10.1. In §10.2 we explicitly describe the local Selmer variety and unipotent Kummer map for $X = E'$ and $n = 3$. In particular, we review results of [Bea17]. In §10.3 we explain why $\Pi$ is semisimple as a motive, which simplifies many calculations.

In §11 we carry out some fundamental computations in shuffle algebras in order to complete the proof of Theorem 1.4.

In §12 we outline some examples we plan to compute explicitly in future work.

1.8. Future Work. Our eventual goal is to extend these methods to all higher genus curves, with a view toward effective Faltings. Nonetheless, the case of a punctured elliptic curve presents many of the features (and complications) of the general case. Some of these include:

1. The need to use Conjecture 2.2
2. The nontriviality of $F^0U_n$
3. A higher-dimensional reductive group in place of $\mathbb{G}_m$
4. The lack of a $\mathbb{Q}$-linear Tannakian category of motives

While our goal in this paper is to focus on an example, we also set up much of the framework necessary to extend to the general case. In particular, see Remarks 4.11, 8.9, and 9.6. Furthermore, in the mixed elliptic case, our computations could be used to compute Chabauty-Kim in all levels.

The biggest obstacle in applying this method to all hyperbolic curves $X$ is that there are not always going to be enough points. We give a brief overview of our plans to overcome this obstacle. See also Remark 10.2.

The issue in fact already appeared in [CDC20], in which $\mathcal{X}(\mathcal{O}_{k,S}) = \emptyset$. The key is to choose a finite extension $k'/k$ and/or a set $S'$ containing the places of $k'$ above those in $S$ for which $\mathcal{X}(\mathcal{O}_{k',S'})$ is large enough. We then use elements of $\mathcal{X}(\mathcal{O}_{k',S'})$ to explicitly understand elements of the coordinate ring

$$\mathcal{O}(\pi_1(\text{Rep}_{\mathbb{Q}_p}(G_{k'}, E))).$$

One then uses an understanding of the inclusion

$$\text{Rep}_{\mathbb{Q}_p}(G_k, E) \subseteq \text{Rep}_{\mathbb{Q}_p}(G_{k'}, E)$$

to work things out over $\mathcal{O}_{k,S}$. If $X(k)$ is infinite, we often keep $k = k'$, while if $X(k)$ is finite, expect to pass to a finite extension field of $k$, possibly without changing $S$.

We mention three approaches to using elements of $\mathcal{X}(\mathcal{O}_{k',S'})$ to determine $\mathcal{X}(\mathcal{O}_{k,S})$:

- Adapt the virtual cocycles of [Bro17a, §10] to a more general context
- Develop a theory of elliptic motivic polylogarithms and use it as in [CDC20]
- Use the “Special Elements” and coproduct formulas of [Pat13]
One major obstacle in defining elliptic motivic polylogarithms is the lack of a canonical de Rham path between any two points in an elliptic curve (and more generally, any higher genus curve). This amounts to the lack of a global fiber functor as in the case of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. Some ideas in this direction are proposed in [Bro17a §11]. In the unpublished note [Cor19], I correct some misstatements in loc.cit. and other articles on the subject and analyze different possible approaches to defining elliptic motivic polylogarithms.

In one ongoing project, Ishai Dan-Cohen, Stefan Wewers, and I have shown how to define a global fiber functor for vector bundles with unipotent connection on a projective curve $X$ depending on a decomposition

$$X = U_1 \cup U_2$$

into affine Zariski opens and a subspace

$$T \subseteq H^0(U_1 \cap U_2; \mathcal{O}_X)$$

mapping isomorphically onto $H^1(X; \mathcal{O}_X)$. Using Deligne’s canonical extension, this provides a fiber functor for the category of vector bundles with unipotent connection on any open subcurve of $X$. For an elliptic curve $E$ given by Weierstrass model, one may take $T$ to be the subspace spanned by $y/x$.

In another approach, one may define elliptic motivic polylogarithms using the proposition in §A.3. A choice of element $\omega \in \mathcal{O}(\Pi)$ along with $z \in X(k)$ (and a basepoint $b \in X(k)$ lurking in the background) would thus define a motivic iterated integral

$$\int_b^z \omega \in \mathcal{O}(U_E)$$

as the pullback of $\omega$ along the cocycle corresponding to $\kappa(z)$.

Either the virtual cocycles approach or the more direct approach of [CDC20] will require a version of Goncharov’s coproduct formula ([Gon05, Theorem 1.2]) for elements of $\mathcal{O}(U_E)$. We expect this to be straightforward, following from a general formula for coproducts of Tannakian matrix coefficients.

A coproduct formula and a conjectural basis of $\mathcal{O}(U_E)$ have already been written down in [Pat13] in terms of the bar construction on cycle complexes. Ongoing work of the author of this paper and Owen Patashnick seeks to use the coproduct formula of loc.cit.

### 1.9. Notation.

When we use the term “motive”, we are thinking of a system of realizations (as in [BK90, Definition 5.5]), but working primarily with the $p$-adic Galois representation realization. Thus when we say “pure motive,” we mean “semisimple $p$-adic Galois representation.” We use \{\text{p}\} to denote the set of places $k$ above a place $p$ of $\mathbb{Q}$; to justify this notation when $p$ is not inert in $k$, one may think of it as the subscheme of $\mathcal{O}_k$ defined by \{p = 0\}. We use $H_f^1$ and $H_g^1$ as in [BK90] and [FPR91], recalled also in §2.3. We write $H_f^1, S$ for the subset of $H_f^1$ unramified/crystalline at all $v \notin S$.

For a $p$-adic representation $V$ of $G_k$, we write

$$h^i(G_k; V) := \dim H^i(G_k; V)$$

and

$$h^i_\bullet(G_k; V) := \dim H^i_\bullet(G_k; V)$$

for $k$ a local or global field and $\bullet \in \{f,g\}$ or $k$ a global field and $\bullet = f, S$. 

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We always work over a number field $k$ and with a chosen rational prime $p$. We write $G_k$ for the absolute Galois group of $k$. If $v$ is a place of $k$, then $k_v$ denotes the completion of $k$ at $v$, $\mathcal{O}_{k_v}$ its integer ring, $G_v$ its absolute Galois group, and $I_v$ the inertia subgroup. We write $\mathcal{O}_{k,S}$ for the subset of $k$ that is integral at all $v \notin S$. For a set $T$ of places, we write $G_{k,T}$ for the Galois group of the maximal extension of $k$ unramified outside $T \cup \{\infty\}$.

If $Y$ is a variety over $k$ and $i$ a non-negative integer, we let $h^i(Y)$ denote the continuous $p$-adic representation of $G_k$ given by $H^i_{\acute{e}t}(Y_k; \mathbb{Q}_p)$, and similarly for $h_i(Y)$.

For a smooth geometrically irreducible variety $Y$ over $k$, we let

$$U(Y) := \pi_1^{\acute{e}t,un}(Y_k)/\mathbb{Q}_p,$$

the $\mathbb{Q}_p$ pro-unipotent completion of the geometric étale fundamental group of $Y$. It has a continuous algebraic action of $G_k$, and $U(Y)^{ab} \cong h_1(Y)$. We always take it relative to basepoint $b \in X(k)$, but we suppress this basepoint in the notation.

We use "∗" to denote the point, in the context of objects of a pointed category.

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2. **Bloch-Kato Groups and Conjectures**

In this section, we go over background on Galois representations and the Bloch-Kato Selmer groups and conjectures.

2.1. **Categories of Galois Representations.** Let $k$ be a number field and $p$ a prime number, and let

$$\text{Rep}^g_{\mathbb{Q}_p}(G_k)$$

denote the category of $p$-adic representations of $G_k$ that are unramified almost everywhere and de Rham at every place of $k$ above $v$. We let

$$\text{Rep}^{sg}_{\mathbb{Q}_p}(G_k)$$
denote the subcategory of representations $V$ that are strongly geometric, meaning $V$ has a finite increasing filtration $W^n V$, known as the motivic weight filtration, for which
\[ \text{Gr}_n W^n V \]
is semisimple and pure of weight $n$ at almost all unramified places in the sense of \cite{Del80}. For an integer $n$, denote by $\text{Rep}_{\mathbb{Q}_p}^g (G_k)_{w \leq n}$ (resp. $\text{Rep}_{\mathbb{Q}_p}^g (G_k)_{w \geq n}$, $\text{Rep}_{\mathbb{Q}_p}^g (G_k)_{w = n}$) the subcategories of objects whose nonzero weight-graded pieces all have weights less than or equal to $n$ (resp. greater than or equal to $n$, equal to $n$).

The conjecture of Fontaine-Mazur states that every irreducible object of $\text{Rep}_{\mathbb{Q}_p}^g (G_k)$ is a subquotient of an étale cohomology group of some variety over $k$. A mixed version of Fontaine-Mazur\footnote{According to B. Mazur, while this was not part of their original conjecture, it may have been ‘in the air’ at the time. Note that \cite[Observation 2]{Kim09} refers to this mixed version rather than the original version.} states that every object of $\text{Rep}_{\mathbb{Q}_p}^g (G_k)$ is such a subquotient. By resolution of singularities and the Weil conjectures, any such subquotient has a motivic weight filtration for which the graded pieces are pure of the appropriate weight. A corollary of this mixed Fontaine-Mazur along with the Grothendieck-Serre semisimplicity conjecture\footnote{This is the conjecture that the étale cohomology of a smooth projective variety over a finitely generated field of characteristic $0$ is semisimple as a Galois representation, and it follows from the Tate conjecture by \cite{Moo19}.} is thus:

**Conjecture 2.1.** The categories $\text{Rep}_{\mathbb{Q}_p}^g (G_k)$ and $\text{Rep}_{\mathbb{Q}_p}^g (G_k)$ are equal\footnote{We really mean equal, not equivalent or isomorphic, as the former is defined as an explicit subcategory of the latter.}\footnote{Not a ring for $n \neq 0$!}

### 2.2. Grothendieck Groups.

In \S7 we will use the Grothendieck ring $K_0(\text{Rep}_{\mathbb{Q}_p}^g (G_k))$, which has a grading with the $n$th graded piece given by the group $K_0(\text{Rep}_{\mathbb{Q}_p}^g (G_k)_{w = n})$. The subring $K_0(\text{Rep}_{\mathbb{Q}_p}^g (G_k)_{w \leq 0})$ is negatively graded, and we denote by $\hat{K}_0(\text{Rep}_{\mathbb{Q}_p}^g (G_k)_{w \leq 0})$ its completion with respect to the ideal $K_0(\text{Rep}_{\mathbb{Q}_p}^g (G_k)_{w \leq -1})$. If $V$ is an object of $\text{Rep}_{\mathbb{Q}_p}^g (G_k)_{w \leq -1}$, then $[\text{Sym} V] = \sum_{n=0}^{\infty} [\text{Sym}^n V]$ and $[TV] = \sum_{n=0}^{\infty} [V^\otimes n]$ are in $\hat{K}_0(\text{Rep}_{\mathbb{Q}_p}^g (G_k)_{w \leq 0})$. We let $\text{pr}_n$ denote projection onto the $n$th component. We thus have $\text{pr}_n([V]) = [\text{Gr}_n W^n V]$.

### 2.3. Bloch-Kato Selmer Groups.

Let $v$ be a place of $k$, $G_v$ a decomposition group of $v$ in $G_k$, and $I_v$ the inertia subgroup. We recall the local and global Selmer groups of \cite{BK90}.

For $v \nmid p$ and a $p$-adic representation $V$ of $G_v$, we set
\[ H^1_g (G_v; V) := H^1 (G_v; V) \]
\[ H^1_f (G_v; V) := H^1_{ur} (G_k; V) := \ker (H^1 (G_v; V) \to H^1 (I_v; V)). \]

For $v \mid p$, let $B_{cris}$ and $B_{dR}$ denote the crystalline and de Rham period rings associated to $k_v$, respectively. We set
\[ H^1_g (G_v; V) := \ker (H^1 (G_v; V) \to H^1 (G_v; V \otimes_{\mathbb{Q}_p} B_{dR})). \]
Let $V$ be a $p$-adic representation of $G_k$. For a place $v$ of $k$, we let $Res_v$ denote the restriction map $H^1(G_k; V) \to H^1(G_v; V)$. We then set
\[
H^1_g(G_k; V) := \{ \alpha \in H^1(G_k; V) \mid Res_v(\alpha) \in H^1_v(G_v; V) \forall v \}.
\]
\[
H^1_f(G_k; V) := \{ \alpha \in H^1(G_k; V) \mid Res_v(\alpha) \in H^1_f(G_v; V) \forall v \}.
\]
As in [FPR94, II.1.3.1], for a set $S$ of places of $k$, we set
\[
H^1_{f,S}(G_k; V) := \{ \alpha \in H^1_f(G_k; V) \mid Res_v(\alpha) \in H^1_f(G_v; V) \forall v \not\in S \}.
\]
Conjecture 2.2 implies the following conjecture of Bloch-Kato:

**Conjecture 2.2 (Bloch-Kato).** For an irreducible geometric $p$-adic representation $V$ of $G_k$ of non-negative weight, the group
\[
H^1_g(G_k; V)
\]
vanishes.

**Remark 2.3.** There are three philosophical reasons behind this conjecture:

1. The mixed version of Fontaine-Mazur predicts that any such extension class comes from geometry and therefore has a weight filtration splitting it.
2. The dimension of the group should be the order of vanishing of an $L$-function in the region of absolute (and hence nonzero) convergence.
3. The group corresponds to an algebraic $K$-theory group in negative degree.

We also need the following consequence of Poitou-Tate duality (c.f. [Be109, Theorem 2.2] or [FPR94, II.2.2.2]):

**Fact 2.4.** For a geometric Galois representation $V$, we have
\[
h^1_f(G_k; V) = h^0_f(G_k; V) + h^1_f(G_k; V^\vee(1)) - h^0_f(G_k; V^\vee(1)) + \sum_{v \mid p} \dim_{\mathbb{Q}_p}(D_{dR} V / D_{dR}^+ V) - \sum_{v \mid \infty} h^0_v(G_v; V).
\]

In terms of the category $\text{Rep}_{\mathbb{Q}_p}(G_k)$, we have
\[
H^1_g(G_k; V) = \text{Ext}^1_{\text{Rep}_{\mathbb{Q}_p}(G_k)}(\mathbb{Q}_p(0), V).
\]

### 2.4. Verifying Cases of the Conjectures

This section is a brief interlude explaining that many cases of interest of Conjecture 2.2 may be verified in specific cases. These verifications require that $E$ be modular, which is known in general when $k$ is totally real.

For a punctured elliptic curve in level 3 (the topic of §11), the only conjecture we need is that
\[
H^1_f(G_k; h^1(E)) = 0.
\]

Kato’s Euler system argument ([Kat04, Theorem 17.4]) combined with a control theorem ([Nek05, §8.10]) may be used in the ordinary case to reduce this statement to non-vanishing of $L_p(E, 2)$. One may then approximate this $L$-value using Mazur-Stickelberger elements defined in terms of Manin symbols ([Wuk14, §3.5]). We have verified that $L_p(E, 2) \neq 0$ when $E = 102a1$ and $p = 5$ and when $E = 128a2$ and $p = 3, 5$ using the following code in SageMath:
def mu(a,p,n,phi):
    return phi(a*p^(-n))*alpha^(-n) - (alpha^(-n-1))*phi(a*p^(-n+1))

def mazur_stickelberger_sum(E,p,prec):
    if E.is_supersingular(p):
        error('curve is supersingular')
    phi = E.modular_symbol(-1)
    K = Qp(p,prec); Fp = GF(p)
    Ep = E.base_extend(Fp); f = Ep.frobenius_polynomial()
    R.<x> = PolynomialRing(K); fp = R(f)
    rs = fp.roots(); alpha = rs[0][0]
    if abs(alpha) < 1:
        alpha = rs[1][0]
    sum = K(0)
    for a in range(p^prec):
        if not p.divides(a):
            sum += K(a*mu(a,p,prec,phi))
    return sum

Computations for higher genus curves, including those outlined in §7.6.1, may require vanishing of
\[ H_1^f(G_k; M_{2,-1}) \]
or
\[ H_1^f(G_k; M_{3,-2}) \]

The former follows in most cases by [Al16], which uses modularity lifting results. The latter may be approached by adapting arguments of Loeffler–Zerbes from [LZ20a] that in turn use the general results of [LZ20b].

2.5. \( S \)-integral Versions. Let \( \mathbf{Rep}_{Q_p}^{f,S}(G_k) \) denote the full subcategory of \( \mathbf{Rep}_{Q_p}^f(G_k) \) of representations that are unramified outside \( S \cup \{p\} \) and crystalline at \( v \) if \( v \notin S \) has residue characteristic \( p \). If \( V \in \mathbf{Rep}_{Q_p}^{f,S}(G_k) \), then
\[
H^1_{f,S}(G_k; V) = \text{Ext}^1_{\mathbf{Rep}_{Q_p}^{f,S}(G_k)}(\mathbb{Q}_p(0), V).
\]

More generally, we might want to consider \( H^1_{f,S} \) even when \( V \) is not in \( \mathbf{Rep}_{Q_p}^{f,S}(G_k) \). For this we make a modification in Remark 4.10.

Let us briefly analyze the extent to which \( H^1_{f,S}(G_k; V) \) depends on \( S \).

**Proposition 2.5.** Let \( S' = S \cup \{v\} \), with \( V \) a \( p \)-adic representation of \( V \) and suppose \( v \notin p \). Then
\[
h^{1}_{f,S'}(G_k; V) = h^{1}_{f,S}(G_k; V) + h^0(G_v; V^\vee(1))
\]
In particular, \( \dim H^1_{f,S}(G_k; V) = \dim H^1_{f,S'}(G_k; V) \) if \( V \big|_{G_v} \) has no quotient isomorphic to \( \mathbb{Q}_p(1) \).

**Proof.** By [FPR94, Proposition III.3.3.1(b)], we have an exact sequence
\[
0 \to H^1_{f,S}(G_k; V) \to \dim H^1_{f,S'}(G_k; V) \to H^1_{q/f}(G_v; V) \to 0,
\]
where $H^1_{g/f}(G_v; V) := H^1_g(G_v; V)/H^1_f(G_v; V)$.

By Tate-Poitou duality and because $v \notin \{p\}$, we have an isomorphism

$$H^1_{g/f}(G_v; V) = H^1(G_v; V)/H^1_f(G_v; V) \cong H^1_f(G_v; V^\vee(1))^\vee.$$  

The inequality then follows by $\dim H^1_f(G_v; V^\vee(1)) = \dim H^0(G_v; V^\vee(1))$.  

2.6. Local Conditions and Weight-Monodromy. Supposing still that $v \nmid p$, we recall the theory of weights for possibly-ramified $p$-adic representations of $G_v$. By Grothendieck’s Monodromy Theorem ([ST68 Appendix]), after restricting to an open subgroup $G'_v$ of $G_v$, the action of inertia is unipotent, so the irreducible pieces in the Jordan-Hölder series of $V|_{I_v}$ are unramified. We may therefore talk about the Frobenius weight of such a piece, and the (Frobenius) weights $W$ of $V|_{G_v}$ is the set of weights that appear in these subquotients.

Let $V \in \Rep_{\Qp}(G_k)$, so that there is a filtration $W_* V$ by motivic weight. By our definition,

$$\Gr^W_i V$$

is then unramified and pure of weight $i$ at almost all places $v$. Let us recall what happens if we ask for a description at all $v$. After restricting to an open subgroup of $I_v$ that acts unipotently, the inertia action may be described by a nilpotent operator $N : V \to V$. By Jacobson-Morosov ([Del80 Proposition 1.6.1]), there is a unique filtration $\Fil^n V$ for which $N(\Fil^N_{i-2} V) \subseteq \Fil^N_{i-2} V$ and $N : \Gr^N_i V \to \Gr^N_{i-1} V$ is an isomorphism for all $i \geq 0$ (c.f. [Sch12 Conjecture 1.13]). If we assume that $V$ comes from geometry (as is implied by the Fontaine-Mazur conjecture), then the weight-monodromy conjecture of Deligne ([Del71]) states that:

Conjecture 2.6. $\Gr^N_i \Gr^W_j V$ is pure of weight $i+j$ as a representation of $V$.

This conjecture is the Weight-Monodromy Theorem of Grothendieck ([ST68 Appendix]) for a Galois representation coming from an abelian variety. As a corollary, it holds for any representation coming from first homology, or even more generally, from the $p$-adic unipotent fundamental group, of any variety. In particular, it holds for all the representations we care about.

If $V$ is potentially unramified at $v$, then $N = 0$, so that the Frobenius weights of $V|_{G_v}$ are the motivic weights of $V$. For example, the set of weights of $\Qp(n)$ is $\{-2n\}$.

Proposition 2.7. If $0$ is not one of the Frobenius weights of $V|_{G_v}$, then $h^0(G_v; V) = 0$. More generally, if $V|_{G_v}$ satisfies the weight-monodromy conjecture at $V$, and

$$\Gr^N_i \Gr^W_j V$$

is trivial whenever $i+j = 0$ and $i \geq 0$.

Remark 2.8. The condition is satisfied if $\Gr^W_i V = 0$ for $i \geq 0$, i.e., if $V \in \Rep_{\Qp}(G_k)_{w \leq -1}$. This corresponds to the condition "(WM<0)" of [BD20 §2.1].

Proof. It suffices to prove this for $V$ of pure motivic weight. By the definition of $\Fil^N_i V$, the kernel of $N$ is contained in $\Fil^N_i V$. Therefore, if $H^0(G_v; V)$ is nontrivial, it projects nontrivially onto $\Gr^N_i \Gr^W_j V$ for some $j \leq 0$. This piece then has a nontrivial Galois-fixed part, so we have $i+j = 0$. Then $i \geq 0$, hence $\Gr^N_i \Gr^W_j V$ is trivial by hypothesis, a contradiction.  

xiiWe write “Frobenius weight” in full when we must distinguish from motivic weight.
We now consider what happens when we dualize and twist. The monodromy operator \( N \) of \( V^\vee \) is the negative dual of the monodromy operator of \( V \); in particular, \( \text{Fil}^N V^\vee \) is dual to \( \text{Fil}^N V \). Finally, \( W \cdot V^\vee \) is dual to \( W \cdot V \). In particular, we find that

\[
\text{Gr}^N_j \text{Gr}^W_i V
\]

is dual to

\[
\text{Gr}^N_{-j} \text{Gr}^{-W}_{-i} V^\vee
\]

as a \( G_v/I_v \)-representation. Since duality negates the Frobenius weights, so that the weight-monodromy conjecture holds for \( V \) iff it does for \( V^\vee \).

A Tate twist shifts the motivic weight filtration down by 2 and does not affect \( \text{Fil}^N \). It also shifts the Frobenius weights down by 2 and thus preserves the truth of the weight-monodromy conjecture. We have

\[
\dim \text{Gr}^N_j \text{Gr}^W_i V = \dim \text{Gr}^N_{-j} \text{Gr}^{-W}_{-i-2} V^\vee (1).
\]

As a corollary of Proposition 2.5 and Proposition 2.7, we get

**Corollary 2.9.** In the notation of Proposition 2.5, if \(-2\) is not one of the Frobenius weights of \( V|_{G_v} \), then

\[
\dim H^1_{f,S}(G_k; V) = \dim H^1_{f,S}(G_v; V).
\]

This is true more generally if \( V|_{G_v} \) satisfies the weight-monodromy conjecture at \( V \), and

\[
\text{Gr}^N_j \text{Gr}^W_i V
\]

is trivial whenever \( i + j = -2 \) and \( i \leq -2 \).

**Remark 2.10.** The Weight-Monodromy Theorem for abelian varieties states more precisely that the weights of the Tate module are contained in \( \{0, -1, -2\} \). The same is therefore true for the first homology of any smooth variety.

### 3. Selmer Varieties

We use this section to precisely specify our notation and definitions and to review the differing notions of Selmer variety or scheme in the literature. Almost none of the material here is new.

Let \( k \) be a number field, \( X/k \) a curve for which \( h_1(X) \) lies in \( \text{Rep}_{\mathbb{Q}_p}^s(G_k, E) \), \( \Pi \) a finite-dimensional Galois-equivariant quotient of \( U = U(X) \) based at \( b \in X(k) \), and \( v \) a place of \( k \). Then \( \text{Lie} \Pi \) is a (Lie algebra) object in

\[
\text{Rep}_{\mathbb{Q}_p}^{f,s}(G_k, E),
\]

while its universal enveloping algebra \( U\Pi \) and coordinate ring \( \mathcal{O}(\Pi) \) are Pro- and Ind-objects, respectively, of that category.

There are local and global unipotent Kummer maps fitting into a diagram

\[
\begin{array}{ccc}
X(k) & \longrightarrow & X(k_v) \\
\downarrow \kappa & & \downarrow \kappa_v \\
H^1(G_k; \Pi) & \longrightarrow & H^1(G_v; \Pi).
\end{array}
\]

\[\text{xiii}\] Also called *unipotent Albanese maps* in some parts of the literature, such as [Kim09] and [KT08].
3.1. **Selmer Schemes as Schemes.** Let \( G = G_v \) or \( G = G_{k,T} \) for a finite set \( T \) of places of \( k \), and let \( \Pi \) be a unipotent group over \( \mathbb{Q}_p \) with continuous action of \( G \).

Kim proves in [Kim09] that

\[
H^1(G; \Pi)
\]

is naturally the set of \( \mathbb{Q}_p \)-points of a scheme over \( \mathbb{Q}_p \). It is characterized by setting it to be the affine space underlying the \( \mathbb{Q}_p \)-vector space \( H^1(G; \Pi) \) when \( \Pi \) is abelian and requiring that

\[
* \to H^1(G; \Pi') \to H^1(G; \Pi) \to H^1(G; \Pi'') \to *
\]

is a short exact sequence of pointed schemes when \( 1 \to \Pi' \to \Pi \to \Pi'' \to 0 \) is a short exact sequence of unipotent groups with \( G \)-action.

**Fact 3.1.** It is furthermore shown in [Kim09, §2] that

- The map \( \text{loc}_v : H^1(G_{k,T}; \Pi) \to H^1(G_v; \Pi) \) is a map of algebraic varieties, and
- The subspace \( H^1_f(G_v; \Pi) \) (Definition 3.3) is an algebraic subvariety of \( H^1(G_v; \Pi) \)

We will not be particularly careful about the difference between a Selmer scheme and its set of \( \mathbb{Q}_p \)-points, and the careful reader may check that it does not pose any problems. By Fact 3.1, all the subsets we define below correspond to algebraic subvarieties.

3.2. **Good Reduction and Local Conditions.**

**Definition 3.2.** We define good reduction as follows:

- For a place \( v \) of \( k \), we say that \( X \) has good reduction at \( v \) if there is a model \( \mathcal{X} \) of \( X \) over \( \mathcal{O}_v \) sitting inside a smooth proper curve \( \overline{\mathcal{X}} \) over \( \mathcal{O}_v \) with étale boundary divisor.\[xv\] We say more precisely that the model \( \mathcal{X} \) has good reduction at \( v \).
- We say that \( X \) has potentially good reduction at \( v \) if there is a finite extension \( l_v/k_v \) for which \( X_{l_v} \) has good reduction at \( v \). We say more precisely that the model \( \mathcal{X} \) has potentially good reduction at \( v \) if \( \mathcal{X}_{l_v} \) is dominated by a good model of \( X_{l_v} \).

**Definition 3.3.** We set

\[
H^1_f(G_v; \Pi) := \text{Ker}(H^1(G_v; \Pi) \to H^1(I_v; \Pi))
\]

for \( v \notin \{p\} \) and

\[
H^1_f(G_v; \Pi) := \text{Ker}(H^1(G_v; \Pi) \to H^1(G_v; \Pi \otimes_{\mathbb{Q}_p} B_{\text{cris}}))
\]

for \( v \in \{p\} \).

If \( \mathcal{X} \) has good reduction at \( v \), the action of \( G_v \) on \( \Pi \) is unramified.

For \( p \in \{p\} \) at which \( \mathcal{X} \) has good reduction, we set

\[
\text{Sel}(\mathcal{X}/\mathcal{O}_p; \Pi) := H^1_f(G_p; \Pi).
\]

**Fact 3.4.** If \( v \notin \{p\} \), then

1. ([BD20, Corollary 2.1.9])

\[
H^1_f(G_k; V) = *
\]

\[xv\] As an example of why the condition on the divisor is necessary, \( \mathbb{P}^1 \setminus \{0, 1, 2, \infty\} \) has bad reduction at 2 even though it has a smooth model over \( \mathbb{Z}_2 \).

\[xv\] In this context, “dominate” means that the good model is the complement in a blowup of an integral compactification of the strict transform of the boundary. In particular, the good model does not necessarily map to \( \mathcal{X}_{\mathcal{O}_v} \) as a scheme over \( \mathcal{O}_{l_v} \).
(2) ([BD20, Proposition 1.2]) If $X$ has potentially good reduction at $v$, then
$$\kappa_v(X(\mathcal{O}_v)) = \tilde{H}^1_f(G_k; V) = \ast$$

If $v \in \{p\}$,

(3) ([Kim09, Theorem 1]) and $X$ has good reduction at $v$, then
$$\kappa_v(X(\mathcal{O}_v))^{\text{zar}} = \tilde{H}^1_f(G_k; V)$$

3.3. The Chabauty-Kim Diagram. Let $S$ denote a finite set of places of $k$ disjoint from $\{p\}$ and $p \in \{p\}$ with $k_p \cong \mathbb{Q}_p$. For a model $X$ of $X$ over $\mathcal{O}_{k,S}$ with good reduction at $p$ and $b \in X(\mathcal{O}_{k,S})$, we will describe a Selmer variety $\text{Sel}(X)_\Pi$ fitting into a diagram

$$\begin{align*}
X(\mathcal{O}_{k,S}) &\xrightarrow{\kappa} X(\mathcal{O}_p) \\
\text{Sel}(X)_\Pi &\xrightarrow{\text{loc}_\Pi} \text{Sel}(X(\mathcal{O}_p)_\Pi)
\end{align*}$$

where $\text{loc}_\Pi$ always refers to $\text{loc}_p$ for the chosen place $p \in \{p\}$ with respect to the fundamental group quotient $\Pi$, and

$$\text{Sel}(X(\mathcal{O}_p)_\Pi) := \tilde{H}^1_f(G_p; \Pi).$$

We set the Chabauty-Kim locus:

$$X(\mathcal{O}_p)_\Pi := \kappa_p^{-1}(\text{Im}(\text{loc}_\Pi)) = \int^{-1}(\text{Im}(\log_{\text{BK}} \circ \text{loc}_\Pi)).$$

The Chabauty-Kim ideal

$$\mathcal{I}_{CK,\Pi}(X)$$

of regular functions vanishing on the image of $\text{loc}_\Pi$ pulls back to a set $\kappa_p^!(\mathcal{I}_{CK,\Pi})$ of functions on $X(\mathcal{O}_p)$ vanishing on $X(\mathcal{O}_{k,S})$ with $X(\mathcal{O}_p)_\Pi$ as its set of common zeroes.

For $\Pi = U_n$, we write $\text{Sel}(X)_n$, $\text{Sel}(X(\mathcal{O}_p)_n)$, $\mathcal{I}_{CK,n}(X)$, and $X(\mathcal{O}_p)_n$.

As described in [Kim09, p.96], when

(3) \quad $\dim \text{Sel}(X)_\Pi < \dim \text{Sel}(X(\mathcal{O}_p)_\Pi),$

we may conclude that $\text{loc}_\Pi$ is non-dominant and therefore that

$$\mathcal{I}_{CK,\Pi}(X)$$

is nonzero, hence by [Kim09, Theorem 1]

$$X(\mathcal{O}_p)_\Pi$$

is finite. The statement of [Kim09, Theorem 2] is that this happens for $k = \mathbb{Q}$ and $n$ sufficiently large if Conjecture 2.2 is true.

In §5-7, we will show how to check (3) when $X$ is mixed elliptic (Definition 4.1).
3.4. **Global Selmer Varieties.** Let $T_0$ denote the set of places at which $X$ does not have potentially good reduction, and let $T' = S \cup \{p\} \cup T_0$. Let $T_1$ denote the set of places at which $X$ has bad but potentially good reduction, and let $T = T' \cup T_1$. Then $X$ has good reduction outside $T$, so the action of $G_k$ on $\Pi$ factors through $G_{k,T}$, and our Selmer varieties will be subvarieties of $H^1(G_{k,T}; \Pi)$.

We first discuss the case in which $T_0 \subseteq S$, which is much simpler and already applies to one of our examples, given by the elliptic curve “128a2”. The general case is in §3.4.2.

3.4.1. **Good Reduction Outside $S$.**

**Definition 3.5.** We suppose $X$ has potentially good reduction at all $v \in \text{Spec} \, \mathcal{O}_{k,S}$. As in [Kim09], we define

\[
\text{Sel}(X)_\Pi := H^1_f(G_k; \Pi) := \{\alpha \in H^1(G_{k,T}; \Pi) \mid \text{loc}_v(\alpha) \in H^1_f(G_v; \Pi) \forall v \notin S\}.
\]

This works, for example, if $X$ is the elliptic curve “128a2”, and $\mathcal{O}_{k,S} = \mathbb{Z}[1/2]$.

If $T_0 \not\subseteq S$, we may expand $S$ to $S' = S \cup T_0$ \cite{xviii}. We may also modify $p$ and $p$ to ensure that $S' \cap \{p\} = \emptyset$. One might then hope to apply the Chabauty-Kim method to compute $X(\mathcal{O}_{k,S'})$, then find the subset $X(\mathcal{O}_{k,S'}) \subseteq X(\mathcal{O}_{k,S'})$ by hand. This is the approach of \cite[xvii]{Kim09}, which is enough to show that Conjecture 2.2 implies finiteness of $X(\mathcal{O}_{k,S})$ when $k = \mathbb{Q}$.

3.4.2. **Bad Reduction Outside $S$.** Nonetheless, it is often more practical to work with a Selmer scheme $\text{Sel}(X)_\Pi$ defined even when $X/\mathcal{O}_{k,S}$ has permanent bad reduction at some $v \in \text{Spec} \, \mathcal{O}_{k,S}$. This is because we often have

\[
\dim \text{Sel}(X)_\Pi < \dim \text{Sel}(X_{\mathcal{O}_{k,S'}})_\Pi,
\]

which means in practice that one may need to pass to a larger $\Pi$ to get finiteness for the Chabauty-Kim locus from $\text{Sel}(X_{\mathcal{O}_{k,S'}})_\Pi$ than from $\text{Sel}(X)_\Pi$. This is especially true when $X$ is proper, for which one hopes to set $S = \emptyset$.

We now recall the definition of Selmer variety from [BDCKW18]:

**Definition 3.6.** Given a model $X$ of $X$ over $\mathcal{O}_{k,S}$, we define $\text{Sel}_{S,\Pi}(X)$ as the subset of $\alpha \in H^1(G_k; \Pi)$ for which

- $\alpha \in \kappa_v(X(\mathcal{O}_v))^{\text{Zar}}$ for all $v \notin S$,

where superscript Zar denotes Zariski closure.

Then, almost by definition, we have a map

\[
\kappa : X(\mathcal{O}_{k,S}) \to \text{Sel}_{S,\Pi}(X).
\]

It follows from Fact 3.4.1 that Definition 3.6 agrees with Definition 3.5 when $X$ has potentially good reduction at all $v \in \text{Spec} \, \mathcal{O}_{k,S}$ and good reduction at all $v \in \{p\}$.

We further assume that $X$ has good reduction at all places in $\{p\}$, not just at $p$.

\[\text{xvi}\] If one is worried about having a single model $X$ over $\mathcal{O}_{k,S}$ with potentially good reduction, or even good reduction, one may, by spreading out, expand $S'$ to ensure this is the case.

\[\text{xvii}\] In [Kim09], it is further assumed that $X$ has good reduction at all $v \in \text{Spec} \, \mathcal{O}_{k,S}$, but this does not appear necessary.
In general, $\text{Sel}_{S, \Pi}(\mathcal{X})$ may differ from $H^1_{f, S}(G_k; \Pi)$ when $T_0 \setminus S \neq \emptyset$. This is because of the fact that

$$\kappa_v(\mathcal{X}(\mathcal{O}_v))$$

may be nontrivial for $v \in T_0 \setminus S$.

Nonetheless, when $v \notin \{p\}$, the image is finite by [KT08, Corollary 0.2]. We therefore assume that $\mathcal{X}$ has good reduction at all $v \in \{p\}$ (previously we assumed this only for $v = p$), i.e. that $(T_0 \cup T_1) \cap \{p\} = \emptyset$. This may be arranged by an appropriate choice of $p$.

We now explain, under the assumptions above (that $\{p\}$ is disjoint from $S \cup T_0 \cup T_1$), why the Selmer variety

$$\text{Sel}_{S, \Pi}(\mathcal{X})$$

is a disjoint union of copies of

$$H^1_{f, S}(G_k; \Pi)$$

indexed by the (finite) image of the map

$$\text{Sel}_{S, \Pi}(\mathcal{X}) \to \prod_{v \in T_0 \setminus S} \kappa_v(\mathcal{X}(\mathcal{O}_v)) = \prod_{v \in T_0 \setminus S} \kappa_v(\mathcal{X}(\mathcal{O}_v))^\text{Zar}.$$ 

As in [BD18a] and [BD18b], let $\alpha_1, \ldots, \alpha_N$ denote a set of representatives in $\text{Sel}_{S, \Pi}(\mathcal{X})$ for this image. For $i = 1, \ldots, N$, let

$$\text{Sel}_{S, \Pi}(\mathcal{X})_{\alpha_i} = \{ \alpha \in \text{Sel}_{S, \Pi}(\mathcal{X}) \mid \text{loc}_v(\alpha) = \text{loc}_v(\alpha_i) \forall v \in T_0 \setminus S \},$$

so that

$$\text{Sel}_{S, \Pi}(\mathcal{X}) = \bigsqcup_{i=1}^N \text{Sel}_{S, \Pi}(\mathcal{X})_{\alpha_i}.$$ 

Then by [BD18a, Lemma 2.6] (c.f. also [BD18b, Lemma 2.1] and [Dog20, Lemma 3.1]), we have a natural isomorphism

$$(4) \quad \text{Sel}_{S, \Pi}(\mathcal{X})_{\alpha_i} \cong H^1_{f, S}(G_k; \Pi^{\alpha_i}),$$

where $\Pi^{\alpha_i}$ denotes the twist of $\Pi$ by the cocycle $\alpha_i$.

In particular, $\text{Sel}_{S, \Pi}(\mathcal{X})_{\alpha_i}$ has the same dimension as $H^1_{f, S}(G_k; \Pi^{\alpha_i})$, so its dimension may be computed by the methods of §5.7.

3.5. Local Geometry at Bad Places. Given $z \in \mathcal{X}(\mathcal{O}_{k, S})$, we may use [BD20, Theorem 1.3.1] to determine the $i$ for which $\kappa(z) \in \text{Sel}_{S, \Pi}(\mathcal{X})_{\alpha_i}$. For $v \in T_0 \setminus S$, let $l_v/k_v$ be a finite extension over which $\mathcal{X}$ has semistable reduction, and let $\mathcal{X}^{\text{-ss}}$ denote a regular semistable integral model of $\mathcal{X}_{l_v}$. Let $\Gamma_v = \Gamma_v(\mathcal{X}^{\text{-ss}})$ denote the reduction graph of $\mathcal{X}$, so that

$$E(\Gamma_v)$$

is the set of irreducible components of the special fiber of $\mathcal{X}^{\text{-ss}}$.

Then there is a natural map $\mathcal{X}^{\text{-ss}}(\mathcal{O}_{l_v}) \to E(\Gamma_v)$, hence a map

$$\text{red}_v : X(k) \to X(l_v) \to \mathcal{X}(l_v) = \mathcal{X}^{\text{-ss}}(\mathcal{O}_{l_v}) \to E(\Gamma_v).$$

**Fact 3.7** ([BD20, Theorem 1.3.1]). Let $\mathcal{X}^{\text{-ss}}$ be an integral model of $X_{l_v}$ equal to the complement in $\mathcal{X}^{\text{-ss}}$ of a horizontal divisor $D$. Let $x, y \in X(k_v) \cap \mathcal{X}^{\text{-ss}}(\mathcal{O}_{l_v}) \subseteq X(l_v)$. Then

1. If $\text{red}_v(x) = \text{red}_v(y)$, then $\kappa_v(x) = \kappa_v(y)$
2. If $\Pi$ dominates $U_3(X)$, and $\kappa_v(x) = \kappa_v(y)$, then $\text{red}_v(x) = \text{red}_v(y)$. 

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Remark 3.8. [BD20, Theorem 1.3.1] also requires the boundary divisor $D$ to be étale over $\mathcal{O}_v$. One may arrange this by blowing up points on the boundary, but this does not change $\text{red}_v|_{\mathcal{X}^{r-ss}(\mathcal{O}_v)}$ and is therefore not strictly necessary.

Remark 3.9. If $x$ or $y$ is not in $\mathcal{X}^{r-ss}(\mathcal{O}_v)$, then the truth of “$\text{red}_v(x) = \text{red}_v(y)$” might depend on the chosen integral model $\overline{\mathcal{X}}^{r-ss}$. An example, suggested to us by A. Betts, is provided by $\mathcal{X} = \mathcal{X}^{r-ss} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $v = v_3$, $x = 2$, and $y = 3$. Then $\text{red}_v(x) = \text{red}_v(y)$ because $|E(\Gamma_v(\overline{\mathcal{X}}^{r-ss}))| = 1$, but $\kappa_v(x) \neq \kappa_v(y)$, at least for $\Pi$ dominating $U_3$.

Remark 3.10. Suppose $x, y \in X(l_v) \setminus \mathcal{X}^{r-ss}(\mathcal{O}_v)$, and $\text{red}_v(x) = \text{red}_v(y)$. Then we still have $\kappa_v(x) = \kappa_v(y)$ for all $\Pi$ factoring through the quotient map

$$U(X) \rightarrow U(\overline{X})$$

but not for general quotients $\Pi$ of $U(X)$.

Relatedly, if $\kappa_v(x) = \kappa_v(y)$ for any $\Pi$ dominating $U_3(\overline{X})$, then we have $\text{red}_v(x) = \text{red}_v(y)$, regardless of whether $x, y$ are integral.

In practice, we will choose a model $\mathcal{X}$ for $X$ over $\mathcal{O}_v$ (or even $\mathcal{O}_{k,S}$) for which it is clear that all elements of $\mathcal{X}(\mathcal{O}_v)$ extend to elements of $\mathcal{X}^{r-ss}(\mathcal{O}_v)$.

In general, one may do this as follows. We suppose that $\mathcal{X}$ is given with a compactification $\mathcal{X} \subseteq \overline{\mathcal{X}}$. First, choose $l_v/k_v$ for which $X$ has semistable reduction. Then we may apply Lipman’s resolution of singularities to $\overline{\mathcal{X}}_{l_v}$ to obtain a regular semistable model $\overline{\mathcal{X}}^{r-ss}$. At each stage, we choose a model of $X_{l_v}$ in our model of $\overline{\mathcal{X}}_{l_v}$ by taking the strict transform (not the preimage) of the boundary divisor.

In one example, $\mathcal{X}$ will be the minimal Weierstrass model of a punctured elliptic curve with semistable reduction at all $v \in T_0 \setminus S = \{3, 17\}$, so that $l_v = k_v$. The model is already regular at 17, but at 3 we obtain a regular model $\overline{\mathcal{X}}^{r-ss}$ by blowing up at the unique singular point. Then the smooth locus of $\overline{\mathcal{X}}^{r-ss}$ is the Néron model of $E = \overline{X}$, and the reduction of a point is its image in the Néron component group.

4. Mixed Elliptic Case

Definition 4.1. We say that a curve $X/k$ is mixed elliptic if the Jacobian $J_X$ of the smooth compactification $\overline{X}$ of $X$ is isogenous to a power of an elliptic curve.

When $X$ is mixed elliptic for an elliptic curve $E$, the Galois representations associated to $U(X)$ and its torsors will lie in a certain subcategory $\text{Rep}_{Q_p}^\text{ss}(G_k, E)$ (Definition 4.4) of $\text{Rep}_{Q_p}^k(G_k)$. We spend the rest of §4 discussing this subcategory and its variants.

Fix a non-CM elliptic curve $E$ over a number field $k$. Since we work only with $p$-adic realizations, we introduce the notation

$$h_1(E) := H_1^\text{ét}(E_{\overline{k}}; \mathbb{Q}_p) = H_1^\text{ét}(E_{\overline{k}}; \mathbb{Q}_p(1)) = T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$ 

We let

$$M_{a,b} := \text{Sym}^a(h_1(E))(b).$$

As $a$ ranges over non-negative integers and $b$ ranges over all integers, these give the set $\text{Irr}_{GL_2}$ of irreducible objects of the Tannakian subcategory $\text{Rep}_{Q_p}^\text{ss}(G_k, E)$ of $\text{Rep}_{Q_p}^k(G_k)$ generated by $h_1(E)$.

It follows from Serre’s Open Image Theorem ([Ser98]) that:
Fact 4.2. The category $\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k, E)$ is a semisimple Tannakian category equivalent to $\text{Rep}_{\mathbb{Q}_p}(\text{GL}_2)$. Furthermore, for a finite extension $l/k$, the functor $\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k, E) \to \text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_l, E)$ induced by restriction along $G_l \hookrightarrow G_k$ is an equivalence.

The weight of $M_{a,b}$ as a Galois representation is

$$w = -a - 2b.$$ 

For $V = M_{a,b}$, we have

$$V^\vee(1) \cong M_{-a-b+1},$$

which has weight $-w - 2$.

Motivated by the desire to check cases of the inequality (3), we would like to determine the conjectured values of

$$d_{a,b} = h_1^f(G_k; M_{a,b}) := \dim_{\mathbb{Q}_p} H_1^f(G_k; M_{a,b})$$

for $M_{a,b} \in \text{Irr GL}_2$ that appear as subquotients of $U(X)$ for $X$ mixed elliptic. In §5 we will describe $h_1^f(G_k; M_{a,b})$ when $M_{a,b} \in \text{Irr}^{\text{ae}}(\text{GL}_2)$, the latter defined as follows:

Definition 4.3. We say that $M_{a,b}$ is anti-effective\footnote{Motives arising from the cohomology of varieties are often called effective. As this is the subring arising from the homology of varieties, we have chosen to call them anti-effective.} if one of the following equivalent conditions holds:

1. $M_{a,b}$ is a subquotient of $h_1(E)^{\otimes w}$
2. $M_{a,b}$ is a subquotient of $\mathbb{Q}_p$-unipotent étale fundamental group $U(X)$ for $X$ hyperbolic and mixed elliptic
3. $M_{a,b}$ is a subquotient of $h_1(Y)$ for an algebraic variety $Y$
4. $b \geq 0$

We denote the set of such $M_{a,b}$ by $\text{Irr}^{\text{ae}}(\text{GL}_2)$.

Definition 4.4. Let

$$\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k, E)$$

denote the thick subcategory of $\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k)$ generated by $\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k, E)$

We then have

$$H_1^f(G_k; M_{a,b}) = \text{Ext}_{\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k, E)}^1(\mathbb{Q}_p(0), M_{a,b})$$

if $w = -a - 2b < 0$.

Remark 4.5. We could instead use $\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k, E)$, defined as the corresponding thick subcategory of $\text{Rep}_{\mathbb{Q}_p}(G_k)$. This is conjecturally the same as $\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k, E)$ by Conjecture 2.1. Then we would know $H_1^f(G_k; M_{a,b}) = \text{Ext}_{\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k, E)}^1(\mathbb{Q}_p(0), M_{a,b})$ for all $a, b$ without assuming Conjecture 2.2. Nonetheless, we find it more convenient in §8 to use the category $\text{Rep}_{\mathbb{Q}_p}^{\text{sg}}(G_k, E)$, as we will make use of the weight filtration. Doing so presents no trouble because the fundamental group of a curve has strictly negative weights, and all path torsors arise from geometry and therefore have a weight filtration.

Remark 4.6. The case $w = -1$ corresponds to $h_1(E) = M_{1,0}$, in which case $d_{a,b}$ is the dimension of the $p$-adic Selmer group, which the BSD conjecture predicts to be equal to the Mordell-Weil rank of $E$.\footnote{Motives arising from the cohomology of varieties are often called effective. As this is the subring arising from the homology of varieties, we have chosen to call them anti-effective.}
Remark 4.7. Conjecturally, there is a category $\mathcal{MM}(k, \mathbb{Q})$ of mixed motives over $k$ with $\mathbb{Q}$-coefficients. It should have a realization functor

$$\mathcal{MM}(k, \mathbb{Q}) \xrightarrow{\text{real}_p} \text{Rep}_{\mathbb{Q}_p}^{sg}(G_k)$$

for which the induced functor $\mathcal{MM}(k, \mathbb{Q}_p) := \mathcal{MM}(k, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \text{Rep}_{\mathbb{Q}_p}^{sg}(G_k)$ \footnote{The tensor product here is base extension from $\mathbb{Q}$-linear categories to $\mathbb{Q}_p$-linear categories; in particular, changes the class of objects as well as the Hom-sets.} is an equivalence. This equivalence amounts to the Fontaine-Mazur, Tate, and Bloch-Kato conjectures. The latter means that the map induced by the realization functor

$$\text{Ext}^1_{\mathcal{MM}(k, \mathbb{Q})}(\mathbb{Q}(0), M_{a,b}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to H^1_d(G_k; M_{a,b})$$

is an isomorphism.

Remark 4.8. The subcategory of $\mathcal{MM}(k, \mathbb{Q})$ corresponding under $\text{real}_p$ to $\text{Rep}_{\mathbb{Q}_p}^{sg}(G_k, E)$ is known as the category of mixed elliptic motives. A candidate for this category was constructed in [Pat13].

Remark 4.9. Nonconjecturally, as mentioned in [Con98], we may define

$$\text{Ext}^1_{\mathcal{MM}(k, \mathbb{Q})}(\mathbb{Q}(0), M_{a,b})$$

as

$$K_{a+2b-1}(E^{(a)})^{(a+b)} \otimes_{\mathbb{Q}} \mathbb{Q},$$

where $E^{(a)}$ denotes the kernel of the addition map $E^{a+1} \to E$, and $\text{sgn}$ denotes the part on which $S_{a+1}$ acts by the sign representation. Note also that

$$K_{a+2b-1}(E^{(a)})^{(a+b)} \otimes \mathbb{Q} \cong CH^{a+b}(E^{(a)}, a+2b-1) \otimes \mathbb{Q} \cong H^{a+1}(E^{(a)}, \mathbb{Q}(a+b))$$

as $S_{a+1}$-representations.

In the notation of loc.cit., our $M_{a,b}$ is $\text{Sym}^a \mathcal{H}(b)$.

4.1. $S$-Integral Elliptic Motives. When $X$ is proper, the question of determining $X(k)$ is the same as that of determining $\mathcal{X}(\mathcal{O}_k)$, so we will use $H^1_f$ to define Selmer varieties in \S3. If $X$ is not proper, we may consider $\mathcal{O}_{k,S}$-points, and we must then replace $H^1_f$ by $H^1_{f,S}$, so we set:

$$d^S_{a,b} := h^1_{f,S}(G_k; M_{a,b}) = \dim H^1_{f,S}(G_k; M_{a,b}).$$

It is well-known that $d^S_{1,0}$ is independent of $S$, while $d^S_{0,1}$ is very much dependent on $S$, as is the rank of $\mathbb{G}_m(\mathcal{O}_{k,S})$.

Let $S' = S \cup \{v\}$ with $v \notin S \cup \{p\}$. As mentioned in \S2.5, we have

$$H^1_{f,S}(G_k; M_{a,b}) = H^1_{f,S'}(G_k; M_{a,b})$$

whenever $\mathbb{Q}_p(1)$ is not a quotient of $M_{a,b}|_{G_v}$, which happens in particular if $-2$ is not one of its weights.

By Remark 2.10, the Frobenius weights of $M_{1,0}|_{G_v}$ are $\{-1\}$ if $E$ has potentially good reduction at $v$ and $\{0, -2\}$ otherwise. As a result, $M_{a,b}|_{G_v}$ has Frobenius weights $\{-a-2b\}$ if $E$ has potentially good reduction at $v$ and weights $\{-2a-2b, -2a-2b+2, \ldots, -2b-2, -2b\}$ otherwise. We thus find by Corollary 2.9 that $d^S_{a,b} = d^S_{a,b}$ whenever.

Remark 4.10. If $E$ has good reduction over $\mathcal{O}_{k,S}$, we set $\mathbf{Rep}_{\mathbb{Q}_p}^{f,S}(G_k; E)$ to be the intersection of $\mathbf{Rep}_{\mathbb{Q}_p}^{s}(G_k; E)$ with $\mathbf{Rep}_{\mathbb{Q}_p}^{f,S}(G_k)$. More generally, for arbitrary $E$ and $\mathcal{O}_{k,S}$, we define $\mathbf{Rep}_{\mathbb{Q}_p}^{f,S}(G_k, E)$ to be the subcategory of $V \in \mathbf{Rep}_{\mathbb{Q}_p}^{s}(G_k, E)$ such that for every place $v \notin S$

- If $v \nmid p$, the weight filtration of $V$ splits as a representation of $I_{k,v}$.
- If $v | p$, the weight filtration of $V \otimes B_{\text{cris}}$ splits as a representation of $G_v$. Then

$$H^1_{f,S}(G_k; M_{a,b}) = \text{Ext}^1_{\mathbf{Rep}_{\mathbb{Q}_p}^{f,S}(G_k, E)}(\mathbb{Q}_p(0), M_{a,b})$$

for $w = -a - 2b < 0$.

Remark 4.11. We may analogously define categories $\mathbf{Rep}_{\mathbb{Q}_p}^{s}(G_k, A)$, $\mathbf{Rep}_{\mathbb{Q}_p}^{s}(G_k, A)$, and $\mathbf{Rep}_{\mathbb{Q}_p}^{f,S}(G_k, A)$ for any abelian variety $A$ in place of $E$. Then $\mathbf{Rep}_{\mathbb{Q}_p}^{s}(G_k, A)$ will be equivalent to the category of representations of the Mumford-Tate group of $A$. To apply Chabauty-Kim to a curve $X$, one may take $A$ to be the Jacobian of $X$. We plan to work with these in the future, the only obstacle being the messiness of the representation theory of reductive groups larger than $\text{GL}_2$.

5. Ranks of Global Selmer Groups

We find formulas for $d_{a,b}$ when $k = \mathbb{Q}$, $w \leq -2$, and $(a, b) \neq (0,1)$. Let $V = M_{a,b}$. Since $V \neq \mathbb{Q}_p(1)$, we have $h^0(G_k; V) = 0$, and since $w \leq -2$, we have $h^1_f(G_k; V^{\vee}(1)) = h^0(G_k; V^{\vee}(1)) = 0$. Thus (2) becomes

$$h^1_f(G_k; V) = \sum_{v \mid p} \dim_{\mathbb{Q}_p} (D_{\text{dR}} V / D^+_{\text{dR}} V) - \sum_{v \mid \infty} h^0(G_v; V)$$

5.1. The Case $k = \mathbb{Q}$. There is one place above $p$ and one place above $\infty$, so we get

$$h^1_f(G_k; V) = \dim D_{\text{dR}} V / D^+_{\text{dR}} V - h^0(G_{k_{\infty}}, V).$$

We first consider $\dim D_{\text{dR}} V / D^+_{\text{dR}} V$. If $b \geq 1$, then all Hodge weights are negative, so we get

$$\dim D_{\text{dR}} V / D^+_{\text{dR}} V = \dim V = a + 1.$$
If $b = 0$, then $D^+_d \ V$ is one-dimensional, so we get
$$\dim D^d \ V / D^+_d \ V = a.$$  

We next consider the term $h^0(G_{k\infty}, V)$. Note that $V$ is isomorphic to the regular representation of $G_{k\infty} \cong C_2$. If $a$ is odd, then $\text{Sym}^a(V) \cong V^@ \oplus \frac{a+1}{2}$ as a $C_2$-representation. Thus, we have $h^0(G_{k\infty}, V) = \frac{a+1}{2}$. If $a$ is even, then $\text{Sym}^a(V)$ has $\frac{a}{2} + 1$ copies of the trivial representation and $\frac{a}{2}$ copies of the sign representation. Therefore, if $b$ is even, then $h^0(G_{k\infty}, V) = \frac{a}{2} + 1$, and if $b$ is odd, then $h^0(G_{k\infty}, V) = \frac{a}{2}$. In summary, we have
$$h^0(G_{k\infty}, V) = \begin{cases} 
\frac{a+1}{2}, & a \text{ odd} \\
\frac{a}{2} + 1, & a \text{ even, } b \text{ even} \\
\frac{a}{2}, & a \text{ even, } b \text{ odd}
\end{cases}.$$  

In summary, we have
$$d_{a,b} = \dim D^d \ V / D^+_d \ V - h^0(G_{k\infty}, V) = \begin{cases} 
\frac{a-1}{2}, & a \text{ odd, } b = 0 \\
\frac{a}{2} - 1, & a \text{ even, } b = 0 \\
\frac{a+1}{2}, & a \text{ odd, } b \geq 1 \\
\frac{a}{2}, & a \text{ even, } b \text{ even and } b \geq 1
\end{cases}.$$  

The relevant values are $d_{1,\geq 1}, d_{2,\geq 1}, d_{3,\geq 0}, d_{4,\geq 0}, d_{5,\geq -1}, d_{6,\geq -1}, d_{7,\geq -2}, \cdots$. We record the values relevant to non-abelian Chabauty up to degree 5:

\[
\begin{align*}
d_{2,0} &= 0 \\
d_{3,0} &= 1 \\
d_{1,1} &= 1 \\
d_{0,2} &= 0 \\
d_{2,1} &= 2 \\
d_{4,0} &= 1 \\
d_{1,2} &= 1 \\
d_{3,1} &= 2 \\
d_{5,0} &= 2
\end{align*}
\]

5.2. The Case of $k$ Imaginary Quadratic. We consider Selmer groups over $G_k$, where $k$ is an imaginary quadratic field, under the assumption that $E$ is defined over $\mathbb{Q}$.

There is one place $v$ above $\infty$, and it is complex, so we simply have $h^0(G_{k\infty}, V) = \dim_{\mathbb{Q}_p} V$. For $M_{a,b}$, this is $a + 1$.

For $v \mid p$, there are two possibilities: 1) $v$ splits into two places 2) $v$ is inert or ramified. However, in either case, the contributions is the same; more specifically it is twice the contribution in the case $k = \mathbb{Q}$. In the first case, this is because we sum over the two $v \mid p$, and in the latter case, it is because $D^d \ V$ outputs a vector space over $K_v$, whose dimension over $\mathbb{Q}_p$ is twice dimension over $K_v$. Thus the contribution is $2a$ for $b = 0$ and $2a + 2$ for $b \geq 0$. 

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We thus have

\[ h_f^1(G_k; V) = \begin{cases} 
  a - 1, & b = 0 \\
  a + 1, & b \geq 1 
\end{cases} \, . \]

In weight \( w \geq -5 \), we list the values:

\[
\begin{align*}
    d_{2,0} &= 1 \\
    d_{3,0} &= 2 \\
    d_{1,1} &= 2 \\
    d_{0,2} &= 1 \\
    d_{2,1} &= 3 \\
    d_{4,0} &= 3 \\
    d_{1,2} &= 2 \\
    d_{3,1} &= 4 \\
    d_{5,0} &= 4 
\end{align*}
\]

### 5.3. \textit{S-Integral Global Selmer Groups.}

We suppose \( S \cap \{p\} = \emptyset \). By the discussion in §4.1 we have

\[
\begin{align*}
    d_{S}^{0,0} &= d_{1,0} \\
    d_{S}^{0,2} &= d_{0,2} \\
    d_{S}^{1,2} &= d_{1,2} \\
    d_{0,1}^S &= |S| \\
    d_{1,1}^S &= 1 + \delta_S(E). 
\end{align*}
\]

Finally, let

\[ \delta_S(E) \]

denote the number of places in \( S \) at which \( E \) has split multiplicative reduction. Then

\[ d_{0,1}^S = |S| \]

For \( k = \mathbb{Q} \), we have

For \( k = \mathbb{Q} \), we have

\[ d_{0,1}^S = |S| \]

In order to check the inequality (8), we need to compute dimensions of local Selmer varieties. To do so, we must compute the ranks of local Selmer groups.

Let \( V = M_{a,b} \) have negative weight, and fix \( \mathfrak{p} \in \{p\} \) of good reduction for \( E \). Then Bel09 Proposition 2.8 (or [FPR94, I.3.3.11]) gives the formula

\[ l_{a,b} := h_f^1(G_{\mathfrak{p}}; V) = \dim_{\mathbb{Q}_p}(D_{dR} V / D_{dR}^+ V). \]

Let us suppose that \( k_{\mathfrak{p}} \cong \mathbb{Q}_p \). In our examples, we take either \( k = \mathbb{Q} \) or \( \mathfrak{p} \) split in an imaginary quadratic field \( k \).

As noted previously, we have

\[ H_f^1(G_{\mathfrak{p}}; V) = \dim_{\mathbb{Q}_p}(D_{dR} V / D_{dR}^+ V) = \begin{cases} 
  a, & b = 0 \\
  a + 1, & b \geq 1 
\end{cases} \, . \]
We record these values in weight $w \geq -5$:

\[
\begin{align*}
l_{1,0} &= 1 \\
l_{0,1} &= 1 \\
l_{2,0} &= 2 \\
l_{3,0} &= 3 \\
l_{1,1} &= 2 \\
l_{0,2} &= 1 \\
l_{2,1} &= 3 \\
l_{4,0} &= 4 \\
l_{1,2} &= 2 \\
l_{3,1} &= 4 \\
l_{5,0} &= 5
\end{align*}
\]

6.1. Difference Between Local and Global. The inequality \([3]\) holds iff the difference between the dimensions of the local and global Selmer varieties is positive. We’re therefore particularly interested in the difference

$$c_{a,b} := l_{a,b} - d_{a,b}.$$  

If $\Pi$ denotes a finite-dimensional Galois-equivariant quotient of $U(X)$, then Chabauty-Kim for $\Pi$ produces a finite set iff the sum of $c_{a,b}$ over all weight-graded pieces of $\Pi$ (counted with multiplicity) is positive.

We let $r$ denote the rank of $E(k)$. We assume that this equals the $p$-Selmer rank $d_{1,0}$, an assumption that may be verified computationally. We also note that $d_{0,1} = 0$ for $k = \mathbb{Q}$ and $k$ imaginary quadratic. We explain the modification necessary for $S$-integral points below.

For $k = \mathbb{Q}$, we have

\[
\begin{align*}
c_{1,0} &= 1 - r \\
c_{0,1} &= 1 \\
c_{2,0} &= 2 \\
c_{3,0} &= 2 \\
c_{1,1} &= 1 \\
c_{0,2} &= 1 \\
c_{2,1} &= 1 \\
c_{4,0} &= 3 \\
c_{1,2} &= 1 \\
c_{3,1} &= 2 \\
c_{5,0} &= 3
\end{align*}
\]
For $k$ imaginary quadratic and $v$ split in $k$, we have
\[
\begin{align*}
    c_{1,0} &= 1 - r \\
    c_{0,1} &= 1 \\
    c_{2,0} &= 1 \\
    c_{3,0} &= 1 \\
    c_{1,1} &= 0 \\
    c_{0,2} &= 0 \\
    c_{2,1} &= 0 \\
    c_{4,0} &= 1 \\
    c_{1,2} &= 0 \\
    c_{3,1} &= 0 \\
    c_{5,0} &= 1.
\end{align*}
\]

For a set $S$ of places of $k$, we set $c^S_{a,b} := l_{a,b} - d^S_{a,b}$. Then
\[
\begin{align*}
    c^S_{1,0} &= c_{1,0} \\
    c^S_{0,2} &= c_{0,2} \\
    c^S_{1,2} &= c_{1,2} \\
\end{align*}
\]

For $k = \mathbb{Q}$, we have
\[
\begin{align*}
    c^S_{0,1} &= 1 - |S| \\
    c^S_{1,1} &= 1 - \delta_S(E)
\end{align*}
\]

7. Motivic Decomposition of Fundamental Groups

Let $X$ be a smooth curve over a number field $k$. To simplify notation, we fix an implicit basepoint $b \in X(k)$ and and set
\[
U = U(X) := \pi^{\text{et}, \text{un}}_1(X, b)_{\mathbb{Q}_p}
\]
and
\[
\begin{align*}
    U^1 &= U \\
    U^{n+1} &= [U, U^n] \\
    U_n &= U/U^{n+1} \\
    U[n] &= U^n/U^{n+1},
\end{align*}
\]
where commutator denotes the closure of the group-theoretic commutator. Notice the short exact sequence
\[
0 \to U[n] \to U_n \to U_{n-1} \to 0.
\]

We let $\Pi$ be a finite-dimensional Galois-equivariant quotient of $U$. In particular, $\Pi$ factors through $U_n$ for some $n$. The main goal of this section is to explain how to use the results of §5-6 to bound $\dim H^1_f(G_k, \Pi)$ and compute $\dim H^1_f(G_{\mathbb{F}_p}, \Pi)$. The key to doing this is
understanding a certain class of $\Pi$ in the ring $K_0(\operatorname{Rep}_{\overline{Q}_p}(G_k))$, explained in §7.1. Doing this will produce an algorithm for checking when the inequality \((3)\) is satisfied.\footnote{More precisely, it allows us to either verify that the inequality is satisfied or that Conjecture \ref{conjecture:crystalline} implies that the inequality is not satisfied.}

Finally, for the purposes of comparing with Quadratic Chabauty, we define

$$U_Q = U_Q(X)$$

to be the quotient of $U_2$ by the maximal subspace of $U[2]$ with no subrepresentation isomorphic to $\overline{Q}_p(1)$.

Remark 7.1. All computations in this section are completely independent of $b$, essentially because the graded pieces $U[n]$ are homological in nature.

7.1. $K_0$ Classes of Unipotent Groups with Galois Action. We define linear functions

$$d, l, d^S : K_0(\operatorname{Rep}_{\overline{Q}_p}(G_k)) \to \mathbb{Z}$$

by $d([M_{a,b}]) = d_{a,b}$ (resp. $l([M_{a,b}]) = l_{a,b}$, $d^S([M_{a,b}]) = d^S_{a,b}$). We set

$$c := l - d$$

and $c^S := l - d^S$.

To such subquotient $\Pi$ of $U$, we associate a class $[\Pi]$ in the ring

$$K_0(\operatorname{Rep}_{\overline{Q}_p}(G_k))$$

defined by requiring

$$[\Pi]_2 = [\Pi]_1 + [\Pi]_3$$

when there is a short exact sequence

(5) \quad 0 \to \Pi_1 \to \Pi_2 \to \Pi_3 \to 0

and that if $\Pi$ is abelian, then $[\Pi]$ is the class of the corresponding Galois representation.

We use the notation

$$d(\Pi)$$

(resp. $l(\Pi)$, $c(\Pi)$, $d^S(\Pi)$, $c^S(\Pi)$) to refer to $d([\Pi])$ (resp. $l([\Pi])$, $c([\Pi])$, $d^S([\Pi])$, $c^S([\Pi])$).

By the short exact sequence for Galois cohomology, we have

(6) \quad \dim H^1_{f, S}(G_k, \Pi) \leq d^S(\Pi)

(7) \quad \dim H^1_f(G_p; \Pi) = l(\Pi),

the latter by the fact that the weights are negative and that crystalline $H^2$ vanishes.

Remark 7.2. In fact, it would follow from Conjecture \ref{conjecture:crystalline} below that

$$\dim H^1_{f, S}(G_k, \Pi) \leq d^S(\Pi).$$

Our goal for the rest of this section is to explain how to compute the class of $\Pi$ in $K_0(\operatorname{Rep}_{\overline{Q}_p}(G_k))$.

For simplicity, we focus on the case $\Pi = U_n$ for a positive integer $n$. The group $U_n$ may be identified via the Lie exponential with its Lie algebra $\operatorname{Lie} U_n$, which has the structure of a $p$-adic Galois representation. Let $W$ denote the weight filtration and $G^W$ the associated graded for the weight filtration.
We suppose \( k \) is a positive integer less than or equal to \( n \), until otherwise specified. The \(-k\)th weight-graded piece is
\[
\text{Gr}_{-k}^W \text{Lie} U_n = U[k].
\]
In particular, we have
\[
[Lie U_n] = [U_n] = \sum_{k=1}^{n} [U[k]] \in K_0(\text{Rep}_{\mathbb{Q}_p}^{sg}(G_k))
\]

7.2. Decomposition of \( U[k] \) in terms of \( U \). We outline a general procedure for computing the class of \( U_n \) in \( K_0(\text{Rep}_{\mathbb{Q}_p}^{sg}(G_k)) \) in terms of \( h_1(X) \).

As \( U_n \) is a unipotent group, we have \( \mathcal{O}(U_n) \cong \text{Sym} \text{Lie} U_n \) as \( \mathbb{Q}_p \)-algebras, in a way that respects Galois action on associated graded. We thus have
\[
\text{Gr}^W_0 \mathcal{O}(U_n) \cong \text{Gr}^W_0 \text{Sym} \text{Lie} U_n.
\]

If we know the structure of \( \text{Gr}^W_0 \mathcal{O}(U_n) \) as a motive, this allows us to inductively compute the structure of \( U^k/U^{k+1} \) as follows. We have
\[
\text{Gr}^W_{-k} \mathcal{O}(U_n) = \text{Gr}^W_{-k} \text{Sym} \text{Lie} U_n = \text{Gr}^W_{-k} \text{Sym} \text{Gr}^W_{\geq -k} \text{Lie} U_n = \text{Gr}^W_{-k} \text{Lie} U_n \oplus \text{Gr}^W_{-k} \text{Sym} \text{Gr}^W_{\geq -k} \text{Lie} U_n
\]
\[
= U[k] \oplus \text{Gr}^W_{-k} \text{Sym}(\oplus_{i=1}^{k-1} U[i])
\]

Note that \( n \) is irrelevant here, as long as \( n \geq k \). We thus get
\[
[U^k/U^{k+1}] = [\text{Gr}^W_{-k} \mathcal{O}(U_n)] - [\text{Gr}^W_{-k} \text{Sym}(\oplus_{i=1}^{k-1} U[i])] = \text{pr}_{-k}[\mathcal{O}(U_n)] - [\text{Sym}(\oplus_{i=1}^{k-1} U[i])]
\]

The term \( \text{pr}_{-k}[\text{Sym}(\oplus_{i=1}^{k-1} U[i])] \) decomposes according to nontrivial partitions of \( k \). We represent a partition by a sequence \( n_1, \cdots, n_k \) for which \( \sum_{i=1}^{k} i n_i = k \), and we call it nontrivial if \( n_k = 0 \). We then have
\[
(8) \quad \text{pr}_{-k}[\text{Sym}(\oplus_{i=1}^{k-1} U[i])] = \sum_{\sum_{i=1}^{k-1} i n_i = k} \prod_{j=1}^{k-1} [\text{Sym}^{n_j}(U[j])].
\]

If \( X \) is affine, then \( U \) is a free pro-unipotent group, and we have
\[
\text{Gr}^W_{-k} \mathcal{O}(U_n) \cong h_1(X)^{\otimes k}.
\]

7.2.1. Projective Case. Suppose \( X \) is projective, and let \( X' \) be the complement of a point in \( X \). We set \( U := U(X) \) and \( U' := U(X') \). Then
\[
\text{Lie} U' \cong \text{FreeLie} h_1(X)
\]
is free on \( h_1(X) = h_1(X') \), while
\[
U
\]
is the quotient of \( U' \) by an element of \( U'^2 \setminus U'^3 \) on which \( \text{GL}_2 \) acts as \( M_{0,1} \). More precisely, this element corresponds to the dual
\[
h_2(X) \to \wedge^2 h_1(X) \cong U(X')[2]
\]
of the intersection pairing \( \wedge^2 h_1(X) \to h^2(X) \).
For $k = 1, 2, 3$, this does nothing more than remove a copy of $h_2(X) \cong \mathbb{Q}_p(1)$ from $U'[2]$ and remove a copy of $h_1(X)(1)$ from $U'[3]$. In the projective case, we prefer to first compute the associated graded of the Lie algebra as if it were affine and then mod out by the appropriate Lie ideal.

7.3. **Elliptic Motive Case.** We suppose for the rest of §7 that $X$ is mixed elliptic.

In this case, every $[\Pi]$ is in the subring

$$K_0(\text{Rep}_{\mathbb{Q}_p}^{ss}(G_k, E)) = K_0(\text{Rep}_{\mathbb{Q}_p}^{s}(G_k, E)) \subseteq K_0(\text{Rep}_{\mathbb{Q}_p}^{ss}(G_k)),$$

which is naturally the free abelian group

$$\bigoplus_{a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}} \mathbb{Z}[M_{a,b}],$$

with product determined by the rule

$$[M_{1,b}][M_{k,b_2}] = [M_{k+1,b_1+b_2}] + [M_{k-1,b_1+b_2+1}],$$

where $[M_{-1,b}] = 0$ by convention. More precisely, it lies in the anti-effective subring

$$K_0(\text{Rep}_{\mathbb{Q}_p}^{ss}(G_k, E))^{\text{ae}} := \bigoplus_{a,b \geq 0} \mathbb{Z}[M_{a,b}].$$

We will need to use the decomposition of tensor powers of $h_1(E)$ in terms of the $M_{a,b}$. One may compute the following using (9):

$$[h_1(E)]^2 = [M_{2,0}] + [M_{0,1}]$$

$$[h_1(E)]^3 = [M_{3,0}] + 2[M_{1,1}]$$

$$[h_1(E)]^4 = [M_{4,0}] + 3[M_{2,1}] + 2[M_{0,2}]$$

$$[h_1(E)]^5 = [M_{5,0}] + 4[M_{3,1}] + 5[M_{1,2}]$$

7.4. **Explicit Decomposition for a Punctured Elliptic Curve.** Let $X = E' = E \setminus \{O\}$ for a non-CM elliptic curve $E$. In this case, we determine $[U[k]]$ for $k = 1, 2, 3, 4$. The cases $k = 1, 2, 3$ will be used for the examples of §10.12.3.

7.4.1. **Level 1.** We have $U_1 = U[1] \cong h_1(X)$. Thus $[U_1] = [h_1(X)] = [M_{1,0}]$.

7.4.2. **Level 2.** We next have $[U[2]] = [h_1(X)^{\otimes 2}] - [\text{Sym}^2 h_1(X)] = [\wedge^2 h_1(X)] = [M_{0,1}]$.

7.4.3. **Level 3.** We get $\text{pr}_{-2}[\text{Sym}(U[1] \oplus U[2])] = [\text{Sym}^3 U[1]] + [U[1]][U[2]] = [M_{3,0}] + [M_{1,1}]$.

We also have $[h_1(E)^{\otimes 3}] = [h_1(E)]^3 = [M_{3,0}] + 2[M_{1,1}]$. Thus

$$[U[3]] = [M_{3,0}] + 2[M_{1,1}] - ([M_{3,0}] + [M_{1,1}]) = [M_{1,1}]$$

7.4.4. **Level 4.** We get

$$\text{pr}_{-4}[\text{Sym}(U_1 \oplus U[2] \oplus U[3])] = [\text{Sym}^4 U_1 + [U_1][U[3]] + [\text{Sym}^2 U_1][U[2]] + [\text{Sym}^2 U[2]]$$

$$= [M_{4,0}] + [M_{1,0}] + [M_{2,0}] + [M_{0,1}] + [M_{0,1}]^2$$

$$= [M_{4,0}] + [M_{2,1}] + [M_{0,2}] + [M_{2,1}] + [M_{0,2}]$$

$$= [M_{4,0}] + 2[M_{2,1}] + 2[M_{0,2}]$$

We thus get

$$[U^4/U^5] = [h_1(E)]^4 - ([M_{4,0}] + 2[M_{2,1}] + 2[M_{0,2}]) = [M_{2,1}].$$
7.4.5. **Dimensions.** We compute $c^S(U_2)$ and $c^S(U_3)$ for $k = \mathbb{Q}$. We have
\[
c^S(U_2) = c^S(U[1]) + c^S(U[2])
\]
\[
= c^S_{1,0} + c^S_{0,1}
\]
\[
= 1 - r + (1 - |S|)
\]
\[
= 2 - r - |S|.
\]

In particular, Quadratic Chabauty may fail when $r + |S| \geq 2$ (note that $U_2 = U_Q$ for $X = E'$). We consider the case $r = |S| = 1$ in the examples of §10.4.3.

On the other hand, we have
\[
c^S(U_3) = c^S(U_2) + c^S(U[3])
\]
\[
= 2 - r - |S| + c^S_{1,1}
\]
\[
= 2 - r - |S| + 1 - \delta_S(E)
\]
\[
= 3 - r - |S| - \delta_S(E).
\]

In particular, if $r = |S| = 1$, and $\delta_S(E) = 0$, then the Chabauty-Kim method will give finiteness in level 3.

7.5. **Formulas for Projective Curves in Levels $\leq 3$.** We suppose that $X = \overline{X}$ is projective, so that
\[
h_1(X) \cong h_1(\overline{X}) \cong h_1(E)^g,
\]
where $g$ is the genus of $X$. We recall from §7.2.1 that $X'$ is $X$ punctured at one point, and $U' = U(X')$.

We have
\[
[U'[1]] = [U[1]] = [h_1(X)] = g[M_{1,0}].
\]

We will similarly find formulas for $[U[2]]$ and $[U[3]]$ in terms of $g$. More specifically, we first find $[U'[2]]$ and $[U'[3]]$ and then apply §7.2.1. Recall that
\[
\text{Gr}_{\mathbb{k}}^W \mathcal{O}(U'_n)^V \cong h_1(X)^{\otimes k}.
\]
for $k \leq n$, so we will need to analyze the $S_k$-action on $h_1(X)^{\otimes k}$. For this, we set
\[
V := H^0(G_k, h_1(X) \otimes h^1(E)),
\]
so that $h_1(X) = h_1(E) \otimes V$. Then $V$ is naturally the rank $g$ trivial object of $\text{Rep}^{\text{sg}}_{\mathbb{Q}_p}(G_k)$.

For a positive integer $k$, we have
\[
h_1(X)^{\otimes k} \cong h_1(E)^{\otimes k} \otimes V^{\otimes k}
\]
as $S_k$-representations, where $S_k$ acts individually on each of the three tensor powers.

7.5.1. **Affine Case in Level 2.** We compute $[U'[2]]$. We have
\[
\text{Sym}^2 h_1(X) \oplus \wedge^2 h_1(X) \cong h_1(X)^{\otimes 2}
\]
\[
\cong (h_1(E)^{\otimes 2} \otimes V^{\otimes 2})
\]
\[
\cong (\text{Sym}^2 h_1(E) \oplus \wedge^2 h_1(E)) \otimes (\text{Sym}^2 V \oplus \wedge^2 V)
\]
\[
\cong \text{Sym}^2 h_1(E) \otimes \text{Sym}^2 V \oplus \text{Sym}^2 h_1(E) \otimes \wedge^2 V
\]
\[
\oplus \wedge^2 h_1(E) \otimes \text{Sym}^2 V \oplus \wedge^2 h_1(E) \otimes \wedge^2 V
\]
Comparing $S_2$-actions on both sides, we get

$$\text{Sym}^2 h_1(X) \cong \text{Sym}^2 h_1(X) \otimes \text{Sym}^2 V \oplus \wedge^2 h_1(E) \otimes \wedge^2 V \cong M_{2,0} \otimes \text{Sym}^2 V \oplus M_{0,1} \otimes \wedge^2 V$$

and

$$\wedge^2 h_1(X) = \text{Sym}^2 h_1(X) \otimes \wedge^2 V \oplus \wedge^2 h_1(E) \otimes \text{Sym}^2 V \cong M_{2,0} \otimes \wedge^2 V \oplus M_{0,1} \otimes \text{Sym}^2 V$$

In particular, we have

$$[U'[2]] = [\wedge^2 V][M_{2,0}] + [\text{Sym}^2 V][M_{0,1}] = \left( \frac{g(g-1)}{2} \right) [M_{2,0}] + \left( \frac{g(g+1)}{2} \right) [M_{0,1}].$$

### 7.5.2. Affine Case in Level 3.

We now turn to $[U'[3]]$.

Let $A$ denote the trivial representation of $S_3$, $B$ the sign representation, and $C$ the standard two-dimensional representation.

We have

$$[h_1(X)^{\otimes 3}] = [h_1(E)]^3[V]^3 = g^3[h_1(E)]^3 = g^3[M_{3,0}] + 2g^3[M_{1,1}].$$

We get

$$\text{pr}_{-3}[\text{Sym}(U_1' + U'[2])] = [\text{Sym}^3 U_1'] + [U'][U'[2]].$$

Note that $\text{Sym}^3 U_1' = \text{Sym}^3 h_1(X) \cong (h_1(E)^{\otimes 3} \otimes V^{\otimes 3})^{S_3}$. We must therefore decompose each of $h_1(X)^{\otimes 3}$ and $V^{\otimes 3}$ as a representation of $S_3 \times \text{GL}_2$.

We know that $h_1(E)^{\otimes 3}$ decomposes as $M_{3,0} \oplus M_{1,1} \oplus M_{1,1}$. Since the $A$-isotypical piece is $\text{Sym}^3 h_1(E) = M_{3,0}$, and $\wedge^3 h_1(E) = 0$, we get that $S_3$ acts on the $M_{1,1} \oplus M_{1,1}$-piece of $h_1(E)^{\otimes 3}$ via two copies of $C$. Thus in total, we find that

$$h_1(E)^{\otimes 3} \cong M_{3,0} \otimes A \oplus M_{1,1} \otimes C$$

as a representation of $\text{GL}_2 \times S_3$.

For $V$, we have $\dim V^{\otimes 3} = g^3$, $a \coloneqq \dim \text{Sym}^3 V = \left( \begin{array} {c} g+2 \end{array} \right) = \frac{g(g+1)(g+2)}{6}$, and $b \coloneqq \dim \wedge^3 V = \left( \begin{array} {c} g \end{array} \right) = \frac{g(g-1)(g-2)}{6}$. We set $c \coloneqq g^3 - a - b = \frac{2g^3 - g}{3}$.

Therefore, as a representation of $S_3$, we get

$$V^{\otimes 3} \cong A^{\oplus a} \oplus B^{\oplus b} \oplus C^{\oplus c}.$$

We therefore find

$$h_1(X)^{\otimes 3} \cong h_1(E)^{\otimes 3} \otimes V^{\otimes 3}$$

$$\cong (M_{3,0} \otimes A \oplus M_{1,1} \otimes C) \otimes (A^{\oplus a} \oplus B^{\oplus b} \oplus C^{\oplus c})$$

$$\cong (M_{3,0} \otimes A)^{\oplus a} \oplus (M_{3,0} \otimes B)^{\oplus b} \oplus (M_{3,0} \otimes C)^{\oplus c} \oplus (M_{1,1} \otimes C)^{\oplus a} \oplus (M_{1,1} \otimes C)^{\oplus b}$$

$$\oplus (M_{1,1} \otimes C)^{\oplus c}$$

$$\cong M_{3,0}^{\oplus a} \otimes A \oplus M_{3,0}^{\oplus b} \otimes B \oplus (M_{3,0}^{\oplus c} \oplus M_{1,1}^{\oplus a+b}) \otimes C \oplus M_{1,1}^{\oplus c} \otimes (C \otimes C)$$

Using the decomposition $C \otimes C \cong A \oplus B \oplus C$, we find that

$$\text{Sym}^3 h_1(X) = (h_1(E)^{\otimes 3} \otimes V^{\otimes 3})^{S_3} \cong M_{3,0}^{\oplus a} \oplus M_{1,1}^{\oplus c},$$

and thus

$$[\text{Sym}^3 h_1(X)] = a[M_{3,0}] + c[M_{1,1}] = \left( \frac{g(g+1)(g+2)}{6} \right) [M_{3,0}] + \left( \frac{g^3 - g}{3} \right) [M_{1,1}].$$
Finally, we have
\[ [U'[3]] = h(X)^3 - \text{pr}_3[\text{Sym}(U'_1 \oplus U'[2])] \]
\[ = [h(X)^3] - [\text{Sym}^3 h(X)] - [U'_1][U'[2]] \]
\[ = g^3[M_{3,0}] + 2g^3[M_{1,1}] - \left[ \left( \frac{g(g+1)(g+2)}{6} \right) [M_{3,0}] + \left( \frac{g^3-g}{3} \right) [M_{1,1}] \right] \]
\[ = \left( g^3 - \frac{g(g+1)(g+2)}{6} - \frac{g^2(g-1)}{2} \right) [M_{3,0}] + \left( 2g^3 - \frac{g^3-g}{3} - g^3 \right) [M_{1,1}] \]
\[ = \left( g^3 - \frac{g}{3} \right) [M_{3,0}] + \left( \frac{2g^3+g}{3} \right) [M_{1,1}] \]

Remark 7.3. I wonder if Schur–Weyl duality could simplify some of the calculations above.

7.5.3. Projective Case. As described in §7.2.1 we may compute \([U[k]]\) in terms of \([U'[k]]\), where the latter is computed as in §7.5.1. We note in particular that
\[ [U[2]] = [U'[2]] - [M_{0,1}] \]
\[ = \left( \frac{g(g-1)}{2} \right) [M_{2,0}] + \left( \frac{g(g+1)}{2} \right) [M_{0,1}] - [M_{0,1}] \]
\[ = \left( \frac{g(g-1)}{2} \right) [M_{2,0}] + \left( \frac{g(g+1)}{2} - 1 \right) [M_{0,1}] \]

and
\[ [U[3]] = [U'[3]] - g[M_{1,1}] \]
\[ = \left( \frac{g^3-g}{3} \right) [M_{3,0}] + \left( \frac{2g^3+g}{3} \right) [M_{1,1}] - g[M_{1,1}] \]
\[ = \left( \frac{g^3-g}{3} \right) [M_{3,0}] + \left( \frac{2g^3-2g}{3} \right) [M_{1,1}] \]

7.6. Explicit Computations for Higher Genus Projective Curves. We now discuss the computation of \([U[k]]\) for \(g = 2\) and \(g = 4\) with attention to particular examples. While we do not need them for the examples of [10][12,3] we do it both to demonstrate in practice the methods of §7.2 and with a view toward future applications.
7.6.1. **Genus 2.** As described in §7.5.3, we have

\[ [U[1]] = g[M_{1,0}] = 2[M_{1,0}], \]

\[ [U[2]] = \left( \frac{g(g - 1)}{2} \right) \] \[ M_{2,0} \] \[ + \left( \frac{g(g + 1)}{2} - 1 \right) [M_{0,1}] = [M_{2,0}] + 2[M_{0,1}], \]

and

\[ [U[3]] = \left( \frac{g^3 - g}{3} \right) \] \[ M_{3,0} \] \[ + \left( \frac{2g^3 - 2g}{3} \right) [M_{1,1}] = 2[M_{3,0}] + 4[M_{1,1}]. \]

For \( k = \mathbb{Q} \) or \( k \) imaginary quadratic, we have

\[ c(U_Q) = c(U[1]) + c(U_Q[2]) \]
\[ = 2c_{1,0} + 2c_{0,1} \]
\[ = 2(1 - r) + 2 \]
\[ = 4 - 2r. \]

In particular, Quadratic Chabauty applies whenever \( r \leq 1 \), or equivalently \( \text{rank}_Q J_X(k) \leq 2 \). Every mixed elliptic genus 2 curve over \( \mathbb{Q} \) on LMFDB satisfies this condition over \( k = \mathbb{Q} \).

Let’s see what happens for \( k \) imaginary quadratic and the full level 2 quotient. We have

\[ c(U_2) = c(U[1]) + c(U[2]) \]
\[ = 2c_{1,0} + 2c_{0,1} + c_{2,0} \]
\[ = 2(1 - r) + 2 + 1 \]
\[ = 5 - 2r. \]

It follows that if \( r = \text{rank}_\mathbb{Q} E(k) = 2 \), then the inequality (3) holds for \( \Pi = U_2 \). (In fact, the same is true for \( k = \mathbb{Q} \), as \( c(U_2) = 6 - 2r \), but we do not have an example of this.)

For the genus 2 curve with LMFDB label 38416.a.614656.1, given by

\[ y^2 = x^6 - 3x^5 - x^4 + 7x^3 - x^2 - 3x + 1, \]

we have \( E \) the curve with Cremona label “196a2”. In this case, \( r = 2 \) for \( k = \mathbb{Q}(\sqrt{-d}) \) and \( d = 1, 2, 5, 6, 10, 14, 17, 21 \).

Its twist by \(-1\) is the genus 2 curve with LMFDB label 614656.a.614656.1, given by

\[ y^2 = -x^6 - 3x^5 + x^4 + 7x^3 + x^2 - 3x - 1, \]

and \( E \) has Cremona label “784i1”. In this case, \( r = 2 \) for \( k = \mathbb{Q}(\sqrt{-d}) \) and \( d = 1, 5, 17, 21, 33, 37, 53 \).

For completeness, we compute \( c(U_3) \), both when \( k = \mathbb{Q} \) and \( k \) is imaginary quadratic. For \( k = \mathbb{Q} \), we have

\[ c(U_3) = c(U_2) + c(U[3]) \]
\[ = 6 - 2r + 2c_{3,0} + 4c_{1,1} \]
\[ = 6 - 2r + 2(2) + 4(1) \]
\[ = 14 - 2r. \]
while for \( k \) imaginary quadratic,
\[
c(U_3) = c(U_2) + c(U[3]) \\
= 5 - 2r + 2c_{3,0} + 4c_{1,1} \\
= 5 - 2r + 2(1) + 4(0) \\
= 7 - 2r.
\]

Notice that if \( r = 3 \), then \([3]\) holds in level 3 but not level 2.

7.6.2. **Explicit Decomposition for Bring’s Curve.** We now demonstrate the decomposition for \( g = 4 \). We were encouraged to do this by B. Mazur, as it applies to Bring’s curve, given by the three homogeneous equations
\[
\sum_{i=0}^{4} x_i^k = 0 \quad k = 1, 2, 3
\]
in \( \mathbb{P}^4 \). When \( X \) is Bring’s curve, there is an elliptic curve \( E \) given by Cremona label ‘50a1’ for which
\[
h_1(X) \cong h_1(E)^{\oplus 4}.
\]

As described in §7.5.3, we have
\[
[U[1]] = g[M_{1,0}] = 4[M_{1,0}], \\
[U[2]] = \left( \frac{g(g - 1)}{2} \right) [M_{2,0}] + \left( \frac{g(g + 1)}{2} - 1 \right) [M_{0,1}] = 6[M_{2,0}] + 9[M_{0,1}],
\]
and
\[
[U[3]] = \left( \frac{g^3 - g}{3} \right) [M_{3,0}] + \left( \frac{2g^3 - 2g}{3} \right) [M_{1,1}] = 20[M_{3,0}] + 40[M_{1,1}].
\]
For \( k \) imaginary quadratic, we have
\[
c(U_Q) = c(U[1]) + c(U_Q[2]) \\
= 4c_{1,0} + 9c_{0,1} \\
= 4(1 - r) + 9 \\
= 13 - 4r.
\]

In particular, Quadratic Chabauty applies whenever \( r \leq 3 \), or equivalently \( \text{rank}_Q J_X(k) \leq 12 \).

8. **Tannakian Selmer Varieties**

The goal of this section is to explicitly understand
\[
H^1_{f,S}(G_k; \Pi)
\]
via the Tannakian categories of §7.3. We refer back to §6 for the relationship between \( H^1_{f,S}(G_k; \Pi) \) and \( \text{Sel}_{S,\Pi}(\mathcal{X}) \).
We have by Remark 4.10 that $H^1_{f,S}(G_k; M_{a,b}) = \text{Ext}^1_{\text{Rep}_{q_p}(G_k,E)}(Q_p(0), M_{a,b})$ and more generally that

$$H^1_{f,S}(G_k; \Pi) = H^1(\text{Rep}_{q_p}^{f_S}(G_k, E); \Pi),$$

where $H^1(\text{Rep}_{q_p}^{f_S}(G_k, E); \Pi)$ denotes the set of torsors under $\Pi$ in the Tannakian category $\text{Rep}_{q_p}^{f_S}(G_k, E)$. In order to express this set as the group cohomology of a pro-algebraic group, we introduce fiber functors.

We recall there is a tensor functor $\text{Rep}_{q_p}^{f_S}(G_k, E) \rightarrow \text{Rep}_{q_p}^{ss}(G_k, E)$ sending a representation $V$ to its associated graded $\text{Gr}^W V$ for the weight filtration.

**Definition 8.1.** We define

$$\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)$$

to be the Tannakian fundamental group of the category $\text{Rep}_{q_p}^{f_S}(G_k, E)$,

defined in Remark 4.10 with respect to the de Rham fiber functor $V \mapsto V^{dR} := D_{dR} V$.

We define

$$\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)^{Gr}$$

to be the Tannakian fundamental group of the same category with respect to the graded de Rham fiber functor

$$V \mapsto V^{Gr,dR} := D_{dR} \text{Gr}_*^W V.$$

The two fiber functors are isomorphic (in a manner compatible with associated graded) by [SR72, IV.2.2.2] and [Zie15, Main Theorem 1.2]. We describe the issue more in §8.2 below.

Note that both fiber functors are canonically isomorphic when restricted to the subcategory $\text{Rep}_{q_p}^{ss}(G_k, E) \subseteq \text{Rep}_{q_p}^{f_S}(G_k, E)$. We denote its Tannakian Galois group by $\pi_1(\text{Rep}_{q_p}^{ss}(G_k, E))$, and it is isomorphic to $\text{GL}_2$.

For a unipotent group $\Pi$ in the Tannakian category $\text{Rep}_{q_p}^{f_S}(G_k, E)$, we denote by $\Pi^{dR}$ the unipotent group over $Q_p$ with $\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)$-action associated to the Lie algebra $(\text{Lie} \Pi)^{dR}$, with its induced Lie algebra structure.

We have

$$\text{Ext}^1_{\text{Rep}_{q_p}^{f_S}(G_k,E)}(Q_p(0), M_{a,b}) = H^1(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E); M_{a,b}^{dR}) = H^1(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)^{Gr}; M_{a,b}^{Gr,dR})$$

and

$$H^1(\text{Rep}_{q_p}^{f_S}(G_k, E); \Pi) = H^1(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E); \Pi^{dR}) = H^1(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)^{Gr}; \Pi^{Gr,dR}).$$

**8.1. Structure of the Graded Tannakian Galois Group.** The inclusion

$$\text{Rep}_{q_p}^{ss}(G_k, E) \hookrightarrow \text{Rep}_{q_p}^{f_S}(G_k, E)$$

induces a map

$$\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)^{Gr} \rightarrow \pi_1(\text{Rep}_{q_p}^{ss}(G_k, E)) \cong \text{GL}_2$$

Note that this equality does not require Conjecture 2.1 because $\Pi$ has negative weights. Furthermore, the semisimplicity condition in the definition of $\text{Rep}_{q_p}^{f_S}(G_k)$ follows from [Fal83]. C.f. Remark 1.5.
that is surjective, with kernel the unipotent radical $U(O_{k,S}, E)^{Gr}$ of $\pi_1^{ME}(O_{k,S}, E)^{Gr}$. We denote the Lie algebra of $U(O_{k,S}, E)^{Gr}$ by $n(O_{k,S}, E)^{Gr}$, its universal enveloping algebra by $U(O_{k,S}, E)^{Gr}$, and its coordinate ring by $A(O_{k,S}, E)^{Gr}$.

The functor $\text{Rep}_{q_p}^{\text{fr}}(G_k, E) \to \text{Rep}_{q_p}^{\text{ss}}(G_k, E)$ defined by $V \mapsto \text{Gr}^W_V$ induces a canonical section

\begin{equation}
(10) \ s: \pi_1(\text{Rep}_{q_p}^{\text{ss}}(G_k, E)) \to \pi_1^{ME}(O_{k,S}, E)^{Gr}.
\end{equation}

We therefore have a canonical semidirect product decomposition

\[ \pi_1^{ME}(O_{k,S}, E)^{Gr} \cong \text{GL}_2 \ltimes U(O_{k,S}, E)^{Gr}, \]

inducing actions of $\text{GL}_2$ on $U(O_{k,S}, E)^{Gr}$, $\Pi^{Gr dR}$, and their associated algebras.

The theory of such extensions is described in §A.2-A.3. In particular, by Theorem A.4, we have the following:

**Corollary 8.2.** We have a natural bijection

\[ H^1_{f,S}(G_k; \Pi) \cong Z^1(U(O_{k,S}, E)^{Gr}; \Pi^{Gr dR})^{\text{GL}_2}. \]

**Proof.** This follows from Theorem A.4 with $G = \pi_1^{ME}(O_{k,S}, E)^{Gr}$, $G = \text{GL}_2$, $U = U(O_{k,S}, E)^{Gr}$, and $\Pi = \Pi^{Gr dR}$ as in the previous sections. \hfill \square

Note that by Proposition A.1, we also have

\[ H^1_{f,S}(G_k; \Pi) \cong Z^1(n(O_{k,S}, E)^{Gr}; \text{Lie} \Pi^{Gr dR})^{\text{GL}_2}. \]

### 8.2. Graded vs. Ungraded

We similarly have a projection

\[ \pi_1^{ME}(O_{k,S}, E) \to \pi_1(\text{Rep}_{q_p}^{\text{ss}}(G_k, E)) \cong \text{GL}_2, \]

with kernel denoted $U(O_{k,S}, E)$, but it is not canonically split. We denote the associated objects by $n(O_{k,S}, E)$, $U(O_{k,S}, E)$, and $A(O_{k,S}, E)$, respectively.

The scheme

\[ \text{Isom}^{\otimes, Gr W}(dR, Gr dR) \]

of isomorphisms from the de Rham fiber functor to the graded de Rham fiber functor inducing the identity on associated graded is a $U(O_{k,S}, E)-U(O_{k,S}, E)^{Gr}$ bitorsor in the fpqc topology over $\mathbb{Q}_p$ by [Zie15, Main Theorem 1.2]. As already mentioned, it is trivial by [SR72, IV.2.2.2]. In particular, the fundamental exact sequence

\[ 1 \to U(O_{k,S}, E) \to \pi_1^{ME}(O_{k,S}, E) \to \pi_1(\text{Rep}_{q_p}^{ss}(G_k, E)) \to 1 \]

splits, although not canonically. In fact, we have

**Proposition 8.3.** The set

\[ \text{Isom}^{\otimes, Gr W}(dR, Gr dR) \]

is naturally in bijection with the set

\[ \text{Sec}(\pi_1^{ME}(O_{k,S}, E)) \]

of sections of the fundamental exact sequence. The $U(O_{k,S}, E)$-torsor structure on $\text{Isom}^{\otimes, Gr W}(dR, Gr dR)$ corresponds to the action of $U(O_{k,S}, E)$ on $\text{Sec}(\pi_1^{ME}(O_{k,S}, E))$ by conjugation.
Proof. Any $\alpha \in \text{Isom}_{\otimes,\text{Gr}}^W(dR, \text{Gr} dR)$ induces an isomorphism $\beta: \pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)^{\text{Gr}} \to \pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)$ over $\pi_1(\text{Rep}^s_{Q_p}(G_k, E))$ defined by $\beta(g) = \alpha^{-1}g\alpha$. The composition

$$\beta \circ s: \pi_1(\text{Rep}^s_{Q_p}(G_k, E)) \to \pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E),$$

with $s$ as in (10), is an element of $\text{Sec}(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E))$. In other words, we have a map:

$$\text{Isom}_{\otimes,\text{Gr}}^W(dR, \text{Gr} dR) \to \text{Sec}(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)).$$

If we replace $\alpha$ by $\alpha \circ u^{-1}$ for $u \in U(\mathcal{O}_{k,S}, E)$, we replace $\beta(g)$ by $u\beta(g)u^{-1}$. In particular, this map intertwines the (left) action of $U(\mathcal{O}_{k,S}, E)$ on the bitorsor with its (left) conjugation action on $\text{Sec}(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E))$.

It therefore suffices to show that the latter is a torsor under $U(\mathcal{O}_{k,S}, E)$. This is true because $\text{GL}_2$ has trivial higher cohomology and because $U(\mathcal{O}_{k,S}, E)^{\text{GL}_2} = 0$ (for any choice of splitting). \hfill \Box

Remark 8.4. Given a section $\text{GL}_2 \cong \pi_1(\text{Rep}^s_{Q_p}(G_k, E)) \to \pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E)$, we may associate a point of $\text{Isom}_{\otimes,\text{Gr}}^W(dR, \text{Gr} dR)$ as follows. For any object $M$ of $\text{Rep}^s_{Q_p}(G_k, E)$, we get from the section a $\text{GL}_2$-action on $M^{dR}$, which then induces an isomorphism

$$M^{\text{Gr}^{dR}} \cong M^{dR}$$

sending $D_{dR}^{\text{Gr}^W} V$ to

$$a + b w \mapsto (M^{dR})^{a,b},$$

where $(M^{dR})^{a,b}$ denotes the $M_{a,b}$-isotypic component of $M^{dR}$.

Remark 8.5. A choice of $\alpha \in \text{Isom}_{\otimes,\text{Gr}}^W(dR, \text{Gr} dR)$ induces an isomorphism

$$H^1(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E); \Pi^{dR}) \cong Z^1(U(\mathcal{O}_{k,S}, E); \Pi^{dR})^{\text{GL}_2}.$$

By Theorem [A.4], this determines a section of the surjection

$$Z^1(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E); \Pi^{dR}) \to H^1(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E); \Pi^{dR}),$$

whose image is

$$\ker(Z^1(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E); \Pi^{dR}) \to Z^1(\text{GL}_2; \Pi^{dR}))$$

and whose projection to $Z^1(U(\mathcal{O}_{k,S}, E); \Pi^{dR})$ is $Z^1(U(\mathcal{O}_{k,S}, E); \Pi^{dR})^{\text{GL}_2}$.

Let’s see what happens if we change $\alpha$. If $c \in Z^1(U(\mathcal{O}_{k,S}, E); \Pi^{dR})$ is $\text{GL}_2$-equivariant with respect to $\alpha$, one may check that

$$u \mapsto u(c(u^{-1}vu))$$

is $\text{GL}_2$-equivariant with respect to $\alpha \circ u^{-1}$.

Remark 8.6. Given $M \in \text{Rep}^f_{Q_p}(G_k, E)$ and an extension

$$1 \to M \to E \to \mathbb{Q}_p(0) \to 1$$

representing an element of $\text{Ext}^1_{\text{Rep}^s_{Q_p}(G_k, E)}(\mathbb{Q}_p(0), M) = H^1(\pi_1^{\text{ME}}(\mathcal{O}_{k,S}, E); M^{dR})$, we may write down a cocycle representing this cohomology class by choosing a lift $1_E \in E$ of $1 \in \mathbb{Q}_p(0)$ and then considering the cocycle

$$u \mapsto u(1_E) - 1_E.$$
It is easy to see that this cocycle is $GL_2$-equivariant if and only if $1_E$ is $GL_2$-invariant. Assuming that $M$ contains no subquotient isomorphic to $\mathbb{Q}_p(0)$, there is a unique lift $1_E$ of $1$ that is $GL_2$-equivariant. The notion of $GL_2$-equivariance of course depends on the choice of $\alpha$, and this $1_E$ is precisely the lift corresponding to the splitting of the weight filtration determined by $\alpha$.

We fix once and for all a point of $\text{Isom}^\otimes,\text{Gr}^W(dR, \text{Gr dR})$.

We thus fix an identification $U(O_{k,S}, E) \cong U(O_{k,S}, E)^{\text{Gr}}$ and thus an identification $H^1_{dR}(G_k; \Pi) \cong Z^1(U(O_{k,S}, E); \text{Lie} \Pi^{dR})^{GL_2}$.

We therefore may mostly ignore the distinction between $dR$ and $\text{Gr dR}$, although we revisit it briefly in §9.3.

Remark 8.7. In fact, if $\Pi$ is semisimple, then $U(O_{k,S}, E)$ acts trivially on $\Pi^{dR}$, so we get a $GL_2$-action on $\Pi^{dR}$. This induces an isomorphism

$$\Pi^{dR} \cong \Pi^{\text{Gr dR}}$$

independent of choice of point of $\text{Isom}^\otimes,\text{Gr}^W(dR, \text{Gr dR})$. In the notation of Remark 8.5 $u(c(u^{-1}vu)) = c(v)$ in this case.

8.3. Structure of the Unipotent Radical. We have

$$n(O_{k,S}, E)^{ab} = U(O_{k,S}, E)^{ab} \cong \prod_{a,b} H^1_{dR}(G_k; M_{a,b})^\vee \otimes_{\mathbb{Q}_p} M_{a,b}^{dR}$$

as an object of $\text{Pro Rep}^{ss}_{\mathbb{Q}_p}(G_k, E)$.

Dually, we have a canonical isomorphism

$$\ker(\Delta': A(O_{k,S}, E) \to A(O_{k,S}, E) \otimes A(O_{k,S}, E)) \cong \bigoplus_{a,b} H^1_{dR}(G_k; M_{a,b}) \otimes_{\mathbb{Q}_p} M_{a,b}^{dR}$$

in $\text{Ind Rep}^{ss}_{\mathbb{Q}_p}(G_k, E)$.

Let us briefly describe this isomorphism explicitly. Let $M_{a,b} \in \text{Irr}(GL_2)$, and let $c \in \text{Ext}^1(\mathbb{Q}_p, M_{a,b})$. Then $c$ is described by an extension

$$0 \to M_{a,b} \to E_c \to \mathbb{Q}_p \to 0.$$ Choose a lift $1_E \in E_c^{dR}$ of $1 \in \mathbb{Q}_p$. Given $v \in (M_{a,b}^{dR})$, let $p_v : E_c^{dR} \to \mathbb{Q}_p$ be the functional given by the projection defined by $1_E$ followed by $v : M_{a,b}^{dR} \to \mathbb{Q}_p$. Then the element $c \otimes v$ of $A(O_{k,S}, E)$ is the Tannakian matrix coefficient ([Bro17b, §2.2])

$$[E_c, 1_E, p_v].$$

Letting $A(O_{k,S}, E)_{>0}$ denote the augmentation ideal, $A(O_{k,S}, E)_{>0} \cdot A(O_{k,S}, E)_{>0}$ is the space of decomposables, and

$$n(O_{k,S}, E)^\vee := A(O_{k,S}, E)_{>0} / A(O_{k,S}, E)_{>0} \cdot A(O_{k,S}, E)_{>0}$$

is the Lie coalgebra. Then $\Delta'$ induces the cobracket on $n(O_{k,S}, E)^\vee$, and

$$\ker(\Delta': A(O_{k,S}, E) \to A(O_{k,S}, E) \otimes A(O_{k,S}, E)) \cong \bigoplus_{a,b} H^1_{dR}(G_k; M_{a,b}) \otimes_{\mathbb{Q}_p} M_{a,b}^{dR}$$

is the kernel of the cobracket.
8.4. **A Free Unipotent Group.** As $GL_2$ is reductive, we may choose a $GL_2$-equivariant splitting of the projection

\[(11) \quad n(O_{k,S}, E) \rightarrow n(O_{k,S}, E)^{ab}.\]

Let $W$ denote the image of $n(O_{k,S}, E)^{ab}$ under this splitting, and set

$$n(W) := \text{FreeLie}_W,$$

and let $U(W), UW, \text{ and } A(W)$ denote the associated pro-unipotent group, universal enveloping algebra, and coordinate ring, respectively.

We have a map

$$\theta: n(W) \rightarrow n(O_{k,S}, E)$$

of Lie algebra objects in $\text{ProRep}^{ss}_{Q_p}(G_k, E)$ that is an isomorphism on abelianizations. In particular, $\theta$ is surjective.

If we knew the following conjecture, then we would know that $\theta$ is an isomorphism:

**Conjecture 8.8** ([FPR94, Conjecture II.3.2.2]). If $V \in \text{Rep}^s_{Q_p}(G_k)$, and $V''$ is a quotient of $V$, then

$$H^1_{f,S}(G_k; V) \rightarrow H^1_{f,S}(G_k; V'')$$

is surjective.

Indeed, Conjecture 8.8 would imply that $\text{Ext}^i_{\text{Rep}^s_{Q_p}(G_k, E)}$ vanishes for $i \geq 2$, and hence that $U(O_{k,S}, E)$ is a free pro-unipotent group over $Q_p$. Then it follows that $\theta$ is an isomorphism.

We nonetheless have an embedding

$$H^1_{f,S}(G_k; \Pi) \cong Z^1(U(O_{k,S}, E); \Pi)^{GL_2} \hookrightarrow Z^1(U(W); \Pi)^{GL_2},$$

conjectured to be an isomorphism.

As an associative algebra in $\text{ProRep}^{ss}_{Q_p}(G_k, E)$, the universal enveloping algebra $UW$ is the tensor algebra on $W$.

Dually, $A(W)$ becomes the free shuffle algebra on the dual object

$$W^\vee = \bigoplus_{M \in \text{Irr GL}_2} H^1_{f,S}(G_k; M) \otimes_{Q_p} M^{\vee dR} = \bigoplus_{a,b} H^1_{f,S}(G_k; M_{a,b}) \otimes_{Q_p} M_{a,b}^{\vee dR}$$

of $\text{IndRep}^{ss}_{Q_p}(G_k, E)$, with coproduct given by deconcatenation and $GL_2$-action given by its action on this vector space. Note that it also has the structure of a non-semisimple motive by §A.2.

**Remark 8.9.** The constructions and results of §8.1-8.4 apply with $E$ replaced by any abelian variety, as in Remark 4.11.

9. **Localization Maps for Selmer Varieties**

The goal of this section is to explicitly understand the map

$$\log_{BK} \circ \text{loc}_{\Pi}: H^1_{f,S}(G_k; \Pi) \rightarrow \Pi/F^0\Pi$$

via the description of $H^1_{f,S}(G_k; \Pi)$ in §8. The results of this section will allow us to find an element of $O(\Pi/F^0)$ vanishing on the image of $\text{loc}_{\Pi}$ by computing a function vanishing on the image of a more explicit map

$$\text{ev}_{\Pi,W/F^0}: Z^1(U(W); \Pi)^{GL_2} \times U(W) \rightarrow \Pi/F^0 \times U(W).$$
The material of this section corresponds to [CDC20, §2.3-2.4].

9.1. Universal Cocycle Evaluation Maps. For any \( \Pi \), we have a universal cocycle evaluation map (c.f. [CDC20, Definition 2.20])

\[
\text{ev}_\Pi : Z^1(U(\mathcal{O}_{k,S}, E); \Pi)^{GL_2} \times U(\mathcal{O}_{k,S}, E) \to \Pi \times U(\mathcal{O}_{k,S}, E).
\]

defined by

\[
(c, u) \mapsto (c(u), u).
\]

Its pullback along \( U(W) \to U(\mathcal{O}_{k,S}, E) \) factors through a map

\[
Z^1(U(\mathcal{O}_{k,S}, E); \Pi)^{GL_2} \times U(W) \to Z^1(U(W); \Pi)^{GL_2} \times U(W) \xrightarrow{\text{ev}_{\Pi,W}} \Pi \times U(W).
\]

Our goal in §9.3-9.4 will be to compute a function on \( \Pi/F_0 \times U(W) \) vanishing on the image of the composition \( \text{ev}_{\Pi,W}/F_0 \) of \( \text{ev}_{\Pi,W} \) with projection from \( \Pi \) to \( \Pi/F_0 \):

\[
\text{ev}_{\Pi,W}/F_0 : Z^1(U(W); \Pi)^{GL_2} \times U(W) \xrightarrow{\text{ev}_{\Pi,W}} \Pi \times U(W) \to \Pi/F_0 \times U(W).
\]

In §9.3-9.4, we show how such a function on \( \Pi/F_0 \times U(W) \) specializes to an element of the Chabauty-Kim ideal. First, we show how to descend \( \text{ev}_{\Pi,W} \) from \( U(W) \) to a scheme of finite type over \( \mathbb{Q}_p \).

9.2. Quotients of the Unipotent Radical. The algebra \( A(W) \) has a weight filtration. However, it is not necessarily finite-dimensional in each degree, even assuming Conjecture 2.2. That’s because given \( w \), there are infinitely many pairs \((a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \) with \( w = -a - 2b \).

Let \( W^{ae} \), the quotient of \( W \) corresponding to \( \prod_{M \in \text{Irr}^{ae} GL_2} H^1_{f,S}(G_k; M)^\vee \otimes_{\mathbb{Q}_p} M^{dR} \).

Then the quotient \( n(W)^{ae} := \text{FreeLie} W^{ae} \) of \( n(W) \) is finite-dimensional in each degree. More generally, for a subset

\[ I \subseteq \text{Irr GL}_2, \]

we define

\[
W^I := \prod_{M \in I} H^1_{f,S}(G_k; M)^\vee \otimes_{\mathbb{Q}_p} M^{dR},
\]

\[
n(W)^I := \text{FreeLie} W^I
\]

as a quotient of \( n(W) \), and

\[
U(W)^I
\]

\[
A(W)^I
\]

the corresponding pro-unipotent group and coordinate ring, respectively. Assuming \( I \subseteq \text{Irr}^{ae} GL_2 \), \( n(W)^I \) and \( A(W)^I \) are finite-dimensional in each degree and are strictly-negatively and positively graded, respectively.

**Proposition 9.1.** If \( I \subseteq \text{Irr GL}_2 \) contains all graded pieces of \( \Pi \), and the action of \( U(W) \) on \( \Pi \) factors through \( U(W)^I \), then

\[
Z^1(U(W)^I; \Pi)^{GL_2} = Z^1(U(W); \Pi)^{GL_2}
\]

⁹² More generally, as long as \( I \) has only representations of negative weight and contains finitely many of each given weight.
Proof. Set
\[ W' := \text{Ker}(W \rightarrow W^I). \]

Then the kernel of
\[ n(W) \rightarrow n(W)^I \]
corresponds under \( \theta \) to the Lie ideal \( n(W)' \) generated by \( W' \). This Lie ideal corresponds to the normal subgroup scheme
\[ U(W)' = \text{Ker}(U(W) \rightarrow U(W)'). \]

Let \( c \in Z^1(n(W); \text{Lie } \Pi)^{GL_2} = Z^1(U(W); \Pi)^{GL_2} \). Note that \( c \) vanishes on \( W' \), since \( \text{Hom}_{GL_2}(W', \text{Lie } \Pi) = 0 \).

It then suffices to show that
\[ \text{Ker } c \cap n(W)' \]
is a Lie ideal in \( n(W) \). For this, suppose \( u \in n(W) \) and \( w \in \text{Ker } c \cap n(W)' \). Then
\[ c([u, w]) = [c(u), c(w)] + u(c(w)) - w(c(v)) = [c(u), 0] + u(0) - w(c(v)) = 0 \]
because \( w \in n(W)' \), which acts trivially on \( \text{Lie } \Pi \).

It follows that
\[ \text{Ker } c = n(W)', \]
so that \( c \in Z^1(U(W)^I; \Pi)^{GL_2} \subseteq Z^1(U(W); \Pi)^{GL_2} \). As \( c \) is arbitrary, we are done. \( \square \)

Remark 9.2. Given \( \Pi \), one may ensure the hypothesis of Proposition 9.1 by taking \( I \) to include the set of irreducible components of \( \text{End}(\text{Lie } \Pi) \) as a \( GL_2 \)-representation.

Suppose that the graded pieces of \( \Pi \) are contained in \( I \subseteq \text{Irr } GL_2 \) and that the action of \( U(W) \) on \( \Pi \) factors through \( U(W)^I \), as in the hypotheses of Proposition 9.1. Then \( \mathfrak{c}u_{\Pi, W}^I \) is just the pullback along \( U(W) \rightarrow U(W)^I \) of a map
\[ \mathfrak{c}u_{\Pi, W}^I : Z^1(U(W)^I; \Pi)^{GL_2} \times U(W)^I \rightarrow \Pi \times U(W)^I. \]

In particular, an element of \( \mathcal{O}(\Pi/F^0 \times U(W)^I) = \mathcal{O}(\Pi/F^0) \otimes A(W)^I \) vanishing on the image of \( \mathfrak{c}u_{\Pi, W}^I/F^0 \) also vanishes on the image of \( \mathfrak{c}u_{\Pi, W}^I/F^0 \). In §11, we will be computing a function vanishing on the image of \( \mathfrak{c}u_{\Pi, W}^I/F^0 \) for \( X = E' \), \( \Pi = U_3 \), and appropriately chosen \( I \).

9.3. \( p \)-adic Periods and Localization. In this section, we describe a \( p \)-adic period map contained in forthcoming work of the author and I. Dan-Cohen. This is a \( \mathbb{Q}_p \)-algebra homomorphism \( A(\mathcal{O}_{k,S}, E) \rightarrow \mathbb{Q}_p \) for \( p \in \text{Spec } \mathcal{O}_{k,S} \) with residue characteristic \( p \) and \( \mathcal{O}_p \cong \mathbb{Z}_p \), compatible with the non-abelian Bloch-Kato map of [Kim09].

Such a map for mixed Tate motives was defined by [CU13] and used in [CDC20, 2.4.1]. A different \( p \)-adic period map for mixed Tate motives was defined in [DG05, 5.28] (c.f. also [Yam10, §3.2] and [Bro17a, 3.4.3]).

Remark 9.3. This map is not logically necessary for the definition of the universal cocycle evaluation map in §9.1 or the calculations in §11. However, the period map motivates these constructions and calculations because it implies that the function derived at the end of §11 specializes to an element of the Chabauty-Kim ideal.

In forthcoming work with I. Dan-Cohen, we prove the following:
**Theorem 9.4.** For \( p \in \text{Spec} \mathcal{O}_{k,S} \) with residue characteristic \( p \) and \( \mathcal{O}_p \cong \mathbb{Z}_p \), there is a point 
\[
\text{per}_p : \text{Spec} \mathbb{Q}_p \to U(\mathcal{O}_{k,S}, E),
\]
satisfying the following property:

Let \( \Pi \) be a unipotent group with negative-weight action of \( \pi_1^\text{ME}(\mathcal{O}_{k,S}, E) \) and 
\[
c \in Z^1(U(\mathcal{O}_{k,S}, E), \Pi)^{\text{GL}_2}.
\]

Then the composition 
\[
c/F^0 \circ \text{per}_p : \text{Spec} \mathbb{Q}_p \xrightarrow{\text{per}_p} U(\mathcal{O}_{k,S}, E) \xrightarrow{c} \Pi \to \Pi/F^0
\]
is \( \log_{\text{BK}}(\text{loc}_{\Pi}(c)) \in \Pi/F^0(\mathbb{Q}_p) \).

**Remark 9.5.** The map \( \text{per}_p \) depends on the choice of element of \( \text{Isom}^{\otimes, \text{Gr} \mathcal{W}(\text{dR})} \) in \( \S 8.2 \), although we omit this from the notation. C.f. Remark 8.5.

**Remark 9.6.** The result Theorem 9.4 (as well as the constructions of \( \S 9 \)) in fact apply to more general categories of Galois representations as in Remarks 4.11 and 8.9.

**9.4. \( p \)-adic Periods and Universal Cocycle Evaluation.** Since the map \( U(W) \to U(\mathcal{O}_{k,S}, E) \) is a surjection of vector spaces, we may lift \( \text{per}_p \in U(\mathcal{O}_{k,S}, E) \) arbitrarily to an element of \( U(W) \). The choice of lift will not matter, so we denote it, by abuse of notation, by \( \text{per}_p \) as well.

We then have the following diagram of schemes over \( \mathbb{Q}_p \):

\[
\begin{array}{ccc}
H^1_{f,S}(G_k; \Pi) & \xrightarrow{\log_{\text{BK}} \circ \text{loc}_H} & \Pi/F^0 \\
\downarrow & & \downarrow \\
Z^1(U(\mathcal{O}_{k,S}; E); \Pi)^{\text{GL}_2} & \xrightarrow{\log_{\text{BK}} \circ \text{loc}_H} & \Pi/F^0 \\
\downarrow & & \downarrow \\
Z^1(U(\mathcal{O}_{k,S}; E); \Pi)^{\text{GL}_2} \times \text{Spec} \mathbb{Q}_p & \xrightarrow{\text{per}_p} & \Pi/F^0 \times \text{Spec} \mathbb{Q}_p \\
\downarrow & & \downarrow \\
Z^1(U(\mathcal{O}_{k,S}; E); \Pi)^{\text{GL}_2} \times U(W) & \xrightarrow{\text{ev}_H} & \Pi/F^0 \times U(\mathcal{O}_{k,S}, E) \\
\end{array}
\]

The commutativity of the diagram follows by Theorem 9.4. More precisely, Theorem 9.4 ensures commutativity of 
\[
Z^1(U(\mathcal{O}_{k,S}; E); \Pi)^{\text{GL}_2} \times \text{Spec} \mathbb{Q}_p \xrightarrow{\text{per}_p} \Pi/F^0 \times \text{Spec} \mathbb{Q}_p,
\]
and the former diagram is commutative because its bottom square is Cartesian.
It follows from the definition of $\text{ev}_{\Pi,W}$ that if $f \in \mathcal{O}(\Pi/F^0 \times U(W)) = \mathcal{O}(\Pi/F^0) \otimes \mathcal{A}(W)$ vanishes on the image of $\text{ev}_{\Pi,W}$, then $f$ also vanishes on the image of $\text{ev}_{\Pi} \times_{U(\mathcal{O}_{k,S},E)} U(W)$. This in turn implies, by the diagram above, that

$$\text{id}_{\mathcal{O}(\Pi/F^0)} \otimes \text{per}_p(f) \in \mathcal{O}(\Pi/F^0) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p = \mathcal{O}(\Pi/F^0)$$

vanishes on the image of $\log_{BK} \circ \text{loc}_{\Pi}$.

10. The Level 3 Quotient for a Punctured Elliptic Curve

We use the notation of §7.4, where $E$ is an elliptic curve, $X = E' = E \setminus \{O\}$, and $U = U(X)$. We set $\mathcal{E} = \overline{X}$ and $\mathcal{E}' = X$ to be the affine and projective curves, respectively, over $\mathcal{O}_{k,S}$, given by the minimal Weierstrass model of $E$. We suppose $E$ is also given by a Weierstrass equation $y^2 = x^3 + ax + b$. We also set $\Pi = U_3(E')$, and we suppose from now on that $|S| = 1$.

Finally, we set

$$e_0 := \frac{dx}{y},$$
$$e_1 := \frac{xdx}{y},$$

and we view words in the set $\{e_0, e_1\}$ as elements of $\mathcal{O}(\Pi)$.

10.1. Coordinates for $\Pi$. We now write down coordinates for $\mathcal{O}(U)$ and $\mathcal{O}(\Pi)$. The coordinate ring $\mathcal{O}(U)$ is filtered by weight, and $\mathcal{O}(\Pi)$ is the subalgebra generated by $W_3 \mathcal{O}(U)$. Furthermore, the choice of differential forms $e_0, e_1$ splits the (motivic) weight filtration. We find bases in each degree. In line with the notation of [CDC20, §3.1], we view $\mathcal{O}(U)$ as the shuffle algebra in words $e_0$ and $e_1$.

A basis of $\mathcal{O}(U)$ in degree $k \geq 0$ is given by the set of words of length $k$ the letters $e_0, e_1$. In degree 0, a basis is

$$\{1\},$$

corresponding to the empty word. In degree 1, a basis is

$$\{e_0, e_1\}.$$

In degree 2, a basis is

$$\{e_0^2, e_0 e_1, e_1 e_0, e_1^2\}.$$

In degree 3, a basis is

$$\{e_0^3, e_0^2 e_1, e_0 e_1 e_0, e_0 e_1^2, e_1 e_0 e_1, e_1^2 e_0, e_1 e_1^2\}.$$

10.2. Local Selmer Variety. We wish to explicitly describe $\mathcal{O}(\Pi/F^0 \Pi)$ in the bases of §10.1.

We have $[\Pi] = [U/U^4] = [M_{1,0}] + [M_{0,1}] + [M_{1,1}]$. Therefore,

$$l(\Pi) = l_{1,0} + l_{0,1} + l_{1,1} = 1 + 1 + 2 = 4.$$

Thus $\mathcal{O}(\Pi/F^0 \Pi)$ is a polynomial ring in 4 variables.
We refer to results of [Bea17] for the coordinate ring. In the notation of loc.cit., we have $\alpha_0 = e_0$, $\alpha_1 = e_1$, $F = \frac{e}{x}$, and $\lambda = -2$. Then by [Bea17] Proposition 5.4, $\mathcal{O}(\Pi/F^0\Pi)$ is generated as a polynomial ring by the four elements

\[
\begin{align*}
J_1 &= e_0 \\
J_2 &= e_0e_1 \\
J_3 &= e_0e_1e_0 \\
J_4 &= e_0e_1e_1 + 2e_1
\end{align*}
\]

Upon choosing a place $\mathfrak{p}$ with $k_\mathfrak{p} \cong \mathbb{Q}_p$ and basepoint $b \in \mathcal{E}'(\mathcal{O}_\mathfrak{p})$, each $J_i$ determines a function $J_i : \mathcal{E}'(\mathcal{O}_\mathfrak{p}) \to \mathbb{Q}_p$.

At the same time, we have $d(\Pi) = d_{1,0} + d_{0,1} + d_{1,1} = r + |S| + 1 = 3$. We therefore expect to have a nontrivial element of the Chabauty-Kim ideal in degree 4.

We will prove the following in \S11.

**Theorem 10.1.** Let $\alpha_1, \ldots, \alpha_N$ be as in \S3.4.2. If $E$ is an elliptic curve over $\mathbb{Q}$ of $p$-Selmer rank 1 for which Conjecture 2.2 holds for $h^1(E)$, then the ideal of functions vanishing on the image of $\text{loc}_\mathfrak{p}$: $H^1_{f,s}(G_k;\Pi)_{\alpha_i} \to \Pi/F^0\Pi$ contains an element of the form

\[
c_1J_4 + c_2J_3 + c_3J_1J_2 + c_4J_1^3 + c_5J_1,
\]

with $c_i \in \mathbb{Q}_p$ arising as periods of elements of $\mathcal{O}(W)$, not all of which are zero.

From this, if one has at least five elements of $\mathcal{E}'(\mathcal{O}_k[1/S])$ mapping to $H^1_{f,s}(G_k;\Pi)_{\alpha_i}$, then one may in theory determine the coefficients $c_i$ by computing Coleman integrals. In \S12 we review two examples where this is the case.

Theorem 10.1 follows from a procedure analogous to the geometric step of [CDC20, \S4.2]. As in loc.cit., we prove more, writing the coefficients $c_i$ as $p$-adic periods of specific elements of $\mathcal{O}(W)$.

**Remark 10.2.** By applying Theorem 9.4 to appropriately chosen $\Pi$ developing the theory of motivic periods more carefully (e.g. as discussed in \S1.8), one will likely be able to determine the $c_i$ even when $\mathcal{E}'(\mathcal{O}_k[1/S])$ is not large enough. This is analogous to [CDC20, \S4.3].

**Remark 10.3.** Work in preparation by A. Betts on weight filtrations of Selmer varieties is expected to prove the weaker statement that this ideal contains a nonzero element of the form

\[
c_1J_4 + c_2J_3 + c_3J_1J_2 + c_4J_1^3 + c_5J_1 + c_6J_2 + c_7J_1^2 + c_8.
\]

In fact, we prove a more refined statement, in that we may write the coefficients in terms of the periods of certain elements of $U(\mathcal{O}_{k,S}, E)$.

In particular, we may find such an equation as long as we have at least 5 sufficiently independent $\mathbb{Z}[1/2]$-points in each fiber of the map

\[
\mathcal{E}'(\mathbb{Z}[1/2]) \to \prod_{\nu \in T_{k,S}} \kappa_\nu(\mathcal{X}(\mathcal{O}_\nu)).
\]

For the elliptic curve with Cremona label “128a2”, this map is trivial (i.e., $N = 1$), and there are thirteen $\mathbb{Z}[1/2]$-points. For the elliptic curve with Cremona label “102a1”, we have $N = 2$, and the two fibers contain nine and ten $\mathbb{Z}[1/2]$-points, respectively.
10.3. The Level 3 Quotient of $U(E')$. We show the motive

$$\text{Lie } \Pi$$

is semisimple. Equivalently, the action of $U(O_{k,S}, E)$ on $\Pi$ is trivial. By Corollary 8.2 this tells us that

$$H^1_{f,s}(G_k; \Pi) \cong Z^1(U(O_{k,S}, E); \Pi)^{GL_2} = \text{Hom}(U(O_{k,S}, E), \Pi)^{GL_2}.$$ 

Furthermore, since $\Pi$ has graded pieces contained in $I = \{M_{1,0}, M_{0,1}, M_{1,1}\}$, this tells us that

$$H^1_{f,s}(G_k; \Pi) = \text{Hom}(U(O_{k,S}, E)^I, \Pi)^{GL_2}.$$ 

This is the specialization of a universal mixed elliptic motive in the sense of [HM20]. It is the weight $\geq -3$ quotient of the extension

$$0 \to (\text{Sym } \mathbb{H}(1))(1) \to \text{Pol}^{\text{all}}_2 \to \mathbb{H}(1) \to 0$$

of [HM20] §14, where $\mathbb{H}$ is the first cohomology of the universal elliptic curve.

In the notation of [HM20], the weight $\geq -3$ part of the extension shows up in

$$\text{Ext}^1(\mathbb{H}(1), M_{0,1}) = \text{Ext}^1(\mathbb{Q}(0), M_{0,1} \otimes \mathbb{H})$$

and

$$\text{Ext}^1(\mathbb{H}(1), M_{1,1}) = \text{Ext}^1(\mathbb{Q}(0), M_{1,1} \otimes \mathbb{H}),$$

where we take Ext in the category $\text{MEM}_*$ for $* = 2, 1, 1$.

Now $M_{0,1} \otimes \mathbb{H}$ is $M_{1,0}$, so the class vanishes for $* = 1, 1$ by [HM20] Theorem 15.1, because it corresponds to $m = 1, r = 1$.

Then $M_{1,1} \otimes \mathbb{H} = M_{1,0} \otimes M_{1,0} = M_{2,0} + M_{0,1}$. That class also vanishes by [HM20] Theorem 15.1 (for $M_{2,0}$ it’s $(m, r) = (2, 2)$ and thus vanishes for all $*$, and for $M_{0,1}$ it’s $(m, r) = (0, 1)$ so vanishes for $* = 1, 2$).

11. Geometric Step

Our goal for this section is to find an element of $O(\Pi/F^0) \otimes A(W)^{IF^0}$ that vanishes on the image of $\text{ev}_{\Pi,W/F^0}$ that vanishes on the image of $\text{ev}_{\Pi,W/F^0}$ (and therefore of $\text{ev}_{\Pi,W/F^0}$, as described in §9.2) for $\Pi = U_3(X)$ and $X = E'$. This is analogous to the “geometric step” of [CDC20, §1.3.4, §4.2]. For such $\Pi$, we may take $I = \{M_{1,0}, M_{0,1}, M_{1,1}\}$ by §10.3. The result will prove Theorem 10.1.

When base-changed to the fraction field $K(W)^I$ of $U(W)^I$, the map $\text{ev}_{\Pi,W/F^0}$ becomes a map of finite-dimensional varieties over a field. Since the left side is three-dimensional, and the right side is four-dimensional, we may use elimination theory to find a nonzero function vanishing on the image of $\text{ev}_{\Pi,W/F^0}$. One may then clear denominators to ensure that the coefficients are in $A(W)^I$ rather than $K(W)^I$.

We perform this elimination theory in §11.4. Before that, we define coordinates on $U(W)^I$ in §11.1 and we write $\text{ev}_{\Pi,W}$ in coordinates in §11.2.
11.1. Basis for $A$. For simplicity of notation, we omit the superscript dR when talking about the realization of a motive. So we may think of $M_{1,0}^\vee$ as a 2-dimensional $\mathbb{Q}_p$-vector space with basis $\{e_0, e_1\}$. We let $A = A(W)^I$. For an integer $n$, we let $A_n$ denote the degree-$n$ part of $A$ for the weight-grading coming from the $\text{GL}_2$-action. We let

$$\text{pr}_n : A \to A$$

denote projection onto $A_n$, and

$$\text{pr}_{m,n} = \text{pr}_m \otimes \text{pr}_n : A \otimes A \to A \otimes A$$

denote projection onto $A_m \otimes A_n$.

We recall from §4.1 that the choice of splitting (11) determines an isomorphism of Hopf algebras with $\text{GL}_2$-action over $\mathbb{Q}_p$ between $A$ and the free shuffle algebra on the vector space

$$(W^I)^\vee = \text{Ext}^1(\mathbb{Q}_p, M_{1,0}) \otimes M_{1,0}^\vee \oplus \text{Ext}^1(\mathbb{Q}_p, M_{0,1}) \otimes M_{0,1}^\vee \oplus \text{Ext}^1(\mathbb{Q}_p, M_{1,1}) \otimes M_{1,1}^\vee.$$

Equivalently, $\mathcal{U}W^I$ is the tensor algebra on $W^I$, and every element of $W^I$ is primitive for the coproduct. We first describe a basis of $W^I$, which is $2 + 1 + 2 = 5$-dimensional. We let $\pi$ denote a generator of $\text{Ext}^1(\mathbb{Q}_p, M_{1,0})^\vee$ corresponding to a chosen generator of $E(\mathbb{Q})$ (to be fixed in each case), $\tau_2$ a generator of $\text{Ext}^1(\mathbb{Q}_p, M_{0,1})^\vee$ corresponding to $\log 2$, and $\sigma$ a generator of $\text{Ext}^1(\mathbb{Q}_p, M_{1,1})^\vee$. We also let $\{e_0^\vee, e_1^\vee\}$ denote a basis of $M_{1,0}$ dual to $\{e_0, e_1\}$. We set:

$$\pi_i := \pi \otimes e_i^\vee$$

$$\tau = \tau_2 \otimes (e_0^\vee e_1^\vee - e_1^\vee e_0^\vee)$$

$$\sigma_i = \sigma \otimes e_i^\vee (e_0^\vee e_1^\vee - e_1^\vee e_0^\vee),$$

so that $\{\pi_0, \pi_1, \tau, \sigma_0, \sigma_1\}$ is a basis of $W^I$. Note that $\pi_0, \pi_1$ have weight 1, $\tau$ has weight 2, and $\sigma_0, \sigma_1$ have weight 3.

This basis determines a basis of $\mathcal{U}W^I$, which is simply the set of words in the set $\{\pi_0, \pi_1, \tau, \sigma_0, \sigma_1\}$. We consider a basis of $A$ dual to this basis of $\mathcal{U}W^I$. For a word $w$, we let $f_w$ denote the dual basis element of $A$. While the notation of subscripted $f$'s may be cumbersome, it helps distinguish between the shuffle product and concatenation product in $A$ (compare [CDC20, §4.1]).

We may take $\{1\}$ as a basis for $A_0$ and $\{f_{\pi_0}, f_{\pi_1}\}$ as a basis for $A_1$.

A basis for $A_2$ is given by $\{f_{\pi_0^2}, f_{\pi_0 \pi_1}, f_{\pi_1 \pi_0}, f_{\pi_1^2}, f_{\tau}\}$.

A basis for $A_3$ is given by $\{f_{\sigma_0}, f_{\sigma_1}, f_{\pi_0^3}, f_{\pi_0^2 \pi_1}, f_{\pi_0 \pi_1 \pi_0}, f_{\pi_0 \pi_1^2}, f_{\pi_1 \pi_0^2}, f_{\pi_1 \pi_0 \pi_1}, f_{\pi_1^2 \pi_0}, f_{\pi_1^2 \pi_1}, f_{\pi_1 \pi_0 \tau}, f_{\pi_1 \tau}, f_{\pi_0 \pi_0 \pi_0}, f_{\pi_1 \pi_1 \pi_1}\}$, and it is $2 + 8 + 4 = 14$-dimensional.

11.2. Coordinates on the Space of Cocycles. Let $c \in Z^1(U(W)^I, U_3)^{\text{GL}_2}$. For a word $\lambda$ in $\{e_0, e_1\}$ and $w$ in $\{\pi_0, \pi_1, \tau, \sigma_0, \sigma_1\}$, we define $\phi^w_\lambda(c)$ so that

$$c^w(\lambda) = \sum_w \phi^w_\lambda(c) f_w.$$
\[
\begin{align*}
  w_2 &:= \phi_{e_0 e_1}^r \\
  w_3 &:= \phi_{e_0 e_1 e_0}^r.
\end{align*}
\]

Note that \(c\) respects the \(\text{GL}_2\)-action and therefore the weight grading. It follows that \(\phi^w_\lambda = 0\) unless \(w\) and \(\lambda\) have the same weight. We let \(\Sigma_n\) denote the set of words of weight \(n\).

Then the \(\phi\)'s generate \(\mathcal{O}(Z^1(U(W)^4; U_3)^{\text{GL}_2})\). We explicitly show that \(\mathcal{O}(Z^1(U(W)^4; U_3)^{\text{GL}_2}) = \mathbb{Q}_p[w_1, w_2, w_3]\); i.e., for each word \(\lambda\) of length at most 3 in \(e_0\) and \(e_1\), we want to write
\[
c^\#(\lambda) \in A
\]
in terms of \(w_1(c), w_2(c),\) and \(w_3(c)\). From now on, we write \(w_1, w_2, w_3\) instead of \(w_1(c), w_2(c), w_3(c)\), with the understanding that they are scalar functions of \(c\).

This also determines the universal cocycle evaluation map
\[
ev_{\Pi, W}: Z^1(U(W); \Pi)^{\text{GL}_2} \times U(W) \to \Pi \times U(W)
\]
in that for \(\lambda \in \mathcal{O}(\Pi)\), we have
\[
ev_{\Pi, W}^\#(\lambda) = \sum_w \phi^w_\lambda f_w \in \mathcal{O}(Z^1(U(W); \Pi)^{\text{GL}_2} \times U(W)).
\]

**Definition 11.1.** Under the basis of \(A\) chosen in §11.1, we let \(\text{pr}_\pi\) denote projection onto the vector subspace spanned by words in the set \(\{\pi_0, \pi_1\}\).

We now prove an analogue of [CDC20, Proposition 3.10].

**Proposition 11.2.** For \(\lambda\) and \(w\) as above, we have
\[
\phi^w_{\lambda e_i} = \phi^w_{\pi_i \lambda} = \delta_{ij}w_1 \phi^w_\lambda.
\]

**Proof.** Suppose \(w\) and \(\lambda\) have weight \(n\) (otherwise \(\phi^w_{\pi_i \lambda} = 0\)). Because \(c\) respects the co-product (a result of the semisimplicity of \(\Pi\)) and commutes with weight projections \(\text{pr}_m\), we have
\[
\begin{align*}
  c^\#(e_i) \otimes c^\#(\lambda) &= (c^\# \otimes c^\#)(e_i \otimes \lambda) \\
  &= (c^\# \otimes c^\#)(\text{pr}_{1,n} \Delta' e_i \lambda) \\
  &= \text{pr}_{1,n} \Delta' c^\#(e_i \lambda) \\
  &= \text{pr}_{1,n} \Delta' \sum_{w' \in \Sigma_{n+1}} \phi^w_{e_i \lambda} f_{w'} \\
  &= \sum_{w' \in \Sigma_{n+1}} \phi^w_{e_i \lambda} \text{pr}_{1,n} \Delta' f_{w'}.
\end{align*}
\]

By \(\text{GL}_2\)-equivariance and Schur’s lemma, we have
\[
\phi^w_{\pi_i} = \delta_{ij} w_1,
\]
so this becomes
\[
(w_1 f_{\pi_i}) \otimes (\sum_{w'' \in \Sigma_n} \phi^w_{\lambda} f_{w''}) = \sum_{w'' \in \Sigma_n} w_1 \phi^w_{\lambda} f_{\pi_i} \otimes f_{w''} = \sum_{w' \in \Sigma_{n+1}} \phi^w_{e_i \lambda} \text{pr}_{1,n} \Delta' f_{w'}.
\]
If \( w' \) begins with a letter other than \( \pi_0 \) or \( \pi_1 \), then \( \text{pr}_{1,n} \Delta'_w f_{w'} \) is zero. Otherwise, if \( w' = \pi_j w \), we have \( \text{pr}_{1,n} \Delta'_w f_{w'} = f_{\pi_j} \otimes f_w \), so that the equation becomes

\[
\sum_{w'' \in \Sigma_n} w_1 \phi^w_{\pi_{w''}} f_{\pi_i} \otimes f_{w''} = \sum_{j=0}^{1} \sum_{w \in \Sigma_n} \phi^w_{\pi_{j}} f_{\pi_j} \otimes f_w.
\]

This implies that if \( j \neq i \), then \( \phi^w_{\pi_{j}} = 0 \). It also implies that if \( j = i \), then

\[
\phi^w_{\pi_{i}} = w_1 \phi^w_{\lambda}.
\]

For \( \phi^w_{\lambda} \), we simply apply the same argument with \( \text{pr}_{n,1} \) in place of \( \text{pr}_{1,n} \). \( \square \)

**Corollary 11.3.** For a word \( \lambda \in \{e_0, e_1\} \), let \( \pi(\lambda) \) denote the word in \( \{\pi_0, \pi_1\} \) obtained by replacing \( e_i \) with \( \pi_i \) for \( i = 0, 1 \). Let

\[
\text{pr}_\pi : A \to A
\]

denote projection onto the subspace generated by words in \( \{\pi_0, \pi_1\} \). Then for any word \( \lambda \) in \( \{e_0, e_1\} \),

\[
\text{pr}_\pi c^\#(\lambda) = f(\pi(\lambda)).
\]

*In other words,*

\[
\phi^w_{\lambda}
\]

is 1 if \( w = \pi(\lambda) \) and 0 if \( w \) is any other word in \( \{\pi_0, \pi_1\} \).

**Proof.** This follows by repeated application of Proposition 11.2. \( \square \)

11.3. **Computing the Universal Cocycle Evaluation Map.** We compute \( c^\#(\lambda) \) in terms of \( W_1(c), w_2(c), w_3(3) \) for \( \lambda = e_0, e_1, e_0e_1, e_1e_0, e_0e_1e_0, e_0e_1e_0e_1^2 \). This will allow us to compute \( c^\#(J_i) \) for \( i = 1, 2, 3, 4 \).

11.3.1. **Degree 1.** As in the proof of Proposition 11.2, \( \text{GL}_2 \)-equivariance and Schur’s Lemma imply that

\[
\phi^w_{\pi_{i}} = \delta_{ij} w_1.
\]

Thus

\[
c^\#(e_0) = w_1 f_{\pi_0}
\]
\[
c^\#(e_1) = w_1 f_{\pi_1}
\]

11.3.2. **Degree 2 Powers.** By Corollary 11.3 and the definition of \( w_2 \), we have

\[
c^\#(e_0e_1) = w_1^2 f_{\pi_0} + w_2 f_\tau.
\]

Applying the matrix

\[
s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2
\]

that switches \( e_0 \) and \( e_1 \), we find

\[
c^\#(e_1e_0) = w_1^2 f_{\pi_1} - w_2 f_\tau.
\]
11.3.3. **Degree 3.** Let’s compute \( c^\pi(e_0e_1e_0) \). Using Proposition 11.2 applied to \( \lambda = e_0e_1 \) and \( \lambda = e_1e_0 \), we may check that
\[
\phi_{e_0e_1e_0}^{\pi} = \phi_{e_0e_1e_0}^\tau = 0,
\]
while
\[
\phi_{e_0e_1e_0}^{\pi} = w_1\phi_{e_0e_1e_0}^\tau = -w_1w_2,
\]
and
\[
\phi_{e_0e_1e_0}^{\pi} = w_1\phi_{e_0e_1e_0}^\tau = w_1w_2.
\]
It follows from this and Corollary 11.3 that
\[
c^\pi(e_0e_1e_0) = w_1^3f_{\pi_0\pi_1\pi_0} + w_1w_2(f_{\pi_0\pi_0\pi_0} - f_{\pi_0\pi_0\pi_0}) + w_3f_{\sigma_0} + af_{\sigma_1}
\]
for some function \( a \) of \( c \). Let \( N := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \), and let \( \text{pr}_a : A_3 \to A_3 \) denote projection onto the subspace spanned by \( f_{\sigma_1} \) and \( f_{\sigma_2} \). Then \( N(e_0e_1e_0) = e_0e_1e_0 + e_0^3 \), and \( \text{pr}_a c^\pi(e_0^3) = \text{pr}_a w_3^2f_{\sigma_0} = 0 \), so
\[
w_3f_{\sigma_0} + af_{\sigma_1} = \text{pr}_a c^\pi(e_0e_1e_0) = \text{pr}_a(N(c^\pi(e_0e_1e_0))) = N(w_3f_{\sigma_0} + af_{\sigma_1}) = (w_3 + a)f_{\sigma_0} + af_{\sigma_1},
\]
which implies that \( a = 0 \).

Applying \( s \) to \( c^\pi(e_0e_1e_0) \) and noting that \( s(f_{\sigma_0}) = -f_{\sigma_1} \), we find that
\[
c^\pi(e_1e_0e_1) = c^\pi(s(e_0e_1e_0)) = s(w_1^3f_{\pi_0\pi_0\pi_0} + w_1w_2(f_{\pi_0\pi_0\pi_0} - f_{\pi_0\pi_0\pi_0}) + w_3f_{\sigma_0}),
\]
\[
= w_1^3f_{\pi_0\pi_0\pi_0} + w_1w_2(-f_{\pi_0\pi_0\pi_0} + f_{\pi_0\pi_0\pi_0}) - w_3f_{\sigma_0}.
\]
We now compute \( c^\pi(e_0e_1e_0^2) \). Notice that \( e_1\text{III}e_0e_1 = e_1e_0e_1 + 2e_0e_1^2 \), so that
\[
2c^\pi(e_0e_1e_0^2) = c^\pi(e_1\text{III}e_0e_1) - c^\pi(e_1e_0e_1)
\]
\[
= c^\pi(e_1)e_0e_1 + \left[ w_1^3f_{\pi_0\pi_0\pi_0} + w_1w_2(-f_{\pi_0\pi_0\pi_0} + f_{\pi_0\pi_0\pi_0}) - w_3f_{\sigma_0} \right]
\]
\[
= (w_1f_{\pi_0\pi_0\pi_0})(w_1^2f_{\pi_0\pi_0\pi_0} + w_2f_{\pi_0\pi_0\pi_0}) - \left[ w_1^3f_{\pi_0\pi_0\pi_0} + w_1w_2(-f_{\pi_0\pi_0\pi_0} + f_{\pi_0\pi_0\pi_0}) - w_3f_{\sigma_0} \right]
\]
\[
= w_1^3(f_{\pi_0\pi_0\pi_0} - f_{\pi_0\pi_0\pi_0}) + w_1w_2(f_{\pi_0\pi_0\pi_0} + f_{\pi_0\pi_0\pi_0} - f_{\pi_0\pi_0\pi_0}) + w_3f_{\sigma_0}
\]
\[
= 2w_1^3f_{\pi_0\pi_0\pi_0} + 2w_1w_2f_{\pi_0\pi_0\pi_0} + w_3f_{\sigma_0}.
\]

11.4. **The Geometric Step.** Let us recall that, as elements of \( \mathcal{O}(U_3) \), we have
\[
J_1 = e_0
\]
\[
J_2 = e_0e_1
\]
\[
J_3 = e_0e_1e_0
\]
\[
J_4 = e_0e_1^2 + 2e_1
\]
By the calculations above, we find
\[
c^\pi(J_1) = w_1f_{\pi_0}
\]
\[
c^\pi(J_2) = w_2^3f_{\pi_0\pi_1} + w_2f_{\pi_0\pi_0\pi_0}
\]
\[
c^\pi(J_3) = w_1^3f_{\pi_0\pi_0\pi_0} + w_1w_2(f_{\pi_0\pi_0\pi_0} - f_{\pi_0\pi_0\pi_0}) + w_3f_{\sigma_0}
\]
Finally, we have
\[
c^\pi(J_4) = c^\pi(e_0e_1^2) + 2c^\pi(e_1) = w_1^3f_{\pi_0\pi_0\pi_0} + w_1w_2f_{\pi_0\pi_0\pi_0} + 2w_3f_{\sigma_0}/2 + 2w_1f_{\pi_0\pi_0\pi_0}.
\]
We now use elimination theory to find a polynomial in $J_1, J_2, J_3, J_4$ with coefficients in $A$ that maps to 0 under $c^\#$.

To eliminate $w_3$, we try
\[ K := f_{\sigma_1}J_3 - 2f_{\sigma_0}J_4 \]
Applying $c^\#$ to this, we get
\[
c^\#(K) = f_{\sigma_1}(w_3^2f_{\pi_0\pi_1\pi_0} + w_1w_2(f_{\pi_0} - f_{\pi_0})) - 2f_{\sigma_0}(w_1^2f_{\pi_0\pi_0}^2 + w_1w_2f_{\pi_1} + 2w_1f_{\pi_1})
= w_3^2(f_{\sigma_1}f_{\pi_0\pi_1\pi_0} - 2f_{\sigma_0}f_{\pi_0\pi_0^2}) + w_1w_2(f_{\sigma_1}(f_{\pi_0} - f_{\pi_0}) - 2f_{\sigma_0}f_{\pi_1}) - 4w_1(f_{\sigma_0}f_{\pi_1})
\]
Now, to eliminate $w_1w_2$, we note
\[
c^\#(J_1J_2) = (w_1f_{\pi_0})(w_1^2f_{\pi_0\pi_1} + w_2f_{\pi_1}) = w_1^3f_{\pi_0}f_{\pi_0\pi_1} + w_1w_2f_{\pi_0}f_{\pi_1}
\]
and then set
\[
L := f_{\pi_0}f_{\pi_1}K - (f_{\sigma_1}(f_{\pi_0} - f_{\pi_0}) - 2f_{\sigma_0}f_{\pi_1})J_1J_2.
\]
Then
\[
c^\#(L) = f_{\pi_0}f_{\pi_1}(w_3^2(f_{\sigma_1}f_{\pi_0\pi_1\pi_0} - 2f_{\sigma_0}f_{\pi_0\pi_0^2}) + w_1w_2(f_{\sigma_1}(f_{\pi_0} - f_{\pi_0}) - 2f_{\sigma_0}f_{\pi_1}) - 4w_1(f_{\sigma_0}f_{\pi_1}))
= w_3^2(f_{\sigma_1}f_{\pi_0\pi_1\pi_0} - 2f_{\sigma_0}f_{\pi_0\pi_0^2}) - (f_{\sigma_1}(f_{\pi_0} - f_{\pi_0}) - 2f_{\sigma_0}f_{\pi_1})(w_1^2f_{\pi_0}\pi_0\pi_1 + w_1w_2f_{\pi_0}f_{\pi_1})
\]

So
\[
L - \frac{f_{\pi_0}f_{\pi_1}(f_{\sigma_1}f_{\pi_0\pi_1\pi_0} - 2f_{\sigma_0}f_{\pi_0\pi_0^2}) - (f_{\sigma_1}(f_{\pi_0} - f_{\pi_0}) - 2f_{\sigma_0}f_{\pi_1})(f_{\pi_0}f_{\pi_0\pi_1})}{f_{\pi_0}^3}J_1^3 + 4f_{\pi_1}f_{\pi_0}f_{\pi_1}J_1
\]
is in the Chabauty-Kim ideal.

In particular, it is a linear combination over $O(W)$ of $J_4, J_3, J_1J_2, J_1^3$, and $J_1$. By the discussion in §9.4 this proves Theorem 10.1

12. Explicit Examples

We describe some examples to be carried out in future work, once we have appropriate code for computing triple integrals on punctured elliptic curves.

12.1. 128a2. We set $X = E'$ to be the affine curve over $\mathbb{Z}[1/2]$ given by minimal Weierstrass equation
\[ y'^2 = x'^3 + x'^2 - 9x' + 7, \]
and $E$ the corresponding projective curve. We let $E'$ and $E$ denote their generic fibers, respectively.

A 2-descent shows that $E(\mathbb{Q})$ has rank 1, with generator
\[ (-1, -4). \]

A simple search reveals 13 elements of $E'(\mathbb{Z}[1/2])$:
\[
(x', y') = (-3, -4), (-3, 4), (-1, -4), (-1, 4), (1, 0), (2, -1), (2, 1),
(3, -4), (3, 4), (29/4, -155/8), (29/4, 155/8), (19, -84), (19, 84)
\]

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We let $P_i$ denote the $i$th element of this list, starting with $P_0 = (-3, -4)$ and ending with $P_{18} = (2738, 141932)$. We let $F = \{ P_i \}_{0 \leq i \leq 12}$. Our goal is to show that $F = \mathcal{E}'(\mathbb{Z}[1/2])$ using the Chabauty-Kim method. More precisely, we set $\Pi = U_3$.

12.1.1. Coleman Integration. We choose $p = 5$ and perform the necessary Coleman integration to determine the coefficients $c_i$.

For the purposes of Coleman integration, it is easier to use a simple Weierstrass model. A simple Weierstrass model for $E$ is given by

$$y^2 = x^3 - 12096x + 470016$$

We thus have $c_0 = dx/y$ and $c_1 = xdy/y$, as in §10.2.

The conversion between the models is given by the transformation $(x', y') \leftrightarrow (x, y) = (u^2x' + r, u^3y' + su^2x' + t)$ for

$$(u, r, s, t) = (6, 12, 0, 0)$$

The inverse is given by

$$(u, r, s, t) = (1/6, -1/3, 0, 0).$$

In particular, these models are isomorphic over $\mathbb{Z}[1/6]$ and thus $\mathbb{Z}_5$. Therefore, we may use it for all local computations at $p = 5$.

Converting the $\mathbb{Z}[1/2]$-points to the simple Weierstrass model, we get

$$(-96, -864), (-96, 864), (-24, -864), (-24, 864), (48, 0), (84, -216), (84, 216), (120, -864), (120, 864), (273, -4185), (273, 4185), (696, -18144), (696, 18144)$$

We choose basepoint

$$b = P_0 = (-96, -864),$$

thus fixing $J_1, J_2, J_3,$ and $J_4$ as functions on $\mathcal{E}'(\mathbb{Z}_5)$.

12.2. **102a1.** We set $\mathcal{X} = \mathcal{E}'$ to be the affine curve over $\mathbb{Z}[1/2]$ given by minimal Weierstrass equation

$$y^2 + x'y' = x'^3 + x'^2 - 2x'$$

and $\mathcal{E}$ the corresponding projective curve. We let $E'$ and $E$ denote their generic fibers, respectively.

A 2-descent shows that $E(\mathbb{Q})$ has rank 1, with generator

$$(-1, -1).$$

A simple search reveals 19 elements of $\mathcal{E}'(\mathbb{Z}[1/2])$:

$$(x', y') = (-2, 0), (-2, 2), (-1, -1), (-1, 2), (-\frac{1}{4}, -\frac{5}{8}), (-\frac{1}{4}, \frac{7}{8}), (0, 0), (1, -1), (1, 0), (\frac{121}{64}, -\frac{1881}{512}),$$

$$(\frac{121}{64}, \frac{913}{512}), (2, -4), (2, 2), (8, -28), (8, 20), (9, -33), (9, 24), (2738, -144670), (2738, 141932)$$

We let $P_i$ denote the $i$th element of this list, starting with $P_0 = (-2, 0)$ and ending with $P_{18} = (2738, 141932)$. We let $F = \{ P_i \}_{0 \leq i \leq 18}$. Our goal is to show that $F = \mathcal{E}'(\mathbb{Z}[1/2])$ using the Chabauty-Kim method.

More precisely, we set $\Pi = U_3$.  

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12.2.1. Local Computation at Bad Primes. We make regular use of the results and notation of §3.4.2-3.5.

We have \( j(E) = \frac{1771561}{612} = 2^{-2} \cdot 3^{-2} \cdot 11^6 \cdot 17^{-1} \). Therefore, the Tamagawa numbers of \( E \) at 3 and 17 are 2 and 1, respectively. The conductor is 102 = 2 ⋅ 3 ⋅ 17, so the curve is semistable at all primes. We have \( T_0 = \{2, 3, 17\} \), so that \( T_0 \setminus S = \{3, 17\} \).

It follows that \( E \) is already a regular semistable model at 17. Its special fiber has one component, so that \( \kappa_{17}(E'(\mathbb{Z}_{17})) = * \).

The special fiber \( E_{\mathbb{Z}_3}' \) contains one singular point, given by \( (x', y') = (2, 2) \). Let \( E'' \) denote the blowup of \( E' \) at this point. Then \( E'' \) is the minimal regular model of \( E' \). Let \( \{\alpha_1, \alpha_2\} = E(T_3(E'')) \), with \( \alpha_1 \) corresponding to the smooth locus of \( E'' \) and \( \alpha_2 \) to the exceptional divisor. Then

\[
\kappa_3(E'(\mathbb{Z}_3)) \subseteq \{\alpha_1, \alpha_2\},
\]

where by abuse of notation we use \( \alpha_1, \alpha_2 \) to refer to the corresponding elements of \( H^1(G_3; \Pi) \).

For \( z \in E'(\mathbb{Z}_3) \), we have

- \( \kappa_3(z) = \alpha_2 \) if \( z \mod 3 = (2, 2) \)
- \( \kappa_3(z) = \alpha_1 \) otherwise

Notice that \( P_0 \mod 3 = (1, 0) \neq (2, 2) \), while \( P_2 \mod 3 = (2, 2) \), so that in fact \( \kappa_3(E'(\mathbb{Z}_3)) = \{\alpha_1, \alpha_2\} \) (in the language of [BD20] Proposition 1.2.1(1)], this is saying that each component contains the reduction of a point in \( X(\mathbb{Z}_p) \)).

We set \( F_1 := \{z \in F \mid \kappa_3(z) = \alpha_1\} \). Then

\[
F_1 = \{P_0, P_1, P_6, P_7, P_8, P_9, P_{10}, P_{15}, P_{16}\}
\]

\[
F_2 = \{P_2, P_3, P_4, P_5, P_{11}, P_{12}, P_{13}, P_{14}, P_{17}, P_{18}\}
\]

12.2.2. Coleman Integration. We choose \( p = p = 5 \) and perform the necessary Coleman integration to determine the coefficients \( c_i \).

For the purposes of Coleman integration, it is easier to use a simple Weierstrass model. A simple Weierstrass model for \( E \) is given by

\[
y^2 = x^3 - 3267x + 45630
\]

We thus have \( c_0 = dx/y \) and \( c_1 = xdx/y \), as in §10.2.

The conversion between the models is given by the transformation \( (x', y') \mapsto (x, y) = (u^2x' + r, u^3y' + su^2x' + t) \) for

\[
(u, r, s, t) = (6, 15, 3, 0)
\]

The inverse is given by

\[
(u, r, s, t) = (1/6, -5/12, -1/2, 5/24).
\]

In particular, these models are isomorphic over \( \mathbb{Z}[1/6] \) and thus \( \mathbb{Z}_5 \). Therefore, we may use it for all local computations at \( p = 5 \).

Converting the \( \mathbb{Z}[1/2] \)-points to the simple Weierstrass model, we get

\[
(x, y) = (-57, -216), (-57, 216), (-21, -324), (-21, 324), (6, -162), (6, 162), (15, 0), (51, -108), (51, 108), (1329/16, -37719/64), (1329/16, 37719/64), (87, -648), (87, 648), (303, -5184), (303, 5184), (339, -6156), (339, 6156), (98583, -30953016), (98583, 30953016)
\]

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We first work with $F_2$. We choose basepoint

$$b_2 = P_2 = (-21, -324),$$

thus fixing $J_1$, $J_2$, $J_3$, and $J_4$ as functions on $\mathcal{E}'(\mathbb{Z}_5)$.

To be continued.

12.3. **Estimation of Zeroes of $p$-adic Power Series.** We describe how the known $\mathbb{Z}[1/2]$-points fit into 5-adic residue discs.

12.3.1. **128a2.** The set $\mathcal{E}'(\mathbb{F}_5)$ is the 7-element set given by

$$(x, y) = (0, 1), (0, 4), (1, 1), (1, 4), (3, 0), (4, 1), (4, 4)$$

We let $R_i$ denote the residue disc of the $i$th point in the list, starting with $R_1$ the residue disc of $(0, 0)$. We note the intersection of $F$ with each residue disc:

$$
\begin{align*}
F \cap R_1 &= \{P_7\} \\
F \cap R_2 &= \{P_8\} \\
F \cap R_3 &= \{P_2, P_{11}\} \\
F \cap R_4 &= \{P_3, P_{12}\} \\
F \cap R_5 &= \{P_4, P_9, P_{10}\} \\
F \cap R_6 &= \{P_0, P_6\} \\
F \cap R_7 &= \{P_1, P_5\}
\end{align*}
$$

For $i \neq 1$, we may choose a constant plus $x$ as an analytic uniformizer on $R_i$.

12.3.2. **102a1.** The set $\mathcal{E}'(\mathbb{F}_5)$ is the 10-element set given by

$$(x, y) = (0, 0), (0, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 4), (4, 1), (4, 4)$$

We let $R_i$ denote the residue disc of the $i$th point in the list, starting with $R_1$ the residue disc of $(0, 0)$. We note the intersection of $F$ with each residue disc:

$$
\begin{align*}
F \cap R_1 &= \{P_6\} \\
F \cap R_2 &= \emptyset \\
F \cap R_3 &= \{P_5, P_7\} \\
F \cap R_4 &= \{P_4, P_8\} \\
F \cap R_5 &= \{P_{11}\} \\
F \cap R_6 &= \{P_{12}\} \\
F \cap R_7 &= \{P_1, P_{13}, P_{18}\} \\
F \cap R_8 &= \{P_0, P_{14}, P_{17}\} \\
F \cap R_9 &= \{P_2, P_{10}, P_{16}\} \\
F \cap R_{10} &= \{P_3, P_9, P_{15}\}
\end{align*}
$$

For $i \neq 1$, we may choose a constant plus $x$ as an analytic uniformizer on $R_i$. 55
Appendix A. Unipotent Groups and Cohomology

A.1. Unipotent Groups. Let $U$ be a pro-unipotent group over a field $K$. Then we have the Lie algebra $\text{Lie } U$ and its universal enveloping algebra $U U$, along with the coordinate ring $O(U)$. The first two are naturally a Lie algebra object and cocommutative Hopf algebra object, respectively, of $\text{ProVect}_K^\text{fin}$, while the latter is a commutative Hopf algebra object of $\text{IndVect}_K^\text{fin}$. The natural duality between $\text{ProVect}_K^\text{fin}$ and $\text{IndVect}_K^\text{fin}$ induces a natural isomorphism

$$O(U) \cong U U^\vee.$$  

If $\Pi$ is a unipotent group with an action of $U$ (equivalently, $\Pi$ is a unipotent group in the Tannakian category $\text{Rep}_K(U)$ in the sense of [Del89, §5]), then $M := U \Pi$ has the structure of a module over $A := U U$. If $\rho: A \otimes M \to M$ denotes the multiplication map, then

$$\rho \circ (\text{id}_A \otimes \text{mult}_M) = \text{mult}_M \circ (\rho \otimes \rho) \circ (\Delta_A \otimes \text{id}_{M \otimes M}): A \otimes M \otimes M \to M,$$

where $\rho \otimes \rho$ sends $u_1 \otimes u_2 \otimes \pi_1 \otimes \pi_2$ to $\rho(u_1 \otimes \pi_1) \otimes \rho(u_2 \otimes \pi_2)$. This implies that

$$\text{Lie } U \subseteq U U = A$$
acts via derivations on $M$, while

$$U \subseteq U U = A$$
acts via automorphisms on $M$. We also have

$$\Delta_M \circ \rho = (\rho \otimes \rho) \circ (\Delta_A \otimes \Delta_M): A \otimes M \to M \otimes M,$$

which implies that the action of $\text{Lie } U \subseteq A$ preserves $\text{Lie } \Pi \subseteq M$, and the action of $U \subseteq A$ preserves both $\text{Lie } \Pi$ and $\Pi$ in $M$, acting via derivations and automorphisms, respectively.

For an object $W$ of $\text{ProVect}_K^\text{fin}$, we denote by

$$\text{FreeLie } W$$
the free pro-nilpotent Lie algebra on $W$.

Proposition A.1. We have

$$Z^1(U; \Pi) = Z^1(n(U); \text{Lie } \Pi),$$

where

$$Z^1(n(U); \text{Lie } \Pi)$$
denotes the space of $K$-linear maps

$$c: n(U) \to \text{Lie } \Pi$$
satisfying the cocycle condition

$$c([g, h]) = [c(g), c(h)] + g(c(h)) - h(c(g)),$$

where $n(U)$ acts on $\text{Lie } \Pi$ by derivations.

\[\text{In fact, they preserve the coalgebra structure, in that elements of Lie } U \text{ act as coderivations, and elements of } U \text{ act as automorphisms of the Hopf algebra.}\]
Proof. Either can easily be seen to be in bijection with the space of $K$-linear maps
\[ c: A \to M \]
satisfying the two conditions
\begin{itemize}
  \item $(c \otimes c) \circ \Delta_A = \Delta_M \circ c: A \to M \otimes M$ (i.e., $c$ is a homomorphism of coalgebras)
  \item $c \circ \text{mult}_A = \text{mult}_M \circ (\text{id}_M \otimes \rho) \circ (c \otimes \text{id}_A \otimes c) \circ (\Delta_A \otimes \text{id}_A): A \otimes A \to M$.
\end{itemize}
\[ \square \]

A.2. Extension of a Reductive Group by a Unipotent Group. Let $G$ be a reductive group acting on $U$, and let $G := G \ltimes U$.

We suppose that the action of $U$ on $\Pi$ extends to an action of $G$ on $\Pi$. The subgroup $G \subseteq G$ acts on $U$ by conjugation and on $\Pi$ via the latter’s action of $G$.

The action of $G$ on $U$ induces an action on $\text{Lie } U$ and on $U$ by Lie and Hopf algebra automorphisms, respectively. We also let $U$ act on $\mathcal{U}(U)$ by left-translation. This is compatible with the action of $G$ on $\mathcal{U}(U)$ in that it extends to an action of $G$ on $\mathcal{U}(U)$.

Remark A.2. For $g \in \text{GL}_2$, $u \in U(O_{k,S}, E)$, and $\pi \in \Pi$, we have
\[ (g(u)) (g\pi) = (gug^{-1})(g\pi) = g(u\pi), \]
implying that the action map
\[ U(O_{k,S}, E) \times \Pi \to \Pi \]
is $\text{GL}_2$-equivariant. The same is true of the associated Lie algebra and universal enveloping algebra actions as in §A.1.

A.3. Cohomology and Cocycles. The main goal of this section is to prove a generalization of [DCW16, Proposition 5.2].

Some computations below use the fact (which follows from the definition of semidirect product) that for $g \in G$, $u \in U$, and $\pi \in \Pi$, we have
\[ g(u)(\pi) = g(u(g^{-1}(\pi))). \]

We recall some facts about nonabelian cocycles. Some of our discussion borrows from [Bro17a, §6.1]. Although we do not denote it in the notation, all points are relative to a $K$-algebra $R$.

For a cocycle $c: G \to \Pi$ and $g \in G$, we denote $c(g)$ by $c_g$. We recall that for a $K$-algebra $R$, we have
\[ Z^1(G; \Pi)(R) := \{ c: G_R \to \Pi_R \mid c_{g_1 g_2} = c_{g_1} c_{g_2} \forall g_1, g_2 \in G \}, \]
where the cocycle condition is imposed on the functor of points. We similarly define $Z^1(U; \Pi)$. The trivial cocycle $1 \in Z^1(G; \Pi)$ (resp. $Z^1(U; \Pi)$) sends all of $G$ (resp. $U$) to the identity of $\Pi$.

Recall also that $\Pi$ acts on $Z^1(G; \Pi)$ (resp. $Z^1(U; \Pi)$) by sending $\pi \in \Pi$ and $c \in Z^1(G; \Pi)$ (resp. $c \in Z^1(U; \Pi)$) to
\[ g \mapsto \pi c_g g(\pi^{-1}) \]
(resp. $u \mapsto \pi_c u (\pi^{-1})$) and that 

$$H^1(G; \Pi) := Z^1(G; \Pi)/\Pi.$$ 

**Lemma A.3.** We have

$$Z^1(G; \Pi) \cong \ker(\text{Hom}(G, \Pi \rtimes G) \to \text{Hom}(G, G)),$$

given by

$$c \mapsto \rho_c,$$

where $\rho_c(g) := c_g g \in \Pi \rtimes G$. The action of $\Pi$ is given by conjugation on $\Pi \rtimes G$.

**Proof.** Since 

$$\rho_c(g_1) \rho_c(g_2) = (c_{g_1}) (c_{g_2}) g_1 g_2 = c_{g_1} c_{g_2} g_1 g_2,$$

while

$$\rho_c(g_1 g_2) = c_{g_1} c_{g_2} g_1 g_2,$$

the homomorphism condition for $c$ is equivalent to the cocycle condition for $\rho$.

To check the action of $\Pi$, note that for $\pi \in \Pi$ and $g \in G$, we have

$$\pi \rho_c(g) \pi^{-1} = \pi (c_g g) \pi^{-1} = \pi c_g g \pi^{-1} = \pi c_g (\pi^{-1}) g = \pi (c) g = \rho_\pi (c)(g),$$

as desired. \hfill \square

Recall that $G$ acts on $Z^1(U; \Pi)$ by

$$g(c)_u = g(c_{g^{-1}(u)}),$$

and the $\Pi$- and $G$-actions on $Z^1(U; \Pi)$ are compatible in that

$$g(\pi(c)) = g(\pi)(g(c)),$$

so that the $G$-action induces an action on 

$$H^1(U; \Pi).$$

The main result of this section is the following generalization of [DCW16, Proposition 5.2]:

**Theorem A.4.** Suppose that $\Pi^G = 1$. Then each equivalence class of cocycles $[c]$ in $H^1(G; \Pi)$ contains a unique representative $c_0$ such that

$$c_0(g) = 1 \text{ for each } g \in G,$$

and its restriction to $U$ is a $G$-equivariant cocycle

$$c_0|_U : U \to \Pi.$$

The map

$$[c] \mapsto c_0|_U$$

defines a bijection

$$H^1(G, \Pi) \cong Z^1(U; \Pi)^G.$$ 

We prove this result via a series of lemmas.
Lemma A.5. The restriction map

\[ Z^1(G; \Pi) \to Z^1(U; \Pi) \]

induces a bijection

\[ \ker(Z^1(G; \Pi) \to Z^1(G; \Pi)) \cong Z^1(U; \Pi)^G \]

Proof. Suppose \( c \in \ker(Z^1(G; \Pi) \to Z^1(G; \Pi)) \). Then for \( u \in U \), we have

\[
g(c)_u = g(c_{g^{-1}(u)}) \\
= g(c_{g^{-1}ug}) \\
= g(c_{g^{-1}ug}^{-1}(u(c_g))) \\
= g(c_{g^{-1}u}) \\
= g(c_{g^{-1}g^{-1}(c_u)}) \\
= g(g^{-1}(c_u)) \\
= c_u
\]

Conversely, suppose \( c \in Z^1(U; \Pi)^G \). We extend \( c \) to an element of \( Z^1(G; \Pi) \) as follows. For \( ug \in G \), we set

\[ c_{ug} = c_u. \]

It is clear by construction that it is trivial when restricted to \( G \). To show that it is a cocycle, suppose we have \( g_1, g_2 \in G \) and \( u_1, u_2 \in U \). Then

\[
c_{u_1g_1u_2g_2} = c_{u_1g_1u_2g_1g_2} \\
= c_{u_1g_1(u_2)g_1g_2} \\
= c_{u_1g_1(u_2)} \\
= c_{u_1u_1}(g_1(c_{u_2})) \\
= c_{u_1u_1}(g_1(c_{u_2})) \\
= c_{u_1g_1u_1}(g_1(c_{u_2})),
\]

as desired.

Given \( c \in Z^1(U; \Pi)^G \), it is clear that the maps are mutual inverses. Given \( c \in \ker(Z^1(G; \Pi) \to Z^1(G; \Pi)) \), note that

\[ c_{ug} = c_u, \]

so the maps are indeed mutual inverses.

\[ \square \]

Lemma A.6. The natural map

\[ H^1(G; \Pi) \to H^1(U; \Pi) \]

induces an isomorphism

\[ H^1(G; \Pi) \cong H^1(U; \Pi)^G \]

Proof. As \( G \) is reductive, all higher group cohomology of \( G \) vanishes, so the Hochschild-Serre spectral sequence for the inclusion \( U \hookrightarrow G \) degenerates. In particular, we get

\[
H^1(G; \Pi) = H^0(G; H^1(U; \Pi)) = H^1(U; \Pi)^G.
\]

\[ \square \]
By Lemma A.5 and Lemma A.6 we have a diagram

\[
\begin{array}{ccccccc}
\text{ker}(Z^1(G; \Pi) \to Z^1(\mathbb{G}; \Pi)) & \to & H^1(G; \Pi) \\
\cong \downarrow & & \cong \downarrow \\
Z^1(U; \Pi)^G & \to & H^1(U; \Pi)^G
\end{array}
\]

It now suffices to prove that either horizontal arrow of the diagram is an isomorphism. We prove this for the top horizontal arrow whenever \(\Pi^G = 0\).

**Lemma A.7.** Suppose \(\Pi^G = 0\). Then the set of splittings of \(\Pi \ltimes G \to G\) forms a pointed torsor under \(\Pi\) acting by conjugation.

**Proof.** To check transitivity, note that the set of splittings modulo \(\Pi\) is the cohomology set \(H^1(\mathbb{G}; \Pi)\). But this vanishes because \(G\) is reductive. The action is simply transitive because the stabilizer of a section is \(\Pi^G\), which is trivial by assumption. The point of the torsor is the trivial splitting \(G \to \Pi \ltimes G\).

**Lemma A.8.** The set

\[
\ker(Z^1(G; \Pi) \to Z^1(\mathbb{G}; \Pi)) \subseteq Z^1(G; \Pi)
\]

maps bijectively onto \(H^1(G; \Pi)\).

**Proof.** By Lemma A.3 under the identification

\[
Z^1(G; \Pi) = \ker(\text{Hom}(G, \Pi \ltimes G) \to \text{Hom}(G, G)),
\]

the action of \(\Pi\) is by conjugation. Via the inclusion \(\mathbb{G} \subseteq G\), we may consider \(\Pi \ltimes G\) as a subgroup of \(\Pi \ltimes G\) (it is not normal, but this does not matter). For any \(\rho \in \ker(\text{Hom}(G, \Pi \ltimes G) \to \text{Hom}(G, G))\), \(\rho|_\mathbb{G}\) is a splitting of \(\Pi \ltimes \mathbb{G} \to \mathbb{G}\).

For \(\pi \in \Pi\), we have \(\rho_{\pi(c)}|_\mathbb{G} = \pi \rho_c|_\mathbb{G} \pi^{-1}\). Therefore, by Lemma A.7 there is a unique \(\pi \in \Pi\) for which \(\rho_{\pi(c)}|_\mathbb{G}\) is the trivial splitting.

Now \(\rho_{\pi(c)}|_\mathbb{G}\) is the trivial splitting iff \(c|_\mathbb{G}\) is trivial, i.e. if \(c \in \ker(Z^1(G; \Pi) \to Z^1(\mathbb{G}; \Pi))\). Therefore, each class \([c] \in H^1(G; U)\) contains a unique representative in \(\ker(Z^1(G; \Pi) \to Z^1(\mathbb{G}; \Pi))\).

**Remark A.9.** One may alternatively show that the bottom horizontal arrow is an isomorphism, as follows. If \(\Pi\) is abelian, this amounts to considering the short exact sequence

\[
0 \to \Pi/\Pi^U \to Z^1(U, \Pi) \to H^1(U, \Pi) \to 0
\]

and then taking \(\mathbb{G}\)-fixed points and using the fact that \(\mathbb{G}\) is reductive and \(\Pi^G = 0\) to get

\[
Z^1(U, \Pi)^G \cong H^1(U, \Pi)^G.
\]

One may replicate this argument for general \(\Pi\) by showing that if \(\Pi\) is a nonabelian group acting on a pointed set (or variety) \(Y\), and both have a compatible action of \(\mathbb{G}\), then there is a long exact sequence of pointed sets

\[
\Pi^G \to Y^G \to (Y/\Pi)^G \to H^1(\mathbb{G}; \Pi).
\]
References

[All16] Patrick B. Allen. Deformations of polarized automorphic Galois representations and adjoint Selmer groups. *Duke Math. J.*, 165(13):2407–2460, 2016.

[BBM16] Jennifer S. Balakrishnan, Amnon Besser, and J. Steffen Müller. Quadratic Chabauty: $p$-adic heights and integral points on hyperelliptic curves. *J. Reine Angew. Math.*, 720:51–79, 2016.

[BD18a] Jennifer S. Balakrishnan and Netan Dogra. Quadratic Chabauty and rational points, I: $p$-adic heights. *Duke Math. J.*, 167(11):1981–2038, 2018. With an appendix by J. Steffen Müller.

[BD18b] Jennifer S. Balakrishnan and Netan Dogra. Quadratic Chabauty and rational points II: Generalised height functions on Selmer varieties, 2018. Preprint. arXiv:1705.00401.

[BD20] L. Alexander Betts and Netan Dogra. The local theory of unipotent kummer maps and refined selmer schemes, 2020. Preprint. arXiv:1909.05734.

[BDCKW18] Jennifer S. Balakrishnan, Ishai Dan-Cohen, Minhyong Kim, and Stefan Wewers. A non-abelian conjecture of Tate-Shafarevich type for hyperbolic curves. *Math. Ann.*, 372(1-2):369–428, 2018.

[Bea17] Jamie Beacom. Computation of the unipotent albanese map on elliptic and hyperelliptic curves, 2017. Preprint. arXiv:1711.03932.

[Bel09] Joel Bellaiche. An introduction to Bloch and Kato’s conjecture. Notes from Lectures at a Clay Summer Institute, 2009.

[BK90] Spencer Bloch and Kazuya Kato. $L$-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[Bro17a] Francis Brown. Integer points on curves, the unit equation, and motivic periods, 2017. Preprint. arXiv:1704.00555.

[Bro17b] Francis Brown. Notes on motivic periods. *Commun. Number Theory Phys.*, 11(3):557–655, 2017.

[CDC20] David Corwin and Ishai Dan-Cohen. The polylog quotient and the Goncharov quotient in computational Chabauty–Kim Theory I. *Int. J. Number Theory*, 16(8):1859–1905, 2020.

[Cha41] Claude Chabauty. Sur les points rationnels des courbes algébriques de genre supérieur à l’unité. *C. R. Acad. Sci. Paris*, 212:882–885, 1941.

[Col85] Robert F. Coleman. Effective Chabauty. *Duke Math. J.*, 52(3):765–770, 1985.

[Cor19] David Corwin. Remarks on the Hodge filtration in non-Abelian Chabauty’s Method. https://math.berkeley.edu/~dcorwin/files/brownkimhodge.pdf, 2019.

[Cor20] David Corwin. From Chabauty’s method to Kim’s non-abelian Chabauty’s method. https://math.berkeley.edu/~dcorwin/files/ChabautytoKim.pdf, 2020.

[CU13] Andre Chatzistamatiou and Sinan Ünver. On $p$-adic periods for mixed Tate motives over a number field. *Math. Res. Lett.*, 20(5):825–844, 2013.

[DC20] Ishai Dan-Cohen. Mixed Tate motives and the unit equation II. *Algebra Number Theory*, 14(5):1175–1237, 2020.

[DCC20] Ishai Dan-Cohen and David Corwin. The polylog quotient and the Goncharov quotient in computational Chabauty–Kim theory II. *Trans. Amer. Math. Soc.*, 373(10):6835–6861, 2020.

[DCW16] Ishai Dan-Cohen and Stefan Wewers. Mixed Tate motives and the unit equation. *Int. Math. Res. Not. IMRN*, (17):5291–5354, 2016.

[Del71] Pierre Deligne. Théorie de Hodge. I. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 425–430. 1971.

[Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[Del89] Pierre Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.

[DG05] P. Deligne and A. B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup. (4)*, 38(1):1–56, 2005.

[Dog20] Netan Dogra. Unlike intersections and the Chabauty-Kim method over number fields, 2020. Preprint. arXiv:1903.05032.
[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983.

[FPR94] Jean-Marc Fontaine and Bernadette Perrin-Riou. Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions $L$. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 599–706. Amer. Math. Soc., Providence, RI, 1994.

[Gon98] Alexander Goncharov. Mixed elliptic motives. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 147–221. Cambridge Univ. Press, Cambridge, 1998.

[Gon05] Alexander B. Goncharov. Galois symmetries of fundamental groupoids and noncommutative geometry. *Duke Math. J.*, 128(2):209–284, 2005.

[Had11] Majid Hadian. Motivic fundamental groups and integral points. *Duke Math. J.*, 160(3):503–565, 2011.

[HM20] Richard Hain and Makoto Matsumoto. Universal mixed elliptic motives. *J. Inst. Math. Jussieu*, 19(3):663–766, 2020.

[Kat04] Kazuya Kato. $p$-adic Hodge theory and values of zeta functions of modular forms. Number 295, pages ix, 117–290. 2004. Cohomologies $p$-adiques et applications arithmétiques. III.

[Kim05] Minhyong Kim. The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel. *Invent. Math.*, 161(3):629–656, 2005.

[Kim09] Minhyong Kim. The unipotent Albanese map and Selmer varieties for curves. *Publ. Res. Inst. Math. Sci.*, 45(1):89–133, 2009.

[KT08] Minhyong Kim and Akio Tamagawa. The $l$-component of the unipotent Albanese map. *Math. Ann.*, 340(1):223–235, 2008.

[LZ20a] David Loeffler and Sarah Livia Zerbes. On the Bloch–Kato conjecture for the symmetric cube, 2020. Preprint. arXiv:2005.04786.

[LZ20b] David Loeffler and Sarah Livia Zerbes. On the Bloch-Kato conjecture for $GSp(4)$, 2020. Preprint. arXiv:2003.05960.

[Moo19] Ben Moonen. A remark on the Tate conjecture. *J. Algebraic Geom.*, 28(3):599–603, 2019.

[MP12] William McCallum and Bjorn Poonen. The method of Chabauty and Coleman. In *Explicit methods in number theory*, volume 36 of *Panor. Synthèses*, pages 99–117. Soc. Math. France, Paris, 2012.

[Nek06] Jan Nekovář. Selmer complexes. *Astérisque*, (310):viii+559, 2006.

[Pat13] Owen Patashnick. A candidate for the Abelian category of mixed elliptic motives. *J. K-Theory*, 12(3):569–600, 2013.

[Poo02] Bjorn Poonen. Computing rational points on curves. In *Number theory for the millennium, III (Urbana, IL, 2000)*, pages 149–172. A K Peters, Natick, MA, 2002.

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[Ser98] Jean-Pierre Serre. Abelian $l$-adic representations and elliptic curves, volume 7 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, 1998. With the collaboration of Willem Kuyk and John Labute. Revised reprint of the 1968 original.

[Sou79] C. Soulé. $K$-théorie des anneaux d’entiers de corps de nombres et cohomologie étale. *Invent. Math.*, 55(3):251–295, 1979.

[SR72] Neantro Saavedra Rivano. *Catégories Tannakiennes*. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York, 1972.

[ST68] Jean-Pierre Serre and John Tate. Good reduction of Abelian varieties. *Ann. of Math. (2)*, 88:492–517, 1968.

[Wut14] Christian Wuthrich. Overview of some Iwasawa theory. In *Iwasawa theory 2012*, volume 7 of *Contrib. Math. Comput. Sci.*, pages 3–34. Springer, Heidelberg, 2014.

[Yam10] Go Yamashita. Bounds for the dimensions of $p$-adic multiple $L$-value spaces. *Doc. Math.*, (Extra volume: Andrei A. Suslin sixtieth birthday):687–723, 2010.

[Zie15] Paul Ziegler. Graded and filtered fiber functors on Tannakian categories. *J. Inst. Math. Jussieu*, 14(1):87–130, 2015.