SPLITTING THEOREMS FOR POISSON AND RELATED STRUCTURES

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Abstract. According to the Weinstein splitting theorem, any Poisson manifold is locally, near any given point, a product of a symplectic manifold with another Poisson manifold whose Poisson structure vanishes at the point. Similar splitting results are known e.g. for Lie algebroids, Dirac structures and generalized complex structures. In this paper, we develop a novel approach towards these results that leads to various generalizations, including their equivariant versions as well as their formulations in new contexts.

1. Introduction

Poisson manifolds and related geometric structures, such as Lie algebroids, Dirac structures and generalized complex structures, display an intricate local theory. The splitting theorems to be discussed in this paper refer to a series of results that provide fundamental local information about these types of geometry.

In each of these contexts, the geometric structure on the given manifold $M$ determines a generalized foliation of $M$, in the sense of Stefan and Sussmann. While the leaves of such a foliated manifold need not be of constant dimension, the Stefan-Sussmann theory shows that they are arranged rather nicely: For every $m \in M$ there is an open neighborhood isomorphic to a product of foliated manifolds $S \times N$, where $S$ has the trivial foliation (with $S$ itself as its only leaf) while $N$ contains the point $m$ as a zero-dimensional leaf. The splitting theorems say that, in each case, one can take this splitting $S \times N$ to be compatible with the given geometric structure. The following are some instances of such results:

(a) Weinstein’s splitting theorem [41] for Poisson manifolds $(M, \pi)$, which asserts the existence of a neighborhood of $m$ that is Poisson diffeomorphic to a product $(S, \pi_S) \times (N, \pi_N)$, where $\pi_S$ is non-degenerate while $\pi_N$ vanishes at $m$;

(b) the splitting theorem for Dirac manifolds [13], obtained by Blohmann [9] (see also Dufour-Wade [17] for related results);

(c) the splitting theorem for Lie algebroids $E \rightarrow M$, due to Dufour [16], Fernandes [19], and Weinstein [42], which gives an isomorphism near $m$ with a product of Lie algebroids $TS \times F$, where the anchor of the Lie algebroid $F \rightarrow N$ vanishes at $m$;

(d) the splitting theorem for generalized complex manifolds [23], due to Abouzaid-Boyarchenko [11], which shows that up to a $B$-field transform, any generalized complex manifold is locally a product $S \times N$ of generalized complex manifolds, where $S$ is ‘of symplectic type’ and $N$ is ‘of complex type’ at $m$.

In this article, we develop a novel approach towards splitting theorems, which allows us to generalize them in various directions and to new contexts. Rather than taking $N$ to be ‘small’, we will allow transverse submanifolds $N \hookrightarrow M$ that may be quite large. Transversality implies that the normal bundle $\nu_N$ inherits a ‘linear approximation’ of the given geometry. Our local
models will give tubular neighborhood embeddings, identifying the geometric structures over
the normal bundle $\nu_N$ and over an open neighborhood of the transversal $N \subseteq M$. (This is not
to be confused with linearization problems around leaves.) In the Poisson case, we recover the
normal form theorem of Frejlich-Mărcuț [20].

Our main technical tool is a linearization lemma for vector fields $X$ vanishing along sub-
manifolds $N \subseteq M$, with linear approximation given by the Euler vector field on the normal
bundle $\nu_N$. Any such ‘Euler-like’ vector field determines a tubular neighborhood embedding,
and the strategy of the proof is to make the vector field, and hence the tubular neighborhood,
compatible with the given geometric data. A key feature of our approach is that constructions
are quite explicit, in the sense that normal forms are fully determined by some specific choices,
with a natural dependence on them. As a result, they have good functorial properties, so that
one obtains the $G$-equivariant versions of the normal form theorems without extra effort. We
remark that it is unclear how to obtain equivariant splitting theorems from the traditional
(induction-based) proofs. Indeed, for Poisson manifolds $(M, \pi)$, a $G$-equivariant Weinstein
splitting theorem was only recently proved by Frejlich-Mărcuț in [20], following partial results
in Miranda-Zung [31]. The argument in [20] towards Weinstein splittings, and more generally
normal forms along cosymplectic transversals $N \subseteq M$, uses ‘Poisson sprays’ and the approach
of Crainic-Mărcuț [13] to symplectic realizations. In contrast, our normal form for Poisson
case is entirely determined by the choice of a 1-form $\alpha \in \Omega^1(M)$ whose image under the map
$\pi^*: T^*M \to TM$ is an Euler-like vector field along $N$.

The structure of this paper is as follows. In Section 2 we discuss the linearization of Euler-like
vector fields and the resulting tubular neighborhood embeddings. In Section 3 we apply this to
anchored vector bundles satisfying an involutivity condition. We obtain a normal form theorem
along transversals, which may be regarded as a version of the Stefan-Sussmann theorem. This
is followed by similar transversal normal form theorems for transversals of Lie algebroids in
Section 4 and Dirac structures in Section 5, which are new in these contexts.

From our result for Dirac structures, we derive as direct consequences the transversality
results for Poisson structures in Section 6 and generalized complex structures in Section 7.
Similar results for generalized complex structures have independently been obtained in recent
work of Bailey-Cavalcanti-Duran [6], using a different approach. Our method also leads to new
results on transverse normal forms for Courant algebroids, but this case is less straightforward
and will be treated separately.

In future work, we plan to generalize some of these techniques to infinite-dimensional settings.
Indeed, one of our inspirations was the proof of Frobenius’ theorem for Banach manifolds, in
the books [2] and [28], and the realization that the geometry behind these proofs involves the
flow of an Euler-like vector field. We expect that similar techniques can be used to prove
versions of the Stefan-Sussmann theorem and other splitting theorems for infinite-dimensional
manifolds.

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2. Euler-like vector fields and tubular neighborhoods

2.1. Notation and conventions. For a manifold $M$, we denote by $\Diff(M)$ the group of diffeomorphisms, and by $\mathfrak{X}(M)$ the Lie algebra of vector fields. For a complete vector field $X \in \mathfrak{X}(M)$, we define its flow to be the 1-parameter group of diffeomorphisms $\Phi_s \in \Diff(M)$ such that $X(f) = \frac{d}{ds}|_{s=0}(\Phi_s^{-1})^*f$ for $f \in C^\infty(M)$. The description also holds for the local flows of incomplete vector fields, taking care of the domains. The flow of a time-dependent vector field $X_s$ is defined in terms of the action on functions by $\frac{d}{ds}\Phi_s^* = -\Phi_s^* \circ X_s$ with $\Phi_0 = \text{id}$.

Given a vector bundle $\pr: E \to M$, we denote by $\Aut(E) \subseteq \Diff(E)$ its group of automorphisms, and by $\aut(E) \subseteq \mathfrak{X}(E)$ the Lie algebra of infinitesimal automorphisms. Any $\tilde{\Phi} \in \Aut(E)$ restricts to a diffeomorphism $\Phi \in \Diff(M)$ with $\pr \circ \tilde{\Phi} = \Phi \circ \pr$; the kernel of this restriction map is denoted $\Gau(E)$. Likewise, any $\tilde{X} \in \aut(E)$ restricts to a vector field $X \in \mathfrak{X}(M)$ with

$$
\tilde{X} \sim_{\pr} X;
$$

the kernel of this restriction map is denoted by $\gau(E)$. According to [22], the elements $\tilde{\Phi} \in \Aut(E)$ are characterized as those diffeomorphisms of the total space of $E$ that commute with the action of the group $\mathbb{R}_{>0}$ by scalar multiplication; it is automatic that such a diffeomorphism preserves fibers and is fiberwise linear. Similarly $\aut(E)$ consists of vector fields on $E$ that are invariant under the action of $\mathbb{R}_{>0}$. Any $\tilde{\Phi} \in \Aut(E)$ determines an invertible linear operator $A: \Gamma(E) \to \Gamma(E)$, taking $\sigma: M \to E$ to $\tilde{\Phi} \circ \sigma \circ \tilde{\Phi}^{-1}$. This has the property

$$
A(f \sigma) = ((\tilde{\Phi}^{-1})^*f) A(\sigma);
$$

conversely, any invertible linear operator with this property corresponds to a unique $\tilde{\Phi} \in \Aut(E)$ lifting $\Phi$. Infinitesimally, for $\tilde{X} \in \aut(E)$ one obtains a ‘Lie derivative’ of sections $D: \Gamma(E) \to \Gamma(E)$, with the derivation property

$$
D(f \sigma) = f D \sigma + X(f) \sigma
$$

for all $f \in C^\infty(M)$. Conversely, any linear operator $D$ with this property corresponds to a unique $\tilde{X} \in \aut(E)$ lifting $X$. If $\tilde{X} \in \gau(E)$ then the corresponding $D$ is a section of the bundle $E^* \otimes E$ of endomorphisms of $E$.

Example 2.1. Let $\kappa_t: E \to E$ denote scalar multiplication by $t \in \mathbb{R}$, and let $\mathcal{E} \in \gau(E)$ be the Euler vector field. In local bundle coordinates on $E$, with $x^i$ the coordinates in the fiber direction and $y^j$ those in the base direction,

$$
\mathcal{E} = \sum_i x^i \frac{\partial}{\partial x^i}.
$$

The flow of $\mathcal{E}$ is $s \mapsto \kappa_{\exp(-s)}$; hence the endomorphism of $E$ corresponding to $\mathcal{E}$ is $D = \kappa_{-1}$.

2.2. Normal bundles and linear approximation. Given a manifold $M$ and a submanifold $N \subseteq M$, let $\nu(M,N) = TM|_N/TN$ be the normal bundle. We write $\nu_N = \nu(M,N)$ if the ambient manifold is clear. Throughout this paper, $p$ and $i$ will denote the following projection
Henrique Bursztyn, Hudson Lima, and Eckhard Meinrenken

and inclusion:

\[ \nu(M, N) \]
\[ \begin{array}{c}
\nu \downarrow \\
\iota \downarrow \\
N \rightarrow M.
\end{array} \]

Given a smooth map of pairs \( \varphi: (M', N') \rightarrow (M, N) \) (that is, \( \varphi: M' \rightarrow M \) is a smooth map with \( \varphi(N') \subseteq N \)), one obtains a vector bundle morphism

\[ \nu(\varphi): \nu(M', N') \rightarrow \nu(M, N) \]

over \( \varphi|_{N'}: N' \rightarrow N \), with the obvious functorial property under composition of such maps. If \( \varphi \) is transverse to \( N \), and \( N' = \varphi^{-1}(N) \), then \( \nu(\varphi) \) is a fiberwise isomorphism.

The normal bundle functor is compatible with the tangent functor: There is a canonical isomorphism

\[ \nu(TM, TN) \cong T\nu(M, N) \]

identifying the structures as vector bundles over \( \nu(M, N) \) and also as vector bundles over \( TN \).

In other words, (6) is an isomorphism between the following two double vector bundles:

\[ \begin{array}{c}
\nu(TM, TN) \rightarrow TN \\
\nu(M, N) \rightarrow N.
\end{array} \]

For any map of pairs \( \varphi: (M', N') \rightarrow (M, N) \), the following diagram commutes:

\[ \begin{array}{c}
\nu(TM', TN') \rightarrow T\nu(M', N') \\
\nu(TM, TN) \rightarrow T\nu(M, N).
\end{array} \]

See Appendix A for a detailed discussion.

Suppose that \( E \rightarrow M \) is a vector bundle, and \( \sigma \in \Gamma(E) \) is a smooth section with \( \sigma|_{N} = 0 \). Then \( \sigma: (M, N) \rightarrow (E, M) \) induces a vector bundle map \( \nu(\sigma): \nu(M, N) \rightarrow \nu(E, M) \). Making use of the natural identification \( \nu(E, M) \cong E \), we obtain a vector bundle map

\[ d^N \sigma: \nu(M, N) \rightarrow E|_{N} \]

referred to as the normal derivative (or intrinsic derivative \[25\]) of \( \sigma \), since it codifies the derivative of \( \sigma \) in directions normal to \( N \). Using a partition of unity, it is clear that every bundle map \( \nu(M, N) \rightarrow E|_{N} \) arises in this way, as the normal derivative \( d^N \sigma \) of some section.

For a diffeomorphism \( \Phi \) of \( M \) preserving \( N \), the map \( \Phi: (M, N) \rightarrow (M, N) \) defines the linear approximation \( \nu(\Phi) \in \text{Aut}(\nu(N)) \). Infinitesimally, for a vector field \( X \in \mathfrak{X}(M) \) tangent to \( N \), the map \( X: (M, N) \rightarrow (TM, TN) \) induces \( \nu(X): \nu(M, N) \rightarrow \nu(TM, TN) \). Using the identification (6) this is a vector field on \( \nu(N) \), called the linear approximation of \( X \):

\[ \nu(X) \in \text{aut}(\nu(N)) \]

Remark 2.2. The linear approximation \( \nu(X) \) can be viewed in alternative ways:
(i) The local flow of $\nu(X)$ is the linear approximation of the local flow of $X$.
(ii) The operator $D: \Gamma(\nu_N) \to \Gamma(\nu_N)$, corresponding to $\nu(X) \in \mathfrak{aut}(\nu_N)$ as in Sec. 2.11 has the following description: If $\tau \in \Gamma(\nu_N)$ is represented by a vector field $Y \in \mathfrak{X}(M)$ (modulo a vector field tangent to $N$), then $D(\tau)$ is represented by the Lie bracket $[X,Y]$.
(iii) If $X|_N = 0$, then $\nu(X) \in \mathfrak{gau}(\nu_N)$ is given by $-d^N X: \nu_N \to TM|_N$, followed by the projection $TM|_N \to \nu_N$.

Recall that the tangent lift $X_T \in \mathfrak{X}(TM)$ of a vector field $X \in \mathfrak{X}(M)$ is obtained by applying the tangent functor to $X$: $M \to TM$ (more precisely, $X_T = J \circ TX$, where $J$ is the canonical involution on $TTM$, see Appendix A). Equivalently, its local flow is the differential $T\Phi_s$ of the local flow $\Phi_s$ of $X$. If $X$ is tangent to $N$, then the infinitesimal version of the identification $\nu(T\Phi_s) = T(\nu(\Phi_s))$ shows that

$$\nu(X_T) = \nu(X)_T$$

as vector fields on $\nu(TM, TN) = T\nu(M, N)$.

2.3. Tubular neighborhood embeddings. Let $N \subseteq M$ be a submanifold, with normal bundle $\nu_N = \nu(M, N)$. We will work with the following strong notion of tubular neighborhood embeddings.

**Definition 2.3.** A tubular neighborhood embedding for $N \subseteq M$ is an embedding $\psi: \nu_N \to M$, taking the zero section of $\nu_N$ to $N$, and such that the map $\nu(\psi)$ induced by $\psi: (\nu_N, N) \to (M, N)$ is the identity map on $\nu_N$.

Here we are making use of the canonical identification $\nu(\nu_N, N) = \nu_N$ given by the vector bundle structure. Note that some authors only require that $\psi|_N$ is the identity, rather than also the linear approximation $\nu(\psi)$. A vector field $X$ tangent to $N$ is called linearizable if there exists a tubular neighborhood embedding $\psi$ such that $\nu(X)$ agrees with $\psi^* X$ on a neighborhood of $N$. We will need linearizability for the following special case.

**Lemma 2.4.** Suppose that $X|_N = 0$, with linear approximation $\nu(X) = E$ the Euler vector field on $\nu_N$. Then $X$ is linearizable.

**Proof.** By choosing an initial tubular neighborhood embedding $\nu_N \hookrightarrow M$, we may assume that $M = \nu_N$, and that the difference

$$Z = E - X \in \mathfrak{X}(M)$$

has linear approximation equal to zero. The family of vector fields

$$Z_t = \frac{1}{t} \nu_t^* Z, \quad t > 0,$$

extends smoothly to $t = 0$. \footnote{In local bundle coordinates on $\nu_N$, with $x^i$ the coordinates in the fiber direction and $y^j$ those in the base direction, we have $Z = \sum_i g^i(x,y) \frac{\partial}{\partial x^i} + \sum_j h^j(x,y) \frac{\partial}{\partial y^j}$, where $x \mapsto g^i(x,y)$ vanishes to second order at $x = 0$ (as a consequence of $\nu(Z) = 0$), and $x \mapsto h^j(x,y)$ vanishes to first order (since $Z|_N = 0$). Hence $Z_t = t^{-1} \sum_i g^i(tx,y) \frac{\partial}{\partial x^i} + \frac{1}{t} \sum_j h^j(tx,y) \frac{\partial}{\partial y^j}$ extends to $t = 0$.} Let $\varphi_t$ be the flow of the time-dependent vector field $Z_t$, with $\varphi_0 = \text{id}$. Since $Z_t|_N = 0$, the set of points $m$ such that the integral curve $\varphi_t(m)$ exists for
time $0 \leq t \leq 1$, is an open neighborhood of $N$ in $\nu_N$. By the scaling property $\kappa_{at}^*Z_t = aZ_{at}$ for $0 < a < 1$, this neighborhood is invariant under $\kappa_t$ for $0 \leq t \leq 1$. Using that $\kappa_t^*\mathcal{E} = \mathcal{E}$, and $t^d \frac{d}{dt} \kappa_t^*Y = \kappa_t^* [\mathcal{E}, Y]$ for all vector fields $Y$, we obtain

$$
\frac{d}{dt} \varphi_t^*(\mathcal{E} - tZ_t) = \frac{d}{dt} \varphi_t^*(\mathcal{E} - \kappa_t^*Z)
= \varphi_t^* \left( - [Z_t, \mathcal{E} - \kappa_t^*Z] - \frac{1}{t} \kappa_t^*[\mathcal{E}, Z] \right)
= \varphi_t^*(-[Z_t, \mathcal{E}] - [\mathcal{E}, Z_t]) = 0.
$$

Hence $\varphi_t^*(\mathcal{E} - tZ_t)$ does not depend on $t$. Equality of the values at $t = 1$ and $t = 0$ gives $\varphi_1^*(X) = \mathcal{E}$. Hence, any tubular neighborhood embedding that agrees with $\varphi_1$ near $N$ will give the desired linearization. \qed

Remark 2.5. The question of linearizability of vector fields is subtle, and has been extensively studied. (See e.g. [8] for a quick overview and recent results.) The classical result of Sternberg [36, 37] gives $C^\infty$-linearizability of vector fields at critical points $m$, provided the endomorphism of $T_mM$ describing this linear approximation has non-resonant eigenvalues. If the linear approximation is the Euler vector field, then this endomorphism is $-id$, and the non-resonance condition is satisfied. Thus, for $N = \{m\}$, Lemma 2.4 reduces to a very special case of Sternberg’s theorem.

Definition 2.6. Let $N \subseteq M$ be a submanifold. A vector field $X \in \mathfrak{x}(M)$ is called Euler-like (along $N$) if it is complete, with $X|_N = 0$, and its linear approximation is the Euler vector field: $\nu(X) = \mathcal{E}$.

Given a tubular neighborhood embedding, the push-forward of $\mathcal{E}$ under $\psi$ is an Euler-like vector field $X$ on the image $U = \psi(\nu_N)$. The tubular neighborhood embedding itself can be recovered from $X$, by using its flow. In fact, we have the following precise result:

Proposition 2.7. Suppose that $X \in \mathfrak{x}(M)$ is Euler-like along $N \subseteq M$. Then there exists a unique tubular neighborhood embedding $\psi: \nu_N \to M$ such that

$$
\mathcal{E} \sim_\psi X.
$$

Given an action of a Lie group $G$ on $M$, preserving the submanifold $N$ and the vector field $X$, then the tubular neighborhood embedding $\psi$ is $G$-equivariant.

Proof. The existence of such a tubular neighborhood embedding is clear from Lemma 2.4. To prove uniqueness, suppose that a tubular neighborhood embedding $\psi$ satisfying $\mathcal{E} \sim_\psi X$ is given.

Let $\Psi_s$ be the flow of $\mathcal{E}$ and $\Phi_s$ the flow of $X$. Recall that $\Psi_s = \kappa_{\exp(-s)}$, where $\kappa_t: \nu_N \to \nu_N$ denotes the scalar multiplication by $t \in \mathbb{R}$. Thus $\kappa_t = \Psi_{-\log(t)}$ for $t > 0$; accordingly we define $\lambda_t = \Phi_{-\log(t)}$. Since $\nu_N$ is invariant under $\kappa_t$ for all $t > 0$, its image $U = \psi(\nu_N)$ is invariant under $\lambda_t$ for all $t > 0$. Furthermore, since $\lim_{t \to 0} \kappa_t$ is the retraction $p$ from $\nu_N$ onto $N \subseteq \nu_N$, we have

$$
U = \{m \in M | \lim_{t \to 0} \lambda_t(m) exists and lies in N \subseteq M\}.
$$
Let \( v \in \nu_N \), with image point \( m = \psi(v) \), and put \( x = \kappa_0(v) = \lambda_0(m) \). Then \( \kappa_t(v) \) is a smooth curve in \( \nu_N \), defined for \( t \geq 0 \), and \( \lambda_t(m) \) its image under \( \psi \). Thus

\[
T_x \psi \left( \frac{d}{dt}_{|t=0} \kappa_t(v) \right) = \left( \frac{d}{dt}_{|t=0} \lambda_t(m) \right).
\]

Since \( \psi \) is a tubular neighborhood embedding, the map \( \nu(\psi) \) on normal bundles is the identity map. Hence

\[
\left( \frac{d}{dt}_{|t=0} \lambda_t(m) \right) \mod T_x N = \left( \frac{d}{dt}_{|t=0} \kappa_t(v) \right) \mod T_x N.
\]

But the element on the right hand side is just \( v \in \nu_N|_x \). Since \( \psi(v) = m \), this shows that

\[
(11) \quad \psi^{-1}(m) = \left( \frac{d}{dt}_{|t=0} \lambda_t(m) \right) \mod T_x N.
\]

Equations (10) and (11) express \( U = \psi(\nu_N) \) and the inverse map \( \psi^{-1}: U \to \nu_N \), hence also \( \psi \) itself, in terms of the flow of the vector field \( X \). This shows that \( \psi \) is unique.

In the \( G \)-equivariant setting, it is immediate that (11) is \( G \)-equivariant, hence so is \( \psi \). \( \square \)

**Remark 2.8.** A result similar to Proposition 2.7 may be found in [22, Theorem 2.2]. (One issue to be pointed out, however, is that the argument in [22], based on Shositaishvili’s theorem on topological normal forms for vector fields, does not apply to the \( C^\infty \)-case.)

**Remark 2.9.** Suppose that \( X|_N = 0 \) with linearization \( \nu(X) = \mathcal{E} \). Then we may multiply \( X \) by a bump function supported on a neighborhood of \( N \), and equal to 1 on a smaller neighborhood, to arrange that \( X \) is complete (and hence Euler-like). Indeed, by Lemma 2.4 there is an open neighborhood of \( N \) consisting of points \( m \) with \( \lim_{s \to \infty} \Phi_s(m) \in N \), and one only needs to take the bump function to be supported in such a neighborhood.

### 2.4. Functoriality

The following functorial property is immediate from the construction. Suppose \( \varphi: (M', N') \to (M, N) \) is a smooth map of pairs, defining a vector bundle morphism \( \nu(\varphi) \) as in (5). Let \( X, X' \) be Euler-like vector fields along \( N, N' \), respectively, with

\[
X' \sim_{\varphi} X.
\]

Then the resulting tubular neighborhood embeddings give a commutative diagram:

\[
\begin{array}{ccc}
\nu(M', N') & \xrightarrow{\psi'} & M' \\
\nu(\varphi) \downarrow & & \downarrow \varphi \\
\nu(M, N) & \xrightarrow{\psi} & M.
\end{array}
\]

**Example 2.10.** If \( X \) is Euler-like along \( N \), then its tangent lift \( X_T \) is Euler-like along \( TN \). Indeed,

\[
\nu(X_T) = \nu(X)_T = \mathcal{E}_T,
\]

the tangent lift of the Euler vector field \( \mathcal{E} \in \mathfrak{X(\nu_N)} \). Letting \( \psi: \nu(M, N) \to M \) and \( \psi_T: T\nu(M, N) \cong \nu(TM, TN) \to TM \) be the tubular neighborhood embeddings, we obtain
a commutative diagram

\[ \begin{array}{ccc}
\nu(TM, TN) & \xrightarrow{\psi_T} & TM \\
\downarrow & & \downarrow \\
\nu(M, N) & \xrightarrow{\psi} & M.
\end{array} \]

But, upon the identification \( \nu(TM, TN) \cong T\nu(M, N) \), we see that \( \mathcal{E}_T \) is just the Euler vector field on \( \nu(TM, TN) \), because the tangent lift of scalar multiplication on \( \nu(M, N) \) is scalar multiplication on \( \nu(TM, TN) \). It follows that \( \psi_T \) is simply the differential \( T\psi \).

3. Anchored vector bundles

As our first application of Euler-like vector fields, we will obtain a normal form theorem for integrable anchored vector bundles. This result may be regarded as a version of the Stefan-Sussmann theorem for generalized distributions on manifolds.

3.1. Basic definitions. A smooth generalized distribution on \( M \), in the sense of Stefan [35] and Sussmann [39], is a collection \( D = \bigcup_{m \in M} D_m \) of subspaces \( D_m \subseteq T_m M \), with the following property: There exists a submodule \( \mathcal{C} \subseteq \mathfrak{X}(M) \) of the \( C^\infty(M) \)-module of vector fields, such that \( D_m \) is the image of \( \mathcal{C} \) under evaluation \( \mathfrak{X}(M) \to T_m M \). If \( m \mapsto \dim(D_m) \) is constant, then \( D \) is a vector subbundle of \( TM \), referred to as a regular distribution.

Given a vector bundle \( E \to M \) equipped with an anchor, i.e., a bundle map \( a: E \to TM \) covering the identity map, the image \( \tilde{D} = a(E) \) is always a smooth generalized distribution with \( \mathcal{C} = a(\Gamma(E)) \). By a result of [15], any smooth generalized distribution arises in this way, though in general there is no canonical choice for the vector bundle and anchor. In many geometric situations, however, vector bundles and anchors are naturally present.

Definition 3.1. An anchored vector bundle is a vector bundle \( E \to M \) together with a vector bundle morphism (called the anchor) \( a: E \to TM \), with base map the identity map. A morphism from an anchored vector bundle \( F \to N \) to an anchored vector bundle \( E \to M \) is a bundle map \( \tilde{\varphi}: F \to E \), with base map \( \varphi: N \to M \), such that the following diagram commutes:

\[ \begin{array}{ccc}
F & \xrightarrow{\tilde{\varphi}} & E \\
\downarrow & & \downarrow \\
TN & \xrightarrow{T\varphi} & TM
\end{array} \]

Here the vertical maps are the anchors.

Anchored vector bundles often arise as parts of more elaborate structures, such as Lie algebroids (see Section 4) or Courant algebroids.

Example 3.2. An anchored vector bundle with injective anchor is the same as a regular distribution.

Example 3.3. A bisubmersion [5] is a manifold \( Q \) with two surjective submersions \( s, t: Q \to M \). Given a bisection \( j: M \to Q \) (that is, \( s \circ j = t \circ j = \text{id}_M \)), the normal bundle \( E = j^*(TQ) / TM \).
has the structure of an anchored vector bundle, with the anchor induced by the difference $T\sigma - T\tau: TQ \to TM$.

**Example 3.4.** Let $(E, a)$ be an anchored vector bundle over $M$, and $F \subseteq E$ an anchored subbundle along a submanifold $N \subseteq M$. Then $\nu(E, F)$ is an anchored vector bundle over $\nu(M, N)$, with anchor $\nu(a): \nu(E, F) \to \nu(TM, TN) \cong T\nu(M, N)$.

**Example 3.5.** Let $\mathfrak{d}$ and $V$ be vector spaces, and $\varrho: \mathfrak{d} \to \text{End}(V)$ a linear map. Then $E = V \times \mathfrak{d}$ is an anchored vector bundle with $a(v, \zeta) = (v, \varrho(\zeta)v) \in TV$. If $(E, a)$ is an anchored vector bundle with $a(E_m) = 0$ at some point $m$, then its linear approximation $\nu(E, E_m)$ at $m$ is of this type, with $V = T_mM$, $\mathfrak{d} = E_m$, and $\varrho$ the normal derivative of the anchor, viewed as a section of the bundle $\text{Hom}(E, TM)$.

The group of automorphisms of an anchored vector bundle $(E, a)$ will be denoted by $\text{Aut}_AV(E)$, and the Lie algebra of infinitesimal automorphisms by $\text{aut}_AV(E)$. Thus $\tilde{X} \in \text{aut}_AV(E)$ are the infinitesimal vector bundle automorphisms satisfying

$$\tilde{X} \sim_a X_T,$$

where $X_T \in \mathfrak{x}(TM)$ is the tangent lift of $X$. Equivalently, the corresponding operator $D$ on sections (cf. (3)) satisfies

$$a(D\tau) = [X, a(\tau)]$$

for all $\tau \in \Gamma(E)$. The local flow defined by $\tilde{X} \in \text{aut}_AV(E)$ is by local automorphisms of the anchored vector bundle $(E, a)$.

**Example 3.6.** Suppose that the anchor map $a$ is injective, defining an inclusion $E \hookrightarrow TM$. Then $a$ determines an isomorphism from $\text{aut}_AV(E) \subseteq \mathfrak{x}(E)$ to $\Gamma(E) \subseteq \mathfrak{x}(M)$; the lift $\tilde{X}$ of $X \in \Gamma(E)$ is the tangent lift $X_T \in \mathfrak{x}(TM)$, restricted to $E \subseteq TM$.

### 3.2. Pull-backs of anchored vector bundles

Suppose that $(E, a)$ is an anchored vector bundle over $M$, and $\varphi: N \to M$ is a smooth map transverse to $a$. Then the fiber product

$$\varphi^* E \longleftarrow E \quad \longrightarrow \quad \varphi^* TN \longrightarrow TM$$

defines an anchored vector bundle $\varphi^* E$ over $N$, such that the diagonal map $\varphi^* E \to E \times TN$ is a morphism of anchored vector bundles. The upper horizontal map is a morphism of anchored vector bundles, with base map $\varphi$. Notable special cases include:

(a) $\varphi^* TM = TN$;
(b) if $N = M \times Q$, with $\varphi$ the projection to $M$, then $\varphi^* E = E \times TQ$;
(c) if $i: N \hookrightarrow M$ is a submanifold transverse to $a$, then $i^* E = a^{-1}(TN)$;
(d) if $a$ is injective, so that $E \subseteq TM$, then $\varphi^* E = (T\varphi)^{-1}(E) \subseteq TN$;
(e) if $\varphi$ is a diffeomorphism, then $\varphi^* E = \varphi^* E$, the usual pull-back as a vector bundle.

Under composition of maps, one has that $\psi^* (\varphi^* E) = (\varphi \circ \psi)^* E$, provided that the appropriate transversality conditions are satisfied.
3.3. Transversals. Let \((E, a)\) be an anchored vector bundle over \(M\).

**Definition 3.7.** A transversal for \((E, a)\) is a submanifold \(i: N \hookrightarrow M\) transverse to the anchor.

Given a transversal, we can form the anchored vector bundle \(i^! E = a^{-1}(TN)\). Its pull-back to the normal bundle \(p: \nu_N \to N\) has the structure of a double vector bundle,

\[
\begin{array}{ccc}
p i^! E & \to & i^! E \\
\downarrow & & \downarrow \\
\nu_N & \to & N \\
\end{array}
\]

Here, the vector bundle structure for the upper horizontal arrow is obtained by restriction from the vector bundle structure on \(i^! E \times T\nu_N \to i^! E \times TN\). In particular, the corresponding Euler vector field is the restriction of \((0, E_T)\) to \(p i^! E \subseteq i^! E \times T\nu_N\).

The following Lemma shows that \(p i^! E\) may be regarded as a linear approximation of \(E\) along \(N\).

**Lemma 3.8.** Given a transversal \(i: N \hookrightarrow M\) for \((E, a)\), there is a canonical isomorphism of double vector bundles

\[
\nu(E, i^! E) \cong p i^! E
\]

**Proof.** The normal bundle functor, applied to \(a: (E, i^! E) \to (TM, TN)\), gives a commutative diagram

\[
\begin{array}{ccc}
\nu(E, i^! E) & \xrightarrow{p i^! E} & i^! E \\
\nu(a) & & \ \downarrow a \\
\nu(TM, TN) & \xrightarrow{PTN} & TN.
\end{array}
\]

It follows from transversality (see the comment after Equation (5)) that the left vertical map is a fiberwise vector bundle isomorphism, with base map the right vertical map. We conclude that (17) is a fiber product diagram. By (6), the lower left corner of the diagram can be replaced with \(T\nu(M, N)\). Then the lower horizontal map becomes \(Tp\), and the left vertical map an anchor map for \(\nu(E, i^! E)\). But the fiber product of \(T\nu(M, N)\) and \(i^! E\) over \(TN\) is exactly \(p i^! E\), by definition. We conclude that

\[
\nu(E, i^! E) \to i^! E \times T\nu_N, \quad \xi \mapsto (p i^! E(\xi), \nu(a)(\xi))
\]

defines an injective morphism of double vector bundles

\[
\begin{array}{ccc}
\nu(E, i^! E) & \to & i^! E \\
\downarrow & & \downarrow \\
\nu_N & \to & N
\end{array}
\quad
\begin{array}{ccc}
i^! E \times T\nu_N & \to & i^! E \times TN \\
\downarrow & & \downarrow \\
N \times \nu_N & \to & N \times N
\end{array}
\]

with image the double vector bundle \(p i^! E \subseteq i^! E \times T\nu_N\) \(\square\)
In the next sections, we formulate a condition under which \((E, a)\) is isomorphic near \(N\) to its linear approximation. The proofs will involve the following fact.

**Lemma 3.9.** Let \((E, a)\) be an anchored vector bundle over \(M\), and \(N \subseteq M\) a transversal. Then there exists a section \(\epsilon \in \Gamma(E)\) with \(\epsilon|_N = 0\), such that \(X = a(\epsilon)\) is Euler-like. Given an action of a Lie group \(G\) by automorphisms of \((E, a)\), such that the action on the base is proper, one can take the section \(\epsilon\) to be \(G\)-invariant.

**Proof.** Consider the exact sequence

\[
0 \to i^* E \to E|_N \to \nu_N \to 0,
\]

where the last map is the anchor map \(E|_N \to TM|_N\) followed by the quotient map. A bundle map \(\nu_N \to E|_N\) defines a splitting of \((19)\) if and only if its composition with the anchor defines a splitting of

\[
0 \to TN \to TM|_N \to \nu_N \to 0.
\]

Choose \(\epsilon \in \Gamma(E)\) with \(\epsilon|_N = 0\), such that the normal derivative of \(\epsilon\) defines a splitting of \((19)\). Then \(X = a(\epsilon)\) satisfies \(X|_N = 0\), and since \(d^N X = d^N a(\epsilon) = a(d^N \epsilon)\), the normal derivative of \(X\) defines a splitting of \((20)\). That is, \(d^N X : \nu_N \to TM|_N\) followed by projection \(TM|_N \to \nu_N\) is the identity. By Remark 2.2 (iii), the linear approximation \(\nu(X)\) equals minus the normal derivative, \(-d^N X\), followed by the projection to \(\nu_N\). Thus \(\nu(X) = -id = \kappa_{-1}\), which agrees with \(E\) by Example 2.1. Multiplying \(\epsilon\) by a bump function, we may arrange that \(X = a(\epsilon)\) is complete (see Remark 2.9).

In the \(G\)-equivariant setting, if the action on the base is proper, choose a \(G\)-equivariant open cover consisting of flow-outs of slices for the action. Over each slice, one can make \(\epsilon\) invariant by averaging (using that the stabilizer groups are compact). This then extends to an invariant section on the flow-out of the slice. Finally, one patches the local definitions by using a \(G\)-invariant partition of unity. \(\square\)

**3.4. Normal form theorem.** One of several versions of the Stefan-Sussmann theorem asserts that if a smooth generalized distribution \(D \subseteq TM\) is spanned by a locally finitely generated submodule \(\mathcal{C} \subseteq \mathfrak{X}(M)\), such that \(\mathcal{C}\) is closed under Lie brackets, then \(D\) defines a generalized foliation. Stefan-Sussmann \cite{35, 39} also gave integrability criteria in terms of the submodule \(\mathcal{D} \subseteq \mathfrak{X}(M)\) of all vector fields tangent to \(D\), but these contain errors; see Balan \cite{7} for counterexamples and corrections. In the case of anchored vector bundles, we take \(\mathcal{C} = a(\Gamma(E))\).

**Definition 3.10.** An anchored vector bundle \((E, a)\) will be called **involutive** if \(a(\Gamma(E))\) is a Lie subalgebra of \(\mathfrak{X}(M)\).

**Example 3.11.** Let \((E, a)\) be an anchored vector bundle equipped with an additional map \([\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)\) such that \(a([\sigma, \tau]) = [a(\sigma), a(\tau)]\), for all \(\sigma, \tau \in \Gamma(E)\). Then \(a(\Gamma(E))\) is a Lie subalgebra, and hence \((E, a)\) is involutive. This applies to Lie algebroids, Courant algebroids, and various kinds of Leibniz algebroids.

**Proposition 3.12.** Suppose that \((E, a)\) is an anchored vector bundle over \(M\), and \(\varphi : N \to M\) is a smooth map transverse to \(a\). If \(E\) is involutive, then the anchored vector bundle \(\varphi^* E\) over \(N\) is involutive.
Proof. By construction, $\varphi^! iE$ is identified with the anchored subbundle $F \subseteq E \times TN$ along the graph of $\varphi$, consisting of elements of $(E \times TN)|_{\text{Gr}(\varphi)}$ whose image under the anchor is tangent to $\text{Gr}(\varphi)$. Given two sections $\sigma_1, \sigma_2$ of $F$, choose extensions to sections $\tau_1, \tau_2 \in \Gamma(E \times TN)$. Their images $Y_1, Y_2$ under the anchor are tangent to $\text{Gr}(\varphi)$, and so is the Lie bracket $Y = [Y_1, Y_2]$. Since $E \times TN$ is involutive, $Y$ lifts to a section $\tau \in \Gamma(E \times TN)$. Its restriction $\sigma = \tau|_{\text{Gr}(\varphi)}$ is a section of $F$, satisfying $a(\sigma) = [a(\sigma_1), a(\sigma_2)]$. □

The main result of this section is the following normal form result, showing that in the involutive case $E$ is locally isomorphic near a given transversal to its linear approximation.

**Theorem 3.13** (Transversals for anchored vector bundles). Let $(E, a)$ be an anchored vector bundle over $M$, and $N \subseteq M$ a transversal.

(a) Suppose $(E, a)$ is involutive. Then there exists $\tilde{X} \in \text{aut}_{AV}(E)$ vanishing along $i^! E$, such that the base vector field $X$ is Euler-like along $N$.

(b) Any $\tilde{X} \in \text{aut}_{AV}(E)$ as in (a) determines an isomorphism of anchored vector bundles

$$\tilde{\psi} : p^! i^! E \to E|_U,$$

with base map a tubular neighborhood embedding $\psi : \nu_N \to U \subseteq M$.

If a Lie group $G$ acts on $(E, a)$ by automorphisms, such that the action on the base $M$ is proper and preserves $N$, then $\tilde{X}$ in (a) can be chosen $G$-invariant, and for any such $\tilde{X}$ the resulting map $\tilde{\psi}$ in (b) is $G$-equivariant.

The proof will be given in Section 3.6, but here is an outline. Using that $N$ is transverse to the anchor, we may choose a section $\epsilon \in \Gamma(E)$ such that $X = a(\epsilon)$ is Euler-like. In Section 3.5 we show that the involutivity of $(E, a)$ implies the existence of a lift $\tilde{a} : \Gamma(E) \to \text{aut}_{AV}(E)$ of the anchor map; we define $\tilde{X} = \tilde{a}(\epsilon)$. We then argue that the vector field $\tilde{X}$ on the total space $E$ is Euler-like along $i^! E$. The map $\tilde{\psi}$ is then obtained as a tubular neighborhood embedding, after identifying $\nu(E, i^! E) = p^! i^! E$.

If the normal bundle is trivial, the normal form in Theorem 3.13 simplifies: For any choice of trivialization $\nu_N = N \times P$ one gets

$$p^! i^! E = i^! E \times TP$$

as anchored vector bundles. As a special case we obtain:

**Corollary 3.14** (Local splitting of anchored vector bundles). Let $(E, a)$ be an involutive anchored vector bundle over $M$. Let $m \in M$, and $N \subseteq M$ a submanifold containing $m$, with $a(E_m) \oplus T_m N = T_m M$. Let $P = a(E_m)$. Then $E$ is isomorphic, near $m$, to the direct product $i^! E \times TP$.

If a compact Lie group $G$ acts by automorphisms of $(E, a)$, such that the action on $M$ fixes $m$ and preserves $N$, then one can take this isomorphism to be $G$-equivariant.

In particular, one obtains a version of the Stefan Sussmann-theorem: the generalized distribution $D = a(E) \subseteq TM$ defined by the involutive anchored vector bundle $(E, a)$ is integrable. Indeed, in the local model of Corollary 3.14 it is immediate that $\{m\} \times P$ is a leaf of the distribution. More generally, the leaves of the singular foliation are seen to have a local product form $L \times P$, where $L$ is a leaf of $a(i^! E) \subseteq TN$. For regular distributions (see Example 3.2) one recovers the (local) Frobenius theorem.
Remark 3.15. The standard proof of the Stefan-Sussmann theorem constructs the leaves by induction, using the flows of vector fields spanning $D \subseteq TM$. See [5, Proposition 1.12] or [18, Section 1.5]. While our approach is not shorter, it has better functorial properties and, being coordinate-free, seems better suited to infinite-dimensional generalizations. Indeed, for regular distributions our argument is similar to the proof of Frobenius’ theorem for Banach manifolds in [28, Chapter VI] and [2, Section 4.4].

3.5. Lift of the anchor map. The space of sections $\sigma \in \Gamma(E)$ such that the vector field $a(\sigma)$ lifts to $\text{aut}_{AV}(E)$, is a $C^\infty(M)$-submodule of $\Gamma(E)$:

**Lemma 3.16.** Suppose that $X = a(\sigma)$ lifts to $\tilde{X} \in \text{aut}_{AV}(E)$ with corresponding operator $D = D_\sigma$, and let $f \in C^\infty(M)$. Then

$$D_{f\sigma} \tau := fD_\sigma \tau - (a(\tau)f)\sigma$$

defines a lift of $fX = a(f\sigma)$ to $f\tilde{X} \in \text{aut}_{AV}(E)$.

**Proof.** By an easy computation, one verifies that $D_{f\sigma}$ satisfies the derivation property (3) and is compatible with the anchor (15). □

The involutivity of an anchored vector bundle $(E, a)$ is equivalent to the existence of lifts of $a(\sigma)$ for all sections $\sigma$.

**Proposition 3.17.** The anchored vector bundle $(E, a)$ is involutive if and only if the anchor map $a : \Gamma(E) \to \mathfrak{X}(M)$ admits a lift $\tilde{a} : \Gamma(E) \to \text{aut}_{AV}(E)$, such that the following diagram commutes:

$$\begin{array}{ccc}
\Gamma(E) & & \text{aut}_{AV}(E) \\
\tilde{a} \downarrow & & \downarrow \\
\mathfrak{X}(M) & & \\
\sigma \mapsto & & a \mapsto \sigma
\end{array}$$

In this case, one can arrange that the operators $D_\sigma : \Gamma(E) \to \Gamma(E)$ defined by the lifts $\tilde{a}(\sigma)$ satisfy

$$D_{f\sigma} \tau = fD_\sigma \tau - (a(\tau)f)\sigma$$

for all $f \in C^\infty(M)$, $\sigma, \tau \in \Gamma(E)$. Given an action of a Lie group $G$ by automorphisms of $(E, a)$, such that the action on the base $M$ is proper, one can take the lift $\tilde{a}$ to be $G$-equivariant.

**Proof.** It is convenient to consider the notion of an $a$-connection, given by a bilinear map

$$\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E), \ (\sigma, \tau) \mapsto \nabla_\sigma \tau$$

that is $C^\infty(M)$-linear in $\sigma$ and satisfies

$$\nabla_\sigma (f\tau) = f\nabla_\sigma (\tau) + (a(\tau)f)\tau$$

for $f \in C^\infty(M)$, $\sigma, \tau \in \Gamma(E)$. (An ordinary vector bundle connection $\nabla'$ on $E$ defines an $a$-connection, by setting $\nabla_\sigma = \nabla'_{a(\sigma)}$.) We define the torsion tensor $\mathcal{T}_\nabla \in \Gamma(\Lambda^2 E^* \otimes TM)$ by

$$\mathcal{T}_\nabla(\sigma, \tau) = a(\nabla_\sigma \tau) - a(\nabla_\tau \sigma) - [a(\tau), a(\sigma)].$$

Suppose that $a(\Gamma(E))$ is a Lie subalgebra. Then the last term in the formula for $\mathcal{T}_\nabla(\sigma, \tau)$ lifts to a section of $E$, and hence $\mathcal{T}_\nabla$ lifts to a tensor $S \in \Gamma(\Lambda^2 E^* \otimes E)$. (One first defines the
lift locally, using a basis of sections, and then uses a partition of unity.) The new a-connection
\( \nabla_{\sigma}\tau = \nabla_{\sigma}\tau - \frac{1}{2} S(\sigma,\tau) \) has vanishing torsion. But for any torsion-free a-connection \( \nabla \), the formula
\[
D_\sigma\tau = \nabla_{\sigma}\tau - \nabla_{\tau}\sigma
\]
has the property \( \text{(23)} \), and defines \( \tilde{a}(\sigma) \in \mathfrak{aut}_A^V(E) \) lifting the vector field \( a(\sigma) \). In the presence of a \( G \)-action on \( (E, a) \) for which the action on the base is proper, one may take \( \nabla \) to be \( G \)-equivariant, resulting in a \( G \)-equivariant lift \( \tilde{a} \).

Conversely, given the lift \( \tilde{a} \), with corresponding operators \( D_\sigma \) on sections, we have that \( a(D_\sigma\tau) = [a(\sigma), a(\tau)] \). Hence \( a(\Gamma(E)) \) is a Lie subalgebra. \( \square \)

**Remark 3.18.** The lift \( \tilde{a} \) in \( \text{(22)} \) is not a \( C^\infty(M) \)-module homomorphism, in general. For example, if \( E = TM \), with \( a \) the identity map, one can take \( \tilde{a} \) to be the tangent lift of vector fields. More generally, for Lie algebroids (treated in the next section) there is a natural lift \( \text{(30)} \) defined by the bracket. These illustrate lifts which are not \( C^\infty(M) \)-linear.

3.6. **Proof of the normal form theorem.**

**Proof of Theorem 3.16**

(a) By Lemma 3.9, we may choose \( \epsilon \in \Gamma(E) \) vanishing on \( N \) and such that \( X = a(\epsilon) \) is Euler-like. By Proposition 3.17, since \( (E, a) \) is involutive, there exists a lift \( \tilde{a} : \Gamma(E) \to \mathfrak{aut}_A^V(E) \) satisfying \( \text{(23)} \). Put \( \tilde{X} = \tilde{a}(\epsilon) \in \mathfrak{aut}_A^V(E) \). We claim that \( \tilde{X} \) vanishes along \( i^*E = a^{-1}(TN) \), as a consequence of property \( \text{(23)} \). Indeed, the condition \( \tilde{X}|_{a^{-1}(TN)} = 0 \) is equivalent to saying that the flow of \( \tilde{X} \) restricts to the identity map on the subbundle \( a^{-1}(TN) \to N \) of \( E \to M \). In terms of the operator \( D_\epsilon : \Gamma(E) \to \Gamma(E) \) corresponding to \( \tilde{X} \), this translates into the following condition: for all \( \tau \in \Gamma(E) \) such that \( \tau|_N \in \Gamma(a^{-1}(TN)) \), we have \( D_\epsilon(\tau)|_N = 0 \). Since \( \epsilon|_N = 0 \), we can assume that \( \epsilon \) is of the form \( f\sigma \), where \( \sigma \in \Gamma(E) \) and \( f \in C^\infty(M) \) vanishes on \( N \). Then
\[
D_\epsilon(\tau) = D_{f\sigma}(\tau) = fD_\sigma(\tau) - (a(\tau)f)\sigma
\]
implies that \( D_\epsilon(\tau)|_N \) vanishes when \( \tau|_N \in \Gamma(a^{-1}(TN)) \).

In the \( G \)-equivariant situation, assuming that the action on \( M \) is proper, one may take the section \( \epsilon \) to be \( G \)-equivariant (cf. Lemma 3.9), and similarly for the lift \( \tilde{a} \). It then follows that \( \tilde{X} \) is \( G \)-invariant.

(b) Suppose that \( \tilde{X} \in \mathfrak{aut}_A^V(E) \) vanishes along \( i^*E \subseteq E \), and is a lift of an Euler-like vector field \( X \in \mathfrak{X}(M) \). Since \( \tilde{X} \) is \( a \)-related to the tangent lift \( X_T \), the linear approximations are related under the bundle map \( \nu(a) \) in \( \text{(17)} \):
\[
\nu(\tilde{X}) \sim_{\nu(a)} \nu(X_T).
\]
By Example 2.10, the tangent lift \( X_T \) is Euler-like, thus \( \nu(X_T) \) is the Euler vector field of \( \nu(TM,TN) \). Since the bundle map \( \nu(a) \) is a fiberwise isomorphism, it follows that \( \nu(\tilde{X}) \) is the Euler vector field for \( \nu(E, i^*E) \). That is, \( \tilde{X} \) is Euler-like. Let \( \Phi_\epsilon \) be the flow of \( X \), and \( \tilde{\Phi}_\epsilon \) the flow of \( \tilde{X} \). Write \( \lambda_t = \Phi_{-\log(t)} \), so that \( \lambda_t \circ \psi = \psi \circ \kappa_t \), where \( \psi : \nu_N \to U \subseteq M \) is the tubular neighborhood embedding determined by \( X \). For all \( t > 0 \), the map \( \tilde{\lambda}_t = \tilde{\Phi}_{-\log(t)} \) restricts to an automorphism of the anchored vector bundle \( E|_U \), with base map the restriction \( \lambda_t|_U \). Since \( \tilde{X} \) is Euler-like, this family extends smoothly to \( t = 0 \). Since \( \tilde{\lambda}_t \) preserves anchors for all \( t > 0 \), the same is true.
Lemma 3.8. Suppose now that \( \tilde{\nu} \) restricts to a morphism \( \nu \) with base map \( (\ref{eq:restriction}) \). Functorial properties.

We take \( \widetilde{\psi} : p^1 i^1 E \to E_0 \) to be the inverse map. In the \( G \)-equivariant case, if the vector field \( \widetilde{\nu} \) is \( G \)-invariant, then all maps in this construction are \( G \)-equivariant, hence so is \( \widetilde{\psi} \).

Remark 3.19. Note that \( \ref{eq:construction} \) relates \( \widetilde{X} \) with the vector field \( (0, X_T) \) on \( i^1 E \times T \nu N \), which therefore restricts to \( p^1 i^1 E \). In turn, the isomorphism \( \nu(E, i^1 E) \cong p^1 i^1 E \) from Lemma \ref{lem:restriction} intertwines this vector field on \( p^1 i^1 E \) with the Euler vector field for \( \nu(E, i^1 E) \). (See Equation \( \ref{eq:regular} \).) We conclude that the isomorphism \( \widetilde{\psi} : p^1 i^1 E \cong \nu(E, i^1 E) \to E_0 \) takes the Euler vector field to \( \widetilde{X} \). It hence follows from the uniqueness part in Proposition \ref{prop:uniqueness} that \( \widetilde{\psi} \) is exactly the tubular neighborhood embedding defined by \( \widetilde{X} \).

Remark 3.20. The maps \( \ref{eq:construction} \) generalize to injective morphisms of anchored vector bundles, for all \( t \geq 0 \),

\[
E_0 \to E_0 \times T \nu N, \quad \xi \mapsto (\lambda_t(\xi), T\psi^{-1}(a(\xi))).
\]

Since \( a(\lambda_0(\xi)) = T\lambda_0(a(\xi)) = T(\lambda_t \circ \psi)(T\psi^{-1}(a(\xi))) \), we see that the image of \( \ref{eq:image} \) is \( \psi^1 \lambda_t^1(E_0) = \kappa^1_t \psi^1 E \). Let

\[
\widetilde{\psi}_t : \kappa^1_t \psi^1 E \to E_0
\]

be the inverse map. Note that \( \widetilde{\psi}_t \) has base map \( \psi \), for all \( t \geq 0 \). For \( t = 0 \) it is the isomorphism \( \tilde{\psi} \) constructed above, noting that \( \psi \circ \kappa_t = i \circ p \). For \( t > 0 \), it can be described as the ‘obvious’ isomorphism \( \kappa^1_t \psi^1 E \to E_0 \) (with base \( \psi \circ \kappa_t = \lambda_t \circ \psi : \nu_N \to U \)), given by the ordinary vector bundle pullback with respect to the diffeomorphism \( \psi \circ \kappa_t = \lambda_t \circ \psi : \nu_N \to U \), followed by the inverse of the map \( \lambda_t : E_0 \to E_0 \) (with base map the inverse of \( \lambda_t \)).

3.7. Functorial properties. Let \( \Phi : (M', N') \to (M, N) \) be a map of pairs, lifting to a morphism of anchored vector bundles \( \tilde{\Phi} : E' \to E \). Suppose that the anchor maps of \( E', E \) are transverse to \( i' : N' \to M', i : N \to M \), respectively. The map \( \tilde{\Phi} \) restricts to a morphism of anchored vector bundles \( i'^1 E' \to i^1 E \), giving rise to a morphism of anchored vector bundles

\[
\nu(\tilde{\Phi}) : \nu(E', i'^1 E') \to \nu(E, i^1 E)
\]

with base map \( \nu(\Phi) : \nu(M', N') \to \nu(M, N) \). On the other hand, the map \( i'^1 E' \times \nu N' \to i^1 E \times \nu N \) restricts to a morphism \( \nu i'^1 E' \to \nu i^1 E \), which coincides with \( \ref{eq:restricted} \) under the isomorphism from Lemma \ref{lem:restriction}. Suppose now that \( \widetilde{X}' \in \text{aut}_{AV}(E') \) is as in the theorem, with \( \widetilde{X}' \sim_{\tilde{\Phi}} \widetilde{X} \). Then the
corresponding tubular neighborhood embeddings give a commutative diagram

\[ p_i^* i^! E' \cong \nu(E', i^! E') \rightarrow E' \]
\[ \downarrow \nu(\tilde{\Phi}) \]
\[ p_i^* i^! E \cong \nu(E, i^! E) \rightarrow E \]

where all maps are morphisms of anchored vector bundles. A similar functorial property holds relative to comorphisms of anchored vector bundles. Recall that a comorphism of (anchored) vector bundles, with base map \( \Phi: M' \rightarrow M \) is given by a bundle map \( \Phi^* E \rightarrow E' \) whose graph in \( E \times E' \) is an (anchored) subbundle along the graph \( \text{Gr}(\Phi) \subseteq M \times M' \). We write \( \tilde{\Phi}: E' \rightarrow E' \) for such a ‘wrong-way’ morphism, and \( \text{Gr}(\tilde{\Phi}) \subseteq E \times E' \) for its graph. By a discussion similar to that for morphisms, one obtains comorphisms \( p_i^* i^! E' \rightarrow p_i^* i^! E \), and in the case of \( \tilde{X} \sim \tilde{\Phi} \tilde{X} \) there is a commutative diagram

\[ p_i^* i^! E' \cong \nu(E', i^! E') \rightarrow E' \]
\[ \downarrow \nu(\tilde{\Phi}) \]
\[ p_i^* i^! E \cong \nu(E, i^! E) \rightarrow E \]

In fact, one can consider the result for comorphisms as a special case of the result for morphisms, applied to the inclusion map \( \text{Gr}(\tilde{\Phi}) \rightarrow E \times E' \). Observe that if \( \tilde{\Phi}: E' \rightarrow E \) is a comorphism, and \( N \subseteq M \) is a transversal for \( E \) which is also transverse to the map \( \tilde{\Phi} \), then it is automatic that \( N' := \Phi^{-1}(N) \) is a transversal for \( E' \). To see this, let \( v' \in TM'|_{N'} \) be given, and let \( v = T\Phi(v') \in TM|_N \) its image. Write \( v = v_1 + v_2 \) where \( v_1 \in TN \) and \( v_2 = a(\xi), \xi \in E|_N \). Let \( \xi' \) be the image of \( \Phi^* \xi \) under the map \( \Phi^* E \rightarrow E' \), with the same base point as \( v' \). Then \( T\Phi(a(\xi')) = a(\xi) \). Putting \( v_2 = a(\xi') \), it follows that \( v' := v' - v_2 \) is tangent to \( N' \).

3.8. Uniqueness of transverse structures. Let \( \psi: N \rightarrow Q \) be a submersion, with fibers \( N_q = \psi^{-1}(q) \). A family of anchored vector bundles \( F_q \rightarrow N_q \) is an anchored vector bundle \( F \rightarrow N \) whose anchor is tangent to \( \ker(T\psi) \), with \( F_q = F|_{N_q} \) the restriction. We will call such a family infinitesimally trivial if every \( Z \in \mathfrak{X}(Q) \) admits a lift \( Y \in \mathfrak{X}(N) \) (i.e., \( Y \sim_{\psi} Z \)) which is the base vector field of an infinitesimal automorphism \( \tilde{Y} \in \mathfrak{aut}_N(F) \). Note that in this case, the (local) flow of \( Y \) preserves the fibers of \( \psi \), and the (local) flow of \( \tilde{Y} \) is by (local) isomorphisms of anchored vector bundles.

We interested in the following situation. Suppose that \( (E, a) \) is an anchored vector bundle over \( M \), and \( \phi: N \rightarrow M \) a smooth map such that all \( i_q = \phi|_{N_q}: N_q \rightarrow M \) are transversals, i.e., transverse to the anchor map of \( E \). Then \( F_q = i_q^* E \) is a family of anchored vector bundles. Here \( F \subseteq \phi^! E \) is the subbundle given as the pre-image of \( \ker(T\psi) \) under the anchor.

**Proposition 3.21.** Suppose that \( (E, a) \) is an involutive anchored vector bundle over \( M \), and that \( i_q: N_q \subseteq M \) is a smooth family of transversals, as above. Then the family of anchored vector bundles \( i_q^* E \) is infinitesimally trivial.

**Proof.** By Proposition 3.12 the bundle \( \phi^! E \) is involutive. We claim that the fibers \( N_q \subseteq N \) are transversals for \( \phi^! E \). Indeed, given \( y \in N \), with image \( x = \phi(y) \), and any \( w \in T_y N \), the
image \( v = T\phi(w) \) can be written as a sum \( v_1 + v_2 \), where \( v_1 \in a(E)_x \) and \( v_2 \in T_x (i_q(N_q)) \). Let \( w_2 \in T_y N_q \) be the pre-image. Then \( w_1 = w - w_2 \) satisfies \( T\phi(w_1) = v - v_2 = v_1 \), hence \( w_1 \in a(\phi^! E)_y \), proving the claim.

The transversality implies that the bundle map \( T\psi \circ a_{\phi^! E} : \phi^! E \to TQ \) is fiberwise onto. Hence, for any given \( Z \in \mathfrak{X}(Q) \) there is a section \( \sigma \in \Gamma(\phi^! E) \) such that its image under the anchor, denoted by \( Y \), satisfies \( Y \sim_\psi Z \). Furthermore, by Proposition 3.17 it admits a lift \( \tilde{Y} \in \text{aut}_{AV}(\phi^! E) \). Since \( \tilde{Y} \) is related under the anchor map to the tangent lift \( Y_T \), and the latter is tangent to \( \ker(T\psi) \) (due to \( Y \sim_\psi Z \)), it is automatic that \( \tilde{Y} \) is tangent to \( F \subseteq \phi^! E \). Hence, \( \tilde{Y} \) restricts to an element of \( \text{aut}_{AV}(F) \).

As a special case, we obtain:

**Corollary 3.22.** Let \((E, a)\) be an involutive anchored vector bundle over \( M \), \( S \) a leaf, and \( m_0, m_1 \in S \). Let \( N_0 \) and \( N_1 \) be transversals of dimension \( \dim M - \dim S \), with inclusions \( i_0 \) and \( i_1 \), and such that \( N_0 \cap S = \{m_0\} \) and \( N_1 \cap S = \{m_1\} \). Then, after replacing the \( N_i \) with smaller neighborhoods of \( m_i \in N_i \), if necessary, there is an isomorphism of the induced anchored vector bundles \( i_0^! E \to i_1^! E \), taking \( m_0 \) to \( m_1 \).

**Proof.** Given a smooth path \( \mathbb{R} \to S \), \( s \mapsto m_s \), taking on the given values at \( s = 0, 1 \), one can find a family of transversals \( i_s : N_s \to M \) with \( N_s \cap S = \{m_s\} \) for all \( s \). That is, the union of \( N_s \subseteq M \times \{s\} \) defines a submanifold \( N \subseteq M \times \mathbb{R} \). The maps \( \phi : N \to M \) and \( \psi : N \to \mathbb{R} \) are given by projections to the two factors. By Proposition 3.21, or rather its proof, there exists a \( \tilde{Y} \in \text{aut}_{AV}(\phi^! E) \) such that the base vector field \( Y \) is tangent to the leaves of \( \phi^! E \) and satisfies \( Y := a(\sigma) \sim_\psi \frac{\partial}{\partial s} \). As argued above, \( \tilde{Y} \) preserves \( F \subseteq \phi^! E \), where \( F_s = i_s^! E \).

The path \( s \mapsto m_s \in N_s \cap S \) defines a section \( \mathbb{R} \to N \) of the submersion \( \psi \). By construction, this is a single leaf of \( \phi^! E \). Since \( Y \) is tangent to the leaves, its restriction to \( \mathbb{R} \subseteq N \) is just \( \frac{\partial}{\partial s} \). In particular, there exists a neighborhood of \( m_0 \) in \( N_0 \) over which the flow of \( Y \) (and hence also of \( \tilde{Y} \)) is defined for time 1. The time 1-flow of \( \tilde{Y} \) gives the desired isomorphism of anchored vector bundles \( i_0^! E \to i_1^! E \) over possibly smaller neighborhoods of \( m_i \) in \( N_i \).

4. Lie Algebroids

Suppose that \((E, a, [,])\) is a Lie algebroid over \( M \). Thus \((E, a)\) is an anchored vector bundle, with a Lie bracket \([,]\) on its space \( \Gamma(E) \) of sections satisfying the compatibility property

\[
[\sigma, f \tau] = f[\sigma, \tau] + (a(\sigma)f) \tau,
\]

for \( \sigma, \tau \in \Gamma(E) \) and \( f \in C^\infty(M) \). As is well-known, this implies that \( a : \Gamma(E) \to \mathfrak{X}(M) \) preserves Lie brackets. We denote by \( \text{Aut}_{LA}(E) \) the automorphisms of \( E \) preserving the Lie algebroid structure, and by \( \text{aut}_{LA}(E) \) the infinitesimal automorphisms, consisting of all \( \tilde{X} \in \text{aut}_{AV}(E) \) such that the corresponding operator \( D \sigma \) on sections is a derivation of the Lie bracket. In particular, the operators \( D\sigma = [\sigma, \cdot] \) define infinitesimal automorphisms \( \tilde{a}(\sigma) \in \text{aut}_{LA}(E) \). The resulting lift

\[
(30) \quad \tilde{a} : \Gamma(E) \to \text{aut}_{LA}(E).
\]

has the property (28).
4.1. Normal form theorem. Given a smooth map \( \varphi: N \to M \) transverse to \( a \), the anchored vector bundle \( \varphi^*E \) over \( N \) inherits a unique Lie algebroid structure, in such a way that the diagonal map \( \varphi^*E \to E \times TN \) is an inclusion as a Lie subalgebroid. See [30, Section 4.3] or [20]. For any transversal \( i: N \to M \), with normal bundle \( p: \nu_N \to N \), we obtain a Lie algebroid \( p^\dagger_i E \to \nu_N \). Theorem 4.1 has the following refinement:

**Theorem 4.1.** Let \( (E, a, [\cdot, \cdot]) \) be a Lie algebroid over \( M \), and let \( N \subseteq M \) be a transversal. Choose \( \epsilon \in \Gamma(E) \) with \( \epsilon|_N = 0 \), such that \( X = a(\epsilon) \) is Euler-like. The choice of \( \epsilon \) determines a tubular neighborhood embedding \( \psi: \nu_N \to M \) with an isomorphism of Lie algebroids

\[
\tilde{\psi}: p^\dagger_i E \to E|_U.
\]

Given a \( G \)-action by Lie algebroid automorphisms of \( E \), and a \( G \)-equivariant choice of \( \epsilon \), the isomorphism \( \tilde{\psi} \) is \( G \)-equivariant.

**Proof.** We use the same construction as in the proof of Theorem 3.13, but with the distinguished lift (30). As discussed in Remark 3.20, the vector field \( \tilde{X} = \tilde{a}(\epsilon) \) determines a family of isomorphisms of anchored vector bundles \( \tilde{\psi}_t: \kappa_t^\dagger \varphi^*E \to E|_U \), for all \( t \geq 0 \). For all \( t > 0 \), these are given by the Lie algebroid automorphisms \( \lambda_t \), and in particular preserve Lie brackets on sections. Hence, by continuity the map \( \psi_0 \) preserves Lie brackets as well. \( \square \)

If the normal bundle is trivial, \( \nu_N = N \times P \), then we obtain the simpler model

\[
p^\dagger_i E = i^\dagger E \times TP
\]
as Lie algebroids. In particular, we obtain:

**Corollary 4.2** (Local splitting of Lie algebroids). Let \( (E, a, [\cdot, \cdot]) \) be a Lie algebroid over \( M \), and \( m \in M \). Let \( i: N \to M \) be a submanifold containing \( m \), such that \( T_m N \) is a complement to \( P = a_m(E_m) \) in \( T_mM \). Then Lie algebroid \( E \) is isomorphic, near \( m \), to the direct product of Lie algebroids \( i^\dagger E \times TP \). If a compact Lie group \( G \) acts on \( E \) by Lie algebroid automorphisms, such that the action on \( M \) fixes \( m \) and preserves \( N \), this isomorphism can be chosen \( G \)-equivariant.

For \( G = \{1\} \) this result is due to Weinstein [42], Fernandes [19], and Dufour [16].

4.2. Functorial properties. The functorial properties of the construction are analogous to those for anchored vector bundles. Of particular interest is the functoriality with respect to Lie algebroid comorphisms \( \Phi: E' \to E \).

**Example 4.3.** Recall that any Poisson structure \( \pi \) on \( M \) makes \( T^*M \) into a Lie algebroid, with anchor \( \pi^\dagger: T^*M \to T^*M' \). Any Poisson map \( M' \to M \) defines a comorphism of Lie algebroids \( T^*M' \to T^*M \).

As remarked in Section 3.7, if \( N \subseteq M \) is a transversal for \( E \) which is also transverse to the map \( \Phi \), then its pre-image \( N' = \Phi^{-1}(N) \) is transversal for \( E' \). Furthermore, if \( \epsilon \in \Gamma(E) \) has the properties in Theorem 4.1, then its image \( \epsilon' \) under the map \( \Gamma(E) \to \Gamma(E') \) determined by the comorphism has similar properties, with respect to \( E' \). Hence, in this situation (20) is a commutative diagram of Lie algebroid comorphisms.
4.3. **Uniqueness of transverse structures.** Just as in Section 3.8 we can consider families of Lie algebroids $F_q \to N_q$, where $N_q$ are the fibers of a submersion $\psi: N \to Q$. For the *infinitesimal triviality* of such a family, one requires that for every $X \in \mathfrak{X}(Q)$ there exists $\tilde{Y} \in \text{aut}_{LA}(E)$ such that $\tilde{Y} \sim_{\psi} X$. The same argument as in Section 3.8 shows that if $i_q: N_q \to M$ is a family of transversals for a Lie algebroid $(E, a)$ then the family of Lie algebroids $i_q^*E$ is infinitesimally trivial. As a consequence, one obtains the natural analogue of Corollary 3.22 for Lie algebroids, which recovers [19, Thm. 1.2].

4.4. **Lie groupoids.** Suppose that $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, with source and target maps $s, t: \mathcal{G} \to M$. The anchor $a$ of its Lie algebroid $E := \nu(\mathcal{G}, M)$ is induced by $T^s - T^t: T\mathcal{G} \to TM$. Using that $\ker(T^s t)$ (the tangent space to the $t$-fiber at $g \in \mathcal{G}$) is spanned by the left-invariant vector fields, one sees that

$$T^g s(\ker(T^g t)) = \text{ran}(a_{t(g)}).$$

Hence, a smooth map $\varphi: N \to M$ is transverse to $a$ if and only if it is transverse to the restriction of $s$ to every $t$-fiber. (Equivalently, it is transverse to the restriction of $t$ to every $s$-fiber.) In this case, there is a well-defined *pull-back Lie groupoid* $^\varphi \mathcal{G} \rightrightarrows N$

where $\varphi^! \mathcal{G} = N \times_M \mathcal{G} \times_M N$ is the fiber product with respect to source and target maps. Here the transversality assumption ensures that the map $N \times_M \mathcal{G} \to M$ induced by the source map is transverse to $\varphi$; hence the second fiber product is well-defined. It also ensures that source and target for $\varphi^! \mathcal{G}$ are surjective submersions. The Lie algebroid of $\varphi^! \mathcal{G}$ is $\varphi^! E$.

A *transversal* for $\mathcal{G}$ is a submanifold $i: N \hookrightarrow M$ such that $i$ is transverse to $a$. In this case, $i^! \mathcal{G} = \mathcal{G}|_N = s^{-1}(N) \cap t^{-1}(N)$. Choose $\epsilon \in \Gamma(E)$ such that $X = a(\epsilon)$ is Euler-like, defining a tubular neighborhood embedding $\psi: \nu_N \to U \subseteq M$. The section $\epsilon$ defines a left-invariant vector field $\epsilon^L \in \mathfrak{X}(\mathcal{G})$ tangent to the $t$-fibers, and a right-invariant vector field $\epsilon^R \in \mathfrak{X}(\mathcal{G})$ tangent to the $s$-fibers. The difference $\tilde{X} = \epsilon^L - \epsilon^R$ is an infinitesimal Lie groupoid automorphism, related to $X$ under both source and target maps, and is Euler-like along $i^{\!\!^!\!^!}_G \subseteq \mathcal{G}$. Similarly to Theorem 4.1, using the flow of $\epsilon^L - \epsilon^R$ one obtains an isomorphism of Lie groupoids

$$\tilde{\psi}: p^! i^! G \xrightarrow{\cong} G|_U$$

where $G|_U = s^{-1}(U) \cap t^{-1}(U)$. In fact, one obtains a family $\tilde{\psi}_t: \kappa^t_! p^! i^! \mathcal{G} \to G|_U$ of groupoid isomorphisms, reducing to $\tilde{\psi}$ at $t = 0$. These are obtained as inverses of the maps

$$G|_U \xrightarrow{\cong} \kappa^t_! p^! i^! \mathcal{G} \subseteq \mathcal{G} \times \nu_N \times \nu_N, \quad g \mapsto (\tilde{\lambda}_t(g), \psi_1(t(g)), \psi_1^{-1}(s(g))).$$

5. **Dirac manifolds**

We next obtain normal form theorems and splitting theorems for Dirac manifolds.

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2The condition is equivalent to transversality of the maps $(t, s): \mathcal{G} \to M \times M$ and $(\varphi, \varphi): N \times N \to M \times M$; hence $\varphi^! \mathcal{G}$ can be also regarded as a fibered product with respect to these two maps. See [10, Appendix A] for a general discussion, including a simplification of [20] Proposition 2.3.1.]
5.1. **Dirac structures.** We begin by recalling the definition of the standard *Courant algebroid* $TM$ over a manifold $M$, with possible twisting by a closed 3-form $\eta \in \Omega^3(M)$. References include Courant’s original paper [13] as well as [29, 32, 33, 34]. We let

$$TM = TM \oplus T^*M$$

be equipped with the symmetric bilinear form $\langle v_1 + \mu_1, v_2 + \mu_2 \rangle = \langle \mu_1, v_2 \rangle + \langle \mu_2, v_1 \rangle$, the anchor $a: TM \to TM$ given by $v + \mu \mapsto v$, and the *Courant bracket* $[,]$ on its space of sections,

$$[X_1 + \alpha_1, X_2 + \alpha_2] = [X_1, X_2] + \mathcal{L}_{X_1} \alpha_2 - \iota_{X_2} d\alpha_1 + \iota_{X_1} \iota_{X_2} \eta$$

for vector fields $X_i$ and 1-forms $\alpha_i$.

A *Dirac structure* on $M$ (relative to $\eta$) is a subbundle $E \subseteq TM$ such that $E = E^\perp$, and such that the space of sections of $E$ is closed under the bracket $[,]$. Dirac structures are Lie algebroids, with the anchor $a_E$ and bracket $[,]$ obtained by restriction from $TM$. If $\eta = 0$, then a Dirac structure with $E \cap TM = \{0\}$ is of the form $E = \text{Gr}(\pi)$, where $\pi \in \Gamma(\wedge^2 TM)$ is a Poisson structure, and $\text{Gr}(\pi)$ is the graph of the bundle map $T^*M \to TM$ defined by $\pi$. For any 2-form $\omega$ we define the *B-field transform* $R_\omega : TM \to TM$,

$$R_\omega(v + \mu) = v + \mu + \iota_v \omega,$$

and put $E^\omega = R_\omega(E)$. Then $E^\omega$ is a Dirac structure relative to the 3-form $\eta + d\omega$. Given a smooth map $\varphi: N \to M$, define

$$\varphi^!E = \{v' + \mu' \in TN| \exists v + \mu \in E : v = T\varphi(v'), \mu' = \varphi^*\mu\}.$$

If $\varphi$ is transverse to $a_E$, then $\varphi^!E \subseteq TN$ is a Dirac structure relative to $\varphi^*\eta$; as a Lie algebroid it coincides with the pull-back Lie algebroid discussed in Section 4. Given a 2-form $\omega$ on $M$, one finds that

$$(\varphi^!E)^{\varphi^*\omega} = \varphi^!(E^\omega).$$

If $\varphi$ is a diffeomorphism, and letting

$$\mathcal{T}\varphi: TN \to TM, \quad v + \mu \mapsto T\varphi(v) + (T\varphi^{-1})^*(\mu)$$

(the sum of tangent and cotangent lifts), we have that $\varphi^!E = \mathcal{T}\varphi^{-1}(E)$.

5.2. **The normal form theorem.** A submanifold $i: N \hookrightarrow M$ is called a *transversal* for the Dirac structure $E$ if it is transverse to the anchor of $E$. In this case, we obtain Dirac structures $i^!E \subseteq TN$ relative to $i^*\eta$ and $p^!i^!E \subseteq T\nu_N$ relative to $p^*i^*\eta$.

**Theorem 5.1** (Normal form for Dirac structures). Let $E \subseteq TM$ be a Dirac structure relative to $\eta$, and $i: N \hookrightarrow M$ a transversal. Choose $\epsilon = X + \alpha \in \Gamma(E)$ with $\epsilon|_N = 0$, such that $X$ is Euler-like along $N$, and let $\psi: \nu_N \to U \subseteq M$ be the resulting tubular neighborhood embedding. Then $\mathcal{T}\psi: T\nu_N \to TM$ restricts to an isomorphism of Dirac structures

$$(p^!i^!E)^\omega \to E|_U,$$

where $\omega \in \Omega^2(\nu_N)$ is the 2-form

$$\omega = \int_0^1 \frac{1}{t} \kappa^* \psi^*(d\alpha + \iota_X \eta) \, dt.$$

Given a proper $G$-action on $M$, preserving $\eta$ and such that its lift to $TM$ preserves $E$, one can choose $\epsilon$ to be $G$-invariant. The resulting $\omega$ is then $G$-invariant, and the isomorphism $\mathcal{T}\psi$ is $G$-equivariant.
We will see that the 2-form $\omega$ is well-defined: the family of 2-forms $\frac{1}{2}\kappa^i_\ast \psi^* (\omega + i_X \eta)$ extends smoothly to $t = 0$. Furthermore, $d\omega = \psi^* \eta - p^! i^*_\eta$. The proof of Theorem 5.1 will be given in Section 5.4; its functorial properties will be discussed in Section 5.5.

**Remark 5.2.** For later reference, we remark that Theorem 5.1 and its proof, extend to complex Dirac structures $E \subseteq \mathbb{T}_C M$ inside the complexified Courant algebroid, provided $\epsilon \in \Gamma(E)$ can be chosen in such a way that its vector field part is real, $X = \overline{X}$.

**Remark 5.3.** Since $E$ is a Dirac structure, the Courant bracket restricts to a Lie-algebroid bracket of $E$. Hence, the section $\epsilon$ defines an isomorphism of Lie algebroids $p^! i^*_E \rightarrow E|_U$, using the approach in Section 4. The Theorem above gives a stronger statement, since it treats $E$ not merely as a Lie algebroid, but as a Dirac structure embedded as a subbundle of $\mathbb{T}M$. Forgetting about this embedding, and identifying $(p^! i^*_E \omega)$ with $p^! i^*_E$ as Lie algebroids, one may verify that the isomorphism from Theorem 5.1 reduces to that of Theorem 1.1.

Theorem 5.1 specializes to local splitting theorems for Dirac manifolds near given points $m \in M$.

**Corollary 5.4.** Let $E \subseteq \mathbb{T}M$ be a Dirac structure relative to the closed 3-form $\eta \in \Omega^3(M)$, and $m \in M$. Let $N \subseteq M$ be a submanifold containing $m$, such that $T_m N$ is a complement to $P = a_m(E_m)$ in $T_m M$. Then the Dirac structure $E$ is isomorphic, near $m$, to a Dirac structure of the form

$$(i^* E \times TP)^\omega \subseteq \mathbb{T}(N \times P).$$

Here $\omega \in \Omega^2(N \times P)$ is a 2-form such that the (local) diffeomorphism of the base manifolds $M$ and $N \times P$ takes $\eta$ to $p^* i^* \eta + d\omega$. Given an action of a compact Lie group $G$ by Dirac automorphisms, fixing $m$ and preserving the submanifold $N$, one obtains a $G$-invariant $\omega$ and a $G$-equivariant isomorphism.

For the case $\eta = 0$, $G = \{1\}$, this result is due to Blohmann [9, Theorem 3.2].

**5.3. Courant automorphisms.** For the $\eta$-twisted Courant algebroid $\mathbb{T}M$, we denote by $\text{Aut}_{CA}(\mathbb{T}M)$ the group of Courant automorphisms, consisting of vector bundle automorphisms preserving the anchor, the symmetric pairing and the bracket. The Lie algebra of infinitesimal Courant automorphisms $\tilde{X}$ is denoted by $\text{aut}_{CA}(\mathbb{T}M)$; the corresponding operators $D$ on sections of $\mathbb{T}M$ preserve the anchor, as well as the bracket and pairing in the sense that

$$\mathcal{L}_X (\sigma_1, \sigma_2) = \langle D(\sigma_1), \sigma_2 \rangle + \langle \sigma_1, D(\sigma_2) \rangle,$$

$$D[\sigma_1, \sigma_2] = [D(\sigma_1), \sigma_2] + [\sigma_1, D(\sigma_2)],$$

for all $\sigma_i = X_i + \alpha_i \in \Gamma(\mathbb{T}M)$. Any section $\sigma = X + \alpha \in \Gamma(\mathbb{T}M)$ defines an infinitesimal Courant automorphism with $D = [\sigma, \cdot]$. The group $\text{Aut}_{CA}(\mathbb{T}M)$ and its Lie algebra $\text{aut}_{CA}(\mathbb{T}M)$ have the following explicit description. The B-field transforms $\mathcal{R}_\omega$ and the bundle maps $\mathbb{T}\Phi$ defined by diffeomorphisms $\Phi$ of $M$ combine into an injective group homomorphism

$$(32) \quad \Omega^2(M) \times \text{Diff}(M) \to \text{Aut}(\mathbb{T}M), \ (\omega, \Phi) \mapsto \mathcal{R}_\omega \circ \mathbb{T}\Phi.$$ 

The image consists of vector bundle automorphisms preserving the anchor and the pairing. Similarly, there is a Lie algebra morphism $\Omega^2(M) \times \mathfrak{X}(M) \to \text{aut}(\mathbb{T}M)$ with the action $(\theta, X)(Y + \beta) = [X, Y] + \mathcal{L}_X \beta - i_Y \theta$. The following is proved in [24, Proposition 2.5] and [27, Section 2], using slightly different sign conventions.
Proposition 5.5. The map \([32]\) restricts to an isomorphism from the group of all \((\omega, \Phi)\) such that
\[(\Phi^{-1})^*\eta - d\omega = \eta\]
on onto \(\text{Aut}_{CA}(TM)\). Similarly, \(\text{aut}_{CA}(TM)\) is isomorphic to the Lie subalgebra of \(\Omega^2(M) \times \mathfrak{X}(M)\) consisting of all \((\partial, X)\) such that \(\mathcal{L}_X\eta + d\partial = 0\).

The formula for the Courant bracket shows that the infinitesimal Courant automorphism \([X + \alpha, \cdot]\) corresponds to \((d\alpha + \iota_X\eta, X) \in \Omega^2(M) \times \mathfrak{X}(M)\). We will also need the following result from [24, Proposition 2.6] and [27, Section 2].

Proposition 5.6. Let \((\partial, X) \in \text{aut}_{CA}(TM)\), where \(X\) is complete with flow \(\Phi_s\). Then the 1-parameter group of automorphisms defined by \((\partial, X)\) is \((\gamma_s, \Phi_s)\), where
\[\gamma_s = \int_0^s (\Phi_u^{-1})^*\partial \, du.\]

If \(E \subseteq TM\) is a Dirac structure, and \(X + \alpha \in \Gamma(E)\), then \((d\alpha + \iota_X\eta, X) \in \text{aut}(TM)\) preserves \(E\), hence so does the resulting flow \((\gamma_s, \Phi_s)\). That is, \(\mathcal{R}_{-\gamma_s}((\mathcal{T}\Phi_s)(E)) = E\), or
\[(\mathcal{T}\Phi_s)(E) = E^{\gamma_s}.\]

5.4. Proof of Theorem 5.1

Proof. Given \(\epsilon \in \Gamma(E)\) as in Theorem 5.1 write \(\epsilon = X + \alpha\) where \(X \in \mathfrak{X}(M)\) is the vector field part and \(\alpha \in \Omega^1(M)\) the 1-form part. By construction, \(X\) is Euler-like along \(N\), hence its flow \(\Phi_s\) defines a tubular neighborhood embedding \(\psi\) such that \(\lambda_t \circ \psi = \psi \circ \kappa_t\), for \(t > 0\), where we write \(\lambda_t = \Phi_{-\log(t)}\). Let
\[\omega_t = \int_t^1 \frac{1}{\tau} \kappa_{\tau}^* \psi^* (d\alpha + \iota_X\eta) \, d\tau,\]
so that \(\omega = \omega_0\). We claim that the family of Dirac structures
\[(\kappa_t^! \psi^! E)^{\omega_t} \subseteq \mathcal{N}_N\]
is independent of \(t \geq 0\). This proves the theorem, because \([34]\) equals \(\psi^! E = (\mathcal{T}\psi)^{-1}(E|_U)\) for \(t = 1\), and \((\kappa_0^! \psi^! E)^{\omega} = (p_1^! E)^{\omega}\) for \(t = 0\).

By continuity, it suffices to prove the \(t\)-independence of \([34]\) for \(t > 0\). By \([33]\), we have
\[\lambda_t^! E = (\mathcal{T} \lambda_t^{-1})(E) = E^{\gamma_s}\]
for \(s = -\log(t)\), where \(\gamma_s \in \Omega^2(U)\) are the 2-forms
\[\gamma_s = \int_0^{-\log(t)} (\Phi_{-u}^{-1})^* (d\alpha + \iota_X\eta) \, du = - \int_t^1 \frac{1}{\tau} \lambda_{\tau}^* (d\alpha + \iota_X\eta) \, d\tau.\]
With \(\psi^* \gamma_s = -\omega_t\), it follows that
\[\kappa_t^! \psi^! E = \psi^! \lambda_t^! E = \psi^! E_{\gamma_s} = (\psi^! E)^{-\omega_t},\]
hence \((\kappa_t^! \psi^! E)^{\omega_t} = \psi^! E\) is independent of \(t\), as desired. \(\square\)
5.5. **Functorial properties of the normal form.** Let \( \Phi : M' \to M \) be a smooth map, and \( E \subseteq T M \) and \( E' \subseteq T M' \) Dirac structures relative to closed 3-forms \( \eta \) and \( \eta' \). Then \( \Phi \) is called a **Dirac morphism** if \( \Phi^* \eta = \eta' \), and for all \( m = \Phi(m') \) and \( v + \mu \in E_m \), there exists a **unique** element \( v' + \mu' \in E_{m'} \) such that \( v = T \Phi(v') + \mu' \). We denote such a Dirac morphism by

\[
\mathbb{T} \Phi : (TM', E') \to (TM, E).
\]

It determines a comorphism of Lie algebroids \( E' \to E \); the corresponding map \( \Phi^* E \to E' \) takes \( v + \mu \in E \) to the unique \( v' + \mu' \in E' \) to which it is related.

Given a transversal \( N \subseteq M \) for the Dirac structure \( E \), and suppose that the map \( \Phi \) is transverse to \( N \). Then \( N' = \Phi^{-1}(N) \) is a transversal for \( E' \), and given a section \( \epsilon \in \Gamma(E) \) such that \( X = \alpha(\epsilon) \) is Euler-like, then its pull-back \( \epsilon' \in \Gamma(E') \) defines an Euler-like vector field \( X' = \alpha' \). Let \( \omega \) be as in Theorem 5.1 and \( \omega' \) its pull-back under \( \nu' \). (Equivalently, \( \omega' \) is given by Equation (12), using \( \alpha' = \Phi^* \alpha \).) We obtain a commutative diagram of Dirac morphisms,

\[
\begin{array}{ccc}
(T \nu, (p^l v^l E')^\omega') & \xrightarrow{\mathbb{T} \nu} & (TM', E') \\
\downarrow \mathbb{T} \nu(\Phi) & & \downarrow \mathbb{T} \Phi \\
(T \nu_N, (p^l v^l E)^\omega) & \xrightarrow{\mathbb{T} \nu} & (TM, E).
\end{array}
\]

5.6. **Uniqueness properties for transversals.** Let \( E \subseteq TM \) be a Dirac structure relative to a closed 3-form \( \eta \in \Omega^3(M) \), and \( i_q : N_q \to M \) a family of transversals labeled by points \( q \in Q \). As in Section 3.8, \( N_q \) are the fibers of a submersion \( \psi : N \to Q \), and \( i_q = \phi \circ j_q \), where \( j_q : N_q \to N \) is the inclusion. Since \( \phi \) is transverse to \( a \), it defines an Dirac structure \( F = \phi^* E \subseteq TN \), and we have \( i_q^* E = j_q^* F \). (Our notation here differs slightly from Section 3.8.)

Given any vector field \( Z \) on \( Q \), we can find a section \( Y + \beta \in \Gamma(F) \) such that \( Y \sim_{\psi} Z \). The section defines \( (d \beta + i_Y \phi^* \eta, Y) \in \mathfrak{aut}_{CA}(TN) \) preserving \( F \). To simplify the discussion, let us assume that \( Z \) and \( Y \) are complete, with flows \( \Phi^Z_s \), \( \Phi^Y_s \) (in the general case, one has to work with local flows). Then the infinitesimal automorphism integrates to a 1-parameter group of automorphisms \( (\gamma_s, \Phi^Y_s) \) where

\[
\gamma_s = \int_0^s \left( \Phi^Y_u \right)^*(d \beta + i_Y \phi^* \eta) \, du \in \Omega^2(N).
\]

As explained above, we have \( F = (\Phi^Y_s)^1(F) \). Applying \( j_q^* \) to both sides and using that \( \Phi^Y_s \circ j_q = j_{\Phi^Z_s(q)} \circ (\Phi^Y_s | N_q) \), we obtain

\[
i^*_q E = j_q^* F = (\Phi^Y_s | N_q)^1(j_{\Phi^Z_s(q)}(F^\gamma_s)) = (\Phi^Y_s | N_q)^1(i_{\Phi^Z_s(q)}^* E)^s,
\]

where

\[
\partial_s = (j_{\Phi^Z_s(q)})^* \gamma_s \in \Omega^2(N_{\Phi^Z_s(q)}).
\]

That is, \( T(\Phi^Y_s | N_q) \) gives an isomorphism of Dirac structures \( i_q^* E \to (i_{\Phi^Z_s(q)}^* E)^s \).

As a special case, given a leaf \( S \subseteq M \) of the Dirac structure, and two transversals \( i_0 : N_0 \to M \) and \( i_1 : N_1 \subseteq M \), intersecting \( S \) in points \( m_0, m_1 \), we can extend to a family of transversals
\[ i_s: N_s \to M \] with \( N_s \cap S = \{m_s\} \), and there is a family of isomorphisms

\[
(38) \quad \alpha_s^i E \to (i_s^* E)^{\alpha_s}.
\]

(Cf. Corollary 3.22)

Remark 5.7. Suppose \( \eta = 0 \), and let \( i: N \to M \) be a transversal through a given point \( m \in M \), with \( T_m M = T_m N \oplus a(E_m) \). Then the Dirac structure \( i^* E \) on \( N \) is in fact a Poisson structure near \( m \). A uniqueness theorem for these transverse Poisson structure was obtained by Dufour-Wade [17, Theorem 4.5]. It can be recovered from our result, using the argument in Remark 6.5 (c) below. (We are grateful to the referee for this remark.)

6. Poisson manifolds

Let \( (M, \pi) \) be a Poisson manifold. We denote by \( \pi^\sharp: T^* M \to TM \) the bundle map defined by \( \pi \), and by \( E = \text{Gr}(\pi) \subseteq TM \) the Dirac structure given as its graph. A submanifold \( i: N \hookrightarrow M \) is a transversal for \( (M, \pi) \) if it is transverse to the map \( \pi^\sharp \). Equivalently, the restriction of \( \pi^\sharp \) to \( \text{ann}(TN) \) is injective. The Poisson bivector restricts to a skew-symmetric bilinear form on the conormal bundle \( \text{ann}(TN) \subseteq T^* M|_N \). The transversal \( N \) has constant corank if this restriction has constant rank; these are special cases of the pre-Poisson submanifolds studied in the work of Cattaneo-Zambon [12] and Calvo-Falceto [11]. If the bilinear form on \( \text{ann}(TN) \) is non-degenerate, then \( N \) is called a cosymplectic transversal; these are discussed in work of Weinstein [41], Xu [43], Cattaneo-Zambon [12], and Frejlich-Mărcuț [20] (under the name of Poisson transversal).

For a cosymplectic transversal, the subbundle \( \nu_N^* \cong \text{ann}(TN) \subseteq T^* M|_N \) is a symplectic vector bundle, with the fiberwise symplectic structure inverse to the restriction of \( \pi|_N \). The range of \( \pi^\sharp: \text{ann}(TN) \to T^* M|_N \) is a complement to \( TN \), identifying \( \pi^\sharp(\text{ann}(TN)) \cong \nu_N \). The non-degeneracy condition is equivalent to the direct sum decomposition

\[
(39) \quad TM|_N = \pi^\sharp(\text{ann}(TN)) \oplus TN,
\]
or dually

\[
(40) \quad T^* M|_N = \text{ann}(TN) \oplus T^* N.
\]

Weinstein [41, Proposition 1.4] showed that any cosymplectic transversal \( N \) inherits a Poisson structure \( \pi_N \). In fact, we have:

Lemma 6.1. A transversal \( i: N \hookrightarrow M \) for a Poisson manifold \( (M, \pi) \) is cosymplectic if and only if the Dirac structure \( i^* E \subseteq TN \) has trivial intersection with \( TN \). In this case, \( i^* E = \text{Gr}(\pi_N) \).

Proof. By definition, \( i^* E \) consists of all \( v' + \mu' \in TN \) such that there exists \( v + \mu \in \text{Gr}(\pi) \) with \( v = Ti(v') \) and \( \mu' = i^* \mu \). Hence, \( i^* E \cap TN = 0 \) holds if and only if the conditions \( i^* \mu = 0 \) and \( \pi^\sharp(\mu) \in TN \) imply that \( \mu = 0 \). But this is exactly the condition \( \pi^\sharp(\text{ann}(TN)) \cap TN = 0 \) for a cosymplectic transversal.

Suppose \( N \subseteq M \) is a cosymplectic transversal. Choose a 1-form \( \alpha \in \Omega^1(M) \) with \( \alpha|_N = 0 \), such that the splitting given by the normal derivative \( d^N \alpha: \nu_N \to T^* M|_N \) coincides with the given inclusion of \( \nu_N \cong \nu_N^* \). Then \( X = \pi^\sharp(\alpha) \) has linearization equal to the Euler-vector field on
\[ \omega = \int_0^1 \frac{1}{\tau} \nu^* \psi^* (\alpha) \, d\tau = \psi^* \int_0^1 \frac{1}{\tau} \nu^* \lambda^* (\alpha) \, d\tau \]

The presymplectic leaves of the Dirac structure \( p^! \text{Gr}(\pi_N) \) are the pre-images under \( p \) of the symplectic leaves of \((N, \pi_N)\), with the 2-forms obtained by pullback under \( p \). The Dirac structure \((p^! \text{Gr}(\pi_N))^\omega\) has the same leaves, but with the pullback of \( \omega \) added to the 2-forms on the leaves. Let us describe the restriction of this 2-form to \( TM|_N = TN \oplus \nu_N \).

**Lemma 6.2.** The restriction of \( \omega \) to to \( T\nu|_N \) has kernel \( TN \), and equals the given symplectic form on \( \nu_N \).

**Proof.** Since \( \alpha|_N = 0 \), with normal derivative taking values in \( \text{ann}(TN) \), the kernel of \( d\alpha|_N \) contains \( TN \). The same is thus true for all \( \frac{1}{\tau} \nu^* \lambda^* \alpha \), and hence for the 2-form \( \omega \). Due to our choice of \( \epsilon \), the differential \( T\psi|_N : T\nu_N|_N \to TM|_N \) respects the decompositions

\[ TM|_N = TN \oplus \nu_N = T\nu_N|_N \]

where we identify \( \pi^*(\text{ann}(TN)) = \nu_N \). Together with the dual decompositions of the cotangent bundles, this means that \( T\psi \) respects the decompositions

\[ TM|_N = TN \oplus (\nu_N \oplus \nu_N^*) = T\nu_N|_N. \]

The subbundle \( \text{Gr}(\pi)|_N \subseteq TM|_N \) splits as the direct sum of \( \text{Gr}(\pi_N) \) and the graph of the symplectic form on \( \nu_N \). Similarly, \( p^! \text{Gr}(\pi_N)|_N \subseteq T\nu_N|_N \) is the direct sum of \( \text{Gr}(\pi_N) \) and \( T\nu_N|_N \). Since \( \omega_N \) has kernel \( TN \), the \( B \)-field transform by \( \omega|_N \) preserves this decomposition, and is trivial on the first summand. On the other hand, by (11) it takes \( p^! \text{Gr}(\pi_N)|_N \) to \( \psi^! \text{Gr}(\pi)|_N \). This means that \( \omega|_N \) is just the given symplectic structure on \( \nu_N \subseteq T\nu_N|_N \).

This allows us to recover the following result.

**Theorem 6.3** (Frejlich-Mărcuț [20]). Let \( N \subseteq M \) be a cosymplectic transversal. Choose a closed 2-form \( \omega \in \Omega^2(\nu_N) \) on the normal bundle, such that \( \omega|_N \) has kernel \( TN \) and restricts to the given symplectic form on \( \nu_N \subseteq T\nu_N|_N \). Then, near the zero section of \( \nu_N \),

\[ (p^! \text{Gr}(\pi_N))^\omega \]

is the graph of a Poisson structure, and there exists a tubular neighborhood embedding \( \psi : \nu_N \to M \), which is a Poisson map on some neighborhood of \( N \).

**Proof.** We have proved the result for a particular \( \omega \) (given by (11)). For the general case, note that it suffices to consider closed 2-forms defined on an open neighborhood \( N \subseteq \nu_N \). Given two 2-form \( \omega, \omega' \) as in the theorem, one has \( \omega' - \omega = d\beta \), where \( \beta \) is a 1-form, with \( \beta|_N = 0 \). (The homotopy operator for the retraction from \( \nu_N \) to \( N \) gives a canonical choice for \( \beta \).) The Moser method for Poisson manifolds (as in, e.g., [4]) gives a Poisson isomorphism between the models over some neighborhood of the zero section.
Remark 6.4. The argument from [20] relies on Crainic-Mărcuț’s approach [14] to symplectic realizations via Poisson sprays on $T^*M$.

Remarks 6.5. (a) The 2-form $\omega$ used in [11] gives a Poisson structure over all of $\nu_N$, not only near the zero section. An alternative choice of $\omega$ uses the ‘minimal coupling’ procedure of Sterbenberg [33] and Weinstein [40], depending on the choice of a symplectic connection on the symplectic vector bundle $\nu_N$.

(b) If the normal bundle is trivial, $\nu_N = N \times \mathbb{R}^{2k}$, we may use the trivial connection. The normal form then says that a neighborhood of $N$ in $M$ is Poisson diffeomorphic to a neighborhood of $N \times \{0\}$ inside the product of Poisson manifolds,

$$(N \times \mathbb{R}^{2k}, \pi_N + \pi_0)$$

where $\pi_0 = -\sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$ is the standard Poisson structure on $\mathbb{R}^{2k}$. In particular, given any $m \in M$, consider a submanifold $N$ through $m$ such that $T_m M = T_m N \oplus \pi^2(T_m^* M)$. Taking $N$ smaller if necessary, this submanifold is a cosymplectic transversal. One refers to $\pi_N$ as the transverse Poisson structure. We recover the Weinstein splitting theorem [11], identifying $U \subseteq M$ with an open neighborhood of the direct product of Poisson manifolds $N \times \mathbb{R}^{2k}$.

(c) Weinstein’s uniqueness result [11, Lemma 2.2] for the transverse Poisson structure can be recovered from the more general result in Section 5.6 applied to $E = \text{Gr}(\pi)$. Consider a symplectic leaf $S$ and two transversals $N_0$ and $N_1$, such that $N_0 \cap S = \{m_0\}$ and $N_1 \cap S = \{m_1\}$. As discussed at the end of Section 5.6, $N_0, N_1$ extend to a smooth family of transversals $i_s: N_s \to M$, with $N_s \cap S = \{m_s\}$, and a family of isomorphisms $i_s^* E \cong (i_s^* E)^{\varphi_s}$, with base maps $\varphi_s: N_0 \to N_s$, for a suitable family of closed 2-forms $\vartheta_s \in \Omega^2(N_s)$. See Equation (33). By the explicit formula (36), the forms $\vartheta_s$ are exact, with a smooth family of primitives $\beta_s$. Equivalently, we obtain a family of isomorphisms

$$(i_0^* E)^{-d\alpha_s} \cong i_s^* E,$$

where $\alpha_s = \varphi_s^* \beta_s$ are the pullbacks with respect to the underlying diffeomorphisms. Since $i_s^* E = \text{Gr}(\pi_s)$ is the graph of the induced Poisson structure on $N_s$, this shows that the diffeomorphism $\varphi_s$ is a Poisson map, up to a gauge transformation of $\pi_0$ by the exact 2-form $-d\alpha_s$. By the Moser argument for Poisson structures (see e.g. [4, Section 3.3] or [3, Section 1.3]), the form $\alpha_s$ defines a time dependent vector field on $N_0$ whose flow $\psi_s$ (defined on a sufficiently small neighborhood of $m_0$) intertwines the gauge transformed Poisson structures. Its composition with $\varphi_s$ gives a family of Poisson diffeomorphisms $(N_0, \pi_0) \to (N_s, \pi_s)$.

We also recover the functorial properties of the normal form, as in [21]. Let $N \subseteq M$ and $\alpha$ as above. Suppose that $(M', \pi')$ is another Poisson manifold, and $\Phi: M' \to M$ is a Poisson map transverse to $N$. Then the pre-image $N' = \Phi^{-1}(N) \subseteq M$ is a Poisson transversal, and the pull-back $\alpha' = \Phi^* \alpha$ defines an Euler-like vector field $X' = \pi'^2 \alpha'$, and hence a tubular neighborhood embedding $\psi'$. Since $X' \sim_\Phi X$, we have that $\psi \circ \nu(\Phi) = \Phi \circ \psi'$. Since $\Phi$ is a Poisson map and since $\psi, \psi'$ are Poisson diffeomorphisms onto their images, it is immediate that the map $\nu(\Phi)$ between models is Poisson. Equivalently, this follows because the 2-forms are related by $\omega' = \nu(\Phi)^* \omega$. 


7. Generalized complex manifolds

Let $\mathbb{T}M$ be equipped with the Courant bracket for the zero 3-form $\eta = 0$. Following Hitchin [26] and Gualtieri [23, 24], one defines a generalized complex structure on $M$ to be a vector bundle automorphism $J \in \text{Aut}(\mathbb{T}M)$ with $J^2 = -\text{id}$, such that $J$ is orthogonal (preserves the metric) and such that its $+\sqrt{-1}$ eigenbundle $E = \ker(J - \sqrt{-1}\text{id}) \subseteq \mathbb{T}_C M$ is a Dirac structure for the complexified Courant bracket and metric. Conversely, a generalized complex structure may be regarded as a complex Dirac structure $E \subseteq \mathbb{T}_C M$ such that $E \cap \overline{E} = 0$. An ordinary complex structure $J$ on $TM$ defines a generalized complex structure of complex type, where $J = J \oplus (J^{-1})^*$. At the opposite extreme, any symplectic form $\omega$ on $M$ defines a generalized complex structure of symplectic type, with $E = \text{Gr}(\sqrt{-1}\omega)$, the graph of the imaginary 2-form $\sqrt{-1}\omega$. If $\gamma \in \Omega^2(M)$ is any closed real 2-form, and $E \subseteq \mathbb{T}_C M$ is a generalized complex structure, then the $B$-field transform $R_{\gamma}(E)$ is again a generalized complex structure.

Any generalized complex structure $J$ determines a Poisson structure $\pi$ on $M$, by

$$\pi^*(\mu) = a(J\mu)$$

for all $\mu \in T^*M \subseteq \mathbb{T}M$. See [24, Section 3.4]. This Poisson structure satisfies

$$\text{ran}(\pi^*)_C = a(E) \cap a(\overline{E}).$$

If $J$ is of complex type, then $\pi = 0$, while for $J$ of symplectic type the Poisson structure is inverse to the given symplectic form.

Lemma 7.1. Suppose that $i: N \hookrightarrow M$ is a cosymplectic transversal with respect to $\pi$. Then $i^!E$ defines a generalized complex structure on $N$.

Proof. Suppose $v + \mu \in \mathbb{T}M$ lies in the intersection $i^!E \cap \overline{i^!E}$. We want to show that $v = 0$ and $\mu = 0$. By treating real and imaginary parts separately, we may assume that $v = \overline{v}$ and $\mu = \overline{\mu}$. By definition of $i^!E$, we have that $v \in TS$, and there exists $\lambda \in T^*_{\mathbb{C}}M$ with $v + \lambda \in E$ and $i^!\lambda = \mu$. Taking the imaginary part of

$$a(J(v + \lambda)) = a(\sqrt{-1}(v + \lambda)) = \sqrt{-1}v$$

we see that $\pi^!(\text{Im}(\lambda)) = v \in TN$. On the other hand, taking the imaginary part of $i^*\lambda = \mu$ we get $i^*\text{Im}(\lambda) = 0$, hence $\text{Im}(\lambda) \in \text{ann}(TN)$. By definition of cosymplectic, this shows that $\text{Im}(\lambda) = 0$. We conclude that $v + \lambda \in E$ is real, and therefore zero. Hence also $v + \mu = 0$, which proves that $i^!E \cap \overline{i^!E} = 0$. □

Letting $p: \nu_N \rightarrow N$ be the bundle projection as before, the pull-back $p^!i^!E$ does not define a generalized complex structure, since it contains the real subbundle $\ker(Tp)$. However, if $\omega$ is a closed 2-form on $\nu_N$ whose restriction to $TM|_N = TN \oplus \nu_N$ has kernel $TN$ and coincides with the given form $\omega_0$ on the symplectic vector bundle $\nu_N$, then the $B$-field transform $(p^!i^!E)^{\sqrt{-1}\omega}$ is a generalized complex structure on some open neighborhood of $N$. Indeed,

$$p^!i^!E|_N = i^!E \oplus \nu_N \subseteq TN \oplus (\nu_N \oplus \nu_N^*),$$

and the gauge transform takes this to $i^!E \oplus \text{Gr}(\sqrt{-1}\omega_0)$.

A version of the following result was independently obtained by Bailey-Cavalcanti-Duran [6, Section 3.2].
Theorem 7.2. Let $E \subseteq T_C M$ be a generalized complex structure, and $N \subseteq M$ a cosymplectic transversal for the underlying Poisson structure $\pi$. Choose a 1-form $\alpha \in \Gamma(TM)$ as in Section 6 defining an Euler-like vector field $X = \pi^!(\alpha)$ with corresponding tubular neighborhood embedding $\psi: \nu_N \to M$. Then $\psi^* E \subseteq T \nu_N$ equals, up to gauge transformation by a closed real 2-form $\gamma \in \Omega^2(\nu_N)$ (defined below), the generalized complex structure

$$(p^! i^! F)^\omega \subseteq T_C \nu_N$$

where $\omega \in \Omega^2(\nu_N)$ is the closed 2-form [42].

Proof. By the results for Poisson manifolds (Section 6), the pre-image of $\text{Gr}(\pi) \subseteq TM$ under $T\psi$ is $(p^! \text{Gr}(\pi_N))\omega \subseteq T \nu_N$. The Euler-like vector field $X = \pi^!(\alpha)$ lifts to a section of $E$, given as

$$\epsilon := (J + \sqrt{-1}\text{id})\alpha = X + \beta + \sqrt{-1}\alpha,$$

where the real 1-form $\beta$ is defined by this equation. (The definition of $\epsilon$ implies that $J\epsilon = \sqrt{-1}\epsilon$, as well as $a(\epsilon) = a(J\alpha) = \pi^!(\alpha) = X$.) Let $\gamma \in \Omega^2(\nu_N)$ be the real 2-form defined similarly to $\omega$ (see Equation [42]), but with $\alpha$ replaced by $\beta$:

$$\gamma = \int_0^1 \frac{1}{\tau} \kappa^* \psi^*(d\beta) \, d\tau.$$

The normal form theorem for Dirac structures (Theorem 5.1) shows that the pre-image of $E$ under the complexified map $T \psi: T_C \nu_N \to T_C M$ is

$$\psi^! E = (p^! i^! E)^\gamma + \sqrt{-1}\omega = ((p^! i^! E)^\omega)^\gamma$$

as subbundles of $T_C \nu_N$. Since $E$ is a generalized complex structure, $(p^! i^! E)^\omega = (\psi^! E)^{-\gamma}$ is one also. \qed

Suppose that the normal bundle $\nu_N$ is trivial. By the Weinstein splitting theorem (cf. Remark 5.5 [11]), one obtains a Poisson isomorphism of a neighborhood of $N$ in $M$ with a neighborhood of $N \times \{0\}$ inside $N \times \mathbb{R}^{2k}$, where $N$ has the Poisson structure $\pi_N$, and $\mathbb{R}^{2k}$ has its standard linear Poisson structure $\pi_0 = -\sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p^i}$, inverse to the symplectic structure $\omega_0 = \sum_i dq_i \wedge dp^i$. Using this model as a starting point, we may take $\alpha = \sum_i (q_i dp^i - p^i dq_i)$. We then obtain $d\alpha = 2\omega_0$, hence $\frac{1}{2} \kappa^* d\alpha = 2\tau \omega_0$, and finally $\omega = \omega_0$. In particular, we recover the splitting theorem for generalized complex manifolds, due to Abouzaid-Boyarchenko [11, Theorem 1.4]:

Corollary 7.3. Let $M$ be a generalized complex manifold, with underlying Poisson structure $\pi$, and $m \in M$. Put $P = \text{ran}(\pi^!_m)$, and let $N \subseteq M$ be a submanifold containing $m$, and such that $T_m M = T_m N \oplus P$. Give $P$ the generalized complex structure corresponding to its symplectic form, and give $N$ the generalized complex structure with corresponding Dirac structure $i^! F$. Up to a $B$-field transform, there is an isomorphism of generalized complex manifolds from a neighborhood of $m$ in $M$ and in the product $N \times P$.

Appendix A. Normal bundles of vector subbundles

For a vector bundle $pr: E \to M$, with a vector subbundle $F \to N$ along a submanifold $N \subseteq M$, the normal bundle $\nu(E, F)$ is a vector bundle over $\nu(M, N)$ with projection $\nu(pr)$:
\( \nu(E, F) \rightarrow \nu(M, N) \). In fact, \( \nu(E, F) \) fits into a double vector bundle

\[
\begin{array}{c}
\nu(E, F) \quad \rightarrow \quad F \\
\nu(\text{pr}) \downarrow \quad \downarrow \\
\nu(M, N) \quad \rightarrow \quad N,
\end{array}
\]

that is, in this diagram both horizontal and vertical arrows are vector-bundle projections, and the horizontal and vertical scalar multiplications commute (see [22] for this characterization of double vector bundles). In particular, for a submanifold \( N \subseteq M \), we have a double vector bundle

\[
\begin{array}{c}
\nu(TM, TN) \quad \rightarrow \quad TN \\
\downarrow \quad \downarrow \\
\nu(M, N) \quad \rightarrow \quad N.
\end{array}
\]

The tangent bundle to \( \nu(M, N) \) also gives rise to a double vector bundle, the so-called tangent prolongation of \( p : \nu(M, N) \rightarrow N \):

\[
\begin{array}{c}
T\nu(M, N) \quad \rightarrow \quad TN \\
\downarrow \quad \downarrow \\
\nu(M, N) \quad \rightarrow \quad N.
\end{array}
\]

**Lemma A.1.** There is a natural map \( \nu(TM, TN) \rightarrow T\nu(M, N) \) which is a vector-bundle isomorphism with respect to the vector-bundle structures over \( TN \) and \( \nu(M, N) \), covering the identity map in each case. In particular, \( \nu(TM, TN) \) and \( T\nu(M, N) \) are identified as double vector bundles.

**Proof.** Let \( \text{pr}_M : TM \rightarrow M \) denote the tangent bundle to \( M \). The iterated tangent bundle \( T(TM) \) is a double vector bundle

\[
\begin{array}{c}
TTM \quad \rightarrow \quad TM \\
\text{pr}_M \downarrow \quad \downarrow \\
TM \quad \rightarrow \quad M.
\end{array}
\]

There is a canonical involution \( J : TTM \rightarrow TTM \) satisfying \( T\text{pr}_M \circ J = \text{pr}_M \) and which interchanges the vertical and horizontal vector bundle structures. See e.g. [30, Section 9.6].

For a submanifold \( N \subseteq M \), the submanifolds \( T(TM|_N) \) and \( (TTM)|_{TN} \) of \( TTM \) are both sub-double vector bundles:

\[
\begin{array}{c}
T(TM|_N) \quad \rightarrow \quad TN \\
\text{pr}_M \downarrow \quad \downarrow \\
TM|_N \quad \rightarrow \quad N,
\end{array}
\]

\[
\begin{array}{c}
(PP)(TTM)|_{TN} \quad \rightarrow \quad TM|_N \\
\text{pr}_M \downarrow \quad \downarrow \\
TN \quad \rightarrow \quad N.
\end{array}
\]
The involution $J : TTM \to TTM$ restricts to an isomorphism $T(TM|_N) \to (TTM)|_{TN}$ between these two double vector bundles. This map also restricts to the canonical involution of $TTN$, viewed as submanifolds of $T(TM|_N)$ and $(TTM)|_{TN}$. In this way, $J$ gives rise to an isomorphism between the two double vector bundles

$$
\begin{align*}
T(\nu(M,N)) & \xrightarrow{T_p} TN \\
\nu(M,N) & \xrightarrow{\nu(p)} (M,N)
\end{align*}
$$

as desired.

\[ \square \]

\section*{References}

1. M. Abouzaid and M. Boyarchenko, \textit{Local structure of generalized complex manifolds}, J. Symplectic Geom. \textbf{4} (2006), no. 1, 43–62.
2. R. Abraham, J. Marsden, and T. Ratiu, \textit{Manifolds, tensor analysis and applications}, Addison-Wesley, Reading, 1983.
3. A. Alekseev and E. Meinrenken, \textit{Linearization of Poisson Lie group structures}, J. Symplectic Geom. \textbf{14} (2016), no. 1, 227–267.
4. \textit{Ginzburg-Weinstein via Gelfand-Zeitlin}, J. Differential Geom. \textbf{76} (2007), no. 1, 1–34.
5. I. Androulidakis and G. Skandalis, \textit{The holonomy groupoid of a singular foliation}, J. Reine Angew. Math. \textbf{626} (2009), 1–37.
6. M. Bailey, G. Cavalcanti, and J. van der Leer Duran, \textit{Blow-ups in generalized complex geometry}, Preprint, 2016, arXiv:1602.02076.
7. R. Balan, \textit{A note about integrability of distributions with singularities}, Boll. Un. Mat. Ital. A (7) \textbf{8} (1994), no. 3, 335–344.
8. J. Basto-Goncalves, \textit{Linearization of resonant vector fields}, Trans. Amer. Math. Soc. \textbf{362} (2010), no. 12, 6457–6476.
9. C. Blohmann, \textit{Removable presymplectic singularities and the local splitting of Dirac structures}, 2014, arXiv:1410.5298. To appear in Int. Math. Res. Notices.
10. H. Bursztyn, A. Cabrera and M. del Hoyo, \textit{Vector bundles over Lie groupoids and algebroids}, Adv. in Math. \textbf{290} (2016), 163–207.
11. I. Calvo and F. Falceto, \textit{Poisson reduction and branes in Poisson sigma models}, Adv. in Math. \textbf{290} (2016), 163–207.
12. A. S. Cattaneo and M. Zambon, \textit{Coisotropic embeddings in Poisson manifolds}, Trans. Amer. Math. Soc. \textbf{361} (2009), no. 7, 3721–3746.
13. T. Courant, \textit{Dirac manifolds}, Trans. Amer. Math. Soc. \textbf{319} (1990), no. 2, 631–661.
14. M. Crainic and I. Marcut, \textit{On the existence of symplectic realizations}, J. Symplectic Geom. \textbf{9} (2011), no. 4, 435–444.
15. L. Drager, J. Lee, E. Park, and K. Richardson, \textit{Smooth distributions are finitely generated}, Annals of Global Analysis and Geometry \textbf{41} (2012), no. 3, 357–369 (English).
16. J.-P. Dufour, \textit{Normal forms for Lie algebroids}, Lie algebroids and related topics in differential geometry (Warsaw, 2000), Banach Center Publ., vol. 54, Polish Acad. Sci. Inst. Math., Warsaw, 2001, pp. 35–41.
17. J.-P. Dufour and A. Wade, \textit{On the local structure of Dirac manifolds}, Compos. Math. \textbf{144} (2008), no. 3, 774–786.
18. J.-P. Dufour and N.T. Zung, \textit{Poisson structures and their normal forms}, Progress in Mathematics, vol. 242, Birkhäuser Verlag, Basel, 2005.
19. R. Fernandes, \textit{Lie algebroids, holonomy and characteristic classes}, Adv. Math. \textbf{170} (2002), no. 1, 119–179.
20. P. Frejlich and I. Marcut, \textit{The local normal form around Poisson transversals}, 2013, arXiv:1306.6055. To appear in Pacific J. Math.
21. _____, Normal forms for Poisson maps and symplectic groupoids around Poisson transversals, 2015, arXiv:1508.05670.
22. J. Grabowski and M. Rotkiewicz, Higher vector bundles and multi-graded symplectic manifolds, J. Geom. Phys. 59 (2009), no. 9, 1285–1305.
23. M. Gualtieri, Generalized complex geometry, Ph.D. thesis, Oxford, 2004, arXiv:math.DG/0401221.
24. _____, Generalized complex geometry, Ann. of Math. (2) 174 (2011), no. 1, 75–123.
25. V. Guillemin and S. Sternberg, Geometric asymptotics, revised ed., Mathematical Surveys and Monographs, vol. 14, Amer. Math. Soc., Providence, R. I., 1990.
26. N. Hitchin, Generalized Calabi-Yau manifolds, Q. J. Math. 54 (2003), no. 3, 281–308.
27. S. Hu, Hamiltonian symmetries and reduction in generalized geometry, Houston J. Math. 35 (2009), no. 3, 787–811.
28. S. Lang, Differential and Riemannian Manifolds, vol. 160, Springer-Verlag, 1995.
29. D. Li-Bland and E. Meinrenken, Courant algebroids and Poisson geometry, International Mathematics Research Notices 11 (2009), 2106–2145.
30. K. Mackenzie, General theory of Lie groupoids and Lie algebroids, London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, 2005.
31. E. Miranda and N. T. Zung, A note on equivariant normal forms of Poisson structures, Math. Res. Lett. 13 (2006), no. 5-6, 1001–1012.
32. D. Roytenberg, Courant algebroids, derived brackets and even symplectic supermanifolds, Thesis, Berkeley 1999. arXiv:math.DG/9910078.
33. P. Ševera, Letters to Alan Weinstein, http://sophia.dtp.fmph.uniba.sk/~severa/letters/, 1998-2000.
34. _____, Poisson Lie T-duality and Courant algebroids, Lett. Math. Phys. 105 (2015), no. 12, 1689–1701.
35. P. Stefan, Integrability of systems of vector fields, J. London Math. Soc. (2) 21 (1980), no. 3, 544–556.
36. S. Sternberg, Local contractions and a theorem of Poincaré, Amer. J. Math. 79 (1957), 809–824.
37. _____, On the structure of local homeomorphisms of Euclidean n-space. II., Amer. J. Math. 80 (1958), 623–631.
38. _____, On minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field, Proc. Nat. Acad. Sci. USA 74 (1977), 5253–5254.
39. H. Sussmann, Orbits of families of vector fields and integrability of distributions, Trans. Amer. Math. Soc. 180 (1973), 171–188.
40. A. Weinstein, A universal phase space for particles in Yang-Mills fields, Lett. Math. Phys. 2 (1978), 417–420.
41. _____, The local structure of Poisson manifolds, J. Differential Geom. 18 (1983), no. 3, 523–557.
42. _____, Almost invariant submanifolds for compact group actions, J. Eur. Math. Soc. (JEMS) 2 (2000), no. 1, 53–86.
43. P. Xu, Dirac submanifolds and Poisson involutions, Ann. Sci. École Norm. Sup. (4) 36 (2003), no. 3, 403–430.

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