TERMINE OF PSEUDO-EFFECTIVE 4-FOLD FLIPS

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Abstract. Let \((X, \Delta)\) be a log canonical 4-fold over an algebraically closed field of characteristic zero. Assume that the \(\mathbb{Q}\)-divisor \(K_X + \Delta\) is pseudo-effective. We prove that any sequence of \((K_X + \Delta)\)-flips terminates.

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Introduction

Two of the main goals of the minimal model program are to prove the existence of flips and termination of any sequence of such birational transformations. The existence and termination of flips for terminal 3-folds was achieved by Mori in [Mor88]. This result was generalized to the log canonical case by Shokurov [Sho96]. All these proofs rely on a careful analysis of the flipping contractions for 3-folds. In dimension 4, Kawamata settled the existence of smooth 4-fold flips in [Kaw89] and this result was generalized to the singular case by many authors (see, e.g., [KMM87]). In [Fuj04], Fujino proved the termination of canonical 4-fold flips by studying an invariant that decreases with flips, the so-called difficulty function. Later, in [AHK07] Alexeev, Hacon, and Kawamata proved termination of many klt 4-fold flips using the same invariant. In [Sho09], Shokurov introduced a special sequence of flips for the minimal model program of a pair, called ordered flips, or flips with scaling, and proved that any sequence of flips with scaling for klt 4-folds terminates. The existence of flips in arbitrary dimension was finally achieved by Birkar, Cascini, Hacon, and McKernan in [BCHM10], where the authors also prove termination of flips with scaling for klt pairs \((X, \Delta)\) with \(\Delta\) a big \(\mathbb{Q}\)-divisor. In [Bir07], Birkar proved termination of any sequence of flips for klt pairs with \(K_X + \Delta \sim_\mathbb{Q} D \geq 0\) assuming termination of flips in dimension \(\dim(X) - 1\) and the ACC conjecture for log canonical thresholds. Such conjecture was proved by Hacon, McKernan, and Xu in [HMX14].

The primary technique in the article [Bir07] is to study the log canonical threshold of the divisor \(D\) with respect to \(K_X + \Delta\), prove that this invariant increases with flips, and eventually it strictly increases in any sequence of flips, up to passing to a quasi-projective variety. Then, the ascending chain condition for log
canonical thresholds shows that the sequence of flips must terminate. In this article, we prove termination of flips for a pseudo-effective log canonical 4-fold, over an algebraically closed field, using a similar invariant. We will consider a generalized log canonical threshold with respect to the pair \((X, \Delta)\) that we denote by \(\text{lct}(K_X + \Delta)\) (see Definition 2.22) and we will call it the log canonical threshold of the pair. We want to prove that it behaves well in a sequence of pseudo-effective Kawamata log terminal 4-fold flips. More precisely, let
\[(X, \Delta) \to (X_1, \Delta_1) \to (X_2, \Delta_2) \to \ldots \to (X_j, \Delta_j) \to (X_{j+1}, \Delta_{j+1}) \ldots\]
be a sequence of flips of klt 4-folds, such that \(K_X + \Delta\) is a pseudo-effective \(\mathbb{Q}\)-divisor. It is straightforward to prove that the inequality
\[\text{lct}(K_{X_j} + \Delta_j) \leq \text{lct}(K_{X_{j+1}} + \Delta_{j+1})\]
holds for every \(j \in \mathbb{Z}_{\geq 1}\) (see Lemma 5.2). We will prove that in any such sequence, after finitely small modifications, the log canonical threshold strictly increases (up to passing to a quasi-projective variety). Then, we use the ACC for generalized log canonical thresholds, concluding the following:

**Theorem 1.** Let \((X, \Delta)\) be a log canonical 4-fold over an algebraically closed field of characteristic zero. Assume that the \(\mathbb{Q}\)-divisor \(K_X + \Delta\) is pseudo-effective. Then, any sequence of \((K_X + \Delta)\)-flips terminates.

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1. **Description of the proof**

Let \((X, \Delta)\) be a Kawamata log terminal 4-fold with \(K_X + \Delta\) a pseudo-effective \(\mathbb{Q}\)-divisor. In Definition 2.22, we will attach an invariant \(c = \text{lct}(K_X + \Delta)\) to the pair \((X, \Delta)\). This invariant, is a generalized log canonical threshold in the sense of [BZ16]. We consider an ample divisor \(A\) on \(X\). For \(\lambda \in \mathbb{Q}_{>0}\) small enough, we will denote by \(G_\lambda\) a general element in the \(\mathbb{Q}\)-linear system \(|K_X + \Delta + c_\lambda G_\lambda|\) and by \(c_\lambda\) the log canonical threshold of \(G_\lambda\) with respect to \(K_X + \Delta\). We prove that the log canonical thresholds \(c_\lambda\) converge to \(c\) when \(\lambda \to 0\). Moreover, we will prove that the non-klt locus of the log canonical pairs \((X, \Delta + c_\lambda G_\lambda)\) stabilize to a subvariety \(W\) for \(\lambda\) sufficiently small.

In Section 3, we will prove an adjunction formula for the \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \((c + 1)(K_X + \Delta)\) to any common minimal log canonical center \(W_i \subseteq W\) of the log canonical pairs \((X, \Delta + c_\lambda G_\lambda)\). More precisely, using Kawamata subadjunction Theorem (see, e.g., [HK10, Theorem 13.13]) we can write

\[(K_X + \Delta + c_\lambda G_\lambda)|_{W_i} \sim_{\mathbb{Q}} K_{W_i} + B^{G_\lambda}_{W_i} + M^{G_\lambda}_{W_i},\]

where \(M^{G_\lambda}_{W_i}\) is a \(\mathbb{Q}\)-divisor which is the push-forward of a nef divisor on a higher birational model of \(W_i\), and \(B^{G_\lambda}_{W_i}\) is an effective \(\mathbb{Q}\)-divisor, such that the pair

\[(W_i, B^{G_\lambda}_{W_i})\]

is log canonical whenever \(M^{G_\lambda}_{W_i}\) is \(\mathbb{Q}\)-Cartier. We aim to define the limit when \(\lambda\) converges to zero of the above subadjunction formula. However, both divisors depend of the choice of \(G_\lambda\) in its \(\mathbb{Q}\)-linear system. So in order to take the limit we need to prove that the divisor

\[B^{\lambda}_{W_i} = \bigwedge_{j \in J} B^{G_\lambda}_{W_i}\]

is a well-defined effective \(\mathbb{R}\)-divisor, where \(J\) is a finite set, and the \(\mathbb{Q}\)-divisors \(G^j_{W_i}\) are general in their \(\mathbb{Q}\)-linear system. Thus, we have a subadjunction formula

\[(K_X + \Delta + c_\lambda G_\lambda)|_{W_i} \sim_{\mathbb{Q}} K_{W_i} + B^{\lambda}_{W_i} + M^{\lambda}_{W_i},\]
where the support of $B^\lambda_{W_i}$ is independent of $\lambda$ and $M_{W_i}^\lambda$ is nef in codimension one. Since the support of $B^\lambda_{W_i}$ is independent of $\lambda$ it makes sense to construct a limit $\mathbb{R}$-divisor $B_{W_i}$ when $\lambda$ converges to zero. This limit divisor will be used to define the desired adjunction formula

$$(c + 1)(K_X + \Delta)|_{W_i} \sim_{q} K_{W_i} + B_{W_i} + M_{W_i}.$$  

Therefore, a sequence of flips for the pair $(X, \Delta)$ which does not contain $W_i$ in a flipped loci will induce a sequence of quasi-flips for the triple $(W_i, B_{W_i} + M_{W_i})$. By construction, this triple is a generalized pair in the sense of [BZ16]. Moreover, we will show that it is a generalized log canonical pair.

In Section 4, we prove that any $(K_X + \Delta)$-flip that intersects $W_i$ non-trivially and does not contain $W_i$ in its flipping locus induces an ample strict quasi-flip for the generalized log canonical pair $(W_i, B_{W_i} + M_{W_i})$. Therefore, in order to prove termination of flips around $W_i$, it suffices to show termination of ample strict quasi-flips for generalized log canonical pairs of dimension at most three. Termination of these quasi-flips in codimension one is proved using the fact that the coefficients of $B_{W_i}$ belong to a DCC set. Then, we prove termination of the weak quasi-flips by using standard arguments of low-dimensional flips.

Once we prove termination around each minimal log canonical center $W_i$ of the log canonical pairs $(X, \Delta + c_iG_X)$ we deduce termination around $W$, meaning that in any sequence of flips eventually all flipping loci are disjoint from the strict transform of $W$ on $X_j$. Replacing the variety $X_j$ with the complement of the strict transform of $W$, and the divisor $\Delta_j$ with the restriction to this quasi-projective subvariety, we achieve that the generalized log canonical pair $(c + 1)(K_X + \Delta_j)$ has strictly less generalized log canonical centers than the pair $(c + 1)(K_X + \Delta)$. Proceeding inductively, we deduce that for $j$ large enough the generalized pair $(c + 1)(K_{X_j} + \Delta_j)$ is Kawamata log terminal, so we have that $\text{lct}(K_X + \Delta) < \text{lct}(K_{X_j} + \Delta_j)$. Thus, we deduce that an infinite sequence of $(K_X + \Delta)$-flips induces a sequence of generalized log canonical thresholds violating the ACC, leading to a contradiction.

2. Preliminaries and notation

In this section, we recall classic results and notation. We will follow the notation of standard references algebraic geometry [Laz04a, Laz04b] and the minimal model program [KM98, HK10, Kol13]. Throughout this paper, we will work over an algebraically closed field $K$ of characteristic zero.

Definition 2.1. Given a projective birational morphism $p: Y \rightarrow X$ from a normal variety $Y$ and a prime divisor $D$ on $Y$, we say that $p(D)$ is the center of $D$ on $X$. In what follows, we may identify the class of prime divisors over $X$ with the class of divisorial valuations of the function field $K(X)$. The center on $X$ of a divisorial valuation of $X$ is just the center of the corresponding prime divisor. The center of the divisorial valuation $E$ on $X$ will be denoted by $c_E(X)$.

2.1. Generalized pairs. In this subsection, we recall the standard definitions of generalized pairs.

Definition 2.2. A generalized sub-pair is a triple $(X, B + M)$, where $X$ is a quasi-projective normal algebraic variety, $K_X + B + M$ is $\mathbb{R}$-Cartier, and $M$ is the push-forward of a nef divisor on a higher birational model of $X$. More precisely, there exists a projective birational morphism $f: X' \rightarrow X$ from a normal quasi-projective variety $X'$ and a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $M'$ such that $M = f_*(M')$. We define $B'$ by the equation

$$K_{X'} + B' + M' = f^*(K_X + B + M).$$

A generalized sub-pair is called a generalized pair if $B$ is an effective divisor. We say that $B$ is the boundary part and that $M$ is the nef part of the generalized pair $(X, B + M)$. We may call the sum $B + M$ a generalized boundary. Observe that $M'$ defines a nef b-Cartier $\mathbb{R}$-divisor in the sense of [Cor07, Definition 1.7.3]. We will say that this is the nef b-divisor associated to the generalized pair.
Remark 2.3. Observe that we can always take a projective birational morphism \( g: X'' \to X' \), such that \((X'', B'' + M'')\) is a log smooth sub-pair, where \( K_{X''} + B'' = g^*(K_{X'} + B') \) and \( M'' = g^*(M') \). Since \( M'' \) is nef, we can always replace \( X' \) by \( X'' \) in the above definition, and therefore we may assume that \((X', B' + M')\) is log smooth.

Definition 2.4. Given a generalized pair \((X, B + M)\) and a projective birational morphism \( f: X'' \to X \), we will say that \( f \) is a log resolution of the generalized pair, if \( X'' \to X \) factors through the variety \( X' \), the exceptional locus \( \text{Ex}(f) \) is a divisor, and \( \text{Ex}(f) + B'' \) is a divisor with simple normal crossing support. We may say that \((X'', B'' + M'')\) is the associated generalized pair on \( X'' \).

Remark 2.5. Given two generalized pairs \((X, B + M)\) and \((X', B' + M')\) such that \( X \) and \( X' \) are birational, we will say that the nef parts \( M \) and \( M' \) are trace of a common nef b-divisor, if there exists a log resolution \((X'', B'' + M'')\) for both generalized pairs with projective birational maps \( \pi: X'' \to X \) and \( \pi': X'' \to X' \), such that \( M = \pi_*(M'') \) and \( M' = \pi'_*(M'') \).

Definition 2.6. Let \((X, B + M)\) be a generalized pair, \( g: Y \to X' \) be a projective birational morphism, and \( E \) a prime divisor on \( Y \). We denote by \( h \) the composition \( f \circ g \), and we define the \( \mathbb{R} \)-divisor \( B_Y \) by the formula
\[ K_Y + B_Y + M_Y = h^*(K_X + B + M), \]
where \( M_Y = g^*(M') \). The generalized discrepancy (resp. generalized log discrepancy) of the generalized pair \((X, B + M)\) along the divisor \( E \) is \(-\text{coeff}_E(B_Y)\) (resp. \(1 - \text{coeff}_E(B_Y)\)). Given \( \epsilon \in (0, 1) \), we say that the generalized pair \((X, B + M)\) is generalized \( \epsilon \)-Kawamata log terminal (resp. generalized log canonical) if generalized log discrepancies with respect to prime divisors over \( X' \) are greater than \( \epsilon \) (resp. non-negative). As usual, we may write klt (resp. lc) to abbreviate Kawamata log terminal (resp. log canonical). Moreover, if \( \epsilon \) is zero in the above definition, we will omit it from the notation.

Given a prime divisor \( E \) over \( X \), we denote the generalized discrepancy of \((X, B + M)\) at \( E \) by \( a_E(X, B + M) \). We also may say that the divisorial valuation \( E \) has generalized discrepancy \( a_E(X, B + M) \) with respect to the generalized pair \((X, B + M)\), or with respect to the \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( K_X + B + M \). If the generalized log discrepancy along \( E \) is non-positive (resp. zero) we say that the image of \( E \) on \( X \) is a generalized non-klt center (resp. generalized log canonical center) of the generalized pair. The union of all the generalized non-klt centers of a generalized pair \((X, B + M)\) is the non-klt locus of the pair.

Remark 2.7. By Remark 2.3, we may assume that \((X', B' + M')\) is itself a log resolution of the generalized pair \((X, B + M)\), and therefore \((X, B + M)\) is generalized klt if and only if its generalized log discrepancies with respect to any prime divisor on \( X' \) are greater than zero, or equivalently, if the coefficients of \( B' \) are less than or equal to one.

Definition 2.8. A divisorial valuation \( E \) over the generalized pair \((X, B + M)\) is called terminal if \( a_E(X, B + M) \in \mathbb{R}_{>0} \), and it is called non-terminal otherwise. A generalized pair \((X, B + M)\) is said to be terminal if all its exceptional divisorial valuations are terminal with respect to \( K_X + B + M \).

Remark 2.9. If \( M = 0 \) in the above definition, then \((X, B)\) is a pair in the usual sense of [KM98]. Conversely, every pair can be considered as a generalized pair with trivial nef part. In what follows, we may denote the boundary part of a pair by \( \Delta \geq 0 \), following the standard notation of [KM98]. If we work with pairs, we will drop the word generalized from the above definitions.

2.2. Quasi-flips. In this subsection, we recall the standard definitions of quasi-flips for generalized pairs.

Definition 2.10. Let \((X, B + M)\) be a generalized log canonical pair. A birational contraction \( \phi: X \to Z \) is said to be a weak contraction for the generalized pair if \(-K_X + B + M\) is nef over \( Z \). A quasi-flip of \( \phi \) is a birational map \( \pi: X \dashrightarrow X^+ \) with a birational contraction \( \phi^+: X^+ \to Z \) such that the following conditions hold:
• The triple \((X^+, B^* + M^+)\) is generalized log canonical,
• the \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(K_{X^+} + B^* + M^+\) is nef over \(Z\),
• the inequality \(\phi^+_j(B^*) \leq \phi_j(B)\) of Weil \(\mathbb{R}\)-divisors on \(Z\) holds, and
• the nef parts \(M\) and \(M^+\) are the trace of a common nef \(b\)-divisor.

As usual, the morphism \(\phi\) is called the \textit{flipping contraction} and the morphism \(\phi^+\) is called the \textit{flipped contraction}. We say that the quasi-flip \(\pi\) is \textit{weak} if both \(\phi\) and \(\phi^+\) are small morphisms, and that the quasi-flip \(\pi\) is \textit{ample} if both \(- (K_X + B + M)\) and \((K_{X^+} + B^* + M^+)\) are ample over \(Z\). A \textit{flip} for a generalized log canonical pair is a weak ample quasi-flip such that both \(\phi\) and \(\phi^+\) have relative Picard rank one. In the case that we have a sequence of quasi-flips, we will further require that all nef parts are trace of the same Cartier \(b\)-divisor.

\textbf{Definition 2.11.} A sequence of quasi-flips for generalized log canonical pairs \((X_j, B_j + M_j)\) is said to be \textit{under a DCC set} if the coefficients of the boundary parts \(B_j\) belong to a DCC set.

\textbf{Definition 2.12.} We say that a quasi-flip \(\pi\) is a \textit{quasi-flop} if \(\pi\) and \(\pi^{-1}\) are quasi-flips. In this case, the flipping contraction (resp. flopped contraction) is called the \textit{flopping contraction} (resp. \textit{flopped contraction}) and the flipping locus (resp. flopped locus) is called the \textit{flopping locus} (resp. \textit{flopped locus}).

\textbf{Definition 2.13.} Given a quasi-flip \(\pi: X \rightarrow X^+\) with flopping and flopped contractions \(\phi\) and \(\phi^+\) respectively, we define the \textit{non-flopping locus} to be the smallest Zariski closed subset \(N\) of \(Z\) such that \(\pi\) is a quasi-flop over \(Z \setminus N\). We say that a quasi-flop is \textit{strict} if the subvariety \(N\) is non-empty.

\textbf{Definition 2.14.} A quasi-flip for generalized log canonical pairs is said to be \textit{klt} if its flipping locus, and therefore its flopped locus, does not intersect the non-klt locus of the generalized pair. A quasi-flop for generalized log canonical pairs is said to be \textit{terminal} if its flipping and flopped locus do not contain the center of a non-terminal valuation.

The following proposition is well-known for pairs, see for example [Sho04, Monotonicity]. The proof in the case of generalized pairs is analogous.

\textbf{Proposition 2.15.} Given a quasi-flip \(\pi: X \rightarrow X^+\) for the generalized pairs \((X, B + M)\) and \((X^+, B^* + M^+)\) over \(Z\), and a prime divisor \(E\) over \(X\), we have that
\[
a_E(X, B + M) \leq a_E(X^+, B^* + M^+)
\]
and such inequality is strict if the center of \(E\) on \(Z\) is contained in the non-flopping locus. Moreover, the non-flopping locus of an ample quasi-flip is the image of the flipping or flopped locus on \(Z\).

2.3. Minimal models. In this subsection, we recall the standard definitions of minimal models.

\textbf{Definition 2.16.} A pair \((X, \Delta)\) is called \textit{divisorially log terminal}, or \textit{dlt} for short, if the coefficients of \(\Delta\) are in the interval \([0, 1]\), there exists a log resolution \(p: Y \rightarrow X\) such that we can write \(K_Y + \Delta_Y + E = p^*(K_X + \Delta)\), where \(\Delta_Y\) is the strict transform of \(\Delta\) on \(Y\), and the coefficients of the prime divisors of \(E\) are less than one.

\textbf{Definition 2.17.} An ample weak contraction \(\phi: X \rightarrow Z\) of relative Picard rank one such that \(K_Z + \phi_*(\Delta)\) is \(\mathbb{R}\)-Cartier is called a \((K_X + \Delta)\)-\textit{divisorial contraction}. Indeed, the \(\mathbb{R}\)-Cartier condition of the divisor \(K_Z + \phi_*(\Delta)\) implies that the exceptional locus of the morphism \(\phi: X \rightarrow Z\) is purely divisorial, and if the divisorial contraction has Picard rank one, then the exceptional divisor must be irreducible (see, e.g., [HK10]).

\textbf{Definition 2.18.} Given a dlt pair \((X, \Delta)\) and a rational map \(\pi: X \dashrightarrow X_{\text{min}}\) such that
• \(X_{\text{min}}\) is a quasi-projective normal variety,
• \(\pi^{-1}\) contracts no divisors, or equivalently, \(\pi\) extracts no divisors,
• \(K_{X_{\text{min}}} + \Delta_{\text{min}}\) is nef \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor, where \(\Delta_{\text{min}}\) is the strict transform of \(\Delta\) on \(X_{\text{min}}\), and
Given a dlt pair \((X, \Delta)\), we say that a sequence of \((K_X + \Delta)\)-divisorial contractions and \((K_X + \Delta)\)-flips is a minimal model program for \((X, \Delta)\) if the composition \(\pi\) of such birational maps is a log terminal model (or minimal model) of \((X, \Delta)\).

**Definition 2.20.** Let \((X, \Delta)\) be a pair and \(p: Y \to X\) a projective birational morphism from a normal variety \(Y\), then we will denote \(K_Y^\Delta/X = K_Y - p^*(K_X + \Delta)\) the relative canonical divisor.

**Definition 2.21.** Given a pseudo-effective klt pair \((X, \Delta)\) with a minimal model \((X_{\min}, \Delta_{\min})\), we can realize \(K_X + \Delta\) as a generalized boundary. Indeed, let \(p: Y \to X\) and \(q: Y \to X_{\min}\) be two projective birational morphisms that give a log resolution of the birational contraction \(\pi: X \dasharrow X_{\min}\), then we can write

\[ p^*(K_X + \Delta) = q^*(K_{X_{\min}} + \Delta_{\min}) + E, \]

where \(q^*(K_{X_{\min}} + \Delta_{\min})\) is a nef \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor and \(E\) is an effective \(q\)-exceptional \(\mathbb{R}\)-divisor. Thus, we have that

\[ K_X + \Delta = p_* (q^*(K_{X_{\min}} + \Delta_{\min})) + p_*(E) \]

is a generalized boundary with nef part \(p_* (q^*(K_{X_{\min}} + \Delta_{\min}))\) and boundary part \(p_*(E)\). Moreover, for every \(\mu \in \mathbb{R}_{\geq 1}\) we can realize the \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(\mu(K_X + \Delta)\) as a generalized pair by writing

\[
\mu(K_X + \Delta) = K_X + \Delta + (\mu - 1)p_*(E) + (\mu - 1)p_*(q^*(K_{X_{\min}} + \Delta_{\min})).
\]

Here the boundary part is

\[ \Delta + (\mu - 1)p_*(E) \]

and the nef part is

\[ (\mu - 1)p_*(q^*(K_{X_{\min}} + \Delta_{\min})). \]

From now on, given a pseudo-effective klt pair \((X, \Delta)\) with a minimal model and \(\mu \in \mathbb{R}_{\geq 1}\) using equation (2.1) we may consider \(\mu(K_X + \Delta)\) as a generalized pair.

**Definition 2.22.** The maximum positive real number \(c\), such that \((c + 1)(K_X + \Delta)\) is a generalized log canonical pair is called the generalized log canonical threshold of \(K_X + \Delta\). The generalized log canonical threshold of \(K_X + \Delta\) will be denoted by \(\text{glt}(K_X + \Delta)\).

**Proposition 2.23.** Let \((X, \Delta)\) be a pseudo-effective pair with a minimal model \((X_{\min}, \Delta_{\min})\). Let \(p: Y \to X\) and \(q: Y \to X_{\min}\) be two projective birational morphisms which give a log resolution of the minimal model program \(\pi: X \dasharrow X_{\min}\), so we can write

\[ p^*(K_X + \Delta) = q^*(K_{X_{\min}} + \Delta_{\min}) + E, \]

for some exceptional divisor \(E\) on \(Y\). Then, the log canonical threshold \(\text{glc}(K_X + \Delta)\) of \(K_X + \Delta\) equals the maximum positive real number \(c\), such that \(K_{Y/X}^\Delta - cE\) has coefficients greater or equal than negative one. Moreover, \(\text{glc}(K_X + \Delta)\) is independent of the minimal model and the resolution.

**Proof.** Observe that we have the following equality

\[
p^*(\mu(K_X + \Delta)) = p^*(K_X + \Delta) + p^*((\mu - 1)(K_X + \Delta)) = p^*(K_X + \Delta) + (\mu - 1)E + (\mu - 1)q^*(K_{X_{\min}} + \Delta_{\min}) = K_Y + (-K_{Y/X}^\Delta) + (\mu - 1)E + (\mu - 1)q^*(K_{X_{\min}} + \Delta_{\min}).
\]

By definition, the generalized pair \((c + 1)(K_X + \Delta)\) is a generalized log canonical pair if and only if the coefficients of \(cE - K_{Y/X}^\Delta\) are greater or equal than negative one. The fact that \(\text{glt}(K_X + \Delta)\) is independent
Corollary 2.24. Let \((X, \Delta)\) be a pseudo-effective pair with a minimal model. Then, the log canonical threshold of \(K_X + \Delta\) is finite if and only if \(K_X + \Delta\) is not nef.

Proof. If \(K_X + \Delta\) is nef, then \(E\) is trivial in the above proof, and the coefficients of \(-K_{Y/X}^\Delta\) are less than one since \((X, \Delta)\) is a klt pair. On the other hand, if \(K_X + \Delta\) is not nef, then by strict monotonicity 2.15, at least one coefficient of \(E\) is non-trivial, so \(\text{let}(K_X + \Delta)\) is finite. \(\square\)

2.4. Notation. In this subsection, we introduce further notation that will be used in the proof of the theorem.

Definition 2.25. Given \(\lambda \in \mathbb{Q}_{>0}\), we will say that \(G_\lambda\) is general in its \(\mathbb{Q}\)-linear system \(|K_X + \Delta + \lambda A|_\mathbb{Q}\) if \(G_\lambda\) is the average of \(k\) general elements of the linear system \(|m(K_X + \Delta + \lambda A)|\) for \(m\) and \(k\) big and divisible enough. Thus, we can write

\[
G_\lambda = \frac{1}{mk} \sum_{j \in J} G_j^\lambda,
\]

where \(J\) is a finite set of cardinality \(k\) and the Cartier divisors \(G_j^\lambda\) are general elements of the linear system \(|m(K_X + \Delta + \lambda A)|\). We will assume that \(G_\lambda\) is general in its \(\mathbb{Q}\)-linear system unless otherwise stated.

Definition 2.26. For every \(\lambda \in \mathbb{Q}_{>0}\), the fixed component or fixed divisor of the \(\mathbb{Q}\)-linear system \(|K_X + \Delta + \lambda A|_\mathbb{Q}\) is the wedge of all the \(\mathbb{Q}\)-divisors in the \(\mathbb{Q}\)-linear system. Since the \(\mathbb{Q}\)-divisor \(K_X + \Delta + \lambda A\) has finitely generated section ring we know that the fixed component is a well-defined effective \(\mathbb{Q}\)-divisor. The fixed component of the \(\mathbb{Q}\)-linear system of a pseudo-effective \(\mathbb{Q}\)-divisor \(D\) will be denoted by \(\text{Fix}(D)\). As usual, the movable part of the \(\mathbb{Q}\)-linear system of a pseudo-effective \(\mathbb{Q}\)-divisor \(D\) is denoted by \(\text{Mov}(D)\).

Notation 2.27. Consider \((X, \Delta)\) a klt pair such that \(K_X + \Delta\) is pseudo-effective, and \(A\) an ample divisor on \(X\). We will denote by \(G_\lambda \in |K_X + \Delta + \lambda A|_\mathbb{Q}\) a general element in the \(\mathbb{Q}\)-linear system. We will write \(c_\lambda\) for the log canonical threshold of the effective divisor \(G_\lambda\) with respect to the klt pair \((X, \Delta)\). Moreover, if the discussion is independent of the chosen divisor \(G_\lambda\) when this is general in its \(\mathbb{Q}\)-linear system, we will just write \(\Delta_\lambda = \Delta + c_\lambda G_\lambda\), and consider the corresponding log canonical pair \((X, \Delta_\lambda)\).

3. Adjunction to the minimal log canonical centers

In this section, we prove an adjunction formula for the \(\mathbb{Q}\)-divisor \((c + 1)(K_X + \Delta)\) to a common log canonical center of the pairs \((X, \Delta_\lambda)\).

3.1. Stabilization of the log canonical places. In this subsection, we prove that there exists a model \(Y\) over \(X\) on which the log canonical places of \((X, \Delta_\lambda)\) stabilize for \(\lambda\) small enough.

Proposition 3.1. There exist \(\lambda_1 \in \mathbb{Q}_{>0}\) and a resolution of singularities \(p: Y \to X\) such that for \(\lambda \in (0, \lambda_1)\) the following statements hold:

1. The set of log canonical places of \((X, \Delta_\lambda)\) on \(Y\) is independent of \(\lambda\),
2. the log canonical centers of \((X, \Delta_\lambda)\) on \(X\) are independent of \(\lambda\),
3. the log canonical threshold \(c_\lambda\) is the inverse of a linear function on \(\lambda\), and
4. we have an equality \(\lim_{\lambda \to 0} c_\lambda = \text{let}(K_X + \Delta)\).
Since the minimality of non-klt centers only depends on the inclusion of such subvarieties we deduce that the set of minimal log canonical centers of \((X, \Delta_0)\) is constant for \(\lambda \in \{0, 1\} \cap \mathbb{Q}\). Let \(p\colon Y \to X\) and \(q\colon Y \to X_{\min}\) be a log resolution of \(\pi\) so we can write

\[
p^*(K_X + \Delta + \lambda A) = q^*(K_{X_{\min}} + \lambda \Delta_{\min} + \lambda A_{\min}) + E_{\lambda},
\]

where \(E_{\lambda}\) is a \(q\)-exceptional effective divisor with simple normal crossing support, and its coefficients at the prime divisors of its support are independent of \(\lambda\). We claim that if \(G_{\lambda}\) is general in its \(\mathbb{Q}\)-linear system \(|K_X + \Delta + \lambda A|_{\mathbb{Q}}\) then we can write \(p^*(G_{\lambda}) = G_{\lambda, Y} + E_{\lambda}\), where \(G_{\lambda, Y}\) is a semiample \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor. Indeed, the \(\mathbb{Q}\)-divisor \(K_{\min} + \lambda \Delta_{\min} + \lambda A_{\min}\) is nef and big, therefore it is semiample by the base point free Theorem.

Hence, we can take \(G_{\lambda, Y}\) general enough, such that for any \(c \in \mathbb{R}_{>0}\) we have that

\[
J((X, \Delta), cG_{\lambda}) = J((Y, K_{Y/X}^\Delta), p^*(cG_{\lambda})) = J((Y, K_{Y/X}^\Delta), cG_{\lambda, Y} + cE_{\lambda}) = J((Y, 0), cG_{\lambda, Y} + cE_{\lambda} - K_{Y/X}^\Delta) = p_*\mathcal{O}_Y([K_{Y/X}^\Delta - cE_{\lambda} - cG_{\lambda, Y}]) = p_*\mathcal{O}_Y([K_{Y/X}^\Delta - cE_{\lambda}]).
\]

We deduce that the log canonical threshold \(c_{\lambda}\) of \(G_{\lambda}\) with respect to the klt pair \((X, \Delta)\) is the supremum of the positive real numbers \(c\), such that the simple normal crossing divisor \(K_{Y/X}^\Delta - cE_{\lambda}\) has coefficients strictly greater than negative one.

Now, we prove the third claim. Let \(E_1, \ldots, E_r\) be the irreducible components of Ex(\(p\)) which appear with non-trivial coefficient on the divisor \(E_{\lambda}\) for some \(\lambda \in (0, 1)\). Denote by \(\alpha_i\) the coefficient of \(K_{Y/X}^\Delta\) at \(E_i\), and \(\beta_i - \gamma_i\) the coefficient of \(E_{\lambda}\) at \(E_i\). By Remark 3.5, we know that \(\beta_i - \gamma_i > 0\), for every \(E_i\) with \(i \in \{1, \ldots, r\}\) and \(\lambda \in (0, 1)\). Then we can define the functions

\[
c_{\lambda, i} = \frac{\alpha_i + 1}{\beta_i - \gamma_i}
\]

and see that

\[
c_{\lambda} = \min \{c_{\lambda, i} \mid i \in \{1, \ldots, r\}\}.
\]

Observe that \(c_{\lambda}\) is the minimum between finitely many multiplicative inverses of linear functions on \(\lambda\), so we deduce that we can take \(\lambda_1 \in \mathbb{Q}_{>0}\) small enough, such that \(c_{\lambda}\) is the inverse of a linear function for \(\{0, \lambda_1\} \cap \mathbb{Q}\).

Now, we turn to prove the first claim. Consider the set \(I_\lambda = \{i \mid c_{\lambda} = c_{\lambda, i}\} \subseteq \{1, \ldots, r\}\). We know that \(I_\lambda\) is constant for \(\lambda \in (0, 1) \cap \mathbb{Q}\) so we may denote it by \(I\). Since

\[
J((X, \Delta), cG_{\lambda}) = p_*\mathcal{O}_Y([K_{Y/X}^\Delta - \lfloor cE_{\lambda}\rfloor]),
\]

we have that the log canonical places of \((X, \Delta_{\lambda})\) on \(Y\) are exactly the divisors \(E_i\) with \(i \in I\).

The second claim is a consequence of the first claim, as we have the following equality

\[
V(J((X, \Delta), c_{\lambda}G_{\lambda})) = p_* \left( \bigcup_{i \in I} E_i \right),
\]

where both sides are taken with reduced scheme structure. Finally, the fourth claim follows from the continuity of the function \(c_{\lambda}\) at the origin, and Proposition 2.23. \(\square\)

**Corollary 3.2.** The log canonical threshold \(c_{\lambda}\) is rational whenever \(\lambda \in \mathbb{R}_{>0}\) is a rational number. Moreover, \(c_{\lambda}\) is a monotone increasing function with respect to \(\lambda \in [0, 1)\).

**Remark 3.3.** Since the minimality of non-klt centers only depends on the inclusion of such subvarieties we deduce that the set of minimal log canonical centers of \((X, \Delta_{\lambda})\) on \(X\) is independent of \(\lambda \in (0, 1) \cap \mathbb{Q}\).
Remark 3.4. Observe that for every $\lambda' < \lambda$ in the interval $(0, \lambda_1)$ we have that $E_{\lambda'} \geq E_{\lambda}$. Indeed, being $A$ an ample divisor we can write $p^*(\lambda A) = q^*(\lambda A_{\min}) - \lambda F$, where $F$ is an effective divisor. Then $E_{\lambda} = E - \lambda F \geq 0$, where $E = p^*(K_X + \Delta) - q^*(K_{X_{\min}} + \Delta_{\min}) \geq 0$ is an effective $\mathbb{Q}$-divisor by the negativity lemma (see, e.g., [KM98, Lemma 3.39]). In particular, we have that $\text{Fix}(p^*(K_X + \Delta + \lambda A)) = E_{\lambda}$ for every $\lambda \in (0, \lambda_1)$.

Remark 3.5. Since the pair $(X, \Delta)$ is assumed to be klt we have that $\alpha_i + 1 > 0$ for every $i \in \{1, \ldots, r\}$. Moreover, the coefficient $\beta_i - \lambda \gamma_i$ is strictly positive for $\lambda \in (0, \lambda_1)$. Indeed, this follows from the definition of minimal model 2.18, and the strict monotonicity of discrepancies 2.15, as $c_{E_i}(X)$ is contained in the exceptional locus of $X \times \rightarrow X_{\min}$.

3.2. Common divisorially log terminal modification. In this subsection, we prove that all the log canonical pairs $(X, \Delta_X)$ share a common divisorially log terminal modification for $\lambda \in \mathbb{Q}_{>0}$ small enough. Recall that dlt modifications exists by [KK10, Theorem 3.1].

Definition 3.6. Let $(X, \Delta)$ be a log canonical pair, we say that a projective birational morphism $p_m: Y_m \rightarrow X$ is a divisorial divisorially log terminal modification of $(X, \Delta)$, or dlt modification for short, if the following conditions hold:

- $Y_m$ is $\mathbb{Q}$-factorial,
- $p_m$ only extracts divisors with log discrepancy $-1$ with respect to $(X, \Delta)$, and
- if $E$ is the sum of the irreducible $p_m$-exceptional divisors and $\Delta_{Y_m}$ is the strict transform of $\Delta$ on $Y_m$, then $(Y_m, \Delta_{Y_m} + E)$ is divisorially log terminal and $K_{Y_m} + \Delta_{Y_m} + E = p_m^*(K_X + \Delta)$.

The following lemma follows from the proof of the existence of dlt modifications for log canonical pairs. For the sake of completeness, we will give a proof of the statement which is not in [KK10, Theorem 3.1].

Lemma 3.7. Let $(X, \Delta)$ be a log canonical pair and $p: Y \rightarrow X$ a log resolution which is composite of blow-ups of centers of codimension at least two. Then, there exists a dlt modification $p_m: Y_m \rightarrow X$ of the pair $(X, \Delta)$ such that the exceptional locus of the rational map $\pi_m: Y \rightarrow Y_m$ is contained in the union of the prime divisors $E \subset Y$ which are exceptional over $X$ and for which $a_E(X, \Delta) > -1$.

Proof. We can write

$$K_Y + \Delta_Y + E_Y^+ + E_Y^0 - E_Y^- = p^*(K_X + \Delta),$$

where $E_Y^+$ denotes the sum of all $p$-exceptional divisors with discrepancy equal to negative one, $E_Y^0$ denotes the sum of all $p$-exceptional divisors with discrepancy in the interval $(-1, 0]$ and $E_Y^-$ is the sum of all $p$-exceptional divisors with discrepancy greater than zero. We will denote by $E_Y^{\text{cr}}$ the sum of the reduced prime divisors on $Y$ with discrepancy zero over $(X, \Delta)$, or equivalently, the components of the support of $E_Y^0$ which appears with coefficient zero in the sum.

Since $p$ is composite of blow-ups of centers of codimension at least two we know that there exists a $p$-exceptional effective divisor $C$ such that $-C$ is $p$-ample. Let $H$ be a sufficiently ample divisor on $X$ such that $-C + p^*(H)$ is ample on $Y$ and the divisor $H' \sim \mathbb{Q} -C + p^*(H)$ intersects $\Delta_Y + E_Y^+ + E_Y^0 + E_Y^{\text{cr}} + E_Y^-$ transversally. Therefore, taking parameters $\epsilon_1, \epsilon_2$ and $\epsilon_3$ in $\mathbb{Q}_{>0}$ which are small enough, we have that the following pair is dlt:

$$\tag{3.1} \left( Y, \Delta_Y + E_Y^+ + (1 + \epsilon_1)E_Y^0 + \epsilon_2E_Y^{\text{cr}} + \epsilon_3H' \right).$$

If we assume that $0 < \epsilon_3 \ll \epsilon_1 \ll 1$ and $0 < \epsilon_3 \ll \epsilon_2 \ll 1$ we can run a minimal model for the pair $(3.1)$ with scaling of an ample divisor over $X$, so we obtain a minimal model $Y_m$ over $X$ which is a dlt modification of $(X, \Delta)$ (see, e.g., [KK10, Theorem 3.1] for the details). Observe that we have

$$K_Y + \Delta_Y + E_Y^+ + (1 + \epsilon_1)E_Y^0 + \epsilon_2E_Y^{\text{cr}} + \epsilon_3H' = \epsilon_1E_Y^+ + \epsilon_2E_Y^{\text{cr}} + E_Y^- + \epsilon_3H' + p^*(K_X + \Delta),$$

where $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ are small enough.
Thus, we get that  
\[ K_Y + \Delta_Y + E_Y^+ + (1 + \epsilon_1)E_Y^0 + \epsilon_2 E_Y^{Cr} + \epsilon_3 H' \sim_{X,Q} \epsilon_1 E_Y^0 + \epsilon_2 E_Y^{Cr} + E_Y^- + \epsilon_3 H'. \]

So the diminished base locus over \( X \) of the \( \mathbb{Q} \)-divisor  
\[ K_Y + \Delta_Y + E_Y^+ + (1 + \epsilon_1)E_Y^0 + \epsilon_2 E_Y^{Cr} + \epsilon_3 H' \]
is contained in the union of the support of the divisors \( E_Y^0, E_Y^{Cr} \) and \( E_Y^- \). Therefore, we conclude that the flipping locus or exceptional divisor of every step of the minimal model program \( \pi_m : Y \to Y_m \) is contained in the strict transform of the union of support of \( E_Y^0, E_Y^{Cr}, \) and \( E_Y^- \). In particular, the exceptional locus of \( \pi_m : Y \to Y_m \) is contained in this locus. \( \square \)

**Lemma 3.8.** The coefficient of the divisor \( p^*(K_X + \Delta_\lambda) \) at any prime divisor on \( Y \), which is exceptional over \( X \), is a monotone function with respect to \( \lambda \) around the origin.

**Proof.** Since \( G_\lambda \) is general in its \( \mathbb{Q} \)-linear system we have that  
\[ \text{Fix}(p^*(K_X + \Delta_\lambda)) = \text{Fix}(p^*(K_X + \Delta + c_\lambda G_\lambda)) = \text{Fix}(p^*(K_X + \Delta + c_\lambda(K_X + \Delta + \lambda A))) = (c_\lambda + 1) \text{Fix}
\left(p^*(K_X + \Delta + \left(\frac{c_\lambda}{c_\lambda + 1}\right) \lambda A)\right) = (c_\lambda + 1)E_{s(\lambda)}, \]
where \( E_{s(\lambda)} \) is defined as in Remark 3.4 and  
\[ s(\lambda) = \left(\frac{c_\lambda}{c_\lambda + 1}\right) \lambda, \]
is a monotone increasing function for \( \lambda \in [0, \lambda_1) \). Indeed, we can compute  
\[ s(\lambda) = \left(\frac{\alpha_i + 1}{\alpha_i + 1 + \beta_i - \lambda \gamma_i}\right) \lambda, \]
where \( \alpha_i + 1 > 0 \) and \( \lambda \mapsto \beta_i - \lambda \gamma_i \) is a positive function which is monotone increasing with respect to \( \lambda \). So we have that  
\[ p^*(K_X + \Delta_\lambda) = \text{Mov}(p^*(K_X + \Delta_\lambda)) + \text{Fix}(p^*(K_X + \Delta_\lambda)) = \text{Mov}(p^*(K_X + \Delta_\lambda)) + (c_\lambda + 1)E_{s(\lambda)} \]
By Remark 3.4, we deduce that  
\[ (c_\lambda + 1)E_{s(\lambda)} = (c_\lambda + 1)(E - s(\lambda)F) = c_\lambda(E - \lambda F) + E, \]
where \( E \) and \( F \) are the effective divisors defined in Remark 3.4. Pick \( E_j \) to be an irreducible divisor on \( Y \) which is exceptional over \( X \). Denote by \( \beta_j \) the coefficient of \( E \) at \( E_j \) and by \( \gamma_j \) the coefficient of \( F \) at \( E_j \). Then, the coefficient of \( p^*(K_X + \Delta_\lambda) \) at the prime divisor \( E_j \) equals  
\[ (\alpha_i + 1) \left(\frac{\beta_j - \lambda \gamma_i}{\beta_i - \lambda \gamma_i}\right) + \beta_j, \]
Recall from Remark 3.5 that \( \beta_i - \lambda \gamma_i > 0 \) for \( \lambda \in (0, \lambda_1) \), so the above function is monotone around the origin. \( \square \)

**Remark 3.9.** From now on, we will assume that \( \lambda_1 \in \mathbb{Q}_{>0} \) is small enough such that the coefficient of the divisor \( p^*(K_X + \Delta_\lambda) \) at any prime divisor on \( Y \) which is exceptional over \( X \) is a monotone function with respect to \( \lambda \in [0, \lambda_1) \).
Remark 3.10. Observe that the prime divisor $E_j$ have coefficient zero in $p^*(X + \Delta_\lambda)$ for some fixed $\lambda_0 \in (0, \lambda_1)$ if and only if $\beta_j = 0$ and $\beta_j - \lambda_0 \gamma_j = 0$, which implies that $\gamma_j = 0$. Assume that $\beta_j = \gamma_j = 0$ for some $E_j$. Since $\gamma_j = 0$ we know that the center of $E_j$ on $X$ is not contained in the exceptional locus of the birational map $X \dasharrow X_{\min}$. Moreover, $\beta_j = 0$ implies that $E_j$ is a crepant divisorial valuation of the pair $(X, \Delta)$. Since $G_\lambda$ is general in its $\mathbb{Q}$-linear system we deduce that $E_j$ is a crepant divisorial valuation of the log canonical pair $(X, \Delta)$ for every $\lambda \in (0, \lambda_1)$.

Proposition 3.11. There exists $\lambda_2 \in (0, \lambda_1)$ and a projective birational morphism $p_m: Y_m \to X$ such that $p_m$ is a $\mathbb{Q}$-factorial dlt modification for every log canonical pair $(X, \Delta)$ with $\lambda \in (0, \lambda_2) \cap \mathbb{Q}$.

Proof. Consider $p: Y \to X$ as in the proof of Proposition 3.1. Since all the klt pairs $(K_X + \Delta + \lambda A)$ with $\lambda \in (0, \lambda_1)$ have the same minimal model, we may assume that the support of the fixed component of $|p^*(K_X + \Delta + \lambda A)|_\mathbb{Q}$ is independent of $\lambda \in (0, \lambda_1) \cap \mathbb{Q}$. Write

$$K_Y + \Delta_{\lambda, Y} + E_{\lambda, Y} = p^*(K_X + \Delta),$$

where $\Delta_{\lambda, Y}$ is the strict transform of $\Delta_\lambda$ on $Y$ and $E_{\lambda, Y}$ is a $p$-exceptional divisor. We consider the following decomposition

$$E_{\lambda, Y} = E_{\lambda, Y}^+ + E_{\lambda, Y}^0 - E_{\lambda, Y}^-,$$

where $E_{\lambda, Y}^+$ is supported on the sum of all $p$-exceptional divisors with discrepancy $-1$, the $\mathbb{Q}$-divisor $E_{\lambda, Y}^0$ is the sum of all $p$-exceptional divisors with discrepancy greater than $-1$ and less than or equal to $0$, and $E_{\lambda, Y}^-$ is the sum of all $p$-exceptional divisors with positive discrepancy. By Lemma 3.8, we can pick $\lambda_2 \in (0, \lambda_1) \cap \mathbb{Q}$ small enough such that the support of the divisors $E_{\lambda, Y}^+, E_{\lambda, Y}^0$, and $E_{\lambda, Y}^-$ are independent of $\lambda \in (0, \lambda_2)$.

Remark 3.10, we know that the set of crepant valuations over $(X, \Delta)$ is independent of $\lambda \in (0, \lambda_1)$ so we will denote the sum of such divisors on $Y$ by $E^\mathbb{Q}_Y$.

We claim that a dlt modification of $(X, \Delta_\lambda)$ is a dlt modification of every log canonical pair $(X, \Delta)$ with $\lambda \in (0, \lambda_2) \cap \mathbb{Q}$. Indeed, let $p_m: Y_m \to X$ be a $\mathbb{Q}$-factorial dlt modification of $(X, \Delta_\lambda)$. Since $Y_m$ is $\mathbb{Q}$-factorial then it suffices to check the second and third conditions of Definition 3.6.

First, we check that $p_m$ only extracts divisors with log discrepancy equal to zero for the pairs $(X, \Delta)$.

By the construction of dlt modifications any divisor extracted by $p_m$ is also extracted by $p$. If a divisor $E$ is extracted by $p_m$ then its log discrepancy with respect to the pair $(X, \Delta_\lambda)$ is zero, so its strict transform on $Y$ is a component of $E^\mathbb{Q}_{\lambda, Y}$, then it is a component of $E_{\lambda, Y}^+$ for any $\lambda \in (0, \lambda_2) \cap \mathbb{Q}$, which means that the log discrepancy of $E$ with respect to $(X, \Delta)$ is zero for every $\lambda \in (0, \lambda_2) \cap \mathbb{Q}$.

Now, it suffices to check that $(Y_m, \Delta_{\lambda, Y_m} + E_{\lambda, Y_m})$ is a dlt pair for every $\lambda \in (0, \lambda_2) \cap \mathbb{Q}$. Observe that the coefficients of $\Delta_{\lambda, Y_m} + E_{\lambda, Y_m}$ are contained in the interval $[0, 1]$. Indeed, all the irreducible components of the divisor $E^0_{\lambda, Y_m}, E^0_{Y_m}$, and $E^-_{\lambda, Y}$ are contracted by the rational map $\pi_m: Y \dasharrow Y_m$ since the coefficients of $\Delta_{\lambda_2, Y_m} + E_{\lambda_2, Y_m}$ are contained in the interval $[0, 1]$. So it suffices to prove that there exists a log resolution $q_Z: Z \to Y$ for the pairs $(Y_m, \Delta_{\lambda, Y_m} + E_{\lambda, Y_m})$ such that the discrepancies of any prime $q_Z$-exceptional divisor with respect to $(Y_m, \Delta_{\lambda, Y_m} + E_{\lambda, Y_m})$ is strictly greater than negative one for every $\lambda \in (0, \lambda_2)$. Indeed, let $p_Z: Z \to Y$ and $q_Z: Z \to Y$ be two projective morphisms that give a log resolution of the minimal model program $\pi_m: Y \dasharrow Y_m$. By Lemma 3.7, we know that the indeterminacy locus of the rational map $\pi_m: Y \dasharrow Y_m$ is contained in the support of the divisor $E^0_{\lambda_2, Y} + E^0_{\lambda_2, Y} + E^-_{\lambda_2, Y}$, so it is contained in the support of the divisor

$$E^0_{\lambda, Y} + E^0_{Y} + E^-_{\lambda, Y},$$

for arbitrary $\lambda \in (0, \lambda_2]$. Thus, we can obtain $Z$ by composite of blow-ups along centers contained in the support of $E^0_{\lambda, Y} + E^0_{Y} + E^-_{\lambda, Y}$. Therefore, using the formula to compute discrepancies over log smooth pairs [KM98, Lemma 2.29] we conclude that any prime divisor on $Z$ which is $q_Z$-exceptional, has positive log discrepancy with respect to $(Y_m, \Delta_{\lambda, Y_m} + E_{\lambda, Y_m})$. \qed
3.3. Divisorial adjunction on the dlt model. In this subsection, we use divisorial adjunction of dlt pairs (see, e.g., [Kol92] and [HK10, Theorem 3.24]) to the log canonical places of \((X, \Delta)\) on \(Y\) which are exceptional divisors over \(X\). We decompose the different divisor which is the divisor induced by the adjunction formula, into a fixed part and a semiample part. From now on, we will always assume that \(\lambda \in (0, \lambda_2) \cap \mathbb{Q}\), where \(\lambda_2\) is constructed in the proof of Proposition 3.11 unless otherwise stated.

Notation 3.12. Observe that we have a commutative diagram
\[
\begin{array}{ccc}
X & \overset{\pi}{\longrightarrow} & Y \\
| & q & | \\
| & | \downarrow q_m \\
| & \downarrow p_m & | \\
Y_m & \overset{\pi_m}{\longrightarrow} & X_{\min}
\end{array}
\]
such that \(p: Y \to X\) and \(q: Y \to X_{\min}\) give a log resolution of the minimal model program \(\pi: X \to X_{\min}\), and \(p_m: Y_m \to X\) is a divisorially log terminal modification of the log canonical pairs \((X, \Delta)\) for every \(\lambda \in (0, \lambda_2) \cap \mathbb{Q}\). Moreover, the rational map \(q_m: Y_m \to X_{\min}\) is defined outside a subvariety of codimension two.

We denote by \(G_1, \ldots, G_k\) the prime divisors of \(X\) which are contracted in the minimal model program \(\pi: X \to X_{\min}\). We claim that for \(\lambda \in \mathbb{Q}_{>0}\) small enough a general divisor \(G_\lambda\) in the \(\mathbb{Q}\)-linear system \(|K_X + \Delta + \lambda A|_{\mathbb{Q}}\) can be written as
\[
G_\lambda = \sum_{i=0}^k G_{\lambda,i},
\]
where \(G_{\lambda,i}\) are divisors supported on the prime divisors \(G_i\) for every \(i \in \{1, \ldots, k\}\) and the strict transform of \(G_{\lambda,0}\) on \(Y\) is a semiample \(\mathbb{Q}\)-divisor. Indeed, for \(\lambda \in (0, \lambda_2) \cap \mathbb{Q}\) we have that the \(\mathbb{Q}\)-divisor \(G_{\lambda,0,X_{\min}}\) belongs to the \(\mathbb{Q}\)-linear system \(|K_{X_{\min}} + \Delta_{\min} + \lambda A_{\min}|_{\mathbb{Q}}\) which is nef and big, therefore \(G_{\lambda,0,X_{\min}}\) is a semiample \(\mathbb{Q}\)-divisor by the base point free Theorem (see, e.g., [HK10, Theorem 5.1]). Thus, for \(\lambda \in (0, \lambda_2) \cap \mathbb{Q}\) we have that the \(\mathbb{Q}\)-divisor \(q^*(G_{\lambda,0,X_{\min}})\) equals \(G_{\lambda,0,Y}\), where \(G_{\lambda,0,Y}\) is the strict transform of \(G_{\lambda,0}\) on \(Y\).

Notation 3.13. Let \(p_m: Y_m \to X\) be the common dlt modification of the log canonical pairs \((X, \Delta)\) constructed in Proposition 3.11. In the proof of Proposition 3.1 we denoted by \(\{E_i \mid i \in \mathcal{I}\}\) the set log canonical places of the pair \((X, \Delta)\) on \(Y\). Observe that such log canonical places may be exceptional divisors over \(X\) or non-exceptional divisors over \(X\). We will denote by \(\mathcal{I}' \subseteq \mathcal{I}\) the set of log canonical places on \(Y_m\) which are exceptional divisors over \(X\). For every \(i \in \mathcal{I}'\) we will denote the prime divisor \(E_{i,Y_m}\) by \(E_i\) in order to abbreviate the notation. Thus, we can write
\[
(3.2) \quad K_{Y_m} + \Delta_{Y_m} + c_\lambda G_{\lambda,Y_m} + \sum_{i \in \mathcal{I}'} E_i = p_m^*(K_X + \Delta + c_\lambda G_\lambda),
\]
where the subscript \(Y_m\) on a divisor denotes its strict transform on the model \(Y_m\). Using divisorial adjunction for the \(\mathbb{Q}\)-divisor \((3.2)\) to the prime divisor \(E_i\) we can write
\[
p_m^*(K_X + \Delta + c_\lambda G_\lambda) |_{E_i} = K_{E_i} + \Phi_{E_i}^{G_\lambda},
\]
where the pair \((E_i, \Phi_{E_i}^{G_\lambda})\) is dlt. The divisor \(\Phi_{E_i}^{G_\lambda}\) is called the different. We use the superscript \(G_\lambda\) in the notation of the different to make explicit that it depends on the choice of \(G_\lambda\) in its \(\mathbb{Q}\)-linear system, even if the latter is chosen to be general in its \(\mathbb{Q}\)-linear system.

Proposition 3.14. Given \(i \in \mathcal{I}'\), there exists \(\lambda_3 \in (0, \lambda_2)\) and a projective birational morphism \(q_Z: E_{i,Z} \to E_i\) over \(W_i\), such that for every \(\lambda \in (0, \lambda_3) \cap \mathbb{Q}\) we can write
\[
K_{E_{i,Z}} + F_{E_{i,Z}} + N_{E_{i,Z}}^Z \sim q_Z^*(K_{E_i} + \Phi_{E_i}^{G_\lambda}),
\]
where the following statements hold:

- The $\mathbb{Q}$-divisor $F_{E_i,Z}^\lambda$ is independent of the choice of $G_\lambda$ in its $\mathbb{Q}$-linear system,
- the support of $F_{E_i,Z}^\lambda$ is independent of $\lambda$, and
- the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $N_{E_i,Z}^{G_\lambda}$ is semiample.

**Proof.** Consider the following commutative diagram

$$
\begin{array}{ccc}
Z & \xymatrix{ & Y \ar[ld]_{p_Z} \ar[rd]^{q_Z} & } & Y_m \\
X & \xymatrix{ & Y_m \ar[rd]^{q_m} & } & \\
& p & \ar@{..}[rrr] & & \lambda & y_{\min} &
\end{array}
$$

where the bottom square is the one introduced in 3.12. The projective birational morphisms $p_Z : Z \to Y$ and $q_Z : Z \to Y_m$ give a log resolution of the minimal model program $\pi_m : Y \dasharrow Y_m$. Thus, we can write

$$
p_Z'(G_{\lambda,0,Y}) = q_Z'(G_{\lambda,0,Y_m}) + D_\lambda,
$$

where $D_\lambda$ is a $q_\lambda$-exceptional anti-effective divisor with coefficients that vary continuously with respect to $\lambda$. Since $p_Z'(G_{\lambda,0,Y})$ is semiample we can take $\lambda_3 \in (0, \lambda_2)$ such that the support of the $\mathbb{Q}$-divisor $D_\lambda$ is independent of $\lambda \in (0, \lambda_3) \cap \mathbb{Q}$. Indeed, we can write

$$
-D_\lambda = \text{Fix}((q \circ q_\lambda)^*G_{\lambda,0})) = \text{Fix}((q \circ q_\lambda)^*(\text{Mov}(K_X + \Delta + \lambda A))),
$$

so the coefficients of $-D_\lambda$ are monotone with respect to $\lambda$ around the origin. We denote by $E_i$ the strict transform of the prime divisor $E_i$ on $Z$ and by abuse of notation we write $q_Z : E_i \to Y_i$ for the restriction of the morphism $q_Z$ to $E_i$. We define $N_{E_i,Z}^{G_\lambda}$ to be the restriction of the semiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $G_{\lambda,0,Z}$ to $E_i$, then we can write

$$
K_{E_i} + F_{E_i}^{G_\lambda} + N_{E_i,Z}^{G_\lambda} = q_Z^*(K_{E_i} + \Phi_{E_i}^{G_\lambda}),
$$

where $F_{E_i}^{G_\lambda}$ is a $\mathbb{Q}$-divisor which is supported on

$$
(3.3) \quad \text{supp}(\Delta_Z) \cup \text{supp}(E^0_{E_i,Z}) \cup \text{supp}(E^-_{E_i,Z}) \cup \text{supp} \left( \sum_{i=1}^k G_{\lambda,i,Z} \right) \cup \text{supp} \left( \sum_{j \neq 1} E_{j,Z} \right) \cap E_{i,Z},
$$

and the divisors $E_{0,i,Z}^0$ and $E_{i,Z}^-$ are defined as in the proof of Proposition 3.11. By Lemma 3.8 and Remark 3.9, we can assume that these two divisors have support independent of $\lambda \in (0, \lambda_2)$. Clearly, the other divisors in the locus 3.3 have support independent of $\lambda \in (0, \lambda_2)$. We deduce that the locus 3.3 is independent of $\lambda$, therefore the support of $F_{E_i}^{G_\lambda}$ is independent of $\lambda \in (0, \lambda_2)$. Finally, we need to argue that $F_{E_i}^{G_\lambda}$ is independent of $G_\lambda$ when the latter is general in its $\mathbb{Q}$-linear system. Indeed, if we choose $G_\lambda$ general in its $\mathbb{Q}$-linear system such that $G_{\lambda,0,Z}$ intersects the locus (3.3) transversally, then the divisor $F_{E_i,Z}^{G_\lambda}$ only depends on the fixed component of the $\mathbb{Q}$-linear system $|G_{\lambda,Z}|_\mathbb{Q}$. \qed

### 3.4. Ambro’s canonical bundle formula

In this subsection, we will use Ambro’s canonical bundle formula (see, e.g., [FG12]) to define a discriminant divisor and a moduli divisor on $W_i$, where $W_i$ is a common minimal non-klt center of the log canonical pairs $(X, \Delta_\lambda)$ of codimension at least two. In particular, $W_i$ is the image of $E_i$ on $X$ for some $i \in \mathcal{T}$. The aim of this subsection is to prove that for every pair $(X, \Delta_\lambda)$ we can construct a discriminant divisor on $W_i$ which is independent of the choice of $G_\lambda$ in its $\mathbb{Q}$-linear system.
and its support is independent of $\lambda$. By abuse of notation, we will denote by $p_m:\mathcal{E}_i \to W_i$ the restriction of the morphism $p_m: Y_m \to X$ to the prime divisor $\mathcal{E}_i$.

**Construction 3.15.** We recall the construction of the *boundary divisor* (see, e.g., [Kol13]). Consider a projective birational morphism $p_m: \mathcal{E}_i \to W_i$ of normal quasi-projective varieties and a $\mathbb{Q}$-divisor $F$ on $\mathcal{E}_i$ such that $(\mathcal{E}_i, F)$ is a sub-pair which is sub-log canonical near the generic fiber of $p_m$. We can define a *boundary divisor* as follows: Given a prime divisor $C \subseteq W_i$, we define the real number

$$
\mu_C (F) = \sup \{ t \in \mathbb{R} \mid (\mathcal{E}_i, F + tp_m(C)) \text{ is sub-log canonical over a neighbourhood of } \eta_C \},
$$

where $\eta_C$ is the generic point of $C$. Observe that the pull-back $p_m^*(C)$ is well-defined over a neighbourhood of $\eta_C$ since $W_i$ is normal. Then we can define the $\mathbb{Q}$-divisor

$$
B_W(F) = \sum_{C \subseteq W_i} (1 - \mu_C (F)) C,
$$

where the sum runs over all the prime divisors $C$ of $W_i$. By [Cor07, Section 8.2], the above sum is finite.

**Notation 3.16.** In 3.2, we constructed a dlt pair $(\mathcal{E}_i, \Phi_{\mathcal{E}_i}^{G})$ for any $i \in I'$ and $\lambda \in (0, \lambda_3) \cap \mathbb{Q}$. Moreover, by construction the divisor $K_{\mathcal{E}_i} + \Phi_{\mathcal{E}_i}^{G}$ is relatively trivial over the base $W_i$. Therefore, using Ambro’s canonical bundle formula we can write

$$
K_{\mathcal{E}_i} + \Phi_{\mathcal{E}_i}^{G} \sim_{\mathbb{Q}} p_m^* (K_{W_i} + B_{W_i}(\Phi_{\mathcal{E}_i}^{G}) + M_{W_i}(\Phi_{\mathcal{E}_i}^{G})),
$$

where $B_{W_i}(\Phi_{\mathcal{E}_i}^{G})$ is called the *boundary divisor* or *discriminant divisor* and $M_{W_i}(\Phi_{\mathcal{E}_i}^{G})$ is called the *moduli divisor* or *j-divisor*. Here, the boundary divisor $B_{W_i}(\Phi_{\mathcal{E}_i}^{G})$ is the one constructed in 3.15 for the pair $(\mathcal{E}_i, \Phi_{\mathcal{E}_i}^{G})$ and the morphism $p_m:\mathcal{E}_i \to W_i$. By construction, the pair $(W_i, B_{W_i}(\Phi_{\mathcal{E}_i}^{G}))$ is log canonical whenever the moduli divisor is $\mathbb{Q}$-Cartier and $W_i$ is normal (see, e.g., [Cor07, Remark 8.6.2]). Observe that both divisors $M_{W_i}(\Phi_{\mathcal{E}_i}^{G})$ and $B_{W_i}(\Phi_{\mathcal{E}_i}^{G})$ depend on the choice of $G_\lambda$ in its $\mathbb{Q}$-linear system. By 3.15, we know that the boundary divisor is uniquely determined by the pair $(\mathcal{E}_i, \Phi_{\mathcal{E}_i}^{G})$. However, the moduli part is only defined up to $\mathbb{Q}$-linear equivalence and the $\mathbb{Q}$-divisor itself depends on the starting choice of a $\mathbb{Q}$-divisor $L$ on $W_i$ such that

$$
K_{\mathcal{E}_i} + \Phi_{\mathcal{E}_i}^{G} \sim_{\mathbb{Q}} p_m^* (L).
$$

**Proposition 3.17.** Let $W_i$ be a common minimal non-klt center of $(X, \Delta_\lambda)$ of codimension at least two for $\lambda \in (0, \lambda_3) \cap \mathbb{Q}$. Then we can write

$$
(K_{X} + \Delta_\lambda)|_{W_i} \sim_{\mathbb{Q}} K_{W_i} + B_{W_i}^{\lambda} + M_{W_i}^{\lambda},
$$

where the following statements hold:

- The effective $\mathbb{Q}$-divisor $B_{W_i}^{\lambda}$ is independent of the choice of $G_\lambda$ in its $\mathbb{Q}$-linear system, and
- the support of $B_{W_i}^{\lambda}$ is independent of $\lambda$.

Moreover, if $K_{W_i} + B_{W_i}^{\lambda}$ is $\mathbb{Q}$-Cartier we have that:

- The pair $(W_i, B_{W_i}^{\lambda})$ is log canonical, and
- The $\mathbb{Q}$-divisor $M_{W_i}^{\lambda}$ is nef.

**Proof.** By Proposition 3.14, for every such $W_i$ we have that $i \in I'$ and there is a commutative diagram
such that
\[ K_{\xi, z} + F^\lambda_{\xi, z} + N^G_{\xi, z} \sim_q q_Z^* (K_{\xi} + \Phi^G_{\xi}), \]
where \( F^\lambda_{\xi, z} \) has support independent of \( \lambda \) and \( N^G_{\xi, z} \) is a semiample \( \mathbb{Q} \)-divisor. Consider \( \pi_i : V_i \to W_i \) to be a projective generically finite morphism from a smooth variety \( V_i \) which factors through the semistable reduction in codimension one for both morphisms \( p_{m, z} \) and \( p_{m} \) (see, e.g., [KKMSD73]). Taking the base change of the above diagram we obtain the following commutative diagram
\[ \begin{array}{ccc} F_{i, z} & \xrightarrow{q_Z} & F_i \\ p_{m, z} \downarrow & & \downarrow p_m \\ V_i & & \to W_i \end{array} \]
where by abuse of notation we use the same symbols for \( p_{m, z}, q_Z \) and \( p_m \) and the corresponding morphisms induced by the base change. Thus, the morphisms of \( p_{m, z} : F_{i, z} \to V_i \) and \( p_m : F_i \to V_i \) have slc fibers in codimension one. We denote by \( N^G_{F_{i, z}} \) the pull-back of the \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( N^G_{\xi, z} \) to \( F_{i, z} \) and by
\[ K_{F_{i, z}} + F^\lambda_{F_{i, z}} \]
the pull-back of the \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( K_{\xi, z} + F^\lambda_{\xi, z} \) to \( F_{i, z} \). Observe that the properties of the divisors \( F^\lambda_{\xi, z} \) and \( N^G_{\xi, z} \) are preserved, meaning that the divisor \( N^G_{\xi, z} \) is a semiample \( \mathbb{Q} \)-divisor and \( F^\lambda_{\xi, z} \) is a \( \mathbb{Q} \)-divisor which is independent of \( G_\lambda \) in its \( \mathbb{Q} \)-linear system and its support is independent of \( \lambda \). Let \( L \) be a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor on \( W_i \) such that
\[ (K_X + \Delta + c_\lambda G_\lambda)|_{W_i} \sim_q L. \]
We will apply Ambro’s canonical bundle formula on \( V_i \) with respect to the \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( \pi_i^*(L) \). By [Amb04, Lemma 2.4], we have the following equality
\[ (3.5) \]
for every \( \mathbb{Q} \)-divisor \( G_\lambda \). We claim that there exists a finite set \( J \) and \( \mathbb{Q} \)-divisors \( G^j_\lambda \) in the \( \mathbb{Q} \)-linear system \( |K_X + \Delta + \lambda A|_\mathbb{Q} \) with \( j \in J \), such that the following equality of \( \mathbb{R} \)-divisors holds
\[ (3.5) \]
Indeed, for every prime divisor \( C \subset V_i \) we can choose \( G^j_\lambda \) general in the \( \mathbb{Q} \)-linear system \( |K_X + \Delta + \lambda A|_\mathbb{Q} \) so that \( N^G_{F_{i, z}} \) intersects the fibers of \( p_{m, z} \) transversally over a neighbourhood of the generic point of \( C \). Therefore, we obtain the following equality of coefficients
\[ \mu_C(F^\lambda_{F_{i, z}}) = \mu_C \left(F^\lambda_{F_{i, z}} + N^G_{F_{i, z}} \right). \]
Since the support of the \( \mathbb{R} \)-divisor
\[ B_{V_i}(F^\lambda_{F_{i, z}} + N^G_{F_{i, z}}) \]
contains finitely many prime divisors for any \( G^j_\lambda \) we conclude that we may take \( J \) to be finite. Putting equation (3.4) and equation (3.5) together we have that
\[ B^\lambda_{V_i} = B_{V_i}(F^\lambda_{F_{i, z}}) \]
contains finitely many prime divisors for any \( G^j_\lambda \) we conclude that we may take \( J \) to be finite. Putting equation (3.4) and equation (3.5) together we have that
\[ B^\lambda_{V_i} = B_{V_i}(F^\lambda_{F_{i, z}}) \]
\[ = \bigwedge_{j \in J} B_{V_i}(N^G_{F_{i, z}} + F^\lambda_{F_{i, z}}) = \bigwedge_{j \in J} B_{V_i}(\Phi^G_{F_{i, z}}) \]
is a $\mathbb{R}$-divisor which is independent of the choice of $G_\lambda$ in its linear system and its support is independent of $\lambda$. Since the pairs $(\mathcal{E}_i, \Phi_{\mathcal{E}_i}^{G_\lambda})$ are dlt for every $\mathbb{Q}$-divisor $G_\lambda$ which is general in its $\mathbb{Q}$-linear system we deduce that the divisors

$$B_{W_i}(\Phi_{\mathcal{E}_i}^{G_\lambda}) = \frac{1}{\deg(\pi_i)}\pi_i(B_{V_i}(\Phi_{\mathcal{E}_i}^{G_\lambda})) \geq 0$$

are effective $\mathbb{Q}$-divisors for any $\lambda \in (0, \lambda_3) \cap \mathbb{Q}$, and therefore

$$B_{W_i}^\lambda = \bigwedge_{j \in J} B_{W_i}(\Phi_{\mathcal{E}_i}^{G_\lambda}) = \frac{1}{\deg(\pi_i)}\pi_i(B_{V_i}^\lambda) \geq 0$$

is an effective $\mathbb{Q}$-divisor which is independent of $G_\lambda$ in its $\mathbb{Q}$-linear system and its support is independent of $\lambda$.

From now on, we assume that $K_{W_i} + B_{W_i}^\lambda$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. By construction, the pair

$$(W_i, B_{W_i}(\Phi_{\mathcal{E}_i}^{G_\lambda})),$$

is log canonical for every $G_\lambda$ general in its $\mathbb{Q}$-linear system, so we have that $(W_i, B_{W_i}^\lambda)$ is log canonical as well. Finally, we claim that the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor

$$M_{V_i}^\lambda = L - K_{V_i} - B_{V_i}^\lambda$$

is nef. Observe that for every $G_\lambda$ general in its $\mathbb{Q}$-linear system the $\mathbb{Q}$-divisor $M_{V_i}^\lambda(\Phi_{\mathcal{E}_i}^{G_\lambda})$ is nef since we have that $\dim(X) = 4$ and $\dim(W_i) \in \{1, 2\}$, so either $\dim(V_i) = 1$ or $\dim(F_i) - \dim(V_i) = 1$ (see, e.g., [Amb04]).

Let $C \subseteq W_i$ be a prime divisor, then by construction we can choose $C_i$ such that the effective $\mathbb{Q}$-divisor

$$B_{V_i}(F_{\mathcal{E}_i}^\lambda + N_{\mathcal{E}_i}^{G_\lambda}) - B_{V_i}^\lambda$$

does not contain $C$ in its support. Moreover, since the $\mathbb{Q}$-divisor

$$M_{V_i}(\Phi_{\mathcal{E}_i}^{G_\lambda}) = M_{V_i}(F_{\mathcal{E}_i}^\lambda + N_{\mathcal{E}_i}^{G_\lambda})$$

is nef, the following inequalities hold

$$M_{V_i}^\lambda \cdot C = (L - K_{V_i} - B_{V_i}(F_{\mathcal{E}_i}^\lambda + N_{\mathcal{E}_i}^{G_\lambda})) \cdot C + (B_{V_i}(F_{\mathcal{E}_i}^\lambda + N_{\mathcal{E}_i}^{G_\lambda}) - B_{V_i}^\lambda) \cdot C = M_{V_i}(\Phi_{\mathcal{E}_i}^{G_\lambda}) \cdot C + (B_{V_i}(F_{\mathcal{E}_i}^\lambda + N_{\mathcal{E}_i}^{G_\lambda}) - B_{V_i}^\lambda) \cdot C \geq 0,$$

proving the claim. Moreover, since $M_{V_i}^\lambda = \pi_i(M_{V_i}^\lambda)$ and $\dim(W_i) \leq 2$, we deduce that $M_{V_i}^\lambda$ is nef. \(\square\)

**Remark 3.18.** From the proof of Proposition 3.17 we can see that the divisor $B_{W_i}^\lambda$ is uniquely determined by the dlt pair $(\mathcal{E}_i, \Phi_{\mathcal{E}_i}^{G_\lambda})$, but the moduli part $M_{W_i}^\lambda$ is only defined up to $\mathbb{Q}$-linear equivalence and different choices of the $\mathbb{Q}$-divisor $L$ induce different moduli divisors.

**Remark 3.19.** By Lemma 3.8, we know that the coefficients of the irreducible components of the fixed divisor of $p^*(K_X + \Delta_\lambda)$ are monotone functions with respect to $\lambda \in [0, \lambda_3)$. In particular, the coefficients of the irreducible components of the fixed divisor of $(p \circ p_Z)^*(K_X + \Delta_\lambda)$ and $p_m^*(K_X + \Delta_\lambda)$ are monotone functions with respect to $\lambda \in [0, \lambda_3)$ as well. We denote by $N_{\mathcal{E}_i}^{G_\lambda}$ the push-forward of the semiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $N_{\mathcal{E}_i}^{G_\lambda}$ by the birational morphism $q_Z$. Therefore, we have an adjunction formula

$$K_{\mathcal{E}_i} + F_{\mathcal{E}_i}^\lambda + N_{\mathcal{E}_i}^{G_\lambda} \sim q_Z p_m^*(K_X + \Delta_\lambda)|_{\mathcal{E}_i},$$

and the coefficients of $F_{\mathcal{E}_i}^\lambda$ are monotone with respect to $\lambda$ around the origin. Therefore, we have a well-defined $\mathbb{R}$-divisor

$$F_{\mathcal{E}_i}^\lambda = \lim_{\lambda \to 0} F_{\mathcal{E}_i}^\lambda.$$
3.5. **Log canonical centers of codimension one.** In this subsection, we use divisorial adjunction of dlt pairs (see, e.g., [Kol92] and [HK10, Theorem 3.24]) to log canonical places of \((X, \Delta_L)\) which are divisors on \(X\), and decompose the different divisor into a fixed component and the push-forward of a nef \(\mathbb{Q}\)-divisor on a higher birational model. From now on, we will always assume that \(\lambda \in (0, \lambda_3) \cap \mathbb{Q}\), where \(\lambda_3\) is constructed in the proof of Proposition 3.14 unless otherwise stated.

**Remark 3.20.** Recall from 3.12 that we denote by \(G_i\) with \(i \in \{1, \ldots, k\}\) the prime divisors that are contracted in the minimal model program \(X \rightarrow X_{\text{min}}\). Observe that every log canonical center of codimension one of \((X, \Delta_L)\) is supported on one of the prime divisors \(G_i\). Indeed, the diminished base locus of \((K_X + \Delta + \lambda A)\) equals the exceptional locus of the rational map \(\pi : X \rightarrow X_{\text{min}}\). Up to permuting the divisors \(G_i\) we can assume that \(G_1, \ldots, G_{k_0}\) are the log canonical centers of codimension one of \((X, \Delta_L)\) for every \(\lambda \in \mathbb{Q}_{>0}\) sufficiently small.

**Proposition 3.21.** Let \(i \in \{1, \ldots, k_0\}\) such that \(G_i\) is normal and \(\lambda \in (0, \lambda_3)\). Then, we can write

\[
(K_X + \Delta + c_\lambda G_{\lambda,i,Y_m})_{|G_i} \sim_\mathbb{Q} K_{G_i} + B_{G_i}^\lambda + M_{G_i}^\lambda,
\]

where the following statements hold:

- The effective \(\mathbb{Q}\)-divisor \(B_{G_i}^\lambda\) is independent of the choice of \(G_{\lambda,i,y}\) in its \(\mathbb{Q}\)-linear system, and
- the support of \(B_{G_i}^\lambda\) is independent of \(\lambda \in (0, \lambda_3) \cap \mathbb{Q}\).

Moreover, if \(K_{G_i} + B_{G_i}^\lambda\) is \(\mathbb{Q}\)-Cartier we have that:

- The pair \((G_i, B_{G_i}^\lambda)\) is log canonical, and
- the \(\mathbb{Q}\)-divisor \(M_{G_i}^\lambda\) is the push-forward of a nef \(\mathbb{Q}\)-divisor on a higher birational model.

In particular, \(M_{G_i}^\lambda\) is nef in codimension one.

**Proof.** Let \(p_m : Y_m \rightarrow X\) be the common dlt modification of the log canonical pairs \((X, \Delta_L)\) constructed in Proposition 3.11. By 3.12, we can write

\[
K_{Y_m} + \Delta_{Y_m} + c_\lambda \left( \sum_{i=1}^k G_{\lambda,i,Y_m} \right) + c_\lambda G_{\lambda,0,Y_m} + \sum_{i \in T'} \mathcal{E}_i = p_m^*(K_X + \Delta + c_\lambda G_{\lambda}),
\]

where the divisors \(G_{\lambda,i,Y_m}\) are the strict transforms on \(Y_m\) of the irreducible components of \(G\), which are contracted by the minimal model program \(X \rightarrow X_{\text{min}}\), and \(G_{\lambda,0,Y_m}\) is the strict transform on \(Y_m\) of the irreducible component of \(G_{\lambda}\) which is not contracted on \(X_{\text{min}}\). Observe that the divisors \(G_1, G_2, \ldots, G_{k_0}\) are exactly those where \(G_{\lambda,i}\) has coefficient one in \(\Delta + c_\lambda G_{\lambda}\). We denote by \(M_{G_{i,z}}^\lambda\) the restriction of the nef \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \((p \circ p_2)^*(K_X + \Delta_L)\) to \(G_{i,Z}\) and by \(M_{G_{i,y}}^\lambda\) the push-forward of \(M_{G_{i,z}}^\lambda\) via the morphism \(q_2 : G_{i,z} \rightarrow G_{i,y_m}\). Therefore, we have a divisorial adjunction formula

\[
K_{G_{i,y_m}} + B_{G_{i,y_m}}^\lambda + M_{G_{i,y_m}}^\lambda \sim_\mathbb{Q} p_m^*(K_X + \Delta_{L})_{|G_{i,Y_m}}.
\]

By pushing-forward via the birational morphism \(p_m : G_{i,y_m} \rightarrow G_i\) we obtain the desired decomposition

\[
K_{G_i} + B_{G_i}^\lambda + M_{G_i}^\lambda \sim_\mathbb{Q} (K_X + \Delta + c_\lambda G_{\lambda})_{|G_i}.
\]

**Remark 3.22.** Arguing similarly as in Remark 3.19, we have that for \(\lambda \in [0, \lambda_3]\) the coefficient at the prime divisor \(G_{\lambda,i,Y_m}\) with \(i \in \{1, \ldots, k_0\}\) of \(p_m^*(K_X + \Delta_L)\) is a monotone function with respect to \(\lambda\) around the origin. Therefore, the coefficient at any prime divisor of \(B_{G_i}^\lambda\) is a monotone function with respect to \(\lambda\) around the origin. In particular, the limit \(B_{G_i} = \lim_{\lambda \rightarrow 0} B_{G_i}^\lambda\) is a well-defined \(\mathbb{R}\)-divisor.
Remark 3.23. The above divisorial adjunction formula (3.6) is a divisorial adjunction of generalized pairs in the sense of [Bir17, Section 3] (see 2.2 for the definition of generalized pair). In [Bir17, Lemma 3.3], Birkar proves that the coefficients of the divisor $B_{G_i}^\lambda$ belong to a DCC set which only depends on the coefficients of $\Delta_\lambda$ and the Cartier index of $M_{G_i,z}^\lambda$.

3.6. Adjunction formula. In this subsection, we prove an adjunction formula for $(c + 1)(K_X + \Delta)$ to each minimal log canonical center of $(X, \Delta_\lambda)$ for $\lambda \in (0, \lambda_3)$.

Proposition 3.24. Let $\lambda \in (0, \lambda_3) \cap \mathbb{Q}$ and $W_i$ a minimal log canonical center of $(X, \Delta_\lambda)$. Then we can write
\[(c + 1)(K_X + \Delta)|_{W_i} \sim_{\mathbb{Q}} K_{W_i} + B_{W_i} + M_{W_i},\]
where the following statements hold:

- The $\mathbb{R}$-divisor $M_{W_i}$ has numerical class in the cone of nef divisors in codimension one of $W_i$, and
- The pair $(W_i, B_{W_i})$ is log canonical whenever $M_{W_i}$ is $\mathbb{R}$-Cartier.

Proof. Being the case $\dim(W_i) = 0$ trivial we need to prove the statement for $\dim(W_i) \in \{1, 2, 3\}$. We prove the case $\dim(W_i) \leq 2$ by using Proposition 3.17. The case $\dim(W_i) = 3$ is analogous by using Proposition 3.21.

We will denote by $L$ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $W_i$ such that
\[((c + 1)(K_X + \Delta))|_{W_i} \sim_{\mathbb{Q}} L,\]
and we aim to prove the existence of an effective $\mathbb{R}$-divisor $B_{W_i}^\lambda$ such that the numerical class of the $\mathbb{R}$-divisor $L - \tilde{K}_{W_i} - B_{W_i}$ is contained in the cone of nef divisors in codimension one of $W_i$. By Proposition 3.17, for every $\lambda \in (0, \lambda_3) \cap \mathbb{Q}$ we may write
\[(K_X + \Delta + c_\lambda G_\lambda)|_{W_i} \sim_{\mathbb{Q}} K_{W_i} + B_{W_i}^\lambda + M_{W_i}^\lambda,\]
where $M_{W_i}^\lambda$ is a $\mathbb{Q}$-divisor whose numerical class is contained in the cone of nef divisors in codimension one of $W_i$ and $B_{W_i}^\lambda$ is an effective divisor such that $(W_i, B_{W_i}^\lambda)$ is log canonical whenever $K_{W_i} + B_{W_i}^\lambda$ is $\mathbb{R}$-Cartier. Moreover, the support of the divisor $B_{W_i}^\lambda$ is independent of $\lambda \in (0, \lambda_3)$. By Remark 3.19 and Remark 3.22, we know that the $\mathbb{R}$-divisor
\[B_{W_i} = \lim_{\lambda \to 0} B_{W_i}^\lambda\]
is well-defined. Moreover, the numerical class of the $\mathbb{R}$-divisor
\[M_{W_i} = L - \tilde{K}_{W_i} - B_{W_i},\]
is the numerical limit of the divisors
\[M_{W_i}^\lambda = L - K_{W_i} - B_{W_i}^\lambda,\]
which are contained in the cone of nef divisors in codimension one of $W_i$. Being the cone of nef divisors in codimension one closed we infer that $M_{W_i}$ is contained in the cone of nef divisors in codimension one. 

Remark 3.25. If the divisor $M_{W_i}$ is $\mathbb{R}$-Cartier, then it is nef in codimension one.

Remark 3.26. By construction, the divisor $K_{W_i} + B_{W_i} + M_{W_i}$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor even if $B_{W_i}$ is just an $\mathbb{R}$-divisor that may not be $\mathbb{R}$-Cartier. In the case $\dim(W_i) = 3$, meaning that $W_i = G_i$ for some $i \in \{1, \ldots, k_0\}$, we know that $M_{G_i}$ is the push-forward to $G_i$ of the restriction of the nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $(q \circ p_2)^* ((c + 1)(K_X^{\text{min}} + \Delta^{\text{min}}))$ to $G_i$.z.

Remark 3.27. The $\mathbb{R}$-divisor $M_{W_i}$ depends on the choice of a starting $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L$ on $W_i$. However, changing the choice of $L$ in its $\mathbb{Q}$-linear system only changes $M_{W_i}$ in its $\mathbb{R}$-linear system.
4. Termination of quasi-flips on the minimal log canonical centers

In this section, we prove that a sequence of flips of a pseudo-effective klt 4-fold \((X, \Delta)\) terminates around any common minimal log canonical center \(W_i\) of the log canonical pairs \((X, \Delta_{\lambda})\) for \(\lambda \in (0, \lambda_3) \cap \mathbb{Q}\).

4.1. Generalized pair on the minimal log canonical center. In this subsection, we prove that the triple \((W_i, B_{W_i} + M_{W_i})\) induced by the adjunction in 3.24, is indeed a generalized pair.

Proposition 4.1. Let \(\lambda \in (0, \lambda_3) \cap \mathbb{Q}\) and let \(W_i\) be a minimal log canonical center of \((X, \Delta_{\lambda})\). Then we can write
\[ ((c + 1)(K_X + \Delta))|_{W_i} \sim_{\mathbb{Q}} K_{W_i} + B_{W_i} + M_{W_i}, \]
where \((W_i, B_{W_i} + M_{W_i})\) is a generalized log canonical pair.

Proof. First, let us assume that \(\dim(W_i) \leq 2\). Consider \(\pi_i: V_i \rightarrow W_i\) to be a resolution of singularities of the generalized pairs \((W_i, B_{W_i} + M_{W_i})\) in the sense of 2.4. We denote by \(p_m: \mathcal{F}_i \rightarrow V_i\) be the morphism induced by \(\mathcal{E}_i \rightarrow W_i\) with respect to the base change \(V_i \to W_i\). For every \(\lambda \in (0, \lambda_3) \cap \mathbb{Q}\) and \(G_{\lambda}\) general in its \(\mathbb{Q}\)-linear system we know that the pair \((\mathcal{E}_i, \Phi^\lambda_{\mathcal{E}_i})\) is dlt, therefore we have that
\[ \operatorname{coeff}(B(V_i(\Phi^\lambda_{\mathcal{E}_i}))) \leq 1. \]
So we conclude that \(\operatorname{coeff}(B(V_i)) \leq 1\), being the latter the wedge of \(\mathbb{R}\)-divisors with coefficients bounded above by one. Thus, we have a resolution of singularities \(p_1: V_i \rightarrow W_i\) of the pairs \((W_i, B_{W_i}^\lambda)\), such that
\[ K_{V_i} + B_{V_i}^\lambda + M_{V_i} = \pi_i^*(K_{W_i} + B_{W_i}^\lambda + M_{W_i}), \]
and such resolution is independent of \(\lambda\). Taking the limit \(\lambda \to 0\) we obtain a divisor \(B_{V_i}\) with \(\operatorname{coeff}(B(V_i)) \leq 1\) and
\[ K_{V_i} + B_{V_i} + M_{V_i} = p_i^*(K_{W_i} + B_{W_i} + M_{W_i}), \]
so the generalized pair \((W_i, B_{W_i} + M_{W_i})\) is log canonical, where we are considering the higher model in 2.2 to be \((V_i, B_{V_i} + M_{V_i})\).

Now, it suffices to show the statement when the log canonical center has codimension one. By Remark 3.20, we know that any such log canonical center equals one of the prime divisors \(G_i\) with \(i \in \{1, \ldots, k_0\}\). In this case, we are in the situation of the proof of Proposition 3.21. For every \(\lambda \in (0, \lambda_3)\) we have a resolution of singularities \(p_2: G_{i,Z} \rightarrow G_i\) such that
\[ K_{G_{i,Z}} + B_{G_{i,Z}} + M_{G_{i,Z}} = p_2^*(K_{G_i} + B_{G_i} + M_{G_i}). \]
Moreover, since the pair
\[ (K_{G_{i,Y_m}} + B_{G_{i,Y_m}} + M_{G_{i,Y_m}}) \]
is dlt we know that \(\operatorname{coeff}(B(G_{i,Z})) \leq 1\). Hence, the statement follows by taking the corresponding limit. \(\square\)

Remark 4.2. Observe that in the above proof, when \(W_i\) is a surface we can take an arbitrary resolution of singularities \(p_1: V_i \rightarrow W_i\) of the pairs \((W_i, B_{W_i}^\lambda)\) to define the generalized pair structure on the triple \((W_i, B_{W_i} + M_{W_i})\).

4.2. Quasi-flips for generalized pairs on the minimal log canonical centers. In this subsection, we introduce quasi-flips for generalized log canonical pairs, and prove that a sequence of \((K_X + \Delta)\)-flips that does not contain the minimal log canonical center \(W_i\) in a flipping locus induces a sequence of quasi-flips for the induced generalized pair. Throughout the remainder of this section, we will consider a single log canonical center so we simply denote it by \(W\).
Proposition 4.3. Let \((X, \Delta)\) be a klt 4-fold with \(K_X + \Delta\) pseudo-effective, and \(W\) a minimal log canonical center of \((X, \Delta_X)\) with \(\lambda \in (0, \lambda_3) \cap \mathbb{Q}\). Consider a sequence of \((K_X + \Delta)\)-flips
\[
(X, \Delta) \overset{\pi_1}{\rightarrow} (X_1, \Delta_1) \overset{\pi_2}{\rightarrow} (X_2, \Delta_2) \overset{\pi_3}{\rightarrow} \cdots \overset{\pi_j}{\rightarrow} (X_j, \Delta_j) \overset{\pi_{j+1}}{\rightarrow} \cdots
\]
which does not contain \(W\) in a flipping locus. Then it induces a sequence of birational transformations
\[
(W, B + M) \overset{\pi_1}{\rightarrow} (W_1, B_1 + M_1) \overset{\pi_2}{\rightarrow} (W_2, B_2 + M_2) \overset{\pi_3}{\rightarrow} \cdots \overset{\pi_j}{\rightarrow} (W_j, B_j + M_j) \overset{\pi_{j+1}}{\rightarrow} \cdots
\]
where \((W, B + M)\) is the generalized log canonical pair obtained by adjunction in 3.24. The map \(\pi_j\) is either a strict ample \((K_W + B_W + M_W)\)-quasi-flip or the identity. In the latter case the flipping locus of \(\pi_j : X_{j-1} \rightarrow X_j\) is disjoint from \(W\).

Proof. It suffices to prove that \(\pi_1\) induces a strict ample \((K_W + B + M)\)-quasi-flip if its flipping contraction and flipped contraction of \(\pi_1\), respectively. By Proposition 4.1, we know that both triples \((W, B + M)\) and \((W_1, B_1 + M_1)\) are generalized lc, so the first condition of 2.10 holds. Moreover, given a curve \(C \subset W_1\) which is being contracted by the flipped contraction, by the adjunction formula 3.24 we have that
\[
(K_{W_1} + B_1 + M_1) \cdot C = (c + 1)(K_X + \Delta_1)|_{W_1} \cdot C < 0,
\]
concluding that the \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(K_{W_1} + B_1 + M_1\) is anti-ample over \(Z\). Analogously, we can check that \(K_W + B + M\) is ample over \(Z\), so the second condition of 2.10 holds. We claim that the projective birational map \(W \dashrightarrow W_1\) is a \((K_W + B + M)\)-negative map, meaning that we can find two projective birational morphisms \(p : V \rightarrow W\) and \(q : V \rightarrow W_1\) which give a log resolution of the above generalized pairs, and we can write
\[
p^*(K_W + B + M) - q^*(K_{W_1} + B_1 + M_1) \geq 0.
\]
Indeed, we can take \(V\) to be the strict transform of \(W\) on a log resolution of the flip \(X \dashrightarrow X_1\) given by the projective birational morphisms \(p_X\) and \(q_X\), so we can write
\[
p^*(K_W + B + M) - q^*(K_{W_1} + B_1 + M_1) = (c + 1)(p_X^*(K_X + \Delta) - q_X^*(K_{X_1} + \Delta_1))|_V = E \geq 0.
\]
Therefore, we have the following inequality of Weil \(\mathbb{R}\)-divisors on \(Z\)
\[
\phi^+_c(B_1) = \phi^+_c(B - E) \leq \phi_*(B),
\]
where \(\phi\) and \(\phi^+_c\) are the flipping and flipped contraction of \(\pi_1\) restricted to \(W\) and \(W_1\), respectively.

Now, we prove that the nef parts \(M_{W_j}\) are the trace of a common Cartier b-divisor. In the case that \(W\) has dimension three, this follows from the divisorial adjunction defining \(M_{W_j}\). On the other hand, if \(W\) has dimension at most two, we will prove that the set of divisorial valuations extracted by the sequence of quasi-flips form a finite set of non-terminal valuations over \(W\). Indeed, by Proposition 3.1 we know that the set of non-terminal valuations of the log canonical pairs \((X, \Delta_X)\) stabilize for \(\lambda\) sufficiently small, therefore by adjunction, the set of non-terminal valuations over \((K_X + \Delta_X)|_W\) stabilize as well. Observe that for every \(j \in \mathbb{Z}_{\geq 1}\), we can find \(\lambda_j \in (0, \lambda_3)\) small enough, such that the sequence of quasi-flips \(\pi_1, \ldots, \pi_j\) is a sequence of \((K_X + \Delta_X)|_W\)-quasi-flips for every \(\lambda \in (0, \lambda_j)\). Moreover, any divisor extracted by the quasi-flip \(\pi_1, \ldots, \pi_j\) is a non-terminal valuation over \(W\) for the klt pairs \((K_X + \Delta_X)|_W\) with \(\lambda \in (0, \lambda_j)\). Thus, we conclude that there exists a smooth variety \(V\) with surjective projective birational morphisms \(p_j : V \rightarrow W_j\) for every \(j \in \mathbb{Z}_{\geq 1}\). Hence, by Remark 4.2 we can define the generalized pair structure of \((W_j, B_j + M_j)\) with a fixed nef divisor on \(V\).

Finally, it suffices to check that the quasi-flip is strict, meaning that its non-flopping locus is non-empty in the sense of 2.13. Observe that the flip \(\pi_1 : X \dashrightarrow X_1\) is ample in the sense of 2.13, therefore by monotonicity
of discrepancies 2.15 the coefficient of
\[(4.1)\]
\[p_X^* (K_X + \Delta) - q_X^* (K_{X_1} + \Delta_1)\]
at any $p_X$-exceptional or $q_X$-exceptional prime divisor with center on the flipping or flipped locus is strictly positive. It suffices to show that there exists a component $E_i$ with non-trivial coefficient in the divisor \[(4.1)\] which intersect $V$ non-trivially. Indeed, by the negativity lemma [KM98, Lemma 3.39 (2)] a fiber of $p$ is either disjoint from the support of \[(4.1)\] or is contained in such support. Therefore, a fiber of $p$ over the intersection of $W$ and the flipping locus of $\pi_1: X \to X_1$ must be contained in the union of prime divisors which have non-trivial coefficient in \[(4.1)\], concluding that there exists a prime divisor $E_i$ with non-trivial coefficient in \[(4.1)\] which intersect $V$ non-trivially. So we deduce that the divisor $E$ is non-trivial, which implies that $\pi_1: W \to W_1$ is a quasi-flop. \qed

In what follows, we will prove that the coefficients of the divisors $B_j$ in the sequence of quasi-flips of Proposition 4.3 belong to a DCC set. To do so, we will construct a generalized boundary part for Ambro’s canonical bundle formula for generalized pairs and we will compare the divisors $B_j$ with such generalized boundary parts. Thus, we can apply the ACC for generalized log canonical thresholds by Birkar and Zhang [BZ16, Theorem 1.5] to prove the statement.

**Construction 4.4.** Consider a projective birational morphism $p_m: E_j \to W_j$ of normal quasi-projective varieties and a generalized boundary $F + N$ on $E_j$, with boundary part $F$ and nef part $N$, such that $(E_j, F + N)$ is a generalized sub-pair which is generalized sub-log canonical near the generic fiber of $p_m$. We can define a boundary part as follows: Given a prime divisor $C \subseteq W_i$, we define the real number
\[\overline{\pi}_C = \sup \{ t \in \mathbb{R} \mid (E_j, F + N + tp_m^*(C)) \text{ is generalized sub-log canonical over a neighbourhood of } \eta_C \},\]
where $\eta_C$ is the generic point of $C$. Observe that the pull-back $p_m^*(C)$ is well-defined over a neighbourhood of $\eta_C$ since $W_j$ is normal. Then we can define the $\mathbb{Q}$-divisor
\[B_{W_j}(F + N) = \sum_{C \subseteq W_j} (1 - \overline{\pi}_C(F)) C,\]
where the sum runs over all the prime divisors $C$ of $W_j$. Arguing as in [Cor07, Section 8.2] we can see that the above sum is finite.

The following lemma is a version of [Amb04, Lemma 2.4] for generalized pairs.

**Lemma 4.5.** Let $q_2: E_{j,Z} \to E_j$ be a log resolution over $W_j$ of the generalized pair $(E_j, F + N)$ and let $(E_{j,Z}, F_Z + N_Z)$ the associated generalized pair on $E_{j,Z}$. Then the following equalities of Weil $\mathbb{R}$-divisors holds
\[B_{W_j}(F + N) = B_{W_j}(F_Z + N_Z) = B_{W_j}(F_Z).\]

**Proof.** Let $C \subseteq W_j$ be an irreducible divisor and $t \in \mathbb{R}$. Consider $E$ a log resolution of the generalized sub-pairs $(E_j, F + tp_m^*(C) + N)$ and $(E_{j,Z}, F_Z + tp_m^*(C) + N_Z)$ with projective birational morphisms $r: E \to E_j$ and $r_Z: E \to E_{j,Z}$. Then, we have that
\[r^*(K_{E_j} + F + tp_m^*(C) + N) = K_E + B_E + t(p_m \circ r)^*(C) + N_E = r_Z^*(K_{E_{j,Z}} + F_Z + tp_m^*(C) + N_Z),\]
where $N_E = r_Z^*(N_Z)$. Therefore, we have that $(E_j, F + N + tp_m^*(C))$ is generalized sub-lc over a neighbourhood of $\eta_C$ if and only if $(E_{j,Z}, F_Z + N_Z + tp_m^*(C))$ is generalized sub-lc over a neighbourhood of $\eta_C$, concluding the first equality. The second equality follows from the fact that
\[K_E + B_E + t(p_m \circ r)^*(C) = r_Z^*(K_{E_{j,Z}} + F_Z + tp_m^*(C)).\] \qed
Proposition 4.6. The sequence of quasi-flips of Proposition 4.3 is a sequence of quasi-flips under a DCC set.

Proof. It suffices to prove that the coefficients of the $\mathbb{Q}$-divisors $B_j$ belong to a DCC set. In order to do so, we will apply Lemma 4.5 and the ACC for generalized log canonical thresholds [BZ16, Theorem 1.5].

Assume that $W_j$ has codimension one on $X_j$ so we are in the situation of Proposition 3.21. By Remark 3.23, the coefficients of $B_j$ are obtained by a divisorial adjunction of generalized pairs whose nef part is the trace of the nef $b$-Cartier divisor induced by (4.3) and whose boundary part has the same coefficients as the $\mathbb{Q}$-divisor

\begin{equation}
K_{Y_m} + \Delta_{Y_m} + c \left( \sum_{i=1}^{k} G_{0,i,Y_m} \right) + \sum_{i \in I'} \mathcal{E}_i.
\end{equation}

Therefore, by [Bir17, Lemma 3.3] the coefficients of the $\mathbb{R}$-divisors $B_j$, with $j \in \mathbb{Z}_{\geq 1}$, belong to a DCC set.

From now on, assume that $W_j$ has dimension at least two on $X_j$ so we are in the situation of Proposition 3.17. Denote by $N^0_{E_j,z}$ the restriction of the nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor

\begin{equation}
(c + 1)p^*_Z(q^*(K_{X_{min}} + \Delta_{min}))
\end{equation}

to $E_{j,z}$ and by $N^0_{E_j}$ the push-forward of $N^0_{E_j,z}$ to $E_j$. Observe that $N^0_{E_{j,z}}$ is the nef part of a generalized boundary in the sense of 2.2 and the Cartier index of the $\mathbb{Q}$-divisor $N^0_{E_{j,z}}$ is independent of $j \in \mathbb{Z}_{\geq 1}$. By Remark 3.19, we know that the $\mathbb{R}$-divisors $F^0_{E_j} = \lim_{\lambda \to 0} F^\lambda_{E_j}$ are well-defined. Moreover, the coefficients of the $\mathbb{R}$-divisors $F^0_{E_j}$ belong to a DCC set, since these coefficients can be obtained by divisorial adjunction of generalized pairs of a divisor whose nef part is the trace of the $\mathbb{Q}$-Cartier divisor induced by (4.3) and whose boundary part has set of coefficients equal to the set of coefficients of the $\mathbb{Q}$-divisor (4.2). Therefore, by the ACC for generalized log canonical thresholds [BZ16, Theorem 1.5] it suffices to prove that

\[ B_{W_j} = \overline{B}_{W_j}(F^0_{E_j} + N^0_{E_j}). \]

Indeed, we have the following equalities

\[ B_{W_j} = \lim_{\lambda \to 0} B_{W_j}(F^\lambda_{E_j,z}) = \lim_{\lambda \to 0} \overline{B}_{W_j}(F^\lambda_{E_j,z} + N^\lambda_{E_j,z}) = \overline{B}_{W_j}(F^0_{E_j,z} + N^0_{E_j,z}) = \overline{B}_{W_j}(F^0_{E_j} + N^0_{E_j}), \]

where the first equality follows from the definition, the second and fourth equalities follows from Lemma 4.5, and the third equality follows from Remark 3.19.

\[ \square \]

Remark 4.7. The coefficients of the divisors $B_j$ belong to a DCC set which only depends on the coefficients of the $\mathbb{Q}$-divisor 4.2 and the Cartier index of the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $K_{X_{min}} + \Delta_{min}$.

4.3. Generalized terminalization and small $\mathbb{Q}$-factorialization. In this subsection, we prove that generalized klt pairs have a generalized $\mathbb{Q}$-factorial terminalization and a generalized small $\mathbb{Q}$-factorialization.

Definition 4.8. A generalized $\mathbb{Q}$-factorial terminalization of a generalized klt pair $(X, B + M)$ is a $\mathbb{Q}$-factorial generalized terminal pair $(Y, B_Y + M_Y)$ together with a projective morphism $p: Y \to X$ such that $K_Y + B_Y + M_Y = p^*(K_X + B + M)$. Moreover, we require that both nef parts $M$ and $M_Y$ are the trace of a common nef $b$-divisor on a higher birational model which dominates $Y$ and $X$.

Definition 4.9. A generalized small $\mathbb{Q}$-factorialization of a generalized klt pair $(X, B + M)$ is a $\mathbb{Q}$-factorial generalized klt pair $(Y, B_Y + M_Y)$ together with a small projective morphism $p: Y \to X$ such that $K_Y + B_Y + M_Y = p^*(K_X + B + M)$. Moreover, we require that both nef parts $M$ and $M_Y$ are the trace of a common nef $b$-divisor on a higher birational model which dominates $Y$ and $X$.

Proposition 4.10. A generalized $\mathbb{Q}$-factorial terminalization and a generalized small $\mathbb{Q}$-factorialization exist for any generalized klt pair $(X, B + M)$. 


Proof. Pick $A$ be an ample effective $\mathbb{Q}$-divisor which contains every center of a divisorial valuation of $(X, B + M)$ of generalized log discrepancy equal to one. If $\epsilon \in \mathbb{R}_{>0}$ is sufficiently small, then $(X, B + \epsilon A + M)$ is a generalized klt pair with boundary part $B + \epsilon A$ and nef part $M$. So replacing $B + M$ with $B + \epsilon A + M$ we may assume that there is no divisorial valuations of generalized log discrepancy equal to one.

Let $(X', B' + M')$ be a log resolution of $(X, B + M)$ so that $M$ is nef and write
\[ K_{X'} + B' + M' = f^*(K_X + B + M) \]
as in the definition of generalized pairs. Hence, we have that
\[ K_{X'} + B'_1 + M' = f^*(K_X + B + M) + B'_2, \]
where $B'_1$ and $B'_2$ are effective divisors with no common components and $f_*(B'_1) = B$. Let $F_1$ be the sum of all the $f$-exceptional divisors which are not irreducible components of $B'_2$. Pick $\epsilon \in \mathbb{R}_{>0}$ sufficiently small such that the generalized pair
\[ K_{X'} + B'_0 + M' = K_{X'} + B'_1 + \epsilon F_1 + M' \]
is a generalized klt pair with boundary part $B'_1 + \epsilon F_1$. By [BZ16, Lemma 4.4 (2)] we can run a minimal model program over $X$, with scaling of an ample divisor, for the generalized klt pair $(X', B'_0 + M')$ to obtain a minimal model $Y$ over $X$. We denote by $\pi: X' \dasharrow Y$ the corresponding minimal model program. The negativity of contractions implies that
\[ K_Y + B_Y + M_Y = f^*(K_X + B + M), \]
where $B_Y = \pi_*(B'_0)$. Therefore, all the irreducible components of $B'_2$ are contracted by the minimal model program $\pi$. Since flips preserve the $\mathbb{Q}$-factorial condition we conclude that $(Y, B_Y + M_Y)$ is a generalized $\mathbb{Q}$-factorial terminalization of $(X, B + M)$.

Analogously, by [BZ16, Lemma 4.4 (2)] we can run a minimal model program with scaling of an ample divisor for the generalized klt pair $(X', B'_1 + M')$ over $X$ to obtain a terminal model $(Y, B_Y + M_Y)$ which is $\mathbb{Q}$-factorial. We denote by $\pi: X' \dasharrow Y$ the corresponding minimal model program. Since $B_Y = \pi_*(B'_1)$ we may denote $E' = \pi_*(B'_2)$ to write
\[ K_Y + B_Y + M_Y = f^*(K_X + B + M) + E'. \]
By negativity of contractions we conclude that $E' = 0$, so we contracted all $f$-exceptional divisors in the minimal model program $\pi$, which means that $(Y, B_Y + M_Y)$ with the projective birational morphism $f: Y \to X$ give a generalized small $\mathbb{Q}$-factorialization of $(X, B + M)$. \hfill \Box

4.4. Generalized difficulty function. In this subsection, we will use a generalized version of the difficulty function introduced in [AHK07] in order to prove the following proposition (compare with [AHK07, Theorem 2.15])

**Proposition 4.11.** Consider a sequence
\[ (W, B + M) \xrightarrow{\pi_1} (W_1, B_1 + M_1) \xrightarrow{\pi_2} (W_2, B_2 + M_2) \xrightarrow{\pi_3} \ldots \xrightarrow{\pi_j} (W_j, B_j + M_j) \xrightarrow{\pi_{j+1}} \ldots \]
of flips for generalized klt pairs. Then it cannot happen infinitely many times that the flipping or flipped locus has a component of codimension 2 in $W_j$ which is contained in $B_j$.

**Definition 4.12.** Let $(X, B + M)$ be a $\mathbb{Q}$-factorial generalized terminal pair. We can pick $f: X' \to X$ a log resolution so that $M'$ is a nef $\mathbb{R}$-divisor. Then, by the negativity lemma we can write $f^*(M) = M' + E$, where $E$ is an effective divisor. Consider $C \subseteq X$ a subvariety of codimension two which is contained in a unique irreducible component $B_i$ of $B$, is not contained in the image of $E$ on $X$, and is not contained in the singular locus of $B_i$. Since the triple $(X, B + M)$ is terminal it is smooth along the generic point of
C. Let $E_1$ be the unique irreducible divisor of the blow up $\pi_1: X_1 = \text{Bl}_c X \to X$ which dominates $C$ and let $C_1 = E_1 \cap \pi_1^{-1}(B_1)$. By induction, for $k \in \mathbb{Z}_{\geq 2}$ we can define $E_k$ to be the unique irreducible divisor of the blow up $\pi_k: X_k = \text{Bl}_{C_{k-1}} X_{k-1} \to X_{k-1}$ which dominates $C_{k-1}$ and $C_k = E_k \cap \pi_k^{-1}(B_k)$. A simple computation shows that $a_{E_k}(X, B + M) = k(1 - b_i)$, where $b_i$ is the coefficient of $B_i$ on $B$. The divisorial valuation corresponding to $E_k$ is called the $k$-th echo of $X$ along $C$. Any such divisorial valuation will be called an echo along a subvariety $C$ of codimension two.

**Lemma 4.13.** Let $(X, B + M)$ be a $\mathbb{Q}$-factorial generalized terminal pair. Then the divisorial valuations $E$ over $X$ with generalized discrepancy $a_E(X, B + M)$ in the interval $(0, 1)$ are the following:

- The echoes of $X$ along subvarieties $C$ of codimension two,
- and finitely many others.

**Proof.** Consider $(X', B' + M')$ a log resolution of $(X, B + M)$ so that $M'$ is nef. Then we can write

$$K_{X'} + B' + M' = f^*(K_X + B + M),$$

where all the prime divisors of $B'$ with positive coefficients are disjoint. Therefore, we conclude using the formula to compute discrepancies over a log smooth pair (see, e.g., [KM98, Lemma 2.29]).

**Corollary 4.14.** Let $(X, B + M)$ be a generalized klt pair. There exists $\epsilon > 0$ and $N \in \mathbb{Z}_{\geq 1}$ such that there are exactly $N$ divisorial valuations $E$ over $X$ with generalized discrepancy $a_E(X, B + M)$ in the interval $(-1, \epsilon)$.

**Proof.** By Proposition 4.10, we can construct a $\mathbb{Q}$-factorial terminalization $(Y, B_Y + M_Y)$ of the generalized pair $(X, B + M)$. Then, by Lemma 4.13 we know that there are finitely many divisorial valuations over $Y$ with generalized discrepancy in the interval $(0, 1)$ which are not echoes along subvarieties of codimension two. Observe that the echoes have generalized discrepancy at least $1 - b$, where $b = \max\{b_i \mid b_i$ is a coefficient of $B_i\}$. Therefore, it suffices to take $0 < \epsilon < 1 - b$ and $N$ the number of divisorial valuations with generalized discrepancy in the interval $(-1, \epsilon)$.

**Definition 4.15.** Given $\alpha \in (0, 1)$, we define the following weight functions $w: \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ to be

- $w^-_\alpha(x) = 1 - x$ for $x \leq \alpha$ and $w^-_\alpha(x) = 0$ for $x > \alpha$, and
- $w^+_\alpha(x) = 1 - x$ for $x < \alpha$ and $w^+_\alpha(x) = 0$ for $x \geq \alpha$.

The summed weight is the function $W: (-\infty, 1) \to \mathbb{R}_{\geq 0}$ defined by the formula

$$W(b) = \sum_{k=1}^\infty w(k(1 - b)),$$

where $w$ is one of the above weight functions. Since the function $w$ have compact support the function $W$ is well-defined.

**Notation 4.16.** Let $\nu: \coprod \widetilde{B}_i \to \bigcup B_i$ be the normalization of the divisor $\text{supp}(B)$. For any irreducible subvariety $C \subset \subset \text{supp}(B)$ the preimage $\widetilde{C} = \nu^{-1}(C)$ splits into a finite union of irreducible components $\widetilde{C}_{i,j} \subset \subset \widetilde{B}_i$.

**Definition 4.17.** Let $(X, B + M)$ be a $\mathbb{Q}$-factorial generalized pair which is terminal. We define

$$\delta(X, B + M) = \sum W(b_i) \rho(\widetilde{B}_i) + \sum_{\text{codim}(E(X)) \neq 2} w(a_E(X, B + M)) + \sum_{C \subset \subset \text{Y irreducible}} \left[ \sum_{\text{codim}(C) = 2} w(a_E(X, B + M)) - \sum_{\widetilde{C}_{i,j}} W(b_1) \right],$$

where $\rho(\widetilde{B}_i)$ is the number of irreducible components $\widetilde{B}_i$.
where $B = \sum_i b_iB_i$ and the $B_i$'s are pairwise different prime divisors. The function $\overline{\delta}(X, B + M)$ will be called the \textit{generalized difficulty function} while the function $\delta(X, B + M) = \delta(X, B)$ will be called the \textit{difficulty function}.

**Remark 4.18.** Since $(X, B + M)$ is $\mathbb{Q}$-factorial and terminal then $(X, B)$ is terminal as a pair. Therefore, by [AHK07, Lemma 2.5] we know that $\delta(X, B + M)$ is a well-defined invariant in the sense that only finitely many summands are giving contribution in the formula (4.4). Moreover, by [AHK07, Lemma 2.6] we know that $\delta(X, B + M) \geq 0$. Observe that for every divisorial valuation $E$ over $X$ we have an inequality $a_E(X, B + M) \leq a_E(X, B)$ and the difference $\overline{\delta}(X, B + M) - \delta(X, B + M)$ is supported in finitely many divisorial valuations over $X$ for which the inequality is strict. Moreover, since $w$ is a decreasing function we have that $\overline{\delta}(X, B + M) \geq \delta(X, B + M) \geq 0$ and that the invariant $\overline{\delta}(X, B + M)$ is finite.

**Lemma 4.19.** Let $f : X' \to X$ be a projective birational morphism, $(X', B' + M')$ and $(X, B + M)$ be two generalized pairs which are $\mathbb{Q}$-factorial and terminal, such that $K_{X'} + B' + M' = f^*(K_X + B + M)$ and the divisors $M$ and $M'$ are the trace of a common nef b-divisor on a higher model. Then $\overline{\delta}(X', B' + M') = \overline{\delta}(X, B + M)$.

**Proof.** Let $B'_i \to \widetilde{B}_i$ be the morphism induced by the birational map $\text{supp}(B') \to \text{supp}(B)$. The difference between $\overline{\delta}(X', B' + M')$ and $\overline{\delta}(X, B + M)$ is produced by the subvarieties $C'_{i,j} \subseteq \widetilde{B}_i$ whose image on $\widetilde{B}_i$ has codimension at least three on $X$. The contribution of such subvarieties is measured by the first and last summands of the formula of the generalized difficulty function (4.4) and they cancel out.

**Definition 4.20.** Let $(X, B + M)$ be a generalized klt pair. We define the \textit{generalized difficulty} of $(X, B + M)$ denoted by $\overline{\delta}(X, B + M)$ to be $\overline{\delta}(Y, B_Y + M_Y)$, where $(Y, B_Y + M_Y)$ is any $\mathbb{Q}$-factorial generalized terminalization of $(X, B + M)$. By Proposition 4.10 and Lemma 4.19 the above is well-defined.

The following proposition is a consequence of the existence of generalized terminalizations 4.10 and the monotonicity properties of the difficulty function proved in [AHK07, Lemma 2.10] and [AHK07, Lemma 2.11]. The monotonicity properties follow from formal conditions satisfied by the weight functions (see, e.g., [AHK07, Conditions 2.1]).

**Proposition 4.21.** In any sequence of flips for generalized klt pairs the generalized difficulty function is eventually decreasing.

**Proposition 4.22.** Fix a number $\alpha \in (-1, 1)$. Then in any sequence of flips for generalized klt pairs there cannot be infinitely many flips for which there exists a divisorial valuation with generalized discrepancy $\alpha$ whose center is in the flipping or flipped locus.

**Proof.** If $\alpha \in (-1, 0)$, then we are done because there are finitely many of such divisorial valuations and after a flip the generalized discrepancy strictly increases. So we may assume that $\alpha \in (0, 1)$. For the flipping locus we can use the weight function $w^{-\alpha}_{\overline{\gamma}}$ as defined in 4.15. After a flip the corresponding generalized discrepancy changes from $a_E(X, B + M) = \alpha$ to $a_E(X^+, B^+ + M^+) > \alpha$ so the corresponding difficulty function decreases at least by $1 - \alpha$. Then, by Remark 4.18 and Proposition 4.21 we know that this cannot happen infinitely many times. A similar argument works for the flipped locus using the weight function $w^{+\alpha}_{\overline{\gamma}}$ instead of $w^{-\alpha}_{\overline{\gamma}}$. 

In order to prove Proposition 4.11 we need to prove that eventually all components of codimension two of the flipping and flipped locus have their generic point contained in the smooth locus of the variety. This statement will be proved in Lemma 4.24 using a surface computation. We will need the following version of the ACC for minimal generalized log discrepancies of generalized pairs of dimension two.
Lemma 4.23. Let $\Lambda$ be a DCC set and $p \in \mathbb{Z}_{>1}$. The set of minimal generalized log discrepancies of generalized klt pairs $(X, B + M)$ of dimension two, such that the coefficients of the boundary part $B$ belong to $\Lambda$ and the Cartier indices of $M$ and $M'$ are bounded above by $p$, satisfies the ACC.

Proof. First, observe that a generalized klt pair $(X, B + M)$ of dimension 2 is $\mathbb{Q}$-factorial. Indeed, the small $\mathbb{Q}$-factorialization constructed in Proposition 4.10 is an isomorphism, therefore the divisor $M$ is $\mathbb{Q}$-Cartier and then $(X, B)$ is a klt pair. Assume that there exists a sequence of generalized klt pairs $(X_j, B_j + M_j)$ of dimension two such that their minimal generalized log discrepancies do not belong to a set with the ACC, meaning that they form an infinite strictly increasing sequence in the interval $(0, 1)$. Let $(X'_j, B'_j + M'_j)$ be a log resolution of $(X_j, B_j + M_j)$, so that $M'_j$ is a nef divisor and let $E_j$ be the prime divisor on $X'_j$ computing the minimal generalized log discrepancy, then we have that

$$a_{E_j}(X_j, B_j + M_j) = a_{E_j}(X_j, B_j) + \text{coeff}_{E_j}(f^*_j(M_j)) - \text{coeff}_{E_j}(M'_j).$$

Observe that the set $a_{E_j}(X_j, B_j)$ holds the ACC (see, e.g., [Ale93]). Moreover, $\text{coeff}_{E_j}(f^*_j(M_j))$ and $\text{coeff}_{E_j}(M'_j)$ belong to a discrete family by the bound $p$ on the Cartier indices of the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $M_j$ and $M'_j$. Thus, we infer that $a_{E_j}(X_j, B_j + M_j)$ holds the ACC as well.

Lemma 4.24. In any sequence of flips for generalized klt pairs, there cannot happen infinitely many times that the flipping or flipped loci contain a component of codimension two which is contained in the singular locus of the generalized pair.

Proof. Let $(X_j, B_j + M_j)$ be the generalized klt pairs in the sequence of flips. Denote by $E_1, \ldots, E_k$ the non-terminal divisorial valuations of $(X_j, B_j + M_j)$ for $j$ large enough. Assume that $X_j$ is generically singular along a codimension 2 component of the flipping or flipped locus. By cutting with two generic hyperplanes we obtain a generalized klt surface $(Z, B_j|_Z + M_j|_Z)$ so that any curve in the terminalization of such surface corresponds to one of the divisors $E_1, \ldots, E_k$. Observe that the Cartier index of $M'_j$ is bounded, so the Cartier index of the b-Cartier divisor associated to the nef part of $(Z, B_j|_Z + M_j|_Z)$ is bounded as well. We claim that the Cartier indices of the divisors $M_j|_Z$ are bounded. Indeed, the coefficients of $M_j|_Z$ belong to a discrete set. Moreover, we may assume that all the $M_j|_Z$ surfaces $(Z, B_j|_Z)$ are $\epsilon$-klt for some $\epsilon > 0$ concluding that $M_j|_Z$ have bounded Cartier index (see, e.g., [Ale93]). Since there are finitely many divisors $E_j$ there is one that appears as the minimal generalized log discrepancy of $(Z, B_j|_Z + M_j|_Z)$ infinitely many times and strictly increasing at every step, leading to a contradiction by Lemma 4.23.

Proof of Proposition 4.11. Let $C$ be a subvariety of codimension two which is contained in $\text{supp}(B_j)$. By Lemma 4.24, we may assume that the generic point of $C$ is contained in the smooth locus of $X_j$. The blow-up of $C$ produces a divisor with generalized discrepancy less than one which has the form $1 - \sum m_i b_i$, where the $m_i$ are positive integers and the $b_i$ are the coefficients of the boundary part $B_j$ of the generalized pair. There are finitely many of such generalized discrepancies in the interval $(-1, 1)$. Thus, we can apply Proposition 4.22 to finish the proof.

4.5. Termination of quasi-flips on the log canonical centers. In this subsection, we prove that any sequence of flips of a pseudo-effective klt 4-fold $(X, \Delta)$ terminates around $W$, where $W$ is a minimal log canonical center of $(X, \Delta)$ for $\lambda \in (0, \lambda_\Delta) \cap \mathbb{Q}$.

By Proposition 4.3, in order to prove termination around $W$ it is enough to show that any sequence of strict ample quasi-flips for generalized log canonical pairs terminates in dimension at most three. We start proving that any such sequence terminates in codimension one in the klt case.

Definition 4.25. We say that a sequence of birational transformations terminates in codimension one if after finitely many birational transformations all maps are isomorphisms in codimension one. In particular,
a sequence of quasi-flips terminates in codimension one, if after finitely many quasi-flips all quasi-flips are weak.

**Lemma 4.26.** Any sequence of strict ample klt quasi-flips for generalized log canonical pairs under a DCC set terminates in codimension one.

*Proof.* We claim that any divisor which is extracted in the sequence of quasi-flips has discrepancy at most zero. Indeed, the generalized discrepancy along the generic point of an irreducible component of the boundary part with coefficient $0 < b_i < 1$ is $-b_i$. Therefore such irreducible divisor is a non-terminal valuation over all previous models in the sequence of quasi-flips. We conclude that there are finitely many divisorial valuations that can be extracted in the sequence of strict ample quasi-flips.

Now, we prove that each of these finitely many divisorial valuations can be extracted at most finitely many times. Indeed, if a non-terminal valuation $E$ is extracted by the quasi-flip $\pi_j$ and $\pi_j$ for $j_1 < j_2$ we have that there exists a quasi-flip between $\pi_{j_1}$ and $\pi_{j_2}$ that is contracting such non-terminal valuation, therefore the generalized log discrepancy of the pairs $(X_j, B_j + M_j)$ strictly increases at $E$, which means that the coefficient at $E$ of $B_{j_1+1}$ is strictly greater than the coefficient at $E$ of $B_{j_2+1}$. Since the sequence of quasi-flips is under a DCC set the coefficients of $B_j$ belong to a DCC set, so we deduce that this can only happen finitely many times, meaning that every non-terminal valuation can be extracted only finitely many times.

We conclude that there are finitely many divisorial valuations that can be extracted in the sequence of quasi-flips, and each of these can be extracted at most finitely many times. Thus, after finitely many quasi-flips all quasi-flips do not extract divisors. By induction on the Picard rank of $X_j$, we conclude that after finitely many quasi-flips both the flipping and flipped contractions are isomorphisms in codimension one.

**Remark 4.27.** Recall from [KMM87, Lemma 5.1.7] that given a flip $\pi_{j+1}: (X_j, \Delta_j) \rightarrow (X_{j+1}, \Delta_{j+1})$ of klt pairs we have that $\dim(\text{Ex}(\phi_{j+1}^+)) + \dim(\text{Ex}(\phi_{j+1}^-)) \geq \dim(X_j) - 1$. In the case that $X_j$ is a 4-fold we infer that the possible choices for the pair $(\dim(\text{Ex}(\phi_{j+1}^+)), \dim(\text{Ex}(\phi_{j+1}^-)))$ are $(2, 1), (1, 2)$, and $(2, 2)$. Observe that we can use Proposition 4.11 to prove that the sequence of quas-flips induced on every log canonical center of codimension one of the pairs $(X, \Delta_X)$ terminates in codimension one. Indeed, any such log canonical center is of the form $G_i$ for some $i \in \{1, \ldots, k_0\}$. The prime divisor $G_i$ appears with coefficient one in the boundary part of the generalized log canonical pair $(c+1)(K_X + \Delta)$, therefore we can apply the proposition to conclude that eventually no component of codimension two of the flipping locus or the flipped locus in contained in the strict transform of $G_i$. In particular, after finitely many flips the generic point of any component of codimension two of the flipping or flipped locus of a flip of type $(2, 2)$ lies in the complement of the strict transform of $G_i$.

**Notation 4.28.** Given a generalized log canonical pair $(X, B + M)$, we denote by $X^0$ the complement of the non-klt locus of $X$. We will write $B^0$ and $M^0$ for the restriction of $B$ and $M$ to $X^0$, respectively. By definition the generalized pair $(X^0, B^0 + M^0)$ is klt. The variety $X^0$ will often be called the *klt locus* of $(X, B + M)$.

**Corollary 4.29.** Any sequence of strict ample quasi-flips for generalized log canonical pairs of dimension two under a DCC set terminates.

*Proof.* By Lemma 4.26, it suffices to show that eventually such quasi-flips are klt in the sense of 2.14. In order to do so, we need to prove a special termination around the generalized log canonical centers (see [Fuj07, Theorem 2.1] for the dlt case, and [Sko04, Corollary 4] for the lc case). If the generalized log canonical center has dimension zero then a quasi-flip is either disjoint from the center or contains such center, so after finitely many quasi-flips we can assume that no quasi-flip contains a generalized log canonical center.
of dimension zero. If the generalized log canonical center has dimension one we can use divisorial adjunction for generalized pairs to induce a generalized pair on this curve, such that the coefficients of its boundary part belong to a DCC set (see, e.g., [Bir17, Lemma 3.3]). By strict monotonicity 2.15, we conclude that every quasi-flip which intersect this curve in a zero-dimensional locus strictly decrease such coefficients. Thus, after finitely many flips a one-dimensional log canonical center is either disjoint from the flipping locus or is contained in it. By finiteness of log canonical centers, we deduce that eventually all flips are disjoint from the non-klt locus of the generalized pair.

\[ \square \]

**Proposition 4.30.** Any sequence of strict ample quasi-flips for generalized log canonical pairs of dimension three under a DCC set terminates.

**Proof.** First, we reduce to the case of klt quasi-flips. We proceed by proving special termination around the generalized log canonical centers. If the log canonical center has dimension zero or one, then the argument is analogous to the one in the proof of Corollary 4.29. If the log canonical center has dimension two then we can use divisorial adjunction for generalized pairs (see [Bir17, Section 3]) to obtain a sequence of strict ample quasi-flips for generalized log canonical surfaces under a DCC set (see [Bir17, Lemma 3.3]). Therefore, termination around such generalized log canonical center follows from Corollary 4.29. Thus, after finitely many quasi-flips all flipping loci are disjoint from the non-klt locus of the generalized pair. So the quasi-flips are klt in the sense of 2.14.

Now, it suffices to prove that a sequence of strict ample klt quasi-flips for generalized lc pairs of dimension three under a DCC set terminates. By Lemma 4.26, we conclude that such sequence terminates in codimension one so we reduce to the case of strict ample weak klt quasi-flips. Consider a strict ample weak klt quasi-flip for generalized log canonical pairs \( \pi : (X, B + M) \rightarrow (X_1, B_1 + M_1) \) with flipping contraction \( \phi : X \rightarrow Z \). By Lemma 4.10, we can take a small \( \mathbb{Q} \)-factorialization \((X', B' + M')\) of the klt locus of \((X, B + M)\) and by [BZ16, Section 4] we can run a relative minimal model program for \((X', B' + M')\) over \(Z\) to produce a minimal model \((X'_1, B'_1 + M'_1)\) over \(Z\), which is a small \( \mathbb{Q} \)-factorialization of the klt locus of \((X_1, B_1 + M_1)\). Moreover, \((X_1, B_1 + M_1)\) is the ample model of \((X', B' + M')\) over \(Z\). Thus, by taking small \( \mathbb{Q} \)-factorializations we reduce to prove termination of klt flips for \( \mathbb{Q} \)-factorial generalized log canonical 3-folds.

We claim that the \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisors \( K_{X_j}^0 + B_j^0 + M_j^0 \) have bounded Cartier index independent of \( j \in \mathbb{Z}_{\geq 1} \). Indeed, by Corollary 4.14 we know that there exists \( \epsilon > 0 \) and \( N \in \mathbb{Z}_{\geq 1} \) such that the \( \mathbb{Q} \)-factorial generalized klt 3-folds \((X_j^0, B_j^0 + M_j^0)\) have at most \( N \) divisorial valuations with generalized discrepancy in the interval \((-1, \epsilon)\). In particular, the \( \mathbb{Q} \)-factorial klt pairs \((X_j^0, B_j^0)\) have at most \( N \) divisorial valuations with generalized discrepancy in the interval \((-1, \epsilon)\). Then, we can apply [Sho04, Lemma 4.4.1] to deduce that the \( \mathbb{Q} \)-divisors \( K_{X_j}^0 + B_j^0 + M_j^0 \) have bounded Cartier index. Therefore, since the flipping locus of \( \pi_j \) is contained in \( X_j^0 \), there exists \( \alpha > 0 \) independent of \( j \in \mathbb{Z}_{\geq 1} \), such that the generalized discrepancy of every divisorial valuation over the generalized pair \((X_j, B_j + M_j)\) increases at least by \( \alpha \) when its center is contained in the flipping or flipped locus.

Now, we reduce to the case of terminal flips. Recall that the generalized pairs \((X_j, B_j + M_j)\) have finitely many non-terminal divisorial valuations over \( X_j^0 \). By the existence of \( \alpha > 0 \), we know that eventually no flip contains the center of a generalized non-terminal valuation on its flipping locus. Thus, the coefficients of the terminalizations of \((X_j^0, B_j^0 + M_j^0)\) stabilize for \( j \) large enough. By Proposition 4.10, we can take a \( \mathbb{Q} \)-factorial terminalization of the klt locus of \((X_j, B_j + M_j)\) to reduce to the case of terminal flips for generalized log canonical 3-folds.
We claim that we may assume that \( B_j = 0 \). Indeed, if the flipping locus intersects \( B_j \) positively then the flipped locus has a component of codimension two which is contained in \( B_{j+1} \). On the other hand, if the flipping locus intersects \( B_j \) non-positively then it is either contained in \( B_j \) or disjoint from \( B_j \). By Proposition 4.11, we deduce that eventually the flipping and flipped loci are disjoint from \( B_j \).

Therefore, we obtain a sequence of terminal flips for \( \mathbb{Q} \)-factorial generalized lc 3-folds \((X_j, M_j)\). By Lemma 4.13 the generalized pair \((X^0_j, M^0_j)\) has finitely many divisorial valuations with generalized discrepancy in the interval \((0,1)\), and every such divisorial valuation increases its generalized discrepancy at least by \( \alpha > 0 \) when its center is contained in the flipping or flipped locus. We conclude that eventually no flip contains the center of a divisorial valuation with generalized discrepancy in the interval \((0,1)\) in its flipping or flipped locus. Since \( a_E(X_j) \geq a_E(X_j, M_j) \) for every divisorial valuation \( E \) over \( X_j \), we deduce that after finitely many flips, no flip contains in its flipping or flipped locus the center of a divisorial valuation with discrepancy in the interval \((0,1)\).

We reduce to the case of \( K_{X_j} \)-flops. Consider a flip \( \pi_{j+1} : (X_j, M_j) \dashrightarrow (X_{j+1}, M_{j+1}) \) which is \( K_{X_j} \)-negative, meaning that \( \pi_{j+1} : X_j \dashrightarrow X_{j+1} \) is a terminal 3-fold flip. By the classification of terminal 3-fold extremal contractions (see [Mor88] or [KM92]), we conclude that the flipping locus of \( \pi_{j+1} \) must contain a terminal singular point of Cartier index \( r > 1 \) and therefore by [CH11, Theorem 2.9] it must contain the center of a divisorial valuation with discrepancy \( 0 < \frac{1}{r} < 1 \) with respect to \( K_{X_j} \), giving a contradiction. If the flip \( \pi_{j+1} \) is \( K_{X_j} \)-positive, meaning that \( \pi_{j+1} : X_{j+1} \dashrightarrow X_j \) is a flip of terminal 3-folds, the same argument applied to the flipped locus leads to a contradiction. Thus, we reduce to termination of terminal flips for \( \mathbb{Q} \)-factorial generalized lc pairs \((X_j, M_j)\) such that all such flips are \( K_{X_j} \)-flops.

Finally, we prove that the sequence of \((K_{X_j} + M_j)\)-flips which are \( K_{X_j} \)-flops terminates. For every divisorial valuation \( E \) over \( X^0_j \) we define the number

\[
\eta_E(X_j, M_j) = a_E(X_j) - a_E(X_j, M_j) \geq 0.
\]

Observe that such number is always positive by the negativity lemma and it decreases at least by \( \alpha > 0 \) when the center of \( E \) on \( X_j \) is contained in the flipping or flipped locus of the flip for the generalized pair. Therefore, a curve \( C \subseteq \text{Bs}_-(M_j) \) which contains the center \( c_E(X_j) \) of a divisorial valuation \( E \) with \( \eta_E(X_j, M_j) < \alpha \) cannot be contained in the flipping locus of a \((K_{X_j} + M_j)\)-flip. Since \( M_j \) is the push-forward of a nef divisor on a higher birational model then its diminished base locus consists of finitely many curves. Recall from [Kol91, Proposition 2.1.12] that a terminal 3-fold \( K_{X_j} \)-flop has the same number of irreducible curves in its flipping and flopped locus, so the number of irreducible components of the diminished base locus of \( M_j \) can only decrease.

We proceed by induction on the number \( k_j = k(X_j, M_j) \) of curves in the diminished base locus of \( M_j \) which are contained in the flipping locus of \( \pi_i \) for some \( i \geq j + 1 \). If \( k_j = 1 \) for some \( j \in \mathbb{Z}_{\geq 1} \) we deduce that the \((K_{X_j} + M_j)\)-flip contains a single curve in the flipping locus and there are no further flips. Observe that \( k_{j+1} = k_j \) if and only if all the curves contained in the flipped locus of \( \pi_{j+1} \) are contained in the flipping locus of \( \pi_i \) for some \( i \geq j + 2 \). Let \( j_0 \in \mathbb{Z}_{\geq 1} \) and assume that \( k_{j_0} \geq 2 \). Consider a curve \( C \subseteq \text{Bs}_-(M_{j_0}) \) contained in the flipping locus of \( \pi_{j_0+1} \). Therefore, we have that \( M_{j_0} \cdot C < 0 \), so there exists a divisorial valuation \( E \) over \( X^0_{j_0} \) such that \( \eta_E(X_{j_0}, M_{j_0}) > 0 \) and \( c_E(X_{j_0}, M_{j_0}) = C \). Since the center of \( E \) on \( X_{j_0} \) is contained in \( C \) we conclude that the center of \( E \) on \( X_{j_0+1} \) is contained in the flipped locus of \( \pi_{j_0+1} \). If \( k_{j_0+1} = k_{j_0} \), we deduce that the center of \( E \) is contained in the flipping locus of \( \pi_{j_1} \) for some \( j_1 \geq j_0 + 2 \). Analogously, if \( k_{j_1+1} = k_{j_1} \), we have that the center of \( E \) is contained in the flipping locus of \( \pi_{j_2} \) for some \( j_2 \geq j_1 + 2 \). Inductively, we produce a sequence of flips \( \pi_{j_l} \) which contain the center of \( E \) in their flipping loci. Since \( \eta_E(X_{j_l}, M_{j_l}) \) is non-negative and decreases at least by \( \alpha > 0 \) when its center is contained in a flipping locus, we conclude that such sequence is finite, meaning that there is \( i \in \mathbb{Z}_{\geq 1} \) such that the center of \( E \) is contained in the flipping locus of \( \pi_i \) but is not contained in the flipping locus of \( \pi_l \) for every \( l \geq j_i + 1 \). Therefore, there exists a curve in the flipped locus of \( \pi_{j_i} \) which contains the center of \( E \) and this curve is
not contained in the flipping locus of $\pi_l$ for every $l \geq j_i + 1$. Thus, we have that $k_{j_i+1} < k_j$, and we obtain termination by the inductive hypothesis. 

\[ \square \]

5. Proof of the theorem

In this section, we prove the main theorem. First, we prove two lemmas which will be used in the proof of the theorem.

**Lemma 5.1.** If $K_X + \Delta$ is not nef, then the log canonical pairs $(X, \Delta)$ have at least one common log canonical center, which is a generalized log canonical center of the generalized log canonical pair $(c+1)(K_X + \Delta)$, where $c = \operatorname{lct}(K_X + \Delta)$.

**Proof.** By Proposition 3.1, we can take a log resolution $Y$ of the minimal model program with scaling of an ample divisor $\pi : X \to X_{\min}$, such that $c_\lambda$ is the maximum positive real number $\mu$ for which $K^\Delta_{Y/X} - \mu E_\lambda$ has coefficients greater or equal than negative one. Moreover, we know that the coefficients at the prime irreducible components of $K^\Delta_{Y/X} - c E_\lambda$ are linear functions on $c$ and $\lambda$. Since $K_X + \Delta$ is not nef, we can use Corollary 2.24 to deduce that $\operatorname{lct}(K_X + \Delta)$ is finite, so that $c_\lambda$ is finite for $\lambda$ small enough, which means that the log canonical pairs $(X, \Delta)$ have at least one common log canonical center. The log canonical place on $Y$ corresponding to such common log canonical center, is an irreducible component with coefficient negative one of $K^\Delta_{Y/X} - c_\lambda E_\lambda$, so we conclude that it is a component with coefficient negative one of $K^\Delta_{Y/X} - c E_\lambda$. By Proposition 2.23, we deduce that such common log canonical center of $(X, \Delta)$ is a generalized log canonical center of the generalized log canonical pair $(c+1)(K_X + \Delta)$. 

\[ \square \]

**Lemma 5.2.** Assume that $(X, \Delta)$ has a minimal model $(X_{\min}, \Delta_{\min})$. Let $\pi_{j+1} : (X_j, \Delta_j) \to (X_{j+1}, \Delta_{j+1})$ be a sequence of flips for $K_X + \Delta$. Then, the sequence $\operatorname{lct}(K_{X_j} + \Delta_j)$ is an increasing sequence satisfying the ACC.

**Proof.** First, we prove that the sequence $\operatorname{lct}(K_{X_j} + \Delta_j)$ is increasing. We can consider a common log resolution $Y$ of $(X_j, \Delta_j), (X_{j+1}, \Delta_{j+1})$ and $(X_{\min}, \Delta_{\min})$, with projective birational morphisms $p_j, p_{j+1}$ and $q$. By monotonicity 2.15, we know that we can write

$$p^*_j(K_{X_j} + \Delta_j) = q^*(K_{X_{\min}} + \Delta_{\min}) + E_j$$

and

$$p^*_{j+1}(K_{X_{j+1}} + \Delta_{j+1}) = q^*(K_{X_{\min}} + \Delta_{\min}) + E_{j+1},$$

where $E_j \geq E_{j+1}$. By Proposition 2.23, we conclude that $\operatorname{lct}(K_{X_j} + \Delta_j) \leq \operatorname{lct}(K_{X_{j+1}} + \Delta_{j+1})$. Now we turn to prove that the sequence $\operatorname{lct}(K_{X_j} + \Delta_j)$ satisfies the ACC. Recall that we can consider $K_{X_j} + \Delta_j$ as the boundary of a generalized pair by writing

$$K_{X_j} + \Delta_j = p^*_j(q^*(K_{X_{\min}} + \Delta_{\min})) + p_j(E_j),$$

where $p_j(E_j)$ is the boundary part. Observe that the Cartier index of the divisor $q^*(K_{X_{\min}} + \Delta_{\min})$ is independent of $j$. We claim that the coefficients of the prime components of $p_j(E_j)$ belong to a finite set which is independent of $j$. Indeed, for every irreducible component $E$ of the exceptional locus of the rational map $X \to X_{\min}$, the coefficient of $p_j(E_j)$ at $E$ equals

$$a_E(X_{\min}, \Delta_{\min}) - a_E(X_j, \Delta_j)$$

and such numbers are independent of $j$ in a sequence of flips. Therefore, we can apply [BZ16, Theorem 1.5] to conclude that the real numbers

$$\operatorname{lct}(K_{X_j} + \Delta_j) = \operatorname{lct}((K_{X_j} + \Delta_j), p_j^*(q^*(K_{X_{\min}} + \Delta_{\min}))) + p_j(E_j)$$

satisfies the ACC. 

\[ \square \]
Proof of Theorem 1. Consider a sequence of \((K_X + \Delta)\)-flips as follows
\[
(X, \Delta) \xrightarrow{\pi_1} (X_1, \Delta_1) \xrightarrow{\pi_2} (X_2, \Delta_2) \xrightarrow{\pi_3} \cdots \xrightarrow{\pi_j} (X_j, \Delta_j) \xrightarrow{\pi_{j+1}} \cdots
\]
By special termination around the log canonical center of the log canonical pair \((X, \Delta)\) (see, e.g., [Sho04, Corollary 4]) we deduce that after finitely many flips all flips are disjoint from the strict transform of the log canonical centers of the pair \((X, \Delta)\). Then, we obtain a sequence of klt flips for log canonical 4-folds \((X, \Delta)\). Hence, without loss of generality we may assume that the pair \((X, \Delta)\) is itself klt. By Lemma 5.1, we may assume that \(\text{let}(K_X + \Delta) = c\) is finite. We claim that for \(j \in \mathbb{Z}_{\geq 1}\) large enough, all flipping loci are disjoint from the generalized non-klt locus of \((c+1)(K_{X_j} + \Delta_j)\). Indeed, by Lemma 5.1 we know that the pairs \((X, \Delta_j)\) have a common minimal log canonical center \(W_i\) which is also a log canonical center of the generalized log canonical pair \((c+1)(K_X + \Delta)\). If \(W_i\) has dimension zero, termination around \(W_i\) follows from monotonicity of discrepancies 2.15. If \(W_i\) has dimension one, then by Proposition 3.24 we can apply adjunction for \((c+1)(K_X + \Delta)\) to \(W_i\) and by Proposition 4.6 we know that the induced boundary part belong to a DCC set. So termination around \(W_i\) follows from the strict monotonicity 2.15. If \(W_i\) has dimension at least two, then by Proposition 4.3 and Proposition 4.6 we obtain an induced sequence of strict ample quasi-flips of generalized log canonical pairs on \(W_i\) under a DCC set. If \(W_i\) has dimension two, by Corollary 4.29 we conclude that such sequence of birational transformations terminates around \(W_i\), on the other hand if \(W_i\) has dimension three, the same statement holds by Proposition 4.30. Thus, all flips are eventually disjoint from the strict transform of \(W_i\) or there is a flip which contains the strict transform of \(W_i\) in its flipping locus. In the latter case the strict transform of \(W_i\) is not a non-klt center of \((c+1)(K_{X_j} + \Delta_j)\) for \(j\) large enough. Thus, we deduce that eventually all flips are disjoint from the strict transform of \(W_i\).

We conclude that after finitely many flips, all flipping loci are disjoint from the strict transform of \(W_i\). Replacing \(X\) with the complement of \(W_i\) on \(X\) we achieve that the number of generalized log canonical centers of \((c+1)(K_X + \Delta)\) strictly decrease after finitely many flips. Meaning that for \(j\) large enough the number of log canonical centers of \((c+1)(K_{X_j} + \Delta_j)\) is strictly less than the number of log canonical centers of \((c+1)(K_X + \Delta)\). Arguing similarly for the generalized log canonical pair \((c+1)(K_{X_j} + \Delta_j)\), we deduce that after finitely many flips the number of log canonical centers of \((c+1)(K_{X_j} + \Delta_j)\) strictly decrease again. Hence, we conclude that \(\text{let}(K_X + \Delta) < \text{let}(K_{X_j} + \Delta_j)\).

Proceeding inductively, we produce a sequence of klt 4-folds \((X_j, \Delta_j)\), such that
\[
\text{let}(K_X + \Delta) < \text{let}(K_{X_1} + \Delta_1) < \cdots < \text{let}(K_{X_j} + \Delta_j) < \cdots
\]
is an ascending sequence of real numbers. By Lemma 5.2, we conclude that such sequence must be finite. Therefore, the sequence of flips is finite. □

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