Vafa-Witten Theory: Invariants, Floer Homologies, Higgs Bundles, a Geometric Langlands Correspondence, and Categorification

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Outline of Talk

- Introduction and Motivation
- Summary of Results
- Main Body of the Talk
- Conclusion
In this talk, we will discuss the Vafa-Witten (VW) twist of a 4d $\mathcal{N} = 4$ SYM gauge theory on $M_4$.

The motivations for doing so are to:

- Derive a novel VW invariant of $M_4$, and relate it to Gromov-Witten (GW) invariants via an $\mathcal{N} = (4, 4)$ $A$-model.

- Derive a novel Vafa-Witten Atiyah-Floer correspondence, and thereby a physical proof and generalization of the Abouzaid-Manolescu conjecture of hypercohomology of perverse sheaves in [1].

- Obtain a Langlands dual of the invariants, Floer homologies and hypercohomology stated hitherto, as well as the quantum and classical Geometric Langlands correspondence [2].

- Obtain a physical framework for higher categorification of VW theory.
This talk is based on

- Tan, Meng-Chwan et al., *Vafa-Witten Theory: Invariants, Floer Homologies, Higgs Bundles, a Geometric Langlands Correspondence, and Categorification*, arXiv preprint arXiv:2203.17115 (2022)

Built on earlier insights in

- Bershadsky, Michael, et al, *Topological reduction of 4D SYM to 2D $\sigma$-models*, Nuclear Physics B 448.1-2, 166-186 (1995).
- Birmingham, Danny, et al, *Topological field theory*, Physics Reports 209.4-5, 129-340 (1991).
- Gukov, Sergei, *Surface operators and knot homologies*, New Trends in Mathematical Physics, 2009 - Springer.
- Kapustin, Anton and Witten, Edward. *Electric-magnetic duality and the geometric Langlands program*, Communications in Number Theory and Physics Volume 1, Number 1, (2007).
Summary of Results

1. If the scalar curvature of $M_4$ and the gauge group $G$ are not simultaneously non-negative and locally a product of $SU(2)$’s, the theory localizes on a **zero-dimensional** moduli space of configurations satisfying the VW equations. The invariant is the partition function

\[ Z_{\text{VW}, M_4}(\tau, G) = \sum_{k} (-1)^{n_k + \tau m_k} \]

2. Compactify VW theory on $M_4 = \Sigma \times C$ along $C$, where both $\Sigma$ and $C$ are closed Riemann surfaces of genus $g \geq 1$ and $g \geq 2$, respectively. We arrive at an $A$-model in complex structure $I$ on $\Sigma$ with $\mathcal{N} = (4, 4)$ supersymmetry and target space $\mathcal{M}_H^G(C)$, the moduli space of Hitchin’s equations on $C$. Topological invariance implies a 4d-2d correspondence

\[ Z_{\text{VW}, M_4}(\tau, G) = Z_{\text{GW}, \Sigma}(\tau, \mathcal{M}_H^G(C)) = \sum_{l} (-1)^{p_l + \tau q_l} \]
Summary of Results

3. Boundary VW theory on $M_4 = M_3 \times \mathbb{R}^+$, with $M_3$ a closed three-manifold in the temporal gauge allows us to recast the 4d theory as 1d supersymmetric quantum mechanics (SQM) on the space of all complexified connections on $M_3$, with potential being the complex Chern-Simons functional. The VW partition function is then

$$Z_{VW,M_4}(\tau,G) = \sum_k \mathcal{F}_{VW}^{G,\tau}(\Psi^k_{M_3}) = \sum_k HF_{VW}^k(M_3,G,\tau) = Z_{VW,M_3}^{\text{Floer}}(\tau,G)$$

4. We then Heegaard split $M_3$ along the Riemann surface $C$ to obtain an equivalent open $A$-model with boundaries given by Lagrangian $(A,B,A)$-branes $L_0, L_1$ in $\mathcal{M}^G_{\text{Higgs}}(C)$, which leads us to a VW Atiyah-Floer correspondence as

$$HF_{VW}^*(M_3,G,\tau) \cong HF_{\text{Lagr}}^*(\mathcal{M}^G_{\text{Higgs}}(C), L_0, L_1, \tau)$$
Summary of Results

5. This allows us to physically realize the Abouzaid-Manolescu conjecture for the hypercohomology $\text{HP}^*(M_3)$ of a perverse sheaf of vanishing cycles in the moduli space of irreducible flat $SL(2,\mathbb{C})$-connections on $M_3$, which can be generalized to

$$\text{HP}^*(M_3) \cong \text{HF}^{\text{inst}}_*(M_3, G_\mathbb{C}, \tau)$$

6. $S$-duality of $\mathcal{N} = 4$ VW theory implies a Langlands duality of the aforementioned invariants and Floer homologies.

7. Also, when we replace $M_3$ with $I \times C$ where $C \to 0$, from $S$-duality, we have a homological mirror symmetry of the category of $A$-branes

$$\text{Cat}_{A\text{-branes}}(\tau, \mathcal{M}_{\text{Higgs}}^G(C)) \leftrightarrow \text{Cat}_{A\text{-branes}}\left(-\frac{1}{n_q \tau}, \mathcal{M}_{\text{Higgs}}^{LG}(C)\right)$$

(1)
8. If $\text{Re}(\tau) = 0$, we obtain a quantum geometric Langlands correspondence

\[
\mathcal{D}_{c-Lh^\vee}^\text{mod}\left(q, \text{Bun}_{G_C}\right) \leftrightarrow \mathcal{D}_{c-Lh^\vee}^\text{mod}\left(- \frac{1}{n_gq}, \text{Bun}_{L G_C}\right)
\]

9. In the “classical” $\tau \to \infty$ limit, this becomes the classical geometric Langlands correspondence

\[
\text{Cat}_{\text{coh}}\left(\mathcal{M}_{\text{flat}}^G(C')\right) \leftrightarrow \mathcal{D}_{c-Lh^\vee}^\text{mod}\left(0, \text{Bun}_{L G_C}\right)
\]

10. One can also observe that by successively adding boundaries to the underlying manifold as we have done, the VW invariant will be categorified as

\[
\begin{align*}
\mathcal{Z}_{VW} \xrightarrow{\text{categorification}} & \text{HF}^V W \xrightarrow{\text{categorification}} \text{Cat}_{A\text{-branes}} \xrightarrow{\text{categorification}} \text{2-Cat}_{\text{mod-cat}}\left(\text{FF-cat}(T^2)\right)
\end{align*}
\]

The last 2-category is that of module categories over the Fukaya- Floer category of $T^2$ when we further let $C = I' \times S^1$ where $S^1 \to 0$, and it is assigned to $S^1$, just like how $\text{Cat}_{A\text{-branes}}$ is assigned to $C$, $\text{HF}^V W$ is assigned to $M_3$, and $\mathcal{Z}_{VW}$ is assigned to $M_4$. 

LET’S EXPLAIN HOW WE GOT THESE RESULTS
Vafa-Witten Theory on \( M_4 \)

- **VW theory has a single scalar supercharge** \( Q \), which BPS equations are obtained by setting the \( Q \)-variation of fermions to zero:

\[
F^+_{\mu\nu} + \frac{1}{2} [B_{\mu\nu}, C] + \frac{1}{4} [B_{\mu\rho}, B_{\lambda\nu}] g^{\rho\lambda} = 0,
\]

\[
\mathcal{D}_\mu C + \mathcal{D}_\nu B^{\nu\mu} = 0. \tag{2}
\]

- We then set \( C = 0 \) to ensure that there are no reducible gauge connections \( A \).

- The 2-form \( B \) need not vanish if the scalar curvature of \( M4 \) and the gauge group \( G \) are not simultaneously non-negative and locally a product of \( SU(2) \)'s, and we will assume this to be the case here.
VW theory is a balanced TQFT (same number of fermion pair zero modes), and the path integral localizes to a zero-dimensional moduli space $\mathcal{M}_{VW}$, whence the only non-vanishing topological invariant is the partition function

$$Z_{VW, M_4}(\tau, G) = \sum_k (-1)^{n_k + \tau m_k}$$

Here, $\tau$ is the complexified gauge coupling, the integer $n_k$ is determined by the corresponding sign of the determinant of an elliptic operator associated with a linearization of the VW equations at the $k^{th}$ point in $\mathcal{M}_{VW}$. The real number $m_k$ is given by the topological term

$$m_k = \frac{1}{4\pi^2} \int_{M_4} \text{Tr} \left( F_{(k)} \wedge F_{(k)} + dB_{(k)} \wedge \star DB_{(k)} + B_{(k)} \wedge d(\star DB_{(k)}) \right).$$
We consider a block diagonal metric $g$ for $M_4 = \Sigma \times C$, of genus $g \geq 1$ and $g \geq 2$, respectively.

$$g = \text{diag}(g_\Sigma, \epsilon g_C), \quad (5)$$

where $\epsilon$ is a small parameter to deform $g_C$.

When $\epsilon \to 0$, in order for the action to remain well-defined, i.e. finite, we obtain the following conditions along $C$:

$$F_C - \varphi \wedge \varphi = D\varphi = D^*\varphi = 0. \quad (6)$$

Here, $A_C$ and a section $\varphi \in \Omega^1(C)$ modulo gauge transformations span Hitchin's moduli space $\mathcal{M}_H^G(C)$.

We get a sigma model on $\Sigma$ with a map $\Phi(X, Y) : \Sigma \to \mathcal{M}_H^G(C)$, where the bosonic scalars $(X, Y)$ on $\Sigma$ correspond to $(A_C, \varphi_C)$.

The BPS equations of the sigma model are holomorphic maps, obtained from the dimensional reduction of (2):

$$\partial_{\bar{z}} X^i = \partial_{\bar{z}} Y^i = 0. \quad (7)$$
\[ \mathcal{N} = (4, 4) \text{ A-model, Higgs Bundles and GW Theory} \]

- This is an \( A \)-model, and it can be further shown that the complex structure is \( I \), whence the target space \( \mathcal{M}_H^G(C) = \mathcal{M}_{\text{Higgs}}^G(C) \).

- The topological invariant is the partition function, a GW invariant:

\[
\mathcal{Z}_{\text{GW}, \Sigma}(\tau, \mathcal{M}_{\text{Higgs}}^G(C)) = \sum_l (-1)^{p_l + \tau q_l}
\]

(8)

The integer \( p_l \) is determined by the corresponding sign of the determinant of an elliptic operator associated with the holomorphic map equations at the \( l^{th} \) point in \( \mathcal{M}_{\text{maps}} \), and the real number \( q_l \) is

\[
q_l = \frac{1}{\pi} \int_{\Sigma} \Phi_i^* (\omega_I).
\]

(9)

- Topological invariance implies a \textbf{4d-2d correspondence}:

\[
\mathcal{Z}_{\text{VW}, M_4}(\tau, G) = \mathcal{Z}_{\text{GW}, \Sigma}(\tau, \mathcal{M}_{\text{Higgs}}^G(C))
\]

(10)
Let $M_4 = M_3 \times \mathbb{R}^+$, where the $M_3$ boundary is a closed three-manifold, and $\mathbb{R}^+$ is the ‘time’ coordinate. Using the temporal gauge $A^0 = 0$, and exploiting the self-duality of the 2-form $B_{\mu\nu}$, VW equations in (2) become

\[
\dot{A}^i + \frac{1}{2} \epsilon^{ijk} (F_{jk} - [B_j, B_k]) = 0, \\
\dot{B}^i + \epsilon^{ijk} (\partial_j B_k + [A_j, B_k]) = 0.
\]  

(11)

If we define a complexified connection $A = A + iB \in \Omega^1(M_3)$, of a $G_{\mathbb{C}}$-bundle on $M_3$, we can rewrite (11) as a flow equation:

\[
\frac{dA^i}{dt} + s g^{ij}_{\mathbb{R}} \frac{\partial V(A)}{\partial A^j} = 0.
\]  

(12)

$\mathcal{A}$ is the space of complexified connections $A$ with metric $g^{ij}_{\mathbb{R}}$, $s$ is a tuneable parameter, and

\[
V(A) = -\frac{1}{4\pi^2} \int_{M_3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).
\]  

(13)
The 4d action of boundary VW theory can be rewritten as a 1d SQM model with target space $\mathcal{A}$:

$$S_{\text{bdry}}^{\text{VW}} = \frac{1}{e^2} \int dt \left( \frac{dA^i}{dt} + s g_{ij} \frac{\partial V(A)}{\partial A^j} \right)^2 + \ldots$$  \hspace{1cm} (14)

The partition function of boundary VW theory will localize onto the configurations that minimize the above term.

That is, it will be an algebraic count of critical points of the complex Chern-Simons functional, corresponding to flat $G_{\mathbb{C}}$-connections on $M_3$.

The complex Chern-Simons functional $V(A)$ is a Morse functional, which critical points generate a Floer complex.

VW flow lines between critical points, as described by the gradient flow equation (12), can be interpreted as Floer differentials, whence the number of outgoing flow lines at each critical point is the degree of the corresponding chain in the complex.
The partition function of boundary VW theory is originally expressed as

$$Z_{VW,M_4}(\tau, G) = \int_{M_{VW}} \mathcal{F}(\Psi_M^3) e^{-S_{VW}^{\text{bdry}}} = \sum_k \left\langle \mathcal{F}_{VW}^G(\Psi_k^3) \right\rangle.$$  \hspace{1cm} (15)

By comparing with the results from the 1d SQM perspective, we have

$$\mathcal{F}_{VW}^G(\Psi_k^3) \in HF_{VW}^{k}(M_3, G, \tau),$$  \hspace{1cm} (16)

where the HF’s are the VW Floer homology classes.

Thus, boundary VW theory allows us to define a novel VW Floer homology assigned to $M_3$ as

$$Z_{VW,M_4}(\tau, G) = \sum_k \mathcal{F}_{VW}^{G,\tau}(\Psi_k^3) = \sum_k HF_{VW}^{k}(M_3, G, \tau) = Z_{VW,M_3}^{\text{Floer}}(\tau, G)$$  \hspace{1cm} (17)

where the second and third expressions are understood to be expectation values of operators.
• We perform a Heegaard split of $M_3 = M_3' \cup_C M_3''$ along $C$ by writing $M_4 = (\mathbb{R}^+ \times I' \times C) \cup_C (\mathbb{R}^+ \times I'' \times C)$:

![Diagram](image)

• When $C \to 0$, we have an open $A$-model in complex structure $I$ on $\mathbb{R}^+ \times I'$ and $\mathbb{R}^+ \times I''$, with target space $\mathcal{M}^G_{\text{Higgs}}(C)$.

• These describe open strings with worldsheets $\mathbb{R}^+ \times I'$ and $\mathbb{R}^+ \times I''$ that propagate in $\mathcal{M}^G_{\text{Higgs}}(C)$ and end on $A$-branes. Specifically, we have a certain $(A, B, A)$-brane that is an $A$-brane in $\mathcal{M}^G_H(C)$ in complex structure $K$, that will correspond to flat $G_C$-connections which can be extended to $M_3', M_3''$, as required.
With two split pieces $M'_4$ and $M''_4$, when $C \to 0$, we have two strings, each ending on pairs of Lagrangian branes $(L_0, L')$ and $(L'', L_1)$.

Gluing the open worldsheets together along their common boundary $L'$ and $L''$ gives a single $A$-model on $\mathbb{R}^+ \times I$, with a single string extending from $L_0$ to $L_1$, which is equivalent to gluing $M'_4$ and $M''_4$ along $C \times \mathbb{R}^+$.

Next, we recast the $A$-model here as an SQM model, where the target space is $\mathcal{P}(L_0, L_1)$, the space of smooth trajectories from $L_0$ to $L_1$ (arising from the interval $I$ that connects them).
• The 2d BPS equation for this open $A$-model can be written as a **gradient flow equation** for the 1d SQM model

\[
\frac{\partial Z^l}{\partial t} + i \frac{\partial Z^l}{\partial s} = 0.
\]  

(18)

• Critical points correspond to **stationary trajectories in** $\mathcal{P}(L_0, L_1)$, i.e., the intersection points of $L_0$ and $L_1$, which generate the chains of the Lagrangian Floer complex.

• Intersection points belong to Lagrangian Floer homology classes

\[
\sum_i (L_0 \cap L_1)_i^n \in \text{HF}^{\text{Lagr}}_n(M^G_{\text{Higgs}}(C), L_0, L_1).
\]  

(19)

• Floer differentials are the **outgoing flow lines at each** $L_0 \cap L_1$, which number would be the degree of the corresponding chain in the complex.
• The partition function of the open $A$-model is then given by

$$Z_{A,L}(\tau, \mathcal{M}^G_{\text{Higgs}}(C)) = \sum_n \text{HF}^\text{Lagr}_n(\mathcal{M}^G_{\text{Higgs}}(C), L_0, L_1, \tau). \quad (20)$$

• Topological invariance gives an equivalence of partition functions, which implies

$$\sum_k \text{HF}^\text{VW}_k(M_3, G, \tau) = \sum_n \text{HF}^\text{Lagr}_n(\mathcal{M}^G_{\text{Higgs}}(C), L_0, L_1, \tau). \quad (21)$$

• It can be shown that we can identify the $k$ and $n$ indices, whence we have

$$\text{HF}^\text{VW}_*(M_3, G, \tau) \simeq \text{HF}^\text{Lagr}_*(\mathcal{M}^G_{\text{Higgs}}(C), L_0, L_1, \tau) \quad (22)$$

This gives a Vafa-Witten Atiyah-Floer Correspondence.
The Abouzaid-Manolescu Conjecture and its Generalization

• \((A, B, A)\)-branes of the open \(A\)-model can be interpreted as Lagrangian branes in \(\mathcal{M}_G^\mathcal{H}(C)\) in complex structure \(K\) i.e., \(\mathcal{M}_{\text{flat}}^{G_C}(C)\), the moduli space of irreducible flat \(G_C\)-connections on \(C\). The VW Atiyah-Floer correspondence is then the Atiyah-Floer correspondence for \(G_C\)-instantons.

\[
\text{HF}_{\text{inst}}^\ast(M_3, G_C, \tau) \cong \text{HF}_{\ast}^\text{Lagr}(\mathcal{M}_{\text{flat}}^{G_C}(C), L_0, L_1, \tau)
\]  

(23)

• \(\text{HP}^\ast(M_3)\) is the hypercohomology of the perverse sheaf of vanishing cycles in \(X_{\text{irr}}(C)\), the moduli space of irreducible flat \(SL(2, \mathbb{C})\)-connections on \(C\).

• There is a one-to-one correspondence between the gradings of \(\text{HP}^\ast(M_3)\) and \(\text{HF}_{\ast}^\text{Lagr}\), and the chains underlying their complexes, whence we can identify

\[
\text{HP}^\ast(M_3) \cong \text{HF}_{\ast}^\text{Lagr}(X_{\text{irr}}(C), L_0, L_1, \tau),
\]  

(24)

in agreement with [3, Remark 6.15].
The Abouzaid-Manolescu Conjecture and its Generalization

- According to (23) and (24), for $G_\mathbb{C} = SL(2, \mathbb{C})$, we have

$$\text{HP}^*(M_3) \cong \text{HF}_{\text{inst}}^*(M_3, SL(2, \mathbb{C}), \tau)$$

(25)

This is just the Abouzaid-Manolescu conjecture in [1].

- The Atiyah-Floer correspondence for $G_\mathbb{C}$ instantons in (23), and a $G_\mathbb{C}$ generalization of (24) in [3, Remark 6.15], imply that a generalization of the Abouzaid-Manolescu conjecture is also possible as

$$\text{HP}^*(M_3, G_\mathbb{C}) \cong \text{HF}_{\text{inst}}^*(M_3, G_\mathbb{C}, \tau)$$

(26)
- $\mathcal{N} = 4$ supersymmetric Yang-Mills theories have an $SL(2, \mathbb{Z})$ (or a subgroup thereof) symmetry on $\tau$ relating a theory to its $S$-dual, giving

$$\mathcal{Z}_{VW, M_4}(\tau, G) \leftrightarrow \mathcal{Z}_{VW, M_4}\left(-\frac{1}{n_g\tau}, LG\right)$$

$$\mathcal{Z}_{GW, \Sigma}(\tau, M^G_{\text{Higgs}}(C)) \leftrightarrow \mathcal{Z}_{GW, \Sigma}\left(-\frac{1}{n_g\tau}, M^{LG}_{\text{Higgs}}(C)\right)$$

$$\text{HF}^VW(M_3, G, \tau) \leftrightarrow \text{HF}^VW(M_3, LG, -1/n_g\tau)$$

$$\text{HF}^{\text{Lagr}}(M^G_{\text{Higgs}}(C), L_0, L_1, \tau) \leftrightarrow \text{HF}^{\text{Lagr}}(M^{LG}_{\text{Higgs}}(C), L_0, L_1, -1/n_g\tau)$$

$$\text{HP}^*(M_3, G_\mathbb{C}, \tau) \leftrightarrow \text{HP}^*(M_3, LG_\mathbb{C}, -1/n_g\tau)$$
A Quantum and Classical Geometric Langlands Correspondence

- If we let $M_4 = I \times \mathbb{R}^+ \times C$ with $C \to 0$, $S$-duality gives a homological mirror symmetry of the category of $A$-branes

$$\text{Cat}_{A\text{-branes}}(\tau, \mathcal{M}^G_{\text{Higgs}}(C)) \leftrightarrow \text{Cat}_{A\text{-branes}}\left(-\frac{1}{n_\mathfrak{g} \tau}, \mathcal{M}^L_G(C)\right)$$

(27)

- When $\text{Re}(\tau) = 0$, the category of $A$-branes can be identified with a category of twisted $D$-modules on $\text{Bun}_{G_C}(C)$ with parameter $q = \tau$ [4], whence we have

$$\mathcal{D}^c_{-h^\vee}\text{-mod}(q, \text{Bun}_{G_C}) \leftrightarrow \mathcal{D}^c_{-L h^\vee}\text{-mod}\left(-\frac{1}{n_\mathfrak{g} q}, \text{Bun}_{L G_C}\right)$$

(28)

a quantum geometric Langlands correspondence [2]. In the “classical limit” of $q \to \infty$, we have the classical correspondence

$$\text{Cat}_{\text{coh}}(\mathcal{M}^G_{\text{flat}}(C)) \leftrightarrow \mathcal{D}^c_{-L h^\vee}\text{-mod}\left(0, \text{Bun}_{L G_C}\right)$$

(29)
• Categorification is naturally realized in our physical framework:

VW theory on $M_4 \rightarrow$ number $\mathbb{Z}_{VW}$
VW theory on $\mathbb{R}^+ \times M_3 \rightarrow$ vector $\text{HF}^\ast_{VW}$
VW theory on $\mathbb{R}^+ \times I \times C \rightarrow$ 1-category $\text{Cat}_{A\text{-branes}}$
VW theory on $\mathbb{R}^+ \times I \times I' \times S^1 \rightarrow$ 2-category $2\text{-Cat}$
VW theory on $\mathbb{R}^+ \times I \times I' \times [0, 1] \rightarrow$ 3-category $3\text{-Cat}$.

(30)

• Notice from our discussion hitherto that as we go down the list, the categories get assigned to $M_3$, $C$, ..., and are determined by the category of boundaries of the effective 1d, 2d, ... theory on $\mathbb{R}^+$, $\mathbb{R}^+ \times I$, ...

• Therefore, the 2-category will be determined by the category of 2d boundaries of the 3d theory on $\mathbb{R}^+ \times I \times I'$ given by VW theory compactified on $S^1$, that is assigned to $S^1$. 
• These 2d boundaries can be realized as surface defects.

• For abelian $G$ and $\text{Re}(\tau) = 0$, the category of such surface defects is the 2-category $\text{2-Cat}_{\text{mod-cat}}(\text{FF-cat}(T^2))$ of module categories over the Fukaya-Floer category of $T^2$ [5].

• $S$-duality of VW theory then implies that they enjoy a Langlands duality

$$\text{2-Cat}_{\text{mod-cat}}(\text{FF-cat}(T^2)) \leftrightarrow \text{2-Cat}_{\text{mod-cat}}(\text{FF-cat}(L T^2))$$  \hspace{1cm} (31)

where $L T^2$ is the dual torus with the radii of the circles inverted.

• Similarly, the 3-Cat will be the 3-category of 3d boundary conditions of VW theory along $\mathbb{R}^+ \times I \times I'$, that is assigned to a point.
Conclusion

- We have physically derived a novel VW invariant of $M_4$.
- We have a 4d-2d correspondence between VW invariants and GW invariants.
- We have recast boundary VW theory as a 1d SQM model, thereby physically deriving a novel Vafa-Witten Floer homology.
- We went further to physically derive a Vafa-Witten Atiyah-Floer correspondence, which in turn allows us to physically realize and generalize the Abouzaid-Manolescu conjecture in [1].
- $S$-duality of VW theory allow us to obtain Langlands duals of all the invariants and Floer homologies. In certain cases, we can also obtain a quantum and classical geometric Langlands correspondence.
- Our physical framework also allows for a higher categorification of VW theory, whence $S'$-duality again implies a Langlands duality of the relevant higher categories.
THANKS FOR LISTENING!
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