Tau functions, infinite Grassmannians and lattice recurrences

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Abstract

The addition formulae for KP $\tau$-functions, when evaluated at lattice points in the KP flow group orbits in the infinite dimensional Sato-Segal-Wilson Grassmannian, give infinite parametric families of solutions to discretizations of the KP hierarchy. The CKP hierarchy may similarly be viewed as commuting flows on the Lagrangian sub-Grassmannian of maximal isotropic subspaces with respect to a suitably defined symplectic form. Evaluating the $\tau$-functions at a sublattice of points within the KP orbit, the resulting discretization gives solutions both to the hyperdeterminantal relations (or Kashaev recurrence) and the hexahedron (or Kenyon-Pemantle) recurrence.

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1 Introduction

The purpose of this work is to formalize and extend the well-known result [25, 29] that evaluation of any KP $\tau$-function $\tau^KP_w(t)$, corresponding to an element $w \in \text{Gr}_{H_+}(H)$ of the Sato-Segal-Wilson [28, 29, 33] infinite dimensional Grassmannian, at suitably defined parametric families of lattice points within the KP flow group orbit, leads to lattice discretizations of the KP hierarchy. The CKP hierarchy, viewed as a restriction to the subgroup of odd parameter KP flows acting upon the Lagrangian sub-Grassmannian [5, 1] $\text{Gr}^L_{H_+}(H,\omega) \subset \text{Gr}_{H_+}(H)$ consisting of maximal isotropic subspaces of the Hilbert space $H$, with respect to a symplectic form $\omega$, may be similarly discretized by evaluation of the corresponding KP $\tau$-function on a suitably defined parametric family of infinite sublattices [1] within the orbit of any given Lagrangian Grassmannian element $w^0 \in \text{Gr}^L_{H_+}(H,\omega)$.

The equations of the discretized KP hierarchy are equivalent to the infinite set of bilinear relations satisfied by the Plücker coordinates of any Grassmannian element $w$. The fact that these are also satisfied by evaluations of KP $\tau$-functions at suitably defined lattice points on the KP flow group orbit follows from the addition formulae ([29, 31], Sec. 3.10 of [9]). Viewed as solutions of lattice recurrence relations, they are generated by the octahedron relations, or discrete Hirota equations [12, 25], which correspond to the short (three-term) Plücker relations.

The discretized CKP hierarchy, on the other hand, consists of an infinite system of quartic equations [30], equivalent to the hyperdeterminantal relations [13, 27]), which are known to be satisfied by the principal minors of any symmetric matrix. These were also introduced by Kashaev [16] in his study of the star-triangle relations satisfied by Boltzmann weights in the Ising model. A further set of recurrence relations, the hexahedron relations, was introduced by Kenyon and Pemantle [17, 18] in their study of double dimer covers and rhombus tilings. These are known to be satisfied by certain of the Plücker coordinates of any Lagrangian Grassmannian element [1] or, equivalently, on the big cell, by the principal and next-to-principal minors of any symmetric matrix.

The addition formulae for KP $\tau$-functions ([29, 31], [9], Sec. 3.10) may be expressed in a form that is equivalent to the full (infinite) set of Plücker relations, which generically follow from the short ones ([9], App. C.9). Since these are also satisfied by suitably normalized evaluations of the corresponding KP $\tau$-function at lattice values, this provides infinite families of solutions to the octahedron recurrence relations. A similar result holds for the quartic hyperdeterminantal relations [1], which are satisfied both by a suitably chosen subset of the Plücker coordinates of an element of the infinite Lagrangian Grassmannian, and by correctly normalized lattice evaluations of the corresponding $\tau$-functions of CKP type.

Subsection 1.1 gives a brief review of the Sato-Segal-Wilson formulation of the KP hierarchy as abelian flows on an infinite dimensional Grassmannian. The fermionic vacuum
expectation value (VEV) representation of the $\tau$-function and its Schur function expansion is recalled in Subsection 1.2. The KP addition theorem [29, 31] is recalled in Subsection 1.3, together with Sato’s interpretation [29] of the latter in terms of Plücker relations. The CKP reduction of the KP hierarchy, which involves restriction to the subgroup of odd flows on the Lagrangian Grassmannian [5, 1, 20] is summarized in Subsection 1.4.

The main new results, stated in Subsections 1.5 and 1.6, and proved in Sections 5 and 6, consist of showing that suitable lattice evaluations of KP $\tau$-functions of the CKP type determine solutions of the Kashaev recursions (or “core” hyperdeterminantal relations) and the Kenyon-Pemantle hexahedron relations.

**Remark 1.1.** An analogous reduction to the sub-Grassmannian $\text{Gr}_0^{\mathcal{H}_+} (\mathcal{H}, Q)$ of maximal isotropic subspaces with respect to a quadratic form $Q$ leads similarly to the BKP (or DKP) hierarchy, but the corresponding $\tau$-functions are defined as Fredholm Pfaffians rather than determinants (see [2], and Sec. 7.1 and App. E of [9]). These satisfy both the BKP analogue of the Hirota bilinear residue equations [5, 7, 14] and an analogous set of addition formulae ([31] and [9], Sec. 7.2). They may again be related to bilinear equations (the Cartan equations [4], [9], App. E2) satisfied by the expansion coefficients in the natural basis for the corresponding projectivized exterior space into which $\text{Gr}_0^{\mathcal{H}_+} (\mathcal{H}, Q)$ is mapped under the Cartan embedding, which is the orthogonal group analogue of the Plücker embedding (see [4] and [9], Sec. 7.1 and App. E). The addition formulae may similarly be interpreted as discretizations of the BKP (or DKP) hierarchy, and solutions obtained by replacing the Cartan coefficients by suitably normalized evaluations of the BKP $\tau$-functions at lattice points within the BKP flow group orbit. The minimal generating set for these relations are the short Cartan relations, which may also be interpreted as a system of recurrence relations, the Miwa relations [25], or cube recursions (see, e.g., [9], Sec. 7.7.2). These will not be further considered here, but will form the subject of a subsequent work.

### 1.1 KP $\tau$-functions and infinite Grassmannians

#### 1.1.1 Hirota residue equation

Solutions of the KP hierarchy of integrable PDE’s are given in terms of an associated $\tau$-function $\tau^{KP}_w (t)$, which depends on an infinite set of KP flow variables

$$ t := (t_1, t_2, \cdots) \quad \text{(1.1.1)} $$

and satisfies the formal Hirota bilinear residue equation [28, 33, 14, 15, 9]

$$ \text{res} \left( e^{\xi (\delta t, z)} \tau^{KP}_w (t + \delta t + [z^{\perp}]) \right) = 0, \quad \text{(1.1.2)} $$
where
\[
\delta t := (\delta t_1, \delta t_2, \ldots), \quad [z] := (z, \frac{z^2}{2}, \ldots, \frac{z^i}{i}, \ldots), \quad \xi(t, z) := \sum_{i=1}^{\infty} t_i z^i.
\]  
(1.1.3)

For any formal power series
\[
f(z) = \sum_{i \in \mathbb{Z}} f_i z^i,
\]
(1.1.4)
the formal residue appearing in (1.1.2) is defined as
\[
\text{res}_\infty(f) := f_{-1}
\]
(1.1.5)
and eq. (1.1.2) is understood as satisfied identically in the parameters \(\delta t = (\delta t_1, \delta t_2, \ldots)\).

1.1.2 Infinite Grassmannians

There are two different approaches to the definition of the infinite Grassmannian appearing in relation to \(\tau\)-functions: one due to Sato [28, 29], the other to Segal and Wilson [33]. The formulation of Segal and Wilson [33], is functional-analytic and geometric in flavour, with the Grassmannian \(\text{Gr}_{\mathcal{H}^+}(\mathcal{H})\) defined as a Banach manifold, and the KP flow variables viewed as additive coordinates on an infinite dimensional abelian transformation group \(\Gamma_+\) acting on it. In the approach of Sato [28, 29, 14, 5] the infinite Grassmannian is identified as a direct limit of finite dimensional Grassmannians. The group is not viewed as acting continuously, but determines the dynamics via a formal expansion of the \(\tau\)-function as a linear combination of Schur functions, with coefficients equal to the Plücker coordinates of the initial point \(w\). (The formal series interpretation is of particular relevance when \(\tau\)-functions are used as generating functions for enumerative combinatorial invariants (see, e.g. [9], Chapts. 13 and 14)).

In the present work, we mainly use the Sato approach, viewing \(\tau\)-functions as formal series in the flow parameters, which are interpreted as indeterminates, and emphasize the coordinate rings, rather than the geometry of flows on the Grassmannian. But for the purposes of this introduction, we begin with the geometrical viewpoint, which corresponds more readily to the intuitive notion of group actions on Grassmannians.

KP \(\tau\)-functions \(\tau_{w}^{KP}(t)\) are parametrized [28, 29, 33] by elements \(w\) of an infinite dimensional Grassmannian manifold \(\text{Gr}_{\mathcal{H}^+}(\mathcal{H})\), which are subspaces \(w \subset \mathcal{H}\) of an infinite dimensional vector space \(\mathcal{H}\), with denumerable basis \(\{e_i\}_{i \in \mathbb{Z}}\) and polarization
\[
\mathcal{H} = \mathcal{H}^+ + \mathcal{H}^-, \quad \mathcal{H}^+ := \text{span}\{e_{-i}\}_{i \in \mathbb{N}^+}, \quad \mathcal{H}^- := \text{span}\{e_{i}\}_{i \in \mathbb{N}},
\]
(1.1.6) that are commensurable with \(\mathcal{H}^+ \subset \mathcal{H}\) (in the sense defined in [33]). \(\text{Gr}_{\mathcal{H}^+}(\mathcal{H})\) is a homogeneous space of the infinite dimensional group \(\text{GL}(\mathcal{H})\) of invertible linear transformations of \(\mathcal{H}\) preserving admissible frames [28, 29, 33]. (Here, \(\mathbb{Z}, \mathbb{N}\) and \(\mathbb{N}^+\) denote the set of
all integers, nonnegative integers and positive integers, respectively.) The KP dynamics may be understood geometrically as the action $\Gamma_+ \times \text{Gr}_{H_+}(H) \to \text{Gr}_{H_+}(H)$ of an infinite abelian subgroup $\Gamma_+ \subset \text{GL}(H)$ on $\text{Gr}_{H_+}(H)$.

**Remark 1.2.** For precise definitions of: *subspaces* $w \subset H$ *commensurable* with $H_+$, *admissible frames* for $w \in \text{Gr}_{H_+}(H)$, the *infinite group* $\text{GL}(H)$ of invertible linear transformation of $H$, its action on the bundle of frames over $\text{Gr}_{H_+}(H)$ and how the determinant (1.1.19) is defined in terms of lifts of the $\Gamma_+$-action to the dual determinantal line bundle $\text{Det}^* \to \text{Gr}_{H_+}(H)$, see [33] and [9], Chapt. 3.

In order to relate the above to the definition of infinite dimensional Grassmannians in terms of inverse limits of finite coordinate sheaves used in Section 3, the spaces $H, H_+, H_-$ must be viewed as suitably defined completions.

### 1.1.3 Fermionic Fock space and Plücker map

We may embed $\text{Gr}_{H_+}(H)$, into the projectivization of an associated fermionic Fock space, which is viewed as a graded space consisting of the semi-infinite wedge product of $H$ with itself,

$$\mathcal{F} := \bigwedge^{\infty/2} H = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n,$$

via the infinite dimensional analogue of the Plücker map [8],

$$\Psi : \text{Gr}_{H_+}(H) \to \mathbb{P}(\mathcal{F})$$

$$\Psi : \text{span}\{w_1, w_2, \ldots\} \mapsto [w_1 \wedge w_2, \wedge \cdots] \in \mathbb{P}(\mathcal{F}). \quad (1.1.8)$$

Here $\mathcal{F}_n \subset \mathcal{F}$ is the graded sector with fermionic charge $n$, which is spanned by orthonormal basis elements

$$|\lambda; n\rangle := e_{l_1} \wedge e_{l_2} \wedge \cdots \quad (1.1.9)$$

labelled by pairs $(\lambda, n)$ of an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ and an integer $n \in \mathbb{Z}$, where the indices $(l_1, l_2, \ldots)$ are the *particle locations* (see [14], [33] and [9], Chapt. 5):

$$l_i := \lambda_i - i + n, \quad (1.1.10)$$

which form a strictly decreasing sequence of integers that eventually saturate at all consecutive decreasing integers. The vacuum vector in each sector $\mathcal{F}_n$ corresponds to the null partition and is denoted

$$|n\rangle := |\emptyset; n\rangle. \quad (1.1.11)$$

For $w \in \text{Gr}_{H_+}(H)$ of *virtual dimension* $n$ (i.e. for which the projection map

$$\Pi_+ : w \to H_+ \quad (1.1.12)$$
along $\mathcal{H}_-$ is Fredholm, with index $n$ [33]), the image $\mathbf{P}(w) \in \mathbf{P}(\mathcal{F}_n)$, of the Plücker map may be expressed as a linear combination of the basis elements

$$\mathbf{P}(w) = \left[ \sum_{\lambda} \pi_\lambda(w) |\lambda; n\rangle \right] \in \mathbf{P}(\mathcal{F}_n),$$

(1.1.13)

where the coefficients $\pi_\lambda(w)$ are (within projectivization) the Plücker coordinates of $w$. Bose-Fermi equivalence ([9], Chapt. 5.6) gives the corresponding expansion of $\tau^\text{KP}_w(t)$ in a basis of Schur functions [23, 29]

$$\tau^\text{KP}_w(t) = \sum_{\lambda} \pi_\lambda(w) s_\lambda(t),$$

(1.1.14)

where the $t_i$'s are interpreted as normalized power sums in a set of auxiliary variables $\{x_a\}_{a \in \mathbb{N}}$

$$t_i = \frac{p_i}{i} := \frac{1}{i} \sum_{a \in \mathbb{N}} x_a^i.$$

(1.1.15)

(The Hirota residue equation (1.1.2) is then equivalent to the fact that the $\pi_\lambda(w)$'s satisfy the full set of Plücker relations (2.1.9). (See eqs. (2.1.7), (2.2.2) and (2.2.5) for detailed definitions of the notations used.)

Equivalently, let

$$\Lambda : e_i \mapsto e_{i-1}, \quad i \in \mathbb{Z},$$

(1.1.16)

be the shift map $\Lambda : \mathcal{H} \to \mathcal{H}$ and denote the infinite abelian group of KP shift flows $\Gamma_+ \subset \text{GL}(\mathcal{H})$ by

$$\Gamma_+ := \{ \gamma_+(t) := e^{\sum_{i=1}^{\infty} t_i \Lambda_i} \}
\quad \text{for} \quad t = (t_1, t_2, \ldots), \quad t_i \in \mathbb{C}, \quad i \in \mathbb{N}^+.$$

(1.1.17)

Applying the elements $\gamma_+(t) \in \Gamma_+$ multiplicatively to the basis elements $\{w_i\}_{i \in \mathbb{N}^+}$ spanning $w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ induces an action of $\Gamma_+$ as commuting flows on the Grassmannian,

$$\Gamma_+ : \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$$

$$\gamma_+(t) : w \mapsto = w(t) := \text{span}\{w_i(t) := \gamma_+(t)w_i\}$$

(1.1.18)

Denoting the projection map from $w(t)$ to $\mathcal{H}_+$ along $\mathcal{H}_-$ as $\pi_+ : w(t) \to \mathcal{H}_+$ and choosing an admissible basis [33] for $w(t) \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$, allowing it to be identified isomorphically with $\mathcal{H}_+$, the KP $\tau$-function may be defined as the determinant of the projection map

$$\tau^\text{KP}_w(t) := \det(\Pi_+ : w(t) \to \mathcal{H}_+).$$

(1.1.19)
1.2 Fermionic representation of KP \( \tau \)-functions

The fermionic representation of the Clifford algebra on \( \mathcal{H} \oplus \mathcal{H}^* \) with respect to the natural scalar product

\[
Q(e_i, e_j^*) = \delta_{ij}, \quad Q(e_i, e_j) = Q(e_i^*, e_j^*) = 0, \tag{1.2.1}
\]

where \( \{e_j^*\}_{j \in \mathbb{Z}} \) is the basis for \( \mathcal{H}^* \) dual to the basis \( \{e_j\}_{j \in \mathbb{H}} \) for \( \mathcal{H} \) as endomorphisms of \( \mathcal{F} \), is generated by the creation and annihilation operators

\[
\psi_i := e_i \wedge \in \text{End}(\mathcal{F}), \quad \psi_i^\dagger := i e_i^\ast \in \text{End}(\mathcal{F}), \quad i \in \mathbb{Z} \tag{1.2.2}
\]

defined, respectively, as outer and inner products with respect to the basis elements. These satisfy the usual fermionic anticommutation relations

\[
[\psi_i, \psi_j]_+ = 0, \quad [\psi_i^\dagger, \psi_j^\dagger]_+ = 0, \quad [\psi_i, \psi_j^\dagger]_+ = \delta_{ij}, \quad i, j \in \mathbb{Z} \tag{1.2.3}
\]

and vacuum annihilation conditions

\[
\psi_i|0\rangle = 0, \quad \psi_i^\dagger|0\rangle = 0, \quad \forall i \in \mathbb{N}^+. \tag{1.2.4}
\]

In terms of the Frobenius index notation for partitions \( \lambda \) (i.e., the “arm” and “leg” lengths in the corresponding Young diagrams [23])

\[
\lambda = (a|b) = (a_1, \ldots, a_r|b_1, \ldots, b_r), \tag{1.2.5}
\]

the basis vectors (1.1.9) are (see [14, 15], or [9], Chapt. 5.1)

\[
|\lambda; n\rangle = (-1)^{\sum_{i=1}^r b_i} \psi_{a_i+n} \psi_{b_i+n-1}^\dagger |n\rangle. \tag{1.2.6}
\]

For an exponential element \( g \in \text{GL}(\mathcal{H}) \)

\[
g = e^A, \quad A := \sum_{ij \in \mathbb{Z}} A_{ij} E_{ij} \in \text{gl}(\mathcal{H}), \quad E_{ij}(e_k) = e_i \delta_{jk}, \tag{1.2.7}
\]

we have the Clifford representation

\[
\hat{g} = e^{\sum_{i,j \in \mathbb{Z}} A_{ij} \psi_i \psi_j^\dagger}, \tag{1.2.8}
\]

where

\[
:\psi_i \psi_j^\dagger := \psi_i \psi_j^\dagger - \langle 0 | \psi_i \psi_j^\dagger |0\rangle \tag{1.2.9}
\]

is the normal ordered product. As indicated in (1.1.13), the Plücker image of an element \( w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \) of virtual dimension \( n \) is in the fermionic charge sector \( \mathcal{F}_n \). In particular, the subspace

\[
\mathcal{H}_+^n := \text{span}\{e_{n-i}, \ldots\}_{i \in \mathbb{N}^+} \tag{1.2.10}
\]
has virtual dimension \( n \), and its Plücker image is (the projectivization of) the vacuum vector
\[
[n] = \mathfrak{P}(\mathcal{H}^+_n) \in \mathbb{P}(\mathcal{F}_n)
\]
in the sector \( \mathcal{F}_n \). Any element \( w \in \text{Gr}_{\mathcal{H}^+_n}(\mathcal{H}) \) with virtual dimension \( n \) may be expressed as
\[
w = g(\mathcal{H}^+_n),
\]
where \( g \in \text{GL}^0(\mathcal{H}) \) is in the identity component of \( \text{GL}(\mathcal{H}) \), and is determined up to right multiplication by an element of the stabilizer of \( \mathcal{H}^+_n \).

The fermionic representation of the KP flow group \( \Gamma_+ \) on \( \mathcal{F} \) is given by
\[
\gamma_+(t) \mapsto \hat{\gamma}_+(t) = e^{\sum_{i=1}^{\infty} t_i J_i} \in \text{End}(\mathcal{F}),
\]
where
\[
J_i := \sum_{j \in \mathbb{Z}} \psi_j \psi^\dagger_{j+i}, \quad i \in \mathbb{N}^+
\]
are the negative Fourier components of the current operator.

From the equivariance of the Plücker map (1.1.8), the image \( \mathfrak{P}(w(t)) \) of the element
\[
w(t) = \gamma_+(t)w
\]
in the \( \Gamma_+ \) orbit of \( w \in \text{Gr}_{\mathcal{H}^+_n}(\mathcal{H}) \) is
\[
\mathfrak{P}(w(t)) = [[\hat{\gamma}_+(t)|w]\]
where
\[
[[w]] := \mathfrak{P}(w).
\]

It follows from the determinantal interpretation of the Plücker coordinates of \( \mathfrak{P}(w(t)) \) and definition (1.1.19), that \( \tau^K_P(w(t)) \) is expressible, within a projective normalization factor, as the fermionic vacuum expectation value
\[
\tau^K_P(w(t)) = \langle n|\gamma_+(t)|w\rangle. \tag{1.2.20}
\]
If \( w \) is expressed as in (1.2.12), we have \( \mathfrak{P}(w) \in \mathbb{P}(\mathcal{F}_n) \) and this becomes
\[
\tau^K_P(w(t)) = \langle n|\gamma_+(t)|\hat{g}|n\rangle. \tag{1.2.21}
\]

The Plücker coordinates are the fermionic matrix elements
\[
\pi_\lambda(w) = \langle \lambda; n|\hat{g}|n\rangle, \tag{1.2.22}
\]
so (1.1.19) is equivalent to
\[
\tau^K_P(w(t)) = \pi_0(\mathfrak{P}(w(t))). \tag{1.2.23}
\]
1.3 KP addition formulae, lattice evaluations and discrete KP

A standard result in the theory of KP $\tau$-functions is that the Hirota residue equation (1.1.2) is also equivalent to an infinite set of multi-parametrically defined addition formulae ([29, 31], [9], Chapt. 3) satisfied by evaluations of $\tau^KP_w(t)$ at lattice of points in the $\Gamma+$ orbit $\{w(t)\}$ of $w$. In the notation of [9], for a finite set of $n$ parameters $(x_1, \ldots, x_n)$, let

$$\zeta_n(t, x_1, \ldots, x_n) := \prod_{i<j} (x_i - x_j) \tau^KP_w(t + \sum_{i=1}^{n} x_i). \quad (1.3.1)$$

We then have ([9], Sec. 3.10).

**Theorem 1.1 (KP addition formulae).** If $\tau(t)$ satisfies the Hirota bilinear equations (1.1.2), then for $k \in \mathbb{N}^+$ and any $2k$ distinct parameters $(x_1, \ldots, x_{k-1}; y_1, \ldots, y_{k+1})$, the following addition formulae hold:

$$\sum_{j=1}^{k+1} (-1)^j \zeta_k(t; x_1, \ldots, x_{k-1}, y_j) \zeta_k(t; y_1, \ldots, \hat{y_j}, \ldots, y_{k+1}) = 0, \quad (1.3.2)$$

where $(y_1, \ldots, \hat{y}_j, \ldots, y_{k+1})$ denotes the sequence of $k$ parameters obtained by omitting $y_j$ from the sequence $(y_1, \ldots, y_{k+1})$.

For an infinite sequence of parameters $x = (x_1, x_2, \ldots)$, we define (cf. eq. (4.1.9)), Subsection 4.1) the following normalized evaluations of $\tau^KP_w(t)$ at lattice points,

$$H^n_w(t) := \prod_{i<j} (x_i - x_j)^{n_i n_j} \tau^KP_w(t + \sum_{i=-\infty}^{+\infty} n_i [x_i]), \quad (1.3.3)$$

where

$$n = \sum_{i \in \mathbb{Z}} n_i \alpha_i \in (\mathbb{Z})^\infty \quad (1.3.4)$$

is any infinite integer lattice element with a finite number of nonzero components $\{n_i\}_{i \in \mathbb{Z}}$ and $\{\alpha_i\}_{i \in \mathbb{Z}}$ are the unit basis elements having 1 in position $i$ and 0 elsewhere. For any choice of lattice element $n$, initial flow group element $t_0$, and partition $\lambda$ with Frobenius indices $(a_1, \ldots, a_r| b_1, \ldots, b_r)$, define $H^n_w^\lambda(t_0)$ as

$$H^n_w^\lambda(t_0) := H^n_w + \sum_{i=1}^{+\infty} \alpha_i - \sum_{i=1}^{-b_i} \alpha_i^{-1}(t_0). \quad (1.3.5)$$

As pointed out by Sato [29], the addition formulae (Theorem 1.1) then imply, for any fixed choice of lattice element $n$ and initial value $t_0$, that the $H^n_w^\lambda(t_0)$’s satisfy the same infinite set of Plücker relations (2.1.9) as the Plücker coordinates $\{\pi_\lambda(w)\}$. This is stated and proved inductively in Theorem 4.8.
In particular, these include the discrete Hirota equations [12, 34] (or octahedron recurrences [25, 32], or $T$-relations [21]), which define a discretization of the KP hierarchy, and may be expressed [25, 26, 34] as

$$a(b-c)\tau_{l+1,m,n} + b(c-a)\tau_{l,m+1,n} + c(a-b)\tau_{l,m,n+1} = 0,$$

(1.3.6)

where

$$\tau_{l,m,n} := \tau^{KP}_{w}(t_0 + l[a] + m[b] + n[c]), \quad l, m, n \in \mathbb{Z}.$$

(1.3.7)

**Remark 1.3.** The recurrences associated to these discretizations take the same form as the short Plücker relations satisfied by the Plücker coordinates of a single element of the Grassmannian $\text{Gr}_{\mathcal{H}+}(\mathcal{H})$. But the relations are associated here to a group of discrete commutative flows on a line bundle over the Grassmannian.

### 1.4 Lagrangian Grassmannians and the CKP hierarchy

A symplectic form $\omega$ may be defined on $\mathcal{H}$ by evaluation on the basis elements $\{e_i\}_{i \in \mathbb{Z}}$

$$\omega(e_i, e_j) = -\omega(e_j, e_i) := (-1)^i \delta_{i,-j-1}, \quad i, j \in \mathbb{Z}.$$

(1.4.1)

The symplectic subalgebra $\mathfrak{sp}(\mathcal{H}, \omega) \subset \mathfrak{gl}(\mathcal{H})$ consists of the elements $X \in \mathfrak{gl}(\mathcal{H})$ satisfying

$$\omega(Xu, v) + \omega(u, Xv) = 0, \quad \forall \ u, v \in \mathcal{H}.$$

(1.4.2)

The symplectic subgroup $\text{Sp}(\mathcal{H}, \omega) \subset \text{GL}(\mathcal{H})$, with Lie algebra $\mathfrak{sp}(\mathcal{H}, \omega)$, consists of elements $g^0 \in \text{GL}(\mathcal{H})$ that preserve $\omega$:

$$\text{Sp}(\mathcal{H}, \omega) = \{g^0 \in \text{GL}(\mathcal{H}) \mid \omega(g^0 u, g^0 v) = \omega(u, v), \quad \forall \ u, v \in \mathcal{H}\}.\quad (1.4.3)$$

The Lagrangian Grassmannian $\text{Gr}^{\mathbb{L}}_{\mathcal{H}+}(\mathcal{H}, \omega) \subset \text{Gr}_{\mathcal{H}+}(\mathcal{H})$ is the sub-Grassmannian consisting of maximal isotropic subspaces $w^0 \in G_{\mathcal{H}+}(\mathcal{H})$; i.e. those on which the restriction of $\omega$ vanishes

$$\omega|_{w^0} = 0, \quad \forall \ w^0 \in \text{Gr}^{\mathbb{L}}_{\mathcal{H}+}(\mathcal{H}, \omega).\quad (1.4.4)$$

This is a homogeneous space of the symplectic group $\text{Sp}(\mathcal{H}, \omega)$, and may be viewed as the orbit of the element $\mathcal{H}+ \subset \mathcal{H}$

$$\text{Gr}^{\mathbb{L}}_{\mathcal{H}+}(\mathcal{H}, \omega) = \text{Sp}(\mathcal{H}, \omega)(\mathcal{H}+).\quad (1.4.5)$$

Restricting to elements $w^0 = g^0(\mathcal{H}+)$ of the Lagrangian Grassmannian, where $g^0 \in \text{Sp}(\mathcal{H}, \omega)$, determines a KP $\tau$-function with fermionic representation

$$\tau^{KP}_{w^0}(t) = \langle 0| \hat{\gamma}_+(t) \hat{g}^0 |0 \rangle.$$

(1.4.6)
which, as shown in [5, 7, 20], must satisfy the following symmetry condition

\[ \tau_{w_0}^{KP}(t) = \tau_{w_0}^{KP}(\tilde{t}), \]  

(1.4.7)

where

\[ \tilde{t} := (t_1, -t_2, t_3, -t_4, \ldots). \]  

(1.4.8)

The full KP flow group \( \Gamma_+ \) does not leave \( \text{Gr}_{H_+}(H, \omega) \) invariant, but the subgroup \( \Gamma'_+ \subset \Gamma_+ \) consisting of purely odd flows

\[ \Gamma'_+ := \Gamma_+ \cap \text{Sp}(H, \omega) = \{ \gamma_+(t'), \ t' := (t_1, 0, t_3, 0, \ldots) \} \]  

(1.4.9)

does. The corresponding Baker function, restricted to vanishing even values

\[ t = t' := (t_1, 0, t_3, 0, \ldots) \]  

(1.4.10)

of the KP flow variables is given by the Sato formula [28, 29]

\[ \Psi_{w_0}(z, t') := e^{\xi(z, t')} \frac{\tau_{w_0}^{KP}(t' - [z^{-1}]')}{\tau_{w_0}^{KP}(t')}, \]  

(1.4.11)

where

\[ \xi(z, t') := \sum_{j=1}^{\infty} t_{2j-1} z^{2j-1}, \quad [z^{-1}]' := \left( z^{-1}, 0, \frac{1}{3} z^{-3}, \frac{1}{5} z^{-5}, 0, \ldots \right), \]  

(1.4.12)

This satisfies the Hirota bilinear residue equation in the form [5, 7]

\[ \text{res}_{z=\infty} (\Psi_{w_0}(z, t')\Psi_{w_0}(-z, t' + \delta t')) \, dz = 0 \]  

(1.4.13)

identically in

\[ \delta t' = (\delta t_1, 0, \delta t_3, 0, \ldots) \]  

(1.4.14)

which gives the reduction of the KP to the CKP hierarchy [5, 6, 7, 20].

1.5 Lattice evaluations of \( \tau \)-functions and hyperdeterminantal relations

Let \( B \) be the infinite sublattice of \((\mathbb{Z})^N\) consisting of infinite sequences \( \mathbf{n'} = (\ldots, n'_i, n'_{i+1}, \ldots) \), \( n'_i \in \mathbb{Z} \), where only a finite number of the entries are nonzero, and these satisfy

\[ n'_{i-1} = -n'_i \quad \forall \ i \in \mathbb{N}^+. \]  

(1.5.1)
Choosing an element \( w^0 \in \text{Gr}^C_{H^+}(\mathcal{H}, \omega) \) and an infinite set of parameters \( \{y_i\}_{i \in \mathbb{N}^+} \) we define, similarly to (1.3.3), a function \( h \) on \( B \) whose values are

\[
h^{n'} := \prod_{1 \leq i < j} (y_i - y_j)^{n_{i-j} + n_{j-i} - i} \prod_{i,j=1}^\infty (-1)^{n_{i-j} - 1}(y_i + y_j)^{n_{i-j} - 1}
\]

\[
\times \tau_{w^0}^{KP} \left( t' + \sum_{i=1}^\infty \left( n'_{i-1}[y_i] + n'_{i-1}[-y_i] \right) \right),
\]

(Cf. eqs. (5.1.6), (6.3.7), (6.3.8)).

Let \( \{\beta_i\}_{i \in \mathbb{N}^+} \) be the generating basis for \( B \) consisting of elements \( \beta_i \) with just two nonvanishing components

\[
n'_j = 1, \quad n'_{-j-1} = -1 \quad \text{for } j = i - 1, \quad \text{and } n_j = 0 \quad \text{for all other } j \in \mathbb{Z}.
\]

For all distinct positive integer \( l \)-tuples \((i_1, \ldots, i_l)\), denote as \( h_{\beta_{i_1}, \ldots, \beta_{i_l}} \) the corresponding lattice functions on \( B \) with values

\[
h_{\beta_{i_1}, \ldots, \beta_{i_l}}^{n'} := h^{n' + \beta_{i_1} + \cdots + \beta_{i_l}}, \quad n' \in B,
\]

given by shifting the argument by such a sum of generators. Proposition 6.8 says that, choosing \( l = 0, 1, 2 \) or 3 in (1.5.4), for all triples \( i, j, k \in \mathbb{N}^+ \), \( i < j < k \), these provide solutions to the system

\[
\begin{align*}
&h^2h_{\beta_{i} + \beta_{j} + \beta_{k}} + h_{\beta_{i} + \beta_{j} + \beta_{k}}^2 + h_{\beta_{j} + \beta_{i} + \beta_{k}}^2 + h_{\beta_{k} + \beta_{i} + \beta_{j}}^2 + h_{\beta_{i} + \beta_{j} + \beta_{k}}^2
\quad - 2hh_{\beta_{i} + \beta_{j} + \beta_{k}}(h_{\beta_{i} + \beta_{j} + \beta_{k}} + h_{\beta_{j} + \beta_{i} + \beta_{k}} + h_{\beta_{k} + \beta_{i} + \beta_{j}}) \\
&\quad - 2(h_{\beta_{i} + \beta_{j} + \beta_{k}} + h_{\beta_{j} + \beta_{i} + \beta_{k}} + h_{\beta_{k} + \beta_{i} + \beta_{j}} + h_{\beta_{i} + \beta_{j} + \beta_{k}})
\quad + 4hh_{\beta_{i} + \beta_{j} + \beta_{k}} = 0,
\end{align*}
\]

known as the \( 2 \times 2 \times 2 \) hyperdeterminantal relations, which are satisfied by the principal minors of any symmetric \( N \times N \) matrix [13, 27]. In the theory of integrable systems, they arise as the Kashaev recurrence [16], and are also interpretable as a discretization of the CKP hierarchy [30, 3]. Proposition 6.8 derives these as a consequence of Theorem 5.6, which gives a homomorphism, as commutative algebras, from the coordinate algebra of the infinite dimensional Lagrangian Grassmannian \( \text{Gr}^C_{H^+}(\mathcal{H}, \omega) \) to a localization of the ring of formal power series in the infinite sequence of odd KP flow variables \( t' := (t_1, 0, t_3, 0, \ldots) \) to which the \( \tau \)-function is restricted.

This provides an infinite lattice extension of Corollary 3.20 of [1], where the hyperdeterminantal relations (1.5.5) are shown satisfied by finite lattice evaluations of the \( \tau \)-function. (Cf. eq. (3.6.43) of [1], where \( \sigma, \sigma_i, \sigma_{ij}, \sigma_{ijk} \) (for \( i, j, k \in \{1, \ldots, N\} \)) are essentially the same as \( h_{\beta_{i}}, h_{\beta_{i} + \beta_{j}}, h_{\beta_{i} + \beta_{j} + \beta_{k}}, \) up to an overall rational normalization factor in the parameters).
1.6 Lagrangian Plücker coordinates hexahedron relations

Kenyon and Pemantle [17, 18] introduced a system of relations that they called the hexahedron recurrence, and applied it to the study of double dimer covers and rhombus tilings. These are defined as follows. Let \( \hat{h}, \hat{h}^{(x)}, \hat{h}^{(y)}, \hat{h}^{(z)} \) be four functions on the 3-dimensional lattice

\[
B_3 := \text{span}_\mathbb{Z}\{\beta_1, \beta_2, \beta_3\}
\]

with basis vectors \( \beta_1, \beta_2, \beta_3 \), and values at any \( n' \in B_3 \) denoted \( \hat{h}^{(n')}, \hat{h}^{(n',x)}, \hat{h}^{(n',y)}, \hat{h}^{(n',z)} \), respectively. For any \( v' \in B_3 \), define four additional functions \( \hat{h}_{v'}, \hat{h}_{v'}^{(x)}, \hat{h}_{v'}^{(y)}, \hat{h}_{v'}^{(z)} \) with values

\[
\hat{h}_{v'}^{(n')} := \hat{h}^{n'+v'}, \quad \hat{h}_{v'}^{(n',x)} := \hat{h}^{n'+v',x}, \quad \hat{h}_{v'}^{(n',y)} := \hat{h}^{n'+v',y}, \quad \hat{h}_{v'}^{(n',z)} := \hat{h}^{n'+v',z},
\]

obtained by translating the evaluation point by \( v' \). These satisfy the hexahedron recurrence if *

\[
\begin{align*}
\hat{h}^{(x)} h_{\beta_1} & = \hat{h} h_{\beta_1} \hat{h}_{\beta_2 + \beta_3} + \hat{h} h_{\beta_1} \hat{h}_{\beta_2} \hat{h}_{\beta_3} + \hat{h}^{(y)} \hat{h}^{(z)}, \\
\hat{h}^{(y)} h_{\beta_2} & = \hat{h} h_{\beta_2} \hat{h}_{\beta_1 + \beta_3} + \hat{h} h_{\beta_1} \hat{h}_{\beta_2} \hat{h}_{\beta_3} + \hat{h}^{(x)} \hat{h}^{(z)}, \\
\hat{h}^{(z)} h_{\beta_3} & = \hat{h} h_{\beta_3} h_{\beta_1 + \beta_2} + \hat{h} h_{\beta_1} h_{\beta_2} h_{\beta_3} + \hat{h}^{(x)} \hat{h}^{(y)}, \\
\hat{h}^{2} h^{(x)} h^{(y)} h_{\beta_1 + \beta_2 + \beta_3} & = (\hat{h}^{(z)} h^{(x)} h^{(y)})^2 \\
& + (\hat{h}^{(z)} h^{(x)} h^{(y)})(2\hat{h} h_{\beta_2} h_{\beta_3} + \hat{h} h_{\beta_1} h_{\beta_2 + \beta_3} + \hat{h} h_{\beta_2} h_{\beta_1 + \beta_3} + \hat{h} h_{\beta_1} h_{\beta_1 + \beta_2}) \\
& + (\hat{h} h_{\beta_1} h_{\beta_2} + \hat{h} h_{\beta_1 + \beta_2})(\hat{h} h_{\beta_1} h_{\beta_3} + \hat{h} h_{\beta_1 + \beta_3})(\hat{h} h_{\beta_2} h_{\beta_3} + \hat{h} h_{\beta_2 + \beta_3}).
\end{align*}
\]

By combining the Plücker relations (2.1.9) with the Lagrangian condition (1.4.4), the hexahedron relations were shown in [1] to hold for a certain subset of the Plücker coordinates of any element of the Lagrangian Grassmannian \( \text{Gr}^{L}_H(\mathcal{H}, \omega) \). The homomorphism of Theorem 5.6 implies Corollary 6.6, which says that, for every element \( w^0 \in \text{Gr}^{L}_H(\mathcal{H}, \omega) \) of the infinite Lagrangian Grassmannian we get a solution of (1.6.3) by normalized evaluation of the \( \tau \)-function of CKP type on a suitably defined sublattice, as localized formal power series in the parameters. Whenever this power series can be evaluated at a given set of values of the parameters, we obtain a complex valued solution of the hexahedron recurrence system (1.6.3).

*To compare with the notation of [18], we have the following correspondence:

\[
\begin{align*}
\hat{h} & \leftrightarrow h, \quad \hat{h}^{(x)} \leftrightarrow h^{(x)}, \quad \hat{h}^{(y)} \leftrightarrow h^{(y)}, \quad \hat{h}^{(z)} \leftrightarrow h^{(z)}, \quad \hat{h}_{\beta_1} \leftrightarrow h_1, \quad \hat{h}_{\beta_2} \leftrightarrow h_2, \quad \hat{h}_{\beta_3} \leftrightarrow h_3, \quad \hat{h}_{\beta_1 + \beta_2} \leftrightarrow h_{12}, \quad \hat{h}_{\beta_1 + \beta_3} \leftrightarrow h_{13}, \quad \hat{h}_{\beta_2 + \beta_3} \leftrightarrow h_{23}, \\
\hat{h}_{\beta_1 + \beta_2 + \beta_3} & \leftrightarrow h_{123}.
\end{align*}
\]
2 Nested finite dimensional Grassmannians

2.1 Finite dimensional Grassmannians and Plücker relations

Let

\[ V_1 \subset V_2 \subset \cdots V_k \subset \cdots \subset V_{k+1} \subset \cdots, \quad k \in \mathbb{N}^+ \quad (2.1.1) \]

be a nested sequence of \( k \)-dimensional complex vector spaces, with \( V_k \) spanned by basis vectors \((e_{-1}, \ldots, e_{-k})\), and \( V_0 \) being the zero dimension space containing as sole element the zero vector \(0\). Denoting the corresponding dual spaces as \( V_k^*\), the dual basis vectors as \((e_0, \ldots, e_{k-1})\), for \( k \geq 1\), with dual pairing

\[ e_i(e_{-j}) = (-1)^i \delta_{i,j-1} \quad i = 0, \ldots, k - 1, \quad j = 1, \ldots, k, \quad (2.1.2) \]

we have

\[ V_1^* \subset V_2^* \subset \cdots V_k^* \subset V_{k+1}^* \subset \cdots \quad (2.1.3) \]

For \( n \in \mathbb{N} \), \( n \geq k \), denote the direct sum of \( V_k \) and \( V_{n-k}^* \) by

\[ H_{k,n} := V_k \oplus V_{n-k}^* \quad (2.1.4) \]

and the Grassmannian of \( k \)-dimensional subspaces of \( H_{k,n} \); i.e., the homogeneous space of the general linear group \( \text{GL}(H_{k,n}) \) consisting of the orbit of \( V_k \subset H_{k,n} \), by \( \text{Gr}_{V_k}(H_{k,n}) \).

Let \( W \) be the \( n \times k \) homogeneous coordinate matrix of an element \( w \in \text{Gr}_{V_k}(H_{k,n}) \), with entries \( \{W_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq k} \), whose column vectors determine a basis \( \{w_j\}_{1 \leq j \leq k} \) for \( w \) defined by

\[ w_j := \sum_{i=1}^{n} W_{ij} e_{i-k-1}, \quad j = 1, \ldots, k. \quad (2.1.5) \]

For \( k \in \mathbb{N}^+ \) and any \( k \)-tuple of indices

\[ L = (L_1, \ldots, L_k), \quad -k \leq L_i \leq n - k - 1, \quad i = 1, \ldots, k. \quad L_i \in \mathbb{Z} \quad (2.1.6) \]

(not necessarily ordered or even distinct).

**Definition 2.1.** If all the \( L_i \)'s are distinct, let \( \text{sgn}(L) \) be the sign of the permutation that puts the \( k \)-tuple \( L = (L_1, \ldots, L_k) \) into increasing order and \( \text{sgn}(L) := 0 \) if \( L \) contains repeated indices.

Denote by \( W_L \) the \( k \times k \) submatrix whose \( j \)th row is the \((L_j + k + 1)\)th row of \( W \), and define

\[ \tilde{\pi}_L(w) := \det(W_L) \quad (2.1.7) \]

as its determinant. These are the *Plücker coordinates* of \( w \in \text{Gr}_{V_k}(H_{k,n}) \). They are defined only within projective equivalence, since any change of basis multiplies them by the determinant of the basis change matrix.

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The quantities \( \{ \tilde{\pi}_L(w) \} \) defined in (2.1.7) satisfy the skew-symmetry conditions
\[
\tilde{\pi}_{L_1, \ldots, L_k} = \text{sgn}(\sigma) \tilde{\pi}_{L_{\sigma_1}, \ldots, L_{\sigma_k}}, \quad \forall \ \sigma \in S_k,
\]
and hence, when \( L \) has repeated indices, \( \tilde{\pi}_L \) vanishes. It follows from (2.1.8) that, for all \( k - 1 \)-tuples \((I_1, \ldots, I_{k-1})\) and \( k + 1 \)-tuples \((J_1, \ldots, J_{k+1})\) of integer indices between \(-k\) and \(n - k - 1\), \( \{ \tilde{\pi}_L \} \) satisfy the Plücker relations [8]
\[
p_{I,J} := \sum_{j=1}^{k+1} (-1)^j \tilde{\pi}_{I_1, \ldots, I_{k-1}, J_j, I_{j+1}, \ldots, I_{k+1}} = 0,
\]
where \( \hat{J}_j \) denotes omission of the element \( J_j \) from the sequence. The \( p_{I,J} \)'s will be called the Plücker quadratic forms.

Alternatively, viewing the symbols \( \{ \tilde{\pi}_L \} \) as indeterminates, define for all pairs \( k, n \in \mathbb{N}^+ \), \( k < n \), the free polynomial ring
\[
\tilde{S}_{k,n} := \mathbb{C}[\tilde{\pi}_{L_1, \ldots, L_k} | -k \leq L_i \leq n - k - 1, \ i \in \{1, \ldots, k\}],
\]
in \( n^k \) indeterminates \( \{ \tilde{\pi}_L \} \) labeled by \( k \)-element multi-indices \( L = (L_1, \ldots, L_k) \) (not necessarily ordered or distinct) with elements in the range indicated. Let \( \tilde{I}_{k,n} \subset \tilde{S}_{k,n} \) denote the ideal generated by the skew-symmetry conditions (2.1.8) and the Plücker relations (2.1.9). Define a \( \mathbb{N} \times \mathbb{N} \) bi-grading on \( \tilde{S}_{k,n} : \) by
\[
\deg(\tilde{\pi}_{L_1, \ldots, L_k}) := \left( 1, \sum_{i=1}^{k} L_i + \frac{k(k+1)}{2} \right).
\]

For a monomial in the generators \( \{ \tilde{\pi}_L \} \), the first component in (2.1.11) is its degree, while the second is the weight \( |\lambda(L)| \) of the associated partition \( \lambda(L) \) defined in eq. (2.2.2) below. It follows from the skew-symmetry relations (2.1.8) and the Plücker relations (2.1.9) that \( \tilde{I}_{k,n} \) is bihomogeneous with respect to this bi-grading. The coordinate ring of the Grassmannian \( \text{Gr}_{V_k}(\mathcal{H}_{k,n}) \) thus has the following presentation in terms of generators and relations
\[
\tilde{S}_{k,n}(\text{Gr}_{V_k}(\mathcal{H}_{k,n})) := \tilde{S}_{k,n} / \tilde{I}_{k,n},
\]
and the quotient ring is also bi-graded.

### 2.2 Plücker map and coordinates \( \pi_\lambda \) labelled by partitions

Geometrically, the Plücker map is defined by
\[
\mathfrak{P}_k^n : \text{Gr}_{V_k}(\mathcal{H}_{k,n}) \rightarrow \mathbb{P}(\Lambda^k \mathcal{H}_{k,n})
\]
\[
\mathfrak{P}_k^n : w \mapsto [w_1 \wedge \cdots \wedge w_k]
\]

(2.2.1)
(where $[v]$ denotes the projective equivalence class of $v \in \Lambda^k(\mathcal{H}_{k,n})$). This embeds $\text{Gr}_{V_k}(\mathcal{H}_{k,n})$ equivariantly, as a projective variety, into the projectivization $\mathbb{P}(\Lambda^k(\mathcal{H}_{k,n}))$ of the $k$th exterior power of $\mathcal{H}_{k,n}$.

Because of the skew-symmetry condition (2.1.8), it is sufficient to only consider $\tilde{\pi}_L$ with $L$ ordered increasingly and no repeated indices. To any such increasing ordered $k$-tuple $L$ there is a uniquely associated partition $\lambda$ whose parts are given by

$$\lambda_i := L_{k-i+1} + i, \quad i = 1, \ldots, k,$$

and whose Young diagram fits into the $k \times (n-k)$ rectangular diagram.

**Definition 2.2.** Denote by

$$\Lambda_{k,n} = \{ \lambda \mid \lambda \subset (n-k)^k \}$$

the set of partitions whose Young diagrams fit into the $k \times (n-k)$ rectangle.

Relative to the basis $\{|\lambda\rangle\}_{\lambda \in \Lambda_{k,n}}$ for $\Lambda^k \mathcal{H}_{k,n}$ defined by

$$|\lambda\rangle := e_{L_k} \wedge \cdots \wedge e_{L_1}, \quad \lambda \in \Lambda_{k,n},$$

these may alternatively be labelled by the corresponding partitions $\{\lambda\}$ defined in (2.2.2):

$$\pi_\lambda(w) := \tilde{\pi}_{L_1,\ldots,L_k}(w).$$

The image of $w$ under the Plücker map is then given by

$$\mathfrak{P}_k^n(w) = \left[ \sum_{\lambda \in \Lambda_{k,n}} \pi_\lambda(w) |\lambda\rangle \right].$$

and the Plücker relations are equivalent to the decomposability of the element $\mathfrak{P}_k^n(w)$ defined by (2.2.1). In terms of $\{\pi_\lambda\}$, the bi-grading (2.1.11) is simply

$$\deg \pi_\lambda = (1, |\lambda|) \in \mathbb{N} \times \mathbb{N},$$

where

$$|\lambda| = \sum_{i=1}^k \lambda_i$$

is the weight of $\lambda$.

We thus have an equivalent presentation of the coordinate ring $S_{k,n}(\text{Gr}_{V_k}(\mathcal{H}_{k,n}))$ in terms of Plücker coordinates labelled by partitions. Denote by

$$S_{k,n} := \mathbb{C}[\pi_\lambda \mid \lambda \in \Lambda_{k,n}]$$

the free polynomial ring generated by indeterminates $\{\pi_\lambda\}$ labelled by partitions $\lambda \in \Lambda_{k,n}$, and let $\mathcal{I}_{k,n} \subset S_{k,n}$ be the ideal generated by the Plücker relations (2.1.9), written in terms of such $\{\pi_\lambda\}$'s. The isomorphism

$$S_{k,n}(\text{Gr}_{V_k}(\mathcal{H}_{k,n})) := S_{k,n}/\mathcal{I}_{k,n} \cong \tilde{S}_{k,n}/\tilde{\mathcal{I}}_{k,n} =: \tilde{S}_{k,n}(\text{Gr}_{V_k}(\mathcal{H}_{k,n})),$$

of these $\mathbb{N} \times \mathbb{N}$-graded rings then follows from (2.1.8) and (2.2.5).
2.3 Embeddings and partial projections of Grassmannians

Define two sequences of embeddings

\[ i_k : \text{Gr}_V(H_{k,n}) \hookrightarrow \text{Gr}_V(H_{k+1,n+1}) \]
\[ i^k : w \mapsto e_{-k-1} \oplus w \subset H_{k+1,n+1}, \quad (2.3.1) \]
\[ i^n : \text{Gr}_V(H_{k,n}) \leftrightarrow \text{Gr}_V(H_{k,n+1}) \]
\[ i^n(w) \mapsto w \subset H_{k,n+1}. \quad (2.3.2) \]

The diagrams

\[
\begin{array}{ccc}
\text{Gr}_V(H_{k,n}) & \overset{i^n}{\leftarrow} & \text{Gr}_V(H_{k,n+1}) \\
i_k & & i_k \\
\downarrow & & \downarrow \\
\text{Gr}_V(H_{k+1,n+1}) & \overset{i^{n+1}}{\rightarrow} & \text{Gr}_V(H_{k+1,n+2})
\end{array}
\]

(2.3.3)

are tautologically commutative.

At the level of coordinate rings, the pull-backs of these embeddings define commuting diagrams of surjective homomorphisms of coordinate rings

\[
\begin{array}{ccc}
\mathcal{S}_{k,n}(\text{Gr}_V(H_{k,n})) & \overset{(i^n)^*}{\leftarrow} & \mathcal{S}_{k,n+1}(\text{Gr}_V(H_{k,n+1})) \\
&(i_k)^* & (i_k)^* \\
\mathcal{S}_{k+1,n+1}(\text{Gr}_V(H_{k+1,n+1})) & \overset{(i^{n+1})^*}{\leftarrow} & \mathcal{S}_{k+1,n+2}(\text{Gr}_V(H_{k+1,n+2}))
\end{array}
\]

(2.3.4)

where \((i_k)^*\) and \((i^n)^*\) act on the Plücker coordinates as follows.

\[
(i_k)^*(\pi_\lambda) = \begin{cases}
\pi_\lambda, & \lambda \in \Lambda_{k,n}, \\
0, & \text{otherwise},
\end{cases} \quad \text{for all } \lambda \in \Lambda_{k+1,n+1}, \quad (2.3.5a)
\]
\[
(i^n)^*(\pi_\lambda) = \begin{cases}
\pi_\lambda, & \lambda \in \Lambda_{k,n}, \\
0, & \text{otherwise},
\end{cases} \quad \text{for all } \lambda \in \Lambda_{k,n+1}. \quad (2.3.5b)
\]

The generators \(\{\pi_\lambda\}\) of the ring \(\mathcal{S}_{k,n}\) are a subset of those of the two larger rings \(\mathcal{S}_{k,n+1}\) and \(\mathcal{S}_{k+1,n+1}\). Denoting the tautological injection maps as

\[
t^n : \mathcal{S}_{k,n} \rightarrow \mathcal{S}_{k,n+1}, \quad t_k : \mathcal{S}_{k,n} \rightarrow \mathcal{S}_{k+1,n+1}
\]
\[t^n(\pi_\lambda) \mapsto \pi_\lambda \\
t_k(\pi_\lambda) \mapsto \pi_\lambda,
\]

(2.3.6)
this determines a commutative diagram of injective ring homomorphisms

\[
\begin{array}{ccc}
S_{k,n} & \xrightarrow{i^n} & S_{k,n+1} \\
\downarrow{i_k} & & \downarrow{i_k} \\
S_{k+1,n+1} & \xrightarrow{i^{n+1}} & S_{k+1,n+2}
\end{array}
\]  

(2.3.7)

Since the Plücker relations that generate \(I_{k,n}\) are also satisfied in the larger Grassmannians, we have two inclusions:

**Lemma 2.1.**

\[i_k(I_{k,n}) \subset I_{k+1,n+1}, \quad i^n(I_{k,n}) \subset I_{k,n+1}.\]  

(2.3.8)

The homomorphisms \(i_k\) and \(i^n\) project to the corresponding quotient rings, providing homomorphisms between the coordinate rings of the Grassmannians. It follows from the definitions that

\[(i_k)^* \circ i_k = \text{Id}_{S_{k,n}(Gr_{V_k}(H_{k,n}))} = (i^n)^* \circ i^n,\]

and therefore:

**Corollary 2.2.** For every pair of integers \((k,n)\) with \(0 \leq k \leq n\), the following is a commutative diagram of injective ring homomorphisms

\[
\begin{array}{ccc}
S_{k,n}(Gr_{V_k}(H_{k,n})) & \xrightarrow{i^n} & S_{k,n+1}(Gr_{V_k}(H_{k,n+1})) \\
\downarrow{i_k} & & \downarrow{i_k} \\
S_{k+1,n+1}(Gr_{V_{k+1}}(H_{k+1,n+1})) & \xrightarrow{i^{n+1}} & S_{k+1,n+2}(Gr_{V_{k+1}}(H_{k+1,n+2}))
\end{array}
\]  

(2.3.10)

### 2.4 Big cell

Relative to the standard complete flag

\[\text{span}\{e_{-k}\} \subset \text{span}\{e_{-k}, e_{-k+1}\} \subset \cdots \subset \text{span}\{e_{-k}, \ldots, e_{n-1}\},\]

(2.4.1)

the *big cell* \(Gr_{V_k}^\emptyset(H_{k,n}) \subset Gr_{V_k}(H_{k,n})\) is defined by the condition that the Plücker coordinate \(\pi_{\emptyset}\) corresponding to the trivial partition is nonzero:

\[Gr_{V_k}^\emptyset(H_{k,n}) := \{w^\emptyset \in Gr_{V_k}(H_{k,n}) \mid \pi_{\emptyset}(w^\emptyset) \neq 0\}.\]  

(2.4.2)

Equivalently, it consists of all elements \(w^\emptyset \in Gr_{V_k}(H_{k,n})\) that can be represented as the graph of a map \(M(w^\emptyset) : V_k \rightarrow V_{n-k}^*\).
Lemma 2.3. Every element \( w^\emptyset \in \text{Gr}_{V_k}^\emptyset (\mathcal{H}_{k,n}) \) can be uniquely represented as the image of the linear injection \( \epsilon_{w^\emptyset} : V_k \hookrightarrow \mathcal{H}_{k,n} \) defined by

\[
\epsilon_{w^\emptyset} : e_{-j} \mapsto e_{-j} + \sum_{i=1}^{n-k} M_{ij}(w^\emptyset)(-1)^{j-1} e_{i-1} \quad \text{for all } 1 \leq j \leq k,
\]

(2.4.3)

where \( \{M_{ij}(w^\emptyset)\}_{i=1,\ldots,n-k; j=1,\ldots,k} \) are the matrix elements of the map \( M(w^\emptyset) \) relative to the basis \((e_{-k}, \ldots, e_{-1})\) for \( V_k \) and the standard dual basis \((e^*_{-1}, \ldots, e^*_{-n+k})\) for \( V_{n-k}^* \).

**Proof.** It follows from formulae (2.1.7), (2.2.2) and (2.2.5) that \( \pi_\emptyset \) corresponds to the determinant of the first \( k \times k \) block of the homogeneous coordinate matrix corresponding to any basis. Multiplying the homogeneous coordinate matrix on the right by the inverse of this block places the identity matrix in the top \( k \times k \) block, and the matrix appearing as the coefficients in the sum in (2.4.3), which is the affine coordinate matrix, as the remaining \( n-k \times k \) block. (The sign factors \((-1)^{j-1}\) follow from the dual basis labelling convention (2.1.2).)

The matrix elements \( \{M_{ij}(w^\emptyset)\}_{i=1,\ldots,n-k; j=1,\ldots,k} \) are uniquely expressed in terms of the Plücker coordinates corresponding to hook partitions \([i, 1^{j-1}] = (i-1|j-1)\) as follows

\[
M_{ij}(w^\emptyset) = \frac{\pi_{(i-1|j-1)}}{\pi_\emptyset}(w^\emptyset) \quad \text{for } 1 \leq i \leq n-k, \ 1 \leq j \leq k.
\]

(2.4.4)

By the generalized Giambelli identity ([9], Sec. 4.10 and App. C.8), the remaining Plücker coordinates \( \pi_\lambda(w^\emptyset) \) on the big cell, corresponding to partitions \( \lambda = (a|b) \) of Frobenius rank \( r > 1 \), can be expressed as determinants of \( r \times r \) submatrices of the affine coordinate matrix via the formula ([9], App.C.8)

\[
\frac{\pi_\lambda}{\pi_\emptyset}(w^\emptyset) = \det \left( \frac{\pi_{(a_i|b_j)}}{\pi_\emptyset}(w^\emptyset) \right)_{1 \leq i,j \leq r}.
\]

(2.4.5)

The big cell \( \text{Gr}_{V_k}^\emptyset (\mathcal{H}_{k,n}) \) may be viewed as an affine variety with polynomial coordinate ring generated by the affine coordinate matrix elements

\[
\mathcal{O}(\text{Gr}_{V_k}^\emptyset (\mathcal{H}_{k,n})) = \mathbb{C}[M_{ij} \mid 1 \leq i \leq n-k, \ 1 \leq j \leq k].
\]

(2.4.6)

**Definition 2.3.** For \( k, n \in \mathbb{N}^+ \), with \( 0 \leq k \leq n \), define the homomorphism

\[
\xi_{k,n} : S_{k,n} \to \mathbb{C}[\pi_\emptyset, \pi_\emptyset^{-1}][\pi_{(i-1|j-1)} \mid 1 \leq i \leq n-k, \ 1 \leq j \leq k],
\]

(2.4.7)

by its values on the generators of the polynomial ring \( S_{k,n} \)

\[
\xi_{k,n} : \pi_\lambda \mapsto \pi_\emptyset \det \left( \frac{\pi_{(a_i|b_j)}}{\pi_\emptyset} \right)_{1 \leq i,j \leq r},
\]

(2.4.8)

where \( r \) is the Frobenius rank of the partition \( \lambda = (a|b) \).
Lemma 2.4. \( \xi_{k,n} \) annihilates all Plücker quadratic forms \( \{ p_{I,J} \} \) in the coordinate ring \( S_{k,n} \), so
\[
\mathcal{I}_{k,n} \subset \ker \xi_{k,n}.
\] (2.4.9)

Proof. Viewing the big cell as an open submanifold of the full Grassmannian
\[
\text{Gr}^\emptyset V_k (H_{k,n}) \hookrightarrow \text{Gr} V_k (H_{k,n})
\] (2.4.10)
gives rise to a surjective homomorphism of their coordinate rings
\[
\zeta_{k,n} : S_{k,n} (\text{Gr} V_k (H_{k,n})) \to \mathcal{O} (\text{Gr}^\emptyset V_k (H_{k,n}))
\]
\[
\pi_\lambda \mapsto \det M_{(a|b)},
\] (2.4.11)
where \( M_{(a|b)} \) denotes the submatrix of the affine coordinate matrix (2.4.4) with row set \( (a_1, \ldots, a_r) \) and column set \( (b_1, \ldots, b_r) \) given by Frobenius indices of the partition \( \lambda = (a|b) \). By construction, the homomorphism \( \zeta_{k,n} \) annihilates all Plücker relations generating \( \mathcal{I}_{k,n} \).

Formula (2.4.4) also determines an injective ring homomorphism
\[
\chi_{k,n} : \mathcal{O} (\text{Gr}^\emptyset V_k (H_{k,n})) \hookrightarrow \mathbb{C} [\pi_\emptyset, \pi^{-1}_\emptyset, \pi_{(i-1)(j-1)} | 1 \leq i \leq n-k, 1 \leq j \leq k]
\] (2.4.12)
generated by
\[
\chi_{k,n} : M_{ij} \mapsto \frac{\pi_{(i-1)(j-1)}}{\pi_\emptyset}.
\] (2.4.13)
Up to the common \( \pi_\emptyset \) factor, the composite map \( \chi_{k,n} \circ \zeta_{k,n} \) acts on the generators \( \pi_\lambda \) in exactly the same way as \( \xi_{k,n} \) in (2.4.8). Therefore all Plücker quadratic forms are annihilated by the map \( \xi_{k,n} \). \( \square \)

Similarly to the inclusions (2.3.10) of coordinate rings, we have a system of inclusions of coordinate rings of the corresponding affine big cells
\[
\tilde{\iota}_k : \mathcal{O} (\text{Gr}^\emptyset V_k (H_{k,n})) \to \mathcal{O} (\text{Gr}^\emptyset V_{k+1} (H_{k+1,n+1})),
\] (2.4.14a)
\[
\tilde{\iota}^n : \mathcal{O} (\text{Gr}^\emptyset V_k (H_{k,n})) \to \mathcal{O} (\text{Gr}^\emptyset V_{k} (H_{k,n+1})),
\] (2.4.14b)
where
\[
\tilde{\iota}_k (M_{ij}) = M_{ij} = \tilde{\iota}^n (M_{ij}) \quad \forall (i,j) \mid 1 \leq i \leq n-k, 1 \leq j \leq k
\] (2.4.15)
defines a pair of inclusions taking the affine coordinate matrix into the larger ones.
Lemma 2.5. For every pair of integers \((k,n)\) with \(0 \leq k \leq n\), we have the following commutative cubic diagram of ring homomorphisms:

\[
\begin{array}{cccc}
\mathcal{O} (\text{Gr}^\emptyset_{V_k} (\mathcal{H}_{k,n})) & \overset{i_n}{\longrightarrow} & \mathcal{O} (\text{Gr}^\emptyset_{V_k} (\mathcal{H}_{k,n+1})) \\
S_{k,n} (\text{Gr}^\emptyset_{V_k} (\mathcal{H}_{k,n})) & \overset{i_k}{\longrightarrow} & S_{k,n+1} (\text{Gr}^\emptyset_{V_k} (\mathcal{H}_{k,n+1})) & \overset{\zeta_{k,n}}{\longrightarrow} \\
S_{k+1,n+1} (\text{Gr}^\emptyset_{V_{k+1}} (\mathcal{H}_{k+1,n+1})) & \overset{i_n+1}{\longrightarrow} & S_{k+1,n+2} (\text{Gr}^\emptyset_{V_{k+1}} (\mathcal{H}_{k+1,n+2})) & \overset{\zeta_{k+1,n+1}}{\longrightarrow} \\
\end{array}
\]

\[S_{k+1,n+1} (\text{Gr}^\emptyset_{V_{k+1}} (\mathcal{H}_{k+1,n+1})) \overset{i_n+1}{\longrightarrow} S_{k+1,n+2} (\text{Gr}^\emptyset_{V_{k+1}} (\mathcal{H}_{k+1,n+2}))\] (2.4.16)

Proof. Take any partition \(\lambda = (a|b) \in \Lambda_{k,n}\) and follow the path of the corresponding generator \(\pi_\lambda \in S_{k,n}\) along any face of the above cubic diagram. \(\square\)

2.5 Nested finite dimensional Lagrangian Grassmannians

Let \(k \in \mathbb{N}^+\) be a positive integer. Define the Lagrangian Grassmannian

\[
\text{Gr}^\emptyset_{V_k} (\mathcal{H}_{k,2k}, \omega_k) \subset \text{Gr}_{V_k} (\mathcal{H}_{k,2k})
\]

as the subvariety consisting of maximal isotropic subspaces of the \(2k\)-dimensional space \(\mathcal{H}_{k,2k}\) with respect to the symplectic form

\[
\omega_k := \omega|_{\mathcal{H}_{k,2k}} = \sum_{i=0}^{k-1} (-1)^i e_i^* \wedge e_{-i-1}^*
\]

Equivalently, denote by

\[
\hat{\omega}_k = \sum_{i=0}^{k-1} (-1)^i \psi_i \psi_{-i-1} \in \text{End}(\Lambda(\mathcal{H}_{k,2k})),
\]

where

\[
\psi_i := e_i \wedge \in \text{End}(\Lambda(\mathcal{H}_{k,2k})), \quad \psi_i^\dagger := i e_i^* \in \text{End}(\Lambda(\mathcal{H}_{k,2k})), \quad j \in \mathbb{Z},
\]

the quadratic Clifford algebra element obtained by taking the exterior product with \(\omega_k\):

\[
\hat{\omega}_k : \Lambda(\mathcal{H}_{k,2k}) \rightarrow \Lambda(\mathcal{H}_{k,2k}) \\
\hat{\omega}_k : \mu \mapsto \omega_k \wedge \mu.
\] (2.5.5)
Then \( w^0 \) is in the Lagrangian Grassmannian \( \text{Gr}^L_{V_k}(\mathcal{H}_{k,2k}) \) if and only if its image \( \mathcal{P}^k_k(w^0) \) under the Plücker map is in the kernel of the restriction of \( \hat{\omega}_{2k} \) to \( \Lambda^k(V_k) \)

\[
\mathcal{P}^k_k(w^0) = \left[ \sum_{\lambda \in \Lambda_{k,2k}} \pi_\lambda |\lambda\rangle \right] \in \ker \hat{\omega}_k |\Lambda^k(\mathcal{H}_{k,2k}) \rangle.
\]  

(2.5.6)

Condition (2.5.6) is equivalent to imposing the system of linear relations

\[
\hat{\omega}_k \left( \sum_{\lambda \in \Lambda_{k,2k}} \pi_\lambda |\lambda\rangle \right) = 0,
\]  

(2.5.7)

on the Plücker coordinates, one for each basis element of \( \hat{\omega}(\Lambda^{k-2}(\mathcal{H}_{k,2k})) \). (See [1], Proposition 2.3 for more details.)

Inclusion of the Lagrangian Grassmannian \( \text{Gr}^L_{V_k}(\mathcal{H}_{k,2k}, \omega_k) \) in \( \text{Gr}^L_{V_{k+1}}(\mathcal{H}_{k+1,2k+2}, \omega_{k+1}) \) is compatible with the corresponding inclusion of the full Grassmannians. For every \( k \in \mathbb{N}^+ \) we thus have a commutative diagram of inclusions

\[
\begin{array}{ccc}
\text{Gr}^L_{V_k}(\mathcal{H}_{k,2k}, \omega_k) & \xrightarrow{i^L_k} & \text{Gr}^L_{V_{k+1}}(\mathcal{H}_{k+1,2k+2}, \omega_{k+1}) \\
\downarrow \ell_k & & \downarrow \ell_{k+1} \\
\text{Gr}_{V_k}(\mathcal{H}_{k,2k}) & \xrightarrow{i^{2k}_k} & \text{Gr}_{V_{k+1}}(\mathcal{H}_{k+1,2k+2})
\end{array}
\]  

(2.5.8)

where

\[
i^{2k}_k := i_k \circ i^{2k}
\]  

(2.5.9)

is defined by composition, \( i^L_k \) is its restriction to \( \text{Gr}^E_{V_k}(\mathcal{H}_{k,2k}, \omega_k) \subset \text{Gr}_{V_k}(\mathcal{H}_{k,2k}) \),

\[
i^L_k := i^{2k}_k |_{\text{Gr}^E_{V_k}(\mathcal{H}_{k,2k}, \omega_k)},
\]  

(2.5.10)

and

\[
\ell_k : \text{Gr}^E_{V_k}(\mathcal{H}_{k,2k}, \omega_k) \hookrightarrow \text{Gr}_{V_k}(\mathcal{H}_{k,2k}, \omega_k)
\]  

(2.5.11)

is inclusion. At the level of homogeneous coordinate rings, by (2.3.5), we get

\[
(i^L_k)^* : S_{k+1,2k+2}(\text{Gr}^E_{V_{k+1}}(\mathcal{H}_{k+1,2k+2}, \omega_{k+1})) \twoheadrightarrow S_{k,2k}(\text{Gr}^E_{V_k}(\mathcal{H}_{k,2k}, \omega_k))
\]

\[
(i^E_k)^* : (\pi_\lambda) \mapsto \begin{cases} 
\pi_\lambda & \text{if } \lambda \in \Lambda_{k,2k}, \\
0 & \text{if } \lambda \notin \Lambda_{k,2k},
\end{cases}
\]  

(2.5.12)

\( \forall \lambda \in \Lambda_{k+1,2k+2} \).

Denote the ideal of \( S_{k,2k} \) generated by the Plücker relations and Lagrange relations (2.5.7) as \( I^E_k \). The Lagrangian Grassmannian is a projective variety with homogeneous coordinate ring

\[
S_{k,2k}(\text{Gr}^E_{V_k}(\mathcal{H}_{k,2k}, \omega)) = S_{k,2k}/I^E_k.
\]  

(2.5.13)
For $k \in \mathbb{N}^+$, define the injective maps $i_{2k}^k : \mathcal{S}_{k, 2k} \rightarrow \mathcal{S}_{k+1, 2k+2}$ by composition:

$$i_{2k}^k := i_k \circ i_{2k}^2.$$ (2.5.14)

**Lemma 2.6.** The defining ideal of (2.5.13) is preserved under the inclusions of the polynomial rings

$$i_{2k}^k : \mathcal{S}_{k, 2k} \rightarrow \mathcal{S}_{k+1, 2k+2}, \quad i_{2k}^k (\mathcal{I}_k^L) \subset \mathcal{I}_{k+1}^L,$$ (2.5.15)

which induces a homomorphism of the quotient rings

$$i_{2k}^L : \mathcal{S}_{k, 2k} (\text{Gr}_{V_k} (\mathcal{H}_{k, 2k}, \omega_k)) \rightarrow \mathcal{S}_{k+1, 2k+2} (\text{Gr}_{V_{k+1}} (\mathcal{H}_{k+1, 2k+2}, \omega_{k+1})).$$ (2.5.16)

satisfying

$$(i_{2k}^L)^* \circ i_{2k}^L = \text{Id}_{\mathcal{S}_{k, 2k} (\text{Gr}_{V_k} (\mathcal{H}_{k, 2k}, \omega_k))}.$$ (2.5.17)

The big cell of the Lagrangian Grassmannian is defined as the intersection

$$\text{Gr}_{V_k}^L (\mathcal{H}_{k, 2k}, \omega_k) := \text{Gr}_{V_k}^L (\mathcal{H}_{k, 2k}, \omega_k) \cap \text{Gr}_{V_k}^0 (\mathcal{H}_{k, 2k}).$$ (2.5.18)

On $\text{Gr}_{V_k}^L (\mathcal{H}_{k, 2k}, \omega_k)$, the Lagrangian condition (2.5.7) is equivalent to the the affine coordinate matrix being symmetric:

$$M_{ij} = M_{ji} \quad 1 \leq i, j \leq k.$$ (2.5.19)

The big cell $\text{Gr}_{V_k}^L (\mathcal{H}_{k, 2k}, \omega_k)$ is thus the same as the variety of symmetric $k \times k$ matrices, with polynomial coordinate ring

$$\mathcal{O}(\text{Gr}_{V_k}^L (\mathcal{H}_{k, 2k}, \omega_k)) = \mathbb{C} [M_{ij}, \ | 1 \leq i \leq j \leq k].$$ (2.5.20)

Similarly to (2.4.11), for all $k \in \mathbb{N}^+$ we have a surjective ring homomorphism

$$\zeta_{2k}^L : \mathcal{S}_{k, 2k} (\text{Gr}_{V_k}^L (\mathcal{H}_{k, 2k}, \omega_k)) \rightarrow \mathcal{O}(\text{Gr}_{V_k}^L (\mathcal{H}_{k, 2k}, \omega_k)),$$ (2.5.21)

generated by

$$\zeta_{2k}^L (\pi_\lambda) = \det \left( M_{\min(a_i, b_j), \max(a_i, b_j)} \right)_{1 \leq i, j \leq r},$$ (2.5.22)

where $r = r(\lambda)$ is the Frobenius rank of partition $\lambda \in \Lambda_{k, 2k}$.

For $n = 2k$, the diagonal slice of diagram (2.4.16) bounded by the upper left edge and lower right edge is mapped surjectively to its Lagrangian counterpart. We therefore have
Lemma 2.7. For all \( k \in \mathbb{N}^+ \) the following diagram is commutative

\[
\begin{array}{ccc}
O(Gr_{V_k}^0(H_{k,2k})) & \overset{\iota_{2k}}\longrightarrow & O(Gr_{V_{k+1}}^0(H_{k+1,2k+1})) \\
S_{k,2k}(Gr_{V_k}(H_{k,2k})) & \overset{\iota_k}\longrightarrow & S_{k+1,2k+1}(Gr_{V_{k+1}}(H_{k+1,2k+2})) \\
S_{k,2k}(Gr_{V_k}^C(H_{k,2k},\omega_k)) & \overset{\iota_k^C}\longrightarrow & S_{k+1,2k+2}(Gr_{V_{k+1}}^C(H_{k+1,2k+2},\omega_{k+1})) \\
\end{array}
\]

(2.5.23)

3 Infinite Grassmannians and direct limits

We start by defining the homogeneous coordinate ring \( S(Gr_{H^+}(H)) \) of the infinite dimensional Grassmannian \( Gr_{H^+}(H) \) as a direct limit of finite dimensional ones, and view \( Gr_{H^+}(H) \) as the homogeneous spectrum of \( S(Gr_{H^+}(H)) \).

3.1 Direct limits of coordinate rings

Recall first some standard definitions regarding directed sets, direct systems and direct limits [24].

Definition 3.1. Let \( \mathcal{I} \) be a set and \( \preceq \subseteq \mathcal{I} \times \mathcal{I} \) a binary relation on \( \mathcal{I} \). A pair \( (\mathcal{I}, \preceq) \) is called a directed set if it has the following properties

1. \( a \preceq a \) for all \( a \in \mathcal{I} \) (reflexivity)
2. For all triples \( a, b, c \in \mathcal{I} \), if \( a \preceq b \) and \( b \preceq c \) then \( a \preceq c \) (transitivity)
3. For every pair \( a, b \in \mathcal{I} \) there exists an element \( c \in \mathcal{I} \) such that \( a \preceq c \) and \( b \preceq c \) (common successor).

The set of finite-dimensional Grassmannians

\[
\mathcal{G} = \{ Gr_{V_k}(H_{k,n}) : 0 \leq k \leq n \}_{n \in \mathbb{N}^+}
\]

(3.1.1)

forms a directed set \( (\mathcal{G}, \preceq) \) with respect to the following partial ordering

\[
Gr_{V_k}(H_{k,n}) \preceq Gr_{V_{k'}}(H_{k',n'}) \text{ iff } k \leq k' \text{ and } n - k \leq n' - k'.
\]

(3.1.2)
**Definition 3.2.** Let \((J, \preceq)\) be a directed set and \(R_J = \{R_a, a \in J\}\) a collection of rings labelled by elements of \(J\) such that for every pair \(a \preceq b\) we have a ring homomorphism

\[
\varphi_{a,b} : R_a \rightarrow R_b.
\] (3.1.3)

Then \((J, \preceq, R_J, \varphi)\) is called a **direct system** of ring homomorphisms over \(J\) if

1. \(\varphi_{a,a} = Id_{R_a}\) for all \(a \in J\),
2. \(\varphi_{b,c} \circ \varphi_{a,b} = \varphi_{a,c}\) whenever \(a \preceq b \preceq c\).

For the case \(\mathcal{G}\) of Grassmannians, where the indexing set consists of pairs of integers \((k, n), 0 \leq k \leq n\), it follows from the commutative diagram (2.3.10) that the homomorphisms \(\iota_k\) and \(\iota_n\) in (2.3.10) generate a direct system of injective ring homomorphisms

\[
\varphi_{(k,n),(k',n')} : S_{k,n}(\text{Gr}_V(H_{k,n})) \hookrightarrow S_{k',n'}(\text{Gr}_V(H_{k',n'}))
\] (3.1.4)

for all pairs \(\text{Gr}(k, n) \preceq \text{Gr}(k', n')\).

**Definition 3.3.** Let \((J, \preceq, R_J, \varphi)\) be a direct system of ring homomorphisms. We say that a pair \((\mathcal{R}, \rho := \{\rho_a\}_{a \in J})\) consisting of a ring \(\mathcal{R}\) and a family of homomorphisms

\[
\rho_a : R_a \rightarrow \mathcal{R}
\] (3.1.5)

is a **homomorphism** from the direct system \((J, \preceq, R_J, \varphi)\) **to the ring** \(\mathcal{R}\) if

\[
\rho_b = \varphi_{a,b} \circ \rho_a \quad \text{for all } a \preceq b.
\] (3.1.6)

Let \(\Lambda\) denote the set of all partitions \(\lambda\) (of any weight \(|\lambda|\) and length \(\ell(\lambda)\)) and

\[
S := \mathbb{C}[\pi_\lambda \mid \lambda \in \Lambda]\n\] (3.1.7)

the polynomial ring in infinitely many indeterminates \(\{\pi_\lambda\}\) labelled by partitions. Consider the ideal \(I \subset S\) generated by all Plücker relations

\[
I := \left\langle \bigcup_{0 \leq k \leq n} I_{k,n} \right\rangle \subset S.
\] (3.1.8)

**Lemma 3.1.** We have a homomorphism \((S/I, \sigma)\) from the direct system (3.1.4) to the ring \(S/I\), generated by

\[
\begin{align*}
\sigma_{k,n} : & S_{k,n}(\text{Gr}_V(H_{k,n})) \rightarrow S/I, \\
\sigma_{k,n} : & \pi_\lambda \mapsto \pi_\lambda.
\end{align*}
\] (3.1.9)
Definition 3.4. Let \((\mathcal{I}, \preceq, \mathcal{R}_3, \varphi)\) be a direct system of homomorphisms. We say a homomorphism \((\mathcal{R}, \rho)\) from this system to a ring \(\mathcal{R}\) is the direct limit of \(\mathcal{R}_3\) if it satisfies the following universal property:

For every homomorphism \((\mathcal{R}', \rho')\) from this system to a ring \(\mathcal{R}\), there exists a unique ring homomorphism \(v : \mathcal{R} \to \mathcal{R}'\) which makes the following diagram commutative

\[
\begin{array}{ccc}
\mathcal{R}_a & \xrightarrow{\rho_a} & \mathcal{R} \\
\downarrow & & \downarrow v \\
\mathcal{R} & \xrightarrow{\mathcal{R}_a} & \mathcal{R}'
\end{array}
\]

for all \(a \in \mathcal{I}\).

It is immediate from the definition that, when it exists, the direct limit is unique up to an isomorphism. When there is no ambiguity, we omit everything but the rings themselves and simply write

\[
\mathcal{R} = \lim_\rightarrow \mathcal{R}_a.
\]

Proposition 3.2.

\[
\mathcal{S}/\mathcal{I} = \lim_\rightarrow \mathcal{S}_{k,n}(\text{Gr}_{\mathcal{V}_k}(\mathcal{H}_{k,n})).
\]

Proof. We need to show that \((\mathcal{S}/\mathcal{I}, \sigma)\) satisfies the universal property. Suppose there is another homomorphism \((\mathcal{R}', \sigma')\). If there is a ring homomorphism \(u : \mathcal{S}/\mathcal{I} \to \mathcal{R}'\) such that

\[
\sigma'_{k,n} = u \circ \sigma_{k,n} \quad \text{for all } 0 \leq k \leq n,
\]

then for all \(0 \leq k \leq n\) we have

\[
u(\pi_\lambda) = \rho'_{k,n}(\pi_\lambda) \quad \text{for all } \lambda \in \Lambda_{k,n}.
\]

The action on the generators (3.1.14) defines a homomorphism \(u : \mathcal{S} \to \mathcal{R}'\) of the polynomial ring which annihilates \(\mathcal{I}_{k,n}\) for all \(0 \leq k \leq n\). Therefore \(u\) annihilates \(\mathcal{I}\) and hence induces a homomorphism of the quotient ring \(u : \mathcal{S}/\mathcal{I} \to \mathcal{R}'\) satisfying (3.1.13). \(\square\)

Definition 3.5. The infinite dimensional Grassmannian is the homogeneous spectrum of the direct limit ring (3.1.12)

\[
\text{Gr}_{\mathcal{H}+}(\mathcal{H}) := \text{Proj}(\mathcal{S}/\mathcal{I}).
\]

By construction, the coordinate ring of the infinite dimensional Grassmannian is

\[
\mathcal{S}(\text{Gr}_{\mathcal{H}+}(\mathcal{H})) := \mathcal{S}/\mathcal{I} = \lim_\rightarrow \mathcal{S}_{k,n}(\text{Gr}_{\mathcal{V}_k}(\mathcal{H}_{k,n})).
\]
Proposition 3.3. The ideal $\mathcal{I} \subset \mathfrak{S}$ is prime, so $\mathfrak{S}/\mathcal{I}$ is an integral domain.

Proof. Let $a, b \in \mathfrak{S}$ be any pair of elements such that $ab \in \mathcal{I}$. From (3.1.8) and (2.3.8) it follows that there exists a pair of integers $0 \leq k \leq n$ large enough such that $ab \in \mathcal{I}_{k,n}$. But since $\mathcal{I}_{k,n}$ is prime we must have either $a \in \mathcal{I}_{k,n} \subset \mathcal{I}$ or $b \in \mathcal{I}_{k,n} \subset \mathcal{I}$. □

Remark 3.1. Relevant homogeneous maximal ideals of $\mathfrak{S}^*(\text{Gr}_H^+(\mathcal{H}))$ are in $1 \rightleftharpoons 1$ correspondence with solutions to the full set of Plücker relations (i.e. closed points). Here, “relevant” means all ideals except the one generated by all elements of positive degree. The reason is that in affine charts the Nullstellensatz still holds.

Remark 3.2. The above construction is different from the so-called Ind-variety ([22]) where one takes the inverse limit of the coordinate rings or, equivalently, the direct limit of the varieties. Whereas our coordinate ring $\mathfrak{S}/\mathcal{I}$ consists of equivalence classes modulo $\mathcal{I}$ of finite polynomials in the Plücker coordinates, the coordinate ring $\widehat{\mathfrak{S}}/\widehat{\mathcal{I}}$ of an Ind-variety is obtained by completion of the ring with respect to the filtration induced by an ascending chain of Grassmannians $\cdots \subseteq \text{Gr}(n,2n) \subseteq \text{Gr}(n+1,2n+2) \subseteq \ldots$. For example, the following formal sum belongs only to the coordinate ring of an Ind-variety

$$\sum_{i=0}^{\infty} \pi_{(i)} \in \left(\widehat{\mathfrak{S}}/\widehat{\mathcal{I}}\right)\setminus(\mathfrak{S}/\mathcal{I}). \quad (3.1.17)$$

On the other hand, points in the Ind-variety always belong to arbitrarily large but finite Grassmannians, while points of $\text{Gr}_H^+(\mathcal{H})$ are solutions of all the infinitely many Plücker relations with possibly infinitely many nonzero Plücker coordinates. For example

$$\pi_{\lambda} = \begin{cases} n^n & \text{if } \lambda = (n) \\ 0 & \text{otherwise} \end{cases} \quad (3.1.18)$$

defines a point in $\text{Gr}_H^+(\mathcal{H})$ which does not belong to any finite Grassmannian.

3.2 Big cell of the infinite Grassmannian $\text{Gr}_H^+(\mathcal{H})$

The homomorphisms $\tilde{\iota}_k$ and $\tilde{\iota}_n$ in (2.4.14) generate a direct system of ring homomorphisms.

Definition 3.6. The big cell of the infinite dimensional Grassmannian $\text{Gr}_H^+(\mathcal{H})$ is the spectrum of the direct limit of coordinate rings of the finite dimensional big cells

$$\text{Gr}^0_{H^+}(\mathcal{H}) := \text{Spec } \lim_{\longrightarrow} \mathcal{O}(\text{Gr}_V^k(\mathcal{H}_{k,n})). \quad (3.2.1)$$

It follows that the coordinate ring of the big cell

$$\mathcal{O}(\text{Gr}^0_{H^+}(\mathcal{H})) = \lim_{\longrightarrow} \mathcal{O}(\text{Gr}_V^k(\mathcal{H}_{k,n})) = \mathbb{C}[M_{i,j} \mid i,j \in \mathbb{N}^+] \quad (3.2.2)$$

is a polynomial ring in countably many variables labelled by pairs of nonnegative integers.
Remark 3.3. Since the cardinality of the base field \(|\mathbb{C}| > |\mathbb{N}^+ \times \mathbb{N}^+|\) is greater than the cardinality of the generating set for (3.2.2), the set of closed points in \(\text{Gr}_{H_+}^0(\mathcal{H})\) is in one-to-one correspondence with \(\mathbb{N}^+ \times \mathbb{N}^+\) matrices of complex numbers. Note, however, that this does not require any finiteness or convergence conditions on the elements, so they are not necessarily matrix representations of bounded or continuous linear operators in the functional analytic sense.

The following are the infinite dimensional counterparts of Definition 2.3 and Lemma 2.4.

Definition 3.7. Let

\[
\xi : \mathcal{S} \to \mathbb{C}[\pi_\emptyset, \pi_\emptyset^{-1}][\pi_{(i-1)|j-1} \mid i, j \in \mathbb{N}^+],
\]

\[
\pi_\lambda \mapsto \pi_\emptyset \det \left( \frac{\pi_{(a_i|b_j)}}{\pi_\emptyset} \right)_{1 \leq i, j \leq r}.
\]

be the homomorphism defined by its action on the generators of the polynomial ring, where \(r\) is the Frobenius rank of the partition \(\lambda = (a|b)\).

Lemma 3.4. All Plücker quadratic forms in \( \mathcal{O}(\text{Gr}_{H_+}(\mathcal{H})) \) are annihilated by \(\xi\), so

\[
\mathcal{I} \subset \ker \xi.
\]

Proof. Note that restriction \(\xi|_{\mathcal{S}_{k,n}} = \xi_{k,n}\) to the subring \(\mathcal{S}_{k,n}\) coincides with the homomorphism (2.4.8). Hence, by Lemma 2.4 we have

\[
\mathcal{I}_{k,n} \subset \ker \xi \quad \text{for all} \quad 0 \leq k \leq n.
\]

Since \(\mathcal{I}\) is generated by \(\mathcal{I}_{k,n}\), it follows that (3.2.4) holds true.

3.3 The infinite Lagrangian Grassmannian and its big cell

Definition 3.8. Let \(\mathcal{I}_\mathcal{L} \subset \mathcal{S}\) denote the homogeneous ideal generated by all the Plücker (2.1.9), and Lagrange (2.5.7) relations (for all \(k\)).

Proposition 3.5. Consider the direct system of ring homomorphisms \(\{\iota_k^\mathcal{L}\}_{k \in \mathbb{N}^+}\) defined in (2.5.16). We have

\[
\mathcal{S}/\mathcal{I}^\mathcal{L} = \lim_{\to} \mathcal{S}_{k,2k}(\text{Gr}_{\mathcal{V}_k}^\mathcal{L}(\mathcal{H}_{k,2k})).
\]

Proof. Similar to Proposition 3.2.
Definition 3.9. The infinite dimensional Lagrangian Grassmannian is the homogeneous spectrum of the direct limit ring (3.3.1)

\[ \text{Gr}_{\mathcal{H}_+}^L (\mathcal{H}, \omega)) := \text{Proj}(S/I^L). \] (3.3.2)

By construction, the coordinate ring of the infinite Lagrangian Grassmannian is

\[ S(\text{Gr}_{\mathcal{H}_+}^L (\mathcal{H}, \omega)) = S/I^L. \] (3.3.3)

Definition 3.10. The big cell of the infinite Lagrangian Grassmannian is the spectrum of the direct limit of coordinate rings of the big cells of finite dimensional Lagrangian Grassmannians.

\[ \text{Gr}_{\mathcal{H}_+}^{L, \emptyset} (\mathcal{H}, \omega)) := \text{Spec} \left( \lim_{\leftarrow} \mathcal{O}(\text{Gr}_{\mathcal{H}_+}^{L, \emptyset} (\mathcal{H}_{2k}, \omega_{2k})) \right). \] (3.3.4)

By construction, the coordinate ring of the big cell of the infinite Lagrangian Grassmannian

\[ \mathcal{O}(\text{Gr}_{\mathcal{H}_+}^{L, \emptyset} (\mathcal{H}, \omega)) = \mathbb{C}[M_{ij} \mid 1 \leq i \leq j] \] (3.3.5)

is a polynomial ring in countably many variables. We have a surjective homomorphism from the coordinate ring of the full big cell to the Lagrangian big cell

\[ \tilde{\ell}^* : \mathcal{O}[\text{Gr}_{\mathcal{H}_+}^{L, \emptyset} (\mathcal{H}, \omega)] \to \mathcal{O}[\text{Gr}_{\mathcal{H}_+}^{L, \emptyset} (\mathcal{H}, \omega)] \]

\[ M_{i,j} \mapsto M_{\min(i,j), \max(i,j)}. \] (3.3.6)

The Lagrangian analogue of Definition 3.7 and Lemma 3.4 is:

Definition 3.11. Define the following homomorphism from the polynomial ring \( S \) to the ring \( \mathbb{C}[\pi_\emptyset, \pi_\emptyset^{-1}] \left[ \pi_{(i-j)} \mid 1 \leq i \leq j \right] \) by its action on the generators

\[ \xi^L : S \to \mathbb{C}[\pi_\emptyset, \pi_\emptyset^{-1}] \left[ \pi_{(i-j)} \mid 1 \leq i \leq j \right] \]

\[ \xi^L : \pi_\lambda \mapsto \pi_\emptyset \det \left( \frac{\pi_{(\min(a,b), \max(a,b))}}{\pi_\emptyset} \right)_{1 \leq i, j \leq r}, \] (3.3.7)

where \( r \) is the Frobenius rank of the partition \( \lambda = (a|b) \).

Lemma 3.6. The homomorphism \( \xi^L \) annihilates all Plücker and Lagrange relations in the infinite dimensional Lagrangian Grassmannian \( \text{Gr}_{\mathcal{H}_+}^L (\mathcal{H}, \omega) \). Hence we have

\[ I^L \subset \ker \xi^L. \] (3.3.8)
Proof. From \( \xi^L = \tilde{\ell}^* \circ \xi \), and Lemma 3.4, it follows that \( \xi \) annihilates all the Plücker quadratic forms, and so does \( \xi^L \). To prove that \( \xi^L \) annihilates all Lagrange relations, consider the restriction to the finitely generated subring

\[
\xi^L|_{S_{k,2k}} = \tilde{\ell}^* \circ \xi_{k,2k} \quad (2.5.23) \quad \xi^L \circ \ell_k^* ,
\]

where we have used the commutativity of the leftmost face in diagram (2.5.23). By construction, \( \ell_k^* \) annihilates all Lagrange relations in \( I_L^k \), and so does the right hand side of (3.3.9).

Since this argument is valid for all \( k \in \mathbb{N}^+ \), it follows that \( \xi^L \) annihilates all Lagrange relations in the defining ideal \( I_L^\mathcal{L} \) of the infinite dimensional Lagrangian Grassmannian.

4 Evaluation of \( \tau \)-functions and lattices in \( \operatorname{Gr}_{\mathcal{H}^{\mathcal{L}}}(\mathcal{H}) \)

4.1 Lattice evaluations of KP \( \tau \)-functions

Now consider two families of indeterminants, \( \{x_j\}_{j \in \mathbb{Z}} \) and \( \{t_i\}_{i \in \mathbb{N}^+} \) labelled by integers and positive integers respectively. Let \( \mathbb{C}[[x,t]] \) be the ring of formal power series in \( \{x_j, t_i\}_{j \in \mathbb{Z}, i \in \mathbb{N}^+} \). Introduce a grading on \( \mathbb{C}[[x, t]] \) by assigning

\[
\deg(x_i) = 1, \quad \deg(t_j) = j, \quad i \in \mathbb{Z}, j \in \mathbb{Z}.
\]

Lemma 4.1. For every \( i \in \mathbb{Z} \), the following maps generate graded ring automorphisms

\[
\delta_i : \mathbb{C}[[x, t]] \to \mathbb{C}[[x, t]], \quad \left\{ \begin{array}{c}
x_i \mapsto x_i, \\
t_j \mapsto t_j + \frac{x^i_j}{j},
\end{array} \right.
\]

that mutually commute

\[
\delta_i \delta_j = \delta_j \delta_i, \quad \forall i, j \in \mathbb{Z}.
\]

In the following, the indeterminates \( t = (t_1, t_2, t_3, \ldots) \) will be viewed as KP flow parameters and \( x \) as a set of additional parameters determining lattice spacings as defined below. When there is no ambiguity, we omit explicitly indicating the dependence on \( x \) and simply write

\[
f(t + \sum_{i \in \mathbb{Z}} n_i [x_i]) := \delta_i^n f(t),
\]

where

\[
n = (\ldots , n_{-1}, n_0, n_1, \ldots , n_i, \ldots) , \quad n_i \in \mathbb{Z}
\]
and

\[ [x_i] := \left(x_i, \frac{x_i^2}{2}, \frac{x_i^3}{3}, \ldots \right) \] (4.1.6)

for any \( f(t) \in \mathbb{C}[[x, t]] \).

Consider the multiplicative subset generated by the elements \( \{x_i - x_j\}_{i \neq j} \)

\[ S_x^x := \langle x_i - x_j \mid i, j \in \mathbb{Z}, i \neq j \rangle \subset \mathbb{C}[[x, t]] \];

(4.1.7)

that is, the set of all finite products of positive powers of \((x_i - x_j)\). Since \( \mathbb{C}[[x, t]] \) is an integral domain, we can introduce its localization \((S_x^x)^{-1} \mathbb{C}[[x, t]]\) with respect to \(S_x^x\), i.e., the ring of equivalence classes consisting of quotients of elements of \( \mathbb{C}[[x, t]] \) by those in \(S_x^x\).

**Remark 4.1.** Note that the generators \( \{x_i - x_j\}_{i \neq j} \) of \(S_x^x\) may be identified with the roots of \(\mathfrak{gl}(\infty)\).

**Corollary 4.2.** For every integer \( i \in \mathbb{Z} \), the graded automorphism \(\hat{\delta}_i\) extends uniquely to an automorphism of the localized ring

\[ \hat{\delta}_i : (S_x^x)^{-1} \mathbb{C}[[x, t]] \rightarrow (S_x^x)^{-1} \mathbb{C}[[x, t]]. \] (4.1.8)

Let \( A \subset \mathbb{Z}^\infty \) be the infinitely generated free abelian subgroup consisting of elements with only a finite number of nonzero entries. This will also be referred to as the infinite-dimensional integer lattice.

Fix an element \( w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \), and let \( \tau^{KP}_w(t) \in \mathbb{C}[[t]] \subset \mathbb{C}[[x, t]] \) denote the corresponding KP \(\tau\)-function. To this we can associate a function \(H_w\) on the infinite-dimensional lattice \( A \), where for each element \( n \in A \) we have

\[ H^n_w(t) := \prod_{i < j} (x_i - x_j)^{n_i n_j} \tau^{KP}_w \left(t + \sum_{i = -\infty}^{\infty} n_i x_i\right) \in (S_x^x)^{-1} \mathbb{C}[[x, t]]. \] (4.1.9)

**Remark 4.2.** For most purposes in what follows, we view \(\tau\)-functions as formal power series. Throughout this section we will be working in the localized ring \((S_x^x)^{-1}\mathbb{C}[[x, t]]\), leaving all convergence issues aside. It is worth noting, however, that in the simplest case, when \( w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \) is chosen so that \( \tau^{KP}_w(t) \) is a polynomial in the \(t\) variables \([10, 19, 11]\), \(H^w(t)\) is also a polynomial in these variables, with coefficients that are rational functions in \(x_i\). Moreover, because the only type of denominators that appear in (4.1.9) are powers of \(x_i - x_j\), we can evaluate the external parameters in this case at any set of pairwise distinct complex numbers.

**Definition 4.1.** For \( i \in \mathbb{Z} \), let \( \alpha_i \in A \) be the basis element with all components 0 except for a single nonzero entry 1 in the \(i\)th position.
The following Lemma describes the change in $H^n_w$ with respect to shifts of the origin by the elements $\alpha_i$.

**Lemma 4.3.** For all $i \in \mathbb{Z}$, we have

$$H^{n+\alpha_i}_w(t) = (-1)^{\sum_{j<i} n_j} \prod_{j \neq i} (x_i - x_j)^{n_j} H^n_w(t + [x_i]).$$  \hspace{1cm} (4.1.10)

**Proof.** This follows from the definition (4.1.9).

**Remark 4.3.** Note that, since $H^n_w(t)$ only differs from $\tau_w(t)$ by a constant translation in the argument $t$ and a $t$-independent multiplicative factor that is rational in the parameters, $H^n_w(t)$ is also a KP $\tau$-function. Since $\tau$-functions are only defined up to multiplication by an arbitrary nonzero constant factor we have, up to projective equivalence,

$$H^n_{\gamma_n+w}(t) \sim H^n_w(t + s)$$  \hspace{1cm} (4.1.11)

and hence

$$H^{n+\alpha_i}_w(t) \sim H^n_w(t + [x_i])) = H^n_{\gamma_+(x_i)+w}(t).$$  \hspace{1cm} (4.1.12)

Eq. (4.1.10) may therefore be viewed as generating a projective action on the space of such $\tau$-functions $H^n_w(t)$, of the infinite discrete abelian subgroup $\Gamma^+_A \subset \Gamma_+$ consisting of evaluations

$$\gamma^n_+ := \gamma_+ \left( \sum_{j \in \mathbb{Z}} n_j [x_j] \right), \hspace{1cm} n = (\ldots, n_i, n_{i+1}, \ldots) \in A$$  \hspace{1cm} (4.1.13)

at lattice points $n \in A$. By eq. (4.1.12), this may be viewed as evaluation of the $\tau$-function at the lattice points

$$\gamma_+ \left( \sum_{j \in \mathbb{Z}} n_j [x_j] \right) w \in \Gamma_+(w)$$  \hspace{1cm} (4.1.14)

embedded within the $\Gamma_+$ orbit $\{w(t)\}$.

The polynomial ring in the $H^{(m)}_w(t)$'s for fixed $w$ and $t$ and varying $n \in A$ has a natural grading by elements of $A$. From Lemma 4.3 it follows that:

**Corollary 4.4.** For every monomial $H^{(1)}_w(t) \ldots H^{(m)}_w(t)$ of total degree

$$n^{(tot)} := n^{(1)} + \ldots + n^{(m)},$$  \hspace{1cm} (4.1.15)

we have

$$H^{n^{(1)}+\alpha_i}_w(t) \ldots H^{n^{(m)}+\alpha_i}_w(t)$$

$$= (-1)^{\sum_{j<i} n^{(tot)}_j} \prod_{j \neq i} (x_i - x_j)^{n^{(tot)}_j} H^{n^{(1)}}_w(t + [x_i]) \ldots H^{n^{(m)}}_w(t + [x_i])$$  \hspace{1cm} (4.1.16)

Thus, all monomials of given homogeneity degree $n^{(tot)}$ scale by the same factor when the origin of the lattice $A$ is shifted.
4.2 Plücker relations satisfied by $H^n_w(t)$

**Proposition 4.5.** Fix an element $w \in \text{Gr}_{n}(\mathcal{H})$ of the infinite Grassmannian and let $k, n \in \mathbb{N}^+$, $k < n$, be an ordered pair of positive integers. The following generates a homomorphism of commutative algebras

$$
\phi_{0,w,k,n} : S(\text{Gr}_{k}(\mathcal{H}^n_w)) \to (S^n_x)^{-1} \mathbb{C}[x, t], \\
\tilde{\pi}_{L_1, \ldots, L_k} \mapsto \text{sgn}(L)H^\alpha_{L_1 + \cdots + L_k}(t).
$$

**Proof.** Let $s$ be the cardinality $s = |I \cap J|$ of the intersection of the multi-indices $I$ and $J$ appearing in the Plücker relation (2.1.9). Because of skew-symmetry, there is no loss of generality in assuming that $I_i = J_i$ for $1 \leq i \leq s$, and all elements $I_{s+1}, \ldots, I_{k-1}, J_{s+1}, \ldots, J_{k+1}$ are distinct, and ordered,

$$
I_{s+1} < I_{s+2} < \cdots < I_{k-1} \quad \text{and} \quad J_{s+1} < J_{s+2} < \cdots < J_{k+1}.
$$

and showing that the Plücker relations (2.1.9) are satisfied by all elements $H^\alpha_{L_1 + \cdots + L_k}(t)$ in this case. Under these assumptions, define

$$
t^I \cap J := t + \sum_{i=1}^{s} [x_{I_i}] = t + \sum_{i=1}^{s} [x_{J_i}].
$$

Applying the KP addition formulae (1.3.2), using

$$
H^{\sum_{i=1}^{k} \alpha_{L_i}}(t) = \text{sgn}(L_1, \ldots, L_k)\zeta_N(t, x_{L_1}, \ldots, x_{L_k}),
$$

we get

$$
\sum_{j=s+1}^{k+1} (-1)^j \text{sgn}(I_{s+1}, \ldots, I_{k-1}, J_j)H^\alpha_{L_{s+1} + \cdots + L_{k-1} + \alpha_j}(t^I \cap J)
$$

$$
\times H^\alpha_{\hat{J}_{s+1} + \cdots + \hat{J}_j + \cdots + \hat{J}_{k+1}}(t^I \cap J) = 0,
$$

where $\hat{\alpha}_{J_j}$ denotes the absence of the term $\alpha_{J_j}$ from the sum. The following therefore holds for all $0 \leq s' \leq s$

$$
\sum_{j=s'+1}^{k+1} (-1)^j \text{sgn}(I_{s'+1}, \ldots, I_{k-1}, J_j)H^\alpha_{L_{s'+1} + \cdots + L_{k-1} + \alpha_j}(t^I \cap J)
$$

$$
\times \text{sgn}(J_{s'+1}, \ldots, J_j, \ldots, J_{k+1})H^\alpha_{\hat{J}_{s'+1} + \cdots + \hat{J}_j + \cdots + \hat{J}_{k+1}}(t^I \cap J) = 0.
$$
This follows by finite induction in $s'$, starting at $s' = s$, and descending. The case $s' = s$ is just (4.2.6), where, according to (4.2.3) we have $\text{sgn}(J_{k+1}, \ldots, J_{k+1}) = 1$. Assuming now that (4.2.7) holds for some $s'$, we show that it also holds for $s' - 1$.

Note that the left-hand side of eq. (4.2.7) is homogeneous with respect to the grading given by elements of $\textbf{A}$. Using Corollary (4.4) for $i = I_{s'} = J_{s'}$ and cancelling the common factor

$$\prod_{j=s'+1}^{k-1} (x_{I_s'} - x_{I_j}) \prod_{j=s'+1}^{k+1} (x_{J_{s'}} - x_{J_j}),$$  

(4.2.8)

we get

$$\sum_{j=s'}^{k+1} (-1)^j \text{sgn}(I_{s'}, \ldots, I_{k-1}, J_j) H_w^{\alpha_{s'} + \cdots + \alpha_{k-1} + \alpha_j} (t + \sum_{i=1}^{s'+1} [x_{I_i}]) \times$$

$$\times \text{sgn}(J_{s'}, \ldots, J_{j-1}, J_{j+1}, \ldots, J_{k+1}) H_w^{\alpha_{s'} + \cdots + \hat{\alpha}_j + \cdots + \alpha_{k+1}} (t + \sum_{i=1}^{s'+1} [x_{J_i}]) = 0,$$

(4.2.9)

where we have used the fact that for $s' \leq s$ we have $I_{s'} = J_{s'}$, and hence the following ratio of signs is independent of $j$:

$$\frac{\text{sgn}(I_{s'}, \ldots, I_{k-1}, J_j) \text{sgn}(J_{s'}, \ldots, \hat{J}_j, \ldots, J_{k+1})}{\text{sgn}(I_{s'+1}, \ldots, I_{k-1}, J_j) \text{sgn}(J_{s'+1}, \ldots, \hat{J}_j, \ldots, J_{k+1})}$$

$$= \left( \prod_{i=s'+1}^{k-1} \text{sgn}(I_{s'}, I_i) \right) \text{sgn}(I_{s'}, J_j) \left( \prod_{i=s'+1}^{k+1} \text{sgn}(J_{s'}, J_i) \right) \text{sgn}(J_{s'}, J_j)$$

(4.2.10)

$$= \prod_{i=s'+1}^{k-1} \text{sgn}(I_{s'}, I_i) \prod_{i=s'+1}^{k+1} \text{sgn}(J_{s'}, J_i).$$

Eq. (4.2.9) shows that (4.2.7) is true for $s' - 1$, and hence, by induction, it holds for all $0 \leq s' \leq s$. In particular, for $s' = 0$ we have

$$\sum_{j=1}^{k+1} (-1)^j \text{sgn}(I_1, \ldots, I_{k-1}, J_j) H_w^{\alpha_1 + \cdots + \alpha_{k-1} + \alpha_j} (t)$$

$$\times \text{sgn}(J_1, \ldots, J_{j-1}, J_{j+1}, \ldots, J_{k+1}) H_w^{\alpha_j + \cdots + \alpha_{j+1} + \cdots + \alpha_{k+1}} (t)$$

$$= \sum_{j=1}^{k+1} (-1)^j \phi^0_{w,k,n} (\pi_{I_1,\ldots,I_{k-1},J_j}) \phi^0_{w,k,n} (\pi_{J_1,\ldots,\hat{J}_j,\ldots,J_{k+1}}) = 0.$$  

(4.2.11)
Proposition 4.6. Fix an element \( w \in \text{Gr}_{\mathcal{H}}(\mathcal{H}) \) and let \( k, n \in \mathbb{N} \), \( k < n \) be an ordered pair of positive integers. For every element \( n \in \mathbf{A} \), we have a homomorphism of commutative algebras

\[
\phi^n_{w,k,n} : S(\text{Gr}_{V_k}(\mathcal{H}^n_k)) \to (S^x)^{-1} \mathbb{C}[[x, t]]
\]  

(4.2.12)
such that

\[
\phi^n_{w,k,n} : \pi_{L_1, \ldots, L_k} \mapsto \text{sgn}(L) H^n_w + \sum_{i=1}^k \alpha_i L_i \cdot w(t).
\]  

(4.2.13)

Proof. We proceed by induction on the lattice \( \mathbf{A} \). The starting point is \( n = 0 \), which is given by Proposition 4.5. Now, assume that \( \phi^n_{w,k,n} \) is a homomorphism for some \( n \in \mathbf{A} \). We will show that this implies \( \phi^{n+\alpha_i}_{w,k,n} \) is also a commutative ring homomorphism for the translation by any \( \alpha_i \).

To do this, let \( I \) be a \( k = 1 \)-tuple of indices and \( J \) a \( k + 1 \) tuple. By the inductive assumption

\[
0 = \sum_{j=1}^{k+1} (-1)^j \phi^n_{w,k,n}(\pi_{I_1, \ldots, I_1, J_j}) \phi^n_{w,k,n}(\pi_{I_1, \ldots, J_j, \ldots, I_{n+1}})
\]

\[
= \sum_{j=1}^{k+1} (-1)^j \text{sgn}(I_1, \ldots, I_{k-1}, J_j) H^n_w + \sum_{i=1}^k \alpha_i L_i \cdot w(t)
\]

\[
\times \text{sgn}(J_1, \ldots, J_j, \ldots, J_{k+1}) H^n_w + \sum_{i=1}^k \alpha_i L_i \cdot \hat{\delta}_i(t) \cdot w(t - [x_i]).
\]  

(4.2.14)

Note that the right hand side of (4.2.14) is homogeneous with respect to the grading given by elements of \( \mathbf{A} \). Corollary 4.4 implies that

\[
0 = \sum_{j=1}^{k+1} (-1)^j \text{sgn}(I_1, \ldots, I_{k-1}, J_j) H^n_w + \sum_{i=1}^k \alpha_i L_i \cdot \hat{\delta}_i(t - [x_i])
\]

\[
\times \text{sgn}(J_1, \ldots, J_j, \ldots, J_{k+1}) H^n_w + \sum_{i=1}^k \alpha_i L_i \cdot \hat{\delta}_i(t - [x_i]).
\]  

(4.2.15)

The right hand side of the latter is the result of applying \( \phi^{n+\alpha_i}_{w,k,n} \circ \hat{\delta}_i \) to the same Plücker relation, where \( \hat{\delta}_i \) is the shift operator introduced in Lemma 4.1 and Corollary 4.2. The same result holds for all Plücker relations, so we conclude that \( \phi^{n+\alpha_i}_{w,k,n} \circ \hat{\delta}_i \) is a homomorphism for all \( i \in \mathbb{Z} \). Since \( \hat{\delta}_i \) is an automorphism of the ring \( \mathbb{C}[[t]] \), it follows that \( \phi^{n+\alpha_i}_{w,k,n} \) must also be a homomorphism.

For consistency with the infinite dimensional case, it is convenient to label Plücker coordinates by partitions. To every \( k \)-element increasingly ordered subset

\[
L = (L_1 \ldots, L_k) \subset \{-k, \ldots, n - k - 1\}
\]  

(4.2.16)
having \( k - r \) negative elements, where \( 0 \leq r \leq k \),

\[
-k \leq L_i \leq -1, \quad i \in \{1, \ldots, k - r\},
\]

(4.2.17)

and \( r \) non-negative elements,

\[
0 \leq L_{k-r+i} \leq n - k - 1, \quad i \in \{1, \ldots, r\},
\]

(4.2.18)

we can associate two increasingly ordered sets of \( r \) positive integers

\[
I = (I_1, \ldots I_r), \quad 1 \leq I_i \leq I_{i+1} \leq n - k,
\]

(4.2.19a)

\[
J = (J_1, \ldots J_r), \quad 1 \leq J_i < J_{i+1} \leq k, \quad i = 1, \ldots, r,
\]

(4.2.19b)

such that

\[
I_i = 1 + L_{k-r+i}, \quad J = \{1, \ldots, k\} \setminus (-L).
\]

(4.2.20)

Let

\[
(a|b) = (a_1 \ldots a_r|b_1 \ldots b_r)
\]

(4.2.21)

be the Frobenius notation for a partition \( \lambda(I, J) \) of Frobenius rank \( r \), whose Young diagram fits into the rectangular one for \( (n - k)^k \), with Frobenius indices defined as

\[
a_i = I_{r-i+1} - 1, \quad b_i = J_{r-i+1} - 1, \quad 1 \leq i \leq r.
\]

(4.2.22)

Introduce the following notation

\[
H_{\lambda}^{n}(w)(t) := H_{w}^{n+\sum_{i=1}^{r} a_i - \sum_{j=1}^{k} b_j - 1}(t),
\]

(4.2.23)

where \( \lambda := \lambda(I, J) = (a|b) \), and define

\[
\Phi_{w,k,n}^{n} := \phi_{w,k,n}^{n+\sum_{j=1}^{k} a_j - j}.
\]

(4.2.24)

We can reformulate Proposition 4.6 in terms of Plücker coordinates labelled by partitions.

**Corollary 4.7.** Fix an element \( w \in \text{Gr}_{H_+}(\mathcal{H}) \) and a lattice point \( n \in A \). The homomorphism (4.2.24) applied to \( \pi_\lambda \) gives

\[
\Phi_{w,k,n}^{n}(\pi_\lambda) = H_{w}^{n,\lambda}.
\]

(4.2.25)

**Proof.** We reduce this to the statement of Proposition 4.6. Let \( J \subset \{1, \ldots, k\} \) and \( I \subset \{1, \ldots, n - k\} \) be a pair of ordered subsets of the same cardinality \( |I| = |J| = r \) and \( \lambda(I, J) \) the corresponding partition of Frobenius rank \( r \). From (4.2.20) we get the expression for the indexing set

\[
L = (-J^c) \cup (I_1 - 1, \ldots, I_r - 1) \subset (-k, \ldots, n - k - 1)
\]

(4.2.26)
where
\[ J^c = \{1, \ldots, k\} \setminus J. \] (4.2.27)
in reversed; i.e., decreasing order) It follows that
\[ m = n + \sum_{i=1}^{r} \alpha_{l_i-1} - \sum_{i=1}^{r} \alpha_{-J_i} = n + \sum_{i=1}^{r} \alpha_{l_i-1} + \sum_{i=1}^{k-r} \alpha_{-J_i} - \sum_{j=1}^{k} \alpha_{-j} \] (4.2.28)
and hence
\[ \Phi_{w,k,n}^n(\pi_\lambda) = H_{w}^{m} = \Phi_{w,k,n}^{n-\sum_{j=1}^{k} \alpha_{-j}}(\pi_{L_1,\ldots,L_k}). \] (4.2.29)

Since this holds for all partitions \( \lambda \) that fit into the \( k \times (n-k) \) rectangle, we arrive at (4.2.25), which concludes the proof.

\subsection*{4.3 Lattice mapping to \( \text{Gr}_{H^+}(\mathcal{H}) \)}

In this section we show that to every KP \( \tau \)-function \( \tau_{KP}^w(t) \) and every choice of evaluations of the indeterminants \( \{x_i\}_{i \in \mathbb{Z}} \), we can associate an infinite dimensional lattice of points in \( \text{Gr}_H^+(\mathcal{H}) \). For every base point \( n \in A \), evaluations of the corresponding \( \tau \)-function \( \tau_{KP}^w \) as in (4.1.9) at a lattice of associated points in the \( \Gamma_+ \) orbit of \( w \) satisfy the Plücker relations of an infinite dimensional Grassmannian and thus can be interpreted as Plücker coordinates of some other Grassmannian element \( \tilde{w} \in \text{Gr}_H^+(\mathcal{H}) \).

\textbf{Theorem 4.8.} Fix an element \( w \in \text{Gr}_H^+(\mathcal{H}) \) and let \( n \in A \) be a lattice point. The following map, evaluated on the Plücker coordinates, viewed as generators of the ring \( S(\text{Gr}_H^+(\mathcal{H})) \), extends to a homomorphism of commutative algebras
\[ \Phi_{w}^{n}: S(\text{Gr}_H^+(\mathcal{H})) \to (S_{X}^{\infty})^{-1} \mathbb{C}[x, t] \]
\[ \pi_\lambda \mapsto H_{w}^{n,\lambda}. \] (4.3.1)

\textbf{Proof.} Every Plücker relation involves only finitely many generators \( \{\pi_\lambda \mid \lambda \in \Lambda\} \) indexed by some finite set \( \Lambda \) of partitions. Let \( n \in \mathbb{N}^+ \) be large enough that all Young diagrams of partitions \( \lambda \in \Lambda \) fit into an \( n \times n \) square. Under these assumptions we have
\[ \Phi_{w}^{n}(\pi_\lambda) = \Phi_{w,n,2n}^{n}(\pi_\lambda), \quad \text{for all} \quad \lambda \in \Lambda. \] (4.3.2)
From Corollary 4.7 it follows that all the Plücker quadratic forms are annihilated by \( \Phi_{w}^{n} \). This is valid for any Plücker relation, and any \( w \in \text{Gr}_H^+(\mathcal{H}) \). \( \square \)
Example 4.1. The octahedron relations (discrete Hirota equations)

Consider the short Plücker relation (2.1.9)
\[ \pi_\emptyset \pi_{(2,2)} - \pi_{(1)} \pi_{(2,1)} + \pi_{(2)} \pi_{(1,1)} = 0. \] (4.3.3)
for \((k,n) = (2,2), \ I_1 = -2, \ (J_1, J_2, J_3) = (-1, 0, 1). \) (4.3.4)

Choose parameter values
\[ x_{-2} = 0, \quad x_{-1} = a, \quad x_0 = b, \quad x_1 = c, \] (4.3.5)
and lattice values
\[ n_{-2} = 1, \quad n_{-1} = l + 1, \quad n_0 = m, \quad n_1 = n, \] (4.3.6)
\[ n_i = 0 \quad \text{if} \quad i \neq -2, -1, 0, 1. \] (4.3.7)
and define the quantities \( H^{n,\emptyset}_w, H^{n,(2,2)}_w, H^{n,(2,1)}_w, H^{n,(1,1)}_w \) as in eq. (4.2.23).

Theorem 4.8 then implies that these also satisfy
\[ H^{n,\emptyset}_w H^{n,(2,2)}_w - H^{n,(1)}_w H^{n,(2,1)}_w + H^{n,(2)}_w H^{n,(1,1)}_w = 0 \] (4.3.8)
for all \( n \). Defining \( \tau_{l,m,n} \) as in eq. (1.3.7), and making the substitutions (4.2.23), we get
\[ a(b-c) \tau_{l+1,m,n} \tau_{l,m+1,n+1} + b(a-c) \tau_{l,m+1,n} \tau_{l+1,m,n+1} + c(a-b) \tau_{l,m,n+1} \tau_{l+1,m+1,n} = 0, \] (4.3.9)
which are the octahedron (or discrete Hirota) relations [12, 25].

4.4 The fraction field \( \mathbb{F}_{x,t} \) and the effective affine big cell

For every element \( w \in \text{Gr}_{H,H^+}(\mathcal{H}) \), the corresponding KP \( \tau \)-function \( \tau^K_{w}(t) \in \mathbb{C}[[x,t]] \) is a nonzero element of the ring of formal power series and so are its normalized shifts
\[ H^n_w(t) = \prod_{i<j} (x_i - x_j)^{n_i n_j} \tau^K_{w} \left( t + \sum_{i=-\infty}^{\infty} n_i [x_i] \right) \neq 0, \] (4.4.1)
for all \( n \in A \). Therefore, for every \( n \in A \), the image of the Plücker coordinate corresponding to the trivial partition
\[ \Phi^n_w(\pi_\emptyset) = H^{n,\emptyset}_w = H^n_w \neq 0. \] (4.4.2)
is a nonzero element.
Recalling that the ring of formal power series \( \mathbb{C}[[x, t]] \) is an integral domain, denote its field of fractions by \( \mathbb{F}_{x, t} \). Lattice evaluations (4.3.1) are elements of the localized ring
\[
\Phi_{w}^{n}(\pi_{\lambda}) = H_{w}^{n,\lambda} \in \left(S_{x}^{\infty}\right)^{-1} \mathbb{C}[[x, t]] \subset \mathbb{F}_{x, t}.
\] (4.4.3)
The localized ring, in turn, is a subring of \( \mathbb{F}_{x, t} \). By (4.4.2), we can thus view \( \Phi_{w}^{n}(\pi_{\emptyset}) \) as invertible elements of the field \( \mathbb{F}_{x, t} \).

The analogue of the determinant formula (2.4.5) for evaluations of the \( \tau \)-function is given by the following.

**Proposition 4.9.** Let \( w \in \text{Gr}_{H+}(H) \) be a fixed point in the infinite Grassmannian and \( n \in A \) a lattice point. For every partition \( \lambda = (a | b) \) of Frobenius rank \( r \), the identity
\[
\frac{H_{w}^{n,\lambda}}{H_{w}^{n}} = \det \left( \frac{H_{w}^{n+\alpha_{a_{i}}-\alpha_{b_{j}}-1}}{H_{w}^{n}} \right)_{1 \leq i, j \leq r}
\] (4.4.4)
holds in the field of fractions \( \mathbb{F}_{x, t} \).

**Proof.** By Proposition 2 in [31] (or Proposition 3.10.4 in [9]), we have the following consequence of the addition formulae (Theorem 1.1, eq. 1.3.2).
\[
\tau_{w}^{KP}(t + \sum_{i=1}^{r} [x_{i}] - \sum_{i=1}^{r} [y_{i}]) \prod_{i<j}(x_{i} - x_{j})(y_{j} - y_{i}) = \det \left( \frac{\tau_{w}^{KP}(t + [x_{i}] - [y_{i}])}{\tau^{KP}(t)} \right)_{1 \leq i, j \leq r}.
\] (4.4.5)
Substituting
\[
\tilde{t} \to t + \sum_{i=-\infty}^{\infty} n_{i}[x_{i}], \quad \tilde{x}_{i} \to x_{a_{i}}, \quad \tilde{y}_{i} \to x_{-b_{i}-1}
\] (4.4.6)
into (4.4.5) we get precisely (4.4.4). \( \square \)

**Remark 4.4.** Viewing lattice evaluations \( H_{w}^{n} \in \mathbb{F}_{x, t} \) as elements of the fraction field of formal power series in the parameters, eq. (4.4.2) implies that, effectively, we are always in the big cell. It is worth noting, however, that when the element \( w \in \text{Gr}_{H+}(H) \) is chosen in such a way that \( \tau_{w}^{KP}(t) \) is a polynomial, [10, 19, 11] its lattice evaluations for particular values of parameters and points of the lattice are allowed to be zero, because \( \tau_{w}^{KP}(t) \) is required to be nontrivial only as a function of \( t \). This corresponds to the fact that although identity (4.4.4) holds for rational functions of \( x \) and \( t \), it does not necessarily translate to evaluations of parameters, even in this simple case.

### 4.5 Plücker coordinates for \( w \in \text{Gr}_{H+}(H) \) vs. evaluation of \( \tau_{w}^{KP}(t) \) on a lattice.

Theorem 4.8 tells us that the “shifted” \( \tau \)-functions \( H_{w}^{n,\lambda} \) are homomorphic images of the Plücker coordinates \( \pi_{\lambda} \) on the Grassmannian \( \text{Gr}_{H+}(H) \), and so satisfy Plücker relations.
Considering just the $H^n$, these are, up to some (important) normalizations, the $\tau$-function evaluated on a lattice in the infinite dimensional Grassmannian $\text{Gr}_{H^+}(H)$ parametrized by $n$. The Plücker relations then translate into recursions on the lattice.

Plücker relations are normally associated to several different coordinates evaluated for a fixed subspace $w$ rather than, as is the case here, a single Plücker coordinate (the $\tau$-function) evaluated on some subset of a lattice of points. It is natural to ask if there is a relation. The answer is in a sense given by Theorem 4.8, but one can see this more directly, at least in a family of simple cases. Our specific example will be based on finite dimensional reductions of the infinite Grassmannian through a suitable subquotient procedure ([9], Chapt. 6).

We fix a finite family of points in the plane $\{x_i\}_{i=1,...,k}$ with $|x_i| > 1$, and establish a correspondence between

- Plücker coordinates, parametrized by a set of indices $\{i_1, ..., i_l\} \subset \{1, ..., k\}$ for $l \leq k$, for a single subspace $w \in \text{Gr}_{H^+}(H)$.
- The single Plücker coordinate $\pi_\emptyset(w(i_1, ..., i_l))$ (i.e. an evaluation of the $\tau$-function) for a corresponding family of elements $w(i_1, ..., i_l) \in \text{Gr}_{H^+}(H)$.

We do this using the procedure for reducing the infinite dimensional Grassmannian to finite dimensional ones given in [9], Secs. 6.3, 6.4. Consider the family $\text{Gr}(w_1, w_2)$ of subspaces $w \subset H$ satisfying

$$w_1 \subset w \subset w_2. \tag{4.5.1}$$

where the subspaces $w_1 \subset w_2 \subset H$ are defined by requiring a specified set of zeroes in the the elements of $w_1 \subset H^+$ and allowing a set of permissible poles in $w_2$:

$$w_1 = r(z)H^+, \quad w_2 := \frac{r(z)}{p(z)}H^+. \tag{4.5.2}$$

Here $r(z)$, $p(z)$ are specified polynomials with roots inside the unit circle $S^1 \subset \mathbb{C}$ centred at the origin, and the virtual dimensions of $w_1$ and $w_2$ are $-\text{deg}(r)$ and $\text{deg}(p) - \text{deg}(r)$, respectively. Our space $\text{Gr}(w_1, w_2)$ is a union over virtual dimensions lying between $-\text{deg}(r)$ and $\text{deg}(p) - \text{deg}(r)$ of finite dimensional Grassmannians of spaces in $H$.

For our example, we choose

$$r(z) := \prod_{i=1}^{k}(1 - x_i z), \quad p(z) := \prod_{i=1}^{k}(1 - x_i z)^2, \tag{4.5.3}$$

with $|x_i| > 1$ and all $x_i$'s distinct. The quotient $w_2/w_1$ has a basis $\{d_{i,m}\}_{i \in \{1, ..., k\}, \ m \in \{1, 2\}}$, given by

$$d_{i,m}(z) := r(z)(1 - x_i z)^{-m}, \quad i \in \{1, ..., k\}, \ m \in \{1, 2\}. \tag{4.5.4}$$

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Recall that the lattice in the preceding sections was given by the action of a discrete subgroup of $\Gamma_+$ defined by evaluating the $t$ variables at translates by sums of integer multiples of the elements

$$[x_i] := \left( x_i, \frac{x_i^2}{2}, \frac{x_i^3}{3}, \ldots \right). \quad (4.5.5)$$

In terms of $\mathcal{H} = L^2(S^1)$, our discrete actions are generated by multiplication by $S_i := \gamma_+( [x_i] ) = e^{\sum_{\ell=1}^{\infty} \frac{1}{\ell} (x_\ell z)^\ell} = \frac{1}{1-x_\ell^2 z}, \quad i = 0, \ldots, k. \quad (4.5.6)$

Consider the multiplicative action of $S_i$ on $\text{Gr}_{\mathcal{H}^+}(\mathcal{H})$ mapping a space $w$ to $(1-x_i z)^{-1} w$. This maps the spaces in $\text{Gr}(w_1, w_2)$ to $\text{Gr}_{\mathcal{H}^+}(\mathcal{H})$ and does not preserve $\text{Gr}(w_1, w_2)$. It does however act on a multi-filtration on $\mathcal{H}/w_1$ defined by the increasing order of poles of elements at the $x_\ell^{-1}, \ell = 1, \ldots, k$, augmenting the order by one for $(x_i)^{-1}$ and leaving the order unchanged at the other $(x_\ell)^{-1}$.

Now consider the action of $S_{i_1} S_{i_2} \ldots S_{i_l}$, with the $i_j$’s distinct, on an element $w \in \text{Gr}(w_1, w_2)$ of virtual dimension $-l$. The resulting plane $w(i_1, \ldots, i_l) = S_{i_1} S_{i_2} \ldots S_{i_l} w$ is of virtual dimension 0. The null partition Plücker coordinate $\pi_0(S_{i_1} S_{i_2} \ldots S_{i_l} w)$ is the coordinate corresponding to the exterior product of all the basis elements $d_{i_1}$ of $w_2/w_1$. This in turn is the Plücker coordinate of $w$ corresponding to the exterior product of the basis elements $d_{i_1}, \ i \notin \{i_1, i_2, \ldots, i_l\}$ of $w_2/w_1$.

For any plane $\hat{w}$ of virtual dimension zero, the $\tau$-function $\tau^K_{\hat{w}}$, evaluated at 0, is the Plücker coordinate (1.2.23) of the subspace $\hat{w} \subset \mathcal{H}$ corresponding to the null partition; that is, up to a projective scaling factor, the determinant (1.1.19) of the projection from $\hat{w} \in \text{Gr}_{\mathcal{H}^+}(\mathcal{H})$ to the subspace $\mathcal{H}^+$. From the definitions, the expressions $H^\alpha w_{\alpha_1 + \cdots + \alpha_l}(0)$ of Proposition 4.5 are also null partition Plücker coordinates, but of an element $w$ shifted by $S_{i_1} \ldots S_{i_l}$, up to a scaling factor.

We have seen that these are Plücker coordinates of the fixed plane $w$, and with the identifications made, it is easy to see to see that the relations implied by the homomorphism (4.2.2) of Proposition (4.5) become Plücker relations for the fixed plane $w$, with suitable rescalings. These are essential; one cannot adjust each Plücker coordinate individually. For the relations to correspond, one needs the appropriate set of scalings by defining $H^w w(t)$ as in (4.1.9).

5 Lattice mappings to $\text{Gr}_{\mathcal{H}^+}(\mathcal{H}, \omega)$

In this section we show that, restricting to KP $\tau$-functions corresponding to the CKP hierarchy and evaluating these on suitably defined lattices within their KP orbits, we can associate to every element $w^0 \in \text{Gr}_{\mathcal{H}^+}(\mathcal{H}, \omega)$ of $\text{Gr}_{\mathcal{H}^+}(\mathcal{H}, \omega)$ a lattice of homomorphisms from the coordinate ring of $\text{Gr}_{\mathcal{H}^+}(\mathcal{H}, \omega)$ to a localization of the ring $\mathbb{C}[[y, t']]$ of formal
power series in a set of indeterminates \((y, t')\), where \(t' = (t_1, 0, t_3, 0 \ldots)\) is identified with the odd KP flow parameters. We also prove the analogue of Theorem 4.8 for the Lagrangian case. For evaluations of the formal power series \(\tau^K_{w_0}\) that are convergent (e.g., when \(\tau^K_{w_0}\) is a polynomial), our result implies that \(\tau^K_{w_0}\) gives rise to a lattice of points in \(Gr_{\mathbb{H} +}^L(H, \omega)\) whose Plücker coordinates are given by normalized evaluation of the \(\tau\)-function.

5.1 Lattice evaluations of Lagrangian KP \(\tau\)-functions

Consider a family of formal parameters \(\{y_i\}_{i \in \mathbb{N}^+}\) labelled by positive integers. Denote by \(C[[y, t']]\) the ring of formal power series in the variables \(y_i\) and the odd KP flow variables \(\{t_{2j-1}\}_{j \in \mathbb{N}^+}\). Similarly to (4.1.9), in order to define the properly normalized evaluation of the \(\tau\)-function we have to include certain denominators involving the variables \(y_i\). To do this, consider the multiplicative subset of the ring \(C[[y, t']]\) of formal power series generated by elements \(\{y_i \pm y_j\}_{i, j \in \mathbb{N}^+}\)

\[
S^x_y = \langle y_i \pm y_j \mid i, j \in \mathbb{N}^+ \rangle \subset C[[y]] \subset C[[y, t']].
\] (5.1.1)

**Remark 5.1.** The generators of \(S^x_x\) correspond to roots of \(sp(\infty)\).

Since \(C[[y, t']]\) is an integral domain, we can introduce its localization \((S^x_x)^{-1} C[[y, t']]\) with respect to \(S^x_x\). Note that \((S^x_x)^{-1} C[[y, t']]\) is itself an integral domain.

The following map of generators defines a surjective homomorphism of graded algebras

\[
T_{sp} : C[[x, t]] \rightarrow C[[y, t']],
\]

\[
t_j \mapsto \begin{cases} 
  t_j, & j \equiv 1 \mod 2, \\
  0, & j \equiv 0 \mod 2,
\end{cases} 
\]

\[
x_i \mapsto \begin{cases} 
  y_{i+1}, & i \geq 0, \\
  -y_{-i}, & i < 0.
\end{cases} 
\] (5.1.2)

**Lemma 5.1.** The homomorphism \(T_{sp}\) extends uniquely to a homomorphism of localized rings

\[
T_{sp} : (S^x_x)^{-1} C[[x, t]] \rightarrow (S^x_y)^{-1} C[[y, t']].
\] (5.1.3)

**Proof.** Consider the multiplicative subset \(S^x_x \subset C[[x, t]]\) introduced in (4.1.7) and note that its image is exactly the multiplicative set (5.1.1)

\[
T_{sp}(S^x_x) = S^x_y.
\] (5.1.4)
In other words, all elements of $T_{sp}(S^\times_x)$ are units of the localized ring $(S^\times_y)^{-1} \mathbb{C}[[y, t']]$. It follows from the Universal Property of localization that there exists a unique homomorphism between localized rings that makes the following diagram commutative

$$
\begin{array}{ccc}
\mathbb{C}[[x, t]] & \xrightarrow{T_{sp}} & (S^\times_y)^{-1} \mathbb{C}[[y, t']] \\
\downarrow & & \exists! \\
(S^\times_x)^{-1} \mathbb{C}[[x, t]] & & 
\end{array}
$$

(5.1.5)

Define the Lagrangian evaluation of the $\tau$-function as the $T_{sp}$-image of (4.1.9) for all $n \in A$

$$h^\mathbf{n}_{w^0} := T_{sp}(H^\mathbf{n}_{w^0}) = \prod_{1 \leq i < j} (y_i - y_j)^{n_{i-1} n_{j-1} + n_{i-j}} \prod_{i, j = 1}^{\infty} (-1)^{n_{i-1} n_{j-1}} (y_i + y_j)^{n_{i-j}}$$

$$\times \tau_{w^0}^{KP} \left( t' + \sum_{i=1}^{\infty} \left( n_{i-1} [y_i] + n_{-i} [-y_i] \right) \right).$$

(5.1.6)

Remark 5.2. We view $h^\mathbf{n}_{w^0} \in (S^\times_y)^{-1} \mathbb{C}[[y, t']]$ here as an element of the localized ring of formal power series. Note, however, that in the simplest special case when the element $w^0 \in \text{Gr}^L_{\mathcal{H}^+}(\mathcal{H}, \omega)$ of the infinite dimensional Lagrangian Grassmannian is such that $\tau_{w^0}^{KP}(t')$ is a polynomial in the KP flow parameters, we can evaluate all parameters in (5.1.6). If we then set $\{t_1, t_3, t_5, \ldots\}$ to be arbitrary complex numbers and the $y_i$'s to be pairwise distinct complex numbers satisfying $y_i + y_j \neq 0$, formula (5.1.6) defines a lattice of complex numbers.

Consider the infinite dimensional sublattice

$$\mathbf{B} = \langle \mathbf{\beta}_i := \alpha_{i-1} - \alpha_{-i} \mid i \in \mathbb{N}^+ \rangle \subset \mathbf{A},$$

(5.1.7)

with generators labeled by the positive integers; i.e., $\mathbf{B} \subset \mathbf{A}$ consists of elements

$$\sum_{i \in \mathbb{Z}} n_i \alpha_i = \sum_{i \in \mathbb{N}^+} n_{i-1} \mathbf{\beta}_i \in \mathbf{A}$$

(5.1.8)

satisfying

$$n_{i-1} = -n_{-i}, \quad \forall i \in \mathbb{N}^+.$$

(5.1.9)

We denote such elements $\mathbf{n}'$. Similarly to the previous section, for every element $\mathbf{n}' \in \mathbf{B}$ of the sublattice and every partition $\lambda = (a|b)$ of Frobenius rank $r$ we introduce the notation

$$h^\mathbf{n'}_{w^0, \lambda} := h^\mathbf{m} = h^\mathbf{n'}_{w^0} + \sum_{i=1}^{r} \alpha_{a_i} - \sum_{i=1}^{r} \alpha_{-b_i-1},$$

(5.1.10)
where
\[ m := n' + \sum_{i=1}^{r} \alpha a_i - \sum_{i=1}^{r} \alpha b_i - 1 \]  
(5.1.11)
and \( n' \in B \) is an arbitrary element of the sublattice, with components \( \{n'_i\}_{i \in \mathbb{Z}} \) satisfying (5.1.9). Note that for a symmetric partition \( \lambda = \lambda^T \), we have \( \{a_i = b_i\}_{i=1,...,r} \), so \( m \) is also in the sublattice \( B \). But for a general partition \( \lambda \), \( m \in A \) is not in \( B \).

For each \( i \in \mathbb{N}^+ \), define the map \( \hat{\delta}_i^L \) which, evaluated on the generators, gives
\[
\begin{align*}
\hat{\delta}_i^L &: y_j \mapsto y_j, \\
\hat{\delta}_i^L &: t_{2j+1} \mapsto t_{2j+1} + \frac{2y_{2j}}{2j+1}.
\end{align*}
\]  
(5.1.12)

**Lemma 5.2.** The maps \( \hat{\delta}_i^L \) extend to homomorphisms
\[ \hat{\delta}_i^L : \mathbb{C}[[y, t]] \to \mathbb{C}[[y, t]], \]  
(5.1.13)
that mutually commute
\[ \hat{\delta}_i^L \hat{\delta}_j^L = \hat{\delta}_j^L \hat{\delta}_i^L, \quad \forall \ i, j \in \mathbb{N}^+. \]  
(5.1.14)

They act on the evaluations of the \( \tau \)-function as shifts along the generators \( \beta_i \in B \) of the sublattice \( B \subset A \). For a given element \( w_0 \in \text{Gr}_{L}^L(H, \omega) \) of infinite dimensional Lagrangian Grassmannian, we have
\[ \hat{\delta}_i^L h_{w_0}^n = h_{w_0}^{n+\beta_i} = \mathcal{T}_{sp} \left( H_{w_0}^{n+\beta_i} \right) = \mathcal{T}_{sp} \left( \hat{\delta}_{i-1} \hat{\delta}_{i-1} H_{w_0}^n \right) \]  
(5.1.15)
for all \( n \in A \) and \( i \in \mathbb{N}^+ \).

### 5.2 Lagrangian condition on the big cell

The analogue of Lagrangian condition on the big cell (2.5.19) for Lagrangian evaluations (5.1.10) of the \( \tau \)-function is given by the following.

**Lemma 5.3.** Let \( w_0 \in \text{Gr}_{L}^L(H, \omega) \) be an element of the infinite Lagrangian Grassmannian. We then have
\[ h_{w_0}^{0, (i|j)} = h_{w_0}^{0, (j|i)}, \quad \forall \ i, j \in \mathbb{N}^+. \]  
(5.2.1)

**Proof.**

\[
\begin{align*}
h_{w_0}^{0, (i|j)} & \overset{(5.1.10)}{=} h_{w_0}^{0, (i|j)} h_{0}^{\alpha_{i-1} - \alpha_{-j}} \\
& \overset{(5.1.6)}{=} -\tau_{w_0}^{KP} (t' + [y_i] - [-y_j]) \\
& \overset{(1.4.7)}{=} \frac{-\tau_{w_0}^{KP} (t' + [y_j] - [-y_i])}{y_j + y_i} = h_{w_0}^{\alpha_{j-1} - \alpha_{-i}} = h_{w_0}^{0, (j|i)}.
\end{align*}
\]  
(5.2.2)

\[ \square \]
Corollary 5.4. Let \( w^0 \in \text{Gr}^L_{\mathcal{H}+} (\mathcal{H}, \omega) \) be a fixed point in the infinite dimensional Lagrangian Grassmannian such that \( h^0_{w^0} \neq 0 \) is a nontrivial formal power series. The following identity holds in the field of fractions \( \mathbb{F}_{y,t'} \)

\[
\frac{h^0_{w^0}}{h^0_{w^0}} = \det \left( \frac{h^0_{w^0} \alpha_{\min(a_i,b_j)} - \alpha_{\max(a_i,b_j)} - 1}{h^0_{w^0}} \right)_{1 \leq i,j \leq r} \quad (5.2.3)
\]

Proof. Apply homomorphism \( T_{sp} \) to both sides of (4.4.4) and use Lemma 5.3.

\[ \square \]

5.3 Lattice points in Lagrangian Grassmannian

Lemma 5.3 gives a condition, valid on the big cell, for a subspace to be Lagrangian; In what follows it is explained why the result on the big cell is sufficient to imply it in general (cf. remark 4.4). Our first step is to show that the evaluation (5.1.6) gives rise to a family of homomorphisms of the coordinate ring identifying the homomorphism that corresponds to the origin \( 0 \in \mathcal{B} \) of the sublattice.

Proposition 5.5. Fix an element \( w^0 \in \text{Gr}^L_{\mathcal{H}+} (\mathcal{H}, \omega) \) of the infinite dimensional Lagrangian Grassmannian such that \( \tau^{KP}_{w^0}(t') \neq 0 \) is a nontrivial formal power series. The following map, defined on the linear basis elements \( \pi_\lambda \in \mathcal{S}(\text{Gr}^L_{\mathcal{H}+} (\mathcal{H}, \omega)) \), generates a homomorphism of commutative algebras

\[
\Psi^0_{w^0} : \mathcal{S}(\text{Gr}^L_{\mathcal{H}+} (\mathcal{H}, \omega)) \to (\mathcal{S}^x_y)^{-1} \mathbb{C}[[y,t']],
\]

\[
\Psi^0_{w^0} : \pi_\lambda \mapsto h^0_{w^0 \alpha_{\lambda}}. \quad (5.3.1)
\]

Proof. Consider the homomorphism \( \tilde{\Psi}^0_{w^0} \) defined by the same action on generators of the polynomial ring \( \mathcal{S} \)

\[
\tilde{\Psi}^0_{w^0} : \mathcal{S} \to (\mathcal{S}^x_y)^{-1} \mathbb{C}[[y,t']],
\]

\[
\tilde{\Psi}^0_{w^0} : \pi_\lambda \mapsto h^0_{w^0 \alpha_{\lambda}}. \quad (5.3.2)
\]

We need to show that \( \Psi^0_{w^0} \) annihilates all Plücker and Lagrange relations. For the Plücker relations, simply note that

\[
\tilde{\Psi}^0_{w^0} = T_{sp} \circ \tilde{\Phi}^0_{w^0}, \quad (5.3.3)
\]

is a composition of two ring homomorphisms. By Theorem 4.8, \( \tilde{\Phi}^0_{w^0} \) annihilates all Plücker relations and, hence, so does the \( \tilde{\Psi}^0_{w^0} \).

It remains to prove that \( \tilde{\Psi}^0_{w^0} \) annihilates all Lagrange relations. Recall that the ring \((\mathcal{S}^x_y)^{-1}\mathbb{C}[[y,t']]\) is a localization of an integral domain, so it must itself be an integral
domain. Denote its field of fractions by $F_{y,t}$. By the assumption of the Proposition, the $t'$-restriction $\tau_{KP}^{t'}(t') \in \mathbb{C}[t'] \subset F_{y,t'}$ of the KP hierarchy $\tau$-function is a nontrivial formal power series, and hence, an invertible element of the field of fractions. By the universal property of localization, we have a unique homomorphism

$$\tilde{\Psi}_{w_0}^0 : \mathbb{C}[\pi_0, \pi_0^{-1}] \left[ \pi_{(i-1)j-1} \mid 1 \leq i \leq j \right] \rightarrow F_{y,t'}$$

(5.3.4)

obtained as a localization of the homomorphism $\tilde{\Psi}_{w_0}^0$ restricted to the subring

$$\mathbb{C}[\pi_0] \left[ \pi_{(i-1)j-1} \mid 1 \leq i \leq j \right] \subset S$$

(5.3.5)

generated by Plücker coordinates corresponding to hook partitions; i.e., those of Frobenius rank 1. Consider the following diagram

$$S = \mathbb{C}[\pi_\lambda \mid \lambda \in \Lambda] \xrightarrow{\tilde{\Psi}_{w_0}^0} (S_y^\infty)^{-1}\mathbb{C}[y, t']$$

(5.3.6)

Its commutativity follows from comparing the actions on the generators $\pi_\lambda$ for all partitions $\lambda = (a|b)$, of any Frobenius rank $r$. This gives

$$\tilde{\Psi}_{w_0}^0(\xi^L(\pi_\lambda)) \overset{(3.3.7)}{=} \tilde{\Psi}_{w_0}^0 \left( \pi_0 \det \left( \frac{\pi_{(\min(a,b),\max(a,b))}}{\pi_0} \right)_{1 \leq i,j \leq r} \right)$$

(5.3.2)

$$= h_{w_0}^0 \det \left( \frac{h_{w_0}^0(\min(a,b),\max(a,b))}{h_{w_0}^0} \right)_{1 \leq i,j \leq r}$$

(5.3.7)

$$= h_{w_0}^0 \det \left( \frac{h_{w_0}^0(\min(a,b),\max(a,b)) - \alpha - \max(a,b)_{-1}}{h_{w_0}^0} \right)_{1 \leq i,j \leq r}$$

(5.3.10)

$$= h_{w_0}^0 \lambda = \tilde{\Psi}_{w_0}^0(\pi_\lambda).$$

By Lemma 3.6, $\xi^L$ annihilates the $\mathcal{I}^L$ defining Lagrangian Grassmannians. Hence

$$\mathcal{I}^L \subset \ker \Delta$$

(5.3.8)

is in the kernel of the diagonal map $\Delta$ of diagram (5.3.6). But $\Delta$ is the composition of an injective map with $\tilde{\Psi}_{w_0}^0$, so

$$\ker \Delta = \ker \tilde{\Psi}_{w_0}^0.$$

(5.3.9)

Combining (5.3.8) with (5.3.9) completes the proof. □
This brings us to the main result of this section.

**Theorem 5.6.** Fix an element $w^0 \in \text{Gr}^L_{\mathcal{H}_+}(\mathcal{H}, \omega)$ of the infinite Lagrangian Grassmannian such that $\tau^{K\Phi}_{w^0}(t') \neq 0$.† To every element $n'_0 \in B$ of the sublattice there is a homomorphism of commutative algebras generated by the following evaluation on linear basis elements

$$\Psi_{w^0}^{n'_0} : S(\text{Gr}^L_{\mathcal{H}_+}(\mathcal{H}, \omega)) \to (S^\times_N)^{-1}\mathbb{C}[y, t'],$$

$$\Psi_{w^0}^{n'_0} : \pi_\lambda \mapsto \pi_{\lambda}^{n'_0 + \beta_i}.$$  \hfill (5.3.10)

**Proof.** We prove this by induction on the sublattice $B \subset A$. The starting point is $n'_0 = 0 \in B$, which is given by Proposition 5.5. Now suppose that the statement of the Theorem holds for $n'_0 \in B$ and let $i \in \mathbb{N}^+$ be a positive integer. Note that for all partitions $\lambda$ we have

$$\Psi_{w^0}^{n'_0 + \beta_i}(\pi_\lambda) = \delta_i(h_{w^0}^{n'_0 + \beta_i, \lambda} \Psi_{w^0}^{n'_0}(\pi_\lambda)).$$  \hfill (5.3.11)

Hence

$$\Psi_{w^0}^{n'_0 + \beta_i} = \hat{\delta}_i \circ \Psi_{w^0}^{n'_0}$$  \hfill (5.3.12)

is a composition of two ring homomorphisms and therefore must itself be a ring homomorphism. By induction on the sublattice $B$, this implies that $\Psi_{w^0}^{n'_0}$ is a homomorphism for all $n'_0 \in B$, concluding the proof.  \hfill \Box

6 Applications to Lattice Integrable Systems

6.1 Kenyon-Pemantle recurrences for $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$

**Lemma 6.1.** For every ordered triple of nonnegative integers $i, j, k \in \mathbb{N}, \ i < j < k$, the cubic

$$\kappa_1 := \pi_{(0)0}\left(\pi_{(ij)}\pi_{(k,j,k,j)} + \pi_{(j,k)}\pi_{(k,i,j,i)}\right) - \pi_{(ij)}\pi_{(ij)}\pi_{(k,k)} + \pi_{(ij)}\pi_{(j,j)}\pi_{(k,k)}. \hfill (6.1.1)$$

belongs to the defining ideal $\mathcal{I}$ of the coordinate ring $S(\text{Gr}_{\mathcal{H}_+}(\mathcal{H})) = S/\mathcal{I}$ of $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$.

†We expect that the condition $\tau^{K\Phi}_{w^0}(t') \neq 0$ is automatically satisfied for all elements of the infinite Lagrangian Grassmannian, whatever the value of $t'$ is. However a proof of this remains to be found. We therefore impose this condition as an additional requirement in the formulation of the Theorem.
Proof. (6.1.1) can be written as

\[ \kappa_1 = \pi_{(j|k)} \rho_1 + \pi_{(\emptyset|\emptyset)} \rho_2, \quad (6.1.2) \]

where

\[ \rho_1 := \pi_{(i|j)} \pi_{(k|i)} - \pi_{(k|j)} \pi_{(i|i)} + \pi_{(\emptyset|\emptyset)} \pi_{(k,i|j,i)}, \quad (6.1.3a) \]
\[ \rho_2 := \pi_{(j|k)} \pi_{(k|j)} - \pi_{(j|j)} \pi_{(k|k)} + \pi_{(\emptyset|\emptyset)} \pi_{(k,j|k,j)} \quad (6.1.3b) \]

are (short) Plücker quadratic forms belonging to \( \mathcal{I} \). Therefore, so does \( \kappa_1 \).

Since the Plücker relations are invariant under transposition of all partitions, (6.1.1) also implies the dual relation:

**Lemma 6.2.** For every ordered triple of nonnegative integers \( i, j, k \in \mathbb{N}^+, \ i < j < k \), the cubic

\[ \kappa_1^T := \pi_{(\emptyset|\emptyset)} \left( \pi_{(i|i)} \pi_{(k,j|k,j)} + \pi_{(k|j)} \pi_{(j,i|k,i)} \right) - \pi_{(i|i)} \pi_{(j|j)} \pi_{(k|k)} + \pi_{(j|i)} \pi_{(k|j)} \pi_{(i|k)} \quad (6.1.4) \]

also belongs to \( \mathcal{I} \).

Since the Plücker ideal is invariant under permutations of basis elements of \( \mathcal{H} \), four more relations of the form (6.1.1), (6.1.4) follow. Let \( i, j, k \in \mathbb{N}^+ \) be an ordered triple of nonnegative integers \( i < j < k \). Consider the permutations \( \sigma_1^{(i,j,k)} \), \( \sigma_2^{(i,j,k)} \) of the basis elements \( \{e_t\}_{t \in \mathbb{Z}} \) of \( \mathcal{H} \) given by

\[ \sigma_1^{(i,j,k)}(e_t) = \begin{cases} 
    e_j, & \text{if } l = i, \\
    e_k, & \text{if } l = j, \\
    e_i, & \text{if } l = k, \\
    e_{-j-1}, & \text{if } l = -i - 1, \\
    e_{-k-1}, & \text{if } l = -j - 1, \\
    e_{-i-1}, & \text{if } l = -k - 1, \\
    e_t, & \text{otherwise},
\end{cases} \quad \sigma_2^{(i,j,k)}(e_t) = \begin{cases} 
    e_{-k-1}, & \text{if } l = i, \\
    e_{-j-1}, & \text{if } l = j, \\
    e_{-i-1}, & \text{if } l = k, \\
    e_k, & \text{if } l = -i - 1, \\
    e_j, & \text{if } l = -j - 1, \\
    e_i, & \text{if } l = -k - 1, \\
    e_t, & \text{otherwise}.
\end{cases} \quad (6.1.5) \]

These induce invertible endomorphisms \( \hat{\sigma}_1^{(i,j,k)}, \hat{\sigma}_2^{(i,j,k)} \in \text{End}(\mathcal{F}) \) of the Fock space, which are just the Clifford representations of these as \( \text{GL}(\mathcal{H}) \) elements. The dual action on the Plücker coordinates gives a pair of automorphisms

\[ \hat{\sigma}_1^{(i,j,k)*} : S \rightarrow S, \quad \hat{\sigma}_2^{(i,j,k)*} : S \rightarrow S \quad (6.1.6) \]

of the polynomial ring \( S \) that preserve the Plücker ideal \( \mathcal{I} \) and thus descend to automorphisms of the coordinate ring \( S(\text{Gr}_{\mathcal{H}+}(\mathcal{H})) = S/\mathcal{I} \):

\[ \hat{\sigma}_1^{(i,j,k)*} : S(\text{Gr}_{\mathcal{H}+}(\mathcal{H})) \rightarrow S(\text{Gr}_{\mathcal{H}+}(\mathcal{H})), \]
\[ \hat{\sigma}_2^{(i,j,k)*} : S(\text{Gr}_{\mathcal{H}+}(\mathcal{H})) \rightarrow S(\text{Gr}_{\mathcal{H}+}(\mathcal{H})). \quad (6.1.7) \]
Proposition 6.3. The following seven elements belong to the defining ideal $\mathcal{I}$ of $\mathcal{S}(\text{Gr}_{H^+}(\mathcal{H}))$.

\[
\begin{align*}
\kappa_1 & := \pi(\emptyset)\left(\pi(i,i)\pi(k,j)\pi(k,j) + \pi(j,i)\pi(i,j)\pi(k,i) - \pi(i,i)\pi(j,j)\pi(k,k) + \pi(i,j)\pi(j,j)\pi(k,k, i)\right) \quad (6.1.8a) \\
\kappa_2 & := \pi(\emptyset)\left(\pi(j,j)\pi(k,i)\pi(k,i) - \pi(k,k)\pi(i,j)\pi(k,k) + \pi(i,j)\pi(j,j)\pi(k,k)\pi(k,i)\right) \quad (6.1.8b) \\
\kappa_3 & := \pi(\emptyset)\left(\pi(k,k)\pi(j,j)\pi(k,i) + \pi(i,i)\pi(j,j)\pi(k,k) + \pi(i,j)\pi(j,j)\pi(k,k)\pi(k,i)\right) \quad (6.1.8c) \\
\kappa_1^T & := \pi(\emptyset)\left(\pi(i,i)\pi(k,j)\pi(k,j) + \pi(k,k)\pi(i,j)\pi(k,k) - \pi(i,j)\pi(j,j)\pi(k,k)\pi(k,i)\right) \quad (6.1.8d) \\
\kappa_2^T & := \pi(\emptyset)\left(\pi(j,j)\pi(k,i)\pi(k,i) - \pi(k,k)\pi(i,j)\pi(k,k) + \pi(i,j)\pi(j,j)\pi(k,k)\pi(k,i)\right) \quad (6.1.8e) \\
\kappa_3^T & := \pi(\emptyset)\left(\pi(k,k)\pi(j,j)\pi(k,i) + \pi(i,i)\pi(j,j)\pi(k,k) - \pi(i,j)\pi(j,j)\pi(k,k)\pi(k,i)\right) \quad (6.1.8f) \\
\kappa_0 & := \pi(\emptyset)\left(\pi(k,j)\pi(k,j)\pi(k,i) + \pi(i,i)\pi(j,j)\pi(k,k)\pi(k,i) - \pi(i,j)\pi(j,j)\pi(k,k)\pi(k,i)\right) \quad (6.1.8g) \\
& \quad + \left(\pi(i,i)\pi(j,j)\pi(k,k)\pi(k,i)\right)\left(\pi(i,i)\pi(j,j)\pi(k,k) - \pi(\emptyset)\left(\pi(i,i)\pi(k,k)\pi(i,j)\pi(k,k)\pi(k,i) + \pi(k,k)\pi(j,j)\pi(k,k)\pi(k,i)\right)\right) \\
& \quad - \left(\pi(i,i)\pi(j,j)\pi(\emptyset)\pi(i,j)\pi(k,k)\pi(k,i) - \pi(\emptyset)\pi(\emptyset)\pi(i,j)\pi(k,k)\pi(k,i)\pi(k,k)\pi(k,i)\right) \\
& \quad \left(\pi(j,j)\pi(k,k) - \pi(\emptyset)\pi(\emptyset)\pi(k,k)\pi(k,k)\pi(k,i)\right) \quad (6.1.8h)
\end{align*}
\]

for every triple $i, j, k \in \mathbb{N}^+$ of nonnegative integers satisfying $i < j < k$.

Proof. Relations (6.1.8a)–(6.1.8c) are the orbit of (6.1.1) under the three-element cyclic group of automorphisms generated by $\hat{\sigma}_1^{(i,j,k)}$ and relations (6.1.8d)–(6.1.8f) are the orbit of (6.1.4) under the same group. To prove (6.1.8h), first note that applying the automorphism $\hat{\sigma}_2^{(i,j,k)}$ to relation (6.1.8a) gives

\[
\tilde{\kappa}_1 := \hat{\sigma}_2^{(i,j,k)}(\kappa_1) = \pi(k,j,i)\pi(k,j,i)\pi(k,j,i)\pi(k,j,i) + \pi(i,j)\pi(i,j)\pi(k,i)\pi(k,i) \\
+ \pi(j,i)\pi(k,i)\pi(k,i)\pi(k,i) - \pi(j,i)\pi(k,i)\pi(k,i)\pi(k,i). \quad (6.1.9)
\]

It follows from Proposition 3.3 that $\mathcal{S}(\text{Gr}_{H^+}(\mathcal{H})) = \mathcal{S}/\mathcal{I}$ is an integral domain. In the corresponding field of fractions we can solve the relations $\kappa_1 = 0$, $\kappa_2 = 0$, $\kappa_3 = 0$ for $\pi(k,j,i), \pi(j,i,k), \pi(i,k,j)$, respectively, and substitute the solutions into relation $\tilde{\kappa}_1 = 0$ to get the following identity in the fraction field

\[
(\pi(i,j)\pi(j,k)\pi(k,i) - \pi(i,i)\pi(j,j)\pi(k,k))\kappa_0 = 0. \quad (6.1.10)
\]

The first factor is nonzero, as can be seen by computing it in terms of determinants of homogeneous coordinates on a sufficiently large finite Grassmannian $\text{Gr}_{V_n}(\mathcal{H}_{n,2n})$ (i.e. with $n \geq k + 1$).

Finally, using the fact that $\mathcal{S}(\text{Gr}_{H^+}(\mathcal{H}))$ is an integral domain, it follows that the relation $\kappa_0 = 0$ holds in this ring and hence $\kappa_0 \in \mathcal{I}$. \hfill □

Now fix an element $w \in \text{Gr}_{H^+}(\mathcal{H})$ and define the infinite family of functions on the sublattice $B \subset A$, consisting of the function

\[
H : B \to \left(S^X_{\mathcal{Y}} \right)^{-1} \mathbb{C}[[x,t]],
\]

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corresponding to the null partition, together with infinitely many functions labelled by nonsymmetric hook partitions \((p|q)\), \(p \neq q\) with values

\[
\begin{align*}
H^{(p|q)}(\mathcal{B}) : (S_{x}^{\infty})^{-1} \mathbb{C}[[x, t]], \\
\mathbf{H}^{(p|q)} : \mathbf{n}' \mapsto H_{w_{v}(p|q)},
\end{align*}
\]

Also define, for \(v' \in \mathcal{B}\) the lattice functions \(\mathbf{H}_{v'}, \mathbf{H}_{v'}^{(p|q)}\) with shifted arguments having values:

\[
\begin{align*}
\mathbf{H}_{v'} : \mathcal{B} \rightarrow (S_{x}^{\infty})^{-1} \mathbb{C}[[x, t]], \\
\mathbf{H}_{v'} : \mathbf{n}' \mapsto H_{w_{v'}^{p|q}}, \\
\mathbf{H}_{v'}^{(p|q)} : \mathcal{B} \rightarrow (S_{x}^{\infty})^{-1} \mathbb{C}[[x, t]], \\
\mathbf{H}_{v'}^{(p|q)} : \mathbf{n}' \mapsto H_{w_{v'}^{p|q}}, \quad \forall v' \in \mathcal{B}, \quad p, q \in \mathbb{N}.
\end{align*}
\]

**Theorem 6.4.** For every ordered triple \((i, j, k)\) of nonnegative integers, \(i < j < k\), these lattice functions satisfy the system of recurrence relations

\[
\begin{align*}
\mathbf{H}^{(i|j)}\mathbf{H}^{(j|i)} = -HH_{\beta_{i}}H_{\beta_{j}+\beta_{k}} + H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}} - H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}}H_{\beta_{i}}, \quad (6.1.14a) \\
\mathbf{H}^{(j|i)}\mathbf{H}^{(i|j)} = HH_{\beta_{i}}H_{\beta_{j}+\beta_{k}} - H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}} + H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}}H_{\beta_{i}}, \quad (6.1.14b) \\
\mathbf{H}^{(i|j)}\mathbf{H}^{(j|i)} = -HH_{\beta_{i}}H_{\beta_{j}+\beta_{k}} + H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}} - H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}}H_{\beta_{i}}, \quad (6.1.14c) \\
\mathbf{H}^{(j|i)}\mathbf{H}^{(i|j)} = HH_{\beta_{i}}H_{\beta_{j}+\beta_{k}} - H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}} + H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}}H_{\beta_{i}}, \quad (6.1.14d) \\
\mathbf{H}^{(i|j)}\mathbf{H}^{(j|i)} = HH_{\beta_{i}}H_{\beta_{j}+\beta_{k}} - H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}} + H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}}H_{\beta_{i}}, \quad (6.1.14e) \\
\mathbf{H}^{(j|i)}\mathbf{H}^{(i|j)} = -HH_{\beta_{i}}H_{\beta_{j}+\beta_{k}} + H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}} - H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}}H_{\beta_{i}}, \quad (6.1.14f) \\
\mathbf{H}^{2}\mathbf{H}^{(i|j)}\mathbf{H}^{(j|i)}\mathbf{H}^{(i|j)} = \left(\mathbf{H}^{(i|j)}\mathbf{H}^{(j|i)}\mathbf{H}^{(i|j)}\right)^{2} - \left(\mathbf{H}^{(i|j)}\mathbf{H}^{(j|i)}\mathbf{H}^{(i|j)}\right) \\
\times \left(2HH_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}} - HH_{\beta_{i}}H_{\beta_{j}+\beta_{k}} + H_{\beta_{i}}H_{\beta_{j}+\beta_{k}} + H_{\beta_{i}}H_{\beta_{j}+\beta_{k}}\right) \\
+ \left(H_{\beta_{i}}H_{\beta_{j}}H_{\beta_{k}} - HH_{\beta_{i}+\beta_{j}}\right)\left(H_{\beta_{i}}H_{\beta_{k}} - HH_{\beta_{i}+\beta_{k}}\right)\left(H_{\beta_{j}}H_{\beta_{k}} - HH_{\beta_{j}+\beta_{k}}\right), \quad (6.1.14g)
\end{align*}
\]

**Proof.** Combine Proposition 6.3 with Theorem 4.8.

Denote the sum of nonvanishing components of \(\mathbf{n}'\) as

\[
\text{height}(\mathbf{n}') := \sum_{j=-\infty}^{\infty} n'_{j}
\]

and suppose that
1. The values of $H$ are known for all $n' \in B$ with $l \leq \text{height}(n') \leq l + 2$.

2. The values of the functions $H^{(p,q)}$ are known for all $n' \in B$ with height$(n') = l$.

If all known values are nonzero, eqs. (6.1.14) allow us to determine the values $H_w^{n'}$ for height$(n') = l + 3$ and the values $H_w^{n'_{(p,q)}}$ for height$(n') = l + 1$ as Laurent polynomials in the known values. However, as written, the recurrence relations (6.1.14) contain negative coefficients.

However, if we restrict to elements $n' \in B_3$ of the 3-dimensional sublattice

$$B_3 := \text{Span}\{\beta_1, \beta_2, \beta_3\} \subset B,$$  \hspace{1cm} (6.1.16)

we can eliminate the $(-)$ signs by defining the following values for all $n' \in B_3$

$$\tilde{H}^{n'} := (-1)^{n_1 + n_2 + n_3 + n_1'n_2 + n_1'n_3 + n_2'n_3} H^{n'},$$

$$\tilde{H}^{n',(x)} := (-1)^{(n_1')^2 + n_2'n_3 + (n_3')^2} H^{n',(1|2)},$$

$$\tilde{H}^{n',(-x)} := (-1)^{(n_1')^2 + n_2'n_3 + (n_3')^2} H^{n',(2|1)},$$

$$\tilde{H}^{n',(y)} := (-1)^{n_1'n_3 + (n_3')^2} H^{n',(2|0)},$$

$$\tilde{H}^{n',(-y)} := (-1)^{n_1'n_3 + (n_3')^2} H^{n',(0|2)},$$

$$\tilde{H}^{n',(z)} := (-1)^{n_1'n_2 + (n_2')^2} H^{n',(0|1)},$$

$$\tilde{H}^{n',(-z)} := (-1)^{n_1'n_2 + (n_2')^2} H^{n',(1|0)}.$$

Denote the lattice translated values of the corresponding seven functions by

$$\tilde{H}^{n'}_v := \tilde{H}^{n'+v'}, \quad \tilde{H}^{n',(i)}_v := \tilde{H}^{n'+v',(i)}, \quad n', v' \in B_3.$$  \hspace{1cm} (6.1.18)

**Corollary 6.5.** For every $w \in \text{Gr}_H(H)$ these lattice functions give solutions of the following system of recurrence relations

$$\tilde{H}^{(x)}_{\beta_1} = \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2 + \beta_3} + \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}(\tilde{H}^{(y)} \tilde{H}^{(z)}),$$  \hspace{1cm} (6.1.19a)

$$\tilde{H}^{(y)}_{\beta_2} = \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}(\tilde{H}^{(y)} \tilde{H}^{(z)}),$$  \hspace{1cm} (6.1.19b)

$$\tilde{H}^{(z)}_{\beta_3} = \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}(\tilde{H}^{(y)} \tilde{H}^{(z)}),$$  \hspace{1cm} (6.1.19c)

$$\tilde{H}^{(x)}_{\beta_1} = \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}(\tilde{H}^{(y)} \tilde{H}^{(z)}),$$  \hspace{1cm} (6.1.19d)

$$\tilde{H}^{(y)}_{\beta_2} = \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}(\tilde{H}^{(y)} \tilde{H}^{(z)}),$$  \hspace{1cm} (6.1.19e)

$$\tilde{H}^{(z)}_{\beta_3} = \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}_{\beta_1} \tilde{H}^{(y)}_{\beta_2} \tilde{H}^{(z)}_{\beta_3} + \tilde{H}^{(x)}(\tilde{H}^{(y)} \tilde{H}^{(z)}),$$  \hspace{1cm} (6.1.19f)

$$\tilde{H}^{2} \tilde{H}^{(x)} \tilde{H}^{(y)} \tilde{H}^{(z)} \tilde{H}_{\beta_1 + \beta_2 + \beta_3} = \left(\tilde{H}^{(x)} \tilde{H}^{(y)} \tilde{H}^{(z)}\right)^2 + \left(\tilde{H}^{(x)} \tilde{H}^{(y)} \tilde{H}^{(z)}\right),$$  \hspace{1cm} (6.1.19g)
Note that, when evaluated at lattice points \( \mathbf{n} \), eq. (6.1.19) can be understood as containing two subcases: the first is when

\[
\tilde{H}'^{\mathbf{n}'}, \begin{cases} 
\tilde{H}'^{\mathbf{n}',(x)}, \tilde{H}'^{\mathbf{n}',(y)}, \tilde{H}'^{\mathbf{n}',(z)}, & \text{and height}(\mathbf{n}') \text{ is even,} \\
\tilde{H}'^{\mathbf{n}',(-x)}, \tilde{H}'^{\mathbf{n}',(-y)}, \tilde{H}'^{\mathbf{n}',(-z)}, & \text{and height}(\mathbf{n}') \text{ is odd,}
\end{cases}
\]

(6.1.20)

the second, when the words even and odd are interchanged. In the next section we show how these results may be reduced to the case of Lagrangian Grassmannians.

### 6.2 Kenyon-Pemantle recurrences for \( \text{Gr}^L_{\mathcal{H}^+}(\mathcal{H}, \omega) \)

We now consider the Lagrangian counterpart of relations (6.1.19), which were named “hexahedron recurrence” relations by R. Kenyon and R. Pemantle [18]. We will show that, for every element \( w^0 \in \text{Gr}^L_{\mathcal{H}^+}(\mathcal{H}) \), suitably normalized evaluations of the KP \( \tau \)-function \( \tau_{w^0}^{KP} \in \mathbb{C}[[t]] \) give rise to solutions of the hexahedron relations. In particular, when the element \( w_0 \in \text{Gr}^L_{\mathcal{H}^+}(\mathcal{H}) \) is such that the lattice evaluations of the formal power series \( \tau_{w_0}^{KP} \) converge (e.g., for polynomial \( \tau \)-functions), we obtain a solution to the hexahedron recurrence in complex numbers.

Fix an element of the Lagrangian Grassmannian \( w^0 \in \text{Gr}^L_{\mathcal{H}^+}(\mathcal{H}) \), and define four functions on the three-dimensional sublattice \( B_3 \subset B \)

\[
\check{h}, \check{h}^{(x)}, \check{h}^{(y)}, \check{h}^{(z)} : B_3 \to (S_y^\infty)^{-1} \mathbb{C}[[y, t]]
\]

(6.2.1a)

where, for \( \mathbf{n}' \in B_3 \), the values are

\[
\check{h}^{\mathbf{n}} := \mathcal{T}_{sp}(\tilde{H}'^{\mathbf{n}}) = (-1)^{n'_1+n'_2+n'_3+n'_4+n'_5+n'_6+n'_7} h_{w^0}^{\mathbf{n},0},
\]

(6.2.1b)

\[
\check{h}'^{\mathbf{n},(x)} := \mathcal{T}_{sp}(\tilde{H}'^{\mathbf{n},(x)}) = (-1)^{(n'_1)^2+n'_2+n'_3+n'_4+n'_5+n'_6+n'_7} h_{w^0}^{\mathbf{n},(1/2)},
\]

(6.2.1c)

\[
\check{h}'^{\mathbf{n},(y)} := \mathcal{T}_{sp}(\tilde{H}'^{\mathbf{n},(y)}) = (-1)^{n'_1+n'_2+(n'_3)^2} h_{w^0}^{\mathbf{n},(0/2)},
\]

(6.2.1d)

\[
\check{h}'^{\mathbf{n},(z)} := \mathcal{T}_{sp}(\tilde{H}'^{\mathbf{n},(z)}) = (-1)^{n'_1+n'_2+(n'_3)^2} h_{w^0}^{\mathbf{n},(0/1)}.
\]

(6.2.1e)

Because of the Lagrangian conditions (2.5.7) (for all \( k \)) we have:

\[
h_{w^0}^{\mathbf{n}',(1/2)} = h_{w^0}^{\mathbf{n}',(2/1)}, \quad h_{w^0}^{\mathbf{n}',(0/2)} = h_{w^0}^{\mathbf{n}',(2/0)}, \quad h_{w^0}^{\mathbf{n}',(1/0)} = h_{w^0}^{\mathbf{n}',(0/1)}
\]

(6.2.2)

so we need not introduce \( h_{w^0}^{\mathbf{n}',(-x)}, h_{w^0}^{\mathbf{n}',(-y)}, h_{w^0}^{\mathbf{n}',(-z)} \) independently. Denote the lattice translates of these functions as

\[
\check{h}_{\mathbf{v}'}, \check{h}_{\mathbf{v}',(x)}, \check{h}_{\mathbf{v}',(y)}, \check{h}_{\mathbf{v}',(z)} : B_3 \to (S_y^\infty)^{-1} \mathbb{C}[[y, t]], \quad \mathbf{v}' \in B_3,
\]

(6.2.3)

with values

\[
\check{h}_{\mathbf{v}'}^{\mathbf{n}'} := \check{h}^{\mathbf{n}'+\mathbf{v}'}, \quad \check{h}_{\mathbf{v}',(x)}^{\mathbf{n}'} := \check{h}^{\mathbf{n}'+\mathbf{v}',(x)}, \quad \mathbf{n}', \mathbf{v}' \in B_3, \quad x = x, y, z.
\]

(6.2.4)

Then as a specialization of Corollary 6.5, we obtain:
Corollary 6.6. For any given element $w^0 \in Gr_H^C(\mathcal{H})$, the functions (6.2.1) satisfy the system of recurrence equations, with positive coefficients:

\[
\begin{align*}
\bar{h}^{(x)}h_{\beta_1} &= \bar{h}h_{\beta_1}h_{\beta_2 + \beta_3} + \bar{h}\beta_1 h_{\beta_2} h_{\beta_3} + h^{(y)}h^{(z)}, \\
\bar{h}^{(y)}h_{\beta_2} &= \bar{h}h_{\beta_2}h_{\beta_1 + \beta_3} + \bar{h}\beta_1 h_{\beta_2} h_{\beta_3} + h^{(x)}h^{(z)}, \\
\bar{h}^{(z)}h_{\beta_3} &= \bar{h}h_{\beta_3}h_{\beta_1 + \beta_2} + \bar{h}\beta_1 h_{\beta_2} h_{\beta_3} + h^{(x)}h^{(z)},
\end{align*}
\]

(6.2.5a)

\[
\begin{align*}
\bar{h}^{(x)}h^{(y)}h^{(z)}h_{\beta_1 + \beta_2 + \beta_3} &= (\bar{h}^{(x)}h^{(y)}h^{(z)})^2 + (h^{(x)}h^{(y)})h^{(z)} \\
&\quad \times (2\bar{h}h_{\beta_1}h_{\beta_2}h_{\beta_3} + hh_{\beta_1}h_{\beta_2 + \beta_3} + hh_{\beta_2}h_{\beta_1 + \beta_3} + hh_{\beta_3}h_{\beta_1 + \beta_2}) \\
&\quad \quad + (h_{\beta_1}h_{\beta_2} + hh_{\beta_1 + \beta_2})(h_{\beta_1}h_{\beta_3} + hh_{\beta_1 + \beta_3})(h_{\beta_2}h_{\beta_3} + hh_{\beta_2 + \beta_3}).
\end{align*}
\]

(6.2.5d)

6.3 Hyperdeterminantal relations

As shown in [18], the hexahedron recurrence implies the hyperdeterminantal relations. Below we show that, for every element $w^0 \in Gr_H^C(\mathcal{H})$ of the Lagrangian Grassmanian, properly normalized evaluations of the KP $\tau$-function $\tau^{KP}_{w^0} \in \mathbb{C}[t]$ give rise to a solution of the hyperdeterminantal relations in any dimensions.

The infinite-dimensional counterpart of the core hyperdeterminantal relations [13, 27, 1] is given by the following.

Lemma 6.7. For all triples $i, j, k \in \mathbb{N}^+$ satisfying $i < j < k$, the following element belongs to the defining ideal of $S(Gr_H^C(\mathcal{H}, \omega))$

\[
\begin{align*}
\varkappa := & \pi^2(0|0)\pi^2(k,j,i|k,j,i) + \pi^2(i|i)\pi^2(k,j,i|k,j) + \pi^2(j,j)\pi^2(k,i|k,i) + \pi^2(k,k)\pi^2(i,j|i,i) \\
&- 2\pi(0|0)\pi(k,j,i|k,j,i) \left( \pi(i|i)\pi(k,j,i|k,j,i) + \pi(j,j)\pi(k,i|k,i) + \pi(k,k)\pi(j,i|i,i) \right) \\
&- 2 \left( \pi(i|i)\pi(j,j)\pi(k,i|k,i) + \pi(j,j)\pi(k,k)\pi(j,i|i,i) \pi(k,i|k,i) + \pi(i|i)\pi(k,k)\pi(j,i|i,i) \pi(k,j|k,j) \right) \\
&+ 4\pi(0|0)\pi(j,i|i,i)\pi(k,i|k,i)\pi(k,j|k,j) + 4\pi(i|i)\pi(j,j)\pi(k,k)\pi(k,j|i,i),
\end{align*}
\]

(6.3.1)

Proof. The element $\varkappa$ can be written as

\[
\varkappa = \varrho_1^2 + 4\pi^2(k,i|k,j)\varrho_2 + 4(\pi(i|i)\pi(j,j) - 4\pi(0|0)\pi(j,i|i,i))\varrho_3,
\]

(6.3.2)

where

\[
\begin{align*}
\varrho_1 &:= 2\pi(i|i)\pi(k,i|k,j) + \pi(0|0)\pi(k,j,i|k,i) \\
&\quad - \pi(i|i)\pi(k,j|k,i) - \pi(j,j)\pi(k,i|k,i) + \pi(k,k)\pi(j,i|i,i),
\end{align*}
\]

(6.3.3a)

\[
\begin{align*}
\varrho_2 &:= \pi^2(j,j) - \pi(i|i)\pi(j,j) + \pi(0|0)\pi(j,i|i,i),
\end{align*}
\]

(6.3.3b)

\[
\begin{align*}
\varrho_3 &:= \pi^2(k,i|k,j) - \pi(k,i|k,i)\pi(k,j|k,j) + \pi(k,k)\pi(k,j|i,i).
\end{align*}
\]

(6.3.3c)
The quadratic forms $\varrho_2$ and $\varrho_3$ coincide with (short) Plücker forms up to linear relations and
\[
\pi_{(i|j)} - \pi_{(j|i)}; \pi_{(k|i;k,j)} - \pi_{(k,j|k,i)} \in \mathcal{I}^c.
\] (6.3.4)
follow from the Lagrangian conditions (2.5.6). To show that $\varrho_1 \in \mathcal{I}^c$, note that
\[
\varrho_1 = P_{i,j} + P_{i,k} + P_{j,k}
\]
\[
+ \pi_{(k,i;k,j)}(\pi_{(i|j)} - \pi_{(j|i)}) + \pi_{(j,i;k,j)}(\pi_{(i|k)} - \pi_{(k|i)})
\]
\[
+ \pi_{(j,i;k,j)}(\pi_{(i|j)} - \pi_{(j|i)}) + \pi_{(j,i;k,j)}(\pi_{(i|k)} - \pi_{(k|i)})
\]
\[
+ \pi_{(i|k)}(\pi_{(k,i;j,j)} - \pi_{(j,i;k,j)}) + \pi_{(i|j)}(\pi_{(k,i;j,k)} - \pi_{(k,j;k,i)}),
\] (6.3.5)
where
\[
P_{i,j} := \pi_{(k|k)}\pi_{(j|i;j,i)} - \pi_{(k|j)}\pi_{(j,i;k,j)} + \pi_{(k,i)}\pi_{(j,i;k,j)} - \pi_{(\emptyset|\emptyset)}\pi_{(k,j,i;k,j,i)},
\]
\[
P_{i,k} := -\pi_{(j|j)}\pi_{(k,i;k,i)} + \pi_{(j|k)}\pi_{(k,i;j,i)} + \pi_{(j|i)}\pi_{(k,i;k,j)} + \pi_{(\emptyset|\emptyset)}\pi_{(k,j,i;k,j,i)},
\]
\[
P_{j,k} := -\pi_{(i|i)}\pi_{(k,j;k,j)} - \pi_{(i|k)}\pi_{(k,j;j,i)} + \pi_{(i|j)}\pi_{(k,j;k,i)} + \pi_{(\emptyset|\emptyset)}\pi_{(k,j,i;k,j,i)}
\] (6.3.6)
are four-term Plücker quadratic forms.

Fix an element $w^0 \in \text{Gr}_H^c(\mathcal{H}, \omega)$ of the infinite Lagrangian Grassmannian and define the function
\[
h : B \rightarrow (S^\infty_y)^{-1} \mathbb{C}[y, t'],
\]
\[
h : n' \mapsto h_{w^0}^{n'}, \quad \forall n' \in B
\] (6.3.7)
on the infinite lattice $B$, where $h_{w^0}^{n'}$ is defined by (5.1.6) for any $n \in A$. Denote the lattice functions with shifted argument by
\[
h_v : B \rightarrow (S^\infty_y)^{-1} \mathbb{C}[y, t'],
\]
\[
h_{v'} : n' \mapsto h_{w^0}^{n'+v'}, \quad \forall n', v' \in B.
\] (6.3.8)

**Proposition 6.8.** For any ordered triple $i, j, k \in \mathbb{N}^+$ of nonnegative integers $i < j < k$, the hyperdeterminantal relations:
\[
h^{2h^1_{\beta_i,\beta_j,\beta_k}} + h^{2h_{\beta_i,\beta_j,\beta_k}} + h^{2h_{\beta_i,\beta_k}} + h^{2h_{\beta_j,\beta_k}}
\]
\[
- 2h_{\beta_i,\beta_j,\beta_k} \left( h_{\beta_i,\beta_j,\beta_k} + h_{\beta_j,\beta_i,\beta_k} + h_{\beta_k,\beta_i,\beta_j} \right)
\]
\[
- 2 \left( h_{\beta_i,\beta_j,\beta_k} + h_{\beta_j,\beta_k,\beta_j} + h_{\beta_i,\beta_j,\beta_k} + h_{\beta_j,\beta_i,\beta_k} + h_{\beta_k,\beta_i,\beta_j} + h_{\beta_k,\beta_j,\beta_i} \right)
\]
\[
+ 4h_{\beta_1,\beta_2,\beta_3} + 4h_{\beta_1,\beta_3,\beta_2} + 4h_{\beta_1,\beta_2,\beta_3} = 0.
\] (6.3.10)

\[\text{To compare with the notations of Proposition 3.19 and Corollary 3.20 in [1], we give the following lexicon}
\]
\[
h^0 = \sigma^0, \quad h^{0}_{\beta_i} = -\sigma^0, \quad h^{0}_{\beta_i,\beta_j} = -\sigma^0, \quad h^{0}_{\beta_i,\beta_j,\beta_k} = -\sigma^0.
\] (6.3.9)
are satisfied.

Proof. This follows from combining Lemma 6.7 with Theorem 5.6.

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References

[1] S. Arthamonov, J. Harnad, and J. Hurtubise. “Lagrangian Grassmannians, CKP hierarchy and hyperdeterminantal relations”, arXiv:2202.13991 (2022).

[2] F. Balogh, J. Harnad and J. Hurtubise, “Isotropic Grassmannians, Plücker and Cartan maps”, J. Math. Phys. 62, 021701 (2021).

[3] A. I. Bobenko and W. K. Schief (2016) “Circle Complexes and the Discrete CKP Equation”, Int. Math. Res. Not., 2016, 1–58 (2016).

[4] E. Cartan, The Theory of Spinors, Dover Publications Inc, Mineola N.Y., 1981.

[5] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, “Transformation groups for soliton equations. VI. KP hierarchies of orthogonal and symplectic type”, J. Phys. Soc. Japan 50, 3813-3818 (1981).

[6] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, “Transformation groups for soliton equations IV. A new hierarchy of soliton equations of KP type”, Physica 4D, 343-365 (1982).

[7] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, “Transformation groups for soliton equations”, in: Nonlinear integrable systems - classical theory and quantum theory, World Scientific (Singapore), eds. M. Jimbo and T. Miwa (1983).

[8] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Chapt. I.5, Wiley-Interscience, John Wiley and Sons, New York, 1978.

[9] J. Harnad and F. Balogh, Tau Functions and their Applications, Monographs on Mathematical Physics series, Cambridge University Press, Cambridge, UK (2021).

[10] J. Harnad and E. Lee, “Symmetric polynomials, generalized Jacobi–Trudi identities and τ-functions”, J. Math. Phys. 59, 091411 (2018).

[11] J. Harnad and A. Yu. Orlov, “Polynomial KP and BKP τ-functions and Correlators” Ann. Henri Poincaré 22, 3025–3049, (2021).

[12] R. Hirota, “Discrete analogue of a generalized Toda equation”, J. Phys. Japan, 50, 3785-91 (1981).
[13] O. Holtz and B. Sturmfels, “Hyperdeterminantal relations among symmetric principal minors”, J. Alg. 316, 634-648 (2007).

[14] M. Jimbo and T. Miwa, T. “Solitons: Differential Equations, Symmetries and infinite dimensional Lie algebras”, (1999), Cambridge Tracts in Mathematics 135, Cambridge University Press, Cambridge U.K. (1999).

[15] M. Jimbo and T. Miwa, “Solitons and infinite-dimensional Lie algebras”, Publ. Res. Inst. Math. Sci., 19 943-1001 (1983).

[16] R. Kashaev, “On discrete three-dimensional equations associated with the local Yang–Baxter equation”, Lett. Math. Phys. 33, 389–397 (1996).

[17] R. Kenyon and R. Pemantle, “Principal minors and rhombus tilings”, J. Phys. A: Math. Theor. 47, 474010 (2014).

[18] R. Kenyon and R. Pemantle, “Double-dimers, the Ising model and the hexahedron recurrence”, J. Combin. Theory, A137, 27-63 (2016).

[19] V F. Kac, N. Rozhkovskaya, and J. van de Leur, “Polynomial tau-functions of the KP, BKP and the s-component KP hierarchies”, J. Math. Phys. 62, 021702 (2021).

[20] I. Krichever and A. Zabrodin, “Kadomtsev-Petviashvili Turning Points and CKP Hierarchy”, Commun. Math. Phys. 386, 643-1683 (2021).

[21] A. Kuniba, T. Nakanishi and J. Suzuki, “T-systems and Y-systems in integrable systems” J. Phys. A: Math. Theor. 44 103001 (2011).

[22] S. Kumar, “Kac-Moody Groups, their Flag varieties and Representation Theory”, Progress in Mathematics bf 204, Birkhauser, Boston (2002).

[23] I. G. Macdonald, Symmetric Functions and Hall Polynomials, Clarendon Press, Oxford, (1995).

[24] S. Mac Lane, “Categories for the working mathematician”, 2nd ed., Chapters. III and IX, Springer Graduate Texts in Mathematics, Science and Business Media (2013).

[25] T. Miwa, “On Hirota’s Difference Equations”, Proc. Japan Acad., 58, Ser. A 9-12 (1982).

[26] J. J. C. Nimmo, “Darboux transformations and the discrete KP equation”, J. Phys. A: Math. Gen. 30, 8693–8704, (1997).

[27] L. Oeding, “Set-theoretic defining equations of the variety of principal minors of symmetric matrices”, Alg. Num. Theor. 5 (1), 75-109 (2011).

[28] M. Sato, “Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold”, Kokyuroku, RIMS 30-46 (1981).

[29] M. Sato and Y. Sato, “Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold”, Lecture Notes in Appl. Anal. 5, 259-271 (1982); North-Holland Mathematics Studies 81, 259-271 (1983).
[30] W. K. Schief, “Lattice Geometry of the Discrete Darboux, KP, BKP and CKP Equations. Menelaus’ and Carnot’s Theorems”, *J. Nonlinear Math. Phys.* **10** Supp. 2, 194-208 (2003).

[31] Y. Shigyo, “On addition formulae of KP, mKP and BKP hierarchies”, *SIGMA*, **9**, 035, (2013).

[32] D.E. Speyer, “Perfect matchings and the octahedron recurrence”, *J Algebr. Comb.* **25**, 309-348 (2007).

[33] G. Segal and G. Wilson, “Loop groups and equations of KdV type”, *Publ. Math. IHÉS* **6**, 5-65 (1985).

[34] A. Zabrodin, “Hirota’s Difference Equations”, *Theor. Math. Phys.*, **113**, 1347-1392 (1997)