Realization of a holonomic quantum computer in a chain of three-level systems

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Holonomic quantum computation is the idea to use non-Abelian geometric phases to implement universal quantum gates that are robust to fluctuations in control parameters. Here, we propose a compact design for a holonomic quantum computer based on coupled three-level systems. The scheme does not require adiabatic evolution and can be implemented in arrays of atoms or ions trapped in tailored standing wave potentials. We demonstrate that perfect transfer of single-qubit states can be accomplished in the holonomic limit of the gates.

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I. INTRODUCTION

Holonomic quantum computation (HQC), first proposed by Zanardi and Rasetti\textsuperscript{1}, is the idea to use non-Abelian geometric phases to implement quantum gates. In the case of adiabatic evolution, this approach allows for universal quantum computation by composing holonomic gates associated with a generic pair of loops in the space of slow control parameters. Adiabatic holonomic gates are insensitive to random fluctuations in the parameters and therefore potentially useful for robust quantum computation\textsuperscript{2}. Physical realizations of adiabatic HQC have been developed in quantum optics\textsuperscript{3}, trapped ions\textsuperscript{4,5} or atoms\textsuperscript{6}, quantum dots\textsuperscript{7,8}, superconducting qubits\textsuperscript{9,10}, and spin chain systems\textsuperscript{11}. The scheme scales linearly with the number of systems and has been demonstrated for other level structures\textsuperscript{12}. Non-adiabatic HQC has been demonstrated for a transmon qubit\textsuperscript{17} and in a nuclear magnetic resonance setup\textsuperscript{18}. Furthermore, the idea of non-adiabatic HQC encoding qubits in the two bare ground state levels of Λ-type systems, has been developed in Ref.\textsuperscript{15}. This scheme was subsequently realized for a transmon qubit\textsuperscript{17} and in a nuclear magnetic resonance setup\textsuperscript{18}. Furthermore, the idea of non-adiabatic HQC has been combined with other methods to achieve resilience to collective errors\textsuperscript{19,20} and has been demonstrated for other level structures\textsuperscript{21}

Here, we demonstrate non-adiabatic universal HQC in a linear chain of interacting three-level systems. The scheme scales linearly with the number of systems and can therefore be used to build a compact holonomic quantum computer with a small overhead of auxiliary systems. Our setup can in principle be implemented for three-level atoms or ions trapped in standing wave potentials.

The outline of the paper is as follows. In the next section, the general idea of non-adiabatic holonomic quantum computation is described. The model system is introduced in Sec. III. We demonstrate a universal set of one- and two-qubit holonomic gates in Sec. IV. We apply the scheme to quantum state transfer in IV and argue that the single-qubit state transfer fidelity reaches unity precisely when the evolution is purely holonomic; a result that makes it possible to test the holonomic nature of the scheme. The paper ends with the conclusions.

II. NON-ADIABATIC HOLOMニック QUANTUM COMPUTATION

Let a computational system be encoded in a subspace $S$ of some Hilbert space $H$. A cyclic evolution of $S$ implements a quantum gate. This gate generally contains a dynamical and a geometric contribution that combine into a unitary transformation acting on $S$. The dynamical part is essentially given by the Hamiltonian $H(t)$ projected onto the evolving computational subspace. The geometric part probe the underlying geometry of the space of subspaces; technically this space is a Grassmannian manifold $G(\dim H; \dim S) \equiv G(N; K)\textsuperscript{22}$.

Non-adiabatic HQC on $S$ is realized when $P(t)H(t)P(t) = \epsilon(t)P(t)$, where $\epsilon(t)$ is the average energy of the subspace at time $t$ and $iP(t) = [H(t), P(t)]$ with $P(0)$ the projection operator onto $S$ (we put $\hbar = 1$ from now on). For a cyclic evolution, i.e., $P(\tau) = P(0)$, the time evolution operator projected onto $S$ becomes unitary, and we find

$$P(0)U(\tau, 0)P(0) = e^{-i \int_0^\tau \epsilon(t)dt} \sum_{a,b=1}^K (P_{a,b}^\dagger f_c A_{a,b}) \langle \zeta_a(0) | \zeta_b(0) \rangle,$$

where $A_{a,b} = i \langle \zeta_a(t) | d\zeta_b(t) \rangle$ the matrix-valued connection one-form with $\{ \zeta_a(t) \}$ any orthonormal basis along the loop $C$ in $G(N; K)$ such that $| \zeta_a(\tau) \rangle = | \zeta_a(0) \rangle$. Here,

$$U(C) \equiv \sum_{a,b=1}^K (P_{a,b}^\dagger f_c A_{a,b}) \langle \zeta_a(0) | \zeta_b(0) \rangle$$

is the holonomic gate associated with $C$. The dynamical phase reduces to an unimportant overall $U(1)$ phase factor $e^{-i \int_0^\tau \epsilon(t)dt}$.
Note that while the holonomy is induced by slow changes of physical control parameters in adiabatic HQC, these parameters play a passive role in the non-adiabatic version. In particular, this means that the non-adiabatic scheme is not restricted to slow evolution and can therefore be made less exposed to decoherence effects by decreasing the run time of the gates [23].

\[ H(t) = \sum_{k=1}^{N} f_k(t) \left[ \sin \frac{\theta_k}{2} \left( \cos \phi_k \lambda_{2k-1}^{(1)} - \sin \phi_k \lambda_{2k-1}^{(2)} \right) - \cos \frac{\theta_k}{2} \lambda_{2k-1}^{(4)} \right] \\
+ \frac{1}{2} \sum_{k=1}^{N-1} g_{k,k+1}(t) \left[ -\cos \frac{\theta_{k,k+1}}{2} \left( \lambda_{2k-1}^{(6)} \lambda_{2k}^{(6)} + \lambda_{2k-1}^{(7)} \lambda_{2k}^{(7)} \right) + \sin \frac{\theta_{k,k+1}}{2} \left( \lambda_{2k}^{(6)} \lambda_{2k+1}^{(6)} + \lambda_{2k}^{(7)} \lambda_{2k+1}^{(7)} \right) \right] \\
= \sum_{k=1}^{N} f_k(t) H_k^{(1)} + \sum_{k=1}^{N-1} g_{k,k+1}(t) H_{k,k+1}^{(3)}, \quad (3) \]

where \( H_k^{(1)}, H_{k,k+1} \) are time independent during each pulse and the corresponding pulse envelopes \( f_k(t), g_{k,k+1}(t) \) are real-valued. The relevant Gell-Mann operators associated with site \( k \) read

\[
\begin{align*}
\lambda_k^{(1)} &= |e\rangle\langle 0| + |0\rangle\langle e|, \\
\lambda_k^{(2)} &= -i|e\rangle\langle 0| + i|0\rangle\langle e|, \\
\lambda_k^{(4)} &= |e\rangle\langle 1| + |1\rangle\langle e|, \\
\lambda_k^{(6)} &= |0\rangle\langle 1| + |1\rangle\langle 0|, \\
\lambda_k^{(7)} &= -i|0\rangle\langle 1| + i|1\rangle\langle 0|,
\end{align*}
\]

where \( |0\rangle, |1\rangle \), and \( |e\rangle \) span the local state space. Each three-level system has a qubit subspace spanned by \( |0\rangle, |1\rangle \) with associated Pauli operators \( \sigma_k^x = |0\rangle\langle 1| + |1\rangle\langle 0| \), \( \sigma_k^y = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \), and \( \sigma_k^z = |0\rangle\langle 0| - |1\rangle\langle 1| \) defining a pseudo-spin-\( \frac{1}{2} \) system. Note that \( \sigma_k^x = \lambda_k^{(6)} \) and \( \sigma_k^y = \lambda_k^{(7)} \).

A logical qubit is encoded in the two-dimensional subspace spanned by \( |0\rangle \) and \( |1\rangle \) of each odd-numbered three-level system. The auxiliary even-numbered systems act as a computational mediator for mediating two-qubit gates, as will be shown below. In this way, \( N \) logical qubits are obtained from the \( 2N - 1 \) systems (see lower panel of Fig. 1). The state space of the \( N \) logical qubits \( \mathcal{H}^{(N)} \) is spanned by the \( 2^N \) states \( \{|n_1 n_2 \ldots n_N \rangle = |n_1 n_2 \ldots n_N \rangle_{L} |n_1, n_2, \ldots, n_N = 0, 1 \rangle \rangle \) defined by setting the state of all auxiliary systems to \( |0\rangle \).

The above Hamiltonian can be implemented in internal energy levels of atoms trapped in a one-dimensional optical lattice and exhibiting the desired XY-type interaction by adjusting the standing wave optical potential of the lattice [24]. Another possible realization consists of ions trapped along a line by off-resonant standing waves. The internal states in this setting can be made to interact via state-dependent Stark shifts that couple the ions to the vibrational degrees of freedom of the trap [25]. In both scenarios, pairs of zero-detuned laser fields couple each qubit level to an excited state, which is unaffected by the trapping fields.

IV. HOLONOMIC QUANTUM COMPUTATION

A. One-qubit gates

A holonomic one-qubit gate acting on qubit \( l \) is implemented by turning on and off \( f_l(t) \) at \( t = 0 \) and \( t = \tau \), respectively, and by simultaneously putting \( g_{l-1,l}(t) = g_{l,l+1}(t) = 0 \). By following Ref. [15], we obtain the gate (see lower panel of Fig. 1)

\[
U_l^{(1)}(C_n) = n_l \cdot \sigma_l \quad (5)
\]

acting on the \( l \)th qubit subspace spanned by \( |0\rangle, |1\rangle \), if the pulse area satisfies \( \int_0^\tau f_l(t) dt = \pi \). Here, \( n_l = (\sin \theta_l, \cos \phi_l \sin \theta_l, \sin \phi_l, \cos \theta_l) \) and \( C_n \) is the loop in the Grassmannian \( G(3; 2) \) traversed by the qubit subspace. The geometric nature is guaranteed by noting that \( H(t) \) vanishes on the qubit subspace on \([0, \tau]\). Two subsequent
that drive the pulses with \( \pi \) which proves the desired one-qubit universality [15].

This is achieved by turning on the coupling between systems 2 \( - U \) gate obtained in a system of 2 \( N - U \) gate this block remain zero. Thus, the \( \pi \) gate \( 0 \) 1, 2 \( \pi \) of the total pseudo-spin operator \( S_z = \frac{1}{2} (\sigma_z^1 + \sigma_z^{2} + \sigma_z^{2'}) \). Thus,

\[
H_{l, l'+1}^{(3)} = \otimes m H_{l, l'+1; m}
\]

where \( H_{l, l'+1; m} \) acts on \( \mathcal{M}_m \). Similarly, the time evolution operator decomposes as

\[
U(t, 0) = \otimes m U_m(t, 0)
\]

which proves the desired one-qubit universality [15].

### B. Two-qubit gates

We now demonstrate how a non-adiabatic holonomic two-qubit gate acting on qubits \( l' \) and \( l' + 1 \) can be implemented. This is achieved by turning on the coupling between systems 2 \( l' - 1, 2l' \), and 2 \( l' + 1 \) at \( t = 0 \) and turning it off at \( t = \pi \), while all other coupling terms involving this block remain zero. Thus, the \( N \)-spin Hamiltonian

\[
H(t) = g_{l', l'+1}(t) H_{l, l'+1}^{(3)}
\]

during this time interval. Let \( \{ |pqr \} \), \( p, q, r = 0, 1 \), span the Hilbert space of the considered three-site block.

We first note that the Hamiltonian \( H_{l, l'+1}^{(3)} \) has four invariant subspaces \( \mathcal{M}_{l, l'}^{-1} = \{ |111 \} \), \( \mathcal{M}_{l, l'}^{-1} = \{ |011 \}, |010 \}, |110 \} \), \( \mathcal{M}_{l, l'}^{+1} = \{ |100 \}, |010 \}, |001 \} \), and \( \mathcal{M}_{l, l'}^{+1} = \{ |000 \} \), each labeled by the quantum number \( m = -\frac{3}{2}, \ldots, +\frac{3}{2} \) of the total pseudo-spin operator \( S_z = \frac{1}{2} (\sigma_z^1 + \sigma_z^{2} + \sigma_z^{2'}) \). Thus,

\[
H_{l, l'+1; m}^{(3)} = \otimes m H_{l, l'+1; m}
\]

where \( H_{l, l'+1; m} \) acts on \( \mathcal{M}_m \). Similarly, the time evolution operator decomposes as

\[
U(t, 0) = \otimes m U_m(t, 0)
\]

where \( U_m(t, 0) = e^{-i \alpha_m H_{l, l'+1; m}^{(3)}} \) with the ‘pulse area’ \( \alpha_t = \int_0^t g_{l', l'+1}(t') dt' \). The computational states (defined by setting the auxiliary second qubit to \( |0 \) \) divide into \( S_1 = \{ |000 \} \equiv |00 \_l \} \subset \mathcal{M}_2 \), \( S_2 = \{ |010 \} \equiv |01 \_l \} \subset \mathcal{M}_2 \), and \( S_3 = \{ |101 \} \equiv |11 \_l \} \subset \mathcal{M}_2 \). Since \( S_1 \) forms an invariant subspace, it undergoes a trivial evolution when exposed to \( H_{l, l'+1}^{(3)} \). The Hamiltonian vanishes on \( S_2 \) and \( S_3 \), i.e., there will be no dynamical phases associated with the evolution of these two subspaces. On the other hand, \( S_2 \) and \( S_3 \) are proper subspaces of the invariant subspaces \( \mathcal{M}_{l, l'}^{+1} \) and may therefore pick up nontrivial holonomies. In this way, the evolution of \( S_1, S_2, S_3 \) is purely geometric and defines holonomic two-qubit gates acting on the first and third qubit. We now demonstrate that these gates are in general sufficient to achieve universality when assisted by the above holonomic one-qubit gates.

The parts \( H_{l, l'+1; \pm \frac{3}{2}}^{(3)} \) of the Hamiltonian, take the form

\[
H_{l, l'+1; \pm \frac{3}{2}}^{(3)} = \sin \frac{\theta_{l', l'+1}}{2} \langle 010 | \langle 001 | - \cos \frac{\theta_{l', l'+1}}{2} \langle 010 | 100 | + \text{H.c.},
\]

\[
H_{l, l'+1; \pm \frac{3}{2}}^{(3)} = \sin \frac{\theta_{l', l'+1}}{2} |101\rangle \langle 110| - \cos \frac{\theta_{l', l'+1}}{2} |101\rangle \langle 011| + \text{H.c.}
\]

Notice that \( H_{l, l'+1; \pm \frac{3}{2}}^{(3)} \) both exhibit a \( \Lambda \)-type configuration with \( |001 \rangle \) and \( |100 \rangle \) playing the role of the two ‘ground state’ levels that couple to the ‘excited state’ \( |010 \rangle \) with relative coupling strength \( -\tan \frac{\theta_{l', l'+1}}{2} \). The action of
reads

tion becomes cyclic and the resulting action

d only when the perturbation tends to zero.

measured by the transition probability) tends to unity

that for \( \gamma \neq 0 \), \( U_d \) and \( U^{(2)}(C') \) do not commute which
means that the action of the time evolution operator
on the computational subspace cannot be written as a sim-
ple product of a dynamical and a geometric part.

Ideally, state transfer is achieved when \( \sin(\vartheta/2) = \frac{1}{\sqrt{2}} \), i.e., \( \vartheta = \pi/2 \). For this \( \vartheta \)-value, we calcu-
late the fidelity of state transfer in terms of the transition probability

\[
P(\ket{10}_L \rightarrow \ket{01}_L) = \left| \bra{001} e^{-i\alpha \tau} \tilde{H}^{(3)} \ket{100} \right|^2
= \cos^4 \gamma \sin^4 \left( \frac{\alpha \tau}{2 \cos \gamma} \right).
\]

The maximal transition probability is

\[
\max P(\ket{10}_L \rightarrow \ket{01}_L) = \cos^4 \gamma
\]

when \( \alpha = \pi \cos \gamma \). Thus, we find that perfect state transfer
when \( \max P(\ket{10}_L \rightarrow \ket{01}_L) = 1 \), which occurs only
in the holonomic limit \( \gamma \rightarrow 0 \); a result that opens up
for the possibility to test the holonomic nature of the
scheme.

VI. CONCLUSIONS

Quantum computation by holonomic means has at-
tracted considerable attention in the past, due to the resil-
ience of quantum holonomies to parameter fluctu-
ations. In this paper, we have introduced a universal scalable scheme
based on non-Abelian holonomies (geo-
metric phases) realized by interacting three-level sys-
tems, each of which exhibiting a \( \Lambda \)-type configuration.
Our scheme provides an architecture for a compact holono-
ic quantum computer based on fast non-adiabatic evolution.
This latter feature opens up for the possibility
to avoid detrimental open systems effects by decreasing
the run time of the gates. We have indicated how
the proposed setup can be implemented in atoms or ions
trapped in tailored pulsed standing wave potentials.
Finally, we have shown that our scheme allows for perfect
state transfer in the holonomic limit.

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