A Linear Programming Method for Finding Orthocomplements in Finite Lattices

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Abstract

A method of embedding partially ordered sets into linear spaces is presented. The problem of finding all orthocomplementations in a finite lattice is reduced to a linear programming problem.

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1 INTRODUCTION AND BASIC DEFINITIONS

We introduce linear algebraic tools for finite lattices. The idea of the proposed method looks as follows. Given a finite lattice $L$, we consider its linear hull $H(L)$ as the collection of all real-valued functions on $L$. Any mapping $f : L \to L$ can be extended to a linear operator $f : H(L) \to H(L)$. The collection $\mathcal{H}$ of all linear operators in the space $H(L)$ is a linear space itself, and we introduce a convex set of precomplements (see exact definitions below) as a subset of $\mathcal{H}$. Then, it turns out that the complements in the lattice $L$ are in 1–1 correspondence with the solutions of a linear programming problem in the space $\mathcal{H}$.

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Now we introduce the basic definitions. Given a finite $L$, let $H = H(L)$ be the set of all real functions on $L$. Fix up a preferred basis in $H$ labeled by the elements of $L$: for any $p \in L$ its counterpart is the delta function $\delta_p$:

$$\delta_p(q) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

The space $H$ possesses the natural structure of commutative algebra since functions can be pointwise multiplied: for any $f, g \in H$ $(f \cdot g)(q) = f(q) \cdot g(q)$. The unit element $I$ of $H$ is $I = \sum_{p \in L} \delta_p$.

The partial order $\leq$ in $L$ is associated with the zeta operator [1] in $H$, whose matrix is the incidence matrix of the partial order $L$:

$$\zeta(p, q) = \begin{cases} 1 & \text{if } p \leq q \\ 0 & \text{otherwise} \end{cases}$$

The operator $\zeta$ has the following properties: its matrix is upper triangle (under an appropriate enumeration of the elements of $L$), and its diagonal entries are equal to 1. Therefore $\zeta$ is always invertible. Its inverse is denoted by $\mu = \zeta^{-1}$. The matrix of the operator $\mu$ considered a function of two variables ranging over $L$ is called the Möbius function of $L$ [2].

2 THE POLYTOPE OF PRECOMPLEMENTS

Two elements $p, q \in L$ are said to be disjoint if $p \land q = 0$, conjoint if $p \lor q = 1$ (where 0, 1 stand for the least and the greatest elements of $L$, respectively) and complemented if they are both disjoint and conjoint. A complement on the lattice $L$ is an idempotent permutation $\alpha$ of the elements of $L$ such that

- $p \leq q$ implies $\alpha q \leq \alpha p$
- any pair $p, \alpha p$ is complemented (it suffices to require them to be disjoint)

With any permutation $\alpha$ on the lattice $L$ we can associate a linear operator $\alpha : H \rightarrow H$ defined on the basis of delta functions as follows: $\alpha(\delta_p) := \delta_{\alpha p}$. When $\alpha$ is a complementation on $L$, the matrix of the operator $\alpha$ is an idempotent orthogonal matrix, that is, satisfying the following conditions:

- $\alpha \geq 0$ — since the entries of $\alpha$ are 0 or 1
• $\alpha I = I$ — since there is only one 1 entry in each row

• $\alpha^T = \alpha$

The next necessary condition for $\alpha$ to be a complement is that it reverses order. In operator form this is expressed as follows

$$\alpha \zeta = \zeta^T \alpha$$

Furthermore, since $\alpha$ is a complement, for any $p, q \in L$ the conditions $q \leq p$ and $q \leq \alpha p$ imply $q = 0$. This means that $\zeta \alpha \delta_p \cdot \zeta \delta_p = \delta_0$, hence

$$\text{Tr}(\zeta^T \zeta \alpha) = \sum_p (\zeta \alpha \delta_p, \zeta \delta_p) = \sum_p (\zeta \alpha \delta_p \cdot \zeta \delta_p, I) = \sum_p (\delta_0, I) = n$$

where $p$ ranges over the elements of $L$ and $n$ is the cardinality of $L$. Now we can introduce the convex subset $\mathcal{P}$ of the space $\text{Mat}_n$ of real $n \times n$ matrices as follows: a matrix $\alpha$ is in $\mathcal{P}$ if and only if the following conditions hold:

$$\begin{align*}
\alpha &\geq 0 \\
\alpha &= \alpha^T \\
\text{Tr} (\zeta^T \zeta \alpha) &= n \\
\alpha I &= I \\
\alpha \zeta &= \zeta^T \alpha
\end{align*}$$

Evidently, all operators $\alpha$ associated with complements are in $\mathcal{P}$; however, $\mathcal{P}$ is a continuous subset of $\text{Mat}_n$, namely, a polytope (since all the equations in (1) are linear. We call the polytope $\mathcal{P}$ defined by (1) the polytope of pre-complements.

### 3 MAIN RESULTS

We present two theorems demonstrating the power of the linear approach to the theory of posets.

**Theorem 1.** The orthocomplements, and only they, are the integer vertices of the polytope $\mathcal{P}$.

**The idea of the proof.** Otherwise the condition $\alpha I = I$ will be broken.
Theorem 2. The orthocomplements of the lattice $L$ are the optimal solutions of the following linear programming problem:

$$\text{Tr} \left( \zeta^T \alpha \right) \rightarrow \min$$
$$\begin{cases} 
\alpha \mathbf{I} = \mathbf{I} \\
\alpha^T = \alpha \\
\alpha \zeta = \zeta^T \alpha 
\end{cases}$$

The idea of the proof. The condition $\alpha \mathbf{I} = \mathbf{I}$ implies $\text{Tr} \left( \zeta^T \alpha \right) \geq n$, then use the previous theorem.

4 CONCLUDING REMARKS

The proposed techniques of linear embedding of finite lattices can be extended to posets. All the results remain valid, but we face a strange effect: in the linear hull $H$ of any poset $L$ the operations $\land$ and $\lor$ are always well-defined, but the results of these operations may bring us beyond the poset $L$ in question and yield a weighted sum of the elements of the poset. This issue needs further investigation.

References

[1] Aigner, M., Combinatorial Theory, Physical Review A49, Springer-Verlag, Berlin (1979)

[2] Stanley, R., Enumerative Combinatorics, Wadsworth and Brooks, Monterrey, California (1986)