Operator Product Expansion for Inclusive Semileptonic Decays in Heavy Quark Effective Field Theory

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Abstract

Inclusive semileptonic decays are discussed in the framework of heavy quark effective field theory by employing the short distance expansion in the effective theory. The lowest order term turns out to be the parton model; the higher order terms may be regarded as correction terms to the parton model result. The first nonvanishing corrections to the parton model result are given and the lepton energy spectrum of inclusive semileptonic decays of heavy mesons is calculated.
1 Introduction

Heavy quark effective field theory \[1, 2\] has turned out to be a useful tool for the description of heavy quark systems. Due to additional symmetries \[1\] emerging in the heavy quark limit several predictions for the decays of heavy hadrons may be obtained which are completely model independent. In particular, the form factors of weak decays involving a heavy to heavy transition are severely constrained by heavy quark symmetry and some predictions may be obtained even for heavy to light decays \[3\].

The heavy quark expansion has also been applied to inclusive decays. The first discussion of inclusive semileptonic decays in this framework has been given by Chay, Georgi and Grinstein in \[4\]. Along similar lines proceeds the work of Bigi, Shifman, Uraltsev and Vainshtein \[5\] and of Blok, Koyrakh, Shifman and Vainshtein \[6\], where inclusive nonleptonic as well as semileptonic decays have been considered.

Usually the inclusive decay rates of heavy hadrons and their spectra are approximated by the decay of the heavy quarks in a parton model approximation. However, this is a model approach and an estimate of its error is difficult. This in is contrast to the methods described in \[4, 5\] and also to the present approach which is a controllable approximation to QCD corresponding to expansions in small parameters.

The method proposed in \[4, 5\] to deal with inclusive processes is to use an operator product expansion, which is justified for heavy hadrons due to its large mass. The lowest dimension operator is a dimension three operator; its matrix elements are normalized due to heavy quark symmetries and yield the parton model result. Corrections enter through operators of dimension five; their matrix elements are parameters to be taken from measurements. These corrections have been calculated for nonleptonic and for semileptonic decays of mesons in \[5\] and for the differential distributions in semileptonic decays in \[6\]; semileptonic decays of baryons have been considered in \[7\].

In the present paper we shall consider only semileptonic decays and formulate a slightly different approach as the one proposed in \[4, 5\]. After switching to heavy quark effective theory the space time dependence of the heavy quark fields in the hadronic tensor is only due to the residual momentum \(k\) of the heavy quark \(Q\) inside the heavy hadron \(H\), i.e. \(k = P_H - m_Q v\). In semileptonic decays there is in addition to the scale set by the heavy mass \(m_Q\) also the momentum transfer to the leptons \(q^2\). After switching to the effective theory the relevant variable becomes \(Q^2 = (q - m_Q v)^2\), which defines in addition to the mass \(m_Q\) of the heavy quark a
second scale in the problem.

In large portions of the phase space $Q^2$ is of the order of $m_Q^2$ and thus a simultaneous expansion in both $1/m_Q$ and $1/Q^2$ is appropriate. This is the approach chosen in [5, 6]. However, one may also consider the region in phase space, where

$$m_Q^2 \gg (q - m_Q v)^2 \gg \Lambda^2_{QCD}$$

and thus an expansion only in powers of $1/Q^2$ is necessary while only the leading term of the $1/m_Q$ is kept.

In the present paper we shall consider the latter approach and compare to the one of [5, 6]. The paper is organized as follows. In the next section we fix our notation by giving the basic formulae for the inclusive decay rates. In section 3 we set up the operator product expansion in the effective theory and discuss the lowest order term and the first nonvanishing correction. The setup given here may be used for baryonic as well as for mesonic decays and in section 4 we give a parametrization for the hadronic matrix elements for meson- and baryon inclusive decays and discuss short distance QCD corrections. Finally, in section 5 we discuss the lepton energy spectrum in inclusive semileptonic decays in some detail.

## 2 Inclusive Decay: Matrix Elements and Rates

In order to set up the notation we shall briefly recall the basic formulae for the description of inclusive semileptonic decays. The inclusive semileptonic decay of a heavy hadron $H$ (either a heavy meson or a heavy baryon) into some hadronic state $X$ and two leptons

$$H(P_H = m_H v) \rightarrow X(P_X) + \ell(k) + \bar{\nu}_\ell(k')$$

(1)

is mediated by an effective Hamiltonian of the form

$$H_{eff} = \frac{G_F}{\sqrt{2}} V_{q\bar{q}}(\bar{Q}\gamma_\mu(1 - \gamma_5)q)(\bar{\ell}\gamma_\mu(1 - \gamma_5)\nu_\ell)$$

(2)

where $Q$ is the heavy quark contained in the in the heavy hadron $H$. The differential decay rate may be written in terms of the hadronic tensor $W_{\mu\nu}(q, v)$

$$W_{\mu\nu}(q, v) = \int_X (2\pi)^4 \delta^4(P_H - q - P_X)$$

$$< H(v) | (\bar{Q}\gamma_\mu(1 - \gamma_5)q) | X > < X | (\bar{q}\gamma_\mu(1 - \gamma_5)Q) | H(v) > .$$

1I thank Mark Wise for a clarifying discussion on this point.

2We shall assume that the flavor of $q$ may be determined from the hadronic state $X$. 
We shall consider only the case where the spins of the final state leptons are not measured; hence the leptonic tensor is given by

$$\Lambda_{\mu\nu} = 8 \left( k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu}(k k') + i\epsilon_{\mu\nu\alpha\beta} k^\alpha k'^\beta \right)$$ \hspace{1cm} (4)

The differential decay rate in the rest frame of the heavy hadron is then

$$d\Gamma = \frac{G_F^2}{4m_H^2} |V_{Qq}|^2 W_{\mu\nu}(k+k', v) \Lambda^{\mu\nu} d(PS)$$ \hspace{1cm} (5)

where \(d(PS)\) is the phase space integral

$$d(PS) = \int \tilde{d}k \tilde{d}k' \delta(\text{observables}) \quad \tilde{d}k = \frac{d^4k}{(2\pi)^3} \delta(k^2 - m_{Lep}^2) \Theta(k_0).$$ \hspace{1cm} (6)

The delta function \(\delta(\text{observables})\) projects out the observables to be considered; for the case of the energy spectrum of the charged lepton in the rest frame of the decaying heavy meson it is \(\delta(\text{observables}) = \delta(E - v k')\), since \(k'\) is the momentum of the charged lepton. Furthermore, \(m_{Lep}\) is the lepton mass which we shall neglect in the following.

The hadronic tensor may be rewritten using standard techniques

$$W_{\mu\nu}(q, v) = \int d^4xe^{iqx} < H(v) | (\bar{Q}(x) \gamma_\mu (1 - \gamma_5) q(x)) (\bar{q}(0) \gamma_\nu (1 - \gamma_5) Q(0)) | H(v) >$$ \hspace{1cm} (7)

where we have explicitly spelled out the \(x\) dependence of the field operators.

The tensor \(W_{\mu\nu}\) may be decomposed into scalar form factors which then depend only on two invariants; these may be conveniently chosen to be \(q^2\) and \(v q\). As discussed in some detail in [4], one may relate these form factors to the discontinuity of the ones defining the time ordered product

$$T_{\mu\nu}(q, v) = \int d^4xe^{iqx}$$

$$< H(v) | T \left[ (\bar{Q}(x) \gamma_\mu (1 - \gamma_5) q(x)) (\bar{q}(0) \gamma_\nu (1 - \gamma_5) Q(0)) \right] | H(v) >$$\hspace{1cm} (8)

For a fixed value of \(q^2\) there are two cuts of (8) in the complex \(v q\) plane. A physical cut extents along the real axis for

$$\sqrt{q^2} < v q < \frac{1}{2m_H} (m_H^2 + q^2 - \mu^2)$$ \hspace{1cm} (9)
where \( \mu \) is the mass of the lightest hadronic final state. In addition, there is a cut extending along the real axis for

\[
vq > \frac{1}{2m_H}((2m_H + \mu)^2 - m_H^2 - q^2)
\]

The problematic region in phase space is the one where the invariant mass of the hadronic final state approaches \( \mu \); here resonances will dominate and perturbative QCD will receive large corrections. However, even in this region the predictions of perturbative QCD will be recovered by a suitable smearing, i.e. by calculating a smooth average along some portion of the cut. This has been discussed in some detail for the case of \( e^+e^- \to \text{hadrons} \) in [8] and applied to the present case in [4]. In particular, one may expect a suitable smearing, if the energy spectrum of the charged lepton is calculated.

In order to calculate such an average one has to perform a contour integration of \( T_{\mu\nu} \) in the complex \( vq \) plane with a suitable weight function. The contour encircles the physical cut in such a way that the discontinuity \( T_{\mu\nu} \) (i.e. the weighted hadronic tensor \( W_{\mu\nu} \)) of the cut is picked up. As has been argued in [4] one may deform the integration contours in the complex \( vq \) plane in such a way that one always stays away from the dangerous region by a distance large compared to \( \Lambda_{QCD} \). This may be achieved provided that \( \mu \gg \Lambda_{QCD} \) because for \( \mu \to 0 \) the cut (9) and (10) pinch the integration contour for \( q^2 = d_{\text{max}}^2 = (m_H - \mu)^2 \). From these arguments one expects that perturbative QCD may be applied after suitable smearing and for \( \mu \gg \Lambda_{QCD} \) which means that we may only study inclusive semileptonic \( b \to c \) transitions in perturbation theory, while \( b \to u \) transitions are likely to receive large, nonperturbative corrections, at least in the region, where \( q^2 \) is close to its maximal value.

Keeping this in mind one may relate the perturbatively calculated time ordered product (8) to the hadronic tensor using the relation

\[
W_{\mu\nu}(q,v) = 2 \text{ Im } T_{\mu\nu}
\]

The matrix element of the time ordered product in (8) may be expanded in a sum of local operators by performing a short distance expansion. However, there are still two large momenta involved, the momentum of the heavy meson which scales with \( m_Q \) and the momentum transfer to the leptons \( q^2 \). In order to disentangle long and short distance contributions it is convenient to switch to the effective theory. In this process the heavy hadron momentum is rescaled and only the small residual
momentum remains. As will become clear below, the relevant momentum transfer variable will be \( Q = q - m Q v \) which we assume to be the only large scale in the problem. Once we are working in the effective theory, an operator product expansion will be an expansion in inverse powers of \( Q^2 \).

## 3 Operator Product Expansion in the Effective Field Theory

Switching to the effective field theory allows to scale out the large parameter \( m_H \). The relevant momentum variable characterizing the initial state is then a small momentum, namely the residual momentum \( k = P_H - m Q v \) of the heavy quark. Matching the effective and the full theory to lowest order in the \( 1/m_Q \) and \( \alpha_s(m_Q) \) expansion one has to replace

\[
Q(x) \rightarrow e^{-i m Q v \cdot x} \frac{1}{2} h_v(x)
\]

where the \( x \) dependence of the static heavy quark field \( h_v \) is due to the residual momentum \( k \).

Injecting this into the hadronic tensor one obtains

\[
W_{\mu \nu}(q, v) = \int d^4 x e^{i x \cdot (q - m Q v)} \left< H(v) | (\bar{h}_v(x) \gamma_\mu (1 - \gamma_5) q(x))(\bar{q}(0) \gamma_\mu (1 - \gamma_5) h_v(0)) | H(v) \right>
\]

An operator product expansion (i.e. a short distance expansion) is justified, if the momentum transfer variable \( Q = q - m Q v \) is large compared to the residual momentum \( k \). In addition, switching to heavy quark effective theory implies an expansion in powers of \( 1/m_Q \). In contrast to \cite{5, 6} we shall consider the portion of phase space in which \( Q^2 \ll m_Q^2 \) and thus the lowest order term in the \( 1/m_Q \) expansion is sufficient. Hence we may use (13) as the starting point.

Using momentum conservation \( P_H = m_Q v + k = q + P_X \) we find that an operator product expansion is possible for \( P_X^2 \gg m_X^2 \) and that this expansion is a power series expansion in powers of \( \Lambda_{QCD}/m_X \), up to logarithmic terms which may be resummed by renormalization group techniques. Note that, if \( m_X \) is much smaller than \( m_H \), the use of the effective theory to lowest order is justified, since higher order corrections to the effective theory will be of the order \( \Lambda_{QCD}/m_H \) and thus will be much smaller.
However, as argued above, we shall only consider $b \rightarrow c$ inclusive transitions and thus $m_X \gg \Lambda_{QCD}$. For this case we may use the framework of perturbative QCD to calculate the coefficients of the operator product expansion.

Thus we are lead to consider the time ordered product of the two currents

$$T_{\mu\nu}(q, v) = \int d^4xe^{ixQ}T[(\bar{h}_v(x)\gamma_\mu(1 - \gamma_5)q(x))(\bar{q}(0)\gamma_\mu(1 - \gamma_5)h_v(0))]$$  \hspace{1cm} (14)

where $Q = q - m_Qv$ is the momentum transfer variable defining the large momentum. We shall define the operator product expansion for the projection of $T_{\mu\nu}$ needed to obtain the inclusive semileptonic rate

$$T(k, k', v) = T_{\mu\nu}(k + k', v)\Lambda^{\mu\nu}$$  \hspace{1cm} (15)

Before we write down the general form of the short distance expansion in the effective theory we shall recall some basic formulae concerning the spin structure in the effective theory. The sixteen covariants built from Dirac matrices reduce to only four when projected into a definite velocity sector. We shall use as a basis the unit matrix and the three spin vectors $s_\mu$ with $v \cdot s = 0$ which correspond in the rest frame $v = (1, 0, 0, 0)$ to the three Pauli matrices. The translation table of the sixteen invariants is given by

$$1 \rightarrow P_+ = \frac{1}{2}(1 + \gamma^\nu) \hspace{1cm} \gamma_\mu \rightarrow P_+\gamma_\mu P_+ = v_\mu P_+$$
$$\gamma_\mu\gamma_5 \rightarrow P_+\gamma_\mu\gamma_5 P_+ = s_\mu \hspace{1cm} \gamma_5 \rightarrow P_+\gamma_5 P_+ = 0$$
$$\sigma_{\mu\nu} \rightarrow P_+\sigma_{\mu\nu} P_+ = v^\alpha\epsilon_{\alpha\mu\nu\beta}s^\beta$$

The spin vectors $s_\mu$ satisfy the relation

$$s_\mu s_\nu = (-g_{\mu\nu} + v_\mu v_\nu)1 + i\epsilon_{\alpha\mu\nu\beta}v^\alpha s^\beta$$  \hspace{1cm} (16)

which is a generalization of the well known relation between the Pauli matrices and which may be used to reduce products of spin vectors.

At tree level one may obtain the operator product expansion up to operators of dimension five by considering the two diagrams depicted in fig.1. The result is

$$T(k, k', v) = \left(\frac{1}{Q^2 - m_t^2 + i\epsilon}\right) 64(k'Q)\bar{h}_v\gamma^\nu(1 - \gamma_5)h_v + \left(\frac{1}{Q^2 - m_t^2 + i\epsilon}\right) 64\bar{h}_v(i\gamma_5)\gamma^\nu(1 - \gamma_5)h_v$$  \hspace{1cm} (17)
\[- \left( \frac{1}{Q^2 - m_i^2 + i\epsilon} \right)^2 128(k'Q)\bar{h}_v(iQD)\bar{k}(1 - \gamma_5)h_v\]
\[- \left( \frac{1}{Q^2 - m_i^2 + i\epsilon} \right)^2 64(k'Q)\bar{h}_v(iD)\bar{k}(1 - \gamma_5)h_v\]
\[- \left( \frac{1}{Q^2 - m_i^2 + i\epsilon} \right)^2 64\bar{h}_v[(iQD),(ik'D)]\bar{k}(1 - \gamma_5)h_v\]
\[- \left( \frac{1}{Q^2 - m_i^2 + i\epsilon} \right)^2 64i\epsilon_{\alpha\beta\rho\xi}Q^\lambda k'^\rho\bar{h}_v(iD^\alpha)(iD^\beta)\bar{k}(1 - \gamma_5)h_v\]
\[- \left( \frac{1}{Q^2 - m_i^2 + i\epsilon} \right)^2 128\bar{h}_v(ik'D)(iQD)\bar{k}(1 - \gamma_5)h_v\]
\[+ \left( \frac{1}{Q^2 - m_i^2 + i\epsilon} \right)^3 256(k'Q)\bar{h}_v(iQD)^2\bar{k}(1 - \gamma_5)h_v\]
+ Operators of Dimension six or higher

We note that the spin structure of all the terms is of the form
\[\bar{h}_v\gamma_\mu(1 - \gamma_5)h_v = v_\mu\bar{h}_v h_v - \bar{h}_v s_\mu h_v \quad (18)\]
reflecting the left handedness of the current.

The differential rate may now be calculated from the imaginary part of the forward matrix element of the operator \(T\)
\[d\Gamma = \frac{G_F^2}{4m_H}|V_{Qq}|^2(2 \text{ Im } < H(v)|T(k, k', v)|H(v) >)d(PS) \quad (19)\]
The relevant hadronic matrix elements will be discussed in the next section.

4 Hadronic Matrix Elements and Short Distance QCD Corrections

We shall first consider the decay of a pseudoscalar meson. The only nonvanishing matrix element of the dimension three operators is
\[< H(v)|\bar{h}_v h_v|H(v) >= 2m_H. \quad (20)\]
which is normalized as given above due to heavy quark symmetries. Inserting this into \((19)\) reproduces the parton model result.

All the matrix elements of the dimension four operators vanish. The general form of the matrix element not involving the spin vector is

\[
< H(v) | \bar{h}_v D^\alpha h_v | H(v) > = B_0 v^\alpha
\]  

and by contraction with \(v\) and using the equations of motion we have \(B_0 = 0\). The corresponding matrix element involving the spin vector vanishes due to parity.

The matrix elements of the dimension five operators may be expressed in terms of two independent parameters. Due to the equations of motion we have

\[
< H(v) | \bar{h}_v (iD^\alpha)(iD^\beta) h_v | H(v) > = -2m_H B_1 (g^{\alpha\beta} - \nu^\alpha \nu^\beta)
\]  

where \(B_1\) is determined by the trace

\[
< H(v) | \bar{h}_v (iD)^2 h_v | H(v) > = -6m_H B_1.
\]  

This trace is related to the expectation value of the residual momentum squared \(k^2\) of the heavy quark in the heavy meson \(<k^2 > \sim 3B_1\) which is expected to be of the order of \(\Lambda^2_{QCD}\). Unfortunately one may not extract a precise value from experimental data and one has to rely on theoretical estimates. An estimate based on QCD sum rules \([10]\) yields rather large value of \(B_1 \sim 0.3\ \text{GeV}^2\). We shall use a default value of \(B_1 \sim 0.2\ \text{GeV}^2\) and discuss the dependence of the result on this parameter.

The other independent matrix element is parametrized as

\[
< H(v) | \bar{h}_v (iD^\alpha)(iD^\beta) s^\rho h_v | H(v) > = 2m_H B_2 \epsilon^{\lambda\alpha\beta\rho} v_\lambda
\]  

where the constant \(B_2\) is given in terms of the scalar matrix element

\[
i \epsilon^{\lambda\alpha\beta\rho} v_\lambda < H(v) | \bar{h}_v (iD_\alpha)(iD_\beta) s^\rho h_v | H(v) > = i g < H(v) | \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v | H(v)>
\]  

where \(g\) is the strong coupling constant. This matrix element is related to the mass splitting between the pseudoscalar meson and its spin symmetry partner vector meson

\[
i g < H(v) | \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v | H(v) > = -12m_H (m_{B_s}^2 - m_B^2)
\]  

\(^3\)This parameter is - up to a factor 3 - the parameter \(\lambda_1\) as defined in the paper by Falk and Neubert \([9]\).
from which we obtain
\[ B_2 = \frac{1}{8}(m_{H^*}^2 - m_H^2) \sim 0.07 \text{ GeV}^2 \] (27)

Thus lowest order corrections in the mesonic sector are given in terms of two parameters, one of which is related to an observable quantity.

The decay of a heavy ground state baryon \( \Lambda_b \) has been considered in [7] in the approach proposed in [5]. For this decay we have two matrix elements of the dimension three operators of which the normalization is known due to heavy quark symmetries. We have
\[
\langle \Lambda_H(v, s)|\bar{h}_v h_v|\Lambda_H(v, s) \rangle = 2m_H \] (28)
\[
\langle \Lambda_H(v, s)|\bar{h}_v s_\mu h_v|\Lambda_H(v, s) \rangle = 2m_H s_\mu \] (29)
where \( s_\mu \) is the direction of the spin of the \( \Lambda_H \) baryon in its rest frame.

As in the case of the mesons the matrix elements of the dimension four operators vanish due to parity and the equations of motion.

Finally, the matrix elements of the dimension five operators are given in terms of only one parameter
\[
\langle \Lambda_H(v, s)|\bar{h}_v(iD_\alpha)(iD_\beta)h_v|\Lambda(v, s) \rangle = -2m_H A_1 (g_{\alpha\beta} - v_\alpha v_\beta) \] (30)
\[
\langle \Lambda_H(v, s)|\bar{h}_v(iD_\alpha)(iD_\beta)s_\mu h_v|\Lambda(v, s) \rangle = -2m_H A_1 s_\mu (g_{\alpha\beta} - v_\alpha v_\beta) \] (31)
since the heavy quark spin symmetry relates these two matrix elements.

The parameters \( A_1, B_1 \) and \( B_2 \) parametrize the long distance QCD contributions to the inclusive semileptonic decay rates. Since we are dealing with an operator product expansion the short distance QCD contributions may be calculated as the scale dependence of the Wilson coefficients, the tree level value of which may be read off from (17).

Short distance corrections may be considered using the usual machinery of the renormalization group. The calculation of the short distance QCD corrections may be found in the literature. The anomalous dimensions of the scalar operators defining the constants \( A_1, B_1 \) and \( B_2 \) has been performed in [11].

Due to reparametrization invariance \[12\] the operator \( O_1 \)
\[ O_1 = \bar{h}_v(iD)^2 h_v \] (32)
has vanishing anomalous dimension and thus \( B_1 \) as well as \( A_1 \) are scale invariant quantities. The second operator \( O_2 \)
\[ O_2 = i g \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v \] (33)
has nonvanishing anomalous dimension but does not mix with any other operator. From the renormalization group equation with the anomalous dimension from \[11\]

\[
\mu^2 \frac{\partial}{\partial \mu^2} B_2(\mu) = -\frac{3}{4\pi} \alpha_s(\mu) B_2(\mu)
\] (34)

one obtains

\[
B_2(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(m_H)} \right)^{9/(33-2n_f)} B_2(m_H).
\] (35)

with \(n_f = 4\) for \(B\) hadron decays. This scale dependence introduces an \(Q^2\) dependence, since this is the relevant momentum transfer variable in the present case. The effect of these short distance corrections will be discussed for the example of the lepton energy spectrum in inclusive semileptonic decays of heavy mesons.

5 An Application: Energy Spectrum in Heavy Meson Decays

As an obvious application we shall consider the lepton energy spectrum of the charged lepton in an inclusive semileptonic decay of heavy mesons. To be specific we shall consider the decay \(\bar{B}^0 \to X_c e^- \bar{\nu}_e\) Here \(X\) is a hadronic final state containing charm.

The differential decay rate is given by

\[
\frac{d\Gamma}{dE} = \frac{G_F^2}{4m_H} |V_{Qq}|^2 \int d\tilde{k} d\tilde{k}' \delta(Q^2 - m_l^2) \delta(E - v k')(2 \text{ Im } < H(v) | T(k, k', v) | H(v) >)
\] (36)

The calculational details of the phase space integration are not very illuminating and are given in an appendix. However, unlike in the usual case of phase space integrals we have due to the relations

\[
2i\text{Im} \left( \frac{1}{Q^2 - m_l^2 + i\epsilon} \right) = -(2\pi)\delta'(Q^2 - m_l^2)
\] (37)

\[
2i\text{Im} \left( \frac{1}{Q^2 - m_l^2 + i\epsilon} \right)^2 = (2\pi)\delta(Q^2 - m_l^2)
\] (38)

\[
2i\text{Im} \left( \frac{1}{Q^2 - m_l^2 + i\epsilon} \right)^3 = -\pi\delta''(Q^2 - m_l^2)
\] (39)
also integrations involving the first and second derivative of the delta function $\delta'$ and $\delta''$. Flipping the derivative yields derivatives of the integrands. Since the QCD short distance corrections imply a dependence of $B_2$ on the kinematic variables one has to differentiate $B_2$ as well. This yields an additive correction term which is proportional to $\alpha_s$ taken at the scale $Q^2$; this scale is fixed to $Q^2 = m_l^2$ due to the delta functions showing that the present approach is indeed limited to $m_l \gg \Lambda_{QCD}$.

The results for the lepton spectrum is displayed in fig.2 and 3. Fig.2 shows the charged lepton energy spectrum for the inclusive semileptonic decay of a $B^0$ or a $B^-$ into charmed final states. The parameters used are $m_B = m_b = 5.28$ GeV, $m_l = m_c = m_D = 1.86$ GeV, $|V_{cb}| = 0.043$ and $B_1 = 0.2$ GeV$^2$. The solid line (a) is the full result, including long- and short distance corrections. This is compared to the spectrum including the corrections only at tree level, i.e. without the QCD short distance contribution (line (b)). Finally, curve (c) is the leading order result which is the spectrum obtained from the parton model. The spectrum obtained this way is slightly shifted to lower energies and the short distance corrections yield a small enhancement at low energies. Qualitatively this result agrees with the one obtained in [5].

Finally, we shall investigate the dependence on the parameter $B_1$ by considering the total rates. In fig.3 we plot the quantity

$$\frac{\Gamma_{corr}(B^0 \to X_c e^- \bar{\nu}_e)}{\Gamma_{parton}(B^0 \to X_c e^- \bar{\nu}_e)} - 1$$

in percent as a function of the value of $B_1$. We chose the same parameters as for fig.2, except that $B_1$ is varied in the range $0.04 \leq B_2 \leq 0.4$. Curve (a) is the full result, including long- as well as short distance contributions, curve (b) contains the tree level corrections only. The corrections reduce the total rate (compared to the parton model rate) for values of $B_1$ above 0.1. This seems to favor large values of $B_1$, since a reduction of the semileptonic branching fraction is phenomenologically welcome. Furthermore, one could in turn try to obtain matrix element corresponding to $B_1$ from a precise measurement of the inclusive lepton spectrum.

## 6 Conclusions

Inclusive semileptonic decays of heavy hadrons may be described by employing an operator product expansion for the operators appearing in the hadronic tensor. The
expansion is performed after switching to heavy quark effective field theory where the large momentum of the heavy hadron is scaled out. The relevant momentum transfer variable is $Q = q - m_Q v$ where $q$ is the momentum transferred to the leptons.

In the present paper the region of phase space has been considered in which $Q^2 \ll m_Q^2$ and heavy quark effective field theory to leading order in the $1/m_Q$ expansion is sufficient. The approach of [5] and [6] is in fact valid in a different portion of phase space, since [5, 6] assume that $Q^2 \sim m_Q^2$ and thus a simultaneous expansion in $1/m_Q$ and $1/Q^2$ is necessary.

The lepton spectrum as obtained in [5, 6] contains delta functions and its derivatives which reflects the failure of the expansion at the edge of phase space $E \sim E_{\text{max}}$. In contrast to that the approach applied here yields a completely smooth lepton energy spectrum without any delta function singularities.

Of course, the limitations of the procedure proposed here (namely $Q^2 \ll m_Q^2$) do show up at some point. If one studies for example the rate differential in the momentum transfer $q^2 = (k + k')^2$ and the total leptonic energy $vq = vk + vk'$ then one finds also delta functions and its derivatives entering the differential rate using the method chosen here. However, after an integration over the total leptonic energy one obtains a smooth $q^2$ spectrum and the only reminder for the breakdown of the $Q^2$ expansion is a behaviour of the correction term of the type

$$\frac{d\Gamma}{dq^2} \sim \frac{1}{\sqrt{(m_Q - m_l)^2 - q^2}}$$

at the endpoint of the leptonic $q^2$ spectrum. This is an integrable singularity and hence a finite total rate may be obtained as expected.

In conclusion, depending on the region of phase space considered one has to choose between the method proposed in [5, 6] and the one presented here. To obtain the lepton spectrum one has to integrate over the phase space and phase space regions contribute where the approximations used become invalid. However, both methods yield finite total rates which means that the singularities in the lepton spectrum are integrable and one may still expect a reasonable lepton energy spectrum from both methods; this expectation is confirmed by the qualitative agreement of the lepton energy spectra obtained from both approaches.
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A Some Details of the Calculation of the Lepton Energy Spectrum

In this appendix we give some of the technical details of the calculation for the energy spectrum of the charged lepton.

In order to perform the phase space integrations in (36) it is convenient to define three integrals of some function $F$ of the three independent scalar products $vk$, $vk'$ and $2kk'$

$$F = F(vk, vk', 2kk').$$

We define the integral

$$I_0[F] = \int \tilde{d}k\tilde{d}k' \delta(Q^2 - m_l^2)\delta(E - vk')F(vk, vk', 2kk')$$

$$= \frac{1}{32\pi^3} \int_\alpha^\beta dk F(k, E, 2m_H(E - E_{max} + k'))$$

where the limits of the integration are given by

$$\alpha = m_H \frac{E_{max} - E}{m_H - 2E} \quad \beta = E_{max} - E$$

By definition $E$ is the energy of the charged lepton in the rest frame of the decaying heavy meson and $E_{max}$ is its maximal energy

$$E_{max} = \frac{1}{2m_H}(m_H^2 - m_l^2)$$

The corrections contain derivatives of delta functions and it is convenient to define two more integrals

$$I_1[F] = \int \tilde{d}k\tilde{d}k' \delta'(Q^2 - m_l^2)\delta(E - vk')F(vk, vk', 2kk')$$

$$I_2[F] = \int \tilde{d}k\tilde{d}k' \delta''(Q^2 - m_l^2)\delta(E - vk')F(vk, vk', 2kk')$$
where $\delta'$ and $\delta''$ denotes the first and second derivative of the delta function. These two integrals may be expressed in terms of the integral $I_0$

$$I_1[F] = -\frac{1}{64\pi^3} \left( \frac{F(\beta, E, 0)}{m_H} - \frac{F(\alpha, E, 4E\alpha)}{m_H - 2E} \right) - I_0[F'] \quad (46)$$

$$I_2[F] = \frac{E}{32\pi} \delta(2m_H(E_{\text{max}} - E)) F(0, E, 0) \frac{1}{m_H(m_H - 2E)} + I_0[F''] \quad (47)$$

where $F'$ and $F''$ denotes the first and second derivative of $F$ with respect to the last argument $2kk'$.

Inserting the operator product expansion (17) one finds that the piece containing the dimension three operators reproduces the parton model result

$$\frac{d\Gamma}{dE} = -32G_F^2 |V_{Qq}|^2 I_0[(vk)(kk' - m_H vk')] \quad (48)$$

At tree level the correction terms originating from the dimension five operators are given by

$$\Delta \frac{d\Gamma}{dE} = 32G_F^2 |V_{Qq}|^2 \left\{ B_1 I_1 [5(vk)(k'Q) - 2(vk)(vk')(vQ)] + B_2 I_1 [2(vk')(kQ) - 2(kk')(vQ)] \right\} \quad (49)$$

$$- 64G_F^2 |V_{Qq}|^2 B_1 I_2 [(k'Q)(vk)(vQ)^2 - Q^2]$$

The QCD short distance contributions may be included by putting in the scale dependence of $B_2$ as discussed in the last section. At the one loop level one finds that aside from the fact that the value of $B_2$ is changed additional terms originating from the derivative in (46) acting on $B_2$ occur, since now $B_2$ depends on $2kk'$. The derivative of $B_2$ is obtained from the renormalization group equation (34). Inserting this we obtain an additional term due to the short distance contribution to $B_2$

$$\Delta_{SD} \frac{d\Gamma}{dE} = -32G_F^2 |V_{Qq}|^2 I_0[2(vk')(kQ) - 2(kk')(vQ)] \frac{3}{4\pi m_l^2} \alpha_s(m_l^2) B_2(m_l^2) \quad (50)$$

Here and in the other terms where it occurs $B_2$ has to be scaled down to $Q^2$ which is due to the delta functions of phase space fixed to the value $m_l^2$.

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Figure Captions

Fig.1 Feynman Diagrams to be evaluated to obtain the Wilson coefficients of the operators up to dimension five. The blob denotes the left handed current and the double line denotes the decaying heavy quark.

Fig.2 The lepton energy spectrum \( d\Gamma /dE \times 10^{15} \) of the inclusive semileptonic decay \( B^0 \rightarrow X_c e^- \bar{\nu}_e \). The parameters used are \( m_b = m_B = 5.28 \text{ GeV}, \ m_c = m_D = 1.86 \text{ GeV}, \ |V_{cb}| = 0.043 \) and \( B_1 = 0.2 \text{ GeV}^2 \). The curves are: (a) Full result including long- and short distance contributions, (b) Result without short distance QCD corrections, (c) leading order result, i.e. the parton model result.

Fig.3 The ratio of total rates \( (\Gamma_{\text{corr}}/\Gamma_{\text{parton}}) - 1 \) in percent for the inclusive decay \( B^0 \rightarrow X_c e^- \bar{\nu}_e \) function of the parameter \( B_1 \). The curves correspond to: (a) full result, (b) corrections only at tree level.
Fig.1: Feynman Diagrams
Fig. 2: Lepton Energy Spectrum in Inclusive $b \rightarrow c$ Decays
Fig. 3: Ratio of Total Rates