ON THE TUBE-OCCUPANCY OF SETS IN $\mathbb{R}^d$

TUOMAS ORPONEN

ABSTRACT. Call a pair $(s, t) \in [0, d] \times [0, d]$ admissible, if there exists a compact set $K \subset \mathbb{R}^d$ and a constant $C > 0$ such that $0 < \mathcal{H}^t(K) < \infty$, and

$$\mathcal{H}_s(K \cap T) \leq C w(T)^t$$

for all tubes $T \subset \mathbb{R}^d$ of width $w(T)$. The purpose of this paper is to show that all pairs $(s, s)$ with $s < 1$ are admissible. Combined with previous results, this settles a question of A. Carbery.

1. INTRODUCTION

In [Ca], A. Carbery asks to determine which pairs $(s, t) \in [0, d] \times [0, d]$ are admissible in the sense that there exists a (compact) set $K \subset \mathbb{R}^d$ of positive and finite $s$-dimensional Hausdorff measure with the property that

$$\mathcal{H}_s(K \cap T) \leq C w(T)^t \tag{1.1}$$

for all tubes $T$ and for some constant $C > 0$. Here and below, $w(T)$ stands for the width of $T$. The known results can be summarised as follows:

- The condition (1.1) is closely related to the projections of the measure $\mathcal{H}^|K$ into $(d - 1)$-dimensional subspaces. In particular, (1.1) implies that the projections of $K$ onto any such subspace are at least $t$-dimensional. This means that only pairs $(s, t)$ with $t \leq \min\{d - 1, s\}$ have a chance of being admissible.
- It follows from the Besicovitch projection theorem that the pair $(d - 1, d - 1)$ is not admissible; for details, see [CSV].
- On the other hand, all pairs $(s, d - 1)$ are admissible for $s > d - 1$. This is due to P. Shmerkin and V. Suomala [SS].
- In [Ca], it was shown that that all pairs $(s, t)$ with $t < d - 1$ and $s > t$ are admissible.

So, the remaining question concerns the pairs $(s, s)$ with $s < d - 1$. We answer this question:

**Theorem 1.2.** All pairs $(s, s)$ are admissible for $s < d - 1$.

We will present an informal overview of the proof right away, while the technical details are given in the third and fourth sections. To prove the theorem, we
need to build a set $K$ with $0 < \mathcal{H}^s(K) < \infty$, satisfying (1.1) with $t = s$. To this end, we use a standard Cantor type construction: for various "generations" $n \in \mathbb{N}$, we seek for collections of closed and disjoint cubes $Q_n$ such that the union of the cubes in $Q_{n+1}$ is contained in the union of the cubes in $Q_n$. For fixed $n \in \mathbb{N}$, the cubes in $Q_n$ will have a common side-length $\ell_n > 0$. As an induction hypothesis, we may assume that every tube $T \subset \mathbb{R}^d$ of width $w(T) = \ell_n$ intersects at most $k$ cubes in $Q_n$ for some large $k \in \mathbb{N}$. Then, to proceed, we need to find the family of cubes $Q_{n+1}$ containing $\sim \ell_{n+1}^{-s}$ members, so that every tube of width $\ell_{n+1}$ meets no more than $k$ of them. Moreover, we have to do this in such a manner that all tubes $T$ of "intermediary" widths $\ell_{n+1} < w(T) \leq \ell_n$ also behave well – that is, do not intersect too many cubes in $Q_{n+1}$.

The first idea would be to throw the centres of the cubes in $Q_{n+1}$ uniformly at random inside the union of the cubes in $Q_n$. This cannot work directly, since random sets contain a certain amount of clustering with high probability (WHP), and (1.1) means practically zero tolerance towards such behaviour. Fortunately, this "certain amount" turns out to be small and easily quantifiable. In particular, WHP, there are only a few clusters of the type where some tube of width of with $w(T) = \ell_{n+1}$ intersects more than $k$ cubes in $Q_{n+1}$. So, we may simply take the clusters one by one and remove points from them, until they are clusters no longer. And, WHP, as it turns out, we can do this so that at least half of the originally selected points remain. A version of this procedure appeared already in [Ca], and, much earlier, a similar idea was used in connection with the Heilbronn triangle problem by Komlós, Pintz and Szemerédi [KPS].

Next, we face the tubes of width $\ell_{n+1} < w(T) \leq \ell_n$. After defining the appropriate notion of "clustering" for tubes of intermediary width, it turns out that there are, again, only a few clusters WHP. So, fixing $w(T) \in (\ell_{n+1}, \ell_n]$ we may remove some further points to get rid of these clusters in the same manner as above. In fact, the expected number of clusters is so negligible that we can get simultaneously rid of all clusters corresponding to tubes of width $w(T) \in (\ell_{n+1}, \eta_{n+1}]$, where $\eta_{n+1} \in (\ell_{n+1}, \ell_n)$ is a certain constant.

For tubes $T$ of width $w(T) \in (\eta_{n+1}, \ell_n]$ the strategy does not seem to work directly. Instead, the number $\eta_{n+1}$ is selected so that every cube of side-length $\eta_{n+1}$ is expected to contain $\sim 1$ of our randomly selected points. In this situation, we can redefine a "cluster" to mean an $A$-element set contained in a single cube of side-length $\eta_{n+1}$, where $A \in \mathbb{N}$ is a large constant. Choosing $A$ large enough, it turns out that there are only a few clusters of this type, and they can be disposed of in a familiar manner. After this is done, we end up with a set, which is well-behaved with respect to thin tubes – those of width $\ell_{n+1} < w(T) \leq \eta_{n+1}$ – and is also very uniformly distributed on scales larger than $\eta_{n+1}$. Finally, it suffices to check that any set, which has the latter uniform distribution property, is well-behaved with respect to thick tubes, namely those of width $\eta_{n+1} < w(T) \leq \ell_n$. This concludes the inductive construction of $Q_{n+1}$. 
Before we start with the technicalities, some quick words on notation and presentation: the letters $Q, R, S$ will be used to denote various cubes (with sides perpendicular to coordinate axes), while $T$ always stands for a tube – a $w(T)/2$-neighbourhood of a line in $\mathbb{R}^d$, where $w(T) > 0$ is the width of $T$.

Given $A, B > 0$, the notation $A \lesssim B$ means that $A \leq CB$ for some constant $C$ depending only on the dimension $d$ of the ambient space (so $d$ is regarded as an absolute constant in our notation). The two-sided inequality $A \lesssim B \lesssim A$ is abbreviated to $A \sim B$. Both cardinality and Lebesgue measure of planar sets will be denoted by $| \cdot |$: the notation will refer to cardinality for finite sets, and to Lebesgue measure otherwise.

Finally, without further remark, we will assume that the various expressions (such as $Cm^s$ or $r/\epsilon$) appearing in the proofs are integers, whenever it is useful for the argument. Often this can be achieved directly by modifying the constants involved – but sometimes the reduction might need a short technical argument, which we simply omit.

2. Acknowledgements

I am grateful to Henna Koivusalo for fruitful discussions during the early stages of the paper’s preparation. I am also thankful to Tony Carbery for suggesting the problem and giving comments on the manuscript.

3. Main lemma

As we outlined above, the set $K$ needed for Theorem 1.2 will be constructed by defining recursively the families of cubes $Q_n$. The initial family will be $Q_0 := \{[0, 1]^d\}$, so all further action will take place inside the unit cube. The cubes have to satisfy certain properties, which are listed in the following main lemma:

Lemma 3.1. Fix $s < d - 1$ and $\delta > 0$. Then, the following holds for large enough $k = k_s, m = m_{\delta,s} \in \mathbb{N}$ (so $k$ depends only on $s$, while $m$ may also depend on $\delta$). Assume that $U \subset [0, 1]^d$ is the union of a family $\mathcal{U}$ of $\delta^{-s}$ disjoint cubes, each of side-length $0 < \delta \leq 1$, such that every tube of width $2\delta$ meets no more than $k$ cubes. Then, there exists a collection of disjoint closed cubes $Q$ with the following properties.

(a) All the cubes in $Q$ have equal side-length $\epsilon \sim 1/m$.

(b) The union of the cubes in $Q$ is contained in the union of the cubes in $\mathcal{U}$. Moreover, every cube in $\mathcal{U}$ contains exactly $(\delta/\epsilon)^s$ cubes in $Q$. In particular, there are $\epsilon^{-s}$ cubes in $Q$ altogether.

(c) An arbitrary cube of side-length $\delta^{(d-s)/d}m^{-s/d}$ intersects at most one cube in $Q$.

(d) Every tube $T$ of width $w(T) = 2\epsilon$ satisfies

$$\text{card}\{Q \in Q : T \cap Q \neq \emptyset\} \leq k,$$

and every tube $T$ of width $\epsilon \leq w(T) \leq \delta$ satisfies

$$\text{card}\{Q \in Q : T \cap Q \neq \emptyset\} \lesssim k \cdot \left(\frac{w(T)}{\epsilon}\right)^s.$$
For the remainder of Section 3, the parameters $\delta, \epsilon, k, m, s, Q, U$ and $\mathcal{U}$ refer to those introduced in the statement of Lemma 3.1. The constant $\eta_{n+1}$ discussed in the introduction is not explicitly mentioned, but it coincides with $\delta^{(d-s)/d}m^{-s/d}$.

The proof idea is to pick $m^s$ points independently and uniformly at random inside $U$, for some large $m \in \mathbb{N}$, and call this random set $P_0$. The set $P_0$ will play the role of the centres of the cubes in $Q$. In the interest of avoiding extra constants, we choose to ignore the issue of the points in $P_0$ being too close to the boundary of $U$ – resulting in the risk that some cubes in $Q$ may not be entirely contained in the union of the cubes of $\mathcal{U}$. The risk could be neutralised by first replacing the cubes in $\mathcal{U}$ by slightly smaller ones and then choosing the points $P_0$ inside the new cubes.

3.1. Preliminary considerations towards (c). Divide each of the $\delta^{-s}$ cubes $R \in U$ into a grid of $(\delta m)^s$ equally sized subcubes using equally spaced hyperplanes perpendicular to the coordinate axes. Denote the collections of cubes so obtained by $S_R, R \in U$.

**Lemma 3.2.** Let $A \in \mathbb{N}$, and for each $R \in U$, consider the random variable

$$X_R := \frac{1}{(\delta m)^s} \sum_{S \in S_R} \left( \frac{|P_0 \cap S|}{A} \right),$$

where $P_0$ is the random $m^s$-element subset of $U$, and we make the usual convention that the binomial coefficient is zero, when $A > |P_0 \cap S|$. Then,

$$\mathbb{P}\left\{ \max_{R \in U} X_R \geq \frac{1}{10} \right\} < \frac{1}{10},$$

if $A$ is large enough (but absolute), and $m \in \mathbb{N}$ is large enough (depending on $\delta$).

**Proof of Lemma 3.2.** The plan is to use the union bound

$$\mathbb{P}\left\{ \max_{R \in U} X_R \geq \frac{1}{10} \right\} \leq \sum_{R \in U} \mathbb{P}\{X_R \geq 1/10\}$$

in the end, so we need to show that $\mathbb{P}\{X_R \geq 1/10\}$ can be pushed smaller than $1/(10 \text{card } U) = \delta^s/10$ by taking $A$ and $m$ large enough. First, we observe that $X_R$ is the average over the random variables

$$Y_S := \left( \frac{|P_0 \cap S|}{A} \right),$$

so we wish to see that (a) the expectation of the variables $Y_S$ can be made small by taking $A$ large, and (b) the average $X_R$ is strongly concentrated around the expectation. The only problem in (b) is that the random variables $Y_S$ are not independent: in fact, if $Y_S$ is large for some particular $S \subset R$, then there are unexpectedly many elements of $P_0$ packed inside $S$, and this makes it less likely that $Y_{S'}$ is large for $S' \neq S$. In fact, this observation is the key to the proof.
Fixing \( R \in \mathcal{U} \), we make the preliminary observation that

\[
\mathbb{P}\left\{ X_R \geq \frac{1}{10} \right\} \leq \mathbb{P}\left\{ |P_0 \cap R| > 2(\delta m)^s \right\} + \mathbb{P}\left\{ X_R \geq \frac{1}{10} \text{ and } |P_0 \cap R| \leq 2(\delta m)^s \right\}.
\]  
(3.3)

Since the random variable \( |P_0 \cap R| \) is distributed \( \text{Bin}(m^s, \delta^s) \), the probability of the first summand tends to zero as \( m \to \infty \). So, we are left to deal with the second summand. This probability can be further expressed as the following sum of conditional probabilities:

\[
(3.3) = \sum_{k=0}^{2(\delta m)^s} \mathbb{P}\left( X_R \geq \frac{1}{10} \mid |P_0 \cap R| = k \right) \mathbb{P}\left\{ |P_0 \cap R| = k \right\}.
\]  
(3.4)

So, we aspire to estimate

\[
\mathbb{P}_k \left\{ X_R \geq \frac{1}{10} \right\}, \quad k \leq 2(\delta m)^s,
\]

where \( \mathbb{P}_k \{ \cdots \} = \mathbb{P}\{ \cdots \mid |P_0 \cap R| = k \} \). The probability \( \mathbb{P}_k \) has the useful interpretation of "throwing \( k \) points independently and uniformly at random inside \( R^n \). Indeed, given the condition that exactly \( k \) points of \( P_0 \) land in \( R \), we can forget that these points come from the \( m^s \)-element subset of \( P_0 \) and think of them as a \( k \)-element random subset \( P_0^k \) of their own. In particular, with a slight abuse of notation, the \( \mathbb{P}_k \)-distributions of the random variables \( Y_S \) and \( X_R \) equal the \( \mathbb{P}_k \)-distributions of the random variables

\[
Y_S^k = \left( \frac{|P_0^k \cap S|}{A} \right) \text{ and } X_R^k = \frac{1}{(\delta m)^s} \sum_{S \subset R} Y_S^k.
\]

In order to avoid overly complicated notation, we will write \( Y_S := Y_S^k \) and \( X_R := X_R^k \) in the sequel.

Now we start to bound the probability \( \mathbb{P}_k[X_R \geq 1/10] \). If \( S \subset S_R \) is a collection of cubes, and \( (r_S)_{S \in \mathcal{S}} \) is a collection of natural numbers, denote by \( E((r_S)_{S \in \mathcal{S}}) \) the event

\[
E((r_S)_{S \in \mathcal{S}}) = \left\{ |P_0^k \cap S| = r_S : S \in \mathcal{S} \right\}.
\]

Then the event \( \{ X_R \geq 1/10 \} \) is contained in the finite disjoint union of events of the form \( E((r_S)_{S \in \mathcal{S}}) \) such that \( r_S \geq A \) for all \( S \in \mathcal{S} \), and

\[
\sum_{S \in \mathcal{S}} \left( \frac{r_S}{A} \right) \geq \frac{(\delta m)^s}{10}.
\]  
(3.5)
Fix one such event $E((r_S)_{S \in S})$, and enumerate the cubes in $S$ by writing $S = \{S_1, \ldots, S_N\}$. Then

$$\mathbb{P}_k[E((r_S)_{S \in S})] = \prod_{n=1}^{N} \mathbb{P}_k[\{|P_0^k \cap S_1| = r_{S_1}\} \cap \ldots \cap \{|P_0^k \cap S_N| = r_{S_N}\}].$$

(3.6)

simply by iterating the definition of conditional probability. Let us fix $n \in \{1, \ldots, N\}$ and study the $n^{th}$ factor of the product. The distribution of the random variable $|P_0^k \cap S_n|$ with respect to the conditional probability measure

$$\mathbb{P}_k[\ldots | \{|P_0^k \cap S_{n+1}| = r_{S_{n+1}}\} \cap \ldots \cap \{|P_0^k \cap S_N| = r_{S_N}\}]$$

is a binomial one, more precisely

$$\text{Bin}\left(k - \sum_{q=n+1}^{N} r_{S_q}, \frac{|S_n|}{|R| - (N-n)|S_n|}\right).$$

Indeed, given the information that $r_{S_{n+1}} + \ldots + r_{S_N}$ points are contained in the cubes $S_{n+1}, \ldots, S_N$, we are left with $k - r_{S_{n+1}} - \ldots - r_{S_N}$ points to choose randomly inside the set

$$R \setminus \bigcup_{q=n+1}^{N} S_q,$$

and the probability that any of these points should land in $S_n$ is exactly the "success probability" of the binomial distribution above. Now, we wish to bound the probability that such a random variable takes the value $r_{S_n}$. This probability is

$$\left(k - \sum_{q=n+1}^{N} r_{S_q}\right) \left(\frac{|S_n|}{|R| - (N-n)|S_n|}\right)^{r_{S_n}} \left(1 - \frac{|S_n|}{|R| - (N-n)|S_n|}\right)^{k - \sum_{q=n+1}^{N} r_{S_q}} \leq \left(e |S_n| \right)^{r_{S_n}} \left(\frac{k - \sum_{q=n+1}^{N} r_{S_q}}{|R| - (N-n)|S_n|}\right)^{r_{S_n}} \leq \left(e |S_n| \right)^{r_{S_n}} \left(\frac{k - (N-n)A}{|R| - (N-n)|S_n|}\right)^{r_{S_n}},$$

(3.7)

using the assumption that $r_S \geq A$ for all $S \in \mathcal{S}$. Next, a simple manipulation shows that

$$\frac{k - (N-n)A}{|R| - (N-n)|S_n|} \leq \frac{k}{|R|}.$$
as soon as $A \geq k|S_n|/|R|$, which is true as soon as $A \geq 2$, since $k \leq 2(\delta m)^a = 2|R|/|S_n|$. Plugging this estimate into (3.7) leads to
\[ (3.7) \leq \frac{k^{r_{S_n}}}{r_{S_n}!} \cdot \left( \frac{e^{|S_n|}}{|R|} \right)^{r_{S_n}} \leq \left( \frac{k}{r_{S_n}} \right)^{r_{S_n}} \cdot \left( \frac{e^{|S_n|}}{|R|} \right)^{r_{S_n}}. \] (3.8)

This form is almost what we were looking for. The final observation is that
\[ \left( 1 - \frac{2e|S_n|}{|R|} \right)^{k-r_{S_n}} \geq \left( 1 - \frac{2e|S_n|}{|R|} \right)^{2(\delta m)^a} \geq \tau \]
for some absolute constant $\tau > 0$, since $|S_n|/|R| = (\delta m)^-a$. Thus, by taking $A$ so large that $2^{r_{S_n}} \geq 2^A \geq 1/\tau$, the estimate in (3.8) can be taken one step further as follows:
\[ (3.8) \leq \left( \frac{k}{r_{S_n}} \right)^{r_{S_n}} \cdot \left( \frac{2e|S_n|}{|R|} \right)^{r_{S_n}} \cdot \left( 1 - \frac{2e|S_n|}{|R|} \right)^{k-r_{S_n}}. \]

Now, we look back to (3.6). The probability of the event $E(\{r_S \in S\})$ was factorised into a product of $N$ numbers, and we have just obtained an estimate for each of these. Consequently,
\[ \mathbb{P}_k[E(\{r_S \in S\})] \leq \prod_{n=1}^N \left( \frac{k}{r_{S_n}} \right)^{r_{S_n}} \cdot \left( 1 - \frac{2e|S_n|}{|R|} \right)^{k-r_{S_n}}. \] (3.9)

The estimate (3.9) is useful once interpreted correctly. To do this, we associate to each cube $S \in S_R$ an independent copy of an abstract $\sim \text{Bin}(k, 2e|S|/|R|)$ distributed random variable, which we denote by $Z_S$. Given any collection of cubes $S = \{S_1, \ldots, S_N\} \subset S_R$, and any natural numbers $r_{S_n} \in \{1, \ldots, k\}$ for $1 \leq n \leq N$, the probability of the event
\[ E_Z(\{r_S \in S\}) = \{Z_{S_n} = r_{S_n} \text{ for all } 1 \leq n \leq N\} \]
equals the right hand side of (3.9). Moreover, assuming that the numbers $r_{S_n}$, $1 \leq n \leq N$ satisfy (3.5), the event $E_Z(\{r_S \in S\})$ is contained in
\[ \left\{ \frac{1}{(\delta m)^a} \sum_{S \in S_R} \left( \frac{Z_S}{A} \right) \geq \frac{1}{10} \right\}. \]

The conclusion is that
\[ \mathbb{P}_k \left\{ X_R \geq \frac{1}{10} \right\} \leq \hat{\mathbb{P}} \left\{ \frac{1}{(\delta m)^a} \sum_{S \in S_R} \left( \frac{Z_S}{A} \right) \geq \frac{1}{10} \right\}, \] (3.10)
where we used $\hat{\mathbb{P}}$ to denote the abstract probability measure associated with the random variables $Z_S$. Let us sum up the argument that lead to this conclusion: the event on the left hand side of (3.10) can be expressed as the union of events of the form $E(\{r_S \in S\})$, where the numbers $r_S$ satisfy $r_S \geq A$ and (3.5). Then,

---

1So, the variable $Z_S$ does not actually have anything to do with the cube $S \in S_R$, but such an indexing is handy nevertheless.
the \( \mathbb{P}_k \)-probability of any such event is bounded by the \( \tilde{\mathbb{P}} \)-probability of the corresponding event \( E_Z(\{r_s\}_{s \in S}) \) by (3.9). Finally, the (disjoint) union of these events is contained in the event on the right hand side of (3.10).

Next, we plan to apply Markov’s inequality to show that for \( A \) large enough but absolute, and for \( m \) large enough depending on \( \delta \), the right hand side probability in (3.10) is less than \( \delta^s/20 \). So, we need to estimate the expectation and variance of the random variables \( (Z^s_A) \). The expectation can be calculated rather explicitly by viewing \( (Z^s_A) \) as the sum of certain other random variables (this may seem complicated, but thinking along these lines will also be useful later on in the paper). Fix a cube \( U \subset R \) of volume \( |U| = 2e|S| \). Then, the \( \mathbb{P}_k \)-distribution of \( |Z^0_k \cap U| \) is \( \sim \text{Bin}(k, |U|/|R|) \), which is the same as the \( \tilde{\mathbb{P}} \)-distribution of \( Z^s_A \). Consequently, the \( \mathbb{P}_k \)-distribution of the random variable

\[
\sum_{P \in \mathcal{P}^A} 1_{\{P \subset U\}} = \left( \frac{|P^0_k \cap U|}{A} \right)
\]

is the same as the \( \tilde{\mathbb{P}} \)-distribution of the random variable \( (Z^s_A) \), where \( \mathcal{P}^A \) stands for the collection of all \( A \)-element subsets of \( P^0_k \). In particular, these random variables have common expectation, which equals

\[
\sum_{P \in \mathcal{P}^A} \mathbb{P}_k\{P \subset U\}.
\]

The \( \mathbb{P}_k \)-probability that any fixed \( A \)-element subset \( P \subset P^0_k \) is contained in \( U \) is

\[
\mathbb{P}_k\{P \subset U\} = \left( \frac{|U|}{|R|} \right)^A = (2e)^A(\delta m)^{-As}.
\]

Since \( \text{card} \mathcal{P}^A = \binom{k}{A} \), we conclude that

\[
\tilde{\mathbb{E}} \left[ \left( \frac{Z^s_A}{A} \right) \right] = \binom{k}{A} (2e)^A(\delta m)^{-As} \leq \left( \frac{e \cdot 2(\delta m)^s}{A} \right)^A (2e)^A(\delta m)^{-As} = \left( \frac{4e^2}{A} \right)^A \leq \frac{1}{100}
\]

for a large enough absolute choice of \( A \in \mathbb{N} \).

Next, we consider the variance of \( (Z^s_A) \). Estimating crudely,

\[
\tilde{\text{Var}} \left[ \left( \frac{Z^s_A}{A} \right) \right] \leq \mathbb{E} \left[ \left( \frac{Z^s_A}{A} \right)^2 \right] \lesssim_A \mathbb{E}[Z^2_A] = \int_0^\infty t^{2A-1} \tilde{\mathbb{P}}\{Z_S \geq t\} \, dt.
\]

Recalling that \( Z_S \sim \text{Bin}(k, 2e|S|/|R|) \), where \( k \leq 2|R|/|S| \), it is easy to check (using directly the formula for the probability density function of \( Z_S \)) that the integral above admits a bound \( C_A < \infty \) depending only on \( A \).

Since the random variables \( (Z^s_A) \) are independent for various \( S \in \mathcal{S}_R \), the variance of their sum is the sum of their variance. Hence, Markov’s inequality is a
useful tool for bounding the probabilities related to the average
\[
\bar{X}_R := \frac{1}{(\delta m)^s} \sum_{S \in S_R} \left( \frac{Z_S}{A} \right).
\]
Recalling that \(\hat{\mathbb{E}}[\bar{X}_R] = \hat{\mathbb{E}}[Z_S] \leq 1/100\) for large enough \(A\), we obtain
\[
\hat{\mathbb{P}}\{\bar{X}_R \geq 1/10\} \leq \hat{\mathbb{P}}\{\bar{X}_R - \mathbb{E}_k[\bar{X}_R] \geq 1/100\} \\
\leq \hat{\mathbb{P}}\{(\bar{X}_R - \mathbb{E}_k[\bar{X}_R])^2 \geq 10^{-4}\} \\
\leq 10^4 \cdot \text{Var}[\bar{X}_R] = \frac{10^4}{(\delta m)^s} \text{Var}[Z_S] \leq \frac{10^4C_A}{(\delta m)^s}.
\]
Taking \(m\) large enough (depending on \(\delta\)) and using (3.10), this gives
\[
\mathbb{P}_k\{X_R \geq 1/10\} \leq \hat{\mathbb{P}}\{\bar{X}_R \geq 1/10\} < \delta^s/20.
\]
This holds uniformly for \(0 \leq k \leq 2(\delta m)^s\), so the sum in (3.4) is also < \(\delta^s/20\). Finally (as discussed above (3.4)), taking \(m\) so large that also \(\mathbb{P}\{|P_0 \cap R| > 2(\delta m)^s\} < \delta^s/20\), we obtain \(\mathbb{P}\{X_R \geq 1/10\} < \delta^s/20\), and finally
\[
\mathbb{P}\left\{\max_{R \in \mathcal{U}} X_R \geq \frac{1}{10}\right\} < \frac{1}{10}.
\]
This concludes the proof of the lemma. \(\square\)

Now we are prepared for the proof of the main lemma.

**Proof of Lemma 3.1.** Informally speaking, the main question to answer is the following: "In expectation, how many large subsets of \(P_0\) land in tubes \(T\) of width \(1/m \leq w(T) \leq \delta?" Now, we set to formalise and answer this question.

Fix a number \(\beta = 2^j\), \(j \geq 0\), such that \(1/m \leq \beta/m \leq \delta/r^2\), where \(r \geq 1\) is an absolute constant, the meaning of which will be clarified soon. If \(T \subset \mathbb{R}^d\) is a tube and \(h > 0\), denote by \(hT\) the tube with the same central line as \(T\) but with \(w(hT) = hw(T)\). We know that any tube \(r^2T\) with \(w(T) = \beta/m\) has width \(\leq \delta\), hence meets at most \(k\) cubes in \(\mathcal{U}\), so
\[
\mathbb{P}\{p \in r^2T\} = \frac{|r^2T \cap U|}{|U|} \leq C \frac{k \cdot \delta \cdot (\beta/m)^{d-1}}{\delta^{d-s}} = Ck\delta^{s-d+1} \cdot \left(\frac{\beta}{m}\right)^{d-1},
\]
where \(C \geq 1\) is a suitable absolute constant.

The purpose of the constant \(r \geq 1\) is the following. Assume that we have already determined the value of \(\epsilon \sim 1/m\) (the side-length of the cubes in \(Q\)), and imagine placing cubes of side-length \(\epsilon\) centred at each point in \(P_0\); denote these cubes by \(Q_0\). We require \(r\) to be so large that the following conditions are satisfied: \(r \geq 2\epsilon m\), and if \(w(T) \geq 1/m\), and \(T\) intersects one the cubes in \(Q_0\), then, \(rT\) contains the centre, a point in \(P_0\). These conditions are satisfied by a large constant \(r\), the size of which depends only on the absolute constants in \(\epsilon \sim 1/m\). As a consequence of the second condition, if \(T\) is a tube of width \(w(T) = \beta/m\), which meets \(> k \cdot (mw(T))^s = k \cdot \beta^s\) cubes in \(Q_0\) for some (large) \(k\), it follows
that \( rT \) must contain a \( k \cdot \beta^s \) element subset of \( P_0 \), and we wish to estimate the probability of this event. For technical reasons, however, we choose to estimate the probability of \( \geq k \cdot \beta^s \) points being contained in \( r^2T \) instead of \( rT \).

Write \( q = k \cdot \beta^s \) and assume that \( q \in \mathbb{N} \). Given any \( q \)-element subset \( \{p_1, \ldots, p_q\} \subset P_0 \), we have

\[
P\{ \{p_1, \ldots, p_q\} \subset r^2T \} \leq (Ck \delta^{s-d+1}q) \cdot \left( \frac{\beta}{m} \right)^{q(d-1)}.
\]

There are no more than essentially \( \sim (m/\beta)^{2(d-1)} \) different tubes of width \( w(T) = \beta/m \) intersecting \( U \subset [0,1]^d \), so\(^2\), taking a slightly larger \( C \) if necessary, we have

\[
P\{ \{p_1, \ldots, p_q\} \subset r^2T \text{ for some } T \text{ with } w(T) = \beta/m \} \leq (Ck \delta^{s-d+1}q) \cdot \left( \frac{\beta}{m} \right)^{(q-2)(d-1)}.
\]

Hence,

\[
\mathbb{E} \left[ \sum_{\{p_1, \ldots, p_q\} \subset P_0} 1_{\{\{p_1, \ldots, p_q\} \subset r^2T \text{ for some } T \text{ with } w(T) = \beta/m\}} \right]
\]

\[
= \sum_{\{p_1, \ldots, p_q\} \subset P_0} \mathbb{P}\{ \{p_1, \ldots, p_q\} \subset r^2T \text{ for some } T \text{ with } w(T) = \beta/m\}
\]

\[
\leq \left( \frac{m^s}{q} \right) (Ck \delta^{s-d+1}q) \cdot \left( \frac{\beta}{m} \right)^{(q-2)(d-1)}
\]

\[
\leq (eCk \delta^{s-d+1}q) \cdot \left( \frac{m^s}{q} \right) \cdot \left( \frac{\beta}{m} \right)^{(q-2)(d-1)}
\]

\[
= (eC \delta^{s-d+1}q) \cdot \left( \frac{m^s}{\beta^s} \right) \cdot \left( \frac{\beta}{m} \right)^{(q-2)(d-1)}
\]

\[
= (eC \delta^{s-d+1}q) \cdot \left( \frac{\beta}{m} \right)^{q(d-1)-2(d-1)}
\]

\[
= D \cdot k \cdot \beta^{s(d-1)-2(d-1)}.
\]

(3.11)

Next, for reasons to become apparent shortly, we wish to estimate the sum of the numbers in (3.11) over \( \beta = 2^j \) such that \( \beta/m \leq \delta^{(d-s)/d}m^{1-s/d} \), that is, for

\[
\beta \leq \delta^{(d-s)/d}m^{1-s/d} \leq m^{1-s/d}.
\]

Observe that then \( \beta/m \leq \delta/r^2 \) for large enough \( m \), which was needed for the estimates above. We are free to choose \( k = k_s \in \mathbb{N} \) at will, and the first requirement

\[\text{More precisely, we may choose a collection of } \sim (m/\beta)^{2(d-1)} \text{ "representative" tubes of width } 2\beta/m \text{ such that the intersection of any tube of width } \beta/m \text{ with } [0,1]^d \text{ is contained in one of these representatives.}\]
we place is that
\[ k(d - 1 - s) - 2(d - 1) > 0. \]
Then also \( k \cdot \beta^s(d - 1 - s) - 2(d - 1) > 0 \) for all \( \beta = 2^j \geq 1 \), so we may plug in the upper bound for \( \beta \) to obtain
\[
\sum_{\beta=2^j=1}^{m^{1-s/d}} D_k \beta^s \left( \frac{\beta}{m} \right)^{k \cdot \beta^s(d-1-s) - 2(d-1)} \leq \sum_{\beta=2^j=1}^{m^{1-s/d}} D_k \beta^s \cdot m^{-sk \cdot \beta^s(d-1-s)/d + 2s(d-1)/d}.
\]

Next, we note that the various numbers \( \beta^s = 2^{js} \) are separated by \( \geq 1 \) for \( j \) large enough (depending on \( s \)), so we may replace the original summation over \( \beta = 2^j \leq m^{s/d} \) by a summation over \( \beta \in \mathbb{N} \), with the gain of replacing \( \beta^s \) by \( \beta \) in the process. This may cost us a multiplicative constant \( C_s \geq 1 \) depending on \( s \). The result is the following geometric sum:
\[
\ldots \leq C_s \sum_{\beta=1}^{\infty} D_k \beta^s \cdot m^{-sk \cdot \beta(d-1-s)/d + 2s(d-1)/d} \leq m^{2s(d-1)/d} \cdot C_s \sum_{\beta=1}^{\infty} (D_k \cdot m^{-sk(d-1-s)/d})^\beta
\]
\[
= \frac{C_s \cdot D_k \cdot m^{-sk(d-1-s)/d}}{1 - D_k \cdot m^{-sk(d-1-s)/d} \cdot m^{2s(d-1)/d}} = \left[ \frac{C_s \cdot D_k \cdot m^{-sk(d-1-s)/d + s(1-2/d)}}{1 - D_k \cdot m^{-sk(d-1-s)/d}} \right] \cdot m^s,
\]
(3.12)

where the upshot is that the factor in front of \( m^s = |P_0| \) can be made arbitrarily small by choosing \( m \) large enough (depending on \( \delta \) via the definition of \( D \)), and taking \( k = k_s \) so large that \( s(1 - 2/d) - sk(d - 1 - s)/d < 0 \).

Let us sum up what we have gained so far. Given \( q \in \mathbb{N} \), denote by \( \mathcal{P}_q \) the collection of all \( q \)-element subsets of \( P_0 \). Choosing \( m \in \mathbb{N} \) large enough depending on \( \delta \), \( k \) large enough depending on \( s \) and \( A \geq 1 \) large enough but absolute, we can now conclude (see explanations below) that the following three events each hold with probability at least 9/10:
\[
\min_{R \in \mathcal{U}} |P_0 \cap R| \geq \frac{(\delta m)^s}{2}, \quad (3.13)
\]
\[
\max_{R \in \mathcal{U}} \sum_{S \in S_R} \sum_{P \in \mathcal{P}_A} 1_{\{P \subseteq S\}} \leq \frac{(\delta m)^s}{8}, \quad (3.14)
\]
and
\[
\sum_{\beta=2^j=1}^{(\delta m)^{1-s/d}} \sum_{P \in \mathcal{P}_{k_s \beta^s}} 1_{\{P \subseteq T \text{ for some } T \text{ with } w(T) = \beta/m\}} \leq \frac{(\delta m)^s}{8}, \quad (3.15)
\]
where \( S_R \) in (3.14) stands for the grid of cubes of side-length \( \delta^{(d-s)/d} m^{-s/d} \) defined above Lemma 3.2. First, (3.13) has high probability (when \( m = m_\delta \) is large
enough) simply because the random variables $|P_0 \cap R|$, $R \in \mathcal{U}$, are distributed $\sim \text{Bin}(m^* \cdot \delta^s)$, and $\text{card}\mathcal{U} = \delta^{-s}$ does not grow as $m \to \infty$. Second, the fact that (3.14) holds with high probability is just another way of writing the conclusion of Claim 3.2, since

$$\sum_{P \in \mathcal{P}_A} 1\{P \subseteq S\} = \binom{|P_0 \cap S|}{A}. $$

Third, the situation that (3.15) has probability $\geq 9/10$ can be reached by taking $m = m_k$ and $k = k_s$ so large that the expectation of the sum in (3.15) is bounded by $(\delta m)^s/100$: this is possible by the bound (3.12).

Since the three events corresponding to (3.13)–(3.15) each hold with probability $\geq 9/10$, all of them hold simultaneously with positive probability. So, we can and will choose a set $P_0$ satisfying all three conditions. Next, to obtain a "regularised" subset $\tilde{P} \subset P_0$, we execute the following point removal process (PRP):

(i) From each $A$-element subset of $P_0$ contained in a cube $S \in \mathcal{S}_R$, for any $R \in \mathcal{U}$, remove one point.

(ii) For all $\beta = 2^j \in \{1, \ldots, (\delta m)^{1-s/d}\}$ and from all $(k \cdot \beta^s)$-element subsets contained in a tube $r^2T$ with $w(T) = \beta/m$, remove one point.

The remaining set is denoted by $\tilde{P}$. The first observation is that $|\tilde{P} \cap R| \geq (\delta m)^s/4$ for every cube $R \in \mathcal{U}$. Indeed, by (3.14), the number of $A$-element subsets contained in some cube $S \in \mathcal{S}_R$ does not exceed $(\delta m)^s/8$ for any $R \in \mathcal{U}$, so PRP(i) deletes at most $(\delta m)^s/8$ points from $P_0$ altogether. Since $|P_0 \cap R| \geq (\delta m)^s/2$ for all $R \in \mathcal{U}$ by (3.13), the claim follows.

3.2. **Conditions (a)–(c).** Next, we will remove some further points from $\tilde{P}$ to ensure that (a)–(c) are satisfied. We already know by PRP(i) that each of the cubes $S \in \mathcal{S}_R$ of side-length $\delta^{(d-s)/d} m^{-s/d}$ contains fewer than $A$ points in $\tilde{P}$. It follows that we may remove points from $\tilde{P}$ until the following two conditions are met: the number of points remaining in each cube $R \in \mathcal{U}$ is $\geq (\delta m)^s/A$, and the pairwise distance between the remaining points is

$$5d \cdot \delta^{(d-s)/d} m^{-s/d}. \tag{3.16}$$

As a final "regularisation", we remove yet more points in order make sure that each cube $R \in \mathcal{U}$ contains the same number, say $N$, of points, and this number satisfies $N \geq (\delta m)^s/A$. The subset of $\tilde{P}$ so obtained is called $P$.

The side-length $\epsilon \sim 1/m$ of the cubes in $\mathcal{Q}$, centred at the points in $P$, is now determined by the requirement (b), saying that each cube $R \in \mathcal{U}$ should contain $(\delta/\epsilon)^s$ cubes in $\mathcal{Q}$. Since each such $R$ contains $N \geq (\delta m)^s/A$ points of $P$, and $A$ is an absolute constant, we may choose $\epsilon \sim 1/m$ so that $(\delta/\epsilon)^s = N$. 
It remains to verify that $Q$ satisfies the requirements (c) and (d). Condition (c), that an arbitrary cube of side-length $\delta^{(d-s)/d}m^{-s/d}$ intersects at most one the cubes in $Q$, follows immediately from (3.16), and the fact that $\epsilon \sim 1/m$ is far smaller than $5d \cdot \delta^{(d-s)/d}m^{-s/d}$ for large $m \in \mathbb{N}$. This also implies that the cubes in $Q$ are all disjoint.

3.3. **Condition (d).** To prove (d), we split into three cases. First, every tube $T$ of width $w(T) = 2\epsilon$ intersects fewer than $k$ cubes $Q$. Otherwise $rT$ would contain $k$ points in $P$ by the choice of $r$, and we could pick a tube $T'$ with $w(T') = 1/m$, $T = (2em)T'$, so that a $k$-element subset of $P$ is contained in $r^2T' \supset rT$ (using $r \geq 2em$). This would contradict the PRP in the case $\beta = 1$.

Second, fix any tube $T$ with $\epsilon \leq w(T) \leq \delta^{(d-s)/d}m^{-s/d}$ and locate $\beta = 2^j \in \{1, \ldots, (\delta m)^{(d-s)/d}\}$ with the property that $\beta \leq w(T)/\epsilon \leq 2\beta$. By PRP(ii), every tube $rT'$ with $w(T') = \beta/m$ can only contain $\leq k \cdot \beta^s$ points in $P$. Since $w(T) \lesssim \beta/m$, it follows that $rT$ can also contain at most $\lesssim k \cdot \beta^s$ points in $P$. Then, by the choice of $r$, the tube $T$ can only meet $\lesssim k \cdot \beta$ cubes in $Q$.

Finally, fix a tube $T$ with $\delta^{(d-s)/d}m^{-s/d} \leq w(T) \leq \delta$. Then, since the cubes $S \in S_R$, $R \in \mathcal{U}$, have side-length $\delta^{(d-s)/d}m^{-s/d} \leq w(T)$, we see that each $S$ intersecting $2T$ is contained in $r_dT$ for a suitable dimensional constant $r_d \geq 1$. Fixing any one cube $R \in \mathcal{U}$, we obtain

$$
\delta^{d-s}m^{-s} \cdot \text{card}\{S \in S_R : S \cap 2T \neq \emptyset\} \\
\leq |S| \cdot \text{card}\{S \in S_R : S \subset r_dT\} \\
\leq |r_dT \cap R| \sim \delta \cdot w(T)^{d-1}.
$$

Since $2T$ can only intersect at most $k$ cubes $R \in \mathcal{U}$, it follows that

$$
\text{card}\left\{S \in \bigcup_{R \in \mathcal{U}} S_R : S \cap 2T \neq \emptyset\right\} \lesssim k \cdot \delta^{s-d+1}m^sw(T)^{d-1} \\
= k \cdot \left(\frac{w(T)}{\delta}\right)^{d-1-s} \cdot (mw(T))^s \\
\lesssim k \cdot \left(\frac{w(T)}{\epsilon}\right)^s.
$$

Now, if $T$ intersects a cube $Q \in Q$, then $2T$ meets the cube $S$ containing the centre of $Q$. On the other hand, each cube $S$ contains only one such centre by (c), and so

$$
\text{card}\{Q \in Q : Q \cap T \neq \emptyset\} \lesssim k \left(\frac{w(T)}{\epsilon}\right)^s.
$$

Thus, the set $P$ satisfies (c) and (d), and the proof of the lemma is complete. \qed

4. **Proof of Theorem 1.2**

With Lemma 3.1 at our disposal, the proof of Theorem 1.2 is straightforward:
Proof of Theorem 1.2. Fix $s < d - 1$ and let $k = k_s$ be the corresponding constant from Lemma 3.1. We will need to construct a compact set $K \subset \mathbb{R}^d$ with $\mathcal{H}^s(K) > 0$ and satisfying the tube condition (1.1) with $t = s$. This is achieved by first defining recursively a sequence of families of closed disjoint cubes $Q_n$. We will maintain the invariant that the families $Q_n$ should always satisfy the hypotheses of Lemma 3.1. Clearly the initial collection of cubes $Q_0 := \{[0, 1]^d\}$ has this property with $\delta = 1$. Applying the lemma with $U = Q_0$, we obtain a family of cubes $Q_1 := Q_0$, which again satisfies the assumptions of the lemma. Proceeding this way, we may define a compact set $K$ by

$$K = \bigcap_{n=0}^{\infty} \bigcup_{Q \in Q_n} Q.$$ 

Given any cube $Q \in Q_n$, the following properties of the cubes in $Q_n$ and $Q_{n+1}(Q) := \{Q' \in Q_{n+1}; Q' \subset Q\}, Q \in Q_n$, are easy consequences of Lemma 3.1:

$$\sum_{Q' \in Q_{n+1}(Q)} d(Q')^s = d(Q)^s,$$

and for any ball $B$ with $d(B) \geq \ell_n$ – the common side-length of the cubes in $Q_n$ – one has

$$\sum_{Q \in Q_n, Q \cap B \neq \emptyset} d(Q)^s \leq C d(B)^s$$

for some absolute constant $C \geq 1$. The first property is precisely Lemma 3.1(b). The second property is a condition far weaker than Lemma 3.1(d), which even implies that the same bound remains valid, if the ball $B$ is replaced by a tube $T \supset B$ of width $w(T) = d(B)$. The two properties have the consequence (see for instance [Ma, §4.12]) that $0 < \mathcal{H}^s(K) < \infty$, and in fact each cube $Q \in Q_n$ has

$$\mathcal{H}^s(K \cap Q) \sim d(Q)^s \sim \ell_n^s$$

with implicit constants independent of $n$; in fact, we only need $\lesssim$ in (4.1), which follows by using the natural covers for $K \cap Q$.

Now we are prepared to prove the estimate $\mathcal{H}^s(K \cap T) \lesssim w(T)^s$ for any given tube $T \subset \mathbb{R}^d$. If $w(T) \geq 1$, there is nothing to show, so assume that $w(T) < 1$ and fix $n \in \mathbb{N}$ such that $\ell_n < w(T) \leq \ell_{n-1}$. Then, Lemma 3.1(d) tells us that $T$ meets no more than

$$\lesssim \left( \frac{w(T)}{\ell_n} \right)^s$$

cubes in $Q_n$. In particular, by (4.1), we have

$$\mathcal{H}^s(K \cap T) \leq \sum_{Q \in Q_n, Q \cap T \neq \emptyset} \mathcal{H}^s(K \cap Q) \lesssim \left( \frac{w(T)}{\ell_n} \right)^s \cdot \ell_n^s = w(T)^s.$$

This completes the proof. 

\qed
5. Open problems

- (Suggested by V. Suomala) The set \( K \) constructed for Theorem 1.2 is far from being Ahlfors \( s \)-regular. Does it have to be so?
- Is it possible to construct a set \( K \subset \mathbb{R}^d \) with \( 0 < \mathcal{H}^{d-1}(K) < \infty \) such that
  \[ \mathcal{H}^{d-1}(K \cap T) \leq C_s w(T)^s \]
  for all tubes \( T \subset \mathbb{R}^d \) and for all \( s < d-1 \) simultaneously? More specifically, what happens with \((d-1)\)-dimensional self-similar sets, which contain "irrationality" in the rotational components of the generating similitudes?

References

[Ca] A. CARBERY: Large sets with limited tube occupancy, J. London Math. Soc. (2) 79 (2009), pp. 529–543

[CSV] A. CARBERY, F. SORIA, A. VARGAS: Localisation and weighted inequalities for spherical Fourier means, J. Anal. Math. 103, Issue 1 (2007), pp. 133–156

[KPS] J. KOMLÓS, J. PINTZ AND E. SZEMERÉDI: A lower bound for Heilbronn’s problem, J. London Math. Soc. (2) 25 (1982), pp. 13–24

[Ma] P. MATTILA: Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability, Cambridge University Press, 1995

[SS] P. SHMERKIN AND V. SUOMALA: Sets which are not tube null and intersection properties of random measures, preprint (2012), arXiv:1204.5883

School of Mathematics, University of Edinburgh

E-mail address: tuomas.orponen@helsinki.fi