A case of entanglement generation between causally disconnected atoms

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We analyze the entanglement generated in a finite time between a pair of space-like separated atoms, one of which emits a photon. As we show to order $e^2$, the origin of entanglement can be traced back to the uncertainty about which one of the atoms emitted the photon. We check this by comparing the time behaviors of the emission processes allowed by energy conservation vs. those forbidden by the same reason. No physical signal propagates between the atoms in the processes considered, however an effective light cone separating non-entangled from entangled regions in space-time emerges from our calculations.

Keywords: Entanglement; non-locality; non-signaling.

Quantum Mechanics may predict correlations between space-like separated systems. These may violate Bell inequalities too [1, 2], in which case they can not be explained by strategies arranged in the past [3]. In this work we show the generation of quantum correlations between two space-like separated parties which do not exchange any signal or any other physical interaction. We will consider a pair of neutral two level atoms $A$ and $B$ separated by a distance $L$ [4, 5, 6, 7, 8, 9]. We study perturbatively, to order $\alpha$, their local interactions with the electromagnetic field during a finite time $T$, and compute the correlations in the final state of the atoms [10, 11] when a lone photon is produced during that time. The bi-atom state shows a finite concurrence that we compute in terms of $L$ and $T$. Our results show the emergence of an effective non-signaling, which is remarkable in a case like this, where nothing at all is exchanged between the atoms. On the one side, there is no phenomenon to whom trace back the change of behavior of the concurrence at $L = cT$. On the other, $T$ is the interaction duration, not the time employed by any propagating signal.

We begin considering that the field is initially in the vacuum state, including in the final state the cases with 0, 1 and 2 photons, to analyze perturbatively the amplitudes to order $\alpha$. We assume that the wavelengths relevant in the interaction with the atoms, and the separation between them, are much longer than the atomic dimensions. The dipole approximation, appropriate to these conditions, permits the splitting of the system Hamiltonian into two parts $H = H_0 + H_I$ that are separately gauge invariant. The first part is the Hamiltonian in the absence of interactions other than the potentials that keep $A$ and $B$ stable, $H_0 = H_A + H_B + H_{\text{field}}$. The second contains all the interaction of the atoms with the field $H_I = -\frac{1}{\epsilon_0} \sum_{N=A,B} d_N(x_N, t) D(x_N, t)$, where $D$ is the electric displacement field, and $d_N = \sum_i e \int d^3x_i \langle E_N | (x_i - x_N) | G_N \rangle$ is the electric dipole moment of atom $N$, that we will take as real and of equal magnitude for both atoms.

In what follows we choose a system given initially by the product state $|\psi\rangle_0 = |EG\rangle \cdot |0\rangle$, in which atom $A$ is in the excited state $|E\rangle$, atom $B$ in the ground state $|G\rangle$, and the field in the vacuum state $|0\rangle$. The system then evolves under the effect of the interaction during a lapse of time $T$ into a state that, to order $\alpha$, can be given in the interaction picture as

\begin{align}
|\text{atom}_1, \text{atom}_2, \text{field}|_T &= \left( (1 + a) |EG\rangle + b |GE\rangle \right) |0\rangle \\
+ (u |GG\rangle + v |EE\rangle) |1\rangle + (f |EG\rangle + g |GE\rangle) |2\rangle
\end{align}

(1)
where
\[ a = -\frac{1}{2} (0|T(S_A S_A + S_B S_B)|0), \quad b = -(0|T(S_B^+ S_A^-)|0) \]
\[ u = -i \langle 1 | S_A^- | 0 \rangle, \quad v = -i \langle 1 | S_B^+ | 0 \rangle \]
\[ f = -\frac{1}{2} (2|T(S_A S_A + S_B S_B)|0), \quad g = -(2|T(S_B^+ S_A^-)|0) \]

and \(|n\rangle, n = 0, 1, 2\) is a shorthand for the state of \(n\) photons with definite momenta and polarizations, i.e., \(|1\rangle = |k, \epsilon\rangle\), etc. Notice that among all the terms that contribute to the final state (1) only \(b\) corresponds to interaction between both atoms. This would change at higher order in \(\alpha\). Here, \(a\) describes intra-atomic radiative corrections, \(u\) and \(v\) single photon emission by one atom, and \(g\) by both atoms, while \(f\) corresponds to two photon emission by a single atom.

Finally, in the dipole approximation the actions \(\hbar S_N^\pm\) in (2) reduce to
\[ S_N^\pm = -\frac{1}{\hbar} \int_0^T dt e^{i\Omega t} d_N E(x_N, t) \]  

where \(\Omega = \omega_E - \omega_G\) is the transition frequency, and we are neglecting atomic recoil. This depends on the atomic properties \(\Omega\) and \(d\), and on the interaction time \(T\). In our calculations we will take \((\Omega|d|/\hbar c) = 5 \times 10^{-3}\), which is of the same order as the \(1s \rightarrow 2p\) transition in the hydrogen atom, consider \(\Omega T \gtrsim 1\), and analyze the cases \((L/cT) \approx 1\) around the light cone. The effective coupling, given by the ratio \((|d|/\hbar L) \approx 10^{-3}\) here, could be larger if \(\Omega T < 1\) entering into the Zeno region (incidentally, the only atom atom interaction \(|b| \propto T^4\) for very small \(T\) as shown in Ref. [12] not to \(T^2\) as is sometimes stated).

Given a definite field state \(|n\rangle\) the pair of atoms is in a pure two qubits state as shown in (1). We will denote these states by \(|A, B, n\rangle\), \(\rho^{(n)}_{AB} = |A, B, n\rangle \langle A, B, n|\), and \(\rho_A^{(n)} = T_{TB} \rho^{(n)}_{AB}\) in the following, and will compute the entropy of entanglement \(S^{(n)}\) [13] and the concurrence \(C^{(n)}\) [14] for them. Our computations will be done for the illustrative case where both dipoles are parallel and orthogonal to the line joining \(A\) and \(B\). This geometrical configuration would correspond to an experimental set up in which the dipoles are induced by suitable external fields.

We first consider that the field state is not detected, that is, we trace over the field degrees of freedom. Then the atomic state is represented by the following density matrix (in the basis \(|EE\rangle, |EG\rangle, |GE\rangle, |GG\rangle\)):
\[ \rho_{AB} = \begin{pmatrix} |v|^2 & 0 & 0 & vu^* \\ 0 & |1 + a|^2 + |f|^2 & (1 + a)b^* + fg^* & 0 \\ 0 & b(1 + a)^* + f^*g & |b|^2 + |g|^2 & 0 \\ v^*u & 0 & 0 & |u|^2 \end{pmatrix} N^{-1} \]  

where \(N = |1 + a|^2 + |b|^2 + |u|^2 + |v|^2 + |f|^2 + |g|^2\). It can be shown that the concurrence associated to this density always vanishes except for a bounded range of small values of \(x = L/cT\). In the inset in Fig. 1 we show the concurrence in this region for several values of \(z = \Omega L/c\). It grows asymptotically as \(x \rightarrow 0\), i.e. when \(T \rightarrow \infty\). Out of this region \(\rho_{AB}\) is a separable state with no quantum correlations, either inside or outside the light cone. The mutual information \(I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})\), which measures the total correlations between both parties, is completely classical in this case. We show this quantity in Fig. 1 for different values of \(z\). In Ref. [15] we showed how the concurrence becomes finite (and the correlations quantum) if specific field states, with \(n = 0, 1\) or 2, are considered.

In what follows we shall focus on the \(n = 1\) case where the atoms excite one photon from the vacuum, jumping to the state \(u|GG\rangle + v|EE\rangle/c_1\), (with \(c_1 = \sqrt{|u|^2 + |v|^2}\)), during the time interval \(T\). The concurrence is
\[ C^{(1)} = 2|f|/c_1^2 \]
which depends explicitly on $L$ comes $\delta$

The inset shows the finite concurrences that are possible only for small values of $x$.

where $l = vu^* = Tr_1 (1 | S^+_B | 0 \rangle \langle 1 | S^-_A | 0 \rangle^* = \langle 0 | S^+_A S^+_B | 0 \rangle$. So, even if this case only describes independent local phenomena attached to the emission of one photon by either atom $A$ or $B$, the concurrence comes from the tangling between the amplitudes $u$ and $v$ which have different loci. The state of the photon emitted by $A$ and the state of $A$ are correlated in the same way as the state of the photon emitted by $B$ with the state of $B$ are. These independent field-atom correlations are transferred to atom-atom correlations when we trace out a photon line with different ends, $A$ and $B$, when computing $vu^*$. In fact, while $|u|^2$ and $|v|^2$ are independent of the distance $L$ between the atoms,

$$l = -\frac{cd_A^d_B}{\hbar \epsilon_0} \{ (\delta_{ij} - \hat{L}_i \hat{L}_j) M''(L) + (\delta_{ij} + \hat{L}_i \hat{L}_j) \frac{M'(L)}{L} \}$$

where

$$M(L) = \int_0^\infty dk \frac{\sin k L}{L} \delta^T(\Omega + ck) \delta^T(\Omega - ck)$$

which depends explicitly on $L$. Above we used $\delta^T(\omega) = \sin(\omega T/2)/\pi \omega$, which becomes $\delta(\omega)$ in the limit $T \to \infty$. In Fig. 2 we represent $C^{(1)}$ in front of $x = L/cT$ for some values of $z = \Omega L/c$. As the Figure shows, there may be a significative amount of concurrence for all finite $x$, indicating that $\rho^{(1)}$ is an entangled state inside and outside the light cone. The peak at $x = 1$ comes from the term with phase $k(L - cT)$ that can be singled out from the linear combination of phasors in the integrand of (7).

Here we have a lone photon whose source we can not tell. It might be $A$ or $B$, with the values of $l$ and $C^{(1)}$ depending on their indistinguishability. Eventually, conservation of energy will forbid the process $G \to E + \gamma$ for large interaction times. Therefore, $v$, $l$ and $C^{(1)}$ will vanish as $T$ grows to infinity, as can be deduced from the vanishing of $\delta^T(\Omega + ck)$ for $T \to \infty$.

The entropy of entanglement gives an alternative description of the situation. Its computation requires tracing over one of the parts $A$ or $B$, so no information is left in $S^{(1)}$ about $L$, but it still gives information about the relative contribution of both participating states $|EE\rangle$ and $|GG\rangle$ to the final state. In terms of $\eta_1 = |v|^2/c_1^2 \in (0, 1)$, we have

$$S^{(1)} = -(1 - \eta_1) \log(1 - \eta_1) - \eta_1 \log \eta_1$$
FIG. 2: Concurrence for one photon final state as a function of \( x = L/cT \) for three values of \( z = \Omega L/c \) when the initial atomic state was \( |EG\rangle \).

Would not be for the difference between \( \Omega^+ck \) and \( \Omega^-ck \), \( v \) should be equal to \( u \), then \( \eta_1 = 0.5 \), and \( S^{(1)} \) would attain its maximum value. Not only this is not the case but, as said above, \( v \) will vanish with time and only \( |GG\rangle \) will be in the final asymptotic state. Notice the result, indistinguishability was swept away because for large \( T \) we know which atom (\( A \)) emitted the photon. Therefore, the entropy will eventually vanish for large interaction times. This is not the case if the initial atomic state is \( |EE\rangle \) or \( |GG\rangle \). In the first case, the final one photon atomic state would be \( (u_A |GE\rangle + u_B |EG\rangle)/c_1' \), where \( u_A \) is the same function of \( x_A \) as \( u_B \) is of \( x_B \), and \( c_1' = \sqrt{2} |u|^2 \). Now, indistinguishability

FIG. 3: Concurrence for one photon final state if \( |EE\rangle \) is the initial state as a function of \( x = L/cT \) for three values of \( z = \Omega L/c \). The values of \( C \) for \( x > 1 \) are of the same order as those displayed in Fig. 2.
persists for large $T$ and so does entropy and concurrence. In particular, entropy attain its maximum value, as commented above. The $L$-dependent concurrence is $C^{(1)} = 2|l'/(c'_1)|^2$, where $l' = \langle 0| S^+_A S^-_B | 0 \rangle$. We represent it in Fig. 3. In the second case, the atoms would be in a state given by $(v_A | E G \rangle + v_B | G E \rangle)/c'_1$, being $c'_1 = \sqrt{2|v|^2}$. The difference is that now both processes are eventually forbidden by energy conservation at large $T$. Entropy, which achieve again its maximum value, is not able to detect this difference because both terms contribute the same (a vanishing amplitude) to the state. The concurrence is now $C^{(1)} = 2|l''/(c''_1)|^2$ with $l'' = \langle 0| S^+_A S^-_B | 0 \rangle$. Indistinguishability, represented by $l''$, persists for large $T$, but for a physical situation whose probability vanishes. Finally, in Fig. 4 we have represented the concurrence for the case where the initial state was $| E E \rangle$ in terms of the inter-atomic distance for three fixed values of time. What we obtain is a shift of the concurrence features to longer $L$ as $T$ grows (so that they appear at the same $(L/cT)$), in such a way that, even if $T$ is just the duration of the interaction, it plays the role of propagation time for the generated correlations. They are negligible small for large $L$, peak at the “light cone” but, on the other hand grow, as we would expect, for larger interaction times.

![FIG. 4: Concurrence for one photon final state if $| E E \rangle$ is the initial state as a function of $z = \Omega L/c$ for three representative values of the time $\Omega T = 8, 10, 12$ with peaks at $z = 8, 10$ and 12 respectively.](image)

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