On numerically pluricanonical cyclic coverings

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Abstract. We investigate some properties of cyclic coverings \( f: Y \to X \) (where \( X \) is a complex surface of general type) branched along smooth curves \( B \subset X \) that are numerically equivalent to a multiple of the canonical class of \( X \). Our main results concern coverings of surfaces of general type with \( p_g = 0 \) and Miyaoka–Yau surfaces. In particular, such coverings provide new examples of multi-component moduli spaces of surfaces with given Chern numbers and new examples of surfaces that are not deformation equivalent to their complex conjugates.

Keywords: numerically pluricanonical cyclic coverings of surfaces, irreducible components of moduli spaces of surfaces.

Introduction

This paper is devoted to the investigation of some properties of cyclic coverings of algebraic surfaces. We recall that there are two equivalent approaches to the definition and study of branched coverings. One of them is based on the Grauert–Remmert–Stein theorem, which asserts that a covering is uniquely determined by its unramified part. In particular, if the base of a branched covering is non-singular, then the branch locus is either the empty set or a divisor, and each unbranched finite covering of the complement of the divisor extends uniquely to a branched covering with a normal covering variety. The other approach, which is also traditional, is preferred in this paper. It uses the canonical equivalence between the finite branched coverings of a given non-singular (or normal) variety \( X \) and the finite extensions of its rational function field \( \mathbb{C}(X) \). A branched covering is called a Galois covering if this extension of fields is a Galois extension. A covering is said to be cyclic if its Galois group is cyclic.

Thus, given a finite cyclic covering \( Y \to X \), one can speak of the action of a finite cyclic group \( G \) on \( Y \) and identify \( X \) with the quotient variety \( X = Y/G \). However, we stress that when speaking of cyclic Galois groups and cyclic coverings, we do not fix an isomorphism between the Galois group and a group of permutations of a given finite set. In particular, Galois coverings \( f_1: Y_1 \to X \) and \( f_2: Y_2 \to X \) are

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\]

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said to be *isomorphic* if there is an isomorphism \( \varphi: Y_1 \rightarrow Y_2 \) such that \( f_2 \circ \varphi = f_1 \) and \( \varphi \) transforms Galois automorphisms into Galois automorphisms.

The cyclic coverings studied in this paper are rather special. Namely, their branch locus \( B \subseteq X \) is non-empty and we forbid the Galois group action to have points in \( Y \) whose stabilizer is a non-trivial proper subgroup of the Galois group. We call such coverings *totally ramified*. (Note that since we are considering only coverings over smooth varieties \( X \), the above restrictions also forbid the Galois group action to have isolated fixed points in \( Y \).)

Applying the assumption of total ramification at the points of the ramification divisor \( R \subseteq Y \), we obtain that \( f^*(B) = dR \), where \( d = \deg f \). Furthermore, a cyclic covering \( f: Y \rightarrow X \) of degree \( d \) is totally ramified (over \( X \)) if and only if \( f^*(B) = dR \) and \( f \) is unramified over \( X \setminus B \). We note that if a surface \( X_1 \) is birationally equivalent to \( X \), then the cyclic covering \( f_1: Y_1 \rightarrow X_1 \) induced by a totally ramified cyclic covering \( f: Y \rightarrow X \) need not be totally ramified over \( X_1 \) in the case when the branch curve \( B \subseteq X \) of \( f \) has singular points.

By a *numerically multi-canonical* cyclic covering we understand a totally ramified cyclic covering whose branch curve is non-singular and numerically equivalent to a multiple of the canonical class.

The main results of the paper can be divided into three groups.

First, we consider the moduli space of surfaces with a fixed square of the canonical class \( K_Y^2 \) and arithmetic genus \( p_g(Y) \) assuming that it contains surfaces given by \( d \)-sheeted totally ramified numerically multi-canonical cyclic coverings \( f: Y \rightarrow X \) of a surface \( X \) of general type. We give a lower bound for the number of connected components of such a moduli space in terms of the number of elements of the torsion group \( \text{Tor} \, H^2(X,\mathbb{Z}) \) (see Theorem 1, Proposition 3, Remark 4 and Corollaries 3, 4).

Second, we investigate the degree of the canonical map \( \varphi_{K_Y}: Y \rightarrow \mathbb{P}^{p_g(Y) - 1} \) for surfaces \( Y \) given by two-sheeted totally ramified numerically multi-canonical cyclic coverings \( f: Y \rightarrow X \) of a surface \( X \) of general type with \( p_g(X) = 0 \) (see Theorem 3, Corollary 1 and Propositions 4–6).

Third, we show that if a Miyaoka–Yau surface \( X \) (that is, a surface of general type with \( c_1^2 = 3c_2 \)) has no anti-holomorphic automorphisms (for example, this holds for all fake projective planes [1]) and if a surface \( Z \) is deformation equivalent to a surface \( Y \) given by a totally ramified numerically multi-canonical cyclic covering \( f: Y \rightarrow X \), then \( Z \) also has no anti-holomorphic automorphisms (see Theorem 7). We note that besides the well-known examples (see [1]–[4]) of pairs of complex surfaces \( (Z, Z') \) that are orientation-preserving diffeomorphic but not deformation equivalent, the surfaces \( Y \) occurring in Theorem 7 and their complex conjugates \( \overline{Y} \) give infinite series of new examples of diffeomorphic but not deformation equivalent surfaces. We also prove that those connected components \( M \) of the moduli space that contain surfaces \( Y \) given by two-sheeted totally ramified numerically multi-canonical cyclic coverings \( f: Y \rightarrow X \) of a Miyaoka–Yau surface \( X \) branched along a curve \( B \) numerically equivalent to \( 2mK_X \) are irreducible varieties of complex dimension \( \dim M = m(2m - 1)K_X^2 + p_g(X) \) and Kodaira dimension \( \kappa(M) = -\infty \). Moreover, if the irregularity \( q(X) \) is equal to zero, then \( M \) is an unirational variety (see Theorem 6 and also Remark 6).
As far as we know, there are very few examples in the literature of complex surfaces of general type with a non-trivial action of a group that is deformed simultaneously with any deformation of the complex structure (compare [5]). As a corollary of the proof of Theorem 6, we construct infinite series of such surfaces with an action of \( \mathbb{Z}/2\mathbb{Z} \) (see Remark 7).

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**§ 1. Totally ramified cyclic coverings: fundamental groups and classification**

We recall that every continuous map \( f : V \to W \) of path connected topological spaces determines a homomorphism \( f_* : \pi_1(V, q) \to \pi_1(W, p) \) of fundamental groups for any pair of points \( q \in V \) and \( p \in W \) with \( q \in f^{-1}(p) \). If such pairs \((q', p')\) and \((q'', p'')\) of base points are connected by paths \( h \) in \( V \) and \( f(h) \) in \( W \), then these homomorphisms are conjugate by the change-of-basepoint homomorphisms \( \beta_h \) and \( \beta_{f(h)} \) in the sense that \( f''_* \circ \beta_h = \beta_{f(h)} \circ f'_* \). Note that all these homomorphisms \( f_* \) induce the same homomorphism

\[
H_1(V, \mathbb{Z}) \cong \pi_1(V, q)/[\pi_1(V, q), \pi_1(V, q)] \to H_1(W, \mathbb{Z})
\]

\[
\cong \pi_1(W, p)/[\pi_1(W, p), \pi_1(W, p)]
\]

of first homology groups. When the choice of the base points \( p \in W \) and \( q \in f^{-1}(p) \) in the fundamental groups \( \pi_1(W, p) \) and \( \pi_1(V, q) \) is irrelevant to the proofs, we do not indicate them and denote the fundamental groups by \( \pi_1(W) \) and \( \pi_1(V) \) respectively.

**Proposition 1.** Let \( X \) be an irreducible smooth projective surface, \( B \subset X \) an irreducible reduced smooth curve divisible by \( d \) as an element of \( \text{Pic}(X) \), and \( f : Y \to X \) a \( d \)-sheeted totally ramified cyclic covering branched along \( B \). If \((B^2)_X > 0\), then the homomorphism \( f_* : \pi_1(Y) \to \pi_1(X) \) induced by \( f \) is an isomorphism.

**Lemma 1.** Let \( B \) be an irreducible reduced smooth curve on an irreducible smooth projective surface \( X \). If \((B^2)_X > 0\), then the kernel \( K \) of the epimorphism \( \pi_1(X \setminus B) \to \pi_1(X) \) is a cyclic group and is contained in the centre of \( \pi_1(X \setminus B) \).

**Proof.** Let \( N \subset X \) be the closure of a tubular neighbourhood of \( B \). Then \( N \) possesses the structure of a locally trivial \( C^\infty \)-fibration over \( B \) with fibre \( D = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \). Deleting one fibre (say, over a point \( b \in B \)) from this fibration, we get a trivial fibration \( N_0 \simeq D \times B_0 \) over \( B_0 = B \setminus \{ b \} \). Consider the section \( B_1 = B_0 \times \{ z = 1 \} \) of the latter fibration and choose a point \( p \in B_1 \).

Let \( \gamma \in G = \pi_1(X \setminus B, p) \) be the element represented by the loop \( \partial D \), where \( \partial D \) is the boundary of the fibre through \( p \). We have an exact sequence of groups

\[
1 \to K \to G \to \pi_1(X, p) \to 1,
\]

where \( K \) is the normal closure of \( \gamma \) in \( G \).

By Nori’s version of the weak Lefschetz theorem (see [6], Proposition 3.27), \( K \) is a finitely generated Abelian group and its centralizer \( C(K) \) in \( G \) is a subgroup of finite index.
On the other hand, we have $\pi_1(N_0 \setminus B_0, p) = \pi_1(B_1, p) \times \{\gamma^n | n \in \mathbb{Z}\}$. Again by Nori’s version of the weak Lefschetz theorem (see [6], Proposition 3.26), the embedding $i : B \hookrightarrow X$ induces an epimorphism $i_* : \pi_1(B) \to \pi_1(X)$. Therefore the elements of $\pi_1(B_1, p)$ generate the group $\pi_1(X, p)$, and the embedding $N_0 \setminus B_0 \hookrightarrow X \setminus B$ induces an epimorphism

$$\pi_1(N_0 \setminus B_0, p) = \pi_1(B_1, p) \times \{\gamma^n | n \in \mathbb{Z}\} \to \pi_1(X \setminus B, p).$$

Hence the normal closure $K$ of the element $\gamma$ in $\pi_1(X \setminus B, p)$ is just the cyclic group generated by $\gamma$. Moreover, $K$ is contained in the centre of $\pi_1(X \setminus B, p)$. □

Proof of Proposition 1. The covering $f : Y \to X$ induces an embedding of fundamental groups $f_* : \pi_1(Y \setminus f^{-1}(B)) \hookrightarrow \pi_1(X \setminus B)$ such that the group $\overline{G} = f_*(\pi_1(Y \setminus f^{-1}(B)))$ is a subgroup of index $d = \deg f$ in $G = \pi_1(X \setminus B)$. As in the proof of Lemma 1, let $\gamma$ be a generator of $K$. Since $f^*(B) = dR$, we have $\gamma^i \notin \overline{G}$ for $1 \leq i \leq d-1$ and $\gamma^d \in \overline{G}$. Therefore $G = \overline{G} \cup \gamma \overline{G} \cup \cdots \cup \gamma^{d-1} \overline{G}$. On the other hand, $\pi_1(Y) \simeq \overline{G}/\langle \gamma^d \rangle$ by Lemma 1. The exact sequence (1) gives rise to an exact sequence

$$1 \to K/\langle \gamma^d \rangle \to G/\langle \gamma^d \rangle \to \pi_1(X) \to 1. \tag{2}$$

The group $\overline{G}/\langle \gamma^d \rangle \simeq \pi_1(Y)$ is naturally embedded in $G/\langle \gamma^d \rangle$. Hence Proposition 1 follows from the exact sequence (2) and the equality

$$G/\langle \gamma^d \rangle = \pi_1(Y) \cup \gamma \pi_1(Y) \cup \cdots \cup \gamma^{d-1} \pi_1(Y).$$

Let $C_1$ and $C_2$ be divisors on a surface $X$. We write $C_1 \sim C_2$ (resp. $C_1 \equiv C_2$) if $C_1$ and $C_2$ are linearly equivalent (resp. numerically equivalent). Let $\text{Tor}_d \text{Pic}(X)$ be the subgroup of the Picard group $\text{Pic}(X)$ consisting of the elements whose order divides $d$. The order of the group $\text{Tor}_d \text{Pic}(X)$ is denoted by $N_{X,d} = |\text{Tor}_d \text{Pic}(X)|$. Given a divisor $B$ whose divisor class is divisible by $d \in \mathbb{N}$, we shall write $(B)_d \subset \text{Pic}(X)$ for the set of divisor classes $\beta$ such that $B \sim d\beta$. Clearly, $(B)_d$ is a principal homogeneous space over $\text{Tor}_d \text{Pic}(X)$.

The following lemma is well known, but we were unable to find an appropriate reference. Therefore we include a proof.

Lemma 2. Let $B \subset X$ be an irreducible reduced curve. If $B$ is divisible by $d$ as an element of $\text{Pic}(X)$ (that is, $B \sim dC$ for some divisor $C$), then there is a natural bijection between $(B)_d$ and the set of isomorphism classes of $d$-sheeted totally ramified cyclic coverings $f_i : Y_i \to X$ branched along $B$. If $[C_i] \in (B)_d$ is the divisor class corresponding to $f_i$ under this bijection, then $[R_i] = f_i^*[C_i]$ and $d[C_i] = [B]$.

If $B$ is a smooth curve, then each of these $Y_i$ is a smooth surface.

Proof. By definition, a finite morphism $f : Y \to X$ is a branched $d$-sheeted cyclic covering of a smooth surface $X$ if $Y$ is a normal surface and $f^* : \mathbb{C}(X) \hookrightarrow \mathbb{C}(Y)$ is a finite Galois extension of fields with Galois group $\text{Gal}(Y/X) \simeq \mathbb{Z}/d\mathbb{Z}$. Let $h^*$ be a generator of $\text{Gal}(Y/X)$. By Hilbert’s theorem 90, the field $\mathbb{C}(Y)$ regarded as a vector space over $\mathbb{C}(X)$ admits a basis $w_0 = 1, w_1, \ldots, w_{d-1}$ over $\mathbb{C}(X)$ such that $h^*(w_i) = \mu^i w_i$, where $\mu$ is a primitive $d$th root of unity. Hence we can put $w = w_1$
and conclude that each branched $d$-sheeted cyclic covering may be regarded as an extension of fields $C(Y) = C(X)(w)$ with $h^*(w) = \mu w$, where $w^d = g \in C(X)$ for some function $g$. Clearly, the converse also holds.

The automorphism $h^*$ determines an automorphism $h: Y \to Y$. Moreover, the branch locus of $f$ consists of those irreducible curves $D_i$ that occur in the principal divisor $(g) = \sum_{i=0}^n a_iD_i \in \text{Div}(X)$, $a_i \in \mathbb{Z}$, with coefficients $a_i \not\equiv 0 \mod d$. Furthermore, $\text{GCD}(a_i, d)$ is equal to the number of points in the inverse image of a generic point of $D_i$. Therefore in our case there is a unique curve $D_i$ (say, $D_0$, and it must be equal to the branch curve $B$) for which $a_0 \not\equiv 0 \mod d$ and, moreover, $a_0$ and $d$ are coprime. Thus we can find an integer $b$ coprime to $d$ such that the divisor $(g^b)$ is equal to $B-dC$ for some $C \in \text{Div}(X)$. Therefore, choosing a function $\tilde{g} \in C(X)$ and replacing $w$ by $w^b \tilde{g}$, $g$ by $\tilde{g}^d g^b$ and $\mu$ by $\mu^b$, we get a representation $C(Y) = C(X)(w)$, where $h^*(w) = \mu w$, $w^d = g \in C(X)$ and the divisor $(g)$ is equal to $B-dC'$ for some (arbitrary) fixed divisor $C'$ linearly equivalent to $C$. This yields a natural bijection between $(B)_d$ and the set of isomorphism classes of $d$-sheeted cyclic coverings of $X$ branched along $B$.

If $R$ is the ramification locus of $f$ with $R = f^{-1}(B)$ and $f^*(B) = dR$ for an irreducible curve $B$, then $(w^d) = (f^*(g)) = dR - df^*(C)$. Therefore $(w) = R - f^*(C)$. □

Remark 1. To enumerate the branched coverings as in Lemma 2, it is often convenient (as in the proof of Lemma 2) to choose a divisor $C$ with $B \sim dC$ for a ‘base point’ and then use the bijection that associates with $\alpha_i \in \text{Tor}_d \text{Pic} X$ the $d$-sheeted cyclic coverings given by the equations $w^d = g_i \in C(X)$, where $(g_i) = B-dC_i$ with $C_i \sim C + \alpha_i$.

Remark 2. Lemma 2 and Remark 1 also hold in the case when the reduced curve $B$ is reducible and $d = 2$.

Let $f: Y \to X$ be a $d$-sheeted totally ramified cyclic covering branched along a smooth irreducible curve $B \subset X$ with $(B^2)_X > 0$. By Proposition 1, $f$ induces an isomorphism between the fundamental groups $\pi_1(Y)$ and $\pi_1(X)$. Therefore it also induces an isomorphism $f_*$ between $H_1(Y, \mathbb{Z})$ and $H_1(X, \mathbb{Z})$ and, in particular, between their torsion subgroups $\text{Tor} H_1(Y, \mathbb{Z})$ and $\text{Tor} H_1(X, \mathbb{Z})$. By the universal coefficient theorem (which guarantees that $\text{Tor} H_1(V, \mathbb{Z}) = \text{Ext}(H_1(V, \mathbb{Z}), \mathbb{Z}) = \text{Tor} H^2(V, \mathbb{Z})$ for every compact manifold $V$) and its functoriality, it follows that $f^*: \text{Tor} H^2(X, \mathbb{Z}) \to \text{Tor} H^2(Y, \mathbb{Z})$ is an isomorphism as well.

In fact, an even stronger statement holds under our assumptions. Indeed, the exponential exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0$$

induces a short exact sequence

$$0 \to \text{Tor} \text{Pic}^0(Y) \to \text{Tor} \text{Pic}(Y) \xrightarrow{\delta} \text{Tor} H^2(Y, \mathbb{Z}) \to 0,$$

where $\text{Pic}^0(Y)$ is the connected component of $0$ in $\text{Pic}(Y)$ and $\delta$ is the connecting homomorphism (the first Chern class). Hence, using the functorial isomorphism
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\[ \text{Pic}^0(V) = H^{0,1}(V)/(H^1(V, \mathbb{Z}))^{0,1} \]
and the five lemma for the diagram formed by the above short exact sequence and its copy written for \( X \), we easily prove that \( f^*: \text{Tor} \text{Pic}(X) \to \text{Tor} \text{Pic}(Y) \) is also an isomorphism.

In what follows we use only the group \( \text{Tor} H^2(V, \mathbb{Z}) \) and write it as \( \text{Tor} \). Moreover, we shall always write \( \delta \) for the Chern homomorphism \( \delta: \text{Pic}(V) \to H^2(V, \mathbb{Z}) \).

§ 2. Numerically multi-canonical cyclic coverings

In this section we fix an integer \( d \geq 2 \) and a smooth irreducible curve \( B \equiv dmK_X \) on a non-singular surface \( X \). A \( d \)-sheeted totally ramified cyclic covering \( f: Y \to X \) branched along \( B \) and given by a divisor class \( C + \alpha, \alpha \in \text{Tor}_d \text{Pic}(X) \) (see Remark 1), is said to be \((d, m)\)-canonical if \( C \sim mK_X \) and \( d\alpha = 0 \), purely \((d, m)\)-canonical if \( C \sim mK_X \) and \( \alpha = 0 \), and numerically \((d, m)\)-canonical if \( C \equiv mK_X \). Note that if \( f \) is \((d, m)\)-canonical, then \( B \in |dmK_X| \).

Given a divisor \( D \) on a surface \( X \), we write \( \varphi_D: X \to \mathbb{P}^\dim |D| \) for the rational map determined by the complete linear system \( |D| \).

Bombieri’s famous theorem \([7]\) states that if \( m \geq 5 \) and \( X \) is a smooth minimal (that is, without \((-1)\)-curves) projective surface of general type, then the \( m \)-canonical map

\[ \varphi_{mK_X}: X \to \mathbb{P}^{P_m - 1} \]

is a birational morphism onto its image, where \( P_m = \dim H^0(X, \mathcal{O}_X(mK_X)) \). Note that the proof of this theorem uses only Ramanujam’s vanishing theorem and numerical properties of the canonical class \( K_X \). Therefore Bombieri’s theorem holds not only for \( m \)-canonical maps, but also for the maps

\[ \varphi_{D_m}: X \to \mathbb{P}^{P_m - 1}, \]

where \( D_m \) is a divisor numerically equivalent to \( mK_X \). In particular, if \( dm \geq 5 \) and \( D \equiv dmK_X \), then a generic curve \( B \in |D| \) is non-singular and irreducible by Bertini’s theorem.

**Proposition 2.** Let \( f: Y \to X \) be a totally ramified numerically \((d, m)\)-canonical cyclic covering of a surface \( X \). Then \( Y \) has the following invariants:

\[ p_a(Y) = dp_a(X) + \frac{d(d - 1)m((2d - 1)m + 3)}{12}K_X^2, \tag{3} \]

\[ K_Y^2 = d(dm - m + 1)^2K_X^2, \tag{4} \]

and \( q(Y) = q(X) \), where \( p_a = p_g - q + 1 \) is the arithmetic genus of a surface.

**Proof.** We have

\[ dK_Y \sim f^*(dK_X + (d - 1)B) \equiv f^*(d(dm - m + 1)K_X). \]

Therefore \( K_Y^2 = d(dm - m + 1)^2K_X^2 \). It follows from Proposition 1 that \( q(Y) = q(X) \). We denote the Euler characteristic of a variety \( V \) by \( e(V) \). By Noether’s formula we have

\[ K_X^2 + e(X) = 12p_a(X), \]
and the adjunction formula yields that

\[-e(B) = 2g(B) - 2 = (B, B + K_X)_X = (dmK_X, (dm + 1)K_X)_X = dm(dm + 1)K_X^2.\]

Since \( f^*(B) = dR \), we have

\[e(Y) = d(e(X) - e(B)) + e(B) = d(12p_a(X) - K_X^2 + dm(dm + 1)K_X^2) - dm(dm + 1)K_X^2 = 12dp_a(X) + d[(d - 1)(dm + 1)m - 1]K_X^2.\]

Hence,

\[e(Y) + K_Y^2 = 12dp_a(X) + d[(d - 1)(dm + 1)m - 1]K_X^2 + d(dm - m + 1)^2K_X^2 = 12dp_a(V) + d[(d - 1)(dm + 1)m - 1 + (dm - m + 1)^2]K_X^2 = 12dp_a(X) + d(d - 1)m[(2d - 1)m + 3]K_X^2\]

and, therefore,

\[p_a(Y) = dp_a(X) + \frac{d(d - 1)m[(2d - 1)m + 3]}{12}K_X^2.\]

By Proposition 2, the invariants \( K_Y^2 \) and \( p_a(Y) \) are the same for all numerically \((d, m)\)-canonical totally ramified cyclic coverings \( f: Y \to X \) with fixed \( d \) and \( m \). We denote them by \( k_{X,d,m} \) and \( p_{X,d,m} \) respectively. Let \( \mathcal{M}_{k,p} \) be the moduli space of surfaces \( Z \) with given invariants \( K_Z^2 = k \) and \( p_a(Z) = p \).

**Theorem 1.** Let \( X \) be a surface of general type. If there is an element \( \alpha \in \text{Tor}_d \text{Pic}(X) \) such that \( \delta(\alpha) \) is not divisible by \( d \) in the group \( \text{Tor}(X) \), then for every integer \( n \geq 1 \) the moduli space \( \mathcal{M}_{k,p} \) with \( k = k_{X,d,dn+1} \) and \( p = p_{X,d,dn+1} \) consists of at least two connected components.

**Proof.** Choose a divisor \( C \sim (dn+1)K_X \) and consider two totally ramified \((d, dn+1)\)-canonical cyclic coverings \( f_i: Y_i \to X, i = 1, 2 \), branched along a smooth irreducible curve \( B \in |d(dn+1)K_X| \), where \( f_1 \) is a purely \((d, dn+1)\)-canonical totally ramified cyclic covering and \( f_2 \) is the \((d, dn+1)\)-canonical totally ramified cyclic covering given by an element \( C + \alpha \), where \( \alpha \in \text{Tor}_d \text{Pic}(X) \) and \( \delta(\alpha) \) is not divisible by \( d \) in \( \text{Tor}(X) \). For \( n \in \mathbb{N} \), the existence of such a curve \( B \) follows from the inequality \( d(dn+1) \geq 6 \).

Consider the covering \( f_1 \). It is given by adding to the field \( \mathbb{C}(X) \) a function \( w_1 \) such that \( w_1^{d} = g_1 \in \mathbb{C}(X) \) and \( (g_1) = B - dC \). Since

\[R_1 \sim f_1^*(C) \sim f^*((dn+1)K_X),\]

it follows from the projection formula for canonical divisors that

\[K_{Y_1} \sim f_1^*(K_X) + (d-1)R_1 \sim f_1^*(K_X) + (d-1)f^*((dn+1)K_X) \sim d(dn-n+1)f^*(K_X).\]

In particular, \( K_{Y_1} \) is divisible by \( d \) in \( \text{Pic}(Y_1) \) and, therefore, its cohomology class is divisible by \( d \) in \( H^2(Y_1, \mathbb{Z}) \).
The covering $f_2$ is given by adding to the field $\mathbb{C}(X)$ a function $w_2$ such that $w_2^3 = g_2 \in \mathbb{C}(X)$ and $(g_2) = B - dC_2$, where $C_2 \sim (dn + 1)K_X + \alpha$. By Lemma 2 we have $R_2 \sim f_2^*(C_2) \sim f_2^*((dn + 1)K_X + \alpha)$. Hence,

$$K_{Y_2} \sim f_2^*(K_X) + (d - 1)R_2 \sim f_2^*(K_X) + (d - 1)f_2^*((dn + 1)K_X + \alpha) \sim d(dn - n + 1)f_2^*(K_X) + (d - 1)f_2^*(\alpha).$$

Since $f_2^*: \text{Tor}(X) \to \text{Tor}(Y_2)$ is an isomorphism and $\delta(\alpha) \in \text{Tor}(X)$ is not divisible by $d$, the cohomology class of $K_{Y_2}$ is not divisible by $d$ in $H^2(Y_2, \mathbb{Z})$.

In view of the different divisibility properties of the cohomology classes of their canonical divisors, the surfaces $Y_1$ and $Y_2$ cannot belong to the same connected component of the moduli space. □

**Remark 3.** Theorem 1 also holds for $n = 0$ if there is a smooth irreducible curve $B \in |dK_X|$. It follows from Theorem 5.2 in [8], Proposition 3 in [9], and Bertini’s theorem that a generic curve $B \in |dK_X|$ is smooth and irreducible when either $d \geq 5$, or $d = 4$ and $K_X^2 \geq 2$, or $d = 3$ and $K_X^2 \geq 3$, or $d = 2$ and $K_X^2 \geq 5$.

**Remark 4.** Note that if $X_1$ and $X_2$ are surfaces of general type such that $K_{X_1} = K_{X_2}$ and $p_a(X_1) = p_a(X_2)$, but $\pi_1(X_1)$ is not isomorphic to $\pi_1(X_2)$, and if $f_1: Y_1 \to X_1$ and $f_2: Y_2 \to X_2$ are totally ramified numerically $(d, m)$-canonical cyclic coverings, then Proposition 1 yields that $Y_1$ and $Y_2$ belong to different connected components of the moduli space $\mathcal{M}_{k,p}$ with $k = k_{X_1,d,m}$ and $p = p_{X_1,d,m}$.

Let $X$ be a Campedelli surface, that is, $X$ is a surface of general type with $p_g = 0$, $K_X^2 = 2$ and $\pi_1(X) \simeq (\mathbb{Z}/2\mathbb{Z})^3$. Every totally ramified numerically $(2, 1)$-canonical cyclic covering $f: Y \to X$ is $(2, 1)$-canonical since no non-zero elements of $\text{Tor}(X) \simeq (\mathbb{Z}/2\mathbb{Z})^3$ are divisible by two. By Proposition 2, totally ramified $(2, 1)$-canonical cyclic coverings $Y$ of $X$ have the following invariants: $K_Y^2 = 16$ and $p_a = 4$. We also note that, by Lemma 2, for every given non-singular curve $B \in |2K_X|$ there are exactly eight different totally ramified $(2, 1)$-canonical cyclic coverings of $X$ branched along $B$.

**Proposition 3.** There are exactly two connected components of the moduli space $\mathcal{M}_{16,4}$ that contain totally ramified $(2, 1)$-canonical cyclic coverings of Campedelli surfaces.

**Proof.** We recall that all Campedelli surfaces $X$ may be obtained as follows (see, for example, [10]). Let $\tilde{L} = L_1 \cup \cdots \cup L_7$ be a line arrangement consisting of seven lines in $\mathbb{P}^2$. We label these lines by the non-zero elements $\alpha_i = (a_{i,1}, a_{i,2}, a_{i,3}) \in (\mathbb{Z}/2\mathbb{Z})^3$ and make two assumptions. First, $\tilde{L}$ has no $r$-fold points with $r \geq 4$. Second, if $\tilde{L}$ has a triple point $p_{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}} = L_{\alpha_{i_1}} \cap L_{\alpha_{i_2}} \cap L_{\alpha_{i_3}}$, then $\alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} \neq 0$.

Consider the Galois covering $\tilde{g}: \tilde{X} \to \mathbb{P}^2$ with Galois group $\text{Gal}(\tilde{X}/\mathbb{P}^2) \simeq (\mathbb{Z}/2\mathbb{Z})^3$, that is, the covering branched along $\tilde{L}$ and given by the automorphism $\varphi: H_1(\mathbb{P}^2 \setminus \tilde{L}, \mathbb{Z}) \to G = (\mathbb{Z}/2\mathbb{Z})^3$ which is determined by the formulae $\varphi(\lambda_i) = \alpha_i$, where $\lambda_i$ is the element of $H_1(\mathbb{P}^2 \setminus \tilde{L}, \mathbb{Z})$ represented by a small circuit around the line $L_{\alpha_i}$. 
The only singular points of $\tilde{X}$ are the points lying over the triple points $p_{\alpha_1, \alpha_2, \alpha_3}$, and the resolution $X$ of the singularities of $\tilde{X}$ is a Campedelli surface. To resolve the singularities of $\tilde{X}$, it suffices to blow up the triple points of $\tilde{L}$. The composite $\sigma : \PP^2 \to \PP^2$ of blow-ups with centres at the triple points of $\tilde{L}$ induces a Galois covering $g : X \to \PP^2$, and we have (see [10])

$$2K_X \sim g^*(L),$$

where $L$ is a line in $\PP^2$. (If $\tilde{L}$ has no triple points, then $g = \tilde{g}$.)

Suppose that the Campedelli surface $X$ is determined by the arrangement of lines $\tilde{L}$ and $f$ is branched along the pre-image $B = g^{-1}(L_8)$ of a line $L_8 \not\subset \tilde{L}$. We can assume without loss of generality that the line arrangement $\mathcal{L} = \tilde{L} \cup L_8$ is generic. Then the fundamental group $\pi_1(\PP^2 \setminus \mathcal{L})$ is Abelian and, therefore, $h = g \circ f : Y \to \PP^2$ is a $(\mathbb{Z}/2\mathbb{Z})^4$-Galois covering of $\PP^2$ which is branched along $\mathcal{L}$ and is determined by an epimorphism $\psi : H_1(\PP^2 \setminus \mathcal{L}, \mathbb{Z}) \to (\mathbb{Z}/2\mathbb{Z})^4$ given by the formulae $\psi(\lambda_i) = \alpha_i = (a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4})$ for some $a_{i,4} \in \mathbb{Z}/2\mathbb{Z}$, $i = 1, \ldots, 7$, and $\psi(\lambda_8) = \alpha_8 = (0, 0, 0, 1)$, where $\lambda_8$ is the element represented by a small circuit around $L_8$. Note that $\psi$ is determined by $h$ uniquely up to automorphisms of $\mathbb{Z}/2\mathbb{Z}^4$ (that is, up to the choice of a basis in $\mathbb{Z}/2\mathbb{Z}^4$). We have $\sum_{i=1}^8 \lambda_i = 0$ in $H_1(\PP^2 \setminus \mathcal{L}, \mathbb{Z})$. Therefore the subset $M = \{ \alpha_i \in \PP^3_{\mathbb{Z}/2\mathbb{Z}} \mid i = 1, \ldots, 8 \}$ of the projective space $\PP^3 = \PP^3_{\mathbb{Z}/2\mathbb{Z}}$ over the field $\mathbb{Z}/2\mathbb{Z}$ is totally even, that is, the following condition holds: for each plane $\PP^2 \subset \PP^3$, the intersection $M \cap \PP^2$ consists of an even number of points.

**Lemma 3.** Up to the action of $\text{PGL}(4, \mathbb{Z}/2\mathbb{Z})$ there are exactly two different totally even $8$-element subsets $M$ of the projective space $\PP^3$ over the field $\mathbb{Z}/2\mathbb{Z}$. These subsets $M$ are symmetric differences of two linear subspaces: either $M = \PP^3 \Delta P$ (type I), or $M = P \Delta l$ (type II), where $P$ is a plane in $\PP^3$ and $l$ is a line transversal to $P$.

**Proof.** Type I corresponds to the case when there is a plane $P \subset \PP^3$ (say, $P$ is given by $a_4 = 0$) such that $M \cap P = \emptyset$. In this case,

$$M = \mathbb{A}^3 = \{ \tilde{\alpha} = (a_1, a_2, a_3, a_4) \in \PP^3 \mid a_4 = 1 \}.$$  

Assume that there are no planes $P$ with $M \cap P = \emptyset$. Given any plane $P_i \subset \PP^3$, we write $n_i = |M \cap P_i|$ for the number of points in the intersection $M \cap P_i$. Then each of the $n_i$ is even, $0 < n_i \leq 6$.

First of all we claim that the condition $n_i = 6$ for some $i$ (say, $n_1 = 6$) is equivalent to the existence of a line $l \subset M$ (and then $|M \cap l| = 3$). Indeed, if $n_1 = 6$, then there is a unique point $\tilde{\alpha}_0 \in P_1$ with $\tilde{\alpha}_0 \notin M$ because $|P_1| = 7$. Therefore for every line $l \subset P_1$ not passing through $\tilde{\alpha}_0$ we have $l \subset M$. Conversely, given a line $l \subset M$, we consider the pencil of planes containing $l$. It consists of three planes (say, $P_1$, $P_2$ and $P_3$). We have $n_i \geq 4$ for $i = 1, 2, 3$ and $(n_1 - 3) + (n_2 - 3) + (n_3 - 3) = 8 - 3 = 5$, that is, $n_1 + n_2 + n_3 = 14$. Up to permutations, this equation has a unique solution in even integers with $4 \leq n_i \leq 6$, namely, $n_1 = 6$ and $n_2 = n_3 = 4$.

We now claim that $M$ is a set of type II if there is a plane $P$ such that $|M \cap P| = 6$. Indeed, there is no loss of generality in assuming that $P$ is given by the equation
$a_4 = 0$ and the points $\alpha_7$ and $\alpha_8$ lying in $M \setminus P$ have coordinates $\alpha_7 = (1, 1, 1, 1)$ and $\alpha_8 = (0, 0, 0, 1)$ and thus belong to the line $l = \{a_1 - a_2 = a_2 - a_3 = 0\}$. Since $|M \cap P| = 6$ and $|P| = 7$, to prove that $M = P \triangle l$ (and hence $M$ is of type II), it suffices to show that the point $\alpha_0 = (1, 1, 1, 0) \in M$. Then there is another point $\alpha_1 = (a_1, a_2, a_3, 0) \notin M$ with $a_{1,i} = 0$ for some $i = 1, 2, 3$. Using a projective transformation that permutes coordinates and leaves $P$ fixed, we can assume without loss of generality that this point is $\alpha_1 = (0, a_{1,2}, a_{1,3}, 0)$. Then $|M \cap l| = 2$, where the line $l \subset \mathbb{P}^3$ is given by the equations $a_1 = a_4 = 0$. Hence $M$ contains only three points whose first coordinate $a_1$ vanishes: the two points lying in $M \cap l$ and the point $(0, 0, 0, 1)$. Therefore $|M \cap P_1| = 3$, where $P_1 = \{\alpha = (a_1, a_2, a_3, a_4) \in \mathbb{P}^3 \mid a_1 = 0\}$, a contradiction.

Let us show that the case when $0 < n_i \leq 4$ for all planes $P_i \subset \mathbb{P}^3$ is impossible. We first assume that $n_i = 4$ for all planes $P_i$. On one hand, the number of planes in $\mathbb{P}^3$ is equal to 15. On the other hand, the number of planes through any point $\alpha \in \mathbb{P}^3$ is equal to 7. The inequality $4 \cdot 15 \neq 8 \cdot 7$ implies that this case is impossible.

We finally assume that $0 < n_i \leq 4$ for all planes $P_i \subset \mathbb{P}^3$ and there is a plane $P_1$ such that $n_1 = |M \cap P_1| = 2$. Then there is a line $l \subset P_1$ such that $M \cap l = \emptyset$. Consider the pencil of planes passing through $l$. It consists of three planes: $P_1, P_2, P_3$. Since $n_1 + n_2 + n_3 = 8$, we have $n_2 + n_3 = 6$. Therefore we can assume that $n_2 = 2$ and $n_3 = 4$. We have $M \cap P_3 = P_3 \setminus l$ and hence every plane $P$ not containing $l$ has two common points with $P_3$ belonging to $M$. On the other hand, let $\{\alpha_1, \alpha_2\} = M \cap P_1$ and $\{\alpha_3, \alpha_4\} = M \cap P_2$. Then the plane $P_4$ passing through the points $\alpha_1, \alpha_2$ and $\alpha_3$, does not contain $l$. Therefore $n_4 \geq 5$, a contradiction. □

We now return to the proof of Proposition 3. We label the lines in a line arrangement $L$ (consisting of eight lines) by the points of a totally even set $M \subset (\mathbb{Z}/2\mathbb{Z})^4$, $L = L_M = \bigcup_{\alpha_i \in M} L_{\alpha_i}$. The line arrangement $L_M$ determines an epimorphism $\psi: H_1(\mathbb{P}^2 \setminus L_M, \mathbb{Z}) \to (\mathbb{Z}/2\mathbb{Z})^4$ by the formulae $\psi(\lambda_{\alpha_i}) = \alpha_i$. The epimorphism $\psi$ in turn induces a $(\mathbb{Z}/2\mathbb{Z})^4$-Galois covering $h: Y \to \mathbb{P}^2$ branched along $L_M$. Since $\psi$ is determined by $h$ only up to an automorphism of $(\mathbb{Z}/2\mathbb{Z})^4$, we can assume by Lemma 3 that either

$$M = \{\alpha = (a_1, a_2, a_3, a_4) \in \mathbb{P}^3 \mid a_4 = 1\}$$

(type I), or

$$M = \{(1, 0, 0, 0), (1, 1, 0, 0), (0, 1, 0, 0), (1, 0, 1, 0), (0, 0, 1, 0), (0, 1, 1, 0), (1, 1, 1, 1), (0, 0, 0, 1)\}$$

(type II). By Remark 3, to complete the proof, it suffices to note that the set of line arrangements $L_M$, where $M$ is of type I (resp. II) is an everywhere-dense Zariski-open (and hence connected) subset of $(\mathbb{P}^2)^8$. □

**Theorem 2.** Let $X$ be a surface of general type, and let $f_1: Y_1 \to X, f_2: Y_2 \to X$ be totally ramified numerically $(d, m)$-canonical cyclic coverings determined by the divisors $C_1 \sim mK_X + \alpha_1$ and $C_2 \sim mK_X + \alpha_2$ respectively, where $\alpha_1$ and $\alpha_2$ are numerically equivalent to zero and $\delta(\alpha_1) = \delta(\alpha_2)$. If $dm \geq 5$, then the surfaces $Y_1$ and $Y_2$ are deformation equivalent.
Proof. Let \( B_i \in |dC_i| \) be the branch curve of the covering \( f_i \), \( i = 1, 2 \). The covering \( f_i \) is obtained by adding to the field \( \mathbb{C}(X) \) a function \( w_i \) such that \( w_i^d = g_i \in \mathbb{C}(X) \) and the divisor \((g_i)\) is equal to \( B_i - dC_i \), where \( C_i \sim mK_X + \alpha_i \). Let \( \Delta = \delta(\alpha_1) = \delta(\alpha_2) \in \text{Tor}(X) \). We put \( \text{Pic}_\Delta(X) = \delta^{-1}(\Delta) \subset \text{Pic}(X) \).

Consider the scheme \( T_{\Delta,dm} \) parametrizing those curves \( B_t \) in \( X \) that are numerically equivalent to \( dmK_X \) and satisfy \( \delta(B_t) = dm\delta(K_X) + d\Delta \) (the scheme \( T_{\Delta,dm} \) is fibred over \( \text{Pic}_\Delta(X) \), \( \gamma_1 : T_{\Delta,dm} \to \text{Pic}_\Delta(X) \), with fibres \( \gamma_1^{-1}(\alpha_t) = \mathbb{P}(H^0(X,\mathcal{O}_X(dmK_X + d\alpha_t))) \) over the points \( \alpha_t \in \text{Pic}_\Delta(X) \); see [11]). Clearly, the subscheme \( U \) consisting of those \( t \in T_{\Delta,dm} \) for which \( B_t \) are smooth curves is a non-empty Zariski-open subset.

Similarly, let \( T_{\Delta,2} \) be the scheme parametrizing those curves \( D_t \) in \( X \) that are numerically equivalent to \( 2K_X \) and satisfy \( \delta(D_t) = 2\delta(K_X) + \Delta \). The scheme \( T_{\Delta,2} \) is also fibred over \( \text{Pic}_\Delta(X) \), \( \gamma_2 : T_{\Delta,2} \to \text{Pic}_\Delta(X) \), with fibres \( \gamma_2^{-1}(\alpha_t) = \mathbb{P}(H^0(X,\mathcal{O}_X(2K_X + \alpha_t))) \) over the points \( \alpha_t \in \text{Pic}_\Delta(X) \). We denote the product of the fibrations \( \gamma_1 \) and \( \gamma_2 \) by \( T = U \times_{\text{Pic}_\Delta(X)} T_{\Delta,2} \) and let \( p_i \), \( i = 1, 2 \), be the projections of \( T \) onto the factors.

Let us fix a divisor \( D \sim (m - 2)K_X \) and associate with each \( t \in T \) a divisor \( C_{p_2(t)} = D_{p_2(t)} + D \sim mK_X + \alpha_{p_2(t)} \). By Lemma 2, the coverings \( f_1 \) and \( f_2 \) are determined by the divisors \( C_{p_2(t_i)} = D_{p_2(t_i)} + D \sim mK_X + \alpha_{p_2(t_i)} \) for some points \( t_i \in T \), \( i = 1, 2 \), and are branched along the curves \( B_{p_1(t_i)} \). The family of divisors \( D_t = B_{p_1(t)} - dC_{p_2(t)} \) gives rise to a divisor \( \tilde{D} \) in \( X \times T \) such that \( \tilde{D} \cap (X \times \{t\}) = D_t \) for every \( t \in T \). By Corollary 6 in [12], there is an invertible sheaf \( \mathcal{L} \) on \( T \) such that \( \mathcal{O}_{X \times T}((\tilde{D})) = p^*(\mathcal{L}) \), where \( p : X \times T \to T \) is the projection. Clearly, we can choose a rational section of \( \mathcal{L} \) such that the support of its divisor \( L \) contains neither of the points \( t_1 \), \( t_2 \). Consider a rational function \( \tilde{g} = g_t \), \( t \in T \), \( t \notin \text{Supp} \mathcal{L} \) such that the divisor \( (\tilde{g}) \) is equal to \( \tilde{D} - p^*(L) \). The function \( \tilde{g} \) determines a cyclic covering \( \tilde{f} : \tilde{Y} \to X \times T \) by the formula \( \tilde{w}^d = \tilde{g} \). This covering may be regarded as a connected family of cyclic coverings \( f_t : Y_t \to X, t \in T \setminus \text{Supp} L \) determined by the divisors \( C_{p_2(t)} = D_{p_2(t)} + D \sim mK_X + \alpha_{p_2(t)} \) and branched along the curves \( B_{p_1(t)} \). Hence \( Y_1 \) and \( Y_2 \) are deformation equivalent. \( \square \)

Remark 5. Theorem 2 holds without the assumption \( dm \geq 5 \) if, for every \( \alpha \in \text{Pic}_\Delta \), a generic curve \( B \sim d(mK_X + \alpha) \) is smooth.

Theorem 3. Let \( X \) be a surface of general type with \( p_g = 0 \), and let \( f : Y \to X \) be a totally ramified numerically \((2,m)\)-canonical cyclic covering of \( X \). Then the rational map \( \varphi_{KY} : Y \to \mathbb{P}p_g(Y) - 1 \) factors through \( f \), that is, there is a rational map \( \psi : X \to \mathbb{P}p_g(Y) - 1 \) such that \( \varphi_{KY} = \psi \circ f \).

Proof. Let \( B \equiv 2mK_X \) be the branch curve of \( f \), and let \( f \) be determined by a divisor \( C \equiv mK_X \). As in the proof of Theorem 1, the projection formula for the canonical divisor yields that

\[
K_Y = f^*(K_X + C). \tag{5}
\]

Being a surface of general type with \( p_g = 0 \), the surface \( X \) is regular (that is, \( q(X) = 0 \)). Hence Mumford’s vanishing theorem and the Riemann–Roch theorem
and 13
In the notation used in the proof of Proposition 1, as in the case of Campedelli surfaces, let a Burniat surface be a regular map of degree four. To complete the proof, we apply Theorem of degree $K_X$ surface with $r$-fold points of $X$. Hence $\deg X = 8$. Theorem 4 now implies that $\varphi_{K_X} = \varphi_{2K_X} \circ f$ and, therefore, $\varphi_{K_Y}$ is a regular morphism of degree $\deg \varphi_{K_Y} = 16$. □

**Proposition 5.** Let $f : Y \to X$ be a totally ramified purely $(2,1)$-canonical cyclic covering of a Burniat surface $X$ with $K_X^2 \geq 3$. Then $\varphi_{K_Y} : Y \to \mathbb{P}^{K_X^2}$ is a morphism of degree 8 over its image, and this image $Z = \varphi_{K_Y}(Y)$ is a del Pezzo surface with $K_Z^2 = K_X^2$ embedded in $\mathbb{P}^{K_X^2}$ by means of the anticanonical map.

Proof. As in the case of Campedelli surfaces, a Burniat surface $X$ is a resolution of singularities of a Galois covering $\tilde{g} : \tilde{X} \to \mathbb{P}^2$ with Galois group $\text{Gal}(\tilde{X}/\mathbb{P}^2) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ branched along a Burniat line arrangement $\tilde{L}$ (see the details in [14] or [10]). To resolve the singularities, it suffices to blow up the $r$-fold points of $\tilde{L}$, $r \geq 3$, and consider the induced Galois covering $g : X \to \mathbb{P}^2$, where $\sigma : \mathbb{P}^2 \to \mathbb{P}^2$ is the composite of blow-ups with centres at the $r$-fold points of $\tilde{L}$, $r \geq 3$. The number of $r$-fold points of $\tilde{L}$ with $r \geq 3$ is equal to $9 - K_X^2$ and, therefore, $\mathbb{P}^2$ is a del Pezzo surface with $K_{\mathbb{P}^2}^2 = K_Z^2$. Moreover, $2K_X = g^*(-K_{\mathbb{P}^2})$ and $\dim H^0(X, \mathcal{O}_X(2K_X)) = \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}))$ (see, for example, [10]). Therefore $\varphi_{2K_X} = \varphi_{-K_{\mathbb{P}^2}} \circ g$ is a regular map of degree four. To complete the proof, we apply Theorem 3 and obtain the equality $\varphi_{K_Y} = \varphi_{2K_X} \circ f$, which implies that $\deg \varphi_{K_Y} = 8$. □
Mendes Lopes and Pardini [15] constructed a six-dimensional family of surfaces $X$ of general type with $p_g = 0$ and $K_X^2 = 3$. For each of these surfaces, which we call Mendes Lopes–Pardini surfaces, the map $\varphi_{2K_X}$ is regular of degree 2 over its image $Z$, which is a singular Enriques surface $Z \subset \mathbb{P}^3$, deg $Z = 6$. Applying Theorem 3 in the same way as above, we get the following result.

**Proposition 6.** Let $f : Y \to X$ be a totally ramified purely $(2,1)$-canonical cyclic covering of a Mendes Lopes–Pardini surface $X$. Then $\varphi_{K_Y} : Y \to \mathbb{P}^3$ is a regular map of degree four over its image $Z$, which is a singular Enriques surface $Z \subset \mathbb{P}^3$, deg $Z = 6$.

§ 3. Cyclic coverings of rigid surfaces

We recall that every Miyaoka–Yau surface $X$, being a holomorphic quotient of a ball, satisfies the Mostow rigidity theorem. This theorem (in one of its well-known formulations) says that each element of the group $\text{Out} \pi_1(X)$ is realized by a holomorphic or antiholomorphic diffeomorphism $g : X \to X$ (the existence part), and this realization is unique (the uniqueness part); compare [16] and [17].

Given a complex projective surface $X$, we write $\overline{X}$ for the surface complex-conjugate to $X$ and denote the group of holomorphic and antiholomorphic automorphisms of $X$ by $\text{Kl}(X)$.

**Theorem 4.** Let $X$ be a Miyaoka–Yau surface, and let $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$ be totally ramified numerically $(d, m)$-canonical cyclic coverings determined respectively by the divisors $C_1 \sim mK_X + \alpha_1$ and $C_2 \sim mK_X + \alpha_2$, where $\alpha_1$, $\alpha_2$ are numerically equivalent to zero. Suppose also that in the case $d \geq 3$ the elements $\delta(\alpha_1)$ and $\delta(\alpha_2)$ have the same order $n$ in the group $\text{Tor}(X)$ and $n$ is coprime to $d - 1$. If $Y_1$ and $Y_2$ are orientation-preserving diffeomorphic, then there is a (holomorphic or antiholomorphic) automorphism $\psi \in \text{Kl}(X)$ such that $\psi^*(\delta(\alpha_2)) = \pm \delta(\alpha_1)$ (we choose the sign + if $\psi$ is holomorphic and − otherwise).

If $Y_1$ and $Y_2$ are deformation equivalent, then there is an automorphism $\psi \in \text{Aut}(X)$ such that $\psi^*(\delta(\alpha_2)) = \delta(\alpha_1)$.

**Proof.** Let $\varphi : Y_1 \to Y_2$ be an orientation-preserving diffeomorphism. We have

$$K_{Y_1} = f_1^*(K_X) + (d - 1)R_1 \sim f_1^*(K_X) + (d - 1)f_1^*(mK_X + \alpha_1)$$

and, similarly,

$$K_{Y_2} \sim (dm - d + 1)f_2^*(K_X) + (d - 1)f_2^*(\alpha_1).$$

Since $Y_1$ and $Y_2$ are coverings of a surface of Kodaira dimension 2, they also have Kodaira dimension 2 and hence are surfaces of general type. They are also minimal because we have $E \cdot K_{Y_i} = f_*(E) \cdot (dm - d + 1)K_X > 0$ for every curve $E \subset Y_i$. Therefore, $\pm \delta(K_{Y_i})$ are the only Seiberg–Witten basic classes in $H^2(Y_i ; \mathbb{Z})$ (see [18] in the case $p_g > 0$ and, for example, [19] in the case $p_g = 0$). It follows that $\varphi^*(\delta(K_{Y_2})) = \pm \delta(K_{Y_1})$.

By Proposition 1, the map $\varphi$ induces an automorphism $\varphi_* : \pi_1(X) = \pi_1(Y_1) \to \pi_1(Y_2) = \pi_1(X)$ of $\pi_1(X)$ (more precisely, an element of $\text{Out} \pi_1(X)$), which is also induced in accordance with Mostow’s rigidity theorem by a holomorphic or
antiholomorphic automorphism \( \psi: X \to X \). By Siu’s rigidity theorem (see [20]), the map \( f_2 \circ \varphi: Y_1 \to X \) is homotopic to a holomorphic or antiholomorphic morphism \( f_1: Y_1 \to X \), and we have \( \tilde{f}_1 = \psi \circ f_1 \) since these two morphisms determine the same element of \( \text{Out} \pi_1(X) \).

If \( \psi \) is holomorphic, then \( \psi^*(K_X) = K_X \). Hence \( \varphi^*(f_2^*(\delta(K_X))) = f_1^*(\delta(K_X)) \) and, therefore, \( \varphi^*((d-1)\delta(\alpha_2)) = (d-1)\delta(\alpha_1) \). We claim that \( \varphi^*(\delta(\alpha_2)) = \delta(\alpha_1) \). Indeed, assume that \( \varphi^*(\delta(\alpha_2)) = \delta(\alpha_1) + \beta \) for some \( \beta \neq 0 \) with \( (d-1)\beta = 0 \). A priori this is possible only when \( d \geq 3 \). But in this case, since \( (n, d-1) = 1 \), the order of the element \( \delta(\alpha_1) + \beta \) must be greater than \( n \). On the other hand, \( \varphi^*(\delta(\alpha_2)) \) has the same order as \( \delta(\alpha_2) \). Therefore \( \beta = 0 \) and \( \psi^*(\delta(\alpha_2)) = \delta(\alpha_1) \).

If \( \psi \) is antiholomorphic, then \( \psi^*(K_X) = -K_X \). Hence \( \varphi^*(f_2^*(\delta(K_X))) = -f_1^*(\delta(K_X)) \). Therefore \( \varphi^*((d-1)\delta(\alpha_2)) = -(d-1)\delta(\alpha_1) \) and we obtain as above that \( \psi^*(\delta(\alpha_2)) = -\delta(\alpha_1) \).

If \( Y_1 \) and \( Y_2 \) are deformation equivalent, then there is an orientation-preserving diffeomorphism \( \varphi: Y_1 \to Y_2 \) such that \( \varphi^*(\delta(K_{Y_2})) = \delta(K_{Y_1}) \). Repeating the arguments used above, we conclude that there is an automorphism \( \psi \in \text{Aut}(X) \) such that \( \psi^*(\delta(\alpha_2)) = \delta(\alpha_1) \). □

**Corollary 2.** Let \( f_1: Y_1 \to X \) and \( f_2: Y_2 \to X \) be totally ramified numerically \((d,m)\)-canonical cyclic coverings as in Theorem 4. Suppose that \( dm \geq 5 \) and there is an orientation-preserving diffeomorphism \( \varphi: Y_1 \to Y_2 \). Then \( Y_2 \) is deformation equivalent to \( Y_1 \) whenever \( \varphi^*(K_{Y_2}) = K_{Y_1} \), and \( Y_2 \) is deformation equivalent to \( Y_1 \) whenever \( \varphi^*(K_{Y_2}) = -K_{Y_1} \).

**Proof.** Let \( B_i \in |dC_i| \) be the branch curve of the covering \( f_i \), \( i = 1, 2 \). The covering \( \tilde{f}_i \) is obtained by adding to the field \( \mathbb{C}(X) \) a function \( w_i \) with \( w_i^d = g_i \in \mathbb{C}(X) \), where \( (g_i) = B_i - dC_i \) and \( C_i \sim mK_{X} + \alpha_i \).

By Theorem 4 there is an automorphism \( \psi \in \text{KL}(X) \) such that \( \psi^*(\alpha_2) = \pm \alpha_1 \) (we choose the sign + if \( \psi \) is holomorphic and – otherwise). If \( \psi \) is antiholomorphic, then we define an automorphism \( \psi^1: \text{Div}(X) \to \text{Div}(X) \) by the formulae \( \psi^1(D) = \psi^{-1}(D) \) for all curves \( D \subset X \).

Let \( \psi \) be a holomorphic automorphism. Then the covering \( \psi^{-1} \circ f_2: Y_2 \to X \) is obtained by adding to the field \( \mathbb{C}(X) \) a function \( \tilde{w}_2 \) such that \( \tilde{w}_2^d = \psi^*(g_2) \) and \( (\psi^*(g_2)) = \psi^*(B_2) - d\psi^*(C_2) \), where \( \psi^*(C_2) \sim mK_X + \psi^*(\alpha_2) \). By Theorem 2, \( Y_1 \) and \( Y_2 \) are deformation equivalent since \( \delta(\psi^*(\alpha_2)) = \delta(\alpha_1) \).

Let \( \psi \) be an antiholomorphic automorphism. Then the covering \( \psi^{-1} \circ f_2: \bar{Y}_2 \to X \) is obtained by adding to the field \( \mathbb{C}(X) \) a function \( \bar{w}_2 \) such that \( \bar{w}_2^d = \psi^*(\bar{g}_2) \) and \( (\psi^*(g_2)) = \psi^1(B) - d\psi^1(C_2) \), where \( \psi^1(C_2) \sim mK_X + \psi^1(\alpha_2) \). By Theorem 2, \( Y_1 \) and \( \bar{Y}_2 \) are deformation equivalent since \( \delta(\psi^1(\alpha_2)) = \delta(-\psi^1(\alpha_2)) = \delta(\alpha_1) \). □

**Corollary 3.** Let \( X \) be a Miyaoka–Yau surface and let \( d \geq 2 \) and \( m \geq 1 \) be integers such that \( dm \geq 5 \) and \( d-1 \) is coprime to the order of the group \( \text{Tor}(X) \). Then the number of those connected components (in the moduli space of surfaces) that contain totally ramified \((d,m)\)-canonical cyclic coverings of \( X \) is equal to the number of orbits of the action of \( \text{Aut}(X) \) on \( \text{Tor}(X) \).

**Proof.** Let \( f_1: Y_1 \to X \), \( f_2: Y_2 \to X \) be totally ramified numerically \((d,m)\)-canonical cyclic coverings given by the divisors \( C_1 \sim mK_X + \alpha_1 \), \( C_2 \sim mK_X + \alpha_2 \).
respectively, where \(\alpha_1\) and \(\alpha_2\) are numerically equivalent to zero. Assume that there is an automorphism \(\psi \in \text{Aut}(X)\) such that \(\psi^*(\delta(\alpha_2)) = \delta(\alpha_1)\). Then \(\psi^{-1} \circ f_2 : Y_2 \to X\) is a totally ramified numerically \((d, m)\)-canonical cyclic covering given by the divisor \(\psi^*(C_2) = mK_X + \psi^*(\alpha_2)\) such that \(\delta(\psi^*(\alpha_2)) = \psi^*(\delta(\alpha_2)) = \delta(\alpha_1)\). Hence, by Theorem 2, the surfaces \(Y_1\) and \(Y_2\) belong to the same connected component of the moduli space. The converse follows from Theorem 4. \(\square\)

**Theorem 5.** Let \(X\) be a Miyaoka–Yau surface, and let \(f_1 : Y_1 \to X\), \(f_2 : Y_2 \to X\) be totally ramified numerically \((d, m)\)-canonical cyclic coverings. If \(Y_1\) and \(Y_2\) are deformation equivalent, then \(\text{Kl}(X) \neq \text{Aut}(X)\).

**Proof.** The covering \(f_2\) also induces a totally ramified numerically \((d, m)\)-canonical cyclic covering \(f_2 : \tilde{Y}_2 \to X\). As in the proof of Theorem 4, we have

\[
K_{Y_1} \equiv (dm - d + 1)f_1^*(K_X), \quad K_{\tilde{Y}_2} \equiv (dm - d + 1)f_2^*(K_X).
\]

Since \(Y_1\) and \(Y_2\) are deformation equivalent, there is an orientation-preserving diffeomorphism \(\varphi : Y_1 \to Y_2\) such that \(\varphi^*(\delta(K_{\tilde{Y}_2})) = -\delta(K_{Y_1})\). Therefore we have

\[
\varphi^*(f_2^*(\delta(K_X))) \equiv -f_1^*(\delta(K_X)). \tag{8}
\]

The arguments used in the proof of Theorem 4 show that the automorphism \(\varphi_* : \pi_1(X) = \pi_1(Y_1) \to \pi_1(Y_2) = \pi_1(X)\) of the group \(\pi_1(X)\) (more precisely, an element of \(\text{Out} \pi_1(X)\)) induces a holomorphic or antiholomorphic automorphism \(\psi : X \to X\) such that the map \(f_2 \circ \varphi : Y_1 \to X\) is homotopic to a holomorphic or antiholomorphic morphism \(\psi \circ f_1 : Y_1 \to X\) because both morphisms determine the same element of \(\text{Out} \pi_1(X)\).

We claim that \(\psi\) is an antiholomorphic automorphism. Indeed, if \(\psi\) is holomorphic, then \(\psi^*(\delta(K_X)) = \delta(K_X)\). Therefore it follows from (8) that

\[
-f_1^*(\delta(K_X)) \equiv \varphi^*(f_2^*(\delta(K_X))) = f_1^*(\psi^*(\delta(K_X))) = f_1^*(\delta(K_X)).
\]

But this is impossible since the element \(f_1^*(\delta(K_X))\) is not numerically equivalent to zero in \(H^2(Y_1, \mathbb{Z})\). \(\square\)

**Corollary 4.** For every pair of positive integers \(d, m\) with \(dm \geq 5\) and \(d \not\equiv 1\) (mod 5) and every surface \(X\) of general type with \((K^2)_X = 333\) and \(e(X) = 111\), the moduli space \(\mathcal{M}_{k_X, d, m, p_X, d, m}\) has at least \(3 \cdot 5^6\) distinct connected components.

**Proof.** Two such surfaces \(X\) were constructed in [1], Examples 4.1, 4.2. We denote them by \(X_1\), \(X_2\) respectively. These surfaces are obtained by resolving the singularities of Abelian \((\mathbb{Z}/5\mathbb{Z})^2\)-coverings \(h_i : \tilde{X}_i \to \mathbb{P}^2\), \(i = 1, 2\), branched along the arrangement \(\tilde{L} = \bigcup_{j=1}^{9} L_j\) of lines dual to the points of inflection of a smooth plane cubic. To resolve the singularities of \(\tilde{X}_i\), we blow up the \(3\)-fold points of \(\tilde{L}\). The composite of these blow-ups is denoted by \(\sigma : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2\). Then we take the normal closure of \(\tilde{\mathbb{P}}^2\) in the field \(\mathbb{C}(\tilde{X}_i)\) and obtain an induced covering \(h_i : X_i \to \tilde{\mathbb{P}}^2\).

The surfaces \(\tilde{X}_i\) can also be described as quotient spaces \(Z/G_i\), where \(G_i \simeq (\mathbb{Z}/5\mathbb{Z})^6\) and \(Z\) is the Abelian \((\mathbb{Z}/5\mathbb{Z})^8\)-covering \(g : Z \to \mathbb{P}^2\) determined by the field extension \(g^* : \mathbb{C}(\mathbb{P}^2) \hookrightarrow \mathbb{C}(Z) = \mathbb{C}(\mathbb{P}^2)(w_1, \ldots, w_8)\) such that \(w_j^5 = l_jl_j^{-1}\) and \(l_j = 0\) are the equations of the lines \(L_j\), \(j = 1, \ldots, 9\). The divisors \(\frac{1}{5}h_i^*(\sigma^*(l_jl_j^{-1}))\) belong to the group \(\text{Tor}_5(X_i)\) and generate in it a subgroup \(G_i \simeq (\mathbb{Z}/5\mathbb{Z})^6\).
It follows from Proposition 2 that the moduli space $M_{kX,d,m,pX,d,m}$ is non-empty since, by Bombieri’s theorem, the surface $X$ of general type contains smooth curves numerically equivalent to $dmK_X$ if $dm \geq 5$.

By Proposition 4.1 in [1] we have $Kl(X_1) = \text{Aut}(X_1) = \text{Gal}(\tilde{X}_1/\mathbb{P}^2)$ and the action of $\text{Aut}(X_1)$ on $G_1$ is trivial since $\text{Aut}(X_1)$ leaves fixed the branch curves $h_1^{-1}(L_j)$, $L_j \subset \tilde{L}$. Therefore, by Theorem 4, defining the surfaces $Y_{1,k}$ as the totally ramified numerically $(d,m)$-canonical coverings $f_k: Y_{1,k} \to X_1$ that are given by the divisors $C_k \sim mK_{\tilde{X}_1} + \alpha_k$, $\alpha_k \in G_1$, we see that the surfaces $Y_{1,k}$ are pairwise non-diffeomorphic in the sense of orientation-preserving diffeomorphisms and, therefore, they belong to distinct connected components of the moduli space $M_{kX,d,m,pX,d,m}$.

Let $Y_1$ and $Y_2$ be the surfaces obtained from the totally ramified numerically $(d,m)$-canonical cyclic coverings $f_1: Y_1 \to X_1$ and $f_2: Y_2 \to \bar{X}_1$. By Theorem 5, $Y_1$ and $Y_2$ cannot be deformation equivalent since $Kl(X_1) = \text{Aut}(X_1)$. Therefore the totally ramified numerically $(d,m)$-canonical cyclic coverings $f_{1,k}: Y_k \to \bar{X}_1$ given by the divisors $C_k \sim mK_{\bar{X}_1} + \alpha_k$, $\alpha_k \in G_1$, determine another $5^6$ distinct connected components of the moduli space $M_{kX,d,m,pX,d,m}$.

Again by Proposition 4.1 in [1], we have $\text{Aut}(X_2) = \text{Gal}(\tilde{X}_2/\mathbb{P}^2)$ and the action of $\text{Aut}(X_2)$ on $G_2$ is trivial. However, $\text{Out}(\pi_1(X_2)) = Kl(X_2) \neq \text{Aut}(X_2)$. Hence the totally ramified numerically $(d,m)$-canonical cyclic coverings $f: \tilde{Y} \to X_2$ give at least $5^6$ further distinct connected components of $M_{kX,d,m,pX,d,m}$ because these surfaces $\tilde{Y}$ are not homeomorphic to the surfaces $Y$ obtained as the totally ramified numerically $(d,m)$-canonical cyclic coverings $f: Y \to X_1$ (the fundamental groups of the surfaces $Y$ and $\tilde{Y}$ have non-isomorphic groups of outer automorphisms).

**Theorem 6.** Let $f_1: Y_1 \to X$ be a totally ramified numerically $(2,m)$-canonical cyclic covering of a Miyaoka–Yau surface $X$, and let $\tilde{Y}_2$ be a surface deformation equivalent to $Y_1$. Then the canonical model $Y_2$ of $\tilde{Y}_2$ can be represented as a totally ramified numerically $(2,m)$-canonical cyclic covering of $X$ branched along a curve $B_2 \subset X$, where $B_2$ is a reduced (not necessarily irreducible) curve with ADE-singularities.

Let $M$ be the connected component of $Y_1$ in the moduli space of surfaces. If $2m \geq 5$, then $M$ is an irreducible variety of dimension $m(2m - 1)K_X^2 + p_g(X)$. The Kodaira dimension $\kappa(M)$ of $M$ is equal to $-\infty$. If the irregularity $q(X)$ is equal to zero, then $M$ is a unirational variety.

**Proof.** Let $f_1: Y_1 \to X$ be the covering determined by the divisor $C \sim mK_X + \alpha_1$ and branched along a curve $B_1 \in |2C|$.

Since $Y_1$ and $\tilde{Y}_2$ are deformation equivalent, there is an orientation-preserving diffeomorphism $\varphi: \tilde{Y}_2 \to Y_1$ with $\varphi^*(K_{Y_1}) = K_{\tilde{Y}_2}$. Hence, by Siu’s rigidity theorem [20], the composite $f_1 \circ \varphi$ is homotopic to a holomorphic map $h: \tilde{Y}_2 \to X$.

We have $\text{deg } h = \text{deg } (f_1 \circ \varphi) = 2$ and, therefore, $h^*: \mathbb{C}(X) \hookrightarrow \mathbb{C}(\tilde{Y}_2)$ is a Galois extension of degree 2. Let $f_2: Y_2 \to X$ be the normalization of $X$ in the field $\mathbb{C}(\tilde{Y}_2)$, and let $\nu: \tilde{Y}_2 \to Y_2$ be a morphism such that $h = f_2 \circ \nu$ (in fact, $\nu$ is the minimal resolution of singularities of $Y_2$).
We claim that $Y_2$ has at most ADE-singularities. Indeed, if $D \subset \tilde{Y}_2$ is an irreducible curve such that $\nu(D)$ is a point, then
\[(K_{\tilde{Y}_2}, D)_{\tilde{Y}_2} = (\varphi^*(K_{Y_1}), D)_{\tilde{Y}_2} = (\varphi^*(f_1^*(D_1)), D)_{\tilde{Y}_2} = (h^*(D_1), D)_{\tilde{Y}_2} = 0,
\]
where $D_1 \equiv mK_X + C \equiv (m + 1)K_X$. It follows by the adjunction formula that $D$ is a rational curve with $(D^2)_{\tilde{Y}_2} = -2$.

Since $f_2 : Y_2 \to X$ is a double covering, its branch curve (to be denoted by $B_2$) has at most ADE-singularities similarly to $Y_2$. We claim that $B_2 \equiv 2mK_X$. Indeed, we have $2K_{\tilde{Y}_2} = h^*(2K_X + B_2)$. On the other hand,
\[2K_{\tilde{Y}_2} = 2\varphi^*(K_{Y_1}) \equiv 2(m + 1)\varphi^*(f_1^*(K_X)) \equiv 2(m + 1)h^*(K_X),
\]
whence $B_2 \equiv 2mK_X$.

By Remark 2, the covering $f_2 : Y_2 \to X$ is given by a divisor $C_2 \sim mK_X + \alpha_2$. A small deformation of $B_2$ into a smooth curve $B_3 \sim B_2$ determines a totally ramified numerically $(2, m)$-canonical covering $f_3 : Y_3 \to X$ branched along $B_3$ and associated with the divisor $C_2 \sim mK_X + \alpha_2$. Since there is a simultaneous resolution of simple singularities, $Y_2$ and $Y_3$ belong to the same irreducible component of the moduli space of surfaces.

By Corollary 2 and Theorem 4 there is an automorphism $\psi \in \text{Aut}(X)$ such that $\psi^*(\delta(\alpha_2)) = \delta(\alpha_1)$. Therefore we can assume that $\delta(\alpha_2) = \delta(\alpha_1)$ (replacing $f_3$ by $\psi \circ f_2$) and, moreover, all surfaces $Y_t$ deformation equivalent to $Y_1$ are totally ramified numerically $(2, m)$-canonical cyclic coverings of $X$ determined by $C_t \sim mK_X + \alpha_t$, where the divisors $\alpha_t$ satisfy $\delta(\alpha_t) = \delta(\alpha_1)$, and branched along the curves $B_t \sim 2(mK_X + \alpha_t)$ with at most ADE-singularities.

Let $\Delta = \delta(\alpha_1) \in \text{Tor}(X)$. We put $\text{Pic}_{\Delta}(X) = \delta^{-1}(\Delta) \subset \text{Pic}(X)$. As in the proof of Theorem 2, we consider the scheme $T_{\Delta, 2m}$ parametrizing the curves $B_t$ that lie on $X$ and are numerically equivalent to $2mK_X$ and satisfy $\delta(B_t) = 2m\delta(K_X) + 2\Delta$. The scheme $T_{\Delta, 2m}$ is fibred over $\text{Pic}_\Delta(X)$, $\gamma_1 : T_{\Delta, 2m} \to \text{Pic}_\Delta(X)$, with fibre $\gamma_{1, t}^{-1}(\alpha_t) = \mathbb{P}(H^0(X, O_X(2mK_X + d\alpha_t)))$ over a point $\alpha_t \in \text{Pic}_\Delta(X)$. Clearly, the subscheme $U$ consisting of those points $t \in T_{\Delta, 2m}$ for which the curves $B_t$ are reduced and have at most ADE-singularities is a non-empty Zariski-open subset.

Similarly, let $T_{\Delta, 2}$ be the scheme parametrizing the curves $D_t$ on $X$ that are numerically equivalent to $2K_X$ and satisfy $\delta(D_t) = 2\delta(K_X) + \Delta$. The scheme $T_{\Delta, 2}$ is also fibred over $\text{Pic}_\Delta(X)$, $\gamma_2 : T_{\Delta, 2} \to \text{Pic}_\Delta(X)$, with fibre $\gamma_{2, t}^{-1}(\alpha_t) = \mathbb{P}(H^0(X, O_X(2K_X + \alpha_t)))$ over a point $\tilde{\alpha}_t \in \text{Pic}_\Delta(X)$. We write $T = U \times_{\text{Pic}_\Delta(X)} T_{\Delta, 2}$ for the fibre product of $\gamma_1$ and $\gamma_2$ and let $p_i, i = 1, 2$, be the projections of $T$ onto the factors.

Note that $T$ is an irreducible variety and it follows from the above considerations and the proof of Theorem 2 that the points of $T$ parametrize all the surfaces deformation equivalent to $Y_1$. Therefore the above considerations show that the connected component $M$ of $Y_1$ in the moduli space of surfaces is irreducible.

To complete the proof of the theorem, we consider the surjective morphism $\mu : T \to M$. It is easy to see that the fibres of $p_1$ are subvarieties of the fibres of $\mu$. We claim that each fibre of $\mu$ is a union of finitely many fibres of $p_1$. Indeed, every totally ramified covering $f : Y \to X$ of degree two branched along a curve
$B = 2mK_X$ determines an extension of fields $\mathbb{C}(X) \hookrightarrow \mathbb{C}(Y)$ of degree two. This extension induces an automorphism $f^* \in \text{Aut}(Y)$ of order two whose set of fixed points is the ramification locus $R$ of $f$ (in the case when $Y$ has no $(-2)$-curves). Conversely, each automorphism $h \in \text{Aut}(Y)$ uniquely determines the set of its fixed points. But the groups $\text{Aut}(X)$ and $\text{Aut}(Y)$ are finite since $X$ and $Y$ are surfaces of general type. Therefore each fibre of $\mu$ is a union of finitely many fibres of $p_1$ and, by the Stein factorization theorem, $\mu$ factors through the finite morphism $\mu_1: U \rightarrow M_1$.

We have

$$\dim M = \dim U = \dim |2mK_X + d\alpha_1| + q(X).$$

By Mumford’s vanishing theorem and the Riemann–Roch theorem,

$$\dim |2(mK_X + \alpha_1)| = \frac{(2(mK_X + \alpha_1), 2(mK_X + \alpha_1) - K_X)_X}{2} + p_g(X) - q(X).$$

Therefore $\dim M = m(2m - 1)K^2_X + p_g(X)$. Moreover, since $U$ has Kodaira dimension $\kappa(U) = -\infty$, we see that $M$ has the same Kodaira dimension. If $q = 0$, then $U$ is a rational variety and, therefore, $M$ is unirational. □

Remark 6. The arguments used in the proof of Theorem 6 show that for every $d \geq 2$ and any surface $Y_2$ deformation equivalent to a surface $Y_1$ obtained as a totally ramified numerically $(d,m)$-canonical cyclic covering of a Miyaoka–Yau surface $X$, there is a morphism $f_2: Y_2 \rightarrow X$ of degree $d$. But if $d \geq 3$, then $f_2$ is not necessarily a cyclic covering. Furthermore, these arguments give rise to a lower bound for the dimension of the irreducible component $M$ of $Y_1$ in the moduli space of surfaces:

$$\dim M \geq \frac{dm(dm - 1)}{2}K^2_X + p_g(X).$$

Remark 7. The proof of Theorem 6 shows that for every surface $Y$ which is a totally ramified numerically $(2,m)$-canonical cyclic covering of a Miyaoka–Yau surface $X$, the action of $\mathbb{Z}/2\mathbb{Z}$ on $Y$ (as the group of deck transformations) deforms simultaneously with any deformation of the complex structure on $Y$.

§ 4. New examples of surfaces having no antiholomorphic automorphisms

In this section, by the Mostow strong rigidity of a compact complex manifold $X$ we mean the following property: every homotopy equivalence $p: X \rightarrow X$ is homotopic to a holomorphic or antiholomorphic map $X \rightarrow X$. By the Mostow rigidity theorem, the fake projective planes and the surfaces in [1], Examples 4.1, 4.2, are Mostow strongly rigid. Moreover, they are $K(\pi,1)$s as topological spaces.

We stress that uniqueness is not required in the definition of Mostow strong rigidity. The reason is that the uniqueness is not used in the proofs below. (However, if the definition had included the uniqueness, then one could replace ‘if’ by ‘if and only if’ in Remark 9.)

Theorem 7. Let $X$ be a Mostow strongly rigid surface of general type. If $X$ is a $K(\pi,1)$-space and the group $\text{Out}(\pi)$ contains no elements of even order, then none of the totally ramified numerically $(d,m)$-canonical cyclic coverings $Y$ of $X$. 

can be deformed to a surface isomorphic to $\overline{Y}$. In particular, none of the surfaces $Y'$ obtained by deformations of such a surface $Y$ have antiholomorphic automorphisms.

Proof. We argue by contradiction. Assume that $Y$ is deformed to a surface $Y'$ which is isomorphic to $\overline{Y}$. We identify $Y'$ and $Y$ as smooth manifolds (using the diffeomorphism induced by the deformation) and consider any antiholomorphic diffeomorphism $c: Y \to Y'$ between $Y$ and $Y'$. The same symbol $c$ will be used for the diffeomorphism of $Y$ induced by $c$ in accordance with our identification of the smooth manifolds underlying $Y$ and $Y'$.

Let $c_* \in \text{Out}(\pi_Y)$ be the element induced by $c$ (and by the deformation equivalence between $Y$ and $Y'$). We have $\pi_1(Y) = \pi_1(X)$ by Proposition 1. Since $X$ is a Mostow strongly rigid surface of general type and a $K(\pi,1)$-space, the element $c_*$ has a finite order (as any element of the group $\text{Out}(\pi_X) = \text{Out}(\pi_Y)$). We denote this order by $n$, $n > 0$. Since $X$ is a $K(\pi,1)$-space, it follows from Proposition 1 that the maps $f: Y' = Y \to X$ and $f \circ c^n$ are homotopic. Since we also have $\delta(K_Y) = -\delta(K_{Y'}) = -\delta(K_Y)$ (the Chern class $\delta(K)$ is integral and therefore remains unchanged under deformations) and $f^*$ transforms $(dm - m + 1)\delta(K_X)$ into $\delta(K_Y)$ modulo torsion, it follows that $n$ is even, a contradiction. □

Remark 8. Literally the same proof shows that under the hypotheses of Theorem 7 the surfaces $Y$ (and $Y'$) have no diffeomorphisms $f: Y \to Y$ with $f^*[K] = -[K]$, $[K] \in H^2(Y; \mathbb{Q})$. Since the canonical class is invariant under deformations in the class of almost complex structures, it follows that the imaginary part $\omega$ of the Kähler structure of $Y$ (regarded as a symplectic structure on the underlying smooth manifold) is not symplectically deformation equivalent to $-\omega$. Thus Theorem 7 and, in particular, its Corollary 5 (see below) give new examples of opposite symplectic structures not equivalent to each other (compare [3]).

Remark 9. If $X$ is a Mostow strongly rigid surface of general type and a $K(\pi,1)$-space, then the fundamental group $\pi_1(X)$ has no automorphisms of even order $> 0$ provided that $X$ has neither antiholomorphic nor holomorphic automorphisms of even order.

Corollary 5. If $X$ is either a fake projective plane or the rigid surface constructed in [1], Example 4.1, then none of the surfaces $Y'$ obtained by deformation of a totally ramified numerically $(d,m)$-canonical cyclic covering $Y$ of $X$ can be deformed to its complex conjugate. In particular, such surfaces $Y$ have no antiholomorphic automorphisms.

Proof. As proved in [1], each of these surfaces $X$ has no antiholomorphic or holomorphic automorphisms of even order $> 0$. Hence the result follows from Theorem 7 and Remark 9. □

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