Higher-dimensional violations of the holographic entropy bound

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The holographic bound, \( S \leq A/4\ell_p^2 \), asserts that the entropy \( S \) of a system is bounded from above by a quarter of the area \( A \) of a circumscribing surface measured in Planck areas. This bound is widely regarded as part of the elusive fundamental theory of nature. In fact, the bound is known to be valid for generic weakly gravitating isolated systems in three spatial dimensions. Nevertheless, the entropy content of a physical system is expected to be an increasing function of the number of spatial dimensions (the more the dimensions, the more ways there are to split up a given amount of energy). Thus, one may expect the challenge to the holographic entropy bound to become more and more serious as the number of spatial dimensions increases. In this paper we explicitly show that thermal radiation in \( D \) flat spatial dimensions with \( D \gtrsim 10^7 \) may indeed violate the holographic entropy bound.

I. INTRODUCTION

The influential holographic principle \([1, 2]\) asserts that there is a deep relation between the physical content of a theory defined in a spacetime and the corresponding content of another theory defined on the boundary of the same spacetime \([3]\). This principle is considered to be an important ingredient of the ultimate physical theory of nature \([3]\).

The holographic principle suggests that the information encoded on the boundary of a physical system should be able to describe the entire set of possible quantum states of the bulk system \([3]\). In light of the correspondence between information and entropy \([4, 5]\), and the well-known entropy-area relation for black holes \([6, 7]\), this requirement has been expressed in the form of the holographic entropy bound \([1, 2]\):

\[
S \leq \frac{A}{4\ell_p^2},
\]

where \( \ell_p^2 = G\hbar/c^3 \) is the Planck area. This bound thus asserts that the entropy \( S \) (or information) that can be contained in a physical system is bounded in terms of the area \( A \) of a surface enclosing it. It also implies that \( (3 + 1) \)-dimensional black holes have the largest possible entropy among (stationary and bounded) physical systems characterized by a given surface area \( A \) (see also \([3, 10]\)).

It should be mentioned that there do exist violations of the holographic bound. For example, a collapsed object already inside its own gravitational radius eventually violates it. The enclosing area can only decrease while the enclosed entropy can only grow \([3, 12]\). Another example is given by a large spherical region of a flat Friedmann universe: its enclosing area grows like radius squared while the enclosed entropy does so like radius cubed. These examples belong to a class of strongly self-gravitating and dynamical systems. The second example also describes an unisolated system. A covariant entropy bound free from such failures was introduced in \([12]\). However, this covariant bound is characterized by a complicated formulation in cases which lack high degree of symmetry \([12]\). Thus, the original holographic bound, which is simpler to apply, would still be very valuable if its range of validity could be defined in a systematic way \([3]\).

Using the generalized second law (GSL) of thermodynamics, it was shown \([8]\) that the holographic bound \([11]\) can indeed be trusted for generic weakly self-gravitating isolated systems in three spatial dimensions. In this paper we address the following question: is the holographic bound also valid for weakly self-gravitating isolated systems in higher-dimensional spacetimes?

One may expect the challenge to the holographic entropy bound to become more and more serious as the number of spatial dimensions increases: the more the dimensions, the more ways there are to split up a given amount of energy between the quantum states of the system \([14]\). Thus, the entropy content of a physical system which is characterized by a given amount of energy is expected to increase with increasing number of spatial dimensions.

II. \((D + 1)\)-DIMENSIONAL RADIATION ENTROPY

Consider in \( D \) flat spatial dimensions a spherical box of radius \( R \) into which we dump energy \( E \) of massless fields. [We shall henceforth use natural units in which \( G = c = k_B = 1 \).] We shall follow the analysis of \([14]\) in order to calculate the system’s entropy in the thermodynamic regime. For the thermodynamic description to be valid, the discreteness of the energy spectrum (due to the finite size of the confining box) should be unnoticeable. We shall therefore require that the characteristic frequency of the thermal radiation be large compared to the energy gaps which characterize the discrete energy spectrum of the bounded system. This would imply that many wavelengths small compared to \( R \) are thermally
excited. Below [see Eqs. (14)-(16)] we shall make this statement more accurate.

The volume of a sphere of radius $R$ in $D$ spatial dimensions is

$$V_D(R) = \frac{2\pi^{D/2}}{D\Gamma(D/2)} R^D,$$

(2)

where $\Gamma(z)$ is the Euler gamma function. Consequently, the volume in frequency space of the shell $(\omega, \omega + d\omega)$ is

$$dV_D(\omega) = D[V_D(\omega)/\omega]d\omega.$$  

(3)

The mean thermal energy in the sphere from one helicity degree of freedom is given by

$$E = V_D(R) \int_0^\infty \frac{\hbar \omega}{(e^{\beta \hbar \omega} \mp 1)(2\pi)^D} d\omega,$$

(4)

where the upper (lower) signs correspond to boson (fermion) fields, and $\beta \equiv 1/T$ is the inverse temperature of the system. We note that the distribution $\omega^D/(e^{\beta \hbar \omega} \mp 1)$ in Eq. (4) peaks at the characteristic frequency

$$\bar{\omega} = \frac{D}{\hbar \beta}[1 \mp e^{-D} + O(e^{-2D})].$$

(5)

From Eqs. (2)-(4) and the relation

$$\int_0^\infty \frac{x^D dx}{e^x + 1} = \zeta(D+1)\Gamma(D+1) \times \begin{cases} 1 & \text{for bosons;} \\ 1 - 2^{-D} & \text{for fermions}, \end{cases}$$

(6)

where $\zeta(z)$ is the Riemann zeta function, one finds that the mean energy of all massless fields is given by

$$E = \frac{2N\zeta(D+1)\Gamma(D+1)}{\pi^{1/2}\Gamma(D/2)\beta^{D+1}\hbar^D}.$$  

(7)

where $N$ is the number of massless degrees of freedom (the number of polarization states). Massless scalars contribute 1 to $N$, massless fermions contribute $1 - 2^{-D}$ to $N$, [14], an electromagnetic field contributes $D - 1$ to $N$, [15], and the graviton contributes $(D + 1)(D - 2)/2$ to $N$. [15]. Solving Eq. (7) for $\beta \hbar / R$ one finds

$$\beta \hbar / R = C_D(N \hbar / RE)^{1/D},$$

(8)

where

$$C_D = \left[ \frac{2\zeta(D+1)\Gamma(D+1)}{\pi^{1/2}\Gamma(D/2)} \right]^{1/D}. $$

(9)

Likewise, one can write the thermal entropy of one helicity degree of freedom as [14]

$$S = V_D(R) \int_0^\infty \left[ \mp \ln(1 \mp e^{-\beta \hbar \omega}) + \frac{\beta \hbar \omega}{e^{\beta \hbar \omega} \mp 1} \right] \frac{dV_D(\omega)}{(2\pi)^D}. $$

(10)

After some algebra we obtain

$$S = \frac{2N(D + 1)\zeta(D + 1)\Gamma(D+1)}{\pi^{1/2}\Gamma(D/2)\beta^{D+1}\hbar^D},$$

(11)

which implies

$$S = \frac{D + 1}{D} \beta E.$$  

(12)

Substituting Eq. (5) into Eq. (12) one finds

$$S = C_D(1 + 1/D)N^{1/D}R \hbar h,$$

(13)

for the $(D + 1)$-dimensional radiation entropy.

### III. THE THERMODYNAMIC CRITERIA

The description of our fixed energy system in terms of temperature (the thermodynamic regime) is valid provided the characteristic thermal frequency $\bar{\omega}$ [see Eq. (5)] is large compared to the characteristic energy gaps in the discrete spectrum of the (finite size) system. [Otherwise, one cannot rely on continuum formulae like Eqs. (4) and (10).] In the familiar case of three spatial dimensions ($D = 3$), this thermodynamic condition implies that the dimensionless ratio $\beta \hbar / R$ should satisfy the simple constraint $\beta \hbar / R \ll 1$. However, as we shall now discuss, the criterions for the validity of the thermodynamic description in $D \gg 1$ spatial dimensions are more involved: they depend on intricate combinations of two dimensionless quantities: $\beta \hbar / R$ and $D$.

For a system in $D$ spatial dimensions confined into a region of length-scale $R$, there are two distinct energy gaps which characterize its discrete energy spectrum [16]:

- The characteristic energy gap between adjacent energy levels is of the order of $\hbar / R$ [16]. The validity of the thermodynamic description requires this energy gap to be small compared to the characteristic thermal frequency. Taking cognizance of Eq. (5) for $\bar{\omega}$, one deduces the thermodynamic condition

$$D / \beta \gg \hbar / R,$$

(14)

- The ground-state has a characteristic energy of the order of $\sqrt{D} \hbar / R$ [16]. The validity of the thermodynamic description requires this ground-state energy to be small compared to the characteristic thermal frequency. Thus, for the thermodynamic description to be valid one should also have

$$D / \beta \gg \sqrt{D} \hbar / R.$$  

(15)

The validity of the thermodynamic description also rests on the assumption that many quanta (of each degree of freedom) are thermally excited in the system:
$E/N \gg \hbar \omega$. Taking cognizance of Eqs. \((\text{5})\) and \((\text{7})\), one finds the condition

$$C_D^D/(R/\beta h)^D \gg D.$$  \((\text{16})\)

For familiar physical systems in three spatial dimensions the three requirements \((\text{14})\) - \((\text{16})\) are basically identical. (This may explain why, in the simple case of three spatial dimensions, the various thermodynamic criteria are usually unified into one requirement, $\beta h/R \ll 1$.) However, for an exceedingly large number of spatial dimensions ($D \gg 1$) the three requirements are not identical – in fact, condition \((\text{16})\) enforces the strongest constraint. Taking cognizance of Eq. \((\text{8})\), one finds that the strongest thermodynamic condition may be cast in the form

$$C_D^D(Nh/RE)^D \ll D^{-1}.$$  \((\text{17})\)

We characterize this constraint by the dimensionless control parameter $\xi$ defined by

$$\xi \equiv C_D^D(Nh/RE)^D \ll 1.$$  \((\text{18})\)

Below we shall come back to this thermodynamic condition.

IV. THE WEAK-GRAVITY CRITERIA

Our analysis is appropriate only for weakly self-gravitating systems. In particular, formula \((\text{13})\) for the system’s entropy can be trusted only in a limited range of parameters (limited range of energies for a given system’s radius $R$) as here stated. The spacetime outside the spherical box (for $D \geq 3$) is described by the $(D+1)$-dimensional Schwarzschild-Tangherlini metric \([17, 18]\) of ADM energy $E$:

$$ds^2 = -H(r)dt^2 + H(r)^{-1}dr^2 + r^2d\Omega^{(D-1)},$$  \((\text{19})\)

with

$$H(r) = 1 - \left(\frac{r_g}{r}\right)^{D-2}.$$  \((\text{20})\)

Here

$$r_g = \left[\frac{16\pi E}{(D-1)A_{D-1}}\right]^{\frac{1}{D-2}}$$  \((\text{21})\)

is the gravitational radius of the box and

$$A_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$  \((\text{22})\)

is the area of a unit $(D-1)$-sphere.

For the system to be weakly self-gravitating, one should impose the criterion $H(r = R) \sim 1$ at the surface of the sphere, or equivalently $(r_g/R)^{D-2} \ll 1$. Taking cognizance of Eqs. \((\text{20})\) - \((\text{21})\), this condition yields the restriction

$$RE \ll \frac{D-1}{16\pi}A,$$  \((\text{23})\)

where $A = A_{D-1}R^{D-1}$ is the surface area of the system. We characterize this restriction by the dimensionless control parameter $\eta$ defined by

$$\eta \equiv \frac{16\pi RE}{(D-1)A} \ll 1.$$  \((\text{24})\)

A similar (but somewhat weaker) constraint can be obtained from the requirement that the magnitude of the system’s interior energy-momentum tensor should be much less than the scalar curvature at the surface. For a physical system confined by a $(D+1)$-dimensional sphere this requirement reads:

$$\frac{E}{V_D(R)} \ll \frac{(D-1)(D-2)}{R^2}.$$  \((\text{25})\)

Using the relation $V_D(R) = A_{D-1}R^{D-1} \times \xi$, one obtains the constraint

$$RE \ll \frac{(D-1)(D-2)}{D}A.$$  \((\text{26})\)

This criterion is in the same spirit of \((\text{23})\).

Taking cognizance of Eqs. \((\text{13})\) and \((\text{24})\), we can write the system’s thermal entropy as

$$S = C_D(1+1/D)\pi^{适合} \left[\frac{\eta(D-1)A}{16\pi h} \right]^\frac{D}{D-2}.$$  \((\text{27})\)

We shall now address the following question: can the $(D+1)$-dimensional radiation entropy exceed the holographic entropy bound \((\text{1})\)?

V. THE HOLOGRAPHIC BOUND

For the system’s entropy \((\text{24})\) to beat the holographic bound (that is, $S > A/4\hbar$), its surface area must be bounded from above according to

$$\frac{A}{h} < [4C_D(1+1/D)]^{D+1}N \left[\frac{\eta(D-1)A}{16\pi h} \right]^\frac{D}{D-2}.$$  \((\text{28})\)

Solving Eqs. \((\text{18})\) and \((\text{24})\) for $RE$, one can express the system’s area as

$$\frac{A}{h} = \frac{16\pi ND^{D+1}}{\eta(D-1)(\xi C_D)^{D+1}}.$$  \((\text{29})\)

Substituting \((\text{24})\) into \((\text{28})\), we realize that a violation of the holographic bound can only occur if the number of spatial dimensions satisfies the inequality \((\text{19})\):

$$D \geq D^* \simeq 4\pi/\eta C_D^{1/D}.$$  \((\text{30})\)
Since $\xi \ll 1$ and $\eta \ll 1$ [see Eqs. (18) and (24)], we learn from (30) that thermal radiation in three spatial dimensions conforms to the holographic bound.

In order to estimate the critical dimension $D^*$ (the minimal value of $D$ above which a violation of the holographic bound can be realized) one may substitute $\eta_{\max} = O(10^{-1})$ and $\xi_{\max} = O(10^{-1})$ for the dimensionless control parameters. We then find that the entropy of our $(D + 1)$-dimensional system may exceed the holographic bound for

$$D \geq D^* = O(10^2) .$$

(31)

It should be recognized that the precise value of the critical dimension $D^*$, Eq. (31), can be challenged. This is a direct consequence of the inherent fuzziness in the numerical values of the control parameters $\eta_{\max}$ and $\xi_{\max}$. This is the price we must pay for not giving our problem a full quantum treatment in curved spacetimes. Nevertheless, it should be clear that there is some critical value for the number of spatial dimensions [see Eq. (30)], probably around $D^* \sim 10^2$, above which the system’s entropy exceeds the holographic bound.

Are there any relevant effects which might change our conclusion? We note that thermal fields restricted to a finite region exert pressure on the regions boundary. This pressure must be balanced by a tensile wall or container, whose mass $E_{\text{box}}$ should be added to the energy $E$ of the confined fields [20]. The container must be sufficiently rigid to withstand the pressure caused by the thermal fields. A lower bound on the surface density $\sigma$ (energy per surface area) of the required container was obtained in [20] for the case of three spatial dimensions:

$$\sigma \geq \frac{p}{K} ,$$

(32)

where $p$ and $K$ are the pressure exerted by the enclosed bulk system on the container and the trace of the extrinsic curvature, respectively. (The only non-vanishing entries of $K_{\alpha\beta}$ are the spatial components tangential to the container [21].) This result can readily be generalized to higher-dimensional spacetimes. Thus, for a spherical box in $D$ spatial dimensions one finds

$$E_{\text{box}} \geq \frac{Ap}{K}$$

(33)

for the energy of the confining box. Substituting $p = \frac{1}{4\pi} \frac{E}{V_D(R)}$ and $K = (D - 1)/R$ in (33), and using the relation $V_D(R) = A \times \frac{R}{D}$, one finds that the minimal container’s mass, $E_{\text{box}}^{\min}$, satisfies the relation

$$E_{\text{box}}^{\min} = \frac{1}{(D - 1)} .$$

(34)

Thus, the contribution of the container’s mass to the total energy of the system can be made negligible in the large $D$ limit [31].

VI. THE GENERALIZED SECOND LAW OF THERMODYNAMICS

It was shown [3] that the holographic bound for generic weakly self-gravitating isolated systems in three spatial dimensions is a direct outcome of the generalized second law (GSL) of thermodynamics. So, does the above violation of the holographic entropy bound (by weakly self-gravitating physical systems in higher-dimensional spacetimes) imply a violation of the GSL?

Consider a gedanken experiment in which a $(D + 1)$-dimensional spherical object with negligible self-gravity is deposited at the horizon of a $(D + 1)$-dimensional black hole with the least possible energy. A violation of the GSL seems to occur unless the system’s entropy $S$ is bounded from above according to the universal entropy bound [8, 12]:

$$S \leq 2\pi RE/\hbar .$$

(35)

Arguing from the GSL, [8] has proposed the existence of the entropy bound (35). The entropy of the swallowed object disappears, but an increase in black-hole entropy occurs which guarantees that the GSL is respected (the total entropy of black hole + object never decreases) provided the object’s entropy $S$ is bounded by (35).

Now, taking cognizance of the thermodynamic condition (18), one realizes that the system’s thermal entropy [13] may be expressed as

$$S = C_D(1 + 1/D)(\xi C_D/D)^{1/D}RE/\hbar .$$

(36)

From Eq. (36) with $\xi \ll 1$ one deduces $S < RE/\hbar$ for all finite values of $D$. In fact, $S \to RE/\hbar$ in the $D \to \infty$ limit. The system’s thermal entropy therefore conforms to the entropy bound (35) regardless of the value of $D$. Thus, the GSL is respected despite the fact that the entropy of the system may exceed the holographic bound [11].

VII. SUMMARY

It is well-established that the holographic bound [11] can be trusted for generic weakly self-gravitating isolated systems in three spatial dimensions. However, in this paper we have shown that this is not a generic property of the holographic bound: weakly self-gravitating isolated thermal radiation in $D \gtrsim 10^2$ spatial dimensions may violate the holographic entropy bound. There is obviously no evidence that the actual number of spatial dimensions in nature is so large. In this sense, the breakdown of the holographic bound is not manifest empirically. In fact, arguing from the holographic principle, one may conjecture that the seeming clash between the holographic bound and an exceedingly large number of spatial dimensions merely tells us that physics is consistent only in a world with a limited number of large spatial dimensions, such as ours.
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