Towards Non-Archimedean Superstrings

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Abstract

An action for a prospect of a $p$-adic open superstring on a target Minkowski space is proposed. The action is constructed for ‘worldsheet’ fields taking values in the $p$-adic field $\mathbb{Q}_p$, but it is assumed to be obtained from a discrete action on the Bruhat-Tits tree. This action is proven to have an analogue of worldsheet supersymmetry and the superspace action is also constructed in terms of superfields. The action does not have conformal symmetry, however it is implemented in the definition of the amplitudes. The tree-level amplitudes for this theory are obtained for $N$ vertex operators corresponding to tachyon superfields and a Koba-Nielsen formula is obtained. Finally, four-point amplitudes are computed explicitly and they are compared to previous work on $p$-adic superstring amplitudes.

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1 Introduction

String theory is a very strong candidate for a quantum theory of gravity. Its non-perturbative formulation known as AdS/CFT correspondence has been successfully applied to many systems of gravity and field theory [1]. Among its most remarkable applications is that of describing the quantum properties of black holes constructed from D-brane configurations. This feature is achieved by counting the corresponding open-string states of the supersymmetric brane configurations. There are other phenomena also described in terms of brane configurations as the loss of information and the emergence of spacetime itself from entangled states in the dual conformal field theory. Some of these considerations are in an early stage and it is observed that in the standard correspondence is quite difficult to find some progress due to the complicated nature of the calculations. Thus, some simpler models that capture some essential features of the AdS/CFT correspondence are very important to be explored in order to achieve some progress. Recently a model was proposed in [2, 3], regarding a $p$-adic version of AdS/CFT correspondence. These $p$-adic models capture the essential features of the usual correspondence. The bulk is described on the Bruhat-Tits tree and its boundary field theory is given in terms of a $p$-adic CFT on the line proposed several years ago [4].

The study of a $p$-adic string theory was introduced many years ago and further developed in [5, 6, 7, 8, 9]. In these references it was studied the proposal of considering the tree-level amplitudes of a bosonic open string theory with the fields on the worldsheet taking values over the $p$-adic number field $\mathbb{Q}_p$. Some reviews on non-Archimedean string amplitudes can be found in Refs. [10, 11, 12, 13].

Later these amplitudes were derived from a $p$-adic worldsheet action defined on the Bruhat-Tits tree [14] or projected out as an effective theory to the boundary of the tree, the $p$-adic line $\mathbb{Q}_p$, in terms of the Vladimirov derivative [15]. Both methods were found to be equivalent in order to derive the mentioned open string amplitudes. In this context the study of the regularization of the amplitudes for non-Archimedean open strings in terms of local zeta functions was discussed in [16]. The rigorous study of open and closed strings on any local field of characteristic zero was considered in [17], and the limit $p \to 1$ was described in [18]. The discussion on the incorporation of a constant NS $B$-field is studied in Ref. [19]. The regularization of the Ghoshal and Kawano amplitudes was carried out in [20]. Very recently the procedure followed in [15, 14] has been studied from a rigorous point of view in [21].

The study of tree-level amplitudes in superstring theory was considered independently in Refs. [22, 23]. Direct analogues for the 4-point superstring amplitudes were considered in [24, 11]. In order to add fermions in the $p$-adic string amplitudes we require of extending the non-Archimedean formalism to include Grassmann numbers. Some further developments of the formalism were carried out in [25].

In the context of $p$-adic AdS/CFT some results concerning the study of non-Archimedean versions of fermionic systems as SYK melonic theories were obtained in [26]. Furthermore a way of introducing the spin was proposed in Ref. [27]. Motivated in part by these works, in [28] was studied the fermionic field theory on the Bruhat-Tits tree and its effective action on the boundary.
In the present article, motivated by all the mentioned works, we propose a world-sheet action containing bosons and fermions on the $p$-adic worldsheet projected on the boundary. We will show that this action is supersymmetric and thus might be considered as a $p$-adic analogue of the worldsheet superstring action in the superconformal gauge [29]. Moreover we will show that this action can be rewritten as an action in a $p$-adic version of the ordinary superspace. Furthermore we compute the tree-level $N$-point open string amplitudes of this superstring action and we obtain the corresponding Koba-Nielsen formula of the well known amplitudes in the NSR formalism [29, 30]. This is carried out explicitly by performing the path integration of this superstring action with $N$ tachyonic vertex operators in the spirit of Refs. [14, 19]. We obtain the amplitudes previously found in Refs. [22, 23].

The article is organized as follows: in Section 2 we introduce the $p$-adic fermionic action and give the details of its construction. Later we study its Green’s function in the $p$-adic Fourier representation. Section 3 is devoted to studying the superstring action. We prove that it satisfies a supersymmetric invariance with the appropriate tools of the $p$-adic construction. The action is also written in terms of $p$-adic superfields. In Section 4 we carry out the explicit computations of the tree-level amplitudes through correlation functions of $N$ vertex operators. We use the path integration of this theory to give a Koba-Nielsen type formula for this amplitude. In Section 5 the 4-point amplitudes are calculated explicitly and are compared with those obtained in Refs. [22, 23]. Section 6 contains the final discussions. Finally, in appendix A, we give a brief overview of the $p$-adic tools in order to introduce the notation and conventions that we will follow in the present article. In appendices B and C we discuss the vertex operators used and a brief discussion of the fermionic propagator is also included, respectively.

2 A $p$-adic fermionic action

In this section, we consider a $p$-adic analogue of a fermionic action corresponding to the fermionic sector of the Archimedean worldsheet superstring action in the superconformal gauge. The action has previously appeared in [23, 26], but a thorough analysis of its properties was not developed. The proposed action is the following

$$S_F[\psi] = \frac{\text{sgn}_r(-1)p}{2} \frac{1}{\alpha'} \int_{\mathbb{Q}_p^2} \psi^\mu(x) \eta_{\mu\nu} \text{sgn}_r(x - y) \frac{\text{sgn}_r(x - y)}{|x - y|_{p}^{s + 1}} \psi^\nu(y) dy dx,$$

(1)

where $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$ is the Minkowski metric, $\psi : \mathbb{Q}_p \to \Lambda$, is a ‘worldsheet’ $p$-adic field valued in the Grassmann number field $\Lambda$, and $\text{sgn}_r$ is the $p$-adic sign function\(^3\). Some comments regarding this proposal are in order. The action (1) requires $\text{sgn}_r(-1) = -1$, otherwise it will vanish identically due to the anti-commutativity of $\psi$. The authors of [28, 27] considered a complex $\psi$, that would imply having $\psi^\dagger$ instead of one of the fields $\psi$ in (1). This eliminates the need for including $\text{sgn}_r$ at all, but whether or not $\text{sgn}_r(-1) = -1$ determines if (1) is symmetric or antisymmetric under

\(^3\)Basically there are 3 distinct non-trivial sign functions determined by $\tau \in \{\epsilon, p, \epsilon p\}$ ($\epsilon$ is a $(p - 1)$-root of unity). One doesn’t always have $\text{sgn}_r(-1) = -1$, such requirement implies the restrictions $\tau \neq \epsilon$ and $p \equiv 3 \text{ mod } 4$. See the appendix A.
the exchange $\psi \leftrightarrow \psi^\ast$. We keep the fields real as they are closer to the Archimedean case.

The action (1) is closely related to a twisted Vladimirov derivative recently studied in \cite{31,32}. It is straightforward to show that

$$
\int_{\mathbb{Q}_p} \psi^{\mu}(x) D^\tau_s \psi^{\nu}(x) dx + \int_{\mathbb{Q}_p} \psi^{\nu}(x) D^\tau_s \psi^{\mu}(x) dx
$$

$$
= (1 - \text{sgn}_r(-1)) \int_{\mathbb{Q}_p^2} \psi^{\mu}(x) \frac{\text{sgn}_r(x - y)}{|x - y|_p^{s+1}} \psi^{\nu}(y) dy dx,
$$

where $D^\tau_s$ is the generalized or twisted Vladimirov derivative defined by

$$
D^\tau_s \psi^{\mu}(x) := \int_{\mathbb{Q}_p} \frac{\psi^{\mu}(y) - \psi^{\mu}(x)}{\text{sgn}_r(x - y)|x - y|_p^{s+1}} dy.
$$

Notice the two terms on the first line of (2) differ only by the exchange of indices $\mu \leftrightarrow \nu$. Thus when we contract with the metric $\eta_{\mu\nu}$ they become equal and we have

$$
\int_{\mathbb{Q}_p} \eta_{\mu\nu} \psi^{\mu}(x) D^\tau_s \psi^{\nu}(x) dx = \frac{1 - \text{sgn}_r(-1)}{2} \int_{\mathbb{Q}_p^2} \psi^{\mu}(x) \eta_{\mu\nu} \frac{\text{sgn}_r(x - y)}{|x - y|_p^{s+1}} \psi^{\nu}(y) dy dx.
$$

We can see then that the fermionic action is almost the same as the bosonic action, except for the inclusion of the sign function and the parameter $s$. To connect with previous work and the Archimedean case, we will eventually make $s = 0$, but for now we leave it general.

Unfortunately, with the inclusion of $\text{sgn}_r$, there is no value of $s$ for which (1) is conformally invariant (invariant under $\text{PGL}(2,\mathbb{Q}_p)$ transformations), although we do have translation invariance. Later in section 4.1 we will implement conformal symmetry directly in the definition of the amplitudes.

### 2.1 Fermionic Green’s function

Action (1) may be rewritten in a simpler quadratic form. Defining a suitable operator $\Delta^\tau_s$, the action is proportional to $\psi \cdot \Delta^\tau_s \psi$. The purpose of this section is to obtain the inverse operator of $\Delta^\tau_s$. The computation is done using Fourier analysis, this is close in spirit to the computation done in \cite{19} for bosonic strings in an external $B$-field. A more rigorous study of Green’s functions for simple Vladimirov derivatives can be found in \cite{33}.

First, we define the function

$$
\mathcal{F}^{\mu\nu}_s(s; x - y) = \eta_{\mu\nu} \mathcal{F}^{\mu\nu}_s(x - y) = \eta_{\mu\nu} \frac{\text{sgn}_r(x - y)}{|x - y|_p^{s+1}}.
$$

The superscript $F$ means we are working with the fermionic sector. $\mathcal{F}^{\mu\nu}_s$ can be regarded as the integration kernel for the operator. Equivalently, we define operator $\Delta^\tau_s$ acting as the convolution with the function $\mathcal{F}^{\mu\nu}_s(\cdot)$,

$$
\Delta^\tau_s \psi^{\mu}(x) = (\mathcal{F}^{\mu\nu}_s * \psi^{\mu})(x).
$$
We also define \( G^{\mu\nu}(s; x - y) = \eta^{\mu\nu} G^{s}_{F}(x - y) \) as the inverse of \( F^{F}_{s} \), such that

\[
\int_{\mathbb{Q}_{p}} F^{F}_{s}(x - z) G^{s}_{F}(z - y) dz = \delta(x - y). \tag{7}
\]

In Fourier space (see A.3), this equation reads

\[
\widehat{G}_{F}^{s}(\omega) = \frac{1}{\widehat{F}^{F}_{s}(\omega)}. \tag{8}
\]

To obtain \( G_{F}^{s} \) we first calculate the Fourier transform of \( F^{F}_{s} \)

\[
\widehat{F}^{F}_{s}(\omega) = \int_{\mathbb{Q}_{p}} \chi(\omega x) \frac{\text{sgn}_{\tau}(x)}{|x|^{s+1}} dx.
\]

\[
= \left\{ \begin{array}{ll}
|\omega|^{s} \text{sgn}_{\tau}(\omega) L(\tau, p) p^{-s-1}, & \tau \neq \epsilon, \\
|\omega|^{s} \text{sgn}_{\tau}(\omega) p^{-s-1} + \frac{1}{p+1}, & \tau = \epsilon, \text{ Re}(s) < 0. 
\end{array} \right\} \tag{9}
\]

Here \( L(\tau, p) = \text{sgn}_{\tau}(p) \sum_{a=1}^{p-1} \text{sgn}_{\tau}(a) \chi(a/p) \), but the exact value is not relevant. Then we have

\[
G_{F}^{s}(x - y) = \int_{\mathbb{Q}_{p}} \chi(-\omega(x - y)) \frac{\text{sgn}_{\tau}(\omega)}{|\omega|^{s} \mathcal{C}(s, \tau)} d\omega
\]

\[
= \left\{ \begin{array}{ll}
\text{sgn}_{\tau}(-1)p|x - y|^{s-1} \text{sgn}_{\tau}(x - y), & \tau \neq \epsilon, \\
\text{sgn}_{\tau}(-1)p|x - y|^{s-1} \text{sgn}_{\tau}(x - y) \frac{1 + p^{-s}}{1 + p^{-s+1}}, & \tau = \epsilon, \text{ Re}(s) < 0, \tag{10}
\end{array} \right\}
\]

where \( \mathcal{C}(s, \tau) \) is the coefficient of \( |\omega|^{s} \text{sgn}_{\tau}(\omega) \) in Eq. (9). We are interested in the case \( \tau \neq \epsilon \). We compensate for the extra factor \( \text{sgn}_{\tau}(-1)p \) by adding it as a coefficient in front of the action (1). In appendix C we show explicitly that, as usual, the Green’s function is the same as the fermion field two-point function.

### 3 A ‘worldsheet’ action for the \( p \)-adic superstring

In this section we propose a prospect of a non-Archimedean superstring action. This action can be compared with the usual Archimedean superstring action in the superconformal gauge. It is also shown that this action satisfies a supersymmetric transformation in the \( p \)-adic context. Moreover a superspace formulation is also provided.

We now propose an action \( I_{S}[X, \psi] = I_{B}[X] + I_{F}[\psi] \) that describes the non-Archimedean superstring, and given by the sum of a bosonic action \( I_{B}[X] \) expressed as

\[
I_{B}[X] = \frac{T_{0}}{2} \int_{\mathbb{Q}_{p}^{2}} \eta_{\mu\nu} \frac{(X^{\mu}(x) - X^{\mu}(y))(X^{\nu}(y) - X^{\nu}(x))}{|x - y|^{2}} dy dx, \tag{11}
\]

and a fermionic action \( I_{F}[\psi] \) given by Eq. (1). Thus \( I_{S}[X, \psi] \) is written as

\[
I_{S}[X, \psi] = \frac{T_{0}}{2} \int_{\mathbb{Q}_{p}^{2}} \eta_{\mu\nu} \frac{(X^{\mu}(x) - X^{\mu}(y))(X^{\nu}(y) - X^{\nu}(x))}{|x - y|^{2}} dy dx
\]
\[
+ \frac{\text{sgn}_\tau(-1)p}{2\alpha'} \int_{\mathbb{R}^3} \psi^\mu(x)\eta_{\mu\nu}\frac{\text{sgn}_\tau(x-y)}{|x-y|^{s+1}} \psi^\nu(y)dydx. \tag{12}
\]

This action can be rewritten as
\[
I_S = -T_0 \int_{\mathbb{R}^p} \eta_{\mu\nu}X^\mu(x)[D^1_\tau X^\nu](x)dx + \frac{\text{sgn}_\tau(-1)p}{2\alpha'} \int_{\mathbb{R}^p} \eta_{\mu\nu}\psi^\mu(x)[\Delta^\tau_\psi \psi^\nu](x)dx, \tag{13}
\]
where \(D^1_\tau\) is the Vladimirov derivative given in Eq. (3) with \(s = 1\) and \(\tau = 1\), and \(T_0 = \frac{p(p-1)}{4(p+1)} \frac{1}{\alpha'}\), (see [14, 15]).

### 3.1 Equations of motion

Now we compute the variation of the action for the bosonic and fermionic fields. Because both fields are real valued, the variation is the same as in the usual Archimedean case. It is given by

\[
I_B[X + \delta X] = -T_0 \int_{\mathbb{R}^p} \eta_{\mu\nu}(X^\mu + \delta X^\mu)(D^1_\tau X^\nu + D^1_\tau \delta X^\nu)
\]

\[
= I_B[X] - 2T_0 \int_{\mathbb{R}^p} \eta_{\mu\nu}\delta X^\mu[D^1_\tau X^\nu] + \mathcal{O}(\delta X^2). \tag{14}
\]

This implies that

\[
\delta I_B[X] = -2T_0 \int_{\mathbb{R}^p} \eta_{\mu\nu}\delta X^\mu(x)[D^1_\tau X^\nu](x)dx. \tag{15}
\]

In the previous computation we used the following fact

\[
\int_{\mathbb{R}^p} f(x)[D^1_\tau g](x)dx = \int_{\mathbb{R}^p} [D^1_\tau f](x)g(x)dx, \tag{16}
\]

which assumes Fubini’s theorem, \(|-1|_p = 1\) and a symmetric \(\eta_{\mu\nu}\). Similarly for the fermionic action we have

\[
I_F[\psi + \delta \psi] = \frac{\text{sgn}_\tau(-1)p}{2\alpha'} \int_{\mathbb{R}^3} \eta_{\mu\nu}\left(\psi^\mu(x) + \delta \psi^\mu(x)\right)\frac{\text{sgn}_\tau(x-y)}{|x-y|^{s+1}}\left(\psi^\nu(y) + \delta \psi^\nu(y)\right)dx dy
\]

\[
= S_F[\psi] + \frac{\text{sgn}_\tau(-1)p}{2\alpha'} \int_{\mathbb{R}^3} \eta_{\mu\nu}\frac{\text{sgn}_\tau(x-y)}{|x-y|^{s+1}}\left(\delta \psi^\mu(x)\psi^\nu(y) + \psi^\mu(x)\delta \psi^\nu(y)\right)dx dy + \mathcal{O}(\delta \psi^2). \tag{17}
\]

This implies that\(^4\)

\[
\delta I_F[\psi] = \frac{\text{sgn}_\tau(-1) - 1}{2\alpha'}p \int_{\mathbb{R}^3} \eta_{\mu\nu}\frac{\text{sgn}_\tau(x-y)}{|x-y|^{s+1}}\delta \psi^\mu(x)\psi^\nu(y)dx dy
\]

\(^4\)We could have started with the equivalent definition of \(S_F \sim \psi \cdot D^\tau_\psi \psi\). If one does this, one would eventually get in the integrand \(\delta \psi \cdot D^\tau_\psi \psi + D^\tau_\psi \psi \cdot \delta \psi + (1 - \text{sgn}_\tau(-1))\delta \psi \cdot \Delta^\tau_\psi \psi\). The first two terms cancel and we end up with the same result as in (18). Notice that this is analogous to a boundary term after integration by parts.
\[
\text{sgn}_r(-1) - 1 \frac{1}{2\alpha'} p \int_{Q_p} \delta\psi^\mu(x)[\Delta^r_{s,\mu\nu}\psi^\nu](x) dx.
\]

These two variations imply the following equations of motion for the bosonic and fermionic fields, \(X\) and \(\psi\)

\[
-2T_0 \eta_{\mu\nu}[D^1_{\mu}X^\nu](x) = 0, \quad \text{sgn}_r(-1) - 1 \frac{1}{2\alpha'} p[\Delta^r_{s,\mu\nu}\psi^\nu](x) = 0.
\]

### 3.2 Supersymmetry transformation

We can see that the proposed action has associated an infinitesimal supersymmetric transformation. From (18) and (15), the variation of the ‘worldsheet’ action \(I_S[X, \psi]\) is written as

\[
\delta I_S[X, \psi] = -2T_0 \int_{Q_p} \eta_{\mu\nu} \delta X^\mu(x)[D^1_{\mu}X^\nu](x) dx + \text{sgn}_r(-1) - 1 \frac{1}{2\alpha'} p \int_{Q_p} \delta\psi^\mu(x)[\Delta^r_{s,\mu\nu}\psi^\nu](x) dx.
\]

Let us insert the following variations:

\[
\delta X^\mu = A\lambda \Delta^r_s\psi^\mu, \quad \delta\psi^\mu = B\lambda D^1_{\mu}X^\mu,
\]

where \(A\) and \(B\) are quantities to determine and \(\lambda\) is a Grassmann parameter of the transformation. Then we have

\[
\delta I_S[X, \psi] = \int_{Q_p} \eta_{\mu\nu} \left(-2T_0 A\lambda[\Delta^r_s\psi^\mu](x)[D^1_{\mu}X^\nu](x) + \text{sgn}_r(-1) - 1 \frac{1}{2\alpha'} pB\lambda[D^1_{\mu}X^\nu](x)[\Delta^r_{s,\mu\nu}\psi^\nu](x) \right) dx
\]

\[
= \int_{Q_p} \left(\lambda[\Delta^r_s\psi] \cdot [D^1_{\mu}X]\right) \left(-2T_0 A + \text{sgn}_r(-1) - 1 \frac{1}{2\alpha'} pB \right) dx,
\]

where we can see that choosing constants \(A\) and \(B\) such that \(-2T_0 A + \text{sgn}_r(-1) - 1 \frac{1}{2\alpha'} pB = 0\), we will get \(\delta I_S[X, \psi] = 0\). Then for example, when \(\text{sgn}_r(-1) = -1\), the transformation

\[
\delta X^\mu = \lambda \Delta^r_s\psi^\mu, \quad \delta\psi^\mu = -\frac{2\alpha'}{p}T_0 \lambda D^1_{\mu}X^\mu,
\]

is an infinitesimal supersymmetric transformation of \(I_S[X, \psi]\).

### 3.3 Superspace description of the \(p\)-adic superstring

It is desirable to have a more efficient and concise approach to describe the \(p\)-adic superstring such as the superspace approach. In [25] a notion of superspace over \(Q_p\) is introduced. Motivated by this, we follow [30] and define a superfield and a superoperator for our model. After some work one can find that the superoperator and the superfield are of the following form

\[
X^\mu(x, \theta) = AX^\mu(x) + B\theta\psi^\mu(x),
\]

\[
D^r_s = a\theta D^1_{\mu} + b\Delta^r_s\partial_{\theta}.
\]

\[7\]
Then
\[ D_s^\tau X^\mu = aA\theta D_1^\tau X^\mu + bB\Delta_s^\tau \psi^\mu, \]
\[ X \cdot D_s^\tau X = aA^2 \theta X \cdot D_1^\tau X + bABX \cdot \Delta_s^\tau \psi + bB^2 \psi \cdot \Delta_s^\tau \psi. \] (25)

In order to write down the action in the following form
\[ I_S[X, \psi] = \int Q_p \int d\theta \eta_{\mu\nu} X^\mu [D_s^\tau X^\nu] dx, \] (26)
we need to choose the constants \(a, b, A, B\), such that
\[ aA^2 = -T_0 \quad \text{and} \quad bB^2 = \text{sgn}_\tau(-1)p/2\alpha'. \] This is underdetermined, however, there is a further constraint that can be considered. We must also require to have the vertex operators given by
\[ \mathcal{V}(k_\ell; y_\ell) = \int d\theta e^{ik_\ell \cdot X_\ell(y_\ell)}. \] (27)

A comparison with Eq. (32) (See below) shows that we need \( A = 1 \) and \( B = -i \). This determines \( a = -T_0 \) and \( b = -\text{sgn}_\tau(-1)p/2\alpha' \). Thus the appropriate choice of constants in Eq. (24) to obtain (26) and (27) is
\[ X^\mu(x, \theta) = X^\mu(x) - i\theta \psi^\mu(x), \] (28)
\[ D_s^\tau = -T_0 \theta D_1^\tau - \frac{\text{sgn}_\tau(-1)p}{2\alpha'} \Delta_s^\tau \partial_\theta. \] (29)

The Green’s function of the differential operator \( D \) is given by
\[ G_s(x - y; \theta, \theta') = \frac{\alpha'}{1-s} \ln \left( |x - y|_p^{1-s} + \text{sgn}_\tau(x - y)(1-s)\theta\theta' \right) \]
\[ = \alpha' \ln |x - y|_p + \alpha' \theta \theta' \frac{\text{sgn}_\tau(x - y)}{|x - y|_p^{1-s}}, \] (30)
and satisfies
\[ \int_{Q_p} D_s^\tau(x - z; \theta) G_s(z - y; \theta, \theta') dz = \delta(x - y)(\theta - \theta'). \] (31)

4 Tree-level amplitudes of the \( p \)-adic superstring

In this section we obtain the tree-level amplitudes of our \( p \)-adic superstring model. They are obtained through the computation of correlation functions of vertex operators.

The \( N \)-point function for this system is given by the insertion of \( N \) vertex operators of the form
\[ \mathcal{V}(y_\ell) = k_\ell \cdot \psi(y_\ell)e^{ik_\ell \cdot X(y_\ell)} = \int d\theta e^{ik_\ell \cdot X(y_\ell) + \theta_\ell k_\ell \cdot \psi(y_\ell)}, \] (32)
where \( y_\ell \) is the insertion point, and \( \theta_\ell \) are auxiliary Grassmann variables. Inserting vertex operators inside the path integral is equivalent to having a generating function with appropriately chosen sources for both fermions and bosons. The integration of the
bosonic part can be obtained in a standard way using the corresponding Green’s function \( G^\mu_{\nu}(x-y) = -\alpha' \log |x-y|_p \) \[19\]. Here we will perform the analogous computation carried out in \[19\], but now for the fermionic sector.

We start by recalling the operator \( \Delta^\tau_{\mu\nu} \) (from now on we will omit the explicit dependance on the parameter \( s \)), and define its inverse \((\Delta^{-1}_\tau)_\mu^\nu \) by

\[
[\Delta^\tau_{\mu\nu} K^\nu](x) = (\mathcal{F}^F_{\mu\nu} * K^\nu)(x) = \int_{Q_p} \mathcal{F}^F_{\mu\nu}(x-y)K^\nu(y)dy \quad (33)
\]

and

\[
[(\Delta^{-1}_\tau)_{\mu\nu} K^\nu](x) = (G^F_{\mu\nu} * K^\nu)(x) = \int_{Q_p} G^F_{\mu\nu}(x-y)K^\nu(y)dy.
\quad (34)
\]

Using Fubini’s theorem it is straightforward to check that

\[
((\Delta^{-1}_\tau)_{\mu}^{\alpha}[\Delta^\tau_{\alpha\nu} K^\nu])(x) = K^\mu(x);
\]

\[(\Delta^\tau_{\mu\nu}[(\Delta^{-1}_\tau)^{\alpha \nu} K_\nu])(x) = K_\mu(x).
\]

Now notice the following identity for general functions \( f \) and \( g \)

\[
\int_{Q_p} f(x)[\Delta^\tau_{\mu\nu} g](x)dx = \int_{Q_p} f(x)\mathcal{F}^F_{\mu\nu}(x-y)g(y)dydx
\]

\[
= \int_{Q_p} \text{sgn}_\tau(-1) \left[ \int_{Q_p} f(x)\mathcal{F}^F_{\mu\nu}(y-x)dx \right] g(y)dy
\]

\[
= \text{sgn}_\tau(-1) \int_{Q_p} [\Delta^\tau_{\mu\nu} f](x)g(x)dx,
\]

where we used \( \mathcal{F}^F_{\mu\nu}(x-y) = \text{sgn}_\tau(-1)\mathcal{F}^F_{\mu\nu}(y-x) \). In words, the operator \( \Delta^\tau_{\mu\nu} \) inside the integral may switch its action to the rest of the integrand at the cost of a sign. From this we can easily get that

\[
\int_{Q_p} [(\Delta^{-1}_\tau)^{\nu \alpha} K_\alpha](x)[\Delta^\tau_{\nu\beta} \psi^\beta](x)dx = \text{sgn}_\tau(-1) \int_{Q_p} K_\alpha(x)\psi_\alpha(x)dx.
\quad (35)
\]

With these results one can verify the following identity

\[
-\frac{1}{2} \int_{Q_p} \psi^\mu(x)[\Delta^\tau_{\mu\nu} \psi^\nu](x)dx + \int_{Q_p} K_\mu(x)\psi^\mu(x)dx
\]

\[
= -\frac{1}{2} \int_{Q_p} \left( \psi^\mu(x) - \text{sgn}_\tau(-1)[(\Delta^{-1}_\tau)^{\nu \alpha} K_\alpha](x) \right) \left[ \Delta^\tau_{\mu\nu} \left( \psi^\nu - \text{sgn}_\tau(-1)[(\Delta^{-1}_\tau)^{\nu \beta} K_\beta] \right) \right](x)dx
\]

\[
+ \frac{1}{2} \int_{Q_p} K_\mu(x)[(\Delta^{-1}_\tau)^{\mu \nu} K_\nu](x)dx,
\quad (36)
\]

that is the fermion analogue of the well known relation

\[
-\frac{1}{2} x^T \cdot A \cdot x + K^T \cdot x = -\frac{1}{2}(x^T - K^T \cdot A^{-1}) \cdot A \cdot (x - A^{-1} \cdot K) + \frac{1}{2} K^T \cdot A^{-1} \cdot K
\]

for the bosonic finite dimensional
case where $A$ is a matrix, $x$ and $K$ are column vectors and $T$ denotes the transpose operation. Thus we are able to obtain the generating function from the path integral with bosonic sources $J^\mu$ and fermionic sources $K^\mu$ in this $p$-adic setting

$$Z[J,K] = \frac{1}{Z} \int D\psi \int DX \exp \left\{ -I_B[X] - I_F[\psi] + \int_{Q_p} J_\mu(x) X^\mu(x) dx + \int_{Q_p} K_\mu(x) \psi^\mu(x) dx \right\}$$

$$= \exp \left\{ \frac{1}{2} \int_{Q_p^2} J^\mu(x) G^{B\mu\nu}(x-y) J^\nu(y) dxdy + \frac{\alpha'}{2 \operatorname{sgn}_r(-1)p} \int_{Q_p^2} K^\mu(x) G^{F\mu\nu}(x-y) K^\nu(y) dxdy \right\}. \quad (37)$$

This generating function is equivalent to the $N$-point function if we use the sources

$$J^\mu(x) = i \sum_{l=1}^N k_l^\mu \delta(x - y_l), \quad K^\mu(x) = \sum_{m=1}^N \theta_m k_m^\mu \delta(x - y_m), \quad (38)$$

and integrate out the $\theta_m$ variables such that

$$\langle V(y_1) \cdots V(y_N) \rangle = \int d\theta_1 \cdots d\theta_N Z[J,K]. \quad (39)$$

The dependence on the insertion points is left implicit in the sources on the right hand side. This is the analogue for the usual basic prescription to obtain $N$-point amplitudes. It is our starting point in order to get explicit expressions for the tree-level amplitudes.

### 4.1 Integral expression

The next step is to obtain a clearer and more explicit expression of amplitudes (39). As mentioned in section 2, we kept $\operatorname{sgn}_r(-1)$ general to exhibit the subtleties that come with it (see Eq. (35)), and so that the previous expressions can also be used if one were to consider complex valued Grassmann fields, for which the action (1) doesn’t identically vanish. However in what follows and for the rest of the paper we will take $\operatorname{sgn}_r(-1) = -1$, otherwise all of the following computations would give identically 0. First we obtain

$$\frac{1}{2} \int_{Q_p^2} K^\mu(x) G^{F\mu\nu}(x-y) K^\nu(y) dxdy = \frac{1}{2} \sum_{m,n=1}^N \theta_m \theta_n k_m^\mu k_n^\nu \int_{Q_p^2} \delta(x-y_m) G^{F\mu\nu}(x-z) \delta(z-y_n) dxdz$$

$$= \frac{1}{2} \sum_{m,n=1}^N \theta_m \theta_n k_m^\mu k_n^\nu G_{\mu\nu}(y_m - y_n)$$

$$= \frac{1}{2} \sum_{m<n} k_m \cdot k_n \left[ \theta_m \theta_n G^{F}(y_m - y_n) + \theta_n \theta_m G^{F}(y_n - y_m) \right]$$

$$= \sum_{m<n} k_m \cdot k_n \theta_m \theta_n G^{F}(y_m - y_n) = \operatorname{sgn}_r(-1)p \sum_{m<n} \theta_m \theta_n k_m \cdot k_n \frac{\operatorname{sgn}_r(y_m - y_n)}{|y_m - y_n|^{1-s}}. \quad (40)$$
Similarly \(^5\)

\[
\frac{1}{2} \int_{\mathbb{Q}_p^2} \mathcal{J}^\mu(x) \mathcal{G}_{\mu\nu}^p(x-y) \mathcal{J}^\nu(y) \, dx \, dy = \alpha' \sum_{m<n} k_m \cdot k_n \log |y_m - y_n|_p. \tag{41}
\]

Using this and the properties of Grassmann variables (see Eq. (74) in Appendix B) we can write

\[
\mathcal{Z}[\mathcal{J}, \mathcal{K}] = \exp \left\{ \sum_{m<n} \alpha' \frac{k_m \cdot k_n}{1-s} \log \left( |y_m - y_n|_p^{1-s} + \text{sgn}_\tau (y_m - y_n)(1-s)\theta_m \theta_n \right) \right\}
\]

\[
= \prod_{m<n} \left( |y_m - y_n|_p^{1-s} + \text{sgn}_\tau (y_m - y_n)(1-s)\theta_m \theta_n \right)^{-\frac{\alpha'}{1-s}}. \tag{42}
\]

We will also make now \(s = 0\) for the rest of the article\(^6\). This is just to make it closer to the Archimedean case and avoid carrying the parameter around. Then finally we define the \(N\)-point amplitudes as follows

\[
\mathcal{A}^{(N)}_p(k) := \mathcal{N} \int_{\mathbb{Q}_p^N} \langle V(k_1; y_1) \cdots V(k_N; y_N) \rangle \prod_{m<n} \text{sgn}_\tau (y_m - y_n) \prod_{i=1}^N dy_i
\]

\[
= \mathcal{N} \int_{\mathbb{Q}_p^N} \prod_{j=1}^N d\theta_j \prod_{m<n} \left( |y_m - y_n|_p + \text{sgn}_\tau (y_m - y_n)\theta_m \theta_n \right)^{\alpha' k_m \cdot k_n} \text{sgn}_\tau (y_m - y_n) \prod_{i=1}^N dy_i, \tag{43}
\]

where \(\mathcal{N}\) is a normalization constant. This is the analogous result of the usual Archimedean result. The factor \(\prod_{m<n} \text{sgn}_\tau (y_m - y_n)\) is added to implement conformal invariance. These amplitudes were also proposed in \([22, 23]\). As we can see it is quite similar to the usual result \([29]\), the only difference is the appearance of the sign functions.

### 4.2 Integrating the Grassmann variables

The purpose of this subsection is to get the tree-level open string amplitudes by carrying out integration (43). If we go back to the first line of (42), we can factorize the

\(^5\)The astute reader may have noticed that we seemingly forgot the case \(m = n\) in the sum. A careful calculation shows that those terms are vanishing. The subtlety lies in the fact that in order to have a well defined integration of the Green’s functions, we must integrate over the \(p\)-adic plane such that \(x \neq y\). This is because \(G^B\) is singular when its argument is 0, and \(G^F\) is ill-defined at 0 (\(\text{sgn}_\tau(0)\) is ill-defined). Then when we integrate one of the deltas in the first line of (4.1), we are left with something proportional to \(\delta(x-y_n)|_{x \neq y_n}\), which is always 0.

\(^6\)Again, a careful reader might doubt about the validity of this value for \(s\), since in (10) we determined that \(\text{Re}(s) < 0\). This is common in physics, when we set \(s = 0\), we mean to take the analytic continuation of the result (10).
exponential and expand the fermionic part
\[
\exp \left\{ \sum_{m<n} \alpha' k_m \cdot k_n \frac{\text{sgn}_r(y_m - y_n)}{|y_m - y_n|_p} \theta_m \theta_n \right\} = \prod_{m<n} \left( 1 + \alpha' k_m \cdot k_n \frac{\text{sgn}_r(y_m - y_n)}{|y_m - y_n|_p} \theta_m \theta_n \right).
\]

(44)

Notice that the terms will always have an even number of \( \theta_n \) variables. Considering that there are \( N \) Grassmann integrations, we conclude that the only nonvanishing amplitudes are for even insertions. From now on we consider to have an even \( N \). Remember that \( \theta_m^2 = 0 \) and terms with less than \( N \) \( \theta \)s will be annihilated by the integrals. Therefore only terms with \( N \) distinct \( \theta_n \) variables survive.

All of these amounts to the amplitudes being composed of \((N - 1)!!\) terms, this is Wick’s theorem. The terms differ in specific permutations of the \( N \) insertion points. We will define these permutations a few lines below, but for now consider them of the form
\[
\theta_{m_1} \theta_{n_1} \cdots \theta_{m_{N/2}} \theta_{n_{N/2}}, \quad m_i < n_i, \ m_i \neq m_j, \ n_i \neq n_j.
\]

(45)

Integrating the \( \theta_m \) variables in (44), we are left with
\[
\alpha' \sum_P (-)^P \prod_{i=1}^{N/2} k_{P(2i-1)} \cdot k_{P(2i)} \frac{\text{sgn}_r(y_{P(2i-1)} - y_{P(2i)})}{|y_{P(2i-1)} - y_{P(2i)}|_p},
\]

(46)

where \( P \) are permutations of the form (45) and \((-)^P\) is its sign. With this the amplitudes (43) can now be written in the Koba-Nielsen form
\[
A_p^{(N)}(k) = \mathcal{N} \int_{Q^N_p} \prod_{m<n} |y_m - y_n|_p^{\alpha' k_m \cdot k_n} \text{sgn}_r(y_m - y_n)
\]

\[
\times \sum_P (-)^P \prod_{i=1}^{N/2} k_{P(2i-1)} \cdot k_{P(2i)} \frac{\text{sgn}_r(y_{P(2i-1)} - y_{P(2i)})}{|y_{P(2i-1)} - y_{P(2i)}|_p} \prod_{i=1}^N dy_i.
\]

(47)

This is made cleaner by defining the following amplitude
\[
A_p^{(N)}(k, P_N) := \mathcal{N} \prod_{m<n} (k_m \cdot k_n)^{q_{mn}(P_N)}
\]

\[
\times \int_{Q^N_p} \prod_{m<n} |y_m - y_n|_p^{\alpha' k_m \cdot k_n - q_{mn}(P_N)} [\text{sgn}_r(y_m - y_n)]^{1+q_{mn}(P_N)} \prod_{i=1}^N dy_i,
\]

(48)

where \( P_N \) is a general permutation of \( N \) elements and \( q_{mn}(P_N) \) is defined as follows
\[
q_{mn}(P) := \begin{cases} 1, & \exists i \in \{1, \ldots, N/2\}; \quad (m, n) = (P(2i-1), P(2i)) \\ 0, & \text{otherwise} \end{cases}
\]

\[
= \sum_{i=1}^{N/2} \delta^P_{m}(2i-1) \delta^P_{n}(2i),
\]

(49)
notice that \( q_{mn} \neq q_{nm} \). It is now a good time to define more rigorously the permutations appearing in the sum in (47). We set \( \tilde{P} \) as the set of permutations of \( N \) elements such that for every \( i \in \{1, \ldots, N/2\} \) we have \( \tilde{P}(2i - 1) < \tilde{P}(2i) \) and \( \tilde{P} \equiv \tilde{P}' \) if \( \tilde{P}(2i - 1) = \tilde{P}'(2j - 1), \tilde{P}(2i) = \tilde{P}'(2j) \) for \( 1 \leq j \leq N/2 \). (All permutations that differ by the exchange of any two pairs of consecutive elements are equivalent.) Notice that we have in total \( \frac{N!}{(N/2)!2^{N/2}} = (N - 1)!! \) elements in \( \tilde{P} \), as we should. Now the amplitudes have a more elegant and deceivingly concise form

\[
A_p^{(N)}(k) = \sum_{\tilde{P}} (-1)^{\tilde{P}} A_p^{(N)}(k, \tilde{P}). \tag{50}
\]

### 4.2.1 Conformal Symmetry

In this section, we proceed to check the conformal invariance of the amplitudes (50). In the usual case, the gauge fixing of the symmetries of worldsheet diffeomorphisms and Weyl transformations can be carried out. However, it is not completely fixed and there is a remnant symmetry on the two-sphere, this is the PSL(2, \( \mathbb{C} \)) symmetry [29]. In the present case, even though the action does not have the conformal symmetry, we will find the conditions under which it can be implemented at the level of the amplitudes. These conditions involved the sign functions. This procedure works for the \( p \)-adic bosonic string where \( k^2 = 2/\alpha' \) [11]. In our case we have \( k^2 = 1/\alpha' \), however, the factors \( |y_m - y_n|_p^{-1} \) coming from the fermionic sector described above save the day. Something similar will happen to the \( \text{sgn}_y \) functions. If you accept that this procedure can be done for our case, go ahead to the next section.

Consider the transformation \( y_m = \frac{a y_m + b}{c y_m + d} \) with \( ad - cb = 1 \) for the integrand in (48) with a given permutation \( \tilde{P} \), then we have

\[
|y_m - y_n|_p^{s_{mn}} = |\bar{y}_m - \bar{y}_n|_p^{s_{mn}} |c\bar{y}_m + d|_p^{-s_{mn}} |c\bar{y}_n + d|_p^{-s_{mn}}, \quad dy_m = |c\bar{y}_m + d|_p^{-2} d\bar{y}_m. \tag{51}
\]

Applying the change of variables, we will encounter the following product

\[
\prod_{m < n} |c\bar{y}_m + d|_p^{-s_{mn}} |c\bar{y}_n + d|_p^{-s_{mn}} = \prod_{m=1}^{N-1} |c\bar{y}_m + d|_p^{-\sum_{n>m} s_{mn}} \prod_{n=2}^{N} |c\bar{y}_n + d|_p^{-\sum_{m<n} s_{mn}} = \prod_{m=1}^{N} |c\bar{y}_m + d|_p^{-\sum_{n \neq m} s_{mn}}. \tag{52}
\]

Of course we must set \( s_{mn} = \alpha' k_m \cdot k_n \). We use the momenta conservation \( \sum_m k_m = 0 \), and that for the open superstrings \( k_m^2 = 1/\alpha' \), to obtain

\[- \sum_{n \neq m} s_{mn} = -\alpha' k_m \cdot \sum_{n \neq m} k_n = -\alpha' k_m \cdot (-k_m) = \alpha' k_m^2 = 1. \tag{53}\]

Then for a fractional linear transformation the integrand of (47) in the new variables will have the extra factor

\[
\prod_{m=1}^{N} |c\bar{y}_m + d|_p \prod_{m=1}^{N} |c\bar{y}_m + d|_p^{-2} \prod_{m=1}^{N} |c\bar{y}_m + d|_p = 1. \tag{54}\]
Here the first product is a result of (52) and (53), the second product comes from the second equality in (51) and third product is the contribution of the fermionic sector (46). We now deal with the sign functions, which is easier. We have

\[
\prod_{m<n} [\text{sgn}_\tau(y_m - y_n)]^{1+q_{mn}} = \prod_{m<n} [\text{sgn}_\tau(\bar{y}_m - \bar{y}_n)]^{1+q_{mn}} \text{sgn}_\tau(c\bar{y}_m + d)\text{sgn}_\tau(c\bar{y}_n + d)
\]

\[\times \prod_{m<n} [\text{sgn}_\tau(c\bar{y}_m + d)]^{q_{mn}} \text{sgn}_\tau(c\bar{y}_n + d)]^{q_{mn}}
\]

\[= \prod_{m<n} [\text{sgn}_\tau(y_m - y_n)]^{1+q_{mn}} \prod_{m=1}^{N} [\text{sgn}_\tau(c\bar{y}_m + d)]^{N-1} \prod_{m=1}^{N} \text{sgn}_\tau(c\bar{y}_m + d)
\]

\[= \prod_{m<n} [\text{sgn}_\tau(y_m - y_n)]^{1+q_{mn}}
\]

(55)

the second product of the third line comes from realizing that for any point \( y_a \), the product \( \prod_{m<n} \text{sgn}_\tau(c\bar{y}_m + d) \) will give us \( [\text{sgn}_\tau(c\bar{y}_a + d)]^{N-a} \). Similarly for \( \prod_{m<n} \text{sgn}_\tau(c\bar{y}_n + d) \) we have \( [\text{sgn}_\tau(c\bar{y}_n + d)]^{a-1} \). Since \( q_{mn} \) is non-zero for each unique pair \( mn \), only one such factor appears per point, this explains the last product in the third line. In the last line we used that \( N \) is even and \( [\text{sgn}_\tau(\cdot)]^2 = 1 \).

With this we have proven the symmetry of the amplitudes. Conformal invariance allows us to fix three insertion points. It is customary to take such points as 0, 1, and \( \infty \). Here is the explicit transformation that does the job

\[x_i = \frac{(y_{N-1} - y_N)(y_1 - y_i)}{(y_1 - y_{N-1})(y_i - y_N)} \Leftrightarrow y_i = \frac{x_i y_N(y_1 - y_{N-1}) + y_1(y_{N-1} - y_N)}{y_N - y_i + x_i(y_1 - y_{N-1})}.
\]

This sends \( x_1 = 0, x_{N-1} = 1, x_N = \infty \). We only transform \( y_i \) for \( i \in \{2, \ldots, N-2\} \). We have

\[y_1 - y_i = \frac{(y_1 - y_N)(y_1 - y_{N-1})}{y_{N-1} - y_N + x_i(y_1 - y_{N-1})} x_i, \quad y_i - y_{N-1} = \frac{(y_1 - y_{N-1})(y_{N-1} - y_N)}{y_N - y_i + x_i(y_1 - y_{N-1})} (1 - x_i)
\]

\[y_i - y_N = \frac{(y_1 - y_N)(y_{N-1} - y_N)}{y_{N-1} - y_N + x_i(y_1 - y_{N-1})}, \quad dy_i = \frac{|y_1 - y_{N-1}|p|y_1 - y_N|p|y_{N-1} - y_N|^2_p}{|y_N - y_i + x_i(y_1 - y_{N-1})|^2_p} dx_i
\]

\[y_i - y_j = \frac{(y_1 - y_{N-1})(y_1 - y_N)(y_{N-1} - y_N)}{(y_{N-1} - y_N + x_i(y_1 - y_{N-1}))(y_{N-1} - y_N + x_j(y_1 - y_{N-1}))} (x_i - x_j).
\]

(56)

Using these expressions one can check that the integrand factorizes into

\[dy_1 dy_{N-1} dy_N \prod_{m=2}^{N-2} |x_m|^{s_1 m + q_1 m} [1 - x_m]^{s_2 (N-1) + q_2 (N-1)} [\text{sgn}_\tau(x_m)]^{1+q_1 m} [\text{sgn}_\tau(1 - x_m)]^{1+q_2 (N-1)} \]

\[\times \prod_{2 \leq m < n \leq N-2} |x_m - x_n|^{s_3 m + q_3 n} [\text{sgn}_\tau(x_m - x_n)]^{1+q_3 m} \prod_{m=2}^{N-2} dx_m.
\]

\(^7\)Had we kept \( s \) arbitrary, we would get \( \prod_{m=1}^{N} |c\bar{y}_m + d|_p^{-s} \) on the right side of (54).
We leave the details to the reader. Having $dy_1dy_{N-1}dy_N$ means that we’ll have the factor $[\text{Vol}(Q_p)]^3$ but this gets canceled after normalizing. Therefore we redefine the amplitudes (48) as
\[
A_p^{(N)}(k, P) = \prod_{m<n} (k_m \cdot k_n)^{q_{mn}}
\]
\[
\times \int_{Q_p^{-2}}^{N-2} \prod_{m=2}^{N-2} |x_m|^{s_{m-1}q_{1m}} |1 - x_m|^{s_{m(N-1)}q_{m(N-1)}} [\text{sgn}_r(x_m)]^{1+q_{1m}} |\text{sgn}_r(1 - x_m)|^{1+q_{m(N-1)}}
\times \prod_{2 \leq m < n \leq N-2} |x_m - x_n|^{s_{mn}q_{mn}} [\text{sgn}_r(x_m - x_n)]^{1+q_{mn}} \prod_{m=2}^{N-2} dx_m.
\]

5 Four-point amplitudes

As an illustrative example, we show the case $N = 4$ in (50). The first ingredients are the permutations $\tilde{P}$, these are $\tilde{P} = \{(1234), (1324) (1423)\}$ with signs $1, -1, 1$. The next step is to determine the non-zero components of $q_{mn}$. They are different depending on the permutation, for example for (1234) only $q_{12}$ and $q_{34}$ are 1 while the other components equal 0.

Then the amplitude (50), after gauge fixing three points as shown in (57), is
\[
A_p^{(4)}(k) = (k_1 \cdot k_2)(k_3 \cdot k_4) \int_{Q_p} |x_2|^{q_{12}k_1k_2^{-1}} |1 - x_2|^{q_{12}k_2k_3}\text{sgn}_r(1 - x_2)dx_2
\]
\[- (k_1 \cdot k_3)(k_2 \cdot k_4) \int_{Q_p} |x_2|^{q_{12}k_1k_2} |1 - x_2|^{q_{12}k_2k_3}\text{sgn}_r(x_2)\text{sgn}_r(1 - x_2)dx_2
\]
\[+ (k_1 \cdot k_4)(k_2 \cdot k_3) \int_{Q_p} |x_2|^{q_{12}k_1k_2} |1 - x_2|^{q_{12}k_2k_3^{-1}}\text{sgn}_r(x_2)dx_2.
\]

Being efficient, we do the following integration
\[
\int_{Q_p} |x|^p [1 - x]^q [\text{sgn}_r(x)]^t_1 = \int_0^1 \int_{Q_p} |x|^p [\text{sgn}_r(x)]^t_1 dx
\]
\[
+ \int_{\mathbb{R}^p} |1 - x|^p [\text{sgn}_r(x)]^t_1 = \int_{\mathbb{R}^p} |x|^p [\text{sgn}_r(x)]^t_1 dx
\]
\[
= \int_{p \mathbb{Z}_p} |x|^p [\text{sgn}_r(x)]^t_1 dx + [\text{sgn}_r(-1)]^p_1 \sum_{a=2}^{p-1} [\text{sgn}_r(a)]^u_1 [\text{sgn}_r(a - 1)]^v
\]
\[
+ [\text{sgn}_r(-1)]^v \int_{p \mathbb{Z}_p} |x|^p [\text{sgn}_r(x)]^t_1 dx
\]
\[
= \begin{cases}
\frac{1-p^{-1}}{p^{v+1}-1} + \frac{1-p^{-1}}{p^{v+1}-1} + 1 - 2p^{-1}, & (t_1, t_2) = (0, 0) \\
\frac{1-p^{-1}}{p^{v+1}-1} - p^{-1} - \frac{1-p^{-1}}{p^{v+1}-1}, & (t_1, t_2) = (0, 1) \\
\frac{1-p^{-1}}{p^{v+1}-1} - p^{-1} = \frac{1-p^{v}}{p^{v+1}-1}, & (t_1, t_2) = (1, 0) \\
p^{-1} - \frac{1-p^{-1}}{p^{v+1}-1} = \frac{p^{v-1}}{p^{v+1}-1}, & (t_1, t_2) = (1, 1)
\end{cases}
\]
With this result, we can easily see that

\[ A_p^{(4)}(k) = (k_1 \cdot k_2)(k_3 \cdot k_4) \left( \frac{1 - p^{\alpha' k_1 k_2 - 1}}{p^{\alpha' k_1 k_2} - 1} \right) - (k_1 \cdot k_3)(k_2 \cdot k_4) \left( \frac{p^{\alpha' k_1 k_3 - 1} - 1}{p^{\alpha' k_1 k_3} - 1} \right) \]

\[ + (k_1 \cdot k_4)(k_2 \cdot k_3) \left( \frac{1 - p^{\alpha' k_2 k_3 - 1}}{p^{\alpha' k_2 k_3} - 1} \right). \]  

This is the same result reported in [23, 22], where it was computed as a direct analogue of the Archimedean expressions. Comparing to the Archimedean result from [30] our amplitudes are very similar in the integral form, and will likely be so for arbitrary points. The main difference is the presence of sign functions, these functions annihilate several terms in the amplitudes when compared to the $p$-adic bosonic string.

6 Final Remarks

In this article we propose a theory of free $p$-adic worldsheet superstrings. An action analogous to the Archimedean case in the superconformal gauge was considered. As usual, the action consists of two terms, a bosonic and a fermionic part. We based our proposal in different works that proposed a fermionic propagator or action. This implies the use of Grassmann valued $p$-adic fields. To prevent the fermionic term from vanishing identically, it is necessary to insert an antisymmetric sign function, i.e. $\text{sgn}_\tau(-1) = -1$. This restricts the possible values of $p$ to roughly half the primes, and $\tau$ to 2 of its non-trivial values. We noticed that the fermionic term is in fact very similar to the bosonic one, the only two differences being the use of a generalized Vladimirov derivative (that includes $\text{sgn}_\tau$) and the order of the derivative is decreased by 1. From this action we were able to find a supersymmetry transformation, and write the action in a superspace formalism, defining a $p$-adic superfield and a derivative superoperator.

Using standard field theory techniques, we obtained the tachyon $N$-point tree amplitudes. We checked that the fermion propagator is equivalent to the corresponding two-point function. This required a functional derivative for $p$-adic fermion fields. Like in the Archimedean case, these amplitudes are non-vanishing only for even $N$. A neat and simple integral form for these amplitudes that is analogous to the Archimedean case can be given, albeit not very useful for computations. Explicit results can be obtained by manipulating the expressions. The procedure is similar to the one of the Archimedean case. One can see that the amplitudes may be obtained as a weighted sum of almost purely bosonic amplitudes, except for the appearance of sign functions in the integrands. Therefore the amplitudes can be integrated using previously developed techniques [20]. They take the form of rational functions with momenta variables as powers of $p$. Previous works have shown that the amplitudes obtained here are integrable and convergent in a certain region of momenta space [16].

Unfortunately, the proposed action is not Möbius invariant, this is independently due to both the sign function, and the order of the Vladimirov derivative in the fermionic term. One may check that the action (1) has translation invariance and if $N$ is a multiple of 4, it is also scale invariant. However this is not sufficient to fix
three points. The guiding principle for $p$-adic theories is the Archimedean counterpart. In that spirit, we choose to implement conformal invariance ($\text{PSL}(2, \mathbb{Q}_p)$ symmetry) by inserting the factor $\prod_{m<n} \text{sgn}_+ (y_m - y_n)$ in the amplitudes, as shown at the end of section 4.1. With this we can gauge fix three points as usual and greatly reduce computations. The 4-point function was obtained explicitly and has crossing symmetry. It coincides with the one obtained in [23].

In terms of Beta functions, the amplitudes are very similar to the Archimedean case. This was exploited in [11, 22] to construct $p$-adic superstring amplitudes. Compared to the $p$-adic bosonic string, the main difference is the appearance of sign functions, their effect is to eliminate several terms after the integration. However, it remains unclear whether these amplitudes can come directly from a Lagrangian. The authors of [19] first pointed out in that having $\text{sgn}_+$ functions breaks the conformal symmetry and therefore one cannot fix three points by symmetry. We find the same conclusion for our proposed Lagrangian. It would be of interest to find a suitable conformally symmetric Lagrangian that directly leads to the amplitudes (43).

As prospects for future work it is worth mentioning that the results obtained here can be generalized in various directions. The work done here was for $\mathbb{Q}_p$, but in principle one can apply it to unramified extensions of the $p$-adic field $\mathbb{Q}_p$. In the spirit of $p$-adic AdS/CFT this would mean having multiple worldsheet coordinates. We have restricted the value of $p$, yet the case $p = 2$ remains to be explored. $\mathbb{Q}_2$ admits antisymmetric sign functions, but they behave very differently from their odd primed partners. Finding more vector amplitudes like the ones proposed in [23] can also be useful to understand better the theory.

Another future direction is including a $B$-field as in [19] in the context of superstrings. Recently it has been explored the idea of the $p$-adic bosonic string as a $p2$-brane defined on the Bruhat-Tits tree [34]. Further generalizations of this idea require the extension to the supersymmetric case. It would be very interesting to pursue the consideration of the results presented in our work to this notion of $p$-branes. Recently, the work [35] found a relation between the bosonic 4- and 5-point amplitudes. It would be interesting to know if this relation can be extended to the superstring case.

Finally, very recently a rigorous study of the $p$-adic bosonic open string amplitudes has been carried out in [21]. In this reference it was shown that in the Euclidean case the limit of a regularized Feynman integral is well defined and the standard non-Archimedean Koba-Nielsen amplitudes are obtained only as the lower term of a series. In this series each term is well defined but the convergence of the series is still an open problem. It would be interesting to implement the procedure followed in [21] for the case of the superstring action considered in the present paper and look for a physical interpretation of those terms in the superstring context.

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A Lightning review of \( p \)-adic numbers

The rational numbers \( \mathbb{Q} \) are topologically incomplete. Sequences of rational numbers exist that do not converge to a rational number (think of subsequently adding all of the digits of \( \pi \) to 3.14). One “fills in the gaps” by adding such limits, this process requires the notion of convergence, that requires a norm. However not all norms are created equal, there are two types, the absolute value \(| \cdot |\), and \( p \)-adic norms, denoted by \(| \cdot |_p\). Throughout this entire paper \( p \) stands for a prime number \((2, 3, 5, \ldots)\). For more details see, for instance, [12].

The field of \( p \)-adic numbers (denoted \( \mathbb{Q}_p \)) is defined as the completion of the rational numbers with respect to the \( p \)-adic norm. Consider a prime number \( p \), and a rational number \( r = \frac{a}{b}p^n \) with \( a \) and \( b \) coprime to \( p \), and \( n \in \mathbb{Z} \). The \( p \)-adic norm is defined as

\[
|r|_p := \begin{cases} 
  p^{-n}, & r \neq 0 \\
  0, & r = 0 \end{cases}
\]

Notice that we have an infinite amount of \( p \)-adic norms, one per prime number. A \( p \)-adic number \( x \neq 0 \) has a unique expansion

\[
x = p^{-v(x)} \sum_{m=0}^{\infty} x_m p^m,
\]

with \( x_m \in \{0, 1, \ldots, p-1\} \); \( x_0 \neq 0 \), \( v(x) \in \mathbb{Z} \) is the valuation or order of \( x \), and now \( |x|_p = p^{v(x)} \). The unit ball is denoted by \( \mathbb{Z}_p := \{ x \in \mathbb{Q}_p; |x|_p \leq 1 \} \). This implies that \( p^k \mathbb{Z}_p = \{ x \in \mathbb{Q}_p; |x|_p \leq p^{-k} \} \), are the balls of radius \( p^{-k} \) centered at 0. The unit circle is \( \mathbb{Z}_p^\times := \{ x \in \mathbb{Q}_p; |x|_p = 1 \} \), that implies \( p^k \mathbb{Z}_p^\times = \{ x \in \mathbb{Q}_p; |x|_p = p^{-k} \} \). We extend the \( p \)-adic norm to \( \mathbb{Q}_p^n \) by taking

\[
|x|_p := \max_{1 \leq i \leq n} |x_i|_p, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.
\]

We define \( v(x) = \min_{1 \leq i \leq n} \{ v(x_i) \} \), then \( ||x||_p = p^{-v(x)} \). The metric space \((\mathbb{Q}_p^n, || \cdot ||_p)\) is a separable complete ultrametric space. \( p \)-Adic balls in multiple dimensions are the product of one-dimensional balls, \( \mathbb{Z}_p^n = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \). However this doesn’t happen for circles, \((\mathbb{Z}_p^2)^\times \neq \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \). From the definition one can see that in fact \((\mathbb{Z}_p^2)^\times = p\mathbb{Z}_p \times \mathbb{Z}_p^\times \sqcup \mathbb{Z}_p^\times \times p\mathbb{Z}_p \sqcup \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \), where \( \sqcup \) is the union of disjoint sets.

A.1 Integration

As a locally compact topological group, \((\mathbb{Q}_p, +)\) has a Haar measure \( dx \), which is invariant under translations, i.e. \( d(x + a) = dx \). If we normalize this measure by the condition \( \int_{\mathbb{Z}_p} dx = 1 \), then \( dx \) is unique. Under scaling the measure behaves as \( d(ax) = |a|_p dx \). The same properties hold for \((\mathbb{Q}_p^n, +)\), where the Jacobian is used for the changes of variable.

As a simple example we do the following integrations

\[
\int_{p^k \mathbb{Z}_p} dx = \int_{\mathbb{Z}_p} d(p^k x) = p^{-k} \int_{\mathbb{Z}_p} dx = p^{-k};
\]
\[ \int_{p^k \mathbb{Z}_p^\times} dx = \int_{p^k \mathbb{Z}_p} dx - \int_{p^{k+1} \mathbb{Z}_p} dx = p^{-k}(1 - p^{-1}). \]

The discrete topology of \( \mathbb{Q}_p \) implies \( p^k \mathbb{Z}_p = \bigcup_{m=k}^{\infty} p^m \mathbb{Z}_p^\times \). From this we can see that

\[ \int_{p^k \mathbb{Z}_p^\times} |x|^a dx = \sum_{m=k}^{\infty} |x|^a \int_{p^m \mathbb{Z}_p^\times} dx = \sum_{m=k}^{\infty} p^{-m(a+1)}(1 - p^{-1}) = \frac{p^{-k(a+1)}}{1 - p^{-(a+1)}}(1 - p^{-1}), \]

if we set \( a = 0 \) the result is the same as in (62).

### A.2 The sign function

We first define the following function known as the Legendre symbol

\[ \left( \frac{a}{p} \right) = \begin{cases} 1 & \text{if } x^2 \equiv a \mod p \text{ has a solution} \\ -1 & \text{if otherwise,} \end{cases} \]

where \( a \) is an integer. This may be thought of as the sign function for the finite field of \( p \) elements \( \mathbb{F}_p \). Now let \( [\mathbb{Q}_p^\times]^2 \) be the multiplicative subgroup of squares in \( \mathbb{Q}_p^\times \), i.e.

\[ [\mathbb{Q}_p^\times]^2 = \{ a \in \mathbb{Q}_p^\times; a = b^2 \text{ for some } b \in \mathbb{Q}_p^\times \}. \]

For \( p \neq 2 \), and \( \epsilon \in \{1, \ldots, p - 1\} \) satisfying \( (\frac{\epsilon}{p}) = -1 \), we have

\[ \mathbb{Q}_p^\times / [\mathbb{Q}_p^\times]^2 = \{1, \epsilon, p, \epsilon p\}. \]

Then any nonzero \( p \)-adic number can be written uniquely as follows

\[ x = \tau a^2, \text{ with } a \in \mathbb{Q}_p^\times \text{ and } \tau \in \mathbb{Q}_p^\times / [\mathbb{Q}_p^\times]^2. \]

Take a fixed \( \tau \in \{\epsilon, p, \epsilon p\} \), and \( x \in \mathbb{Q}_p^\times \), the usual definition of the sign function is given by

\[ \text{sgn}_\tau(x) := \begin{cases} 1 & \text{if } x = a^2 - \tau b^2 \text{ for } a, b \in \mathbb{Q}_p \\ -1 & \text{otherwise.} \end{cases} \quad (63) \]

All the possible \( p \)-adic sign functions are better summarized in the following table (see [27]):

| \( p \equiv 1 \mod 4 \) | \( p \equiv 3 \mod 4 \) |
|------------------------|------------------------|
| \( \text{sgn}_\epsilon(x) = (-1)^{v(x)} \) | \( \text{sgn}_\epsilon(x) = (-1)^{v(x)} \) |
| \( \text{sgn}_p(x) = \left( \frac{x_0}{p} \right) \) | \( \text{sgn}_p(x) = (-1)^{v(x)} \left( \frac{x_0}{p} \right) \) |
| \( \text{sgn}_\epsilon p(x) = (-1)^{v(x)} \left( \frac{x_0}{p} \right) \) | \( \text{sgn}_\epsilon p(x) = \left( \frac{x_0}{p} \right) \) |
A.3 Fourier Transform

Fourier analysis is very similar to the usual Archimedean case. The $p$-adic Fourier transform of a locally constant function $\phi(x)$ is defined as

$$\tilde{\phi}(\omega) = \int_{\mathbb{Q}_p} \chi(\omega x) \phi(x) dx,$$

where $\chi(x) = e^{2\pi i \{x\}_p}$, with $\{x\}_p$ being the fractional part of $x$, i.e. the terms with negative powers of $p$ in (61). One can show the following

$$\int_{p^k \mathbb{Z}_p} \chi(\omega x) dx = \begin{cases} p^{-k}, & |\omega|_p \leq p^k \\ 0, & |\omega|_p \geq p^{k+1} \end{cases},$$

which is used to prove that

$$\delta(x) = \int_{\mathbb{Q}_p} \chi(\omega x) d\omega.$$ 

With this we can obtain the inverse transformation

$$\phi(x) = \int_{\mathbb{Q}_p} \chi^*(x\omega) \tilde{\phi}(\omega) d\omega. \quad (66)$$

B Vertex operators

We briefly review the process for the basic tree amplitudes in the Archimedean superstrings done in Ref. [29]. Consider the vertex operator

$$V(k; X, \psi) = k \cdot \psi : e^{ik \cdot X} = \int d\theta e^{ik \cdot X + \theta \psi}, \quad (67)$$

where $\theta$ is an auxiliary Grassmann variable. It is important to note that the second equality above is a consequence of the Grassmann variables properties.

Now we use the two-point functions [29]

$$\langle X^{\mu}(y_i)X^{\nu}(y_j) \rangle = -\eta^{\mu\nu} \log(y_i - y_j), \quad (68)$$

$$\left\langle \frac{\psi^{\mu}(y_i)}{\sqrt{y_i}} \frac{\psi^{\nu}(y_j)}{\sqrt{y_j}} \right\rangle = \frac{\eta^{\mu\nu}}{y_i - y_j}, \quad (69)$$

and we get that

$$\left\langle \frac{V(k_i; y_i)}{\sqrt{y_i}} \frac{V(k_j; y_j)}{\sqrt{y_j}} \right\rangle = \int d\theta_id\theta_j e^{-k_i \cdot k_j \log(y_i - y_j) - \frac{\theta_i \theta_j}{y_i - y_j}} = \int d\theta_id\theta_j (y_i - y_j - \theta_i \theta_j)^{k_i \cdot k_j}. \quad (70)$$

Now for multiple vertex operators we have

$$\prod_{l=1}^{N} V(k_l; y_l) = \int d\theta_1 \cdots d\theta_N \exp \left\{ \sum_{l=1}^{N} k_l \cdot (iX(y_l) + \theta_l \psi(y_l)) \right\}. \quad (71)$$
Then
\[
\left\langle \prod_{l=1}^{N} \frac{V(k_l; y_l)}{\sqrt{y_l}} \right\rangle = \int d\theta_1 \cdots d\theta_N \\
\times \exp \left\{ \sum_{l,m=1}^{N} k_l \cdot k_m \left( -\left\langle X^\mu(y_l)X^\nu(y_m) \right\rangle - \theta_l \theta_m \left\langle \frac{\psi^\mu(y_l) \psi^\nu(y_m)}{\sqrt{y_l} \sqrt{y_m}} \right\rangle \right) \right\} \\
= \int d\theta_1 \cdots d\theta_N \exp \left\{ \sum_{l,m=1}^{N} k_l \cdot k_m \left( \log(y_l - y_m) - \frac{\theta_l \theta_m}{y_l - y_m} \right) \right\} \\
= \int d\theta_1 \cdots d\theta_N \prod_{l<m} (y_l - y_m - \theta_l \theta_m)^{k_l \cdot k_m}.
\] (72)

This derivation demanded only Fubini’s theorem and changes of variables, both are well defined over the p-adics. Thus in the non-Archimedean setting, we can follow this same path, the main difference would be the two point function, that is described in the main text.

As a side note, the last equality of (72) used the following identity for Grassmann variables \(\theta_i\)
\[
\log(y_i - y_j) + \frac{\theta_i \theta_j}{y_i - y_j} = \log(y_i - y_j) + \log \left( 1 + \frac{\theta_i \theta_j}{y_i - y_j} \right) \\
= \log(y_i - y_j + \theta_i \theta_j).
\] (73)

This is actually quite general, in fact one can easily check that for constants \(A, B\) and \(s\), the following is true
\[
A \log |y_i - y_j| + B \frac{\theta_i \theta_j}{|y_i - y_j|^s} = A \left( \log |y_i - y_j|^s + \frac{Bs}{A} \frac{\theta_i \theta_j}{|y_i - y_j|^s} \right) \\
= \frac{A}{s} \log \left( |y_i - y_j|^s + \frac{Bs}{A} \theta_i \theta_j \right).
\] (74)

This more general identity is used in the non-Archimedean case.

C Functional derivatives

In this appendix we define in more detail a functional derivative for p-adic fermion fields. It is done in a very similar way to the usual bosonic variables. We also use it to obtain the fermion propagator as a two point function. Even though we are using Grassmann variables, commutativeness issues do not arise because we use only pairs of Grassmann variables.

We define the functional derivative for the Grassmann field \(K\)
\[
\frac{\delta Z[K]}{\delta K}(y) = \int d\theta \lim_{\varepsilon \to 0} \frac{Z[K + \varepsilon \theta \delta \cdot (\cdot - y)] - Z[K]}{\varepsilon},
\] (75)
where θ is a Grassmann variable. The dots indicate a missing argument in the deltas. Consider the following partition function with a propagator \( G_{\mu\nu}(x) \) that is antisymmetric (it satisfies \( G_{\mu\nu}(−x) = −G_{\mu\nu}(x) \))

\[
Z[K] = \exp \left\{ \frac{1}{2} \int_{\mathcal{Q}_p^2} K^\mu(x)G_{\mu\nu}(x−y)K^\nu(y)dx\,dy \right\}. \tag{76}
\]

Now let’s first see that

\[
Z[K + \varepsilon \theta \delta_\alpha \delta(-z)] = \exp \left\{ \frac{1}{2} \int_{\mathcal{Q}_p^2} (K^\mu(x) + \varepsilon \theta \delta_\alpha^\mu \delta(x - z))G_{\mu\nu}(x−y)(K^\nu(y) + \varepsilon \theta \delta_\alpha^\nu \delta(y - z))dx\,dy \right\}
\]

\[
= \exp \left\{ \frac{1}{2} \int_{\mathcal{Q}_p^2} K^\mu(x)G_{\mu\nu}(x−y)K^\nu(y)dx\,dy \right\}
\]

\[
\times \exp \left\{ \frac{1}{2} \varepsilon \int_{\mathcal{Q}_p^2} [K^\mu(x)G_{\mu\nu}(x−y)\theta \delta_\alpha^\nu \delta(y - z) + \theta \delta_\alpha^\mu \delta(x - z)G_{\mu\nu}(x−y)K^\nu(y)]dx\,dy \right\}
\]

\[
= Z[K] \exp \left\{ \frac{1}{2} \varepsilon \int_{\mathcal{Q}_p} G_{\mu\alpha}(x - z) [K^\mu(x)\theta - \theta K^\mu(x)] \, dx \right\}
\]

\[
= Z[K] \exp \left\{ \varepsilon \int_{\mathcal{Q}_p} G_{\mu\alpha}(z - x)K^\mu(x) \, dx \right\}
\]

\[
= Z[K] \left[ 1 + \varepsilon \int_{\mathcal{Q}_p} G_{\mu\alpha}(z - x)K^\mu(x) \, dx + \mathcal{O}(\varepsilon^2) \right]. \tag{77}
\]

Thus, we can now obtain the functional derivative

\[
\frac{\delta Z[K]}{\delta K^\mu(y)} = \int d\theta \lim_{\varepsilon \to 0} \left[ \theta \int_{\mathcal{Q}_p} G_{\mu\alpha}(z - x)K^\mu(x) \, dx Z[K] + \mathcal{O}(\varepsilon) \right]
\]

\[
= \int_{\mathcal{Q}_p} G_{\mu\alpha}(y - x)K^\mu(x) \, dx Z[K]. \tag{78}
\]

Our functional derivative (75) follows the Leibniz rule, one can easily check this. Now we can obtain the two point function

\[
\frac{\delta^2 Z[K]}{\delta K^\mu(y_1) \delta K^\nu(y_2)} \bigg|_{K=0} = \left[ G_{\mu\nu}(y_1 - y_2) + \int_{\mathcal{Q}_p} G_{\mu\alpha}(y_1 - x)K^\alpha(x) \, dx \right.
\]

\[
\times \left. \int_{\mathcal{Q}_p} G_{\nu\beta}(y_2 - x)K^\beta(x) \, dx \right] Z[K] \bigg|_{K=0} = G_{\mu\nu}(y_1 - y_2). \tag{79}
\]

One also can easily check that

\[
\frac{\delta}{\delta K^\mu(y)} \exp \left\{ \int_{\mathcal{Q}_p} K^\nu(x)\psi^\nu(x) \, dx \right\} \bigg|_{K=0} = \psi^\mu(x). \tag{80}
\]
Looking at the fermionic part in (37) coming from the action $I_F[\psi]$. We see that indeed

$$\langle \psi^\mu(x) \psi^\nu(y) \rangle = \frac{\alpha'}{\text{sgn}_r(-1)p} G_F^{\mu\nu}(x - y).$$

(81)
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