1. INTRODUCTION

Recall that an automorphism \( \nu \) of a group \( H \) is called involutory if \( \nu \neq id \) and \( \nu^2 = id \). The automorphism \( \nu \) is called almost regular if \( C_H(\nu) \) is finite. Recall that a group \( U \) is uniquely 2-divisible if for each \( u \in U \) there exists a unique \( v \in U \) such that \( v^2 = u \). Note that in particular a uniquely 2-divisible group contains no involutions (i.e. elements of order 2).

The purpose of this paper is to use the techniques introduced in the impressive paper [Sh] of Shunkov, where he proves that a periodic group that admits an almost regular involutory automorphism is virtually solvable (i.e. it has a solvable subgroup of finite index). We prove

**Theorem 1.1.** Let \( U \) be a uniquely 2-divisible group. If \( U \) admits an involutory almost regular automorphism, then \( U \) is solvable.

Our main motivation for dealing with automorphisms of uniquely 2-divisible groups comes from questions about the root groups of special Moufang sets, and those tend to be uniquely 2-divisible; see, e.g., [S]. Indeed, using Theorem 1.1 it immediately follows that

**Corollary 1.2.** Let \( \mathcal{M}(U, \tau) \) be a special Moufang set. If the Hua subgroup contains an involution \( \nu \) such that \( C_U(\nu) \) is finite, then \( U \) is abelian.

Proof. If \( U \) contains involutions, then \( U \) is abelian by [DST] Theorem 5.5, p. 782. If \( U \) does not contain involutions, then by [DS] Proposition 4.6, p. 5840, \( U \) is uniquely 2-divisible, and then by Theorem 1.1 and by the main theorem of [SW], \( U \) is abelian.

The proof of Theorem 1.1 is obtained as follows. First note that if \( U \) is finite, then \( U \) has odd order, so by the Feit-Thompson theorem \( U \) is solvable. Hence we may assume that \( U \) is infinite.

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We let $A$ be a maximal abelian subgroup of $U$ (with respect to inclusion) inverted by $\nu$ (i.e. each element of $A$ is inverted by $\nu$). In Lemma 3.1(2) we show that we can take $A$ to be infinite. We then show that for elements $u_1, \ldots, u_n \in U$, the involutions $u_1\nu u_1^{-1}, \ldots, u_n\nu u_n^{-1}$ in the semi-direct product $U \rtimes \langle \nu \rangle$ invert a subgroup $D \leq A$ with $|A : D| < \infty$ (Proposition 3.3). The next step is to show that $C_U(D)/D$ is finite and solvable (Lemma 3.5). Since $K := \langle \nu u_1\nu u_1^{-1}, \ldots, \nu u_n\nu u_n^{-1} \rangle \leq C_U(D)$, the subgroup $K$ is solvable and $K/Z(K)$ is finite.

Next let $S := \{x \in U \mid x^\nu = x^{-1}\}$. It is easy to see that an element $y \in U$ is in $S$ iff $y = \nu uu^{-1}$, for some $u \in U$, so by the above each finitely generated subgroup $H$ of $R := \langle S \rangle$ is solvable and satisfies the fact that $H/Z(H)$ is finite. It follows that $R'$ is periodic (Proposition 3.6). Using the above mentioned result of Shunkov, we see that $R'$ is solvable, so $R$ is solvable.

As is well known (see [K]), $U = RC_U(\nu)$ and $R \not\leq U$. Since $C_U(\nu)$ is finite and uniquely 2-divisible it has odd order. By the Feit-Thomson theorem, $C_U(\nu)$ is solvable, and this at last shows that $U$ is solvable.

We remark that it is possible that with the aid of the Theorem on page 286 of [HM], one can get even more delicate information on $U$. However, we do not need that, so we do not pursue this avenue further.

2. NOTATION AND PRELIMINARY RESULTS

Notation 2.1. (1) Throughout this paper $U$ is an infinite uniquely 2-divisible group and $\nu \in \text{Aut}(U)$ is an involutory automorphism which is almost regular.

(2) We denote by $G$ the semi-direct product of $U$ by $\nu$, and we identify $U$ and $\nu$ with their images in $G$. We let $\text{Inv}(G)$ denote the set of involutions of $G$.

(3) We let $S := \{x \in U \mid x^\nu = x^{-1}\}$.

(4) The letter $A$ always denotes a fixed infinite maximal (with respect to inclusion) abelian subgroup of $U$ which is inverted by $\nu$ (i.e. all of whose elements are inverted by $\nu$). The existence of $A$ is guaranteed by Lemma 3.1(2) and by Zorn’s lemma.

(5) For each $u \in U$ we denote by $A_u$ the subgroup of $A$ inverted by $\nu uu^{-1}$.

Remark 2.2. (1) Note that for any non-empty subset $T \subseteq U$, the centralizer $C_U(T)$ is a uniquely 2-divisible subgroup of $U$.

(2) Notice that $A$ is uniquely 2-divisible. Also, for any $u \in U$, the subgroup $A_u$ is uniquely 2-divisible.

(3) It is easy to check that $S = \{\nu x^\nu \mid x \in U\}$.

Lemma 2.3 ([K], Lemma 4.1, p. 239). Let the group $H$ be the union of finitely many, let us say $n$, cosets of subgroups $C_1, C_2, \ldots, C_n$:

$$H = \bigcup_{i=1}^n C_i g_i.$$

Then the index of (at least) one of these subgroups in $H$ does not exceed $n$.

Corollary 2.4. Let the group $H$ be the union of finitely many, let us say $n$, subsets $S_1, S_2, \ldots, S_n$:

$$H = \bigcup_{i=1}^n S_i.$$

For each $i$ set $C_i := \langle ab^{-1} \mid a, b \in S_i \rangle$. Then the index of (at least) one of the subgroups $C_1, \ldots, C_n$ in $H$ does not exceed $n$. 
Proof. For each \( i = 1, \ldots, n \), pick an arbitrary \( g_i \in S_i \). Notice that \( S_i \subseteq C_i g_i \) for all \( i \), so \( H = \bigcup_{i=1}^n C_i g_i \) and Corollary 2.4 follows from Lemma 2.3. \( \square \)

**Lemma 2.5.**

1. All involutions in \( G \) are conjugate;
2. \( S = \{ \nu \tau \mid \tau \in \text{Inv}(G) \} \).

Proof. Let \( \tau \in \text{Inv}(G) \). Then \( \tau = x \nu \) for some \( x \in U \). Since \( \tau \) is an involution, \( x \in S \). Let \( y \in U \) be the unique element with \( y^2 = x \). Then \( y \in S \) and \( \tau = y \nu = y \nu y^{-1} \). This shows (1). Part (2) is Remark 2.2(3). \( \square \)

**Lemma 2.6.** Let \( D \) be an abelian uniquely 2-divisible subgroup of \( U \). Then the following hold:

1. \( C_U(D)/D \) is a uniquely 2-divisible group.
2. If \( D \) is inverted by \( \nu \), then \( \nu D \) is an almost regular involutory automorphism of \( C_U(D)/D \).
3. Assume that \( D \) is inverted by \( \nu \) and let \( E/D \) be a subgroup of \( C_U(D)/D \) which is inverted by \( \nu D \). Then \( E \) is inverted by \( \nu \), so, in particular, \( E \) is abelian.

Proof. (1) Set \( C := C_U(D) \). Assume that \( a, b \in C \) and \( a^2 D = b^2 D \). Let \( x, y \in D \) with \( a^2 x = b^2 y \) and let \( u, v \in D \) with \( u^2 = x \) and \( v^2 = y \). Then \( a^2 u^2 = b^2 v^2 \), and since \( a, b \) commute with \( u, v \) we see that \( (au)^2 = (bv)^2 \). Hence \( au = bv \), so \( aD = bD \).

Furthermore let \( aD \subseteq C/D \). Let \( b \in U \) with \( b^2 = a \). Then \( b \in C \) and \( bD \) is the square root of \( aD \) in \( C/D \).

(2) Clearly \( \nu D \) is an involutory automorphism of \( C/D \) (acting via conjugation). Assume that \( aD \in C/D \) centralizes \( \nu D \). Then \( \nu a = \nu d \) for some \( d \in D \). Let \( x \in D \) with \( x^2 = d \). Then \( \nu \) inverts \( x \), and we see that \( \nu a = \nu x^2 \) and \( ax^{-1} \in C_U(\nu) \). It follows that \( C_{C/D}(\nu D) = C_{C(\nu)D}/D \), and since \( \nu \) is almost regular, so is \( \nu D \).

(3) Let \( xD \in C/D \) be an element inverted by \( \nu D \). Then \( \nu x = x^{-1} d \) for some \( d \in D \), and conjugating by \( \nu \) we see that \( x = x^{-1} d^{-1} \), which implies that \( x\nu x^{-1} = x^{-1} d^{-1} \). Thus \( d = d^{-1} \), so \( d = 1 \).

Now let \( e \in E \). Then, by hypothesis, \( eD \) is inverted by \( \nu D \), so \( e\nu = e^{-1} \). \( \square \)

3. The proof of Theorem 1.1

**Lemma 3.1.** Let \( D \) be an abelian subgroup of \( U \) (we allow \( D = 1 \)) such that \( D \) is inverted by \( \nu \) and such that \( C_U(D) \) is infinite. Assume that

\[
(S \cap C_U(D)) \setminus D \neq \emptyset.
\]

Then the following hold:

1. There exists an element \( w \in C_U(D) \setminus D \) which is inverted by \( \nu \) and such that \( C_U((D, w)) \) is infinite.
2. There exists an infinite abelian subgroup of \( U \) which is inverted by \( \nu \).

Proof. (1) Set \( V := C_U(D) \). Then \( V \) is an infinite uniquely 2-divisible group, and \( \nu \) acts on \( V \), so without loss we may assume that \( U = V \) and that \( D \leq Z(U) \).

Pick \( b \in S \setminus D \) (note that \( b \) exists by hypothesis), and write \( b = \nu \tau \) with \( \tau \in \text{Inv}(G) \). Let

\[
u \in U \) with \( u^{-2} = \nu \tau,
\]

\[\]
and note that since $u$ is inverted by both $\nu$ and $\tau$, we have

$$\nu = \tau^u.$$ 

We claim that there exists $h \in C_U(\tau)$ such that $hu$ is inverted by infinitely many involutions of $G$. Suppose for a moment that the claim holds. Note that $hu \notin D$, for all $h \in C_U(\tau)$. Indeed, if $h = 1$, then $hu = u$, and since $b \notin D$ also $u \notin D$. Otherwise if $hu \in D$ and $h \neq 1$, then

$$u^{-1}h^{-1} = (hu)^\tau = h^\tau u^\tau = hu^{-1},$$

and it follows that $u$ inverts $h$, which is not possible in a uniquely 2-divisible group.

Since all involutions in $G$ are conjugate, conjugating $hu$ by an appropriate element we may assume that $\nu$ inverts $hu$, and since $hu$ is inverted by infinitely many involutions, we see that $C_U(hu)$ is infinite. Taking $w = hu$, we are done.

It remains to show the existence of $h$. For each $a \in S$, let

$$s_a := \nu^\tau a$$

and let $\ell_a = s_a^{-2} = s_a$. It is easy to check that since $\ell_a$ is inverted by $\nu$ and $\tau^a$, we have $\tau^{a\ell_a} = \nu$. Hence $\nu = \tau^u$, and hence $h_a := a\ell_a u^{-1} \in C_U(\tau)$.

It follows that $\ell_a = a^{-1}h_a u$. Since both $\ell_a$ and $a$ are inverted by $\nu$ we get after conjugating by $\nu$ that $\ell_a^{-1} = a(h_a u)^\nu = (h_a u)^{-1}$. Notice now that $a\nu \in \text{Inv}(G)$, and it follows that

$$(h_a u)^{au} = (h_a u)^{-1}.$$ 

By hypothesis the set $\{h_a \mid a \in S\}$ is finite since it is contained in $C_U(\tau)$. Further, the set $S$ is infinite. This implies the existence of $h \in C_U(\tau)$ such that the number of involutions $\nu a$ that invert $hu$ is infinite. This proves (1).

(2) If $D$ is finite and $C_U(D)$ is infinite, then, since $\nu$ is almost regular, $(S \cap C_U(D)) \setminus D \neq \emptyset$ (because $S \cap C_U(D)$ is infinite; see Remark 2.2(3)). Hence (2) follows from (1) by starting with $D = 1$ and iterating the process as long as the subgroup $\langle D, w \rangle$ is finite.

\begin{lemma}
Let $x \in U$ and let $s, \nu \in U$ be the unique element such that $s^{-2} = \nu x^{-1} \nu x$. Then $xs \in C_U(\nu)$.
\end{lemma}

\begin{proof}
Notice that $s$ is inverted by $\nu$ and $\nu^x$. Hence

$$1 = s^2 \nu x = s^{-2} \nu x = \nu s^{-1} \nu x,$$

so the lemma holds.
\end{proof}

\begin{proposition}
Let $A$ be as in Notation 2.1(4) and let $u \in U$. Let $A_u$ be as in Notation 2.1(5). Then $[A : A_u] < \infty$.
\end{proposition}

\begin{proof}
Fix $a \in A$ and consider the element

$$\nu^a u.$$ 

This element is in $U$. Let $s \in U$ with $s^{-2} = \nu^a u$. By Lemma 3.2 we get that

$$v_a := aus \in C_U(\nu).$$

Now set

$$\mathcal{M}_a := \{b \in A \mid v_b = v_a\}.$$ 

Notice that since $|C_U(\nu)| < \infty$,

$$\text{the set } \{\mathcal{M}_c \mid c \in A\} \text{ is finite and } A = \bigcup_{c \in A} \mathcal{M}_c.$$
By equation (3.1) we get \( s^{-1} = v_a^{-1}au \), and conjugating by \( \nu \) and noticing that \( \nu \) inverts \( a \) and \( s \) and centralizes \( v_a \), we see that \( s^{-1} = u^{-\nu}av_a \). So we get the equality

\[
v_a^{-1}au = u^{-\nu}av_a,
\]

from which it follows that

\[
u^{-1}bv_a^{-1} = \nu v_a^{-1}b, \quad \forall b \in \mathcal{M}_a.
\]

Let \( c \in \mathcal{M}_a \). Then as in equation (3.3) we get that \( u^{-1}\nu c v_a^{-1} = \nu v_a^{-1}c \), and this together with equation (3.3) yields

\[
u^{-1}c^{-1}bv_a^{-1} = c^{-1}b, \quad \forall b, c \in \mathcal{M}_a.
\]

Since \( \nu \) inverts \( c^{-1}b \in A \), it follows that \( uv_a^{-1}\nu c v_a^{-1} = uv_a^{-1}c \) inverts \( c^{-1}b \). We thus can conclude that

\[
u \nu^{-1} \text{ inverts } \langle bc^{-1} \mid b, c \in \mathcal{M}_a \rangle, \quad \forall a \in A.
\]

By equation (3.2) and by Corollary 2.4, one of the subgroups \( \langle bc^{-1} \mid b, c \in \mathcal{M}_a \rangle \) has finite index in \( A \), so \( |A : A_s| < \infty \) as asserted. \( \Box \)

**Lemma 3.4.** Let \( B \) be a finitely generated abelian subgroup of \( U \) which is inverted by \( \nu \). Then \( A \) contains a subgroup \( A_1 \) of finite index such that \( \langle A_1, B \rangle \) is abelian.

**Proof.** Recall the definition of \( A \) from Notation 2.1(4) and for \( b \in B \) the definition of \( A_b \) from Notation 2.1(5). Let \( \mathcal{B} \) be a finite set of generators for \( B \) and set \( A_1 := \bigcap_{b \in \mathcal{B}} A_b \). By Proposition 3.3 and since \( B \) is finite, \( |A : A_1| < \infty \). Further, for each \( b \in \mathcal{B} \), \( \nu \) and \( \nu b b^{-1} \) invert \( A_1 \), so \( b^2 = \nu b b^{-1} \nu \in C_U(A_1) \) (recall that \( \nu \) inverts \( b \)). Since \( U \) is uniquely 2-divisible, \( b \in C_U(A_1) \). Hence \( B \leq C_U(A_1) \) and the lemma holds. \( \Box \)

**Lemma 3.5.** Let \( D \) be a uniquely 2-divisible subgroup of \( A \) of finite index. Then \( C_U(D)/D \) is finite and solvable.

**Proof.** Set \( C := C_U(D) \) and \( \overline{C} := C/D \). Assume that \( \overline{C} \) is infinite. By Lemma 2.6(1), \( \overline{C} \) is uniquely 2-divisible, and by hypothesis \( A := A/D \) is a finite subgroup of \( \overline{C} \).

Let \( \overline{A} \) be an infinite maximal abelian subgroup of \( \overline{C} \) inverted by \( \nu D \). The existence of \( \overline{A} \) is guaranteed by Lemma 2.6(2) and by Lemma 3.3(2) (with \( \overline{C} \) in place of \( U \)). By Lemma 3.3 (with \( \overline{C} \) in place of \( U \) and \( \overline{A} \) in place of \( B \)), there exists a finite index \( \overline{A}_1 \leq \overline{A} \) such that \( \overline{A}_2 := (\overline{A}_1, \overline{A}) \) is abelian. Note that \( \overline{A}_2 \) is inverted by \( \nu D \), so by Lemma 2.6(3) the inverse image \( A_2 \) of \( \overline{A}_2 \) in \( C_U(D) \) is an abelian subgroup inverted by \( \nu \). Clearly \( A_2 \) properly contains \( A \). This contradicts the maximality of \( A \) and shows that \( \overline{C} \) is finite.

Suppose \( tD \in \overline{C} \) is an involution. Then \( t^2 \in D \), so also \( t \in D \), and we see that \( \overline{C} \) has odd order. By the Feit-Thompson theorem, \( \overline{C} \) is solvable, and the proof of the lemma is complete. \( \Box \)

**Proposition 3.6.** Let \( R := \langle S \rangle \). Then:

1. \( R' \) is a periodic group;
2. \( R \) is solvable.
Proof. (1) We first show that for elements $u_1, \ldots, u_n \in U$ the subgroup $K := \langle \nu u_1 \nu u_1^{-1}, \ldots, \nu u_n \nu u_n^{-1} \rangle$ is solvable and $K/Z(K)$ is finite. By Remark 2.2(3), this will show that

(*) if $H$ is a finitely generated subgroup of $R$, then $H$ is solvable, and $H/Z(H)$ is finite.

Let $D := \bigcap_{i=1}^{n} A_{u_i}$. By the definition of $A_{u_i}$ and by Proposition 3.3 $|A : D| < \infty$ and $D$ is inverted by $\nu, u_1 \nu u_1^{-1}, \ldots, u_n \nu u_n^{-1}$. Also, by Remark 2.2(2), $D$ is uniquely 2-divisible. By Lemma 3.5 $C_U(D)/D$ is finite and solvable, so since $K \leq C_U(D)$, we see that $K/Z(K)$ is finite and solvable. Hence (*) holds.

Next let $g \in R'$. Then there exists a finitely generated subgroup $H$ of $R$ such that $g \in H'$. By (*) and by [A, (33.9), p. 168], $H'$ is finite, so the order of $g$ is finite. This completes the proof of part (1).

(2) By (1), $R'$ is a periodic group, and since $R$ is $\nu$-invariant, $\nu$ is an almost regular automorphism of $R'$. By the main result of Shunkov in [Sh], $R'$ is virtually solvable. But by (*), $R'$ is also locally solvable, so this shows that $R'$ is solvable and hence so is $R$. □

Proof of Theorem 1.1. By Proposition 3.6 $\langle S \rangle$ is solvable. By [K, (3.4), p. 281] (see also [S, Lemma 2.1(1) and Lemma 2.2(1)]), $U = \langle S \rangle C_U(\nu)$ and $\langle S \rangle \leq U$. Since $C_U(\nu)$ is a finite unique 2-divisible group, it has odd order. By the Feit-Thompson theorem it is solvable. Hence $U$ is solvable. □

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