Functional Itô calculus and martingale representation formula for integer-valued random measures

P. Blacque-Florentin
R. Cont
August 4, 2015

Abstract

We develop a calculus for functionals of integer-valued measures, which extends the Functional Itô calculus to functionals of Poisson random measures in a pathwise sense. We show that smooth functionals in the sense of this pathwise calculus are dense in the space of square-integrable (compensated) integrals with respect to a large class of integer-valued random measures. As a consequence, we obtain an explicit martingale representation formula for all square-integrable martingales with respect to the filtration generated by such integer-valued random measures. Our representation formula extends beyond the Poisson framework and allows for random and time-dependent compensators.

Keywords: Martingale representation formula, Functional Itô calculus, Integer-valued measures, Jump processes, Square-integrable martingales

1 Introduction

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $(\mathcal{F}_t)_{t \geq 0}$ generated by a continuous martingale $X$ and a jump measure $J$ with compensator $\mu$. We say that $(\mathcal{F}_t)_{t \geq 0}$ has the martingale representation property if for any square integrable martingale $Y$, there exist $\phi$ and $\psi$ predictable such that:

$$Y(t) = Y(0) + \int_0^t \phi(s) dX(s) + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \psi(s,y)(J - \mu)(dsdy).$$

Moreover, $\phi$ and $\psi$ are essentially unique. It is well known that such a result holds if the filtration is generated by a Brownian motion and a Poisson measure for example, but it also holds for general diffusions and a much larger class of jump measures. See e.g. Cohen [6] for sufficient conditions.

The martingale representation theorem, however, is an existence result and does not provide an explicit representation for the integrands $\phi$ and $\psi$.

In the case of continuous martingales, i.e. $\psi \equiv 0$, the Malliavin calculus provides a characterisation of $\phi$:

$$\phi(s) = E[D_sY(t)|\mathcal{F}_s],$$

where $D$ represents the Malliavin derivative. This result is the well-known Clark-Ocone formula. See e.g. Nualart for background in the continuous case [23].

In the presence of jumps, the problem of finding an explicit representation appears in many applications such as hedging, control of jump processes, or BSDEs with jumps, and has been approached through various methods in the literature.

For the jump part, $\psi$ takes the form

$$\psi(s,y) = pE[D_{s,y}Y(t)|\mathcal{F}_s],$$

where $pE[\cdot|\mathcal{F}_s]$ is the predictable projection with respect to $\mathcal{F}_s$, and $D$ is an appropriate Malliavin-type operator, for which many constructions have been proposed. Bismut [4] constructs $D$ as a perturbation of the probability measure, which is essentially the same as perturbing the intensity of an infinity of jumps. Løkka [21] makes use of chaos expansion and shows that this approach is equivalent to a Picard “addition of mass” operator. Jacod-Méleard and Protter [14] use Markov semigroup theory. Léon et al [20] introduce a quotient operator. All these operators are different, but their predictable projections all coincide. Also, note that in all of these approaches, the jump component is either Poisson or Lévy.
Functional Itô calculus, introduced by Dupire [12] and inspired by Föllmer’s pathwise stochastic integration [33], has been used in the continuous case by Cont and Fournié [7] to provide a direct pathwise expression for the integrand, rather than constructing an operator and taking predictable projections.

Here, we extend this approach to include a jump component. In order to do so, we introduce a pathwise calculus on the space of σ-finite integer-valued measures, and then use it to provide an explicit version of the martingale representation formula for functionals of integer-valued measures, in passing extending the functional Itô calculus framework to integer-valued measures.

Section 2 defines the framework by introducing the spaces of measures of interest, and defining the functionals on such measures.

We use this framework in section 3 to obtain a martingale representation formula for pure-jump martingales. Notice that we do not require the jump measure generating the filtration to be Poisson or Lévy. All that is required is the absolute continuity of its compensator with respect to time.

The more general form of the theorem, where a diffusion part is allowed, is given in Section 4. Section 5 provides a short note on the influence of changes of measures on the pathwise operator, and 6 provides some examples of application.

Section 7 provides a proof of the density of simple processes in \( L^2_\mu \), where \( \mu \) is only assumed to be absolutely continuous with respect to time.

2 Functionalities of integer-valued measures

In the present section, we introduce the definitions relative to functional Itô calculus on integer-valued measures.

2.1 Definitions on measures

In the rest of the paper, we shall use \( \mathcal{B}(A) \) to denote the Borel σ-algebra on the set \( A \). Also \( \mathbb{R}^d \) denotes the space \( \mathbb{R}^d \) without the origin.

**Definition 2.1.** (Space of σ-finite integer-valued measures) Denote by \( \mathcal{M}([0, T] \times \mathbb{R}^d_0) \) the space of σ-finite simple integer-valued measures on \([0, T] \times \mathbb{R}^d_0\). For a measure \( j : \mathcal{B}([0, T] \times \mathbb{R}^d_0) \to \mathbb{N} \cup \{ +\infty \} \),

\[
j \in \mathcal{M}([0, T] \times \mathbb{R}^d_0) \Leftrightarrow j(\cdot) = \sum_{i=0}^{\infty} \delta_{(t_i, z_i)}(\cdot) \text{ and is finite on compacts,}
\]

with \((t_i)_{i\in\mathbb{N}} \in [0, T]^\mathbb{N}\) not necessarily distinct nor ordered, and \((z_i)_{i\in\mathbb{N}} \in (\mathbb{R}^d_0)^\mathbb{N}\). For convenience, we denote \( \mathcal{M}([0, T] \times \mathbb{R}^d_0) \) by \( \mathcal{M}_T \) throughout this article. We equip this space with a σ-algebra \( \mathcal{F} \) such that the mapping \( j \mapsto j(A) \) is measurable for all \( AB([0, T] \times \mathbb{R}^d_0) \), the Borel σ-algebra on \([0, T] \times \mathbb{R}^d_0\).

**Definition 2.2.** (Stopped measure) For any \((t, j) \in [0, T] \times \mathcal{M}([0, T] \times \mathbb{R}^d_0)\), we define the stopped measure

\[
j(\cdot) := j(\cdot \cap [0, t] \times \mathbb{R}^d_0).
\]

Similarly, we write

\[
j(\cdot) := j(\cdot \cap ([0, t] \times \mathbb{R}^d_0)).
\]

**Definition 2.3.** (Space \( \Omega \) of processes and canonical process) We identify the space of processes

\[
\Omega := \mathcal{M}_T
\]

equipped with the σ-algebra \( \mathcal{F} \) as in Definition 2.1 and we define a measure-valued process \( Y \) on \((\Omega, \mathcal{F})\) with values in \((\mathcal{M}_T, \mathcal{F})\) as a family \((Y(t))_{t \geq 0}\) of mappings

\[
Y : [0, T] \times \Omega \to \mathcal{M}_T.
\]

We define the canonical measure-valued process \( J \) as follows: for any \( \omega \in \Omega \),

\[
J(t, \omega, \cdot) := \omega(\cdot) = j(\cdot),
\]

i.e. the measure \( j \) stopped at time \( t \).

**Definition 2.4.** (Filtration generated by \( J \)) We define the filtration generated by the canonical process

\[
J : t, j, \cdot \mapsto j(t)(\cdot) : \mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]} \text{ on } \mathcal{M}_T \text{ as the increasing sequence of } \sigma\text{-algebras}
\]

\[
\mathcal{F}_t = \sigma(J_s(\cdot), s \in [0, t]).
\]
Now, we define non-anticipative functionals on the space:

**Definition 2.5.** (Non-anticipative functional process) A non-anticipative functional $F$ is a map $F : [0, T] \times \mathcal{M}_T \to \mathbb{R}$ such that $F(t, j) = F(t, j_t)$, such that $F$ is measurable with respect to the product $\sigma$-algebra $\mathcal{B}([0, T]) \times \mathcal{F}$, and such that for all $t \in [0, T], F(t, \cdot)$ is $\mathcal{F}_t$-measurable. We denote $\mathcal{O}$ the space of such functionals.

**Definition 2.6.** (Predictable functional process) A predictable functional $F$ is a non-anticipative functional such that $F(t, j) = F(t, j_t) = F(t, j_{t-})$, and so $F(t, \cdot)$ is $\mathcal{F}_t$-measurable. We write $\mathcal{P}$ the space of predictable functional processes, and we have $\mathcal{P} \subset \mathcal{O}$.

**Definition 2.7.** (Functional fields) A non-anticipative functional field $\Psi$ is a map $\Psi : [0, T] \times \mathbb{R}_d^0 \times \mathcal{M}_T \to \mathbb{R}$ such that $\Psi(t, z, j) = \Psi(t, j_t)$, such that $\Psi$ is measurable with respect to the product $\sigma$-algebra $\mathcal{B}([0, T] \times \mathbb{R}_d^0) \times \mathcal{F}_T$, and for all $(t, z) \in [0, T] \times \mathbb{R}_d^0, \Psi(t, \cdot)$ is $\mathcal{F}_t$-measurable. We denote $\mathcal{O}_f$ the space of such functionals.

Similarly, we call predictable functional field any $\Psi \in \mathcal{O}_f$ such that $\Psi(t, z, j) = \Psi(t, j_{t-})$, and we denote by $\mathcal{P}_f$ the space of such predictable functional fields.

**Example 1.** Any integral functional $F(t, j) = \int_0^t \int_{\mathbb{R}_d^0} f(s, z) j(dsdz)$, with $f : [0, T] \times \mathbb{R}_d^0 \to \mathbb{R}$ having compact support in $[0, T] \times \mathbb{R}_d^0$ is well-defined and non-anticipative.

**Definition 2.8.** The operator $\nabla_{j, z}$ is defined on non-anticipative functional processes as

$$\nabla_{j, z} F(t, j_t) = F(t, j_{t-} + \delta(t, z)) - F(t, j_{t-}).$$

We also define the operator $\nabla_p$ as follows:

**Definition 2.9.** The operator $\nabla_p$ that maps functional processes to predictable functional fields is defined as

$$\nabla_p : \mathcal{O} \to \mathcal{P}_f,$$

$$F \mapsto \nabla_p F$$

where

$$(\nabla_p F)(t, z, j) = \nabla_{j, z} F(t, j) = F(t, j_{t-} + \delta(t, z)) - F(t, j_{t-}).$$

### 2.2 Compensated integral functionals and simple predictable functionals

**Definition 2.10.** (\(\sigma\)-finite predictable measure) We call \(\sigma\)-finite predictable measure any \(\sigma\)-finite measure $\mu : \mathcal{B}([0, T] \times \mathbb{R}_d^0) \times \mathcal{M}_T \to \mathbb{R}^+$ which satisfies for $A \in \mathcal{B}([0, T] \times \mathbb{R}_d^0)$:

$$_{t \leq} \mu(A, j_t) = \mu(A, j_{t-}).$$

**Definition 2.11.** (Stopping time) A stopping time $\tau$ is a non-anticipative mapping $\tau : \mathcal{M}_T \to [0, T]$ such that for any $t \in [0, T]$

$$\mathbb{I}_{\tau(j) \leq t} = \mathbb{I}_{\tau(j_t) \leq t}.$$

Moreover, $\tau$ is a predictable stopping-time if

$$\mathbb{I}_{\tau(j) \leq t} = \mathbb{I}_{\tau(j_{t-}) \leq t}.$$
Example 2. For all \( \epsilon > 0 \), \( Z \in \mathcal{B}(\mathbb{R}^d_0), 0 \not\in \overline{Z} \) (the closure of \( Z \)), and a \( \sigma \)-finite predictable measure \( \mu \),

\[
\tau^*(j, Z) = \inf\{t \in [0, T] | \mu([0, t] \times Z, j) \geq \epsilon\}
\]

is a stopping time. If \( \mu \) is absolutely continuous with respect to time, \( \tau \) is also predictable.

For convenience, write

\[
C^\alpha_t(j) = j([s] \times \{ \frac{1}{\alpha} < |z| \leq \alpha \}).
\]

for \( \alpha \geq 1 \).

As a typical integrand for functional integrals, we introduce the following space of simple predictable functionals:

Definition 2.12. (Set \( \mathcal{S} \) of simple predictable functionals) The functional \( \psi : [0, T] \times \mathbb{R}^d_0 \times \mathcal{M}_T \rightarrow \mathbb{R} \) belongs to \( \mathcal{S} \), the space of simple predictable functionals, if

- for any \((t, z) \in [0, T] \times \mathbb{R}^d_0\), \( \psi(t, z, \cdot) \) is \( \mathcal{F}_t \)-measurable.
- there exists \( I \) grids \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = T \) such that

\[
\psi(t, z, j_1) = \sum_{i=0}^{I} \psi_{ik}(j_{\tau_i}) \mathbb{1}_{(\tau_i, \tau_{i+1}-]}(t) \mathbb{1}_{A_k}(z)
\]

with \( A_k \in \mathcal{B}([0, T] \times \mathbb{R}^d), 0 \not\in \overline{A_k} \), the \( \tau_i \) are predictable stopping times (allowed to depend on the \( Z_k \)) and

\[
\psi_{ik}(j_{\tau_i}) = g_{ik}(\{(C_t^m(j))_{t \in [1..p, m \in 1..k]}\}.
\]

where \( g_{ik} : \mathbb{R}^p \times \mathbb{R}^k \rightarrow \mathbb{R}, 0 \leq t_1 \leq \cdots \leq t_p \leq \tau_i \) and \( 1 \leq \epsilon_1 \leq \cdots \leq \epsilon_k \) is Borel-measurable.

Proposition 2.13. For \( \psi \in \mathcal{O}_{\mathcal{F}_c} \), the subset of elements of \( \mathcal{O}_I \) with compact support in \([0, T] \times \mathbb{R}^d_0\), the compensated integral functional

\[
F(t, j_t) = \int_0^t \int_{\mathbb{R}^d_0} \psi(s, y, j_{s-})(j - \mu)(ds \ dy)
\]

has finite value and

\[
\nabla_{j, z} F(t, j_t) = \psi(t, z, j_{t-}),
\]

hence

\[
(\nabla_P F)(t, z, j_t) = \psi(t, z, j_{t-}),
\]

defines a predictable functional field. In particular, if \( \psi \in \mathcal{P}_{\mathcal{F}_c} \), the set of predictable fields with compact support, then \( \nabla_P \) is the inverse of the Lebesgue-Stieltjes integral operator defined as:

\[
\int_{\mathcal{I}_{LS}} : \mathcal{P}_{\mathcal{F}_c} \rightarrow \mathcal{I}(\mathcal{P}_{\mathcal{F}_c}) \subset \mathcal{O},
\]

\[
\psi \mapsto \int_0^t \int_{\mathbb{R}^d_0} \psi(s, z, j_{s-})(j - \mu)(ds \ dy),
\]

with \( \mathcal{I}(\mathcal{P}_{\mathcal{F}_c}) \) the image of \( \mathcal{P}_{\mathcal{F}_c} \) through \( \int_{\mathcal{I}_{LS}} \).

Proof. Let us first check that \( F \) has finite value: notice that \( \psi \) has compact support in \( \mathbb{R}^d_0, 0 \not\in \text{supp}(\psi) \) and so \( j \) can only have a finite number of jumps on the support of \( \psi \). Moreover, since \( \mu \) is absolutely continuous, the integral of \( \psi \) with respect to \( j - \mu \) is finite. Using the pathwise predictability of \( \psi(s, y, j_{s-}) \) and \( \mu \), we have

\[
F(t, j_{t-} + \delta_{(t, z)}) = F(t, j_{t-}) + \psi(t, z, j_{t-}).
\]

So for all \( t \) and \( z \),

\[
\nabla_{j, z} F(t, j_t) = \psi(t, z, j_{t-}).
\]

In particular, if \( t, z, j \mapsto \psi(t, z, j_t) \) is predictable, then \( \psi(t, z, j_{t-}) = \psi(t, z, j_t) \) and \( \nabla \) is indeed the inverse of \( \int_{\mathcal{I}_{LS}} \) on \( \mathcal{I}(\mathcal{P}_{\mathcal{F}_c}) \).

So \( \nabla_P \) is an operator that maps a non-anticipative functional to a predictable functional, and if \( F \) is the functional in Proposition 2.13 then \( \nabla_P F \) recovers the integrand.
3 Martingale representation formula: purely discontinuous case

In this section, we will focus on the vertical perturbation operator $\nabla$ which we lift from the functional framework to the space of processes in order to obtain a martingale representation formula. Consider the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{P})$, where $(\Omega, \mathcal{F})$ is the measurable space of $\sigma$-finite integer-valued measures as defined in the previous section, $J$ is the canonical process on $(\Omega, \mathcal{F})$ and

1. $\mathbb{P}$ is a probability measure on the space $(\Omega, \mathcal{F})$ such that the sum of the squared amplitudes of $J$ is finite a.s. $\int_0^T \int_{\mathbb{R}_0^d} |z|^2 J(dsdz) < \infty$ a.s.

2. $\mathcal{F} := (\mathcal{F}_t^0)_{t \in [0, T]}$ is the filtration defined by the canonical process $J$, completed by the $\mathbb{P}$-null sets.

**Remark 3.1.** $J$ now defines a random $\sigma$-finite integer-valued measure. We denote by $\mu$ its compensator.

We denote by $T_k$ and $Z_k$ the jump times and jump amplitudes of the atoms of $J$ respectively. So

$$J = \sum_{i=1}^{\infty} \delta_{T_k, Z_k} (\cdot).$$

So one has that

$$\mathcal{F}_t^0 = \sigma \{ (T_k, Z_k), T_k \leq t \},$$

completed with the $\mathbb{P}$-null sets.

**Assumption 1.** We assume absolute continuity with respect to time of the compensator $\mu$:

$$\mu(ds\ dy, \omega) << ds.$$

**Remark 3.2.** Recall that the compensator is a predictable measure by definition.

In the following, we denote by $\hat{J}$ the compensated random measure $J - \mu$.

We introduce the following two spaces:

$$L^2_\mathbb{P}(\mu) := \left\{ \psi : [0, T] \times \mathbb{R}_0^d \times \Omega \to \mathbb{R} \text{ measurable} \middle| \mathcal{E} \int_0^T \int_{\mathbb{R}_0^d} \psi(s, y)^2 \mu(ds\ dy) < \infty \right\},$$

equipped with the norm

$$\|\psi\|_{L^2_\mathbb{P}(\mu)}^2 := \mathcal{E} \int_0^T \int_{\mathbb{R}_0^d} \psi(t, z)^2 \mu(dt dz),$$

and

$$M^2_\mathbb{P}(\mu) := \left\{ Y : [0, T] \times \Omega \to \mathbb{R} \middle| Y(t) = \int_0^t \int_{\mathbb{R}_0^d} \psi(s, y)\hat{J}(ds\ dy), \psi \in L^2_\mathbb{P}(\mu) \right\},$$

equipped with the norm

$$\|Y\|_{M^2_\mathbb{P}(\mu)}^2 := \mathcal{E} \|Y(T)\|^2.$$

In this setting, the compensated integral operator has the precise following definition:

$$I_{\mu : L^2_\mathbb{P}(\mu) \to M^2_\mathbb{P}(\mu)},$$

$$\psi \mapsto \int_0^t \int_{\mathbb{R}_0^d} \psi(s, y)\hat{J}(ds\ dy).$$

**Definition 3.3.** We denote by $I(\mathcal{O}_{fc})$ the subspace of $M^2_\mathbb{P}(\mu)$ such that its elements can be represented as

$$Y(t) = F(t, J_t),$$

with $F$ a non-anticipative functional of the form

$$F(t, j) = \int_0^t \int_{\mathbb{R}_0^d} \psi(s, y, j_{s-})(j(ds\ dy) - \mu(dsdy, j_{s-})), \quad \psi \in \mathcal{O}_{fc}.$$
Remark 3.4. Recall that for \( \phi \in I(O_{fc}) \), by Proposition 2.13 we can rewrite \( \nabla_p \) more specifically:

\[
\nabla_p : I(O_{fc}) \to L^2_p(\mu),
\]

\[
F(t, J_t) = \int_0^t \int_{\mathbb{R}} \psi(s, y, J_{t-}) \tilde{J}(ds \, dy) \mapsto \nabla_p F(t, J_{t-}) = \psi(t, z, J_{t-}).
\]

So far, \( \nabla_p \) is defined on the space of processes that are integrals of fields in \( O_{fc} \). However, the following lemma allows us to close \( \nabla_p \) to the whole \( M^2_p(\mu) \) space.

Lemma 3.5. The set of cylindrical random variables

\[
f((C_{k_j}^j)_{i \in 1..n, j \in 1..p}) \quad (10)
\]

with \( f : \mathbb{R}^n \to \mathbb{R} \) bounded, and the \( C \) as in equation (5), is dense in \( L^2(F^0_t, \mathbb{P}) \) (the space of random variables with finite second moment).

Proof. Let \((t_i)_{i \in \mathbb{N}}\) be a dense subset of \([0, T]\), and \((k_j)_{j \in \mathbb{N}}\) a dense subset of \([1, \infty)\). Denote

\[
\mathcal{F}^n,m = \sigma((T_k, Z_k) | T_k \leq t_n, \frac{1}{k_m} < Z_k \leq k_m),
\]

i.e. the (completed) filtration generated by the \((C_{k_j}^j (J_i))_{i \in 1..n, k \in 1..m}\). One has

\[
\mathcal{F}^n,m \subset \mathcal{F}^{n+1,m}
\]

Moreover, \( F^0_T \) is the smallest \( \sigma \)-algebra containing all the \( \mathcal{F}^n,m \).

For \( g \in L^2(F^0_t, \mathbb{P}) \),

\[
g = E[g | F^0_t].
\]

The martingale convergence theorem yields

\[
g = \lim_{m \to \infty} \lim_{n \to \infty} E[g | \mathcal{F}^n,m],
\]

Moreover, for each \( n \) and \( m \), there exists a \( \mathcal{F} \)-measurable random variable \( h_{nm} \) (i.e. a random variable measurable with respect to the non-completed filtration) such that, \( \mathbb{P} \)-a.s.,

\[
E[g | \mathcal{F}^n,m] = h_{nm}.
\]

(see e.g. Lemma 1.2. in Crauel [9].)

Finally, the Doob-Dynkin lemma ([15], p. 7) yields that for each couple \((n, m)\), there exists a Borel-measurable \( g_{nm} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) with

\[
E[g | \mathcal{F}^n,m] = g_{nm}((C_{k_j}^j (J_i))_{i \in 1..n, j \in 1..m}), \mathbb{P} \text{-} a.s.
\]

(13)

Lemma 3.6. The space \( I(S) \) is dense in \( M^2_p(\mu) \).

Proof. The set of simple random fields of the form

\[
\psi(t, z, \omega) = \sum_{i,k=1}^{I,K} \psi_{ik}(\omega) \mathbb{1}_{(t_i,t_{i+1}]}(t) \mathbb{1}_{Z_k}(z),
\]

with \( Z_k \) Borel-sets of finite measure, the \( t_i \) \( F^0 \)-measurable stopping-times and the \( \psi_{ik}(\omega) \) \( F^0_{t_i} \)-measurable, is dense in \( L^2_p(\mu) \) (see Subsection [7]).

Moreover, from lemma [3.5] the \( \psi_{ik} \) can be approximated by some sequence of functionals

\[
f^n_{np}(\omega) := f_{np}((C_{k_j}^j)_{i \in 1..n, j \in 1..p}).
\]
Hence, \( Y = \psi \). Then, by the Itô isometry on the stochastic integral, in the sense of the following integration by parts:

\[
E \left[ \sum_{i,k=1}^{I,K} (\psi_{ik}(\omega) - f_{ik}(\omega))^2 \right] \mu([t_i, t_{i+1}] \times Z_k, \omega),
\]

using the disjointness of the time and space intervals. Since the term in the expectation is bounded, it tends to zero by the dominated convergence theorem.

Moreover, by Lemma 1.2. in Crauel [9], there exists \( \mu^- \) and \( (t_i^-)_{i=1..n} \), that are measurable with respect to the non-completed filtration and such that, \( \mathbb{P} \)-a.s.:

\[
\mu = \mu^-, t_i = t_i^-,
\]

and so

\[
\lim_{n \to \infty} \mu \left( \int_{t_i^-}^{t_i} |\nabla \psi\rangle d\mu \right) = 0.
\]

So \( \mathcal{S} \) is dense in \( L^2(\mu) \). Since the stochastic integral operator defines a bijective isometry (hence a continuous bijection) between \( \mathcal{M}_2(\mu) \) and \( L^2(\mu) \), \( I(\mathcal{S}) \) is dense in \( \mathcal{M}_2(\mu) \).

**Lemma 3.7.** The operator \( \nabla_\mu : I(\mathcal{S}) \to L^2(\mu) \) is closable in \( \mathcal{M}_2(\mu) \), and its closure \( \nabla_\mu \) is the adjoint of the stochastic integral, in the sense of the following integration by parts:

\[
< Y, I_\mu(\psi) >_{\mathcal{M}_2(\mu)} := E \left[ Y(T) \int_0^T \int_{\mathbb{R}^d} \psi(s, y) \nabla_\mu \psi(s, y) \mu(ds, dy) \right] = E \left[ \int_0^T \int_{\mathbb{R}^d} \nabla_\mu Y(s, y) \psi(s, y) \mu(ds, dy) \right] =: < \nabla_\mu Y, \psi >_{L^2(\mu)}.
\]

**Proof.** By definition, for any \( Y \in \mathcal{M}_2(\mu) \), there exists \( \psi \in L^2(\mu) \) such that

\[
Y(t) = \int_0^t \int_{\mathbb{R}^d} \psi(t, z) \nabla_\mu \psi(t, z) dt dz.
\]

For any \( Z \in I(\mathcal{S}) \), we have

\[
E [Y(T)Z(T)] = E \left[ \int_0^T \int_{\mathbb{R}^d} \psi(t, y) \nabla_\mu Z(t, y) dt dy \right] = E \left[ \int_0^T \int_{\mathbb{R}^d} \nabla_\mu Y(t, y) \nabla_\mu Z(t, y) dt dy \right].
\]

Then, by the Itô isometry on \( \mathcal{M}_2(\mu) \),

\[
E [Y(T)Z(T)] = E \left[ \int_0^T \int_{\mathbb{R}^d} \psi(s, y) \nabla_\mu Z(s, y) d\mu(ds, dy) \right].
\]

(17)

uniquely characterises \( \psi d\mu \times d\mathbb{P}\)-a.e. For if \( \eta \) is any either solution, and for all \( Z \in I(\mathcal{S}) \):

\[
< Y - I_\mu(\eta), Z >_{\mathcal{M}_2(\mu)} = E [(Y - I_\mu(\eta))(T)Z(T)] = 0
\]

Hence, \( Y - I_\mu(\eta) = 0 \) \( \mathbb{P} \)-a.s. on \( \mathcal{M}_2(\mu) \) by density of \( Z \in I(\mathcal{S}) \) in \( \mathcal{M}_2(\mu) \) and so \( \nabla_\mu Y = \eta \) in \( L^2(\mu) \). So \( \psi \) is essentially unique.

Now, let \( Y \in \mathcal{M}_2(\mu) \), and \( (Y^n)_{n \in \mathbb{N}} \) a sequence of \( I(\mathcal{S}) \) converging to \( Y \) in \( \mathcal{M}_2(\mu) \). Then

\[
0 = \lim_{n \to \infty} E[|Y^n(T) - Y(T)|^2],
\]

\[
= \lim_{n \to \infty} E[|\int_0^T \int_{\mathbb{R}^d} \nabla_\mu Y^n(t, z) - \psi(t, z) \nabla_\mu \psi(t, z) dt dz|^2],
\]

\[
= \lim_{n \to \infty} E[\int_0^T \int_{\mathbb{R}^d} |\nabla_\mu Y^n(t, z) - \psi(t, z)|^2] d\mu(ds, dy),
\]

by the Itô isometry. So for any sequence \( Y^n \) of \( I(\mathcal{S}) \) converging to \( Y \), the limit of \( \nabla_\mu Y^n \) is the same and equal to \( \psi d\mu \times d\mathbb{P}\)-a.e. So \( \nabla_\mu \) is closable in \( \mathcal{M}_2(\mu) \), and we can identify \( \psi \) as \( \nabla_\mu Y \).
This gives the following representation formula:

**Theorem 3.8** (Martingale representation formula). If \( J - \mu \) has the martingale representation property, then any adapted square-integrable martingale has the representation

\[
Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}^d} (\nabla \mu Y)(s, z)(J - \mu) \, ds \, dz,
\]

where \( \nabla \mu \) is the closure in \( \mathcal{L}^2(\mu) \) of the pathwise operator introduced in Definition 2.13 for functionals with a regular functional representation.

### 3.1 Change of probability measure and extension to semimartingales

Consider two equivalent probability measures \( \mathbb{P} \) and \( \mathbb{Q} \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \), and take \( Y \) a square-integrable \( \mathbb{P} \)-martingale. Then, from above, \( Y \) can be written

\[
Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}^d} \nabla \mu Y(s, z)(J - \mu) \, ds \, dz.
\]

Now, since \( \mathbb{P} \) and \( \mathbb{Q} \) are equivalent, \( Y \) is also a \( \mathbb{Q} \)-semimartingale and has the following decomposition:

\[
Y(t) = X(t) + M(t),
\]

with \( X(t) \) a predictable finite variation process, and \( M \) a square-integrable \( \mathbb{Q} \)-martingale starting at zero. By the martingale representation formula,

\[
M(t) = \int_0^T \int_{\mathbb{R}^d} \nabla \nu M(s, z)(J - \nu) \, d\mathbb{Q} \, d\nu - a.s.
\]

**Theorem 3.9.** Take two equivalent probability measures \( \mathbb{P} \) and \( \mathbb{Q} \). For \( Y \) a \( \mathbb{P} \)-square-integrable martingale with decomposition \( Y = X + M \) with \( X \) a finite variation process and \( M \) a \( \mathbb{Q} \)-square-integrable martingale, the following hold:

1. \( \nabla \mu Y = \nabla \mu M \) in \( \mathcal{L}^2(\mu) \) and \( \mathcal{L}^2(\nu) \).
2. \( \nabla \mu X = \nabla \nu X = \nabla \mu X = 0 \).
3. For a sequence \( (Y^n)_{n \in \mathbb{N}} \) in \( L^2(\mathcal{F}) \) with \( Y^n \to Y \) in \( L^2(\mathbb{P}) \), \( \nabla \nu Y^n \) converges in \( \mathcal{L}^2(\nu) \) to a random field \( \nabla \nu Y \) verifying \( \nabla \nu Y = \nabla \nu M \, d\mathbb{Q} \times d\nu \) a.e.

**Proof.** 1. Subtracting (20) from (21) gives

\[
X(t) = X(0) + \int_0^T \int_{\mathbb{R}^d} \nabla \mu M(s, z)(J - \nu) \, ds \, dz = 0.
\]

Rewriting:

\[
X(t) - X(0) = \int_0^T \int_{\mathbb{R}^d} \nabla \mu M(s, z)(\mu - \nu) \, ds \, dz = 0,
\]

so the finite-variation part and the stochastic integral part must both vanish. In particular,

\[
\nabla \mu Y = \nabla \mu M, d\mathbb{P} \times d\mu - a.e.
\]

and

\[
X(t) = Y(0) + \int_0^T \int_{\mathbb{R}^d} \nabla \nu M(\nu - \mu) \, ds \, dz.
\]

Reworking equation (22) into

\[
X(t) = \int_0^T \int_{\mathbb{R}^d} \nabla \mu Y(s, z)(\mu - \nu) \, ds \, dz = 0,
\]

we also have that

\[
\nabla \mu Y = \nabla \mu M, d\mathbb{Q} \times d\nu - a.e.
\]

(and so \( d\mathbb{P} \times d\nu \)-a.e. since the probability measures are equivalent).
2. Since $X$ is predictable, $\nabla_\mu X = \nabla_\nu X = \nabla_p X = 0$.

3. We know that there exists a sequence of processes $Y^n \in I_\mu(S)$ converging to $Y$ in $\mathcal{M}^2_\mu(\mu)$, with $\nabla_\mu Y^n = \nabla_\nu Y^n$. Let us define

$$M^n(t) = X - Y^n.$$ 

Since $Y^n \to Y$ in the $\mathcal{M}^2_\mu(\mu)$-norm, $M^n \to M$ in the $\mathcal{M}^2_\nu(\nu)$-norm. Also,

$$\nabla_p M^n(t) = \nabla_p X - \nabla_p Y^n. \quad (23)$$

Now, $\nabla_p M^n \to \nabla_\nu M \, dQ \times d\nu$ a.e. So the right-hand side hand side of equation (23) converges, and since $\nabla_\nu M = 0$ by (ii):

$$\nabla_\nu M = \nabla_\nu (X - Y) = \lim_{n \to \infty} \nabla_p X - \nabla_p Y^n = \lim_{n \to \infty} \nabla_p Y^n = \nabla_\nu Y,$$

with $\nabla_\nu Y$ the limit of $\nabla_p Y^n$ in $L^2_Q(\nu)$. This gives

$$\nabla_\nu M = \nabla_\nu Y dQ \times d\nu \text{ a.e.}.$$ 

But since $\nabla_\nu M = \nabla_\nu M \, dQ \times d\nu$ a.e.,

$$\nabla_\nu Y = \nabla_\nu Y dQ \times d\nu \text{ a.e.}.$$ 

The equality in $L^2_\mu(\mu)$ is obtained by interchanging $Y$ and $M$ above and taking the limits in $L^2_\mu(\mu)$ instead.

The above theorem states that closure of the finite difference operator actually does not depend on what probability measure we consider (as long as they are equivalent). This behaviour is in some sense orthogonal to the Bismut-Malliavin calculus, where the perturbation operator precisely consists in an equivalent change of probability measure.

Another consequence is that one can extend the $\nabla_\mu$ operator to any square-integrable semimartingale $S$, as long as there exists an equivalent martingale measure under which $S$ would be a square-integrable martingale. In that case, for any square-integrable semimartingale $S$ with predictable finite variation part $S^{\text{FV}}$ and square-integrable martingale part $S^m$, $\nabla_\mu S$ is defined as

$$\nabla_\mu S = \nabla_\mu S^m.$$ 

4 Martingale Representation Formula: general case

In this section, we write $\mathcal{D}_T := D([0,T], \mathbb{R}^d)$ for the space of càdlàg functions from $[0,T]$ to $\mathbb{R}^d$.

**Definition 4.1** (Stopped trajectory). For $x \in \mathcal{D}_T$, we write

$$x_t(u) = \begin{cases} x(u) & \text{if } u \leq t, \\ x(t) & \text{if } u > t. \end{cases}$$

Note that $x_t$ is an element of $\mathcal{D}_T$.

We redefine the notions of non-anticipative functionals in a way that accomodates for the new variable $x$.

**Definition 4.2.** (Space $\Omega^p$ of processes and canonical process) We identify the space of processes

$$\Omega^p := \mathcal{D}_T \times \mathcal{M}_T$$

equipped with the $\sigma$-algebra $\mathcal{F}^p$ defined as the product Borel $\sigma$-algebra on the product space.

**Remark 4.3.** If $\mathcal{D}_T$ is equipped with the Skorohod topology and $\mathcal{M}_T$ with the topology of weak convergence, both are separable, and there is no difference between the product Borel $\sigma$-algebra and the product of the Borel $\sigma$-algebras defined respectively on $\mathcal{D}_T$ and $\mathcal{M}_T$. 

We define a process $Y$ on $(\Omega^p, F^p)$ with values in $D_T \times M_T$ as a family $(Y(t))_{t \geq 0}$ of mappings
\[ Y : [0, T] \times \Omega^p \to D_T \times M_T. \]
We define the canonical process $(X, J)$ as follows: for any $\omega := (x, j) \in \Omega^p$,
\[ (X, J)(\omega, t) := \omega_t = (x_t(\cdot), j_t(\cdot)), \]
i.e. the couple consisting in the trajectory and the measure $j$ both stopped at time $t$.

**Definition 4.4.** (Filtration generated by $(X, J)$) For a given $(j_t)_{t \in [0, T]}$, we define the filtration generated by the canonical process $(X, J)$: $\mathcal{F}^p := \{ F^p_t \}_{t \in [0, T]}$ on $(\Omega^p, F^p)$ as the increasing sequence of $\sigma$-algebras
\[ \mathcal{F}^p_t = \sigma(X_s(\cdot), J_s(\cdot), s \in [0, t]). \]

Now, we define non-anticipative functionals:

**Definition 4.5.** (Non-anticipative functional process) A non-anticipative functional $F$ is a map
\[ F : [0, T] \times \Omega^p \to \mathbb{R} \]
such that
\[ F(t, x, j) = F(t, x_t, j_t), \]
such that $F$ is measurable with respect to the product $\sigma$-algebra $\mathcal{B}([0, T]) \times F^p$. We denote $\mathcal{O}^p$ the space of such functionals.

**Definition 4.6.** (Predictable functional process) A predictable functional $F$ is a non-anticipative functional such that
\[ F(t, x, j) = F(t, x_t, j_t) = F(t, x_{t-}, j_{t-}), \]
and so $F(t, \cdot)$ is $\mathcal{F}^p_{t-}$-measurable. We write $\mathcal{P}^p$ the space of predictable functional processes, and we have $\mathcal{P}^p \subset \mathcal{O}^p$.

**Definition 4.7.** (Functional fields) A non-anticipative functional field $\Psi$ is a map
\[ \Psi : [0, T] \times \mathbb{R}^d \times \Omega^p \to \mathbb{R} \]
such that $\Psi(t, z, x, j) = \Psi(t, z, x_t, j_t)$, such that $\Psi$ is measurable with respect to the product $\sigma$-algebra $\mathcal{B}([0, T] \times \mathbb{R}^d) \times F^p$, and for all $(z, t) \in [0, T] \times \mathbb{R}^d$, $\Psi(t, z, \cdot)$ is $\mathcal{F}^p_t$-measurable. We denote $\mathcal{O}^p_\Psi$ the space of such functionals.

Similarly, we call predictable functional field any $\Psi \in \mathcal{O}^p_\Psi$ such that
\[ \Psi(t, z, x, j) = \Psi(t, z, x_{t-}, j_{t-}), \]
and we denote by $\mathcal{P}^p_\Psi$ the space of such predictable functional fields.

**Definition 4.8.** (Vertical perturbation in $x$). The vertical perturbation of a càdlàg function $x \in D_t$, $t < T$ is given by
\[ x^h(t)(\cdot) = x_t(\cdot) + h\mathbb{I}_{[t, \infty)}(\cdot). \]

**Definition 4.9.** (Vertical derivative). A non-anticipative functional process is said to be vertically differentiable if, for $(e_i)_{i \in \mathbb{N}}$ the canonical basis of $\mathbb{R}^d$, and $h > 0$ real, the limit
\[ \lim_{h \to 0} \frac{F(t, x_t^{h e_i}, j_t) - F(t, x_t, j_t)}{h} \]
is well defined for all $t$ and all $i$. The resulting vector is called the vertical derivative of $F$ at $t$ with respect to $x$, and is noted $\nabla_x F(t, x_t, j_t)$.

We now equip the space $(\Omega^p, F^p)$ with a probability measure $P$ such that $X$ defines a continuous martingale and $J$ a jump-measure such that $\int_0^T \int_{\mathbb{R}^d} |z|^2 J(dsdz) < \infty$ a.s., with compensator $\mu$ absolutely continuous in time. Moreover, we complete the filtration $\mathcal{F}^p$ by the $P$-null sets. We now introduce the following spaces:
\[ L^2_p([X], \mu) = \{ (\phi, \psi) \mid \phi : [0, T] \times \Omega^p \to \mathbb{R}^d \text{ and } \psi : [0, T] \times \mathbb{R}^d \times \Omega^p \to \mathbb{R}^d \text{ both predictable and } \}
\]
\[ \| (\phi, \psi) \|^2_{L^2_p([X], \mu)} := E \left[ \int_0^T \phi^2(s)d[X](s) + \int_0^T \int_{\mathbb{R}^d} \psi^2(s, y)\mu(ds\,dy) \right] < \infty \}, \]
and
\[ \mathcal{M}_2^\phi([X], \mu) = \bigg\{ Y = \int_0^1 \phi(s, \omega) dX + \int_0^1 \int_{\mathbb{R}^d} \psi(s, y) d\nu X(\omega) : (\phi, \psi) \in \mathcal{L}^2(X, \mu) \bigg\}, \]
equipped with the norm
\[ \|Y\|_{\mathcal{M}_2^\phi([X], \mu)}^2 = E \left[ \|Y(T)\|^2 \right]. \]

Then
\[ \mathcal{L}_2^\phi([X], \mu) = \mathcal{L}_2^\phi([X]) \oplus \mathcal{L}_2^\psi(\mu) \]
and
\[ \mathcal{M}_2^\phi([X], \mu) = \mathcal{M}_2^\phi([X]) \oplus \mathcal{M}_2^\psi(\mu), \]
with
\[ \mathcal{L}_2^\phi([X]) := \left\{ \phi : [0, T] \times \Omega^p \to \mathbb{R} \text{ predictable} \bigg| \|\phi\|_{\mathcal{L}_2^\phi(X)} := E \left[ \int_0^T \phi^2(t) d[X](t) \right] < \infty \right\}, \]
and
\[ \mathcal{M}_2^\phi([X]) := \left\{ Y : [0, T] \times \Omega^p \to \mathbb{R} \bigg| Y(t) = \int_0^t \phi(z) dX(t) \right\}, \]
equipped with the norm
\[ \|Y\|_{\mathcal{M}_2^\phi([X])} := E[|Y|^2]. \]

**Definition 4.10.** (Couples of simple functionals on the product space) We consider the space \( \mathcal{S}_2 \) of couples \((\phi, \psi)\) such that
\[ \phi : [0, T] \times \mathcal{D}_T \times \mathcal{M}_T \to \mathbb{R}^d, \] (24)
\[ \psi : [0, T] \times \mathbb{R}^d_0 \times \mathcal{D}_T \times \mathcal{M}_T \to \mathbb{R}^d \] (25)
and
1. \( \phi \) has the following form
   \[ \phi(t, x_1, j_1) = \sum_{i=1}^I \phi_i(x_{t_i}, j_{t_i}) \mathbb{I}_{(t_i, t_{i+1}]}(t) \]
   with the \( t_i \) predictable stopping times.
2. Moreover, there exist \( I \) grids \( 0 \leq t_1^i \leq t_2^i \leq \cdots \leq t_{n(i)}^i = T \) such that for any \( i \in 1..I \)
   \[ \phi_i(j_{t_i}) = f_i(\{x(t_u^i, C_{t_i}^u(j)) : u \in \{1, T \}, v \in \{1..V\}\}, \]
   where \( f_i : \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R} \) is Borel-measurable, \( 0 \leq t_1^i \leq \cdots \leq t_{U^i}^i \leq t_i \) and \( 1 \leq \epsilon_1 \leq \cdots \leq \epsilon_V \). \( (C_{t_i}^u) \) is defined as in Equation (25).)
3. \( \psi \) has the following form
   \[ \psi(t, z, j_1) = \sum_{m=0}^{M, K} \psi_{mk}(x_{k_m}, j_{k_m}) \mathbb{I}_{(k_m, k_{m+1}]}(t) \mathbb{A}_m(z) \]
   with \( \mathbb{A}_m \in \mathcal{B}([0, T] \times \mathbb{R}^d), 0 \not\in \overline{\mathbb{A}_m} \) (the closure of the \( \mathbb{A}_m \)), the \( k_m \) are predictable stopping times (allowed to depend on the \( Z_k \)).
4. Moreover, there exist \( M \) grids \( 0 \leq t_1^m \leq t_2^m \leq \cdots \leq t_{n(m)}^m = T \) such that for any \( m \in 1..M \):
   \[ \psi_{mk}(x_{k_m}, j_{k_m}) = g_{mk}(\{x(t_p^m, C_{t_p}^m(j)) : p \in \{1, P \}, q \in \{1..Q\}\}), \]
   where \( g_{mk} : \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R} \) is Borel-measurable, \( 0 \leq t_1^m \leq \cdots \leq t_P^m \leq k_m \) and \( 1 \leq \epsilon_1 \leq \cdots \leq \epsilon_Q \).
As in the previous section, we write \( I(\mathcal{S}_2) \) for the set of processes that are stochastic integrals of processes in \( \mathcal{S}_2 \), and \( I(\mathcal{S}_2) \).

The stochastic integral operator is defined as

\[
I_{[X]} : \mathcal{L}_2^p([X], \mu) \to \mathcal{M}_p^2([X], \mu),
\]

\[
(\phi, \psi) \mapsto \int_0^t \phi(s)dX(s) + \int_0^t \int_{\mathbb{R}^d} \psi(s,y)d\gamma(dy).
\]

**Definition 4.11.** The operator \( \nabla_{x,j} : (\nabla_x, \nabla_p) \), is defined pathwise on \( I(\mathcal{S}_2) \) as

\[
\nabla_{x,j} : I(\mathcal{S}_2) \to \mathcal{L}_2^2([X], \mu),
\]

\[
F(t, X_t, J_t) = \int_0^t \phi(s)dX(s) + \int_0^t \int_{\mathbb{R}^d} \psi(s,y)d\gamma(dy) \mapsto (\nabla_x F(t, X_t, J_t), \nabla_p F(t, X_t, J_t))
\]

\[
= (\phi(t, J_{t-})_{t \in [0,T]}, \psi(t, z, J_{t-})_{z \in \mathbb{R}^d}).
\]

**Lemma 4.12.** The set of random variables

\[
f((X(t_i), C^i_t (J))_{i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}}
\]

with \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) bounded, and the \( C \) as in equation (5), is dense in \( \mathcal{L}_2^2(F_T^p, \mathbb{P}) \) (the space of random variables with finite second moment).

**Proof.** Let \((t_i)_{i \in \mathbb{N}} \) be a dense subset of \([0, T]\), and \((k_j)_{j \in \mathbb{N}} \) a dense subset of \([1, \infty)\). Denote

\[
\mathcal{F}^{n,m} = \sigma(X(t_i), (T_k, Z_k)|t_i \leq n, T_k \leq n, T_k \leq t, \frac{1}{k_m} < Z_k \leq k_m),
\]

–where the \((T_k, Z_k)\) denote the jump times and sizes of the jump measure \( J \)– i.e. the (completed) filtration generated by the \((X(t_i), C^i_t (J))_{i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}}\). One has

\[
\mathcal{F}^{n,m} \subset \mathcal{F}^{n,m+1}
\]

\[
\mathcal{F}^{n+1,m} \subset \mathcal{F}^{n+1,m+1}
\]

Moreover, \( \mathcal{F}_T^p \) is the smallest \( \sigma \)-algebra containing all the \( \mathcal{F}^{n,m} \). For \( g \in \mathcal{L}_2^2(F_T^p, \mathbb{P}) \),

\[
g = E[g|\mathcal{F}_T^p].
\]

The martingale convergence theorem yields

\[
g = \lim_{m \to \infty} \lim_{n \to \infty} E[g|\mathcal{F}^{n,m}],
\]

Moreover, for each \( n \) and \( m \), there exists a \( \mathcal{F} \)-measurable random variable \( h_{nm} \) (i.e. a random variable measurable with respect to the non-completed filtration) such that, \( \mathbb{P} \)-a.s.,

\[
E[g|\mathcal{F}^{n,m}] = h_{nm}.
\]

(see e.g. Lemma 1.2. in Crâuel[9].)

Finally, the Doob-Dynkin lemma ([15], p. 7) yields that for each couple \((n, m)\), there exists a Borel-measurable \( g_{nm} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) with

\[
E[g|\mathcal{F}^{n,m}] = g_{nm}((X(t_i), C^i_t (J))_{i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}}, \mathbb{P} \)-a.s.
\]

**Lemma 4.13.** The processes of the form \((\nabla_x Y, \nabla_p Y)\) are dense in \( \mathcal{L}_2^p([X] \otimes \mu) \).
Proof. Let \((\phi, \psi)\) be some element of \(S_2\), and consider a continuous process \(Y\) of the form
\[
Y^c(t) = \int_0^t \phi(s)dX(s).
\]
Notice that the integral is well defined in a pathwise sense, as this is just a Riemann sum:
\[
Y^c(t, \omega) = F(t, X_t, J_t),
\]
with
\[
F(t, x_t, j_t) = \int_0^t \phi(s, x_{s-}, j_{s-})dx(s) = \sum_{i=1}^I \phi_i(x_{\tau_i}, j_{\tau_i})\mathbb{1}_{(\tau_{i+1}, \tau_i)}(t)(x(t) - x(\tau_i)).
\]
Hence,
\[
\nabla_x F(t, x_t, j_t) = \phi(t, x_t, j_t).
\]
So these processes have the form
\[
\nabla_x Y^c(t) = \phi((X(t_i), C_{r_i}^{(j)}(j))_{i \in \{1, \ldots, n\}}, \mu)\mathbb{1}_{t > t_n},
\]
such \(\nabla_x Y^c\) define a total set in \(L_p^2([X])\) (see Cont-Fournié [7] and Lemma 4.12). Similarly, the processes of the form
\[
Y^d = \int_0^t \int_{\mathbb{R}^d} \psi(s, y, J_{s-})\tilde{J}(ds
dy),
\]
have the form
\[
(\nabla_p Y^d)(t, z) = \nabla_p G(t, X_t, J_t) = \psi(t, z, J_{t-}),
\]
with \(G\) the following regular functional representation of \(Y^d:\)
\[
G(t, x_t, j_t) = \int_0^t \int_{\mathbb{R}^d} \psi(s, y, j_{s-})(ds
dy) - \mu(ds
dy, j_{s-})).
\]
Moreover, such \(\psi\) are dense in \(L_p^2(\mu)\) (by Section 7 and Lemma 4.12). \(\Box\)

Remark 4.14. Notice that \(\nabla_x Y^d \equiv 0\). Similarly, \(\nabla_p Y^c \equiv 0\).

Corollary 4.15. The space \(I(S_2)\) is dense in \(M_p^2([X], \mu)\).

Proof. This follows by the previous lemma, as the stochastic integral
\[
I_{\tilde{X}, \tilde{J}} : L_p^2([X], \mu) \to M_p^2([X], \mu)
\]
\[
(\phi, \psi) \mapsto \int_0^t \phi(s, \omega)dX(s) + \int_0^t \int_{\mathbb{R}^d} \psi(s, y)\tilde{J}(ds, dy)
\]
defines a bijective isometry (hence a continuous bijection) between \(L_p^2([X], \mu)\) and \(M_p^2([X], \mu)\). \(\Box\)

Theorem 4.16. (Extension of \(\nabla_X, J\) to \(M_p^2([X], \mu)\)) The operator \(\nabla_X, J : I_{\tilde{X}, \tilde{J}}(S_2) \to L_p^2([X], \mu)\) is closable in \(M_p^2(X, \mu)\) and its closure defines a bijective isometry between these two spaces:
\[
\nabla_X, J : M_p^2([X], \mu) \to L_p^2([X], \mu),
\]
\[
F(t, X_t, J_t) := \int_0^t \phi(s)dX(s) + \int_0^t \int_{\mathbb{R}^d} \psi(s, y)\tilde{X}(ds\ dy) \mapsto (\nabla_x F, (\nabla_p F)) = (\phi, \psi).
\]
In particular, \(\nabla_X, J\) is the adjoint of the stochastic integral
\[
I_{\tilde{X}, \tilde{J}} : L_p^2(X, \mu) \to M_p^2(X, \mu),
\]
\[
(\phi, \psi) \mapsto \int_0^t \phi(s)dX(s) + \int_0^t \int_{\mathbb{R}^d} \psi(s)\tilde{J}(ds
dy).
\]
in the sense that for all $\psi \in L^2_\mathbb{F}(X, \mu)$ and for all $Y \in M^2_\mathbb{F}(X, \mu)$:

$$
< Y, I_{X,J}(\phi, \psi) >_{M^2_\mathbb{F}(X, \mu)} := E \left[ Y(T) \left( \int_0^T \phi(s) dX(s) + \int_0^T \psi(s, y) \tilde{J}(ds \, dy) \right) \right]
$$

$$
= E \left[ \int_0^T \nabla_x Y(s) \phi(s) dX(s) + \int_0^T \int_{\mathbb{R}^d} \psi(s, y) Y(s, y) \rho(\mu)(ds \, dy) \right]
$$

$$
=: < \nabla X, J Y, (\phi, \psi) >_{L^2_\mathbb{F}(X \otimes \mu)}.
$$

**Proof.** By definition of $M^2_\mathbb{F}(X, \mu)$, we know that there exists $(\phi, \psi) \in L^2_\mathbb{F}([X], \mu)$ such that

$$
Y(t) = \int_0^t \phi(s, \omega) dX + \int_0^t \int_{\mathbb{R}^d} \psi(s, y) \tilde{J}(ds \, dy, \omega)
$$

Then, by the Itô isometry on $M^2([X], \mu)$, with $Z \in I(S_2)$,

$$
E[Y(T)Z(T)] = E \left[ \int_0^T \phi(s) \nabla_x Z(s) dX_s + \int_0^T \int_{\mathbb{R}^d} \psi(s, y) \nabla_x Z(s, y) \rho(\mu)(ds \, dy) \right]
$$

Moreover, (39) uniquely characterises $\phi \, d\mathbb{P} \times d[X]$-a.e., and $\psi \, d\mathbb{P} \times d\mu$-a.e. For if $(\eta, \rho)$ is any other solution of (39). Then

$$
< Y - I_{X,J}(\eta, \rho), Z >_{M^2_\mathbb{F}(X, \mu)} = E \left[ (Y - I_{X,J}(\eta))Z(t) \right] = 0
$$

for all $Z \in I(S_2)$. Hence, $Y - I_{X,J}(\eta, \rho) = 0 \, \mathbb{P}$-a.s. on $M^2(X, \mu)$ by density of $I(S_2)$ in $M^2(X, \mu)$. So $\psi = \eta \, d[X] \times d\mathbb{P}$-a.e. and $\rho = \rho \, d\mu \times d\mathbb{P}$-a.e. and so $(\phi, \psi)$ is essentially unique.

Now, for any $Y \in M^2_\mathbb{F}(X, \mu)$ and let $(Y^n)_{n \in \mathbb{N}}$ a sequence of $I(S_2)$ that converges to $Y$ in $I(S_2)$.

$$
0 = \lim_{n \to \infty} E[|Y^n(t) - Y(T)|^2],
$$

$$
= \lim_{n \to \infty} E\left[ \int_0^T \nabla_x Y^n(t) - \phi(t) dX(t) + \int_0^T \int_{\mathbb{R}^d} |\nabla_x Y^n(t, z) - \psi(t, z)\tilde{J}(dtdz)|^2 \right],
$$

$$
= \lim_{n \to \infty} E\left[ \int_0^T \int_{\mathbb{R}^d} |\nabla_x Y^n(t, z) - \psi(t, z)|^2 \rho(\mu)(ds \, dz) \right] + \lim_{n \to \infty} E\left[ \int_0^T |\nabla_x Y^n(t) - \phi(t)|^2 d[X](s) \right]
$$

and the last term is zero by the Itô isometry. Consequently, for any approximating sequence $Y^n$, $(\nabla_x Y^n, \nabla_p Y^n)$ converges to $(\phi, \psi)$. So $\nabla X, J$ is closable and we can write $(\phi, \psi) = (\nabla_x Y, \nabla_p Y)$

This gives:

**Theorem 4.17.** (Martingale representation theorem for square integrable martingales) For any square integrable $(\mathbb{F}, \mathbb{P})$-martingale, $\mathbb{P}$-almost surely,

$$
Y(t) = Y(0) + \int_0^t \nabla_x Y(s) dX(s) + \int_0^t \int_{\mathbb{R}^d} \nabla_p Y(s, z) \tilde{J}(ds \, dy).
$$

**Remark 4.18.** When taking the closure in $L^2_\mathbb{F}(\mu)$ of $\nabla_p$, one loses the pathwise interpretation.

Note that this approach treats the continuous and jump parts in a similar fashion. If the filtration is generated by a continuous martingale $X$ and jump measure $J$ (with compensator $\mu$), then one can construct the following martingale $S$:

$$
S(t) = \int_0^t \int_{\mathbb{R}^d} (\mathbb{1}_{z=0} \, dX(s) + z\tilde{J}(ds \, dz)),
$$
and the martingale representation formula can then be rewritten for any square-integrable martingale as

\[ Y(t) = Y(0) + \int_0^t \nabla_{X,J} Y(s,y) \otimes dS(s), \]

with the element-wise quotient operator \( D_{t,z} = \nabla_X \mathbb{1}_{z=0} + \frac{1}{z} \nabla_{j,z} \in \mathbb{R} \).

So \( \nabla_X \) is the limiting quotient operator when \( z \to 0 \) of the operator \( \frac{1}{z} \nabla_{j,z} \), and the continuous and jump integrands are treated in a similar way.

## 5 Comparison with other Malliavin calculi for jumps

In this section, we review the functional Itô approach against other Malliavin calculus frameworks for jump processes.

### 5.1 Change of measure

As we saw in Subsection 3.1, the functional Itô perturbation’s behaviour is invariant under change of equivalent probability measure, and is consequently intrinsically different from Bismut’s approach to the Malliavin calculus [4] (see also [3]). There, the perturbation actually takes the form of an equivalent change of measure: Bismut’s approach consists in changing the intensity of the process – in other words, perturbing infinitely many jumps – and the direction in which one is allowed to perturb is the space of equivalent changes of measure.

### 5.2 Comparison with chaos expansions and mass addition

As mentioned in the introduction, another approach to Malliavin calculus with jumps is chaos expansions on the Poisson space (see e.g. Øksendal et al. [10]). Here one decomposes a random variable satisfying some integrability conditions as a series of iterated integrals:

\[ F = \sum_{n=0}^{\infty} I_n(f_n), \]

where the \( I_n \) are iterated integrals of a (symmetric) function \( f_n \) with respect to a Poisson process.

The Malliavin derivative is then defined as the operator that bring this decomposition down by one level:

\[ D : \text{Dom}(D) \to L^2(\lambda \times \nu \times P), \]

\[ F \mapsto D_{t,z} := \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot,t,z)), \]

and \( D \) can actually be closed in the whole set of square-integrable random variables.

It was in fact already noted that in the finite-activity case, a pathwise interpretation of the chaos expansion operator is possible, as mentioned in Last-Penrose [19]. Løkka [21] extends the results of Nualart-Vives [24] from the Poisson to the Lévy case, showing that this chaos expansion approach expands to the pure-jump Lévy setting, and is equivalent to Picard operator, which consists in putting an extra weight locally on the jump measure; define the annihilation and creation operators \( \epsilon^- \) and \( \epsilon^+ \) on measures by

\[ \epsilon^-_{t,z} m(A) = m(A \cap \{(t,z)\}^c), \]

\[ \epsilon^+_{t,z} m(A) = \epsilon^-_{t,z} m(A) + \mathbb{1}_A(t,z). \]

Then, for functionals defined on \( \Omega^p \), the space of general measures from \([0,T] \times \mathbb{R}^d \) to \( \mathbb{R} \), the “Malliavin-type” operator is defined as

\[ \tilde{D}_{t,z} : \Omega^p \times [0,T] \times \mathbb{R}^d \to \mathbb{R}, \]

\( (F,t,z) \mapsto F \circ \epsilon^+_{t,z} - F. \)

This “addition of mass” approach through a creation and annihilation operator appears in other approaches such as the “lent-particle method” (Bouleau-Denis [5]).
Hence, we consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) of the perturbed process. Moreover, there is no need to restrict the setting to a Poisson or Lévy space.

We directly perturb the predictable projection of the process, rather than taking the predictable projection of the perturbed process. Moreover, there is no need to restrict the setting to a Poisson or Lévy space.

These approaches consisting in adding mass to the measure look in some aspects similar to the one we have taken here. There is, however, a fundamental difference: in our pathwise approach, we directly perturb the predictable projection of the process, rather than taking the predictable projection of the perturbed process. However, there is no need to restrict the setting to a Poisson or Lévy space.

The relationship between \(\nabla_j\) and the different operators \(D_{t,z}\), can be summarised as follows, which is a jump counterpart to the Cont-Fournié lifting theorem [7]. Since all the operators defined above give rise to the following type of martingale representation:

\[
Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}_0^d} p E[D_{s,y} Y_s | \mathcal{F}_s] (J - \mu)(ds \ dy),
\]

we have the following “lifting diagram”:

\[
\begin{align*}
\mathcal{M}^2_{\mathbb{P}} & \quad \overset{(\mathcal{P} E[ |\mathcal{F}_s|]_{s \in [0,T]} \quad \nabla_j} \quad \overset{\mathcal{L}^2_{\mathbb{P}}(\mu)} \quad \mathcal{P} E[ |\mathcal{F}_s|]_{s \in [0,T]} \quad \mathcal{L}^2([0,T] \times \Omega^p) \quad \nabla_{j}.
\end{align*}
\]

6 Examples

6.1 Kunita-Watanabe decomposition

We consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), on which a Brownian motion \(W\) and a jump measure \(J\), with compensator \(\mu\), such that \(\mu(dt dz) = \nu(dz)dt\), generate the filtration \(\mathbb{F}\).

We write

\[
X(t) := \sigma W(t) + \int_0^t \int_{\mathbb{R}_0} z \tilde{J}(ds \ dz),
\]

with \(\tilde{J}\) the compensated jump measure.

The Kunita-Watanabe decomposition (see e.g. [8]) states that for a martingale \(X\) and \(Y \in \mathcal{L}_2^2(X)\), there exists a unique \(\tilde{Y}\) with

1. \(\tilde{Y} = E[Y] + \int_0^t \psi(s)dX(s)\),
2. \(E[(Y - \tilde{Y})M] = 0\) for all \(M = \int_0^t \xi(s)dX(s)\).

One can then compute the Kunita-Watanabe decomposition, such as in [2], extending it from the Malliavin space \(\mathbb{D}^{1,2}\) to the whole \(\mathcal{M}^2_{\mathbb{P}}([X], \mu)\).

\[
Y_t = E[Y] + \int_0^t \nabla W Y \sigma dW(s) + \int_0^T \int_{\mathbb{R}_0} \nabla_j Y(s, z) \tilde{J}(ds \ dz).
\]

Hence

\[
Y^\omega := Y - \tilde{Y} = \int_0^t (\nabla W Y(s) - \psi(s))\sigma dW(s) + \int_0^t \int_{\mathbb{R}_0} (\nabla_j Y(s, z) - z \psi(s)) \tilde{J}(ds \ dz),
\]

\[\text{16}\]
The orthogonality condition then becomes:

\[
E[Y^n M] = E\left[\int_0^t (\nabla_W Y(s) - \psi(s))\xi(s)s^2 ds + \int_0^T \int_{\mathbb{R}_0} (\nabla_j Y(s, z) - z\psi(s)).z\xi(s)\mu(ds \, dz)\right]
\]

\[
= E\left[\int_0^T \xi(s) \left((\nabla_W Y(s) - \psi(s))s^2 + \int_{\mathbb{R}_0} z(\nabla_j Y(s, z) - z\psi(s))\right) ds\right],
\]

using that \(\int_0^t \xi(s) dX(s)\) is a martingale, the Itô isometries and the orthogonality relations between continuous and pure jump parts. This implies that

\[
(\nabla_W Y(s) - \psi(s))s^2 + \int_{\mathbb{R}_0} z(\nabla_j Y(s, z) - z\psi(s))\nu(dz) = 0.
\]

and so that

\[
\psi(s) = \left(\frac{s^2 \nabla_W Y(s)}{(s^2 + \int_{\mathbb{R}_0} |z|^2 \nu(dz))}\right)^{-1}.
\]

### 6.2 Doléans-Dade exponential for pure-jump Lévy processes

In this example, we show how one can recover the SDE satisfied by a stochastic exponential that is a martingale. Consider a probability space \((\Omega, \mathcal{F}, P)\), where a jump measure \(J\) such that \(\int_0^T \int_{\mathbb{R}_0^2} |z|^2 J(ds \, dz) < \infty\) a.s., and with absolutely-continuous compensator \(\mu\) generates the filtration \((\mathcal{F}_t)_{t\in[0,T]}\). As in the previous sections, we write \(\tilde{J}\) for the compensated jump measure \(J - \mu\). Then, we can write the Doléans-Dade process:

\[
E_t = e^{\int_0^t \int_{\mathbb{R}_0} z(J - \mu)(ds \, dz)} \prod_{s\in[0,t]} (1 + \int_{\mathbb{R}_0} zJ(\{s\} \times dz)e^{-\int_0^d zJ(s) \times dz}).
\]

Let us introduce the cut-off Doléans-Dade process:

\[
E^n_t = e^{\int_0^t \int_{(\mu, \infty)^2} z(J - \mu)(ds \, dz)} \prod_{s\in[0,t]} (1 + \int_{(\mu, \infty)} zJ(\{s\} \times dz)e^{-\int_0^d zJ(s) \times dz}).
\]

Notice that the functional

\[
F^n(t, j_t) = e^{\int_0^t \int_{(\mu, \infty)^2} z(j(ds \, dz) - \mu(ds \, dz), j_{t-})} \prod_{s\in[0,t]} (1 + \int_{(\mu, \infty)^2} zj(\{s\} \times dz))e^{-\int_0^d zj(s) \times dz},
\]

is well defined, since the compensated integral is simply a Lebesgue-Stieltjes integral. It is straightforward to compute that \(\nabla_{j,s} F^n(t, j_t) = zF^n(t, j_{t-})\). Now, since \(E^n_t\) tends to \(E_t\) in \(\mathcal{M}_F(\mu)\),

\[
\nabla_j E^n_t \quad \nabla_j E_t
\]

in \(L^2_p(\mu)\). But

\[
\nabla_{\mu} E^n_t = zF^n(t, j_{t-}) \quad \nabla_{\mu} E_t
\]

in the \(L^2_p(\mu)\) sense. So by uniqueness of the integrand in the martingale representation formula,

\[
E_t = E_0 + \int_0^T E_{t-}dX(t)
\]

with \(X(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{J}(ds \, dz)\) a purely discontinuous Lévy martingale. We recover the classical SDE satisfied by \(E\), the stochastic exponential of the martingale \(X\).
6.3 Application to the Kella-Whitt martingale

The Kella-Whitt martingale, introduced in [17], is a process that appears in queueing theory and modelling storage processes (see also Kella-Boxma [16] and Kyprianou [18]). On a probability space \((\Omega, F, P)\), where a jump measure \(J\) such that \(\int_0^T \int_{\mathbb{R}_0} |z|^2 J(ds, dz) < \infty\) a.s. and with absolutely-continuous compensator \(\mu\) generates the filtration \((F_t)_{t \in [0, T]}\), consider \(X\) a pure-jump spectrally negative Lévy process, i.e. whose jumps belong to \((0, \infty)\) a.s. \(X\) can be written as

\[ X(t) = \gamma t + \int_0^t \int_{(-\infty, 0)} z \tilde{J}(ds, dz), \]

with \(\tilde{J}\) the compensated jump measure of \(X\). Recall that, for a Lévy process, the compensator \(\mu\) can be written as

\[ \mu(dsdy) = \nu(dy)ds. \]

We assume also \(\int_{(-\infty, 0)} e^{2\alpha x} \nu(dx) < \infty\). We now introduce the so-called Kella-Whitt martingale \(M\):

\[ M(t) = \psi(\alpha) \int_0^t e^{-\alpha Z(s)} ds + 1 - e^{-\alpha Z(t)} - \alpha \overline{X(t)}, \]

with \(\overline{X(\cdot)}\) the running maximum of \(X\), \(Z(t) = \overline{X(t)} - X(t)\), \(\alpha > 0\) and

\[ \psi(\alpha) = \gamma t + \int_{(-\infty, 0)} (e^{-\alpha x} - 1 - \alpha x I_{\{x < 1\}}) \nu(dx). \]

Then \(M_t\) is a martingale (see e.g. Kyprianou [18], Chap. 3, Section 5.) Moreover,

\[ [M, M]_t = \sum_{0 < s < t, \Delta \overline{X(s)} \neq 0} e^{-\alpha \overline{X(s)}} |\Delta \overline{X(s)}|^2 \leq \sum_{0 < s < t, \Delta \overline{X(s)} \neq 0} e^{-\alpha \overline{X(s)}} |\Delta \overline{X(s)}|^2. \]

By hypotheses on \(\nu\), this quantity is finite, and using Protter ([25], Cor. 3 p. 73) \(M\) is a square-integrable martingale. Moreover, the functional Itô calculus approach allows us to obtain the following representation:

**Theorem 6.1.** The Kella-Whitt martingale \(M\) has the following martingale representation formula:

\[ M(t) = E[M(t)] + \int_0^t \int_{(-\infty, 0)} e^{-\alpha \overline{X(s)}} (1 - e^{\alpha y}) \tilde{J}(ds, dy). \]  

(55)

**Proof.** Write

\[ M^n(t) := \psi^n(\alpha) \int_0^t e^{-\alpha \overline{X^n(s)} - X^n(s)} ds + 1 - e^{-\alpha \overline{X^n(t)} - X^n(t)} - \alpha X^n(t), \]

\((M^n\) is \(M\) without the small jumps) where

\[ \psi^n(\alpha) = \gamma t + \int_{(-\infty, -\infty)} (e^{-\alpha x} - 1 - \alpha x I_{\{|x| < 1\}}) \nu(dx), \]

and

\[ X^n(t) = \gamma t + \int_0^t \int_{(-\infty, 0)} z \tilde{J}(ds, dz). \]

Then \(M^n\) is a square-integrable martingale, and \(M^n\) converges to \(M\) in \(M^2(\mu)\), i.e.

\[ E \left[ |M^n(t) - M(t)|^2 \right] \rightarrow_\infty 0. \]

Noticing that \(M^n\) has finite variation and is therefore well defined as a pathwise integral, and that since \(X\) being spectrally negative, it never reaches its maximum when it jumps, we compute

\[ (\nabla_p M^n)(t, z) = e^{-\alpha \overline{X^n(t)} - X^n(t)} (1 - e^{az}), \]

where \(\nabla_p\) is the pathwise derivative.
which — by using the martingale representation formula — yields:

\[ M^n(t) = \int_0^t \int_{(-\infty,0)} e^{-a(X^n(t^-)-X^n(t^+))} (1-e^{az}) J(dsdz). \]

To continue: when using functional Itô calculus, pathwise computations — when available — are fairly straightforward. But the price one has to pay for that is to be able to justify the convergence in \( M^2(t) \) of an approximating martingale sequence to the desired one. In the rest of this example, we shall therefore justify such a convergence. We have:

\[ E[(M(T) - M^n(T))^2] \leq E[(\psi(\alpha) - \psi^n(\alpha))^2 \int_0^T e^{-\alpha(X(s^-)-X(s^+))} ds]^2 \]

(56)

\[ + E[^2 \int_0^T e^{-\alpha(X(s^-)-X(s^+))} ds]^2 \]

(57)

\[ + E[^2 \int_0^T e^{-\alpha(X(s^-)-X(s^+))} ds]^2 \]

(58)

\[ + E[\int e^{-2\alpha X(t)} e^{\alpha X(t)} ds] \]

(59)

\[ + E[\int e^{-2\alpha X(t)} e^{\alpha X(t)} ds] \]

(60)

\[ + \alpha^2 E[(X(t) - X^n(t))^2]. \]

(61)

We shall now show that all terms on the left-hand side tend to zero. Results in Dia \[11\] on small-jump truncations approximations prove useful here, and we use several of them. Term \( 59 \) tends to zero: the proof relies on noticing the residual \( X(t) - X^n(t) \) is a martingale and using Doob’s martingale inequality for the sup (see Dia \[11\], proof of proposition 2.10). In term \( 56 \), notice that the integrand is always less than 1. Hence

\[ E[(\psi(\alpha) - \psi^n(\alpha))^2 \int_0^T e^{-\alpha(X(s^-)-X(s^+))} ds]^2 \leq T^2 E[(\psi(\alpha) - \psi^n(\alpha))^2] \]

\[ = E[\int_{(-\frac{1}{2},0)} e^{ax} - 1 - axI_{|x|<1}^2 dx]^2. \]

and the integral is deterministic, so the expectation vanishes, and this term tends to zero. Taking term \( 59 \),

\[ E[\int e^{-2\alpha X(t)} e^{\alpha X(t)} ds]^2 \leq e^{-\alpha X(t)} E[\int e^{\alpha X(t)} - e^{\alpha X^n(t)} ds]^2 \]

\[ = e^{-\alpha X(t)} E[\int e^{2\alpha X(t)} + e^{2\alpha X^n(t)} - 2e^{\alpha X(t)} e^{\alpha X^n(t)} ds]. \]

Moreover, by Proposition 2.2 in Dia \[11\], \( e^{2\alpha X^n(t)} \) converges to \( e^{2\alpha X(t)} \) in the following norm:

\[ E[|e^{2\alpha X(t)} - e^{2\alpha X^n(t)}|] \longrightarrow 0. \]

Also

\[ -E[e^{\alpha X^n(t)} e^{\alpha X(t)}] \leq -E[e^{\alpha X^n(t)}] E[e^{\alpha X(t)}] \]

and by the same proposition again, \( e^{\alpha X^n(t)} \to e^{\alpha X(t)} \) in \( T^2 \). Hence, the nonnegative term \( 59 \) is bounded from above by a quantity tending to zero.

Concerning term \( 56 \),

\[ E[e^{2\alpha X^n(t)} e^{-\alpha X(t)} - e^{-\alpha X^n(t)} e^{\alpha X(t)}] \leq E[e^{4\alpha X^n(t)}] E[e^{-4\alpha X^n(t)}] \]

\[ = e^{2\alpha X^n(t)} E[e^{-4\alpha X^n(t)}] \]

by Cauchy-Schwarz and the definition of the characteristic exponent. Moreover

\[ E[e^{-4\alpha X(t)}] - 4e^{-3\alpha X(t)} - 3e^{-2\alpha X(t)} - e^{-\alpha X(t)} - 4e^{-\alpha X(t)} - 3e^{-2\alpha X(t)} - e^{-3\alpha X(t)} + 4e^{-4\alpha X(t)}]. \]
Also
\[ -4E[e^{3X(t)}] \leq -4E[e^{X(t)}E[X^3(t)]] \quad \text{and} \quad -4E[e^{-3X(t)}] \leq -4E[e^{-X(t)}E[X^3(t)]] \]
and these two terms tend to \(-4E[e^{-X(T)}]\) using Proposition 2.2 in Dia [11] once more. Finally
\[ 6E[e^{-2X(t)}] \leq 6E[e^{-4X(t)}] + E[e^{-4X(t)}] \]
which tends to \(3E[e^{-4X(t)}]\) . Summing up, term (60) tends to zero.

Regarding (57), by the mean value theorem:
\[ E[(\psi^n(\alpha))^2] = \left(\int_0^T e^{-\alpha X(s)} e^{\alpha X(s)} ds\right)^2 = (\psi^n(\alpha))^2 E[T^2 e^{-\alpha X(t)} e^{\alpha X(t)}] \]
for some \(t_0\) in \([0, T]\), and we conclude by the same argument as in term (59).

Using the mean-value theorem on term (58) in a similar fashion and carrying on as in (60), we conclude that \(M^n \to M\) in \(M^2(\mu)\).

Notice in passing that \(\nabla_p M^n\) converges in \(L^2(\mu)\) to
\[ \nabla_p M(t, z) := e^{-\alpha X(t)}(1 - e^{\alpha z}); \]
the proof follows exactly the same lines as terms (57) and (58). This yields the following martingale representation formula for the Kella-Whitt martingale:
\[ M(t) = E[M(t)] + \int_0^t \int_{(-\infty, 0)} e^{-\alpha X(s)}(1 - e^{\alpha y})\tilde{J}(ds\, dy). \]  
(62)

### 6.4 Supremum of a Lévy process

A martingale representation for the supremum of Brownian motion can be proved using Clark’s formula (see e.g. [27]). In the case of a Lévy process, a representation was provided by Shiryaev and Yor [28] and the proof relies on the Itô formula. Recently, Rémiarmand and Renaud [26] reproved the result using Malliavin calculus. In this section, we derive the representation using the Functional Itô operators instead.

Let us introduce the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where the filtration is generated by a Brownian motion \(W\) and a Poisson measure \(J\). Let us then consider \(X\), the square-integrable Lévy process defined by,
\[ X(t) = X(0) + \mu t + \sigma W(t) + \int_0^t \int_{|z| > 1} zJ(ds\, dz) + \int_0^t \int_{|z| < 1} zJ(ds\, dz) \]

For \(T > 0\), we are interested in finding a martingale representation for its supremum at \(T\), denoted by \(\bar{X}(T) := \sup_{0 \leq s \leq T} X_s\).

**Theorem 6.2.**
\[ \bar{X}(T) = E[\bar{X}(T)] + \int_0^T \nabla W P(t) dW(t) + \int_0^T \int_{R^2} \nabla_p P(t, z) J(dt\, dz). \]  
(63)

with
\[ \nabla_p P(t, z) = \frac{\bar{X}(T) - X(t)}{\bar{X}(T) - z} F_{T-t}(u) du \]
\[ \nabla W P(t) = \sigma F_{T-t}(\bar{X}(T) - X(t)) \]

To prove the above theorem, we consider the process
\[ P(t) = E[\bar{X}_T | \mathcal{F}_t]. \]
We start from the same point as Shiryaev-Yor and Rémillard-Renaud. Using the properties of Lévy processes, one can show that:

\[ P(t) = \overline{X}(t) + \int_0^\infty F_{T-t}(u) \, du, \]  

where \( F_{T-t}(u) = \mathbb{P}(\overline{X}(T-t) \leq u) \).

As in the previous example, we focus first on the computations with a process \( X^n \) that corresponds to \( X \) with all the jumps of size less than 1/\( n \) truncated:

\[ X^n(t) = X^n(0) + \mu t + \sigma W(t) + \int_0^t \int_{z \in (\frac{1}{n},1]} z \tilde{J}(dsdz) + \int_0^t \int_{|z| \geq 1} z J(dsdz) \]

and introduce

\[ P^n(t) := E[\overline{X^n_T} | F_t], \quad \overline{X^n}(t) + \int_0^\infty \overline{F}_{T-t}(u) \, du, \]  

where \( \overline{F}_{T-t}(u) = \mathbb{P}(\overline{X}(T-t) \leq u) \).

One sees that \( P^n \) has a functional representation that is not vertically differentiable at the points where \( X^n \) reaches its supremum, because the supremum itself is not vertically differentiable at these points. To remedy this, we introduce the following Laplace softsup approximation defined below.

**Lemma 6.3.** For a càdlàg function \( f \), the associated Laplace softsup

\[ L(f, t) := \frac{1}{a} \log(\int_0^t e^{af(s)} \, ds), \]

satisfies

\[ \lim_{a \to \infty} L^a(f, t) = \sup_{0 \leq s \leq t} f(s) \]

**Proof.** This result can be found for continuous functions in Mörters-Peys (22, Lemma 7.30). The proof is similar in the càdlàg case.

Using continuities of the exponential and logarithm functions. Having \( a \to \infty \) yields the “\( \leq \)” inequality. For the converse inequality, let us consider two cases.

In the first case, let us assume that the supremum is attained at a certain point, i.e. there exists \( t_0 \) such that \( f(t_0) = \max_{0 \leq s \leq t} f(s) \). Then, by right-continuity of \( f \), for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( f(r) \geq f(t_0) - \epsilon \) for \( r \in (t_0, t_0 + \delta) \). So

\[ \frac{1}{a} \log(\int_0^t e^{af(s)} \, ds) \leq \frac{1}{a} \log(t) \sup_{0 \leq s \leq t} e^{af(s)} = \sup_{0 \leq s \leq t} f(s) + \frac{\log(t)}{a}, \]

Taking the limit \( a \to \infty \) yields the result, as \( \epsilon \) is arbitrary.

In the second case where the sup is not reached, the càdlàg property of \( f \) entails that there exists \( t_1 \) such that

\[ \sup_{0 \leq s \leq t} f(s) = \lim_{u \uparrow t} f(u) =: f(t_1). \]

Then, for any \( \epsilon \), there exists \( \delta > 0 \) such that \( f(r) \geq f(t_1) - \epsilon \) for \( r \in (t_1 - \delta, t_1) \), since \( f \) is làg. Using the same computations as in the first case, this yields the result.

**Lemma 6.4.** For a Lévy process \( X \), and any \( t > 0 \), one has

\[ \lim_{a \to \infty} E[|L^a(X, t) - \overline{X}(t)|^2] = 0. \]

**Proof.**

\[ E[|L^a(X, t) - \overline{X}(t)|^2] = E[|L^a(X, t)|^2] + E[|\overline{X}(t)|^2] + 2E[L^a(X, t)]E[\overline{X}(t)]. \]
Moreover,
\[
L^a(X,T) \leq \overline{X}(T) + \frac{\log(T)}{a} \leq \overline{X}(T) + (\log(T))^+, 
\]
entailing, by dominated convergence
\[
\lim_{a \to \infty} E[L^a(X,T)] = \overline{X}(T).
\]
and
\[
\lim_{a \to \infty} E[|L^a(X,T)|^2] = E[|\overline{X}(T)|^2],
\]
thus yielding convergence.

Going back to \(P^n\), we introduce the process \(Y^{a,n}(t)\):
\[
Y^{a,n}(t) := L^a(X^n,t) + \int_0^\infty \mathbb{E}_{X^n} \left[ Y^{a,n}(t) - \mathbb{E}_{X^n}(t) \right] F_{T-t}(u) du.
\]

**Lemma 6.5.** For any \(t\), \(Y^{a,n}(t)\) is square integrable, and
\[
\lim_{a \to \infty} E[|Y^{a,n}(t) - P^n(t)|^2] = 0
\]

**Proof.** The square-integrability of \(Y^{a,n}\) stems from the previous lemma, using that \(L^a(X^n,t) \leq \overline{X^n}(t) + (\log(T))^+\). Now,
\[
E[|Y^{a,n}(t) - P^n(t)|^2] \leq 2E[|L^a(X^n,t) - \overline{X^n}(t)|^2] + 2E[\int \mathbb{E}_{X^n} \left[ Y^{a,n}(t) - \mathbb{E}_{X^n}(t) \right] F_{T-t}(u) du]^2]
\]
We get the following inequality
\[
\int \mathbb{E}_{X^n} \left[ Y^{a,n}(t) - \mathbb{E}_{X^n}(t) \right] F_{T-t}(u) du \leq E[|L^a(X^n,t) - \overline{X^n}(t)|^2]
\]
by bounding the integrand by 1 in the left-hand side. So
\[
E[|Y^{a,n}(t) - P^n(t)|^2] \leq 4E[|L^a(X^n,t) - \overline{X^n}(t)|^2]
\]
Taking the limit \(a \to \infty\) and using Lemma 6.4 yields
\[
\lim_{a \to \infty} E[|Y^{a,n}(t) - P^n(t)|^2] = 0;
\]

For fixed \(t \in [0,T]\) let us now introduce the following family of square-integrable martingales
\[
(Z^{a,n}(s,t))_{s \in [0,t]} = E[Y^{a,n}(t) \mid \mathcal{F}_s].
\]

**Lemma 6.6.**
\[
\lim_{a \to \infty} E[|Z^{a,n}(s,t) - E[P^n_s \mid \mathcal{F}_s]|^2] = 0
\]

**Proof.**
\[
E[Z^{a,n}(s,t) - E[P^n_s \mid \mathcal{F}_s] = E[|E[Y^{a,n}(t) - P(t) \mid \mathcal{F}_s]|^2],
\]
which by Jensen’s inequality,
\[
\leq E[E[|Y^{a,n}(t) - P(t)|^2 \mid \mathcal{F}_s]] = E[Y^{a,n}(t) - P(t)]^2,
\]
which converges to zero by Lemma 6.5. \(\square\)
In particular, this entails that for every \( t \), \( \lim_{\epsilon \to 0} Z(t, t, a) = P^n_t \). Let us now compute the functional Itô operators of \( Z^n(t, t, a) \). At time \( s = t \), \( Z^n(t, t, a) = Y^{a,n}(t) \), and \( Y^{a,n} \) has a pathwise functional representation, allowing to compute explicitly the operators.

\[
\nabla_p(Z^n)(t, t, z) = \int_{L^n(X^n(t) - X^n(t))} F_{T-t}(u) du,
\]

and

\[
\nabla_W(Z^n)(t, t) = \lim_{h \to 0} \frac{1}{h} \int_{L^n(X^n(t) - X^n(t)) - \sigma h} F_{T-t}(u) du = F_{T-t}(L^n(X^n(t) - X^n(t))).
\]

In a way similar to Lemma 6.3 we can show that
\[
\lim_{a \to \infty} \nabla_p(Z^n)(t, t, z) = \int_{X^n - X^n(t)} F_{T-t}(u) du =: \nabla_p P^n(t, z)
\]

\[
\lim_{a \to \infty} \nabla_W(Z^n)(t, t) = \sigma F_{T-t}(X^n(t) - X^n(t))
\]

and these quantities must equate to \( \nabla_p P^n \) and \( \nabla_W P^n \) respectively. We can conclude that at time \( T \)

\[
P^n(T) = \overline{X^n}(T) = E[P^n(T)] + \int_0^T \nabla_W P(t) dW(t) + \int_0^T \int_{(0, \infty)} \nabla_p P(t, z) \tilde{J}(dtdz)
\]

(67)

with the integrands defined as above.

All that remains to do is to remove the truncation of the small jumps. In this case, however, this is straightforward, as

\[
E[|P^n(T) - P(T)|^2] = E[|\overline{X^n}(T) - \overline{X}(T)|^2],
\]

which tends to zero, using the result of Dia [11], p11. This yields the convergence of \( \nabla_W P^n \) and \( \nabla_p P^n \) to \( \nabla_W P \) and \( \nabla_p P \) respectively, and so the final result for the martingale representation of the supremum of a Lévy process:

\[
P(T) = \overline{X}(T) = E[P(T)] + \int_0^T \nabla_p (t) dW(t) + \int_0^T \int_{(0, \infty)} \nabla_p P(t, z) \tilde{J}(dtdz)
\]

(68)

with

\[
\nabla_p P(t, z) = \int_{X - X(t)} F_{T-t}(u) du
\]

\[
\nabla_W P(t) = \sigma F_{T-t}(X(T) - X(t))
\]

7 A density result

In this subsection, we prove that the simple random fields are dense in \( L^2(\mu) \) for a fairly general measure \( \mu \). This is a common result in the continuous case (see Steele [30] that extends to Lévy compensators using some isometry properties between Hilbert spaces (Applebaum [1]). To the best of our knowledge, such a result is not proved when one allows the compensator to be random and/or time-inhomogeneous. For completeness, we should deal with it here. Throughout this section, \( f \) denotes a random field

\[
f : [0, T] \times \mathbb{R}_0^d \times \Omega \to \mathbb{R}
\]

Before stating the theorem, let us first make some assumptions:

**Assumption 2.** (Absolute continuity in time) The measure

\[
\mu : \mathcal{B}([0, T] \times \mathbb{R}_0^d) \times \Omega \to \mathbb{R}
\]

satisfies the hypotheses of “absolute continuity in time”: \( \mu(ds,d\omega) << ds \) for all \( \omega \in \Omega \).

**Assumption 3.** (\( \sigma \)-finiteness) For all \( \omega \in \Omega \), \( \mu \) is finite on every Borel set \( A \times B \in \mathcal{B}([0, T] \times \mathbb{R}_0^d) \) such that \( 0 \notin \overline{B} \) (\( \overline{B} \) denoting the closure of \( B \)).
Theorem 7.1. The set of simple random fields \( \mathcal{R} \) of the form
\[
\psi(t, z, \omega) = \sum_{i,k=1}^{n,m} \psi_{ik}(\omega) \mathbf{1}_{(t^n_i(Z_k, \omega), t^n_{i+1}(Z_k, \omega))}(t) \mathbf{1}_{Z_k}(z),
\]
with the \( \psi_{ij} \) \( \mathcal{F}_t \)-measurable, the \( Z_k \) disjoint Borel sets such that \( 0 \not\in Z_k \), is dense in \( L^2_\mu(\mu) \) and the \( (t^n_i)_{i=0}^{2^n} \) finite stopping times given by
\[
t^n_i(\omega, K) = \inf\{t \in [0, T] | \mu([0, t] \times K, \omega) \geq i2^{-n}\} \land T \land \inf\{t \in [0, T] | \mu([0, t] \times K, \omega) \geq 2^n\}
\]
for any Borel set \( K \) of \( \mathbb{R}^d \) such that \( 0 \not\in K \). We shall just write \( t^n_i \) for \( t^n_i(\omega, Z) \) to alleviate the notation when there is no risk of confusion.

The rest of the appendix is devoted to proving the above theorem.

Let us consider a random field \( f : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d, f \in L^2_\mu(\mu) \). Let us first assume that \( f \) bounded. Let \( (Z_j)_{j \in \mathbb{N}} \) be a sequence of sets of \( \mathbb{R}^d \), \( 0 \not\in Z_j \), such that for all \( j \) and all \( \omega \in \Omega \), \( \mu([0, T] \times Z_j) < \infty \).

This is possible because of the \( \sigma \)-finiteness of \( \mu \) stemming from Assumption 3.

Remark 7.2. Notice that if \( \mu \) is a predictable measure, then the stopping times defined above are also predictable.

We introduce the following family of operators, for all integers \( n, m \geq 1 \):
\[
A_{nm}(f) = \sum_{i=1}^{2^{n-1}} \sum_{k=1}^{m} \mathbf{1}_{\left(\int_{t^n_{i-1}(Z_k, \omega)}^{t^n_i(Z_k, \omega)} f(s, y, \omega) \mu(ds dz, \omega)\right)}
\]
with the convention that \( 0/0 = 0 \), which occurs when \( t^n_{i-1} = t^n_i \). Notice that for all \( n, m \), \( A_{nm}(f) \) belongs to \( \mathcal{R} \). Moreover, it satisfies the following useful properties:

Lemma 7.3. For \( f \) bounded and \( L^2_\mu(\mu) \)-integrable,
1. \( \|A_{nm}(f)\| \leq \|f\|\infty \);
2. \( \|A_{nm}(f)\|L^2_\mu(\mu) \leq \|f\|L^2_\mu(\mu) \).

Proof. 1. For all \( \omega \), there are three cases to be considered:
(a) for all \( z \not\in \bigcup_{k=1}^m Z_k \), then \( A_{nm}(f) = f = 0 \);
(b) for all \( t \in (T \land 2^{n-1}, T) \), \( A_{nm}(f) = 0 \leq |f| \);
(c) otherwise, for all \( \omega \), \( A_{nm}(f)(t, z, \omega) = \frac{1}{\mu([t_{i-1}, t_i) \times Z_k, \omega]} \int_{(t_{i-1}, t_i]} f(s, y, \omega) \mu(ds dz, \omega) \), for a unique couple \( (i, k) \) with \( i^* \in 0..2^{n-1} - 1 \) and \( k^* \in 1..m \) such that \( (t, z) \in (t_{i^*}, t_{i^*+1}] \times Z_{k^*} \). That is, \( A_{nm}(f) \) is the average of \( f \) over \( (t_{i-1}, t_i) \times Z_k \). So by the very definition of the average, there exists a point \( (t_0, z_0) \), with \( t_0 \in (t_{i-1}, t_i) \times Z_k \), such that the value of \( A_{nm}(f) \) is lower than the value of \( \|f\| \) at that point.

This implies that \( \|A_{nm}(f)\| \leq \|f\|\infty \).

2. To alleviate the notation, let us write
\[
c_{ik} = \frac{1}{\mu([t_{i-1}, t_i) \times Z_k, \omega]} \int_{(t_{i-1}, t_i]} f(s, y, \omega) \mu(ds dz, \omega) .
\]

Thus, for all \( \omega \)
\[
c_{ik}^2 \leq \frac{1}{\mu([t_{i-1}, t_i) \times Z_k, \omega]} \int_{(t_{i-1}, t_i]} f^2(s, y, \omega) \mu(ds dy, \omega) \tag{69}
\]
by Cauchy-Schwarz.

Since the time intervals are pairwise disjoint, as well as the \( Z_k \),
\[
A_{nm}^2(f) = \sum_{i=1}^{2^{n-1}} \sum_{k=1}^{m} c_{ik}^2 \mathbf{1}_{(t_i, t_{i+1})}(t) \mathbf{1}_{Z_k}(z).
\]
Integrating,
$$E[\int_{[0,T] \times \mathbb{R}_0^n} A_{mn}(f)(s,y,\omega) \mu(dsdy,\omega)]$$
$$= E[\sum_{i=1}^{2^n-1} \sum_{j=1}^{m} \mu([t_{i}^{n}, t_{i+1}^{n}] \times Z_j) \int_{(t_{i-1}^{n}, t_{i}^{n}] \times Z_j} f(s, y, \omega) \mu(dsdz, \omega)]$$.

Now, we have two cases to consider:

1. for a given $k$, if $t_{2^n}^{n}(Z_k, \omega) < T$, then by construction, for all $i$,
$$\mu([t_{i}^{n}, t_{i+1}^{n}] \times Z_k) = \mu([t_{i-1}^{n}, t_{i}^{n}] \times Z_k) = 2^{-n};$$

2. otherwise, there exists an index $i^*(k)$ such that $t_{i^*(k)}^{n} < T$ and $t_{i}^{n} = T$ for all $i \geq i^*(k)$. Hence for a given $k$, all the terms of time index greater than $i^*(k)$ are null. Besides, this implies two things:
$$\forall i < i^*(k), \mu([t_{i}^{n}, t_{i+1}^{n}] \times Z_k) = \mu([t_{i-1}^{n}, t_{i}^{n}] \times Z_k) = 2^{-n}, \quad (70)$$
$$\mu([t_{i^*(k)}^{n}, t_{i^*(k)+1}^{n}] \times Z_k) = \mu([t_{i^*(k)-1}^{n}, t_{i^*(k)}^{n}] \times Z_k) = 2^{-n}. \quad (71)$$

In any case, we obtain
$$\|A_{mn}^2(f)\|_{L^2(\mu)} \leq E[\sum_{i=1}^{2^n-1} \sum_{j=1}^{m} \int_{(t_{i-1}^{n}, t_{i}^{n}] \times Z_j} f^2(s, y, \omega) \mu(dsdz, \omega)] \leq \|f\|_{L^2(\mu)}.$$\[
\textbf{Remark 7.4.} We can see why we needed to define the $t_{i}^{n}$ as a random partition: it is so that the operator $A_{mn}(f)$ defines a contraction in $L^2(\mu)$. In fact, this is actually the only reason; in the case where $\mu$ is a Lévy compensator for example, the fact that $\mu$ is time homogeneous and independent of $\omega$ to take the partition deterministic and Lebesgue-equidistant in time. If $\mu$ is deterministic but time-inhomogeneous, then the $t_{i}^{n}$ can be taken deterministic and $\mu$-equidistant, since we know the value of $\mu([0,T] \times B)$ for any measurable set $B$ from the start.
\]

We are now ready to prove the important part:

\textbf{Lemma 7.5.}
$$\lim_{n \to \infty} \|A_{mn}(f) - f\|_{L^2(\mu)} = 0.$$\[
\textbf{We start by introducing the operator}
$$B_{mn}(f) := \sum_{i=1}^{2^n} \sum_{k=1}^{m} \mathbb{1}_{[t_{i}^{n}, t_{i+1}^{n}]}(t) \mathbb{1}_{Z_k \in Z_k} \mu([t_{i-1}^{n}, t_{i}^{n}] \times Z_k, \omega) \left( \int_{(t_{i-1}^{n}, t_{i}^{n}] \times Z_k} f(s, y, \omega) \mu(dsdz, \omega) \right).$$

Notice that $B_{mn}(f)$ is not a simple predictable process: it is actually anticipative. However,
$$(B_{mn}(f)(\cdot, \cdot, \omega))_{n \geq 0}$$
is a martingale, when defined as a discrete-time martingale on the right space.

\textbf{Lemma 7.6.} Fix $\omega$ and define a new probability space $(\Omega', \mathcal{F}_m', \mathbb{Q}_{\omega,m})$ with
$$\Omega'_m = \{\omega\} \times [0,T] \times \cup_{j=1}^{m} Z_k,$$
$$\mathcal{F}_m' = \{ (\omega, A \times B), A \in B([0,T]), B \in B(\cup_{j=1}^{m} Z_k) \}, \quad (72)$$
$$\mathbb{Q}_{\omega,m}(A \times B) = \frac{\mu(A \times B, \omega)}{\mu([0,T] \times \cup_{j=1}^{m} Z_k, \omega)}, \quad (73)$$
which we equip with the smallest filtration $(\mathcal{F}_m')_{n \in \mathbb{N}}$ that makes the functions
$$t, z \mapsto \sum_{i=1}^{2^n} c_i \mathbb{1}_{[t_{i}^{n}, t_{i+1}^{n}]}(t) \mathbb{1}_{Z_k}(z)$$
measurable for all $k$ in $1..m$. Then
$$(B_{mn}(f)(\cdot, \cdot, \omega))_{n \geq 0}$$
is a martingale on this space.
Proof. Notice that in general \( f \) is not \( F_{\omega,m} \)-measurable. However, \( f \mathbf{1}_{U_{k=1}^m Z_k}(z) \) is. Then, since conditional expectation is just orthogonal projection,

\[
E[f(t,z)\mathbf{1}_{U_{k=1}^m Z_k}(z)|F_{\omega,m}^n](t,z) = \sum_{i=1}^{2^n-1} \sum_{k=1}^m \sum_{p=1}^{2^l} \mathbf{1}_{(t_i^{(p-1)+1}, t_i^p]}(t) \mathbf{1}_{Z_k}(z) \int_{(t_i^{(p-1)+1}, t_i^p]} f(s,y,\omega)\mu(ds\ dy,\omega)
\]

\[
= B_{mn}(f)(t,z).
\]

So \( B_{mn}(f) \) is indeed a \( F_{\omega,m}^n \)-martingale.

\( \square \)

**Lemma 7.7.** Let \( f \) be bounded and \( L_2^2(\mu) \)-integrable. Then

\[
\lim_{n \to \infty} \|B_{mn}(f) - f \mathbf{1}_{U_{k=1}^m Z_k}(\cdot)\|_{L_2^2(\mu)} = 0.
\]

Proof. For fixed \( \omega \), \( \{B_{mn}(f)(\cdot,*,\omega), n \geq 0\} \) being a martingale (by Lemma 7.6) allows us to apply the \( L_2(Q) \)-bounded martingale convergence theorem to conclude that \( B_{mn}(f)(\cdot,*,\omega) \) converges in \( L_2(Q) \) except perhaps on a \( \mu \)-null set. We note this limit \( B_{m\infty}(f) \).

Moreover, since \( f \) is bounded, dominated convergence gives

\[
\lim_{n \to \infty} \int_{A \times B} B_{mn}(f)(s,y,\omega)\mu(ds\ dy,\omega) = \int_{A \times B} B_{m\infty}(f)(s,y,\omega)\mu(ds\ dy,\omega)
\]

for all \( A \times B \in \mathcal{B}([0,T] \times U_{k=1}^m Z_k) \).

Also, by definition of the operator \( B_{mn}(f) \), we know that

\[
\int_{A \times B} B_{mn}(f)(s,y,\omega)\mu(ds\ dy,\omega) = \int_{A \times B} f(s,y,\omega)\mu(ds\ dy,\omega)
\]

for all \( A \times B \) such that \( (\omega, A \times B) \in F_{\omega,m}^n (n \in \mathbb{N}) \) and all \( n \geq a \). Lévy’s zero-one law gives \( B_{m\infty} = f \mathbf{1}_{U_{k=1}^m Z_k} \)

Finally, by dominated convergence, we can take the limit out of the integral:

\[
\lim_{n \to \infty} \int_{[0,T] \times U_{k=1}^m Z_k} |B_{mn}(f)(s,y,\omega) - f(s,y,\omega)(\mathbf{1}_{U_{k=1}^m Z_k}(z))|^2\mu(ds\ dy,\omega).
\]

The result follows on taking expectations and applying dominated convergence once more. \( \square \)

**Lemma 7.8.** Let \( f \) be bounded and \( L_2^2(\mu) \)-integrable. For any \( m \) and any fixed \( l \),

\[
\lim_{n \to \infty} A_{mn}(B_{ml}(f))(t,z,\omega) = B_{ml}(f)(t,z,\omega), \mu \text{-a.e.}
\]

Proof. For that purpose, we expand \( A_{mn}(B_{ml}(f)) \), with \( n \geq l 

\[
A_{mn}(B_{ml}(f)) \quad (77)
\]

\[
= \sum_{i=1}^{2^n-1} \sum_{k=1}^m \sum_{p=1}^{2^l} \mathbf{1}_{(t_i^{(p-1)+1}, t_i^p]}(t) \mathbf{1}_{Z_k}(z) \int_{(t_i^{(p-1)+1}, t_i^p]} f(s,y,\omega)\mu(ds\ dy,\omega).
\]

We now note two things: first, for \( q \neq k \),

\[
\mu((t_i^{(p-1)+1}, t_i^p) \times Z_k) \cap ((t_q^{(p-1)+1}, t_q^p) \times Z_q)) = 0.
\]

Moreover, recall that the time grid is refining. This means that since \( n \geq l \), the \( (t_i^p)_{1 \leq i \leq 2^n} \) are a subset of \( (t_i^p)_{1 \leq i \leq 2^l} \). So

\[
\mu((t_i^{(p-1)+1}, t_i^p) \times Z_k) \cap ((t_q^{(p-1)+1}, t_q^p) \times Z_q))
\]

is either equal to \( \mu((t_i^{(p-1)+1}, t_i^p) \times Z_k) \) or zero. So [77] is

\[
\sum_{i=1}^{2^n-1} \sum_{k=1}^m \sum_{p=1}^{2^l} \mathbf{1}_{(t_i^{(p-1)+1}, t_i^p]}(t) \mathbf{1}_{Z_k}(z) \int_{(t_i^{(p-1)+1}, t_i^p]} f(s,y,\omega)\mu(ds\ dy,\omega).
\]
By the above, this is almost $B_{ml}(f)$. In fact

$$A_{mn}(B_{ml}(f))(t, z, \omega) = B_{ml}(f)(t, z, \omega)$$

$$= B_{ml}(f)(t, z, \omega) I_{t \in \bigcup_{l=1}^{n} [t_{l}, t_{l}^{*} + 2^{n}(m-l)_{+}]} + \sum_{i=1}^{\infty} B_{ml}(f)(t - 2^{n}(m-i)_{+}, z, \omega) I_{t \in [t_{i}, t_{i}^{*} + 2^{n}(m-i)_{+}]}$$

We note two things. First,

$$\lim_{m \to \infty} A_{mn}(B_{ml}(f))(t, z, \omega) = B_{ml}(f)(t, z, \omega)$$

(79)

for all $(t, z, \omega)$ such that $t \neq i2^{-m}T$, $0 \leq i \leq T)$. Second, for $n \geq 1$

$$|A_{mn}(B_{ml}(f))(t, z, \omega)| \leq |B_{ml}(f)(t, z, \omega)| + |B_{ml}(f)(t - 2^{-m}T, z, \omega)|.$$

Moreover,

$$|A_{mn}(B_{ml}(f))(t, z, \omega)| = |A_{mn}(B_{ml}(f))(t, z, \omega) - B_{ml}(f)(t, z, \omega) + B_{ml}(f)(t, z, \omega)|$$

$$\geq |A_{mn}(B_{ml}(f))(t, z, \omega) - B_{ml}(f)(t, z, \omega)| - | - B_{ml}(f)(t, z, \omega)|$$

$$\geq |A_{mn}(B_{ml}(f))(t, z, \omega) - B_{ml}(f)(t, z, \omega)| - |B_{ml}(f)(t, z, \omega)|.$$

Since $\|B_{ml}(f)\|_{L^{2}(\mu)} < \infty$, we finally obtain

$$\lim_{n \to \infty} \|A_{mn}(B_{ml}(f)) - B_{ml}(f)\|_{L^{2}(\mu)} = 0,$$

(80)

since $\mu$ is absolutely continuous with respect to time.

Proof. (Proof of Lemma 7.5) For $f$ bounded, we are now ready to prove that

$$\lim_{n \to \infty} \|A_{mn}(f) - f\|_{L^{2}(\mu)} = 0.$$  

(81)

We have

$$\|A_{mn}(f) - f\|_{L^{2}(\mu)} \leq \|A_{mn}(f - f I_{E_{k}} z_{k})\|_{L^{2}(\mu)} + \|A_{mn}(f I_{E_{k}} z_{k}) - B_{ml}(f)\|_{L^{2}(\mu)}$$

(82)

$$+ \|A_{mn}(B_{ml}(f)) - f\|_{L^{2}(\mu)}$$

$$\leq \|f - f I_{E_{k}} z_{k}\|_{L^{2}(\mu)} + \|f I_{E_{k}} z_{k} - B_{ml}(f)\|_{L^{2}(\mu)}$$

(83)

$$+ \|A_{mn}(B_{ml}(f)) - f\|_{L^{2}(\mu)},$$

(84)

as $A_{mn}$ is a contraction. So by Lemmas 7.7 and 7.8 for all $l$,

$$\lim_{n \to \infty} \|A_{mn}(B_{ml}(f)) - B_{ml}(f)\|_{L^{2}(\mu)}$$

(85)

$$\leq \|f - f I_{E_{k}} z_{k}\|_{L^{2}(\mu)} + 2 \|f I_{E_{k}} z_{k} - B_{ml}(f)\|_{L^{2}(\mu)}.$$  

(86)

Now, since $l$ is arbitrary, the previous expression yields

$$\lim_{n \to \infty} \|A_{mn}(f) - f\|_{L^{2}(\mu)} \leq \|f - f I_{E_{k}} z_{k}\|_{L^{2}(\mu)}$$

(87)

$$\lim_{n \to \infty} \|A_{mn}(f) - f\|_{L^{2}(\mu)} \leq \|f - f I_{E_{k}} z_{k}\|_{L^{2}(\mu)}$$

(88)

since we know $B_{ml}(f) \to f I_{E_{k}} z_{k}$ when $l \to \infty$. This holds for any $m$. Letting $m \to \infty$, dominated convergence gives

$$\|f - f I_{E_{k}} z_{k}\|_{L^{2}(\mu)} \to 0,$$

which concludes the proof for $f$ bounded.
Lemma 7.5 proves the theorem of approximation of any random field $f$ by simple ones in the $L^2(\mu)$. To finish proving theorem 7.1 we consider an unbounded $f$. We can approximate $f$ by bounded $f^n$ as follows. Write

$$f^n(s, y, \omega) = f(s, y, \omega) 1_{|f(s, y, \omega)| \leq n}. \quad (89)$$

We now prove that $f^n$ converges to $f$ pointwise everywhere.

$$\mathbb{P} \circ \mu\left( \bigcup_{\epsilon \in \mathbb{R} \cap \mathbb{Q}} \bigcap_{n_0 \in \mathbb{N}} \bigcup_{n \geq n_0} \{ (t, z, \omega) : |f_n(t, z, \omega) - f(t, z, \omega)| > \epsilon \} \right) \quad (90)$$

$$\mathbb{P} \circ \mu\left( \bigcap_{n_0 \in \mathbb{N}} \bigcup_{n \geq n_0} \{ (t, z, \omega) : |f(t, z, \omega)| > n \} \right) \quad (91)$$

$$\mathbb{P} \circ \mu(\bigcap_{n_0 \geq 1} \bigcup_{n \geq n_0} \{ (t, z, \omega) : |f(t, z, \omega)| > n \}). \quad (92)$$

On the other hand,

$$\sum_{n=1}^{\infty} (\mathbb{P} \circ \mu)(|F(t, z, \omega)| > n) \leq \sum_{n \geq 1} \frac{\|f(t, z, \omega)\|^2_{n^2}},$$

using the Chebyshev-Markov inequality. In particular, the series converges. This implies that

$$\inf_{N \geq 1} \sum_{n \geq N} (\mathbb{P} \circ \mu)(|f(t, z, \omega)| > n) = 0.$$

Hence

$$\mathbb{P}(\bigcap_{n_0 \geq 1} \bigcup_{n \geq n_0} \{ (t, z, \omega) : |f(t, z, \omega)| > n \}) \leq \inf_{N \geq 1} (\mathbb{P} \circ \mu)(|f(t, z, \omega)| > n) \quad (93)$$

$$\leq \inf_{N \geq 1} (\mathbb{P} \circ \mu)(\bigcup_{n \geq 1} |f(t, z, \omega)| > n) \quad (94)$$

$$\leq \inf_{N \geq 1} \sum_{n \geq N} (\mathbb{P} \circ \mu)(|f(t, z, \omega)| > n) = 0 \quad (95)$$

Remark 7.9. In case $\mu$ is a measure that charges only $\{0\}$, i.e.

$$m(ds \ dy, \omega) = 1_0(dy) n(\{s\}) ds,$$

it is possible to carry out the same demonstration as before, except that we do not need to be concerned with the $Z_k$’s. By doing so, we end up recovering the density of the processes

$$\phi(s, \omega) = \sum_{i=0}^{I} \phi_i(\omega) 1_{(t_i(\omega), t_{i+1}(\omega)]}(t),$$

in $L^2_p(\text{Leb}([0, T]))$, with $\phi_i \mathcal{F}_{t_i}$ measurable and the $t_i$ stopping times.

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