SPECTRUM OF THE RELATIVISTIC PARTICLES IN VARIOUS POTENTIALS

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We extend the notion of Dirac oscillator in two dimensions, to construct a set of potentials. These potentials becomes exactly and quasi-exactly solvable potentials of non-relativistic quantum mechanics when they are transformed into a Schrödinger-like equation. For the exactly solvable potentials, eigenvalues are calculated and eigenfunctions are given by confluent hypergeometric functions. It is shown that, our formulation also leads to the study of those potentials in the framework of the supersymmetric quantum mechanics.

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INTRODUCTION

The solution of the (2 + 1)-dimensional Dirac equation are of special interest, because of the rapid growth in nanofabrication technology that has made possible to confine laterally two-dimensional (2D) electron systems. These quantum confined electron systems are referred to as artificial atoms where the potential of the nucleus, in the non-relativistic case, is replaced by an effective potential of the form \( V = \frac{1}{2}r^2 \) which is often used as realistic approximation \([1, 2, 3]\). Although the parabolic potential appears to be a good approximation for artificial atom structures, their modelling with various potential profiles will be interesting from the theoretical point of view as well as from its practical applications.

In order to analyze relativistic effects on the spectrum of such physical systems one should construct Dirac equation including adequate potentials and obtain its solution. For the relativistic case, the spectrum and properties of such systems can be determined by using two dimensional Dirac oscillator \([2, 4, 5]\). Relativistic extensions of the various exactly and quasi-exactly solvable (QES) potentials have also turned out to be of importance in the description of 2D phenomena \([2, 7, 8, 9, 10]\). Different condensed matter physics phenomena point to the existence of (2 + 1)-dimensional systems whose spectrum determined by Dirac equation Hamiltonian including various potentials. It is well known that the dirac equation is used for the description of spin-1/2 relativistic particle. Meanwhile we mention here that the Hamiltonian in the form of the Klein-Gordon equation, so called Fesbach-Villars equation, has been constructed in a two component form for spinless particles \([11]\) and in an eight-component form for spin-1/2 particles \([12, 13, 14]\). Regrettably, the Dirac equation is exactly solvable only in a very restricted potentials. It is the purpose of the present article to construct a Dirac equation including a class of potentials whose spectrum can exactly be determined. For this purpose we transform the Dirac equation into two Schrödinger-like equation, because there exist a large number of papers discussing the 2D, few electron systems, most of which are tackling the problem in the framework of the Schrödinger-like equation.

A Dirac equation with an interaction linear in coordinates was considered long ago \([15]\) and recently rediscovered in the context of the relativistic many body theories \([16]\). The equation is named Dirac oscillator, since in the non-relativistic limit it becomes a harmonic oscillator with a very strong spin-orbit coupling term. Dirac oscillator has attracted much attention and the concept gave rise to a large number of papers concerned with its different aspects \([16, 17, 18, 21, 22, 23, 24]\). Analogous to the Dirac equation, with a modified momentum operator, which is a usual Schrödinger equation. As we have already noted the Dirac equation including various potentials might attract much attention because it may have some physical applications, particularly in the condensed matter physics. It seems that one can present more realistic models for the artificial atoms using the procedure given here.

In order to solve Dirac equation, in this paper, we use functional approach which have been applied to solve Schrödinger equation for a exactly or QES potential profile. For a QES potential it is possible to determine algebraically a part of spectrum but not whole spectrum \([21, 26, 27]\). Our approach also gives a hint to the solution of the problem in the framework of the SUSYQM.

The method presented here consist of the followings. In section 2, we introduce (2 + 1)-dimensional Dirac equation.
Using the structures worked for the Dirac oscillator we develop a method to construct a class of exactly and QES potential profile. We transform the Dirac equation into the form of the Schrödinger equation. In section 3, we construct a Dirac equation including, harmonic oscillator, Coulomb and Morse potentials. We obtain the corresponding eigenvalues and eigenfunctions. In section 4 a class of QES potentials are constructed and their ground state wave functions are explicitly determined. We conclude our results in section 5.

METHOD

The \((2 + 1)\)-dimensional Dirac equation for free particle of mass \(m\) in terms of two-component spinors \(\psi\), can be written as

\[
E\psi = \left[ E - \sigma_{0j}p_j + \beta mc^2 \right] \psi 
\]

Since we are using only two component spinors, the matrices \(\beta\) and \(\beta\gamma_i\) are conveniently defined in terms of the Pauli spin matrices which satisfy the relation \(\sigma_i\sigma_j = \delta_{ij} + \varepsilon_{ijk}\sigma_k\), given by

\[
\beta\gamma_1 = \sigma_1; \quad \beta\gamma_2 = \sigma_2; \quad \beta = \sigma_3.
\]

In \((2 + 1)\)-dimensions, the momentum operator \(p_i\) is two component differential operator, \(p = -i\hbar(\partial_x, \partial_y)\), for free particle. In the presence of the magnetic field it is replaced by \(p \rightarrow p - eA\), where \(A\) is the vector potential, and the 2D Dirac oscillator can be constructed by changing the momentum \(p \rightarrow p - i\hbar\sigma_3\hat{r}\). We are now seeking for a certain form of the momentum operator that can be interpreted as exactly solvable Schrödinger equation in the non-relativistic limit. For this purpose we introduce the following momentum operator

\[
p \rightarrow p - eA + i\sigma_3 v(r)\hat{r}.
\]

Then Dirac equation (1) with the momentum operator (3) takes the form

\[
\left[ E - \sigma_0 mc^2 \right] \psi = c\sigma_+ \left[ p_x + ip_y - e(A_x + iA_y) - i(v_x + iv_y) \right] \psi + c\sigma_- \left[ p_x - ip_y - e(A_x - iA_y) + i(v_x - iv_y) \right] \psi
\]

where \(p_i, A_i,\) and \(v_i\) are the \(i^{th}\) components of the momentum, vector potential and the potential in Cartesian coordinate system, respectively. In polar coordinate, \(x = r\cos\phi, \quad y = r\sin\phi\), with the choices of the vector potential \(A_x = -A(r)\sin\phi, \quad A_y = A(r)\cos\phi\) and the potential \(v_x = v(r)\cos\phi, \quad v_y = v(r)\sin\phi\) the 2D Dirac equation (1) takes the form

\[
E_\pm \psi_\pm = ce^{i\phi} \left[ -ih\frac{\partial}{\partial r} + \hbar \frac{\partial}{r \partial \phi} - i(eA(r) + v(r)) \right] \psi_\pm
\]

where \(E_\pm = E \pm mc^2\) and \(\psi_\pm = \psi_\pm(r, \phi)\) are the upper and lower components of the spinor \(\psi\). The substitution of the wave functions

\[
\psi_\pm(r, \phi) = \frac{e^{-i(eA(r) + v(r))}}{\sqrt{r}} f_\pm(r)
\]

leads to the following set of coupled differential equations

\[
E_- f_+(r) = -ie \left( \frac{h}{r} \frac{\partial}{\partial r} + \frac{\hbar}{r} + (eA(r) + v(r)) \right) f_-(r) \quad \text{(7a)}
\]

\[
E_+ f_-(r) = -ie \left( \frac{h}{r} \frac{\partial}{\partial r} - \frac{\hbar}{r} - (eA(r) + v(r)) \right) f_+(r). \quad \text{(7b)}
\]

Our task is now to transform (7a) and (7b) in the form of the Schrödinger-like equation. Substitution of

\[
W(r) = \frac{\hbar}{r} + \frac{(eA(r) + v(r))}{\hbar}, \quad E_\pm = \pm ich\varepsilon_\pm; \quad \varepsilon^2 = \varepsilon_+\varepsilon_-,
\]

Using the structures worked for the Dirac oscillator we develop a method to construct a class of exactly and QES potential profile. We transform the Dirac equation into the form of the Schrödinger equation. In section 3, we construct a Dirac equation including, harmonic oscillator, Coulomb and Morse potentials. We obtain the corresponding eigenvalues and eigenfunctions. In section 4 a class of QES potentials are constructed and their ground state wave functions are explicitly determined. We conclude our results in section 5.
into (7a) and (7b), leads to the following expressions

\[
\varepsilon f_+(r) = \left( \frac{\partial}{\partial r} + W(r) \right) f_+(r) \tag{9a}
\]

\[
\varepsilon f_-(r) = \left( -\frac{\partial}{\partial r} + W(r) \right) f_-(r). \tag{9b}
\]

Notice that the result (9a) and (9b) which seem to hint at a supersymmetric treatment of the Dirac equation, because the supersymmetric operators can be expressed as \( A^\pm = \left( \mp \frac{\partial}{\partial r} + W(r) \right) \). It is obvious that the functional form of the superpotential \( W(r) \) and \( v(r) \) are the same except that the radial function \( \frac{\partial}{\partial r} \). Thus the superpotential of the non-relativistic quantum mechanics can be recognized as the potential of the relativistic quantum mechanics. It is obvious that the expressions (9a) and (9b) can be written in the form of the Schrödinger-like equation, by eliminating \( f_+ (r) \) and/or \( f_- (r) \) between (9a) and (9b) provides the following expressions

\[
\begin{align*}
-\partial^2 f_-(r) - W^2(r) + W'(r) + \varepsilon^2 &= 0 \tag{10a} \\
-\partial^2 f_+(r) + W^2(r) + W'(r) - \varepsilon^2 &= 0. \tag{10b}
\end{align*}
\]

This is indeed an interesting result for the potentials \( V^\pm = W^2(r) \pm W'(r) \) leading to a common spectrum, thus forming an isospectral system.

In the following we analyze the solutions and the energy spectrum of the \((2+1)\)-dimensional Dirac equation including various potentials.

**EXACTLY SOLVABLE POTENTIALS**

Appropriate choices of the superpotential \( W(r) \) permits the construction of the equations having exactly solvable potentials. In this section we illustrate that for a large class of potentials, the Dirac equation in the form of Schrödinger type equations of (10a) and (10b) possess exact solutions.

**Harmonic oscillator**

Let us start with the well known problem, namely Dirac oscillator. The Dirac oscillator can be constructed with the choices of the superpotential \( W(r) = \frac{m}{\hbar} \omega_T r - \frac{\ell + 1}{r} \). Thus, the Schrödinger type equations (10a) and (10b) takes the form:

\[
\begin{align*}
-\partial^2 f_-(r) - W^2(r) + W'(r) + \varepsilon^2 &= 0 \tag{11a} \\
-\partial^2 f_+(r) + W^2(r) + W'(r) - \varepsilon^2 &= 0. \tag{11b}
\end{align*}
\]

In this case the vector potential \( A(r) \) and scalar potential \( v(r) \) are given by

\[
A(r) = \frac{1}{2} Br, \quad v(r) = \frac{m}{\hbar} \omega_T r - \frac{2\ell + 1}{r}, \tag{12}
\]

and the frequency \( \omega_T \) can be expressed in terms of the Larmor frequency as follows

\[
\omega_T = \omega + \omega_L = \omega + \frac{eB}{2m} \tag{13}
\]

In order to solve (11a), we change the variable \( z = \frac{m}{\hbar} \omega_T r^2 \), and introduce the wave function

\[
f_-(z) = C z^{\frac{\ell + 1}{2}} e^{-\frac{z}{2}} g_-(z) \tag{14}
\]
where $C$ is the normalization constant, then (11a) takes the form:

$$\left[ z \frac{\partial^2}{\partial z^2} + \left( \ell + \frac{3}{2} - z \right) + n \right] g_-(z) = 0. \quad (15)$$

The natural number $n$ and energy satisfy the relation

$$\varepsilon^2 = \frac{E^2 - m^2 c^4}{\hbar^2 c^2} = 4n \frac{m}{R} \omega_T. \quad (16)$$

Note that the non-relativistic limit of the energy is obtained by setting $E = E_{nr} + mc^2$ and considering $E_{nr} \ll mc^2$, we obtain the non relativistic energy

$$E_{nr} = 4n \hbar \omega_T.$$  

We now investigate the dependence of the energy on the spin. In order to analyze spin effects, it is worth to obtain non-relativistic form of the (11a):

$$(- \frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \frac{\ell(\ell + 1)\hbar^2}{2mr^2} + \frac{1}{2} m \omega_T^2 r^2 + \hbar \omega_T \left( \ell + \frac{3}{2} \right) - E_{nr}) f_-(r) = 0. \quad (17)$$

The corresponding Hamiltonian is the Hamiltonian of a harmonic oscillator with an additional spin dependent term, $\hbar \omega_T \left( \ell + \frac{3}{2} \right)$.

To complete our analysis we turn our attention to the normalization of the wave function. It is easy to see that the solution of (15) is the associated Laguerre polynomials, $L_{\ell + \frac{1}{2}}^n(z)$, then $f_-(z)$ can be written as

$$f_-(z) = Cz^{\frac{\ell + 1}{2}} e^{-\frac{z}{2}} L_{\ell + \frac{1}{2}}^n(z). \quad (18)$$

The upper component, $f_+(z)$, of spinor $\psi(r)$, can be obtained from the relation (9a) and it is given by

$$f_+(z) = -2C \sqrt{\frac{\omega_T}{\varepsilon_+}} z^{\frac{\ell + 1}{2}} e^{-\frac{z}{2}} L_{\ell + \frac{1}{2}}^n(z).$$

Normalization condition in polar coordinate is given by

$$\langle \psi | \psi \rangle = \int_0^{\infty} \left( |f_+(r)|^2 + |f_-(r)|^2 \right) dr = 1. \quad (19a)$$

Associated Laguerre polynomials satisfy the orthogonality condition:

$$\int_0^{\infty} z^\ell e^{-z} L_n^\ell(z) L_k^\ell(z) dr = \frac{\Gamma(n + \ell + 1)}{n!} \delta_{nk}. \quad (19b)$$

Thus, we finally obtain an expression for the normalization constant $C$:

$$C = \left[ \frac{\sqrt{\omega_T \varepsilon_+}}{n \omega_T + \varepsilon_+} \frac{\Gamma(n + 1)}{\Gamma(n + \ell + \frac{3}{2})} \right]^{\frac{1}{2}}. \quad (20)$$

We have solved Dirac oscillator in two dimensional space which leads to a series of interesting results. The Dirac oscillator has various physical applications particularly in semiconductor physics.

**Coulomb Potential**

Another well known examples of the exactly solvable Dirac equation is that relativistic Hydrogen atom. The problem can be solved exactly when the magnetic field is zero, $A(r) = 0$ and in the presence of the Coulomb interaction.
\( \nu(r) = \frac{m e^2}{4 \pi \varepsilon_0 h^2} - \frac{\hbar (2 \ell + 1)}{r} \). In this case the superpotential is \( W(r) = \frac{m e^2}{4 \pi \varepsilon_0 h^2} - \frac{\epsilon \ell}{r} \), and then the Schrödinger-like equations (10a) and (10b) takes the form

\[
\left( -\frac{\partial^2}{\partial r^2} + \frac{\ell (\ell + 1)}{r^2} - \frac{m e^2}{2 \pi \varepsilon_0 h^2} \right) f_-(r) = 0 \tag{21a}
\]

\[
\left( -\frac{\partial^2}{\partial r^2} + \frac{(\ell + 1)(\ell + 2)}{r^2} - \frac{\epsilon^2}{r} + \frac{m e^2}{4 \pi \varepsilon_0 h^2} \right) f_+(r) = 0 \tag{21b}
\]

Following the similar developments used in the construction of the harmonic oscillator problem, the equation (21a) can be transformed in the form:

\[
\left[ z \frac{d^2}{dz^2} + (2 \ell + 2 - z) + n \right] g_-(z) = 0 \tag{22}
\]

by changing the variable \( r = 2 \pi \varepsilon_0 h^2 (n + \ell + 1) z / m e^2 \) and the wave function

\[
f_-(z) = C z^{\ell+1} e^{-\frac{z}{2}} g_-(z). \tag{23}\]

Natural number \( n \) and energy of the Hamiltonian satisfy the relation

\[
\epsilon^2 = \frac{E^2 - m^2 e^4}{h^2 c^2} = \left( \frac{m e^2}{4 \pi \varepsilon_0 h^2 (\ell + 1)} \right)^2 - \left( \frac{m e^2}{4 \pi \varepsilon_0 h^2 (n + \ell + 1)} \right)^2. \tag{24}\]

Solution of the (22) leads to the following expression for the wave function

\[
f_-(z) = C z^{\ell+1} e^{-\frac{z}{2}} L^{2\ell+1}_n(z) \tag{25a}\]

and from the relation (10a) we obtain

\[
f_+(z) = - C \frac{\ell+1}{\ell} z^{\ell+1} e^{-\frac{z}{2}} \left[ 2 (\ell + 1) L^{2\ell+2}_{n-1}(z) + (\ell + 1) - \frac{m e^2}{2 \pi \varepsilon_0 h^2} (2\ell + 1) L^{2\ell+1}_n(z) \right]. \tag{26}\]

Using the identity

\[
\int_0^{\infty} z^{\ell+1} e^{-z} L_n^\ell(z) L_n^\ell(z) dr = (2n + \ell + 1) \ell + 2 \Gamma(n+\ell+1) \frac{\Gamma(n+2\ell+2)}{n!}, \tag{27}\]

after some straightforward calculation we obtain the normalization constant

\[
C = \left[ \frac{2 \pi \varepsilon_0 h^2}{m e^2} \left( \frac{4(\ell + 1)^2}{\epsilon^2 \Gamma(n)} + (2n + 2\ell + 2)^{2\ell+3} K \right) \Gamma(n+2\ell+2) \right]^{-\frac{1}{2}} \tag{28}\]

where \( K \) is given by

\[
K = \left( 1 + \frac{1}{\ell^2 n!} \left( \ell + 1 - \frac{m e^2}{2 \pi \varepsilon_0 h^2 (2\ell + 1)} \right)^2 \right)^{\frac{1}{2}} \tag{29}\]

Due to the recent interest in the 2D field theory in the condensed matter physics, the 2D Coulomb potential is physically relevant and the results obtained in 2D exhibit some new features [6, 35].

**Morse Potential**

The Morse oscillator is exactly solvable quantum mechanical problem and it is used to model the interaction of the atoms in the diatomic molecules. In order to obtain its relativistic form we apply the standard procedure. The
choices of the parameters \( v(r) = -\frac{\hbar}{r} - hae^{-\alpha r} + \hbar b, A(r) = 0 \) and \( W(r) = \hbar - ae^{-\alpha r} \), provides the following potential

\[
\left( \frac{-\partial^2}{\partial r^2} + a^2 e^{-2\alpha r} - a(\alpha + 2b) e^{-\alpha r} + b^2 - \varepsilon^2 \right) f_-(r) = 0 \quad (30a)
\]

\[
\left( -\frac{-\partial^2}{\partial r^2} + a^2 e^{-2\alpha r} + a(\alpha - 2b) e^{-\alpha r} + b^2 - \varepsilon^2 \right) f_+(r) = 0. \quad (30b)
\]

In order to solve (30a) we change the variable \( e^{-\alpha r} = \frac{\alpha}{2\beta} z \) and then introduce the wave function

\[
f_-(r) = z^{\frac{\beta}{\alpha} - n} e^{-\frac{z^2}{2}} g_-(z) \quad (31)
\]

then we obtain

\[
\left[ z \frac{\partial^2}{\partial z^2} + \left( \frac{2b}{\alpha} - 2n + 1 - z \right) + n \right] g_-(z) = 0. \quad (32)
\]

Energy expression for the Morse oscillator is given by

\[
\varepsilon^2 = \alpha n(\alpha n - 2b).
\]

The wave functions can be obtained from the equations (32) and (9a) and are given by

\[
f_-(r) = C z^{\frac{\beta}{\alpha} - n} e^{-\frac{z^2}{2}} L_n^{\frac{2\alpha}{\alpha}}(z) \quad (33a)
\]

\[
f_+(z) = \frac{C \alpha}{\varepsilon} z^{\frac{\beta}{\alpha} - n} e^{-\frac{z^2}{2}} \left[ nL_n^{\frac{2\alpha}{\alpha} - 2n}(z) + zL_{n-1}^{\frac{2\alpha}{\alpha} - 2n+1}(z) \right] \quad (33b)
\]

Normalization condition yields the following expression for the normalization constant

\[
C = \left[ \frac{\varepsilon^2 \Gamma(n) \Gamma(n - 1)}{\Gamma \left( \frac{2\alpha}{\alpha} + 1 - n \right) \left( \varepsilon^2 \Gamma(n) + 2\alpha^2 \Gamma(n + 1) \right)} \right]^{-\frac{1}{2}} \quad (34)
\]

The potentials we have derived here is significant from both physical and mathematical points of view. For instance the relativistic quark model requires the solution of the dirac equation containing single quark potential. The potential behaving like \( r \) at a large distances and \( 1/r \) at a short distances has been recently treated by Muci [30]. We mention that the Morse oscillator potential can also be used as a large distance potential in some various physical systems. Consequently we have constructed relativistic version of the three well known potentials whose eigenfunctions are associated to the confluent hypergeometric functions. In the following section we construct QES potentials.

**QES POTENTIALS**

The formulation given in section 2 is also useful to construct the relativistic version of the QES potentials. In this section we construct three of them. At this stage we mention that the underlying idea behind the quasi-exact solvability is the existence of a hidden algebraic structure. Our task is now to demonstrate by appropriate choices of relativistic potential we can construct QES Dirac equations. Our examples includes anharmonic oscillator potential, radial sextic oscillator potential and perturbed Coulomb potential.

**Anharmonic Oscillator potential**

The anharmonic oscillator potential have been widely used in many physical and chemical applications. In order to construct anharmonic oscillator potential we introduce

\[
v(r) = -\frac{\hbar}{r} + h\omega r + hbr^2 + ha; \quad A(r) = \hbar Br; \quad W(r) = a + \omega Tr + cr^2. \quad (35)
\]
With these choices we obtain the following Schrödinger-like equations

\[
\begin{align*}
-\frac{\partial^2}{\partial r^2} + V_+ + \omega_T + a^2 + \epsilon^2 \bigg) & \bigg) f_+(r) = 0 \quad (36a) \\
-\frac{\partial^2}{\partial r^2} + V_- - \omega_T + a^2 + \epsilon^2 \bigg) & \bigg) f_-(r) = 0 \quad (36b)
\end{align*}
\]

where \( V_\pm \) are given by

\[
V_\pm = 2(\omega_T \pm b)r + (2ab + \omega_T^2)r^2 + 2b\omega_T r^3 + b^2 r^4
\]

and its groundstate wave function is given by

\[
f_-^{(0)}(r) = \exp \left( -\frac{1}{3} b r^3 - \frac{1}{2} \omega_T r^2 - a r \right)
\]

A part of the spectrum of the anharmonic oscillator potential can be obtained in the framework of the QES problem or its approximate solution can be obtained by using perturbation theory.

**Sextic Oscillator potential**

Another well known QES potential is the sextic oscillator potential. Its relativistic form can be obtained by the following choices of functions:

\[
v(r) = -\frac{2\hbar \ell}{r} + \hbar \omega_T + \hbar r^3; A(r) = \hbar Br; W(r) = -\frac{\ell}{r} + \omega_T r + br^3.
\]

In this case we obtain the following equations:

\[
\begin{align*}
-\frac{\partial^2}{\partial r^2} + \frac{\ell(\ell + 1)}{r^2} + V_+ + \epsilon^2 \bigg) & \bigg) f_+(r) = 0 \quad (40a) \\
-\frac{\partial^2}{\partial r^2} + \frac{\ell(\ell - 1)}{r^2} + V_- + \epsilon^2 \bigg) & \bigg) f_-(r) = 0 \quad (40b)
\end{align*}
\]

where

\[
V_\pm = (\omega_T^2 \pm b(2\ell \mp 3))r^2 + 2b\omega_T r^4 + b^2 r^6 - \omega_T (2\ell \mp 1)
\]

Its groundstate wave function is given by

\[
f_-^{(0)}(r) = r^\ell \exp \left( -\frac{1}{2} \omega_T r^2 - \frac{1}{4} br^4 \right)
\]

**Deformed Coulomb potential**

Our last example is the deformed Coulomb potential which can be obtained by introducing

\[
v(r) = \frac{e^2 \hbar}{2(\ell + 1)} - \frac{\hbar(2\ell + 1)}{r}; A(r) = \hbar Br; W(r) = \frac{e^2}{2(\ell + 1)} - \frac{\ell + 1}{r}.
\]

In this case the equations take the forms

\[
\begin{align*}
-\frac{\partial^2}{\partial r^2} - \frac{(\ell + 1)(\ell + 2)}{r^2} + V_+ - \omega_T (2\ell + 1) + \frac{e^4}{4(\ell + 1)^2} + \epsilon^2 \bigg) & \bigg) f_+(r) = 0 \quad (44a) \\
-\frac{\partial^2}{\partial r^2} - \frac{\ell(\ell + 1)}{r^2} + V_- - \omega_T (2\ell + 3) + \frac{e^4}{4(\ell + 1)^2} + \epsilon^2 \bigg) & \bigg) f_-(r) = 0 \quad (44b)
\end{align*}
\]
\[ V_\pm = -\frac{e^2}{r} + \omega_T r^2 + \frac{e^2 \omega_T}{\ell + 1} r + \frac{e^4}{4(\ell + 1)^2} \] 

(45)

Its groundstate eigenfunction are given by

\[ f_{\pm}^{(0)}(r) = r^{\ell + 1} \exp\left(\frac{-\omega_T r^2}{2} - \frac{e^2 r}{2(\ell + 1)}\right). \] 

(46)

The non-relativistic two dimensional Hamiltonian described Coulomb interaction between charged particles, i.e. the interaction between conduction electron and donor impurity center when a constant magnetic field is applied perpendicular to the plane of motion has been discussed in the literature\textsuperscript{28, 31, 32, 33}. We have obtained relativistic interaction and it can be solved quasi-exactly. Recently its spectrum and wavefunction has been computed numerically\textsuperscript{1}.

CONCLUSION

We have obtained analytical solutions of the \((2 + 1)\)-dimensional Dirac equation for a set of potentials in two dimensions with the hope that they could be useful in the low dimensional field theory and condensed matter physics. The potentials for Dirac equation have been obtained by extending the notion of the Dirac oscillator. In a similar manner one can construct the exactly and QES Dirac equation including hyperbolic and trigonometric potentials. It has been shown that the \((2 + 1)\)-dimensional Dirac equation can be transformed in the form of a Schrödinger-like equation and for all exactly and QES Schrödinger equations one can find potentials of the Dirac equation which are also exactly solvable or QES.

Therefore, besides its importance as a new treatment of the construction of the various potentials for Dirac equation, the potentials obtained here might be relevant to model some physical problems. Before ending this work a remark is in order. It is expected that the models presented here to construct Dirac equation including various potentials may provide a good starting point for the study of more realistic models for the low dimensional field theory and condensed matter physics.

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