Abstract. We develop a general framework based on splines to understand the interpolation properties of overparameterized neural networks. We prove that minimum “norm” two-layer neural networks (with appropriately chosen activation functions) that interpolate scattered data are minimal knot splines. Our results follow from understanding key relationships between notions of neural network “norms”, linear operators, and continuous-domain linear inverse problems.

Key words. splines, neural networks, inverse problems

AMS subject classifications. 41A25, 46E27, 47A52, 68T05, 82C32, 94A12

1. Introduction. Contradicting classical statistical wisdom, recent trends in data science and machine learning have shown that overparameterized models that interpolate the training data perform surprisingly well on new, unseen data. This phenomenon is very often seen in the generalization performance of overparameterized neural network models. These models are typically trained to zero training error, i.e., they interpolate the training data, yet they predict very well on new, test data. Prior work has tried to understand this phenomenon from a statistical perspective by studying the statistical properties of such interpolating models [16, 4, 5, 3]. To control the complexity of overparameterized neural networks, solutions with minimum $\ell^2$-norm of the network weights are often encouraged in optimization using regularization (often referred to as “weight decay” in Stochastic Gradient Descent (SGD) steps). Motivated by this practice, the functional mappings generated by two-layer, infinite-width neural networks with Rectified Linear Unit (ReLU) activation functions were studied in [20].

Such networks that interpolate data subject to minimizing the $\ell^2$-norm of the network weights were shown to correspond to linear spline interpolation of the data. However, in this work we build off these results and develop a general framework based on splines to understand what function spaces can be learned by neural networks. We study more general two-layer neural networks that interpolate data while minimizing certain “norms”. We relate this optimization to an equivalent optimization over functions that live in the native space of a particular linear operator. Both the neural network activation function and the “norm” are tailored to the linear operator.

Our key contribution is understanding key relationships among neural networks “norms”, linear operators, and continuous-domain linear inverse problems. With these connections we turn to the recent work of [25] who introduce a framework for L-splines in which particular type of spline is a solution to a continuous-domain linear inverse problem involving a linear operator $L$. We prove that minimizing the “norm” of a specific neural network architecture (where the architecture and “norm” are determined by the choice of the linear operator $L$)
subject to interpolating scattered data exactly solves this inverse problem. By noticing and understanding these connections we bridge the gap between spline theory and understanding the interpolation properties of overparameterized neural networks.

In particular, we prove that overparameterized two-layer neural networks mapping $\mathbb{R} \rightarrow \mathbb{R}$ can learn functions that interpolate scattered data with a large class of L-splines, including polynomial splines, fractional splines, and many exponential splines. This result is not only interesting to the data science and machine learning communities, but also to the spline community as a new, and perhaps unconventional, way to compute splines. Additionally, since our result follows from neural networks indirectly solving continuous-domain linear inverse problems, this is also interesting to the inverse problem community as a new way to solve such inverse problems as opposed to more standard multiresolution or grid-based approaches [13, 9]. Our work may also be relevant to recent work on developing an infinite-dimensional theory of compressed sensing [1, 2] as well as other inverse problems set in the continuous domain [6].

This paper is organized as follows. In section 2 we provide relevant background from spline theory and introduce the framework of L-splines from [25]. In section 3 we introduce the neural networks we’ll be interested in as well as the notion of neural network “norms”. In section 4 we present our main theoretical results establishing connections among neural networks, continuous-domain linear inverse problems, and splines. In section 5 we get into detail of which L-splines a neural network can learn. In section 6, we discuss various neural network “norms” that result in minimum “norm” neural networks learning splines. In section 7 we verify that the theory developed in section 4 holds with some empirical validation.

2. Splines and Continuous-Domain Linear Inverse Problems. In this section we state results from spline theory that make powerful associations between splines and operators. Specifically, we review L-splines and their connections to continuous-domain linear inverse problems with generalized total-variation regularization [25]. The main result of interest is a representer theorem that describes the solution sets of the aforementioned inverse problems.

2.1. Preliminaries. To state the main result we require some technical notation (see Section 3 of [25] for a more thorough discussion). Let $S'(\mathbb{R})$ denote the space of tempered distributions, which is the continuous dual of the Schwartz space $S(\mathbb{R})$ of smooth and rapidly decaying test functions on $\mathbb{R}$ [11, 19]. In this work, we’ll be interested in the space $\mathcal{M}(\mathbb{R})$ of finite Radon measures. For each $\mu \in \mathcal{M}(\mathbb{R})$ we can associate a tempered distribution $w^1$, so it becomes convenient to think that $\mathcal{M}(\mathbb{R}) \subset S'(\mathbb{R})$.

The Riesz–Markov–Kakutani representation theorem says that $\mathcal{M}(\mathbb{R})$ is the continuous dual of $C_0(\mathbb{R})$, the space of continuous functions vanishing at infinity. Since $C_0(\mathbb{R})$ is a Banach space when equipped with the uniform norm, we have the dual norm

$$
\|w\|_{\mathcal{M}(\mathbb{R})} := \sup_{\|\varphi\|_{L^\infty(\mathbb{R})} = 1} \left\langle w, \varphi \right\rangle = \sup_{\|\varphi\|_{L^\infty(\mathbb{R})} = 1} \int_{\mathbb{R}} w(x) \varphi(x) \, dx
$$

1In the sense that $d\mu(x) = w(x) \, dx$. Note that since we’re working with tempered distributions this is a slight abuse of notation when the measure associated with $w$ is not absolutely continuous with respect to the Lebesgue measure.
and so
\[ M(\mathbb{R}) := \{ w \in S'(\mathbb{R}) : \| w \|_{M(\mathbb{R})} < \infty \} \]
A key observation is that \( M(\mathbb{R}) \) is a space larger than \( L^1(\mathbb{R}) \) that includes absolutely integrable tempered distributions. Indeed, for any \( f \in L^1(\mathbb{R}) \) we have
\[ \| f \|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| \, dx = \| f \|_{M(\mathbb{R})} \]
and we remark that \( L^1(\mathbb{R}) \) is dense in \( M(\mathbb{R}) \). We’re interested in the space \( M(\mathbb{R}) \) since the Dirac deltas \( \delta(\cdot - x_0) \not\in L^1(\mathbb{R}) \) for \( x_0 \in \mathbb{R} \), but \( \delta(\cdot - x_0) \in M(\mathbb{R}) \) with \( \| \delta(\cdot - x_0) \|_{M(\mathbb{R})} = 1. \)

**Definition 2.1 ([25, Definition 1]).** A linear operator \( L : S'(\mathbb{R}) \to S'(\mathbb{R}) \) is called spline-admissible if it satisfies the following properties
- it is translation-invariant, i.e., \( L \tau_y = \tau_y L \), where \( \tau_y f(x) = f(x - y) \) is the translation operator;
- there exists a function \( \sigma_L : \mathbb{R} \to \mathbb{R} \) such that \( L \sigma_L = \delta \), i.e., \( \sigma_L \) is a Green’s function of \( L \);
- the kernel \( \ker L = \{ p : L p = 0 \} \) has finite-dimension \( N_0 \geq 0 \).

**Definition 2.2 ([25, Definition 2]).** Let \( L \) be spline-admissible in the sense of Definition 2.1. Then, a function \( f : \mathbb{R} \to \mathbb{R} \) is called a non-uniform \( L \)-spline with spline knots \( (x_1, \ldots, x_K) \) and weights \( (a_1, \ldots, a_K) \) if
\[ L f = \sum_{k=1}^{K} a_k \delta(\cdot - x_k) \]
\( L f \) is called the innovation of the spline \( f \).

**Remark 2.3.** Associating a spline with an operator \( L \) captures many common splines including
- the well studied polynomial splines of order \( N_0 \) by choosing \( L = D^{N_0} \) where \( D = \frac{d}{dx} \) is the derivative operator\(^2\) [21].
- the fractional splines of order \( \gamma \) by choosing \( L = D^\gamma \), where \( D^\gamma \) is the fractional derivative and \( \gamma \in \mathbb{R}_{\geq 0} \) [23].
- the exponential splines by choosing \( L = D^{N_0} + a_{N_0-1} D^{m-1} + \cdots + a_1 D + a_0 I \), where \( I \) is the identity operator [8, 24].

**2.2. Continuous-Domain Representer Theorem.** With the preliminaries out of the way we can now state the relevant representer theorem result of [25].

**Theorem 2.4 ([25, Based on Theorems 1 and 2]).** Let \( L \) be a spline-admissible operator in the sense of Definition 2.1 and consider the problem of interpolating the scattered data \( \{(x_m, y_m)\}_{m=1}^{M} \). Then, the extremal points of the general constrained minimization problem
\[
\min_{f \in M_L(\mathbb{R})} \| L f \|_{M(\mathbb{R})} \quad \text{s.t.} \quad f(x_m) = y_m, \quad m = 1, \ldots, M
\]
\(^2\)Here and in the rest of this paper, derivatives are understood in the distributional sense.
are necessarily non-uniform \( L \)-splines of the form

\[
(2.3) \quad s(x) = \sum_{k=1}^{K} a_k \sigma_L(x - x_k) + \sum_{n=1}^{N_0} c_n p_n(x)
\]

with the \( K \leq M - N_0 \) knots and where \( \sigma_L \) is a Green’s function of \( L \) and \( \text{span}\{p_n\}_{n=1}^{N_0} = \text{ker} \ L \).
Here, \( M_L(\mathbb{R}) \) is the native space of \( L \) defined by

\[
M_L(\mathbb{R}) := \{ f \in S'(\mathbb{R}) : L f \in M(\mathbb{R}) \}
\]

The full solution set of (2.2) is the convex hull of the extremal points.

Remark 2.5. This theorem provides a powerful result since this problem is defined over a continuum and hence has an (uncountably) infinite number of degrees of freedom, yet the solutions are intrinsically sparse since they can be represented with only \( K \leq M - N_0 \) coefficients.

3. Neural Network Architectures and “Norms”. We’ll consider two-layer \( \mathbb{R} \to \mathbb{R} \) networks with activation function \( \sigma \)

\[
(3.1) \quad \varphi_{w,v,b,c}(x) = \left[ \sum_{n=1}^{N} v_n \sigma(w_n x + b_n) \right] + c(x)
\]

where \( c \in \mathcal{C} \) is a generalized bias term\(^3\) and

\[
\begin{align*}
\mathbf{w} &:= \begin{bmatrix} w_1 \\
\vdots \\
w_N \end{bmatrix}, & \mathbf{v} &:= \begin{bmatrix} v_1 \\
\vdots \\
v_N \end{bmatrix}, & \mathbf{b} &:= \begin{bmatrix} b_1 \\
\vdots \\
b_N \end{bmatrix}
\end{align*}
\]

where \( w_n, v_n, b_n \in \mathbb{R} \) for \( n = 1, \ldots N \).

Put \( \theta := (w, v, b, c) \). We will often write \( \varphi_{\theta} \) or \( \varphi \) for \( \varphi_{w,v,b,c} \). Let \( \Theta \) be the parameter space, i.e.,

\[
\Theta := \left\{ \theta = (w, v, b, c) \mid \begin{array}{l} w \in \mathbb{R}^N, \\
v \in \mathbb{R}^N, \\
b \in \mathbb{R}^N, \\
c \in \mathcal{C} \end{array} \right\}
\]

Remark 3.1. Infinite-width networks are the same as the above, but we consider the continuum limit of the number of neurons.

\(^3\)We will see in section 4 that depending on how we choose the activation function \( \sigma \), the generalized bias term \( c(x) \) is a constant or “simple” function.
3.1. “Norm” of a Neural Network. Let $\mathcal{C}(\theta)$ be a non-negative function that measures the “size” of $\theta$. We refer to this as the “norm” of the neural network parameterized by $\theta \in \Theta$. Here and in the rest of the paper we write “norm” in quotes since $\mathcal{C}(\cdot)$ may not be a true norm. However, the “norms” are non-negative and increasing functions of the weight magnitudes.

Let $f$ be any function that can be represented by a neural network $\varphi_\theta$ for some parameter $\theta \in \Theta$. The minimum “norm” neural network is defined by the following constrained minimization

$$\min_{\varphi_\theta : \theta \in \Theta} \mathcal{C}(\theta) \quad \text{s.t.} \quad \varphi_\theta = f$$

where $\Theta_c := \{ \theta \in \Theta : \mathcal{C}(\theta) < \infty \}$ is the parameter space of bounded “norm” neural networks. Note that by considering overparameterization in the limit, universal approximation essentially says we can represent any continuous function exactly with only mild conditions on the activation function [7, 14, 12]. Thus, we will first consider overparameterization in the limit and consider infinite-width networks.

Remark 3.2. We will see later (in Theorem 4.4) that since our goal is to establish an equivalence between minimum “norm” neural networks and splines for the scattered data interpolation problem, we only require a sufficiently wide network.

Remark 3.3. As will become clear later, neural networks are only capable of learning a subset of spline-admissible operators that satisfy a few key properties. We refer to this subset of spline-admissible operators as neural-network-admissible operators and define them as follows.

Definition 3.4. A linear operator $L : S'(\mathbb{R}) \to S'(\mathbb{R})$ is called neural-network-admissible if it satisfies the following properties

- it is spline-admissible in the sense of Definition 2.1;
- it commutes with the reflection operator up to a complex constant of magnitude 1, i.e., if $R$ is the reflection operator$^4$, $L R = \alpha_L R L$ for some $\alpha_L \in \mathbb{C}$ such that $|\alpha_L| = 1$;
- for a Green’s function $\sigma_L : \mathbb{R} \to \mathbb{R}$ of $L$, we have for any $y \in \mathbb{R}$,

$$\sigma_L(\cdot - y) - \alpha_L \sigma_L(y - \cdot) \in \ker L.$$

We will now state some relevant facts about these neural-network-admissible operators, which follow by direct calculations.

Fact 3.5. It is easy to verify that the third property in Definition 3.4 is implied by the second. Moreover, if the second bullet in Definition 3.4 did not enforce that $|\alpha_L| = 1$, then the third bullet in Definition 3.4 would never hold.

Fact 3.6. If a neural-network-admissible $L$ admits a causal$^5$ Green’s function $\sigma_L^{(\text{causal})}$, then it also admits a non-causal Green’s function

$$\sigma_L^{(\text{non-causal})} := \frac{\sigma_L^{(\text{causal})} + \alpha_L R \sigma_L^{(\text{causal})}}{2}.$$ 

$^4$Recall that a function $s$ is called causal if $s(t) = 0$ for $t < 0$. 

$^5$Recall that a function $s$ is called causal if $s(t) = 0$ for $t < 0$. 

$Rf(x) = f(-x)$. 

$R^4f(x) = f(-x)$. 

$\sigma_L^{(\text{causal})}$.
with the property
\[ \sigma_L^{(non-causal)}(\cdot - y) - \alpha_L \sigma_L^{(non-causal)}(y - \cdot) = 0 \]

**Remark 3.7.** The reason we need to consider a subset of spline-admissible operators is because the “shape” of a neural network is not the same as the “shape” of a spline. In particular, in a neural network representation, reflections of the activation functions occur (since the weights can be negative) which forces us to require the second bullet in Definition 3.4.

With the understanding of what kind of neural networks we are considering and what a “norm” of a neural network should look like, to establish a connection between minimum “norm” neural networks and splines there are three questions that need to be answered:

**Question 3.8.** What is the activation function \( \sigma \)?

**Question 3.9.** What is the space of generalized bias functionals \( C \)?

**Question 3.10.** What is the neural network “norm” \( C(\theta) \)?

We can answer Question 3.8 and Question 3.9 immediately since they follow directly from comparing (2.3) and (3.1). Notice that L-splines (resp. two-layer \( \mathbb{R} \to \mathbb{R} \) neural networks) have the “shape” of linear combinations of Green’s functions (resp. activation functions) plus a term in ker \( L \) (resp. generalized bias term). Hence, given a neural-network-admissible operator \( L \), the answer to

- **Question 3.8** is to choose \( \sigma \) to be a Green’s function of \( L \).
- **Question 3.9** is to choose

\[
(3.2) \quad c(x) = \sum_{n=1}^{N_0} \kappa_n p_n(x)
\]

with \( \kappa_1, \ldots, \kappa_n \in \mathbb{R} \). As in Theorem 2.4, \( \text{span}\{p_n\}_{n=1}^{N_0} = \ker L \).

**Remark 3.11.** If \( L \) admits a non-causal Green’s function as in Fact 3.6, then we can simply take \( c \equiv 0 \), as shown later in the proof of Lemma 4.1.

**Remark 3.12.** If the constant function is in ker \( L \) and \( L \) admits a non-causal Green’s function as in Fact 3.6, we can simply take \( c \equiv \kappa \in \mathbb{R} \) meaning we can use the “standard” feedforward neural network architecture. Many commonly used splines, e.g., polynomial splines, admit such a Green’s function.

Before answering Question 3.10, we need to develop some connections between neural networks and splines. We will answer Question 3.10 in section 6.

**4. Neural Networks and Splines.** When considering infinite-width \( \mathbb{R} \to \mathbb{R} \) networks it becomes convenient to work with the integral representation

\[
\psi(x) := \int_{\mathbb{R} \times \mathbb{R}} \sigma(wx + b) \, d\mu(w, b) + c(x)
\]

where \( \mu \) is a signed measure over the weights and biases \((w, b)\) and \( c \in C \) as in (3.1). With this representation, the measure \( \mu \) corresponds exactly to \( \nu \), the last layer weights in a finite-width
network, when \( \mu \) corresponds to a finite-width network (i.e., \( \mu \) is a finite linear combination of of Dirac measures) and we have the equality

\[
\| \mu \|_{TV} = \| v \|_{\ell^1}
\]

where given a measurable space \((X, \Sigma)\) and any signed measure \(\nu\) defined on the \(\sigma\)-algebra \(\Sigma\), the total variation of \(\nu\) is defined by

\[
\| \nu \|_{TV} := \| \nu^+ + \nu^- \|_{X}
\]

where \(\nu^+\) and \(\nu^-\) are the Jordan decomposition of \(\nu\) [11].

When working with \(\mathbb{R} \to \mathbb{R}\) infinite-width networks we have

\[
\psi = \psi_{(\mu, c)} =: \psi^c
\]

is our new parameter space.

We will first prove the following lemma about a subset of infinite-width neural networks that are easier to work with analytically. Following [20], we consider the parameter space \(\tilde{\Xi} \subset \Xi:\)

\[
\tilde{\Xi} := \{ (\mu, c) \in \Xi \mid |\mu|_{TV} < \infty, \quad c \text{ chosen according to (3.2)} \}
\]

of networks where the first layer weights are constrained in absolute value to be 1.

**Lemma 4.1.** Suppose we want to represent a function \(f: \mathbb{R} \to \mathbb{R}\) with an infinite-width network with parameter space \(\tilde{\Xi}\) and activation function \(\sigma\) chosen to be the Green’s function of a neural-network-admissible operator \(L\). Then, the following is true

\[
\min_{\xi = (\mu, c) \in \Xi} \| \mu \|_{TV} \quad \text{s.t.} \quad \psi^c = f \quad = \quad \| L f \|_{M(\mathbb{R})}
\]

**Proof.** Write

\[
\psi^c(x) = \int_{\{-1, +1\} \times \mathbb{R}} \sigma(wx + b) \, d\mu(w, b) + c(x)
\]

\[
\overset{(\star)}{=} \int_{\mathbb{R}} \sum_{w \in \{-1, +1\}} \sigma(wx + b) g(w, b) \, db + c(x)
\]

\[
= \int_{\mathbb{R}} g(+1, b) \sigma(x + b) + g(-1, b) \sigma(b - x) \, db + c(x)
\]

where in (\(\star\)), \(g\) is a distribution such that \(d\mu = g \, d((\delta_{-1} + \delta_{+1}) \times m)\), where \(\delta_x\) is the Dirac measure and \(m\) is the Lebesgue measure.

---

\(6\)When \(X = \mathbb{R}\) and \(\Sigma\) is the Borel sigma algebra on \(\mathbb{R}\), \(\| \cdot \|_{TV}\) is exactly the same as \(\| \cdot \|_{M(\mathbb{R})}\) defined in (2.1). We could generalize (2.1) to general measurable spaces \((X, \Sigma)\) and write \(\| \cdot \|_{M(X)}\) instead of \(\| \cdot \|_{TV}\).
Next, from the constraint $\psi_\xi = f$, 

$$
\mathbf{L} f(x) = \mathbf{L} \psi_\xi(x) = \int_\mathbb{R} g(+1, b) \mathbf{L} \sigma(x + b) + g(-1, b) \mathbf{L} \sigma(b - x) \, db + \mathbf{L} c(x)
$$

$$
\overset{(\ast)}{=} \int_\mathbb{R} g(+1, b) \delta(x + b) + \alpha_L g(-1, b) \delta(b - x) \, db
$$

$$
= g(+1, -x) + \alpha_L g(-1, x)
$$

where $\ast$ holds by the second bullet in Definition 3.4. Put 

$$
\eta(x) := g(+1, -x) + \alpha_L g(-1, x) = \mathbf{L} f(x)
$$

$$
\zeta(x) := g(+1, -x) - \alpha_L g(-1, x)
$$

Hence, 

$$(4.4) \quad g(+1, b) = \frac{1}{2} [\eta(-b) + \zeta(-b)] \quad \text{and} \quad g(-1, b) = \frac{\alpha_L}{2} [\eta(b) - \zeta(b)]$$

From (4.3), 

$$
f(x) = \int_\mathbb{R} \left\{ \frac{1}{2} [\eta(-b) + \zeta(-b)] \right\} \cdot \sigma(x + b) + \left\{ \frac{\alpha_L}{2} [\eta(b) - \zeta(b)] \right\} \cdot \sigma(b - x) \, db + c(x)
$$

$$
= \frac{1}{2} \int_\mathbb{R} [\eta(b) + \zeta(b)] \sigma(x - b) + \alpha_L [\eta(b) - \zeta(b)] \sigma(b - x) \, db + c(x)
$$

$$
= \frac{1}{2} \int_\mathbb{R} \eta(b) [\sigma(x - b) + \alpha_L \sigma(x - b)] + \zeta(b) [\sigma(b - x) - \alpha_L \sigma(b - x)] \, db + c(x)
$$

By the third bullet in Definition 3.4 and Fact 3.6, $\zeta$ either controls a term in $\ker \mathbf{L}$ or a term that is 0. Thus, regardless of what $\zeta$ is, we can always adjust $c(x)$ to establish the constraint $\psi_\xi = f$. Then, since $\eta = \mathbf{L} f$, we see that representing $f$ while minimizing $\|\mu\|_{TV}$ only depends on $\eta$.

Since 

$$
\|\mu\|_{TV} = \int_\mathbb{R} |g(+1, b)| + |g(-1, b)| \, db
$$

we have from (4.4) that the minimization in the lemma statement as 

$$
\min_{\xi=(\mu, c) \in \Xi} \|\mu\|_{TV} \quad \overset{(\ast)}{=} \quad \min_{\xi} \frac{1}{\mathbf{L}} \int_\mathbb{R} |\mathbf{L} f(b) + \zeta(b)| + |\mathbf{L} f(b) - \zeta(b)| \, db
$$

$$
\overset{\text{s.t.} \quad \psi_\xi = f}{=} \int_\mathbb{R} |\mathbf{L} f(b)| \, db = \|\mathbf{L} f\|_{\mathcal{M}(\mathbb{R})}
$$

where $\ast$ holds since $\mu$ completely determines $g$ and vice-versa, $\zeta$ is completely determined by $g$, by the decomposition in (4.4), and by the argument of the previous paragraph, we have that the value of $\zeta$ has no effect on being able to represent $f$ and so the only free term in controlling $g$ is $\zeta$; thus we can simply minimize over the choice of $\zeta$. ■
Remark 4.2. Definition 3.4 provides both necessary and sufficient conditions for the analysis in the proof of Lemma 4.1 to hold.

Corollary 4.3. Consider the problem of interpolating the scattered data \( \{(x_m, y_m)\}_{m=1}^M \). We have the following equivalence

\[
\min_{\xi=(\mu, c) \in \tilde{\Xi}} \|\mu\|_{TV} \quad \text{s.t.} \quad \psi_\xi(x_m) = y_m, \quad m = 1, \ldots, M
\]

\[
\min_{f \in M_L(\mathbb{R})} \|L_f\|_{M(\mathbb{R})} \quad \text{s.t.} \quad f(x_m) = y_m, \quad m = 1, \ldots, M
\]

Proof. From Lemma 4.1, given \( \psi_{(\mu, c)} \), \( (\mu, c) \in \tilde{\Xi} \), we have

\[
\|\mu\|_{TV} = \|L \psi_{(\mu, c)}\|_{M(\mathbb{R})}
\]

By Theorem 2.4, we know that a minimal knot L-spline, say, \( g \), is a solution to the right-hand side of (4.5). It suffices to show that \( g \) is also a solution to the left-hand side of (4.5). Clearly we can construct \( g = \psi_{(\mu, c)} \) with a neural network parameterized by \( \tilde{\Xi} \) (simply take \( \mu \) to be a finite linear combination of Dirac measures at the spline knots), so \( g \) is feasible. We will now proceed by contradiction. Suppose there exists a \( \tilde{g} = \psi_{(\tilde{\mu}, \tilde{c})} \) that interpolates \( \{(x_m, y_m)\}_{m=1}^M \) with \( \|\tilde{\mu}\|_{TV} < \|\mu\|_{TV} \). Hence by (4.6),

\[
\|L \tilde{g}\|_{M(\mathbb{R})} < \|L g\|_{M(\mathbb{R})}
\]

which contradicts the optimality of \( g \) for the right-hand side of (4.5).

Theorem 4.4. Let \( \varphi_\theta : \mathbb{R} \rightarrow \mathbb{R}, \ \theta \in \Theta \), be a feedforward neural network architecture with activation function \( \sigma \) chosen to be a Green’s function of a neural-network-admissible operator \( L \) and last layer bias \( c(x) \) chosen according to (3.2), i.e.,

\[
\varphi_\theta(x) = \left[ \sum_{n=1}^N v_n \sigma(w_n x + b_n) \right] + c(x)
\]

Consider the problem of interpolating the scattered data \( \{(x_m, y_m)\}_{m=1}^M \). We have the following equivalence

\[
\min_{\varphi_{\theta}: \Theta} \|\varphi\|_{L^1} \quad \text{s.t.} \quad \varphi(x_m) = y_m, \quad m = 1, \ldots, M
\]

\[
\min_{f \in M_L(\mathbb{R})} \|L f\|_{M(\mathbb{R})} \quad \text{s.t.} \quad f(x_m) = y_m, \quad m = 1, \ldots, M
\]

so long as the number of neurons is \( N \geq M - N_0 \). The solutions of the above minimizations are minimal knot, non-uniform L-splines.

Proof. By Theorem 2.4, we know the extremal points to the right-hand side optimization has \( K \leq M - N_0 \) knots and by noting that each neuron can create exactly one knot, \( N \geq M - N_0 \) is a necessary and sufficient number of neurons. It’s then a matter of using the equality in (4.1) and invoking Corollary 4.3 and Theorem 2.4 to prove the theorem.

\footnote{An L-spline with \( K \leq M - N_0 \) knots.}
Remark 4.5. For a neural network to learn an $L$-spline, we must solve the left-hand side optimization in (4.7). The constraints $|w_n| = 1$ for $n = 1, \ldots, N$ pose a challenge, but we will see in section 6 that we can rewrite the left-hand side optimization in (4.7) in the form

$$\min_{\varphi_{\theta} : \theta \in \Theta} \mathcal{C}(\theta)$$

s.t. $\varphi_{\theta}(x_m) = y_m$, $m = 1, \ldots, M$

for some neural network “norm” $\mathcal{C}$, and thus answer Question 3.10. Before answering this question, we will first see (in section 5) some examples of neural-network-admissible operators.

Remark 4.6. In all our optimizations, we considered the setting of ideal sampling, i.e., we are optimizing subject to interpolating given points. From [25], all our results hold for generalized sampling where instead of the constraints

$$f(x_m) = \langle \delta(\cdot - x_m), f \rangle = y_m, \quad m = 1, \ldots, M$$

we can replace the sampling functionals $\delta(\cdot - x_m)$, $m = 1, \ldots, M$, with any weak*-continuous linear measurement operator $\nu : f \mapsto (\langle \nu_1, f \rangle, \ldots, \langle \nu_M, f \rangle)$

and instead consider the constraints

$$\langle \nu_m, f \rangle = y_m, \quad m = 1, \ldots, M$$

This allows us to bring our framework into the context of infinite-dimensional compressed sensing and other more general inverse problems [1, 2, 6].

5. Neural-Network-Admissible Operators. The framework we have developed in section 4 relies on our definition of neural-network-admissible operators (Definition 3.4). This definition captures many common splines used in practice. The prominent examples include:

- the polynomial splines of order $N_0$ by choosing $L = D^{N_0}$. Then $\alpha_L = (-1)^{N_0}$. The operator $D^{N_0}$ admits two obvious Green’s functions

$$\sigma^{(\text{causal})}_{D^{N_0}}(x) = \max\{0, x\}^{N_0-1} \frac{(N_0 - 1)!}{N_0!}$$

and

$$\sigma^{(\text{non-causal})}_{D^{N_0}}(x) = \frac{|x|^{N_0-1}}{2(N_0 - 1)!} \cdot \begin{cases} 1, & \text{if } N_0 \text{ is even} \\ \text{sgn}(x), & \text{if } N_0 \text{ is odd} \end{cases}$$

Remark 5.1. When $L = D^2$, $\sigma^{(\text{causal})}_{D^2}$ is precisely the ReLU. Thus our framework captures ReLU neural networks which correspond to linear spline interpolations.

- the fractional splines of order $\gamma$ by choosing $L = D^\gamma$, where $D^\gamma$ is the fractional derivative and $\gamma \in \mathbb{R}_{\geq 0}$. Then $\alpha_L = (-1)^{\gamma}$. The operator $D^\gamma$ admits the obvious causal Green’s function

$$\sigma^{(\text{causal})}_{D^\gamma}(x) = \max\{0, x\}^{\gamma-1} \frac{(\gamma - 1)!}{\Gamma(\gamma)}$$
where \( \Gamma \) is Euler’s Gamma function. Just as before, we can use Fact 3.6 to also find a non-causal Green’s function.

- many of the exponential splines. Specifically,
  - the even exponential splines by choosing
    \[
    L = D^{N_0} + \sum_{n=0}^{N_0-1} a_n D^n
    \]
    with \( N_0 \) being even. Then \( \alpha_L = 1 \).
  - the odd exponential splines by choosing
    \[
    L = D^{N_0} + \sum_{n=0}^{N_0-1} a_n D^n
    \]
    with \( N_0 \) being odd. Then \( \alpha_L = -1 \).

Exponential splines are of interest from a systems theory perspective since they easily model cascades of first-order linear and translation-invariant systems. For a full treatment of exponential splines we refer to \([24, 22]\).

To compute the Green’s functions for these operators it becomes convenient to work with the transfer function of \( L \). We follow a similar computation as in \([24]\). Since \( L \) is neural-network-admissible, it’s linear and translation-invariant, i.e., \( L \) is a convolution operator. Thus there exists an \( h \in S'(\mathbb{R}) \) such that \( L f = h \ast f \). In particular, \( h = L \delta \), the impulse response of \( L \).

For both even and odd exponential splines, the bilateral Laplace transform of \( h \) will be a monic polynomial of degree \( N_0 \). Thus in both cases we can write

\[
H(s) := \mathcal{L}\{h\}(s) = \prod_{n=1}^{N_0} (s - \alpha_n)
\]

where \( \{\alpha_n\}_{n=1}^{N_0} \) are the roots of the polynomial \( H(s) \). This can also be written as

\[
H(s) = \prod_{d=1}^{D} (s - \beta_d)^{n_d}
\]

where \( \{\beta_d\}_{d=1}^{D} \) are the distinct roots of \( H(s) \) and \( n_d \) is the multiplicity of the root \( \beta_d \).

Thus we have \( n_1 + n_2 + \cdots + n_D = N_0 \). Since we want to find \( \sigma \) such that \( h \ast \sigma = \delta \), it follows that

\[
\sigma = \mathcal{L}^{-1}\left\{ \frac{1}{H} \right\} = \mathcal{L}^{-1}\left\{ \prod_{d=1}^{D} \frac{1}{(s - \beta_d)^{n_d}} \right\} = \mathcal{L}^{-1}\left\{ \sum_{d=1}^{D} \sum_{n=1}^{n_d} \frac{c_{d,n}}{(s - \beta_d)^n} \right\}
\]

\[\text{8Here and in the rest of the paper we will use capital letters to denote the bilateral Laplace transform of their lower-case counterparts.}\]
where the last equality follows from a partial fraction decomposition which imposes the coefficients $c_{d,n}$. Finally, by taking the inverse transform we find the causal Green’s function
\[
\sigma^{(\text{causal})}_L(x) = \sum_{d=1}^{D} \sum_{n=1}^{n_d} c_{d,n} \max\{0, x\}^{n-1} \frac{\beta d x}{(n-1)!}.
\]

One can then use Fact 3.6 to find a non-causal Green’s function $\sigma^{(\text{non-causal})}_L$.

\begin{figure}[h]
\centering
\begin{subfigure}{.4\textwidth}
\centering
\includegraphics[width=\textwidth]{causal_polynomial}
\caption{Causal activations for polynomial splines}
\end{subfigure}\hspace{1cm}
\begin{subfigure}{.4\textwidth}
\centering
\includegraphics[width=\textwidth]{non-causal_polynomial}
\caption{Non-causal activations for polynomial splines}
\end{subfigure}
\begin{subfigure}{.4\textwidth}
\centering
\includegraphics[width=\textwidth]{causal_fractional}
\caption{Causal activations for fractional splines}
\end{subfigure}\hspace{1cm}
\begin{subfigure}{.4\textwidth}
\centering
\includegraphics[width=\textwidth]{causal_exp}
\caption{Causal activations for exponential splines}
\end{subfigure}
\caption{Examples of neural-network-admissible activation functions.}
\end{figure}

Some examples of what these activation functions may look like can be seen in Figure 1.

We can also characterize whether or not a given $L$ fits into our framework based on its transfer function (so long as it exists). Since $L$ is neural-network-admissible and hence a convolution operator, we have $L f = h \ast f$ for $h = L \delta$ so the second bullet in Definition 3.4 is equivalent to saying
\[
(5.1) \quad h \ast (R f) = \alpha_L R(h \ast f)
\]
Taking the bilateral Laplace transform of (5.1), we find
\[ H(s)F(-s) = \alpha_L H(-s)F(-s) \]
In other words, for \( L \) to be neural-network-admissible we require its transfer function satisfies
\[ (5.2) \quad H(s) = \alpha_L H(-s) \]
for some \( \alpha_L \in \mathbb{C} \) with \( |\alpha_L| = 1 \). This immediately implies that any \( L \) with even or odd transfer function is neural-network-admissible (choose \( \alpha_L = +1 \) or \(-1\), respectively). With (5.2) in hand, we have a simple test of whether or not a given \( L \) fits under our framework.

It would also be useful to characterize whether or not a given activation function fits into our framework. This is equivalent to, for a given \( \sigma \), finding \( h \) such that \( h \ast \sigma = \delta \), then checking if (5.2) holds. Given \( \sigma \), we have
\[ (5.3) \quad H(s) = \frac{1}{\mathcal{L}\{\sigma\}(s)} \]
So as long as \( \mathcal{L}\{\sigma\} \) exists, we can find \( H \) with (5.3) and check it against (5.2).

**Remark 5.2.** The sigmoid activation function
\[ \sigma_c(x) = \frac{1}{1 + e^{-cx}} \]
fits into our framework. Indeed,
\[ \mathcal{L}\{\sigma_c\}(s) = \int_{\mathbb{R}} \frac{e^{-sx}}{1 + e^{-cx}} dx = \frac{1}{c} \int_{1}^{\infty} \frac{(t-1)^{s/c-1}}{t} dt = \frac{\pi}{c} \csc\left(\frac{\pi s}{c}\right) \]
where (*) holds by the substitution \( t := 1 + e^{-cx} \). Then,
\[ (5.4) \quad H_c(s) = \frac{1}{\frac{\pi}{c} \cdot \csc\left(\frac{\pi s}{c}\right)} = \frac{c \sin\left(\frac{\pi s}{c}\right)}{\pi} \]
\( H_c \) is odd and thus the sigmoid activation fits into our framework. Thus we can consider this as a notion of a “sigmoidal spline”. Moreover,
\[ (5.5) \quad \lim_{c \to \infty} \sigma_c(x) = u(x) \]
where \( u \) is the unit step function, i.e., \( u \) is a Green’s function of \( D \). From (5.4) we have
\[ \lim_{c \to \infty} H_c(s) = \lim_{c \to \infty} \frac{c \sin\left(\frac{\pi s}{c}\right)}{\pi} = s \]
which is exactly the transfer function of \( D \). Thus, in the limit, the “sigmoidal spline” recovers the polynomial spline of order one as we’d expect from (5.5).

**Remark 5.3.** The activation functions \( \arctan \) and \( \tanh \) do not fit into our framework since their bilateral Laplace transforms do not exist.
Remark 5.4. The Gaussian activation function
\[ \sigma(x) = e^{-x^2} \]
fits into our framework. Indeed,
\[
\mathcal{L} \{ \sigma \}(s) = \int_{\mathbb{R}} e^{-sx} e^{-x^2} \, dx = \sqrt{\pi} e^{s^2/4}
\]
Then,
\[
H(s) = \frac{1}{\sqrt{\pi} e^{s^2/4}} = \frac{e^{-s^2/4}}{\sqrt{\pi}}
\]
which is even and thus fits into our framework. A two-layer neural network with Gaussian
activation functions can be thought of as a generalized kernel machine where the bandwidth
of the kernel is now a trainable parameter.

6. Neural Network “Norms”. In this section we will answer Question 3.10. To begin, we
state the following general proposition about training \( \mathbb{R}^d \to \mathbb{R} \) networks is our generalization
of Theorem 1 from [17]. Theorem 1 from [17] relates minimizing the \( \ell^2 \)-norm of the network
weights in a two-layer ReLU network to minimizing the \( \ell^1 \)-norm of the last layer of network
while constraining the weights of the first layer. Our result introduces a generalized notion
of the “\( \ell^2 \)-norm” of the weights that holds for neural networks with activation functions
that satisfy a homogeneity condition. Moreover, our proof is completely constructive unlike
Theorem 1 from [17].

Proposition 6.1. Suppose we want to represent a function \( f : \mathbb{R}^d \to \mathbb{R} \) with a finite-width
network with activation \( \sigma \) that is \( k \)-non-negative homogeneous, i.e., \( \sigma(\gamma x) = \gamma^k \sigma(x) \)
for all \( \gamma \geq 0 \). Then, the following is true
\[
(6.1) \quad \min_{\theta \in \Theta} \frac{1}{2} \sum_{n=1}^N \left( |v_n|^2 + \|w_n\|^2 \right) = \min_{\theta \in \Theta} \|v\|_{\ell^1}
\]
s.t. \( \varphi_\theta = f \)
\[
\|w_n\|_{\ell^2} = 1, \quad n = 1, \ldots, N
\]

Proof. See Appendix A.

Remark 6.2. The polynomial splines of order \( N_0 \), \( L = D^{N_0} \), have Green’s functions that
are \((N_0 - 1)\)-non-negative homogeneous (as we saw in section 5), so for the special case of
polynomial splines we have the corollary to Theorem 4.4 that
\[
\min_{\varphi_\theta : \theta \in \Theta} \frac{1}{2} \sum_{n=1}^N \left( |v_n|^2 + |w_n|^{2N_0 - 2} \right)
\]
s.t. \( \varphi_\theta(x_m) = y_m, \quad m = 1, \ldots, M \)
has solutions that are minimal knot, non-uniform polynomial splines of order \( N_0 \). Written
differently, we have
\[
\min_{\varphi_\theta : \theta \in \Theta} \mathcal{C}(\theta)
\]
s.t. \( \varphi_\theta(x_m) = y_m, \quad m = 1, \ldots, M \)
where we have the neural network “norm”

\[(6.2) \quad \mathcal{C}(\theta) := \frac{1}{2} \sum_{n=1}^{N} \left( |v_n|^2 + |w_n|^{2N_0-2} \right) \]

which recovers the $\ell^2$-norm of the weights when $N_0 = 2$, which corresponds exactly to ReLU activations by choosing $\sigma = \sigma_{D^2}^{(\text{causal})}$ and also corresponds to training a neural network with weight decay. This says perhaps “non-linear” notions of weight decay in SGD should be considered when training neural networks.

**Remark 6.3.** In the event we do not have the homogenity property as in Proposition 6.1, we can always use regularization have the corollary to Theorem 4.4 that

\[
\min_{\varphi: \theta \in \Theta} \|v\|_{\ell_1} + \frac{1}{\gamma} \sum_{n=1}^{N} \left( |w_n|^2 - 1 \right)^2
\]

s.t. $\varphi_\theta(x_m) = y_m$, $m = 1, \ldots, M$

where $\gamma$ is a small constant, has solutions that are minimal knot, non-uniform L-splines. In other words, we have the regularized neural network “norm”

\[(6.3) \quad \mathcal{C}_\gamma(\theta) := \|v\|_{\ell_1} + \frac{1}{\gamma} \sum_{n=1}^{N} \left( |w_n|^2 - 1 \right)^2 \]

In practice, one could take $\gamma \sim 10^{-5}$.

**7. Empirical Validation and Discussion.** To verify that our theory developed in section 4 holds, we verify that empirically neural networks actually do learn splines. In our empirical results, we consider regularized problems of the form

\[
\min_{\varphi: \theta \in \Theta} \sum_{m=1}^{M} \ell(\varphi_\theta(x_m), y_m) + \lambda \mathcal{C}(\theta)
\]

where $\ell : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ is the squared error loss with the regularization parameter $\lambda = 10^{-5}$. We verify our theory holds with linear and cubic splines. We found empirically that minimizing the “norm” (6.3) or the “norm” (6.2) made no difference in the learned interpolations. We used PyTorch\(^9\) to implement the neural networks and used AdaGrad \(^10\) as the optimization procedure with a step size of $10^{-2}$. To compute spline interpolations with standard methods we used SciPy\(^10\) \([15]\).

In Figure 2 we compute a linear spline using standard methods and also compute it by training a neural network while minimizing the “norm” (6.2) using both a causal and non-causal Green’s function of $D^2$. The neural network interpolations have more knots than the “connect the dots” linear spline, though it’s clear that both solutions are still minimizers to Theorem 4.4.

\(^9\)https://pytorch.org/
\(^10\)Specifically, we used scipy.interpolate.InterpolatedUnivariateSpline.
Figure 2: Here $L = D^2$. In (a), (b), and (c), we have $\|D^2 f\|_{\mathcal{M}(\mathbb{R})}$, $C_{D^2}^{(causal)}(\theta)$, and $C_{D^2}^{(non-causal)}(\theta) \approx 137$. The neural network interpolations (b) and (c) are not the “connect the dots” linear spline, but have extra knots. Both are clearly valid solutions to both the minimizations in Theorem 4.4.

Figure 3: Here $L = D^4$. In (a), (b), and (c), we have $\|D^4 f\|_{\mathcal{M}(\mathbb{R})}$, $C_{D^4}^{(causal)}(\theta)$, and $C_{D^4}^{(non-causal)}(\theta) \approx 79$. The neural network interpolations (b) and (c) are exactly the same (up to floating point precision) as the cubic spline.

In Figure 3 we compute a cubic spline using standard methods and also compute it by training a neural network while minimizing the “norm” (6.2) using both a causal and non-causal Green’s function of $D^4$. In this case, the neural network interpolations learn the exact same function as the standard cubic spline. Thus we see that neural networks are indeed capable of learning splines.

In Figure 4 we show that explicit regularization can be needed to learn the spline interpolations. We see that if we have no regularization while training a neural network with a causal Green’s function of $D^4$, the learned interpolation is not the standard cubic spline, but the moment we include the proper regularization, the neural network learns the cubic spline function exactly.
8. Conclusions and Future Work. We have developed a general framework based on the theory of splines for understanding the interpolation properties of sufficiently wide minimum “norm” neural networks that interpolate scattered data. We have proven that neural networks are capable of learning a large class of L-splines and thus overparameterized neural networks do in fact learn “nice” interpolations of data. To the data science and machine learning communities, this gives intuition as to why overparameterized and interpolating models generalize well on new, unseen, data. To the spline and inverse problems communities, we have shown that by simply training a neural network (a discrete object) to zero error with appropriate regularization, we can exactly solve various continuous-domain linear inverse problems. Our current results hold for two-layer $\mathbb{R} \rightarrow \mathbb{R}$ neural networks. Future work will be directed towards developing a theory based on splines for both deep and multivariate neural networks.

Deep architectures. It remains an open question about what kind of functions do minimum “norm” deep networks learn. Working with the framework of L-splines required that our “building blocks” be Green’s functions of operators. With two-layer networks, this works by simply letting the activation function be the desired Green’s function. With deep networks, due to function compositions, it becomes very unclear what exactly the “building blocks” are, and what would be a reasonable “norm”. In the case of Green’s functions of $D^{N_0}$ operators, it would make sense that the function composition simply increases the order of the polynomial spline. Since the Fourier transform of a piecewise polynomial with pieces of degree $n$ is $\sim 1/\omega^{n+1}$, we conjecture that a fairly deep network with a Green’s function of $D^{N_0}$ activation function should learn something approaching a bandlimited interpolation.

Multivariate functions. Our theory only holds for $\mathbb{R}^d \rightarrow \mathbb{R}$ functions in the $d = 1$ case. Extending our results for the multivariate case would require showing that minimum “norm” neural networks that interpolate scattered data solve some multivariate continuous-domain linear inverse problem whose solutions are some type of spline. Recent work has examined the minimum $\ell^2$-norm of all the network weights for multivariate ReLU networks subject to representing a particular function, but has not made any connections to splines [18]. They show that in the multivariate case, there even exist continuous piecewise linear functions that
two-layer networks cannot represent with a finite $\ell^2$-norm of the network weights.

Appendix A. Proof of Proposition 6.1.

Proof. Let $\theta^* = (W^*, v^*, b^*, c^*)$ be an optimal solution for the left-hand side in (6.1) and let $\tilde{\theta}^* = (\tilde{W}^*, \tilde{v}^*, \tilde{b}^*, \tilde{c}^*)$ be an optimal solution for the right-hand side in (6.1). We will prove the claim by using the optimal solution for the left-hand side to construct a feasible solution for the right-hand side and vice-versa. We will then use the constructed feasible solutions to show

$$\frac{1}{2} \sum_{n=1}^{N} \left( |v_n^*|^2 + \|w_n^*\|^2 \right) \geq \|v^*\|_{\ell^1}$$

and

$$\frac{1}{2} \sum_{n=1}^{N} \left( |v_n^*|^2 + \|w_n^*\|^2 \right) \leq \|\tilde{v}^*\|_{\ell^1}$$

Starting with (A.2), for $n = 1, \ldots, N$ put

$$w_n^* = \|w_n^*\|_{\ell^2}, \quad v_n^* = \|v_n^*\|_{\ell^2}, \quad b_n^* = \|b_n^*\|_{\ell^2}, \quad \text{and } \tilde{c} = c^*$$

Clearly this constructed solution is feasible for the right-hand side of (6.1) since for $n = 1, \ldots, N$ we have $\|\tilde{w}_k\|_{\ell^2} = 1$ by construction and

$$\varphi_{\tilde{\theta}}(x) = \sum_{n=1}^{N} \tilde{v}_n \sigma \left( \frac{\tilde{w}_n (x) + \tilde{b}_n}{\tilde{c}} \right) + \tilde{c}(x)$$

$$= \sum_{n=1}^{N} \|w_n^*\|_{\ell^2} v_n^* \sigma \left( \left\langle \frac{w_n^*}{\|w_n^*\|_{\ell^2}}, x \right\rangle + \frac{b_n^*}{\|w_n^*\|_{\ell^2}} \right) + c^*(x)$$

$$= \sum_{n=1}^{N} v_n^* \sigma \left( \frac{w_n^*}{\|w_n^*\|_{\ell^2}}, x + b_n^* \right) + c^*(x)$$

$$= \varphi_{\theta^*}(x) = f(x)$$

where the third equality holds since $\sigma(\gamma t) = \gamma \sigma(t)$ for $\gamma \geq 0$ combined with the fact that $\|\cdot\|_{\ell^2} \geq 0$. Next,

$$\|\tilde{v}\|_{\ell^1} = \sum_{n=1}^{N} |\tilde{v}_n| = \sum_{n=1}^{N} |v_n^*| \|w_n^*\|_{\ell^2} \leq \sum_{n=1}^{N} \frac{1}{2} \left( |v_n^*|^2 + \|w_n^*\|^2 \right)$$

where the last line holds by the inequality of arithmetic and geometric means. Using the optimality of $\tilde{\theta}^*$ we have $\|\tilde{v}^*\|_{\ell^1} \leq \|\tilde{v}\|_{\ell^1}$. Combining this with the above display we find

$$\|\tilde{v}^*\|_{\ell^1} \leq \frac{1}{2} \sum_{n=1}^{N} \left( |v_n^*|^2 + \|w_n^*\|^2 \right)$$

Moving to (A.4), for $n = 1, \ldots, N$ put

$$w_n := \|w_n^*\|_{(2k)} \tilde{w}_n^*, \quad v_n := \text{sgn}(\tilde{v}_n^*) |\tilde{v}_n^*|^{1/2}, \quad b_n := \|\tilde{v}_n^*\|_{(2k)} \tilde{b}_n^*, \quad \text{and } c := \tilde{c}^*$$
Clearly this constructed solution is feasible for the left-hand side of (6.1) since
\[ \varphi_{\theta}(x) = \left[ \sum_{n=1}^{N} v_n \sigma(\langle w_n, x \rangle + b_n) \right] + c(x) \]
\[ = \left[ \sum_{k=n}^{N} \text{sgn}(\tilde{\nu}^*_n) |\tilde{\nu}^*_n|^{1/2} \sigma(\langle |\tilde{\nu}^*_n|^{1/(2k)} \tilde{w}^*_n, x \rangle + |\tilde{\nu}^*_n|^{1/(2k)} \tilde{b}^*_n) \right] + \tilde{c}^*(x) \]
\[ = \left[ \sum_{n=1}^{N} \text{sgn}(\tilde{\nu}^*_n) |\tilde{\nu}^*_n| \sigma(\langle \tilde{w}^*_n, x \rangle + \tilde{b}^*_n) \right] + \tilde{c}^*(x) \]
\[ = \varphi_{\tilde{\theta}^*}(x) = f(x) \]
where the third equality holds since \( \sigma(\gamma t) = \gamma^k \sigma(t) \) for \( \gamma \geq 0 \) combined with the fact that \(| \cdot | \geq 0\). Next,
\[ \frac{1}{2} \sum_{n=1}^{N} \left( |v_n|^2 + \|w_n\|_{\ell^2}^{2k} \right) = \frac{1}{2} \sum_{n=1}^{N} \left( |\tilde{\nu}^*_n| + |\tilde{\nu}^*_n| \|\tilde{w}^*_n\|_{\ell^2}^{2k} \right) = \frac{1}{2} \sum_{n=1}^{N} \left( |\tilde{\nu}^*_n| + |\tilde{\nu}^*_n| \|\tilde{\nu}^*_n\|_{\ell^1} \right) = \sum_{n=1}^{N} |\tilde{\nu}^*_n| = \|\tilde{\nu}^*\|_{\ell^1} \]
where the second equality holds since \( \|\tilde{w}^*_n\|_{\ell^2} = 1 \) for \( n = 1, \ldots, N \) by the optimality (and hence feasibility) of \( \theta^* \). Using the optimality of \( \theta^* \) we have
\[ \frac{1}{2} \sum_{n=1}^{N} \left( |v_n|^2 + \|w_n\|_{\ell^2}^{2k} \right) \leq \frac{1}{2} \sum_{n=1}^{N} \left( |v_n|^2 + \|w_n\|_{\ell^2}^{2k} \right) \]
Combining this with the above display we find
\[ (A.4) \frac{1}{2} \sum_{n=1}^{N} \left( |v_n|^2 + \|w_n\|_{\ell^2}^{2k} \right) \leq \|\tilde{\nu}^*\|_{\ell^1} \]
Combining the inequalities (A.2) and (A.4) proves the proposition. \[ \blacksquare \]

**Remark A.1.** Our proof of Proposition 6.1 provides explicit constructions of an optimal solution for the right-hand side from and optimal solution for the left-hand side and vice-versa in (A.1) and (A.3), respectively.

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