STRICHARTZ ESTIMATES FOR THE KINETIC TRANSPORT EQUATION

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Abstract. We study the range of validity of the Strichartz estimates for the kinetic transport equation. We are able to give the full range in spatial dimension $n = 1$ while in higher dimensions we leave open some inhomogeneous estimates. Our results extend the previous works by Castella and Perthame [6], Keel and Tao [12] and Guo and Peng [9]. Based on a geometric interpretation of the kinetic transport propagator we construct counterexamples over Besicovitch sets to various endpoint Strichartz estimates containing $L^\infty$-norms. We also give an application to a nonlinear kinetic model of bacterial chemotaxis where we extend the result of Bournaveas et al. [5].

1. Introduction

1.1. Historical Remarks. In this paper we prove Strichartz estimates for the kinetic transport (KT) equation

$$\partial_t u(t, x, v) + v \cdot \nabla_x u(t, x, v) = F(t, x, v), \quad (t, x, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n,$$

(1.1)

to which we prescribe initial data

$$u(0, x, v) = f(x, v).$$

(1.2)

Strichartz estimates for the KT equation were first proved by Castella and Perthame [6] in 1996. The following year appeared the seminal paper by Keel and Tao [12] where they introduce some novel techniques in the abstract setting. Although they improve the range of validity of the homogeneous Strichartz estimates compared to [6], the authors point out the intractability of the endpoint Strichartz estimate (7.1) with the current methods. The next result appeared in 2007 when Guo and Peng [9] demonstrated the expected failure of the endpoint estimate (6.1) in spatial dimension $n = 1$. Interestingly enough, they also showed that the estimate holds if one replaces the $L^\infty$-norm in $x$ by the BMO norm.

In the meantime, several authors extended the results of [12] in the inhomogeneous setting, see Foschi [8], Vilela [20], Taggart [17], Ovcharov [13]. We shall apply their methods to the context of the KT equation to derive inhomogeneous Strichartz estimates for non-admissible exponents.

A recent survey article to the field of nonlinear kinetic equations with references to (1.1) and the estimates in [6] can be found in Perthame [14]. An application of the Strichartz estimates to prove existence of global solutions to a kinetic model of
chemotaxis appeared in Bournaveas et al. [5]. In section 8 we extend their result to dimensions \( n \geq 3 \).

1.2. Scope of the paper. We try to give a full and systematic study of the Strichartz estimates for the KT equation with the known methods to us from the literature. We thus vastly improve the range of the known estimates compared to \([6],[12]\), in several directions.

1. We prove all (homogeneous and inhomogeneous) Strichartz estimates for admissible exponents apart from the homogeneous endpoint \((7.1)\) in \( n > 1 \) and the inhomogeneous double endpoint in \( n > 1 \), see theorem 2.1 and the remark thereafter.

2. We also consider homogeneous estimates for mixed \( L^p \)-data. For comparison, these estimates correspond to homogeneous estimates for \( L^p \)-data for the Schrödinger equation. We find their full range in \( n = 1 \) and leave open only some endpoint estimates in higher dimensions.

3. We apply the techniques of Foschi [8] to study the inhomogeneous estimates for non-admissible exponents. We find the full range in \( n = 1 \) and leave open some estimates in higher dimensions. An analogous set of estimates is unresolved in the context of the Schrödinger equation too. Note that our result is not contained in the abstract estimates of Taggart [17]. The reason for that is twofold. One is that the real method of interpolation used there does not interact well with mixed \( L^p \)-spaces and obtains estimates in non-\( L^p \) spaces. The other reason is that (1.1) has a specific invariance, see (1.3) below, that is not taken into account in the abstract setting.

4. We devise a number of counterexamples to show sharpness to our results. We present a new approach to treat the various endpoint estimates that contain \( L^\infty \)-norms by considering them over Besicovitch sets, thus including and extending the counterexamples of Guo and Peng [9].

In the final section of this paper we consider the Othmer-Dunbar-Alt kinetic model of chemotaxis \([8.1]-[8.3]\) and prove the existence of global weak solutions for small data in dimensions \( n \geq 3 \) by making use of the Strichartz estimates of theorem 2.1.

1.3. Basic Facts and Properties of the KT equation. The KT equation is a very interesting model for studying Strichartz estimates. First, the kinetic transport propagator \( U(t) \) has a simple explicit form

\[
U(t)f = f(x - tv, v).
\]

Second, the homogeneous equation \( (1) \) is invariant to the transformation

\[
(1.3) \quad f \mapsto f^\alpha, \quad U(t)f \mapsto (U(t)f)^\alpha,
\]

which allows us to derive from a known Strichartz estimate a new estimate with different exponents. The exponents transform according to the rule

\[
(1.4) \quad (q, r, p, a) \mapsto (\alpha q, \alpha r, \alpha p, \alpha a), \quad 0 < \alpha < \infty.
\]

But on the other hand, the KT equation enjoys a rather complex vector-valued dispersive inequality

\[
(1.5) \quad \|U(t)f\|_{L^q_t L^r_v} \lesssim \frac{1}{|t|^a} \|f\|_{L^p_t L^\infty_v}.
\]

\(^1\)That is equation (1.1) with \( F(t) = 0 \).
This fact makes the endpoint Strichartz estimate intractable with the current methods, see section 6.5 for an example in support of that claim.

Let us fix the notation for the remainder of the paper. By \( W(t) \) we shall denote the operator

\[
W(t)F = \int_{-\infty}^{t} U(t - \tau)F(\tau)\,d\tau
\]

that gives the solution to the inhomogeneous equation (1.1) that decays as \( t \to -\infty \). In particular, when \( \text{supp } F \subseteq [0, \infty) \) this coincides with Duhamel’s formula, i.e. \( W(t) \) is the operator that solves (1.1), (1.2) with zero initial conditions. The estimates for \( U(t) \) have the form

(1.6) \[
\|U(t)f\|_{L_t^q L_x^r L_z^p} \lesssim \|f\|_{L_{t,x,z}^{q,r,p}},
\]

where the subscripts indicate integration over the whole space and the shortcut \( L_t^q L_x^r L_z^p \) stands for \( L^q(\mathbb{R}; L^r(\mathbb{R}^n); L^p(\mathbb{R}^n)) \). Note that the global in time estimate (1.0) trivially implies the local estimate

(1.7) \[
\|U(t)f\|_{L_t^q([0,T]; L_x^r L_z^p)} \lesssim \|f\|_{L_{x,z}^{q,r,p}},
\]

if one considers the localized operator \( U(t)\chi_{[0,T]} \), where \( \chi_{[0,T]} \) is the characteristic function of the interval \( [0,T] \). The last assertion is due to the fact that both \( U(t) \) and \( U(t)\chi_{[0,T]} \) satisfy the same basic assumptions. We shall also consider estimates for \( U(t) \) for mixed \( L^p \)-data.

(1.8) \[
\|U(t)f\|_{L_t^q L_x^r L_z^p} \lesssim \|f\|_{L_{t,x,z}^\infty}.
\]

The inhomogeneous estimates have the form

(1.9) \[
\|W(t)F\|_{L_t^q L_x^r L_z^p} \lesssim \|F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}} L_z^{\tilde{p}}},
\]

for some exponent triplets \( (q, r, p) \) and \( (\tilde{q}, \tilde{r}, \tilde{p}) \), the precise range of which shall be subject of theorem 2.3. The “\( \lesssim \)” sign means less or equal up to a constant that does not depend on the function \( f \) or \( F \) in the estimates (1.6), (1.9), etc. We shall say that two non-negative quantities \( a \) and \( b \) are comparable, and write \( a \sim b \), if \( \exists C > 0, C=\text{const: } a \leq Cb \) and \( b \leq Ca \). We shall say that two functions \( f \) and \( g \), lying in the same normed space with norm \( \|\cdot\| \) are approximately equal and write \( f \approx g \), if \( \|f - g\| \) is small enough.

1.4. Some definitions. Following Keel and Tao [12], we shall call the quadruplet of exponents that appear in estimate (1.6) admissible if (1.6) holds for all \( f \in L^p \). The range of validity of the inhomogeneous estimates (1.9) is in some sense larger than that of the homogeneous estimates (1.6). As in Foschi [8], we shall call the exponents for which (1.6) holds acceptable. The precise definitions are given next.

We denote by \( \text{HM}(p, r) \) the harmonic mean of \( p \) and \( r \), i.e. \( a = \text{HM}(p, r) \) if and only if

\[
\frac{1}{a} = \frac{1}{2} \left( \frac{1}{r} + \frac{1}{p} \right).
\]

Definition 1.1. Set

(1.10) \[
\begin{cases}
p^*(a) = \frac{na}{n + 1}, & r^*(a) = \frac{na}{n - 1}, \quad \text{if } \frac{n+1}{n} \leq a \leq \infty, \\
p^*(a) = 1, & r^*(a) = \frac{n+1}{n}, \quad \text{if } 1 \leq a \leq \frac{n+1}{n}.
\end{cases}
\]

When \( a = 2 \) we simply write \( p^* = p^*(2), r^* = r^*(2) \).
Definition 1.2. We say that the exponent quadruplet \((q, r, p, a)\) is KT-admissible if
\[
\frac{1}{q} + \frac{n}{r} = \frac{n}{a}, \quad a = \text{HM}(p, r),
\]
(1.11)
\[
1 \leq a \leq \infty, \quad a \leq r \leq r^*(a), \quad p^*(a) \leq p \leq a,
\]
(1.12)
apart from the case \(n = 1, (q, r, p, a) = (a, \infty, a/2, a), 2 \leq a < \infty\).

The boundaries \(r^*(a)\) and \(p^*(a)\) are chosen to ensure that the Lebesgue exponents \(a, p, q, r\) remain in the range \([1, \infty]\). Note that when \(a \geq (n + 1)/n\) this is always so and in this case definition 1.2 gives the range of all possible homogeneous estimates (1.6) in the \(L^p\)-norms (up to an endpoint), see theorem 2.1. In view of the power invariance (1.3), we can generalize these estimates to any \(0 < a < \infty\). However, the proof of the inhomogeneous estimates (1.9) relies on the duality relations of the \(L^p\)-spaces, for \(1 \leq p \leq \infty\). This restriction is reflected in the second condition in (1.10). We next give an equivalent definition to 1.2, which is more convenient in the context of the inhomogeneous estimates.

Definition 1.3. We say that the exponent triplet \((q, r, p)\) is KT-admissible if
\[
\frac{1}{q} = \frac{n}{2} \left(\frac{1}{p} - \frac{1}{r}\right), \quad a \overset{\text{def}}{=} \text{HM}(p, r),
\]
(1.13)
\[
1 \leq p, q, r \leq \infty, \quad a \leq r \leq r^*(a), \quad p^*(a) \leq p \leq a,
\]
(1.14)
apart from the case \(n = 1, (q, r, p) = (a, \infty, a/2)\).

A consequence of the above definition is the fact that if \((q, r, p)\) is KT-admissible then \(a \leq q \leq \infty\) and \(p \leq r\). Triplets of the form \((q, r, p) = (a, r^*(a), p^*(a))\), for \((n + 1)/n \leq a < \infty\), shall be called endpoint. When \(a = 1\) the only admissible triplet is \((\infty, 1, 1)\), and similarly when \(a = \infty\) the only admissible triplet is \((\infty, \infty, \infty)\).

Definition 1.4. We say that the exponent 5-tuple \((q, r, p, b, c)\) is mixed KT-admissible if
\[
\frac{1}{q} + \frac{1}{r} = \frac{n}{b}, \quad \text{HM}(p, r) = \text{HM}(b, c) \overset{\text{def}}{=} a \in [1, \infty],
\]
(1.15)
\[
1 \leq r < r^*(a), \quad p^*(a) < p \leq a, \quad p < b < a < c < r,
\]
(1.16)
or if \(a = b = c\) and \((q, r, p, a)\) is KT-admissible.

Definition 1.5. We say that the exponent triplet \((q, r, p)\) is KT-acceptable if
\[
\frac{1}{q} < n \left(\frac{1}{p} - \frac{1}{r}\right), \quad 1 \leq q \leq \infty,
\]
(1.17)
\[
1 \leq p < r \leq \infty,
\]
(1.18)
or if \(q = \infty, 1 \leq p = r \leq \infty\).

Note that every KT-admissible triplet is KT-acceptable too. We introduce the following definition.

Definition 1.6. We say that the two KT-acceptable exponent triplets \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) are jointly KT-acceptable if
\[
\frac{1}{q} + \frac{1}{\tilde{q}} = n \left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right), \quad \frac{1}{q} + \frac{1}{\tilde{q}} \leq 1,
\]
(1.19)
\[
\text{HM}(p, r) = \text{HM}(\tilde{p}', \tilde{r}') = a,
\]
and if further the exponents satisfy the following restrictions

- $r < \infty$, unless the point $(1/q, 1/r, 1/p, 1/q, 1/r, 1/p) \in \Sigma_1 \cup B$,
  \[ \Sigma_1 = \{(\mu, 0, \kappa, \nu, 1 - \kappa, 1) : 0 < \mu, \nu < 1, 0 < \kappa < 1/n, \mu + \nu = n\kappa \}, \]
  \[ B = (0, 0, 0, 1, 1). \]

- $\tilde{r} < \infty$, unless the point $(1/q, 1/r, 1/p, 1/q, 1/r, 1/p) \in \Sigma_2 \cup C$,
  \[ \Sigma_2 = \{(\mu, 1 - \kappa, \nu, 0, \kappa) : 0 < \mu, \nu < 1, 0 < \kappa < 1/n, \mu + \nu = n\kappa \}, \]
  \[ C = (0, 1, 1, 0, 0, 0). \]

- if $\tilde{q} = \infty$, then $a \leq r < r^*(a)$,
- if $q = \infty$, then $a < \tilde{r} < r^*(a)$.

## 2. Main Results

Throughout this section we suppose that $1 \leq q, r, p, a \leq \infty$.

**Theorem 2.1** (Strichartz estimates for admissible exponents). Let $u$ be the solution to the IVP for (1.1), (1.2). Then the estimate

\[
\|u(t, x, v)\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L^\infty} + \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}},
\]

holds for all $f \in L^q(\mathbb{R}^{2n})$, $F \in L_t^{q'} L_x^{r'} L_v^{p'}$ if and only if $(q, r, p)$ and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two KT-admissible exponent triplets and $a = \text{HM}(p, r) = \text{HM}(\tilde{p}, \tilde{r})$, apart from the case when $n > 1$ and $(q, r, p)$ being an endpoint triplet, which remains open.

**Remark 2.2.** Note that in theorem 2.1 the KT-admissible exponent triplet $(\tilde{q}, \tilde{r}, \tilde{p})$ is allowed to be endpoint in $n > 1$ as long as the other KT-admissible exponent triplet $(q, r, p)$ remains non-endpoint. In such case $(q, r, p)$ and $(\tilde{q}, \tilde{r}, \tilde{p})$ are jointly acceptable and satisfy the conditions of the next theorem.

**Theorem 2.3** (Global inhomogeneous estimates). The estimate

\[
\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}}
\]

holds for all $F \in L_t^{q'} L_x^{r'} L_v^{p'}$ in $n = 1$ if and only if $(q, r, p)$ and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two jointly KT-admissible exponent triplets. If $n > 1$ we suppose that $(q, r, p)$ and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two jointly KT-admissible exponent triplets and consider the cases

- \( q, \tilde{q} < \infty, 1/q + 1/\tilde{q} < 1, \) and $2 \leq r, \tilde{r} < \infty$: then estimate (2.2) holds if additionally
  \[ \frac{n - 1}{p'} < \frac{n}{r}, \quad \frac{n - 1}{\tilde{p}'} < \frac{n}{\tilde{r}}. \]
- $r, \tilde{r} = \infty$: estimate (2.2) holds only under the conditions of definition 1.6, that is if $(1/q, 1/r, 1/p, 1/q, 1/r, 1/p) \in \Sigma_1 \cup B$,
  $(1/q, 1/r, 1/p, 1/q, 1/r, 1/p) \in \Sigma_2 \cup C$, respectively.
- $\tilde{q} = \infty$: then we have the estimate
  \[
  \|W(t)F\|_{L_t^{q'} L_x^{r'} L_v^{p'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}}.
  \]
  Whenever $q \geq \tilde{p}$ (2.3) implies (2.2).
- $q = \infty$: then we have the estimate
  \[
  \|W(t)F\|_{L_t^{q'} L_x^{r'} L_v^{p'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}}.
  \]
  Whenever $\tilde{q} < p$ (2.2) implies (2.2).
Conversely, if estimate (2.3) holds for all $F \in L^q_x L^r_y L^p_z$ in some dimension $n > 1$, then $(q, r, p)$ and $(\tilde{q}, \tilde{r}, \tilde{p})$ must be two jointly KT-acceptable exponent triplets.

Remark 2.4. Apparently, theorem 2.3 leaves open the existence of some inhomogeneous estimates (2.2) in $n > 1$. The analogous estimates for the Schrödinger equation are posed as unresolved in Foschi [8] (section Open Questions). The latter work addressed the observation by Keel and Tao [12] (and others) that the full range of validity of the inhomogeneous Strichartz estimates for a given dispersive equation is larger than the set of those that correspond to admissible exponent pairs (triplets) only. We hope that our results could be beneficial to the general question by providing a perspective to a specific context.

**Theorem 2.5** (Global homogeneous estimates). Suppose that $(q, r, p, b, c)$ is a mixed KT-admissible exponent 5-tuple. Then the estimate

$$
(2.6) \quad \|U(t)f\|_{L^q_x L^r_y L^p_z} \lesssim \|f\|_{L^b_x L^c_y}
$$

holds for all $f \in L^b_x L^c_y$. Conversely, if estimate (2.0) holds for all $f \in L^b_x L^c_y$, then $(q, r, p, b, c)$ must be a mixed KT-admissible exponent 5-tuple.

Remark 2.6. The appearance of Lorentz norms in some of the estimates above is not a great obstacle to applications. For example, if we restrict to finite time intervals $[0, T]$, we have the continuous embeddings

$$
L^{q,r}([0, T]) \hookrightarrow L^p([0, T]), \quad q > p, 1 \leq q, p \leq \infty,
$$

$$
L^p([0, T]) \hookrightarrow L^{q,r}([0, T]), \quad p > q, 1 \leq q, p \leq \infty,
$$

see [1] p. 217. Let us recall also the continuous embedding $L^{q,c}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ whenever $q \geq c$. For example, let $(\infty, r, p)$ and $(\tilde{q}, \tilde{r}, \tilde{p})$ be such that estimate (2.4) holds and let $1 \leq \tilde{Q} < \tilde{q}$. Then we have the local inhomogeneous estimate

$$
\|W(t)F\|_{L^q([0, T]; L^r_x L^p_y)} \lesssim_T \|F\|_{L^{Q'}([0, T]; L^{r'}_x L^{p'}_y)}.
$$

3. Preliminaries

**Lemma 3.1** (The dispersive estimate [14]). The kinetic transport propagator $U(t)$ enjoys the estimate

$$
(3.1) \quad \|U(t)f\|_{L^p_x L^q_y} \leq \frac{1}{|t|^n} \|f\|_{L^p_x L^q_y},
$$

for all $f \in L^1_x L^\infty_y$.

**Proof.**

$$
\int_{\mathbb{R}^n} |U(t)f| \, dv = \int_{\mathbb{R}^n} |f(x-tv, v)| \, dv \leq \int_{\mathbb{R}^n} \sup_{y \in \mathbb{R}^n} |f(x-ty, y)| \, dv 
$$

$$
\leq \frac{1}{|t|^n} \int_{\mathbb{R}^n} \sup_{y \in \mathbb{R}^n} |f(z, y)| \, dz = \frac{1}{|t|^n} \|f\|_{L^1_x L^\infty_y}.
$$

\[\square\]

\[2\]With $L^{q,c}$ replaced by $L^q$.

\[3\]However, the exact upper bound on $r$ is not known in the case $n > 1$, $b \neq c$, see section 6.3 and we cannot discard the possibility of omissions of some estimates in that case.
Lemma 3.2 (The transport estimate). The kinetic transport propagator $U(t)$ enjoys the estimate
\begin{equation}
\|U(t)f\|_{L^a_{x,v}L^a_v} \leq \|f\|_{L^a_{x,v}}, \quad 0 < a \leq \infty,
\end{equation}
for all $f \in L^a_{x,v}$.

Proof. Trivial. □

Corollary 3.3 (The decay estimate). The kinetic transport propagator $U(t)$ enjoys the estimate
\begin{equation}
\|U(t)f\|_{L^p_{x,v}L^r_v} \leq \frac{1}{|t|^{\alpha(\frac{1}{p} - \frac{1}{r})}} \|f\|_{L^p_{x,v} L^r_v}, \quad 1 \leq p \leq r \leq \infty,
\end{equation}
for all $f \in L^p_{x,v}L^r_v$.

Proof. Complex interpolation between the dispersive estimate (1.5) and the two transport estimates (3.2) with $a = 1$ and $a = \infty$. □

Lemma 3.4. The formal adjoint to $U(t)$ is the operator $U(t)^* = U(-t)$.

Proof. We denote by $\langle \cdot, \cdot \rangle$ the scalar product on $L^2(\mathbb{R}^n)$. Thus,
\begin{align*}
\langle U(t)f, g \rangle & = \int_{-\infty}^{\infty} f(x - tv, v)g(x, v)dx dv \\
& = \int_{-\infty}^{\infty} f(x, v)g(y + tv, v)dy dv = \langle f, U(-t)g \rangle,
\end{align*}
where we have made the substitution $y = x - tv$. □

Lemma 3.5 (Scaling properties of $U(t)$ and $W(t)$). The evolution operators $U(t)$ and $W(t)$ enjoy the following scaling properties
\begin{align*}
U(t)f_\lambda & = f(x/\lambda - tv/\lambda, v) = U(t/\lambda, x/\lambda, v) f, \\
& \quad \text{where } f_\lambda(x, v) = f(x/\lambda, v), \\
U(t)f_\lambda & = f(x/\lambda - tv/\lambda, v/\lambda) = U(t, x/\lambda, v/\lambda) f, \\
& \quad \text{where } f_\lambda(x, v) = f(x/\lambda, v/\lambda), \\
W(t)F_\lambda & = \lambda \int_0^{t/\lambda} F(s, x/\lambda - (t/\lambda - s)v, v) ds = \lambda W(t/\lambda, x/\lambda, v) F, \\
& \quad \text{where } F_\lambda(t, x, v) = F(t/\lambda, x/\lambda, v), \\
W(t)F_\lambda & = \int_0^{t} F(s, x/\lambda - (t - s)v/\lambda, v/\lambda) ds = W(t, x/\lambda, v/\lambda) F, \\
& \quad \text{where } F_\lambda(t, x, v) = F(t, x/\lambda, v/\lambda).
\end{align*}

Proof. Direct inspection. □

Lemma 3.6 (Christ-Kiselev, see lemma 3.1 of [19], or [18]). Suppose that the integral operator
\begin{equation}
T[F](t) = \int_{-\infty}^{\infty} K(t, s)F(s)ds
\end{equation}
is bounded from $L^p(\mathbb{R}; B_1)$ to $L^q(\mathbb{R}; B_2)$ for some Banach spaces $B_1$, $B_2$ and $1 \leq p < q \leq \infty$. The operator-valued kernel $K(t, s)$ maps $B_1$ to $B_2$ for all $t, s \in \mathbb{R}$.
Assume also that $K$ is regular enough to ensure that (3.4) makes sense as a $B_2$-valued Bochner integral for almost all $t \in \mathbb{R}$. Then the operator
\[
\hat{T}[F](t) = \int_{-\infty}^{t} K(t,s)F(s)ds
\]
is also bounded on the same spaces.

**Proposition 3.7** (Hörmander, see [10]). Whenever a (non-trivial) linear and bounded operator maps a vector-valued $L^p$-space to another vector-valued $L^q$-space, $1 \leq p, q \leq \infty$, and additionally this operator is translation invariant, then we must have that $p \leq q$.

### 3.1. The TT*-principle

Consider the operator $T : L^2_{x,v} \rightarrow L^q_t L^r_x L^p_v$, given by
\[
T[f](t,x,v) = f(x - tv, v), \quad \forall f \in L^2_{x,v}.
\]
Its formal adjoint $T^* : L^q_t L^r_x L^p_v \rightarrow L^2_{x,v}$ has the form of an $L^2$-valued integral
\[
T^*[F](x,v) = \int_{-\infty}^{\infty} F(s,x + sv,v)ds, \quad \forall F \in L^q_t L^r_x L^p_v.
\]
Then the composition of the two $TT^* : L^q_t L^r_x L^p_v \rightarrow L^q_t L^r_x L^p_v$ has the form
\[
TT^*[F](t,x,v) = \int_{-\infty}^{\infty} F(s,x - (t-s)v,v)ds, \quad \forall F \in L^q_t L^r_x L^p_v.
\]
By the TT*-principle, $T$ and $TT^*$ are equally bounded with $\|T\|^2 = \|TT^*\|$. Thus, the two estimates are equivalent
\[
\|Tf\|_{L^q_t L^r_x L^p_v} \leq C \|f\|_{L^2_{x,v}}, \quad \forall f \in L^2_{x,v},
\]
\[
\|TT^*F\|_{L^q_t L^r_x L^p_v} \leq C^2 \|F\|_{L^q_t L^r_x L^p_v}, \quad \forall F \in L^q_t L^r_x L^p_v,
\]
where $\|T\| \leq C$. Moreover, if $(q,r,p)$ and $(\tilde{q},\tilde{r},\tilde{p})$ are two exponent triplets for which (3.5) holds, then by the factorization property of $TT^*$ we obtain the consequence
\[
\|TT^*F\|_{L^q_t L^r_x L^p_v} \leq C^2 \|F\|_{L^q_t L^r_x L^p_v}, \quad \forall F \in L^q_t L^r_x L^p_v.
\]
By duality, (3.6) is equivalent to
\[
\int_{-\infty}^{\infty} \langle TT^*[F](t),G(t) \rangle dt \leq C^2 \|F\|_{L^q_t L^r_x L^p_v} \|G\|_{L^q_t L^r_x L^p_v}, \quad \forall F, \forall G \in L^q_t L^r_x L^p_v,
\]
which in turn simplifies to
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle (U(s))^* F, (U(t))^* G \rangle dsdt \leq C^2 \|F\|_{L^q_t L^r_x L^p_v} \|G\|_{L^q_t L^r_x L^p_v},
\]
Consider the bilinear form
\[
B(F,G) = \int \int \langle U(s)^* F, U(t)^* G \rangle dsdt.
\]
By symmetry, the last inequality simplifies to
\[
\|B(F,G)\| \leq C^2 \|F\|_{L^q_t L^r_x L^p_v} \|G\|_{L^q_t L^r_x L^p_v}, \quad \forall F, \forall G \in L^q_t L^r_x L^p_v.
\]
Analogously, we have that the inhomogeneous estimate

\[ \|W(t)F\|_{L^q_t L^r_x L^p_y} \lesssim \|F\|_{L^q_t L^{r/2}_x L^{p'}_y}, \quad \forall F \in L^q_t L^{r/2}_x L^{p'}_y, \]

is equivalent to the estimate

\begin{align}
\|B(F, G)\| \lesssim & \|F\|_{L^q_t L^{r/2}_x L^{p'}_y} \|G\|_{L_t^{q'} L_x^{r'} L_y^{p'}}, \\
& \forall F \in L^q_t L^{r/2}_x L^{p'}_y, \forall G \in L_t^{q'} L_x^{r'} L_y^{p'}.
\end{align}

In our proofs of the Strichartz estimates below we shall work with estimate (3.6) and (3.7) to treat the case of admissible exponents, and with the bilinear formulation (3.9) in the case of acceptable exponents.

4. Estimates for admissible exponents

4.1. Estimates for \( L^p \)-data. We prove here the sufficient part of theorem 2.4. In particular, we prove separately the homogeneous and the inhomogeneous estimate

\begin{align}
(2.1b) & \quad \|U(t)f\|_{L^q_t L^r_x L^p_y} \lesssim \|f\|_{L^2}, \quad 1 \leq a < \infty, \\
(2.1b') & \quad \|W(t)F\|_{L^q_t L^r_x L^p_y} \lesssim \|F\|_{L^q_t L^{r/2}_x L^{p'}_y},
\end{align}

respectively, where \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) are two non-endpoint KT-admissible triplets subject to the scaling condition \(a = \text{HM}(p, r) = \text{HM}(\tilde{p}', \tilde{r}')\).

Proof. It suffices to consider only estimate

\[ \|U(t)f\|_{L^q_t L^r_x L^p_y} \lesssim \|f\|_{L^2}, \]

since the general case of (2.1b) will follow by the power invariance (1.3). By the TT*-principle, the last inequality is equivalent to (3.6).

In view of the decay estimate

\[ \|U(t)f\|_{L^q_t L^r_x L^p_y} \lesssim \frac{1}{|t|^{\beta(r)}} \|f\|_{L_t^{q'} L_x^{r'} L_y^{p'}}, \quad 2 \leq r \leq \infty, \]

where \(\beta(r) = n(1 - 2/r)\), cf. corollary 3.3 we obtain the following estimate for \(TT^*F\)

\[ \|TT^*F\|_{L^q_t L^r_x L^p_y} \lesssim \int_{-\infty}^{\infty} \|F(t-s)\|_{L^q_t L^p_y} ds \lesssim \int_{-\infty}^{\infty} \frac{\|F(s)\|_{L_t^{q'} L_x^{r'} L_y^{p'}}}{|t-s|^{\beta(r)}} ds. \]

We take the \(L^q\)-norm in \(t\) and in view of the Hardy-Littlewood-Sobolev (HLS) theorem of fractional integration, see [1] pp. 228-229], [15], we obtain

\[ \|TT^*F\|_{L^q_t L^r_x L^p_y} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_y^{p'}}, \]

whenever \(0 < \beta(r) < 1, 1 + 1/q = 1/q' + \beta(r)\). The latter conditions are equivalent to \(2 < r < r^*, 1/q + n/r = n/2\). The left endpoint \(r = 2\) follows trivially from the transport estimate (3.2). The right endpoint \(r = r^*\) remains an open problem.

The inhomogeneous estimate (2.1b) is a consequence of estimate (2.1b'), the factorization (3.7), and the Christ-Kiselev lemma 3.6. Note that the latter is applicable only in the non-endpoint case \(1/q + 1/\tilde{q} < 1\). \(\square\)
Proof of remark 2.2. To complete the proof of the sufficient part of theorem 2.1 we only need to consider the case when \((q, r, p)\) is a non-endpoint admissible triplet while \((\tilde{q}, \tilde{r}, \tilde{p})\) is an endpoint admissible triplet. It is easy to check that the two triplets are jointly acceptable with \(1 < q \leq \tilde{q} < a\). Suppose that both \(q, \tilde{q} < \infty\). Then condition (2.3) is satisfied by the mere construction of the proof of theorem 2.3. We can give a direct proof of that claim as well. For example, the inequality \((n-1)/p' < n/r\) is equivalent to \(a'/p' < r^*(a')/r = 1\), i.e. to \(p < a\). But if the last inequality is violated, i.e. \(p = a\), then \(p = r\) and \(q = \infty\), contradiction. Similarly, \((n-1)/p' < n/r\) is congruent with \(a/p' < r^*(a)/r\). For the left hand side we have that \(a/p' < 1\), while for the right hand side \(r^*(a)/r > 1\). Thus, we are left only with the case when either \(q\) or \(\tilde{q}\) is equal to infinity. Suppose, for example, that \(q = \infty\). Then \(r = p = a\), \(\tilde{q} = a\) and hence estimate (2.5) implies the desired claim. Finally, the case when \(\tilde{q} = \infty\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) is endpoint is only possible if \(\tilde{q} = \tilde{r} = \tilde{p} = \infty\). Then \(p' = 1\) and estimate (2.4) implies the desired claim. □

4.2. Estimates for mixed \(L^p\)-data. We prove here theorem 2.5 that is estimate (2.6)
\[
\|U(t)f\|_{L^q L^r L^p} \lesssim \|f\|_{L^q_{-\theta} L^r_{-\theta} L^p_{-\theta}}.
\]
To that end we recall two results from real interpolation that shall be used in the proof. By \(L^p = L^p(A)\) and \(L^{p,q} = L^{p,q}(A)\) we denote the \(L^p\)-space and the Lorentz space, respectively of functions that take their values in a fixed Banach space \(A\). If \(A_0\) and \(A_1\) are two Banach spaces that are compatible, we denote by \(A\) the interpolation couple \((A_0, A_1)\).

Proposition 4.1 (see the Appendix of [7]). For every \(1 \leq p_0, p_1 < \infty, 0 < \theta < 1, 1/p = (1 - \theta)/p_0 + \theta/p_1\) and \(p \leq q\) we have \(L^p(A_0) \hookrightarrow (L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}\).

Proposition 4.2 (see [2], p. 113). Suppose that \(0 < p_0, p_1, q_0, q_1 \leq \infty, 0 < \theta < 1,\) and \(p_0 \neq p_1\). Then \((L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q} = L^{p,q}\), where \(1/p = (1 - \theta)/p_0 + \theta/p_1\).

Now we are ready to commence in earnest.

Proof of theorem 2.5. We interpolate between the estimates
\[
\|U(t)f\|_{L^q L^r L^p} \lesssim \|f\|_{L^q_{-\theta} L^r_{-\theta} L^p_{-\theta}}, \tag{4.1}
\]
\[
\|U(t)f\|_{L^q L^r L^p} \lesssim \|f\|_{L^q_{-\theta} L^r_{-\theta} L^p_{-\theta}}, \quad 1/\xi = n (1/p - 1/r), \tag{4.2}
\]
where \((\tilde{q}, \tilde{r}, p, a)\) is an admissible exponent quadruplet. In view of the real method with interpolation functor \(K_{\theta,c}\) and propositions 4.1–4.2 we obtain (2.6), where
\[
0 < \theta < 1, \quad \frac{1}{c} = \frac{1}{a} - \frac{\theta}{r},
\]
\[
\frac{1}{a} = \frac{1 - \theta}{a} + \frac{\theta}{r},
\]
\[
\frac{1}{b} = \frac{1 - \theta}{a} + \frac{\theta}{p}.
\]
We can eliminate $\theta$ to obtain that estimate (2.6) holds under the conditions
\[
\frac{1}{q} + \frac{n}{r} = \frac{n}{b}, \quad \text{HM}(p, r) = \text{HM}(b, c) = a,
\]
\[a < r < r^*(a), \quad p^*(a) < p < a, \quad p < b < a < c < r,
\]
or if $a = b = c$ and $(q, r, p, a)$ is an admissible exponent quadruplet (corresponds to $\theta = 0$).

We next prove the two cases of theorem 2.3 when either $\tilde{q} = \infty$ or $q = \infty$, that is estimates (2.4)
\[
\|W(t)F\|_{L^q_t L^r_t L^\infty_x} \lesssim \|F\|_{L^1_t L^{\tilde{r}'}_t L^{\tilde{p}'}_x},
\]
(2.5)
\[
\|W(t)F\|_{L^q_t L^r_t L^\infty_x} \lesssim \|F\|_{L^{q'}_t L^{r'}_x L^{\tilde{p}'}_x},
\]
where $(q, r, p)$ and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two jointly acceptable exponent triplets.

Proof. The homogeneous estimate (2.6) implies the inhomogeneous estimate (2.4), see [13, theorem 2.9] for proof. We need only to verify that the range of validity of (2.4) is precisely described by the requirement that $(q, r, p)$ and $(\tilde{q}, \tilde{r}, \tilde{p})$ be two jointly acceptable exponent triplets. In the table below we give the conditions of a mixed KT-admissible 5-tuple in the second column and we reformulate them as in the definition of two jointly acceptable exponent triplets in the first column.

| $\frac{1}{q} + \frac{1}{\infty} = n\left(1 - \frac{1}{r} - \frac{1}{r}\right)$, | $\frac{1}{q} + \frac{n}{r} = \frac{n}{b}$, |
| $\frac{1}{q} < n\left(\frac{1}{p} - \frac{1}{r}\right)$, | $\frac{1}{b} < \frac{1}{p}$, |
| $q = \infty$, $1 \leq p = r \leq \infty$, | $a = b = c = p = r$, |
| $\text{HM}(p, r) = \text{HM}(\tilde{p}', \tilde{r}')$, \quad $\text{HM}(p, r) = \text{HM}(b, c)$, | $\tilde{q} = \infty$, $(q, r, p)$ is KT-admissible |
| $a < r < r^*(a)$ | $a = b = c$, |
| $p^*(a) < p < a$, | $p < b < a < c < r$. |

Estimate (2.5) follows from (2.4) by duality, see [13, Duality lemma] for proof. Note also that the tricky cases which cannot be treated by duality, e.g. the dual estimate of (2.4), where $\tilde{p}' = \infty$, $\tilde{r}' < \infty$, are precluded by our definition of joint acceptability in the case when $\tilde{q} = \infty$.

5. Estimates for Acceptable Exponents

5.1. Local Inhomogeneous Estimates. Following Foschi [8], we want to find the range of local estimates for $W(t)$ that are invariant to the scaling
\[
(5.1) \quad \|W(t)[\chi_{\lambda I} F]\|_{L^q_t(\lambda I; L^r_t L^\infty_x)} \lesssim \lambda^{\frac{1}{q} + \frac{1}{q} - n\left(\frac{1}{r} - \frac{1}{r}\right)} \|F\|_{L^{q'}(\lambda J; L^{r'}_t L^{\tilde{p}'}_x)}, \quad \forall \lambda > 0,
\]
where $I$ and $J$ are two unit intervals separated by a unit distance and $\chi_{\lambda I}$ is the characteristic of the scaled-out interval $\lambda I$. Note that
\[
\frac{1}{q} + \frac{1}{q} - n\left(\frac{1}{1/r} - \frac{1}{1/r}\right) = \frac{1}{q} + \frac{n}{q} - \frac{n}{2} \left(\frac{1}{p} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r}\right),
\]
due to the scaling condition \( \text{HM}(p, r) = \text{HM}(\tilde{p}', \tilde{r}') \). We set

\[
\beta(q, r, \tilde{q}, \tilde{r}) = \frac{1}{q} + \frac{1}{\tilde{q}} - n \left( 1 - \frac{1}{r} - \frac{1}{\tilde{r}} \right).
\]

**Lemma 5.1.** Estimate \((5.1)\) holds for any two non-endpoint KT-admissible triplets \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) with \( a = \tilde{a}' \).

**Proof.** The proof follows trivially from theorem 2.1 due to the fact that \( \beta(q, r, \tilde{q}, \tilde{r}) = 0 \) under the hypothesis of the lemma. \(\square\)

**Lemma 5.2.** Estimate \((5.1)\) holds with \((q, r, p) = (\infty, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p}) = (\infty, \tilde{r}', \tilde{r}')\), where \( 1 \leq p \leq r \leq \infty \).

**Proof.** Due to the decay estimate \((5.3)\) we have that

\[
\sup_{t \in \mathcal{M}} \| W(t) [\chi_{\lambda J} F] \|_{L^q L^r} \lesssim \sup_{t \in \mathcal{M}} \int_{\lambda J} \| F(\tau) \|_{L^q L^r}^{\frac{1}{q} + \frac{1}{\tilde{q}} - n \left( 1 - \frac{1}{r} - \frac{1}{\tilde{r}} \right)} d\tau
\]

\[
\lesssim \lambda^{\beta(\infty, r, \infty, r')} \| F \|_{L^1(\lambda J; L^q L^r)}.
\]

\(\square\)

**Lemma 5.3.** Whenever \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) are exponent triplets for which estimate \((5.1)\) holds, we have that \((5.1)\) also holds with \((Q, r, p)\) and \((\tilde{Q}, \tilde{r}, \tilde{p})\), where \( 1 \leq Q \leq q, 1 \leq \tilde{Q} \leq \tilde{q} \).

**Proof.** A trivial application of Hölder’s inequality

\[
\| W(t) [\chi_{\lambda J} F] \|_{L^q L^r(\lambda J; L^q L^r)} \lesssim \lambda^{\frac{1}{q} - \frac{1}{r}} \| W(t) [\chi_{\lambda J} F] \|_{L^q(\lambda J; L^q L^r)}
\]

\[
\lesssim \lambda^{\beta(Q, r, \tilde{q}, \tilde{r})} \| F \|_{L^q(\lambda J; L^q L^r)} \lesssim \lambda^{\beta(Q, r, \tilde{q}, \tilde{r})} \| F \|_{L^q(\lambda J; L^q L^r)}.
\]

\(\square\)

Let us define the range of validity of the local estimates \((5.1)\) as the set \( E \) in \( \mathbb{R}^6 \). Each point in \( E \) corresponds to a 6-tuple of exponents \((1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})\). Below we find the convex hull \( E^* \) (\( E^* \subset E \)) of the points in \( \mathbb{R}^6 \) that correspond to the estimates in the three lemmas above. We shall call any point or collection of points in \( E \) acceptable.

**Lemma 5.4** (Local inhomogeneous estimates). Estimate \((5.1)\) holds whenever the exponent triplets \((q, r, p)\), \((\tilde{q}, \tilde{r}, \tilde{p})\) satisfy the following conditions

\[(5.2)\quad 0 \leq \frac{1}{q} + \frac{1}{r} + \frac{1}{s}, \quad \frac{1}{p} \leq \frac{1}{r} + \frac{1}{\tilde{r}}, \quad \text{HM}(p, r) = \text{HM}(\tilde{p}', \tilde{r}'), \]

\[(5.3)\quad \frac{1}{r} - \frac{1}{\tilde{r}} - \frac{1}{r} + \frac{1}{p} \leq \frac{2}{na}, \quad \frac{1}{r} - \frac{1}{\tilde{r}} + \frac{1}{p} < \frac{2}{na},
\]

\[(5.4)\quad \frac{n - 1}{p' r} < \frac{n}{r}, \quad \frac{n - 1}{p' r} < \frac{n}{r},
\]

or if the point \((1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})\) lies inside one of the cubes in \( \mathbb{R}^6 \) below

\[(5.5)\quad (\kappa, 0, \mu, 1 - \mu, 1), \quad 0 \leq \kappa, \mu, \nu \leq 1,
\]

\[(5.5)\quad (\kappa, 1 - \mu, 1, \nu, 0, \mu), \quad 0 \leq \kappa, \mu, \nu \leq 1.
\]
Proof. We apply the Riesz-Thorin convexity theorem to interpolate between the already proven local estimates. In essence, we find the convex hull of the locally acceptable sets associated with lemmas 5.1 and 5.2 and then expand that set by the rule given in lemma 5.3.

The set of acceptability $S_1$ of the local estimates in lemma 5.1 is given by the system
\begin{align*}
0 &\leq \frac{1}{q}, \frac{1}{\tilde{q}}, \frac{1}{p}, \frac{1}{\tilde{p}}, \frac{1}{r}, \frac{1}{\tilde{r}} \leq 1, \\
\frac{1}{q} & = \frac{n}{2} \left( \frac{1}{p} - \frac{1}{r} \right), \quad \frac{1}{\tilde{q}} = \frac{n}{2} \left( \frac{1}{\tilde{p}} - \frac{1}{\tilde{r}} \right), \\
\frac{1}{r} + \frac{1}{p} + \frac{1}{\tilde{r}} + \frac{1}{\tilde{p}} & = 2, \\
\frac{n-1}{p} & < \frac{n+1}{r}, \quad \frac{n-1}{\tilde{p}} < \frac{n+1}{\tilde{r}},
\end{align*}
(5.6)

or if $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \{B, C\}$, $B = (0, 0, 0, 1, 1), C = (0, 1, 1, 0, 0)$.

Note that $S_1$ is a convex polyhedron in $\mathbb{R}^6$ and the two points $B$ and $C$ lie on its boundary. The set of acceptability $S_2$ of the local estimates in lemma 5.2 is the convex hull (in fact a triangle) of the three points
\begin{align*}
A & = (0, 0, 1, 0, 0, 1), \\
B & = (0, 0, 0, 0, 1, 1), \\
C & = (0, 1, 1, 0, 0, 0).
\end{align*}
(5.10)

Vertices $B$ and $C$ are already included in $S_1$, thus it would suffice to take only the vertex $A$. Hence, we obtain the following set
\begin{align*}
\frac{1}{Q} & = \frac{\theta}{q}, \quad \frac{1}{R} = \frac{\theta}{r}, \\
\frac{1}{Q} & = \frac{\theta}{\tilde{q}}, \quad \frac{1}{R} = \frac{\theta}{\tilde{r}}, \\
\frac{1}{P} & = 1 - \theta + \frac{\theta}{p}, \\
\frac{1}{\tilde{P}} & = 1 - \theta + \frac{\theta}{\tilde{p}}, \quad 0 \leq \theta \leq 1,
\end{align*}
where $(1/Q, 1/R, 1/P, 1/\tilde{Q}, 1/\tilde{R}, 1/\tilde{P})$ are the coordinates of the new set $S_3$ written in terms of $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$ and $\theta$. Of course, to $S_3$ we must also add the line segments $[A, B]$ and $[A, C]$. We shall treat this case separately at the end.

Finally, we apply the rule given in lemma 5.3 and thus we replace the equations for $Q$ and $\tilde{Q}$ above with the following inequalities
\begin{align*}
1 &\geq \frac{1}{Q} \geq \frac{\theta}{q}, \\
1 &\geq \frac{1}{\tilde{Q}} \geq \frac{\theta}{\tilde{q}},
\end{align*}
plus the restrictions
\begin{align*}
\frac{1}{r} &\leq \frac{1}{p}, \quad \frac{1}{\tilde{r}} \leq \frac{1}{\tilde{p}},
\end{align*}
(5.11)

which were implicitly assumed in (5.7).

1. We first eliminate $q$ and $\tilde{q}$ from the system for $S_1$ to obtain
\begin{align*}
\frac{1}{Q} &\geq \frac{n}{2} \left( \frac{\theta}{p} - \frac{\theta}{r} \right), \quad \Leftrightarrow \quad \frac{1}{Q} \geq \frac{n}{2} \left( \theta - 1 + \frac{1}{P} - 1 \frac{1}{R} \right), \\
\Leftrightarrow \quad \theta &\leq \frac{1}{P^\nu} + \frac{1}{R} + \frac{2}{nQ}.
\end{align*}

Similarly,
\begin{align*}
\theta &\leq \frac{1}{P^\nu} + \frac{1}{R} + \frac{2}{nQ}, \quad \frac{1}{Q}, \frac{1}{\tilde{Q}} \leq 1.
\end{align*}
2. As expected, condition \((5.8)\) is invariant
\[
\frac{1}{R} + \frac{1}{P^r} + \frac{1}{R} + \frac{1}{P} = 2.
\]

3. Reworking condition \((5.9)\), we obtain
\[
\theta < n + \frac{1}{1 - n} + \frac{1}{P^r}, \quad \theta < n + \frac{1}{1 - n} + \frac{1}{P^r}.
\]

4. Condition \((5.11)\) is replaced by
\[
\frac{1}{P^r} + \frac{1}{R^r} \leq \theta, \quad \frac{1}{P^r} + \frac{1}{R^r} \leq \theta.
\]

5. Finally, conditions \((5.10)\) are transformed into
\[
\frac{1}{P^r}, \frac{1}{P^r} \leq \theta, \quad 0 \leq \frac{1}{Q}, \frac{1}{Q}, \frac{1}{P^r}, \frac{1}{R}, \frac{1}{P} \leq 1.
\]

6. We group all conditions obtained in the previous 5 steps according to their type
\[(5.12)\]
\[
0, \frac{1}{P^r}, \frac{1}{P^r}, \frac{1}{P^r}, \frac{1}{P^r} + \frac{1}{R} + \frac{1}{P} \leq \theta.
\]
\[(5.13)\]
\[
\theta \leq \frac{1}{R} + \frac{1}{P^r} + \frac{2}{nQ}, \quad \frac{1}{R} + \frac{1}{P^r} + \frac{2}{nQ}, \quad \frac{n + 1}{n - 1 R} + \frac{1}{P^r}, \quad \frac{n + 1}{n - 1 R} + \frac{1}{P^r} = 1.
\]
\[(5.14)\]
\[
0 \leq \frac{1}{Q}, \frac{1}{Q}, \frac{1}{P^r}, \frac{1}{R}, \frac{1}{P^r} \leq 1, \quad \frac{1}{P^r} + \frac{1}{R} + \frac{1}{P} = 2.
\]

7. We discard the redundant conditions like
\[
0, \frac{1}{P^r}, \frac{1}{P^r}, \frac{1}{P^r} \leq \theta,
\]
which are all weaker than the other two in \((5.12)\).

There exists \(\theta\) solving all inequalities in \((5.12), (5.13)\), if and only if every quantity in \((5.12)\) is bounded from above by any quantity in \((5.13)\). Thus we form all possible combinations between the quantities in the two types of (reduced) inequalities to obtain the following set of conditions
\[
\frac{1}{R} + \frac{1}{P^r} \leq \frac{1}{R} + \frac{1}{P^r} + \frac{2}{nQ}, \quad \frac{1}{R} + \frac{1}{P^r} \leq \frac{1}{R} + \frac{1}{P^r} + \frac{2}{nQ},
\]
\[
\frac{1}{R} + \frac{1}{P^r} \leq \frac{n + 1}{n - 1 R} + \frac{1}{P^r}, \quad \frac{1}{R} + \frac{1}{P^r} \leq \frac{n + 1}{n - 1 R} + \frac{1}{P^r},
\]
\[
\frac{1}{R} \leq \frac{1}{P}, \quad \frac{1}{R} \leq \frac{1}{P^r}.
\]

8. We apply the rule given in lemma \((5.9)\) to the two line segments \([A, B]\) and \([A, C]\) to obtain the following two cubes in \(\mathbb{R}^6\)
\[(5.15)\]
\[
(\mu, 0, \nu, 1 - \kappa, 1), \quad 0 \leq \mu, \nu, \kappa \leq 1,
\]
\[(\mu, 1 - \kappa, 1, \nu, 0, \kappa), \quad 0 \leq \mu, \nu, \kappa \leq 1.
\]

Hence, the computation of the set \(\mathcal{E}^*\) is finished. \(\square\)
5.2. **Global Inhomogeneous Estimates.** The key point in the proof of the global inhomogeneous estimates is to decompose the bilinear operator $B(F,G)$ into a series of time-localized operators to which we can apply the local inhomogeneous estimates. We denote by $\Omega$ the region $\{(t,s)|s<t\}$ on the $t,s$-coordinate plane.

**Definition 5.5.** We call any positive integer that is a power of two a dyadic number. Furthermore, we call a square $Q$ in $\mathbb{R}^2$ dyadic if its side length is a dyadic number and the coordinates of its vertices are integer multiples of dyadic numbers.

Applying Whitney’s decomposition to partition $\Omega$, we obtain the family $\mathcal{O}$ of disjoint dyadic squares $Q$ (overlapping on the sides is not excluded) such that the distance between any square $Q \in \mathcal{O}$ and the boundary of $\Omega$ ($\{(t,s)|t=s\}$) is approximately proportional to the diameter of $Q$, see figure 1. By $\mathcal{O}_\lambda$ we denote the collection of all squares in $\mathcal{O}$ whose side length is $\lambda$.

![Figure 1. Whitney’s decomposition for the region $s<t$](image)

Thus we obtain the representations

$$
\Omega = \bigcup_{\lambda} \bigcup_{Q \in \mathcal{O}_\lambda} \Omega, \quad B(F,G) = \sum_{\lambda} \sum_{Q \in \mathcal{O}_\lambda} B_Q(F,G),
$$

where

$$
B_Q(F,G) = \int \int_Q \langle U^*(s)F(s), U^*(t)G(t) \rangle dsdt.
$$

The benefit of the above decomposition is that whenever $Q = J \times I$ and $Q \in \mathcal{O}_\lambda$ we have

$$
\lambda = |I| = |J| \sim \text{dist}(\Omega, \partial \Omega) \sim \text{dist}(I, J).
$$

This allows us to obtain the estimate

$$
|B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \|F\|_{L^q(J;L^\rho I)} \|G\|_{L^\rho(I;L^\rho J)},
$$

whenever $(q,r,p)$ and $(\tilde{q},\tilde{r},\tilde{p})$ are such that we have the scaling (5.1).

**Lemma 5.6.** Suppose $\frac{1}{p} + \frac{1}{\tilde{p}} \geq 1$. Then

$$
\sum_{Q \in \mathcal{O}_\lambda, Q = J \times I} \|f\|_{L^p(J)} \|g\|_{L^p(I)} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^p(\mathbb{R})}.
$$
Proof. The lemma follows directly from the inequality
\[ \sum_j |a_j b_j| \leq \left( \sum_j |a_j|^p \right)^{\frac{1}{p'}} \left( \sum_j |b_j|^q \right)^{\frac{1}{q'}}, \]
which holds in the range \( \frac{1}{p'} + \frac{1}{q'} \geq 1 \), and the fact that for each dyadic interval \( I \) there are at most two dyadic squares in \( \mathcal{O}_\lambda \) with side \( I \).

In the next estimate we relate the scaling argument with the dyadic decomposition.

**Corollary 5.7.** If \( \frac{1}{q} + \frac{1}{q'} \leq 1 \), then
\[ \sum_{Q \in \mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \|F\|_{L^{q'}(R;L^{r'}_x L^{r''}_v)} \|G\|_{L^{q'}(R;L^{r'}_x L^{r''}_v)}. \]  

**Proof.** In view of (5.18)
\[ \sum_{Q \in \mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \sum_{Q \in \mathcal{O}_\lambda} \|F\|_{L^{q'}(J;L^{r'}_x L^{r''}_v)} \|G\|_{L^{q'}(J;L^{r'}_x L^{r''}_v)}. \]
An application of lemma 5.6 concludes the proof. \( \square \)

We denote by \( l^p_s \) the space of number sequences with a norm
\[ \|\{a\}_{j \in \mathbb{Z}}\|_{l^p_s} = (2^{js} |a_j|^p)^{1/p}, \quad 1 \leq p < \infty, \]
\[ \|\{a\}_{j \in \mathbb{Z}}\|_{l^\infty_s} = \sup_{j \in \mathbb{Z}} 2^{js} |a_j|, \quad p = \infty. \]

**Lemma 5.8** (See theorem 5.6.1 in [2]). We have the identity
\[ (l^{s_0}_s, l^{s_1}_s)_{\theta,1} = l^1_s, \]
where \( s_0, s_1 \in \mathbb{R}, \) \( s_0 \neq s_1 \) and \( s = (1 - \theta)s_0 + \theta s_1. \)

**Lemma 5.9** (See pp. 76-77 in [2]). Suppose that \((A_0, A_1), (B_0, B_1), (C_0, C_1)\) are interpolation couples and that the bilinear operator \( T \) acts as a bounded transformation as indicated below:
\[ T : A_0 \times B_0 \rightarrow C_0, \]
\[ T : A_0 \times B_1 \rightarrow C_1, \]
\[ T : A_1 \times B_0 \rightarrow C_1. \]
If \( \theta_0, \theta_1 \in (0, 1) \) and \( p, q, r \in [1, \infty] \) such that \( 1/p + 1/q \geq 1 \), then \( T \) also acts as a bounded transformation in the following way:
\[ T : (A_0, A_1)_{\theta_0,p,r} \times (B_0, B_1)_{\theta_1,q,r} \rightarrow (C_0, C_1)_{\theta_0 + \theta_1, r}. \]

After this preparatory work, we can now begin with the crux of the main argument of this section. Consider the bilinear operator \( T : L^{q'}_1 \times L^{r'}_x L^{r''}_v \rightarrow l^\infty_s \), defined by the formula
\[ T(F,G) = \{b_\lambda\}_{\lambda \in 2^s} = \left\{ \sum_{Q \in \mathcal{O}_\lambda} |B_Q(F,G)| \right\}_{\lambda \in 2^s}. \]
Our aim is to find the range of exponent triplets \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) where we have the estimate
\[
\|{b_\lambda}\|_2 \lesssim \|F\|_{L^q_t L^r_x L^p_v} \|G\|_{L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v}, \quad \forall F \in L^q_t L^r_x L^p_v, \forall G \in L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v.
\]

Suppose that \((1/q, 1/\tilde{q}) \in \Delta\), where \(\Delta = \{1/q > 0, 1/\tilde{q} > 0, 1/q + 1/\tilde{q} < 1\}\), and that \((q, r, p), (\tilde{q}, \tilde{r}, \tilde{p}) \in \mathcal{E}^*\). Then in virtue of corollary 5.7 we have the estimate
\[
|b_\lambda| \lesssim \lambda^{\beta(q, r; \tilde{q}, \tilde{r})} \|F\|_{L^q_t L^r_x L^p_v} \|G\|_{L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v},
\]
or in other words \(\{b_\lambda\} \in l^{\infty}_{\beta(q, r; \tilde{q}, \tilde{r})}\). Since \(\Delta\) is an open set (triangle) on the \((1/q, 1/\tilde{q})\)-coordinate plane, we can always find a small enough open neighborhood of points in \(\Delta\) around \((1/q, 1/\tilde{q})\). Let us set
\[
1/q_0 = 1/q + \epsilon, \quad 1/q_1 = 1/q - 3\epsilon, \quad 1/q_1 = 1/\tilde{q} - 3\epsilon.
\]

Suppose that \(\epsilon > 0\) is small enough so that \((1/q_0, 1/q_0), (1/q_1, 1/q_1) \in \Delta \cap \mathcal{E}^*\). Suppose also that
\[
1/q + 1/\tilde{q} = n (1 - 1/r - 1/\tilde{r}). \tag{5.20}
\]

Then we have that \(\beta(q_0, r, \tilde{q}_0, \tilde{r}) = 2\epsilon\), and \(\beta(q_1, r, \tilde{q}_1, \tilde{r}) = \beta(q_0, r, \tilde{q}_0, \tilde{r}) = -2\epsilon\). Thus we obtain the maps
\[
T : L^q_t L^r_x L^p_v \times L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v \to l^{\infty}_{2\epsilon},
\]
\[
T : L^q_t L^r_x L^p_v \times L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v \to l^{\infty}_{2\epsilon},
\]
\[
T : L^q_t L^r_x L^p_v \times L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v \to l^{\infty}_{2\epsilon},
\]
are bounded. In virtue of lemma 5.9 we have that the map
\[
T : (L^q_t L^r_x L^p_v, L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v)_{1/4, \tilde{q}} \times (L^q_t L^r_x L^p_v, L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v)_{1/4, \tilde{q}} \to (l^{\infty}_{2\epsilon}, l^{\infty}_{2\epsilon})_{1/2, 1}
\]
is also bounded. Finally, in view of the well-known interpolation identities of the Lorentz spaces and that of lemma 5.8 this simplifies to
\[
T : L^q_t L^r_x L^p_v \times L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v \to l^1,
\]
QED.

Now let us recapitulate all conditions that we have imposed so far on the exponents. We have the conditions of the local estimates (set \(\mathcal{E}^*\)) plus the scaling condition (5.20). Note that conditions (5.8) together with (5.20) are equivalent to \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) being KT-acceptable. Let us also note that the two locally acceptable cubes in (5.15) give rise to the two globally acceptable sets (cubic sections) \(\Sigma_1\) and \(\Sigma_2\) in definition 4.3.

Thus, in view of the two inhomogeneous estimates that we proved in section 4.2 all estimates in theorem 2.3 are now proven.

6. Sharpness of the Estimates

In this section we present some counterexamples to the Strichartz estimates (1.10), (1.13) for \(U(t)\) and the inhomogeneous estimates (1.19) for \(W(t)\). Thus we show sharpness of the estimates that we stated in the three theorems of section 2.
6.1. **Geometric interpretation of the Strichartz estimates for** $U(t)$. Let us begin with the case $n = 1$. We consider the velocity averages $A$ of the kinetic transport propagator $U(t)$, that is

$$A[f](t,x) := \int_{-\infty}^{\infty} f(x - tv, v) dv.$$ 

It is not hard to see that

$$A[f] = \frac{1}{\sqrt{1 + t^2}} \int_{\gamma_{x,t}} f(l) dl,$$

where in the above line integral we integrate along the straight line $\gamma_{x,t}$ on the plane $Oxv$, passing through the point $X = (x,0)$ and having a gradient $-1/t$. Let $f(x,v) = \chi_Q$, where $\chi_Q$ is the characteristic of the measurable subset $Q$ of the plane $Oxv$.

Let us interpret $\|U(t)\chi_Q\|_{L^q_tL^p_xL^\infty_v}$ geometrically. By the above we have

$$\|U(t)\chi_Q\|_{L^q_tL^p_xL^\infty_v} = \frac{1}{\sqrt{1 + t^2}} \sup_{l \in \mathbb{R}} \int_{\gamma_{x,t}} \chi_Q(l) dl,$$

or in other words $\|U(t)\chi_Q\|_{L^q_tL^p_xL^\infty_v}$ is the supremum of the (line) measure of all intersections of $Q$ with straight lines in a fixed direction (times a factor in $t$).

**Lemma 6.1.** There exist an open set $Q$ on the plane of arbitrary small positive measure $\epsilon$ that contains in its interior a unit line segment in every direction. The sets $Q(\epsilon)$ can be chosen to be uniformly bounded with respect to $\epsilon$.

**Proof.** We repeat the construction of a Besicovitch set, see Besicovitch [3]. As we do not need to turn the line segment continuously, we drop from the construction the joints between the triangles. This allows us to keep $Q$ bounded regardless of $\epsilon$. We iterate a finite number of times, thus the resulting set is a polygon, a modification of the so called Perron-Schoenberg tree. For $Q$ we choose any open set that properly contains the latter so that the measure of their difference is small enough. We shall also call the set $Q$ a Besicovitch set. \hfill $\Box$

It is now clear that the one-dimensional endpoint

$$\|U(t)f\|_{L^q_tL^p_xL^\infty_v} \lesssim \|f\|_{L^a_xL^b_v}, \tag{6.1}$$

fails on the characteristic functions of Besicovitch sets (thus we recover the result of [9] by different means).

**Lemma 6.2.**

- The Strichartz estimate

$$\|U(t)f\|_{L^q_tL^p_xL^c_v} \lesssim \|f\|_{L^a_xL^b_v}, \tag{1.0}$$

where $0 < a < \infty$, $1 \leq q, p \leq \infty$, fails for some $f \in L^a(\mathbb{R}^{2n})$.

- The Strichartz estimate

$$\|U(t)f\|_{L^q_tL^p_xL^c_v} \lesssim \|f\|_{L^b_xL^c_v}, \tag{1.8}$$

where $0 < q, p, b, c \leq \infty$, fails for some $f \in L^b_xL^c_v$, apart from the trivial case $q = p = b = c = \infty$. 

The Strichartz estimate

\[ \|W(t)F\|_{L_t^q L_x^r L_v^s} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{s'}} , \]

where \( 1 \leq q, \tilde{q}, p, \tilde{r} \leq \infty, 1 < \tilde{p} \), fails for some \( F \in L_t^q L_x^r L_v^s \). Moreover, if \( \tilde{p} = 1 \), estimate (1.5) holds for all \( F \in L_t^q L_x^r L_v^s \) if and only if \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) are two jointly KT-acceptable triplets, that is if and only if \((1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \Sigma_1 \cup B \), see definition 1.6 when \( n = 1 \), and the same up to an endpoint in \( n > 1 \). The only unresolved estimate of this kind, therefore, is (1.22) in \( n > 1 \).

The Strichartz estimate

\[ \|W(t)F\|_{L_t^1 L_x^1 L_v^\infty} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{s'}}, \]

where \( 1 \leq q, \tilde{q}, p, \tilde{r} \leq \infty, 1 < p \), fails for some \( F \in L_t^q L_x^r L_v^s \). Moreover, if \( p = 1 \), estimate (1.6) holds for all \( F \in L_t^q L_x^r L_v^s \) if and only if \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) are two jointly KT-acceptable triplets, that is if and only if \((1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \Sigma_2 \cup C \), see definition 1.6 when \( n = 1 \), and the same up to an endpoint in \( n > 1 \). The only unresolved estimate of this kind, therefore, is (1.23) in \( n > 1 \).

Proof. We begin with the case \( n = 1 \). Let us first consider the homogeneous estimates (1.6) and (1.8). We set \( \chi = \chi_Q \), where \( Q \) is a Besicovitch set on the plane \( Oxv \) of measure \( \epsilon > 0 \). It suffices to consider the case \( c < \infty \), the case \( c = \infty \) is trivial since then we must have \( q = p = b = c = \infty \), see section 6.3. By Hölder, \( \|\chi\|_{L_t^q L_x^r L_v^\infty} \lesssim \|\chi\|_{L_t^q L_x^r L_v^\infty} = \epsilon^{1/c} \), thus disproving (1.8) is reduced to disproving (1.6).

In view of the power invariance (1.3), we can always assume that \( p = 1 \) in (1.6). But then we can use the argument that we produced to disprove (6.1) above.

To construct a counterexample for \( W(t) \) we consider an inhomogeneous term \( F(t, x, v) = \chi(t \in [0, 1]; (x, v) \in Q) \), where \( Q \) is a Besicovitch set on the plane \( Oxv \) of measure \( \epsilon > 0 \). Then \( F \) is a characteristic function of a set in \( \mathbb{R}^{2+1} \) with measure \( \epsilon \). Therefore, as above estimate (1.4) fails whenever \( \tilde{p} > 1 \). The case when \( \tilde{p} = 1 \) has two subcases. The first is when \( 1/q + 1/\tilde{q} < 1 \), then estimate (1.4) holds in view of theorem 2.3 set \( \Sigma_1 \). In the endpoint case \( 1/q + 1/\tilde{q} = 1 \) we have that \( r = \tilde{r} = \infty, p = \tilde{p} = 1 \). These estimates would fail if we can find an explicit \( F \neq 0 \) satisfying

\[ \|W(t)F\|_{L_t^\infty L_x^1 L_v^\infty} \gtrsim \int_0^t \frac{1}{t-s} \|F(s)\|_{L_t^\infty L_x^\infty} ds = \infty, \]

\[ \text{supp } \|F(s)\|_{L_t^\infty L_x^\infty} \subseteq [0, 1], \]

for example. For simplicity we can choose \( F \) to be of product type, i.e. \( F(t, x, v) = \phi(t) \psi(x) g(v) \) and set \( \phi(t) = \chi_{[0,1]}, g(v) \equiv 1 \in L^\infty(\mathbb{R}) \), and choose any \( \psi \in L^1 \), \( \psi \geq 0 \). We have

\[ \int_{-\infty}^{\infty} W(t)Fdv = \int_0^t \int_{-\infty}^{\infty} \phi(s) \psi(x - (t-s)v) dv ds \]

\[ \geq \|g\|_{L^\infty} \|\psi\|_{L^1} \int_0^t \frac{1}{t-s} \phi(s) ds, \]

which proves the desired inequality. Note that the change of order of integration above is crucial for this argument. Consequently, the argument works only for \( p = 1 \).
By duality, see the duality lemma in [13], estimate (1.9) is equivalent to estimate (1.9'). And finally, when in \( n > 1 \) we can reproduce the same arguments by replacing \( Q \) with the product set \( Q^n = Q \times Q \times \ldots Q \).

\[ \square \]

Remark 6.3. Note that the problem of finding all possible Strichartz estimates for the operators \( U(t) \) and \( W(t) \) is now completely solved in \( n = 1 \) in view of counterexamples in the previous lemma that covered all endpoint cases in that dimension, and the counterexamples in the following subsections, which cover all other non-endpoint cases.

Let us now try finding a geometric interpretation of \( \| U(t) \chi_Q \|_{L^q_t L^r_x L^v} \), where \( 1 \leq r < \infty \). It is not hard to see that

\[
\int_{-\infty}^{\infty} \int_{\gamma_{x,t}} \chi_Q(l) dldx = \sqrt{1 + t^2} \int_{\mathbb{R}^2} R(\pi/2 - \theta) \chi_Q(l, m) dldm,
\]

where by \( R(\psi) \) we denote rotation by an angle \( \psi \) and \( \cot \theta = -t \). Therefore, if we denote

\[
N_r(\theta, Q) = \| R(\pi/2 - \theta) \chi_Q \|_{L^L_x L^V_t},
\]

we obtain the representation

\[
\| U(t) \chi_Q \|_{L^q_t L^r_x L^v} = \left\| \left( \frac{1}{\sqrt{1 + t^2}} \right)^{1/r'} N_r(\theta, Q) \right\|_{L^q_t\mathbb{R}}.
\]

Note that when the exponent triplet \( (q, r, 1) \) is KT-admissible in \( n = 1 \), that is \( 1/q = 1/2(1/p - 1/r) \) we have that

\[
\| U(t) \chi_Q \|_{L^q_t L^r_x L^v} = \| N_r(\theta, Q) \|_{L^q(0, \pi)}.
\]

We can generalize the above idea to any dimension \( n \) by considering the product \( Q^n = Q \times Q \times \ldots Q \). By repeating the argument above we obtain

\[
\| U(t) \chi_Q^n \|_{L^q_t L^r_x L^v} = \left\| \left( \frac{1}{\sqrt{1 + t^2}} \right)^{n/r'} N_r^n(\theta, Q) \right\|_{L^q_t\mathbb{R}}.
\]

Let \( Q \) be the unit circle in the plane \( O x_1 v_1 \). Then obviously \( N_r(\theta, Q^n) = \text{const} \) for all \( \theta \in [0, \pi] \). Consequently, the norm above is finite only if \( r \geq 1, \ qn/r' > 1 \).

By the power invariance (1.3), we generalize to any admissible \( (q, r, p) \) and obtain

the restrictions

\[
(6.2) \quad r \geq a, \quad \frac{1}{q} < n \left( \frac{1}{p} - \frac{1}{r} \right), \quad \text{or} \quad q = \infty, \quad 1 \leq p = r \leq \infty,
\]

to the range of validity of estimate (1.6). The restrictions above trivially imply that \( p \leq r \).

6.2. Homogeneous Estimates. By scaling, see lemma 3.5, estimate (1.6)

\[
\| U(t) f \|_{L^q_t L^r_x L^v} \lesssim \| f \|_{L^a_x v}, \quad \forall f \in L^a_x v,
\]

can only hold if

\[
\frac{1}{q} + \frac{n}{r} = n/a, \quad a = \text{HM}(p, r).
\]
By the $TT^*$-principle, estimate \((1.6)\) with $a = 2$ is equivalent to

$$
\|TT^* F\|_{L_t^1 L_x^\infty L_v^c} \lesssim \|F\|_{L_t^b L_x^c L_v^c}, \quad \forall F \in L_t^q L_x^r L_v^s.
$$

By translation invariance in $x$, see proposition \((5.7)\) $r \geq r'$, or equivalently $r \geq 2$. By the power invariance \((1.3)\) we generalize the case $a = 2$ to any $0 < a < \infty$, the respective condition is $r \geq a$.

We now show the upper bound on $r$, that is $r \leq r^*(a)$. To that end, we first consider again the special case $a = 2$. Note that the condition $q \geq 2$ equivalent to the one at hand. By translation invariance in $t$, see proposition \((5.7)\) $q \geq q'$, or equivalently $q \geq 2$. By the power invariance \((1.3)\) we generalize the case $a = 2$ to any $0 < a < \infty$, the respective condition is $q \geq a$, or equivalently $r \leq r^*(a)$. Thus in view of \((6.2)\), $p^*(a) \leq p \leq a \leq r \leq r^*(a)$.

### 6.3. Homogeneous estimates for mixed $L^p$-data

If we consider the mixed $L^p$-data estimate

\begin{equation}
\|U(t)f\|_{L_t^1 L_x^b L_v^c} \lesssim \|f\|_{L_t^b L_x^c L_v^c}, \quad \forall f \in L_t^b L_x^c L_v^c
\end{equation}

then most of the preceding section applies as well. By scaling, we have that the conditions

\begin{equation}
\frac{1}{q} + \frac{n}{r} = \frac{n}{b}, \quad \text{HM}(p, r) = \text{HM}(b, c) \overset{\text{def}}{=} a,
\end{equation}

are necessary. The conditions $p \leq a \leq r$ and $a \leq r$ carry over from the preceding section. We do not have, however, a suitable counterexample giving that exact upper bound $r^*(b, c)$ to $r$ in estimate \((1.8)\) in $n > 1$. The last condition we need to verify is that $b \leq c$. Estimate \((1.8)\) is equivalent to

$$
\|TT^* F\|_{L_t^1 L_x^b L_v^c} \lesssim \|F\|_{L_t^1 L_x^b L_v^c}, \quad \forall F \in L_t^1 L_x^b L_v^c.
$$

By duality, it is equivalent to

$$
\|TT^* F\|_{L_t^\infty L_x^c L_v^c} \lesssim \|F\|_{L_t^q L_x^c L_v^c}, \quad \forall F \in L_t^q L_x^c L_v^c.
$$

By the restrictions on the inhomogeneous estimates, see section \((6.4)\), we obtain $b' \geq c'$ or equivalently $b \leq c$. Thus we have that $p < b \leq a \leq c < r$ (follows from \((6.2)\) and \((6.3)\)) or $a = b = c = p = r$ (and $q = \infty$).

### 6.4. Global inhomogeneous estimates

In this section we present some counterexamples to the validity of the inhomogeneous estimates

\begin{equation}
\|W(t)F\|_{L_t^1 L_x^b L_v^c} \lesssim \|F\|_{L_t^q L_x^c L_v^c}.
\end{equation}

By scaling, see lemma \((3.5)\) we obtain that the conditions

\begin{equation}
\frac{1}{q} + \frac{1}{q'} = n \left(1 - \frac{1}{r} - \frac{1}{r'}\right), \quad \text{HM}(p, r) = \text{HM}(b', r') \overset{\text{def}}{=} a,
\end{equation}

are necessary.

Consider $F(t, x, v) = \chi (0 \leq t \leq 1, |x| \leq 1, |v| \leq 1)$. When $t \gg 1$ we have that

$$
TT^* F \approx W(t) F \approx \chi \left(|v - \frac{x}{t}| \leq \frac{1}{t}, |v| \leq 1\right) \approx \chi \left\{ v \sim \frac{1}{t}, x \sim t \right\}.
$$

Hence,

$$
\|W(t)F\|_{L_t^q L_x^b L_v^c} \sim t^{\frac{q}{q'} - \frac{n}{r'}}, \quad t \gg 1.
$$
It follows that \( \|W(t)F\|_{L_t^q L_x^r L_z^\sigma} < \infty \) only if
\[
(6.4) \quad \left(\frac{n}{r} - \frac{n}{p}\right) q < -1, \quad \text{or if } q = \infty, r = p.
\]

By duality, see the duality lemma of [13], (6.4) also applies to the dual exponents \((\hat{q}, \hat{r}, \hat{p})\). Thus we have that the conditions \( p \leq r \) and \( \hat{p} \leq \hat{r} \) are necessary for the validity of estimate (1.9). Indeed, that follows from the translation invariance of \( W(t) \) in \( t \) and \( x \) and proposition 3.7. We check this fact only for \( t \). Consider \( F_r(t) = F(t - \tau) \) and \( W(t)F_r \). We have
\[
\int_{-\infty}^{t} U(t - s) F(s - \tau) ds = \int_{\infty}^{t} U(t - \tau - \sigma) F(\sigma) d\sigma,
\]
or in other words \( W(t)F_r = W(t - \tau)F \), QED.

Thus we have fully verified the necessity of the condition that \((q, r, p)\) and \((\hat{q}, \hat{r}, \hat{p})\) be two jointly KT-acceptable exponent triplets (apart from some boundary cases, e.g. \( q = \infty \), etc.)

We do not have a suitable counterexample showing the necessity of condition (2.3)
\[
(2.3) \quad \frac{n - 1}{p'} < \frac{n}{r}, \quad \frac{n - 1}{\tilde{r}'} < \frac{n}{r}.
\]
However, we can show that the similar condition
\[
(2.3b) \quad \frac{n}{p'} < \frac{1}{q} + \frac{n}{\tilde{r}'}, \quad \frac{n}{\tilde{r}'} < \frac{1}{q} + \frac{n}{r},
\]
is sharp. Indeed, (2.3b) is a direct consequence of the two conditions (1.17), (1.19). Condition (2.3b) implies (2.3) whenever \( p' \leq \tilde{q} \) and \( \tilde{p}' \leq q \). Thus, if there are some other global inhomogeneous estimates for \( W(t) \) not included in theorem 2.3, they must belong to the range \( q < p' \) or \( q < \tilde{p}' \).

6.5. Local inhomogeneous estimates. In this section we show that the condition \( \text{HM}(p, r) = \text{HM}(\tilde{p}', \tilde{r}') \) is necessary for the validity of the local inhomogeneous estimates, e.g.
\[
(6.5) \quad \|W(t)F\|_{L_t^q(\mathbb{R}^n; L_x^r L_z^\sigma)} \lesssim \|F\|_{L_t^{q'}(\mathbb{R}^n; L_x^{r'} L_z^\sigma)}.
\]
This implies that we cannot perturbate the exponents \( p, r, \tilde{p}, \tilde{r} \) freely. Consequently, the method of Keel and Tao [12] that was successful in treating the endpoints for the wave and the Schrödinger equations does not apply in the present context.

Let us set \( F(t, x, v) = \chi(t \in [0, 1], (x, v) \in Q_R) \), where by \( Q_R \) we denote the cube of side length \( 2R \) with center at the origin of \( \mathbb{R}^{2n} \), that is
\[
Q_R = \{(x, v): \|x\|_\infty \leq R, \|v\|_\infty \leq R\},
\]
where \( \|x\|_\infty = \sup_{1 \leq i \leq n} |x_i| \), for \( x = (x_1, \ldots, x_n) \). Hence,
\[
\|F\|_{L_t^{q'}(\mathbb{R}^n; L_x^{r'} L_z^\sigma)} \sim R^{\frac{\tilde{p}'}{\tilde{r}'} + \tilde{\sigma}}.
\]
We set $\tau = t - s$, and consider the set $Q_R(\tau)$ given by
\[
\|x - (t - s)v\|_\infty \leq R, \quad \|v\|_\infty \leq R.
\]
Then, for $t \in [2, 3]$, $s \in [0, 1]$, and therefore $\tau \in [1, 3]$, we have the inclusions
\[
Q_{R/4} \subset Q_R(\tau) \subset Q_4R.
\]
Hence,
\[
\|W(t)F\|_{L^q_t([2,3];L^r_xL^p_v)} \sim R^{n/p + \frac{n}{\tilde{r}} - \frac{n}{p}}.
\]
We conclude that condition
\[
\frac{1}{r} + \frac{1}{p} = \frac{1}{\tilde{r}} + \frac{1}{p'}
\]
is necessary for the validity of the local estimates (6.5).

7. SOME OPEN QUESTIONS

The following estimates/questions remain unresolved.

- In $n > 1$, the endpoint
  \begin{equation}
  \|U(t)f\|_{L^q_tL^r_xL^{\tilde{p}}_v} \lesssim \|f\|_{L^1_x}.
  \end{equation}

- In $n > 1$, inhomogeneous endpoints of the type
  \begin{equation}
  \|W(t)F\|_{L^q_tL^r_xL^{\tilde{p}}_v} \lesssim \|F\|_{L^q_tL^r_xL^\infty_v}, \quad 1/q + 1/\tilde{q} = 1, \quad p = n.
  \end{equation}

- In $n > 1$, inhomogeneous endpoints of the type
  \begin{equation}
  \|W(t)F\|_{L^q_tL^r_xL^\infty_v} \lesssim \|F\|_{L^q_tL^r_xL^\infty_v}, \quad 1/q + 1/\tilde{q} = 1, \quad p = n.
  \end{equation}

- In $n > 1$, the full range of acceptability for the estimate
  \[
  \|W(t)F\|_{L^q_tL^r_xL^\infty_v} \lesssim \|F\|_{L^q_tL^r_xL^\infty_v}.
  \]

- In $n > 1$, the upper bound $r^*(b,c)$ to $r$ in the mixed $L^p$-data estimate
  \[
  \|U(t)f\|_{L^q_tL^r_xL^\infty_v} \lesssim \|f\|_{L^\infty_xL^c_v}.
  \]

Is $r^*(b,c) = r^*(a)$ or is $r^*(b,c) > r^*(a)$?

We would also like to ask the following questions

- Is there any improvement to the above estimates for spherically symmetric $f$ or $F$?
- Is there any improvement to the above estimates in the case when (1.1) is defined over a finite velocity space, (the only physically relevant setting), that is $v \in V$ for some compact subset $V$ of $\mathbb{R}^n$?
- What happens with the estimates of lemma 6.2 that we showed to fail, if one replaces the $L^\infty$-norm by the BMO-norm?
- What is the full range of acceptability of the local estimates, that is the set $\mathcal{E}$? The answer to this question is unknown even in the context of the Schrödinger equation, see Foschi [8] sect. Open questions].
8. Applications

Below we make an application of the estimates that we have derived above to a nonlinear kinetic model of bacterial chemotaxis. The model describes a population of bacteria in motion in a field of a chemoattractant, see \[5\], \[4\], and the references therein. So we consider the following system

\[
\partial_t u(t, x, v) + v \cdot \nabla_x u(t, x, v) = \int_{v' \in V} T[S](t, x, v', v)u(t, x, v')dv',
\]

\[
- \int_{v' \in V} T[S](t, x, v', v)u(t, x, v)dv', \quad t > 0, \ x \in \mathbb{R}^n, \ v \in V
\]

(8.1) 

\[
- \Delta S + S = \rho(t, x) \text{ def} = \int_{v \in V} u(t, x, v)dv,
\]

(8.2) 

\[
u(0, x, v) = f(x, v) \geq 0,
\]

(8.3) 

where by \(u(t, x, v)\) we denote the cell density in phase space and we assume that the space of admissible velocities \(V \in \mathbb{R}^n\) is bounded. The cell density in physical space is denoted by \(\rho(t, x)\), where \(t\) and \(x\) are a time and a space coordinate respectively. The free transport operator \(\partial_t u(t, x, v) + v \cdot \nabla_x u(t, x, v)\) describes the free runs of the bacteria which have velocity \(v \in V\). The nonlinear terms in the right hand side of (8.1) denote a scattering operator that expresses the reorientation process occurring during bacterial migration towards regions of high concentration in chemoattractant \(S(t, x)\). In this model the second integral term is always positive and will be omitted in our estimates below. Based on biologically realistic assumptions, the turning kernel \(T[S](t, x, v, v')\) is assumed to satisfy the following bound

\[
\|T[S](t, \cdot, \cdot, \cdot)\|_{L^1_t L^{p_1}_v L^{p_2}_v} \lesssim |V|, \|S(t, \cdot, \cdot)\|_{L^r} + \|\nabla S(t, \cdot, \cdot)\|_{L^r},
\]

whenever \(r \geq p_1, p_2\).

Following the bootstrap argument in theorem 3 of \[5\], we shall extend their result to dimensions \(n \geq 3\).

**Theorem 8.1.** Suppose that the initial data \(f \in L^1(\mathbb{R}^{2n}) \cap L^{n/2}(\mathbb{R}^{2n})\), where \(n \geq 3\) and \(\|f\|_{L^{n/2}(\mathbb{R}^{2n})}\) is sufficiently small and the turning kernel \(T\) satisfies condition (8.4). Then the IVP (8.1) - (8.3) has a global weak solution \(u \in L^q([0, \infty]; L^r(\mathbb{R}^{2n}; L^p(V)))\), where

\[
q = n, \quad r = \frac{n^2}{2n - 1}, \quad p = \frac{n^2}{2n + 1}.
\]

**Proof.** Applying estimate (2.1) to (8.1) we have

\[
\|u\|_{L^q_t L^p_x L^p_v} \lesssim \|f\|_{L^{n/2}_x}, \quad \int_V T[S]u'dv' \lesssim \|T[S]\|_{L^p_v L^p_v L^p_v},
\]

(8.5) 

where we have set \(u' = u(t, x, v')\) and assumed that the Lebesgue exponents above are subject to the conditions of theorem 2.1. To the integral term above we first apply Hölder’s inequality to get

\[
\int_V T[S](t, x, v, v')u(t, x, v')dv' \leq \|T[S](t, x, v, \cdot)\|_{L^p_{v'}} \|u(t, x, \cdot)\|_{L^p_v}.
\]
We next take the $L^p_t$-norm in $v$ and then take the $L^\infty_x$-norm in $x$ using Hölder’s inequality with $\frac{1}{p'} = \frac{1}{p} + \frac{1}{r}$ to get
\[
\left\| \int_V T[S] u(t', x', v') dv' \right\|_{L^p_t L^r_x} \lesssim \|T[S](t, \cdot, \cdot)\|_{L^p_t L^{p'}_v} \|u(t, \cdot, \cdot)\|_{L^r_x L^\infty_v}.
\]
Assuming that $w \geq \tilde{p}'$, $p'$, we can use condition (8.3) to get
\[
\|T[S](t, \cdot, \cdot)\|_{L^p_t L^{p'}_v} \lesssim \|G \star \rho\|_{L^p_x} + \|\nabla G \star \rho\|_{L^r_x},
\]
where $G$ is the Bessel potential
\[
G(x) = \int_0^\infty e^{-\pi \frac{it^2}{s}} \frac{a}{s} \frac{ds}{s},
\]
and we have used the representation formula $S = G \star \rho$. We recall the following two estimates on $G$, see [16], $G \in L^b(\mathbb{R}^n)$, for $1 \leq b < \frac{n}{n-1}$, and $\nabla G(x) \lesssim \frac{1}{|x|^{n-1}}$ for all $x$. We use Young’s inequality to estimate the $G$-term and the HLS inequality to estimate the $\nabla G$-term to get
\[
\|G \star \rho\|_{L^p_x} \lesssim \|G\|_{L^{p/(n-1)}_x} \|\rho\|_{L^r_x},
\]
\[
\|\nabla G \star \rho\|_{L^r_x} \lesssim \|\nabla G\|_{L^{p/(n-1)}_x} \|\rho\|_{L^r_x},
\]
where $1 + \frac{1}{w} = \frac{n-1}{n} + \frac{1}{r}$ and $n \geq 2$. Noting that $\|\rho\|_{L^r_x} \lesssim \|u(t, \cdot, \cdot)\|_{L^q_t L^\infty_v}$ and setting $\tilde{q}' = q/2$ we obtain
\[
\left\| \int_V T[S] u(t') dv' \right\|_{L^q_t L^\infty_v L^\infty_x} \lesssim \left\| u(t, \cdot, \cdot) \right\|_{L^q_t L^\infty_v}^2 \lesssim \|u\|_{L^q_t L^\infty_v L^\infty_x}^2.
\]
Hence, we obtain the following a-priori estimate for $u$
\[
(8.6) \quad \|u\|_{L^q_t L^\infty_v L^\infty_x} \lesssim \|f\|_{L^{q,v}_t x} + \|u\|_{L^q_t L^\infty_v L^\infty_x}^2;
\]
whenever the following system of conditions
\[
\text{HM}(r, p) = \text{HM}(\tilde{p}', \tilde{p}') = a,
\]
\[
\frac{1}{q} = \frac{n}{2} \left(\frac{1}{p'} - \frac{1}{r}\right), \quad \frac{1}{\tilde{q}'} = \frac{n}{2} \left(\frac{1}{\tilde{p}'} - \frac{1}{\tilde{r}}\right),
\]
\[
\tilde{q}' = q/2, \quad w \geq \tilde{p}', p', \quad n \geq 2
\]
\[
p^*(a) < p \leq a, \quad a \leq r < r^*(a),
\]
\[
p^*(a') \leq \tilde{p} \leq a', \quad a' \leq \tilde{r} \leq r^*(a'),
\]
\[
1 + \frac{1}{w} = \frac{n-1}{n} + \frac{1}{r},
\]
admits a solution. One could check that the condition $w \geq p'$ is equivalent to $a \geq n/2$. Therefore, we shall look for a solution with the mildest regularity assumption $a = n/2$. Thus, $a' = n/(n-2)$ and $p^*(a) = n^2/2(n+1), r^*(a) = n^2/2(n-1), p^*(a') = n^2/(n+1)(n-2), r^*(a') = n^2/(n-1)(n-2)$ in $n \geq 3$. Since this section serves for illustrative purposes only, we shall try to guess a simple solution without making any attempt to find out the full (two parameter) solution space. An educated guess can be $q = n, r = n^2/(2n-1), p = n^2/(2n+1)$, and $a = n/2$. This implies $\tilde{q} = n/(n-2), \tilde{r} = n^2/(n-1)(n-2), \tilde{p} = n^2/(n+1)(n-2)$, and $n \geq 3$. Note that $(q, r, p)$ is a non-endpoint KT-admissible triplet while $(\tilde{q}, \tilde{r}, \tilde{p})$ is an endpoint KT-admissible triplet. The condition $w \geq p'$ holds as we have that
\[ w = \frac{n^2}{(n-1)} \] and \[ \tilde{\beta'} = \frac{n^2}{(n+2)}. \] Apparently, all other conditions in the system above are satisfied. Hence estimate (8.5) holds with the chosen exponents due to theorem 2.1, see also remark 2.2.

As a final comment we mention that the a-priori estimate (8.6) is used in a standard bootstrap argument to finish the proof of the theorem, see remark 4.2 in [5]. □

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