Gauge Theoretic Chaology. *

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Abstract

We review here the work done on the occurrence of chaotic configurations in systems derived from Gauge theories. These include Yang-Mills and associated field theories with modifications including Chern Simons and Higgs fields.

1 Introduction.

The upsurge of interest in studies of Chaos in various systems has all the ingredients of a ‘scientific revolution’ as defined by Thomas Kuhn in his pioneering work ‘The Structure of Scientific Revolution’. Kuhn describes a scientific revolution as the result of a dramatic shift of ”paradigm”. A paradigm is an ”accepted example of actual scientific practice from which spring coherent traditions of scientific research”. Transformation of paradigms lead to scientific revolutions, which in turn lead to cross fertilization between different fields, giving rise to new areas of research. The chaos revolution represents a shift from calculus and a smooth deterministic description of dynamical systems to a highly non-linear, often fractal and unpredictable one. Thus, in some sense, chaos is a revolution from a deterministic world view to an unpredictable picture of the universe resulting from non-linearity. Since non-linearity of physical and biological systems is more of a rule than an exception, the possibility of applications is enormous.

This mathematical revolution was paralleled in physics by another revolution with the advent of quantum field theory and the postulation of a "fundamental theory of particles"

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based on the "paradigm" of gauge symmetry. This field, the modern avatar of which is "string theory", has lead to a new era in applied mathematics, in which not only have fundamental mathematics been applied to solve physical problems, but physical concepts have contributed to solving problems in pure mathematics and generating new areas of mathematics. Of particular relevance to this review, are Yang-Mills theories and their topological generalizations vis-a-vis Chern Simons terms and the profound impact on mathematics they have had since the work of Atiya-Drinfeld-Hitchin-Manin and that of Witten and Donaldson recently. Yang-Mills theories, central to quantum theories of elementary particles, became of interest to dynamicists with the recognition of the fact that the Self Dual Yang Mills theories serve as some sort of a "universal integrable system" from which a large class of known integrable systems can be derived by reduction.

The belief that chaotic phenomena play an important role in the study of fundamental particles goes as far back as Heisenberg, before the advent of gauge theories, in a seminal paper describing meson production in terms of turbulent fields. As the modern theory of fundamental particles is based on Yang-Mills Field theories which are highly non-linear and evolve chaotically in space-time, work in the dynamical structure of these theories was initiated in the eighties by Matinyan, Saviddy and their collaborators [6],[9] in an attempt to study the role of chaotic phenomena in quark confinement. Since then a whole spectrum of work has been done in chaos in Yang-Mills theories and its various extensions, including topological ones, and contributions of chaos to particle production processes as well as confinement have been studied. This review is intended to survey this field, highlighting the work done, its implications and future studies that can be done. Indeed, it represents an area in which the two hitherto parallel revolutions in Mathematics and Physics cease being parallel and converge.

In our recent papers [1],[2],[3],[4],[5], we have continued these studies into chaos in Yang-Mills and associated field theories. Of special interest in the modern context and in our work is the role of topological Chern-Simons "effects" in dynamical systems theory. The Chern Simons term is the metric independent "topological" term that can be added to Yang Mills theories in odd-dimensions. It is a different way to get finite mass other than the Higgs mechanism. This mass is hence called the topological mass. We have shown, as will be described in the review, that a field theory described by a Lagrangian of the type \( L = L_{YM} + L_{CS} + L_{Higgs} \) admits order chaos transitions as a function of energy and the three parameters (the gauge coupling, the "topological mass", and the "Higgs coupling") of the theory. The dynamical systems are obtained from the field equations of \( L \) through an assumption of spatial homogeneity. Earlier workers in the field had placed additional restrictions and reduced the dynamics to two degrees of freedom. In our work, we have maintained the full complement 9 degrees of freedom for the SU(2) gauge group. The results therefore are mainly numerical out of necessity, though a thorough Painleve analysis, which will be described has been done. We find from our numerical results that the order chaos transition in this system implies a constant creation and destruction of KAM torii in phase space. Thus far this phenomenon has not been seen in Hamiltonian systems and warrants an analytical study.

In addition to physical applications the study of the Chern-Simons term has the additional benefits in dynamical systems study, in that, being topological it admits topological invariants in its quantum analysis. Since this topological theory lies in three dimensions, it yields topological information on 3-manifolds, in particular on knots and links embedded in three dimensions. It has been conjectured that, viewed as a dynamical system, the resulting ODE's derived from Chern-Simons field theory may be used to generate knots and links by searching for periodic orbits and/or strange attractors in the system. An alternative, more automated
Table 1: Translation of Terminology from Physics to Mathematics

| Physics.                  | Mathematics.                                      |
|---------------------------|---------------------------------------------------|
| Global gauge              | Principal Co-ordinate Bundle                      |
| Gauge Type                | Principal Fibre Bundle                            |
| Gauge Potential $A^\mu_\lambda$ | Connection on a Principal Fibre Bundle           |
| Field Strength $F_{\mu\nu}$ | Curvature                                       |
| Electromagnetism          | Connection on a $U(1)$ Bundle                     |
| SU(2) Yang Mills          | Connection on an SU(2) bundle                     |
| Dirac Monopole            | Classification of $U(1)$ bundle according to first Chern Class. |

method would be to generate time-series data from the ODE and use simple recurrence to find periodic orbits and the Ruelle-Takens method to search for a strange attractors in the data. The strange attractor would then allow a knot template to be constructed and the associated embedded knots to be enumerated and examined. These are just some of the directions in mathematical physics that a study of the dynamical behaviour of topological systems can lead us to. The review presented here is a summary of the work already done in this field.

We begin this review with an overview of the work done on Yang-Mills dynamical systems and their extensions. We present the analytical and numerical tools for the study of these dynamical systems and then sketch the work done which extends to the entire field theory by considering perturbations around vortex solutions of the field theory. We conclude by highlighting the salient results and highlighting the avenues for future research.

A useful table of identification of Gauge theory terminology with Fibre Bundle Terminology due to C.N. Yang is given in Table 1.

2 Motivation and History of Problem.

In considering a quantum field theory of non-Abelian fields, quantization of fields proceeds by examining fluctuations around a particular classical solution and building up a Hilbert space on which the operators of the quantum theory then act. Quantum theory is probabilistic in nature and therefore, it is necessary that the classical solution chosen for quantization be of lowest energy otherwise, there would be a finite, non-zero probability of the quantum solution decaying into the state of lower energy. For Yang-Mills theories, in particular when the gauge group is non-Abelian, such as SU(3) (for QCD), a lowest energy solution has been a problem. It is found that all quantizations around background fields are prone to decay. Indeed, the simplest such background is that of a uniform "magnetic" field. The reason for studying such backgrounds for QCD, describing quarks, the (ultimate?) constituents of all matter, is that it does not allow the quarks to live independently which therefore cannot be seen—a property called confinement. Their existence is only inferred. It has been suggested that the QCD vacuum is a colour superconductor causing the color electric fields to be expelled in analogy with the Meissner effect causing magnetic flux to be expelled in an ordinary superconductor (leading to confinement of quarks). To be able to understand such phenomena, it is important to study the effects of background magnetic fields on the QCD vacuum. The uniform background field
causes the QCD vacuum to be unstable. Attempts to render the vacuum stable have focussed on what may be called randomization of the background field. Such a scenario is possible since most of the gauge theories of interest allow vortex solutions\cite{7},\cite{10}. The background is then to be viewed as a random configuration of vortices and the quantization is carried around this background. However this is not an easy task and is yet to be carried out in its full glory through effective action methods. Our approach to this problem is to utilize the fact that chaos exists in non-Abelian gauge theories leading to the possibility of searching for an attractor in the phase space of the gauge theory. Such an attractor may allow us to find a background configuration around which to construct a quantum theory.

Therefore it is very important to study and understand the solution spaces of Yang-Mills theories (including topological theories such as Chern-Simons). The purely mathematical work of Donaldson does precisely this for pure Yang-Mills theories.

Pure Yang Mills is described by an action:

\[ S_{YM} = -\frac{1}{2} \int_M \text{Trace}(F_{\mu\nu} F_{\mu\nu}). \]  

and the equations of motion are:

\[ \nabla_\mu F_{\mu\nu} = 0, \]  

with

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \]  

These are a set of highly non-linear PDE’s for the gauge potentials \( A_\mu \).

Until 1974, the YM Action was thought to be the most general action that could be written down (modulo boundary integrals) for the YM equations in \( \text{dim}(M) \). This changed when Chern and Simons produced a purely differential geometric result in 1974, showing the existence of an odd-dimensional secondary characteristic class. Coupled with the geometric interpretation of YM fields, it follows that in odd dimensions, i.e. for odd \( \text{dim}(M) \), the YM action allows an additional term (now called the Chern-Simons term or CS form), modifying the Yang Mills equations. Inclusion of this term in the action leads to extra non-linearities in the gauge fields

\[ S_{\text{odd}} = \frac{m}{2} \epsilon^{\mu\nu\rho} \text{Trace}(F_{\mu\nu} A_\rho - \frac{2}{3} i A_\mu A_\nu A_\rho) \]  

"Matter" fields in interaction with the gauge fields are included giving a set of functions \( (f_1(x), \ldots, f_m(x)) \) on a space time manifold transforming as an \( m \)-dimensional representation of the gauge group G. An action is constructed by including the \( f(x) \)'s. Consider, for example, a set of scalar functions \( \phi(x) \) transforming under some irreducible representation of G. Such fields are called scalar or Higgs fields. A typical action for the \( \phi(x) \)'s in interaction with gauge fields with gauge group G is given by:

\[ S = \int -\frac{1}{2} \text{Tr}(F_{\mu\nu} F_{\mu\nu}) + (D_\mu) ^\dagger (D_\mu \phi) - m^2 \phi \phi ^\dagger + V(\phi) \]  

where

\[ (D_\mu)_{ij} = \delta_{ij} \partial_\mu - i A_\mu^a T^a_{ij} ; \quad ij = 1, \ldots, m. \]  

\( V(\phi) \) is a potential term which takes care of the non-linear interactions of \( \phi \)'s amongst themselves.

The procedure adopted to study Dynamical systems require the reduction of these PDE’s to ODE’s. This reduction is through an assumption of spatial homogeneity of the gauge potentials i.e.

\[ A_\mu(x) = A_\mu(t), \]  

with

\[ \nabla_\mu F_{\mu\nu} = 0, \]  

and

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \]
The gauge potentials are assumed to be functions depending only on the time coordinate while still taking values in the Lie Algebra L(G). Further simplifications occur by assuming G to be the simplest non-Abelian group: SU(2), and, through trivial dependence on the group indices. Writing the gauge potential as $A^a_\mu(t)$; $a=(1,2,3)$ is the group index for $G=SU(2)$, the following is assumed:

$$gA^1_\mu = x(t); gA^2_\mu = y(t); gA^3_\mu = z(t),$$

for $\mu = 1, 2, 3$, while

$$A^a_\mu = 0 \forall a.$$  

This last condition is a simple gauge condition. With these definitions, $H_{YM}$ becomes

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}(x^2 y^2 + y^2 z^2 + z^2 x^2)$$

and the corresponding equations of motion are:

$$\ddot{x} = -x(y^2 + z^2)$$
$$\ddot{y} = -y(x^2 + z^2)$$
$$\ddot{z} = -z(y^2 + x^2)$$

Thus pure Yang Mills classical dynamics corresponds to a system of coupled oscillators via quartic potentials. While the pure YM system is mixing, the addition of Higgs fields to the YM system modifies the behaviour dramatically. The Higgs fields tend to make the motion more regular. An additional parameter, $\lambda$, allows altering the strength of the nonlinear coupling of the Higgs fields.

Further, when the Higgs fields are introduced, there is a transition from regular motion to chaos, when the ratios of the Higgs coupling to the gauge coupling and energy of the system are varied [2]. It has been conjectured by Nikolaevskii and Shur[8] that if chaos is present in the dynamics of spatially homogeneous fields then it is present in the full field theory [3]. This was confirmed in the Yang-Mills field theory by considering its classical solutions, including the Wu-Yang monopole and reducing them to a Fermi-Pasta-Ulam problem of coupled anharmonic oscillators [4]. The analysis of the Yang-Mills Higgs system with 't-Hooft-Polyakov monopole solutions has also been carried out along similar lines. It has been demonstrated that there is a transition from regularity to chaos in this system [5]. Thus the pure Yang-Mills and Yang-Mills Higgs theories exhibit ‘field theoretic’ or ‘spatio-temporal’ chaos, thereby providing examples of field theories which possess two extremes of non-linear behaviour- solitons and chaos. The Abelian Higgs model (which is the relativistic extension of the Ginzburg - Landau theory) is another example where both chaos and vortices exist.

The Maxwell Chern- Simons Higgs system (MCSH) has been receiving considerable attention lately as it is a potential candidate for an effective field theory of High- $T_c$ superconductors. It has been shown that the MCSH system (which includes the kinetic energy term for the gauge field) and the pure Chern-Simons Higgs (CSH) system (which is the low-energy limit of MCSH) both admit charged vortex solutions [7]. We examine here, the possibility of these systems exhibiting chaos as in the examples cited above.

For the CSH theory we will see below that for the case of spatially homogeneous solutions, the field theory reduces to a dynamical system with two degrees of freedom, which is integrable. A Painlevé analysis of MCSH shows that the addition of the Maxwell term destroys the integrability of the system.
The Lagrangian density of the (2+1 dimensional) CSH system is given by:
\[ L_0 = \frac{m}{2} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + |(\partial_\mu - i e A_\mu \phi)|^2 - V(\phi) \] (12)

By choosing the gauge \( A_0 = 0 \) and considering spatially homogeneous solutions \( \partial_i \phi = \partial_j A_j = 0, (i, j = 1, 2) \), we reduce the problem to a mechanical system. Writing \( A_1 = C \cos(\chi) \) and \( A_2 = C \sin(\chi) \) implies:
\[ \dot{\chi} = -\frac{e^2}{m} \phi^2 \] (13)
\[ \ddot{\phi} + e^2 C^2 \phi = -\frac{1}{2} \frac{\partial V}{\partial \phi} \] (14)

Therefore, the CSH theory is reduced to a dynamical system with two degrees of freedom \( \phi \) and \( \chi \). The corresponding Hamiltonian:
\[ H = \frac{p_\phi^2}{4} - \frac{e^2}{m} p_\phi \phi^2 + e^2 C^2 \phi^2 + V(\phi) \] (15)

\( p_\chi \) is clearly a constant of motion. Hence, the CSH system with two degrees of freedom, \( \phi \) and \( \chi \), is integrable with two integrals of motion \( H \) and \( p_\chi \).

The MCSH Lagrangian is of the form:
\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + |(\partial_\mu - i e A_\mu \phi^2)|^2 - V(\phi) \] (16)

Considering the spatially homogeneous solutions: \( A_1 = A_1(t); A_2 = A_2(t); \phi = \phi(t) = |\phi| e^{i \xi} \) and choosing the gauge \( A_0 = 0 \), we find once again, that \( \phi \) can be chosen to be real and that the equations of motion can be found from the following Hamiltonian:
\[ H = \frac{1}{2} [(p_1 - mA_2)^2 + (p_2 + mA_1)^2] + \frac{p_\xi^2}{4} + e^2 (A_1^2 + A_2^2) \phi^2 + V(\phi) \]

Using the quartic Higgs potential \( V(\phi) = \frac{1}{4}(\phi^2 - v^2)^2 \) and defining \( A_1 = ACos\zeta \) and \( A_2 = ASin\zeta \). Then the variables are \( A, \phi \) and \( \zeta \) and the Hamilton’s equations of motion are:
\[ \dot{A} = p_\phi \] (17)
\[ \dot{p}_A = -2e^2 \phi^2 A + \frac{(p_\xi)^2}{A^2} - m^2 A \] (18)
\[ \dot{\phi} = \frac{p_\phi}{2} \] (19)
\[ \dot{p}_\phi = -2e^2 A^2 \phi - \lambda \phi(\phi^2 - v^2) \] (20)
\[ \dot{\zeta} = m - \frac{p_\xi^2}{A^2} \] (21)
\[ \dot{p}_\zeta = 0. \] (22)

Thus we see that there are two constants of motion \( H \) and \( p_\xi \), but three degrees of freedom. Now \( \zeta \) is a cyclic coordinate whose dynamics is determined by the other co-ordinates and we
need not consider it further. We use the integral of motion \( p_\zeta \) to reduce these equations to the second order differential equations:

\[
\ddot{\phi} = -e^2 \phi A^2 - \frac{\lambda}{2} \phi (\phi^2 - v^2) \tag{23}
\]

\[
\ddot{A} = -m^2 A + \frac{p_\zeta^2}{A^3} - 2e^2 A^2 \phi^2 \tag{24}
\]

The Lagrangian for the non-Abelian (SU(2)) Chern Simons Higgs (NACSH) system in 2+1 dimensions in Minkowski space is given by:

\[
L = \frac{m}{2} \epsilon^{\mu\nu\lambda} [F^a_{\mu\nu} A^a_\lambda - \frac{g}{3} f_{abc} A^a_\mu A^b_\nu A^c_\lambda] + D_\mu \phi_a \bar{D}^\mu \phi_a - V(\phi) \tag{25}
\]

where:

\[
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \tag{26}
\]

\( f_{abc} \) are the structure constants of SU(2) Lie algebra and

\[
D_\mu \phi_a = (\partial_\mu - igT^a_\mu) \phi_a \tag{27}
\]

Where: \( T^a \) are the generators of the SU(2) algebra, so that \( tr[T^a T^b] = \lambda \delta^{ab} \). The equations of motion can be described with three parameters:

\[
\ddot{A}^1 = [\ddot{A}^2 \phi^2 - \phi (A^2 \cdot \phi)] \tag{28}
\]

\[
\ddot{A}^2 = -[\ddot{A}^1 \phi^2 - \phi (A^1 \cdot \phi)] \tag{29}
\]

\[
\ddot{\phi} = -[(\ddot{A}^2 + \ddot{A}^2)\phi] - (A^1 \cdot \phi A^1 + A^2 \cdot \phi A^2)] - \frac{\kappa}{2} \phi (\phi^2 - v^2) \tag{30}
\]

with \( \kappa = \frac{\lambda m}{g^2} \). We shall set the scaled \( v \) to be one without loss of generality.

The Yang-Mills Chern-Simons Higgs System (YMCSH) Lagrangian is:

\[
L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{m}{2} \epsilon^{\mu\nu\alpha} [F^a_{\mu\nu} A^a_\alpha - \frac{g}{3} f_{abc} A^a_\mu A^b_\nu A^c_\alpha] + D_\mu \phi_a \bar{D}^\mu \phi_a - V(\phi) \tag{31}
\]

The YMCSH dynamical equations in the gauge \( A_0 = 0 \) and in the spatially homogeneous case are:

\[
\frac{1}{m} \ddot{A}^1 + 2 \ddot{A}^2 + 2(\ddot{A}^2 \phi^2 - \ddot{\phi} \ddot{A}^1 \cdot \ddot{\phi}) + \frac{1}{m} (A^1 \ddot{A}^1 \cdot \ddot{A}^1 + \ddot{A}^2 \cdot \ddot{A}^2) = 0 \tag{32}
\]

\[
\frac{1}{m} \ddot{A}^2 - 2 \ddot{A}^1 + 2(\ddot{A}^1 \phi^2 - \ddot{\phi} \ddot{A}^2 \cdot \ddot{\phi}) + \frac{1}{m} (A^2 \ddot{A}^1 \cdot \ddot{A}^1 + \ddot{A}^2 \cdot \ddot{A}^2) = 0 \tag{33}
\]
and

\[ \ddot{\phi} = -\left( [\vec{A}_1^2 + \vec{A}_2^2] \phi - (\vec{A}_1 \cdot \vec{A}_1 + \vec{A}_2 \cdot \vec{A}_2) \right) - \frac{\kappa}{2} \phi (\phi^2 - 1). \]  

(34)

It is interesting to note that while in the NACSH system the Yang-Mills parameter \( g \), the Higgs parameter \( \lambda \) and the Chern-Simons parameter \( m \) could all be combined into the parameter \( \kappa \), this is not possible for the YMCSH system where we are left with both \( \kappa \) and \( m \) appearing explicitly. The energy function is:

\[ E = \frac{1}{2m} (\dot{\vec{A}}_1^2 + \dot{\vec{A}}_2^2) + \phi^2 + \frac{1}{2m} [\vec{A}_1^2 \vec{A}_2^2 - (\vec{A}_1 \cdot \vec{A}_2)^2] + \left[ (\vec{A}_1^2 + \vec{A}_2^2) \phi^3 - (\vec{A}_1 \cdot \vec{A}_2)^2 \right] - (\vec{A}_2 \cdot \vec{A}_2)^2 + \frac{\kappa}{4} (\phi^2 - 1)^2. \]  

(35)

This completes the description of the dynamical systems which we shall be studying.

### 2.1 Painlevé Analysis.

The Painlevé analysis for the CSH system is instructive in illustrating the Painlevé property and serves as a useful precursor to test the integrability property of the more complicated MCSH theory.

The Painlevé test for integrability is usually stated as follows[11],[12]: Consider the system of differential equations:

\[ \frac{d^{n_i} x_i}{dt^{n_i}} = f_i[t; x_i, \dot{x}_i, x_2, \dot{x}_2, \cdots] \]  

(36)

and continue \( x_i(t) \) from real to complex times. Then, the Painlevé conjecture asserts that if the singularities of \( x_i(t) \) are no more than poles or branch points and if the Laurent expansion around the leading singularity has a sufficient number of arbitrary constants warranted by the set of differential equations, then the system is integrable. The actual Painlevé analysis proceeds in three steps:

1) Determine the leading singularity. If the leading singularity is a pole (branch point) it indicates a strong (weak) Painlevé property. If neither is the case, then the system is non-integrable.

2. If the leading singularity is a pole or a branch point, a formal Laurent series expansion of the solution around the singularity \( t_0 \) is carried out. The powers of \( (t - t_0) \) in the series expansion, for which the corresponding coefficients become arbitrary, are determined. These are called the Kowaleskaya exponents or resonances.

3.) Next we verify that at the resonance values sufficient number of arbitrary constants exist.

For the CSH system the equation of motion can be written as:

\[ \ddot{x} = -x - A x(x^2 - 1). \]  

(37)

To find the dominant term we continue \( t \) to the complex plane and substitute: \( x = a(t - t_0)^{-\alpha} \). We find that \( \alpha = 1 \) and \( a^2 = 2 - 2/A \). Thus the leading singularity is a pole.
Including the next to leading term: \( x = a(t - t_0)^{-1} + p(t - t_0)^{r-1} \) and balancing the terms linear in \( p \), we find that the roots are \( r = -1 \) and \( r = 4 \). Carrying out the Laurent series expansion about the leading singularity,

\[
x = \sum_{i=0}^{\infty} a_i(t - t_0)^i
\]

we find that all the \( a_i \)'s are determined except \( a_4 \), which is arbitrary. Thus the second order differential equation has a solution with two arbitrary constants \( t_0 \) and \( a_4 \). Hence, the system is integrable.

We can now check the integrability (or lack of it) of the MCSH system by carrying out the Painlevé test. The equations of motion we examine are the second order differential equations written above. In terms of the rescaled variables \( x = \frac{\phi}{v} \), \( y = \frac{A}{v} \), \( t' = \epsilon v t \), \( C = \frac{\nu^2}{v} \), \( A = \frac{1}{2\epsilon^2} \), \( B = \frac{m^2}{\epsilon^4 \nu^2} \), we get the equations:

\[
\ddot{x} = -xy^2 - Ax(x^2 - 1) \quad (39)
\]
\[
\ddot{y} = -By + C \frac{y^3}{y^3} - 2x^2y \quad (40)
\]

where the differentiation is w.r.t \( t' \). In the subsequent analysis we drop the prime. The dominant singularity structure is found by substituting:

\[
x = a(t - t_0)^{\alpha} \quad \text{and} \quad y = b(t - t_0)^{\beta},
\]

where \( \alpha \) and \( \beta \) are assumed to be less than zero. Balancing the singularity at \( t_0 \) gives:

\[
a\alpha(\alpha - 1)(t - t_0)^{\alpha-2} = -ab^2(t - t_0)^{\alpha+2\beta}
\]
\[
-Aa^3(t - t_0)^{3\alpha} + Aa(t - t_0)^\alpha
\]
\[
b\beta(\beta - 1)(t - t_0)^{\beta-2} = -Bb(t - t_0)^{\beta}
\]
\[
-2a^2b(t - t_0)^{\alpha+2\beta}
\]

These equations immediately reveal that there are two possibilities for the leading order:

- Case 1. \( \alpha = -1, \beta = -1 \)
  \[ a^2 = -1 \quad \text{and} \quad b^2 = A - 2 \]
- Case 2. \( \alpha = -1 \) and \( \beta = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 16/A} \); \( a^2 = -\frac{2}{A} \) and \( b \) arbitrary.

Both cases must be tested for the Painlevé property.

The resonance analysis is carried out for both cases below.

Case 1:

\[
x = a(t - t_0)^{-1} + p(t - t_0)^{r-1} \quad (41)
\]
\[
y = b(t - t_0)^{-1} + q(t - t_0)^{r-1} \quad (42)
\]

with \( a^2 = -1 \) and \( b^2 = A - 2 \) where \( p \) and \( q \) are arbitrary parameters. From the above equations, we find that the resonances occur at \( r = -1, 1, 3 \pm \frac{\sqrt{3}}{2} \). Reality of the roots together with the non-leading nature of the resonance terms gives the condition \( \frac{7}{8} \leq A \leq 2 \).

For this case to have the Painlevé property we require \( r \) to be a non-negative integer (except for \( r = -1 \), which is the root corresponding to the movable singularity). This gives rise to two possibilities:
• case 1(a).
\( \sqrt{8A-7} = 1 ; \ A = 1 ; \ r = -1, 1, 2, 4. \) (43)

• case 1(b).
\( \sqrt{8A-7} = 3 ; \ A = 2 ; \ r = -1, 4, 3, 0. \) (44)

We check the Painlevé property of cases 1(a) and 1(b) by substituting
\[
x = \sum_{j=0}^{\infty} a_j(t - t_0)^{j-1} \quad (45)
\]
and
\[
y = \sum_{j=0}^{\infty} b_j(t - t_0)^{j-1} \quad (46)
\]

For case 1(a), our resonance analysis indicates that the indeterminate coefficients in these Laurent expansions should occur at \( j=1,2 \) and 4. If \( a_1/b_1, a_2/b_2, a_4/b_4 \) are arbitrary, then the system is integrable. When we carry out the detailed analysis for case 1(a), we find:
\[
b_2 = 4a_2^2, \quad a_2^2 = (B + 1)/9
\]
Thus \( a_1 \) and \( b_1 \) are determined and the Painlevé property fails. Furthermore, for \( j \leq 4 \) all the \( a_j \)'s and \( b_j \)'s are determined except \( a_4 \) and \( b_4 \), between which there is a linear relation. For \( j \geq 4 \), the coefficients can be determined from the \( a_j, b_j \ldots j = 1, \ldots 4 \). Thus there is only one arbitrary constant among the \( a_j \)'s and the \( b_j \)'s. But in order for case 1(a) to have the Painlevé property there must be at least 3 arbitrary parameters beside \( t_0 \). Thus, case 1(a) fails the Painlevé test.

For case 1(b), \( r=0 \) is a root. This implies that \( a_0 \) and \( b_0 \) should be arbitrary. However, in this case, we find that \( a_0^2 = -1 \) and \( b_0^2 = A - 2 = 0 \), i.e. they are determined. This, therefore, leads to a contradiction and the system does not pass the Painlevé test.

Case 2:
In this we find that the resonances occur at \( r=1, 0, 4, 1 - 2\beta \). In order that \( r > 0 \), we have the condition that \( 1 - 2\beta > 0 \). As \( \beta > -1 \) and \( 1 - 2\beta = m \), \( m = 0, 1, 2 \).

Thus, we have three cases.

• Case 2a.
\[ m=0; \quad \beta = \frac{1}{2}; \quad 1 + \frac{16}{A} = 0 \quad \text{or} \quad A = -\frac{1}{16}. \]
This value of \( A \) is unphysical as it corresponds to an unbounded Higgs potential.

• Case 2b.
\[ m=1; \quad \beta = 0; \quad A = \infty. \]
Again, this is not physically interesting as for \( A = \infty \) the Higgs potential is infinite.

• Case 2c.
\[ m=2; \quad \beta = -1/2. \] Hence, we examine the system for the ‘weak Painlevé property’.
In this case, \( A = \frac{16}{3} \) and \( r=-1,0,2,4 \).
We now carry out the consistency check of the full resonance analysis by substituting:
\[
x = \sum_{j=0}^{\infty} a_j(t - t_0)^{j-1} \quad (47)
\]
\[
y = \sum_{j=0}^{\infty} b_j(t - t_0)^{j-\frac{1}{2}} \quad (48)
\]
We find that $a_0^2 = \frac{1}{\beta}$ and that $a_1$ and $b_1$ are determined in terms of $b_0$. In the second order $b_0$ is determined in terms of $B$. But, as $r=0$ is a root, one of the two coefficients $a_0$ or $b_0$ must be arbitrary. Hence, case 2c also fails the Painlevé test.

We have thus established that for all possible cases the MCSH theory is non-integrable.

3 Order-Chaos Transitions

Is there a sharp order to chaos transition in the parameter space of these theories? In the context of Abelian Higgs theories Kawabe [7] has reported transition from order to chaos within a certain range of the Higgs coupling constant and energy. The onset of chaos is remarkably different qualitatively from the corresponding transition in the YMCS system where the existence of an interesting ‘fractal’ structure appears in the phase transition region. This aspect of chaos in YM systems studied in the non-Abelian CSH (NACSH) and the YMCSH systems with an SU(2) symmetry group. A comparative analysis is done to see the role of the kinetic term, the Higgs term and the CS term in the transition.

We have computed the Lyapunov exponents for various values of the parameters $\lambda$, $e^2$, $m$ and $v$. In each case the Lyapunov exponent [12] converges to some positive value. We examined the variation of the maximal Lyapunov exponent as the two NACSH parameters energy and $\kappa$ are varied. This clearly shows us regions of regular behaviour (where the exponent goes to zero) and regions of chaotic behaviour (where the exponent is positive). These calculations were carried out for a wide range of initial conditions. The initial conditions that were chosen were in turn dictated by the dynamical systems themselves. Being derived from the equations of motion the field variables are required to satisfy the Gauss’ law constraint. Fig. 1 shows the comparison of maximal Lyapunov exponent versus $x$ in a two-variable initial ansatz for YMCSH for different values of $\kappa$, energy and $m$. The graph shows that for large $\kappa$ (where either the Higgs coupling $\lambda$ is large or the YM coupling $g$ is small) the system exhibits more regularity for low energies. Here, we see that for the YMCSH system such transitions to regularity are seen for small energies as $\kappa$ increases while the Lyapunov exponent increases almost linearly with energy for the large energy regime.

A striking feature that emerges in our studies that is counter-intuitive is that, in general, it is not true that increase in $\kappa$ ‘regularizes’ the gauge term at all energies. An increase in $\kappa$ could either be due to an increase in the Higgs coupling $\lambda$ or a decrease in the gauge coupling $g$, for a fixed $m$. Whereas in the former case the regularity is expected to increase, in the latter case, it is not established that for all small non-zero $g$ more regular islands appear. An increase in $\kappa$ produces more regularity only for small values of energy. This can be understood better if we realize that it is not just the value of $\kappa$ that determines the appearance of regular islands, but also the available phase space as well.

Another aspect is that as the energy increases, the maximal Lyapunov exponent increases in magnitude in the YMCSH system. This shows that the YM terms takes over for large energies and the CS term produces the ‘oscillatory effect’. The effect of the CS term is reminiscent of the ‘fractal’ structure observed in YMCS systems where in various energy windows, order-chaos-order transitions are observed.

The final picture which emerges bears out the fact that in a complex dynamical system with a large phase space (in contrast to the wide class of Hamiltonian systems with two degrees of freedom) curious interplay between different coupling constants and the rich structure of phase space itself can lead to novel results- some of them quite counter-intuitive and surprising.
A step towards field theoretic chaos in these systems has been the subject of our current studies. We consider the dynamics of the NACSH and YMCSH systems by perturbing around initial vortex solutions. Initial studies have shown that such an exercise holds promise in yielding results of the Fermi-Pasta Ulam type for coupled harmonic oscillators. However, we have to note that the $\frac{1}{\hbar}$ singularity of the vortex solution has been a cause for concern in the dynamical evolution of the solutions vis-a-vis the energy conservation. We hope to solve this particular aspect of the problem using more refined PDE solution tools. Another direction for further research is the study of quantum chaos in these systems and the distribution of energy level spacings in the quantum mechanical system.

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