Occurrences of consecutive patterns of length 3 in 3-1-2-avoiding permutations

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Abstract We exploit Krattenthaler’s bijection between the set $S_n(3\!-\!1\!-\!2)$ of permutations in $S_n$ avoiding the classical pattern 3-1-2 and Dyck $n$-paths to study the distribution of every consecutive pattern of length 3 on the set $S_n(3\!-\!1\!-\!2)$. We show that these consecutive patterns split into 3 equidistribution classes, by means of an involution on Dyck paths due to E. Deutsch. In addition, we state equidistribution theorems concerning triplets of statistics relative to the occurrences of the consecutive patterns of length 3 in a permutation.

Keywords: Restricted permutations, consecutive patterns, Dyck paths.

1 Introduction

Let $\sigma \in S_n$ and $\tau \in S_k$, $k \leq n$, be two permutations. We say that $\sigma$ contains the pattern $\tau$ if $\sigma$ contains a subsequence order-isomorphic to $\tau$. We say that $\sigma$ avoids $\tau$ if such a subsequence does not exist. The subject of pattern avoiding permutations was initiated by Simion and Schmidt [15], and, after that, a large literature on this topic has blossomed. Problems treated include counting permutations avoiding a pattern or a set of patterns, or containing patterns a specified number of times.

More recently, Babson and Steingrímsson [2] introduced generalized permutation patterns where two adjacent letters in a pattern may be required to be adjacent in the permutation. A number of interesting results on generalized patterns were obtained by several authors in recent years (for an extensive survey, see [17]).

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One particular case of generalized patterns are consecutive patterns. For a subsequence of a permutation to be an occurrence of a consecutive pattern, its elements have to appear in adjacent positions of the permutation. We write a “classical” pattern with dashes between symbols, while a consecutive pattern will be written without dashes, accordingly with the most common notation (see [17]). A number of results for the enumeration of permutations by consecutive patterns have recently been obtained (see, e.g., [1], [6], [7], [8], [12], [13], [18], and [19]).

Many well known integer sequences arise in enumerative problems concerning permutations avoiding a pattern \( \tau \) or containing it a fixed number of times. In particular, in [15] it has been shown that the number of permutations avoiding any classical pattern \( \tau \in S_3 \) equals the \( n \)-th Catalan number \( C_n \). Afterward, Krattenthaler [9] described a bijection between Dyck paths of semilength \( n \) and the set \( S_n(3\text{-}1\text{-}2) \) of permutations avoiding 3\text{-}1\text{-}2.

In this paper, we study the distribution of the five non-trivial consecutive patterns of length 3 on the set of 3\text{-}1\text{-}2-avoiding permutations. More precisely, for every consecutive pattern \( \tau \) of length 3, \( \tau \neq 312 \), we study the bivariate generating function

\[
A^{\tau}(t, z) = \sum_{n,k \geq 0} a^{\tau}_{n,k} t^n z^k,
\]

where \( a^{\tau}_{n,k} \) is the number of permutations in \( S_n(3\text{-}1\text{-}2) \) containing \( k \) occurrences of the consecutive pattern \( \tau \). First of all, we prove that each occurrence of such a \( \tau \) in a permutation \( \sigma \) correspond bijectively to a peculiar configuration in the Dyck path associated to \( \sigma \) by Krattenthaler’s bijection. This correspondence allows us to show that the five non-trivial patterns of length 3 split into 3 classes (i.e., \( \{213\} \), \( \{123, 321\} \), and \( \{132, 231\} \)), so that two patterns in the same class are equidistributed on \( S_n(3\text{-}1\text{-}2) \), namely, the corresponding bivariate generating functions coincide. These equidistribution results are obtained applying an involution on Dyck paths due to Deutsch [4].

For each one of these classes we can choose a representative such that the distribution of the corresponding Dyck path configuration has been determined (see [5] and [14]). Hence, we get the bivariate generating function of the distribution of each one of the five non-trivial consecutive patterns. This allows us to get an explicit expression for the coefficients of the power series \( A^{\tau}(t, z) \) for every consecutive pattern \( \tau \) of length 3.
As a fallout, we obtain a formula for the number of permutations in $S_n(3\text{-}1\text{-}2)$ that avoid any consecutive pattern $\tau \in S_3$. In the two cases $\tau = 321$ and $\tau = 123$, we get the Motzkin numbers, and present two bijections $\nu : S_n(3\text{-}1\text{-}2, 321) \to M_n$ and $\mu : S_n(3\text{-}1\text{-}2, 123) \to M_n$, where $M_n$ denotes the set of Motzkin $n$-paths.

In the last section, we show how previous results yield equidistribution theorems concerning triplets of statistics that associate the number of occurrences of a given consecutive pattern with each permutation in $S_n(3\text{-}1\text{-}2)$.

## 2 Preliminaries

### 2.1 Lattice paths

A **Dyck path** of semilength $n$ (or Dyck $n$-path) is a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the $x$-axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. A **Motzkin path** of length $n$ (or Motzkin $n$-path) is a lattice path starting at $(0, 0)$, ending at $(n, 0)$, and never going below the $x$-axis, consisting of up steps $U = (1, 1)$, horizontal steps $H = (1, 0)$, and down steps $D = (1, -1)$.

Dyck paths of semilength $n$ are counted by the $n$-th Catalan number $C_n$, while Motzkin paths of length $n$ are counted by the $n$-th Motzkin number $M_n$.

A Dyck path can be regarded as a word over the alphabet $\{U, D\}$ such that any prefix contains at least as many symbols $U$ as symbols $D$. A **subword** of a Dyck path $P$ is a subsequence of consecutive steps in $P$.

An **irreducible** Dyck path is a Dyck path that does not touch the $x$-axis except for the origin and the final destination. An **irreducible component** of a Dyck path $P$ is a maximal irreducible Dyck subpath of $P$.

We list some notions on Dyck paths that will be used in the following. A **run** (respectively **fall**) of a Dyck path is a maximal subword consisting of up (resp. down) steps. A **return** of a Dyck path is a down step landing on the $x$-axis.

We now describe an involution $\Delta$ on Dyck paths due to Deutsch [4]. Consider a Dyck path $P$ and decompose it according to its first return as $P = U A D B$, where $A$ and $B$ are (possibly empty) Dyck paths. Then, the Dyck path $\Delta(P)$
is recursively determined by the following rules (see Figure 1):

- if $P$ is empty, so is $\Delta(P)$;
- otherwise, $\Delta(P) = U \Delta(B) D \Delta(A)$.

It is easily checked that the two paths $P$ and $\Delta(P)$ have the same length. In Figure 2 we show how the involution $\Delta$ acts on Dyck paths of semilength 3.

![Figure 1: The involution $\Delta$.](image)

### 2.2 Restricted permutations

Let $\sigma \in S_n$, and $\pi \in S_k$, $k \leq n$, be two permutations. The permutation $\sigma$ contains the pattern $\pi$ if there exists a subsequence $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ that is order-isomorphic to $\pi$. Moreover, if $i_1, i_2, \ldots, i_k$ are consecutive integers - i.e. $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)$ is a subword of $\sigma$ - then we say that $\sigma$ contains the consecutive pattern $\pi$. In order to avoid confusion, we write a "classical" pattern with dashes between symbols, while a consecutive pattern will be written without dashes.
For example, the permutation $\sigma = 4 \, 3 \, 1 \, 7 \, 2 \, 5 \, 6$ contains 5 occurrences of the pattern 3-1-2 (namely, 412,312,725,726,756), but only one occurrence of the consecutive pattern 312 (namely, 725).

The permutation $\sigma$ avoids the (consecutive) pattern $\pi$ if $\sigma$ does not contain $\pi$. For example, the permutation $\sigma = 5 \, 2 \, 1 \, 3 \, 4$ avoids the consecutive pattern 312, but it does not avoid the pattern 3-1-2, since it contains, for instance, the subsequence 513.

We denote by $S_n(\tau_1, \ldots, \tau_k)$ the set of permutations in $S_n$ that avoid simultaneously the patterns $\tau_1, \ldots, \tau_k$.

We say that two permutation statistics $f$ and $g$ are equidistributed on a set $A \subseteq S_n$, if

$$\sum_{\sigma \in A} x^{f(\sigma)} = \sum_{\sigma \in A} x^{g(\sigma)}.$$

In the present paper we are interested in studying the distribution of an arbitrary consecutive pattern of length 3 on the set $S_n(3-1-2)$. It is well known that the permutation $\sigma$ avoids 3-1-2 if and only if it can be written as follows:

$$\sigma = m_1 \, w_1 \, m_2 \, w_2 \ldots \, m_k \, w_k,$$

where

- the integers $m_i$ are the left-to-right maxima of $\sigma$ (where a left-to-right
maximum of a permutation $\sigma$ is an integer $\sigma(i)$ such that $\sigma(i) > \sigma(j)$ for every $j < i$; 

- for every symbol $a$ appearing in one of the words $w_i$, consider the suffix $\sigma_a$ of $\sigma$ starting with $a$. Then, $a$ is the greatest symbol in $\sigma_a$ among those that are less than $m_i$.

Some results in this direction can be found in [10], where the author exhibits the generating functions for the number of permutations on $n$ letters avoiding 1-3-2 (or containing 1-3-2 exactly once) and an arbitrary generalized pattern $\tau$ on $k$ letters, or containing $\tau$ exactly once.

We describe the bijection $K$ between 3-1-2-avoiding permutations and Dyck paths due to Krattenthaler [9].

Give a permutation $\sigma \in S_n(3-1-2)$, $\sigma = m_1 w_1 m_2 w_2 \ldots m_k w_k$ the Dyck path $K(\sigma)$ of semilength $n$ is constructed as follows: start with $m_1$ up steps followed by $|w_1| + 1$ down steps. Then add $m_2 - m_1$ up steps followed by $|w_2| + 1$ down steps, and so on.

For example, the 3-1-2-avoiding permutation $\sigma = 43652781$ is mapped to the Dyck path in Figure 3.

Figure 3: The Dyck path $K(43652781)$.

### 3 Consecutive patterns

In this section we describe the bivariate generating function

$$A^\tau(t, z) = \sum_{n,k \geq 0} a^\tau_{n,k} t^n z^k,$$
where \( a_{n,k}^{\tau} \) is the number of permutations in \( S_n(3\text{-}1\text{-}2) \) containing \( k \) occurrences of the consecutive pattern \( \tau \), for every \( \tau \in S_3 \). Of course, we do not consider the case \( \tau = 312 \).

### 3.1 The pattern 213

We recall that a 3\text{-}1\text{-}2-avoiding permutation can be written as

\[
\sigma = m_1 \, w_1 \, m_2 \, w_2 \ldots \, m_k \, w_k,
\]

where the integers \( m_i \) are the left-to-right maxima of \( \sigma \) and the (possibly empty) subwords \( w_j \) are decreasing. We note that occurrences of the consecutive pattern 213 in \( \sigma \) correspond bijectively to nonempty subwords \( w_i \), \( i < k \). In fact, if \( |w_i| > 0, i < k \), then \( \sigma \) contains the 213-subword \( \, a \, m_{i+1} \), where \( a \) is the last element in \( w_i \) and \( b \) is either the second last element in \( w_i \), or \( b = m_i \), when \( |w_i| = 1 \). It is easily seen that each one of such subwords \( w_i \), in turn, corresponds to an occurrence of \( DDU \) in the Dyck path \( K(\sigma) \). The distribution of the subword \( DDU \) on Dyck paths is well known (see [16] seq. A091894). More precisely, in [5], the author determines a functional equation satisfied by the bivariate generating function of this distribution. Hence, we deduce the following expression for the generating function \( A_{213}(t, z) \):

**Theorem 1** We have

\[
A_{213}(t, z) = \frac{1 - 2t + 2tz - \sqrt{(1 - 2t)^2 - 4tz^2}}{2tz}
\]

that yields

\[
a_{n,k}^{213} = 2^{n-2k-1} C_k \binom{n-1}{2k}
\]

where \( C_k \) is the \( k \)-th Catalan number.

\( \diamond \)

In particular:

**Proposition 2** The number of permutations in \( S_n \) that avoid both the pattern 3\text{-}1\text{-}2 and the consecutive pattern 213 is

\[
|S_n(3\text{-}1\text{-}2, 213)| = 2^{n-1}.
\]

\( \diamond \)
3.2 The pattern 321

Consider a 3-1-2-avoiding permutation

$$\sigma = m_1 w_1 m_2 w_2 \ldots m_k w_k.$$  

The consecutive pattern 321 occurs in $\sigma$ if and only if at least one among the subwords $w_j$ has length greater than one. More precisely, if $|w_j| = t > 0$, the subword $m_j w_j$ contains $t - 1$ occurrences of 321, since the elements in $w_j$ appear in decreasing order. Note that a subword $w_j$ of length $t$ corresponds to a fall $F_j$ of length $t + 1$ of the Dyck path $K(\sigma)$. This implies that occurrences of 321 in $m_j w_j$ correspond bijectively to occurrences of $DDD$ in $F_j$. The distribution of $DDD$ on Dyck paths is well known (see [16] seq. A092107). In [14], the author deduces a functional equation satisfied by the bivariate generating function of this distribution. These considerations imply that:

**Theorem 3** We have

$$A^{321}(t, z) = \frac{1 - t + tz - \sqrt{1 - 2t - 3t^2 + tz(tz + 2t - 2)}}{2t(t + z - tz)}$$

that yields

$$a_{n,k}^{321} = \frac{1}{n + 1} \sum_{j=0}^{k} (-1)^{k-j} \binom{n+j}{n} \binom{n+1}{k-j} \sum_{i=j}^{\left[\frac{n+i}{2}\right]} \binom{n+j+1-k}{i+1} \binom{n-i}{i-j}.$$  

In particular:

**Proposition 4** The number of permutations in $S_n$ that avoid both the pattern 3-1-2 and the consecutive pattern 321 is

$$|S_n(3\text{-}1\text{-}2, 321)| = M_n,$$

where $M_n$ is the $n$-th Motzkin number.

In fact, a bijection $\nu$ between permutations in $S_n(3\text{-}1\text{-}2, 321)$ and Motzkin paths of length $n$ can be obtained as the composition of the map $\hat{K}$ with the well-known bijection between Dyck $n$-paths with no $DDD$ and Motzkin $n$-paths, defined by replacing each $UDD$ with $D$ and each remaining $UD$ with a horizontal step $H$ (see e.g. [3]).
3.3 The pattern 231

First of all, note that occurrences of the consecutive pattern 231 in the 3-1-2 avoiding permutation

\[ \sigma = m_1 w_1 m_2 w_2 \ldots m_k w_k \]

correspond bijectively to those indices \( i \) such that:

1. \( m_{i+1} = m_i + 1 \);
2. \( w_{i+1} \) is nonempty.

In fact, consider the subword \( m_i w_i m_{i+1} w_{i+1} \) and its subword \( a m_{i+1} b \), where \( a \) is the rightmost symbol in \( w_i \) (or \( m_i \) if \( w_i \) is empty) and \( b \) is the leftmost symbol in \( w_{i+1} \). Then, \( a m_{i+1} b \) is order isomorphic to 231 if and only if \( a > b \). This happens whenever the two conditions above hold. Each one of these occurrences corresponds to an occurrence of \( DUDD \) in the Dyck path \( K(\sigma) \).

The distribution of \( DUDD \) on Dyck paths was deeply studied (see [11], [14], and [16] seq. A116424). In [14], the author deduces a functional equation satisfied by the bivariate generating function of this distribution. Then:

**Theorem 5** We have

\[
A^{231}(t, z) = \frac{1 - (1 - z)t^2 - \sqrt{(1 - z)t^2 + 1} - 4t}{2t(1 - (1 - z)t)}
\]

that yields

\[
d_{n,k}^{231} = \sum_{j=k}^{\frac{n-1}{2}} \frac{(-1)^{j-k}}{n-j} \binom{j}{k} \binom{n-j}{j} \binom{2n-3j}{n-j+1}.
\]
In particular:

**Proposition 6** The number of permutations in $S_n$ that avoid both the pattern 3-1-2 and the consecutive pattern 231 is

$$|S_n(3\text{-}1\text{-}2, 231)| = \sum_{j=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{(-1)^j}{n-j} \binom{n-j}{j} \binom{2n-3j}{n-j+1}.$$ 

3.4 The pattern 123

A 3-1-2-avoiding permutation

$$\sigma = m_1 w_1 m_2 w_2 \ldots m_k w_k$$

contains an occurrence of the consecutive pattern 123 if and only if it contains two adjacent left-to-right maxima $m_i m_{i+1}$, with $i > 1$. In fact, in this case, $\sigma$ contains the 123-subword $a m_i m_{i+1}$, where $a$ is the symbol preceding $m_i$.

This consideration implies that occurrences of 123 in $\sigma$ correspond bijectively to occurrences of $DU^t DU$, with $t > 0$, in the Dyck path $K(\sigma)$.

**Proposition 7** The two statistics "number of occurrences of DDD" and "number of occurrences of DU$^t$DU", $t > 0$, are equidistributed on Dyck n-paths.

**Proof** Consider Deutsch’s involution $\Delta$ described in Section 2. Denote by $f(P)$ the number of occurrences of the subword DDD in a given Dyck path $P$ and by $g(P)$ the number of occurrences of $DU^t DU$, $t > 0$, in $P$.

We prove that $g(P) = f(\Delta(P))$ by induction on the semilength $n$ of the Dyck path $P$. The assertion is trivially true for $n = 0$. Fix an integer $n > 0$ and assume that the assertion holds for all Dyck paths of semilength less than $n$.

Let $P$ be a Dyck $n$-path, and $P = U A D B$ its first return decomposition. It is easy to verify that:

$$f(P) = \begin{cases} f(A) + f(B) + 1 & \text{if } A \text{ ends with } DD \\ f(A) + f(B) & \text{otherwise} \end{cases}$$

$$g(P) = \begin{cases} g(A) + g(B) + 1 & \text{if } B \text{ begins with } U^t DU \\ g(A) + g(B) & \text{otherwise} \end{cases}.$$
Since the semilengths of the two Dyck paths $A$ and $B$ are strictly less than $n$, by induction hypothesis we have $g(A) = f(\Delta(A))$ and $g(B) = f(\Delta(B))$. It is sufficient to show that the involution $\Delta$ acts as follows: if a Dyck path $P$ begins with a subword of type $U^t D U$, then $\Delta(P)$ ends with two down steps. Consider a Dyck path $P$ starting with $U^t D U$. Such a Dyck path can be decomposed into $U^{t-1} U D M W$, where $M$ is an irreducible (and nonempty) Dyck path. In this case, the Dyck path $\Delta(P)$ decomposes into $\Delta(P) = W' U \Delta(M) D$ (see Figure 5). The path $\Delta(M)$ is nonempty, hence $\Delta(P)$ must end with two down steps, as desired. Since $\Delta$ is an involution, we have also $f(P) = g(\Delta(P))$. This proves that $\Delta$ maps every occurrence of the subword $DU^tDU$ into an occurrence of the subword $DDD$, and conversely.

\[\Delta\]

Figure 5: The involution $\Delta$ maps a Dyck path starting with $U^t D U$ to a Dyck path ending with two down steps.

As an immediate consequence, we have:

**Theorem 8** The two generating functions $A^{123}(t, z)$ and $A^{321}(t, z)$ coincide.
Hence, by Proposition 4, we can state the following

**Proposition 9** The number of permutations in $S_n$ that avoid both the pattern 3-1-2 and the consecutive pattern 123 is

$$|S_n(3\text{-}1\text{-}2,123)| = M_n.$$
3.5 The pattern 132

A 3-1-2-avoiding permutation

$$\sigma = m_1 \, w_1 \, m_2 \, w_2 \, \ldots \, m_k \, w_k$$

contains an occurrence of the consecutive pattern 132 if and only if there exists an index $i$, $i < k$, such that the subword $m_i \, w_i \, m_{i+1} \, w_{i+1}$ verify the following conditions:

1. $m_{i+1} - m_i > 1$;

2. $w_{i+1}$ is nonempty.

In fact, in this case, $\sigma$ contains the 132-subword $a \, m_{i+1} \, b$, where $a$ is the symbol preceding $m_{i+1}$ in $\sigma$ and $b = m_{i+1} - 1$.

These remarks imply that occurrences of 132 in $\sigma$ correspond bijectively to occurrences of $DU^tDD$, with $t > 1$, in the Dyck path $K(\sigma)$.

**Proposition 11** The two statistics "number of occurrences of $DUDD$" and "number of occurrences of $DU^tDD$, $t > 1$, are equidistributed on Dyck $n$-paths.

**Proof** Denote by $h(P)$ the number of occurrences of the subword $DUDD$ in a given Dyck path $P$ and by $l(P)$ the number of occurrences of $DU^tDD$, $t > 1$, in $P$.

We prove that $l(P) = h(\Delta(P))$, where $\Delta$ is Deutsch's involution, by induction on the semilength $n$ of the Dyck path $P$. The assertion is trivially true for $n = 0$. Fix an integer $n > 0$ and assume that the assertion holds for all Dyck paths of semilength less than $n$. Let $P$ be a Dyck $n$-path, and $P = U \, A \, D \, B$ its first return decomposition. It is easy to verify that:

$$h(P) = \begin{cases} h(A) + h(B) + 1 & \text{if } A \text{ ends with } UD \\ h(A) + h(B) & \text{otherwise} \end{cases}$$
\[ l(P) = \begin{cases} l(A) + l(B) + 1 & \text{if } B \text{ begins with } U^t D D, t > 1 \\ l(A) + l(B) & \text{otherwise} \end{cases} \]

Since the semilengths of the two Dyck paths \( A \) and \( B \) are strictly less than \( n \), by induction hypothesis we have \( l(A) = h(\Delta(A)) \) and \( l(B) = h(\Delta(B)) \). It is sufficient to show that the involution \( \Delta \) maps a Dyck path \( P \) that begins with a subword of type \( U^t D D, t > 1 \), to a path \( \Delta(P) \) ending with \( UD \). Consider a Dyck path \( P \) starting with the subword \( U^t D D, t > 1 \). In this case, the last step of the recursive procedure defining the map \( \Delta \) maps the first peak \( UD \) of \( P \) to the last irreducible component of the Dyck path \( \Delta(P) \) (see Figure 7). Since \( \Delta(UD) = UD \), the Dyck path \( \Delta(P) \) ends with \( UD \), as desired.

Recalling that \( \Delta \) is an involution, we have also \( h(P) = l(\Delta(P)) \). Hence, \( \Delta \) maps every occurrence of the subword \( DU^t DD, t > 1 \), into an occurrence of the subword \( DUDD \), and vice versa.

\[ \diamond \]

Figure 7: The involution \( \Delta \) maps a Dyck path starting with \( U^t D D, t > 1 \), to a Dyck path ending with \( U D \).

Hence, we have:
Theorem 12 The two generating functions $A^{132}(t, z)$ and $A^{231}(t, z)$ coincide.

4 Joint distributions

The bijection $\Delta$ on Dyck paths induces an involution $\hat{\Delta} = K^{-1} \circ \Delta \circ K$ on the set $S_n(3\text{-}1\text{-}2)$. In particular, $\hat{\Delta}$ acts on $S_3(3\text{-}1\text{-}2)$ as follows:

$$321 \overset{\hat{\Delta}}{\leftrightarrow} 123$$
$$231 \overset{\hat{\Delta}}{\leftrightarrow} 132$$
$$213 \overset{\hat{\Delta}}{\leftrightarrow} 213$$

In this section we prove that the action of $\hat{\Delta}$ on $S_3(3\text{-}1\text{-}2)$ reveals to be paradigmatic for the general case: the involution $\hat{\Delta}$ maps an occurrence of a consecutive pattern $\tau \in S_3$ to an occurrence of the consecutive pattern $\hat{\Delta}(\tau)$.

First of all, the statements of Propositions 7 and 11 can be reformulated in terms of $\hat{\Delta}$ as follows:

Proposition 13 Let $\sigma \in S_n(3\text{-}1\text{-}2)$. Then:

1. $\sigma$ contains $k$ occurrences of the consecutive pattern $123$ $\iff$ $\hat{\Delta}(\sigma)$ contains $k$ occurrences of the consecutive pattern $321$;

2. $\sigma$ contains $k$ occurrences of the consecutive pattern $132$ $\iff$ $\hat{\Delta}(\sigma)$ contains $k$ occurrences of the consecutive pattern $231$.

We now determine the behavior of the map $\hat{\Delta}$ with respect to the distribution of the pattern 213:

Proposition 14 The two permutations $\sigma$ and $\hat{\Delta}(\sigma)$ have the same number of occurrences of the consecutive pattern 213.
Proof Let \( \sigma \in S_n(3\cdot1\cdot2) \). As remarked in Subsection 3.1, each occurrence of 213 in \( \sigma \) corresponds to an occurrence of DDU in the Dyck path \( K(\sigma) \). Hence, it is sufficient to prove that, for any \( P \), the two Dyck paths \( P \) and \( \Delta(P) \) have the same number of occurrences of DDU.

Denote by \( r(P) \) the number of occurrences of the subword DDU in \( P \). We prove that \( r(P) = r(\Delta(P)) \) by induction on the semilength \( n \) of the Dyck path \( P \). The assertion is trivially true for \( n = 0 \). Fix an integer \( n > 0 \) and assume that the assertion holds for all Dyck paths of semilength less than \( n \). Let \( P \) be a Dyck \( n \)-path, and \( P = U A D B \) its first return decomposition. It is easy to verify that:

\[
    r(P) = \begin{cases} 
        r(A) + r(B) + 1 & \text{if both } A \text{ and } B \text{ are nonempty} \\
        h(A) + h(B) & \text{otherwise}
    \end{cases}
\]

Since the semilengths of the two Dyck paths \( A \) and \( B \) are strictly less than \( n \), by induction hypothesis we have \( r(A) = r(\Delta(A)) \) and \( r(B) = r(\Delta(B)) \). Noting that both \( A \) and \( B \) are nonempty if and only if \( \Delta(A) \) and \( \Delta(B) \) are nonempty, we get the assertion.

\[ \diamond \]

Let \( \text{occ}_\tau(\sigma) \) be the number of occurrences of the pattern \( \tau \) in the permutation \( \sigma \). The preceding results can be restated as follows:

**Theorem 15** The triplets of statistics

- \( (\text{occ}_{321}, \text{occ}_{132}, \text{occ}_{213}) \) and \( (\text{occ}_{123}, \text{occ}_{231}, \text{occ}_{213}) \)
- \( (\text{occ}_{321}, \text{occ}_{231}, \text{occ}_{213}) \) and \( (\text{occ}_{123}, \text{occ}_{132}, \text{occ}_{213}) \)

are equidistributed on \( S_n(3\cdot1\cdot2) \), namely,

\[
    \sum_{\sigma \in S_n(3\cdot1\cdot2)} x^{\text{occ}_{321}(\sigma)} y^{\text{occ}_{132}(\sigma)} z^{\text{occ}_{213}(\sigma)} = \sum_{\sigma \in S_n(3\cdot1\cdot2)} x^{\text{occ}_{123}(\sigma)} y^{\text{occ}_{231}(\sigma)} z^{\text{occ}_{213}(\sigma)},
\]

\[
    \sum_{\sigma \in S_n(3\cdot1\cdot2)} x^{\text{occ}_{321}(\sigma)} y^{\text{occ}_{231}(\sigma)} z^{\text{occ}_{213}(\sigma)} = \sum_{\sigma \in S_n(3\cdot1\cdot2)} x^{\text{occ}_{123}(\sigma)} y^{\text{occ}_{132}(\sigma)} z^{\text{occ}_{213}(\sigma)}.
\]

\[ \diamond \]
References

[1] S.Avgustinovich, S.Kitaev, On uniquely k-determined permutations, Discrete Math., 308 (2008), no. 9, 1500-1507.

[2] E.Babson, E.Steingrimsson, Generalized permutation patterns and a classification of the Mahonian statistics, Sém. Lothar. Combin., 44 (2000), art. B44b, 18 pp. (electronic).

[3] A.Claesson, Generalized pattern avoidance, European J. Combin., 22 (2001) no. 7, 961-971.

[4] E.Deutsch, An involution on Dyck paths and its consequences, Discrete Math., 204 (1999), 163-166.

[5] E.Deutsch, Dyck path enumeration, Discrete Math., 204 (1999), 167-202.

[6] S.Elizalde, Asymptotic enumeration of permutations avoiding generalized patterns, Adv. in Appl. Math., 36 (2006), no. 2, 138-155.

[7] S.Elizalde, M.Noy, Consecutive patterns in permutations, Formal Power Series and Algebraic Combinatorics (Scottsdale, AZ, 2001), Adv. in Appl. Math., 30 (2003), no. 1-2, 110-125.

[8] S.Kitaev, T.Mansour, P.Séébold, Patrice Counting ordered patterns in words generated by morphisms, Integers, 8 (2008), A03, 28 pp (electronic).

[9] C.Krattenthaler, Permutations with restricted patterns and Dyck paths, Adv. in Appl. Math., 27 (2001), 510-530.

[10] T.Mansour, Restricted 1-3-2 permutations and generalized patterns, Ann. Comb., 6 (2002), no. 1, 65-76.

[11] T.Mansour, Statistics on Dyck paths, J. Integer Seq., 9 (2006), no. 1, Article 06.1.5, 13 pp. (electronic).

[12] A.Mendes, J.Remmel, Permutations and words counted by consecutive patterns, Adv. in Appl. Math., 37 (2006), no. 4, 443-480.
[13] D. Rawlings, The $q$-exponential generating function for permutations by consecutive patterns and inversions, *J. Combin. Theory Ser. A* 114 (2007), no. 1, 184-193.

[14] A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, *Discrete Math.*, 307 (2007), 2909-2924.

[15] R. Simion, F. W. Schmidt, Restricted permutations, *Europ. J. Combin.*, 6 (1985), 383-406.

[16] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, [http://www.research.att.com/~njas/sequences/](http://www.research.att.com/~njas/sequences/).

[17] E. Steingrímsson, Generalized permutations - a short survey, [arXiv:0801.2412](https://arxiv.org/abs/0801.2412).

[18] R. Warlimont, Permutations avoiding consecutive patterns, *Ann. Univ. Sci. Budapest. Sect. Comput.*, 22 (2003), 373-393.

[19] R. Warlimont, Permutations avoiding consecutive patterns II, *Arch. Math. (Basel)*, 84 (2005), no. 6, 496-502.