The existence of periodic solution for infinite dimensional Hamiltonian systems

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Abstract: In this paper, we will consider a kind of infinite dimensional Hamiltonian system (HS), by the method of saddle point reduction, topology degree and the index defined in \cite{11}, we will get the existence of periodic solution for (HS).

Keywords: infinite dimensional Hamiltonian systems; periodic solution; variational methods

1 Introduction and main results

1.1 Introduction of a kind of infinite dimensional Hamiltonian system

In this paper, we will consider the following infinite dimensional Hamiltonian system

\[
\begin{align*}
\partial_t u - \Delta_x u &= H_v(t, x, u, v), \\
-\partial_t v - \Delta_x v &= H_u(t, x, u, v),
\end{align*}
\]

\forall (t, x) \in \mathbb{R} \times \Omega, \quad (HS)

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with smooth boundary $\partial \Omega$ and $H : \mathbb{R} \times \Omega \times \mathbb{R}^{2m} \to \mathbb{R}$ is a $C^1$ function, $\partial_t := \frac{\partial}{\partial t}$, $\Delta_x := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$, $H_u := \frac{\partial H}{\partial u}$ and $H_v := \frac{\partial H}{\partial v}$. System like (HS) are called unbounded Hamiltonian system, cf. Barbu \cite{1}, or infinite dimensional Hamiltonian system, cf. \cite{2,4,5}. This systems arises in optimal control of systems governed by partial differential equations. See, e.g, Lions \cite{8}, where the combination of the model $\partial_t - \Delta_x$ and its adjoint $-\partial_t - \Delta_x$ acts as a system for studying the control. Brézis and Nirenberg \cite{3} considered a special case of the system (HS):

\[
\begin{align*}
\partial_t u - \Delta_x u &= -v^5 + f, \\
-\partial_t v - \Delta_x v &= u^3 + g,
\end{align*}
\]

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where \( f, g \in L^\infty(\Omega) \), subject to the boundary condition \( z(t, \cdot)|_{\partial \Omega} = 0 \) on variable \( x \) and the periodicity condition \( z(0, \cdot) = z(T, \cdot) = 0 \) on variable \( t \) for a given \( T > 0 \), where \( z = (u, v) \). They obtained a solution \( z \) with \( u \in L^4 \) and \( v \in L^6 \) by using Schauder’s fixed point theorem. Clément, Felemér and Mitidieri considered in [4] and [5] the following system which is also a special case of (HS):

\[
\begin{cases}
\partial_t u - \Delta_x u = |v|^{q-2}v, \\
-\partial_t v - \Delta_x v = |u|^{p-2}u,
\end{cases}
\]

(1.2)

with \( \frac{N}{N+2} < \frac{1}{p} + \frac{1}{q} < 1 \). Using their variational setting of Mountain Pass type, they proved that there is a \( T_0 > 0 \) such that, for each \( T > T_0 \), (1.2) has at least one positive solution \( z_T = (u_T, v_T) \) satisfying the boundary condition \( z_T(t, \cdot)|_{\partial \Omega} = 0 \) for all \( t \in (-T, T) \) and the periodicity condition \( z_T(T, \cdot) = z_T(-T, \cdot) \) for all \( x \in \overline{\Omega} \). Moreover, by passing to limit as \( T \to \infty \) they obtained a positive homoclinic solution of (1.2). If the Hamiltonian function \( H \) in (HS) can be displayed in the following form

\[
H(t, x, u, v) = F(t, x, u, v) - V(x)uv,
\]

(1.3)

where \( V \in C(\Omega, \mathbb{R}) \), \( H \in C^1(\mathbb{R} \times \overline{\Omega} \times \mathbb{R}^m, \mathbb{R}) \), the system (HS) will be rewritten as

\[
\begin{cases}
\partial_t u + (-\Delta_x + V(x))u = F_u(t, x, u, v), \\
\partial_t v + (-\Delta_x + V(x))v = F_u(t, x, u, v).
\end{cases}
\]

(HS.1)

Bartsch and Ding [2] dealt with the system (HS.1). They established existence and multiplicity of homoclinic solutions of the type \( z(t, x) \to 0 \) as \( |t| + |x| \to \infty \) if \( \Omega = \mathbb{R}^N \) and the type of \( z(t, x) \to 0 \) as \( |t| \to \infty \) and \( z(t, \cdot)|_{\partial \Omega} = 0 \) if \( \Omega \) is bounded. Recently, there are several results on system (HS) and (HS.1), cf. [6, 9–11, 13–16].

### 1.2 Introduction of relative Morse index \((\mu_L(M), \upsilon_L(M))\)

In [11], we developed the so called relative Morse index \((\mu_L(M), \upsilon_L(M))\) for (HS). Let \( I_m \) the identity map on \( \mathbb{R}^m \) and

\[
J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},
\]

(1.4)

\[
L := J\partial_t - N\Delta_x,
\]

(1.5)

and denoted by \( \nabla_z \) the gradient operator on variable \( z = (u, v)^T \), then (HS) with \( T \)-periodic and Dirichlet boundary conditions can be rewritten as

\[
\begin{cases}
Lz = \nabla_z H(t, x, z), \\
z(t, x) = z(t+T, x), \quad \forall (t, x) \in \mathbb{R} \times \Omega. \\
z(t, \partial \Omega) = 0,
\end{cases}
\]

(HS)
Let \( H := L^2(S^1 \times \Omega, \mathbb{R}^{2m}) \), where \( S^1 = \mathbb{R}/T \mathbb{Z} \). Then \( L \) is a self-adjoint operator acting in \( H \) with domain \( D(L) \). The linearized system of the nonlinear system in (HS) at a solution \( z = z(t, x) \) is the following system

\[
L y = M(t, x)y
\]

with \( M(t, x) = \nabla^2_h H(t, x, z(t, x)) \) and \( \nabla^2_h H \) the Hessian of \( H \) on the variable \( z \). Denote by \( SM^{2m} \) the set of all symmetric \( 2m \times 2m \) matrixes and \( M := C(S^1 \times \bar{\Omega}, SM^{2m}) \). Denote by \( L_s(H) \) the set of all bounded self-adjoint operators on \( H \). For any \( M \in M \), it is easy to see \( M \) determines a bounded self-adjoint operator on \( H \), by

\[
 z(t) \mapsto M(t, x)z(t), \quad \forall z \in H,
\]

we still denote this operator by \( M \). Thus, we have \( M \subset L_s(H) \). In [11], for any \( B \in L_s(H) \), we defined the relative Morse index pair

\[
(\mu_L(B), \nu_L(B)) \in \mathbb{Z} \times \mathbb{Z}^*, \tag{1.7}
\]

where \( \mathbb{Z} \) and \( \mathbb{Z}^* \) denote the set of all integers and non-negative integers respectively. Then, we got the relationship between the index \( \mu_L(B) \) and other indexes. Spectrally, with the relationship between the index \( \nu_L(B) \) and spectral flow, we have the following equality which will be used in this paper. For \( B_1, B_2 \in L_s(H) \), \( B_1 \leq B_2 \) means that \( B_2 - B_1 \) is semi-positive definite. Then, if \( B_1 \leq B_2 \), we have

\[
\mu_L(B_2) - \mu_L(B_1) = \sum_{s \in [0, 1)} \nu_L(sB_2 + (1 - s)B_1). \tag{1.8}
\]

By assuming some twisted conditions of the asymptotically linear Hamiltonian function, we studied the existence and multiplicity of (HS) in [11].

### 1.3 Main results

In this paper, we don’t need the Hamiltonian function \( H \) to be \( C^2 \) continuous and without assuming \( H \) satisfying the twisted conditions, by the method of topology degree, saddle point reduction and the index \( (i_L(B), \nu_L(B)) \) defined in [11], we have the following results.

**Theorem 1.1.** Assume \( H \) satisfies the following conditions.

\( (H_1) \) \( H \in C^1(S^1 \times \bar{\Omega} \times \mathbb{R}^{2m}, \mathbb{R}) \) and there exists \( l_H > 0 \), such that

\[
|H_z'(t, x, z + y) - H_z'(t, x, z)| \leq l_H |y|, \quad \forall (t, x) \in S^1 \times \bar{\Omega}, \quad z, y \in \mathbb{R}^{2m}.
\]
There exists $M_1, M_2, K > 0, B \in \mathcal{M}$, such that

$$H'(t, x, z) = B(t, x)z + r(t, x, z),$$

with

$$|r(t, x, z)| \leq M_1, \quad \forall (t, x, z) \in S^1 \times \bar{\Omega} \times \mathbb{R}^{2m},$$

and

$$\pm (r(t, x, z), z)_{\mathbb{R}^{2m}} \geq M_2 |z|_{\mathbb{R}^{2m}}, \quad \forall (t, x) \in S^1 \times \bar{\Omega}, \quad \|z\|_{\mathbb{R}^{2m}} > K.$$ (1.9)

Then (HS) has at least one solution.

In Theorem 1.1, we don’t need $B$ to be non-degenerate. If we assume some non-degenerate property of $B$, the rest item $r$ can be relaxed and we have the following result.

**Theorem 1.2.** Assume $H$ satisfying condition $(H_1)$ and the following condition

$(H_3)$ There exists $B \in C(S^1 \times \bar{\Omega} \times \mathbb{R}^{2m}, SM^{2m})$ such that

$$H'(t, z) = B(t, x, z)z + r(t, x, z), \quad \forall (t, x, z) \in S^1 \times \bar{\Omega} \times \mathbb{R}^{2m},$$

with

$$r(t, x, z) = o(z), \quad \text{uniformly for} |z| \to \infty.$$

$(H_4)$ There exist $B_1, B_2 \in \mathcal{M}$ satisfying

$$i_L(B_1) = i_L(B_2), \quad \nu_L(B_2) = 0,$$

and

$$B_1(t, x) \leq B(t, x, z) \leq B_2(t, x), \quad \forall (t, x, z) \in S^1 \times \bar{\Omega} \times \mathbb{R}^{2m}.$$ Then (HS) has at least one solution.

# Preliminarys and the proof of our main results

Before the proof of Theorem 1.1 and Theorem 1.2, we need some preliminarys. Firstly, we need the following Lemma.

**Lemma 2.1.** [11, Lemma 2.1] For simplicity, let $T = 2\pi$ and $\sigma(-\Delta) = \{\mu_l\}_{l \geq 1}$, we have

$$\sigma(L) = \sigma_p(L) = \{\pm (k^2 + \mu_l^2)^{1/2}\}_{k \in \mathbb{Z}, l \in \mathbb{N}},$$

where $\sigma_p(L)$ denotes the eigenvalue set of $L$ on $H$. That is to say $L$ has only eigenvalues. More over every eigenvalue in $\sigma_p(L)$ has $2m$ dimensional eigenspace.
Secondly, since $H$ satisfies condition ($H_1$), the map
\[ z \mapsto \int_{S^1 \times \bar{\Omega}} H(t, x, z(t, x))dtdx, \forall z \in H, \]
define a functional on $H$, without confusion, we still denote it by $H$. It is easy to see $H \in C^1(H, \mathbb{R})$, with
\[ (H'(z), y)_H = \int_{S^1 \times \bar{\Omega}} \left( (H'_1(t, x, z(t, x)), y(t))dtdx, \forall z, y \in H, \right. \]
and $H'$ is Lipschitz continuous with
\[ \|H'(z + y) - H'(z)\| \leq l_H \|y\|_H, \forall, y, z \in H. \tag{2.1} \]
Thus (HS) can be regard as an operator equation on $H$.

**Proof of Theorem 1.1.** Now, we consider the case of ($H_2^-$). From Lemma 2.1, $L$ has compact resolvent, since $B \in \mathcal{M} \subset L_s(H)$, 0 is at most an isolate point spectrum with finite dimensional eigenspace, that is to say there exists $\varepsilon_0 > 0$ and small enough, such that $(-\varepsilon_0, 0) \cap \sigma(L - B) = \emptyset$. For any $\varepsilon \in (0, \varepsilon_0)$ and $\lambda \in [0, 1]$, consider the following two-parameters equation
\[ (\varepsilon \cdot I + L - B)z = \lambda r(t, x, z), \quad (HS_{\varepsilon, \lambda}) \]
with $I$ the identity map on $H$. If $\varepsilon = 0$ and $\lambda = 1$, it is (HS). We divide the following proof into four steps.

**Step 1.** There exists a constant $C$ independent of $\varepsilon$ and $\lambda$, such that if $z_{\varepsilon, \lambda}$ is a solution of $(HS_{\varepsilon, \lambda})$,
\[ \varepsilon \|z_{\varepsilon, \lambda}\|_H \leq C, \forall (\varepsilon, \lambda) \in (0, \frac{\varepsilon_0}{2}) \times [0, 1]. \]

Since $(-\varepsilon_0, 0) \cap \sigma(L - B) = \emptyset$, we have $(\varepsilon - \varepsilon_0, \varepsilon) \cap \sigma(\varepsilon \cdot I + L - B) = \emptyset$. Consider the orthogonal splitting
\[ H = H_{\varepsilon \cdot I + L - B}^- \oplus H_{\varepsilon \cdot I + L - B}^+, \]
where $\varepsilon \cdot I + L - B$ is negative definite on $H_{\varepsilon \cdot I + L - B}^-$, and positive define on $H_{\varepsilon \cdot I + L - B}^+$. Thus, if $z \in H$, we have the splitting
\[ z = x + y, \]
with $x \in H_{\varepsilon \cdot I + L - B}^-$ and $y \in H_{\varepsilon \cdot I + L - B}^+$. If $z_{\varepsilon, \lambda}$ is a solution of $(HS_{\varepsilon, \lambda})$ with its splitting $z_{\varepsilon, \lambda} = x_{\varepsilon, \lambda} + y_{\varepsilon, \lambda}$ defined above, then we have
\[ ((\varepsilon \cdot I + L - B)z_{\varepsilon, \lambda}, y_{\varepsilon, \lambda} - x_{\varepsilon, \lambda})_H = \lambda (r(t, z_{\varepsilon, \lambda}), y_{\varepsilon, \lambda} - x_{\varepsilon, \lambda})_H. \]
Since \((\varepsilon - \varepsilon_0, \varepsilon) \cap \sigma(\varepsilon \cdot I + L - B) = \emptyset\), we have
\[
((\varepsilon \cdot I + L - B)z_{\varepsilon, \lambda}, y_{\varepsilon, \lambda} - x_{\varepsilon, \lambda})_H \geq \min\{\varepsilon_0 - \varepsilon, \varepsilon\} \|z_{\varepsilon, \lambda}\|_H^2.
\]
Since \(r\) is bounded, for \((\varepsilon, \lambda) \in (0, \varepsilon_0^2) \times [0, 1]\), we have
\[
C \|z_{\varepsilon, \lambda}\|_H \geq \lambda(r(t, x, z_{\varepsilon, \lambda}), y_{\varepsilon, \lambda} - x_{\varepsilon, \lambda})_H \geq \varepsilon \|z_{\varepsilon, \lambda}\|_H^2.
\]
Therefore, we have
\[
\varepsilon \|z_{\varepsilon, \lambda}\|_H \leq C, \quad \forall (\varepsilon, \lambda) \in (0, \varepsilon_0^2) \times [0, 1].
\]

**Step 2.** For any \((\varepsilon, \lambda) \in (0, \varepsilon_0^2) \times [0, 1]\), \((HS_{\varepsilon, \lambda})\) has at least one solution. Here, we use the topology degree theory. Since \(0 \notin \sigma(\varepsilon \cdot I + L - B)\), \((HS_{\varepsilon, \lambda})\) can be rewritten as
\[
z = \lambda(\varepsilon \cdot I + L - B)^{-1}r(t, x, z).
\]
Denote by \(f(\varepsilon, \lambda, z) := \lambda(\varepsilon \cdot I + L - B)^{-1}r(t, x, z)\) for simplicity. From the compactness of \((\varepsilon \cdot I + L - B)^{-1}\) and condition \((H^-_2)\), Leray Schauder degree theory can be used to the map
\[
z \mapsto z - f(\varepsilon, \lambda, z), \quad z \in H.
\]
From the result received in Step 1, we have
\[
deg(I - f(\varepsilon, \lambda, \cdot), B(R(\varepsilon), 0), 0) \equiv deg(I - f(\varepsilon, 0, \cdot), B(R(\varepsilon), 0), 0) = deg(I, B(R(\varepsilon), 0), 0) = 1,
\]
where \(R(\varepsilon) > \frac{C}{\varepsilon}\) is a constant only depends on \(\varepsilon\), and \(B(R(\varepsilon), 0) := \{z \in H||z||_H < R(\varepsilon)\}\).

**Step 3.** For \(\lambda = 1, \varepsilon \in (0, \varepsilon_0/2)\), denote by \(z_\varepsilon\) one of the solutions of \((HS_{\varepsilon, 1})\). We have \(\|z_\varepsilon\|_H \leq C\). In this step, \(C\) denotes various constants independent of \(\varepsilon\).

From the boundedness received in Step 1, we have
\[
\|(L - B)z_\varepsilon\|_H = \|\varepsilon z_\varepsilon - r(t, x, z_\varepsilon)\|_H \leq C. \quad (2.2)
\]
Now, consider the orthogonal splitting
\[
L = H^0_{L-B} \oplus H^\perp_{L-B},
\]
where \(L - B\) is zero definite on \(H^0_{L-B}\), and \(H^\perp_{L-B}\) is the orthonormal complement space of \(H^0_{L-B}\). Let \(z_\varepsilon = u_\varepsilon + v_\varepsilon\) with \(u_\varepsilon \in H^0_{L-B}\) and \(v_\varepsilon \in H^\perp_{L-B}\). Since \(0\) is an isolated point in \(\sigma(L - B)\), from \((2.2)\), we have
\[
\|v_\varepsilon\|_H \leq C. \quad (2.3)
\]
Additionally, since \( r(t, x, z_e) \) and \( v_e \) are bounded in \( H \), we have

\[
(r(t, x, z_e), z_e)_{H} = (r(t, x, z_e), v_e)_{H} + (r(t, x, z_e), u_e)_{H} \\
= (r(t, x, z_e), v_e)_{H} + (z_e + (L - B)z_e, u_e)_{H} \\
= (r(t, x, z_e), v_e)_{H} + (u_e, u_e)_{H} \\
\geq C. \tag{2.4}
\]

On the other hand, from (1.9) in \((H_2^-)\), we have

\[
(r(t, x, z_e), z_e)_{H} = \int_{S^1 \times \bar{\Omega}} (r(t, x, z_e), z_e) dtdx + \int_{S^1 \times \bar{\Omega} / S^1 \times \Omega(K)} (r(t, x, z_e), z_e) dtdx \\
\leq -M_2 \int_{S^1 \times \bar{\Omega}(K)} |z_e|^2 dtdx + C \\
\leq -M_2 \|z_e\|_{H}^2 + C, \tag{2.5}
\]

where \( S^1 \times \bar{\Omega}(K) := \{(t, x) \in S^1 \times \bar{\Omega} | |z_e| > K\} \). From (2.4) and (2.5), we have proved the boundedness of \( \|z_e\|_{H} \).

**Step 4. Passing to a sequence of \( \varepsilon_n \rightarrow 0 \), there exists \( z \in H \) such that**

\[
\lim_{\varepsilon_n \rightarrow 0} \|z_{\varepsilon_n} - z\|_{H} = 0.
\]

Here, we will use the method of saddle point reduction. Since \( \sigma(L) \) has only isolate finite dimensional eigenvalues and from condition \((H_1)\), we can assume \( \pm l_H \notin \sigma(L) \). That is to say there exists \( \delta > 0 \) such that

\[
(-l_H - \delta, -l_H + \delta) \cap \sigma(L) = (l_H - \delta, l_H + \delta) \cap \sigma(L) = \emptyset.
\]

Denote \( E_L \) the spectrum measure of \( L \) and definite the projections on \( H \) by

\[
P_{L,l_H}^0 := \int_{-l_H}^{l_H} dE_L(z), \quad P_{L,l_H}^1 := I - P_{L,l_H}^0,
\]

where \( I \) is the identity map on \( H \). Correspondingly, consider the splitting of \( H \) by

\[
H = H_{L,l_H}^0 \oplus H_{L,l_H}^1 \tag{2.7}
\]

with \( H_{L}^* := P_{L,l_H}^* H (\star = 0, \perp) \). Without confusion, we rewrite \( P^* := P_{L,l_H}^* \) and \( H^* := H_{L,l_H}^* \) for simplicity \((\star = 0, \perp)\). Denote \( L^* = L|_{H^*} \) and \( z^* = P^* z \), for all \( z \in H \). thus we have \( L^\perp \) has bounded inverse on \( H^\perp \) and

\[
\| (L^\perp)^{-1} \| \leq \frac{1}{l_H + \delta}.
\]
Let \( \varepsilon' := \min\{\varepsilon_0, \delta\} \), for \( \varepsilon \in (0, \frac{\pi}{2}) \), denote by \( L_\varepsilon := \varepsilon + L \). Then \( L_\varepsilon \) has the same invariant subspace with \( L \), so we can also denote by \( L'_\varepsilon := L_\varepsilon|_H \) (\( \ast = 0, \perp \)), and we have

\[
\|(L'_\varepsilon)^{-1}\| \leq \frac{1}{l_H + \delta/2}.
\] (2.8)

Since \( z_\varepsilon \) satisfies \((HS_{\varepsilon,1})\), so we have

\[
L'_\varepsilon z_\varepsilon = P^\perp H'(z_\varepsilon^\perp + z_\varepsilon^0),
\]

and

\[
z_\varepsilon^\perp = (L'_\varepsilon)^{-1} P^\perp \Phi'(z_\varepsilon^\perp + z_\varepsilon^0).
\] (2.9)

Since \( H^0 \) is a finite dimensional space and \( \|z_\varepsilon\|_H \leq C \), there exists a sequence \( \varepsilon_n \to 0 \) and \( z^0 \in H^0 \), such that

\[
\lim_{n \to \infty} z^0_{\varepsilon_n} = z^0.
\]

For simplicity, we rewrite \( z_n^\ast := z_{\varepsilon_n}^\ast (\ast = \perp, 0) \), \( L_n := \varepsilon_n + L \) and \( L_n^\perp := L_{\varepsilon_n}^\perp \). So, we have

\[
\|z_n^\perp - z_m^\perp\|_H = \|(L_n^\perp)^{-1} P^\perp \Phi'(z_n^\perp) - (L_m^\perp)^{-1} P^\perp \Phi'(z_m^\perp)\|_H
\]

\[
\leq \|(L_n^\perp)^{-1} P^\perp (\Phi'(z_n^\perp) - \Phi'(z_m^\perp))\|_H + \|(L_n^\perp)^{-1} - (L_m^\perp)^{-1}\| P^\perp \Phi'(z_m^\perp)\|_H
\]

\[
\leq \frac{l_H}{l_H + \delta/2} \|z_n^\perp - z_m^\perp\|_H + \|(L_n^\perp)^{-1} - (L_m^\perp)^{-1}\| P^\perp \Phi'(z_m^\perp)\|_H.
\]

Since \((L_n^\perp)^{-1} - (L_m^\perp)^{-1} = (\varepsilon_n - \varepsilon_n)(L_n^\perp)^{-1}(L_m^\perp)^{-1} \) and \( z_n \) are bounded in \( H \), we have

\[
\|(L_n^\perp)^{-1} - (L_m^\perp)^{-1}\| = o(1), \quad n, m \to \infty.
\]

So we have

\[
\|z_n^\perp - z_m^\perp\|_H \leq \frac{2l_H}{\delta} \|z_n^0 - z_m^0\|_H + o(1), \quad n, m \to \infty,
\]

therefor, there exists \( z^\perp \in H^\perp \), such that \( \lim_{n \to \infty} \|z_n^\perp - z^\perp\|_H = 0 \). Thus, we have

\[
\lim_{n \to \infty} \|z_{\varepsilon_n}^\perp - z\|_H = 0,
\]

with \( z = z^\perp + z^0 \). Last, let \( n \to \infty \) in \((HS_{\varepsilon,1})\), we have \( z \) is a solution of \((HS)\). \( \square \)

Before the proof of Theorem 1.2, we need the following Lemma.

**Lemma 2.2.** Let \( B_1, B_2 \in \mathcal{L}_s(H) \) with \( B_1 \preceq B_2 \), \( \mu_L(B_1) = \mu_L(B_2) \), and \( \nu_L(B_2) = 0 \), then there exists \( \varepsilon > 0 \), such that for all \( B \in \mathcal{L}_s(H) \) with

\[
B_1 \preceq B \preceq B_2,
\]

8
we have
\[ \sigma(L - B) \cap (-\varepsilon, \varepsilon) = \emptyset. \]

**Proof.** For the property of \( \mu_L(B) \), we have \( \nu_L(B_1) = 0 \). So there is \( \varepsilon > 0 \), such that\[ \mu_L(B_{1,\varepsilon}) = \mu_L(B_1) = \mu_L(B_2) = \mu_L(B_{2,\varepsilon}), \]
with \( B_{*,\varepsilon} = B_* + \varepsilon \cdot I, (\ast = 1, 2) \). Since \( B_{1,\varepsilon} \leq B - \varepsilon I < B + \varepsilon I \leq B' \). It follows that\[ \mu_L(B - \varepsilon I) = \mu_L(B + \varepsilon I). \]

Note that by (1.8)
\[ \sum_{-\varepsilon < t \leq \varepsilon} \nu_L(B - t \cdot I) = \mu_L(B + \varepsilon I) - \mu_L(B - \varepsilon I) = 0. \]

We have \( 0 \notin \sigma(L - B - \eta), \forall \eta \in (-\varepsilon, \varepsilon) \), thus the proof is complete. \( \square \)

**Proof of Theorem 1.2.** Consider the following one-parameter equation
\[ Lz = (1 - \lambda)B_1 z + \lambda H'(z), \quad (HS_\lambda) \]
with \( \lambda \in [0, 1] \). Denote by
\[ \Phi_\lambda(z) = \frac{1 - \lambda}{2}(B_1 z, z)_L + \lambda \Phi(z), \forall z \in L. \]
Since \( H \) satisfies condition \( (H_1) \) and \( B_1 \in C(S^1 \times \bar{\Omega}, \mathbf{SM}^{2m}) \), we have \( \Phi_\lambda : H \to H \) is Lipschitz continuous, and there exists \( l' > 0 \) independent of \( \lambda \) such that \( l' \notin \sigma(L) \) and
\[ \| \Phi'_\lambda(z + h) - \Phi'_\lambda(z) \|_H \leq l' \| h \|_H, \forall z, h \in L, \lambda \in [0, 1]. \]

Now, replace \( l_H \) by \( l' \) in (2.6), we have the projections \( P_{\lambda, l'}^* \) \((\ast = \perp, 0)\) and the splitting
\[ H = H_{L, l'}^\perp \oplus H_{L, l'}^0, \]
with \( H_{L, l'}^\perp = P_{\lambda, l'}^* H(\ast = \perp, 0) \). Thus \( L^\perp \) has bounded inverse on \( H_{L, l'}^\perp \) with
\[ \| (L^\perp)^{-1} \| < \frac{1}{l' + c}, \]
for some \( c > 0 \). Without confusion, we still use \( z^\perp \) and \( z^0 \) to represent the splitting
\[ z = z^\perp + z^0, \]
with \( z^* \in H_{L, l'}^\ast \) \((\ast = \perp, 0)\). Now, we derive the following proof into three steps and \( C \) denotes various constants independent of \( \lambda \).
Step 1. If \( z \) is a solution of \((HS_\lambda)\), then we have \( \|z^+(z^0)\|_H \leq C\|z^0\|_H + C \)

Since \( Lz = \Phi'_\lambda(z) \), we have

\[
\|z^+(z)\|_H = \|(L^\pm)^{-1}P^\pm_L\Phi'_\lambda(z^+(z^0) + z^0)\|_H \\
\leq \frac{1}{\nu + c}\|\Phi'_\lambda(z^+(z^0) + z^0)\|_H \\
\leq \frac{1}{\nu + c}\|\Phi'_\lambda(z^+(z^0) + z^0) - \Phi'_\lambda(0)\|_H + \frac{1}{\nu + c}\|\Phi'_\lambda(0)\|_H \\
\leq \frac{\nu'}{\nu + c}(\|z^+(z^0)\|_H + \|z^0\|_H) + \frac{1}{\nu + c}\|\Phi'_\lambda(0)\|_H.
\]

So we have \( \|z^+(z^0)\|_H \leq \frac{\nu'}{\nu + c}\|z^0\|_H + \frac{1}{\nu + c}\|\Phi'_\lambda(0)\|_H \). Thus, we have proved this step.

Step 2. We claim that the set of all the solutions \((z, \lambda)\) of \((HS_\lambda)\) are a priori bounded.

If not, assume there exist \(\{(z_n, \lambda_n)\}\) satisfying \((HS_\lambda)\) with \(\|z_n\|_H \to \infty\). Without lose of generality, assume \(\lambda_n \to \lambda_0 \in [0,1]\). From step 1, we have \(\|z^0_n\|_L \to \infty\). Denote by

\[
y_n = \frac{z_n}{\|z_n\|_H},
\]

and \(\tilde{B}_n := (1 - \lambda_n)B_1 + \lambda_nB(t, z_n)\), we have

\[
Ay_n = \tilde{B}_ny_n + \frac{O(\|z_n\|_H)}{\|z_n\|_H}. \quad (2.10)
\]

Decompose \(y_n = y^+_n + y^-_n\) with \(y^*_n = z^*_n/\|z_n\|_H\), we have

\[
\|y^0_n\|_H = \|z^0_n\|_H/\|z_n\|_H \\
\geq \frac{\|z^0_n\|_H}{\|z^0_n\|_H + \|z^+_n\|_H} \\
\geq \frac{\|z^0_n\|_H}{C\|z^0_n\|_H + C}.
\]

That is to say

\[
\|y^0_n\|_H \geq C > 0, \quad (2.11)
\]

for \(n\) large enough. Since \(B_1(t) \leq B(t, z) \leq B_2(t)\), we have \(B_1 \leq \tilde{B}_n \leq B_2\). Let \(H = H^+_{L-B_n} \bigoplus H^-_{L-B_n}\) with \(L - \tilde{B}_n\) is positive and negative define on \(H^+_{L-B_n}\) and \(H^-_{L-B_n}\) respectively. Re-decompose \(y_n = \tilde{y}^+_n + \tilde{y}^-_n\) respect to \(H^+_{L-B_n}\) and \(H^-_{L-B_n}\). From \((H_4)\) and
(2.10), we have
\[
\|y^0_n\|_H^2 \leq \|y_n\|_H^2
\leq C((A - \bar{B}_n)y_n, \bar{y}^+_n + \bar{y}^-_n)_H
\leq \frac{o(\|z_n\|_H)}{\|z_n\|_H}\|y_n\|_H.
\]
(2.12)

Since \(\|z_n\|_H \to \infty\) and \(\|y_n\|_H = 1\), we have \(\|y^0_n\|_H \to 0\) which contradicts to (2.11), so we have \(\{z_n\}\) is bounded.

**Step 3. By Leray-Schauder degree, there is a solution of (HS).**
Since the solutions of \((HS_\lambda)\) are bounded, there is a number \(R > 0\) large enough, such that all of the solutions \(z_\lambda\) of \((HS_\lambda)\) are in the ball \(B(0, R) := \{ z \in L|\|z\|_H < R\}\). So we have the Leray-Schauder degree
\[
deg(I - (L - B_1)^{-1}(\Phi'(z) - B_1z), B(0, R)_\cap, 0) = deg(I, B(0, R), 0) = 1.
\]
That is to say (HS) has at least one solution.

### 3 Further results

In the system of (HS), Lemma 2.1 played an important role to keep the Leray-Schauder degree valid, if we change the Dirichlet boundary condition \(z(t, \partial \Omega) = 0\) in (HS) to Neumann boundary condition \(\frac{\partial z}{\partial n}(t, \partial \Omega) = 0\), we will also have Lemma 2.1, thus Theorem 1.1 and Theorem 1.2 will also be true for Neumann boundary condition.

What we want to say in this section is \(\Omega = \mathbb{R}^N\). Generally, the operator \(-\Delta_x\) on \(L^2(\mathbb{R}^N, \mathbb{R})\) doesn’t have compact inversion, then the results in Lemma 2.1 will not be true. Thus our Maslov type index theory defined in [11] will not work. But if the Hamiltonian function \(H\) can be displayed in the following form
\[
H(t, x, u, v) = F(t, x, u, v) - V(x)uv,
\]
then system (HS) will be rewritten as systems
\[
\begin{align*}
\partial_t u + (-\Delta_x + V(x))u &= F_v(t, x, u, v), \\
-\partial_t v + (-\Delta_x + V(x))v &= F_u(t, x, u, v).
\end{align*}
\]
\((HS.1)\)

We have the following result.
Lemma 3.1. [7, Lemma 6.10]. If the function $V(x)$ satisfies the following conditions:

(V$_1$) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^N} V(x) > 0$.

(V$_2$) There exists $l_0 > 0$ and $M > 0$ such that

$$\lim_{|y| \to \infty} \text{meas}\{x \in \mathbb{R}^N : |x - y| \leq l_0, V(x) \leq M\} = 0,$$

where $\text{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$. Then we have

$$\sigma_e(-\Delta x + V(x)) \subset [M, +\infty),$$

where $\sigma_e(A)$ denotes the essential spectrum of operator $A$.

If we redefine the operator $L$ as

$$L := J\partial_t - N(\Delta x - V(x)).$$

We have the following result.

Lemma 3.2. If $\sigma_e(-\Delta x + V(x)) \subset [M, +\infty)$, then

$$\sigma_e(L) \cap (-M, M) = \emptyset.$$

Proof. Let $E(z)$ be the spectrum measure of $-\Delta x + V(x)$, for any $\delta > 0$ small enough, define the following projection on $L^2(\mathbb{R}^N, \mathbb{R})$,

$$P_{-\Delta x + V(x), \delta} = \int_0^{M-\delta} dE(z).$$

We have the following orthogonal splitting

$$L^2(\mathbb{R}^N, \mathbb{R}) = L^2(\delta) \oplus (L^2(\delta)),$$

where $L^2(\delta) := P_{-\Delta x + V(x), \delta} L^2(\mathbb{R}^N, \mathbb{R})$ and $(L^2(\delta))$ is its orthogonal complement. So we have

$$-\Delta x + V(x)|_{L^2(\delta)} \leq M - \delta, \quad -\Delta x + V(x)|_{(L^2(\delta))} \geq M - \delta.$$

Since

$$L^2(S^1 \times \mathbb{R}^N, \mathbb{R}^{2m}) = L^2(S^1, L^2(\mathbb{R}^N, \mathbb{R}^{2m})) = L^2(S^1, L^2(\mathbb{R}^N, \mathbb{R})^{2m}) = L^2(S^1, (L^2(\delta) \oplus L^2(\delta))^{2m}) = L^2(S^1, L^2(\delta)^{2m} \oplus L^2(\delta)^{2m}) = L^2(S^1, L^2(\delta)^{2m}) \oplus L^2(S^1, L^2(\delta)^{2m}),$$
$L^2(S^1, L^2(\delta)^{2m})$ and $L^2(S^1, L^{2,1}(\delta)^{2m})$ are invariant subspaces of $L$, let

$$L_1 := L|_{L^2(S^1, L^2(\delta)^{2m})}, \quad L_2 := L|_{L^2(S^1, L^{2,1}(\delta)^{2m})}.$$ 

Corresponding to the splitting of $L^2(S^1 \times \mathbb{R}^N, \mathbb{R}^{2m})$, we have

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$ 

So we have

$$\sigma(L) = \sigma(L_1) \cup \sigma(L_2).$$ 

With the similarly method in Lemma 2.1, we can prove $\sigma_e(L_1) = \emptyset$, so

$$\sigma_e(L) = \sigma_e(L_2).$$

Now, we will prove $\sigma(L_2) \cap (-M, M) = \emptyset$. For any $\lambda \in \sigma(L_2)$, we have $z_n \in L_2$ with $\|z_n\| = 1$, such that

$$\|L_2 z_n - \lambda z_n\| \to 0.$$ 

So we have

$$(Lz_n - \lambda z_n, Nz_n) \to 0,$$ 

that is to say

$$(-J \frac{\partial}{\partial t} z_n - N(\Delta - V(x))z_n, Nz_n) - \lambda(z_n, Nz_n) \to 0.$$ 

Since $(-J \frac{\partial}{\partial t} z_n, Nz_n) \equiv 0$ and from $z_n \in L_2$ we have $(-N(\Delta - V(x))z_n, Nz_n) \geq M$, so we have $|\lambda| \geq M$. Thus, we have finished the proof. \hfill \Box

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