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To cite this version:
van Duong Dinh. On instability of standing waves for the mass-supercritical fractional nonlinear Schrödinger equation. 2018. hal-01936071
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Abstract. We consider the focusing $L^2$-supercritical fractional nonlinear Schrödinger equation

$$i\partial_t u - (-\Delta)^s u = -|u|^\alpha u, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where $d \geq 2$, $\frac{d}{2s-1} \leq s < 1$ and $\frac{d}{s} < \alpha < \frac{d+1}{2s-2}$. By means of the localized virial estimate, we prove that the ground state standing wave is strongly unstable by blow-up. This result is a complement to a recent result of Peng-Shi [J. Math. Phys. 59 (2018), 011508] where the stability and instability of standing waves were studied in the $L^2$-subcritical and $L^2$-critical cases.

Keywords. Fractional nonlinear Schrödinger equation; Standing wave; Instability; Localized virial estimate; Blow-up.

1. Introduction

In recent years, there has been a great deal of interest in studying the nonlinear fractional Schrödinger equation, namely

$$i\partial_t u - (-\Delta)^s u = f(u), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where $0 < s < 1$ and $f(u)$ is the nonlinearity. The fractional differential operator $(-\Delta)^s$ is defined by $(-\Delta)^s u = F^{-1} \left[|\xi|^{2s} F u\right]$, where $F$ and $F^{-1}$ are the Fourier transform and inverse Fourier transform, respectively. The fractional nonlinear Schrödinger equation was first discovered by N. Laskin [19, 20] owing to the extension of the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. The fractional nonlinear Schrödinger equation also appears in the continuum limit of discrete models with long-range interactions (see [18]) and in the description of Boson stars as well as in water wave dynamics (see e.g. [13] or [15]).

In this paper, we consider the Cauchy problem for the focusing fractional nonlinear Schrödinger equation

$$\begin{cases}
    i\partial_t u - (-\Delta)^s u = -|u|^\alpha u, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\
    u(0) = u_0,
\end{cases} \quad (1.2)$$

where $u$ is a complex valued function defined on $\mathbb{R}_+ \times \mathbb{R}^d$, $d \geq 1$, $0 \leq s < 1$ and $0 < \alpha < \alpha^*$ with

$$\alpha^* := \begin{cases}
    \frac{d}{2s} & \text{if } d > 2s, \\
    \infty & \text{if } d \leq 2s.
\end{cases} \quad (1.3)$$

Throughout the sequel, we call a standing wave a solution of (1.2) of the form $e^{i\omega t}\phi_\omega$, where $\omega \in \mathbb{R}$ is a frequency and $\phi_\omega \in H^s$ is a non-trivial solution to the elliptic equation

$$(-\Delta)^s \phi_\omega + \omega \phi_\omega - |\phi_\omega|^\alpha \phi_\omega = 0. \quad (1.4)$$

We are interested in the instability of standing waves for (1.2). Before stating our main result, let us recall known results of orbital stability and instability of standing waves for (1.1). In the case of Hartree-type nonlinearity $f(u) = -((|x|^{-\gamma} * |u|^2)u$, Wu [24] showed the existence of stable standing waves for $d \geq 1$, $0 < s < 1$ and $0 < \gamma < \min\{2s, d\}$. Zhang-Zhu [25] extended the result of Wu and showed the strong instability of standing waves for $d \geq 2$, $0 < s < 1$ and $\gamma = 2s$. Recently, Cho-Fall-Hajaiej-Markowich-Trabelsi [4] studied the existence of stable standing waves for more general Hartree-type nonlinearities. In
the case of Choquard nonlinearity $f(u) = -(I_\beta|u|^p |u|^{p-2}u$ with $d \geq 3$, $0 < s < 1$, $1 + \frac{\beta}{d} < p < 1 + \frac{2s + \beta}{d}$ and
\[
I_\beta := A(\beta)|x|^{-(d-\beta)}, \quad A(\beta) = \frac{\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{d}{2}\right) \pi^{\frac{d}{2}} 2^\beta}, \quad 0 < \beta < d,
\]
Feng-Zhang [10] established the stability of standing waves under an assumption on the local well-posedness of the Cauchy problem (1.1). In the case of combined power-type and Choquard nonlinearities $f(u) = -|u|^\alpha u - (I_\beta|u|^p |u|^{p-2}u$, Bhattachar [2] proved the existence of stable standing waves for $d \geq 2$, $0 < s < 1$, $0 < \alpha < \frac{4s}{d}$ and $2 \leq p < 1 + \frac{2s+\beta}{d}$. Recently, Feng-Zhang [9] showed the stability of standing waves for $d \geq 2$, $0 < s < 1$, $\alpha = \frac{4s}{d}$ and $1 + \frac{2s}{d} < p < 1 + \frac{2s + \beta}{d}$ under an assumption on the local theory of the Cauchy problem (1.1) and an assumption on the initial data $\|u_0\|_{L^s} < \|Q\|_{L^s}$, where $Q$ is the ground state of
\[
(-\Delta)^s Q + |Q|^{\frac{4s}{d}} Q = 0. \tag{1.5}
\]
In the case of combined power-type nonlinearities $f(u) = -|u|^\alpha u - |u|^\beta u$, Guo-Huang [14] showed the existence of stable standing waves for $0 < \alpha_1 < \alpha_2 < \frac{4s}{d}$. Cho-Hwang-Hajaiej-Ozawa [5] proved the stability of standing waves for more general subcritical nonlinearities. For $0 < \alpha_1 < \alpha_2 = \frac{4s}{d}$, Zhu [26] showed the existence of stable standing waves with $\|u_0\|_{L^s} < \|Q\|_{L^s}$, where $Q$ is the ground state of (1.5). In the case of logarithmic nonlinearity $f(u) = -u \log(|u|^2)$, Ardila [1] proved the existence of stable standing waves for $d \geq 2$ and $0 < s < 1$.

In the case of a single power-type nonlinearity (1.2), Peng-Shi [22] recently established the existence of stable standing waves for $d \geq 2$, $0 < s < 1$ and $0 < \alpha < \frac{4s}{d}$. They also proved the strong instability of standing waves for $d \geq 2$, $\frac{1}{2} < s < 1$ and $\alpha = \frac{4s}{d}$. Note that the local well-posedness is not considered in [22]. Due to loss of regularity in Strichartz estimates for the unitary group $e^{i(-\Delta)^s t}$, there are restrictions on the local well-posedness in $H^s$ for (1.2) with non-radial initial data (see e.g. [16] or [6]). One can overcome the loss of derivatives in Strichartz estimates by considering radial initial data. However, it leads to another restriction on the validity of $d$ and $s$, that is, $d \geq 2$ and $\frac{d}{2d-1} \leq s < 1$. We refer the reader to [7] for the local well-posedness for (1.2) with $H^s$ radial initial data.

The main purpose of this paper is to show the strong instability of ground state standing waves for (1.2) in the mass-supercritical and energy-subcritical case $\frac{4s}{d} < \alpha < \frac{4s}{d-2}$. In order to state our main result, let us introduce the notion of ground states related to (1.4).

**Definition 1.1 (Ground states).** A non-zero, non-negative $H^s$ solution $\phi_\omega$ to (1.4) is called a **ground state related to (1.4)** if it is a minimizer of the Weinstein’s functional
\[
J(v) := \left[\|v\|^2_{H^s} + \|v\|^{\alpha+2}_{L^{\alpha+2}}\right]^{\frac{2}{\alpha+2}} \|v\|^s_{L^{\alpha+2}}, \tag{1.6}
\]
that is,
\[
J(\phi_\omega) = \inf \{J(v) : v \in H^s \setminus \{0\}\}.
\]

Similarly, a non-zero, non-negative $H^s$ solution $\phi$ to the elliptic equation
\[
(-\Delta)^s \phi + \phi |\phi|^{\alpha} \phi = 0 \tag{1.7}
\]
called a ground state related to (1.7) if it is a minimizer of the Weinstein’s functional (1.6). Note that for $d \geq 1$, $0 < s < 1$ and $0 < \alpha < \alpha^*$, the existence and uniqueness (up to translation) of ground states related to (1.7) were established recently in [11, 12]. Moreover, the ground state related to (1.7) can be chosen to be radially symmetric, strictly positive and strictly decreasing in $|x|$.

Now, let $\phi$ be the ground state related to (1.7). It is easy to see that for $\omega > 0$, the scaling $\phi_\omega(x) := \omega^\frac{s}{2} \phi \left(\omega^\frac{s}{2} x\right)$ maps a solution of (1.7) to a solution of (1.4). We thus get a non-zero, non-negative $H^s$ solution to (1.4). On the other hand, a direct computation shows that
\[
\|\phi_\omega\|^2_{L^2} = \omega^{\frac{4s}{d} - \frac{d}{2}} \|\phi\|^2_{L^2}, \quad \|\phi_\omega\|_{H^s}^2 = \omega^{\frac{4s}{d} + s - \frac{d}{2}} \|\phi\|^2_{H^s}, \quad \|\phi_\omega\|_{L^{\alpha+2}}^{\alpha+2} = \omega^{\frac{4s}{d}(\alpha+2) - \frac{d}{2}} \|\phi\|_{L^{\alpha+2}}^{\alpha+2}.
\]
It follows that
\[
J(\phi_\omega) = \left[\|\phi_\omega\|^2_{H^s} \|\phi_\omega\|_{L^{\alpha+2}}^{\alpha+2} \right]^{\frac{2}{\alpha+2}} \|\phi_\omega\|_{L^{\alpha+2}}^{s} = J(\phi).
\]
Thus, $\phi_\omega$ is a minimizer of the Weinstein’s functional. By definition, $\phi_\omega$ is a ground state related to (1.4). Similarly, we can construct ground states related to (1.7) from ground states related to (1.4). This implies the existence and uniqueness (up to translation) of ground states related to (1.4) when $\omega > 0$. Moreover, the unique (up to translation) ground state related to (1.4) can be chosen to be radially symmetric, strictly positive and strictly decreasing in $|x|$.
It is easy to see that (1.4) can be written as $S_\omega'(\phi_\omega) = 0$, where
\[
S_\omega(v) := E(v) + \frac{\omega}{2}\|v\|_{L^2}^2
= \frac{1}{2}\|v\|_{H^s}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{1}{\alpha + 2}\|v\|_{L^{\alpha+2}}^{\alpha+2}
\]
is the action functional. We also define the Nehari functional
\[
K_\omega(v) := \partial_\lambda S_\omega(\lambda v)|_{\lambda = 1} = \|v\|_{H^s}^2 + \omega\|v\|_{L^2}^2 - \|v\|_{L^{\alpha+2}}^{\alpha+2}.
\]
From now on, we denote the functional
\[
H_\omega(v) := \|v\|_{H^s}^2 + \omega\|v\|_{L^2}^2.
\]
It is easy to see that for $\omega > 0$ fixed,
\[
H_\omega(v) \sim \|v\|_{H^s}^2.
\]
Let us start with the following observation concerning the ground state related to (1.4).

**Proposition 1.2.** Let $d \geq 1$, $0 < s < 1$, $0 < \alpha < \alpha^*$ with $\alpha^*$ as in (1.3) and $\omega > 0$. Let $\phi_\omega$ be the ground state related to (1.4). Then
\[
S_\omega(\phi_\omega) = \inf \{ S_\omega(v) : v \in H^s \backslash \{0\}, \ K_\omega(v) = 0 \}. \quad (1.9)
\]
We refer the reader to Section 3 for the proof of the above result. We next recall the definition of the strong instability of standing waves.

**Definition 1.3 (Strong instability).** A standing wave $e^{i\omega t}\phi_\omega$ is strongly unstable if for any $\epsilon > 0$, there exists $u_0 \in H^s$ such that $\|u_0 - \phi_\omega\|_{H^s} < \epsilon$ and the solution $u(t)$ to (1.2) with initial data $u_0$ blows up in finite time.

Our main result in this paper reads as follows.

**Theorem 1.4.** Let $d \geq 2$, $\frac{d}{2s-1} \leq s < 1$, $\frac{4s}{d} < \alpha < \frac{4s}{d-4s}$, $\alpha < 4s$, $\omega > 0$ and $\phi_\omega$ be the ground state related to (1.4). Then the ground state standing wave $e^{i\omega t}\phi_\omega$ is strongly unstable.

Note that the condition $\alpha < 4s$ is technical due to the localized virial estimate (see Remark 4.8). However, this only leads to a restriction in the two dimensional case, that is, $\frac{2}{3} \leq s < 1$ and $2s < \alpha < 4s$.

To our knowledge, the usual strategy to show the strong instability of standing waves is to use the variational characterization of the ground states as minimizers of the action functional and the virial identity, namely
\[
\frac{d}{dt}\|xu(t)\|_{L^2}^2 = 4\text{Im} \left( \int \bar{u}(t)x \cdot \nabla u(t) dx \right).
\]
However, in the case $0 < s < 1$, there is no such virial identity. To overcome this difficulty, we use localized virial estimates for radial solutions of (1.2). These localized virial estimates were proved by Boulenger-Himmelsbach-Lenzmann [3] to show the existence of radial blow-up solutions for (1.2) in the mass-critical and mass-supercritical cases. As far as we know, this paper seems to be the first one dealing with the strong instability of standing waves for the fractional nonlinear Schrödinger equation with power-type nonlinearity in the mass-supercritical and energy-subcritical regime. The method used to prove Theorem 1.4 is robust, and can be applied to show the strong instability of standing waves for the fractional nonlinear Schrödinger equation with other type of nonlinearities, such as Hartree, Choquard, combined power-type,... A similar approach based on localized virial estimates is used in [8] to show the strong instability of radial standing waves for the nonlinear Schrödinger equation with inverse-square potential in the mass-supercritical and energy-subcritical case.

The paper is organized as follows. In Section 2, we recall basic tools needed in the sequel such as the sharp Gagliardo-Nirenberg inequality, the Pohozaev’s identities and the profile decomposition. In Section 3, we show the characterization of the ground state related to (1.4) given in Proposition 1.2. We will give the proof of the strong instability of the ground state standing waves given in Theorem 1.4 in Section 4.

## 2. Preliminaries

Let us recall some basic tools related to (1.2) which are needed in this paper. Let us start with the sharp Gagliardo-Nirenberg inequality.
Lemma 2.1 (Sharp Gagliardo-Nirenberg inequality [3, 11, 12]). Let $d \geq 1$, $0 < s < 1$, $0 < \alpha < \alpha^*$ with $\alpha^*$ as in (1.3). Then for any $u \in H^s$,
\[
\|u\|_{L_2^{\alpha+2}}^{\alpha+2} \leq C_{\text{opt}} \|u\|_{H^s}^{\frac{d}{d^*}} \|u\|_{L_2^{\alpha+2}}^{\alpha-\frac{4}{d^*}},
\]
where the optimal constant $C_{\text{opt}}$ is given by
\[
C_{\text{opt}} = \left(\frac{2s(\alpha+2) - d\alpha}{d\alpha}\right)^{\frac{d}{d^*}} \frac{2s(\alpha+2)}{2s(\alpha+2) - d\alpha} \frac{1}{\|Q\|_{L_2^2}^{\alpha+2}}.
\]
Here $Q$ is the unique (up to translation) positive radial solution to the elliptic equation
\[-(\Delta)^s Q + |Q|^{\alpha} Q = 0.
\]
Moreover, the following Pohozaev’s identities hold true:
\[
\|Q\|_{L_2^2}^{\frac{2}{d^*}} = \frac{4s - (d - 2s)\alpha}{d\alpha} \|Q\|_{H^s}^{\frac{d}{d^*}} = \frac{4s - (d - 2s)\alpha}{2s(\alpha+2)} \|Q\|_{L_2^{\alpha+2}}^{\alpha+2}.
\]

We refer the reader to [11], [12, Proposition 3.1, Theorem 3.4] and [3, Appendix] for the proof of the above result. In the same spirit as (2.2), we have the following Pohozaev’s identities associated to (1.4):
\[
\omega\|\phi_\omega\|_{L_2^2}^{\frac{2}{d^*}} = \frac{4s - (d - 2s)\alpha}{d\alpha} \|\phi_\omega\|_{H^s}^{\frac{d}{d^*}} = \frac{4s - (d - 2s)\alpha}{2s(\alpha+2)} \|\phi_\omega\|_{L_2^{\alpha+2}}^{\alpha+2}.
\]

We next recall the profile decomposition of bounded $H^s$ sequences.

Lemma 2.2 (Profile decomposition [7, 27]). Let $d \geq 1$ and $0 < s < 1$. Let $(v_n)_{n \geq 1}$ be a bounded sequence in $H^s$. Then there exist a subsequence of $(v_n)_{n \geq 1}$ (still denoted $(v_n)_{n \geq 1}$), a family $(x_n^j)_{n \geq 1}$ of sequences in $\mathbb{R}^d$ and a sequence $(V_j^j)_{j \geq 1}$ of $H^s$ functions such that
\begin{itemize}
  \item for every $k \neq j$,
  \[
  |x_n^k - x_n^j| \to \infty,
  \]
  \end{itemize}
as $n \to \infty$;
\begin{itemize}
  \item for every $l \geq 1$ and every $x \in \mathbb{R}^d$,
  \[
v_n(x) = \sum_{j=1}^{l} V_j^j(x - x_n^j) + v_n^j(x),
  \]
  \end{itemize}
with
\[
\limsup_{n \to \infty} \|v_n^j\|_{L^q} = 0,
\]
as $l \to \infty$ for every $q \in (2, 2 + \alpha^*)$ with $\alpha^*$ as in (1.3).

Moreover, for every $l \geq 1$, the following expansions hold true:
\[
\|v_n\|_{L_2^2}^2 = \sum_{j=1}^{l} \|V_j^j\|_{L_2^2}^2 + \|v_n^j\|_{L_2^2}^2 + o_n(1),
\]
\[
\|v_n\|_{H^s}^2 = \sum_{j=1}^{l} \|V_j^j\|_{H^s}^2 + \|v_n^j\|_{H^s}^2 + o_n(1),
\]
\[
\|v_n\|_{L_2^{\alpha+2}}^{\alpha+2} = \sum_{j=1}^{l} \|V_j^j\|_{L_2^{\alpha+2}}^{\alpha+2} + \|v_n^j\|_{L_2^{\alpha+2}}^{\alpha+2} + o_n(1),
\]
as $n \to \infty$.

We refer the reader to [27, Proposition 2.3] or [7, Theorem 3.1] for the proof of this result which is similar to the one proved by Hmidi-Keraani [17, Proposition 3.1].

Remark 2.3. The number of non-zero terms in $(V_j^j)_{j \geq 1}$ may be one, finite or infinite, which may correspond to compactness, dichotomy, and vanishing, respectively, in the concentration compactness principle proposed by Lions [21]. The profile decomposition given in Lemma 2.2 may look as another equivalent description of the concentration-compactness principle. However, there are two major advantages: one is that we can inject the decomposing expression (2.5) into our aim functionals, and the other is that the decomposition is orthogonal by (2.4) and norms of $(v_n)_{n \geq 1}$ have similar decompositions (2.7) – (2.9). These properties are useful in the calculus of variational methods.
3. Characterization of the ground state

In this section, we give the proof of the characterization of the ground state given in Proposition 1.2. The proof is done by several steps.

**Step 1.** We first show that the minimizing problem

\[ d(\omega) := \inf \{ S_\omega(v) : v \in H^s \setminus \{0\}, \ K_\omega(v) = 0 \} \]  

is attained and \(d(\omega) > 0\). The later fact is easy to see. Indeed, let \(v \in H^s \setminus \{0\}\) be such that \(K_\omega(v) = 0\). By the Sobolev embedding, the equivalent norm \((1.8)\) and the fact \(H_\omega(v) = \|v\|_{L_{s+2}}^{\alpha+2}\), we have

\[ \|v\|_{L_{s+2}}^2 \leq C_1 \|v\|_{H^s}^2 \leq C_2 H_\omega(v) = C_2 \|v\|_{L_{s+2}}^{\alpha+2}, \]

for some \(C_1, C_2 > 0\). This implies that

\[ S_\omega(v) = \frac{1}{2} H_\omega(v) + \frac{1}{\alpha + 2} \|v\|_{L_{s+2}}^{\alpha+2} = \frac{\alpha}{2(\alpha + 2)} \|v\|_{L_{s+2}}^{\alpha+2} \geq \frac{\alpha}{2(\alpha + 2)} \left( \frac{1}{C_2} \right)^{\frac{\alpha+2}{\alpha}}. \]

Taking the infimum over \(v\), we get \(d(\omega) > 0\). We now show the minimizing problem \((3.1)\) is attained. Indeed, let \((v_n)_{n \geq 1}\) be a minimizing sequence of \(d(\omega)\), i.e. \(v_n \in H^s \setminus \{0\}\), \(K_\omega(v_n) = 0\) and \(S_\omega(v_n) \to d(\omega)\) as \(n \to \infty\). Since \(K_\omega(v_n) = 0\), we have \(H_\omega(v_n) = \|v_n\|_{L_{s+2}}^{\alpha+2}\) for any \(n \geq 1\). The fact \(S_\omega(v_n) \to d(\omega)\) as \(n \to \infty\) implies that

\[ \frac{\alpha}{2(\alpha + 2)} \|v_n\|_{L_{s+2}}^{\alpha+2} = \frac{\alpha}{2(\alpha + 2)} H_\omega(v_n) \to d(\omega), \]

as \(n \to \infty\). We infer that there exists \(C > 0\) such that

\[ H_\omega(v_n) \leq \frac{2(\alpha + 2)}{\alpha} d(\omega) + C, \]

for all \(n \geq 1\). By \((1.8)\), it follows that \((v_n)_{n \geq 1}\) is a bounded sequence in \(H^s\). Thanks to the profile decomposition given in Lemma 2.2, there exist a subsequence still denoted by \((v_n)_{n \geq 1}\), a family \((x_n^j)_{n \geq 1}\) of sequences in \(\mathbb{R}^d\) and a sequence \((V^j)_{j \geq 1}\) of \(H^s\)-functions such that for every \(l \geq 1\) and every \(x \in \mathbb{R}^d\),

\[ v_n(x) = \sum_{j=1}^l V^j(x - x_n^j) + v_n^l(x), \]

and \((2.6) - (2.9)\) hold. We have from \((2.7)\) and \((2.8)\) that

\[ H_\omega(v_n) = \sum_{j=1}^l H_\omega(V^j) + H_\omega(v_n^l) + o_n(1), \]

as \(n \to \infty\). This implies that

\[ K_\omega(v_n) = H_\omega(v_n) - \|v_n\|_{L_{s+2}}^{\alpha+2} \]

\[ = \sum_{j=1}^l H_\omega(V^j) + H_\omega(v_n^l) - \|v_n\|_{L_{s+2}}^{\alpha+2} + o_n(1) \]

\[ = \sum_{j=1}^l K_\omega(V^j) + \sum_{j=1}^l \|V^j\|_{L_{s+2}}^{\alpha+2} - \|v_n\|_{L_{s+2}}^{\alpha+2} + H_\omega(v_n^l) + o_n(1), \]

Since \(K_\omega(v_n) = 0\), \(\|v_n\|_{L_{s+2}}^{\alpha+2} \to \frac{2(\alpha + 2)}{\alpha} d(\omega)\) as \(n \to \infty\) and \(H_\omega(v_n^l) \geq 0\) for all \(n \geq 1\), we infer that

\[ \sum_{j=1}^l K_\omega(V^j) + \sum_{j=1}^l \|V^j\|_{L_{s+2}}^{\alpha+2} - \frac{2(\alpha + 2)}{\alpha} d(\omega) \leq 0, \]

or equivalently,

\[ \sum_{j=1}^l H_\omega(V^j) - \frac{2(\alpha + 2)}{\alpha} d(\omega) \leq 0. \]

On the other hand, by \((2.6)\) and \((2.9)\), we have

\[ \frac{2(\alpha + 2)}{\alpha} d(\omega) = \lim_{n \to \infty} \|v_n\|_{L_{s+2}}^{\alpha+2} = \sum_{j=1}^\infty \|V^j\|_{L_{s+2}}^{\alpha+2}. \]
Combining (3.2), (3.3) and (3.4), we obtain
\[
\sum_{j=1}^{\infty} K_\omega(V^j) \leq 0, \quad \sum_{j=1}^{\infty} H_\omega(V^j) \leq \frac{2(\alpha + 2)}{\alpha} d(\omega). \tag{3.5}
\]
We claim that \( K_\omega(V^j) = 0 \) for all \( j \geq 1 \). Indeed, suppose that there exists \( j_0 \geq 1 \) such that \( K_\omega(V^{j_0}) < 0 \). Set
\[
\lambda_0 := \left( \frac{H_\omega(V^{j_0})}{\|V^{j_0}\|^2_{L^{\alpha+2}}} \right)^{\frac{1}{2}}.
\]
Since \( K_\omega(V^{j_0}) < 0 \), we see that \( \lambda_0 \in (0, 1) \). Moreover, for \( \lambda > 0 \), we have
\[
K_\omega(\lambda V^{j_0}) = \lambda^2 H_\omega(V^{j_0}) - \lambda^{\alpha+2} \|V^{j_0}\|^2_{L^{\alpha+2}}.
\]
By the choice of \( \lambda_0 \), we see that \( K_\omega(\lambda_0 V^{j_0}) = 0 \). Thus,
\[
d(\omega) \leq S_\omega(\lambda_0 V^{j_0}) = \frac{\alpha}{2(\alpha + 2)} H_\omega(\lambda_0 V^{j_0}) = \frac{\alpha \lambda_0^2}{2(\alpha + 2)} H_\omega(V^{j_0}) < \frac{\alpha}{2(\alpha + 2)} H_\omega(V^{j_0}).
\]
Thanks to the second inequality of (3.5), we get
\[
d(\omega) < \frac{\alpha}{2(\alpha + 2)} H_\omega(V^{j_0}) \leq d(\omega),
\]
which is absurd. We next claim that there exists exactly one \( j \) such that \( V^j \) is non-zero. Indeed, if there exists \( V^{j_1} \) and \( V^{j_2} \) non-zero, then the second inequality of (3.5) implies that both \( H_\omega(V^{j_1}) \) and \( H_\omega(V^{j_2}) \) are strictly smaller than \( \frac{2(\alpha + 2)}{\alpha} d(\omega) \). However, since \( K_\omega(V^{j_1}) = 0 \), we learn from the definition of \( d(\omega) \) that
\[
\frac{2(\alpha + 2)}{\alpha} d(\omega) \leq \frac{2(\alpha + 2)}{\alpha} S_\omega(V^{j_1}) = H_\omega(V^{j_1}) < \frac{2(\alpha + 2)}{\alpha} d(\omega),
\]
which is again absurd.

Without loss of generality, we may assume that the only non-zero profile is \( V^1 \). We have from (3.4) that
\[
\|V^1\|^2_{L^{\alpha+2}} \leq \frac{2(\alpha + 2)}{\alpha} d(\omega),
\]
which implies \( V^1 \neq 0 \). On the other hand, by the first estimate of (3.5), we have \( K_\omega(V^1) \leq 0 \). By the same argument as above, we get \( K_\omega(V^1) = 0 \). Therefore, \( V^1 \) is a minimizer of \( d(\omega) \), and the minimizing problem (3.1) is attained.

**Step 2.** We will show that \( V^1 \) is a ground state related to (1.4). Since \( V^1 \) is a minimizer of \( d(\omega) \), there exists a Lagrange multiplier \( \mu \in \mathbb{R} \) such that \( S'_\omega(V^1) = \mu K'_\omega(V^1) \). We thus have
\[
0 = K_\omega(V^1) = \langle S'_\omega(V^1), V^1 \rangle = \mu \langle K'_\omega(V^1), V^1 \rangle.
\]
It is easy to see that
\[
K'_\omega(V^1) = 2(-\Delta)^s V^1 + 2\omega V^1 - (\alpha + 2)|V^1|^\alpha V^1.
\]
Hence,
\[
\langle K'_\omega(V^1), V^1 \rangle = 2H_\omega(V^1) - (\alpha + 2)\|V^1\|^2_{L^{\alpha+2}} = -\alpha\|V^1\|^\alpha_{L^{\alpha+2}} < 0.
\]
It follows that \( \mu = 0 \) and \( S'_\omega(V^1) = 0 \). In particular, \( V^1 \) is a solution to (1.4).

We will show that \( V^1 \) is a minimizer of the Weinstein functional (1.6). Let \( v \in H^s \setminus \{0\} \). It is easy to see that \( K_\omega(\lambda_0 v) = 0 \), where
\[
\lambda_0 := \left( \frac{H_\omega(v)}{\|v\|^2_{L^{\alpha+2}}} \right)^{\frac{1}{2}} > 0. \tag{3.6}
\]
By the definition of \( d(\omega) \), we have
\[
S_\omega(V^1) \leq S_\omega(\lambda_0 v). \tag{3.7}
\]
On the other hand, we have
\[
S_\omega(\lambda v) = \frac{\lambda^2}{2} H_\omega(v) - \frac{\lambda^{\alpha+2}}{\alpha + 2} \|v\|^2_{L^{\alpha+2}},
\]
and
\[
\partial_\lambda S_\omega(\lambda v) = \lambda H_\omega(v) - \lambda^{\alpha+1} \|v\|^\alpha_{L^{\alpha+2}}.
\]
Hence $\partial_t S_{\omega}(\lambda_0 v) = 0$, or $\lambda_0 v$ is a solution to (1.4). It follows that both $V^1$ and $\lambda_0 v$ satisfy the following_Pohozaev’s identities (see e.g. [3, Appendix]):

$$\omega \|V^1\|_{L^4}^4 = \frac{4s - (d - 2s) \alpha}{2s(\alpha + 2)} \|V^1\|_{L^{s+2}}^{s+2} = \frac{4s - (d - 2s) \alpha}{d \alpha} \|V^1\|_{H^s}^4,$$

$$\omega \|\lambda_0 v\|_{L^4}^4 = \frac{4s - (d - 2s) \alpha}{2s(\alpha + 2)} \|\lambda_0 v\|_{L^{s+2}}^{s+2} = \frac{4s - (d - 2s) \alpha}{d \alpha} \|\lambda_0 v\|_{H^s}^4.$$ 

On one hand, by (3.7) and the fact $K_{\omega}(V^1) = K_{\omega}(\lambda_0 v) = 0$, we get

$$\|V^1\|_{L^{s+2}}^{s+2} \leq \|\lambda_0 v\|_{L^{s+2}}^{s+2}.$$ 

On the other hand, using Pohozaev’s identities, we have

$$J(v) = J(\lambda_0 v) = \left(\|\lambda_0 v\|_{H^s}^{d \alpha} \|\lambda_0 v\|_{L^{s+2}}^{s+2 - \frac{d \alpha}{s}}\right) \div \|\lambda_0 v\|_{H^{s+2}}^{2}$$

$$= \left(\frac{d \alpha}{2s(\alpha + 2)}\right)^{\frac{d \alpha}{s}} \left(\frac{4s - (d - 2s) \alpha}{2s(\alpha + 2)\omega}\right)^{\frac{s+2}{2} - \frac{d \alpha}{s}} \left(\|\lambda_0 v\|_{L^{s+2}}^{s+2}\right)^{\frac{2}{s}}$$

$$= J(V^1).$$

This implies that $J(V^1) \leq J(v)$ for any $v \in H^s \setminus \{0\}$, or $V^1$ is a minimizer of the Weinstein functional (1.6). Since $J(|V^1|) \leq J(V^1)$, we can choose $V^1$ to be non-negative. Therefore, $V^1$ is a ground state related to (1.4).

**Step 3.** Conclusion. By the uniqueness (up to translation) of ground states related to (1.4), we obtain $V^1 = \phi_{\omega}$ (up to translation), hence $S_{\omega}(V^1) = S_{\omega}(\phi_{\omega})$. This proves (1.9) and the proof is complete. □

### 4. Strong instability of standing waves

The main purpose of this section is to give the proof of Theorem 1.4. Let us start with the local well-posedness of (1.2).

The local well-posedness for (1.2) in the energy space $H^s$ was first studied by Hong-Sire in [16] (see also [6]). The proof is based on Strichartz estimates and the contraction mapping argument. Note that for non-radial data, Strichartz estimates have a loss of derivatives. Fortunately, this loss of derivatives can be compensated for by using Sobolev embedding. However, it leads to a weak local well-posedness in the energy space compared to the well-known nonlinear Schrödinger equation ($s = 1$). We refer the reader to [16, 6] for more details. One can remove the loss of derivatives in Strichartz estimates by considering radially symmetric data. However, it needs a restriction on the validity of $s$, namely $\frac{d}{d+1} \leq s < 1$. More precisely, we have the following local well-posedness for (1.2) with radial $H^s$ initial data.

**Proposition 4.1 (Local well-posedness [7]).** Let $d \geq 2$, $\frac{d}{d+1} \leq s < 1$ and $0 < \alpha < \frac{4s}{d - 2s}$. Let

$$p = \frac{4s(\alpha + 2)}{d(\alpha - 2s)}, \quad q = \frac{d(\alpha + 2)}{d + \alpha s}.$$ 

Then for any $u_0 \in H^s$ radial, there exist $T \in (0, +\infty)$ and a unique solution to (1.2) satisfying

$$u \in C([0, T), H^s) \cap L^p_{loc}([0, T), W^{s,q}).$$

Moreover, the following properties hold:

- If $T < +\infty$, then $\|u(t)\|_{H^s} \to +\infty$ as $t \uparrow T$;

- $u \in L^p_{loc}([0, T), W^{s,b})$ for any $(a, b)$ fractional admissible pair, i.e.

$$a \in [2, \infty], \quad b \in [2, \infty), \quad (a, b) \neq \left(2, \frac{4d - 2}{2d - 3}\right), \quad \frac{2s}{a} + \frac{d}{b} = \frac{d}{2};$$

- There is conservation of mass and energy,

$$\text{Mass} \quad M(u(t)) = \int |u(t, x)|^2 dx = M(u_0),$$

$$\text{Energy} \quad E(u(t)) = \frac{1}{2} \int |(-\Delta)^{s/2} u(t, x)|^2 dx - \frac{1}{\alpha + 2} \int |u(t, x)|^{\alpha + 2} dx = E(u_0),$$

for all $t \in [0, T)$. 

We refer the reader to [7, Proposition 2.5] for the proof of this result. Now, let us denote
\[ I(v) := s\|v\|_{H^s}^2 - \frac{da}{2(\alpha + 2)}\|v\|_{L^{\alpha + 2}}^{\alpha + 2}. \] (4.1)
Note that if we take
\[ v^\lambda(x) := \lambda^{\frac{d}{2}}v(\lambda x), \] (4.2)
then a simple computation shows
\[ \|v^\lambda\|_{L^2} = \|v\|_{L^2}, \quad \|v^\lambda\|_{H^s} = \lambda^s\|v\|_{H^s}, \quad \|v^\lambda\|_{L^{\alpha + 2}} = \lambda^{\frac{\alpha s}{\alpha - 2}}\|v\|_{L^{\alpha + 2}}. \]
We also have
\[ S_\omega(v^\lambda) = \frac{1}{2}\|v^\lambda\|_{H^s}^2 + \frac{\omega}{2}\|v^\lambda\|_{L^2}^2 - \frac{1}{\alpha + 2}\|v^\lambda\|_{L^{\alpha + 2}}^{\alpha + 2} = \frac{\lambda^{2s}}{2}\|v\|_{H^s}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{\lambda^{\frac{\alpha s}{\alpha - 2}}}{\alpha + 2}\|v\|_{L^{\alpha + 2}}^{\alpha + 2}. \]
It is easy to see that
\[ I(v) = \partial_\lambda S_\omega(v^\lambda)|_{\lambda = 1}. \]
We have the following characterization of the ground state in the mass-supercritical and energy-subcritical case.

**Lemma 4.2.** Let \( d \geq 1, \ 0 < s < 1, \ \frac{d}{d-2} < \alpha < \alpha^* \) with \( \alpha^* \) as in (1.3) and \( \omega > 0 \). Let \( \phi_\omega \) be the ground state related to (1.4). Then
\[ S_\omega(\phi_\omega) = \inf \{ S_\omega(v) : v \in H^s \setminus \{0\}, \ I(v) = 0 \}. \] (4.3)

**Proof.** Denote \( m := \inf \{ S_\omega(v) : v \in H^s \setminus \{0\}, \ I(v) = 0 \}. \) By Pohozaev’s identities, it is easy to check that \( I(\phi_\omega) = K_\omega(\phi_\omega) = 0 \). By definition of \( m \), we have
\[ S_\omega(\phi_\omega) \geq m. \] (4.4)
Let \( v \in H^s \setminus \{0\} \) be such that \( I(v) = 0 \). If \( K_\omega(v) = 0 \), then \( S_\omega(v) \geq S_\omega(\phi_\omega) \). Assume \( K_\omega(v) \neq 0 \). We have
\[ K_\omega(v^\lambda) = \lambda^{2s}\|v\|_{H^s}^2 + \omega\|v\|_{L^2}^2 - \lambda^{\frac{\alpha s}{\alpha - 2}}\|v\|_{L^{\alpha + 2}}^{\alpha + 2}, \]
where \( v^\lambda \) is as in (4.2). Since \( \frac{da}{2} > 2s \), we see that \( \lim_{\lambda \to 0} K_\omega(v^\lambda) = \omega\|v\|_{L^2}^2 > 0 \) and \( \lim_{\lambda \to +\infty} K_\omega(v^\lambda) = -\infty \). It follows that there exists \( \lambda_0 > 0 \) such that \( K_\omega(v^\lambda) = 0 \). By (1.9), we get \( S_\omega(v^\lambda) \geq S_\omega(\phi_\omega) \). On the other hand,
\[ \partial_\lambda S_\omega(v^\lambda) = s\lambda^{2s-1}\|v\|_{H^s}^2 - \frac{da}{2(\alpha + 2)}\lambda^{\frac{\alpha s}{\alpha - 2}-1}\|v\|_{L^{\alpha + 2}}^{\alpha + 2} = \frac{I(v^\lambda)}{\lambda}. \]
It is easy to see that the equation \( \partial_\lambda S_\omega(v^\lambda) = 0 \) admits a unique non-zero solution
\[ \left( \frac{\|v\|_{H^s}^2}{2s(\alpha + 2)}\|v\|_{L^{\alpha + 2}}^{\alpha + 2} \right)^{\frac{2}{\alpha s - 2}} = 1. \]
The last equality comes from the fact that \( I(v) = 0 \). This implies that
\[ \begin{cases} \partial_\lambda S_\omega(v^\lambda) > 0 & \text{if } \lambda \in (0, 1), \\ \partial_\lambda S_\omega(v^\lambda) < 0 & \text{if } \lambda \in (1, \infty). \end{cases} \]
In particular, we have \( S_\omega(v^\lambda) < S_\omega(v) \) for any \( \lambda > 0, \lambda \neq 1 \). Since \( \lambda_0 > 0 \), we have \( S_\omega(v^{\lambda_0}) \leq S_\omega(v) \). Thus, \( S_\omega(\phi_\omega) \leq S_\omega(v) \) for any \( v \in H^s \setminus \{0\} \) satisfying \( I(v) = 0 \). It follows that
\[ S_\omega(\phi_\omega) \leq m. \] (4.5)
Combining (4.4) and (4.5), the proof is complete. \( \square \)

Let \( \phi_\omega \) be the ground state related to (1.4). We define
\[ C_\omega := \{ v \in H^s \setminus \{0\} : S_\omega(v) < S_\omega(\phi_\omega), \ I(v) < 0 \}. \]

**Lemma 4.3.** Let \( d \geq 1, \ 0 < s < 1, \ \frac{d}{d-2} < \alpha < \alpha^* \) with \( \alpha^* \) as in (1.3), \( \omega > 0 \) and \( \phi_\omega \) be the ground state related to (1.4). Then the set \( C_\omega \) is invariant under the flow of (1.2), that is, if \( u_0 \in C_\omega \), then the solution \( u(t) \) to (1.2) with initial data \( u_0 \) belongs to \( C_\omega \) for any \( t \) in the existence time.
Lemma 4.4. Let $d ≥ 1$, $0 < s < 1$, $\frac{2s}{d} < \alpha < \alpha^*$ with $\alpha^*$ as in (1.3), $\omega > 0$ and $\phi_\omega$ be the ground state related to (1.4). If $v \in \mathcal{C}_\omega$, then

$$I(v) ≤ 2s(S_\omega(v) - S_\omega(\phi_\omega)).$$

Proof. Denote

$$f(\lambda) := S_\omega(v^\lambda) = \frac{\lambda^{2s}}{2} ||v||^2_{L^2} + \frac{\omega}{2} ||v||^2_{L^2} - \frac{\lambda^{\frac{2s}{\alpha}}}{\alpha + 2} ||v||^2_{L^{\alpha + 2}}.$$ 

We have

$$f'(\lambda) = s\lambda^{2s-1} ||v||^2_{L^2} - \frac{d\alpha}{2(\alpha + 2)} \lambda^{\frac{2s}{\alpha} - 1} ||v||^2_{L^{\alpha + 2}}.$$ 

We also have

$$(\lambda f'(\lambda))' = 2s^2 \lambda^{2s-1} ||v||^2_{L^2} - \frac{d^2\alpha^2}{4(\alpha + 2)} \lambda^{\frac{2s}{\alpha} - 1} ||v||^2_{L^{\alpha + 2}}$$

$$= 2s \left( s \lambda^{2s-1} ||v||^2_{L^2} - \frac{d\alpha}{2(\alpha + 2)} \lambda^{\frac{2s}{\alpha} - 1} ||v||^2_{L^{\alpha + 2}} \right) - \frac{d(\alpha - 4s)}{4(\alpha + 2)} \lambda^{\frac{2s}{\alpha} - 1} ||v||^2_{L^{\alpha + 2}}.$$ 

Since $d\alpha > 4s$, we thus get

$$(\lambda f'(\lambda))' ≤ 2sf'(\lambda),$$

for all $\lambda > 0$. Note that since $I(v) < 0$, we see that the equation $\partial_t S_\omega(v^\lambda) = 0$ admits a unique non-zero solution

$$\lambda_0 = \left( \frac{||v||^2_{L^2}}{2s(\alpha + 2) ||v||^2_{L^{\alpha + 2}}} \right)^{\frac{\alpha}{\alpha + 2}} ∈ (0, 1).$$

It follows that $I(v^{\lambda_0}) = \lambda_0 \partial_t S_\omega(v^\lambda)|_{\lambda = \lambda_0} = 0$. Taking integration (4.9) over $(\lambda_0, 1)$ and using (4.8), we obtain

$$I(v) - I(v^{\lambda_0}) ≤ 2s(S_\omega(v) - S_\omega(v^{\lambda_0})) ≤ 2s(S_\omega(v) - S_\omega(\phi_\omega)).$$

Here the last inequality follows from (4.3) and the fact $I(v^{\lambda_0}) = 0$. The proof is complete. □

We next recall the localized virial estimate related to (1.2) which is the main ingredient in the proof of the strong instability of the ground state standing wave. The localized virial estimate was used by Boulenger-Himmelsbach-Lenzmann [3] to show the existence of finite time blow-up radial solutions to (1.2) in the mass-critical and mass-supercritical cases. Let us start with the following estimate.

Lemma 4.5. Let $d ≥ 1$ and $\varphi : \mathbb{R}^d → \mathbb{R}$ be such that $\nabla \varphi \in W^{1, \infty}$. Then for all $u \in H^{1/2}$,

$$\left| \int \overline{\varphi}(x) \nabla \varphi(x) \cdot \nabla u(x) dx \right| ≤ C \left( ||\nabla||^{1/2} u||^2_{L^2} + ||u||_{L^2} ||\nabla||^{1/2} u||_{L^2} \right),$$

for some $C > 0$ depending only on $||\nabla \varphi||_{W^{1, \infty}}$ and $d$.

Now, let $d ≥ 1$, $1/2 ≤ s < 1$ and $\varphi : \mathbb{R}^d → \mathbb{R}$ be such that $\nabla \varphi \in W^{3, \infty}$. Assume $u ∈ C([0, T), H^s)$ is a solution to (1.2). Note that in [3], Boulenger-Himmelsbach-Lenzmann assume $u ∈ C([0, T), H^{2s})$ due to the lack of local theory at that time. Thanks to the local theory (see e.g. [6, 7, 16]), one can recover $H^s$-valued solutions by an approximation argument (see [3, Section 2]). The localized virial action of $u$ is defined by

$$M_\varphi(u(t)) := 2 \int \nabla \varphi(x) \cdot \text{Im}(\overline{\varphi}(t, x) \nabla u(t, x)) dx.$$ 

We see that $M_\varphi(u(t))$ is well-defined. Indeed, by Lemma 4.5,

$$|M_\varphi(u(t))| ≤ C(\varphi)||u(t)||_{H^{1/2}}^2 ≤ C(\varphi)||u(t)||_{H^s}^2 < \infty.$$
In order to study the time evolution of $M_\varphi(u(t))$, we need to introduce the following auxiliary function
\begin{equation}
 u_m(t, x) := c_s \frac{1}{-\Delta + m} u(t, x) = c_s \mathcal{F}^{-1} \left( \frac{\hat{u}(t, \xi)}{|\xi|^2 + m} \right), \quad m > 0,
\end{equation}
where
\[ c_s := \sqrt{\frac{\sin \pi s}{\pi}} \]
is the normalization factor. Remark that since $u(t) \in H^s$, the smoothing operator $(-\Delta + m)^{-1}$ implies that $u_m(t) \in H^{s+2}$.

We have the following time evolution of $M_\varphi(u(t))$ (see [3, Lemma 2.1]).

**Lemma 4.6 (Time evolution $M_\varphi(u(t))$ [3]).** Let $d \geq 1, 1/2 < s < 1$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ be such that $\nabla \varphi \in W^{3, \infty}$. Assume that $u \in C([0, T), H^s)$ is a solution to (1.2). Then for any $t \in [0, T)$, it holds that
\[
\frac{d}{dt} M_\varphi(u(t)) = -\int_0^\infty m^s \int \Delta^2 \varphi |u_m(t)|^2 dx dm + 4 \sum_{j,k=1}^d \int_0^\infty m^s \int \partial_j \varphi \partial_j \eta_m(t) \partial_k u_m(t) dx dm
- \frac{2\alpha}{\alpha + 2} \int \Delta \varphi |u(t)|^{\alpha+2} dx,
\]
where $u_m(t)$ is as in (4.10).

**Remark 4.7.** Using Plancherel’s and Fubini’s theorems, it follows that
\[
\int_0^\infty m^s \int |\nabla u_m| dx dm = \int \left( \frac{\sin \pi s}{\pi} \right) \int_0^\infty m^s dm \int |\xi|^2 |\hat{u}(\xi)|^2 d\xi
= \int (s|\xi|^{2s-2}) |\xi|^2 |\hat{u}(\xi)|^2 d\xi = 8\|u(t)\|_{H^s}^2.
\]
If we make a formal substitution and take the unbounded function $\varphi(x) = |x|^2$, then by Lemma 4.6 and (4.11), we find formally the virial identity
\[
\frac{d}{dt} M_{\varphi^2}(u(t)) = 8s\|u(t)\|_{H^s}^2 - \frac{4\alpha s}{\alpha + 2} \|u(t)\|^s_{L^{s+2}}
= 4\alpha E(u(t)) - 2(\alpha - 4s)\|u(t)\|^2_{H^s}
= 8I(u(t)),
\]
where $I$ is given in (4.1).

We now recall localized virial estimates for radial $H^s$ solutions related to (1.2). Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be as above. We assume in addition that $\varphi$ is radially symmetric and satisfies
\[
\varphi(r) := \begin{cases} r^2 & \text{if } r \leq 1, \\
\text{const.} & \text{if } r \geq 10, \quad \text{and } \varphi''(r) \leq 2 \text{ for } r \geq 0.
\end{cases}
\]
The precise constant here is not important. For $R > 1$ given, we define the scaled function $\varphi_R : \mathbb{R}^d \to \mathbb{R}$ by
\[
\varphi_R(x) = \varphi(r/R), \quad r = |x|.
\]
It is easy to check that
\[
2 - \varphi''_R(r) \geq 0, \quad 2 - \frac{\varphi''_R(r)}{r} \geq 0, \quad 2d - \Delta \varphi_R(x) \geq 0, \quad \forall r \geq 0, \forall x \in \mathbb{R}^d.
\]
Using Lemma 4.6, we have the following localized virial estimate for the time evolution of $M_{\varphi_R}(u(t))$.

**Lemma 4.8 (Localized virial estimate [3]).** Let $d \geq 2$, $\frac{d}{2(d-1)} \leq s < 1$, $0 < \alpha < \frac{4s}{d-2s}$, $\varphi_R$ be as in (4.13). Let $u \in C([0, T), H^s)$ be a radial solution to (1.2). Then for any $t \in [0, T)$,
\[
\frac{d}{dt} M_{\varphi_R}(u(t)) \leq 8s\|u(t)\|_{H^s}^2 - \frac{4\alpha s}{\alpha + 2} \|u(t)\|^s_{L^{s+2}} + O \left( R^{-2s} + R^{-\frac{2(d-1)}{2} - \epsilon} \|u(t)\|^s_{H^{s+\epsilon}} \right)
\]
\[
= 4\alpha E(u(t)) - 2(\alpha - 4s)\|u(t)\|^2_{H^s} + O \left( R^{-2s} + R^{-\frac{2(d-1)}{2} - \epsilon} \|u(t)\|^s_{H^{s+\epsilon}} \right)
\]
\[
= 8I(u(t)) + O \left( R^{-2s} + R^{-\frac{2(d-1)}{2} + \epsilon} \|u(t)\|^s_{H^{s+\epsilon}} \right),
\]
for any $0 < \epsilon < \frac{(2s-1)s}{2s}$. Here the implicit constant depends only on $\|u_0\|_{L^2}, d, \epsilon, \alpha$ and $s$.

We refer the reader to [3, Lemma 2.2] for the proof of the above result.
Remark 4.9. • The restriction \( \frac{d}{2^{d-1}} \leq s < 1 \) follows from the local well-posedness of radial \( H^s \) solutions for (1.2) given in Proposition 4.1.

• In practice, we need the exponent \( \frac{\alpha}{2^{d}} + \epsilon \) to be smaller than or equal to 2. This leads to the restriction \( \alpha < 4s \).

We are now able to prove our main result—Theorem 1.4.

Proof of Theorem 1.4. Let \( \epsilon > 0 \). Since \( \phi^{\lambda}_{\omega} \to \phi_{\omega} \) in \( H^s \) as \( \lambda \to 1 \). There exists \( \lambda_0 > 1 \) such that \( \| \phi_{\omega} - \phi^{\lambda_0}_{\omega} \|_{H^s} < \epsilon \). By decreasing \( \lambda_0 \) if necessary, we claim that \( \phi^{\lambda_0}_{\omega} \in C_{\omega} \). Indeed, a direct computation shows that

\[
S_{\omega}(\phi^{\lambda_0}_{\omega}) = \frac{\lambda^{2s}}{2} \| \phi_{\omega} \|^2_{H^s} + \frac{\omega}{2} \| \phi_{\omega} \|_{L^2}^2 - \frac{\lambda \omega}{\alpha + 2} \| \phi_{\omega} \|^{\alpha+2}_{L^\omega_{\alpha+2}},
\]

and

\[
\partial_\lambda S_{\omega}(\phi^{\lambda}_{\omega}) = s \lambda^{2s-1} \| \phi_{\omega} \|^2_{H^s} - \frac{da}{2(\alpha + 2)} \lambda^{\omega - 1} \| \phi_{\omega} \|^{\alpha+2}_{L^\omega_{\alpha+2}} = \frac{I(\phi^{\lambda}_{\omega})}{\lambda}.
\]

It is not hard to see that the equation \( \partial_\lambda S_{\omega}(\phi^{\lambda}_{\omega}) = 0 \) admits a unique non-zero solution

\[
\left( \frac{\| \phi_{\omega} \|^2_{H^s}}{\frac{da}{2(\alpha + 2)} \| \phi_{\omega} \|^{\alpha+2}_{L^\omega_{\alpha+2}}} \right)^{\frac{1}{\alpha+2}} = 1.
\]

Note that the last equality comes from the fact that \( I(\phi_{\omega}) = 0 \), which follows from the Pohozaev’s identities (2.3). This implies that

\[
\begin{cases}
\partial_\lambda S_{\omega}(\phi^\lambda_{\omega}) > 0 & \text{if } \lambda \in (0,1), \\
\partial_\lambda S_{\omega}(\phi^\lambda_{\omega}) < 0 & \text{if } \lambda \in (1, \infty).
\end{cases}
\]

We thus get \( S_{\omega}(\phi^\lambda_{\omega}) < S_{\omega}(\phi_{\omega}) \) for any \( \lambda > 0, \lambda \neq 1 \). Since \( I(\phi^\lambda_{\omega}) = \lambda \partial_\lambda S_{\omega}(\phi^\lambda_{\omega}) \), we have

\[
\begin{cases}
I(\phi^\lambda_{\omega}) > 0 & \text{if } \lambda \in (0,1), \\
I(\phi^\lambda_{\omega}) < 0 & \text{if } \lambda \in (1, \infty).
\end{cases}
\]

We thus obtain

\[
S_{\omega}(\phi^{\lambda_0}_{\omega}) < S_{\omega}(\phi_{\omega}), \quad I(\phi^{\lambda_0}_{\omega}) < 0.
\]

It follows that \( \phi^{\lambda_0}_{\omega} \in C_{\omega} \).

By Proposition 4.1, we see that under the assumption \( d \geq 2, \frac{d}{2^{d-1}} \leq s < 1 \) and \( \frac{4s}{d} < \alpha < \frac{4s}{d-2s} \), there exists a unique solution \( u \in C([0,T), H^s_{radal}) \) to (1.2) with initial data \( u_0 = \phi^{\lambda_0}_{\omega} \), where \( T > 0 \) is the maximal time of existence. Note that since (1.2) is invariant under the space translation and the ground state related to (1.4) is unique up to the space translation, we can assume that \( \phi_{\omega} \) is radially symmetric, so is \( \phi^{\lambda_0}_{\omega} \). We will show that the solution \( u \) blows up in finite time. It is done by several steps.

**Step 1.** We claim that there exists \( a > 0 \) such that

\[
I(u(t)) \leq -a, \quad \forall t \in [0,T).
\]

Indeed, since \( C_{\omega} \) is invariant under the flow of (1.2) with \( d \geq 2, \frac{d}{2^{d-1}} \leq s < 1 \) and \( \frac{4s}{d} < \alpha < \frac{4s}{d-2s} \), we have \( u(t) \in C_{\omega} \) for all \( t \in [0,T) \). By Lemma 4.4, we have

\[
I(u(t)) \leq 2s(S(\phi_{\omega}) - S_{\omega}(\phi_{\omega})) = 2s(S_{\omega}(\phi^{\lambda_0}_{\omega}) - S_{\omega}(\phi_{\omega})).
\]

This proves (4.15) with \( a = 2s(S_{\omega}(\phi_{\omega}) - S_{\omega}(\phi^{\lambda_0}_{\omega})) > 0 \).

**Step 2.** We next claim that there exists \( b > 0 \) such that

\[
\frac{d}{dt} M_{\phi_{\omega}}(u(t)) \leq -b \| u(t) \|_{H^s}^2,
\]

\[
\| u(t) \|_{H^s} \geq 1,
\]

for all \( t \in [0,T) \), where \( \varphi_R \) is as in (4.13). Let us first prove (4.17). Assume that (4.17) is not true. Then there exists \( (t_n)_{n \geq 1} \) a time sequence in \([0,T)\) such that \( \| u(t_n) \|_{H^s} \to 0 \) as \( n \to \infty \). By the sharp Gagliardo-Nirenberg inequality (2.1), we have

\[
\| u(t_n) \|_{L_{\alpha+2}^\omega}^2 \leq C_{opt} \| u(t_n) \|_{H^s}^{\frac{4s}{\alpha+2}} \| u(t_n) \|_{L_{\alpha+2}^2}^{\alpha+2-\frac{4s}{\alpha+2}} \to 0,
\]

as \( n \to \infty \). Here we use the conservation of mass to get the last convergence. It follows that

\[
I(u(t_n)) = s \| u(t_n) \|_{H^s}^2 - \frac{da}{2(\alpha + 2)} \| u(t_n) \|_{L_{\alpha+2}^\omega}^{\alpha+2} \to 0,
\]
as \( n \to \infty \), which contradicts to (4.15). We now prove (4.16). Since \( u(t) \) is radially symmetric, we apply Lemma 4.8 to have
\[
\frac{d}{dt} M_{\varphi}(u(t)) \leq 4d\alpha E(u(t)) - 2(d\alpha - 4s)\|u(t)\|_{H^s}^2 + O \left( R^{-2s} + R^{\frac{\alpha(4s-d)}{4s - \alpha - 2s}} \right),
\]
for any \( t \in [0, T) \) and any \( R > 1 \). Thanks to the assumption \( \alpha < 4s \), we can apply the Young inequality to get for any \( \eta > 0 \),
\[
R^{-\frac{\alpha(4s-d)}{4s - \alpha - 2s}} \|u(t)\|_{H^s}^2 + \eta \|u(t)\|_{H^s}^2 \leq \eta \|u(t)\|_{H^s}^2 + \eta^{-\frac{\alpha(4s-d)}{4s - \alpha - 2s}} R^{-2s} + R^{\frac{\alpha(4s-d)}{4s - \alpha - 2s}}.
\]
We thus get
\[
\frac{d}{dt} M_{\varphi}(u(t)) \leq 4d\alpha E(u(t)) - 2(d\alpha - 4s)\|u(t)\|_{H^s}^2 + C\eta \|u(t)\|_{H^s}^2 + O \left( R^{-2s} + \eta^{-\frac{\alpha + 2s}{4s - \alpha - 2s}} R^{-\frac{2(\alpha(4s-d)-2s)}{4s - \alpha - 2s}} \right),
\]
for any \( t \in [0, T) \), any \( \eta > 0 \), any \( R > 1 \) and some constant \( C > 0 \).

Now, we fix \( t \in [0, T) \) and denote
\[
\mu := \frac{4d\alpha E(u_0)}{d\alpha - 4s} + 2.
\]
We consider two cases.

**Case 1.**
\[
\|u(t)\|_{H^s}^2 \leq \mu.
\]
Since \( 4d\alpha E(u(t)) - 2(d\alpha - 4s)\|u(t)\|_{H^s}^2 = 8I(u(t)) \leq -8\alpha \) for all \( t \in [0, T) \), we have
\[
\frac{d}{dt} M_{\varphi}(u(t)) \leq -8\alpha + C\eta \mu + O \left( R^{-2s} + \eta^{-\frac{\alpha + 2s}{4s - \alpha - 2s}} R^{-\frac{2(\alpha(4s-d)-2s)}{4s - \alpha - 2s}} \right).
\]
By choosing \( \eta > 0 \) small enough and \( R > 1 \) large enough depending on \( \eta \), we see that
\[
\frac{d}{dt} M_{\varphi}(u(t)) \leq -4\alpha \leq -4\frac{\alpha}{\mu} \|u(t)\|_{H^s}^2.
\]

**Case 2.**
\[
\|u(t)\|_{H^s}^2 \geq \mu.
\]
In this case, we have
\[
4d\alpha E(u_0) - 2(d\alpha - 4s)\|u(t)\|_{H^s}^2 \leq 4d\alpha E(u_0) - (d\alpha - 4s)\mu - (d\alpha - 4s)\|u(t)\|_{H^s}^2 \leq -2 - (d\alpha - 4s)\|u(t)\|_{H^s}^2.
\]
Thus,
\[
\frac{d}{dt} M_{\varphi}(u(t)) \leq -2 - (d\alpha - 4s)\|u(t)\|_{H^s}^2 + C\eta \|u(t)\|_{H^s}^2 + O \left( R^{-2s} + \eta^{-\frac{\alpha + 2s}{4s - \alpha - 2s}} R^{-\frac{2(\alpha(4s-d)-2s)}{4s - \alpha - 2s}} \right).
\]
Since \( d\alpha - 4s > 0 \), we choose \( \eta > 0 \) small enough so that
\[
d\alpha - 4s - C\eta \geq \frac{d\alpha - 4s}{2}.
\]
We next choose \( R > 1 \) large enough depending on \( \eta \) so that
\[
-2 + O \left( R^{-2s} + \eta^{-\frac{\alpha + 2s}{4s - \alpha - 2s}} R^{-\frac{2(\alpha(4s-d)-2s)}{4s - \alpha - 2s}} \right) \leq 0.
\]
We thus obtain
\[
\frac{d}{dt} M_{\varphi}(u(t)) \leq -\frac{d\alpha - 4s}{2} \|u(t)\|_{H^s}^2.
\]
In both cases, the choices of \( \eta > 0 \) and \( R > 1 \) are independent of \( t \). Therefore, (4.16) follows with \( b = \min \left\{ \frac{4\alpha}{\mu}, \frac{d\alpha - 4s}{2} \right\} > 0 \).

**Step 3.** We are now able to show that the solution \( u \) blows up in finite time. Assume by contradiction that \( T = +\infty \). By (4.16) and (4.17), we see that \( \frac{d}{dt} M_{\varphi}(u(t)) \leq -C \) for some \( C > 0 \). Integrating this bound, it yields that \( M_{\varphi}(u(t)) < 0 \) for all \( t \geq t_0 \) with some \( t_0 \gg 1 \) large enough. Taking integration over \([t_0, t]\) of (4.16), we obtain
\[
M_{\varphi}(u(t)) \leq -b \int_{t_0}^{t} \|u(\tau)\|_{H^s}^2 d\tau,
\]
for all \( t \geq t_0 \). On the other hand, by Lemma 4.5 and the conservation of mass, we have
\[
|M_{\varphi}(u(t))| \leq C(\varphi R) \left( \|u(t)\|_{H^s}^2 + \|u(t)\|_{H^s}^2 \right),
\]
(4.19)
where we have used the interpolation estimate \( \|u\|_{H^{1/2}} \leq \|u\|_{L^{2}}^{1/2} \|\nabla u\|_{L^2}^{1/2} \). Combining (4.17) and (4.19), we get
\[
|\mathcal{M}_\varphi(u(t))| \leq C(\varphi_R)\|u(t)\|_{H^{1/2}}^{\frac{3}{2}}.
\]
(4.20)

It follows from (4.18) and (4.20) that
\[
\mathcal{M}_\varphi(u(t)) \leq -A \int_{\tau_0}^{t} |\mathcal{M}_\varphi(u(\tau))|^{2s}d\tau,
\]
(4.21)

for all \( t \geq t_0 \) with some constant \( A = C(b, R) > 0 \). Set \( z(t) := \int_{\tau_0}^{t} |\mathcal{M}_\varphi(u(\tau))|^{2s}d\tau \) for \( t \geq t_0 \) and fix some time \( t_1 > t_0 \). We see that \( z(t) \) is strictly increasing and non-negative. Moreover,
\[
z'(t) = |\mathcal{M}_\varphi(u(t))|^{2s} \geq A^{2s}z^{2s}(t),
\]

for some \( A = C(b, R) > 0 \). Integrating the above inequality on \([t_1, t]\), we get
\[
z(t) \geq \frac{z(t_1)}{(1 - (2s - 1)A^{2s}|z(t_1)|^{2s-1}(t - t_1))^{\frac{1}{2s-1}}},
\]

for all \( t \geq t_1 \). It follows that
\[
z(t) \to +\infty \text{ as } t \uparrow t_* := t_1 + \frac{1}{(2s - 1)|z(t_1)|^{2s-1}} > t_1.
\]

By (4.21), we obtain
\[
\mathcal{M}_\varphi(u(t)) \leq -Az(t) \to -\infty \text{ as } t \uparrow t_*.
\]

Therefore, the solution cannot exist for all time \( t \geq 0 \) and consequently we must have \( T < +\infty \). The proof is complete. \( \square \)

Acknowledgments

The author would like to express his deep gratitude to his wife - Uyen Cong for her encouragement and support. He would like to thank his supervisor Prof. Jean-Marc Bouclet for the kind guidance and constant encouragement. He also would like to thank the reviewer for his/her helpful comments and suggestions.

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