Helical $\alpha$-dynamos as twisted magnetic flux tubes in Riemannian space

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Abstract

Analytical solution of $\alpha$-dynamo equation representing strongly torsioned helical dynamo is obtained in the thin twisted Riemannian flux tubes approximation. The $\alpha$ factor possesses a fundamental contribution from torsion which is however weaken in the thin tubes approximation. It is shown that assuming that the poloidal component of the magnetic field is in principle time-independent, the toroidal magnetic field component grows very fast in time, actually it possesses a linear time dependence, while the poloidal component grows under the influence of torsion or twist of the flux tube. The toroidal component decays spatially with as $r^{-2}$ while vorticity may decay as $r^{-5}$ (poloidal component) where $r$ represents the radial distance from the magnetic axis of flux tube. Toroidal component of vorticity decays as $r^{-1}$. In turbulent dynamos unbounded magnetic fields may decay at least as $r^{-3}$. PACS numbers:02.40.Hw-Riemann geometries

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I  Introduction

Despite of the success of the application of the numerical simulations to the dynamo problem [1] in plasma astrophysics [2] and in the stretch-twist-fold (STF) Vainshtein-Zeldovich [2] mechanism, recently new dynamo analytical solutions have been found [3] by using the conformal mapping technique in Riemannian manifolds from old Arnold cat dynamo metric [2]. Earlier T. Kambe [4] found simultaneous vortical and magnetohydrodynamic (MHD) solutions. In this paper an helical dynamo [8] solution of self-induction is obtained in vortical strongly torsioned thin twisted magnetic flux tubes in Riemannian space [5] where the MHD equations are linear in the magnetic field and nonlinear in the velocity flow. Assuming that the poloidal is time-independent the toroidal component of the magnetic field grows fast in time, actually it grows linear and not exponential. The rate of growing of the toroidal component depending on the inverse squared of the radial distance of the magnetic axis which possesses curvature and torsion. Recently, Hanasz and Lesch [6] have used also a conformal Riemannian metric in $E^3$ to the galactic dynamo magnetic flux tubes. Pioneering work on the magnetic flux tubes as dynamos was done earlier by Schussler [7], however in his work tubes were untwisted and straight. The main advantage of the investigation of the isolated flux tube dynamo is that one is able to investigate the curvature and twist contributions of the tube to the dynamo action. Twist is actually related to the torsion of the magnetic axis of the tube, which makes the words strong torsion equivalent to strong twist, which physically is important to the twist-kink relation investigated by Alfven [8]. Helical dynamo here is understood as the one where the flow describes a circular helix where torsion and curvature are constants and equal. The paper is organized as follows: In section II the dynamo solution in the Riemann metric representing flux rope (twisted tubes) is obtained. In section III the approximate solution is presented. In section IV conclusions are given.

II  Helical dynamos in Riemannian space

In this section we shall be concerned with solving the MHD equations in the curved coordinates of a thin twisted magnetic flux tube of Riemann metric

$$ds^2 = dr^2 + r^2 d\theta_R^2 + K^2(s) ds^2$$

(1)
which represents a Riemannian line element
\[ ds^2 = g_{ij}dx^i dx^j \] (2)
if the tube coordinates are \((r, \theta_R, s)\) [1] where \(\theta(s) = \theta_R - \int \tau ds\) and \(\tau\) is the Frenet torsion of the tube axis, \(K(s)\) is given by
\[ K^2(s) = [1 - r\kappa(s)\cos\theta(s)]^2 \] (3)
Since we are considered thin magnetic flux tubes, this expression shall be taken as \(K \approx 1\) in future computations. In curvilinear coordinates the Riemannian Laplacian operator \(\nabla^2\) [1] is
\[ \nabla^2 = \frac{1}{\sqrt{g}} \partial_i [\sqrt{g} g^{ij} \partial_j] \] (4)
where \(\partial_j := \frac{\partial}{\partial x^j}\) and \(g := det g_{ij}\) where \(g_{ij}\) are the covariant components of the Riemann metric of flux rope. Let us now start by considering the MHD field equations
\[ \nabla \cdot \vec{B} = 0 \] (5)
\[ \frac{\partial}{\partial t} \vec{B} = \nabla \times [\alpha \vec{B}] = \alpha \nabla \times \vec{B} + \nabla \alpha \times \vec{B} \] (6)
called the \(\alpha\)-dynamo equation [?]. Sometimes the \(\alpha := \langle \vec{v} \cdot \vec{\omega} \rangle\) parameter is constant but here we shall be considering the more general case where it depends on the radial and poloidal coordinate. Here \(\omega := \nabla \times \vec{v}\) is the vorticity of the dynamo flow. Equation (3) represents the self-induction equation. The vectors \(\vec{t}\) and \(\vec{n}\) along with binormal vector \(\vec{b}\) form the Frenet holonomic frame, which obeys the Frenet-Serret equations
\[ \vec{t}' = \kappa \vec{n} \] (7)
\[ \vec{n}' = -\kappa \vec{t} + \tau \vec{b} \] (8)
\[ \vec{b}' = -\tau \vec{n} \] (9)
where the dash represents the ordinary derivation with respect to coordinate \(s\), and \(\kappa(s, t)\) is the curvature of the curve, where \(\kappa = R^{-1}\). Here \(\tau\) represents the Frenet torsion. The gradient operator is
\[ \nabla = \vec{t} \frac{\partial}{\partial s} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_r \frac{\partial}{\partial r} \] (10)
Now we shall consider the analytical solution of the self-induction magnetic equation which represents a non-dynamo thin magnetic flux tube. Before the derivation of this result is obtained, we would like to point it out that it is not trivial, since the Zeldovich antidynamo theorem states that the two dimensional magnetic fields do not support dynamo action. Here, as is shown bellow, the flux tube axis possesses not only Frenet curvature, but torsion as well, and this last one vanishes in planar curves. The magnetic field does not possess a radial component and the magnetic field can be split inti its toroidal and poloidal components as

\[ \vec{B}(r, s, t) = B_\theta(t, r, \theta(s)) + B_s(r) \vec{t} \]  \hspace{1cm} (11)

Now let us substitute the definition of the poloidal plus toroidal magnetic fields into the self-induction equation, along with expressions

\[ \vec{e}_\theta = -\vec{n} \sin \theta + \vec{b} \cos \theta \]  \hspace{1cm} (12)

and

\[ \vec{e}_r = \vec{n} \cos \theta + \vec{b} \sin \theta \] \hspace{1cm} (13)

\[ \partial_t \vec{e}_\theta = \omega_\theta \vec{e}_r - \partial_t \vec{n} \sin \theta + \partial_t \vec{b} \cos \theta \] \hspace{1cm} (14)

Considering the equations for the time derivative of the Frenet frame given by the hydrodynamical absolute derivative

\[ \dot{\vec{X}} = \partial_t \vec{X} + [\vec{v}, \nabla] \vec{X} \] \hspace{1cm} (15)

where \( \vec{X} = (\vec{t}, \vec{n}, \vec{b}) \) is used into the expressions for the total derivative of each Frenet frame vectors

\[ \dot{\vec{t}} = \partial_t \vec{t} + [\kappa' \vec{b} - \kappa \tau \vec{n}] \] \hspace{1cm} (16)

\[ \dot{\vec{n}} = \kappa \tau \vec{t} \] \hspace{1cm} (17)

\[ \dot{\vec{b}} = -\kappa' \vec{t} \] \hspace{1cm} (18)

therefore leading to the following values of respective partial derivatives of the Frenet frame

\[ \partial_t \vec{t} = -\tau \kappa [1 - \kappa \tau^{-2} \frac{\nu_\theta}{r}] \vec{n} \] \hspace{1cm} (19)
\[
\begin{align*}
\partial_t \vec{n} &= \tau \kappa [1 - \kappa \vec{t}^{-2} \frac{v_\theta}{r}] \vec{t} + \frac{v_\theta}{r} \vec{b} \\
\partial_t \vec{b} &= \kappa \tau^{-1} \frac{v_\theta}{r} \vec{n}
\end{align*}
\]  

(20)  

(21)

where use has been made of the hypothesis that \( \vec{b} = 0 \) or \( \kappa'(t, s) = 0 \), which means that the curvature only depends on time. Substitution of these vectorial expressions into expression (14) yields

\[
\partial_t \vec{e}_\theta = -\omega \vec{e}_r + \gamma [\tau_0 \sin \theta (1 - \tau_0) \vec{t} + \tau_0 \cos \theta \vec{n} - \tau_0 \sin \theta \vec{b}] 
\]

(22)

where \( \gamma := (v_s - \frac{1}{r} \tau_0^{-1} v_\theta) \). The other dynamical equation for the Frenet holonomic frame is

\[
\partial_t \vec{e}_\theta = [\tau_0 \sin \theta \vec{t} - [\theta + \tau_0] \cos \theta \vec{n} - [\omega + \tau_0] \sin \theta \vec{b}] 
\]

(23)

note that in the mean field dynamo case, where \( \vec{v} = \vec{v}(\vec{B}) \), equation (6) is an eigenvalue problem equation. Dynamo operators and eigenvalue of dynamos in compact Riemannian manifolds have been previously investigated by Chicone and Latushkin [9]. Substitution of previous equations into equation (6) and splitting these equations along the components of the Frenet frame \((\vec{t}, \vec{n}, \vec{b})\) yields the following three scalar equations

\[
\begin{align*}
\partial_t B_s + [\sin \theta \gamma - \tau_0] \tau_0 B_\theta &= \partial_t (\alpha B_\theta) \\
-\partial_t B_\theta \sin \theta - B_\theta \omega \cos \theta + (\cos \theta B_\theta \gamma - \tau_0 B_s) \tau_0 &= A \cos \theta - C \sin \theta \\
\partial_t B_\theta \cos \theta - B_\theta (\omega + \tau_0) \sin \theta &= A \sin \theta + C \cos \theta
\end{align*}
\]

(24)  

(25)  

(26)

where functions \( A \) and \( C \) are given respectively by

\[
A := \frac{1}{r} (\partial_\theta \alpha) B_s - (\partial_s \alpha) B_\theta - \alpha \partial_s B_\theta
\]

(27)

and

\[
C := (\partial_r \alpha) B_s + \frac{\alpha}{r} \partial_r B_r
\]

(28)

In the next section we shall solve find an approximate solution for strong torsioned \( \alpha - dynamos \) tubes. Helical dynamo hypothesis of \( \kappa_0 = constant = \tau_0 \) has been taken throughout these computations.
III Analytical solution of helical dynamos in flux tubes

From the solenoidal equation for the magnetic field one obtains

$$\partial_s B_\theta = B_\theta \tau_0^2 r \sin \theta$$  \hspace{1cm} (29)

one obtains the value for the poloidal component as

$$B_\theta = B_0 \exp(\tau_0 r \cos \theta)$$  \hspace{1cm} (30)

with this expression and the expressions for the value of $\alpha$ which can be obtained as

$$\alpha = v_\theta \omega_\theta + v_s \omega_s$$  \hspace{1cm} (31)

To compute this important factor $\alpha$ one computes the vorticity $\vec{\omega}$ as

$$\omega_r = -\partial_s v_\theta = \kappa_0 \tau_0 r v_\theta \sin \theta$$  \hspace{1cm} (32)

where we have used in this equation the physical assumption of the incompressibility of the dynamo flow. The remaining vorticity expressions are

$$\omega_\theta = -\partial_r v_s$$  \hspace{1cm} (33)

$$\omega_s = \partial_r v_\theta + \frac{1}{r} v_\theta$$  \hspace{1cm} (34)

This allows us to obtain the following value for $\alpha$

$$\alpha = \frac{1}{r} v_0 \exp(-r \tau_0 \cos \theta)$$  \hspace{1cm} (35)

By making use of thin tube approximation where we consider we are close to the magnetic flux tube axis ($r = 0$), one may use the following approximation

$$\alpha = \frac{1}{r} v_0$$  \hspace{1cm} (36)

which yields

$$\partial_s \alpha = -\frac{1}{r^2} v_0$$  \hspace{1cm} (37)
which upon substitution in equation (ref1) yields

\[
\partial_t B_s = \left[ \frac{\tau_0^2}{r^2} - \frac{v_0}{r^2} \right] B_\theta
\]  
(38)

\[
\partial_t B_s = -\frac{v_0}{r^2} B_0
\]  
(39)

which simply solves to

\[
B_s = -\frac{v_0}{r^2} B_0 t
\]  
(40)

which shows that the toroidal field grows linearly in time which close to the magnetic axis may represent a very fast dynamo. Comparison between the poloidal and toroidal field one obtains

\[
\frac{B_\theta}{B_s} \approx \frac{v_0}{r^2} t
\]  
(41)

In regions not so close to the magnetic axis where torsion may dominate one obtains an alternative solution in which however the magnetic toroidal field still grows in time as

\[
B_s \approx \tau_0^2 t
\]  
(42)

Substitution of these equations in the remaining dynamo equation yields the poloidal component of vorticity as

\[
\omega_\theta = \frac{v_0 t}{r^5}
\]  
(43)

which implies that

\[
v_s = \frac{v_0 t}{4r^4}
\]  
(44)

and the other vorticity components are

\[
\omega_r \approx -\tau_0^2 r v_0 \exp(\tau_0 r \cos \theta)
\]  
(45)

and \(\omega_r = 0\) at the magnetic axis. Finally

\[
\omega_s = \frac{v_0}{r} \exp(\tau_0 r \cos \theta)
\]  
(46)

which decays slower as we go away from the magnetic axis of flux tube dynamo.
IV Conclusions

In conclusion, we have used an approximate method of strong torsion to find near the magnetic axis of the dynamo flux tube to analytical (non-numerical) solutions of \(\alpha\)-dynamo equation. Previously Ruzmaikin et al [10] have found solutions of turbulent dynamos where unbounded magnetic field could decay at least as \(r^{-3}\) which is distinct to the bounded case of twisted magnetic flux tube helical dynamo we have found here. The general equations of the helical dynamo found in section II could be used to find out more general solutions, with less degree of approximation that we used here, for example by letting the magnetic poloidal component vary with time and also considered the case of untwisted tubes. This can be done elsewhere.

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