Quantization of $U_q[so(2n+1)]$ with deformed para-Fermi operators

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Abstract. The observation that $n$ pairs of para-Fermi (pF) operators generate the universal enveloping algebra of the orthogonal Lie algebra $so(2n+1)$ is used in order to define deformed pF operators. It is shown that these operators are an alternative to the Chevalley generators. On this background $U_q[so(2n+1)]$ and its "Cartan-Weyl" generators are written down entirely in terms of deformed pB operators.

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The very idea of the present paper is much along the line of the one, developed in [1], where we have quantized the orthosymplectic Lie superalgebra $osp(1/2n)$ in terms of deformed para-Bose operators. Here we solve the same problem for the Lie algebra (LA) $so(2n+1)$. More precisely, we define deformed pF operators $a^\pm_1, \ldots, a^\pm_n$ and show that the quantized universal enveloping algebra $U_q[so(2n+1)]$ can be defined entirely in terms these operators. In other words $U_q[so(2n+1)]$ appears as a Hopf algebra with the pF operators being its free generators. It is a deformation of the universal enveloping algebra (UEA) $U[so(2n+1)]$ of $so(2n+1)$ with a deformation parameter $q$. At $q = 1$ one obtains the nondeformed algebra $U[so(2n+1)]$.

We wish to stress that we do not give any new deformation of $U[so(2n+1)]$. The deformation is the same as the one obtained in terms of the Chevalley generators and this will be explicitly shown. The only difference is that in our case the generating elements are deformed pF operators instead of deformed Chevalley generators.

Soon after the parastatistics was invented [2], it was shown that any $n$ pairs $\hat{a}^\pm_1, \ldots, \hat{a}^\pm_n$ of pF operators generate the simple Lie algebra $so(2n+1)$ [3], whereas $n$ pairs of para-Bose operators generate a Lie superalgebra [4], which is isomorphic to the basic Lie superalgebras $osp(1/2n) \equiv B(0/n)$ [5]. Purely algebraically the pF operators are defined as operators, which satisfy the relations ($\xi, \eta, \epsilon = \pm$ or $\pm 1$, $i, j, k = 1, 2, \ldots, n ; [x, y] = xy - yx$)

$$[[\hat{a}^\xi_i, \hat{a}^\eta_j], \hat{a}^\epsilon_k] = \frac{1}{2}(\epsilon - \eta)^2\delta_{jk}\hat{a}^\xi_i - \frac{1}{2}(\epsilon - \xi)^2\delta_{ik}\hat{a}^\eta_j.$$  \hspace{1cm} (1)

Let $pF(n)$ be the pF algebra, i.e., the free associative unital algebra with generators $\hat{a}^\pm_1, \ldots, \hat{a}^\pm_n$ and

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relations (1). Then \( pF(n) \) is also a Lie algebra with respect to the natural commutator \([x, y] = xy - yx\), \( x, y \in pF(n) \). Its subspace \( L(n) = \text{lin.env.} \{[\hat{\alpha}_i^k, \hat{\alpha}_j^k], \hat{\alpha}_k^k \mid \xi, \eta, \epsilon = \pm, i, j, k = 1, 2, \ldots, n \} \), is a subalgebra of the LA \( pF(n) \), isomorphic to \( so(2n + 1) \) [3]. The commutation relations in \( L(n) = so(2n + 1) \) are completely defined from the triple relations (1). Therefore from the very definition of an UEA of a LA one concludes that \( pF(n) \) is the UEA of \( so(2n + 1) \).

**PROPOSITION 1** [6]. The para-Fermi algebra \( pF(n) \) is (isomorphic to) the universal enveloping algebra of \( so(2n + 1) \). The basis in the Cartan subalgebra of \( so(2n + 1) \) can be chosen in such a way that the \( pF \) creation (resp. annihilation) operators are negative (resp. positive) root vectors.

The relations between the \( pF \) operators and the Chevalley generators \( \hat{e}_i, \hat{f}_i, \hat{h}_i \), \( i = 1, \ldots, n \) of \( so(2n + 1) \) can be easily written down (\( i = 1, \ldots, n - 1 \)):

\[
\begin{align*}
\hat{e}_n &= \frac{1}{\sqrt{2}} \hat{a}_n^- \quad \hat{e}_i = \frac{1}{2} [\hat{a}_i^-, \hat{a}_{i+1}^+], \\
\hat{f}_n &= \frac{1}{\sqrt{2}} \hat{a}_n^+ \quad \hat{f}_i = \frac{1}{2} [\hat{a}_{i+1}^-, \hat{a}_i^+], \\
\hat{h}_n &= \frac{1}{2} (\hat{a}_n^-, \hat{a}_n^+), \\
\hat{h}_i &= \frac{1}{2} (\hat{a}_i^-, \hat{a}_i^+) - \frac{1}{2} [\hat{a}_{i+1}^-, \hat{a}_{i+1}^+].
\end{align*}
\]

(2)

The inverse relations, namely the expressions of the \( pB \) operators in terms of the Chevalley generators, read (\( i = 1, \ldots, n - 1 \)):

\[
\begin{align*}
\hat{a}_i^- &= \sqrt{2} \hat{e}_i, \quad [\hat{e}_{i+1}, \hat{e}_{i+2}, \ldots, \hat{e}_{n-1}, \hat{e}_n] = 0, \quad \hat{a}_n^- = \sqrt{2} \hat{e}_n, \\
\hat{a}_i^+ &= \sqrt{2} \hat{f}_i, \quad \hat{f}_n = [\hat{f}_{n-1}, \hat{f}_{n-2}, \ldots, \hat{f}_{i+2}, \hat{f}_{i+1}, \hat{f}_i], \quad \hat{a}_n^+ = \sqrt{2} \hat{f}_n.
\end{align*}
\]

(3)

Following [7] we proceed first to introduce the deformed UEA \( U_q[so(2n + 1)] \equiv U_q \) in terms of its Chevalley generators. The Cartan matrix (\( \alpha_{ij} \)) is a \( n \times n \) symmetric matrix with \( \alpha_{nn} = 1, \alpha_{ii} = 2, i = 1, \ldots, n - 1, \alpha_{j+1} = -1, j = 1, \ldots, n - 1 \), and all other \( \alpha_{ij} = 0 \). Then \( U_q \) is the free associative superalgebra with Chevalley generators \( e_i, f_i, k_i = q^{h_i}, \bar{k}_i = k_i^{-1} = q^{-h_i}, i = 1, \ldots, n \), which satisfy the Cartan relations

\[
k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad k_i e_j = q^{\alpha_{ij}} e_j k_i, \quad k_i f_j = q^{-\alpha_{ij}} f_j k_i, \quad [e_i, f_j] = \delta_{ij} k_i - \bar{k}_i q^{-1} = \bar{k}_i q^{-1}.
\]

(4)

and the Serre relations (\( \bar{q} \equiv q^{-1} \))

\[
[e_i, e_j] = 0, \quad [f_i, f_j] = 0, \quad |i - j| > 1,
\]

(5)

\[
[e_i, e_{i \pm 1}]_q = 0, \quad [f_i, f_{i \pm 1}]_q = 0.
\]

(6)

\[
[e_n, e_n]_q = 0, \quad [f_n, f_n]_q = 0.
\]

(7)

Here and throughout the paper \([a, b]_q = ab - q^ba\) and it is assumed that the deformation parameter \( q \) is any complex number except \( q = 0, q = 1 \) and \( q^2 = 1 \). The eqs. (4)-(7) are invariant with respect to the antiinvolution

\[
(e_i)^* = f_i, \quad (k_i)^* = k_i^{-1} = \bar{k}_i, \quad (q)^* = q^{-1} = \bar{q}.
\]

(8)
We do not write here the explicit forms of the coproduct, the counit and the antipode, since we shall not use them. They are the same as in [1], eqs.(13)-(15).

Having in mind the expressions (3) we define the deformed PF operators as follows:

\[ a_i^- = \sqrt{q} [e_i, [e_{i+1}, e_{i+2}, \ldots, [e_{n-2}, e_{n-1}, e_n]]_{q} \ldots]_{q} = [e_i, a_{i+1}]_{q}, \quad a_n^- = \sqrt{q} e_n, \]

\[ a_i^+ = \sqrt{q} [[f_{i}, f_{i+1}, q], f_{i+2}]_{q}, f_{i+1}]_{q} = [a_{i+1}, f_i]_{q}, \quad a_n^+ = \sqrt{q} f_n. \]  \hspace{1cm} \text{(9)}

From the definition (9), the Cartan and the Serre relations (4)-(6) one obtains:

\[ [e_i, a_j^+] = -q \delta_{ij} a_{i+1}^+ k_i = -\delta_{ij} k_i a_{i+1}^+, \quad [a_j^+, f_i] = -q \delta_{ij} a_{i+1}^- \bar{k}_i = -q \delta_{ij} \bar{k}_i a_{i+1}^-, \quad i \neq n. \]  \hspace{1cm} \text{(10)}

\[ [e_i, a_j^-] = 0, \quad [a_j^+, f_i] = 0, \quad j < i \text{ or } i + 1 < j, \quad i \neq n, \]

\[ [e_i, a_{i+1}^-]_{q} = a_i^-, \quad [a_{i+1}^+, f_i]_{q} = a_i^+, \quad i \neq n, \]

\[ [e_i, a_i^-]_{q} = 0, \quad [a_i^+, f_i]_{q} = 0, \quad i \neq n. \]  \hspace{1cm} \text{(11)}

The derivation of these equations is based on identities like

1° If \( [a, b] = 0 \), then \( [[a, c]_{q}, b]_{p} = [a, [c, b]]_{p} \).

2° If \( [a, b] = 0 \), then \( [[a_{1}, [a_{2}, c]]_{q}], b]_{p} = [a_{1}, [a_{2}, [c, b]]_{p}]_{q} \).

3° If \( [a, c] = 0 \), then \( (q + q^{-1}) [b, [a, [b, c]]_{q}] = [a, [b, [b, c]]_{q}] - [b, [b, c]]_{q} \).

We mention some of the steps.

(a) From (4) and (5) one easily derives \( [e_i, a_j^+] = 0 \) for \( i \neq j, \quad i \neq n. \) Therefore, \( [e_i, a_j^+] = [e_i, [a_{i+1}^+, f_i]]_{q} = [a_{i+1}^+, [e_i, f_i]]_{q} = \frac{1}{q-q'} [a_{i+1}^+, k_i - \bar{k}_i]_{q} = -qa_{i+1}^- k_i; \) hence \( [e_i, a_j^+] = -q \delta_{ij} a_{i+1}^+ k_i. \)

(b) For \( i \neq n \) one has from (5) \( [e_i, a_j^-] = 0 \) for \( i < j - 1 \); by definition \( [e_i, a_{i+1}^-]_{q} = a_i^-; \) from (6)

\[ [e_i, a_i^-]_{q} = [e_i, e_i, [e_i, a_{i+1}^+]_{q}]_{q} = [[e_i, [e_i, a_{i+1}^+]_{q}], a_{i+2}^-]_{q} = 0 \]

\[ [e_i, a_{i+1}^-] = [e_i, e_i, [e_i, a_{i+1}^+]_{q}] = [a_i^+, e_i, a_{i+1}^+]_{q} = q a_{i+1}^- k_i, \]

\[ a_{i+2}^- = q a_{i+1}^- k_i. \]  \hspace{1cm} \text{(12)}

(c) The r.h.s. equations (10), (11), i.e., those involving \( f_i, \) are obtained from the l.h.s. equations by means of the antiinvolution (8) and a consequence of it, namely \( (a_i^+)^* = a_i^- \).

PROPOSITION 2. The deformed PF operators (9) together with the "Cartan" operators \( k_1, \ldots, k_n \) generate (in a sense of an associative algebra) \( U_q[so(2n + 1)]. \)

Proof. The proof is an immediate consequence of the relations:

\[ [a_i^-, a_j^+] = 2 k_n k_{n-1} \ldots k_{i-1} - k_n \bar{k}_{n-1} \ldots \bar{k}_i q - \bar{q}, \quad i = 1, \ldots, n, \]

\[ [a_i^-, a_{i+1}^-] = 2 k_n k_{n-1} \ldots k_{i+1} e_i, \quad [a_i^+, a_{i+1}^+] = 2 f_i k_n k_{n-1} \ldots \bar{k}_{i+1}, \quad i = 1, \ldots, n - 1. \]  \hspace{1cm} \text{(13)}

These equations are proved by induction on \( i. \) For \( i = n \) (12) holds. Suppose that for a certain \( i + 1 \)

\[ [a_i^-, a_{i+1}^-] = 2 k_n k_{n-1} \ldots k_i - k_n \bar{k}_{n-1} \ldots \bar{k}_{i+1} q - \bar{q}, \quad i = 1, \ldots, n - 1. \]  \hspace{1cm} \text{(14)}
Then \([a_i^-, a_{i+1}^+] = [e_i, a_{i+1}^-], a_{i+1}^+] = \{e_i, [a_{i+1}^-, a_{i+1}^+]\}\) and from (14) \([a_i^-, a_{i+1}^+] = \frac{2}{q - q_0} [e_i, k_n k_{n-1} \ldots k_{i+1} - \bar{k}_n \bar{k}_{n-1} \ldots \bar{k}_{i+1}]\) and the Cartan relations (4), (15) and the Cartan relations (4); (C) is the same as (12). Replacing (14) with relations (17).

Therefore eqs.(13) hold. Thus, if eq. (14) holds, then eqs. (13) are fulfilled. From here and eqs. (11) we compute \([a_i^-, a_{i+1}^+] = [a_i^-, a_{i+1}^-], f_i] = [a_i^-, [a_{i+1}^-, f_i]] = [2k_nk_{n-1} \ldots k_{i+1} e_i, f_i]/q - [a_{i+1}^-, a_{i+1}^-] f_i) = 2k_nk_{n-1} \ldots k_{i+1} e_i + [a_{i+1}^-, a_{i+1}^-] f_i = \frac{2}{q - q_0} (k_n k_{n-1} \ldots k_i - k_i k_{n-1} \ldots k_i).

This proves the validity of eqs. (12), (13) and, hence, of Proposition 2.

Let

\[L_i = k_i k_{i+1} \ldots k_n, \quad i = 1, \ldots, n; \quad k_i = L_i L_{i+1}, \quad i = 1, \ldots, n - 1.\] (15)

Following the terminology, introduced in [8], we call the operators

\[a_i^+, \quad L_i \quad i = 1, \ldots, n\] (16)

pre-oscillator generators of \(U_q[so(2n + 1)]\).

PROPOSITION 3. The defining relations (4)-(7) of \(U_q[so(2n + 1)]\) in terms of its Chevalley generators \(e_i, f_i, k_i = q^h, \ i = 1, \ldots, n\), hold if and only if the pre-oscillator generators (16) satisfy the relations:

\[\quad [L_i, a_j^+] = 0, \quad i \neq j = 1, \ldots, n, \quad (A)\]

\[\quad [L_i, a_i^+]_{q+1} = 0, \quad i = 1, \ldots, n, \quad (B)\]

\[\quad [a_i^-, a_i^+] = 2\frac{L_i - L_{i+1}}{q - q_0}, \quad i = 1, \ldots, n, \quad (C)\]

\[\quad [[a_i^-, a_{i+1}^-], a_j^+] = 2\delta_{j,i+1} L_j^{\pm} a_i^-, \quad \eta = \pm, \quad (D)\]

\[\quad [a_i^+, [a_n^-, a_{n-1}^\eta]] = 0, \quad \xi = \pm. \quad (E)\]

Therefore \(U_q[so(2n + 1)]\) can be viewed as a free associative unital algebra of the pre-oscillator generators (16) with relations (17).

Proof. We sketch the proof.

1) Necessity. Let the equations for the Chevalley generators (4)-(7) hold. (A) and (B) are simple consequence from the definitions (9), (15) and the Cartan relations (4); (C) is the same as (12). Replacing in (11) \(e_i, f_i\) from eqs. (13) and rearranging the terms, one obtains after a long, but simple calculations (D). Inserting from (9) \(e_n\) and \(e_{n-1}\) in the Serre relations (7), one ends with

\[\quad [e_n, [e_n, [e_n, e_{n-1}]]] = -\frac{q}{2\sqrt{2}} [a_n^-, [a_n^-, a_{n-1}^-]]\] (18)

Hence eqs. (E) hold.

2) Sufficiency. Assume that the pre-oscillator generators, defined with (9), (15), satisfy eqs. (17). From (9) and (13) we have

\[\quad e_n = \frac{1}{\sqrt{2}} a_n, \quad f_n = \frac{1}{\sqrt{2}} a_n^+, \quad e_i = \frac{1}{\sqrt{2}} L_{i+1} [a_i^-, a_{i+1}^+] = \frac{1}{2} L_{i+1} [a_i^-, a_{i+1}^+] L_{i+1}, \quad i \neq n. \quad (19)\]

Using (15), (17) and (18) it is easy to derive the Cartan relations (3).
In order to show that one of the bilinear Serre relations (5) hold, consider \( i < j - 1 \). Then from (17)

\[
[e_i, e_n] = \frac{1}{2\sqrt{2}}\{\bar{L}_{i+1}[a_i^-, a_i^{+1}], a_n^+\} = \frac{1}{2\sqrt{2}}\bar{L}_{i+1}[a_i^-, a_i^{+1}], a_n^+\] = 0;
\]

\[
[e_i, e_j] = \frac{1}{2\sqrt{2}}\bar{L}_{i+1}\bar{L}_{j+1}\{[a_i^-, a_j^+], a_i^+\} = \frac{1}{2\sqrt{2}}\bar{L}_{i+1}\bar{L}_{j+1}\{[a_i^-, a_j^+], a_i^+\} = 0, j \neq n.
\]

Therefore \([e_i, e_j] = 0\), if \(|i - j| > 1\).

As an example of a triple Serre relation we consider \([e_i, [e_i, e_i-1]]\). In this case \( i \neq n \). First we derive from (19)

\[
[e_i, e_i-1]_q = \frac{q}{4}\bar{L}_{i+1}\bar{L}_{i}\{[a_i^-, a_i^{+1}], a_i^+\} = \frac{q}{4}\bar{L}_{i+1}[a_i^-, a_i^{+1}].
\]

Therefore from (17)

\[
[e_i, e_i-1]_q = \frac{q}{4}\bar{L}_{i+1}[a_i^-, a_i^{+1}], a_i^+\] = 0
\]

where

\[
F = [\bar{L}_{i+1}[a_i^-, a_i^{+1}], a_i^+\] = 0.
\]

The validity of the other triple Serre relations (6) is proved in the same way.

The last Serre relations follow from the eq.(18) and its conjugate. This completes the proof.

Since

\[
L_i = q^{H_i}, \quad H_i = h_i + h_{i+1} + \ldots + h_n,
\]

in the limit \( q \to 1 \) the equations (17) reduce to

\[
[[a_i^-, a_j^\eta], a_k^\eta]] = 2\delta_{jk}\hat{a}_i^\eta, \quad |i - j| < 2, \quad \eta = \pm,
\]

(21)

\[
[[\hat{a}_i^\xi, \hat{a}_j^\xi], \hat{a}_i^\xi]] = 0, \quad \xi = \pm.
\]

(22)

We came to an interesting conclusion, which is new even for the undeformed pF operators. The point is that the eqs.(21-22) are only a small part of all eqs.(1), initially used to define the pF operators [2]. Nevertheless they define completely the para-Fermi statistics. In a certain sense (21-22) give the minimal set of relations, defining the pF operators. Elsewhere we shall write down the complete set of quantum relations, namely the quantum analog of eqs.(1), which is a more difficult task. The relevance of the complete set of relations stems from the following proposition, which we only formulate.

PROPOSITION 4. The operators \( L_i, a_i^\pm, [a_i^-, a_j^+], [a_k^\xi, a_p^\xi], i \neq j, i, j, p, q = 1, \ldots, n \) are an analogue of the Cartan-Weyl generators for \( so(2n + 1) \). In terms of these generators one can introduce a basis in \( U_q[so(2n + 1)] \).

For the deformed para-Bose operators the complete set of relations was given in [1], whereas the question about the minimal set of relations was settled in [9].
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