Recovering the effective cosmological constant in extended gravity theories

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Abstract

In the framework of extended gravity theories, we discuss the meaning of a time dependent “cosmological constant” and give a set of conditions to recover asymptotic de Sitter behaviour for a class of cosmological models independently of initial data. To this purpose we introduce a time–dependent (effective) quantity which asymptotically becomes the true cosmological constant. We will deal with scalar–tensor, fourth and higher than fourth–order theories.

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1 Introduction

The determination of cosmological constant has become one of the main issues of modern physics since by fixing its value, one could contribute to select self-consistent models of fundamental physics and cosmology. Briefly, its determination should provide the gravity vacuum state [1], should make to understand the mechanism which led the early universe to the today observed large scale structures [2],[3], and to predict what will be the fate of the whole universe (no-hair conjecture) [4].

From the cosmological point of view, the main feature of inflationary models is the presence of a finite period during which the expansion is de Sitter (or quasi-de Sitter or power law): this fact implies that the expansion of the scale factor $a(t)$ is superluminal (at least $a(t) \sim t$, in general $a(t) \sim \exp H_0 t$ where $H_0$ is the Hubble parameter nearly constant for a finite period) with respect to the comoving proper time $t$. Such a situation arises in presence of an effective energy-momentum tensor which is approximately proportional (for a certain time) to the metric tensor and takes place in various gravitational theories: i.e. the Einstein gravity minimally coupled with a scalar field [2],[3], fourth or higher-order gravity [5],[6],[7],[8] scalar-tensor gravity [9],[10].

Using conformal transformations (by which higher-order geometric terms and non-minimally coupled fields are reduced to the Einstein gravity plus (non)interacting scalar fields [11],[12],[13],[14],[15]) all of these approaches can furnish dynamical models where some scalar fields are displaced from their equilibrium states (false vacuum states) and then evolve sufficiently slow toward the minima of a potential, in general toward new equilibrium states (true vacuum states). If more than one scalar field undergo such a phenomenology, one can get multiple inflation [7],[16].

However, in all these schemes, we have to provide the solution of the so called ”flatness”, ”monopole” and ”horizon” problems [2],[3] and, besides, mechanisms able to give a natural explanation of the observed small inhomogeneities in the large scale structure of the universe [17].

Several inflationary models are affected by the shortcoming of ”fine tuning” [18], that is inflationary phase proceeds from very special initial conditions, while a natural issue would be to get inflationary solutions as attractors for a large set of initial conditions. Furthermore, the same situation should be achieved also in the future: if a remnant of cosmological constant is today observed, the universe should evolve toward a final de Sitter stage. A more precise formulation of such a conjecture is possible for a restricted class of cosmological models, as discussed in [19]. We have to note that the conjecture holds when any ordinary matter field, satisfies the dominant and strong energy conditions [20]. However it is possible to find models which explicitly violate such conditions but satisfies no-hair theorem requests. Precisely, this fact happens if extended gravity theories are involved and matter is in the form of scalar fields, besides the ordinary perfect fluid matter [21].

In any case, we need a time variation of cosmological constant to get successful inflationary models, to be in agreement with observations, and to obtain a de Sitter stage.
toward the future. In other words, this means that cosmological constant has to acquire a great value in early epoch (de Sitter stage), has to undergo a phase transition with a graceful exit (in order to recover the observed Friedman stage of present epoch) and has to result in a small remnant toward the future \[22\].

In this context, a fundamental issue is to select the classes of gravitational theories and the conditions which "naturally" allow to recover an effective time–dependent cosmological constant without considering special initial data.

This paper is devoted to this problem. We take into consideration extended gravity theories and try to select conditions to obtain effective time–dependent cosmological constant. The main request is that such effective cosmological constants evolve (at least asymptotically) toward the actual cosmological constant which means that the de Sitter behaviour has to be recovered.

In Sect.2, we discuss the effective cosmological constant and the properties of the de Sitter space–times. Sect.3 is devoted to the general discussion of extended gravity theories involving higher–order corrections to the Einstein–Hilbert action and scalar–tensor couplings. In Sects.4,5,6,7, we take into account specific realizations of such a theories that is scalar–tensor, fourth–order, fourth–order–scalar tensor and higher than fourth–order gravity theories, respectively, and the conditions to obtain de Sitter. Some cosmological models, as examples, are outlined in Sect.8.

Discussion and conclusions are drawn in Sect.9.

2 The effective cosmological constant

Before starting with our analysis, it is worthwhile to spend some words on what we mean by "effective cosmological constant". The no–hair conjecture \[4\] claims that if there is a positive cosmological constant, all the expanding universes will approach a de Sitter behaviour. In other words, if a cosmological constant is present, the universe will become homogeneous and isotropic with any initial conditions. However, there is no general proof of such a conjecture and there are counter–examples of initially expanding and then recollapsing universes which never become de Sitter \[23\].

A simplified version of the conjecture can be proved. It is:

All Bianchi cosmologies (except IX), in presence of a positive cosmological constant, asymptotically approach the de Sitter behaviour \[19\].

It is worthwhile to note that here the cosmological constant is a true constant (put by hands) and the contracted Bianchi identity is not used, then the proof is independent of the evolution of matter. In order to extend no–hair conjecture to generalized theories of gravitation, we have to introduce different sets of conditions (respect to those given in \[19\]) since the cosmological constant is not introduced \textit{a priori}, but it can be "recovered" from dynamics of scalar fields (considering as a sort of "scalar fields" also higher–order geometric terms in the gravitational Lagrangian \[7, 14\]). Such conditions must not use the "energy conditions" \[20\], but they have to allow the introduction of a sort of "effective cosmological constant" which asymptotically becomes the de Sitter constant.
This feature is due to the fact that, in an expanding universe, all the contributions to the energy density and to the Ricci tensor has to decay as some power of the scale factor $a(t)$. The cosmological constant is the only term that does not decrease with time. Hence, in an expanding universe, $\Lambda$ is the asymptotically dominant term in the Einstein equations (i.e. the $(0,0)$ Einstein equation becomes $H^2 = \frac{\Lambda}{3}$ with $H$ the Hubble parameter); giving rise to a de Sitter spacetime. Actually, the effective cosmological constant is time–dependent but, at the end, it has to coincide with the de Sitter one (the real gravitational vacuum state). Then, given any extended theory of gravity, it could be possible, in general, to define an effective time varying cosmological constant which becomes the ”true” cosmological constant if and only if the model asymptotically approaches de Sitter (that is only asymptotically no–hair conjecture is recovered). This fact will introduce constraints on the choice of the gravitational couplings, scalar field potentials and higher–order geometrical terms which combinations can be intended as components of the effective stress–energy tensor.

3 The extended gravity theories and cosmology

There is no a priori reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar $R$ minimally coupled with matter \[ [24] \]. Additionally, we have to note that, recently, some authors have taken into serious consideration the idea that there are no ”exact” laws of physics but that the Lagrangians of physical interactions are ”stochastic” functions with the property that local gauge invariances (i.e. conservation laws) are well approximated in the low energy limit and physical constants can vary \[ [6] \]. This scheme was adopted in order to treat the quantization on curved spacetimes and the result was that the interactions among quantum scalar fields and background geometry or the gravitational self–interactions yield corrective terms in the Einstein–Hilbert Lagrangian \[ [24] \]. Furthermore, it has been realized that such corrective terms are inescapable if we want to obtain the effective action of quantum gravity on scales closed to the Planck length \[ [29] \]. They are higher–order terms in curvature invariants as $R^2$, $R_{\mu\nu}R^{\mu\nu}$, $R^{\rho\sigma\alpha\beta}R_{\rho\sigma\alpha\beta}$, $R\Box R$, or $R\Box^k R$, or nonminimally coupled terms between scalar fields and geometry as $\phi^2 R$. Terms of these kinds arise also in the effective Lagrangian of strings and Kaluza–Klein theories when the mechanism of dimensional reduction is working \[ [27] \].

From a completely different point of view, these alternative theories become interesting when one try to incorporate the Mach principle in gravity and to consider the concept of ”inertia” in connection to the various formulations of equivalence principle. For example, the Brans–Dicke theory is a serious attempt of alternative theory to the Einstein gravity: it takes into consideration a variable Newton gravitational constant whose dynamics is governed by a scalar field nonminimally coupled with geometry. In such a way, the Mach principle is better implemented \[ [10], [28], [29] \].

Besides fundamental physics motivations, all these theories have acquired a huge
interest in cosmology due to the fact that they “naturally” exhibit inflationary behaviours and that the related cosmological models seem very realistic [5], [9]. Furthermore, it is possible to show that, via conformal transformations, the higher–order and nonminimally coupled terms (Jordan frame) always corresponds to the Einstein gravity plus one or more than one minimally coupled scalar fields (Einstein frame) [3], [11], [14], [15], [30]. More precisely (in the Jordan frame), the higher–order terms appear always as an enhancement of order two in the equations of motion. For example, a term like $R^2$ gives fourth order equations [31], $R \Box R$ gives sixth order equations [30], [32], $R \Box^2 R$ gives eighth order equations [33] and so on. By the conformal transformation, any 2–orders give a scalar field: for example, fourth–order gravity gives Einstein plus one scalar field, sixth order gravity gives Einstein plus two scalar fields and so on [7], [30]. This feature results very interesting if we want to obtain multiple inflationary events since an early stage could select “very” large scale structures (clusters of galaxies today), while a late stage could select “small” large scale structures (galaxies today) [32]. The philosophy is that each inflationary era is connected with the dynamics of a scalar field [16]. Furthermore, these extended schemes naturally could solve the problem of “graceful exit” bypassing the shortcomings of former inflationary models [9], [34].

Here we want to consider such theories, in general, and to ask for recovering the de Sitter behaviour in the related cosmological models.

Let us start with the most general class of higher–order–scalar–tensor theories in four dimensions. They can be assigned by the action

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ F(R, \Box R, \Box^2 R, \ldots, \Box^k R, \phi) - \frac{\epsilon}{2} g^{\mu\nu} \phi,_{\mu} \phi,_{\nu} + \mathcal{L}_m \right],$$

(1)

where $F$ is an unspecified function of curvature invariants and of a scalar field $\phi$. The term $\mathcal{L}_m$ is the minimally coupled ordinary matter contribution. We shall use physical units $8\pi G = c = \hbar = 1$; $\epsilon$ is a constant which specifies the theory.

The field equations are obtained by varying (1) with respect to the metric $g_{\mu\nu}$. We get

$$G^{\mu\nu} = \frac{1}{G} \left[ T^{\mu\nu} + \frac{1}{2} g^{\mu\nu} (F - \mathcal{G} R) + (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma}) \mathcal{G}_{,\lambda\sigma} \right. + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{i} \left( g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\lambda} g^{\nu\sigma} \right) \left( \Box^{j-i} \right)_{,\sigma} \left( \Box^{i-j} \frac{\partial F}{\partial \Box^{i} R} \right)_{,\lambda} \left. \right]$$

$$- g^{\mu\nu} g^{\lambda\sigma} \left( \Box^{i-j} \right)_{,\sigma} \left( \Box^{j-i} \frac{\partial F}{\partial \Box^{i} R} \right)_{,\lambda},$$

(2)

where

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

is the Einstein tensor and

$$\mathcal{G} \equiv \sum_{j=0}^{n} \Box^{j} \left( \frac{\partial F}{\partial \Box^{j} R} \right).$$

(4)
The differential Eqs. (2) are of order \((2k + 4)\). The stress–energy tensor is due to the kinetic part of the scalar field and to the ordinary matter:

\[
T_{\mu\nu} = T_{\mu\nu}^{(m)} + \frac{\epsilon}{4} \left[ \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} \phi^{\alpha\beta} \phi_{,\alpha \beta} \right] .
\]  

The (eventual) contribution of a potential \(V(\phi)\) is contained in the definition of \(F\). From now on, we shall indicate by a capital \(F\) a Lagrangian density containing also the contribution of a potential \(V(\phi)\) and by \(f(\phi)\), \(f(R)\), or \(f(R, \Box R)\) a function of such fields without potential.

By varying with respect to the scalar field \(\phi\), we obtain the Klein–Gordon equation

\[
\epsilon \Box \phi = -\frac{\partial F}{\partial \phi} .
\]

Several approaches can be used to treat such equations. For example, as we said, by a conformal transformation, it is possible to reduce an extended theory of gravity to a (multi)scalar–tensor theory of gravity \([7],[14],[30],[35]\). Here we want to discuss under what conditions it is possible to obtain asymptotic de Sitter behaviour from (2) considering some cases of physical interest. Our discussion will be in Jordan frame. For a detailed exposition of the differences between the Jordan and the Einstein frames, see e.g. \([15],[36]\): the debate of which of them is the true physical frame is still open.

4 Scalar–tensor gravity

The scheme which we adopt to find the conditions for an asymptotic no–hair theorem is outlined, for scalar–tensor gravity, in \([37],[38]\) and in \([39]\). Here, for the sake of completeness, we shall carry the same discussion and enlarge it to other extended gravity theories.

With the choice

\[
F = f(\phi) R - V(\phi) , \quad \epsilon = -1 ,
\]

we recover the scalar–tensor gravity in which a scalar field \(\phi\) is nonminimally coupled with the Ricci scalar \([11],[32]\). Here, we do not fix the coupling \(f(\phi)\) and the potential \(V(\phi)\) but we ask for recovering (in general) the de Sitter behaviour by restoring the cosmic no–hair theorem \([37]\). As we shall see, this request will fix a class of couplings and potentials.

The action \([1]\) now becomes

\[
\mathcal{A} = \int d^4 x \sqrt{-g} \left[ f(\phi) R + \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) + \mathcal{L}_m \right] ,
\]

while the Einstein equations are

\[
G_{\mu\nu} = \tilde{T}_{\mu\nu} = -\frac{1}{2f(\phi)} T_{\mu\nu} ,
\]
with stress–energy tensor defined as
\[ T_{\mu \nu} = T_{\mu \nu}^{(\phi)} + T_{\mu \nu}^{(m)} \, ; \]  
and the scalar field part
\[ T_{\mu \nu}^{(\phi)} = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu \nu} \phi_{,\alpha} \phi_{,\alpha} + g_{\mu \nu} V(\phi) + 2 g_{\mu \nu} \Box f(\phi) - 2 f(\phi) \phi_{,\mu} \phi_{,\nu} \, , \]  
in which we have assembled also the terms coming from the coupling \( f(\phi) \) which were outside \( T_{\mu \nu} \) in (2). Here \( G = f(\phi) \). The standard Newton gravitational constant is replaced by the effective coupling
\[ G_{eff} = -\frac{1}{2f(\phi)} \, . \]  
Einstein gravity is restored when \( f(\phi) \) assumes the value \(-1/2\).

The Klein–Gordon equation is
\[ \Box \phi - R f'(\phi) + V'(\phi) = 0 \, , \]  
where the prime means the derivative with respect to \( \phi \). The derivation of such an equation from the contracted Bianchi identity for \( T_{\mu \nu} \) is discussed in [39]. As a general feature, the models described by (8) are singularity free [21]; then, there are no restrictions on the interval of time on which the scale factor \( a(t) \) and the scalar field \( \phi(t) \) are defined. As we shall see in this context, it is possible to introduce a sort of time dependent (effective) cosmological constant and this will be the goal for any extended gravity theory which we shall take into consideration.

For the sake of simplicity, we develop our considerations in a FRW–flat spacetime, but the results can be easily extended to any homogeneous cosmological model including also Kantowski–Sachs models [37], [38], [40]. To get our goal, we shortly sketch the scheme already presented in [37].

From (9), using a Friedman–Robertson–Walker (FRW) flat metric
\[ ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2) \, , \]  
where \( a = a(t) \) is the scale factor of the universe, we get the (cosmological) Einstein equations
\[ H^2 + \frac{j}{f} \dot{H} + \frac{\rho}{6f} + \frac{\rho_m}{6f} = 0 \, , \]  
\[ \dot{H} = - \left( H^2 + \frac{V}{6f} \right) - H \frac{j}{2f} \frac{\dot{\phi}^2}{6f} - \frac{1}{2} \frac{j}{f} \frac{\ddot{\phi}}{6f} + \frac{3p_m + \rho_m}{12f} \, . \]  
where
\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \, , \]
\( \rho_m \) is the ordinary matter density and the equation of state

\[ p_m = (\gamma - 1) \rho_m, \quad (18) \]

is assumed.

Eq. (13) can be rewritten as:

\[ \mathcal{P}(H) \equiv (H - \Lambda_{eff,1})(H - \Lambda_{eff,2}) = -\frac{\rho_m}{6f}. \quad (19) \]

\( \mathcal{P}(H) \) is a second degree polynomial in \( H \), and

\[ \Lambda_{eff,1,2} = -\frac{\dot{f}}{2f} \pm \sqrt{\left( \frac{\dot{f}}{2f} \right)^2 - \frac{\rho_\phi}{6f}}, \quad (20) \]

\( \Lambda_{eff,1,2} \) have to be real. Let us assume, for large \( t \),

\[ \frac{\dot{f}}{f} \rightarrow \Sigma_0, \quad \frac{\rho_\phi}{6f(\phi)} \rightarrow \Sigma_1; \quad (21) \]

where \( \Sigma_{0,1} \) are two constants depending on the parameters present in the coupling and the potential. From these two hypotheses, \( \Lambda_{eff,1,2} \) asymptotically go to constants. Vice versa, if for large \( t \), \( \Lambda_{eff,i} \rightarrow \Lambda_i \), where \( \Lambda_i \) are constants, \( f/f \) and \( \rho_\phi/6f \) become constants. Then hypotheses (21) are necessary and sufficient conditions since \( \Lambda_{eff,1,2} \) are asymptotically constants.

If, asymptotically, the sign of \( f(\phi) \) is constant (this is a natural assumption), we have two cases: \( f(t \gg 0) \leq 0 \) and \( f(t \gg 0) \geq 0 \). Then being also \( f/f \) asymptotically constant, each of the above cases has two subcases related to the sign of \( \dot{f} \).

The case \( f(t \gg 0) \leq 0 \) is physically relevant while the other one (repulsive gravity) tells us that recovering a de Sitter asymptotic behaviour for \( a(t) \) is not connected to the sign of gravity.

Let us now consider the case \( f(t \gg 0) \leq 0 \) and \( \dot{f}(t \gg 0) \leq 0 \): from (21) we have \( \Sigma_0 \geq 0 \). Eq. (19) gives

\[ \mathcal{P}(H) \geq 0, \quad (22) \]

then we have \( H \geq \Lambda_1, \, H \leq \Lambda_2 \). For the two \( \Lambda_i \), we obtain the asymptotic expressions:

\[ \Lambda_{1,2} = -\frac{\Sigma_0}{2} \pm \sqrt{\left( \frac{\Sigma_0}{2} \right)^2 + |\Sigma_1|}, \quad (23) \]

Considering Eq. (16), if

\[ H^2 \geq \frac{V}{6|f|}, \quad (24) \]

we obtain

\[ \dot{H} \leq 0. \quad (25) \]
In other words, from the two disequalities on $\mathcal{P}(H)$ and on $\dot{H}$ we find that $H(t)$ has a horizontal asymptote, or, equivalently, $H$ goes to a constant (see [37]). Then the universe, for large $t$, has a de Sitter behaviour; (i.e. $a(t) \sim \exp(\alpha t)$, where $\alpha$ is a constant). Due to the conditions (24), the constant asymptotic sign of $f(\phi(t))$ and the condition (24), the universe, for large $t$, expands as de Sitter, even if it is not fixed the parameter which specifies such an expansion, i.e. the effective cosmological constant. If we compare the conditions in [19] with ours, we have:

(Wald’s conditions) (our asymptotic conditions)

\begin{align*}
  i) & \quad \left(H - \sqrt{\frac{\Lambda}{3}}\right) \left(H + \sqrt{\frac{\Lambda}{3}}\right) \geq 0 & \iff & \quad A) & (H - \Lambda_1)(H - \Lambda_2) \geq 0, \\
  ii) & \quad \dot{H} \leq \frac{\Lambda}{3} - H^2 \leq 0 & \iff & \quad B) & \dot{H} \leq 0.
\end{align*}

The hypothesis (24), when $\phi \to$ constant is nothing else but $H^2 \geq \frac{\Lambda}{3}$ (in our unit $G_{eff} \to G_N$ if $f(\phi) \to -\frac{1}{2}$); that is we recover the standard case where $\frac{V}{6|f|} = const$ can be interpreted as the cosmological constant. By some algebra, it is easy to show that such a hypothesis is equivalent to

$$\left(\frac{1}{12f}\right) \dot{\phi}^2 \geq \left(\frac{f'}{f}\right)^2 = \left(\frac{G_{eff}'}{G_{eff}}\right)^2,$$

where $G'_{eff} = dG_{eff}/d\phi$. That is we have a constraint on the minimal value of the (effective) ratio of the kinetic energy and the potential energy of the scalar field given by $G_{eff}$ and its derivative $G_{eff}'$.

Since $a(t)$ behaves like de Sitter for large $t$, we have to see if it is possible to fix $\alpha$ in order to recover the "true" cosmological constant. To this purpose, the Bianchi contracted identity for matter is needed (we have not used any Bianchi identity to find the asymptotic behaviour of $a(t)$). We get $\rho_m = Da^{-3\gamma}$ (by the state equation $p_m = (\gamma-1)\rho_m$ with $1 \leq \gamma \leq 2$; $D$ is an integration constant). Introducing this result in Eq.(13), for large $t$, we have

$$\left(H - \Lambda_1\right)(H + |\Lambda_2|) = \frac{D}{|f_0|} e^{-(3\gamma\alpha + \Sigma_0)t},$$

being $3\gamma\alpha + \Sigma_0 \geq 0$. Then we get $(H - \Lambda_1)(H + |\Lambda_2|) \to 0$, i.e. $H \to \Lambda_1$. The (effective) matter content, $\rho_m/6f(\phi)$, tells us how much $H$ is "distant" from the true de Sitter behaviour given by the cosmological constant $\Lambda_1$. In other words, we do not use the Bianchi identity for finding the type of expansion, we only use it to select (asymptotically) the specific value of the "cosmological constant". In any case, we have to note that, for $\rho_m = 0$, $H = \Lambda_{eff,1}$ is a solution for any $t$. Actually the effective cosmological constant that we have obtained via such a procedure will depend on the parameters of the effective gravitational coupling $f(\phi)$ and the potential $V(\phi)$. 

8
In a certain sense, the approach followed in [19] is reversed: there, \( \Lambda \) (constant) is introduced \textit{a–priori} and this leads, under certain hypotheses, to a de Sitter expansion. Here, the de Sitter expansion is recovered under different hypotheses, and this (together with the contracted Bianchi identity for matter) selects the effective cosmological constant. Moreover, we have obtained such a result without assuming to recover the standard gravity (i.e. we do not need that \( G_{\text{eff}} \to G_N \)). If we now consider also the Klein–Gordon equation, from the conditions (21), we get, for large \( t \), that \( \dot{\phi}^2/f(\phi) \) goes to a constant. Being \( f(\phi(t \gg 0)) \leq 0 \), such a constant has to be negative: this request implies \( |\Sigma_1| \geq 2\Sigma_0^2 \) [37]. By this last condition and (21), we get also that the potential has to be (asymptotically) non–negative. In the case \( \Sigma_0 = 0 \), we get that only \( V/6f \) is different from zero, giving rise to the expression \( V/6f(t \gg 0) = -\Sigma_1^2 \) which identifies the cosmological (asymptotic) constant [37].

Let us now consider the case \( f(\phi(t \gg 0)) \leq 0 \), that is \( \dot{f}(\phi(t \gg 0)) \geq 0 \). Here \( \Sigma_0 \leq 0 \) while everything else is the same as above. In particular, the signs of the asymptotic values of \( \Lambda_{1,2} \) are the same. From the compatibility of all the hypotheses we made with the Klein–Gordon equation we get \( \dot{\phi}^2/f(\phi) \geq 0 \), being \( \Sigma_0 \leq 0 \). Then the compatibility between (21) and the Klein–Gordon equation implies, for large \( t \), that the scalar field has to go to a constant. In our units, \( f \to -1/2 \), and \( \Lambda \to \sqrt{V(t \gg 0)/3} \).

Finally, let us consider the case of asymptotically repulsive gravity, that is \( f(\phi(t \gg 0)) \geq 0 \). Also here we have two subcases, \( \dot{f}(\phi(t \gg 0)) \leq 0 \) and \( f(\phi(t \gg 0)) \geq 0 \). This unphysical situation tells us that the (asymptotic) de Sitter behaviour and the recovering of standard (attractive) gravity are not necessarily related. Of course, the condition on the reality of \( \Lambda \) has to be carefully considered. The most interesting subcase is \( \dot{f} \leq 0 \). Here, we have two (asymptotic) positive cosmological constants, that is \( \Lambda_{\text{eff},1,2} \to \Lambda_{1,2} \geq 0 \). Being \( -\rho_m/6f \leq 0 \), we have \( \Lambda_1 \leq H \leq \Lambda_2 \). Then, it is crucial to know the sign of \( \dot{H} \): if \( H \geq 0 \) the effective \( \Lambda \) is given by the \( \max (\Lambda_1, \Lambda_2) \); viceversa, if \( \dot{H} \leq 0 \), \( \Lambda \) is given by the minimum between them.

In conclusion, in scalar–tensor theories, it is possible to extend asymptotically the no–hair theorem if an effective cosmological constant is introduced and, asymptotically, it becomes the true cosmological constant. Starting from these results, we enlarge the discussion to fourth–order, fourth–order–scalar tensor, and higher than fourth–order theories by applying the same scheme.

5 Fourth-order gravity

The approach we are discussing works also if the gravitational Lagrangian is nonlinear in the Ricci scalar (and, in general, in the curvature invariants). In this case, dynamics, \( \text{i.e.} \) the Einstein equations, is of order higher than second (for this reason such theories are often called \textit{higher–order gravitational theories}). Physically, they are interesting since higher–order terms in curvature invariants appear when one performs a one–loop
renormalization of matter and gravitational fields in curved background (see for example [25],[11]).

In cosmology, such theories can furnish inflationary behaviours (see e.g. [4],[12],[43],[44]) but the usual inflaton $\phi$ has to be replaced by its geometric counterpart, the Ricci scalar $R$, called *scalaron*.

As we have discussed in Sect. 2, higher–order theories can be reduced to minimally coupled scalar–tensor ones, and *vice–versa*, by a conformal transformation [11] so that it is reasonable that the approach we are dealing with can work in such a context. Here, we take into account the simplest case, a function $f(R)$.

Let us start from the action

$$A = \int d^4x \sqrt{-g} [f(R) + \mathcal{L}_m] ,$$

where, as usual, $R$ is the Ricci scalar. It is recovered from the extended action (1) with the choice

$$F = f(R), \quad \epsilon = 0.$$  

By varying Eq.(28), we obtain the field equations

$$f'(R)R_{\alpha\beta} - \frac{1}{2}f(R)g_{\alpha\beta} = f'(R)^{\mu\nu} (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) + T^{(m)}_{\mu\nu} ,$$

which are fourth order equations, due to $f'(R)^{\mu\nu}$. The prime indicates now the derivative with respect to $R$ (standard Einstein vacuum equations are immediately recovered if $f(R) = R$). Eq.(31) can be written in the above Einstein form $G_{\mu\nu} = \tilde{T}_{\mu\nu}$ by defining

$$\tilde{T}_{\mu\nu} = \frac{1}{f'(R)} \left\{ \frac{1}{2}g_{\mu\nu} \left[ f(R) - Rf'(R) \right] + f'(R)_{;\mu\nu} - g_{\mu\nu} \Box f'(R) + T^{(m)}_{\mu\nu} \right\} .$$

The standard (minimally coupled) matter has the same role discussed above, *i.e.* it gives no contribution to dynamics when we consider the asymptotic behaviour of system and, eventually, tells us how much $H$ is "distant" from the exact de Sitter behaviour. For the sake of simplicity, we discard its contribution (*i.e.* $\mathcal{L}_m = 0$) from now on, taking in mind, however, the previous discussion.

As before, we adopt a FRW metric considering that the results can be extended to any Bianchi model. What we want show is that there exists a formal analogy (without performing conformal transformations) between a scalar–tensor theory and a fourth–order theory which allows us to use the same above conditions in order to recover the de Sitter behaviour.

In a FRW metric, the action (28) can be written as

$$A = \int \mathcal{L}(a, \dot{a}, R, \dot{R}) dt ,$$

considering $a$ and $R$ as canonical variables. Such a position seems arbitrary, since $R$ is not independent of $a$ and $\dot{a}$, but it is generally used in canonical quantization of higher
order gravitational theories [4], [10], [14]. In practice, the definition of \( R \) by \( \ddot{a}, \dot{a} \) and \( a \) introduces a constraint which eliminates the second and higher order derivatives in (32), then this last one produces a system of second order differential equations in \( \{a, R\} \). In fact, using a Lagrange multiplier \( \lambda \), we have that the action can be written as

\[
\mathcal{A} = 2\pi^2 \int dt \left\{ f(R)a^3 - \lambda \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\} .
\]  

(33)

In order to determine \( \lambda \), we have to vary the action with respect to \( R \), that is

\[
\dot{a}^3 \frac{df(R)}{dR} \delta R - \lambda \delta R = 0 ,
\]  

(34)

from which

\[
\lambda = a^3 f'(R) .
\]  

(35)

Substituting into (33) and integrating by parts, we obtain the Lagrangian [10]

\[
\mathcal{L} = a^3 \left[ f(R) - R f'(R) \right] + 6\dot{a}^2 a f'(R) + 6a^2 \ddot{a} R f''(R) - 6akf'(R) .
\]  

(36)

Then the equations of motion are

\[
\left( \frac{\ddot{a}}{a} \right) f'(R) + 2 \left( \frac{\ddot{a}}{a} \right) f''(R) \dot{R} + f''(R) \ddot{R} + f'''(R) \dot{R}^2 - \frac{1}{2} [R f'(R) + f(R)] = 0 ,
\]  

(37)

and

\[
R = -6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) .
\]  

(38)

The \((0,0)\)-Einstein equation, implying the energy condition \( E_L = 0 \), is

\[
6\dot{a}^2 a f'(R) - a^3 \left[ f(R) - R f'(R) \right] + 6a^2 \ddot{a} R f''(R) + 6akf'(R) = 0 .
\]  

(39)

Let us now define the auxiliary field

\[
p \equiv f'(R) ,
\]  

(40)

so that the Lagrangian (36) can be recast in the form

\[
\mathcal{L} = 6a\dot{a}^2 p + 6a^2 \dot{a} \dot{p} - 6akp - a^3 W(p) ,
\]  

(41)

where

\[
W(p) = h(p)p - r(p) ,
\]  

(42)

with

\[
r(p) = \int f'(R)dR = \int pdR = f(R) , \quad h(p) = R ,
\]  

(43)

such that \( h = (f')^{-1} \) is the inverse function of \( f' \).
Considering the FRW pointlike Lagrangian derived from the action (8), we have \[ L = 6a\dot{a}^2 f(\phi) + 6a\dot{a}^2 f'(\phi) - 6akf(\phi) + a^3 \left[ \frac{1}{2} \dot{\phi}^2 - V(\phi) \right] ; \] (44)
so that we get the formal analogy between a fourth–order pointlike Lagrangian and a nonminimally coupled pointlike Lagrangian in FRW spacetime. The only difference is that in fourth-order Lagrangian there is no kinetic term, as \( \frac{1}{2} \dot{\phi}^2 \), for the field \( \phi \). In this sense, the above considerations, which hold for nonminimally coupled theories, work also in fourth–order gravity. A Lagrangian like (41) is a special kind of the so called Helmholtz Lagrangian [36].

Dynamical system (37)–(39) becomes
\[ 6 \left[ \left( \frac{\ddot{a}}{a} \right) + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] = - \frac{dW(p)}{dp} , \] (45)
\[ \ddot{p} + 2 \left( \frac{\dot{a}}{a} \right) \dot{p} + \left[ \left( \frac{\dot{a}}{a} \right)^2 + 2 \left( \frac{\ddot{a}}{a} \right) + \frac{k}{a^2} \right] p = - \frac{1}{2} W(p) , \] (46)
\[ 6 \left( \frac{\dot{a}}{a} \right)^2 p + 6 \left( \frac{\dot{a}}{a} \right) \dot{p} + \frac{6k}{a^2} p = - W(p) . \] (47)
We want, also in this case, to obtain an effective cosmological constant. For semplicity, let us assume \( k = 0 \). Eq.(47) becomes
\[ H^2 + \left( \frac{\dot{p}}{p} \right) H + \frac{W(p)}{6p} = 0 , \] (48)
which can be recast, as above,
\[ (H - \Lambda_{eff,1})(H - \Lambda_{eff,2}) = 0 . \] (49)
Note that now \( \rho_m = 0 \), but we can easily consider theories with \( \rho_m \neq 0 \). The results are the same of previous section. The effective cosmological constant can be formally defined as
\[ \Lambda_{eff,1,2} = - \frac{\dot{p}}{2p} \pm \sqrt{ \left( \frac{\dot{p}}{2p} \right)^2 - \frac{W(p)}{6p} } . \] (50)
We have to note that Eq.(49) defines the exact solutions \( H(t) = \Lambda_{eff,1,2} \) which, respectively, separate the region with expanding universes (\( H > 0 \)) from the region with contracting universes (\( H < 0 \)). See the discussion in previous section with \( \rho_m \neq 0 \).

In order to restore the asymptotic de Sitter behaviour, we rewrite Eq.(48), by using (47), as
\[ \dot{H} = - \frac{1}{2} \left( H^2 + \frac{W(p)}{6p} \right) - \frac{1}{2} \left( \frac{\dot{p}}{p} \right)^2 - \frac{1}{2} \frac{d}{dt} \left( \frac{\dot{p}}{p} \right) . \] (51)
The effective $\Lambda_{\text{eff},1,2}$ becomes an asymptotic constant for $t \to \infty$, if the conditions

$$\frac{\dot{p}}{p} \to \Sigma_0, \quad \frac{W(p)}{6p} \to \Sigma_1,$$

(52)

hold. From (51), we get $\dot{H} \leq 0$ if

$$H^2 \geq -\frac{W(p)}{6p}.$$

(53)

$\Lambda$, obviously, is real if

$$\left(\frac{\dot{p}}{2p}\right)^2 \geq \frac{W(p)}{6p}.$$

(54)

Conditions (52) gives the asymptotic behaviour of field $p$ and potential $W(p)$. By a little algebra, we obtain that asymptotically must be

$$\Sigma_0 = 0, \quad f(R) = f_0(R + 6\Sigma_1);$$

(55)

where $f_0$ is an arbitrary constant. The asymptotic solution is then

$$H^2 = \Sigma_1, \quad p = p_0, \quad \dot{H} = 0.$$

(56)

From Eq.(47), or, equivalently, from the constraint (38), we get

$$R = -12H^2 = -12\Sigma_1.$$

(57)

Also here the no–hair theorem is restored without using Bianchi identities (i.e. the Klein–Gordon equation). The de Sitter solution of Einstein gravity is exactly recovered if

$$\Sigma_1 = \frac{\Lambda}{3}.$$

(58)

It depends on the free constant $f_0$ in (55) which is assigned by introducing ordinary matter in the theory. This means that, asymptotically,

$$f(R) = f_0(R + 2\Lambda).$$

(59)

The situation is not completely analogue to the scalar–tensor case since the request that asymptotically $a(t) \to \exp(\Lambda t)$, univocally ”fixes” the asymptotic form of $f(R)$. Inversely, any fourth order theory which asymptotically has de Sitter solutions, has to assume the form (59).

We have to stress the fact that it is the $a \text{ priori}$ freedom in choosing $f(R)$ which allows to recover an asymptotic cosmological constant (which is not present in the trivial case $f(R) = R$, unless it is put by hand) so that de Sitter solution is, in some sense, intrinsic in higher–order theories [6], [42].
Fourth–order–scalar–tensor gravity

Several effective actions of fundamental physics imply higher–order geometric terms non-minimally coupled with scalar fields [14], [27], [26], [46]. Such theories have cosmological realizations which, sometimes, allow to bypass the shortcomings of inflationary models as that connected with the "graceful exit" and bubble nucleation (see for example [34]). Then it is interesting to ask for the recovering of de Sitter asymptotic behaviour also for these theories.

With the choice
\[ F = F(R, \phi), \quad \text{any } \epsilon, \quad \mathcal{L}_m = 0, \] (60)
we obtain the action
\[ \mathcal{A} = \int d^4x \sqrt{-g} \left[ F(R, \phi) - \frac{\epsilon}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} \right], \] (61)
which was extensively studied in [11].

We have put \( \mathcal{L}_m = 0 \) for simplicity as above. Also here, the considerations of Sect.4 hold.

The Einstein equations are
\[ G_{\mu\nu} = \frac{1}{G} \left\{ T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} (F - G R) + [G_{;\mu\nu} - g_{\mu\nu} \Box G] \right\}, \] (62)
where
\[ G \equiv \frac{\partial F}{\partial R}. \] (63)
and \( T_{\mu\nu} \) is just the expression
\[ T_{\mu\nu} = \frac{\epsilon}{2} [\phi_{;\mu} \phi_{;\nu} - \frac{1}{2} \phi^{\alpha} \phi_{;\alpha}]. \] (64)

The (eventual) contribution of a potential \( V(\phi) \) is contained in the definition of \( F \). By varying with respect to the scalar field \( \phi \) we obtain the Klein–Gordon equation of the form (6).

A pointlike FRW Lagrangian can be recovered by the technique already used. In fact, using the Lagrange multiplier \( \lambda \), we have
\[ \mathcal{A} = 2\pi^2 \int dt \left\{ F(R, \phi)a^3 - \frac{\epsilon}{2} a^3 \dot{\phi}^2 - \lambda \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\}. \] (65)

In order to determine \( \lambda \), we have to vary the action with respect to \( R \), that is
\[ a^3 \frac{\partial F(R, \phi)}{\partial R} \delta R - \lambda \delta R = 0, \] (66)
from which
\[ \lambda = a^3 \frac{\partial F(R, \phi)}{\partial R}. \] (67)
Substituting into (65) and integrating by parts, we obtain

\[ \mathcal{L} = a^3 \left[ F(R, \phi) - R \frac{\partial F(R, \phi)}{\partial R} \right] + 6\dot{a}^2 a \frac{\partial F(R, \phi)}{\partial R} + 6a^2 a \ddot{R} \frac{\partial^2 F(R, \phi)}{\partial R^2} + \\
+ 6a^2 \dot{a} \frac{\partial^2 F(R, \phi)}{\partial \phi^2} - 6ak \frac{\partial F(R, \phi)}{\partial R} - \frac{\epsilon}{2} a^3 \dot{\phi}^2. \]  

(68)

To get a formal analogy with previous results, we define

\[ p \equiv \frac{\partial F(R, \phi)}{\partial R}, \]  

(69)

and

\[ \dot{p} = \frac{d}{dt} \frac{\partial F}{\partial R} = \frac{\partial^2 F}{\partial R^2} \dot{R} + \frac{\partial^2 F}{\partial R \partial \phi} \dot{\phi}, \]  

(70)

so that we have again a Helmholtz point–like Lagrangian.

\[ \mathcal{L} = 6a\dot{a}^2 p + 6a^2 \dot{a} \dot{p} - 6akp - \frac{\epsilon}{2} a^3 \dot{\phi}^2 - a^3 W(p, \phi), \]  

(71)

where the potential \( W(p, \phi) \) corresponds to \( \left[ R \frac{\partial F(R, \phi)}{\partial R} - F(R, \phi) \right] \). Even if (71) describes the dynamics of geometry and two scalar fields \((p, \phi)\) it is formally similar to (41) and (44) so that above considerations work also here. Assuming \( k = 0 \), the cosmological equations of motion are

\[ H^2 + \left( \frac{\dot{p}}{p} \right) H - \frac{\rho}{6p} = 0, \]  

(72)

\[ \left[ 2\dot{H} + 3H^2 \right] p + \ddot{p} + 2H \dot{p} = -\frac{1}{2} W(p, \phi) - \frac{1}{4} \epsilon \dot{\phi}^2, \]  

(73)

The Klein–Gordon equations (one for each scalar field) are

\[ \frac{\partial W(p, \phi)}{\partial p} = -6 \left( \dot{H} + 2H^2 \right), \]  

(74)

and

\[ \epsilon [\ddot{\phi} + 3H \dot{\phi}] = \frac{\partial W(p, \phi)}{\partial \phi}. \]  

(75)

The “energy–density” in (72) depends on two fields and it is

\[ \rho = \frac{\epsilon}{2} \dot{\phi}^2 - W(p, \phi). \]  

(76)

As usual, we recast Eq.(72) as

\[ (H - \Lambda_{\text{eff},1})(H - \Lambda_{\text{eff},2}) = 0, \]  

(77)
and then

$$\Lambda_{\text{eff},1,2} = -\frac{\dot{p}}{2p} \pm \sqrt{\left(\frac{\dot{p}}{2p}\right)^2 + \frac{\rho}{6p}}. \quad (78)$$

Eq. (73) can be rewritten as

$$\dot{H} = -\frac{1}{2} \left( H^2 + \frac{W(p, \phi)}{6p} \right) - \frac{1}{2} \left( \frac{\dot{p}}{p} \right)^2 - \frac{1}{2} \frac{d}{dt} \left( \frac{\dot{p}}{p} \right) - \frac{5}{24} \frac{\dot{\phi}^2}{p}. \quad (79)$$

The effective \( \Lambda_{\text{eff},1,2} \) become asymptotically constants for \( t \to \infty \), if the conditions

$$\frac{\dot{p}}{p} \rightarrow \Sigma_0, \quad \frac{\rho}{6p} \rightarrow \Sigma_1, \quad (80)$$

hold. From (79), we get \( \dot{H} \leq 0 \) when

$$H^2 \geq -\frac{W(p, \phi)}{6p}, \quad \frac{\epsilon}{p} \geq 0. \quad (81)$$

The quantities \( \Lambda_{\text{eff},1,2} \) converge to real constants if

$$\left( \frac{\dot{p}}{2p} \right)^2 \geq -\frac{\rho}{6p}. \quad (82)$$

In conclusion, the situation is very similar to the fourth–order and scalar–tensor cases. However, we have to stress that the quantities \( W(p, \phi) \) and \( \Lambda_{\text{eff},1,2} \) are functions of two fields and this fact increase the number of conditions needed to get the asymptotic de Sitter behaviour (e.g. Eq.(81)).

7 Higher than fourth–order gravity

A pure higher than fourth–order gravity theory is recovered, for example, with the choice

$$F = f(R, \Box R), \quad \epsilon = 0, \quad L_m = 0, \quad (83)$$

which is, in general, an eighth–order theory. If \( F \) depends only linearly on \( \Box R \), we have a sixth–order theory. With this consideration in mind, we shall take into account the action (47) which becomes

$$A = \int d^4x \sqrt{-g} f(R, \Box R). \quad (84)$$

The Einstein field equations are now

$$G_{\mu\nu} = \frac{1}{G} \left\{ \frac{1}{2} g_{\mu\nu} [f - GR] + G_{\mu\nu} - g_{\mu\nu} \Box G - \frac{1}{2} g_{\mu\nu} [\mathcal{F}_{\gamma R} R^\gamma + \mathcal{F} \Box R] + \right. \right.$$

$$\left. + \frac{1}{2} [\mathcal{F}^{\mu R}_{\nu} + \mathcal{F}^{\nu R}_{\mu}] \right\}, \quad (85)$$
where
\[ G = \frac{\partial f}{\partial R} + \Box \frac{\partial f}{\partial \Box R}, \quad \mathcal{F} = \frac{\partial f}{\partial \Box R}. \] (86)

As above, we can get a FRW pointlike Lagrangian with the position
\[ \mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R}, \Box R, (\Box R)) . \] (87)

Also here, we consider \( R \) and \( \Box R \) as two independent fields and use the method of Lagrange multipliers to eliminate higher derivatives than one in time. The action is
\[ \mathcal{A} = 2\pi^2 \int dt \left\{ f(R, \Box R)a^3 - \lambda_1 \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] - \lambda_2 \left[ \Box R - \ddot{R} - 3H\dot{R} \right] \right\} . \] (88)

In order to determine \( \lambda_{1,2} \), we have to vary the action with respect to \( R \), and \( \Box R \) so that
\[ \lambda_1 = a^3 \left[ \frac{\partial f}{\partial R} + \Box \frac{\partial f}{\partial \Box R} \right], \] (89)
\[ \lambda_2 = a^3 \frac{\partial f}{\partial \Box R}. \] (90)

Substituting into (88) and integrating by parts, we obtain the Helmholtz–like Lagrangian
\[ \mathcal{L} = a^3 \left[ f - RG + 6H^2 \mathcal{G} + 6H \dot{\mathcal{G}} \right. \]
\[ \left. \left.- \frac{6k}{a^2} \mathcal{G} - \Box R \mathcal{F} - \ddot{R} \dot{\mathcal{F}} \right] . \] (91)

The equations of motion, for \( k = 0 \), are
\[ \dot{H} + \frac{3}{2} H^2 + H \left( \frac{\dot{G}}{\mathcal{G}} \right) + \frac{1}{2} \left( \frac{\ddot{G}}{\mathcal{G}} \right) + \frac{\chi}{4G} + \frac{\dot{R} \dot{\mathcal{F}}}{2G} = 0 ; \] (92)
\[ R = -6[\dot{H} + 2H^2], \] (93)
\[ \Box R = \ddot{R} + 3H \dot{R}, \] (94)

where (93) and (94) have the role of Klein–Gordon equations for the fields \( R \) and \( \Box R \) and are also ”constraints” for such fields. The (0,0)–Einstein equation is
\[ H^2 + H \left( \frac{\dot{G}}{\mathcal{G}} \right) + \frac{\chi}{6G} = 0 ; \] (95)

while the quantity \( \chi \) is defined as
\[ \chi = RG + \mathcal{F} \Box R - f - \dot{R} \dot{\mathcal{F}}. \] (96)

It is interesting to note that \( \chi \) has a role similar to that of the energy density in previous theories.
As usual, we can define an effective cosmological constant as

$$\Lambda_{\text{eff},1,2} = -\frac{\dot{G}}{2G} \pm \sqrt{\left(\frac{\dot{G}}{2G}\right)^2 - \frac{\chi}{6G}}.$$  

(97)

Now, the role of the coupling $f(\phi)$ is played by the function $G = G(R, \Box R)$. By substituting Eq. (95) into Eq. (92), we get

$$\dot{H} = -\frac{1}{2} \left( H^2 - \frac{\chi}{6G} \right) - \frac{1}{2} \left( \frac{\dot{G}}{G} \right)^2 - \frac{1}{2} \frac{d}{dt} \left( \frac{\dot{G}}{G} \right) - \frac{\dot{R} \ddot{F}}{2G}.$$  

(98)

The quantities $\Lambda_{\text{eff},1,2}$ becomes asymptotically constants if

$$\frac{\dot{G}}{G} \to \Sigma_0, \quad \frac{\chi}{6G} \to \Sigma_1.$$  

(99)

From (98), we have $\dot{H} \leq 0$ if

$$H^2 \geq \frac{\chi}{6G}, \quad \frac{\dot{R} \ddot{F}}{G} \geq 0.$$  

(100)

The cosmological constant is real if

$$\left(\frac{\dot{G}}{2G}\right)^2 \geq -\frac{\chi}{6G}.$$  

(101)

This case is analogous to the previous fourth–order–scalar tensor: There the fields involved where $p, \phi$ (or $R, \phi$), now they are $R, \Box R$. In fact, the quantities $\chi, G, \Lambda_{\text{eff}}$ are functions of two fields and the de Sitter asymptotic regime select particular surfaces \{\text{R, } \Box \text{R}\}.

8 Examples

The above discussion can be realized on specific cosmological models. Now, as in [37], we want to give examples where, by fixing the scalar–tensor or the higher–order theory, the asymptotic de Sitter regime is restored in the framework of our generalization of no–hair theorem. The presence of standard fluid matter can be implemented by adding the term $L_m = Da^{3(1-\gamma)}$ into the FRW–pointlike Lagrangian [21]. It is a sort of pressure term. We can restrict to the case $\gamma = 1$, (dust) that is $L_m = D$, since we are considering asymptotic regimes, but, in any case the presence of standard fluid matter is not particularly relevant.

1. Let us consider a generic coupling $f(\phi)$ and the potential $V(\phi) = \Lambda$. Using the Noether Symmetry Approach [10, 33], we get $f(\phi) = \frac{1}{12} \phi^2 + F'_0 \phi + F_0$, where $F'_0$ and $F_0$ are two generic parameters. We have already discussed such a case in [37] where we show that an asymptotic de Sitter regime is restored as soon as $G_{\text{eff}} \to G_N$. 

18
2. In the case $f(\phi) = k_0 \phi^2$, $V(\phi) = \lambda \phi^2$, $\gamma = 1$, where $k_0 < 0$ and $\lambda > 0$ are free parameters, the de Sitter regime is recovered even if solutions do not converge toward standard gravity. The coupling $f(\phi)$ is always negative, whereas $V(\phi)$ is always positive and $\dot{f}(\phi(t \gg 0)) < 0$.

3. Both the above cases can be translated in the fourth–order formalism and the same results are found if we take into consideration a theory as $f(R) = R + \alpha R^2$ (see [12] for the discussion of the case and [36] for the physical equivalence).

4. The conditions for the existence and stability of de Sitter solutions for fourth–order theories $f(R)$ are widely discussed in [9]. In particular, it is shown that, for $R$ covariantly constant (i.e. $R = R_0$), as recovered in our case for $R \to -12 \Sigma_1$ (see Eq.(57), the field equations (30) yield the existence condition

$$R_0 f'(R_0) = 2f(R_0).$$

Thus, given any $f(R)$ theory, if there exists a solution $R_0$ of (102) then the theory contains a de Sitter solution. From our point of view, any time that the ratio $\dot{f}(R(t))/f(R(t))$ converges to a constant, a de Sitter (asymptotic) solution exists.

On the other hand, given, for example, a theory of the form

$$f(R) = \sum_{n=0}^{N} a_n R^n,$$

the condition (102) is satisfied if the polynomial equation

$$\sum_{n=0}^{N} (2 - n) a_n R_0^n = 0,$$

has real solutions. Examples of de Sitter asymptotic behaviours recovered in this kind of theories are given in [43].

5. Examples of theories higher than fourth–order in which asymptotic de Sitter solutions are recovered are discussed in [8], [33]. There is discussed under which circumstances the de Sitter space–time is an attractor solution in the set of spatially flat FRW models. Several results are found: for example, a $R^2$ non–vanishing term is necessarily required (i.e. a fourth–order term cannot be escaped); the models are independent of dimensionality of the theory; more than one inflationary phase can be recovered.

Reversing the argument from our point of view, a wide class of cosmological models coming from higher–order theories, allows to recover an asymptotic cosmological constant which seems an intrinsic feature if Einstein–Hilbert gravitational action is modified by higher–order terms. In this sense, and with the conditions given above, the cosmological no–hair theorem is extended.
We conclude the discussion of these examples stressing, again, that it appears clear that the (asymptotic) cosmological constant, as introduced in our approach, depends on the parameters appearing into the functions $f(\phi), V(\phi), f(\phi, R)$, or $f(R, \Box R)$. Furthermore, it depends on the order of higher–order theory and on the possibility that the condition $\dot{H} \leq 0$ is restored.

9 Discussion and conclusions

We have discussed the cosmic no–hair theorem in the framework of extended theories of gravity by introducing a time dependent cosmological ”constant”. Such an effective cosmological ”constant” has been reconstructed by $\dot{G}_{\text{eff}}/G_{\text{eff}}$ and by $\rho_0/6f(\phi)$ but such quantities assume different roles in accordance with the theory used (higher–order or scalar–tensor). It is interesting to stress that $R, \Box R$ and $\phi$ can be all treated as “scalar fields” in the construction of $\Lambda_{\text{eff}}$, i.e. all of them give rise to extra–terms in the field equations which contribute to the construction of an effective stress–energy tensor $T^{(\text{eff})}_{\mu\nu}$. Actually $\Lambda_{\text{eff}}$ has been introduced only in the case of homogeneous–isotropic flat cosmologies but it is not difficult to extend the above considerations to Bianchi models (see [38],[37]). The way we have followed to reconstruct the no–hair theorem is opposite of that usually adopted: instead of introducing by hands a cosmological constant and then searching for the conditions to get an asymptotic de Sitter behaviour, we find the conditions to get such an asymptotic behaviour, and then we define an effective cosmological ”constant” (actually function of time), which becomes a (true) constant for $t \gg 0$. Of course, the time behaviour of $\Lambda_{\text{eff}}$ can be of any type with respect to the asymptotic constant value [47]. Under the hypotheses we used, the de Sitter asymptotic regime is obtained and this is not necessarily connected with the recovering of standard Einstein gravity (which is restored, in our units, for the value $f(\phi)_{\infty} = -1/2$ of the coupling). In other words, the cosmic no–hair theorem holds even if we are not in the Einstein regime (it is not even necessary that the right (attractive gravity) sign of the coupling is recovered). Furthermore, the role of the Bianchi contracted identity for the (standard) matter is to fix (only) the specific value of $\Lambda$, not the kind of the (de Sitter) asymptotic behaviour of $a(t)$. It is interesting to stress that, by this mechanism, the ”amount of $\Lambda$” is strictly related to the matter content of the universe. This is worthwhile in connection to the $\Omega$ problem since it seems that cold dark matter models, with non trivial amount of cosmological constant, have to be taken into serious consideration for large scale structure formation [48]. In conclusion, we want to make two final remarks. The first concerns an important question which we have only mentioned. The way we have followed to introduce the (effective) cosmological ”constant” seems to confine its meaning only to the cosmological arena. In the standard way used to define such a quantity, this problem does not exists since it is a true constant of the theory and then it is defined independently of any cosmological scenario. We believe that this question can be solved stressing that cosmology has to be taken into account in any other specific physical situation in relativity. Then the effective time–dependent cosmological constant we have
introduced gets a role of the same kind of the standard $\Lambda$. From this point of view, the question we are discussing can be answered still using the (standard) way to define the cosmological constant, i.e. (the cosmological) $T_{00}$. This is what we actually have done and what we believe to be the ingredient to use for understanding the role of (effective) cosmological "constant" also in different contexts than cosmology. Finally, in our construction of $\Lambda$, there is a contribution given by the (relative) time variation of the effective gravitational coupling: this implies that it would be possible to compute it, for example, via the density contrast parameter.

A final comment concerns the fact that all extended theories of gravity can be treated under the same standard of no–hair conjecture. In this sense, the determination of the effective dynamics of cosmological constant could be a test on which of them actually works.

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