An Uneven Vacuum Energy Fluid as $\Lambda$, Dark Matter, MOND and Lens

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Various TeVeS-inspired and f(R)-inspired theories of gravity have added an interesting twist to the search for dark matter and vacuum energy, modifying the landscape of astrophysics day by day. These theories can be together called a Non-uniform Dark Energy fluid (a Nu-Lambda fluid or a $V\Lambda$ fluid); a common thread of these theories, according of an up-to-date summary by HZL [1], is a non-uniform vector field, describing an uneven vacuum energy fluid. The so-called "alternative" gravity theories are in fact in the standard GR gravity framework except that the cosmological "constant" is replaced by a non-trivial non-uniform vacuum energy, which couples the effects of Dark Matter and Dark Energy together by a single field. Built initially bottom-up rather than top-down as most gravity theories, TeVeS-inspired theories are healthily rooted on empirical facts. Here I attempt a review of some sanity checks of these fast-developing theories from galaxy rotation curves, gravitational lensing and cosmic acceleration. I will also discuss some theoretical aspects of the vacuum energy, and point out some analogies with electromagnetism and the Casimir effect.

1. The three pillars of the standard $\Lambda$CDM cosmology

The standard cosmological paradigm is built on three pillars: Einsteinian gravity, a cosmological constant or vacuum energy density about $10^{-10}$ erg/cm$^3$ due to unknown physics, and a thermal relic of Cold Dark Matter due to physics at the TeV scale. While the independent experimental basis of each of the three is debatable on astronomical scales, but their synergy (characterised by the cosmological pie diagram) has proven amazingly successful at describing the Universe especially on large scale.

Despite its apparently enticing simplicity, the paradigm leaves much to be understood and is challenged by observations on galaxy scale. For example, the experimentally undetected dark matter is generally thought to be Minimal SuperSymmetry Model (MSSM) particles, and is predicted to be cold and clump in scale-free fashion, while observations of dwarf galaxies suggest the particles are warm with
HongSheng Zhao

a kpc-scale, below which DM is smoothed out by free-streaming of the thermal motion.

Most embarrassingly is that there is no physics for the cosmological constant; The MSSM physics at TeV scale fails to explain the tiny vacuum energy of the universe by 120 orders of magnitude. This is regarded by many theoreticians as evidence for new physics at low energy scales.

1.1. A characteristic scale for both Dark Matter and Dark Energy

As an important puzzle about dark matter, it has long been noted that on galaxy scales dark matter and baryonic matter (stars plus gas) have a remarkable correlation, and respect a mysterious acceleration scale $a_0 \sim 1$ Angstrom per second squared.\(^2\)\(^3\)\(^4\)\(^5\)

The Newtonian gravity of the known matter (baryons, neutrinos, eletrons, etc.) $g_K$ and the dark matter gravity $g_{DM}$ are correlated through an empirical relation\(^6\) such that the light-to-dark ratio, experimentally determined to fit rotation curves, satisfies a very simple relation

$$g_{DM} \approx \sqrt{g_K a_0}, \quad a_0 \equiv 1\text{Angstrom sec}^{-2}$$

(1)

where $a_0$ is the fore-mentioned gravity scale, below which DM and DE phenomena start to surface. This DM-to-baryon relation fits rotation curves of faint and bright spiral galaxies fairly well (cf. Fig. 1).

Such a tight correlation is difficult to understand in a galaxy formation theory where dark matter and baryons interactions enjoy huge degrees of freedom. This spiral galaxy based empirical relation is also consistent with some elliptical galaxies and gravitational lenses.

It is also hard to explain from fundamental physics why vacuum energy starts to dominate the Universe density only at the present epoch, hence marking the present as the turning point for the universe from de-acceleration to acceleration.

The puzzles of DM and DE are related by the fact that

$$a_0 \sim \sqrt{\Lambda} \sim cH_0.$$  \hspace{1cm} (2)

Somehow dark energy and dark matter are tuned to shift dominance when the energy density falls below $\frac{a^2}{8\pi G} \sim 10^{-10}$ erg/cm\(^3\). These empirical facts should not be completely treated as random coincidences of the fundamental parameters of the universe. The explanation with standard paradigm has been unsatisfactory.

The problems of $\Lambda$CDM have led some to believe the paradigm is an effective theory, e.g., a 4D projection of a more fundamental 5D brane world theory. Some also question the Einsteinian gravity since its associated equivalence principles, remain untested on galaxy scale and cosmological scale. A less drastic approach is to keep the framework of the Einsteinian gravity, but design the Lagrangian for the dark energy field to have the effect of dark matter as well. An example of the latter approach is the Vector-for-$\Lambda$ model or the V$\Lambda$ model of Zhao\(^7\), where a
photon-like but massive vector field is speculated to exist even in vacuum. A careful choice of the dark energy field can replace the role of dark matter too, i.e., the DM and DE parts of the cosmic pie diagram are in fact two aspects of a single species of dark fluid.

2. Energy Density of the Uneven Vacuum

A common way to probe dark matter in galaxies is gravitational lensing. The amount of light bending is an indicator of the non-flatness of the space-time metric, hence constraining the matter distribution. However, light bending is a general property of propagation of E&M waves following Fermat’s principle, or the geodesics. The amount of bending can be an indicator of the non-uniformness of the propagated medium, e.g., in the case of atmospherical seeing. Light could be bent even in the vacuum because the vacuum is not empty, and can be a fluid of certain energy density.

The energy density in the vacuum can vary with space and time as well. It is interesting that the Casimir effect predicts in principle a pressure \( \frac{\hbar c^2}{2\pi^2} \sim 10^{-10} \) erg/cm\(^3\) for two neutral metal plates separated by a distance \( \Delta \sim 0.01 \) cm. This pressure can drive the plates closer and closer, because the zero-point of the vacuum energy density due to electromagnetic waves between the plates is lower than outside the plates; as the plates close in the pressure goes up as \( \Delta^{-4} \). The Casimir effect is indeed observed experimentally when the plates are separated by \( \Delta = 100 \) nanometer or closer. The vacuum energy density could fluctuate spatially, too.\(^{4}\)

Likewise, for very different physics, the zero-point of the vacuum could fluctuate spatially or evolve time-wise due to gravitational physics. E.g., the universal vacuum energy density during inflation is much higher than the vacuum energy density today. Any spatial variation of the vacuum energy density would generate more curvature in some patches of space-time, creating a dark-matter-like effect. The vacuum in this case appears as a dark fluid with fluctuations. The effects of fluctuation might manifest as a temporal or spatial change of the gravitational coupling factor\(^{5}\) \( G_{\text{eff}} = G/\mu(t,r) \), where \( G \) is the usual gravitational constant determined in earth-based labs, and \( \mu \) is some kind of dielectric-like parameter, which can determined in a Gedanken experiment by \( \frac{\ddot{r}}{\mu} \equiv \frac{|\ddot{r}_1 - \ddot{r}_2|}{(m_1 + m_2)|r_1 - r_2|} \), where one measures the relative acceleration \( |\ddot{r}_1 - \ddot{r}_2| \) of two neutral test particles of \( m_1 \) and \( m_2 \) slightly separated by a distance \( |r_1 - r_2| \) in a table-top Cavendish-type experiment near the space-time coordinate \((t,r)\) in the intergalactic space.

\(^{4}\)E.g., the vacuum energy due to electromagnetism would not be uniform if many Casimir plates were randomly distributed, or if these Casimir plates were replaced by a distribution of polarisable neutral atoms in the universe. An analogous situation (although with a different physics from the Casimir effect) happens in solid-state physics, where the effective dielectric "constant" \( \epsilon \) can be spatially varying. As an effect, e.g., the normally \( r^{-2} \) repulsive force between two electrons becomes a complicated function of their separation if they are inside a lattice of polarisable neutral atoms, and can even change the sign in special cases.\(^{5}\)
The gravity at a typical place in a galaxy is very weak, is about a factor $4 \times 10^4$ smaller than the solar gravity on Pluto. E.g., the Sun’s acceleration around the Galaxy

$$g \sim \frac{(200 \text{ km/s})^2}{10 \text{kpc}} \sim \frac{\text{LightSpeed}}{10 \times \text{HubbleTime}} \sim \frac{1 \text{m}}{\text{day}^2} \sim \frac{1 \text{Angstrom}}{\text{sec}^2}. \quad (3)$$

The gravitational energy density associated with 1 Angstrom per second squared gravitational field is about $10^{-10} \text{ erg/cm}^3$. This is roughly the scale of the cosmological constant, yet $10^{-12}$ smaller than the current experimental sensitivity in the Casimir pressure. New physics on such weak scale is allowed as far as experiments are concerned.

A mundane example of 1 Angstrom per second squared gravity is the mutual Newtonian gravity of two nearly parallel sheets of printing papers approximately. The gravitational attraction of two sheets of paper could depend on environment. Consider a Gedanken experiment with a gravitationally torquing pendulum made by two misaligned suspended sheets of paper. If one could measure the period of the torquing pendulum not only here on Earth (as in free-fall experiments in an Einstein tower), but also take the table-top experiments to the edge of the solar system (where Pioneer 10/11 probes are), in the interstellar space (where galactic stars orbit) and in the expanding void between galaxies, then one could measure how $G_{\text{eff}}$ changes with space and time.

3. TeVeS-like modified gravity: motivations and challenges

Modifying gravity is a recurring exercise which started ever since the general acceptance of Einsteinian gravity, which was itself a revolutionary modification to Newtonian gravity. Many theories modify the Einstein-Hilbert action to introduce a new scalar field which manifests itself only through the extra bending of space-time, but its coupling to the metric is different from the simple coupling of massive particles with the space-time metric.

By construction, the theories would respect Special Relativity prescription of metric co-variance, and preserve conservations of momentum and energy. They do allow for a table-top Cavendish-type experiment with a torquing pendulum to measure an effective gravitational constant $G_{\text{eff}}(t, x)$ which varies with time and environment of the experiment. For example, the recent $F(R)$ models are motivated to replace the cosmological constant with a vacuum energy density depending on the curvature of space-time, hence evolving with the cosmic time in a way to drive the acceleration of the universe at late time.

However, among two dozen theories proposed after GR, very few survive the precise tests on SEP in the solar system and the well-studied binary pulsars. Even fewer are motivated and succeeded in addressing both astronomical dark matter and cosmological constant.

Bekenstein’s TeVeS[10] is a first effort in the direction of solving outstanding problems. Its partial success has spurred several variations of the theory, including
Sanders’ Bi-Scalar-Tensor-Vector theory [11], Zlosnik et al.’s generalized Einstein-Aether theory [12], and Zhao’s Vector-for-Λ model [7]. These hold the promise of explaining both dark matter and cosmological constant by relaxing the SEP (strong equivalence principle) only in untested weak gravity environments like in galaxies, but respecting the SEP to high accuracy in the solar system.

Crudely speaking, such theories have an aether-like field with an quadratic kinetic term in its Lagrangian density, so the $G_{\text{eff}}$ can be made a function of the strength of gravitational energy $|g|^2 / 8\pi G$, such that $G_{\text{eff}}$ is constant within $10^{-16}$ anywhere in the solar system, yet varies by a factor of 10 in galaxies. Enhancing the $G_{\text{eff}}$ mimics the effects of adding dark matter. The effects resemble dielectric. E.g., in the $f(K_4)$ model of $VA$, the Poisson equation around a static galaxy of a baryonic density $\rho$ becomes:

$$\nabla \cdot (E - P) = \rho,$$

where $E = \sum_{\alpha} \Phi_{\alpha}$ is the rescaled gravity, reminiscence of an electric field and $P = \lambda(|E| / \Pi_0)E$ is a polarisation-like field with a susceptibility $\lambda$ being a function of the field strength $|E|$ and a characteristic column density constant $\Pi_0$ comparable to that of a sheet of paper.

4. DM and DE as two faces of the same coin: Uneven Dark Energy fluid

4.1. Vector or scalar, modified or not?

TeVeS-like theories, as GR, are single-metric theories. They can often be casted to the GR framework with a sophisticated Vacuum Energy term, all in physical metric [13]. To see this, let $g_{\mu\nu}$ being the physical metric, then near a quasi-static system like a galaxy, the physical space-time is only slightly curved, and can be written as in terms of $x_0 = ct$, and cartesian coordinates $(x_1, x_2, x_3)$ centred on the galaxy as

$$-c^2 dt^2 \approx -\exp \left( \frac{-2\Phi}{c^2} \right) dx_0^2 + \exp \left( \frac{2\Phi}{c^2} \right) dl^2,$$

where $|\Phi| / c^2 \ll 1$. To show that $\Phi$ takes the meaning of a gravitational potential, we note that a non-relativistic massive particle moves along the geodesic equation (or Lagrangian equation)

$$\sum_{\beta=0}^{3} \frac{d^2 (g^{\alpha\beta} x_{\beta})}{d\tau^2} = \sum_{\beta,\gamma=0}^{3} \frac{\partial g^{\beta\gamma}}{2\partial x_{\alpha}} \frac{dx_{\beta}}{d\tau} \frac{dx_{\gamma}}{d\tau},$$

which approximates to the non-relativistic equation of motion,

$$\frac{d^2 x_i}{dt^2} \approx -\frac{\partial g^{00}}{2\partial x_i} c^2 \approx -\partial_i \Phi,$$

where $dx_0 = cdt \approx c d\tau$, and $g^{00} \approx -(1 - 2\Phi / c^2)$. 

The vector field is more fundamental than the scalar field in TeVeS-like theories. Any time-like vector field with four components can be approximated as

\[ A_\alpha \approx -e^{-\phi_{DE} + \Phi}(1, 0, 0, 0) \]  

(8)

\[ g^{\alpha\beta}A_\alpha \approx e^{-\phi_{DE} - \Phi}(1, 0, 0, 0), \]  

(9)

where we avoid the ambiguous notation of the upper index for the vector field, and we omit the \( c^2 \) factor for more contact notations. Here \( \phi_{DE} \) is a scalar field, describing the modulus of the vector field \( A^2 \equiv -g_{\alpha\beta}A_\alpha A_\beta \equiv e^{-2\phi_{DE}} \).

So the scalar field can be described by the physical metric \( g^{\alpha\beta} \) and vector field \( A_\beta \) alone. The original proposal of Bekenstein contains two metrics; the other metric (called Einstein metric \( \tilde{g} \), where notations of tildes are opposite of Bekenstein) is fully described by the relation

\[ \tilde{g}_{\mu\nu} - A_\mu A_\nu = e^{2\phi_{DE}} g_{\mu\nu} - A_\mu A_\nu(2 - e^{4\phi_{DE}}), \]  

(11)

The work of [13] shows that the TeVeS theory is equally described by a single physical metric \( g^{\mu\nu} \), whose geodesics particles and light will follow. All the effects of the vector field potential \( A \) can be lumped together as a sophisticated Dark Energy like term. E.g., the vector field contributes an E&M-like Lagrange density \( F_{\alpha\beta}F^{\alpha\beta} \), where from the covariant derivative of the vector potential \( A \), one can form the Maxwell tensor field \( F_{\alpha\beta} \)

\[ \tilde{g}_{\alpha\gamma}F_{\beta}^\gamma = F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha, \]  

(12)

similar to the electric and magnetic field in electromagnetism. This makes TeVeS in the similar framework as dark energy theories. From this perspective, one has not modified gravity. One simply have a sophisticated energy term to replace the cosmological constant in the GR framework.

4.2. Uneven Dark Energy fluid as Cosmological constant and as galaxy Dark Matter

While Bekenstein’s original TeVeS Lagrangian is able to yield reasonable fits to CMB [14], there is an intrinsic discontinuity in its original proposal. Zhao & Famaey [15] proposed to modify TeVeS Lagrangian to ensure a smooth transition between galaxies and cosmology.

In the ZF proposal, the total action is that of the matter action \( S_m \) plus Einstein-Hilbert action \( S_{EH} \) plus the "cosmological constant"-like action for the vector field \( A_\alpha \)

\[ S = S_m + S_{EH} + \int d^4x \sqrt{-g} \frac{\Lambda}{8\pi G} - \frac{\Lambda}{8\pi G} \equiv \left[ \int_0^L \frac{\mu_3 d\xi}{8\pi G_1} + \frac{1}{16\pi G_2} F_{\alpha\beta} F^{\alpha\beta} \right], \]  

(13)
where the cosmological "constant" \( \Lambda \) is replaced with uneven Dark Energy fields

\[
\mu_s \equiv \frac{f}{1 - \alpha f}, \quad f \equiv \frac{\sqrt{T}}{a_1}, \quad \mathcal{L} \equiv (\tilde{g}_{\mu\nu} - A_\mu A_\nu) \nabla^\mu \phi_{DE} \nabla^\nu \phi_{DE},
\]

(14)

where \( \phi_{DE} = -\frac{1}{2} \ln(-g^{\mu\nu} A_\mu A_\nu) = \frac{1}{2} \ln \sqrt{-\tilde{g}} \) is a scalar field which depends on the physical metric and the vector. Let the parameter \( \alpha = 0 \) and adjust the constant parameters \( a_1, G_1, G_2 \), the model is able to fit approximately (cf. Fig. 2) the late time acceleration from SNe without explicitly introducing a cosmological constant, and can explain the horizon scale angular size at recombination without explicitly introducing Dark Matter. This uneven DE fluid also satisfies the BBN constraints at \( z \sim 10^9 \), and the solar system constraints (see also [16] for details).

This Lagrangian can also fit galaxy rotation curves at the present epoch without dark matter. To see this, one can take variations of the action with respect to \( A_\alpha \) and the metric \( g_{\alpha\beta} \) respectively. We get the vector field equation of motion for this theory, and the Einstein equation for the dynamics of the metric tensor respectively. The latter equation has the form

\[
\frac{G_{\alpha\beta}}{8\pi G} = T^K_{\alpha\beta} + T^\alpha_{\alpha\beta},
\]

(15)

(16)

where the left-hand side is proportional to the Einstein tensor \( G_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \) and on rhs the 1st term is the stress-energy tensor of known matter, the 2nd term is the stress-energy tensor for the vector field \( T_{\alpha\beta} \), which is a non-linear function of derivatives of the field \( A_\beta \). Near a galaxy, \( G_{00} = 2 \nabla \nabla \Phi \). Note that the vector field stress tensor creates the mirage of additional matter.

4.3. TeVeS scalar field as effective dark matter

In TeVeS, the galaxy potential \( \Phi \) comes from two parts,

\[
\Phi = \Phi_K + \phi_{DE}
\]

(17)

where the known Newtonian gravitational potential \( \Phi_K(x) \) of known matter of density \( \rho_K(x) \) satisfies

\[
\nabla \cdot \nabla \Phi_K = 4\pi G \rho_K
\]

(18)

and the added scalar field satisfies

\[
\nabla [\mu_s \nabla \phi_{DE}] = 4\pi G \rho_K.
\]

(19)

Our Lagrangian free-function (eq. 14) corresponds to the \( \mu \)-function proposed by [18] that

\[
\mu_s = \frac{f}{1 - \alpha f}, \quad f = \left| \frac{\nabla \phi_{DE}}{a_0} \right|.
\]

(20)

This one-parameter \( \alpha \) family of functions recovers Bekenstein’s [10] toy model and the simple model of Zhao & Famaey [15] if setting \( \alpha = 0 \) and \( \alpha = 1 \) respectively.
Note if \( \alpha = 0 \)

\[
\mu_s = \left| \frac{\nabla \phi_{DE}}{a_0} \right|, \quad |\nabla \phi_{DE}| = \sqrt{g_K a_0}
\]  

(21)

This way we recover the observed DM effects in eq. (1) and the classical MOND effect, i.e., the gravity \( |\nabla \Phi| \) drops as \( \sqrt{GMa_0/r} \) far away from a point mass \( M \).

Indeed this \( \mu \) functions is able to fit rotation curves of faint and bright spiral galaxies approximately (cf. Fig. 1), although models with \( \alpha = 1 \) fit better. 17

The picture to keep in mind is that the scalar field replaces the usual role of the potential of the Dark Matter. The vector field \( A \) is fully specified once \( \phi_{DE} \) and \( \Phi \) are given.

4.4. Different interpolating functions: MOND vs TeVeS

The gravitational potential in the classical MOND theory satisfies a modified Poisson’s equation,

\[
\nabla [\mu \nabla \Phi] = 4\pi G \rho_K
\]  

(22)

where the \( \rho_K \) is the density of all known matter, where \( \mu \) is a function of total gravity. This is different from TeVeS, where the total potential is the sum of Newtonian potential \( \Phi_N \) and a potential due to a scalar field \( \phi_{DE} \). TeVeS \( \mu_s \) is a function of the scalar field strength \( g_s = |\nabla \phi_{DE}| \), and is derived from a free function in the action of the scalar field. In spherical symmetry, the two interpolation functions are related by

\[
\mu = \frac{\mu_s}{1 + \mu_s}, \quad g_{DE} = |\nabla \phi_{DE}| = \frac{|\nabla \Phi_K|}{\mu_s} = (1 - \mu)|\nabla \Phi| = |\nabla \Phi - \nabla \Phi_K| \quad (23)
\]

where \( g_{DE} \) is the effective Dark Matter gravity due to a non-uniform Dark Energy (DE) field.

The standard MOND interpolating function \( \mu(x) = \frac{r}{\sqrt{1 + x^2}} \) is often used in fitting rotation curves. But Zhao & Famaey 15 argued that this function has undesirable features in TeVeS. For spherical systems our Lagrangian corresponds to a MOND function

\[
\mu(x) = \frac{2x}{1 + (2 - \alpha x) + \sqrt{(1 - \alpha x)^2 + 4x}, \quad x = \left| \frac{\nabla \Phi}{a_0} \right|.
\]

(24)

5. Light Bending in Slightly Curved Space Time

Light rays trace the null geodesics of the space time metric. Lensing, or the trajectories of light rays in general, are uniquely specified once the metric is given. In this sense light bending works exactly the same way in any relativistic theory as in GR.

Near a quasi-static system like a galaxy, the physical space-time is only slightly curved. Consider lensing by the galactic potential \( \Phi(r) \). A light ray moving with a constant speed \( c \) inside follows the null geodesics \( dt = \sqrt{-g_{ij}dx^i dx^j} \). An observed light
ray travels a proper distance $l_{os} = l_{ls} + l_{ol}$ from a source to the lens and then to an observer. Hence it arrives after a time interval (seen by an observer at rest with respect to the lens) $\int dt = \int_0^{l_{os}} \frac{dl}{c} - \int_0^{l_{ol}} \frac{2\Phi(x)}{c^2} dl$ containing a geometric term and a Shapiro time delay term due to the $\Phi$ potential of a galaxy.

In fact, gravitational lensing in TeVeS recovers many familiar results of Einstein gravity in (non-)spherical geometries. Especially an observer at redshift $z = 0$ sees a delay $\Delta t_{obs}$ in the light arrival time due to a thin deflector at $z_l = l_{ol}/(1 + z_l)$:

$$\frac{c\Delta t_{obs}(R)}{(1 + z_l)} \approx \frac{D_s}{2D_lD_{ls}}(R - R_s)^2 - \int_{-\infty}^{\infty} dt \frac{2\Phi(R, l)}{c^2},$$

as in GR for a weak-field thin lens, $\Phi/c^2 \ll 1$. A light ray penetrates the lens with a nearly straight line segment (within the thickness of the lens) with the 2-D coordinate, $R = D_l \theta$, perpendicular to the sky, where $D_l(z_l) = l_{ol}/(1 + z_l)$ is the angular diameter distance of the lens at redshift $z_l$, $D_s$ is the angular distances to the source, and $D_{ls}$ is the angular distance from the lens to the source. The usual lens equation can be obtained from the gradient of the arrival time surface with respect to $R$, i.e.,

$$x - \frac{D_l D_{ls}}{D_s} \alpha_x(x, y) = x_s \quad \alpha_x = \int_{-\infty}^{\infty} dt \frac{2\partial_x \Phi(x, y, l)}{c^2},$$

$$y - \frac{D_l D_{ls}}{D_s} \alpha_y(x, y) = y_s \quad \alpha_y = \int_{-\infty}^{\infty} dt \frac{2\partial_y \Phi(x, y, l)}{c^2},$$

and the convergence $\kappa$ is related to the deflection $(\alpha_x, \alpha_y)$ by

$$\kappa = \frac{D_l D_{ls}}{2D_s} (\partial_x \alpha_x + \partial_y \alpha_y).$$

Likewise we get standard formulae for the shear and amplification: $\gamma_2 = D_l \partial_y \alpha_x$ and $\gamma_1 = D_l \partial_x \alpha_y - \partial_y \alpha_y$ and for the amplification $A^{-1} = (1 - \kappa)^2 - \gamma_1^2 - \gamma_2^2$.

6. Differences in lensing by uneven DE fluid and by DM halo

An interesting point is that in GR $\kappa$ is proportional to the projected surface density of known matter. This is not the case for a non-linear theory of gravity, nor for GR but with an even DE fluid. We can express $\kappa$ into the critical density as follows,

$$\kappa = \frac{\bar{\Sigma}(x, y)}{\Sigma_{crit}}, \quad \Sigma_{crit}^{-1} \equiv 4\pi GD_lD_{ls}D_s c^2,$$

where we define an effective projected density as follows,

$$\bar{\Sigma}(x, y) \equiv \int_{-\infty}^{\infty} dl \bar{\rho}(x, y, l),$$

note the integrand is NOT the true matter volume density at $(x, y, l)$, rather

$$\bar{\rho}(x, y, l) \equiv \frac{\nabla^2 \Phi(x, y, l)}{4\pi G} = \rho_K + \rho_{DE} > \rho_K$$
because $\Phi$ is the addition of two fields, and we have a non-uniform Dark Energy (DE) fluid from the $\phi_{DE}$ field,

$$\rho_{DE} = \frac{\nabla^2 \phi_{DE}(x, y, l)}{4\pi G}. \quad (31)$$

The DE fluid tracks the known matter $\rho_K$, because the TeVeS $\phi_{DE}$ field is determined by non-linearly with $\rho_K$

$$\rho_K = \frac{\nabla [\mu_s \nabla \phi_{DE}(x, y, l)]}{4\pi G}. \quad (32)$$

There are some important differences between lensing in TeVeS and in GR and between lensing a DE fluid and real DM halo: the potential is different. To demonstrate this explicitly, let’s consider a special non-spherical case, e.g., a Kuzmin disk lens. Here one can solve the TeVeS Poisson equations analytically. Consider an edge-on razor-thin disk lens of the Kuzmin profile of a typical length $b$. In TeVeS theory with $\mu_s = |\nabla \phi_{DE}|/a_0$, the Kuzmin disk would acquire a potential

$$\Phi(x, y, z) = \Phi_K + \phi_{DE} = -\frac{GM}{r_1} + \sqrt{GM a_0 \ln r_1}, \quad r_1 \equiv \sqrt{(b + |y|)^2 + x^2 + z^2}. \quad (33)$$

where the effective halo $\phi_{DE}(x, y, z)$ is non-spherical; its gradient has a sudden jump across the plane $y = \pm 0$, meaning that there is a razor thin layer of Dark Energy fluid. The effective halo would yield a non-zero non-axisymmetric convergence

$$\kappa_{DE}(x, y) = \frac{\pi \sqrt{GM a_0}}{c^2} \frac{D_l D_{ls}}{D_s} \frac{D_l D_{ls}}{D_s} \sqrt{(b + |y|)^2 + x^2}. \quad (34)$$

In GR an edge-on disk without dark halo would have zero convergence. We could add a spherical halo of real Dark Matter

$$\phi_{DM}(x, y, z) = \sqrt{GM a_0} \ln \sqrt{b^2 + y^2 + x^2 + z^2}, \quad (35)$$

centered on the origin $(x, y, z) = (0, 0, 0)$ such that the GR model produces identical potential $\phi_{DM}(x, 0, z) = \phi_{DE}(x, 0, z)$, hence identical rotation curve in the equator, as the TeVeS model. The corresponding axisymmetric convergence

$$\kappa_{DM}(x, y) = \frac{\pi \sqrt{GM a_0}}{c^2} \frac{D_l D_{ls}}{D_s} \sqrt{b^2 + y^2 + x^2}. \quad (36)$$

which is slightly bigger than that of the TeVeS $\kappa_{DE}$; e.g., for a line of sight with an impact parameter $(x, y) = (0, b)$, we find $\kappa_{DE} = \kappa_{DM}/\sqrt{2} = \frac{\pi \sqrt{GM a_0}}{c^2} \frac{D_l D_{ls}}{2D_s}$. Note that the lensing time delay between a pair of images satisfies the scaling

$$H_0 \Delta t_{\text{obs}} \propto 1 - \kappa, \quad (37)$$

so the smaller convergence in TeVeS could predict a larger $H_0$ to fit the same time delay data than in CDM model. Hence an uneven DE fluid offers a new way to bring consistency of $H_0 \sim 70\text{km/s/Mpc}$ from Hubble Key project and $H_0 \sim 50\text{km/s/Mpc}$ from CDM fits to the lensing time delay measurements.\[19]
The vertical force $\partial_y \phi_{DE}$ and the deflection angle $\alpha_y$ along the $x = 0$ line of sight are also bigger in real spherical DM halos than in TeVeS effective halo. These differences suggest that a combined lensing and kinematics study of a lens could decide whether the DM effects are due to real dark matter or an effective halo of uneven dark energy fluid.

7. Lensing and Other Sanity Tests of TeVeS-like theories

For lenses with almost co-linear double images in the CASTLES survey, Zhao, Bacon, Taylor, Horne conducted a detailed fit using spherical point or Hernquist profile lenses. Cares have been taken in including the K-correction, the luminosity evolution with redshift, and the possibility of significant gas and extinction from dust. They applied two methods, using the image positions only, and using the image amplifications. They found that the mass-to-$M_*$ ratios calculated using the two independent methods closely agree, and most of the lenses are found to have $M/M_*$ between 0.5 and 2. This shows that TeVeS is a sensible theory for doing gravitational lensing, in agreement with statistical analysis of a larger sample of lenses.

Nevertheless, I caution that there are several lenses (cf. Fig. 1), typically in galaxy clusters, which require extreme $M/L$, e.g.,. Outliers can occasionally be caused by photometry errors since the lens galaxy is barely resolved, and its total luminosity is subject to the uncertain subtraction of the much brighter quasar images around it. On the modeling side, all previous models are spherical while the flattening and the external shear of real lenses are not taken into account. Also the cluster environment makes prediction highly uncertain: the cluster gas increases the total baryonic material in the lens, but the whole lens accelerates in the cluster, and this so-called external acceleration decreases the MOND effect of the lens. In general, our non-linear Poisson equation for $\phi_{DE}$ can be solved by adapting the numerical code of e.g., the Bologna or the Paris group.

Less model-dependent one can ask if the gravity in stars are correlated with the image-splitting power of the lens; such a correlation is expected if TeVeS is correct. Indeed Fig. 1 shows such a correlation. The horizontal axis is proportional to the critical density $\Sigma_{crit} = c^2/(4\pi GD)$ needed to split images, where $D = D_l D_{ls}/D_s$. Most strong lenses are elliptical galaxies, and the projected surface density within the Einstein radii is much higher than $\frac{\alpha_0}{2\pi G}$, hence the MOND effect is very mild, and the MOND effective halo is sub-dominant within the Einstein ring. The vertical axis is proportional to the mean density of stars within the Einstein radii $\Sigma_* = M_*/(\pi R_E^2)$, assuming $M_* / L_* = 4$ (circles); effects of raising/lowering $M_* / L_*$ by a factor of 2 (solid) or a factor of 4 (dotted) are shown by little vertical rods for each lens. There are, however, quite a few lenses whose gravitation appear uncorrelated with its baryonic mass.

For complicated lens geometry, one can also model lensing by starting with a reasonable guess for the 3D potential, and find the density by taking appropriate
derivatives. E.g., Angus et al. \cite{Angus2008} model the Bullet Cluster by a double-peaked potential and find that the lensing peaks of the Bullet Cluster could be explained by adding neutrinos as part of the known density $\rho_K$ in a TeVeS-like modified gravity; there is also a tentative evidence from galaxy rotation curves in the Ursa Major cluster \cite{Carroll2003}. The phase space density of neutrinos at the lensing peaks requires at least 2eV mass for neutrinos in order not to violate exclusion principle for fermions.

Sanity checks from the solar system to large scale have also been done in recent papers. For example, TeVeS is found to be broadly consistent with galaxy dynamics of early-type galaxies and disk galaxies (see references in \cite{TeVeS1,TeVeS2}), and observations of vertical force, escape velocity and microlensing in the Milky Way (see references in \cite{vertical,escape}). It is possible to build self-consistent triaxial elliptical galaxies using the Schwarzschild method \cite{Schwarzschild}. It is possible to explain the rotation curve of tidal dwarf galaxies, which are hard to understand in CDM framework \cite{rotation}. Structures and CMB anisotropy can form from linear perturbations (see references in \cite{anisotropy}). In general, a non-uniform dark energy fluid can mimic many effects of Dark Matter \cite{fluid}.

Nevertheless, TeVeS-like theories are by no means a firmly established paradigm since many comparisons of the theories with observations are still unknown. While this is normal for a new theory, the Bullet Cluster and some outliers among gravitational lensing galaxies are worrying. Also such theories face challenge to explain why globular clusters and dwarf galaxies of the same baryonic mass shows very different gravitational mass \cite{clusters,dwarfs} unless the dark energy fluid is allowed to condense on sub-kpc scale. In the process of understanding and falsifying TeVeS-like theories, we hope to learn to design more clever and robust emulators for dark matter effects.

It is worth stressing that a common goal of both the standard approach and alternative approach is to understand the detailed physics of the vacuum energy. The (scalar or vector) fields in the vacuum might ultimately hold the answers to both DE and DM mysteries and the answers to many fundamental questions in particle physics.

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It is difficult to write a review and not to be outdated in less than a year in this rapidly developing field, where ideas are constantly being falsified, and remerged with other ideas for synergy. I apologize for my incomplete survey of the vast literatures involved. The aim of this review is partially to stimulate better theories to emerge. I thank Benoit Famaey, Sean Carroll, Eugene Lim, Alan Kostelecky, Ted Jacobson, Jacob Bekenstein, Constantinos Skordis, David Mota, Pedro Ferreira, Tom Zlosnik, Glenn Starkman, Andy Taylor, Luca Ciotti, Carlo Nipoti, Françoise Combes, J-P Bruneton, Daming Chen, Baojiu Li, Garry Angus, Martin Feix and many other colleagues for stimulating discussions on cosmology, dynamics and lensing. HSZ acknowledges partial support of PPARC Advanced Fellowship and National Natural Science Foundation of China (NSFC under grant No. 10428308).
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Fig. 1. **Upper panel**: Shows TeVeS baryon-only fits (solid) to rotation curves of a gas-rich dwarf galaxy NGC1560 ($M_*/L_*= 1.3$) and a gas-poor larger spiral galaxy NGC4157 ($M_*/L_*= 0.6$), and adopting $\alpha_0 = 1.2 \times 10^{-8}$, $\alpha = 0$; the Newtonian $V_{\text{circ}}$ by baryons is also shown (dashed).

**Lower panel**: Shows the scatter of two measurements of gravity near Einstein radii of about 50 CASTELS multi-imaged lenses. The gravity due to stars (vertical axis) and the gravity observed (horizontal axis) appear correlated around a straight line for many lenses, as expected in TeVeS. A few outliers are labeled, consistent with the recent analysis of Ferreras et al.:
Fig. 2. **Upper panel:** compares ΛCDM (dashed) with a TeVeS flat cosmologies without Λ (solid) assuming zero mass for neutrinos and a μ-function with α = 0. Shown are the co-moving distance $D_{\text{com}}$ vs. the physical scale factor $a$ in log-log diagram overplotted with SNeIa data (small symbols) up to redshift 2. Likewise shows the horizon, the Hubble parameter $H$ in units of ($\text{Mpc}^{-1} c$) in two theories. The evolution of the Dark Energy scalar field $\phi_{DE}$ and μ can be inferred from (thin solid lines) $a' = a \exp(\phi_{DE})$ and $a = \mu^{-1}$. **Lower panel:** Shows an enlarged view of the TeVeS fits to co-moving distance to the SNe data points.
