Topological Vector Potentials Underlying One-dimensional Nonlinear Waves

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(Dated: February 23, 2021)

We reveal intrinsic topological vector potentials underlying the nonlinear waves governed by one-dimensional nonlinear Schrödinger equations by investigating the Berry connection of the linearized Bogoliubov-de-Gennes (BdG) equations in an extended complex coordinate space. Surprisingly, we find that the density zeros of these nonlinear waves exactly correspond to the degenerate points of the BdG energy spectra and can constitute monopole fields with a quantized magnetic flux of elementary π. Such a vector potential consisting of paired monopoles with opposite charges can completely capture the essential characteristics of nonlinear wave evolution. As an application, we investigate rogue waves and explain their exotic property of “appearing from nowhere and disappearing without a trace” by means of a monopole collision mechanism. The maximum amplification ratio and multiple phase steps of a high-order rogue wave are found to be closely related to the number of monopoles. Important implications of the intrinsic topological vector potentials are discussed.

Introduction—Real-space topology has been found to be useful for describing some important physical effects, such as the Aharonov-Bohm effect [1], as well as some topological structures of vortices [2, 3], skyrmions [4], and knots [5, 6]. Topologies can also emerge in parameter space, as in the case of the Berry phase and virtual monopole theory [7], or in momentum space, as in topological energy band theory [8–10], revealing bizarre virtual particles and characterizing new forms of matter, including topological insulators [11] and Weyl fermion semimetals [12], and even facilitating quantum computing [13, 14].

Topological issues are mainly discussed in the context of systems with two or more dimensions. The topological properties associated with one-dimensional (1D) nonlinear waves such as solitons, rogue waves and breathers are seldom discussed, even though they exist widely in atomic gases [15], plasma [16], water waves [17], and ferromagnetic materials [18] and play important roles in both integrability theory [19] and optical communications [20]. It is usually assumed that the phase variation of wavefunctions is a sign of topological excitation; since 1D nonlinear excitations such as rogue waves (RWs), dark solitons and breathers always contain abundant phase information [21–24]. However, in contrast to the case of 2D or 3D topological structures such as vortices, the topological charge defined according to the phase variations is no longer valid for characterizing the topology of such 1D nonlinear waves [25, 26].

In this paper, we present a theory to reveal intrinsic vector potentials underlying 1D nonlinear waves that can completely capture the topological properties hidden in their phase variations. By investigating the Berry connection of the linearized Bogoliubov-de-Gennes (BdG) equations in an extended complex coordinate space, we surprisingly find that the density zeros of these nonlinear waves exactly correspond to the degenerate points of the BdG energy spectra and can constitute monopole fields with a quantized magnetic flux of elementary π. Taking nonlinear RWs as an example, we obtain analytical expressions for the topological vector potentials and find that the exotic dynamic evolution of an RW solution can be explained by a topological vector potential reconnection mechanism, analogous to magnetic field reconnection in astrophysics [27].

Virtual magnetic monopole and topological vector potential—We choose one of the simplest models, i.e., the scalar nonlinear Schrödinger equation (NLSE) \(i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + g|\psi|^2\psi\), to demonstrate our theory. The 1D NLSE describes the dynamics of nonlinear waves in optical fibers [28], water wave tanks [29], and plasma systems [30]. It has a nonlinear wave solution \(\psi_0(x, t)\), which can be a dark soliton (for \(g > 0\)), a bright soliton, an RW, a breather (for \(g < 0\)), or some superposition thereof. We will attempt to uncover the intrinsic topological vector potentials of these 1D nonlinear waves.

For a linear system, the Berry phase theory for its quantum adiabatic evolution predicts a virtual monopole topological potential that is closely related to the degenerate points of the energy spectrum [8–10]. For nonlinear systems, the adiabatic condition of an eigenstate depends on the BdG excitation spectrum rather than the level spacings between the eigenstates [31]; therefore, we investigate the BdG excitation spectra of nonlinear waves by considering the degenerate points. Introducing an arbitrary Fourier mode perturbation of the wave solution, \(\psi = \psi_0(x, t) + f_1(t) \exp[ikx] + f_2(t) \exp[-ikx]\), we obtain

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a Hamiltonian for BdG excitation [32]:

\[
H = \left( \frac{k^2}{2} + 2g|\psi_0(x,t)|^2 - \frac{g^2}{2}\psi_0(x,t)^*2 - \frac{k^2}{2} - 2g|\psi_0(x,t)|^2 \right)
\]  

(1)

Because the Hamiltonian is non-Hermitian, the left eigenvector of each branch is no longer equal to the transposed conjugate of its right eigenvector [33]. We obtain instantaneous eigenvalues of \( E_\pm = \pm \sqrt{k^4/4 + 3g^2|\psi_0|^4 + 2k^2g|\psi_0|^2} \). The instantaneous eigenvectors \( |V^R_\pm| \) and \( |V^L_\pm| \) are given by \( H|V^R_\pm| = E_\pm|V^R_\pm| \) and \( \langle V^L_\pm|H = E_\pm(V^L_\pm|, \) and they take the following forms:

\[
|V^R_\pm| = \left( \frac{k^2}{2} + 2g|\psi_0|^2 + E_\pm \right), \quad \langle V^L_\pm| = -\overline{g}\psi_0^2 \left( -\frac{k^2}{2} - 2g|\psi_0|^2 + E_\pm \right).
\]  

(2)

Here, we can see that \( \langle V^L_\pm|V^R_\pm| = 0 \).

We consider the mode of \( k = 0 \), and the energy degeneracy corresponds to density zeros. The BdG spectra are usually not degenerate in real space. Nevertheless, new phenomena would remain hidden if one were to restrict one's attention to real physical parameters [34]. Two famous examples are the Lee-Yang zeros [35] and Fisher zeros [36] reported for imaginary magnetic fields and imaginary temperatures, respectively. Inspired by these, we extend the real coordinate variable \( x \) to the imaginary variable \( z = x + yi \) to explore the density zeros and underlying monopole topological potentials. Thus, we introduce the parameters \( C = (x,y) \).

The Berry connection is then calculated in the parameter space:

\[
\mathcal{A}_\pm = \frac{i\langle V^L_\pm(C)|\nabla_C|V^R_\pm(C)\rangle}{\langle V^L_\pm(C)|V^R_\pm(C)\rangle} = \frac{i\langle V^L_\pm|\partial_x|V^R_\pm\rangle}{\langle V^L_\pm|V^R_\pm\rangle} (e_x + ie_y).
\]  

(4)

We note that there are always two branches of BdG excitations with positive and negative energies, corresponding to a pair of quasiparticles with opposite energies [37]. The effective vector potential should be the average over the two branches, which is

\[
\mathcal{A}_{eff} = \frac{\mathcal{A}_+ + \mathcal{A}_-}{2} = F(x)(e_x + ie_y),
\]  

(6)

with \( F(z) = -\frac{f}{|\psi_0|^2} \frac{\partial_x|\psi_0|^2}{|\psi_0|^2} dx \). To obtain Eq. (6), we have used the fluid conservation law \( \partial_t|\psi_0|^2 + \partial_x|\psi_0|^2 \partial_x \phi = 0 \) and ignored the purely imaginary term of \( i\partial_x|\psi_0|^2/|\psi_0|^2 \). Here, \( |\psi_0| \) and \( \phi \) are the amplitude and phase of the wavefunction, respectively.

The existence of the density zero points \( (|\psi_0(z_N)|^2 = 0) \) implies that \( \mathcal{A}_{eff} \) might have \( N \) singularities on the complex plane (denoted by \( z_N = x_N + iy_N \)). According to the Cauchy integral formula, a meromorphic function can be expressed in terms of these singularities [38], that is, \( \mathcal{A}_{eff} = \sum_{N} \text{Res} \left[ \frac{F(z_N)}{z - z_N} \right] e_x + i e_y \), where \( \text{Res}[F(z_N)] = \Omega/2\pi i \) is the residue.

The real part of the above Berry connection constitutes a 2D vector potential in the following explicit form:

\[
\mathbf{A} = \text{Re}[\mathcal{A}_{eff}] = \sum_{N} \frac{\Omega}{2\pi} \left[ \frac{1}{(x - x_N)^2 + (y - y_N)^2} \right] e_x.
\]  

(7)

Here, \( \mathbf{A} \) takes the form of a monopole's topological vector potential in a 2D case [39, 40]: the corresponding magnetic field will be zero everywhere except at those singular points, that is, \( \mathbf{B} = \Sigma_{N} \Omega \delta(\mathbf{r} - \mathbf{r}_N) \). Interestingly, we find that the magnetic flux is \( \Phi = \pm \pi \), corresponding to a monopole with a charge of \( \pm 1/2 \), for all known 1D nonlinear waves [41].

In particular, the line integral of the vector potential along the real axis can predict the phase variations of such a nonlinear wave. From Eq. (6), we can also find that \( \mathcal{A}_{eff}(x,y) = \partial_x \phi(x_0 + i e_y) \). By reducing the problem to the real axis, we obtain \( \phi(x) = \int \mathbf{A}(x,y = 0) \cdot e_x dx \).

It is interesting to compare the above results with the Aharonov-Bohm effect [1], which predicts a topological phase when an electron moves on a close path around a solenoid. A 1D nonlinear wave moving on the real axis cannot see the magnetic fields scattered on the complex plane; however, it will acquire a phase due to the presence of the vector potential. The evolution of such a nonlinear wave can be understood from the transformed equation based on the topological vector potential, \( i \partial_t \psi = \bar{\psi} \left( \frac{\partial_x + \mathbf{A}(x,y = 0) \cdot e_x}{2} \right)^2 + \frac{\partial}{\partial t} \mathbf{A}(x,y = 0) \cdot e_x dx \psi \), with a transformation \( \tilde{\psi} = \psi_0 \exp \left[ -i \int \mathbf{A}(x,y = 0) \cdot e_x dx \right] \). The effective magnetic field and electric field can be derived as \( \mathbf{B} = \nabla \times \mathbf{A} \) and \( \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \frac{\partial}{\partial t} \mathbf{A}(x,y = 0) \cdot e_x dx \), respectively [42]. In this sense, the phase variations of these nonlinear waves can be viewed as a 1D counterpart to the Aharonov-Bohm phase. Usually, the topological vector potential exhibits time dependence, which provides an alternative way to understand the dynamics of such nonlinear waves.

**Topological vector potentials for rogue waves**—The above NLSE with \( g = -1 \) has a fundamental RW (FRW) solution on a uniform background [43, 44], \( \psi_0 = [1 - \left( \frac{x}{a_0 + \sqrt{2\pi t}} \right)] e^{it} \). The temporal evolution of the RW amplitude depicted in Fig. 1 (a) shows that the wave amplitude remains almost constant until \( t = -5 \), after which sudden growth occurs. At \( t = 0 \), the amplitude amplification ratio (defined as the peak amplitude divided by the background amplitude) reaches its maximum value of 3. Subsequently, the RW quickly decays, and the wave density recovers to be nearly constant. The amplitude peak is located at \( x = 0 \), and on either side of the peak, there are two valleys at \( x = \pm a_0 \) (\( a_0 = \sqrt{\frac{6}{\pi t}} \)). Interestingly, phase jumps accompany the rise in amplitude. In Fig. 1 (b), we see that there is a \( \pi \) phase jump corresponding
to each amplitude valley [23, 24], and the direction of the phase jump at \( x = -a_0 \) suddenly inverts to \(-\pi\) slightly after the moment when the maximum amplitude peak emerges.

The characteristic of “appearing from nowhere and disappearing without a trace” of RWs [44, 45] is believed to be the cause of many ocean disasters and therefore has attracted much attention [46, 47]. Modulational instability can well explain the rapid growth of RWs [45–49]; however, it fails to explain the decay process. Moreover, analytical studies have indicated that the amplitude amplification ratios for RWs of different orders are subject to certain limits [44, 50], but no physical mechanism for these ceiling values has been discovered. The \( \pi \) phase jumps and abrupt inversion discussed above are also not fully understood [23]. Here, we attempt to elucidate these issues with the help of our developed topological vector potential theory.

According to Eq. (6), the vector potential underlying the FRW takes an explicit form of \( A_{eff}(x, y) = \frac{64t^2}{16t^2 + 8t^2(12t^2 + 5) + (12t^2 + 5)^2} \left( e_x + ie_y \right) = F(z)(e_x + ie_y) \). It has four singularities, i.e., \( z_{1,2} = \pm (a + ib) \) and \( z_{3,4} = \pm (a - ib) \), where \( a = \frac{4}{\sqrt{4t^2 + \sqrt{16t^2 + 64t^2 + 3}} - 3} \) and \( b = \frac{\sqrt{4t^2 + \sqrt{16t^2 + 64t^2 + 3}} - 3}{2\sqrt{2}} \). Thus, we have

\[
A_{eff}(x, y) = \sum_{N=1, \ldots, 4} \text{Res}[F(z_N)] \left( e_x + ie_y \right),
\]

where the residue is \( \text{Res}[F(z_N)] = \lim_{z \to z_N} (z - z_N)F(z) = \pm \frac{\pi}{4} \). According to Eq. (7), the vector potential \( A \) is composed of two pairs of monopoles, and in each pair, the two monopoles have charges of \( \pm 1 \) with opposite signs. The temporal evolution of the potential is shown in Fig. 1 (c-f). The monopoles with opposite charges in each pair approach each other (see Fig. 1 (c-d)), collide elastically in the vertical direction at time \( t = 0 \) with speeds of \( dB/dt = \sqrt{11/6} \) and \( dA/dt = 0 \), and then bounce back after exchanging their charges (see Fig. 1 (e-f)). As \( t \to \pm 0, a \to \pm a_0 \) and \( b \to 0 \). The vector potential on the real axis then takes the following form:

\[
\lim_{t \to \pm 0} A = \lim_{b \to 0} \left\{ \begin{array}{l}
-\pi [ (x \pm a_0) e_y + b e_x ] - \frac{\pi [ (x \pm a_0)^2 + b^2 ]}{2\pi [ (x \pm a_0)^2 + b^2 ]} \\
+ \frac{\pi [ (x \pm a_0) e_y - b e_x ]}{2\pi [ (x \pm a_0)^2 + b^2 ]} \\
- \frac{\pi [ (x \pm a_0) e_y + b e_x ]}{2\pi [ (x \pm a_0)^2 + b^2 ]} \\
+ \frac{\pi [ (x \pm a_0) e_y - b e_x ]}{2\pi [ (x \pm a_0)^2 + b^2 ]}
\end{array} \right\} \\
= (- \pi \delta [x \pm a_0] + \pi \delta [x \mp a_0]) e_x.
\]

The line integral of the above vector potential can explain the \( \pi \) phase jumps shown in Fig. 1 (b). The phase gradient determines the density flow, and the change in the phase distribution can provide an understanding of the growth and decay of RWs [23]. The collision of the monopole charges after collision can well explain the striking phase reversal that induces the RW’s rapid decay.

Higher-order RWs admit higher amplitude peaks, more density valleys (or humps) and multiple phase steps, as shown in Fig. 2 [23, 44, 50, 51]. For a second-order RW, the maximum amplitude amplification ratio is 5. There are five phase steps distributed symmetrically with respect to the \( x = 0 \) axis, each of which is associated with a phase jump of \( \pm \pi \) (see Fig. 2 (a)). The vector field is composed of six pairs of monopoles, as shown in Fig. 2 (b). They can be divided into two classes: the four pairs that are closer to the \( x \)-axis collide with each other on the \( x \)-axis, leading to four \( \pi \) phase jumps, whereas the other two pairs (upper and lower pairs) collide on the imaginary axis, mainly contributing to a sudden rise in the wave amplitude. For a third-order RW, the maximum amplitude amplification ratio is 7, and there are seven phase steps distributed symmetrically with respect to the \( x = 0 \) axis (see Fig. 2 (c)). The vector field is composed of twelve pairs of monopoles, as shown in Fig. 2 (d). The paired monopoles collide and merge when the RW

\[
(\text{a}) \text{ Amplitude distribution, (b) phase distribution, and (c-f) evolution of the topological vector potential for a fundamental rogue wave (FRW).}
\]
reaches its highest peak. Among them, six pairs collide and merge on the real axis, and the other six pairs collide on the imaginary axis or in other locations on the complex plane. The monopole pairs that do not collide on the real axis do not contribute to multiple phase jumps, but they do influence the peak amplitude and the energy transfer involving RWs.

Energy transfer and topological vector potential reconnection—The energy of an RW can be written as $\int_{-\infty}^{+\infty} \left( \frac{1}{2} |\partial_x \psi|^2 - \frac{1}{2} (|\psi|^2)^2 \right) dx$, where the first term is the kinetic energy $E_k$ and the second term is the interaction energy $E_{int}$. For an FRW, the time-dependent kinetic energy is $E_k = \frac{1}{(4n^2+1)^{1/2}}$. Because the total energy is conserved, the amplitude amplification of the RW corresponds to the energy transfer process from interaction energy to kinetic energy, as indicated in Fig. 3 (a).

Quantitatively, the total energy transfer can be evaluated as the time integral of the kinetic energy ($\int_{-\infty}^{+\infty} E_k dt$). We have numerically calculated the integrals for RWs whose orders are up to 10 and have found that they are equal to the sum of the absolute magnetic flux of the monopoles (see Fig. 3 (b)). From the vector field perspective, we know that monopole collisions induce the conversion of interaction energy into kinetic energy. This process is analogous to the magnetic field reconnection process identified in astrophysics, in which magnetic field energy is transformed into the kinetic energy of a plasma [27].

We have also investigated the relation between the peak amplitude of an RW and the number of monopoles and found that they are closely related. According to our discussion above, an FRW admits 4 monopoles. Because the $n$th-order RW solution is a nonlinear superposition of $\frac{n(n+1)}{2}$ FRWs [50], it will contain $N = 2n(n+1)$ monopoles on the complex plane. Among them, there are $4n$ monopoles colliding on the real axis that are responsible for the multiple phase steps, whereas the other $2n(n-1)$ monopoles will collide in other locations on the complex plane. On the other hand, the square of the maximum amplitude amplification ratio $P$ for an $n$th-order RW can be calculated to be $(2n+1)^2$ [44, 52]. We thus obtain the following explicit relation: $P = 2N + 1$.

Based on this observation, we can predict that the amplitude amplification ratio of a high-order RW will be subject to a certain limit because an RW contains only a limited number of monopoles due to the finite number of valleys in its geometric configuration [50, 51].

Conclusion and discussion—Theoretically, we reveal an intrinsic topological vector potential that is composed of paired monopoles hidden in the phase variations of 1D nonlinear waves by investigating the Berry connection of BdG equations in an extended complex coordinate space, which is in contrast to the topological genus on a Riemann surface of complex parameters [53–56]. As an application, we demonstrate that an $n$th-order RW solution contains $n(n + 1)$ pairs of monopoles with opposite charges and that the collision of these monopoles and the reconnection of the corresponding vector field lead to energy conversion from interaction energy to kinetic energy and are responsible for the exotic property of “appearing from nowhere and disappearing without a trace” of RWs. Our theory is also applicable to other 1D nonlinear models [57–62] and thus may inspire new research in directions related to the topological fluid field representation of 1D nonlinear systems. It can be further extended to study the semiclassical dynamics of BdG excitations in Bose-Einstein condensation under the action of intrinsic topological potentials [63].

Acknowledgments This work was supported by the
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We have checked all known 1D nonlinear waves, such as dark solitons, W-shaped solitons, Akhmediev breathers, and rogue waves, the residue of each singularity of $F(z)$ is found to be $\pm i/2$. Theoretically, we make analysis on the simplest BdG system of

$$
\begin{pmatrix}
  r & -p - iw \\
  p - iw & -r
\end{pmatrix}
$$

and find the effective vector potential $A_{eff} = A_\pm + A_\mp 2$ (where $A_\pm$ is the Berry connection in $(r,p,w)$ space for the two energy branches) admits a monopole with the charge of $\pm 1/2$.

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