A role of the renormalization group invariance in calculations of the ground state energy for models with confined fermion fields is discussed. The case of the (1+1)D MIT bag model with the massive fermions is studied in detail.

1. Introduction

The calculations of the Casimir energy for quantized fields under nontrivial boundary conditions encounter usually a number of difficulties (for the most recent review on the Casimir energy see the ref.[1]). A majority of them are connected with ambiguities in results obtained by means of different regularization and renormalization methods. Physically more interesting problem is the dependence of the (renormalized) energy from an additional mass parameter, which emerges unavoidably in any regularization scheme. For example, in the widely used $\zeta$-function regularization, the mass $\mu$ must be introduced in order to restore the correct dimension of the sum:

$$E = - \sum \omega_n \to E_{\text{reg}}(\mu, \varepsilon) = -\mu^\varepsilon \sum \omega_n^{1-\varepsilon}.$$  \hspace{1cm} (1)

Whatever renormalization procedure one applies, the finite part of the energy would contain a $\mu$-dependent contribution (recently this problem was addressed and studied from the RG point of view in ref. [2]). Of course, there are several situations, for which this dependence is canceled due to some geometrical, or other, features of the given configuration. However, it would be very useful and interesting to investigate more general case.

In the present paper we consider the renormalization of the Casimir energy from the point of view of the convenient quantum field theory, and assume that variations of the mass scale $\mu$ must not yield any physical consequences. This requirement naturally leads to a sort of the renormalization group equation, the
solution of which allows to conclude that some of the parameters of the “classical” mass formula have to be considered as running constants. This may be important, e. g., in some phenomenological applications, such as the quark bag models, since it may provide an incite into the relations between fundamental and effective aspects of the investigations of the hadronic structure.

2. Confined Massless Fermion Field in a Spherical Cavity: the MIT Bag Model

Let us illustrate the general ideas of the method with the simple example. Consider the free fermion field confined to the spherical volume of the radius $R$ under the MIT-bag boundary conditions:

$$(in_{\alpha}\gamma^\alpha + 1)\Psi(R) = 0 , \quad n_{\alpha} = \frac{r_{\alpha}}{R} .$$

(2)

These conditions provide the absence of the quark flow through the surface of the bag.

As is well-known, the Casimir energy of such configuration defined as the $\zeta$-regularized sum (1), where $\omega_n$ are the eigenvalues of the free Dirac Hamiltonian $H = -i\alpha\partial + \beta m$, contains the terms singular in the limit $\varepsilon \to 0$, that are proportional to the powers of $R$ from $R^3$ to $R^{-1}$. These singularities have to be absorbed into the definitions of the corresponding constants in the “classical” mass formula

$$E_{cl} = B\frac{4}{3}\pi R^3 + \sigma 4\pi R^2 + fR + \Lambda + hR^{−1} .$$

(3)

In the case of a massless field, the only divergent term survives that is proportional to $R^{-1}$, and the renormalized total energy of the MIT bag without valence quarks reads (for the sake of simplicity, we keep only the volume part in the classical energy):

$$E_{MIT} = \frac{1}{R}(h - h_1\ln\mu R) + \frac{4}{3}\pi R^3 B ,$$

(4)

where $B$ is the bag constant, $h_1$ can be calculated by means of, e. g., heat-kernel technique (it had been shown that $h_1 = \frac{1}{63\pi}$), and $\mu$ is the additional arbitrary mass scale.

The condition of the independence of $E_{MIT}$ from a choice of the value of $\mu$ leads to the functional equation

$$\mu \frac{d}{d\mu} E_{MIT} = 0 ,$$

(5)

what means that $h$ should be considered as a “running constant”:

$$h(\mu) = h_1\ln\frac{\mu}{\mu_0} ,$$

(6)

where $\mu_0$ is the normalization point. Therefore, the bag energy equals

$$E_{MIT}(R, \mu_0) = -\frac{1}{R} h_1\ln\mu_0 R + \frac{4}{3}\pi R^3 B .$$

(7)
If we want to consider this bag as a model of a stable composite object, like hadron, we need its energy to have a minimum at a certain value of the radius $R$. So, the condition of the RG invariance for the energy (5) must be supplied with the equation

$$ \frac{d}{dR} E_{MIT}(R, \mu_0) = 0 , $$

what yields

$$ \ln \mu_0 R = - \left( \frac{8 \pi R^4 B}{3 \hbar_1} + 1 \right) . $$

Equations (5) and (8) give the minimal radius $R_{min}$ as a function of the energy scale

$$ R_{min} = R_{min}(\mu_0) . $$

It’s clearly seen that $R_{min}$ decreases when $\mu_0$ increases, what corresponds to the idea of the bag model as an effective approach to the investigation of the hadronic structure at some low energy scale.

3. (1+1)D MIT Bag Model with Massive Fermions

The presence of a mass may lead, in general, to some new divergences that have to be subtracted. Consider in detail the (1+1)D MIT bag model with the massive fermions. The Lagrangian of this system

$$ L_{MIT} = i \bar{\psi} \gamma \psi - \bar{\psi} \psi (m \theta(|x| < R) + M \theta(|x| > R)) $$

describes (in the limit $M \to \infty$) the fermion field confined to the segment $[-R, R]$ under the (1+1)D boundary condition:

$$ (\pm i \gamma^1 + 1) \psi(\pm R) = 0 . $$

The exact spectrum of the elementary fermionic excitations reads:

$$ \omega_n = \sqrt{\left( \frac{\pi}{2R} n + \frac{\pi}{4R} \right)^2 + m^2} . $$

Here we will be interesting only in the small mass $m$ limit, so we drop out all terms of the order $m^4$ and higher. Then the eigenvalues $\omega_n$ can be written as

$$ \omega_n = \Omega_1 n + \Omega_0 + \frac{m^2}{2(\Omega_1 n + \Omega_0)} + O(m^4) , $$

where

$$ \Omega_1 = \frac{\pi}{2R} , \quad \Omega_0 = \frac{\pi}{4R} . $$

In order to analyze the singularities in the Casimir energy, we use the expansion for $n > 0$:

$$ \omega_n = \Omega_1 n + \Omega_0 + \frac{\Omega_1}{n} + O(n^{-2}) , $$
where $\Omega_{-1} = m^2 R/\pi$, and assume the lowest valence state with $\omega_0 = \Omega_0 + 2\Omega_{-1}$ to be filled.

It can be shown, that the $\zeta$-regularized sum (1) reads:

$$E_{\text{reg}} = -\Omega_{-1} \left( \frac{1}{\varepsilon} + \gamma_E \right) + \frac{\Omega_0}{12} + \frac{\Omega_0}{2} + \frac{\Omega_0^2}{2\Omega_1} + \Omega_{-1} \left( \ln \frac{\Omega_1}{\mu} + 1 \right),$$

(17)

where $\gamma_E = 0.5772...$ is the Euler constant. It's interesting to note, that the regularization by the exponential cutoff gives the equivalent result\(^5\).

The divergent part of (17) can be extracted in the form:

$$E_{\text{div}} = -\frac{m^2 R}{\pi} \left( \frac{1}{\varepsilon} + \gamma_E - \ln \frac{\pi}{8} \right).$$

(18)

We include in $E_{\text{div}}$ the pole $\varepsilon^{-1}$ as well as the transcendent numbers $\gamma_E$ and $\ln \frac{\pi}{8}$ in analogy to the widely used scheme $\overline{\text{MS}}$ in QFT, but we should mention that this analogy is only formal one, since \(^\text{(18)}\) has nothing to do with the singularities appearing in the conventional field theory since it depends on the geometrical parameter $R$. This kind of divergences containing the dependence from a dimensional parameter is close to the so-called cusp singularity in the Wilson loops renormalization, and may be treated on the same basis\(^6\).

The renormalization of (17) is performed by the absorption of $E_{\text{div}}$ (18) into the definition of the “classical” bag constant $B$, which is introduced in the mass formula and characterizes the energy excess inside the bag volume as compared to the energy of nonperturbative vacuum outside\(^3\).

The finite renormalized energy of our bag with one valence fermion on the lowest energy level is

$$E(R, \mu) = 2B_0 R + \frac{11\pi}{48R} + \frac{3m^2 R}{\pi} - \frac{m^2 R}{\pi} \ln \mu R,$$

(19)

where $B_0$ is the renormalized bag constant. Taking into account the RG equation (5), we find that $B_0$ should be treated as the running constant

$$B_0(\mu) = \frac{m^2}{2\pi} \ln \frac{\mu}{\mu_0},$$

(20)

with the normalization point $\mu_0$. Therefore, the total energy is

$$E_{\text{MIT}} = -\frac{m^2 R}{\pi} \ln \mu_0 R + \frac{11\pi}{48R} + \frac{3m^2 R}{\pi}.$$  

(21)

The condition of stability (8) may be written as

$$\ln \mu_0 R = 2 - \frac{11\pi^2}{48m^2 R^2},$$

(22)

and hence we obtain the radius of a stable bag $R_{\text{min}}$ as a function of the energy scale, decreasing with growth of $\mu_0$.  

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\(^5\)~\cite{Eichler1975}

\(^6\)~\cite{Jaffe1975}
4. Conclusion

We have analyzed the consequences of the condition of renormalization group invariance in the Casimir energy calculations on the simple examples related to the quark bag models. It is shown, that the requirement of the stability supplied with the RG invariance leads to the dependence of the bag’s size $R_{\text{min}}$ on the mass renormalization point $\mu_0$ which characterizes the energy scale, and therefore, the validity of the bag approximation.

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