Discrete Bessel functions and transform

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Abstract

We present a straightforward discretization of the Bessel functions $J_n(x)$ to discrete counterparts $B_n(x_m)$, of $N$ integer orders $n$ on $N$ integer points $x_m = m$, that we call discrete Bessel functions. These are built from a Bessel integral generating function, restricting the Fourier transform over the circle to $N$ points. We show that the discrete Bessel functions satisfy several linear and quadratic relations, particularly Graf’s product-displacement formulas, that are exact analogues of well-known relations between the continuous functions. It is noteworthy that these discrete Bessel functions approximate very closely the values of the continuous functions in ranges $n + |m| < N$. For fixed $N$, this provides an $N$-point transform between functions of order and of position, $f_n$ and $\tilde{f}_m$, which is efficient for the Fourier analysis of finite decaying signals.

1 Introduction: Discrete Bessel functions

Fourier-Bessel analysis originates from the radial part of two-dimensional Fourier analysis. Solutions of the wave equation in cylindrical coordinates yield the functions known collectively as cylinder functions. Of these, we shall be particularly interested in the Bessel functions of the first kind and of integer order, defined from a plane-wave decomposition that provides their generating function [1, KU120(13)],

$$e^{ix\sin \varphi} = J_0(x) + 2 \sum_{k=1}^{\infty} J_{2k}(x) \cos(2k\varphi) + 2i \sum_{k=0}^{\infty} J_{2k+1}(x) \sin((2k+1)\varphi),$$

(1)

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for $x \in \mathbb{R}$ real and $\varphi \in S^1$ the circle. Seen as Fourier sine and cosine series, this provides an expression for the coefficient functions, as

$$J_k(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \exp(i x \sin \varphi) \left[ C_k \cos(k\varphi) - iS_k \sin(k\varphi) \right],$$

where we used

$$C_k := |\cos(\frac{1}{2}k\pi)| = \begin{cases} 1, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases} \quad S_k := |\sin(\frac{1}{2}k\pi)| = \begin{cases} 0, & k \text{ even}, \\ 1, & k \text{ odd}. \end{cases}$$

In Fourier analysis, a well-known strategy to discretize the Fourier integral transform over a circle to an $N$-point cyclic finite Fourier transform, is to replace integrals by finite sums over $N = 2j + 1$ equidistant points $\varphi_k$ on the circle, through

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi f(\varphi) \rightarrow \frac{1}{2j+1} \sum_{k=-j}^{j} f(\varphi_k),$$

$$\varphi_k := \frac{2\pi k}{2j+1} \quad \text{so} \quad \Delta \varphi = \varphi_{k+1} - \varphi_k = \frac{2\pi}{2j+1}. \quad (5)$$

We consider $j$ to be integer, and thus $N$ odd.

In Ref. [2] the authors proposed to discretize the Bessel function from its integral definition (1) to the $N$-point sum; however, their results are incomplete for not having respected the difference between the even and odd orders, with values over different sets of points over the circle. We thus propose here to expand the definition of discrete Bessel functions as

$$B_n^{(N)}(x_m) := \frac{1}{2j+1} \sum_{k=-j}^{j} \exp(ix_m \sin \varphi_k) \left[ C_n \cos(n\varphi_k) - iS_n \sin(n\varphi_k) \right],$$

$$= \frac{1}{2j+1} \sum_{k=-j}^{j} \exp(im \sin \varphi_k) \times \begin{cases} \cos n\varphi_k, & n \text{ even}, \\ -i \sin n\varphi_k, & n \text{ odd}, \end{cases} \quad (6)$$

where $n$ and $m \equiv x_m$ are integers; their range, initially the set of all integers, can be reduced to $n \in \{0, 1, \ldots, N-1 = 2j\}$, or to $m \in \{-j, -j+1, \ldots, j\}$, due to the symmetries

$$B_n^{(N)}(m) = (-1)^n B_{-n}^{(N)}(m) = (-1)^n B_n^{(N)}(-m) \quad \text{real},$$

$$B_n^{(N)}(0) = \delta_{n,0}. \quad (8)$$
In Sect. 2 we show that beyond superficial similarities, the discrete Bessel functions $B_{n}^{(N)}(m)$ exhibit various other properties that are exact counterparts of those satisfied by the continuous Bessel functions $J_{n}(x)$. This includes linear relations that are proven straightforwardly, and the quadratic relation known as Graf’s formula [3] in its various forms and special cases.

The feature that initially caught attention is shown in Sect. 3, where we compare the actual numerical values of the discrete and continuous Bessel functions. Although it is clear from the beginning that a properly written limit $\lim_{N \to \infty} B_{n}^{(N)}(m)$ should return the continuous Bessel function $J_{n}(x)$, the approximation provided by the discrete Bessel function is surprisingly close in a region of the integer grid of indices $(n,m)$. The differences between the two for $0 \leq n+m < j \approx \frac{1}{2}N$ are of the order of $\Delta_{n}^{(0)} < 10^{-16}$ for $N = 161$; they are smaller when farther from the upper edge of that region.

We shall call the $N \times N$-matrix $B^{(N)} = \|B_{n}^{(N)}(x_m)\|$ in (6)–(7) simply as the discrete Bessel transform kernel. Note that among Bessel functions at $x = 0$, only $J_{0}(0) = 1 \neq 0$; the discrete Bessel matrix also has at its apex $B_{0}^{(N)}(0) = 1$, with the rest of its column $B_{n>0}^{(N)}(0) = 0$. This discrete Bessel matrix is nonsingular, so in Sect. 4 we can define a discrete Bessel transform between an $N$-point function $f_{m} \equiv f(x_m)$ and its partial Bessel coefficients $\tilde{f}_{n}$. The $N \rightarrow \infty$ limit of this transform is not granted, however.

The quest for discrete analogues of the cylinder functions has both computational and analytical interest, and has been approached in several ways, from an early definition in Ref. [4], to recent work based on difference equations postulated as analogues to the Bessel differential equation [5, 6]. The resulting definitions are not equivalent to that of Ref. [2] nor the one we study here. Therefore we emphasize in the concluding Sect. 5 that, beyond the many analytic properties and computational applications that cylinder functions have enjoyed, the present discrete analogue and its associated transform may have not yet been regarded.

2 Linear and Graf discrete Bessel identities

The discrete Bessel functions $B_{0}^{(2j+1)}(m)$ in [6] obey analogues of several well-known identities satisfied by the continuous Bessel functions $J_{n}(x)$. The Fourier series transform over the circle can be straightforwardly discretized to the transform over $2j+1$ points in linear expressions. For brevity, we shall henceforth omit the upper index, understanding that $N = 2j + 1$; formulas
will also simplify upon the introduction of the so-called Neumann factor:

\[
\varepsilon_n := 2 - \delta_{n,0} = \begin{cases} 1, & n = 0, \\ 2, & n \neq 0. \end{cases}
\]  

(9)

It is then straightforward to write and prove the sum of even orders of the discrete Bessel functions, as

\[
j \sum_{n=-j}^{j} B_{2n}(m) = \sum_{n=0}^{j} \varepsilon_n B_{2n}(m) = B_0(m) + 2 \sum_{n=1}^{j} B_{2n}(m) = 1,
\]  

(10)

which can be compared with the corresponding equation in \[1\] Eq. WA44, p. 934. This is a particular case (for \(\varphi_k = 0\)) of the linear discrete Bessel summations for odd and even orders,

\[
\sum_{n=0}^{j} \varepsilon_n B_{2n}(m) \cos(2n\varphi_k) = \cos(m \sin \varphi_k),
\]  

(11)

\[
\sum_{n=0}^{j} B_{2n+1}(m) \sin((2n+1)\varphi_k) = \frac{1}{2} \sin(m \sin \varphi_k),
\]  

(12)

which has been proven for the four cases of \(k\) and \(n\) even or odd, using trigonometric sum identities. They can be compared with \[1\], Eqs. KU120(14) and (15), on p. 935, respectively.

Regarding quadratic expressions, a sum found by Neumann in 1867 for integer orders, was extended by Graf in 1893 to all real orders, and known since as Graf’s formula \[3\] Sec. 7.6.2, Eq. (6)]. Its group-theoretic origin is the linear transformation of spherical harmonics \(Y_{\ell,m}(\theta,\phi)\) by Wigner-\(d\) functions under rotations around the \(y\)-axis, contracted for \(\ell \to \infty\) \[7\]. In that limit both spherical harmonics and Wigner \(d\)-functions become Bessel functions and particularly yield

\[
\sum_{n=-\infty}^{\infty} J_n(x) J_{n'-n}(x') = J_{n'}(x + x'),
\]  

(13)

which can be seen as a displacement and convolution of arguments and indices.

The discrete Bessel functions \(B_n(m)\) in \[6\]–\[7\] satisfy a corresponding formula for \(N = 2j+1\), that is

\[
j \sum_{n=-2j}^{2j} B_n(m) B_{n'-n}(m') = B_{n'}(m + m').
\]  

(14)

4
To prove this relation, we directly replace the discrete functions from (6),
keeping in mind the parity and periodicity properties (8), which allow the
sum to be over the \( N \) terms, as \( \sum_{-j}^{j} \) or \( \sum_{0}^{2j} \). The left-hand side of this
discrete Graf formula is

\[
\sum_{n=-2j}^{2j} B_n(m) B_{n'-n}(m') = \frac{1}{(2j+1)^2} \sum_{n=-2j}^{2j} \sum_{k,k'=-j}^{j} \exp(i(m \sin \varphi_k + m' \sin \varphi_{k'})) \\
\times \left[ C_n \cos(n \varphi_k) - i S_n \sin(n \varphi_k) \right],
\]

(15)

The sum over \( n \) shifts over to the two factors that house the parities in (3),
which then separate into two cases, for \( n' \) even or odd, to reconstruct the
right-hand side of the discrete Graf formula.

For \( n' \) even, the sum over \( n \) becomes,

\[
\sum_{n=-2j}^{2j} \left[ C_n C_{n'-n} \cos((n'-n) \varphi_{k'}) + S_n S_{n'-n} \sin((n'-n) \varphi_{k'}) \right]
= (2j+1) \cos(n' \varphi_k) \delta_{k,k'},
\]

(16)

while for \( n' \) odd the sum yields

\[
-i \sum_{n=-2j}^{2j} \left[ C_n S_{n'-n} \cos((n'-n) \varphi_{k'}) + S_n C_{n'-n} \sin((n'-n) \varphi_{k'}) \right]
= -i (2j+1) \sin(n' \varphi_k) \delta_{k,k'}.
\]

(17)

This brings the left-hand side of (14) closer to that of \( B_{n'}(m+m') \),

\[
\frac{1}{2j+1} \sum_{k,k'=-j}^{j} \exp(i(m \sin \varphi_k + m' \sin \varphi_{k'})) \left[ \tilde{C}_{n',k'} \cos(n' \varphi_k) - i \tilde{S}_{n',k'} \sin(n' \varphi_k) \right],
\]

(18)

where (16) and (17) contribute to the coefficients

\[
\tilde{C}_{n',k'} = \begin{cases} 0, & n' \text{ odd}, \\ \delta_{k,k'}, & n' \text{ even}, \end{cases} \quad \tilde{S}_{n',k'} = \begin{cases} \delta_{k,k'}, & n' \text{ odd}, \\ 0, & n' \text{ even}. \end{cases}
\]

(19)

These factors placed in the double sum (15) reduce terms to those with \( k = k' \)
into a single sum, where the two exponents join to render \( (m + m') \sin \varphi_k \),
and the left-hand side of (14) has become indeed

\[
\sum_{k=-j}^{j} \exp(i(m+m')\sin\varphi_k)\left[C_{n'} \cos(n'\varphi_k) - iS_{n'} \sin(n'\varphi_k)\right] = B_{n'}(m+m').
\]

(20)

The symmetries (8) indicate that the \(4j + 1\) summands in \(\sum_{n=-2j}^{2j}\) can be reduced to \(N = 2j + 1\) summands by introducing the Neumann factor \(\varepsilon_n\) in (9). Thus, for \(n' = 0\) and \(m = -m'\), Eq. (14) can be written in the forms

\[
\sum_{n=-2j}^{2j} B_n(m)^2 = \sum_{n=0}^{2j} \varepsilon_n B_n(m)^2 = \left[B_0(m)\right]^2 + 2\sum_{n=1}^{2j} B_n(m)^2 = 1,
\]

(21)

that correspond to well-known formulas for Bessel functions with infinite sums. The basic Graf formula that is valid for the discrete Bessel functions yields several of its versions under the symmetries listed in (8). We should mention that all formulas in this Section have also been verified numerically.

3 The discrete-to-continuous approximations

Figure 1 shows the values of the discrete and the continuous Bessel functions for various values of \(j\), \(n\) and \(m\). In this section we indicate the approximate equalities by \(B_n(m) \equiv J_n(m)\) and report estimates for ranges in the integer grid \((n,m)\) adjacent to the origin. For \(N = 2j+1\) differences, we measure the merit of the approximation through the mean quadratic error,

\[
\Delta_n^{(N)} := \frac{1}{N} \sum_{m=0}^{N-1} \left(J_n(m) - B_n^{(N)}(m)\right)^2.
\]

(22)

The space of a discrete system of \(N = 2j + 1\) points \(\{x_m\}_{m=0}^{2j}\) is spanned by a basis set of \(N\) independent functions \(J_n(x_m)\). The graphs in Fig. 1 contain intervals of \(m\) beyond \([0, 2j]\), and up to \(4j = 2N - 2\). Good matches between discrete and continuous Bessel values in the grid \((n,m)\) are seen to lie in the first quadrant for \(n+m \leq 2j\); in the interval \([0, 2j]\), the mean square error between values is \(\Delta_n^{(321)} < 10^{-16}\), for \(j = 160\). In the next Section we shall use this feature to define a discrete Bessel transform between functions of position \(m\) and mode \(n\) for any given integer \(j\).
Figure 1: Comparison of values of the discrete Bessel functions $B_{n}^{(2j+1)}(m)$ (open circles) and the continuous Bessel functions $J_{n}(m)$ (lines) in the same ranges. We show the cases for $j \in \{10, 30, 50\}$, (i.e., $N \in \{21, 61, 101\}$ discrete points), for Bessel orders $n \in \{0, 10, 30, 50\}$, over the ranges $m \in [0, 4j=2N-2]$ of their argument. The continuous lines are gray where the difference between the discrete interpolation and the continuous Bessel values is less than $10^{-16}$ and replaced by heavy black lines where it is greater.

When we enlist other well known formulas that are valid for continuous Bessel functions, and replace them by their discrete version we also find matches with similar approximations. Among them we find

\[ 2 \sum_{n=0}^{j} (-1)^{n} B_{2n+1}(m) = \sin(m), \]  \hspace{1cm} (23)

\[ \sum_{n=0}^{j} \varepsilon_{n}(-1)^{n} B_{2n}(m) = \cos(m), \]  \hspace{1cm} (24)

that can be compared with Eqs. [1, WH(1,2), p. 934]. We point to the
Figure 2: Comparison of the left-hand sides of Eqs. (23) and (24) for integer \( m \in [0, 100] \) (open circles), and \( \sin(m) \) and \( \cos(m) \) for continuous \( m \) (line) in the same range; here, \( j = 50 \) and \( N = 101 \).

fact that in (23)–(24), the argument of sine and cosine is integer \( m \in \{ -j, -j+1, \ldots, j \} \). To compare these functions of discrete \( m \) with the continuous functions of \( m \), we show both in Fig. 2. The mean square errors there are of the order \( \approx 10^{-6} \).

For other expressions in Fig. 3 we show, for \( n = 1 \) and \( m \in [-j, j] \), the approximations

\[
\sum_{k=0}^{j} B_1(m \cos \varphi_k) \cos \varphi_k = \frac{\sin m}{m} =: \text{sinc} m, \tag{25}
\]

\[
\sum_{k=0}^{j} B_1(m \cos \varphi_k) = 1 - \frac{\cos m}{m} =: \text{csc} m. \tag{26}
\]

that also hold with a mean square error less than \( 10^{-6} \).

4 Discrete Bessel transform and inverse

In Section 1 we introduced the \( N \times N \) discrete Bessel matrix \( B = \|B_{n,m}\| \), \( B_{n,m} := B_{n}(m) \). Any finite set of \( N \) linearly independent vectors can be used to define an \( N \)-dimensional vector space; although we cannot prove linear independence here, numerical verifications of \( \text{det}B \neq 0 \) support this very plausible conclusion. Hence, given a function of \( N \) positions \( f_m \), the matrix \( B \) will transform this into a function \( f_n \) of \( N \) modes. The inverse matrix
Figure 3: Comparison of values of the discrete and continuous Bessel functions, left- and right-hands of Eqs. (25) and (26), as before, with open circles and continuous lines, in the same range for $j = 50$, $N = 101$.

$C := B^{-1}$ then recuperates the original function of positions,

$$\tilde{f}_n := \sum_{m=0}^{N-1} B_{n,m} f_m, \quad f_m = \sum_{n=0}^{N-1} C_{m,n} \tilde{f}_n, \quad CB = 1. \quad (27)$$

The elements of the matrix $B$ closely approximate the values of $J_n(m)$ on the integer grid region $0 \leq n + m \leq N - 1$. We saw that $B_{n,0} = \delta_{n,0} = J_n(0)$, and we know that also for $0 \leq m \leq n-1$, both the continuous $J_n(m)$ and the discrete Bessel functions are very small. They start oscillating with a maximal amplitude just beyond $m \approx n$, and decrease as $\sim m^{-1/2}$ along the position $m$-axis. The matrix $B$ is thus effectively upper-triangular and the value of its determinant will be approximately given by the product of its $N$ diagonal elements,

$$D_N := \det B \approx \prod_{n=0}^{N-1} J_n(n).$$

This determinant quickly becomes very small: for $N = \{5, 11, 21, 51\}$, one finds $D_N < \{10^{-2}, 2 \times 10^{-6}, 7 \times 10^{-14}, 10^{-39}\}$. Although this does not negate the existence of the discrete Bessel transform $\{27\}$, it effectively will reduce the numerical stability of computations for larger $N$’s, and will preclude the existence of an $N \to \infty$ limit.

We should remark here that series or integral transforms with kernels involving Bessel functions, are quite distinct from $\{27\}$. The Hankel-$n$ transform kernel is $\sim (pq)^{1/2} J_n(pq)$, between function spaces $f(q)$ and $\tilde{f}(p)$; the two-dimensional Fourier-Bessel series with the drum harmonics has the kernel $\sim J_n(k_{n,m} r) e^{in\phi}$ in polar coordinates, with integer $n, m$ and $k_{n,m}$ being frequencies allowed by circular boundary conditions. Further series detailed...
in Watson’s treatise [8] are the Neumann-\(c\) series with kernel \(\sim J_{n+c}(r)\), the Kapteyn-\(c\) series with \(\sim J_{n+c}(n+c)r\), and the Schlömilch-\(\mu\) series with \(J_{n}(\mu r)\), to transform between functions \(f_n\) with \(n\) integer and \(f(r)\) on continuous \(r\). As can be seen, none has the simple structure of the discrete, \(N \times N\) Bessel matrix in [27].

## 5 Concluding remarks

After trigonometric functions, cylinder—and in particular Bessel—functions can be seen as the most important in mathematical physics. These Bessel functions also have important group-theoretic properties as irreducible representation matrix elements for the Euclidean groups of rotations and translations [9].

The trigonometric functions are the basis for three distinct transforms: the Fourier integral transforms, Fourier series, and the finite Fourier transforms. The three relate through continuous limits and discretizations of the real line \(\mathbb{R}\), the integers \(\mathbb{Z}\), and on \(\mathbb{Z}_N\), the integers modulo \(N\). Such a discretization of the generating functions was used here for \(J_n(m)\) on \(\mathbb{Z}\) to define \(B_n(m)\) on \(\mathbb{Z}_N\). In fact, the position \(m\) need not be integer; as occurs in the finite Fourier case, the \(N\) phases on the circle can be shifted freely. The values of the discrete Bessel functions in the figures can be similarly defined on an \(m\)-line in \([0, N-1]\) or extended beyond. On the other hand, the order \(n\) cannot but be integer, because for non-integer \(n\), \(|J_n(0)|\) is infinite.

The interest on the discrete Bessel functions [7] as proposed in Ref. [2] is that the discretization of decaying, or radial wave propagation, or scattering in two or more dimensions, could be profitably seen as a superposition of Bessel normal modes, whose starting location and decay give physical meaning to those modes. Cylinder functions other than the Bessel functions of the first kind will be investigated elsewhere.

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