AN \( L^p \)- PRIMAL-DUAL WEAK GALERKIN METHOD FOR DIV-CURL SYSTEMS

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Abstract. This paper presents a new \( L^p \)-primal-dual weak Galerkin (PDWG) finite element method for the div-curl system with the normal boundary condition for \( p > 1 \). Two crucial features for the proposed \( L^p \)-PDWG finite element scheme are as follows: (1) it offers an accurate and reliable numerical solution to the div-curl system under the low \( W^{\alpha,p} \)-regularity (\( \alpha > 0 \)) assumption for the exact solution; (2) it offers an effective approximation of the normal harmonic vector fields on domains with complex topology. An optimal order error estimate is established in the \( L^q \)-norm for the primal variable where \( \frac{1}{p} + \frac{1}{q} = 1 \). A series of numerical experiments are presented to demonstrate the performance of the proposed \( L^p \)-PDWG algorithm.

1. Introduction

In this paper we shall develop a new \( L^p \)-primal-dual weak Galerkin (PDWG) methods for the div-curl system with the normal boundary condition. To this end, we consider the model problem: Find a vector field \( \mathbf{u} = \mathbf{u}(x) \) such that

\[
\begin{align*}
\nabla \cdot (\varepsilon \mathbf{u}) &= f, \quad \text{in } \Omega, \\
\nabla \times \mathbf{u} &= g, \quad \text{in } \Omega, \\
\varepsilon \mathbf{u} \cdot n &= \phi_1, \quad \text{on } \Gamma,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^3 \) is an open, bounded and connected polyhedral domain, and \( \Gamma = \partial \Omega \) is the boundary of \( \Omega \). Assume that \( \Gamma \) is the union of a finite number of disjoint surfaces \( \Gamma = \bigcup_{i=0}^L \Gamma_i \) with \( \Gamma_0 \) being the exterior boundary of \( \Omega \) and \( \Gamma_i \) (\( i = 1, \cdots, L \)) being the other connected components with finite surface areas. Note that \( L \) is equal to the number of holes in the domain \( \Omega \) geometrically which is known as the second Betti number of \( \Omega \) or the dimension of the second de Rham cohomology group of \( \Omega \). Assume the coefficient matrix \( \varepsilon = \{\varepsilon_{ij}(x)\}_{3 \times 3} \) is symmetric and uniformly positive definite in \( \Omega \) with \( \varepsilon_{ij} \) (\( i, j = 1, 2, 3 \)) being in \( L^\infty(\Omega) \).

The solution uniqueness for the div-curl system \((1.1a)-(1.1c)\) depends on the topology of the domain \( \Omega \). It is well-known that the solution uniqueness holds true for simply connected \( \Omega \), while the solution is unique up to a normal \( \varepsilon \)-harmonic function in \( W^{\varepsilon,n,p}_0(\Omega) \) defined in

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for the case that the domain $\Omega$ is not simply connected. The dimension of $W_{0}^{n,p}(\Omega)$ is the first Betti number of $\Omega$ which is the rank of the first homology group of $\Omega$.

The div-curl system (1.1a)-(1.1b) arises in many applications in science and engineering such as electromagnetic fields and fluid mechanics. Computational electro-magnetics plays an important role in many areas such as radar, satellite, antenna design, waveguides, optical fibers, medical imaging and design of invisible cloaking devices [22]. In linear magnetic fields, the function $f(x)$ vanishes, $u$ represents the magnetic field intensity and $\varepsilon(x)$ is the inverse of the magnetic permeability tensor. In fluid mechanics fields, the coefficient matrix $\varepsilon(x)$ is diagonal with diagonal entries being the local mass density. In electrostatics fields, $\varepsilon(x)$ is the permittivity matrix.

There have been several numerical methods proposed and analyzed for the div-curl system (1.1a)-(1.1c). A covolume method was developed by the employment of the Voronoi-Delaunay mesh pairs in three dimensional space [27]. [4] developed a least-squares finite element method for two types of boundary value problems. The least-squares method was proposed in [3] for the div-curl problem based on discontinuous elements on nonconvex polyhedral domains. In [5], a classical numerical method was introduced for solving the magnetostatic problem by employing a scalar or vector potential. The control volume method [26] was proposed directly for planar div-curl problems. [15] proposed a discrete duality finite volume method for div-curl problems on almost arbitrary polygonal meshes. A mixed finite element method was introduced in [14] for three dimensional axisymmetric div-curl systems through a dimension reduction technique based on the cylindrical coordinates in simply connected and axisymmetric domains. The mimetic finite difference scheme [6, 24] was introduced for the magnetostatic problems on general polyhedral partitions. The numerical algorithm [28] was designed to construct a finite element basis for the first de Rham cohomology group of the computational domain, which was further used for a numerical approximation of the magnetostatic problem. [32] proposed a weak Galerkin finite element method for the div-curl system with either normal or tangential boundary conditions. Another weak Galerkin scheme was introduced in [23] by using a least-squares approach for the div-curl problem. [25, 13] developed primal-dual weak Galerkin finite element methods for the div-curl system with tangential boundary condition and normal boundary condition respectively and proved that the schemes work well for the exact solution with low-regularity assumptions.

There are two main challenges in the approximation of the div-curl system (1.1a)-(1.1c): (1) the low-regularity of the exact solution $u$ limiting the stability and accuracy of the numerical solutions, and (2) the non-uniqueness of the solution $u$ on domains with complex topology. The later one can be relaxed to certain extent by seeking a particular solution orthogonal to the space of normal $\varepsilon$-harmonic vector space $W_{0}^{n,p}(\Omega)$, but with an immediate obstacle lying in the determination of the space $W_{0}^{n,p}(\Omega)$ or an effective approximation of this space. To address these challenges, we shall devise a new $L^p$ primal-dual weak Galerkin (PDWG) scheme for (1.1a)-(1.1c) by following the framework developed in [13]. It should be noted that the $L^p$-PDWG framework was originated in [9] for convection-diffusion equations. Our $L^p$-PDWG numerical method for (1.1a)-(1.1c) has two prominent features over the existing numerical methods: (1) it offers an effective approximation for the normal $\varepsilon$-harmonic vector space $W_{0}^{n,p}(\Omega)$ regardless of the topology of the domain $\Omega$; and (2) it provides an accurate and reliable numerical solution for the div-curl system (1.1a)-(1.1c) with low $W^{\alpha,p}$-regularity ($\alpha > 0$) assumption for the exact solution $u$. 
The paper is organized as follows. In Section 2, we introduce the notation and derive the weak formulation for the div-curl system \([1.1a]-[1.1c]\). In Section 3, a \(L^p\)-PDWG algorithm for both the div-curl problem and the discrete normal \(\varepsilon\)-harmonic vector fields is proposed. The solution existence and uniqueness for the \(L^p\)-PDWG scheme is discussed in Section 4. The convergence theory for the \(L^p\)-PDWG approximation is established in Section 5. Finally, several test examples are demonstrated to illustrate the performance of the \(L^p\)-PDWG algorithm in Section 6.

2. Weak Formulations

2.1. Notations. We follow the usual notations for Sobolev spaces and norms \([12, 20]\). Let \(D \subset \mathbb{R}^3\) be an open bounded domain with Lipschitz continuous boundary. Denote by \(W^{\text{div},p}(D)\) the closed subspace of \([L^p(D)]^3\) such that \(\nabla \cdot (\varepsilon v) \in L^p(D)\). Denote \(W^{\text{div},p}(D)\) by \(W^{\text{div},p}(\varepsilon I)\) when \(\varepsilon = I\). An analogously, we use \(W^{\text{curl},p}(D)\) to denote the closed subspace of \([L^p(D)]^3\) so that \(\nabla \times v \in [L^p(D)]^3\). Denote by \(W^{\text{curl},p}_0(D)\) the closed subspace with vanishing tangential boundary values, i.e.,

\[
W^{\text{curl},p}_0(D) := \{ v \in W^{\text{curl},p}(D), \ v \times n = 0 \text{ on } \partial D \}.
\]

Denote by \(\langle \cdot, \cdot \rangle_{\Gamma_i}\) the inner product in \(L^2(\Gamma_i)\). We introduce the following Sobolev space

\[
W_\varepsilon(\Omega) = \{ v \in W^{\text{curl},p}_0(\Omega) \cap W^{\text{div},p}(\Omega), \ \nabla \cdot (\varepsilon v) = 0, \ \langle \varepsilon v \cdot n, 1 \rangle_{\Gamma_i} = 0, \ i = 1, \ldots, L \}.
\]

A vector field \(v \in [L^p(\Omega)]^3\) is defined to be \(\varepsilon\)-harmonic in \(\Omega\) if it is \(\varepsilon\)-solenoidal and irrotational in \(\Omega\). Denoted by \(W^{\text{curl},p}_\varepsilon(\Omega)\) the space of normal \(\varepsilon\)-harmonic vector fields that consists of all \(\varepsilon\)-harmonic vector fields satisfying vanishing normal boundary condition, i.e.,

\[
W^{\text{curl},p}_\varepsilon(\Omega) = \{ v \in [L^p(\Omega)]^3 : \ \nabla \cdot v = 0, \ \varepsilon \cdot (\varepsilon v) = 0, \ v \cdot n = 0 \text{ on } \Gamma \}.
\]

Denote by \(W^{\text{curl},p}_\varepsilon(\Omega)\) by \(W^{\text{curl},p}_0(\Omega)\) for \(\varepsilon = I\). Similarly, denoted by \(W^{\text{curl},p}_\varepsilon(\Omega)\) the space of tangential \(\varepsilon\)-harmonic vector fields that consists of all \(\varepsilon\)-harmonic vector fields satisfying vanishing tangential boundary condition, i.e.,

\[
W^{\text{curl},p}_\varepsilon(\Omega) = \{ v \in [L^p(\Omega)]^3 : \ \nabla \times v = 0, \ \varepsilon \cdot (\varepsilon v) = 0, \ v \times n = 0 \text{ on } \Gamma \}.
\]

2.2. A Weak Formulation. Testing \([1.1a]\) by any \(\varphi \in W^{1,p}(\Omega)\) and using the normal boundary condition \([1.1c]\) yields

\[
(u, \varepsilon \nabla \varphi) = \langle \phi_1, \varphi \rangle - (f, \varphi), \quad \forall \varphi \in W^{1,p}(\Omega).
\]

Testing \([1.1b]\) by any \(u \in W^{\text{curl},p}_0(\Omega)\) gives

\[
(u, \nabla \times w) = (g, w), \quad \forall w \in W^{\text{curl},p}_0(\Omega).
\]

Combining with the equations \([2.2]\) and \([2.3]\), we obtain a weak solution \(u \in [L^q(\Omega)]^3\) \((\frac{1}{p} + \frac{1}{q} = 1)\) of the div-curl system with normal boundary condition \([1.1a]-[1.1c]\) satisfying

\[
(u, \varepsilon \nabla \varphi + \nabla \times \psi = (g, \psi) - (f, \varphi) + \langle \phi_1, \varphi \rangle,
\]

for all \(\varphi \in W^{1,p}(\Omega)\) and \(\psi \in W^{0,\text{curl},p}_0(\Omega)\).

As discussed in \([13]\), the solution to the variational problem \([2.4]\) is non-unique in general. The homogeneous version of \([2.4]\) is to seek a \(u \in [L^q(\Omega)]^3(\frac{1}{p} + \frac{1}{q} = 1)\) satisfying

\[
(u, \varepsilon \nabla \varphi + \nabla \times \psi = 0 \quad \forall \varphi \in W^{1,p}(\Omega), \ \forall \psi \in W^{0,\text{curl},p}_0(\Omega).
\]

Note that the solution could be any \(\varepsilon\)-harmonic function in \(W^{\text{curl},p}_0(\Omega)\) which is non-unique provided that the \(\varepsilon\)-harmonic space \(W^{\text{curl},p}_\varepsilon(\Omega)\) has a positive dimension. The solution to the
div-curl system \(1.1a\sim 1.1c\) is unique provided that the solution is \(\varepsilon\)-weighted \(L^2\) orthogonal to \(W_0^{r,n,p}(\Omega)\).

2.3. An Extended Weak Formulation. In this subsection, we slightly modify the weak formulation \(2.5\) to ensure that the solution to the homogeneous version of \(2.4\) is unique.

We first denote by \(W_0^{1,p}(\Omega)\) the subspace of \(W^{1,p}(\Omega)\) with vanishing value on \(\Gamma_0\) and constant values on other connected components of the boundary; i.e.,

\[
W_0^{1,p}(\Omega) = \{ \phi \in W^{1,p}(\Omega) : \phi|_{\Gamma_0} = 0, \phi|_{\Gamma_i} = \alpha_i, \ i = 1, \ldots, L \}. 
\]

Define the following bilinear form:

\[
B(u, s; \varphi, \psi) := (u, \varepsilon \nabla \varphi + \nabla \times \psi) + (\psi, \varepsilon \nabla s).
\]

Now the extended weak formulation for the div-curl system \(1.1a\sim 1.1c\) seeks \((u, s) \in \{L^q(\Omega)\}^3 \times W_0^{1,p}(\Omega)\) satisfying

\[
B(u, s; \varphi, \psi) = F(\varphi, \psi), \quad \forall \varphi \in W^{1,p}(\Omega), \forall \psi \in W_0^{\text{curl},p}(\Omega),
\]

where

\[
F(\varphi, \psi) = (g, \psi) - (f, \varphi) + \langle \phi_1, \varphi \rangle.
\]

The homogeneous dual problem \(2.7\) seeks \((\lambda, q) \in W^{1,p}(\Omega)/\mathbb{R} \times W_0^{\text{curl},p}(\Omega)\) such that

\[
B(v, r; \lambda, q) = 0, \quad \forall v \in [L^p(\Omega)]^3, \forall r \in W_0^{1,p}(\Omega).
\]

It has been proved in [13] that the solution to the homogeneous dual problem \(2.9\) is unique.

3. \(L^p\)-PDWG Scheme

To design a \(L^p\)-PDWG scheme for the div-curl system \(1.1a\sim 1.1c\), we first briefly review the definitions of discrete weak gradient and discrete weak curl [13] and then introduce some finite element spaces, which shall be used in our later algorithm.

Denote by \(T_h\) a finite element partition of the domain \(\Omega\) that consists of shape-regular polyhedra [33]. Denote by \(\mathcal{E}_h\) and \(\mathcal{E}^0_h = \mathcal{E}_h \setminus \partial \Omega\) the set of all faces and the set of all interior faces in \(T_h\) respectively. Let \(h_T\) be the diameter of the element \(T \in T_h\) and \(h = \max_{T \in T_h} h_T\) be the meshsize of the partition \(T_h\).

Let \(T \in T_h\) be a polyhedral domain with boundary \(\partial T\). We define the space of scalar-valued weak functions on \(T\) as follows

\[
W(T) = \{ v = \{v_0, v_b\} : v_0 \in L^p(T), v_b \in L^p(\partial T) \},
\]

where \(v_0\) and \(v_b\) represent the values of \(v\) in the interior and on the boundary of \(T\) respectively.

Similarly, the space of vector-valued weak functions on \(T\) is defined by

\[
V(T) = \{ v = \{v_0, v_b\} : v_0 \in [L^p(T)]^3, v_b \in [L^p(\partial T)]^3 \}.
\]

Let \(P_j(T)\) be the polynomial space on \(T\) with total degree no more than \(j\). Denote by \(n\) an unit outward normal direction on \(\partial T\). For any \(v \in W(T)\), the discrete weak gradient \(\nabla_{w,j,T} v\) is defined as the unique vector-valued polynomial in \([P_j(T)]^3\) such that

\[
(\nabla_{w,j,T} v, \varphi)_T = -(v_0, \nabla \cdot \varphi)_T + \langle v_b, \varphi \cdot n \rangle_{\partial T}, \quad \forall \varphi \in [P_j(T)]^3.
\]

Similarly, for any \(v \in V(T)\), the discrete weak curl \(\nabla_{w,j,T} \times v\) is defined as the unique vector-valued polynomial in \([P_j(T)]^3\) such that

\[
(\nabla_{w,j,T} \times v, \varphi)_T = (v_0, \nabla \times \varphi)_T - \langle v_b \times n, \varphi \rangle_{\partial T}, \quad \forall \varphi \in [P_j(T)]^3.
\]
For a given non-negative integer $k$, the finite element spaces are defined as follows

\[ V_h = \{ v : v|_T \in [P_k(T)]^3 \mid \forall T \in T_h \}, \]
\[ S_h = \{ \{ s_0, s_b \} : s_0|_T \in P_k(T), s_b|_{\partial T} \in P_k(\partial T), \forall T \in T_h, s_0|_{\Gamma_0} = 0, s_b|_{\Gamma} \text{ is a constant} \}, \]
\[ M_h = \{ \{ \varphi_0, \varphi_b \} : \varphi_0|_T \in P_k(T), \varphi_b|_{\partial T} \in P_k(\partial T), \forall T \in T_h, \int_{\Omega} \varphi_0 = 0 \}, \]
\[ W_h = \{ \psi = \{ \psi_0, \psi_b \} : \psi_0|_T \in [P_k(T)]^3, \psi_b|_{\partial T} \in G_k(\partial T), \forall T \in T_h, \psi_b|_{\Gamma} = 0 \}, \]

where $G_k(\partial T) := [P_k(\tau)]^3 \times \mathbf{n}$ is the space of polynomials of degree $k$ in the tangent space of $\partial T$.

Algorithm 1 (L$^p$-PDWG Algorithm). The L$^p$-PDWG finite element method for the div-curl system (1.1a)-(1.1c) seeks a vector field $\eta$ such that

\[ (\nabla \eta|_T)^T = (-\operatorname{curl} \eta|_T)^T, \quad \forall \eta \in S_h. \]

Similarly, for any $q \in W_h$, denote by $\nabla \times q$ the discrete weak curl $\nabla w, k, T \times q$ computed by (3.2) on $T$, i.e.,

\[ (\nabla \times q)|_T = \nabla w, k, T \times (q|_T), \quad \forall q \in W_h. \]

With the discrete weak gradient and discrete weak curl, an approximation of the bilinear form $B(\cdot, \cdot)$ is thus given by

\[ B_h(\psi, r, \varphi, \psi) = (\psi, \varepsilon \nabla \varphi + \nabla \times \psi) + (\psi_0, \varepsilon \nabla \varphi_0, r), \forall (\psi, r, \varphi, \psi) \in V_h \times S_h \times M_h \times W_h. \]

Now we are ready to present the L$^p$-PDWG finite element method for the div-curl system (1.1a)-(1.1c).

**Algorithm 1 (L$^p$-PDWG Algorithm).** The L$^p$-PDWG finite element method for the div-curl system (1.1a)-(1.1c) seeks a vector field $u_h \in V_h$, together with three auxiliary variables $s_h \in S_h$, $\lambda_h \in M_h$, $q_h \in W_h$, such that

\[ (3.3) \quad \left\{ \begin{array}{l}
 s_1(\lambda_h, q_h; \varphi, \psi) + B_h(u_h, s_h; \varphi, \psi) = F(\varphi, \psi), \quad \forall \varphi \in M_h, \psi \in W_h, \\
 -s_2(s_h, r) + B_h(v, r; \lambda_h, q_h) = 0, \quad \forall v \in V_h, r \in S_h.
 \end{array} \right. \]

Here $F(\cdot, \cdot)$ is given in (2.8), and the L$^p$ stabilizer $s_1$ is defined by

\[ s_1(\lambda_h, q_h; \varphi, \psi) = \rho_1 \sum_{T \in T_h} \int_{\partial T} h_T^{1-p} |\lambda_0 - \lambda_b|^{p-1} \operatorname{sgn}(\lambda_0 - \lambda_b)(\varphi_0 - \varphi_b)ds \]
\[ + \rho_2 \sum_{T \in T_h} h_T^{1-p} \int_{\partial T} |q_0 \times n - q_b \times n|^{p-1} \operatorname{sgn}(q_0 \times n - q_b \times n)(\psi_0 \times n - \psi_b \times n)ds, \]

and the L$^2$ stabilizer $s_2$ is defined accordingly in the space $M_h$ as follows

\[ s_2(s_h, r) = \rho_3 \sum_{T \in T_h} h_T^{1-q} \int_{\partial T} |s_0 - s_b|^{q-1} \operatorname{sgn}(s_0 - s_b)(r_0 - r_b)ds, \]

where $p > 1, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\rho_i > 0$ for $i = 1, 2, 3$ are parameters with values at user’s discretion.

The above L$^p$-PDWG scheme (3.3) also offers an approximation of the normal $\varepsilon$-harmonic vector fields $\mathbb{V}_0^{\varepsilon,p}$. Our later theoretical result (see Theorem 5.2) demonstrates that the difference $\eta_h = \mathbb{Q}_h u - u_h$ is sufficiently close to a true normal $\varepsilon$-harmonic vector field $\eta$. Here $\mathbb{Q}_h$ denote the L$^2$ projection operator onto the finite element space $V_h$, and $u_h$ is the solution of (3.4) for the div-curl system (1.1a)-(1.1c). Consequently, a vector field
\[ \eta_h \in V_h \] is said to be a discrete normal \( \varepsilon \)-harmonic function if there exists a vector field \( u \in W^{\text{div},p}(\Omega) \cap W^{\text{curl},p}(\Omega) \) satisfying \( \eta_h = Q_h u - u_h \).

### 4. Solution Existence and Uniqueness

This section is dedicated to the study of solution existence and uniqueness of the \( L^p \)-PDWG scheme \([3.4]\). For simplicity, we assume that \( \varepsilon \) is piecewise constant with respect to the partition \( T_h \) respectively. Note that all the results can be generalized to piecewise smooth \( \varepsilon \) without any difficulty.

We define the following two semi-norms; i.e.,

\begin{align*}
\| (\lambda_h, q_h) \| &= \left( s_1(\lambda_h, q_h; \lambda_h, q_h) \right)^{\frac{1}{2}} , \quad \lambda_h \in M_h , \ q_h \in W_h , \\
\| s_h \| &= \left( s_2(s_h; s_h) \right)^{\frac{1}{2}} , \quad s_h \in S_h .
\end{align*}

Let \( Q_h \) be the projection operator onto the weak finite element space \( S_h \) or \( M_h \) such that

\[ (Q_h w)|_T = \{ Q_0 w|_T , Q_h w|_\partial T \} , \]

where \( Q_0 \) and \( Q_h \) are the \( L^2 \) projection operators onto \( P_k(T) \) and \( P_k(\tau) \) on each face \( \tau \in \partial T \). Similarly, denote by \( Q_0, Q_h \) and \( Q_{\partial}\) the \( L^2 \) projection operators onto \( \{ P_k(T) \}^3 \), \( G_k(\tau) = [P_k(\tau)]^3 \times n_\tau , \) and \( W_h \), respectively.

**Lemma 4.1.** \([38]\) The \( L^2 \) projections \( Q_h \) and \( Q_{\partial} \) satisfy the commutative property

\begin{align*}
(4.3) & \quad \nabla_w (Q_h w) = Q_h (\nabla w) \quad \forall w \in W^{1,p}(T) , \\
(4.4) & \quad \nabla_w \times (Q_h \psi) = Q_h (\nabla \times \psi) \quad \forall \psi \in W^{\text{curl},p}(T) .
\end{align*}

**Theorem 4.2.** \([38]\) (Helmholtz Decomposition) For any vector-valued function \( u \in [L^p(\Omega)]^3 \), there exists a unique \( \psi \in W_0^{\text{curl},p}(\Omega) \), \( \phi \in W^{1,p}(\Omega)/\mathbb{R} \), and \( \eta \in W_0^{\text{div},p}(\Omega) \) such that

\begin{align*}
(4.5) & \quad u = \varepsilon^{-1} \nabla \times \psi + \nabla \phi + \eta , \\
(4.6) & \quad \nabla \cdot (\varepsilon \psi) = 0 , \quad (\varepsilon \psi \cdot n_i)_\Gamma = 0 , \quad i = 1, \ldots, L .
\end{align*}

In addition, there holds

\begin{equation}
(4.7) \quad \| \psi \|_{W_0^{\text{curl},p}(\Omega)} + \| \nabla \phi \|_{L^p(\Omega)} \lesssim \| \varepsilon^{-\frac{1}{2}} u \|_{L^p(\Omega)} .
\end{equation}

**Theorem 4.3.** The kernel of the matrix of the \( L^p \)-PDWG method \([3.4]\) is given by

\[ K_h = \{(v_h, s_h = 0, \lambda_h = 0, q_h = 0) : \ v_h \in V_h \cap W_0^{\text{curl},p}(\Omega) \} . \]

In other words, the kernel of the matrix of the \( L^p \)-PDWG scheme \([3.4]\) is isomorphic to the subspace of \( W_0^{\text{curl},p}(\Omega) \) consisting of harmonic functions that are piecewise polynomial of degree \( k \).

**Proof.** Let \( (u_h^{(1)}, s_h^{(1)}, \lambda_h^{(1)}, q_h^{(1)}) \) and \( (u_h^{(2)}, s_h^{(2)}, \lambda_h^{(2)}, q_h^{(2)}) \) be two different solutions of \([3.4]\). This gives, for \( i = 1, 2 \),

\begin{align*}
(4.8) & \quad s_1(\lambda_h^{(i)}, q_h^{(i)}; \varphi, \psi) + B_h(u_h^{(i)}, s_h^{(i)}; \varphi, \psi) = F(\varphi, \psi) , \quad \forall \varphi \in M_h , \ \psi \in W_h , \\
(4.9) & \quad -s_2(s_h^{(i)}; r) + B_h(v_h, s_h^{(i)}; \lambda_h^{(i)}, q_h^{(i)}) = 0 , \quad \forall v_h \in V_h , \ r \in S_h .
\end{align*}
Given any $j = 1, 2$, taking $(\varphi, \psi) = (\lambda_h^{(j)}, q_h^{(j)})$ in (4.8) and using (4.9), we easily get
\begin{equation}
\begin{aligned}
&\left(\lambda_h^{(j)} \cdot q_h^{(j)}; \lambda_h^{(j)} \cdot q_h^{(j)}\right) + s_2(s_h^{(j)}, s_h^{(j)}) = F(\lambda_h^{(j)}, q_h^{(j)}), \quad \forall i, j = 1, 2.
\end{aligned}
\end{equation}
Consequently, for $j = 1, 2$,
\begin{equation}
\begin{aligned}
&\left(\lambda_h^{(1)} \cdot q_h^{(1)}; \lambda_h^{(1)} \cdot q_h^{(1)}\right) + s_2(s_h^{(1)}, s_h^{(1)}) = s_1(\lambda_h^{(2)} \cdot q_h^{(2)}; \lambda_h^{(2)} \cdot q_h^{(2)}) + s_2(s_h^{(2)}, s_h^{(2)}).
\end{aligned}
\end{equation}
Choosing $j = 1$ in (4.11) and using the Young’s inequality $|AB| \leq \frac{|A|^p}{p} + \frac{|B|^q}{q}$ yields that
\begin{equation}
\begin{aligned}
s_1(\lambda_h^{(1)} \cdot q_h^{(1)}; \lambda_h^{(1)} \cdot q_h^{(1)}) + s_2(s_h^{(1)}, s_h^{(1)}) &\leq \frac{s_1(\lambda_h^{(2)} \cdot q_h^{(2)}; \lambda_h^{(2)} \cdot q_h^{(2)})}{p} + \frac{s_2(s_h^{(1)}, s_h^{(1)})}{q} + \frac{s_2(s_h^{(2)}, s_h^{(2)})}{p},
\end{aligned}
\end{equation}
which leads to
\begin{equation}
\begin{aligned}
s_1(\lambda_h^{(1)} \cdot q_h^{(1)}; \lambda_h^{(1)} \cdot q_h^{(1)}) + s_2(s_h^{(1)}, s_h^{(1)}) &\leq s_1(\lambda_h^{(2)} \cdot q_h^{(2)}; \lambda_h^{(2)} \cdot q_h^{(2)}) + s_2(s_h^{(2)}, s_h^{(2)}).
\end{aligned}
\end{equation}
Similarly, we take $j = 2$ in (4.11) and use the Young’s inequality again to derive
\begin{equation}
\begin{aligned}
s_1(\lambda_h^{(2)} \cdot q_h^{(2)}; \lambda_h^{(2)} \cdot q_h^{(2)}) + s_2(s_h^{(2)}, s_h^{(2)}) &\leq s_1(\lambda_h^{(1)} \cdot q_h^{(1)}; \lambda_h^{(1)} \cdot q_h^{(1)}) + s_2(s_h^{(1)}, s_h^{(1)}).
\end{aligned}
\end{equation}
Combining the last two inequality leads to
\begin{equation}
\begin{aligned}
&\left(\lambda_h^{(1)} \cdot q_h^{(1)}; \lambda_h^{(1)} \cdot q_h^{(1)}\right) + s_2(s_h^{(1)}, s_h^{(1)}) = s_1(\lambda_h^{(2)} \cdot q_h^{(2)}; \lambda_h^{(2)} \cdot q_h^{(2)}) + s_2(s_h^{(2)}, s_h^{(2)}).
\end{aligned}
\end{equation}
Note that for any two real numbers $A$ and $B$, there holds
\begin{equation}
\begin{aligned}
\left| \frac{A + B}{2} \right|^p \leq \frac{|A|^p + |B|^p}{2},
\end{aligned}
\end{equation}
and the equality holds true if and only if $A = B$. This follows that
\begin{equation}
\begin{aligned}
s_1(\lambda_h^{(1)} + \lambda_h^{(2)}; q_h^{(1)} + q_h^{(2)}) + s_2(s_h^{(1)} + s_h^{(2)}) &\leq \frac{1}{2} \left( s_1(\lambda_h^{(1)} \cdot q_h^{(1)}; \lambda_h^{(1)} \cdot q_h^{(1)}) + s_1(\lambda_h^{(2)} \cdot q_h^{(2)}; \lambda_h^{(2)} \cdot q_h^{(2)}) + \frac{1}{2} \left( s_2(s_h^{(1)}, s_h^{(1)}) + s_2(s_h^{(2)}, s_h^{(2)}) \right) \right).
\end{aligned}
\end{equation}
On the other hand, a direct calculation from (4.11)-(4.12) yields
\begin{equation}
\begin{aligned}
&\left(\lambda_h^{(1)} \cdot q_h^{(1)}; \lambda_h^{(1)} \cdot q_h^{(1)}\right) + s_2(s_h^{(1)}, s_h^{(1)}) \quad = \frac{1}{2} \left( s_1(\lambda_h^{(1)} \cdot q_h^{(1)}; \lambda_h^{(1)} \cdot q_h^{(1)}) + s_2(s_h^{(1)}, s_h^{(1)}) + s_1(\lambda_h^{(2)} \cdot q_h^{(2)}; \lambda_h^{(2)} \cdot q_h^{(2)}) + s_2(s_h^{(1)}, s_h^{(2)}) \right) \\
&= \frac{1}{2} \left( s_1(\lambda_h^{(1)} + \lambda_h^{(2)}; q_h^{(1)} + q_h^{(2)}) + s_2(s_h^{(1)} + s_h^{(2)}) \right).
\end{aligned}
\end{equation}
Using the Young’s inequality again, we get
\begin{equation}
\begin{aligned}
&\left(\lambda_h^{(1)} \cdot q_h^{(1)}; \lambda_h^{(1)} \cdot q_h^{(1)}\right) + s_2(s_h^{(1)}, s_h^{(1)}) \quad \leq s_1(\lambda_h^{(1)} + \lambda_h^{(2)}; q_h^{(1)} + q_h^{(2)}) + s_2(s_h^{(1)} + s_h^{(2)}).
\end{aligned}
\end{equation}
Then we conclude from (4.13) and (4.12)

\[
s_1\left(\frac{\lambda_1^h - \lambda_2^h}{2}, \frac{q_1^h + q_2^h}{2}; \frac{\lambda_1^h + \lambda_2^h}{2}, \frac{q_1^h + q_2^h}{2}\right) + s_2\left(\frac{s_1^h + s_2^h}{2}, \frac{s_1^h + s_2^h}{2}\right)
\]


= \(s_1(\lambda_1^h, q_1^h; \lambda_1^h, q_1^h) + s_2(s_1^h, s_1^h) = s_1(\lambda_2^h, q_2^h; \lambda_2^h, q_2^h) + s_2(s_2^h, s_2^h)\).

The above equation holds true if and only if

\[
\lambda_0^{(1)} - \lambda_0^{(2)} = \lambda_0^{(2)} - \lambda_0^{(2)}, \quad \text{on } \partial T;
\]

\[
q_0^{(1)} \times n - q_0^{(2)} \times n = q_0^{(2)} \times n - q_0^{(2)} \times n, \quad \text{on } \partial T;
\]

\[
s_0^{(1)} - s_0^{(1)} = s_0^{(2)} - s_0^{(2)}, \quad \text{on } \partial T;
\]

Denoting \(\epsilon_h = \lambda_0^{(1)} - \lambda_0^{(2)} = \{\epsilon_0, \epsilon_b\}, \epsilon_h = q_0^{(1)} - q_0^{(2)} = \{\epsilon_0, \epsilon_b\}, \epsilon_h = s_0^{(1)} - s_0^{(2)} = \{\epsilon_0, \epsilon_b\}, \)

we have

\[
\epsilon_0 = \epsilon_0, \quad \text{on } \partial T, \quad \epsilon_0 \times n = \epsilon_0 \times n, \quad \text{on } \partial T, \quad \epsilon_0 = \epsilon_0, \quad \text{on } \partial T.
\]

Since \(s_0^{(1)} - s_0^{(1)} = s_0^{(2)} - s_0^{(2)} \) on \(\partial T\), there holds

\[
s_2(s_1^h, r) = s_2(s_2^h, r), \quad \forall r \in S_h,
\]

which, combined with (4.9), gives

\[
B_h(v, r; \lambda_0^{(1)} q_0^{(1)}) = B_h(v, r; \lambda_0^{(2)} q_0^{(2)}), \quad \forall v \in V_h, r \in S_h,
\]

or equivalently,

\[
B_h(v, r; \epsilon_h, e_h) = 0, \quad \forall v \in V_h, r \in S_h,
\]

i.e.,

\[
(e_0, \nabla \cdot \epsilon h) + (v, \nabla \cdot \epsilon h + \nabla \times \epsilon h) = 0, \quad \forall v \in V_h, r \in S_h.
\]

It follows from (4.17) that \(\epsilon_0 \in C(\Omega), \epsilon_0 \in C(\Omega)\) and \(e_0 \in H_0(curl; \Omega)\), which indicates

\[
\nabla \epsilon_0 = \nabla_w \epsilon_h, \quad \nabla \epsilon_0 = \nabla_w \epsilon_h.
\]

Letting \(r = 0\) and varying \(v\) in (4.18), we get

\[
\epsilon_0 = \nabla_w \epsilon h + \nabla_w \times \epsilon h = 0,
\]

which together with (4.19), gives

\[
\epsilon_0 = \nabla \epsilon h + \nabla \times \epsilon h = 0.
\]

Using \(e_0 \in H_0(curl; \Omega)\), we have

\[
\epsilon_0 = \nabla \epsilon h + \nabla \times \epsilon h
\]

which, from (4.20), implies \(\epsilon_0 = 0\), and hence \(\epsilon_0 \equiv 0\) as a function with mean value 0. This further leads to \(\epsilon_0 \equiv 0\). Thus, from (4.20) we have

\[
\nabla \times \epsilon_0 = 0, \quad \text{in } \Omega.
\]

Note that \(\epsilon_0\) satisfies

\[
(\epsilon_0, \nabla \cdot r) = 0, \quad \forall r \in S_h.
\]

This leads to \(\epsilon_0 \in H(div; \Omega)\) and

\[
\nabla \cdot (\epsilon_0) = 0, \quad \langle \epsilon_0 \cdot n_i, 1 \rangle_{\Gamma_i} = 0, i = 1, 2, \cdots L.
\]
This, together with \( \nabla \times e_0 = 0 \) and \( e_0 \in H_0(\text{curl}; \Omega) \), indicates that \( e_0 \equiv 0 \), and further \( e_h = n \times (e_h \times n) = n \times 0 = 0 \).

Using (4.14)-(4.16), we have
\[
s_1(\lambda_h^{(1)}, q_h^{(1)}; \varphi, \psi) = s_1(\lambda_h^{(2)}, q_h^{(2)}; \varphi, \psi),
\]
which yields, together with (4.8),
\[
B_h(u_h^{(1)}, s_h^{(1)}; \varphi, \psi) = B_h(u_h^{(2)}, s_h^{(2)}; \varphi, \psi), \quad \forall \varphi \in M_h, \, \psi \in W_h.
\]
Denote \( e_{u_h} = u_h^{(1)} - u_h^{(2)} \). The above equality is equivalent to
\[
0 = B_h(e_{u_h}, e_h; \varphi, \psi) = (e_{u_h}, \varepsilon \nabla_w \varphi + \nabla_w \times \psi) + (\psi, \varepsilon \nabla_w e_h), \quad \forall \varphi \in M_h, \, \psi \in W_h.
\]
Now we have, from the Helmholtz decomposition (4.3),
\[
e_{u_h} = \varepsilon^{-1} \nabla \times \tilde{\psi} + \nabla \tilde{\phi} + \tilde{\eta},
\]
where \( \tilde{\eta} \in \mathbb{W}^p_0(\Omega) \) and \( \tilde{\psi} \in \mathbb{W}^p_0(\Omega) \) satisfying \( \nabla \cdot (\varepsilon \tilde{\psi}) = 0 \) and \( (\varepsilon \tilde{\psi} \cdot n_i)_{T_i} = 0 \) for \( i = 1, \ldots, L \). It follows from \( e_0 = e_h \) on \( \partial T \) for each element \( T \in T_h \) that \( e_0 \in W^p(\Omega) \). This leads to \( \nabla_w e_h = \nabla e_0 \). If the dimension of \( \mathbb{W}^p_0(\Omega) \) is 0, we have \( \tilde{\eta} = 0 \). Letting the test functions \( \phi \) and \( \psi \) in (4.21) be the \( L^2 \) projections of the corresponding function in the Helmholtz decomposition gives rise to
\[
0 = (e_{u_h}, \varepsilon \nabla_w \tilde{\psi} + \nabla_w \times Q_h \tilde{\eta}) = (e_{u_h}, Q_h \varepsilon \nabla \tilde{\phi} + Q_h \nabla \times \tilde{\psi}) + (\tilde{\psi}, \varepsilon \nabla e_0)
\]
which leads to \( e_{u_h} - \tilde{\eta} = 0 \), i.e., \( e_{u_h} \) is a harmonic function. As a harmonic function in the form of piecewise polynomial of degree \( k \), the first term on the right-hand side of (4.21) is zero for any test functions \( \varphi \in M_h \) and \( \psi \in W_h \), which further implies that \( \nabla_w e_h = 0 \). Using (4.17) gives \( \nabla e_0 = \nabla_w e_h = 0 \). Therefore we obtain \( e_0 \equiv 0 \) and further \( e_h \equiv 0 \).

This completes the proof of the theorem. 

Our main result for the solution existence and uniqueness of the numerical scheme (3.4) is stated as follows.

**Theorem 4.4.** The \( L^p \)-PDWG finite element scheme (3.4) has a unique solution for \( s_h, \lambda_h \) and \( q_h \). The solution \( u_h \) is unique up to a harmonic function \( \eta_h \in \mathbb{W}^p_0(\Omega) \) which is a piecewise polynomial of degree \( k \).

**Remark 4.5.** For the lowest order \( k = 0 \) of the \( L^p \)-PDWG scheme (3.4), any \( \eta_h \) in the kernel \( K_h \) of the matrix of the \( L^p \)-PDWG method is a piecewise constant vector field. \( \eta_h \) is thus continuous across each interior element interface and has vanishing value on the domain boundary along the normal direction. This leads to \( \eta_h \equiv 0 \). Therefore, the \( L^p \)-PDWG finite element scheme (3.4) has a unique solution for \( u_h \) in the case of the lowest order element.
5. $L^p$-Error Analysis for the Primal Variable

In this section, we shall establish the $L^q$ error estimates for primal variable $u_h$ in the $L^p$-PDWG scheme \[(\ref{3.4})\]. Denote the error functions by $e_u = Q_h u - u_h$, $e_s = Q_h s - s_h$, $e_\lambda = Q_h \lambda - \lambda_h$, $e_q = Q_h q - q_h$.

We begin with the study of error equations for the $L^p$-PDWG scheme \[(\ref{3.4})\] developed for the div-curl system \[(\ref{1.1a})-(\ref{1.1c})\].

5.1. Error Equations. For the exact solution $\{u, s, \lambda, q\}$ of the div-curl system, recalling the definition of $B_h(\cdot, \cdot)$ in \[(\ref{3.3})\] and using \[(\ref{3.1})\] and \[(\ref{3.2})\], we have

\[
B_h(Q_h u, Q_h s; \varphi, \psi) = (Q_h u, \varepsilon \nabla \varphi_0 + \nabla \times \psi_0) + (Q_h u, \varepsilon n (\varphi_0 - \varphi_b) + (\psi_0 - \psi_b) \times n) \varphi_h
\]

\[
+ (Q_h u, \varepsilon n (\varphi_0 - \varphi_b) + (\psi_0 - \psi_b) \times n) \varphi_h + (\varphi_1, \varphi_b)_{\partial \Omega}
\]

\[
= (\varphi_1, \varphi_b)_{\partial \Omega} - (f, \varphi_0) + (g, \psi_0) + (u - Q_h u, \varepsilon n (\varphi_0 - \varphi_b) + (\psi_0 - \psi_b) \times n) \varphi_h.
\]

Noticing that $\lambda = 0$ and $q = 0$, then

\[
B_h(e_u, e_s; \varphi, \psi) = (u - Q_h u, \varepsilon n (\varphi_0 - \varphi_b) + (\psi_0 - \psi_b) \times n) \varphi_h.
\]

Similarly, we conclude from the fact $s = 0$, $q = 0$, $\lambda = 0$ that

\[
-s_1 (e_\lambda, e_q) + B_h(v, e_s, \lambda, q) = 0.
\]

The above two equations \[(\ref{5.1})-(\ref{5.2})\] are the error equations for the $L^p$-PDWG scheme \[(\ref{3.4})\], which will be frequently used in our error estimates. 

5.2. Error estimates for the dual variables. Recall that $T_h$ is a shape-regular finite element partition of the domain $\Omega$. For any $T \subset T_h$ and $\nabla w \in L^q(T)$ with $q > 1$, the following trace inequality holds true:

\[
||w||^q_{L^q(T)} \leq Ch^{-1}_T ||w||^q_{L^q(\Omega)} + h^q_T ||\nabla w||^q_{L^q(T)}.
\]

By using the Cauchy-Schwarz inequality and the trace inequality, we get

\[
|\langle w, v \rangle_{\varphi_h}| \leq \left( \sum_{T \in T_h} ||w||^q_{L^q(T)} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} ||v||^p_{L^p(\partial T)} \right)^{\frac{1}{2}}
\]

\[
\leq Ch^{-\frac{1}{2}} ||w||^q_{L^q(T)} + h ||\nabla w||^q_{L^q(T)} \left( \sum_{T \in T_h} ||v||^p_{L^p(\partial T)} \right)^{\frac{1}{2}}.
\]

Now we are ready to present the error estimates for the dual variables.

**Theorem 5.1.** Assume the solution of the div-curl system \[(\ref{1.1a})-(\ref{1.1c})\] satisfies $u \in [W^{k+\theta,q}(\Omega)]^3$ for $\theta \in (1/2, 1]$. For the numerical solution $u_h, s_h, \lambda_h, q_h$ arising from the $L^p$-PDWG scheme \[(\ref{3.4})\], there holds

\[
||e_u|| \leq Ch^{k+\theta} ||\nabla^{k+\theta} u||^q_{L^q(\Omega)},
\]

\[
||e_s|| \leq Ch^{k+\theta} ||\nabla^{k+\theta} u||^q_{L^q(\Omega)}.
\]
By using the Helmholtz decomposition (5.10), we have

\[ s_1(e_\lambda, e_q; e_\lambda, e_q) + s_2(e_\sigma, e_\sigma) = (u - Q_h u, \varepsilon n(e_\lambda,0 - e_\lambda,b) + (e_q,b - e_q,0) \times n)_{E_h}. \]

In light of (5.4) and using the approximation property of \( Q_h \), we get

\[ \| (u - Q_h u, \varepsilon n(e_\lambda,0 - e_\lambda,b) + (e_q,b - e_q,0) \times n)_{E_h} \| \leq Ch^{k+\theta - \frac{1}{2}} \| \nabla^{k+\theta} u \|_{L^p} \left( \sum_{T \in T_h} \left( \| \varepsilon n(e_\lambda,0 - e_\lambda,b) \|_{L^p(T)}^p + \| (e_q,b - e_q,0) \times n \|_{L^p(T)}^p \right) \right)^{\frac{1}{p}} \]

\[ \leq Ch^{k+\theta} \| \nabla^{k+\theta} u \|_{L^q(E_h)} \| (e_\lambda, e_q) \|_{L^p}. \]

Substituting the above inequality into (5.7) and using the Cauchy-Schwarz inequality yields

\[ \| (e_\lambda, e_q) \|_{L^p}^p + \| e_\sigma \|_{L^q}^q \leq C_1 h^{(k+\theta)q} \| \nabla^{k+\theta} u \|_{L^q(E_h)}. \]

Then (5.5)-(5.6) follow directly. \( \square \)

### 5.3. \( L^q \) error estimates for the primal variable \( u_h \).

To derive the \( L^q \)-estimate for the error function \( e_u \), we need employ the Helmholtz decomposition (4.5) for any function \( v \), such that

\[ v = \varepsilon^{-1} \nabla \times \tilde{\psi} + \nabla \tilde{\phi} + \tilde{\eta}, \]

where \( \tilde{\phi} \in W^{1,p}(\Omega) \), \( \tilde{\psi} \in W_0^{curl,p}(\Omega) \), and \( \tilde{\eta} \in W^{n,p}_0(\Omega) \). We assume the \( W^{\alpha,p} \)-regularity holds true for some fixed \( \alpha \in (1/2, 1] \):

\[ \| \nabla \tilde{\phi} \|_{L^p} + \| \tilde{\phi} \|_{L^p} \leq C \| v - \tilde{\eta} \|_{0,p}. \]

The main convergence result of this paper is stated as follows.

**Theorem 5.2.** Let \( u \) be a solution of the div-curl system (1.1a)-(1.1c) such that \( u \in [W^{k+\theta, q}(\Omega)]^3 \) for \( \theta \in (1/2, 1] \). Assume that the Helmholtz decomposition (5.10) has the \( W^{\alpha,p} \)-regularity estimate (5.11). For a numerical solution \( u_h, s_h, \lambda_h, q_h \) arising from \( L^p \)-PDWG scheme (3.4), there exists a harmonic function \( \tilde{\eta} \in W^{n,p}_0(\Omega) \) such that

\[ \| e_u + Q_h u_h - Q_h u \|_{L^q(\Omega)} \leq Ch^{k+\theta + \alpha - 1} \| \nabla^{k+\theta} u \|_{L^q(\Omega)}. \]

**Proof.** Given any function \( v \), let \( \tilde{\phi} \in W^{1,p}(\Omega) \), \( \tilde{\psi} \in W_0^{curl,p}(\Omega) \), and \( \tilde{\eta} \in W^{n,p}_0(\Omega) \) satisfy (5.10). Taking \( \varphi = Q_h \tilde{\phi} \) and \( \psi = Q_h \tilde{\psi} \) in \( B_h(e_u, e_s; \varphi, \psi) \) and using Lemma 4.1 gives

\[ B_h(e_u, e_s; \varphi, \psi) = (e_u, \varepsilon \nabla_w Q_h \tilde{\phi} + \nabla_w \times Q_h \tilde{\psi}) + (Q_0 \tilde{\psi}, \varepsilon \nabla_w e_s) \]

\[ = (e_u, \varepsilon Q_h \nabla \tilde{\phi} + Q_h \nabla \times \tilde{\psi}) + (Q_0 \tilde{\psi}, \varepsilon \nabla_w e_s) \]

\[ = (e_u, \varepsilon \nabla \tilde{\phi} + \nabla \times \tilde{\psi}) + (Q_0 \tilde{\psi}, \varepsilon \nabla_w e_s). \]

By using the Helmholtz decomposition (5.10), we have

\[ B_h(e_u, e_s; \varphi, \psi) = (\varepsilon e_u, v - \tilde{\eta}) + (\varepsilon Q_0 \tilde{\psi}, \nabla_w e_s) \]

\[ = (\varepsilon (e_u - \tilde{\eta}), v - \tilde{\eta}) + (\varepsilon Q_0 \tilde{\psi}, \nabla_w e_s). \]
From the definition of the weak gradient, we have
\[
(\varepsilon Q_0 \tilde{\psi}, \nabla_w e_s) = (\varepsilon Q_0 \tilde{\psi}, \nabla e_{s,0}) + (\varepsilon Q_0 \tilde{\psi} \cdot n, e_{s,b} - e_{s,0}) e_h
\]
\[
= (\varepsilon \tilde{\psi}, \nabla e_{s,0}) + (\varepsilon Q_0 \tilde{\psi} \cdot n, e_{s,b} - e_{s,0}) e_h
\]
\[
= - (\nabla \cdot (\varepsilon \tilde{\psi}), e_{s,0}) + (\varepsilon \tilde{\psi} \cdot n, e_{s,0}) e_h + (\varepsilon Q_0 \tilde{\psi} \cdot n, e_{s,b} - e_{s,0}) e_h
\]
\[
= (\varepsilon \tilde{\psi} \cdot n, e_{s,0} - e_{s,b}) e_h + (\varepsilon Q_0 \tilde{\psi} \cdot n, e_{s,b} - e_{s,0}) e_h
\]
\[
= (\varepsilon \tilde{\psi} - Q_0 \tilde{\psi}) \cdot n, e_{s,0} - e_{s,b}) e_h.
\]

Substituting the above into (5.3) yields
\[
(\varepsilon (e_u - \tilde{\eta}), v - \tilde{\eta}) = B(e_u, e_s; \varphi, \psi) - (\varepsilon \tilde{\psi} \cdot n, e_{s,0} - e_{s,b}) e_h
\]
\[
= (u - Q_h u, \varepsilon n(\varphi_0 - \varphi_h) + (\psi_0 - \psi_h) \times n) e_h - s_1(e_{\lambda}, e_q; \varphi, \psi)
\]
\[
= I_1 + I_2 + I_3,
\]
where we used the first error equation (5.1), $I_i(i = 1, \cdots, 3)$ are defined accordingly.

As to $I_1$, using the same argument as what we did for (5.8), we get
\[
|I_1| \leq Ch^{k+\theta} \|\nabla^k \psi\|_{L^p(\Omega)} \|\psi\|_{L^p(\Omega)}.
\]
As to $I_2$, recalling the definition of $s_1$ and using the Cauchy-Schwarz inequality, we have
\[
|I_2| \leq C \left( \sum_{T \in T_h} h_T^{1-p} \int_{\partial T} |c_{\lambda,0} - e_{\lambda,b}|^{q(p-1)} ds \right)^{\frac{1}{q}} \left( \sum_{T \in T_h} h_T^{1-p} |\varphi_0 - \varphi_h|^p ds \right)^{\frac{1}{p}}
\]
\[
+ C \left( \sum_{T \in T_h} h_T^{1-p} \int_{\partial T} |e_{q,0} \times n - e_{q,b} \times n|^{q(p-1)} ds \right)^{\frac{1}{q}} \left( \sum_{T \in T_h} \int_{\partial T} h_T^{1-p} |\psi_0 \times n - \psi_h \times n|^p ds \right)^{\frac{1}{p}}
\]
\[
\leq C \|e_{\lambda,0} - e_{\lambda,b}\|_{L^q(\Omega)} \|\psi_0 \times n - \psi_h \times n\|_{L^p(\Omega)}.
\]
Similarly, we use (5.4) and the approximation property of $Q_0$ to get that
\[
|I_3| \leq Ch^{k+\frac{\theta}{p}} \|\nabla^k \tilde{\psi}\|_{L^p(\Omega)} \|e_s\| = Ch^\alpha \|\nabla^k \tilde{\psi}\|_{L^p(\Omega)} \|e_s\|.
\]

It is easy to check
\[
\|\varphi, \psi\| \leq Ch^{\alpha-1} (\|\tilde{\varphi}\|_{L^p(\Omega)} + \|\tilde{\psi}\|_{L^p(\Omega)}).
\]

Substituting (5.15), (5.17) into (5.14), and using (5.5), (5.6), (5.11) and (5.18), this gives
\[
|\varepsilon(e_u - \tilde{\eta}, v - \tilde{\eta})| \leq C h^{k+\theta} \|\nabla^k \psi\|_{L^p(\Omega)} \|\psi\| + Ch^\alpha \|\nabla^k \tilde{\psi}\|_{L^p(\Omega)} \|e_s\|
\]
\[
\leq C h^{k+\theta+\alpha-1} \|\nabla^k \psi\|_{L^p(\Omega)} \|v - \tilde{\eta}\|_{L^p(\Omega)}.
\]

It follows that
\[
\varepsilon \|e_s\| \leq C h^{k+\theta+\alpha-1} \|\nabla^k \psi\|_{L^p(\Omega)},
\]
which gives rise to the error estimate (5.12). This completes the proof of the theorem. \qed
6. Numerical Experiments

In this section, we present some numerical examples to test the performance and accuracy of the PDWG method proposed in [5,4]. In our numerical experiments, the computation domain is first partitioned into cubes, and then each cube is divided into 6 tetrahedra of equi-volume. We choose the discontinuous piecewise constant vector fields to approximate the exact solution $u$. That is, the finite element spaces are given as follows:

$$V_h = \{ v : v|_T \in [P_0(T)]^3, \forall T \in T_h \},$$

$$S_h = \{ s : s|_T = \{ s_0, s_b \} \in \{ P_0(T), \Pi^1_{0,1} P_0(F_i) \}, \forall T \in T_h, F_i \in \partial T \},$$

$$M_h = \{ \varphi : \varphi|_T = \{ \varphi_0, \varphi_b \} \in \{ P_0(T), \Pi^1_{0,1} P_0(F_i) \}, \forall T \in T_h, F_i \in \partial T \},$$

$$W_h = \{ \psi : \psi|_T = \{ \psi_0, \psi_b \} \in \{ [P_0(T)]^3, T_0(\partial T) \}, \forall T \in T_h \},$$

where $T_0(\partial T)$ is the the tangent space of $\partial T$ given by

$$T_0(\partial T) = \{ \psi : \psi_{i,j} \in [P_0(F_i)]^3 \times n_{F_i}, F_i \in \partial T, i = 1, 2, 3, 4, j = 1, 2 \}.$$

Here $n_{F_i}$ denotes the outer unit normal vector to face $F_i$.

To solve the system of nonlinear equation (3.4), we adopt an iterative scheme similar to that for the $L^1$ minimization problem in [20]. Specifically, given an approximation $(u_h^m, \lambda_h^m, S_h^m, q_h^m) \in V_h \times M_h \times S_h \times W_h$ at step $m$, the scheme shall compute a new approximate solution $(u_h^{m+1}, \lambda_h^{m+1}, S_h^{m+1}, q_h^{m+1}) \in V_h \times M_h \times S_h \times W_h$ such that

$$s_1(\lambda_h, q_h; \varphi, \psi) + B_h(u_h, s_h; \varphi, \psi) = F(\varphi, \psi), \quad \forall \varphi \in M_h, \psi \in W_h,$$

$$-s_2(s_h, r) + B_h(v, r; \lambda_h, q_h) = 0, \quad \forall v \in V_h, \rho \in S_h,$$

where

$$s_1(\lambda_h, q_h; \varphi, \psi) = \rho_1 \sum_{T \in T_h} h_T^{p-1} \int_{\partial T} (|\lambda_h^m - \lambda_h^m| + \epsilon_0)^{p-2}(\lambda_h^{m+1} - \lambda_h^{m+1})(\varphi_0 - \varphi_b)ds,$$

$$+ \rho_2 \sum_{T \in T_h} h_T^{p-1} \int_{\partial T} (|q_h^m - q_h^m| + \epsilon_0)^{p-2}(q_h^{m+1} - q_h^{m+1})(\psi_0 - \psi_b)ds,$$

and

$$s_2(s_h; r) = \rho_3 \sum_{T \in T_h} h_T^{p-1} \int_{\partial T} (|s_h^m - s_h^m| + \epsilon_0)^{q-2}(s_h^{m+1} - s_h^{m+1})(r_0 - r_b)ds.$$

Here $\epsilon_0$ is a small, but positive constant, and $\rho_i, i \leq 3$ are some positive stabilization parameters.

We would like to point out that although $\rho_i, i \leq 3$ in the algorithm (3.4) could be arbitrary, the iterative scheme (6.1) is not convergent for any positive $\rho_i, i \leq 3$. Our numerical experiments indicate that $\rho_i, i = 1, 2$ should be taken large enough to ensure the convergence of the iterative scheme.

In our experiments, we test various problems in which the exact solution $u$ has different regularities and the computational domain includes convex, non-convex polyhedral regions and cavities. We shall evaluate the errors for both $u_h$ and the auxiliary variables $\lambda_h, s_h, q_h$, including the $L^p$ error for $e_h := u_h - u$ and $e_{Q} := Q_h u - u_h$, and the errors $\|e_h\|_p$ and $\|s_h\|$ defined in (4.1) and (4.2). We test different values of $p > 1$ with $p = 2, 3, 4, 5$. The right-hand side function, the boundary condition are calculated from the exact solution. The coefficient $\epsilon_0$ in (6.1) is taken as $\epsilon = 10^{-6/(p-1)}$, $\rho_3 = 1$, and $\rho_i, i = 1, 2$ are carefully taken according to different problems. We stop our iterative procedure when the maximum error between the $m$-th step and the $(m+1)$-th step reaches the accuracy $10^{-5}$. AN L^p- PRIMAL-DUAL WEAK GALERKIN METHOD FOR DIV-CURL SYSTEMS 13
Example 6.1. In this test, we consider the model problem \([1.1]\) in the domain \(\Omega = (0, 1)^3\) with \(\varepsilon = \text{diag}(3, 2, 1)\). The right-hand side function and the boundary condition are chosen such that the exact solution to this problem is

\[
\mathbf{u}(x, y, z) = \begin{pmatrix}
\sin(\pi x) \cos(\pi y) \\
-\sin(\pi y) \cos(\pi x) \\
0
\end{pmatrix} + \begin{pmatrix}
x \\
y \\
z
\end{pmatrix}.
\]

It is easy to see that \(\mathbf{u} \in [H^1(\Omega)]^3\).

The problem is solved by \([6.1]\) with the coefficients \(\rho_1 = \rho_2 = 1\) for \(p = 2\) and \(\rho_1 = \rho_2 = 9 \times 10^{p-1}\) for \(p \geq 2\). Table 1 illustrates the approximation error and the rate of convergence for the primal variable \(\mathbf{u}_h\) and the auxiliary variables \(\lambda_h, s_h, q_h\) with \(p = 2, \ldots, 5\). We observe a convergence rate of \(O(h)\) for the error \(\|\varepsilon^1 \mathbf{q}_e_h\|_{L^q}\). For the dual variables \(\lambda_h, q_h\), the table suggests a \(p\)-dependence rate of the convergence, i.e., \(O(h)\) for \(p = 2\), \(O(h^{0.55})\) for \(p = 4\) and \(O(h^{0.4})\) for \(p = 5\). Note that the convergence rate of \(\|\langle \varepsilon, \mathbf{e}_q \rangle\|\) is slightly higher than the theoretical result \(O(h^2)\) in \([5.3]\). As for the dual variable \(s_h\), we observe a better convergence rate than the theoretical finding \(O(h)\) given in \([5.6]\), which indicates a superconvergence result.

**Table 1.** Numerical error and rate of convergence for the \(L^p\)-PDWG method for Example 1.

| \(p\) | \(1/h\) | \(\|\varepsilon^1 \mathbf{q}_h\|_{L^q}\) | rate | \(\|\langle \varepsilon, \mathbf{e}_q \rangle\|\) | rate | \(\|s_h\|\) | rate |
|------|--------|------------------|------|------------------|------|--------|------|
| 2    | 4      | 1.52e-01         | -    | 3.27e-02         | -    | 1.52e-03| -    |
| 4    | 8      | 7.67e-02 0.99    | 0.84 | 3.05e-04 2.32    |      |        |      |
| 8    | 16     | 3.82e-02 1.00    | 0.96 | 5.10e-05 2.58    |      |        |      |
| 16   | 32     | 1.91e-02 1.00    | 0.99 | 9.29e-06 2.46    |      |        |      |
| 2    | 4      | 1.73e-01         | -    | 2.84e-01         | -    | 2.74e-03| -    |
| 4    | 8      | 8.65e-02 1.00    | 0.52 | 3.20e-04 2.95    |      |        |      |
| 8    | 16     | 4.27e-02 1.02    | 0.61 | 3.78e-05 3.08    |      |        |      |
| 16   | 32     | 2.17e-02 0.98    | 0.70 | 5.03e-06 2.91    |      |        |      |
| 2    | 4      | 1.97e-01         | -    | 4.26e-01         | -    | 2.30e-03| -    |
| 4    | 8      | 1.02e-01 0.95    | 0.48 | 2.31e-04 3.31    |      |        |      |
| 8    | 16     | 5.51e-02 0.88    | 0.56 | 1.80e-05 3.68    |      |        |      |
| 16   | 32     | 2.92e-02 0.92    | 0.55 | 1.59e-06 3.50    |      |        |      |
| 2    | 4      | 2.13e-01         | -    | 4.16e-01         | -    | 5.79e-04| -    |
| 4    | 8      | 1.21e-01 0.82    | 0.46 | 4.29e-05 3.76    |      |        |      |
| 8    | 16     | 6.38e-02 0.92    | 0.43 | 2.89e-06 3.89    |      |        |      |
| 16   | 32     | 3.20e-02 1.00    | 0.40 | 3.11e-07 3.22    |      |        |      |

Example 6.2. The domain in this test case is the \(L\)-shape domain, which is given by \(\Omega = (0, 1)^3 \setminus \Omega_1\) with \(\Omega_1 = [0, 1] \times [-1, 0] \times [0, 1]\). We take the coefficient \(\varepsilon = \text{diag}(1,1,1)\) and the singular solution in \([H^{2/3-\delta}(\Omega)]^3\):

\[
\mathbf{u} = \nabla \times (0, 0, r^{2/3} \sin(\frac{2}{3} \theta)).
\]

Here \(r = \sqrt{x^2 + y^2}\) and \(\theta = \arctan(y/x) + c\) are the cylindrical coordinates. We take \(c\) such that \(\mathbf{u} \in H(\text{div}) \cap H(\text{curl})\).
We take the iterative scheme (6.1) to solve this paper with the same coefficient choice of $\rho_1, \rho_2$ as Example 1. Numerical error and rate of convergence for the $L^p$-PDWG method are listed in Table 2, from which we observe an optimal convergence rate $\mathcal{O}(h^{3/2})$ for the errors $\|\varepsilon_h\|_{L^q}$ and $\|(e_\lambda, e_q)\|$ for $p = 2$. While, as $p$ increases, the convergence rate for $u_h$ is improved from $\mathcal{O}(h^{3/3})$ to $\mathcal{O}(h)$ (for $p = 5$). Like in Example 1, the numerical convergence for the dual variables is faster than the theory predicted in Theorem 5.1.

Example 6.3. In this example, we test a singular solution in the following vector potential form on a toroidal domain with 2 holes: $\Omega = \left((-1, \frac{3}{2})^2 \setminus \{\Omega_1 \cup \Omega_2\}\right)$ with $\Omega_1 = [-\frac{1}{2}, 0]^2 \times [0, \frac{1}{2}]$ and $\Omega_2 = [\frac{1}{2}, 1] \times [-\frac{1}{2}, 0] \times [0, \frac{1}{2}]$. We take $\epsilon = \text{diag}(1, 1, 1)$ and $u = \nabla \times (0, 0, r_1^{\gamma_1} \sin(2\theta_1) + r_2^{\gamma_2} \sin(2\theta_2))$, where $(r_i, \theta_i), i = 1, 2$ are the cylindrical coordinates centered at a nonconvex corner of the $i$-th hole. That is, $r_1 = \sqrt{x^2 + y^2}, \theta_1 = \arctan(y/x) + c_1, \ r_2 = \sqrt{(x-1)^2 + y^2}, \ \theta_2 = \arctan(y/x) + c_2$.

In our numerical experiments, we choose $\gamma_1 = 1/2, \gamma_2 = 2/3$ such that the vector field is singular near the nonconvex corners of both holes. Note that $u \in H^{5/2-\delta}$ in a neighborhood of the edge $\{x = 0, y = 0\}$, and $u \in H^{7/2-\delta}$ in a neighborhood of the edge $\{x = 1, y = 0\}$.

From Table 3, we observe a convergence rate of $\mathcal{O}(h^{3/2})$ for the error $\|\varepsilon_h\|_{L^q}$, and $\mathcal{O}(h^{5/2})$ for the error $\|(e_\lambda, e_q)\|$ with $p = 2, \ldots, 5$. Again, it seems that the convergence rate for $\|s_h\|$ is better than the theory predicted in Theorem 5.1.
Table 3. Numerical error and rate of convergence for the $L^p$-PDWG method for Example 3.

| $p$ | $1/h$ | $\|\frac{\tau}{h^2} e_h\|_{L^q}$ | rate | $\|(e_\lambda, e_q)\|$ | rate | $\|s_h\|$ | rate |
|-----|-------|---------------------------------|------|--------------------------|------|----------------|------|
| 2   | 1.49e-00 | 3.48e-00 | 5.16e-01 | 0.96 |
| 4   | 1.02e-00 | 0.55 | 2.60e-00 | 0.42 | 1.01e-01 | 1.39 |
| 8   | 6.58e-01 | 0.57 | 1.86e-00 | 0.48 | 1.01e-01 | 1.39 |
| 16  | 4.69e-01 | 0.55 | 1.30e-00 | 0.51 | 3.49e-02 | 1.54 |
|     | 2.17e-00 | – | 6.45e-01 | – |
| 3   | 1.21e-00 | 0.57 | 5.14e-01 | 0.33 | 1.24e-04 | 1.60 |
| 4   | 8.12e-01 | 0.57 | 4.02e-01 | 0.35 | 4.14e-05 | 1.59 |
| 8   | 5.70e-01 | 0.51 | 3.12e-01 | 0.37 | 1.28e-05 | 1.69 |
| 16  | 3.49e-01 | 0.51 | 2.42e-01 | 0.37 | 3.94e-05 | 1.70 |

Example 6.4. In this test, we consider a singular solution in the following vector potential form on a toroidal domain with 1 hole: $\Omega = \left[(-1, \frac{1}{2})^2 \times [0, \frac{1}{2}] \setminus \Omega_1 \right]$ with $\Omega_1 = \left[-\frac{1}{2}, 0\right] \times [0, \frac{1}{2}]^2$. We take $\epsilon = \text{diag}(1,1,1)$ and $u = \nabla \times (0,0,r^\gamma \sin(2\theta))$.

We consider three cases: $\gamma = 5/4, 1, 2/3$, where the regularity of the exact solution ranges from smooth to singular.

The coefficients $\rho_1, \rho_2$ in (6.1) are chosen as following: $\rho_1 = \rho_2 = 1$ for $p = 2$, $\rho_1 = \rho_2 = 3 \times 10^3$ for $p = 3$ and $\rho_1 = \rho_2 = 3 \times 10^4$ for $p \geq 4$. Numerical error and rate of convergence for the $L^p$-PDWG method with different $\gamma$ are listed in Tables 4 and 5. We observe the following result:

- Regular solution, i.e., $\gamma = 5/4$, $\|\frac{\tau}{h^2} e_h\|_{L^q}$ has optimal convergence rate $O(h)$ for all $p \geq 2$, $\|(e_\lambda, e_q)\|$ has optimal convergence rate $O(h)$ for $p = 2$, and $p$-dependency rate when $p \geq 3$, which is slightly better than the theoretical rate $O(h^{\frac{1}{2}})$. As for $s_h$, a superconvergent order still observed in this cases.

- Singular case $\gamma = 1$, where $u \in H^{1-\delta}$ and $u \notin H(\text{curl})$. An asymptotical rate of $O(h^{1-\delta})$ is observed for the error $\|\frac{\tau}{h^2} e_h\|_{L^q}$ with all $p \geq 2$. The convergence behavior for the auxiliary variable are the same as that for the regular case $\gamma = 5/4$.

- Singular case $\gamma = 2/3$, where $u \in H^{\frac{2}{3}-\delta}$ and $u \notin H(\text{curl})$. $\|\frac{\tau}{h^2} e_h\|_{L^q}$ has an optimal convergence rate $O(h^{\frac{2}{3}})$ for all $p \geq 2$, and $\|(e_\lambda, e_q)\|$ shows a rate of $O(h^{0.6})$ for $p = 2$ and $O(h^{\frac{2}{3}})$ for all $p \geq 3$.

Example 6.5. In this test, we reveal some computational results for a test problem where the existence of a harmonic vector field has effect on convergence. We consider the
Table 4. Numerical error and rate of convergence for the $L^p$-PDWG method with $\gamma = 5/4$ for Example 4.

| $p$ | $1/h$ | $\|\tilde{\varepsilon}_h\|_{L^q}$ rate | $\|(c_{\lambda}, c_q)\|$ rate | $\|s_h\|$ rate |
|-----|-------|---------------------------------|-----------------|----------------|
| 2   | 3.96e-01 | -                              | 9.07e-01 -       | 6.51e-02 -     |
| 4   | 2.07e-01 | 0.93                           | 5.01e-01 0.86   | 3.23e-02 1.01  |
| 8   | 1.06e-01 | 0.97                           | 2.67e-01 0.91   | 9.48e-03 1.77  |
| 16  | 5.38e-02 | 0.98                           | 1.39e-01 0.95   | 2.17e-03 2.12  |
| 2   | 4.01e-01 | -                              | 4.14e-01 -       | 4.17e-04 -     |
| 4   | 2.39e-01 | 0.95                           | 2.88e-01 0.53   | 1.05e-04 1.99  |
| 8   | 1.27e-01 | 0.91                           | 1.94e-01 0.57   | 2.20e-05 2.25  |
| 16  | 6.65e-02 | 0.94                           | 1.25e-01 0.63   | 4.23e-06 2.38  |
| 2   | 4.92e-01 | -                              | 5.92e-01 -       | 5.23e-04 -     |
| 4   | 2.67e-01 | 0.88                           | 4.29e-01 0.46   | 1.05e-04 2.31  |
| 8   | 1.41e-01 | 0.92                           | 3.04e-01 0.50   | 1.78e-05 2.57  |
| 16  | 7.23e-02 | 0.97                           | 2.16e-01 0.50   | 2.74e-06 2.70  |
| 2   | 5.33e-01 | -                              | 8.43e-01 -       | 3.56e-03 -     |
| 4   | 2.93e-01 | 0.86                           | 6.50e-01 0.38   | 6.15e-04 2.53  |
| 8   | 1.50e-01 | 0.97                           | 4.98e-01 0.39   | 1.04e-04 2.56  |
| 16  | 7.55e-02 | 0.99                           | 3.80e-01 0.39   | 1.54e-05 2.75  |

Table 5. Numerical error and rate of convergence for the $L^p$-PDWG method with $\gamma = 1$ for Example 4.

| $p$ | $1/h$ | $\|\tilde{\varepsilon}_h\|_{L^q}$ rate | $\|(c_{\lambda}, c_q)\|$ rate | $\|s_h\|$ rate |
|-----|-------|---------------------------------|-----------------|----------------|
| 2   | 5.33e-01 | -                              | 1.22e-00 -       | 1.14e-01 -     |
| 4   | 3.01e-01 | 0.83                           | 7.27e-01 0.74   | 5.65e-02 1.01  |
| 8   | 1.63e-01 | 0.88                           | 4.15e-01 0.81   | 1.79e-02 1.66  |
| 16  | 8.86e-02 | 0.88                           | 2.30e-01 0.85   | 4.62e-03 1.96  |
| 2   | 6.02e-01 | -                              | 5.84e-01 -       | 1.22e-03 -     |
| 4   | 3.29e-01 | 0.87                           | 3.98e-01 0.55   | 2.55e-04 2.26  |
| 8   | 1.87e-01 | 0.82                           | 2.78e-01 0.52   | 5.63e-05 2.18  |
| 16  | 1.03e-01 | 0.86                           | 1.88e-01 0.56   | 1.22e-05 2.21  |
| 2   | 6.50e-01 | -                              | 7.57e-01 -       | 1.26e-03 -     |
| 4   | 3.80e-01 | 0.77                           | 5.68e-01 0.41   | 2.52e-04 2.32  |
| 8   | 2.14e-01 | 0.83                           | 4.19e-01 0.44   | 4.91e-05 2.36  |
| 16  | 1.13e-01 | 0.93                           | 3.07e-01 0.45   | 9.16e-06 2.42  |
| 2   | 6.88e-01 | -                              | 8.31e-01 -       | 1.60e-03 -     |
| 4   | 4.14e-01 | 0.73                           | 6.57e-01 0.34   | 3.09e-04 2.38  |
| 8   | 2.26e-01 | 0.87                           | 5.16e-01 0.35   | 5.79e-05 2.41  |
| 16  | 1.19e-01 | 0.93                           | 4.04e-01 0.35   | 1.01e-05 2.52  |

problem on a toroidal domain with 2 holes, which is the same as that in Example 6.3. We
Table 6. Numerical error and rate of convergence for the $L^p$-PDWG method with $\gamma = 2/3$ for Example 4.

| p  | $1/h$ | $\|\varepsilon_h\|_{L^p}$ | rate | $\|(e_{\lambda}, e_q)\|$ | rate | $\|s_h\|$ | rate |
|----|-------|--------------------------|------|-----------------|------|----------|------|
| 2  | 4     | 5.77e-01                 | 0.62 | 1.45e-00        | 0.52 | 1.38e-01 | 1.01 |
| 8  | 4     | 3.62e-01                 | 0.68 | 9.67e-01        | 0.59 | 4.92e-02 | 1.49 |
| 16 | 4     | 2.29e-01                 | 0.66 | 6.26e-01        | 0.63 | 1.55e-02 | 1.67 |
| 2  | 4     | 8.87e-01                 | –    | 2.09e-00        | –    | 2.77e-01 | –    |
| 4  | 3.94e-01 | 0.64 | 4.37e-01        | 0.40 | 1.39e-04        | 1.82 |
| 16 | 2.34e-01 | 0.67 | 6.26e-01        | 0.63 | 1.55e-02        | 1.67 |
| 2  | 4     | 8.04e-01                 | –    | 2.07e-00        | –    | 2.77e-01 | –    |
| 4  | 7.10e-01 | 0.59 | 8.64e-01        | 0.30 | 1.13e-03        | 1.98 |
| 8  | 4.63e-01 | 0.62 | 6.93e-01        | 0.32 | 2.86e-04        | 1.98 |
| 16 | 2.90e-01 | 0.67 | 5.53e-01        | 0.32 | 7.19e-05        | 1.99 |
| 2  | 8     | 3.62e-01                 | 0.68 | 9.67e-01        | 0.59 | 4.92e-02 | 1.49 |
| 4  | 3.94e-01 | 0.64 | 4.37e-01        | 0.40 | 1.39e-04        | 1.82 |
| 8  | 2.34e-01 | 0.67 | 6.26e-01        | 0.63 | 1.55e-02        | 1.67 |
| 2  | 4     | 8.04e-01                 | –    | 2.07e-00        | –    | 2.77e-01 | –    |
| 4  | 7.10e-01 | 0.59 | 8.64e-01        | 0.30 | 1.13e-03        | 1.98 |
| 8  | 4.63e-01 | 0.62 | 6.93e-01        | 0.32 | 2.86e-04        | 1.98 |
| 16 | 2.90e-01 | 0.67 | 5.53e-01        | 0.32 | 7.19e-05        | 1.99 |

The coefficients $\rho_1, \rho_2$ in (6.1) are chosen as following: $\rho_1 = \rho_2 = 1$ for $p = 2$, $\rho_1 = \rho_2 = 5 \times 10^4$ for $p \geq 3$. The plot of the vector field $\eta_h$ is provided in Figure [7] and the errors and rates of convergence of the $L^p$-PDWG method for the primal variable and the dual variables are given in Table [7]. As indicated by Theorem [5.2], the numerical solution $u_h$ approximates the exact solution $u$, up to a harmonic field. As we may observe from Figure [1], the vector field $\eta_h$ is an approximate harmonic field with normal boundary condition. Furthermore, due to the presence of the harmonic field vector, the error $\eta_h$ or $e_h$ may not exhibit a convergence. Our numerical result in Table [7] verifies this point. We do not observe any convergence for the vector field $u$. It is noteworthy that although the vector field $u_h$ is not convergent to $u$ while our iterative algorithm is still convergent. We list in Table [7] that the iterative number used in the iterative procedure. As for the dual variable $\lambda_h, q_h$, we observe a rate of $O(h^\frac{3}{4})$ for $p \geq 3$. The numerical performance is in consistency with our theory as established in Theorem [5.1] for the convergence of $e_\lambda, e_q$. The convergence rate for $s_h$ is still better than the one given in (5.6).

**Example 6.6.** In this test, we consider the problem on a toroidal domain with 1 holes, which is the same as that in Example 6.4. We take $\epsilon = \text{diag}(1, 1, 1)$ and

$$u = \nabla \times (0, 0, r_{11}^1 \sin(\theta_1) + r_{12}^2 \sin(\theta_2)) + \beta \begin{pmatrix} e^y \sin(z) \\ e^x \sin(z) \\ z \end{pmatrix}$$

with $(\gamma_1, \gamma_2, \beta) = \left(\frac{4}{5}, \frac{2}{3}, \frac{1}{40}\right)$.

The coefficients $\rho_1, \rho_2$ in (6.1) are chosen as following: $\rho_1 = \rho_2 = 1$ for $p = 2$, $\rho_1 = \rho_2 = 5 \times 10^4$ for $p \geq 3$. The plot of the vector field $\eta_h$ is provided in Figure [1] and the errors and rates of convergence of the $L^p$-PDWG method for the primal variable and the dual variables are given in Table [7]. As indicated by Theorem [5.2], the numerical solution $u_h$ approximates the exact solution $u$, up to a harmonic field. As we may observe from Figure [1], the vector field $\eta_h$ is an approximate harmonic field with normal boundary condition. Furthermore, due to the presence of the harmonic field vector, the error $\eta_h$ or $e_h$ may not exhibit a convergence. Our numerical result in Table [7] verifies this point. We do not observe any convergence for the vector field $u$. It is noteworthy that although the vector field $u_h$ is not convergent to $u$ while our iterative algorithm is still convergent. We list in Table [7] that the iterative number used in the iterative procedure. As for the dual variable $\lambda_h, q_h$, we observe a rate of $O(h^\frac{3}{4})$ for $p \geq 3$. The numerical performance is in consistency with our theory as established in Theorem [5.1] for the convergence of $e_\lambda, e_q$. The convergence rate for $s_h$ is still better than the one given in (5.6).
Figure 1. Plot of the vector fields $\eta_h$ calculated by the $L^p$-PDWG method of Example 5 with different values of $p$.

with $(\gamma, \beta) = (\frac{2}{3}, \frac{1}{8})$.

We take the iterative parameters as those in Example 6.5 and plot the vector field $\eta_h$ in Figure 1 and present in Table 8 the errors and rates of convergence for the primal variable and the dual variables approximation. Just the same as that in Example 6.5, the numerical results do not demonstrate any convergence for the vector field $u$, while show a rate of $O(h^{\frac{2}{p}})$ for $\|(e_{\lambda}, e_q)\|$ and a superconvergence rate $O(h^2)$ for $\|s_h\|$ when $p \geq 3$. Again we observe that the iterative scheme is still convergent and the vector field $\eta_h$ is an approximate harmonic field with normal boundary condition.

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Table 7. Numerical error and rate of convergence for the $L^p$-PDWG method for Example 5.

| $p$ | $1/h$ | $\|\tilde{\varepsilon}e_h\|_{L^p}$ rate | $\|\tilde{\varepsilon}_{\eta h}\|_{L^p}$ rate | $\|(e_{\lambda}, e_q)\|$ rate | $\|s_{h}\|$ rate | It. |
|-----|-------|----------------------------------|----------------------------------|---------------------------------|-----------------|-----|
| 2   | 2.60e-01 | 1.90e-01 – 3.79e-01 | 5.83e-02 | 1 |
| 4   | 2.03e-01 | 0.36 1.69e-01 0.17 2.55e-01 0.57 2.93e-02 0.99 1 |
| 8   | 1.70e-01 | 0.26 1.54e-01 0.13 1.67e-01 0.61 1.03e-02 1.51 1 |
| 16  | 1.53e-01 | 0.15 1.46e-01 0.08 1.07e-01 0.64 3.26e-03 1.66 1 |
| 2   | 2.78e-01 | 2.16e-01 – 3.35e-01 | 1.52e-05 | 14 |
| 4   | 2.28e-01 | 0.36 2.02e-01 0.09 9.93e-02 0.44 4.98e-06 1.60 15 |
| 8   | 1.98e-01 | 0.20 1.85e-01 0.11 7.11e-02 0.48 1.20e-06 2.06 18 |
| 16  | 1.80e-01 | 0.14 1.76e-01 0.09 5.06e-02 0.49 2.88e-07 2.06 19 |
| 2   | 3.12e-01 | 2.55e-01 – 3.9e-01 | 1.60e-04 | 0 18 |
| 4   | 2.62e-01 | 0.25 2.40e-01 0.09 3.12e-01 0.35 4.54e-05 1.82 28 |
| 8   | 2.25e-01 | 0.22 2.15e-01 0.15 2.45e-01 0.35 1.08e-05 2.07 28 |
| 16  | 1.99e-01 | 0.18 1.95e-01 0.14 1.93e-01 0.34 2.60e-06 2.05 26 |
| 2   | 3.42e-01 | 2.87e-01 – 6.12e-01 | 8.62e-04 | 29 |
| 4   | 2.84e-01 | 0.27 2.63e-01 0.13 5.05e-02 0.28 2.32e-04 1.89 23 |
| 8   | 2.37e-01 | 0.26 2.28e-01 0.20 4.19e-01 0.27 5.59e-05 2.06 24 |
| 16  | 2.08e-01 | 0.19 2.04e-01 0.16 3.48e-01 0.27 1.31e-05 2.09 30 |

Table 8. Numerical error and rate of convergence for the $L^p$-PDWG method for Example 6.

| $p$ | $1/h$ | $\|\tilde{\varepsilon}e_h\|_{L^p}$ rate | $\|\tilde{\varepsilon}_{\eta h}\|_{L^p}$ rate | $\|(e_{\lambda}, e_q)\|$ rate | $\|s_{h}\|$ rate | It. |
|-----|-------|----------------------------------|----------------------------------|---------------------------------|-----------------|-----|
| 2   | 2.18e-01 | 1.61e-01 – 2.98e-01 | 5.18e-02 | 1 |
| 4   | 1.75e-01 | 0.32 1.45e-01 0.15 2.16e-01 0.46 2.67e-02 0.95 1 |
| 8   | 1.47e-01 | 0.25 1.33e-01 0.13 1.47e-01 0.56 9.49e-03 1.50 1 |
| 16  | 1.33e-01 | 0.15 1.26e-01 0.07 9.61e-02 0.61 3.04e-03 1.64 1 |
| 2   | 2.14e-01 | 1.67e-01 – 1.14e-01 | 1.19e-05 | 14 |
| 4   | 1.81e-01 | 0.24 1.60e-01 0.06 8.85e-02 0.37 4.29e-06 1.48 16 |
| 8   | 1.58e-01 | 0.19 1.49e-01 0.10 6.51e-02 0.44 1.05e-06 2.03 18 |
| 16  | 1.45e-01 | 0.13 1.41e-01 0.08 4.72e-02 0.47 2.58e-07 2.03 19 |
| 2   | 2.28e-01 | 1.85e-01 – 3.56e-01 | 1.27e-04 | 18 |
| 4   | 1.99e-01 | 0.20 1.81e-01 0.03 2.89e-01 0.30 3.89e-05 1.7 30 |
| 8   | 1.73e-01 | 0.20 1.66e-01 0.13 2.30e-01 0.33 9.44e-06 2.04 28 |
| 16  | 1.54e-01 | 0.17 1.51e-01 0.13 1.84e-01 0.33 2.33e-06 2.02 26 |
| 2   | 2.42e-01 | 2.01e-01 – 5.63e-01 | 7.06e-04 | 18 |
| 5   | 2.11e-01 | 0.20 1.94e-01 0.05 4.75e-01 0.24 1.99e-04 1.83 24 |
| 8   | 1.79e-01 | 0.23 1.72e-01 0.17 3.99e-01 0.25 4.86e-05 2.03 24 |
| 16  | 1.58e-01 | 0.18 1.55e-01 0.15 3.35e-01 0.26 1.16e-05 2.06 31 |

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Figure 2. Plot of the vector fields $\eta_h$ calculated by the $L^p$-PDWG method of Example 5 with different values of $p$.

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