Resilience of Complex Networks

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Abstract—This article determines and characterizes the minimal number of actuators needed to ensure structural controllability of a linear system under structural alterations that can sever the connection between any two states. We assume that initially the system is structurally controllable with respect to a given set of controls, and propose an efficient system-synthesis mechanism to find the minimal number of additional actuators required for resilience of the system w.r.t such structural changes. The effectiveness of this approach is demonstrated by using standard IEEE power networks.

I. INTRODUCTION

Consider a controlled linear system

\[ \dot{x}(t) = Ax(t) + Bu(t), \]

where \( x(t) \in \mathbb{R}^d \) are the states and \( u(t) \in \mathbb{R}^m \) are the control actions at time \( t \), and \( A \in \mathbb{R}^{d \times d} \) and \( B \in \mathbb{R}^{d \times m} \) are the given state and control matrices, respectively. We assume that (I.1) is controllable, and let it under structural alterations. These alterations may be of different kinds, e.g.,

(i) connections between certain states may be severed,
(ii) connections between some of the control inputs and some of the states may be severed, or
(iii) some of the control inputs may become dysfunctional.

Structural changes of the type (i) reflect in the system (I.1) in the form of certain non-zero entries of \( A \) being set to 0, (ii) reflect as certain elements of the control matrix \( B \) being set to 0, and (iii) reflect as certain columns of \( B \) being set to 0.

The structural alterations (i)-(ii)-(iii) above may be consequences of natural causes such as ageing/malfunctioning of system components [2], or due to malicious external attacks that are designed to adversely affect normal operations of the systems [15], [14], [11], [3].

Against the backdrop of the possibility of (I.1) undergoing structural changes of the types (i)-(ii)-(iii), it is natural to ask whether certain fundamental system-theoretic properties of the altered system are preserved. For instance: What is the structure of \( B \) such that if a certain number of actuators in (I.1) fail, then the resulting system is still controllable? Moving one step further, is it possible to identify conditions on \( A \) together with a class of structural change of type (i) such that the altered system

\[ \dot{x}(t) = A'x(t) + Bu(t), \]

is still controllable? Such questions, in general, turn out to be difficult from a complexity-theoretic standpoint [8], typically requiring a large number of computations (depending on the size of the system) to be performed to arrive at the answers.

In this article we study the simplest of such problems:

Given \( A \), identify a suitably “minimal” \( B \) such that

\[ (I.1) \text{ is controllable even after the connection } \ (P) \text{ between any two system states is severed.} \]

The problem \( (P) \) may look deceptively simple, but it is a computationally difficult problem: indeed, it can be recast in terms of a well-known hard combinatorial problem — see Remarks II.1, II.2, and III.5 for details.

We will resort to structural systems theory to solve \( (P) \). Structural systems theory deals with system-theoretic properties that depend on the sparsity pattern of the interconnections between the system states and control inputs. More precisely, the locations of zeroes in the system and control matrices of (I.1) provide crucial information about controllability and other system-theoretic properties. This approach turns out to be very useful in the context of \( (P) \) since it is naturally fine-tuned to observing whether the entries of \( A \) and \( B \) are zeros.

The literature on structural system theory is comprehensive: The key concepts have been explored in several articles, e.g., [9], [16], [17], [18], [13], [20], [19]. Over the past three decades, several verification results have been proposed for \( (P) \); e.g., in [21] the authors explored the impact of directed link failures on structural controllability of the system, [22] analysed the connection between controllability and standard network parameters such as topological transitivity and degree, etc. Our problem \( (P) \) is closely related to the problems treated in [5] and [7], our work differs from others in the sense that we do not restrict to a special class of systems but are interested in solving \( (P) \) a general class of systems by using an efficient system-synthesis mechanism; see Remark (II.3) for a detailed discussion. To the best of our knowledge, our approach to solve \( (P) \) is novel, and the advantage of our system-synthesis mechanism lies in that it can be effectively adapted to solve other similar problems. We provide illustrations of our approach on standard benchmark IEEE power networks to establish its effectiveness.

The rest of the article is organised as follows: §II reviews certain concepts and results from discrete mathematics that will be used in this article. The precise problem statement and our main results are presented in §III. An illustrative example is presented in §IV. We conclude with a summary of this article and a set of future directions in §V.

For us controllability of (I.1) is equivalent to the rank of the matrix

\[ \begin{pmatrix} B & AB & \cdots & A^{d-1}B \end{pmatrix} \] being \( d \).
II. Background

Structural system theory starts with the representation of (I.1) as a directed graph \( G(A, B) \): Let \( V_A = \{v_1, v_2, \ldots, v_d\} \), and \( V_B = \{r_1, r_2, \ldots, r_m\} \) be the state vertices and control/input vertices corresponding to the states \( x(t) \in \mathbb{R}^d \) and the control \( u(t) \in \mathbb{R}^m \) of the system (I.1). Similarly, let \( E_A = \{(v_j, v_i) | a_{ij} \neq 0\} \) and \( E_B = \{(r_j, u_i) | b_{ij} \neq 0\} \) where \( a_{ij} \) and \( b_{ij} \) are the elements of the matrix \( A \) and \( B \). The directed graph \( G(A, B) \) is represented as \((V, E)\), where \( V = V_A \cup V_B \) and \( E = E_A \cup E_B \) where \( \square \) represents the disjoint union. In \( G(A, B) \), \( E_A \) symbolizes the set of edges between the state vertices, and \( E_B \) symbolizes the set of edges from the control vertices to the state vertices. In the similar manner, we can define \( G(A) = (V_A, E_A) \) as a directed graph which considers the state vertices and the edges in between them.

A directed graph \( G_1 = (V_s, E_s) \) with \( V_s \subset V_A \) and \( E_s \subset E_A \) is called a subgraph of \( G(A) \). When \( U \subset V(G(A)) \), the induced subgraph \( G[U] \) consists of \( U \) and all the edges whose endpoints are contained in \( U \). A sequence of edges \( \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\} \) where \( (v_i, v_j) \in E_A \) with all the vertices distinct, is called a directed path from \( v_1 \) to \( v_k \). The directed graph \( G(A) \) is said to be strongly connected if there exists a directed path between every pair of vertices in it. A strongly connected component (SCC) of \( G(A) \) is a maximal subgraph such that for every \( v, w \in V_A \) in the subgraph there exist a directed path from \( v \) to \( w \) and a directed path from \( w \) to \( v \). A Directed acyclic graph (DAG) of \( G(A) \) represents each SCC as a node and a directed edge exists between two nodes iff there exists a directed edge connecting the corresponding SCCs in \( G(A) \). The DAG associated with \( G(A) \) can be efficiently generated in \( O(|V_A| + |E_A|) \). For a set \( X \subset V_A \), the set of edges of \( G(A) \) entering \( X \) is termed as incut of \( X \). Similarly the set of edges of \( G(A) \) leaving \( X \) is called outcut of \( X \). The size of the incut and outcut associated with a set of vertices are denoted by \( d^+(X) \) and \( d^-(X) \), and is termed as in-degree and out-degree of \( X \) denoted by \( d^+(X) \) and \( d^-(X) \). \( \Delta^+(G(A)) \) and \( \Delta^-(G(A)) \) represents the maximum out-degree and in-degree of \( G(A) \). Let \( v \in V_A \), the in-degree \( d^-(v) \) is the number of edges terminating at \( v \). The out-degree \( d^+(v) \) is the number of edges leaving \( v \).

The directed graph \( G(A) \) can be represented by a undirected bipartite graph in the following standard fashion: \( H(A) := (V_A^1 \cup V_A^2, \Gamma) \) where \( V_A^1 := \{v_1^1, v_2^1, \ldots, v_d^1\} \), \( V_A^2 := \{v_1^2, v_2^2, \ldots, v_d^2\} \), and \( \Gamma = \{(v_i^1, v_i^2) | a_{ij} \neq 0\} \) as portrayed in Fig. 1.

A matching \( M \) in \( H(A) \) is a subset of edges in \( \Gamma \) that do not share vertices. A maximum matching \( M \) in \( H(A) \) is defined as a matching \( M \) that has largest number of edges among all possible matchings. A vertex is said to be matched if it belongs to an edge in the matching \( M \); otherwise, it is unmatched. A matching \( M \) in \( H(A) \) is said to be perfect if all the vertices in \( H(A) \) are matched. A maximum matching \( M \) can be found in \( H(A) \) in polynomial time \( O(\sqrt{|V_A^1| \cdot |V_A^2|}) \). Note that a maximum matching \( M \) may not be unique.

Remark II.1. The problem of finding the family of all maximum matchings in a digraph is a sharp P-complete problem [10], which is extremely difficult to solve.

Remark II.2. We note that (P) can be transformed into a set-cover problem [7], [8], which is known to be NP-hard. This clearly depicts the complexity associated with solving (P).

Fig. 1: A digraph and its bipartite representation.

For the directed graph \( G(A, B) \) corresponding to the system (I.1)

- A vertex \( v \) is said to be accessible from the control vertices if there exists a directed path terminating at \( v \) starting from at least one of the control vertices; otherwise it is said to be inaccessible from the control vertices.
- For a subset \( S \subset V_A \), the neighbourhood of \( S \) is the set \( \{N^+(S) = v_j | (v_j, v_i) \in E_A \cup E_B, v_i \in S\} \). Each vertex in \( N^+(S) \) is termed as an in-neighbour of \( S \). The directed graph \( G(A, B) \) is said to have a dilation if there exists a set \( S \subset V_A \) such that \( |N^+(S)| < |S| \).

A fundamental connection between the system theoretic property of structural controllability and certain structural properties of \( G(A, B) \) is given by

Theorem II.4 ([6]). The following are equivalent:

\begin{enumerate}[(a)]
  \item The pair \((A, B)\) is structurally controllable.
  \item The directed graph \( G(A, B) \) derived from (I.1) as described in §II has all the state vertices accessible from the control vertices, and \( G(A, B) \) has no dilation.
\end{enumerate}

Definition 1: Given a \( \delta \in \{0, 1\}^d \), let \( R := \{v_i : v_i \in V(G(A)) \text{ and } \delta_i = 1\} \) then, \( R \) is termed the root set and \( v_i \in R \) is called a root vertex. Those directed edges terminating at one of the root vertices are termed as root edges.

The following more recent structural result will also be needed in the sequel:
Theorem II.5 ([16]). Let \( G(A) \) be the state digraph and \( H(A) \) be the associated bipartite graph, then the following statements are equivalent:

(a) The set \( R \) is a dedicated input configuration.
(b) There exists a subset \( \mathcal{U}(M) \subset R \) corresponding to the set of unmatched vertices of some maximum matching \( M \) of \( H(A) \), and a subset \( A \subset R \) consisting of one state variable from SCC of \( G(A) \) that has no incoming edge.

A root set \( R \) obtained via Theorem II.5 ensures structural controllability of \( G(A) \).

III. Main Results

We catalogue some important notions specific to digraphs:

Definition 2: The minimum number of non-root edges whose removal makes \( G \) not structurally controllable is referred as the edge-controllability index of the digraph \( G \), and is denoted by \( ec_R(G) \) w.r.t root set \( R \). The digraph \( G \) is said to be \( k \)-edge-controllable if its edge-controllability index is at most \( k \).

Let the assume that a digraph \( G = (V, E) \) is structurally controllable w.r.t \( R \) then we introduce the notion of critical edges in the digraph.

Definition 3: Given a digraph \( G = (V, E) \) and a root set \( R \), an edge \((v_i, v_j) \in E \) is said to be critical/sensitive if the digraph \( G_{v_i} \) obtained by deleting edge \((v_i, v_j) \) is not structurally controllable with respect to \( R \).

For each edge \( e \in E(G) \), the criticality of \( e \) is examined by analysing structural controllability of digraph obtained by deleting \( e \) from \( G \). Therefore the problem of finding critical edges can be achieved in polynomial time as it is equivalent to analysing structural controllability of \([E]\) digraphs [12]. In view of Theorem II.4, deletion of a critical edge creates either input inaccessibility or dilation or both with respect to \( R \).

In other words, removal of a critical edge \( e \) creates either of the two sets:

(a) An SCC \( X_e \) with in-degree 0, i.e., \( d^-(X_e) = 0 \). As \( X_e \cap R = \emptyset \), every vertex \( v \in X_e \) is inaccessible from \( R \). 
(b) A minimal set \( S_e \) such that \( |S_e| > |N^-(S_e)| \) and \( S_e \cap R = \emptyset \).

Figure (2) displays the critical edges, and sets \( X_e \) and \( S_e \) corresponding to them, in an illustrative example.

Remark III.1. If \( G \) is 2-edge controllable w.r.t \( R \), then \( G' \) obtained by adding an edge between any two vertices is also 2-edge controllable w.r.t \( R \).

Let \( \bar{A} \) and \( B \) represent the structured/sparsity pattern of \( A \) and \( B \).  

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2 In this article \( B \) is considered as \( B := \text{diag}(\delta) \), where \( \delta \in \{0, 1\}^d \). Here, if \( \delta_i = 1 \) state vertex \( v_i \) receives an input, while if \( \delta_i = 0 \), it receives no input.
3 Location of zeroes and non-zeroes in the system and control matrices of (1.1) of \( A, B \).
A proof of Proposition III.2 is presented in §VI.

Remark III.3. It is not necessary that total number of vertices in all the subgraphs encountered in Algo 1 is equal to the number of vertices in $G$.

Consider all the $X_i$’s obtained by removal of edges, say $\{X_i\}_{i \in I}$ with $I = \{1, 2, 3, \ldots, n\}$ and $Z = \{v_1, v_2, \ldots, v_d\}$.

Define the family of subsets $F_j = \{i \in I : v_j \in X_i\}$ for $j = 1, 2, \ldots, d$. If there exists an $L \subset \{1, 2, \ldots, d\}$ such that

$$I \subset \bigcup_{j \in L} F_j$$

then the family $\{F_j\}_{j \in L}$ covers $I$ and

$$\tilde{R} = \bigcup_{j \in L} \{v_j\} \cup R,$$

where $\tilde{R}$ is the minimal root set such that each subgraph is robust w.r.t one edge failure. There exists a greedy approximation algorithm that provides a set cover that is in $n$ larger than an optimal set cover [4].

The second step involves addition of vertices to the subgraphs to retrieve the original $G(\tilde{A})$ such that $G(\tilde{A})$ is also resilient to an edge failure w.r.t $\tilde{R}$.

Step 2: To obtain the original graph $G(\tilde{A})$, vertices are added to the subgraphs. Let $G_1$ be a subgraph obtained from step 1 above. Suppose $z$ is added to $G_1$. Let $G_2$ be the new subgraph obtained by adding $z$ and its edges to $G_1$.

If $S \subset V(G_1)$ be a set satisfying the following conditions:

1) $|S| = |N^-(S)|$, and
2) even after removal of any one edge $e = (x, y)$, where $x \in N^-(S)$ and $y \in S$, condition 1) holds,

then $S$ is termed as a critical set. The following theorem ensures that $G_1$ obtained by adding a vertex $z$ to the subgraph $G_1$ is also robust w.r.t a failure with the same root set $\tilde{R}$.

Theorem III.4. Let $z$ and its edges are added to a subgraph $G_1$ and $S \subset V(G_1)$ be a critical set. If $G_1$ is 2-edge controllable w.r.t $\tilde{R}$, then $G_1 = G_1 \cup \{z\}$ obtained by adding a new vertex $z$ is also 2-edge controllable w.r.t $\tilde{R}$, if the following conditions are satisfied:

(a) $z$ has at least two in-neighbours from $G_1$, and
(b) if $N^-(z) \cap N^-(S) \neq \emptyset$, then $z$ has at least two in-neighbours not contained in $N^-(S)$.

We provide a proof in §VI. Theorem (III.4) allows us to add vertices to the graph consecutively such that robustness of graph w.r.t failure is preserved. This completes the two-step procedure involved to obtain the root set $\tilde{R}$ for $G(\tilde{A})$ such that $ec(\tilde{R}) \geq 2$.

Remark III.5. The minimal root set $\tilde{R}$ obtained in step 1 depends on the initial root set $R$ and the way the subgraphs have been constructed there from the original graph $G(\tilde{A})$.

IV. Illustrative Example

Let us consider the network topology $G$ of the IEEE 14-bus system [1] depicted in Fig. (3). Each undirected edge between the two vertices denotes bidirectional edges between them. Let $R = \{v_9, v_{10}\}$ be the initial root set such that $G$ is structurally controllable w.r.t $R$. Two subgraphs $G_1 = \{v_1, v_2, v_3, v_4, v_7, v_8\}$ and $G_2 = \{v_6, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ are obtained such that $G_1 \cup G_2$ is structurally controllable w.r.t $R$ as shown in Fig (4). The critical edges corresponding to $R$ are: $e_1 = (v_2, v_1),$ $e_2 = (v_3, v_2),$ $e_3 = (v_4, v_3),$ $e_4 = (v_7, v_4),$ $e_5 = (v_8, v_7),$ $e_6 = (v_10, v_{11}),$ $e_7 = (v_{11}, v_6),$ $e_8 = (v_6, v_{12}),$ $e_9 = (v_12, v_{13}),$ $e_{10} = (v_{13}, v_{14}).$

Each critical edge creates an $X_e$ corresponding to it. $X_{e_1} = \{v_1\},$ $X_{e_2} = \{v_1, v_2\},$ $X_{e_3} = \{v_1, v_2, v_3\},$ $X_{e_4} = \{v_1, v_2, v_3, v_4\},$ $X_{e_5} = \{v_1, v_2, v_3, v_4, v_7\},$ $X_{e_6} = \{v_1, v_2, v_3, v_4, v_7, v_{11}\},$ $X_{e_7} = \{v_6, v_{12}, v_{13}, v_{14}\},$ $X_{e_8} = \{v_{12}, v_{13}, v_{14}\},$ $X_{e_9} = \{v_{13}, v_{14}\},$ and $X_{e_{10}} = \{v_{14}\}.$ By using greedy approximation algorithm for set cover as in step 1, the additional root vertices required for ensuring robustness w.r.t an arbitrary edge failure is computed. The solution of the set cover is $\tilde{R} = \{v_1, v_8, v_{10}, v_{14}\}$ displayed in Fig. (5). Hence, $\tilde{R}$ is a root set such that each subgraph is 2-edge controllable.

Now $\{v_9\}$ is added to the subgraph $G_2$ with its undirected edges $(v_9, v_{10})$ and $(v_3, v_{14})$ to obtain $G_2^*$ as shown in Fig. (6). As each undirected edge between the two vertices represent two bidirectional edges between them, four edges corresponding to $v_9$ are added. Since $v_9$ has two in-neighbours $v_{10}$ and $v_{14}$ condition(a) of Theorem III.4 is satisfied. Also $G_2$ has no critical set in it. Hence, $ec(\tilde{R}) \geq 2$. Similarly, $\{v_5\}$ and its edges $(v_5, v_1)$ and $(v_5, v_6)$ is added which connects $G_1$ and $G_2$ as shown in Fig (7). Let the new graph obtained by adding $v_5$ be $G^*$, $G^*$ has no critical set and $v_5$ has two in-neighbours $v_1$ and $v_6$. As both the conditions of Theorem III.4 are satisfied, $ec(\tilde{R}) \geq 2$. The remaining edges corresponding to $v_5$ and $v_9$ are added to retrieve the original graph $G$ of IEEE 14-bus power system depicted in Fig. (8). In view of Remark III.1, $ec(\tilde{R}) \geq 2$. Therefore, $G$ is resilient w.r.t to an arbitrary edge failure with $\tilde{R} = \{v_1, v_8, v_{10}, v_{14}\}$.

V. Conclusions and future directions

In this article we presented the problem of finding the minimal number of actuators that ensure that the system remains structurally controllable under external malicious attack. The scenario considered here is the loss of physical connection between the two system states. Due to the combinatorial nature of this problem, we proposed an efficient system synthesis mechanism to obtain robustness with respect to an edge failure. The effectiveness of our technique is illustrated by using a standard IEEE bus power system. In future we aim to study and develop this technique further to tackle
Fig. 3: Network topology of IEEE-14 bus power system $G$.

Fig. 4: The brown colored vertices $\{v_8, v_{10}\}$ represent the initial root set $R$. $G_1 = \{v_1, v_2, v_3, v_4, v_7, v_8\}$ and $G_2 = \{v_6, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ are the subgraphs.

Fig. 5: The dashed gray vertices $\{v_1, v_{14}\}$ represent the vertices where additional inputs are applied.

interesting combinatorial problems. A natural extension would be to determine the applicability of this technique to multiple edge failures occurring simultaneously.

VI. Appendix A

Lemma VI.1. Given a digraph $G = (V, E)$ structurally controllable w.r.t root set $R$ such that $\Delta^+(G) \leq 2$ then, for each nonempty $S_{e_j}$, there exists a nonempty $X_{e_i}$ where $e_j, e_i \in E$ are the critical edges.

Proof. Consider a digraph $G$ structurally controllable w.r.t $R$. Here we will show that if $X_{e_i} = \emptyset$, then $S_{e_j} = \emptyset$. 
Suppose $S_{e_j} \neq \emptyset$. We know that it is obtained by removal of a critical edge $e_j = (w, y)$, where $w, y \in V(G)$ and $y \in S_{e_j}$. Note that the case $|S_{e_j}| = 1$ is trivial as $N^-(S_{e_j}) = 0$. Consider $S_{e_j}$ such that $|S_{e_j}| \geq 2$.

As $S_{e_j}$ is a minimal set it satisfies the following criterion:

1. $|S_{e_j}| = |N^-(S_{e_j})| + 1$
2. Every vertex in $N^-(S_{e_j})$ is adjacent to at least two vertices in $S_{e_j}$.

As $\Delta^+(G) \leq 2$, every vertex in $N^-(S_{e_j})$ is adjacent to exactly two vertices in $S_{e_j}$.

**Claim:** There exists a vertex $v \in S_{e_j}$ with $d^-(v) = 1$ and $v \neq y$. Firstly it is proved that there exists at least one vertex belonging to $S_{e_j}$ whose in-degree is at most 1. Consider the induced subgraph of $S_{e_j}$ and $N^-(S_{e_j})$.

Suppose that every vertex in $S_{e_j}$ has in-degree at least 2, then

$$\sum_{v \in S_{e_j}} d^-(v) \geq 2 |S_{e_j}|$$

and

$$\sum_{v \in N^-(S_{e_j})} d^+(v) = 2 |N^-(S_{e_j})|$$

$$= 2(|S_{e_j}| - 1)$$

$$< \sum_{v \in S_{e_j}} d^-(v)$$

which is a contradiction. Therefore there exists at least one vertex whose in-degree is at most 1. For any vertex $v \in S_{e_j}$, if $d^-(v) = 0$ then it contradicts the minimality of $S_{e_j}$. This confirms that there exist at least one vertex, say $z$ with $d^-(z) = 1$

Secondly we need to prove that $z \neq y$. This can also be proved using the same argument as above. Suppose that $y \in S_{e_j}$ is the only vertex with in-degree 1 and rest of vertices in $S_{e_j}$ has in-degree at least 2 in the induced subgraph of $S_{e_j}$ and $N^-(S_{e_j})$, then

$$\sum_{v \in S_{e_j}} d^-(v) \geq 1 + 2(|S_{e_j}| - 1)$$

$$\geq 1 + 2|N^-(S_{e_j})|$$

$$\geq 1 + \sum_{v \in N^-(S_{e_j})} d^+(v)$$

which is a contradiction. This proves that there exists at least one vertex $z \neq y$ in $S_{e_j}$ s.t $d^-(z) = 1$. Therefore the edge $e_i = (m, z)$ corresponding to $z$, where $m \in N^-(S_{e_j})$, is a critical edge as its removal makes $z$ inaccessible from the root set $R$. Therefore, $X_{e_i} = \{z\}$ which is non-empty. Contradiction.

**Proof of Proposition III.2.** For each subgraph $G_i$, step 2a deals with finding the critical edges that can be achieved in polynomial time as it is equivalent to analysing structural controllability of $|E(G_i)|$ digraphs. It has complexity of $O(\sqrt{V(G_i)}|E(G_i)|^2)$. Step 2c computes the SCCs corresponding to each critical edge in $G_i$. The SCCs can be obtained by using DAG with complexity $O(|V(G_i)| + |E(G_i)|)$. Similar steps are followed for each subgraph in the for loop. Hence, Algorithm 1 has polynomial-time complexity.

**Proof of Theorem III.4.** We know that every vertex in $G_1$ is input accessible from the root set $R$. Condition (a) ensures that $z$ added to the subgraph $G_1$ is also accessible from $R$. Since in-degree of $z$ is at least 2, removal of an edge terminating at $z$ does not result in input-inaccessibility w.r.t $R$.

We will prove condition (b) by analysing the following cases:

1) Suppose $z$ has all its in-neighbours contained in $N^-(S)$. By condition (a), $z$ has at least two in-neighbours. Since $S$ is a critical set, $|S| = |N^-(S)|$. Addition of $z$ creates a new set $S_z$ that satisfies the following criterion:

- $|S_z| = |S| + 1$ as $z$ is added to $G_1$.
- $|N^-(S_z)| = |N^-(S)|$

This shows that $|N^-(S_z)| = |S_z| - 1$. So, $G_1$ has dilation in it.

2) Suppose $z$ has only one in-neighbour, say $k \in V(G_1)$, not contained in $N^-(S)$. Let $e = (k, z)$. Suppose $e$ is removed from $G_1$. Then all the in-neighbours of $z$ lie in $N^-(S)$, which again lead to dilation after removal of the edge $e$ from $G_1$.

3) Suppose $z$ has at least two in-neighbours. Let $k$ and $m$ are the two in-neighbours of $z$ not contained in $N^-(S)$. Let $e_1 = (k, z)$ and $e_2 = (m, z)$. Addition of $z$ creates a new set $S_z$ with $|N^-(S_z)| \geq |N^-(S)| + 2$. Therefore, removal of any one edge does not result in dilation.

This proves that if $z$ has at least two in-neighbours not contained in $N^-(S)$, then it does not introduce dilation in $G_1$ after removal of an edge.

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