PERIOD AND SIGNAL RECONSTRUCTION FROM THE CURVE OF TRAINS OF SAMPLES

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ABSTRACT
A finite sequence of equidistant samples (a sample train) of a periodic signal can be identified with a point in a multi-dimensional space. Such a point depends on the sampled signal, the sampling period, and the starting time of the sequence. If the starting time varies, then the corresponding point moves along a closed curve. We prove that such a curve, i.e., the set of all sample trains of a given length, determines the period of the sampled signal, provided that the sampling period is known. This is true even if the trains are short, and if the samples comprising trains are taken at a sub-Nyquist rate. The presented result is proved with a help of the theory of rotation numbers developed by Poincaré. We also prove that the curve of sample trains determines the sampled signal up to a time shift, provided that the ratio of the sampling period to the period of the signal is irrational. Eventually, we give an example which shows that the assumption on incommensurability of the periods cannot be dropped.

Index Terms — signal sampling, period estimation, topological data analysis, dynamical system, rotation number, periodic time-series data, time delay embedding

1. INTRODUCTION
A short train of samples taken at sub-Nyquist frequency rarely allows for determining the period of a continuous signal. Clearly, it is not possible unless some sparsity conditions are imposed on the class of the considered signals. Intuitively, any extra sample train should help in finding a better estimate for the unknown period even if the time offsets between the trains are not known. The aim of this paper is to augment the intuition by showing that the period of the signal is uniquely determined by the set of all sample trains of a given length, provided that the sampling period is known. For this, we embed the sample trains into a multi-dimensional space:

\[ s_{d,\tau}(t) = [s(t), s(t + \tau), \ldots, s(t + (d - 1)\tau)] \in \mathbb{R}^d, \quad (1) \]

where \( s \) is the signal being sampled, \( d \) is the length of the train, \( \tau \) is the inter-sample distance, i.e., the sampling period, and \( t \) is the starting time of the train. Such an embedding was proposed by Packard et al. \cite{1} and by Takens \cite{2} in order to study the dynamics of nonlinear systems. It has also been used in topological data analysis proposed by Carlsson \cite{3} to identify qualitative properties of the data-sets. We denote the set of all points \((1)\) by \( S_{d,\tau} \), i.e.,

\[ S_{d,\tau} = \{ s_{d,\tau}(t) : t \in \mathbb{R} \} \subset \mathbb{R}^d. \quad (2) \]

If signal \( s \) is continuous and periodic, then \( S_{d,\tau} \) is a closed curve. We show that if a \( T \)-periodic signal \( s \) satisfies some regularity conditions, then curve \( S_{d,\tau} \subset \mathbb{R}^d \) defines a homeomorphism \( R_\tau \) of curve \( S_{d-1,\tau} \subset \mathbb{R}^{d-1} \) which is formed by sample trains of length \( d - 1 \). Poincaré \cite{4} showed that each homeomorphism of a closed curve has a topological invariant called the rotation number. In this paper, we prove that the rotation number of the homeomorphism \( R_\tau \) is strictly related to the ratio \( \tau/T \). This relationship can be used to find signal period \( T \) if curve \( S_{d,\tau} \) and sampling period \( \tau \) are known.

Works of Rader \cite{5} and Choi et al. \cite{6} showed that \( T \)-periodic signals can be reconstructed from infinite trains of their samples taken with any sampling period \( \tau < \frac{T}{2} \) such that \( \tau/T \) is irrational. In this paper, we prove that a similar reconstruction is possible from the infinite set of finite-length sample trains, i.e., we may reconstruct a periodic signal from its curve \( S_{d,\tau} \). In \cite{7} and \cite{8} the author demonstrated that periodic signals can be reconstructed from a probabilistic distribution defined on their curves \( S_{d,\tau} \). The results presented in this paper imply that such a distribution is not needed if the sampling period and the signal period are incommensurable.

The next two sections introduce the concepts of a periodic map and a rotation number, respectively. Section \cite{9} shows how to reconstruct the period of the signal from the curve of sample trains, and the following section deals with the reconstruction of signals up to a time shift. The paper is concluded in Section \cite{10}.
2. PERIODIC COVERING MAPS

In this and the following sections we make use of some concepts of algebraic topology and dynamical systems, e.g., a homeomorphism, a covering map, a lift and a rotation number. Because of the scope and the form of this paper, we reduce the presentation of the required mathematical concepts to a bare minimum. A more comprehensive treatment of these concepts can be found in, e.g., [9, 10].

Definition. A $T$-periodic mapping $s : \mathbb{R} \rightarrow \mathbb{R}^d$ is called a covering map if $s$ restricted to any interval of length smaller than $T$ is a homeomorphism, i.e., if each such restriction is continuous and has continuous inverse function.

If a $T$-periodic mapping $s : \mathbb{R} \rightarrow \mathbb{R}^d$ is a covering map, then the image of $s$ does not have self-intersections and thus this image is a closed curve which is homeomorphic to a circle. The covering map carries the natural orientation of real line (from smaller to bigger numbers) to one of the two possible orientations of this closed curve, see Fig. 1.

The following example shows that whether or not a mapping $s_{d, \tau}$ is a covering map may depend on the sample train length $d$.

Example 1. Let $s : \mathbb{R} \rightarrow \mathbb{R}$ be a 1-periodic signal defined as

$$s(t) = (\sin \pi t)^2 \sin (2\pi t (1 + t)), \quad \text{for } t \in [0, 1].$$

(3)

Figure 2 shows the graph of signal $s$. Mapping $s_{2,0.2}$ is not a covering map because curve $S_{2,0.2}$ has intersections as shown in Fig. 3a. However, $s_{4,0.2}$ becomes a covering map for $d = 3$ (see Fig. 3b), and it stays a covering map for all $d > 3$.

The last statement of the above example results from the following lemma, which is a direct consequence of (1) and the definition of a periodic covering map.

Lemma 1. If $s$ is a signal for which $s_{d, \tau}$ is a $T$-periodic covering map, then $s_{d+1, \tau}$ is also a $T$-periodic covering map.

3. ROTATION NUMBER

Let a curve $S$ be the image of a $T$-periodic covering map $s : \mathbb{R} \rightarrow \mathbb{R}^d$ and let $R : S \rightarrow S$ be a homeomorphism.

Definition. A mapping $F : \mathbb{R} \rightarrow \mathbb{R}$ is called a lift of $R$ across $s$ if

$$R \circ s = s \circ F,$$

(4)

where symbol $\circ$ denotes function composition.

If $R$ preserves the orientation of $S$ induced by $s$, then the lift of $R$ across $s$ is an increasing function and

$$F(x + kT) = F(x) + kT,$$

(5)

for every real $x$ and integer $k$. Similarly, if $R$ reverses the orientation of $S$, then $F$ is decreasing and $F(x + kT) = F(x) - kT$.

If $s_{d, \tau}$ is a $T$-periodic covering map, then there exists a natural homeomorphism $R_{\tau}$ of curve $S_{d, \tau}$:

$$R_{\tau} : S_{d, \tau} \rightarrow S_{d, \tau}, \quad R_{\tau} (s_{d, \tau} (t)) = s_{d, \tau} (t + \tau).$$

(6)

Translation of real line by $\tau$:

$$T_{\tau} : \mathbb{R} \rightarrow \mathbb{R}, \quad T_{\tau} (x) = x + \tau$$

(7)

is a lift of $R_{\tau}$ across $s_{d, \tau}$, i.e.,

$$R_{\tau} \circ s_{d, \tau} = s_{d, \tau} \circ T_{\tau}.$$

(8)

Definition. Rotation number of an orientation-preserving homeomorphism $R$ of an oriented closed curve is defined as the limit

$$\rho (R) = \lim_{n \rightarrow \infty} \frac{F^n (x) - x}{nT} \mod 1,$$

(9)
where \( x \in \mathbb{R} \) is an arbitrary point, \( F: \mathbb{R} \to \mathbb{R} \) is a lift of \( \mathcal{R} \) across a \( T \)-periodic covering map \( s \) which induces the orientation of the curve, and where \( F^n \) stands for \( n \)-fold composition of \( F \) with itself.

Poincaré [4] showed that the choice of point \( x \), lift \( F \) and covering map \( s \) does not affect the value of (9) (see also [10] for a comprehensive treatment of the subject). He also showed that the rotation number is invariant under topological conjugacy, i.e., if \( \mathcal{R} \) and \( \mathcal{Q} \) are two homeomorphisms of the same closed curve, then the rotation number of \( \mathcal{R} \) is the same as the rotation number of \( \mathcal{Q}^{-1} \circ \mathcal{R} \circ \mathcal{Q} \). Note that the rotation number depends on the orientation of the curve, i.e., for a given homeomorphism \( \mathcal{R} \) the rotation number changes from \( \rho \) to \( 1 - \rho \) if the considered orientation of the curve is reversed.

Substitution \( s = s_{d, \tau}, \ F = T_\tau \) and \( \mathcal{R} = \mathcal{R}_\tau \) into (9) yields

\[
\rho(\mathcal{R}_\tau) = \frac{\tau}{T} \mod 1,
\]

provided that the chosen orientation of curve \( S_{d, \tau} \) happens to be the same as the orientation induced by \( s_{d, \tau} \). In the opposite case, we get

\[
1 - \rho(\mathcal{R}_\tau) = \frac{\tau}{T} \mod 1.
\]

We conclude the above consideration with the following lemma.

\[\textbf{Lemma 2.}\] If \( s \) is a \( T \)-periodic signal such that \( s_{d, \tau} \) is a covering map and \( \tau < \frac{T}{2} \), then

\[
\frac{\tau}{T} = \min\left(\rho(\mathcal{R}_\tau), 1 - \rho(\mathcal{R}_\tau)\right),
\]

where \( \mathcal{R}_\tau \) and \( \rho(\mathcal{R}_\tau) \) are defined by (8) and (9), respectively.

4. \textbf{PERIOD RECONSTRUCTION}

Let \( s \) be a periodic signal. If its curve \( S_{d+1, \tau} \) is given, then we can recover curve \( S_{d, \tau} \) by projecting \( S_{d+1, \tau} \) onto the first \( d \) or the last \( d \) coordinates, i.e.,

\[
S_{d, \tau} = \{ \mathbf{p} \mid \mathbf{p} \in S_{d+1, \tau} \} = \{ \mathbf{p} \mid \mathbf{p} \in S_{d+1, \tau} \},
\]

where for any point \( \mathbf{p} = [x_1, \ldots, x_{d+1}] \in \mathbb{R}^{d+1} \), the projections are defined as

\[
\mathbf{p}_x = [x_1, \ldots, x_d] \quad \text{and} \quad \mathbf{p}_y = [x_2, \ldots, x_{d+1}].
\]

If \( s_{d, \tau} \) happens to be a covering map, then homeomorphism \( \mathcal{R}_\tau: S_{d, \tau} \to S_{d, \tau} \) is determined by curve \( S_{d+1, \tau} \) with some abuse of notation we denote by the same symbol \( \mathcal{R}_\tau \) all the homeomorphisms defined by (6) independently of the value of \( d \). The above fact is implied by Lemma 1 and the observation stated below as Lemma 3.

\[\textbf{Lemma 3.}\] If \( s \) is a periodic signal such that \( s_{d, \tau} \) is a covering map, then the homeomorphism \( \mathcal{R}_\tau: S_{d, \tau} \to S_{d, \tau} \) defined by (6), is uniquely determined by \( S_{d+1, \tau} \):

\[
\mathcal{R}_\tau(\mathbf{p}_x) = \mathbf{p}_y, \quad \text{for each } \mathbf{p} \in S_{d+1, \tau}.
\]

Lemmas 2 and 3 imply the following theorem, which provides conditions under which curve \( S_{d+1, \tau} \) determines the period of a periodic signal \( s \).

\[\textbf{Theorem 1.}\] Let \( s \) be a \( T \)-periodic signal such that \( s_{d, \tau} \) is a covering map. If \( \tau < \frac{T}{2} \), then curve \( S_{d+1, \tau} \) determines period \( T \) as

\[
T = \frac{\tau}{\min\left(\rho(\mathcal{R}_\tau), 1 - \rho(\mathcal{R}_\tau)\right)}.
\]
where homeomorphism $\mathcal{R}_\tau$ is defined by (15).

Notice that without condition $\tau < \frac{T}{2}$ in the assertion of the above theorem, all that could be deduced from $S_{d,\tau}$ is that

$$T = \frac{\tau}{n + \rho(\mathcal{R}_\tau)} \quad \text{or} \quad T = \frac{\tau}{n + 1 - \rho(\mathcal{R}_\tau)},$$

(17)

for some non-negative integer $n$. In other words, we may replace condition $\tau < \frac{T}{2}$ in Theorem 1 with the requirement that

$$\tau \in \left( \frac{T}{2}, (n+1)\frac{T}{2} \right),$$

(18)

for any non-negative integer $n$, provided that this integer is known.

5. SIGNAL RECONSTRUCTION

In the previous section, we showed that the period of a periodic signal may be recovered from its curve $S_{d+1,\tau}$ formed by all sample trains of length $d+1$ which are taken with a fixed sampling period $\tau$. An interesting question is whether it is also possible to reconstruct the signal $s$ from that curve. The answer to this question depends on the rotation number of the homeomorphism $\mathcal{R}_\tau$.

**Theorem 2.** Let $s$ be a $T$-periodic signal such that $s_{d,\tau}$ is a covering map for some $\tau < \frac{T}{2}$, and let $\mathcal{R}_\tau : S_{d,\tau} \to S_{d,\tau}$ be the homeomorphism defined by (15). If rotation number $\rho(\mathcal{R}_\tau)$ is irrational, then curve $S_{d+1,\tau}$ determines signal $s$ up to a time shift.

**Proof.** By Theorem 1 we can assume that period $T$ is known. Let $q$ be a point lying in $S_{d,\tau}$. This point equals $s_{d,\tau}(t_0)$ for some $t_0 \in [0, T)$. We set

$$s_1(0) = \pi_1(q) = s(t_0),$$

(19)

where $\pi_1$ denotes the projection to the first coordinate, i.e., $\pi_1([x_1, \ldots, x_d]) = x_1$. Without loss of generality, we may assume that the orientation of $S_{d,\tau}$ is so that $\rho(\mathcal{R}_\tau) = \tau/T$.

Since the rotation number $\rho(\mathcal{R}_\tau)$ is irrational, multiples of $\tau$ form a dense subset of interval $[0, T)$ when considered modulo $T$ (10). Therefore, for any $t \in \mathbb{R}$ there exists a sequence $k_1, k_2, \ldots$ of positive integers such that

$$\lim_{n \to \infty} k_n \tau = t \mod T.$$

(20)

We set

$$s_1(t) = \pi_1 \left( \lim_{n \to \infty} \mathcal{R}_{k_n}^{-1}(q) \right) = s(t_0 + t).$$

(21)

Thus, we obtain a signal $s_1 : \mathbb{R} \to \mathbb{R}$, which differs from $s$ by a time shift only.

If the rotation number of homeomorphism $\mathcal{R}_\tau$ is rational, then $S_{d,\tau}$ is not enough to recover signal $s$ up to a time shift, what is shown in the following example.

Example 2. Let $\tau = p/q$ for some positive integers $p < q$. Signal $s(t) = \sin 2\pi t$ is a 1-periodic signal and $s_{d,\tau}$ is a covering map for each $d \geq 2$ ($S_{2,\tau}$ is an ellipse). Let $r$ be a homeomorphism of $\mathbb{R}$ defined as

$$r(t) = t + \frac{1}{4\pi q} \sin (2\pi qt).$$

(22)

Signal $s'$ defined as $s' = s \circ r$ is another 1-periodic signal and $s'_{d,\tau} = s_{d,\tau} \circ r$ is a covering map for each $d \geq 2$. Signals $s$ and $s'$ do not differ by a time shift only (see Fig. 4). However, curves $S_{d,\tau}$ defined for signal $s$ are the same as the corresponding curves $S'_{d,\tau}$ constructed for $s'$ because the image of $s'_{d,\tau}$ is the same as the image of $s_{d,\tau}$ for each $d \geq 2$.

It is worth to note that, although $S_{d,\tau}$ is not enough to recover the underlying signal $s$, it is possible to reconstruct signal $s$ from the probability distribution on $S_{d,\tau}$ which results from uniform probability distribution of the starting times of the sample trains (7, 8).

6. CONCLUSION

We showed that a curve $S_{d,\tau}$ formed by sample trains of length $d \geq 3$ carries valuable information on the sampled signal $s$. If mapping $s_{d-1,\tau}$ is a covering map, then the period $T$ of the signal $s$ can be recovered from curve $S_{d,\tau}$, provided that $2\tau/T < 1$ (Theorem 1). The same holds if $2\tau/T \in (k, k+1)$ for any positive integer $k$, provided that $k$ is known prior to the recovery of $T$. Moreover, if ratio $\tau/T$ is irrational then $S_{d,\tau}$ determines signal $s$ up to a time shift (Theorem 2). By providing a counterexample we showed that the same statement does not hold when $\tau/T$ is rational.

The results presented in this paper establish a link between sampling theory, the theory of dynamical systems, and algebraic topology. We believe that the results will also form a basis for further research on construction of consistent period estimators from finite number of trains of samples.

![Fig. 4. Signals $s$ and $s'$ considered in Example 2](image-url)
7. REFERENCES

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