Deformed integrable models from holomorphic Chern-Simons theory

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We study the approaches to two-dimensional integrable field theories via a six-dimensional (6D) holomorphic Chern-Simons theory defined on twistor space. Under symmetry reduction, it reduces to a 4D Chern-Simons theory, while under solving along fibres it leads to a four-dimensional (4D) integrable theory, the anti-self-dual Yang-Mills or its generalizations. From both 4D theories, various two-dimensional integrable field theories can be obtained. In this work, we try to investigate several two-dimensional integrable deformations in this framework. We find that the $\lambda$-deformation, the rational $\eta$-deformation, and the generalized $\lambda$-deformation can not be realized from the 4D integrable model approach, even though they could be obtained from the 4D Chern-Simons theory. The obstacle stems from the incompatibility between the symmetry reduction and the boundary conditions. Nevertheless, we show that a coupled theory of the $\lambda$-deformation and the $\eta$-deformation in the trigonometric description could be obtained from the 6D theory in both ways, by considering the case that (3,0)-form in the 6D theory is allowed to have zeros.

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1 Introduction

One interesting issue on integrable systems is to find a unifying way to organize various kinds of integrable models. It was found [1] that many two-dimensional (2D) integrable models could be organized by four-dimensional (4D) anti-self-dual Yang-Mills (ASDM) equations via symmetry reductions, by viewing the Lax equation as a zero curvature condition. By the Penrose-Ward transformation, the ASDYM connections are identified with holomorphic vector bundles on the twistor space [2, 3] and the spectral parameter of an integrable system is interpreted as a coordinate on the twistor bundle from this geometric point of view. Quite recently, another geometric perspective on 2D integrable field theories has been proposed in ref. [4]. In this new approach, the 2D integrable field theories can be systematically constructed by inserting 2D defects into a novel four-dimensional Chern-Simons (4DCS) theory [5] which is specified by a holomorphic 1-form $\omega$, generalizing the study on lattice integrable models in refs. [6-8]. This 4D gauge theory approach has attracted much attention and has been under intense study since its proposal [9-21].

The relationship between these two geometric approaches was investigated in refs. [17, 22] from the perspective of a
six-dimensional holomorphic Chern-Simons theory (6DhCS) which is specified by a meromorphic (3, 0)-form $\Omega$. Starting from the 6DhCS theory, one can either perform symmetry reduction on the twistor space to get the 4DCS, or insert defects, dubbed solving along fibres in ref. [17], to obtain a 4D integrable theory which is a generalization of ASDYM. An essential feature is that appropriate boundary conditions on the fields have to be imposed at the locations of the poles of $\omega$ and $\Omega$ to ensure the gauge invariance. After imposing the boundary conditions, the field equations can be solved uniquely up to a gauge transformation. Schematically the relationship is illustrated in the diagram shown in Figure 1.

In ref. [17], the authors focused on the simplest example, the 2D principal chiral model\(^1\), in which case the resulting 4D integrable model is a WZW model [23] (the Chalmers-Siegel action for ASDYM [24]). In this case, the above diagram turns out to be commutative. It would be interesting to explore if other 2D integrable models can be fitted into the above diagram, as suggested in ref. [17].

In this work, we would like to pick two important classes of deformed integrable models and consider their constructions from the 6DhCS. From the 4DCS theory, various integrable deformations of 2D integrable models have been constructed by inserting more general defects which are described by Manin triples [10-12, 14, 15]. Among them two important examples are the $\lambda$-deformed principal chiral model [25] and the Yang-Baxter $\sigma$-model [26, 27]. One compelling question is to investigate if these models can be read from the 4D ASDYM and more generally what the diagram in Figure 1 would look like when more general defects are inserted into the 6DhCS. In this paper, we show that when we consider more general defects the diagram in Figure 1 is not completely commutative. This loss of commutativity originates from the fact that the symmetry reduction process may not be compatible with the boundary conditions. The symmetry reduction from the 6DhCS to the 4DCS suggests the matching conditions between the 6D gauge connection and the 4D gauge connection. Using this matching conditions, we can obtain the boundary conditions in one theory from the ones in the other. However, the induced boundary conditions are often problematic in the sense that they cannot remove the gauge freedoms. On the other hand, even if we discard the matching conditions and consider appropriate boundary conditions to remove gauge freedoms, it is still hard to find interesting 4D integrable deformations, whose symmetry reduction would give rise to the 2D deformed model we want. In particular, it is impossible to construct a non-trivial 4D $\lambda$-deformed WZW model from the 6DhCS.

We find that the resulting 2D model is either undeformed or with the deformation parameter being restricted to specific values. This is because the $(3, 0)$-form $\Omega$ in the 6DhCS is too restrictive to allow non-trivial defects. Therefore, we consider the situation that $\Omega$ has zeros such that non-trivial defects are allowed, and we discard the matching conditions at the same time. We will investigate two cases, one being that $\Omega$ has zeros and a fourth-order pole, the other being that $\Omega$ has zeros and two double poles. The former case leads to the 2D $\lambda$-deformed model coupled with an additional field, while the latter one leads to the trigonometric description of the Yang-Baxter deformation [28, 29]. In both cases, we can generate a diagram like Figure 1.

This paper is organized as follows. In sect. 2 we review the holomorphic Chern-Simons theory on the twistor space and the constructions in Figure 1 described in ref. [17] with a particular emphasis on the matching conditions and the commutativity of the diagram. In sect. 3 we consider the $\lambda$-deformation and show that a direct lift of the boundary condition associated with the poles of $\omega$ in the 4DCS to the boundary condition associated with the poles of $\Omega$ in the 6DhCS through the matching conditions is problematic. In sect. 4 we extend our analysis to other possible defects and show that in general they do not lead to desired deformations. In sect. 5, we consider the case that $\Omega$ has zeros and a fourth-order pole, and obtain a coupled $\lambda$-deformation. In sect. 6 we show that the Yang-Baxter deformation in the trigonometric description can actually be constructed in both ways. In this case both the 4D and 2D version of deformed theory are obtained but with a necessary violation of the matching condition. In sect. 7 we briefly discuss the reality conditions. For reader's convenience, we collect some notions on the spinors and the twistor space in Appendix A1, and list our notations in this work in Appendix A2.

### 2 Integrable field theories from the holomorphic Chern-Simons theory

In this section we will review the constructions\(^2\) depicted by

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1) The possible trigonometric deformation and a coupled $\sigma$-model were also considered in ref. [17].

2) We will adopt the convention used in ref. [17].
the diagram shown in Figure 2.

2.1 The holomorphic Chern-Simons theory on the twistor space

The starting point is the holomorphic Chern-Simons theory on the twistor space\(^3\). The action of the theory is of the form:

\[
S_{\Omega}[\mathcal{A}] = \frac{1}{2\pi i} \int_{\mathbb{P}_T} \Omega \wedge hCS(\mathcal{A}),
\]

(1)

where \(hCS(\mathcal{A})\) is of the form of the Chern-Simons action:

\[
hCS(\mathcal{A}) = tr \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right),
\]

(2)

with the Dolbeault operator \(\bar{\partial} = \partial^{\mathbb{C}} + \bar{\partial}_A\) defined on the twistor space, and \(\Omega\) is a meromorphic (3,0)-form which may contain both zeros and poles. The (3,0)-form \(\Omega\) can be rewritten as:

\[
\Omega = D^3 Z \otimes \Phi,
\]

(3)

where

\[
D^3 Z = \frac{(d\tau \nu) \wedge d^2 \pi^A \pi_B}{2}
\]

(4)

is the canonical holomorphic 3-form on \(\mathbb{CP}^3\) in the spinor notation and \(\Phi\) is a meromorphic section of \(\mathcal{O}(-4) \to \mathbb{P}_T\) which encodes the analytic properties of \(\Omega\). We restrict to the case in which \(\Phi\) only depends on the fibre direction of the twistor space \(\mathbb{P}_T \to \mathbb{CP}^1\), i.e., \(\Phi\) depends only on the coordinates \(\pi^A\). To derive the principal chiral model, we should take \(\Phi = (\tau \nu (\tau \nu))^{-2}\) which was first proposed by Costello [22] in a seminar and was concretely realized by Bittleston and Skinner [17].

The partial gauge connection \(\mathcal{A}\) in eq. (2) can be decomposed in terms of bases of (0,1)-forms on the twistor space:

\[
\mathcal{A} = \partial \mathcal{A}_0 + \bar{\partial} \mathcal{A}_A,
\]

(5)

where \(\partial\) is in the direction of \(\mathbb{CP}^1\) fibre over \(\mathbb{B}^4\) and \(\partial A\) are in the directions of \(\mathbb{B}^4\). At the poles of \(\Omega\), the variation of the action \(S_{\Omega}[\mathcal{A}]\) generates the boundary terms\(^4\):

\[
\delta S_{\Omega}[\mathcal{A}]_{\text{bdy}} = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\mathbb{P}_T} d(\Omega \wedge \text{tr}(d\mathcal{A} \wedge \mathcal{A})),
\]

(6)

where \(\nu\) is the set of poles of \(\Omega\) and \(S_{\epsilon}^{1/2}\) is a circle of radius \(\epsilon\) around the pole \(\pi = z\). Using the following identity:

\[
d^2 \pi^A \pi_B \wedge \partial A \wedge \partial B = -2e^{AB} \text{vol}_4,
\]

we find that

\[
\delta S_{\Omega}[\mathcal{A}]_{\text{bdy}} = -\frac{1}{2\pi i} \int_{\mathbb{P}_T} \text{vol}_4 e^{AB} \text{tr}(d\mathcal{A} \wedge \mathcal{A})),
\]

(8)

which should be set to 0 by choosing appropriate boundary conditions in order to make the variation well defined.

2.2 Symmetry reduction to the 4D Chern-Simons theory

Firstly we follow the left route in Figure 2. It was shown in ref. [17] that the 4DCS theory can be obtained by performing a symmetry reduction by a 2D group of translations \(H\) in the holomorphic Chern-Simons theory. The generators of \(H\) are chosen to be the translations along complex null vectors:

\[
X = k^\lambda \bar{\mu}^\lambda \partial_{AA^\prime} = \partial_z, \quad \bar{X} = k^\lambda \mu^\lambda \partial_{AA^\prime} = \partial_{\bar{z}},
\]

(9)

where we have introduced double-null complex coordinates on the 4D Euclidean space \(\mathbb{B}^4\) as\(^5\):

\[
\begin{align*}
\bar{z} &= x^{AA^\prime} \kappa_A \mu_{A^\prime}, \\
\bar{w} &= x^{AA^\prime} \kappa_A \mu_{A^\prime}, \\
\bar{\xi} &= x^{AA^\prime} \kappa_A \kappa_{A^\prime}.
\end{align*}
\]

(10)

For future convenience we also define

\[
\begin{align*}
\partial_w &= k^\lambda \bar{\mu}^\lambda \partial_{AA^\prime}, \\
\bar{\partial}_w &= -k^\lambda \mu^\lambda \partial_{AA^\prime}.
\end{align*}
\]

(11)

The invariance condition of the connection is

\[
\mathcal{L}_X(D_z) = D_\mathcal{L}(X),
\]

(11)

where \(x \in \Gamma(E)\) and \(E\) is a vector bundle over an open subset \(\mathcal{U}\) of the twistor space \(\mathbb{P}_T\), \(D\) is the partial covariant derivative with respect to the partial connection \(\mathcal{A}\) and the symbol

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\(^3\) For a brief introduction to twistor space, see sect. A1.2.

\(^4\) Note that \(\mathbb{P}_T\) here denotes the \(\mathbb{CP}^1/D^*_\epsilon\) bundle over \(\mathbb{B}^4\) where \(D^*_\epsilon\) is the union of the disks of radius \(\epsilon\) around the poles of \(\Omega\).

\(^5\) Here we take \(|\bar{z}|^2 = |\bar{w}|^2 = 1\).
$\mathcal{V}_\chi$ denotes the Lie derivative which is defined by lifting the action of $H$ on $\mathcal{U}$ to $\mathcal{E}$. The action of translation group $H$ is necessarily free such that it is always possible to find an invariant gauge where the invariant condition eq. (11) becomes

$$\mathcal{V}'_\chi \mathcal{A} = \mathcal{V}'_\chi \mathcal{A} = 0,$$  \hspace{1cm} (12)

where $\mathcal{V}'$ denotes the ordinary Lie derivative operator acting on differential forms. In this gauge choice, the residual gauge freedom consists of gauge transformations generated by $\chi$ and $\tilde{\chi}$.

The Lagrangian density of the holomorphic Chern-Simons theory is invariant under the action of $H$ as well so one is able to do the dimensional reduction by integrating out $H$ directions. However it is more convenient to do this reduction by contracting the bi-vector $\chi \wedge \tilde{\chi}$ with the Lagrangian density of the holomorphic Chern-Simons theory. Performing this contraction directly to eqs. (2) and (3) gives

$$t_\chi \Omega = (d\tau \mathcal{V}) \wedge (dw(\mu \gamma) - dz(\tilde{\mu} \gamma))(\mu \gamma) \Phi,$$

$$t_{\tilde{\chi}} \Omega = (d\tau \mathcal{V}) \wedge (d\tilde{z}(\mu \gamma) + dw(\tilde{\mu} \gamma))(\mu \gamma) \Phi,$$

$$t_{\chi} t_{\tilde{\chi}} \mathcal{A} = (d\tau \mathcal{V})(\mu \gamma) \Phi,$$

and

$$t_\chi \text{hCS}(\mathcal{A}) = 2 \text{tr} \left( (t_\chi \mathcal{A}) \Phi \right),$$

$$t_{\tilde{\chi}} \text{hCS}(\mathcal{A}) = 2 \text{tr} \left( (t_{\tilde{\chi}} \mathcal{A}) \Phi \right),$$

$$t_{\chi} t_{\tilde{\chi}} \text{hCS}(\mathcal{A}) = -2 \text{tr} \left( (t_\chi \mathcal{A})(\tilde{\partial} + \mathcal{A})(t_{\tilde{\chi}} \mathcal{A}) \right) + 2 \text{tr} \left( (t_\chi \mathcal{A})(t_{\tilde{\chi}} \mathcal{A}) \mathcal{A} \right).$$

Thus we have

$$t_{\chi} t_{\tilde{\chi}} \left( \chi \wedge \text{hCS}(\mathcal{A}) \right) = (d\tau \mathcal{V})(\tau \tilde{\mu}) \Phi \text{hCS}(\mathcal{A})$$ \hspace{1cm} (13)

with

$$\mathcal{A} = \mathcal{V} \mathcal{A}_0 + \left( t_\chi \mathcal{A} - \frac{\langle \tau \tilde{\mu} \rangle}{\langle \tau \mu \rangle} t_{\tilde{\chi}} \mathcal{A} \right) dw + \left( t_\chi \mathcal{A} + \frac{\langle \tau \mu \rangle}{\langle \tau \tilde{\mu} \rangle} t_{\tilde{\chi}} \mathcal{A} \right) dw,$$ \hspace{1cm} (14)

and

$$\text{phCS}(\mathcal{A}) = \text{tr} \left( A d' d + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right),$$ \hspace{1cm} (15)

where $d' = \mathcal{V} \mathcal{A}_0 + dw(\mu) + dw(\tilde{\mu}) = \mathcal{A} \mathcal{C}_4 + d\mathcal{C}_6$. In deriving eq. (13), we have used the following identity:

$$\text{hCS}(X + Y) = \text{hCS}(X) + 2 \text{tr}(\mathcal{V}(X)Y) - \mathcal{V}\text{tr}(XY) + 2 \text{tr}(XY^2) + \text{hCS}(Y).$$ \hspace{1cm} (16)

After the symmetry reduction, we end up with the 4DCS theory with action

$$S_\omega [\mathcal{A}] = \frac{1}{2\pi i} \int_{\mathcal{E} \times \mathbb{C}P^1} \omega \wedge \text{phCS}(\mathcal{A}),$$ \hspace{1cm} (17)

where $\omega = (d\tau \mathcal{V})(\tau \mu)(\tau \tilde{\mu}) \Phi$ is a meromorphic one-form. The 4D gauge connection $(\mathcal{A}_w, \tilde{\mathcal{A}}_w)$ along $\mathbb{E}^2$ is related to the holomorphic gauge connection $\mathcal{A}$ by

$$\tilde{\mathcal{A}}_w = i_{\partial_w} \mathcal{A} - \frac{\langle \tau \tilde{\mu} \rangle}{\langle \tau \mu \rangle} t_{\tilde{\chi}} \mathcal{A} = -\frac{k \mathcal{A}}{\langle \tau \mu \rangle},$$ \hspace{1cm} (18)

$$\mathcal{A}_w = i_{\partial_w} \mathcal{A} + \frac{\langle \tau \mu \rangle}{\langle \tau \tilde{\mu} \rangle} t_{\chi} \mathcal{A} = \frac{k \mathcal{A}}{\langle \tau \tilde{\mu} \rangle},$$ \hspace{1cm} (19)

which we will refer to as the matching condition. It relates the 4D gauge connection to the 6D gauge connection, where two theories are related simply by the symmetry reduction. Then it is natural to expect that the boundary conditions on the 4D gauge connection can be transferred to the ones on the 6D gauge connection.

### 2.3 Solving the 4DCS theory reduced from the holomorphic Chern-Simons

Taking $\Phi = ((\mu \gamma)(\tilde{\mu} \gamma))^{-2}$, the resulting one-form $\omega$ of the 4D CS theory is

$$\omega = \frac{(d\tau \mathcal{V})(\tau \mu)(\tau \tilde{\mu})}{(\tau \mu)^2(\tau \tilde{\mu})^2},$$ \hspace{1cm} (20)

where we choose a pair of constant left-handed spinors $\alpha_\beta, \beta_\gamma$ with the normalization $\langle \alpha \beta \rangle = 1$. Now we can follow the procedure proposed in ref. [4] to derive the 2D integrable field theories. At the two double poles, the boundary term eq. (3) can be set to zero by taking the Dirichlet boundary conditions:

$$\mathcal{A}_w |_{\tau = \infty} = 0 = \mathcal{A}_w |_{\tau = \beta}.$$ \hspace{1cm} (21)

The Lax connection $\mathcal{L}$ of the 2D integrable field theory is related to $\mathcal{A}$ by a gauge transformation:

$$\mathcal{A} = \sigma^{-1} \mathcal{L}' \mathcal{A} + \sigma^{-1} \mathcal{L}' \mathcal{A}.$$ \hspace{1cm} (22)

The choice of $\sigma$ has the gauge symmetry $\sigma \rightarrow h \sigma g^{-1}$ where $h : \mathbb{E}^2 \rightarrow G$ and $g : \mathbb{E}^2 \times \mathbb{C}P^1 \rightarrow G$. The freedom in $h$ corresponds to the gauge freedom in $\mathcal{L}$, and the freedom in $g$ corresponds to the gauge freedom in $\mathcal{A}$. These gauge freedoms can be removed by fixing

$$\sigma |_{\tau = \infty} = \sigma, \hspace{1cm} \sigma |_{\tau = \beta} = \text{id}.$$ \hspace{1cm} (23)

Substituting eqs. (23) and (22) into the boundary condition eq. (21), one finds that

$$\sigma^{-1} d\mathcal{C}_6 \sigma + \sigma^{-1} \mathcal{L}' |_{\tau = \infty} \sigma = 0.$$ \hspace{1cm} (24)

$$\mathcal{L}' |_{\tau = \beta} = 0.$$ \hspace{1cm} (25)
At the positions of the zeros of the one-form $\omega$, one has to insert the defects which describe the pole structures of the 4D gauge fields such that the poles of the gauge field cancel the zeros of the one-form [4]. The cancellation is necessary for the gauge field to have a non-degenerate propagator. Considering the condition eq. (25), the Lax connection has to be of the form:

$$ L_w = \frac{(\tau \beta)}{(\tau \mu)} U_w, \quad L_w = \frac{(\tau \beta)}{(\tau \mu)} U_w, \quad \text{(26)} $$

where $U_{w,w}$ does not depend on $\tau^4$. Putting eq. (26) into eq. (24) and solving the equation, one gets

$$ L_w = -\frac{(\tau \beta)}{(\tau \mu)} (\alpha \beta) \partial_w \sigma^{-1}, \quad L_w = -\frac{(\tau \beta)}{(\tau \mu)} (\alpha \beta) \partial_w \sigma^{-1}. \quad \text{(27)} $$

Substituting eq. (27) into eq. (22) by using eqs. (15) and (17), one ends up with the action of the 2D WZW model:

$$ S_{\Omega}[\tilde{A}] = \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{C}^1} \omega \wedge \mathrm{phCS} (\tilde{A}) $$

$$ = \int_{\mathbb{R}^4} \mathrm{tr} (j_{\omega} j_{\mu} + k \int_{\mathbb{R}^2 \times [0,1]} \mathrm{tr} (j^3)), \quad \text{(28)} $$

where $j_{\omega} = -d \omega \sigma^{-1}$, $j_{\mu} = -d \sigma \partial \sigma^{-1}$, and $k = (\alpha \beta)(\mu \beta) + (\alpha \beta)(\mu \beta)$.

### 2.4 Solving along fibres for the holomorphic Chern-Simons theory

Next we derive the same 2D action eq. (28) following the right route shown in Fig. 2. At the two double poles, the boundary conditions of the 6D gauge connection can be obtained directly by the boundary conditions of the 4DCS (eqs. (24) and (25)) through the matching condition (eq. (14))

$$ [\kappa \tilde{A}|_{\tau = 0}] = 0 = [\kappa \tilde{A}|_{\tau = +}] . \quad \text{(29)} $$

The $(3,0)$-form $\Omega$ here does not have a zero, thus the 6D gauge connection does not have any pole6 in $\mathbb{C}^2$ and $\tilde{A}_1 = \pi \bar{A}_A$, where $A_{\bar{A}}$ does not depend on the coordinates $\tau^4$. By applying a formal gauge transformation the dynamical field $\tilde{A}$ can be rewritten as:

$$ \tilde{A} = \partial^{-1} \partial \partial + \partial^{-1} \partial \sigma, \quad \text{(30)} $$

where $\tilde{A} = \partial \tilde{A}$ is the 4D analogue of the Lax connection. Similar to what we did in the 4DCS case, we can fix the gauge freedom by requiring that

$$ \tilde{A}|_{\tau = 0} = \sigma, \quad \partial|_{\tau = 0} = \text{id}, \quad \text{(31)} $$

and solve the boundary condition eq. (29):

$$ \tilde{A} = \partial \tilde{A} = - (\tau \beta) \partial \partial \partial^{-1} \sigma^{-1}. \quad \text{(32)} $$

Thus the resulting 4D action is given by

$$ S_{\Omega}[\tilde{A}] = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Omega \wedge h\mathrm{CS}(\tilde{A}) $$

$$ = \frac{1}{4\pi} \sum_{e \to \hat{c}} \int_{S^3} \left( \frac{(\partial \sigma \partial \partial^{-1})}{(\tau \mu)^2} \right) $$

$$ \times \int_{\mathbb{R}^4} d^2 x \delta \left( \partial \partial \partial^{-1} \wedge \left( \hat{b} \wedge \hat{a} \right) \right) $$

$$ + \frac{1}{12\pi} \int_{\mathbb{R}^4} \left( \partial \partial \partial^{-1} \wedge d^2 x \delta \left( \partial \partial \partial^{-1} \wedge \left( \hat{b} \wedge \hat{a} \right) \right) \right), \quad \text{(33)} $$

where $\hat{b} = - \partial \partial \partial^{-1}$ and $\hat{a} = - \partial \partial \partial^{-1}$. Without losing of generality, we fix the normalization such that $\alpha \beta - \beta \alpha = e^{2\sigma}$ then eq. (33) becomes

$$ S_{\Omega}[\tilde{A}] = \frac{1}{2} \int_{\mathbb{R}^2} \mathrm{tr} (\partial \partial \partial^{-1} \wedge \left( \hat{b} \wedge \hat{a} \right) \wedge \left( \partial \partial \partial^{-1} \wedge \left( \hat{b} \wedge \hat{a} \right) \right)$$

$$ - \frac{1}{4} \int_{\mathbb{R}^2 \times [0,1]} \mu_{\sigma \beta} \wedge \mathrm{tr} (\hat{b}) \wedge \mathrm{tr} (\hat{a}) \wedge \mathrm{tr} (\hat{b} \wedge \hat{a}), \quad \text{(34)} $$

where $\mu_{\sigma \beta} = d^2 x \delta \left( \partial \partial \partial^{-1} \wedge \left( \hat{b} \wedge \hat{a} \right) \right)$ and $\hat{b} = - \partial \partial \partial^{-1}$ with $\hat{a}$ being a smooth homotopy from $\sigma$ to $\text{id}^7$. Note that $\hat{b}$ is defined to be the exterior derivative on $\mathbb{R}^4 \times [0,1]$.

### 2.5 Symmetry reduction to the 2D theory

The next step is to apply the same symmetry reduction along $H$ to this 4D action eq. (34) as before to get the action of the 2D integrable model. The actions of $H$ on $\mathbb{R}^4 \times [0,1]$ are generated by the vectors $X$ and $\bar{X}$ defined by the pullback:

$$ \pi_X = X, \quad \pi_{\bar{X}} = \bar{X}. \quad \text{(35)} $$

Since $X$ and $\bar{X}$ do not depend on $\mathbb{C}^2$, the expressions for $X$ and $\bar{X}$ are the same as before, i.e.,

$$ X = \kappa \mu \tilde{A} \partial \tilde{A} = \partial , \quad \bar{X} = \kappa \mu \tilde{A} \partial \tilde{A} = \partial. \quad \text{(36)} $$

Now we impose the constraint that the dynamical field $\sigma$ for this 4D effective model has the symmetry

$$ \Omega_X \sigma = \Omega_{\bar{X}} \sigma = 0. \quad \text{(37)} $$

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6 In this case there is no need to insert any defect. But to be compatible with our later discussion, here we abuse the terminology by saying that we insert a trivial defect.

7 This means that $\sigma$ is a smooth function such that $\sigma|_{\tau = 0} = \sigma$ and $\sigma|_{\tau = 1} = \text{id}$. Similar to the case of Wess-Zumino term, this term depends only on the four-dimensional field $\sigma$. 
Introducing the following vector fields:

\[ \partial_u = \kappa \beta^u \partial_{AA'}, \quad \partial_{\bar{u}} = -\kappa \beta^u \partial_{AA'}, \]
\[ \partial_c = \kappa \beta^c \partial_{AA'}, \quad \partial_{\bar{c}} = \kappa \beta^c \partial_{AA'}, \]

the symmetry reduction of eq. (34) can be easily performed as in the 6D case. The resulting action is

\[ \int_E \text{tr} (j_u j_{\bar{u}}) \text{d}w \wedge \text{d}\bar{w} + k \int_{E \times [0,1]} \text{tr} (j^3), \]

which is exactly the same as what we got by directly solving the reduced 4DCS theory. Moreover, we find that the Lax connections derived from the two different routes indeed match with each other:

\[ L_u = -\frac{\kappa \tilde{A}^u}{\partial \bar{u}} A^\alpha A^\sigma \sigma^{-1} \hat{A}^u = \frac{\partial \bar{u}}{\partial \bar{u}} \hat{A}^u, \]
\[ L_{\bar{u}} = -\frac{\kappa \tilde{A}^u}{\partial \bar{u}} A^\alpha A^\sigma \sigma^{-1} \hat{A}^u = \frac{\partial \bar{u}}{\partial \bar{u}} \hat{A}^u. \]

This concludes the commutativity of the diagram in Figure 2.

It is very instructive to derive the Lax connection of the 2D model directly from the ASDYM:

\[ e^{AB} [\nabla_{AA'}, \nabla_{BB'}] = 0. \]

Rewriting the 4D gauge Lax connection eq. (32) as \( A_{AA'} = \beta_{A'AA'}, \) where \( A_A = \beta_{A'AA'}, \) and \( J_{AA'} \equiv -\partial_{AA'} \sigma \sigma^{-1}, \) then the left hand side of eq. (42) becomes

\[ E_{AB'} = e^{AA'} (\beta_{A'AA'} A_B - \beta_{A'BB} A_A + \beta_{A'BB'} [A_A, A_B]). \]

Since \( \langle \alpha \beta \rangle = 1 \) we can choose \( \langle \alpha \beta' \rangle \) as the dyads. Then eq. (43) is equivalent to three equations obtained by contracting it with \( \alpha \beta \), \( \alpha \beta' \), and \( \beta \beta' \) separately

\[ \beta' \beta' E_{AA'} = 0, \]
\[ \alpha' \beta E_{AA'} = e^{AA'} (\alpha' \partial_{AA'} A_B - \alpha' \beta B \partial_{BB'} A_A + [A_A, A_B]) = 0, \]
\[ \alpha' \beta E_{AA'} = e^{AA'} \beta A' A_B = 0. \]

Performing symmetry reduction on eq. (46) gives the equation of motion eq. (39) of 2D theory:

\[ e^{AB} \beta' A_B = \beta_{A'AA'} (\alpha' \partial_{AA'} j_u - \beta_{A'BB'} j_u) + \beta_{A'BB'} [A_A, A_B] = 0, \]
\[ \delta (\bar{u}) (\bar{u}) (\bar{u}) = -k. \]

Similarly, the symmetry reduction of eq. (45) gives the flatness condition:

\[ \partial_u j_u - \partial_{\bar{u}} j_{\bar{u}} + [j_u, j_{\bar{u}}] = 0. \]

Therefore the 2D Lax connection can be obtained by contracting \( \langle \alpha A' + \beta B' \rangle (\alpha B' + \beta B') \) with eq. (42) and doing the symmetry reduction:

\[ \langle \alpha A' + \beta B' \rangle (\alpha B' + \beta B') E_{AA'} = 0. \]

where \( z \in \mathbb{C} \) is the spectral parameter. Comparing eqs. (40) and (41), one can match in the following way:

\[ \alpha A' + \beta B' = -\frac{\pi'}{\langle \pi \alpha \rangle} \Rightarrow z = -\frac{\langle \pi \alpha \rangle}{\langle \pi \beta \rangle}, \]

to get the agreement.

3 The \( \lambda \)-deformed principal chiral model from the holomorphic Chern-Simons

In this section, we will show that the diagram shown in Figure 2 can not be simply generalized to the \( \lambda \)-deformed theory. Instead, we find the following construction summarized in the diagram shown in Figure 3.

The \( \lambda \)-deformed principal chiral model is derived from the 4DCS with a 1-form:

\[ \omega = \frac{\langle \pi \alpha \rangle \langle \pi \alpha \rangle}{\langle \pi \alpha \rangle \langle \pi \beta \rangle}. \]

One should note that the spinor \( \beta \) introduced here has nothing to do with the \( \beta \) we used in sect. 2 while the spinor \( \mu \) and its conjugate come from the generators of symmetry group we introduced in eq. (9). The relation eq. (13) between \( \omega \) and \( \Omega \) implies that the (3, 0)-form \( \Omega \) should be

\[ \Omega = \frac{\langle \pi \alpha \rangle \langle \pi \alpha \rangle}{2 \langle \pi \alpha \rangle \langle \pi \beta \rangle}. \]
As we discuss in last section, varying \( S_{\Omega} \) gives rise to the following boundary terms:

\[
\delta S_{\Omega}_{\text{bdy}} = - \frac{1}{2\pi i} \sum_{x \in \partial} \lim_{e \to 0} \int_{S^{x}_{e}} \left( \frac{\partial \psi}{\partial x} \right) \left[ \frac{\partial}{\partial x} \left( \chi_{\Omega} \frac{\partial}{\partial x} \right) \int_{\mathbb{R}^{3}} \operatorname{vol}_{4} A_{A_{H}} \operatorname{tr}(\delta A_{A_{H}} A_{G}) \right],
\]

which should vanish by imposing appropriate boundary conditions. We will follow the left route first, and then demonstrate that the matching conditions for eqs. (18) and (19) will not lead to well defined boundary conditions for the 6DhCS such that the diagram shown in Figure 3 is not commutative.

### 3.1 Solving the 4D CS theory reduced from the holomorphic Chern-Simons

In eq. (52), we assume that the residues at the two simple poles are opposite, i.e.,

\[
(\alpha_{\beta}, \mu)(\alpha_{\bar{\beta}}, \bar{\mu}) = - (\alpha_{\bar{\beta}}, \bar{\mu})(\alpha_{\beta}, \mu),
\]

then the 1-form coincides with the choice of 1-form made in ref. [9] to derive the \( \lambda \)-deformation. At the boundary associated with the double pole, we still impose the Dirichlet boundary condition. The boundary conditions at the two simple poles are more tricky. The vanishing condition of the boundary terms at the two simple poles is given by

\[
\operatorname{Res}_{\partial} \omega \left( \frac{\operatorname{tr}(\delta A_{A_{H}})}{\partial x} \right)_{x_{0}} = 0,
\]

where we have used \( \operatorname{Res}_{\partial} \omega = - \operatorname{Res}_{\partial} \omega \). This condition can be solved in general by requiring that the pair \( (\hat{A}_{A_{H}}, \hat{A}_{A_{H}}) \) take values in a Lagrangian subalgebra of this two-copy algebra \((g, g) \equiv b\), where \( g \) is the algebra of the gauge group \( G \). To derive the \( \lambda \)-deformation, the Lagrangian subalgebra is \( g^{0} = (x, x), x \in \mathbb{R} \). If \( g \) is a semi-simple real algebra, then the complementary part \( g_{b} \) in \( b \) is also a Lagrangian subalgebra and the triplet \((b, g^{0}, g_{b})\) forms a Manin triple. Therefore in this case, the boundary conditions should be

\[
\hat{A}_{A_{H}} \big|_{x = x_{0}} = \hat{A}_{A_{H}} \big|_{x = x_{0}}, \quad \hat{A}_{A_{H}} \big|_{x = x_{0}} = 0.
\]

The Lax connection \( L \) for 2D integrable field theories is related to \( A \) by the gauge transformation eq. (30). As shown in ref. [9], the gauge parameter \( \sigma \) satisfies the so-called archipelago conditions, of which we omit the details here. The important fact about the archipelago conditions is that \( \sigma \) can be chosen to be the identity almost everywhere except for the neighbors around each poles, where they should be

\[
\hat{\sigma}_{x_{0}} = \sigma, \quad \hat{\sigma}_{x_{0}} = \sigma, \quad \hat{\sigma}_{y} = \sigma_{y}.
\]

Next using the gauge symmetry \( \sigma \rightarrow h \sigma h^{-1}, h : \mathbb{R}^{3} \rightarrow G \) one can set \( \sigma_{g} = id \), and using the residue gauge symmetry at the two simple poles \( \sigma \rightarrow g^{-1}, g \in (G, G) \) one can set \( \sigma_{g} = id \). After fixing all the gauge freedoms, the boundary conditions eqs. (56) and (57) are equivalently rewritten as:

\[
(\sigma_{x}^{-1} d_{\sigma} \sigma + \sigma_{x}^{-1} L_{|_{x_{0}}}, \sigma) = L_{|_{x_{0}}},
\]

\[
\mathcal{L}_{|_{x_{0}}} = 0.
\]

Since the 1-form has two zeros, the Lax connection has to be of the form eq. (26). The boundary condition eq. (59) becomes

\[
\begin{align*}
(\alpha_{\beta}, \mu)(\alpha_{\bar{\beta}}, \bar{\mu}) A_{A_{H}} & = \frac{1}{\lambda} \delta_{x_{0}} \sigma^{-1}, \\
(\alpha_{\beta}, \mu)(\alpha_{\bar{\beta}}, \bar{\mu}) A_{A_{H}} & = - \delta_{x_{0}} \sigma^{-1},
\end{align*}
\]

which can be simply solved and the resulting 2D Lax connection is just

\[
\mathcal{L}_{x_{0}} = \left( \frac{\partial}{\partial x} \right) \left( \frac{1}{\lambda} \delta_{x_{0}} \sigma^{-1} \right) A_{A_{H}}.
\]

Substituting eq. (63) into eq. (22) by using eqs. (15) and (17), we end up with the action of \( \lambda \)-deformed principal chiral model:

\[
S_{\lambda} = \frac{1}{2 \lambda} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \omega \wedge \operatorname{phCS}(\hat{A})
\]

\[
= \frac{1}{2 \lambda} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \delta_{x_{0}} \sigma^{-1} \frac{1}{\lambda} \delta_{x_{0}} \sigma^{-1} \frac{1}{\lambda} \delta_{x_{0}} \sigma^{-1} A_{A_{H}} \wedge \frac{1}{\lambda} \delta_{x_{0}} \sigma^{-1} A_{A_{H}}
\]

\[
= \frac{1}{2 \lambda} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \delta_{x_{0}} \sigma^{-1} \frac{1}{\lambda} \delta_{x_{0}} \sigma^{-1} \frac{1}{\lambda} \delta_{x_{0}} \sigma^{-1} A_{A_{H}} \wedge \frac{1}{\lambda} \delta_{x_{0}} \sigma^{-1} A_{A_{H}}
\]

and the second term is just the topological term. The deformation parameter is defined to be

\[
\lambda \equiv \frac{(\alpha_{\beta}, \mu)(\alpha_{\bar{\beta}}, \bar{\mu})}{(\alpha_{\beta}, \mu)(\alpha_{\bar{\beta}}, \bar{\mu})} = \frac{(\alpha_{\beta}, \mu)(\alpha_{\bar{\beta}}, \bar{\mu})}{(\alpha_{\beta}, \mu)(\alpha_{\bar{\beta}}, \bar{\mu})}.
\]

---

9) Note that the notation \( \lambda \)-deformed principal chiral model

\[ X = \sigma X \sigma^{-1} \] for \( X \in g \).
3.2 Boundary conditions for the holomorphic Chern-Simons theory

If one assumes that the commutativity in Figure 1 still holds, then the boundary conditions eq. (56) for the 4DCS theory can be lifted to the boundary conditions for the 6DhCS via the matching conditions (18) and (19) which implies that the boundary condition on the 6D gauge connection $\mathcal{A}$ should satisfy

$$\langle \alpha, \mu \rangle |_{\mathcal{A}} = \langle \alpha, \mu \rangle |_{\mathcal{A}} = 0,$$

(68)

$$\langle \alpha, \mu \rangle |_{\mathcal{A}} = \langle \alpha, \mu \rangle |_{\mathcal{A}} = 0.$$  

(69)

This boundary condition itself is well defined in the sense that it makes the boundary terms eq. (53) vanish. But it is problematic to construct a 4D theory via solving along fibres. Because of the appearance of the extra factors, the pair $(\mathcal{A}_{Ab}, \mathcal{A}_{Ac})$ for $A = 0, 1$ is not in the Lagrangian subalgebra and it does not belong to proper boundary conditions which are classified in ref. [9]. Therefore if we use this boundary condition, the residue gauge symmetry is\(^{10}\)

$$\tilde{\sigma} \rightarrow \tilde{\sigma} \gamma^{-1}, \quad \mathcal{A} \rightarrow g^{-1} \mathcal{A} g + g^{-1} d g g,$$  

(70)

where the gauge parameter $g^{-1} d g g$ must satisfy the same boundary conditions (68) and (69), thus $g$ is not in $G^2$. The residue gauge freedoms in eq. (58) can not be removed completely\(^{11}\), for example we can simply set $\sigma_+ = \text{id}$ any more. Therefore the resulting 4D theory and 2D theory would be some field theories with gauge symmetries, rather than the $\lambda$-deformed theories that we expect. To get a boundary condition without such problem, we have to set

$$\langle \alpha, \mu \rangle |_{\mathcal{A}} = \langle \alpha, \mu \rangle |_{\mathcal{A}} = 0.$$

(71)

However this implies that $\alpha_+$ and $\alpha_+$ collide to a double pole and thus eq. (51) reduces to eq. (20), which leads to an un-deformed theory. Given the failure of the matching condition one may wonder how about directly imposing $(\mathcal{A}_{Ab}, \mathcal{A}_{Ac})$ in $g^2$ for $A = 0, 1$. Next we will show that this indeed leads to a deformation but only with $\lambda = \pm 1$.

3.3 Solving along fibres for the holomorphic Chern-Simons theory

Recall that the $(3, 0)$-form $\Omega$ is

$$\Omega = \frac{2(\pi i \kappa \alpha) \alpha A B \pi A B}{2(\pi i \kappa \alpha)(\pi i \kappa \alpha)^2}.$$  

(72)

To impose $(\mathcal{A}_{Ab}, \mathcal{A}_{Ac}) \in g^2$, we need to set

$$\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle = 0,$$  

(73)

such that the residues at the two simple poles are opposite. Therefore the boundary conditions are

$$\mathcal{A}_{|b} = 0, \quad \mathcal{A}_{|a} = \mathcal{A}_{|c}, \quad \lambda = 0, 1.$$  

(74)

The 4D Lax connection is given by eq. (30) and can be written as:

$$\mathcal{A}_A = \pi A \mathcal{A} \mathcal{A}^A.$$  

(75)

The gauge parameter $\tilde{\sigma}$ now can be fixed to be

$$\tilde{\sigma} |_{\mathcal{A}} = \tilde{\sigma} = \text{id}, \quad \tilde{\sigma} |_{\mathcal{A}} = \sigma.$$  

(76)

Substituting it into the first boundary condition in eq. (74), we find

$$\beta^A \mathcal{A} \mathcal{A}^A = 0, \quad \Rightarrow \mathcal{A} \mathcal{A}^A \sim \beta \mathcal{A}^A,$$

which implies

$$\mathcal{A}_A = \langle \pi B \rangle \mathcal{A}.$$  

(77)

The second boundary condition in eq. (74) then gives the relation:

$$\langle \alpha, \beta \rangle \mathcal{A}_A = \sigma^{-1} \alpha_+^A \partial A A \sigma + \langle \alpha, \beta \rangle \sigma^{-1} \alpha_+ A \sigma,$$

from which we obtain

$$\mathcal{A}_A = - \frac{\alpha_+^A \partial A A \sigma^{-1}}{\langle \alpha, \beta \rangle} \left( 1 - \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \beta \rangle} \right).$$  

(78)

Substituting them into the 6D action and defining $\mathcal{F} \equiv - \delta \tilde{\sigma} \mathcal{F}^{-1}$, we have

$$S_{\Omega}(\mathcal{A}) = \frac{1}{2 \pi i} \int_{S^4} \Omega \wedge hCS(\mathcal{A})$$

$$= \frac{1}{4 \pi i} \sum_{\kappa \in \mathbb{Z}} \lim_{\rho \rightarrow 0} \int_{S^3} \left( \frac{d\Omega}{(\pi i \kappa \alpha)(\pi i \kappa \alpha)^2} \right)$$

$$\times \int_{S^1} d^2 \mathcal{A}^B \pi A B A \partial A A \wedge \text{tr} \left( \mathcal{F} \wedge \mathcal{A} \right)$$

$$+ \frac{1}{12 \pi i} \int_{S^4} \left( \frac{d\Omega}{\pi \kappa \alpha} \right) \wedge d^2 \mathcal{A}^B \pi A B A \partial A A \text{tr} \left( \mathcal{F}^3 \right).$$  

(79)

The kinetic term in eq. (79) is

$$- \frac{\alpha_+^A \alpha_+^B}{2(\pi i \kappa \alpha)(\pi i \kappa \alpha)^2} \epsilon_{\alpha \beta}^{AB} \int_{S^4} \text{vol}_4 \left( \frac{1}{1 - \lambda A A \sigma} \right) \mathcal{F}^{AB},$$

(80)

\(^{10}\) See eq. (58).

\(^{11}\) Probably the gauge freedoms can be removed by introducing auxiliary edge modes as described in ref. [16].
where we use the identity eq. (7) and \( \lambda = \frac{(\gamma_\sigma)}{(\gamma_\sigma + \gamma_\mu)} \), with
\[
\lambda^2 = 1. \tag{81}
\]

The topological term in the action eq. (79) becomes
\[
\frac{1}{12\pi} \int_{\mathcal{M}} \frac{1}{\pi(\gamma_\sigma \gamma_\mu, \gamma_\sigma \gamma_\mu)} \text{tr} \left( \frac{1}{x^\sigma x^\mu} \right) \nabla \left( \frac{1}{\gamma_\sigma \gamma_\mu} \right)
\]
\[
= \frac{1}{6} \left( \frac{\mu_{\gamma, \gamma}}{(\gamma_\sigma \gamma_\mu, \gamma_\sigma \gamma_\mu)} \right) \int_{\mathcal{M}} \mu_{\gamma, \gamma} \text{tr} \left( \frac{1}{\gamma_\sigma \gamma_\mu} \right), \tag{82}
\]
where \( \mu_{\gamma, \gamma} \equiv d^2 x^\sigma \sigma^\gamma \sigma^\gamma \). Combining eq. (80) with eq. (82), we get the 4D \( \lambda \)-deformed WZW model. Now imposing the symmetry \( H \) to the 4D field \( \sigma \) as we did in the last section, we can get the 2D action with the kinetic term:
\[
\frac{1}{\lambda^2} \text{tr} \left( \frac{1}{\gamma_\sigma \gamma_\mu, \gamma_\sigma \gamma_\mu} \right) \int_{\mathcal{M}} \left( \frac{\mu_{\gamma, \gamma}}{(\gamma_\sigma \gamma_\mu, \gamma_\sigma \gamma_\mu)} \right) \text{d}w \wedge \text{d}v, \tag{83}
\]
and the 3D topological term
\[
\int_{\mathcal{M}} \left( d^2 x^\sigma \sigma^\gamma \sigma^\gamma \right) = 2 \left( (\gamma_\sigma \gamma_\mu, \gamma_\sigma \gamma_\mu) \right)^3. \tag{84}
\]

However the parameter \( \lambda \) is not a continuous parameter anymore, instead \( \lambda = \pm 1 \), which makes the deformation less interesting. What is worse is that the operator \( 1 - A_{d, e} \) may not be invertible considering \( A_{d, e} \).

4 Other possible deformations

In this section, we study other boundary conditions which have been used in the 4DCS to construct 2D integrable field theories, including the \( \eta \)-deformation, the generalized \( \lambda \)-deformation [30] and the deformation of a coupled theory. In first two cases, because there is no zero in the (3, 0)-form \( \Omega \), only trivial defects are inserted. In none of these two cases, we get our desired deformation since the deformation parameters are strictly restricted, compared with those studied in the literature.

4.1 \( \eta \)-deformation

When we discuss the \( \lambda \)-deformation in last section, we have used one of the Lagrangian subalgebra \( g^4 \) in the Manin triple \((g, g^4, \mathfrak{g})\). In the context of the 4DCS, the other Lagrangian subalgebra \( g_0 \) will lead to the \( \eta \)-deformation (the Yang-Baxter deformation). This choice of Lagrangian subalgebra requires the following boundary condition at the simple poles,
\[
(R - 1) \tilde{A}_{d, e} = (R + 1) \tilde{A}_{d, e}, \tag{85}
\]
where the skew-symmetric operator \( R \) satisfies the (split type) modified classical Yang-Baxter equation:
\[
[R_x, R_y] = - R ([R_x, y + [x, R_y]]) = - [x, y]. \tag{86}
\]

(x, y \in \mathfrak{g}, \ R \in \text{End} \mathfrak{g}).

In this case the gauge freedom can be removed by fixing \( \bar{\sigma} \) to
\[
\bar{\sigma}_{|_{\gamma = 0}} = \gamma, \quad \bar{\sigma}_{|_{\gamma = 3}} \equiv \text{id}. \tag{87}
\]

Thus, after assuming eq. (75), we get the following equations:
\[
\tilde{A}_A = (\gamma \bar{\sigma}) A_A, \quad (R - 1)(-\tilde{J}_A^\sigma + (\gamma \bar{\sigma}) A_A) = (R + 1)(-\tilde{J}_A^\sigma + (\gamma \bar{\sigma}) A_A), \tag{88}
\]
where \( \bar{\sigma} = \gamma R \sigma^{-1} \) and \( J_A^\sigma \equiv -\gamma \sigma^\alpha \partial \sigma^\alpha \sigma^{-1} \). The solution of these equations is
\[
A_A = \frac{R (J_A^\sigma - J_A^\sigma) + (J_A^\sigma + J_A^\sigma)}{-R \gamma (\gamma \bar{\sigma}) + (\gamma \bar{\sigma})}, \quad \gamma \bar{\sigma} \equiv \gamma \bar{\sigma} + \gamma \bar{\sigma}. \tag{89}
\]

To avoid the singularity of \( R^{-1} \), we set \( (\gamma \bar{\sigma}) = 0 \) and leave \((\gamma \bar{\sigma})\) to be arbitrary. Substituting eq. (87) into eq. (79), we find that the action of the 4D theory is
\[
S_{4D}[\sigma] = -\frac{e^{AB}}{2(\gamma \bar{\sigma})^2} \int_{\mathcal{M}} \text{tr} \left( \frac{J_A^\sigma J_B^\sigma - J_A^\sigma R \sigma^{-1} J_B^\sigma}{J_A^\sigma + J_B^\sigma} \right)
\]
\[
+ \frac{1}{6} \int_{\mathcal{M}} \text{tr} \left( \frac{\mu_{\gamma, \gamma}}{(\gamma \bar{\sigma})^2} \right), \tag{90}
\]
where \( \mu_{\gamma, \gamma} \equiv d^2 x^\sigma \sigma^\gamma \sigma^\gamma \). This 4D theory is integrable in the sense that its equation of motion can be cast into the form of ASDYM. Without loss of generality we fix the normalization \((\gamma \bar{\sigma}) \equiv 1 \) then perform the symmetry reduction according to eqs. (36) and (37). The resulting action of the 2D theory is
\[
S_{2D}[\sigma] = \int_{\mathcal{M}} \text{tr} \left( j_\sigma (1 + \eta R \sigma) j_\sigma \right) + \frac{k}{3} \int_{\mathcal{M}} \text{tr} \left( \frac{1}{\gamma \bar{\sigma}} \right), \tag{91}
\]
where we have dropped the overall factor $1/\langle \alpha, \beta \rangle^2$ and defined
\begin{equation}
\kappa = \langle \alpha, \mu \rangle \langle \alpha, \bar{\mu} \rangle - \langle \alpha, \mu \rangle \langle \alpha, \bar{\mu} \rangle, \quad \eta = -\langle \gamma, \mu \rangle \langle \gamma, \bar{\mu} \rangle. \tag{90}
\end{equation}

Unfortunately it is not the action of the Yang-Baxter model, and moreover it has been shown in ref. [31] that the theory eq. (89) is not integrable for non-vanishing $\eta$. From the perspective of our construction, the non-integrability is due to the failure of the matching conditions such that eq. (89) can not be obtained from the 4DCS theory. It is a kind of puzzling that under symmetry reduction the 4D Lax connection can not be pushed forward to the Lax connection of a 2D integrable system. To understand the failure of the pushing forward, let us contract the ASDYM with $\beta^A \beta^B$, $\beta^A \beta^B$ and $\beta^A \beta^B$, and obtain
\begin{equation}
\beta^A \beta^B \mathcal{E}_{\alpha\beta} = 0, \tag{91}
\end{equation}
\begin{equation}
\beta^A \beta^B \mathcal{E}_{\alpha\beta} = \mathcal{A} \mathcal{B} \left( \beta^A \partial_{\alpha} A_B - \beta^B \partial_{\beta} A_A - [A_A, A_B] \right) = 0, \tag{92}
\end{equation}
\begin{equation}
\beta^A \beta^B \mathcal{E}_{\alpha\beta} = \mathcal{A} \mathcal{B} \partial_{\alpha} \mathcal{K}_B - \partial_{\beta} \mathcal{K}_A = 0. \tag{93}
\end{equation}

As before, the symmetry reduction of eq. (93) gives the equation of motion of the 2D theory eq. (89):
\begin{equation}
\mathcal{E} \mathcal{B} \partial_{\alpha} \mathcal{A}_B = (\beta \mu) \partial_{\alpha} \mathcal{K}_B + (\beta \bar{\mu}) \partial_{\alpha} \mathcal{K}_B = 0, \tag{94}
\end{equation}
with
\begin{align*}
\mathcal{K}_B &= \langle \gamma, \mu \rangle \langle \gamma, \bar{\mu} \rangle \langle \gamma, \mu \rangle \langle \gamma, \bar{\mu} \rangle R_x \mathcal{J}_B = (1 + k + \eta R) \mathcal{J}_B, \\
\mathcal{K}_B &= -\langle \gamma, \mu \rangle \langle \gamma, \bar{\mu} \rangle \langle \gamma, \mu \rangle \langle \gamma, \bar{\mu} \rangle R_x \mathcal{J}_B = (1 - k - \eta R) \mathcal{J}_B,
\end{align*}
where we have used the fact that $\langle \gamma, \beta \rangle = 0$, $\gamma \sim \beta$. Similarly, the symmetry reduction of eq. (92) leads to
\begin{equation}
\partial_{\alpha} \mathcal{K}_B - \partial_{\beta} \mathcal{K}_B + \mathcal{K}_B [\mathcal{K}_B, \mathcal{K}_B] + x (\partial_{\alpha} \mathcal{K}_B + \partial_{\beta} \mathcal{K}_B) = 0, \tag{95}
\end{equation}
with
\begin{equation}
x = \langle \alpha, \beta \rangle \left( \langle \beta \mu \rangle \langle \gamma, \bar{\mu} \rangle + \langle \beta \bar{\mu} \rangle \langle \gamma, \mu \rangle \right). \tag{96}
\end{equation}

However the flatness of $\mathcal{K}_B$ is incompatible with the flatness of $\mathcal{J}_B$ when $\eta \neq 0$. Consequently, the symmetry reduction of $A_{\alpha \beta}$ is not a 2D Lax connection anymore.

4.2 $\eta$-deformation with the WZ term

In the previous construction, the resulting 2D theory is less interesting. Comparing with the construction of $\eta$-deformation from 4D CS, it seems that the constraint eq. (73) is too strong thus we have to set the coefficient of $R_{\gamma}$ to zero in the denominator of eq. (87) to get an integrable theory. Another interesting observation is that the topological term which leads to a non-trivial WZ term in the resulting 2D theory is non-vanishing even though we have imposed the conditions (73). The origin of this non-vanishing topological term eq. (73) is due to the fact that the topological term does not depend on the residue only, thus if we start with the 6DhCS, it seems hard to eliminate the WZ term in the resulting 2D theory. This suggests that it may be better to construct the integrable $\eta$-deformation with an WZ term by relaxing the constraint and see whether the commutativity holds. However as a cost we have to modify the Lagrangian subalgebra
\begin{equation}
\mathcal{A}_1 \rightarrow \mathcal{A}_2 \tag{97}
\end{equation}
with
\begin{equation}
R^3 = \mathcal{C}^2 R, \quad [R_x, R_y] = (c^2 x, y) \equiv -c^2 [x, y]. \tag{98}
\end{equation}

\begin{equation}
\theta = -\langle \gamma, \beta \rangle \langle \gamma, \bar{\mu} \rangle \tag{99}
\end{equation}

Accordingly the boundary condition at the pair of simple poles becomes
\begin{equation}
(\mathcal{R} + c) \mathcal{A}_B |_{\alpha} = (\mathcal{R} - c) \mathcal{A}_B |_{\alpha}. \tag{100}
\end{equation}
The Dirichlet boundary condition at the double pole $\beta$ still leads to $\mathcal{A}_B = (\gamma \beta) |_{\alpha}$. Then using eq. (30) one can solve
\begin{equation}
\mathcal{A}_B = \frac{1}{c (\gamma, \beta)} \left( \langle \gamma, \beta \rangle + c \right) \frac{c}{2} \left( \frac{c R_x}{c R_x} \right) - \frac{1}{\mathcal{A}_A} \left( \frac{c R_x}{c R_x} \right) - \frac{1}{\mathcal{A}_A} \left( \frac{c R_x}{c R_x} \right). \tag{101}
\end{equation}

Substituting eq. (101) into eq. (79) will give the 4D action. The kinetic term of the Lagrangian density is
\begin{equation}
-\frac{\mathcal{E}_{\alpha \beta}}{c (\gamma, \beta)} \text{Tr} \left( \frac{1}{\mathcal{A}_B} \left( \frac{c R_x}{c R_x} A_B \right) - \frac{1}{\mathcal{A}_B} \langle \alpha \beta \rangle \mathcal{A}_B \right) = -\frac{\mathcal{E}_{\alpha \beta}}{c (\gamma, \beta)} \text{Tr} \left( \frac{c R_x}{c R_x} A_B \right) \left( \frac{c R_x}{c R_x} \right) + \frac{1}{2} \frac{c R_x}{c R_x} \mathcal{A}_B. \tag{102}
\end{equation}

The topological term can be obtained similarly as before
\begin{equation}
\frac{1}{b (\alpha, \beta)} \int_{\mathcal{B} \times [0, 1]} \left( \langle \alpha, \beta \rangle \mathcal{A}_B \right) \wedge (\mathcal{J})^3. \tag{103}
\end{equation}
Next we perform the same symmetry reduction as before to derive the 2D theory. Using the identity:
\[ e^{\hat{A}_2} j_B^2 M_B^2 = (\mu\phi_2) \mu\phi j_B M_B^2 - (\mu\phi_2) \mu\phi j_B M_B^2, \]
(104)
one can obtain the action of the 2D theory\(^{12}\)
\[ S_{2D}[\sigma] = \int_{\mathbb{R}^2} \text{tr} \left( j_{\sigma} (1 + \eta R_\sigma) j_{\bar{\sigma}} \right) + \frac{k}{3} \int_{\mathbb{R}^2 \times [0,1]} \text{tr} \left( \hat{J}^2 \right), \]
(105)
which is in the same form as eq. (89) but with the parameters being
\[ \eta = \frac{1}{c(\alpha, \alpha_\sigma)} \left( \frac{(\alpha, \beta)}{(\alpha, \beta)} \alpha + \frac{(\alpha, \beta)}{(\alpha, \beta)} \alpha \right), \]
(106)
\[ k = \frac{(\alpha, \beta)}{(\alpha, \beta)} \alpha \alpha - \frac{(\alpha, \beta)}{(\alpha, \beta)} \alpha \alpha, \]
(107)
\[ [\alpha, \beta] \equiv (\mu\phi)(\mu\phi). \]
(108)
This 2D theory is integrable only when \( \eta = 0 \) and one possible solution of this integrability locus is \( \mu = \alpha, \mu = \alpha \).

Next we derive the 2D theory from the other route. The symmetry reduction will lead to the 1-form eq. (51). To construct a Lagrangian subalgebra of \( \mathfrak{b} \), we should fix \( \theta \) to particular value. The condition that \( \mathfrak{g}_\theta \) being isotropic in \( \mathfrak{b} \) gives that
\[ \frac{(\alpha, \beta)(\alpha, \beta)}{(\alpha, \beta)^2} (\theta - c)^2 = \frac{(\alpha, \beta)(\alpha, \beta)}{(\alpha, \beta)^2} (\theta + c)^2, \]
(109)
which is the analogue of eq. (98). We can solve the relation and find \( \theta \) to be
\[ \theta = c - \frac{2c}{(\alpha, \beta)(\alpha, \beta)} (\alpha, \beta), \]
(110)
Note that we have discarded the solution which is singular when \( \alpha = \alpha \). Then we can take the boundary conditions to be
\[ (\hat{R} + c) \hat{A}_{\mu\nu} |_{\nu=\alpha} = (\hat{R} - c) \hat{A}_{\mu\nu} |_{\nu=\alpha}, \]
(111)
\[ \hat{A}_{\mu\nu} |_{\nu=\beta} = 0. \]
(112)
We can fix the gauge freedom of \( \hat{\sigma} \) such that
\[ \hat{\sigma} |_{\nu=\alpha} = \sigma, \hat{\sigma} |_{\nu=\beta} = \text{id}. \]
(113)
Then the boundary conditions can be rewritten as:
\[ (\hat{R} + c)(\sigma^{-1} \partial_{\nu\sigma} \sigma + \sigma^{-1} \mathcal{L}_{\nu\sigma} \sigma) |_{\nu=\alpha}, \]
(114)
\[ \mathcal{L}_{\nu=\beta} = 0. \]
(115)
We can take the Lax connection to be of the same form as eq. (26), i.e.,
\[ \mathcal{L}_w = \frac{\langle \gamma_B \rangle}{\langle \gamma_B \rangle} U_w, \quad \mathcal{L}_{\bar{w}} = \frac{\langle \gamma_B \rangle}{\langle \gamma_B \rangle} U_{\bar{w}}, \]
(116)
where \( U_w, U_{\bar{w}} \) do not depend on \( \pi \). Putting this into the boundary conditions we will have
\[ \left[ \left( \frac{(\alpha, \beta)}{(\alpha, \mu)} + \frac{(\alpha, \beta)}{(\alpha, \mu)} \right) c + \left( \frac{(\alpha, \beta)}{(\alpha, \mu)} - \frac{(\alpha, \beta)}{(\alpha, \mu)} \right) \hat{R}_\sigma \right] U_w = 2c j_w, \]
(117)
\[ \left[ \left( \frac{(\alpha, \beta)}{(\alpha, \mu)} + \frac{(\alpha, \beta)}{(\alpha, \mu)} \right) c + \left( \frac{(\alpha, \beta)}{(\alpha, \mu)} - \frac{(\alpha, \beta)}{(\alpha, \mu)} \right) \hat{R}_{\bar{w}} \right] U_{\bar{w}} = 2c j_{\bar{w}}. \]
(118)
For future convenience, we can define
\[ r_+ := \frac{(\alpha, \beta)}{(\alpha, \mu)} + \frac{(\alpha, \beta)}{(\alpha, \mu)}, \quad r_- := \frac{(\alpha, \beta)}{(\alpha, \mu)} - \frac{(\alpha, \beta)}{(\alpha, \mu)}, \]
(119)
\[ \hat{r}_+ := \frac{(\alpha, \beta)}{(\alpha, \mu)} + \frac{(\alpha, \beta)}{(\alpha, \mu)}, \quad \hat{r}_- := \frac{(\alpha, \beta)}{(\alpha, \mu)} - \frac{(\alpha, \beta)}{(\alpha, \mu)}. \]
(120)
Then solutions for eqs. (117) and (118) can be written as:
\[ U_w = 2c (a_1 + a_2 R_\sigma + a_3 R_{\sigma}^2) j_w, \]
(121)
\[ U_{\bar{w}} = 2c (\hat{a}_1 + \hat{a}_2 R_\sigma + \hat{a}_3 R_{\sigma}^2) j_{\bar{w}}, \]
(122)
where \( R_\sigma = \sigma R_\sigma^{-1} \) and we have used the following shorthand notations:
\[ a_1 = \frac{1}{c r_+ + c \hat{r}_-}, \]
(123)
\[ a_2 = -\frac{1}{2[c^2 - 1] c r_+ - c r_+ + r_-} + \frac{1}{2[c^2 - 1] c r_+ - c r_+ + r_+}, \]
(124)
\[ a_3 = -\frac{1}{2[c^2 - 1] c r_+ - c r_+ + r_-} - \frac{1}{2[c^2 - 1] c r_+ - c r_+ + r_+}, \]
(125)
\[ \hat{a}_1 = \frac{1}{c \hat{r}_+ + c \hat{r}_-}, \]
(126)
\[ \hat{a}_2 = -\frac{1}{2[c^2 - 1] c \hat{r}_+ - c \hat{r}_+ + \hat{r}_-} + \frac{1}{2[c^2 - 1] c \hat{r}_+ - c \hat{r}_+ + \hat{r}_+}, \]
(127)
\[ \hat{a}_3 = -\frac{1}{2[c^2 - 1] c \hat{r}_+ - c \hat{r}_+ + \hat{r}_-} - \frac{1}{2[c^2 - 1] c \hat{r}_+ - c \hat{r}_+ + \hat{r}_+}. \]
(128)
Then we can write down the action for the resulting 2D theory:
\[ \int_{\mathbb{R}^2} \text{tr} \left( j_{\sigma} (c_0 + c_1 R_\sigma + c_2 R_{\sigma}^2) j_{\bar{\sigma}} \right) + \frac{k}{3} \int_{\mathbb{R}^2 \times [0,1]} \text{tr} \left( \hat{J}^2 \right), \]
where
\[ c_0 = \frac{2c}{(\alpha, \alpha)} \left( \frac{(\alpha, \beta)(\alpha, \beta)}{(\alpha, \beta)^2} \hat{a}_1 - \frac{(\alpha, \beta)(\alpha, \beta)}{(\alpha, \beta)^2} a_1 \right). \]
(129)
\(^{12}\) Where we have dropped the overall factor \((\alpha, \beta)/\alpha\beta\).
\[ c_1 = \frac{2c}{(\alpha, \alpha_0)} \left( \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \beta _2^2 + \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \beta _2^2 \right), \quad (130) \]

\[ c_2 = \frac{2c}{(\alpha, \alpha_0)} \left( \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \beta _3^2 - \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \beta _3^2 \right), \quad (131) \]

\[ k = \frac{1}{(\alpha, \alpha_0)} \left( \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \beta _2^2 - \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \beta _2^2 \right). \quad (132) \]

As shown in ref. [31], this model is integrable if and only if the coefficients satisfy

\[ c_1^2 = c_2 c_0 (c_0 - c_0 c_2 - k^2). \quad (133) \]

Obviously, the diagram in Figure 1 is not commutative, and relaxing the constraint eq. (73) is not helpful.

### 4.3 Generalized α-deformation

The 2D generalized α-deformed models have been successfully constructed from the 4DCS in ref. [10]. In this case, the 1-form \( \omega \) has only pairs of simple poles. This suggests us to consider the (3, 0)-form \( \Omega \) with four simple poles in the 6DhCS:

\[ \Omega = \frac{\langle \alpha, \beta \rangle \wedge d^2 x^{AB} \pi_{\alpha} \pi_{\beta}}{2(\pi_{\alpha} - \pi_{\beta}) (\pi_{\alpha} - \pi_{\beta})}, \quad (134) \]

with the conditions

\[ \langle \alpha, \beta \rangle \langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle \langle \alpha, \beta \rangle, \quad (135) \]

These conditions lead to

\[ \text{Res}_{\alpha} \Omega + \text{Res}_{\beta} \Omega = \Omega = 0. \quad (136) \]

Parameterizing the spinors as\(^{13}\):

\[ \alpha_+ = (0, 1), \quad \alpha_- = (1, 0), \quad \beta_+ = (b_1, b_2), \quad \beta_- = (b_3, b_4), \quad (137) \]

we can solve the conditions eq. (135) by

\[ \beta_+ = (a, b), \quad \beta_- = \pm (b, a). \quad (138) \]

Moreover we impose the requirement

\[ a \neq 0, \quad b \neq 0, \quad |a| \neq |b|, \quad (139) \]

such that the four poles do not collide. At the four simple poles, we impose the boundary conditions:

\[ \mathcal{A}_{\lambda} |_{\alpha} = \mathcal{A}_{\lambda} |_{\beta}, \quad (R - 1) \mathcal{A}_{\lambda} |_{\alpha} = (R + 1) \mathcal{A}_{\lambda} |_{\beta}. \quad (140) \]

and in order to remove the gauge freedom we fix

\[ \mathcal{A}_{\lambda} = \sigma |_{\alpha}, \quad \mathcal{A}_{\lambda} = \sigma |_{\beta}. \quad (141) \]

With this choice of boundary condition eq. (140), one might expect to obtain a 4D analogue of generalized λ-deformed models [30] which contain more than one deformation parameter. However, we will show that the resulting 4D theories actually have only one deformation parameter.

Assuming \( \mathcal{A}_{\lambda} = \pi^{\lambda} A_{\lambda} \alpha \) and using eq. (30), we arrive at the equations\(^{14}\):

\[ \text{Ad}_{\lambda} \alpha \lambda = -f_{\lambda}^{\alpha} + A_{\lambda}^{\alpha}, \quad (R - 1) A_{\lambda}^{\alpha} = (R + 1) A_{\lambda}^{\alpha}, \quad (142) \]

from which one can solve

\[ A_{\lambda}^{\alpha} = \frac{f_{\lambda}^{\alpha}}{\delta N - \text{Ad}_{\lambda}}, \quad A_{\lambda}^{\alpha} = \frac{f_{\lambda}^{\alpha}}{1 - \delta \text{Ad} N^{-1}}, \quad (143) \]

with

\[ \delta = \pm 1, \quad N = \eta R + 1, \quad \eta = \frac{a - \delta b}{a + \delta b}. \quad (144) \]

Therefore we find

\[ \epsilon_{AB} \text{tr} \left( f_{\lambda}^{\alpha} A_{\lambda}^{\beta} \right) = \frac{\epsilon_{AB}}{2} \text{tr} \left( f_{\lambda}^{\alpha} \left( (1 - \delta \text{Ad}_{\lambda} N^{-1})^{-1} \right) \left( (1 - \delta N^{-1} \text{Ad}_{\lambda} N^{-1})^{-1} f_{\lambda}^{\alpha} \right) \right). \quad (145) \]

The symmetry reduction along \( H \) leads to

\[ (\alpha, \mu) \langle \alpha, \mu \rangle \text{tr} \left( J_{\alpha} \left( (1 - \delta \text{Ad}_{\lambda} N^{-1})^{-1} \right) \left( (1 - \delta N^{-1} \text{Ad}_{\lambda} N^{-1})^{-1} J_{\alpha} \right) \right) = \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \text{tr} \left( J_{\alpha} J_{\alpha} + 2 J_{\alpha} \frac{1}{\delta N \text{Ad}_{\lambda} N^{-1} - 1} J_{\alpha} \right). \quad (146) \]

which reproduces the action of the generalized \( \lambda \)-deformed PCM [30] with \( \lambda = \delta N \).

(147)

However, like the situation of \( \lambda \)-deformation, there is only one free parameter.

### 5 Coupled \( \lambda \)-deformation: An example with a fourth-order pole

In the study of \( \lambda \)-deformation, we noticed that the failure in constructing the 4D deformed model could be due to the absence of zero in the (3, 0)-form \( \Omega \) and thus no nontrivial defect is inserted. It would be interesting to consider the (3, 0)-form \( \Omega \) with zeros. The simplest (3, 0)-form with zeros and even numbers of simple poles is

\[ \Omega = \frac{\langle \alpha, \beta \rangle \wedge d^2 x^{AB} \pi_{\alpha} \pi_{\beta}}{2(\pi_{\alpha} - \pi_{\beta}) (\pi_{\alpha} - \pi_{\beta})}. \quad (148) \]

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\(^{13}\) One can choose a more general parameterization but it will not change the conclusion.

\(^{14}\) Here we introduce the notations \( A_{\lambda}^{\alpha} = \alpha_{\lambda} A_{\lambda} \alpha \) and \( \beta_{\lambda}^{\alpha} = \beta_{\lambda}^{\alpha} A_{\lambda} \alpha \).
In this case, there is a fourth-order pole. In next section, we will consider another $3, 0$-form $\Omega$ with zeros and two double poles. As before we require $\text{Res}_x \Omega + \text{Res}_y \Omega = 0$, i.e.,
\[
\langle \alpha, \mu \rangle \langle \alpha, \mu \rangle = \frac{\langle \alpha, \mu \rangle \langle \alpha, \mu \rangle}{\langle \alpha, \beta \rangle^2}
\]
(149)

To avoid the singularity at the position of the fourth-order pole in the action eq. (1), the gauge connection there should be proportional to $(\nabla \phi)^2$. Therefore the proper boundary conditions are
\[
\mathcal{A}^{\beta}_{\sigma x} = (\nabla \phi)^2, \quad \mathcal{A}^{\beta}_{\sigma y} = \mathcal{A}^{\beta}_{\sigma x}, \quad A = 0, 1
\]
(150)

Because the gauge transformation (30) should be compatible with this boundary condition we can not use the archipelago condition to set $\sigma = \text{id}$. Instead, it was shown in ref. [17] that the proper archipelago condition at the fourth-order pole is $\sigma = \exp\left(-\frac{\langle \nabla \phi \rangle^2}{||\nabla \phi||^2}\phi\right)$, where $\phi$ is a regular function valued in $g$. At the two simple poles, we can remove the residue gauge freedom as before. Therefore, we have
\[
\partial_\sigma |_{V_{\alpha_x}} = \sigma, \quad \partial_\sigma |_{V_{\alpha_y}} = \text{id}, \quad \partial_\sigma |_{V_{\beta}} = \exp\left(-\frac{\langle \nabla \phi \rangle^2}{||\nabla \phi||^2}\phi\right)
\]
(151)

where $V_{\alpha_x}, V_{\alpha_y}, V_{\beta}$ denote small neighborhoods of $\alpha_x, \alpha_y, \beta$, respectively.

Combining eq. (151) with eq. (150) and considering the analytic structure of the $(3, 0)$-form $\Omega$, we expect that the 4D connection $\mathcal{A}$ should be proportional to $(\nabla \phi)$ and have two simple poles at $\mu_x$ and $\mu_y$. Thus we take the following ansatz:
\[
\mathcal{A} = (\nabla \phi) \left( \kappa_\alpha \frac{\langle \pi \mathcal{A}_\alpha \rangle}{\langle \pi \mu_x \rangle} + \kappa_\beta \frac{\langle \pi \mathcal{A}_\beta \rangle}{\langle \pi \mu_y \rangle} \right),
\]
(152)

where $\kappa_\alpha$ is a spinor of norm one, i.e., $||\kappa|| = 0$. Note that $\mathcal{A}_{\sigma \alpha}$ and $\mathcal{A}_{\sigma \alpha}$ do not depend on $\pi^A$ here. For later convenience, we expand $\mathcal{A}_{\alpha}$ and $\mathcal{A}_{\sigma \alpha}$ in the following form:
\[
\mathcal{A}_A = \beta A_\alpha A^\alpha + \beta A_\alpha A^\beta, \quad \mathcal{A}_{\alpha \beta} = \beta A_\alpha A^\beta + \beta A_\beta A^\beta.
\]
(153)

The first boundary condition in eq. (150) is equivalent to
\[
\kappa_\alpha \frac{\langle \beta \mathcal{A}_\alpha \rangle}{\langle \beta \mu_x \rangle} - \kappa_\beta \frac{\langle \beta \mathcal{A}_\beta \rangle}{\langle \beta \mu_y \rangle} + \beta A_\alpha \partial_{AA} \phi = 0.
\]
(154)

Putting the decomposition eq. (153) into these equations and solving the resulting equations, we get
\[
A_\alpha^\beta = -\langle \beta \mu_x \rangle \kappa_\beta \beta A^\alpha \partial_{AA} \phi, \quad A_\beta^\beta = -\langle \beta \mu_x \rangle \kappa_\beta \beta A^\alpha \partial_{AA} \phi.
\]
(155)

Substituting the ansatz eq. (152) in the second boundary equation in eq. (150) gives
\[
\langle \alpha, \beta \rangle \kappa_\alpha \frac{\langle \alpha \mathcal{A}_\alpha \rangle}{\langle \alpha \mu_x \rangle} - \kappa_\beta \frac{\langle \alpha \mathcal{A}_\beta \rangle}{\langle \alpha \mu_x \rangle} = \langle \alpha, \beta \rangle \kappa_\beta \frac{\langle \alpha \mathcal{A}_\beta \rangle}{\langle \alpha \mu_x \rangle} - \kappa_\beta \frac{\langle \alpha \mathcal{A}_\beta \rangle}{\langle \alpha \mu_x \rangle}.
\]
(156)

By contracting it with $\kappa_\alpha$ and $\kappa_\beta$, we find $A_\alpha^\beta$ and $A_\beta^\beta$.

The action $S_{\Omega}[\mathcal{A}]$ in this case get contributions from the poles of $\Omega$ both at $\alpha_x$ and $\beta$, i.e.,
\[
S_{\Omega}[\mathcal{A}] = \frac{1}{2m} \text{lim}_{\epsilon \to 0} \int \mathcal{F}_{\sigma x} \frac{\langle \pi \mathcal{A}_\alpha \rangle}{\langle \pi \mu_x \rangle} \left( \frac{d\pi}{\langle \pi \mathcal{A}_\alpha \rangle} \left( \frac{d\pi}{\langle \pi \mu_x \rangle} \right) \right) \int_{B_{\sigma x}} \text{vol}_4 \epsilon^{AB} \text{tr}(\mathcal{F}_{\sigma x} \mathcal{A}^\alpha \mathcal{A}_\beta)
\]
\[
- \frac{1}{2m} \text{lim}_{\epsilon \to 0} \int \mathcal{F}_{\sigma y} \frac{\langle \pi \mathcal{A}_\beta \rangle}{\langle \pi \mu_y \rangle} \left( \frac{d\pi}{\langle \pi \mathcal{A}_\beta \rangle} \left( \frac{d\pi}{\langle \pi \mu_y \rangle} \right) \right) \int_{B_{\sigma y}} \text{vol}_4 \epsilon^{AB} \text{tr}(\mathcal{F}_{\sigma y} \mathcal{A}^\alpha \mathcal{A}_\beta)
\]
\[
+ \frac{1}{12m} \int_{\mathbb{T}} \left( \frac{d\pi}{\langle \pi \mathcal{A}_\alpha \rangle} \wedge d^2 \varepsilon^A \epsilon^{\beta A} \left( \frac{d\pi}{\langle \pi \mathcal{A}_\alpha \rangle} \wedge \frac{d\pi}{\langle \beta \mu_x \rangle} \right) \right) \text{tr}(\mathcal{F}_{\sigma x} \mathcal{A}^\alpha \mathcal{A}_\beta)
\]
(160)

The contribution from the pole at $\alpha_x$ is simply
\[
- \langle \alpha, \beta \rangle \langle \alpha, \beta \rangle \int \left( \kappa_\beta \kappa_\alpha \beta A^\alpha \partial_{AA} \sigma \kappa_\alpha \beta A^\alpha \partial_{AA} \sigma \right) \left( 1 - \frac{2}{1 - \lambda A_{\sigma \alpha}} \right) A_\alpha^\beta \kappa_\alpha \beta A^\alpha \partial_{BB} \phi
\]
\[
- \langle \alpha, \beta \rangle \langle \alpha, \beta \rangle \int \left( \kappa_\beta \kappa_\alpha \beta A^\alpha \partial_{AA} \sigma \kappa_\alpha \beta A^\alpha \partial_{AA} \sigma \right) \left( 1 - \frac{2}{1 - \lambda A_{\sigma \alpha}} \right) A_\alpha^\beta \kappa_\alpha \beta A^\alpha \partial_{BB} \phi
\]
\[
+ \langle \alpha, \beta \rangle \langle \alpha, \beta \rangle \int \left( \kappa_\beta \kappa_\alpha \beta A^\alpha \partial_{AA} \sigma \kappa_\alpha \beta A^\alpha \partial_{AA} \sigma \right) \left( 1 - \frac{2}{1 - \lambda A_{\sigma \alpha}} \right) A_\alpha^\beta \kappa_\alpha \beta A^\alpha \partial_{BB} \phi
\]
(161)

with $A_\alpha^\beta$ and $A_\beta^\beta$ given in eq. (155). On the other hand, the contribution from the pole at $\beta$ is
\[
\langle \beta \mu_x \rangle \langle \beta \mu_x \rangle \left( \frac{d\pi}{\langle \beta \mu_x \rangle} \right) \left( \frac{d\pi}{\langle \beta \mu_x \rangle} \right) \int_{B_{\beta \mu_x}} \text{vol}_4 \epsilon^{AB} \text{tr}(\mathcal{F}_{\beta \mu_x} \mathcal{A}^\alpha \mathcal{A}_\beta)
\]
\[
- \frac{1}{2m} \text{lim}_{\epsilon \to 0} \int \mathcal{F}_{\beta \mu_x} \frac{\langle \pi \mathcal{A}_\beta \rangle}{\langle \beta \mu_x \rangle} \left( \frac{d\pi}{\langle \beta \mu_x \rangle} \right) \int_{B_{\beta \mu_x}} \text{vol}_4 \epsilon^{AB} \text{tr}(\mathcal{F}_{\beta \mu_x} \mathcal{A}^\alpha \mathcal{A}_\beta)
\]
\[
+ \frac{1}{12m} \int_{\mathbb{T}} \left( \frac{d\pi}{\langle \beta \mu_x \rangle} \wedge d^2 \varepsilon^A \epsilon^{\beta A} \left( \frac{d\pi}{\langle \beta \mu_x \rangle} \wedge \frac{d\pi}{\langle \beta \mu_x \rangle} \right) \right) \text{tr}(\mathcal{F}_{\beta \mu_x} \mathcal{A}^\alpha \mathcal{A}_\beta)
\]
(162)

\text{er}
\[ + \kappa^4 \rho^4 \partial_{AA} \phi \frac{A_{(\mu)}^B}{\beta \mu \lambda} - \kappa^4 \rho^4 \partial_{AA} \phi \frac{A_{(\mu)}^B}{\beta \mu \lambda} \].

(162)

The topological term can be obtained as before so we do not repeat the analysis here. By adding eqs. (161) and (162) up, we get the action of the 4D theory. Note that the first line of eq. (161) is very similar to \( \lambda \)-deformed principal chiral model in 2D (one can further check it by doing symmetry reduction). However, we also get some extra terms which couple the \( \sigma \) field to an new scalar field \( \phi \). Thus this model should not be viewed as 4D analogue of \( \lambda \)-deformed principal chiral model. It has not been fully studied in the literature.

One can also do the symmetry reduction directly on the twister space, which yields the 4DCS theory on \( \mathbb{R}^2 \times \mathbb{C}^2 \) with 1-form:

\[ \omega = \frac{\langle \pi \partial \rho \rangle \langle \rho \pi \mu \rangle \langle \pi \mu \rangle \langle \pi \pi \mu \rangle}{\langle \pi \partial \rho \rangle \langle \rho \pi \rangle \langle \pi \mu \rangle \langle \pi \pi \mu \rangle} \].

(163)

The second boundary condition in eq. (150) will not be compatible with the matching condition. In order to proceed, we impose the new condition and boundary condition:

\[ \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle = \frac{\langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle}{\langle \alpha, \mu \rangle \langle \alpha, \mu \rangle} \]  

(164)

\[ \tilde{A}_{w, \mu | \pi \lambda - \pi \lambda} = \tilde{A}_{w, \mu | \pi \lambda - \pi \lambda} \]  

(165)

Then we can again fix the gauge as eq. (151). Considering that there are four zeros in the 1-form eq. (163), the 2D Lax connection should be of the form:

\[ L_\omega = \langle \pi \beta \rangle \left( \frac{\beta_{\mu \lambda}}{\langle \rho \pi \mu \rangle} U_\omega + \frac{\beta_{\mu \lambda}}{\langle \rho \pi \rangle} V_\omega \right), \]  

(166)

\[ L_\omega = \langle \pi \beta \rangle \left( \frac{\beta_{\mu \lambda}}{\langle \rho \pi \mu \rangle} U_\omega + \frac{\beta_{\mu \lambda}}{\langle \rho \pi \rangle} V_\omega \right), \]  

(167)

where \( U \) and \( V \) are regular function on \( \mathbb{R}^2 \). Thanks to the relation of \( U \) and \( V \), the boundary conditions of the gauge fields at the fourth-order pole and at the two simple poles are equivalent to

\[ U_\omega + V_\omega = - \partial_\omega \phi, \quad U_\omega + V_\omega = - \partial_\omega \phi, \]

\[ \partial_\omega \sigma^{-1} + \langle \alpha, \beta \rangle \left( \frac{\beta_{\mu \lambda}}{\langle \rho \pi \mu \rangle} U_\omega + \frac{\beta_{\mu \lambda}}{\langle \rho \pi \rangle} V_\omega \right) \]

\[ = A_{\omega} \langle \alpha, \beta \rangle \left( \frac{\beta_{\mu \lambda}}{\langle \rho \pi \mu \rangle} U_\omega + \frac{\beta_{\mu \lambda}}{\langle \rho \pi \rangle} V_\omega \right), \]  

(168)

from which one can solve

\[ U_\omega = \frac{\dot{\omega}_w - D_{\omega} \partial_\omega \phi}{N_{\omega}(1 - \lambda A_{\omega}^{-1})}, \quad U_\omega = \frac{\dot{\omega}_w - D_{\omega} \partial_\omega \phi}{N_{\omega}(1 - \lambda A_{\omega}^{-1})} \]  

(169)

with

\[ \lambda \equiv \langle \alpha, \beta \rangle^2 \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle / \langle \alpha, \beta \rangle^2 \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \]

(170)

\[ N_{\omega} = \langle \alpha, \beta \rangle \left( \frac{\beta_{\mu \lambda}}{\langle \rho \pi \mu \rangle} - \frac{\beta_{\mu \lambda}}{\langle \rho \pi \rangle} \right), \]  

(171)

\[ D_{\omega} = \langle \alpha, \beta \rangle \left( \frac{\beta_{\mu \lambda}}{\langle \rho \pi \mu \rangle} - \frac{\beta_{\mu \lambda}}{\langle \rho \pi \rangle} \right) A_{\omega}, \]  

(172)

\[ D_{\omega} = \langle \alpha, \beta \rangle \left( \frac{\beta_{\mu \lambda}}{\langle \rho \pi \mu \rangle} - \frac{\beta_{\mu \lambda}}{\langle \rho \pi \rangle} \right) A_{\omega}. \]

Note that the definition of \( \lambda \) here is different from eq. (159). To derive the 2D action one only need to substitute the 2D Lax connection into the action of the 4DCS and integrate over \( \mathbb{C}^2 \) with residue theorem\(^{\text{16}}\). The contribution to the Lagrangian density from the pole at \( \alpha, \) is

\[ \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \langle \alpha, \mu \rangle \left( \frac{\dot{\omega}_w}{\langle \alpha, \beta \rangle^2} \right) \left( \frac{\dot{\omega}_w}{\langle \alpha, \beta \rangle^2} \right) \frac{\dot{\omega}_w}{\langle \alpha, \beta \rangle^2} \left( \dot{\omega}_w \right) \]

(173)

Similarly the contribution from the pole at \( \beta \) is

\[ \langle \beta_{\mu \lambda} \rangle \langle \beta_{\mu \lambda} \rangle \langle \beta_{\mu \lambda} \rangle \langle \beta_{\mu \lambda} \rangle \left( \frac{\dot{\omega}_w}{\langle \alpha, \beta \rangle^2} \right) \left( \dot{\omega}_w \right) \]

(174)

As expected, the 2D model describes a \( \lambda \)-deformed model coupled with an additional field \( \phi \). It would be very interesting to understand how to construct this coupled integrable model from only 2D point of view.

6 \( \eta \)-deformation from the trigonometric description

In sect. 4.1 we have shown that we can not construct the \( \eta \)-deformation by solving along fibre first from the 6DhCS due to the fact that there is no zero in the \((3, 0)\)-form \( \Omega \) and no

\(^{16}\) It is also straightforward but tedious to obtain the topological term as before so we will not repeat the analysis here.
non-trivial defect is inserted. In this section we will show that the \( \eta \)-deformation can be constructed if one considers its trigonometric description. The \( \eta \)-deformation in the trigonometric description has been studied from the point of view of the 4DCS in ref. [11]. An important observation made in ref. [11] is that in the trigonometric description the numbers of zeros and double poles of the corresponding 1-form \( \omega \) are doubled comparing with the ones in the rational description such that the resulting 2D field theory has an additional \( \mathbb{Z}_2 \) symmetry, and different types of Yang-Baxter models can be obtained by gauging the \( \mathbb{Z}_2 \) symmetry differently. The constructions in this section can be summarized in Figure 4.

From the 6DhCS point of view, if we want to double the numbers of the double poles we also need to introduce two zeros so that the \( (0,0) \)-form \( \Omega \) becomes

\[
\Omega = \frac{(d \Omega) \wedge d^2 A^\mu y_+ \tau_+ \tau_- (\tau_+ \tau_-)}{2(\tau_+ \tau_-)^2 (\Omega^2)^2}.
\]  

At the two simple poles we still require that the residues are opposite, which is equivalent to set

\[
\frac{\langle \alpha, \mu_+ \rangle \langle \alpha, \mu_- \rangle}{\langle \alpha, \beta \rangle^2 \langle \alpha, \beta \rangle^2} = \frac{\langle \alpha, \mu_+ \rangle \langle \alpha, \mu_- \rangle}{\langle \alpha, \beta \rangle^2 \langle \alpha, \beta \rangle^2}.
\]

6.1 Solving along fibres for the holomorphic Chern-Simons theory

Let us first follow the right route in Figure 4. Similar to eq. (83), the proper boundary conditions should be

\[
\mathcal{A}\big|_{\tau=1} = 0 = \mathcal{A}\big|_{\tau=-1}, \quad (R-1)\mathcal{A}\big|_{\tau=\tau_0} = (R+1)\mathcal{A}\big|_{\tau=\tau_0}, \quad A = 1, 2.
\]  

Recall that the 4D Lax connection \( \mathcal{A} \) is related to the 6D gauge connection through the formal gauge transformation:

\[
\tilde{\mathcal{A}} = -\tilde{\sigma} \mathcal{A} + \sigma, \quad \tilde{\mathcal{A}} \big|_{\tau=1} = \mathcal{A} \big|_{\tau=1}, \quad \tilde{\mathcal{A}} \big|_{\tau=-1} = -\mathcal{A} \big|_{\tau=-1}, \quad \tilde{\mathcal{A}} \big|_{\tau=\tau_0} = -\mathcal{A} \big|_{\tau=\tau_0}.
\]

We may use the gauge freedom \( \tilde{\sigma} \rightarrow \tilde{\sigma} g^{-1} \), where \( g = \exp(i x), x \in \mathbb{R} \) to fix \( \sigma_1 = 1 = \sigma_2 \), then we use the gauge freedom \( \tilde{\sigma} \rightarrow \tilde{h} \tilde{\sigma} \), where \( h : \mathbb{R}^4 \rightarrow G \) to set \( \sigma_1 = 1 \). The gauge connections must have poles at the simple zeros of \( \Omega \), so we make the following ansatz:

\[
\mathcal{A}_k = -\kappa_k [\mathcal{A}] + \kappa_k [\kappa \mathcal{A}],
\]

\[
[k \mathcal{A}] = \frac{\langle \pi \mathcal{A} \rangle \langle \mathcal{A} \rangle}{\langle \pi \mathcal{A} \rangle} \mathcal{U}^{(1)}_k + \langle \pi \mathcal{A} \rangle \bar{U}_k + \langle \pi \bar{U} \rangle \mathcal{U}_k,
\]

\[
[k \kappa \mathcal{A}] = \frac{\langle \pi \mathcal{A} \rangle \langle \mathcal{A} \rangle}{\langle \pi \mathcal{A} \rangle} \mathcal{U}^{(1)}_k + \langle \pi \mathcal{A} \rangle \bar{U}_k + \langle \pi \bar{U} \rangle \mathcal{U}_k,
\]

where \( \mathcal{U}^{(1)}_k U_k, \bar{U}_k \) do not depend on \( \mathbb{C} P^1 \). It is crucial that the gauge connections have weight \( (1, 0) \) even though it is not linear in \( \pi \) anymore. Substituting the ansatz into the boundary conditions (177) gives

\[
U_k = \kappa^A \beta^A \partial_{AA} \sigma \pi^{-1} \equiv F_k^A, \quad \bar{U}_k = -\kappa^A \beta^A \partial_{AA} \sigma \pi^{-1} \equiv \bar{F}_k^A
\]

\[
U_k = \kappa^A \beta^A \partial_{AA} \tilde{\sigma} \pi^{-1} \equiv F_k^A, \quad \bar{U}_k = -\kappa^A \beta^A \partial_{AA} \tilde{\sigma} \pi^{-1} \equiv \bar{F}_k^A
\]

\[
(R - 1) \left( \frac{\langle \alpha, \beta \rangle \langle \alpha, \beta \rangle}{\langle \alpha, \mu_- \rangle} \mathcal{U}^{(1)}_k + \langle \alpha, \beta \rangle \mathcal{U}_k + \langle \alpha, \beta \rangle \bar{U}_k \right) = (R + 1) \left( \frac{\langle \alpha, \beta \rangle \langle \alpha, \beta \rangle}{\langle \alpha, \mu_- \rangle} \mathcal{U}^{(1)}_k + \langle \alpha, \beta \rangle \mathcal{U}_k + \langle \alpha, \beta \rangle \bar{U}_k \right),
\]

\[
(R - 1) \left( \frac{\langle \alpha, \mu_- \rangle \langle \alpha, \mu_- \rangle}{\langle \alpha, \mu_- \rangle} \mathcal{U}^{(1)}_k + \langle \alpha, \mu_- \rangle \mathcal{U}_k + \langle \alpha, \mu_- \rangle \bar{U}_k \right) = (R + 1) \left( \frac{\langle \alpha, \mu_- \rangle \langle \alpha, \mu_- \rangle}{\langle \alpha, \mu_- \rangle} \mathcal{U}^{(1)}_k + \langle \alpha, \mu_- \rangle \mathcal{U}_k + \langle \alpha, \mu_- \rangle \bar{U}_k \right),
\]

which are solved by

\[
U^{(1)}_k = -(\eta + 1) \frac{\langle \alpha, \mu_- \rangle}{\langle \alpha, \beta \rangle} \left( \frac{\gamma_+ \gamma_- + \gamma_+ \gamma_-}{\eta R + 1} \right) \mathcal{U}_k + (\gamma_+ \gamma_- + \gamma_+ \gamma_-) \bar{U}_k,
\]

\[
U^{(1)}_k = (\eta + 1) \frac{\langle \alpha, \mu_- \rangle}{\langle \alpha, \beta \rangle} \left( \frac{\gamma_+ \gamma_- + \gamma_+ \gamma_-}{\eta R - 1} \right) \mathcal{U}_k + (\gamma_+ \gamma_- + \gamma_+ \gamma_-) \bar{U}_k,
\]

where we have defined the following quantities:

\[
\gamma_+ = \langle \alpha, \beta \rangle \pm \langle \alpha, \mu \rangle, \quad \gamma_- = \langle \alpha, \beta \rangle \pm \langle \alpha, \beta \rangle
\]

\[
\eta = \frac{1 - \langle \alpha, \beta \rangle \langle \alpha, \mu_- \rangle}{\langle \alpha, \mu_- \rangle} \frac{1 - \langle \alpha, \beta \rangle \langle \alpha, \mu_+ \rangle}{\langle \alpha, \beta \rangle \langle \alpha, \beta \rangle}
\]

Figure 4 A summary of the construction of \( \eta \)-deformation in the trigonometric description.
To derive the 4D action, we need to substitute the 4D Lax connection just obtained into the action of the 6DhCS:

\[
S_{\Omega}[\mathcal{A}] = \frac{1}{2\pi i} \oint_{\mathcal{J}T} \Omega \wedge hCS(\mathcal{A}) \equiv \frac{1}{4\pi i} \sum_{\sigma} \lim_{\epsilon \to 0} \int_{\mathcal{J}T} \left[ \frac{(d\mu)^2}{(\partial \mu)(\partial \nu)} \frac{(\partial \nu)(\partial \tau)(\partial \sigma)}{(\partial \tau)(\partial \sigma)(\partial \rho)^2} \right] \frac{1}{2\pi i} \cdot \left[ d^2x^{\mathcal{A}} \frac{\pi}{4\pi i} \pi(\mathcal{A}) \pounds_{\mathcal{A}}(\mathcal{A}) \right] \frac{1}{2\pi i} \cdot \left[ d^2x^{\mathcal{A}} \frac{\pi}{4\pi i} \pi(\mathcal{A}) \pounds_{\mathcal{A}}(\mathcal{A}) \right]
\]

(192)

Because \( \hat{\sigma} \) is not trivial at both \( \hat{\beta} \) and \( \hat{\beta} \), eq. (192) is given by

\[
\frac{1}{2\pi i} \sum_{\sigma} \lim_{\epsilon \to 0} \int_{\mathcal{J}T} \left[ \frac{(d\mu)^2}{(\partial \mu)(\partial \nu)} \frac{(\partial \nu)(\partial \tau)(\partial \sigma)}{(\partial \tau)(\partial \sigma)(\partial \rho)^2} \right] \frac{1}{2\pi i} \cdot \left[ d^2x^{\mathcal{A}} \frac{\pi}{4\pi i} \pi(\mathcal{A}) \pounds_{\mathcal{A}}(\mathcal{A}) \right] \frac{1}{2\pi i} \cdot \left[ d^2x^{\mathcal{A}} \frac{\pi}{4\pi i} \pi(\mathcal{A}) \pounds_{\mathcal{A}}(\mathcal{A}) \right]
\]

(193)

To evaluate this contour integral, we need to expand \( \Omega^{\mathcal{A}}_A \mathcal{A}_B \) in powers of \( (\mu \phi) \) and \( (\phi \phi) \). The expansion of \( \mathcal{A} \) is

\[
\hat{\mathcal{A}} = \prod_{\mathcal{A}} \frac{\partial \mathcal{A}}{\partial \sigma} \hat{\sigma}^{-1} \quad = \frac{\partial \mathcal{A}}{\partial \sigma} \hat{\sigma}^{-1} + \frac{\partial \mathcal{A}}{\partial \rho} \hat{\rho} \frac{\partial \mathcal{A}}{\partial \rho} \hat{\rho}^{-1}
\]

(194)

Then we can easily find that the contribution from the terms proportional to \( (\mu \phi) \) is \( -e^{AB} tr \left( \omega_0 \mathcal{A}_A \mathcal{A}_B - \omega_0 \mathcal{A}_A \mathcal{A}_B \right) \), where

\[
\frac{\langle \mu_\mu \rangle}{\langle \mu_\mu \rangle} = \omega_0 + \omega_1 (\mu \phi) + o((\mu \phi)^2),
\]

\[
K_A = -\frac{k_A}{(\beta \mu)} U_k^{(1)} + \frac{k_A}{(\beta \mu)} U_k^{(1)}.
\]

And the contribution from the terms proportional to \( (\phi \phi) \) is \( -e^{AB} tr \left( \hat{\omega}_0 \mathcal{A}_A \mathcal{A}_B + \omega_0 \mathcal{A}_A \mathcal{A}_B \right) \), where

\[
\frac{\langle \nu_\nu \rangle}{\langle \nu_\nu \rangle} = \omega_0 + \omega_1 (\phi \phi) + o((\phi \phi)^2),
\]

\[
K_A = -\frac{k_A}{(\beta \mu)} U_k^{(1)} + \frac{k_A}{(\beta \mu)} U_k^{(1)}.
\]

Adding them up and dropping the vanishing terms, we get the Lagrangian density of the resulting 4D theory:

\[
L_{4D} = e^{AB} tr \left( \omega_0 \mathcal{A}_A \mathcal{A}_B - \omega_0 \mathcal{A}_A \mathcal{A}_B \right) - \omega_0 \mathcal{A}_A \mathcal{A}_B + \omega_0 \mathcal{A}_A \mathcal{A}_B ,
\]

(195)

with

\[
\omega_0 = \frac{(\beta \mu)(\beta \mu)}{(\beta \mu)(\beta \mu)}, \quad \hat{\omega}_0 = \frac{(\beta \mu)(\beta \mu)}{(\beta \mu)(\beta \mu)}.
\]

(196)

This Lagrangian density is invariant under the \( \mathbb{Z}_2 \) symmetry \(^{17}\)

\[
\beta \to \hat{\beta}, \quad \hat{\beta} \to -\beta; \quad J \to J.
\]

(197)

To get the 4D analogue of the usual Yang-Baxter deformation, one may impose the simplest involution relation \( J = J \).

The Lagrangian density then reduces to

\[
L'_{4D} = e^{AB} tr \left( \omega_0 \mathcal{A}_A \mathcal{A}_B - \omega_0 \mathcal{A}_A \mathcal{A}_B \right) + L_{top},
\]

(198)

where the topological terms \( L_{top} \) are similar to eq. (82), and come from the two double poles:

\[
\frac{1}{3} \int_{J_D(0)} \left( k_1 \mu_\mu \beta_\beta + k_2 \mu_\beta \beta_\mu + k_3 \mu_\beta \beta_\beta \right) + tr(J^3),
\]

(199)

where

\[
k_1 = \omega_0 \left( \frac{(\beta \mu)(\beta \mu)}{(\beta \mu)(\beta \mu)} + \frac{(\mu_\mu)(\mu_\mu)}{(\mu_\mu)(\mu_\mu)} \right),
\]

\[
k_2 = \omega_0 \left( \frac{(\beta \beta)(\beta \beta)}{(\beta \beta)(\beta \beta)} + \frac{(\beta \mu)(\beta \mu)}{(\beta \mu)(\beta \mu)} \right),
\]

\[
k_3 = 2(\omega_0 - \omega_0).
\]

Note that these topological terms can not be set to zero at the same time. They lead to a topological term for the 2D theory with level

\[
k = 2k_1 (\beta \mu)(\beta \mu) + 2k_2 (\beta \mu)(\beta \mu) + k_3 (\beta \mu)(\beta \mu) + (\beta \mu)(\beta \mu).
\]

(200)

which can be set to zero by choosing proper \( \mu_\mu, \mu_\beta, \beta \) and \( \hat{\beta} \).

6.2 Symmetry reduction to 2D theory

Now we are ready to do the symmetry reduction. Imposing the symmetry eq. (37) and introducing eq. (38), we can find

\[
\kappa^A \gamma^A \partial_{\mathcal{A}A} \sigma = k^A \langle (\mu \phi)(\phi \phi) \rangle \partial_{\mathcal{A}A} \sigma = -\langle \mu \phi \rangle \partial_{\sigma} \sigma,
\]

(201)

17) The \( \mathbb{Z}_2 \) operation on \( \beta \) coincides with the complex conjugation of the spinors.
\[
\begin{align*}
\partial_{A\kappa} \sigma & = \partial^A (\langle \eta \mu \rangle \mu^N - \langle \eta \mu \rangle \mu^N) \partial_{A\kappa} \sigma = -\langle \eta \mu \rangle \partial_u \sigma.
\end{align*}
\]

(202)

Thus the right hand side of eq. (195) becomes

\[
\begin{align*}
\text{tr} \left[ \left( \langle \beta \mu \rangle \langle \beta \mu \rangle \right)_{\langle \beta \alpha \rangle} \left( \langle \beta \mu \rangle \langle \beta \mu \rangle \right)_{\langle \beta \alpha \rangle} \partial_u \sigma & - \langle \beta \mu \rangle \langle \beta \mu \rangle \partial_u \sigma \partial \sigma - \langle \beta \mu \rangle \langle \beta \mu \rangle \partial_u \sigma \partial \sigma - \\
& + \langle \beta \mu \rangle \langle \beta \mu \rangle - \langle \beta \mu \rangle \langle \beta \mu \rangle \partial_u \sigma \partial \sigma - \langle \beta \mu \rangle \langle \beta \mu \rangle \partial_u \sigma \partial \sigma - \\
& + \langle \beta \mu \rangle \langle \beta \mu \rangle \partial_u \sigma \partial \sigma - \langle \beta \mu \rangle \langle \beta \mu \rangle \partial_u \sigma \partial \sigma - \\
& - \omega_0 \langle \beta \mu \rangle \langle \beta \mu \rangle j_\mu U^{(1)}_k + \omega_0 \langle \beta \mu \rangle \langle \beta \mu \rangle j_\mu U^{(1)}_k
\end{align*}
\]

(203)

where \( U^{(1)}_k \) are given by the same expressions as eqs. (188) and (189) but with the symmetry reduced version of \( U_{\kappa,\kappa} \).

\[
\begin{align*}
U^{(1)}_k = -\langle \beta \mu \rangle j_\mu, & \quad \hat{U}^{(1)}_k = -\langle \beta \mu \rangle j_\mu, \\
U_k = \langle \beta \mu \rangle j_\mu, & \quad \hat{U}_k = \langle \beta \mu \rangle j_\mu,
\end{align*}
\]

(204)

On the other hand, contracting both sides of

\[
d^2 x^A \mathcal{F} \partial_{A\kappa} \alpha_{\beta} \wedge \alpha_{\gamma} \wedge \alpha_{\delta} \wedge \alpha_{\epsilon} = -2 \text{vol}_{4} \left( \mathcal{A}_{\kappa} \mathcal{A}_{\alpha} \mathcal{A}_{\beta} \mathcal{A}_{\gamma} \mathcal{A}_{\delta} \mathcal{A}_{\epsilon} \right)
\]

with \( \mathcal{E} \mathcal{R}_{\mathcal{D}}^{\mathcal{D}'} \) and acting \( \iota_{\gamma,\kappa} \) on both sides of the resulting equation, we have

\[
-\iota_{\gamma,\kappa} \text{vol}_{4} = dw \wedge dw.
\]

Therefore the resulting expression for the symmetry reduced 2D version of eq. (195) is

\[
L_{2D} = \text{tr} \left[ \langle \beta \mu \rangle \langle \beta \mu \rangle \left( \langle \beta \mu \rangle \langle \beta \mu \rangle \right)_{\langle \beta \alpha \rangle} \left( \langle \beta \mu \rangle \langle \beta \mu \rangle \right)_{\langle \beta \alpha \rangle} \right]
\]

(205)

which also enjoys the \( Z_2 \) symmetry eq. (197). Setting \( j = \hat{j} \), we will obtain the following action:

\[
L_{2D} = \text{tr} \left[ \omega_0 \langle \beta \mu \rangle \langle \beta \mu \rangle j_\mu U^{(1)}_k - \omega_0 \langle \beta \mu \rangle \langle \beta \mu \rangle j_\mu U^{(1)}_k \right]
\]

(206)

with the condition eq. (176). If we use the matching conditions eqs. (40) and (41) then the boundary condition of the 4DCS would be

\[
\langle \alpha \mu \rangle (R - 1) \hat{A}_w |_{\eta = \omega} = (\alpha \mu) (R + 1) \hat{A}_w |_{\eta = \omega}, \quad \langle \alpha \mu \rangle (R - 1) \hat{A}_w |_{\eta = \omega} = (\alpha \mu) (R + 1) \hat{A}_w |_{\eta = \omega},
\]

(211)

\[
\langle \alpha \mu \rangle (R - 1) \hat{A}_w |_{\eta = \omega} = 0, \quad \hat{A}_w |_{\eta = \omega} = 0.
\]

(212)

(213)

We encounter the same problem that we revealed in the example of \( \lambda \)-deformation. To proceed, we should not insist on the exact matching of the boundary conditions in the 6Dcs...
and the 4DCS, and that is why we used dotted line in the diagram in Figure 4. Therefore we just start from eq. (210) and introduce a new condition:

\[
\frac{(x_\mu)(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)(x_\mu)} = \frac{(x_\mu)(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)(x_\mu)},
\]

(214)

to ensure that the residues at the two simple poles are opposite. Correspondingly the boundary conditions should be chosen to be

\[
(R - 1)\bar{A}_{\omega}|_{\omega=\infty} = (R + 1)\bar{A}_{\omega}|_{\omega=\infty},
\]

(215)

\[
\bar{A}_{\omega}|_{\rho=0} = 0, \quad \bar{A}_{\omega}|_{\rho=\rho'} = 0.
\]

(216)

In particular the boundary condition is not equivalent to eqs. (177) and (178) which one can check easily using eq. (14). Nevertheless, since the gauge symmetry is the same, we can still use a similar argument above eq. (180) to remove the gauge freedoms by fixing

\[
\partial_\rho \beta \equiv \sigma, \quad \partial_\rho \gamma \equiv \xi, \quad \partial_\rho \alpha = id, \quad \partial_\rho \alpha' \equiv id.
\]

(217)

Eqs. (178), (183), and (14) imply that the 2D Lax connection has to be of the form:

\[
\mathcal{L}_w = \frac{\langle \gamma \beta \rangle}{\langle \gamma \mu \rangle} V^{(1)}_w + \frac{\langle \gamma \beta \rangle}{\langle \gamma \mu \rangle} V_w + \frac{\langle \gamma \beta \rangle}{\langle \gamma \mu \rangle} \bar{V}_w,
\]

(218)

\[
\mathcal{L}_w = \frac{\langle \gamma \beta \rangle}{\langle \gamma \mu \rangle} V^{(1)}_w + \frac{\langle \gamma \beta \rangle}{\langle \gamma \mu \rangle} V_w + \frac{\langle \gamma \beta \rangle}{\langle \gamma \mu \rangle} \bar{V}_w,
\]

(219)

where \( V^{(1)}_w, V_w, \) and \( \bar{V}_w \) do not depend on the coordinates \( \pi \) and \( \eta \) of \( \mathbb{CP}^1 \). Substituting the ansatz eqs. (218) and (219) into the boundary conditions (215) and (216) gives

\[
V_w = -(\beta_\mu) \partial_\eta \sigma^{-1} \equiv -(\beta_\mu) j_w, \quad \bar{V}_w = -(\beta_\mu) \partial_\eta \sigma^{-1} \equiv -(\beta_\mu) j_w,
\]

(220)

\[
V_w = -(\beta_\mu) \partial_\eta \sigma^{-1} \equiv -(\beta_\mu) j_w, \quad \bar{V}_w = -(\beta_\mu) \partial_\eta \sigma^{-1} \equiv -(\beta_\mu) j_w,
\]

(221)

and

\[
(R - 1) \left( \frac{(x_\mu)(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)(x_\mu)} + \frac{(x_\mu)(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)(x_\mu)} \right)
\]

(222)

\[
(R - 1) \left( \frac{(x_\mu)(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)(x_\mu)} + \frac{(x_\mu)(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)(x_\mu)} \right)
\]

(223)

The solution of our ansatz is

\[
V^{(1)}_w = -(\eta' + 1) \frac{(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)},
\]

(224)

\[
V^{(1)}_w = -(\eta' + 1) \frac{(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)},
\]

(225)

where we have introduced the following quantities:

\[
\gamma_w \equiv (x_\mu)(x_\mu)(x_\mu)(x_\mu), \quad \bar{\gamma}_w \equiv (x_\mu)(x_\mu)(x_\mu)(x_\mu),
\]

(217)

\[
\eta' = \frac{(x_\mu)(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)(x_\mu)}.
\]

Finally substituting the 2D Lax connection into the 4DCS action, we end up with

\[
S = \frac{1}{2 \pi} \int_{\mathbb{CP}^1} \omega \wedge phCS(\hat{A})
\]

(226)

\[
S = \frac{1}{2 \pi \eta} \int_{\mathbb{CP}^1} \left[ \frac{d \pi}{\eta} \frac{d \mu}{\eta} \frac{d \nu}{\eta} \frac{d \tau}{\eta} \frac{d \psi}{\eta} \right] \int_{\mathbb{CP}^1} \delta^2 tr(j \wedge L)
\]

(226)

where \( j \equiv d' \sigma^{-1} \). There are also two contributions to the action coming from the terms linear in \( \langle \gamma \beta \rangle \) and the terms linear in \( \langle \gamma \beta \rangle \) in the expansion of \( \langle \gamma \mu \rangle \langle \gamma \mu \rangle \langle \gamma \mu \rangle \langle \gamma \mu \rangle \) \( tr(j \wedge L) \). From those terms proportional to \( \langle \gamma \beta \rangle \), the contribution to the Lagrangian is

\[
tr \left( \omega_1 j_w \frac{(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)} j_w - \omega_1 j_w \frac{(x_\mu)(x_\mu)(x_\mu)}{(x_\mu)(x_\mu)(x_\mu)} j_w \right)
\]

(226)

\[
tr \left( \omega_1 \frac{1}{(x_\mu)(x_\mu)(x_\mu)} j_w j_w - \omega_1 \frac{1}{(x_\mu)(x_\mu)} j_w j_w \right)
\]

(226)

\[
tr \left( \omega_1 \frac{1}{(x_\mu)(x_\mu)(x_\mu)} j_w j_w + \omega_1 \frac{1}{(x_\mu)(x_\mu)(x_\mu)} j_w j_w \right).
\]
where
\[
\frac{\langle \tau \mu \rangle \langle \tau \mu \rangle \langle \tau \mu \rangle \langle \tau \mu \rangle}{\langle \tau \sigma \rangle \langle \tau \sigma \rangle} = \omega'_0 + \omega'_1 \langle \tau \phi \rangle + O((\langle \tau \phi \rangle)^2)
\]
and the contribution from the term proportional to $\langle \tau \dot{\phi} \rangle$ is
\[
\text{tr} \left( \omega'_0 \dot{j}_w \frac{1}{\langle \beta \mu \rangle} \dot{j}_w + \omega'_0 \dot{j}_w \frac{1}{\langle \beta \mu \rangle} \dot{j}_w \right)
- \omega'_0 \dot{j}_w \frac{1}{\langle \beta \mu \rangle} \dot{j}_w - \omega'_1 \dot{j}_w \frac{1}{\langle \beta \mu \rangle} \dot{j}_w + V^{(1)}_w)
- \text{tr} \left( \omega'_0 \frac{1}{\langle \beta \mu \rangle} \dot{j}_w \right) \dot{j}_w + \omega'_0 \frac{1}{\langle \beta \mu \rangle} \dot{j}_w + j_w \dot{j}_w)
- \omega'_0 \frac{1}{\langle \beta \mu \rangle} \dot{j}_w \dot{j}_w \dot{j}_w + \omega'_1 \dot{j}_w \dot{j}_w \dot{j}_w
+ \frac{1}{\langle \beta \mu \rangle} \dot{j}_w - \frac{1}{\langle \beta \mu \rangle} \dot{j}_w + \frac{1}{\langle \beta \mu \rangle} \dot{j}_w
\]
\[
\frac{\langle \tau \mu \rangle \langle \tau \mu \rangle \langle \tau \mu \rangle \langle \tau \mu \rangle}{\langle \tau \sigma \rangle \langle \tau \sigma \rangle} = \omega'_0 + \omega'_1 \langle \tau \phi \rangle + O((\langle \tau \phi \rangle)^2)
\]
Combining these two contributions together, we get the Lagrangian density of the 2D integrable theory
\[
\tilde{L}_{2D} = \text{tr} \left( \frac{\langle \beta \mu \rangle \langle \beta \mu \rangle}{\langle \beta \sigma \rangle \langle \beta \sigma \rangle} \left( -\langle \beta \mu \rangle \langle \beta \mu \rangle \langle j \rangle \dot{j} + \langle \beta \mu \rangle \langle \beta \mu \rangle \dot{j} \dot{j} + j_w \dot{j}_w \right)
+ \frac{1}{\langle \beta \mu \rangle} \dot{j}_w \dot{j}_w \dot{j}_w + \frac{1}{\langle \beta \mu \rangle} \dot{j}_w \dot{j}_w \dot{j}_w
\]
\[
\omega'_0 = \frac{\langle \beta \mu \rangle \langle \beta \mu \rangle \langle \beta \mu \rangle \langle \beta \mu \rangle}{\langle \beta \sigma \rangle \langle \beta \sigma \rangle}, \quad \omega'_1 = \frac{\langle \beta \mu \rangle \langle \beta \mu \rangle \langle \beta \mu \rangle \langle \beta \mu \rangle}{\langle \beta \sigma \rangle \langle \beta \sigma \rangle}
\]
Note that this is almost the same as eq. (206) but with only $U^{(1)}_w$, $U^{(1)}_w$ replaced by $-V^{(1)}_w$ and $-V^{(1)}_w$, so this 2D theory is still invariant under the $Z_2$ symmetry eq. (197). Imposing the involution $j = \dot{j}$ up to a overall factor, we reproduce the familiar action of the Yang-Baxter model:
\[
\tilde{L}_{2D} = \text{tr} \left( \frac{\omega'_0 \dot{j}_w \dot{j}_w}{\langle \beta \mu \rangle} + \frac{\omega'_0 \dot{j}_w \dot{j}_w}{\langle \beta \mu \rangle} + \frac{\omega'_0 \dot{j}_w \dot{j}_w}{\langle \beta \mu \rangle} + \frac{\omega'_0 \dot{j}_w \dot{j}_w}{\langle \beta \mu \rangle} \right)
= N \text{tr} \left( j \dot{j}_w \right)
\]
where
\[
N = (\eta' + 1)
\]
This concludes the “commutativity” of the diagram in Figure 4.

7 Conclusions

In this work we studied the relations between the 6DHCs, the 4DCS, and 2D integrable theories for more general cases. We found that the diagram in Figure 1 proposed in ref. [17] is not always commutative due to the fact that the matching condition between the 6D and 4D theory is not always compatible with the boundary conditions. More precisely, the ascending or descending of the boundary conditions in one theory to the boundary conditions in the other could be problematic in the sense that the ascended or descended boundary condition can not remove the gauge freedom and the resulting theory has gauge symmetry and is not the theory we want. Even if we discard the matching condition and impose appropriate boundary conditions to remove the gauge freedoms, the resulting 2D models in two approaches are often different. For the $\lambda$-deformation, the deformation parameter of the 2D model obtained from 4D WZW model is either trivial or restricted to specific value. For the rational $\eta$-deformation, even though the 4D WZW-like model is integrable, its symmetry reduction to 2D is not integrable anymore even when we relax the constraint to allow the WZ term. Furthermore, we investigated two more cases where $\Omega$ are of more general forms including zeros, without insisting on the matching condition. In particular, we managed to construct $\eta$-deformation in the trigonometric description from both two routes in the diagram in Figure 4. What’s more, we also obtained a coupled version of $\lambda$-deformation following both paths by considering higher-order pole in $\Omega$.

It is well-known that the $\lambda$-deformation is related to the $\eta$-deformation through Poisson-Lie-T-duality and analytic continuation. If there is a trigonometric description for the $\lambda$-deformation, then it would be possible to construct a 4D $\lambda$-deformation from the 6DHCs. It will be also interesting to study the 4D $\lambda$-deformation from the point of view of the 4D WZW model directly following the original construction of $\lambda$-deformation in 2D [25]. In refs. [14, 12-15], the coset models and supercoset models have been successfully constructed from the 4DCS, then it is interesting to investigate their constructions from the 6DHCs theory.

One remark is on the reality condition. In our construction we mainly consider Euclidean signature and real gauge
group. The real structure of the 6DhCS had been studied generally in ref. [17]. The real gauge group can be understood as the real form of a complex gauge group $G^c$ with an involution $\Theta: G^c \to G^c$ which induces a map on the Lie algebra $\theta: g^c \to g^c$. Recall that the spinor conjugation on $\mathbb{R}^8$ is defined as $\mathcal{C}: (x, \pi) \to (x, \bar{\alpha})$ so we should impose the reality condition

$$\mathcal{C}^* \bar{\alpha} = \theta(\bar{\alpha}),$$

(231)

and the gauge transformation $g$ should satisfy

$$\mathcal{C}^* g = \Theta(g).$$

(232)

If we start from the real gauge group, then this condition is trivially satisfied, and the reality condition of the action eq. (1) is [17]

$$\mathcal{C}^* \Omega = \Omega.$$ 

(233)

For the simplest choice of $\Omega$ which leads to the 4D WZW, the reality condition is solved by fixing $\beta = \bar{\alpha}$. Similarly for the $\Omega$ eq. (175) in sect. 6 we can fix

$$\mu_- = \mu_+, \quad \alpha_- = \bar{\alpha}_+.$$ 

(234)

However, for eqs. (51) and (148) there is no solutions for the reality condition. This also happens for the choice $\Omega = D^\theta Z/(\tau \alpha)^4$ which was used to generate Chalmers-Siegel action [17]. One possible way out is to consider the ultrahyperbolic signature in which the spinor conjugation is $\mathcal{C}: (x, \pi) \to (x, \bar{\pi})$ which has the fixed points so the reality condition is less restrictive.

Another closely related model is the affine Gaudin model [32] which also can be used to construct 2D integrable field theories systematically. The relation between the affine Gaudin model and the 4DhCS has been studied in ref. [33]. It would be interesting to see how the affine Gaudin model fits in the diagram in Figure 1.

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**Appendix**

**A1 Spinor and twistor space**

In this part we will review some basic facts and notations about the spinors and the twistor space used in this article. This part mainly follows refs. [17, 34].

**A1.1 Spinor**

The spin group of complexified Minkowski space is $SO(4, \mathbb{C})$, which is locally isomorphic to $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. 
Due to this fact, a vector in $CM^4$ is related to the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation of the group $SL(2,\mathbb{C}) \ltimes SL(2,\mathbb{C})$. We use the primed capital $A', B', C', ...$ to denote the indices of $\left(\frac{1}{2}, 0\right)$ representation of $SL(2,\mathbb{C}) \ltimes SL(2,\mathbb{C})$ and the un-primed capital $A, B, C, ...$ to denote the indices of $\left(0, \frac{1}{2}\right)$. Just as we use the metric $g_{ab}$ and its inverse $g^{ab}$ to lower and raise the indices of fundamental representation of $SO(4,\mathbb{C})$, the spinor indices can be raised and lowered using the two dimensional Levi-Civita symbols, which are $SL(2,\mathbb{C})$-invariant. We use the following convention:

$$
\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

and

$$
\alpha_A := \epsilon_{ab} \alpha^b, \quad \beta^A := \beta_c g^{ba}
$$

(1a)

to lower and raise the indices for un-primed version of spinors, and use similar convention for the primed ones. Besides, we use the Greek alphabet $\alpha, \beta, ...$ to denote the spinors.

For future convenience, we also introduce the shorthand notation for the $SL(2,\mathbb{C})$-invariant, skew-symmetric inner products on the spinors:

$$
(k\omega) := k^A \omega_A = k^A \omega^B \epsilon_{AB}, \quad \bar{k}\bar{\omega} := \bar{k}^A \bar{\omega}_A = \bar{k}^A \omega_B \epsilon_{AB}.
$$

(1b)

### 1.2 Basic notions about twistor space

The twistor space $\mathbb{P}T$ of complexified space-time $\mathbb{C}M^4$ is the complex manifold $\mathbb{CP}^3 \setminus \mathbb{CP}^1$, which can be viewed as an open subset of $\mathbb{CP}^3$. The four homogeneous coordinates $z^a$ of $\mathbb{CP}^3$ can be divided into two different Weyl spinors in the following way:

$$
Z^a = \left(\omega^a, \pi^a\right).
$$

(1c)

Note that the $\mathbb{CP}^1$ with homogeneous coordinate of the form $(\omega^a, 0)$ is removed from the $\mathbb{CP}^3$ space. There is a nontrivial relationship between $\mathbb{P}T$ and $\mathbb{C}M^4$:

$$
\omega^a = x^{AB} \pi_B^a,
$$

(1d)

where $x^{AB}$ are the coordinates in $\mathbb{C}M^4$. This is called the incidence relation. With this equation, one finds that the conformal structure of space-time is encoded by the holomorphic structure of $\mathbb{CP}^1$ in $\mathbb{P}T$.

We focus on the Euclidean signature in this paper. In the following, we will introduce the notions of complex conjugation on the space of the spinors to match the reality conditions of the Euclidean real slice $\mathbb{E}^4$ of $\mathbb{C}M^4$. We define the conjugation on the spinors to be

$$
\omega^a \rightarrow \tilde{\omega}^a = (-\omega^1, \omega^0),
$$

$$
\pi^a \rightarrow \tilde{\pi}^a = (-\pi^1, \pi^0).
$$

(1e)

Thus the complex conjugation acting on the twistor space is

$$
Z^a = (\omega^a, \pi^a) \rightarrow \tilde{Z}^a = (\tilde{\omega}^a, \tilde{\pi}^a).
$$

(1f)

There is no invariant point in the twistor space under this conjugation, but there are invariant lines. These lines are uniquely determined by a point and its conjugate. One knows from the incidence relation eq. (1a) that such a line in the twistor space corresponds to a point on $\mathbb{E}^4$, thus every point in twistor space $\mathbb{P}T$ is uniquely associated with a point in $\mathbb{E}^4$. Combining these facts together, one learns that there is a fibration induced by the Euclidean reality conditions $\mathbb{P}T \rightarrow \mathbb{E}^4$. The fibres are $\mathbb{CP}^1$ subspace of $\mathbb{P}T$ related to the points in $\mathbb{E}^4$ by the incidence relation eq. (1a).

Due to the identification between the twistor space of $\mathbb{E}^4$ and the projective spinor bundle $\mathbb{PS} \cong \mathbb{E}^4 \times \mathbb{CP}^1$, one can use the coordinates $(\chi^{AA'}, \pi_{AA'})$ to replace the original homogeneous coordinates eq. (1a) introduced before. To find out the bases of this bundle, one should note that the complexified tangent bundle $TM_C$ can be decomposed into subspaces of holomorphic vector fields (also called (1, 0)-vector fields) and anti-holomorphic vector fields (also called (0, 1)-vector fields), i.e.,

$$
TM_C = T_M^{(1,0)} \oplus T_M^{(0,1)}.
$$

(1g)

This induces the decomposition of the $k$-form fields:

$$
\Omega^k(M)_C = \bigoplus_{\nu+k=\nu} \Omega^\nu \eta(M),
$$

(1h)

where a section of $\Omega^\nu \eta(M)$ has the following form:

$$
\omega = \omega_a dx^a_1 \wedge \cdots \wedge dx^a_k \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^l, \quad \omega \in \Omega^\nu \eta(M).
$$

(1i)

Note that $(z^a, \bar{z}^a)$ are local complex coordinates on $M$. As an open subset of $\mathbb{CP}^1$, the (0, 1)-forms on $\mathbb{P}T$ have natural bases, which can be written in terms of the coordinates of projective spinor bundle

$$
\tilde{e}_0 = \frac{dz}{||\tilde{\eta}||^2}, \quad \tilde{e}_a = \frac{dx^{AA'} \tilde{\eta}_{AA'}}{||\tilde{\eta}||^2},
$$

(1j)

with $||\omega||^2 = \omega^a \tilde{\omega}_a$ and $||\tilde{\eta}||^2 = \tilde{\eta}^{AA'} \tilde{\eta}_{AA'}$. The bases of (0, 1)-vectors are

$$
\tilde{\xi}_0 = ||\tilde{\eta}||^2 \tilde{\xi}_{AA'}, \quad \tilde{\xi}_A = \tilde{\eta}^{AA'} \tilde{\xi}_{AA'}
$$

(1k)

and the Dolbeault operator which define the complex structure on the twistor space can be expressed as

$$
\tilde{\partial} = \tilde{e}^0 \partial_0 + \tilde{e}^A \partial_A.
$$

(1l)

### 2 List of notations

For better legibility, we summarize some notations used in this paper in Table A1.
| Classes | Notations | Description |
|---------|-----------|-------------|
| Derivatives | \( \tilde{\partial} \) | \( \tilde{\partial} = \partial^* \partial_0 + \partial^* \partial_A \) acting on \( \mathbb{R}^4 \) |
| | \( \partial' \) | \( \partial' = \partial^* \partial_0 + d \omega + d \partial \sigma \) acting on \( \mathbb{C} \mathbb{P}^1 \times \mathbb{R}^2 \) |
| | \( d_{\text{cl}} \) (or \( d \)) | \( d_{\text{cl}} = d \omega + d \partial \sigma \) acting on \( \mathbb{R}^3 \times [0,1] \) |
| | \( d' \) | the exterior derivative acting on \( \mathbb{R}^3 \times [0,1] \) |
| Gauge fields | \( \mathcal{A} \) | \( \mathcal{A} = \mathcal{A}_0 + \mathcal{A}_A \) defined on \( \mathbb{R}^4 \) |
| | \( \mathcal{A}_0 \) | component of \( \mathcal{A} \) on \( \mathbb{C} \mathbb{P}^1 \) fibre direction |
| | \( \mathcal{A}_A \) | component of \( \mathcal{A} \) on \( \mathbb{R}^2 \) base space direction |
| | \( A_{\alpha \nu}^A \) | the part of \( \mathcal{A}_A \) without \( \pi^A \) dependence defined by \( \mathcal{A}_A = \pi^A A_{\alpha \nu}^A \) |
| | \( A_{\alpha}^A \) | \( A_{\alpha}^A = \sigma^A A_{\alpha \nu}^A \) for arbitrary left-handed spinor \( \sigma^A \) |
| Currents | \( J \) | \( J = -\partial \bar{\sigma} \bar{\sigma}^{-1} \) where \( \bar{\sigma} \) is a group-valued field defined on \( \mathbb{R}^4 \) |
| | \( J_0 \) | component of \( J \) on \( \mathbb{C} \mathbb{P}^1 \) fibre direction |
| | \( J_A \) | component of \( J \) on \( \mathbb{R}^2 \) base space direction |
| | \( J_{\alpha \nu} \) | \( J_{\alpha \nu} = -\partial_{\bar{\sigma}} \bar{\sigma} \sigma^{-1} \) for group-valued field \( \sigma \) defined on \( \mathbb{R}^4 \) |
| | \( J_A^\alpha \) | \( J_A^\alpha = \sigma^A J_{\alpha \nu}^A \) for arbitrary left-handed spinor \( \sigma^A \) |
| | \( \bar{J} \) | \( \bar{J} = -\partial J \bar{\sigma}^{-1} \) is the current defined on \( \mathbb{R}^4 \times [0,1] \) |
| | \( j \) | \( j = -d_{\text{cl}} \bar{\sigma} \) is the current defined on \( \mathbb{R}^3 \times [0,1] \) |
| | \( \bar{j} \) | \( \bar{j} = -d' \bar{\sigma} \) is the current defined on \( \mathbb{R}^3 \times [0,1] \) |