Volatility and dynamic dependence modeling: Review, applications, and financial risk management

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Abstract
Since the introduction of ARCH models close to 40 years ago, a wide range of models for volatility estimation and prediction have been developed and integrated into asset allocation, financial derivative pricing, and financial risk management. Research has also been very active in extending volatility modeling to dependence modeling and in developing our understanding of risk and uncertainty in financial systems. This paper presents a review on the statistical modeling on volatility and dynamic dependence of financial returns. In addition, we present a real data example using a time-varying copula model to estimate the dynamic dependence of stock returns. Research on volatility and dynamic dependence modeling will continue to encounter statistical and computational challenges; it is necessary to persist in dealing with the 3H (high dimension, high frequency, high complexity) paradigm in modeling.

This article is categorized under:
- Statistical Learning and Exploratory Methods of the Data Sciences > Modeling Methods
- Statistical Models > Nonlinear Models
- Data: Types and Structure > Time Series, Stochastic Processes, and Functional Data

KEYWORDS
copula, GARCH, high-frequency data, risk management, stochastic volatility, the 3H paradigm

1 | INTRODUCTION

The research on volatility (or standard deviation in financial returns) and correlation modeling has been very active in the fields of economics and finance. Traditionally, volatility—a measure of financial assets' variation—is one of the key components of portfolio allocation (Markowitz, 1991), options pricing (Black & Scholes, 1973), and financial risk management (Christoffersen & Diebold, 2000). Due to the specific roles of volatility in quantifying uncertainty in financial markets, many studies have been devoted to the estimation and prediction of volatility. The introduction of the Auto-regressive Conditional Heteroscedastic (ARCH) model by Engle (1982) marked a breakthrough in volatility modeling. The ARCH model is one of the earliest models to explicitly take into account an empirical stylized fact—namely...
volatility clustering—in describing the time series evolution of volatility. As seen in Figure 1a, which presents the 20-day historical volatility of three stocks, Alphabet Inc. (GOOGL), the Bank of America Corporation (BAC), and the Coca-Cola Company (KO), from 26 December 2017 to 31 December 2020, the volatility of a stock changes over time, and may drastically spike during a market crash. Different from the family of ARCH models, stochastic volatility (SV) models (Harvey et al., 1994) assume that volatility follows a stochastic process, allowing volatility to evolve stochastically in addition to enabling dependence on lagged data. A key difference between ARCH and SV models is that the former uses the concept of conditional variance to estimate volatility, whereas the latter estimates a latent variable from the data. An obvious inconvenience when applying SV models is the parameter estimation, which is highly non-trivial compared with that in ARCH models (Shephard, 1996). Through computational advancements, recent revolutionary works on volatility modeling have improved statistical inference and have better enabled the use of risk measures, such as value at risk (VaR) and expected shortfall management (Jorion, 2007). Financial risk is related to uncertainty in investment gain due to market variation. Naturally, the probabilistic assessment of financial risk through volatility modeling is vital for risk management and decision making.

Volatility modeling has been a growing research field since the introduction of ARCH and SV models. In economics, research is mainly interested in modeling macroeconomic time series, such as the inflation and unemployment rate, while, in finance, many studies have applied volatility models to time series, such as the returns of financial instruments, to estimate investment risk. Advances in univariate modeling have largely focused on extending classic ARCH and SV models to explain the empirical properties of financial returns, such as the extension to GARCH models made by Bollerslev (1986); works on volatility asymmetry by Harvey and Shephard (1996), So et al. (1998), and Chen and So (2006); developments in heavy-tailed behavior by Chen et al. (2008), and Nakajima and Omori (2009); research in long memory effects by Ding et al. (1993) and So (2000). Multivariate volatility models began to gain popularity as financial analysts recognized the importance of time-varying correlations in financial time series. As seen in the historical correlations for stock returns among the aforementioned stocks (GOOGL, BAC, and KO) in Figure 1b, these correlations can drastically change within days, implying substantial changes in portfolio risk. Applications can be found in finance, where accurate estimation of correlation among stocks is crucial for assessing portfolio risk, hedging effectiveness, and stress testing (So et al., 2013). Attempts to extend univariate ARCH models to multivariate ARCH models started with the bivariate ARCH model proposed by Engle et al. (1984). SV models have also received multivariate extensions introduced by Harvey et al. (1994). A major statistical issue in multivariate volatility modeling is enabling scalability in high-dimensional contexts.

The advancement of computer storage technology allows companies and institutions to obtain and store real-time transactional data in information systems. More importantly, faster networks allow data transmission to occur at near real-time speeds. This leads to the availability of so-called high-frequency data, which usually contain a large number of observations with small time intervals, such as minutes, seconds, or even milliseconds. This calls for better modeling techniques to model volatility at a high-frequency level, as traditional ARCH and SV models struggle to capture intraday behavior in financial returns. Realized volatility (Andersen et al., 2003) was considered as an additional source of

![Figure 1](https://via.placeholder.com/150)

**Figure 1** Moving-window 20-day time-varying volatility and dynamic correlations of GOOGL, BAC and KO from 26 December 2017 to 31 December 2020.
information in the prediction of daily volatility. Volatility models incorporating high-frequency data have been applied (Hansen et al., 2012; McAleer & Medeiros, 2008; Takahashi et al., 2009) to data regarding stock price movement at very small time intervals, enabling more accurate risk management in short-term horizon trading.

Financial markets have become more “chaotic” and unpredictable, particularly during financial catastrophes, for example the worldwide crash in 1987, the financial tsunami in 2007–2008, and the COVID-19 pandemic in 2020. Extreme and highly unexpected events can cause financial return distributions to become highly complex and nonlinear. Another direction of the development in financial risk management is in the introduction of copula models to volatility modeling. Copula modeling gained popularity after the occurrence of extreme market crashes particularly after the 2007 subprime mortgage crisis. Traditional multivariate volatility models, such as multivariate ARCH and multivariate SV models can only measure the linear correlations between time series. In recent developments in risk management, a focus of correlation estimation is placed on tail dependence, which is the correlation at the tail of two distributions. Copula models, which can capture tail dependence and other nonlinear forms of dependencies, are particularly useful. Examples of such copula volatility models can be found in Dias et al. (2004), Jondeau and Rockinger (2006), Liu and Luger (2009), Lee and Long (2009), and So et al. (2020). Recently, there have also been attempts to incorporate vine copula models into volatility modeling (So & Yeung, 2014), offering a simpler and more flexible way of modeling when using high-dimensional data.

In this paper, we review the historical development of volatility and dynamic dependence modeling starting from ARCH (Engle, 1982) and encompassing recent models that incorporate copulas in modeling, as well as their extensions. The rest of this paper is organized as follows. Section 2 gives a brief overview of univariate models. Section 3 moves on to multivariate extensions of volatility models. Section 4 focuses on volatility models utilizing high-frequency data. Section 5 presents the recent volatility models that incorporate copula ideas. Section 6 provides a real data application, in which the vine copula GARCH model (So & Yeung, 2014) is applied to data regarding the three aforementioned stocks. Section 7 concludes the paper.

2 | UNIVARIATE MODELS

The modeling of volatility in financial markets plays an important role in risk management. The volatility (or the variability) of a financial security return variable allows us to assess the uncertainty of changes in asset values or profit of an investment. In conventional applications, financial security returns at time $t$ can be defined as the change in logarithmic prices; that is, $y_t = \ln p_t - \ln p_{t-1}$, where $p_t$ is a security price at time $t$. Volatility models were initially developed as a method of preserving unconditional mean and variance while allowing for time-varying conditional mean and volatility. We define $\bar{y}_t = (y_1, ..., y_t)$ as the price information up to time $t$ and the set of publicly known information up to time $t$ as $\psi_t$. The most commonly adopted approach in financial econometrics for modeling volatility formulates a time series structure for the conditional variance of $y_t$, given $\bar{y}_{t-1}$ or $\psi_{t-1}$.

2.1 | Autoregressive conditional heteroscedastic model and its variants

A seminal work in market volatility modeling is the ARCH model, pioneered by Engle (1982). A fundamental property of the ARCH model is its ability to incorporate the well-known volatility clustering feature, as observed in Figure 1a, a stylized fact in financial time series. We refer to the ARCH model as the basic linear form of the model introduced by Engle (1982) as follows:

$$y_t = \mu_t + \epsilon_t, \quad \epsilon_t = h_t^{1/2}z_t, \quad z_t \sim N(0, 1),$$

where

$$\mu_t = \phi_0 + \sum_{i=1}^{r} \phi_i y_{t-i} = \phi_0 + \phi_1 y_{t-1} + \cdots + \phi_r y_{t-r},$$

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_p \epsilon_{t-p}^2.$$
where \(\alpha_i \geq 0, \forall i\), to ensure positive conditional variances. In Equations (1) and (2), \(\mu_t = E[y_t|\psi_{t-1}]\) and \(h_t = \text{var}(y_t|\psi_{t-1})\), are the mean and variance of \(y_t\) conditional on \(\psi_{t-1}\), respectively. The volatility at time \(t\) is usually specified as \(h_t^{1/2}\).

Note that \(e_t = y_t - \mu_t\) is the residual at time \(t\). The conditional variance \(h_t\) is modeled as a linear combination of past squared residuals \(e_{t-i}^2\) while the conditional mean of \(y_t\), \(\mu_t\), is modeled as an autoregressive (AR) process. The generalized ARCH (GARCH) model was proposed by Bollerslev (1986) as an extension to the ARCH model. It introduces an additional autoregressive component to the volatility, altering \(h_t\) in Equation (2) to:

\[
h_t = a_0 + \sum_{i=1}^{p} a_i e_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}.
\]

An alternative way to write Equation (3) is \((1 - \alpha(B) - \beta(B))e_t^2 = a_0 + (1 - \beta(B))v_t\), where \(v_t = e_t^2 - h_t\), \(\alpha(B) = \sum_{i=1}^{p} a_i B^i\), \(\beta(B) = \sum_{j=1}^{q} \beta_j B^j\), where \(B\) denotes the backshift operator. The ARCH\((p)\) and GARCH\((p,q)\) models are often used in place of each other regarding model extensions and are considered to be the same overarching branch of volatility model.

It is almost immediately evident that the simple GARCH\((p,q)\) model is fairly restrictive. To ensure positive conditional variance, the model parameters of GARCH models are restricted to be positive. Furthermore, \(h_t\) in Equation (3) is modeled as a sum of squared past residuals, leading to problems with short-term memory (Ding et al., 1993) and symmetric volatility (Nelson, 1991), which are not always desirable (Black, 1976). The exponential GARCH (EGARCH) model was developed in Nelson (1991) to combat these issues. The model alters the volatility form without changing its functionality a great deal, by introducing a minor adjustment to Equation (3):

\[
\ln h_t = a_0 + \sum_{i=1}^{p} a_i g(z_{t-i}) + \sum_{j=1}^{q} \beta_j \ln h_{t-j}, \quad g(z_t) = \theta z_t + \gamma(|z_t| - E[|z_t|]).
\]

Building a volatility equation through \(\ln h_t\) allows for negative model parameters and, in addition, it allows function \(g(\cdot)\) to take any form that accounts for both size and sign, releasing the symmetry constraint of the previously proposed function of linear sum of residual squares in Equations (2) and (3). The EGARCH model and the quadratic GARCH (QGARCH) model, posited by Sentana (1995), allow asymmetric responses to positive and negative shocks in determining \(h_t\) to capture the asymmetry phenomenon in volatility. In addition, the threshold-type of GARCH models used by Glosten et al. (1993), Li and Li (1996), Chen and So (2006), Chiang et al. (2007), Chen et al. (2006), and Chen et al. (2011) were also utilized to capture asymmetric volatility properties in financial time series. Regime switching and threshold nonlinearity have been of interest when modeling market cycles and phases. Piecewise-linear models were first popularized by Tong (1978) and were then extended to the ARCH family as double threshold ARCH (DTARCH) model by Li and Li (1996). They were further developed into multi-regime models, with threshold parameters dependent on both endogenous and exogenous data (Chen & So, 2006; Magtanggol III De Guzman & So, 2018).

Another limitation of the GARCH model is that the effect of past squared returns decays exponentially over time. For applications such as the study of long daily stock market price series, it is observed that such data have positive serial correlations over long lags (Ding et al., 1993; Taylor, 2008). The asymmetric power ARCH (A-PARCH) model was proposed to incorporate the long-term memory observed in many financial return series. Building from the GARCH model, Ding et al. (1993) investigated the effects of the power parameter of past absolute returns, \(d\), on volatility \(h_t\) through utilizing \(|y_t|^d\) with \(d \in [1,2]\), in the A-PARCH model. Another approach to capturing long memory effects is the fractionally integrated GARCH (FIGARCH) model (Baillie et al., 1996). The autoregressive moving average form of Equation (3) was extended to an ARFIMA specification, allowing for fractional integration of an autoregressive polynomial, such that \((1 - \alpha(B) - \beta(B))(1 - B)^d e_t^2 = a_0 + (1 - \beta(B))v_t\). A more general conditional heteroskedastic model was developed by Tsay (1987).

Heavy-tailed data are not uncommon in econometrics and finance. With stationarity properties, the GARCH model was extended to the preserve unconditional mean and variance, which allows a \(t\)-distribution to be easily slotted into place (de Vries, 1991). The extensions by Wagner and Marsh (2005), and So et al. (2008) would allow for higher kurtosis in residuals than a normal distribution. For applications that value inference on the tail over the whole distribution, an interesting approach was proposed by So and Chung (2015). Conditional quantiles were modeled instead and showed promising results. The expected and median shortfalls were utilized together with conditional quantiles by So and Wong (2012) to estimate tail risk.
2.2 Stochastic volatility model

A separate class of volatility model was developed by incorporating a separate stochastic process for volatility. This class of model is known as the stochastic volatility (SV) model (Harvey & Shephard, 1996; Taylor, 1994). The philosophy behind these models is that volatility should be driven by economic or market forces, rather than by the movement of prices. Following this philosophy, SV models were developed in which volatility evolves stochastically over time. We refer to the following form as the SV model:

\[ y_t = \varepsilon_t h_t^{1/2}, \quad \ln h_{t+1} = \alpha + \phi \ln h_t + \eta_t, \quad (\varepsilon_t, \eta_t) \sim \mathcal{N}(0, \Sigma), \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\eta^2 \end{pmatrix}, \quad t = 1, \ldots, T. \tag{4} \]

A key difference between GARCH and SV models is that \( h_t \) is deterministic given \( \psi_{t-1} \) in GARCH models, whereas, in SV models, \( h_t \) is a latent variable and is unobservable even when the information set \( \psi_{t-1} \) is known. Similar to GARCH models, we can interpret \( h_t^{1/2} \) as the volatility at time \( t \). For simplicity, the return in the SV model in Equation (4) is assumed to be demeaned, but we can also specify a mean equation similar to Equation (1) of the GARCH model. The basic form of the SV model assumes \( \varepsilon_t \) and \( \eta_t \) to be independently and normally distributed, described by \( \Sigma \) in Equation (4). To address the issue of asymmetry, Jacquier et al. (2004) proposed an alternate specification for \( \Sigma \):

\[ (\varepsilon_t, \eta_t) \sim \mathcal{N}(0, \Sigma), \quad \Sigma = \begin{pmatrix} 1 & \rho \sigma_\eta \\ \rho \sigma_\eta & \sigma_\eta^2 \end{pmatrix}. \]

Notably, when \( \rho < 0 \), a decrease in \( \varepsilon_t \) will be associated with an increase in \( \eta_t \) and, subsequently, the volatility through \( h_t \), allowing us to model the leverage effect observed by Black (1976), where a negative return is associated with an increase in volatility.

Multiregime-type extensions of the SV model in Equation (4) include the Markov switching SV (MSSV) model in So et al. (1998) and the threshold SV (THSV) in So et al. (2002). These two models mark different approaches to modeling phases and cycles, with MSSV focusing on stepwise state changes, based on a first-order Markov transition matrix. Threshold-SV (THSV) and the further extended THSV model in Chen et al. (2008) explore the more familiar threshold-based regime switching introduced for the ARCH family. These models were shown to be capable of fully incorporating exogenous data in both the autoregression and threshold equations. Uncertainty from the threshold parameter and delay parameter, which determines the amount of lagged terms, is also modeled and accounted for. By formulating a fractionally integrated process for \( \ln h_t \) in Equation (4), Breidt et al. (1998) developed a long memory SV model. So (1999) and So (2002) performed quasi-MLE and Bayesian analyses respectively on a class of long memory SV model. The SV model likewise suffers from problems with outliers and heavy-tailed data. This was intuitively solved by introducing Student’s \( t \)-distributions and inverse gamma distributions, to model the error variables, and was shown to produce good improvements (de Vries, 1991; Jacquier et al., 2004). We can interpret the change as letting \( \varepsilon_t = \sqrt{\lambda_t} z_t \) with \( z_t \) uncorrelated i.i.d. from \( \mathcal{N}(0, 1) \) with \( \lambda_t \) being an inverse gamma random variable. By introducing a new error variable \( \lambda_t \), the new moving part allows the model to not always associate changes in return directly with volatility; it hence allows for better fitting of extreme data. However, this change also causes the model to have less shock persistence compared to the SV model, as volatility is less likely to account for sudden changes in return. Similar methods to alter the error term with leptokurtic distributions in Asai and McAleer (2011) and the further extended generalized hyperbolic distribution (GH) (Barndorff-Nielsen, 1977) in Nakajima and Omori (2012) were also shown to be applicable to SV models.

2.3 Parameter estimation

To estimate GARCH models, maximum likelihood estimation (MLE) was first proposed by Engle (1982), using an iterative score algorithm of the second-order derivative of the likelihood. Then, with the A-PARCH model, a Box–Cox power transformation was introduced to help linearize the asymmetric absolute return, estimated using the Berndt–Hall–Hall–Hausman algorithm (Ding et al., 1993). Ordinary least square (OLS) regression was also
used when distributional assumptions, such as conditional Gaussianity, are less applicable (Sentana, 1995). Under some regularity conditions, Li and Li (1996) showed that the iterative weighted least squares algorithm (Li & Mak, 1994) can be used to estimate the parameters of DTARCH. Aside from the aforementioned frequentist approaches, Chen et al. (2003) proposed a Markov chain Monte Carlo (MCMC) method with which to estimate the double threshold autoregressive GARCH (DTAR-GARCH) model. The MCMC method also allows the time lag and threshold variables to be estimated.

Stochastic volatility models were relatively less popular compared with GARCH models in analyzing financial time series, despite their attractive properties. Due to the intractability of the likelihood function, it is not easy to obtain MLE for unknown parameters. Generalized method of moments (GMM) implementations was attempted by Andersen and Serensen (1996). A quasi maximum likelihood (QML) method was proposed by Harvey and Shephard (1996). Kim et al. (1998) showed that the log\( \chi^2 \) distribution of \( \epsilon_t \) can be approximated by a mixture of seven normal distributions to match its first four moments, from which the statistical inference of the SV models can be acquired via a state space form (So, 2006; West & Harrison, 2006). Other methods and algorithms were developed utilizing MCMC, such as that by Jacquier et al. (2004) and the Monte Carlo maximum likelihood approach in Sandmann and Koopman (1998), Gaussian state data augmentation in So et al. (2002), truncated normal Metropolis–Hastings (MH) in Omori et al. (2007), block sampling in Omori and Watanabe (2008), and the other MCMC methods that have been extensively compared (Chib et al., 2002). Non-Gaussian nonlinear state models were also attempted, applying Kalman filter and the quadratic hill-climbing method, to derive a posterior mode estimator for \( h_t \) (So, 2003).

### 2.4 Dynamic tail risk calculations

An important aspect of volatility modeling is to assess the tail risk of an investment. Among various tail risk measures, VaR is commonly used. VaR can be defined as the maximum loss over a horizon \( h \) with a predetermined probability \( 1 - p \), where \( p \) is small. In statistical terms, VaR is the negative of the 100\( p \)-th percentile of the return distribution. Under the general GARCH formation in Equation (1) with a distributional assumption on \( \epsilon_t \) and when \( h = 1 \), the predicted VaR for the risk position at time \( t + 1 \) given information up to time \( t \) is given by \(- \tau_p \sqrt{h_{t+1}}\), where \( \tau_p \) is the 100\( p \)-th percentile of the distribution of \( \epsilon_t \). So and Yu (2006) compared different GARCH models based on their performance regarding VaR estimation and found that the heavy-tailed distributional assumption in \( \epsilon_t \) is crucial to accurate VaR estimation.

A more challenging problem is the multiple-period VaR estimation; that is, when \( h > 1 \). Given the portfolio’s current market value \( C \) at time \( t \), the \( h \)-period VaR is given by: \( \text{VaR}_{h,p} = -CV_{h,p} \), where \( V_{h,p} \) is the 100\( p \)-th percentile of the aggregate return \( Y_{t,h} = y_{t+1} + \ldots + y_{t+h} \). As noted in Wong and So (2003), \( V_{h,p} \) has to be determined from the aggregate return distribution given information up to time \( t \), which is also dependent on the conditional kurtosis of \( Y_{t,h} \) given \( y_t \). According to Wong and So (2003), good estimates of \( V_{h,p} \) can be obtained based on the conditional kurtosis. Furthermore, So and Wong (2012) proposed estimators for multiple-period expected shortfall:

\[
\text{ES}_{h,p} = \frac{1}{p} \int_0^p \text{VaR}_{h,u} du = \frac{1}{p} \int_0^p -CV_{h,u} du = \frac{-C}{p} \int_0^p V_{h,u} du,
\]

which is a coherent risk measure (Artzner et al., 1999). With the VaR and expected shortfall estimators, we are able to conduct model comparisons between various GARCH models (and maybe even SV models) among a range of values for \( p \) to investigate the accuracy regarding tail risk estimation.

### 3 Multivariate Models

In many financial applications, we have to analyze multiple return variables simultaneously. High-dimensional model building for multiple return variables allows for more general purposes and is extensively used in risk management, optimal portfolio selection, forecasting VaR and estimating optimal capital charges with respect to the Basel Accord. We denote multivariate return \( y_t \) at time \( t \) as an \( N \)-dimensional vector, such that \( y_t = (y_{1t}, \ldots, y_{Nt})' \). We also let \( h_t = (h_{1t}, \ldots, h_{Nt})' \) be the corresponding vector of conditional variances in the GARCH framework or the vector of
stochastic variances in the stochastic volatility framework. In multivariate modeling, a focus is placed on the statistical learning of how correlations between returns change over time. In other words, our main interest is the study of dynamic correlations or, more generally, dynamic dependence in financial time series.

### 3.1 Multivariate GARCH models

Multivariate GARCH research stems from the early work of Engle et al. (1984). For a bivariate ARCH model, necessary conditions for a conditional covariance matrix to be positive definite were presented. However, when scaled to higher dimensions, it leads to overparameterization and quickly becomes intractable. The general setup for multivariate GARCH (MGARCH) models is as follows:

\[
y_t = \mu_t + \epsilon_t, \quad \epsilon_t = H_t^{1/2} z_t, \quad z_t \sim N(0, I),
\]

where \( \mu_t \) is the conditional mean and \( H_t \), a positive definite \( N \times N \) matrix, is the conditional covariance matrix of \( y_t \) given \( \psi_{t-1} \). A computational challenge is to calculate the \( N(N-1)/2 \) dynamic correlations in \( H_t \) while ensuring the positive definite property. A general form for \( H_t \) can be written as:

\[
H_t = D_t R_t D_t, \quad \text{With} \quad D_t = \text{diag} \left( \sqrt{h_{ii}} \right).
\]

In this form, \( R_t \) is the conditional correlation matrix, \( D_t^2 \) is the conditional variance diagonal matrix with elements \( h_{ii} \), which are analogous to the univariate \( h_t \) though they can be functions of variables from other systems. Constant conditional correlation MGARCH (CCC-MGARCH) was one of the first variants proposed by Bollerslev (1990). Its conditional correlation is time-invariant such that \( R_t = R \) for all \( t \). Although the model was shown to be empirically reasonable in some analyses (Baillie & Bollerslev, 1990; Schwert & Seguin, 1990), Figure 1b illustrates that the conditional correlations can be time-varying. The Vec representation of MGARCH was introduced by Engle and Kroner (1995). This representation allows for simplification to the repeated off-diagonal parameters in \( H_t \) as well as the introduction of the innovative Baba–Engle–Kraft–Kroner (BEKK) specification (Engle & Kroner, 1995). The BEKK specification shows that covariance is positive definite if it can be written in the form

\[
H_t = d_t^2 \alpha_t^0 + \sum_{i=1}^p d_t^2 \alpha_t^{i-1} \epsilon_{t-i}^2 \epsilon_t^2 \alpha_t^i + \sum_{j=1}^q d_t^2 \beta_t^{i-j} \epsilon_{t-j}^2 \beta_t^j,
\]

where \( \alpha_t^i, \beta_t^j \) for \( i, j \geq 1 \) are unrestricted \( N \times N \) matrices and \( \alpha_t^0 \) is an upper triangular matrix. Though ensuring positive definiteness, the specification still suffers from overparameterization when \( N \) is large.

To relieve the restrictive constant correlation assumption, dynamic conditional correlation MGARCH (DCC-MGARCH) (Engle, 2002) and varying correlation MGARCH (VC-MGARCH) (Tse & Tsui, 2002) were proposed by extending the CCC-MGARCH model. A core development of the DCC-MGARCH and VC-MGARCH models is the exponential smoother estimator used by RiskMetrics (Mina et al., 2001; Morgan et al., 1997), which is

\[
[R_t]_{i,j} = \rho_{i,j,t} = \frac{\sum_{i=1}^{l-1} \lambda_t \epsilon_{t-i} \epsilon_{t-j} \epsilon_{t-s}}{\sqrt{\left( \sum_{i=1}^{l-1} \lambda_t \epsilon_{t-i}^2 \right) \left( \sum_{j=1}^{l-1} \lambda_t \epsilon_{t-j}^2 \right)}}.
\]

In the DCC-MGARCH model, the exponential smoothing is done via

\[
Q_t = (1 - \gamma_1 - \gamma_2) Q_t + \gamma_1 Q_{t-1} + \gamma_2 \epsilon_{t-1} \epsilon_{t-1}',
\]

where \( Q_t \) is interpreted as the variance of \( \epsilon_t \) given \( \psi_{t-1} \), and \( Q_t \) can be obtained as the sample average \( \frac{1}{T} \sum_{t=1}^{T} \epsilon_t \epsilon_t' \). \( Q_t \) is then standardized to achieve the time-varying correlation matrix \( R_t \). Similarly, the VC-MGARCH allows correlations to evolve parsimoniously:
Here, \( \gamma_1, \gamma_2 \) are the average constants such that \( 0 \leq \gamma_1, \gamma_2 \leq 1 \) and \( \gamma_1 + \gamma_2 \leq 1 \). \( \mathbf{R}_c \) is a time-invariant positive definite matrix with a unit diagonal and \( \mathbf{Q}_{t-1} \) is a function of lagged standardized residuals \( (\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots) \). A simple rolling average of all the past standardized residuals was used to define \( \mathbf{Q}_{t-1} \). By obtaining a weighted average of past covariance/correlation matrices and other positive definite matrices, the DCC-MGARCH model in Equation (6) and the VC-MGARCH model in Equation (7) ensure that the resulting correlation matrix is positive definite, given that \( \mathbf{Q}_0 \) and \( \mathbf{R}_0 \) are positive definite. However, the work of Billio et al. (2003) suggests that different systems can influence the current value differently. A generalization from So and Yip (2012) to accommodate this feature. The clustered correlation MGARCH (CC-MGARCH) model in So and Yip (2012) introduced \( d \) clusters, selected via a Bayesian model selection method. Each cluster uses pairs of weights \( \mathbf{w}_{1}^{(s_i)}, \mathbf{w}_{2}^{(s_i)} \) where \( s_i \) takes values \( 1, \ldots, d \), determining which cluster the correlation belongs to. Combining this with the weight parameters \( \mathbf{w}_{1}, \mathbf{w}_{2} \), we form matrices \( \mathbf{Q}_{1} \) and \( \mathbf{Q}_{2} \) respectively such that:

\[
\mathbf{R}_t = (\mathbf{1} - \mathbf{Q}_1 - \mathbf{Q}_2) \ast \mathbf{R}_c + \mathbf{Q}_1 \ast \mathbf{R}_{t-1} + \mathbf{Q}_2 \ast \mathbf{Q}_{t-1},
\]

where \( \mathbf{1} \) is a column vector of one. Here, we preserve the weighted average form from Equation (7). Since the weight parameters \( \mathbf{Q}_1 \) and \( \mathbf{Q}_2 \) are no longer scalar, we take the Hadamard product, denoted by \( \ast \). To accommodate for tail modeling, a MGARCH model in So and Tse (2009) was proposed as a modification of the VC-MGARCH model and, consistent with the univariate approach, a multivariate t-distribution was used in place of the multivariate normal distribution.

Another class of extended MGARCH models is based on factor structures (Alexander & Chibumba, 2009; Van der Weide, 2002). Assume that there is an underlying latent layer explaining the cause of the economic process. This layer provides suitable modeling constraints through which to improve ease of both interpretation and estimation, allowing for inference even on large dimensions through dimension reduction (Alexander, 2002; Van der Weide, 2002). Notably, an orthogonal MGARCH (O-MGARCH) model in Alexander and Chibumba (1997) implicitly assumes that the observed data can be transformed into a system of uncorrelated unobserved economic components via a linear transformation governed by an orthogonal matrix. To further extend the model’s flexibility, a generalized O-MGARCH model in Van der Weide (2002) can use any possible invertible matrix while keeping the computational cost relatively affordable. The model can essentially be thought of as a transformation of \( \mathbf{x}_t = \mathbf{Zy}_t \), where \( \mathbf{x}_t \) is an observed process and \( \mathbf{y}_t \) is an unobserved process that can be modeled as the diagonal-MGARCH model in Bollerslev et al. (1988) and Ding and Engle (2001), where the off-diagonals of the covariance matrix are restricted to zero.

### 3.2 Multivariate SV

A multivariate extension of the SV model was proposed by Harvey et al. (1994). The multivariate SV (MSV) naturally extends the SV model from Equation (4) to:

\[
\mathbf{y}_t = \varepsilon_t \mathbf{V}_t^{1/2}, \quad \mathbf{h}_t = \mathbf{\Phi} \mathbf{h}_t + \eta_t, \quad \mathbf{V}_t = \text{diag}(h_{11}, \ldots, h_{NN}), \quad t = 1, \ldots, T,
\]

with

\[
\begin{pmatrix}
\varepsilon_t \\ \eta_t
\end{pmatrix} \sim \mathcal{N}_{2N}(\mathbf{0}, \Sigma), \quad \Sigma = \begin{pmatrix}
\Sigma_{\varepsilon\varepsilon} & 0 \\
0 & \Sigma_{\eta\eta}
\end{pmatrix},
\]

where \( \mathbf{h}_t = (\ln h_{11}, \ldots, \ln h_{NN})' \). Extensions to the model can be classified into three notable subbranches: constant conditional correlation (Harvey et al., 1994; So et al., 1997), dynamic conditional correlation (Asai et al., 2006; Asai & McAleer, 2009), and latent factors (Quintana & West, 1987; Ray & Tsay, 2000). Following the spirit of CCC-MGARCH, Harvey et al. (1994) proposed the use of similar constraints on the MSV to simplify the computation process. This is achieved by restricting \( \mathbf{\Phi} \) to be diagonal, such that the volatility of each system is affected only by its own evolution.
Dynamic conditional correlation was incorporated into the MSV framework to allow for separate modeling of stochastic variance and correlation. Asai and McAleer (2009) investigated two dynamic correlation MSV (DC-MSV) models. The first DC-MSV model can be seen as the MSV counterpart to VC-MGARCH, where Equation (7) is modified to the MSV as follows:

\[ Q_t = (1 - \gamma)Q_{t-1} + \gamma Q_t + \Omega_t, \quad \Omega_t \sim \mathcal{W}_N(\kappa, \Lambda), \]

where the correlation matrix of \( y_t \) is given by \( Q_t^{-1/2}Q_t^{-1/2} \) and \( Q_t = \text{diag}(Q_t) \). The matrix \( \Omega_t \) follows a Wishart distribution \( \mathcal{W}_N(\kappa, \Lambda) \) with a scale matrix \( \Lambda \) and the degrees of freedom \( \kappa \). The second DC-MSV model also allows the stochastic correlation matrix to evolve where the positive definiteness condition is ensured via a Wishart process. Other direct methods involving transformations were applied to model the dynamic correlation matrix \( R_t \) through mathematical constraints. A Cholesky decomposition on the matrix \( \Sigma_p \) was suggested by Tsay (2005) to fit its lower triangular elements as independent AR(1) processes (Asai et al., 2006). Matrix exponential transformation was applied to MSV (Kawakatsu, 2006), where \( \Gamma_t \) is directly modeled. Since \( \Gamma_t \) is a positive definite matrix, there exists a real symmetric matrix \( \Lambda_t \) such that \( \Gamma_t = \exp(\Lambda_t) \). Then, \( \Lambda_t \) can be modeled as an AR(1) process after vectorization. Ishihara et al. (2016) extended the matrix exponential idea to accommodate cross-leverage in stochastic covariances (Chib et al., 2009). The matrix exponential process in stochastic covariance was further developed to explain long memory and asymmetry in a high-dimensional context (Asai & So, 2015). To model for cross-leverage effect without over-parameterization, a pseudo DC-MSV was proposed. Kurose and Omori (2016) proposed the dynamic equicorrelation MSV (DEC-MSV), where \( \Sigma_{\varepsilon} \) in Equation (10) has time-varying equicorrelations:

\[
\Sigma_{\varepsilon} = \begin{pmatrix} 1 & \rho_t & \ldots & \rho_t \\
\rho_t & 1 & \ldots & \ldots \\
\vdots & \ddots & \ddots & \ddots \\
\rho_t & \ldots & \rho_t & 1 \end{pmatrix}.
\]

Under these specifications, a parameter \( \gamma_t \) was defined and is allowed to evolve similarly to how SV volatility \( h_t \) would, with \( \rho_t = \frac{\exp(g_t)}{\exp(g_t + 1)} \), forming the dynamic equicorrelation structure. In addition, So and Choi (2009) introduced the threshold factor MSV (TFMSV), where the basic additive K-factor MSV (Chib et al., 2006) was generalized. Models involving the principle component concept can also be found in Hu and Tsay (2014) and Matteson and Tsay (2011).

Using dimension reduction ideas, the K-factor MSV models proposed in Jacquier et al. (1995) and Aguilar and West (2000) can be represented by \( y_t = L_t f_t + \zeta_t \), where \( L_t \) is the factor loading matrix and the elements of \( f_t \) follow independent SV models. Philipov and Glickman (2006) further improved the factor component by allowing the factor covariance matrices to evolve as a Wishart process. The multiplicative factor model was introduced by Quintana and West (1987) as the stochastic discount factor model. The model is a one-factor model, decomposing the return into two multiplicative components: the common scalar factor \( f_t \) and the vector of noise \( \varepsilon_t \). Long memory MSV models were developed independently by Ray and Tsay (2000) and So and Kwok (2006). A more complex model allowing for tail modeling and a Bernoulli series specific jump was proposed by Chib et al. (2006), as MSVJ. It extends the additive factor MSV form to \( y_t = Lf_t + K_t q_t + \zeta_t \), where the factor loading matrix \( L \) is kept time invariant. \( K_t q_t \) are the jump components, with \( K_t = \text{diag}(k_{t1}, \ldots, k_{tN}) \), and \( q_t \) is the vector of independent Bernoulli processes, with time-invariant probability \( p(q_{it}) = 1 = k_i \) for each Bernoulli series. The vector of innovation \( \zeta_t \) elements \( \zeta_{it} \) following independent Student’s t- distributions.

4 | Modeling with High-Frequency Data

As improvements to streaming speeds and reductions in the cost of memory are implemented, time intervals between data points have drastically reduced. However, when modeling high-frequency data, there have been issues with parameterizing the assets return process; specifically, existing volatility models fail to capture the particular behavior across such fine time levels (So & Xu, 2013). Under the assumption of a continuous stochastic process, realized volatility
estimators were proposed. Traditionally, the stochastic process of continuous price \( p(s) \) was assumed to follow the Itô process, such that \( dp(s) = \mu(s)ds + \sigma(s)dw(s) \), where \( \mu(s) \) is the drift coefficient and \( \sigma(s)^2 \) is the spot or instantaneous variance which is independent of the standard Brownian motion \( w(s) \). The instantaneous variance is assumed to have a square integrable sample path and therefore gives us the discretized integrated variance \( h_t = \int_{t-\tau}^{t} \sigma^2(s)ds \) with the star denoting the integration from the assumed true continuous process. To estimate the integrated variance, realized variance,\( \sigma_t^2 \), is a function of \( \int_{t-\tau}^{t} \sigma^2(s)ds \) with the star denoting the integration from the assumed true continuous process. To estimate the integrated variance, realized variance, \( \sigma_t^2 \) has been proposed. For each \( M \)-intraday return during each day \( t \), the realized variance \( R_h_t \) is the squared sum of the intraday returns \( y_{t,m} \) over day \( t \)

\[
R_{h_t} = \sum_{m=1}^{M} y_{t,m}^2, \tag{12}
\]

and \( \sqrt{R_{h_t}} \) can be described as realized volatility. This estimator converges to the integrated variance as the time interval tends towards zero or as \( M \to \infty \). Despite this, the continuous stochastic process assumption has been shown to be incorrect by contextual factors such as unevenly spaced data (Takahashi et al., 2009) and microstructure noise (discreteness of pricing, bid-ask spread, etc.) (Bai, 2000; Hasbrouck, 2007; O’Hara, 1997). Asset prices have trajectories that are piecewise constant and jumps at discrete time points. This contradicts the quadratic variation assumed by the Itô process, where the diffusion process is assumed to have a continuous sample path (Frey & Runggaldier, 2001). There are various ways of modifying Equation (12), including the methods used by Hansen and Lunde (2005), Bandi and Russell (2008) and Zhang et al. (2005, 2006), to accommodate more empirical features in high-frequency trading and data. In what follows, we review recent advancements in the incorporation of high-frequency data in volatility modeling.

### 4.1 Realized GARCH models

With the availability of high-frequency data, some studies have incorporate realized volatility or realized variance in volatility modeling and prediction. Hansen et al. (2012) proposed the simultaneous modeling of financial returns and realized variance as follows:

\[
y_t = h_t^{1/2} z_t, \quad h_t = \alpha_0 + \beta_1 h_{t-1} + \gamma R_{h_{t-1}}, \quad R_{h_t} = \xi_0 + \xi_1 h_t + g(z_t) + a_t,
\]

where \( g(z_t) \) is a function of \( z_t \) influencing changes in the realized variance. Watanabe (2012) also developed GARCH models with realized volatility and applied a realized GARCH model to conditional quantile forecasting for risk management. We define \( y_{t,m} \) and \( R_{h_{t,m}} \) as the intraday return and realized variance calculated in the \( m \)-th intraday intervals. The length of the intraday intervals can be as short as 30 minutes or 1 hour. To capture the intraday risk of financial trading, So and Xu (2013) proposed modeling the intraday returns \( y_{t,m} \) and intraday volatility using a GARCH-RV model with realized volatility. Seasonality was modeled in GARCH-RV via a multiplicative seasonality term \( S(m) \) such that \( \prod_{i=1}^{m} S(i) = 1 \) as:

\[
y_{t,m} | \psi_{t,m-1} = g_{t,m}^{1/2} z_{t,m}, \quad g_{t,m} = S(m) \times h_{t,m}, \quad h_{t,m} = \alpha_0 + \alpha_1 y_{t,m-1}^2 + \beta_1 h_{t,m-1} + \gamma R_{h_{t,m-1}}^2, \tag{13}
\]

where \( \psi_{t,m} \) represents the information up to the \( m \)-th intraday interval at time \( t \) and \( h_{t,m} \) is the corresponding conditional variance. It was shown to successfully identify intraday seasonality, subsequently improving volatility estimation and inference on fine-scale trends (So & Xu, 2013). An exponential GARCH structure with realized volatility was proposed in Hansen and Huang (2016) as a way of including the information regarding \( R_{h_t} \) for volatility prediction.

### 4.2 Realized SV models

To model both the return and realized volatility simultaneously in the stochastic volatility framework, Takahashi et al. (2009) introduced the SV-RV model as below:
\[
y_t = e^{1 \ln h_t}, \quad \ln h_{t+1} = \alpha + \phi \ln h_t + \eta_t, \quad \ln R h_t = \xi + \ln h_t + \epsilon_t,
\]

with
\[
\begin{pmatrix}
\epsilon_t \\
\eta_t \\
\end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix}
1 & \rho \sigma_\eta \\
0 & \sigma_\eta^2 \\
\rho \sigma_\eta & \sigma_\eta^2 \\
\end{pmatrix}\right),
\]

The SV-RV model in Equation (13) contains the basic SV equation as in Equation (4) and a measurement equation for \(\ln h_t\) to estimate the latent volatility \(h_t\). Combining information from \(y_t\) and \(R h_t\), the benefits of both measurement equations improve the volatility estimation. To correct for the bias attributed to nontrading hours and microstructure noise, an extra time invariant term \(\xi\) was introduced. Asymmetry and leverage were easily introduced into the SV-RV model through the parameter \(\rho\).

Using high-frequency intraday data, capturing seasonality and long-range dependence over a day becomes possible. In Koopman and Scharth (2012), long memory properties of volatility were incorporated into the modeling via superposition. Another approach is to model \(\ln h_t\) as an ARFIMA process (Shirota et al., 2014). Accommodation for extreme cases has also been investigated by many researchers, through the generalized hyperbolic (GH) skewed-\(t\) distribution (Nakajima & Omori, 2012). The \(\epsilon_t\) term was modified to fit the GH skewed-\(t\) distribution and to allow for better tail and quantile estimation (Takahashi et al., 2016).

### 4.3 Realized covariance and multivariate models

The high-dimensional generalization of the realized volatility in Equation (12) is straightforward. We can define the realized covariance at time \(t\) as \(C_t = \frac{1}{M} \sum_{m=1}^{M} y_{t,m} y_{t,m}'\), where \(y_{t,m}\) is the \(m\)-th intraday return vector at time \(t\) (or day \(t\)). Through a Wishart process, the covariance is ensured to be positive definite. Although, previously, attempts were made to use it on the conditional correlation matrix component, the restrictions required for it in regard to diagonal unity proved cumbersome in the previously mentioned Wishart process DC-MSV (Asai & McAleer, 2009). The Wishart process factor for multivariate stochastic volatility in Philipov and Glickman (2006) was extended by Asai and So (2013) without the factor dimension reduction, as a stochastic covariance model:

\[
y_t \mid y_{t-1}, C_t \sim \mathcal{N}(\mathbf{0}, C_t), \quad C_t \mid A_{t-1} \sim \mathcal{IW}_N(k, A_{t-1}), \quad A_{t-1} = (k - N - 1)C_{t|t-1},
\]

where \(\mathcal{IW}_N(k, A)\) is an inverse Wishart process with \(k\) degrees of freedom, scale matrix \(A\), and \(C_{t|t-1}\) is the stochastic conditional covariance matrix, \(\text{var}(y_t | y_{t-1})\). To improve the flexibility of the model, a mixture stochastic covariance model was studied by So et al. (2017). The model adds an extra layer to the basic model in Asai and So et al. (2013) through the introduction of a latent categorical variable \(Z_t\). The variable takes discrete values to indicate which mixture component the observed realized covariance \(C_t\) belongs to and the generating process can be thought of as:

\[
y_t \mid y_{t-1}, C_t, \mu_t \sim \mathcal{N}(\mu_t, C_t), \quad C_t \mid y_{t-1}, \{Z_t = j\} \sim \mathcal{IW}_N(k_j, A_{t-1,j}), \quad Z_t \mid \mathbf{p} \sim \text{Categorical}(\mathbf{p}), \quad \mathbf{p} \sim \text{Dir}(\alpha).
\]

Shirota et al. (2017) extended the work of Takahashi et al. (2009) to incorporate the realized covariance information in volatility modeling using Cholesky decomposition. In a recent research study, Yamauchi and Omori (2020) used pairwise realized correlations to make the model scalable and to handle problems that occur due to nonsynchronous trading. The literature generally shows that improvements can be achieved when realized covariance information is taken into account appropriately in stochastic covariance modeling.
5 | TIME-VARYING COPULA MODELS

5.1 | Copulas for extremal dependence

The management of both catastrophic and systemic risks plays an important role in modern risk management. Incidents in these risks generally lead to large scale damages and severely impact a multitude of financial systems. As such, recent developments in risk management have placed heavier emphasis on market behavior associated with extreme market losses. Although infrequent, the occurrence of market turmoil has had devastating effects on the market and its participants, with the 2008 subprime mortgage crisis and the recent COVID-19 pandemic being key examples. There are numerous studies showing that financial data are heavy-tailed (Creal & Tsay, 2002; Jondeau & Rockinger, 2006; Longin & Solnik, 1995). In particular, tail behavior exhibits asymmetric correlation between appreciation and depreciation of exchange rates (Patton, 2009). Models with Gaussian or multivariate-t assumptions have already been thoroughly discussed. However, these elliptical distribution models place a heavy constraint in regards to tail symmetry, causing the model to become too rigid to accommodate for stylized characteristics such as asymmetric tail behavior in the data. Limitations of classical multivariate distributions lead to the recent rise in popularity of copula modeling. Following conventions, we denote \( F_n (\cdot) \) as the marginal CDF of \( y_n \), \( F_y (\cdot) \) the joint CDF of \( y \) and \( C(\cdot) \) a copula function. Then, through probability integral transformation, any multivariate distribution can be described by the marginal distributions of \( y_n \in y = (y_1, ..., y_N)' \) and a suitable copula function \( C(\cdot) \) (Schweizer & Sklar, 1983), more formally written as:

\[
F_y (y) = C(F_1(y_1), F_2(y_2), ..., F_N(y_N)).
\]

Note that \( F_n (y_n) \sim \text{Unif}(0,1) \) and by differentiating the above, other auxiliary forms can be obtained:

\[
f_y (y) = c(F_1(y_1), F_2(y_2), ..., F_N(y_N)) \prod_{n=1}^{N} f_n (y_n),
\]

where \( f_y (y) \) is the joint probability density function of \( y \), \( c(\cdot) \) is the copula density and \( f_n (\cdot) \) is the probability density function of \( y_n \). Although copula theories accommodate \( N \geq 2 \) cases, there are many more pairwise copulas (\( N = 2 \)) than copulas with dimension \( N \geq 3 \). In many financial applications, elliptical copulas were adopted, since direct extensions of other bivariate copulas to multivariate versions require nontrivial work. Although there have been many attempts to construct multivariate Archimedean and general copulas (Fischer et al., 2009; McNeil et al., 2009), the construction of multivariate copulas for real data applications can be complicated. One interesting approach has recently drawn attention in financial econometrics research. Shown by Bedford and Cooke (2001, 2002), high-dimensional copulas can be constructed from pairwise copulas via vine decomposition.

5.2 | Vine-copula GARCH models

Classical portfolio theory was pioneered by the work of Markowitz (1952) and can be implemented through dynamic covariance modeling (Engle, 2002; Tse & Tsui, 2002). However, covariance-based methods are unable to account for nonlinear dependencies (Embrechts et al., 1999). Integration of copulas in GARCH-type models comes naturally when accounting for nonlinearity, conditional heteroskedasticity, and extremal dependence in financial time series. To capture the time series dependence inherent in financial returns, it is important to adopt time-dependent copulas. In copula-GARCH frameworks (Jondeau & Rockinger, 2006; So & Yeung, 2014), it is necessary to define \( C^t (\cdot) \) and \( c^t (\cdot) \), the copula function at time \( t \) and its corresponding copula density, which are functions of information up to time \( t - 1 \).

Vine copulas stem from the building blocks of pairwise copula (Bedford & Cooke, 2002; Joe, 1996). An advantage of vine decomposition in copulas is the usage of the wider range of available bivariate copulas, where high-dimensional density can be represented by a product of pairwise copulas such that:
\[ f_y(y) = \prod_{n=1}^{N} f_{1,n}^{[n]}(y_n) \prod_{j=1}^{N-1} \prod_{i=1}^{N-j} c_{i+i+1,\ldots,i+j-1}^{[i+j]} \bigg\{ F_{1}^{[i]}(y_i|y_{i+1},\ldots,y_{i+j-1}), F_{1}^{[j]}(y_{i+j}|y_{i+1},\ldots,y_{i+j-1}) \bigg\}, \tag{16} \]

where the key element equates to the conditional bivariate copulas, \( c_{i+i+1,\ldots,i+j-1}^{[i+j]}(\cdot) \). The way the pairs are organized is by the concept of regular vines (Bedford & Cooke, 2001) on \( N \) variables. There are a total of \( N-1 \) trees, \( \{T_1, T_2,\ldots,T_{N-1}\} \), which can be thought of as layers. For each tree \( T_i \), there are \( N-1+i \) nodes connected by \( N-2+i \) edges. Conventionally, the D-vine (Kurowicka & Cooke, 2004) and the canonical vine are used; the D-vine has nodes connected to, at most, two nodes, while the canonical vine emphasizes a node, such that all nodes at each layer are connected to a single pilot node. See Figures 2 and 3 for two examples of vine structures. Simplifications of Equation (16) can be achieved by assuming conditional independence. For example, if \( Y_1 \) and \( Y_3 \) are independent given \( Y_2 \), then \( c_{13|2}^{[1]} F(Y_1|Y_2), F(Y_3|Y_2) = 1 \), allowing for reductions in some of the pair copula terms, depending on the tree structure. Although the number of pair copulas explodes exponentially with dimensionality, several inference methods have been proposed, such as partial correlation studied by Kurowicka and Cooke (2006) and introduced by Aas et al. (2009), a maximum pseudo likelihood methodology alongside a suitable model selection framework that avoid numerical comparison between all possible combinations. A popular inference method for integrating copulas into GARCH models is via a two-step estimation approach (Dias et al., 2004; Jondeau & Rockinger, 2006; Liu & Luger, 2009). These models generally involve first estimating the univariate GARCH and then, using those GARCH parameter estimates, the copula function parameters are estimated (Patton, 2006). We illustrate the two-step method used to estimate the vine-copula GARCH model by So and Yeung (2014) in the next section.

6 | REAL DATA ANALYSIS

6.1 | Vine-copula with dynamic conditional dependence

We illustrate the vine-copula GARCH idea using the stock data shown in Figure 1. We use the three stocks, BAC, GOOGL, and KO, listed on the U.S. stock markets, setting dimension \( N \) as three. For the sake of simplicity, we ignore the other attributes, as they are irrelevant to model building in this section. The data will be further explained in Section 6.2. We follow the procedure in So and Yeung (2014) to perform the two-step estimation. Let \( y_{it}, i = 1, 2, 3 \), be
the three stock returns. According to Equation (16), the conditional distribution of \( y_{1t}, y_{2t}, \) and \( y_{3t} \) given \( \psi_{t-1} \), the information up to time \( t-1 \), can be decomposed as the following:

\[
\begin{align*}
f^{(i)}(y_{1t}, y_{2t}, y_{3t}) &= f^{(i)}_1(y_{1t}) \cdot f^{(i)}_2(y_{2t}) \cdot f^{(i)}_3(y_{3t}) \\
&\quad - c_{12}^{(i)} \left\{ F^{(i)}_1(y_{1t}), F^{(i)}_2(y_{2t}) \right\} \\
&\quad - c_{13|2}^{(i)} \left\{ F^{(i)}(y_{1t}, y_{2t}), F^{(i)}(y_{3t}, y_{2t}) \right\}.
\end{align*}
\]

Note that the decomposition in Equation (17) is unique up to the relabeling of the variables. Altogether, there are six different permutations for \( y_1, y_2, \) and \( y_3, \) and only three give different decompositions (Aas et al., 2009). When \( N = 3 \), all decompositions are both a canonical vine and a D-vine; the corresponding tree is shown in Figure 4.

In a typical vine-copula GARCH model, there are two sets of parameters: the GARCH parameter \( \vartheta \), and the copula parameter \( \phi \). Using Equation (17), the log likelihood function for all \( T \) observations, \( y_{it}, \ i = 1, 2, 3 \) and \( t = 1, \ldots, T \) is:

\[
l(\vartheta, \phi | y_T) = \sum_{t=1}^{T} \log \left( f^{(i)}(y_{1t}, y_{2t}, y_{3t}) \right) \\
= \frac{3}{T} \sum_{i=1}^{T} \log \left( f^{(i)}_1(y_{1t}) \right) + \sum_{i=1}^{T} \log \left( c_{12}^{(i)} \left\{ F^{(i)}_1(y_{1t}), F^{(i)}_2(y_{2t}) \right\} \right) \\
+ \sum_{i=1}^{T} \log \left( c_{13|2}^{(i)} \left\{ F^{(i)}(y_{1t}, y_{2t}), F^{(i)}(y_{3t}, y_{2t}) \right\} \right) \\
= l_C(\vartheta | y_T) + l_C(\phi | \vartheta, y_T).
\]

This formulation shows the motivation of the two-step estimation method where the GARCH log likelihood, \( l_C(\vartheta | y_T) \), is first maximized, producing the GARCH model parameter \( \hat{\vartheta} \) estimate. In this example, they correspond to the parameters of the GARCH model in Equation (3). In the second step, the log likelihood of the copula, \( l_C(\phi | \vartheta, y_T) \) is maximized, where \( \vartheta \) is replaced by its estimate from the first step. In practice, we can choose the functional form for the bivariate copulas, \( c_{12}^{(i)}, c_{13|2}^{(i)} \) and \( c_{13|2}^{(i)} \) for financial data analysis. In particular, if we set all bivariate copulas to be t-copulas, we can write down the following from So and Yeung (2014):

\[
C^{(i)}_{12}(u_{11}, u_{22} | \vartheta, \phi) = T_{\psi_{12}^{(i)}} \left( t_{1|2}^{-1}(u_{11}), t_{2|1}^{-1}(u_{22}) \right)
\]

\[
c^{(i)}_{12}(u_{11}, u_{22} | \vartheta, \phi) = \frac{\Gamma \left( \psi_{12}^{(i)} + 2 \right) \Gamma \left( \frac{\psi_{12}^{(i)}}{2} \right)}{2 \left( \Gamma \left( \frac{\psi_{12}^{(i)} + 1}{2} \right) \right)^2} \prod_{i=1}^{2} \left( 1 + \frac{t_{i|12}(u_{11})^2}{\psi_{i2}^{(i)}} \right)^{\frac{\psi_{12}^{(i)} + 1}{2} \sqrt{1 - \psi_{12}^{(i)}}} \times \left( 1 + \frac{t_{1|2}^{-1}(u_{11})^2 + t_{2|1}^{-1}(u_{22})^2 - 2 \psi_{12}^{(i)} t_{1|2}^{-1}(u_{11}) t_{2|1}^{-1}(u_{22})}{\psi_{12}^{(i)} \left( 1 - \psi_{12}^{(i)} \right)^2} \right)^{\frac{\psi_{12}^{(i)} + 2}{2}},
\]

where \( \psi_{12}^{(i)} \) is a time-varying dependence parameter in the t-copula (So & Yeung, 2014). We write the multivariate t-distribution with degrees of freedom \( \nu \) as \( T_{\nu} \) and the corresponding univariate Student’s \( t \)-distribution as \( t_{\nu} \). The degrees of freedom from fitting the GARCH-t(1,1) models are carried into transforming variables, \( u_{11} \) and \( u_{22} \), which follow uniform distributions. Similarly, the conditional copula function can be written as \( C^{(i)}_{13|2}(u_{11|2}, u_{3|2} | \vartheta, \phi) = T_{\psi_{13|2}^{(i)}} \left( t_{1|2}^{-1}(u_{11|2}), t_{2|3}^{-1}(u_{3|2}) \right) \),

![Figure 4](image-url)  

**Figure 4**: Canonical vine and D-vine for \( N = 3 \)
and subsequently substituted with conditional transformed returns instead. The dynamic conditional dependence parameter $\varphi_{13|2}^{[t]}$ was proposed by So and Yeung (2014) and takes the form of dynamic conditional linear correlation, as:

$$\varphi_{13|2}^{[t]} = (1 - a_{13|2} - b_{13|2}) \varphi_{13|2} + a_{13|2} \xi_{13|2,t-1} + b_{13|2} \varphi_{13|2}^{[t-1]},$$  \hspace{1cm} (20)

where $0 \leq a_{13|2}, b_{13|2} \leq 1$ and $0 \leq a_{13|2} + b_{13|2} \leq 1$. Equation (20) gives a weighted average of $\varphi_{13|2}^{[t-1]}$ and the sample conditional dependence $\xi_{13|2,t-1}$, which is the rolling-sample correlation between the conditional returns for the past $m$ values. We denote $y_{1|2t} = t_{13|2}^{-1}(F(t_{1|2t})|y_{1|2t})$. Then, we have

$$\xi_{13|2,t-1} = \frac{\sum_{i=1}^{m} \left( \tilde{y}_{1|2,t-i} \tilde{y}_{3|2,t-i} \right)}{\sqrt{\left( \sum_{i=1}^{m} \tilde{y}_{1|2,t-i}^2 \right) \left( \sum_{j=1}^{m} \tilde{y}_{3|2,t-j}^2 \right)}}.$$  \hspace{1cm} (21)

Note that the conditional probability distributions can be estimated by using the h-function method in copula theories (So & Yeung, 2014):

$$F^{[t]}(y_{1|2t}|y_{2t}) = h_{12}^{[t]}(F^{[t]}(y_{1|2t}), F^{[t]}(y_{2t})) = \frac{\partial C_{12}^{[t]}(F^{[t]}(y_{1|2t}), F^{[t]}(y_{2t}))}{\partial F^{[t]}(y_{2t})},$$

with $h_{12}^{[t]}(u_{1t}, u_{2t}) = t_{\psi_{12t}^{-1}} \left( \left( t_{\psi_{12t}^{-1}}(u_{1t}) - \varphi_{13|2}^{[t]} t_{\psi_{12t}^{-1}}(u_{2t}) \right) \sqrt{v^{[12]} + 1} \right) \sqrt{\left( v^{[12]} + \left( t_{\psi_{12t}^{-1}}(u_{2t}) \right)^2 \right) \left( 1 - \varphi_{13|2}^{[t]} \right)}.$$

The two-step algorithm is summarized as follows:

1. Fit the univariate GARCH-t(1,1) to produce the ML estimate $\tilde{\sigma}$.
2. For each tree layer, $j = 1, 2$:
   a. Identify the distributions, conditional or unconditional, used in the pair copulas in the layer and estimate as follows:
      i. Unconditional: Use the fitted volatility to estimate $u_{it} = F^{[t]}(y_{it})$.
      ii. Conditional: Evolve using the estimates from the previous layer via Equation (22).
   b. For each copula in the tree layers:
3. Calculate the sample correlations $\xi_{12,t-1}$, $\xi_{23,t-1}$ and $\xi_{13|2,t-1}$ for all $t$.
   a. Determine the dependence parameters $\varphi_{12}^{[t]}, \varphi_{23}^{[t]}$ and $\varphi_{13|2}^{[t]}$ for all $t$.
   b. Maximize $l_\mathcal{C}(\theta|\tilde{\sigma}, \psi, T)$ in Equation (18) with respect to $\theta$.

For decompositions with more trees ($N \geq 4$), Step 2 will be repeated until all trees are estimated.

### 6.2 Real data results

We used the stock prices of the three companies (the Bank of America Corporation, Alphabet Inc., and the Coca-Cola Company) identified by their respective NASDAQ ID (BAC, GOOGL, and KO, respectively), to demonstrate the vine-copula GARCH modeling. Taking daily stock prices from 26 December 2017 to 31 December 2020, we have obtained a sample size of 759 daily stock prices per stock. The two-step method illustrated above is applied using the regular vine structure shown in Figure 4. Given the $i$-th daily price $p_{it}$ at time $t$, the daily return is defined as $y_{it} = 100\% \times \ln(p_{it}/p_{it-1})$. The fitted GARCH model parameters are listed in Table 1. The individual GARCH-t(1,1) agrees with the usual result that the volatility persistence reflected by $\alpha_1 + \beta_1$ is close to one. Low $\alpha_1$ suggests relatively less impact from
the squared return at time \( t - 1 \) on the conditional variance at time \( t \), and the high \( \beta_1 \) indicates that the current conditional variance is highly dependent on its past conditional variances.

To estimate the copula parameters, we apply parameter constraints with the unrestricted parameters \( \lambda_\alpha, \lambda_\beta, \lambda_\phi \) and the reparameterization, \( a = \lambda_\alpha^2 / (1 + \lambda_\alpha^2 + \lambda_\beta^2) \), \( b = \lambda_\beta^2 / (1 + \lambda_\alpha^2 + \lambda_\beta^2) \), \( \nu = 2 + \lambda_\alpha^{-2} \) and \( \varphi = 2/((\lambda_\phi^2 + 1) - 1) \). Table 2 presents the parameter estimates for \( a, b \) in Equation (20), and \( \nu, \varphi \) of the bivariate \( t \)-copula in Equation (18). The \( t \)-copula’s degree of freedom \( \nu \) is relatively close to 2, indicating heavy-tailed behavior given the \( \nu \geq 2 \) constraint. This heavy-tailed behavior indicates extremal dependence among the three returns. The sum of \( a \) and \( b \) is around 0.7 in all three copulas, indicating mild persistence in the dynamic conditional dependence in Equation (20). Regarding the bivariate copula for BAC and GOOGL, the copula model weight \( \xi_{13,24} \) is significantly higher than \( \phi_{13/2} \), whereas in the other two copulas, the reverse is true. Figure 5 shows the time series plot of the conditional dependence, \( \phi_{12}^{[r]} \) and \( \phi_{23}^{[r]} \), which are above 0.4 most of the time. It is interesting to see that the dependence between BAC and GOOGL drops to about zero in mid-2020. In practice, estimating the time series dependence can help financial analysts to quantify and monitor the risk of their portfolio investment.

### 7 CONCLUSION

In this paper, we present a review of volatility and dynamic dependence modeling, with a focus on financial applications. Many volatility models stem from the family of GARCH and SV models, including multivariate extensions, models developed to make use of high-frequency data, and copula models designed for capturing nonlinear and tail dependence. From incorporating empirical stylized facts in financial markets to building better probabilistic models for
volatility, to integrating more available information such as transactional data due to advances in technology, modified models have been shown to be extremely valuable and effective to various applications. A few other research directions for risk management that have recently been or are currently being explored shows promising results. For example, financial network analysis (Billio et al., 2012; Diebold & Yilmaz, 2014; Ng et al., 2021; So et al., 2021), applications of artificial neural networks in volatility modeling (Liu & So, 2020), risk assessment in robo-advising (So, 2021) and incorporating textual information for risk management (Mitra & Mitra, 2011). Research on volatility and dynamic dependence modeling will continue, and statistical and computational challenges still persist in dealing with the 3H (high dimension, high frequency, high complexity) paradigm in modeling.

**CONFLICT OF INTEREST**
The authors have declared no conflicts of interest.

**AUTHOR CONTRIBUTIONS**
Mike K. P. SO: Conceptualization; formal analysis; funding acquisition; investigation; methodology; project administration; supervision; writing-original draft; writing-review & editing. Amanda M. Y. Chu: Conceptualization; formal analysis; methodology; writing-review & editing. Cliff C. Y. Lo: Formal analysis; visualization; writing-original draft. Chun Yin Ip: Formal analysis; visualization; writing-review & editing.

**DATA AVAILABILITY STATEMENT**
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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