UNIFORM ESTIMATES ON MULTI-LINEAR OPERATORS WITH MODULATION SYMMETRY

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ABSTRACT. In a previous paper [20] in this series, we gave $L^p$ estimates for multi-linear operators given by multipliers which are singular on a non-degenerate subspace of some dimension $k$. In this paper we give uniform estimates when the subspace approaches a degenerate region in the case $k = 1$, and when all the exponents $p$ are between 2 and $\infty$. In particular we recover the non-endpoint uniform estimates for the Bilinear Hilbert transform in [12].

1. Introduction

We are concerned with $n$-linear forms mapping $n$ Schwartz functions on the real line to a complex number. We shall assume these forms are invariant under a simultaneous translation of all $n$ functions. Dually, these forms can be viewed as operators mapping $n-1$ functions to a distribution. Our goal is to prove $L^p$ regularity for these forms and operators.

By the Schwartz kernel theorem, we can identify such an $n$-linear form with a distribution in $\mathbb{R}^n$. Translation invariance implies that the Fourier transform of this distribution lives on the hyperplane

$$\Gamma := \{ \xi \in \mathbb{R}^n : \xi_1 + \ldots + \xi_n = 0 \}.$$

Since we are interested in $L^p$ regularity, we may restrict attention to the case that this distribution is a function $m(\xi_1, \ldots, \xi_n)$ on this hyperplane.

Then the associated multi-linear form $\Lambda := \Lambda_m$ is given by

$$\Lambda_m(f_1, \ldots, f_n) := \int \delta(\xi_1 + \ldots + \xi_n)m(\xi)\hat{f}_1(\xi_1)\ldots\hat{f}_n(\xi_n) \, d\xi$$

where $\xi =: (\xi_1, \ldots, \xi_n)$ and $\delta$ denotes the Dirac delta. Dually, the associated multi-linear operator $T := T_m$ is given by

$$T_m(f_1, \ldots, f_{n-1})(-\xi_n) := \int \delta(\xi_1 + \ldots + \xi_n)m(\xi)\hat{f}_1(\xi_1)\ldots\hat{f}_{n-1}(\xi_{n-1}) \, d\xi_1 \ldots d\xi_{n-1},$$

the relationship between $T$ and $\Lambda$ is given by

$$\Lambda(f_1, \ldots, f_n) = \int T(f_1, \ldots, f_{n-1})(x)f_n(x) \, dx.$$

Examples of such objects include the pointwise product operator

$$T_1(f_1, \ldots, f_{n-1}) := f_1 \ldots f_{n-1},$$

which occurs when the multiplier $m$ is identically 1. Operators from classical paraproduct theory studied in [2]-[7], [16], [14] also fall into this category and are given by multipliers
satisfying symbol estimates as in (3) below with $\Gamma' = \{0\}$. When $n = 2$ these operators are just linear Fourier multipliers. More recently, multipliers satisfying (3) with nontrivial subspaces $\Gamma'$ have been studied. Observe that (3) is invariant under translations in direction of $\Gamma'$, so the class of multipliers satisfying this condition is translation invariant.

Taking the Fourier transform we obtain a class of forms and operators which has modulation symmetry. In case the multiplier is invariant under translations in direction of $\Gamma'$, the operator itself has a modulation symmetry. Hence the title of this paper. Such forms and operators have been discussed in [20], [10] (also [25], [23], [22], [9], [11]).

Given a multiplier $m$, the main interest in the subject is to obtain $L^p$ estimates for $\Lambda_m$ of the form

$$|\Lambda_m(f_1, \ldots, f_n)| \leq C \prod_{i=1}^n \|f_i\|_{p_i}$$

when $1 \leq p_i \leq \infty$. By duality this is equivalent to $T_m$ having the mapping property

$$T_m : L^{p_1} \times \ldots \times L^{p_{n-1}} \to L'_{p_n}$$
on test functions, where $p'$ is given by $1/p + 1/p' := 1$. When (3) obtains, we say that $\Lambda_m$ is of strong type $(1/p_1, \ldots, 1/p_n)$. In the dual formulation for $T_m$ one may also consider the case $p_{n'} < 1$, but we shall not do so in this paper.

In the pointwise product case $m \equiv 1$ one has strong type whenever $1 \leq p_i \leq \infty$ and one has the scaling condition

$$\sum_{i=1}^n \frac{1}{p_i} = 1.$$  (4)

These estimates (with the exception of some of the endpoint estimates) also generalize to paraproducts and the bilinear Hilbert transform. We cite the following multiplier theorem, proven\footnote{Actually, a more general theorem was proven in [20] in which some $1/p_i$ were allowed to be zero or negative, but we will not discuss this further here. Also, the $n = 3$ case of Theorem 1.1 was first proven in [10], [11].} in [20]:

**Theorem 1.1.** [20] Let $\Gamma'$ be a subspace of $\Gamma$ of dimension $k$ where

$$0 \leq k < n/2.$$  (5)

Assume that $\Gamma'$ is non-degenerate in the sense that for every $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, the space $\Gamma'$ is a graph over the variables $\xi_{i_1}, \ldots, \xi_{i_k}$. Suppose that $m$ satisfies the estimates

$$|\partial_\xi^\alpha m(\xi)| \leq C \text{dist}(\xi, \Gamma')^{-|\alpha|}$$

for all partial derivatives $\partial_\xi^\alpha$ on hyperplane up to some sufficiently large finite order. Then (3) holds whenever (4) holds and $1 < p_i < \infty$ for all $i = 1, \ldots, n$.

Following up on the work begun in [20], we consider in this paper the problem of obtaining uniform estimates when the subspace $\Gamma'$ becomes increasingly degenerate. In order to do so one must modify the condition (3). To illustrate this, suppose that $n = 3$, $k = 1$, and $\Gamma'$ has degenerated completely to

$$\Gamma' = \{(\xi_1, \xi_2, \xi_3) \in \Gamma : \xi_1 = 0\}.$$
Let $m$ denote the multiplier

$$m(\xi_1, \xi_2, \xi_3) := \psi(\xi_1)a(\xi_2)$$

where $\psi$ is a bump function on $[1, 2]$ and $a$ is any smooth function satisfying the estimates $\|\partial^k a\|_\infty \leq C_k$ for all integers $k$. This multiplier $m$ satisfies (4) for the given degenerate subspace $\Gamma'$. Then one has

$$\Lambda_m(f_1, f_2, f_3) = \int T_\psi(f_1)T_a(f_2)f_3$$

where $T_m$ is the Fourier multiplier corresponding to $m$. This operator can be quite badly behaved because $a$ need not be an $L^p$ multiplier for any $p \neq 2$. Indeed, it is easy to construct examples which show unboundedness of this operator whenever $p_2 > 2 > p_3$ or $p_2 < 2 < p_3$.

The main result of this paper is the following $k = 1$ result.

**Theorem 1.2.** Let $n \geq 3$, and let $\Gamma' = \text{span}(v)$ be a one-dimensional subspace of $\Gamma$, where $v = (v_1, \ldots, v_n)$ and $v_1, \ldots, v_n$ are non-zero real numbers which sum to zero. Define the metric $d_v(x, y)$ on $\Gamma$ by

$$d_v(x, y) := \sup_{1 \leq i \leq n} \frac{|x_i - y_i|}{|v_i|}$$

and write

$$d_v(x, \Gamma') := \inf_{y \in \Gamma'} d_v(x, y).$$

Suppose that $m$ satisfies the estimates

$$|\partial^\alpha m(\xi)| \leq C \prod_{i=1}^n (v_i d_v(\xi, \Gamma'))^{-\alpha_i}$$

for all partial derivatives $\partial^\alpha$ on $\Gamma$ up to some finite order. Then (3) holds whenever (4) holds and $2 < p_i < \infty$ for all $i = 1, \ldots, n$, with the bounds uniform in the choice of $v_i$.

We do not know how to modify (3) in the higher rank case $k > 1$, mainly because we have no natural analogue of the metric $d_v$. In analogy with [20], [13] one expects that one should be able to go beyond the case $2 < p_i < \infty$, but we do not pursue these matters here. Restraining ourselves to the case $2 < p_i < \infty$ gives us some considerable technical simplification.

In the non-degenerate case, when all the $v_i$ have comparable magnitudes, this theorem is a corollary of Theorem [11]. A more careful examination of the proof of this Theorem in [20] would reveal that the constant given for (3) would grow polynomially in the ratio between the largest and smallest magnitudes of $v_i$.

A special case of this theorem occurs when $n = 3$, $v = (\beta_2 - \beta_3, \beta_3 - \beta_1, \beta_1 - \beta_2)$, for some distinct real numbers $\beta_1, \beta_2, \beta_3$, and $m(\xi_1, \xi_2, \xi_3) = \text{sgn}(\beta_1 \xi_1 + \beta_2 \xi_2 + \beta_3 \xi_3)$. This corresponds to the (essentially one-parameter) family of bilinear Hilbert transforms

$$\Lambda_m(f_1, f_2, f_3) = C \int \int_3 f_i(x - \beta_it) \frac{dt}{t} dx.$$
In [17], [18] the estimate (3) was proven for all $1 < p_i < \infty$ obeying (4), but with the bound growing polynomially when two of the $\beta_i$ approached each other. In [26] the weak-type estimate $T_m : L^2 \times L^2 \to L^{1,\infty}$ was shown with uniform control on the bounds when $\beta_1$ or $\beta_2$ approached $\beta_3$, but not when $\beta_1$ approached $\beta_2$. (This weak-type bound is already sufficient to prove a version of Calderón’s conjecture which is strong enough to recover the boundedness of the Cauchy integral on Lipschitz curves. See [26] for further discussion). More recently, (3) was established whenever $2 < p_i < \infty$ obeyed (4), with bounds uniform over all values of $\beta$ in [12]. Thus Theorem 1.2 is already known in this case. Also, estimates beyond the $2 \leq p_i \leq \infty$ case have been obtained in [13].

Our argument follows the standard approach to the modulation invariant setting of [17], [18], [26], [20] (see also [8], [12], [13]). We perform dyadic decompositions in both space and frequency, which has the effect of decomposing the phase plane into an overlapping set of tiles. We then split $\Lambda_m$ into pieces associated to various $n$-tuples of tiles. By using tree selection arguments and an orthogonality argument based on Bessel’s inequality for tiles as in the above references, we can reduce matters to estimating the contribution of a single tree of tiles. After eliminating the spatial cutoffs, the problem now becomes that of obtaining uniform estimates for paraproducts. This estimate may be of independent interest, it has been shown in the prequel [21] of this paper.

We quote it in the form that is needed here:

**Proposition 1.3.** Let $n \geq 2$, and let $M_1, \ldots, M_n$ be real numbers. For each $1 \leq i \leq n$ and $j \in \mathbb{Z}$, let $\pi_{j,i}$ be a Fourier multiplier whose symbol is a bump function adapted to $\{\xi : |\xi| \leq 2^j + M_i\}$. Suppose that for each $k \in \mathbb{Z}$ there exists at least one $1 \leq i \leq n$ such that the symbol of $\pi_{j,i}$ vanishes at the origin. Then one has the estimate

$$\sum_j |\int \prod_{i=1}^n \pi_{j,i} f_i| \leq C_{(p_i),n} \prod_{i=1}^n \|f_i\|_{p_i}$$

(9)

for all $1 < p_i < \infty$ obeying (3), where the constant $C_{(p_i),n}$ depends on the $p_i$, and $n$ but is independent of the $M_i$ and $f_i$.

Here we say a bump function is adapted to an interval $I$ if it is supported in this interval and its $k$-th derivative is bounded by $|I|^{-k}$ for all $k$ up to some sufficiently large power, which may depend on $n$ and $(p_i)$. Proposition 1.3 may be viewed as a lacunary version of Theorem 1.2 in the same way that paraproduct estimates are a lacunary version of the results in [17], [18], [20] and the theory of maximal truncated Hilbert transforms are a lacunary version of the Carleson-Hunt theorem. Indeed, it is possible to derive Proposition 1.3 as a special case of Theorem 1.2 at least in the $2 < p_i < \infty$ case.

One of the main innovations in this paper lie in using phase space projections to obtain the tree estimate from Proposition 1.3. Such phase plane projections were previously only known and utilized in the Walsh case [27]. The fact that we are restricted to the $2 < p_i < \infty$ case allows for some simplifications in the argument; most notably the argument works with a uniform spatial localization of all functions involved, unlike in [20]. Moreover, one can replace all the $f_i$ by characteristic functions. Also, since all functions are locally in $L^2$, one does not need to remove any exceptional set (which needs to be done for the $p_i < 2$ theory).

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2. Discretization in the frequency space

The first step in the proof of Theorem 1.2 is a Whitney decomposition away from $\Gamma'$ of the multiplier $m$, which is defined as a function in frequency variables. The Whitney pieces will be smooth functions and compactly supported in the frequency variables. While we would like to think of these pieces as parameterized by dyadic intervals, it will not be possible to use the standard dyadic grid to parameterize them. As in [17], we will spend some effort in constructing a grid structure akin to the dyadic structure. We thought it desirable to not having to do this technical step, which involves a strong interaction of intervals in $i$- and $i'$-coordinates for $i \neq i'$, but we have not been able to proceed without it.

We fix $n$. All constants that follow (typically denoted by $C$) may depend on $n$. We also fix $p_1, \ldots, p_n$ as in Theorem 1.2, and all constants are also allowed to depend on $p_1, \ldots, p_n$. In particular, we choose a large $N$ depending on these exponents, which will describe decay of functions in physical space.

We shall need the constants $C_0 = 2^{100n}$ and $J = 2^{C_0}$. These measure the fineness of our Whitney decomposition and separation in scales respectively. All constants $C$ can be viewed as dependent on $C_0$ and $J$.

If $Q$ is a box in $\mathbb{R}^n$ with sides parallel to the axes, we use $Q_1, \ldots, Q_n$ to denote the intervals comprising $Q$, thus $Q = Q_1 \times \ldots \times Q_n$. All our intervals and boxes will be closed, but when we say two intervals are disjoint we mean disjoint up to possibly points at the boundary. For $A > 0$, we denote by $AQ$ the box with the same center as $Q$ and $A$ times the sidelength.

In the non-degenerate case treated in [20], we used the standard Euclidean Whitney decomposition of $\mathbb{R}^n \setminus \Gamma'$ into cubes. This decomposition does not work well in the near-degenerate setting; following [26] we shall instead decompose the multiplier into boxes which are adapted to the subspace $\Gamma'$. To do this, it is convenient to perform a rescaling to convert the near-degenerate space $\Gamma'$ into a non-degenerate one, which we choose to be the diagonal

$$\tilde{\Gamma}' = \{ (\xi, \ldots, \xi) : \xi \in \mathbb{R} \} \subset \mathbb{R}^n.$$

We shall use tildes to denote quantities that are defined in the rescaled setting.

Fix $v_1, \ldots, v_n$. By symmetry we may assume $|v_1| \geq \cdots \geq |v_n|$. By rescaling we may assume $|v_n| = 1$. It is important that all our constants $C$ will be independent of the $v_i$; indeed, this is the whole point of this paper. We also define $M_i$ and $m_i$ by $2^{M_i} = |v_i|$ and $Jm_i = M_i$.

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation

$$L(x_1, \ldots, x_n) := (v_1x_1, \ldots, v_nx_n).$$

The space $L^{-1}(\Gamma')$ is thus the diagonal $\tilde{\Gamma}'$ in $\mathbb{R}^n$. 
One can easily verify from (4) that one can find a smooth function \( \tilde{m} : \mathbb{R}^n \setminus \tilde{\Gamma}' \to \mathbb{R} \) which agrees with \( m \circ L \) on \( L^{-1}(\Gamma) \) and which satisfies the symbol estimates
\[
|\partial_\xi^\alpha \tilde{m}(\xi)| \leq C \text{dist}(\xi, \tilde{\Gamma}')^{-|\alpha|}
\]
for all \( \alpha \) up to some large finite order \( N^5 \), where dist is now Euclidean distance.

Let \( Q \) denote the set of all cubes \( \tilde{Q} \) which have side-length \( 2^j \) for some integer \( j \), whose center lies in the lattice \( 2^{j-10} \mathbb{Z}^n \), and which obey the Whitney conditions
\[
C_0 \tilde{Q} \cap \tilde{\Gamma}' = \emptyset \quad (10)
\]
\[
4C_0 \tilde{Q} \cap \tilde{\Gamma}' \neq \emptyset. \quad (11)
\]

The sets \( \frac{1}{10} \tilde{Q} \) form a finitely overlapping cover of \( \mathbb{R}^n \setminus \tilde{\Gamma}' \), and so one may decompose
\[
\tilde{m} = C \sum_{\tilde{Q} \in \tilde{Q}} \tilde{m}_{\tilde{Q}}
\]
where each \( \tilde{m}_{\tilde{Q}} \) is a bump function adapted to \( \frac{1}{2} \tilde{Q} \) with degree of regularity \( N^4 \). I.e., \( \tilde{m}_{\tilde{Q}} \) is supported in \( \frac{1}{2} \tilde{Q} \) and satisfies
\[
|\partial_\xi^\alpha \tilde{m}_{\tilde{Q}}(\xi)| \leq \text{diam}(\tilde{Q})^{-|\alpha|}
\]
for all \( |\alpha| \leq N^4 \). It thus suffices to show that
\[
|\sum_{\tilde{Q} \in \tilde{Q}} \int \delta(\xi_1 + \ldots + \xi_n) \tilde{m}_{\tilde{Q}}(L^{-1}(\xi)) \hat{f}_1(\xi_1) \ldots \hat{f}_n(\xi_n) \, d\xi| \leq C \prod_{i=1}^{n} \|f_i\|_{p_i}.
\]

For each \( \tilde{Q} \in \tilde{Q} \), we may use a Fourier series and the smoothness of \( \tilde{m}_{\tilde{Q}} \) to decompose \( \tilde{m}_{\tilde{Q}} \) into tensor products
\[
\tilde{m}_{\tilde{Q}} = C \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{-10n} \prod_{i=1}^{n} \tilde{m}_{\tilde{Q},i,k}(\xi_i)
\]
where the \( \tilde{m}_{\tilde{Q},i,k} \) are bump functions adapted to the interval \( \tilde{Q}_i \) with degree of regularity \( N^3 \) uniformly in \( k \). It thus suffices to show that
\[
|\sum_{\tilde{Q} \in \tilde{Q}} \int \delta(\xi_1 + \ldots + \xi_n) \prod_{i=1}^{n} \tilde{m}_{\tilde{Q},i,k}(v_i^{-1} \xi_i) \hat{f}_i(\xi_i) \, d\xi| \leq C \prod_{i=1}^{n} \|f_i\|_{p_i} \quad (12)
\]
for each \( k \). We fix \( k \) once and for all and for notational convenience drop the index \( k \), i.e., write \( \tilde{m}_{\tilde{Q},i} \) instead of \( \tilde{m}_{\tilde{Q},i,k} \).

Next, we will make the collection of cubes sparser.

**Definition 2.1.** We call a collection \( \tilde{Q}' \subseteq \tilde{Q} \) sparse, if we have for any \( \tilde{Q}, \tilde{Q}' \in \tilde{Q}' \) and any \( 1 \leq i \leq n \) the following properties
\[
|\tilde{Q}_i| < |\tilde{Q}'_i| \implies |\tilde{Q}_i| \leq 2^j |\tilde{Q}'_i|, \quad (13)
\]
\[
|\tilde{Q}_i| = |\tilde{Q}'_i|, \quad \tilde{Q}_i \neq \tilde{Q}'_i \implies \text{dist}(\tilde{Q}_i, \tilde{Q}'_i) \geq 2^j |\tilde{Q}_i|, \quad (14)
\]
\[
\tilde{Q}_i = \tilde{Q}'_i \implies \tilde{Q} = \tilde{Q}'. \quad (15)
\]
Thanks to the fact that at given size the cubes in \( \tilde{Q} \) form essentially a one dimensional set in direction of the diagonal, we may decompose \( Q \) into a bounded number of sparse sets. Thus it suffices to prove (12) under the assumption that \( \tilde{Q} \) is now a sparse subset of the original \( \tilde{Q} \). For convenience of notation we shall call this sparse set again \( \tilde{Q} \). By a standard limiting argument, we can also assume that \( Q \) is finite, as long as the estimates do not depend on the set \( Q \).

We now begin to introduce a structure akin to a dyadic grid. We shall be interested in the enlarged cubes 1000\( \tilde{Q} \). For each interval 1000\( \tilde{Q} \), we shall define a slightly (by at most one percent on either side) increased interval \( \overline{Q}_i \supset 1000\tilde{Q} \), so that these increased intervals have good dyadic properties as formulated in Lemma 2.2.

By (13) the possible sizes of cubes in \( \tilde{Q} \) come in discrete quantities, each separated by a large factor. We shall refer to these different sizes as different scales.

We shall define the intervals \( \overline{Q}_i \), successively beginning with the smallest scale in \( \tilde{Q} \). For \( \tilde{Q} \in \tilde{Q} \) of the smallest scale we simply set \( \overline{Q}_i = 1000\tilde{Q}_i \).

Observe that the distance between \( \tilde{Q}_i \) and \( \tilde{Q}_j \) for \( i \neq j \) is less than \( 8C_0|\tilde{Q}_i| \), because the cartesian product of \( 4C_0\tilde{Q}_i \) and \( 4C_0\tilde{Q}_j \) contains a point on the diagonal by (11). Thus the diameter of the convex hull \( h(\overline{Q}_i) \) of all 10\( \overline{Q}_i \), \( 1 \leq i \leq n \) is less than \( 10C_0|\tilde{Q}_i| \). If \( \tilde{Q}' \) is another cube of the smallest scale in \( \tilde{Q} \), then the convex hull \( h(\overline{Q}_i) \) has distance at least \( j|\tilde{Q}_i| \) from \( h(\overline{Q}_i) \) by (14). Hence every interval of length \( 20C_0|\tilde{Q}_i| \) contains at least one interval of length \( |\tilde{Q}_i| \) which does not intersect \( h(\overline{Q}_i) \) for any \( \tilde{Q}' \) at the smallest scale.

Now consider a cube \( \tilde{Q} \) at the second smallest scale. We may increase each interval 1000\( \tilde{Q}_i \) by at most 1 percent on either side, thus obtaining an interval \( \overline{Q}_i \), so that for any cube \( \tilde{Q}' \) at the smallest scale we either have \( h(\overline{Q}_i) \subseteq \tilde{Q}_i \) or \( h(\overline{Q}_i) \cap \tilde{Q}_i = \emptyset \). All we need is that the endpoints of the enlarged interval are not contained in the convex hull \( h(\overline{Q}_i) \) for any \( \tilde{Q}' \) at the smallest scale, which can clearly be achieved from (13) and the previous discussion.

Then we may define the convex hull \( h(\overline{Q}) \) as before, and the separation properties of these convex hulls discussed for the smallest scale also hold for cubes at the second smallest scale.

By successively passing to larger scales, we can increase each interval 1000\( \tilde{Q}_i \) by at most 1 percent on either side so that it contains either all of \( h(\overline{Q}_i) \) for any \( \tilde{Q}' \) at a smaller scale or is disjoint from \( h(\overline{Q}_i) \). Namely each interval of length \( |\tilde{Q}_i| \) contains an interval of length comparable to the sidelength of cubes at the next smaller scale which does not intersect \( h(\overline{Q}_i) \) for any \( \tilde{Q}' \) at the next smaller scale, and so on passing to smaller and smaller scales, so we can find an endpoint for \( \overline{Q}_i \) which is not contained in \( h(\overline{Q}_i) \) for any \( \tilde{Q}' \) of any smaller scale.

We can now formulate the good dyadic properties that the intervals \( \overline{Q}_i \) have:

**Lemma 2.2.** Let \( \tilde{Q}, \tilde{Q}' \in \tilde{Q} \). If \( \text{diam}(\tilde{Q}) < \text{diam}(\tilde{Q}') \), then 10\( \overline{Q}_i \cap \overline{Q}_j = \emptyset \) for some 1 \( \leq i, j \leq n \) implies 10\( \overline{Q}_i \subseteq \overline{Q}_j \) for all 1 \( \leq i' \leq n \).

**Proof** The proof is clear by construction.

Let \( Q \) denote the set of boxes

\[
Q := \{ L(\tilde{Q}) : \tilde{Q} \in \tilde{Q} \}.
\]
We let $j_Q$ denote the rational number such that $2^{j_Q} = |Q_n|$, and refer to $j_Q$ as the frequency parameter of $Q$. By (13) the set of $j_Q$ is a discrete set so that any two elements have at least distance 1. For purely notational convenience we shall assume that $j_Q$ is an integer for each $Q \in \mathbb{Q}$. This is only a special case of the general situation, in which one may assume the $j_Q$ belong to a fixed shift of the integer lattice, but the proof is the same in the general situation with some additional notation.

For each $Q \in \mathbb{Q}$ define $\pi_Q$ to be the Fourier multiplier

\[ \hat{\pi}_Q f(\xi) := m_{L^{-1}(Q)}(v_i^{-1} \xi) \hat{f}(\xi). \]

The notation $\pi_Q$ is a bit sloppy: to be more precise one should write $\pi_{Q,i}$ or $\pi_{Q,j}$. However, the index $i$ will always either appear in the subscript of $\pi$ or otherwise be clear from the context. Note that the symbol of $\pi_Q$ is a bump function adapted to $Q_i$.

By Plancherel, we can then rewrite the desired estimate in physical space as

\[ |\sum_{Q \in \mathbb{Q}} \int \prod_{i=1}^{n} \pi_Q f_i(x) \, dx| \leq C \prod_{i=1}^{n} \| f_i \|_{p_i}. \]

This completes our frequency space decomposition. Observe that we have returned to a notation that does not explicitly involve any quantities with tilde accent. We shall not need the tilde accent anymore to denote quantities under the transformation $L^{-1}$, and thus be free to use the tilde accent in other contexts.

3. Discretization in the physical space

For fixed $Q$, the projections $\pi_{Q,i}$ are Fourier multipliers supported on intervals of length ranging from $|Q_n|$ to $2^{M_i} |Q_n|$. The Heisenberg uncertainty principle then suggests that one needs to consider several spatial scales, from the coarse scale of $|Q_n|^{-1}$ to the fine scale of $2^{-M_i} |Q_n|^{-1}$. This multiplicity of scales causes much technical difficulty in [23], [12]. A key difference and simplification in our approach is that we only decompose in the coarsest scale $|Q_n|^{-1}$, so that we will sometimes localize less than the uncertainty principle suggests. Unfortunately we have only been able to make this simplification work in the $2 < p < \infty$ case, which is why this paper is restricted to this range of exponents.

In frequency space we have used compactly supported cutoff functions to decompose the multiplier. Consequently, we will continue to use truncations which have compact support in frequency space and merely satisfy rapid decay estimates in physical space. Because of this, we can use the standard dyadic grid to partition physical space as opposed to the carefully constructed grid we use for frequency space. An interval is called dyadic if it is of the form $[2^k n, 2^k (n+1)]$ with integers $k$ and $n$.

Following standard procedure, we shall index the space-frequency decomposition using tiles.

**Definition 3.1.** Let $1 \leq i \leq n$. An $i$-tile is a rectangle $P = I_P \times \omega_P$ with area $2^{M_i}$, $I_P$ a dyadic interval, and $\omega_P$ an interval in the mesh $\{Q_i : Q \in \mathbb{Q}\}$. A multi-tile is an $n$-tuple $\vec{P} = (P_1, \ldots, P_n)$ such that each $P_i$ is an $i$-tile, the interval $I_{P_i} = I_{\vec{P}}$ is independent of $i$, and such that the frequency box $Q_{\vec{P}} := \prod_{i=1}^{n} \omega_{P_i}$ of $\vec{P}$ is an element of $\mathbb{Q}$. The frequency parameter $j_{\vec{P}}$ of a multi-tile is defined by $j_{\vec{P}} := j_{Q_{\vec{P}}}$. In particular, we have $|I_{\vec{P}}| = 2^{-j_{\vec{P}}}$. 
A multi-tile $\vec{P}'$ is called a translate of $\vec{P}$, if $Q_{\vec{P}} = Q_{\vec{P}'}$. If $P = I_P \times \omega_P$ is an $i$-tile and $\omega_P = Q_i$, we write $\omega_P$ for $Q_i$.

Since we assumed $|Q_n|$ to be an integral power of $2^J$, we also have that $|I_P|$ is an integral power of $2^J$ for all tiles $P$.

Note that $n$-tiles have area 1. In the Walsh analogue [27] they correspond to one dimensional subspaces of $L^2(\mathbb{R})$, and will thus be easier to handle than the other tiles. For instance, we shall be able to obtain good $L^p$ bounds on these tiles in addition to $L^2$ bounds thanks to Lemma 5.4. The $i$-tiles have area larger than one, and due to our goal to prove uniform estimates in the $M_i$, we do not have any control over the area. In the Walsh model these tiles correspond to high dimensional subspaces of $L^2(\mathbb{R})$, and we can do little more than considering good $L^2$ estimates on them.

We proceed to define the cutoff operators in physical space. Let $\eta$ denote a fixed positive function with total $L^1$-mass 1 and with Fourier transform supported in $[-2^{-2J}, 2^{-2J}]$, satisfying the pointwise estimates

$$C^{-1}(1 + |x|)^{-N^2} \leq \eta(x) \leq C(1 + |x|)^{-N^2}. \quad (16)$$

Here $N$ is the previously chosen constant which controls decay in physical space.

Let $\eta_j$ denote the function $\eta_j(x) := 2^{-Jj}\eta(2^{-Jj}x)$. For any subset $E$ of $\mathbb{R}$, denote by $\chi_E$ the characteristic function of $E$ and define the smoothed out characteristic function $\chi_{E,j}$ by

$$\chi_{E,j} := \chi_E * \eta_j.$$  \quad (17)

Note that

$$\chi_{\bigcup_{\alpha \in A} E_{\alpha,j}} = \sum_{\alpha \in A} \chi_{E_{\alpha,j}}.$$  \quad (18)

Informally, $\chi_{E,j}$ is a frequency-localized approximation to $\chi_E$. In fact we have the pointwise estimate

$$|\chi_{E,j}(x) - \chi_E(x)| \leq C(1 + 2^{Jj}\text{dist}(x, \partial E))^{-N^2+1}, \quad (19)$$

where $\partial E$ is the topological boundary of $E$; this is easily verified by checking the cases $x \in E$ and $x \in \mathbb{R \setminus E}$ separately.

Now let $\vec{P}_0$ denote the space of all multi-tiles. From (18) we have the identity

$$\sum_{\vec{Q} \in \mathcal{Q}} \prod_{i=1}^n \int \pi_{Q_i} f_i(x) \, dx = \sum_{\vec{P} \in \vec{P}_0} \int \chi_{I_P \times J_P}(x) \prod_{i=1}^n \pi_{\omega_P_i} f_i(x) \, dx.$$  \quad (20)

It thus suffices to show that

$$| \sum_{\vec{P} \in \vec{P}_0} \int \chi_{I_P \times J_P} \prod_{i=1}^n \pi_{\omega_P_i} f_i | \leq C \prod_{i=1}^n \|f_i\|_{p_i}.$$  \quad (21)
Again, by a standard limiting argument it suffices to prove this estimate where \( \vec{P}_0 \) has been replaced by the set of all tiles \( \vec{P} \in \vec{P}_0 \) such that \( I_P \subseteq [-2^{jk}, 2^{jk}] \) for some fixed large \( k \), provided the constant in the estimate does not depend on \( k \). We shall fix such \( k \) and denote this subset by \( \vec{P}_1 \). Thus \( \vec{P}_1 \) is a finite set of tiles (recall that the set \( \vec{Q} \) of possible frequency boxes had been restricted to a finite set). We thus are aiming to show

\[
| \sum_{\vec{P} \in \vec{P}_1} \int_{\pi \omega_{\vec{P}, j} P} \prod_{i=1}^{n} |f_i| \leq C \prod_{i=1}^{n} \|f_i\|_{p_i}. \tag{20}
\]

4. The Geometry of Tiles and Trees

The following definition establishes an order relation on the set of tiles.

**Definition 4.1.** Let \( P, P' \) be i-tiles for some \( 1 \leq i \leq n \). We say that \( P \leq P' \) if \( I_P \subseteq I_{P'} \) and \( \vec{\omega}_P \supseteq \vec{\omega}_{P'} \). If \( \vec{P} \) and \( \vec{P}' \) are two multi-tiles, we say that \( \vec{P} \leq \vec{P}' \) if there exists a \( 1 \leq i \leq n \) such that \( P_i \leq P_{i}' \).

Clearly the order of i-tiles is transitive. However, by the good dyadic property of Lemma 2.2, we also have

**Lemma 4.2.** The order on multi-tiles is transitive, i.e., \( \vec{P} \leq \vec{P}' \) and \( \vec{P}' \leq \vec{P}'' \) imply \( \vec{P} \leq \vec{P}'' \).

**Proof** Assume \( \vec{P} \neq \vec{P}' \) and \( \vec{P}' \neq \vec{P}'' \). Then \( \vec{P}_i \leq \vec{P}'_i \) and \( \vec{P}'_i \leq \vec{P}''_i \) imply \( \vec{Q}_i \supseteq \vec{Q}'_i \) and \( \vec{Q}'_i \supseteq \vec{Q}''_i \) with strict containment by (14). By Lemma 2.2 we conclude \( \vec{Q}_i \supseteq \vec{Q}'_i \) and \( \vec{Q}'_i \supseteq \vec{Q}''_i \). This gives \( \vec{P} \leq \vec{P}'' \) as desired. \( \blacksquare \)

As is standard in the theory, the main argument shall consist in splitting the set \( \vec{P}_1 \) into smaller subsets, for which the name tree has become standard. We call a dyadic interval \( J \)-dyadic, if it has length \( 2^{j} \) for some integer \( j \). If \( \xi \in \Gamma' \), \( 1 \leq i \leq n \), and \( I \) is a \( J \)-dyadic interval, we define

\[
\omega_{i, \xi, I} := [\xi_i - \frac{1}{2}M_i |I|^{-1}, \xi_i + \frac{1}{2}M_i |I|^{-1}]
\]

and

\[
\vec{\omega}_{\xi, I} := [L^{-1}(\xi)_i - 500|I|^{-1}, L^{-1}(\xi)_i + 500|I|^{-1}] .
\]

Observe that the right hand side of the last display does not depend on \( i \), because \( L^{-1}(\xi) \) is on the diagonal. We shall not need any good dyadic properties for the intervals \( \vec{\omega}_{\xi, I} \).

**Definition 4.3.** Let \( \xi \in \Gamma' \), let \( I \) be a \( J \)-dyadic interval, and let \( T \) be a set of multi-tiles. The triple \( (T, \xi, I) \) is called a tree, if \( T \) is non-empty, \( I_P \subseteq I \) for each \( \vec{P} \in T \), and for each \( \vec{P} \in T \) there is a \( 1 \leq i \leq n \) such that \( \vec{\omega}_{\xi, I} \subseteq \vec{\omega}_{P_i} \).

We shall write \( \omega_{i, T} \) and \( \vec{\omega}_T \) for \( \omega_{i, \xi, I} \) and \( \vec{\omega}_{\xi, I} \). The data \( (\xi, I) \) are called the top data of the tree.

We will frequently call the set \( T \) itself a tree, with the understanding that top data \( (\xi_T, I_T) \) are associated to \( T \). If \( T \) is a tree, we define the box set to be the set \( Q_T := \{ \vec{Q}_P : \vec{P} \in T \} \). For each \( Q \in Q_T \), we define the support \( E_{Q, T} \) of \( Q \) to be the set

\[
E_{Q, T} := \bigcup \{ I_P : \vec{P} \in T, Q_{\vec{P}} = Q \}. \tag{21}
\]
We say that two trees $T, T'$ are distinct if $T$ and $T'$ have no tiles in common, that is $T \cap T' = \emptyset$. (We are reserving the term disjointness for a stronger property, that the tiles in $T$ and $T'$ do not overlap).

If we say that one tree $T$ is contained in another tree $T'$, $T \subseteq T'$, this simply means that the multi-tiles in $T$ are also in $T'$, and does not imply any additional relation between the top data $(\xi_T, I_T)$ and $(\xi_{T'}, I_{T'})$.

A main observation for trees is that the box set of a tree can be parameterized by the frequency parameter. In this sense trees make the connection to Littlewood Paley theory, in which frequency boxes are essentially parameterized by their scale.

**Lemma 4.4.** If $T$ is a tree, then for each $j$ there is at most one $Q \in Q_T$ such that $j = j_Q$.

**Proof.** Let $Q, Q' \in Q_T$, then there are $i$ and $i'$ such that $\mathcal{Q}_T \subseteq \overline{Q}_i$ and $\mathcal{Q}_T \subseteq \overline{Q}_{i'}$. Thus $Q_i \cap Q_{i'} \neq \emptyset$. This implies $Q = Q'$ or $j_Q \neq j_{Q'}$. \hfill □

If $T$ is a tree, we define the scale set

$$J_T := \{ P j_Q : Q \in Q_T \}.$$ 

For $j \in J_T$ we define $Q^j$ to be the $Q \in Q_T$ for which $j_Q = j$.

The main tree selection algorithm will consist of a tree selection process as follows:

**Definition 4.5.** A tree selection process shall consist of choosing a tree $T_1$ from $\tilde{P}_1$, then choosing a tree $T_2$ from $\tilde{P}_1 \setminus T_1$ and so on. I.e., at the $k$-th step we choose a tree $T_k$ from $\tilde{P}_1 \setminus (T_1 \cup \cdots \cup T_{k-1})$. We shall refer to the trees $T_k$ as the selected trees.

For two reasons it will be necessary that at each step these trees be as large as possible.

**Definition 4.6.** Consider a subset $\tilde{P}$ of $\tilde{P}_1$ and top data $(\xi, I)$ as in Definition 4.3. Then the maximal tree $T^*$ in $\tilde{P}$ with top data $(\xi_{T^*}, I_{T^*}) = (\xi, I)$ is the set of all $\tilde{P} \in \tilde{P}$ such that $I_{\tilde{P}} \subseteq I$ and $\mathcal{Q}_T \subseteq \mathcal{Q}_{\tilde{P}}$.

A tree selection process is called greedy, if at the $k$-th step the tree $T_k$ is maximal in $\tilde{P}_1 \setminus (T_1 \cup \cdots \cup T_{k-1})$.

One of the reasons to run a greedy selection process is to gain a nesting property described by the following lemma:

**Lemma 4.7.** Let $T$ be one of the selected trees of a greedy tree selection process. If $Q, Q' \in Q_T$ and $j_Q < j_{Q'}$, then $E_{Q,T} \supset E_{Q',T}$.

**Proof.** Suppose under the assumptions of the lemma we had $E_{Q,T} \not\supset E_{Q',T}$, then there was a $\tilde{P}' \in T$ with $Q_{\tilde{P}_T} = Q'$ such that $I_{\tilde{P}_T} \not\subseteq E_{Q,T}$. Pick any $\tilde{P} \in T$ such that $Q_{\tilde{P}_T} = Q$, and let $\tilde{P}''$ be a translate of $\tilde{P}$ such that $I_{\tilde{P}_T} \subseteq I_{\tilde{P}''}$. Clearly $\tilde{P}''$ is an element of $\tilde{P}_1$.

The multi-tile $\tilde{P}''$ cannot be in the tree $T$ because of $I_{\tilde{P}_T} \not\subseteq E_{Q,T}$, but its geometry would qualify it to be in the tree, namely, $I_{\tilde{P}} \subseteq I_T$ and $\mathcal{Q}_T \subseteq \overline{\mathcal{Q}_{\tilde{P}}}$ for some $i$. Thus it must have been selected for a tree $T''$ at a previous stage of the selection process. However, the geometry of $\tilde{P}''$ qualifies it to be in the same tree $T''$, namely, $I_{\tilde{P}} \subseteq I_{\tilde{P}''} \subseteq I_T$ and $\mathcal{Q}_T \subseteq \overline{\mathcal{Q}_{\tilde{P}'}}$ because $\overline{\mathcal{Q}_{\tilde{P}''}} \subseteq \overline{\mathcal{Q}_{\tilde{P}'}}$ for some $j$ and $\overline{\mathcal{Q}_{\tilde{P}'}} \subseteq \overline{\mathcal{Q}_{\tilde{P}''}}$ by Lemma 2.2 and the fact that $\overline{\mathcal{Q}_{\tilde{P}'}}$ and $\overline{\mathcal{Q}_{\tilde{P}''}}$ intersect. This gives a contradiction to the maximality of $T''$. \hfill □

The nesting property of the above lemma implies the following bound on the cardinality of the finite sets $\partial E_{Qj,T}$. 

Lemma 4.8. Let $T$ be a selected tree of a greedy selection process. Then

$$
\sum_{j \in J_T} 2^{-T_j} \# \partial E_{Q_j,T} \leq C |I_T|.
$$

This should be compared with the trivial bound $\# \partial E_{Q_j,T} \leq 2^{T_j} |I_T|$.

**Proof** Since $E_{Q_j,T}$ is a finite union of intervals, there are two types of points in $\partial E_{Q_j,T}$: those that are the left endpoint of connected components in $E_{Q_j,T}$, and those that are right endpoints. Clearly it suffices to prove the bound for left endpoints only.

For each $j \in J_T$ and each left endpoint $x$ of $E_{Q_j,T}$, consider the interval $(x - 2^{-T_j}, x - 2^{-T_j-1})$. We claim that these intervals are pairwise disjoint. This claim implies the conclusion of the lemma because these intervals are contained in $3I_T$.

To prove the claim, assume $(x - 2^{-T_j}, x - 2^{-T_j-1})$ and $(x' - 2^{-T_j'}, x - 2^{-T_j'-1})$ have nonempty intersection. If $j = j'$, we necessarily have $x = x'$ because then both $x$ and $x'$ are endpoints of dyadic intervals of length $2^{-T_j}$. Thus we can assume $j' < j$. Then $x$ is contained in the interior of the dyadic interval $I'$ of length $2^{-T_j'}$ with right endpoint $x'$. However, the interior of $I'$ is disjoint from $E_{Q_j',T}$ and $x$ is contained in $E_{Q_j',T}$, a contradiction to Lemma 4.7. This proves the claim.

We will use Lemma 4.8 to replace the spatial truncations in (20) by certain variants of themselves when we sum over the multi-tiles in a selected tree.

It will be convenient to replace the sets $E_{Q_j,T}$ by variants $\tilde{E}_j$ which have better regularity properties:

**Definition 4.9.** Let $T$ be a tree, and let $I_T$ be the collection of all maximal $J$-dyadic intervals $I \subseteq I_T$ which have the property that $3I$ does not contain any of the intervals $I_P$ with $P \in T$. For an integer $j$ with $2^{-T_j} \leq |I_T|$ let $\tilde{E}_j$ be the union of all intervals $I$ in $I_T$ such that $|I| < 2^{-T_j}$ (We emphasize that we have strict inequality here, which will make the index $j$ most natural, as we can see for example in the following lemma). For an integer $j$ with $2^{-T_j} > |I_T|$ we define $\tilde{E}_j = \emptyset$.

The sets $\tilde{E}_j$ obviously depend on the tree $T$, but we suppress this dependence. The construction of the sets $E_j$ appears implicitly in (19).

Clearly the intervals in $I_T$ form a partition of $I_T$. The nice regularity properties are stated in the following lemma:

**Lemma 4.10.** Any two neighboring intervals in $I_T$ differ by at most a factor $2^J$ in length.

The set $\tilde{E}_j$ is a union of dyadic intervals of length $2^{-T_j}$ and contains $E_{Q_j,T}$ if $j \in J_T$.

**Proof** To prove the first statement, let $I$ and $I'$ be two neighboring intervals of $I_T$ and assume $I$ is larger. Let $I''$ be the dyadic interval which contains $I'$ and has $2^{-T_j}$ times the length of $I$. We have to prove $I'' \subseteq I'$. However, $3I''$ is contained in $3I$, and thus does not contain any interval $I_P$ with $P \in T$. By maximality of $I'$ we have $I'' \subseteq I'$. This proves the first statement of the lemma.

To prove the second statement, let $I \subseteq I_T$ be any $J$-dyadic interval of length $2^{-T_j}$ and observe the dichotomy that either $I$ is contained in an interval of $I_T$, or $I$ is partitioned into intervals of $I_T$ which are strictly smaller than $I$. The latter is necessarily the case if $I \cap \tilde{E}_j$ is nonempty or if $I = I_P$ for some $P \in T$. This proves the second statement.
Lemma 4.11. With the notation as above, if $I_0$ is a $J$-dyadic interval of length $2^{-J_{j_0}}$ such that $3I_0 \cap \tilde{E}_{j_0} \neq \emptyset$, then there is a multi-tile $\vec{P} \in T$ with $|I_0| \leq |I_0|$ such that $I_0 \subseteq 10I_0$

Proof. There is a dyadic interval $I_1$ of same length as $I_0$ which is contained in $3I_0$ and $\tilde{E}_{j_0}$. By definition of $E_{\omega_0}$, $3I_1$ contains an $I_0$ for some multi-tile $\vec{P} \in T$. This together with (13) proves the lemma. 

The sets $\tilde{E}_j$ are obviously nesting, hence we have the following analogue of Lemma 4.8.

Lemma 4.12. Let $T$ be any tree. Then with the notation as above,

$$\sum_{j \in J} 2^{-j} \# \partial \tilde{E}_j \leq C |I_T| .$$

For each $j \in J$, let $\Omega_j$ be the collection of connected components of $\tilde{E}_j$; thus $\Omega_j$ is a finite collection of intervals. For each $I \in \Omega_j$, let $x^l_j$ and $x^r_j$ denote the left and right endpoints of $I$, and let $I^l_j$ and $I^r_j$ denote the intervals

$$I^l_j := (x^l_j - 2^{-J(j+m_j)-1}, x^l_j - 2^{-J(j+m_j)-2})$$

$$I^r_j := (x^r_j + 2^{-J(j+m_j)-2}, x^r_j + 2^{-J(j+m_j)-1}).$$

Then the intervals $I^l_j$ are disjoint as $j$ varies in the integers with $2^{-J_j} \leq |I_T|$ and $I$ varies in $\Omega_j$.

Moreover, if $I^l_j$ is an interval in the above collection, then the distance to the next interval $I^l_{j'}$ is at least $2^{-J(j+2)}$.

Similar statements hold for the $I^r_j$.

Proof. Most of the proof is exactly as in the proof of Lemma 4.8, the only new statement is the one on the distance between two neighboring intervals.

Let $I^l_j$ be such an interval. It suffices to show every different interval $I^l_{j'}$ with $j' \geq j$ has distance at least $2^{-J(j+2)}$ from $I^l_j$. The case $j' \leq j$ then follows by symmetry.

The case $j = j'$ is easy, since the distance between $I^l_j$ and $I^l_{j'}$ is a multiple of $2^{-J_j}$.

Thus consider $j' > j$. Let $\tilde{I}$ be the dyadic interval of length $2^{-J_j}$ which contains $I^l_j$ and let $\tilde{I}'$ be the dyadic interval of length $2^{-J(j'+1)}$ whose left endpoint is equal to the right endpoint of $I$. Clearly $I$ is disjoint from $\tilde{E}_j$, whereas $I'$ is contained in $\tilde{E}_j$. The crucial observation is that $I'$ cannot be contained in $\tilde{E}_{j+1}$, because two neighboring intervals in $I_T$ differ by at most a factor $2^j$.

Thus the distance from $I^l_j$ to any point in $\tilde{E}_{j+1}$ is larger than $2^{-J(j+1)}$, which proves the claim.

While we will only consider trees selected by a greedy selection process, we remark that the sets $\tilde{E}_j$ can be defined and satisfy the above lemmata for arbitrary trees, which do not necessarily come from a greedy selection process and thus may not satisfy a nesting property as in Lemma 4.7.

The regularity properties of the sets $\tilde{E}_j$ will be used to construct phase plane projection operators. It is worth mentioning that a similar notion of regularity called convexity was used in [27] to construct phase plane projections in the Walsh setting, although technically
the use of this type of convexity to construct phase plane projections is quite different in
the current paper.

We introduce the notion of lacunarity in a tree:

**Definition 4.13.** An element $\vec{P} \in T$ is called $i$-lacunary if $(\xi_T)_i \not\in 2 \omega_{P_i}$, and it is called $i$-non-lacunary if $(\xi_T)_i \in 2 \omega_{P_i}$. If $A$ is a subset of $\{1, \ldots, n\}$, then $T_A$ is defined to be the set of all elements in $T$ which are $i$-lacunary for all $i \in A$ and $i$-non-lacunary for all $i \not\in A$.

Clearly, a tree can be written as the disjoint union of subsets $T_A$ parameterized by all subsets $A \subseteq \{1, \ldots, n\}$. Each of the sets $T_A$ is either empty or again a tree with the same top data as $T$.

We have

**Lemma 4.14.** If $T$ is a tree and $A \subseteq \{1, \ldots, n\}$, then for each $Q$ in $Q_{T_A}$ we have $Q \in Q_T$ and $E_{Q,T_A} = E_{Q,T}$.

**Proof** This follows easily from the fact that lacunarity depends only on the frequency intervals. ■

As a consequence of this lemma, if $T$ is a selected tree as in Lemma 4.7, then whenever $T_A$ is non-empty and thus a tree, it satisfies the analogues of Lemmata 4.7 and 4.8.

There is always at least one lacunary index:

**Lemma 4.15.** If $T$ is a tree, then $T_\emptyset$ is empty.

**Proof** If $\vec{P} \in T_\emptyset$, then $L^{-1}(\xi_T) \in 2Q_{\vec{P}}$. Since $\xi_T \in \Gamma'$ this contradicts (10). ■

We return to the second reason for running a greedy selection process. It gives a certain strong disjointness property of multi-tiles of selected trees, as described by the following lemmata:

**Lemma 4.16.** Let $T$ and $T'$ be two selected trees of a greedy selection process and assume that $T$ has been selected prior to $T'$. Let $\vec{P} \in T$ and $\vec{P}' \in T'$ be such that we have

$$10 \omega_{P_i} \cap 10 \omega_{P_i'} \neq \emptyset$$

and

$$|\omega_{P_i}| < |\omega_{P_i'}|$$

for some $i$. Then $I_{\vec{P}_i} \cap I_T = \emptyset$.

**Proof** Assume to get a contradiction that $I_{\vec{P}_i} \cap I_T \neq \emptyset$. By size comparison we then necessarily have $I_{\vec{P}_i} \subseteq I_T$.

By (23) and (13) we have $100|\omega_{P_i}| < |\omega_{P_i'}|$. Hence (22) implies $20 \omega_{P_i} \subseteq 20 \omega_{P_i'} \neq \emptyset$, which implies $\omega_{P_i} \subseteq \omega_{P_i'}$ (We made a point of not using Lemma 2.2 to conclude this).

Hence the geometry of the multi-tile $\vec{P}'$ qualifies it to be in the tree $T$. But it is not, because different selected trees have no multi-tiles in common, so by maximality of $T$ it must be an element of a tree that was selected prior to $T$. This contradicts the assumed order of selection of $T$ and $T'$.

The above lemma requires information about the order in which trees have been selected. In our application this information will be provided in the form described by the
next lemma. If $\vec{P}$ is a multi-tile, $s$ a real number, $\xi \in \mathbb{R}^n$ and $1 \leq i \leq n$, define the interval
\[
\omega_{\vec{P},\xi,i,s}^+ := [\xi_i + 2^{-s}5000C_0|\omega_{P_i}|, \xi_i + 2^{-s}5000C_0|\omega_{P_i}|].
\]

**Lemma 4.17.** Let $1 \leq i \leq n$. Let $T$ be a subset of the set of selected trees of a greedy selection process such that $T, T' \in T$ and $(\xi_T)_i > (\xi_{T'})_i$ imply that $T$ has been selected prior to $T'$. Let $s$ be some real number. Let $T, T'$ be two (not necessarily different) trees in $T$ and assume $\vec{P} \in T$ and $\vec{P}' \in T'$ such that
\[
10\omega_P \cap 10\omega_P' \neq \emptyset,
\]
\[
|\omega_P| < |\omega_{P'}|,
\]
\[
\omega_{\vec{P},\xi,i,s}^+ \cap \omega_{\vec{P}',\xi',i,s}^+ \neq \emptyset.
\]
Then $I_{P'} \cap I_T = \emptyset$ and in particular $I_{P'} \cap I_P = \emptyset$ and the trees $T$ and $T'$ are indeed different.

**Proof** By the previous lemma we only need to prove that $T$ has been selected prior to $T'$. However, (23) together with
\[
100|\omega_P| < |\omega_{P'}|
\]
implies $(\xi_T)_i > (\xi_{T'})_i$, which in turn implies that $T$ has been selected prior to $T'$. ■

Similarly, we can define
\[
\omega_{\vec{P},\xi,i,s}^- := [\xi_i - 2^{-s}5000C_0|\omega_{P_i}|, \xi_i - 2^{-s}5000C_0|\omega_{P_i}|]
\]
and then have an analogous lemma to 4.17 which we do not state explicitly.

We shall need another variant of this theme. For a tree $T$ define
\[
\omega_{i,T,s}^+ := [(\xi_T)_i + 2^{-s}10|\omega_{i,T}|, (\xi_T)_i + 2^{-s}10|\omega_{i,T}|].
\]

**Lemma 4.18.** Let $1 \leq i \leq n$. Let $T$ be a subset of the set of selected trees of a greedy selection process such that $T, T' \in T$ and $(\xi_T)_i > (\xi_{T'})_i$ imply that $T$ has been selected prior to $T'$.

Let $s$ be some real number. Let $T_0, T_1, T_2, T_3 \in T$ and assume we have for $j = 1, 2, 3
\[
10\omega_{i,T_0} \cap 10\omega_{i,T_j} \neq \emptyset,
\]
\[
|\omega_{i,T_0}| \leq |\omega_{i,T_j}|,
\]
\[
\omega_{i,T_0}^+ \cap \omega_{i,T_j}^+ \neq \emptyset.
\]
If $T_1, T_2, T_3$ are all different, then $I_{T_1} \cap I_{T_2} \cap I_{T_3} = \emptyset$.

**Proof** As before, we can use (23) to conclude that if $|I_{T'}| > |I_{T''}|$, for $T', T'' \in \{T_1, T_2, T_3\}$, then $T'$ has been selected prior to $T''$. Thus we may assume $|I_{T_1}| \geq |I_{T_2}| \geq |I_{T_3}|$ and $T_1, T_2, T_3$ have been selected in this order.

Assume to get a contradiction that $I_{T_1} \cap I_{T_2} \cap I_{T_3} \neq \emptyset$, then we have by dyadicity $I_{T_1} \supset I_{T_2} \supset I_{T_3}$. Let $\vec{P}$ be a multi-tile in $T_1 \cup T_2 \cup T_3$ for which $|\omega_P|$ is minimal. Let $T \in \{T_1, T_2, T_3\}$ be the tree so that $\vec{P} \in T$. If there is another multi-tile $\vec{P}'$ for which
|ω_P'| = |ω_P|, then we conclude by (14) that ω_P' = ω_P. Hence ̃P and ̃P' are in the same tree T.

Now let ̃P' ∈ T' for some T' ∈ {T_1, T_2, T_3} with |ω_P'| > |ω_P|. We aim to show ̃P' ∈ T_1, which will prove the lemma, because then one of the trees T_2 and T_3 has to be empty, a contradiction.

Observe that by (26) and (27) we have ω_i,T_0 ⊆ 20ω_i,T_1. By a similar argument for T' we have 20ω_i,T_1 ∩ 20ω_i,T' ≠ ∅. By assumption on the size of tree tops we have ω_i,T_1 ⊆ 40ω_i,T'. If |ω_i,T_1| < |ω_i,T'|, then this implies ̃ω_T_1 ⊆ ̃ω_T', which in turn implies ̃P' ∈ T_1.

We may thus assume |ω_i,T_1| = |ω_i,T'|. Then we can merely conclude ̃ω_T_1 ∩ ̃ω_T' ≠ ∅ and ̃ω_T_1 ⊆ 3̃ω_T'. By a similar argument with T' in place of T_1 we have ̃ω_T' ∩ ̃ω_T ≠ ∅ and ̃ω_T' ⊆ 3̃ω_T. But ̃ω_T ⊆ ̃ω_P_j for some index 1 ≤ j ≤ n. Hence ̃ω_T' ⊆ 3̃ω_P_j. However, for some possibly different index l we have ̃ω_T' ⊆ ̃ω_P_l. Hence 3̃ω_P_j and ̃ω_P_l have non-empty intersection, and thus by Lemma 2.2 we have 10̃ω_P_j ⊆ ̃ω_P_l. However, we have seen before ̃ω_T_1 ⊆ 3̃ω_T' and we have 3̃ω_T' ⊆ 9̃ω_P_j. Hence ̃ω_T_1 ⊆ ̃ω_P_l, which proves the claim.

5. A FEW REMARKS ON SMOOTH TRUNCATIONS AND WEIGHTS

We pause to prove a few lemmata on weight functions and smooth truncations of functions, which are best separated from the main string of arguments so as to not slow the main argument down by these technical lemmata.

Given an interval I, we define the approximate cutoff function ˜χ_I as

\[ ˜χ_I(x) := (1 + \frac{\text{dist}(x, I)}{|I|})^{-1}. \]

We shall need the following lemma, which may be of independent interest.

**Lemma 5.1.** Let I be a finite collection of disjoint intervals, and suppose that for each I one has an L^2 function f_I. Then

\[ \| \sum_{I \in \mathbf{I}} |I|^{1/2} \tilde{x}_I f_I \|_2 \leq C \left( \sum_{I \in \mathbf{I}} |I| \right)^{1/2} \sup_{I \in \mathbf{I}} \| f_I \|_2. \]

Note that this lemma would be automatic from Cauchy-Schwarz if \( \sum_I \tilde{x}_I^2 \) was uniformly bounded. Unfortunately, this function is only in BMO, so one needs to work a little harder. This lemma is a special case of a phase space Bessel inequality (28) that we will need in the sequel.

**Proof** We may assume that the f_I are real and positive. By estimating ˜χ_I by a weighted sum of \( \chi_{AI} \) for dyadic A ≥ 1 (as before AI has the same center as I but A times its length), it suffices by the triangle inequality to show

**Lemma 5.2.** Let I be as above, f_I be real and positive, and let A ≥ 1. Then there exists an absolute constant M > 0 such that

\[ \| \sum_{I \in \mathbf{I}} |I|^{1/2} \chi_{A I} f_I \|_2 \leq MA^{1/2} \left( \sum_{I \in \mathbf{I}} |I| \right)^{1/2} \sup_{I \in \mathbf{I}} \| f_I \|_2. \] (29)

**Proof** Fix A. Let \( M_A \) be the best constant M for which (29) obtains for all subsets of \( \mathbf{I} \) (in place of \( \mathbf{I} \) itself) and all \( f_I \); our objective is to show that \( M_A \) is bounded uniformly in A and \( \mathbf{I} \).
Since \( \chi \) and an analoguous estimate for the term \( \sum \) consistent with (2) if we lift \( m \) by the triangle inequality it suffices to prove for each \( k \) one only has a contribution if \( J \subseteq 5AI \) or \( I \subseteq 5AJ \). By symmetry and positivity we thus have

\[
\sum_{I} \sum_{J: J \subseteq 5AI} \langle |I|^{1/2} \chi_{AI}f, |J|^{1/2} \chi_{AJ}f \rangle \gtrsim M_{A}^{2}A(\sum_{I} |I|) \sup_{I} \| f_{I} \|^{2}.
\]

Moving the \( J \) summation inside the inner product, and using Cauchy-Schwarz and the definition of \( M_{A} \), we may estimate the left-hand side by

\[
\sum_{I} |I|^{1/2} \| f_{I} \|^{2} M_{A} A^{1/2}(\sum_{J: J \subseteq 5AI} |J|) \sup_{J} \| f_{J} \|^{2}.
\]

On the other hand, by disjointness we have \( \sum_{J: J \subseteq 5AI} |J| \leq 5A|I| \). Inserting this into the previous estimates we obtain \( M_{A} \leq C_{1} \) as desired.

Lemma 5.1 will be used through the following variant:

**Lemma 5.3.** Let \( I' \) be an interval and let \( I \) be a collection of disjoint intervals such that \( |I| \leq |I'| \) for each \( I \in I \). For each \( I \in I \) consider a positive \( L^{2} \) function \( f_{I} \). Then

\[
\| \tilde{\chi}_{I'} \sum_{I \in I} |I|^{1/2} \tilde{\chi}_{I} f_{I} \|_{2} \leq C|I'|^{1/2} \sup_{I} \| f_{I} \|_{2}.
\]

**Proof**

We may write

\[
\tilde{\chi}_{I'} \leq C\chi_{I'} + \sum_{k \geq 0} 2^{-k} \chi_{2^{k+1}I' \setminus 2^{k}I'}
\]

By the triangle inequality it suffices to prove for each \( k \geq 0 \)

\[
\| \sum_{I \in I: I \cap 2^{k+1}I' \setminus 2^{k}I' \neq \emptyset} |I|^{1/2} \tilde{\chi}_{I} f_{I} \|_{2} \leq C' 2^{k/2} |I'|^{1/2} \sup_{I} \| f_{I} \|_{2}.
\]

and an analoguous estimate for the term \( \chi_{I'} \), which is clearly an easy variant of the above. Since \( |I| \leq |I'| \) for all \( I \in I \) and the intervals \( I \) are pairwise disjoint, we have

\[
\sum_{I \in I: I \cap 2^{k+1}I' \setminus 2^{k}I' \neq \emptyset} \leq C2^{k} |I'|.
\]

The claim now follows by Lemma 5.1.

For any symbol \( m \) on \( \mathbb{R} \), we let \( T_{m} \) be the associated Fourier multiplier. (This is consistent with (2) if we lift \( m \) from \( \mathbb{R} \) to \( \{(\xi, -\xi) : \xi \in \mathbb{R}\} \) in the obvious manner).
Let $w$ be a positive function on $\mathbb{R}$ and $r > 0$ be a number. We say that $w$ is essentially constant at scale $r$ if one has
\[(1 + \frac{|x - y|}{r})^{-100} \leq \frac{w(x)}{w(y)} \leq C(1 + \frac{|x - y|}{r})^{-100}\]
for all $x, y \in \mathbb{R}$. In particular, the weights $\tilde{\chi}_I^\alpha$ are essentially constant at any scale $|I|$ or less when $|\alpha| \leq 100$.

We shall need the following weighted version of Bernstein’s inequality.

**Lemma 5.4.** Let $f$ be a function whose Fourier transform is supported on an interval $\omega$ of width $O(2^{Jj})$ for some integer $j$. Then we have
\[\|w f\|_\infty \leq C 2^{j/2} \|w f\|_2\]
for all weights $w$ which are essentially constant at scale $2^{-Jj}$.

**Proof** We can write $f = T_m f$ where $m$ is a suitable bump function adapted to $2\omega$. From the decay of the kernel of $T_m$ we thus have the pointwise estimate
\[|f(x)| = |T_m f(x)| \leq C 2^{j} \int \frac{f(y)}{(1 + 2^{j} |x - y|)^N} \, dy\]
and the claim easily follows. \[
\]

Let $T$ be a (possibly vector-valued) convolution operator and $r > 0$ be a number. We say that $T$ is essentially local at scale $r$ if the convolution kernel $K(x)$ satisfies the bounds
\[|K(x)| \leq C r^N |x|^{-N}\]
for $|x| \gg r$.

**Lemma 5.5.** Let $T$ be a convolution operator which is bounded on $L^2$ and which is essentially local at some scale $r > 0$. Then one has
\[\|w T f\|_2 \leq C \|w f\|_2\]
for all weights $w$ which are essentially constant at scale $r$.

**Proof** We can truncate $T$ so that the kernel is supported on the interval $\{|x| \leq Cr\}$; from (30) it is easy to see that this does not affect the $L^2$ boundedness of $T$ or (31). The claim then follows by partitioning space into intervals of length $r$ and applying the $L^2$ boundedness hypothesis to each interval separately. \[
\]

6. Phase space norms and the size of a tree

The general approach to proving an estimate such as (20) is to prove the estimate first in the easier case when the summation goes only over a tree rather than the whole set $\mathbf{\tilde{P}}_1$. The point being that this easier estimate is a matter of standard Littlewood-Paley theory without modulation invariance. The top of the tree fixes a frequency, which after a modulation we can think of as being the zero frequency in standard Littlewood-Paley theory. In our situation this Littlewood-Paley estimate is Proposition 1.3.

The second step is to organize the whole set into trees by a greedy selection process, so as to sum the tree estimates. In this organization, the notion of size of a tree plays a crucial role. In this section we shall introduce this notion.
Definition 6.1. In the following definitions $1 \leq i \leq n$, and $f_i$ is an $L^2$ function.

If $P_i$ is an $i$-tile and $\xi_i \in \mathbb{R}$, we define the semi-norm $\|f_i\|_{P_i, \xi_i}$ by

$$\|f_i\|_{P_i, \xi_i} := \sup_{m_{P_i}} \|\tilde{\chi}_{I_{P_i}}^{10} T_{m_{P_i}}(f_i)\|_2$$

(32)

where $m_{P_i}$ ranges over all smooth functions supported on $10\omega_{P_i}$ which satisfy the estimates

$$|\partial_{\xi}^k m_{P_i}(\xi)| \leq |\xi - \xi_i|^{-k} \frac{|\xi - \xi_i|}{|\omega_{P_i}|}$$

(33)

for all $\xi \in \mathbb{R}$ and $0 \leq k \leq N^2$. In particular, $m_{P_i}$ vanishes at $\xi_i$.

If $T$ is a tree, we define the $i$-size $\text{size}_i(T)$ of $T$ by

$$\text{size}_i(T) := \left(\frac{1}{|T|} \sum_{P_i \in T} \|f_i\|_{P_i, (\xi_{P_i})}^2\right)^{1/2} + |I_T|^{-\frac{1}{2}} \sup_{m_i, T} \|\tilde{\chi}_{I_T}^{10} T_{m_i, T}(f_i)\|_2.$$  

(34)

where $m_i, T$ ranges over all smooth functions supported on $10\omega_i, T$ which satisfy the estimates

$$|\partial_{\xi}^k m_{i, T}(\xi)| \leq |\xi - (\xi_{T_i})_i|^{-k} \frac{|\xi - (\xi_{T_i})_i|}{|\omega_{T_i}|}$$

(35)

for all $\xi \in \mathbb{R}$ and $0 \leq k \leq N^2$.

If $\tilde{P}$ is any collection of multi-tiles, we define the maximal size $\text{size}_i^*(\tilde{P})$ of $\tilde{P}$ to be

$$\text{size}_i^*(\tilde{P}) := \sup_{(T, \xi, I) : T \subseteq \tilde{P}} \text{size}_i(T)$$

(36)

where $(T, \xi, I)$ ranges over all trees with $T \subseteq \tilde{P}$.

We remark, that for $|\xi - \xi_i| \gg |\omega_{P_i}|$ we have the following estimates for $m_{P_i}$ which are stronger than (33):

$$|\partial_{\xi}^k m_{P_i}(\xi)| \leq C |\xi - \xi_i|^{-k} \left(\frac{|\xi - \xi_i|}{|\omega_{P_i}|}\right)^{1-l}.$$  

(37)

for all $0 \leq l, k < N^2/2$. These estimates can be obtained from (33) and the support condition on $m_{P_i}$ by integrating the $m_{P_i}$ over its support $l$ times. Thus $m_{P_i}$ is forced to be rather small if $\xi_i$ is far away from its support. This observation shall however be of technical importance, since in our applications $\xi_i$ will always be within $C\omega_{P_i}$ for some moderate constant $C$.

Thus, heuristically, $\|f_i\|_{P_i, \xi_i}$ is the $L^2$ norm of $f_i$ when restricted to the portion of $P_i$ which is away from the frequency $\xi_i$. The tree size $\text{size}_i(T)$ is heuristically the $L^2$ norm of $f_i$ when restricted to the region $\bigcup_{P_i \in T} P_i \cup (I_T \times \omega_i, T)$ in phase space. As a gross caricature, one has the very approximate relationship

$$\text{size}_i(T) \approx^{\text{osc}} T_{I_T}\text{osc}_{I_T}(e^{-2\pi i (\xi_{P_i})_i} : f_i)$$

although this caricature does not fully capture the phase space localization to $T$ in (34).

Here we have written $\text{osc}_I(f)$ for the $L^2$ mean oscillation given by

$$\text{osc}_I(f) := \left(\frac{1}{|I|} \int_I |f - \frac{1}{|I|} \int_I f|^{2}ight)^{1/2}.$$  

(38)

If $\tilde{P}$ is a tree, the maximal size $\text{size}_i^*(\tilde{P})$ is a strengthened version of the tree size $\text{size}_i(\tilde{P})$. The relationship between the two is analogous to the relationship between the
BMO norm on an interval $I$ and the $L^2$ mean oscillation on that interval $I$. This analogy
is particularly accurate when $M_i = 0$ and $(\xi_T)_i = 0$.

The freedom to choose $I$ in (30) independently of the set $T$ adds a useful “Hardy-
Littlewood maximal function”-component to the size definition. For example this is used
in the following lemma:

**Lemma 6.2.** We have
\[
\|\tilde{\chi}_{I'}^{10} T_{m'}(f_i)\|_2 \leq C\|f_i\|_{p_i, \xi_i} \leq C|I_{\tilde{P}}|^{1/2}\text{size}^*(T) \tag{39}
\]
for all indices $1 \leq i \leq n$, trees $T$, multi-tiles $\tilde{P} \in T$, frequencies $\xi_i \in \mathbb{R}$, and symbols $m_{P_i}$ supported on $10\omega_{P_i}$ which obey (33).

Moreover,
\[
\|\tilde{\chi}_{I'}^{10} T_{m_i}(f_i)\|_2 \leq C|I|^{1/2}\text{size}^*(T) \tag{40}
\]
for all indices $1 \leq i \leq n$, trees $T$, all $i$-non-lacunary multi-tiles $\tilde{P} \in T$, all $J$-dyadic
intervals $I$ with $I_{\tilde{P}} \subseteq 10I$ and all symbols $m_{i, I}$ supported on $5\omega_i, \xi_T, I$ and satisfying
\[
|\partial_{\xi}^k m_i (\xi) | \leq |\omega_i, \xi_T, I|^{-k}
\]
for all $\xi \in \mathbb{R}$ and $0 \leq k \leq N^2$.

**Proof** We first consider (39). The first inequality is just by definition. By the remark
just after Definition 3.1, we only have to prove the second inequality for $\xi_i \in 100\omega_{P_i}$. Then
this inequality follows from (34), (33) since $T$ contains the singleton tree $\{\tilde{P}\}$ with
top data $\xi', I'$ where $\xi'$ is defined by $\xi_i = (\xi')_i$ and $I'$ is the $J$-dyadic interval of length
$2^J|I_{\tilde{P}}|$ which contains $I_{\tilde{P}}$. It is easily verified that these top data indeed turn $\{\tilde{P}\}$ into a
tree.

Now we consider (40). Observe first of all that $|I_{\tilde{P}}| \leq |I|$ because both intervals are
$J$-dyadic. By translating $I$, we may as well assume $I_{\tilde{P}} \subseteq I$. Namely, we have to translate
$I$ by at most ten times its length, so that $\tilde{\chi}_I$ stays the same up to some bounded factor.

We consider the two cases $|I_{\tilde{P}}| = |I|$ and $|I_{\tilde{P}}| < |I|$. Assume first $|I_{\tilde{P}}| = |I|$.

Observe that by $i$-non-lacunarity, $\omega_{P_i}$ is contained in $5\omega_i, \xi_T, I$. Hence $5\omega_i, \xi_T, I$ is contained
in $9\omega_{P_i}$. Pick $\xi' \in \Gamma'$ so that $(\xi')_i$ is an endpoint of $10\omega_{P_i}$. Then
\[
\|\tilde{\chi}_{I'}^{10} T_{m_i}(f_i)\|_2 \leq C\|f_i\|_{p_i, (\xi')_i},
\]
by definition of the right hand side, because the multiplier $m_{i, I}$ is supported in $10\omega_{P_i}$ and
satisfies (33), possibly with a constant. Now the claim follows from (33), which proves (40)
in the case $|I_{\tilde{P}}| = |I|$.

Now assume $|I_{\tilde{P}}| < |I|$. We consider again a singleton tree $\{\tilde{P}\}$ with top data $\xi$, $I$ so
that $\xi_i$ is an endpoint of $5\omega_i, \xi_T, I$. Again, by $i$-non-lacunarity of $\tilde{P}$ we see that these top
data indeed turn $\{\tilde{P}\}$ into a tree. Then the multiplier $m_{i, I}$ satisfies (33) with respect to
this singleton tree as one can easily see, possibly with some constant, and (40) follows by
definition of the tree size.

Inequality (40) will be mainly applied in connection with Lemma 4.1.

Note that our definition of size is $L^2$-based (but normalized to have an $L^\infty$ scaling,
see the proposition below). In principle one can define $L^p$ based notions of size by using
the $L^p$ norm instead of the $L^2$ norm in (32) and the properly adjusted normalizations thereafter. In the $M_i = 0$ case the $L^2$-based notion of size is essentially equivalent to the $L^p$-based notion thanks to Lemma 5.4 and the fact that the $P_i$ then have area 1. Only if $n = i$ this will be guaranteed, which is why this case will have special treatment.

However when $M_i \gg 1$ the $L^2$-based notion of size is not equivalent to an $L^p$ based notion. In order to make the Bessel inequality (98) work we need $L^2$-based sizes, and this is one of the reasons for the restriction $2 \leq p_i < \infty$ in Theorem 1.2. Presumably one would need arguments such as those in [26] to remove this restriction.

We conclude this section with the observation that the size is always controlled by the $L^\infty$ norm.

**Proposition 6.3.** For all $1 \leq i \leq n$ and arbitrary functions $f_i$, we have

$$\text{size}^*_i(\vec{P}_1) \leq C\|f_i\|_\infty.$$  \hspace{1cm} (41)

**Proof** Fix $i$. It suffices to show that

$$\text{size}_i(T) \leq C\|f_i\|_\infty$$

for all trees $T$ in $\vec{P}_1$.

Fix $T$. We have to estimate both summands in (34). The second summand is immediate since $T_{m_i,T}$ is a universally bounded operator in $L^\infty$.

We consider the first summand in (34). By frequency modulation invariance we may assume that $\xi_T = 0$. From (34), (32) it suffices to show that

$$\sum_{\vec{P} \in T} \|\tilde{\chi}_{\vec{P}}^{10} T_{m_i}(f_i)\|_2^2 \leq CC|I_T|\|f_i\|_2^2$$

whenever $m_{P_i}$ is supported on $10\omega_{P_i}$ and obeys (33) with $\xi_i = 0$.

First suppose that $f_i$ vanishes on $3I_T$. Then a simple computation following Lemma 5.5 shows that

$$\|\tilde{\chi}_{\vec{P}}^{10} T_{m_i}(f_i)\|_2^2 \leq C|I_{\vec{P}}|\left|\frac{|\vec{P}|}{|I_{\vec{P}}|}\right|^2\|f_i\|_\infty^2$$

for all $\vec{P} \in T$, and the claim follows by summing in $\vec{P}$. Thus we may assume that $f_i$ is supported in $3I_T$. It then suffices to show that

$$\sum_{\vec{P} \in T} \|\tilde{\chi}_{\vec{P}}^{10} T_{m_i}(f_i)\|_2^2 \leq CC\|f_i\|_2^2.$$

By Khinchin's inequality, we may estimate the left-hand side by

$$\|\sum_{\vec{P} \in T} \epsilon_{\vec{P}} \tilde{\chi}_{\vec{P}}^{10} T_{m_i}(f_i)\|_2^2$$

for a suitable choice of signs $\epsilon_{\vec{P}} \in \{+1, -1\}$. But then the claim follows since the expression inside the norm is simply a pseudo-differential operator of order 0 in the symbol class $S^{1, -1}$ applied to $f_i$, with bounds uniform in $T, M_i, m_{P_i},$ and $\epsilon_{\vec{P}}$ (see [24]).

We remark that one can improve the bound from 1 to $|E_i|/|E_k|$ for some other $1 \leq k \leq n$, if one is willing to remove from $E_k$ the exceptional set where the Hardy-Littlewood maximal function of $E_i$ is $\gg |E_i|/|E_k|$, and also to remove from $\vec{P}_1$ all multi-tiles whose spatial interval is contained in this exceptional set (cf. [20] Lemma 7.8; see also [26], [18], [22]).
Such improved estimates will not be needed here, because we restrict attention to the case $p_i > 2$.

7. Tree estimates

To follow the approach outlined at the beginning of Section 6, it is necessary to estimate the sum on the left hand side of (20), where the summation over $\vec{P}_1$ is replaced by a summation over a tree, by the various $i$-sizes of the tree. In doing so we may assume that the tree has been selected by a greedy selection process, because all our trees will be selected this way. This tree estimate will be Proposition 7.1 below.

Proposition 7.1. Let $T$ be a selected tree of a greedy selection process, and let $f_1, \ldots, f_n$ be test functions on $\mathbb{R}$ which satisfy the normalization
\[ \|f_i\|_\infty \leq 1 \] (42)
for all $1 \leq i \leq n$. Then we have
\[ \left| \sum_{\vec{P} \in T} \int \chi_{I_{\vec{P}}} \prod_{i=1}^n \pi_{\omega_{P_i}, f_i} \right| \leq C(\theta_i) |I_T| \prod_{i=1}^n \text{size}_i^*(T)^{\theta_i} \] (43)
whenever $\theta_n = 1$, and $0 < \theta_1, \ldots, \theta_{n-1} < 1$ are such that $\sum_{i=1}^{n-1} \theta_i < 2$, and the implicitly used constant $N$ is sufficiently large depending on $(\theta_i)$.

In spirit this estimate is a version of Proposition 1.3 with additional localization in physical space. It would be very tedious to adapt the proof of Proposition 1.3 to the localized setting, we shall therefore follow a different approach. The idea is to replace the $f_i$ by functions (phase plane projections) which are very close to $f_i$ near the phase plane region of the tree, but essentially vanish outside this region. Then we shall apply Proposition 1.3 to these phase plane projections directly and recover the result of Proposition 7.1 from there: the tree sizes on the right hand side of (43) will essentially be the suitably normalized $L^p$-norms of the phase plane projections.

The parameters $\theta_i$ will be chosen depending on the exponents $p_i$ in Theorem 1.2. In the following we shall not write explicitly $(\theta_i)$-dependence of our constants, in the same way as we do not write $p_i$-dependence explicitly.

In analogy to [17, 18, 20] one might expect a bound of $|I_T| \prod_{i=1}^n \text{size}_i^*(T)$ on the right-hand side of (13). This bound is achievable in the non-degenerate case $m_i = O(1)$; however, in general only the $n^{th}$ size $\text{size}_n^*(T)$ can be recovered with a full power $\theta_n = 1$. This gain in the case $i = n$ will be crucial in the rest of the paper. One should probably be able to obtain the endpoint $\sum_{i=1}^{n-1} \theta_i = 2$ of this result, but we shall not attempt to do so here.

We will prove Proposition 7.1 in this and the next section. We remark that knowledge of the proof of (43) will not be needed in the later sections.

Proof. Fix $T, \theta_i, f_i$. We shall exploit scale invariance to reduce to the case $|I_T| = 1$. By a frequency modulation leaving $\Gamma$ and $\Gamma'$ invariant, we may as well assume $\xi_T = 0$.

By the triangle inequality it suffices to prove estimate (13) where the summation goes over the subset $T_A$ instead of $T$ for any subset $A \subseteq \{1, \ldots, n\}$. Fix such $A$ and let $B$ denote the set of non-lacunary indices, i.e., $\{1, \ldots, n\} = A \uplus B$. We may assume $T_A$ is non-empty and thus a tree with top data $(\xi_T, I_T)$. By Lemma 4.13 we have that $A$
is non-empty. Changing notation we may as well assume that all multi-tiles in $T$ have lacunarity type $A$ and thus $T = T_A$. Recall that $T_A$ also satisfies the crucial nesting property of Lemma 4.7.

Let $J$ denote the set $J := \{ j_Q : Q \in Q_T \}$; its smallest element is greater or equal 0. Recall that by Lemma 4.4 the map $Q \mapsto j_Q$ is one-to-one.

From (17) we may estimate the left-hand side of (43) as

$$\sum_{j \in J} | \int \tilde{\chi}_j \prod_{i=1}^n \tilde{\pi}_j f_i | \tag{44}$$

where the spatial cutoff $\tilde{\chi}_j$ and Fourier cutoff $\tilde{\pi}_j$ are defined as

$$\tilde{\chi}_j := \chi_{E Q_i, T, j} = \sum_{\tilde{P} \in T : \tilde{Q} = Q_i} \chi_{\tilde{P} \cap E Q_i, j} \tag{45}$$

and

$$\tilde{\pi}_j := \pi_{Q_i}$$

for $j \in J$. In particular, $\tilde{\chi}_j$ has Fourier support in the region $\{ \xi : |\xi| \leq 2^{Jj-\theta} \}$. We remark that our notation is sloppy here, the operator $\tilde{\pi}_j$ also depends on the parameter $i$. We will always write $\tilde{\pi}_j$ in combination with a function $f_i$, and the omitted index is always the one of the function $f_i$.

Let $2 < p_i < \infty$ be chosen so that $p_i < 2/\theta_i$ for $1 \leq i \leq n - 1$ and $\sum_{i=1}^n 1/p_i = 1$; this can be done by the hypotheses on $\theta$ stated in Proposition 7.1. We observe the simple bounds

**Lemma 7.2.** For all $1 \leq i \leq n$, $j \in J$, and intervals $I$ of length $|I| = 2^{-Jj}$ we have

$$\| \tilde{\chi}^{1/2n}_j \tilde{\pi}_j f_i \|_{L^{p_i}(I)} \leq C |I|^{1/p_i} \text{size}^*_i (T)^{\theta_i}.$$  

**Proof** By interpolation and Proposition 6.3 it suffices to prove the bounds

$$\| \tilde{\chi}^{1/2n}_j \tilde{\pi}_j f_i \|_{L^2(I)} \leq C |I|^{1/2} \text{size}^*_i (T),$$

and (in the $i = n$ case only)

$$\| \tilde{\chi}^{1/2n}_j \tilde{\pi}_j f_i \|_{L^\infty(I)} \leq C \text{size}^*_i (T).$$

The second estimate is immediate from the boundedness of the $f_i$, while the third follows from the first and Lemma 5.4 since $|v_n| = 1$ and thus $m_n = 0$. Thus it suffices to prove the first inequality.

Fix $i, j, I$. From (46) we see that there exists $\tilde{P} \in T$ with $|\tilde{P}| = |I|$ such that we have the pointwise estimate

$$\tilde{\chi}_j(x)^{1/2n} \leq C \tilde{\chi}_{I_{\tilde{P}}}^{10}$$

on $I$. It thus suffices to show that

$$\| \tilde{\chi}_{I_{\tilde{P}}}^{10} \tilde{\pi}_j f_i \|_{L^2} \leq C |I|^{1/2} \text{size}^*_i (T).$$

This however was observed in (49).
From this Lemma and Hölder we have
\[ \int_I |\tilde{\chi}_j \prod_{i=1}^n \tilde{\pi}_j f_i| \leq C \|\tilde{\chi}_j\|_{L^\infty(I)}^{1/2} |I| \prod_{i=1}^n \text{size}_i^*(T)^{\theta_i} \]
for all \( j \in J \) and \(|I| = 2^{-J_j}\). Summing in \( I \), we see that we may bound each summand in (44) by the right-hand side of (43). Thus the main difficulty is to obtain summability in \( j \).

In the non-degenerate case \( m_i = O(1) \) this summability is obtained by noting there must be at least two lacunary indices \( j \) in \( A \), estimating those indices in \( L^2 \) in space (and \( l^2 \) in \( j \)), and taking all other indices in \( L^\infty \). See e.g. [17], [18], [20] and the discussion in the introduction of [21]. This however is not feasible in the general case for two reasons. Firstly, there might only be one lacunary index; and secondly, one can only get good \( L^\infty \) bounds for the \( i \) index when \( m_i = O(1) \). Thus we will have to invoke Proposition 1.3 as outlined before.

We shall need to replace \( \chi_j \) in (44) by a product of cutoff functions. If \( \chi_j \) was a characteristic function, we could simply write it as a power of itself. However, it is an approximate characteristic function. From (19) we have:
\[ |\tilde{\chi}_j(x) - \chi_{E_{Q_j,T}}| \leq C (1 + 2^{J_j} \text{dist}(x, \partial E_{Q_j,T}))^{-N^2+1}. \] (46)

The right hand side can be handled by Lemma 4.8.

**Lemma 7.3.** Let \( 1 \leq k \leq n \), then
\[ \sum_{j \in J} \left| \int (\tilde{\chi}_j - \tilde{\chi}_j^k) \prod_{i=1}^n \tilde{\pi}_j f_i \right| \leq C |I_T| \prod_{i=1}^n \text{size}_i^*(T)^{\theta_i}. \]

**Proof** By the triangle inequality, we can estimate this sum by
\[ \sum_{j \in J} \sum_{I: |I| = 2^{-J_j}} \int_I |\tilde{\chi}_j - \tilde{\chi}_j^k| \prod_{i=1}^n |\tilde{\pi}_j f_i|. \]

By Hölder and Lemma 7.2, this is bounded by
\[ \sum_{j \in J} \sum_{I: |I| = 2^{-J_j}} |I| \| (\tilde{\chi}_j - \tilde{\chi}_j^k) \tilde{\chi}_j^{-\frac{1}{2}} \|_{L^\infty(I)} \prod_{i=1}^n \text{size}_i^*(T)^{\theta_i}. \]

On the other hand, from (46) and the lower bound in (19) we have
\[ |I| \| (\tilde{\chi}_j - \tilde{\chi}_j^k) \tilde{\chi}_j^{-\frac{1}{2}} \|_{L^\infty(I)} \leq C \int_I (1 + 2^{J_j} \text{dist}(x, \partial E_{Q_j,T}))^{-N}. \]

Summing this in \( I \) and evaluating the integral, we estimate the previous by
\[ C \sum_{j \in J} 2^{-J_j} \# \partial E_{Q_j,T} \]
and the claim follows from Lemma 4.8.

---

\(^2\)This is due to the fact that the cubes 10Q^j must intersect \( \Gamma \) in order to have a non-zero contribution to (44).
Instead of estimating \((\Pi)\), it suffices by the above Lemma to to show that
\[
\sum_{j \in J} \left| \int \prod_{i=1}^{n} \tilde{\chi}_j \tilde{\pi}_j f_i \right| \leq C |I_T| \prod_{i=1}^{n} \text{size}_i^*(T)^{\theta_i}.
\] (47)

To do this we shall now introduce the crucial tree projection operators.

**Proposition 7.4.** Let the notation and assumptions be as above. Then for each \(1 \leq i \leq n\) there exists a function \(\Pi_i(f_i)\) with the following properties.

- **(Control by size)** We have
  \[
  \|\Pi_i(f_i)\|_{p_i} \leq C |I_T|^{1/p_i} \text{size}_i^*(T)^{\theta_i}.
  \] (48)

- **(\(\Pi_i(f_i)\) approximates \(f_i\) on \(T\): lacunary case)** For all \(i \in A\) and \(j \in J\), we have
  \[
  \tilde{\chi}_j \tilde{\pi}_j f_i = S_{j+m_i} \Pi_i(f_i)
  \] (49)
  where \(S_{j+m_i}\) is a suitable Littlewood-Paley projection to the frequency region \(\{\xi : \frac{1}{10}2^{J(j+m_i)} \leq |\xi| \leq 5000C_02^{J(j+m_i)}\}\).

- **(\(\Pi_i(f_i)\) approximates \(f_i\) on \(T\): non-lacunary case)** For all \(i \in B, j_0 \in J\), and all intervals \(I_0\) of length \(2^{-J_0}\), we have the bounds
  \[
  \|\tilde{\chi}_{j_0}^{1/2n} \pi_{j_0} (f_i - \Pi_i(f_i))\|_{L^{p_i}(I_0)} \leq C \text{size}_i^*(T)^{\theta_i} |I_0|^{1/p_i - 1} \int \tilde{\chi}_{I_0}^2 \mu_{j_0}
  \] (50)
  where \(\mu_j\) is the function
  \[
  \mu_j(x) := \sum_{0 \leq j'} 2^{-|j' - j|/100} \sum_{y \in \partial \tilde{E}_j} (1 + 2^{j |x - y|})^{-100}
  \] (51)
  and the sets \(\tilde{E}_j\) have been defined in Definition 4.9. Also, for all \(1 \leq i \leq n, j \in J\), and intervals \(I\) of length \(|I| = 2^{-J_i}\) we have
  \[
  \|\tilde{\chi}_j^{1/2n} \pi_j \Pi_i(f_i)\|_{L^{p_i}(I)} \leq C |I|^{1/p_i} \text{size}_i^*(T)^{\theta_i}.
  \] (52)

One can construct the \(\Pi_i\) to be linear operators, but we shall not use this. Roughly speaking, \(\Pi_i(f_i)\) is the projection of \(f_i\) to the region \(\bigcup_{P \in T} P_i\) of phase space. Although such a description can easily be made rigorous in a Walsh model, and is not too difficult in a lacunary Fourier model, it is substantially more delicate in the Fourier setting in the non-lacunary case.

Note that, the function \(\mu_j\) can be controlled by Lemma 4.12.

We prove this rather technical Proposition in Section 8. For now, we see why this proposition, combined with Proposition 1.3, gives \((43)\).

To exploit the fact that \(\Pi_i(f_i)\) approximates \(f_i\) on \(T\), we use \((49)\) and the triangle inequality to estimate \((47)\) by the main term
\[
\sum_{j \in J} \left| \int \prod_{i \in A} \tilde{\chi}_j \tilde{\pi}_j f_i \prod_{i \in B} \tilde{\chi}_j \tilde{\pi}_j \Pi_i(f_i) \right| \tag{53}
\]
plus \(#B\) error terms of the form
\[
\sum_{j \in J} \left| \int \tilde{\chi}_j \tilde{\pi}_j (f_{i_0} - \Pi_{i_0}(f_{i_0})) \prod_{i \in A \text{ or } i < i_0} \tilde{\chi}_j \tilde{\pi}_j f_i \prod_{i \in B : i > i_0} \tilde{\chi}_j \tilde{\pi}_j \Pi_i(f_i) \right| \tag{54}
\]
where $i_0 \in B$.

Let us first control the contribution of a term (54). Fix $i_0$. By the triangle inequality we may estimate (54) by

$$
\sum_{j \in J} \sum_{I: |I| = 2^{-J}} \int_I |\tilde{\chi}_j \tilde{\pi}_j (f_{i_0} - \Pi_{i_0}(f_{i_0}))| (\prod_{i \in A \text{ or } i < i_0} |\tilde{\chi}_j \tilde{\pi}_j f_i|) \prod_{i \in B: i > i_0} |\tilde{\chi}_j \tilde{\pi}_j \Pi_i(f_i)|.
$$

By Hölder, we may estimate the previous by

$$
\sum_{j \in J} \sum_{I: |I| = 2^{-J}} \|\tilde{\chi}_j \tilde{\pi}_j (f_{i_0} - \Pi_{i_0}(f_{i_0}))\|_{L^{p_{i_0}}(I)} \times \prod_{i \in A \text{ or } i < i_0} \|\tilde{\chi}_j \tilde{\pi}_j f_i\|_{L^{p_i}(I)} \prod_{i \in B: i > i_0} \|\tilde{\chi}_j \tilde{\pi}_j \Pi_i(f_i)\|_{L^{p_i}(I)}. \tag{55}
$$

The first factor we estimate using (50). The second group of factors we estimate using Lemma 7.2. For the last group we use (52). Combining all these estimates and using (4), we see that we may estimate (55) by

$$
C \sum_{j \in J} \sum_{I: |I| = 2^{-J}} \left( \prod_{i = 1}^n \text{size}_i(T)^{\theta_i} \right) \int |\tilde{\chi}_j|^2 \mu_j
$$

which of course sums to

$$
C \left( \prod_{i = 1}^n \text{size}_i^*(T)^{\theta_i} \right) \sum_j \int \mu_j.
$$

Expanding out $\mu_j$, we may estimate this by

$$
C \left( \prod_{i = 1}^n \text{size}_i^*(T)^{\theta_i} \right) \sum_j \sum_y (1 + 2^{J_j} |x - y|)^{-100} \, dx;
$$

computing the integral, we thus obtain

$$
C \left( \prod_{i = 1}^n \text{size}_i^*(T)^{\theta_i} \right) \sum_j 2^{-J_j} \# \partial E_j
$$

and the claim follows from Lemma 4.12.

It remains to estimate (53). By repeating the proof of Lemma 7.3 (but using (52) in place of Lemma 7.2 when $i \in B$) it suffices to estimate

$$
\sum_{j \in J} \left| \int \left( \prod_{i \in A} \tilde{\chi}_j \tilde{\pi}_j f_i \right) \left( \prod_{i \in B} \tilde{\pi}_j \Pi_i(f_i) \right) \right|
$$

which we rewrite using (49) as

$$
\sum_{j \in J} \left| \int \left( \prod_{i \in A} S_{j+m_i} \Pi_i(f_i) \right) \left( \prod_{i \in B} \tilde{\pi}_j \Pi_i(f_i) \right) \right|.
$$

By Proposition 1.3, we may estimate this by

$$
C \prod_{i = 1}^n \|\Pi_i(f_i)\|_{p_i},
$$
and the claim follows from (18). Observe that in applying Proposition 1.3 we have used $A \neq \emptyset$, which had been a consequence of Lemma 4.15.

This concludes the proof of Proposition 7.1, except for the proof of Proposition 7.4 which shall be done in the next section.

8. Proof of Proposition 7.4

We now prove Proposition 7.4. The proof of this Proposition is rather involved, and the techniques used here are not needed elsewhere in the paper; readers who are interested in the general shape of the proof of Theorem 1.2 may wish to skip this section on a first reading.

We begin with the lacunary case $i \in A$, which is substantially easier.

Fix $i \in A$. We define $\Pi_i(f_i)$ as

$$
\Pi_i(f_i) := \sum_{j \in J} \tilde{\chi}_j \tilde{\pi}_j f_i
$$

in this case. From the Fourier support of $\tilde{\chi}_j$ and $\tilde{\pi}_j$ we see that (49) is obeyed. It remains to show (48).

By interpolation and Proposition 6.3 it suffices to prove the estimates

$$
\|\Pi_i(f_i)\|_2 \leq C|IT|^{1/2}\text{size}^*_i(T) \quad (57)
$$

$$
\|\Pi_i(f_i)\|_{\text{BMO}} \leq C \quad (58)
$$

with the additional estimate

$$
\|\Pi_i(f_i)\|_{\text{BMO}} \leq C\text{size}^*_i(T) \quad (59)
$$

when $i = n$. Here we read BMO as $J$-dyadic BMO (see the proof below).

We first prove (57). By orthogonality and (45) we have

$$
\|\Pi_i(f_i)\|_2^2 = \sum_{j \in J} \sum_{\vec{P} \in T: Q_{\vec{P}} = Q_j} |\chi_{\vec{P},j} \tilde{\pi}_j f_i|_2^2.
$$

Since we have the pointwise bound

$$
\sum_{\vec{P} \in T: Q_{\vec{P}} = Q_j} |\chi_{\vec{P},j} \tilde{\pi}_j f_i| \leq C
$$

we can bound the previous using Cauchy-Schwarz by

$$
C \sum_{j \in J} \sum_{\vec{P} \in T: Q_{\vec{P}} = Q_j} \||\chi_{\vec{P},j}|^{1/2} \tilde{\pi}_j f_i\|_2^2.
$$

But by (18) this is bounded by

$$
C \sum_{j \in J} \sum_{\vec{P} \in T: Q_{\vec{P}} = Q_j} \|f_i\|_{P_{j,0}}^2
$$

and the claim follows from (18).

The claim (58) follows immediately from the observation that the linear operator $\Pi_i$ is a pseudo-differential operator of order 0 in the symbol class $S^{1,-1}$, and therefore maps $L^\infty$ to BMO (see e.g. [24]), so it remains to show (59).
We need to show that
\[
\text{osc}_I(\Pi_n(f_n)) \leq C\text{size}_n^*(T)
\] (60)
for all $J$-dyadic intervals $I$, which is what we mean by $J$-dyadic BMO.

Fix $I$. We expand the left-hand side of (60) using (45) as
\[
\text{osc}_I\left(\sum_{\vec{P} \in T} \chi_{I_{\vec{P},j\vec{P}}} \pi_{\omega_{P_n}} f_n \right).
\]

First consider the contribution of the coarse scales, when $|I_{\vec{P}}| > |I|$. By the sub-linearity of $\text{osc}_I()$, it suffices to show that
\[
\sum_{\vec{P} \in T : |I_{\vec{P}}| > |I|} \text{osc}_I(\chi_{I_{\vec{P},j\vec{P}}} \pi_{\omega_{P_n}} f_n) \leq C\text{size}_n^*(T).
\] (61)

We use the easily verified Poincare inequality
\[
\text{osc}_I(f) \leq C(|I| \int_I |\nabla f|^2)^{1/2}.
\]
The Fourier multipliers $|I_{\vec{P}}|\nabla \pi_{\omega_{P_n}}$ and $\pi_{\omega_{P_n}}$ have symbols adapted to $10\omega_{P_n}$ which vanish at the origin, whereas $\chi_{I_{\vec{P},j\vec{P}}}$ and $I_{\vec{P}}\nabla \chi_{I_{\vec{P},j\vec{P}}}$ are dominated by $\tilde{\chi}_{I_{\vec{P}}}$. By the previous and (39), we thus have
\[
\text{osc}_I(\chi_{I_{\vec{P},j\vec{P}}} \pi_{\omega_{P_n}} f_n) \leq C\frac{|I|}{|I_{\vec{P}}|} (1 + \text{dist}(I, I_{\vec{P}}))^{-100}\text{size}_n^*(T).
\]
Summing over all $\vec{P}$ such that $|I_{\vec{P}}| > |I|$ we see that this contribution is acceptable.

It remains to consider the contribution of the fine scales, i.e. those $\vec{P}$ in the set
\[
T_I := \{ \vec{P} \in T : |I_{\vec{P}}| \leq |I| \}.
\]
For this contribution we will shall use the estimate
\[
\text{osc}_I(f) \leq C \frac{1}{|I|^{1/2}} \| f \chi_{I,jI} \|_2,
\]
where $2^{jI} := |I|$. It thus suffices to show that
\[
\| \sum_{\vec{P} \in T_I} \chi_{I_{\vec{P},j\vec{P}}} \chi_{I_{\vec{P},j\vec{P}}} \pi_{\omega_{P_n}} f_n \|_2 \leq C|I|^{1/2}\text{size}_n^*(T).
\] (62)

For fixed $j_{\vec{P}}$, the expression inside the norm has Fourier support in the region $|\xi| \sim 2^{j_{\vec{P}}}$ (lacunary supports). By orthogonality we may thus estimate the left-hand side of (62) by
\[
C(\sum_{j \in J, j \geq j_{\vec{P}}} \| \sum_{\vec{P} \in T_I : j_{\vec{P}} = j} \chi_{I_{\vec{P},j\vec{P}}} \chi_{I_{\vec{P},j\vec{P}}} \pi_{\omega_{P_n}} f_n \|_2^2)^{1/2}.
\]
For fixed $j_{\vec{P}} = j$, the $\chi_{I_{\vec{P},j\vec{P}}}(x)$ are uniformly summable in $I_{\vec{P}}$ and $x$. By Cauchy-Schwarz we may thus estimate the previous by
\[
C(\sum_{j \in J, j \geq j_{\vec{P}}} \sum_{\vec{P} \in T_I : j_{\vec{P}} = j} \| \chi_{I_{\vec{P},j\vec{P}}} \chi_{I_{\vec{P},j\vec{P}}} \pi_{\omega_{P_n}} f_n \|_2^2)^{1/2}.
\]
On the other hand, from (39) we see that
\[ \|\chi_{I,j} x_{P,j,\tilde{\rho}} \tilde{\pi}_n f_n\|_2 \leq C(1 + \frac{\text{dist}(I, I_{\tilde{P}})}{|I|})^{-100} \|f_n\|_{P_n,0}. \]

Thus we may estimate the left-hand side of (62) by
\[ C\left( \sum_{\tilde{P}} (1 + \frac{\text{dist}(I, I_{\tilde{P}})}{|I|})^{-200} \|f_n\|_{P_n,0}^2 \right)^{1/2}. \]

We now break up this as a sum over sub-trees of \( T \). Let \( T_I \) denote those multi-tiles in \( T \) for which the interval \( I \) is maximal. Clearly the \( I \) are disjoint as \( \tilde{P} \) varies over \( T_I \).

We can thus rewrite the previous as
\[ C\left( \sum_{\tilde{P}} (1 + \frac{\text{dist}(I, I_{\tilde{P}})}{|I|})^{-200} \sum_{\tilde{P} \in T : I_{\tilde{P}} \subseteq I_{\tilde{P}}} \|f_n\|_{P_n,0}^2 \right)^{1/2}. \]

Since \( |I_{\tilde{P}}| \leq |I| \), we can estimate this by
\[ C\left( \sum_{\tilde{P}} (1 + \frac{\text{dist}(I, I_{\tilde{P}})}{|I|})^{-200} \sum_{\tilde{P} \in T : I_{\tilde{P}} \subseteq I_{\tilde{P}}} \|f_n\|_{P_n,0}^2 \right)^{1/2}. \]

For each \( \tilde{P}' \in T_I \), the collection \( \{\tilde{P} \in T : I_{\tilde{P}} \subseteq I_{\tilde{P}'}\} \) is a subtree of \( T \) with top data \((\xi, I_{\tilde{P}'})\). By (33) we can thus estimate the previous by
\[ C\left( \sum_{\tilde{P}'} (1 + \frac{\text{dist}(I, I_{\tilde{P}'})}{|I|})^{-200} \sum_{\tilde{P} \in T : I_{\tilde{P}} \subseteq I_{\tilde{P}'} \cap T} \|f_n\|_{P_n,0}^2 \right)^{1/2}. \]

Since the \( I_{\tilde{P}} \) are disjoint and of size \( \leq |I| \), we can bound this by
\[ C \text{size}^*_n(T) \left( \int (1 + \frac{\text{dist}(I, x)}{|I|})^{-200} \, dx \right)^{1/2} \]
and the claim follows.

It remains to handle the more difficult non-lacunary case. Fix \( i \in B \).

For each real number \( j \), let \( T_j \) be a Fourier multiplier (defined, say, by dilations of a fixed multiplier) whose symbol is adapted to the frequency region \( \{ |\xi| \leq 2^{2+Jj}\} \), whose symbol equals 1 for \( \{ |\xi| \leq 2^{1+Jj}\} \), and let \( S_j \) be the associated Littlewood-Paley projections \( S_j := T_j - T_{j-1} \). We may assume that the kernels of \( T_j \) and \( S_j \) are real and even.

In (54), one used lacunary Fourier multipliers followed by smooth spatial cutoffs to construct \( \Pi_i(f_i) \). In the non-lacunary case these smooth spatial cutoffs are not desirable as they interfere with the ability to decompose non-lacunary multipliers as a telescoping series of lacunary ones. To avoid this difficulty we are forced to use rough spatial cutoffs instead.

A first guess as to the construction of \( \Pi_i(f_i) \) would be
\[ \tilde{\Pi}_i(f_i) := \chi_{E_0} T_{j(x)} + m_i f_i, \]

\(^3\)Of course, one could try to control this error with commutator estimates, however one does not seem able to recover the crucial \( 2^{-j}j_{10}^{-1/100} \) decay in (54) by this approach.
where for each \( x \in \tilde{E}_0 \) we define the integer-valued function \( j(x) \) by
\[
j(x) := \max\{0 \leq j : x \in \tilde{E}_j\}
\]

One can expand \( \tilde{\Pi}_i(f_i) \) as a telescoping series:
\[
\tilde{\Pi}_i(f_i) = \chi_{\tilde{E}_0} T_m f_i + \sum_{1 \leq j} \chi_{\tilde{E}_j} S_{j+m} f_i.
\]

This proposed projection turns out to obey (68), but does not obey (50) due to the poor frequency localization properties of the characteristic functions \( \chi_{\tilde{E}_j} \) in (63). Specifically, the cutoffs destroy the vanishing moments of the \( S_{j+m} f_i \), and this will cause a difficulty when trying to sum in \( j \) because the projection \( \bar{\pi}_{J0} \) is non-lacunary.

To get around this problem we shall modify each term of \( \tilde{\Pi}_i(f_i) \) (except for the first term \( \chi_{\tilde{E}_0} T_m f_i \)) to have a zero mean. In order that these modifications do not collide with each other, we shall place them in disjoint intervals.

We recall the intervals \( I_j^* \) and \( I_j^l \) and the collections \( \Omega_j \) introduced in Lemma 4.12. Let \( \phi_{l,j}^I \) and \( \phi_{r,j}^I \) be bump functions adapted to \( I_j^* \), \( I_j^l \) with total mass
\[
\int \phi_{l,j}^I = \int \phi_{r,j}^I = 2^{-J(j+m_i)}.
\]

For each \( j \geq 1 \) and \( I \in \Omega_j \), decompose \( \chi_I \) as \( \chi_I = H_I^f + H_I^r \), where \( H_I^f(x) := H(x-x_I^f) \) and \( H_I^r(x) := -H(x-x_I^r) \) are shifted Heaviside functions.

For each \( j \geq 1 \) and \( I \in \Omega_j \), define the quantities \( c_{l,j}^I \) and \( c_{r,j}^I \) by
\[
c_{l,j}^I := 2^{J(j+m_i)} \int H_I^f S_{j+m} f_i, \quad (64)
\]
\[
c_{r,j}^I := 2^{J(j+m_i)} \int H_I^r S_{j+m} f_i.
\]

A basic estimate on \( c_{l,j}^I \) is

**Lemma 8.1.** Let \( j \geq 0 \) and \( I \in \Omega_j \). Then we have the estimate
\[
|c_{l,j}^I| \leq C 2^{J(j+m_i)} \int \frac{S_{j+m} f_i(x) \ dx}{(1 + 2^{J(j+m_i)}|x-x_I^f|)^{100}}. \quad (65)
\]

In particular, we have
\[
|c_{l,j}^I| \leq C 2^{Jm_i/2size^*_{I}(T)} \quad (66)
\]

and
\[
|c_{r,j}^I| \leq C. \quad (67)
\]

**Proof**

From (64) we may write
\[
c_{l,j}^I = 2^{J(j+m_i)} \int (\tilde{S}_{j+m} H_I^f) S_{j+m} f_i.
\]

Here \( \tilde{S}_{j+m} \) is a Littlewood Paley projection whose multiplier is supported in \( 5\omega_{i,T} \), is constant 1 for \( \xi \in 4\omega_{i,T} \setminus 2J^{-1}\omega_{i,T} \), and vanishes on \( J^{-1}\omega_{i,T} \). The claim (65) then follows by using repeated integration by parts to obtain pointwise bounds on \( \tilde{S}_{j+m} H_I^f \).
There exists a $J$-dyadic interval $I'$ of length $2^{-J}$ whose left endpoint coincides with $x_I$, because $\tilde{E}_j$ and hence $I$ is a union of dyadic intervals of length $2^{-J}$ by Lemma 4.10. By Lemma 4.11 there is a tile $\tilde{P} \in T$ with $I_{\tilde{P}} \subseteq 10I'$.

The bounds (66), (67) then follow from (65), (40), (42).

We can now define the corrected projections $\Pi_i(f_i)$ as

$$\Pi_i(f_i) := \tilde{\Pi}(f_i) - \sum_{1 \leq j} \sum_{I \in \Omega_j} (c^l_{I,j} \phi_{I,j} + c^r_{I,j} \phi^r_{I,j}).$$

We first verify (48). We first observe that

$$\|\Pi_i(f_i)\|_\infty \leq C.$$  (69)

Indeed, from (42) we see that $T_{j_0 + m} f_i$ and $\Pi_i(f_i)$ are bounded. The remaining terms of (69) then follow from (67) and the disjointness of the $\phi_{I,j}$, and similarly for $c^l_{I,j}$ and $\phi^r_{I,j}$.

In light of (69) it suffices by interpolation and Proposition 5.3 to prove (57), together with the bound

$$\|\Pi(f_i)\|_\infty \leq C 2^{jm_i/2} \text{size}^*_i(T).$$  (70)

In fact, to prove (48) we only need the bound (70) for $i = n$, but the general case will be useful later.

We now prove (70). From (66) the contribution of the $c^l_{I,j}$ is acceptable. Similarly for $c^r_{I,j}$. Thus it remains to show that

$$\|\Pi_i(f_i)\|_\infty \leq C 2^{jm_i/2} \text{size}^*_i(T),$$

or in other words that

$$|T_{j(x) + m} f_i(x)| \leq C 2^{jm_i/2} \text{size}^*_i(T)$$

for all $x \in \tilde{E}_0$.

Fix $x$, and define $j := j(x)$. From the definition of $j(x)$ there exists a $J$-dyadic interval $I' \subseteq \tilde{E}_j$ of length $2^{-J}$ which contains $x$. It thus suffices to show that

$$\|T_{j + m} f_i\|_{L^\infty(I')} \leq C 2^{jm_i/2} \text{size}^*_i(T).$$

By Lemma 4.11 there is a multi-tile $\tilde{P} \in T$ with $I_{\tilde{P}} \subseteq 10I'$. Hence the desired estimate follows from (40) and Lemma 5.4.

This completes the proof of (70).

It remains to prove (57). From (68), the triangle inequality, and the disjointness and size of the $\phi^l_{I,j}$ (and of the $\phi^r_{I,j}$) it suffices to show that

$$\|\tilde{\Pi}(f_i)\|_2 \leq C |I_j|^{1/2} \text{size}^*_i(T)$$

and

$$\left(\sum_{0 \leq j} \sum_{I \in \Omega_j} |c^l_{I,j}|^2 2^{-J(j + m_j)}\right)^{1/2} \leq C |I_j|^{1/2} \text{size}^*_i(T)$$

(72)

together with a similar bound for the $c^r_{I,j}$.

The bound (72) follows from (66) and Lemma 4.12.
To prove (71) we expand
\[ \tilde{\Pi}(f_i) = \sum_{0 \leq j} \chi_{\tilde{E}_j \setminus \tilde{E}_{j+1}} T_{j+m} f_i = \sum_{0 \leq j} \sum_{I \cap \tilde{E}_j \setminus \tilde{E}_{j+1} \neq \emptyset; \lvert I \rvert = 2^{-j}} \chi_{I \cap \tilde{E}_j \setminus \tilde{E}_{j+1}} T_{j+m} f_i. \]

As \( j \) and \( I \) vary in the above sum, the sets \( I \cap \tilde{E}_j \setminus \tilde{E}_{j+1} \) are pairwise disjoint, hence it satisfies to show
\[ \sum_{0 \leq j} \sum_{I \cap \tilde{E}_j \setminus \tilde{E}_{j+1} \neq \emptyset; \lvert I \rvert = 2^{-j}} \lVert T_{j+m} f_i \rVert_{L^2(I)}^2 \leq C \lvert I \rvert \text{size}_i^*(T)^2 \]

for all \( j, I \) in the above sum. But for such \( j, I \) we can find a multi-tile \( \tilde{P} \in T \) with \( I_\tilde{P} \subseteq 10I \) by Lemma 4.11. The proof of (48) is now complete.

The estimate (52) will follow from (50), Lemma 7.2, the triangle inequality, and the fact that the \( \mu_j \) are uniformly bounded. Thus it only remains to verify (50).

Fix \( j_0 \geq 0 \) and \( I_0 \) such that \( \lvert I_0 \rvert = 2^{-j_0} \). From the frequency support of \( \tilde{\pi}_{j_0} \) we may replace \( f_i - \Pi_i(f_i) \) with \( T_{j_0+m_0} f_i - \Pi_i(f_i) \). By interpolation and Proposition 3.3 it suffices to show the bounds
\[ \lVert \tilde{\chi}_{j_0}^{1/2n} \tilde{\pi}_{j_0}(T_{j_0+m_0} f_i - \Pi_i(f_i)) \rVert_{L^\infty(I_0)} \leq C \frac{1}{\lvert I_0 \rvert} \int \tilde{\chi}_{I_0}^2 \mu_{j_0} \]  
(73)

and
\[ \lVert \tilde{\chi}_{j_0}^{1/2n} \tilde{\pi}_{j_0}(T_{j_0+m_0} f_i - \Pi_i(f_i)) \rVert_{L^2(I_0)} \leq C \text{size}_i^*(T) \frac{1}{\lvert I_0 \rvert} \int \tilde{\chi}_{I_0}^2 \mu_{j_0} \]  
(74)

with the additional bound
\[ \lVert \tilde{\chi}_{j_0}^{1/2n} \tilde{\pi}_{j_0}(T_{j_0+m_0} f_i - \Pi_i(f_i)) \rVert_{L^\infty(I_0)} \leq C \text{size}_i^*(T) \frac{1}{\lvert I_0 \rvert} \int \tilde{\chi}_{I_0}^2 \mu_{j_0} \]  
(75)

when \( i = n \).

We first show (73). For this estimate the only bound we use on \( f_i \) is (42).

First suppose that \( I_0 \) is outside \( \tilde{E}_{j_0} \). Then the claim follows from (42), (69), the decay of \( \tilde{\chi}_{j_0} \) and the estimate
\[ (1 + 2^{j_0} \text{dist}(I_0, \partial \tilde{E}_{j_0}))^{-N} \leq C \frac{1}{\lvert I_0 \rvert} \int \tilde{\chi}_{I_0}^2 \mu_{j_0}. \]  
(76)
It remains to consider the case when \( I_0 \) is inside \( \tilde{E}_{j_0} \). In this case the cutoff \((\tilde{\chi}_{j_0})^{1/2n}\) is useless and will be discarded. We now decompose

\[
T_{j_0+m_i} f_i - \Pi_i(f_i) = \chi_{\mathbb{R}\setminus\tilde{E}_{j_0}} T_{j_0+m_i} f_i \tag{77}
\]

\[
- \chi_{\mathbb{R}\setminus\tilde{E}_{j_0}} \Pi_i(f_i) \tag{78}
\]

\[
+ \chi_{\mathbb{R}\setminus\tilde{E}_{j_0}} \sum_{1 \leq j \leq j_0} \sum_{I \in \Omega_j} c_{I,j}^l \phi_{I,j}^l \tag{79}
\]

\[
+ \chi_{\mathbb{R}\setminus\tilde{E}_{j_0}} \sum_{1 \leq j \leq j_0} \sum_{I \in \Omega_j} c_{I,j}^r \phi_{I,j}^r \tag{80}
\]

\[
- \sum_{j_0 < j} \sum_{I \in \Omega_j} H_{S_j+m_i} f_i - c_{I,j}^l \phi_{I,j}^l \tag{81}
\]

\[
- \sum_{j_0 < j} \sum_{I \in \Omega_j} H_{I,j} S_j+m_i f_i - c_{I,j}^r \phi_{I,j}^r. \tag{82}
\]

From (12) and the separation between \( I_0 \) and \( \mathbb{R}\setminus\tilde{E}_{j_0} \) and the decay of the kernel of \( \tilde{\chi}_{j_0} \) we can bound the contribution of (77) by (76) as desired. The terms (78), (79), (80) are similar, although (79), (80) use (67) instead of (12).

Now consider (81). The idea is to interact the smoothing properties of \( \tilde{\chi}_{j_0} \) with the moment property

\[
\int H_{S_j+m_i}^l f_i - c_{I,j}^l \phi_{I,j}^l = 0 \tag{83}
\]

coming from the construction of the \( c_{I,j}^l \).

By the triangle inequality and the definition of \( \mu_j \) it suffices to show that

\[
\| \tilde{\pi}_{j_0}(H_{S_j+m_i}^l f_i - c_{I,j}^l \phi_{I,j}^l) \|_{L^\infty(I_0)} \leq C2^{-(j-j_0)/100}2^{-J(j-j_0)}(1 + \frac{\text{dist}(I_0, x_i^l)}{|I_0|})^{-30} \tag{84}
\]

for all \( j > j_0 \) and \( I \in \Omega_j \).

Fix \( j, I \). We first assume \( x_i^l \in 3I_0 \), in which case we may replace the right-hand side by \( C2^{-(j-j_0)/100}2^{-J(j-j_0)} \). Let \( K(x) \) denote the convolution kernel of \( \tilde{\chi}_{j_0} \). By (83) it suffices to show that

\[
| \int (K(\rho) - K(\rho - x_i^l))(H_{S_j+m_i} f_i(x) - c_{I,j}^l \phi_{I,j}^l(x)) \, dx | \leq C2^{-(j-j_0)/100}2^{-J(j-j_0)}
\]

for all \( x_0 \in I_0 \).

Fix \( x_0 \in I_0 \). The contribution of \( c_{I,j}^l \phi_{I,j}^l \) can be controlled using (77), the fact that the support of \( \phi_{I,j}^l \) is within a distance of \( C2^{-J(j+m_i)} \) from \( x_i^l \), and the bound

\[
|K(x_0 - x) - K(x_0 - x_i^l)| \leq C2^{2J(j+m_i)}|x - x_i^l|. \tag{85}
\]

To deal with the \( H_{S_j+m_i} f_i \) term we rewrite it as

\[
| \int (K(\rho) - K(\rho - x_i^l))(H_{S_j+m_i} f_i(x)) \, dx |
\]
with the commutator $[H^j_2, S_{j+m}]$ of $H^j_2$ and $S_{j+m}$. Here we have used that $\pi_{j_0}S_{j+m} = 0$ and the kernel of $S_{j+m}$ has mean zero. However, we have the easily verified estimate
\[
||[H^j_2, S_{j+m}] f_i(x)|| \leq C(1 + 2^{j_0+j+m})(x-x_i^j))^{-200} ||f_i||_{\infty}.
\]
The desired bound for the $H^j_2 S_{j+m} f_i$ term now follows again from \((83)\).

It remains to consider the case $x_i^j \notin 3I_0$. This is done similarly to the previous case, however using instead of \((85)\) the kernel bound
\[
|K(x_0 - x) - K(x_0 - x_i^j)| \leq C2^{2j_0}(1 + 2^{j_0+j})|x_0 - x_i^j|^{-200}|x - x_i^j|
\]
for $2|x - x_i^j| < |x - x_0|$.

The treatment of \((82)\) is similar. This concludes the proof of \((73)\) in all cases.

The inequality \((77)\) follows from \((74)\) and Lemma \(3.4\). Thus it only remains to show \((74)\).

Now we show \((74)\). This will be a reprise of the proof of \((73)\), except that we shall rely on \((10)\) instead of \((12)\).

We turn to the details. We shall only concern ourselves with the case when $5I_0$ intersects $E_{j_0}$. The case when $5I_0$ is disjoint from $E_{j_0}$ is done with similar arguments as those below; one loses some powers of the separation between $I_0$ and $E_{j_0}$, whenever one applies \((39)\), but this is more than compensated for by the decay of $\tilde{\chi}_{j_0}$, which also gives the additional factor of \((36)\). We omit the details.

Discarding the cutoff $\tilde{\chi}_{j_0}$, we reduce to
\[
||\tilde{\pi}_{j_0}(T_{j_0+m} f_i - \Pi_i(f_i))||_{L^2(I_0)} \leq C \text{size}^*_i(T) |I_0|^{1/2} \int \tilde{\chi}_{I_0}^2 \mu_{j_0}.
\]

Since $I_0$ is near $E_{j_0}$, we can find a multi-tile $\tilde{P} \in T$ such that $\tilde{P} \subseteq 10I_0$ by Lemma \(11\). From \((10)\) we have
\[
||\tilde{\chi}_{I_0}^{10}T_{j_0+m} f_i||_2 \leq C |I_0|^{1/2} \text{size}^*_i(T).
\]

Decompose $T_{j_0+m} f_i - \Pi_i(f_i)$ as \((77) - (78) + (79) + (80) - (81) - (82)\) as before.

First consider the contributions \((77), (78), (79), (80)\) which come from outside $\tilde{E}_{j_0}$. Let us first examine the non-local portion of these contributions, or more precisely
\[
||\tilde{\pi}_{j_0}(\chi_{\tilde{P}\setminus 3I_0}((77) - (78) + (79) + (80)))||_{L^2(I_0)}.
\]
From the rapid decay of the kernel of $\tilde{\pi}_{j_0}$ we may estimate this by
\[
2^{-100/j_0}||\tilde{\chi}_{I_0}^{100}((77) - (78) + (79) + (80)))||_2.
\]
But each of these terms is acceptable thanks to \((70)\) (or more precisely, the arguments used to prove \((70)\) but applied to \((77), (78), (79), (80)\) rather than $\Pi_i(f_i)$).

It remains to estimate the local contribution
\[
||\tilde{\pi}_{j_0}(\chi_{3I_0}((77) - (78) + (79) + (80)))||_{L^2(I_0)}.
\]

Of course, these contributions are non-zero only when $3I_0$ is not contained in $\tilde{E}_{j_0}$, so that $I_0$ is near the boundary of $\tilde{E}_{j_0}$. We can then discard the projection $\tilde{\pi}_{j_0}$ and the $\mu$-factor on the right hand side and reduce to showing that
\[
||((77) - (78) + (79) + (80)||_{L^2(3I_0)} \leq C |I_0|^{1/2} \text{size}^*_i(T).
\]

The contribution of \((77)\) is acceptable by \((86)\).
Now consider (78). The set \((\tilde{E}_{0} \setminus \tilde{E}_{j_{0}}) \cap 3I_{0}\) is the union of at most three intervals \(I_{1}\) of length \(2^{-J_{j_{0}}}\). On each of these intervals \(I_{1}\), the function \(j(x)\) is equal to \(j_{0} - 1\) by Lemma 4.10.

For each \(I_{1}\), the claim then follows from (77), Lemma 5.5, and the identity \(T_{j_{0}-1+\mathbf{m}} = T_{j_{0}-1+\mathbf{m}} T_{j_{0}+\mathbf{m}}\).

Now consider (79). By Lemma 4.12, there are a bounded number of functions \(\phi_{j_{1},j}\) with \(j \leq j_{0}\) which have support near \(3I_{0}\). Also, since \(I_{0}\) is near the boundary of \(\tilde{E}_{j_{0}}\), we have \(j \geq j_{0} - 2\) for all these functions \(\phi_{j_{1},j}\). Thus it suffices to show

\[
|c_{j_{1},j}| 2^{J(j - \mathbf{m})/2} \leq C|I_{0}|^{1/2} \text{size}_{j_{0}+1/2}(T)
\]

for each of the coefficients \(c_{j_{1},j}\) involved.

From (65) and Cauchy-Schwarz we see that

\[
|c_{j_{1},j}| \leq C 2^{J(j + \mathbf{m})/2} \|
\]

Since \(j_{0} - 2 \leq j \leq j_{0}\) we may replace \(I_{0}\) on the right hand side by the \(J\)-dyadic interval of length \(2^{-J_{j}}\) which contains \(I_{0}\). The desired estimate follows now from Lemma 4.11 and (10).

The treatment of (80) is similar to (79), which concludes the discussion of the contributions (77), (78), (79), (80) which come from outside \(\tilde{E}_{j_{0}}\).

We turn to (81). As with the corresponding treatment of (81) in (73), we shall extract a gain by interacting the smoothing of \(\tilde{\pi}_{j_{0}}\) with the moment condition (83).

By (51) and the triangle inequality it suffices to show that

\[
\| \tilde{\pi}_{j_{0}}(H_{j}^{l} S_{j + \mathbf{m}} f_{i} - c_{j_{1},j} \phi_{j_{1},j}^{l}) \|_{L^{2}(I_{0})} \leq C|I_{0}|^{-1/2} \text{size}_{j_{0}+1/2}(T) \int \tilde{\chi}_{I_{0}}(x) |x - l|^{-100}
\]

for all \(j_{0} < j\) and \(I \in \Omega_{2}\). Evaluating the integral, this becomes

\[
\| \tilde{\pi}_{j_{0}}(H_{j}^{l} S_{j + \mathbf{m}} f_{i} - c_{j_{1},j} \phi_{j_{1},j}^{l}) \|_{L^{2}(I_{0})} \leq C \text{size}_{j_{0}+1/2}(T) 2^{J_{j_{0}}/2} 2^{-J_{j} 2^{-j} \chi_{I_{0}}(x)^{2}}
\]

Fix \(j, I\). Observe from Fourier support considerations that

\[
\tilde{\pi}_{j_{0}}((T_{j_{0}+\mathbf{m}} H_{j}^{l}) S_{j + \mathbf{m}} f_{i}) = 0;
\]

in particular, \((T_{j_{0}+\mathbf{m}} H_{j}^{l}) S_{j + \mathbf{m}} f_{i}\) has mean zero. It thus suffices to show that

\[
\| \tilde{\pi}_{j_{0}} F_{j_{1},l,i} \|_{L^{2}(I_{0})} \leq C \text{size}_{j_{0}+1/2}(T) 2^{J_{j_{0}}/2} 2^{-J_{j} 2^{-j} \chi_{I_{0}}(x)^{2}}
\]

where

\[
F_{j_{1},l,i} := [(1 - T_{j_{0}+\mathbf{m}}) H_{j}^{l}] S_{j + \mathbf{m}} f_{i} - c_{j_{1},j} \phi_{j_{1},j}^{l}.
\]

From the construction of \(c_{j_{1},j}\), we see that \(F_{j_{1},l,i}\) has mean zero, and thus has a primitive \(\nabla^{-1} F_{j_{1},l,i}\) which goes to zero at \(\pm \infty\). We thus may write

\[
\| \tilde{\pi}_{j_{0}} F_{j_{1},l,i} \|_{L^{2}(I_{0})} = 2^{J_{j_{0}+\mathbf{m}}} \| 2^{-J_{j_{0}+\mathbf{m}}} \nabla \tilde{\pi}_{j_{0}}(\nabla^{-1} F_{j_{1},l,i}) \|_{L^{2}(I_{0})}.
\]

The multiplier \(2^{-J_{j_{0}+\mathbf{m}}} \nabla \tilde{\pi}_{j_{0}}\) is bounded in \(L^{2}\) and is essentially local at scale \(2^{-J_{j_{0}+\mathbf{m}}},\) and hence at scale \(2^{-J_{j_{0}}}\). Thus we may estimate the previous by

\[
2^{J_{j_{0}+\mathbf{m}}} \| \nabla \tilde{\pi}_{j_{0}}(\nabla^{-1} F_{j_{1},l,i}) \|_{L^{2}(I_{0})}.
\]

\[\text{(87)}\]

\[\text{(87)}\]

\[\text{(87)}\]
We now claim the pointwise bound
\[ |F_{j,1,i}(x)| \leq C \text{size}^*_i(T) 2^{j\text{size}_i(T)/2} (1 + 2^{j+m_j}) |x - x^I_j|^5. \] (88)

Assuming (88) for the moment, we may integrate it (using the mean zero condition to integrate from either $+\infty$ or $-\infty$, depending on which one gives the more favorable estimate) to obtain
\[ |\nabla^{-1} F_{j,1,i}(x)| \leq C \text{size}^*_i(T) 2^{-j(m_j+1)} 2^{j\text{size}_i(T)/2} (1 + 2^{j(m_j+1)}) |x - x^I_j|^{-50}. \]

We can thus estimate (87) by
\[ C^2 2^{j(m_j+1)} \lambda_{I_0}(x^I_j)^{100} \text{size}^*_i(T) 2^{-j(m_j+1)} 2^{j\text{size}_i(T)/2} 2^{-j(m_j+1)/2}, \]
which is acceptable.

It remains to show (88). The contribution of $e_{j,1,j}^{-1} g_{I,j}$ is acceptable from (10), so it remains to consider $[(1 - T_{j+m,1}) H_{I,j}] S_{j+m,i} f_i$. From repeated integration by parts we have the pointwise estimate
\[ |(1 - T_{j+m,1}) H_{I,j}(x)| \leq C (1 + 2^{j(m+1)}) |x - x^I_j|^{100} \]
so it suffices to show that
\[ |S_{j+m,i} f_i(x)| \leq C 2^{j\text{size}_i(T)/2} (1 + 2^{j(m+1)}) |x - x^I_j|^{50}. \]

Let $I'$ be a dyadic interval of length $2^{-J} j$ which is adjacent to the left endpoint of $I$. By Lemma 4.11 we can find a multi-tile $\vec{P}' \in T$ with $I_{\vec{P}} \subseteq 10'I'$. From (11) we thus have
\[ \| \lambda_{I'}^{10} S_{j+m,i} f_i \|_2 \leq C 2^{-J/2} \text{size}^*_i(T). \]

From Lemma 5.4 we thus have
\[ \| \lambda_{I'}^{10} S_{j+m,i} f_i \|_\infty \leq C 2^{j\text{size}_i(T)/2} \]
and the desired bound follows. This completes the treatment of (81).

The treatment of (82) is similar to (81). This completes the proof of (74), and therefore of (54). The proof of Proposition 7.4 is thus (finally!) complete.  

9. Deducing Theorem 1.2 from Proposition 7.4

In this section we state standard Propositions which will allow us to deduce (24) and hence Theorem 1.2 from Proposition 7.4.

The idea is to break the multi-tile set $\vec{P}_1$ into trees $T$, such that one has control on the $i$-sizes $\text{size}^*_i(T)$ and on the total tree width $\sum T |I_T|$. This will be accomplished using Proposition 6.3 and the counting function estimate of Proposition 7.2 on trees of a given size.

The selection of the trees is done by a greedy selection process, which will be defined in various steps. We need the following definition:

**Definition 9.1.** Call a tree convex, if it is a selected tree in a greedy selection process. In particular, convex trees satisfy Lemma 4.7. Call a subset $\vec{P} \subseteq \vec{P}_1$ convex, if it is of the form $\vec{P}_1 \setminus (T_1 \cup \cdots \cup T_k)$ where $T_1, \ldots, T_k$ are the selected trees of a greedy selection process.
**Proposition 9.2.** Let \(1 \leq i \leq n\), \(m \in \mathbb{Z}\), and suppose that \(\vec{P}\) is a convex collection of multi-tiles such that
\[
\text{size}^*_i(\vec{P}) < \|f_i\|_2 2^{m/2}.
\] (89)

Then there exists a collection \(T\) of distinct convex trees in \(\vec{P}\) such that
\[
\sum_{T \in T} |I_T| \leq C 2^{-m}
\] (90)
and the remainder set \(\vec{P}' := \vec{P} - \bigcup_{T \in T} T\) is convex and satisfies
\[
\text{size}^*_i(\vec{P}') < \|f_i\|_2^{(m-1)/2}.
\] (91)

We prove this Proposition in Section 10; it is the main step in our tree selection algorithm.

We now aim to prove (20). By varying the \(p_i\) slightly and using Marcinkiewicz interpolation \([15]\) it suffices to prove this estimate under the assumption that \(f_i = \chi_{E_i}\) are characteristic functions.

Starting with \(m\) large and working downward, applying Proposition 9.2 for each \(1 \leq i \leq n\) for each \(m\), we obtain

**Corollary 9.3.** For every integer \(m\) there exists a collection \(T_m\) of distinct convex trees in \(\vec{P}_1\) such that we have the size estimate
\[
\text{size}^*_i(T_m) < |E_i|^{1/2} 2^{m/2}
\] (92)
for all \(1 \leq i \leq n\) and \(m \in \mathbb{Z}\), the total tree width estimate
\[
\sum_{T \in T_m} |I_T| \leq C 2^{-m}
\] (93)
for all \(m \in \mathbb{Z}\), and the partitioning
\[
\vec{P}_1 = \vec{P}_2 \cup \bigcup_{m \in \mathbb{Z}} \bigcup_{T \in T_m} T.
\] (94)
where \(\vec{P}_2\) is a subset of \(\vec{P}_1\) with \(\text{size}^*_i(\vec{P}_2) = 0\) for all \(1 \leq i \leq n\).

In the \(i = n\) case we apply Proposition 6.3, (92), and the fact that \(f_n\) is a characteristic function to obtain
\[
\text{size}^*_n(T_m) \leq C \min(2^{m/2}|E_n|^{1/2}, 1)
\] (95)
This is in fact true for all \(1 \leq i \leq n\), but we shall only exploit it for \(i = n\).

Applying (94) we may estimate the left-hand side of (20) by
\[
\sum_{m \in \mathbb{Z}} \sum_{T \in T_m} \left| \sum_{\vec{p} \in T} \chi_{I_{\vec{p}, \vec{p}}} \prod_{i=1}^n \pi_{\omega_{p_i}} f_i \right|.
\]
(Observe that the set \(\vec{P}_2\) gives no contribution, e.g. by an appropriate application of Proposition tree-est-prop.)
We may apply Proposition 7.1 with \( \theta_i := \frac{2}{p_i} \) for \( 1 \leq i \leq n - 1 \), and estimate the previous expression by

\[
C \sum_{m \in \mathbb{Z}} \sum_{T \in T_n} |I_T| \left( \prod_{i=1}^{n-1} \text{size}^*_i(T)^{2/p_i} \right) \text{size}^*_n(T).
\]

By (92), (95) and then (93), we may estimate this by

\[
C \sum_{m \in \mathbb{Z}} 2^{-m} \left( \prod_{i=1}^{n-1} \left( 2^{m/2} |E_i|^{1/2} \right)^{2/p_i} \right) \min\left(2^{m/2} |E_n|^{1/2}, 1\right).
\]

By (4) this simplifies to

\[
C \left( \prod_{i=1}^{n-1} |E_i|^{1/p_i} \right) \sum_{m \in \mathbb{Z}} \min\left(2^{m/2} |E_n|^{1/2}, 2^{-m/p_n}\right).
\]

Performing the \( m \) summation we obtain the desired estimate

\[
\left| \sum_{\vec{P} \in \vec{P}_1} \int \chi_{I_{\vec{P}, J_{\vec{P}}}} \prod_{i=1}^{n} \pi_{\omega_{P_i}} f_i \right| \leq C \prod_{i=1}^{n} |E_i|^{1/p_i}.
\]

and conclude Theorem 1.2.

It remains only to prove Proposition 9.2.

10. Proof of Proposition 9.2

Fix \( 1 \leq i \leq n \), \( f_i \) and \( \vec{P} \). We shall need to split our notion of size \( \text{size}^*_i(\vec{P}) \) into upper and lower components.

For any \( \xi_i \) and any sign \( \pm \), let \( H_{\xi_i}^\pm \) denote the Riesz projection to the half-line \( \{ \xi \in \mathbb{R} : \pm(\xi - \xi_i) > 0\} \) in frequency space. This Riesz projection is a linear combination of a modulated Hilbert transform and the identity.

Observe that

\[
\|f_i\|_{P_i, \xi_i} \sim \|H_{\xi_i}^+ f_i\|_{P_i, \xi_i} + \|H_{\xi_i}^- f_i\|_{P_i, \xi_i}
\]

for any tile \( P_i \) and any \( \xi_i \). Thus if we define

\[
\text{size}_{i, \pm}(T) := \left( \frac{1}{|I_T|} \sum_{\vec{P} \in T} \|H_{(\xi_i,T)} \pm f_i\|_{P_i, (\xi_i,T)}^2 \right)^{1/2} + |I_T|^{-\frac{1}{2}} \sup_{m_i,T} \|\tilde{\chi}_{I_T} T_{m_i,T} (H_{(\xi_i,T)}^\pm f_i)\|_2,
\]

where \( m_i,T \) is a multiplier in the range defined in Definition 6.1, and

\[
\text{size}^*_{i, \pm}(\vec{P}) := \sup_{(T, \xi, I): T \subseteq \vec{P}} \text{size}_{i, \pm}(T),
\]

then we have

\[
\text{size}^*_i(\vec{P}) \sim \text{size}^*_{i, +}(\vec{P}) + \text{size}^*_{i, -}(\vec{P}).
\]

Proposition 9.2 will then follow from a finite number of applications of

**Proposition 10.1.** Let \( \vec{P} \) be a convex collection of multi-tiles. Let \( \pm \) be a sign, and let \( m \in \mathbb{Z} \) be such that

\[
\text{size}^*_{i, \pm}(\vec{P}) < \|f_i\|_2 2^{m/2}.
\]
Then there exists a collection $\mathbf{T}$ of distinct convex trees in $\mathbf{\tilde{P}}$ such that
\[ \sum_{T \in \mathbf{T}} |I_T| \leq C 2^{-m} \tag{98} \]
and the remainder set $\mathbf{\tilde{P}}' := \mathbf{\tilde{P}} - \bigcup_{T \in \mathbf{T}} T$ is convex and satisfies
\[ \text{size}_{i,+}'(\mathbf{\tilde{P}}') < \| f_i \|_2^2 (m-1)/2. \tag{99} \]

Estimate (98) is a variant of Bessel’s inequality, expressing that the distinct trees in this proposition correspond to almost orthogonal components of $f_i$.

**Proof**

We shall prove Proposition 10.1 for the sign $+$; the other sign follows by applying the frequency reflection $\xi \to -\xi$ and conjugating $f_i$.

The idea is to remove maximal trees in a greedy selection process from $\mathbf{\tilde{P}}$ until (99) is obeyed for the remainder set. This procedure shall be given by iteration. If (99) holds, we terminate the iteration. If (99) does not hold, then there exists a tree for which
\[ \text{size}_{i,+}(T) \geq \| f_i \|_2^2 (m-1)/2. \tag{100} \]

Since the set of all possible trees $(T, \xi, I)$ obeying (100) is compact, we may select $T$ so that $(\xi_T)_i$ is maximal. By retaining the top data but adding further multi-tiles if necessary, we may assume that $T$ is maximal in the sense of Definition 4.6. We then add this tree $T$ to $\mathbf{T}$. Then, we remove all the multi-tiles in $T$ from $\mathbf{\tilde{P}}$. We then repeat this iteration until (99) holds.

Since $\mathbf{\tilde{P}}$ is finite, this procedure halts in finite time and yields a collection $\mathbf{T}$ of mutually distinct convex trees. Note that trees with a larger value of $\xi_T$ will be selected before trees with a smaller value of $\xi_T$. The property (99) holds by construction, so it only remains to show (98).

As usual, we shall use the $TT^*$ method to prove this orthogonality estimate. One may think of this Lemma as a phase space version of Lemma 5.1, which was set entirely in physical space, and we shall need Lemma 5.1 in the proof of this estimate.

Write $X := \text{size}_{i,+}(\mathbf{\tilde{P}})$. Observe that
\[ X/2 \leq \text{size}_{i,+}(T) \leq X \tag{101} \]
for all $T \in \mathbf{T}$ by construction.

It suffices to prove (98) separately for the set of all $T \in \mathbf{T}$ which satisfy
\[ \left( \frac{1}{|I_T|} \sum_{\rho \in T} \| H^+_{(\xi_T)_i, f_i\rho} \|_{I_{(\xi_T)_i}}^2 \right)^{1/2} \geq X/4 \tag{102} \]
and the set of all $T \in \mathbf{T}$ which satisfy
\[ |I_T|^{-1/2} \sup_{m_{i,T}} \| x^{(10)}_{\tilde{T}}T_{m_{i,T}}(H^+_{(\xi_T)_i}, f_i) \|_2 \geq X/4 \tag{103} \]
for appropriate $m_{i,T}$. We first consider the set of trees which satisfy (102). For simplicity of notation we may assume this set is equal to $\mathbf{T}$.

From (101), (99), (32) we may associate to each $T \in \mathbf{T}$ and $\mathbf{\tilde{P}} \in T$ a multiplier $m_{\tilde{P}}$ supported on the interval
\[ \omega_{\tilde{P}}^+ := \{ \xi \in 10\omega_{\tilde{P}} : \xi \geq (\xi_T)_i \} \]
obeying \((33)\) with \(\xi_i = (\xi_T)_i\) such that
\[
\sum_{\vec{P} \in T} c_{\vec{P}}^2 \sim X^2 |I_T|,
\]
where \(c_{\vec{P}}\) is the non-negative quantity
\[
c_{\vec{P}} := \|\tilde{\chi}_{I_{\vec{P}}^T}^T m_\vec{P} f_i\|_2.
\]
From the signed version of \((33)\) we have
\[
c_{\vec{P}} \leq CX|I_{\vec{P}}|^{1/2}.
\]
Summing \((104)\) in \(T\), we obtain
\[
\sum_{T \in \mathcal{T}} \sum_{\vec{P} \in T} c_{\vec{P}} c_{\vec{P}} \sim X^2 \sum_{T \in \mathcal{T}} |I_T|.
\]
On the other hand, from the definition of \(c_{\vec{P}}\) and duality we can find for each \(\vec{P}\) an \(L^2\)-normalized function \(g_{\vec{P}}\) such that
\[
c_{\vec{P}} = \langle f_i, T_{m_{\vec{P}}}(\tilde{\chi}_{I_{\vec{P}}^T}^T g_{\vec{P}}) \rangle.
\]
We thus have
\[
\sum_{T \in \mathcal{T}} \sum_{\vec{P} \in T} c_{\vec{P}} \langle f_i, T_{m_{\vec{P}}}(\tilde{\chi}_{I_{\vec{P}}^T}^T g_{\vec{P}}) \rangle \sim X^2 \sum_{T \in \mathcal{T}} |I_T|.
\]
We can write the left-hand side as
\[
\langle f_i, \sum_{T \in \mathcal{T}} \sum_{\vec{P} \in T} c_{\vec{P}} T_{m_{\vec{P}}}(\tilde{\chi}_{I_{\vec{P}}^T}^T g_{\vec{P}}) \rangle.
\]
By the Cauchy-Schwarz inequality we thus have
\[
X^2 \sum_{T \in \mathcal{T}} |I_T| \leq C \|f_i\|_2 \sum_{T \in \mathcal{T}} \sum_{\vec{P} \in T} c_{\vec{P}} T_{m_{\vec{P}}}(\tilde{\chi}_{I_{\vec{P}}^T}^T g_{\vec{P}}) \|_2.
\]
To prove \((98)\) it suffices to show that
\[
\| \sum_{T \in \mathcal{T}} \sum_{\vec{P} \in T} c_{\vec{P}} T_{m_{\vec{P}}}(\tilde{\chi}_{I_{\vec{P}}^T}^T g_{\vec{P}}) \|_2 \leq CX \left( \sum_{T \in \mathcal{T}} |I_T| \right)^{1/2}.
\]
It will be necessary to dyadically decompose the operator \(T_{m_{\vec{P}}^i}\) around the base frequency \((\xi_T)_i\). For each \(\vec{P} \in T \in \mathcal{T}\), decompose
\[
T_{m_{\vec{P}}^i} = \sum_{s=0}^{\infty} 2^{-s} T_{m_{\vec{P}}^i,s}
\]
where \(m_{\vec{P}}^i,s\) is a bump function adapted to the region
\[
\omega_{\vec{P},\xi_T,i,s}^+ := [(\xi_T)_i + 2^{-s}5000C_0|\omega_{\vec{P}}|, (\xi_T)_i + 2^{2-s}5000C_0|\omega_{\vec{P}}|]
\]
and supported in \(10\omega_{\vec{P}}\).
We can then decompose
\[
T_{m_{\vec{P}}^i}(\tilde{\chi}_{I_{\vec{P}}^T}^T g_{\vec{P}}) = \sum_{s=0}^{\infty} 2^{-s} h_{\vec{P},s},
\]
where

\[ h_{\vec{P},s} := T_{m_{P,s}}(\tilde{\gamma}_{10}^{10} g_{\vec{P}}). \] (106)

Thus \( h_{\vec{P},s} \) has Fourier support in \( \omega_{\vec{P},t,i,s}^+ \), is bounded in \( L^2 \), and is rather weakly localized in physical space near \( I_{\vec{P}} \). For \( s \leq M_i \) the multiplier \( T_{m_{P,s}} \) has good spatial localization properties, but for \( s > M_i \) the multiplier \( T_{m_{P,s}} \) begins to spread \( h_{\vec{P},s} \) along a wider interval than \( I_{\vec{P}} \). However in the case \( s > M_i \) we have the easily verified pointwise estimate

\[ |h_{\vec{P},s}(x)| \leq C 2^{s-M_i} |I_{\vec{P}}|^{-1/2} \tilde{X}_{2^{s-M_i}I_{\vec{P}}}(x)^5 \] (107)

from kernel bounds on \( T_{m_{P,s}} \).

By the triangle inequality it thus suffices to show that

\[ \sum_{\vec{P} \in \mathbf{T}} c_{\vec{P}} |h_{\vec{P},s} h_{\vec{P}',s}| \leq C X^{2s/100} \sum_{\vec{P} \in \mathbf{T}} |I_{\vec{P}}|. \]

for all \( s \geq 0 \).

Fix \( s \). We square this as

\[ \sum_{\vec{P} \in \mathbf{T}} \sum_{\vec{P}' \in \mathbf{T}} c_{\vec{P}} c_{\vec{P}'} \langle h_{\vec{P},s}, h_{\vec{P}',s} \rangle \leq C X^{2s/50} \sum_{\vec{P} \in \mathbf{T}} |I_{\vec{P}}|. \]

By symmetry it suffices to show that

\[ \sum_{\vec{P} \in \mathbf{T}} \sum_{\vec{P}' \in \mathbf{T}} c_{\vec{P}} c_{\vec{P}'} |\langle h_{\vec{P},s}, h_{\vec{P}',s} \rangle| \leq C X^{2s/50} \sum_{\vec{P} \in \mathbf{T}} |I_{\vec{P}}|. \] (108)

We first consider the contribution when \( |\omega_{\vec{P}}| = |\omega_{\vec{P}'}| \). It suffices to show that

\[ \sum_{\vec{P} \in \mathbf{T}, \vec{P}' \in \mathbf{T}: |\omega_{\vec{P}}| = |\omega_{\vec{P}'}| = 2^{j_m}} c_{\vec{P}} c_{\vec{P}'} |\langle h_{\vec{P},s}, h_{\vec{P}',s} \rangle| \leq C 2^{s/50} \sum_{\vec{P} \in \mathbf{T}: |\omega_{\vec{P}}| = 2^{j_m}} c_{\vec{P}}^2 \]

for all integers \( m \), since the claim then follows by summing in \( m \) and applying (104).

Fix \( m \). By Schur’s test (i.e. estimating \( c_{\vec{P}} c_{\vec{P}'} \leq \frac{1}{2} (c_{\vec{P}}^2 + c_{\vec{P}'}^2) \)) and symmetry it suffices to show that

\[ \sum_{\vec{P}' \in \mathbf{T}: |\omega_{\vec{P}'}| = |\omega_{\vec{P}}|} |\langle h_{\vec{P},s}, h_{\vec{P}',s} \rangle| \leq C 2^{s/50} \]

for all \( \vec{P} \in \mathbf{T} \).

Fix \( \vec{P} \). From (104) we conclude that \( \langle h_{\vec{P},s}, h_{\vec{P}',s} \rangle \neq 0 \) implies \( \vec{P} = \vec{P}' \). For those values one has a bound of

\[ |\langle h_{\vec{P},s}, h_{\vec{P}',s} \rangle| \leq C 2^{-J(s-M_i)+}(1 + 2J_{\vec{P}'+(s-M_i)+}\text{dist}(I_{\vec{P}}, I_{\vec{P}'})^{-2}, \]

as can be easily verified from (106) when \( s \leq M_i \) and (107) when \( s > M_i \). The claim then follows by summing.

It remains to control the contribution when \( |\omega_{\vec{P}}| < |\omega_{\vec{P}'}| \). It suffices to show that

\[ \sum_{\vec{P}' \in \mathbf{T}} \sum_{|\omega_{\vec{P}'}| < |\omega_{\vec{P}}|} |I_{\vec{P}'}|^{1/2} |I_{\vec{P}}|^{1/2} |\langle h_{\vec{P},s}, h_{\vec{P}',s} \rangle| \leq C |I_{\vec{P}}| (1 + \frac{\text{dist}(2^{s-M_i}I_{\vec{P}}, \mathbb{R}\setminus I_{\vec{P}})}{|I_{\vec{P}}|})^{-2} \] (109)
for all \( T \in T \) and \( \tilde{P} \in T \), since we may then sum in \( \tilde{P} \) to obtain
\[
\sum_{\tilde{P} \in T} \sum_{T' \in T} \sum_{\tilde{P}' \in T'} |I_{\tilde{P}}|^{1/2} |I_{\tilde{P}}|^{1/2} |\langle h_{\tilde{P},s}, h_{\tilde{P}',s} \rangle| \leq C 2^{s/50} |I_T|,
\]
and (108) follows by summing in \( T \) and applying (105).

It remains to show (109). Fix \( \tilde{P}, T \). Let \( \tilde{P}(\tilde{P}, s) \) denote the set of all \( \tilde{P}' \) which make a non-zero contribution to (109). We now make the key observation that by Lemma 4.17 we have intervals \( I_{\tilde{P}_s} \) are disjoint from \( I_T \). Namely, non-vanishing of \( \langle h_{\tilde{P},s}, h_{\tilde{P}',s} \rangle \) implies hypotheses (24) and (25). Moreover, by similar reasoning, if \( \tilde{P}' \) and \( \tilde{P}'' \) are in \( \tilde{P}(\tilde{P}, s) \) and \( \tilde{P}' \neq \tilde{P}'' \), then \( I_{\tilde{P}_s} \) and \( I_{\tilde{P}_s}' \) are disjoint, as one can see from (14) if \( |I_{\tilde{P}}| = |I_{\tilde{P}_s}| \) and from (a slight variant of) Lemma 4.17 if \( |I_{\tilde{P}}| \neq |I_{\tilde{P}_s}| \). For future reference we summarize:

**Observation 10.2.** The intervals \( I_{\tilde{P}_s} \) are disjoint from each other and from \( I_T \).

We first verify (109) in the case \( s > M_i \). In this case we see from (107) and a calculation that
\[
|\langle h_{\tilde{P},s}, h_{\tilde{P}',s} \rangle| \leq C |I_{\tilde{P}_s}|^{-1/2} |I_{\tilde{P}}|^{-1/2} 2^{-(s-M_i)} \int_{I_{\tilde{P}}'} \tilde{\chi}_{2^{s-M_i}}(I_{\tilde{P}}),
\]
so by Observation 10.2 the inequality (109) reduces to
\[
\int_{\mathbb{R} \setminus I_T} 2^{-(s-M_i)} \tilde{\chi}^{2^{s-M_i}}(I_{\tilde{P}}) \leq C |I_{\tilde{P}}|(1 + \frac{\text{dist}(2^{s-M_i}I_{\tilde{P}}, \mathbb{R} \setminus I_T)}{|I_{\tilde{P}}|})^{-2},
\]
which is easily verified.

Now consider the case \( s \leq M_i \). By (106) we have
\[
|\langle h_{\tilde{P},s}, h_{\tilde{P}',s} \rangle| \leq |F, \tilde{\chi}_{10}^{I_{\tilde{P}}}, \chi_{10}^{I_{\tilde{P}}}| g_{\tilde{P}'}|
\]
where
\[
F := \tilde{\chi}_{10}^{I_{\tilde{P}}}, \sup_{\tilde{P}' | \tilde{P} | \leq |I_{\tilde{P}}|} |T^{*}_{m_{\tilde{P}_s}} T_{m_{\tilde{P}_s}}(\tilde{\chi}_{10}^{I_{\tilde{P}}})|.
\]

From the decay of the kernel of \( T^{*}_{m_{\tilde{P}_s}} T_{m_{\tilde{P}_s}} \), we can control \( F \) pointwise by the Hardy-Littlewood maximal function:
\[
F(x) \leq CM g_{\tilde{P}}(x).
\]

From the previous (109) reduces to
\[
\langle M g_{\tilde{P}}, \tilde{\chi}_{10}^{I_{\tilde{P}}}, \sum_{\tilde{P}' \in \tilde{P}(\tilde{P}, s)} |I_{\tilde{P}}|^{1/2} \tilde{\chi}_{10}^{I_{\tilde{P}}}, g_{\tilde{P}'}| \leq C |I_{\tilde{P}}|^{1/2}(1 + \frac{\text{dist}(I_{\tilde{P}}, \mathbb{R} \setminus I_T)}{|I_{\tilde{P}}|})^{-2},
\]
so by Cauchy-Schwarz and the Hardy-Littlewood maximal inequality it suffices to show that
\[
\|\tilde{\chi}_{10}^{I_{\tilde{P}}}, \sum_{\tilde{P}' \in \tilde{P}(\tilde{P}, s)} |I_{\tilde{P}}|^{1/2} \tilde{\chi}_{10}^{I_{\tilde{P}}}, g_{\tilde{P}'}\|_2 \leq C |I_{\tilde{P}}|^{1/2}(1 + \frac{\text{dist}(I_{\tilde{P}}, \mathbb{R} \setminus I_T)}{|I_{\tilde{P}}|})^{-2}.
\]

Let us first consider the portion of the \( L^2 \) norm in the region
\[
\Omega := \{ x \in I_T : \text{dist}(x, \mathbb{R} \setminus I_T) \geq |I_{\tilde{P}}| + \frac{1}{2} \text{dist}(I_{\tilde{P}}, \mathbb{R} \setminus I_T) \}.
\]
In this region we have from the $L^2$ boundedness of $g_{\vec{p}'}$ and the decay of $\tilde{\chi}_{\vec{P}'}^{10}$ that
\[
\|I_{\vec{P}'}\|^{1/2} \tilde{\chi}_{\vec{P}'}^{10} \|g_{\vec{P}'}\|_{L^2(\Omega)} \leq C|I_{\vec{P}'}|^{-1/2} \int_{I_{\vec{P}'}} (1 + \frac{\text{dist}(x, \Omega)}{|I_{\vec{P}'|})^{-5}} dx
\]
(in fact one can get much better bounds than this, especially if $|I_{\vec{P}'}| \ll |I_{\vec{P}'|}$) so by Cauchy-Schwarz and the above key observation (pairwise disjointness of $I_{\vec{P}'}$ and disjointness from $I_T$) again this contribution to (110) is bounded by
\[
C|I_{\vec{P}'}|^{-1/2} \int_{\mathbb{R}\setminus I_T} (1 + \frac{\text{dist}(x, \Omega)}{|I_{\vec{P}'|})^{-5}} dx
\]
which is acceptable.

It remains to estimate the contribution outside of $\Omega$. Since we have
\[
\|\tilde{\chi}_{\vec{P}'}^5 \chi_{\mathbb{R}\setminus I_\Omega}\|_{\infty} \leq C(1 + \frac{\text{dist}(I_{\vec{P}'}, \mathbb{R}\setminus I_T)}{|I_{\vec{P}'|})^{-2}},
\]
this follows simply from Lemma 5.3.

This completes the proof of (98) for the trees which satisfy (102).

Now we consider the set of trees which satisfy (103). For simplicity of notation we may again assume this set is equal to $T$. The proof is a reprise of the previous case.

We may associate to each $T \in T$ a multiplier $m_{i,T}$ supported on the interval
\[
\omega_{i,T}^+ := \{\xi \in 10\omega_{i,T} : \xi \geq (\xi_T)\}
\]
obeying (B5) such that
\[
c_T^2 \sim X^2|I_T|
\]
(111)
where $c_T$ is the non-negative quantity
\[
c_T := \|\tilde{\chi}_{I_T}^{10} T_{m_{i,T}} f_i\|_2.
\]

By duality we can find for each $T \in T$ an $L^2$-normalized function $g_T$ such that
\[
\langle f_i, T_{m_{i,T}}(\tilde{\chi}_{I_T}^{10} g_T) \rangle
\]

We thus have
\[
\sum_{T \in T} c_T \langle f_i, T_{m_{i,T}}(\tilde{\chi}_{I_T}^{10} g_T) \rangle \sim X^2 \sum_{T \in T} |I_T|.
\]
By the Cauchy-Schwarz inequality we have
\[
X^2 \sum_{T \in T} |I_T| \leq C\|f_i\|_2 \sum_{T \in T} c_T T_{m_{i,T}}(\tilde{\chi}_{I_T}^{10} g_T)\|_2.
\]
To prove (98) it thus suffices to show that
\[
\|\sum_{T \in T} c_T T_{m_{i,T}}(\tilde{\chi}_{I_T}^{10} g_T)\|_2 \leq CX(\sum_{T \in T} |I_T|)^{1/2}.
\]

We dyadically decompose the operator $T_{m_{i,T}}$ around the base frequency $(\xi_T)_i$:
\[
T_{m_{i,T}} = \sum_{s=0}^{\infty} 2^{-s} T_{m_{i,T,s}}
\]
where $m_{i,T,s}$ is a bump function adapted to the region
\[ \omega_{i,T,s}^+ := \left( \left. \xi_T \right| + 2^{-s} 10 |\omega_{i,T}|, \left. \xi_T \right| + 2^{-s} 10 |\omega_{i,T}| \right) \]
and supported in $10 \omega_{i,T}$.

Define
\[ h_{T,s} := T_{m_{i,T,s}}(\tilde{\chi}_{10}^{10} I_T g_T) . \]  
(112)

By the triangle inequality it thus suffices to show that
\[ \| \sum_{T \in T} c_T h_{T,s} \|_2 \leq C X 2^{s/2} (\sum_{T \in T} |I_T|)^{1/2} \]
for all $s \geq 0$. (While a better power of $s$ can be achieved, we shall not be ambitious here to do so.) Fix $s$. By squaring and using symmetry it suffices to show that
\[ \sum_{T' \in T} \sum_{T' \in T \cap |\omega_{i,T'}| \leq |\omega_{i,T'}|} c_T c_{T'} |\langle h_{T,s}, h_{T',s} \rangle| \leq C X 2^s \sum_{T \in T} |I_T| . \]

Using (111) we see that it suffices to prove for each $T \in T$
\[ \sum_{T' \in T \cap |\omega_{i,T'}| \leq |\omega_{i,T'}|} |I_{T'}|^{1/2} |\langle h_{T,s}, h_{T',s} \rangle| \leq C 2^s |I_T|^{1/2} . \]

Fix $T$. Observe
\[ \langle h_{T,s}, h_{T',s} \rangle = \langle T_{m_{i,T',s}}^{*} T_{m_{i,T,s}} \tilde{\chi}_{10}^{10} I_T g_T, \tilde{\chi}_{10}^{10} I_{T'} g_{T'} \rangle \]
and the pointwise bound
\[ |T_{m_{i,T',s}}^{*} T_{m_{i,T,s}} \tilde{\chi}_{10}^{10} I_T g_T(x)| \leq C \tilde{\chi}_{2(s-M_{i})}^{5} I_T(x) M g_T(x) \]
with the Hardy Littlewood maximal function $M g_T$. The latter followed from the kernel bound
\[ |K(x, y)| \leq C |I_T|^{-12-s-M_{i}} (1 + |I_T|^{-12-s-M_{i}} |x - y|)^{-100} \]
for the kernel $K$ of the operator $T_{m_{i,T',s}}^{*} T_{m_{i,T,s}}$.

Let $T(T)$ be the set of all $T' \in T$ such that $|\omega_{i,T'}| \leq |\omega_{i,T'}|$ and $\langle h_{T,s}, h_{T',s} \rangle \neq 0$. By the above and the Hardy Littlewood maximal theorem it suffices to show
\[ \| \sum_{T' \in T(T)} |I_{T'}|^{1/2} \tilde{\chi}_{2(s-M_{i})}^{5} I_{T'} \tilde{\chi}_{10}^{10} I_{T'} g_{T'} \|_2 \leq C 2^{(s-M_{i})} |I_T|^{1/2} \]
This however follows from Lemma 5.3 provided we can show $T(T)$ can be split into two subsets, each of which has the property that the intervals $I_{T'}$ with $T'$ in the subset are pairwise disjoint. This however follows from Lemma 4.18.

This completes the proof of (98) for the trees which satisfy (103).
References

[1] Calderon, C. *On commutators of singular integrals*, Studia Math. 53, pp. 139–174 [1975].
[2] Coifman, R. R and Meyer, Y. *On commutators of singular integrals and bilinear singular integrals*, Trans. AMS 212, pp. 315–331 [1975].
[3] Coifman, R. R and Meyer, Y. *Commutateurs d’integrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) 28, pp. 177-202 [1978].
[4] Coifman, R. R and Meyer, Y. *Fourier analysis of multilinear convolutions, Calderón’s theorem, and analysis of Lipschitz curves*, Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md.), pp. 104–122, Lecture Notes in Math., 779 [1979].
[5] Coifman, R. R and Meyer, Y. *Au delà des opérateurs pseudo-différentiels*, Astérisque 57, Société Mathématique de France, Paris 1978.
[6] Coifman, R. R and Meyer, Y. *Non-linear harmonic analysis, operator theory and P.D.E*, Beijing Lectures in Analysis, Annals of Math. Studies 112, 3-46 [1986]
[7] Coifman, R. R and Meyer, Y. *Ondelettes et opérateurs III, Opérateurs multilinéaires*, Actualités Mathématiques, Hermann, Paris 1991
[8] Fefferman, C. *Pointwise convergence of Fourier series*, Ann. of Math. (2) 98, pp. 551–571 [1973].
[9] J. Gilbert, A. Nahmod, *Hardy spaces and a Walsh model for bilinear cone operators*, Trans. Amer. Math. Soc. 351, pp. 3267–3300 [1999]
[10] J. Gilbert, A. Nahmod, *Bilinear operators with non-smooth symbol*, preprint.
[11] J. Gilbert, A. Nahmod, *L^p-boundedness for time-frequency paraproducts*, preprint.
[12] Grafakos, L. and Li, X., *Uniform bounds for the bilinear Hilbert transforms, I.*, preprint.
[13] Li, X., *Uniform bounds for the bilinear Hilbert transforms, II.*, preprint.
[14] Grafakos, L. and Torres, R. *On multilinear singular integrals*, preprint
[15] Janson, S., *On interpolation of multilinear operators*, in Function spaces and applications (Lund 1986), Lecture Notes in Math. 1302, Springer, Berlin-New York, 1988
[16] Kenig, C. and Stein, E. *Multilinear estimates and fractional interpolation*, Math. Res. Lett. 6, pp. 1–15 [1999]
[17] Lacey, M. and Thiele, C. *L^p estimates on the bilinear Hilbert transform for 2 < p < ∞*, Ann. Math. 146, pp. 693-724, [1997]
[18] Lacey, M. and Thiele, C. *On Calderon’s conjecture*, Ann. Math. 149, pp. 475-196, [1999]
[19] Lacey, M. and Thiele, C. *A proof of boundedness of the Carleson operator*, Math. Res. Lett. 7, pp. 361-370, [2000]
[20] Muscalu, C., Tao, T., and Thiele, C., *Multi-linear operators given by singular multipliers*, to appear, J. Amer. Math. Soc.
[21] Muscalu, C., Tao, T., and Thiele, C., *Uniform estimates for paraproducts*, preprint
[22] Muscalu, C., Tao, T., and Thiele, C., *L^p estimates for the biest I. The Walsh case*, preprint
[23] Muscalu, C., Tao, T., and Thiele, C., *L^p estimates for the biest II. The Fourier case*, preprint
[24] Stein, E. *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, Princeton, [1993]
[25] Tao, T. *Multilinear weighted convolution of L^2 functions, and applications to non-linear dispersive equations*, to appear, Amer. J. Math.
[26] Thiele, C. *On the Bilinear Hilbert transform*, Universität Kiel, Habilitationsschrift [1998]
[27] Thiele, C. *The quartile operator and almost everywhere convergence of Walsh-Fourier series*. Trans. AMS 352 (no. 12) pp. 5745-5766 (2000)
[28] Thiele, C. *A uniform estimate for the quartile operator*, to appear in Rev. Mat. Iberoam.
