Topological aspects in the photonic crystal analog of single-particle transport in quantum Hall systems

Luca Esposito¹ and Dario Gerace¹

¹Department of Physics, University of Pavia, via A. Bassi 6, I-27100 Pavia, Italy

We present a perturbative approach to derive the semiclassical equations of motion for the two-dimensional electron dynamics under the simultaneous presence of static electric and magnetic fields, where the quantized Hall conductance is known to be directly related to the topological properties of translationally invariant magnetic Bloch bands. In close analogy to this approach, we develop a perturbative theory of two-dimensional photonic transport in gyrotropic photonic crystals to mimic the physics of quantum Hall systems. We show that a suitable permittivity grading of a gyrotropic photonic crystal is able to simulate the simultaneous presence of analog electric and magnetic field forces for photons, and we rigorously derive the topology-related term in the equation for the electromagnetic energy velocity that is formally equivalent to the electronic case. A possible experimental configuration is proposed to observe a bulk photonic analog to the quantum Hall physics in graded gyromagnetic photonic crystals.

I. INTRODUCTION

Since its first phenomenological observation more than thirty years ago [1][2], the physics of the quantum Hall effects has spurred a wealth of groundbreaking theoretical achievements, which have eventually clarified the generality of the topological aspects at the heart of this fascinating problem [3][4]. It is now understood that the dynamical properties of the two-dimensional (2D) electron motion under the simultaneous presence of electric and magnetic fields are determined by a topological invariant of the Bloch bands, an integer known as the Chern number [5], which is different from zero only after time-reversal symmetry (TRS) is broken by the external magnetic field perpendicular to the plane of motion. As a consequence, the semiclassical equations of motion for the electron group velocity depend on a topological term related to the non-vanishing Berry curvature [6][8]. The relevance of such topological theories is twofold. On one hand, the generality of geometrical properties has been extensively used to explain a number of physical phenomena in condensed matter, from the anomalous Hall effect [9] to the existence of topological superconductors and insulators [10]. On the other, since the topological invariant is a global property of the energy eigenstates of the system, it is intrinsically robust against system perturbations, such as lattice distortions and disorder. As a typical example, in a quantum Hall system the transverse conductance is a multiple of the Chern invariant of the gauge bundle [9][11], for which its value is extremely stable against structural characteristics of the system, and it is measured with accuracies of one part on hundred million [12]. As a further consequence, topologically non-trivial systems possess chiral ballistic edge states at the border of a finite sample [13][14]. Such states, induced by the spatial boundary between systems with distinct topological phases, allow uni-directional and nonreciprocal electronic transport [15], and they are intrinsically immune to back-scattering.

The analogies between photonic band dispersion in artificially periodic electromagnetic systems, known as photonic crystals [16], and the electron band theory in crystalline solids have recently motivated the idea that TRS breaking allows non-trivial topological properties of the photonic modes in such systems [17][18]. Typically, Faraday-active elements arranged in a periodic lattice produce the required breaking of symmetry, necessary to induce a non-vanishing Chern number for photonic bands [19]. Following these early proposals, propagation of back-scattering immune photonic edge states has been observed at the interface between a magneto-optical photonic crystal and a topologically trivial photonic medium [20]. Clearly, these features could be very important for future applications in integrated photonic circuits, because of the possibility to exploit uni-directional channels of electromagnetic energy transport that are intrinsically insensitive to disorder in the sample, just like electronic transport in quantum Hall systems. More recently, several theoretical works have elaborated on the topological nature of one-way photonic edge modes in specific gyroelectric [21][24] photonic crystals, TRS breaking in microwave circuits [25][26], or the generation of artificial gauge fields for photons in coupled cavity arrays [27][29]. The photonic analog of topological insulators have also been recently proposed [30] and observed [31], along the same lines of previous works [19][20]. However, the theoretical problem of recovering the effective photon dynamics in TRS broken photonic systems, in full analogy to the electron transport theory, has been not fully explored in the literature, to our knowledge. A few early attempts to derive a topological-based photon dynamical theory were mostly focussed on systems without TRS breaking [32][33], i.e. with a strict analogy with the classical Hall
transport properties. A rigorous derivation of the topological terms in the semiclassical equations of motion for photonic transport starting from a direct analogy between Bloch-Floquet photonic modes and the magnetic Bloch electron states is still lacking.

Here we go beyond previous works in analyzing the analogies between electronic and photonic formalisms for TRS broken 2D crystals. To this end, we will first present a perturbative approach to obtain the equations of motion for the electron transport in quantum Hall systems, re-deriving the well known result that the semiclassical electron dynamics is described by where \( k \) is the wave vector, \( n \) is the band index, \( \mathbf{v}_n(k) \) is the group velocity associated with the magnetic Bloch band energy \( E_n(k) \), \( \mathcal{E} \) the applied electric field, and \( \Omega_n(k) \) the Berry curvature of the gauge bundle constructed on the Brillouin zone. Essentially, TRS breaking results in a topological correction, given by the Berry curvature, to the standard equations of motion for the electron in the periodic potential of crystalline solids (see, e.g., [34] for a textbook-like formulation). We will then apply the same formalism to Maxwell equations in periodic meta-materials with gyrotropic components and weak grading along one direction, rigorously obtaining the equation for the electromagnetic mode velocity containing an analogous topological correction, as already conjectured in [17–19]. As a final remark, we point out that the present work is mainly concerned with the link between linear photonic crystal theory and the topological aspects of single-electron transport in quantum Hall systems, while we are not dealing with the interesting problem of mimicking manybody quantum states, such as the ones leading to the fractional quantum Hall phenomenology [2], with in strongly nonlinear photonic systems [35].

The paper is organized as follows. In the first part, Sec. II, we present a perturbative approach to derive the known results of a topological term in the single-electron semiclassical equations of motion in quantum Hall systems. In the second part, Sec. III, we explicitly treat photonic crystals on an analog footing, by applying the same perturbative concepts from Sec. II to Maxwell equations. We will then show that a combination of gyrotropic materials and weak grading of the photonic crystal permittivity along the propagation direction are able to closely mimic the semiclassical single-electron dynamics in quantum Hall systems also from a topological perspective. Finally, in Sec. IV we give some conclusive remarks, by proposing a possible experimental setting where these geometrical aspects could be probed through photon transmission.

\[
v_{nk} = \frac{1}{\hbar} \nabla_k E_{nk} - \mathbf{k} \times \Omega_{nk} \tag{1}
\]

\[
\mathbf{k} = -\frac{e\mathcal{E}}{\hbar} \tag{2}
\]

where \( \mathbf{k} \) is the wave vector, \( n \) is the band index, \( \mathbf{v}_n(k) \) is the group velocity associated with the magnetic Bloch band energy \( E_n(k) \), \( \mathcal{E} \) the applied electric field, and \( \Omega_n(k) \) the Berry curvature of the gauge bundle constructed on the Brillouin zone. Essentially, TRS breaking results in a topological correction, given by the Berry curvature, to the standard equations of motion for the electron in the periodic potential of crystalline solids (see, e.g., [34] for a textbook-like formulation). We will then apply the same formalism to Maxwell equations in periodic meta-materials with gyrotropic components and weak grading along one direction, rigorously obtaining the equation for the electromagnetic mode velocity containing an analogous topological correction, as already conjectured in [17–19]. As a final remark, we point out that the present work is mainly concerned with the link between linear photonic crystal theory and the topological aspects of single-electron transport in quantum Hall systems, while we are not dealing with the interesting problem of mimicking manybody quantum states, such as the ones leading to the fractional quantum Hall phenomenology [2], with in strongly nonlinear photonic systems [35].

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The effects of a static electric field (the Hall field, $\mathbf{E}$) on the single-particle dynamics can be described by using a perturbative approach. The perturbed Hamiltonian will be a sum of the zero-order Hamiltonian, Eq. (3), and a perturbation term given by

$$V_p = e\mathbf{E} \cdot \mathbf{r}.$$  \hspace{1cm} (7)

Up to first order in perturbation theory, the eigenvalues $E_n(k)$ and eigenvectors $|\psi_{nk}\rangle$ read

$$\tilde{E}_{nk} \simeq E_{nk} + \langle \psi_{nk} | V_p | \psi_{nk} \rangle,$$  \hspace{1cm} (8)

$$|\tilde{\psi}_{nk}\rangle \simeq |\psi_{nk}\rangle + \sum_{m \neq n} |\psi_{mk}\rangle \langle \psi_{mk} | V_p | \psi_{nk} \rangle / (E_{nk} - E_{mk}).$$  \hspace{1cm} (9)

We notice that there is no mixing in $|k\rangle$ (horizontal mixing) in Eq. (6), since $V_p$ is an electric dipole term with a static electric field, which does not produce mixing of different states within the first Brillouin zone. To proceed with the calculation of the conductivity in this system, we first calculate the expectation value of the group velocity in a perturbed state, within the framework of the Hellmann-Feynman (HF) theorem [40], whose validity is guaranteed by using the completeness relation

$$\langle \tilde{\psi}_{nk} | \tilde{\psi}_{nk} \rangle = 1,$$

from which, using Eq. (15) we get

$$\tilde{v}_{nk} = \frac{1}{\hbar} \nabla_k E_{nk} +$$

$$+ \sum_{m' \neq n} \sum_{m \neq n} \frac{i e E_H}{\hbar} \langle m|\nabla_k E_{nk} | n \rangle \langle n | x | m \rangle +$$

$$+ \langle n|\nabla_k m' \rangle \langle m'|x|n \rangle,$$ \hspace{1cm} (10)

valid for $m \neq n$. By using Eq. (14) in Eq. (13), and assuming (without loss of generality) that the Hall field is directed along $x$, $\mathbf{E} = E_H \hat{x}$ ($\hat{x}$ indicates the unit vector in the $x$ direction), we have

$$\tilde{v}_{nk} = \frac{1}{\hbar} \nabla_k E_{nk} +$$

$$+ \sum_{m' \neq n} \sum_{m \neq n} \frac{i e E_H}{\hbar} \langle m|\nabla_k n | n \rangle |x | m \rangle +$$

$$- \langle n|\nabla_k m' \rangle \langle m'|x|n \rangle,$$ \hspace{1cm} (11)

from which, using Eq. (15) we get

$$\tilde{v}_{nk} = \frac{1}{\hbar} \nabla_k E_{nk} +$$

$$+ \sum_{m' \neq n} \sum_{m \neq n} \frac{i e E_H}{\hbar} \langle m|\nabla_k n | n \rangle |x | m \rangle +$$

$$- \langle n|\nabla_k m' \rangle \langle m'|x|n \rangle.$$ \hspace{1cm} (12)

where, in the spirit of $\mathbf{k} \cdot \mathbf{p}$ theory, we have defined

$$|\tilde{u}_{nk}\rangle \equiv |\tilde{\psi}_{nk}\rangle,$$ \hspace{1cm} (13)

Redefining for ease of notation $|\tilde{u}_{nk}\rangle \equiv |\tilde{n}\rangle$, from Eqs. (4), (5), and (6) the expectation value for the group velocity of the electron on the state $|\tilde{\psi}_{nk}\rangle$ is

$$\tilde{v}_{nk} = \langle \tilde{n} | \frac{1}{\hbar} \mathbf{p} + \frac{\hbar}{\hbar} | \tilde{n} \rangle = \langle \tilde{n} | \nabla_k H_k | \tilde{n} \rangle,$$ \hspace{1cm} (14)

from which, using Eq. (11), we get (neglecting higher order terms)

$$\tilde{v}_{nk} \simeq \langle n | \frac{1}{\hbar} \mathbf{p} + \frac{\hbar}{\hbar} | n \rangle +$$

$$+ \sum_{m \neq n} \langle m | \frac{1}{\hbar} \mathbf{p} + \frac{\hbar}{\hbar} | n \rangle \langle n | e \tilde{\mathbf{E}} \cdot \mathbf{r} | m \rangle / (E_{nk} - E_{mk}) +$$

$$+ \sum_{m' \neq n} \langle n | \frac{1}{\hbar} \mathbf{p} + \frac{\hbar}{\hbar} | m' \rangle \langle m' | e \tilde{\mathbf{E}} \cdot \mathbf{r} | n \rangle / (E_{nk} - E_{m'k}).$$ \hspace{1cm} (15)

With this notation, the HF equations read

$$\langle m | \frac{1}{\hbar} \mathbf{p} + \frac{\hbar}{\hbar} | n \rangle = \frac{E_{nk} - E_{mk}}{\hbar} \langle m|\nabla_k n \rangle,$$ \hspace{1cm} (16)

and defining the Berry curvature $\Omega_{nk} = i (\mathbf{rot} \langle n | \nabla_k n \rangle)$ we finally get

$$\tilde{v}_{nk} = \frac{1}{\hbar} \nabla_k E_{nk} + \frac{e}{\hbar} \times \Omega_{nk}.$$ \hspace{1cm} (17)
which can be recast in the more familiar and well known expression \([6,8]\), Eq. (1), by using the semiclassical equation of motion, Eq. (2).

The importance of the topological term defined by the Berry curvature in Eq. (23) emerges after calculating the Hall conductance for this system. We briefly report here this calculation, for the sake of completeness. In the simplest thermodynamical case in which the temperature of the system is \(T = 0\), the contribution of a given magnetic Bloch band to the drift velocity is

\[
v_{d,n} = \frac{V}{4\pi^2} \int_{B.Z.} d^2k \tilde{v}_{nk} = \frac{V}{4\pi^2} \frac{e\mathcal{E}_H\hat{y}}{\hbar} \int_{B.Z.} d^2k (\Omega_{nk} \cdot \hat{z}) ,
\]

(24)

where \(V\) is the volume of the primitive cell, \(\Omega_{nk} \cdot \hat{z}\) is the component of the Berry curvature along \(z\), and the integral is performed over the first Brillouin zone for which, using Eq. (1), the term \(\nabla_k E_{nk}\) does not contribute. From the last equation we straightforwardly get the current density contributed by the given band

\[
J_n = -\frac{1}{4\pi^2} \frac{e^2\mathcal{E}_H\hat{y}}{\hbar} \int_{B.Z.} d^2k (\Omega_{nk} \cdot \hat{z}) ,
\]

(25)

from which the transverse conductivity (in 2D, the Hall conductance) is quantized and given by integer multiples of the quantum of conductance, \(e^2/h\), as

\[
\sigma_n^{xy} = -\frac{1}{4\pi^2} \frac{e^2}{\hbar} \int_{B.Z.} d^2k (\Omega_{nk} \cdot \hat{z}) = -\frac{e^2}{\hbar} C_n ,
\]

(26)

where \(C_n = \frac{1}{2\pi} \int_{B.Z.} d^2k (\Omega_{nk} \cdot \hat{z})\) is exactly the well known expression for the Chern number \([3,41]\), which we have independently obtained here.

### III. THEORY OF PHOTONIC TRANSPORT IN GYROTROPIC 2D PHOTONIC CRYSTALS

Time-reversal symmetry (TRS) breaking is responsible for the topological nature of the integer quantum Hall phenomenology, which is a strong indication that an analogous effect must exist in photonic band gap media with broken TRS, as pointed out by Haldane and Raghu \([17,18]\). In order to rigorously check the deep connections between electronic and photonic semiclassical dynamics, we hereby develop a bulk topological theory for weakly perturbed photonic crystals with broken TRS, which will lead to an equation for the velocity of the electromagnetic mode containing a topological term formally equivalent to Eq. (1), thus enforcing the analogies between Schrödinger and Maxwell equations.

In the most general case where the dielectric permittivity \(\varepsilon\) and the magnetic permeability \(\mu\) are second-order tensors, Maxwell equations in photonic crystals can be written in the form of a generalized eigenvalue problem (see, e.g., Ref. [16])

\[
\Pi_e E(r) = \omega^2 \varepsilon(r) E(r)
\]

(27)

\[
\Pi_m H(r) = \omega^2 \mu(r) H(r) ,
\]

(28)

where \(E(r)\) and \(H(r)\) are the electric and the magnetic fields, respectively, and \(\omega\) is the oscillation frequency, while

\[
\Pi_e = \nabla \times \left( \mu^{-1}(r) \nabla \times \right)
\]

(29)

\[
\Pi_m = \nabla \times \left( \varepsilon^{-1}(r) \nabla \times \right)
\]

(30)

are linear operators of the generalized eigenvalue problem. Such eigenvalue problem can be recast in a standard one by using the following basis states \([12,44]\)

\[
F_e(r) = \varepsilon^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}}(r) E(r)
\]

(31)

\[
F_m(r) = \mu^{\frac{1}{2}} \mu^{-\frac{1}{2}}(r) H(r) ,
\]

(32)

which allow to obtain the eigenvalue equations

\[
\Theta_e F_e(r) = \omega^2 F_e(r)
\]

(33)

\[
\Theta_m F_m(r) = \omega^2 F_m(r) ,
\]

(34)

where the hermitian operators are defined as

\[
\Theta_e = \varepsilon^{-\frac{1}{2}}(r) \nabla \times \left( \mu^{-1}(r) \nabla \times \right) \varepsilon^{\frac{1}{2}}(r)
\]

(35)

\[
\Theta_m = \mu^{-\frac{1}{2}}(r) \nabla \times \left( \varepsilon^{-1}(r) \nabla \times \right) \mu^{\frac{1}{2}}(r)
\]

(36)

Normalization of the fields is well defined by the notion of scalar product, \((F_{e,m}|F_{e,m}) = \int d^3r F^*_{e,m}(r) F_{e,m}(r)\) and the physical requirement that the electromagnetic energy density be finite in the system \([16]\). Since the two eigenvalue equations are perfectly specular with each other, we will focus on the equation for the electric field henceforth. Following the proposal in \([19]\), we allow TRS breaking in the system by using 2D gyrotropic photonic crystals. For practical purposes, we assume a 2D square lattice of YIG (Yttrium iron garnet) rods in air \([19]\), without loss of generality of the formalism. The permittivity and the permeability of this system can be explicitly represented as

\[
\varepsilon^+ = \begin{bmatrix}
\varepsilon(r) & 0 & 0 \\
0 & \varepsilon(r) & 0 \\
0 & 0 & \varepsilon(r)
\end{bmatrix},
\]

(37)

\[
\mu^+ = \begin{bmatrix}
\mu(r) & i\gamma(r) & 0 \\
-\gamma^*(r) & \mu(r) & 0 \\
0 & 0 & \mu_0
\end{bmatrix},
\]

(38)

and the inverse of \(\mu^+\) is

\[
\mu^{-1} = \begin{bmatrix}
\bar{\mu}^{-1}(r) & i\eta(r) & 0 \\
-\eta^*(r) & \bar{\mu}^{-1}(r) & 0 \\
0 & 0 & \mu_0^{-1}
\end{bmatrix},
\]

(39)
where $\tilde{\mu}^{-1}(r) = \frac{\mu(r)}{\mu^{-1}(r) - \gamma(r)}$, and $\eta(r) = \frac{\gamma(r)}{\mu^{-1}(r) - \gamma(r)}$.

In the following, and in full analogy to the electron dynamics reported above, we will assume a 2D photon dynamics, where mirror symmetry with respect to the propagation plane allows to define even (transverse-electric, TE) and odd (transverse-magnetic, TM) modes, respectively [10]. Moreover, as it can be seen from Eq. (39), we have introduced a magnetic “Faraday mixing” only in the $xy$ plane, which means that we can restrict our analysis to the TM modes only, i.e. $(H_x, H_y, E_z)$ field components different from zero. This assumption is realistic for the cases usually considered for 2D photonic crystals with gyrotropic constituents (see also discussion in Sec. IV), where no mixing of the two polarization eigenstates occurs. An eigenvalue equation for the scalar problem is then explicitly derived as (see App. A for the derivation details)

$$\Theta F_z = \omega^2 F_z,$$

(40)

where $F_z$ is the $z$ component of the vector $F$ (we have dropped the subscript $\epsilon$ for easier notation), and the operator is explicitly given by

$$\Theta = \left[ -\varepsilon^{-1}\tilde{\mu}^{-1}\nabla^2 + \right.$$

$$- (\tilde{\mu}^{-1}\nabla\varepsilon^{-1} + \varepsilon^{-1}\nabla\tilde{\mu}^{-1} + i\varepsilon^{-1}(\hat{z} \times \nabla\eta)) \cdot \nabla +$$

$$- \frac{1}{2} \nabla\tilde{\mu}^{-1} \cdot \nabla \varepsilon^{-1} - \frac{1}{2} \tilde{\mu}^{-1}\nabla^2 \varepsilon^{-1} +$$

$$+ \frac{1}{4} \tilde{\mu}^{-1}\varepsilon (\nabla \varepsilon^{-1})^2 - \frac{1}{2} i (\hat{z} \times \nabla\eta) \cdot \nabla \varepsilon^{-1}\right].$$

(41)

The operator in Eq. (41) has translational symmetry, so its eigenvectors satisfy the Bloch-Floquet theorem [16], and are given by an expression similar to Eq. (4). We can rewrite the eigenvalue problem for the periodic part of $F_z$, which we define $u_{nk}$ to keep the analogy with the electronic case, as in Eq. (6)

$$\Theta_{nk} u_{nk} = \omega_{nk}^2 u_{nk}.$$  

(42)

In order to apply the perturbative approach described in the previous section, we introduce a photonic perturbation mimicking the effect of an electric field as a dragging force, which is achieved by adding a weak modulation $\Delta \varepsilon$ to the periodic permittivity, imposing the following conditions:

1. $\frac{\Delta \varepsilon}{\varepsilon} \ll 1$;

2. $\frac{\Delta \varepsilon}{\varepsilon}$ is slowly varying on the scale determined by the lattice constant, $a$;

3. $\frac{\Delta \varepsilon}{\varepsilon}$ is a linear function of $x$.

We notice that the $x$ axis is chosen here just to preserve the connection with the treatment given for the electron dynamics in Sec. II. As an explicit example and without loss of generality, we can assume $\frac{\Delta \varepsilon}{\varepsilon} = \lambda \frac{\Delta}{a}$, where $\lambda$ is a small constant. With this slow grading of the permittivity, the perturbed operator $\Theta$ takes the form (see App. B for the explicit derivation)

$$\Theta = \Theta - \lambda \frac{\Delta}{a} \Theta = \Theta + V^p,$$

(43)

where we have implicitly defined

$$V^p = -\lambda \frac{\Delta}{a} \Theta.$$  

(44)

Using the perturbation theory up to the first order, we get the perturbed eigenvectors and eigenstates

$$\tilde{\omega}_{nk}^2 \simeq \omega_{nk}^2 + \langle F_{nk}\rangle V^p F_{nk},$$

(45)

$$|\tilde{F}_{nk}\rangle \simeq |F_{nk}\rangle + \sum_{m \neq n} |F_{mk}\rangle \frac{\langle F_{nk}|V^p F_{mk}\rangle}{\omega_{nk}^2 - \omega_{mk}^2},$$

(46)

where by $F_{nk}$ we mean the $z$-component of the Bloch eigenfunction, $F$. Exactly as done in the previous section we then write

$$|\tilde{u}_{nk}\rangle \simeq |u_{nk}\rangle + \sum_{m \neq n} |u_{mk}\rangle \frac{\langle u_{nk}|V^p u_{mk}\rangle}{\omega_{nk} - \omega_{mk}}$$

(47)

$$V^p_k \simeq e^{-ikr} V^p e^{ikr}.$$  

(48)

In this framework, we notice that we are conceptually exploiting an adaptation of the $k \cdot p$ theory [45][47]. To avoid mathematical issues at degeneracy points in the first Brillouin zone, we are assuming non-degenerate photonic bands here throughout the manuscript.

The dynamical properties will be given by calculating the electromagnetic field velocity. However, a note of warning is worth here. In fact, while the physical velocity of the electromagnetic field velocity. However, a note of warning is worth here. In fact, while the physical velocity of the electromagnetic field $\vec{V}$ is a constant of the motion, the electromagnetic energy flux from the Poynting vector, $S_{nk} = \frac{1}{2} \text{Re} \{E_{nk} \times H_{nk} \}$, is given by

$$v_{nk}^{(e)} = \frac{\int d^3r \ S_{nk}}{U_{nk}},$$

(49)

where the electromagnetic energy density is expressed as $U_{nk} = U_{nk}^e + U_{nk}^m$, with $U_{nk}^e = \frac{1}{4} \int d^3r (\vec{E}_{nk} \cdot \vec{E}_{nk}^* + \vec{H}_{nk} \cdot \vec{H}_{nk}^*)$ and $U_{nk}^m = \frac{1}{4} \int d^3r (\vec{E}_{nk} \cdot \vec{H}_{nk}) + \frac{1}{i} (\vec{E}_{nk} \times \vec{H}_{nk}) \cdot \hat{n}$, the group velocity of the mode is actually given by

$$v_{nk}^{(g)} = \nabla_{nk}^\omega = \frac{1}{2\omega_{nk}} \frac{\langle u_{nk}|\nabla_{rk}\Theta_k|u_{nk}\rangle}{\langle u_{nk}|u_{nk}\rangle},$$

(50)

where the last equality is the photonic crystal version of Eq. (12), as in Ref. 10 (see also App. C). In an ideal photonic crystal made of non-dispersive constituents, one can show that $v_{nk}^{(e)} = v_{nk}^{(g)}$ [16][38], as it has been specifically shown for generic 2D photonic crystals in a photonic $k \cdot p$ framework [49]. Even if the equality between energy and group velocity is not generally fulfilled in perturbed systems, it can be shown (see App. D) that in the case
of non-dispersive media (i.e., for frequency-independent permittivity and permeability tensors) the energy velocity of the mode can be defined as
\[
\tilde{v}^{(c)}_{nk} = \frac{1}{2\omega_{nk}} \frac{\langle \hat{u}_{nk} | \nabla_k \Theta_k | \hat{u}_{nk} \rangle}{\langle \hat{u}_{nk} | \hat{u}_{nk} \rangle},
\] (51)
where we are implicitly assuming that, up to first order in perturbation theory, we can approximate \( \langle \hat{u}_{nk} | \hat{u}_{nk} \rangle \approx \langle \hat{u}_{nk} | \hat{u}_{nk} \rangle \) in the denominator (as we have done throughout App. D).

Using now Eq. (47), and redefining \( |\tilde{u}_{nk} \rangle \equiv |\hat{n} \rangle \) and \( |u_{nk} \rangle \equiv |n \rangle \) for ease of notation, Eq. (51) can be written as
\[
\tilde{v}^{(c)}_{nk} = \frac{1}{2\omega_{nk}} \langle n | n \rangle \left[ \left( |n\rangle + \sum_{m \neq n} \langle m | | \nabla_k \Theta_k | m \rangle \frac{\langle V_k^p n | m \rangle}{\omega_{nk}^2 - \omega_{nk}^2} \right) \nabla_k \Theta_k \right.
\]
\[
+ \left. \left( |n\rangle + \sum_{m' \neq n} \langle m' | | \nabla_k \Theta_k | m' \rangle \frac{\langle V_k^p n | m' \rangle}{\omega_{nk}^2 - \omega_{nk}^2} \right) \right],
\] (52)
from which, taking into account only the first order terms, we get
\[
\tilde{v}^{(c)}_{nk} = \frac{1}{2\omega_{nk}} \langle n | n \rangle \left[ \langle n | \nabla_k \Theta_k | n \rangle + \right.
\]
\[
\left. \sum_{m' \neq n} \langle n | \nabla_k \Theta_k | m' \rangle \frac{\langle | V_k^p n | m' \rangle}{\omega_{nk}^2 - \omega_{nk}^2} + \right]
\]
\[
\left. \sum_{m \neq n} \langle m | \nabla_k \Theta_k | n \rangle \frac{\langle V_k^p n | m \rangle}{\omega_{nk}^2 - \omega_{nk}^2} \right].
\] (53)
Using now the first of the photonic HF equations (see App. C for details)
\[
\langle u_{nk} | \nabla_k \Theta_k | u_{nk} \rangle = \left( \omega_{nk}^2 - \omega_{nk}^2 \right) \langle u_{nk} | \nabla_k u_{nk} \rangle,
\] (54)
Eq. (53) takes the form
\[
\tilde{v}^{(c)}_{nk} = \frac{1}{2\omega_{nk}} \langle n | n \rangle \left[ \langle n | \nabla_k \Theta_k | n \rangle + \right]
\]
\[
\left. \sum_{m' \neq n} \langle n | \nabla_k m' \rangle \frac{\langle m' | V_k^p n | m' \rangle}{\omega_{nk}^2 - \omega_{nk}^2} + \right]
\]
\[
\left. \sum_{m \neq n} \langle m | \nabla_k n \rangle \frac{\langle V_k^p n | m \rangle}{\omega_{nk}^2 - \omega_{nk}^2} \right].
\] (55)
From Eq. (44), we can write that
\[
\langle m | V_k^p n | n \rangle = -\langle m | x | \Theta_k n \rangle
\]
\[
= \sum_s -\langle m | x | s \rangle \frac{\lambda}{\alpha} | \Theta_k n \rangle
\]
\[
= \sum_s \frac{\lambda}{\alpha} \langle m | x | s \rangle \omega_{nk}^2 \delta_{sn}
\]
\[
= -\frac{\lambda}{\alpha} \langle m | x | n \rangle \omega_{nk}^2,
\] (56)
and using what we have shown in Eq. (55), we get
\[
\tilde{v}^{(c)}_{nk} = \frac{1}{2\omega_{nk}} \langle n | n \rangle \left[ \langle n | \nabla_k \Theta_k | n \rangle + \right]
\]
\[
\left. \sum_{m' \neq n} \frac{\lambda}{\alpha} \langle m' | V_k^p n | m' \rangle + \right]
\]
\[
\left. \sum_{m \neq n} \frac{\lambda}{\alpha} \langle m | V_k^p n | m \rangle \right].
\] (57)
Using now the second photonic HF equation, and the fact that the operator \(-i \nabla_k\) is self-adjoint, Eq. (57) becomes
\[
\tilde{v}^{(c)}_{nk} = \frac{1}{2\omega_{nk}} \langle n | n \rangle \left[ \langle n | \nabla_k \Theta_k | n \rangle + \right]
\]
\[
\left. \sum_{m' \neq n} \frac{\lambda}{\alpha} \langle m' | V_k^p n | m' \rangle + \right]
\]
\[
\left. \sum_{m \neq n} \frac{\lambda}{\alpha} \langle m | V_k^p n | m \rangle \right].
\] (58)
from which exploiting the completeness relation we straightforwardly obtain
\[
\tilde{v}^{(c)}_{nk} = \frac{1}{2\omega_{nk}} \langle n | n \rangle \left[ \langle n | \nabla_k \Theta_k | n \rangle + \right]
\]
\[
\left. \sum_{m' \neq n} \frac{\lambda}{\alpha} \langle m' | V_k^p n | m' \rangle + \right]
\]
\[
\left. \sum_{m \neq n} \frac{\lambda}{\alpha} \langle m | V_k^p n | m \rangle \right]
\] (59)
At last, in close analogy to the perturbative approach described for the electronic transport in the previous section, the last line in Eq. (59) gives a null contribution, and with little algebraic effort we obtain
\[
\tilde{v}^{(c)}_{nk} = \frac{1}{2\omega_{nk}} \langle n | n \rangle \left[ \langle n | \nabla_k \Theta_k | n \rangle - \frac{\lambda}{\alpha} \omega_{nk}^2 \hat{\Omega}_n \cdot \hat{z} \right]
\]
\[
= \tilde{v}^{(c)}_{nk} - \frac{\lambda}{\alpha} \omega_{nk}^2 \hat{\Omega}_n \cdot \hat{z}.
\] (60)
If we now define the generalized wave vector equation
\[
\kappa = -\frac{\lambda}{\alpha} \omega_{nk}^2 \hat{\Omega}_n \cdot \hat{z},
\] (61)
we can recast Eq. (60) in the compact and familiar form
\[
\tilde{v}^{(c)}_{nk} = \tilde{v}^{(c)}_{nk} - \kappa \times \Omega_{nk},
\] (62)
which exactly represents the photonic analog of Eq. (1). In fact, from Eq. (59), a “photonic” Berry curvature can be explicitly defined by a formally similar expression to the electronic case, \( \Omega_{nk} \equiv i \langle \text{rot} \langle n | \nabla_k n \rangle \rangle \). We stress that the derivation of Eq. (62) is strictly valid only within
the assumptions made throughout this section, namely considering 2D photonic crystals made of non-dispersive gyrotropic materials in which no TE/TM mixing occurs. We notice that also here the Berry curvature is interpreted as a geometric property of the gauge bundle, referred to a given photonic band. The relevance of this topological property in photonic systems has been already discussed in the literature \cite{17, 19}. In particular, when this quantity is different from zero, the gauge bundle is non-trivial and the topology of the bundle affects the dynamics of light propagation in the system. As remarked by Raghu and Haldane \cite{17}, a necessary condition to obtain a non-trivial bundle is TRS breaking, as in the gyrotropic 2D photonic crystal system assumed here. In general, when the degeneracy of photonic bands along high-symmetry direction in reciprocal space is removed by TRS breaking, the associated bundle can “twist” giving rise to non-trivial topological features, which can manifest themselves, e.g., as ballistic one-way edge states \cite{19}.

Despite the formal similarities between electronic and photonic periodic systems, there is a considerable number of physical and mathematical differences that should be carefully taken into account, such as the vectorial nature of operators and fields or the re-formulation of HF theorem in the general case. However, it is interesting to emphasize that the detection of such topological features appears easier in the photonic case, since electromagnetic edges mode can be observed by direct injection of light into the system, as it is further discussed in the next section.

**IV. DISCUSSION AND PHYSICAL REALIZATION**

Equation (62) is the central result of this paper. It rigorously shows that by perturbing a gyrotropic photonic crystal with a linear grading of the permittivity (permeability), the expression for the energy velocity of an electromagnetic mode contains, in addition to a zero-order term, a geometrical term depending on the Berry curvature, in full analogy with the semiclassical equation for the electron velocity in quantum Hall systems. Within this framework, the bulk dynamics of light propagation in a gyrotropic photonic crystal is deeply connected to the recent observation of back-scattering immune edge states \cite{20}, which is now rigorously explained in the photonic context by the bulk-edge correspondence \cite{13} (also known as holographic principle). In fact, although the presence of such edge states in systems with broken TRS is a clear indication of non-trivial topological properties, a bulk theory is always needed to rigorously justify and fully understand their physics.

A possible experimental scheme to demonstrate the bulk analog of the quantum Hall effect is proposed in Fig. 2. It shows a weak (e.g., $\lambda \sim 10^{-3}$) linear grading of the dielectric permittivity along the propagation direction of a 2D YIG-rods photonic crystal. Indeed, grading of the refractive index can be technologically achieved in different ways today \cite{50}. A light beam propagating along the grading direction would experience a topology-related bending within the photonic crystal region, according to Eq. (62), that would not be present in the absence of TRS breaking.

Finally, we would like to stress that a rigorous reformulation of the semiclassical equation of motion, Eq. (2), expressing the time derivative of the crystal momentum $k$, is out of the range of application of the present model. In fact, even if we can speculate that the equation for $k$ should have the same explicit expression as in Eq. (61), the demonstration that $k = \kappa$ has to be found independently of the theoretical framework presented so far, which goes beyond the scopes of the present work.

**V. CONCLUSIONS**

In summary, we have derived a perturbative theory of the photonic transport in two-dimensional, non-dispersive photonic crystals with gyrotropic constituents and a weak permittivity grading. Time-reversal symmetry is broken by the gyrotropic nature of the metamaterials employed, in analogy to the magnetic field in the electron transport, while the role of the dragging force induced by the electric field in quantum Hall systems is played here by the weak permittivity grading along the propagation direction. The specularity of the theoretical formulation between electric and magnetic fields, in terms of a generalized eigenvalue problem from Maxwell equations, allows a direct transfer of these results to two-dimensional photonic crystals made of gyroelectric materials with a grading of the magnetic permeability.

Under the assumption made, we have found that a complete formal analogy exists between the semiclassical equations of motion for an electron in a quantum Hall system and the electromagnetic energy transport in
such a bulk photonic crystal, where the energy velocity of a given photonic mode is corrected by a topology-related term that has the meaning of a Berry curvature. We have rigorously derived the explicit expression of the Berry curvature in terms of differential operators derived from Maxwell equations, with a formally analogous procedure to the electronic case. Thanks to the bulk-edge correspondence, this work gives a fully rigorous theoretical account of the recent experimental results obtained for electromagnetic energy transport through back-scattering immune chiral edge states.

Moreover, these results allow to design possible experimental configurations where the direct photonic analog of the quantum Hall effect can be probed in a bulk two-dimensional photonic crystal, instead of edge transport. In fact, the two-dimensional propagation of an electromagnetic beam in the photonic crystal region should be strongly influenced by the topological term, and a measurement of the beam deviation from the expected direction should give a direct measure of the Berry curvature in such a topological photonic insulator system.

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Appendix A: Explicit form of the photonic eigenvalue equation

Here we explicitly obtain the eigenvalue problem for the field component $F_z$ starting from Eq. (A3)

$\varepsilon^{-\frac{1}{2}} \nabla \times \left[ \mu^{-1} \nabla \times \left( \varepsilon^{-\frac{1}{2}} F \right) \right] = \omega^2 F_e.$ \hfill (A1)

First, we impose the condition

$F = F_e = (0, 0, F_z),$ \hfill (A2)

with obvious definitions of $\zeta_x$ and $\zeta_y.$ The second step is to calculate

$$\varepsilon^{-\frac{1}{2}} \nabla \times \left( \varepsilon^{-\frac{1}{2}} F \right) = \mu^{-1} \nabla \times (\varepsilon^{-\frac{1}{2}} F_e) = \mu^{-1} \zeta \nabla \times (\varepsilon^{-\frac{1}{2}} F_e) = (\mu^{-1} \zeta_x \hat{x} + i \mu^{-1} \zeta_y \hat{y}) = \theta_x \hat{x} + \theta_y \hat{y}.$$ \hfill (A3)

Moreover, these results allow to design possible experimental configurations where the direct photonic analog of the quantum Hall effect can be probed in a bulk two-dimensional photonic crystal, instead of edge transport. In fact, the two-dimensional propagation of an electromagnetic beam in the photonic crystal region should be strongly influenced by the topological term, and a measurement of the beam deviation from the expected direction should give a direct measure of the Berry curvature in such a topological photonic insulator system.

from which, using the relation $\partial_z F_z = 0$, which derives from the transversality condition $\nabla \cdot (\varepsilon \mu E)$, and the relations $\partial_z \varepsilon = \partial_z \mu = \partial_z \eta = 0$, due to the symmetry of the system, we get

$$\nabla \times \mu^{-1} \nabla \times (\varepsilon^{-\frac{1}{2}} F) = (\partial_x \theta_y - \partial_y \theta_x) \hat{z}.$$ \hfill (A4)

Multiplying Eq. (A3) by $\varepsilon^{-\frac{1}{2}}$, we obtain the eigenvalue problem

$$\omega^2 F_z = \left[ -\varepsilon^{-1} \mu^{-1} \nabla \nabla + \varepsilon^{-1} \left( \mu^{-1} \varepsilon^{-1} \nabla \varepsilon - \nabla \mu^{-1} - i (\hat{z} \times \nabla \eta) \right) \nabla + \frac{1}{2} \varepsilon^{-2} \left( \nabla \mu^{-1} \cdot \nabla \varepsilon - \frac{3}{2} \mu^{-1} \varepsilon^{-1} (\nabla \varepsilon)^2 + i (\hat{z} \times \nabla \eta) \cdot \nabla \varepsilon + \mu^{-1} \nabla \varepsilon \right) \right] F_z.$$ \hfill (A5)

Using the relations

$$\varepsilon^{-2} \nabla \varepsilon = -\nabla \varepsilon^{-1},$$ \hfill (A7)

$$\varepsilon^{-2} \nabla^2 \varepsilon = 2 \varepsilon (\nabla \varepsilon^{-1})^2 - \nabla \varepsilon^{-1},$$ \hfill (A8)

to point out the role of $\varepsilon^{-1}$ with respect to $\varepsilon$, Eq. (A6) assumes exactly the same expression as in Eq. (40) with the operator in Eq. (41).
which demonstrates the formal expression given in Eq. (43).

Appendix C: Formulation of Hellmann-Feynman equations in photonic crystal context

In this appendix we show how the HF theorem is easily reformulated in the photonic crystal context, and explicitly derive the two HF equations that are the photonic crystal analog of Eqs. (14) and (15), respectively. We begin by considering the parametric eigenvalue problem for the periodic part of the Bloch function

\[ \Theta_k |u_{nk}\rangle = \omega_{nk}^2 |u_{nk}\rangle, \quad (C1) \]

Taking the derivative with respect to \( k \) and multiplying both sides by \( \langle u_{nk} | \), we obtain

\[ \langle u_{nk} | \nabla_k \Theta_k | u_{nk} \rangle = 2 \omega_{nk} \langle u_{nk} | \nabla_k \omega_{nk} | u_{nk} \rangle, \quad (C2) \]

from which, given the definition of group velocity as \( v_{nk}^{(g)} = \nabla_k \omega_{nk} \), we get

\[ v_{nk}^{(g)} = \frac{1}{2\omega_{nk}} \frac{\langle u_{nk} | \nabla_k \Theta_k | u_{nk} \rangle}{\langle u_{nk} | u_{nk} \rangle}, \quad (C3) \]

which is the photonic formulation of the HF theorem reported in Eq. (12). In the same way, by differentiating Eq. (C1) with respect to \( k \), and then multiplying by \( \langle u_{nk} | \), we get

\[ \langle u_{nk} | \nabla_k \Theta_k | u_{nk} \rangle = \langle u_{nk} | \nabla_k \omega_{nk} | u_{nk} \rangle, \quad (C4) \]

which is the photonic formulation of the first HF equation reported in Eq. (14).

Unfortunately, the tensorial nature of the operators makes the photonic reformulation of the second HF equation, Eq. (15) rather difficult to demonstrate explicitly in the general case. However, we show here a demonstration for the particular case considered, i.e. applying the photonic operator to the scalar component of the field. From Eq. (40) with Eq. (41) we have

\[ \Theta = \left[ -\varepsilon e^2 + \mu e^2 \right] \Theta, \quad (B3) \]
\[ \Theta_{k}u_{nk} = \left[ -\varepsilon^{-1}\mu^{-1}\nabla \mathbf{r} - \left( 2i\varepsilon^{-1}\mu^{-1}\mathbf{k} + \mu^{-1}\nabla\varepsilon^{-1} + \varepsilon^{-1}\mu^{-1}\nabla\mathbf{r} \right) \cdot \mathbf{r} + \varepsilon^{-1}\mu^{-1}\mathbf{k} \cdot \nabla\varepsilon^{-1} - i\varepsilon^{-1}\mu^{-1}\mathbf{k} \cdot \nabla\mu^{-1} + \varepsilon^{-1}\mu \cdot \left( \mathbf{z} \times \nabla\eta \right) \right] \cdot \mathbf{r} \] 

where \( u_{nk} \) is the Bloch part of the component \( F_{z} \), and hence

\[ \nabla_{k}\Theta_{k} = \nabla_{k} \left[ \varepsilon^{-1}\mu^{-1}( -i\nabla + \mathbf{k})^{2} - i\mu^{-1}\nabla\varepsilon^{-1} + \right. 
\left. -i\varepsilon^{-1}\mu^{-1} + \xi^{-1} \left( \mathbf{z} \times \nabla\eta \right) \right] = -2i\varepsilon^{-1}\mu^{-1}\nabla + 2\varepsilon^{-1}\mu^{-1}\mathbf{k} - i \left( \mu^{-1}\varepsilon^{-1} + 
\right. 
\left. +\varepsilon^{-1}\mu^{-1} + \xi^{-1} \left( \mathbf{z} \times \nabla\eta \right) \right) . \] 

Using now the commutation relations

\[ \begin{align*}
[\nabla, \mathbf{r}] &= 1, \\
[\nabla^{2}, \mathbf{r}] &= 2\nabla,
\end{align*} \] 

it is straightforward to show

\[ \nabla_{k}\Theta_{k} = -2\varepsilon^{-1}\mu^{-1}\nabla + 2\varepsilon^{-1}\mu^{-1}\mathbf{k} - i \left( \mu^{-1}\varepsilon^{-1} + 
\right. 
\left. +\varepsilon^{-1}\mu^{-1} + \xi^{-1} \left( \mathbf{z} \times \nabla\eta \right) \right) = i\nabla_{k}\Theta_{k} . \] 

Using the last relation in Eq. (C4), we obtain

\[ i \langle \mu_{mk} | \Theta_{k} | \mu_{nk} \rangle = (\omega_{nk}^{2} - \omega_{mk}^{2}) \langle \mu_{nk} | \nabla_{k} \mu_{nk} \rangle \] 

from which

\[ i \langle \mu_{mk} | \Theta_{k} | \mu_{nk} \rangle - r \Theta_{k} | \mu_{nk} \rangle = i \langle \mu_{mk} | r | \mu_{nk} \rangle \left( \omega_{nk}^{2} - \omega_{mk}^{2} \right) = (\omega_{nk}^{2} - \omega_{mk}^{2}) \langle \mu_{nk} | \nabla_{k} \mu_{nk} \rangle \] 

which finally gives

\[ \langle \mu_{nk} | \mathbf{r} | \mu_{nk} \rangle = i \langle \mu_{nk} | \nabla_{k} \mu_{nk} \rangle , \] 

i.e. exactly the photonic crystal analog of Eq. (15). 

**Appendix D: Demonstration of an expression for the energy velocity**

In the perturbed two-dimensional photonic crystal considered, where separation of TE/TM modes occurs and the permittivity/permeability tensors are assumed non-dispersive, from Eq. (49) we can write Eq. (51) as

\[ \nu^{(e)}_{nk} = \frac{\text{Re} \int d^{3}r \, \hat{E}^{(z)} \times \hat{H}^{(x,y)}_{nk}}{\int d^{3}r \, \varepsilon |E_{nk}^{(z)}|^{2}} , \] 

where \( E_{nk}^{(z)} \) and \( H_{nk}^{(x,y)} \) are the perturbed electric and magnetic fields, respectively. In Eq. (D1) we have implicitly kept only the unperturbed product in the denominator, and we considered twice the electric contribution to the total electromagnetic energy density, i.e. \( U_{nk} = 2\mu_{nk}^{2} \) as it is true for harmonic modes (see, e.g., page 16 of Ref. [16]). Using the Maxwell equation

\[ \nabla \times \mathbf{E}(\mathbf{r}) = i\omega \mu^{(e)}(\mathbf{r})\mathbf{H}(\mathbf{r}) , \] 

we can rewrite Eq. (D1) as

\[ \nu^{(e)}_{nk} = \frac{\text{Re} \int d^{3}r \, \varepsilon |E_{nk}^{(z)}|^{2}}{\text{Re} \int d^{3}r \, \varepsilon |E_{nk}^{(z)}|^{2}} \times \hat{E}^{(z)}_{nk} \] 

From Eq. (46) we straightforwardly obtain the relation

\[ |E_{nk}^{(z)}| = |E_{nk}^{(z)}| + \sum_{m \neq n} |E_{nk}^{(z)}| \text{Re} \left[ \frac{\hat{E}^{(z)}_{nk} \cdot \mathbf{v} \times \hat{E}^{(z)}_{nk} \cdot \mathbf{v}}{\omega_{nk}^{2} - \omega_{mk}^{2}} \right] \] 

which, once inserted in Eq. (D3), gives

\[ \nu^{(e)}_{nk} = \nu^{(e)}_{nk} + \frac{1}{\omega \int d^{3}r \, \varepsilon |E_{nk}^{(z)}|^{2}} \times \hat{E}^{(z)}_{nk} \] 

\[ \times \text{Re} \int d^{3}r \, \sum_{m \neq n} \left[ -i\mu^{-1}J_{mn}^{(e)} \hat{E}^{(z)}_{nk} - \eta J_{mn}^{(e)} \hat{E}^{(z)}_{nk} \cdot (\mathbf{z} \times \nabla \hat{E}^{(z)}_{nk}) + \eta J_{mn}^{(e)} \hat{E}^{(z)}_{nk} (\mathbf{z} \times \nabla \hat{E}^{(z)}_{nk}) \right] . \]
Integrating the second and the forth term in the square brackets by parts, the last equation becomes

\[
\tilde{v}_{nk}^{(e)} = v_{nk}^{(e)} + \frac{1}{\omega} \int d^3r \sum_{\mu \neq n} \left[ -2\mu^{-1} \text{Re} \left( iJ_{mn}^{*} E_{mk}^{(e)} \nabla E_{nk}^{(e)} \right) + \right.
\]

\[ + i\nabla \mu^{-1} E_{mn}^{(e)} E_{nk}^{(e)} + \]

\[ - 2i\eta \text{Im} \left( J_{mn}^{*} u_{mn}^{(e)} \left( \hat{z} \times \nabla E_{nk}^{(e)} \right) \right) + \]

\[ + J_{mn} E_{mn}^{(e)} E_{nk}^{(e)} \left( \hat{z} \times \nabla \eta \right) \].

(D6)

Except for the first term, all the other terms in the square bracket of Eq. (D6) give a null contribution. In fact, after observing that \( J_{mn} \) does not depend on spatial variables, we can see that the third term is purely imaginary, while the second and the fourth are odd functions \([51]\). Thus, we can write

\[
\tilde{v}_{nk}^{(e)} = v_{nk}^{(e)} - \frac{2}{\omega} \int d^3r \sum_{\mu \neq n} \tilde{\mu}^{-1} \text{Re} \left( iJ_{mn}^{*} E_{mk}^{(e)} \nabla E_{nk}^{(e)} \right) .
\]

(D7)

Now we have to verify if the following equality is correct

\[
\tilde{v}_{nk}^{(e)} = \frac{1}{2\omega} \left\langle \tilde{u}_{nk} \left| \nabla_k \Theta_k \right| \tilde{u}_{nk} \right\rangle ,
\]

where as, before, \( \tilde{u}_{nk} \) is the Bloch part of the perturbed field component \( \tilde{E}_{nk} = \tilde{E}_{nk}^{(e)} \). To this end, the second member of Eq. (D8), which we define \( S \), can be re-written using the relations

\[
S = \frac{1}{2\omega} \left\langle \tilde{F}_{nk} \left| \nabla_k \Theta_k \right| \tilde{F}_{nk} \right\rangle
\]

\[ = \frac{1}{2\omega} \left\langle \frac{\varepsilon^{(e)}}{\tilde{E}_{nk}^{(e)}} \left| \nabla_k \Theta_k \right| \frac{\varepsilon^{(e)}}{\tilde{E}_{nk}^{(e)}} \right\rangle ,
\]

where we have defined the operator \( \nabla_k \Theta \) as

\[
\nabla_k \Theta \left| F_{nk} \right\rangle = \nabla_k \Theta \left| u_{nk} \right\rangle ,
\]

(D10)

which explicitly gives

\[
\nabla_k \Theta = -2i\varepsilon^{-1} \tilde{\mu}^{-1} \nabla - i \left( \tilde{\mu}^{-1} \varepsilon^{-1} + \right.
\]

\[ + \varepsilon^{-1} \tilde{\mu}^{-1} + i\varepsilon^{-1} \left( \hat{z} \times \nabla \eta \right) \). \]

(D11)

Using Eqs. (D9) and (D11), we obtain

\[
S = \frac{1}{2\omega} \left\langle \tilde{E}_{nk}^{(e)} \left| \varepsilon E_{nk}^{(e)} \right\rangle \right.
\]

\[ \cdot \int d^3r \left[ \tilde{E}_{nk}^{(e)} \left( -i\varepsilon^{-1} \tilde{\mu}^{-1} \varepsilon \tilde{E}_{nk}^{(e)} - 2i\mu^{-1} \varepsilon \tilde{E}_{nk}^{(e)} + \right. \right.
\]

\[ -i\tilde{\mu}^{-1} \varepsilon \tilde{E}_{nk}^{(e)} - i\tilde{\mu}^{-1} \tilde{E}_{nk}^{(e)} + (\hat{z} \times \nabla \eta) \tilde{E}_{nk}^{(e)} \right] ,
\]

(D12)

from which, observing that \( \varepsilon^{-1} = -\varepsilon^{-2} \varepsilon \), we get

\[
S = \frac{1}{2\omega} \left\langle \tilde{E}_{nk}^{(e)} \left| \varepsilon E_{nk}^{(e)} \right\rangle \right.
\]

\[ \cdot \int d^3r \left[ -2i\mu^{-1} \tilde{E}_{nk}^{(e)} \nabla \tilde{E}_{nk}^{(e)} + \right.
\]

\[ -i\varepsilon^{-1} \tilde{E}_{nk}^{(e)} \right| \tilde{E}_{nk}^{(e)} \right\rangle \right. ,
\]

(D13)

In the last equation, we can use the same considerations as before regarding odd functions, from which we get

\[
S = \tilde{v}_{nk}^{(e)}(k) - \frac{4}{2\omega} \left\langle \varepsilon E_{nk}^{(e)} \right| \varepsilon E_{nk}^{(e)} \right\rangle .
\]

(D14)

which compared to Eq. (D7) finally gives Eq. (51).

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