Lotka–Volterra competition-diffusion system: the critical competition case

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ABSTRACT
We consider the reaction-diffusion competition system in the so-called critical competition case. The associated ODE system then admits infinitely many equilibria, which makes the analysis intricate. We first prove the nonexistence of ultimately monotone traveling waves by applying the phase plane analysis. Next, we study the large time behavior of the solution of the Cauchy problem with a compactly supported initial datum. We not only reveal that the “faster” species excludes the “slower” one (with a known spreading speed), but also provide a sharp description of the profile of the solution, thus shedding light on a new bump phenomenon.

1. Introduction
We consider the Lotka–Volterra competition-diffusion system
\begin{equation}
\begin{cases}
    u_t = u_{xx} + u(1 - u - v), & t > 0, x \in \mathbb{R}, \\
    v_t = dv_{xx} + rv(1 - v - u), & t > 0, x \in \mathbb{R},
\end{cases}
\end{equation}
which is critical among systems in the form of (1.5). The main difficulty is that the underlying ODE competition system
\begin{equation}
\begin{cases}
    u' = u(1 - u - v), & t > 0, \\
    v' = rv(1 - v - u), & t > 0,
\end{cases}
\end{equation}
adopts infinitely many (nontrivial) equilibria: the whole line $u + v = 1$. Because of that, there are very few available mathematical results on system (1.1). In the present paper, we fill this gap by proving the nonexistence of ultimately monotone traveling waves, and giving a very precise description of the large time behavior of the solution starting from a compactly supported initial datum, thus revealing a new bump phenomenon.

In the absence of one species, system (1.1) reduces to the reaction-diffusion equation
\begin{equation}
    u_t = du_{xx} + ru(1 - u), \quad t > 0, x \in \mathbb{R},
\end{equation}
introduced by Fisher [1] and Kolmogorov, Petrovskii, and Piskunov [2] as a population genetics model to investigate the propagation of a dominant gene in a homogeneous environment. The KPP equation (1.3) has two main properties. Firstly, nonnegative traveling waves, corresponding to the ansatz $u(t, x) = U(x - ct)$ and solving
\[
\begin{align*}
  dU'' + cU' + rU(1 - U) &= 0 \quad \text{in } \mathbb{R}, \\
  U(-\infty) &= 1, \quad U(\infty) = 0,
\end{align*}
\]
exist if and only if their speeds $c \geq c^* := 2\sqrt{dr}$. Secondly, the solution of (1.3) starting from a nonnegative (nontrivial) compactly supported initial datum, satisfies
\[
\begin{align*}
  \lim_{t \to \infty} \sup_{|x| \geq ct} u(t, x) &= 0, \quad \text{for all } c > c^*, \\
  \lim_{t \to \infty} \sup_{|x| \leq ct} |1 - u(t, x)| &= 0, \quad \text{for all } c < c^*,
\end{align*}
\]
see [3]. In other words, the minimal speed $c^*$ of traveling wave solutions corresponds to the spreading speed of the solution of the Cauchy problem with a compactly supported initial datum.

The general Lotka–Volterra competition-diffusion system is written
\[
\begin{align*}
  u_t &= u_{xx} + u(1 - u - av), \quad t > 0, x \in \mathbb{R}, \\
  v_t &= dv_{xx} + rv(1 - v - bu), \quad t > 0, x \in \mathbb{R}.
\end{align*}
\]
Here $u = u(t, x)$ and $v = v(t, x)$ represent the population densities of two competing species, $d > 0$ and $r > 0$ stand for the diffusion rate and intrinsic growth rate of $v$ (while those of $u$ have been normalized), and $a > 0$ and $b > 0$ represent the strength of $v$ and $u$, respectively, as competitors. The parameters $a$ and $b$ determine the behavior of the underlying ODE system (see below) but, once fixed, the outcomes for system (1.5) are highly dependent on the parameters $r$, $d$ and the initial datum. The situation is therefore very rich and we refer to the works mentioned below for more details and references.

The so-called weak competition case corresponds to $a < 1$ and $b < 1$. Nontrivial solutions of the underlying ODE system tend to the co-existence equilibrium. For the diffusion system, it was proved by Tang and Fife [4] that there exists a minimal speed $c^* > 0$ such that a monotone traveling wave solution connecting the co-existence equilibrium to the null state $(0, 0)$ exists if and only if $c \geq c^*$, which is comparable to the Fisher-KPP equation mentioned above. Concerning the large time behavior of the Cauchy problem, some first estimates were obtained by Lin and Li [5]. More recently, Liu, Liu, and Lam [6, 7] obtained some rather complete results.

The so-called strong competition case corresponds to $a > 1$ and $b > 1$. Since the co-existence equilibrium is unstable and the equilibria $(1, 0)$ and $(0, 1)$ are both stable for the underlying ODE system, this case corresponds to a bistable situation. For the diffusion system, it was proved by Kan-On [8], see also [9], that there exists a unique traveling wave solution connecting $(1, 0)$ to $(0, 1)$. The sign of the speed of this wave determines the “winner” between $u$ and $v$, and thus is very relevant for applications, see the review [10]. We refer to [11–13] for some results on this delicate issue. As far as the large time behavior of the Cauchy problem is concerned, we refer to the recent
work of Carrère [14] revealing the possibility of propagating terraces, see [15,16]. Very recently, Peng, Wu, and Zhou [17] provided refined estimates of both the spreading speed and the profile of the solution.

Last, the so-called strong-weak competition case corresponds to $a < 1 < b$. Nontrivial solutions of the underlying ODE system tend to the state $(1, 0)$ meaning that “$u$ excludes $v$.” For the diffusion system, the traveling wave solutions were constructed by Kan-On [18]. Among others, the important issue of the linear determinacy of the minimal traveling wave speed was first studied by Lewis, Li, and Weinberger [19,20]. Concerning the large time behavior of the solution of the Cauchy problem, Girardin and Lam [21] recently studied the spreading speed of solutions with an initial datum that is null (or exponentially decaying) on the right half line. They obtained a rather complete understanding of the spreading properties, revealing in particular the possibility of a so-called nonlocal pulling phenomenon (see the Appendix of the present paper for more details).

In the present paper, our goal is to complete the above picture. More precisely, we aim at understanding the long time behavior of the Cauchy problem in the so-called critical competition case $a = b = 1$, corresponding to system (1.1). Let us notice that the outcomes are far from being obvious. Indeed, assume for instance $dr > 1$. Observe that, by the comparison principle, the case $a = b = 1$ under consideration is “sandwiched” between the case $b < 1 < a$ (the faster is the stronger) and the case $a < 1 < b$ (the faster is the weaker). As revealed in [21], these two strong-weak competition cases are qualitatively different: when $b < 1 < a$ the faster and stronger species $v$ excludes $u$ and spreads with a speed $2 \sqrt{\sqrt{dr}}$ given by the single KPP equation; when $a < 1 < b$, the faster and weaker species $v$ spreads with a speed $2 \sqrt{\sqrt{dr}}$ and is chased by the slower and stronger species $u$, whose speed is more delicate to identify (see the Appendix for details), thus leading to “stacked fronts.” What happens in the critical case $a = b = 1$ is thus not “predictable” and deserves a sharp analysis.

2. Main results

A traveling wave solution of system (1.1) is defined as follows.

**Definition 2.1** $(\alpha, \beta)$-traveling wave. Let $0 \leq \alpha, \beta \leq 1$ be given with $\alpha \neq \beta$. Then an $(\alpha, \beta)$-traveling wave solution of (1.1) is a triplet $(c, U, V)$, where $c \in \mathbb{R}$ is the traveling wave speed and $(U, V)$ two nonnegative profiles, solving

$$
\begin{align*}
U'' + cU' + U(1 - U - V) &= 0, \\
\sigma V'' + cV' + rV(1 - V - U) &= 0, \\
(U, V)(-\infty) &= (\alpha, 1 - \alpha), \\
(U, V)(+\infty) &= (\beta, 1 - \beta).
\end{align*}
$$

As mentioned above, for both the strong competition case and the strong-weak competition case, monotone traveling waves connecting $(1, 0)$ to $(0, 1)$ are known to exist. In the critical competition case under consideration, our first main result is that there is no ultimately monotone $(\alpha, \beta)$-traveling wave connecting any two different nonnegative steady states on the line $u + v = 1$. 
Definition 2.2 (Ultimately monotone \((x, \beta)\)-traveling wave). Let \(0 \leq x, \beta \leq 1\) be given with \(x \neq \beta\). Then an ultimately monotone \((x, \beta)\)-traveling wave solution is an \((x, \beta)\)-traveling wave for which there further exist \(-\infty < z_0 \leq z_0^* < +\infty\) such that
\[
U'(z)V'(z) \neq 0, \quad \text{for all } z \in (-\infty, z_0] \cup [z_0^*, +\infty).
\] (2.2)

Remark 2.3. Obviously, if \((c, U(z), V(z))\) is an (ultimately monotone) \((x, \beta)\)-traveling wave then \((-c, U(-z), V(-z))\) is a (ultimately monotone) \((\beta, x)\)-traveling wave.

In other words, we do not require the \((x, \beta)\)-traveling wave to be monotone on , but only to be monotone in some neighborhoods of both \(-\infty\) and \(+\infty\). This reinforces our nonexistence result which states as follows.

Theorem 2.4 (Nonexistence of ultimately monotone \((x, \beta)\)-traveling waves). Let \(0 \leq x, \beta \leq 1\) be given with \(x \neq \beta\). Then, there is no ultimately monotone \((x, \beta)\)-traveling wave for system (1.1).

The above theorem is proved in Section 3. The starting point consists in transforming system (2.1) into a first order system of four ODEs. Then, by a phase plane analysis, we prove that \(U + V - 1\) has to “oscillate” in a neighborhood of \(-\infty\) or \(+\infty\), from which we get a contradiction.

Remark 2.5. As easily seen from the proof, to exclude the existence of an \((x, \beta)\)-traveling wave with speed \(c > 0\) (resp. \(c < 0\)), it is enough to assume that (2.2) holds in a neighborhood of \(-\infty\) (resp. \(+\infty\)). In other words, there is no \((x, \beta)\)-traveling wave for which the invading state is monotonically reached. The existence of an \((x, \beta)\)-traveling wave for which the invading state is not monotonically reached remains an open issue. Last, as seen from Section 3.1, the nonexistence of \((x, \beta)\)-standing waves \((c = 0)\) does not require any ultimately monotonicity assumption.

Regarding the framework of entire solutions for competition-diffusion systems, we would like to mention the recent works [22–24] and the references cited therein. For the critical case under consideration in the present work, we believe such issues would deserve further explorations.

Our second main focus is concerned with the large time behavior of the solution of system (1.1) starting from a nonnegative (nontrivial) compactly supported initial datum. In both the strong competition case [14, 17], and the strong-weak competition case [21], the monotone traveling wave solutions of the entire system play a key role in studying the large time behavior of the solution of the Cauchy problem. However, for the critical competition case, such traveling wave solutions do not exist.

Recalling the issue raised in the last paragraph of Section 1 (say in the case \(dr > 1\)), Theorem 2.4 on the nonexistence of ultimately monotone \((x, \beta)\)-traveling waves is a first indication that the long time behavior of the Cauchy problem should not be “stacked fronts” (which occurs when \(a < 1 < b\)). In other words, we suspect that the critical case \(a = b = 1\) should behave more like the \(b < 1 < a\) case.

This shows the relevance of the following tools: we define the (minimal) Fisher-KPP traveling wave solution \((c_u, U_{KPP})\) as

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\[ c_u := 2, \quad \begin{cases} U'_{\text{KPP}} + c_u U_{\text{KPP}} + U_{\text{KPP}}(1 - U_{\text{KPP}}) = 0, \\ U_{\text{KPP}}(-\infty) = 1, U_{\text{KPP}}(\infty) = 0, \end{cases} \]  

and, similarly, \((c_v, V_{\text{KPP}})\) as

\[ c_v := 2\sqrt{d}r, \quad \begin{cases} dV'_{\text{KPP}} + c_v V_{\text{KPP}} + rV_{\text{KPP}}(1 - V_{\text{KPP}}) = 0, \\ V_{\text{KPP}}(-\infty) = 1, V_{\text{KPP}}(\infty) = 0. \end{cases} \]

Let us recall that both \(U_{\text{KPP}}\) and \(V_{\text{KPP}}\) are uniquely defined “up to a shift.” Note that, \(c_u\) (resp. \(c_v\)) also represents the spreading speed of \(u\) (resp. \(v\)) in the absence of \(v\) (resp. \(u\)).

We can now state our first result on the Cauchy problem.

**Theorem 2.6** (Propagation phenomenon). Let \((u, v) = (u, v)(t, x)\) be the solution of system (1.1) starting from an initial datum \((u_0, v_0) = (u_0, v_0)(x)\) satisfying

\[ u_0 \text{ and } v_0 \text{ are continuous, nontrivial, compactly supported, and } 0 \leq u_0, v_0 \leq 1. \]

Then the following holds.

i. Assume \(dr > 1\) (i.e. \(c_v > c_u\)). Then

\[ \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |v(t, x) - V_{\text{KPP}}(|x| - c_v t + \frac{3d}{c_v} \ln t + \eta_v(t))| + \sup_{x \in \mathbb{R}} u(t, x) = 0, \]

where \(\eta_v\) is a bounded function on \([0, \infty)\).

ii. Assume \(dr < 1\) (i.e. \(c_v < c_u\)). Then

\[ \lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}} |u(t, x) - U_{\text{KPP}}(|x| - c_u t + \frac{3}{c_u} \ln t + \eta_u(t))| + \sup_{x \in \mathbb{R}} v(t, x) \right) = 0, \]

where \(\eta_u\) is a bounded function on \([0, \infty)\).

The above theorem is proved in Section 4. The main difficulty consists in showing that “the slow species” uniformly goes to extinction. While in the strong or strong-weak competition cases, adequate sub- or super-solutions can be constructed by only using traveling waves of some single equations, in the critical case under consideration, we need a precise (and new) combination of the Fisher-KPP traveling waves defined in (2.3), (2.4), and solutions to some heat equations.

Let us briefly comment on Theorem 2.6, say in the case \(dr > 1\). First of all the “faster species,” namely \(v\), excludes the “slower one,” namely \(u\), and imposes its spreading speed, see (2.6). Furthermore, we find that the profile of the solution uniformly converges to the corresponding minimal KPP traveling wave solution, and this up to an identified logarithmic Bramson correction, see (2.6) again. Regarding the issue raised in the last paragraph of Section 1, Theorem 2.6 thus reveals that the conclusion in the critical case \(a = b = 1\) “corresponds” to the case \(b < 1 < a\).
On the other hand, for the most critical case $dr = 1$, it is difficult to decide which species leads the invading front, and the behavior of the solution is highly depending on both the parameters and the shape of the initial datum. For the case $d = r = 1$, for any initial datum satisfying (2.5), a coexistence phenomenon happens. However, for the case $dr = 1$ but $d \neq 1$, the behavior of the solution is much more intricate. Indeed, in this case, the logarithmic phase drifts for $u$ and $v$ are different and there is a narrow region of width $O(\ln t)$ where the behaviors of $u$ and $v$ are difficult to “anticipate.” This may cause some subtle phenomena (both species driving the front or one excluding the other) and makes the mathematical analysis quite involved. We hope to address these issues in a future work.

Our second result on the Cauchy problem deals with the region “$|x| \leq \varepsilon_s t$,” where the profile of the solution is more of the “heat equation type.”

**Theorem 2.7** (Bump phenomenon). Let $(u, v) = (u, v)(t, x)$ be the solution of system (1.1) starting from an initial datum $(u_0, v_0) = (u_0, v_0)(x)$ satisfying (2.5). Denote

$$k^* := \min\left(\frac{1}{2d} \frac{d}{2}\right), \quad d^* := \max(1, d).$$

Then the following holds.

(i) Assume $dr > 1$ (i.e. $c_v > c_u$). Then for $\varepsilon > 0$ small enough and $0 < \theta < \frac{1}{2}$ there exist $C_2 > C_1 > 0$ and $T > 0$ such that both

$$C_1 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \leq u(t, x) \leq C_2 t^{-k^*} e^{-\frac{x^2}{4t}},$$

$$\max\left(C_1 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} - t^{-(1 + \theta)}, 0\right) \leq 1 - v(t, x) \leq C_2 t^{-k^*} e^{-\frac{x^2}{4t}},$$

hold for any $t \geq T$, $|x| \leq \varepsilon_s t$.

(ii) Assume $dr < 1$ (i.e. $c_v < c_u$). Then for $\varepsilon > 0$ small enough and $0 < \theta < \frac{1}{2}$ there exist $C_4 > C_3 > 0$ and $T > 0$ such that both

$$C_3 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \leq v(t, x) \leq C_4 t^{-k^*} e^{-\frac{x^2}{4t}},$$

$$\max\left(C_3 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} - t^{-(1 + \theta)}, 0\right) \leq 1 - u(t, x) \leq C_4 t^{-k^*} e^{-\frac{x^2}{4t}},$$

hold for any $t \geq T$, $|x| \leq \varepsilon_s t$.

The upper estimate in Theorem 2.7 comes from the super-solution constructed in Section 4 while the lower estimate is proved in Section 5. Since in sublinear regions the solutions do not converge to 0 or 1 exponentially, we propose a new construction of super- and sub-solutions, by the combination of traveling waves of single equations and some solutions of heat equations. Particularly, a solution of the heat equation with moving boundaries is used to construct the sub-solution.

Let us briefly comment on Theorem 2.7, say in the case $dr > 1$. As revealed by (2.8) and (2.9), the solution converges to $(0, 1)$ exponentially in regions of the form $|x| \geq st$ with $\varepsilon > 0$, but only algebraically in “sublinear regions” of the form $|x| \leq \sqrt{t}$. We call
this a bump phenomenon, see Figure 1. Such a phenomenon does not occur in the strong competition case [17]. As far as the strong-weak competition case is concerned, the results as stated in [21] are not sufficient to decide if it occurs or not, but we assert it does not, as proved in the forthcoming work [25]. Therefore, the present paper is the first one revealing a bump phenomenon in the context of competition-diffusion systems. We believe such a phenomenon is reserved for the critical case \(a = b = 1\), and is rare to happen in the context of homogeneous reaction-diffusion equations.

While, in the case \(d = 1\), the upper and the lower estimates are both of order \(\mathcal{O}(t^{-\frac{1}{2}})\) in sublinear regions, it is much more challenging to find the correct order of convergence in the case \(d \neq 1\), see Remark 5.6.

3. Nonexistence of \((x, \beta)\)-traveling waves

This section is devoted to the proof of Theorem 2.4 on the nonexistence of ultimately monotone \((x, \beta)\)-traveling waves for system (1.1).

3.1. Preliminary results and observations

In this subsection, the ultimately monotonicity assumption is not required, and thus an \((x, \beta)\)-traveling wave is understood in the sense of Definition 2.1. Let us recall that \(a \neq \beta\) is always assumed so that, e.g., taking \(U\) a Fisher-KPP traveling wave solution and \(V \equiv 0\) does not provide a \((x, \beta)\)-traveling wave in the sense of Definition 2.1.

We start with the following a priori estimates for an \((x, \beta)\)-traveling wave.

**Lemma 3.1.** Any \((x, \beta)\)-traveling wave has to satisfy \(0 < U < 1, 0 < V < 1\), and \(U'(\pm \infty) = V'(\pm \infty) = 0\).

**Proof.** The positivity of the profiles follows from the strong maximum principle. If \(U \leq 1\) is not true, then \(U\) has to reach a maximum value strictly larger than 1 at some point, and evaluating the \(U\)-equation at this point gives a contradiction. Hence \(U \leq 1\) and, from the strong maximum principle, \(U < 1\). Similarly, one has \(V < 1\).

We now prove the limit behavior \(U'(\infty) = 0\), the other ones being proved similarly. Denote the set of accumulation points of \(U'\) in \(+\infty\) by \(\mathcal{A}\). Since \(U\) is bounded, \(0 \in \mathcal{A}\). Let \(\ell \in \mathcal{A}\). Then there exists a sequence \(z_n \to +\infty\) such that \(U'(z_n) \to \ell\) as \(n \to +\infty\). Then \((U_n, V_n)(z):=(U, V)(z + z_n)\) solves

\[
U''_n + cU'_n = -U_n(1 - U_n - V_n).
\]

Since the \(L^\infty\) norm of the right hand side term is uniformly bounded with respect to \(n\), the interior elliptic estimates imply that, for all \(R > 0\) and \(1 < p < \infty\), the sequence

![Figure 1. The asymptotic profile of the solution, in the case \(dr > 1\).](image-url)
(\(U_n\)) is bounded in \(W^{2,p}(-R,R)\). From Sobolev embedding theorem we have that, up to a subsequence, \(U_n\) converges to some \(U_\infty\) in \(C^1_{loc}(\mathbb{R})\). The boundary condition \(U(+\infty) = \beta\) thus enforces \(U_\infty = \beta\) and \(U'_\infty = 0\). As a result, \(U'(z_n) = U'_n(0) \to U'_\infty(0) = 0\), and thus \(\ell = 0\). Hence \(A = \{0\}\), which concludes the proof. 

We conclude this subsection by showing the nonexistence of \((x, \beta)\)-traveling wave solutions for two special cases.

**Proposition 3.2** (Nonexistence of \((x, \beta)\)-standing waves). There is no \((x, \beta)\)-standing wave, i.e. \((x, \beta)\)-traveling wave with speed \(c = 0\), for system (1.1).

**Proof.** Assume \(c = 0\). By adding the both sides of the \(U\)-equation and the \(V\)-equation, we find that \(W := U + V\) satisfies

\[
W'' + \left(U + \frac{r}{d}V\right)(1 - W) = 0, \quad W(\pm \infty) = 1.
\]

If \(W \neq 1\), then \(W\) reaches either a maximum value strictly larger than 1 or a minimum value in \((0, 1)\), which is impossible from the above equation (recall that \(U + \frac{r}{d}V > 0\)). As a result \(W = U + V \equiv 1\). Going back to the original equations we have \(U'' = V'' = 0\). Since \(U\) and \(V\) are bounded, \(U\) and \(V\) must be constant, which cannot happen since \(x \neq \beta\).

**Proposition 3.3** (Nonexistence of \((x, \beta)\)-traveling waves when \(d = 1\)). Assume \(d = 1\). Then there is no \((x, \beta)\)-traveling wave for system (1.1).

**Proof.** Again, by adding the both sides of the \(U\)-equation and \(V\)-equation, we see that \(W := U + V\) satisfies

\[
W'' + cW' + (U + rV)(1 - W) = 0, \quad W(\pm \infty) = 1,
\]

so that, as in Proposition 3.2, we have \(W \equiv 1\), and thus \(U'' + cU' = 0\). From Lemma 3.1, by integrating both sides from \(-\infty\) to \(+\infty\), we have \(c(\beta - x) = 0\), which yields a contradiction since \(c \neq 0\) and \(x \neq \beta\).

### 3.2. Proof of Theorem 2.4

We now consider the case of ultimately monotone \((x, \beta)\)-traveling waves.

**Lemma 3.4.** Let \(0 \leq x, \beta \leq 1\) be given with \(x \neq \beta\). Let \((c, U, V)\) be an ultimately monotone \((x, \beta)\)-traveling wave solution. Then (2.2) is refined in

\[
U'(z)V'(z) < 0, \quad \text{for all } z \in (-\infty, z_0] \cup [z_0', +\infty). \tag{3.1}
\]

**Proof.** We only deal with the behavior around \(-\infty\). If the conclusion is false, we may assume that \(U'(z) > 0\) and \(V'(z) > 0\) for all \(z \in (-\infty, z_0]\), the case \(U'(z) < 0\) and \(V'(z) < 0\) being treated similarly. From the boundary conditions \((U, V)(-\infty) = (x, 1 - x)\) and \((U, V)(+\infty) = (\beta, 1 - \beta)\), there must exist a point \(z_1 > z_0\) such that

\[
U'(z) > 0, V'(z) > 0, \quad \text{for all } z < z_1, \quad U'(z_1) = 0 \text{ or } V'(z_1) = 0.
\]
Assume w.l.o.g. that \( U'(z_1) = 0 \). In particular \( U''(z_1) \leq 0 \). From the \( U \)-equation, this enforces \( (U + V)(z_1) \leq 1 \) which contradicts to \( (U + V)(-\infty) = 1 \) and \( (U + V)' > 0 \) on \((-\infty, z_1)\). 

We now prove, in the case \( d \neq 1 \), the nonexistence of ultimately monotone \((\alpha, \beta)\)-traveling waves with speed \( c \neq 0 \). In view of Section 3.1, this is enough to complete the proof of Theorem 2.4.

Completion of the proof of Theorem 2.4. For \( d \neq 1 \), let us consider \((c, U, V)\) an ultimately monotone \((\alpha, \beta)\)-traveling wave with \( c \neq 0 \). In the sequel we only deal with the case \( c > 0 \) for which we perform a phase plane analysis around \(-\infty\) (for the case \( c < 0 \), one has to perform a phase plane analysis around \(+\infty\) with similar arguments). We define \( W := \alpha - U, P := U', R := V - 1 + \alpha, Q := V' \). Then we can rewrite (2.1) as

\[
\begin{align*}
W' &= -P, \\
P' &= -cP - (\alpha - W)(W - R), \\
R' &= Q, \\
Q' &= -\frac{c}{d}Q - \frac{r}{d}(R + 1 - \alpha)(W - R), \\
(W, P, R, Q)(-\infty) &= (0, 0, 0, 0), \\
(W, P, R, Q)(+\infty) &= (\alpha - \beta, 0, \alpha - \beta, 0).
\end{align*}
\]

(3.2)

Assume that \( W - R \) is ultimately nonnegative, that is

\[
\exists z^* < z_0, \forall z \leq z^*, (W - R)(z) \geq 0.
\]

(3.3)

From (3.1), we know that it holds either \( P > 0 \) or \( Q > 0 \) on \((-\infty, z_0)\). Moreover, from Lemma 3.1, \( \alpha - W = U \geq 0 \) and \( R + 1 - \alpha = V \geq 0 \). If \( P > 0 \) on \((-\infty, z_0)\), then from the \( P \)-equation in (3.2), we have \( P' < 0 \) on \((-\infty, z^*\) which contradicts to \( P > 0 \) on \((-\infty, z_0)\) and \( P(-\infty) = 0 \). If \( Q > 0 \) on \((-\infty, z_0)\), we similarly get a contradiction from the \( Q \)-equation. Hence (3.3) does not hold.

Assume that \( W - R = 1 - U - V \) is ultimately nonpositive, that is

\[
\exists z^* < z_0, \forall z \leq z^*, (W - R)(z) \leq 0.
\]

(3.4)

From (3.1) again, we know that it holds either \( P < 0 \) or \( Q < 0 \) on \((-\infty, z_0)\). If \( P < 0 \) on \((-\infty, z_0)\), then from the \( P \)-equation in (3.2), we have \( P' > 0 \) on \((-\infty, z^*\) which contradicts to \( P < 0 \) on \((-\infty, z_0)\) and \( P(-\infty) = 0 \). If \( Q < 0 \) on \((-\infty, z_0)\), we similarly get a contradiction from the \( Q \)-equation. Hence (3.4) does not hold.

As a result, since \( W - R = 1 - U - V \) is neither ultimately nonnegative nor ultimately nonpositive, we can find a local maximum point \( z_1 < z_0 \) and a local minimum point \( z_2 < z_0 \) such that

\[
(W - R)(z_1) > 0, (W - R)'(z_1) = 0, (W - R)''(z_1) \leq 0; \\
(W - R)(z_2) < 0, (W - R)'(z_2) = 0, (W - R)''(z_2) \geq 0.
\]

Note that

\[
(W - R)'' = cP + \frac{c}{d}Q + \left( \alpha - W + \frac{r}{d}(R + 1 - \alpha) \right)(W - R).
\]

From (3.1), it holds either \( Q > 0 \) or \( Q < 0 \) on \((-\infty, z_0)\). Let us first consider the case \( d < 1 \). If \( Q > 0 \) on \((-\infty, z_0)\), since \( (W - R)'(z_1) = 0 \) means \( (P + Q)(z_1) = 0 \), we have
\((cP + \frac{\gamma}{\delta}Q)(z_1) > 0\). Therefore, from the above equation, \((W - R)''(z_1) > 0\), which is a contradiction. On the other hand, if \(Q < 0\) on \((-\infty, z_0]\), since \((P + Q)(z_2) = 0\), we have \((cP + \frac{\gamma}{\delta}Q)(z_2) < 0\). Therefore, from the above equation, \((W - R)''(z_2) < 0\), which is a contradiction. Last, the case \(d > 1\) can be treated similarly.

Therefore, we conclude that system (1.1) does not admit any ultimately monotone \((x, \beta)\)-traveling wave. \(\square\)

4. The Cauchy problem

In this section, we consider system (1.1) with a compactly supported initial datum, and prove the propagation result, namely Theorem 2.6.

4.1. Preliminaries

Let us start by briefly recalling the competitive comparison principle. Define the operators

\[ N_1[u, v] := u_t - u_{xx} - u(1 - u - v) \quad \text{and} \quad N_2[u, v] := v_t - dv_{xx} - rv(1 - v - u). \]

Consider a domain \(\Omega := (t_1, t_2) \times (x_1, x_2)\) with \(0 \leq t_1 < t_2 \leq +\infty\) and \(-\infty \leq x_1 < x_2 \leq +\infty\). A (classical) super-solution is a pair \((\bar{u}, \bar{v})\) \(\in C^1((t_1, t_2), C^2((x_1, x_2))) \cap C_0(\Omega)\)

satisfying

\[ N_1[\bar{u}, \bar{v}] \geq 0 \quad \text{and} \quad N_2[\bar{u}, \bar{v}] \leq 0 \quad \text{in} \quad \Omega. \]

Similarly, a (classical) sub-solution \((u, v)\) requires \(N_1[u, v] \leq 0\) and \(N_2[u, v] \geq 0\).

Proposition 4.1 (Comparison principle). Let \((\bar{u}, \bar{v})\) and \((u, v)\) be a super-solution and sub-solution of system (1.1) in \(\Omega\), respectively. If

\[ \bar{u}(t_1, x) \geq u(t_1, x) \quad \text{and} \quad \bar{v}(t_1, x) \leq v(t_1, x), \quad \text{for all} \quad x \in (x_1, x_2), \]

and, for \(i = 1, 2\),

\[ \bar{u}(t, x_i) \geq u(t, x_i) \quad \text{and} \quad \bar{v}(t, x_i) \leq v(t, x_i), \quad \text{for all} \quad t \in (t_1, t_2), \]

then, it holds

\[ \bar{u}(t, x) \geq u(t, x) \quad \text{and} \quad \bar{v}(t, x) \leq v(t, x), \quad \text{for all} \quad (t, x) \in \Omega. \]

If \(x_1 = -\infty\) or \(x_2 = +\infty\), the hypothesis on the corresponding boundary condition can be omitted.

Denote \((u, v) = (u, v)(t, x)\) as the solution of (1.1) starting from \((u_0, v_0) = (u_0, v_0)(x)\) satisfying (2.5). Obviously, \((1, 0)\) is a super-solution while \((0, 1)\) is a sub-solution. It thus follows from (2.5), the comparison principle and the strong maximum principle that

\[ 0 < u(t, x) < 1 \quad \text{and} \quad 0 < v(t, x) < 1, \quad \text{for all} \quad t > 0, x \in \mathbb{R}. \quad (4.1) \]

Actually, the comparison principle also holds for the so-called generalized sub- and super-solutions. This is a rather well-known fact, and we refer to the clear exposition in [21, Section 2.1] for more details. In particular, if \((\bar{u}_1, \bar{v})\) and \((\bar{u}_2, \bar{v})\) are both classical sub-solutions, then \((\max(\bar{u}_1, \bar{u}_2), \bar{v})\) is a generalized sub-solution. Also, if \((u, \bar{v}_1)\) and
(\(u, v_2\)) are both classical sub-solutions, then \((u, \min(v_1, v_2))\) is a generalized sub-solution.

We now start the proof of Theorem 2.6. Observe that, by changing the variables as \(x = \sqrt{t} y\) and \(t = \frac{1}{q} s\), system (1.1) can be rewritten as

\[
\begin{align*}
\begin{cases} 
 v_s = v_{yy} + v(1 - v - u), & s > 0, y \in \mathbb{R}, \\
 u_s = d^{-1} u_{xx} + r^{-1} u(1 - u - v), & s > 0, y \in \mathbb{R}.
\end{cases}
\end{align*}
\]

Therefore, without loss of generality, we assume from now that

\[dr > 1 \quad (so \ that \ c_v > c_u),\]

and shall prove the statement (i) in Theorem 2.6.

Since \(c_v > c_u\) and \(u\) cannot propagate faster than \(c_u\), the behavior of the solution in the region \(|x| > c_u t\) is rather well-understood.

**Proposition 4.2** (Estimates in the region \(|x| > c_u t\)). We have

\[
\limsup_{t \to \infty, |x| \geq ct} u(t, x) = 0, \quad \text{for all } c > c_u, \quad (4.2)
\]

\[
\limsup_{t \to \infty, |x| \geq ct} v(t, x) = 0, \quad \text{for all } c > c_v, \quad (4.3)
\]

and

\[
\lim_{t \to \infty, c_1 t \leq |x| \leq c_2 t} \sup_{t} |1 - v(t, x)| = 0, \quad \text{for all } c_u < c_1 < c_2 < c_v. \quad (4.4)
\]

**Proof.** Without loss of generality, we only deal with the case \(x \geq 0\). Define

\[
\bar{u}(t, x) := C_1 e^{-\frac{u_0}{C_2}(x - c_0 t)} \quad \text{and} \quad \bar{v}(t, x) := C_2 e^{-\frac{v_0}{C_1}(x - c_1 t)},
\]

where \(C_1 > 0\) and \(C_2 > 0\) are chosen large enough so that \(\bar{u}(0, \cdot) \geq u_0\) and \(\bar{v}(0, \cdot) \geq v_0\). We can easily check that \((\bar{u}, 0)\) is a super-solution while \((0, \bar{v})\) is a sub-solution. As a result, we have

\[
0 < u(t, x) \leq \min\left\{1, C_1 e^{-\frac{u_0}{C_2}(x - c_0 t)}\right\} \quad \text{and} \quad 0 < v(t, x) \leq \min\left\{1, C_2 e^{-\frac{v_0}{C_1}(x - c_1 t)}\right\}, \quad (4.5)
\]

from which (4.2) and (4.3) follow.

Next, let \(c_u < c_1 < c_2 < c_v\) be given. Select \(0 < a < b\) and consider \((u^*, v^*)\) the solution of the strong-weak competition system

\[
\begin{align*}
\begin{cases} 
 u_t^* = u_{xx}^* + u^*(1 - u^* - av^*), \\
 v_t^* = dv_{xx}^* + rv^*(1 - v^* - bu^*),
\end{cases}
\end{align*}
\]

starting from \((u_0, v_0)\). Obviously, \((u^*, v^*)\) is a super-solution for system (1.1), and thus \(v^*(t, x) \leq v(t, x) \leq 1\) for all \(t \geq 0\) and \(x \in \mathbb{R}\). Since the statements (2) and (3) in [21, Theorem 1.1] imply

\[
\lim_{t \to \infty, c_1 t \leq x \leq c_2 t} \sup_{t} |1 - v^*(t, x)| = 0,
\]

the same conclusion holds for \(v\). \(\square\)
4.2. Construction of the super-solution

The goal of this subsection is to construct an adequate super-solution in, roughly speaking, the region $|x| < c_u t$. More precisely, let $\frac{1}{d} < r_1 < r$ be given and define $c^*_v := 2\sqrt{dr_1} < c_v$. In the sequel, we introduce $V_1$ as a traveling wave solution with speed $c^*_v = 2\sqrt{d}$ solving
\[
\begin{cases}
    dV''_1 + c^*_v V'_1 + r_1 V_1(1 - V_1) = 0, \\
    V_1(-\infty) = 1, \quad V_1(\infty) = 0.
\end{cases}
\] (4.7)

As well-known, $V'_1 < 0$ and there are $\lambda_1 > 0$ and $M_1 > 0$ such that
\[1 - V_1(\xi) \sim M_1 e^{\lambda_1 \xi} \quad \text{as} \quad \xi \to -\infty.\] (4.8)

Let us fix $c_u < c_1 < c^*_v$. For $T > 0$, we will work in the domain (which is “expanding in time”)
\[\Omega_1(T) := \{(t, x) \in (T, \infty) \times \mathbb{R} : |x| < c_1 t\}.\] (4.9)

It turns out that the construction of the super-solution is highly dependent on the value of $d$.

- The case $d \leq 1$. We introduce $s = s(t, x)$ as the solution of the Cauchy problem
\[
\begin{cases}
    s_t = s_{xx}, \\
    s(0, x) = s_0(x) := B_1 e^{-g|x|},
\end{cases}
\] (4.10)

and look for a super-solution $(\bar{u}, \bar{v})$ in the form
\[
\begin{cases}
    \bar{u}(t, x) := t^{-\frac{d}{1+d}}(1 - e^{-\tau})s(t, x), \\
    \bar{v}(t, x) := V_1(x - c^*_v t) + V_1(-x - c^*_v t) - 1 - \bar{u}(t, x).
\end{cases}
\] (4.11)

All parameters that will be determined below (namely $B_1, q$ and $\tau$) are positive, and $q < 1$.

- The case $d \geq 1$. We introduce $s = s(t, x)$ as the solution of the Cauchy problem
\[
\begin{cases}
    s_t = ds_{xx}, \\
    s(0, x) = s_0(x) := B_1 e^{-q|x|},
\end{cases}
\] (4.12)

and look for a super-solution $(\bar{u}, \bar{v})$ in the form
\[
\begin{cases}
    \bar{u}(t, x) := t^{-\frac{d}{1+d}}(1 - e^{-\tau})s(t, x), \\
    \bar{v}(t, x) := V_1(x - c^*_v t) + V_1(-x - c^*_v t) - 1 - \bar{u}(t, x).
\end{cases}
\] (4.13)

All parameters that will be determined below (namely $B_1, q$ and $\tau$) are positive, and $q < \frac{1}{d}$.

Obviously, (4.10)–(4.11) and (4.12)–(4.13) coincide when $d = 1$.

Proposition 4.3 (Super-solutions). The following holds.

i. Assume $d \leq 1$. Let $0 < q < 1$ and $0 < \tau < \lambda_1(c^*_v - c_1)$ be given. Then there exists $T^* > 0$ such that, for all $B_1 > 0$, $(\bar{u}, \bar{v})$, given by (4.10)–(4.11), is a super-solution in the domain $\Omega_1(T^*)$ as defined in (4.9).
ii. Assume $d \geq 1$. Let $0 < q < \frac{1}{d}$ and $0 < \tau < \lambda_1(c^*_v - c_1)$ be given. Then there exists $T^* > 0$ such that, for all $B_t > 0$, $(\bar{u}, \bar{v})$, given by (4.12)-(4.13), is a super-solution in the domain $\Omega^+_1(T^*)$ as defined in (4.9).

**Proof.** Since our super-solutions are even functions, it is enough to deal with $x \geq 0$. In other words, we work for $t \geq T$ (with $T > 0$ to be selected) and $0 \leq x < c_1 t$, with $c_u < c_1 < c^*_v$. For ease of notations, we shall use the shortcuts $\xi_\pm := \pm x - c^*_v t$. Since $\xi_- < -c^*_v t$ and $\xi_+ \leq -(c^*_v - c_1) t$, it follows from $V'_1 < 0$ and (4.8) that there exist $C_- > 0$ and $C_+ > 0$ such that, for $T > 0$ large enough,

$$1 - V_1(\xi_-) \leq C_- e^{-\lambda_1 c^*_v t} \quad \text{and} \quad 1 - V_1(\xi_+) \leq C_+ e^{-\lambda_1 (c^*_v - c_1)t}, \quad \text{for all } (t,x) \in \Omega^+_1(T),$$

where

$$\Omega^+_1(T) := \Omega_1(T) \cap (T, \infty) \times [0, \infty).$$

Moreover, up to enlarging $T > 0$ if necessary, there exists $0 < \rho < \frac{1}{\tau}$ such that

$$0 < 1 - V_1(\xi_\pm) \leq \rho, \quad \text{for all } (t,x) \in \Omega^+_1(T). \quad (4.15)$$

We first assume $d \leq 1$. Some straightforward computations combined with (4.10) yield

$$N_1[\bar{u}, \bar{v}] = t^{1-d} (1 - e^{-\tau t}) s \left( \frac{1 - d}{2} t^{-1} + \frac{\tau e^{-\tau t}}{1 - e^{-\tau t}} - 2 + V_1(\xi_+) + V_1(\xi_-) \right).$$

In view of (4.14), by choosing $\tau < \lambda_1(c^*_v - c_1)$, we deduce that, for $T > 0$ large enough, $N_1[\bar{u}, \bar{v}] \geq 0$ in $\Omega^+_1(T)$. On the other hand, some straightforward computations combined with (4.10) and (4.7) yield

$$N_2[\bar{u}, \bar{v}] = J_1 + J_2 + J_3,$$

where

$$J_1 := t^{1-d} (1 - e^{-\tau t}) s \left( r \left( 2 - V_1(\xi_+) - V_1(\xi_-) - \frac{\tau e^{-\tau t}}{r(1 - e^{-\tau t})} \right) - \frac{1 - d}{2} t^{-1} - (1 - d) \frac{s_t}{s} \right),$$

$$J_2 := (r_1 - r) V_1(\xi_+) (1 - V_1(\xi_+)),$n$$

$$J_3 := (1 - V_1(\xi_-)) ((r_1 - r)(1 - \rho) + r(2 - 2 V_1(\xi_+))).$$

Since $r_1 < r$ and $0 < V_1 < 1$, we have $J_2 \leq 0$. Next, from (4.15), we have

$$J_3 \leq (1 - V_1(\xi_-)) ((r_1 - r)(1 - \rho) + r(2 - 2 V_1(\xi_+))),$$

which, in view of (4.14), is nonpositive up to enlarging $T > 0$ if necessary. Last, from the “heat kernel expression” of $s(t, x)$, namely

$$s(t, x) = (G(t, \cdot) * s_0)(x), \quad \text{where} \quad G(t, x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}},$$

we can check that $s(t, x) \geq -\frac{1}{2t} s(t, x)$. As a result, since $d \leq 1$, we have

$$J_1 \leq t^{1-d} (1 - e^{-\tau t}) s r \left( 2 - V_1(\xi_+) - V_1(\xi_-) - \frac{\tau e^{-\tau t}}{r(1 - e^{-\tau t})} \right).$$
In view of (4.14) and \( \tau < \lambda_1(c_0^* - c_1) \), we have \( J_1 \leq 0 \) up to enlarging \( T > 0 \) if necessary. As a result, \( N_2[\tilde{u}, \tilde{v}] \leq 0 \) in \( \Omega_1^+(T) \).

Next, we assume \( d \geq 1 \). Some straightforward computations combined with (4.12) yield
\[
N_1[\tilde{u}, \tilde{v}] = t^{\frac{d}{d-1}}(1 - e^{-\tau t})s \left( \frac{d-1}{2d} t^{-1} + \frac{d-1}{d} s_t + \frac{\tau e^{-\tau t}}{1 - e^{-\tau t}} - 2 + V_1(\xi_+) + V_1(\xi_-) \right).
\]
As above, since \( d \geq 1 \), \( s_t(t, x) \geq -\frac{1}{2} s(t, x) \) implies
\[
N_1[\tilde{u}, \tilde{v}] \geq t^{\frac{d}{d-1}}(1 - e^{-\tau t})s \left( \frac{\tau e^{-\tau t}}{1 - e^{-\tau t}} - 2 + V_1(\xi_+) + V_1(\xi_-) \right).
\]
In view of (4.14) and \( \tau < \lambda_1(c_0^* - c_1) \), we deduce that, for \( T > 0 \) large enough, \( N_1[\tilde{u}, \tilde{v}] \geq 0 \) in \( \Omega_1^+(T) \). On the other hand, some straightforward computations combined with (4.12) and (4.7) yield
\[
N_2[\tilde{u}, \tilde{v}] = J_1 + J_2 + J_3,
\]
where
\[
J_1 := t^{\frac{d}{d-1}}(1 - e^{-\tau t})s \left( r(2 - V_1(\xi_+) - V_1(\xi_-)) - \frac{\tau e^{-\tau t}}{1 - e^{-\tau t}} - \frac{d-1}{2d} t^{-1} \right),
\]
\[
J_2 := (r_1 - r)V_1(\xi_+)(1 - V_1(\xi_+)),
\]
\[
J_3 := (1 - V_1(\xi_-))((r_1 - r)V_1(\xi_-) + r(2 - 2V_1(\xi_+))).
\]
By applying the same argument as that for \( d \leq 1 \), we get \( N_2[\tilde{u}, \tilde{v}] \leq 0 \) in \( \Omega_1^+(T) \).

Note that, time \( T^* \) in Proposition 4.3 is independent on \( B_1 > 0 \), which leaves “some room” to enlarge \( B_1 \) so that the “order on the boundary of the domain” and the “initial order” are suitable for the comparison principle to be applicable. In particular, we emphasize that choosing \( B_1 > 0 \) large enough will insure that \( \tilde{u} \geq 1 \) and \( \tilde{v} \leq 0 \) at the initial time \( T^* \).

**Proposition 4.4** (First estimate on \((u, v)\)). There exist \( 0 < q < \min(1, \frac{1}{d}) \), \( T^{**} > 0 \) and \( B_1 > 0 \) such that
\[
u(t, x) \leq \tilde{u}(t, x) \quad \text{all} \quad \forall (t, x) \leq v(t, x), \quad \text{for all} \quad t \geq T^{**}, |x| \leq c_1 t,
\]
where \((\tilde{u}, \tilde{v})\) is given by (4.10)–(4.11) when \( d \leq 1 \), and by (4.12)–(4.13) when \( d \geq 1 \).

**Proof.** We aim at applying the comparison principle in \( \Omega_1(T) \), as defined in (4.9), with a well-chosen \( T > 0 \). Select \( 0 < q \leq \min(1, \frac{1}{d}) \) small enough so that
\[
\max(qc_1 - q^2, qc_1 - dq^2) < c_1 - c_\nu.
\]
From Proposition 4.3, for any \( T \geq T^* \), we are equipped with a super-solution \((\tilde{u}, \tilde{v})\) for which \( B_1 > 0 \) is arbitrary. We only deal with the case \( d \leq 1 \), the case \( d \geq 1 \) being similar.

We first focus on \( x = c_1 t, t \geq T^* \) (the case \( x = -c_1 t, t \geq T^* \) being similar). Let us prove that, up to enlarging \( T^* \) if necessary, it holds
\[
u(t, c_1 t) \leq \tilde{u}(t, c_1 t), \quad \text{for all} \quad t \geq T^*.
\]
From the proof of Proposition 4.2, we know that \( u(t, c_1 t) \leq C_1 e^{-(c_1 - c_u)t} \) (recall that \( c_u = 2 \)). Recalling \( s_0(x) = B_1 e^{-q|x|} \), we have

\[
s(t, x) = \frac{B_1}{\sqrt{4\pi t}} \left( \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\sigma^2 t}} e^{-qy} dy + \int_{0}^{+\infty} e^{-\frac{(x-y)^2}{4\sigma^2 t}} e^{-qy} dy \right),
\]

which can be recast, after some elementary computations,

\[
s(t, x) = \frac{B_1}{\sqrt{\pi}} \left( e^{t-q|x|} \int_{\frac{2q-x}{2\sqrt{t}}}^{+\infty} e^{-w^2} dw + e^{t+q|x|} \int_{\frac{2q+x}{2\sqrt{t}}}^{+\infty} e^{-w^2} dw \right). \tag{4.18}
\]

In particular, since \( 2q < 2 < c_1 \), we have, by enlarging \( T^* > 0 \) if necessary,

\[
\bar{u}(t, c_1 t) \geq \frac{1}{2} s(t, c_1 t) \geq \frac{B_1}{4} e^{-(q c_1 - q^2)t} \geq \frac{B_1}{4} e^{-(c_1 - c_u)t}.
\]

The last inequality holds from the choice (4.16). Thus \( B_1 > 4C_1 \) is enough to get (4.17).

Let us recall that \( \nu \geq \nu^* \) where \( (\nu^*, \nu^*) \) is the solution of the strong-weak competition system (4.6) with the same initial datum \((u_0, v_0)\). From Lemma A.1(ii), up to enlarging \( T^* \) if necessary, there exist \( \mu > 0 \) and \( K > 0 \) such that \( \nu^*(t, c_1 t) \geq 1 - Ke^{-\mu t} \) for all \( t \geq T^* \).

On the other hand, the construction of \( \nu \) implies that \( \nu(t, c_1 t) \leq 1 - \bar{u}(t, c_1 t) \). Therefore, by choosing \( q c_1 - q^2 < \mu \), up to enlarging \( T^* > 0 \) if necessary, we have

\[
\nu(t, c_1 t) \leq \nu(t, c_1 t), \quad \text{for all } t \geq T^*. \tag{4.19}
\]

Now, \( q > 0 \) and \( T^* > 0 \) are fixed from the above discussion. We focus on the initial datum, namely \( t = T^* \), \( |x| \leq c_1 T^* \). As above, we deduce from (4.18) that

\[
\inf_{|x| \leq c_1 T^*} \bar{u}(T^*, x) \geq \frac{1}{2} s(T^*, c_1 T^*) \geq \frac{B_1}{4} e^{-(q c_1 - q^2)T^*} \geq \frac{1}{4} \sup_{t > 0, x \in \mathbb{R}} u(t, x),
\]

provided that \( B_1 \geq 4e^{(q c_1 - q^2)T^*} \). On the other hand,

\[
\sup_{|x| \leq c_1 T^*} \nu(T^*, x) \leq 1 - \inf_{|x| \leq c_1 T^*} \bar{u}(T^*, x) \leq 1 - \frac{B_1}{4} e^{-(q c_1 - q^2)T^*} \leq 0 \leq \inf_{t > 0, x \in \mathbb{R}} \nu(t, x),
\]

provided that \( B_1 \geq 4e^{(q c_1 - q^2)T^*} \).

As a consequence, the comparison principle can be applied in \( \Omega_1(T^*) \), which concludes the proof of Proposition 4.4.

\[\square\]

### 4.3. Proof of Theorem 2.6

From the discussion above, we are now in the position to obtain the following spreading speed result.

**Proposition 4.5 (Spreading speed).** Let \( (u, v) = (u, v)(t, x) \) be the solution of (1.1) starting from an initial datum \((u_0, v_0) = (u_0, v_0)(x)\) satisfying (2.5). Then the following holds.
i. Assume $dr > 1$ (i.e. $c_r > c_u$). Then, for any $0 < c_1 < c_r < c_2$,
\[
\lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}} u(t,x) + \sup_{|x| \leq c_1 t} |1 - v(t,x)| + \sup_{|x| \geq c_2 t} v(t,x) \right) = 0.
\] (4.20)

ii. Assume $dr < 1$ (i.e. $c_r < c_u$). Then, for any $0 < c_3 < c_r < c_4$,
\[
\lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}} v(t,x) + \sup_{|x| \leq c_3 t} |1 - u(t,x)| + \sup_{|x| \geq c_4 t} u(t,x) \right) = 0.
\] (4.21)

**Proof.** Let us prove (i). The result on $u$ in (4.20) is obtained by combining (4.2) and Proposition 4.4. Next, for a given $0 < c_1 < c_r$, we select $c_1 < c^*_r < c_r$. Then, Proposition 4.4 yields $\sup_{|x| \leq c_1 t} |1 - v(t,x)| \to 0$ as $t \to \infty$. The last part of (4.20) is nothing else than the estimate (4.3).

We are now in the position to complete the proof of Theorem 2.6.

**Proof of Theorem 2.6(i).** Since the proof for $x \leq 0$ follows from the same argument, we only deal with $x \geq 0$. Let us prove (2.6). For a given $m \in (0,1)$, we define $E_m(t)$ as the $m$-level set of $v(t,\cdot)$, namely
\[
E_m(t) := \{x > 0 : v(t,x) = m\}.
\]

We claim that there exist $M > 0$ and $T > 0$ such that
\[
c_r t - \frac{3d}{c_r} \ln t - M \leq \min E_m(t) \leq \max E_m(t) \leq c_r t - \frac{3d}{c_r} \ln t + M, \quad \text{for all } t \geq T.
\] (4.22)

The upper bound in (4.22) is obtained by using the solution of $\hat{v}_t = d\hat{v}_{xx} + r\hat{v}(1 - \hat{v})$, starting from $\hat{v}(0,x) = v_0(x)$, as a super-solution. We refer to [17, Lemma 4.1], see also [26, 27]. As for the lower bound in (4.22), it follows from [17, Lemma 4.5] which is based on an idea of [27]. A sketch of the proof is as follows. Let us fix a small $\varepsilon > 0$ (this is necessary because of the bump phenomenon). By combining (4.5), Proposition 4.4 and (4.18),\footnote{From which one can straightforwardly deduce that $\sup_{x \geq ct} s(t,x) = s(t,ct) = 0\left(\frac{\varepsilon}{\sqrt{ct}}\right)$.} we see that there exist $C > 0$, $\mu > 0$ and $T > 0$ such that
\[
\sup_{|x| \geq ct} u(t,x) \leq Ce^{-\mu t}, \quad \text{for all } t \geq T.
\] (4.23)

As a result, by setting $C_0 = rC$, we have
\[
v_t \geq dv_{xx} + v(r - rv - C_0 e^{-\mu t}), \quad \text{for all } t > 0, x > ct.
\]

The key idea, borrowed from [27], is then to linearize the above equation, and to consider
\[
w_t = dw_{xx} + w(r - C_0 e^{-\mu t}), \quad t > 0, x \geq \Gamma(t) := c_r t - \frac{3d}{c_r} \ln (t + t_0),
\] (4.24)

together with the Dirichlet boundary conditions $w(t,\Gamma(t)) = 0$ and a compactly supported initial datum $w(0,\cdot)$. Then, one can exactly reproduce the technical arguments of [17, Lemmas 4.3 and 4.4], mainly borrowed from [27], to obtain the lower bound in
(4.22). Last, by applying (4.22), we can reproduce the argument of [27, Section 4], see also [17, Proof of Theorem 2], to conclude that there exists a bounded function \( \eta_\ast : [0, \infty) \rightarrow \mathbb{R} \) such that

\[
\lim_{t \to \infty} \sup_{x \geq 0} \left| v(t, x) - V_1 \left( x - c_v t + \frac{3d}{c_v^2} \ln t + \eta_\ast(t) \right) \right| = 0,
\]

which, combined with (4.20), concludes the proof of (2.6). \( \square \)

5. The bump phenomenon

In this section, we will provide a lower estimate for the solution of system (1.1) starting from a compactly supported initial datum, and prove Theorem 2.7 on the bump phenomenon.

5.1. Construction of the sub-solution

The goal of this subsection is to construct an adequate sub-solution in, roughly speaking, the region \(|x| < c_v t\). More precisely, let \( r_2 > r \) be given and define \( c_v^* := 2\sqrt{dr_2} > c_v \). Let us fix \( c_v < c_2 < c_v^* \). For \( T > 0 \), we will work in the domain (which is “expanding in time”)

\[
\Omega_2(T) := \{(t, x) \in (T, \infty) \times \mathbb{R} : |x| < c_2 t\}. \tag{5.1}
\]

Let us start with the following formal observation. From Theorem 2.6, in the region \(|x| < (c_v - \varepsilon)t\) with some small \( \varepsilon > 0 \), we have \( u + v \approx 1 \) so that

\[
u_t \approx u_{xx}, \quad |x| \leq (c_v - \varepsilon)t.
\]

Moreover, we expect that \( u \) converges to 0 exponentially in the region \((c_u + \varepsilon)t < |x| < (c_v - \varepsilon)t\), so that, for some small \( \rho > 0 \),

\[
u(t, x) \leq e^{-\rho t}, \quad |x| = (c_v - \varepsilon)t.
\]

We thus introduce \( f = f(t, x) \) and \( h = h(t, x) \) the solutions of the Cauchy problems

\[
\begin{align*}
    f_t &= f_{xx}, \\
    f(0, x) &= f_0(x) := B_2 \mathbf{1}_{(-1,1)}(x), \\
    h_t &= h_{xx}, \\
    h(0, x) &= h_0(x) := B_3 e^{-k|x|},
\end{align*}
\]

and look for a sub-solution \((u, \tilde{v})\) in the form

\[
\begin{align*}
u(t, x) &= g(t)f(t, x) - h(t, x), \\
\tilde{v}(t, x) &= V_2(x - c_v^* t - \zeta_0) + V_2(-x - c_v^* t - \zeta_0) - 1 - u(t, x) + \frac{1}{t^{1+\theta}},
\end{align*}
\]

where

\[
g(t) := \exp \frac{1}{\delta(1 + t)^{\delta}}.
\]

Here, all parameters that will be determined below (namely \( B_2, B_3, k, \zeta_0, \theta, \delta \)) are positive, \( B_2 < 1, B_3 < 1 \), while \( V_2 \) is the traveling wave solution with speed \( c_v^* = 2\sqrt{dr_2} \) satisfying
\[
\begin{align*}
&\left\{ \begin{array}{l}
\frac{dV''_2}{dt} + c''_2 V'_2 + r_2 V_2 (1 - V_2) = 0, \\
V_2(-\infty) = 1, V_2(\infty) = 0.
\end{array} \right.
\end{align*}
\] (5.4)

It is well-known that \( V'_2 < 0 \) and there exist \( \lambda_2 > 0 \) and \( M_2 > 0 \) such that
\[1 - V_2(\zeta) \sim M_2 e^{\lambda_2 \zeta} \quad \text{as} \quad \zeta \to -\infty. \] (5.5)

Next, we shall provide some estimates which are based on the “heat kernel expressions” of the solutions \( f \) and \( h \) of (5.2). Note that, in Lemma 5.1, \( 0 < B_2 < 1 \) and \( 0 < B_3 < 1 \) can be relaxed to \( B_2 > 0 \) and \( B_3 > 0 \).

**Lemma 5.1.** Let \( \delta > 0 \) and \( k > 0 \) be given, and set \( B_3 = \gamma B_2 \) with some \( \gamma > 0 \). Then the following holds.

i. For any given \( 0 < j < k \),
\[h(t,x) \leq \frac{B_3}{\sqrt{\pi}} \frac{k}{k^2 - j^2} t^{-\frac{j}{2}} e^{-\frac{x^2}{4t}}, \quad \text{for all} \quad t > 0, |x| \leq 2jt.
\]

ii. For any given \( 0 < j < k \) and \( T > 0 \), there exists \( \gamma_1 > 0 \) such that, if \( 0 < \gamma \leq \gamma_1 \), then, for all \( B_2 > 0 \),
\[g(t)f(t,x) - h(t,x) > 0, \quad \text{for all} \quad t \geq T, |x| \leq 2jt.
\]

iii. There is \( T^0 > 0 \) such that, for all \( B_2 > 0 \),
\[g(t)f(t,x) - h(t,x) \leq 0, \quad \text{for all} \quad t \geq T^0, |x| = 2kt.
\]

**Proof.** Since \( f(t,\cdot) \) and \( h(t,\cdot) \) are even functions, it is enough to deal with \( x \geq 0 \). Recalling that \( h_0(x) = B_3 e^{-k|x|} \), we have
\[h(t,x) = \frac{B_3}{\sqrt{4\pi t}} \left( \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} e^{ky} dy + \int_{0}^{+\infty} e^{-\frac{(x-y)^2}{4t}} e^{-ky} dy \right),
\]
which can be recast, after some elementary computations,
\[h(t,x) = \frac{B_3}{\sqrt{\pi}} \left( e^{k^2 t - kx} \int_{2\sqrt{t}}^{+\infty} e^{-w^2} dw + e^{k^2 t + kx} \int_{-\frac{2\sqrt{t}}{k^2}}^{+\infty} e^{-w^2} dw \right). \] (5.6)

Now, recalling that \( \int_{X}^{+\infty} e^{-w^2} dw \leq \frac{e^{X^2}}{2X} \) for any \( X > 0 \), the above expression implies that, for any \( 0 \leq x \leq 2jt < 2kt \),
\[h(t,x) \leq \frac{B_3}{\sqrt{\pi}} \frac{e^{\frac{x^2}{8t}}}{2kt - x} \left( \sqrt{t} + \frac{\sqrt{t}}{2kt + x} \right) \leq \frac{B_3}{\sqrt{\pi}} \frac{k}{k^2 - j^2} t^{-\frac{j}{2}} e^{-\frac{x^2}{4t}},
\]
which proves (i).

Recalling that \( f_0(x) = B_2 \mathbf{1}_{(-1,1)}(x) \), we have
\[f(t,x) = \frac{B_2}{\sqrt{4\pi t}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{4t}} dy = \frac{B_2}{\sqrt{\pi}} \int_{-\frac{1}{2\sqrt{t}}}^{\frac{1}{2\sqrt{t}}} e^{-w^2} dw. \] (5.7)
Lemma 5.3. There exists \( C \) which is nonpositive, up to increasing 
\( B \left( \| j \|_{\infty} t^{-1} e^{-|x|^2 / 4} - \gamma \frac{k}{k^2 - j^2} t^{-1} e^{-x^2 / 4} \right) \)
\[\geq \frac{B_2}{\sqrt{\pi}} t^{-1} e^{-x^2 / 4} \left( e^{-\frac{1}{2} \| j \|_{\infty} t^{-1} e^{-|x|^2 / 4}} - \gamma \frac{k}{k^2 - j^2} t^{-1} e^{-x^2 / 4} \right),\]  
which is enough to prove (ii).

From (5.6), we have \( h(t, 2kt) \geq \frac{B_1}{\sqrt{\pi}} \int_0^{t/\sqrt{2}} e^{-w^2} dw = \frac{B_1}{\sqrt{\pi}} e^{-k^2t} t^{1/2} \). Hence, from \( B_3 = \gamma B_2 \) and (5.7), we deduce that, for \( t \geq T^0 := \frac{1}{2k} \),
\[g(t)f(t, 2kt) - h(t, 2kt) \leq \frac{B_2}{\sqrt{\pi}} \left( \| g \|_{\infty} t^{-1} e^{-|x|^2 / 4} - \gamma \frac{\sqrt{\pi}}{2} e^{-k^2t} \right) \]
\[\leq \frac{B_2}{\sqrt{\pi}} e^{-k^2t} \left( \| g \|_{\infty} e^k t^{-1} - \gamma \frac{\sqrt{\pi}}{2} \right),\]
which is nonpositive, up to increasing \( T^0 \) if necessary. The proof of (iii) is complete. \( \square \)

Remark 5.2. The statement (ii) in Lemma 5.1 guarantees that \( \bar{u}(t, x) = g(t)f(t, x) - h(t, x) \) is not a trivial sub-solution if \( \gamma \) is small enough.

Lemma 5.3. There exists \( C = C(k) > 0 \) such that
\[|f(t, x)| + |h(t, x)| \leq C t^{-1}, \quad \text{for all } t > 0, x \in \mathbb{R},\]  
and
\[|f_t(t, x)| + |h_t(t, x)| \leq C t^{-1}, \quad \text{for all } t > 0, x \in \mathbb{R}.\]  

Proof. This proof is very classical. Denoting \( G(t, x) := \frac{1}{\sqrt{4\pi t}} e^{-x^2 / 4} \), we have \( f(t, x) = (G(t, \cdot) * f_0)(x) \), and thus
\[|f(t, x)| \leq \| G(t, \cdot) \|_{\infty} \| f_0 \|_1 \leq C_G t^{-\frac{1}{2}} \| f_0 \|_1,\]
with some \( C_G > 0 \). Also, we have \( f_t(t, x) = (G_t(t, \cdot) * f_0)(x) \), and thus
\[|f_t(t, x)| \leq \| G_t(t, \cdot) \|_{\infty} \| f_0 \|_1 \leq C'_G t^{-\frac{1}{2}} \| f_0 \|_1,\]
with some \( C'_G > 0 \). Note that, \( \| f_0 \|_1 = 2B_2 \), which implies \( C \) is independent on \( B_2 < 1 \) in (5.9) and (5.10).

Since \( h_0 \in L^1(\mathbb{R}) \), similar estimates hold for \( h(t, x) \) and \( h_t(t, x) \), and \( C = C(k) \) since \( \| h_0 \|_1 = \frac{2B_2}{k} \leq \frac{3}{k} \).

We are now in the position to complete the construction of the sub-solution \((\bar{u}, \bar{v})\) in the form (5.3).

Proposition 5.4 (Sub-solutions). Let \( 0 < \delta < \theta < \frac{1}{2} \) be given. Let us fix \( k > 0 \), and set \( B_3 = \gamma B_2 \) with \( 0 < \gamma < 1 \).

Then there exists \( T^* > 0 \) such that, for all \( 0 < B_2 < 1 \) and \( \zeta_0 > 0 \), \((\bar{u}, \bar{v})\) is a sub-solution in the domain \( \Omega_2(T^*) \) as defined in (5.1).
Proof. Since $u(t, \cdot)$ and $\bar{v}(t, \cdot)$ are even functions, it is enough to deal with $x \geq 0$. In other words, we work for $t \geq T$ (with $T > 0$ to be selected) and $0 \leq x < c_2 t$, with $c_2 < c^{\ast}_{v}$. For simplicity of notations, we shall use the shortcuts $\bar{\xi}_\pm := \mp x - c^{\ast}_{v} t - \zeta_0$. Since $\bar{\xi}_- \leq -c^{\ast}_{v} t$ and $\bar{\xi}_+ \leq -(c^{\ast}_{v} - c_2) t$, it follows from $V_2 < 0$ and (5.5) that there exist $C_- > 0$ and $C_+ > 0$ such that, for $T > 0$ large enough,

$$1 - V_2(\bar{\xi}_-) \leq C_- e^{-\frac{c_2}{2} t} \quad \text{and} \quad 1 - V_2(\bar{\xi}_+) \leq C_+ e^{-\frac{c_2}{2} (c^{\ast}_{v} - c_2) t}, \quad \text{for all } (t, x) \in \Omega^+_2(T), \quad (5.11)$$

where $\Omega^+_2(T) := \Omega_2(T) \cap (T, \infty) \times [0, \infty)$. Moreover, up to enlarging $T > 0$ if necessary, there exists $0 < \rho < \frac{1}{3}$ such that

$$0 < 1 - V_2(\bar{\xi}_-) \leq \rho, \quad \text{for all } (t, x) \in \Omega^+_2(T). \quad (5.12)$$

Some straightforward computations combined with (5.2) yield

$$N_1[y, \bar{v}] = \frac{g'f - (gf - h)(2 - V_2(\bar{\xi}_+) - V_2(\bar{\xi}_-) - t^{-(1+\delta)})}{-\frac{1}{2} (1 + \delta) t^{-(1+\delta)}} + \frac{gft^{-(1+\delta)}}{2} + h\left(2 - V_2(\bar{\xi}_+) - V_2(\bar{\xi}_-) - t^{-(1+\delta)}\right),$$

since $2 - V_2(\bar{\xi}_+) - V_2(\bar{\xi}_-) > 0$. Thus, it follows from (5.11) that

$$N_1[y, \bar{v}] \leq \left(-\frac{1}{2} (1 + \delta) t^{-(1+\delta)} + t^{-(1+\delta)} + \frac{C_- e^{-\frac{c_2}{2} t} + C_+ e^{-\frac{c_2}{2} (c^{\ast}_{v} - c_2) t} - t^{-(1+\delta)}}{t^{-(1+\delta)}}.\right)$$

Since $\delta < \theta$, it follows that, for $T > 0$ large enough, $N_1[y, \bar{v}] \leq 0$ in $\Omega^+_2(T)$.

Next, some straightforward computations combined with (5.2) and (5.4) yield

$$N_2[y, \bar{v}] = r_2 V_2(\bar{\xi}_+)(1 - V_2(\bar{\xi}_+)) + r_2 V_2(\bar{\xi}_-)(1 - V_2(\bar{\xi}_-)) + (1 + t)^{(1+\delta)} \frac{gf}{t^{-(1+\delta)}}$$

$$+ \frac{(d-1)(gf - h)}{t^{-(1+\delta)}} - (1 + \frac{\delta}{r} t^{-(1+\delta)})$$

$$- r(V_2(\bar{\xi}_+) + V_2(\bar{\xi}_-) - 1 - gf + h + t^{-(1+\delta)}(2 - V_2(\bar{\xi}_+) - V_2(\bar{\xi}_-) - t^{-(1+\delta)})$$

$$= I_1 + \cdots + I_5,$$

where

$$I_1 := r g f \left(2 - V_2(\bar{\xi}_+) - V_2(\bar{\xi}_-) + \frac{1}{r} (1 + t)^{-(1+\delta)}\right),$$

$$I_2 := (1 - V_2(\bar{\xi}_-)) \left((r_2 - r) V_2(\bar{\xi}_-) + r \left(2 - 2 V_2(\bar{\xi}_+) - t^{-(1+\delta)} - h\right)\right),$$

$$I_3 := (1 - V_2(\bar{\xi}_+)) \left((r_2 - r) V_2(\bar{\xi}_+) - rh\right),$$

$$I_4 := (d-1)(gf - h) t^{-(1+\delta)},$$

$$I_5 := r t^{-(1+\delta)} \left(V_2(\bar{\xi}_-) + V_2(\bar{\xi}_+) - 2 + t^{-(1+\delta)} - gf + h - \frac{1 + \theta}{r} t^{-(1+\delta)}\right).$$

Since $0 < V_2 < 1$, we have $I_1 \geq 0$. From $r_2 > r$, (5.9) and (5.12), we have

$$I_2 \geq (1 - V_2(\bar{\xi}_-)) \left((r_2 - r)(1 - \rho) - r t^{-(1+\delta)} - r C \ell^{-\frac{1}{2}}\right) \geq 0,$$

up to enlarging $T > 0$ if necessary. Similarly, we obtain $I_3 \geq 0$. Last, from (5.12) and Lemma 5.3, we obtain
\[ I_4 + I_5 \geq r^{-(1+\theta)} \left( 1 - 3\rho - \|g\|_{\infty} C t^{-\frac{1}{2}} - \frac{1+\theta}{r} t^{-1} \right) - C |d - 1| (\|g\|_{\infty} + 1) t^{-\frac{3}{2}}. \]

Since \( \theta < \frac{1}{2} \) and \( 0 < \rho < \frac{1}{3} \), we have \( I_4 + I_5 \geq 0 \) up to enlarging \( T > 0 \) if necessary. As a result, for \( T > 0 \) large enough, \( N_2[u, v] \geq 0 \) in \( \Omega_+^0(T) \), and the proof of Proposition 5.4 is complete.

Note that, time \( T^* \) in Proposition 5.4 is independent on \( 0 < B_2 < 1 \) and \( \zeta_0 > 0 \), which leaves “some room” to reduce \( B_2 \) and to enlarge \( \zeta_0 \) so that the “initial order” is suitable for the comparison principle to be applicable. We shall also need the suitable “order on the boundary of the domain,” which will be obtained by choosing an appropriate \( k \) and Lemma 5.1(iii). More precisely, the following holds.

**Proposition 5.5** (Second estimate on \( (u, v) \)). Let \( 0 < \delta < \theta < \frac{1}{2} \) be given. Let us fix \( k := \frac{\delta}{2} > 0 \), and set \( B_3 = \gamma B_2 \) with \( 0 < \gamma < 1 \).

Then there exist \( T^{**} > 0 \), \( 0 < B_2 < 1 \) and \( \zeta_0 > 0 \) such that
\[
u(t, x) \leq u(t, x) \quad \text{and} \quad v(t, x) \leq \tilde{v}(t, x), \quad \text{for all} \ t \geq T^{**}, |x| \leq c_2 t,
\]
where \( (u, \tilde{v}) \) is given by (5.3).

**Proof.** We aim at applying the comparison principle in \( \Omega_2(T) \), as defined in (5.1), with a well-chosen \( T > 0 \). From Proposition 5.4, for any \( T \geq T^* \), we are equipped with a sub-solution \( (u, \tilde{v}) \) for which \( 0 < B_2 < 1 \) and \( \zeta_0 > 0 \) are arbitrary.

We now focus on \( |x| = c_2 t \), \( t \geq T \), that is, with a slight abuse of language, the boundary of \( \Omega_2(T) \). We now set \( T^{**} := \max(T^*, T^0) \), where \( T^0 > 0 \) is provided by Lemma 5.1(iii). In particular this implies (recall the choice \( k = \frac{\delta}{2} \)) that
\[ u(t, \pm c_2 t) \geq 0 \leq u(t, \pm c_2 t), \quad \text{for all} \ t \geq T^{**}. \]

Next, recalling that \( c_v < c_2 < c_v^{**} \), it follows from Proposition 4.2 that \( v(t, \pm c_2 t) \to 0 \) as \( t \to \infty \). On the other hand, for any \( t \geq T^{**} \),
\[ v(t, \pm c_2 t) \geq V_2(\pm (c_v^{**} - c_2) t) + V_2(-c_2 t) - 1. \]

As a result, up to enlarging \( T^{**} \) if necessary, one has
\[ v(t, \pm c_2 t) \leq \tilde{v}(t \pm c_2 t), \quad \text{for all} \ t \geq T^{**}. \]

Last we focus on the initial data, namely \( t = T^{**} \) and \( |x| \leq c_2 T^{**} \). From (4.1), there exists \( \varepsilon > 0 \) such that
\[ \min \left( \inf_{|x| \leq c_2 T^{**}} u(T^{**}, x), \inf_{|x| \leq c_2 T^{**}} 1 - v(T^{**}, x) \right) \geq \varepsilon > 0. \]

We now select \( 0 < B_2 < \frac{\varepsilon}{2 \|g\|_{\infty}} \). From this choice, we have
\[ u(T^{**}, x) \leq \|g\|_{\infty} B_2 \leq \frac{\varepsilon}{2} \leq u(T^{**}, x), \quad \text{for all} \ |x| \leq c_2 T^{**}, \]
and, for all \( |x| \leq c_2 T^{**} \),
\[ \tilde{v}(T^{**}, x) \geq 2V_2(-\zeta_0) - 1 - \|g\|_{\infty} B_2 \geq 2V_2(-\zeta_0) - 1 - \frac{\varepsilon}{2} \geq 1 - \varepsilon, \]
by selecting \( \zeta_0 > 0 \) large enough, which implies
\[ v(T^{**}, x) \leq \bar{v}(T^{**}, x), \quad \text{for all } |x| \leq c_2 T^{**}. \]

As a consequence, the comparison principle can be applied in \( \Omega_2(T^{**}) \), which concludes the proof of Proposition 5.5.

\[ \square \]

### 5.2. Proof of Theorem 2.7

It remains to prove the bump phenomenon which, as explained in Section 2, is reserved to the critical competition case under consideration. Let \( 0 < \varepsilon < c_u \) be given and let us prove (2.8) and (2.9). Let us set \( T^{**} > 0 \) such that both Proposition 5.5 and Proposition 4.4 apply. In the sequel, we always consider \( t \geq T^{**} \) and \( 0 \leq x \leq \varepsilon t \).

In particular, one has

\[ g(t)f(t, x) - h(t, x) = u(t, x) \leq u(t, x) \leq \bar{u}(t, x) = t^{-\left(k^* - \frac{1}{2}\right)}(1 - e^{-t})s(t, x). \]

This estimate and (5.8) (with \( j = \frac{d}{2} \)) yield the lower estimate in (2.8). On the other hand, Lemma 5.1(i) (with \( j = \frac{d}{2} \)) provides an upper bound of the form \( t^{-\frac{1}{2}}e^{-\frac{d}{2}} \) in the case \( d \leq 1 \) (since then \( s_t = s_{xx} \)) and of the form \( t^{-\frac{1}{2}}e^{-\frac{d}{2}} \) in the case \( d \geq 1 \) (since then \( s_t = ds_{xx} \)). This gives the upper estimate in (2.8).

Similarly, one obtains

\[
\begin{align*}
\bar{u}(t, x) - t^{-\left(1+\theta\right)} &\leq 1 - \bar{v}(t, x) \leq 1 - v(t, x) \\
&\leq 1 - v(t, x) \\
&\leq 2 - V_1(-c_v^* t) - V_1(-(c_v^* - \varepsilon)t) + \bar{u}(t, x),
\end{align*}
\]

which gives the lower estimate in (2.9). On the other hand, we deduce from (4.8) that there exist \( C_- > 0 \) and \( C_+ > 0 \) such that, for all \( 0 \leq x \leq \varepsilon t \),

\[ 1 - v(t, x) \leq C_- e^{-\theta t} + C_+ e^{-\theta t}(-c_v^* - \varepsilon)^t + \bar{u}(t, x) \leq Ct^{-k^*} e^{-\frac{d}{2}t} + \bar{u}(t, x), \]

with some \( C > 0 \) provided \( \varepsilon > 0 \) is chosen sufficiently small. This gives the upper estimate in (2.9) and concludes the proof of Theorem 2.7.

**Remark 5.6.** Note that, in Theorem 2.7, the orders of the upper and lower estimates coincide if and only if \( d = 1 \). In order to get a sharp estimate in the case \( d \neq 1 \), we believe that one should improve the super-solution (4.11), or (4.13). However the attempt \( \bar{u}(t, x) = (1 - e^{-rt})s(t, x) \) is unlikely to work. Indeed, one should then estimate the term \( (d - 1)s_t \). The problem is that the constant \( C \) appearing in (5.10) depends on the \( L^1 \) norm of the initial datum, and thus on \( B_1 \). As a result the time \( T^* \) that would appear in Proposition 4.3 could not be chosen independent on \( T^* \), which would prevent the use of comparison arguments in the proof of Proposition 4.4.

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Appendix: A result on the strong-weak competition system

In this appendix, we consider the strong-weak competition system \((0 < a < 1 < b)\)

\[
\begin{aligned}
  u_t &= u_{xx} + u(1 - u - av), \\
  v_t &= dv_{xx} + rv(1 - v - bu),
\end{aligned}
\]

supplemented with an initial datum \((u_0, v_0)\) satisfying (2.5), for which we need a technical result, namely Lemma A.1, which is inspired by [17, Lemma 2.8] and the forthcoming work [25].

When \(c_r = 2\sqrt{r/d} > c_s = 2\), as proved in [21], the spreading properties are rather subtle: the rapid competitor \(v\) invades first at speed \(c_r\) and is then replaced by the strong competitor \(u\) at a speed \(C\) which can take two different values. To make this clear, we quote the following from [21] to which we refer for more details. First, the strong-weak system admits a minimal monotone traveling wave solution \((c_{LLW}, U, V)\) with speed \(2\sqrt{1 - a} \leq c_{LLW} \leq 2\), defined as

\[
\begin{aligned}
  U'' + c_{LLW} U' + U(1 - U - aV) &= 0, \\
  dV'' + c_{LLW} V' + rV(1 - V - bU) &= 0, \\
  (U, V)(-\infty) &= (1, 0), \\
  (U, V)(\infty) &= (0, 1), \\
  U' < 0, V' > 0.
\end{aligned}
\]

Next define the decreasing function \(f : [2\sqrt{1 - a}, +\infty) \to (2\sqrt{a}, 2(\sqrt{1 - a} + \sqrt{a})]\)

\[
f(c) := c - \sqrt{c^2 - 4(1 - a) + 2\sqrt{a}} \quad \text{so that} \quad f^{-1}(\tilde{c}) := \frac{\tilde{c}}{2} - \sqrt{\tilde{a} + \frac{2(1 - a)}{\tilde{c} - 2\sqrt{a}}}.
\]
If \(2\sqrt{rd} \in (2, f(c_{LLW}))\), then define
\[
e_{nlp} := f^{-1}(2\sqrt{rd}) = \sqrt{rd} - \sqrt{a} + \frac{1 - a}{\sqrt{rd} - \sqrt{a}} \in (c_{LLW}, 2).
\]
Then we can precisely state
\[
C = \begin{cases} 
e_{nlp} & \text{if } 2 < c_v < f(c_{LLW}), \
\text{(nonlocal pulling phenomenon),} & \\
c_{LLW} & \text{if } c_v \geq f(c_{LLW}).
\end{cases}
\]

We now state the result used in the proof of Proposition 4.4.

**Lemma A.1.** Assume \(dr > 1\) (i.e. \(c_v > c_u\)). Let \((u, v) = (u, v)(t, x)\) be the solution of the strong-weak competition system (A.1) starting from an initial datum \((u_0, v_0) = (u_0, v_0)(x)\) satisfying (2.5). Then the following holds.

i. For any \(c > c_v\), there exist \(C_1 > 0, \nu_1 > 0, T_1 > 0\) such that
\[
\sup_{|x| \geq c} u(t, x) \leq C_1 e^{-\nu_1 t}, \quad \text{for all } t \geq T_1.
\]

ii. For any \(c_1, c_2\) with \(c_v < c_1 < c_2 < c_v\), there exist \(C_2 > 0, \nu_2 > 0, T_2 > 0\) such that
\[
\sup_{c_1 \leq |x| \leq c_2} v(t, x) \geq 1 - C_2 e^{-\nu_2 t}, \quad \text{for all } t \geq T_2.
\]

**Proof.** Let us briefly start with (i). If \(c > c_u\) then the conclusion is clear by the same argument as in Proposition 4.2. Since \(c_v > c_u\) it thus suffices to consider the case \(c_v < c < c_v\). In this case, the conclusion is already included in [21, Proposition 1.5] and the proof of [21, Section 3.2.3, Theorem 1.1]. We do not present the full details but only emphasize that a key tool is, for any small \(\delta > 0\), the minimal monotone traveling wave of the perturbed system

\[
\begin{aligned}
U'' + cU' + U(1 + \delta - U - aV) &= 0, \\
dV'' + cV' + rV(1 - 2\delta - V - bU) &= 0, \\
(U, V)(-\infty) &= (1 + \delta, 0), (U, V)(\infty) = (0, 1 - 2\delta), \\
U' &= 0, V' > 0.
\end{aligned}
\]

Let us now turn to (ii) (which is the estimate that we need in the proof of Proposition 4.4) for which the above perturbation argument seems unsipplicable. Let \(C < c_1 < c_2 < c_v\) be given. We only deal with \(x \geq 0\). From [21, Theorem 1.1] we know
\[
\lim_{t \to \infty} \sup_{c_1 t \leq x \leq c_2 t} (u(t, x) + |1 - v(t, x)|) = 0.
\]

From this and (i), one can choose \(\varepsilon > 0\) small enough and \(T_0 \gg 1\) such that
\[
0 < u(t, x) \leq C_1 e^{-\nu_1 t}, \quad v(t, x) > 1 - \varepsilon, \quad \text{for all } t \geq T_0, x \in [c_1 t, c_2 t].
\]

From the \(v\)-equation in (A.1), we have
\[
v_t \geq d_{xx} + r(1 - v)(1 - v) - rbC_1 v e^{-\nu_1 t}, \quad \text{for all } t \geq T_0, x \in [c_1 t, c_2 t]. \tag{A.2}
\]

Defining
\[
\bar{v}(t, x) := v(t, x + \bar{c}t), \quad \bar{c} := \frac{c_1 + c_2}{2}, \tag{A.3}
\]

it follows from (A.2) that
\[
\bar{v}_t \geq d \bar{v}_{xx} + \bar{c} \bar{v}_x + r(1 - \bar{v})(1 - \bar{v}) - rbC_1 \bar{v} e^{-\nu_1 t}, \quad \text{for all } t \geq T_0, x \in [-c_3 t, c_3 t],
\]
where \(c_3 := \frac{\alpha - c_1}{2} \).
To estimate $\tilde{v}$, for any $T \geq T_0$, we define
\[
\varphi(t) := 1 + \frac{bC_1}{1-e^{-\nu_1 t}}e^{-\nu_1 t(T-t)}, \quad \text{for all } t \geq 0.
\]
Up to enlarging $T > 0$ if necessary, we may assume $\varphi(0) < \frac{1}{1-e}$. Now, let us first consider the auxiliary problem
\[
\begin{align*}
\varphi_t &= d\varphi_{xx} + \varphi_t + r(1-\varepsilon)[1 - \varphi(t)]\varphi, \quad t > 0, -c_3 T < x < c_3 T, \\
\varphi(t, \pm c_3 T) &= 1 - \varepsilon, \quad t > 0, \\
\varphi(0, x) &= 1 - \varepsilon, \quad -c_3 T \leq x \leq c_3 T.
\end{align*}
\] (A.4)

Letting
\[
\Phi(t, x) := Q(t) := r(1-\varepsilon)\varphi(t) - \frac{rbC_1}{\nu_1}e^{-\nu_1 t(T-t)},
\]
so that $Q(t) = r(1-\varepsilon)\varphi(t)$, it follows from (A.4) that
\[
\begin{align*}
\Phi_t &= d\Phi_{xx} + \Phi_t + r(1-\varepsilon)e^{-Q(t)}(1 - (1-\varepsilon)\varphi(t)), \quad t > 0, -c_3 T < x < c_3 T, \\
\Phi(t, \pm c_3 T) &= 0, \quad t > 0, \\
\Phi(0, x) &= 0, \quad -c_3 T \leq x \leq c_3 T.
\end{align*}
\] (A.5)

Up to a rescaling, we may assume $d = 1$ so that (A.5) is very comparable to [28, problem (3.12)] on which we now rely. Denoting $G_1(t, x, x_1)$ the Green function of [28, p. 53] (with obvious changes of constants), we obtain the analogous version of [28, (3.14)], namely
\[
\begin{align*}
\Phi(t, x) &\geq r(1-\varepsilon)\int_0^t e^{-Q(t)}(1 - (1-\varepsilon)\varphi(s)) \left( \int_{-c_3 T}^{c_3 T} G_1(t - s, x, x_1) dz \right) ds,
\end{align*}
\]
for all $t > 0, -c_3 T < x < c_3 T$. Next, for any small $0 < \delta < 1$, we define
\[
D_\delta := \{ (t, x) \in \mathbb{R}^2 : 0 < t < \delta c_3 T, |x| < (1-\delta)c_3 T \}.
\]

From the same process used in [28, pp. 54–55], there exist $C_3, C_4 > 0$ such that the following lower estimate holds
\[
\Phi(t, x) \geq r(1-\varepsilon)(1 - (1-\varepsilon)\varphi(0))(1 - C_3e^{-C_4 t}) \int_0^t e^{-Q(t)} ds, \quad \text{for all } (t, x) \in D_\delta,
\]
resulting in
\[
\varphi(t, x) \geq \Psi(t)(1 - C_3e^{-C_4 t})(1 - (1-\varepsilon)\varphi(0)) + 1 - \varepsilon, \quad \text{for all } (t, x) \in D_\delta,
\] (A.6)
where $\Psi(t) := r(1-\varepsilon)e^{-Q(t)}\int_0^t e^{-Q(s)} ds$. Denoting $K = \frac{rbC_1}{\nu_1}$, we have
\[
\begin{align*}
\Psi(t) &\geq r(1-\varepsilon)e^{-Q(t)} \int_0^t e^{(1-\varepsilon)s}e^{-K e^{-\nu_1 t}} ds \\
&= e^{-r(1-\varepsilon)t}e^{K e^{-\nu_1 t}} \int_0^t e^{(1-\varepsilon)s} ds \\
&= e^{K e^{-\nu_1 t}(e^{-\nu_1 t-1})} (1 - e^{-r(1-\varepsilon)t}).
\end{align*}
\]

Inserting this into (A.6) and using $e^y \geq 1 + y$ for all $y \in \mathbb{R}$, we have, for all $(t, x) \in D_\delta,$
\[
\begin{align*}
\varphi(t, x) &\geq e^{K e^{-\nu_1 t}(e^{-\nu_1 t-1})} (1 - e^{-r(1-\varepsilon)t})(1 - C_3e^{-C_4 t})(1 - (1-\varepsilon)\varphi(0)) + 1 - \varepsilon \\
&\geq (1 - Ke^{-\nu_1 t})(1 - e^{-r(1-\varepsilon)t})(1 - C_3e^{-C_4 t})(1 - (1-\varepsilon)\varphi(0)) + 1 - \varepsilon \\
&\geq (1 - Ke^{-\nu_1 t})(1 - e^{-r(1-\varepsilon)t})(1 - C_3e^{-C_4 t})(1 - (1-\varepsilon)\varphi(0)) + 1 - \varepsilon.
\end{align*}
\]

Letting
we get
\[ \phi(t,x) \geq I_1 I_2(t) I_3 + (1 - e)(1 - I_1 I_2(t) I_3 x(0)), \quad \text{for all } (t,x) \in D_\delta. \]

Now observe that \( I_1 I_2(t) I_3 \leq I_1 I_2(\delta^2 c_3 T) I_3 \). Furthermore some straightforward computations show that, if
\[ r(1 - e)\delta^2 c_3 < \nu_1, \quad (A.7) \]
then \( I_1 I_2(\delta^2 c_3 T) I_3 x(0) \leq 1 \) up to enlarging \( T > 0 \) if necessary. As a result, for all \((t,x) \in D_\delta\),
\[ \phi(t,x) \geq I_1 I_2(t) I_3 \geq 1 - K_1 e^{-\nu_1 T} - K_2 e^{-r(1-e)t}, \]
with some \( K_1, K_2 > 0 \). The last inequality holds since we can always choose \( \nu_1 < C_4 \). As a conclusion, we have
\[ \phi(t,x) \geq 1 - K_1 e^{-\nu_1 T} - K_2 e^{-r(1-e)t}, \quad \text{for all } (t,x) \in D_\delta, \quad (A.8) \]
provided that \( \delta > 0 \) is sufficiently small for (A.7) to hold and \( T > 0 \) is sufficiently large.

In particular (A.8) implies that, for all \(|x| \leq (1 - \delta)c_3 T\),
\[ \phi(\delta^2 c_3 T, x) \geq 1 - K_1 e^{-\nu_1 T} - K_2 e^{-r(1-e)\delta^2 c_3 T} \geq 1 - (K_1 + K_2) e^{-r(1-e)\delta^2 c_3 T}, \quad (A.9) \]
in virtue of (A.7). On the other hand, we know from the comparison principle that \( \tilde{v}(t + T, x) \geq \phi(t, x) \) for \( t \geq 0 \) and \(|x| \leq c_3 T\), which together with (A.9) implies that
\[ \tilde{v}(\delta^2 c_3 T + T, x) \geq 1 - (K_1 + K_2) e^{-r(1-e)\delta^2 c_3 T}, \quad \text{for all } |x| \leq (1 - \delta)c_3 T. \]

We further take \( t = (\delta^2 c_3 + 1) T \), which yields
\[ \tilde{v}(t, x) \geq 1 - C_2 e^{-\nu_2 t}, \quad \text{for all large } t \text{ and } |x| \leq \frac{(1 - \delta)c_3}{1 + c_3 \delta^2} t, \]
where \( C_2 := K_1 + K_2 \) and \( \nu_2 := \frac{r(1-e)\delta^2 c_3}{1 + c_3 \delta^2} > 0 \). Recalling that \( \tilde{v}(t, x) = v(t, x + \tilde{c} t) \) with \( \tilde{c} = \frac{\bar{c} + \tilde{c}^*}{2} \),
that \( c_3 = \frac{\bar{c} + \tilde{c}^*}{2} \) and since \( \delta > 0 \) can be chosen arbitrarily small, the above estimate completes the proof of (ii).