A positivity preserving numerical scheme for the mean-reverting alpha-CEV process.

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Abstract
In this article, we present a method to construct a positivity preserving numerical scheme for both the jump-extended CEV process and jump-extended CIR process, whose jumps are governed by a (compensated) spectrally positive $\alpha$-stable process with $\alpha \in (1, 2)$. The proposed scheme is obtained by making the diffusion coefficient partially implicit and then finding the appropriate adjustment factor. We show that the proposed scheme converges and theoretically achieves a strong convergence rate of at least $\frac{\alpha}{2} \left( \frac{\alpha}{\alpha - 2} \wedge \frac{1}{2} \right)$, where the constant $\alpha_- < \alpha$ can be chosen arbitrarily close to $\alpha \in (1, 2)$. Finally, to support our result, we present some numerical simulations which suggest that the optimal rate of convergence is $\frac{\alpha}{2}$.

Key words and phrases. Euler-Maruyama scheme, positivity preserving implicit scheme, CBI process, alpha-CIR process and alpha-CEV process.

AMS 2000 subject classification.
Introduction

The Cox-Ingersoll-Ross (CIR) process and the constant elasticity of variance (CEV) process have been used extensively in financial applications such as the modelling of interest rates, default rates, and stochastic volatility, e.g., Duffie et al. [12, 13] and Alfonsi and Brigo [3]. As a result, the study of strong approximation schemes and positivity preserving strong approximation schemes for CIR and CEV processes has received a great deal of attention in the literature. We mention here the works of Alfonsi [11, 2], Berkouki et al. [5], Alfonsi and Brigo [3], Dereich et al. [11], Neuenkirch and Szpruch [28], and more recently, Bossy and Olivero [8], Chassagneux et al. [9], Hefter and Herzzwurm [20], Hefter and Jentzen [21], and Cozma and Reisinger [10]. The literature on this topic is vast and interested readers can refer to the references within the above-mentioned works.

In this article, we study a positivity preserving strong approximation scheme of the so-called alpha-CEV process and the alpha-CIR process. This class of models was first studied in the context of continuous state branching processes with interaction or/and immigration, see, for example, Li and Mytnik [20], Fu and Li [16]. The alpha-CIR process was recently introduced to mathematical finance literature in Jiao et al. [22, 23, 24] to model sovereign interest rates, power and energy markets and stochastic volatility. The authors in [22, 23, 24] argues that this family of models can capture persistent low interest rate, self-exciting, and large jump behaviours exhibited by sovereign interest rates and power markets.

The alpha-CEV process and the alpha-CIR process extend the classic diffusion CEV and CIR processes by introducing jumps which are governed by a compensated spectrally positive $\alpha$-stable Lévy process for $\alpha \in (1, 2)$. To be specific, given a positive initial point $x_0$ and $\gamma \in (\frac{1}{2}, 1)$, we consider the non-negative solution to the following stochastic differential equation (SDE),

$$dX_t = (a - kX_t) dt + \sigma_1(X^+_t)^\gamma dW_t + \sigma_2(X^-_t)^{\frac{\gamma}{2}} dZ_t,$$

where $a, \sigma_1, \sigma_2$ are non-negative and $k \in \mathbb{R}$. The diffusion coefficient and the jump coefficient are given by $g(x) = \sigma_1(x^+)^\gamma$ and $b(x) = \sigma_2(x^-)^{\frac{\gamma}{2}}$ respectively. The process $W$ is a Brownian motion and $Z$ is a compensated spectrally positive $\alpha$-stable process, independent of $W$, of the form

$$Z_t = \int_0^t \int_0^\infty z \tilde{N}(dz, ds),$$

where $\tilde{N}$ is a compensated Poisson random measure with its Lévy measure denoted by $\nu$. In other words, the process $Z$ is a Lévy process with the characteristic triple $(0, \nu, \gamma_0)$, where $\nu(dx) = x^{1-\alpha-1}1_{(0, \infty)}(x) dx$ and $\gamma_0 := -\int_x^\infty x \nu(dx)$. In general, under some monotonicity conditions on the jump coefficient, the above SDE will have a unique non-negative strong solution for any integrable compensated spectrally one-sided Lévy process $Z$, see [10, 20]. Here we focus our attention on the $\alpha$-stable case for $\alpha \in (1, 2)$.

In the current literature, numerical schemes for jump-extended CEV and jump-extended CIR models have started to receive increasing attention, we refer to Yang and Wang [32], Fatemion Aghdas [14] and Stamatiu [29]. However, to the best of our knowledge, for the jump-extended CEV process and the jump-extended CIR process, the existing results all focused on the case of Poisson jumps (finite activity jumps) and results on positivity preserving strong approximation schemes in the case of infinite activity jumps have only appeared in our previous work, Li and Taguchi [20]. In the case of the alpha-CIR process, we stress that although the derived scheme in [3] might appear similar to the one given in Alfonsi [11] and Li and Taguchi [20], it was initially not clear how such a scheme can be obtained. Due to the presence of infinite activity jumps, jump-adapted schemes devised through combining the Lamperti transform and the backward Euler scheme (see [31]) are not feasible. Although Alfonsi [11] obtained a scheme for the CIR process ($\gamma = \frac{1}{2}$), he explicitly mentioned on page 4 that such a scheme can only be obtained if $\gamma = \frac{1}{2}$ or 1. Hence there was a need to find a method of simulation that does not make use of transformations. The key idea in the derivation of the scheme is to make the diffusion coefficient partially implicit and to identify the appropriate adjustment factor. In this way, we can obtain a positivity preserving scheme, by solving quadratic equations, for both the alpha-CIR and the alpha-CEV.

Numerically speaking, the advantage of the scheme proposed in [3] is similar to that in the cases of the CIR process in Alfonsi [11] and the alpha-CIR process in Li and Taguchi [20], that the scheme is obtained by solving, at each step, a quadratic equation. Therefore, even in the diffusion case ($\sigma_2 = 0$), we do not
need to solve for the positive root of a non-linear equation as done in [2] [11] [28], where the scheme was obtained through combining the Lamperti transform and the backward Euler scheme. We mention also that the symmetrized scheme developed in Berkaoui et al. [5], Bossy and Olivier [8], Bossy and Diop [6] and Bossy et al. [7] can potentially be applied here and the symmetrized scheme also has the advantage of not having to solve a non-linear equation at each step. However, in the infinite activity case, the local time techniques used in the proof of convergence do not appear to translate well into our setting. This is because, in addition to the continuous local time, one needs to compensate the reflected jumps and it appears that one quickly faces integrability issues. Of course, this is outside the scope of the current work.

In the analysis of the proposed numerical scheme, we make use of a combination of continuous time and discrete time techniques to obtain the required moment estimates. We first show in Lemma 1.5 that \( k > 0 \) is a sufficient condition for the alpha-CEV to be strictly positive. Then we compute in Lemma 1.6 an inverse moment estimate of the alpha-CIR and alpha-CEV processes, which allowed us to extend a technique from Berkaoui et al. [5] to obtain a rate of convergence which is independent of \( \gamma \). In particular, this enabled one to improve, under the Feller condition, the convergence rate for the alpha-CIR from logarithmic to polynomial. Another technical difficulty, which we dealt with in Lemma 1.7, is to show the existence of the \( \beta \)-moment for the numerical scheme for \( \beta \in [1, \alpha) \). This result is fundamental in removing the boundedness assumption on the jump coefficient \( h \) in [18, 19, 26, 27] and thus the removal of the truncation step and the restriction that \( \alpha > \sqrt{2} \) in [20]. Even in the case of the Euler-Maruyama scheme, this integrability issue was only studied recently in Frikha and Li [15], where a mean-field extension of the equation from Li and Mytnik [25], together with the corresponding propagation of chaos property and Euler-Maruyama scheme were studied. In Theorem 2.1, the derivation of the convergence rate is done through a careful splitting of the scheme, making use of the martingale representation theorem in the Lévy filtration and applying the Yamada-Watanabe approximation technique. We stress that the way the scheme is split in (5) is crucial in achieving a higher rate of convergence rate in Theorem 2.1.

The structure of the paper is as follows. We first present the derivation of our scheme in section 1 and give some auxiliary lemmas and estimates in subsection 1.2. Then in section 2, for \( k > 0 \), we present in Theorem 2.1 our main result on the strong rate of convergence for the alpha-CEV where we show that the strong convergence rate is at least \( \frac{\alpha}{2} \left( \frac{\alpha}{\alpha - 1} \wedge \frac{1}{2} \right) \), which is faster compared to those obtained for the Euler-Maruyama scheme in [15]. We point out that, although not explicitly stated, for the alpha-CIR one can obtain a polynomial rate which improves upon the logarithmic rate obtained in a preliminary investigation in [20]. Finally, we provide some numerical simulations in section 3 which suggest that the optimal rate of convergence is \( \frac{\alpha}{2} \). For the reader’s convenience, we include an appendix in section 4.

1 The Numerical Scheme

In the following, we consider the equal discounted grid \( \pi : 0 = t_0 < t_1 < \ldots < t_n = T \) with grid size \( \Delta t = T/n \). The design of this scheme is inspired by Alfonsi’s work on the diffusion CIR process in [1] which was later extended to the alpha-CIR in Li and Taguchi [20]. We describe below the main idea behind our scheme. To this end, we start with the Euler-Maruyama scheme

\[
\Delta X_{t_i} = (a - kX_{t_i})\Delta t_i + \sigma_1(X_{t_i}^{+})^{\gamma} \Delta W_{t_i} + \sigma_2(X_{t_i}^{+})^{\frac{\beta}{2}} \Delta Z_{t_i}.
\]

To proceed, we make the diffusion coefficient partially implicit, and consider for \( i = 0, 1, 2, \ldots, n - 1 \),

\[
\Delta X_{t_i} = (a - kX_{t_i})\Delta t_i + \sigma_1(X_{t_{i+1}}^{+})^{\frac{1}{2}}(X_{t_i}^{+})^{\gamma - \frac{1}{2}} \Delta W_{t_i} + \sigma_2(X_{t_i}^{+})^{\frac{\beta}{2}} \Delta Z_{t_i} - \sigma_1((X_{t_{i+1}}^{+})^{\frac{1}{2}} - (X_{t_i}^{+})^{\frac{1}{2}})((X_{t_i}^{+})^{\gamma - \frac{1}{2}} \Delta W_{t_i}.
\]

Then by summing over \( i \) and supposing that the scheme is positive, we see that the last term or the adjustment factor is given by

\[
\sum_{i=0}^{n} \sigma_1((X_{t_{i+1}}^{+})^{\frac{1}{2}} - (X_{t_i}^{+})^{\frac{1}{2}})((X_{t_i}^{+})^{\gamma - \frac{1}{2}} \Delta W_{t_i} \approx \sigma_1 \int X_{\gamma - \frac{1}{2}}^{\gamma - \frac{1}{2}} d\langle \sqrt{\gamma}, W \rangle_s = \frac{\sigma_1^2}{2} \int_0^T X_{\gamma - 1}^{\gamma - 1} ds.
\]

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The above computations suggest that we consider the following implicit scheme, for \( X_{t_0} = x_0 \)

\[
\Delta X_t = (a - kX_{t_i})\Delta t_i + \sigma_1(X_{t_i})^\gamma \frac{\Delta W_{t_i}}{2} + \sigma_2(X_{t_i})^\frac{1}{2} \Delta Z_{t_i} - \frac{\sigma_1^2}{2} X_{t_i}^{\gamma - 1} \Delta t_i. \tag{2}
\]

For every \( i = 0, 1, \ldots, n - 1 \), by setting \( \sqrt{X_{t_i+1}} = x \) and rearranging equation (2), one can obtain a quadratic equation in \( x \):

\[
(1 + k\Delta t_i)x^2 - \sigma_1X_{t_i}^{\gamma - \frac{1}{2}} \Delta W_{t_i} x - (X_{t_i} + (a - \frac{\sigma_1^2}{2} X_{t_i}^{2\gamma - 1}) \Delta t_i + \sigma_2 X_{t_i}^\frac{1}{2} \Delta Z_{t_i}) = 0.
\]

It is clear that, in the diffusion case, the discriminant of the above quadratic equation is non-negative if the condition \( a - \sigma_1^2/2 > 0 \) (and \( 1 - \frac{\gamma}{2} \sigma_1^2 \Delta t > 0 \), for \( \gamma > \frac{1}{2} \)) is satisfied. In the jump-extended case, the support of the discriminant is bounded below only in the case where \( Z \) has finite activity jumps (Type A) or is of finite variation and has infinite activity jumps (Type B). In the case where \( Z \) is of infinite variation (Type C) and therefore has infinite activity jumps, there is no hope of finding a set of conditions on the parameters \( a, k, \sigma_1, \sigma_2, \alpha \) and the grid size such that the discriminant is non-negative. We refer to Remark 1.2 in [26] for more detailed discussion.

Therefore to keep the discriminant of the quadratic equation non-negative, we further modify the quadratic equation by taking the absolute value of the constant term in the quadratic equation. This ensures the existence of a unique positive root at each step by the Descartes’ Sign Rule. To this end, we propose the following positivity preserving numerical scheme \( X^n \):

\[
X^n_{t_{i+1}} = \left[ \frac{\sigma_1(X^n_{t_i})^\gamma \frac{\Delta W_{t_i}}{2} + \sqrt{\frac{\sigma_1^2(X^n_{t_i})^{2\gamma - 1}}{2(1 + k\Delta t_i)}}((\Delta W_{t_i})^2 + 4(1 + k\Delta t_i)D_{t_i+1})}{2} \right]^2, \quad X^n_{t_0} = x_0,
\]

\[
D_{t_{i+1}} = X^n_{t_i} + \left( a - \frac{\sigma_1^2}{2}(X^n_{t_i})^{2\gamma - 1} \right) \Delta t_i + \sigma_2(X^n_{t_i})^\frac{1}{2} \Delta Z_{t_i}. \tag{3}
\]

As far as we are aware, the above method of making the diffusion coefficient partially implicit and modifying the adjustment factor in the case \( \gamma \in \left( \frac{1}{2}, 1 \right) \) appears to be new, even for the classic diffusion CEV process.

Before proceeding, we point out that the idea of making terms in the coefficients partially implicit can give rise to other positivity preserving numerical schemes. For example, suppose the coefficients \( g \) and \( h \) are such that \( X \) is the positive solution to the SDE in (1), and we consider the Euler-Maruyama scheme:

\[
\Delta X_{t_i} = (a - kX_{t_i})\Delta t_i + g(X_{t_i})\Delta W_{t_i} + h(X_{t_i})\Delta Z_{t_i}.
\]

To obtain a positivity preserving scheme we can multiply the constant \( a \) in the drift by \( \frac{X_{t_i}}{X_{t_i}} \) and replacing \( a \frac{X_{t_i}}{X_{t_i}} \) with \( a \frac{X_{t_i}}{X_{t_i+1}} \). More specifically, we consider

\[
\tilde{X}_{t_{i+1}} = \tilde{X}_{t_i} + \left( a \frac{\tilde{X}_{t_i}}{\tilde{X}_{t_{i+1}}} - k\tilde{X}_{t_i} \right) \Delta t_i + g(\tilde{X}_{t_i})\Delta W_{t_i} + h(\tilde{X}_{t_i})\Delta Z_{t_i},
\]

which after some algebraic manipulation gives

\[
\tilde{X}_{t_{i+1}} - \tilde{X}_{t_{i+1}}(\tilde{X}_{t_i}(1 - k\Delta t_i) + g(\tilde{X}_{t_i})\Delta W_{t_i} + h(\tilde{X}_{t_i})\Delta Z_{t_i}) - a\tilde{X}_{t_i} \Delta t_i = 0.
\]

Let \( B_{t_{i+1}} = (\tilde{X}_{t_i}(1 - k\Delta t_i) + g(\tilde{X}_{t_i})\Delta W_{t_i} + h(\tilde{X}_{t_i})\Delta Z_{t_i}) \) and \( C_{t_{i+1}} = a\tilde{X}_{t_i} \Delta t_i \). Then a positive valued scheme \( \tilde{X} \) can be obtained if we consider the positive root to the above quadratic equation, that is \( \tilde{X}_0 = x_0 \),

\[
\tilde{X}_{t_{i+1}} = \frac{B_{t_{i+1}} + \sqrt{B_{t_{i+1}}^2 + 4C_{t_{i+1}}}}{2}, \quad i = 0, \ldots, n - 1.
\]

In fact, one can obtain a family of schemes using this technique. That is for \( a > 0 \) and \( \zeta > 0 \), one can multiply the constant \( a \) by \( X^n_{t_{i+1}} \), and by setting \( x = X^n_{t_{i+1}} \) we arrive at the equation

\[
x^{\zeta/2} - a x^{-\zeta/2} \tilde{X}_{t_i} \Delta t = \tilde{X}_{t_i}(1 - k\Delta t_i) + g(\tilde{X}_{t_i})\Delta W_{t_i} + h(\tilde{X}_{t_i})\Delta Z_{t_i}.
\]

\[
\tilde{X}_{t_{i+1}} = \frac{B_{t_{i+1}} + \sqrt{B_{t_{i+1}}^2 + 4C_{t_{i+1}}}}{2}, \quad i = 0, \ldots, n - 1.
\]
which for \( \tilde{X}_{t_i} > 0 \) has an unique positive root. This family of schemes mimics the Lamperti transform and backward Euler scheme, used in Dereiche et al. [11], Neuenkirch and Szpruch [28] and Alfonsi [2], without actually doing the Lamperti transform. The study of convergences for these schemes are likely difficult as they require inverse moment estimates of the scheme and therefore are left for future works.

The aim in the rest of the article is to compute the strong convergence rate for the scheme in (3). From this point forward we suppose that the following assumption holds.

**Assumption 1.1.** The parameters \( \alpha, \gamma, a, k, n, \sigma_1 \) are such that \( \alpha \in (1,2) \), \( 2 \gamma < \alpha \), \( a - \sigma_1^2/2 > 0 \) and \( \Delta t = 1/n \) is small enough so that \( 1 + \Delta t > 0 \) and \( 1 - \frac{1}{2} \sigma_1^2 \Delta t > 0 \).

We point out that in the case of the classic diffusion CIR, by using more advanced techniques, c.f. Hefter and Herzwurm [20], it is possible to relax the Feller condition \( a - \sigma_1^2/2 > 0 \). However, it is not clear if these techniques can be translated to the jump-extended setting. For the alpha-CEV, the Feller condition is only used in controlling the probability that the discriminant is negative in Lemma 1.2 and is assumed for the convenience of the proof, and it might be possible to remove the Feller condition by working with an adaptive scheme or study in more detail the (conditional) law of \( D \). Our main result Theorem 2.1 states for \( \gamma > 0 \), but we point out that, for \( \gamma < 0 \), the scheme can still be shown to converge with a slower rate which is dependant on \( \gamma \).

From this point forward, we denote the lower bound of \( 1 + k \Delta t \) by \( \kappa_0 \), that is \( \kappa_0 \in (0, 1) \) and \( 1 + k \Delta t \geq \kappa_0 \). We also use \( C, C', C'', C_0, C_1, C_2, C_T, c \), etc. to denote constants, which may change from line to line.

### 1.1 Continuous Time Dynamic of the Scheme

By expanding the quadratic in (3), rearranging and collecting the appropriate terms, we obtain the following expression for the scheme:

\[
X_{t_{i+1}}^n = X_{t_i}^n + (a - k_n X_{t_i}^n) \Delta t_i + \sigma_1 (X_{t_i}^n)^\gamma \Delta W_{t_i} + \sigma_2 (X_{t_i}^n)^{1/2} \Delta Z_{t_i} + \Delta R_{t_i}^n,
\]

where \( k_n := k/(1 + k \Delta t) \) and the remainder is given by

\[
\Delta R_{t_i}^n = -\sigma_2 (X_{t_i}^n)^{1/2} \Delta Z_{t_i} + \frac{\sigma_2^2 (X_{t_i}^n)^{2\gamma - 1}}{2} \left[ \frac{(\Delta W_{t_i})^2}{(1 + k \Delta t)^2} - \frac{\Delta t}{1 + k \Delta t} \right] + a \Delta t \left[ \frac{1}{1 + k \Delta t} - 1 \right]
\]

\[- \sigma_1 (X_{t_i}^n)^\gamma \Delta W_{t_i} + \sigma_2 (X_{t_i}^n)^{1/2} \Delta Z_{t_i} + \Delta M_{t_i}^n + \frac{2}{1 + k \Delta t} \Delta D_{t_i}^n.
\]

where the term \( \Delta M_{t_i}^n \) is given by

\[
\Delta M_{t_i}^n = \frac{\sigma_1 (X_{t_i}^n)^{\gamma - 1/2} \Delta W_{t_i}}{2(1 + k \Delta t)^2} \sqrt{\sigma_2^2 (X_{t_i}^n)^{2\gamma - 1}(\Delta W_{t_i})^2 + 4(1 + k \Delta t)^2 \Delta D_{t_i}^n}.
\]

The semimartingale decomposition of \( \Delta R_{t_i}^n \) is given by

\[
\Delta R_{t_i}^n = \Delta \tilde{M}_{t_i}^n + \Delta \tilde{M}_{t_i}^n + \Delta \tilde{M}_{t_i}^n + A_{t_i} \Delta t_i
\]

with

\[
\Delta \tilde{M}_{t_i}^n := \frac{\sigma_1^2 (X_{t_i}^n)^{2\gamma - 1}}{2(1 + k \Delta t)^2}((\Delta W_{t_i})^2 - \Delta t_i), \quad \Delta \tilde{M}_{t_i}^n := \Delta M_{t_i}^n - \sigma_1 (X_{t_i}^n)^\gamma \Delta W_{t_i},
\]

\[
\Delta \tilde{M}_{t_i}^n := \frac{2 \Delta M_{t_i}^D}{1 + k \Delta t} - k_n \Delta t \sigma_2 (X_{t_i}^n)^{1/2} \Delta Z_{t_i}, \quad A_{t_i} \Delta t_i := (\Delta t)^2 \left[ -ak_n - \frac{k \sigma_1^2 (X_{t_i}^n)^{2\gamma - 1}}{2(1 + k \Delta t)^2} \right] + \frac{2 \mathbb{E}(D_{t_{i+1}}^n | F_{t_i})}{1 + k \Delta t},
\]

where \( \Delta M_{t_i}^D := D_{t_{i+1}}^n - \mathbb{E}(D_{t_{i+1}}^n | F_{t_i}) \). Finally, we can extend the discrete scheme (4) to continuous time and write \( X_t^n = \tilde{X}_t^n + \tilde{R}_t^n \) where

\[
\tilde{X}_t^n = x_0 + \int_0^t (a - k_n X_{\eta(s)}^n) ds + \sigma_1 \int_0^t (X_{\eta(s)}^n)^\gamma dW_s + \sigma_2 \int_0^t (X_{\eta(s)}^n)^{1/2} dZ_s + \tilde{M}_t^n + \tilde{\tilde{M}}_t^n,
\]

\[
\tilde{R}_t^n = \tilde{M}_t^n + \int_0^t A_{\eta(s)} ds,
\]

where for \( t \in (t_i, t_{i+1}) \), \( i = 0, \ldots, n - 1 \), we set \( \eta(t) := t_i \) and \( \tilde{M}_t^n := \mathbb{E}(M_{t_{i+1}}^n | F_t) \), the continuous time extension of \( \tilde{M} \) and \( \tilde{\tilde{M}} \) are similarly defined.
Lemma 1.1. For $\beta \in [1, \alpha)$ there exist some positive constant $C_0$ such that

$$\sup_{t \leq T} \mathbb{E}[X_t^\beta] \leq \frac{C_0}{(\alpha - \beta)} e^{C_0 T/(\alpha - \beta)}$$

Lemma 1.2. For $\gamma \in [\frac{1}{2}, 1]$ and $\alpha \in (1, 2)$

$$\max_{i=1, \ldots, n} \mathbb{P}[D_{i,t} < 0] \leq \exp \left( - K_{\alpha_1, \alpha_2}^\alpha (\Delta t)^{-\frac{2-\gamma}{2-\alpha}} \right)$$

where $K_{\alpha_1, \alpha_2}^\alpha$ is a positive constant depending on $a, \alpha, \sigma_1, \sigma_2$.

Lemma 1.3. For $\alpha \in (1, 2)$, we have

$$\sup_n \max_{i=0, 1, \ldots, n} \mathbb{E}[X_{i,t}^\alpha] < \infty.$$  

Lemma 1.4. For $\beta \in [1, \alpha)$ there exist positive constants $C_1, C_2$ such that

$$\max_{i=1, \ldots, n} \mathbb{E}[(D_{i,t})^\beta] \leq C_1 (\Delta t)^{\beta} \exp \left( - C_2 (\Delta t)^{-\frac{2-\gamma}{2-\alpha}} \right).$$

Remark 1.2. By adapting a technique from Szpruch et al. [30] used for the Ait-Sahalia type interest rate process, we show below in Lemma 1.5 that for $k > 0$ the alpha-CEV process is strictly positive. For the strict positivity of the alpha-CIR under the Feller condition $a - \frac{\sigma^2}{2} > 0$ (and without the restriction $k > 0$) we refer to Proposition 3.7 in Jiao et al. [23]. Using the fact that $X$ is strictly positive, we can obtain in Lemma 1.6 the required inverse moment estimate of $X$ on the whole time horizon $[0, T]$. This is needed in improving the rate of convergence on the whole time horizon in Theorem 2.1.

Lemma 1.5. Given $x_0 > 0$ and $\gamma \in (\frac{1}{2}, 1]$, if $k > 0$ then $\mathbb{P}(X_t \in (0, \infty), \forall t > 0) = 1$.

Proof. For $m \geq 0$ we set $\tau_m := \inf\{t \geq 0 : X_t \notin (m^{-1}, m)\}$. The aim is to show that for any fixed $T > 0$, $\lim_{m \to \infty} \mathbb{P}(\tau_m \leq T) = 0$. To do this, for a fixed $0 < \beta < 1$, we consider the $C^2(\mathbb{R}_+)$ function

$$V(x) = x^\beta - 1 - \beta \ln x.$$  

From the Itô formula applied to $V(X_t^\gamma)$, we note that the generator is of the form

$$V'(x)(a - kx) + \frac{1}{2} V''(x)\sigma^2 x^2 + \int_0^\infty [V(x + \sigma x^{1/2} z) - V(x) - \sigma x^{1/2} z V'(x)] \nu(dz),$$

where $V'(x) = \beta(x^{\beta-1} - x^{-1})$ and $V''(x) = \beta(\beta-1)x^{\beta-2} + \beta x^{-2}$. Hence the generator associated with the diffusion part of $X$ is given by

$$a_\alpha x^{-(1-\beta)} - k_\alpha x^\beta - a_\beta x^{-1} + k_\beta \frac{\sigma_\alpha^2 (1-\beta)}{2} x^{\beta-2(1-\gamma)} + \frac{\sigma_\beta^2 \beta}{2} x^{-2(1-\gamma)}.$$
From the fact that $0 < \beta < 1$ and $\frac{1}{2} < \gamma < 1$, we see that for $k > 0$ the terms with the smallest and largest power are $-a\beta x^{-1}$ and $-k\beta x^\beta$ respectively, both of which have negative constant coefficients.

On the other hand, for the integral against the Lévy measure we can apply Taylor’s expansion to obtain, for $(x, y, z) \in \mathbb{R}_+^3$, the following useful equalities

$$V(x + \sigma_x x^\frac{1}{2} z) - V(x) - \sigma_x x^\frac{1}{2} z V'(x) = \sigma_x^2 x^\frac{3}{2} z^2 \int_0^1 (1 - \theta) V''(x + \theta \sigma_x x^\frac{1}{2} z) d\theta, \quad (6)$$

$$V(x + \sigma_x x^\frac{1}{2} z) - V(x) - \sigma_x x^\frac{1}{2} z V'(x) = \sigma_x x^\frac{1}{2} z \int_0^1 [V'(x + \theta \sigma_x x^\frac{1}{2} z) - V'(x)] d\theta. \quad (7)$$

For $0 < z < 1$, by using (6) we obtain

$$\int_0^1 [V(x + \sigma_x x^\frac{1}{2} z) - V(x) - \sigma_x x^\frac{1}{2} z V'(x)] d\nu(dz) = \sigma_x^2 x^\frac{3}{2} \int_0^1 z^2 \int_0^1 (1 - \theta) [\beta(\beta - 1)(x + \theta \sigma_x x^\frac{1}{2} z)^{\beta - 2} + \beta(x + \theta \sigma_x x^\frac{1}{2} z)^{-2}] d\theta \nu(dz)$$

$$\leq \sigma_x^2 x^\frac{3}{2} \int_0^1 z^2 \int_0^1 (1 - \beta) x^\beta x^{-2} d\theta d\nu(dz) = \frac{\beta \sigma_x^2}{2(2 - \alpha)} x^{\beta - 2}. \quad \text{For } \int_0^1 [V(x + \sigma_x x^\frac{1}{2} z) - V(x) - \sigma_x x^\frac{1}{2} z V'(x)] d\nu(dz)$$

For $z \geq 1$, by using (7) we obtain

$$\int_1^\infty [V(x + \sigma_x x^\frac{1}{2} z) - V(x) - \sigma_x x^\frac{1}{2} z V'(x)] d\nu(dz) = \beta \sigma_x x^\frac{1}{2} \int_1^\infty z \int_0^1 ((x + \theta \sigma_x x^\frac{1}{2} z)^{\beta - 1} - (x + \theta \sigma_x x^\frac{1}{2} z)^{-1} - x^{\beta - 1} + x^{-1}) d\theta d\nu(dz)$$

$$\leq \beta \sigma_x x^\frac{1}{2} \int_1^\infty z \int_0^1 (x^{\beta - 1} + x^{-1}) d\theta d\nu(dz) = \frac{\beta \sigma_x^2}{\alpha - 1} (x^{\frac{1}{2} \beta - 1} + x^{\frac{1}{2} - 1}).$$

By combining the above computations, we observe that

$$V'(x)(a - kx) + \frac{1}{2} V''(x) \sigma_x^2 x^{2\gamma} + \int_0^\infty [V(x + \sigma_x x^\frac{1}{2} z) - V(x) - \sigma_x x^\frac{1}{2} z V'(x)] d\nu(dz)$$

$$\leq \left( a\beta x^{-(1 - \beta)} - k\beta x^\beta - a\beta x^{-1} + k\beta - \frac{\sigma_x^2 (1 - \beta)}{2} x^{\beta - 2(1 - \gamma)} + \frac{\sigma_x^2 \beta}{2} x^{-2(1 - \gamma)} \right)$$

$$+ \frac{\beta \sigma_x^2}{2(2 - \alpha)} x^{-(1 - \frac{1}{2})} + \frac{\beta \sigma_x^2}{\alpha - 1} (x^{-(\frac{1 - \beta}{2})} + x^{-(\frac{1}{2} - 1)}). \quad (8)$$

Using the fact that $1 < \alpha < 2$, we see that the terms with the smallest and largest power are again $-a\beta x^{-1}$ and $-k\beta x^\beta$ respectively, both of which have negative constant coefficients. Therefore, the right hand side of (8) is a continuous function which is bounded from above. That is we can find a constant $C > 0$ such that for all $0 < x < \infty$

$$V'(x)(a - kx) + \frac{1}{2} V''(x) \sigma_x^2 x^{2\gamma} + \int_0^\infty [V(x + \sigma_x x^\frac{1}{2} z) - V(x) - \sigma_x x^\frac{1}{2} z V'(x)] d\nu(dz) \leq C.$$
Lemma 1.6. For $\gamma = \frac{1}{2}$ there exists a positive constant $C_f$ such that for $0 < p < \frac{2\gamma}{\alpha} - 1$

$$\sup_{t \leq T} \mathbb{E}[X_t^{-p}] \leq (x_0^{-p} + C_f T)e^{Tk}. $$

For $\gamma \in \left(\frac{1}{2}, 1\right)$ and $k > 0$ the above estimate is true for all $p > 0$.

Proof. In the following, we apply the Itô formula to estimate the inverse moments of $X$. We extend the argument in Lemma 4.1 of Bossy and Diop [8]. Let us consider a sequence of stopping times $(\tau_m)_{m \in \mathbb{N}^+}$ given by $\tau_m = \inf\{s \leq T : X_s \leq m^{-1}\}$. Then by applying the Itô formula to the stopped process $(X_{\tau_m}^{-p})$, we obtain

$$(X_{t \wedge \tau_m}^{-p}) = x_0^{-p} + M_{t \wedge \tau_m}^i + I_{t \wedge \tau_m}^i + J_{t \wedge \tau_m}^i + K_{t \wedge \tau_m}^i,$$

where we set

$$M_t^i := -\sigma \int_0^t X_s^{p-1} dW_s + \int_0^t \left\{ (X_{s-} + \sigma_2 (X_{s-})^\frac{\gamma}{2} z)^{-p} - (X_{s-})^{-p} \right\} \tilde{N}(ds, dz),$$

$$I_t^i := -p \int_0^t X_s^{-p-1} (a - kX_s) ds,$$

$$J_t^i := \frac{\sigma_1^2}{2} p(p+1) \int_0^t X_s^{\gamma p-2} ds,$$

$$K_t^i := \int_0^t \left\{ (X_{s-} + \sigma_2 (X_{s-})^\frac{\gamma}{2} z)^{-p} - (X_{s-})^{-p} + \sigma_2 (X_{s-})^\frac{\gamma}{2} z p(X_{s-})^{-p-1} \right\} \nu(dz) ds.$$

We consider now the term $K_{t \wedge \tau_m}^i$. For $z \in (0,1)$ we deduce, from the first-order Taylor expansion of the map $\theta \mapsto (y + \theta xz)^{-p}$, the following inequality for every positive $x, y$ and $z$,

$$(y + xz)^{-p} - y^{-p} + pxz y^{-p-1} = p(p+1)(xz)^{2} \int_0^1 (y + \theta xz)^{-p-2} (1 - \theta) d\theta \leq \frac{p(p+1)(xz)^2}{y^{p+2}}.$$

For $z \geq 1$, since $x, y, z$ are positive we have $(y + xz) \geq y$ and

$$(y + xz)^{-p} - y^{-p} + pxz y^{-p-1} \leq pxz y^{-p-1}.$$

Hence we have the estimate

$$\mathbb{E}[K_{t \wedge \tau_m}^i] \leq \frac{\sigma_1^2}{2} p(p+1) \mathbb{E}\left[ \int_0^{t \wedge \tau_m} \int_0^1 X_s^{\frac{\gamma}{2} p-2} x^2 \nu(dz) ds \right] + p\sigma_2 \mathbb{E}\left[ \int_0^{t \wedge \tau_m} \int_0^\infty X_s^{\frac{1}{2} p-1} z \nu(dz) ds \right]$$

$$= \frac{\sigma_1^2}{2} p(p+1) \mathbb{E}\left[ \int_0^{t \wedge \tau_m} X_s^{\frac{\gamma}{2} p-2} ds \right] + \frac{p\sigma_2}{\alpha - 1} \mathbb{E}\left[ \int_0^{t \wedge \tau_m} X_s^{\frac{1}{2} p-1} ds \right].$$

Similarly for $I_{t \wedge \tau_m}$ and $J_{t \wedge \tau_m}$ we have

$$\mathbb{E}[I_{t \wedge \tau_m}] = -p \mathbb{E}\left[ \int_0^{t \wedge \tau_m} X_s^{-p-1} (a - kX_s) ds \right] = -ap \mathbb{E}\left[ \int_0^{t \wedge \tau_m} X_s^{-p-1} ds \right] + pk \mathbb{E}\left[ \int_0^{t \wedge \tau_m} X_s^p ds \right],$$

$$\mathbb{E}[J_{t \wedge \tau_m}] = \frac{\sigma_1^2}{2} p(p+1) \mathbb{E}\left[ \int_0^{t \wedge \tau_m} X_s^{\frac{\gamma}{2} p-2} ds \right].$$

Thus by combining the above estimates, we obtain

$$\mathbb{E}[(X_{t \wedge \tau_m}^{-p})] = x_0^{-p} + \mathbb{E}[I_{t \wedge \tau_m}] + \mathbb{E}[J_{t \wedge \tau_m}] + \mathbb{E}[K_{t \wedge \tau_m}]$$

$$\leq x_0^{-p} + pk \mathbb{E}\left[ \int_0^{t \wedge \tau_m} X_s^p ds \right] +$$

$$\mathbb{E}\left[ \int_0^{t \wedge \tau_m} \left( -apX_s^{-1} + \frac{\sigma_1^2}{2} p(p+1)X_s^{\frac{\gamma}{2} p-2} + \frac{\sigma_1^2}{2} p(p+1)X_s^{\frac{1}{2} p-1} \right) ds \right].$$

$$\tag{9}$$
In the case $\gamma > \frac{1}{2}$, we see, in inequality (9), that $(-p - 1)$ is the smallest among the negative powers given by $(-p - 1), (2\gamma - p - 2), (2/\alpha - p - 2)$ and $(1/\alpha - p - 1)$. Also, the term $X_{\tau_m}^{-p}$ is the only term with a negative coefficient. Thus, for $f : (0, \infty) \mapsto \mathbb{R}$ defined by

$$f(x) := -apx^{-p-1} + \frac{\sigma^2(p+1)}{2}x^{2\gamma-p-2} + \frac{\sigma^2(p+1)}{2-\alpha}x^{\frac{2}{\alpha}-p-2} + \frac{\sigma^2}{\alpha-1}x^{\frac{1}{\alpha}-p-1},$$

there exists a positive constant $C_f$ such that $f(x) \leq C_f$ for all $x > 0$. Therefore we obtain

$$\mathbb{E}[(X_{\tau_m}^{-p})] \leq x_0^{-p} + C_fT + pk\int_0^T \mathbb{E}[(X_{s}^{-p})]ds,$$

and finally Gronwall’s inequality gives us

$$\mathbb{E}[(X_{\tau_m}^{-p})] \leq (x_0^{-p} + C_fT) \exp(pkT). \tag{10}$$

In the case $\gamma = \frac{1}{2}$, inequality (9) becomes

$$\mathbb{E}[(X_{\tau_m}^{-p})] \leq x_0^{-p} + pk\int_0^{\tau_m} X_{s}^{-p}ds + \mathbb{E}\left[\int_0^{\tau_m} ((-a + \frac{\sigma^2}{2}(p+1))X_{s}^{-p-1} + \frac{\sigma^2(p+1)}{2-\alpha}X_{s}^{\frac{2}{\alpha}-p-2} + \frac{\sigma^2}{\alpha-1}X_{s}^{\frac{1}{\alpha}-p-1})ds\right],$$

and we require the extra condition $p < 2a/\sigma^2 - 1$ to guarantee a negative coefficient for the term $X_{\tau_m}^{-p}$. One can then proceed similarly to the case $\gamma > \frac{1}{2}$ to obtain (10).

Lastly, since the constant $C_f$ is independent of $m$, the upper bound (10) for $\mathbb{E}[(X_{\tau_m}^{-p})]$ is independent of the localising sequence $(\tau_m)_{m \in \mathbb{N}^+}$ and by Lemma 1.5 we can conclude by applying the monotone convergence theorem and letting $m \to \infty$. \hfill \square

Next we prove the key result of the article. That is for $\beta \in (1, \alpha)$ we show that the $\beta$-moment of the discretization scheme is finite. To proceed, we decompose the Lévy process $Z$ into small jumps which are less than one and large jumps which are larger than one and express the scheme as

$$X^n_{t_{i+1}} = x_0 + \int_0^{t_{i+1}} (a - k_nX^n_{\eta(s)}) - \frac{\sigma^2}{\alpha-1}(X^n_{\eta(s)})^{\frac{1}{\alpha}}ds + \int_0^{t_{i+1}} \sigma_1(X^n_{\eta(s)})^\gamma dW_s + \hat{M}^n_{t_{i+1}} + \hat{M}^n_{t_{i+1}} + \hat{M}^n_{t_{i+1}} + \int_0^{t_{i+1}} A^n_{\eta(s)}ds + \int_0^{t_{i+1}} \int_0^1 \sigma_2(X^n_{\eta(s)})^{\frac{1}{\alpha}}zN(dz, ds) + \int_0^{t_{i+1}} \int_1^\infty \sigma_2(X^n_{\eta(s)})^{\frac{1}{\alpha}}zN(dz, ds).$$

For notational simplicity, in the following we set $N^n := \hat{M}^n + \hat{M}^n + \hat{M}^n$ and $b(x) = a - k_nx - \frac{\sigma^2}{\alpha-1}x^{\frac{1}{\alpha}}$.

**Lemma 1.7.** For $\beta \in (1, \alpha)$ the $\beta$-moment of the scheme $X^n$ is finite, that is

$$\sup_n \mathbb{E}\left[\max_{i=0,1,\ldots,n} (X^n_{t_i})^\beta\right] < \infty.$$

**Proof.** For simplicity, we set $V_i := \int_0^1 \int_1^\infty \sigma_2(X^n_{\eta(s)})^{\frac{1}{\alpha}}zN(dz, ds)$, and $I_j := \{0, 1, 2, \ldots, n\}$ for $j = 0, 1, \ldots, n$. The goal here is to localise the discretisation scheme and apply the the Grönwall inequality. Before proceeding further, we note that for any continuous time process $Y$ observed on the time grid $\pi$, when $Y$ is stopped at a discrete random time $\tau$ which takes value on the grid $\pi$, we have

$$Y^n_{t_i} - Y_0 = \sum_{j=0}^{i-1} 1_{(t_j < \tau)} \Delta Y_{t_j} + \sum_{j=0}^{i-1} 1_{(t_j \leq \tau)} \Delta Y_{t_j} = \int_0^T 1_{(s \leq \tau)} 1_{(s \leq t_i)} dY_s,$$

where by convention $\sum_{j=0}^{-1} = 0$. In addition, we observe that $Y_i = Y^n_{t_i}$ on the set $\{t_i \leq \tau\}$ and $\{t_i < \tau\} = \{t_i \leq \eta(\tau)\}$ which implies that $Y_i = Y^n_{t_i(\tau)}$ on the set $\{t_i < \tau\}$.
For a fixed $N \in \mathbb{N}$, let $\tau_N := \min\{t_i \leq T : X^n_{t_i} \geq N\}$ be a localising sequence of stopping times. Then from the Jensen inequality, we have
\begin{align*}
\max_{i \in I_n} (X^n_{t_i \wedge \eta(\tau_N)})^\beta &\leq \max_{i \in I_n} (X^n_{t_i \wedge \tau_N})^\beta \\
&\leq C_T \left[ x_0^\beta + \max_{i \in I_n} \sum_{j=0}^{i-1} 1_{\{t_j < \tau_N\}} b(X^n_{t_j})^\beta + \max_{i \in I_n} \sum_{j=0}^{i-1} 1_{\{t_j < \tau_N\}} A^n_{t_j}^\beta \right] \\
&
+ \max_{i \in I_n} \sum_{j=0}^{i-1} 1_{\{t_j < \tau_N\}} \sigma_1(X^n_{t_j})^\beta \Delta W_{t_j}^\beta + \max_{i \in I_n} \sum_{j=0}^{i-1} \int_0^1 1_{\{t_j < \tau_N\}} \sigma_2(X^n_{t_j})^\beta \frac{1}{2} \tilde{N}(dz, \Delta t_j)^\beta
\end{align*}
(11)
\begin{align*}
&+ \max_{i \in I_n} \left[ V_{t_i \wedge \tau_N}^\beta + \max_{i \in I_n} |N_{t_i \wedge \tau_N}|^\beta \right].
\end{align*}
(13)

As the rest of proof is long and somewhat repetitive, we explain here the main idea by first focusing on the term with $b(x)$ in $\{1\}$. To estimate this term, we note that $1_{\{t_j < \tau_N\}} b(X^n_{t_j}) = 1_{\{t_j < \tau_N\}} b(X^n_{t_j \wedge \eta(\tau_N)})$ and by using linear growth condition on $b$ and the Jensen equality, we obtain
\begin{align*}
\max_{i \in I_n} \sum_{j=0}^{i-1} 1_{\{t_j < \tau_N\}} (X^n_{t_j})^\beta \Delta t_j \Delta t_j^\beta &\leq C \sum_{j=0}^{n-1} \frac{1}{|1 + \max(X^n_{t_j \wedge \eta(\tau_N)})^\beta|} |\Delta t_j|^\beta.
\end{align*}
(14)

From the above, we see that one is ready to apply the discrete Grönwall inequality as we have appropriately stopped the summand at $\eta(\tau_N)$ to make sure that it is bounded.

**Step 1:** Consider the terms in $\{1\}$, that is
\begin{align*}
\max_{i \in I_n} \sum_{j=0}^{i-1} 1_{\{t_j < \tau_N\}} b(X^n_{t_j})^\beta \Delta t_j^\beta.
\end{align*}

For the process $A^n$, we have by the Jensen inequality that
\begin{align*}
\mathbb{E} \left[ \max_{i \in I_n} \sum_{j=0}^{i-1} 1_{\{t_j < \tau_N\}} A^n_{t_j}^\beta \Delta t_j^\beta \right] &\leq n^{\beta-1} \sum_{j=0}^{n-1} \mathbb{E} [ |A^n_{t_j} |^\beta ].
\end{align*}

One can then apply Lemma 1.3 and Lemma 1.4 to obtain
\begin{align*}
\mathbb{E} [ |A^n_{t_j} \Delta t_j |^\beta ] &\leq C((\Delta t)^{2\beta} + \mathbb{E} [ |D_{t_{j+1}} | |F_{t_j} |^\beta ])
&\leq C((\Delta t)^{2\beta} + (\Delta t)^{\frac{\alpha}{\alpha - 1}} \exp \left( - C'((\Delta t)^{\frac{\alpha}{\alpha - 2}}) \right)) \leq C_T n^{-2\beta}.
\end{align*}

This gives the estimate
\begin{align*}
\mathbb{E} \left[ \max_{i \in I_n} \sum_{j=0}^{i-1} 1_{\{t_j < \tau_N\}} A^n_{t_j} \Delta t_j^\beta \right] &\leq C_T n^{-\beta}.
\end{align*}
(15)

**Step 2:** Given any discrete martingale $M$, for any $\beta > 1$, the Doob maximal inequality gives
\begin{align*}
\mathbb{E} \left[ \max_{i \in I_n} |M^{\beta}_{t_i} |^\beta \right] &\leq C_T \mathbb{E} [ |M_{t_i} |^\beta ].
\end{align*}

In addition, if the martingale $M$ is square integrable then by the discrete time Burkholder-Davis-Gundy inequality we obtain
\begin{align*}
\mathbb{E} \left[ \max_{i \in I_n} |M_{t_i} |^\beta \right] &\leq C_T \mathbb{E} [ |M_{t_i} |^{\beta/2} ] = C_T \mathbb{E} [ \left\{ \sum_{j=0}^{n-1} 1_{\{t_j < \tau\}} (\Delta M_{t_j})^\beta \right\}^{\frac{\beta}{2}} ].
\end{align*}
(16)

To this end, we recall that $X^n = \tilde{M}^n + \tilde{M}^n + \tilde{M}^n$, where $\tilde{M}^n, \tilde{M}^n$ are $L^2$-martingales, $\tilde{M}^n$ an $L^\beta$-martingale, with increments on the grid given by
\begin{align*}
\Delta M^n_{t_i} := \frac{2 \Delta M^n_{t_i} \Delta t_{t_i}}{1 + k \Delta t} - k \Delta t \sigma_2(X^n_{t_i})^\frac{1}{2} \Delta Z_{t_i},
\Delta \Delta M^n_{t_i} := \frac{\sigma_2(X^n_{t_i})^{2\gamma - 1}}{2(1 + k \Delta t)^2} ((\Delta W_{t_i})^2 - \Delta t_i),
\Delta M^n_{t_i} := \Delta M^n_{t_i} - \sigma_1(X^n_{t_i})^\gamma \Delta W_{t_i}.
\end{align*}
We first consider $\tilde{M} n$ and let $\kappa n := 1 + k\Delta t \geq \kappa 0$. From direct estimates, we have

$$
(\Delta \tilde{M} n)^2 = \left| \frac{\sigma_1 (X^n_t)^{2-1} \Delta W_t}{2 \kappa n} \sqrt{\sigma_1^2 (X^n_t)^{2\gamma-1} (\Delta W_t \kappa t)^2} + 4 \kappa n |D_{t+1}^n| - \sigma_1 (X^n_t)^{2\gamma} \Delta W_t \kappa t \right|^2
$$

$$
= \frac{\sigma_1^2 (X^n_t)^{2\gamma-1} (\Delta W_t \kappa t)^2}{4 \kappa n^2} \left| \sqrt{\sigma_1^2 (X^n_t)^{2\gamma-1} (\Delta W_t \kappa t)^2} + 4 \kappa n |D_{t+1}^n| - 2 \kappa n (X^n_t)^{2\gamma} \right|^2
$$

$$
\leq \frac{\sigma_1^2 (X^n_t)^{2\gamma-1} (\Delta W_t \kappa t)^2}{4 \kappa n^2} \times
$$

$$
\left| \sigma_1^2 (X^n_t)^{2\gamma-1} (\Delta W_t \kappa t)^2 + 4 \kappa n \left(1 - \kappa n^2\right) X^n_t + 2 D_{t+1}^{-1} + \left(\alpha - \frac{\sigma_1^2}{2} (X^n_t)^{2\gamma-1}\right) \Delta t + \sigma_2 (X^n_t)^{2\gamma} \Delta Z_t \right|
$$

$$
\leq C \left( (X^n_t)^{2\gamma-2} (\Delta W_t) \kappa t^4 + n^{-1} (X^n_t)^{2\gamma} (\Delta W_t) \kappa t^2 + (X^n_t)^{2\gamma-1} (\Delta W_t) \kappa t^2 D_{t+1}^{-1} \right.
$$

$$
+ n^{-1} (X^n_t)^{2\gamma-1} (\Delta W_t) \kappa t^2 + (X^n_t)^{2\gamma-1} (\Delta W_t) \kappa t^2 |\Delta Z_t| \left. \right]^{2} .
$$

(17)

By making use of the indicator function $1_{t_j = \Gamma T_N} \kappa t$ in (16), all $X^n_t \kappa t$ terms in (17) can be stopped at $\eta (\tau N) \kappa t$. We then proceed similarly to Lemma 3.2 of Gyöngy and Rásonyi [17] and take out $\max_{t \in I_n} X_{t_j \wedge \Gamma T_N} \kappa t$ from the sum, which gives us

$$
\left| \sum_{j=0}^{n-1} 1_{t_j < \Gamma T_N} \kappa t (\Delta \tilde{M} n_j)^2 \right|^2
$$

$$
\leq C \left[ \frac{1}{CK (1 - \frac{\beta}{2}) \kappa t \kappa n \max_{t \in I_n} X_{t_j \wedge \Gamma T_N} \kappa t} \right]^{\kappa t \kappa n} \left| \sum_{j=0}^{n-1} \left[ (X^n_{t_j \wedge \Gamma T_N})^{2\gamma-2-\kappa \theta} (\Delta W_t) \kappa t^4 + n^{-1} (X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1-\kappa \theta} (\Delta W_t) \kappa t^2 D_{t+1}^{-1} \right. \right.
$$

$$
+ n^{-1} (X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1-\kappa \theta} (\Delta W_t) \kappa t^2 \left. \right] \left[ (X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1-\kappa \theta} (\Delta W_t) \kappa t^2 + (X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1} (\Delta W_t) \kappa t^2 |\Delta Z_t| \right] \right|^{\kappa \theta} .
$$

(18)

where $K > 0$ is a constant which can be freely chosen, and $\theta \in (2\gamma - \beta, 2\gamma - 1)$ is a constant chosen in such a way that all powers of $X^n_t \kappa t$ in the above are positive and smaller than $\beta$. For explicit example, the fact that $\theta < 2 - \beta$ ensures that $\theta \beta / (2 - \beta) < \beta$ in the first term of (18). On the other hand, in the sum, the largest power is $2\gamma$, and the condition $\theta > (2\gamma - \beta)$ guarantees that $2\gamma - \theta < \beta$, while for the smallest power $2\gamma - 1$, we will have $2\gamma - 1 - \theta > 0$. Then by applying the Young inequality with $p = 2 / (2 - \beta)$ and $q = 2 / \beta$ to the right hand side of (18), we obtain

$$
\left| \sum_{j=0}^{n-1} 1_{t_j < \Gamma T_N} \kappa t (\Delta \tilde{M} n_j)^2 \right|^2 \leq \frac{1}{K \kappa n \kappa t \kappa n \max_{t \in I_n} X_{t_j \wedge \Gamma T_N} \kappa t} \left[ \left( X_{t_j \wedge \Gamma T_N} \right)^{2\gamma-1-\theta} (\Delta W_t) \kappa t^4 + n^{-1} (X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1-\theta} (\Delta W_t) \kappa t^2 D_{t+1}^{-1} \right.
$$

$$
+ n^{-1} (X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1-\theta} (\Delta W_t) \kappa t^2 \left. \right] \left[ (X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1} (\Delta W_t) \kappa t^2 + (X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1} (\Delta W_t) \kappa t^2 |\Delta Z_t| \right] \right|^{\kappa \theta} .
$$

(19)

The next step is to take the expectation in equation (19). To do that, we first evaluate the term involving $D_{t+1}^{-1}$ in the above. By independence we have

$$
E[(X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1-\theta} (\Delta W_t) \kappa t^2 D_{t+1}^{-1}] = n^{-1} E[(X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1-\theta} D_{t+1}^{-1}],
$$

and then since $2\gamma - \theta < \beta < \alpha$, we can apply Hölder’s inequality as such

$$
E[(X^n_{t_j \wedge \Gamma T_N})^{2\gamma-1-\theta} D_{t+1}^{-1}] \leq E[(X^n_{t_j \wedge \Gamma T_N})^{2\gamma-\theta} D_{t+1}^{-1}]^{2\gamma / 2 - \kappa \theta} \left[ \frac{1}{2 - \kappa \theta} E[(D_{t+1}^{-1})^{2\gamma-1}] \right]^{\kappa \theta / (2 - \kappa \theta)}
$$

\[ \leq C n^{-1} \exp \left( - c \frac{2 - \kappa \theta}{\alpha - \kappa \theta} \right) \left( 1 + E[(X^n_{t_j \wedge \Gamma T_N})^{2\gamma-\theta}] \right), \]
where we also used Lemma [14] and the linear growth condition. Then we can take the mathematical expectation of \( \{1\} \), and using independence among \( X_{t_i}, \Delta W_{t_i}, \Delta Z_{t_i} \), we obtain

\[
E\left[ \sum_{j=0}^{n-1} 1_{\{t_j < \tau_N\}} (\Delta \dot{M}_{t_j}^n)^2 \right] \leq \frac{1}{K} \max_{i \in t_n} \left( X_{t_i}^{\tau_N} \right)^{2\beta} + C \beta \sum_{j=0}^{n-1} \left[ E\left( |X_{t_j}^{\tau_N}|^{4\gamma - 2\gamma} \right) \right] n^{-2} + E\left[ |X_{t_j}^{\tau_N}|^{2\gamma - \theta} \right] n^{-2} + 2 \exp \left( -c n \right) \left( 1 + E\left( |X_{t_j}^{\tau_N}|^{2\gamma - \theta} \right) \right) + E\left( |X_{t_j}^{\tau_N}|^{2\gamma - \theta} \right) n^{-2} + E\left[ |X_{t_j}^{\tau_N}|^{4\gamma - 2\gamma - 2\theta} \right] n^{-2} + E\left[ |X_{t_j}^{\tau_N}|^{2\gamma - 1 + \frac{\theta}{2}} \right] n^{-1 - \frac{\theta}{2}}. \]

Note that given our choice of \( \theta \), all the powers of \( X_{t_j}^{\tau_N} \) in the right hand side above are positive and smaller than \( \beta \), so using the linear growth condition, the above can then be bounded by:

\[
\frac{1}{K} \max_{i \in t_n} \left( X_{t_i}^{\tau_N} \right)^{\beta} + C T + \frac{CE}{n} \int_0^T \left( X_{t_i}^{\tau_N} \right)^{\beta} ds. \]

This shows that

\[
E\left[ \max_{i \in t_n} \left( \dot{M}_{t_i}^{\tau_N} \right)^{\beta} \right] \leq \frac{1}{K} \max_{i \in t_n} \left( X_{t_i}^{\tau_N} \right)^{\beta} + C T + \frac{CE}{n} \int_0^T \left( X_{t_i}^{\tau_N} \right)^{\beta} ds. \tag{20} \]

For the martingale \( \dot{M}^n \), we again make use of \([10]\) and proceed similarly to \([18]\). That is we have

\[
\left| \sum_{j=0}^{n-1} 1_{\{t_j < \tau_N\}} (\Delta \dot{M}_{t_j}^n)^2 \right|^{\frac{2\beta}{2}} = \left| \sum_{j=0}^{n-1} 1_{\{t_j < \tau_N\}} \frac{\sigma^n_{\alpha} (X_{t_j}^{\tau_N})^{4\gamma - 2\gamma} - (\Delta W_{t_j})^2 - \Delta t_j^2 \right|^{\frac{\beta}{2}} \]

\[
\leq \frac{\sigma^n_{\alpha} (X_{t_j}^{\tau_N})^{4\gamma - 2\gamma} - (\Delta W_{t_j})^2 - \Delta t_j^2 \right|^{\frac{\beta}{2}} \]

\[
\leq \frac{\sigma^n_{\alpha} (X_{t_j}^{\tau_N})^{4\gamma - 2\gamma} - (\Delta W_{t_j})^2 - \Delta t_j^2 \right|^{\frac{\beta}{2}} \]

for some \( K > 0 \) and \( \theta \in ((4\gamma - 2 - \beta)^+, 4\gamma - 2 \wedge (2 - \beta)) \), which is a constant that was chosen so that all powers of \( X_{t_j}^{\tau_N} \) in the right hand side are positive and smaller than \( \beta \). Let \( C = \sigma^n_{\alpha} / 2\kappa^n_{\alpha} \), we rewrite the right hand side above into

\[
C \left( \frac{1}{CK (1 - \frac{\beta}{2})} \right) \max_{i \in t_n} \left( X_{t_i}^{\tau_N} \right)^{\frac{4\gamma - 2\gamma}{2\beta}} \left| \left( \Delta W_{t_j} \right)^2 - \Delta t_j^2 \right|^{\frac{\beta}{2}}, \]

and by applying the Young inequality with \( p = 2/(2 - \beta) \) and \( q = 2/\beta \) we obtain

\[
\left| \sum_{j=0}^{n-1} 1_{\{t_j < \tau_N\}} (\Delta \dot{M}_{t_j}^n)^2 \right|^\frac{\beta}{2} \]

\[
\leq \frac{1}{K} \max_{i \in t_n} \left( X_{t_i}^{\tau_N} \right)^{\frac{4\gamma - 2\gamma}{2\beta}} + \frac{C \beta}{2} \right( \frac{1}{CK (1 - \frac{\beta}{2})} \right) \left( \Delta W_{t_j} \right)^2 - \Delta t_j^2 \right|^{\frac{\beta}{2}}. \]

Again given the choice of \( \theta \), both powers of \( X_{t_j}^{\tau_N} \) in the above are positive and smaller than \( \beta \), and by making use of the linear growth condition, we have

\[
E\left[ \left| \sum_{j=0}^{n-1} 1_{\{t_j < \tau_N\}} (\Delta \dot{M}_{t_j}^n)^2 \right|^{\frac{2\beta}{2}} \right] \leq \frac{1}{K} \max_{i \in t_n} \left( X_{t_i}^{\tau_N} \right)^{\frac{4\gamma - 2\gamma}{2\beta}} + C T + C \int_0^T \left( X_{t_i}^{\tau_N} \right)^{\beta} ds. \]

Thus for \( \dot{M}^n \) we have the following estimate

\[
E\left[ \max_{i \in t_n} \left( \dot{M}_{t_i}^{\tau_N} \right)^{\beta} \right] \leq \frac{1}{K} \max_{i \in t_n} \left( X_{t_i}^{\tau_N} \right)^{\beta} + C T + C \int_0^T \left( X_{t_i}^{\tau_N} \right)^{\beta} ds. \tag{21} \]
To estimate the $L^\beta$-martingale $\hat{M}^n$, we apply the Doob maximal inequality and the Jensen inequality to obtain that
\[
E\left[ \max_{i \in I^n} |\hat{M}^n_{t_i \wedge \tau_N}|^\beta \right] \leq (1 - \beta^{-1})^{-\beta} E\left[ \sum_{j=0}^{n-1} 1_{(t_j < \tau_N)} |\Delta \hat{M}^n_{t_j}|^\beta \right]
\leq (1 - \beta^{-1})^{-\beta} n^{-\frac{\beta}{2}} \frac{\alpha}{\beta(\alpha + 1)} E[|\Delta \hat{M}^n_{t_0}|^\beta].
\]

By using Lemma 13 independence and Lemma 14, the summand can be estimated as follows
\[
E[|\Delta \hat{M}^n_{t_0}|^\beta] \leq C_T \left( n^{-\frac{\beta}{2}} \exp(-C_{2n}^{\frac{2}{\beta(\alpha + 1)}}) + n^{-\frac{\beta}{2}} E[|X^n_{t_0}|^\frac{\beta}{2}] \right)
\leq C_T \left( n^{-\frac{\beta}{2}} \exp(-C_{2n}^{\frac{2}{\beta(\alpha + 1)}}) + n^{-\frac{\beta}{2}} \right),
\]
which gives the estimate
\[
E\left[ \max_{i \in I^n} |\hat{M}^n_{t_i \wedge \tau_N}|^\beta \right] \leq C_T n^{-\frac{\beta}{2}}.
\tag{23}
\]

**Step 3:** We compute the stochastic integrals against $L^2$-martingales in \([22]\). By using the set inclusion \(\{s \leq \tau_N\} \subset \{\eta(s) \leq \eta(\tau_N)\}\), we can ease the computation by consider the continuous extension of the integrals and applying the Burkholder-Davis-Gundy inequality to obtain to obtain the following
\[
E\left[ \max_{i \in I^n} \int_0^{t_i \wedge \tau_N} \sigma_1(X^n_{\eta(s) \wedge \eta(\tau_N)})^\beta dW_s \right] \leq C E\left[ \left| \int_0^T |X^n_{\eta(s) \wedge \eta(\tau_N)}|^2 ds \right|^\frac{\beta}{2} \right] + CT^\frac{\beta}{2},
\]
where we have used the linear growth condition. Similarly, we have
\[
E\left[ \max_{i \in I^n} \int_0^{t_i \wedge \tau_N} \int_0^1 \sigma_2(X^n_{\eta(s) \wedge \eta(\tau_N)})^\frac{\beta}{2} z N(dz, ds) \right] \leq C E\left[ \left| \int_0^T \int_0^1 (X^n_{\eta(s) \wedge \eta(\tau_N)})^\frac{\beta}{2} z^2 N(dz) ds \right|^\frac{\beta}{2} \right]
\leq C_{\alpha, \beta} E\left[ \left| \int_0^T |X^n_{\eta(s) \wedge \eta(\tau_N)}|^2 ds \right|^\frac{\beta}{2} \right] + C_{\alpha, \beta} T^\frac{\beta}{2}.
\]

To this end, we can apply similar techniques to those in \([18] and [19] from Step 2. That is we first write
\[
\int_0^T |X^n_{\eta(s) \wedge \eta(\tau_N)}|^2 ds \leq \int_0^T \left( |X^n_{\eta(s) \wedge \eta(\tau_N)}|^\frac{\beta}{2} \right)^{\frac{2(1 - \frac{\beta}{2})}{\beta}} |X^n_{\eta(s) \wedge \eta(\tau_N)}|^{\frac{\beta}{2}} ds \leq \frac{1}{K(1 - \frac{\beta}{2})} \max_{i \in I^n} \left| X^n_{\eta(s) \wedge \eta(\tau_N)} \right|^{\frac{2(1 - \frac{\beta}{2})}{\beta}} \int_0^T |X^n_{\eta(s) \wedge \eta(\tau_N)}|^2 ds \frac{\beta}{2},
\]
and then apply Young’s inequality with \(p = \frac{2}{\beta - 2}\) and \(q = \frac{2}{\beta}\) to obtain
\[
\int_0^T |X^n_{\eta(s) \wedge \eta(\tau_N)}|^2 ds \leq \frac{1}{K} \max_{i \in I^n} \left| X^n_{t_i \wedge \eta(\tau_N)} \right|^{\frac{\beta}{2}} + \frac{\beta}{2} \left( K(1 - \frac{\beta}{2}) \right)^{\frac{2(1 - \frac{\beta}{2})}{\beta}} \int_0^T |X^n_{\eta(s) \wedge \eta(\tau_N)}|^2 ds,
\]
and finally, we have
\[
E\left[ \int_0^T |X^n_{\eta(s) \wedge \eta(\tau_N)}|^2 ds \right]^\frac{\beta}{2} \leq \frac{1}{K} E\left[ \max_{i \in I^n} |X^n_{t_i \wedge \eta(\tau_N)}|^\beta \right] + C_{\beta, K} \int_0^T E[|X^n_{\eta(s) \wedge \eta(\tau_N)}|^\beta] ds, \tag{24}
\]
where again the constant \(K > 0\) can be freely chosen.

**Step 4:** The last term to be computed is the Poisson integral in \([13]\) given by
\[
V_{t_i} = \sigma_2 \int_0^{t_i} \int_1^\infty (X^n_{\eta(s)})^{\frac{\beta}{2}} z N(dz, ds).
\]
We note that the continuous extension of $V$ is positive and from the Itô formula and the Jensen inequality
\[
\max_{i \in I_n} |V_{t_i}^{\gamma N}|^\beta \leq \sup_{t \in T} |V_t^{\gamma N}|^\beta = \int_0^T \int_1^\infty ((V_{t_i}^{\gamma N} + \sigma_2(X_{t_i}^{\gamma N})^{\frac{\beta}{2}} z) - (V_{t_i}^{\gamma N})^\beta) N(dz, ds)
\]
\[
\leq C \int_0^T \int_1^\infty ((|V_{t_i}^{\gamma N}|^\beta + |X_{t_i}^{\gamma N}|^{\frac{\beta}{2}} z)^\beta) N(dz, ds).
\]
The integrand in the above is bounded and predictable, therefore by Lemma 1.3
\[
\mathbb{E}\left[ \sup_{t \leq T} |V_t^{\gamma N}|^\beta \right] \leq C \int_0^T \left( \mathbb{E}\left[ |V_s^{\gamma N}|^\beta \right] + \mathbb{E}\left[ |X_{t_i}^{\gamma N}(s)|^{\frac{\beta}{2}} \right] \right) ds \leq C \int_0^T \mathbb{E}\left[ \sup_{u \leq s} |V_u^{\gamma N}|^\beta \right] ds + C_T.
\]
We can now apply Grönwall’s inequality to obtain
\[
\mathbb{E}\left[ \sup_{t \leq T} |V_t^{\gamma N}|^\beta \right] \leq C_T. \quad (*5)
\]

**Step 5:** By combining the estimates (14), (15), (20), (21), (23), (24) and (25) to obtain
\[
\mathbb{E}\left[ \max_{i \in I_n}(X_{t_i}^{\gamma N}(s))^\beta \right] \leq C_T \left( 1 + n^{-\frac{\alpha}{2}} + n^{-\beta} + \sum_{j=0}^{n-1} \mathbb{E}\left[ \max_{i \in I_n}(X_{t_i}^{\gamma N}(s))^\beta \right] \Delta t_j \right) + \frac{5}{K} \mathbb{E}\left[ \max_{i \in I_n}(X_{t_i}^{\gamma N}(s))^\beta \right],
\]
and then by choosing, for example $K = 10$, we obtain
\[
\mathbb{E}\left[ \max_{i \in I_n}(X_{t_i}^{\gamma N}(s))^\beta \right] \leq C_T \left( 1 + \sum_{j=0}^{n-1} \mathbb{E}\left[ \max_{i \in I_n}(X_{t_i}^{\gamma N}(s))^\beta \right] \Delta t_j \right).
\]
From the above one can conclude by applying the discrete Grönwall inequality and letting $N \uparrow \infty$. □

**Lemma 1.8.** (i) For $\beta \in [1, 2]$, we have
\[
\max_{i=0,1,\ldots,n-1} \mathbb{E}[\Delta \hat{M}_{t_i}^\beta] \leq C n^{-\beta} \quad \text{and} \quad \max_{i=0,1,\ldots,n-1} \mathbb{E}[\Delta \hat{\bar{M}}_{t_i}^\beta] \leq C n^{-\frac{\beta}{2} - \frac{\beta}{2}},
\]
and for any $t \in [0, T]$,
\[
\mathbb{E}[\hat{M}_t^\beta] \leq C n^{-\frac{\beta}{2}} \quad \text{and} \quad \mathbb{E}[\hat{\bar{M}}_t^\beta] \leq C n^{-\frac{\beta}{2}}.
\]
(ii) For $\beta \in [1, \alpha)$, we have
\[
\max_{i=0,1,\ldots,n-1} \mathbb{E}[\Delta \hat{M}_{t_i}^\beta] \leq C n^{-\beta - \frac{\beta}{2}} \quad \text{and} \quad \max_{i=0,1,\ldots,n-1} \mathbb{E}[\Delta \hat{\bar{M}}_{t_i}^\beta] \leq C n^{-2\beta}
\]
and for any $t \in [0, T]$,
\[
\mathbb{E}[\hat{M}_t^\beta] \leq C n^{-\frac{\beta}{2}} \quad \text{and} \quad \mathbb{E}[\int_{(0,t]} A_{n(s)}^\beta ds^\beta] \leq C n^{-\beta}.
\]

**Remark 1.3.** We note that $\hat{M}$ has the slowest rate, but it is also square integrable. The proof of strong convergence is designed so that we make use of the square integrability whenever possible.

**Proof.** From the Jensen inequality, we have
\[
\mathbb{E}\left[ \int_{(0,t]} A_{n(s)}^\beta ds^\beta \right] \leq n^{\beta-1} \sum_{i=0}^{n} \mathbb{E}[\hat{M}_{t_i}^\beta] \leq C_T n^{-\beta},
\]
where in the second inequality we have used Lemma 1.3, Jensen’s inequality and Lemma 1.4 to obtain the fact that $\mathbb{E}[\Delta \hat{M}_{t_i}^\beta] \leq C_T n^{-2\beta}$.
For the martingale $\hat{M}^n$, we apply Hölder’s inequality, inequality (17) and Lemma 1.7 to obtain
\[
E[|\Delta \hat{M}^n_t|^\beta] \leq C \left\{ n^{-2}E[|X^n_{t_i}|^{\gamma - 2}] + n^{-2}E[|X^n_{t_i}|^{\gamma}] + E[|X^n_{t_i}|^{2\gamma - 1}] n^{-\frac{1}{\gamma}} \exp \left( -C_n \frac{2}{\gamma - 2} \right) + n^{-2}E[|X^n_{t_i}|^{2\gamma - 1}] \right\}^\frac{\beta}{2} 
\]
\[
\leq C n^{-\frac{\beta}{2} - \frac{\beta}{\gamma}}.
\]
Then, similar to (16), we apply the discrete time Burkholder-Davis-Gundy inequality and Jensen’s inequality to obtain that
\[
E[|\tilde{M}^{n}_{j+1}|^\beta] \leq CE \left\{ \sum_{i=0}^{n-1} |\Delta \tilde{M}^n_t|^2 \right\}^\frac{\beta}{2} \leq C n^{-\frac{\beta}{2}}.
\]
For the martingale $\hat{M}^n$, we have
\[
E[|\Delta \hat{M}^n_t|^\beta] = CE[|X^n_{t_i}|^{\beta(2\gamma - 1)}] E[|\Delta W_t|^2] \leq C n^{-\beta},
\]
and also $E[|\Delta \hat{M}^n_t|^2] \leq n^{-2}$ since $4\gamma - 2 < \alpha$. Then we have
\[
E[|\tilde{M}^{n}_{j+1}|^\beta] \leq CE \left\{ \sum_{i=0}^{n-1} |\Delta \tilde{M}^n_t|^2 \right\}^\frac{\beta}{2} \leq C n^{-\frac{\beta}{2}}.
\]
For the martingale $\tilde{M}^n$, from (22) we see that
\[
E[|\Delta \tilde{M}^n_t|^\beta] \leq C n^{-\beta - \frac{\beta}{\gamma}},
\]
and then by the Doob maximal inequality and the Jensen inequality together give us the following
\[
E[|\tilde{M}^{n}_{j+1}|^\beta] \leq C n^{-\beta - 1} \sum_{j=0}^{n-1} E[|\Delta \tilde{M}^n_t|^\beta] \leq C n^{-\frac{\beta}{2}}.
\]

Lemma 1.9. For $\beta \in [1, \alpha)$ there exists a positive constant $C$ that
\[
\sup_{t \leq T} E[|\tilde{X}^n_t - X^n_{\eta(t)}|^\beta] \leq C n^{-\frac{\beta}{2}}.
\]

Proof. From (24), we see that by the Jensen inequality $E[|\tilde{X}^n_t - X^n_{\eta(t)}|^\beta]$ can be upper estimated by
\[
E[|\tilde{X}^n_t - X^n_{\eta(t)}|^\beta] \leq (1 + |K|^\beta E[|X^n_{\eta(t)}|^\beta]) n^{-\beta} + C_1 E[|X^n_{\eta(t)}|^\beta] n^{-\frac{\beta}{2}} + \sigma_1^\beta E[|X^n_{\eta(t)}|^\beta] n^{-\frac{\beta}{2}} + \sigma_2^\beta E[|X^n_{\eta(t)}|^\beta] n^{-\frac{\beta}{2}} + E[|\tilde{M}^n_t - \tilde{M}^n_{\eta(t)}|^\beta] + E[|\hat{M}^n_t - \hat{M}^n_{\eta(t)}|^\beta] + E[|\hat{M}^n_t - \hat{M}^n_{\eta(t)}|^\beta] + E[|\tilde{M}^n_t - \tilde{M}^n_{\eta(t)}|^\beta] + E[|\tilde{M}^n_t - \tilde{M}^n_{\eta(t)}|^\beta].
\]
The above can be further estimated by using the Hölder inequality, Lemma 1.7 and Lemma 1.8 giving us
\[
E[|\tilde{X}^n_t - X^n_{\eta(t)}|^\beta] \leq C n^{-\frac{\beta}{2}}.
\]

2 Strong Rate of Convergence

In this section, we state our main result on the strong rate of convergence. We present below only the case of the alpha-CEV and for $k > 0$. The reason is that the rate which we obtain in the case $k < 0$ and also in the case of the alpha-CIR are far from what we believe is optimal. We mention that the proof of convergence for those two cases are identical to that of Theorem 2.1 and the rate of convergence for the alpha-CIR can be improved from logarithmic to be polynomial. The sub-optimal rate for the alpha-CIR is due to the parameter constraints on the existence of inverse moments in Lemma 1.6.
Theorem 2.1. Suppose Assumption \[1.1\] hold, \( \gamma \in (1/2, 1] \) and \( k > 0 \), then there exists some constant \( C_T > 0 \) such that

\[
\sup_{t \leq T} \mathbb{E}[|X_t - X^n_t|] \leq C_T n^{-\frac{1}{2}q(\alpha)}
\]

where \( q(\alpha) \) is given by

\[
q(\alpha) = \begin{cases} 
\frac{\alpha}{\gamma} & \alpha \in (1, \sqrt{2}] \\
\frac{1}{2} & \alpha \in (\sqrt{2}, 2) 
\end{cases}
\]

and the constant \( \alpha_- \in (1, \alpha) \) can be chosen arbitrarily close to \( \alpha \).

Remark 2.1. We point out that the assumption \( k > 0 \) is only used in Lemma \[1.6\] to obtain inverse moment estimates for \( X \) on the whole time horizon \([0, T]\. For \( k \leq 0 \), one can obtain a slower polynomial rate of convergence which depends on \( \gamma \) and is comparable to the rate obtained for the Euler-Maruyama scheme studied in Frikha and Li \[12\].

Remark 2.2. The rate \( \frac{1}{2}q(\alpha) = \frac{1}{2} \left( \frac{\alpha}{\gamma} \wedge \frac{1}{2} \right) \) obtained in Theorem \[2.1\] is likely very close to the optimal rate, as numerical simulations in section \[3\] appears to suggest a rate of \( \frac{1}{2} \).

To proceed, we set \( \bar{\sigma}_n \) for square integrable martingales in a Lévy filtration (see e.g. Applebaum \[4\]) we know that there where each of the terms above are given by

\[
\epsilon \sup_{t \leq T} \mathbb{E}[|X_t - X^n_t|] \leq C_T n^{-\frac{1}{2}q(\alpha)}
\]

\( \bar{\sigma}_n \) is the case of the Brownian motion taken to the power of \( \alpha \) and the rate \( \frac{1}{2} \) comes from the increment of the Brownian motion constraint in computing \[29\], that is \( |X^n - X^n_s| \) has only moments up to \( \alpha \) and the rate \( \frac{1}{2} \) comes from the increment of the Brownian motion taken to the power of \( \alpha \). Without going into the technical details, the term \( \frac{1}{2} \) in \( q(\alpha) \) derives from the increment of the alpha-stable process in the remainder process \( \bar{R}^n \). While, the appearance of the term \( \frac{\alpha}{\gamma} \) is due to (i) an integrability constraint in computing \[29\], that is \( |X^n - X^n_s| \) has only moments up to \( \alpha \) and the rate \( \frac{1}{2} \) comes from the increment of the Brownian motion taken to the power of \( \alpha \) and, (ii) without the inverse moment estimate of the scheme \( X^n \), we were not able to improve the estimate of the term \( |X^n - X^n_s|^{\alpha/2} \) from the difference of the jump coefficients in \[31\] and \[36\]. In the case of the diffusion C\( CEV \), we do not encounter the above-mentioned issues. That is one can take \( \alpha_- = 2 \) and the term \( \frac{1}{2} \) will not appear, which gives us a rate of \( \frac{1}{2} \). For more details see Corollary \[2.7\].

Proof. We consider the decomposition \( X - X^n = X - \bar{X}^n - \bar{R}^n \). From the proof of Lemma \[1.8\] we can easily obtain that \( \mathbb{E}[|\bar{R}^n_t|] \leq C n^{-\frac{1}{2}} \), and we focus on estimating \( |X - \bar{X}^n| \). From the martingale representation theorem for square integrable martingales in a Lévy filtration (see e.g. Applebaum \[4\]) we know that there exist predictable and square-integrable processes \( F \) and \( G \) such that for \( t \geq 0 \) we have

\[
\bar{M}^n_t + \bar{M}^n_t = \int_0^t F(s) dW_s + \int_0^t \int_0^\infty G(s, z) \tilde{N}(dz, ds).
\]

To proceed, we set \( \bar{Y}^n := X - \bar{X}^n \) and denote its jumps by \( \Delta \bar{Y}^n(z) := \sigma_2 [X^n_z - (X^n_{\bar{N}(s)})^\gamma] z - G(t, z) \). Suppose \( \epsilon \in (0, 1) \) and \( \delta > 1 \), by using \[40\] and the Ito formula, we have the estimate

\[
|\bar{Y}^n_t| \leq \epsilon + \phi_{\delta, \epsilon}(\bar{Y}^n_t) = \epsilon + M^n_{t, \delta, \epsilon} + I^n_{t, \delta, \epsilon} + J^n_{t, \delta, \epsilon} + K^n_{t, \delta, \epsilon},
\]

where each of the terms above are given by

\[
M^n_{t, \delta, \epsilon} := \int_0^t \phi_{\delta, \epsilon}(\bar{Y}^n_{s-}) (\sigma_1 [X^n_{s-} - (X^n_{\bar{N}(s)})^\gamma] - F(s)) dW_s,
\]

\[
I^n_{t, \delta, \epsilon} := \int_0^t \int_0^\infty \phi_{\delta, \epsilon}(\bar{Y}^n_{s-}) \{ \phi_{\delta, \epsilon}(\bar{Y}^n_s + \Delta \bar{Y}^n_s(z)) - \phi_{\delta, \epsilon}(\bar{Y}^n_{s-}) \} \tilde{N}(ds, dz),
\]

\[
J^n_{t, \delta, \epsilon} := \int_0^t \phi_{\delta, \epsilon}(\bar{Y}^n_{s-}) \{ -k X_{s-} + k \sigma_1 [X^n_{s-} - (X^n_{\bar{N}(s)})^\gamma] \} ds,
\]

\[
K^n_{t, \delta, \epsilon} := \frac{1}{2} \int_0^t \phi_{\delta, \epsilon}(\bar{Y}^n_{s-}) \{ \sigma_1 [X^n_{s-} - (X^n_{\bar{N}(s)})^\gamma] - F(s) \}^2 ds,
\]

Using the standard localisation arguments, the martingale term \( M^n_{t, \delta, \epsilon} \) can be eliminated by taking expectation, and we then focus on the upper estimates for \( I^n_{t, \delta, \epsilon} \), \( J^n_{t, \delta, \epsilon} \) and \( K^n_{t, \delta, \epsilon} \).
Estimates for $I_{n,k}^{r,k,e}$, $J_{n,k}^{r,k,e}$ and $K_{n,k}^{r,k,e}$: Let us first consider the term $I_{n,k}^{r,k,e}$. Note that the quantity $|k_n|^{-1}$ is of order $n^{-1}$ and we can apply Lemma 1.1 and Lemma 1.9 to obtain
\[
E[I_{n,k}^{r,k,e}] \leq \frac{\sigma_n^2}{2} \int_0^\tau \phi_{k_n}^n(\bar{Y}_s) \left((X_s)^r - (X_n^{n})^r\right)^2 ds \\
\leq C \int_0^\tau \phi_{k_n}^n(\bar{Y}_s) \left((X_s)^r - (X_n^{n})^r\right)^2 ds + C \int_0^\tau \phi_{k_n}^n(\bar{Y}_s) F(s)^2 ds.
\] (27)
The second term in (27) can be estimated using (42) and the fact that $E[z^2] \leq C_{n^{-\frac{1}{3}}}$. To estimate the first term in (27), we use the inequality that, for $x, y \in \mathbb{R}_+$ and $\kappa > \gamma \geq 0$,\[
|x^\gamma - y^\gamma| \leq |x^\kappa - y^\kappa| |x^{-(\kappa - \gamma)}|.
\] (28)Then for any $\alpha_{-}$ such that $2\gamma < \alpha_{-} < \alpha$, we have\[
\int_0^\tau \phi_{k_n}^n(\bar{Y}_s) \left((X_s)^\gamma - (X_n^{n})^\gamma\right)^2 ds \leq C \int_0^\tau \phi_{k_n}^n(\bar{Y}_s) \left((X_s)^\gamma - (X_n^{n})^\gamma\right)^2 ds \leq C \int_0^\tau \phi_{k_n}^n(\bar{Y}_s) \left((X_s)^\gamma - (X_n^{n})^\gamma\right)^2 ds + C \int_0^\tau \phi_{k_n}^n(\bar{Y}_s) F(s)^2 ds.
\] (27)
where the second inequality is obtained by using again (28) and triangular inequality. By using (42) and the inverse moment estimate of $X$ in Lemma 1.6, the expected value of the above can be further upper estimated by\[
C \left\{ \varepsilon + \frac{\delta}{\varepsilon \log \delta} E[\int_0^\tau (X_s^n - X_n^{n})^{\alpha - (\alpha - 2\gamma)} ds] \right\}.
\] where the constant $\alpha_{-} < \alpha$ can be chosen to be arbitrarily close to $\alpha$. To proceed, we let $p > 1$ so that $p \alpha_{-} < \alpha$ and $q > 1$ is such that $p^{-1} + q^{-1} = 1$. Next, by applying the Hölder inequality with $p, q$, we obtain\[
\frac{C \delta}{\varepsilon \log \delta} E[\int_0^\tau (X_s^n - X_n^{n})^{\alpha - (\alpha - 2\gamma)} ds] \leq \frac{C \delta}{\varepsilon \log \delta} E[\int_0^\tau (X_s^n - X_n^{n})^{p \alpha_{-}} ds] \leq \frac{C \delta}{\varepsilon \log \delta} n^{-\frac{\alpha_{-}}{p - 2}}.
\] Note that although the quantity $q$ can be very large, we can still control the inverse moments using Lemma 1.6 and obtain an upper estimate which is independent of $p$ and $q$. More explicitly, we know from Lemma 1.6 that for $\gamma \in (1/2, 1)$\[
\sup_{t \in T} E[|X_t|^q] \leq (x_0^{-\gamma} + C \delta) \exp\{T(q + 1)/\alpha\} \leq C_1 \exp\{C_2 q\},
\] which implies that $E[|X_t|^q]^{\frac{1}{q}} \leq C \exp(C)$ for some constant $C$. Therefore, by combining all the above estimates, we obtain\[
E[I_{n,k}^{r,k,e}] \leq C \left\{ \frac{\varepsilon}{\log \delta} + \frac{\delta}{\varepsilon \log \delta} (n^{-\frac{\alpha_{-}}{p - 2}} + n^{-\frac{1}{q}}) \right\}.
\] (29)It is important to point out here that the worst rate in the remainder $R$ is coming from the square integrable martingale $\bar{M}$. This observation motivated our choice of the proxy process $\bar{X}^n$. In this way the process $\bar{M}$
is embedded in $\int_0^t F(s) dW_s$, which is taken to the power of two. This allows one to achieve a better rate than if the continuous time extension $X^n$ or the forward Euler scheme was used as a proxy.

Before proceeding to the estimate of the jump terms, we point out that the above estimates are purely associated with the diffusion part and, although some later computations might not be optimal, we will not strive to improve them as long as the obtained rates are greater or equal to those obtained in (26) and (29).

Next, we estimate the term $K_t^{n,\delta,\varepsilon}$. In the following, we make use of Lemma 4.1, Lemma 4.2 and, for $\alpha_0 \geq \alpha$, the quantities

$$ I_{t_1} := x^{\alpha_0} \int_0^x z \nu(dz) = \frac{x^{\alpha_0} - x}{2 - \alpha} \text{ and } J_{t_1} := x^{\alpha_0} \int_x^\infty z \nu(dz) = \frac{x^{\alpha_0} - x}{\alpha - 1}. $$

We first decompose $K_t^{n,\delta,\varepsilon}$ into $K_t^{n,\delta,\varepsilon,1} + K_t^{n,\delta,\varepsilon,2} + K_t^{n,\delta,\varepsilon,3}$ where

$$ K_t^{n,\delta,\varepsilon,1} := \int_0^t \int_0^\infty \left( \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^- + \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}]z) - \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^-) - \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}]z\phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^-) \right) \nu(dz)ds, $$

$$ K_t^{n,\delta,\varepsilon,2} := \int_0^t \int_0^\infty \left( \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^- + \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_{n(s)})^+)\frac{\delta}{\varepsilon}]z) - \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^- + \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_{n(s)})^+)\frac{\delta}{\varepsilon}]z) \right) \nu(dz)ds, $$

$$ K_t^{n,\delta,\varepsilon,3} := \int_0^t \int_0^\infty \left( \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^- + \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_{n(s)})^+)\frac{\delta}{\varepsilon}]z - G(s, z)) \right) - \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^- + \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_{n(s)})^+)\frac{\delta}{\varepsilon}]z) \right) \nu(dz)ds. $$

To estimate the term $K_t^{n,\delta,\varepsilon,1}$, we apply Lemma 4.1 with $y = Y_{n,s}^- = X_{n,s}^- - \bar{X}^n_s$ and $x = \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}]$.

Since $xy \geq 0$, we have for any $y > 0$

$$ \int_0^\infty \left( \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^- + \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}]z) - \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^-) - \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}]z\phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^-) \right) \nu(dz) $$

$$ \leq \frac{C}{\log \delta} \int_0^\infty \left( \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^- + \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}]z) - \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^-) - \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}]z\phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^-) \right) \nu(dz) $$

$$ \leq C_T \varepsilon \left( X_{n,s}^{-2(1-\alpha)} \right) \int_0^\infty \left( \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^- + \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}]z) - \phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^-) - \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}]z\phi_{\delta,\varepsilon}(\bar{Y}_{n,s}^-) \right) \nu(dz) $$

where $\alpha_1 \in [\alpha, 2]$. Then we can set $u = \log(\delta)$ to balance the two terms and the above is upper estimated by

$$ C_T \varepsilon \log(\delta)^{1-\alpha} \left( X_{n,s}^{-2(1-\alpha)} I_{\log(\delta)}^{\alpha_1} + X_{n,s}^{-1(1-\alpha)} J_{\log(\delta)}^{\alpha_1} \right). $$

Then by using Lemma 1.6 we obtain

$$ \mathbb{E}[|K_t^{n,\delta,\varepsilon,1}|] \leq C_T \varepsilon \log(\delta)^{1-\alpha} \left( I_{\log(\delta)}^{\alpha_1} + J_{\log(\delta)}^{\alpha_1} \right). \quad (30) $$

To estimate the term $K_t^{n,\delta,\varepsilon,2}$, we apply Lemma 4.2 with $y = Y_{n,s}^- = X_{n,s}^- - \bar{X}^n_s$, $x = \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_{n(s)})^+)\frac{\delta}{\varepsilon}]$, and $x' = \sigma_2[\bar{X}_{n,s}^- - ((\bar{X}^n_{n(s)})^+)\frac{\delta}{\varepsilon}]$. Since $x'y \geq 0$, we have for $\alpha_2 \in [\alpha, 2]$:}

$$ |K_t^{n,\delta,\varepsilon,2}| \leq C \left[ \frac{\delta}{\log \delta} + 1 \right] \left( ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon} - (\bar{X}_{n(s)}^n)^+)\frac{\delta}{\varepsilon} \right) \left( \frac{1}{|\bar{Y}_{n,s}^-|} \wedge \frac{\delta}{\varepsilon} \right) |\bar{X}_{n,s}^- - ((\bar{X}^n_s)^+)\frac{\delta}{\varepsilon}| \int_0^\infty z \nu(dz) + \int_0^\infty z \nu(dz). \right). $$

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By using (28), the right hand side above can be further estimated by
\[
C\left[ \left( \frac{\delta}{\varepsilon \log \delta} + 1 \right) |\bar{X}^n_{s} - X^n_{\eta(s)}|^{\frac{2}{n}} + |\bar{X}^n_{s} - X^n_{\eta(s)}|^{\frac{1}{n}} \right.
\]
\[+ |\bar{X}^n_{s} - X^n_{\eta(s)}|^{\frac{1}{n}} \left\{ \frac{\delta}{\varepsilon \log \delta} \int_0^u z^2 \nu(dz) + \int_u^{\infty} z \nu(dz) \right\} \]
\[\leq C\left( \frac{\delta}{\varepsilon \log \delta} + 1 \right) |\bar{X}^n_{s} - X^n_{\eta(s)}|^{\frac{2}{n}} + |\bar{X}^n_{s} - X^n_{\eta(s)}|^{\frac{1}{n}} \right)
\]
\[+ |\bar{X}^n_{s} - X^n_{\eta(s)}|^{\frac{1}{n}} \left\{ \frac{\delta}{\varepsilon \log \delta} u^{-2-\alpha_3} I_{u}^{\alpha_3} + u^{-(\alpha_3-1)} J_{u}^{\alpha_3} \right\} \left( X_s^{-(1-\eta)} + 1 \right) \]

where \( \alpha_3 \geq \alpha \) is chosen later. We choose \( u = \delta^{-1} \log(\delta) \), and obtain that
\[
\left| \frac{\delta}{\varepsilon \log \delta} u^{-2-\alpha_3} I_{u}^{\alpha_3} + u^{-(\alpha_3-1)} J_{u}^{\alpha_3} \right| \leq C \left( \frac{\delta}{\varepsilon \log \delta} \right)^{\alpha_3-1}
\]

where we note that \( \log(\delta)/\delta < 1 \) for \( \delta > 1 \). In order to compute the expected value we need to further impose the constraint \( \alpha_2 < \alpha^2 \), and by writing \( \bar{X}^n_{s} - X^n_{\eta(s)} = \bar{X}^n_{s} - X^n_{\eta(s)} - \bar{R}^n_{s} \), we obtain from Lemma 1.8 and Lemma 1.9 that
\[
\mathbb{E}[|\bar{X}^n_{s} - X^n_{\eta(s)}|^{\frac{2}{n}}] \leq \mathbb{E}[|\bar{X}^n_{s} - X^n_{\eta(s)}|^{\frac{2}{n}}] + \mathbb{E}[|\bar{R}^n_{s}|^{\frac{2}{n}}] \leq C n^{-\frac{\alpha_2}{n}}.
\]

Therefore, by applying Hölder inequality whenever needed, we obtain
\[
\mathbb{E}[|K_t^{n,\delta,\varepsilon,2}|] \leq C_T \left[ \left( \frac{\delta}{\varepsilon \log \delta} + 1 \right) n^{-\frac{3}{2n}} + \tilde{n}^{-\frac{3}{2n}} \right] \leq C_T \left[ \frac{\delta}{\varepsilon \log \delta} n^{-\frac{3}{2n}} + \left( \frac{\delta}{\varepsilon \log \delta} \right)^{\alpha_3-1} + 1 \times \tilde{n}^{-\frac{3}{2n}} \right].
\]

With the constraints \( \alpha_2 \in [\alpha, \alpha/2] \cap (0, \alpha^2) \) and \( \alpha_3 \geq \alpha \).

To estimate the term \( K_t^{n,\delta,\varepsilon,3} \), we further decompose it into \( K_t^{n,\delta,\varepsilon,3,1} + K_t^{n,\delta,\varepsilon,3,2} + K_t^{n,\delta,\varepsilon,3,3} \) which are given by
\[
K_t^{n,\delta,\varepsilon,3,1} := \int_0^t \int_0^\infty \left\{ \phi_{\delta,\varepsilon} (\bar{Y}^n_{s-} + \sigma_2 [X^\frac{1}{n}_{s-} - (X^n_{\eta(s)})^{\frac{1}{n}}] z) - \phi_{\delta,\varepsilon} (\bar{Y}^n_{s-} + \sigma_2 [X^\frac{1}{n}_{s-} - (X^n_{\eta(s)})^{\frac{1}{n}}] z) \right\} \nu(dz)ds,
\]
\[
K_t^{n,\delta,\varepsilon,3,2} := \int_0^t \int_0^\infty G(s,z) \left( \phi_{\delta,\varepsilon} (\bar{Y}^n_{s-} + \sigma_2 [X^\frac{1}{n}_{s-} - (X^n_{\eta(s)})^{\frac{1}{n}}] z) \right) \nu(dz)ds,
\]
\[
K_t^{n,\delta,\varepsilon,3,3} := \int_0^t \int_0^\infty G(s,z) \left( \phi_{\delta,\varepsilon} (\bar{Y}^n_{s-} + \sigma_2 [X^\frac{1}{n}_{s-} - (X^n_{\eta(s)})^{\frac{1}{n}}] z) \right) \nu(dz)ds.
\]

By using the second-order Taylor expansion, the first term \( K_t^{n,\delta,\varepsilon,3,1} \) can be estimated by
\[
G(s,z)^2 \int_0^1 (1-\theta) \phi_{\delta,\varepsilon} (\bar{Y}^n_{s-} + \sigma_2 [X^\frac{1}{n}_{s-} - (X^n_{\eta(s)})^{\frac{1}{n}}] z) \nu(dz)ds \leq \frac{\delta G(s,z)^2}{\varepsilon \log \delta},
\]
and by Lemma 1.8, we obtain
\[
\mathbb{E}[|K_t^{n,\delta,\varepsilon,3,1}|] \leq \frac{\delta}{\varepsilon \log \delta} E \left[ \int_0^t \int_0^\infty G(s,z)^2 \nu(dz)ds \right] \leq \frac{\delta}{\varepsilon \log \delta} E \left[ \left| \bar{X}^n_{s} - X^n_{\eta(s)} \right|^2 \right] \leq C \frac{\delta}{\varepsilon \log \delta} n^{-\frac{\alpha_3}{n}}.
\]

To estimate the second term \( K_t^{n,\delta,\varepsilon,3,2} \), we consider separately the sets \( \{ z \leq u \} \) and \( \{ z > u \} \) for \( u > 0 \). On the set \( \{ z \leq u \} \) we apply the first-order Taylor expansion and use the fact that \( \bar{X}^n_{s-} - (X^n_{\eta(s)})^{\frac{1}{n}} \geq 0 \) to obtain
\[
\mathbb{E}[|K_t^{n,\delta,\varepsilon,3,2}|] \leq C_T \frac{\delta}{\varepsilon \log \delta} n^{-\frac{\alpha_3}{n}} + C_T \frac{\delta}{\varepsilon \log \delta} \int_0^u z^2 \nu(dz),
\]
(33)
where we’ve used the Young’s inequality at the second last line. While on the set \( \{ z > u \} \), by applying the Hölder’s inequality we obtain

\[
\mathbb{E} \left[ \left| \int_0^u \int_u^\infty G(s, z) \left\{ \phi_{\delta, \epsilon} (Y^n_s) - \phi_{\delta, \epsilon} (Y^n_{s-}) + \sigma_2 [X^n_s - ((X^n_y)^+) \frac{z}{\delta}] \right\} \nu(dz) ds \right| \right]
\]

\[
\leq \mathbb{E} \left[ \left| \int_0^u \int_u^\infty G(s, z)^2 \nu(dz) ds \right|^{\frac{1}{2}} \mathbb{E} \left[ \left( \int_0^u \int_u^\infty |\phi_{\delta, \epsilon} (Y^n_s) - \phi_{\delta, \epsilon} (Y^n_{s-}) + \sigma_2 [X^n_s - ((X^n_y)^+) \frac{z}{\delta}]| \nu(dz) ds \right)^{\frac{1}{2}} \right] \right].
\]

To estimate the second term in the above we use \([41]\), \([42]\) and the first-order Taylor expansion to obtain

\[
|\phi_{\delta, \epsilon} (Y^n_s) - \phi_{\delta, \epsilon} (Y^n_{s-}) + \sigma_2 [X^n_s - ((X^n_y)^+) \frac{z}{\delta}]| \leq 1_{(0, \epsilon)}(|Y^n_s|) \frac{4\sigma_2}{\log \delta} \left( \frac{1}{Y^n_s} \right)^{\delta} \frac{\delta}{\epsilon} |z| Y^n_s^{-1} \frac{1}{1+\frac{1}{2}}.
\]

By Lemma 1.6 we have the following estimates

\[
\mathbb{E}[|K_{t, \delta, \epsilon, 3, 3}|^2] \leq C \left( \frac{\delta}{\epsilon \log \delta} n^{-\frac{1}{3}} + \frac{\epsilon}{\log \delta} \int_0^u z^2 \nu(dz) \right) \left( \frac{1}{\log \delta} \int_0^\infty z \nu(dz) \right)^{\frac{1}{2}} n^{-\frac{1}{3}},
\]

\[
\leq C \left( \frac{\delta}{\epsilon \log \delta} n^{-\frac{1}{3}} + \left( I^{\alpha_4}_u + (J^{\alpha_4}_u)^\frac{1}{2} \right) \left( \frac{\epsilon}{\log \delta} \int_0^\infty z^2 \nu(dz) \right)^{\frac{1}{2}} n^{-\frac{1}{3}} \right).
\]

Then by choosing \( u = (n^{-\frac{1}{3}} \epsilon^{-2} \log \delta) \frac{\sqrt{\alpha_4}}{\sqrt{\alpha_4}} \), we balance the two quantities related to \( u \) to obtain

\[
\mathbb{E}[|K_{t, \delta, \epsilon, 3, 3}|^2] \leq C \left( \frac{\delta}{\epsilon \log \delta} n^{-\frac{1}{3}} + \left( \frac{\epsilon}{\log \delta} \int_0^\infty z^2 \nu(dz) \right)^{\frac{1}{2}} n^{-\frac{1}{3}} \right).
\]

Finally, to estimate the term \( K_{t, \delta, \epsilon, 3, 3} \), we proceed similarly to the proof of Lemma 4.2 that is we consider the two cases \( |X^n_s - X_{n(1)}^\alpha| > 1 \) and \( |X^n_s - X_{n(0)}^\alpha| \leq 1 \). In the case \( |X^n_s - X_{n(1)}^\alpha| > 1 \) and \( z > 1 \), we apply the Hölder’s inequality, the first-order Taylor expansion and the Young’s inequality to obtain

\[
\int_0^t \int_1^\infty G(s, z) \left\{ \phi_{\delta, \epsilon} (Y^n_s) - \phi_{\delta, \epsilon} (Y^n_{s-}) + \sigma_2 [X^n_s - ((X^n_y)^+) \frac{z}{\delta}] \right\} \nu(dz) ds
\]

\[
\leq C \left( \int_0^t \int_1^\infty G(s, z)^2 \nu(dz) ds \right)^{\frac{1}{2}} \left( \int_0^t \int_1^\infty \frac{\delta}{\epsilon \log \delta} z \left| \left( (X^n_y)^+ \frac{z}{\delta} - (X_{n(1)}^\alpha) \frac{z}{\delta} \right) \right| \nu(dz) ds \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \frac{\delta}{\epsilon \log \delta} \int_0^t \int_1^\infty G(s, z)^2 \nu(dz) ds \right)^{\frac{1}{2}} \left( \int_0^t \int_1^\infty z \left| \left( (X^n_y)^+ \frac{z}{\delta} - (X_{n(1)}^\alpha) \frac{z}{\delta} \right) \right| \nu(dz) ds \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \frac{\delta}{\epsilon \log \delta} \int_0^t \int_1^\infty G(s, z)^2 \nu(dz) ds \right)^{\frac{1}{2}} \left( \int_0^t \int_1^\infty \left| X^n_s - X_{n(1)}^\alpha \right| \frac{z^2 (\alpha_4 - \alpha)}{\alpha} \nu(dz) ds \right).
\]

On the other hand, for \( z \leq 1 \), by the first-order Taylor expansion, \([12]\) and the Young’s inequality, we have

\[
\int_0^t \int_0^u G(s, z) \left\{ \phi_{\delta, \epsilon} (Y^n_s) - \phi_{\delta, \epsilon} (Y^n_{s-}) + \sigma_2 [X^n_s - ((X^n_y)^+) \frac{z}{\delta}] \right\} \nu(dz) ds
\]

\[
\leq \int_0^t \int_0^u |G(s, z)| \left( \frac{2\sigma_2 z}{\epsilon \log \delta} \right)^{\frac{1}{2}} \left( \left( (X^n_y)^+ \frac{z}{\delta} - (X_{n(1)}^\alpha) \frac{z}{\delta} \right) \right)^{\frac{1}{2}} \nu(dz) ds
\]

\[
\leq C \left( \frac{\delta}{\epsilon \log \delta} \int_0^t \int_0^u \left( z \left( (X^n_y)^+ \frac{z}{\delta} - (X_{n(1)}^\alpha) \frac{z}{\delta} \right) \right)^2 \nu(dz) ds \right)^{\frac{1}{2}}.
\]

\[
\leq C \left( \frac{\delta}{\epsilon \log \delta} \int_0^t \int_0^u G(s, z)^2 \nu(dz) ds \right)^{\frac{1}{2}} \left( \int_0^t \int_0^u \left| X^n_s - X_{n(1)}^\alpha \right| \frac{z^2 (\alpha_4 - \alpha)}{\alpha} \nu(dz) ds \right).
\]
Then by taking the expected value of the estimates on \( \{ z \leq 1 \} \) and \( \{ z > 1 \} \), we obtain
\[
E[|K_t^{n,δ,ε,3,3}||] \leq C_T \left( \frac{δ}{ε \log δ} (n^{-\frac{1}{4}} + n^{-\frac{α}{2}}) + n^{-\frac{1}{2}} \right).
\]

In the case \(|X^n_u - X^n_{n(ε)}| > 1\), we again consider upper estimates for \( z \geq u \) and \( z < u \) where we set \( u = |X^n_u - X^n_{n(ε)}|^\frac{1}{δ} < 1 \). For \( z \geq u \), by the Young inequality, \([11]\) and the first-order Taylor expansion
\[
\int_0^t \int_u^∞ G(s, z)(\phi_{δ,ε}(Y^n_s) + σ_2[X^n_s - (X^n_s)^\frac{1}{δ}])z - \phi_{δ,ε}(Y^n_s) + σ_2[X^n_s - (X^n_{n(ε)})^\frac{1}{δ}])z)\nu(dz)ds
\]
\[
\leq C \left( \frac{δ}{ε \log δ} \int_0^t \int_0^∞ G(s, z)^2\nu(dz)ds + \int_0^t \int_u^∞ z|X^n_s|^\frac{1}{δ} \nu(dz)ds \right)
\]
\[
\leq C \left( \frac{δ}{ε \log δ} \int_0^t \int_0^∞ G(s, z)^2\nu(dz)ds + \frac{δ}{ε \log δ} \int_0^t |X^n_s - X^n_{n(ε)}|^\frac{α_5}{15} ds \right).
\]
The expected value of the above is bounded by \( C(δ/ε \log δ)(n^{-\frac{1}{4}} + n^{-\frac{α}{2}}) \) where \( α_5 \in [0, 2] \cap (0, α^2) \). In the case where \( z < u \), we apply the first-order Taylor expansion and the Young inequality to obtain
\[
\int_0^t \int_0^u G(s, z)(\phi_{δ,ε}(Y^n_s) + σ_2[X^n_s - (X^n_s)^\frac{1}{δ}])z - \phi_{δ,ε}(Y^n_s) + σ_2[X^n_s - (X^n_{n(ε)})^\frac{1}{δ}])z)\nu(dz)ds
\]
\[
\leq C \left( \frac{δ}{ε \log δ} \int_0^t \int_0^u G(s, z)^2\nu(dz)ds + \frac{δ}{ε \log δ} \int_0^t |X^n_s|^\frac{1}{δ} \nu(dz)ds \right)
\]
\[
\leq C \left( \frac{δ}{ε \log δ} \int_0^t \int_0^u G(s, z)^2\nu(dz)ds + \frac{δ}{ε \log δ} \int_0^t |X^n_s - X^n_{n(ε)}|^\frac{α_5}{15} ds \right).
\]
The expected value of the above is bounded by \( C(δ/ε \log δ)(n^{-\frac{1}{4}} + n^{-\frac{α}{2}}) \) where \( α_5 \in [0, 2] \cap (0, α^2) \). To this end, the expected value of the term \( K_t^{n,δ,ε,3,3} \) can be upper estimated by
\[
E[|K_t^{n,δ,ε,3,3}||] \leq C \left( \frac{δ}{ε \log δ} (n^{-\frac{1}{4}} + n^{-\frac{α}{2}}) + n^{-\frac{1}{2}} \right).
\]

where \( α_5 \in [0, 2] \cap (0, α^2) \).

**Optimising the convergence rate:** We can combine the calculations above to estimate \( E[|X_t - X^n_t||] \). For \( 1/2 < γ < 1 \), we have from \([26], [29], [30], [31], [32], [33], \) and \([36]\)
\[
E[|X_t - X^n_t||] \leq C n^{-\frac{1}{4}} + ε + C_T \left\{ \int_0^t E[|X_s - X^n_s||]ds + n^{-1/2} + \frac{ε}{e \log δ} + \frac{δ}{ε \log δ} (n^{-1/α} + n^{-α/2}) \right.
\]
\[
+ ε \log(δ)^{α-1} (I_{\text{log}(δ)}^{1/ε})^α + \frac{δ}{ε \log δ} n^{-\frac{2α}{3}} + \left( \frac{δ}{ε \log δ} \right)^{α_3-1} + \frac{δ}{ε \log δ} n^{-\frac{1}{2}} + \frac{δ}{ε \log δ} n^{-\frac{1}{2}} + \frac{δ}{ε \log δ} \left( n^{-\frac{1}{2}} + n^{-\frac{α}{2}} + n^{-\frac{α_3}{2}} + n^{-\frac{α_4}{2}} \right) \right\}
\]
Then, we can choose \( δ = 2 \) and apply Grönwall’s inequality to obtain:
\[
E[|X_t - X^n_t||] \leq C_T \left\{ n^{-\frac{1}{4}} + ε + n^{-1/2} + ε + ε^{-1} (n^{-\frac{α}{2}} + n^{-\frac{1}{4}}) + ε + ε^{-1} n^{-\frac{2α}{3}} + n^{-\frac{1}{2}} \right.
\]
\[
+ ε^{-1} n^{-\frac{1}{2}} + ε + ε^{-1} \left( n^{-\frac{α}{2}} + n^{-\frac{1}{4}} \right) \frac{2α_4}{3} + ε^{-1} \left( n^{-\frac{1}{2}} + n^{-\frac{α}{2}} + n^{-\frac{α_3}{2}} + n^{-\frac{α_4}{2}} \right) + n^{-\frac{1}{2}} \right\}
\]
\[
\leq C_T \left\{ n^{-\frac{1}{4}} + ε + ε^{-1} (n^{-\frac{α}{2}} + n^{-\frac{1}{4}} + n^{-\frac{2α}{3}} + n^{-\frac{α_3}{2}} + n^{-\frac{α_4}{2}}) + ε (ε^{-2} n^{-\frac{1}{2}} + \frac{2α_4}{3}) \right\}
\]
(37)
where we have chosen \( α_1 = α_3 = α \). To optimise the convergence rate in \([37]\), we must select \( α_2, α_4 \) and \( α_5 \) with the constraints \( α_2 \in [0, 2] \cap (0, α^2) \), \( α_4 \in [0, 2] \), \( α_5 \in [0, 2] \cap (0, α^2) \).
Clearly we want to select $\alpha_2$ and $\alpha_3$ to be as large as possible. Since both $\alpha_2$ and $\alpha_3$ has the same constraint for its upper value, we can combine them into one term and write
\[
E[|X_1 - X^n_1|] \leq C_T \left\{ n^{-\frac{1}{2\alpha}} + \epsilon + \epsilon^{-1}(n^{-q(\alpha)} + n^{-\frac{4}{2\alpha}}) + \epsilon^{\frac{1}{2}} n^{-\frac{3}{2\alpha}} \right\},
\]
with $q(\alpha) = \frac{\alpha}{2} \wedge \frac{1}{\alpha}$, $\alpha_2 \in [\alpha, 2] \cap (0, \alpha^2)$ and $\alpha_4 \in [\alpha, 2]$. To further simplify the third term in the above, we note that $q(\alpha) = \frac{\alpha}{2} \wedge \frac{1}{\alpha} = \frac{2\epsilon^{\frac{1}{2}} - \alpha}{2\epsilon}$ which is the best rate $n^{-q(\alpha)}$ can achieve by selecting $\alpha_2 \in [\alpha, 2] \cap (0, \alpha^2)$.

To this end, to find the optimal choice of $\epsilon$, we further write the above as
\[
E[|X_1 - X^n_1|] \leq C_T \left( n^{-\frac{1}{2\alpha}} + n^{r} + n^{-(\gamma_2 - r)} + n^{-(\gamma_3 + r)} \right),
\]
where we set $\epsilon = n^{-r}$ and
\[
\gamma_1 = \frac{1}{2\alpha}, \quad \gamma_2 = \frac{\alpha}{2} \wedge \frac{1}{\alpha}, \quad \gamma_3 = \frac{2 - \alpha_4}{3 - \alpha_4}, \quad \gamma_4 = \frac{\alpha_4 - 1}{3 - \alpha_4}
\]
Consider the term $n^{-(\gamma_3 + r)}$. We can choose optimal values of $\alpha_4$ depending on the values of $r$, in order to maximise $(\gamma_3 + r\gamma_4)$. We see that
\[
(\gamma_3 + r\gamma_4) = \frac{(\alpha_4 - 1)}{(3 - \alpha_4)} + \frac{2 - \alpha_4}{\alpha_4(3 - \alpha_4)} = -r + \left( 2r - \frac{1}{\alpha_4} \right) \frac{1}{3 - \alpha_4} - \frac{1}{\alpha_4}
\]
which is a monotone function in $\alpha_4 \in [\alpha, 2]$. Then by maximizing with respect to $\alpha_4$, we obtain the following piecewise function
\[
f(r) = \begin{cases} \frac{(\alpha - 1)}{(3 - \alpha)} r + \frac{2 - \alpha}{\alpha(3 - \alpha)}, & r < \frac{1}{2\alpha}, \\ r, & r \geq \frac{1}{2\alpha}, \end{cases}
\]
which is greater than $r$. Therefore the optimal choice of $r$ is
\[
r^* = \arg \max_{r \geq 0} \left( r \wedge (\gamma_2 - r) \wedge f(r) \right) = \arg \max_{r \geq 0} \left( r \wedge (\gamma_2 - r) \right)
\]
which is given by the intercept of $r$ and $\gamma_2 - r$, that is $r^* = \frac{1}{2} q(\alpha) = \frac{1}{2} \left( \frac{\alpha}{2} \wedge \frac{1}{\alpha} \right) \leq \frac{1}{2\alpha}$. This shows that the strong convergence rate is given by
\[
E[|X_1 - X^n_1|] \leq C_T n^{-\frac{1}{2} \left( \frac{\alpha}{2} \wedge \frac{1}{\alpha} \right)} = \begin{cases} C_T n^{-\frac{3}{2\alpha}}, & \alpha \in (1, \sqrt{2}) \\ C_T n^{-\frac{1}{2\alpha}}, & \alpha \in [\sqrt{2}, 2) \end{cases}
\]
where $\alpha_\epsilon < \alpha$ can be chosen arbitrarily close to $\alpha$.

**Corollary 2.1.** Suppose Assumption [1] hold, $\gamma \in (1/2, 1]$, $k > 0$ and $\sigma_2 = 0$, then there exists some constant $C_T > 0$ such that
\[
\sup_{t \leq T} E[|X_t - X^n_t|] \leq C_T n^{-\frac{1}{2}}.
\]

**Proof.** Again we write $E[|X_t - X^n_t|] + E[|\tilde{M}_t^n|]$. By using the fact that $\sigma_2$ is zero, we have $\tilde{M}_t^n = 0$ and $E[|\tilde{M}_t^n|] \leq C_T n^{-1}$. The estimate of $f^{n, 0, \epsilon}$ in [26] remain the same and we need only to modify the estimate in [29]. Since all processes involved are square integrable one can take $\alpha_\epsilon = 2$ in [29] and one can show, by modifying Lemma 1.5.8 that $E[|\tilde{M}_t|^2] + E[|\tilde{M}_t|^2] = E[0_t^{T} (P^2(s)ds) = n^{-1}$. By combining these estimates, we obtain from the Gronwell inequality
\[
E[|X_t - X^n_t|] \leq C_T \left\{ n^{-1} + \epsilon + n^{-\frac{\epsilon}{2}} + \frac{n^{-1}}{\epsilon} \right\},
\]
Finally by setting $\epsilon = n^{-r}$ and noticing that the optimal $r$ is $r = \frac{1}{2}$, we obtain a convergence rate of $\frac{1}{2}$. \qed
Remark 2.3. We demonstrate here that the rate obtained in Theorem 2.1 is an improvement over the rate obtained for the Euler-Maruyama scheme in the recent work of Frikha and Li [15]. To do this, we note that for \( \alpha \in (1, 2) \),

\[
\frac{\alpha - 4}{4} > \frac{1}{2} q(\alpha) > \frac{(2 - \alpha)}{(3 - \alpha)} q(\alpha),
\]

and we show in the following that \( n^{-\frac{(2-\alpha)}{(3-\alpha)} q(\alpha)} \) is faster than the order of convergence obtained in Theorem 2.11 of [15] which, in the case of the alpha-CEV, is given by \( n^{-\frac{\eta}{2}} + \varepsilon_n \) where

\[
\varepsilon_n = \begin{cases} 
  n^{-\eta/2} + n^{-(\eta - \frac{\alpha - 4}{2})}, & \text{if } \alpha \in [1, \frac{2(1-\gamma)}{1-\eta}] \text{ and } \eta \leq \gamma, \\
  n^{-(\gamma - \frac{1}{2})}, & \text{if } \alpha \in [1, \frac{2(1-\gamma)}{1-\eta}] \text{ and } \eta > \gamma, \\
  n^{-\eta(1 - \frac{1}{2(1-\eta)}}), & \text{if } \alpha \in (\frac{2(1-\gamma)}{1-\eta}, 2], 
\end{cases}
\]

and \( \eta = \frac{2}{3} \). Note that we do not consider the case \( \alpha = 1 \) or \( \alpha = 2 \) as they are not covered by Theorem 2.1.

In the first case where \( \alpha \in (1, 2(1-\gamma)/(1-\eta)] \) and \( \eta \leq \gamma \), we first write

\[
\eta - \frac{\eta}{2\gamma} = \frac{\eta(2\gamma - 1)}{(2\gamma - 1) + 1}.
\]

By assumption we have \( 2\gamma < \alpha \), from which one can deduce that \( \eta \leq \gamma < \alpha/2 \), and hence \( \alpha^2 > 2 \). On the other hand, we can similarly write

\[
\frac{2 - \alpha}{3 - \alpha} = \frac{\eta(2 - \alpha)}{(2 - \alpha) + 1}.
\]

Then using the condition \( \alpha \leq 2(1-\gamma)/(1-\eta) \) we must have \( 2 - \alpha \geq 2\gamma - 1 \). Together with the fact that \( x \mapsto \eta x/(x + 1) \) is an increasing function on \( \mathbb{R}_+ \), we deduce that \( \frac{2 - \alpha}{3 - \alpha} \) must be greater than \( \frac{2 - \alpha}{2} \). In the second case where \( \alpha \in (1, 2(1-\gamma)/(1-\eta)] \) and \( \eta > \gamma \), we deduce that \( \alpha^2 \leq 2 \) and one can similarly write

\[
\gamma - \frac{1}{2} = \frac{\gamma(2\gamma - 1)}{(2\gamma - 1) + 1} \quad \text{and} \quad \frac{2 - \alpha}{3 - \alpha} = \frac{\alpha - (2 - \alpha)}{2(2 - \alpha) + 1}.
\]

Then we use the fact that \( 2 - \alpha \geq 2\gamma - 1 \), \( \gamma < \eta \) and \( \gamma < \frac{\alpha}{2} \) we deduce that

\[
\frac{\gamma(2\gamma - 1)}{(2\gamma - 1) + 1} \leq \frac{\gamma(2 - \alpha)}{(2 - \alpha) + 1} < \frac{\alpha - (2 - \alpha)}{2(2 - \alpha) + 1}.
\]

In the third case where \( \alpha \in (2(1-\gamma)/(1-\eta), 2] \), it is necessary that \( 2(1-\gamma)/(1-\eta) < 2 \), which implies that \( \eta < \gamma \) and \( \alpha^2 > 2 \). Thus the two convergence rates in this case are equal, that is \( \frac{2 - \alpha}{\alpha(3 - \alpha)} = \eta(1 - \frac{1}{2(1-\eta)}). \)

3 Numerical Simulations

To numerically verify our result in Theorem 2.1 we present here some numerical illustrations for the alpha-CEV in the cases \( \sigma_2 = 0 \) and \( \sigma_2 \neq 0 \). The goal is to numerically study the rate of convergence of our scheme as a function of the parameter \( \alpha \). To estimate the rate of convergence we adapt the method proposed in Alfonsi [12] and compute the strong error at \( T = 1 \), by computing the quantity

\[
S_n := \mathbb{E}[|X_1^n - X_1^n|] \approx \frac{1}{N} \sum_{i=1}^N |X_1^{2n,i} - X_1^{n,i}|,
\]

on a grid of size \( n \) and performing \( N = (10n)^2 \) number of simulations. Below we assume that \( S_n \sim C/n^r \), where \( r \) denotes the strong convergence rate. Then the strong rate of convergence \( r \) can be estimated through \( \log_{10}(S_n) - \log_{10}(S_{10n}) \approx r \). The simulation of the \( i \)-th sample \( X_1^{n,i} \) or more specifically the pair \( (X_1^{2n,i}, X_1^{n,i}) \) is performed in MATLAB and we have included the pseudo-code in the subsection 4.3 of
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the appendix. We also mention that increasing (decreasing) the size of $\sigma_1$ and/or $\sigma_2$ will decrease (increase) the estimated rate, but this will only cause a parallel shift in the graphs, i.e. the relationship with $\alpha$ remain the same and for $\sigma_2 \neq 0$ the convergence rate appears to be $\alpha/4$ or $\alpha/4 + c$ for some $c \in \mathbb{R}$.

Figure 1: The estimated convergence rate in the diffusion case is plotted against different $\alpha$ values. The red and yellow lines are the reference lines $1/2$ and $1/\alpha$ respectively. The graph is generated with parameters $n = 2^5$, $\gamma = 0.54$, $\sigma_1 = 1$, $\sigma_2 = 0$, $a = 1.05$, $k = 2$, $x_0 = 1$.

Figure 2: The estimated convergence rate in the jump-extended case is plotted against different $\alpha$ values. The red and yellow lines are the reference lines $1/2$ and $\alpha/4$ respectively. The graph is generated with parameters $n = 2^6$, $\gamma = 0.54$, $\sigma_1 = 0.37$, $\sigma_2 = 0.37$, $a = 1.05$, $k = 2$, $x_0 = 1$. 
4 Appendix

4.1 Yamada-Watanabe Function

Here we introduce the Yamada-Watanabe approximation technique (see for example Yamada and Watanabe [31], Gyöngy and Rásonyi [17], Li and Mytnik [25], or Li and Taguchi [27]). For each $\delta \in (1, \infty)$ and $\varepsilon \in (0, 1)$ we select a continuous function $\psi_{\delta, \varepsilon} : \mathbb{R} \to \mathbb{R}^+$ with support $[\varepsilon/\delta, \varepsilon]$, and the function $\psi_{\delta, \varepsilon}$ satisfies that

$$\int_{\varepsilon/\delta}^{\varepsilon} \psi_{\delta, \varepsilon}(z)dz = 1, \quad \text{and} \quad 0 \leq \psi_{\delta, \varepsilon}(z) \leq \frac{2}{z \log \delta}, \quad \forall z > 0.$$ 

Define a function $\phi_{\delta, \varepsilon} \in C^2(\mathbb{R}; \mathbb{R})$ by setting

$$\phi_{\delta, \varepsilon}(x) := \int_0^{|x|} \int_0^y \psi_{\delta, \varepsilon}(z)dzdy.$$ 

Then this $C^2$ function $\phi_{\delta, \varepsilon}$ satisfies the following useful properties:

$$|x| \leq \varepsilon + \phi_{\delta, \varepsilon}(x), \quad \text{for any} \quad x \in \mathbb{R};$$

$$0 \leq |\phi_{\delta, \varepsilon}'(x)| \leq 1, \quad \text{for any} \quad x \in \mathbb{R};$$

$$\phi_{\delta, \varepsilon}'(x) \geq 0 \quad \text{for} \quad x \geq 0; \quad \phi_{\delta, \varepsilon}'(x) \leq 0 \quad \text{for} \quad x \leq 0;$$

$$\phi_{\delta, \varepsilon}''(\pm |x|) = \psi_{\delta, \varepsilon}(|x|) \leq \frac{2}{|x| \log \delta} 1_{[\varepsilon/\delta, \varepsilon]}(|x|) \leq \frac{2 \delta}{\varepsilon \log \delta}, \quad \text{for any} \quad x \in \mathbb{R} \setminus \{0\}.$$

In addition, we quote the following lemma from Li and Taguchi [27] and Frika and Li [15].

Lemma 4.1 (Lemma 1.3 of Li and Taguchi [27]). Suppose the Lévy measure $\nu$ satisfies $\int_0^\infty (z \wedge z^2)\nu(dz) < \infty$. Let $\delta \in (1, \infty)$ and $\varepsilon \in (0, 1)$. Then for any $x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}$ with $xy \geq 0$ and $u > 0$, it holds that

$$\int_0^\infty \{\phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y) - xz\phi_{\delta, \varepsilon}'(y)\}\nu(dz)$$

$$\leq 2 \cdot 1_{[0, \varepsilon]}(|y|) \left\{ \frac{|x|^2}{\log \delta} \left( \frac{1}{|y|} \wedge \frac{\delta}{\varepsilon} \right) \int_0^u z^2\nu(dz) + |x| \int_u^\infty z\nu(dz) \right\}.$$ 

Lemma 4.2 (Lemma 4.3 of Frika and Li [15]). Consider the $\alpha$-stable Lévy measure $\nu$, that $\nu(x) = x^{-\alpha - 1} 1_{(0, \infty)}(x)dx$, with $\alpha \in (1, 2)$. Let $\delta \in (1, \infty)$ and $\varepsilon \in (0, 1)$. Then for any $\alpha_0 \in [\alpha, 2]$ there exists a positive constant $C$ such that for any $u \in (0, \infty)$ and any $x, x', y \in \mathbb{R}$ satisfying $x'y \geq 0$ we have

$$\int_0^\infty |\phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y + x'z) - \phi_{\delta, \varepsilon}(x'z)\phi_{\delta, \varepsilon}'(y)\nu(dz)$$

$$\leq C \left[ \frac{\delta}{\varepsilon \log \delta} + 1 \right] \left| x - x' \right|^{\alpha_0} + \left| x - x' \right| + \left| x - x' \right| \left\{ \frac{1_{[0, \varepsilon]}(|y|)}{\log \delta} \left( \frac{1}{|y|} \wedge \frac{\delta}{\varepsilon} \right) x' \int_0^u z^2\nu(dz) + \int_u^\infty z\nu(dz) \right\}.$$ 

Proof. The proof is identical to that of Lemma 4.3 of Frika and Li [15] except that here $\alpha_0$ is allowed to take value $\alpha$ since the form of the Lévy measure is explicit.

4.2 Proofs of Auxiliary Estimates

Proof of Lemma 4.1. First let $(\tau_m)_{m \in \mathbb{N}^+}$ be a localizing sequence of stopping times so that when stopped at $\tau_m$, all local martingales are martingales. Then apply Itô’s formula to $X^\beta$:

$$(X_{t \wedge \tau_m})^\beta = x_0^\beta + M_{t \wedge \tau_m} + I_{t \wedge \tau_m} + J_{t \wedge \tau_m} + K_{t \wedge \tau_m},$$

(43)
where we set
\[ M_t := \sigma_1 \beta \int_0^t (X_s)^{\beta-(1-\gamma)} dW_s + \int_0^t \int_0^\infty \{(X_{s-} + \sigma_2 (X_{s-})^\frac{1}{2}) z\} \tilde{N}(ds, dz), \]
\[ I_t := \beta \int_0^t (X_s)^{\beta-1} (a - kX_s) ds \]
\[ J_t := \sigma_1^2 \beta (\beta - 1) \int_0^t (X_s)^{\beta-2} ds \]
\[ K_t := \int_0^t \int_0^\infty \{(X_{s-} + \sigma_2 (X_{s-})^\frac{1}{2}) z\} \tilde{N}(ds, dz). \]

Here we don’t distinguish between \( X \) and \( X^+ \) in the coefficients above, since SDE (1) admits a unique positive strong solution. The martingale term \( M_{t \wedge \tau_m} \) can be removed by taking the expectation, and we then consider \( K_{t \wedge \tau_m} \). As in (26), we have the following two inequalities for \( z \in (0, 1) \) and \( z \in [1, \infty) \) via Taylor’s Theorem and mean value theorem
\[ (y + xz)^\beta - y^\beta - \beta xz y^{\beta-1} \leq \frac{\beta(\beta - 1) |xz|^2}{2y^{\beta-2}}, \]
\[ |(y + xz)^\beta - y^\beta - \beta xz y^{\beta-1}| \leq \beta |xz|^\beta, \]
so \( \mathbb{E}[|K_{t \wedge \tau_m}|] \) can be bounded by
\[ \frac{\sigma_1^2 \beta (\beta - 1)}{2} \mathbb{E}[\int_0^{t \wedge \tau_m} \int_0^1 |X_s|^{2/\alpha + \beta - 2} z^2 \nu(dz) ds] + \sigma_2^2 \beta \mathbb{E}[\int_0^{t \wedge \tau_m} \int_1^\infty |X_s|^{\beta/2} z^\beta \nu(dz) ds]. \]

By linear growth condition, there exists a constant \( C > 0 \) such that
\[ \max \{ |x|^\beta, |x|^\beta-1, |x|^{\beta - 2(1-\gamma)}, |x|^\frac{\beta}{2} + \beta, |x|^\frac{\beta}{2} \} \leq C(1 + |x|^{\beta}). \]

Then by taking expectation of (13) we see there exists \( C_0 > 0 \) such that
\[ \mathbb{E}[|X_{t \wedge \tau_m}|^\beta] \leq C_0 (\alpha - \beta)^{-1} e^{C_0 (\alpha - \beta)^{-1} t}, \]
and monotone convergence theorem completes the proof. \( \square \)

**Proof of Lemma 1.3** By using the Laplace transform of \( \Delta Z_{t_1} \) (see subsection 2.1 of [22]) for any \( m > 0 \) and the fact that \( x^{2\gamma - 1} \leq 1 + x \) for \( x \geq 0 \), we obtain
\[ \mathbb{E}[e^{-mD_{t_1+1}} | \mathcal{F}_{t_1}] = \exp \left( -mX^a_{t_1} - m \left( a - \frac{\sigma_1^2}{2} (X^a_{t_1})^{2\gamma-1} \right) \Delta t \right) \mathbb{E}[\exp (-m\sigma_2 x^a \Delta Z_{t_1})] \bigg|_{x=X^a_{t_1}} \]
\[ = \exp \left( -mX^a_{t_1} - m \left( a - \frac{\sigma_1^2}{2} (X^a_{t_1})^{2\gamma-1} \right) \Delta t \right) \exp \left( \frac{m^\alpha \Delta t \sigma_2^2 (X^a_{t_1})^{\alpha-1}}{\sin(\pi (\alpha - 1)/2)} \right) \]
\[ \leq \exp \left( - \left( a - \frac{\sigma_1^2}{2} \right) m \Delta t \right) \exp \left( \frac{m^{\alpha-1} \Delta t \sigma_2^2 (X^a_{t_1})^{\alpha-1}}{\sin(\pi (\alpha - 1)/2)} - \left( 1 - \frac{1}{2} \sigma_2^2 \Delta t \right) \right) mX^a_{t_1}. \]

Then, since \( \eta = \frac{1}{\alpha} \), we can eliminate the term \( X^a_{t_1} \) by choosing \( m \) such that
\[ \frac{m^{\alpha-1} \Delta t \sigma_2^2}{\sin(\pi (\alpha - 1)/2)} - \left( 1 - \frac{1}{2} \sigma_2^2 \Delta t \right) = 0. \]
This gives the upper estimate
\[ \mathbb{P}[D_{t_1+1} < 0 | \mathcal{F}_{t_1}] \leq \exp \left( -K^{\alpha,a}_{\sigma_1,\sigma_2} (\Delta t)^{-\frac{2}{2 - \gamma}} \right) \]
where the positive constant \( K^{\alpha,a}_{\sigma_1,\sigma_2} \) is given by
\[ K^{\alpha,a}_{\sigma_1,\sigma_2} := \left( a - \frac{\sigma_1^2}{2} \right) \left( 1 - \frac{\sigma_1^2}{2} \Delta t \right) \sin \left( \frac{\pi (\alpha - 1)}{2} \right) \left( \frac{1}{\sigma_2^2} \right)^{-\frac{1}{\alpha - 1}}. \]
The proof is complete after taking the expectation of both hand side. \( \square \)
Proof of Lemma 1.3} By taking expectation of the scheme (4) we obtain that for some $C > 0$

$$
\mathbb{E}[X^n_{i+1}] = \mathbb{E}[X^n_i] + (a - \kappa_n \mathbb{E}[X^n_i]) \Delta t + \mathbb{E}[A^n_i] \leq C \Delta t + (1 + C \Delta t) \mathbb{E}[X^n_i] + \frac{2}{\kappa_0} \mathbb{E}[D^n_{i+1}].
$$

Next, by using the inequality $x^{2\gamma-1} \leq 1 + x$ for $x \geq 0$, the assumptions that $a - \sigma_i^2/2 > 0$ and $1 - \sigma_i^2 \Delta t/2 > 0$, we see that

$$
\mathbb{E}[D^n_{i+1} \mid F_i] = -(X^n_i + (a - \sigma_i^2 (X^n_i)^{2\gamma-1}/2) \Delta t) \mathbb{E}[1_{\{D^n_{i+1} > 0\}} \mid F_i] - \mathbb{E}[\sigma_i (X^n_i)^{1/2} \Delta Z_i 1_{\{D^n_{i+1} > 0\}} \mid F_i] \\
\leq -\mathbb{E}[\sigma_i (X^n_i)^{1/2} \Delta Z_i 1_{\{D^n_{i+1} < 0\}} \mid F_i] \\
\leq \sigma_i \mathbb{E}[(X^n_i)^{\alpha/2} | \Delta Z_i | 1_{\{D^n_{i+1} < 0\}} \mid F_i].
$$

Then by taking the expectation of the above, applying the Hölder inequality with $1/p + 1/q = 1$ and $p > 1$, $q \in (1, \alpha)$, the Jensen inequality since $q < \alpha$, and the fact that $|x^\gamma| \leq (1 - \eta) + |x|\eta$, we obtain

$$
\mathbb{E}[D^n_{i+1}] \leq C(1 + \mathbb{E}[X^n_i]) (\Delta t)^{\frac{\beta}{2}} \mathbb{E}[1_{\{D^n_{i+1} < 0\}}]^{\frac{1}{2}}. \tag{44}
$$

Then by Lemma 1.2

$$
\mathbb{E}[D^n_{i+1}] \leq C'(1 + \mathbb{E}[X^n_i]) \Delta t,
$$

for some $C' > 0$, and hence we have the recursive equation $1 + \mathbb{E}[X^n_{i+1}] \leq (1 + C'' \Delta t)(1 + \mathbb{E}[X^n_i])$ for some $C'' > 0$. This gives $1 + \mathbb{E}[X^n_i] \leq (1 + x)(1 + C'' \Delta t)^n \leq (1 + x)e^{C'' \Delta t}$.

Proof of Lemma 1.4} For $\beta = 1$, from (44) we see that there exists constants $C > 0$ and $p > 1$ such that

$$
\mathbb{E}[D^n_{i+1}] \leq C(1 + \mathbb{E}[X^n_i]) (\Delta t)^{\frac{\beta}{2}} \mathbb{P}[D^n_{i+1} < 0]^{\frac{1}{2}},
$$

which when combined with the results of Lemma 1.2 and Lemma 1.3 proves the assertion.

For $\beta > 1$, similar to the proof of Lemma 1.3 we first recall the assumptions $a - \sigma_i^2/2 > 0, 1 - \sigma_i^2 \Delta t/2 > 0$, and the fact that $x^{2\gamma-1} \leq 1 + x$ for $x > 0$. From these we can obtain the following inequality

$$
D^n_{i+1} = -(X^n_i + (a - \sigma_i^2 (X^n_i)^{2\gamma-1}/2) \Delta t + \sigma_i (X^n_i)^{1/2} \Delta Z_i) 1_{\{D^n_{i+1} < 0\}} \\
\leq -\sigma_i (X^n_i)^{1/2} \Delta Z_i 1_{\{D^n_{i+1} < 0\}} \\
= \sigma_i (X^n_i)^{1/2} |\Delta Z_i| 1_{\{D^n_{i+1} < 0\}}.
$$

To proceed, we choose a constant $p > 1$ small enough such that $\beta p < \alpha$, and another constant $q > 1$ satisfying $1/p + 1/q = 1$. Then by using Hölder’s inequality we have

$$
\mathbb{E}[(D^n_{i+1})^{\beta}] \leq (\sigma_i)^{\beta} \mathbb{E}[(X^n_i)^{\frac{\beta}{2}} |\Delta Z_i|^{\beta} 1_{\{D^n_{i+1} < 0\}}] \\
\leq (\sigma_i)^{\beta} \mathbb{E}[(X^n_i)^{\frac{\beta p}{2}} |\Delta Z_i|^{\beta p} 1_{\{D^n_{i+1} < 0\}}]\mathbb{P}[D^n_{i+1} < 0]^{\frac{1}{2}} \\
\leq (\sigma_i)^{\beta} \mathbb{E}[(X^n_i)^{\frac{\beta p}{2}}]^{\frac{1}{2}} (\Delta t)^{\frac{\beta}{2}} \mathbb{P}[D^n_{i+1} < 0]^{\frac{1}{2}} \\
\leq C_1 (\Delta t)^{\frac{\beta}{2}} \exp \left(-C_2(\Delta t)^{-2}\right)
$$

where we have used independent increments in the second inequality and Lemma 1.2 in the last inequality.
4.3 MATLAB algorithm

Algorithm 1: Simulation of a \((X_{1}^{2n,(i)}, X_{1}^{n,(i)})\) pair

Function `ImplicitEM(n, alpha, a, K, sigma1, sigma2, q, dt, dW, dL, x0)

\[
\begin{align*}
y(1) &= x0; \\
\text{for } i = 1:n \text{ do} \\
d(i) &= y(i) + (a - sigma1^2 * y(i)^{(2*q-1)/2}) * dt(i) + sigma2 * y(i)^{1/alpha} * dL(i); \\
k(i) &= (sigma1^2 * dW(i))^2 * y(i)^{(2*q-1)} + 4 * (1 + K * dt(i)) * abs(d(i)); \\
y(i+1) &= \left( (sigma1 * y(i)^{(q-1/2)} * dW(i) + sqrt(k(i))) / (2 * (1 + K * dt(i))) \right)^2;
\end{align*}
\]
\text{return } y;

Function `merge2(x)

\[
\begin{align*}
\text{if } n \text{ needs to be even;} \\
\text{n=}\text{length(x);} \\
\text{for } i = 1:(n/2) \text{ do} \\
y(i) &= x(2*i - 1) + x(2*i); \\
\text{end} \\
\text{return } y;
\end{align*}
\]

\[
\begin{align*}
dt &= \text{ones(1,n)*1/n}; \\
dt2 &= \text{merge2(dt);} \\
dL2 &= \text{stablernd(alpha, 1, (2*n)^(-1/alpha), 0, 2*n, 1);} \\
dL &= \text{merge2(dL2);} \\
dW2 &= \text{random(‘Normal’, 0, sqrt(1/(2*n)), 2*n, 1);} \\
dW &= \text{merge2(dW2);} \\
x2 &= \text{ImplicitEM(2*n, alpha, a, K, sigma1, sigma2, q, dt2, dW2, dL2, x0);} \\
x &= \text{ImplicitEM(n, alpha, a, K, sigma1, sigma2, q, dt, dW, dL, x0);} \\
\end{align*}
\]

Result: \((x2(end), x(end))\) will be the \((X_{1}^{2n,(i)}, X_{1}^{n,(i)})\) pair simulated.

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