ON THE GEOMETRY OF THE SECOND FUNDAMENTAL FORM
OF THE TORELLI MAP

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Abstract. In this paper we give a geometric interpretation of the second fundamental form of the period map of curves and we use it to improve the upper bounds on the dimension of a totally geodesic subvariety $Y$ of $A_g$ generically contained in the Torelli locus obtained in [3], [7]. We get $\dim Y \leq 2g - 1$ if $g$ is even, $\dim Y \leq 2g$ if $g$ is odd. We also study totally geodesic subvarieties $Z$ of $A_g$ generically contained in the hyperelliptic Torelli locus and we show that $\dim Z \leq g + 1$.

1. Introduction

The aim of this paper is to study the local geometry of the immersion given by the period map of curves. Let $M_g$ be the moduli space of smooth complex projective curves of genus $g$, $A_g$ the moduli space of principally polarized abelian varieties of dimension $g$. Denote by $j: M_g \to A_g$ the period map or Torelli map and by $T_g$ the Torelli locus, that is the closure of $j(M_g)$ in $A_g$. We consider $A_g$ endowed with the (orbifold) metric induced by the symmetric metric on the Siegel space $H_g$ of which $A_g$ is a quotient by the symplectic group $Sp(2g, \mathbb{Z})$. We want to relate the Torelli locus to the geometry of $A_g$ considered as a locally symmetric variety (see also [13], [2], [3], [6] for motivation and related problems). More precisely, we are interested in studying totally geodesic subvarieties $Y$ of $A_g$ which are generically contained in $T_g$, i.e. such that $Y$ is contained in $T_g$ and $Y \cap j(M_g) \neq \emptyset$. A totally geodesic subvariety $Y$ of $A_g$ is an algebraic subvariety which is the image of a totally geodesic submanifold $\tilde{Y} \subset H_g$. One expects that there exist very few totally geodesic subvarieties of $A_g$ generically contained in $T_g$, at least for high genus $g$. This is related with a conjecture of Coleman and Oort according to which, for $g$ sufficiently high, there should not exist Shimura subvarieties of $A_g$ generically contained in $T_g$ ([13]). In fact Shimura subvarieties of $A_g$ are those totally geodesic subvarieties of $A_g$ admitting a point with complex multiplication ([11]).

To study totally geodesic subvarieties of $A_g$ generically contained in the Torelli locus, we compute the second fundamental form of the Torelli map. This also allows to study totally geodesic subvarieties that are not necessarily algebraic. The computation of the second fundamental form of the Torelli map was initiated in [4] and continued in [3] and [7]. In [3] a bound for the maximal dimension of a germ of a totally geodesic subvariety $Y$ of $A_g$ contained in $T_g$ is given in terms of the gonality of a point $[C] \in M_g$ such that $j([C]) \in Y$. From this one obtains a bound for the maximal dimension of a germ of a totally geodesic subvariety $Y$ of $A_g$ contained in $T_g$ for $g \geq 4$, which only depends on the genus: $\dim Y \leq \frac{5}{2}(g - 1)$. This bound was recently improved in [7], where it is proven that $\dim Y \leq \frac{7}{2}(g - 2)/3$, using the Fujita decomposition of the Hodge bundle of a (real one-dimensional) family.
of abelian varieties given by a geodesic in $H_g$ and results on the Massey products obtained in [15], [8].

In this paper we further improve such a bound by developing the techniques used in [3].

We give a geometric interpretation of one of the main results of [3] where the second fundamental form was expressed in terms of a multiplication by an intrinsic double differential of the second kind on the product $C \times C$. This form appears also in an unpublished book of Gunning [9].

In this way we are able to use a family of quadrics of rank 4 containing the canonical curve whose images via the second fundamental form give quadrics that can be simultaneously diagonalised on a suitable linear subspace of $H^1(T_C)$. This allows to give a better estimate on the rank of the second fundamental form.

The main results are the following

**Theorem 1.1.** If $C$ is a smooth curve of genus $g > 5$, gonality $k$, it has no involutions and is not a smooth plane curve, then any totally geodesic subvariety generically contained in the Torelli locus and passing through $j(C)$ has dimension

$$m \leq \frac{3(g - 1)}{2} + k.$$

From this, recalling that the gonality $k \leq \lceil (g + 3)/2 \rceil$, we obtain the following estimate (the cases $g = 4, 5$ follow from [3 Thm.4.4]).

**Theorem 1.2.** Let $Y$ be a germ of a totally geodesic submanifold generically contained in the Torelli locus $T_g$, $g > 5$, then $\dim Y \leq 2g - 1$ if $g$ is even, $\dim Y \leq 2g$ if $g$ is odd.

These techniques are also used to give a good estimate on the dimension of totally geodesic subvarieties $Y$ of $A_g$ generically contained in the hyperelliptic Torelli locus $TH_g$, i.e. contained in the closure of the image of the hyperelliptic locus $j(HE_g)$ and such that $Y \cap j(HE_g) \neq \emptyset$.

The analogous of the Coleman Oort conjecture for Shimura subvarieties generically contained in the hyperelliptic Torelli locus was considered by Lu and Zuo in [10] where they proved that for $g > 7$ there do not exist Kuga curves generically contained in the hyperelliptic Torelli locus. In particular for $g > 7$ there do not exist Shimura curves generically contained in the hyperelliptic Torelli locus.

Our techniques are different, since they are local, hence allow to study totally geodesic subvarieties which are not necessarily algebraic and of any dimension.

In fact recall that the Torelli map is an immersion outside the hyperelliptic locus but also if restricted to the hyperelliptic locus ([14]), so it is possible to study the second fundamental form of the restriction of the Torelli map to the hyperelliptic locus (see also [2]).

We prove the following

**Theorem 1.3.** Let $Y$ be a germ of a totally geodesic submanifold of $A_g$, contained in $TH_g$ then $\dim Y \leq g + 1$.

For $g = 3$ we get a better bound, namely

**Proposition 1.4.** Let $Y$ be a germ of a totally geodesic submanifold of $A_3$, contained in the hyperelliptic Torelli locus, then $\dim Y \leq 3$.

Notice that for low genus there are examples of Shimura (hence totally geodesic) subvarieties of $A_g$ contained in $TH_g$, namely the ones given by families (8), (22), (36), (39) of Table 2 of [5] (see also Tables 1 of [12] for family (8), Table 2 of [13] for family (22)). Family (8) yields a 2-dimensional Shimura subvariety generically
The coboundary map gives an injection $\delta : H^0(C, T_C(p)|_p) \hookrightarrow H^1(C, T_C)$. A Schiffer variation at $p$ is a generator of $\delta(H^0(C, T_C(p)|_p))$. Choose a local coordinate $(U, z)$ at $p$ and $b$ a bump function which is equal to 1 in a neighbourhood of $p$. Then the form $\theta := \frac{\partial b}{\partial z} \in A^{0,1}(T_C)$ is a Dolbeault representative of a Schiffer variation at $p$. One can easily check that the map

$$\xi : T_p C \to H^1(C, T_C), \ u := \lambda \frac{\partial}{\partial z}(p) \mapsto \xi_u := \lambda^2[\theta]$$

is independent of the choice of the local coordinate $z$.

Consider a curve $C$ of genus $g \geq 2$, and take a point $p \in C$. The space $H^0(C, K_C(2p))$ of meromorphic 1-forms on $C$ with a double pole on $p$ injects into $H^1(C \setminus \{p\}, \mathbb{C}) \cong H^1(C, \mathbb{C})$. Denote by $j_p : H^0(C, K_C(2p)) \to H^1(C, \mathbb{C})$ this injection.
Observe that $h^0(C, K_C(2p)) = g + 1$ and $H^0(C, K_C) \subset H^0(C, K_C(2p))$ is mapped by $j_p$ onto $H^{1,0}(C)$, hence the preimage $j_p^{-1}(H^{0,1}(C))$ has dimension 1. Fix a local coordinate $(U, z)$ centered in $p$. Then there exists a unique element $\phi$ in this line whose expression on $U \setminus \{p\}$ is

$$\phi := \left( \frac{1}{z^2} + h(z) \right) dz,$$

where $h$ is a holomorphic function. The form $\eta_p$ is defined as follows:

$$\eta_p : T_pC \to H^0(C, K_C(2p)),$$

$$u = \lambda \frac{\partial}{\partial z}(p) \mapsto \eta_p(u) = \lambda \phi.$$

One can easily prove that $\eta_p$ is independent of the choice of the local coordinate.

Denote by $\mu_2 : I_2(K_C) \to H^0(4K_C)$ the second gaussian map of the canonical bundle [16].

Let us now state a result of [4].

**Theorem 2.1** (Colombo, Pirola, Tortora [4]). Let $C$ be a non-hyperelliptic curve of genus $g \geq 4$. Let $p, q \in C$ and $u \in T_pC$, $v \in T_qC$. Then:

$$\rho(Q)(\xi_u \otimes \xi_v) = -4\pi i \eta_p(u)(v)Q(u, v),$$

$$\rho(Q)(\xi_u \otimes \xi_u) = -2\pi i \mu_2(Q)(u \otimes v).$$

Theorem 2.1 is used in [2] to compute the holomorphic sectional curvature of curvature $M_\eta$ with respect to the Siegel metric along the Schiffer variations.

Let us now recall a more intrinsic description of the form $\eta_p$ obtained in [3].

Let $S := C \times C$, let $\Delta$ denote the diagonal and let $p, q : S \to C$ be the projections $p(x, y) = x$, $q(x, y) = y$. Then $K_S = p^*K_C \otimes q^*K_C$. Consider the line bundle $L := K_S(2\Delta)$ on $S$ and set $V := p_*q^*K_C(2\Delta)$, $E := p_*L \cong K_C \otimes V$ by the projection formula. We have $H^0(p^{-1}(x), q^*K_C(2\Delta)) \cong H^0(C, K_C(2x))$, hence $V$ is a holomorphic vector bundle on $C$ with fibre $V_x \cong H^0(C, K_C(2x))$ and the map $x \mapsto \eta_x$ is a section of $E$ that we call $\eta$. Since $E = p_*L$ there is an isomorphism $H^0(C, E) \cong H^0(S, L)$ and $\eta$ corresponds to a global section $\hat{\eta} \in H^0(S, L)$ such that for $u \in T_xC$ and $v \in T_yC$ with $x \neq y$, we have $\eta_x(u)(v) = \hat{\eta}(u, v)$.

**Proposition 2.2** ([3]). The section $\eta$ of $E$ is holomorphic. Moreover the form $\hat{\eta}$ is symmetric, i.e. $\hat{\eta}(u, v) = \hat{\eta}(v, u)$.

With the identification of $H^0(C, K_C) \otimes H^0(C, K_C)$ with $H^0(S, K_S)$, $I_2(K_C)$ can be seen as a subspace of $H^0(S, K_S(-\Delta))$. Since elements of $I_2(K_C)$ are symmetric, they are in fact contained in $H^0(S, K_S(-2\Delta))$. So if $Q \in I_2(K_C)$, the section $Q \cdot \hat{\eta}$ lies in $H^0(S, 2K_S) \cong H^0(C, K_C(2\Delta)) \otimes H^0(C, 2K_C).

**Theorem 2.3** ([3]). With the above identifications, if $C$ is non-hyperelliptic and of genus $g \geq 4$, then $\rho : I_2(K_C) \to S^2H^0(C, 2K_C)$ is the restriction to $I_2(K_C)$ of the multiplication map

$$H^0(S, K_S(-2\Delta)) \to H^0(S, 2K_S) \quad Q \mapsto Q \cdot \hat{\eta}.$$  

The form $\hat{\eta}$ is very mysterious, locally one has

$$\hat{\eta} = \frac{dz \wedge dw}{(z - w)^2} + f(z, w)dz \wedge dw$$

in coordinates near the diagonal and $f(z, w) = f(w, z)$ is smooth. It appears also in an unpublished book of Gunning under the name of "intrinsic double differential of the second kind" [9]. It seems very hard to give an actual computation of $\hat{\eta}$. But we have the following geometric argument that gives information. Take a quadric
Theorem 2.4. If \( Q \) has rank 4, we project form the kernel of divisor of the image of the second Gaussian map. In particular, if the quadric \( Q \) is isomorphic to \( \mathbb{P}^2 \), then the projection form the kernel of \( Q \).

Consider the linear subspace \( P \) composition of the canonical map with the projection to one of the two factors of \( P \). Then if \( s \) is a line bundle on \( Z \), the restriction of \( s \) is diagonal with non zero diagonal entries.

By Theorem 2.3 we have

\[
Z(Q) = Z(Q) \cup Z(\hat{q}).
\]

Then if \( p \neq q \), we have

\[
Q(p,q) = 0 \Rightarrow Q'(p,q) = 0,
\]

but this time \( Q'(p,p) \neq 0 \) in general (the intersection with the diagonal is the divisor of the image of the second Gaussian map). In particular, if the quadric \( Q \) has rank 4, we project form the kernel of \( Q \) on \( \mathbb{P}^3 \) to get a quadric \( Q_p \) of \( \mathbb{P}^3 \) which is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \). Call \( f : C \to \mathbb{P}^1 \) the composition of the canonical map with the projection form the kernel of \( Q \) and finally with one of the two projections from \( \mathbb{P}^1 \times \mathbb{P}^1 \) to \( \mathbb{P}^1 \). For \( y \in \mathbb{P}^1 \), set \( f^{-1}(y) = \{ P_1 \ldots P_d \} \). If \( y \) is general we get for \( i \neq j \)

\[
Q(P_i, P_j) = Q'(P_i, P_j) = 0,
\]

but \( Q'(P_i, P') \neq 0 \).

Theorem 2.4. If \( Q \) has rank 4 the quadric \( Q' \) has rank \( \geq \deg f \) where \( f \) is the composition of the canonical map with the projection to one of the two factors of \( \mathbb{P}^1 \times \mathbb{P}^1 \).

Proof. Consider the linear subspace \( V = \langle \xi_{P_1}, ..., \xi_{P_d} \rangle \subset H^1(T_C) \). It is easy to see that \( \dim V = d \) (see e.g. the proof of Theorem 4.1 of [3]). The quadric \( Q' \in S^2 H^1(T_C) \) and the image of the points \( P_i \) via the bicanonical map are the points \( [\xi_{P_i}] \in \mathbb{P}(H^1(T_C)) \). Hence the above argument shows that the matrix associated to the restriction of \( Q' = \rho(Q) \) to \( V \) in the basis given by the Schiffer variations at the points \( P_i \) is diagonal with non zero diagonal entries.

Recall that quadrics \( Q \in I_2(K_C) \) of rank 4 correspond to \( \{ L, K_C(-L), U, W \} \) where \( L \) is a line bundle on \( C \), \( U \subset H^0(C, L) \) and \( W \subset H^0(K_C(-L)) \) are 2-dimensional subspace of \( H^0(K_C(-L)) \) correspond to \( \{ L, K_C(-L), U, W \} \). The quadric \( Q \) has rank 3 if and only if \( 2L = K_C \) and \( U = W \). In the case \( rank(Q) = 4 \), if \( U = \langle s_0, s_1 \rangle \) and \( W = \langle t_0, t_1 \rangle \), \( Q = s_0 t_0 \odot s_1 t_1 - s_0 t_1 \odot s_1 t_0 \). So, if we denote by \( f, g : C \to \mathbb{P}^1 \) the two maps given by the two pencils \( U \) and \( W \), we see that the zero locus of \( Q \) in \( S \) is given by

\[
\{ (p, q) \in S \mid f(p) = f(q) \} \cup \{ (p, q) \in S \mid g(p) = g(q) \} \cup (B_f \cup B_g) \cup (C \times (B_f \cup B_g)),
\]

where \( B_f \) and \( B_g \) are the base loci of the pencils \( f \) and \( g \).

If \( rank(Q) = 3 \) and \( U = W = \langle s_0, s_1 \rangle \), then \( Q = s_0^2 \odot s_1^2 - s_0 s_1 \odot s_0 s_1 \). If we denote by \( h : C \to \mathbb{P}^1 \) the map given by the pencil \( U = W \), the zero locus of \( Q \) in \( S \) is \( 2R_h + 2F \) where \( F = (B_h \times C) \cup (C \times B_h) \) with \( B_h \) the base locus of \( h \) and

\[
R_h := \{ (p, q) \in S \mid h(p) = h(q) \}.
\]

Now we give another application

Proposition 2.5. If \( Q \) is non zero the rank of \( Q' \) is \( \geq 2 \). If \( rank(Q) = 3 \) then the rank of \( Q' \) is \( \geq 3 \).
Theorem 2.7. Assume that \( C \) is a \( k \)-gonal curve of genus \( g \) with \( g \geq 4 \) and \( k \geq 3 \). Let \( Y \) be a germ of a totally geodesic submanifold of \( A_g \) which is contained in the jacobian locus and passes through \([C]\). Then \( \dim Y \leq 2g + k - 4 \).

From this, using that the gonality is at most \([(g + 3)/2]\), they get the following

Theorem 2.7 ([3]). If \( g \geq 4 \) and \( Y \) is a germ of a totally geodesic submanifold of \( A_g \) contained in the jacobian locus, then \( \dim Y \leq \frac{5}{2}(g - 1) \).

The strategy used in [3] to prove Theorem 2.6 is to construct a rank 4 quadric \( Q \in I_2(K_C) \) such that the quadric \( \rho(Q) \in S^2H^0(2K_C) \cong S^2H^1(T_C)^\vee \) has rank at least \( 2g - 2k - 1 \). This was done in Theorem 4.1 of [3] and now follows from Theorem 2.6. Then the proof of Theorem 2.6 follows by observing that the tangent space to a totally geodesic submanifold contained in the Torelli locus and passing through \( j(C) \) is a linear subspace of \( H^1(T_C) \) which is isotropic with respect to \( \rho(Q) \), hence its dimension is at most \( 3g - 3 - (2g - 2k + 1) + \frac{2g - 2k - 1}{2} = 2g + k - \frac{7}{2} \).

3. Gonality

Let \( C \) be a curve of genus \( g \geq 5 \) assume that the gonality is \( k > 2 \), \( (C \) non-hyperelliptic), we have \( k < g - 1 \). Let \( L \) be line a bundle of degree \( k \) and such that \( h^0(L) = 2 \). Set \( M = K_C - L \) where \( K_C \) is the canonical bundle. From Riemann Roch we get \( h^0(M) = 2g - 2 - k + 2 - (g - 1) = g + 1 - k > 2 \). We have that the Clifford index is either \( k - 2 \) (computed by \( L \)) or \( k - 3 \). We have

Proposition 3.1. Let \( B \) be the base locus of \( M = K_C - L \). Then

1. \( B \) is either empty or \( B = P \) is a point.
2. If \( B = P \) is a point \( h^0(L(P)) = 3 \), the map \( f : C \to |L(P)| \) is birational onto its image which is a smooth curve.
(3) If $B = \emptyset$, and $D > 0$ a divisor such that $h^0(M(-D)) = h^0(M) - 1$, then $\deg D \leq 3$.

(4) If $B = \emptyset$, and $D, E > 0$ are divisors such that $h^0(M(-D)) = h^0(M(-E)) = h^0(M) - 1$, $\deg D = \deg(E) = 3$, then $D = E$. In particular the map $g : C \to |M|$ is either birational on its image or has degree 2.

Proof. If $B \neq \emptyset$ is the base locus of $M$, we have that $h^0(M(-B)) = h^0(M)$. Then by Riemann Roch, $h^0(L(B)) = 2 + \deg B$, the Clifford index of $L(B)$ is $k + \deg B - 2(1 + \deg B) = k - 2 - \deg B \geq k - 3$, hence $B = P$ is a point, and $h^0(L(P)) = 3$. Consider the map $f : C \to |L(P)|$. It is birational onto its image which is smooth. In fact, if we set $s = \deg f(C)$ and $d = \deg(f)$, we have $sd = k + 1$. Considering the composition of $f$ with the projection from a point $q \in f(C)$ of multiplicity $m_q$, we get a map $C \to \mathbb{P}^1$ of degree $d(s - m_q) = k + 1 - dm_q \geq k$ and then $dm_q = 1$, $d = 1$ that is $f$ birational and for any $q \in f(C)$, $m_q = 1$, hence $q$ is a smooth point.

Assume now that $B = \emptyset$. Let $D > 0$ be a divisor of degree $\deg(D) = s$ such that $h^0(M) = 1 + h^0(M(-D))$, then again from Riemann Roch we have $h^0(L(D)) = 2+s-1 = s+1$, $\deg(L(D)) = k+s$, hence its Clifford index is $k+s-2s = k-s \geq k-3$, so $s \leq 3$.

Assume that there are two divisors $D$ and $E$ of degree 3 such that $h^0(M(-D)) = h^0(M(-E)) = h^0(M) - 1$. If $F = D \cap E$ (this is the MCD of the divisors) we would like to show that $F = \emptyset$. In fact, let $G = D + E - F$ and consider the exact sequence:

$$0 \to M(-G) \to M(-D) \oplus M(-E) \to M(-F) \to 0.$$ 

If by contradiction we assume $F \neq \emptyset$ we have $h^0(M-E) \leq h^0(M-F) < h^0(M)$ then $h^0(M-F) = h^0(M) - 1$. It follows that $h^0(M-G) \geq 2(h^0(M) - 1) - (h^0(M) - 1) = h^0(M) - 1$, but $\deg G > 3$ and we find a contradiction. So we have $F = \emptyset$. Consider the exact sequence:

$$0 \to L \to L(D) \oplus L(E) \to L(D+E) \to 0$$

that in cohomology gives $0 \to H^0(L) \to H^0(L(D)) \oplus H^0(L(E)) \to H^0(L(D+E))$. Since $h^0(L(D)) = h^0(L(E)) = 4$, we obtain $H^0(L(D+E)) \geq 6$ and its Clifford index $\leq k + 6 - 10 = k - 4 < k - 3$ and we find a contradiction.

From this we immediately get the following

**Corollary 3.2.** Let $C$ be a projective curve of genus $g$, gonality $k = \deg L$ where $h^0(L) = 2$. Then if $C$ is not a smooth plane curve and has no involutions, the linear system $|M| = |K_C - L|$ is base point free and $C \to |M|$ is birational onto its image.

4. **Divisors and quadrics**

4.1. **The quadrics and the second fundamental form.** Here we follow [3]. Let $C$ be a smooth projective curve, $L$ a line bundle on $C$ and $M = K_C(-L)$. In practice we will consider the case where $L$ computes the gonality, but for the moment we only assume

1. $h^0(L) \geq 2$.
2. $h^0(M) = r + 1 \geq 3$.
3. $M$ base point free.
4. $f : C \to |M| = \mathbb{P}^r$ is birational onto its image.

Set $\deg M = d$ and fix now and for all two independent sections $x, y$ of $L$. Let $\omega = \mu_{1,L}(x \land y) \in H^0(K_C(L^2))$ be the image of the first gaussian (or Wahl) map. The zero divisor $Z$ of $\omega$ is $Z = B + 2F$ where $B$ is the branch divisor of $h = x/y$ and $F$ is the fixed divisor (that in the application will be empty). For any global section
Consider the induced injection \( p : M \rightarrow N \) and denote by \( \delta_r \) the linear isomorphism which has at most \( r \) distinct points \( p_1, \ldots, p_r \) and \( p_i \neq p_j \) if \( i \neq j \).

The last condition follows for instance from the uniform lemma of Castelnuovo (see e.g. [1, Ch.3]) since \( f : C \rightarrow \mathbb{P}^r \) is birational onto its image.

Consider the exact sequence induced by \( f \)

\[
0 \rightarrow \mathcal{O}_C(t) \rightarrow M \rightarrow M_D \rightarrow 0.
\]

We get

\[
M_D = \sum M_{p_i} \cong \sum \mathcal{C}_{p_i}
\]

where the last isomorphism follows from the choice of local trivializations of \( M \).

Let \( W \subset H^0(M) \) be complementary to \( t : H^0(M) = (t) \oplus W \), so that \( \dim W = r \).

Consider the induced injection \( j : W \rightarrow H^0(\mathbb{C}^{p_i}) = \mathbb{C}^d \).

We can rewrite the linear uniform condition. For any \( s \in W \) \( s \neq 0 \) then the vector \( j(s) = (a_1, \ldots, a_i, \ldots, a_d) \) has at most \( r - 1 \) coordinates that are zero. Let \( I_2(K_C) \subset S^2 H^0(K_C) \) be the vector space of quadrics that contain the canonical image of \( C \).

For any \( s \in W \) set \( \omega_1 = x, \omega_2 = y, \omega_3 = x, \omega_4 = y \).

We define the rank 4 quadric \( Q_s \in I_2 \)

\[
Q_s = \omega_1 \omega_4 - \omega_2 \omega_3.
\]

Denote by \( \mu_{1,M} : \Lambda^2 H^0(M) \rightarrow H^0(K_C \otimes M^2) \) the first Gaussian map of \( M \).

Let \( V := \langle \xi_{p_1}, \ldots, \xi_{p_d} \rangle \subset H^1(T_C) \) be the subspace generated by the Schiffer variations \( \xi_{p_i} \) of the points \( p_i \).

To be more precise we choose a coordinate \( z_i \) around any point of \( p_i \) and a trivialization \( \sigma \) of \( L \) such that \( t_i = z_i \sigma \).

We then let

\[
\xi_i = \delta(1/\overline{z}_i \partial/\partial z_i)
\]

under the coboundary map \( \delta : H^0(T(p_i)) \rightarrow H^1(T_C) \).

Dually we have a surjection \( H^0(K_C^2) \rightarrow V^* \).

We also define

\[
\pi : W \rightarrow S^2 V^*, \quad \pi(s)(v \oplus w) = \rho(Q_s)(v \oplus w), \forall v, w \in V.
\]

The following fundamental result has been proved in [3] Theorem 4.1, with the variant here that all the quadrics \( Q_s, s \in W \), are taken into account. It also immediately follows from Theorem 2.4

**Theorem 4.1.** Let \( x_1, \ldots, x_d \) be the basis of \( V^* \) dual to the basis \( \{ \xi_{p_1}, \ldots, \xi_{p_d} \} \) of \( V \) given by the Schiffer variations. Then

\[
\pi(s) = \lambda \sum_{i=1}^d a_i x_i^2
\]

where \( \lambda \neq 0 \) is a constant independent on \( s \), \( s \neq 0 \) and \( j : W \rightarrow \mathbb{C}^d \) is the evaluation map. The quadrics \( \pi(s) \) are diagonalized at the same time and for any \( s \neq 0 \), \( \text{rank}(\pi(s)) \geq d - r + 1 \).

**Proof.** The fact that all the quadrics are in diagonal form follows from [3], Theorem 4.1, or from Theorem 2.4. Recall formula [5] \( \rho(Q_s) \xi_{p_i} \xi_{p_i} = -2 \pi i \mu_{2}(Q_s)(u_i^2) = -2 \pi i \mu_{1}\xi_{p_i} \cdot \mu_{1,M}(u_i^2) \) where \( u_i = \partial z_i \partial z_i \) and \( z_i \) is the coordinate centred at \( p_i \). Since \( \mu_{1,M}(u_i^2) \) does not depend on \( s \) we have to evaluate \( \mu_{1,M}(u_i^2) \). Now \( \mu_{1,M}(u_i^2) = (t^s - s^t)(p_i) = (t^s)(p_i) \) since \( t(p_i) = 0 \) by construction. Then \( a_i \) is the evaluation of \( s \) at \( p_i \).
4.2. **Le zero locus of the quadrics.** For an element \( z \in V \), write \( z = \sum_{i=1}^{d} z_i \xi_{p_i} \) and denote by \([z] := [z_1, ..., z_d] \in \mathbb{P}^{d-1} \cong \mathbb{P}(V)\). Consider the locus

\[
Z = \{[z] \in \mathbb{P}^{d-1} : \pi(s)(z \odot z) = \rho(Q_s)(z \odot z) = 0, \ \forall s \in W\}.
\]

First we have

**Lemma 4.2.** Set \( H = \{[z] \in \mathbb{P}^{d-1} : z_i = 0, \ i > r\} \), then \( Z \cap H = \emptyset \), therefore \( \dim Z = d - r - 1 \).

**Proof.** Notice that by the uniform position (see e.g. [1, Ch.3]), we know that for all \( i \in \{1, ..., r\} \), there is exactly a section \( s_i \in W \) such that \( s_i(p_j) = 0, \ \forall j \in \{1, ..., r\} \), \( j \neq i \), \( s_i(p_i) \neq 0 \), \( s_i(p_k) \neq 0 \), \( \forall k > r \), hence by Theorem 4.1 we get

\[
(12) \quad \pi(s_i) = a_{i,i}x_i^2 + \sum_{j>r} a_{i,j}x_j^2,
\]

with \( a_{i,i} \neq 0, \ a_{i,j} \neq 0, \ \forall j > r \). Take \([z] = [z_1, ..., z_r, 0, ..., 0] \in H\) such that \( \pi(s)(z \odot z) = \rho(Q_s)(z \odot z) = 0 \), \( \forall s \in W \). Set \( Q_i := Q_{s_i} \), then \( \rho(Q_i)(z \odot z) = a_{i,i}z_i^2 = 0 \) \( \forall i \) if and only if \( z_i = 0, \ \forall i = 1, ..., r \), which is impossible, since \([z] \in H\). Therefore we have \( H \cap Z = \emptyset \) and then \( \dim Z \leq d - 1 - r \). Notice that \( Z = \{[v] \in \mathbb{P}^{d-1} : \rho(Q_i)(v \odot v) = 0, \ i = 1, ..., r\} \), so \( \dim Z = d - 1 - r \).

\[ \square \]

4.3. **Estimate.** We need to estimate the dimension of a linear space \( \Pi \subset Z \). Denote by \( T \) the linear subspace of \( V \) corresponding to \( \Pi \).

Consider the maps \( \pi_i : V \to V, \ \pi_i(x_1, ..., x_i, x_{i+1}, ..., x_d) = (0, ..., 0, x_{i+1}, ..., x_d) \)

The restriction of \( \pi_i \) to \( T \) is injective for \( i \leq r \) since \( \Pi \subset Z \). By formula (12) we can see \( \pi(s_r) \) as a quadric in \( \pi_{r-1}(V) \).

We have the inclusion

\( \pi_{r-1}(T) \subset \{v \in \pi_{r-1}(V) \mid \pi(s_r)(v \odot v) = \rho(Q_r)(v \odot v) = 0\} \).

Since \( \pi(s_r) \) has rank \( d - r + 1 \), \( \dim(\pi_{r-1}(V)) = d - r + 1 \) and \( \pi_{r-1} : T \to V \) is injective, we get:

**Proposition 4.3.** With the previous notation, let \( \Pi \) be a linear subspace contained in \( Z \), and let \( T \) be the corresponding subspace of \( V \). Then

\( \dim T \leq \frac{d - r + 1}{2} \).

5. **Application.**

We assume that \( C \) is a curve of genus \( g > 5 \), gonality \( k \) computed by \( L \) and assume that \( C \) has no involutions and is not a smooth plane curve. Then we can apply to the general section of \( M = K_C - L \) the estimate of the previous section.

We have: \( \deg M = 2g - 2 - k \) and \( h^0(M) = g + 1 - k \), that is \( r = g - k \). Then if \( t \) is the general section of \( M \) and \( V \) is generated by the Schiffer variations of the zeroes of \( t \), by Proposition 4.3 we get that a linear space \( T \subset V \) contained in the zero locus of the quadrics \( \rho(Q_s) \), for \( s \in W \) has dimension

\[
(13) \quad n \leq \frac{g - 1}{2}.
\]

Then we obtain the following

**Theorem 5.1.** If \( C \) is a smooth curve of genus \( g \geq 5 \), gonality \( k \), it has no involutions and is not a smooth plane curve, then any totally geodesic subvariety generically contained in the Torelli locus and passing through \( j(C) \) has dimension

\[
m \leq \frac{3(g - 1)}{2} + k.
\]
Proof. Let $S$ be tangent space at $[C]$ of a totally geodesic subvariety. Let $V$ be as above, then $T = S \cap V$ is a linear subspace where all the quadrics vanish, so by we get: $\dim(S \cap V) \leq \frac{2g - 1}{2}$. Then $\dim S + \dim V \leq 3g - 3 + \dim(S \cap V)$, hence

$$\dim S \leq 3g - 3 - (2g - k - 2) + \frac{g - 1}{2} = \frac{3(g - 1)}{2} + k.$$  

$\square$

**Theorem 5.2.** Let $Y$ be a germ of a totally geodesic submanifold generically contained in the Torelli locus $T_g$, $g > 5$, then $\dim Y \leq 2g - 1$ if $g$ is even, $\dim Y \leq 2g$ if $g$ is odd.

**Proof.** For $g > 5$ the result follows by Theorem 5.1 recalling that $k \leq [(g + 3)/2]$. For $g = 4, 5$ the result follows from [3, Thm.4.4]. $\square$

### 6. The hyperelliptic locus

Assume that $C$ is a hyperelliptic curve of genus $g \geq 3$. Denote by $L$ the line bundle giving the $g^1_2$, $H^0(L) = \langle x, y \rangle$, call $M = K_C(-L)$, denote by $\pi : C \to \mathbb{P}^1$ the map induced by $|L|$. Call $\nu_n : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ the $n$th Veronese embedding. The canonical map is the composition $\nu_{g-1} \circ \pi$, so $K_C \cong \pi^*(\mathcal{O}_{\mathbb{P}^1}(g-1)) \cong L^{\otimes (g-1)}$ and $M \cong L^{\otimes (g-2)}$. Then $H^0(C, M) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-2))$ that has dimension $r + 1 = g - 1$, so $r = g - 2$. Denote by $\sigma$ the hyperelliptic involution and write $H^0(C, 2K_C) \cong H^0(C, 2K_C)^+ \oplus H^0(C, 2K_C)^-$ the decomposition of $H^0(C, 2K_C)$ in invariant and anti-invariant subspaces by the action of $\sigma$. By the projection formula one gets: $H^0(C, 2K_C)^+ \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g - 2))$, $H^0(C, 2K_C)^- \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 3))$. Denote by $HE_g$ the hyperelliptic locus in $M_g$ and by $j_h : HE_g \to A_g$ the restriction of the Torelli map to $HE_g$. Then $j_h$ is an orbifold immersion and we have the following tangent bundle exact sequence

\[0 \to T_{HE_g} \to T_{A_g|HE_g} \to N_{HE_g/A_g} \to 0\]

Denote by

\[\rho_{HE} : N_{HE_g/A_g}^* \to S^2 T_{HE_g}\]

the dual of the second fundamental form of $j_h$.

At a point $[C] \in HE_g$, the dual of $[14]$ is

\[0 \to I_2(K_C) \to S^2 H^0(K_C) \xrightarrow{m} H^0(2K_C)\]

and $I_2(K_C)$ can be identified with the set of quadrics containing the rational normal curve.
We have \( \forall Q \in I_2(K_C), \forall v, w \in H^1(T_C)^+, \rho_{HE}(Q)(v \circ w) = \rho(Q)(v \circ w) \).

Take \( t \in H^0(M) \) a generic section and denote by \( D(t) \) its divisor. It has degree \( 2g - 4 \) and is invariant by the action of \( \sigma \), hence \( D(t) = q_1 + \sigma(q_1) + \ldots + q_{g-2} + \sigma(q_{g-2}) \). Write as above \( H^0(M) = \langle t \rangle \oplus W \). Since \( H^0(C, M) \cong H^0(O_{\mathbb{P}^1}(g-2)) \), the section \( t \in H^0(C, M) \) corresponds to a section \( s_i \in W \) such that \( s_i(r_j) = 0 \) for all \( j \neq i \) and \( s_i(r_i) \neq 0 \). The section \( s_i \) corresponds to a section \( s_i \in W \) such that \( s_i(q_j) = 0 \) for all \( j \neq i \) and \( s_i(q_i) \neq 0 \).

For every section \( s \in W \) consider as above the quadric

\[
Q_s = (xt) \circ (ys) - (xs) \circ (yt) \in I_2(K_C),
\]

and the tangent vectors \( \nu_i := \xi_{q_i} + \xi_{\sigma(q_i)}, i = 1, \ldots, g - 2 \). Clearly \( \nu_i \in H^1(T_C)^+ \), hence by the \( \sigma \)-invariance, and using \([5]\), if \( i \neq j \) we have

\[
\rho_{HE}(Q_s)(\nu_i \circ \nu_j) = \rho(Q_s)(\nu_i \circ \nu_j) = 2\rho(Q_s)(\nu_i \circ \nu_j) + 2\rho(Q_s)(\nu_i \circ \nu_j) = 8\pi i(\delta s(q_i, q_j)\eta_{q_i}(q_j) + Q_s(q_i, \sigma(q_j))\eta_{q_i}(\sigma(q_j))) = 0,
\]

since \( q_i, q_j \) are in the zero locus of \( t \). On the other hand we have

\[
\rho_{HE}(Q_s)(\nu_i \circ \nu_i) = \rho(Q_s)(\nu_i \circ \nu_i) = 2\rho(Q_s)(\nu_i \circ \nu_i) = 2(\rho(Q_s)(\xi_{q_i} \circ \xi_{q_i}) + 2\rho(Q_s)(\xi_{q_i} \circ \xi_{q_i})) = 8\pi i(2\mu_2(Q_s)(q_i) + Q_s(q_i, \sigma(q_i))\eta_{q_i}(\sigma(q_i))) = 16\pi i\mu_2(Q_s)(q_i) = 16\pi i\mu_1\mu_1(x \wedge y)(q_i)\mu_1, \lambda, s \wedge t(q_i) = 16\pi i a_i,
\]

where \( a_i = 0 \) if and only if \( s(q_i) = 0 \).

So if so we denote by \( V = \langle v_1, \ldots, v_{g-2} \rangle \), we have shown that \( \rho_{HE}(Q_s)|_V \) is diagonal with diagonal entries equal to \( ca_i, c = 16\pi i \).

**Proposition 6.1.** Assume that \( T \subset H^1(T_C)^+ \) is a linear subspace which is isotropic with respect to all the quadrics \( \rho_{HE}(Q_s), \forall s \in W \), then \( T \cap V = \{0\} \).

**Proof.** If \( v = z_1v_1 + \ldots + z_{g-2}v_{g-2} \in V \), then \( \rho_{HE}(Q_s)(v \circ v) = ca_i z_i^2 \), with \( a_i \neq 0 \). Hence \( \rho_{HE}(Q_s)(v \circ v) = 0 \) if and only if \( z_i = 0 \). \( \square \)

**Theorem 6.2.** Let \( Y \) be a germ of a totally geodesic submanifold of \( \mathbb{A}_3 \) contained in the hyperelliptic locus, then \( \dim Y \leq g + 1 \).

**Proof.** If \( Y \) is a germ of a totally geodesic submanifold of \( \mathbb{A}_3 \) passing to \([C]\), its tangent space at \([C]\) is isotropic with respect to all the quadrics \( \rho_{HE}(Q), Q \in I_2(K_C) \), hence to all the quadrics \( \rho_{HE}(Q_s) \), therefore \( T_{[C]}Y \cap V = \{0\} \) by Proposition \(6.1\). Thus \( \dim T_{[C]}Y + \dim V = \dim Y + g - 2 \leq \dim H^1(T_C)^+ = 2g - 1 \), so \( \dim Y \leq g + 1 \). \( \square \)

**Proposition 6.3.** Let \( Y \) be a germ of a totally geodesic submanifold of \( \mathbb{A}_3 \), contained in the hyperelliptic locus, then \( \dim Y \leq 3 \).

**Proof.** If \( g = 3 \) then the dimension of the space of quadrics containing the rational normal curve is one and this space is generated by the rational normal curve, which is a smooth conic \( Q \). By Proposition \(6.5\), the rank of \( \rho(Q) \) is at least 3, hence \( \dim(Y) \leq 5 - 3 + \frac{3}{2} = \frac{7}{2} \). \( \square \)

**Remark 6.4.** Families (8) and (22) of Table 2 of \([5]\) (see also Tables 1 and 2 of \([13]\)) yield respectively a two-dimensional and a one-dimensional Shimura (hence totally geodesic) subvariety of \( \mathbb{A}_3 \) generically contained in the hyperelliptic Torelli locus.

Family (36) of Table 2 of \([5]\) yields a one-dimensional Shimura (hence totally geodesic) subvariety of \( \mathbb{A}_4 \) generically contained in the hyperelliptic Torelli locus.

Family (39) of Table 2 of \([5]\) yields a one-dimensional Shimura (hence totally geodesic) subvariety of \( \mathbb{A}_5 \) generically contained in the hyperelliptic Torelli locus.

**Proof.** See \([5]\) 4.6. \( \square \)
REFERENCES

[1] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J. *Geometry of algebraic curves, Vol. I*, Grundlehren der Mathematischen Wissenschaften, 267. Springer-Verlag, New York, 1985.

[2] E. Colombo and P. Frediani. Siegel metric and curvature of the moduli space of curves. *Trans. Amer. Math. Soc.*, 362(3):1231–1246, 2010.

[3] E. Colombo, P. Frediani, and A. Ghigi. On totally geodesic submanifolds in the Jacobian locus. *International Journal of Mathematics*, 26(01):1550005, 2015.

[4] E. Colombo, G. P. Pirola, and A. Tortora. Hodge-Gaussian maps. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 30(1):125–146, 2001.

[5] P. Frediani, A. Ghigi and M. Penegini. Shimura varieties in the Torelli locus via Galois coverings. *Int. Math. Res. Not.*, 2015, no. 20, 10595-10623.

[6] A. Ghigi. On some differential-geometric aspects of the Torelli map. *Boll. Unione. Mat. Ital.*, 12 (2019), no. 1-2, 133-144.

[7] A. Ghigi, P. Pirola, S. Torelli. Totally geodesic subvarieties in the moduli space of curves. \[arXiv:1902.06095\]

[8] V. González-Alonso, L. Stoppino, and S. Torelli. On the rank of the flat unitary factor of the Hodge bundle. *ArXiv: 1709.05670*. To appear in *Transactions of the AMS*.

[9] R.C. Gunning. Some topics in the function theory of compact Riemann surfaces. https://web.math.princeton.edu/~gunning/book.pdf.

[10] X. Lu, K. Zuo. The Oort conjecture on Shimura curves in the Torelli locus of hyperelliptic curves. *J. Math. Pures Appl. (9)* 108 (2017), no. 4, 532-552.

[11] B. Moonen. Linearity properties of Shimura varieties. I. *J. Algebraic Geom.*, 7(3):539–567, 1998.

[12] B. Moonen. Special subvarieties arising from families of cyclic covers of the projective line. *Doc. Math.*, 15:793–819, 2010.

[13] B. Moonen and F. Oort. The Torelli locus and special subvarieties. In *Handbook of Moduli: Volume II*, pages 549–94. International Press, Boston, MA, 2013.

[14] F. Oort and J. Steenbrink. The local Torelli problem for algebraic curves. In *Journées de Géometrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 157–204. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[15] G. P. Pirola, S. Torelli. Massey products and Fujita decompositions on fibrations of curves. \[arXiv:1710.02825v2\]. To appear in *Collectanea Mathematica*. https://doi.org/10.1007/s13348-019-00247-4.

[16] J. Wahl. Introduction to Gaussian maps on an algebraic curve. *Complex projective geometry (Trieste and Bergen, 1989)*, London Math. Soc. Lecture Note Ser., 179, pp. 304-323, Cambridge Univ. Press, Cambridge, 1992.

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