A discrete analogue of odd Weibull–G family of distributions: properties, classical and Bayesian estimation with applications to count data

M. El-Morshedy, M. S. Eliwa and Abhishek Tyagi

Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj, Saudi Arabia; Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt; Department of Statistics, Chaudhary Charan Singh University, Meerut, India

ABSTRACT
In the statistical literature, several discrete distributions have been developed so far. However, in this progressive technological era, the data generated from different fields is getting complicated day by day, making it difficult to analyze this real data through the various discrete distributions available in the existing literature. In this context, we have proposed a new flexible family of discrete models named discrete odd Weibull-G (DOW-G) family. Its several impressive distributional characteristics are derived. A key feature of the proposed family is its failure rate function that can take a variety of shapes for distinct values of the unknown parameters, like decreasing, increasing, constant, J-, and bathtub-shaped. Furthermore, the presented family not only adequately captures the skewed and symmetric data sets, but it can also provide a better fit to equi-, over-, under-dispersed data. After producing the general class, two particular distributions of the DOW-G family are extensively studied. The parameters estimation of the proposed family, are explored by the method of maximum likelihood and Bayesian approach. A compact Monte Carlo simulation study is performed to assess the behavior of the estimation methods. Finally, we have explained the usefulness of the proposed family by using two different real data sets.

KEYWORDS
Discrete distributions; dispersion index; Bayesian method; maximum likelihood method; L-moment statistics; odd Weibull-G family; simulation

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
62E15; 62F10; 62F15

1. Introduction
The Weibull (W) model is a well-known continuous distribution and has been used extensively over the last several decades for modelling data in many areas, especially in engineering, reliability, and biological fields. It is commonly applied for modellign monotone hazard rates. The cumulative distribution function (CDF) of W distribution can be presented as

\[ F(x; \lambda, \beta) = 1 - e^{-\lambda x^{\beta}}, \quad x \geq 0, \quad (\lambda, \beta) > 0, \]

CONTACT M. El-Morshedy m.elmorshedy@psau.edu.sa

© 2021 Informa UK Limited, trading as Taylor & Francis Group
where $\lambda$ and $\beta$ are the scale and shape parameters, respectively. Because of the flexibility of the W distribution, many statisticians proposed several modifications by introducing one or more parameters to the baseline model in Equation (1). For instance, Lai et al. [43], Pal et al. [52], Lee et al. [44], Silva et al. [62], Pinho et al. [55], Almalki and Yuan [7], Sarhan and Apaloo [61], Nofal et al. [51], Al-Babtain et al. [4], Abd EL-Baset and Ghazal [1], and references cited therein.

Alzaatreh et al. [8] developed a new approach in which they used a general form to produce a new family, called the transformed-transformer family. Bourguignon et al. [12] utilized this technique to introduce a new flexible family of continuous distributions and it is known as odd Weibull-G (OW-G) family. The CDF of OW-G family is given by

$$F(x; \lambda, \beta; \Phi) = 1 - e^{-\lambda \left( \frac{G(x; \Phi)}{1 - G(x; \Phi)} \right)^\beta}; \quad x \in \mathbb{R},$$

where $\Phi$ is a vector of model parameters $(1 \times j, j = 1, 2, 3, \ldots, k)$ and $G(x; \Phi)$ is the CDF of the baseline distribution. For more details about the generator $G(x; \Phi)/(1 - G(x; \Phi))$, see Cooray [17]. Due to the flexibility of this generator, several authors used it for producing flexible families which can be utilized for modelling different types of data. For instance, Tahir et al. [63], Korkmaz and Genç [40], Korkmaz [37], Hamedani et al. [30], Korkmaz et al. [38], El-Morshedy and Eliwa [24], Korkmaz et al. [38], Reyad et al. [58], Bakouch et al. [9], Nascimento et al. [47], Alizadeh et al. [5,6], Korkmaz et al. [39], Afify and Alizadeh [2], Cordeiro et al. [16], and references cited therein.

The concept of discretization generally emerges when it becomes impossible or inconvenient to record the lifetime of a product/device on a continuous scale. These circumstances may appear when the life length needs to be recorded on a discrete scale rather than on a continuous analogue. For example, in survival analysis, the survival times for those having the diseases like lung cancer, or the period from remission to relapse may be recorded as a number of days/weeks; in engineering systems, the number of successful cycle prior to the failure when device work in cycle, the number of times a device is switched on/off. Moreover, in many practical problems, the count phenomenon occurs as, for example, the number of occurrences of earthquakes in a calendar year, the number of absences, the number of accidents, the number of kinds of species in ecology, the number of insurance claims, and so on. Therefore, we can easily infer that it is rational and appropriate to model such situations by a suitable discrete distribution.

Since classical discrete models like Binomial, Poisson, Negative Binomial, and Geometric were not sufficient to analyze different types of discrete data, therefore, Roy [59] proposed a new technique to develop a new discrete distribution by using the survival function of any continuous distribution. This approach is named as survival discretization method (Chakraborty [13]). The one of the important virtue of this methodology is that, the developed discrete model keeps the same form of the survival function as that of its continuous counterpart. Due to this feature many reliability characteristics of the distribution remain unchanged. In recent years this method has been widely used to develop several discrete models. For instance, Roy [60], Inusah and Kozubowski [35], Ghitany and Al-Mutairi [27], Krishna and Pundir [41], Gómez-Déniz [28], Gómez-Déniz and Calderín-Ojeda [29], Bebbington et al. [11], Nekoukhou et al. [48], Bakouch et al. [10], Nekoukhou and Bidram [49,50], Alamatsaz et al. [3], Chandrakant et al. [14], Kus et al. [42], Tyagi et al.
In view of the existing literature, we found that although several discrete models have been proposed over the past few decades, we are still facing a lack of more admirable discrete distributions that adequately capture the diversity of real datasets. Such phenomenon motivates us to provide a flexible family of discrete models for analyzing a broad spectrum of discrete real-world data sets. Therefore, in present article, we propose a family of discrete uni-variate distributions with two additional parameters using survival discretization method. The main objectives of proposing DOW-G family are as follows:

• To produce discrete models that not only fit a positively skewed, a negatively skewed, or a symmetric shaped data set, they are also capable enough for fitting equi-, under-, and over-dispersed real data.
• To develop discrete distributions whose failure rate functions can take diverse shapes for different values of parameters (eg. increasing, decreasing, constant, J- and bathtub-shaped).
• To generate models for modelling probability distribution of count data.
• To produce consistently superior fits than other developed discrete distributions with the same baseline model and other popular discrete distributions in the existing literature.

The remaining parts of this article are as follows: Section 2 introduces the DOW-G family. Some statistical characteristics of the DOW-G family are derived in Section 3. Two special models of the proposed family are extensively studied in Section 4. The method of maximum likelihood and the Bayesian approach is used to estimate the unknown parameters of the DOW-G family in Section 5. An extensive simulation study is conducted to investigate the behaviour of different estimation methods in Section 6. Section 7 contains two applications to real data sets. Finally, some important remarks about the presented study are discussed in Section 8.

2. Synthesis of the family

The random variable (RV) $X$ is said to follow the DOW-G family if its CDF is of the form

$$F_X(x; p, \beta, \Phi) = 1 - p \left( \frac{G(x+1; \Phi)}{1-G(x+1; \Phi)} \right)^\beta; \quad x \in \mathbb{N}_0,$$

where $p = e^{-\lambda}$, $p \in (0, 1)$, $\beta \in (0, \infty)$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. The reliability function (RF) of the DOW-G family is given by

$$R_X(x; p, \beta, \Phi) = p \left( \frac{G(x+1; \Phi)}{1-G(x+1; \Phi)} \right)^\beta; \quad x \in \mathbb{N}_0.$$

Suppose $X_1, X_2, \ldots, X_n$ be independent and identically distributed (IID) integer valued RVs and $Z = \min(X_1, X_2, \ldots, X_n)$, then $Z \sim \text{DOW-G}(x; p^n, \beta, \Phi)$ provided $X_i(i =
1, 2, \ldots, n) \sim \text{DOW-G}(z; p, \beta, \Phi) \) family where

\[ R_{Z}(x; p, \beta, \Phi) = \prod_{i=1}^{n} \Pr[X_i \geq x] = p^n \left( \frac{G(x+1; \Phi)}{1-G(x+1; \Phi)} \right)^\beta; \quad x \in \mathbb{N}_0 \]  

(5)

The probability mass function (PMF) and hazard rate function (HRF) corresponding to Equation (3) can be expressed as

\[ f_{X}(x; p, \beta, \Phi) = p \left( \frac{G(x; \Phi)}{1-G(x; \Phi)} \right)^\beta - p \left( \frac{G(x+1; \Phi)}{1-G(x+1; \Phi)} \right)^\beta; \quad x \in \mathbb{N}_0 \]  

(6)

and

\[ h(x; p, \beta, \Phi) = 1 - p \left( \frac{G(x; \Phi)}{1-G(x; \Phi)} \right)^\beta - \left( \frac{G(x+1; \Phi)}{1-G(x+1; \Phi)} \right)^\beta; \quad x \in \mathbb{N}_0, \]  

(7)

respectively.

3. Distributional properties of the DOW-G family

3.1. Quantile function (QF)

Under the DOW-G family, the \( q \)th QF, say \( x_q \), is the solution of

\[ F_{X}(x_q; p, \beta, \Phi) = q; \quad x_q > 0, \]  

then

\[ x_q = G^{-1} \left( \frac{1}{\ln(1-q) - \ln(p)} \right)^{-1/\beta} + 1, \]  

(8)

where \( q \in (0, 1) \) and \( G^{-1} \) denotes the baseline QF. By putting \( q = 0.5 \), we can obtain the median of the proposed family. To study the effect of shape parameters on skewness and kurtosis, one can use \( x_q \). The Bowley skewness and Moors kurtosis based on the quantiles can be obtained as

\[ \text{Bowley Skewness} = \frac{\frac{x_3}{4} + \frac{x_1}{4} - 2\frac{x_1}{2}}{\frac{x_1}{4} - \frac{x_1}{4}}, \]  

and Moors kurtosis

\[ x_3 - x_1 + x_7 - x_5. \]

3.2. Moments, dispersion index, skewness and kurtosis

Suppose a RV \( X \sim \text{DOW-G}(p, \beta, \Phi) \), then the \( r \)th moment of the RV \( X \) can be proposed by

\[ \mu'_r = \sum_{x=1}^{\infty} (x^r - (x-1)^r) p \left( \frac{G(x; \Phi)}{1-G(x; \Phi)} \right)^\beta; \quad x \in \mathbb{N}_0, \]  

(9)

where \( \mu' = \sum_{x=0}^{\infty} x^r f_{X}(x; p, \beta, \Phi) \). Using the expression of Equation (9), the mean (\( \mu'_1 \)) and variance can be, respectively, formulated as

\[ \mu'_1 = \sum_{x=1}^{\infty} p \left( \frac{G(x; \Phi)}{1-G(x; \Phi)} \right)^\beta \]  

and variance

\[ = \sum_{x=1}^{\infty} (2x - 1) p \left( \frac{G(x; \Phi)}{1-G(x; \Phi)} \right)^\beta - (\mu'_1)^2. \]  

(10)

The dispersion index (DSI) is defined as the variance / |mean|, it determines that whether a given distribution is suitable for equi-, under- or over- dispersed data sets. The DSI is
widely used in ecology as a standard measure for measuring repulsion (under dispersion) or clustering (over dispersion). If $D_{SI} < 1$ ($D_{SI} > 1$) the distribution is under-dispersed (over-dispersed), whereas the distribution is equi-dispersed at $D_{SI} = 1$.

The moment generating function (MGF) of the proposed family can be formulated as

$$M_X(t) = \sum_{x=0}^{\infty} \sum_{l=0}^{\infty} \frac{(xt)^l}{l!} \left( p \left( \frac{G(x; \Phi)}{1-G(x; \Phi)} \right)^\beta - p \left( \frac{G(x+1; \Phi)}{1-G(x+1; \Phi)} \right)^\beta \right); \quad x \in \mathbb{N}_0.$$  (11)

The first four partial derivatives of $M_X(t)$, with respect to $t$ at $t = 0$, produce the first four raw moments about the origin, i.e. $\mu'_r = \frac{d^r}{dt^r} M_X(t)|_{t=0}$. Furthermore, by using Equations (9) or (11), the skewness and kurtosis based on moments can be computed as skewness $= (\mu'_3 - 3\mu'_2\mu'_1 + 2\mu'_1^3)/(\text{variance})^{3/2}$ and kurtosis $= (\mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'_1^2 - 3\mu'_1^4)/(\text{variance})^2$.

### 3.3. Mean of excess over threshold and tail value at risk

Considerable attention has been paid to measuring operational risk over the past few decades. In this segment, we obtain few risk measures such as mean excess over the threshold (MEOT) and tail value at risk (TVAR) for the DOW-G family. The MEOT of the RV $X$ following DOW-G family is given as

$$m(x) = E(X - x | X \geq x) = p^{-\left( \frac{G(x; \Phi)}{1-G(x; \Phi)} \right)^\beta} \sum_{j=x}^{\infty} \left( \frac{G(j+1; \Phi)}{1-G(j+1; \Phi)} \right)^\beta; \quad x \in \mathbb{N}_0.$$  (12)

Based on an arbitrary choice of $G(x; \Phi)$, we can get $m(0) \leq m(x)$, then the DOW-G family belongs to a class of discrete distributions having new worse than used in expectation, for more detail refer Marshall and Proschan [45]. Hence, the DOW-G family plays an vital role in the reliability theory. Regarding the TVAR, for any quantile $x_q$, the TVAR of DOW-G family can be derived as

$$\text{TVAR}_q(X) = E(X | X \geq x_q) = x_q + \frac{1}{1 - F_X(x_q; p, \beta, \Phi)} \sum_{x=0}^{x_q-1} (x - x_q) f_X(x; p, \beta, \Phi)$$

$$= x_q + \frac{1}{1 - F_X(x_q; p, \beta, \Phi)} \left( \mu'_1 - x_q + \sum_{x=0}^{x_q-1} (x_q - x) f_X(x; p, \beta, \Phi) \right)$$

$$= x_q + p^{-\left( \frac{G(x_q+1; \Phi)}{1-G(x_q+1; \Phi)} \right)^\beta} \left( \mu'_1 - x_q + \sum_{x=0}^{x_q-1} \left[ p \left( \frac{G(x; \Phi)}{1-G(x; \Phi)} \right)^\beta - p \left( \frac{G(x+1; \Phi)}{1-G(x+1; \Phi)} \right)^\beta \right] \right);$$

$$x \in \mathbb{N}_0.$$  (13)
### 3.4. Rényi and Shannon entropies

The amount of uncertainty associated with a RV \( X \) is referred to as Entropy. It is widely applicable in many areas such as information theory, econometrics, survival analysis, computer science and quantum information, for more details, see Rényi [57]. The Rényi and Shannon entropies of the RV \( X \) can be formulated as

\[
I_\delta(X) = \frac{1}{1 - \delta} \log \sum_{x=0}^{\infty} \left( p \left( \frac{G(x; \Phi)}{1 - G(x; \Phi)} \right)^\delta - p \left( \frac{G(x+1; \Phi)}{1 - G(x+1; \Phi)} \right)^\delta \right); \quad x \in \mathbb{N}_0
\]

and

\[
I(X) = -\sum_{x=0}^{\infty} \left\{ p \left( \frac{G(x; \Phi)}{1 - G(x; \Phi)} \right)^\beta - p \left( \frac{G(x+1; \Phi)}{1 - G(x+1; \Phi)} \right)^\beta \right\} \log \left\{ p \left( \frac{G(x; \Phi)}{1 - G(x; \Phi)} \right)^\beta - p \left( \frac{G(x+1; \Phi)}{1 - G(x+1; \Phi)} \right)^\beta \right\}; \quad x \in \mathbb{N}_0,
\]

respectively, where \( \delta \in (0, \infty) \) and \( \delta \neq 1 \). It is notable that the Shannon entropy can be derived as a particular case of the Rényi entropy if \( \delta \to 1 \), i.e. \( I(X) = \lim_{\delta=1} I_\delta(X) \).

### 3.5. Order statistics and L-moment statistics

Order statistics (ORST) play an important role in various fields of statistical theory and practice. Suppose \( X_1, X_2, \ldots, X_n \) be a random sample (RS) from the DOW-G(\( \rho, \beta, \Phi \)), and let \( X_{1:n}, X_{2:n}, \ldots, X_{n:n} \) be their corresponding ORST. Then, the CDF of the \( i \)th ORST \( X_{i:n} \) for an integer value of \( x \) is proposed as

\[
F_{i:n}(x; \rho, \beta, \Phi) = \sum_{k=i}^{n} \binom{n}{k} \left[ F_i(x; \rho, \beta, \Phi) \right]^k \left[ 1 - F_i(x; \rho, \beta, \Phi) \right]^{n-k}
\]

\[
= \sum_{k=i}^{n} \sum_{j=0}^{n-k} (-1)^j \binom{n}{k} \binom{n-k}{j} \left[ F_i(x; \rho, \beta, \Phi) \right]^{k+j}
\]

\[
= \sum_{k=i}^{n} \sum_{j=0}^{n-k} \sum_{m=0}^{k+j} \Delta_{(n,k)}^{(m,j)} F_i(x; p^m, \beta, \Phi),
\]

where \( \Delta_{(n,k)}^{(m,j)} = (-1)^{j+m} \binom{n}{k} \binom{n-k}{j} (k+j+m) \). The PMF of the \( i \)th ORST can be formulated as

\[
f_{i:n}(x; \rho, \beta, \Phi) = \sum_{k=i}^{n} \sum_{j=0}^{n-k} \sum_{m=0}^{k+j} \Delta_{(n,k)}^{(m,j)} f_i(x; p^m, \beta, \Phi).
\]

The \( u \)th moment of \( Z_{i:n} \) is given by

\[
\Psi_{i:n}^u = E(Z_{i:n}^u) = \sum_{z=0}^{\infty} \sum_{k=i}^{n} \sum_{j=0}^{n-k} \sum_{m=0}^{k+j} \Delta_{(n,k)}^{(m,j)} z^u f_i(x; p^m, \beta, \Phi).
\]
L-moments (L-MS) are linear combinations of ORSTs and are used to study the shape of a probability distribution. Hosking and Wallis [32] has proposed L-MS to summaries theoretical distribution and observed samples. Let \( X(i|n) \) be ith largest observations based on a sample of size \( n \), then the L-MS is

\[
\Lambda_r^* = \frac{1}{r} \sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} E(X_{r-s:r}).
\]

Using Equation (19), we get

\[
\Lambda_1^* = E(X_{1:1}), \quad \Lambda_2^* = \frac{1}{2} E(X_{2:2} + X_{1:2}), \quad \Lambda_3^* = \frac{1}{3} \{ E(X_{3:3} - X_{2:3}) - E(X_{2:3} + X_{1:3}) \} \quad \text{and} \quad \Lambda_4^* = \frac{1}{4} \{ E[(X_{4:4} - X_{3:4}) + (X_{2:4} - X_{1:4})] - 2E(X_{3:4} - X_{2:4}) \},
\]

and consequently, we can define some descriptive statistics like L-M of mean, L-M coefficient of variation, L-M coefficient of skewness and L-M coefficient of kurtosis as \( \Lambda_1^*, \frac{\Lambda_2^*}{\Lambda_1^*}, \frac{\Lambda_3^*}{\Lambda_2^*} \), and respectively.

### 4. Special models

In this segment, we study two particular distributions of the DOW-G family to demonstrate its viability. Some other main objectives of providing these models are: to evaluate the aforementioned characteristics for particular models of the presented family; to investigate the behaviour of various estimation methods for estimation of unknown parameters (in Section 6); to express the practicality of the developed family in real data analysis through two special models (in Section 7).

#### 4.1. The DOW-Geometric (DOWGeo) distribution

Consider the CDF of the Geo distribution. Then, the PMF of the DOWGeo distribution can be formulated as

\[
f_x(x; \beta, \theta) = p(\theta^{x-1})^\beta \cdot (\theta^{-(x+1)})^\beta; \quad x \in \mathbb{N}_0,
\]

where \( \beta > 0 \) and \( 0 < p, \theta < 1 \). Figures 1 and 2 show the PMF (a: \( p = 0.9, \beta = 0.5, \theta = 0.9 \). b: \( p = 0.9, \beta = 0.5, \theta = 0.5 \). c: \( p = 0.5, \beta = 0.5, \theta = 0.9 \). d: \( p = 0.9, \beta = 0.5, \theta = 0.9 \)) and HRF (a: \( p = 0.9, \beta = 0.5, \theta = 0.5 \). b: \( p = 0.9, \beta = 0.5, \theta = 0.1 \). c: \( p = 0.1, \beta = 0.5, \theta = 0.9 \). d: \( p = 0.9, \beta = 0.1, \theta = 0.9 \)) of the DOWGeo model, respectively.

The PMF can be either unimodal or bimodal and can be used to analyze both a positively skewed and a negatively skewed data set. Furthermore, the HRF can be either increasing, constant, increasing-constant-, J-, and bathtub-shaped. Therefore, the parameters of the DOWGeo model can be fixed to fit most data sets.

Regarding the \( r \)th moment, it is difficult to express it in an explicit form, and consequently, Maple software is utilized to interpret some of the descriptive statistics of the DOWGeo model. Table 1 lists some descriptive statistics using the DOWGeo model for different values of the parameters \( p, \beta \) and \( \theta \).

From Table 1, it can be easily observed that the DOWGeo model is appropriate of modelling over- and (under)-dispersed data; for some selected values of the model parameters \( DsI=1 \), so it can be used as an alternative model to Poisson distribution; it is suitable
for modelling negative and positive skewed as well as symmetric data sets; and DOWGeo model can be used to analyze either leptokurtic (kurtosis > 3) or platykurtic (kurtosis < 3) data sets. Tables 2–4 reports some numerical computations of MEOT, TVAR and Rényi entropy at \( x = 5 \) based on DOWGeo parameters when \( \delta = 2 \) and \( q = 0.5 \).

From Tables 2–4, it is clearly seen that: the MEOT and Rényi entropy increase whereas the TVAR can take bathtub-shaped for some fixed values of \( \beta \) and \( \theta \) with \( p \to 1 \); the MEOT and Rényi entropy increase whereas the TVAR have inverse-J-shaped for some fixed values of \( \beta \) and \( p \) with \( \theta \to 1 \); and the MEOT and Rényi entropy decrease whereas the TVAR increases for some fixed values of \( \theta \) and \( p \) with \( \beta \) grows.

### 4.2. The DOW-Inverse Weibull (DOWIW) distribution

Consider the CDF of the inverse Weibull (IW) model. Then, the PMF of the DOWIW distribution can be defined as

\[
f_x(x; p, \beta, \theta, \alpha) = p (\theta^{-x-\alpha-1})^{-\beta} - p (\theta^{-(x+1)-\alpha-1})^{-\beta}; \quad x \in \mathbb{N}_0 - \{0\},
\]

where \( 0 < p, \theta < 1, \alpha > 0 \) and \( \beta > 0 \). Figures 3 and 4 show the PMF (a : \( p = 0.9, \beta = 0.5, \theta = 0.1, \alpha = 3.8 \). b : \( p = 0.9, \beta = 0.9, \theta = 0.9, \alpha = 0.2 \). c : \( p = 0.5, \beta = 1.3, \theta = 0.1 \),
$\alpha = 1.2$. $d : p = 0.5, \beta = 2.3, \theta = 0.1, \alpha = 1.2$) and HRF ($a : p = 0.9, \beta = 0.5, \theta = 0.1, \alpha = 2.8$. $b : p = 0.9, \beta = 0.9, \theta = 0.1, \alpha = 2.8$. $c : p = 0.9, \beta = 0.9, \theta = 0.9, \alpha = 0.9$. $d : p = 0.5, \beta = 0.3, \theta = 0.1, \alpha = 1.2$) of the DOWIW distribution, respectively. The shape of the PMF is only unimodal, whereas the HRF can be increasing, increasing-constant-, or decreasing.

Like the DOWGeo model, the DOWIW distribution can be applied to model skewed data set. Moreover, it is appropriate for modelling over-, equi- or under-dispersed data sets.

5. Estimation

5.1. Maximum likelihood estimation

In this segment, unknown parameters of the DOW-G family are estimated by maximum likelihood estimation (MLE). Let $X_1, X_2, \ldots, X_n$ be a RS from the proposed family. Then, the likelihood function ($l$) can be expressed as

$$l(\underline{x}; p, \beta, \Phi) = \prod_{i=1}^{n} \left( p^{[\Psi(x_i; \Phi)]^{\beta}} - p^{[\Psi(x_i+1; \Phi)]^{\beta}} \right),$$

(22)
Table 1. Some useful descriptive statistics of the DOWGeo distribution.

| Parameter | Measure |
|-----------|---------|
| $\beta$  | Mean   | Variance | Skewness | Kurtosis | DsI     |
| 0.5       | 0.00100 | 0.00099  | 31.5753  | 998.0020 | 0.99900 |
| 0.3       | 0.03034 | 0.03074  | 5.84226  | 18.27296 | 1.26718 |
| 0.7       | 0.38322 | 0.7623   | 2.93334  | 10.70328 | 5.57875 |
| 0.9       | 0.87829 | 10.478   | 1.44336  | 4.38707  | 1.13074 |
| 0.5       | 1.05464 | 1.80762  | 1.22931  | 3.80505  | 1.71396 |
| 0.7       | 2.38896 | 7.29211  | 1.11821  | 3.54407  | 3.05241 |
| 0.9       | 9.05357 | 86.8501  | 1.05653  | 3.41683  | 9.59291 |
| 0.1       | 1.11533 | 0.71572  | 0.13437  | 2.07950  | 9.59291 |
| 0.3       | 2.54887 | 2.63579  | $-0.01709$ | 2.09458  | 1.03410 |
| 0.9       | 4.77186 | 8.02525  | $-0.07148$ | 2.11899  | 1.68178 |
| 0.7       | 9.72264 | 30.52447 | $-0.09810$ | 2.13726  | 3.13952 |
| 0.9       | 34.0694 | 351.4762 | $-0.11116$ | 3.53164  | 10.3164 |
| 5.5       | 1.0 $\times 10^{-171147}$ | 1.0 $\times 10^{-177147}$ | 3.2 $\times 10^{880573}$ | 1.0 $\times 10^{177147}$ | 1.00000 |
| 0.3       | 3.9 $\times 10^{-52627}$ | 3.9 $\times 10^{-92627}$ | 1.5 $\times 10^{56413}$ | 2.5 $\times 10^{92626}$ | 1.00000 |
| 0.5       | 2.7 $\times 10^{-333227}$ | 2.7 $\times 10^{-53327}$ | 1.9 $\times 10^{36663}$ | 3.6 $\times 10^{53326}$ | 1.00000 |
| 0.7       | 3.8 $\times 10^{-27441}$ | 3.8 $\times 10^{-27441}$ | 1.6 $\times 10^{13720}$ | 2.6 $\times 10^{27440}$ | 1.00000 |
| 0.9       | 1.5 $\times 10^{-8106}$ | 1.5 $\times 10^{-8106}$ | 7.9 $\times 10^{10552}$ | 6.3 $\times 10^{8105}$ | 1.00000 |
| 0.1       | 0.100000 | 0.090000 | 2.66666 | 8.11111 | 0.90000  |
| 0.3       | 0.300000 | 0.210000 | 0.87287 | 1.76190 | 0.70000  |
| 0.5       | 0.500000 | 0.250000 | 0.00000 | 1.00000 | 0.50000  |
| 0.7       | 0.700000 | 0.210000 | $-0.87287$ | 1.76190 | 0.30000  |
| 0.9       | 0.900000 | 0.210000 | $-0.87287$ | 1.76190 | 0.30000  |
| 0.1       | 5.00111 | 0.90089 | 2.66666 | 8.11111 | 0.10000  |
| 0.3       | 5.50660 | 1.01886 | $-0.54209$ | 3.37346 | 0.18502  |
| 0.5       | 5.96232 | 1.12473 | $-0.56793$ | 3.42405 | 0.18863  |
| 0.7       | 6.54227 | 1.26243 | $-0.59542$ | 3.47461 | 0.19296  |
| 0.9       | 7.69496 | 1.53654 | $-0.64448$ | 3.58922 | 0.19968  |

Table 2. The MEOT, TVAR and entropy at $\beta = 2.5$, $\theta = 0.9$ and $p \to 1$.

| Measure ↓ Parameter → | $p$ |
|-----------------------|-----|
|                       | 0.05 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.95 |
| MEOT                  | 1.23741 | 1.36315 | 1.81559 | 2.37393 | 3.26988 | 5.56188 | 7.23255 |
| TVAR                  | $1.3 \times 10^{+0129544}$ | $1.6 \times 10^{+05277}$ | $1.7 \times 10^{+150}$ | $2.2 \times 10^{+6}$ | $8.56188$ | $15.97817$ |
| Rényi entropy         | 1.69774 | 1.76243 | 1.91188 | 2.02716 | 2.15060 | 3.58922 | 2.41787 |

Table 3. The MEOT, TVAR and entropy at $\beta = 1.5$, $p = 0.5$ and $\theta \to 1$.

| Measure ↓ Parameter → | $\theta$ |
|-----------------------|-----|
|                       | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.95 |
| MEOT                  | 0.99974 | 0.99998 | 1.00000 | 1.00001 | 3.56188 | 9.89159 |
| TVAR                  | $1.3 \times 10^{-58009544}$ | $1.6 \times 10^{-55277}$ | $1.7 \times 10^{150}$ | $2.2 \times 10^{6}$ | $8.56188$ | $15.97817$ |
| Rényi entropy         | $1.4 \times 10^{-8}$ | 0.16816 | 0.74610 | 1.30292 | 2.48811 | 3.20516 |

Furthermore, the log-likelihood function ($L$) can be obtained as

$$L(x; \beta, \Phi) = \sum_{i=1}^{n} \log \left( p \left[ \Psi(x_i; \Phi_j) \right]^{\beta} - p \left[ \Psi(x_i; \Phi_j) \right]^{\beta} \right),$$

(23)
Table 4. The MEOT, TVAR and entropy at $\theta = 0.5$, $p = 0.5$ and $\beta$ grows.

| Measure ↓ Parameter → | $\beta$ | 0.1 | 0.5 | 1.5 | 2.5 | 3.0 | 5.5 |
|-----------------------|---------|-----|-----|-----|-----|-----|-----|
| MEOT                  |         | 8.81406 | 1.10315 | 1.00000 | 0.99999 | 0.99998 | 0.77272 |
| TVAR                  |         | 15.01324 | 258.4995 | $1.7 \times 10^{150}$ | $1.1 \times 10^{9483}$ | $2.2 \times 10^{75271}$ | $1.6 \times 10^{2371279759}$ |
| Rényi entropy         |         | 1.34251 | 1.13804 | 0.74610 | 0.69318 | 0.69314 | 0.69313 |

Figure 3. The PMF of the DOWIW distribution.

where $\Psi (\cdot; \Phi_j) = \frac{G(\cdot; \Phi_j)}{1 - G(\cdot; \Phi_j)}$. The maximum likelihood estimators of the unknown parameters $p$, $\beta$ and $\Phi$ can be achieved by solving the non-linear normal equations obtained by differentiating (23) with respect to the family parameters. The components of the score vector, $B(p, \beta, \Phi) = (\frac{\partial L}{\partial p}, \frac{\partial L}{\partial \beta}, \frac{\partial L}{\partial \Phi})^T$, are

$$B_p = \frac{1}{p} \sum_{i=1}^n \frac{[\Psi(x_i; \Phi_j)]^\beta p[\Psi(x_i; \Phi_j)]^\beta - [\Psi(x_i + 1; \Phi_j)]^\beta p[\Psi(x_i + 1; \Phi_j)]^\beta}{p[\Psi(x_i; \Phi_j)]^\beta - p[\Psi(x_i + 1; \Phi_j)]^\beta}, \quad (24)$$
Figure 4. The HRF of the DOWIW distribution.

\[ B_\beta = \ln(p) \sum_{i=1}^{n} \frac{\left[ \Psi(x_i; \Phi_j) \right]^{\beta} p[\Psi(x_i; \Phi_j)]^{\beta} \ln(\Psi(x_i; \Phi_j)}{p[\Psi(x_i; \Phi_j)]^{\beta} - p[\Psi(x_i+1; \Phi_j)]^{\beta}} \]  

(25)

and

\[ B_{\Phi_j} = \ln(p) \beta \sum_{i=1}^{n} \frac{\left[ \Psi(x_i; \Phi_j) \right]^{\beta-1} p[\Psi(x_i; \Phi_j)]^{\beta} \left[ \Psi(x_i; \Phi_j) \right]_{\Phi_j}}{p[\Psi(x_i; \Phi_j)]^{\beta} - p[\Psi(x_i+1; \Phi_j)]^{\beta}} \]  

(26)

where \( [\Psi(\cdot; \Phi_j)]_{\Phi_j} = \frac{\partial \Psi(\cdot; \Phi_j)}{\partial \Phi_j}; j = 1, 2, 3, \ldots, k \). By putting Equations (24)–(26) equal to zero and solving them, immediately produces the MLEs of unknown parameters of the DOW-G family. Since the analytical solution to the above normal equations is difficult to obtain, therefore, we can solve these equations through an iterative method such as Newton-Raphson.
5.2. Bayesian estimation

When the experimenter has some prior knowledge (in the form of prior distribution) regarding the unknown parameters involved in the model, Bayesian estimation becomes crucial. In such a context, we discuss the estimation of unknown parameters of the DOW-G family under the Bayesian framework. Here, we assume the independent prior densities as \( p \sim \text{Beta}(a_1, b_1) \), \( \beta \sim \text{Gamma}(a_2, b_2) \), and \( \Phi_j \sim d_j(\Phi_j, \tau_j) \); \( j = 1, 2, \ldots, k \), where \( d_j(\Phi_j, \tau_j) \) and \( \tau_j \) are the prior density and set of associated hyperparameters (i.e., parameters involved in the prior distribution), respectively, for the \( j^{th} \) member of the vector parameter \( \Phi \). Then, the joint prior distribution of \( (p, \beta, \Phi) \) is

\[
\pi(p, \beta, \Phi) = \frac{1}{\text{Beta}(a_1, b_1)} p^{a_1-1} (1 - p)^{b_1-1} \cdot \frac{b_2^{a_2}}{\Gamma(a_2)} \beta^{a_2-1} e^{-b_2 \beta} \cdot \prod_{j=1}^{k} d_j(\Phi_j, \tau_j),
\]

where, \( a_1, b_1, a_2, b_2 > 0 \), and the domain of hyperparameters in \( \tau_j \) depends on the arbitrary choice of prior density \( d_j \). If we choose \( a_1, b_1, a_2, b_2 \) and \( \tau_j \) such that Equation (27) has minimal or no effect on posterior distribution, we can perform Bayesian study under non-informative priors.

In view of the likelihood function (22), and the joint prior distribution (27), the unnormalized joint posterior density of \( (p, \beta, \Phi) \) given \( X \) can be written as

\[
\Pi(p, \beta, \Phi | x) \propto l(x; p, \beta, \Phi) \cdot \pi(p, \beta, \Phi).
\]

Loss functions (LFs) have an important role in statistical decision inference and Bayesian theory. Therefore, in order to develop the Bayes estimators (BEs) of unknown parameters (or any of their functions), we ought to consider the question of which type of LF will be used. Here, we use a popular LF, named as a squared error loss function (SELF). It is widely applicable due to its symmetric nature (i.e., it equally penalized negative and positive error). The SELF has the following form

\[
\text{Loss}(\xi, \hat{\xi}) \propto (\hat{\xi} - \xi)^2,
\]

where \( \hat{\xi} \) is any estimate of the parameter \( \xi \), then the BE of \( \xi \) under SELF is \( \hat{\xi}^* = E_{\xi}(\xi | x) \), here the expectation is carried out under the posterior distribution of the unknown parameter \( \xi \). Thus, in our case, the BE of \( g(p, \beta, \Phi) \) (any function of parameters \( p, \beta, \Phi \) with SELF, can be derived as

\[
\hat{g}_{\text{BE}}(p, \beta, \Phi | x) = E_{p, \beta, \Phi | x}(g(p, \beta, \Phi)) = \int_{0}^{1} \int_{0}^{\infty} \cdots \int_{\Phi_{k}} g(p, \beta, \Phi) \cdot \Pi(p, \beta, \Phi | x) \, d\Phi \cdots d\Phi_1 \, dp. \quad (29)
\]

Since the analytical solution of the expectation in Equation (29) is difficult to derive due to the non-closure form of the joint posterior density (28), therefore we need numerical approximation methods. Here, we utilize two important Markov Chain Monte Carlo (MCMC) approaches, known as Gibbs sampling [25] and Metropolis–Hastings (MH) algorithm [31, 46] to generate the RSs from the posterior density and computing posterior
quantities of interest. The work of the Gibbs algorithm is to decompose the joint density (28) into full conditional posterior densities of $p$, $\beta$, $\Phi$, so that these densities can be further used to simulate posterior RSs and to achieve the BEs of the unknown parameters.

To implement the Gibbs algorithm, the unnormalized conditional posterior densities of the parameters $p$, $\beta$, and $\Phi_{j}$; $j = 1, 2, \ldots, k$ are as follows

$$
\Pi_1(p | x, \beta, \Phi) \propto l(x; p, \beta, \Phi) \cdot p^{a_1-1}(1-p)^{b_1-1},
$$

$$
\Pi_2(\beta | x, p, \Phi) \propto l(x; p, \beta, \Phi) \cdot \beta^{a_2-1}e^{-b_2\beta},
$$

$$
\Pi_{\Phi_j}(\Phi_j | x, p, \beta, \Phi) \propto l(x; p, \beta, \Phi) \cdot d_j(\Phi_j, \tau_j); \quad j = 1, 2, \ldots, k,
$$

where, $l(x; p, \beta, \Phi)$ is the likelihood function in Equation (22). Since the posterior distributions (30)–(32) are difficult to simulate directly by conventional techniques of generating RSs, therefore, we employ MH algorithm to generate posterior RSs for every element of the parameter vector $\Theta = (p, \beta, \Phi_1, \Phi_2, \ldots, \Phi_k)$ from their respective densities given in Equations (30)–(32).

The detailed steps of MH with in Gibbs algorithm for obtaining BE of $\Theta_l$ ($l$th element of $\Theta$), $l = 1, 2, \ldots, k + 2$, are as follows:

Step I. Initialize with some starting guess of $\Theta_l$ as $\Theta_l^{(0)}$ and set $i = 1$.

Step II. Obtain the proposal point $\Theta_l^{*(i)}$ from the proposal distribution $q(\Theta_l^{(i-1)}, \Theta_l^{*(i)}) = N(\hat{\Theta}_l^{(i)}, \text{Var}(\hat{\Theta}_l^{(i)}))$ and $u$ from Uniform distribution $U(0, 1)$. Here, we utilize $\hat{\Theta}_l^{(i)} = \Theta_l^{(0)}$, and Var$(\hat{\Theta}_l^{(i)})$ is suitably chosen.

Step III. If $u < \rho(\Theta_l^{(i-1)}, \Theta_l^{*(i)}) = \min\left(\frac{\Pi_l(\Theta_l^{*(i)} | x)}{\Pi_l(\Theta_l^{(i-1)} | x)} \times \frac{q(\Theta_l^{*(i)}, \Theta_l^{(i-1)})}{q(\Theta_l^{(i)}, \Theta_l^{*(i)})}, 1\right)$, then $\Theta_l^{(i)} = \Theta_l^{*(i)}$, otherwise $\Theta_l^{(i)} = \Theta_l^{(i-1)}$. Here, the notation $\rho(\Theta_l^{(i-1)}, \Theta_l^{*(i)})$ represents the acceptance probability and $\Pi_l$ has been defined in Equations (30)–(32).

Step IV. Set $i = i + 1$.

Step V. Rerun steps 2–4, say $M$ times (a large number of times), and store $\Theta_l^{(i)}$; $i = 1, 2, \ldots, m$.

Step VI. Under SELF, the BE of $\Theta_l$, say $\hat{\Theta}_l$ is obtained as $\hat{\Theta}_l = \frac{1}{M-m_0} \sum_{i=m_0+1}^{M} \Theta_l^{(i)}$, where $m_0$ is the burn-in-iterations of the Markov Chain.

6. A Monte Carlo simulation study

This section is devoted for an extensive simulation study to examine the behaviour of the various estimation methods discussed in Section 5. For this purpose, we generate samples of different sizes, i.e. $n = 25, 50,$ and $100$ from DOWGeo and DOWIW distributions. To draw the required RSs, we assume the parametric values of DOWGeo distribution as $(0.5, 0.7, 0.8)$ and $(0.9, 1.3, 0.5)$, while for DOWIW distribution, the model parameters are taken to be as $(0.6, 0.8, 0.3, 1.2)$ and $(0.9, 1.5, 0.5, 0.7)$. In classical setup, we compute the MLEs of the unknown parameters with their associated standard errors (Ses). The MLEs and Ses of the unknown parameters of DOWGeo distribution are presented in Table 5, whereas Table 6 holds the MLEs and Ses in case of DOWIW distribution.

For Bayesian analysis of DOWGeo distribution, $Beta(a_1, b_1)$, $Gamma(a_2, b_2)$, and $Beta(a_3, b_3)$ are utilized as the independent informative priors for the parameters $p$, $\beta$, and $\Phi$. 
Table 5. MLE and Bayes estimates for unknown parameters of DOWGeo distribution.

| Sample size | Parameters ↓ | (p, β, θ) = (0.5, 0.7, 0.8) | (p, β, θ) = (0.9, 1.3, 0.5) |
|-------------|--------------|-----------------------------|-----------------------------|
|             | MLE          | SE  | BE  | PSe | MLE          | SE  | BE  | PSe |
| 25          |              |     |     |     |              |     |     |     |
| p           | 0.5287       | 0.3740 | 0.4906 | 0.0683 | 0.8807       | 0.3218 | 0.9183 | 0.0315 |
| β           | 0.7236       | 0.3349 | 0.7103 | 0.0624 | 1.2667       | 1.2353 | 1.3290 | 0.1243 |
| θ           | 0.8303       | 0.1584 | 0.7875 | 0.0289 | 0.5235       | 0.5427 | 0.5129 | 0.0373 |
| 50          |              |     |     |     |              |     |     |     |
| p           | 0.5117       | 0.2879 | 0.4946 | 0.0501 | 0.9150       | 0.1628 | 0.9108 | 0.0274 |
| β           | 0.7168       | 0.2628 | 0.7112 | 0.0445 | 1.3118       | 0.8405 | 1.3107 | 0.1103 |
| θ           | 0.8122       | 0.1180 | 0.7819 | 0.0261 | 0.5112       | 0.4107 | 0.4956 | 0.0345 |
| 100         |              |     |     |     |              |     |     |     |
| p           | 0.5078       | 0.1894 | 0.5059 | 0.0479 | 0.8966       | 0.1143 | 0.9076 | 0.0204 |
| β           | 0.6962       | 0.1576 | 0.7023 | 0.0314 | 1.3092       | 0.7469 | 1.3012 | 0.0649 |
| θ           | 0.7961       | 0.0892 | 0.8003 | 0.0113 | 0.5070       | 0.3319 | 0.4989 | 0.0314 |

Table 6. MLE and Bayes estimates for unknown parameters of DOWIW distribution.

| Sample size | Parameters ↓ | (p, β, θ, α) = (0.6, 0.8, 0.3, 1.2) | (p, β, θ, α) = (0.9, 1.5, 0.5, 0.7) |
|-------------|--------------|---------------------------------------|---------------------------------------|
|             | MLE          | SE  | BE  | PSe | MLE          | SE  | BE  | PSe |
| 25          |              |     |     |     |              |     |     |     |
| p           | 0.5476       | 0.9712 | 0.5873 | 0.0526 | 0.9123       | 0.5718 | 0.9111 | 0.0271 |
| β           | 0.8369       | 2.4693 | 0.8263 | 0.1934 | 1.5739       | 4.8323 | 1.4857 | 0.1297 |
| θ           | 0.2889       | 0.6648 | 0.3124 | 0.0454 | 0.5348       | 0.9743 | 0.4874 | 0.0433 |
| α           | 1.2332       | 4.0627 | 1.2261 | 0.1274 | 0.6792       | 2.3019 | 0.6917 | 0.0715 |
| 50          |              |     |     |     |              |     |     |     |
| p           | 0.5679       | 0.8459 | 0.5919 | 0.0413 | 0.8869       | 0.4573 | 0.9049 | 0.0205 |
| β           | 0.8194       | 1.8607 | 0.8079 | 0.0992 | 1.5349       | 2.9876 | 1.4943 | 0.1203 |
| θ           | 0.2914       | 0.3369 | 0.2941 | 0.0374 | 0.4819       | 0.7260 | 0.4889 | 0.0406 |
| α           | 1.2204       | 3.2751 | 1.2219 | 0.1227 | 0.6898       | 1.7737 | 0.7096 | 0.0672 |
| 100         |              |     |     |     |              |     |     |     |
| p           | 0.6109       | 0.6257 | 0.6071 | 0.0397 | 0.9079       | 0.2951 | 0.8996 | 0.0183 |
| β           | 0.8102       | 1.1781 | 0.7989 | 0.0907 | 1.5104       | 2.0373 | 1.5030 | 0.1126 |
| θ           | 0.2975       | 0.3123 | 0.3039 | 0.0314 | 0.4989       | 0.5813 | 0.5011 | 0.0327 |
| α           | 1.2100       | 2.7216 | 1.2041 | 0.1146 | 0.6997       | 1.3602 | 0.7029 | 0.0612 |

and θ, respectively, while in case of DOWIW distribution, the variability of the unknown parameters p, β, and α is measured by assuming independent informative priors as Beta(a₄, b₄), Gamma(a₅, b₅), Beta(a₆, b₆), and Gamma(a₇, b₇), respectively. In these prior densities, the hyperparameters have been selected such that the mean of the prior density is approximately equal to the corresponding predetermined value of the parameter. Under this setting, in order to calculate the Bayes estimates of unknown parameters, we generate 51,000 items for each parameter by utilizing MH within Gibbs sampling. The starting 1000 burn-in iterations for each of the chains have been discarded for removing the effects of initial values of the parameters. Also, every 25th observation is stored to neutralize the auto-correlation of successive draws. The convergence diagnostics of each generated chain is investigated by Geweke’s [26] test at a 95% credibility level. This diagnostics is also examined by plotting posterior densities, MCMC runs, and auto-correlation functions. The above plots can be seen in Figures S1, S2, S3, and S4 for DOWGeo(0.5, 0.7, 0.8), DOWGeo (0.9, 1.3, 0.5), DOWIW(0.6, 0.8, 0.3, 1.2), and DOWIW (0.9, 1.5, 0.5, 0.7) distributions, respectively.

After the convergence diagnostics, we have used these generated values to obtain Bayes estimates with their associated posterior standard errors (PSes) under SELF. The Bayes estimates and their PSes for the unknown parameters of DOWGeo distribution are given in Table 5, whereas Table 6 consists the Bayes estimates and PSes for the parameters
of DOWIW distribution. All numerical computations are performed using open-source software \textit{R}.

From this simulation study, we have observed that for each pair of sample size and true parametric value, both estimation methods work satisfactorily, but Bayesian method outperforms the MLE with respect to the estimation errors. Also, it is noticed that the estimation error associated with an estimate for both estimation procedures, tends to decrease as we increase the sample size.

### 7. Real data illustration

In this segment, we illustrate the applications of the DOWGeo and DOWIW models by using two real data sets. The fitting of the models are compared using some well-known measures, namely, $-L$, Akaike information criterion (AkIC), correct Akaike information

### Table 7. The competitive models.

| Distribution                      | Abbreviation | Author(s)          |
|-----------------------------------|--------------|--------------------|
| Geometric                         | Geo          | –                  |
| Generalized geometric             | GGeo         | Gómez-Déniz [28]   |
| Discrete Rayleigh                 | DR           | Roy [60]           |
| Discrete inverse Rayleigh         | DIR          | Hussain and Ahmad [33] |
| Discrete inverse Weibull          | DIW          | Jazi et al. [36]   |
| One parameter discrete Lindley    | DLI-I        | Gómez-Déniz and Calderín-Ojeda [29] |
| Two parameters discrete Lindley   | DLI-II       | Hussain et al. [34] |
| Three parameters discrete Lindley | DLI-III      | Eliwa et al. [19]  |
| One parameter discrete flexible model | DFx-I        | Eliwa and El-Morshedy [22] |
| Negative Binomial                 | NeBi         | –                  |
| Poisson                           | Poi          | Poisson [56]       |
| Discrete Bilal                    | DBL          | Eliwa et al. [20]  |
| Discrete Pareto                    | DPa          | Krishna and Pundir [41] |
| Discrete Burr                      | DB           | Krishna and Pundir [41] |
| Discrete Burr-Hatke               | DBH          | El-Morshedy et al. [22] |
| Discrete log-logistic             | DLogL        | Para and Jan [53]  |
| Discrete Lomax distribution       | DLo          | Para and Jan [54]  |

### Table 8. The MLEs with their corresponding Se for data set I.

| Parameter → | $\rho$  | $\beta$ | $\theta$ |
|-------------|---------|---------|---------|
| Model ↓      | MLE     | Se      | MLE     | Se      | MLE     | Se      |
| DOWGeo       | 0.181   | 0.152   | 0.441   | 0.119   | 0.813   | 0.147   |
| GGeo         | –       | –       | 0.188   | 0.089   | 0.800   | 0.064   |
| Geo          | –       | –       | –       | 0.582   | 0.030   |
| DR           | –       | –       | –       | –       | 0.901   | 0.009   |
| DIR          | –       | –       | –       | –       | 0.554   | 0.049   |
| DIW          | –       | –       | 1.049   | 0.146   | 0.581   | 0.048   |
| NeBi         | 0.812   | 0.045   | 0.322   | 0.074   | –       | –       |
| Poi          | 1.390   | 0.112   | –       | –       | –       | –       |
| DFx-I        | 0.623   | 0.031   | –       | –       | –       | –       |
| DLI-I        | 0.436   | 0.026   | –       | –       | –       | –       |
| DLI-II       | 0.581   | 0.045   | 0.001   | 0.058   | –       | –       |
| DLI-III      | 0.582   | 0.005   | 358.728 | 11863.37| 0.001   | 20.698  |
| DLogL        | 0.780   | 0.136   | 1.208   | 0.159   | –       | –       |
| DB           | 0.278   | 0.045   | 1.053   | 0.167   | –       | –       |
| DLo          | 0.152   | 0.098   | 1.830   | 0.952   | –       | –       |
| DPa          | 0.268   | 0.034   | –       | –       | –       | –       |
The competitive distributions are provided in Table 7.

Table 9. Various fitted distributions with GOFS for data set I based on one parameter competitive models.

| ExFr | DOWGeo | Geo | DR | DIR | Poi | DFx-I | DL-I | DPa |
|------|--------|-----|----|-----|-----|-------|------|-----|
| 0    | 65     | 45.980 | 10.890 | 60.888 | 27.398 | 45.256 | 40.286 | 65.842 |
| 1    | 14     | 26.760 | 26.618 | 33.99 | 29.094 | 29.834 | 18.357 | 8.164 |
| 2    | 10     | 15.575 | 29.448 | 8.123 | 26.468 | 16.508 | 18.357 | 8.164 |
| 3    | 6      | 9.064  | 22.296 | 3.004 | 26.848 | 12.264 | 8.893 | 5.232 |
| 4    | 4      | 5.275  | 12.629 | 1.420 | 4.262 | 4.703 | 5.523 | 2.820 |
| 5    | 2      | 3.070  | 5.539  | 0.779 | 1.185 | 2.489 | 2.851 | 1.909 |
| 6    | 2      | 1.787  | 1.914  | 0.473 | 0.274 | 1.335 | 1.437 | 1.368 |
| 7    | 2      | 1.039  | 0.526  | 0.054 | 0.731 | 0.711 | 1.022 |
| 8    | 1      | 0.605  | 0.116  | 0.009 | 0.409 | 0.347 | 0.789 |
| 9    | 1      | 0.352  | 0.020  | 0.001 | 0.231 | 0.167 | 0.626 |
| 10   | 1      | 0.205  | 0.003  | 0.112 | 0.137 | 0.079 | 0.506 |
| 11   | 2      | 0.288  | 0.001  | 0.539 | 0.211 | 0.072 | 4.174 |
| Total| 110    | 110   | 110 | 110 | 110 | 110 | 110 | 110 |

\[ L = 166.605, \text{AkIC} = 339.209, \text{CAkIC} = 339.436, \text{HQIC} = 342.495, \chi^2 = 0.596, \text{DF} = 2, \text{P}-value = 0.742 \]\n
Table 10. Various fitted distributions with GOFS for data set I based on competitive models with two or three parameters.

| ExFr | DOWGeo | GGeo | NeBi | DL-II | DL-III | DLogL | DB | DLo | DIW |
|------|--------|------|------|-------|--------|-------|----|-----|-----|
| 0    | 65     | 62.738 | 56.520 | 46.026 | 46.008 | 63.192 | 64.743 | 61.615 | 63.910 |
| 1    | 14     | 19.665 | 15.885 | 26.768 | 26.765 | 20.101 | 19.177 | 21.023 | 20.699 |
| 2    | 10     | 9.436  | 9.173  | 15.568 | 15.570 | 8.644  | 8.484  | 9.687  | 8.053 |
| 3    | 6      | 5.436  | 6.203  | 9.054  | 9.058  | 4.656  | 4.632  | 5.275  | 4.234 |
| 4    | 4      | 3.463  | 4.502  | 5.266  | 5.269  | 2.864  | 2.863  | 3.197  | 2.599 |
| 5    | 2      | 2.349  | 3.400  | 3.062  | 3.066  | 1.921  | 1.920  | 2.088  | 1.754 |
| 6    | 2      | 1.663  | 2.635  | 1.783  | 1.783  | 1.368  | 1.365  | 1.441  | 1.261 |
| 7    | 2      | 1.213  | 2.079  | 1.036  | 1.037  | 1.019  | 1.013  | 1.038  | 0.949 |
| 8    | 1      | 0.904  | 1.663  | 0.602  | 0.604  | 0.786  | 0.777  | 0.773  | 0.739 |
| 9    | 1      | 0.685  | 1.344  | 0.350  | 0.351  | 0.623  | 0.613  | 0.592  | 0.592 |
| 10   | 1      | 0.525  | 1.095  | 0.204  | 0.204  | 0.494  | 0.463  | 0.463  | 0.485 |
| 11   | 2      | 1.920  | 5.501  | 0.283  | 0.285  | 4.322  | 3.919  | 2.808  | 4.725 |
| Total| 110    | 110   | 110 | 110 | 110 | 110 | 110 | 110 | 110 |

\[ L = 168.544, \text{AkIC} = 340.088, \text{CAkIC} = 341.225, \text{HQIC} = 343.2788, \chi^2 = 2.444, \text{DF} = 2, \text{P}-value = 0.742 \]
7.1. Data set I: count cysts of kidneys using steroids

This data set is taken from the study of Chan et al. [15]. Here, we examine the fitting capability of the DOWGeo model with some other competitive distributions like: GGeo, Geo, DR, DIR, DIW, NeBi, Poi, DFx-I, DLI-I, DLI-II, DLI-III, DLogL, DB, DLo, and DPa. The MLEs with their associated standard errors (Se) are tabulated in Table 8, whereas Tables 9 and 10 list the goodness of fit statistics (GOFs). In Tables 9 and 10, ExFr and ObFr represent the expected and observed frequencies, respectively.

From Tables 9 and 10 it is noted that, the DPa, GGeo, NeBi, DLogL, DB, DLo and DIW distributions work quite satisfactory for analyzing data set I besides the DOWGeo distribution under condition (P-value > 0.05). Depending on $-\log L$, AkIC, CAkIC, HQIC, $\chi^2$ and

![Figure 5. The observed and expected PMFs for data set I.](image_url)

| Table 11. Some useful descriptive statistics for data set I. |
| --- | --- | --- | --- | --- |
| Type | Measures | Mean | Variance | Dsl | Skewness | Kurtosis |
| --- | --- | --- | --- | --- | --- | --- |
| Theoretical | 1.39493 | 5.98717 | 4.29207 | 2.34691 | 9.11467 |
| Empirical | 1.39090 | 6.11184 | 4.39416 | 2.29259 | 8.17345 |
P-value, it is easily visible that the DOWGeo model provides the best fit among all other tested models since it has the smallest value of \(-L\), AkIC, CakIC, HQIC, and \(\chi^2\), whereas it achieve the highest P-value associated to the \(\chi^2\) test. Figure S5 depicts the profile of \(L\) for each parameter based on data set I, which indicates that the estimators of the parameters are unimodal. Figures 5 and S6 support the results of Tables 9 and 10, and they conclude that data set I follows the DOWGeo, DPa, GGeo, NeBi, DLogL, DBX-II, DLo and DIW models. But, the DOWGeo model is better than other rival models.

Table 11 lists some theoretical and empirical descriptive statistics for data set I. From Table 11, it is clear that the theoretical measures are closed to empirical ones. Most of the distribution is at the left and platykurtic. Further, the data considered herein suffering from over-dispersion phenomena.

7.2. Data set II: COVID-19 in South Korea

This data set can be viewed at (https://www.worldometers.info/coronavirus/country/south-korea/) and consists the daily new deaths in South Korea for COVID-19 from 15

| Model      | MLE   | Se    | MLE   | Se    | MLE   | Se    | MLE   | Se    |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|
| DOWIW      | 0.997 | 0.0002| 1.064 | 0.069 | 0.991 | 0.003 | 1.083 | 0.102 |
| DIW        | –     | –     | –     | –     | 0.271 | 0.025 | 1.411 | 0.083 |
| DLogL      | 1.716 | 0.095 | 1.878 | 0.107 | –     | –     | –     | –     |
| DB         | 0.591 | 0.031 | 2.466 | 0.248 | –     | –     | –     | –     |
| DIR        | –     | –     | –     | –     | 0.229 | 0.023 | –     | –     |
| DBl        | 0.707 | 0.010 | –     | –     | 0.229 | 0.023 | –     | –     |
| DPa        | 0.377 | 0.021 | –     | –     | 0.229 | 0.023 | –     | –     |
| DBH        | 0.904 | 0.020 | –     | –     | 0.229 | 0.023 | –     | –     |
| Poi        | 1.901 | 0.079 | –     | –     | 0.229 | 0.023 | –     | –     |

Table 13. Various fitted distributions with GOFS for data set II.

| No. ECB | O. Fr | DOWIW | DIW | DLogL | DB | DIR | DBl | DPa | DBH | Poi |
|---------|-------|-------|-----|-------|----|-----|-----|-----|-----|-----|
|         |       |       |     |       |    |     |     |     |     |     |
| 0       | 89    | 89.040| 82.351| 80.931| 92.887| 69.890| 63.089| 149.356| 166.600| 45.408|
| 1       | 79    | 74.528| 103.702| 92.775| 97.788| 140.613| 89.133| 50.500| 54.598| 86.336|
| 2       | 51    | 51.862| 44.390| 51.431| 42.676| 47.685| 64.868| 25.478| 26.667| 82.074|
| 3       | 29    | 34.059| 22.317| 27.336| 21.174| 19.125| 39.578| 15.379| 15.540| 52.014|
| 4       | 19    | 21.595| 12.926| 15.559| 12.172| 9.333| 22.307| 10.312| 10.017| 24.725|
| 5       | 17    | 13.351| 8.248 | 9.518 | 7.754 | 5.198 | 12.037| 7.387 | 6.885 | 9.402 |
| 6       | 9     | 8.092 | 5.636 | 6.193 | 5.308 | 3.162 | 6.328 | 5.533 | 4.953 | 2.979 |
| 7       | 6     | 4.825 | 4.052 | 4.247 | 3.831 | 2.098 | 3.273 | 4.408 | 3.682 | 0.809 |
| 8       | 6     | 2.387 | 3.031 | 3.061 | 2.879 | 1.429 | 1.674 | 3.466 | 2.808 | 0.192 |
| 9       | 1     | 4.261 | 17.347| 12.949| 17.531| 5.467 | 1.713 | 32.181| 12.25 | 0.061 |
| Total   | 304   | 304   | 304  | 304   | 304 | 304 | 304 | 304 | 304 | 304 |

| −L       | 564.625| 586.855| 577.011| 587.652| 606.870| 575.338| 633.530| 620.466| 621.0976|
| AIC      | 1137.250| 1177.711| 1158.023| 1179.304| 1215.740| 1152.676| 1269.061| 1242.932| 1244.195|
| CAIC     | 1137.384| 1177.751| 1158.063| 1179.344| 1215.754| 1152.689| 1269.075| 1242.945| 1244.208|
| BIC      | 1152.118| 1185.145| 1165.457| 1186.738| 1219.457| 1156.393| 1272.777| 1246.649| 1247.912|
| HQIC     | 1143.198| 1180.684| 1160.997| 1182.278| 1217.227| 1154.163| 1270.548| 1244.419| 1245.682|
| \(\chi^2\) | 2.792 | 41.868 | 25.019 | 44.784 | 92.204 | 28.203 | 128.631 | 109.333 | 115.896 |
| D\(F\)   | 3     | 6     | 6     | 6     | 6     | 6     | 7     | 6     | 4     |
| P-value   | 0.425 | ≤ 0.001| ≤ 0.001| ≤ 0.001| ≤ 0.001| ≤ 0.001| ≤ 0.001| ≤ 0.001| ≤ 0.001|
February to 12 Dec 2020. Here, we examine the fitting capability of the DOWIW model with some other rival distributions like DIW, DLogL, DB, DIR, DBL, DPa, DBH, and Poi. The MLEs with their associated Se are reported in Table 12, whereas Table 13 lists the GOFS.

Based on the fitting measures in Table 13, we can announce that the DOWIW model is the best fitted model. Figure S7 shows the profiles of $L$ for each parameter based on data set II, and it shows that every estimator has a unimodal function. Figures 6 and S8 support the results of Table 13, and we can say that the data set II follows the DOWIW model.

Some theoretical and empirical descriptive statistics based on data set II can be viewed in Table 14.

From Table 14, the empirical mean, variance, Dsl and skewness are closed to theoretical ones. Data set II suffering from over-dispersion phenomena, and most of the distribution is at the left with platykurtic.

![Figure 6. The observed and expected PMFs for data set II.](image)

| Type        | Mean  | Variance | Dsl   | Skewness | Kurtosis |
|-------------|-------|----------|-------|----------|----------|
| Theoretical | 1.9013| 4.1222   | 2.1681| 1.2638   | 4.0864   |
| Empirical   | 1.8567| 3.9978   | 2.1531| 1.3657   | 4.2367   |

*Table 14. Some useful descriptive statistics for data set II.*
Table 15. Bayes estimates for DOWGeo and DOWIW distributions under real data sets I and II.

| Parameter | DOWGeo distribution under data set I | DOWIW distribution under data set II |
|-----------|-------------------------------------|-------------------------------------|
| Parameter | Model BE PSe | Model BE PSe |
| $p$       | 0.1804 0.0105 | 0.9970 0.0002 |
| $\beta$   | 0.4390 0.0285 | 1.1105 0.0427 |
| $\theta$  | 0.8141 0.0235 | 0.9869 0.0024 |
| $\alpha$  | -- -- | 1.0135 0.0728 |

7.3. Bayesian estimation for real data sets I and II

Here, we compute the BEs with their PSes for the unknown parameters of the DOWGeo and DOWIW distributions using a similar manner discussed in Section 5. Since there is no prior information available regarding the unknown parameters of the special distributions, therefore, we have used non-informative priors for the unknown parameters of the models under study. The calculated estimates with the related estimation errors of DOWGeo and DOWIW distributions are jointly given in Table 15.

From the comparison of Bayes estimates (in Table 15) and MLEs (in Tables 8 and 12), we can easily observe that the result obtained in case of simulation study do hold in case of real-life data sets as well.

8. Concluding remarks

In the present research article, we have introduced a new family of discrete distributions called DOW-G family. Its several important statistical characteristics have been investigated. Two particular distribution of the proposed family are studied in detail. These special models provides good flexibility in terms of shapes for the PDFs. Further, we have noticed that the proposed family can model a positively skewed, a negatively skewed or a symmetric shaped data set. In addition, the DOW-G family can be used quite effectively for modelling a wide variety of failure data because its HRF can take diverse shapes (increasing, decreasing, constant, J-, and bathtub-shaped). Moreover, it is suitable for modelling under-, equi- and over-dispersed data set. In classical and non-classical setups, the method of maximum likelihood and Bayesian approach have been utilized to estimate the unknown parameters of particular models.

An extensive Monte Carlo simulation analysis has been conducted to evaluate the behaviour of above stated estimation methods. The results of simulation studies declare that these two estimation procedures perform quite satisfactorily in estimating unknown parameters of the model, but the Bayesian method dominates the method of maximum likelihood in terms of estimation errors. The flexibility of the DOW-G family has also been exemplified by using two distinctive real data sets. Hence, it is reasonable to say that the special distributions of the developed family can serve as an alternative model to the existing discrete models for modelling count or failure data in various fields including reliability, insurance, medicine, economics, and demography, etc.

It is worth noting that the above research can be expanded into several dimensions, for example- with different types of censoring schemes, we can study the particulars models
of the proposed family (see Tyagi et al. [65]); In reliability inference, we can analyze stress-strength reliability or load share systems under special distributions of the DOW-G family. Also, various characteristics of order statistics from the proposed family could be explored and to infer the bi-variate data, a bi-variate discrete family may also be studied.

Acknowledgments

The authors thankfully acknowledge the critical suggestions and comments from the associate Editor and referees which greatly helped us in the improvement of the paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

M. El-Morshedy http://orcid.org/0000-0002-7585-5519
M. S. Eliwa http://orcid.org/0000-0001-5619-210X

References

[1] A.A. Abd EL-Baset and M.G.M. Ghazal, Exponentiated additive Weibull distribution, Reliab. Eng. Syst. Saf. 193 (2020), p. 106663.
[2] A.Z. Afify and M. Alizadeh, The odd dagum family of distributions: Properties and applications, J. Appl. Probab. Stat. 15 (2020), pp. 45–72.
[3] M.H. Alamatsaz, S. Dey, T. Dey, and S.S. Harandi, Discrete generalized Rayleigh distribution, Pakistan J. Statist. 32 (2016), pp. 1–20.
[4] A. Al-Babtain, A.A. Fattah, A.H.N. Ahmed, and F. Merovci, The Kumaraswamy-transmuted exponentiated modified Weibull distribution, Commun. Stat. Simul. Comput. 46 (2017), pp. 3812–3832.
[5] M. Alizadeh, A.Z. Afify, M.S. Eliwa, and S. Ali, The odd log-logistic Lindley-G family of distributions: Properties, Bayesian and non-Bayesian estimation with applications, Comput. Stat. 35 (2020), pp. 281–308.
[6] M. Alizadeh, M. Emadi, M. Doostparast, G.M. Cordeiro, E.M. Ortega, and R.R. Pescim, A new family of distributions: The Kumaraswamy odd log-logistic, properties and applications, Hacettepe J. Math. Statist. 44 (2015), pp. 1491–1512.
[7] S.J. Almalki and J Yuan, A new modified Weibull distribution, Reliab. Eng. Syst. Saf. 111 (2013), pp. 164–170.
[8] A. Alzaatreh, C. Lee, and F Famoye, A new method for generating families of continuous distributions, Metron 71 (2013), pp. 63–79.
[9] H.S. Bakouch, C. Chesneau, and M.N. Khan, The extended odd family of probability distributions with practice to a submodel, Filomat 33 (2019), pp. 3855–3867.
[10] H.S. Bakouch, M. Aghababaei, and S. Nadarajah, A new discrete distribution, Statistics 48 (2014), pp. 200–240.
[11] M. Bebbington, C.D. Lai, M. Wellington, and R Zitikis, The discrete additive Weibull distribution: A bathtub-shaped hazard for discontinuous failure data, Reliab. Eng. Syst. Saf. 106 (2012), pp. 37–44.
[12] M. Bourguignon, R.B. Silva, and G.M. Cordeiro, The Weibull-G family of probability distributions, J. Data. Sci. 12 (2014), pp. 53–68.
[13] S. Chakraborty, Generating discrete analogues of continuous probability distributions – a survey of methods and constructions, J. Stat. Distrib. Appl. 2 (2015), pp. 1–30.
[14] K. Chandrakant, M.T. Yogesh, and K.R. Manoj, On a discrete analogue of linear failure rate distribution, Am. J. Math. Manage. Sci. 36 (2017), pp. 229–246.
[15] S.K. Chan, P.R. Riley, K.L. Price, F. McElduff, P.J. Winyard, S.J. Welham, A.S. Woolf, and D.A. Long, Corticosteroid-induced kidney dysmorphogenesis is associated with deregulated expression of known cystogenic molecules, as well as Indian hedgehog, Am. J. Physiology-Renal Physiol. 298 (2010), pp. F346–F356.

[16] G.M. Cordeiro, E. Altun, M.C. Korkmaz, R.R. Pescim, A.Z. Afify, and H.M. Yousof, The xgamma family: Censored regression modelling and applications, REVSTAT–Statist. J. 18 (2020), pp. 593–612.

[17] K. Cooray, Generalization of the Weibull distribution: The odd Weibull family, Stat. Model. 6 (2006), pp. 265–277.

[18] M.S. Eliwa, Z.A. Alhussain, and M. El-Morshedy, Discrete Gompertz-G family of distributions for over-and under-dispersed data with properties, estimation, and applications, Mathematics 8 (2020a), p. 358.

[19] M.S. Eliwa, E. Altun, M. El-Dawoody, and M. El-Morshedy, A new three-parameter discrete distribution with associated INAR(1) process and applications, IEEE Access 8 (2020b), pp. 91150–91162. doi:10.1109/ACCESS.2020.2993593.

[20] M.S. Eliwa, M. El-Morshedy, and E. Altun, A study on discrete Bilal distribution with properties and applications on integer-valued autoregressive process, Revstat – Statist. J. (2020c), pp. 1–30.

[21] M.S. Eliwa and M. El-Morshedy, The odd flexible Weibull-H family of distributions: Properties and estimation with applications to complete and upper record data, Filomat 33 (2019), pp. 2635–2652.

[22] S.Geman and D Geman, Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images, IEEE. Trans. Pattern. Anal. Mach. Intell. 6 (1984), pp. 721–741.

[23] J. Geweke, Evaluating the accuracy of sampling-based approaches to the calculation of posterior moments Minneapolis, MN, Federal Reserve Bank of Minneapolis, Research Department, 196, 1991.

[24] M.E. Ghitany and D.K. Al-Mutairi, Estimation methods for the discrete Poisson Lindley distribution, J. Stat. Comput. Simul. 79 (2009), pp. 1–9.

[25] W.K Hastings, Monte Carlo sampling methods using Markov chains and their applications, Biometrika 57 (1970), pp. 97–109.

[26] T. Hussain and M Ahmad, Discrete inverse Rayleigh distribution, Pakistan J. Statist. 30 (2014), pp. 203–222.

[27] T. Hussain, M. Aslam, and M. Ahmad, A two-parameter discrete Lindley distribution, Revista Colombiana De Estadistica 39 (2016), pp. 45–61.

[28] S. Inusah and T.J. Kozubowski, A discrete analogue of the Laplace distribution, J. Stat. Plann. Inference 136 (2006), pp. 1090–1102.

[29] M.A. Jazi, C.D. Lai, and M.H. Alamatsaz, A discrete inverse Weibull distribution and estimation of its parameters, Stat. Methodol. 7 (2010), pp. 121–132.
[37] M. Ç. Korkmaz, A new family of the continuous distributions: The extended Weibull-G family, Commun. Faculty Sci. Univ. Ankara Ser. A1 Math. Statist. 68 (2018), pp. 248–270.

[38] M. Ç. Korkmaz, E. Altun, H.M. Yousof, and G.G. Hamedani, The odd power Lindley generator of probability distributions: Properties, characterizations and regression modelling. Int. J. Stat. Probab. 8 (2019), pp. 70–89.

[39] M. Ç. Korkmaz, E. Altun, H.M. Yousof, and G.G. Hamedani, The Hjorth’s IDB generator of distributions: Properties, characterizations, regression modelling and applications, J. Statist. Theory Appl. 19 (2020), pp. 59–74.

[40] M. Ç. Korkmaz and A. İ. Genç, A new generalized two-sided class of distributions with an emphasis on two-sided generalized normal distribution, Commun. Statist.-Simul. Comput. 46 (2017), pp. 1441–1460.

[41] H. Krishna and P.S. Pundir, Discrete Burr and discrete Pareto distributions, Stat. Methodol. 6 (2009), pp. 177–188.

[42] C. Kus, Y. Akdoğan, A. Asgharzadeh, I. Kinaci, and K. Karakaya, Binomial-discrete Lindley distribution, Commun. Faculty Sci. Univ. Ankara Ser. A1-Math. Statist. 68 (2018), pp. 401–411.

[43] C.D. Lai, M. Xie, and D.N.P. Murthy, A modified Weibull distribution, IEEE Trans. Reliab. 52 (2003), pp. 33–37.

[44] C. Lee, F. Famoye, and O. Olumolade, Beta-Weibull distribution: Some properties and applications to censored data, J. Modern Appl. Statist. Methods 6 (2007), p. 17.

[45] A.W. Marshall and F. Proschan, Classes of distributions applicable in replacement with renewal theory implications, Sixth Berkeley Symposium I, 1970, pp. 395–416.

[46] N. Metropolis and S. Ulam, The Monte Carlo method, J. Amer. Statist. Assoc. 44 (1949), pp. 335–341.

[47] A.D. Nascimento, K.F. Silva, G.M. Cordeiro, M. Alizadeh, H.M. Yousof, and G.G. Hamedani, The odd Nadarajah-Haghighi family of distributions: Properties and applications, Studia Scientiarum Mathematicarum Hungarica 56 (2019), pp. 185–210.

[48] V. Nekoukhou, M.H. Alamatsaz, and H. Bidram, Discrete generalized exponential distribution of a second type, Statistics 47 (2013), pp. 876–887.

[49] V. Nekoukhou and H. Bidram, The exponentiated discrete Weibull distribution, Statist. Oper. Res. Trans. 39 (2015a), pp. 127–146.

[50] V. Nekoukhou and H. Bidram, A new four-parameter discrete distribution with bathtub and unimodal failure rate, J. Appl. Statist. 42 (2015b), pp. 2654–2670.

[51] Z.M. Nofal, A.Z. Afify, H.M. Yousof, D.C. Granzotto, and F. Louzada, Kumaraswamy transmuted exponentiated additive Weibull distribution, Int. J. Statist. Probab. 5 (2016), pp. 78–99.

[52] M. Pal, M.M. Ali, and J. Woo, Exponentiated weibull distribution, Statistica 66 (2006), pp. 139–147.

[53] B.A. Para and T.R. Jan, Discrete version of log-logistic distribution and its applications in genetics, Int. J. Modern Math. Sci. 14 (2016a), pp. 407–422.

[54] B.A. Para and T.R. Jan, On discrete three parameter burr type XII and discrete Lomax distributions and their applications to model count data from medical science, Biom. Biostatist. Int. J. 4 (2016b), pp. 1–15.

[55] L.G.B. Pinho, G.M. Cordeiro, and J.S. Nobre, The gamma-exponentiated Weibull distribution, J. Statist. Theory Appl. 11 (2012), pp. 379–395.

[56] S.D. Poisson, Probabilité des jugements en matière criminelle et en matière civile, précédées des règles générales du calcul des probabilités, Paris, Bachelier, 1837, pp. 206–207.

[57] A Rényi, On measures of entropy and information, Math. Statist. Probab. 1 (1961), pp. 547–561.

[58] H. Reyad, M.Ç. Korkmaz, A.Z. Afify, G.G. Hamedani, and S. Othman, The Fréchet topp Leone-G family of distributions: Properties, characterizations and applications, Ann. Data Sci. 8 (2019), pp. 345–366.

[59] D. Roy, The discrete normal distribution, Commun. Statist. Theory Methods 32 (2003), pp. 1871–1883.

[60] D. Roy, Discrete Rayleigh distribution, IEEE Trans. Reliab. 53 (2004), pp. 255–260.

[61] A.M. Sarhan and J Apaloo, Exponentiated modified Weibull extension distribution, Reliab. Eng. System Saf. 112 (2013), pp. 137–144.
[62] G.O. Silva, E.M. Ortega, and G.M. Cordeiro, *The beta modified Weibull distribution*, Lifetime Data. Anal. 16 (2010), pp. 409–430.

[63] M.H. Tahir, G.M. Cordeiro, M. Alizadeh, M. Mansoor, M. Zubair, and G.G. Hamedani, *The odd generalized exponential family of distributions with applications*, J. Statist. Distrib. Appl. 2 (2015), p. 1.

[64] A. Tyagi, N. Choudhary, and B. Singh, *Discrete additive Perks–Weibull distribution: Properties and applications*, Life Cycle Reliab. Saf. Eng. 8 (2019), pp. 183–199.

[65] A. Tyagi, N. Choudhary, and B. Singh, *A new discrete distribution: Theory and applications to discrete failure lifetime and count data*, J. Appl. Probab. Statist. 15 (2020), pp. 117–143.

[66] A. Tyagi, N. Choudhary, and B. Singh, *Inferences on discrete Rayleigh distribution under type-II censored data using a tie-run approach*, Int. J. Agricult. Stat. Sci. 16 (2020), pp. 939–951.