The bounds of the spectral radius of general hypergraphs in terms of clique number *

Cunxiang Duan$^{a,b}$, Ligong Wang$^{a,b,†}$

$^a$School of Mathematics and Statistics, Northwestern Polytechnical University,
Xi’an, Shaanxi 710129, P.R. China.
$^b$Xi’an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University,
Xi’an, Shaanxi 710129, P.R. China.

E-mail: cxduanmath@163.com; lgwangmath@163.com

Abstract

The spectral radius (or the signless Laplacian spectral radius) of a general hypergraph is the maximum modulus of the eigenvalues of its adjacency (or its signless Laplacian) tensor. In this paper, we firstly obtain a lower bound of the spectral radius (or the signless Laplacian spectral radius) of general hypergraphs in terms of clique number. Moreover, we present a relation between a homogeneous polynomial and the clique number of general hypergraphs. As an application, we finally obtain an upper bound of the spectral radius of general hypergraphs in terms of clique number.

Key Words: spectral radius, clique number, general hypergraphs

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1 Introduction

Let $G$ be a general hypergraph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \ldots, e_m\}$, where $e_1, e_2, \ldots, e_m \subseteq V(G)$. If $V' \subseteq V$ and $E' \subseteq E$, then $H = (V', E')$ is called a subhypergraph of $G$. The rank (resp., co-rank) of $G$ is $\text{rank}(G) = \max\{|e|: e \in E\}$ (resp., $\text{corank}(G) = \min\{|e|: e \in E\}$). If the rank and the

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†Corresponding author.
co-rank of a hypergraph $G$ are equal to $k$, then $G$ is called $k$-uniform. Obviously, a graph is a 2-uniform hypergraph. For two vertices $u, v \in V$, if there exists an edge $e$ such that $u, v \in e \in E$, then vertices $u$ and $v$ are said to be adjacent. Otherwise, two vertices $u$ and $v$ are said to be nonadjacent. We use $E_v$ to denote the set of edges containing $v$ of $G$. The degree of a vertex $v$ in $G$ is $d_v = |E_v|$. In addition, we use $R(v)$ to denote the multiset of edge types in $E_v$ for a vertex $v$.

Let $G = (V, E)$ be a general hypergraph. The set $R = \{|e|: e \in E\}$ is called the set of edge types of $G$. Then $G$ is also called an $R$-graph. For a set $S$ and a positive integer $i$, let $\binom{S}{i} = \{T \subseteq S: |T| = i\}$. A complete $R$-graph on $n$ vertices is an $R$-graph with vertex set $[n]$ and edge set $\cup_{e \in R} \binom{e}{n}$, where $[n] = \{1, 2, \ldots, n\}$. A clique of a $R$-graph $G$ is a complete $R$-subgraph of $G$. A maximal clique is a clique that cannot extended to a larger clique, and a maximal clique is a clique which covers as many vertices as possible. The clique number $\omega(G)$ of a hypergraph $G$ is the number of vertices in a maximum clique of $G$. If $E(G) = \emptyset$, then we define $\omega(G) = 1$.

In 2005, the concept of tensor eigenvalues and the spectra of tensors were independently introduced by Qi \cite{Qi2005} and Lim \cite{Lim2005}. A tensor $A = (a_{i_1i_2\ldots i_k})$ with order $k$ and dimension $n$ is a multidimensional array, where $a_{i_1i_2\ldots i_k} \in \mathbb{C}$ and $1 \leq i_1, i_2, \ldots, i_k \leq n$. For an $n$-dimensional complex vector $x = (x_1, x_2, \ldots, x_n)^T$, $Ax^{k-1}$ is defined as an $n$-dimensional complex vector whose $i$-th component is the following

$$
(Ax^{k-1})_i = \sum_{i_2, \ldots, i_k=1}^n a_{i_1i_2\ldots i_k} x_{i_2} \cdots x_{i_k}, \text{ for any } i \in [n].
$$

Let $x^{[k-1]} = (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1})^T \in \mathbb{C}^n$. Then a number $\lambda \in \mathbb{C}$ is called an eigenvalue of the tensor $A$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$
Ax^{k-1} = \lambda x^{[k-1]},
$$

and in this case, $x$ is called an eigenvector of $A$ corresponding to the eigenvalue $\lambda$. The spectral radius of $A$ is $\rho(A) = \max\{|\lambda|: \lambda \text{ is an eigenvalue of } A\}$. In particular, if $x$ is real, then $\lambda$ is called an $H$-eigenvalue. If $x \in \mathbb{R}^n_+$, then $\lambda$ is called an $H^+$-eigenvalue. If $x \in \mathbb{R}^n_{++}$, then $\lambda$ is called an $H^{++}$-eigenvalue \cite{He2009}.

**Definition 1.1.** \cite{He2009} Let $G = (V, E)$ be a general hypergraph with $n$ vertices, $m$ edges and rank($G$) = $k$. The adjacency tensor $A(G)$ of $G$ is

$$
A(G) = (a_{i_1i_2\ldots i_k}), \quad 1 \leq i_1, i_2, \ldots, i_k \leq n.
$$

For all edges $e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E$ of cardinality $s \leq k$,

$$
a_{i_1i_2\ldots i_k} = \frac{s}{\alpha(s)}, \text{ where } \alpha(s) = \sum_{k_1+\ldots+k_s=k, k_1, \ldots, k_s \geq 1} \frac{k!}{k_1!k_2!\cdots k_s!},
$$

and $i_1, i_2, \ldots, i_k$ are chosen in all possible ways from $L = \{l_1, \ldots, l_s\}$ with at least once for each element of the set $L$, while each $k_i$ represents the times that $l_i$ appears in $\{i_1, i_2, \ldots, i_k\}$. The other positions of the tensor are zero.
The eigenvalues of a general hypergraph $G$ refer to the eigenvalues of the adjacency tensor of $G$. And we use $\rho(G)$ to denote the spectral radius of a general hypergraph $G$.

Let $G$ be a general hypergraph with $n$ vertices and $\text{rank}(G) = k$, and $x = (x_1, x_2, \ldots, x_n)^T$ be a vector. For an edge $e = \{i_1, i_2, \ldots, i_s\} \in E$, we have

$$x^T (A(e)x^{k-1}) = \frac{s}{\alpha(s)} \sum_{k_1, \ldots, k_s \geq 1, k_1 + \cdots + k_s = k} \frac{k!}{k_1! \cdots k_s!} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s}.$$  

And we also have

$$x^T (A(G)x^{k-1}) = \sum_{e \in E(G)} x^T (A(e)x^{k-1}).$$

Let $D(G)$ be a degree diagonal tensor with order $k$ and dimensional $n$, and its diagonal element $d_{ii,i}$ (or simply $d_i$) be the degree of a vertex $v_i$ of $G$, for all $i \in \{1, 2, \ldots, n\}$. Then $Q(G) = D(G) + A(G)$ is the signless Laplacian tensor of the hypergraph $G$. The signless Laplacian eigenvalues of a hypergraph $G$ refer to the eigenvalues of the signless Laplacian tensor of $G$. We use $q(G)$ to denote the signless Laplacian spectral radius of $G$.

Similarly, Let $G$ be a hypergraph with $n$ vertices and $\text{rank}(G) = k$, and $x = (x_1, x_2, \ldots, x_n)^T$ be a vector. For an edge $e = \{i_1, i_2, \ldots, i_s\} \in E$, we have

$$x^T (Q(e)x^{k-1}) = \sum_{j=1}^{s} x_{ij}^k + \frac{s}{\alpha(s)} \sum_{k_1, \ldots, k_s \geq 1, k_1 + \cdots + k_s = k} \frac{k!}{k_1! \cdots k_s!} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s}.$$  

And we also have

$$x^T (Q(G)x^{k-1}) = \sum_{e \in E} x^T (Q(e)x^{k-1}).$$

In 1965, Motzkin and Straus [12] defined a homogeneous polynomial $L(G, x)$ of a graph $G$.

**Definition 1.2.** ([12]) Let $G = (V, E)$ be a graph with $n$ vertices, and let $x = (x_1, x_2, \ldots, x_n)$ be a nonnegative real vector such that $\sum_{i=1}^{n} x_i = 1$. Then

$$L(G, x) = \sum_{\{i,j\} \in E} x_i x_j.$$  

Moreover, Motzkin and Straus [12] established a remarkable connection between the clique number and a homogeneous polynomial $L(G, x)$ of a graph $G$, where $S = \{x: x \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i = 1\}$.

In 2020, Hou et al. [11] first defined a homogeneous polynomial $L(G, x)$ for a general hypergraph $G$, which generalized the definition $L(G, x)$ from a graph $G$ to a general hypergraph $G$. 

3
Definition 1.3. (11) Let $G = (V, E)$ be a general hypergraph with $n$ vertices and rank$(G) = k$, and let $x = (x_1, x_2, \ldots, x_n)$ be a nonnegative real vector such that $\sum_{i=1}^{n} x_i = 1$. Then

$$L(G, x) = \sum_{e \in E} \frac{1}{\alpha(s)} \sum_{\substack{k_1 + \cdots + k_s = k, \ \ k_1, \ldots, k_s \geq 1}} \frac{k!}{k_1!k_2! \cdots k_s!} x_1^{k_1}x_2^{k_2} \cdots x_s^{k_s},$$

where $s$ is the number of vertices of $e$ and $\alpha(s) = \sum_{\substack{k_1 + \cdots + k_s = k, \ \ k_1, \ldots, k_s \geq 1}} \frac{k!}{k_1!k_2! \cdots k_s!}$.

Similarly, for a general hypergraph $G$,

$$L(G) = \max_{x \in S} L(G, x),$$

where $S = \{x : x \in R^n_+, \sum_{i=1}^{n} x_i = 1\}$.

In 1986, Wilf [20] gave a lower bound on the spectral radius of a graph in terms of clique number, which was inspired by a result of Motzkin and Straus [12]. In 2007, Bollobás and Nikiforov [2] obtained a number of relations between the number of cliques of a graph $G$ and the spectral radius. In 2007, Lu et al. [16] presented some lower and upper bounds for the independence number and the clique number involving the Laplacian eigenvalues of a graph $G$. In 2008, Liu and Liu [15] obtained some lower and upper bounds for the independence number and the clique number involving the signless Laplacian eigenvalues of a graph $G$. In 2009, Rota and Pelillo [3] gave some new upper and lower bounds on the clique number of graphs. In 2009, Nikiforov [17] gave some new bounds for the clique and independence numbers of a graph in terms of its eigenvalues. In 2013, He et al. [10] gave the bounds on the signless Laplacian spectral radius of graphs in terms of clique number, which disproved the two conjectures on the signless Laplacian spectral radius in [9].

Recently, spectral hypergraph theory develops rapidly. There are many work about the spectral theory of hypergraphs [4, 5, 6, 14, 21, 22]. However, there are still few studies the relation between the clique number and the (signless Laplacian) spectral radius of hypergraphs. In 2015, Xie and Qi [23] mainly researched some inequality relations between the signless (Laplacian) $H$-eigenvalues and the clique (coclique) numbers of uniform hypergraphs. In 2020, Hou et al. [11] first defined a homogeneous polynomial for a general hypergraph, and gave a Motzkin-Straus type result for $\{k-1, k\}$-graphs. And they gave some lower and upper bounds on the spectral radius of $\{k-1, k\}$-graphs in terms of clique number. In this paper, motivated by [11] and [23], we mainly generalize the results of Hou et al. [11] form $\{k-1, k\}$-graphs to $R$-graphs.

This paper is organized as follows. In Section 2, some necessary lemmas and inequalities are given. In Section 3, we give a bound on the signless Laplacian spectral radius of general hypergraphs in terms of clique number. We also present a relation between a homogeneous polynomial and the clique number of general hypergraphs. Moreover, we also obtain a lower and an upper bound on the spectral radius of general hypergraphs in terms of clique number, respectively.
2 Preliminaries

In this section, we mainly give some useful lemmas and inequalities.

Lemma 2.1. (Maclaurin’s inequality [3]) Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and

$$S_k = \frac{\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2} \cdots x_{i_k}}{\binom{n}{k}}, \text{ for } 1 \leq k \leq n.$$  

Then

$$S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \cdots \geq \sqrt[n]{S_n},$$

and the equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

By Lemma 2.1, we have $S_1 \geq \sqrt[k]{S_k}$ for any $k \in [n]$. That is,

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2} \cdots x_{i_k} \leq \binom{n}{k} \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^k.$$

Lemma 2.2. ([24])

(1). Let $G$ be a general hypergraph, then $\rho(G)$ is an $H^+$-eigenvalue of $G$.

(2). If $G$ is connected, then $\rho(G)$ is the unique $H^{++}$-eigenvalue of $G$, with the unique eigenvector $x \in R^n_{++}$, up to a positive scaling coefficient.

If $G$ is a connected hypergraph, by Lemma 2.2, then there exists a unique positive eigenvector $x = (x_1, x_2, \ldots, x_n)^T$ corresponding to $\rho(G)$ such that $\sum_{i=1}^{n} x_i^{k} = 1$, where $x$ is called the principal eigenvector of $A(G)$. For a principal eigenvector $x$ of $A(G)$, we have

$$\rho(G) = x^T(A(G)x^{k-1}) = \sum_{e \in E(G)} x^T(A(e)x^{k-1}).$$

Note that $Q(G)$ is a weakly irreducible tensor. By the Perron-Frobenius theorem for nonnegative tensors [7], there exists a unique positive eigenvector $x = (x_1, x_2, \ldots, x_n)^T$ corresponding to $q(G)$ such that $\sum_{i=1}^{n} x_i^{k} = 1$, where $x$ is called the principal eigenvector of $Q(G)$. Similarly, for a principal eigenvector $x$ of $Q(G)$, we have

$$q(G) = x^T(Q(G)x^{k-1}) = \sum_{e \in E(G)} x^T(Q(e)x^{k-1}).$$

Lemma 2.3. ([11]) Let $G$ be an $R$-graph with clique number $\omega$. Then

$$\rho(G) \geq \sum_{s \in R} \binom{\omega - 1}{s - 1},$$

and the equality holds if and only if $G$ is a complete $R$-graph.
3 Main results

The following Theorem 3.1 presents a lower bound on the signless Laplacian spectral radius of a general hypergraph in terms of clique number. This result generalizes Theorem 3.1 of Xie and Qi [23] from uniform hypergraphs to general hypergraphs.

**Theorem 3.1.** Let $G$ be an $R$-graph with clique number $\omega$. Then

$$q(G) \geq 2 \sum_{s \in R} \left( \frac{\omega - 1}{s - 1} \right),$$

and the equality holds if and only if $G$ is a complete $R$-graph.

**Proof.** Assume that $G'$ is the maximum complete $R$-subgraph of $G$ with $\omega$ vertices and $\text{rank}(G) = k$. Let $x$ be a vector such that $x_i = \frac{1}{\sqrt{\omega}}$ for $i \in V(G')$ and $x_i = 0$ for otherwise. It is obvious that $\|x\|_k^k = 1$. By the definition of the signless Laplacian spectral radius of general hypergraphs, we have

$$q(G) \geq x^T (Q(G)x)^{k-1}$$

$$= \sum_{s \in R} \left( \frac{1}{\omega} + \frac{1}{\omega} + \cdots + \frac{1}{\omega} + \frac{s}{\alpha(s)} \sum_{k_1 + \cdots + k_s = k} \frac{k!}{k_1!k_2! \cdots k_s!} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s} \right)$$

$$= \sum_{s \in R} \left( \frac{s}{\omega} + \frac{1}{\omega} \alpha(s) \right)$$

$$= \sum_{s \in R} \sum_{|e| = s} \frac{2s}{\omega} = \sum_{s \in R} \frac{2s}{\omega} \left( \frac{\omega}{s} \right)$$

$$= 2 \sum_{s \in R} \left( \frac{\omega - 1}{s - 1} \right). \quad (1)$$

If $q(G) = 2 \sum_{s \in R} \left( \frac{\omega - 1}{s - 1} \right)$, by Inequality (1), then we have $q(G) = x^T Q(G)x$. Since Lemma 2.2 holds for $q(G)$ of a general hypergraph $G$, we know $x$ is a positive vector. Therefore, we have $\omega = n$. That is, $G$ is a complete $R$-graph.

If $G$ is a complete $R$-graph, then we have $\omega = n$. By the definition of the signless Laplacian spectral radius of general hypergraphs, we have $q(G) = 2 \sum_{s \in R} \left( \frac{\omega - 1}{s - 1} \right)$. □

If $G$ be a $k$-uniform hypergraph with clique number $\omega$, by Lemma 2.3 then we have $\rho(G) \geq \left( \frac{\omega - 1}{k - 1} \right)$. Therefore, we only consider $R$-graphs such that $|R| > 1$ in the following Theorem 3.2.
Theorem 3.2. Let $G$ be an $R$-graph with clique number $\omega$, $\text{rank}(G) = k$ and $\text{corank}(G) = c$. Then
\[
\rho(G) \geq \frac{(\omega - c + 1)^{c-1}(\omega - k + 1)^{k-c}}{(k-1)!} + \frac{(\omega - c + 1)^{c-1}}{(c-1)!}.
\]

Proof. By Lemma 2.3 we have
\[
\rho(G) \geq \sum_{s \in R} \binom{\omega - 1}{s - 1}
\]
\[
\geq \binom{\omega - 1}{k - 1} + \binom{\omega - 1}{c - 1}
\]
\[
= \frac{(\omega - 1)!}{(k-1)!(\omega-k)!} + \frac{(\omega - 1)!}{(c-1)!(\omega-c)!}
\]
\[
= \frac{(\omega - 1)!}{(c-1)!(\omega-c)!} + \binom{\omega - 1}{c - 1}
\]
\[
= \frac{1}{(k-1)!(\omega-k)!} + \binom{\omega - 1}{c - 1}
\]
\[
\geq \frac{(\omega - c + 1)^{c-1}}{(k-1)!(\omega-k)!} + \binom{\omega - c + 1}{c - 1}
\]
\[
= \frac{(\omega - c + 1)^{c-1}}{(k-1)!}(\omega-c)\cdots(\omega-k+1) + \binom{\omega - c + 1}{c - 1}
\]
\[
\geq \frac{(\omega - c + 1)^{c-1}(\omega - k + 1)^{k-c}}{(k-1)!} + \frac{(\omega - c + 1)^{c-1}}{(c-1)!},
\]
where
\[
\binom{\omega - 1}{c - 1} = \frac{(\omega - 1)!}{(c-1)!(\omega-c)!} = \frac{(\omega - 1)\cdots(\omega-c+1)}{(c-1)!} \geq \frac{(\omega - c + 1)^{c-1}}{(c-1)!}.
\]

Remark. If $c = k - 1 > 1$, by Theorem 3.2 then we have
\[
\rho(G) \geq \frac{(\omega - k + 2)^{k-2}(\omega - k + 1)}{(k-1)!} + \frac{(\omega - k + 2)^{k-2}}{(k-2)!}
\]
\[
= \frac{(\omega - k + 2)^{k-2}}{(k-2)!} \left( \frac{(\omega - k + 1)}{k-1} + 1 \right)
\]
\[
= \omega \frac{(\omega - k + 2)^{k-2}}{(k-1)!}
\]
\[
> \frac{(\omega - k + 2)^{k-1}}{(k-1)!}.
\]

Thus, we know Theorem 3.2 improves Theorem 3.2 of Hou et al. [11].

The following Theorem 3.3 generalizes Theorem 3.3 of Hou et al. [11] from $\{k-1,k\}$-graphs to $R$-graphs.
Theorem 3.3. Let $G = (V, E)$ be an $R$-graph with clique number $\omega$ and rank($G$) = $k$. If $G$ is a complete $R$-graph, or if there exists two nonadjacent vertices $v$ and $v'$ in $G$ such that $R(v) = R(v')$, then

$$L(G) = \sum_{s \in R} \frac{\omega^k}{s} \binom{\omega}{s}.$$ 

Proof. Assume that $G'$ is the maximum complete $R$-subgraph of $G$ with $\omega$ vertices. Let $x$ be an $n$-dimensional vector such that $x_i = \frac{1}{\omega}$ for $i \in V(G')$ and $x_i = 0$ for otherwise. It is obvious that $\|x\|_1 = 1$. By the definition of $L(G)$, we have

$$L(G) \geq L(G, x)$$

$$= \sum_{e = \{i_1, i_2, \ldots, i_s\} \in E(G')} \frac{1}{\alpha(s)} \sum_{s \in R} \frac{k!}{k_1!k_2! \cdots k_s!} x_{i_1}^k x_{i_2}^k \cdots x_{i_s}^k$$

$$= \sum_{e = \{i_1, i_2, \ldots, i_s\} \in E(G')} \frac{1}{\omega^k} \frac{1}{\alpha(s)} \sum_{s \in R} \frac{k!}{k_1!k_2! \cdots k_s!}$$

$$= \sum_{e = \{i_1, i_2, \ldots, i_s\} \in E(G')} \frac{1}{\omega^k} \frac{1}{\alpha(s)}$$

$$= \sum_{s \in R} \sum_{e \in E(G')} \frac{1}{\omega^k}$$

$$= \sum_{s \in R} \frac{1}{\omega^k} \binom{\omega}{s}.$$ 

In the following, we prove

$$L(G) \leq \sum_{s \in R} \frac{1}{\omega^k} \binom{\omega}{s}$$

(2)

by induction on $n$.

If $n \leq c - 1$, then the edge set of $G$ is empty. It is obvious that $L(G) \leq \sum_{s \in R} \frac{1}{\omega^k} \binom{\omega}{s}$ holds. Assume that Inequality (2) holds for less than $n$ vertices. Without loss of generality, supposed that $n$ is sufficient large and $x$ is a vector such that $L(G) = L(G, x)$. And $x_1 \geq x_2 \geq \cdots \geq x_{t+1} > x_{t+1} > \cdots > x_n = 0$.

If $t < n$, then we have $x_n = 0$. We can obtain $G_1$ from $G$ by deleting the vertex $n$ and the edges containing the vertex $n$. Let $G_1$ be a general hypergraph with clique number $\omega_1$ and $\omega_1 \leq \omega$. By the induction hypothesis, we have

$$L(G) = L(G_1) \leq \sum_{s \in R} \frac{1}{\omega_1^k} \binom{\omega_1}{s} \leq \sum_{s \in R} \frac{1}{\omega^k} \binom{\omega}{s},$$

where $\left(\frac{1}{\omega_1}\right)^k \binom{\omega_1}{s}$ is an monotonically increasing function of $\omega_1$. 

8
If $t = n$, then we have $x_1 \geq x_2 \geq \cdots \geq x_n > 0$.

**Case 1.** If $G$ is a complete $R$-graph, then we have $\omega = n$. By the definition of $L(G)$ and triangle inequality, we have

$$L(G) = \max_{x \in S} L(G, x)$$

$$= \max_{x \in S} \sum_{e = (1_t, t_2, \ldots, t_s) \in E(G)} \frac{1}{\alpha(s)} \sum_{k_{1} + k_{2} + \cdots + k_{s} = k} \frac{k!}{k_{1}!k_{2}! \cdots k_{s}!} x_{i_{1}}^{k_{1}} x_{i_{2}}^{k_{2}} \cdots x_{i_{s}}^{k_{s}}$$

$$\leq \max_{x \in S} \sum_{e \in E(G)} \frac{1}{\alpha(c)} \sum_{k_{1} + k_{2} + \cdots + k_{c} = k} \frac{k!}{k_{1}!k_{2}! \cdots k_{c}!} x_{i_{1}}^{k_{1}} x_{i_{2}}^{k_{2}} \cdots x_{i_{c}}^{k_{c}} + \cdots + \max_{x \in S} \sum_{e \in E(G)} x_{i_{1}} x_{i_{2}} \cdots x_{i_{k}}. \quad (3)$$

Without loss of generality, assume that there exist two integers $d \geq 1$ and $u \geq 0$ such that $k = sd + u$ for any $s \in R$. We know that

$$\alpha(s) = \sum_{k_{1}, \ldots, k_{s} \geq 1, \ k_{1} + \cdots + k_{s} = k} \frac{k!}{k_{1}!k_{2}! \cdots k_{s}!}$$

$$= a_{1} \left( \frac{s}{1} \right) + a_{2} \left( \frac{s}{2} \right) + a_{3} \left( \frac{s}{3} \right) + \cdots + a_{b} \left( \frac{s}{u} \right)$$

$$= a_{1} \left( \frac{s}{1} \right) + a_{2} \left( \frac{s}{2} \right) + a_{3} \left( \frac{s}{3} \right) + \cdots + a_{b} \left( \frac{s}{u} \right)$$

$$= s! \left( \frac{a_{1}}{s-1)!} + \frac{a_{2}}{(s-2)!} + \frac{a_{3}}{(s-3)!} + \cdots + \frac{a_{b}}{(s-u)!} \right), \quad (4)$$

where

$$a_{1} = \frac{k!}{(k-s+1)! \cdots 1!}, \quad a_{2} = \left\{ \begin{array}{ll}
\frac{k!}{(k-s)!2! \cdots 1!}, & \text{if } k - s = 2, \\
\frac{k!}{(k-s)!2! \cdots 1!}, & \text{if } k - s \neq 2,
\end{array} \right.$$  

and

$$a_{3} = \left\{ \begin{array}{ll}
\frac{k!}{(k-s-1)!3! \cdots 1!}, & \text{if } k - s - 1 = 3, \\
\frac{k!}{(k-s)!3! \cdots 1!}, & \text{if } k - s - 1 \neq 3,
\end{array} \right. \quad a_{b} = \frac{k!}{(d+1)! \cdots (d+1)! d \cdots d!} \quad \text{u times}$$

Since $G$ is a complete $R$-graph, we also have

$$\sum_{e = (1_t, t_2, \ldots, t_s) \in E(G)} \frac{k!}{k_{1}!k_{2}! \cdots k_{s}!} x_{i_{1}}^{k_{1}} x_{i_{2}}^{k_{2}} \cdots x_{i_{s}}^{k_{s}}$$
\[
= \frac{k!}{(k-s+1)!1! \cdots 1!} \sum_{e=(i_1,i_2, \ldots, i_s) \in E(G)} x_{i_1}x_{i_2} \cdots x_{i_s} \left( x_{i_1}^{k-s} + \cdots + x_{i_s}^{k-s} \right) \\
+ \frac{k!}{(k-s)!2!1! \cdots 1!} \sum_{e=(i_1,i_2, \ldots, i_s) \in E(G)} x_{i_1}x_{i_2} \cdots x_{i_s} \left( x_{i_1}^{k-s-1}x_{i_2} + \cdots + x_{i_{s-1}}x_{i_s}^{k-s-1} \right) + \cdots \\
+ \frac{k!}{(d+1)! \cdots (d+1)!d! \cdots d!} \sum_{e=(i_1,i_2, \ldots, i_s) \in E(G)} x_{i_1}^d x_{i_2}^d \cdots x_{i_s}^d \left( x_{i_1} \cdots x_{i_u} + \cdots + x_{i_{s-u}+1} \cdots x_{i_s} \right) \\
= \frac{k!}{(k-s+1)!1!} \left[ \sum_{e=(1,i_2, \ldots, i_s) \in E(G), 2 \leq \iota_2 < \cdots < \iota_s \leq n} x_{i_1}x_{i_2} \cdots x_{i_s} + \cdots + x_{i_{n-k+1}}x_{i_{n-k+2}} \cdots x_{i_{n-s}} \right] \\
+ \frac{k!}{(k-s)!2!} \left[ \sum_{e=(1,2,i_3, \ldots, i_s) \in E(G), 3 \leq \iota_3 < \cdots < \iota_s \leq n} x_{i_1}x_{i_2}x_{i_3} \cdots x_{i_s} + \cdots + x_{i_{n-1}}x_{i_n}x_{i_{n-k+1}} \cdots x_{i_{n-s}} \right] \\
+ \cdots + \frac{k!}{(d+1)! \cdots (d+1)!d! \cdots d!} \left[ x_1^d \cdots x_s^d \sum_{e=(1, \ldots, s-u, i_{s-u+1}, \ldots, i_s) \in E(G), \sum_{1 \leq \iota_1 < \cdots < \iota_{s-u+1} < \iota_s \leq n} x_{i_1}^d x_{i_2}^d \cdots x_{i_{s-u}}^d \sum_{e=(i_1, 1 \ldots, n-s+u, \ldots, i_s) \in E(G), \sum_{1 \leq \iota_1 < \cdots < \iota_{n-s+u+1} < \iota_s \leq n} x_{i_1}^d x_{i_2}^d \cdots x_{i_{n-s+u}}^d x_{i_{n-s+u+1}} \cdots x_{i_{n-s}} \right].
\]

Thus, by Maclaurin’s inequality, we have

\[
\sum_{e=(1,i_2, \ldots, i_s) \in E(G), k_1+k_2+\cdots+k_s=k, k_s \geq 1} k! k_1! k_2! \cdots k_s! x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s} \\
\leq \frac{k!}{(k-s+1)!1!} \left[ \left( \sum_{2 \leq \iota \leq n} x_{i} \right)^{s-1} \left( \frac{n-1}{s-1} \right) + \cdots + x_{n}^{k-1} \left( \sum_{1 \leq \iota \leq n-1} x_{i} \right)^{s-1} \left( \frac{n-1}{s-1} \right) \right] \\
+ \frac{k!}{(k-s)!2!} \left[ \left( \sum_{3 \leq \iota \leq n} x_{i} \right)^{s-2} \left( \frac{n-2}{s-2} \right) + \cdots + x_{n}^{k-2} \left( \sum_{2 \leq \iota \leq n-2} x_{i} \right)^{s-2} \left( \frac{n-2}{s-2} \right) \right] \\
+ \cdots + \frac{k!}{(d+1)! \cdots (d+1)!d! \cdots d!} \left[ x_1^d \cdots x_s^d \left( \sum_{1 \leq \iota \leq n-s+u} x_{i} \right)^u \left( \frac{n-s+u}{u} \right) + \cdots + \left( \sum_{s-n+u+1 \leq \iota \leq n} x_{i} \right)^u \left( \frac{n-s+u}{u} \right) \right].
\]

The equality holds if and only if \( x_1 = x_2 = \cdots = x_n = \frac{1}{n} \).

For any \( 1 \leq i \leq s \), we have

\[
\binom{n-i}{s-i} \binom{n}{i} = \binom{s}{i} = \frac{n!}{(n-s)! (s-i)! i!}.
\]

Let \( x_1 = x_2 = \cdots = x_n = \frac{1}{n} \). Therefore, we have

\[
\sum_{e=(1,i_2, \ldots, i_s) \in E(G), k_1+k_2+\cdots+k_s=k, k_s \geq 1} k! k_1! k_2! \cdots k_s! x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s}
\]
\[ a_1 \left( \frac{1}{n} \right)^k \left( \frac{n-1}{s-1} \right) \left( \frac{n}{1} \right) + a_2 \left( \frac{1}{n} \right)^k \left( \frac{n-2}{s-2} \right) \left( \frac{n}{2} \right) + \cdots + a_b \left( \frac{1}{n} \right)^k \left( \frac{n-s+u}{u} \right) \left( \frac{n}{u} \right) \]
\[ = \left( \frac{1}{n} \right)^k \frac{n!}{(n-s)!} \left( \frac{a_1}{(s-1)!} + \frac{a_2}{(s-2)!} + \cdots + \frac{a_b}{u!(s-u)!} \right). \]  

(5)

By (1) and (5), we have

\[ \frac{1}{\alpha(s)} \sum_{e=(i_1,i_2,\ldots,i_s) \in E(G), k_1+k_2+\cdots+k_s=k, k_1,k_2,\ldots,k_s \geq 1} \frac{k!}{k_1!k_2!\cdots k_s!} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s} \]
\[ \leq \left( \frac{1}{n} \right)^k \frac{n!}{(n-s)!} = \left( \frac{1}{n} \right)^n \left( \frac{n}{s} \right). \]  

(6)

By (3) and (6), we have

\[ L(G) \leq \sum_{s \in R} \left( \frac{1}{n} \right)^k \left( \frac{n}{s} \right). \]

**Case 2.** If there exist two nonadjacent vertices v and v' in G such that \( R(v) = R(v') \), then we prove that there exists a vector y such that \( L(G,y) \geq L(G,x) \). By the definition of \( L(G) \), we denote \( L^v(G) \) as

\[ L^v(G) = \sum_{e \in E_v, \left| e \right| = s \in R} \frac{1}{\alpha(s)} \frac{k!}{k_1!k_2!\cdots k_s!} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_s}^{k_s}, \]

for any a vertex \( v \in V(G) \).

Let y be a vector such that \( y_i = x_i \) for \( i \neq v \), \( y_v = x_v + x_{v'} \) and \( y_v' = 0 \). And let z be a vector such that \( z_i = x_i \) for \( i \neq v \) and \( z_v = x_{v'} \). It is obvious that \( y, z \in R^+ \). For any positive integer k, we have \( (x_v + x_{v'})^k \geq x_v^k + x_{v'}^k \). Hence, we have

\[ L^v(G,y) \geq L^v(G,x) + L^v(G,z). \]

Moreover, without loss of generality, assume that \( L^v(G,z) \geq L^v(G,x) \). Then we have

\[ L(G,y) - L(G,x) = L^v(G,y) - L^v(G,x) - L^v(G,z) \geq L^v(G,z) - L^v(G,x) \geq 0. \]

Thus, we can obtain \( G_1 \) from G by deleting the vertex v' and the edges containing v'. Similarly, by the induction hypothesis, we have

\[ L(G) = L(G,x) \leq L(G,y) \leq L(G_1) \leq \sum_{s \in R} \left( \frac{1}{\omega} \right)^k \left( \frac{\omega}{s} \right). \]

Therefore, we have

\[ L(G) = \sum_{s \in R} \frac{1}{\omega^k} \left( \frac{\omega}{s} \right), \]
where $G$ is a complete $R$-graph, or there exists two nonadjacent vertices $v$ and $v'$ in $G$ such that $R(v) = R(v')$. □

The following Theorem 3.4 generalizes Theorem 3.4 of Hou et al. [11] from $\{k-1, k\}$-graphs to $R$-graphs.

**Theorem 3.4.** Let $G = (V, E)$ be an $R$-graph with clique number $\omega$ and $\text{rank}(G) = k$. If $G$ is a complete $R$-graph, or if there exists two nonadjacent vertices $v$ and $v'$ in $G$ such that $R(v) = R(v')$, then

$$\rho(G) \leq \sum_{s \in R} k \left( \frac{U}{\omega} \right)^k \binom{\omega}{s},$$

where $U$ is the sum of entries of the principal eigenvector.

**Proof.** Let $x$ be the principal eigenvector of $G$ with $n$ vertices, and $y_i = \frac{x_i}{\omega}$, for $1 \leq i \leq n$. It is obvious that $\sum_{i=1}^{n} y_i = 1$. By Theorem 3.3 and the definition of $\rho(G)$, we have

$$\frac{\rho(G)}{U^k} = \sum_{e=(i_1, i_2, \ldots, i_s) \in E(G), s \in R} \frac{s}{\alpha(s)} \sum_{k_1 + \cdots + k_s = k, k_1, \ldots, k_s \geq 1} \frac{k!}{k_1! k_2! \cdots k_s!} y_{i_1}^{k_1} y_{i_2}^{k_2} \cdots y_{i_s}^{k_s} \leq \sum_{s \in R} k \left( \frac{U}{\omega} \right)^k \binom{\omega}{s}.$$  

Therefore, we have

$$\rho(G) \leq \sum_{s \in R} k \left( \frac{U}{\omega} \right)^k \binom{\omega}{s}. \quad \square$$

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