The topology of chaotic iterations

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Abstract

Chaotic iterations have been introduced on the one hand by Chazan, Minker [6] and Miellou [10] in a numerical analysis context, and on the other hand by Robert [12] and Pellegrin [11] in the discrete dynamical systems framework. In both cases, the objective was to derive conditions of convergence of such iterations to a fixed state. In this paper, a new point of view is presented, the goal here is to derive conditions under which chaotic iterations admit a chaotic behaviour in a rigorous mathematical sense. Contrary to what has been studied in the literature, convergence is not desired.

More precisely, we establish in this paper a link between the concept of chaotic iterations on a finite set and the notion of topological chaos [9], [7], [8]. We are motivated by concrete applications of our approach, such as the use of chaotic boolean iterations in the computer security field. Indeed, the concept of chaos is used in many areas of data security without real rigorous theoretical foundations, and without using the fundamental properties that allow chaos. The wish of this paper is to bring a bit more mathematical rigour in this field. This paper is an extension of [3], and a work in progress.

1 Introduction

Let us consider the system \( B^2 = \{0; 1\}^2 \), in which each of the two cells \( c_i \) is characterized by a boolean state \( e_i \). An evolution rule is, for example,

\[
f : \quad \begin{array}{c} B^2 \\ (e_1, e_2) \end{array} \quad \rightarrow \quad \begin{array}{c} B^2 \\ (e_1 + e_2, e_1) \end{array}
\]

These cells can be updated in a serial mode (the elements are iterated in a sequential mode, at each time only one element is iterated), in a parallel mode (at each time, all the elements are iterated), or by following a sequence \((S^n)_{n \in \mathbb{N}}\): the \( n^{th} \) term \( S^n \) is constituted by the block components to be updated at the \( n^{th} \) iteration. This is the chaotic iterations, and \( S \) is called the strategy. Let us notice that serial and parallel modes are particular cases of chaotic iterations. Until now, only the conditions of convergence have been studied.
A priori, the chaotic adjective means “in a disorder way”, and has nothing to do with the mathematical theory of chaos, studied by Li-Yorke [9], Devaney [7], Knudsen [8], etc. We asked ourselves what it really was.

In this paper we study the topological evolution of a system during chaotic iterations. To do so, chaotic iterations have been written in the field of discrete dynamical system:

$$\begin{cases} x^0 \in X \\
 x^{n+1} = f(x^n) \end{cases}$$

where \((X, d)\) is a metric space (for a distance to be defined), and \(f\) is continuous.

Thus, it becomes possible to study the topology of chaotic iterations. More exactly, the question: “Are the chaotic iterations a topological chaos?” has been raised.

This study is the first of a series we intend to carry out. We think that the mathematical framework in which we are placed offers interesting new tools allowing the conception, the comparison and the evaluation of new algorithms where disorder, hazard or unpredictability are to be considered.

The rest of the paper is organised as follows. The first next section is devoted to some recalls on the domain of topological chaos and the domain of discrete chaotic iterations. In third section is defined the framework of our study. Fourth section presents the first results concerning the topology (compacity) of the chaotic iterations. Fifth and sixth sections constitute the study of the chaotic behaviour of such iterations. In section 7 the computer and so the finite set of machine numbers is considered. The paper ends with some discussions and future work.

2 Basic recalls

This section is devoted to basic notations and terminologies in the fields of topological chaos and chaotic iterations.

2.1 Chaotic iterations

In the sequel \(S^n\) denotes the \(n^{th}\) term of a sequence \(S\), \(V_i\) denotes the \(i^{th}\) component of a vector \(V\), and \(f^k = f \circ \ldots \circ f\) denotes the \(k^{th}\) composition of a function \(f\). Finally, the following notation is used: \([1; N] = \{1, 2, \ldots, N\}\).

Let us consider a system of a finite number \(N\) of cells, so that each cell has a boolean state. Then a sequence of length \(N\) of boolean states of the cells corresponds to a particular state of the system.

A strategy corresponds to a sequence \(S\) of \([1; N]\). The set of all strategies is denoted by \(S\).

**Definition 1** Let \(S \in S\). The shift function is defined by

\[
\sigma : S^n \rightarrow S^{n+1} : (S^n)_{n \in \mathbb{N}} \mapsto (S^{n+1})_{n \in \mathbb{N}}
\]
and the *initial function* is the map which associates to a sequence, its first term

\[ i : \mathbb{S} \rightarrow [1; N] \]

\[ (S^n)_{n \in \mathbb{N}} \mapsto S^0. \]

The set \( \mathbb{B} \) denoting \( \{0, 1\} \), let \( f : \mathbb{B}^N \rightarrow \mathbb{B}^N \) be a function, and \( S \in \mathbb{S} \) be a strategy. Then, the so called *chaotic iterations* are defined by

\[
x^0 \in \mathbb{B}^N, \quad \forall n \in \mathbb{N}^*, \forall i \in [1; N], x^n_i = \begin{cases} 
x^{n-1}_i & \text{if } S^n \neq i \\
(f(x^n))_{S^n} & \text{if } S^n = i. 
\end{cases}
\]

(1)

In other words, at the \( n^{th} \) iteration, only the \( S^n \)-th cell is “iterated”. Note that in a more general formulation, \( S^n \) can be a subset of components, and \( f(x^n)_{S^n} \) can be replaced by \( f(x^k)_{S^n} \), where \( k \leq n \), modeling for example delay transmission (see *e.g.* [2]). For the general definition of such chaotic iterations, see, e.g. [12].

### 2.2 Chaotic properties of dynamical systems

Consider a metric space \((\mathcal{X}, d)\), and a continuous function \( f : \mathcal{X} \rightarrow \mathcal{X} \).

**Definition 2** \( f \) is said to be *topologically transitive* if, for any pair of open sets \( U, V \subset \mathcal{X} \), there exists \( k > 0 \) such that \( f^k(U) \cap V \neq \emptyset \).

**Definition 3** \((\mathcal{X}, f)\) is said to be *regular* if the set of periodic points is dense in \( \mathcal{X} \).

**Definition 4** \( f \) has *sensitive dependence on initial conditions* if there exists \( \delta > 0 \) such that, for any \( x \in \mathcal{X} \) and any neighbourhood \( V \) of \( x \), there exists \( y \in V \) and \( n \geq 0 \) such that \( |f^n(x) - f^n(y)| > \delta \).

\( \delta \) is called the *constant of sensitivity* of \( f \).

**Definition 5** \( f \) is said to have the property of *expansivity* if

\[
\exists \varepsilon > 0, \forall x \neq y, \exists n \in \mathbb{N}, d(f^n(x), f^n(y)) > \varepsilon.
\]

Then, \( \varepsilon \) is the *constant of expansivity* of \( f \). We also say \( f \) is \( \varepsilon \)-expansive.

**Remark 1** A function \( f \) has a constant of expansivity equals to \( \varepsilon \) if an arbitrary small error on any initial condition is amplified till \( \varepsilon \).

### 3 A topological approach for chaotic iterations

In this section we will put our study in a topological context by defining a suitable metric set.
3.1 The iteration function and the phase space

Let us denote by $\delta$ the discrete boolean metric, $\delta(x, y) = 0 \iff x = y$, and define the function

$$F_f : [1; N] \times \mathbb{B}^N \longrightarrow \mathbb{B}^N$$

$$(k, E) \longrightarrow (E_j \delta(k, j) + f(E)_k \delta(k, j))_{j \in [1; N]},$$

where + and . are boolean operations.

Consider the phase space

$$\mathcal{X} = [1; N]^R \times \mathbb{B}^N,$$

and the map

$$G_f (S, E) = (\sigma(S), F_f (i(S), E))$$

Then one can remark that the chaotic iterations defined in (1) can be described by the following iterations

$$\begin{cases}
X^0 \in \mathcal{X} \\
X^{k+1} = G_f (X^k).
\end{cases}$$

The following result can be easily proven, by comparing $S$ and $R$.

**Theorem 1** The phase space $\mathcal{X}$ has the cardinality of the continuum.

Note that this result is independent on the number of cells.

3.2 A new distance

We define a new distance between two points $(S, E), (\hat{S}, \hat{E}) \in \mathcal{X}$ by

$$d((S, E); (\hat{S}, \hat{E})) = d_e (E, \hat{E}) + d_s (S, \hat{S}),$$

where

$$\begin{cases}
d_e (E, \hat{E}) = \sum_{k=1}^{N} \delta(E_k, \hat{E}_k), \\
d_s (S, \hat{S}) = \frac{9}{N} \sum_{k=1}^{\infty} \frac{|S^k - \hat{S}^k|}{10^k}.
\end{cases}$$

It should be noticed that if the floor function $\lfloor d(X, Y) \rfloor = n$, then the strategies $X$ and $Y$ differs in $n$ cells and that $d(X, Y) - \lfloor d(X, Y) \rfloor$ gives a measure on how the strategies $S$ and $\hat{S}$ diverge. More precisely,

- This floating part is less than $10^{-k}$ if and only if the first $k^{th}$ terms of the two strategies are equal.
- If the $k^{th}$ digit is nonzero, then the $k^{th}$ terms of the two strategies are different.
3.3 The topological framework

It can be proved that,

**Theorem 2** \(G_f\) is continuous on \((\mathcal{X}, d)\).

**Proof** We use the sequential continuity (we are in a metric space).

Let \((S^n, E^n)_{n \in \mathbb{N}}\) be a sequence of the phase space \(\mathcal{X}\), which converges to \((S, E)\). We will prove that \((G_f(S^n, E^n))_{n \in \mathbb{N}}\) converges to \(G_f(S, E)\). Let us recall that for all \(n, S^n\) is a strategy, thus, we consider a sequence of strategy \((i.e.\ a\ sequence\ of\ sequences)\).

As \(d((S^n, E^n); (S, E))\) converges to 0, each distance \(d_s(E^n, E)\) and \(d_s(S^n, S)\) converges to 0. But \(d_s(E^n, E)\) is an integer, so \(\exists n_0 \in \mathbb{N}, d_s(E^n, E) = 0\) for any \(n \geq n_0\).

In other words, there exists threshold \(n_0 \in \mathbb{N}\) after which no cell will change its state:

\[
\exists n_0 \in \mathbb{N}, n \geq n_0 \implies E^n = E. 
\]

In addition, \(d_s(S^n, S) \to 0\), so \(\exists n_1 \in \mathbb{N}, d_s(S^n, S) < 10^{-1}\) for all indices greater than or equal to \(n_1\). This means that for \(n \geq n_1\), all the \(S^n\) have the same first term, which is \(S_0:\)

\[
\forall n \geq n_1, S^n_0 = S_0. 
\]

Thus, after the \(max(n_0, n_1)\)-th term, states of \(E^n\) and \(E\) are the same, and strategies \(S^n\) and \(S\) start with the same first term. Consequently, states of \(G_f(S^n, E^n)\) and \(G_f(S, E)\) are equal, then distance \(d\) between this two points is strictly less than 1 (after the rank \(max(n_0, n_1)\)).

We now prove that the distance between \((G_f(S^n, E^n))\) and \((G_f(S, E))\) is convergent to 0. Let \(\varepsilon > 0\).

- If \(\varepsilon \geq 1\), then we have seen that the distance between \((G_f(S^n, E^n))\) and \((G_f(S, E))\) is strictly less than 1 after the \(max(n_0, n_1)\)th term (same state).

- If \(\varepsilon < 1\), then \(\exists k \in \mathbb{N}, 10^{-k} \geq \varepsilon \geq 10^{-(k+1)}\). But \(d_s(S^n, S)\) converges to 0, so \(\exists n_2 \in \mathbb{N}, \forall n \geq n_2, d_s(S^n, S) < 10^{-(k+2)}\), after \(n_2\), the \(k + 2\) first terms of \(S^n\) and \(S\) are equal.

As a consequence, the \(k + 1\) first entries of the strategies of \(G_f(S^n, E^n)\) and \(G_f(S, E)\) are the same (because \(G_f\) is a shift of strategies), and due to the definition of \(d_s\), the floating part of the distance between \((S^n, E^n)\) and \((S, E)\) is strictly less than \(10^{-(k+1)} \leq \varepsilon\).

In conclusion, \(G_f\) is continuous,

\[
\forall \varepsilon > 0, \exists N_0 = max(n_0, n_1, n_2) \in \mathbb{N}, \forall n \geq N_0, d(G_f(S^n, E^n); G_f(S, E)) \leq \varepsilon. 
\]

Then chaotic iterations can be seen as a dynamical system in a topological space. In the next section, we will study the compacity of such a topological space, with a view to prove the expansive chaos in section 6.3.
4 Compacity

To prove that \((X, G_f)\) is a compact topological space, we have to check whether it is separate or not. Then, the sequential characterisation of the compacity for the metric spaces will be used to obtain the result.

4.1 Separated spaces

This section starts with some basic recalls...

Definition 6 A topological space \((X, \tau)\) is said to be a separated space if for any points \(x \neq y \in X\), there exist two open sets \(\omega_x, \omega_y\) such that \(x \in \omega_x, y \in \omega_y\) and \(\omega_x \cap \omega_y = \emptyset\).

Theorem 3 \((X, G_f)\) is a separated space.

Proof Let \((E, S) \neq (\hat{E}, \hat{S})\).

1. If \(E \neq \hat{E}\), then the intersection between the two balls \(B((E, S), \frac{1}{2})\) and \(B((\hat{E}, \hat{S}), \frac{1}{2})\) is empty.

2. Else, it exists \(k \in \mathbb{N}\) such that \(S^k \neq \hat{S}^k\), then the balls \(B((E, S), 10^{-(k+1)})\) and \(B((\hat{E}, \hat{S}), 10^{-(k+1)})\) can be chosen.

4.2 Compact spaces

Definition 7 A topological space \((X, \tau)\) is said to be compact if it is a separated space, and if each of its open covers has a finite subcover.

Definition 8 Let \((X, \tau)\) be a topological space, and \(A\) a subset of \(X\). \(a \in A\) is an accumulation point if \(\forall V \in \mathcal{V}_a, V \cap A \neq \emptyset\), and \(V \cap A \neq \{a\}\).

Let us now recall the sequential characterisation of the compacity for the metric spaces:

Theorem 4 Let \((E, d)\) be a metric space, and \(K \subset E\). The following properties are equivalents:

1. \(K\) is a compact space.

2. For any sequence of \(K\), it can be possible to extract another sequence which converge in \(K\).

3. Any sequence of \(K\) has an adherence value in \(K\).

4. Any infinite subset of \(K\) has an accumulation point in \(K\).
4.3 Compacity result

**Theorem 5** \((X, d)\) is a compact metric space.

**Proof** First, \((X, d)\) is a separate space.

Let \((E^n, S^n)_{n \in \mathbb{N}}\) be a sequence of \(X\).

1. A state \(E^k\) which appears an infinite number of time in this sequence can be found. Let
   \[ I = \{(E^n, S^n)|E^n = E^k\} \]
   For all \((E, S) \in I, S^n[0] \in [1, N]\), and \(I\) is an infinite set. Then it can be found \(k \in [1, N]\) such that an infinite number of strategies of \(I\) start with \(k\).
   Let \(n_0\) be the smallest integer such that \(E^n = E^k\) and \(S^n[0] = k\).

2. The set
   \[ I' = \{(E^n, S^n)|E^n = E^{n_0} et S^n[0] = S^{n_0}[0]\} \]
   is infinite, then one of the element of \([1, N]\) will appear an infinite number of times in the \(S^n[1]\) of \(I'\): let us call it \(\bar{l}\).
   Let \(n_1\) be the smallest \(n\) such that \((E^n, S^n) \in I'\) and \(S^n[1] = \bar{l}\).

3. The set
   \[ I'' = \{(E^n, S^n)|E^n = E^{n_0}, S^n[0] = S^{n_0}[0], S^n[1] = S^{n_1}[1]\} \]
   is infinite, etc.

Let \(l = (E^{n_k}, (S_{n_k}[k])_{k \in \mathbb{N}})\), then the subsequence \((E^{n_k}, S^{n_k})\) converge to \(l\).

5 Topological chaotic properties

To prove that we are in the framework of topological chaos, we have to check some topological conditions.

5.1 Regularity

**Theorem 6** Periodic points of \(G_f\) are dense in \(X\).

**Proof** Let \((S, E) \in X\), and \(\varepsilon > 0\). We are looking for a periodic point \((S', E')\) satisfying \(d((S, E):(S', E')) < \varepsilon\).

We choose \(E' = E\), and we reproduce enough entries from \(S\) to \(S'\) so that the distance between \((S', E)\) and \((S, E)\) is strictly less than \(\varepsilon\): a number \(k = [\log_{10}(\varepsilon)] + 1\) of terms is sufficient.

After this \(k^{th}\) iterations, the new common state is \(E\), and strategy \(S'\) is shifted of \(k\) positions: \(\sigma^k(S')\).

Then we have to complete strategy \(S'\) in order to make \((E', S')\) periodic (at least for sufficiently large indices). To do so, we put an infinite number of 1 to the strategy \(S'\).

Then, either the first state is conserved after one iteration, so \(E\) is unchanged and we obtain a fixed point. Or the first state is not conserved, then: if the first
state is not conserved after a second iteration, then we will be again in the first
case above (due to the fact that a state is a boolean). Otherwise the first state
is conserved, and we have indeed a fixed (periodic) point.

Thus, there exists a periodic point into every neighbourhood of any point,
so \((X, G_f)\) is regular, for any map \(f\).

5.2 Transitivity

Contrary to the regularity, the topological transitivity condition is not automa-
tically satisfied by any function

\((f = \text{Identity} \text{ is not topologically transitive})\). Let us denote by \(\mathcal{T}\) the set of
maps \(f\) such that \((X, G_f)\) is topologically transitive.

**Theorem 7**  \(\mathcal{T}\) is a nonempty set.

**Proof** We will prove that the vectorial logical negation function \(f_0\)

\[
\begin{align*}
 f_0 : \quad \mathbb{B}^N & \longrightarrow \mathbb{B}^N \\
 (x_1, \ldots, x_N) & \longmapsto (\overline{x}_1, \ldots, \overline{x}_N)
\end{align*}
\]

is topologically transitive.

Let \(B_A = B(X_A, r_A)\) and \(B_B = B(X_B, r_B)\) be two open balls of \(X\), where
\(X_A = (S_A, E_A)\), and \(X_B = (S_B, E_B)\). Our goal is to start from a point of \(B_A\)
and to arrive, after some iterations of \(G_{f_0}\) in \(B_B\).

We have to be close to \(X_A\), then the starting state \(E\) must be \(E_A\); it remains
to construct the strategy \(S\). Let \(S^n = S^n_A, \forall n \leq n_0\), where \(n_0\) is chosen in such
a way that \((S, E_A) \in B_A\), and \(E'\) be the state of \(G_{f_0}(S_A, E_A)\).

\(E'\) differs from \(E_B\) by a finite number of cells \(c_1, \ldots, c_{n_1}\). Let \(S^{n_0+n} = c_n, \forall n \leq n_1\).
Then the state of \(G_{f_0}^{n_0+n_1}(S, E)\) is \(E_B\).

Last, let \(S^{n_0+n_1+n} = S^n_B, \forall n \leq n_2\), where \(n_2\) is chosen in such a way that
\(G_f^{n_0+n_1+n}(S, E)\) is at a distance less than \(r_B\) from \((S_B, E_B)\). Then, starting from
a point \((S, E)\) close to \(X_A\), we are close to \(X_B\) after \(n_0 + n_1\) iterations: \((X, G_f)\)
is transitive.

**Remark 1** If, in the preceeding proof, the strategy were completed using \(S_B\),
then it can be proved that there exist a point \(X\) close to \(X_A\), and \(k_0 \in \mathbb{N}\), such
that \(G_f^{k_0}(X) = X_B\): this property is called **strong transitivity**.

**Remark 2** The question of the characterisation of \(\mathcal{T}\) will be discussed in
another paper.

5.3 Sensitive dependence on initial conditions

**Theorem 8** \((X, G_{f_0})\) has sensitive dependence on initial conditions, and its
constant of sensitiveness is equal to \(\mathbb{N}\).

**Proof** Let \((S, E) \in X\), and \(\delta > 0\). A new point \((S', E')\) is defined by:
\(E' = E,\)
\(S'^n = S^n, \forall n \leq n_0,\)
where \(n_0\) is chosen in such a way that \(d((S, E); (S', E')) < \delta,\)
and \(S'^{n_0+k} = k, \forall k \in [1; \mathbb{N}]\). Then the point \((S', E')\) is as close as we want than \((S, E),\)
and systems of \(G_f^{k+N}(S, E)\) and \(G_{f_0}^{k+N}(S', E')\) have no cell presenting the same state:
distance between those two points is greater or equal than \(\mathbb{N}\).
Remark 2 This sensitive dependence could be stated as a consequence of regularity and transitivity (by using the theorem of Banks [4]). However, we have preferred proving this result independently of regularity, because the notion of regularity must be redefined in the context of the finite set of machine numbers (see section 7.2).

5.4 Expansivity

Theorem 9 \((\mathcal{X}, G_{f_0})\) is an expansive chaotic system. Its constant of expansivity is equal to 1.

Proof If \((S, E) \neq (\hat{S}; \hat{E})\), then:

- Either \(E \neq \hat{E}\), and then at least one cell is not in the same state in \(E\) and \(\hat{E}\). Then the distance between \((S, E)\) and \((\hat{S}; \hat{E})\) is greater or equal to 1.
- Or \(E = \hat{E}\). Then the strategies \(S\) and \(\hat{S}\) are not equal. Let \(n_0\) be the first index in which the terms \(S\) and \(\hat{S}\) differ. Then
  \[\forall k < n_0, G^k_{f_0}(S, E) = G^k_{f_0}(\hat{S}, \hat{E}),\]
  and \(G^{n_0}_{f_0}(S, E) \neq G^{n_0}_{f_0}(\hat{S}, \hat{E})\), then as \(E = \hat{E}\), the cell which has changed in \(E\) at the \(n_0\)-th iterate is not the same than the cell which has changed in \(\hat{E}\), so the distance between \(G^{n_0}_{f_0}(S, E)\) and \(G^{n_0}_{f_0}(\hat{S}, \hat{E})\) is greater or equal to 2.

Remark 3 It can be easily proved that \((\mathcal{X}, G_{f_0})\) is not \(A\)-expansive, for any \(A > 1\).

In the next section, we will show that chaotic iterations are a case of topological chaos, in the sense of Knudsen, Devaney [7] and expansion.

6 Discrete chaotic iterations and topological chaos

6.1 Knudsen’s chaos

Definition 9 A discrete dynamical system is said to be \(K\)-chaotic if:

1. it possesses a dense orbit,
2. it has sensitive dependence on initial conditions.

Theorem 10 If \(\mathcal{X}\) is a compact space, then being regular and transitive implies being \(K\)-chaotic.

Theorem 11 \((\mathcal{X}, G_f)\) is chaotic in the sense of Knudsen, \(\forall f \in T\).

Proof \(\mathcal{X}\) is a compact space, and \((\mathcal{X}, G_f)\) is regular and transitive, then \((\mathcal{X}, G_f)\) is \(K\)-chaotic.
6.2 Devaney’s chaos

Let us recall the definition of a chaotic topological system, in the sense of Devaney [7]:

**Definition 10** $f : X \to X$ is said to be $D$-chaotic on $X$ if $(X, f)$ is regular, topologically transitive, and has sensitive dependence on initial conditions.

If $f \in T$, then $(X, G_f)$ is topologically transitive, regular and has sensitive dependence on initial conditions. Then we have the result.

**Theorem 12** $\forall f \in T \neq \emptyset$, $G_f$ is a chaotic map on $(X, d)$ in the sense of Devaney.

6.3 Expansive chaos

**Definition 11** A discrete dynamical system is said to be $E$-chaotic if it has transitive, regular and expansive properties.

**Theorem 13** $\forall f \in T, (X, G_f)$ is $E$-chaotic.

**Proof** $(X, G_f)$ is $D$-chaotic, and has the expansive property, then $(X, G_f)$ is $E$-chaotic.

We have proven that under the transitivity condition of $f$, chaotic iterations generated by $f$ can be described by a chaotic map on a topological space in different senses.

We have considered a finite set of states $B^\mathbb{N}$ and a set $S$ of strategies composed by an infinite number of infinite sequences. In the following section we will discuss the impact of these assumptions in the context of the finite set of machine numbers.

6.4 Topological entropy

6.4.1 Recalls

Let $(X, d)$ be a compact metric space and $f : XX$ be a continuous map. For each natural number $n$, a new metric $d_n$ is defined on $X$ by

$$d_n(x, y) = \max \{ d(f^i(x), f^i(y)) : 0 \leq i < n \}.$$

Given any $\varepsilon > 0$ and $n \geq 1$, two points of $X$ are $\varepsilon$-close with respect to this metric if their first $n$ iterates are $\varepsilon$-close.

This metric allows one to distinguish in a neighborhood of an orbit the points that move away from each other during the iteration from the points that travel together. A subset $E$ of $X$ is said to be $(n, \varepsilon)$-separated if each pair of distinct points of $E$ is at least $\varepsilon$ apart in the metric $d_n$. Denote by $H(n, \varepsilon)$ the maximum cardinality of an $(n, \varepsilon)$-separated set.

**Definition 12** The topological entropy of the map $f$ is defined by (see e.g. [1] or [5])

$$h(f) = \lim_{\varepsilon \to 0} \left( \limsup_{n \to \infty} \frac{1}{n} \log H(n, \varepsilon) \right).$$
6.4.2 Result

Theorem 14  Entropy of \((X, G_f)\) is infinite.

Proof Let \(E, \bar{E} \in \mathbb{B}^N\) such that \(\exists i_0 \in [1, N], E_{i_0} \neq \bar{E}_{i_0}\). Then, \(\forall \mathcal{S}, \bar{\mathcal{S}} \in \mathcal{S}\),
\[
d((E, \mathcal{S}); (\bar{E}, \bar{\mathcal{S}})) \geq 1
\]
But the cardinal \(c\) of \(\mathcal{S}\) is infinite, then \(\forall n \in \mathbb{N}, c > e^{n^2}\).

Then for all \(n \in \mathbb{N}\), the maximal number \(H(n, 1)\) of \((n, 1)\)-separated points is greater than or equal to \(e^{n^2}\), so
\[
h_{top}(G_f, 1) = \lim_{n \to \infty} \frac{1}{n} \log \left( H(n, 1) \right) > \lim_{n \to \infty} \frac{1}{n} \log \left( e^{n^2} \right) = \lim_{n \to \infty} (n) = +\infty
\]
But \(h_{top}(G_f, \varepsilon)\) is an increasing function when \(\varepsilon\) is decreasing, then
\[
h_{top}(G_f) = \lim_{h \to 0} h_{top}(G_f, \varepsilon) > h_{top}(G_f, 1) = +\infty
\]

We have proven that it is possible to find \(f\), such that chaotic iterations generated by \(f\) can be described by a chaotic and entropic map on a topological space in the sense of Devaney. We have considered a finite set of states \(\mathbb{B}^N\) and a set \(\mathcal{S}\) of strategies composed by an infinite number of infinite sequences. In the following section we will discuss the impact of these assumptions in the context of the finite set of machine numbers.

7 The case of finite strategies

7.1 A new definition for the periodicity

In the computer science framework, we also have to deal with a finite set of states of the form \(\mathbb{B}^N\) and the set \(\mathcal{S}\) of sequences of \([1; N]\) is infinite (countable), so in practice the set \(X\) is also infinite. The only difference with respect to the theoretical study comes from the fact that the sequences of \(\mathcal{S}\) are of finite but not fixed length in the practice.

The proof of the continuity, the transitivity and the sensitivity conditions are independent of the finitude of the length of strategies (sequences of \(\mathcal{S}\)), so even in the case of finite machine numbers, we have the two fundamental properties of chaos: sensitivity and transitivity, which respectively implies unpredictability and indecomposability (see [7, p.50]). The regularity property has no meaning in the case of finite systems because of the notion of periodicity.

We propose a new definition in order to bypass the notion of periodicity in practice.

Definition 13 A strategy \(S = (S^1, ..., S^L)\) is said cyclic if a subset of successive terms is repeated from a given rank, until the end of \(S\). A point of \(X\) that admits a cyclic strategy is called a cyclic point.

For example,
- \((1, 3, 2, 4, 1, 2, 1, 2)\) and \((1, 3, 2, 4, 1, 2, 2, 2)\) are cyclic,
but (1, 3, 2, 4, 1, 2) and (1, 3, 2, 1, 3) are not cyclic.

This definition can be interpreted as the analogous of periodicity on finite sets. Then, following the proof of regularity (section 5.1), it can be proved that the set of cyclic points is dense on \( X \), hence obtaining a desired element of regularity in finite sets, as quoted by Devaney ([7], p.50): two points arbitrary close to each other could have different behaviours, the one could have a cyclic behaviour as long as the system iterates while the trajectory of the second could "visit" the whole phase space.

It should be recalled that the regularity was introduced by Devaney in order to counteract the transitivity and to obtain such a property: two points close to each other can have fundamental different behaviours.

7.2 How it is concretely possible to deal with infinite length strategies

It is worthwhile to notice that even if the set of machine numbers is finite, we deal with strategies that have a finite but unbounded length. Indeed, it is not necessary to store all the terms of the strategy in the memory, only the \( n^{th} \) term (an integer less than or equal to \( N \)) of the strategy has to be stored at the \( n^{th} \) step, as it is illustrated in the following example.

Let us suppose that a given text is input from the outside world in the computer character by character, and that the current term of the strategy is given by the ASCII code of the current stored character. Then, as the set of all possible texts of the outside world is infinite and the number of their characters is unbounded, we have to deal with an infinite set of finite but unbounded strategies.

Of course, the preceding example is a simplistic illustrating example. A chaotic procedure should to be introduced to generate the terms of the strategy from the stream of characters.

In conclusion, even in the computer science framework our previous theory applies.

8 Discussion and future work

We proved that discrete chaotic iterations are a particular case of topological chaos, in sense of Devaney, Knudsen and expansivity, if the iteration function is topologically transitive, and that the set of topologically transitive functions is non void.
This theory has a lot of applications, because of the high number of situations that can be described with the chaotic iterations: neural networks, cellular automata, multi-processor computing, and so on. If this system is requested to evolve in an apparently disorderly manner, e.g. for security reasons (encryption, watermarking, pseudo-random number generation, hash functions, etc.), our results could be useful.

More concretely, for example, any medium (text, image, video, etc.) can be considered to be an aggregation of elementary cells (respectively: character, pixel, image). Thus a digital watermarking of this medium can be describe as an insertion of cells of a watermark into some cells of a carrier image (the state of the system), in a deterministic but unpredictable manner carried by a strategy.

Moreover, the theory brings another way to compare two given algorithms concerned by disorder (evaluation of theirs constants of sensitivity, expansivity, etc.), which can be seen as a complement of existing statistical evaluations.

In future work, other forms of chaos (such as Li-York chaos [9]) will be studied, other quantitative and qualitative tools such as entropy (see e.g. [1] or [5]) will be explored, and the domain of applications of our theoretical concepts will be enlarged.

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