A note on a Caro-Wei bound for the bipartite independence number in graphs

Shimon Kogan

Department of Computer Science and Applied Mathematics
Weizmann Institute, Rehovot 76100, Israel
shimon.kogan@weizmann.ac.il

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Abstract

A bi-hole of size $t$ in a bipartite graph $G$ is a copy of $K_{t,t}$ in the bipartite complement of $G$. Given an $n \times n$ bipartite graph $G$, let $\beta(G)$ be the largest $k$ for which $G$ has a bi-hole of size $k$. We prove that

$$\beta(G) \geq \left\lfloor \frac{1}{2} \cdot \sum_{v \in V(G)} \frac{1}{d(v)+1} \right\rfloor.$$

Furthermore we prove the following generalization of the result above. Given an $n \times n$ bipartite graph $G$, let $\beta_d(G)$ be the largest $k$ for which $G$ has a $k \times k$ $d$-degenerate subgraph. We prove that

$$\beta_d(G) \geq \left\lfloor \frac{1}{2} \cdot \sum_{v \in V(G)} \min \left(1, \frac{d+1}{d(v)+1}\right) \right\rfloor.$$

Notice that $\beta_0(G) = \beta(G)$.

1 Introduction

A bi-hole of size $t$ in a bipartite graph $G$ is a copy of $K_{t,t}$ in the bipartite complement of $G$. Given an $n \times n$ bipartite graph $G$, let $\beta(G)$ be the largest $k$ for which $G$ has a bi-hole of size $k$. The following Theorem is proven in [EMR20].
Theorem 1.1. Given an $n \times n$ bipartite graph $G$ with average degree $d$ we have
\[ \beta(G) \geq \frac{n}{d+1} - 2 \]

We generalize and improve the theorem above by proving the following.

Theorem 1.2. Given an $n \times n$ bipartite graph $G$ we have
\[ \beta(G) \geq \left\lfloor \frac{1}{2} \cdot \sum_{v \in V(G)} \frac{1}{d(v) + 1} \right\rfloor. \]

Notice that Theorem 1.1 follows from Theorem 1.2 by Jensen’s inequality. We note that better results exist when the average degree is large. In [FK10] the following theorem is proven.

Theorem 1.3. For any $0 < \epsilon < 1$ there is a constant $d_0$ such that the following holds. Given an $n \times n$ bipartite graph $G$ with average degree $d \geq d_0$ and $n \geq (1 + \epsilon)d$ we have
\[ \beta(G) \geq \frac{\epsilon}{2} \cdot \frac{n \ln d}{d}. \]

Theorems similar to Theorem 1.3 were proven in [ASSW20] and [EMR20].

Now we shall discuss a generalization of Theorem 1.2. A graph $H$ is $d$-degenerate if every non-empty subgraph of it contains a vertex of degree at most $d$. Thus 0-degenerate graphs are independent sets and 1-degenerate graphs are forests. Let $\alpha_d(G)$ denote the maximum number of vertices of an induced $d$-degenerate subgraph of $G$. The following Theorem was proven in [AKS87].

Theorem 1.4. Let $G = (V, E)$ be a graph. Then
\[ \alpha_d(G) \geq \sum_{v \in V(G)} \min\left(1, \frac{d + 1}{d(v) + 1}\right). \]

The special case $d = 0$ of Theorem 1.4 is the well known Caro-Wei bound [Car79, Wei81]. We prove an analogous theorem to Theorem 1.4 in bipartite graphs. Given an $n \times n$ bipartite graph $G$, let $\beta_d(G)$ be the largest $k$ for which $G$ has a $k \times k$ $d$-degenerate subgraph. We prove the following.

Theorem 1.5. Given an $n \times n$ bipartite graph $G$ and an integer $d \geq 0$ we have
\[ \beta_d(G) \geq \left\lfloor \frac{1}{2} \cdot \sum_{v \in V(G)} \min\left(1, \frac{d + 1}{d(v) + 1}\right) \right\rfloor. \]

Notice that Theorem 1.2 is the special case of $d = 0$ in Theorem 1.5. We prove Theorem 1.2 and Theorem 1.5 in the next two sections.
2 Proof of Theorem 1.2

We shall prove the following slightly stronger result. Given an \( n \times n \) bipartite graph \( G(A, B, E) \) where \( \Delta_A \) is the maximum degree in \( G \) of the vertices of \( A \) and \( \Delta_B \) is the maximum degree in \( G \) of the vertices of \( B \), we have

\[
\beta(G) \geq \frac{1}{2} \left( \frac{1}{\Delta_A + 1} + \frac{1}{\Delta_B + 1} + \sum_{v \in A \cup B} \frac{1}{d(v) + 1} \right) - 1.
\]

Define the potential function \( f(d) = \frac{1}{d+1} \). And let the degree sequence of graph \( G \) be \( d_1, d_2, \ldots, d_{2n} \). Hence we need to prove that \( \beta(G) \geq S \) where

\[
S = \frac{1}{2} \left( f(\Delta_A) + f(\Delta_B) + \sum_{i=1}^{2n-2} f(d_i) \right) - 1.
\]

We prove this claim by induction on \( n \). The base case is \( n = 1 \) is trivially correct. Furthermore we may assume that \( \Delta_A \geq 1 \) and \( \Delta_B \geq 1 \) for otherwise the graph is an independent set and the claims follows once again trivially.

Now we consider two cases.

**Case 1:** There is a vertex \( a \in A \) such that \( d(a) = \Delta_A \) and a vertex \( b \in B \) such that \( d(b) = \Delta_B \) and there is no edge between vertices \( a \) and \( b \).

Let graph \( H(A', B', E') \) be the \( (n - 1) \times (n - 1) \) bipartite graph formed from \( G \) by removing vertices \( a \) and \( b \), and Let \( d_1', d_2', \ldots, d_{2n-2}' \) be the degree sequence of graph \( H \). Let

\[
Q = \frac{1}{2} \left( f(\Delta_A') + f(\Delta_B') + \sum_{i=1}^{2n-2} f(d_i') \right) - 1.
\]

Where \( \Delta_A' \) is the maximum degree in \( H \) of the vertices of \( A' \) and \( \Delta_B' \) is the maximum degree in \( H \) of the vertices of \( B' \). Now notice that

\[
Q \geq S - \frac{1}{2} (f(\Delta_A) + f(\Delta_B) - \Delta_A(f(\Delta_B - 1) - f(\Delta_B)) - \Delta_B(f(\Delta_A - 1) - f(\Delta_A))) = S - \frac{1}{2} \left( \frac{1}{\Delta_A + 1} + \frac{1}{\Delta_B + 1} - \frac{\Delta_A}{\Delta_B(\Delta_B + 1)} - \frac{\Delta_B}{\Delta_A(\Delta_A + 1)} \right) = S + \frac{1}{2} \frac{(\Delta_A - \Delta_B)^2(\Delta_A + \Delta_B + 1)}{\Delta_A(\Delta_A + 1)\Delta_B(\Delta_B + 1)} \geq S
\]
And thus we are done by applying the induction hypothesis to graph $H$.

**Case 2:** Each vertex $a \in A$ such that $d(a) = \Delta_A$ and each vertex $b \in B$ such that $d(b) = \Delta_B$ are connected.

Pick an arbitrary vertex $a \in A$ such that $d(a) = \Delta_A$ and an arbitrary vertex $b \in B$ such that $d(b) = \Delta_B$. Note that there is an edge between $a$ and $b$. Let graph $H'(A', B', E')$ be the $(n-1) \times (n-1)$ bipartite graph formed from $G$ by removing vertices $a$ and $b$, and let $d'_1, d'_2, \ldots, d'_{2n-2}$ be the degree sequence of graph $H'$. Let

$$Q = \frac{1}{2} \left( f(\Delta_A') + f(\Delta_B') + \sum_{i=1}^{2n-2} f(d'_i) \right) - 1.$$  

Where $\Delta_A'$ is the maximum degree in $H'$ of the vertices of $A'$ and $\Delta_B'$ is the maximum degree in $H'$ of the vertices of $B'$. Notice that by the definition of case 2 we have $\Delta_A' \leq \Delta_A - 1$ and $\Delta_B' \leq \Delta_B - 1$. Hence we have

$$Q \geq S - \frac{1}{2} \left( f(\Delta_A) + f(\Delta_B) - (\Delta_A - 1)(f(\Delta_B - 1) - f(\Delta_B)) - (\Delta_B - 1)(f(\Delta_A - 1) - f(\Delta_A)) \right)$$

$$+ \frac{1}{2} \left( (f(\Delta_A') - f(\Delta_A)) + (f(\Delta_B') - f(\Delta_B)) \right)$$

$$\geq S - \frac{1}{2} \left( \frac{1}{\Delta_A + 1} + \frac{1}{\Delta_B + 1} - \frac{\Delta_A}{\Delta_B(\Delta_B + 1)} - \frac{\Delta_B}{\Delta_A(\Delta_A + 1)} \right)$$

$$= S + \frac{1}{2} \frac{(\Delta_A - \Delta_B)^2(\Delta_A + \Delta_B + 1)}{\Delta_A(\Delta_A + 1)\Delta_B(\Delta_B + 1)}$$

$$\geq S$$

And thus we are done by applying the induction hypothesis to graph $H$.

### 3 Proof of Theorem 1.5

We shall prove the following slightly stronger result. Set a fixed integer $d \geq 1$ (the $d = 0$ case is Theorem 1.2). Define the potential function

$$f(x) = \min \left( 1, \frac{d + 1}{x + 1} \right).$$

Let the degree sequence of the $n \times n$ bipartite graph $G(A, B, E)$ be $d_1, d_2, \ldots, d_{2n}$. Furthermore let $\Delta_A$ be the maximum degree in $G$ of the vertices of $A$ and $\Delta_B$ be the maximum degree in $G$
of the vertices of $B$. We claim that $\beta_d(G) \geq S$ where

$$S = \frac{1}{2} \left( f(\Delta_A) + f(\Delta_B) + \sum_{i=1}^{2n} f(d_i) \right) - 1.$$ 

We prove this claim by induction on $n$. The base case is $n = 1$ is trivially correct. Now we assume that the claim holds for all graphs on bipartite graphs on $(n-1) \times (n-1)$ vertices and prove for bipartite graphs on $n \times n$ vertices. We do it by induction on the number of edges in the bipartite graph $G$ on $n \times n$ vertices. The base case when $G$ has no edges follows trivially from the fact that such a graph is an independent set. Now if $G$ contains a vertex $v$ such that $1 \leq d(v) \leq d$ then we delete the edges incident to $v$ and apply the edge induction hypothesis on the resulting graph $G'$. Hence $G'$ contains a $k \times k$ $d$-degenerate subgraph $H$ such that $k \geq S$. If $H$ does not contain vertex $v$ then $H$ is also a $k \times k$ $d$-degenerate subgraph of $G$. If $H$ contains vertex $v$ then we add back the edges that are incident to $v$ in $G$ and are inside $H$. The resulting graph $H'$ is a $d$-degenerate subgraph of $G$ since we took a vertex $v$ of degree 0 in a $d$-degenerate subgraph and added to it at most $d$ edges.

Hence we can assume that the minimum degree of each vertex which is not of degree 0 in $G$ is at least $d+1$ and furthermore $\Delta_A \geq d+1$ and $\Delta_B \geq d+1$. The rest of the proof is identical to the proof of Theorem 1.2 and thus omitted (in particular we do the same cases analysis of the two cases in the proof of Theorem 1.2 but with potential function $f(x) = \min \left( 1, \frac{d+1}{x+1} \right)$).

4 Concluding remarks

It would be interesting to improve Theorem 1.2 for bipartite graphs without cycles of length 4. Furthermore it would be interesting to generalize the results for $k$-Partite graphs where $k \geq 3$.

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