On the Robustness of Decision Tree Learning under Label Noise

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Abstract

In most practical problems of classifier learning, the training data suffers from the label noise. Hence, it is important to understand how robust is a learning algorithm to such label noise. Experimentally, Decision trees have been found to be more robust against label noise than SVM and logistic regression. This paper presents some theoretical results to show that decision tree algorithms are robust to symmetric label noise under the assumption of large sample size. We also present some sample complexity results for this robustness. Through extensive simulations we illustrate this robustness.

Keywords: Robust learning, Decision trees, Label noise

1. Introduction

Decision tree is among the most widely used machine learning approaches in many applications (Wu et al., 2007). Interpretability, applicability to all types of features, less demands on data pre-processing and scalability to big data scenario are some of the reasons for its popularity. In general, decision tree is learnt in a top down greedy fashion where at each node a split rule is learnt by minimizing an appropriate objective function.

For learning a decision tree classifier, we make use of labeled training data. When the class labels in the training data may be incorrect, it is referred to as label noise. Subjectivity and other errors in human labeling, measurement errors, insufficient feature space are some of the main reasons behind label noise. In many large data problems, labeled samples are often obtained through crowd sourcing and the unreliability of labels obtained like this can be viewed as label noise. Learning from positive and unlabeled samples can also be cast as a problem of learning under label noise (du Plessis et al., 2014). Thus, learning classifiers in the presence of label noise is an important problem (Frénay and Verleysen, 2014).

It is generally accepted that among all the classification methods, decision tree is probably closest to ‘off-the-shelf’ method which has all the desirable properties including robustness to outliers (Hastie et al., 2005). While there are many results about generalization bounds for decision trees (Mansour and McAllester, 2000; Kearns and Mansour, 1998), not
many theoretical results are known about the robustness of decision tree learning in presence of label noise. In this paper, we study such robustness properties of decision trees.

Recently, Nettleton et al (Nettleton et al., 2010) empirically studied robustness of different classifiers under label noise. While decision tree learning is better than SVM or logistic regression in terms of robustness to label noise, it is also seen that naive Bayes is more robust than decision trees. It is also observed that label noise in training data increases size of the learnt tree and consequently results in over-fitting; detecting and removing noisy examples improves the learnt tree (Brodley and Friedl, 1999).

Recently, there have been many analytical results reported on robust learning of classifiers, using the general framework of risk minimization, when training data is corrupted with label noise. The robustness or noise tolerance of risk minimization depends on the loss function used. It is shown that some of the standard convex losses are not robust to symmetric label noise while the 0-1 loss is robust (Manwani and Sastry, 2013). This can, for example, explain why naive Bayes has good robustness. A general sufficient condition on the loss function for risk minimization to be robust is derived in (Ghosh et al., 2015). The 0-1 loss, sigmoid loss and ramp loss are shown to satisfy this sufficient condition while convex losses such as hinge loss (used in SVM) and the logistic loss do not satisfy this condition. Long and Servedio (2010) proved that any convex potential loss is not robust to uniform or symmetric label noise. It is also noted by du Plessis et al. (2014) that convex surrogates losses are not good for learning from positive and unlabeled data. Interestingly, it is possible to have a convex loss (which is not a convex potential) that satisfies the sufficient condition for noise tolerance and the corresponding risk minimization essentially amounts to a highly regularized SVM (van Rooyen et al., 2015). Provably robust risk minimization strategies under the so called class-conditional (or non-symmetric) label noise are also proposed (Natarajan et al., 2013; Scott et al., 2013), assuming the noise rates are known. Some sufficient conditions for robustness of risk minimization under 0-1 loss, ramp loss and sigmoid loss when the training data is corrupted with most general non-uniform label noise are also presented in (Ghosh et al., 2015). None of these results are applicable for decision trees because the popular decision tree learning algorithms cannot be captured under the risk minimization framework.

In this paper, we analyze learning of decision trees under label noise. We consider the usual method of top down induction of decision trees where, at each node, one learns a split rule to minimize an objective function. We consider many popular objective functions such as those based on gini index, misclassification rate and also the so called twoing rule. We show, under symmetric or uniform label noise, that the split rule that minimizes the objective function with noise-free data is same as that under noisy data. We show this under large sample assumption (where the fraction of noisy examples at a node is assumed same as the expected fraction). We show how this results in the algorithm learning same decision tree in the noisy and noise-free case. We also derive some sample complexity bounds to indicate how large a sample we need at a node. We also derive some generalization error bounds for trees learnt under label noise. We present empirical results to show that trees learnt with noisy data give accuracies that are comparable with those learnt with noise-free data. (This robustness may not hold when the number of samples at a node is small). We also show empirically that the random forests algorithm is far more robust to label noise.
This is also explained by our theoretical results because the large sample effect is obtained through bagging in the random forests algorithm.

2. Label Noise and Decision Tree Robustness

In this paper, we only consider binary decision trees for binary classification. In this section we introduce our notation and formally define our notion of noise tolerance of a learning algorithm. We use the same notion of noise tolerance as discussed in (Manwani and Sastry, 2013; van Rooyen et al., 2015).

2.1. Label Noise

Let $\mathcal{X} \subset \mathbb{R}^d$ be the feature space from which the examples are drawn and let $\mathcal{Y} = \{1, -1\}$ be the class labels. Let $S = \{(x_1, y_{x_1}), (x_2, y_{x_2}), \ldots, (x_N, y_{x_N})\} \in (\mathcal{X} \times \mathcal{Y})^N$ be the ideal noise-free data drawn iid from a fixed but unknown distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$. However, the learning algorithm does not have access to this data.

The noisy training data given to learner is $S^\eta = \{(x_i, \hat{y}_{x_i}), i = 1, \cdots, N\}$, where $\hat{y}_{x_i} = y_{x_i}$ with probability $(1 - \eta_{x_i})$ and $\hat{y}_{x_i} = -y_{x_i}$ with probability $\eta_{x_i}$. We use $\mathcal{D}^\eta$ to denote the joint probability distribution of $x$ and $\hat{y}_{x_i}$. We use $(\cdot)^\eta$ to denote the noisy version of a variable $(\cdot)$.

We say that the noise is uniform or symmetric if $\eta_{x_i} = \eta$, $\forall x_i$. Noise is said to be class conditional if $\eta_{x_i} = \eta_+$, $\forall x_i \in C_+$ and $\eta_{x_i} = \eta_-$, $\forall x_i \in C_-$. In general, when noise rate $\eta_{x_i}$ is a function of $x$, it is termed as non-uniform noise.

2.2. Criteria for Learning Split Rule at a Node of Decision Trees

Most decision tree learning algorithms grow the tree in top down fashion starting with all training data at the root node. At any node, the algorithm selects a split rule to optimize a criterion and uses that split rule to split the data into the left and right children of this node; then the same process is recursively applied to the children nodes till the node satisfies the criterion to become a leaf. Let $F$ denote a set of split rules. Suppose, a split rule $f \in F$ at a node $v$, sends a fraction $a$ of the samples at $v$ to the left child $v^l$ and the remaining fraction $(1 - a)$ to the right child $v^r$. Then many algorithms select a $f \in F$ to maximize a criterion

$$C(f) = G(v) - (aG(v^l) + (1 - a)G(v^r))$$

where $G(\cdot)$ is a so called impurity measure. There are many such impurity measures. Of the samples at any node $v$, suppose a fraction $p$ are of positive class and a fraction $q = (1 - p)$ are of negative class. Then the gini impurity is defined by $G_{\text{Gini}} = 2pq$ (Breiman et al., 1984); entropy based impurity is defined as $G_{\text{Entropy}} = -p \log p - q \log q$ (Quinlan, 1986); and misclassification impurity is defined as $G_{\text{MC}} = \min\{p, q\}$. Often the criterion $C$ is called the gain. Hence, we also use gain$_{\text{Gini}}$ to refer to $C(f)$ when $G$ is $G_{\text{Gini}}$ and similarly for other impurity measures.

A split criterion different from impurity is twoing rule. Consider a split rule $f$ at a node $v$. Let $p_l, q_l$ be the fraction of positive and negative class examples at the left child $v_l$. Also,
let \( p_r \) and \( q_r \) be the fraction of positive and negative examples at the right child \( v_r \). (We have, \( p_l + q_l + p_r + q_r = 1 \)). Then twoing rule selects \( f \in \mathcal{F} \) which maximizes

\[
G_{\text{Twoing}}(f) = \frac{a(1 - a)}{4} \left[ \left| \frac{p_l}{p_l + q_l} - \frac{p_r}{p_r + q_r} \right| + \left| \frac{q_r}{p_l + q_l} - \frac{q_r}{p_r + q_r} \right| \right]^2
\]

2.3. Noise Tolerance of Decision Tree

One way of achieving noise robustness is if the decision tree learning algorithm learns the same tree in presence of label noise as it would learn with noise free data.\(^1\) To achieve such robustness for a decision tree learning algorithm, it is required that the split criterion should be robust to noise. Which means that minimization of the split criterion should lead to same split rule with and without noise. We would also require that the criterion for labeling leaf nodes is also robust in the same sense.

**Definition 1** A split criterion \( C \) is said to be noise-tolerant if

\[
\arg\min_{f \in \mathcal{F}} C(f) = \arg\min_{f \in \mathcal{F}} C^n(f)
\]

where \( C(f) \) and \( C^n(f) \) denote the values of the split criterion \( C \) for a split rule \( f \in \mathcal{F} \) applied on noise free and noisy data respectively.

**Definition 2** A decision tree learning algorithm \( \text{LearnTree} \) is said to be noise-tolerant if

\[
P_D(\text{LearnTree}(S)(x) \neq y_x) = P_D(\text{LearnTree}(S^n)(x) \neq y_x)
\]

Note that for the above to hold it is sufficient if \( \text{LearnTree}(S) \) is same as \( \text{LearnTree}(S^n) \).

In the next section, we show that split criterion based on gini impurity, misclassification impurity and twoing rule are robust. We also prove the robustness of labeling rules at the leaf node. We extend the robustness results to random forest.

3. Theoretical Results

As discussed in the previous section, the robustness of a decision tree requires robustness of the labeling rule at each leaf node and the robustness of the split criterion at each non-leaf node. We show each of these in turn. Then we discuss the robustness of the decision tree in general. We derive all results under large sample assumptions. That is, we assume that the number of samples at any node large enough so that we can take the actual fraction of noisy examples to be the expected fraction.

3.1. Robustness of Labeling Rule at Leaf Nodes

A popular approach for labeling is to take majority vote at the leaf node. We prove that, majority voting is robust to symmetric label noise.

**Theorem 3** Let \( \eta_x < 0.5, \forall x \). Then, majority voting at a leaf node is robust to (a) symmetric label noise, (b) nonuniform label noise if all the points at the leaf node belong to one of the class in the noise free data.

\(^1\) For simplicity, we do not consider pruning of the tree.
Proof Let $p$ and $q = 1 - p$ be the fraction of positive and negative samples at leaf node $v$.

- (a) Under symmetric label noise, fraction of positive and negative samples become, $p^\eta = (1 - \eta)p + \eta q$ and $q^\eta = (1 - \eta)p + \eta p$. Thus, $p^\eta - q^\eta = (1 - 2\eta)(p - q)$. Since $\eta < 0.5$, $(p^\eta - q^\eta)$ will have the same sign as $(p - q)$. This proves robustness of the majority voting under symmetric label noise.

- (b) Let the leaf node $v$ contains all the points from the positive class. Thus, $p = 1$, $q = 0$. Under non-uniform noise (with $\eta_x < 0.5, \forall x$),

$$
\eta \sum_{i}^n \mathbb{1}_{x_i \in v \cap C_+} (1 - \eta_{x_i}) > 0.5 \sum_{x_i \in v} 1 = 0.5
$$

Thus, the majority vote will assign positive label to the leaf node $v$. This proves the second part of the theorem.

3.2. Robustness Of Split Rules

Here, we analyze the robustness of different split criterion. That is, whether the optimal split rule for a split criterion remains same when there is label noise.

**Theorem 4** Splitting criterion based on gini impurity, mis-classification rate and twining rule are tolerant to symmetric label noise given $\eta \neq 0.5$.

**Proof**

For impurity based methods, assume $f^*$ is the optimal splitting at a node $v$. Let $p$ and $q$ be the fractions of positive and negative class at node $v$. For any arbitrary split $f$, let $a$ be the fraction of points at the left child ($v_l$). For the right child ($v_r$), the fraction of points is $(1 - a)$.

- **Gini Impurity** For a node $v$, the gini impurity is $G_{\text{Gini}}(v) = 2pq$. Under symmetric label noise, the gini impurity becomes,

$$
G_{\text{Gini}}^\eta(v) = 2p^\eta q^\eta = 2((1 - 2\eta)p + \eta q)((1 - 2\eta)q + \eta) \\
= 2pq(1 - 2\eta)^2 + (\eta - \eta^2) = G_{\text{Gini}}(v)(1 - 2\eta)^2 + (\eta - \eta^2)
$$

Note that, after splitting the noisy data, same split, $f$, has same fraction of points in each child. Thus, the impurity gain in presence of noise can be written as

$$
gain_{\text{Gini}}^\eta(f) = G_{\text{Gini}}^\eta(v) - [a G_{\text{Gini}}^\eta(v_l) + (1 - a)G_{\text{Gini}}^\eta(v_r)] \\
= (1 - 2\eta)^2[G_{\text{Gini}}(v) - a G_{\text{Gini}}(v_l) - (1 - a)G_{\text{Gini}}(v_r)] \\
= (1 - 2\eta)^2 gain_{\text{Gini}}(f)
$$

Thus for any $\eta \neq 0.5$, if $gain_{\text{Gini}}(f^*) > gain_{\text{Gini}}(f)$, then $gain_{\text{Gini}}^\eta(f^*) > gain_{\text{Gini}}^\eta(f)$. Which means that the maximizer of the impurity gain based on gini index does not change in presence of symmetric label noise.
• **Misclassification rate** For a node \( v \), misclassification impurity is defined as, \( G_{MC}(v) = \min\{p, q\} \). Under symmetric label noise with \( \eta < 0.5 \), impurity becomes,

\[
G_{MC}^\eta(v) = \min\{p^\eta, q^\eta\} = \min\{(1 - 2\eta)p + \eta, (1 - 2\eta)q + \eta\} = (1 - 2\eta)G_{MC}^\eta(v) + \eta
\]

In presence of symmetric label noise, impurity gain for a split \( f \) can be written as

\[
gain_{MC}^\eta(f) = G_{MC}^\eta(v) - \left[ a G_{MC}^\eta(v_l) + (1 - a)G_{MC}^\eta(v_r) \right] = (1 - 2\eta)\left[ G_{MC}^\eta(v) - a G_{MC}^\eta(v_l) - (1 - a)G_{MC}^\eta(v_r) \right] = (1 - 2\eta)\gain_{MC}(f)
\]

where we note that \((1 - 2\eta) > 0\) because we are considering the case \( \eta < 0.5 \). When \( \eta > 0.5 \), one can similarly show that \( \gain_{MC}^\eta(f) = (2\eta - 1)\gain_{MC}(f) \). Thus, the maximizer of the impurity gain does not change in presence of symmetric label noise given \( \eta \neq 0.5 \).

• **Twoing rule** Let \( p_l \) and \( q_l \) be the fractions of positive and negative class points at \( v_l \). Also, let \( p_r \) and \( q_r \) be the fractions of positive and negative class points at \( v_r \). Note that, \( p_l + q_l + p_r + q_r = 1 \). Let \( \frac{p_l}{p_l + q_l} = m_l \) and \( \frac{p_r}{p_r + q_r} = m_r \). Twoing criterion, for a split \( f \), can be rewritten as

\[
G_{Twoing}(f) = \frac{(p_l + q_l)(p_r + q_r)}{4} \left[ \frac{p_l}{p_l + q_l} - \frac{p_r}{p_r + q_r} \right]^2 = \frac{(p_l + q_l)(p_r + q_r)}{4} \left[ |m_l - m_r| + |1 - m_l - 1 + m_r|^2 \right]
\]

When there is symmetric label noise, \( m_l^\eta = (1 - 2\eta)m_l + \eta \) and \( m_r^\eta = (1 - 2\eta)m_r + \eta \)

\[
G_{Twoing}^\eta(f) = \frac{(p_l + q_l)(p_r + q_r)}{4} [m_l^\eta - m_r^\eta]^2 = (p_l + q_l)(p_r + q_r)(1 - 2\eta)^2 [m_l - m_r]^2 = (1 - 2\eta)^2 G_{Twoing}(f)
\]

Thus, the maximizer of twoing rule does not change when there is symmetric label noise. This proves the robustness of twoing rule.

We proved impurity gain (using Gini, twoing and misclassification) based splitting criterion are robust under symmetric label noise.

**Remark 5 Impurity based on entropy** Splitting criterion based on entropy may not be robust. However, we emphasize that, ordering change of two split rules based on entropy happens rarely as we observed through experimentation on large number of splits. If one class is extreme in a child node, this behavior can be observed. We give one such counter-example.
Robust Decision Trees

Let's take split rule $f_1$ which has $p_l = 0.05$, $q_l = 0.45$, $p_r = 0.25$, $q_r = 0.25$ where $p_l$, $q_l$, $p_r$, $q_r$ are defined as in the description of twoing rule. For another split rule $f_2$, fractions are, $p_l = 0.003$, $q_l = 0.297$, $p_r = 0.297$, $q_r = 0.403$. It can be easily verified that, $\text{gain}_E(f_1) < \text{gain}_E(f_2)$. However, under symmetric label noise, $\eta = 40\%$, $\text{gain}_E(f_2) < \text{gain}_E(f_1)$.

Remark 6 Robustness under class conditional noise: In the risk minimization framework, class conditional noise can be taken care when the noise rates are known (Natarajan et al., 2013; Scott et al., 2013; Ghosh et al., 2015). However, in the context of decision trees, knowledge of noise rate makes the problem trivial. True fraction of each class can be computed with the noisy fraction and noise rate.

3.3. Robustness of Decision Tree Learning : Large Sample Analysis

We have proved that the split criteria based on gini or misclassification impurity as well as the twoing rule are noise-tolerant and hence the same split rule would be learnt at any node irrespective of whether the labels come from noise-free data or noisy data. (Here we assume for simplicity that there is a unique split rule maximizing the criterion at each node. In general we need some prefixed rule to break ties among all the split rules that maximize the criterion). We also showed that under majority rule a leaf node would get the same label under noisy or noise-free data. To conclude that we learn the same tree, we need to examine the rule for deciding when a node becomes a leaf. If this is determined by the depth of the node then it is easy to see that the same tree would be learnt with noisy and noise-free data. In many algorithms one makes a node as leaf if no split rule gives positive value to the gain. This will also lead to learning of the same tree with noisy samples as with noise-free samples, because we showed that the gain under noisy case is a linear function of the gain under noise-free case. Thus the above results establish robustness of the these decision tree learning algorithms. We emphasize again that all this is proved under the large sample limit where we have taken the fraction of noisy samples at any node to be the expected fraction.

3.4. Sample complexity under noise

We established robustness of decision tree learning algorithms under large sample limit. Hence an interesting question is that of how large the sample size should be for our assertions about robustness of majority voting and split criteria to hold with a large probability. We provide some such sample complexity bounds in this subsection.

Lemma 7 Assume there are $N$ samples at a leaf node $v$. Under symmetric label noise with $\eta < 0.5$, majority voting will not fail with probability at-least $1 - \delta$ when $N \geq \frac{2}{\rho^2 (1 - 2\eta)^2} \ln(\frac{1}{\delta})$, where $\rho$ denotes the difference between fraction of positive and negative samples in the noise-free case.

Proof is presented in Appendix A.

It is easy to see that the samples needed increases with increasing $\eta$, which is intuitive. It also increases with decreasing $\rho$. The value of $\rho$ tells us the ‘margin of majority’ in the
noise-free case and hence when $\rho$ is small we should expect to need more examples in the noisy case.

**Lemma 8** We assume there are $N$ samples at a arbitrary non-leaf node $v$ and given two splits $f_1$ and $f_2$, gain from split rule (gini, misclassification, twoing rule) $f_1$ is higher than $f_2$. Under symmetric label noise with $\eta \neq 0.5$, gain from $f_1$ will be higher with probability $1 - \delta$ when $N \geq O(\frac{1}{\rho^2(1 - 2\eta) \ln(\frac{1}{\delta})})$, where $\rho$ denotes the difference between gain of the two splits in the noise-free case.

Proof is given in Appendix B.

### 3.5. Consistency and Generalization Bounds

All the decision tree learning algorithms that we considered are greedy algorithms that determine the split rule at each node using some criterion function. None of these are provably consistent in the sense that as the number of samples go to infinity the decision tree would become the optimal classifier for the given distribution. However, it has been shown recursive partition can be consistent if the some condition holds (Devroye et al., 2013; Lugosi et al., 1996). Any decision tree algorithm ultimately comes up with a partitioning of the feature space. We can term each partition a cell. The final class label for all feature vectors in a cell would be the majority class of all training samples in that cell. For consistency we essentially need that, as number of samples go to infinity, the cell size goes to zero, the number of samples in a cell go to infinity and the fraction of samples in any cell goes to zero. These are essentially same as the conditions needed for consistency of nonparametric estimates. In this paper, our concern is only with noise robustness. In any consistent recursive partition based algorithm, the final classification for any new pattern is based on the majority class in that cell. We have already showed that, under large sample results, majority voting is robust in the sense that the same label would be assigned to the cell whether we are given noisy samples or noise-free samples. Thus, our results also show that such consistent decision tree learning algorithms are also robust to label noise.

We proved that decision tree learning is robust to symmetric label noise. We end this section by discussing generalization bounds for decision tree learning under label noise. To derive generalization bounds we need to assume the algorithm is consistent. That is it converges and is searching over a space of finite VC dimension (which would be true, e.g., if the algorithm uses a finite depth bound). We emphasize that we are not analyzing generalization bounds for the greedy tree learning algorithms. We are addressing the issue of generalization bound under label noise assuming we have a generalization bound under noise-free case.

**Theorem 9** Let $g_n$ be a binary classifier robust to noise. If $g_n$ has finite vc dimension and the finite sample classifier error rate uniformly converge to the true error rate of the minimizer, under noisy sample also, uniform convergence can be guaranteed.

**Proof**

We use risk for 0-1 loss. We denote error rate as $R(g)$ for classifier $g$ under noise-free case and as $R^\eta(g)$ under the noisy case. For the finite sample case we write this as $\hat{R}$. Let $g_n^*$ be the minimizer for finite $n$ sample case and let $g^*$ be the true minimizer. Superscript of $\eta$ on these denotes these quantities for the noisy case.
Since the noise-free learning is assumed consistent, we have,

\[ R(g_n^*) \leq \hat{R}(g_n^*) + \mathcal{O}(n, vc), \quad R(g_n^*) \leq R(g^*) + \mathcal{O}(n, vc) \]

where \( \mathcal{O}(n, vc) \) is a term that goes to zero as \( vc/n \) where \( n \) is the sample size and \( vc \) is the VC dimension (Vapnik, 1995). Under symmetric label noise \( \eta \), we can derive the risk for any \( g \) as (taking \( L \) to be the 0-1 loss)

\[
R^\eta_L(g) = (1 - \eta) \int_X L(g(x), y_x)dp(x) + \eta \int_X L(g(x), -y_x)dp(x)
\]

\[
= (1 - \eta) \int_X L(g(x), y_x)dp(x) + \eta \int_X (1 - L(g(x), y_x))dp(x)
\]

\[
= R_L(g)(1 - 2\eta) + \eta
\]

Since we assume learning algorithm is robust, \( R(g^{\eta}_n) = R(g^*) \). Thus we have the following,

\[
R^\eta(g_n^*) - R^\eta(g^{\eta}_n) = (R(g_n^*) - R(g^{\eta}_n))(1 - 2\eta) \leq \mathcal{O}(n, vc)
\]

\[
R(g_n^*) - R(g^{\eta}_n) = R(g_n^*) - R(g^*) \leq \frac{\mathcal{O}(n, vc)}{1 - 2\eta}
\]

Thus proves error for finite noisy sample robust classifier is uniformly bounded from the true classifier.

### 3.6. Noise Robustness in Random Forest

A random forest (Breiman, 2001) is a collection of randomized tree classifiers. We can represent the set of trees as \( g_n = \{g_n(x, \pi_1), \ldots, g_n(x, \pi_m)\} \). Here \( \pi_1, \ldots, \pi_m \) are iid random variables, conditioned on data, which are used for partitioning the nodes. Finally, majority vote is taken among the random tree classifiers for prediction. We denote this classifier as \( \hat{g}_n \) (averaged classifier).

In a purely random forest classifier, partitioning does not depend on the label \( y \). At each step, a node is chosen randomly and a feature is selected randomly for the split. A split threshold is chosen uniformly randomly from the interval of the selected feature. This procedure is done \( k \) times.

A greedily grown random forest classifier is a set of randomized tree classifiers taking a greedy step each time for partitioning using the labels \( y \). Each tree is grown greedily improving impurity with some randomization. At each node, a subset of random features are chosen. Tree is grown by computing the best split among those random features only. Breiman’s random forest classifier use gini impurity gain (Breiman, 2001).

Consistency of random forests has been studied in Biau et al. (2008). We can argue when large sample assumption holds, random forest will also be robust to symmetric label noise.

**Remark 10** A purely random forest classifier/ greedily grown random forest, \( \hat{g}_n \), is robust to symmetric label noise with \( \eta < 0.5 \) under large sample assumption.
We need to prove each randomized tree is robust to label noise in both cases. In purely random forest, randomization is on the partitions and the partitions does not depend on label (probably noisy). Thus we need robustness in leaf nodes. We proved robustness of majority vote at leaf nodes under symmetric label noise. Thus, for a purely uniform random forest, $\overline{g}^* = g^*$. Similarly for a greedily grown trees with gini impurity measure, we showed that each tree is robust because of both split rule robustness and majority voting robustness. Thus when large sample assumption holds, greedily grown random forest will be also robust to symmetric label noise.

4. Empirical Illustration

4.1. Dataset Description

We used two synthetic datasets in 2D. These are $2 \times 2$ and $4 \times 4$ checker board patterns (CB). We also present results for 10 UCI datasets (Lichman, 2013) which are described in Table 1. We used the pre-processing steps used in libsvm tool (Chang and Lin, 2011).

4.2. Experimental Setup

We used decision tree implementation available in scikit learn library (Pedregosa et al., 2011). We present results only with gini impurity based decision tree classifier. Experimentally, we found that decision tree learnt using twoing rule and misclassification rate perform similar even in presence of label noise. Hence, we do not present results with other split criterion. We also show results using random forest classifier (RF) using scikit learn library. Number of trees in random forest was set to 100. Except for minimum leaf size, there was no constraint.

For synthetic datasets, minimum samples in leaf node was restricted to 250. For UCI datasets, it was restricted to 50 (except for breast cancer, four-class, liver datasets where it was set to 20 because these data sets are small). We used 20% data for testing and 20% for validation. For validation, we only tuned with leaf sample size. Symmetric label noise (introduced in the data) was varied from 0% – 40%. For synthetic datasets and two real world datasets (where classes are almost separable), we also experimented with class conditional noise (a special type of non-uniform noise). We fixed the class conditional noise rates (CC2) as $\eta_+ = 20\%$ and $\eta_- = 40\%$. For UCI datasets,
we experimented with smaller class-conditional noise (CC1) where we fixed $\eta_+ = 10\%$ and $\eta_- = 20\%$. In all experiments, noise was introduced only on training and validation data. Test set was noise free.

### 4.3. Simulation Results

In this section, we discuss the experimental results. The average accuracy and standard deviation over 10 runs on different data sets are shown in Table 2-3.

For synthetic datasets the sample sizes are large and hence we expect good robustness. For $2 \times 2$ checkerboard, accuracy of a single tree decreases to 98.5% (under 40% noise) from 99.7% (under noise free case). For random forest, the decrease in the accuracy is very small with the increase in noise rate. As can be seen that both decision tree and RF are robust to class conditional noise also. We showed that majority voting at leaf nodes is robust to non-uniform noise if the node is pure under noise-free data. Hence, we can expect robustness to class conditional noise when the classes are separable. As both the synthetic datasets are separable, the drop in accuracy is very small due to label noise. For $2 \times 2$ CB data, accuracy dropped to 99.25% under 20%-40% class conditional noise. For $4 \times 4$ CB dataset, the accuracy drops to 97% when there is class conditional noise. Random forest shows better robustness results against label noise.

Among UCI datasets, Skin-segmentation and Mushroom datasets are almost separable. We tried higher level of class-conditional noise (CC2) here. Results are similar to synthetic datasets. Upto 30% symmetric label noise, accuracy of a single tree does not drop more than 1% for both the datasets. Under 40% noise, accuracy of a tree drops only 1.5% and 3% for Skin and Mushroom datasets respectively. Under 40% noise, accuracy drop of random forest is only 0.2% and 1.5% for Skin and Mushroom datasets. It is expected because of Lemma 7. Under CC2, accuracy drops by less than 2% for all the datasets.

For Adult, German, Diabetes, Splice and Covtype datasets, the minimum leaf size was restricted to 50. For Adult dataset, accuracy remains similar for all levels of noise. For German, diabetes, splice, covtype, accuracy drops drastically only under 40% noise for a single decision tree. This is expected, as higher noise rates requires higher samples in each leaf node for the algorithm to be robust. Also for all these datasets, small class-
Table 3: Comparison Results on UCI datasets

| Data      | Method | $\eta = 0\%$ | $\eta = 10\%$ | $\eta = 20\%$ | $\eta = 30\%$ | $\eta = 40\%$ |
|-----------|--------|--------------|--------------|--------------|--------------|--------------|
| Adult     | Gini   | 83.95±0.39   | 83.7±0.36    | 83.37±0.39   | 82.96±0.42   | 81.02±0.64   | 83.26±0.38   |
|           | RF     | 83.88±0.28   | 83.92±0.3    | 83.71±0.39   | 83.79±0.44   | 83.12±0.27   | 83.04±0.37   |
| Breast cancer | Gini   | 92.26±2.12   | 92.48±2.47   | 90.95±2.02   | 89.85±2.74   | 86.35±5.1    | 92.26±1.96   |
|            | RF     | 95.91±1.88   | 95.77±1.81   | 96.28±1.2    | 95.4±1.09    | 94.31±2.35   | 94.89±1.99   |
| German    | Gini   | 71.7±2.52    | 71.0±2.53    | 69.95±3.59   | 69.55±2.81   | 63.15±7.34   | 71.8±2.79    |
|           | RF     | 70.75±3.04   | 70.95±2.9    | 70.8±2.78    | 70.8±2.71    | 71.45±2.78   | 70.75±3.04   |
| Liver     | Gini   | 62.61±7.36   | 62.75±4.94   | 62.03±9.59   | 55.07±7.78   | 56.96±6.35   | 60.29±5.98   |
|           | RF     | 67.68±7.64   | 64.93±7.09   | 65.22±7.39   | 60.0±10.89   | 52.32±8.21   | 64.93±7.93   |
| Fourclass | Gini   | 92.49±3.36   | 91.68±3.19   | 90.64±5.28   | 81.79±6.86   | 72.14±7.54   | 89.77±4.53   |
|           | RF     | 93.06±2.87   | 93.29±2.15   | 90.58±3.39   | 86.53±4.53   | 76.76±6.61   | 89.19±3.88   |
| Diabetes  | Gini   | 72.47±4.06   | 73.25±3.27   | 71.82±4.38   | 69.16±6.07   | 65.19±7.41   | 70.65±6.06   |
|           | RF     | 75.78±2.65   | 74.81±2.94   | 73.51±3.58   | 72.47±3.92   | 70.45±2.99   | 74.81±3.41   |
| Splice    | Gini   | 91.8±1.29    | 91.21±1.13   | 90.93±2.41   | 88.41±2.3    | 70.98±4.53   | 91.17±0.95   |
|           | RF     | 94.9±1.04    | 94.69±0.78   | 93.92±1.56   | 92.9±0.88    | 80.8±5.6     | 91.78±1.01   |
| Covtype   | Gini   | 88.58±0.13   | 88.07±0.2    | 86.97±0.35   | 83.09±0.23   | 76.69±0.37   | 87.09±0.2    |
|           | RF     | 86.45±0.23   | 86.43±0.24   | 86.43±0.17   | 85.95±0.13   | 82.9±0.24    | 86.02±0.12   |

conditional noise (CC1) does not affect the accuracy much. We observe that random forest performs significantly better than a single tree. As number of independent tree grows from randomization, we expect that the random forest would be robust even if leaf sample size is small.

For Breast-cancer, Four class and Liver datasets, we restricted minimum leaf size to be 20. In all these datasets, learning is robust up to 20% noise for a single decision tree. After that, accuracy starts to drop significantly. Again, this happens due to the small sample size at the leaf nodes.

5. Conclusion

In this paper, we have addressed the robustness properties of decision tree learning under label noise. In many current applications one needs to take care of label noise in training data. hence, it is very desirable to have learning algorithms that are not affected by label noise. Since most impurity based top-down decision tree algorithms learns split rules based on fractions of positive and negative samples at a node, one can expect that they should have some robustness. We proved that decision tree algorithms based on gini or misclassification impurity and the twoing rule algorithm are all robust to symmetric label noise. We showed that at any node they would learn the same split rule with noise-free data as with noisy data. We also showed that majority rule at leaf node does not change the class label under symmetric label noise. We showed all this under large sample limit. We also provided some sample complexity results for the robustness. Through extensive empirical investigations
we illustrated the robustness of a single decision tree using gini impurity based algorithm as well as for random forest algorithm.

Top-down decision tree approach is very popular classifier techniques in many practical applications. Hence, the robustness results presented in this paper are interesting. All the results we proved are for symmetric noise. Extending these results to class conditional and non-uniform noise is an important research direction.

References

Gérad Biau, Luc Devroye, and Gábor Lugosi. Consistency of random forests and other averaging classifiers. *The Journal of Machine Learning Research*, 9:2015–2033, 2008.

L. Breiman, J. Friedman, R. Olshen, and C. Stone. Classification and Regression Trees. Wadsworth and Brooks, Monterey, CA, 1984.

Leo Breiman. Random forests. *Machine learning*, 45(1):5–32, 2001.

Carla E. Brodley and Mark A. Friedl. Identifying mislabeled training data. *Journal of Artificial Intelligence Research*, pages 131–167, 1999.

Chih-Chung Chang and Chih-Jen Lin. LIBSVM: A library for support vector machines. *ACM Transactions on Intelligent Systems and Technology*, 2:27:1–27:27, 2011. Software available at http://www.csie.ntu.edu.tw/~cjlin/libsvm.

Luc Devroye, László Györfi, and Gábor Lugosi. *A probabilistic theory of pattern recognition*, volume 31. Springer Science & Business Media, 2013.

Marthinus C du Plessis, Gang Niu, and Masashi Sugiyama. Analysis of learning from positive and unlabeled data. In *Advances in Neural Information Processing Systems*, pages 703–711, 2014.

Benoît Frénay and Michel Verleysen. Classification in the presence of label noise: a survey. *Neural Networks and Learning Systems, IEEE Transactions on*, 25(5):845–869, 2014.

Aritra Ghosh, Naresh Manwani, and PS Sastry. Making risk minimization tolerant to label noise. *Neurocomputing*, 160:93–107, 2015.

Trevor Hastie, Robert Tibshirani, Jerome Friedman, and James Franklin. The elements of statistical learning: data mining, inference and prediction. *The Mathematical Intelligencer*, 27(2):83–85, 2005.

Michael J Kearns and Yishay Mansour. A fast, bottom-up decision tree pruning algorithm with near-optimal generalization. In *ICML*, volume 98, pages 269–277. Citeseer, 1998.

M. Lichman. UCI machine learning repository, 2013. URL http://archive.ics.uci.edu/ml.

Philip M Long and Rocco A Servedio. Random classification noise defeats all convex potential boosters. *Machine Learning*, 78(3):287–304, 2010.
Gábor Lugosi, Andrew Nobel, et al. Consistency of data-driven histogram methods for density estimation and classification. *The Annals of Statistics*, 24(2):687–706, 1996.

Yishay Mansour and David A McAllester. Generalization bounds for decision trees. In *COLT*, pages 69–74. Citeseer, 2000.

Naresh Manwani and PS Sastry. Noise tolerance under risk minimization. *Cybernetics, IEEE Transactions on*, 43(3):1146–1151, 2013.

Nagarajan Natarajan, Inderjit S Dhillon, Pradeep K Ravikumar, and Ambuj Tewari. Learning with noisy labels. In *Advances in neural information processing systems*, pages 1196–1204, 2013.

David F Nettleton, Albert Orriols-Puig, and Albert Fornells. A study of the effect of different types of noise on the precision of supervised learning techniques. *Artificial intelligence review*, 33(4):275–306, 2010.

Fabian Pedregosa, Gaël Varoquaux, Alexandre Gramfort, Vincent Michel, Bertrand Thirion, Olivier Grisel, Mathieu Blondel, Peter Prettenhofer, Ron Weiss, Vincent Dubourg, et al. Scikit-learn: Machine learning in python. *The Journal of Machine Learning Research*, 12:2825–2830, 2011.

J. Ross Quinlan. Induction of decision trees. *Machine learning*, 1(1):81–106, 1986.

Clayton Scott, Gilles Blanchard, and Gregory Handy. Classification with asymmetric label noise: Consistency and maximal denoising. In *COLT 2013 - The 26th Annual Conference on Learning Theory, June 12-14, 2013, Princeton University, NJ, USA*, pages 489–511, 2013.

Brendan van Rooyen, Aditya Menon, and Robert C Williamson. Learning with symmetric label noise: The importance of being unhinged. In *Advances in Neural Information Processing Systems*, pages 10–18, 2015.

Vladimir N. Vapnik. *The Nature of Statistical Learning Theory*. Springer-Verlag New York, Inc., New York, NY, USA, 1995. ISBN 0-387-94559-8.

Xindong Wu, Vipin Kumar, J. Ross Quinlan, Joydeep Ghosh, Qiang Yang, Hiroshi Motoda, Geoffrey J. McLachlan, Angus Ng, Bing Liu, Philip S. Yu, Zhi-Hua Zhou, Michael Steinbach, David J. Hand, and Dan Steinberg. Top 10 algorithms in data mining. *Knowledge and Information Systems*, 14(1):1–37, 2007.

**Appendix A. Sample Complexity Bound for Voting Rule**

**Proof** [of Lemma 7]

Let $N_p$ and $N_n$ denote the positive and negative samples at the node under noise-free case. (Note $N = N_p + N_n$). Without loss of generality assume that positive class is in majority and let $N_p = N_n + \rho N$. Recall that this $\rho > 0$ is the difference in the fractions of positive and negative samples in the noise-free case. Let $\tilde{N}_p$ and $\tilde{N}_n$ be the positive
and negative samples under the noisy case. Since under symmetric noise we assume that each example has its label corrupted independently, we can think of the generation of noisy samples as follows. We toss a coin, whose probability of heads is \((1 - \eta)\), \(N\) times. The \(\tilde{N}_p\) would be \(N_p\) minus the number of tails in the first \(N_p\) plus the number of tails in the next \(N_n\) tosses. Similarly for \(\tilde{N}_n\). Hence we can bound \(\text{Prob}[\tilde{N}_p - \tilde{N}_n < 0]\) using the Hoeffding bound. The details can be worked out as follows.

Let \(X_i, i = 1, \ldots, N_p\) be random variables with \(\text{Prob}[X_i = 1] = 1 - \text{Prob}[X_i = 0] = \eta\). Let \(X_i, i = N_p + 1, \ldots, N\) be random variables with \(\text{Prob}[X_i = -1] = 1 - \text{Prob}[X_i = 0] = \eta\). Let \(S_N = \sum_{i=1}^{N} X_i\). Then, it is easy to see that \(\tilde{N}_p - \tilde{N}_n = (N_p - N_n) - 2S_N = \rho N - 2S_N\). Also, note that \(E S_N = \eta N_p - \eta N_n = \eta \rho N\). Now we have

\[
\text{Prob}[\tilde{N}_p - \tilde{N}_n < 0] = \text{Prob}[\rho N - 2S_N < 0] = \text{Prob}[2S_N - 2ES_N > \rho N(1 - 2\eta)] \leq \exp\left(-\frac{\rho^2 N(1 - 2\eta)^2}{2}\right)
\]

where the last line follows from hoeffding’s inequality. If we want this probability to be less than \(\delta\) then we would need \(N > \frac{2}{\rho^2 (1 - 2\eta)^2} \ln(\frac{1}{\delta})\). This completes the proof.

**Appendix B. Sample Complexity Bounds for Splitting Rule**

**Proof** [Of Lemma 8]

We want to bound how finite sample estimates of different impurity gain differs from the large sample assumption. We can bound each node’s positive fraction with \(\epsilon\) accuracy using Hoeffding bound. We assume split rule \(f\) divides a node \(v\). Let \(a\) be the fraction of positive class at node \(v\). Let \(p_l\) and \(q_l\) be the fraction of positive and negative class points at the left child. Similarly, let \(p_r\) and \(q_r\) be the fraction of positive and negative class points at the right child \(v_r\). Let \(p_l + q_l + p_r + q_r = 1\). Thus, \(p_l+p_r=a\) and \(q_l+q_r=1-a=b\). Here, we have two set of nodes, left having \(l \times (p_l + q_l)\) samples and the right child has \(l \times (p_r + q_r)\) samples. Under noise, number of samples in each child node does not change.

- Gini Impurity based split criterion: For a node \(v\), we can write finite sample estimate of impurity after some simplification as,

\[
|\hat{G}_{\text{Gini}}(v) - G_{\text{Gini}}(v)| \leq 2|\epsilon(a - b)|
\]

Thus we can bound finite noisy sample gain from gini impurity as,

\[
|\text{gain}_{\text{Gini}}^i(f) - \text{ain}_{\text{Gini}}^i(f)| \leq 2|\epsilon_1(a' - b')| + 2(p'_l + q'_l)|\epsilon_2(p'_l - q'_l)| + 2(p'_r + q'_r)|\epsilon_3(p'_r - q'_r)|
\]

\[
\leq 2(1 - 2\eta)[|\epsilon_1(a - b)| + (p_l + q_l)|\epsilon_2(p_l - q_l)| + (p_r + q_r)|\epsilon_3(p_r - q_r)|]
\]

\[
\leq 2(1 - 2\eta)[|\epsilon_1(a - b)| + |\epsilon_2(p_l + q_l)| + |\epsilon_3(p_r + q_r)|]
\]

\[
\leq 2(1 - 2\eta)[|\epsilon(a - b)| + |\epsilon\sqrt{p_l + q_l}| + |\epsilon\sqrt{p_r + q_r}|]
\]

\[
\leq 6(1 - 2\eta)\epsilon
\]
Where we set $\epsilon_1 = \epsilon$, $\epsilon_2 = \epsilon/\sqrt{p_l + q_l}$ and $\epsilon_3 = \epsilon/\sqrt{p_r + q_r}$. The probability can be lower bounded as, $1 - 6\exp(-2\epsilon^2)$ using hoeffding bound. Note that, this probability does not depend on any split anymore and can be applied to any arbitrary split. Under noise free case, we assume the difference of gini gain between two splits is $\rho$. Under noise corrupted signal label, large sample difference is $\rho^2 = (1 - 2\eta)^2 \rho$.

Setting $\epsilon = \rho^2/12(1 - 2\eta) = \rho(1 - 2\eta)/12$ for both the splits, we get the probability of ordering change as,

$$= 12\exp(-l\rho^2(1 - 2\eta)^2/72)$$

• Misclassification Impurity: Similarly we show for misclassification impurity. We have

$$|\hat{G}_{MC}(v) - G_{MC}(v)| \leq |\epsilon|$$

Thus we can bound finite noisy sample gain from misclassification impurity as,

$$|\text{gain}_{MC}(f) - \text{gain}_{MC}(f)| \leq |\epsilon_1| + (p_l + q_l)|\epsilon_2| + (p_r + q_r)|\epsilon_3|$$

$$\leq |\epsilon| + |\epsilon\sqrt{p_l + q_l}| + |\epsilon\sqrt{p_r + q_r}|$$

$$\leq 3\epsilon$$

Similar to gini impurity, we set $\epsilon_1, \epsilon_2, \epsilon_3$. Also if $\rho$ is the difference in gain in noise free case, under noise, difference in gain becomes, $\rho(1 - 2\eta)$. Thus we can set $\epsilon = \rho(1 - 2\eta)/6$ for both of the splits to get the probability bound.

• Twoing Rule: Similarly we bound positive fraction inside left and right node. Thus we get, after simplifying,

$$|\hat{G}_{Twoing}(f) - G_{Twoing}(f)| \leq (p_l + q_l)(p_r + q_r)(|\epsilon_2 - \epsilon_3|)(|\frac{p_l^\eta}{p_l^\eta + q_l^\eta} - \frac{p_r^\eta}{p_r^\eta + q_r^\eta}|)$$

$$\leq (p_l + q_l)(p_r + q_r)(|\frac{\epsilon}{\sqrt{p_l + q_l}}| + |\frac{\epsilon_2}{\sqrt{p_r + q_r}}|)(|\frac{p_l^\eta}{p_l^\eta + q_l^\eta} - \frac{p_r^\eta}{p_r^\eta + q_r^\eta}|)$$

$$\leq (1 - 2\eta)(|\epsilon(p_r + q_r)\sqrt{p_l + q_l}| + |\epsilon(p_l + q_l)\sqrt{p_r + q_r}|)(|\frac{p_l}{p_l + q_l} - \frac{p_r}{p_r + q_r}|)$$

$$\leq \frac{(1 - 2\eta)}{2}(|\epsilon| + |\epsilon|) \leq (1 - 2\eta)\epsilon$$

Note $\sqrt{p_l + q_l}\sqrt{p_r + q_r} \leq 1/2$ and $\epsilon_2, \epsilon_3$ are defined as before. Under noise, difference of gain becomes $(1 - 2\eta)^2 \rho$. Here we can set $\epsilon = \rho(1 - 2\eta)/2$ to get the probability of ordering change.