Reconstructing the Gravitational Lensing Potential from the Lyman-α Forest

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ABSTRACT

We demonstrate a method for reconstructing the weak lensing potential from the Lyman-α forest data. We derive an optimal estimator for the lensing potential on the sky based on the correlation between pixels in real space. This method effectively deals with irregularly spaced data, holes in the survey, missing data and inhomogeneous noise. We demonstrate an implementation of the method with simulated spectra and weak lensing. It is shown that with a source density of \( \sim 0.5 \) per square arcminutes and \( \sim 200 \) pixels in each spectrum \((\lambda/\Delta\lambda = 1300)\) the lensing potential can be reconstructed with high fidelity if the relative absorption in the spectral pixels is signal dominated. When noise dominates the measurement of the absorption in each pixel the noise in the lensing potential is higher, but for reasonable numbers of sources and noise levels and a high fidelity map the lensing potential is obtainable. The lensing estimator could also be applied to lensing of the Cosmic Microwave Background (CMB), 21 cm intensity mapping (IM) or any case in which the correlation function of the source can be accurately estimated.

Key words. cosmology: observations

1. Introduction

Weak gravitational lensing (Bartelmann & Schneider 2001; Bartelmann & Maturi 2017) describes the distortion of background sources of light by foreground matter in the regime where statistical averaging is needed to detect the effect. As any object at cosmological distances can be lensed, a variety of background sources apart from galaxies (Vald é s et al. 1983; Kaiser 1992; Troxel et al. 2018), such as radio sources (Chang et al. 2004; Harrison et al. 2016), or even already-lensed images themselves (Birrer et al. 2018) can be studied with weak lensing techniques. These examples are discrete objects and statistical measures of weak lensing (e.g., Miralda-Escude 1991), that are applied to those observations treat them as such. This includes estimators of the foreground lensing matter distribution (e.g., Kaiser & Squires 1993; Jeffrey et al. 2020). On the other hand, a continuous background source field at cosmological distances can also be lensed, and a whole class of estimators has been developed to deal with this case, starting with the Cosmic Microwave Background (CMB) as the source field (e.g., Bernardeau 1997; Metcalf & Silk 1997, 1998; Zaldarriaga & Seljak 1999; Hu & Okamoto 2002; Schaan & Ferraro 2019). In the continuous case, it is the statistical properties of the field which change as it is lensed, with the both the scale and isotropy of correlations imparting information on the foreground matter (see reviews by Lewis & Challinor 2006; Hanson et al. 2010). In the present paper we develop and test the first estimator appropriate to a particular cosmological field for which lensing has not yet been observed, the Lyman-α (Lyα) forest.

After the CMB, another continuous field which has been studied in this context is the emission from neutral hydrogen in the Universe close to the epoch of reionization, investigated using the 21 cm radio line (Madau et al. 1997; Furlanetto et al. 2006). In this case the field is in three dimensions as the use of a discrete line yields redshift information, and datacubes of line intensity are known as intensity maps (Wyithe & Morales 2007). Even at lower redshifts where galaxies act as more discrete sources of 21 cm emission, intensity maps can be made without identifying individual sources, and trace out large-scale structure more efficiently (Chang et al. 2010; Anderson et al. 2018). Weak lensing using estimators appropriate to 21 cm intensity maps have been developed (Zahn & Zaldarriaga 2006a; Metcalf & White 2007a; Pourtsidou & Metcalf 2015; Foreman et al. 2018, see also Pourtsidou & Metcalf 2014b for the low redshift case). They are generalizations of the estimators applied to the CMB, and as in that case act on a continuous lensed source field, usually employing Fourier techniques.

The neutral hydrogen density field also absorbs radiation, and the Lyα line can be used to observe it in the spectra of background galaxies and quasars (which we call “sources” here). The absorption signatures are the Lyman-α forest (Rauch 1998; Prochaska 2019), and are good tracers of the overall cosmological density field on scales larger than the gas pressure smoothing scale (of order 0.1 Mpc, Peeples et al. 2010). Clustering of the Lyα forest (see e.g. Croft et al. 1998; Hui et al. 1999; McDonald & Miralda-Escudé 1999, for some early work) is now a mainstream cosmological probe, for example having used to measure the Baryon Oscillation scale at \( z = 2-3 \) (Busca et al. 2013a; Slosar et al. 2013a; de Sainte Agathe et al. 2019), the power spectrum of mass fluctuations (e.g., Chabanier et al. 2019), and constrain the properties of dark matter (e.g., Iršič...
et al. 2017)). By including high redshift galaxies as well as quasars a higher source density can be reached at the expense of more telescope time. Lee et al. (2018) and Newman et al. (2020) have used this higher density to construct 3 dimensional maps of the neutral atomic hydrogen (HI).

The Lyα forest is well understood, and can be simulated in the context of CDM models (e.g., Bolton et al. 2017; Rogers et al. 2018; Chabanier et al. 2020). McQuinn & White (2011, 2015) have studied the problem of measuring the power spectrum from the Lyα forest. Many significant observational datasets either exist (Ahumada et al. 2019; Lee et al. 2018; Newman et al. 2020), are planned or are in progress (DESI Collaboration et al. 2016; Dalton et al. 2018; Flagey et al. 2018) The forest is also the first weak lensing probe for which data exists that has full spectroscopic information, thus bypassing the difficulties of galaxy lensing associated with photometric redshifts (Bordoloi et al. 2010; Cunha et al. 2014). As such it represents a useful source field for weak lensing, and Croft et al. (2018) and Metcalf et al. (2018) explored what will be possible, but without developing an estimator appropriate for the specific geometry of the forest.

The Lyα forest requires development of a new lensing estimator because although it samples a continuous three-dimensional field, the sampling is not uniform or space-filling. The absorption is imprinted in the spectra of sources (galaxies or quasars), which have an angular extent which is much smaller than the separation between them. The angular coverage is therefore effectively a set of delta functions, although the spectra are continuous along the line of sight (excepting masked contaminated regions). There is also noise in the pixel values in these spectra, including CCD readout noise, Poisson photon noise, flux calibration errors and sky subtraction errors (see e.g., Bautista et al. 2015), and also potentially large-scale correlated noise from uncertainty in the zero level (continuum fitting, Blomqvist et al. 2015). The angular positions of the sources are usually determined to a much smaller uncertainty (< 0.05 arcsec, Ahn et al. 2012) than the mean lensing deflection angle of a few arcmins (Lewis & Challinor 2006) however, and are not a source of noise in themselves.

We are therefore not able to apply the usual quadratic estimators (e.g., Hu & Okamoto 2002; Metcalf & White 2009) to forest lensing, as these rely on uniform coverage with a regular grid. In our previous work reconstructing the lensing potential from simulated Lyα forest sightlines (Croft et al. 2018), we worked with a toy model mock survey, with all sources on an evenly spaced grid, a situation not applicable to observational data. In practice, the angular distribution of quasar and galaxy sources is not only irregular, but can be extremely sparse, with the data sets with the forest is very irregularly sampled. The flux at observed wavelength or pixel $\lambda$ in the spectra of the qth source can be written (Busca et al. 2013b; Slosar et al. 2011, 2013b)

$$f_{q,\lambda} = C_q(\lambda) \bar{F}(\lambda) (1 + \delta_{q,\lambda}) + n_{q,\lambda},$$

(1)

where $C_q(\lambda)$ is the unabsorbed spectrum of qth source, $\bar{F}(\lambda)$ is the mean transmission in the forest at $\lambda$ and $n_{q,\lambda}$ is uncorrelated pixel noise. Other factors such as the sky background, spectrograph transmissivity and PSF are either negligible or can be absorbed into these quantities. The quantities $\delta_{q,\lambda}$ are the fluctuations in the absorption which to first order are proportional to fluctuations in the column density of HI. For the purposes of this paper however, the details of the relationship between $\delta_{q,\lambda}$ and the HI density or the dark matter density are not important. It will only be assumed that the statistical distribution of $\delta_{q,\lambda}$ is isotropic and that its correlation function can be measured or modeled.

A two parameter model for the unabsorbed spectra, $C_q(\lambda)$, the $\delta_{q,\lambda}$’s and their covariance matrix can be found using different methods, for example those of Busca et al. (2013b) and Slosar et al. (2011, 2013b), or by fitting template spectra in the case of sources that are galaxies (Newman et al. 2020; Lee et al. 2018). One method is to find the maximum likelihood solution for $\delta_{q,\lambda}$ by brute force minimization and then approximate the covariance matrix as the Fisher matrix found numerically at the maximum likelihood solution. $\bar{F}(\lambda)$ is found by averaging across all sources in the data or perhaps from another survey with higher signal to noise and higher resolution spectra (e.g., Faucher-Giguère et al. 2008). The fitting of these quantities will induce covariance between pixels that would not otherwise be there solely due to observational noise.

In this paper we will assume that the modeling of $C_q(\lambda)$ and $\bar{F}(\lambda)$ has already been done to find $\delta_{q,\lambda}$ and the covariance matrix $\Sigma$. It may also be advantageous to transform the data using a function that makes the distribution of $\delta_{q,\lambda}$ values strictly Gaussian (e.g., Croft et al. 1998), but we do not investigate that here.

3. The effect of lensing on the Lyα forest

As light passes through structures in the Universe it will be deflected from a geodesic of the unperturbed, homogeneous metric.
One way of representing this is to imagine many planes that are perpendicular to the unperturbed light ray. At each plane the ray is deflected. In the Born approximation the perpendicular position at redshift \( z \) can be written as

\[
x_{\perp}(z) = \left[ \theta D(z) - \sum_{m<z} D(z_{\perp m}, z) \hat{a}_{m}(\theta) \right] \]  

(2)

\[
x_{\perp} = \theta - \sum_{m<z} \alpha_{m}(\theta) D(z) \]  

(3)

where \( \hat{a}_{m}(\theta) \) is the deflection at the \( m \)th plane and the sum is over all planes at lower redshift (Petkova et al. 2014; Schneider et al. 1992). The angular position of the image is \( \theta \), the angular diameter distance between redshifts \( z_1 \) and \( z_2 \) is \( D(z_1, z_2) \), and \( D(z) = D(0, z) \). The usual scaled deflection angle \( \alpha_{m}(\theta) \) is obtained by rescaling all deflection angles to redshift \( z \). The total deflection for a light ray originating at redshift \( z \) is hence the sum over all lensing planes below \( z \).

\[
\alpha(\theta, z) = \sum_{m<z} \alpha_{m}(\theta). \]  

(4)

In addition for this paper we will make the approximation that all the deflections of significance take place at a redshift smaller than the redshift of the Ly\( \alpha \) absorption that we will consider. This is a good first approximation which can be relaxed when the data warrants it.

In the weak lensing limit, the deflection field is curl free so a lensing potential can be defined, the gradient of which is the deflection

\[
\alpha(\theta, z) = \nabla \phi(\theta, z). \]  

(5)

The gradient \( \nabla \) here is taken with respect to the angular coordinates \( \theta \). The potential is related to the convergence, \( \kappa(\theta) \), by a Poisson equation

\[
\nabla^2 \phi(\theta, z) = 2 \kappa(\theta, z). \]  

(6)

The convergence is the trace of the magnification matrix and in the weak lensing context can be interpreted as a weighted surface density on the sky,

\[
\kappa(\theta) = \frac{3}{2} \frac{\Omega_m H_0}{c^2} \int_{0}^{\chi} d\chi \left[ \frac{d\alpha(\chi) d\alpha(\chi, \chi_{\perp})}{d\alpha(\chi)} \right] \delta(\theta, \chi) \]  

(7)

where \( \delta(\theta, \chi) \) is the density contrast at the radial coordinate, comoving distance \( \chi, d\alpha(\chi) \) is the comoving angular size distance, \( \Omega_m \) is the matter density parameter and \( H_0 \) is the Hubble parameter.

Gravitational lensing moves the apparent position of a source on the sky. In the plane sky approximation (this approximation is relaxed in Appendix A.4) the observed absorption \( \delta_{\alpha}(x) \) at the radial position \( r_\perp \) and perpendicular position \( r_\perp \) is, to first order,

\[
\delta_{\alpha}(x_\perp, x_\perp) = \delta(x_\perp, x_\perp - D_\alpha(\theta)) = \delta(x) - D_\alpha(\theta) \cdot \nabla_\perp \delta(x) + \ldots \]  

\[
= \delta(x) - D \nabla \phi(\theta) \cdot \nabla_\perp \delta(x) + \ldots \]  

(8)

where \( \delta(\theta) \) is the intrinsic absorption, as it would have appeared without lensing. To first order in the deflection, the observed absorption correlation function will be changed by

\[
\langle \delta_{\alpha}(x) \delta_{\alpha}(x') \rangle = \langle \delta(x) \delta(x') \rangle - D_\alpha(\theta) \cdot \nabla_\perp \langle \delta(x) \delta(x') \rangle \]  

(9)

\[
= \xi(r) + [D \alpha(\theta) - D_\alpha(\theta')] \cdot \nabla_\perp \xi(r_\perp, r_\perp) \]  

(10)

\[
= \xi(r) + [D \alpha(\theta') - D_\alpha(\theta')] \cdot \nabla_\perp \xi(r_\perp, r_\perp) \]  

(11)

\[
= \xi(r) + [d(z) \alpha(\theta) - d(z') \alpha(\theta')] \cdot \gamma \begin{pmatrix} D_\perp \partial_\perp \xi \end{pmatrix} \]  

(12)

\[
= \xi(r) + [d(z) \alpha(\theta) - d(z') \alpha(\theta')] \cdot \gamma \xi \]  

(13)

\[
= \xi(r) + [d(z) \alpha(\theta) - d(z') \alpha(\theta')] \cdot \gamma \xi(r_\perp, r_\perp) \]  

(14)

where \( r = (|x_{\perp} - x_{\perp}||x_{\perp} - x_{\perp}'|) \), \( r_\perp \leq D_\perp, D_\perp' \). This approximation can be relaxed if necessary, but is probably a good one. Line (14) serves to define the function \( \xi(D, r_\perp) \). We have written the intrinsic correlation function as a function of both \( r_\perp \) and \( r_\perp \), to allow for redshift distortions caused by the radial velocities of the II.

Our proposed method is based on modelling the deflection field \( \phi(\theta) \) in a form that is linear in the coefficients \( \phi_\ell \) for a set of basis functions \( f_\ell(\theta) \),

\[
\phi(\theta) = \sum_\ell \phi_\ell f_\ell(\theta). \]  

(15)

The deflection field \( \alpha = \nabla \phi \) at angular position \( \theta \) is then related to the model parameters \( \phi_\ell \) as

\[
\alpha^\ell(\theta) = \frac{\partial}{\partial \phi_\ell} \phi(\theta) = \sum_\ell \mathcal{R}^\ell \phi_\ell. \]  

(16)

The noise correlation matrix \( \mathcal{N}_{ij} \), the correlation (14) between pixels \( \delta_i \) and \( \delta_j \) can be written

\[
\langle \delta_i \delta_j \rangle = \xi(r_{ij}) + \mathcal{N}_{ij} + \sum_\ell P_{ij}^\ell \phi_\ell \]  

(17)

\[
= C_{ij} + \sum_\ell P_{ij}^\ell \phi_\ell \]  

(18)

where

\[
P_{ij}^\ell = \sum_r (\mathcal{R}^\ell_r - \mathcal{R}^\ell_r) \gamma_r \xi(D, r_{ij}). \]  

(19)

and \( \mathbf{C} \) is the correlation matrix between pixels including intrinsic correlations, noise and correlations that come from the modeling procedure outlined in section 2. The \( \mathcal{P}^\ell \) matrix will differ depending on the model for the potential and on how its derivatives are estimated.
For example, if the potential is modelled by a discrete Fourier transform (DFT) with coefficients \( \hat{\phi}_\ell \) and \( \ell = (\ell_1, \ell_2) \), we can compute the deflection field, \( \alpha(\theta) = \nabla \phi(\theta) = \sum_\ell \hat{\phi}_\ell \mathbf{e}^{i \ell \cdot \theta} \), and the \( \alpha \) matrices (19),

\[
P^{ij}_\ell = 2 \left( \ell \cdot \theta_{ij} \right) \sin \left( \frac{\ell \cdot \theta_{ij}}{2} \right) e^{-i \ell \cdot \theta_{ij}} G(D, r_{ij}),
\]

where \( \theta_{ij} = \theta_i - \theta_j \) and \( \theta_{ij} = (\theta_j + \theta_i)/2 \).

More details for the implementation of this and other expansions are given in Appendix A. We use the Legendre expansion, section A.1, for our numerical demonstrations for reasons that will become clear later. We have also implemented and tested a bilinear, finite difference expansion on a grid and a Chebychev expansion, section A.2. The advantage of the Legendre and Chebyshev expansions are that they do not impose any boundary conditions and, unlike the bilinear case, they have continuous derivatives so that the deflection field is continuous.

Note that two pixels \((i, j)\) in the spectrum of a single source will have the same angular position and so \( P^{ij}_\ell \) will be zero. Two pixels from different spectra will have uncorrelated noise, so \( N_{ij} \) will be zero, except perhaps through the estimate (1) of \( \hat{F}(i) \). The intrinsic correlation between the absorption must be isotropic in angle, so \( \xi(r_{ij}) \) will be a function of only the absolute angular separation between the pixels and their radial separation.

4. Quadratic Estimator

We can express equation (18) in matrix form

\[
(\Delta) = \mathbf{C} + \mathbf{P} \hat{\phi},
\]

with the definition

\[
[\Delta]_{i,j} \equiv \delta_i \delta_j.
\]

Assuming that the \( \delta \)-field is Gaussian the Fisher matrix for the parameters \( \hat{\phi} \) evaluated at \( \hat{\phi} = 0 \) is

\[
F^{\mu\nu} = \frac{\partial^2 \ln \mathcal{L}}{\partial \hat{\phi}_\mu \partial \hat{\phi}_\nu} = \frac{1}{2} \text{tr} \left[ \mathbf{P} \mathbf{C}^{-1} \mathbf{P}^{\nu} \mathbf{C}^{-1} \right].
\]

This is the standard result for a multivariate Gaussian and easily derived.

Now consider the quantity

\[
\left\{ \mathbf{C}^{-1} \mathbf{P}^{\nu} \mathbf{C}^{-1} \right\} = \left\{ \text{tr} \left[ \mathbf{C}^{-1} \mathbf{P}^{\nu} \mathbf{C}^{-1} \right] \right\}
\]

\[
= \text{tr} \left[ \mathbf{C}^{-1} \mathbf{P}^{\nu} \mathbf{C}^{-1} (\mathbf{C} + \mathbf{P} \hat{\phi}_0) \right]
\]

\[
= \text{tr} \left[ \mathbf{C}^{-1} \mathbf{P}^{\nu} \mathbf{C}^{-1} \right] + \text{tr} \left[ \mathbf{C}^{-1} \mathbf{P}^{\nu} \mathbf{C}^{-1} \mathbf{P} \hat{\phi}_0 \right]
\]

\[
= \text{tr} \left[ \mathbf{C}^{-1} \mathbf{P}^{\nu} \right] + 2 \text{tr} \mathbf{P}^{\nu} \hat{\phi}_0
\]

where (24) and the invariance of the trace under cyclic permutations was used. We can therefore construct the unbiased estimator that can be written in several different forms

\[
\hat{\phi}_\mu = \frac{1}{2} \mathbf{F}^{\mu\nu} \left( \mathbf{C}^{-1} \mathbf{P}^{\nu} \right) \mathbf{C}^{-1} \hat{\phi}_0
\]

\[
= \frac{1}{2} \mathbf{F}^{\mu\nu} \left( \text{tr} [\mathbf{C}^{-1} \mathbf{P}^{\nu} \mathbf{C}^{-1}] - \text{tr} [\mathbf{C}^{-1} \mathbf{P}^{\nu}] \right)
\]

\[
= \frac{1}{2} \mathbf{F}^{\mu\nu} \left( \mathbf{C}^{-1} \mathbf{P}^{\nu} (\mathbf{C}^{-1} \mathbf{A} - \mathbf{I}) \right)
\]

\[
= \frac{1}{2} \mathbf{F}^{\mu\nu} \left( \mathbf{C}^{-1} \mathbf{P}^{\nu} \mathbf{A} - \text{tr} [\mathbf{C}^{-1} \mathbf{P}^{\nu}] \right)
\]

where \( \mathbf{z} = \mathbf{C}^{-1} \hat{\phi}_0 \)

The covariance of this estimator is

\[
\left\{ \hat{\phi}_\mu \hat{\phi}_\nu \right\} = \frac{1}{4} \mathbf{F}^{-1} \mathbf{F}^{\mu\nu} \left( \mathbf{C}^{-1} \mathbf{P}^{\nu} \mathbf{C}^{-1} \mathbf{P}^{\mu} \mathbf{C}^{-1} \right) + \left( \Delta_{\mu \nu} \Delta_{\mu \nu} \right)
\]

\[
= \text{tr} \left[ \mathbf{C}^{-1} \mathbf{P}^{\mu} \mathbf{C}^{-1} \mathbf{P}^{\nu} \right]
\]

Now we will assume that the distribution of the pixel values pre-lensing is Gaussian. This allows us to calculate the fourth order moment

\[
\left\{ \Delta_{\mu \nu} \Delta_{\mu \nu} \right\} = \left( \delta_{\mu \nu} \delta_{\mu \nu} \right) C_{\mu \nu} C_{\mu \nu} + \left( \delta_{\mu \nu} \delta_{\mu \nu} \right) C_{\mu \nu} C_{\mu \nu}
\]

(35)

Plugging this into (33) gives

\[
\left\{ \hat{\phi}_\mu \hat{\phi}_\nu \right\} = F_{\alpha \beta}^{-1}
\]

(36)

This is the Cramér-Rao lower limit on the covariance of an unbiased estimator so estimator (29) is an efficient estimator, i.e. the best that can be done given the assumptions.

In the above we have made the approximation that there is just one \( \phi(\theta) \) (or \( \alpha(\theta) \)) field for all pixels. The deflection field should be a function of the pixel’s redshift because of the distance factors and because some lensing of the high redshift pixels will be caused by structures that are at the redshift of the lower redshift pixels (we call this self-lensing). The above formalism can be easily expanded so that there are a series of potential fields, \( \phi(z, z') \), so that the range of angular size distances for the pixels is explicitly taken into account. When the data can support more free parameters a slightly more complicated treatment is needed. For now we will consider the \( \phi(\theta) \) field to be averaged over the redshifts of the Ly\( \alpha \) forest.

To calculate the estimator (29) one needs to find the inverse of the covariance matrix \( \mathbf{C} \). The data vector \( \mathbf{z} = \mathbf{C}^{-1} \hat{\phi} \) could be calculated by solving \( \mathbf{Cz} = \hat{\phi} \) without finding \( \mathbf{C}^{-1} \), but it is still needed to calculate the Fisher matrix and bias term. This can be very numerically burdensome in realistic cases.

4.1. Unconstrained Modes & Degeneracies

Only differences in the deflection between points appear in the estimator so the potential can only be constrained up to an additive constant and a constant gradient. Physically the gradient is a uniform shift in the sky which does not change the relative positions of the sources.

In addition to these degeneracies the data may not constrain some other properties of our reconstruction depending on the expansion and the distribution of sources on the sky. For example, if the parameters of the expansion each have a limited range of influence on the deflection field, as in a gridded representation of the potential, then regions that have no data in them will be unconstrained, i.e. holes in the map or regions off the edge of the survey. In the extreme case where the number of model parameters exceeds the number of pairs so that the problem is under determined there will clearly be combinations of parameters that are unconstrained, but it is not necessary for the number of parameters to be very high to have some unconstrained combinations. These unconstrained combinations correspond to eigenvectors of \( \mathbf{F} \) that have zero eigenvalues so they can be identified.

The physical degeneracies (the offset and gradient) do not necessarily correspond perfectly to some combination of the expansion parameters. For example the gradient over a small field cannot be expressed as a finite sum of discrete Fourier modes or spherical harmonics. In these expansions the above procedure
will not remove the gradient. In these cases large artificial gradients can appear if they are not removed in a separate step. In other expansions, such as the Legendre expansion used later, the three degenerate modes are easily identified and removed.

Treating the deflection as a potential field necessarily removes the curl or B-mode component of the deflection field. This component is expected to be quite small in the weak lensing regime and so we will ignore it here. If the B-modes ever become of interest, they could be reconstructed by solving for the scalar potential and the pseudo-scalar B-mode potential (Stebbins 1996) in a very similar way to what is done here.

4.2. Choice of basis

Since the individual correlations between spectra give limited information on the deflection field it makes sense to try to reconstruct a smoothed version of the deflection field. Using a grid with a resolution higher than the density of sources would clearly make the problem under determined. As shown above, one possible basis is that of discrete Fourier modes. The appearance of an artificial gradient as discussed in section 4.1 is a significant drawback to this basis.

It is also possible to use the spherical harmonic coefficients as free parameters (appendix A.4). This is probably not advantageous unless the data covers almost all of the sky. Otherwise there will be many unconstrained combinations of spherical harmonic modes, making it difficult to obtain a solution for the potential field. This approach might be advantageous for recovering the power spectrum of the potential, but this will not be considered in this paper.

In what follows we use the Legendre expansion in a square patch on the sky (appendix A.1). This has the advantages that the degenerate modes can be easily removed, the deflection is continuous, there is no implicit periodicity and the expansion can be cut off at a particular angular scale as with a DFT. The Legendre polynomial of order $n$ has $n$ zeros so the effective angular scale for a mode is the field size divided by $n$.

4.3. Sparse case & approximation

If the sources are clumped into groups that have relatively little cross-correlation in the absorption and noise between them, the computational cost of the quadratic estimator can be significantly reduced. In this case the covariance matrix is block-diagonal

$$
C = \begin{pmatrix}
C^{(1)} & 0 & \cdots & 0 \\
0 & C^{(2)} & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & C^{(n)}
\end{pmatrix}
$$

(37)

where $C^{(k)}$ is the covariance matrix for the sources in the $k$th of $n$ groups. Likewise

$$
P^\dagger = \begin{pmatrix}
P^{(1)}_1 \\
P^{(1)}_2 \\
\vdots \\
P^{(n)}_n
\end{pmatrix}
$$

(38)

The Fisher matrix (24) becomes

$$
F^{\alpha\beta} = \frac{1}{2} \sum_{k=1}^{n} \text{tr} \left[ P^{(k)} C^{(k)} \left( P^{(k)} \right)^\dagger C^{(k)} \right] = \sum_{k=1}^{n} F^{\alpha\beta}_{(k)}
$$

(39)

and the estimator (29) becomes

$$
\hat{\phi}_\mu = \frac{1}{2} F^{\mu\nu} \sum_{k=1}^{n} \text{tr} \left[ C^{(k)} P^{(k)} (C^{(k)} \Delta_\mu - I) \right] = \frac{1}{2} F^{\mu\nu} \sum_{k=1}^{n} \hat{\phi}^k_\nu.
$$

(40)

(41)

Calculating this estimator is faster by a factor of $\sim 1/n^2$, uses less memory by a factor of $\sim 1/n^2$ if each term is calculated in series and has less numerical error than in the general case if the sources are evenly distributed among the groups, but if there is significant cross-correlation between the groups the estimator will be less than optimal.

The groupings could be in positions on the sky and/or in redshift bins. To reduce computational cost the data could be broken up into redshift bins, the quantities $F^{\mu\nu}_{(k)}$ and $\hat{\phi}^k_\nu$ calculated, and then combined with equation (41) to get an overall estimate of the potential. This ignores correlations between bins, both intrinsic and in the noise, but might not entail a significant reduction in the signal-to-noise.

5. Simulation method

To simulate the observations and test out implementation of the estimator we need to simulate the absorption and the lensing potential. Here we describe how we carry this out.

5.1. Simulating the Ly$\alpha$ forest

We have used two methods for simulating the absorption field. The first is to go directly from the covariance matrix $C$ for the pixels to a simulated realization. The correlation matrix is calculated using the methods discussed in appendix C. The Cholesky decomposition of the covariance matrix is

$$
C = LL^T
$$

(42)

where $L$ is a lower triangular matrix. This decomposition is done using the linear algebra package Eigen\(^1\).

We generate $n$ normally distributed numbers:

$$
\delta_i = Lx_i
$$

(43)

where the $x_i$’s are uncorrelated normally distributed numbers with variance 1, white noise. This data set, $\delta_i$, will have the desired covariance and be statistically equivalent to a sample taken from a Gaussian random field.

Another method is to directly simulate a Gaussian random field in 3D and then sample the field at the locations of the pixels. This would require a much larger array of simulated pixels than data voxels since it must resolve the pixels and contain enough modes to fully represent the fluctuations on the scale of the voxels. This is very computationally expensive and will give the same results to the extent that the absorption field is Gaussian random.

5.2. Simulating the lensing potential

We simulate the lensing potential by generating a two-dimensional Gaussian random field that is several times as large as the intended field size using the standard method in Fourier space.

\(^1\) http://eigen.tuxfamily.org
The field is then cropped to the desired size. This avoids imposing periodic boundary conditions and includes Fourier modes that are large than the field size. The resolution of the cropped map is $512 \times 512$. The deflection is found by finite difference method in configuration space. The power spectrum for the lensing potential is calculated with the CAMB (Lewis et al. 2000)\textsuperscript{2} and CosmoSIS (Zuntz et al. 2015)\textsuperscript{3} software. As discussed already, the constant and linear modes are not measurable so these are subtracted from the simulated fields before comparing them with the reconstructed field.

The parameters that are actually measured are the Legendre coefficients so the expected signal can be characterized by finding the standard deviation of these coefficients. This is done by simulating fields and calculating their coefficients by numerical integration. Figure 1 shows the standard deviation of the first 13 Legendre coefficients of 1000 simulations as a function of field size excluding the degenerate modes - $(0,0)$, $(1,0)$ and $(0,1)$.

![Fig. 1. The standard deviation of the Legendre lensing potential modes (appendix A.1) as a function of field size. Each standard deviation is estimated using 1000 simulated potential fields.](https://bitbucket.org/joezuntz/cosmosis)

6. Demonstrations

To test how well the algorithm can reconstruct the lensing potential we carried out a series of simulations. The source image positions are chosen randomly within the field. The lensing potential is simulated as described in section 5.2 and the deflection at each of the source positions is calculated. The spectroscopic pixels are displaced according to the deflection and a realization of the absorption in each pixel is created as described in section 5.1 and appendix C. Then the estimator is applied to these pixels to recover the Legendre coefficients of the lensing potential. An image of the reconstructed potential is created using these coefficients that can be compared to the original. This is done many times with new realizations of the Lyα forest to calculate the variance in the recovered coefficients.

The parameters that characterize each simulation run are: 1) the angular size of the field over which the potential is reconstructed, 2) the pixel length which is the length of the spectroscopic bin that is used to measure $\delta$ (this could be the spectrograph’s pixel size or an average over multiple pixels, but we will still call it a “pixel” for simplicity.) This length is expressed in comoving Mpc using the the conversion appropriate to the background cosmology,) 3) the number of sources within the field 4) number of pixels in each spectrum (in real data this will vary among sources, but for the simulations it is the same for all sources,) 5) the redshift of the lowest redshift pixel in each spectrum, which is taken to be the same for all sources, 6) the noise in each pixel, $\sigma_\delta$, which for these tests we take to be uncorrelated between pixels, 7) the number of coefficients in the expansion of the potential that are used.

Spectra used for studying the Lyα forest typically consist of hundreds of pixels and the number of sources we expect to be in the hundreds to tens of thousands. This results in $\sim 10^4$ to $\sim 10^6$ pixel pairs. This makes the storage and manipulation of the matrices involved difficult. Fortunately, as we will show in section 6.1, the correlation between pixels at different redshifts (given the adopted pixel lengths) are small and so, for the purposes of these simulations, they can be considered independent. Using this property, we reconstruct the lensing potential using only two consecutive pixels in each spectrum. These pixels are at the same redshifts for all the sources. The Lyα simulation within this redshift slice contains all the correct correlations between pixels. The optimal way to combine statistically independent redshift slices is to simply average them (see section 4.3).

Using this property, we find the final potential reconstruction by averaging the reconstructed lens maps from 100 redshift slices, each with different realizations of the forest. The result should have the same noise properties as the case with 200 pixels in each of the spectra. We do not simulate gaps in the spectra caused by masking, but this poses no fundamental problem. This procedure greatly reduces the computational cost.

Table 1 gives the parameter values for each of the simulations that will be discussed and some measures of the fidelity of the reconstruction are given in the last four columns. The signal-to-noise, $S/N$, is calculated by dividing the expected variance in each coefficient due to signal (from Monte Carlo as in figure 1) and by the variance expected for the estimator. This is done for each mode and the range is given in columns 9 and 10.

The $\chi^2/n$ column in table 1 is calculated for all of the coefficients (22 of them) that are measured for the particular random realization simulated with the null hypothesis being that the data is only the expected noise without any lensing signal. The last column of of table is the p-value for the $\chi^2$ in the previous column. This p-value represents the probability of getting a $\chi^2$ larger than what is measured if there were no lensing signal. Small values signify stronger detections.

Figure 2a shows the input and reconstructed potential fields for case AA, 5000 sources in one square degree. The initial potential field has structure in it on a smaller scale than is represented by the finite Legendre series that is used in the reconstruction. In the central panel of Figure 2a (and the others like it) we show the potential constructed by calculating only the Legendre coefficients of the input field that we are trying to estimate and filtering the others out. It can be seen that it lacks some of the detail of the original input fields. The right hand panel of Figure 2a (and the others like it) is the reconstructed field using the estimated Legendre coefficients. By chance this field has relatively mild structure in the potential. (No attempt has been made to select better looking cases.) Never the less, in the AA case the reconstructed image broadly reproduces many of the features seen in the input field.

Figure 3a shows the results for case AA more quantitatively. The upper panel shows predictions for the average signal and noise in each Legendre coefficient. It can be seen there that the analytic predictions given in section 4 for the variance of the estimator agree well with the variances estimated from the simulations. It can also be seen that the noise in this case is well below

\textsuperscript{2} http://camb.info

\textsuperscript{3} https://bitbucket.org/joezuntz/cosmosis
In the lower panel of figure 3a are the actual input and output values for the particular potential field used in the simulation. The p-value in table 1 is very small indicating the a strong detection would be expected in this case.

Simulation run DD is shown in figure 2b and 3b. This case is for a smaller area (0.5 x 0.5 deg) with a higher source density as compared to run AA. The potential seems better recovered than in the AA run, but this appears to be only because the structure in the potential happens to be more robust. The upper panel of 3b shows that the signal-to-noise would typically be smaller in this case than for AA although still above one. Note that the scale in figure 3b is different than that in 3a. The expected fluctuations on a smaller field are smaller. The $\chi^2/n = 5.9$ in this case which gives a p-value that is too small to calculate accurately meaning that the detection would be very strong.

Simulation run EE is for the same size field as DD, but with fewer sources, 500 compared to 2000. Figure 3c shows that the signal-to-noise is smaller, but still apparently good enough to recover the major features in the field shown in figure 2c and yields a very small p-value. Run FF is the same, but with even fewer sources, 200. Here it seems that the potential cannot be accurately recovered (figures 2d and 3d). The $\chi^2/n = 1.0$, consistent with noise.

Run CC is the same as AA, but with one fifth the number of sources. From figure 2e it appears that a local peak in the potential is being detected, but not much more detail is present. Figure 3e confirms the impression that the signal-to-noise is around one or a bit smaller for the individual coefficients. The p-value very small indicating that a signal is strongly detected.

Run BB is with a larger field than AA and the same number of sources. Figures 2f and 3f, and the $\chi^2$ show that we should not expect to be able to reconstruct the potential with source densities this low despite the variance of the Legendre coefficients being larger on this scale than it is for smaller fields.

In the first six simulation runs listed in table 1 the pixel noise was taken to be zero. This represents the optimistic case were the intrinsic variation in the absorption in each pixel is larger than than the noise, signal dominated. This is not the case for the majority of sources in current surveys that reach the kind of source densities we are considering. For example CLAMATO (Lee et al. 2018) and LATIS (Newman et al. 2020) both have median pixel noise $\sigma_p \sim 0.58$ which makes them noise dominated for individual pixels ($\sigma_{\text{igm}} \sim 0.28$ in our case) although the bright quasar sources are signal dominated. For comparison LATIS expects to have a source density of 2,000 deg$^{-2}$ over 1.7 deg$^2$ and CLAMATO about half the density over 1 deg$^2$.

In the final simulations we added pixel noise that is taken to be constant and uncorrelated. In this case $N = \sigma_p^2 I$. Run GG has 2000 deg$^{-2}$ and $\sigma_p = 0.6$. The results are shown in figures 2g and 3g. The signal takes a significant hit and although some structure does seem to be recovered the p-value is 0.1 which signifies a marginal detection. The noise was reduced somewhat in run JJ to $\sigma_p = 0.5$ (figures 2h and 3h). The image of the potential is better recovered, but the signal-to-noise for the individual coefficients is always below one. The $\chi^2$ is larger than for GG probably due to chance.

Run JJ (figures 2i and 3i) has $\sigma_p = 0.6$, but 2.5 times higher source density than GG. The fidelity is much improved and the p-value quite low. Finally, run KK (figures 2j and 3j) has smaller spectral pixels ($L_{\text{pix}} = 1$ Mpc), but the same total range in wavelength as the other runs. The other parameters are as in case GG. The fidelity is similarly to GG indicating that the signal-to-noise is not sensitive to $L_{\text{pix}}$, at least in this regime.

Figure 4 shows the normalized covariance between estimated Legendre coefficients in case AA. These are estimated from the repeated simulations with random Ly$\alpha$ forests. It can be seen here that the estimates of the different coefficients are not strongly correlated and thus can be considered independent measurements. This plot looks very similar for all the other cases.

### 6.1. Noise scaling

Figures 5, 6 and 7 show the standard deviation of the estimator as a function of number of spectral pixels per source, size of the field on the sky and number of sources respectively. These are calculated directly from the expected error given in Section 4. It can be seen in Figure 5 that the variance scales as $N^{-1}$, indicating that each map of the Ly$\alpha$ forest at a constant redshift constitutes essentially an independent measurement of the lensing potential, at least for a pixel length of 2 Mpc. This means that a calculation of the noise with one or two pixels per source can be easily scaled to find the noise for more pixels and that the potential estimates can be "stacked" as was done in the previous section (Note that Section 4.3 shows that the optimal weighting is not the simple average of the estimates except in the cases where the noise, pixels positions and intrinsic correlations are identical in each redshift bin.)
Fig. 2. Images of the lensing potential from the simulations of Table 1. The left panels of each case show the random realization of the input lensing potential with the mean and gradient subtracted. The centre panels show only the reconstructed modes of the input field. The right panels show the reconstructed field using 100 (or, for KK, 200) stacked redshift layers.
Fig. 3. In the top panel of each case are the standard deviations of the Legendre modes of the lensing potential and the expected errors in the estimator. The blue bars are the variance of the coefficients calculated from a thousand realizations of the lensing potential. The orange dots in the top panels are the expected standard deviation of the estimate of the coefficients according to the analytic formula (36). The smaller red dots are the variance among the 100 simulations. They are in good agreement in all cases. The green dots are the errors expected if 100 (200 for case KK) sets of the simulation are stacked as if they are at different redshifts. By comparing the green dots to the blue bars some idea of the signal-to-noise can be gained. In this case the noise levels for 200 pixel spectra are below the typical signal values expected. In the lower panels are the coefficients for the particular realization used in the simulation (blue bars) and the estimated values of those coefficients after stacking 100 (200 for case KK) realizations of the Ly$\alpha$ forest. (orange dots). The modes are listed without the ones that are not measured ($mn = 00, 10$ and $01$).
Fig. 3. Continued

Fig. 4. The normalized covariance between the coefficient estimates in simulation run AA (table 1). The estimates are consistent with being uncorrelated.

Fig. 5. The standard deviation of the Legendre lensing potential modes (appendix A.1) computed using the estimator as a function of the number of pixels in each of the spectra for a 0.5 deg x 0.5 deg field with 500 sources.
The standard deviation of the Legendre lensing potential modes (appendix A.1) computed using the estimator as a function of the width of a square field with 1000 sources and a depth of 200 pixels.

Figure 6 shows that the scaling with the size of the field is not quite a power-law. Nevertheless the error in each coefficient scales approximately as

$$\sigma_{nm} \approx N_{\text{spec}}^{\gamma} L^\beta N_{\text{source}}^{\alpha}$$

where $\gamma \approx -0.5$, $\beta \approx 2.6$ and $\alpha \sim 0.8$ for the modes and ranges of parameters that we have investigated.

7. Sensitivity to assumption about the correlation function

It should be noted that the quadratic estimator contains the correlation function of the Ly$\alpha$ forest which must be estimated in some way and thus might be in error. If the assumed covariance $C$ is incorrect the estimator will be biased. In the simplest case where the assumed correlation function is off by a normalization, $C = \alpha \langle \Delta \rangle$ where $\alpha$ is a constant, the average of the estimator will be

$$\langle \delta \phi \rangle = \alpha \phi + \frac{(a - 1)}{2} F_{\mu}^{-1} \text{tr} \left[ C^{-1} \mathbf{p}^\nu \right]$$

so that there will be an erroneous scaling and bias. Given that the correlation function can be assumed to be isotropic and a smooth function of distance it seems reasonable that it can be constrained well enough to allow for a good measurement of the lensing. However this is a problem that has not yet been investigated thoroughly.

8. Discussion

We have derived a minimum variance estimator for the gravitational lensing potential using the Ly$\alpha$ forest and found that it can recover an image of the the potential on ~ 1 degree scales if the source density is high and the pixel noise is low. We carried out simulations that showed that our implementation functions as expected. At the source densities and noise levels currently available (Lee et al. 2018; Newman et al. 2020) the signal-to-noise in the recovered map is expected to be marginal, but with improvements in either of these a high fidelity map could be recovered.

The ELT (Extremely Large Telescope) should be able to reach a magnitude fainter than CLAMATO or LATIS which will mean source densities that are an order of magnitude greater ($\eta \sim 10^3 \text{deg}^{-2}$) and a reduction in the pixel noise to point where it is subdominant for a much larger fraction of the pixels (Ellis & Dawson 2019; Schlegel et al. 2019). This should be a better case than even our most optimistic simulations so high fidelity lensing maps should be possible.

The estimator proposed here is computationally expensive. The storage is $\propto N_{\text{pix}}^2$ and the computational time $\propto N_{\text{pix}}^3$. This could pose a limitation, but we believe this can be significantly reduced through some methods in development. A remaining question is whether the correlation function of the Ly$\alpha$ forest, that is needed for the estimator, is or can be determined to high enough accuracy to not bias the lensing potential estimate. We do not believe that this possess a significant problem in the long run, but it has still to be determined precisely how well this can be done.

No attempt has been made here to recover the lensing power spectrum from the Ly$\alpha$ forest. For the small field surveys studied here any such estimate would be dominated by sample variance, but it might be possible for larger, sparser surveys (see Metcalf et al. 2017). Estimating the power spectrum will require a different approach and will be the subject of a future paper.

The method used by the Planck collaboration to map the lensing using the CMB is essentially to Wiener filter the temperature and polarization maps to fill in masked regions of the sky and then use an estimator based in spherical harmonic space (Planck Collaboration et al. 2014, 2016). This is not an optimal method and is not applicable to the Ly$\alpha$ forest because any map of the sky that took full advantage of the accuracy of the source positions would have such high resolution and so few pixels containing sources that it would not be practical. Some effort has been made to develop a real-space lensing estimator for the CMB (Bucher et al. 2012). Our method could be applied to the CMB by treating each unmasked pixel as a source and it would take into account window functions and inhomogeneous noise in an optimal way. For the whole Planck map this might be computationally onerous, but it is certainly possible for smaller CMB surveys in its present form.

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Appendix A: Expansions

In this appendix are the details for several possible expansions or parameterizations of the lensing potential.

Appendix A.1: Legendre expansion

The two dimensional Legendre expansion of the potential will be

$$\phi(\theta) = \sum_{m=0}^{N_x} \sum_{n=0}^{N_y} a_{mn} F_{nm}(\theta)$$

(A.1)

where

$$F_{nm}(\theta) = P_n(x)P_m(y)$$

(A.2)

are the Legendre polynomials and

$$x \equiv \frac{2\theta_1 - \theta_0}{\Delta \theta_x} - 1 \quad y \equiv \frac{2\theta_2 - \theta_0^2}{\Delta \theta_y} - 1$$

(A.3)

for the range $-1 \leq x \leq 1$. The derivative of them is given by

$$\frac{d}{dx} T_n(x) = n U_{n-1}(x)$$

(A.9)

where $U_n(x)$ are the Chebyshev polynomials of the second kind are given by

$$\begin{align*}
U_0(x) &= 1 \\
U_1(x) &= 2x \\
U_{n+1}(x) &= 2x U_n(x) - U_{n-1}(x)
\end{align*}$$

(A.10)

The two dimensional Chebyshev expansion is defined like in the Legendre case.

$$\phi(\theta) = \sum_{m=0}^{N_x} \sum_{n=0}^{N_y} a_{mn} F_{mn}(\theta)$$

(A.11)

where

$$F_{mn}(\theta) = T_m(x)T_n(y).$$

(A.12)

Using (A.9), the deflection is

$$\alpha(\theta) = \nabla \phi(\theta) = \sum_{m=0}^{N_x} \sum_{n=0}^{N_y} 2a_{mn} \left( \frac{m U_{m-1}(x) \Delta \theta_x}{n T_m(x) U_{n-1}(y) \Delta \theta_y} \right)$$

(A.13)

Using the orthogonality and normalization of these polynomials we can find the coefficients given a potential field

$$a_{mn} = e_m e_n \int_{-1}^{1} dx \int_{-1}^{1} dy \frac{\phi(x,y) T_m(x) T_n(y)}{\sqrt{1 - x^2} \sqrt{1 - y^2}}$$

(A.14)

$$\epsilon_i = \frac{1}{\pi} \left\{ \begin{array}{ll} 1 & i = 0 \\
2 & i > 0 \end{array} \right.$$

(A.15)

As in the Legendre case, modes $a_{00}$, $a_{01}$ and $a_{10}$ are not measurable because they represent a constant potential and two linear, or constant deflection, components.

Appendix A.3: Discrete Fourier Transform

The potential field in the flat-sky approximation can be expanded as a $N_1 \times N_2$ pixel map $\phi(n_1, n_2) 0 \leq n_1 \leq N_1 - 1$, using the Discrete Fourier Transform (DFT),

$$\phi[n] = \sum_{k_{1}=0}^{N_1-1} \sum_{k_{2}=0}^{N_2-1} \hat{\phi}_k e^{2\pi i n_1 k_1 / N_1 + 2\pi i n_2 k_2 / N_2},$$

(A.16)

where $0 \leq n < N$, $0 \leq k < N$ are multi-indices, and $\hat{\phi}_k$ are the $N_1 \times N_2$ complex-valued modes of the expansion. The discrete field values correspond to the continuous field at the grid points $\theta_i \equiv n \Delta$, where $\Delta$ is the pixel size of the discrete map. We can interpolate the DFT (A.16) to obtain a continuous potential field over the field of view,

$$\phi(\theta) = \sum_{k_{1}=0}^{N_1-1} \sum_{k_{2}=0}^{N_2-1} \hat{\phi}_k e^{i \theta k},$$

(A.17)

where the frequencies $k = (l_1, l_2)$ are chosen to produce a real-valued interpolation,

$$l_k \equiv \frac{2\pi}{N \Delta} \begin{cases} k_1, & 2k_1 < N_1, \\
2k_2 > N_2, & 0, \end{cases}$$

(A.18)
where the last equality now expresses everything in terms of the truncated quantities. Expanding \( \mathbf{P}' \) and \( \mathbf{\phi}' \) into real and imaginary parts \( \mathbf{P}' = \mathbf{R} + i \mathbf{Q} \) and \( \mathbf{\phi}' = \mathbf{u} + i \mathbf{v} \), the relation is evidently still linear in \( \mathbf{u} \) and \( \mathbf{v} \).

\[
\langle \delta(\mathbf{\theta}) \rangle = \langle \delta(\mathbf{\theta}) \rangle \mathbf{u} + i \langle \delta(\mathbf{\theta}) \rangle \mathbf{v} = \delta(\mathbf{\theta}) + \alpha(\mathbf{\theta}) \cdot \mathbf{\nabla} \delta(\mathbf{\theta})
\]

Averaging over the ensemble of \( \langle \delta(\mathbf{\theta}) \rangle \) fields gives

\[
\langle \delta(\mathbf{\theta}) \rangle \delta(\mathbf{\theta}) \rangle = \langle \delta(\mathbf{\theta}) \rangle \delta(\mathbf{\theta}) \rangle + \mathbf{\nabla} \delta(\mathbf{\theta}) \rangle + \mathbf{\nabla} \delta(\mathbf{\theta}) \rangle
\]

The potential will be expanded in spherical harmonics

\[
\phi(\mathbf{\theta}, \mathbf{\phi}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{m \ell} \mathbf{Y}_{\ell}^{m}(\mathbf{\theta}, \mathbf{\phi})
\]

so that the gradient of potential is

\[
\mathbf{\nabla} \phi(\mathbf{\theta}, \mathbf{\phi}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{m \ell} \mathbf{\Psi}_{\ell}^{m}(\mathbf{\theta}, \mathbf{\phi})
\]

where \( \mathbf{\Psi}_{\ell}^{m}(\mathbf{\theta}, \mathbf{\phi}) \) are the vector spherical harmonics.

Isotropy requires that the correlation function be a function of only the angular separation between pixels, \( \gamma_{ij} \), and their radial positions. The angular separation is given by

\[
\cos(\gamma_{ij}) = \sin(\theta_{i}) \sin(\theta_{j}) \cos(\phi_{i} - \phi_{j}) + \cos(\theta_{i}) \cos(\theta_{j})
\]
The gradient of the correlation function with respect to the position of point $i$ is

$$\nabla_{\xi ij} = \nabla_{\xi}(\gamma_{ij}, r_i, r_j) = \frac{\partial \xi_{ij}}{\partial \theta_i} + \frac{1}{\sin \theta_i} \frac{\partial \xi_{ij}}{\partial \phi_i}$$

(A.35)

$$= \left[ \frac{\partial \xi_{ij}}{\partial \theta_i} \hat{\theta} + \frac{1}{\sin \theta_i} \frac{\partial \xi_{ij}}{\partial \phi_i} \right] \frac{\partial \xi(\gamma, r_i, r_j)}{\partial \gamma}$$

(A.36)

$$= \nabla_{\gamma} \frac{\partial \xi(\gamma, r_i, r_j)}{\partial \gamma}$$

(A.37)

Using this the $P$ matrix can be calculated as

$$P^{\ell \mu}_{ij} = \left[ \nabla_{\gamma} \Psi^{\mu} (\theta_i, \phi_i) + \nabla_{\gamma} \Psi^{\mu} (\theta_j, \phi_j) \right] \frac{\partial \xi(\gamma, r_i, r_j)}{\partial \gamma}$$

(A.38)

We will make the approximation that pairs of points that are correlated enough to contribute significantly to the potential estimate will be separated by a small angle ($\gamma \ll \pi$). To a good approximation we can take

$$\frac{\partial \xi(\gamma, r_i, r_j)}{\partial \gamma} \approx \tilde{D}_{ij} \frac{\partial \xi(\gamma, r_i, r_j)}{\partial \sigma}$$

(A.39)

where $\tilde{D}_{ij}$ is the average comoving size distance to the pixels $i$ and $j$.

**Appendix B: Lyα Forest Power spectrum**

For the power spectrum of the Lyα flux we use the model of McDonald (2003)

$$P_{\text{Lyα}}(k) = T^2(k, \mu)P_{\text{lin}}(k, z)$$

(B.1)

where $P_{\text{lin}}(k, z)$ is the linear power spectrum of matter and

$$T^2(k, \mu) = b^2 f(\mu)^2 E(k, \mu)$$

(B.2)

is a transfer function with

$$f(\mu) = 1 + \beta \mu^2$$

(B.3)

$$E(k, \mu) = \exp \left[ \left( \frac{k}{k_\text{rad}} \right)^{\alpha_\mu} - \left( \frac{k}{k_p} \right)^{\alpha_\mu} - \left( \frac{k}{k_\nu} \right)^{\alpha_\nu} \right]$$

(B.4)

$$k_p = k_{\nu}\left( 1 + \frac{k}{k_{\nu}} \right)^{\alpha_\nu}$$

(B.5)

$$\mu = |k_\theta|/|k|.$$  

(B.6)

The $f(\mu)$ factor is a result of redshift distortion and $E(k, \mu)$ comes from a combination of nonlinear structure formation, thermal broadening and gas pressure effects.

In our simulations we use the following parameter values

$$k_{\text{rad}} = 6.77 \text{ hMpc}^{-1} \quad k_p = 15.9 \text{ hMpc}^{-1}$$

$$k_{\nu} = 0.917 \text{ hMpc}^{-1} \quad k_\mu = 0.819 \text{ hMpc}^{-1}$$

$$a_\mu = 0.55 \quad a_p = 2.12 \quad a_\nu = 1.5 \quad a_\mu = 0.528 \quad b^2 = 0.0173 \quad \beta = 1.58$$

as suggested by McDonald (2003).

McDonald (2003) does not consider any evolution in the transfer function with redshift. We evolve the power spectrum only through the evolution in the linear power spectrum $P_{\text{lin}}(k, z)$ according to the usual linear theory. A more sophisticated model can be used in the future, but this seems adequate for the purposes of this paper.

We need to find the correlation function for pixels (or voxels) that are very narrow in the directions perpendicular to the line-of-sight but finite in length along the line-of-sight. The variation in the relative flux within a pixel of comoving length $L_{\text{pix}}$, or a frequency interval translated into a comoving length, will be

$$\delta(x) = \frac{1}{L_{\text{pix}}} \int_{-L_{\text{pix}}/2}^{L_{\text{pix}}/2} dx' \delta(x - x')$$

(C.1)

$$= \int \frac{d^3k}{(2\pi)^3} \delta(k) \int_0^{L_{\text{pix}}/2} \left( \frac{L_{\text{pix}}k_0}{2} \right) P(k) e^{i x' k}$$

(C.2)

where $j_0(x)$ are the spherical Bessel functions and $\delta(x)$ is the relative density fluctuation, not the delta function. The correlation function is

$$\xi(s) \equiv \left\langle \delta(x) \delta(x+s) \right\rangle = \int \frac{d^3k}{(2\pi)^3} j_0 \left( \frac{L_{\text{pix}}k_0}{2} \right) P(k) e^{i s \cdot k}$$

(C.3)

$$= \int dk_0 dk_{\perp} \int_0^{L_{\text{pix}}/2} j_0 \left( \frac{L_{\text{pix}}k_0}{2} \right) 2 J_0(s k_{\perp}) P(k_{\perp}) e^{i s k_{\perp}}$$

(C.4)
where $J_0(x)$ is the Bessel function of the first kind.

For calculating the correlation function it is useful to expand it in Legendre polynomials (Hamilton 1993)

$$
\xi(s,\alpha) = \sum_\ell \xi_\ell(s) P_\ell(\cos(\alpha)) \quad (C.5)
$$

where $\mu = s_\parallel / |s| = \cos(\alpha)$ and

$$
\xi_\ell(s) = \left(\frac{2\ell + 1}{2}\right) \int_0^\alpha d\alpha \sin(\alpha) P_\ell(\cos(\alpha)) \xi(s,\alpha) \quad (C.6)
$$

$$
= \left(\frac{2\ell + 1}{2}\right) \int_{-1}^1 d\mu P_\ell(\mu) \xi(s,\mu) \quad (C.7)
$$

Symmetry requires that $\xi_\ell(s) = 0$ for odd $\ell$. Inserting (C.4) and using the integral

$$
\int_0^\pi d\theta \sin(\theta) P_\ell(\cos(\theta)) J_\ell(sk \sin(\theta)) \exp(i sk \parallel \cos(\theta)) = 2\ell P_\ell(\cos(\alpha)) j_\ell(sk) \quad (C.8)
$$

(Neves et al. 2006; Cregg & Svedlindh 2007) gives

$$
\xi_\ell(s) = (-1)^{\ell/2} \frac{2\ell + 1}{(2\pi)^2} \int_0^\infty dk \, k^2 \, j_\ell(sk) \
\times \int_{-1}^1 d\mu \, j_\ell \left(\frac{L_{pix} k}{2\mu}\right)^2 T^2(k,\mu) P_\ell(\mu) \quad \ell \text{ even} \quad (C.9)
$$

Substituting the Ly$\alpha$ power spectrum (B.1) gives

$$
\xi_\ell(s) = \int_0^\infty dk \, \frac{2\ell + 1}{(2\pi)^2} \, k^2 \, j_\ell(sk) P_{lin}(k) I_\ell(k) \quad (C.10)
$$

with

$$
I_\ell(k) = (-1)^{\ell/2}(2\ell + 1) \int_{-1}^1 d\mu \, j_\ell \left(\frac{L_{pix} k}{2\mu}\right)^2 T^2(k,\mu) P_\ell(\mu) \quad (C.11)
$$

For small $k$ the prefactor, $T^2(k,\mu)$, becomes independent of $k$ ($E(k,\mu) \to 1$) so $I_\ell(k)$ will be constant for large scales. The first few $I_\ell(k)$ are shown in figure C.1 for the power spectrum discussed in section B. The first five non zero $\xi_\ell(s)$ are plotted in figure C.2.