THE LICHNEROWICZ–OBATA THEOREM FOR THE KOHN
LAPLACIAN IN THREE DIMENSIONS

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Abstract. We prove rigidity for the Lichnerowicz-type eigenvalue estimate
for the Kohn Laplacian on strictly pseudoconvex three-manifolds with non-
negative CR Paneitz operator and positive Webster curvature.

1. Introduction

The classical Lichnerowicz–Obata Theorem [12, 13] states that if $(M^n, g)$ is a
closed Riemannian manifold with $\text{Ric} \geq (n - 1)\kappa > 0$, then the first nonzero
eigenvalue $\lambda_1(-\Delta)$ of the Laplacian on functions satisfies $\lambda_1(-\Delta) \geq n\kappa$, with equality if
and only if $(M^n, g)$ is isometric to the round $n$-sphere of constant sectional curvature $\kappa$. The contribution of Lichnerowicz [12] is to prove the estimate of $\lambda_1(-\Delta)$, while the contribution of Obata is to prove rigidity; i.e. the characterization of equality.

Chanillo, Chiu and Yang [3] proved an analogue of the Lichnerowicz eigenvalue
estimate for the Kohn Laplacian $\Box_b := 2\overline{\partial}_b \partial_b$. Specifically, they proved that any
closed pseudohermitian three-manifold with nonnegative CR Paneitz operator and positive Webster curvature $R$ satisfies $\lambda_1(\Box_b) \geq \inf R$, where $\lambda_1(\Box_b)$ denotes the infimum of the positive eigenvalues of $\Box_b$. There are two subtle differences in comparison with the Riemannian case. First, $\ker \Box_b$ is generally infinite-dimensional, as it consists of the space of CR functions. Second, the assumption on the CR Paneitz operator — which is a fourth-order CR invariant operator — cannot be removed; this can be seen by considering the Rossi spheres [3,14]. Using an embeddability result of Kohn [7], it follows that any closed strictly pseudoconvex three-manifold with positive CR Yamabe constant and nonnegative CR Paneitz operator — all CR invariant assumptions — is globally embeddable.

Li, Son and Wang [10] observed that the argument of Chanillo, Chiu and Yang [3]
extends to higher dimensions, where a fourth-order assumption is no longer required. More precisely, they proved that if $(M^{2n+1}, T^{1,0}, \theta)$, $n \geq 2$, is a closed pseudohermitian manifold for which the Ricci curvature of the Tanaka–Webster connection satisfies $\text{Ric} \geq \kappa > 0$, then $\lambda_1(\Box_b) \geq \frac{2n}{n+1}\kappa$. They also proved rigidity: If $\lambda_1(\Box_b) = \frac{2n}{n+1}\kappa$, then $(M^{2n+1}, T^{1,0}, \theta)$ is isometric to the round CR $(2n + 1)$-
sphere with $\text{Ric} = \kappa$. Their proof relies crucially on the fact that $\ker B = \ker \overline{\partial}_b \partial_b B$ when $n \geq 2$, where $Bf := dd^c f$ is the second-order operator whose kernel coincides with the space of CR pluriharmonic functions [9]. When $n = 1$, the corresponding
statement one needs is that the kernel of the CR Paneitz operator coincides with the space of CR pluriharmonic functions. It is presently unknown whether there exists a closed CR three-manifold for which the kernel of the CR Paneitz operator is strictly larger than the space of CR pluriharmonic functions.

Takeuchi [15] recently showed that on embeddable strictly pseudoconvex three-manifolds, the kernel of the CR Paneitz operator coincides with the space of CR pluriharmonic functions. Since embeddability follows from the assumptions of positive Webster scalar curvature and nonnegative CR Paneitz operator, the obstacle encountered by Li–Son–Wang does not arise. As such, one expects a Lichnerowicz–Obata-type theorem for the Kohn Laplacian in dimension three. The purpose of this note is to realize this expectation:

**Theorem 1.1.** Let \((M^3, T^{1,0}, \theta)\) be a closed pseudohermitian three-manifold with nonnegative CR Paneitz operator and Webster curvature \(R \geq \kappa > 0\). Then \(\lambda_1(\Box_b) \geq \kappa\), with equality if and only if \((M^3, T^{1,0}, \theta)\) is isometric to the standard CR three-sphere of constant Webster curvature \(\kappa\).

In particular, Theorem [1.1] and the result of Li, Son and Wang [10] establish CR analogues of the Lichnerowicz–Obata Theorem for the Kohn Laplacian in all dimensions. Note that CR analogues of the Lichnerowicz–Obata Theorem for the sub-Laplacian are also known [6,11].

This note is organized as follows. In Section 2 we define the CR Paneitz operator and discuss some of its important properties, especially how it arises in the work of Chanillo, Chiu and Yang [3]. In Section 3 we prove Theorem 1.1. Indeed, our argument is much simpler than the Li–Son–Wang argument, in part because the Webster curvature is a scalar in dimension three.

## 2. Background

A **CR three-manifold** is a pair \((M^3, T^{1,0})\) of an orientable three-dimensional (real) manifold \(M^3\) and a rank one distribution \(T^{1,0} \subset TM \otimes \mathbb{C}\) such that \(T^{1,0} \cap T^{0,1} = \{0\}\), where \(T^{0,1} := \overline{T^{1,0}}\). Since \(M\) is orientable and \(H := \text{Re} T^{1,0} \subset TM\) is a rank two distribution which is oriented by its complex structure, there is a nonvanishing (real) one-form \(\theta\) on \(M\) with \(\ker \theta = H\). Such a one-form is called a **contact form** for \((M^3, T^{1,0})\). Note that if \(\theta, \bar{\theta}\) are two contact forms for \((M^3, T^{1,0})\), then there is a smooth (real-valued) function \(\Upsilon \in C^\infty(M)\) such that \(\bar{\theta} = e^\Upsilon \theta\).

A **strictly pseudoconvex three-manifold** is a CR three-manifold \((M^3, T^{1,0})\) which admits a contact form \(\theta\) for which the Levi form, \(L_\theta(U, V) := -i \, d\theta(U, \overline{V})\), is positive definite on \(T^{1,0}\). Note that if \(L_\theta\) is positive definite on \(T^{1,0}\), then so too is \(L_{\bar{\theta}}\) for all contact forms \(\bar{\theta}\) on \((M^3, T^{1,0})\).

A **pseudohermitian three-manifold** is a triple \((M^3, T^{1,0}, \theta)\) of a strictly pseudoconvex three-manifold \((M^3, T^{1,0})\) and a contact form \(\theta\) for \((M^3, T^{1,0})\). The **Reeb vector field** \(T\) is the unique (real) vector field on \(M\) such that \(\theta(T) = 1\) and \(d\theta(T, \cdot) = 0\). There is a unique connection, the **Tanaka–Webster connection** [10][18], associated to such a structure, defined as follows: Let \(Z_1\) be a local frame of \(T^{1,0}\) and set \(Z_1 := \overline{Z_1}\). Then \(\{\theta, \theta^1, \theta^1\}\), the coframe dual to \(\{T, Z_1, Z_1\}\), is an **admissible coframe**; it satisfies \(d\theta = i h_{11} \theta^1 \wedge \theta^1\) for some positive (real-valued) function \(h_{11}\). We also refer to \(h_{11}\) as the **Levi form**. Denote by \(h^{11}\) the multiplicative inverse of \(h_{11}\). The
connection form $\omega_1^1$ is uniquely determined by the equations
\[
d\theta^1 = \theta^1 \wedge \omega_1^1 + h^{1i} A_{11} \theta \wedge \theta^1,
\]
\[
dh_{1\bar{1}} = 2 \text{Re} \omega_1^{1} h_{1\bar{1}},
\]
where $A_{11}$ is a complex-valued function, the pseudohermitian torsion. The Tanaka–Webster connection is determined from $\nabla T := 0$ and $\nabla Z_1 := \omega_1^1 \otimes Z_1$ by linearity and conjugation. The structure equation for the Tanaka–Webster connection is
\[
d\omega_1^1 = R \theta^1 \wedge \theta^1 \mod \theta,
\]
where $R$ is a real-valued function, the Webster curvature.

Given a pseudohermitian three-manifold $(M^3, T^{1,0}, \theta)$, we use the subscripts 1, $\bar{1}$, and 0 to denote components of a covariant tensor field with respect to a given admissible coframe $\{\theta, \theta^1, \theta^\bar{1}\}$. We raise subscripts using $h^{11}$. Subscripts following a semicolon denote components of a covariant derivative, though the semicolon will be omitted when denoting covariant derivatives of a function. For example, the exterior derivative $df$ of a complex function $f \in C^\infty(M; \mathbb{C})$ may be written
\[
df = f_1 \theta^1 + f_{\bar{1}} \theta^\bar{1} + f_0 \theta.
\]

A $(0, q)$-form on a CR three-manifold $(M^3, T^{1,0})$ is a complex-valued $q$-form which annihilates $T^{1,0}$. Denote by $\Lambda^{0,q}$ the space of $(0, q)$-forms. When $M$ is closed and strictly pseudoconvex, define Hermitian inner products on $C^\infty(M; \mathbb{C})$ and $\Lambda^{0,1}$ by
\[
\langle f, g \rangle := \int_M f \overline{g} \theta \wedge d\theta,
\]
\[
\langle \rho, \eta \rangle := \int_M h^{11} \rho_1 \overline{\eta_1} \theta \wedge d\theta,
\]
respectively. We define $\overline{\partial}_b : C^\infty(M; \mathbb{C}) \to \Lambda^{0,1}$ by
\[
\overline{\partial}_b f := f_1 \theta^1 + i f_{\bar{1}} \theta.
\]
It follows from the transformation formula for the Tanaka–Webster connection under change of contact form [5] Equation (2.7); [8] Lemma 5.6] that $\overline{\partial}_b$ is CR invariant (cf. [13] Lemma 3.1]): i.e. $\overline{\partial}_b$ is independent of the choice of contact form. It follows from Equations (2.2) and (2.3) that $\overline{\partial}_b : \Lambda^{0,1} \to C^\infty(M; \mathbb{C})$, the adjoint of $\overline{\partial}_b$, is given by
\[
\overline{\partial}_b^* \rho = -\rho_{1\bar{1}}.
\]
The Kohn Laplacian (on functions), $\Box_b : C^\infty(M; \mathbb{C}) \to C^\infty(M; \mathbb{C})$, is defined by $\Box_b f := 2\overline{\partial}_b \overline{\partial}_b f$.

A (complex-valued) function $f \in C^\infty(M; \mathbb{C})$ is CR if $\overline{\partial}_b f = 0$. A (real-valued) function $u \in C^\infty(M)$ is CR pluriharmonic if locally $u = \text{Re} f$ for some CR function $f$. There is an alternative characterization of CR pluriharmonic functions as the kernel of a third-order $(0, 2)$-form-valued differential operator: Define $P_1 : C^\infty(M) \to \Lambda^{0,2}$ by
\[
P_1 u := (u_{1\bar{1}}^1 - i A^1_1 u_1) \theta \wedge \theta^1.
\]
One easily checks that $P_1 u$ is the $(0, 2)$-part of $dd^c_b u$, where $d^c_b := \frac{i}{2}(\overline{\partial}_b - \partial_b)$, and hence $P_1$ is CR invariant. From this Lee concluded that $\ker P_1$ is the space of CR pluriharmonic functions [8 Proposition 3.4].
The CR Paneitz operator on \((M^3,T^{1,0})\) is the operator \(P : C^\infty(M) \to C^\infty(M)\) defined by \(Pu \theta \wedge d\theta = -i d(P_1 u)\). It follows from the previous paragraph that the CR Paneitz operator is a real-valued CR invariant fourth-order differential operator. It is immediate from Equation (2.5) and the definition of \(P\) that the CR Paneitz operator is formally self-adjoint.

Note that we may equivalently define \(P := -\overline{\partial}_b P_1\), where \(P_1\) is the \((0,1)\)-form-valued differential operator \(P_1 u := (u_1 \overline{1} - i A_1 \overline{u_1}) \theta^3\). Also, we may extend both \(P_1\) and \(P\) to differential operators on \(C^\infty(M; \mathbb{C})\) by linearity. It is this perspective that we will take. In particular, \(P : C^\infty(M; \mathbb{C}) \to C^\infty(M; \mathbb{C})\) is formally self-adjoint. We say that a closed strictly pseudoconvex three-manifold \((M^3,T^{1,0})\) has nonnegative CR Paneitz operator if

\[ \langle Pf, f \rangle \geq 0 \]

for all \(f \in C^\infty(M; \mathbb{C})\), where the left-hand side is interpreted using Equation (2.11).

The key observation underlying the Lichnerowicz-type theorem for the Kohn Laplacian is the following Bochner-type formula of Chanillo, Chiu and Yang [3] Proposition 2.1:

**Proposition 2.1.** Let \((M^3,T^{1,0},\theta)\) be a pseudohermitian three-manifold. Then

\[-\frac{1}{2} \Box_b |\partial_b f|^2 = |f^{11}|^2 + \frac{1}{4} |\overline{\partial}_b f|^2 - \langle \overline{\partial}_b \Box_b f, \overline{\partial}_b f \rangle - \frac{1}{2} \langle \overline{\partial}_b f, \overline{\partial}_b \overline{\partial}_b f \rangle + R|\overline{\partial}_b f|^2 - \langle P_1 f, \overline{\partial}_b f \rangle \]

for all \(f \in C^\infty(M; \mathbb{C})\).

### 3. Rigidity of the eigenvalue estimate

Our proof of Theorem 1.1 differs from the higher-dimensional proof given by Li, Son and Wang [10] in two ways. First, as already noted in the introduction, we need a result of Takeuchi [15] to conclude that the kernel of the CR Paneitz operator coincides with the complexified space of CR pluriharmonic functions. Second, we give a simple and direct proof that if \(\lambda_1(\Box_b) = \kappa\), then the Webster curvature equals \(\kappa\) and the pseudohermitian torsion vanishes identically. The classification of such pseudohermitian manifolds then implies the conclusion.

**Proof of Theorem 1.1** We begin by recalling the argument of Chanillo, Chiu and Yang [3]: Recall that the spectrum of the Kohn Laplacian in \((0, \infty)\) consists only of point eigenvalues [2] Theorem 1.3]. Let \(\lambda > 0\) be a nonzero eigenvalue of \(\Box_b\) and let \(f \in C^\infty(M; \mathbb{C})\) be a nontrivial function such that \(\Box_b f = \lambda f\). Set \(\kappa := \inf R > 0\). Proposition 2.1 implies that

\[ 0 = \langle Pf, f \rangle + \int_M \left( |f^{11}|^2 - \frac{1}{2} |\Box_b f|^2 + R|\overline{\partial}_b f|^2 \right) \theta \wedge d\theta \]

\[ \geq \frac{1}{2} \int_M \lambda(\kappa - \lambda) |f|^2 \]

with equality if and only if \(f^{11} = 0\) and \(\langle Pf, f \rangle = 0\) and \(R = \kappa\) on \(\text{supp} \, |\overline{\partial}_b f|^2\). In particular, \(\lambda \geq \kappa\), and hence \(\lambda_1(\Box_b) \geq \kappa\). It follows immediately that \(\Box_b\) has closed range in \(L^2\). A result of Kohn [7] then implies that \((M^3,T^{1,0})\) is embeddable.

We now consider the case of rigidity. That is, suppose that \(\lambda_1(\Box_b) = \kappa\) and let \(f \in C^\infty(M; \mathbb{C})\) be a nontrivial function such that \(\Box_b f = \kappa f\).

First observe that \(f \in \ker P_1 \cap \ker P_1\). Indeed, from the first paragraph it holds that \(\langle Pf, f \rangle = 0\). Let \(u\) and \(v\) be the real and imaginary parts, respectively, of \(f\). Since \(P\) is a formally-self adjoint real operator, \(\langle Pu, u \rangle = \langle Pv, v \rangle = 0\). Therefore \(u\)
and \( v \) are CR pluriharmonic functions [13]. Since \( P_1 \) and \( \overline{P}_1 \) are linear, we conclude that \( f \in \ker P_1 \cap \ker \overline{P}_1 \).

Next observe that the set \( U := \{ p \in M \mid \partial_b f(p) \neq 0 \} \) is dense in \( M \). Indeed, let \( V \) be the interior of \( M \setminus U \) and suppose that \( V \neq \emptyset \). Since \( \Box_b f = 0 \) on \( V \), we have that \( f = 0 \) on \( V \) as well. Thus \( \Re f \) and \( \Im f \) both vanish on \( V \). By the above paragraph, \( \Re f \) and \( \Im f \) are CR pluriharmonic functions on \( V \). Since \( M \) is embeddable, the unique continuation property for holomorphic functions [11] Chapter VII implies that \( \Re f = 0 \) and \( \Im f = 0 \) on \( M \), contradicting the assumption \( f \neq 0 \). In particular, since \( R = \kappa \) on \( U \), it holds that \( (M^3, T^{1,0}, \theta) \) has constant Webster curvature \( \kappa \).

Now, since \( P_1 f = 0 \) and \( \Box_b f = \kappa f \), it holds that

\[ (3.1) \quad iA_{11} f^1 = \frac{\kappa}{2} f_1. \]

Combining Equation (3.1) with the fact \( f^{11} = 0 \) yields

\[ if^1 A_{11;11} = \frac{\kappa}{2} f^{11}. \]

Since \( P_1 f = 0 \), we then deduce that

\[ if^1 A_{11;11} = \frac{i\kappa}{2} A^{11} f_1. \]

Applying Equation (3.1) again yields

\[ iA_{11;11} f^1 = -|A_{11}|^2 f^1. \]

Since \( U \) is dense, we conclude that \( iA_{11;11} = -|A_{11}|^2 \) on \( M \). Integrating over \( M \) yields \( A_{11} = 0 \).

Finally, since \( (M, T^{1,0}, \theta) \) has vanishing pseudohermitian torsion and constant positive scalar curvature \( \kappa \), it is isometric to a quotient of the round CR three-sphere \( S^3_\kappa \) with constant Webster scalar curvature \( \kappa \) [17]. It is known [4] that the eigenspace \( \{ f \in C^\infty(S^3_\kappa) \mid \Box_b f = \kappa f \} \) is spanned by \( z \) and \( w \), the holomorphic coordinates on \( C^2 \). Since there is no nontrivial linear combination of \( z \) and \( w \) which descends to a quotient of \( S^3_\kappa \), we conclude that \( (M^3, T^{1,0}, \theta) \) is isometric to \( S^3_\kappa \).

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References

[1] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, Real submanifolds in complex space and their mappings, Princeton Mathematical Series, vol. 47, Princeton University Press, Princeton, NJ, 1999. MR1668103
[2] D. M. Burns and C. L. Epstein, Embeddability for three-dimensional CR-manifolds, J. Amer. Math. Soc. 3 (1990), no. 4, 809–841. MR1071115 (93b:32024)
[3] S. Chanillo, H.-L. Chiu, and P. Yang, Embeddability for 3-dimensional Cauchy-Riemann manifolds and CR Yamabe invariants, Duke Math. J. 161 (2012), no. 15, 2909–2921. MR2999315
[4] G. B. Folland, The tangential Cauchy-Riemann complex on spheres, Trans. Amer. Math. Soc. 171 (1972), 83–133. MR309156
[5] A. R. Gover and C. R. Graham, CR invariant powers of the sub-Laplacian, J. Reine Angew. Math. 583 (2005), 1–27. MR2146851 (2006f:32049)
[6] S. Ivanov and D. Vassilev, The Lichnerowicz and Obata first eigenvalue theorems and the Obata uniqueness result in the Yamabe problem on CR and quaternionic contact manifolds, Nonlinear Anal. 126 (2015), 262–323. MR3688882
[7] J. J. Kohn, Estimates for $\partial_b$ on pseudoconvex CR manifolds, Pseudodifferential operators and applications (Notre Dame, Ind., 1984), 1985, pp. 207–217. MR812292
[8] J. M. Lee, The Fefferman metric and pseudo-Hermitian invariants, Trans. Amer. Math. Soc. 296 (1986), no. 1, 411–429. MR837820 (87j:32063)
[9] ———, Pseudo-Einstein structures on CR manifolds, Amer. J. Math. 110 (1988), no. 1, 157–178. MR926742 (89f:32034)
[10] S.-Y. Li, D. N. Son, and X. Wang, A new characterization of the CR sphere and the sharp eigenvalue estimate for the Kohn Laplacian, Adv. Math. 281 (2015), 1285–1305. MR3368867
[11] S.-Y. Li and X. Wang, An Obata-type theorem in CR geometry, J. Differential Geom. 95 (2013), no. 3, 483–502. MR3128992
[12] A. Lichnerowicz, Géométrie des groupes de transformations, Travaux et Recherches Mathématiques, III. Dunod, Paris, 1958. MR0124009 (23 #A1329)
[13] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan 14 (1962), 333–340. MR0142086 (25 #5479)
[14] H. Rossi, Attaching analytic spaces to an analytic space along a pseudoconcave boundary, Proc. Conf. Complex Analysis (Minneapolis, 1964), 1965, pp. 242–256. MR0176106 (31 #381)
[15] Y. Takeuchi, Non-negativity of the CR Paneitz operator for embeddable CR manifolds, preprint. arXiv:1908.07672.
[16] N. Tanaka, A differential geometric study on strongly pseudo-convex manifolds, Kinokuniya Book-Store Co. Ltd., Tokyo, 1975. Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9. MR0399517 (53 #3361)
[17] S. Tanno, Sasakian manifolds with constant $\phi$-holomorphic sectional curvature, Tohoku Math. J. (2) 21 (1969), 501–507. MR0251667
[18] S. M. Webster, Pseudo-Hermitian structures on a real hypersurface, J. Differential Geom. 13 (1978), no. 1, 25–41. MR520599

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