Griffiths Inequalities for Ising Spin Glasses on the Nishimori Line

Hidetsugu Kitatani

Department of General Education, Nagaoka University of Technology, Nagaoka, Niigata 940-2188

The Griffiths inequalities for Ising spin glasses are proved on the Nishimori line with various bond randomness which includes Gaussian and $\pm J$ bond randomness. The proof for Ising systems with Gaussian bond randomness has already been carried out by Morita et al, which uses not only the gauge theory but also the properties of the Gaussian distribution, so that it cannot be directly applied to the systems with other bond randomness. The present proof essentially uses only the gauge theory, so that it does not depend on the detail properties of the probability distribution of random interactions. Thus, the results obtained from the inequalities for Ising systems with Gaussian bond randomness do also hold for those with various bond randomness, especially with $\pm J$ bond randomness.

KEYWORDS: spin glass, Griffiths inequality, Ising model, Nishimori line, gauge theory, thermodynamic limit

1. Introduction

The Griffiths inequalities make significant contributions to the understanding of phase transition for ferromagnetic Ising models.\(^1\) For one formulation, the Griffiths inequalities are written as

$$\frac{dP}{dJ_B} \geq 0,$$

and

$$\frac{d\langle S_C \rangle}{dJ_B} \geq 0,$$

where $J_B$ is a positive interaction and $S_C = \prod_{i \in C} S_i (S_i = \pm 1)$ is a product of Ising variables for arbitrary subset of the sites of the system. The above inequalities state that the pressure, $P(= \log Z)$, and the correlation function, $\langle S_C \rangle$, are monotonic increasing functions of the strength of any interaction, $J_B$. Using the two inequalities, various results can be proved, for example, the existence of the thermodynamic limit of the pressure per unit volume and correlation functions. They also give significant insights for the existence of ferromagnetic phase transition.

For random Ising systems which have both ferromagnetic and antiferromagnetic interactions, inequalities analogous to the first Griffiths inequality have been proved with various bond randomness.\(^2-4\) In these cases, $J_B$ is a random variable, so that it has been proved that the pressure is a monotonic increasing function with respect to some parameter which controls the effect of an interaction term. Similar to the ferromagnetic cases, the inequalities
may, for example, be used for the proof of the existence of the thermodynamic limit of the pressure density.

As far as we know, however, the inequality analogous to the second Griffiths inequality has only been proved on the Nishimori line, which is a restricted region in the phase diagram of random bond systems. Morita et al. first proved the inequalities analogous to both the first and second Griffiths inequalities on the Nishimori line for Ising systems with Gaussian bond randomness.\(^5\)\(^,\)\(^6\) Using the two inequalities, the existence of the thermodynamic limit for the pressure per unit volume and correlation functions has been proved under various boundary conditions. Relations between the location of multicritical points for various lattices have also been derived. The proof, however, uses not only the gauge theory, but also the properties of the Gaussian distribution, so that the proof cannot be directly applied to the systems with other bond randomness, for example, \(\pm J\) bond randomness.

Almost all the rigorous results obtained on the Nishimori line are proved, in essential, only using the properties of the gauge transformation, so that they hold not only for any lattice structure, dimension of the lattice, and range of the interaction, but also for various bond randomness.\(^7\)\(^,\)\(^8\) For example, related to the second Griffiths inequality, the present author proved that the following inequality holds for finite \(\pm J\) Ising systems, \(A\) and \(B\), on the Nishimori line:

\[
\langle S_0 S_r \rangle_A \geq \langle S_0 S_r \rangle_B,
\]

when the system \(A\) is obtained from the system \(B\) by adding random bonds.\(^9\) The above proof was originally carried out for the \(\pm J\) Ising models, and used for deriving an inequality about the location of multicritical points of various lattices. However, the same argument can be applied to the systems with other bond randomness, since the proof only uses the gauge theory.

Therefore, it is a natural question whether the existence of the two Griffiths inequalities on the Nishimori line is a special property of Ising systems with Gaussian bond randomness, or they hold without the detail property of the probability distribution of random interactions. The main results of the present paper is that the two Griffiths inequalities do hold on the Nishimori line for Ising systems with various bond randomness which includes both Gaussian and \(\pm J\) bond randomness, where the proof essentially uses only the properties of the gauge transformation. Thus, the results obtained from the two inequalities for Ising systems with Gaussian bond randomness\(^4\)\(^,\)\(^5\) do also hold for Ising systems with bond randomness considered in the present paper, especially, for the systems with \(\pm J\) bond randomness.
2. The Model and the Bond Randomness

Let us first define several quantities. We treat the Ising spin system described by the Hamiltonian

\[ H = - \sum_{A \subseteq V} J_A S_A, \] (4)

where

\[ S_A = \prod_{i \in A} S_i. \] (5)

Here \( V \) is the set of sites, and the sum over \( A \) runs over all subsets of \( V \) among which interactions exist. The number of sites in \( A \) is arbitrary and may be different from subset to subset. The lattice structure is assumed to be reflected in the choice of \( A \) for which \( J_A \neq 0 \). The partition function of this system is written as

\[ Z = \sum_{\{S_i\}} \exp(\sum_{A \subseteq \Omega} \beta_A J_A S_A), \] (6)

where we introduce local inverse temperature, \( \beta_A \). The reason is that we use local inverse temperature, \( \beta_A \), as a parameter which controls the effect of the interaction term, \( -J_A S_A \), along the Nishimori line, as will be shown later. To investigate the property of a physical quantity at inverse temperature, \( \beta \), we must, of course, set all the local inverse temperature to \( \beta_A = \beta \). The following argument, however, does not depend on the value of each local inverse temperature, \( \beta_A \).

Then the pressure of the system is written as

\[ P = [\log Z], \] (7)

where the configurational average over the distribution of bond randomness is written as \([\cdots]\).

The probability distribution of the random interaction \( J_A \) is denoted by \( P(J_A) \). In this paper, we investigate the case that the probability distribution, \( P(J_A) \), satisfies the following two conditions:

\[ P(-J_A) = P(J_A) \exp(-2\beta_{p,A} J_A) \] (8)

and

\[ \frac{\partial P(J_A)}{\partial \beta_{p,A}} = (J_A - [J_A])P(J_A), \] (9)

where \( \beta_{p,A} \) is a parameter which characterizes \( P(J_A) \).

Most of the probability distributions of random interactions investigated in the spin glass problems may satisfy the above two conditions. In the case of \( \pm J \) distribution with the ferromagnetic bond concentration, \( p_A (p_A > 1/2) \), defining \( \beta_{p,A} \) as

\[ \exp(2\beta_{p,A} J) = \frac{p_A}{1 - p_A}, \quad (J > 0) \] (10)
we may rewrite $P(J_A)$ as
\[ P(J_A) = p_A \delta(J_A - J) + (1 - p_A) \delta(J_A + J) \]
\[ = \frac{\exp(\beta_{p,A} J)}{2 \cosh(\beta_{p,A} J)} \delta(J_A - J) + \frac{\exp(-\beta_{p,A} J)}{2 \cosh(\beta_{p,A} J)} \delta(J_A + J) \quad (11) \]
\[ = \frac{1}{2 \cosh(\beta_{p,A} J)} (\delta(J_A - J) + \delta(J_A + J)) \exp(\beta_{p,A} J_A). \]

For the Gaussian distribution with average, $J_{0,A}$, and variance, $\sigma_A$, defining $\beta_{p,A}$ as
\[ \beta_{p,A} = \frac{J_{0,A}}{\sigma_A^2} \quad (12) \]
we have
\[ P(J_A) = \frac{1}{\sqrt{2\pi\sigma_A}} \exp \left( -\frac{(J_A - J_{0,A})^2}{2\sigma_A^2} \right) \quad (13) \]
\[ = \frac{1}{\sqrt{2\pi\sigma_A}} \exp \left( -\frac{\sigma_A^2 \beta_{p,A}^2}{2} \right) \exp \left( -\frac{J_A^2}{2\sigma_A^2} \right) \exp(\beta_{p,A} J_A). \]

In both cases, we can easily see that $P(J_A)$ satisfies two conditions, eqs. (8) and (9), by direct calculations. It is noted that, in the case of $\pm J$ distribution, $[J_A] = J \tanh(\beta_{p,A} J)$.

In the present notation, the interaction term, $-J_A S_A$, satisfies the Nishimori condition when
\[ \beta_{p,A} = \beta_A. \quad (14) \]

Namely, when eq. (14) is satisfied for all the subsets $\{A\}$, we may use the properties on the Nishimori line which have originally been proved by Nishimori, using the local gauge transformation.\(^7,8\)

Here, we show two properties of the probability distribution, $P(J_A)$, which are often used in the rest of the paper. When the function, $f(J_A)$, is an odd function of $J_A$, namely, $f(-J_A) = -f(J_A)$, we obtain
\[ [f(J_A)] = [\tanh(\beta_{p,A} J_A) f(J_A)]. \quad (15) \]
Also, when the function, $f(J_A)$, is an even function of $J_A$, and monotonic increasing function of $|J_A|$, we have
\[ [J_A \tanh(\beta_A J_A) f(J_A)] \geq [J_A \tanh(\beta_A J_A)][f(J_A)] \quad (16) \]
The proofs of eq. (15) and ineq. (16) are shown in Appendices A and B, respectively.

3. The inequalities on the Nishimori line

Let us clarify the situation. We assume that all the interaction terms satisfy the Nishimori condition, eq. (14). In the following, we investigate the change of the pressure and the correlation functions with respect to arbitrary $\beta_B$, and prove the following two inequalities:
\[ \frac{d}{d\beta_B} [P] \geq 0, \quad (17) \]
and
\[ \frac{d}{d\beta_B} \langle S_C \rangle \geq 0, \tag{18} \]
where we denote the thermal average by angular brackets as \( \langle \cdots \rangle \). The above inequalities state that the configurational average of the pressure and the correlation functions are monotonic increasing functions of \( \beta_B \) on the Nishimori line. Here, we explain the role of \( \beta_B \). When \( \beta_B \) is zero, there is no interaction term, \( -J_B S_B \), in the system. Increasing the value of \( \beta_B \) makes the effect of the interaction term, \( -J_B S_B \), larger, though we consider the restricted case that \( \beta_B \) always satisfies the Nishimori condition, \( \beta_B = \beta_{p,B} \). Namely, increasing \( \beta_B \) for the present system corresponds to increasing the strength of an interaction for ferromagnetic Ising models. Thus, on the Nishimori line, ineqs. (17) and (18), play the same role as the two Griffiths inequalities do for ferromagnetic Ising models.

Next, we briefly explain the procedures of the proof, since the proof of two inequalities can be carried out similarly.

3.1 The basic transformation of the physical quantity

First, using the identity,
\[ \exp(\beta_B J_B S_B) = \cosh(\beta_B J_B)(1 + \tanh(\beta_B J_B)S_B), \tag{19} \]
we rewrite the physical quantity, \( Q \), so that the dependence of \( \beta_B \) may be seen explicitly. Corresponding to this procedure, we denote the configurational average over the distribution of bond randomness except \( J_B \) as \( \langle \cdots \rangle' \). The thermal average without the Boltzmann factor, \( \exp(\beta_B J_B S_B) \), is also denoted as \( \langle \cdots \rangle' \).

3.2 The physical quantity on the Nishimori line

Next, we rewrite the physical quantity, \( [Q] \), on the Nishimori line, using the local gauge transformation. Since we investigate the change of the physical quantity, \( [Q] \), in the situation where all the interaction terms including \( J_B \) always satisfy the Nishimori condition, we may use the properties on the Nishimori line, for example, \( \langle (S_C) \rangle = \langle (S_C)^2 \rangle \), in any step of the proof. Actually, we perform the local gauge transformation for all the spin variables, and interactions except \( J_B \). This can be done, since, in this step, there is no Boltzmann factor, \( \exp(\beta_B J_B S_B) \), and the dependence with respect to \( J_B \) is explicitly seen, as will be shown later.

3.3 The change of the physical quantity on the Nishimori line

Finally, we consider the change of the physical quantity, \( [Q] \), with respect to \( \beta_B \). Using eq. (9), we rewrite the total derivative of \( [Q] \) by \( \beta_B \) as
\[ \frac{d}{d\beta_B} [Q] = \frac{d}{d\beta_B} \int_{-\infty}^{\infty} dJ_B P(J_B)' [Q]' \]
\[ = \int_{-\infty}^{\infty} dJ_B \left( \frac{\partial}{\partial\beta_B} P(J_B) \right)' [Q]' + \int_{-\infty}^{\infty} dJ_B P(J_B) \left[ \frac{\partial}{\partial\beta_B} Q \right]' \]
\[= [(J_B - |J_B|)Q] + \left[ \frac{\partial}{\partial \beta} Q \right]. \quad (20)\]

We evaluate the second term of rhs by direct calculations. For the first term of rhs, when \(Q\) is an even function of \(J_B\) and monotonic increasing function of \(|J_B|\), we have
\[[(J_B - |J_B|)Q] = (J_B \tanh(\beta_B J_B) - |J_B| \tanh(|\beta_B J_B|))Q \geq 0, \quad (21)\]
where we use eq. (15) and ineq. (16). It must be noted that, for the \(\pm J\) Ising models, \(Q\) mentioned above becomes constant with respect to \(J_B\), since \(J_B^2 = J^2\), so that eq. (21) takes the value, zero.

4. The proof of the first inequality

In this section, we prove the first inequality (17) following the procedures shown in the previous section.

4.1 The basic transformation of the pressure

Using the identity (19), we rewrite the pressure \([P]\) as
\[[P] = [\log Z] = [\log Z'] + [\log(\cosh(\beta B J_B))] + \left[ \log(1 + \tanh(\beta_B J_B) \langle S_B \rangle) \right]. \quad (22)\]

Here, \(Z'\) denotes the partition function without the Boltzmann factor, \(\exp(\beta_B J_B S_B)\).

4.2 The pressure on the Nishimori line

We expand the third term of rhs of eq.(22) as:
\[\left[ \log(1 + \tanh(\beta_B J_B) \langle S_B \rangle) \right] = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \left[ \tanh^n(\beta_B J_B) \langle S_B \rangle \right]. \quad (23)\]

On the Nishimori line, using the local gauge transformation and eq. (15), we can rewrite each odd term of the series as
\[\left[ \tanh^{2n-1}(\beta_B J_B) \langle S_B \rangle^{(2n-1)} \right] = \left[ \tanh^{2n}(\beta_B J_B) \langle S_B \rangle^{2n} \right], \quad (24)\]

where, the local gauge transformation is performed for all the spin variables and all the interactions except \(J_B\). Then, we can further rewrite the third term of rhs of eq.(22) as
\[\left[ \log(1 + \tanh(\beta_B J_B) \langle S_B \rangle) \right] = \left[ \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} \tanh^{2n}(\beta_B J_B) \langle S_B \rangle^{2n} \right] = [F(J_B)], \quad (25)\]

were we define \(F(J_B)\) as
\[F(J_B) = \frac{1}{2} \left( (1 + \tanh(\beta_B J_B) \langle S_B \rangle) \log(1 + \tanh(\beta_B J_B) \langle S_B \rangle) \right) \]
\[+ \frac{1}{2} \left( (1 - \tanh(\beta_B J_B) \langle S_B \rangle) \log(1 - \tanh(\beta_B J_B) \langle S_B \rangle) \right). \quad (26)\]

By direct calculation, we have
\[F(-J_B) = F(J_B). \quad (27)\]
We also obtain
\[ \frac{\partial}{\partial J_B} F(J_B) = \frac{\beta_B \langle S_B \rangle'}{2 \cosh^2(\beta_B J_B)} \log \left( \frac{1 + \tanh(\beta_B J_B) \langle S_B \rangle'}{1 - \tanh(\beta_B J_B) \langle S_B \rangle'} \right) \geq 0, \quad (J_B > 0) \] (28)
regardless of the sign of \( \langle S_B \rangle' \). Namely, \( F(J_B) \) is an even function of \( J_B \) and monotonic increasing function of \( |J_B| \).

4.3 The change of the pressure on the Nishimori line

Now, we calculate the total derivative of the pressure, \( [P] \), by \( \beta_B \) on the Nishimori line. First, we have
\[ \frac{d}{d\beta_B} [P] = \frac{d}{d\beta_B} [\log(\cosh(\beta_B J_B))] + \frac{d}{d\beta_B} \left[ \log(1 + \tanh(\beta_B J_B) \langle S_B \rangle') \right]. \] (29)
The first term of rhs of eq. (29) is directly calculated as
\[ \frac{d}{d\beta_B} [\log(\cosh(\beta_B J_B))] = [(J_B - [J_B]) \log(\cosh(\beta_B J_B))] + [J_B \tanh(\beta_B J_B)] \geq 0, \] (30)
where we use ineq. (21), since \( \log(\cosh(\beta_B J_B)) \) is an even function of \( J_B \), and monotonic increasing function of \( |J_B| \).

Similarly, the second term of rhs of eq. (29) is calculated as
\[ \frac{d}{d\beta_B} \left[ \log(1 + \tanh(\beta_B J_B) \langle S_B \rangle') \right] = [(J_B - [J_B]) F(J_B)] + \left[ \frac{\partial}{\partial \beta_B} F(J_B) \right]. \] (31)
Using, ineq. (21), it yields that
\[ [(J_B - [J_B]) F(J_B)] \geq 0 \] (32)
For the second term of rhs of eq. (31), a direct calculation gives
\[ \left[ \frac{\partial}{\partial \beta_B} F(J_B) \right] = \left[ \frac{J_B \langle S_B \rangle'}{2 \cosh^2(\beta_B J_B)} \log \left( \frac{1 + \tanh(\beta_B J_B) \langle S_B \rangle'}{1 - \tanh(\beta_B J_B) \langle S_B \rangle'} \right) \right] \geq 0, \] (33)
since it is easily seen that each term in the square brackets of rhs of eq. (33) is nonnegative regardless of the sign of \( J_B \langle S_B \rangle' \). Thus, we obtain the first inequality (17).

5. The proof of the second inequality

We can similarly prove the second inequality (18), which states that the correlation function is a monotonic increasing function of \( \beta_B \).

5.1 The basic transformation of the correlation function

Using the identity (19), we rewrite the correlation function as
\[ \langle S_C \rangle = \frac{\langle S_C \rangle' + \tanh(\beta_B J_B) \langle S_B S_C \rangle'}{1 + \tanh(\beta_B J_B) \langle S_B \rangle'}. \] (34)
In this subsection, for simplicity, we define \( x_B \) and \( x_{p,B} \) as
\[ x_B = \tanh(\beta_B J_B), \] (35)
and

\[ x_{p,B} = \tanh(\beta_{p,B} J_B). \]

We further rewrite the correlation function as

\[
\langle S_C \rangle = \langle S_C \rangle' + \frac{x_B(\langle SBSC \rangle' - \langle SB \rangle' \langle SC \rangle') - x_B^2 \langle SBSC \rangle' \langle SB \rangle'}{1 - x_B^2 \langle SB \rangle'^2} \\
= \langle S_C \rangle' + \frac{x_B(\langle SBSC \rangle' - \langle SB \rangle' \langle SC \rangle') - x_B^2(\langle SBSC \rangle' - \langle SB \rangle' \langle SC \rangle') \langle SB \rangle'}{1 - x_B^2 \langle SB \rangle'^2}.
\]

(37)

Then, the configurational average of the correlation function can be written as

\[
[\langle S_C \rangle] = [\langle S_C \rangle'] + \left[ \frac{(\langle SBSC \rangle' - \langle SB \rangle' \langle SC \rangle')(x_B - x_B^2 \langle SB \rangle')}{1 - x_B^2 \langle SB \rangle'^2} \right].
\]

(38)

where we use eq. (15) for the second term of rhs.

5.2 The correlation function on the Nishimori line

On the Nishimori line, since \( \beta_{p,B} = \beta_B \), and \( x_{p,B} = x_B \), the correlation function can be rewritten as

\[
[\langle S_C \rangle] = [\langle S_C \rangle'] + \left[ \frac{x_B^2(\langle SBSC \rangle' - \langle SB \rangle' \langle SC \rangle') (1 - \langle SB \rangle')}{1 - x_B^2 \langle SB \rangle'^2} \right].
\]

(39)

Performing the local gauge transformation for all the spin variables and interactions except \( J_B \), we obtain

\[
\left[ \frac{x_B^2 \langle SBSC \rangle'}{1 - x_B^2 \langle SB \rangle'^2} \right] = \left[ \frac{x_B^2 \langle SBSC \rangle'^2}{1 - x_B^2 \langle SB \rangle'^2} \right],
\]

(40)

\[
\left[ \frac{x_B^2 \langle SB \rangle' \langle SC \rangle'}{1 - x_B^2 \langle SB \rangle'^2} \right] = \left[ \frac{x_B^2 \langle SBSC \rangle' \langle SB \rangle'}{1 - x_B^2 \langle SB \rangle'^2} \right] = \left[ \frac{x_B^2 \langle SBSC \rangle' \langle SB \rangle' \langle SC \rangle'}{1 - x_B^2 \langle SB \rangle'^2} \right],
\]

(41)

and

\[
\left[ \frac{x_B^2 \langle SB \rangle'^2 \langle SC \rangle'}{1 - x_B^2 \langle SB \rangle'^2} \right] = \left[ \frac{x_B^2 \langle SBSC \rangle'^2}{1 - x_B^2 \langle SB \rangle'^2} \right].
\]

(42)

Thus, the second term of rhs of eq. (39) becomes

\[
\left[ \frac{x_B^2(\langle SBSC \rangle' - \langle SB \rangle' \langle SC \rangle') (1 - \langle SB \rangle')}{1 - x_B^2 \langle SB \rangle'^2} \right] = \left[ \frac{x_B^2(\langle SBSC \rangle' - \langle SB \rangle' \langle SC \rangle')^2}{1 - x_B^2 \langle SB \rangle'^2} \right].
\]

(43)

Hence, we obtain

\[
[\langle S_C \rangle] = [\langle S_C \rangle'] + \left[ \frac{\tanh^2(\beta_B J_B)(\langle SBSC \rangle' - \langle SB \rangle' \langle SC \rangle')^2}{1 - \tanh^2(\beta_B J_B) \langle SB \rangle'^2} \right],
\]

(44)

where, for the second term of rhs of eq. (44), it is easily seen that the term in the square brackets is an even function of \( J_B \) and monotonic increasing function of \( | J_B | \).
5.3 The change of the correlation function on the Nishimori line

The total derivative of the correlation function by $\beta_B$ on the Nishimori line can be written as

$$
\frac{d}{d\beta_B} \langle S_C \rangle = \left[ (J_B - [J_B]) \frac{\tanh^2(\beta_B J_B)(\langle S_B S_C \rangle' - \langle S_B \rangle' \langle S_C \rangle')^2}{1 - \tanh^2(\beta_B J_B)\langle S_B \rangle'^2} \right] + \left[ \frac{\partial}{\partial \beta_B} \left( \frac{\tanh^2(\beta_B J_B)(\langle S_B S_C \rangle' - \langle S_B \rangle' \langle S_C \rangle')^2}{1 - \tanh^2(\beta_B J_B)\langle S_B \rangle'^2} \right) \right] (45)
$$

Using ineq. (21), the first term of rhs of eq. (45) becomes nonnegative. For the second term of rhs of eq. (45), a direct calculation gives

$$
= \left[ \frac{2J_B \tanh(\beta_B J_B)(\langle S_B S_C \rangle' - \langle S_B \rangle' \langle S_C \rangle')^2}{\cosh^2(\beta_B J_B)(1 - \tanh^2(\beta_B J_B)\langle S_B \rangle'^2)} \right] \geq 0 (47)
$$

Thus, we obtain the second inequality $(18)$.

6. Summary and discussion

We have proved two inequalities, ineqs. (17) and (18), for Ising spin glasses on the Nishimori line with various bond randomness which includes Gaussian and $\pm J$ bond randomness, where the probability distribution of random interactions must satisfy two conditions, eqs. (8) and (9). The two inequalities, which correspond to the Griffiths inequalities for ferromagnetic Ising models, state that, along the Nishimori line, the pressure and the correlation functions are monotonic increasing functions of any $\beta_B$ which controls the effect of the interaction term, $-J_B S_B$. The present results are a generalization of those by Morita et al for Ising systems with Gaussian bond randomness, where the proof uses not only the gauge theory but also the properties of the Gaussian distribution, so that it cannot be directly applied to the systems with other bond randomness. The present proof essentially uses only the gauge theory, so that it holds without the detail property of the probability distribution of random interactions.

Using the present proof, the results obtained from the two inequalities for Ising system with Gaussian bond randomness $^{5,6}$ can be also derived for the systems with various bond randomness. In the research of the Ising spin glass problems, however, most studies have been carried out for the systems with Gaussian or $\pm J$ bond randomness. Thus, we may insist that the most important physical consequence of the present paper is that it is found that the results obtained for Ising systems with Gaussian bond randomness $^{5,6}$ do also hold for Ising systems with $\pm J$ bond randomness.

Let us briefly explain several physical consequences on the Nishimori line for regular lattices which can be derived from the two inequalities, though they have already been explained by Morita et al for Ising systems with Gaussian bond randomness $^{5}$.
From the first inequality (17), we can show that the pressure has the well known super-additivity$^{2,3}$ namely
\[ [P]_V \geq \sum_i [P]_{V_i}, \] (48)
where $[P]_V$ denotes the pressure of the set of sites, $V$, and
\[ V = \sum_i V_i, \] (49)
since $[P]_V$ is obtained from $\sum_i [P]_{V_i}$ by adding random bonds among $V_i$ and $V_j$. Thus, we can show the existence of the thermodynamic limit of the pressure density on the Nishimori line under free boundary conditions, assuming invariance by translation with respect to random interactions and the stability boundedness.$^{2,3}$ For $\pm J$ Ising spin glasses with short range interactions, it is easily seen that the pressure has a definite stable boundedness.

From the second inequality (18), we can show the existence of the thermodynamic limit of the correlation functions on the Nishimori line under free and fixed boundary conditions. For free boundary conditions, when we consider two finite sets of sites, $V$ and $V'$ so that $V' \supset V$, we get
\[ \langle SC \rangle_{V'} \geq \langle SC \rangle_V, \] (50)
since $V'$ is obtained from $V$ by adding random bonds. Thus, we may say that the correlation function is a monotonic increasing function with the system size. With the fact that the correlation function is bounded by unity, we can assert the existence of the thermodynamic limit of the correlation function under free boundary conditions. We can also prove the existence of the thermodynamic limit of the correlation functions under fixed boundary conditions by similar procedure. (See Ref. 5 for details.)

From the second inequality, we can also obtain the relation between the location of the multicritical points. When the lattice $L_1$ is obtained from the lattice $L_2$ by adding random bonds, we get the following inequality for the magnetization
\[ \langle S_i \rangle_{L_1} \geq \langle S_i \rangle_{L_2}, \] (51)
from which, we have
\[ T_{c,L_1} \geq T_{c,L_2}, \] (52)
where $T_{c,L_1}(T_{c,L_2})$ is the temperature of the multicritical point of the lattice, $L_1(L_2)$.

Finally, we mention about the relation between ineq. (3) and the second ineq. (18) of the present paper including the work by Morita et al.$^5$ Inequality (3) only states that, for finite systems, the correlation function of some positive $\beta_B$ is larger than that of $\beta_B = 0$, and cannot give the information between the values of the correlation functions of two different positive $\beta_B$. Compared to the above fact, from eq. (44), we can explicitly see how the value of the correlation function increases as $\beta_B$ increases. Thus, ineq. (3) has clearly less information.
than ineq. (18), which comes from the derivative of eq. (44) by $\beta_B$. However, ineq. (50), for example, can be derived using only ineq. (3), though it was not explicitly mentioned in Ref. 9. In the proof of ineq. (3), the condition for the probability distribution of random interactions is only one condition, namely, eq. (8), which states that the system has the Nishimori line itself. Thus, following the argument executed in Ref. 5, the existence of the thermodynamic limit of the correlation function on the Nishimori line under free boundary conditions may be proved for all the systems which have the Nishimori line in the phase diagram. The relation between the location of multicritical points of various lattices can also be derived for the same systems.

Acknowledgement

We thank Satoshi Morita for useful comments.

Appendix A: Derivation of eq. (15)

In this appendix we explain the derivation of eq. (15). Through Appendices A and B, we denote the configurational average over the distribution of bond randomness except $J_A$ as $\langle \cdots \rangle$.

Using eq. (8) and the fact that $f(J_A)$ is an odd function of $J_A$, we have

$$
\langle \exp(-\beta_{p,A}J_A) f(J_A) \rangle = \int_{-\infty}^{\infty} dJ_A P(J_A) \exp(-\beta_{p,A}J_A) [f(J_A)]' = \int_{-\infty}^{\infty} dJ_A P(-J_A) \exp(\beta_{p,A}J_A) [f(J_A)]' \n$$

$$
= -\int_{-\infty}^{\infty} dJ_A P(J_A) \exp(-\beta_{p,A}J_A) [f(-J_A)]' = -\int_{-\infty}^{\infty} dJ_A P(J_A) \exp(-\beta_{p,A}J_A) [f(J_A)]' \n$$

which implies

$$
\langle \exp(-\beta_{p,A}J_A) f(J_A) \rangle = 0. \quad \text{(A-1)}
$$

Thus, we also obtain

$$
\frac{\exp(-\beta_{p,A}J_A)}{\exp(\beta_{p,A}J_A) + \exp(-\beta_{p,A}J_A)} f(J_A) = 0, \quad \text{(A-2)}
$$

since $(1/(\exp(\beta_{p,A}J_A) + \exp(-\beta_{p,A}J_A)) f(J_A))$ is also an odd function of $J_A$. Therefore, it yields that

$$
\langle f(J_A) \rangle = \langle f(J_A) \rangle - 2 \frac{\exp(-\beta_{p,A}J_A)}{\exp(\beta_{p,A}J_A) + \exp(-\beta_{p,A}J_A)} f(J_A) \n$$

$$
= \langle \tanh(\beta_{p,A}J_A) f(J_A) \rangle \quad \text{(A-4)}
$$
Appendix B: Derivation of ineq. (16)

When two functions, \(f(J_A)\) and \(J_A \tanh(\beta_A J_A)\), are both even functions of \(J_A\) and monotonic increasing functions of \(|J_A|\), we have

\[
[(J_A \tanh(\beta_A J_A) - J_0 \tanh(\beta_A J_0))(f(J_A) - f(J_0))] \geq 0,
\]

for any constant value, \(J_0\), from which, it yields that

\[
[J_A \tanh(\beta_A J_A) f(J_A)] \geq J_0 \tanh(\beta_A J_0)[f(J_A)] + ([J_A \tanh(\beta_A J_A)] - J_0 \tanh(\beta_A J_0))[f(J_0)]'.
\]

Here, we can choose the value, \(J_0\), so that it satisfies

\[
[J_A \tanh(\beta_A J_A)] = J_0 \tanh(\beta_A J_0).
\]

Substituting eq. (B.3) into eq.(B.2), we obtain ineq. (16).
References

1) R. B. Griffiths: J. Math. Phys. 8 (1967) 478, 484.
2) F. Guerra and F. Toninelli: Commun. Math. Phys. 230 (2002) 71.
3) P. Contucci and S. Graffi: J. Stat. Phys. 115 (2004) 581.
4) P. Contucci and J. Lebowitz: Ann. Henri Poincare 8 (2007) 1461.
5) S. Morita, H. Nishimori and P. Contucci: J. Phys. A: Math. Gen. 37 (2004) L203.
6) P. Contucci, S. Morita and H. Nishimori: J. Stat. Phys. 122 (2006) 303.
7) H. Nishimori: Prog. Theor. Phys. 66 (1981) 1169.
8) H. Nishimori: Statistical Physics of Spin Glasses and Information Processing: An Introduction, Oxford University Press (Oxford, 2001).
9) H. Kitatani: J. Phys. Soc. Jpn. 63 (1994) 2070.