Electroweak radiative corrections to $e^+e^- \rightarrow WW \rightarrow 4\text{ fermions}$

in double-pole approximation — the RACOONWW approach

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Abstract:

We calculate the complete $\mathcal{O}(\alpha)$ electroweak radiative corrections to $e^+e^- \rightarrow WW \rightarrow 4f$ in the electroweak Standard Model in the double-pole approximation. We give analytical results for the non-factorizable virtual corrections and express the factorizable virtual corrections in terms of the known corrections to on-shell W-pair production and W decay. The calculation of the bremsstrahlung corrections, i.e. the processes $e^+e^- \rightarrow 4f\gamma$ in lowest order, is based on the full matrix elements. The matching of soft and collinear singularities between virtual and real corrections is done alternatively in two different ways, namely by using a subtraction method and by applying phase-space slicing. The $\mathcal{O}(\alpha)$ corrections as well as higher-order initial-state photon radiation are implemented in the Monte Carlo generator RACOONWW. Numerical results of this program are presented for the W-pair-production cross section, angular and W-invariant-mass distributions at LEP2. We also discuss the intrinsic theoretical uncertainty of our approach.

June 2000
1 Introduction

At present, the focus of electroweak Standard Model tests lies on W-boson-pair production at LEP2, \( e^+ e^- \rightarrow WW \rightarrow 4f \) [1, 2, 3]. While the invariant-mass distributions of the final-state fermion pairs allow for precise measurements of the W-boson mass, the total cross section and the angular distributions can be used to obtain information on the non-abelian couplings of the W bosons. LEP2 provides us with \( \mathcal{O}(10^4) \) W-boson pairs, thus leading to a typical experimental accuracy of 1\% [4]. At a future \( e^+ e^- \) linear collider, the W-boson yield could be increased by two orders of magnitude [4], and the experimental accuracy will reach the level of some 0.1\%.

The Monte Carlo generators used in the past [5] for W-pair production are based on the lowest-order matrix elements and typically include only the universal electroweak corrections. These comprise the running of the electromagnetic coupling, corrections associated with the \( \rho \) parameter, the Coulomb singularity close to threshold, and photonic corrections in leading-logarithmic approximation. For the total cross section these generators have a precision at the level of 1–2\% at LEP2 energies [6, 7, 8]; for distributions the accuracy in general is worse. While this was sufficient for the analysis of LEP2 data in the past, it becomes already insufficient for the present analysis and will be definitely inadequate for the final LEP2 analysis. Since the non-universal corrections increase with energy, the accuracy of generators including only universal corrections is at the level of 5–20\% [8] in the total cross section and thus not adequate at linear collider energies.

In this paper we aim at a theoretical accuracy of \( \lesssim 0.5\% \) at not too large energies, which is sufficient for LEP2 and reasonable for a 500 GeV linear collider. This requires, on the one hand, to include the complete set of lowest-order diagrams for \( e^+ e^- \rightarrow 4f \), which is already implemented in many existing generators. Moreover, the complete \( \mathcal{O}(\alpha) \) corrections for \( e^+ e^- \rightarrow WW \rightarrow 4f \) have to be taken into account, and care must be taken that also the leading higher-order corrections are included.

The full treatment of the processes \( e^+ e^- \rightarrow 4f \) at the one-loop level is of enormous complexity. Nevertheless, there is ongoing work in this direction [9]. Moreover, the requirement to include the finite width of the W bosons poses severe theoretical problems with gauge invariance.

An economic approach for the calculation of the \( \mathcal{O}(\alpha) \) corrections to \( e^+ e^- \rightarrow WW \rightarrow 4f \) consists in using the double-pole approximation (DPA). In DPA, only those terms are kept that are enhanced by two resonant W propagators and thus by a factor \( M_W/\Gamma_W \) with respect to all other contributions. With respect to the leading contributions, the corrections to the non-doubly-resonant contributions are typically of the order \( \alpha/\pi \times \Gamma_W/M_W \). Even when taking into account a conservative safety factor, this approximation is sufficient for our precision tag of 0.5\% as long as the doubly-resonant contributions dominate the cross section. This is the case at LEP2 energies sufficiently above threshold if forward-scattered electrons or positrons in the final state (single-W production) and, for processes involving the neutral current, fermion–antifermion pairs with low invariant masses are excluded. At higher energies, where diagrams without two resonant W bosons

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1 Above 0.5–1 TeV at least the leading electroweak logarithms at the two-loop level should be taken into account.
2 A discussion of state-of-the-art calculations for single-W production can be found in Ref. [10].
become more sizeable, it may be necessary to suppress those by appropriate cuts, e.g. invariant-mass cuts, on the final-state fermions. These types of cuts must be applied anyhow in order to extract the physically interesting WW signal.

The DPA has several advantages as far as the virtual corrections are concerned. Besides reducing the number of diagrams drastically, it naturally provides a gauge-invariant answer. In contrast to the corrections to the full process $e^+e^- \rightarrow 4f$, the virtual corrections in DPA depend on the fermion flavours of the final state in a universal way, i.e. there is only one generic set of diagrams for all final states. Moreover, the invariant functions that build up the corrections do not depend on the decay angles but only on the production angle of the W bosons, thus allowing to speed up their calculation in the Monte Carlo generators considerably. Finally, the virtual factorizable corrections in DPA can be directly obtained from the existing results for on-shell W-pair production [10, 11] and W-boson decay [12, 13]. The virtual non-factorizable corrections, i.e. those virtual corrections that cannot be associated with either W-pair production or W-boson decay, have a simple structure [14, 15, 16, 17].

On the other hand, the definition of the DPA is problematic for the bremsstrahlung corrections. Photon radiation from the initial and from the final state leads to W propagators that become resonant at different locations in phase space. The situation depends on the regime of the photon energy. If the radiated photons are hard ($E_\gamma \gg \Gamma_W$), these locations are well separated, and the photon radiation can be unambiguously assigned to the W-pair production subprocess (at least theoretically) or to one of the W-decay subprocesses. For soft photons ($E_\gamma \ll \Gamma_W$), the resonances coincide, and the DPA is identical to the one without photon. However, for semi-soft photons ($E_\gamma \sim \Gamma_W$), the resonant propagators overlap, and it is not obvious how the DPA can be applied without omitting or double-counting doubly-resonant contributions. For these reasons, we have preferred to employ the exact $e^+e^- \rightarrow 4f\gamma$ matrix element for the real bremsstrahlung corrections and the exact $e^+e^- \rightarrow 4f$ matrix element for the soft and collinear singular virtual corrections, and to apply the DPA only to the non-leading part of the virtual corrections. Furthermore, all observables for real photons are based on a full lowest-order calculation for $e^+e^- \rightarrow 4f\gamma$.

A possible strategy for a DPA for the $O(\alpha)$ corrections to pair production of unstable particles was already proposed in Ref. [18]. Recently different versions of DPAs have been used in the literature.

A first complete calculation of the $O(\alpha)$ corrections for off-shell W-pair production, including a numerical study of leptonic final states, was presented in Ref. [19]. In that work the DPA is applied both to the virtual and real corrections using a semi-analytical approach. Moreover, also the phase space is treated in DPA. The proposed DPA for the bremsstrahlung process is based on the distinction between hard, semi-soft, and soft photons, without a common treatment of photons of these domains for all observables. A realistic photon recombination for collinear photon emission is not considered there. For instance, the invariant masses of the W bosons are reconstructed from the momenta of the corresponding decay fermions alone, without recombining soft and collinear photons. The resulting large shifts in the peak position of the W line shape, namely $-20\text{ MeV}$, $-39\text{ MeV}$, and $-77\text{ MeV}$ for $\tau^+\nu_\tau$, $\mu^+\nu_\mu$, and $e^+\nu_e$ final states at a centre-of-mass (CM) energy of

\[^3\text{A survey of state-of-the-art calculations for } e^+e^- \rightarrow 4f\gamma \text{ is also contained in Ref. [3].}\]
184 GeV, respectively, result from mass-singular logarithms of the form \( \alpha \ln(m_l/M_W) \) and are due to the absence of collinear photons in the definition of the invariant masses.

\( \mathcal{O}(\alpha) \) corrections for W-pair production in DPA have also been implemented in the Monte Carlo generator YFSWW3 \[20, 21\]. In Ref. \[20\] the \( \mathcal{O}(\alpha) \) corrections to on-shell W-pair production have been combined with the exponentiation of photonic corrections from the initial and intermediate (WW) states. In Ref. \[21\], the final-state corrections have been added in the leading-logarithmic approximation. The normalization of the W decays is fixed using the corrected branching ratios. The non-factorizable corrections are not fully included but only as an approximation in terms of the screened Coulomb ansatz \[22\]. The spin correlations between the two W decays have been neglected in the contributions of the non-leading corrections. In Ref. \[21\] the results of Ref. \[19\] have been confirmed qualitatively. Moreover, it has been found that the shifts of the peaks in the W-invariant-mass distributions are reduced by taking into account collinear photons in the definition of the invariant masses, which effectively replaces the mass-singular logarithms by logarithms of a minimum opening angle for collinear photon emission.

Another approximate version of the DPA was presented in Ref. \[23\] and discussed for a purely hadronic final state. Both virtual and real corrections were treated within DPA, but non-factorizable corrections were neglected. The main emphasis was put on linear-collider energies, in particular on the comparison to a high-energy approximation \[24\] for the one-loop correction.

In this paper we present the first complete calculation of the \( \mathcal{O}(\alpha) \) corrections for off-shell W-pair production in the DPA that has been implemented in a Monte Carlo generator. Numerical results of this generator, which is called RACOONWW, have already been presented in Refs. \[25, 3\] and \[26\] for LEP2 and linear-collider energies, respectively. The lowest-order cross section is calculated including the complete diagrams for any-four-fermion final state or in the CC03 approximation, i.e. including only the doubly-resonant W-pair production diagrams. The complete virtual corrections to W-pair production and W decay and the virtual non-factorizable corrections are included in the DPA. As far as the real corrections are concerned, the matrix elements for the minimal gauge-invariant subset of \( e^+e^- \to 4f\gamma \) including the doubly-resonant contributions corresponding to W-pair production (the CC11 subset) are included. The real corrections and the virtual corrections are matched in such a way that soft and collinear singularities cancel as far as they should. The treatment of these singularities has been implemented in two different ways, one of which uses the subtraction method described in Refs. \[27, 17\], the other one uses phase-space slicing. We use the exact four-fermion phase space throughout. Finally, we note that all parts of the calculations have been performed in two independent ways.

The paper is outlined as follows: in Section 2 we describe the general strategy of our calculation. In Section 3 details on the virtual factorizable and non-factorizable corrections in DPA are given. In Section 4 we explain the matching of soft and collinear singularities between virtual and real corrections, both for the subtraction method and for phase-space slicing. The inclusion of higher-order initial-state radiation is described in Section 5, and Section 6 contains a discussion of QCD corrections. Numerical results are presented in Section 7, including predictions for the W-pair production cross section, angular and W invariant-mass distributions at LEP2, a detailed discussion of the intrinsic ambiguities of the proposed DPA, and comparisons to results of other authors. Our
conclusions are presented in Section 8, and the appendices provide some useful explicit results.

2 Strategy of the calculation

We consider the process

\[ e^+(p_+, \sigma_+) + e^-(p_-, \sigma_-) \rightarrow W^+(k_+, \lambda_+) + W^-(k_-, \lambda_-) \]
\[ \rightarrow f_1(k_1, \sigma_1) + \bar{f}_2(k_2, \sigma_2) + f_3(k_3, \sigma_3) + \bar{f}_4(k_4, \sigma_4). \]  

(2.1)

The arguments label the momenta \( p_{\pm}, k_i \) and helicities \( \sigma_i = \pm 1/2, \lambda_j = 0, \pm 1 \) of the corresponding particles. We often use only the signs to denote the helicities. The fermion masses are neglected whenever possible. As a consequence, the helicities of the incoming electrons and positrons are related as \( \sigma_+ = \sigma_- = -\sigma_+ \) (in the absence of collinear photon emission). For the W-pair-mediated diagrams, and more generally for the graphs of the CC11 class, the helicities of the outgoing fermions are fixed, \( \sigma_{1,3} = -\sigma_{2,4} = -1/2 \), owing to the left-handed coupling of the W bosons. In general this does not hold for other background diagrams.

The lowest-order cross section is calculated using the complete lowest-order matrix elements \( M_{\text{Born}}^{e^+ e^- \rightarrow 4f} \) of Ref. [28].

The radiative corrections to (2.1) consist of virtual corrections, resulting from loop diagrams, as well as of real corrections, originating from the process

\[ e^+(p_+, \sigma_+) + e^-(p_-, \sigma_-) \rightarrow W^+(k_+, \lambda_+) + W^-(k_-, \lambda_-) ( + \gamma) \]
\[ \rightarrow f_1(k_1, \sigma_1) + \bar{f}_2(k_2, \sigma_2) + f_3(k_3, \sigma_3) + \bar{f}_4(k_4, \sigma_4) + \gamma(k, \lambda_\gamma). \]  

(2.2)

Both have to be combined properly in order to ensure necessary cancellations of soft and collinear singularities.

2.1 Virtual corrections

We treat the non-leading virtual corrections in double-pole approximation (DPA), i.e. we take only those terms into account that are enhanced by two resonant W propagators (doubly-resonant corrections).

2.1.1 Virtual factorizable corrections

In the DPA, there are two types of contributions, the \emph{factorizable} and the \emph{non-factorizable} ones. The former are the ones that can be associated to one of the production or decay subprocesses, the latter are the ones that connect these subprocesses. This splitting cannot be completely done on the basis of Feynman diagrams; otherwise the splitting would not be gauge-invariant. To define this splitting, we start by inspecting only those graphs that retain two resonant W propagators outside the loops, i.e. all diagrams of the generic structure shown in Figure II. Since the loop corrections in such graphs can be associated either to the production of the W-boson pair or to the decay of
one of the $W$ bosons, this subset of graphs can be associated with the virtual factorizable corrections. The corresponding amplitude is of the form

$$\mathcal{M} = \frac{R(k_+^2, k_-^2, \theta)}{(k_+^2 - M_W^2 + iM_W\Gamma_W)(k_-^2 - M_W^2 + iM_W\Gamma_W)},$$

where $k_\pm^2$ are the invariant masses of the virtual $W$ bosons, $\theta$ represents all other kinematical variables, $M_W$ is the renormalized $W$-boson mass, for which we take the on-shell mass, and $\Gamma_W$ is the width of the $W$ boson.

The diagrams of Figure 1 do not form a gauge-invariant subset. However, in DPA a gauge-invariant contribution can be extracted by replacing the numerator with the gauge-invariant residue [29],

$$\mathcal{M}_{\text{DPA}}^{e^+e^-\rightarrow WW\rightarrow 4f} = \frac{R(M_W^2, M_W^2, \theta)}{(k_+^2 - M_W^2 + iM_W\Gamma_W)(k_-^2 - M_W^2 + iM_W\Gamma_W)},$$

In principle, the amplitude should be expanded about the complex poles in $k_\pm^2$. Replacing the complex pole position by $M_W^2 - iM_W\Gamma_W$ with on-shell mass and width introduces an error of $\mathcal{O}(\alpha^2)$ (see App. D of Ref. [30]). The neglect of the width in the numerator introduces an error of order $\Gamma_W/M_W$. Since we apply the DPA only to the corrections, this leads to an error of $\mathcal{O}(\alpha\Gamma_W/M_W)$ which is the order of the uncertainty of our calculation. Note that additional infrared (IR) singularities appear in the factorizable corrections when taking the on-shell limit in the numerator.

In lowest order, the matrix element in DPA factorizes into the one for the production of the two on-shell $W$ bosons, $\mathcal{M}_{\text{Born}}^{e^+e^-\rightarrow W^+W^-}$, the (transverse parts of the) propagators of these bosons, and the matrix elements for the decays of these on-shell bosons, $\mathcal{M}_{\text{Born}}^{W^+\rightarrow f_1\bar{f}_2}$ and $\mathcal{M}_{\text{Born}}^{W^-\rightarrow f_3\bar{f}_4}$:

$$\mathcal{M}_{\text{Born,DPA}}^{e^+e^-\rightarrow WW\rightarrow 4f} = \sum_{\lambda_+\lambda_-} \frac{\mathcal{M}_{\text{Born}}^{e^+e^-\rightarrow W^+W^-} \mathcal{M}_{\text{Born}}^{W^+\rightarrow f_1\bar{f}_2} \mathcal{M}_{\text{Born}}^{W^-\rightarrow f_3\bar{f}_4}}{K_+ K_-},$$

where we introduced the abbreviations

$$K_\pm = k_\pm^2 - M^2, \quad M^2 = M_W^2 - iM_W\Gamma_W$$

5
for the off-shellness of the W bosons. Equation (2.3) contains the coherent sum over the physical polarizations $\lambda_{\pm}$ of the $W^{\pm}$ bosons. Note that the polarizations have to be defined in the same way for the production and the decay matrix elements.

The definition of the DPA for factorizable diagrams (2.4) implies that the squared momenta of the W-boson legs of the production and decay vertex functions, which are hidden in the shaded blobs of Figure 1, have to be set to their on-shell values $M_{W}^{2}$. The W self-energies, marked by open blobs, have to be expanded about $M_{W}^{2}$, and the corresponding residues have to be distributed equally to the production and decay parts. Finally, we can express the factorizable doubly-resonant corrections by the product of gauge-independent on-shell matrix elements for W-pair production and W decays, and the (transverse parts of the) W propagators,

$$\delta M^{\mu+e^{-}\rightarrow W W \rightarrow 4 f}_{\text{virt, fact, DPA}} = \sum_{\lambda_{+}, \lambda_{-}} \frac{1}{K_{+}K_{-}} \left( \delta M^{\mu+e^{-}\rightarrow W^{+}W^{-}}_{\text{Born}} M^{W^{+}\rightarrow f_{1}f_{2}}_{\text{Born}} M^{W^{-}\rightarrow f_{3}f_{4}}_{\text{Born}} ight.$$

$$+ M^{\mu+e^{-}\rightarrow W^{+}W^{-}}_{\text{Born}} \delta M^{W^{+}\rightarrow f_{1}f_{2}}_{\text{Born}} M^{W^{-}\rightarrow f_{3}f_{4}}_{\text{Born}}$$

$$+ M^{\mu+e^{-}\rightarrow W^{+}W^{-}}_{\text{Born}} M^{W^{+}\rightarrow f_{1}f_{2}}_{\text{Born}} \delta M^{W^{-}\rightarrow f_{3}f_{4}}_{\text{Born}} \right), \quad (2.7)$$

where $\delta M$ denote one-loop contributions. Details on the actual calculation of $\mathcal{M}^{\mu+e^{-}\rightarrow W W \rightarrow 4 f}_{\text{virt, fact, DPA}}$ can be found in Section 3.1.

2.1.2 Virtual non-factorizable corrections

All loop diagrams that are not of the generic form of Figure 1 belong to the virtual non-factorizable corrections. Simple power counting [16] reveals that only diagrams with photon exchange in the loop can give rise to doubly-resonant (virtual) non-factorizable corrections. The considered set of diagrams is not gauge-invariant even in the DPA. Therefore, we have defined the non-factorizable doubly-resonant corrections by subtracting the factorizable doubly-resonant corrections from the complete doubly-resonant corrections [16]. The so-defined non-factorizable corrections do not only receive contributions from diagrams in which the photon links the production and decay subprocesses, i.e. manifestly non-factorizable diagrams [graphs like (a), (b), (c) of Figure 2], but also from factorizable diagrams [graphs like (d), (e), (f), (g) of Figure 2]. The latter contributions arise by subtracting the factorizable contributions of those graphs, which contain the artificial IR singularities mentioned above, from the complete graphs in DPA, in which the IR singularities are replaced by logarithms of the form $\ln K_{\pm}$. A representative set of diagrams contributing to the non-factorizable corrections to $e^{+}e^{-}\rightarrow WW \rightarrow 4 f$ is shown in Figure 2.

Note that the non-factorizable corrections do not contain a product of two independent W propagators and that these corrections involve logarithms of the form $\ln(K_{\pm})$, which have to be kept exactly. Therefore, it is not possible to define a residue similar to $R(M_{W}^{2}, M_{W}^{2}, \theta)$ for the double resonance as done for the factorizable corrections. The resonance structure of the non-factorizable corrections has the form of a homogeneous polynomial of order two in $K_{+}$ and $K_{-}$ in the denominator.

The non-factorizable corrections yield a simple correction factor $\delta^{\text{virt, DPA}}_{\text{nfact}}$ to the lowest-order cross section. Its explicit form is given in Section 3.2.
Figure 2: A representative set of diagrams contributing to the virtual non-factorizable corrections
2.1.3 On-shell projection

The gauge-invariant definition of the factorizable corrections requires the introduction of an associated phase-space point with on-shell W bosons for each phase-space point with general off-shell W bosons. In our formulas, we leave the form of the on-shell projection open and mark all momenta of the associated on-shell point by carets. In particular, the on-shell projections of the momenta \( k_+ = k_1 + k_2 \) and \( k_- = k_3 + k_4 \) of the W bosons obey

\[
\hat{k}_+ = \hat{k}_1 + \hat{k}_2, \quad \hat{k}_- = \hat{k}_3 + \hat{k}_4, \quad \hat{k}_\pm^2 = M_W^2.
\]  

(2.8)

The actual form of this on-shell projection is not uniquely determined, but involves some freedom. However, this intrinsic ambiguity leads to differences of the order of \( \alpha \Gamma_W / (\pi M_W) \) for different versions of the projection, i.e. the ambiguity remains below the desired level of accuracy for the DPA. The explicit on-shell projection used in the numerics is given in App. A. For the on-shell kinematics we define the Mandelstam variables \( s \) and \( \hat{t} \) as usual,

\[
s = (p_+ + p_-)^2 = (\hat{k}_+ + \hat{k}_-)^2,
\]

\[
\hat{t} = (p_+ - \hat{k}_+)^2 = (p_- - \hat{k}_-)^2 = M_W^2 - \frac{s}{2} (1 - \beta \cos \theta),
\]

(2.9)

where \( \beta = \sqrt{1 - 4M_W^2/s} \) and \( \theta \) are the velocity and the scattering angle of the outgoing W bosons, respectively.

2.2 Real photonic corrections

A substantial part of the \( \mathcal{O}(\alpha) \) corrections to four-fermion production is due to real photon emission. Since detectors are not able to detect photons that are soft or collinear to fermions (other than muons), we have events with visible and invisible photons. The former are events in which a photon of finite energy is separated from all other particles in the detector, the latter correspond to events in which no separate photon is seen. Events with visible photons can be taken into account via the processes \( e^+e^- \to 4f\gamma \), for invisible photons a careful treatment of soft and collinear singularities is required in addition.

2.2.1 The bremsstrahlung process \( e^+e^- \to 4f\gamma \)

For the bremsstrahlung process \( e^+e^- \to 4f\gamma \) the complete lowest-order matrix elements \( \mathcal{M}^{e^+e^-\to4f\gamma} \) are taken into account. More precisely, we employ the matrix elements for the \( 4f + \gamma \) final states of the CC11 class, which is the minimal gauge-invariant subset of graphs containing all doubly-resonant diagrams corresponding to W-pair production. We use the explicit results presented in Ref. [28].

2.2.2 Invisible photons

The emission of invisible photons contributes to the \( \mathcal{O}(\alpha) \) corrections to \( e^+e^- \to 4f \). They include soft and collinear singularities, which are regularized by an infinitesimal photon mass \( \lambda \) and small fermion masses \( m_f \), respectively. Since the singularities of real and virtual corrections are related to each other, these corrections have to be combined carefully, in order to avoid a mismatch in the singularity structure. We use two different
procedures of treating soft and collinear photon emission: one is based on a subtraction method, the other on phase-space slicing. The precise implementation of these procedures is discussed in Section 4.

2.3 The master formula

The total cross section is composed of the following contributions

\[
\int d\sigma = \int d\sigma^{e^+e^-\rightarrow 4f}_{\text{Born}} + \int d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt}} + \int d\sigma^{e^+e^-\rightarrow 4f\gamma}_{\text{DPA}}. \tag{2.10}
\]

Here \(d\sigma^{e^+e^-\rightarrow 4f}_{\text{Born}}\) is the full lowest-order cross section to \(e^+e^-\rightarrow 4f\),

\[
d\sigma^{e^+e^-\rightarrow 4f}_{\text{Born}} = \frac{1}{2s} d\Phi_{4f} |\mathcal{M}^{e^+e^-\rightarrow 4f}_{\text{Born}}|^2 \tag{2.11}
\]

with the corresponding matrix element \(\mathcal{M}^{e^+e^-\rightarrow 4f}_{\text{Born}}\) and the four-particle phase-space element \(d\Phi_{4f}\). Similarly, \(d\sigma^{e^+e^-\rightarrow 4f\gamma}_{\text{DPA}}\), which describes the real corrections, is the full lowest-order cross section to \(e^+e^-\rightarrow 4f\gamma\), and \(d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt}}\) denotes the virtual one-loop corrections.

Both the virtual and the real corrections involve soft and collinear singularities. These singularities are extracted by separating the cross sections into finite and singular parts:

\[
d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt}} = d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,finite}} + d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,sing}}; \quad d\sigma^{e^+e^-\rightarrow 4f\gamma}_{\text{DPA}} = d\sigma^{e^+e^-\rightarrow 4f\gamma}_{\text{finite}} + d\sigma^{e^+e^-\rightarrow 4f\gamma}_{\text{sing}}. \tag{2.12}
\]

The singular part of the virtual corrections factorizes into the lowest-order cross section and a simple correction factor. Moreover, the singularities in the real part, \(d\sigma^{e^+e^-\rightarrow 4f\gamma}_{\text{sing}}\), can be split off, and the five-particle phase-space element \(d\Phi_{4f\gamma}\) can be decomposed into \(d\Phi_{4f}\) and the phase-space element of the photon \(d\Phi_{\gamma}\). The integration over \(d\Phi_{\gamma}\) can be partially performed, and the result can be written as a convolution of a structure function with the lowest-order cross section \(d\sigma^{e^+e^-\rightarrow 4f}_{\text{Born}}\). When adding both contributions, all soft singularities and all collinear singularities associated with the final state cancel, and the resulting “singular” cross section,

\[
d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt+real,sing}} = d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,sing}} + d\sigma^{e^+e^-\rightarrow 4f\gamma}_{\text{sing}}, \tag{2.13}
\]

contains, apart from finite terms, only collinear singularities associated with the initial state, i.e. leading logarithms of the form \(\ln(s/m_e^2)\), at least for inclusive enough observables.

In RACOONWW, the DPA is only applied to the finite part of the virtual corrections,

\[
d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,finite}} \rightarrow d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,finite,DPA}} = d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,DPA}} - d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,sing,DPA}}, \tag{2.14}
\]

with \(d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,sing,DPA}}\) defined in Section 4.3. The doubly-resonant virtual corrections are composed as follows,

\[
d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,DPA}} = \frac{1}{2s} d\Phi_{4f} \left[ 2 \text{Re} \left( (\mathcal{M}^{e^+e^-\rightarrow 4f}_{\text{Born,DPA}}) * \delta \mathcal{M}^{e^+e^-\rightarrow 4f}_{\text{virt,fact,DPA}} \right) + |\mathcal{M}^{e^+e^-\rightarrow 4f}_{\text{Born,DPA}}|^2 \delta_{\text{fact,DPA}} \right], \tag{2.15}
\]
where $\mathcal{M}_{\text{virt, fact, DPA}}^{e^+e^-\rightarrow WW\rightarrow 4f}$ denotes the matrix element for the factorizable virtual corrections (Section 3.1) and $\delta_{\text{nfact, DPA}}$ is the factor describing the non-factorizable virtual corrections (Section 3.2).

Finally, we arrive at the master formula for the cross section:

$$\int d\sigma = \int d\sigma_{\text{Born}}^{e^+e^-\rightarrow 4f} + \int d\sigma_{\text{virt, finite, DPA}}^{e^+e^-\rightarrow WW\rightarrow 4f} + \int d\sigma_{\text{virt+real, sing}}^{e^+e^-\rightarrow 4f} + \int d\sigma_{\text{finite}}^{e^+e^-\rightarrow 4f\gamma}. \quad (2.16)$$

Since the contribution $d\sigma_{\text{virt+real, sing}}^{e^+e^-\rightarrow 4f}$ is not treated in DPA, the leading-logarithmic photonic corrections resulting from initial-state radiation (ISR) are treated exactly in our approach.

The separation of the singularities can be done in different ways. In RACOONWW, we have implemented two possibilities to extract the singularities from the real corrections, one is based on a subtraction method and discussed in Section 4.1, the other is based on the phase-space-slicing method and described in Section 4.2. Both procedures are equivalent. For the extraction of the singularities from the virtual corrections, also two choices have been implemented. One is motivated by the subtraction method, the other is inspired by the definition of a U(1)-gauge-invariant virtual photon part following Ref. [31] (cf. Section 4.3). Since the definition of a finite part of the virtual corrections differs by finite terms between both cases, and since the singular parts are treated exactly but the finite parts in DPA, both approaches differ by terms of the order of the intrinsic uncertainty of the DPA. These differences are due to finite terms that are either taken into account in DPA or exactly.

## 3 Virtual corrections in double-pole approximation

### 3.1 Factorizable corrections

We have calculated the contribution of the virtual corrections to the transition matrix elements in DPA, $\delta\mathcal{M}_{\text{virt, fact, DPA}}^{e^+e^-\rightarrow WW\rightarrow 4f}$, in two different ways, the results of which are in perfect numerical agreement. In a first approach we made use of the results of Refs. [10] and [13] for the pair production and decay of on-shell W bosons, respectively. The virtual corrections in DPA are given by (2.7) in this approach. When using the existing on-shell results, particular care is needed concerning the polarization of the intermediate W bosons. In Refs. [10, 13] the polarization vectors for the W bosons were introduced in a way convenient for each process. In order to keep the spin correlation between production and decay subprocesses, it is, however, necessary to take one and the same polarization vectors for the production and the decay of a W boson and to perform the summation over the W polarizations coherently, i.e. at the level of the unsquared matrix element. Hence, the results for on-shell W bosons cannot be taken as black boxes, but have to be carefully combined.

In a second approach we have performed a completely new and independent calculation of all relevant one-loop diagrams for the pair production and the decay of on-shell W bosons. The Feynman graphs have been generated with FeynArts [32]. The actual calculation has been done twice using our own Mathematica [33] routines, once in the ‘t Hooft–Feynman gauge using the Feynman rules and the renormalization scheme of Ref. [34] and once in the background-field formalism using the results of Ref. [35].
evaluation of the one-loop diagrams follows the methods described in Ref. [34] in both calculations, and the two results are in perfect numerical agreement. In the following we briefly sketch the strategy of these calculations which exploits the factorization property of the amplitudes.

3.1.1 Structure of the matrix elements

The DPA matrix elements are decomposed into a set of so-called standard matrix elements (SMEs) \( \mathcal{M}_n^\sigma \), which contain the spin structure of the external particles, and invariant coefficients \( F_n^\sigma \). Since the relevant diagrams factorize into W-pair production and W decay, the SMEs for W-pair production can be chosen as (\( \sigma = \pm \))

\[
\begin{align*}
\mathcal{M}_1^\sigma &= \bar{v}(p_+)\gamma^\sigma \hat{k}_+ \cdot \omega u(p_-), \\
\mathcal{M}_2^\sigma &= \bar{v}(p_+)\frac{1}{2}(\hat{k}_+ - \hat{k}_-)\omega u(p_-)(\epsilon^+_+ \epsilon^-_-), \\
\mathcal{M}_3^\sigma &= \bar{v}(p_+)\gamma^\sigma \omega u(p_-)(\epsilon^+_+ \hat{k}_+) - \bar{v}(p_+)\gamma^\sigma \omega u(p_-)(\epsilon^+_+ \hat{k}_-), \\
\mathcal{M}_4^\sigma &= \bar{v}(p_+)\gamma^\sigma \omega u(p_-)(\epsilon^-_+ \hat{k}_-) - \bar{v}(p_+)\gamma^\sigma \omega u(p_-)(\epsilon^+_+ \hat{k}_+), \\
\mathcal{M}_5^\sigma &= \bar{v}(p_+)\frac{1}{2}(\hat{k}_+ - \hat{k}_-)\omega u(p_-)(\epsilon^-_+ \hat{k}_+) \\
\mathcal{M}_6^\sigma &= \bar{v}(p_+)\frac{1}{2}(\hat{k}_+ - \hat{k}_-)\omega u(p_-)(\epsilon^+_+ \hat{k}_+)(\epsilon^-_+ \hat{k}_-) \\
\mathcal{M}_7^\sigma &= \bar{v}(p_+)\frac{1}{2}(\hat{k}_+ - \hat{k}_-)\omega u(p_-)[(\epsilon^+_+ \hat{k}_-)(\epsilon^-_+ \hat{k}_+) + (\epsilon^+_+ \hat{k}_+)(\epsilon^-_+ \hat{k}_+)]
\end{align*}
\]

with “effective W-polarization vectors”

\[
\begin{align*}
\epsilon^+_+ &= \frac{e}{\sqrt{2} s_w} \frac{1}{K_+} \bar{u}(\hat{k}_1) \gamma^\mu \omega_- v(\hat{k}_2), \\
\epsilon^-_+ &= \frac{e}{\sqrt{2} s_w} \frac{1}{K_-} \bar{u}(\hat{k}_3) \gamma^\mu \omega_- v(\hat{k}_4).
\end{align*}
\]

The 14 SMEs given in (3.1) are exactly those defined in Ref. [34], where \( \bar{v}(p_+) \) and \( u(p_-) \) are the Dirac spinors of the massless e\(^+\)e\(^-\) initial state. The chirality projectors are given by \( \omega_\pm = \frac{1}{2}(1 \pm \gamma_5) \). The effective W-polarization vectors of (3.2) are a formal shorthand for the W propagators and the tree-level decay matrix elements. Of course, the off-shellness of the W bosons has to be kept in these W propagators, where \( k_+ = k_1 + k_2 \) and \( k_- = k_3 + k_4 \) are inserted, but the W decays are calculated with the on-shell kinematics, i.e. the Dirac spinors \( \bar{u}(\hat{k}_i) \) and \( v(\hat{k}_i) \) depend on the on-shell-projected momenta. Owing to the consistent neglect of fermion masses, the polarization vectors are transverse,

\[
\hat{k}_\pm \epsilon^\pm_+ = 0.
\]

This transversality and \( \hat{k}_+^2 = M_W^2 \) ensure that the algebraic reduction of the \( \mathcal{O}(\alpha) \) corrections to the W-pair production subprocess to SMEs and invariant coefficients can be performed in the same way as for pure on-shell pair production, i.e. without knowing any details of the W decays. Explicitly, the decomposition of DPA matrix elements reads

\[
\mathcal{M}_{\text{virt, fact, DPA}}^{e^+e^-\rightarrow W+W-\rightarrow 4f}\sigma(p_+, p_-, \hat{k}_+, \hat{k}_-, k_+^2, k_-^2) = \sum_{n=1}^{7} F_n^\sigma(s, t) \mathcal{M}_n^\sigma(p_+, p_-, \hat{k}_+, \hat{k}_-, k_+^2, k_-^2).
\]

11
It is important to realize that the invariant coefficients $F_\sigma^\sigma(s, \hat{t})$ depend only on the scalar products of the momenta of the on-shell-projected W bosons, but not on the momenta of their decay products. The explicit expressions of the SMEs are listed in App. B.

A few remarks on the choice of SMEs are appropriate. The choice (3.1) with (3.2) is obtained by taking all products of SMEs for the subprocesses $e^+e^- \to W^+W^-$, $W^+ \to f_1\bar{f}_2$, and $W^- \to f_3\bar{f}_4$ for massless fermions (cf. (5.11) and (5.14) of Ref. [34]), and by subsequently replacing $\varepsilon_{\pm\mu}\varepsilon_{\pm\nu}$ by the W propagators. Owing to the purely left-handed coupling of the W boson and the use of massless fermions, there is only one SME for each decay subprocess. Moreover, we have included the lowest-order coupling factor for the decays so that the invariant functions $F_\sigma^\sigma$ for the lowest-order amplitudes and the loop corrections for the production process are normalized in the same way as for on-shell W-pair production. In particular, the non-vanishing coefficients $F_{n,\text{Born}}\sigma(s, \hat{t})$ of the lowest-order diagrams (see Figure 3) in DPA are given by

$$F_{1,\text{Born}}^\sigma(s, \hat{t}) = \frac{e^2}{2s_w^2 \hat{t}},$$

$$F_{3,\text{Born}}^\sigma(s, \hat{t}) = -F_{2,\text{Born}}^\sigma(s, \hat{t}) = \frac{2e^2}{s} - \frac{2e^2}{s - M_Z^2} \left(1 - \frac{\delta_{\sigma-}}{2s_w^2}\right). \quad (3.5)$$

The chosen set of 14 SMEs is not minimal; more precisely, only 12 SMEs are independent. The reduction to independent SMEs can be performed by making use of two relations between several SMEs that follow from the four-dimensionality of space time (cf. (11.8) of Ref. [34]).

The contributions of the virtual corrections to the W-pair production subprocess to the invariant coefficients $F_\sigma^\sigma(s, \hat{t})$ can be directly read off from the results of Ref. [10]. The contributions of the W decays are obtained by multiplying the corrections to the decay matrix elements, which can be extracted for instance from Ref. [13], with the lowest-order coefficients $F_{n,\text{Born}}\sigma(s, \hat{t})$ of (3.3). Note that care has to be taken that the relevant phases are taken into account properly.

### 3.1.2 Evaluation of coefficient functions

Finally, we add some remarks on the actual numerical evaluation of the invariant coefficients $F_\sigma^\sigma(s, \hat{t})$. The expressions for the one-loop contributions to these coefficients are rather involved (see Refs. [10, 11]) so that the corresponding computer codes are lengthy and slow. Moreover, the employed Passarino–Veltman reduction [36] of tensor integrals to scalar integrals breaks down at the boundary of phase space ($\cos \theta = \pm 1$),
necessarily leading to numerical instabilities in the very forward and backward directions of the W-production angle $\theta$. In order to solve these problems of CPU time and numerical instabilities, we have made use of the fact that all invariant coefficients $F_n^\sigma(s, \hat{t})$ depend only on $\theta$ for a fixed scattering energy and fixed input parameters, such as masses and couplings. Before starting the multi-dimensional Monte Carlo integration over the four-particle phase space, we calculate a set of generalized Fourier coefficients

$$c_{n,l}^\sigma(s) = \frac{2l + 1}{2} \int_{-\pi}^{\pi} d\cos \theta \, \hat{t} F_n^\sigma(s, \hat{t}) P_l(\cos \theta)$$

by performing the integrals with the Legendre polynomials

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} [x^l - 1], \quad l = 0, 1, \ldots$$

numerically. Using, for instance, Gaussian integration for these integrations yields sufficiently precise results without the need to enter the region of numerical instabilities in the coefficients $F_n^\sigma(s, \hat{t})$. During the Monte Carlo integration, the coefficients are numerically reconstructed by making use of the generalized Fourier series

$$F_n^\sigma(s, \hat{t}) = \sum_{l=0}^{\infty} \frac{1}{l} c_{n,l}^\sigma(s) P_l(\cos \theta)$$

which involves only trivial algebra. Thus, its evaluation is extremely fast. In practice, relatively few generalized Fourier coefficients $c_{n,l}^\sigma(s)$ are needed; for example, taking $l = 0, \ldots, 20$ reproduces the full calculation of $F_n^\sigma(s, \hat{t})$ within roughly six digits for moderate scattering angles and LEP2 energies. Note also that the generalized Fourier series remains stable in the forward and backward directions where the original evaluation of $F_n^\sigma(s, \hat{t})$ breaks down. The explicit factor $\hat{t}$ in (3.6) and (3.8) was included in order to account for a $t$-channel pole in some of the $F_n^\sigma(s, \hat{t})$; without this factor the expansion is less efficient and requires much more terms in the sum over $l$ in (3.8).

3.2 Non-factorizable corrections

In Refs. [14, 15, 37, 16] the non-factorizable corrections have been discussed in detail. In these references, however, the virtual non-factorizable corrections have been combined with their counterparts involving real photon emission, in order to obtain an IR-safe correction that can be discussed separately. In our present approach we do not separate the non-factorizable parts from the full real corrections so that we need the virtual non-factorizable corrections separately. Here also those graphs must be taken into account that are cancelled by their real counterparts [graphs (c), (e), (f), and (g) of Figure 2] in the approach of Refs. [14, 15, 37, 16]. The complete expression for the virtual non-factorizable corrections is given below closely following the notation and conventions of Ref. [17].

The virtual non-factorizable corrections can be written in terms of a correction factor $\delta_{\text{virt,DPA}}$ to the lowest-order cross section. Analogously to Ref. [15], $\delta_{\text{virt,DPA}}$ is decomposed into contributions that are associated with a pair of final-state fermions,

$$\delta_{\text{virt,DPA}} = \sum_{a=1,2} \sum_{b=3,4} (-1)^{a+b+1} Q_a Q_b \frac{\alpha}{\pi} \text{Re}\{\Delta_{\text{virt}}(p_+, p_--; k_a, k_b; p_-, k_a)\}$$

The complete expression for the virtual non-factorizable corrections can also be found in Ref. [17].
and we only give \( \Delta_{\text{virt}} = \Delta_{\text{virt}}(p_+, p_-, k_+, k_2; k_-, k_3) \), since the other terms follow by obvious substitutions. The function \( \Delta_{\text{virt}} \) receives contributions from diagrams that have been classified into seven different types in Figure 2.

\[
\Delta_{\text{virt}} = \Delta_{\text{mf}} + \Delta_{\text{if}} + \Delta_{\text{im}} + \Delta_{\text{mm}} + \Delta_{\text{mf}} + \Delta_{\text{im}} + \Delta_{\text{mm}}. \tag{3.10}
\]

We note that in contrast to the sum of the virtual and real contributions given in Refs. [14, 15, 37, 16] the pure virtual non-factorizable corrections depend both on the final and initial state of the reaction. The contributions of the different types of diagrams are given by

\[
\Delta_{\text{mf}} \sim -(s_{23} + s_{24})K_0D_0(-k_-, k_+, k_2, 0, M, M, m_2) - (s_{13} + s_{23})K_0D_0(-k_3, -k_-, k_+, 0, m_3, M, M), \tag{3.11}
\]

\[
\Delta_{\text{if}} \sim -s_{23}K_0K_0E_0(-k_3, -k_-, k_+, k_2, \lambda, m_3, M, M, m_2), \tag{3.12}
\]

\[
\Delta_{\text{im}} \sim t_2K_0D_0(p_+, k_+, k_2, \lambda, m_3, M, m_2) + u_2K_0D_0(p_-, k_+, k_2, \lambda, m_3, M, m_2) - t_3K_0D_0(p_-, k_+, k_3, \lambda, m_3, M, m_3) + u_3K_0D_0(p_+, k_+, k_3, \lambda, m_3, M, m_3), \tag{3.13}
\]

\[
\Delta_{\text{mm}} \sim (2M_W^2 - s)\left\{ C_0(k_+, -k_-, 0, M) - \left[ C_0(k_+, -k_-, \lambda, M_W, M_W) \right]_{k_+^2 = M_W^2} \right\}, \tag{3.14}
\]

\[
\Delta_{\text{im}} \sim -(t - M_W^2)\left\{ C_0(p_+, k_+, 0, M) - \left[ C_0(p_+, k_+, \lambda, M_W, M_W) \right]_{k_+^2 = M_W^2} \right\}
+ C_0(p_-, k_-, 0, m_3) - \left[ C_0(p_-, k_-, \lambda, m_3, M_W, M_W) \right]_{k_-^2 = M_W^2}
+ (u - M_W^2)\left\{ C_0(p_+, k_-, 0, M) - \left[ C_0(p_+, k_-, \lambda, M_W, M_W) \right]_{k_-^2 = M_W^2} \right\}
+ C_0(p_-, k_+, 0, m_3) - \left[ C_0(p_-, k_+, \lambda, M_W, M_W) \right]_{k_+^2 = M_W^2}, \tag{3.15}
\]

\[
\Delta_{\text{mf}} \sim M_W^2\left\{ C_0(k_+, k_2, 0, M, m_2) - \left[ C_0(k_+, k_2, \lambda, M_W, m_2) \right]_{k_+^2 = M_W^2}
+ C_0(k_-, k_3, 0, M, m_3) - \left[ C_0(k_-, k_3, \lambda, M_W, m_3) \right]_{k_-^2 = M_W^2} \right\}, \tag{3.16}
\]

\[
\Delta_{\text{mm}} \sim 2M_W^2\left\{ \frac{B_0(k_+^2, 0, M) - B_0(M_W^2, 0, M)}{k_+^2 - M^2} + \frac{B_0(k_-^2, 0, M) - B_0(M_W^2, 0, M)}{k_-^2 - M^2}
- 2B_0(M_W^2, \lambda, M_W) \right\}, \tag{3.17}
\]

with the kinematical invariants

\[
s_{ij} = (k_i + k_j)^2, \quad i, j = 1, 2, 3, 4,
\]

\[
t_{+i} = (p_+ - k_i)^2, \quad u_{-i} = (p_- - k_i)^2, \quad i = 1, 2,
\]

\[
t_{-i} = (p_- - k_i)^2, \quad u_{+i} = (p_+ - k_i)^2, \quad i = 3, 4,
\]

\[
t = (p_+ - k_-)^2, \quad u = (p_+ - k_-)^2. \tag{3.18}
\]
The sign “∼” in (3.11)–(3.17) indicates that the limits \( k^2_+ \to M_W^2 \) and \( \Gamma_W \to 0 \) are implicitly understood whenever possible. The definition of the scalar integrals \( B_0, C_0, D_0, E_0 \) and of their arguments can be found in Refs. [34, 16]. The explicit expressions of these functions have been given in Ref. [16] for the \( mf, ff, \) and \( mm' \) parts; the ones of the remaining scalar integrals are listed in App. C.

In order to facilitate the evaluation of \( \Delta_{\text{virt}}^\text{mm} \) as much as possible, we insert the explicit expressions for the scalar integrals into the different contributions. For \( \Delta_{\text{mm}}^\text{virt} \), we can simply take over the result of Ref. [16]; specifically, the combination of \( C_0 \) functions in \( \Delta_{\text{mm}}^\text{virt} \) is given there in (C.1) and (C.2). For \( \Delta_{\text{mm}}^\text{virt} \) we obtain

\[
\Delta_{\text{mm}}^\text{virt} \sim 2 \ln \left( \frac{\lambda M_W}{-K_+} \right) + 2 \ln \left( \frac{\lambda M_W}{-K_-} \right) + 4
\]

(3.19)

using (C.1) of the appendix below. For the remaining parts it is convenient to add up all contributions, resulting in

\[
\Delta_{\text{mf}}^\text{virt} + \Delta_{\text{ff}}^\text{virt} + \Delta_{\text{if}}^\text{virt} + \Delta_{\text{in}}^\text{virt} + \Delta_{\text{mf}}^\text{virt}
\sim - \frac{K_+ K_- s_{23} \det(Y_0)}{\det(Y)} D_0(-k_4, k_+ + k_3, k_2 + k_3, 0, M, M, 0)
- \frac{K_+ \det(Y_3)}{\det(Y)} F_3 - \frac{K_- \det(Y_2)}{\det(Y)} F_2 + \ln \left( \frac{\lambda^2}{M_W^2} \right) \ln \left( - \frac{s_{23}}{M_W^2} - i \epsilon \right)
+ 2 \ln \left( \frac{-K_+}{\lambda M_W} \right) \ln \left[ \frac{u_{-2}(M_W^2 - t)}{t_{-2}(M_W^2 - u)} \right] + 2 \ln \left( \frac{-K_-}{\lambda M_W} \right) \ln \left[ \frac{u_{+3}(M_W^2 - t)}{t_{-3}(M_W^2 - u)} \right]
+ \text{Li}_2 \left( 1 + \frac{M_W^2 - t}{t_{+2}} \right) + \text{Li}_2 \left( 1 + \frac{M_W^2 - t}{t_{-3}} \right)
- \text{Li}_2 \left( 1 + \frac{M_W^2 - u}{u_{-2}} \right) - \text{Li}_2 \left( 1 + \frac{M_W^2 - u}{u_{+3}} \right),
\]

(3.20)

where the determinants \( \det(Y), \det(Y_i) \) and the functions \( D_0(\ldots), F_i \) are given in Ref. [16] [see (3.36), (4.10), and (C.3) there]. Finally, we note that in \( \delta_{\text{fact.DPA}}^\text{virt} \) all fermion-mass singularities of the initial- and final-state fermions drop out, although some non-factorizable graphs contain such fermion-mass singularities.

4 Treatment of soft and collinear photon emission

In the following we describe two procedures of treating soft and collinear photon emission: one is based on a subtraction method, the other on phase-space slicing. In both cases soft and collinear singularities are regularized by an infinitesimal photon mass and small fermion masses, respectively.

For convenience, we introduce a second, generic notation for the momenta and helicities of the external particles:

\[
e^+(q_1, \kappa_1) + e^-(q_2, \kappa_2) \to f_1(q_3, \kappa_3) + \bar{f}_2(q_4, \kappa_4) + f_3(q_5, \kappa_5) + \bar{f}_4(q_6, \kappa_6) \left( + \gamma(k, \lambda \gamma) \right)
\]

(4.1)

in addition to (2.1) and (2.2). Moreover, the masses of the external fermions are denoted by \( m_i \) \( (q_i^2 = m_i^2 \to 0) \), and we use the invariants \( s_{ij} = 2q_i q_j \) in the following.
4.1 The dipole subtraction approach

The idea of so-called subtraction methods is to subtract a simple auxiliary function from the singular integrand of the bremsstrahlung integral and to add this contribution back again after partial analytic integration. This auxiliary function has to be chosen in such a way that it cancels all singularities of the original integrand so that the phase-space integration of the difference can be performed numerically, even over the singular regions of the original integrand, which is $|M_{e^+e^-\rightarrow 4f\gamma}|^2$ in our case. In this difference $\mathcal{M}_{e^+e^-\rightarrow 4f\gamma}$ can be evaluated without regulators for soft or collinear singularities, i.e. we can make use of the results of Ref. [28] for $\mathcal{M}_{e^+e^-\rightarrow 4f\gamma}$ with massless fermions. The auxiliary function has to be simple enough so that it can be integrated over the singular regions analytically, when the subtracted contribution is added again. This part contains the singular contributions and requires regulators, i.e. photon and fermion masses have to be reintroduced there. In RACOONWW we have applied the dipole subtraction formalism, which is a process-independent approach that was first proposed [38] within QCD for massless unpolarized partons and subsequently generalized to photon radiation of massive polarized fermions [27]. We only need the limit of small fermion masses [27, 17] in which the application of the method is relatively simple. In order to keep the description of the method transparent, we describe only the basic structure of the individual terms in (2.16) explicitly and refer to Refs. [27, 17] for the details.

In the dipole subtraction formalism the subtraction function is constructed from contributions that are labelled by ordered pairs $ij$ of charged fermions, so-called “dipoles”. The fermions $i$ and $j$ are called emitter and spectator, respectively, since by construction only the kinematics of the emitter $i$ leads to collinear singularities. The auxiliary functions $|M_{\text{sub,}ij}|^2$ that are subtracted from the original integrand $|M_{e^+e^-\rightarrow 4f\gamma}|^2$ are given by

$$|M_{\text{sub,}ij}(\Phi_{4f\gamma})|^2 = -(-1)^{i+j}Q_iQ_j e^2 \sum_{\tau=\pm} g_{ij,\tau}^{(\text{sub})}(q_i, q_j, k) |M_{\text{Born}}^{e^+e^-\rightarrow 4f}(\Phi_{4f,ij}, \tau\kappa_i)|^2,$$

where $\tau\kappa_i$ denotes the helicity of the emitter. Here and in the following the indices $i$, $j$ run over $1, \ldots, 6$ if not stated otherwise. Note that $|M_{\text{sub,}ij}|^2$ is a function on the entire $4f\gamma$ phase space $\Phi_{4f\gamma}$, while the lowest-order matrix element $M_{\text{Born}}^{e^+e^-\rightarrow 4f}$ on the r.h.s. requires momenta of the non-radiative $4f$ phase space $\Phi_{4f}$, i.e. $\Phi_{4f}$ has to be embedded in $\Phi_{4f\gamma}$ by an appropriate mapping. This mapping is chosen differently for different $ij$ pairs, as indicated by $\Phi_{4f,ij}$ in the argument of $M_{\text{Born}}^{e^+e^-\rightarrow 4f}$. The mappings have to ensure that the kinematics in $\Phi_{4f\gamma}$ and $\Phi_{4f,ij}$ asymptotically approach each other in the soft limit,

$$q_i \sim_{k\rightarrow 0} \tilde{q}_i, \quad q_j \sim_{k\rightarrow 0} \tilde{q}_j,$$

and in the collinear limits,

$$q_i - k_{\sim_{q_i\rightarrow 0}} \tilde{q}_i, \quad q_j \sim_{q_j\rightarrow 0} \tilde{q}_j, \quad i = 1, 2, \quad (4.4)$$

$$q_i + k_{\sim_{q_i\rightarrow 0}} \tilde{q}_i, \quad q_j \sim_{q_j\rightarrow 0} \tilde{q}_j, \quad i = 3, \ldots, 6, \quad (4.5)$$

5The subtraction functions of Ref. [27] and Ref. [17] are slightly different. As default, the subtraction function of Ref. [27] are implemented in RACOONWW.

6We suppress the polarization arguments as far as possible.
as required by the factorization theorems for mass singularities. The momenta \( \tilde{q}_i, \tilde{q}_j \) have to respect momentum conservation and mass-shell conditions everywhere. The process-independent radiator functions \( g_{ij;x}^{(sub)} \) behave in the soft limit as

\[
\frac{2(q_iq_j)}{(q_i)(q_i + q_jk)} - \frac{m_i^2}{(q_i)(q_i + q_jk)^2},
\]

and in the collinear limits as

\[
g_{ij,+}^{(sub)}(q_i, q_j, k) \sim \frac{1}{q_i k} \left[ \frac{1}{x_i} P_{ff}(x_i) \right],
\]

\[
g_{ij,-}^{(sub)}(q_i, q_j, k) \sim \frac{(1 - x_i)^2}{x_i} \frac{m_i^2}{2(q_i)(q_i + q_jk)^2}, \quad x_i = 1 - \frac{k_0}{q_i}, \quad i = 1, 2,
\]

\[
g_{ij,+}^{(sub)}(q_i, q_j, k) \sim \frac{1}{q_i k} \left[ P_{ff}(z_i) - \frac{1 + z_i^2}{z_i} \frac{m_i^2}{2q_i k} \right],
\]

\[
g_{ij,-}^{(sub)}(q_i, q_j, k) \sim \frac{(1 - z_i)^2}{z_i} \frac{m_i^2}{2(q_i)(q_i + q_jk)^2}, \quad z_i = \frac{q_i^0}{q_i^0 + k^0}, \quad i = 3, \ldots, 6,
\]

where \( P_{ff}(y) \) is the usual splitting function,

\[
P_{ff}(y) = \frac{1 + y^2}{1 - y}.
\]

Taking into account charge conservation, it is easy to show that the sum of the subtraction functions \( |\mathcal{M}_{sub,ij}|^2 \) and \( \sum_{\lambda} |\mathcal{M}^{e^+e^- \rightarrow 4f\gamma}|^2 \) are asymptotically equal in the singular limits,

\[
\sum_{\lambda} |\mathcal{M}^{e^+e^- \rightarrow 4f\gamma}(\Phi_{4f\gamma})|^2 \sim \sum_{i,j=1}^{6} |\mathcal{M}_{sub,ij}(\Phi_{4f\gamma})|^2 \quad \text{for} \quad k \to 0 \quad \text{or} \quad q_i k \to 0,
\]

so that the integral of their difference becomes integrable in those regions and the photon and fermion masses can be neglected.

Now we are able to define the finite part of the real corrections,

\[
\int d\sigma_{\text{finite}}^{e^+e^- \rightarrow 4f\gamma} = \frac{1}{2s} \int d\Phi_{4f\gamma} \left[ \sum_{\lambda} |\mathcal{M}^{e^+e^- \rightarrow 4f\gamma}|^2 \Theta(\Phi_{4f\gamma}) - \sum_{i,j=1}^{6} |\mathcal{M}_{sub,ij}|^2 \Theta(\tilde{\Phi}_{4f,ij}) \right],
\]

where the photon recombination procedure and separation cuts are included in the observable defined by \( \Theta \). We define \( \Theta(\Phi_{4f,ij}) = 1 \) if the event passes the separation cuts after eventual photon recombination and \( \Theta(\Phi_{4f,ij}) = 0 \) otherwise. In contrast to the first term, the observable in the second term depends on the 4f phase space, \( \tilde{\Phi}_{4f,ij} \), and is independent of the photon recombination procedure. The reason for the different arguments is that the final-state mass singularities included in the subtraction functions have to match the corresponding singularities of the virtual corrections exactly. On the other hand, the different arguments of the terms require that all photons have to be combined with the
nearest charged fermion in the collinear limits in order to obtain \( \Theta(\Phi_{4f\gamma}) = \Theta(\Phi_{4f,ij}) \) in the soft and collinear limits [see (4.3) and (4.4)].

The calculation of distributions is similar to the application of cuts, since a histogram of a distribution is nothing but a series of cuts. Hence, the histogram routine that generates the desired distribution during the Monte Carlo integration has to handle each column of the histogram in the same way as a cutted contribution to the integrated cross section. Note that in this procedure the original differential cross section and the subtraction functions may contribute to different columns of the histogram for one and the same event. The final result for each column is nevertheless finite, because such events are in general far away from the singular regions.

In addition to the finite part of the real corrections, which is given in (4.11), we have to determine the singular part of the real corrections, \( d\sigma_{\text{sing}}^{e^+e^\to4f\gamma} \). Therefore, we have to evaluate the phase-space integral of all contributions \(|M_{\text{sub},ij}|^2\) at least over the singular regions for finite mass regulators \( m_i \) and \( \lambda \). To this end, we split the five-particle phase space into the four-particle phase space and the remaining photonic parts:

\[
\int d\Phi_{4f\gamma} = \int_0^1 dx \int d\Phi_{4f,ij}(x) \int d\Phi_{\gamma,ij}. \tag{4.12}
\]

If \( i \) and/or \( j \) are initial-state particles, the momentum of one of these is reduced by a factor \( x \), i.e. \( ˜q_1 = xq_1 \) or \( ˜q_2 = xq_2 \). This is indicated by the argument \( x \) in \( ˜\Phi_{4f,ij}(x) \). (For final-state emitter and spectator \( d\Phi_{\gamma,ij} \) includes the function \( \delta(1-x) \) which fixes \( x = 1 \).) Then, the singular part of the real corrections can be written as

\[
\int d\sigma_{\text{sing}}^{e^+e^\to4f\gamma} = -\frac{\alpha}{2\pi} \sum_{i,j=1}^6 \sum_{\tau=\pm} (-1)^{i+j} Q_i Q_j \tag{4.13}
\]

\[
\times \int_0^1 dx \int d\Phi_{4f,ij}(x) G_{ij,\tau}^{(\text{sub})}(s_{ij}, x) \frac{1}{2xs} |M_{\text{Born}}^{e^+e^\to4f}(\Phi_{4f,ij}(x), \tau \kappa_1)|^2 \Theta(\Phi_{4f,ij}(x)),
\]

where the functions \( G_{ij,\tau}^{(\text{sub})} \) originate from the photonic phase-space integral over the radiator functions \( g_{ij,\tau}^{(\text{sub})} \),

\[
G_{ij,\tau}^{(\text{sub})}(\tilde{s}_{ij}, x) = 8\pi^2 \int d\Phi_{\gamma,ij} x g_{ij,\tau}^{(\text{sub})}(q_i, q_j, k), \tag{4.14}
\]

and \( \tilde{s}_{ij} = 2\tilde{q}_i \tilde{q}_j \). The momenta \( \tilde{q}_i, \tilde{q}_j \) result from mapping the momenta \( q_i, q_j, k \) to \( \Phi_{4f,ij} \).

The functions \( G_{ij,\tau}^{(\text{sub})} \) involve mass singularities from the initial state,

\[
G_{ij,\tau}^{(\text{sub})}(\tilde{s}_{ij}, x)|_{\text{sing}} = P_{ff}(x) \ln \left( \frac{\tilde{s}_{ij}}{m_c^2} \right), \quad i = 1, 2, \tag{4.15}
\]

which are associated to the splitting function. These singular terms are exactly the well-known universal logarithms of the structure-function approach for leading-logarithmic ISR.

\[\text{At the edges of the histogram columns this can also occur for "singular events". The finiteness of such contributions is guaranteed by the suppression of phase space for those events.}\]
For both emitter and spectator from the final state \((i, j = 3, \ldots, 6)\) the convolution over \(x\) in (4.13) is absent, so that the integrand depends on the \(4f\) kinematics of the original CM system in this case. For the other cases, where at least the emitter or the spectator are from the initial state, the soft singularities appear at the endpoint \(x \to 1\) in the convolution over \(x\). This IR-sensitive endpoint contributions can be separated from the convolution as follows,

\[
\int d\sigma_{\text{sing}}^{e^+e^-\to 4f} = -\frac{\alpha}{2\pi} \sum_{i,j=1}^{6} \sum_{\tau=\pm} (-1)^{i+j} Q_i Q_j \int_0^1 dx \int d\Phi_{4f,ij}(x) \mathcal{G}_{ij,\tau}^{(sub)}(\tilde{s}_{ij}, x) \mathcal{M}^{e^+e^-\to 4f}(\Phi_{4f,ij}(x), \tau \kappa_i)^2 \Theta(\Phi_{4f,ij}(x)) \]

\[
- \int d\Phi_{4f,ij}(1) \mathcal{G}_{ij,\tau}^{(sub)}(\tilde{s}_{ij}, x) \mathcal{M}^{e^+e^-\to 4f}(\Phi_{4f,ij}(1), \tau \kappa_i)^2 \Theta(\Phi_{4f,ij}(1)) \]

\[
- \frac{\alpha}{2\pi} \sum_{i,j=1}^{6} \sum_{\tau=\pm} (-1)^{i+j} Q_i Q_j \frac{1}{2s} \int d\Phi_{4f} \mathcal{G}_{ij,\tau}^{(sub)}(s_{ij}) \mathcal{M}^{e^+e^-\to 4f}(\Phi_{4f}, \tau \kappa_i)^2 \Theta(\Phi_{4f}) .
\]

If both \(i > 2\) and \(j > 2\), \(\mathcal{G}_{ij,\tau}^{(sub)}\) is proportional to \(\delta(1-x)\) and thus does not contribute to the second and third line of (4.16). The second term within square brackets in (4.16) represents the endpoint contribution (with opposite sign) at \(x \to 1\) that is split off. All the endpoint contributions, which are contained in the last term of (4.16), obey the \(4f\) kinematics of the original CM system, since \(\Phi_{4f} = \Phi_{4f,ij}(1)\). The functions \(\mathcal{G}_{ij,\tau}^{(sub)}\) originate from the integral of the functions \(\mathcal{G}_{ij,\tau}\) over \(x\),

\[
\mathcal{G}_{ij,\tau}^{(sub)}(\tilde{s}_{ij}) = \int_0^1 dx \mathcal{G}_{ij,\tau}^{(sub)}(\tilde{s}_{ij}, x) .
\]

As mentioned above, the photon mass \(\lambda\) and the fermion masses \(m_i\) have to be kept finite, in order to regularize soft and collinear singularities in (4.17). Among these integrals of course only the non-spin-flip parts receive singular contributions,

\[
\mathcal{G}_{ij,+}^{(sub)}(\tilde{s}_{ij}) = \mathcal{L}(\tilde{s}_{ij}, m_i^2) + C_{ij} - \frac{1}{2} ,
\]

\[
\mathcal{G}_{ij,-}^{(sub)}(\tilde{s}_{ij}) = \frac{1}{2} ,
\]

where

\[
\mathcal{L}(\tilde{s}_{ij}, m_i^2) = \ln\left(\frac{m_i^2}{\tilde{s}_{ij}}\right) \ln\left(\frac{\lambda^2}{\tilde{s}_{ij}}\right) + \ln\left(\frac{\lambda^2}{\tilde{s}_{ij}}\right) - \frac{1}{2} \ln^2\left(\frac{m_i^2}{\tilde{s}_{ij}}\right) + \frac{1}{2} \ln\left(\frac{m_i^2}{\tilde{s}_{ij}}\right) .
\]

These are exactly the soft and collinear singularities of the virtual \(\mathcal{O}(\alpha)\) corrections with opposite sign. The constants \(C_{ij}\), which are specific for the subtraction terms of Ref. [27], read

\[
C_{ab} = -\frac{\pi^2}{3} + 2 , \quad C_{ak} = \frac{\pi^2}{6} - 1 , \quad C_{ka} = -\frac{\pi^2}{2} + \frac{3}{2} , \quad C_{kl} = -\frac{\pi^2}{3} + \frac{3}{2} ,
\]

with \(a, b = 1, 2\) and \(k, l = 3, \ldots, 6\).

Owing to their kinematical structure, the sum of all endpoint contributions, i.e. the last line of (4.16), can be directly combined with the virtual corrections before integrating over the \(4f\) phase space.
4.2 The phase-space-slicing approach

The idea of the phase-space-slicing method is to divide the $4f\gamma$ phase space into singular and non-singular regions, then to evaluate the singular regions analytically and to perform an explicit cancellation of the arising soft and collinear singularities against their counterparts in the virtual corrections. The finite remainder can be evaluated by using the usual Monte Carlo techniques. For the actual implementation of this well-known procedure (see e.g. Ref. [39]) we closely follow the approaches of Refs. [40, 41]. We divide the five-particle phase space into soft and collinear regions by introducing the cut-off parameters $\delta_s$ and $\delta_c$, respectively. We decompose the real corrections as

$$d\sigma_{\text{soft}} + d\sigma_{\text{coll}} = d\sigma_{\text{finite}}^{e^+e^-\rightarrow 4f\gamma}. \quad (4.21)$$

Here $d\sigma_{\text{soft}}$ describes the contribution of the soft photons, i.e. of photons with energies $E_\gamma < \delta_s \sqrt{s}/2 = \Delta E$ in the CM frame, and $d\sigma_{\text{coll}}$ describes real photon radiation outside the soft-photon region ($E_\gamma > \Delta E$) but collinear to a charged fermion. We define the collinear region by $1 > \cos \theta_{qf} > 1 - \delta_c$, where $\theta_{qf}$ is the angle between the charged fermion and the emitted photon in the CM frame. The remaining part, which is free of singularities, is denoted by $d\sigma_{\text{finite}}^{e^+e^-\rightarrow 4f\gamma}$.

In the soft and collinear regions, the squared matrix element $|M^{e^+e^-\rightarrow 4f\gamma}|^2$ factorizes into the leading-order squared matrix element $|M^{\text{Born}}_{e^+e^-\rightarrow 4f\gamma}|^2$ and a soft or collinear factor. Also the five-particle phase space factorizes into a four-particle and a soft or collinear part, so that the integration over the photon phase space can be performed analytically.

In the soft-photon region, we apply the soft-photon approximation to $|M^{e^+e^-\rightarrow 4f\gamma}|^2$, i.e. the photon four-momentum $k$ is omitted everywhere but in the IR-singular propagators. Since we neglect $k$ also in the resonant gauge-boson propagators we have to assume $E_\gamma < \Delta E \ll \Gamma_W$. In this region $d\sigma_{\text{soft}}^{e^+e^-\rightarrow 4f\gamma}$ can be written as [33, 34]

$$d\sigma_{\text{soft}} = \frac{\alpha}{4\pi^2} \sum_{i=1}^{6} \sum_{j=i+1}^{6} (-1)^{i+j} Q_i Q_j \int_{E_\gamma < \Delta E} \frac{d^3k}{E_\gamma (kq_i + kq_j)} \left( \frac{q_i^\mu}{kq_i} - \frac{q_j^\mu}{kq_j} \right)^2 \quad (4.22)$$

with the basic integrals

$$I_{ij} = \frac{1}{2\pi} \int_{E_\gamma < \Delta E} \frac{d^3k}{E_\gamma (kq_i + kq_j)} q_i q_j. \quad (4.23)$$

The explicit expression for the integrals $I_{ij}$ can be found in Refs. [42, 43]. For our purpose it is sufficient to keep the fermion masses only as regulators for the collinear singularities ($E_i \gg m_i$). In this limit we obtain

$$d\sigma_{\text{soft}} = \frac{\alpha}{2\pi} \sum_{i=1}^{6} \sum_{j=i+1}^{6} (-1)^{i+j} Q_i Q_j \times \left\{ 2 \ln \left( \frac{2\Delta E}{\lambda} \right) \left[ 2 - \ln \left( \frac{s_{ij}^2}{m_i^2 m_j^2} \right) \right] - 2 \ln \left( \frac{4E_i E_j}{m_i m_j} \right) \right\}$$

20
\[ + \frac{1}{2} \ln^2 \left( \frac{4E_i^2}{m_f^2} \right) + \frac{1}{2} \ln^2 \left( \frac{4E_j^2}{m_f^2} \right) + \frac{2\pi^2}{3} + 2 \text{Li}_2 \left( 1 - \frac{4E_i E_j}{s_{ij}} \right) \right]. \quad (4.24) \]

Now we turn to the collinear singularities. In the collinear region, we consider an initial-state (final-state) fermion with momentum \( q_i \) being split into a collinear photon and a fermion with the resulting momentum \( \tilde{q}_i \) after (before) photon radiation, i.e. for initial-state radiation

\[ e(q_i) \rightarrow \gamma(k = (1 - x_i)q_i) + e(\tilde{q}_i = x_iq_i), \quad i = 1, 2, \]

and for final-state radiation

\[ f(\tilde{q}_i = q_i/z_i) \rightarrow \gamma(k = q_i(1 - z_i)/z_i) + f(q_i), \quad i = 3, \ldots, 6. \]

In the asymptotic limit, \( |\mathcal{M}^{e^+e^-\rightarrow 4f}e\gamma|^2 \) factorizes into the leading-order squared matrix element \( |\mathcal{M}^{e^+e^-\rightarrow 4f}|^2 \) and a collinear factor describing collinear initial-state and final-state radiation, respectively, as long as \( \delta_c \) is sufficiently small. In the collinear region also the five-particle phase space factorizes into a four-particle phase space and a collinear factor, so that the cross section for hard photon radiation \( (E_\gamma > \Delta E) \) in the collinear region \( 1 > \cos \theta_{\gamma f} > 1 - \delta_c \) reads

\[ \text{d}\sigma_{\text{coll}} = \text{d}\sigma_{\text{coll}}^{\text{initial}} + \text{d}\sigma_{\text{coll}}^{\text{final}} \] \quad (4.25)

with \( E_i \gg m_i, \theta_{\gamma f} = \mathcal{O}(m_i/E_i) \) and

\[ \text{d}\sigma_{\text{coll}}^{\text{initial}} = \sum_{i=1,2} \frac{\alpha}{2\pi} \int_0^{1-\delta_i} dx_i \left\{ \text{d}\sigma_{\text{Born}}^{e^+e^-\rightarrow 4f}(x_iq_i, +\kappa_i) P_{ff}(x_i) \left[ \ln \left( \frac{\hat{s}_c \delta_c}{2m_f^2 \frac{1}{x_i}} \right) - 1 \right] \right. \]

\[ + \left. \text{d}\sigma_{\text{Born}}^{e^+e^-\rightarrow 4f}(x_iq_i, -\kappa_i) (1 - x_i) \right\} \quad (4.26) \]

and

\[ \text{d}\sigma_{\text{coll}}^{\text{final}} = \sum_{i=3}^{6} \frac{\alpha}{2\pi} Q_i^2 \left\{ \text{d}\sigma_{\text{Born}}^{e^+e^-\rightarrow 4f}(q_i, +\kappa_i) \int_0^{1-\frac{\Delta E}{E_i}} dz_i \right. \]

\[ P_{ff}(z_i) \left[ \ln \left( \frac{4E_i^2 \delta_c}{m_i^2 \frac{1}{z_i}} \right) - 1 \right] \right. \]

\[ + \left. \text{d}\sigma_{\text{Born}}^{e^+e^-\rightarrow 4f}(q_i, -\kappa_i) \int_0^{1} dz_i (1 - z_i) \right\} \]

\[ = \sum_{i=3}^{6} \frac{\alpha}{2\pi} Q_i^2 \left\{ \text{d}\sigma_{\text{Born}}^{e^+e^-\rightarrow 4f}(q_i, +\kappa_i) \right. \]

\[ \times \left( \frac{3}{2} + 2 \ln \left( \frac{\Delta E}{E_i} \right) \right) \left[ 1 - \ln \left( \frac{4E_i^2 \delta_c}{m_i^2 \frac{1}{2}} \right) \right] + \frac{5}{2} - \frac{2\pi^2}{3} \right) \]

\[ + \left. \text{d}\sigma_{\text{Born}}^{e^+e^-\rightarrow 4f}(q_i, -\kappa_i) \frac{1}{2} \right\} \quad (4.27) \]

with the splitting function \( P_{ff} \) of \([1,9]\). In the case of initial-state radiation, the four-particle phase space is generated in the CM system of the hard scattering process after the emission of the collinear photon, i.e. in the system with CM energy \( \hat{s} = x_{1,2} s \), and the
four-momenta are then Lorentz-boosted back to the laboratory frame. In the case of final-state radiation the CM energy before and after collinear photon radiation are the same, so that \( d\sigma_{e^+e^-\rightarrow 4f} \) does not depend on \( z_i \). Thus, the integration over \( z_i \) can be performed analytically, as done in the last equation of (4.27). Note that this procedure implicitly assumes that photons within small cones collinear to charged final-state fermions will never be separated from those collinear fermions.

Subtracting the soft and collinear cross sections (4.24) and (4.25) from the cross section of the process \( e^+e^- \rightarrow 4f \gamma \) (4.21) yields the finite cross section \( d\sigma_{e^+e^-\rightarrow 4f\gamma} \). As usual in the phase-space-slicing approach, this subtraction is done in practice by imposing cuts on the \( 4f\gamma \) phase space, i.e. a photon-energy cut, \( E_\gamma > \Delta E = \delta_s\sqrt{s}/2 \), and a cut on the angles between the photon and charged fermions, \(-1 < \cos\theta_{\gamma f} < 1 - \delta_c \), but by using the exact matrix elements for \( e^+e^- \rightarrow 4f\gamma \) of Ref. \[ 28 \]. The collinear and soft cross sections are added,

\[
d\sigma^{e^+e^-\rightarrow 4f\gamma} = d\sigma_{\text{soft}} + d\sigma_{\text{coll}},
\]

and combined with the singular part of the virtual cross section \( d\sigma_{\text{sing}}^{e^+e^-\rightarrow 4f} \) of Section 4.3, as described in Section 2.3.

The full radiative corrections are split into a 4f part with no or an invisible photon, \( d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,finite,DPA}} + d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt+real,sing}} \), and a 4f\gamma part with a visible photon, \( d\sigma^{e^+e^-\rightarrow 4f\gamma}_{\text{finite}} \). Both contributions depend on the cut-off parameters \( \delta_s, \delta_c \). The dependence on these technical cuts cancels in the sum when the cut-off parameters are chosen to be small enough so that the soft-photon and leading-pole approximations apply. In Figure 4 we display the cut-off dependence of the total cross sections to both parts separately and also illustrate the cancellation of the cut-off dependence in their sum.

### 4.3 Definition of finite virtual corrections

Next, we turn to the extraction of the singular contributions from the virtual corrections, i.e. the definition of \( d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,sing}} \) in (4.28). Here we can make use of the well-known cancellation for soft and collinear singularities between real and virtual corrections (KLN theorem \[ 13 \]). For inclusive photons, the only uncanceled singularities are the collinear singularities originating from the continuum part of initial-state radiation \((x_i \neq 1)\). In other words, the singularities of the virtual corrections are exactly given by the endpoint parts defined in Section 4.1 within the subtraction approach, but with opposite sign. Therefore, we can define the singular part of the virtual corrections by

\[
d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt,sing,sub}} = d\sigma^{e^+e^-\rightarrow 4f}_{\text{Born}} \frac{\alpha}{2\pi} \sum_{i=1}^{6} \sum_{j=i+1}^{6} (-1)^{i+j} Q_i Q_j \left( L(s_{ij}, m_i^2) + L(s_{ij}, m_j^2) + C_{ij} + C_{ji} \right)
\]

with \( L(s_{ij}, m_i^2) \) and \( C_{ij} \) defined in (4.19) and (4.20) of Section 4.1 respectively. Note that we do not include the polarization-dependent contributions of the endpoint parts [the terms \( \pm 1/2 \) in (4.18)] in this definition. Since the polarization-dependent terms are non-singular, the simple formulae of this section hold also for the polarized case. Subtracting (4.29) from the virtual corrections yields the finite virtual cross section in the subtraction-method-inspired approach. We note that it is gauge-invariant by construction, since only the Born cross section enters without modification.
The definition of the virtual singular cross section is a matter of convention as far as finite contributions are concerned, since finite terms can be redistributed between singular and finite parts. Another possibility is to define a finite virtual photon contribution by following the approach of Ref. \[31\]. In this YFS-inspired approach, the IR-singular contributions are extracted with the help of an explicitly U(1)-gauge-invariant current as follows,

\[
d\sigma_{e^+e^\rightarrow 4f} = d\sigma_{\text{Born}} \frac{\alpha}{2\pi} \sum_{i=1}^{6} \sum_{j=i+1}^{6} (-1)^{i+j} Q_i Q_j \times \int \frac{d^4k}{(2\pi)^4} \left[ \frac{(k - 2\theta_i q_i)^\mu}{k^2 - 2\theta_i q_i k + i\epsilon} \right] \left[ \frac{(k + 2\theta_j q_j)^\mu}{k^2 + 2\theta_j q_j k + i\epsilon} \right]^2 \frac{1}{k^2 - \lambda^2 + i\epsilon} \right],
\]

where \(\theta_i = +1(-1)\) if \(i\) is a final-state (initial-state) fermion. Again we keep the fermion masses only as regulators for the collinear singularities. As can easily be seen from (4.30), the U(1) gauge invariance is guaranteed by the existence of a conserved current. This YFS factor takes care only of the IR-singular logarithms and the double-logarithmic mass...
singularities. Since the single-logarithmic mass-singular terms build a separately gauge-invariant subset we can simply add them to the YFS factor as follows \[44, 7\],
\[
\sigma_{\text{virt\_sing\_YFS}} = \sigma_{\text{soft\_YFS}} - \sigma_{\text{Born}} \frac{\alpha}{2\pi} \sum_{i=1}^{6} \sum_{j=i+1}^{6} (-1)^{i+j} Q_i Q_j \ln \left( \frac{s_{ij}}{m_i m_j} \right). \tag{4.31}
\]

This defines the singular virtual cross section in the YFS-inspired approach.

While the requirement of the cancellation of the soft and collinear singularities in \[(2.14)\] unambiguously fixes the logarithmic terms, the two definitions of the virtual singular contribution differ by finite, non-logarithmic terms
\[
\sigma_{\text{virt\_sing\_sub}} - \sigma_{\text{virt\_sing\_YFS}} = \frac{\alpha}{2\pi} \sum_{i=1}^{6} \sum_{j=i+1}^{6} (-1)^{i+j} Q_i Q_j \left[ \frac{\pi^2}{3} + \theta(\theta_i \theta_j) \pi^2 - 2 + C_{ij} + C_{ji} \right],
\tag{4.32}
\]
as can be seen by comparing \[(1.29)\] and \[(1.31)\]. Since the singular term \(\sigma_{\text{virt\_sing\_(sub\_YFS)}}\) is treated once in DPA, when subtracted from the doubly-resonant virtual corrections \(\sigma_{\text{virt\_DPA}}\) [see \[(2.14)\]], and once exactly, when included in \(\sigma_{\text{virt\_real\_sing}}\), the cross sections calculated with the subtraction-method-inspired and in the YFS-inspired approach differ by non-doubly-resonant terms
\[
\Delta = \left( \sigma_{\text{virt\_Born}} - \sigma_{\text{ Born\_DPA}} \right) \times \frac{\alpha}{2\pi} \sum_{i=1}^{6} \sum_{j=i+1}^{6} (-1)^{i+j} Q_i Q_j \left[ \frac{\pi^2}{3} + \theta(\theta_i \theta_j) \pi^2 - 2 + C_{ij} + C_{ji} \right]. \tag{5.1}
\]

As can be seen from \[(1.33)\], the ambiguity induced by these finite terms is of the order of the uncertainty of the DPA. A numerical discussion of this ambiguity for observables can be found in Section \[7.3\].

While the two different definitions of the virtual singular corrections are inspired by the subtraction method and the YFS treatment of IR singularities, their use is independent of the method employed for the treatment of singularities in the bremsstrahlung contribution. This means that for the splitting between singular and finite virtual corrections both definitions \(\sigma_{\text{virt\_sing\_(sub\_YFS)}}\) can be used in the subtraction and phase-space-slicing approaches.

### 5 Higher-order initial-state radiation

The emission of photons collinear to the incoming electrons or positrons leads to corrections that are enhanced by large logarithms. In order to achieve an accuracy at the few 0.1% level, the corresponding higher-order contributions, i.e. contributions beyond \(\mathcal{O}(\alpha)\), must be taken into account. This can be done in the structure-function method \[15, 7\]. According to the mass-factorization theorem, the leading-logarithmic (LL) initial-state QED corrections can be written as a convolution of the lowest-order cross section with structure functions, and the corresponding differential cross section reads
\[
\int d\sigma^{\text{LL}} = \int_0^1 dx_1 \int_0^1 dx_2 \Gamma_{\text{ee}}^{\text{LL}}(x_1, Q^2) \Gamma_{\text{ee}}^{\text{LL}}(x_2, Q^2) \int d\sigma_{\text{Born}}^{e^+e^-\to 4f}(x_1 p_+, x_2 p_-). \tag{5.1}
\]
Here $x_1$ and $x_2$ denote the fractions of the longitudinal momentum carried by the incoming electron and positron momenta just before the hard scattering process occurs. This means that the incoming momenta $p_\pm$ before emission of the collinear photon are rescaled by $x_1, x_2$, and the CM frame of the hard scattering process with the lowest-order cross section $d\sigma^{e^+e^{-}\rightarrow A}_\text{Born}(x_1 p_+, x_2 p_-)$ is boosted along the beam axis. The LL structure function including $\mathcal{O}(\alpha^3)$ terms is given by [7]

$$
\Gamma^{\text{LL}}_{ee}(x, Q^2) = \frac{\exp\left(-\frac{1}{2} \beta_e^2 \gamma_E + \frac{3}{8} \beta_e \right)}{\Gamma\left(1 + \frac{1}{2} \beta_e \right)} \cdot \frac{\beta_e}{2} \left(1 - x\right) - \frac{\beta_e}{4} \left(1 + x\right) \\
- \frac{\beta_e^2}{32} \left\{ \frac{1 + 3x^2}{1 - x} \ln(x) + 4(1 + x) \ln(1 - x) + 5 + x \right\} \\
- \frac{\beta_e^3}{384} \left\{ (1 + x) \left[ 6 \text{Li}_2(x) + 12 \ln^2(1 - x) - 3\pi^2 \right] \\
+ \frac{1}{1 - x} \left[ \frac{3}{2} (1 + 8x + 3x^2) \ln(x) + 6(x + 5)(1 - x) \ln(1 - x) \\
+ 12(1 + x^2) \ln(x) \ln(1 - x) - \frac{1}{2} (1 + 7x^2) \ln^2(x) \\
+ \frac{1}{4} (39 - 24x - 15x^2) \right] \right\} \\
(5.2)
$$

with

$$
\beta_e = \frac{2\alpha}{\pi} (L - 1),
(5.3)
$$

and the leading logarithm

$$
L = \ln \frac{Q^2}{m_e^2}.
(5.4)
$$

Note that the scale $Q^2$ is not fixed within LL approximation, but has to be set to a typical scale of the underlying process; for the numerics we use $Q^2 = s$. In (5.2) $\gamma_E$ is the Euler constant and $\Gamma(y)$ the gamma function, which should not be confused with the structure functions. Note that some non-leading terms are incorporated, taking into account the fact that the residue of the soft-photon pole is proportional to $L - 1$ rather than $L$ for the initial-state photon radiation.

We add the cross section (5.4) to the one-loop result and subtract the lowest-order and one-loop contributions $d\sigma^{\text{LL},1}$ already contained within this formula,

$$
\int d\sigma^{\text{LL},1} = \int_0^1 dx_1 dx_2 \left[ \delta(1 - x_1)\delta(1 - x_2) + \Gamma^{\text{LL},1}_{ee}(x_1, Q^2)\delta(1 - x_2) \\
+ \delta(1 - x_1)\Gamma^{\text{LL},1}_{ee}(x_2, Q^2) \right] \int d\sigma^{e^+e^{-}\rightarrow A}_\text{Born}(x_1 p_+, x_2 p_-),
(5.5)
$$
in order to avoid double counting. The one-loop contribution to the structure function reads

$$
\Gamma^{\text{LL},1}_{ee}(x, Q^2) = \frac{\beta_e}{4} \left( \frac{1 + x^2}{1 - x} \right)_+ \\
= \frac{\beta_e}{4} \lim_{\epsilon \to 0} \left[ \delta(1 - x) \left( \frac{3}{2} + 2 \ln \epsilon \right) + \theta(1 - x - \epsilon) \frac{1 + x^2}{1 - x} \right].
(5.6)
$$
Note that the uncertainty that is connected with the choice of $Q^2$ enters now in $\mathcal{O}(\alpha^2)$, since all $\mathcal{O}(\alpha)$ corrections, including constant terms, are taken into account.

6 QCD corrections

QCD corrections enter the processes $e^+e^- \rightarrow WW \rightarrow 4f$ in two different places. On the one hand, the hadronic $W$ width receives a QCD correction, namely a factor $(1+\alpha_s/\pi)$ \cite{12, 13} in $\mathcal{O}(\alpha_s)$. This affects the total $W$ width $\Gamma_W$ in the resonant $W$ propagators. On the other hand, each hadronically decaying $W$ boson receives a QCD correction to the $Wq\bar{q}'$ vertex. If the full phase space for gluon emission is integrated over, this correction reduces to a multiplicative correction factor $(1+\alpha_s/\pi)$ for each hadronically decaying $W$ boson in $\mathcal{O}(\alpha_s)$. The application of this inclusive factor to distributions is usually called "naive QCD correction". In the total cross section these naive QCD factors cancel against the corresponding factors in the $W$ width in the resonant $W$ propagators. Note that one-gluon exchange\footnote{Multiple gluon exchange, which is intrinsically connected to colour reconnection, is not considered here. It should be included in the hadronization simulation.} between quarks from different $W$ bosons vanishes exactly for pure charged-current (CC) reactions, because of the colour structure.

If the gluon phase space is not integrated over, QCD corrections have to be calculated from the virtual vertex corrections and from real gluon emission with the matrix elements $e^+e^- \rightarrow 4f + g$. This option is also supported by RACOONWW for four-fermion final states of the CC11 class. The calculation is technically similar to the one for the photonic corrections \cite{28}.

7 Numerical results

For the numerical results, with the exception of the comparison to the results of other groups, we used (as in Refs. \cite{25, 26, 3}) the following parameters:

$$
G_\mu = 1.16637 \times 10^{-5} \text{GeV}^{-2}, \quad \alpha = 1/137.0359895,
$$
$$
M_W = 80.35 \text{GeV}, \quad \Gamma_W = 2.08699\ldots \text{GeV},
$$
$$
M_Z = 91.1867 \text{GeV}, \quad \Gamma_Z = 2.49471 \text{GeV},
$$
$$
m_t = 174.17 \text{GeV}, \quad M_H = 150 \text{GeV},
$$
$$
m_e = 510.99907 \text{keV}.
$$

(7.1)

We work in the fixed-width scheme and fix the weak mixing angle by $c_w = M_W/M_Z$, $s_w^2 = 1 - c_w^2$. The parameter set (7.1) is over-complete but self-consistent. Instead of $\alpha$ we use $G_\mu$ to parametrize the lowest-order matrix element, i.e. we use the effective coupling

$$
\alpha_{G_\mu} = \sqrt{2}G_\mu M_W^2 s_w^2 \pi
$$

(7.2)

in the lowest-order matrix element. This parametrization has the advantage that all higher-order contributions associated with the running of the electromagnetic coupling from zero to $M_W^2$ and the leading universal two-loop $m_t$-dependent corrections are already absorbed in the Born cross section. In the relative $\mathcal{O}(\alpha)$ corrections, on the other hand,
we use $\alpha = \alpha(0)$, since in the real corrections and in the mass-singular virtual corrections, which yield the bulk of the remaining corrections, the scale of the real or virtual photon is zero. The W-boson width given above is calculated including the electroweak and QCD one-loop corrections with $\alpha_s = 0.119$.

As default, we use the following set of separation and recombination cuts:

1. All photons within a cone of 5 degrees around the beams are treated as invisible, i.e. their momenta are disregarded when calculating angles, energies, and invariant masses.

2. Next, the invariant masses $M_{f\gamma}$ of the photon with each of the charged final-state fermions are calculated. If the smallest $M_{f\gamma}$ is smaller than a certain cutoff $M_{\text{rec}}$ or if the energy of the photon is smaller than 1 GeV, the photon is combined with the charged final-state fermion that leads to the smallest $M_{f\gamma}$, i.e. the momenta of the photon and the fermion are added and associated with the momentum of the fermion, and the photon is discarded.

3. Finally, all events are discarded in which one of the charged final-state fermions is within a cone of 10 degrees around the beams. No other cuts are applied.

In Refs. [25, 26] we have already presented a short survey of numerical results for $\mathcal{O}(\alpha)$ corrections to $e^+e^- \rightarrow WW \rightarrow 4f$ obtained with RACOONWW for LEP2 and linear-collider energies. In the meantime we extended RACOONWW by taking into account higher-order ISR and by considering QCD one-loop corrections as described in Section 4.1 and Section 4.2, respectively. Here we concentrate on LEP2 energies and provide results for the total cross sections and distributions obtained with this extended version of RACOONWW. Additional numerical results can also be found in Ref. [3]. Unless stated otherwise, for the lowest-order contributions only the CC03 diagrams are taken into account. Our “best” results comprise the CC03 Born cross sections and the corrected cross sections including all the radiative corrections described in this paper, i.e. electroweak one-loop corrections in DPA, exact $\mathcal{O}(\alpha)$ photon radiation based on the full matrix element for $e^+e^- \rightarrow 4f\gamma$, higher-order ISR up to $\mathcal{O}(\alpha^3)$ and “naive” QCD $\mathcal{O}(\alpha_s)$ corrections (see Section 6).

As mentioned before, RACOONWW involves two branches for the treatment of soft and collinear singularities, one following the subtraction (see Section 4.1) and one the phase-space-slicing method (see Section 4.2). While these two branches use the same matrix elements, the Monte Carlo integration is performed completely independently, thus providing us with a powerful numerical check of RACOONWW. In the following we present numerical results of both branches of RACOONWW, starting with the total cross sections to $e^+e^- \rightarrow WW \rightarrow 4f$ at LEP2 CM energies. If not stated otherwise, the finite virtual corrections that are treated in DPA are defined by the subtraction-method-inspired approach (1.29).

9 Except for the 1 GeV cut, the described cut and recombination procedure coincides with the one used in Refs. [25, 26].
7.1 The total W-pair production cross section at LEP2

In Tables 1 and 2 we list the predictions of RacoonWW for the different W-pair production channels as well as for the total CC03 W-pair production cross section for LEP2 CM energies. Note that for CC03 and negligible fermion masses the results are independent of the final state within these channels. In the calculation of these numbers no cuts have been applied. The only difference between the results shown in the two tables lies in the ISR beyond $O(\alpha)$: while Table 1 includes only the $O(\alpha^2)$ contributions of (5.2), Table 2 includes the $O(\alpha^3)$ contributions in addition. There are no significant differences between both tables, i.e. the effect of the $O(\alpha^3)$ contributions is smaller than the integration error, which is at the level of 0.1%. The given errors are purely statistical. The errors for the total cross sections were obtained by adding the (statistically correlated) errors of the various channels linearly.

Figure 5 shows a comparison of the results of RacoonWW as presented in Table 1 (see also Ref. [3]) and of other Monte Carlo predictions with recent LEP2 data, as given by the LEP Electroweak Working Group [46] for the Winter 2000 conferences.

The data are in good agreement with the predictions of RacoonWW and YFSWW3 [20, 21]. At the time of the conferences in Winter 2000 the predictions of YFSWW3 were about 0.5–0.7% larger than those of RacoonWW, which is somewhat larger than the intrinsic DPA ambiguity. Meanwhile, however, the main source of this discrepancy was found, and the improved YFSWW3 results differ from the ones of RacoonWW only by about 0.3% at LEP2 energies. More details on the conceptual differences of the
| √s/GeV | σ_{leptonic}/fb | σ_{semileptonic}/fb | σ_{hadronic}/fb | σ_{WW}/pb |
|-------|-----------------|---------------------|-----------------|----------|
| 172.086 | 142.088(71) | 442.50(36) | 1376.14(67) | 12.0934(76) |
| 176.000 | 160.076(78) | 498.03(25) | 1550.04(75) | 13.6171(67) |
| 182.655 | 180.697(89) | 562.22(28) | 1749.48(86) | 15.3708(76) |
| 188.628 | 190.882(96) | 594.31(55) | 1848.07(92) | 16.2420(111) |
| 191.583 | 194.271(118) | 604.12(31) | 1880.19(94) | 16.5187(85) |
| 195.519 | 197.320(123) | 614.11(31) | 1911.45(97) | 16.7910(88) |
| 199.516 | 199.497(103) | 620.53(33) | 1931.28(99) | 16.9670(89) |
| 201.624 | 200.200(104) | 622.65(33) | 1937.94(100) | 17.0254(89) |
| 205.000 | 200.910(107) | 624.95(33) | 1945.00(103) | 17.0876(91) |

Table 1: RacoonWW predictions for W-pair production cross sections for LEP2 CM energies including $O(\alpha^2)$ ISR (results for the Winter conferences 2000)

| √s/GeV | σ_{leptonic}/fb | σ_{semileptonic}/fb | σ_{hadronic}/fb | σ_{WW}/pb |
|-------|-----------------|---------------------|-----------------|----------|
| 168.000 | 115.639(57) | 359.86(18) | 1120.03(55) | 9.8392(49) |
| 172.086 | 142.043(70) | 442.36(36) | 1375.72(67) | 12.0896(76) |
| 176.000 | 160.192(78) | 498.39(24) | 1551.17(75) | 13.6271(66) |
| 180.000 | 173.511(85) | 539.96(27) | 1679.35(82) | 14.7585(72) |
| 182.655 | 180.669(89) | 562.13(28) | 1749.21(86) | 15.3684(76) |
| 185.000 | 185.474(92) | 576.88(29) | 1794.93(88) | 15.7716(78) |
| 188.628 | 190.959(95) | 594.55(55) | 1848.84(91) | 16.2486(111) |
| 191.583 | 194.271(118) | 604.13(31) | 1880.21(94) | 16.5188(85) |
| 195.519 | 197.435(123) | 614.47(31) | 1912.58(97) | 16.8009(87) |
| 199.516 | 199.640(103) | 620.98(33) | 1932.66(99) | 16.9791(88) |
| 201.624 | 200.271(104) | 622.65(33) | 1937.94(100) | 17.0254(89) |
| 205.000 | 200.742(105) | 624.88(33) | 1943.80(103) | 17.0316(89) |
| 208.000 | 200.973(106) | 626.33(33) | 1945.37(102) | 17.0942(90) |
| 210.000 | 200.888(107) | 624.88(33) | 1944.79(103) | 17.0858(91) |
| 215.000 | 200.334(107) | 623.28(33) | 1938.88(103) | 17.0378(91) |

Table 2: RacoonWW predictions for W-pair production cross sections for LEP2 CM energies including $O(\alpha^3)$ ISR (results for the Summer conferences 2000)
### Table 3: Total CC03 cross sections when using the subtraction method and the phase-space-slicing method at $\sqrt{s} = 200$ GeV with bare cuts. The numbers in parentheses are statistical errors corresponding to the last digits.

| final state | program | $\sigma_{\text{tot}}$ [fb] |
|-------------|---------|--------------------------|
| $\nu_\mu\mu^+\tau^-\bar{\nu}_\tau$ | slicing | 211.166(36) | 191.847(91) |
| | subtraction | 211.034(39) | 191.686(46) |
| | (sub–sli)/sli | $-0.06(3)\%$ | $-0.08(5)\%$ |
| $u\bar{d}\mu^-\bar{\nu}_\mu$ | slicing | 627.38(11) | 591.11(25) |
| | subtraction | 627.22(12) | 590.94(14) |
| | (sub–sli)/sli | $-0.03(3)\%$ | $-0.03(5)\%$ |
| $u\bar{d}s\bar{c}$ | slicing | 1864.79(32) | 1821.72(66) |
| | subtraction | 1864.28(35) | 1821.16(43) |
| | (sub–sli)/sli | $-0.03(3)\%$ | $-0.03(4)\%$ |

two generators, as well as a detailed comparison of numerical results, can be found in Ref. [3]. A brief summary is given in Section 7.4.2. Figure 5 also includes the prediction provided by GENTLE [47], which is 2–2.5% larger than those from RACOONWW and YFSWW3. This difference is due to the neglect of non-leading, non-universal $O(\alpha)$ corrections in GENTLE. The 1–2% reduction in the $W$-pair production cross section by these effects could already be seen from the results of Refs. [6, 7, 8], where such corrections to on-shell $W$-pair production were considered. In summary, the comparison between predictions of the electroweak Standard Model and LEP2 data reveals empirical evidence of non-leading electroweak radiative corrections beyond the level of universal effects.

### 7.2 Distributions for $e^+e^- \rightarrow u\bar{d}\mu^-\bar{\nu}_\mu$ at LEP2

Now we turn to results obtained with the cut and recombination procedure described at the beginning of Section 4. We consider the cases of a tight recombination cut $M_{\text{rec}} = 5$ GeV (“bare”) and of a loose recombination cut $M_{\text{rec}} = 25$ GeV (“calo”).

In Table 3 we provide the total cross sections for leptonic, semileptonic and hadronic processes at the CM energy $\sqrt{s} = 200$ GeV with the bare cuts applied. The corrections are $-9.1\%$, $-5.8\%$, and $-2.3\%$ for the leptonic, semileptonic and hadronic channel, respectively. The results obtained when using the phase-space-slicing method agree well with those of the subtraction method, i.e. within the statistical errors of about $0.05\%$. For calo cuts we find the same good agreement between both branches. The differences between the cross sections for bare and calo cuts are below $0.05\%$; of course, the lowest-order results are not affected by the recombination procedure.

In the following we show a variety of distributions for the semi-leptonic process $e^+e^- \rightarrow u\bar{d}\mu^-\bar{\nu}_\mu$ at $\sqrt{s} = 200$ GeV at lowest order and when taking into account radiative
Figure 6: Distributions in the $W^+$ (l.h.s.) and $W^-$ (r.h.s.) invariant masses for a calorimeter setup for $e^+e^- \rightarrow u\bar{d}\mu^-\bar{\nu}_\mu$ at $\sqrt{s} = 200$ GeV.
Figure 7: Relative corrections to the $W^+$ invariant-mass distributions for the bare (l.h.s.) and calo (r.h.s.) setup for $e^+e^- \rightarrow ud\mu^-\bar{\nu}_\mu$ at $\sqrt{s} = 200$ GeV

Figure 8: Relative corrections to the $W^-$ invariant-mass distributions for the bare (l.h.s.) and calo (r.h.s.) setup at for $e^+e^- \rightarrow ud\mu^-\bar{\nu}_\mu$ at $\sqrt{s} = 200$ GeV
Figure 9: Distributions in the cosine of the $W^+$ production angle with respect to the $e^+$ beam (l.h.s.) and the relative corrections (r.h.s.) for $e^+e^- \rightarrow ud\mu^-\bar{\nu}_\mu$ at $\sqrt{s} = 200$ GeV.

Figure 10: Distributions in the cosine of the $\bar{d}$ decay angle with respect to the $W^+$ direction (l.h.s.) and the relative corrections (r.h.s.) for $e^+e^- \rightarrow ud\mu^-\bar{\nu}_\mu$ at $\sqrt{s} = 200$ GeV.
agree with each other within integration errors, i.e. in general to better than 0.5%. The increase of the integration errors for large decay angles is due to the smallness of the corresponding cross section.

Finally, we consider distributions in azimuthal angles. In Figure 11 we show the distributions over the azimuthal decay angle $\phi_{W^+}$ of the $W^+$ boson, i.e. the angle between the decay plane of the $W^+$ and the plane of $W$-pair production,

$$\cos \phi_{W^+} = \frac{(k_+ \times p_+)(k_+ \times k_1)}{|k_+ \times p_+||k_+ \times k_1|}, \quad \text{sgn}(\sin \phi_{W^+}) = \text{sgn}\left\{ k_+ \cdot ([k_+ \times p_+] \times (k_+ \times k_1)) \right\}.$$  

(7.3)

In this distribution the corrections vary only weakly with the angle.

The distributions over the angle $\phi$ between the two planes spanned by the momenta of the two fermion pairs in which the $W$ bosons decay, i.e. (note that $k_+ = -k_-$ for non-photonic events)

$$\cos \phi = \frac{(k_+ \times k_1)(-k_- \times k_3)}{|k_+ \times k_1||-k_- \times k_3|}, \quad \text{sgn}(\sin \phi) = \text{sgn}\left\{ k_+ \cdot ([k_+ \times k_1] \times (-k_- \times k_3)) \right\},$$  

(7.4)

are presented in Figure 12. The corrections are about $-5\%$ except for angles $\phi$ about $0^\circ$ or $180^\circ$, i.e. if the two decay planes coincide. This can be understood as follows: for non-photonic events the angle $\phi$ is related to the azimuthal decay angles $\phi_{W^\pm}$ by $\phi_{W^+} \pm \phi_{W^-}$ plus some constant (depending on the precise definition of $\phi_{W^-}$). Since the virtual corrections depend on $\phi_{W^\pm}$ only via $\sin \phi_{W^\pm}$ and $\cos \phi_{W^\pm}$, which are rather smooth functions, the peaks for $\phi = 0^\circ$ and $\phi = 180^\circ$ cannot originate from the virtual corrections. If, however, a hard photon is emitted, the finite angle between $k_+$ and $-k_-$ leads to a finite angle between the two decay planes except for the rare situation where $k_1$ and $k_3$ are in the plane spanned by $k_+$ and $-k_-$. Thus, for hard photonic events $\phi = 0^\circ$ and $\phi = 180^\circ$ are suppressed, and the large negative corrections at these angles are remnants of the virtual counterparts of the collinear singularities of the ISR, which are only partially compensated by real photon emission. This interpretation is supported by the numerical observation that the peaks are mainly contained in the convolution over $x$ in $d\sigma^{e^+e^-\rightarrow 4f}_{\text{virt+real,sing}}$.

If we had used $k_+$ instead of $-k_-$ in the definition of $\phi$, these large corrections would be absent.

Theoretically the shown azimuthal-angle distributions are of particular interest, since they are sensitive to the imaginary parts of the one-loop corrections. As can be deduced from App. A of Ref. [19], the contributions of the imaginary parts of the one-loop corrections always involve a factor $\sin \phi_{W^+}$ or a factor $\sin \phi_{W^-}$ together with symmetric functions in these angles. Consequently, they average to zero if the azimuthal angles of the decay fermions are integrated over. This is obviously the case for all quantities discussed so far except for the $\phi$ and $\phi_{W^+}$ distributions. While we do not find significant effects of imaginary loop parts in the $\phi$ distribution, we find a small impact on the $\phi_{W^+}$ distribution. In Figure 13 we compare the relative corrections to the $\phi_{W^+}$ distribution with (“imag”) and without (“def”) imaginary parts.
Figure 11: Distributions in the azimuthal $W^+$ angle (l.h.s.) and the relative corrections (r.h.s.) for $e^+e^- \rightarrow u\bar{d} \mu^- \bar{\nu}_\mu$ at $\sqrt{s} = 200$ GeV

Figure 12: Distributions in the azimuthal $\phi$ angle (l.h.s.) and the relative corrections (r.h.s.) for $e^+e^- \rightarrow u\bar{d} \mu^- \bar{\nu}_\mu$ at $\sqrt{s} = 200$ GeV
Figure 13: Relative corrections to the distribution in the azimuthal angle $\phi_{W^+}$ with ("imag") and without ("def") imaginary parts of the loop integrals for $e^+e^- \rightarrow ud\mu^-\bar{\nu}_\mu$ at $\sqrt{s} = 200$ GeV

7.3 Discussion of intrinsic ambiguities and of other options

In order to investigate the accuracy of the DPA quantitatively, we have modified the implementation of the DPA and compared the obtained results. The differences should be of the order of $\alpha \Gamma_W/\Gamma_W$. Recall that in RACOONWW only the finite (i.e. non-mass-singular) virtual corrections are treated in DPA, while real photon emission is based on the full $e^+e^- \rightarrow 4f\gamma$ matrix element with the exact five-particle phase space. Thus, only finite virtual corrections are affected by the following modifications. Specifically, we consider four types of options:

- Different on-shell projections:
  As explained in Section 2.1.3, one has to specify a projection of the physical momenta to a set of momenta for on-shell W-pair production and decay in order to define a DPA. This can be done in an obvious way by fixing the direction of one of the W bosons and of one of the final-state fermions originating from either W boson in the CM frame of the incoming $e^+e^-$ pair. The default in RACOONWW, which is explicitly specified in App. A, is to fix the directions of the momenta of the fermions (not of the anti-fermions) resulting from the $W^+$ and $W^-$ decays ("def"). A different projection is obtained by fixing the direction of the anti-fermion from the $W^+$ decay ("proj") instead of the fermion direction.

- Different definitions of finite virtual corrections:
  In Section the matching of soft and collinear singularities between virtual and real corrections has been explained in detail. Recall that the redistribution of these

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10 This option only illustrates the effect of different on-shell projections in the four-particle phase space; if real photonic corrections were treated in DPA the impact of different projections could be larger.
singular parts fixes only the universal, singular parts, while the redistribution of non-singular parts is a mere convention. Owing to the asymmetric treatment of the corrections (finite virtual in DPA, real and mass-singular virtual from full matrix elements), different redistributions of non-singular contributions change the result by terms of the order \((\alpha/\pi)(\Gamma_W/M_W)\), which is beyond the accuracy of the DPA. As discussed in Section 4, \textsc{RacoonWW} allows to choose between two possible definitions of the finite part of the virtual photonic corrections to which the DPA is applied. As default, the subtraction-method-inspired approach (4.29) is chosen ("def") and compared to the YFS-inspired approach ("eik") (4.31). The two treatments differ in the finite parts of the contribution \(d\sigma_{\text{virt,sing}}^{e^+e^\rightarrow 4f}\) (see Section 4.3), which is subtracted from the virtual corrections in DPA and added in its exact form to the singular part of the real photon corrections. As can be seen from the difference \(\Delta\) of (4.33), the ambiguity originates from terms of the form \((\alpha/\pi)^2\pi^2\times O(1)\) which are either multiplied with the DPA or with the full off-shell Born cross sections. 

- **On-shell versus off-shell Coulomb singularity:**
  The Coulomb singularity is (up to higher orders) fully contained in the virtual \(O(\alpha)\) correction. Performing the on-shell projection of the DPA to the full virtual correction leads to the on-shell Coulomb singularity, which is a simple factor of \(\alpha\pi/(2\beta)\). However, since the Coulomb singularity is an important correction in the LEP2 energy range and is also known beyond DPA [49], \textsc{RacoonWW} includes these extra off-shell parts of the Coulomb correction as default. This replacement of the Coulomb singularity is performed by adding and subtracting the corresponding contributions in the virtual non-factorizable corrections, as described in Ref. [14]. Switching the extra off-shell parts of the Coulomb correction off ("Coul"), yields an effect of the order of the uncertainty of the DPA. Note that for CM energies close to the W-pair-production threshold, the on-shell Coulomb singularity is not adequate, and the difference between on-shell and off-shell Coulomb singularity cannot be viewed as a measure of the theoretical uncertainty.

- **Imaginary parts of virtual corrections:**
  As default the imaginary parts of the loop integrals are neglected in \textsc{RacoonWW}. However, \textsc{RacoonWW} contains an option ("imag") that takes into account the imaginary parts. Since all contributions of these imaginary parts are proportional to the sine of the azimuthal decay angle of one of the W bosons and otherwise involve only symmetric functions of these angles, the imaginary parts drop out in distributions where these decay angles are integrated over. The numerical check of this cancellations can serve as a consistency check of \textsc{RacoonWW}. Moreover, this option might be useful for observables where the imaginary parts do not drop out.

In the following table and figures the total cross section and various distributions to \(e^+e^\rightarrow u\bar{d}\mu^-\bar{\nu}_\mu(\gamma)\) are compared for the different versions of the DPA defined above in the calo setup, i.e. with the loose recombination cut \(M_{\text{rec}} = 25\text{ GeV}\). We consider the CM energies 172 GeV, 200 GeV, and 500 GeV. The used lowest-order cross section is based on the full 4\(f\) matrix element. "Naive" QCD correction factors and ISR corrections beyond \(O(\alpha)\) are not included in the results of this section.
Table 4: RacoonWW predictions for the total cross section to $e^+e^- \rightarrow u\bar{d}\mu^-\bar{\nu}_\mu(\gamma)$ for the calo setup at different CM energies in various versions of the DPA and the relative differences $\delta = \sigma/\sigma_{def} - 1$

| Energy | $\sigma$/pb | $\delta$/% |
|--------|-------------|------------|
| $\sqrt{s} = 172$ GeV | | |
| $\sigma$/pb | def | proj | eik | Coul |
| 400.39(22) | 400.27(22) | 400.03(22) | 403.54(22) |
| 0 | -0.03 | -0.09 | 0.79 |
| $\sqrt{s} = 200$ GeV | | |
| $\sigma$/pb | def | proj | eik | Coul |
| 570.10(37) | 569.93(37) | 570.04(37) | 570.85(37) |
| 0 | -0.03 | -0.01 | 0.13 |
| $\sqrt{s} = 500$ GeV | | |
| $\sigma$/pb | def | proj | eik | Coul |
| 190.30(20) | 190.28(20) | 190.45(20) | 190.31(20) |
| 0 | -0.01 | 0.08 | 0.01 |

The results for the total cross section are shown in Table 4. We find relative differences at the level of 0.1%. As expected, the prediction that is based on the on-shell Coulomb correction is somewhat higher than the exact off-shell treatment, since off-shell effects screen the positive Coulomb singularity. As mentioned above, for the low CM energy of 172 GeV the difference between on-shell and off-shell Coulomb singularity, which is quite large (0.79%), cannot be viewed as a measure of the theoretical uncertainty. While the difference between the two on-shell projections is below 0.03%, the effect of including different finite terms in the DPA (“eik”) is at the level of 0.1% in the considered energy range. When switching on the imaginary parts in the corrections we find the same results as without imaginary parts (“def”).

In Figures 14, 15, and 16 we show the differences of the “proj”, “eik”, “Coul”, and “imag” modifications to the default version of the DPA for some distributions at $\sqrt{s} = 200$ GeV. For the distribution in the cosines of the W-production angle $\theta_W$ and in the W-decay angle $\theta_{W+d}$ (Figure 14) the relative differences are of the order of 0.1% for all angles, which is of the expected order for the intrinsic DPA uncertainty. For the $\mu$-energy distribution, shown on the l.h.s. of Figure 15, the differences are typically of the same order, as long as $E_\mu$ is in the range for W-pair production, which is $20.2$ GeV $< E_\mu < 79.8$ GeV at $\sqrt{s} = 200$ GeV. Outside this region, the four-fermion process is not dominated by the W-pair diagrams, and the DPA is not reliable anymore, which is also indicated by large intrinsic ambiguities. The r.h.s. of Figure 15 shows the DPA ambiguities for the ud invariant-mass distribution. Within a window of $2\Gamma_W$ around the W resonance the relative differences between the considered modifications are also at the level of 0.1–0.3%. The differences grow with the distance from the resonance point. As expected, we find that the contribution of the imaginary parts to the production-angle and decay-angle distributions and to the invariant-mass distributions are compatible with zero.

Finally, we show in Figure 16 the difference between different versions of the DPA for the distributions in the azimuthal angle between the decay planes of the two W bosons and in the azimuthal decay angle of the W$^+$ boson. The ambiguities resulting
Figure 14: Differences between different versions of the DPA for distributions in the cosines of the production $\theta_{W^+}$ (l.h.s.) and decay $\theta_{W^+\bar{d}}$ (r.h.s.) angles for $e^+e^- \rightarrow u\bar{d}\mu^-\bar{\nu}_\mu(\gamma)$ at $\sqrt{s} = 200$ GeV, as described in the text.

Figure 15: Differences between different versions of the DPA for distributions in the $\mu$ energy (l.h.s.) and in the $u\bar{d}$ invariant mass (r.h.s.) for $e^+e^- \rightarrow u\bar{d}\mu^-\bar{\nu}_\mu(\gamma)$ at $\sqrt{s} = 200$ GeV, as described in the text.
Figure 16: Differences between different versions of the DPA for distributions in the azimuthal angles $\phi$ (l.h.s.) and $\phi_{W^+}$ (r.h.s.) for $e^+e^- \rightarrow u\bar{d}\mu^-\bar{\nu}_\mu(\gamma)$ at $\sqrt{s} = 200$ GeV, as described in the text.

from “proj”, “eik”, and “Coul” are of the same size as for the production- and decay-angle distributions considered above. While for the $\phi$ distribution no significant impact of the imaginary parts is visible, the effect influences the $\phi_{W^+}$ distribution by a relative contribution that is roughly given by $0.7\% \times \sin \phi_{W^+}$.

The energy dependence of the ambiguities in the distributions reflects the one of the total cross section, in particular if the deviations are flat as for the production-angle distribution. The differences for the distributions in the W-decay angle $\theta_{W^+\bar{d}}$ and in the u\bar{d} invariant mass, which are not flat, are illustrated in Figures 17 and 18 at $\sqrt{s} = 172$ GeV and 500 GeV, respectively. As already stated above, for the low CM energy of $\sqrt{s} = 172$ GeV the difference between on-shell and off-shell Coulomb singularity cannot be viewed as an ambiguity. On the other hand, the effect of the Coulomb singularity is completely negligible for $\sqrt{s} = 500$ GeV. Note also that the largest ambiguities, which can become of the order of 0.5%, appear when the cross section becomes small. The large differences between the two different on-shell projections in the $\theta_{W^+\bar{d}}$ distribution at 500 GeV are due to the increasing size and the larger angular variations of the relative corrections at higher energies. This uncertainty could be reduced by treating the dominant corrections at high energies, which are due to Sudakov-type corrections of the form $(\alpha/\pi) \ln^2(s/M_W^2)$ [24], not in DPA but in connection with the full $e^+e^- \rightarrow 4f$ cross sections.

The discussed results illustrate that the intrinsic ambiguities of the DPA, as applied in RACOONWW, are at the level of a few per mil, whenever resonant W-pair production dominates the considered observable.
Figure 17: Differences between different versions of the DPA for distributions in the cosine of the decay angle $\theta_{W+d}$ (l.h.s.) and in the ud invariant mass (r.h.s.) for $e^+e^- \rightarrow ud\mu^-\bar{\nu}_\mu(\gamma)$ at $\sqrt{s} = 172$ GeV, as described in the text.

Figure 18: Differences between different versions of the DPA for distributions in the cosine of the decay angle $\theta_{W+d}$ (l.h.s.) and in the ud invariant mass (r.h.s.) for $e^+e^- \rightarrow ud\mu^-\bar{\nu}_\mu(\gamma)$ at $\sqrt{s} = 500$ GeV, as described in the text.
7.4 Comparison with existing results

In Ref. [3] we performed a detailed, tuned comparison of RACOONWW with a semi-analytical calculation of the complete $\mathcal{O}(\alpha)$ corrections to $e^+e^- \rightarrow WW \rightarrow 4\text{leptons}$ in DPA by Beenakker, Berends and Chapovsky [19] (called BBC in the following) and with the Monte Carlo generator YFSWW3 [20, 21]. Here we restrict ourselves to a brief summary of the outcome and refer to Ref. [3] for more details. Moreover, we present a brief comparison of the real-photonic corrections obtained by Jegerlehner and Kolodziej [48] (JK) based on an exact matrix-element calculation.

7.4.1 Comparison with Berends, Beenakker and Chapovsky

The numerical comparison with BBC has been done for the purely leptonic channel $e^+e^- \rightarrow \nu_\mu\mu^+\tau^-\bar{\nu}_\tau$. The conceptual differences between RACOONWW and the BBC approach are as follows: BBC treat both the virtual and real corrections in DPA, while the complete real bremsstrahlung corrections and the universal leading-logarithmic part of the $\mathcal{O}(\alpha)$ ISR correction are based on the exact CC11 matrix elements for $e^+e^- \rightarrow 4f\gamma$ and $e^+e^- \rightarrow 4f$ in RACOONWW. The virtual corrections are treated in DPA in both programs, but RACOONWW employs the full off-shell phase space with an appropriate on-shell projection, whereas BBC have chosen the on-shell phase space with decoupled invariant masses of the $W$ bosons. These differences are (at least formally) beyond the accuracy of the DPA. Thus, if the complete photonic phase space is integrated over and virtual corrections are added, both approaches yield results in DPA accuracy.

For the total cross section, the differences between the two approaches should be of the naively expected accuracy of the DPA of 0.5% or better, whenever the DPA is applicable. We found that both calculations agree very well above 185 GeV. Below this energy the differences in the implementation of the DPA become visible. The main effect originates probably from the different treatment of the $\mathcal{O}(\alpha)$ ISR and the phase space. While the uncertainty arising from the DPA is enhanced by leading ISR logarithms in the BBC approach, this enhancement is absent in RACOONWW.

For distributions unavoidable differences arise from the definition of the phase-space variables in the presence of photon recombination. When defining the momenta of the $W$ bosons for angular distributions, BBC choose to assign the photon to one of the production/decay subprocesses. The angles are then determined from the resulting $W$ momenta and the original fermion momenta. In RACOONWW, all angles are defined from the fermion momenta after eventual photon recombination. The two different angle definitions lead to a redistribution of events in the angular distributions, which arises, in particular, from hard photon emission. For the considered $W$-production-angle and $W$-decay-angle distributions the differences between BBC and RACOONWW turned out to be of the order of 1% at a typical LEP2 energy, with a tendency to increase with increasing energy.

Invariant-mass distributions of the $W$ bosons depend crucially on the treatment of real photons, which is done in a fundamentally different way in RACOONWW compared to the treatment of BBC. Therefore, it does not make sense to compare these distributions between the two programs. Specifically, BBC define the $W$ invariant masses from the fermion momenta only, i.e. without photon recombination at all. The resulting shifts in
the maxima of the distributions are negative, up to \(-77\) MeV for \(e^\pm\) in the final state for \(\sqrt{s} = 184\) GeV, and dominated by mass singularities \([19]\). On the other hand, studies with RACOONWW show that these shifts are in general positive and of the order of several 10 MeV \([25, 26]\) if photon recombination is taken into account.

Finally, we compared the photon-energy spectrum \(E_\gamma(d\sigma/dE_\gamma)\). Since the RACOONWW prediction for the \(E_\gamma\) spectrum is not based on a DPA, but on the full matrix element for \(4f + \gamma\) production, this comparison can serve as a consistency check of the DPA for the real-photonic corrections as used by BBC. We find an agreement between the two approaches within \(\sim 10\%\), which is of the order of the naive expectation for the DPA error of \(O(\Gamma_W/\Delta E)\). Although these results illustrate the reliability of the naive error estimate of the DPA used by BBC for the real corrections at least for LEP2 energies, this observation cannot serve as a proof that a DPA for real-photonic corrections will work for any observable in any kinematical situation, as e.g. for higher energies.

7.4.2 Comparison with YFSWW3

In contrast to the comparison with BBC, where some improvements of the DPA were switched off in RACOONWW, in the comparison with YFSWW3 only the input parameters, cuts, and recombination procedures are chosen to be the same, but no further modifications of the codes are made. Both event generators produce their “best” results, which can also be directly compared to the LEP2 data.

We compared the total CC03 cross sections for leptonic, semileptonic, and hadronic W-pair production channels and various distributions for \(e^+e^- \rightarrow u\bar{d}\mu^-\bar{\nu}_\mu\) at \(\sqrt{s} = 200\) GeV for the same setup as used in this paper.

The “best” results for the total cross sections obtained with RACOONWW and YFSWW3 when no cuts, bare or calo cuts have been applied are consistent with each other. The relative differences amount to only 0.3\%, which is within the expected DPA accuracy of 0.5\%.

The comparison of the distributions has been done for both, calo and bare, recombination schemes. We found that the W-production-angle and W-invariant-mass distributions are statistically compatible with each other, i.e. they agree within 1\%. We also compared several photon observables, such as the distributions of the photon energy, of the photon production angle, and of the angle between the photon and the nearest charged fermion and found relative differences at the 10\% level. This is not surprising since visible photons are treated quite differently in RACOONWW and YFSWW3: while the predictions for photon observables by RACOONWW are based on the exact lowest-order matrix element for \(e^+e^- \rightarrow 4f\gamma\), in YFSWW3 multi-photon radiation to W-pair production (within the YFS scheme) is combined with \(O(\alpha^2)\) LL photon radiation in the W decays (done by PHOTOS).

7.4.3 Comparison with Jegerlehner and Kołodziej

In Ref. \([48]\) JK have evaluated some cross sections and distributions for semi-leptonic channels \(e^+e^- \rightarrow 4f\gamma\) of the CC11 class. The calculation is based on the exact matrix element with finite fermion masses, and finite gauge-boson widths are introduced in the so-called fixed-width scheme. The presented bremsstrahlung corrections to the total cross
sections of $e^+e^- \to 4f$, which have been regularized with a small photon mass $\lambda$, can be compared with predictions made with RACOONWW. To this end, the virtual corrections are omitted in RACOONWW and consequently no DPA is needed. The real correction provided by RACOONWW corresponds to the full $e^+e^- \to 4f\gamma$ cross section in the successive asymptotic limits $\lambda \to 0$ and $m_f \to 0$. The relative difference to the results of JK should thus be of the order of $m_f/M_W$, where $m_f$ is the largest fermion mass of the process. Table 5 shows a comparison of results for the process $e^+e^- \to u\bar{d}\mu\bar{\nu}_\mu$ with $\lambda = 10^{-6}$ GeV, as obtained by JK [48] and RACOONWW.

8 Conclusions

A proper treatment of electroweak radiative corrections to the W-pair production process $e^+e^- \to WW \to 4f$ is mandatory in order to account for the experimental accuracy at LEP2 and future linear colliders. A practical calculation of the $O(\alpha)$ corrections to the full $e^+e^- \to 4f$ process is beyond present possibilities. However, as long as W-pair channels dominate, the combination of full lowest-order predictions for the four-fermion processes $e^+e^- \to 4f$ with those corrections that are enhanced by two resonant W bosons approximates the full $O(\alpha)$-corrected cross sections within a relative accuracy of 0.5%, at least for not too high energies. A strategy for such a double-pole approximation (DPA) for the $O(\alpha)$ corrections and its implementation in the Monte Carlo event generator RACOONWW are described in this paper.

In RACOONWW only the non-leading virtual $O(\alpha)$ corrections are treated in DPA, while real photonic corrections are based on full matrix elements for the radiative processes $e^+e^- \to 4f\gamma$. In this way, potential problems in the definition of a DPA for real photon
emission are avoided, and possible DPAs for the real corrections can be controlled. We have shown explicitly how existing results for the corrections to on-shell $W$-pair production and $W$ decay can be employed in the DPA; they define the class of so-called factorizable corrections. For the remaining, non-factorizable contributions, which are due to photon exchange between the various production and decay subprocesses, we have completed the results already given in the literature. Since virtual and real corrections are not treated on equal footing, particular care is needed in the treatment of soft and collinear singularities. The proper cancellation of these singularities has been discussed in detail; in practice two different methods (a subtraction method and phase-space slicing) have been applied. Finally, the leading contributions from initial-state radiation beyond $\mathcal{O}(\alpha)$ have been included in RACOONWW, rendering this program a state-of-the-art calculation.

For LEP2 energies at detailed discussion of numerical results is presented, including total cross sections, angular and invariant-mass distributions. The consistency of RACOONWW is shown by comparing the results obtained with the subtraction method with the ones of the slicing approach. Moreover, the reliability of the applied DPA is demonstrated by various modifications of the DPA within its intrinsic freedom. The obtained intrinsic ambiguity supports the expectation that the theoretical uncertainty of the used DPA is of the order of 0.5% for energies between 170 GeV and 500 GeV. Above 500–1000 GeV, leading-logarithmic electroweak corrections of higher orders have to be taken into account.

As a first phenomenological application of RACOONWW the predictions for the total $W$-pair production at LEP2 have been compared with LEP2 data and older “improved Born predictions” of GENTLE, which were previously used in the data analyses. The RACOONWW results agree with the results of the new generator YFSWW3 but lie 2–2.5% below the GENTLE predictions. The fact that the experimental LEP2 data favour predictions of RACOONWW and YFSWW3 can be viewed as empirical evidence for non-leading electroweak radiative corrections beyond the level of universal effects.

Acknowledgement

We thank all the authors of Refs. [19, 21, 23, 48] for helpful information and discussions about their results.

Appendix

A Explicit form of an on-shell projection for the four-fermion final state

When performing the projection to on-shell $W$-boson momenta, care should be taken that the projected momenta lie in the physical region. This can be ensured by fixing angles while performing the limit $k^2 \rightarrow M^2_W$. An obvious way is to fix the direction of one of the $W$ bosons and of one of the final-state fermions originating from either $W$ boson in the CM frame of the incoming $e^+e^-$ pair. We fix the direction of the $W^+$ boson, of the fermion $f_1$, and of the fermion $f_3$. Thus, we find for the on-shell-projected momenta

$$\hat{k}_{+0} = \frac{1}{2} \sqrt{s}, \quad \hat{k}_{+} = \frac{k_{+}}{|k_{+}|} \beta \frac{\sqrt{s}}{2}, \quad \hat{k}_{-} = p_{+} + p_{-} - \hat{k}_{+},$$

where $s$ is the center-of-mass energy squared and $\beta$ is the Lorentz factor. The projection $\hat{k}_{+0}$ is the on-shell projection of the center-of-mass energy, $\hat{k}_{+}$ is the on-shell projection of the $W^+$ momentum, and $\hat{k}_{-}$ is the on-shell projection of the fermion $f_3$ momentum. This allows for a precise calculation of the final-state fermion momenta in the on-shell frame.

45
\[ \hat{k}_1^\mu = \hat{k}_1^\mu \frac{M_W^2}{2k_+ k_1}, \quad \hat{k}_2^\mu = \hat{k}_1^\mu - \hat{k}_1^\mu, \]
\[ \hat{k}_3^\mu = \hat{k}_3^\mu \frac{M_W^2}{2k_- k_3}, \quad \hat{k}_4^\mu = \hat{k}_3^\mu - \hat{k}_3^\mu \] (A.1)

with \( \beta = \sqrt{1 - 4M_W^2/s} \).

B Explicit expressions for standard matrix elements

The SMEs \( \mathcal{M}_n^\pm \), which have been defined in Section 3.4, have been calculated in the Weyl–van der Waerden spinor formalism using the conventions of Ref. [50]. Since we consistently neglect fermion masses, the final-state fermions are polarized according to \( \sigma_1 = -\sigma_2 = \sigma_3 = -\sigma_4 = - \), and the index \( \sigma \) determines the polarization of the incoming \( e^\pm \) according to \( \sigma = \sigma_- = -\sigma_+ \). The explicit expressions for the two sets \( \mathcal{M}_n^\pm \) read

\[ \mathcal{M}_1^+ = -4A \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_3 \rangle \left( \langle p_+ \hat{k}_4 \rangle^* \langle p_- \hat{k}_1 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_2^+ = 2A \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle \left( \langle p_+ \hat{k}_1 \rangle^* \langle p_- \hat{k}_1 \rangle + \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_3^+ = 2A \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_1 \rangle \left( \langle \hat{k}_1 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_2 \hat{k}_3 \rangle \right), \]
\[ \mathcal{M}_4^+ = -2A \langle p_+ \hat{k}_4 \rangle^* \langle p_- \hat{k}_3 \rangle \left( \langle \hat{k}_2 \hat{k}_3 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_4 \rangle \right), \]
\[ \mathcal{M}_5^+ = 2A \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_4 \rangle^* \langle p_- \hat{k}_3 \rangle, \]
\[ \mathcal{M}_6^+ = -2A \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_4 \rangle^* \langle p_- \hat{k}_1 \rangle \langle p_- \hat{k}_3 \rangle, \]
\[ \mathcal{M}_7^+ = A \langle \hat{k}_2 \hat{k}_3 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_4 \rangle \left( \langle \hat{k}_1 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_2 \hat{k}_3 \rangle \right) \times \left( \langle p_+ \hat{k}_1 \rangle^* \langle p_- \hat{k}_1 \rangle + \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_8^+ = A \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_1 \rangle^* \langle p_+ \hat{k}_1 \rangle^* \langle p_- \hat{k}_3 \rangle \left( \langle p_+ \hat{k}_1 \rangle^* \langle p_- \hat{k}_1 \rangle + \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_9^+ = A \langle p_+ \hat{k}_4 \rangle^* \langle p_- \hat{k}_3 \rangle \left( \langle \hat{k}_2 \hat{k}_3 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_4 \rangle \right) \times \left( \langle p_+ \hat{k}_1 \rangle^* \langle p_- \hat{k}_1 \rangle + \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_{10}^+ = A \langle p_+ \hat{k}_2 \rangle^* \langle p_+ \hat{k}_1 \rangle \left( \langle \hat{k}_1 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_2 \hat{k}_3 \rangle \right) \times \left( \langle p_+ \hat{k}_1 \rangle^* \langle p_- \hat{k}_1 \rangle + \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_1^- = -4A \langle p_- \hat{k}_1 \rangle^* \langle p_+ \hat{k}_1 \rangle \left( \langle p_+ \hat{k}_2 \rangle^* \langle p_- \hat{k}_3 \rangle - \langle \hat{k}_1 \hat{k}_2 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle \right), \]
\[ \mathcal{M}_2^- = 2A \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle \left( \langle p_- \hat{k}_1 \rangle^* \langle p_+ \hat{k}_1 \rangle + \langle p_- \hat{k}_2 \rangle^* \langle p_+ \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_3^- = 2A \langle p_- \hat{k}_2 \rangle^* \langle p_+ \hat{k}_1 \rangle \left( \langle \hat{k}_1 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_2 \hat{k}_3 \rangle \right), \]
\[ \mathcal{M}_4^- = -2A \langle p_- \hat{k}_4 \rangle^* \langle p_- \hat{k}_3 \rangle \left( \langle \hat{k}_2 \hat{k}_3 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_4 \rangle \right), \]
\[ \mathcal{M}_5^- = 2A \langle p_- \hat{k}_2 \rangle^* \langle p_- \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle, \]
\[ \mathcal{M}_6^- = -2A \langle p_- \hat{k}_2 \rangle^* \langle p_- \hat{k}_4 \rangle^* \langle p_- \hat{k}_1 \rangle \langle p_- \hat{k}_3 \rangle, \]
\[ \mathcal{M}_7^- = A \langle \hat{k}_2 \hat{k}_3 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_4 \rangle \left( \langle \hat{k}_1 \hat{k}_4 \rangle^* \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle^* \langle \hat{k}_2 \hat{k}_3 \rangle \right) \times \left( \langle p_- \hat{k}_1 \rangle^* \langle p_+ \hat{k}_1 \rangle + \langle p_- \hat{k}_2 \rangle^* \langle p_+ \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_g^- = A\langle p_+ \hat{k}_2 \rangle \langle p_- \hat{k}_4 \rangle \langle p_+ \hat{k}_1 \rangle \langle p_- \hat{k}_3 \rangle \left( \langle p_- \hat{k}_1 \rangle \langle p_+ \hat{k}_1 \rangle + \langle p_- \hat{k}_2 \rangle \langle p_+ \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_0^- = A\langle p_- \hat{k}_4 \rangle \langle p_- \hat{k}_3 \rangle \left( \langle \hat{k}_2 \hat{k}_3 \rangle \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle \langle \hat{k}_1 \hat{k}_4 \rangle \right) \times \left( \langle p_- \hat{k}_1 \rangle \langle p_+ \hat{k}_1 \rangle + \langle p_- \hat{k}_2 \rangle \langle p_+ \hat{k}_2 \rangle \right), \]
\[ \mathcal{M}_{10}^- = A\langle p_+ \hat{k}_2 \rangle \langle p_+ \hat{k}_1 \rangle \left( \langle \hat{k}_1 \hat{k}_4 \rangle \langle \hat{k}_1 \hat{k}_3 \rangle + \langle \hat{k}_2 \hat{k}_4 \rangle \langle \hat{k}_2 \hat{k}_3 \rangle \right) \times \left( \langle p_- \hat{k}_1 \rangle \langle p_+ \hat{k}_1 \rangle + \langle p_- \hat{k}_2 \rangle \langle p_+ \hat{k}_2 \rangle \right), \]
where \( \langle pk \rangle \) denotes the spinor product for two light-like momenta \( p^\mu \) and \( k^\mu \), and \( A \) is the global factor
\[ A = \frac{e^2}{2s_w^2} \frac{1}{k_+^2 - M_W^2} + iM_W \Gamma_W \frac{1}{k_-^2 - M_W^2} + iM_W \Gamma_W, \]

(C.1)

C Scalar integrals for the virtual non-factorizable corrections

In Section 3.2 we have given the virtual non-factorizable corrections in terms of scalar one-loop integrals. In this appendix we list the explicit expressions for those scalar integrals that are not already given in Ref. [16]:

\[ D_0(p_\pm, k_\pm, k_i, \lambda, m_e, M, m_i) \sim \frac{1}{t \mp M_W^2} \left\{ 2 \ln \left( \frac{m_e m_i}{-t_{\pm i}} \right) \ln \left( \frac{-K_\pm}{-t_{\pm i}} \right) - \ln^2 \left( \frac{m_e M_W}{M_W^2 + t} \right) - \ln^2 \left( \frac{m_i}{M_W} \right) - \frac{\pi^2}{3} - \text{Li}_2 \left( 1 - \frac{t - M_W^2}{t_{\pm i}} \right) \right\}, \]
\[ D_0(p_\pm, k_\mp, k_i, \lambda, m_e, M, m_i) \sim D_0(p_\pm, k_\pm, k_i, \lambda, m_e, M, m_i)|_{t \to u, t_\pm \to u_\pm, K_\pm \to K_\mp}, \]
\[ C_0(p_\pm, k_\mp, 0, m_e, M) - \left[ C_0(p_\pm, k_\pm, \lambda, m_e, M_W) \right]_{k_\pm = M_W^2} \]
\[ \sim \frac{1}{t - M_W^2} \left\{ \ln \left( \frac{m_e M_W}{M_W^2 - t} \right) \ln \left( \frac{-K_\pm}{M_W^2 - t} \right) + \ln \left( \frac{-K_\pm}{\lambda^2} \right) \right\} \left( \frac{m_e}{M_W} \right) + \frac{\pi^2}{6}, \]
\[ C_0(p_\pm, k_\mp, 0, m_e, M) - \left[ C_0(p_\pm, k_\pm, \lambda, m_e, M_W) \right]_{k_\pm = M_W^2} \]
\[ \sim \left\{ C_0(p_\pm, k_\pm, 0, m_e, M) - \left[ C_0(p_\pm, k_\pm, \lambda, m_e, M) \right]_{k_\pm = M_W^2} \right\}_{t \to u, K_\pm \to K_\mp}, \]
\[ C_0(k_\pm, k_i, 0, M, m_i) - \left[ C_0(k_\pm, k_i, \lambda, M_W, m_i) \right]_{k_\pm = M_W^2} \]
\[ \sim -\frac{1}{M_W^2} \left\{ \ln \left( \frac{m_i^2}{M_W^2} \right) \ln \left( \frac{-K_\pm}{\lambda M_W} \right) + \ln^2 \left( \frac{m_i}{M_W} \right) + \frac{\pi^2}{6} \right\}, \]
\[ \frac{B_0(k_\pm^2, 0, M) - B_0(M_W^2, 0, M)}{k_\pm^2 - M^2} - B'_0(M_W^2, \lambda, M_W) \sim \frac{1}{M_W^2} \left\{ \ln \left( \frac{\lambda M_W}{-K_\pm} \right) + 1 \right\}. \]

Recall that all expressions are given for \( k_\pm^2 \to M_W^2 \) and \( \Gamma_W \to 0 \).

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