SHARP GLOBAL WELL-POSEDNESS OF THE BBM EQUATION IN L^p TYPE SOBOLEV SPACES

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Abstract. The global well-posedness of the BBM equation is established in $H^{s,p}(\mathbb{R})$ with $s \geq \max\{0, \frac{1}{p} - \frac{1}{2}\}$ and $1 \leq p < \infty$. Moreover, the well-posedness results are shown to be sharp in the sense that the solution map is no longer $C^2$ from $H^{s,p}(\mathbb{R})$ to $C([0,T]; H^{s,p}(\mathbb{R}))$ for smaller $s$ or $p$. Finally, some growth bounds of global solutions in terms of time $T$ are proved.

1. Introduction. In this paper we are concerned with the Cauchy problem for the Benjamin-Bona-Mahony (BBM) equation

$$u_t + u_x - uu_x + uu_x = 0, \quad u(0,x) = u_0(x) \quad (1)$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a real-valued function, $u_0$ is a given initial data. The BBM equation was studied in [4] for modeling the propagation of unidirectional, one-dimensional, small-amplitude long waves in nonlinear dispersive media.

The well-posedness and ill-posedness of Eq. (1) are studied extensively in references [5, 7, 19, 21, 27]. In [5], Bona and Tzvetkov showed that the BBM equation is globally well-posed in $H^s(\mathbb{R})$ with $s \geq 0$. The same result holds in periodic case, see [21, 27]. The well-posedness is sharp in the sense that Eq. (1) is ill-posed in $H^s(\mathbb{R})$ if $s < 0$. In fact, Bona and Tzvetkov [5] also illustrated that the solution map $u_0 \mapsto u(t)$ of Eq. (1) is not $C^2$ from $H^s(\mathbb{R})$ to $C([0,T]; H^s(\mathbb{R}))$ for smaller $s$ and $p$. Later, Panthee [19] improved this to that the solution map is not continuous at the origin from $H^s(\mathbb{R})$ to even $\mathcal{D}'(\mathbb{R})$ at any fixed $t > 0$ small enough. Very recently, Carvajal and Panthee [7] considered the following generalized BBM equation

$$u_t + u_x + Lu_t + (u^{k+1})_x = 0; \quad u(0,x) = u_0(x) \quad (2)$$

where $L$ is defined by Fourier transform $\hat{L}u(\xi) = |\xi|^\alpha \hat{u}(\xi)$, $\alpha > 0$. They proved that Eq. (2) is locally well-posed in $H^s(\mathbb{R})$ for some $s$ depending on $\alpha$ and $k$. Moreover, Eq. (2) is ill-posed in $H^s(\mathbb{R})$ if $s < \max\{0, \frac{1}{2} - \frac{\alpha}{k}\}$ in the sense that the solution map is not $C^{k+1}$-differentiable at the origin from $H^s(\mathbb{R})$ to $C([0,T]; H^s(\mathbb{R}))$ for any $T > 0$. In the case $\alpha = 1$, some global well-posedness results of (2) with periodic boundary condition were obtained in [27].

Observe that the well-posedness and ill-posedness in [5, 7, 19, 21, 27] are established in $L^2$-based Sobolev spaces $H^s$. Now we recall some well-posedness results

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in $L^p$–type Sobolev spaces. Consider the generalized BBM equation in an open domain $\Omega \subset \mathbb{R}^n$

$$u_t - \Delta u_t + \text{div}(\phi(u)) = 0; \quad u(0, x) = u_0(x)$$ \hspace{1cm} (3)

where $\Delta = \sum_{j=1}^n \partial^2_j/\partial x_j^2$, $\phi : \mathbb{R} \mapsto \mathbb{R}^n$ is a smooth vector satisfying some polynomial-like growth bounds. Goldstein and Wichnoski [11] proved that Eq.(3) is locally well-posed in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for $p > n$. Later, Avrin and Goldstein [1] showed that the solution is global. Further, Avrin [2] obtained that Eq.(3) is locally well-posed in $W^{1,p}(\mathbb{R}^n)$ for $p > 1$. In particular, Avrin’s result implies that Eq. (1) is locally well-posed in $W^{1,p}(\mathbb{R})$ with $p \geq 1$. It’s well known that the energy of Eq. (1) is conservative, namely

$$\int_{\mathbb{R}} u_t^2(t) + u^2(t)dx = \int_{\mathbb{R}} u_t^2(0) + u^2(0)dx, \quad t > 0.$$ \hspace{1cm} (E)

As a consequence, one concludes that Eq. (1) is globally well-posed in $H^1(\mathbb{R})$. However, the conservation law is no longer valid if the solution is not regular enough, say the solution only belongs to $H^{s,p}(\mathbb{R})$ with $s < 1$ and $p < 2$. Here and below, the inhomogeneous Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$, $1 \leq p < \infty, s \in \mathbb{R}$, are defined as the completion of Schwartz class with respect to the norm

$$\|f\|_{H^{s,p}(\mathbb{R}^n)} := \|\Lambda^s f\|_{L^p(\mathbb{R}^n)},$$

where $\Lambda^s$ is the fractional differential operator defined by the symbol $(1 + |\xi|^2)^{s/2}$. Thus, a natural question arises, is Eq. (1) globally well-posed in $H^{s,p}(\mathbb{R})$ for such $s$ and $p$? We answer the question in the following theorem.

**Theorem 1.1.** Let $p \in [1, \infty)$ and $s \geq \max\{0, \frac{1}{p} - \frac{1}{2}\}$, then Eq.(1) is globally well-posed in $H^{s,p}(\mathbb{R})$.

The most interesting fact given by Theorem 1.1 is that Eq. (1) is globally well-posed in low regularity spaces, in particular it allows $s < 1$ and $p < 2$. A nice way to prove global well-posedness in low regularity spaces is the $I$–method, which is introduced in [9] and widely used to establish global well-posedness in rough spaces for many dispersive equations, see e.g. [16, 25, 28]. The $I$–method is very efficient when the phase space is $L^2$–based Sobolev spaces $H^s$. But we are working in $L^p$ type Sobolev spaces, it’s not obvious how to use the $I$–method directly. To overcome this difficulty, we split $u = v + w$, where $v$ satisfies

$$v_t + v_x - v_{xxx} + vv_x = 0, \quad v(0, x) = (1 - \chi_k(x))u_0(x),$$ \hspace{1cm} (4)

and $w$ solves

$$w_t + w_x - w_{xxx} + ww_x + (wv)_x = 0, \quad w(0, x) = \chi_k(x)u_0(x).$$ \hspace{1cm} (5)

Here $\chi_k(x)$ is a smooth function such that $\chi = 1$ if $|x| \leq k$ and $\chi = 0$ if $|x| \geq 2k$. To illustrate our ideas, we consider the simple case $u_0 \in L^p(\mathbb{R}), p \in (2, \infty)$. On one hand, since $\|v(0, \cdot)\|_{L^p}$ goes to zero as $k$ goes to infinity, one can show that, for any $T > 0$, Eq. (4) has a unique solution $v \in C([0, T]; L^p(\mathbb{R}))$ provided $k$ is large enough (see Lemma 3.3). On the other hand, for any $k > 0$ fixed, $w(0, x)$ belongs to $L^p(\mathbb{R}) \cap L^2(\mathbb{R})$, then one can adapt the $I$–method to show that the solution $w$ of Eq.(5) does not blow up in $L^p(\mathbb{R})$ norm at any $T > 0$. Combining these together gives the global well-posedness of Eq.(1) in $L^p(\mathbb{R})\,(p > 2)$. The more general case follows similarly, see Section 3 for details.

In section 4, we shall show that theorem 1.1 is sharp in the sense that Eq. (1) is ill-posed in $H^{s,p}$ for smaller $s$. Precisely, we have
Theorem 1.2. Let $p \in [1, \infty)$ and $s < \max\{0, \frac{1}{p} - \frac{1}{2}\}$, then Eq. (1) is ill-posed in $H^{s,p}(\mathbb{R})$.

The definition of ill-posedness is given in section 4. The ill-posedness extends the corresponding results in [5, 7, 19] to an $L^p$ setting. The proof of Theorem 1.2 is essentially the same to disproving

$$
\|\varphi(D)f^2\|_{H^{s,p}} \lesssim \|f\|^2_{H^{s,p}}
$$

with the same $s$ and $p$, where $\varphi(D)$ is the Fourier multiplier with symbol $\xi/(1 + \xi^2)$. The idea is to find a sequence $\{f_n\}$ of smooth functions such that $\|f_n\|_{H^{s,p}}$ goes to zero as $n$ goes to infinity, and at the same time $f_n^2 \to \delta$ in distribution sense, where $\delta$ is the Dirac function. Then the right hand side of (6) goes to zero but the left hand side will never be trivial as $n$ goes to infinity. In Section 4, we make the formal argument rigorous and adapt the approach to obtain the ill-posedness.

In Section 5, we are devoted to the growth of norm of solutions to Eq. (1). Precisely, we are interested in the estimates of $\|u(T)\|_{H^{s,p}}$ in terms of $T$ as $T$ grows. It turns out that the picture of our results in the case $p \leq 2$ and $p > 2$ are different. We shall show a polynomial growth bound of $\|u(T)\|_{H^{s,p}}$ for all $s \geq \frac{1}{p} - \frac{1}{2}$ if $p \in [1, 2]$. Note that the range of $s$ coincides to that of well-posedness, thus the study in the case $p \leq 2$ is more or less complete. However, if $u_0 \in L^p(\mathbb{R})$ with $p > 2$, no growth bounds of $\|u(T)\|_{L^p}$ are available so far due to technical reasons in this paper. Nevertheless, an exponential growth bound of $\|u(T)\|_{L^p}$ can be proved if we assume further some decay of $u_0$ at infinity, say $u_0$ belongs to the weighted Lebesgue space $L^p((x)\alpha dx)$, $\alpha > 0$. Finally, we investigate the persistence of the BBM equation in weighted spaces $L^p((x)\alpha dx)$.

Notations. We denote by $s + (-)$ that a constant equals $s$ plus (minus) a small enough positive number, $A \lesssim B$ means $A \leq CB$ for some absolute constant $C$, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$, and $A \gg B$ means $A/B$ is very big, say $A/B \geq 1000$.

2. Preliminaries.

2.1. Some facts from harmonic analysis. We start with the classical theory of Fourier multipliers. Let $m$ be a measurable function on $\mathbb{R}^n$ and define the operator

$$
m(D)f = \mathcal{F}^{-1}(m(\xi)\hat{f}(\xi)), \quad \forall f \in \mathcal{S}(\mathbb{R}^n),
$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space, $\hat{f}(\xi) = \mathcal{F}f$, and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse transform, respectively. We shall call $m(\xi)$ the symbol of the operator $m(D)$. Let $1 \leq p \leq \infty$, we say $m$ is an Fourier multiplier on $L^p(\mathbb{R}^n)$ if $m(D)$ can be extended to a bounded operator on $L^p(\mathbb{R}^n)$, namely

$$
\|m(D)\|_{L^p(\mathbb{R}^n), L^p(\mathbb{R}^n)} \leq C.
$$

The set of all $L^p(\mathbb{R}^n)$ multipliers is denoted by $\mathcal{M}_p(\mathbb{R}^n)$. We list some basic facts on $\mathcal{M}_p(\mathbb{R}^n)$ as follows:

- $\mathcal{M}_2(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$;
- $\mathcal{M}_p(\mathbb{R}^n) = \mathcal{M}_q(\mathbb{R}^n)$ if $\frac{1}{p} + \frac{1}{q} = 1$;
- $\mathcal{M}_p(\mathbb{R}^n) \subset \mathcal{M}_q(\mathbb{R}^n)$ if $1 \leq p \leq q \leq 2$.

The following theorem is a powerful tool to determine if a bounded function is an $L^p(\mathbb{R}^n)$ multiplier. From now on, we use the notation $\langle x \rangle = (1 + |x|^2)^{1/2}$.
Theorem 2.1. Let \( k \in \mathbb{N}, k > \frac{n}{2} + 1, m \in C^k(\mathbb{R}^n \setminus \{0\}) \), and there exists \( \rho \geq 0 \) such that for any \(|\alpha| \leq k\),
\[
|D^\alpha m(\xi)| \leq C_\alpha (\xi)^{-\rho - |\alpha|}, \quad \forall \xi \in \mathbb{R}^n.
\]
- If \( \rho = 0 \), then \( m \in \mathcal{M}_p(\mathbb{R}^n), 1 < p < \infty \).
- If \( \rho > 0 \), then \( m \in \mathcal{M}_1(\mathbb{R}^n) \).

Proof. The case \( \rho = 0 \) is Mihlin multiplier theorem [24], while the case \( \rho > 0 \) is a consequence of Bernstein theorem, see e.g. [10].

It follows from the Theorem 2.1 that \( \langle \xi \rangle^{-\rho} \), \( \rho > 0 \), belongs to \( \mathcal{M}_1(\mathbb{R}^n) \). It turns out that \( \mathcal{M}_1(\mathbb{R}^n) \not\subset \bigcup_{1 < p < \infty} \mathcal{M}_p(\mathbb{R}^n) \). A typical example is the Riesz potential operator \( R_j \) with symbol \( \xi_j/|\xi| \). In fact, \( R_j \) is bounded on \( L^p(\mathbb{R}^n) \) for all \( 1 < p < \infty \) but not on \( L^1(\mathbb{R}^n) \). It should be noted that \( R_j \) is bounded from \( L^1(\mathbb{R}^n) \) to \( L^{1,\infty}(\mathbb{R}^n) \), where the weak-\( L^p \) space \( (0 < p < \infty) \) is defined as a set of all measurable functions \( f \) such that
\[
\|f\|_{L^{p,\infty}} = \inf \left\{ C > 0 : \text{mes}\{x \in \mathbb{R}^n : |f(x)| > \alpha\} \leq \frac{C}{\alpha^p} \text{ for all } \alpha > 0 \right\}
\]
is finite.

We now recall the Sobolev embedding theorem.

Theorem 2.2. (a) Let \( 0 < sp < n \). Then
\[
\|f\|_{L^q} \lesssim \|f\|_{H^{s,p}}
\]
holds for \( q \in [p, \frac{np}{n-sp}] \) if \( 1 < p < \infty, q \in [p, \frac{np}{n-sp}] \) if \( p = 1 \).

(b) Let \( sp = n, 1 \leq p < \infty \). Then (7) holds for all \( q \in [p, \infty) \).

Proof. If \( sp < n \), it follows from Corollary 6.1.6 and Theorem 6.2.4 of [12] that the operator
\[
\Lambda^{-s} : \begin{cases} 
L^p(\mathbb{R}^n) \rightarrow L^{\frac{np}{n-sp}}(\mathbb{R}^n), & p = 1 \\
L^p(\mathbb{R}^n) \rightarrow L^{\frac{np}{n-sp}}(\mathbb{R}^n), & p > 1
\end{cases}
\]
is bounded. Also, for all \( p \geq 1 \)
\[
\Lambda^{-s} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)
\]
is bounded. Then by Marcinkiewicz interpolation theorem
\[
\Lambda^{-s} : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \begin{cases} 
p = 1, \quad q \in [p, \frac{np}{n-sp}) \\
p > 1, \quad q \in [p, \frac{np}{n-sp}]
\end{cases}
\]
is bounded. This proves (a). Since (b) is a consequence of (a), the proof is complete.

The following fractional Leibniz rule will be used in this paper, see [15] for \( 1 < p < \infty \) and [3, 13] for \( p = 1 \), respectively.

Proposition 1. If \( s \geq 0, 1 \leq p < \infty \), then for all \( f, g \in \mathcal{S}(\mathbb{R}^n) \)
\[
\|\Lambda^s(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}}\|g\|_{H^{s,q_1}} + \|f\|_{H^{s,q_2}}\|g\|_{L^{q_2}}
\]
with \( p_1, p_2, q_1, q_2 \in (1, \infty) \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} \).
2.2. Bilinear estimates. Now we prove some a priori estimates that will be used in this paper.

**Proposition 2.** If $1 \leq p < \infty$, $s \geq \max\{\frac{1}{p} - \frac{1}{2}, 0\}$, then 
\[ \| \partial_x (1 - \partial_x^2)^{-1} (fg) \|_{H^{s,p}(\mathbb{R})} \lesssim \| f \|_{H^{s,p}(\mathbb{R})} \| g \|_{H^{s,p}(\mathbb{R})} \]
provided that the right hand side is finite.

**Proof.** We first consider the case $p \geq 2$. Write
\[ \partial_x (1 - \partial_x^2)^{-1} (fg) = \partial_x \Lambda^{\frac{1}{2} - 2} \cdot \Lambda^{-\frac{1}{p}} (fg). \]
Since the symbol of $\partial_x \Lambda^{\frac{1}{2} - 2}$ is $i \xi \langle \xi \rangle^{-\frac{1}{2} - 2}$, by Theorem 2.1 it is an $L^1$ multiplier, and of course an $L^p$ multiplier. Moreover, by Sobolev embedding, $\Lambda^{-\frac{1}{p}} : L^2 \to L^p$ is bounded. Thus
\[ \| \partial_x (1 - \partial_x^2)^{-1} (fg) \|_{H^{s,p}} = \| \partial_x (1 - \partial_x^2)^{-1} \Lambda^s (fg) \|_{L^p} \]
\[ \lesssim \| \Lambda^s (fg) \|_{L^2} \lesssim \| f \|_{H^{s,p}} \| g \|_{H^{s,p}} \]
where we used Proposition 1 in the last step.

Now we turn to the case $1 \leq p < 2$. Write
\[ \partial_x (1 - \partial_x^2)^{-1} \Lambda^s (fg) = \partial_x \Lambda^{-1-s} \cdot \Lambda^{-\frac{1}{p} - 1} \cdot \Lambda^{\frac{s}{p} - \frac{1}{p}} (fg). \]
By Theorem 2.1 and 2.2 again, we find $\partial_x \Lambda^{-1-s}$ is bounded on $L^p$ and $\Lambda^{-\frac{1}{p} - 1}$ is bounded from $L^1$ to $L^p$. Then
\[ \| \partial_x (1 - \partial_x^2)^{-1} (fg) \|_{H^{s,p}} \lesssim \begin{cases} \| fg \|_{L^1}, & s < \frac{1}{p} \\ \| \Lambda^{s - \frac{1}{p} +} (fg) \|_{L^1}, & s \geq \frac{1}{p}. \end{cases} \]
By Proposition 1 again, the above quantity is bounded by
\[ \begin{cases} \| f \|_{L^2} \| g \|_{L^2}, & s < \frac{1}{p} \\ \| \Lambda^{s - \frac{1}{p} +} f \|_{L^2} \| g \|_{L^2} + \| \Lambda^{s - \frac{1}{p} +} g \|_{L^2} \| f \|_{L^2}, & s \geq \frac{1}{p}. \end{cases} \]
We have used the fact $s \geq \frac{1}{p} - \frac{1}{2}$ in the first case of the last step. This completes the proof. \qed

From the proof of Proposition 2, we have the following variants of bilinear estimates. These inequalities will be used to obtain the global well-posedness of (1) and growth of norms of solutions.

**Lemma 2.3. (a)** If $2 \leq p < \infty$, $s \geq 0$, then
\[ \| \partial_x (1 - \partial_x^2)^{-1} f^2 \|_{H^{s,p}} \lesssim \| f \|_{H^{s,p}} \| f \|_{L^p} \]
and
\[ \| \partial_x (1 - \partial_x^2)^{-1} f^2 \|_{H^{s,p}} \lesssim \| f \|_{H^{s,p}}^2. \]
(b) If $1 \leq p < 2$, $\frac{1}{p} - \frac{1}{2} \leq s < \frac{1}{p}$, then
\[ \| \partial_x (1 - \partial_x^2)^{-1} f^2 \|_{H^{s,p}} \lesssim \| f \|_{H^{s,p}}^2. \]
(c) If $1 \leq p < 2$, $s \geq \frac{1}{p}$, then
\[ \| \partial_x (1 - \partial_x^2)^{-1} f^2 \|_{H^{s,p}} \lesssim \| f \|_{H^{s - \frac{1}{p} +}} \| f \|_{L^2}. \]
Now we introduce the $I$–operator used in this paper. Let $0 \leq s < 1$, $N \gg 1$. Define the $I$–operator
\[ I : H^s (\mathbb{R}) \hookrightarrow H^1 (\mathbb{R}), \quad \phi \mapsto I\phi, \]
where $\hat{I}\phi (\xi) = m(\xi)\hat{\phi}(\xi)$, $m(\xi)$ is a positive smooth even function satisfying
\[
 m(\xi) = \begin{cases} 1, & |\xi| \leq N; \\ \left(\frac{|\xi|}{N}\right)^{s-1}, & |\xi| \geq 2N. \end{cases}
\]
It’s easy to show that the following assertions hold:
\[
\|u\|_{H^s} \leq \|I u\|_{H^1} \leq CN^{1-s} \|u\|_{H^s}. \tag{8}
\]

**Proposition 3.** Let $0 \leq s < 1$, $2 \leq p < \infty$, then
\[
\|I (fg)\|_{L^2 (\mathbb{R})} \lesssim \|f\|_{L^p (\mathbb{R})} \|g\|_{H^s (\mathbb{R})}
\]
provided that the right hand side is finite, the implicit constant is independent of $N$.

**Proof.** By a limit process, it suffices to prove the proposition for $f, g \in \mathscr{S} (\mathbb{R})$. We first observe that the inequality is equivalent to
\[
\|I (f^{-1}A^{-1}g)\|_{L^2} \lesssim \|f\|_{L^p} \|g\|_{L^2}.
\]
By Parseval identity and duality, it suffices to show
\[
|A| \lesssim \|f\|_{L^p} \|g\|_{L^2} \|G\|_{L^2} \tag{9}
\]
for all $G \in L^2$, where $A = \int d\xi \int_{\xi = \xi_1 + \xi_2} \frac{\hat{f}(\xi_1)\hat{g}(\xi_2)\chi(\xi_2)\hat{G}(\xi)m(\xi)}{m(\xi_1)\xi_2)}$.

In order to estimate $A$, we write
\[
A = A_1 + A_2 + A_3
\]
where
\[
A_1 = \int d\xi \int_{\xi = \xi_1 + \xi_2} \frac{\hat{f}(\xi_1)\hat{g}(\xi_2)\chi(\xi_2)\hat{G}(\xi)m(\xi)}{m(\xi_1)\xi_2)},
\]
\[
A_2 = \int d\xi \int_{\xi = \xi_1 + \xi_2} \frac{\hat{f}(\xi_1)\hat{g}(\xi_2)(1 - \chi(\xi_2))\hat{G}(\xi)m(\xi)\chi(\xi)}{m(\xi_1)\xi_2)},
\]
\[
A_3 = \int d\xi \int_{\xi = \xi_1 + \xi_2} \frac{\hat{f}(\xi_1)\hat{g}(\xi_2)(1 - \chi(\xi_2))\hat{G}(\xi)m(\xi)(1 - \chi(\xi))}{m(\xi_1)\xi_2)},
\]
where $\chi$ is a smooth even function such that
\[
\chi(x) = \begin{cases} 1, & |x| \leq 2N; \\ 0, & |x| \geq 2N + 1 \end{cases}
\]
and $|\partial_x^\alpha \chi| \leq C_\alpha$ for $\alpha \leq 2$, here $C_\alpha$ is independent of $N$.

For $A_1$, by properties of Fourier transform and convolution
\[
A_1 = (\hat{f}(\xi) * \hat{g}(\xi)\chi(\xi)\hat{G}(\xi)m(\xi)) \tag{10}
\]
\[
= (\frac{\hat{g}(\xi)\chi(\xi)}{m(\xi)\xi_2}) * \hat{G}(\xi)m(\xi), \hat{f}(\xi))
\]
\[
= (\mathcal{F}^{-1}\frac{\hat{g}(\xi)\chi(\xi)}{m(\xi)\xi_2}) \mathcal{F}^{-1}\hat{G}(\xi)m(\xi), \mathcal{F}^{-1}\hat{f}(\xi))
\]
where $*$ denotes the convolution operator, $(\cdot, \cdot)$ the inner product of $L^2$, $\varphi(x) = \varphi(-x)$. By Hölder inequality, we have

$$|A_1| \lesssim \|f\|_{L^p} \|G\|_{L^2} \|\mathcal{F}^{-1} \frac{\tilde{g}(\xi)}{m(\xi)} \chi(\xi)\|_{L^q}$$

with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Since $q > 2$, it follows from Hausdorff-Young’s inequality that

$$\|\mathcal{F}^{-1} \frac{\tilde{g}(\xi)}{m(\xi)} \chi(\xi)\|_{L^q} \lesssim \|\frac{\tilde{g}(\xi)}{m(\xi)} \chi(\xi)\|_{L^q'}$$

holds for $\frac{1}{q} + \frac{1}{r} = 1$. Since the support of $\chi$ is contained in $\{\xi : |\xi| \leq 2N + 1\}$, where $m(\xi) \gtrsim 1$, we get

$$\|\frac{\tilde{g}(\xi)}{m(\xi)} \chi(\xi)\|_{L^q'} \lesssim \|\tilde{g}(\xi)\|_{L^q'} \lesssim \|\tilde{g}\|_{L^2} \|\chi\|_{L^q}^{-1} \|\chi\|_{L^q}^{-1} \lesssim \|g\|_{L^2}. \leqno(13)$$

Thanks to (11)-(13), we find that the contribution of $A_1$ is bounded by the right hand side of (9).

For $A_2$, similar to $A_1$, we have

$$|A_2| \leq \|f\|_{L^p} \|\mathcal{F}^{-1} m(\xi) \chi(\xi) \tilde{G}\|_{L^q} \|\mathcal{F}^{-1} \frac{\tilde{g}(\xi)(1 - \chi(\xi))}{m(\xi)}\|_{L^r}$$

with the same $q$ as in (11). Since the support of $1 - \chi$ is contained in $\{\xi : |\xi| \geq 2N\}$, where $m(\xi) \gtrsim N$, then

$$\|\mathcal{F}^{-1} \frac{\tilde{g}(\xi)(1 - \chi(\xi))}{m(\xi)}\|_{L^r} \lesssim N^{-1} \|g\|_{L^2}. \leqno(15)$$

Moreover, it follows from Hausdorff-Young’s inequality again that

$$\|\mathcal{F}^{-1} m(\xi) \chi(\xi) \tilde{G}\|_{L^q} \lesssim \|m(\xi) \chi(\xi) \tilde{G}\|_{L^{q'}} \lesssim \|m(\xi) \chi(\xi)\|_{L^p} \|\tilde{G}\|_{L^2} \lesssim N^{\frac{1}{2}} \|G\|_{L^2}. \leqno(16)$$

Note that $N > 1$, it follows from (14)-(16) that

$$|A_2| \lesssim N^{\frac{1}{2} - 1} \|f\|_{L^p} \|g\|_{L^2} \|G\|_{L^2} \lesssim \|f\|_{L^p} \|g\|_{L^2} \|G\|_{L^2}.\leqno(17)$$

For $A_3$, using the formula similar to (10), by Hölder inequality

$$|A_3| \leq \|f\|_{L^p} \|\mathcal{F}^{-1} m(\xi) (1 - \chi(\xi)) \tilde{G}\|_{L^{p_1}} \|\mathcal{F}^{-1} \frac{\tilde{g}(\xi)(1 - \chi(\xi))}{m(\xi)}\|_{L^{p_2}}$$

where $p_1, p_2$ are given by

$$\begin{cases}
p_2 = 2, \frac{1}{p} + \frac{1}{p_2} = \frac{1}{2}, & \text{if } 0 \leq s \leq \frac{1}{2}, \\
p_1 = 2, \frac{1}{p} + \frac{1}{p_1} = \frac{1}{2}, & \text{if } \frac{1}{2} < s < 1.
\end{cases}$$

In the case $0 \leq s \leq \frac{1}{2}$, write

$$m(\xi) (1 - \chi(\xi)) = \langle \xi \rangle^{1-s} m(\xi) (1 - \chi(\xi)) \cdot \langle \xi \rangle^{s-1}.$$
Combining this and
\[ \|\mathcal{F}^{-1} \tilde{g}(\xi)(1 - \chi(\xi)) \|_{L^2} \lesssim N^{-1} \|g\|_{L^2} \]
implies that
\[ |A_3| \lesssim \|f\|_{L^p} \|g\|_{L^2} \|G\|_{L^2}. \tag{18} \]

Since the proof of the case \( \frac{1}{2} < s < 1 \) is similar, we give only a sketch of it. In fact, write \( \frac{1-\chi(\xi)}{m(\xi)\xi} = \frac{1-\chi(\xi)}{m(\xi)\xi} \cdot (\xi)^{-s} \), use the fact that \( \frac{1-\chi(\xi)}{m(\xi)\xi} \in M_{p_2} \) and \( H^s \hookrightarrow L^{p_2} \), then we obtain (18).

Finally, the desired conclusion follows from the estimates of \( A_1, A_2, A_3 \). \( \square \)

**Corollary 1.** It holds that
\[ \| \partial_x (1 - \partial_x^2)^{-1} I(f^2) \|_{\dot{H}^1(\mathbb{R})} \lesssim \| If \|_{\dot{H}^1(\mathbb{R})} \]
provided that the right hand side is finite, the implicit constant is independent of \( N \).

**Proof.** It follows from Proposition 3 in the case \( p = 2 \) that
\[ \| \partial_x (1 - \partial_x^2)^{-1} I(f^2) \|_{\dot{H}^1} \lesssim \| If \|_{L^2} \lesssim \| f \|_{L^2} \| If \|_{\dot{H}^1}. \]
Moreover, in light of (8), we find
\[ \| f \|_{L^2} \lesssim \| If \|_{H^s} \lesssim \| If \|_{\dot{H}^1}. \]
Thus the conclusion follows. \( \square \)

**Remark 1.** Corollary 1 also holds on torus \( \mathbb{T} \), and the proof is more direct, see Lemma 2.4 of [27].

3. Well-posedness for BBM.

3.1. **Local well-posedness.** Applying \((1 - \partial_x^2)^{-1}\) on both side of (1) yields that
\[ \begin{cases} u_t = -i\varphi(D)u - \frac{i}{2}\varphi(D)u^2 \\ u(0, x) = u_0(x). \end{cases} \tag{19} \]
Here \( \varphi(D) \) is defined as the Fourier multiplier with \( \varphi(\xi) = \frac{\xi}{1 + \xi^2} \). Let \( S(t) \) be the unitary group generated by \( -i\varphi(D) \), namely
\[ S(t)u_0 = e^{-it\varphi(D)}u_0. \]
Note that \( S(t) \) can be also understood as a Fourier multiplier with the symbol \( \exp\{-it\xi/(1 + \xi^2)\} \).

**Lemma 3.1.** Let \( 1 \leq p < \infty, s \in \mathbb{R}, \) then
\[ \| e^{-it\varphi(D)} \|_{\dot{H}^{-s}(\mathbb{R}), \dot{H}^{s}(\mathbb{R})} \lesssim \langle t \rangle^{2|\frac{1}{2} - \frac{1}{p}|}. \]

**Proof.** Since \( \Lambda^s \) commutes with \( e^{-it\varphi(D)} \), it suffices to show
\[ \| e^{-it\varphi(D)} \|_{L^p, L^p} \lesssim \langle t \rangle^{2|\frac{1}{2} - \frac{1}{p}|}. \tag{20} \]
Let \( m_0 = e^{-it\varphi(D)} - 1 \). It follows from Bernstein theorem that
\[ \|m_0(D)\|_{L^1, L^1} \lesssim \|m_0(\xi)\|_{L^2}^{\frac{3}{2}} \|\partial_\xi m_0(\xi)\|_{L^2}^{\frac{1}{2}}. \tag{21} \]
By Euler’s formula and Leibniz rule, we obtain
\[ |m_0(\xi)| \lesssim |t| |\xi|^{-1}, \quad |\partial_\xi m_0(\xi)| \lesssim |t| |\xi|^{-2}. \tag{22} \]
respectively. It follows from (21)-(22) that for all $t \in \mathbb{R}$
\[
\|m_0(D)\|_{L^1,L^1} \lesssim |t|.
\]
Hence
\[
\|e^{-it\varphi(D)}\|_{L^1,L^1} \lesssim \langle t \rangle.
\] (23)
Clearly,
\[
\|e^{-it\varphi(D)}\|_{L^2,L^2} \lesssim 1.
\] (24)
Now (20) follows from (23)-(24), an interpolation and a duality argument.

It’s convenient to rewrite (19) in an integral form as
\[
\begin{aligned}
    u(t) &= S(t)u_0 - \frac{i}{2} \int_0^t S(t-\tau)\varphi(D)u^2(\tau)d\tau.
\end{aligned}
\] (25)

Now we can state the main result in this subsection.

**Theorem 3.2.** Let $u_0 \in H^{s,p}(\mathbb{R})$ with $1 \leq p < \infty$, $s \geq \max\{\frac{1}{p} - \frac{1}{2}, 0\}$, then problem (1) has a unique solution $u \in C([0, T]; H^{s,p}(\mathbb{R}))$, the lifespan $T$ satisfies that
\[
T\langle T \rangle^{\frac{1}{2} - \frac{1}{p}}\|u_0\|_{H^{s,p}} \lesssim 1.
\]
Moreover, the solution map $\Xi(t) : u_0 \mapsto u(t)$ is real analytic from $H^{s,p}(\mathbb{R})$ to $C([0, T]; H^{s,p}(\mathbb{R}))$.

**Proof.** Let $\Gamma$ be the map defined by (25), namely
\[
\begin{aligned}
    u(t) &\mapsto \Gamma u = S(t)u_0 - \frac{i}{2} \int_0^t S(t-\tau)\varphi(D)u^2(\tau)d\tau.
\end{aligned}
\]
The strategy is to prove $\Gamma$ has a fixed point in the ball
\[
\mathcal{B} = \{u \in X_T : \|u\|_{X_T} \leq 2C\langle T \rangle^{\frac{1}{2} - \frac{1}{p}}\|u_0\|_{H^{s,p}}\}
\]
where $C$ is the implicit constant of Lemma 3.1, and $X_T$ is the space of bounded functions on $[0, T]$ with values in $H^{s,p}$ equipped with norm
\[
\|u\|_{X_T} := \sup_{0 \leq t \leq T} \|u(t)\|_{H^{s,p}}.
\]
Let $u, v \in \mathcal{B}$, it follows from Lemma 3.1 and Proposition 2 that
\[
\|\Gamma u\|_{X_T} \leq C\langle T \rangle^{\frac{1}{2} - \frac{1}{p}}\|u_0\|_{H^{s,p}}(1 + \frac{C'}{2}TC\langle T \rangle^{\frac{1}{2} - \frac{1}{p}}\|u_0\|_{H^{s,p}}),
\]
Moreover,
\[
\|\Gamma u - \Gamma v\|_{X_T} \leq \frac{C'}{2}TC\langle T \rangle^{\frac{1}{2} - \frac{1}{p}}\|u_0\|_{H^{s,p}}\|u - v\|_{X_T}.
\]
Thus, $\Gamma$ is a contraction on $\mathcal{B}$ if
\[
\frac{C'}{2}TC\langle T \rangle^{\frac{1}{2} - \frac{1}{p}}\|u_0\|_{H^{s,p}} \leq \frac{1}{2},
\]
This proves the existence of solution in $\mathcal{B}$. The solution, thanks to (19), satisfies further that $\|u_t\|_{X_T} < \infty$. Thus the solution is continuous from $[0, T]$ to $H^{s,p}(\mathbb{R})$. Finally, the analytic property of the solution map $\Xi(t)$ follows from a standard argument, see e.g. [5].
3.2. Global well-posedness. Let $u$ be the solution of (1), we split $u = v + w$ where $v$ satisfies that
\[
\begin{aligned}
v_t + v_x - v_{xxx} + vv_x &= 0 \\
v(0, x) &= (1 - \chi_k(x)) u_0(x)
\end{aligned}
\tag{26}
\]
while the reminder $w$ solves
\[
\begin{aligned}
w_t + w_x - w_{xxx} + wv_x &= 0 \\
w(0, x) &= \chi_k(x) u_0(x).
\end{aligned}
\tag{27}
\]
Here $\chi_k(x) = \chi(\frac{x}{k})$, $k \gg 1$, $\chi$ is a smooth function such that $\chi = 1$ if $|x| \leq 1$ and $\chi = 0$ if $|x| \geq 2$.

**Lemma 3.3.** Let $u_0 \in H^{s,p}(\mathbb{R}), 0 \leq s < 1, 2 \leq p < \infty$. Then for any $T > 0$, problem (26) has a unique solution $v \in C([0, T]; H^{s,p}(\mathbb{R}))$ such that
\[
\sup_{0 \leq t \leq T} \|v(t)\|_{H^{s,p}} \lesssim (T)^{-1}
\]
provided that $k$ is large enough.

**Proof.** The lemma follows from Theorem 3.2 if
\[
\|(1 - \chi_k)u_0\|_{H^{s,p}} \lesssim (T)^{-2}.
\]
This is possible for $k$ large enough since the support of $(1 - \chi_k)u_0$ is contained in $\{x \in \mathbb{R} : |x| \geq k\}$.

**Remark 2.** For any $\varepsilon > 0$, we may obtain a solution $v$ satisfying
\[
\sup_{0 \leq t \leq T} \|v(t)\|_{H^{s,p}} \leq \varepsilon/(T)^2
\]
provided that $k$ is large enough depending on $T$ and $\varepsilon$.

Now we deal with the $w$ part of the solution. Applying $I-$operator on both sides of (27) yields that
\[
\begin{aligned}
(Iw)_t + (Iw)_x - (Iw)_{xxx} + (Iw)v_x &= 0 \\
(Iw)(0, x) &= I(\chi_k(x)u_0(x)).
\end{aligned}
\tag{28}
\]
The equivalent integral equation form of (28) is
\[
Iw(t) = S(t)Iw_0 - \frac{i}{2} \int_0^t S(t - \tau) \varphi(D)I(w^2)d\tau - i \int_0^t S(t - \tau) \varphi(D)I(wv)d\tau
\tag{29}
\]
where $S(t)$ is the free evolution as before. Note that the support of $\chi_ku_0$ is contained in $\{x \in \mathbb{R} : |x| \leq 2k\}$, by H"older inequality and (8), we have
\[
\|Iw(0)\|_{H^1} \lesssim N^{1-s}\|\chi_ku_0\|_{H^s} \lesssim N^{1-s}k^{\frac{1}{2} - \frac{1}{p}}\|u_0\|_{H^{s,p}}
\tag{30}
\]
if $p \geq 2, 0 \leq s < 1$. This allows us to solve (29), as an equation of $Iw$, in $H^1$. Similar to Theorem (3.2), using Proposition 3 and Corollary 1 instead, we find that (29) has a unique solution $Iw \in C([0, \tau]; H^1)$ satisfying the bound
\[
\sup_{0 \leq t \leq \tau} \|Iw(t)\|_{H^1} \leq 2\|Iw_0\|_{H^1}
\tag{31}
\]
with life span $\tau \sim (\|Iw_0\|_{H^1} + 1)^{-1}$.
In order to extend the local solution to a global one, we need to prove a bound of \( \sup_{0 \leq t \leq T} \| I(w(t)) \|_{H^1} \) for any \( T > 0 \). To this end, we decompose

\[
[0, T] = \bigcup_{0 \leq j \leq n-1} [j\tau, (j + 1)\tau],
\]

estimate \( \| I(w(t)) \|_{H^1} \) on each interval and then add those estimates together. Multiplying (28) with \( Iw \) and integrating we find for \( t \in (0, \tau) \)

\[
\frac{1}{2} \frac{d}{dt} \| Iw(t) \|_{H^1}^2 = -(Iw, I(ww_x)) - (Iw, I(wv)_x).
\]

To proceed, we need to deal with the terms on right hand side of (32).

**Lemma 3.4.** There exists a constant \( C \) independent of \( N \) such that

\[
|\langle (I\partial_x(w^2), Iw) | \leq CN^{-\frac{3}{2}} \| Iw \|_{H^1}^3.
\]

**Proof.** The lemma has been proved in [26], we reproduce the proof here for the reader’s convenience. One can use Fourier transform to write

\[
(I\partial_x(w^2), Iw) = \int_{\xi_1 + \xi_2 + \xi_3 = 0} m(\xi_1)m(\xi_2 + \xi_3)i(\xi_2 + \xi_3)\langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle\langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle.
\]

Note that \( \xi_1 + \xi_2 + \xi_3 = 0 \) and \( m(\xi) \) is an even function, we have

\[
-(I\partial_x(w^2), Iw) = \int_{\xi_1 + \xi_2 + \xi_3 = 0} i\xi_1m^2(\xi_1)\langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle\langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle.
\]

It’s easy to see that the identity still holds if we replace \( \xi_1m^2(\xi_1) \) by \( \xi_2m^2(\xi_2) \) or \( \xi_3m^2(\xi_3) \). Therefore,

\[
-(I\partial_x(w^2), Iw) = \int_{\xi_1 + \xi_2 + \xi_3 = 0} i\xi_1m^2(\xi_1)\langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle\langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle.
\]

The lemma will be proved if we can show

\[
\left| \int_{\xi_1 + \xi_2 + \xi_3 = 0} \frac{\langle \xi_1m^2(\xi_1) + \xi_2m^2(\xi_2) + \xi_3m^2(\xi_3) \rangle}{\langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle m(\xi_1)m(\xi_2)m(\xi_3)} \langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle \langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle \right| \lesssim N^{-\frac{3}{2} + \| u \|_{L^2}^3}.
\]

By the symmetry of the integral, we can assume that \( |\xi_1| \leq |\xi_2| \leq |\xi_1| \) without loss of generality. In the case \( |\xi_1| \leq N \), then \( m(\xi_1) = m(\xi_2) = m(\xi_3) = 1 \), and the integral is zero since \( \xi_1 + \xi_2 + \xi_3 = 0 \). So let \( |\xi_1| \geq N \). Since \( |\xi_2| \ll N \) is impossible, we assume \( |\xi_2| \geq N \). One can check that ([9], Lemma 4.3)

\[
|\xi_1m^2(\xi_1) + \xi_2m^2(\xi_2) + \xi_3m^2(\xi_3) | \lesssim |\xi_3|m^2(\xi_3).
\]

Thus, it suffices to bound

\[
\left| \int_{\xi_1 + \xi_2 + \xi_3 = 0} \frac{|\hat{w}(\xi_1)||\hat{w}(\xi_2)||\hat{w}(\xi_3)|}{\langle \xi_1 \rangle\langle \xi_2 \rangle\langle \xi_3 \rangle m(\xi_1)m(\xi_2)} \right|.
\]

Since \( \langle \xi_i \rangle m(\xi_i) \gtrsim N \), the quantity can be controlled by

\[
CN^{-\frac{3}{2}} \int_{\xi_1 + \xi_2 + \xi_3 = 0} \frac{|\hat{w}(\xi_1)||\hat{w}(\xi_2)||\hat{w}(\xi_3)|}{\langle \xi_1 \rangle^{\frac{1}{2} + \frac{3}{2}}} \leq CN^{-\frac{3}{2} + \| u \|_{L^2}^3}
\]

as desired. \( \square \)
It follows from Lemma 3.4 that
\[- (I_w, I(ww_x)) \lesssim N^{-\frac{3}{2}+p} \|Iw\|_{H^1}^3.\] (33)
By Proposition 3, we have
\[- (I_w, I(wv_x)) \lesssim \|v\|_{L^6} \|Iw\|_{H^1}^2.\] (34)
Let \(V_0 = C \sup_{t \in [0, T]} \|v(t)\|_{H^{s,p}},\) \(C\) is the bigger implicit constant in (33) and (34), \(v\) is the solution obtained in Lemma 3.3. It follows from (32)-(34) that for \(t \in (0, \tau)\)
\[\frac{d}{dt} \|Iw(t)e^{-V_0 t}\|_{H^1}^2 \leq 2CN^{-\frac{3}{2}+p} \|Iw\|_{H^1}^3 e^{-2V_0 t}.\] (35)
Using Lemma 3.3 and (31), (35) becomes
\[\frac{d}{dt} \|Iw(t)e^{-V_0 t}\|_{H^1}^2 \leq 16CN^{-\frac{3}{2}+p} \|Iw(0)\|_{H^1}^3 e^{-2V_0 \tau}.\] (36)
Integrating (36) over \([0, \tau],\) we obtain
\[G(\tau) - G(0) \leq 16C\tau N^{-\frac{3}{2}+p} \|Iw(0)\|_{H^1} G(0).\]
where \(G(t) = \|Iw(t)e^{-V_0 t}\|_{H^1}^2.\) Similarly, we have for \(1 \leq j \leq n - 1\)
\[G(j\tau) - G((j-1)\tau) \leq 16C\tau N^{-\frac{3}{2}+p} \|Iw((j-1)\tau)\|_{H^1} G((j-1)\tau).\]
Taking the sum for \(j,\) we find
\[G(T) = G(n\tau) \leq 2G(0)\] (37)
provided that
\[16Cn\tau N^{-\frac{3}{2}+p} \|Iw(0)\|_{H^1} \leq 1\]
which is always possible if we choose
\[N \sim \langle T\|u_0\|_{H^{s,p}} \rangle^{\frac{1}{2\frac{1}{2}+p}}.\]
It follows from (37) that
\[\sup_{0 \leq t \leq T} \|Iw(t)\|_{H^1} \leq 2e^{V_0 T} \|Iw(0)\| \leq 2e \|Iw(0)\|_{H^1}.\] (38)
Thus, we can choose the same \(\tau\) in the bootstrapping argument. Combining (30), (38) implies that
\[\sup_{0 \leq t \leq T} \|w(t)\|_{H^{s,p}} \lesssim k^{\frac{1}{2} - \frac{1}{p}} \langle T\|u_0\|_{H^{s,p}} \rangle^{\frac{1}{2\frac{1}{2}+p}} \|u_0\|_{H^{s,p}}\] (39)
for \(p \geq 2, 0 \leq s < 1.\)
To obtain a bound of \(\sup_{0 \leq t \leq T} \|w(t)\|_{H^{s,p}},\) we rewrite (27) as
\[w(t) = S(t)w(0) - \frac{i}{2} \int_0^t S(t-\tau)\varphi(D)w^2 d\tau - i \int_0^t S(t-\tau)\varphi(D)(vw) d\tau\] (40)
where \(S(t)\) is the free group. Thanks to Lemma 3.1, we deduce from (40) that
\[\sup_{0 \leq t \leq T} \|w(t)\|_{H^{s,p}} \leq C(T) \|w(0)\|_{H^{s,p}} + C(T)^2 \sup_{0 \leq t \leq T} \|\varphi(D)w^2\|_{H^{s,p}} + C(T)^2 \sup_{0 \leq t \leq T} \|\varphi(D)(vw)\|_{H^{s,p}}.\] (41)
It follows from Lemma 2.3 (a) that
\[\|\varphi(D)w^2\|_{H^{s,p}} \lesssim \|w\|_{H^s}^2.\] (42)
Moreover, by Lemma 2 and Remark 2, we have

\[ C(T)^2 \sup_{0 \leq t \leq T} \| \varphi(D)(vw) \|_{H^{s,p}} \leq CC'(T)^2 \sup_{0 \leq t \leq T} \| v \|_{H^{s,p}} \| w \|_{H^{s,p}} \leq \frac{1}{2} \sup_{0 \leq t \leq T} \| w \|_{H^{s,p}} \]  

(43)

provided that \( k \) is large enough.

Now inserting (42) and (43) into (41) yields that

\[ \sup_{0 \leq t \leq T} \| w(t) \|_{H^{s,p}} \leq C(T) \| w(0) \|_{H^{s,p}} + C''(T)^2 \sup_{0 \leq t \leq T} \| w \|_{H^{s,p}}^2 + \frac{1}{2} \sup_{0 \leq t \leq T} \| w \|_{H^{s,p}} \]  

(44)

Absorbing the last term of (44) in the left hand side, and using (39), we obtain

\[ \sup_{0 \leq t \leq T} \| w(t) \|_{H^{s,p}} \leq C(\| u_0 \|_{H^{s,p}}, T, k) < \infty \]

for \( 2 \leq p < \infty, 0 \leq s < 1 \), where \( k \) depends on \( u_0, T \).

Thus, \( u = v + w \) does not blow up on the interval \([0, T]\) in \( H^{s,p}(\mathbb{R}) \). Since \( T \) can be arbitrary large, we obtain the following theorem.

**Theorem 3.5.** Let \( 0 \leq s < 1, 2 \leq p < \infty \), problem (1) is globally well-posed in \( H^{s,p}(\mathbb{R}) \).

Based on this result, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** If \( p \geq 2 \), then by Theorem 3.5, it suffices to consider the subcase \( s \geq 1 \). Since \( u_0 \in H^{s,p}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R}) \), by Theorem 3.5 again, for any \( T > 0 \), there exists a unique solution \( u \) of (1) such that

\[ \sup_{0 \leq t \leq T} \| u(t) \|_{L^p} < \infty. \]

Now using integral equation (25), Lemma 2.3 (a) and Lemma 3.1, we have for all \( t \in [0, T] \)

\[ \| u(t) \|_{H^{s,p}} \lesssim \langle T \rangle \| u_0 \|_{H^{s,p}} + \langle T \rangle \sup_{0 \leq s \leq T} \| u(t) \|_{L^p} \int_0^t \| u(\tau) \|_{H^{s,p}} d\tau. \]

Then it follows from Gronwall’s lemma that

\[ \sup_{0 \leq t \leq T} \| w(t) \|_{H^{s,p}} < \infty. \]

This completes the proof in case \( p \geq 2 \).

The case \( 1 \leq p < 2 \) will be proved in a similar manner if one notes the embedding \( H^{s,p}(\mathbb{R}) \hookrightarrow H^{s-(1/p - 1/2)}(\mathbb{R}) \) and uses Lemma 2.3 (b) instead.

---

4. **Ill-posedness for BBM.** In this section, we will show that the BBM equation is ill-posed in \( H^{s,p}(\mathbb{R}) \) if the restrictions \( s \geq \max\{0, \frac{1}{2} - \frac{1}{p}\}, 1 \leq p < \infty \) are violated. This implies the sharpness of global well-posedness obtained in section 3. We begin with the definition of ill-posedness discussed in this paper.

**Definition 4.1.** We say that the Cauchy problem (1) is ill-posed in \( H^{s,p}(\mathbb{R}) \) if, for any \( T > 0 \), the solution map \( u_0 \mapsto u(t) \) is not \( C^2 \) from \( H^{s,p}(\mathbb{R}) \) to \( C([0, T]; H^{s,p}(\mathbb{R})) \).
Following [5, 7, 19], the ill-posedness may reduce to disproving a bilinear estimate. Let \( \varepsilon > 0 \), consider the Cauchy problem

\[
\begin{align*}
    u_t + uu_x - u_{txx} + uu_x &= 0, \\
    u(0, x) &= \varepsilon u_0(x).
\end{align*}
\]  

(45)

Rewriting (45) into an integral equation form, we obtain

\[
    u(\varepsilon, t, x) = \varepsilon S(t)u_0 - i \int_0^t S(t - \tau) \varphi(D)(u^2(\varepsilon, \tau, x))d\tau
\]

where \( S(t) \) denotes the free evolution \( \exp\{-it\varphi(D)\} \) as before. Then a formal calculation yields that

\[
\begin{align*}
    \frac{\partial u}{\partial \varepsilon}(0, t, x) &= S(t)u_0 := U(t, x) \\
    \frac{\partial^2 u}{\partial \varepsilon^2}(0, t, x) &= -i \int_0^t S(t - \tau) \varphi(D)(U(\tau, x))^2 d\tau.
\end{align*}
\]

Thus, we obtain the following sufficient condition for ill-posedness of Cauchy problem (1) in \( H^{s,p}(\mathbb{R}) \).

**Proposition 4.** For any \( T > 0 \), there exists a \( u_0 \in H^{s,p}(\mathbb{R}) \) such that

\[
    \sup_{0 \leq t \leq T} \left\| \int_0^t S(t - \tau) \varphi(D)(S(\tau)u_0)^2 d\tau \right\|_{H^{s,p}} \lesssim \| u_0 \|_{H^{s,p}}^2
\]

fails, and consequently the Cauchy problem (1) is ill-posed in \( H^{s,p}(\mathbb{R}) \).

In the sequel, we shall need the following more general approximate identity than that in Rudin [22].

**Definition 4.2.** We say a sequence \( \{f_j(t, x)\} \in L^\infty(0, T; L^1(\mathbb{R})) \), \( j = 1, 2, \cdots \), is a \( t \)-uniform approximate identity if the following hold:

1. \( f_j(t, x) \geq 0 \), for any \( (t, x) \in [0, T] \times \mathbb{R} \);
2. \( \int_\mathbb{R} f_j(t, x)dx = 1 \), for any \( t \in [0, T] \);
3. for any \( \tau > 0 \), it holds uniformly for \( t \in [0, T] \) that

\[
    \lim_{j \to \infty} \int_{|x| \leq \tau} f_j(x)dx = 1.
\]

**Remark 3.** It’s easy to see that if \( \{f_j(t, x)\} \) is a \( t \)-uniform approximate identity then for any \( \tau > 0 \)

\[
    \lim_{j \to \infty} \int_{|x| \leq \tau} f_j(t, x)dx = 1
\]

holds uniformly for \( t \in [0, T] \).

The following property of \( t \)-uniform approximate identities is important in the proof of our ill-posedness results.

**Proposition 5.** Let \( 1 \leq p < \infty \), \( g \in L^p(\mathbb{R}) \), and \( \{f_j(t, x)\} \in L^\infty(0, T; L^1(\mathbb{R})) \) be a \( t \)-uniform approximate identity. Then we have

\[
    f_j * g \to g \quad \text{in} \quad L^p(\mathbb{R})
\]

uniformly for \( t \in [0, T] \), where * denotes the convolution operator. In other words,

\[
    \lim_{j \to \infty} \sup_{0 \leq t \leq T} \| f_j(t, \cdot) * g - g \|_{L^p} = 0.
\]
Proof. Since translations are continuous on $L^p(\mathbb{R})$, for any $\varepsilon > 0$, there exists $\tau > 0$ such that for $|y| \leq \tau$
\[ \|g(\cdot - y) - g(\cdot)\|_{L^p} \leq \varepsilon. \]
Now by Minkowski inequality we find
\[ \|f(t, \cdot) * g - g\|_{L^p} \leq \int_{\mathbb{R}} f_j(t, y)\|g(\cdot - y) - g(\cdot)\|_{L^p} dy \]
\[ = (\int_{|y| \leq \tau} + \int_{|y| \geq \tau}) f_j(t, y)\|g(\cdot - y) - g(\cdot)\|_{L^p} dy. \]
By Definition 4.2 and Remark 3 we obtain
\[ \sup_{0 \leq t \leq T} \|f_j(t, \cdot) * g - g\|_{L^p} \leq \varepsilon + 2\|g\|_{L^p} \int_{|y| \geq \tau} f_j(t, y) dy \leq C\varepsilon \]
provided that $j$ is large enough. $\square$

4.1. The case $p \geq 2$. The main result in this subsection is the following theorem.

Theorem 4.3. Let $2 \leq p < \infty$. Then the Cauchy problem (1) is ill-posed in $H^{s,p}(\mathbb{R})$ for any $s < 0$.

The idea of the proof is to construct a sequence $\{u_{0j}\}$ such that the right hand side of (47) goes to zero but the left hand side has a positive lower bound. Let $u_{0j}$ be smooth functions defined by Fourier transform as
\[ \hat{u}_{0j} = (2\gamma)^{-1/2}(1_{[-j-\gamma,-j]} + 1_{[j,j+\gamma]}) \]
where $j > 1$, $\gamma = \ln j$ and $1_A$ denotes the characteristic function of $A$. It follows from Plancherel theorem that
\[ \|u_{0j}\|_{L^2} = 1. \]
Since $\hat{u}_{0j}$ is even, $u_{0j}$ is real. Moreover, using Fourier inversion, we arrive at
\[ u_{0j} = (\gamma/2)^{-1/2} \sin \gamma \pi x \cos(2j + \gamma) \pi x. \]
Then it is easy to see that for $1 < p < \infty$
\[ \|u_{0j}\|_{L^p} \lesssim \gamma^{\frac{1}{2} - \frac{1}{p}}. \] (48)
Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\chi(x) = 0$ if $|x| \leq \frac{1}{2}$ or $|x| \geq 2$ and $\chi(x) = 1$ if $1 \leq |x| \leq \frac{3}{2}$. Set $\chi_j = \chi(\cdot/j)$. Note that the support of $\hat{u}_{0j}$ is contained in $[-\frac{3j}{2}, -j] \cup [j, \frac{3j}{2}]$, where $\chi_j = 1$, we find
\[ \chi_j(D) u_{0j} = u_{0j}. \]
Thus, for any $s \in \mathbb{R}$ and $p \in [2, \infty)$
\[ \|u_{0j}\|_{H^{s,p}} = \|\Lambda^s \chi_j(D) u_{0j}\|_{L^p} \lesssim j^s \gamma^{\frac{1}{2} - \frac{1}{p}}. \] (49)
Here we have used the fact that $j^{-s}(\xi)^s \chi_j(\xi)$ is an $L^p$ multiplier, namely
\[ \|j^{-s} \Lambda^s \chi_j(D)\|_{L^p, L^p} \lesssim 1 \]
with an upper bound independent of $j$, which can be verified by Theorem 2.1.

It follows from Lemma 3.1 that for $1 < p < \infty$
\[ \|S(t) u_{0j}\|_{L^p} \lesssim \langle t \rangle^{\frac{1}{2} - \frac{1}{p}} \gamma^{\frac{1}{2} - \frac{1}{p}}. \] (50)
In particular, we find $S(t) u_{0j} \in L^\infty(0, T; L^2)$. Moreover, we have
Lemma 4.4. For any $T > 0$, $(S(t)u_{0j})^2 \in L^\infty(0,T; L^1(R))$ is a $t$-uniform approximate identity.

Proof. Since $S(t)$ is a unitary group on $L^2(R)$, we find for all $t \in [0,T]$

$$\int_R (S(t)u_{0j})^2 dx = \|S(t)u_{0j}\|_{L^2}^2 = \|u_{0j}\|_{L^2}^2 = 1.$$ 

So it suffices to show that $(S(t)u_{0j})^2$ satisfies (3) of Definition 4.2. Note that $S(t)u_{0j}$ is the solution of the linear equation

$$U_t - U_{txx} + U_x = 0, \quad U(0) = u_{0j}.$$ 

Since $-i\varphi(D)$ is bounded on $L^p(R)$, using (50) we obtain for $1 < p < \infty$

$$\|U_t\|_{L^\infty(0,T; L^p)} \leq \| - i\varphi(D)U\|_{L^\infty(0,T; L^p)} \leq (T)^{\frac{1}{2} - \frac{1}{p}} \|\gamma\|^{1 - \frac{1}{p}}.$$  (51)

Let $\psi : R \mapsto R$ be a smooth function such that $\psi(x) = 1$ for $|x| \geq 1$ and $\psi(x) = 0$ for $|x| \leq \frac{1}{2}$. Set $\psi_\tau = \psi(\cdot \tau)$ and $U_\tau = \psi_\tau U$. Then $U_\tau$ satisfies the following equation

$$\begin{cases}
(U_\tau)_t - (U_\tau)_{txx} + (U_\tau)_x = -2\partial_x(\psi'_\tau U_t) + \psi''_\tau U_t + \psi'_\tau U \\
U_\tau(0) = \psi_\tau u_{0j}(x).
\end{cases}$$

By Duhamel principle we find

$$U_\tau(t) = S(t)\psi_\tau u_{0j}(x) + \int_0^t S(t - \tau)(2i\psi'_\tau U_t)d\tau + \int_0^t S(t - \tau)(1 - \partial_x^2)^{-1}(\psi''_\tau U_t + \psi'_\tau U)d\tau.$$  (52)

Thanks to Lemma 3.1, (51) and the fact that $\varphi(D)$ is bounded from $L^\frac{4}{\tau}$ to $L^2$, we have

$$\sup_{0 \leq t \leq T} \|\int_0^t S(t - \tau)\varphi(D)(2i\psi'_\tau U_t)d\tau\|_{L^2} \lesssim \sup_{0 \leq t \leq T} \int_0^t \|\varphi(D)(\psi'_\tau U_t)\|_{L^2}d\tau \lesssim \tau \|U_t\|_{L^\infty(0,T; L^\frac{4}{\tau})} \lesssim (T)^{\frac{1}{2}} \tau \gamma^{-\frac{1}{2}} \to 0$$  (53)

as $j \to \infty$. Similarly, we obtain

$$\sup_{0 \leq t \leq T} \|\int_0^t S(t - \tau)(1 - \partial_x^2)^{-1}(\psi''_\tau U_t + \psi'_\tau U)d\tau\|_{L^2} \lesssim (T)^{\frac{1}{2}} (\tau)^2 \gamma^{-\frac{1}{2}} \to 0$$  (54)

as $j \to \infty$. Moreover,

$$\sup_{0 \leq t \leq T} \|S(t)\psi_\tau u_{0j}(x)\|_{L^2} = \|\psi_\tau u_{0j}(x)\|_{L^2} \lesssim \gamma^{-1/2} \tau^{-1} \to 0$$  (55)

as $j \to \infty$. Combining (52)-(55) implies

$$\sup_{0 \leq t \leq T} \int_{|x| \geq \tau} (S(t)u_{0j})^2 dt \leq \sup_{0 \leq t \leq T} \|U_\tau\|_{L^2}^2 \to 0$$

as $j \to \infty$. This completes the proof. \hfill \Box

Proof of Theorem 4.3. Since the Green function of $(1 - \partial_x^2)^{-1}$ is $e^{-|x|}$, one can show that (see [14])

$$\varphi(D)f = K(x) \ast f$$
for all \( f \in C_0^\infty \) with \( K(x) = -\text{sgn}(x)e^{-|x|} \), where \( \text{sgn}(x) \) is the standard sign function. Then it follows from Proposition 5 and Lemma 4.4 that
\[
\lim_{j \to \infty} \sup_{0 \leq t \leq T} \| \varphi(D)(S(\tau)u_{0j})^2 - K(x) \|_{L^p} = 0. \tag{56}
\]

For \( s < 0 \), we have
\[
\sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau)\varphi(D)(S(\tau)u_{0j})^2 - K(x) \right\|_{H^{s,p}} \\
\leq \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau)\varphi(D)(S(\tau)u_{0j})^2 - K(x) \right\|_{L^p} \\
\lesssim \sup_{0 \leq t \leq T} \int_0^t \left\| (t-\tau)^{2\frac{1}{2} - \frac{1}{p}} \|\varphi(D)(S(\tau)u_{0j})^2 - K(x) \|_{L^p} \right\| d\tau \\
\lesssim \langle T \rangle^{1+2\frac{1}{2} - \frac{1}{p}} \sup_{0 \leq \tau \leq T} \|\varphi(D)(S(t)u_{0j})^2 - K(x) \|_{L^p}.
\]

which gives, in light of (56), that
\[
\lim_{j \to \infty} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau)\varphi(D)(S(\tau)u_{0j})^2 - K(x) \right\|_{H^{s,p}} = 0. \tag{57}
\]

Since the Fourier transform of \( K(x) \) is \( \varphi(\xi) \), it is easy to check that
\[
\mathcal{F} \left( \int_0^t S(t-\tau)K(x) d\tau \right) = 1 - e^{-it\varphi(\xi)} \neq 0.
\]

Thus, for any \( T > 0 \), there exists a constant \( c_T > 0 \) depending only on \( T \) such that
\[
\sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau)K(x) d\tau \right\|_{H^{s,p}} \geq c_T. \tag{58}
\]

Collecting (57) and (58) gives
\[
\lim_{j \to \infty} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau)\varphi(D)(S(\tau)u_{0j})^2 d\tau \right\|_{H^{s,p}} \geq c_T.
\]

On the other hand, since \( s < 0 \), by (49) we find \( \| u_{0j} \|_{H^{s,p}} \to 0 \) as \( j \) goes to infinity. Hence
\[
\sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau)\varphi(D)(S(\tau)u_{0j})^2 d\tau \right\|_{H^{s,p}} \lesssim \| u_{0j} \|^2_{H^{s,p}}
\]
fails for \( j \) large enough. Then the theorem follows from Proposition 4. \( \square \)

4.2. **The case** \( 1 \leq p < 2 \).

**Theorem 4.5.** If \( 1 \leq p < 2 \) and \( s - \frac{1}{p} + \frac{1}{2} < 0 \), then problem (1) is ill-posed in \( H^{s,p}(\mathbb{R}) \).

In particular, let \( s = 0 \) in Theorem 4.5 we obtain the following

**Corollary 2.** Problem (1) is ill-posed in \( L^p(\mathbb{R}) \) if \( 1 \leq p < 2 \).

In order to prove Theorem 4.5, we need the following interpolation inequalities between Bessel potential spaces \( H^{s,p}(\mathbb{R}) \).
Lemma 4.6. Let \( s' < s < s'' \in \mathbb{R} \). If \( 1 < p < \infty \), then
\[
\| f \|_{H^{s'}, p} \lesssim \| f \|_{H^{s', \infty}, p} \lesssim \| f \|_{H^{s'', p}}.
\]
If \( p = 1 \), then
\[
\| f \|_{H^{s}, 1} \lesssim \| f \|_{H^{s', \infty}, 1} \lesssim \| f \|_{H^{s'', 1}}.
\]

Proof. We only prove the case \( p = 1 \), the case \( 1 < p < \infty \) is similar. Let \( \psi : \mathbb{R} \to \mathbb{R} \) be a smooth function such that \( \psi(x) = 1 \) for \( |x| \geq 1 \) and \( \psi(x) = 0 \) for \( |x| \leq \frac{1}{2} \). Set \( \psi_N = \psi(\cdot / N) \). Then using Theorem 2.1 one obtains that
\[
\| \psi_N(D) \Lambda^{s-s'} \|_{L^1, L^1} \lesssim N^{s-s'}
\]
and
\[
\| (1- \psi_N(D)) \Lambda^{s-s''} \|_{L^1, L^1} \lesssim N^{s-s''}.
\]
Then
\[
\| f \|_{H^{s, 1}} = \| \Lambda^s f \|_{L^1} \lesssim N^{s-s'} \| \Lambda^{s'} f \|_{L^1} + N^{s-s''} \| \Lambda^{s''} f \|_{L^1}.
\]
Choosing \( N = (\| \Lambda^{s'} f \|_{L^1}/\| \Lambda^{s''} f \|_{L^1})^{1/(s''-s')} \) yields the desired conclusion.

Let \( \phi \geq 0 \) be a smooth function with compact support such that \( \| \phi \|_{L^2} = 1 \). For \( j \gg 1 \), we set
\[
u_{0j} = j^{1/2} \phi(j \cdot).
\]
Then
\[
\| \nu_{0j} \|_{L^2} = 1
\]
and we have the following

Lemma 4.7. If \( 1 \leq p < 2 \) and \( s - \frac{1}{p} + \frac{1}{2} < 0 \), then
\[
\lim_{j \to \infty} \| \nu_{0j} \|_{H^{s, p}} = 0.
\]

Proof. It’s easy to see that for \( \alpha = 0, 1, 2 \),
\[
\| D^\alpha \nu_{0j} \|_{L^p} \lesssim j^{\frac{1}{2} - \frac{1}{p} + \alpha}.
\]
Then
\[
\| \nu_{0j} \|_{H^{2, p}} \lesssim j^{\frac{1}{2} - \frac{1}{p}}.
\]
An application of Lemma 4.6 implies that
\[
\| \nu_{0j} \|_{H^{2, p}} \lesssim \| \nu_{0j} \|_{L^p} \lesssim j^{\frac{1}{2} - \frac{1}{p}} \to 0
\]
as \( j \) goes to infinity for \( 1 < p < 2 \).

If \( p = 1 \), then the restriction on \( s \) becomes \( s < \frac{1}{2} \). By Lemma 4.6 again we have for any \( \varepsilon > 0 \)
\[
\| \nu_{0j} \|_{H^{1, 1}} \lesssim \| \nu_{0j} \|_{L^1} \lesssim j^{\frac{1}{2} - \frac{1}{p} - \varepsilon} \nu_{0j} \|_{L^1} \lesssim j^{s - \frac{1}{2} + 2\varepsilon}.
\]
Now choosing \( \varepsilon = \frac{1}{4} (1 - s) \) yields that
\[
\| \nu_{0j} \|_{H^{1, 1}} \lesssim j^{\frac{1}{2}(s - \frac{1}{2})} \to 0
\]
as \( j \) goes to infinity.

\[\square\]
Proof of Theorem 4.5. Similar to Lemma 4.4, one can show that for any $T > 0$, $(S(t)u_0)^2 \in L^\infty(0,T;L^1(R))$ is a $t$-uniform approximate identity. Similar to the proof of Theorem 4.3

$$\lim_{j \to \infty} \sup_{0 \leq \tau \leq T} \left\| \int_0^t S(t-\tau)\varphi(D)(S(\tau)u_0)^2 d\tau \right\|_{H^s, p} \geq c_T$$

for some $c_T > 0$. Combining this and Lemma 4.7, we find

$$\sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau)\varphi(D)(S(\tau)u_0)^2 d\tau \right\|_{H^s, p} \lesssim \|u_0\|_{H^s, p}^2$$

fails for $j$ large enough. By Proposition 4 again, Theorem 4.5 follows.

4.3. Proof of Theorem 1.2. Theorem 1.2 follows easily from Theorem 4.3 and 4.5 proved in subsection 4.1 and 4.2, respectively.

5. Growth of norms in Sobolev spaces. In section 3, we established the global well-posedness of the BBM equation in Sobolev spaces $H^{s,p}(R)$. In particular, for any $t > 0$ the norms $\|u(t)\|_{H^{s,p}}$ are finite if $u(t)$ is the solution of (1) associated with initial data $u_0 \in H^{s,p}(R)$. However, the growth of $\|u(t)\|_{H^{s,p}}$ in terms of time $t$ is not known. This is an interesting topic since the growth of norms quantify the transfer of energy from low to high frequencies, see Bourgain [6] and Sohinger [23]. Due to some technical reasons, we divide our exploration in two cases.

5.1. The case $1 \leq p \leq 2$. We start with growth of norms in $H^s(R)$, $0 \leq s \leq 1$. The growth of norm in this case is a byproduct of proving global well-posedness of (1) by $I-$method.

Theorem 5.1. Let $0 \leq s \leq 1$, $u_0 \in H^s(R)$, then the solution $u(t)$ of (1) satisfies that for any $T > 0$

$$\|u(T)\|_{H^s} \lesssim \langle T \rangle^{\frac{s}{2} + \frac{1}{p}} \|u_0\|_{H^s}$$

where the implicit constant depends only on $s$ and $\|u_0\|_{H^s}$.

Proof. The case $s = 1$ follows from the conservation law $(E)$. So suppose $0 \leq s < 1$. One can proceed with the same argument in Section 3 dealing with the $w-$part, and find that (39) becomes

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \lesssim \langle T \|u_0\|_{H^s} \rangle^{\frac{s}{2} + \frac{1}{p}} \|u_0\|_{H^s}.$$ 

This implies the desired conclusion.

Now we turn to the growth of norms in $H^s(R)$, $s > 1$. The strategy is different from that in Theorem 5.1. We follow the idea of Bourgain [6] and Sohinger [23] to deduce an iteration bound as

$$\|u(t + \tau)\|_{H^s} \leq \|u(t)\|_{H^s} + C\|u(t)\|_{H^s}^{1-r}$$ (59)

which holds for all times $t$ and some constants $r \in (0,1]$. Here $\tau, C$ depend only on $s$ and the initial data. Then (59) implies that

$$\|u(T)\|_{H^s} \lesssim \langle T \rangle^{\frac{1}{2}}.$$ (60)
Theorem 5.2. Let $s > 1$, $u_0 \in H^s(\mathbb{R})$, then the solution $u(t)$ of (1) satisfies that for any $T > 0$

$$\|u(T)\|_{H^s} \lesssim \begin{cases} \langle T \rangle, & 1 < s < \frac{3}{2}, \\
\frac{(2s-1)}{2s-\frac{3}{2}}, & \frac{3}{2} \leq s < 2, \\
\frac{(2s-1)}{s}, & s \geq 2.
\end{cases}$$

where the implicit constant depends only on $s$ and $\|u_0\|_{H^s}$.

Proof. Let $D$ be the operator defined by the symbol $|\xi|$. Multiplying (1) with $D^{2s-2}u$ and integrating yield that

$$\frac{d}{dt}\|D^s u\|_{L_2}^2 + (\partial_x u^2, D^{2s-2}u) = 0. \tag{61}$$

Let $s - 1 = m + \alpha$ where $m \geq 0$ is an integer and $0 \leq \alpha < 1$. Then

$$(\partial_x u^2, D^{2s-2}u) = (D^m \partial_x^{m+1}u^2, D^m \partial_x^m u)$$

$$= (2D^m(u\partial_x^{m+1}u), D^m \partial_x^m u) + \sum_{\substack{j_1+j_2=m+1, \\
0 \leq j_1, j_2 \leq m}} c_{j_1,j_2}(D^\alpha(\partial_x^{j_1} u \partial_x^{j_2} u), D^\alpha \partial_x^m u)$$

$$=: I_1 + I_2.$$

One can rewrite $I_1$ as

$$2(uD^\alpha \partial_x^{m+1}u, D^m \partial_x^m u) + 2(D^\alpha(u\partial_x^{m+1}u) - uD^\alpha \partial_x^{m+1}u, D^m \partial_x^m u).$$

On one hand, applying integration by parts, Hölder inequality and Theorem 2.2 we find

$$|2(uD^\alpha \partial_x^{m+1}u, D^m \partial_x^m u)| \lesssim |(\partial_x u, (D^\alpha \partial_x^m u)^2)|$$

$$\lesssim \|\partial_x u\|_{L^2}\|D^\alpha \partial_x^m u\|_{L^4}^2 \lesssim \|u\|_{H^1}\|u\|_{H^{m+\alpha}}^2$$

$$\lesssim Q(\|u\|_{H^1})\|u\|_{H^s}^{\max\{0, 2(1 - \frac{3}{4(s-1)})\}}.$$

Here and below $Q(\cdot)$ is an increasing function which may be changed in different places. On the other hand, thanks to a commutator estimate (see Appendix, Theorem A.12 [15])

$$|2(D^\alpha(u\partial_x^{m+1}u) - uD^\alpha \partial_x^{m+1}u, D^m \partial_x^m u)|$$

$$\lesssim \|((D^\alpha(u\partial_x^{m+1}u) - uD^\alpha \partial_x^{m+1}u)\|_{L^2}\|D^\alpha \partial_x^m u\|_{L^2}$$

$$\lesssim \|\partial_x^{m+1}u\|_{L^2}\|D^\alpha u\|_{L^\infty}\|D^\alpha \partial_x^m u\|_{L^2}$$

$$\lesssim \|u\|_{H^{m+1}}\|u\|_{H^{m+\alpha}}^{\frac{3}{2} + \frac{\alpha}{s-1}}\|u\|_{H^s}^{\max\{0, \frac{3}{2(1 - \frac{3}{4(s-1)})}\}}$$

$$\lesssim Q(\|u\|_{H^1})\|u\|_{H^s}^{\max\{0, \frac{3}{2(1 - \frac{3}{4(s-1)})}\}}.$$ 

Observe that both $\frac{2m+\alpha-1}{s-1}$ and $\frac{2m+\alpha-1}{s-1} + \frac{3}{4(s-1)}$ are less than $2(1 - \frac{3}{4(s-1)})$. Thus,

$$I_1 \lesssim Q(\|u\|_{H^1})\|u\|_{H^s}^{\max\{0, \frac{3}{2(1-\frac{3}{4(s-1)})}\}}. \tag{62}$$
For $I_2$, it follows from Proposition 1 that
\[
I_2 \lesssim \sum_{j_1, j_2 = m + 1, 0 \leq j_1 \leq j_2 \leq m} \| D^{j_1} (\partial_x^{j_2} u \partial_x^{j_2} u) \|_{L^2} \| D^{-m} \partial_x^m u \|_{L^2} \\
\lesssim \sum_{j_1, j_2 = m + 1, 0 \leq j_1 \leq j_2 \leq m} \| A^{j_1} (\partial_x^{j_2} u \partial_x^{j_2} u) \|_{L^2} \| D^{-m} \partial_x^m u \|_{L^2} \\
\lesssim \sum_{j_1, j_2 = m + 1, 0 \leq j_1 \leq j_2 \leq m} (\| \partial_x^{j_1} u \|_{H^{\alpha + 4}} \| \partial_x^{j_2} u \|_{L^4} + \| \partial_x^{j_2} u \|_{H^{\alpha + 4}} \| \partial_x^{j_1} u \|_{L^4}) \| D^{-m} \partial_x^m u \|_{L^2} \\
\lesssim \sum_{j_1, j_2 = m + 1, 0 \leq j_1 \leq j_2 \leq m} \left( \| u \|_{H^{j_1 + \alpha + \frac{1}{4}}} \| u \|_{H^{j_2 + \frac{1}{4}}} + \| u \|_{H^{j_1 + \frac{1}{4}}} \| u \|_{H^{j_2 + \frac{1}{4}}} \right) \| u \|_{H^{m + \alpha}} \tag{63}
\]
where we have used $\xi^j \langle \xi \rangle^{-j} \in \mathcal{M}_4$ and embedding $H^{j + \frac{1}{4}} \hookrightarrow H^{j, 4}$ in the last step.

The restrictions on $j_1$ and $j_2$ in the sum imply that $1 < j + \frac{1}{4} \leq j + \alpha + \frac{1}{4} < s$ for $j = j_1, j_2$. This allows us to use the following interpolation inequalities
\[
\| u \|_{H^{j_1 + \alpha + \frac{1}{4}}} \lesssim Q(\| u \|_{H^{j_1}}) \| u \|_{H^{s}}^{j_1 + j_2 + \alpha - \frac{2}{s}} \| u \|_{H^{s}}^{\max{0, \frac{1}{\alpha}, 2(1 - \frac{s}{3(s - 1)})}} \\
\| u \|_{H^{j_1 + \frac{1}{4}}} \lesssim Q(\| u \|_{H^{j_1}}) \| u \|_{H^{s}}^{j_1 + j_2 + \alpha - \frac{2}{s}} \| u \|_{H^{s}}^{\max{0, \frac{1}{\alpha}, 2(1 - \frac{s}{3(s - 1)})}} \tag{64}
\]
Inserting these into (63) yields that
\[
I_2 \lesssim \sum_{j_1, j_2 = m + 1, 0 \leq j_1 \leq j_2 \leq m} Q(\| u \|_{H^{j_1}}) \| u \|_{H^{s}}^{j_1 + j_2 + \alpha - \frac{2}{s}} \| u \|_{H^{s}}^{\max{0, \frac{1}{\alpha}, 2(1 - \frac{s}{3(s - 1)})}} \\
\lesssim Q(\| u \|_{H^{j_1}}) \| u \|_{H^{s}}^{\max{0, \frac{1}{\alpha}, 2(1 - \frac{s}{3(s - 1)})}}. \tag{64}
\]
It follows from (62) and (64) that
\[
| (\partial_x u^2, D^{2s-2} u) | \lesssim Q(\| u \|_{H^{j_1}}) \| u \|_{H^{s}}^{\max{0, \frac{1}{\alpha}, 2(1 - \frac{s}{3(s - 1)})}, 1 - \frac{s}{3(s - 1)}, 2(1 - \frac{s}{3(s - 1)})}.
\]
After calculating the maximum value of those exponents, we obtain
\[
| (\partial_x u^2, D^{2s-2} u) | \lesssim Q(\| u \|_{H^{j_1}}) \| u \|_{H^{s}}^{A(s)} \tag{65}
\]
where $A(s)$ is given by
\[
A(s) = \begin{cases} 
0, & 1 < s < \frac{3}{2}, \\
\frac{1 - \frac{3}{2}}{s - 1} + \frac{2}{s} & \frac{3}{2} < s < 2, \\
2(1 - \frac{3}{4(s - 1)}), & s \geq 2.
\end{cases}
\]
For any $T \geq 0$, substituting (65) into (61) and integrating over $[T, T + 1]$ imply that
\[
\| D^s u(T + 1) \|_{L^2}^2 \leq \| D^s u(T) \|_{L^2}^2 + Q(\| u(T) \|_{H^{j_1}}) \| u(T) \|_{H^{s}}^{A(s)}.
\]
Note that $\| u(T) \|_{H^{j_1}}$ is bounded independent of $T$, then the above inequality becomes
\[
\| u(T + 1) \|_{H^{s}}^2 \leq \| u(T) \|_{H^{s}}^2 + C \| u(T) \|_{H^{s}}^{A(s)}, \tag{66}
\]
for all $T \geq 0$, here $C$ is a constant depends only $s$ and $\| u_0 \|_{H^{s}}$.

Then the desired conclusion follows from (66) and (60).
It should be noted that the main part of the proof for Theorem 5.2 is devoted to obtain nice upper bounds. In fact, if one naively deduces that
\[ |(uu_x, D^{2(s-1)}u)| = |(\partial_x D^{s-2}u^2, D^s u)| \lesssim \|D^{s-1}u^2\|_{L^2} \|D^s u\|_{L^2} \lesssim \|D^{s-1}u\|_{L^2} \|u\|_{L^\infty} \|u\|_{H^s} \lesssim Q(\|u\|_{H^s}) \|u\|^{2(1-\frac{2}{p}+)}_{H^s} \]
which in turn implies that
\[ \|u(T)\|_{H^s} \lesssim \langle T \rangle^{2s+}. \]

Compared to Theorem 5.2, this is obviously a worse bound.

**Theorem 5.3.** Let \(1 \leq p \leq 2, s \geq \frac{1}{p} - \frac{1}{2}, u_0 \in H^{s,p}(\mathbb{R})\), then the solution \(u(t)\) of (1) satisfies that for any \(T > 0\)
\[ \|u(T)\|_{H^{s,p}} \lesssim \begin{cases} \langle T \rangle^{2(\frac{1}{p}+2)}, & \frac{1}{p} - \frac{1}{2} \leq s < \frac{1}{p} + 1, \\ \langle T \rangle^{2(\frac{1}{p} - \frac{1}{2})}, & s \geq \frac{1}{p} + 1 \end{cases} \]
where the implicit constant depends only on \(s, p\) and \(\|u_0\|_{H^s}\).

**Proof.** Using integral equation (25) and Lemma 3.1, we find that
\[ \|u(T)\|_{H^{s,p}} \lesssim \langle T \rangle^{2(\frac{1}{p} - \frac{1}{2})} + \int_0^T \langle T - \tau \rangle^{2(\frac{1}{p} - \frac{1}{2})} \|\varphi(D)u^2(\tau)\|_{H^{s,p}} d\tau. \] (67)

Absorbing the first term by an upper bound of the second term on RHS of (67), and applying Lemma 2.3 imply that
\[ \|u(T)\|_{H^{s,p}} \lesssim \begin{cases} \langle T \rangle^{\frac{2}{p}} \sup_{0 \leq \tau \leq T} \langle \|u(t)\|_{L^2}\rangle^2, & s < \frac{1}{p}, \\ \langle T \rangle^{\frac{2}{p}} \sup_{0 \leq \tau \leq T} \langle \|u(t)\|_{L^2} \|u(t)\|_{H^{s-\frac{1}{p}+}}\rangle, & s \geq \frac{1}{p}. \end{cases} \] (68)

Since \(u_0 \in H^{s,p}(\mathbb{R}) \hookrightarrow H^{s-\left(\frac{1}{p} - \frac{1}{2}\right)}(\mathbb{R})\), it follows from Theorem 5.1 and 5.2 that
\[ \|u(T)\|_{L^2} \lesssim \langle T \rangle^{2}, \quad s < \frac{1}{p}, \] (69)
\[ \|u(T)\|_{L^2}, \|u(T)\|_{H^{s-\frac{1}{p}+}} \lesssim \langle T \rangle^{2}, \quad \frac{1}{p} \leq s < \frac{1}{p} + 1, \] (70)
\[ \|u(T)\|_{L^2} \lesssim 1, \|u(T)\|_{H^{s-\frac{1}{p}+}} \lesssim \langle T \rangle^{\frac{2}{p}(s-\frac{1}{2})}, \quad s \geq \frac{1}{p} + 1. \] (71)

Inserting (69)-(72) into (68) implies the desired conclusion. \(\square\)

**Remark 4.** One can also obtain an upper bound with an exponent in terms of \(s\) in the case \(\frac{1}{p} - \frac{1}{2} \leq s < \frac{1}{p} + 1\). However, we shall not dwell on this issue further.

5.2. **The case** \(p > 2\). Let \(u_0 \in L^p(\mathbb{R}), 2 < p < \infty\), it follows from Theorem 3.5 that \(\sup_{0 \leq t \leq T} \|u(t)\|_{L^p}\) is finite for any \(T > 0\). The proof relies heavily on the fact that
\[ \int_{|x| \geq k} |u_0|^p dx \to 0 \] (72)
as \(k\) goes to infinity. Since (72) may go to zero arbitrary slowly, no quantitative bounds of \(\sup_{0 \leq t \leq T} \|u(t)\|_{L^p}\) are obtained. Thus we shall assume further some decay
conditions on \( u_0 \) at infinity. Let \( \alpha > 0 \), denote by \( L^p((x)\alpha \, dx) \) the space of functions satisfying that
\[
\|u\|_{L^p((x)\alpha \, dx)} := \left( \int_{\mathbb{R}} |u(x)\alpha \, dx \right)^{1/p} < \infty.
\]

**Theorem 5.4.** Let \( 2 < p < \infty, \alpha > 0, u_0 \in L^p((x)\alpha \, dx) \), then the solution of (1) satisfying the bound
\[
\|u(T)\|_{L^p} \lesssim \langle T \rangle^{\frac{2\alpha}{p} \left( \frac{1}{2} - \frac{1}{p} \right) \left( 1 + 4 \left( \frac{1}{2} - \frac{1}{p} \right) \right) + 2 \left( \frac{1}{2} - \frac{1}{p} \right) + 5},
\]
for all \( T > 0 \), and the implicit constant depends only on \( \|u_0\|_{L^p((x)\alpha \, dx)} \).

**Proof.** The theorem can be proved in the same way as Theorem 3.5. Split \( u = v + w \), where \( v, w \) solves (26) and (27), respectively. It follows from (40) and Lemma 3.1 that
\[
\sup_{0 \leq t \leq T} \|u(t)\|_{L^p} \leq \langle T \rangle^{\frac{2\alpha}{p} \left( \frac{1}{2} - \frac{1}{p} \right) \left( 1 + 2 \left( \frac{1}{2} - \frac{1}{p} \right) \right)} \sup_{0 \leq t \leq T} \|\varphi(D)v\|_{L^p} + \langle T \rangle^{\frac{1}{2} - \frac{1}{p}} \sup_{0 \leq t \leq T} \|\varphi(D)w\|_{L^p}.
\]
Absorbing the first term by the second term on RHS of (73), applying Lemma 2.3 yield that
\[
\sup_{0 \leq t \leq T} \|u(t)\|_{L^p} \leq C\langle T \rangle^{\frac{1}{2} - \frac{1}{p}} \sup_{0 \leq t \leq T} \|w(t)\|_{L^2}^2 + C\langle T \rangle^{\frac{1}{2} - \frac{1}{p}} \sup_{0 \leq t \leq T} \|w(t)\|_{L^p} \|v(t)\|_{L^p}.
\]
Thanks to Remark 2, one can choose \( k \sim \langle T \rangle^{\frac{3}{2} \left( 1 + 4 \left( \frac{1}{2} - \frac{1}{p} \right) \right)} \) such that
\[
\langle T \rangle^{\frac{1}{2} - \frac{1}{p}} \sup_{0 \leq t \leq T} \|v(t)\|_{L^p} \leq \frac{1}{2}.
\]
Then (74) and Theorem 5.1 give
\[
\sup_{0 \leq t \leq T} \|u(t)\|_{L^p} \lesssim k^{2\left( \frac{1}{2} - \frac{1}{p} \right)} \langle T \rangle^{5 + 2\left( \frac{1}{2} - \frac{1}{p} \right)}.
\]
This implies the desired conclusion. \( \square \)

**Remark 5.** The upper bound obtained in Theorem 5.4 goes to infinity as \( \alpha \to 0 \) or \( p \to \infty \). An explicit upper bound in case \( \alpha = 0 \) or \( p = \infty \) are still unknown.

In order to obtain a growth bound of solutions in \( L^p(\mathbb{R}) \), the assumption \( u_0 \in L^p(\langle x \rangle^\alpha \, dx) \) is imposed in Theorem 5.4. An interesting problem is that whether the BBM equation persists in weighted spaces \( L^p((x)\alpha \, dx) \). In other words, does the solution \( u(t) \) of (1) belongs to \( L^p((x)\alpha \, dx) \) for any \( t > 0 \)? We give an affirmative answer to this question in the following theorem.

**Theorem 5.5.** Let \( 2 \leq p < \infty, \alpha > 0, u_0 \in L^p((x)\alpha \, dx) \), then the solution of (1) satisfies
\[
\|u(T)\|_{L^p((x)\alpha \, dx)} \lesssim \exp \left( \langle T \rangle^{7 \alpha \frac{3}{2}} \right)
\]
for all \( T > 0 \), the implicit constant depends only on \( \|u_0\|_{L^p((x)\alpha \, dx)} \).

Before proving Theorem 5.5, we give two lemmas.
Lemma 5.6. Let $1 \leq p < \infty$, $\alpha \geq 0$, then
\[
\|\partial_x (1 - \partial_x^2)^{-1/2}\|_{L^p(\langle x \rangle^\alpha \, dx), L^p(\langle x \rangle^\alpha \, dx)} \lesssim 1.
\]

Proof. Let $\phi(D)$ be a Fourier multiplier with symbol $\phi(\xi)$, then it’s easy to check that
\[
\phi(D) = (1 - \partial_x^2)^{-1/2}(D^\alpha(D) + \partial_x^2)^{-1/2} = (\partial_x^{-1})^{-1}(D^\alpha(D) + \partial_x^2)^{-1/2}
\]
Using (75) repeatedly, we find for some constants $c_j$
\[
x^k \phi(D) = \sum_{j=0}^k c_j D^j \phi(D) x^{2k-j}, \quad k = 1, 2, \ldots.
\]
It follows that
\[
\|x^{2k} \phi(D) f\|_{L^p} \lesssim \sum_{j=0}^{2k} \|D^j \phi(D) x^{2k-j} f\|_{L^p} \lesssim \sum_{j=0}^{2k} \|D^j \phi(D)\|_{L^p, L^p} \|f\|_{L^p(\langle x \rangle^{2k} \, dx)}
\]
which implies that
\[
\|\phi(D)\|_{L^p(\langle x \rangle^{2k} \, dx), L^p(\langle x \rangle^{2k} \, dx)} \lesssim \sum_{j=0}^{2k} \|D^j \phi(D)\|_{L^p, L^p}.
\]

Lemma 5.7. Let $2 \leq p < \infty$, $\alpha \geq 0$, then
\[
\|\partial_x (1 - \partial_x^2)^{-1/2}f\|_{L^p(\langle x \rangle^\alpha \, dx)} \lesssim \|f\|_{L^p} \|f\|_{L^p(\langle x \rangle^\alpha \, dx)}.
\]

Proof. By Lemma 5.6, it suffices to show
\[
\|(1 - \partial_x^2)^{-1/2}f\|_{L^p(\langle x \rangle^\alpha \, dx)} \lesssim \|f\|_{L^p} \|f\|_{L^p(\langle x \rangle^\alpha \, dx)}.
\]
According to Proposition 6.1.5 of [12], we have $(1 - \partial_x^2)^{-1/2}g = K(x) * g$ with
\[
0 \leq K(x) \sim \begin{cases} e^{-|x|}, & |x| \geq \frac{1}{2} \\ \ln \frac{1}{|x|}, & 0 < |x| \leq \frac{1}{2}. \end{cases}
\]
For any $\beta \geq 0$, using $\langle x \rangle \lesssim \langle x-y \rangle + \langle y \rangle$, we find
\[
\langle x \rangle^\beta (1 - \partial_x^2)^{-1/2} f^2 \lesssim \langle x-y \rangle^\beta + \langle y \rangle^\beta \int R K(x-y)f^2(y)dy
\]
\[
= \langle x \rangle^\beta K(x) * f^2 + K(x) * \langle x \rangle^\beta f^2.
\]
By Young’s inequality, note that $\langle x \rangle^\beta K(x) \in L^{\frac{p}{p-1}}$, we have
\[
\|\langle x \rangle^\beta (1 - \partial_x^2)^{-1/2} f^2\|_{L^p} \lesssim \|\langle x \rangle^\beta f^2\|_{L^{\frac{p}{p-1}}} \lesssim \|f\|_{L^p} \|\langle x \rangle^\beta f\|_{L^p}.
\]
Now letting $\beta = \frac{\alpha}{p}$, we obtain (77). \qed
Proof of Theorem 5.5. It’s convenient to rewrite (1) as
\[ u_t + i\varphi(D)(u + \frac{u^2}{2}) = 0. \]
Thus
\[ u(t) = u(0) - i\int_0^t \varphi(D)(u + \frac{u^2}{2})d\tau. \]
Taking \( L^p(\langle x \rangle^\alpha dx) \) norm on both sides, using Lemma 5.6-5.7 yield that
\[ \|u(t)\|_{L^p(\langle x \rangle^\alpha dx)} \lesssim \|u_0\|_{L^p(\langle x \rangle^\alpha dx)} + \int_0^t (1 + \|u(\tau)\|_{L^p}) \|u(\tau)\|_{L^p(\langle x \rangle^\alpha dx)}d\tau. \] (78)
It follows from Gronwall’s inequality that
\[ \|u(t)\|_{L^p(\langle x \rangle^\alpha dx)} \lesssim \exp\left(\int_0^t (1 + \|u(\tau)\|_{L^p})d\tau\right). \] (79)
Thanks to Theorem 5.4, we have
\[ \|u(t)\|_{L^p} \lesssim (t)^{6+\frac{2\alpha}{p}}. \] (80)
Substituting (80) into (79) implies the desired conclusion.

As a direct consequence of Theorem 5.5, we have the following results.

**Corollary 3.** Let \( u(t) \) be the solution of Eq. (1) with initial data \( u_0 \in L^2(\langle x \rangle^\alpha dx) \), \( \alpha > 0 \). Then \( u(t) \in L^2(\langle x \rangle^\alpha dx) \) for any \( t > 0 \).

**Remark 6.** Nahas proved in [17] that if \( u(t, x) \) is the solution of
\[ u_t + u_{xxx} + (u^3)_x = 0, \quad u(0, x) = u_0(x) \]
with \( u_0 \in H^s \cap L^2(\langle x \rangle^\alpha dx) \), where \( s \geq \max\{\frac{1}{2}, \alpha\} \), then \( u(t) \in H^s \cap L^2(\langle x \rangle^\alpha dx) \) for all \( t > 0 \) in the lifespan of \( u \). Comparing Corollary 3 with the persistence of KdV equation in weighted Sobolev spaces, the smoothing assumptions on solutions are removed. This reflects the regularized property of the BBM equation to some extent.

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