Convergence of the homotopy analysis method

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Abstract

The homotopy analysis method is studied in the present paper. The question of convergence of the homotopy analysis method is resolved. It is proven that under a special constraint the homotopy analysis method does converge to the exact solution of the sought solution of nonlinear ordinary or partial differential equations. An optimal value of the convergence control parameter is given through the square residual error. An error estimate is also provided. Examples, including the Blasius flow, clearly demonstrate why and on what interval the corresponding homotopy series generated by the homotopy analysis method will converge to the exact solution.

Key words: Nonlinear equations, Approximate solution, Homotopy analysis method, Convergence

1. Introduction

The search for a better and easy to use tool for the solution of nonlinear equations illuminating the nonlinear phenomena of our life keeps continuing.

A variety of methods therefore were proposed to find approximate solutions. One of the most recent popular technique is the homotopy analysis method, which is a combination of the classical perturbation technique and homotopy concept as used in topology. In the homotopy analysis method, which requires neither a small parameter nor a linear term in a differential equation, a homotopy with an embedding parameter $p \in [0, 1]$ is constructed. In [1] a basic idea of homotopy analysis method for solving nonlinear differential equations was presented. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions. A numerous nonlinear problems in science, finance and engineering were recently treated by the method, see at least [2], [3],
Particularly, a few new solutions of some nonlinear problems were found by means of the method [11], which were neglected by other analytic methods and even by numerical techniques. All of these show the potential of the homotopy analysis method for strongly nonlinear problems. Recently an optimal homotopy analysis approach for strongly nonlinear differential equations were proposed in [12]. Even though a great deal of equations were solved using the homotopy analysis method, the question of convergence of the method is yet to be answered.

We in the present paper investigate the homotopy analysis technique from a mathematical point of view. The aim is to analyze the method and to show that under certain circumstances the homotopy analysis method converges to the exact solution desired, without a prior knowledge of the exact solution. An optimal value of the convergence control parameter is defined through the square residual error concept. Our another emphasis is to address the error estimate of the approximate solution. The given theorem is justified exemplifying it by basic examples from ordinary and partial differential equations from the literature. The presented theory not only gives the convergence, but it also provides the information about the interval of convergence of the homotopy series.

In the rest of the paper, §2 lays the basis of homotopy analysis method. A theory is outlined in §3 for the convergence and error estimate. Illustrative examples in §4 are followed by the conclusions in §5.

2. The Homotopy Analysis Method

Liao [1] described the early form of the homotopy analysis method in 1992. The essential idea of this method is to introduce a homotopy parameter, say \( p \), which varies from 0 to 1 and a nonzero auxiliary parameter so-called the convergence control parameter \( h \). At \( p = 0 \), the system of equations usually has been reduced to a simplified form which normally admits a rather simple solution. As \( p \) gradually increases continuously toward 1, the system goes through a sequence of deformations, and the solution at each stage is close to that at the previous stage of the deformation. Eventually at \( p = 1 \), the system takes the original form of the equation and the final stage of the deformation gives the desired solution. To illustrate the basic ideas of this method, consider the nonlinear boundary value problem

\[
N(u(r)) = 0; \quad r \in \Omega, \quad B(u(r), \frac{du}{dn}) = 0; \quad r \in \Gamma
\]  

(2.1)

where \( u(r) \) defined over the region \( \Omega \) is the function to be solved under the boundary constraints in \( B \) defined over the boundary \( \Gamma \) of \( \Omega \). The homotopy analysis technique defines a homotopy \( u(r, p) : R \times [0, 1] \rightarrow R \) so that

\[
H(u, p) = (1 - p)[L(u) - L(u_0)] + phN(u),
\]

(2.2)
where \( L \) is a suitable auxiliary linear operator, \( u_0 \) is an initial approximation of equation (2.1) satisfying exactly the boundary conditions. It is obvious from equation (2.2) that
\[
H(u, 0) = L(u) - L(u_0), \quad H(u, 1) = N(u).
\tag{2.3}
\]
As \( p \) moves from 0 to 1, \( u(r, p) \) moves from \( u_0(r) \) to \( u(r) \). In topology, this called a deformation and \( L(u) - L(u_0) \) and \( N(u) \) are said to be homotopic. Our basic assumption is that the solution of equation (2.2) when equated to zero can be expressed as a power series in \( p \)
\[
u(r, p) = u_0(r) + pu_1(r) + p^2u_2(r) + \cdots = \sum_{k=0}^{\infty} u_k(r)p^k.
\tag{2.4}
\]
The appropriate solutions of the coefficients \( u_k(r) \) in (2.4) can be found from the homotopy deformation equations, see [13]. Hence, the approximate solution of equation (2.1) can be readily obtained as
\[
u(r) = \lim_{p \to 1} u(r, p) = \sum_{k=0}^{\infty} u_k(r).
\tag{2.5}
\]
It was found that the auxiliary parameter \( h \) can adjust and control the convergence region and rate of homotopy series solutions (2.4). Whenever the series (2.4) is known to be convergent, then (2.5) represents the exact solution of (2.1), as proved in [13].

3. A convergence Theorem and error estimate

Using the methodology underlined above, the number of problems treated by the homotopy analysis method approaches a couple of hundreds now. However, the very basic question of why the series obtained by setting \( p = 1 \) in (2.4) should be convergent remains unanswered till today. To remedy this issue up to a point, we provide the subsequent theorems here. It should be noted that how to find a proper convergence control parameter \( h \), or even better, to get a faster convergent one, to be used in Theorem 1 will be clarified later in this section.

**Theorem 1.** Suppose that \( A \subset R \) be a Banach space donated with a suitable norm \( \| \| \) (depending on the physical problem considered), over which the sequence \( u_k(t) \) of (2.1) is defined for a prescribed value of \( h \). Assume also that the initial approximation \( u_0(t) \) remains inside the ball of the solution \( u(t) \). Taking \( r \in R \) be a constant, the following statements hold true:

(i) If \( \| v_{k+1}(t) \| \leq r \| v_k(t) \| \) for all \( k \), given some \( 0 < r < 1 \), then the series solution \( u(t, p) = \sum_{k=0}^{\infty} u_k(t)p^k \), defined in (2.4) converges absolutely at \( p = 1 \) to (2.5) over the domain of definition of \( t \),
(ii) If \( \|v_{k+1}(t)\| \geq r\|v_k(t)\| \) for all \( k \), given some \( r > 1 \), then the series solution \( u(t, p) = \sum_{k=0}^{\infty} u_k(t)p^k \), defined in (2.4) diverges at \( p = 1 \) over the domain of definition of \( t \).

**Proof.** (i) In compliance with the ratio test for the power series in \( p \), the proof is clear. However, in order to give an estimate to the truncation error of homotopy analysis method, we shortly give the whole proof here. If \( S_n(t) \) denote the sequence of partial sum of the series (2.5), we need to show that \( S_n(t) \) is a Cauchy sequence in \( A \). For this purpose, consider,

\[
\|S_{n+1}(t) - S_n(t)\| = \|u_{n+1}(t)\| \leq r\|u_n(t)\| \leq r^2\|u_{n-1}(t)\| \leq \cdots \leq r^{n+1}\|u_0(t)\|. \tag{3.6}
\]

It should be remarked that owing to (3.6), all the approximations produced by the homotopy method (2.2) will lie within the ball of \( u(t) \). For every \( m, n \in \mathbb{N} \), \( n \geq m \), making use of (3.6) and the triangle inequality successively, we have,

\[
\|S_n(t) - S_m(t)\| = \|(S_n(t) - S_{n-1}(t)) + (S_{n-1}(t) - S_{n-2}(t)) + \cdots + (S_{m+1}(t) - S_m(t))\| \tag{3.7}
\]

\[
\leq \frac{1 - r^{n-m}}{1 - r} \|u_0(t)\|.
\]

Since \( 0 < r < 1 \), we get from (3.7)

\[
\lim_{n,m \to \infty} \|S_n(t) - S_m(t)\| = 0. \tag{3.8}
\]

Therefore, \( S_n(t) \) is a Cauchy sequence in the Banach space \( A \), and this implies that the series solution (2.5) is convergent. This completes the proof (i). \( \diamond \)

The proof of (ii) follows from the fact that under the hypothesis supplied in (ii), there exist a number \( l \), \( l > r > 1 \), so that the interval of convergence of the power series (2.4) is \( |p| < 1/l < 1 \), which obviously excludes the case of \( p = 1 \). \( \diamond \)

**Remark 1.** Since the finite number of terms does not affect the convergence, Theorem 1 is equally valid if the inequalities stated in (i-ii) are true for sufficiently large \( k \)'s. Thus, it is sufficient to keep track of magnitudes of the ratios \( \text{rat}_k \) defined by

\[
\text{rat}_k = \frac{\|v_{k+1}(t)\|}{\|v_k(t)\|}, \tag{3.9}
\]

and whether they remain less than unity.

**Remark 2.** By enforcing the ratio in (i) to hold true in the infinite limit, the validity region of \( t \) for the series solution can also be constructed, see Example 1 below.

**Theorem 2.** If the series solution defined in (2.5) is convergent, then it converges to an exact solution of the nonlinear problem (2.1).
Proof. The proof can be found in [13].

Theorem 3. Assume that the series solution \( \sum_{n=0}^{\infty} u_n(t) \), defined in (2.5), is convergent to the solution \( u(t) \) for a prescribed value of \( h \). If the truncated series \( \sum_{n=0}^{M} u_n(t) \) is used as an approximation to the solution \( u(t) \) of problem (2.1), then an upper bound for the error, \( E_M(t) \), is estimated as

\[
E_M(t) \leq \frac{r^{M+1}}{1-r} \| u_0(t) \|. \tag{3.10}
\]

Proof. Making use of the inequality (3.7) of Theorem 1, we immediately obtain

\[
\| u(t) - S_M(t) \| \leq \frac{1 - r^{n-M}}{1-r} r^{M+1} \| u_0(t) \|, \tag{3.11}
\]

and taking into account \( (1-r^{n-M}) < 1 \), (3.11) leads to the desired formula (3.10). This completes the proof. \( \diamond \)

Remark 3. An optimal value of the convergence control parameter \( h \) can be found by means of the exact square residual error integrated in the whole region of interest \( \Gamma \), at the order of approximation \( M \), that is,

\[
Res(h) = \int_{\Gamma} \left[ N \left( \sum_{k=0}^{M} u_k(r) \right) \right]^2 dr. \tag{3.12}
\]

Obviously, the more quickly \( Res(h) \) in (3.12) decreases to zero, the faster the corresponding homotopy series solution (2.5) converges. So, at the given order of approximation \( M \), the corresponding optimal value of the convergence control parameter \( h \) is given by the minimum of \( Res(h) \), corresponding to a nonlinear algebraic equation of the form

\[
\frac{dRes}{dh} = 0. \tag{3.13}
\]

Therefore, the convergence control parameter obtained via (3.12-3.13) can be supplied into the Theorem 1. However, it is unfortunate that the exact square residual error \( Res(h) \) defined by (3.12) needs too much CPU time to calculate even if the order of approximation is not very high, and thus is often useless in practice. To avoid the time-consuming computation, Liao in [13] suggested to investigate the convergence of some special quantities, which often have important physical meanings. For example, one can consider the convergence of \( u'(0) \) and \( u''(0) \) of a nonlinear differential equation (2.1), if they are unknown. It is found by the homotopy analysis researchers that there often exists such a region that
certain values of $h$ give a convergent series solution of such kind of quantities. Besides, such a region can be found, although approximately, by plotting the curves of these unknown quantities versus $h$. These curves are called $h$-curves or curves for convergence-control parameter, which have been successfully applied in many nonlinear problems as cited herein. This approach constitutes another way of finding proper values of $h$ for the Theorem 1.

**Remark 4.** Similar to the constant $h$--curves idea of Liao, an approximate interval of convergence for $h$ can be determined by application of Theorem 1 to some certain physical quantities, see Example 3 below.

### 4. Illustrative Examples

To illustrate the validity of the Theorems outlined, we take into account the following examples taken from the homotopy analysis studies in the literature.

**Example 1.** Consider the first-order nonlinear differential equation

$$u' + u^2 = 1, \quad u(0) = 0,$$

(4.14)

that governs the steady free convection flow over a vertical semi-infinite flat plate which is embedded in a fluid saturated porous medium of ambient temperature [14] and also the steady-state boundary-layer flows over a permeable stretching sheet [15]. In accordance with Theorem 1, choosing $u_0(t) = 1 - e^{-2t}$ and $L = \frac{d}{dt} + 2$, the homotopy series solution via the homotopy approach (2.2) can be straightforwardly constructed. Employing the $L^2$ norm in $R$, then Theorem 1 assures the convergency of (4.14) provided that

$$\frac{\|u_{n+1}(t)\|}{\|u_n(t)\|} < 1.$$  

(4.15)

Table 1 presents evolution of the ratio (4.15) for a variety of the convergence control parameter $h$ for the problem (4.14). Additionally, Table 1 gives the error, see the last column, defined by

$$err = \int_0^\infty |u_e(t) - u(t)|dt,$$

(4.16)

where $u_e(t)$ is the solution of (4.14) calculated numerically. Data displayed in Table 1 clearly explains why the homotopy analysis method generates completely convergent series solution to the problem (4.14) for the chosen parameters $h$.

The optimal value of convergence control parameter $h$ is further calculated from equations (3.12-3.13) only for $M = 7$ as $h = 1.30405$, which strongly indicates the reason of the faster converge of the homotopy series near this value, see
Table 1: The evolution of the ratio \((4.15)\) for the equation \((4.14)\). Last column corresponds to the error defined by \((4.16)\).

Table 1. Moreover, the uniform validity region of the homotopy series solution \((2.5)\) can also be analytically evaluated for the special value of \(h = 1\). In this particular case, the terms in the homotopy series \((2.5)\) satisfy the ratio

\[
\frac{u_{n+1}(t)}{u_n(t)} = \frac{1}{2}(1 - e^{-2t}).
\]

Hence, regarding \((4.17)\), Theorem 1 assures the convergence of the corresponding homotopy series \((2.5)\) for all values of \(t\) valid in the interval \(t > -\frac{\ln 3}{2}\).

**Example 2.** Consider now the second-order nonlinear differential equation

\[
2u'' + u - u^2 = 0, \quad u(0) = 0, \quad u(\infty) = 1,
\]

that governs the steady mixed convection flow past a plane of arbitrary shape under the boundary layer and Darcy-Boussinesq approximations [16]. With the choices of \(L = \frac{d^2}{dx^2} - 1\) and \(u_0(t) = 1 - e^{-t}\), the homotopy \((2.2)\) generates a homotopy series \((2.5)\) whose ratios of successive terms are tabulated in Table 2 at several order of approximations for some selected values of \(h\). As observed from the Table, \(h = 1.2\) generates divergent homotopy solution, and also the convergence of the homotopy series to the exact solution takes place at a considerably slow rate near \(h = 1\) as compared to the smaller values of \(h\).

Table 2: The ratio of successive terms in the homotopy series corresponding to equation \((4.18)\). Last column corresponds to the error defined by \((4.16)\).
The optimal value of convergence control parameter $h$ is calculated from equations (3.12, 3.13) only for $M = 7$ as $h = 0.73258$, which explains why the homotopy series (2.5) should converge faster near this value to the solution of (4.18), see Table 2.

**Example 3.** Consider now the nonlinear partial differential Burger’s equation

$$u_t + uu_x = u_{xx}, \quad u(x, 0) = 2x,$$  \hspace{1cm} (4.19)

that has been found to describe various kind of phenomena, such as a mathematical model of turbulence and the approximate theory of the flow through a shock wave traveling in a viscous fluid [17]. Equation (4.19) admits an exact solution given by

$$u(x, t) = \frac{2x}{1 + 2t}. \hspace{1cm} (4.20)$$

To approximate the exact solution (4.20), we choose the auxiliary parameters as $u_0(x, t) = 2x$ and $L = \frac{\partial}{\partial t}$. Then, the homotopy (2.2) turns out to be

$$\left(1 + \frac{1 - p}{hp}\right) u_t(x, t, p) + u(x, t, p)u_x(x, t, p) - u_{xx}(x, t, p) = 0, \quad u(x, 0, p) = 2(x). \hspace{1cm} (4.21)$$

Equation (4.21) produces the below homotopy series for the solution of (4.19)

$$u(x, t) = 2x - 4htx + 4ht(-1 + h + 2ht)x - 4ht(-1 + h + 2ht)^2x + 4ht(-1 + h + 2ht)^3x - 4ht(-1 + h + 2ht)^4x + \cdots, \hspace{1cm} (4.22)$$

whose convergence to the exact solution (4.20) takes place for the values satisfying $\lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = |1 - h(1 + 2t)| < 1$. Thus, in accordance with Theorem 1, this holds exactly for the values of $-\frac{1}{2} < t < \frac{2 - h}{2h}$ together with $0 < h < 2$. It is easy to demonstrate that the case $h = 1$ corresponds to the traditional Taylor series expansion of the solution (4.20), which is only valid in the region $-\frac{1}{2} < t < \frac{1}{2}$.

**Example 4.** Consider now the following fourth-order parabolic partial differential equation arising in the study of the transverse vibrations of a uniform flexible beam [18]

$$u_{tt} + \left(\frac{y + z}{2 \cos x} - 1\right)u_{xxxx} + \left(\frac{z + x}{2 \cos y} - 1\right)u_{yyyy} + \left(\frac{x + y}{2 \cos z} - 1\right)u_{zzzz} = 0, \hspace{1cm} (4.23)$$

$$u(x, y, z, 0) = -u_t(x, y, z, 0) = x + y + z - (\cos x + \cos y + \cos z),$$

whose exact solution is given by

$$u(x, t) = (x + y + z - \cos x - \cos y - \cos z)e^{-t}. \hspace{1cm} (4.24)$$
To approximate the exact solution (4.24), if we choose the auxiliary parameters respectively, $u_0(x, t) = (x + y + z - \cos x - \cos y - \cos z)(1 - t)$ and $L = \frac{\partial^2}{\partial t^2}$, the homotopy (2.2) for $h = 1$ then generates the subsequent homotopy series

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} (x + y + z - \cos x - \cos y - \cos z) \left( \frac{t^{2n}}{(2n)!} - \frac{t^{2n+1}}{(2n+1)!} \right)$$

which leads to $\lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = 0$. So, the homotopy series (4.25) converges to the exact solution of (4.23) for all $t$.

**Example 5.** Let us consider the linear partial differential equation

$$u_t + u_x - 2u_{xxt} = 0, \quad u(x, 0) = e^{-x}, \quad (4.26)$$

whose exact separable solution is given by

$$u(x, t) = e^{-x-t}. \quad (4.27)$$

To approximate the exact solution (4.27), if we choose the auxiliary parameters $u_0(x, t) = e^{-x}$ and $L = \frac{\partial}{\partial t}$ respectively, then the homotopy (2.2) turns out to be

$$(1 - h)u_t(x, t, p) + ph(u_t(x, t, p) + u_x(x, t, p) - 2u_{xxt}(x, t, p)) = 0, \quad u(x, 0, p) = e^{-p} \quad (4.28)$$

When $h = 1$, it appears that the homotopy series solution is convergent only for $t \leq 0$, that is out of physical interest. On the other hand, for $h = -1$, the successive ratio yields

$$\left| \frac{u_{n+1}(t)}{u_n(t)} \right| = \frac{|t|}{n+1},$$

whose limit gives rise to zero. This explains why the homotopy series solution (2.5) in this case represents the real physical solution over $t \geq 0$ for $h = -1$, as also explained in [8].

**Example 6.** As a final example, we consider the classical Blasius flat-plate flow problem of fluid mechanics governed by the nonlinear initial-value of third-order [19]

$$y''' + \frac{yy''}{2} = 0, \quad y(0) = y'(0) = y'(-\infty) = 1 = 0, \quad \eta \in [0, \infty). \quad (4.29)$$

With the transformations

$$y = u, \quad t = \lambda \eta,$$

where $\lambda$ is a scaling parameter taken here as 4, system (4.29) is converted into

$$u''' + \frac{uu''}{2\lambda^2} = 0, \quad u(0) = u'(0) = u'(-\infty) = 1 = 0, \quad t \in [0, \infty). \quad (4.30)$$
Taking $L = \frac{d^3}{dt^3} + \frac{d^2}{dt^2}$ with $u_0(t) = -1 + t + e^{-t}$, the values for the convergence control parameters $h$ from the homotopy (2.2) are shown in Table 3 along with the corresponding residual errors. It is seen that the optimal value for the physical problem considered is $h = -3/2$. Figures 1(a-d) demonstrate the ratios (3.9) in Theorem 1 which were computed from the homotopy series (2.5) for the values of $h$ given in Table 3. The convergence is assured further by the values of the ratios less than unity, suggesting that limit tends to 0.83, 0.77, 0.76 and 0.78, for the values of $h$ shown.

Table 3: The square residual errors $Res(h)$ for the homotopy series solutions corresponding to equation (4.30) at several values of convergence control parameters $h$.

| $h$ | $M = 1$ | $M = 10$ | $M = 20$ | $M = 30$ |
|-----|---------|----------|----------|----------|
| 1.6 | 1.258621 | 1.201 x 10^{-3} | 2.463 x 10^{-7} | 1.032 x 10^{-9} |
| 1.5 | 1.027450 | 1.372 x 10^{-4} | 3.602 x 10^{-7} | 2.573 x 10^{-9} |
| 1.4 | 0.820771 | 1.399 x 10^{-4} | 7.621 x 10^{-7} | 5.454 x 10^{-9} |
| 1.0 | 0.238796 | 7.621 x 10^{-4} | 1.511 x 10^{-5} | 4.540 x 10^{-7} |

5. Concluding remarks

In this paper, the homotopy analysis method has been analyzed with an aim to investigate the conditions which result in the convergence of the generated homotopy solutions of the nonlinear ordinary and partial differential equations. The theorems outlined in the paper have proved that if specific values are assigned to the auxiliary parameters in the homotopy analysis method, then the approximate homotopy results successfully converge to the exact solution. An optimal value approach for the convergence control parameter has also been given. Examples have been provided to verify the theory. Via the theorems provided here, not only the question of the convergence of the homotopy series is answered, but also the region of validity of the space variable ensuring the convergence is determined. The traditional flat-plate boundary layer flow problem has been finally treated by the convergence theorem.

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Figure 1: A list plot of the ratios $rat$ from the theorem to reveal the convergence of the HAM solutions for the Blasius equation (4.30) for different choices of $h$. (a) $h = -1$, (b) $h = -1.4$, (c) $h = -1.5$ and (d) $h = -1.6$.

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