Decision theory in an algebraic setting

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Abstract

In decision theory an act is a function from a set of conditions to the set of real numbers. The set of conditions is a partition in some algebra of events. The expected value of an act can be calculated when a probability measure is given. We adopt an algebraic point of view by substituting the algebra of events with a finite distributive lattice and the probability measure with a lattice valuation. We introduce a partial order on acts that generalizes the dominance relation and show that the set of acts is a lattice with respect to this order. Finally we analyze some different kinds of comparison between acts, without supposing a common set of conditions for the acts to be compared.

Keywords: Decision theory, lattice theory, partitions, Allais paradox.

1 Classical acts

The concept of an act is at the basis of decision theory, in fact decision making under risk can be reduced to the choice among different acts on the basis of their expected value. Loosely speaking, we can say that an act is a function from a set of conditions to a set of consequences. In this paragraph we introduce the intuitive framework for decisions, grounded on the concept of probability space, but in the following paragraphs we shall adopt a more algebraic point of view, based on the concept of valued lattice.

The consequence of an act can be any kind of thing, but we confine ourselves to elements of \( R \), the set of real numbers. We only observe that in an economical framework real numbers can represent any definite amount of goods, money and so on, but in a psychological framework they may also represent degrees of satisfaction, subjective feelings of pain and pleasure and so on.

The conditions of an act are events from a probability space. A probability space is a triple \((S, \mathcal{C}_S, p)\) where \( S \) is a sample space, \( \mathcal{C}_S \) a field of sets over \( S \) and \( p : \mathcal{C}_S \rightarrow [0, 1] \) a function satisfying Kolmogoroff’s axioms: 1) \( p(A) = 1 \), 2) \( p(X \cup Y) = p(X) + p(Y) \), when \( X \cap Y = \emptyset \). In this way probability is seen as the measure of an event represented by a set. In the following we shall limit ourselves to finite sample spaces, so the algebra of events \( \mathcal{C}_S \) will coincide with \( \mathcal{P}(S) \), the Boolean algebra of all subsets of \( S \). The conditions of an act must
satisfy a fundamental property: they must be a partition of $S$. We say that a subset $E$ of $P(S)$ is a \textit{partition} of $S$ when the following three conditions are satisfied:

1. $\bigcup E = S$,
2. $e_2 \cap e_2 = \emptyset$, for all $e_2, e_2 \in E$ with $e_2 \neq e_2$,
3. $e \neq \emptyset$, for all $e \in E$.

One of the possible partitions is given by the set of all atoms in $\mathcal{P}(S)$, i.e. the set of all singletons $\{s\}$ with $s \in S$.

Now we can define an \textit{act on $E$} as a function $\alpha : E \to R$, where $E$ is a partition of $S$. We denote with $A(E)$ the set of all acts on $E$. The elements of $\alpha(E)$, the range of $\alpha$, are the consequences (rewards, payoffs) of $\alpha$: intuitively, $\alpha(e)$ is the consequence of $\alpha$ when the event $e$ happens. The choice of a partition as the domain of an act reflects a relevant aspect of real life acts, where we are confronted with a set of alternative and exclusive conditions, represented by events $e_1, ..., e_n$, leading to consequences $\alpha(e_1), ..., \alpha(e_n)$. This amounts to say that the domain $\{e_1, ..., e_n\}$ of an act is a partition of $S$. For every state of the world, for every experimental outcome, one and only one event $e_i$ of the partition $E$ takes place leading to a single consequence $\alpha(e_i)$.

A central problem of decision theory is to define a preference relation on the set of acts. When $\alpha, \beta \in A(E)$, $\beta$ is obviously preferred to $\alpha$ when it gives a better or equal reward for all conditions and in this case we say that $\beta$ \textit{dominates} $\alpha$. So we define $\alpha \preceq E \beta$, iff $\alpha(e) \leq \beta(e)$ for all $e \in E$. The relation $\preceq_E$ is a partial order on $A(E)$, but not every pair of acts on $A(E)$ can be compared in $\preceq_E$, so it is not complete. If $\alpha \in A(E)$ and $\beta \in A(D)$, where $E$ and $D$ are different sets of conditions, the relation of dominance is not defined.

When we choose a dominant act, we ignore all questions about the probability of the relevant conditions, the elements of $E$ involved in the decision process. So we define another preference relation that takes in account the probability of events/conditions. Given a probability measure $p : \mathcal{P}(S) \to [0, 1]$, we define the \textit{expected value of $\alpha$ with respect to $p$ setting}

$$\exp(\alpha, p) = \sum \{\alpha(e)p(e) : e \in E\}$$

and we define $\alpha \succeq_{\exp} \beta$ iff $\exp(\alpha, p) \leq \exp(\beta, p)$. The relation $\succeq_{\exp}$ is reflexive, transitive and complete, i.e. is a total preorder. ($\succeq_{\exp}$ is not antisymmetric, so it is not a partial order.) The choice among different acts, with respect to a given probability measure, is accomplished by ranking acts by their expected value: this is the most important rule of choice in the field of decision under risk. In general, we observe that we may have $\exp(\alpha, p) \leq \exp(\beta, p)$ even if $\alpha(x) \leq \beta(x)$ holds in a single case ($\alpha(x) \leq \beta(x)$ if $x = e$ and $\alpha(x) > \beta(x)$ if $x \neq e$), because the relevance of the single condition $x \in E$ in establishing $\alpha \succeq_{\exp} \beta$ depends on its probability value $p(x)$. A huge value $p(e)$ may rule out all other conditions $x \in E$. Some critical remarks against the ranking of acts
based on expected utility are due to Allais and Ellsberg, in [1] and [5]. A way out to Allais Paradox is sketched in Appendix B, where the notion of intrinsic expected value is introduced as the ratio between the expected value of $\alpha$ and the total sum of all possible rewards of $\alpha$.

The comparison of acts with respect to expected utility is generally confined to acts on the same set of conditions. This is clear when the representation of decisions is based on decision matrices. (See, for instance, [6].) We underline, however, that every couple of acts $\alpha : E \rightarrow R$ and $\beta : D \rightarrow R$ can be compared in $\leq_{\text{exp}}$, as far as the conditions $E$ and $D$ are partitions of the same algebra of events. In this way we can rely on a common probability measure $p$ and so we can compute the expected value of $\alpha$ and $\beta$ with respect to $p$. From the point of view of expected value, the ranking of $\alpha$ and $\beta$ is reduced to the comparison of two real numbers $\exp(\alpha, p)$ and $\exp(\beta, p)$, leaving out every consideration regarding the very nature of the events/conditions involved in the decision process, even if the result may be somewhat unnatural. In this work we introduce a partial order $\preceq_v$ in which acts based on different set of conditions can be compared, as it happens with expected value. The comparison of acts in $\preceq_v$ is a generalization of dominance, taking in account the partial order of the partitions involved (see Appendix A) and the probability of the relevant conditions.

The plan of this work is the following. In the second paragraph we analyze the connections between acts and lotteries or gambles. In the third paragraph we adopt an algebraic standpoint: the algebra of events $\mathcal{P}(S)$, the Boolean algebra of all subsets of $S$, is substituted by a finite distributive lattice $\mathcal{A}$, the set of conditions of an act becomes an algebraic partition of $\mathcal{A}$ and an act is a function from such a partition to the set of real numbers. Then we introduce a partial order $\preceq_v$ on acts and in the fourth paragraph we show that the set of all acts on $\mathcal{A}$ is a lattice with respect to $\preceq_v$. In the fifth paragraph we discuss some different ways of comparing acts.

2 Acts and lotteries

The process of decision is often described in the literature as a choice between lotteries or gambles: the relationships between acts and lotteries are sketched in this paragraph. For all finite set $X$, we say that a function $f : X \rightarrow [0, 1]$ is a distribution on $X$ when $\sum \{ f(x) : x \in X \} = 1$. Given a finite set $Z = \{ z_1, ..., z_n \}$ of real numbers, the outcomes or rewards, a lottery on $Z$ is a distribution $l : Z \rightarrow [0, 1]$. The expected value of $l$ is defined as follows:

$$\text{Exp}(l) = \sum \{ zl(z) : z \in Z \}. $$

The concepts of lottery and act are connected but not equivalent. To every pair constituted by an act $\alpha : E \rightarrow R$ and a probability measure $p : \mathcal{P}(S) \rightarrow [0, 1]$ we can associate a lottery $l_{\alpha, p}$ on $\alpha[E]$ with the same expected value, defined as follows: for all $x \in \alpha[E]$, we set

$$l_{\alpha, p}(x) = p(\bigcup \alpha^{-1}(x)).$$
We show that \( l_{\alpha,p} \) is a lottery by verifying that \( l_{\alpha,p} \) is a distribution on \( \alpha[E] \). If \( x \in \alpha[E] \) then
\[
l_{\alpha,p}(x) = \sum \{ p(e) : e \in \alpha^{-1}(x) \}
\]
because the events in \( E \) belong to a partition and are pairwise disjoint. We have
\[
\sum \{ l_{\alpha,p}(x) : x \in \alpha[E] \} = \sum \{ \sum \{ p(e) : e \in \alpha^{-1}(x) \} : x \in \alpha[E] \} = \sum \{ p(e) : e \in E \} = 1.
\]

Now we prove that \( \text{Exp}(\alpha, p) = \text{Exp}(l_{\alpha,p}) \). In fact,
\[
\text{Exp}(\alpha, p) = \sum \{ \alpha(e)p(e) : e \in E \}
= \sum \{ \sum \{ \alpha(e)p(e) : e \in \alpha^{-1}(x) \} : x \in \alpha[E] \}
= \sum \{ \sum \{ xp(e) : e \in \alpha^{-1}(x) \} : x \in \alpha[E] \}
= \sum \{ x \sum \{ p(e) : e \in \alpha^{-1}(x) \} : x \in \alpha[E] \}
= \sum \{ xl_{\alpha,p}(x) : x \in \alpha[E] \}
= \text{Exp}(l_{\alpha,p}).
\]

In the other direction, we cannot immediately associate an act to a lottery, because a lottery \( l \) is only a finite sequence \( (l(z_1), \ldots, l(z_n)) \) of probability values adding to 1, without any reference to a sample space, an algebra of events and a probability measure on this algebra. So, given a lottery \( l \) on \( Z \), we must supply a finite Boolean algebra \( \mathcal{P}(S) \) and a partition \( E \) before defining the act \( \alpha_l \) associated to \( l \). If \( Z \) contains \( n \) rewards, we choose \( S = \{ s_1, \ldots, s_n \} \). Then \( E = \{ \{ s_i \} : 1 \leq i \leq n \} \) is a partition of \( S \) and we define an act \( \alpha_l : E \to R \) setting \( \alpha_l(\{ s_i \}) = z_i \). Finally, we define a probability measure \( p \) on \( \mathcal{P}(S) \) starting with \( p(\{ s_i \}) = l(z_i) \): as every event can be expressed as a disjoint union of singletons, we can extend \( p \) to the whole of \( \mathcal{P}(S) \). Now we show that \( \exp(l) = \exp(\alpha_l, p) \) where \( \alpha_l \) and \( p \) are defined as above:
\[
\exp(\alpha_l, p) = \sum \{ \alpha_l(\{ s_i \})p(\{ s_i \}) : \{ s_i \} \in E \}
= \sum \{ z_i l(z_i) : z_i \in Z \}
= \exp(l)
\]
where the second line follows because there is a bijection between \( Z \) and \( E \).

### 3 Acts in finite distributive lattices

The framework for acts introduced so far can be formulated in algebraic terms as follows. Let \( A \) be a finite distributive lattice, we introduce an algebraic counterpart of the set theoretic notion of partition as follows: we say that \( E \subseteq A \) is an algebraic partition of \( A \) if
1. \( \sqrt{E} = 1 \),
2. \( e_2 \land e_2 = 0 \), for all \( e_2, e_2 \in E \) with \( e_2 \neq e_2 \),
3. \( e \neq 0 \), for all \( e \in E \).

We speak of a partition \( E \) of \( A \) (\( E \) on \( A \)) when an algebraic (set-theoretical) partition is intended. We denote with \( \Pi(A) \) (\( \Pi(A) \)) the set of all algebraic (set-theoretical) partitions of \( A \). The aim of 3) is to avoid redundancies. On one side, if \( E \) is a partition and \( 0 \in E \), then \( E \) is redundant because \( E - \{ 0 \} \) is a partition too. On the other side, if \( E \) is a redundant partition and so, for some \( e \in E \), \( E - \{ e \} \) is a partition too, then we can easily see that \( e = 0 \). In fact, \( e = e \land \sqrt{(E - \{ e \})} = 0 \). To make life easier we have collected some basic results about partitions in Appendix [A].

Given a partition \( E \) of \( A \), an act on \( E \) is a function \( \alpha : E \to R \). We denote with \( A(E) \) the set of all acts on \( E \). An act of \( A \) is an act \( \alpha : E \to R \), for some partition \( E \) of \( A \). We denote with \( A(A) \) the set of all acts of \( A \), i.e. \( \bigcup \{ A(E) : E \in \Pi(A) \} \). It can be easily seen that the intuitive notion of an act, as introduced in the first paragraph, is only a particular case of the algebraic notion. Finally, the concept of a probability measure \( p \) on \( P(S) \) is to be generalized to the concept of a valuation \( v \) on \( A \), thus obtaining a valued lattice \( (A, v) \). To make our exposition self-contained, we introduce some basic facts about valued lattices. (See [2] Chapter X.)

When \( A \) is a lattice we say that a function \( v : A \to R \) is a valuation on \( A \) if
\[
v(a \lor b) = v(a) + v(b) - v(a \land b).
\]

If \( x \leq y \) implies \( v(x) \leq v(y) \), we say that \( v \) is isotone; \( v \) is strictly isotone if we can substitute \( \leq \) with \( < \). (In [2] a strictly isotone valuation is called a positive one.) In the following we will confine ourselves to non-negative valuations, i.e. valuations such that \( 0 \leq v(a) \), for all \( a \in A \). A valued lattice is a pair \( (A, v) \) where \( A \) is a lattice and \( v \) a valuation on \( A \).

If \( A \) is a bounded lattice, we say that \( v \) is a bounded lattice valuation if \( v \) is a valuation on \( A \) and \( v(0) = 0 \) and \( v(1) = 1 \). If \( v \) is an isotone valuation on a bounded lattice \( A \), then \( v[A] \subseteq [0, 1] \). A valued bounded lattice is a pair \( (A, v) \) where \( A \) is a bounded lattice and \( v \) a bounded lattice valuation on \( A \).

If \( A \) is a Boolean algebra, we say that \( v \) is a Boolean valuation if \( v \) is a bounded lattice valuation on \( A \). A valued Boolean algebra is a pair \( (A, v) \), where \( A \) is a Boolean algebra and \( v \) a Boolean valuation on \( A \). We can give an equivalent definition of a Boolean valuation as follows. In a lattice with 0, a function \( f : A \to [0, 1] \) is said to be additive iff \( f(a \lor b) = f(a) + f(b) \), whenever \( a \land b = 0 \). It can be easily proved that, if \( A \) is a Boolean algebra and \( v : A \to [0, 1] \), then \( v \) is a Boolean valuation iff \( v(1) = 1 \) and \( v \) is additive. So a probability space \( (A, C_A, p) \), where \( A \) is a finite sample space, \( C_A \) a field of sets on \( A \) and \( p \) a probability measure satisfying Kolmogoroff’s axioms with finite additivity, is a particular case of valued Boolean algebra. It can be easily proved that: if \( (A, v) \) is a Boolean valued algebra then \( v \) is isotone and \( v(\neg a) = 1 - v(a) \).
As \( v \) is isotone, every Boolean valuation takes its values in \([0, 1]\) (see \([7]\), par. 2).

When \( \mathcal{A} \) is a finite distributive lattice, \( E \) is a partition of \( \mathcal{A} \) and \( v \) is an isotone valuation on \( \mathcal{A} \), we can define the expected value of an act \( \alpha : E \to R \) with respect to \( v \) setting

\[
\exp(\alpha, v) = \sum \{ \alpha(e)v(e) : e \in E \}.
\]

When the valuation \( v \) is clear from the context, we simply write \( \exp(\alpha) \). We underline that we confine ourselves to isotone valuations, so that \( v(a) \in [0, 1] \) for all \( a \in A \). As in the preceding paragraph, acts can be ranked on the basis of their expected value so we define, for all \( \alpha \) and \( \beta \) in \( A(\mathcal{A}) \), \( \alpha \leq_{\exp} \beta \) iff \( \exp(\alpha, v) \leq \exp(\beta, v) \). The relation \( \leq_{\exp} \) is reflexive, transitive and complete.

For acts having the same domain, it is natural to define an order pointwise. Suppose \( \alpha, \beta \in A(E) \), then we define \( \alpha \leq_{E} \beta \) iff \( \alpha(e) \leq \beta(e) \), for all \( e \in E \). In this case, we say that \( \beta \) dominates \( \alpha \). It can be easily shown that \( A(E) \) is a partial order with respect to \( \leq_{E} \) and in particular a lattice where, for all \( e \in E \),

\[
\inf(\alpha, \beta)(e) = \min(\alpha(e), \beta(e)) \quad \text{and} \quad \sup(\alpha, \beta)(e) = \max(\alpha(e), \beta(e)).
\]

We leave a direct proof to the reader, but we observe that it is only a particular case of the following proposition:

For all lattice \( B \) and all set \( E \), the power \( B^{E} \), where the order relation is defined pointwise, is a lattice. If \( B \) is distributive, bounded, complemented, so is \( B^{E} \).

In fact, the axioms involved are equational and then are preserved by direct products and powers (see, for instance, \([3]\) par. 6.2). As real numbers with their natural order are a lattice, so is \( A(E) \) when acts are ordered pointwise (i.e. by dominance).

The preference relations \( \leq_{E} \) and \( \leq_{\exp} \) are inspired by different points of view. In \( \alpha \leq_{E} \beta \) acts are compared with respect to their conditions and this is possible because \( \alpha \) and \( \beta \) have \( E \) as a common domain. When we assert \( \alpha \leq_{E} \beta \) we know that the payoff of \( \beta \) is better or equal to the payoff of \( \alpha \) for all conditions \( e \in E \). In the case of expected value, an overall valuation of the performances of \( \alpha \) and \( \beta \) is given by separately calculating the weighted average of payoffs of each one of them, for all conditions. There is no need of a common set of conditions \( E \). But even if there is such an \( E \), we don’t know whether the payoff of \( \beta \) is better or equal to the payoff of \( \alpha \) for all conditions \( e \in E \), we know only that it is so for some \( e \) and in particular for \( e \) with an high probability value.

Now we introduce a preference relation \( \preceq_{v} \) on acts in \( A(\mathcal{A}) \) that borrows from \( \leq_{E} \) the comparison of conditions and from \( \leq_{\exp} \) the reference to probabilities. For all \( E, D \in \Pi(\mathcal{A}) \), we say that \( E \) is a refinement of \( D \), in symbols \( E \leq B \) when, for all \( e \in E \), there is a \( d_{e} \in D \) such that \( e \leq d_{e} \). Such an element of \( D \) is unique (see Appendix \([A]\) lemma \([13]\) and this is why we denote it by \( d_{e} \). If \( \alpha, \beta \in A(\mathcal{A}) \), where \( \alpha : E \to R \) and \( \beta : D \to R \), and \( v \) is an isotone valuation on \( \mathcal{A} \), we say that \( \beta \) is preferred to \( \alpha \) with respect to \( v \), in symbols \( \alpha \preceq_{v} \beta \), when the following conditions are satisfied:

1. \( E \leq B \),
2. for all \( e \in E \), \( \alpha(e) \leq \beta(d_e) \frac{v(e)}{v(d_e)} \).

As in the case of \( \preceq_E \beta \), the performances of \( \alpha \) and \( \beta \) are compared with respect to the single conditions of the acts involved. As in the case of \( \preceq_{\exp} \beta \), the comparison of \( \alpha \) and \( \beta \) depends on \( v \), i.e. on the probability values of the relevant conditions. In fact, we cannot compare directly \( \alpha(e) \) with \( \beta(d_e) \), as we did with the relation of dominance, because \( e \) and \( d_e \) belong to different sets of conditions, so we compare \( \alpha(e) \) with \( \beta(d_e) \frac{v(e)}{v(d_e)} \). As \( \frac{v(e)}{v(d_e)} < 1 \), because \( e \preceq d_e \), implies \( v(e) \leq v(d_e) \), we compare \( \alpha(e) \) with a reduced \( \beta(d_e) \). This reduction can be justified as follows. The value \( \beta(d) \), for any \( d \in D \), can be smeared on the set \( E_d = \{ x \in E : x \leq d \} \) as the set \( \{ \beta(d_e) \frac{v(e)}{v(d_e)} : x \in E_d \} \). In fact, we have

\[
\sum \{ \beta(d) \frac{v(x)}{v(d)} : x \in E_d \} = \frac{\beta(d)}{v(d)} \sum \{ v(x) : x \in E_d \}
= \frac{\beta(d)}{v(d)} v(\vee E_d)
= \frac{\beta(d)}{v(d)} v(d)
= \beta(d),
\]

where the second line follows because \( x \wedge x' = 0 \) for all \( x, x' \in E \) and the third line because \( \vee E_d = d \) by theorem 17 of Appendix A. So we can compare \( \alpha(e) \) with \( \beta(d_e) \frac{v(e)}{v(d_e)} \) on \( E \) and this is just clause 2.

The following theorem shows that \( \preceq_E \) is the restriction to \( A(E) \) of \( \preceq_v \) defined on \( A(A) \).

**Theorem 1** For all \( \alpha, \beta \in A(E) \), \( \alpha \preceq_v \beta \) iff \( \alpha \preceq_E \beta \).

**Proof.** We have \( \alpha \preceq_v \beta \) iff, for all \( e \in E \), \( \alpha(e) \leq \beta(e_e) \frac{v(e)}{v(e_e)} \), where \( e_e \) is the only \( x \in E \) such that \( e \preceq x \). But \( e_e = e \) so \( \alpha \preceq_v \beta \) iff for all \( e \in E \), \( \alpha(e) \leq \beta(e) \frac{v(e)}{v(e)} \) iff \( \alpha(e) \leq \beta(e) \) iff \( \alpha \preceq_E \beta \). □

The following theorem shows the relationship between \( \preceq_{\exp} \) and \( \preceq_v \).

**Theorem 2** For all \( \alpha, \beta \in A(A) \), if \( v \) is an isotone valuation then \( \alpha \preceq_v \beta \) implies \( \alpha \preceq_{\exp} \beta \).

**Proof.** We suppose that \( \alpha : E \to R \) and \( \beta : D \to R \). We have

\[
\exp(\alpha, v) = \sum \{ \alpha(e)v(e) : e \in E \}
\leq \sum \{ \sum \{ \alpha(x)v(x) : x \in E_d \} : d \in D \}
= \sum \{ \beta(d)v(d) : d \in D \}
= \exp(\beta, v),
\]

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where the second line follows because \( \{ E_d : d \in D \} \) is a set theoretic partition on \( E \) (see lemma \[15\] of Appendix \( \mathbb{A} \)) and the third line because

\[
\sum \{ \alpha(x)v(x) : x \in E_d \} \leq \sum \{ \alpha(x)v(d) : x \in E_d \} \\
\leq \sum \{ \beta(d)v(x) : x \in E_d \} \\
= \beta(d)\sum \{ v(x) : x \in E_d \} \\
= \beta(d)v(d),
\]

where the first line follows because \( x \leq d \) implies \( v(x) \leq v(d) \), as \( v \) is isotone, and the second line follows because \( \alpha \preceq_v \beta \) by hypothesis. In fact, \( \alpha \preceq_v \beta \) implies \( \alpha(x) \leq \beta(d)\frac{v(x)}{v(d)} \), for all \( x \in E_d \), so \( \alpha(x)v(d) \leq \beta(d)v(x) \). The last line follows because \( \sum \{ v(x) : x \in E_d \} = v(\bigvee E_d) = v(d) \) (see theorem \[17\] of Appendix \( \mathbb{A} \)). \( \blacksquare \)

Of course, we cannot substitute implication with equivalence in the preceding theorem, because \( \exp(a,v) \leq \exp(\beta,v) \) may hold between acts \( \alpha \) and \( \beta \) whose domains \( E \) and \( D \) are such that \( E \notin D \) and \( D \notin E \). We observe that, as a consequence of the two preceding theorems, \( \alpha \preceq \beta \) implies \( \alpha \preceq \exp \beta \).

**Theorem 3** \( (A(A), \preceq_v) \) is a partially ordered set.

**Proof.** Reflexivity. For all act \( \alpha : E \to R \), we have: 1) \( E \leq E \), because \( \leq \) is a partial order on the set of all partitions on \( A \) (see theorem \[19\] of Appendix \( \mathbb{A} \)); 2) for all \( e \in E \), \( \alpha(e) \leq \alpha(e) \frac{v(e)}{v(e)} \) because, for all \( e \in E \), \( e_e = e \) (the only \( x \in E \) such that \( e \leq x \) being \( e \) itself), so \( \frac{v(e)}{v(e)} = 1 \). This proves that \( \alpha \preceq_v \alpha \).

Transitivity. We assume that \( \alpha \preceq_v \beta \) and \( \beta \preceq_v \gamma \), where \( \alpha : E \to R \), \( \beta : D \to R \) and \( \gamma : G \to R \). 1) By hypothesis, \( E \leq D \) and \( D \leq G \), so \( E \leq G \), because \( \leq \) is a partial order on partitions. 2) By hypothesis, for all \( e \in E \), we have \( \alpha(e) \leq \beta(d_e) \frac{v(e)}{v(d_e)} \) and for all \( d \in D \) we have \( \beta(d) \leq \gamma(g_d) \frac{v(d)}{v(g_d)} \). In particular, \( \beta(d_e) \leq \gamma(g_{d_e}) \frac{v(d_e)}{v(g_{d_e})} \). By definition, \( e \leq d_e \leq g_{d_e} \) and \( e \leq g_{d_e} \), so \( g_{d_e} = g_e \), because there is only one \( x \in G \) such that \( e \leq x \), so \( \beta(d_e) \leq \gamma(g_e) \frac{v(d_e)}{v(g_e)} \). Then we have

\[
\alpha(e) \leq \beta(d_e) \frac{v(e)}{v(d_e)} \leq \gamma(g_e) \frac{v(d_e)}{v(g_e)} \frac{v(e)}{v(d_e)} = \gamma(g_e) \frac{v(e)}{v(g_e)}. 
\]

This proves that \( \alpha \preceq_v \gamma \).

Antisymmetry. We assume that \( \alpha \preceq_v \beta \) and \( \beta \preceq_v \alpha \). 1) By hypothesis, \( E \leq D \) and \( D \leq E \), so \( E = D \), because \( \leq \) is a partial order on partitions. 2) By hypothesis, for all \( e \in E \), we have \( \alpha(e) \leq \beta(d_e) \frac{v(e)}{v(d_e)} \) and for all \( d \in D \) we have \( \beta(d) \leq \alpha(e_d) \frac{v(d)}{v(e_d)} \). As \( E = D \), \( \alpha(e) \leq \beta(e) \frac{v(e)}{v(e)} = \beta(e) \). For the same reason, we have \( \beta(e) \leq \alpha(e) \frac{v(e)}{v(e)} = \alpha(e) \), so \( \alpha(e) = \beta(e) \). This proves that \( \alpha = \beta \). \( \blacksquare \)
4 The lattice of acts

Given an act $\beta : D \rightarrow R$ and a partition $E$ such that $E \leq D$, we can downgrade $\beta$ to an act $\beta_E : E \rightarrow R$ setting, for all $e \in E$

$$\beta_E(e) = \beta(d_e) \frac{v(e)}{v(d_e)}.$$  

We have $\beta_E \leq_v \beta$ by definition of $\beta_E$. In fact, $\beta_E$ is the best approximation from below to $\beta$ in $A(E)$, as shown in corollary 2.

**Lemma 4** $(A(A), \leq_v)$ is closed with respect to $\inf$.

**Proof.** We know that $\Pi(A)$ is a lattice, by theorem 18 of Appendix A. Given $\alpha : E \rightarrow R$ and $\beta : D \rightarrow R$, we define $\phi : E \land D \rightarrow R$ setting, for all $z \in E \land D$,

$$\phi(z) = \min(\alpha_E(z), \beta_E(z)).$$

We show that $\phi = \inf(\alpha, \beta)$.

1. $\phi \leq_v \alpha$, $\beta$. By definition, we have $E \land D \leq E$. Then we have, for all $z \in E \land D$,

$$\phi(z) = \min(\alpha(e_z) \frac{v(z)}{v(e_z)}, \beta(d_z) \frac{v(z)}{v(d_z)}) \leq \alpha(e_z) \frac{v(z)}{v(e_z)},$$

so $\phi \leq_v \alpha$. In the same way, we can prove that $\phi \leq_v \beta$.

2. We prove that, for all $\gamma : G \rightarrow R$, if $\gamma \leq_v \alpha$ and $\gamma \leq_v \beta$, then $\gamma \leq_v \phi$.

By hypothesis $G \leq E$ and $G \leq D$, so $G \leq E \land D$. We have to show that, for all $g \in G$, $\gamma(g) \leq \phi(z_g) \frac{v(g)}{v(z_g)}$ where $z_g$ is the element of $E \land D$ such that $g \leq z_g$. By definition of $\phi$, this amounts to prove that

$$\gamma(g) \leq \min(\alpha_E(z_g), \beta_E(z_g)) \frac{v(g)}{v(z_g)}.$$

$$= \min(\alpha(e_{z_g}) \frac{v(z_g)}{v(e_g)}, \beta(d_{z_g}) \frac{v(z_g)}{v(d_g)}) \frac{v(g)}{v(z_g)}$$

$$= \min(\frac{\alpha(e_{z_g})}{v(e_g)}, \frac{\beta(d_{z_g})}{v(d_g)}) v(g),$$

where $z_g \leq e_{z_g}$ and $z_g \leq d_{z_g}$. By hypothesis, for all $g \in G$, we have $\gamma(g) \leq \alpha(e_g) \frac{v(g)}{v(e_g)}$ and $\gamma(g) \leq \beta(d_g) \frac{v(g)}{v(d_g)}$, so

$$\gamma(g) \leq \min(\frac{\alpha(e_g)}{v(e_g)}, \frac{\beta(d_g)}{v(d_g)}) v(g).$$

We conclude the proof by observing that, from $g \leq e_g$ and $g \leq z_g \leq e_{z_g}$, we can derive $e_g = e_{z_g}$, because $G$ and $E$ are partitions. In the same way, from $g \leq d_g$ and $g \leq z_g \leq d_{z_g}$, we have $d_g = d_{z_g}$. This proves that $\phi$ is the greatest lower bound of $\alpha$ and $\beta$. ■
Given an act \( \beta : D \rightarrow R \) and a partition \( G \) such that \( D \leq G \), we can upgrade \( \beta \) to an act \( \beta^G : G \rightarrow R \) setting, for all \( g \in G \),

\[
\beta^G(g) = \max(\beta(x) \frac{v(g)}{v(x)} : x \in D_g).
\]

We have \( \beta \preceq_v \beta^G \). In fact \( d \in D_{g_d} \) so \( \beta(d) \frac{v(g)}{v(d)} \leq \beta^G(g) \) and then \( \beta(d) \leq \beta^G(g) \frac{v(d)}{v(g)} \), thus proving that \( \beta \preceq_v \beta^G \). From corollary\(^2\) we can see that \( \beta^G \) is the best approximation from above to \( \beta \) in \( A(G) \).

**Lemma 5** \( (A(\mathcal{A}), \preceq_v) \) is closed with respect to sup.

**Proof.** Given \( \alpha : E \rightarrow R \) and \( \beta : D \rightarrow R \), we define \( \phi : E \vee D \rightarrow R \) setting, for all \( z \in E \vee D \),

\[
\phi(z) = \max(\alpha^{E \vee D}(z), \beta^{E \vee D}(z)).
\]

We show that \( \phi = \sup(\alpha, \beta) \).

1. \( \alpha, \beta \preceq_v \phi \). In order to show that \( \alpha \preceq_v \phi \), we must prove that, for all \( e \in E \), \( \alpha(e) \leq \phi(z_e) \frac{v(e)}{v(z_e)} \).

   \[
   \phi(z_e) = \max(\alpha^{E \vee D}(z_e), \beta^{E \vee D}(z_e))
   \]

   \[
   = \max(v(z_e) \max(\frac{\alpha(x)}{v(x)} : x \in E_{z_e}), v(z_e) \max(\frac{\beta(x)}{v(x)} : x \in D_{z_e}))
   \]

   \[
   \geq v(z_e) \max(\frac{\alpha(x)}{v(x)} : x \in E_{z_e})
   \]

   so

\[
\phi(z_e) \frac{v(e)}{v(z_e)} \geq v(z_e) \max(\frac{\alpha(x)}{v(x)} : x \in E_{z_e}) \frac{v(e)}{v(z_e)}
\]

\[
= v(e) \max(\frac{\alpha(x)}{v(x)} : x \in E_{z_e})
\]

\[
\geq v(e) \frac{\alpha(e)}{v(e)}
\]

\[
= \alpha(e),
\]

where the third line follows because \( e \in E_{z_e} \) and then \( \max(\frac{\alpha(x)}{v(x)} : x \in E_{z_e}) \geq \frac{\alpha(e)}{v(e)} \). In the same way we can prove that \( \beta \preceq_v \phi \).

2. For all \( \gamma : G \rightarrow R \), if \( \alpha \preceq_v \gamma \) and \( \beta \preceq_v \gamma \), then \( \phi \preceq_v \gamma \). By hypothesis \( E \leq G \) and \( D \leq G \), so \( E \vee D \leq G \). We have to show that, for all \( z \in E \vee D \), \( \phi(z) \leq \gamma(g_z) \frac{v(z)}{v(g_z)} \). By hypothesis we have, for all \( e \in E \), \( \alpha(e) \leq \gamma(g_e) \frac{v(e)}{v(g_e)} \) so

\[
\alpha(e) \frac{v(g_e)}{v(e)} \leq \gamma(g_e).
\]

In particular, for all \( x \in E_z \) we have \( \alpha(x) \frac{v(g_z)}{v(x)} \leq \gamma(g_z) \). When \( x \in E_z \) we have also \( x \preceq z \leq g_z \) and \( x \preceq g_x \), so \( g_z = g_x \) because \( G \) and \( E \) are partitions. Then we can conclude that, for all \( x \in E_z \), \( \alpha(x) \frac{v(g_z)}{v(x)} \leq \gamma(g_z) \) and so

\[
\max(\alpha(x) \frac{v(g_z)}{v(x)} : x \in E_z) \leq \gamma(g_z)
\]
and 
\[ v(z) \max(\frac{\alpha(x)}{v(x)} : x \in E_z) \leq \gamma(g_z, \frac{v(z)}{v(g_z)}) \]

In the same way, from the hypothesis for all \( d \in D, \beta(d) \leq \gamma(g_d, \frac{v(d)}{v(g_d)}) \), we can prove that 
\[ v(z) \max(\frac{\beta(x)}{v(x)} : x \in D_z) \leq \gamma(g_z, \frac{v(z)}{v(g_z)}) \]

so we can conclude that 
\[
\gamma(g_z, \frac{v(z)}{v(g_z)}) \geq \max(v(z) \max(\frac{\alpha(x)}{v(x)} : x \in E_z), v(z) \max(\frac{\beta(x)}{v(x)} : x \in D_z)) = \phi(z).
\]

\[ \square \]

**Theorem 6** \((A(A), \preceq_v)\) is a lattice with \((A(E), \preceq_E)\) as a sublattice.

**Proof.** \((A(A), \preceq_v)\) is a lattice by the preceding lemmas. We prove that \( \inf (A(E), \preceq_E) \) is a sublattice of \((A(A), \preceq_v)\). In the first place, we show that for all \( \alpha, \beta \in A(E), \inf_{E}(\alpha, \beta) = \inf_{v}(\alpha, \beta) \), where \( \inf_{E} \) denotes \( \inf \) in \( A(E) \) and \( \inf_{v} \) denotes \( \inf \) in \( A(A) \). We observe that \( \inf_{E}(\alpha, \beta) \) is a lower bound of \( \{\alpha, \beta\} \) in \((A(A), \preceq_v)\) because \( \inf_{E}(\alpha, \beta) \preceq_{E} \alpha, \beta \) implies \( \inf_{E}(\alpha, \beta) \preceq_{v} \alpha, \beta \), by theorem \( \square \) We can see that \( \inf_{E}(\alpha, \beta) \) is the greatest lower bound of \( \{\alpha, \beta\} \) in \((A(A), \preceq_v)\) as follows: we suppose that \( \xi \preceq_v \alpha, \beta \), where \( \xi \in A(A) \) is an act \( \xi : G \rightarrow R \) where \( G \leq E \), and we show that \( \xi \preceq_{E} \inf_{E}(\alpha, \beta) \). So we must show that, for all \( g \in G, \xi(g) \leq \inf_{E}(\alpha, \beta)(e_g)\frac{v(g)}{v(e_g)} = \min(\alpha(e_g), \beta(e_g))\frac{v(g)}{v(e_g)} \). (We remember that \( \alpha \) and \( \beta \) are functions \( E \rightarrow R \) and \( \inf_{E} \) is defined pointwise.) By hypothesis, \( \xi(g) \leq \alpha(e_g)\frac{v(g)}{v(e_g)} \) and \( \xi(g) \leq \beta(e_g)\frac{v(g)}{v(e_g)} \), so 
\[
\xi(g) \leq \min(\alpha(e_g), \beta(e_g))\frac{v(g)}{v(e_g)},
\]
where the second line follows because \( \min(ax, bx) = \min(a, b)x \) when \( x \geq 0 \).

Finally, we show that \( \sup_{E}(\alpha, \beta) = \sup_{v}(\alpha, \beta) \). On one side, \( \sup_{E}(\alpha, \beta) \) is an upper bound of \( \{\alpha, \beta\} \) in \((A(A), \preceq_v)\), because \( \alpha, \beta \preceq_E \sup_{E}(\alpha, \beta) \) and so \( \alpha, \beta \preceq_v \sup_{E}(\alpha, \beta) \) by theorem \( \square \) On the other side, for all \( \xi \in A(A) \) such that \( \alpha, \beta \preceq_v \xi \), we can show that \( \sup_{E}(\alpha, \beta) \preceq_v \xi \). In fact, \( \xi : G \rightarrow R \) for some \( G \geq E \), so by hypothesis we have \( \alpha(e) \leq \xi(g_e)\frac{v(e)}{v(g_e)} \) and \( \beta(e) \leq \xi(g_e)\frac{v(e)}{v(g_e)} \). Then
\[
\sup_{E}(\alpha, \beta)(e) = \max(\alpha(e), \beta(e)) \leq \xi(g_e)\frac{v(e)}{v(g_e)},
\]
thus proving that \( \sup_{E}(\alpha, \beta) \preceq_v \xi \). \( \square \)
Corollary 7

1. If $\beta : D \to R$ and $E \subseteq D$, then $\beta_E = \bigvee\{\xi \in A(E) : \xi \preceq_v \beta\}$, where $\bigvee$ is taken in $A(A)$.

2. If $\beta : D \to R$ and $D \subseteq G$, then $\beta^G = \bigwedge\{\xi \in A(G) : \beta \preceq_v \xi\}$, where $\bigwedge$ is taken in $A(A)$.

Proof. 1. On one side, we show that $\beta_E$ is an upper bound of $\{\xi \in A(E) : \xi \preceq_v \beta\}$. If $\xi : E \to R$ is such that $\xi \preceq_v \beta$ then, for all $e \in E$, $\xi(e) \leq \beta(d_e)v(e)\xi(d_e)\beta(e)$. So $\xi \preceq_E \beta_E$ and then $\xi \preceq_v \beta_E$ by theorem 1. On the other side, let $\delta : E \to R$ be an upper bound of $\{\xi \in A(E) : \xi \preceq_v \beta\}$, then $\beta_E \preceq_v \delta$ because $\beta_E \preceq_v \beta$, by definition of $\beta_E$.

2. On one side, we show that $\beta^G$ is a lower bound of $\{\xi \in A(G) : \beta \preceq_v \xi\}$. If $\xi : G \to R$ is such that $\beta \preceq_v \xi$ then, for all $d \in D$, $\beta(d) \leq \xi(g(d))v(d)\xi(g(d))$ and $\beta(d)\xi(g(d)) \preceq_v \xi(g(d))$. We observe that, for all $g \in G$, $g = g(d)$ holds for all $d \in D_g$, so $\beta(d)\xi(g(d)) \preceq_v \xi(g)$ holds for all $d \in D_g$ and then

$$\max(\beta(x)\frac{v(g)}{v(x)} : x \in D_g) \leq \xi(g).$$

In this way we have shown that, for all $g \in G$, $\beta^G(g) \leq \xi(g)$ and so $\beta^G \preceq_G \xi$.

By theorem 1 we can conclude that $\beta^G \preceq_v \xi$. On the other side, let $\delta : G \to R$ be a lower bound of $\{\xi \in A(G) : \beta \preceq_v \xi\}$, then $\delta \preceq_v \beta^G$ because $\beta \preceq_v \beta^G$, as we have shown before the preceding lemma.

In the preceding corollary, $\bigvee$ can indifferently be taken in $A(E)$ and $\bigwedge$ in $A(G)$.

Corollary 8 If $\alpha : E \to R$ and $\beta : D \to R$ then $\inf(\alpha, \beta) = \inf(\alpha_{E \land D}, \beta_{E \land D})$ and $\sup(\alpha, \beta) = \sup(\alpha_{E \lor D}, \beta_{E \lor D})$.

Proof. We denote with $\inf$ the greatest lower bound taken in $(A(A), \preceq_v)$ and with $\inf_{E \land D}$ the greatest lower bound taken in $(A(E), \preceq_{E \land D})$. As $\alpha_{E \land D}$ and $\beta_{E \land D}$ are acts in $A(E \land D)$, for all $z \in E \land D$ we have $\inf_{E \land D}(\alpha_{E \land D}, \beta_{E \land D})(z) = \min(\alpha_{E \land D}(z), \beta_{E \land D}(z))$ because $\inf_{E \land D}$ is defined pointwise. But $\inf(\alpha, \beta)(z) = \min(\alpha_{E \land D}(z), \beta_{E \land D}(z))$ by definition, so $\inf(\alpha, \beta) = \inf(\alpha_{E \land D}, \beta_{E \land D})$. Finally, we have $\inf_{E \land D}(\alpha_{E \land D}, \beta_{E \land D}) = \inf(\alpha_{E \land D}, \beta_{E \land D})$ by theorem 6. The same kind of proof works for the least upper bound.

5 The comparison of acts

We can summarize the different ways of comparing acts introduced so far as follows. Given $\alpha : E \to R$ and $\beta : D \to R$, we can always compare $\alpha$ with $\beta$ in $\preceq_{\exp}$. If $\alpha$ and $\beta$ have the same domain, $E = D$, they can also be compared in $\preceq_E$. We know that $\alpha \preceq_E \beta$ implies $\alpha \preceq_{\exp} \beta$. (We have $\alpha \preceq_E \beta$ implies
\( \alpha \preceq \beta \) by theorem [1] and \( \alpha \preceq \beta \) implies \( \alpha \preceq_{\exp} \beta \) by theorem [2]. If \( \alpha \) and \( \beta \) have different, but comparable, domains, i.e. \( E \leq D \) or \( D \leq E \), then we can compare \( \alpha \) with \( \beta \) in \( \preceq_v \). We know that \( \alpha \preceq_v \beta \) implies \( \alpha \preceq_{\exp} \beta \) by theorem [2]. If \( E \) and \( D \) are incomparable, we can resort to the best approximations from below to \( \alpha \) and \( \beta \) in \( A(E \land D) \). So we define a preference relation \( \alpha \prec \beta \) iff \( \alpha_{E \land D} \preceq E \land D \beta_{E \land D} \). An intuitive meaning may be attached to \( \prec \) if we observe how the set of conditions \( E \land D \) arises from \( E \) and \( D \) by meet. As an example, we set \( E = \{ \uparrow \$, \downarrow \$ \} \) and \( D = \{ \uparrow \£, \downarrow \£ \} \), where \( \uparrow \) means ‘rises’ and \( \downarrow \) means ‘sinks’. Then \( E \land D = \{ \uparrow \$ & \uparrow \£, \uparrow \$ \downarrow \£, \downarrow \$ & \uparrow \£, \downarrow \$ \downarrow \£ \} \) is a natural common set of conditions for \( \alpha_{E \land D} \) and \( \beta_{E \land D} \) where every condition \( e \in E \) splits in the different cases \( \{ e \land d : d \in D \} \).

We can easily prove that \( \prec \) is reflexive transitive and antisymmetric. Firstly, we observe that \( \alpha_{E \land D} \preceq E \land D \beta_{E \land D} \) iff for all \( e \land d \) in \( E \land D \), \( \frac{\alpha(e)}{v(e)} \leq \frac{\beta(d)}{v(d)} \). In fact,

\[
\alpha_{E \land D}(e \land d) \leq \beta_{E \land D}(e \land d) \text{ iff } \alpha(e) \frac{v(e \land d)}{v(e)} \leq \beta(e) \frac{v(e \land d)}{v(d)} \text{ iff } \frac{\alpha(e)}{v(e)} \leq \frac{\beta(d)}{v(d)}.
\]

Now we can easily see that \( \prec \) is a partial order. The following theorem shows that \( \prec \) can be seen as a generalization of \( \leq_E \) and \( \leq_v \).

**Lemma 9** If \( \alpha : E \rightarrow R \) then \( \alpha_E = \alpha \) and \( \alpha^E = \alpha \).

**Proof.** For all \( e \in E \), \( \alpha_E(e_e) = \alpha(e) \frac{v(e)}{v(e_e)} = \alpha(e) \), because \( e_e = e \). For all \( e \in E \), \( \alpha^E(e) = \max\{\alpha(x) \frac{v(e)}{v(x)} : x \in E_e\} = \alpha(e) \), because \( E_e = \{e\} \).

**Theorem 10** If \( \alpha : E \rightarrow R \) and \( \beta : D \rightarrow R \) then

1. \( E = D \) implies \( \alpha \prec \beta \) iff \( \alpha \leq_E \beta \),

2. \( \alpha \leq_v \beta \) implies \( \alpha \prec \beta \); \( \alpha \prec \beta \) and \( E \leq D \) imply \( \alpha \leq_v \beta \).

**Proof.** 1. If \( E = D \) then \( E \land D = E \) and so \( \alpha \prec \beta \) iff \( \alpha_E \leq E \beta_E \) iff \( \alpha \leq E \beta \), as \( \alpha_E = \alpha \) and \( \beta_E = \beta_D = \beta \), by the lemma.

2. We must show that \( \alpha_{E \land D} \preceq E \land D \beta_{E \land D} \). By our hypothesis \( \alpha \leq_v \beta \), \( E \leq D \) holds, so we can reduce ourselves to prove that \( \alpha_E \leq E \beta_E \) i.e. \( \alpha \leq E \beta_E \).

So we have to prove that \( \alpha(e) \leq \beta_E(e) = \beta(d_e) \frac{v(e)}{v(d_e)} \), what follows from our hypothesis. Now we assume \( \alpha \prec \beta \) and \( E \leq D \), then \( \alpha_{E \land D} \preceq E \land D \beta_{E \land D} \) and \( \alpha \leq E \beta_E \), so for all \( e \in E \), \( \alpha(a) \leq \beta(d_e) \frac{v(e)}{v(d_e)} \) and \( \alpha \leq \beta \) follows. ■

Dually, we can define a preference relation setting \( \alpha \preceq \beta \) iff \( \alpha^E \preceq E \land D \beta^{E \land D} \). Now \( \alpha^{E \lor D} \) and \( \beta^{E \lor D} \) are the best approximation to \( \alpha \) and \( \beta \) from above in \( A(E \lor D) \). The join of the set of conditions \( E \) and \( D \) is trivial in the example above. In general, if \( |E| = |D| = 2 \), then \( E = \{e, \neg e\} \) and \( D = \{d, \neg d\} \), where \( \neg \) denotes the complement operation, so \( E \lor D = 1 \), the top element of the lattice \( \Pi(\mathcal{A}) \). We can give a non-trivial example of \( E \lor D \) as follows. We consider the interval \([0, 1] \) as the price range of a good and we define five
subintervals $a = [0, 0.2)$, $b = [0.2, 0.4)$, $c = [0.4, 0.6)$, $d = [0.6, 0.8)$, $e = [0.8, 1)$. Let $A = \mathcal{P}(A)$, where $A = \{a, b, c, d, e\}$. We identify each $x \in A$ with the singleton $\{x\}$ and denote with $a|b|c|d|e$ the least partition in $\Pi(A)$. There are 52 partitions in $\Pi(A)$, but we can fix our attention on the four-elements lattice of the following figure, where juxtaposition denotes set-union (i.e. $abc = a \cup b \cup c$).

We set $E = ac|b|ed$ and $D = ab|c|ed$. Let $\alpha : E \to R$ and $\beta : D \to R$ be two acts. As we suppose that $[0, 1]$ be the price range of a good, then $\alpha(b)$ represents the payoff of $\alpha$ when the price is in $[0.2, 0.4)$. The same holds for $\beta$. Now $E \wedge D$ is the set of conditions containing all non-empty meets of conditions in $E$ with conditions in $D$ and $E \vee D$ is the set of all (minimal) common-joins from $E$ and $D$ (joins of conditions of $\alpha$ that arise also as joins of conditions of $\beta$).

We can easily prove that $\blacktriangleleft$ is reflexive and antisymmetric. Firstly, we observe that $\alpha^{E\wedge D} \leq_{E\wedge D} \beta^{E\wedge D}$ iff for all $w \in E \wedge D$, $\alpha^{E\wedge D}(w) \leq \beta^{E\wedge D}(w)$ and this happens iff $\max\{\alpha(x) : x \in E_w\} \leq \max\{\beta(x) : x \in D_w\}$. Transitivity fails, as can be easily seen by a counterexample.

The following theorem shows that $\blacktriangleleft$ can be seen as a generalization of $\leq_E$ and $\leq_v$.

**Theorem 11** If $\alpha : E \to R$ and $\beta : D \to R$ then

1. $E = D$ implies $\alpha \blacktriangledown \beta$ iff $\alpha \leq_E \beta$,
2. $\alpha \leq_v \beta$ implies $\alpha \angleleft \beta$; $\alpha \blacktriangledown \beta$ and $E \leq D$ imply $\alpha \leq_v \beta$.

**Proof.** 1. If $E = D$ then $E \vee D = E$ so $\alpha \blacktriangledown \beta$ iff $\alpha^{E} \leq_E \beta^{E}$ iff $\alpha^{E} \leq_E \beta$, the the lemma above.

2. We assume $\alpha \leq_v \beta$ so $E \leq D$ and, for all $e \in E$, $\alpha(a) \leq \beta(d_e) \frac{v_a(e)}{v(d_e)}$ and so $\alpha(a) \frac{v(d_e)}{v(e)} \leq \beta(d_e)$. We must show that $\alpha^{E\wedge D} \leq_{E\wedge D} \beta^{E\wedge D}$, i.e. $\alpha^{D} \leq_{D} \beta$. So we can reduce ourselves to prove that, for all $d \in D$, $\alpha^{D}(d) \leq \beta(d)$, i.e.

$max\{\alpha(x) \frac{v(d)}{v(x)} : x \in E_d\} \leq \beta(d)$. If $x \in E_d$ then $d_x = d$, so by our hypothesis we have $\alpha(x) \frac{v(d)}{v(x)} \leq \beta(d)$ for all $d \in D$ and the result follows. We assume $\alpha \blacktriangledown \beta$ and $E \leq D$, then $\alpha^{E\wedge D} \leq_{E\wedge D} \beta^{E\wedge D}$, i.e. $\alpha^{D} \leq_{D} \beta$ and so $max\{\alpha(x) \frac{v(d)}{v(x)} : x \in E_d\} \leq \beta(d)$, i.e. $\alpha^{D}(d) \leq \beta(d)$.

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$E_d \leq \beta(d)$ for all $d \in D$. If $x \in E_d$ then $\alpha(x) \frac{v(d)}{v(x)} \leq \beta(d)$ and $\alpha(x) \leq \beta(d) \frac{v(x)}{v(d)}$.

But for all $e \in E$, we have $e \in E_d$, so $\alpha(e) \leq \beta(d) \frac{v(e)}{v(d)}$ and $\alpha \leq v \beta$ follows.

### A Partitions in finite distributive lattices

The set theoretic notion of partition, introduced in the first paragraph, can be generalized as follows. Let $A$ be a finite distributive lattice, we say that $E \subseteq A$ is a partition of $A$ if

1. $\bigvee E = 1$,
2. $e_2 \wedge e_2 = 0$, for all $e_2, e_2 \in E$ with $e_2 \neq e_2$,
3. $e \neq 0$, for all $e \in E$.

We denote with $\Pi(A)$ the set of all partitions of $A$. Of course, every set theoretic partition $\{X_i : i \in I\}$ is also an algebraic partition of $P(X)$, the Boolean algebra of all subsets of $X$. In the following we speak generically of partitions, leaving to the context to decide whether algebraic or set theoretical partitions are involved. In general, we speak of a partition on (a set) $X$ when a set theoretical partition is intended, and speak of a partition of (a lattice) $A$ when an algebraic partition is intended.

**Lemma 12** For all partition $E$ of $A$ and all $e \in E$, $E - \{e\}$ is not a partition of $A$

**Proof.** We set $E' = E - \{e\}$ and suppose that $E'$ is a partition, then $\bigvee E' = 1$ and so

$$e = e \wedge \bigvee E' = \bigvee \{e \wedge e' : e' \in E'\} = 0,$$

because $e, e' \in E$ and $e \neq e'$. But $e \neq 0$, because $E$ is a partition. ■

We define a relation on partitions setting $E \leq D$ iff for all $e \in E$ there is a $d \in D$ such that $e \leq d$. In this case, we say that $E$ is a refinement of (or is finer than) $D$. The following lemma shows that there is only one $d$ of this kind, so we can speak of the $d \in D$ such that $e \leq d$ and denote it with $d_e$.

**Lemma 13** If $E \leq D$ then, for all $e \in E$, there is only one $d \in D$ such that $e \leq d$.

**Proof.** We suppose that, for some $e \in E$, there are $d$ and $d'$ in $D$ such that $d \neq d'$ and $e \leq d, d'$. Then $e \leq d \wedge d' = 0$, but this is absurd because $E$ is a partition. ■

The following lemma shows that $d_e$, as a function $E \rightarrow D$, is surjective.

**Lemma 14** If $E \leq D$ then, for all $d \in D$, there is $e \in E$ such that $e \leq d$. 

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Proof. As $E \subseteq D$, for all $e \in E$ there is a $d \in D$ such that $e \leq d$. We suppose that there is $\bar{d} \in D$ such that, for all $e \in E$, $e \not\leq \bar{d}$, then we have $1 = \bigvee E \leq \bigvee (D - \{\bar{d}\})$. Then for all $x, y \in D - \{\bar{d}\}$, we have $x \land y = 0$. Finally, for all $x \in D - \{\bar{d}\}$, we have $x \neq 0$. So $D - \{\bar{d}\}$ is a partition of $A$, but this is absurd by lemma $12$. ■

If $E \subseteq D$ then, for all $d \in D$, we define $E_d = \{x \in E : x \leq d\}$: we shall prove that $d = \bigvee E_d$.

**Lemma 15** If $E \subseteq D$ then:

1. $\{E_x : x \in D\}$ is a (set theoretic) partition on $E$.
2. $\bigvee \{E_x : x \in D\}$ is an (algebraic) partition of $A$.

**Proof.** 1. Firstly, we prove that $\bigcup \{E_x : x \in D\} = E$. On one side we have $\bigcup \{E_x : x \in D\} \subseteq E$, because $E_x \subseteq E$ for all $x \in D$. On the other side, $E \subseteq \bigcup \{E_x : x \in D\}$ because, for all $e \in E$, there is a $x \in D$ such that $e \leq x$ and $e \in E_x$. Then we prove that, for all $x, x' \in D$, $x \neq x'$, we have $E_x \cap E_x' = 0$, because $y \in E_x \cap E_x'$ implies $y \leq x$ and $y \leq x'$ so that $y \leq x \land x' = 0$, that is absurd. Finally, we have $E_x \neq \emptyset$, for all $x \in D$, by lemma $14$.

2. Firstly, we have

$$\bigvee \{\bigvee E_x : x \in D\} = \bigvee \bigcup \{E_x : x \in D\} = \bigvee E = 1$$

because, by point 1), we have $\bigcup \{E_x : x \in D\} = E$. Then we have, for all $x, y \in D$, $x \neq y$,

$$(\bigvee E_x) \land (\bigvee E_y) = \bigvee \{a \land b : a \in E_x, b \in E_y\} = 0$$

because $E_x \cap E_y = \emptyset$, by point 1). Finally, for all $x \in D$, we have $\bigvee E_x \neq 0$, because $E_x \neq \emptyset$ by point 1). ■

**Lemma 16** If $E \subseteq D$ then for all $x, y \in D$, if $x \neq y$ then $x \land \bigvee E_y = 0$.

**Proof.** We have $x \land \bigvee E_y = \bigvee \{x \land z : z \in E_y\} = 0$, because $z \leq y$ implies $x \land z \leq x \land y = 0$. ■

**Theorem 17** If $E \subseteq D$ then, for all $d \in D$, $d = \bigvee E_d$.

**Proof.** We observe that

$$d \leq 1 = \bigvee \{\bigvee E_x : x \in D\},$$

by point 2) of lemma $15$. So

$$d = d \land \bigvee \{\bigvee E_x : x \in D\}$$

$$= \bigvee \{d \land \bigvee E_x : x \in D\}$$

$$= \bigvee E_d.$$  

The last line follows because $d \land \bigvee E_x = 0$ when $d \neq x$, by lemma $16$ and $d \land \bigvee E_x = \bigvee E_d$ when $x = d$, as $\bigvee E_d \leq d$ ($d$ is an upper bound for $E_d$). ■
Lemma 18 For all $E$, $D \in \Pi(A)$, for all $e \in E$ there is a $d \in D$ such that $e \land d \neq 0$.

Proof. We have $e \land \bigvee D = e \neq 0$, then $\bigvee \{e \land d : d \in D\} \neq 0$ so there is a $d \in D$ such that $e \land d \neq 0$. ■

Theorem 19 For all distributive finite lattice $A$, $\Pi(A)$ is a bounded lattice with respect to $\leq$. If $A$ is a Boolean algebra, then $At(A)$ is the bottom element of $\Pi(A)$.

Proof. Firstly we prove that $\Pi(A)$ is partially ordered by $\leq$. Reflexivity and transitivity of $\leq$ are immediate. As for antisimmetry, we suppose $E \leq D$ and $D \leq E$ and prove that $E = D$. If $e \in E$ then there is $d \in D$ such that $e \leq d$ and $e' \in E$ such that $d \leq e'$: so $e \leq e'$. As $e, e' \in E$, if $e \neq e'$ then $e \land e' = 0$, but $e \land e' = e$ and $e \neq 0$, so we conclude that $e = e'$. As $d$ is sandwiched between $e$ and $e'$, we have $d = e$, so $e \in D$. In the same way we prove that $D \leq E$, so $E = D$.

Secondly we prove that $\Pi(A)$ is a bounded lattice with respect to $\leq$. $\Pi(A)$ has a greatest element $\{1\}$, where $1$ is the top element of $A$. $\Pi(A)$ contains the greatest lower bound $E \land D$ for all $E, D \in \Pi(A)$. For every $E, D \in \Pi(A)$, we set

$$H = \{e \land d : e \in E, d \in D, e \land d \neq 0\}$$

We observe that $H$ is not empty, by the above lemma, and we prove that $H = E \land D$. In the first place we prove that $E \land D$ is a partition of $A$. In fact, we have

$$\bigvee \{e \land d : e \in E, d \in D\} = \bigvee \{\bigvee \{e \land d : d \in D\} : e \in E\}$$

$$= \bigvee \{e \land \bigvee \{d : d \in D\} : e \in E\}$$

$$= \bigvee \{e : e \in E\} \land \bigvee \{d : d \in D\}$$

$$= 1,$$

and we have $(e \land d) \land (e' \land d') = 0$ whenever $e, e' \in E'$ and $d, d' \in D$. Now we can easily see that $H$ is $E \land D$. On one side, $H \leq E, D$ because $e \land d \leq e$ and $e \land d \leq d$, for all $e \land d \in H$. On the other side, for all $Z \in \Pi(A)$ such that $Z \leq E, D$, we have $Z \leq H$, because for all $z \in Z$ there are $e \in E$ and $d \in D$ such that $z \leq e$ and $z \leq d$ and then $z \leq e \land d$. As $A$ is finite, the bottom element of $\Pi(A)$ is $\bigwedge \Pi(A)$. The existence $E \lor D$ follows by theorem 2.31 of [4].)

Now we suppose that $A$ is a finite Boolean algebra. Firstly, we prove that $At(A)$, the set of all atoms in $A$, is a partition of $A$. By definition of atom, we have $a \land a' = 0$ for all $a, a' \in At(A)$. Then we remember that, in a finite Boolean algebra $A$, we have $a = \bigvee \{x \in At(A) : x \leq a\}$ for all $a \in A$, (see lemma 5.4 of [4]) so $1 = \bigvee At(A)$. Now we can prove that, for all $E \in \Pi(A)$, $At(A) \leq E$: in fact, for all $a \in At(A)$ we have $a \leq \bigvee E = 1$ an so there is an $e \in E$ such that $a \leq e$, by lemma 5.11 (iii) of [4]. ■
In the above theorem $E \lor D$ is described as $\bigwedge \{ Z \in \Pi(A) : E \leq Z$ and $D \leq Z \}$. As a result of this definition from above, we have no idea of the inner constitution of $E \lor D$. The following two theorems are devoted to this scope. For all $X \subseteq A$, we define $[X]$ as the least subalgebra of $A$ including $X$, i.e. the intersection of all $B \subseteq A$ such that $X \subseteq B$. We denote with $[X]_\lor$ the least subset of $A$ that is closed with respect to finite (even empty) joins.

**Theorem 20** If $A$ is a finite distributive lattice and $E$ partition of $A$, then

1. $[E]_\lor = [E]$,
2. $[E]_\lor$ is closed with respect to complement, i.e. $[E]_\lor$ is a Boolean algebra.

**Proof.** 1. We show that $[E]_\lor$ is the least subalgebra of $A$ including $E$. Firstly we show that $[E]_\lor$ is a subalgebra of $A$. We observe that $1 \in [E]_\lor$ because $\bigvee E = 1$ and $0 \in [E]_\lor$ because $\bigvee \emptyset = 0$. Obviously, $[E]_\lor$ is closed with respect to $\lor$ by definition. We prove that if $a, b \in [E]_\lor$ then $a \land b \in [E]$. If $a = 0$ or $b = 0$ then $a \land b \in [E]$. Then we suppose $a \neq 0$ and $b \neq 0$. By hypothesis, there are some non empty subsets $X, Y \subseteq E$ such that $a = \bigvee X$ and $b = \bigvee Y$. So

$$a \land b = \left( \bigvee X \right) \land \left( \bigvee Y \right) = \bigvee \{x \land y : x \in X, y \in Y\}.$$  

As $E$ is a partition, $x \land y = 0$ when $x \neq y$ and $x \land y = x$ when $x = y$, so

$$a \land b = \left\{ \begin{array}{ll} 0 & \text{if } X \cap Y = \emptyset, \\ \bigvee (X \cap Y) & \text{if } X \cap Y \neq \emptyset. \end{array} \right.$$  

In both cases, $a \land b \in [E]_\lor$, as $X \cap Y \subseteq E$. Trivially $E \subseteq [E]_\lor$. Minimality follows because, for all $B \subseteq A$ such that $E \subseteq B$, $[E]_\lor \subseteq B$.

2. We prove that for all $a \in [E]_\lor$ there is an element $\neg a$ that is the complement of $a$ in $[E]_\lor$. We know that, by hypothesis, $a = \bigvee X$ for some $X \subseteq E$, so we set $\neg a = \bigvee (E - X)$. Then we have

$$a \lor \neg a = \bigvee X \lor \bigvee (E - X) = \bigvee E = 1$$

and

$$a \land \neg a = \bigvee X \land \bigvee (E - X) = \bigvee \{x \land y : x \in X, y \in E - X\} = 0,$$

as $X \cap (E - X) = \emptyset$ and $x \land y = 0$ when $x \neq y$. ■

**Theorem 21** If $E, D \in \Pi(A)$ then $E \leq D$ iff $[D] \subseteq [E]$.

**Proof.** We assume $E \leq D$. If $x \in [D]$ then $x = \bigvee D'$ for some $D' \subseteq D$. For all $d' \in D'$, we have $d' = \bigvee E_{d'}$ by (17) so we have $x = \bigvee \{\bigvee E_{d'} : d' \in D'\}$ and $x \in [E]$. We assume $[D] \subseteq [E]$. Then $D \subseteq [E]$ and $\bigvee D \in [E]$, so for all $e \in E$ we have $e \leq 1 = \bigvee D$. As every $e \in E$ is $\lor$-irreducible in $[E]$, there is a $d \in D$ such that $e \leq d$, by lemma 5.11 of [4], so $E \leq D$. ■

Now we can give a more constructive description of $E \lor D$. 

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Theorem 22 If \( A \) is a finite distributive lattice and \( E, D \) are partitions of \( A \), then \( E \lor D = \text{At}([E] \cap [D]) \).

Proof. Firstly we observe that \([E], [D]\) are subalgebras of \( A \), so \([E] \cap [D]\) is also a subalgebra of \( A \). As \([E], [D]\) are Boolean algebras, by [20] so is \([E] \cap [D]\) and we can speak of \( \text{At}([E] \cap [D]) \). As \( \text{At}([E] \cap [D]) \) is a partition of \([E] \cap [D]\) and \([E] \cap [D]\) is a subalgebra of \( A \), \( \text{At}([E] \cap [D]) \) is a partition of \( A \). Firstly, we show that \( E \leq \text{At}([E] \cap [D]) \). In fact, \( [\text{At}([E] \cap [D])] = [E] \cap [D] \subseteq [E] \) so, by theorem [21] we can conclude that \( E \leq \text{At}([E] \cap [D]) \). In the same way we prove that \( D \leq \text{At}([E] \cap [D]) \). Finally, we prove that, for all \( G \) such that \( E \leq G \) and \( D \leq G \), we have \( \text{At}([E] \cap [D]) \leq G \). From our hypothesis, by theorem [21] \( [G] \subseteq [E] \) and \( [G] \subseteq [D] \), so \( [G] \subseteq [E] \cap [D] = [\text{At}([E] \cap [D])] \) and by theorem [21] we have \( \text{At}([E] \cap [D]) \leq G \). ■

The following theorem shows that every element of \([E]\), different from 0, is uniquely generated by \( \lor \) from \( E \).

Theorem 23 For all \( E \in \Pi(A) \), if \( X, Y \subseteq E \) and \( \lor X = \lor Y \), then \( X = Y \).

Proof. If \( x \in X \) then \( x \leq \lor X = \lor Y \). If for all \( y \in Y \) we have \( x \land y = 0 \), then \( x = x \land \lor Y = \lor \{x \land y : y \in Y\} = 0 \), but this is absurd because \( x \in E \) and \( E \) is a partition, so there is \( y \in Y \) such that \( x \land y \neq 0 \). As \( x, y \in E \), this implies \( x = y \), so \( x \in Y \) and \( X \subseteq Y \). In the same way we prove that \( Y \subseteq X \). ■

B Allais Paradox and intrinsic expected value

A decision problem leading to a somewhat paradoxical conclusion has been presented by Maurice Allais in 1953 (see [1]). The acts involved are \( \alpha, \alpha', \beta \) and \( \beta' \), all having a common domain \( E = \{e_1, e_2, e_3\} \). The probabilities of the three conditions are \( p(e_1) = 0.01, p(e_2) = 0, 1 \) and \( p(e_3) = 0.89 \). The rewards, in dollars, are:

|     | \( e_1 \) | \( e_2 \) | \( e_3 \) |     | \( e_1 \) | \( e_2 \) | \( e_3 \) |
|-----|----------|----------|----------|-----|----------|----------|----------|
| \( \alpha \) | 500000   | 500000   | 500000   | \( \beta \) | 500000   | 500000   | 0        |
| \( \alpha' \) | 0        | 2500000  | 500000   | \( \beta' \) | 0        | 2500000  | 0        |

The decision maker must choose between \( \alpha \) and \( \alpha' \) and between \( \beta \) and \( \beta' \): if he maximizes expected utility, then \( \alpha' \) is better than \( \alpha \) and \( \beta' \) is better then \( \beta \), but empirical evidence shows that a great many people prefer \( \alpha \) to \( \alpha' \) and \( \beta' \) to \( \beta \). So maximizing expected value cannot be considered as an universal rule of choice between acts. This situation is generally explained by observing that the choice between \( \alpha \) and \( \alpha' \) is a decision problem qualitatively different from the choice between \( \beta \) and \( \beta' \). The choice of \( \alpha \) stems from risk aversion, because \( \alpha \) is a constant function that banishes every aleatory aspect, so the decision maker leaves out any question about probability and expected value. On the other side, \( \beta \) and \( \beta' \) are both risky acts and consideration of expected value is appropriate. Before going farther in the analysis of Allais Paradox, we introduce some concepts of general character.
When $X$ and $Y$ are partially ordered sets, we say that $f : X \to Y$ is an order embedding when $x \leq x'$ in $X$ iff $f(x) \leq f(x')$ in $Y$. (It can be easily seen that every order embedding is injective.) If $f : A(E) \to A(E)$ is an order embedding, then any decision problem about acts in $A(E)$ can be reduced to a decision problem about acts in $f[A(E)]$, in the following sense. If $\leq$ represents desirability of acts and we are asked if $\alpha \leq \beta$, then we can shift the problem to desirability of $f(\alpha)$ and $f(\beta)$: if we find that $f(\alpha) \leq f(\beta)$, then also $\alpha \leq \beta$. We focus on the family of order embeddings associated to positive affine transformations of $R$. For all pair of real numbers $h > 0$ and $k$, $\tau(x) = hx + k$ is the positive affine transformation associated to $(h, k)$. Now we consider the function $f : A(E) \to A(E)$ such that $f(\alpha) = \tau \circ \alpha$: we have, for all $e \in E$, $f(\alpha)(e) = \tau(\alpha(e)) = h\alpha(e) + k$.

**Theorem 24** If $f : A(E) \to A(E)$, where $f(\alpha) = \tau \circ \alpha$ and $\tau$ is the positive affine transformations $\tau(x) = hx + k$, then

1. $f$ is an order embedding of $A(E)$ in itself,
2. for all $\alpha \in A(E)$ and all valuation $v : A \to [0, 1]$, $\exp(\tau \circ \alpha, v) = \tau(\exp(\alpha, v))$,
3. $\exp(\alpha, v) \leq \exp(\alpha', v)$ iff $\exp(f(\alpha), v) \leq \exp(f(\alpha'))$.

**Proof.** 1. We have $\alpha \leq \beta$ iff, for all $e \in E$, $\alpha(e) \leq \beta(e)$ iff $h\alpha(e) + k \leq h\beta(e) + k$ iff $f(\alpha) \leq f(\beta)$.

2. We have

\[
\exp(\tau \circ \alpha, v) = \sum \{(h\alpha(e) + k)v(e) : e \in E\} = \sum \{h\alpha(e)v(e) + kv(e) : e \in E\} = h \sum \{\alpha(e)v(e) : e \in E\} + k \sum \{v(e) : e \in E\} = h \exp(\alpha, v) + k.
\]

3. Trivial. ■

The same kind of order embedding can be defined from $A(A)$ to $A(A)$ and a similar theorem can be proved.

Now we can give an equivalent formulation of Allais Paradox by defining four acts as follows:

| $f(\alpha)$ | $e_1$ | $e_2$ | $e_3$ |
|-------------|-------|-------|-------|
| $f(\alpha')$ | 1 | 1 | 1 |

| $f(\beta)$ | $e_1$ | $e_2$ | $e_3$ |
|-------------|-------|-------|-------|
| $f(\beta')$ | 0 | 5 | 0 |

The transformation involved is $\tau(x) = 1/500000x$. As a consequence of point 1) in the above theorem, we have $\exp(f(\alpha), v) \leq \exp(f(\alpha'), v)$ and $\exp(f(\beta), v) \leq \exp(f(\beta'), v)$, but we may still prefer $f(\alpha)$ to $f(\alpha')$ by risk aversion, as in the original formulation of Allais Paradox. We underline that only an equivalence of
mathematical character is discussed here: $f$ preserves $\leq$, but may not preserve the psychological impact of acts.

We can give a more abstract formulation of Allais Paradox with the following acts:

|   | $e_1$ | $e_2$ | $e_3$ |
|---|-------|-------|-------|
| $\alpha$ | $x$   | $y$   | $x$   |
| $\alpha'$ | $0$   | $y$   | $0$   |
| $\beta$  | $x$   | $x$   | $0$   |
| $\beta'$ | $0$   | $y$   | $0$   |

We assume $x, y > 0$. The particular case above is obtained setting $x = 1$ and $y = 5$, but not every choice of $x$ and $y$ gives place to an instance of Allais Paradox.

By definition of the four acts, we have $\exp(\alpha, v) \leq \exp(\alpha', v)$ iff $\exp(\beta, v) \leq \exp(\beta', v)$, but in particular we should also have $\exp(\alpha, v) < \exp(\alpha', v)$. We observe that

\[
\exp(\alpha, v) < \exp(\alpha', v) \quad \text{iff} \quad x \cdot 0.01 + x \cdot 0.1 + x \cdot 0.89 < y \cdot 0.1 + x \cdot 0.89
\]

so we can conclude that $\alpha'$ is better than $\alpha$ as far as $y > x + \frac{1}{10}x$. Thus Allais Paradox can be sharpened by choosing a value of $y$ just a little bigger than $x + \frac{1}{10}x$: if risk aversion works when $y$ is five times $x$, it should also be at work with a lesser value of $y$. As for $\beta$ and $\beta'$, there is no risk aversion because both are risky acts. But there is still another aspect of acts that can be considered in the analysis of Allais Paradox.

The consequences of acts are real numbers and we can rather naively say that big numbers correspond to big rewards, but the meaning of ‘big’ is dependent from the context. We assume that the context of an act is the total sum of rewards, so a reward is a really big one if it is a relevant part of this total. For all act $\alpha : E \to R$, we define $T(\alpha) = \sum \{ \alpha(e) : e \in E \}$ and we call $T(\alpha)$ the total of $\alpha$. When $T(\alpha) > 0$, we define an act $\bar{\alpha} : E \to R$ setting $\bar{\alpha}(e) = \frac{1}{T(\alpha)} \alpha(e)$: we call $\bar{\alpha}$ the standardization of $\alpha$. Clearly, $T(\bar{\alpha}) = 1$, because $\frac{1}{T(\alpha)} \sum \{ \bar{\alpha}(e) : e \in E \} = 1$. If $\alpha(e) \geq 0$, for all $e \in E$, then $\bar{\alpha}[E] \subseteq [0, 1]$. Finally, we define

\[
\exp(\alpha, v) = \exp(\bar{\alpha}, v)
\]

and call $\exp(\alpha, v)$ the intrinsic expected value of $\alpha$. In the original form of Allais Paradox, we have $\exp(\alpha, v) > \exp(\alpha', v)$ and $\exp(\beta, v) < \exp(\beta', v)$: if the decision maker maximizes the intrinsic expected value, then $\alpha$ is preferred to $\alpha'$ and $\beta'$ to $\beta$. So the paradox can be explained non only by risk aversion, but also by a different valuation of the performance of an act, based on the ratio between the expected value and the total amount of possible rewards. As a consequence of point 3) of the following theorem, this kind of solution works for all the instances of the original form of the Allais Paradox obtained by a similarity, i.e. a positive affine transformation $\tau(x) = hx + k$ with $k = 0$.

**Theorem 25** If $f : A(E) \to A(E)$, where $f(\alpha) = \tau \circ \alpha$ and $\tau(x) = hx$, with $k > 0$, then:
1. \( \exp(\bar{\alpha}, v) = \frac{1}{T(\alpha)} \exp(\alpha, v) \),

2. \( \exp(\tau \circ \alpha, v) = \exp(\alpha, v) \),

3. \( \exp(\alpha, v) \leq \exp(\alpha', v) \) iff \( \exp(f(\alpha), v) \leq \exp(f(\alpha')) \).

**Proof.**

1. We have

\[
\sum \{ \bar{\alpha}(e)v(e) : e \in E \} = \frac{1}{T(\alpha)} \sum \{ \bar{\alpha}(e)v(e) : e \in E \} = \frac{1}{T(\alpha)} \exp(\alpha, v).
\]

2. We have \( T(\tau \circ \alpha) = \sum \{ h\alpha(e) : e \in E \} = h \sum \{ \alpha(e) : e \in E \} = hT(\alpha) \). So

\[
\exp(\tau \circ \alpha, v) = \exp(\tau \circ \alpha, v) \frac{1}{T(\tau \circ \alpha)} \exp(\tau \circ \alpha, v) = \frac{1}{hT(\alpha)} h \exp(\alpha, v)
\]

\[
= \frac{1}{T(\alpha)} \exp(\alpha, v) = \exp(\bar{\alpha}, v).
\]

3. Trivial. ■

If we take in account the abstract form of Allais Paradox, we have

\[
\exp(\alpha', v) < \exp(\alpha', v) \quad \text{iff} \quad \frac{1}{T(\alpha')} \exp(\alpha', v) < \frac{1}{T(\alpha)} \exp(\alpha, v)
\]

\[
\quad \text{iff} \quad \frac{1}{x+y} (y \cdot 0.1 + x \cdot 0.89) < \frac{1}{3}
\]

\[
\quad \text{iff} \quad 2.3857 x < y,
\]

while on the other side we have always \( \exp(\beta, v) < \exp(\beta', v) \).

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