Abstract

We propose two new Q-learning algorithms, Full-Q-Learning (FQL) and Elimination-Based Half-Q-Learning (HQL), that enjoy improved efficiency and optimality in the full-feedback and the one-sided-feedback settings over existing Q-learning algorithms. We establish that FQL incurs regret $\tilde{O}(H^2\sqrt{T})$ and HQL incurs regret $\tilde{O}(H^3\sqrt{T})$, where $H$ is the length of each episode and $T$ is the total number of time periods. Our regret bounds are not affected by the possibly huge state and action space. Our numerical experiments using the classical inventory control problem as an example demonstrate the superior efficiency of FQL and HQL, and shows the potential of tailoring reinforcement learning algorithms for richer feedback models, which are prevalent in many natural problems.

1 Introduction

Motivated by one of the most fundamental problems in supply chain optimization–inventory control, we customize Q-learning to more efficiently solve operations research questions where we are often given richer feedback than usual. Q-learning is a popular reinforcement learning method that estimates the state-action value functions without estimating the huge transition matrix in a large MDP (see [WD92], [JJS93]). This paper is concerned with devising Q-learning algorithms that leverage the natural full-feedback/one-sided-feedback structures that exist in many problems in operations research and finance, as an analog of the corresponding studies in the bandit literature. By contrast, most reinforcement learning literature considers settings with little feedback, while the study of single-stage online learning for bandits has a history of considering a plethora of graph-based feedback models. We are particularly interested in the full-feedback and the one-sided-feedback models because of their natural prevalence in many famous problems, such as inventory control, online auctions, airlines’ overbook policy, and portfolio management.

• We propose two provably more efficient Q-learning algorithms, Full-Q-Learning and Elimination-Based Half-Q-Learning, for the full-feedback and the one-sided-feedback settings, respectively.
• Our numerical experiments, using the classical inventory control problem as an example, demonstrate the improved efficiency of our algorithms over existing Q-learning algorithms.
We consider the setting of an episodic Markov decision process, MDP. We use \( \pi \) to denote a policy. The goal is to maximize the total reward accrued in each episode.

At the beginning of each episode, the agent observes state \( x \) \( \in S \), picks an action \( y \) \( \in A \), receives a reward \( r_h(x, y, h) \) and transitions to the next state \( x' \). At the final stage \( H \), the episode terminates after the agent takes action \( y_H \) and receives reward \( R_H \). Next episode begins. Let \( K \) denote the number of episodes, and \( T \) denote the length of the horizon: \( T = H \times K \). The goal is to maximize the total reward accrued in each episode.

A policy \( \pi \) of an agent is a collection of functions \( \{ \pi_h : S \rightarrow A \}_{h \in [H]} \). We use \( V_h^\pi : S \rightarrow \mathbb{R} \) to denote the value function at stage \( h \) under policy \( \pi \), so that \( V_h^\pi(x) \) gives the expected sum of remaining rewards under policy \( \pi \) until the end of the episode, starting from state \( x_h = x \):

\[
V_h^\pi(x) := \mathbb{E}\left[ \sum_{h'=h}^H r_{h'}(x_{h'}, \pi_{h'}(x_{h'})) \middle| x_h = x \right].
\]

We use \( Q_h^\pi : S \times A \rightarrow \mathbb{R} \) to denote the Q-value function at stage \( h \), so that \( Q_h^\pi(x, y) \) gives the expected sum of remaining rewards under policy \( \pi \) until the end of the episode, starting from state \( x_h = x, y_h = y \):

\[
Q_h^\pi(x, y) := \mathbb{E}\left[ r_h(x_h, y) + \sum_{h'=h+1}^H r_{h'}(x_{h'}, \pi_{h'}(x_{h'})) \middle| x_h = x, y_h = y \right].
\]

Let \( \pi^* \) denote an optimal policy in the MDP which gives the optimal value functions \( V_h^{\pi^*}(x) = \sup_\pi V_h^\pi(x) \) for any \( x \in S \) and \( h \in [H] \). Recall the Bellman equations:

\[
\begin{align*}
V_h^{\pi^*}(x) &= \max_{y \in A} \left\{ r_h(x, y) + \sum_{h'=h+1}^H \mathbb{E}[V_{h'}^{\pi^*}(x_{h'}) \mid x_h = x, y_h = y, \pi_{h'}(x_{h'})] \right\} \\
Q_h^{\pi^*}(x, y) &= r_h(x, y) + \mathbb{E}[V_{h+1}^{\pi^*}(x_{h+1}) \mid x_h = x, y_h = y, \pi_{h+1}(x_{h+1})].
\end{align*}
\]

### 2 Prior Work

The most relevant literature to us is [JAZBJ18], who prove the optimality of Q-learning with Upper-Confidence-Bound bonus and Bernstein-style bonus in tabular MDPs. The recent work of [DRZ19] improves upon [JAZBJ18] when an aggregation of the state-action pairs with known error is given beforehand. [DRZ19] makes sense for a proper subset of the problems that [JAZBJ18] can solve. Our algorithms substantially improve the regret bounds (see Table 1) by catering to the full-feedback/one-sided-feedback structures of many problems such as inventory control. The set of problems our Q-learning algorithms can solve optimally overlap with the set of problems [JAZBJ18] can solve optimally, but neither set is a proper subset of the other. Because our regret bounds are unaffected by the cardinality of the state and action space, our Q-learning algorithms are able to deal with huge state-action space, and even continuous state space in some cases (see Remark 1). Section 7 uses inventory control as an example problem where [JAZBJ18] and [DRZ19] obtain much larger regrets than our algorithms.

### 3 Preliminaries

We consider the setting of an episodic Markov decision process, MDP \((S, A, H, P, r)\), where \( S \) is the set of states with \(|S| = S\), \( A \) is the set of actions with \(|A| = A\), \( P \) is the transition matrix that gives the distribution over states if some action \( y \) is taken at some state \( x \) at step \( h \in [H] \), and \( r_h : S \times A \rightarrow [0, 1] \) is the reward function at stage \( h \) that depends on the environment randomness \( D_h \). In each episode, an initial state \( x_1 \) is picked arbitrarily by an adversary. Then, at each stage \( h \), the agent observes state \( x_h \in S \), picks an action \( y_h \in A \), receives a realized reward \( r_h(x_h, y_h) \), and then transitions to the next state \( x_{h+1} \), which is determined by \( x_h, y_h, D_h \). At the final stage \( H \), the episode terminates after the agent takes action \( y_H \) and receives reward \( R_H \). Then next episode begins. Let \( K \) denote the number of episodes, and \( T \) denote the length of the horizon: \( T = H \times K \).

The goal is to maximize the total reward accrued in each episode.

A policy \( \pi \) of an agent is a collection of functions \( \{ \pi_h : S \rightarrow A \}_{h \in [H]} \). We use \( V_h^\pi : S \rightarrow \mathbb{R} \) to denote the value function at stage \( h \) under policy \( \pi \), so that \( V_h^\pi(x) \) gives the expected sum of remaining rewards under policy \( \pi \) until the end of the episode, starting from state \( x_h = x \):

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We use \( Q_h^\pi : S \times A \rightarrow \mathbb{R} \) to denote the Q-value function at stage \( h \), so that \( Q_h^\pi(x, y) \) gives the expected sum of remaining rewards under policy \( \pi \) until the end of the episode, starting from state \( x_h = x, y_h = y \):

\[
Q_h^\pi(x, y) := \mathbb{E}\left[ r_h(x_h, y) + \sum_{h'=h+1}^H r_{h'}(x_{h'}, \pi_{h'}(x_{h'})) \middle| x_h = x, y_h = y \right].
\]

Let \( \pi^* \) denote an optimal policy in the MDP which gives the optimal value functions \( V_h^{\pi^*}(x) = \sup_\pi V_h^\pi(x) \) for any \( x \in S \) and \( h \in [H] \). Recall the Bellman equations:

\[
\begin{align*}
V_h^{\pi^*}(x) &= \max_{y \in A} \left\{ r_h(x, y) + \sum_{h'=h+1}^H \mathbb{E}[V_{h'}^{\pi^*}(x_{h'}) \mid x_h = x, y_h = y, \pi_{h'}(x_{h'})] \right\} \\
Q_h^{\pi^*}(x, y) &= r_h(x, y) + \mathbb{E}[V_{h+1}^{\pi^*}(x_{h+1}) \mid x_h = x, y_h = y, \pi_{h+1}(x_{h+1})].
\end{align*}
\]

\( ^1 \)Here \( M \) is the number of aggregate state-action pairs; \( \epsilon \) is the largest difference between any pair of optimal state-action values associated with a common aggregate state-action pair.
We let $\text{Regret}_{MDP}(k)$ denote the expected cumulative regret against the optimal policy on the MDP up to the end of episode $k$. Let $\pi_k$ denote the policy the agent chooses before starting the $k$th episode.

$$\text{Regret}_{MDP}(K) = \sum_{k=1}^{K} \left[ V_1^*(x^k_1) - V_1^{\pi_k}(x^k_1) \right]$$ (1)

### 3.1 Full-Feedback

For the full-feedback setting, whenever we take an action at any stage $h$, once the environment randomness $D_h$ becomes realized, we can learn not only the reward of the action we take, but also what the reward and next state would have been for all the other state-action pairs. Some example problems are inventory control (backlogged model) and portfolio management in finance.

### 3.2 One-Sided-Feedback

The one-sided-feedback setting requires that the action space is a compact subset of $\mathbb{R}$ (see extension to a totally ordered finite set in Appendix B), and that the Q-values only depend on the actions, not the states. Whenever we take an action $y$ at stage $h$, once the environment randomness becomes realized, we can learn what the rewards and next states would have been for all the actions that lie on one side of our action, i.e., all $y'$ s.t. $y' \leq y$ (or all $y' \geq y$ depending on the problem). Some example problems are inventory control (lost-sales model), airline’s overbook policy and online auctions.

**Remark 1** We can work directly with continuous action space if the Q-values depend only on the actions (see Appendix E,F). Meanwhile, if the reward function is Lipschitz, we can discretize with high resolution and our regret has only a log dependence on $|S|$ and $|A|$.

### 4 Algorithms

#### 4.1 Full-Q-Learning

Algorithm Full-Q-Learning (FQL) is an adaption of [JAZBJ18] to the full-feedback setting. A second version FQL* caters to Remark 1 (see Appendix E). We define constants $\alpha_k = (H+1)/(H+k), \forall k \in [K]$. We use $x^k_{h+1}(x, y, \hat{D}^k_h)$ to denote the next state we would be in given $x$, $y$ and $\hat{D}^k_h$.

**Algorithm 1** Full-Q-Learning.

Initialization: $Q_h(x, y) \leftarrow H, \forall (x, y, h) \in S \times A \times [H]$.

for $k = 1, \ldots, K$ do

- Receive $x^k_1$;

for $h = 1, \ldots, H$ do

- Take action $y^k_h \leftarrow \arg \max_{\text{feasible}} y$ given $x^k_h Q_h(x^k_h, y)$; and observe realized $\hat{D}^k_h$;

for $x \in S$ do

- for $y \in A$ do

- Update $V_h(x, y) \leftarrow (1 - \alpha_k)Q_h(x, y) + \alpha_k \left[ r_h(x, y, \hat{D}^k_h) + V_{h+1}(x_{h+1}, y, \hat{D}^k_{h+1}) \right]$;

- Update $V_h(x) \leftarrow \max_{\text{feasible}} y$ given $x Q_h(x, y)$;

- Update $x^k_{h+1} \leftarrow x^k_{h+1}(x^k_h, y^k_h, \hat{D}^k_h)$;

- **Main Idea of Algorithm** Whenever we make an action at any stage $h$, we can update the Q-values of all state-action pairs, once we observe the realized environment randomness $\hat{D}^k_h$.  

3
4.2 Elimination-Based Half-Q-Learning

Without loss of generality, we assume that we get the lower one-sided-feedback, i.e., we can observe the rewards of all actions \( y^* \leq y \) when we take action \( y \). Elimination-Based Half-Q-Learning has four versions: HQL, HQL\(^*\), HQL-concave and HQL-concave\(^*\).

Algorithm\(^2\) HQL assumes all the actions for stage \( h \) are always accessible at stage \( h \). An example application is online auctions. If the application is such that some actions sometimes become inaccessible depending on the state, then we use Algorithm\(^3\) HQL-concave, assuming that the optimal value functions are concave and not dependent on the state, and that the feasible action set at any time is an interval \( \mathcal{A} \cap [a, \infty) \) for some \( a = a_h(x_h) \). We also assume for HQL-concave that given \( D_h \), the next state \( x_{h+1}() \) is increasing (decreasing) in \( y_h \), and \( a_h() \) is increasing (decreasing) in \( x_h \) for the lower (higher) one-sided-feedback problem. The lost-sales inventory control problem in Section 7 is one example that satisfies these assumptions. HQL\(^*\) extends the bandit result from [ZC19], while for HQL-concave we devise more intricate techniques to obtain the regret bound. The other two versions, HQL\(^*\) and HQL-concave\(^*\), are for continuous state space as in Remark 1 (see Appendix F).

For convenience of proof, we use a “Confidence Interval” of \( \frac{\sqrt{2}}{\sqrt{K-1}}(\sqrt{H}T) \) defined in Algorithm 2 and 3 where \( T = 9 \log(AT) \), but for real applications, the confidence interval can be optimized. Throughout the paper, we use \( r_{h,h'} \) to denote the cumulative reward from stage \( h \) to stage \( h' \).

Algorithm 2 Elimination-Based Half-Q-Learning.

Initialization: \( Q_h(y) \leftarrow 0 \), \( \forall (y, h) \in \mathcal{A} \times [H] \).
Set \( A_h^0 \leftarrow \mathcal{A}, \forall h \in [H] \) and \( A_{H+1}^k \leftarrow \mathcal{A}, \forall k \in [K] \);
for \( k = 1, \ldots, K \) do
  Receive \( x^k \);
  for \( h = 1, \ldots, H \) do
    Take action \( y^k_h \leftarrow \max \{ A^k_h \} \); and observe \( \hat{D}_h^k \); and update \( x^k_{h+1} \leftarrow x^k_{h+1}(x^k_h, y^k_h, \hat{D}_h^k) \);
  for \( h = H, \ldots, 1 \) do
    for \( y \in A^k_h \) do
      Update \( Q_h(y) \leftarrow (1 - \alpha_k)Q_h(y) + \alpha_k[r_h(x, y, \hat{D}_h^k) + V_{h+1}(x^k_{h+1}(x, y, \hat{D}_h^k))] \);
      Update \( V_h(x) \leftarrow \max \{ \text{feasible } y \text{ given } x \} Q_h(y) \);
      Update \( y^k_h \leftarrow \arg \max_{y \in A^k_h} Q_h(y) \);
      Update \( A^k_{h+1} \leftarrow \{ y \in A^k_h : |Q_h(y^{k*}) - Q_h(y)| \leq \text{Confidence Interval} \} \);

Main Idea of Algorithm 2 & 3 At any episode \( k \), we have a “running set” \( A^k_h \) of all the actions that are possibly the best action for stage \( h \). Whenever we take an action, we update the Q-values for all the actions in \( A^k_h \). To maximize the utility of the lower one-sided feedback, we always select the largest action in \( A^k_h \), which lets us observe the most feedback. Since \( A^k_h \) are decreasing sets in \( k \), we have updated all actions in it \( k \) times by the end of episode \( k \).

For HQL-concave, we might be in a state where we cannot choose from \( A^k_h \). Then we take the closest feasible action to \( A^k_h \) (which is the smallest feasible action in the lower one-sided-feedback case). Assuming the optimal value functions are concave, and the feasible action sets are in the form \( \mathcal{A} \cap [a, \infty) \) for some \( a = a_h(x_h) \), this is w.h.p. the optimal action in this state. By monotonicity of \( x_{h+1}(\cdot) \) and \( a_h(\cdot) \), we are always able to observe all the rewards and next states for actions in the running set. During the episode, we act in real-time and keep track of the realized environment randomness. At the end of the episode, we simulate all the trajectories as if we had taken each action in \( A^k_h \), and update the corresponding value functions, so that we can keep shrinking the running sets.

5 Main Results

Theorem 1. For any \( p \in (0, 1) \), the total regret of Full-Q-learning is at most \( O(H^2\sqrt{T}) \) on the episodic MDP problems in the full-feedback setting.
Algorithm 3 Elimination-Based Half-Q-learning Concave.

Initialization: \( Q_h(y) \leftarrow H, \forall (y, h) \in A \times [H] \).

Set \( A_h^0 \leftarrow A, \forall h \in [H] \) and \( A_{h+1}^0 \leftarrow A, \forall k \in [K] \);

for \( k = 1, \ldots, K \) do
  
  Initiate the list of realized environment randomness to be empty \( D_k = [] \); Receive \( x^k \);
  
  for \( h = 1, \ldots, H \) do
    
    if \( \max \{ A_h^k \} \) is not feasible then
      
      Take action \( y_h^k \leftarrow \) closest feasible action to \( A_h^k \);
    
    else
      
      Take action \( y_h^k \leftarrow \max \{ A_h^k \} \);
    
    Observe realized environment randomness \( D_h^k \), append it to \( D_k \);
    
    Update \( x_{h+1}^k \leftarrow x_{h+1}^k(x_h^k, y_h^k, D_h^k) \);
  
  for \( h = H, \ldots, 1 \) do
    
    for \( y \in A_h^k \) do
      
      Simulate trajectory \( x_{h+1}^i, \ldots, x_{T_h^k}^i(x, y) \) as if we had chosen \( y \) at stage \( h \) using \( D_k \) until we find \( \tau_h^k(x, y) \), which is the next time we are able to choose from \( A_h^k(x, y) \);
      
      Update \( Q_h(y) \leftarrow (1 - \alpha_k)Q_h(y) + \alpha_k [\tau_h^k(x, y) + V_{h+1}(x_{h+1}^k(x_h^k, y_h^k, D_h^k))] \);
    
    Update \( y_h^k \leftarrow \arg \max_{y \in A_h^k} Q_h(y) \);
    
    Update \( A_h^{k+1} \leftarrow \{ y \in A_h^k : |Q_h(y_h^k) - Q_h(y)| \leq \text{Confidence Interval} \};
  
  Update \( V_h(x) \leftarrow \max_{\text{feasible} \ y \text{ given } x} Q_h(y) \);

Theorem 2. For any \( p \in (0, 1) \), the total regret of Elimination Based Half-Q-learning is at most \( O(H^2 \sqrt{TK}) \) in the one-sided-feedback setting, and also the full-feedback setting as an easy corollary.

Theorem 3. For any (randomized or deterministic) algorithm, there exists a full-feedback episodic MDP problem that has expected regret \( \Omega(\sqrt{HT}) \), even if the Q-values are independent of the state.

Corollary 3.1. The same lower-bound applies to the one-sided-feedback setting.

6 Proof Sketches

The proof sketches here are for HQL-concave, HQL and FQL. See other proofs in Appendix B, C, D. From now on, we denote by \( Q_h^k, V_h^k \) respectively the \( Q_h, V_h \) functions at the beginning of episode \( k \).

6.1 Proof for HQL-concave

Recall \( \alpha_k = (H + 1)/(H + k) \). As in [JAZBJ18] and in [DRZ19], we define weights \( \alpha_h^0 := \prod_{j=1}^{H} (1 - \alpha_j) \), and \( \alpha_h^1 := \alpha_h \prod_{j=1+1}^{H} (1 - \alpha_j) \). Lemma 4 provides some useful properties of \( \alpha_h^1 \).

Lemma 4. The following properties hold for \( \alpha_h^1 \):

1. \( \sum_{t=1}^{\infty} \alpha_t^1 = 1 \) and \( \alpha_0^1 = 0 \) for \( t \geq 1 \); \( \sum_{t=1}^{\infty} \alpha_t^1 = 0 \) and \( \alpha_0^1 = 1 \) for \( t = 0 \).
2. \( \max_{i \in [H]} \alpha_t^i \leq \frac{2H}{t} \) and \( \sum_{t=1}^{\infty} (\alpha_t^i)^2 \leq \frac{4H}{t} \) for every \( t \geq 1 \).
3. \( \sum_{t=1}^{\infty} \alpha_t^i = 1 + \frac{i}{H} \) for every \( i \geq 1 \).
4. \( \frac{1}{\sqrt{t}} \leq \sum_{t=1}^{\infty} \frac{\alpha_t^i}{\sqrt{t}} \leq \frac{1+i}{\sqrt{t}} \) for every \( t \geq 1 \).

The following lemma is a popular technique for regret analysis of reinforcement learning policies.
Lemma 5. (shortcut decomposition) For any policy $\pi$ and any $k$, the per-episode regret:

$$
(V^*_1 - V^{\pi_h}_1)(x^1_h) = \mathbb{E}_\pi \left[ \sum_{t=1}^{H} \left( \max_{y \in A} Q^*_h(x^t_h, y) - Q^*_h(x^t_h, y^k_h) \right) \right].
$$

Shortfall decomposition lets us calculate the regret of our policy by simply summing up the difference between Q-values of the action taken at each step by our policy and of the action OPT would have taken if it was in the same state as us. We need to then take expectation of this random sum, but we get around it by finding high-probability upper-bounds on the random sum as follows:

Recall for any $(x, h, k) \in S \times [H] \times [K]$, and for any $y \in A_h^k$, $\tau^k_h(x, y)$ is the next time stage after $h$ in episode $k$ that allows us to take an action in the running set $A^k_{\tau^k_h(x,y)}$. The time steps in between are "skipped" in the sense that we do not perform Q-value updating or V-value updating during those time steps when we take $y$ at time $(h, k)$. Over all the $h' \in [H]$, we only update Q-values and V-values while it is feasible to choose from the running set. E.g. if no skipping happened, then $\tau^k_h(x, y) = h + 1$. Therefore, $\tau^k_h(x, y)$ is a stopping time. Using the general fact that $\mathbb{E}[M_{\tau^k_h(x,y)}] = M_0$ for any stopping time $\tau$ and martingale $M$, which is a form of optional stopping, our Bellman equation becomes

$$Q^*_h(y) = \mathbb{E}_{\tau^k_h(x,y) \sim \tau^k_h(x,y) \sim \mathcal{D}(\pi)} \left[ r^k_{\tau^k_h(x,y)}(x^\tau^k_h(x,y)) + V^*_h(x^\tau^k_h(x,y)) \right]$$

where again $r_{\tau^k_h,x'}$ denotes the cumulative reward from stage $h$ to $h'$. For notation, we use $\tau^k_h = \tau^k_h(x, y)$ when there is no confusion.

Combined with Lemma 5, this Bellman equation says that under the optimal policy the quantity $V^*_h(x_h) + \sum_{y < h} r^*_y$ is a martingale with respect to the total reward within each episode.

On the other hand, through computing the simulated paths, our algorithm updates the Q functions in the following way for any $x \in S$ and $y \in A_h^k$ at any stage $h$ in any episode $k$:

$$Q^{k+1}_h(y) \leftarrow (1 - \alpha_k)Q^*_h(y) + \alpha_k \left[ \sum_{i=1}^{\infty} \mathbb{E}[r^i_{\tau^k_h(x,y)}(x^i_{\tau^k_h(x,y)}) + V^{k+1}_h(x^i_{\tau^k_h(x,y)})] \right]$$

since we update backward $h = H, \ldots, 1$. Then by Equation 6 and the definition of $\alpha_k$’s, we have

$$Q^*_h(y) = \alpha^0_{k-1}H + \sum_{i=1}^{k-1} \alpha^i_{k-1} \left[ \sum_{x \in S} \mathbb{E}[r^i_{\tau^k_h(x,y)}(x^i_{\tau^k_h(x,y)}) + V^{i+1}_h(x^i_{\tau^k_h(x,y)})] \right],$$

which naturally gives us Lemma 6. For notation, we use $\tau^k_h = \tau^k_h(x, y)$ when there is no confusion.

Lemma 6. For any $(x, h, k) \in S \times [H] \times [K]$, and for any $y \in A_h^k$, we have

$$Q^*_h(y) = \alpha^0_{k-1}H + \sum_{i=1}^{k-1} \alpha^i_{k-1} \left[ \sum_{x \in S} \mathbb{E}[r^i_{\tau^k_h(x,y)}(x^i_{\tau^k_h(x,y)}) + V^{i+1}_h(x^i_{\tau^k_h(x,y)})] \right].$$

By identifying the martingales in the right-hand side of Lemma 6, we are able to bound the difference between our Q-value estimates and the optimal Q-values:

Lemma 7. For any $(x, h, k) \in S \times [H] \times [K]$, and for any $y \in A_h^k$, let $\iota = 9 \log(\lambda)$, we have:

$$
Q^*_h(y) - Q^*_h(y) \leq \alpha^0_{k-1}H + \sum_{i=1}^{k-1} \alpha^i_{k-1} \left[ \sum_{x \in S} \mathbb{E}(V^{i+1}_h - V^*_h)(x^i_{\tau^k_h(x,y)}) + r^i_{\tau^k_h(x,y)}(x^i_{\tau^k_h(x,y)}) \right] + \mathbb{E}[H^3T]^{\frac{1}{k-1}}
$$

with probability at least $1 - 1/(\lambda^8)$, and we can choose $\delta = 2/\sqrt{2}$.

We define $(\delta_h)_{h=1}^{H+1}$ to be a list of values that satisfy the recursive relationship

$$\delta_h = H + (1 + 1/H)\delta_{h+1} + cH^{3/4}, \text{ for any } h \in [H],$$

where again $c$ is a constant.
where $c$ is the same constant as in Lemma \[7\] and $\delta_{H+1} = 0$. Obviously, $\delta_h$ is a decreasing sequence. We will use the fact that each $\delta_h$ can be upper bounded by $4\sqrt{H/5}$. Proof is also in Appendix \[8\].

Now we connect $\{\delta_h\}_{h=1}^{H+1}$ to our upper bound of the difference between our Q-value estimates and the optimal Q-values using Lemma \[7\].

**Lemma 8.** For any $(h, k) \in [H] \times [k]$, $\{\delta_h\}_{h=1}^{H+1}$ is a sequence of values that satisfy
\[
\max_{y \in \mathcal{A}_h^k}(Q_h^k - Q_h^*) (y) \leq \frac{\delta_h}{\sqrt{k - 1}} \quad \text{with high probability at least } 1 - 1/(AT)^5.
\]

Knowing Lemma \[8\] we obtain the validity of our running sets $A_h^k$‘s in the following two lemmas:

**Lemma 9.** For any $(h, k) \in [H] \times [k]$, the optimal action $y_h^*$ is in the running set $A_h^k$ with probability at least $1 - 1/(AT)^5$.

**Lemma 10.** Anytime we can play in $A_h^k$, the optimal Q-value of our action is within $3\delta_h/\sqrt{k - 1}$ of the optimal Q-value of OPT's action, with high probability at least $1 - 2/(AT)^5$.

However, since we might end up not being able to choose from the running set, we also need to show the validity of our action to order no new inventory in that case.

**Lemma 11.** Anytime we cannot play in $A_h^k$, our action that is the feasible action closest to the running set is the optimal action for the state $x$ with high probability at least $1 - 1/(AT)^5$.

Naturally, we want to partition the stages $h = 1, \ldots, H$ in each episode $k$ into two sets, $\Gamma_A^k$ and $\Gamma_B^k$, where $\Gamma_A^k$ denotes the set of all the stages $h$ where we are able to choose from the running set, and $\Gamma_B^k$ denotes the set of all the stages $h$ where we are unable to choose from the running set. So $\Gamma_A^k \cup \Gamma_B^k = [H], \forall k \in [K]$.

Now we can prove Theorem \[2\]. By Lemma \[5\] we have that
\[
V_h^* - V_h^{\pi_k} = \mathbb{E} \left[ \sum_{h=1}^{H} \left( \max_{y \in \mathcal{A}} Q_h^k (y) - Q_h^* (y^k) \right) \right] \leq \mathbb{E} \left[ \sum_{h=1}^{H} \max_{y \in \mathcal{A}} \left( Q_h^k (y) - Q_h^* (y^k) \right) \right] \leq \mathbb{E} \left[ \sum_{h \in \Gamma_A^k} \max_{y \in \mathcal{A}} \left( Q_h^k (y) - Q_h^* (y^k) \right) \right] + \mathbb{E} \left[ \sum_{h \in \Gamma_B^k} \max_{y \in \mathcal{A}} \left( Q_h^k (y) - Q_h^* (y^k) \right) \right].
\]

By Lemma \[11\] the second term is upper bounded by
\[
0 \cdot (1 - \frac{1}{A^5T^5}) + \sum_{h \in \Gamma_B^k} H \cdot \frac{1}{A^5T^5} \leq \sum_{h \in \Gamma_B^k} H \cdot \frac{1}{A^5T^5}.
\]

By Lemma \[8\] the first term is upper-bounded by
\[
\mathbb{E} \left[ \sum_{h \in \Gamma_A^k} \mathcal{O} \left( \frac{\delta_h}{\sqrt{k - 1}} \right) \right] \mathbb{P} \left( \max_{y \in \mathcal{A}} \left( Q_h^k (y) - Q_h^* (y^k) \right) \leq \frac{\delta_h}{\sqrt{k - 1}} \right) + \sum_{h \in \Gamma_B^k} H \cdot \mathbb{P} \left( \max_{y \in \mathcal{A}} \left( Q_h^k (y) - Q_h^* (y^k) \right) > \frac{\delta_h}{\sqrt{k - 1}} \right) \leq \mathcal{O} \left( \sum_{h \in \Gamma_A^k} \frac{\delta_h}{\sqrt{k - 1}} \right) + \mathcal{O} \left( \sum_{h \in \Gamma_B^k} \frac{H}{A^5T^5} \right).
\]

Then the expected cumulative regret between HQL-concave and the optimal policy is:
\[
\text{Regret}_{\text{MDP}}(K) = \sum_{k=1}^{K} (V_1^* - V_1^{\pi_k}) (x_1^k) + \sum_{k=2}^{K} (V_1^* - V_1^{\pi_k}) (x_1^k) \leq \sum_{k=2}^{K} \mathcal{O} \left( \frac{\sqrt{K} + H^2}{k - 1} \right) \leq \mathcal{O} (H^3 \sqrt{T}) \quad \square
\]
The main idea is that, instead of using shortfall decomposition (Lemma 5), we use \(HQL\)-concave value functions. The feasible action sets are in the form of a high probability upper-bound on \(\left( V^k_1 - V^{\pi_k}_1 \right) (x^k_1) \), so that

\[
\text{Regret}(K) = \sum_{k=1}^{K} (V^k_1 - V^{\pi_k}_1) (x^k_1) \leq \sum_{k=1}^{K} (V^k_1 - V^{\pi_k}_1) (x^k_1)
\]

and then upper-bound the right-hand side using martingale properties and recursion.

Because \(FQL\) leverages the full-feedback structure, the extra information allows us to shrink our concentration bounds much more swiftly than existing algorithms, resulting in a significantly improved regret bound. See detailed proof in Appendix E.

### 6.2 Proof for \(HQL\)

\(HQL\) is a special case of \(HQL\)-concave. For \(HQL\), since actions for each stage \(h\) are always accessible at stage \(h\), we can always select from the running set and no skipping happens. Thus, we do not need to assume concavity of the value-functions, or that the feasible action set is an interval, or the monotonicity of \(x_{h+1}(\cdot)\) and \(a_h(\cdot)\). The same proof for \(HQL\)-concave suffices for \(HQL\). Using just Lemmas 5, 7, 9 and 10 we obtain the same bound for \(HQL\).

### 6.3 Proofs for \(FQL\)

For this proof, we adopt similar notations and flow of the proof in [JAZBJ18] (but adapted to our full-feedback setting) to facilitate quick comprehension for readers who are familiar with [JAZBJ18]. The main idea is that, instead of using shortfall decomposition (Lemma 5), we use \(\left( V^k_1 - V^{\pi_k}_1 \right) (x^k_1) \) as a high probability upper-bound on \(\left( V^*_1 - V^{\pi_k}_1 \right) (x^k_1) \), so that

\[
\text{Regret}(K) = \sum_{k=1}^{K} (V^*_1 - V^{\pi_k}_1) (x^k_1) \leq \sum_{k=1}^{K} (V^*_1 - V^{\pi_k}_1) (x^k_1)
\]

and then upper-bound the right-hand side using martingale properties and recursion.

### 7 Example Applications: Inventory Control and More

Inventory control is one of the most fundamental problems in supply chain optimization (see [Zip00], [SLCB14]). Past literature typically studies under the assumption that the demands at all time-steps are i.i.d. ([AJ19], [ZCS18], etc.). Unprecedented, our algorithms solve optimally the episodic version of the problem where the demand distributions are arbitrary within each episode (up to an \(O(\text{poly}(H)\sqrt{\log T})\) factor in regret bound). At the beginning of each stage, the retailer sees the inventory and places an order to raise the inventory level to a certain level. Then an independently distributed random demand is realized. We use the replenished inventory that arrives instantly (with 0 lead time) to satisfy the demand. The remaining inventory becomes the starting inventory for the next stage. We pay a holding cost for each unit of left-over inventory. For the backlogged model, for each unit of excess demand, we pay a backlogging cost and fulfill them in the following stage. For the lost-sales model, the excess demand is lost. We pay a penalty for each unit of lost demand, and the starting inventory for next stage is 0. The base-stock policies are known to be optimal for these models. Inventory control literature typically considers a continuous action space \([0, M]\) and continuous demand space. We discretize the action space \(A\) with step-size \(\frac{M}{K}\), so \(A = T^2\). Discretization incurs additional regret: \(\text{Regret}_{\text{total}} = \text{Regret}_{\text{MDDP}} + \text{Regret}_{\text{gap}}\). By Lipschitzness, it is easy to see \(\text{Regret}_{\text{gap}} \leq O\left(\frac{M}{K^2} \cdot HT\right) = O\left(\frac{M}{K}\right)\). We state the following facts: the backlogged model can be formulated as a full-feedback problem; the lost-sales model can be formulated as a one-sided-feedback problem; for both, the Q-values only depend on the action not the state; the optimal value functions are concave; the feasible action sets are in the form \(A \cap [a, \infty)\) for some \(a = a_h(x_h)\); and \(x_{h+1}(\cdot)\) and \(a_h(\cdot)\) are increasing. See detailed description and literature in Appendix C.

For the backlogged model, \(FQL\) gives \(O(H^2 \sqrt{T \log T})\) regret. Alternatively, \(HQL\)-concave* gives \(O(H^2 \sqrt{T \log T})\) regret. For the lost-sales model, \(HQL\)-concave* gives \(O(H^2 \sqrt{T \log T})\) regret. We can also apply \(FQL\) or \(HQL\)-concave to get the same bounds by discretizing the state space \(S\) with high resolution because the cost functions are Lipschitz and our regret has only a log dependence on the cardinality of the state-action space. Our proof for Theorem 3 in Appendix A shows that for any algorithm, the expected regret for the backlogged model (and the lost-sales model) is lower-bounded by \(\Omega(\sqrt{T \log T})\). Thus, our \(FQL\) and \(HQL\)-concave are optimal up to an \(O(\text{poly}(H) \log T)\) factor. Our results can be generalized to more complex models (with nonzero lead times etc.).

### Comparison with existing Q-learning algorithms

Neither [JAZBJ18] nor [DRZ19] can be directly applied to stochastic inventory control because of the continuous action and state space. Suppose we discretize the action space and the state space optimally for [JAZBJ18] and for [DRZ19].
then applying [JAZBJ18] to the backlogged model gives a regret bound of $O(T^{3/4} \sqrt{\log T})$. Applying [DRZ19] to the easiest episodic backlogged inventory model with optimized aggregation gives us $O(T^{2/3} \sqrt{\log T})$. More details in Appendix D.

Other Examples of Applications Another control problem that HQL can solve is online second-price auctions where the auctioneer needs to decide the reserve price for its product at each round. Each bidder draws a value from its unknown distribution and only submits the bid if the bid is no lower than the reserve price. The auctioneer observes the bids, gives the item to the highest bidder if any, and collects the second highest bid price (including the reserve price) as profits. This is a one-sided-feedback problem where all actions are always accessible to the auctioneer, so this can be solved efficiently and optimally by HQL. Other examples include portfolio management in finance, airlines’ overbook policy, service wait-time forecast, full-feedback robotic control, etc.

8 Numerical Experiments

We compare FQL and HQL-concave on the backlogged episodic inventory control problem against 3 benchmarks: the clairvoyant optimal policy (OPT) that knows the demand distributions beforehand and minimizes the cost in expectation, QL-UCB from [JAZBJ18], and Aggregated QL from [DRZ19]. We assume discrete state and action spaces.

For Aggregated QL from [DRZ19] and for QL-UCB from [JAZBJ18], we optimize by taking the Q-values to be only dependent on the action, thus reducing the state-action pair space. A caveat of Aggregated QL from [DRZ19] is that a good aggregation of the state-action pairs needs to be known beforehand, which is usually unavailable for online problems. We further aggregate the state and actions to be multiples of 1 for [DRZ19] in Table 2.

For HQL, we previously used a large confidence interval (CI) for ease of proof. We do not fine-tune the CI in our experiments, but use a general formula $\sqrt{\frac{H \log(HKA)}{k}}$ for all settings. In real applications, the CI can be optimized. We do not fine-tune the USB-bonus in QL-UCB either.

Below is a summary list for the experiment settings. See more experiments with different settings and models in Appendix C. Each experimental point is run 300 times for statistical significance.

- **Episode length**: $H = 1, 3, 5$.
- **Number of episodes**: $K = 100, 500, 2000$.
- **Demands**: $D_h \sim (10 - h)/2 + U[0, 1]$.
- **Holding cost**: $o_h = 2$.
- **Backlogging cost**: $b_h = 10$.
- **Action space**: $[0, \frac{1}{20}, \frac{2}{20}, \ldots, 10]$.

|          | OPT  | FQL  | HQL-concave | Aggregated QL | QL-UCB |
|----------|------|------|-------------|--------------|--------|
|          | mean | SD   | mean        | mean         | mean   |
|          | mean | SD   | mean        | mean         | mean   | mean   |
| $H$      |      |      | $K$         | $H$          | $K$    |
| 1        | 100  | 88.2 | 103.4       | 6.6          | 125.9  |
|          | 500  | 437.2| 453.1       | 6.6          | 528.9  |
|          | 2000 | 1688.9| 1709.5     | 5.8          | 1929.2 |
| 3        | 100  | 257.4| 313.1       | 7.6          | 433.1  |
|          | 500  | 1274.6| 1336.3     | 10.5         | 1660.2 |
|          | 2000 | 4965.6| 5048.2     | 13.3         | 5700.6 |
| 5        | 100  | 421.2| 528.0       | 10.4         | 752.6  |
|          | 500  | 2079.0| 2204.0     | 13.1         | 2735.1 |
|          | 2000 | 8285.7| 8444.7     | 16.4         | 9514.4 |

As discussed in Section 7, the backlogged model is a full-feedback problem, which allows us to compare all algorithms at once; we present similar results for the lost-sales model in Appendix C. Table 2 shows that FQL and HQL-concave both perform very promisingly in the episodic inventory control problems, with significant advantage over the other two existing algorithms. FQL stays consistently very close to the clairvoyant optimal, while HQL-concave catches up rather quickly to FQL and OPT. Even though HQL-concave is not designed specifically for the backlogged model, it nevertheless obtains very close-to-optimal performance using only one-sided feedback speedily.
9 Conclusion

In this paper, we propose two new Q-learning algorithms that are provably more efficient than existing algorithms in the full-feedback and one-sided-feedback settings. We achieve optimal regret bounds (up to an $O(\text{poly}(H) \log T)$ factor). The fact that our regret bounds are not affected by the possibly huge cardinality of the state-action space gives us not only efficiency, but also a lot more flexibility in the formulation of the MDP. Unprece dentedly, our algorithms obtain $O(\sqrt{T})$ regret for the episodic inventory control problem, and perform promisingly in the numerical experiments. Our work leverages the feedback structures to improve Q-learning, and can be applied to many natural control problems in operations management, finance, robotics and artificial intelligence. Most of the reinforcement learning literature so far has only considered settings with little feedback, while single-stage online learning for bandits considers a plethora of graph-based feedback models. We hope to provoke more efforts to utilize the varied feedback structures intrinsic in real-world problems to augment reinforcement learning. We expect that future extensions of our methods to more varied feedback settings, such as graph-based feedback models, could be very fruitful.

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Appendices

A Proof for Lower Bounds

We construct here an easy instance of the episodic inventory control problem (as in Section 7), for which the regret of any algorithm must be at least $\Omega(\sqrt{HT})$. Our instance work for both Theorem 3 and Corollary 5.1.

Proof. Suppose for any $h$ time step inside an episode, the demand distribution is $h + 100$ units w.p. $0.5 + \frac{1}{\sqrt{K}}$, and $h + 200$ units w.p. $0.5 - \frac{1}{\sqrt{K}}$. Suppose the unit costs for holding, backlogging, and lost-sales penalty are all the same. Suppose we generously allow the algorithm to have the correct prior that the best base stock level is one of these two actions, and the other actions are worse than these two actions. Then for each time stage $h$, our problem of estimating the $Q$ values very well degenerates to the stochastic full-feedback online bandit problem.

It is a well-known result that in this case, each stage $h$ will incur at least a $\Omega(\sqrt{K})$ regret across the $K$ episodes. In particularly, at any time step of any episode, the probability of any algorithm choosing the wrong action is lower-bounded by $\frac{1}{2}$: see Corollary 2.10 in [Sli19]. Then at each time step, the algorithm incur a $\Omega(\frac{1}{\sqrt{K}})$ expected regret. This regret at stage $h$ across the $K$ episodes sum up to $\Omega(\sqrt{K})$ expected regret. Since there are $H$ time steps with demand independent from each other, we have that the regret of this example is lower bounded by $\Omega(H\sqrt{K}) = \Omega(\sqrt{HT})$ regret. Note that even though we assume the algorithm receives full information feedback at each time step, Corollary 2.10 in [Sli19] still applies by scaling the time horizon by a factor of 2, which does not affect the regret bound. Then we put back the $\Theta(M \cdot \max(|\alpha_k|, |b_k|))$ factor (or $\Theta(M \cdot \max(|\alpha_k|, |p_k|))$ factor) because in the Preliminaries we scaled the unit costs down by $\Theta(M)$ to have the reward for each time period bounded by 1.

B Missing Proofs for $HQL$-concave

Proof. (Lemma 4) We prove number (4) by induction. For the base case $t = 1$, we have $\sum_{i=1}^{t} \frac{\alpha_i}{\sqrt{t}} = \alpha_1 = 1$ so the statement holds. For $t \geq 2$, by the relationship $\alpha_i = (1 - \alpha_{t}) \alpha_{t-1}$ for $i = 1, \ldots, t - 1$ we have

$$\sum_{i=1}^{t} \frac{\alpha_i}{\sqrt{t}} = \alpha_1 + (1 - \alpha_{t}) \sum_{i=1}^{t-1} \frac{\alpha_i}{\sqrt{t}}$$

(8)

Assuming the inductive hypothesis holds, on the one hand,

$$\frac{\alpha_t}{\sqrt{t}} + (1 - \alpha_{t}) \sum_{i=1}^{t-1} \frac{\alpha_i}{\sqrt{t}} \geq \frac{\alpha_t}{\sqrt{t}} + \frac{1 - \alpha_t}{\sqrt{t-1}} \geq \frac{\alpha_t}{\sqrt{t}} + \frac{1 - \alpha_t}{\sqrt{t}} = \frac{1}{\sqrt{t}}$$

(9)

where the first inequality holds by the inductive hypothesis. On the other hand,

$$\frac{\alpha_t}{\sqrt{t}} + (1 - \alpha_{t}) \sum_{i=1}^{t-1} \frac{\alpha_i}{\sqrt{t}} \leq \frac{\alpha_t}{\sqrt{t}} + \frac{1 + 1/H}{\sqrt{t-1}} \leq \frac{H + 1}{\sqrt{t}(H + t)} + \frac{(1 + 1/H)\sqrt{t-1}}{H + t}$$

(10)

where the first inequality holds by the inductive hypothesis.

This is a tighter bound than the bound in [JAZBJ18]. For rest of the lemma, see Lemma 4.1 in [JAZBJ18].

The following proof for shortfall decomposition is adapted from Benjamin Van Roy’s reinforcement learning notes for the class MS 338 at Stanford University.

Proof. (Lemma 5) For any policy $\pi_k$, let $y_{h}^{k}$ denote the action the policy $\pi_k$ takes at stage $h$ of episode $k$. Let $R_h$ denote the expected reward of $y_{h}^{k}$.

$$E_n \left[ Q^* \left( x_{h}, y_{h}^{k} \right) \right] = E_n \left[ Z_{h+1} \right]$$

where $Z_{h+1} = \begin{cases} R_h + \max_{y} Q^* \left( x_{h+1}, y \right) & \text{if } h < H \\ R_h & \text{if } h = H \end{cases}$
Therefore,
\[
V_1^* - V_1^\pi_k = \mathbb{E}_n \left[ \max_{a \in A} Q^*(x_1^k, a) - \sum_{h=1}^H R_h \right]
= \mathbb{E}_n \left[ \max_{a \in A} Q^*(x_1^k, a) - \sum_{h=1}^H \left( R_h - Z_{h+1} + Q^* \left( x_{h+1}^k, y_h^k \right) \right) \right]
= \mathbb{E}_n \left[ \sum_{h=1}^H \left( \max_{a \in A} Q^* \left( x_h^k, a \right) - Q^* \left( x_h, y_h^k \right) \right) \right]
\]

Proof. (Lemma 6) From the Bellman optimality equation (3), and the fact that \( \sum_{i=0}^{k-1} \alpha_{k-1} = 1 \), we have

\[
Q_i^k(y) = \alpha_i^0 Q_i^k(y) + \sum_{i=1}^{k-1} \alpha_{k-1} \left[ \mathbb{E}_{x_i', r_i'(y)} \left[ \mathbb{P}(|x_i, y) | V_{i+1}^* \left( x_i', y \right) + V_{i+1}^* \left( x_i', y \right) \right] \right]
\]

Subtracting Equation 5 from this equation, and adding some of the middle terms that cancel with themselves gives us Lemma 6.

Proof. (Lemma 7) Since we assume that given a fixed value \( D_h \), the next state \( x_{h+1}^k(y_h) \) is increasing in \( y_h \), and \( a_h(x) \) is increasing in \( x_h \) for the lower one-sided-feedback problem, we conclude that the (deterministic given \( D_h \)) dynamics are monotone with respect to any simulation starting point \( x_h \). Since the algorithm chooses at least the maximal action in \( A_h^k \) at all times, this implies it can observe the simulated trajectory started from any \( x_h \in A_h^k \) for any \( h, k \in [K] \times [H] \).

Let \( \mathcal{F}_h^i \) be the \( \sigma \)-field generated by all the random variables until episode \( i \), stage \( h \). Then for any \( \tau \in [K] \), \( \left( V_{i+1}^* \left( x_{i+1}^k, y \right) + V_{i+1}^* \left( x_{i+1}^k, y \right) \right) \mathbb{E}_{x_i', r_i'(y)} \mathbb{P}(|x_i, y) | V_{i+1}^* \left( x_i', y \right) + V_{i+1}^* \left( x_i', y \right) ) \) is a martingale difference sequence w.r.t. the filtration \( \{ \mathcal{F}_h^i \}_{i \geq 0} \). Then by Azuma-Hoeffding Theorem, we have that with probability at least \( 1 - (1/AT)^6 \):

\[
\left| \sum_{i=1}^{k-1} \alpha_{k-1} \cdot \left( V_{i+1}^* \left( x_{i+1}^k, y \right) + V_{i+1}^* \left( x_{i+1}^k, y \right) \right) \mathbb{E}_{x_i', r_i'(y)} \mathbb{P}(|x_i, y) | V_{i+1}^* \left( x_i', y \right) + V_{i+1}^* \left( x_i', y \right) ) \right| \leq c \frac{H^3 t}{k - 1}
\]

for any constant \( c \geq 2 \sqrt{2} \).

By union bound, we have with probability at least \( 1 - (1/AT)^6 \) that for any \( x, h, k, y \in A_h^k \),

\[
\left| \sum_{i=1}^{k-1} \alpha_{k-1} \left( V_{i+1}^* \left( x_{i+1}^k, y \right) + V_{i+1}^* \left( x_{i+1}^k, y \right) \right) \mathbb{E}_{x_i', r_i'(y)} \mathbb{P}(|x_i, y) | V_{i+1}^* \left( x_i', y \right) + V_{i+1}^* \left( x_i', y \right) ) \right| \leq c \sqrt{\frac{H^3 t}{k - 1}}
\]

Then Lemma 7 follows immediately this equation and Lemma 6.

Proof. (Upper Bound on \( \delta_h \)'s) We set \( \delta_h = (\delta_h) \cdot \left( 1 + \frac{1}{H} \right) \) and observe that the recurrence implies

\[
d_h = d_{h+1} + H + 2\sqrt{2H^3 t}
\]

(12)

Then from this recursion we see \( d_h \leq H^2 + 2\sqrt{2H^3 t} \) for all \( h \). Since \( d_h, \delta_h \) differ by a constant factor \( (1 + \frac{1}{H}) \), we have \( \delta_h = \frac{H^2 + 2\sqrt{2H^3 t}}{(1 + \frac{1}{H})} \leq 4\sqrt{H^3 t} \).

Proof. (Lemma 8) We prove by backward induction. Note that all of our statements below hold with high probability. In particular, we will use Azuma-Hoeffding no more than \( AT \) times in the below, with each use holding with probability at least \( 1/(AT)^3 \). Under the assumption that each use of Azuma-Hoeffding holds we will obtain the statement of the Lemma. Our proof goes by induction; for the base case \( \delta_{H+1} = 0 \) satisfies the Inequality 8 (actually equality here) with probability 1 based on Bellman equations.
We can bound $\alpha_1$ with high probability at least $\tau$ if we cannot play in the running set, then the running set, and hence w.h.p. the true optimal action, is contained $Q$ the feasible action set at any time is an interval of the form running set, which is inaccessible. Then recall the assumptions that the value functions are concave and that $Q$ Proof.

Now suppose inequality $8$ is true for any $\alpha_1$ such that $\alpha_1 \leq \tau$ and bound $\alpha_0$ by $\frac{1}{\sqrt{k}}$, and bound $\sum_{i=1}^{k-1} \alpha_i \cdot \frac{\delta_{\alpha_i}(x,a)}{\sqrt{i}}$ by $\frac{1}{\sqrt{k}} \delta_{\alpha}(x,a)$ using Lemma $3$.

\[
\max_{y \in A_k^h} \left| (Q_h^k - Q_h^k)(y) \right| \leq \max_{y \in A_k^h} \left[ \left( V_{i+1}^k \tau_{i+1}(x,a) - V_{i+1}^h \tau_{i+1}(x,a) \right) \left( x_{i+1}^k \tau_{i+1}(x,a) - x_{i+1}^h \tau_{i+1}(x,a) \right) \right]
\]

Then

\[
\max_{y \in A_k^h} \left| (Q_h^k - Q_h^k)(y) \right| \leq \max_{y \in A_k^h} \left[ \frac{\alpha_0}{\sqrt{k}} H + \sum_{i=1}^{k-1} \alpha_i \cdot \frac{\delta_{\alpha_i}(x,a)}{\sqrt{i}} + c \sqrt{\frac{H^2}{k}} \right]
\]

where the second inequality is because $\tau_i(x,a) \geq h + 1$ and $\delta_h$’s is a decreasing sequence. The last equality is true based on the recursive definition of $\delta_h$.

\[
\max_{y \in A_k^h} \left| (Q_h^k - Q_h^k)(y) \right| \leq \frac{1}{\sqrt{k}} H + \frac{1 + H/\sqrt{k}}{\sqrt{k-1}} \delta_{\alpha}(x,a) + c \sqrt{\frac{H^2}{k}}
\]

Proof. (Lemma $9$) Recall for any $(x, h, k), y_h^k = \arg \max_{y \in A_k^h} Q_h^k(y)$ in HQL-concave. Suppose $y_h \not\in A_k^h$, then $Q_h^k(y_h^k) - Q_h^k(y_h^k) - \frac{\delta_h}{\sqrt{k-1}}$. Then we need either $Q_h^k(y_h^k) < Q_h^k(y_h^k) - \frac{\delta_h}{\sqrt{k-1}}$ or $Q_h^k(y_h^k) > Q_h^k(y_h^k) + \frac{\delta_h}{\sqrt{k-1}}$. Thus by Lemma $8$ Prob($y_h \not\in A_k^h$) $\leq \frac{1}{(AT)^3}$.

Proof. (Lemma $10$) Lemma $8$ says for any $y \in A_k^h$, our estimated $Q_h^k(y)$ differs from the optimal value $Q_h^k(y)$ by at most $\frac{\delta_h}{\sqrt{k-1}}$ with high probability at least $1 - \frac{1}{(AT)^3}$. Therefore, the optimal Q-value of the OPT’s action $Q^*(y_h^k)$ is at most $\frac{\delta_h}{\sqrt{k-1}}$ more than the estimated Q-value of our estimated best arm $Q_h^k(y_h^k)$, with high probability at least $1 - \frac{1}{(AT)^3}$. Any action we take in $A_k^h$ has an estimated Q-value no more than $\frac{8\sqrt{H^2}}{\sqrt{k-1}} = \frac{3\delta_h}{\sqrt{k-1}}$, lower than $Q_h^k(y_h^k)$ base on our algorithm. Therefore, the optimal Q-value of OPT’s action $Q^*(y_h^k)$ is at most $\frac{3\delta_h}{\sqrt{k-1}}$ more than the estimated Q-value of any action $y \in A_k^h$, with high probability at least $1 - \frac{1}{(AT)^3}$. Then again, by Lemma $8$ we know that the optimal Q-value of OPT’s action $Q^*(y_h^k)$ is at most $\frac{4\delta_h}{\sqrt{k-1}}$ more than the optimal Q-value of any action in $A_k^h$, with high probability at least $1 - \frac{2}{(AT)^3}$.

Proof. (Lemma $11$) From Lemma $9$ we know that with with probability at least $1 - \frac{1}{(AT)^3}$, OPT is in the running set, which is inaccessible. Then recall the assumptions that the value functions are concave and that the feasible action set at any time is an interval of the form $A \cap [a, \infty)$ for some $a$ dependent on the state. So if we cannot play in the running set, then the running set, and hence w.h.p. the true optimal action, is contained in $(-\infty, a]$. By concavity, this implies that the closest feasible action to the running set is optimal in this case w.p. at least $1 - \frac{1}{(AT)^3}$.

Extension to a Totally-Order Set Instead of restricting the action set to be a compact subset of $\mathbb{R}$, we can have the action set be any finite totally-ordered set. Then instead of assuming concavity on the value functions, we assume that the value functions are unimodal (monotonically non-decreasing and then monotonically non-increasing). Then the rest of the proofs are unchanged.
C Missing Proofs for Inventory Control

We gave a more detailed description of the backlogged model and the lost-sales model of the episodic stochastic inventory control problems.

Backlogged At the beginning of each stage $h$, the retailer sees the inventory $x_h \in \mathbb{R}$ and places an order to raise the inventory level up to $y_h > x_h$. Without loss of generality, we assume the purchasing cost is 0 (see proof in Appendix C). Replenishment of $y_h - x_h$ units arrive instantly. Then an independently distributed random demand $D_h$ from unknown distribution $F_h$ is realized. We use the replenished inventory $y_h$ to satisfy demand $D_h$. At the end of stage $h$, the remaining inventory becomes the starting inventory for the next time period $x_{h+1} = y_h - D_h$. When $D_h$ exceeds $y_h$, the additional demand is backlogged, and we pay a backlogging cost $b_h > 0$ for each unit of demand we fail to meet, and then we try to fulfill them in the following stages. For each unit of left-over inventory at the end of stage $h$, we pay a holding cost $o_h > 0$. The reward for period $h$ is the negative cost:

$$r_h(x_h, y_h) = -C_h(x_h, y_h) = -(c_h(y_h - x_h) + o_h(y_h - D_h)^+ + p_h(D_h - y_h)^+)$$

Note that the backlogged model is a full-feedback problem.

Lost-Sales The only difference is that when $D_h > y_h$, the additional demand is lost instead of backlogged. We pay a penalty of $p_h > 0$ for each unit of lost demand. Then the starting inventory for next time period is $x_{h+1} = 0$. The reward for period $h$ is the negative cost:

$$r_h(x_h, y_h) = -C_h(x_h, y_h) = -(c_h(y_h - x_h) + o_h(y_h - D_h)^+ + p_h(D_h - y_h)^+)$$

One caveat is we cannot observe the realized reward at each time period because the lost demand $(D_h - y_h)^+$ is unobserved for the lost-sales model. However, we can use a pseudo-cost $C_h(x_h, y_h) = o_h(y_h - D_h)^+ + p_h \min(y_h, D_h)$ that will leave the regret of any policy against the optimal policy unchanged (See [AI9] [YLS19]). Then we can observe our reward and the rewards of all the actions smaller than the action $y_h$ we take, which is consistent with the (lower) one-sided feedback setting.

As we have discussed in Section 7, we discretize the action space of taking base-stock levels in $[0, M]$ with step-size $\frac{M}{T}$, and leave the state space continuous as it is (or we can discretize it further if we want). Since the base-stock policy is to decide an inventory level to order up to, assuming the new level is no lower than the current level, it is easy to see that the reward at any time only depends on the action, not the state. The feasible action set at any time step is an interval $A \cap [a, \infty)$ for some $a = a_h(x_h)$. And the next state $x_{h+1}(c)$ and $a_h(\cdot)$ are monotonely non-decreasing.

Lemma 12. For any $h \in [H]$, the optimal $V$-value function $V^*_h(x)$ is concave in $x$, and the optimal $Q$-value function $Q^*_h(y)$ is concave in $y$. This is true for the lost sales and the backlogged models.

Proof. (Lemma 12) We prove this by backward induction. The base case is $Q^*_H(x, y)$ and $V^*_H(x)$. Since $Q^*_H(y)$ is just the expectation of a one time reward for the last period, we know that it is $Q^*_H(x, y) = r_H(x, y, D_H) = [-o_H(y - D_H)^+ + p_H \min(y, D_H)]$. This function is obviously concave in $y$. Note that the Q-values are not affected by $x$ for the inventory control problems. Since $V^*_H(x) = \max_{y \geq x} Q^*_H(x, y)$, the graph of $V^*_H(x)$ is constant on the left side of $x = \arg \max_y Q^*_H(x, y)$, and then goes down with a slope of $o_H$ on the right side of $x = \arg \max_y Q^*_H(x, y)$. So $V^*_H(x)$ is obviously also concave.

Now suppose $Q^*_h(x, y)$ and $V^*_h(x)$ are concave. It remains to show concavity of $Q^*_h(x, y)$ and $V^*_h(x)$.

We know $Q^*_h(x, y) = \mathbb{E}[V^*_{h+1}(y - D_h) + r_h(x, y, D_h)]$. We know $r_h(x, y, D_h)$ is concave in $y$ for the same reason that $Q^*_H(x, y)$ is concave. We know that $V^*_h(x)$ is concave in $x$ from our induction hypothesis, which means $V^*_{h+1}(y - D_h)$ is concave in $y$ for any value of $D_h$. Therefore, $\mathbb{E}[V^*_{h+1}(y - D_h) + r_h]$ is also concave, being a weighted average of concave functions. So we know $Q^*_h(x, y)$ is also concave in $y$. Then again $V^*_h(x) = \max_{y \geq x} \mathbb{E}[Q^*_h(x, y)]$ is concave for the same reason why $V^*_H(x)$ is concave.

Proof. (Assumption of 0 Purchasing Costs) We want to show that for the episodic lost-sales (and similarly for the backlogged) model, we can amortize the unit purchasing costs $c_h$ into unit holding costs $o_h$ and unit lost-sales penalty $p_h$, so that without loss of generality we can assume 0 purchasing costs.

$$\forall h \geq 2, y_h - x_h = y_h - D_h + D_h - x_h = (y_h - D_h)^+ - (D_h - y_h)^+ + D_h - (y_{t-1} - D_{t-1})^+ \tag{15}$$

Let $c_h$ denote the unit purchasing cost, then the total sum of costs starting from stage 2 is

$$\sum_{h=2}^{H} \left( c_h(y_h - x_h) + o_h(y_h - D_h)^+ + p_h(D_h - y_h)^+ \right) = \sum_{h=2}^{H} \left( c_h D_h - c_h(y_h - x_{h-1})^+ + (o_h + c_h)(y_h - D_h)^+ + (p_h - c_h)(D_h - y_h)^+ \right)$$
And the cost of stage 1 is equal to \( o_1(y_1 - D_1)^+ + p_1(D_1 - y_1)^+ + c_1((y_1 - D_1)^+ - (D_1 - y_1)^+ + D_1 - x_1) \).

Let \( c_{H+1} \geq 0 \) denote the salvage price at which we sell the remaining inventory \((y_H - D_H)^+\) at the end of each episode. Then the total sum of costs from stage 1 to \( H \) is

\[
\sum_{h=2}^{H} \left( c_h D_h - c_h(y_{h-1} - D_{h-1})^+ + (o_h + c_h)(y_h - D_h)^+ + (p_h - c_h)(D_h - y_h)^+ \right) \\
+ c_1(y_1 - D_1)^+ - c_1(D_1 - y_1)^+ + c_1 D_1 - c_1 x_1 + o_1(y_1 - D_1)^+ + p_1(D_1 - y_1)^+ - c_{H+1}(y_H - D_H)^+
\]

\[
= \sum_{h=2}^{H} \left( c_h D_h - c_h(y_{h-1} - D_{h-1})^+ + (o_h + c_h)(y_h - D_h)^+ + (p_h - c_h)(D_h - y_h)^+ \right) \\
+ c_1(y_1 - D_1)^+ - c_1(D_1 - y_1)^+ + c_1 D_1 - c_1 x_1 + o_1(y_1 - D_1)^+ + p_1(D_1 - y_1)^+ - c_{H+1}(y_H - D_H)^+
\]

\[
= \sum_{h=1}^{H} c_h D_h + \sum_{h=1}^{H} \left( (o_h + c_h - c_{h+1})(y_h - D_h)^+ + (p_h - c_h)(D_h - y_h)^+ \right) - c_1 x_1
\]

Since \( \sum_{h=1}^{H} c_h D_h \) and \(-c_1 x_1\) are fixed costs independent of our action, we can take them out of our consideration. Then we can effectively consider the cost of each stage \( h \) is just \( o'_h(y_h - D_h)^+ + p'_h(D_h - y_h)^+ \), where \( o'_h = o_h + c_h - c_{h+1} \) is the adjusted holding cost, and \( p'_h = p_h - c_h \) is the adjusted lost-sales penalty.

\[ \square \]

## D Comparison with Existing Q-Learning Algorithms

For \([JAZBJ18]\), suppose we discretize the state and action space optimally with step-size \( \epsilon_1 \) to apply \([JAZBJ18]\) to the backlogged/lost-sales episodic inventory control problem with continuous action and state space. Then the \( \text{Regret}_{\text{gap}} \) we get is \( \epsilon_1 T \). Applying the results of \([JAZBJ18]\), their \( \text{Regret}_{\text{MDP}} \) is \( O(\sqrt{H^3SAT}) = O(\sqrt{\frac{1}{\epsilon_1^3} \cdot T - 1}) \). To minimize \( \text{Regret}_{\text{total}} \), we balance the \( \text{Regret}_{\text{MDP}} \) and \( \text{Regret}_{\text{gap}} \) by setting \( \frac{1}{\epsilon_1^3} \cdot T = \epsilon_1 T \), which gives \( \epsilon_1 = \frac{1}{\sqrt{H^3 \log T}} \), giving us an optimized regret bound of \( O(T^{\frac{1}{2}} \sqrt{H^3 \log T}) \).

For \([DRZ19]\), suppose we discretize the state and action space optimally with step-size \( \epsilon_2 \) to apply \([DRZ19]\) to the backlogged/lost-sales episodic inventory control problem. We also optimize aggregation using the special property of these inventory control problems that the Q-values only depend on the action not the state, so we aggregate all the state-action pairs \((x_1, y), (x_2, y)\) into one aggregated state-action pair. This 0-error aggregation helps reduce the aggregated state-action space. Then the optimized regret bound in \([DRZ19]\) is \( O(\sqrt{H^4 T \log T + cT}) \). We minimize \( \text{Regret}_{\text{total}} \) by balancing the two terms and take \( \epsilon = \frac{1}{\sqrt{H^4 \log T}} \), obtaining an optimized regret bound of \( O(T^{\frac{1}{2}} \sqrt{H^4 \log T}) \).

## E Missing Proofs for FQL and FQL*

### Proof for Theorem 1

The not-“cheap” proofs for FQL and FQL* are the same. We state FQL* below as Algorithm 4 which applies when the Q-values only depend on the action not the state. FQL* is the same as FQL except that it works with continuous state space directly for problems like inventory control, so we don’t need to discretize the state space.

For the proof for FQL and FQL*, we adopt similar notations and flow of the proof in \([JAZBJ18]\) (but adapted to our full-feedback setting) to facilitate quick comprehension for readers who are familiar with \([JAZBJ18]\).

Like \([JAZBJ18]\), we use \([P_h V_{h+1}] (x, y) := E_{x' \sim \mathcal{D}(x, y)} V_{h+1} (x') \). Then the Bellman optimality equation becomes \( Q_h(x, y) = (r_h + \mathbb{P}_h V_{h+1} (x, y)) \).

Similar to Equation 4 but without “skipping”, FQL updates the Q functions in the following way for any \((x, y) \in A\) at any time step:

\[
Q_{h+1}^k (x, y) \leftarrow (1 - \alpha_k)Q_{h+1}^k (x, y) + \alpha_k [r_{h+1}^k (x, y) + V_{h+1}^k (x_{h+1})]
\]

(16)

Then by the definition of weights \( \alpha_h^k \), we have

\[
Q_{h+1}^k (x, y) = \alpha_{k-1}^{h+1} H + \sum_{j=1}^{k-1} \alpha_{k-1}^j \left[ r_h^j (x, y) + V_{h+1}^j \left( x_{h+1}^j \right) \right]
\]

(17)

The following two lemmas are variations of Lemma 6 and Lemma 7.
Algorithm 4 Full-Q-Learning*.

Initialization: $Q_h(y) \leftarrow H, \forall(y, h) \in A \times [H].$

for $k = 1, \ldots, K$ do

Receive $x^*_i$:

for $h = 1, \ldots, H$ do

Take action $y^*_h \leftarrow \arg \max_{y \text{ feasible}} Q_h(y)$; and observe realized $\hat{D}_h^k$.

for $y \in A$ do

Update $V_{h+1}(x'_{h+1}(y, \hat{D}_h^k)) \leftarrow \max_{y \text{ feasible}} Q_{h+1}(y)$;

Update $Q_h(y) \leftarrow (1 - \alpha_k)Q_h(y) + \alpha_k \left[ r_h(x, y, \hat{D}_h^k) + V_{h+1}(x'_{h+1}(x, y, \hat{D}_h^k)) \right]$;

Update $x_{h+1}^k \leftarrow x'_{h+1}(x^*_h, y^*_h, \hat{D}_h^k)$;

end for

end for

end for

Lemma 13. For any $(x, y, h, k) \in \mathcal{S} \times A \times [H] \times [K]$, we have

$$Q_h^k - Q_h^k(x, y) = \alpha_{k-1}^h (H - Q_h^k(x, y)) + \sum_{i=1}^{k-1} \alpha_{k-1}^i \left[ (V_{h+1}^i - V_{h+1}^i) (x_{h+1}^i) + r_h^i - \mathbb{E}[r_h^i] \right]$$

Proof. From the Bellman optimality equation $Q_h^k(x, y) = \mathbb{E}[r_h(x, y)] + \mathbb{P}_h V_{h+1}^k(x, y)$, our notation $[\hat{P}_h V_{h+1}] (x, y) := V_{h+1}(x_{h+1}^i)$, and the fact that $\sum_{i=1}^{k-1} \alpha_{k-1}^i = 1$, we have

$$Q_h^k(x, y) = \sum_{i=1}^{k-1} \alpha_{k-1}^i \left[ (V_{h+1}^i - V_{h+1}^i) (x_{h+1}^i) + r_h^i - \mathbb{E}[r_h^i] \right]$$

Subtracting Equation (17) from this equation gives us Lemma [13].

Lemma 14. For any $p \in (0, 1)$, with probability at least $1 - p$, for any $(x, y, h, k) \in \mathcal{S} \times A \times [H] \times [K]$, let $\iota = \log(SAT/p)$, we have for some absolute constant $c$:

$$0 \leq \left( Q_h^k - Q_h^k \right)(x, y) \leq \alpha_{k-1}^h H + \sum_{i=1}^{k-1} \alpha_{k-1}^i \left[ (V_{h+1}^i - V_{h+1}^i) (x_{h+1}^i) \right] + c\sqrt{\frac{H^3}{k - 1}}$$

(18)

Proof. For any $i \in [k]$, recall that episode $i$ is the episode where the state-action pair $(x, y)$ was updated at stage $h$ for the $i$th time. Let $\mathcal{F}_i$ be the $\sigma$-field generated by all the random variables until episode $i$, stage $h$. Then for any $\tau \in [K]$, $\{([\hat{P}_h - \mathbb{P}_h] V_{h+1}^i |(x, y) + r_h^i - \mathbb{E}[r_h^i]) \}_{i=1}^{\tau}$ is a martingale difference sequence w.r.t. the filtration $\{\mathcal{F}_i\}_{i \geq 0}$. Then by Azuma-Hoeffding Theorem, we have that with probability at least $1 - p/SAT$:

$$\sum_{i=1}^{k-1} \alpha_{k-1}^i \left[ (\hat{P}_h - \mathbb{P}_h) V_{h+1}^i \right] (x, y) + r_h^i - \mathbb{E}[r_h^i] \leq \frac{cH}{2} \sqrt{\sum_{i=1}^{k-1} \alpha_{k-1}^i} \leq c\sqrt{\frac{H^3}{k - 1}}$$

(19)

for some constant $c$.

Now we union bound over states, actions and times, we see that with probability at least $1 - p$, we have

$$\sum_{i=1}^{k-1} \alpha_{k-1}^i \left[ (\hat{P}_h - \mathbb{P}_h) V_{h+1}^i \right] (x, y) + r_h^i - \mathbb{E}[r_h^i] \leq c\sqrt{\frac{H^3}{k - 1}}$$

(20)

Then the right-hand side of Lemma [14] follows from Lemma [13] and Inequality [20]. The left-hand side also follows from Lemma [13] and Inequality [20] using induction on $h = H, H - 1, \ldots, 1$.

Proof. (Theorem) Define $\Delta_h^k := (V_h^k - V_h^k) (x_h^k)$ and $\phi_h^k := (V_h^k - V_h^k) (x_h^k)$.
By Lemma 19 with 1 − p probability, $Q_k^h \geq Q_h^*$ and thus $V_k^h \geq V_h^*$. Thus the total regret can be upper bounded:

$$\text{Regret}(K) = \sum_{k=1}^{K} \left( V^*_1 - V^*_{i_k} \right) (x^k_1) \leq \sum_{k=1}^{K} \left( V^*_1 - V^*_{i_k} \right) (x^k_1) = \sum_{k=1}^{K} \Delta^k_1$$

The main idea of the rest of the proof is to upper bound $\sum_{k=1}^{K} \Delta^k_1$ by the next step $\sum_{k=1}^{K} \Delta^k_{h+1}$, which gives a recursive formula to obtain the total regret. Let $y_h^k$ denote the base stocks taken at stage $h$ of episode $k$, which means $y_h^k = \max_Q Q_h^k(y')$.

$$\Delta^k_1 = \left( V^*_1 - V^*_{i_k} \right) (x^k_1) \leq \left( Q^*_h - Q_h^k \right) (x^k_h, y_h^k)$$

$$= \left( Q^*_h - Q_h^k \right) (x^k_h, y_h^k) + (Q^*_h - Q^*_{i_k}) (x^k_h, y_h^k)$$

$$\leq \alpha^0_{k-1} H + \sum_{i=1}^{k-1} \alpha^i_{k-1} \phi^i_{h+1} + c \sqrt{\frac{H y^k}{k-1}} + \left[ (\mathbb{P}_h (V^*_{i_{h+1}} - V^*_{h+1})) \right] (x^k_h, y_h^k)$$

$$= \alpha^0_{k-1} H + \sum_{i=1}^{k-1} \alpha^i_{k-1} \phi^i_{h+1} + c \sqrt{\frac{H y^k}{k-1}} + \left[ (\mathbb{P}_h - \hat{\mathbb{P}}_h) (V^*_{i_{h+1}} - V^*_{h+1}) \right] (x^k_h, y_h^k)$$

$$\leq \alpha^0_{k-1} H + \sum_{i=1}^{k-1} \alpha^i_{k-1} \phi^i_{h+1} + c \sqrt{\frac{H y^k}{k-1}} - \phi^k_{h+1} + \Delta^k_{h+1} + \xi^k_{h+1}$$

(21)

where $\xi^k_{h+1} := \left[ (\mathbb{P}_h - \hat{\mathbb{P}}_h) (V^*_{i_{h+1}} - V^*_{h+1}) \right] (x^k_h, y_h^k)$ is a martingale difference sequence. Inequality (1) holds because $V^*_1 (x^k_1) \leq \max_{x \text{ feasible}} (x^k_h, y^k_h) = Q^*_h (x^k_h, y^k_h)$, and Inequality (2) holds by Lemma 14 and the Bellman equations. Inequality (3) holds by definition $\Delta^k_{h+1} = \phi^k_{h+1} = (V^*_{i_{h+1}} - V^*_{h+1}) (x^k_{h+1})$.

In order to compute $\sum_{k=1}^{K} \Delta^k_1$, we need to first bound the first term in Equation 21. Since $\alpha^0_k = 0, \forall k \geq 1$, we know that $\sum_{k=1}^{K} \alpha^0_{k-1} H \leq H$.

Now we bound the sum of the second term in Equation 8 over the episodes by regrouping:

$$\sum_{k=1}^{K} \sum_{i=1}^{k-1} \alpha^i_{k-1} \phi^i_{h+1} \leq \sum_{i=1}^{\infty} \phi^i_{h+1} \sum_{i=1}^{k-1} \alpha^i_{k-1} \leq \sum_{i=1}^{\infty} \phi^i_{h+1} \sum_{i=1}^{\infty} \alpha^i_{k-1} \leq \left( 1 + \frac{1}{H} \right) \sum_{k=1}^{K} \phi^k_{h+1}$$

(22)

where the last inequality uses $\sum_{i=1}^{\infty} \alpha^i_{k-1} = 1 + \frac{1}{H}$ for every $i \geq 1$ from Lemma 14.

Plugging the above Equation 22 and $\sum_{k=1}^{K} \alpha^0_k H \leq H$ back into Equation 8 we have:

$$\sum_{k=1}^{K} \Delta^k_1 \leq H + \sum_{k=2}^{K} \Delta^k_1 \leq H + H + \left( 1 + \frac{1}{H} \right) \sum_{k=1}^{K} \phi^k_{h+1} - \sum_{k=1}^{K} \Delta^k_{h+1} + \sum_{k=1}^{K} \alpha^i_{k-1} \sum_{i=1}^{\infty} \alpha^i_{k-1} \sum_{k=1}^{K} \phi^i_{h+1} + \sum_{k=1}^{K} \Delta^k_{h+1} + \sum_{k=2}^{K} \sqrt{\frac{H y^k}{k-1}} + \sum_{k=2}^{K} \xi^k_{h+1}$$

$$\leq 2H + \phi^k_{h+1} + \frac{1}{H} \sum_{k=2}^{K} \phi^k_{h+1} + \sum_{k=2}^{K} \Delta^k_{h+1} + \sum_{k=2}^{K} \sqrt{\frac{H y^k}{k-1}} + \sum_{k=2}^{K} \xi^k_{h+1}$$

$$\leq 3H + \left( 1 + \frac{1}{H} \right) \sum_{k=2}^{K} \Delta^k_{h+1} + \sum_{k=2}^{K} \sqrt{\frac{H y^k}{k-1}} + \sum_{k=2}^{K} \xi^k_{h+1}$$

where the last inequality uses $\phi^k_{h+1} \leq \Delta^k_{h+1}$. By recursing on $h = 1, 2, \ldots, H$, and because $\Delta^K_{h+1} = 0$:

$$\sum_{h=1}^{H} \sum_{k=1}^{K} \Delta^k_{h+1} \leq \mathcal{O} \left( \sum_{h=1}^{H} \sum_{k=1}^{K} \left( c \sqrt{\frac{H y^k}{k-1}} + \xi^k_{h+1} \right) \right)$$

where $\sum_{h=1}^{H} \sum_{k=1}^{K} \sqrt{\frac{H y^k}{k-1}} = \mathcal{O}(H \sqrt{H^3 \log(SAT/p)} \sqrt{R}) = \tilde{\mathcal{O}}(\sqrt{HT^3})$.

On the other hand, by Azuma-Hoeffding inequality, with probability 1 − p, we have

$$\left| \sum_{h=1}^{H} \sum_{k=1}^{K} \xi^k_{h+1} \right| = \left| \sum_{h=1}^{H} \sum_{k=1}^{K} \left[ (\mathbb{P}_h - \hat{\mathbb{P}}_h) (V^*_{i_{h+1}} - V^*_{h+1}) \right] (x^k_h, y^k_h) \right| \leq cH \sqrt{T} \leq \tilde{\mathcal{O}}(\sqrt{HT^3})$$

(24)

which establishes $\sum_{k=1}^{K} \Delta^k_1 \leq \tilde{\mathcal{O}}(H^2 \sqrt{T})$. □
For $FQL^*$, essentially the same proof goes through. We can optimize the log factor by taking $\epsilon = 9 \log(AT)$ as in $HQL$ and $HQL$-concave instead of $\epsilon = 9 \log(SAT)$ because the Q-values do not depend on the state and we don’t need to union bound over the state space.

**Extension to More Complex Inventory Models** Our algorithms can actually achieve (near) optimal regret for many more complex models for inventory control beyond the single-product backlogged model with no lead time that we have discussed so far.

We can actually manage the inventory of multiple types of products for the backlogged model. There can be a fixed joint ordering cost if any amount of product of any type is ordered. The products arrive after some lead time instead of immediately. The lead times can be different for different types of products. There can also be an order capacity constraint that dictates an upper bound on the amount ordered at each time period. All of these constraints can vary over time within each episode. We can apply $FQL^*$ where each action is to decide how much new inventory to order.

If there is one single product, then $HQL$-concave$^*$ can be applied to optimally solve the backlogged model with nonzero lead times or with an order capacity limit, because the (transformed) base-stock policies are optimal for these models (See [SLCB14], [CSL19]). The cost/reward for a feasible action of taking base stock level $y_i \geq x_i$ at each period does not depend on the state $x_i$ (See [SLCB14], [CSL19]). E.g., for the basic model with positive lead times, the inventory position $x_i$ is defined to be the inventory on-hand $I_i$ plus the inventory in transit, and the model can be transformed into a zero lead time backlogged model (See [SLCB14]).

**F The $HQL^*$ and $HQL$-concave$^*$ cases of Theorem 2**

We do not write out the pseudo-code and proofs for $HQL^*$ and $HQL$-concave$^*$ because they are the same as $HQL$ and $HQL$-concave, except that they are designed for problems where the Q-values only depend on the actions not the states (such as inventory control). $HQL^*$ and $HQL$-concave$^*$ can be obtained by adapting $HQL$ and $HQL$-concave exactly as we went from $FQL$ to $FQL^*$. They can be proved with the same proofs for $HQL$-concave and $HQL$ in Section 6.

**G More Numerical Experiments**

We show more numerical experiment results to demonstrate the performance of $FQL$ and $HQL$-concave. In Table 3 we use again the backlogged model to compare $FQL$ and $HQL$-concave against $OPT$, Aggregated $QL$ and $QL$-UCB, but with a different set of parameter than in Section 8. In Table 4 and 5 we use the lost-sales model to compare $HQL$-concave against $OPT$, Aggregated $QL$ and $QL$-UCB.

For Tables 3 and 4 we make the demand distribution less adversarial: with each step in the episode, we have demands that are increasing in expectation. However, we let the upper bound of base-stock levels increase with the episode length $H$, which is more adversarial. For Table 5 we use the same demand distributions and base-stock upper bound as in Table 2 in Section 8.

We run each experimental point 300 times for statistical significance.

- **Episode length**: $H = 1, 3, 5$.
- **Number of episodes**: $K = 100, 500, 2000$.
- **Demands**: $D_h \sim U[0, 1] + h$.
- **Holding Cost**: $c_h = 2$.
- **Backlogging Cost**: $b_h = 10$.
- **Action Space**: $[0, \frac{1}{20}, \frac{2}{20}, \ldots, 2H]$.

Table 3: Comparison of cumulative costs for backlogged episodic inventory control with less adversarial demands and increasing base-stock upper bounds

|     | OPT  | FQL  | HQL-concave | Aggregated QL | QL-UCB |
|-----|------|------|-------------|---------------|--------|
| **H** | **K** | mean | SD | mean | SD | mean | SD | mean | SD | mean | SD |
| 1   | 100  | 89.1 | 3.8 | 97.1 | 5.5 | 117.3 | 16.8 | 160.1 | 8.3 | 327.5 | 18.8 |
| 1   | 500  | 420.2 | 4.2 | 431.2 | 4.2 | 507.8 | 45.6 | 732.7 | 22.1 | 825.4 | 10.9 |
| 1   | 2000 | 1669.8 | 4.8 | 1691.2 | 6.6 | 1883.6 | 99.7 | 2546.2 | 32.6 | 2952.1 | 19.9 |
| 3   | 100  | 253.0 | 6.6 | 304.6 | 9.6 | 423.8 | 15.4 | 510.9 | 14.4 | 1712.0 | 19.1 |
| 3   | 500  | 1252.4 | 7.0 | 1314.3 | 11.9 | 1611.0 | 43.9 | 1703.2 | 16.1 | 4603.7 | 101.6 |
| 3   | 2000 | 5056.2 | 6.5 | 5128.7 | 10.2 | 5702.8 | 104.7 | 6188.0 | 14.1 | 15088.6 | 132.0 |
| 5   | 100  | 415.9 | 6.4 | 543.6 | 11.0 | 762.4 | 30.0 | 3011.8 | 1294.6 | 6101.9 | 357.6 |
| 5   | 500  | 2077.1 | 12.7 | 2224.6 | 15.6 | 2746.3 | 113.7 | 10277.1 | 6888.5 | 11763.6 | 2982.5 |
| 5   | 2000 | 8394.3 | 6.2 | 8557.2 | 11.1 | 9630.4 | 356.6 | 30489.8 | 31232.4 | 39873.8 | 7210.1 |
Again for Aggregated QL from [DRZ19] and for QL-UCB from [JAZBJ18], we optimize by taking the Q-values to be only dependent on the action, thus reducing the state-action pair space.

As in Section 8 we do not fine-tune the confidence interval for HQL-concave for different settings, but use a general formula $\sqrt{\frac{H \log(HKA)}{k}}$ as the confidence interval for all settings. We also do not fine-tune the UCB bonus defined in QL-UCB (see [JAZBJ18]).

A caveat of Aggregated QL from [DRZ19] is that we need to know a good aggregation of the state-action pairs beforehand, which is usually unavailable for online problems. For using Aggregated QL in Table 3 and 4 we further aggregate the state and actions to be multiples of $\frac{1}{2}$. For using Aggregated QL in Table 5 (and also in Section 8) we further aggregate the state and actions to be multiples of 1.

Table 4: Comparison of cumulative costs for lost-sales episodic inventory control with less adversarial demands and increasing base-stock upper bounds

| H   | K     | OPT mean | HQL-concave mean | Aggregated QL mean | QL-UCB mean |
|-----|-------|----------|------------------|--------------------|-------------|
| 100 | 3     | 89.1     | 117.3            | 201.7              | 291.7       |
| 1   | 500   | 420.2    | 507.8            | 1002.8             | 1452.8      |
| 2000| 6.4   | 1669.8   | 1883.6           | 4012.1             | 5812.1      |
| 100 | 4.4   | 253.0    | 443.8            | 1902.8             | 2071.4      |
| 3   | 1253.5| 7.0      | 1730.7           | 9534.0             | 10375.7     |
| 2000| 6.5   | 5056.2   | 6163.4           | 38139.6            | 41504.9     |
| 100 | 6.4   | 415.9    | 780.6            | 5716.6             | 5902.8      |
| 5   | 7.2   | 2077.1   | 2926.0           | 28510.7            | 29385.1     |
| 2000| 12.7  | 8394.3   | 10560.1          | 114010.7           | 117481.6    |

Table 5: Comparison of cumulative costs for lost-sales episodic inventory control with the original demands and base-stock upper bounds

| H   | K     | OPT mean | HQL-concave mean | Aggregated QL mean | QL-UCB mean |
|-----|-------|----------|------------------|--------------------|-------------|
| 100 | 3     | 88.2     | 125.9            | 705.4              | 895.4       |
| 1   | 500   | 437      | 528.9            | 3506.1             | 4456.1      |
| 2000| 6.4   | 1688.9   | 1929.2           | 14005.6            | 17805.6     |
| 100 | 3.2   | 257.4    | 448.4            | 2405.6             | 2975.6      |
| 3   | 6.1   | 1274.6   | 1746.7           | 12009.3            | 14859.3     |
| 2000| 8.3   | 4965.6   | 6111.2           | 47926.4            | 59326.4     |
| 100 | 3.3   | 421.2    | 774.6            | 4497.4             | 5447.4      |
| 5   | 8.2   | 2079.0   | 2973.9           | 22478.5            | 27228.5     |
| 2000| 8.3   | 8285.7   | 10701.1          | 89929.7            | 108929.7    |

As we can see in all of our experiments, FQL and HQL-concave both perform very promisingly with significant advantage over the other two existing algorithms. FQL stays consistently very close to the clairvoyant optimal in both the more adversarial and less adversarial settings for the backlogged model. HQL-concave catches up rather quickly to OPT in all the settings for both the backlogged model and the lost-sales model.