About a conjecture for uniformly isochronous polynomial centers

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Abstract
We study a specific family of uniformly isochronous polynomial systems. Our results permit to solve a problem about centers of such systems.

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1. Consider the planar autonomous system of ordinary differential equations
\begin{align*}
\dot{x} &= -y + xH(x, y), \\
\dot{y} &= x + yH(x, y),
\end{align*}
where $H(x, y)$ is a polynomial in $x$ and $y$ of degree $n$, and $H(0, 0) = 0$. This system has only one singular point at $O(0,0)$ which is the center of the linear part of the system. The solutions of this system move around the origin with constant angular speed, and the origin is so a uniformly isochronous singular point.

The problem of characterizing uniformly isochronous centers has attracted attention of several authors; see \cite{1}–\cite{4} and references therein. In particular, the following problem was posed:

It is true that all centers for uniformly isochronous polynomial systems are either reversible or admit a nontrivial polynomial commuting system?

The problem first appeared in \cite{2} and it was mentioned as one of the open questions in \cite{3}. It was also marked by the reviewers in Zbl. Math. 1037.34024 and in MR1963468 (2004b:34090). We prove the following proposition which permits to give a negative answer to the problem.

**Theorem 1.** Let a uniformly isochronous polynomial system has the form
\begin{align*}
\dot{x} &= -y + xQ(x, y) \sum_{i=0}^{m} a_i(x^2 + y^2)^i, \\
\dot{y} &= x + yQ(x, y) \sum_{i=0}^{m} a_i(x^2 + y^2)^i,
\end{align*}

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\cite{4}
where $Q(x, y)$ is a homogeneous polynomial in $x, y$ of degree $k$ and

$$
\int_0^{2\pi} Q(\cos \vartheta, \sin \vartheta) d\vartheta = 0. \tag{3}
$$

Then the origin is a center of (2). The center is of type $B^\nu$ with $\nu \leq k$, and a "generic" center is of type $B^1$ when $k$ is odd or of type $B^2$ when $k$ is even.

**Proof.** System (2) can be written as a single separable equation

$$
\frac{d\varrho}{d\vartheta} = \varrho^{k+1} Q(\cos \vartheta, \sin \vartheta) R(\varrho), \tag{4}
$$

with $\varrho, \vartheta$ polar coordinates and $R(\varrho) = \sum_{i=0}^m a_i \varrho^{2i}$

Equation (4) has a solution $\varrho \equiv 0$ which is defined for all $\vartheta$. Therefore every solution $\varrho(\vartheta)$ with the initial value $\varrho(0) = \varrho_0$ where $\varrho_0 > 0$ is small enough is defined for $\vartheta \in [0, 2\pi]$ and satisfies the condition

$$
\int_0^\vartheta Q(\cos \varphi, \sin \varphi) d\varphi = \int_0^{\varrho(\vartheta)} \frac{dr}{r^{k+1} R(r)}. \tag{5}
$$

From (3) we conclude that the solution is $2\pi$-periodic, so that the origin is a center. The first part of the theorem is proved.

By (3), the center of (2) is of type $B^\nu$, and the boundary of the center region is the union of $\nu$ open unbounded trajectories with $\nu \leq \nu = k + 2m$.

The circles $x^2 + y^2 = \varrho_i^2$ where $\varrho_i$ are roots of the equations $R(\varrho) = 0$ are trajectories of (4). All of them lie in the center region.

Unbounded trajectories of (2) correspond to unbounded solutions of (4). Studying the behaviour of solution curves of (4) for large $\varrho$, it can be shown that for every null isocline $\vartheta = \vartheta^*$ where solutions have a maximum there exist two solutions $\varrho_1^* (\vartheta), \varrho_2^* (\vartheta)$, for which the isocline is a vertical asymptote

$$
\lim_{\vartheta \to \vartheta^* - 0} \varrho_1^*(\vartheta) = +\infty, \lim_{\vartheta \to \vartheta^* + 0} \varrho_2^*(\vartheta) = +\infty,
$$

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In this situation, there is a relevant equilibrium point at infinity in the intersection of the equator of the Poincaré sphere with the ray \( x = \rho \cos \vartheta^*, y = \rho \sin \vartheta^*, \rho > 0 \). The point has one hyperbolic sector with two separtrices which correspond to the solutions \( q_1^*(\vartheta), q_2^*(\vartheta) \) (see Fig. 2). The boundary of the center region consists of such separatrices. The number \( \nu \) of these equilibrium points coincides with the number of the null isoclines of direction field (4) where solutions has a maximum for large \( \rho \). These isoclines are vertical lines \( \vartheta = \vartheta^*_i \), where the values of \( \vartheta^*_i \) are determined from the conditions \( Q(\cos \vartheta, \sin \vartheta) = 0, 0 \leq \vartheta < 2\pi \). Hence we have the estimate \( \nu \leq k \) when we describe type \( B^\nu \) of the center of system (2). The upper bound \( k \) can be attained by \( \nu \). As an example we can consider system (2) with \( Q(\cos \vartheta, \sin \vartheta) = \sin k\vartheta \) and \( a_i \) arbitrary real numbers.

In a \( \text{\textit{generic}} \) situation, (4) has no solution for which two different null isoclines are its asymptotes. Therefore in such a situation the solution curve which separates bounded and unbounded solutions has a minimum number of discontinuity points within \([0, 2\pi]\), it has one point when \( k \) is odd, and it has two points when \( k \) is even.

Hence a \( \text{\textit{generic}} \) center is of type \( B^1 \) when \( k \) is odd or type \( B^2 \) when \( k \) is even (see Fig. 3).

![Fig. 3](image)

The theorem is proved.

**Remark.** Theorem 2.1 from [5] about centers of homogeneous systems is a particular case of our theorem for \( m = 0, a_0 = 1 \).

The functions

\[
f_1(x, y) = x^2 + y^2, f_2(x, y) = \sum_{i=0}^{m} a_i(x^2 + y^2)^i
\]

\footnote{If \( k \) is even our trigonometric polynomial \( Q(\cos \vartheta, \sin \vartheta) \) has a period equal to \( \pi \) (but not \( 2\pi \) as it takes place for odd \( k \)). Therefore (4) has an even number of the blocks discussed above and the relevant equilibrium points are disposed at the diameters of the Poincaré sphere. It may be noted that system (4) is \( O \)-symmetric in this case.}
are invariants for (2) with respective cofactors
\[ K_1(x, y) = 2Q(x, y) \sum_{i=0}^{m} a_i(x^2 + y^2)^i, \]
\[ K_2(x, y) = 2Q(x, y) \sum_{i=0}^{m} ia_i(x^2 + y^2)^i. \]

We have
\[ \frac{k+2}{2} K_1(x, y) + K_2(x, y) = \text{div}, \]
where \( \text{div} \) is the divergence of \( \mathcal{F} \). In this case the function \( \mu(x, y) = f_1^{(k+2)/2} f_2 \) is a reciprocal integrating factor of Darboux form \( \mathcal{F} \). The factor gives information about our system. For instance, it may be used to find a first Darboux integral of \( \mathcal{F} \). A first integral of \( \mathcal{F} \) may be found from (5) also.

It is obvious that \( \mathcal{F} \) commutes with the system
\[ \dot{x} = x(x^2 + y^2)^{k/2} \sum_{i=0}^{m} a_i(x^2 + y^2)^i, \]
\[ \dot{y} = y(x^2 + y^2)^{k/2} \sum_{i=0}^{m} a_i(x^2 + y^2)^i. \]  \((6)\)

If \( k \) is even \( (6) \) gives a polynomial commuting system without a linear part. If \( k \) is odd we have a non-polynomial commuting system. Nevertheless a polynomial commuting system may exist in the case of odd \( k \). For example, if \( \mathcal{F} \) is homogeneous \( (m = 0, a_0 = 1) \) then there exists a polynomial system which commutes with \( \mathcal{F} \).

Using Theorem 1, we may construct an example of an uniformly isochronous system which is not reversible and commutes with no polynomial system.

If a system is reversible then its trajectories are symmetric with respect a common symmetric line. If a symmetric line of system \( \mathcal{F} \) is \( x \sin \vartheta^* - y \cos \vartheta^* = 0 \), then the vertical line \( \vartheta = \vartheta^* \) is the symmetric axis of the graph of the trigonometric polynomial \( Q(\cos \vartheta, \sin \vartheta) \) and it is the symmetric axis of solution curves of vector field \( \mathcal{F} \).

The problem about conditions for the existence of polynomial commuting systems for uniformly isochronous polynomial systems was considered in \( \mathcal{F} \). In particular, it was proved that system \( \mathcal{F} \) commutes with a polynomial system if and only if the function \( H(x, y) \) satisfies one of the following two conditions:
1) \( H(x, y) = P_2(x, y) \sum_{j=0}^{r} a_j(x^2 + y^2)^j \) \( \text{(7)} \)
2) there are homogeneos polynomials \( \alpha_l, \beta_l \) of order \( l \) \( (l \leq n, l \text{ divides } n) \), verifying \( x\partial_y \beta_l - y\partial_x \beta_l = l\alpha_l \), such that
\[ H(x, y) = \alpha_l \sum_{k=0}^{n/l-1} a_k \beta_l^k. \]  \( \text{(8)} \)

About algebraic invariants and Darboux’s method of integration see \( \mathcal{F} \), for example.

Necessary and sufficient conditions for reversibility of planar analytic vector fields were derived in \( \mathcal{F} \).
So, to construct our example it is sufficient to take a system of the form
\[ (2) \]
where the homogeneous polynomial \( Q(x, y) \) is of an odd degree \( 4 \), the graph of the trigonometric polynomial \( Q(\cos \vartheta, \sin \vartheta) \) has no symmetric axes and the numbers \( m, a_i \) are such that the function
\[ H(x, y) = Q(x, y) \sum_{i=0}^{m} a_i (x^2 + y^2)^i \]
is not of the form \( (8) \).

Let us set
\[ Q(x, y) = y^3 - 3xy^2 + 2x^2y = y(x - y)(2x - y), m = 1, a_0 = a_1 = 1. \]

Then we have
\[
\dot{x} = -y + x(y^3 - 3xy^2 + 2x^2y)(1 + x^2 + y^2), \\
\dot{y} = x + y(y^3 - 3xy^2 + 2x^2y)(1 + x^2 + y^2).
\] (9)

According to Theorem 1, \( (9) \) has a center (isochronous) at the origin.

The function
\[ I(x, y) = \frac{r^6}{(1 - 3r^2 - 4x^3 - 3xy^2 - 3y^3 - 3r^3 \arctan r)^2}, \]
is a first integral of \( (9) \) obtained from \( (3) \).

It is evident that the graph of \( Q(\cos \vartheta, \sin \vartheta) \) has no symmetric axes, and therefore system \( (9) \) is non-reversible.

It easy to verify that system \( (9) \) may fail to commute with any nonproportional polynomial systems. This fact follows from the impossibility to present the function
\[ H(x, y) = (y^3 - 3xy^2 + 2x^2y)(1 + x^2 + y^2) \equiv H_3(x, y) + H_5(x, y) \]
in the form \( (8) \) but we can also prove it immediately.

Indeed, let system \( (9) \) commutes with a polynomial system of degree \( n \)
\[
\dot{x} = R(x, y) \equiv R_1(x, y) + R_2(x, y) + \ldots + R_n(x, y), \\
\dot{y} = S(x, y) \equiv S_1(x, y) + S_2(x, y) + \ldots + S_n(x, y),
\] (11)

\[ \text{In this case (11) is fulfilled and the function } H(x, y) \text{ is not of the form (8).} \]

\[ \text{It may be shown that if the graph of the homogeneous trigonometric polynomial of degree } 3 \]
\[ T_3(\vartheta) = a_1 \cos \vartheta + a_3 \cos 3\vartheta + b_1 \sin \vartheta + b_3 \sin 3\vartheta \]
has a symmetric axes then its coefficients satisfy the condition
\[ a_1b_3(a_1^2 - 3b_1^2) = a_3b_1(3a_1 - b_1^2). \]
where $R_i(x, y), S_i(x, y)$ are homogeneous polynomials of degree $i$.

Then the Lie bracket between vector fields $\overset{\circ}{\text{(3)}}, \overset{\circ}{\text{(4)}}$ is equal to zero:

$$\left[(-y + xH(x, y), x + yH(x, y))^T, (R(x, y), S(x, y))^T\right] = (0, 0)^T.$$

In particular, we have that terms of highest degree are equal to zero:

$$\left[(xH_5(x, y), yH_5(x, y))^T, (R_n(x, y), S_n(x, y))^T\right] = (0, 0)^T.$$

After transformations taking into account Euler’s theorem for homogeneous functions this equality may be written in the form

$$(xH_5(x, y)x(x, y) + (1 - n)H_5(x, y))R_n(x, y) + xH_5(y)(yH_5(x, y) + (1 - n)H_5(x, y))S_n(x, y) = 0,$$

$$xyH_5(x, y)H_5(y, x) = 0.$$

The linear system for determining the polynomials $R_n(x, y), S_n(x, y)$ has a nontrivial solution if its determinant $\Delta$ is equal to zero:

$$\Delta \equiv (xH_5(x, y) + (1 - n)H_5(x, y))(yH_5(x, y) + (1 - n)H_5(x, y)) - xyH_5(x, y)H_5(y, x) = 0,$$

Taking into account the fact that $xH_5(x, y) + yH_5(x, y) = 5H_5(x, y)$, we have

$$\Delta = (1 - n)(6 - n)H_2^2(x, y) = 0.$$

Therefore we must have that $n = 6$ or $n = 1$. Straightforward calculations using the software package Mathematica show that in this case the commuting system $\overset{\circ}{\text{(3)}}$ is proportional to system $\overset{\circ}{\text{(4)}}$.

Hence system $\overset{\circ}{\text{(3)}}$ has a center but it is non-reversible and it commutes with no polynomial system nonproportional to it.

We derive that the answer to the question from $\overset{\circ}{\text{[2]}}$, $\overset{\circ}{\text{[3]}}$ is negative.

2. We can generalize the first part of Theorem 1.

**Theorem 2.** Let the polynomial $H(x, y)$ in (1) has the form

$$H(x, y) = q(x, y)h(x^2 + y^2, p(x, y)),$$

where $h(u, v)$ is a polynomial, $p(x, y), q(x, y)$ are homogeneous polynomials of the same degree $k$ and $q(x, y) = c(xp_y(x, y) - yp_x(x, y))$. Then the origin is a center of $\overset{\circ}{\text{(1)}}$.

**Proof.**

In the case under study system $\overset{\circ}{\text{(1)}}$ can be written as a single equation of the form

$$\frac{d\vartheta}{d\varrho} = c\varrho^{k+1}h(\varrho^2, \varrho^k f(\vartheta))f'(\vartheta).$$

According with $\overset{\circ}{\text{[6]}}$ system $\overset{\circ}{\text{(7)}}$ commutes with the system

$$\dot{x} = x(x^2 + y^2)\sqrt{x^2 + y^2(1 + x^2 + y^2)}, \dot{y} = y(x^2 + y^2)\sqrt{x^2 + y^2(1 + x^2 + y^2)}.$$
with $\rho, \vartheta$ polar coordinates and $f(\vartheta) = \rho(\cos \vartheta, \sin \vartheta)$.

It is clear that solutions of this equation are functions of $f(\vartheta)$. The function $f(\vartheta)$ is $2\pi$-periodic. Then solutions with initial values which are small enough are $2\pi$-periodic functions also. So, the origin is a center.

**References**

[1] J. Chavarriga, M. Sabatini; *A survey of isochronous centers*, Qualitative Theory of Dynamical Systems. 1999. Vol. 1. No. 1. P. 1–70.

[2] A. Algaba, M. Reyes, and A. Bravo; *Geometry of the uniformly isochronous centers with polynomial commutators*, Differential Equations Dynam. Systems. 2002. Vol. 10. No. 3-4. P. 257–275.

[3] A. Algaba, M. Reyes; *Centers with degenerate infinity and their commutators*, J. Math. Anal. Appl. 2003. Vol. 78. No. 1. P. 109–124.

[4] A. Algaba, M. Reyes; *Computing center conditions for vector fields with constant angular speed*, J. Comput. Appl. Math. 2003. Vol. 154. No. 1. P. 143–159.

[5] R. Conti, *Uniformly isochronous centers of polynomial systems in $\mathbb{R}^2$*, Elworthy K. D. (ed.) et al., Differential equations, dynamical systems, and control science. New York: Marcel Dekker. Lect. Notes Pure and Appl. Math. 152, 21–31 (1994).

[6] J. Chavarriga, H. Giacomini, and J. Giné; *The null divirgence factor*, Publ. Mat., Barc. 1997. Vol. 41. No. 1. P. 41–56.

[7] J. M. Pearson, N. G. Lloyd, and C.J. Christopher; *Algorithmic derivation of centre conditions*, SIAM Review. 1996. Vol. 38. No. 4. P. 619–636.

[8] D. Schlomiuk, *Algebraic and geometric aspects of the theory of polynomial vector fields*, Schlomiuk, Dana (ed.), Bifurcations and periodic orbits of vector fields. Proceedings of the NATO Advanced Study Institute and Séminaire de Mathématiques Supérieures, Montréal, Canada, July 13-24, 1992. Dordrecht: Kluwer Academic Publishers. NATO ASI Ser., Ser. C, Math. Phys. Sci. 408, 429-467 (1993).

[9] L. Mazzi, M. Sabatini; *Commutators and linearizations of isochronous centers*, Atti Acad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 2000. Vol. 11. No 2. P. 81–98.

[10] C. B. Collins, *Poincaré’s reversibility conditions*, J. Math. Anal. Appl. 2001. Vol. 259. No. 1. P. 168–187.