SU(3)-STRUCTURES ON SUBMANIFOLDS OF A SPIN(7)-MANIFOLD

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Abstract. Local SU(3)-structures on an oriented submanifold of Spin(7)-manifold are determined and their types are characterized in terms of the shape operator and the type of Spin(7)-structure. An application to Bryant [5] and Calabi [10] examples is given. It is shown that the product of a Cayley plane and a minimal surface lying in a four-dimensional orthogonal Cayley plane with the induced complex structure from the octonions described by Bryant in [5] admits a holomorphic local complex volume form exactly when it lies in a three-plane, i.e. it coincides with the example constructed by Calabi in [10]. In this case the holomorphic (3, 0)-form is parallel with respect to the unique Hermitian connection with totally skew-symmetric torsion.

Keywords: Spin(7)-structure, SU(3)-structure, (special) almost Hermitian structure, G-structures, intrinsic torsion, G-connections, submanifold, normal connection, shape tensor, minimal submanifold.

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1. Introduction

A Spin(7)-structure on an eight-dimensional manifold is by definition a reduction of the structure group of the tangent bundle to Spin(7). An eight-dimensional manifold equipped with a Spin(7)-structure is called Spin(7)-manifold. Moreover, associated with a Spin(7)-structure, there exists a nowhere vanishing four-form Φ, called the fundamental form, which determines a Riemannian metric ⟨·, ·⟩ and a volume form due to the fact that Spin(7) is the maximal compact subgroup of SO(8). Likewise, choosing a vector of unit length as unity, the
tangent vector space on each point of a Spin(7)-manifold can be identified with the octonion algebra $\mathbb{O}$.

Decomposing the space $\{\nabla \Phi\}$ of covariant derivatives of $\Phi$ with respect to the Levi-Civita connection $\nabla$ into Spin(7)-irreducible components, Fernández [19] classified Spin(7)-manifolds and obtained four classes, namely, $W_0$ (parallel), $W_1$ (balanced), $W_2$ (locally conformal parallel) and the whole class $\mathring{W}$. Studying the obstruction of a Spin(7)-structure to be parallel, we find (Theorem 3.2) an expression for the intrinsic torsion of a Spin(7)-structure in terms of the exterior derivative $d\Phi$ which explicitly expresses $\nabla \Phi$ in terms of $d\Phi$. Note, that a formula of $\nabla \Phi$ in terms of $d\Phi$ was given in [37]. The existence of such an explicit formula is an implicit consequence of the fact, noted by Bryant [6] (see [19, 46]), that the Riemannian holonomy group of a Spin(7)-manifold is contained in Spin(7) iff the form $\Phi$ is closed.

If $M^6$ is an orientable six-dimensional submanifold of a Spin(7)-manifold $(M^8, \Phi, \langle \cdot, \cdot \rangle)$, Gray [30] showed that there is on $M^6$ an almost Hermitian structure (U(3)-structure) naturally induced from the Spin(7)-structure on $M^8$. When $M^8$ is a parallel Spin(7)-manifold, Gray derived conditions in terms of the shape operator of $M^6$ characterizing types of almost Hermitian structure on $M^6$.

In the present paper, we define local SU(3)-structures on $M^6$ inherited from the Spin(7)-structure on $M^8$. Note that in general there is not a global SU(3)-structure on $M^6$ induced from the Spin(7) structure on $M^8$, since the stabilizer of an oriented two-plane in Spin(7) is the group U(3) [5]. We show the existence of local complex volume forms naturally induced from the fundamental four-form $\Phi$ and the choice of a local oriented orthonormal frame $N_1, N_2$ of the normal bundle of $M^6$. We present relations between the Spin(7)-structure on the ambient manifold $M^8$, the induced local SU(3)-structure and the shape operator on $M^6$ (Proposition 4.2). Consequently, we characterize the types of the local SU(3)-structures on $M^6$ in terms of the fundamental four-form $\Phi$ and the shape operator (Theorem 4.3, Theorem 4.5, Theorem 4.6). In particular, we recover Gray’s results in [30] in an alternative way.

In Section 5 we study the problem when there exists a closed local SU(3)-structure on $M^6 \subset M^8$, which in particular, implies that the almost complex structure is integrable due to the considerations in [35]. We focus our attention to the case $M^8 = \mathbb{O}$ studied in detail by Bryant in [5]. In this case (even more general, when the Spin(7)-structure of the ambient manifold is parallel), Gray [30] showed that the Lee form of the submanifold is always zero. When the almost complex structure is integrable, then it is balanced (type $\mathcal{W}_3$) and the submanifold is necessarily minimal. The properties of submanifolds with balanced Hermitian structure are investigated by Bryant in [5]. He shows that if $M^6 \subset \mathbb{O}$ inherits complex and non-Kähler structure, then $M^6$ is foliated by four-planes in $\mathbb{O}$ in a unique way, he calls this foliation asymptotic ruling. He obtains that if the asymptotic ruling is parallel, then $M^6$ is a product of a fixed associative four-plane $Q^4$ in $\mathbb{O}$ with a minimal surface in the orthogonal four-plane. He shows that the Calabi examples, described in [10], are exactly those complex $M^6$ with parallel asymptotic ruling contained in $\text{Im}\mathbb{O} \subset \mathbb{O}$, i.e. the minimal surface lies in an associative three-plane in $\text{Im}\mathbb{O}$.

We investigate when there exists a local holomorphic SU(3)-structures in the case of parallel asymptotic ruling. We show that there exists a holomorphic local SU(3)-structure on $M^6$ exactly when the minimal surface lies in a three-plane (Theorem 5.3). We also prove that the corresponding Bismut connection (the unique Hermitian connection with totally
skew-symmetric torsion) preserves the holomorphic volume form having holonomy contained in $SU(3)$. Therefore, the structure is Calabi-Yau with torsion (CYT). CYT structures are attractive in heterotic string theory as a possible solution to the heterotic string model proposed by Str"ominger [47]. Consequently, we derive that the compact complex non-K"ahler six-manifold with vanishing first Chern class constructed by Calabi in [10, Theorem 7] has holomorphically trivial canonical bundle and the $SU(3)$-structure constructed by Calabi is a CYT-structure (Theorem 5.4).

Recently, Bryant discussed in [7] a generalization of the notion of holomorphic vector bundles on complex manifold to the almost complex case and, consequently, a generalization of the notion of Hermitian-Yang-Mills connection. He referred the class of almost complex six-manifolds admitting such non-trivial bundles as quasi-integrable. An important subclass is the class of strict quasi-integrable structures which is defined as quasi-integrable structures with nowhere vanishing Nijenhuis tensor. He introduced the notion of quasi-integrable $U(3)$-structure, pointing out that this class of almost Hermitian six-manifold coincides with the class $W_1 \oplus W_3 \oplus W_4$ according to Gray-Hervella classification [31], i.e. the class where the Nijenhuis tensor is totally skew-symmetric. The case of nearly K"ahler structures is also investigated in details in [7]. Following our approach, in Example 6.1, we describe a strict quasi-integrable non-nearly K"ahler $SU(3)$-structures on $S^3 \times S^3$ compatible with the standard product metric on $S^3 \times S^3$. Four of these structures are half-flat in the class $W_1 \oplus W_3$. These four structures are also left-invariant on the group $SU(2) \times SU(2) \cong S^3 \times S^3$.

**Remark 1.1.** We note that another compact example of strict quasi-integrable non-nearly K"ahler half-flat $SU(3)$-structure of type $W_1 \oplus W_3$ tensor on nil-manifold has been constructed in [38], Section 6.2.

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2. **General properties of $SU(3)$ and Spin(7)-structures**

In this section we recall necessary properties of $SU(3)$ and Spin(7)-structures.

First we recall some notions relative to $G$-structures, where $G$ is a subgroup of the linear group $GL(m, \mathbb{R})$. If $M$ possesses a $G$-structure, then there always exists a $G$-connection defined on $M$. Moreover, if $(M^m, \langle \cdot, \cdot \rangle)$ is an orientable $m$-dimensional Riemannian manifold with associated Levi-Civita connection $\nabla$ and $G$ is a closed and connected subgroup of $SO(m)$, then there exists a unique metric $G$-connection $\nabla^G$ such that $\xi^G_x = \nabla^G_x - \nabla_x$ takes its values in $g^+$, where $g^+$ denotes the orthogonal complement in $so(m)$ of the Lie algebra $g$ of $G$ [46, 16]. The tensor $\xi^G$ is called the intrinsic torsion of the $G$-structure and $\nabla^G$ is referred as the minimal $G$-connection.

2.1. **$SU(3)$-structures.** Here we give a brief summary of the properties of $SU(3)$-structures on six-dimensional manifolds which are also called special almost Hermitian six-manifolds. For more detailed and exhaustive information see [13, 44].
An almost Hermitian manifold is a 2n-dimensional manifold \( M \) with a \( U(n) \)-structure. This means that \( M \) is equipped with a Riemannian metric \( \langle \cdot, \cdot \rangle \) and an orthogonal almost complex structure \( J \). Each fibre \( T_{m}M \) of the tangent bundle can be considered as complex vector space letting \( ix = Jx \). The Kähler form \( \omega \) is defined by \( \omega(x, y) = \langle x, Jy \rangle \).

Convention. For a \((0, s)\)-tensor \( B \), we write
\[
J_{(j)}B(X_1, \ldots, X_j, \ldots, X_s) = -B(X_1, \ldots, JX_j, \ldots, X_s),
\]
\[
JB(X_1, \ldots, X_s) = (-1)^s B(JX_1, \ldots, JX_s).
\]
The Lee form \( \theta \) of an almost Hermitian structure is defined by \( \theta = Jd^* \omega \), where \( d^* \) denotes the codifferential. Also we will consider the natural extension of the metric \( \langle \cdot, \cdot \rangle \) to \( \Lambda^p T^*M \) given by
\[
\langle \alpha, \beta \rangle = \frac{1}{p!} \sum_{i_1, \ldots, i_p = 1}^{2n} \alpha(e_{i_1}, \ldots, e_{i_p})\beta(e_{i_1}, \ldots, e_{i_p}),
\]
where \( \{e_1, \ldots, e_{2n}\} \) is an orthonormal basis for vectors.

A special almost Hermitian manifold is a 2n-dimensional manifold \( M \) with an \( SU(n) \)-structure. This means that \((M, \langle \cdot, \cdot \rangle, J)\) is an almost Hermitian manifold equipped with a complex volume form \( \Psi = \Psi_+ + i\Psi_- \), i.e. \( \Psi \) is an \((n, 0)\)-form such that \( \langle \Psi, \overline{\Psi} \rangle = 1 \), where \( \langle \cdot, \cdot \rangle \) denotes the natural extension of the metric on (complex) forms and \( \overline{\Psi} \) is the conjugated \((0, n)\)-form. Note that \( J_{(j)} \Psi_+ = \Psi_- \).

In general, an almost Hermitian manifold admits a linear connection preserving the almost Hermitian structure and having totally skew-symmetric torsion exactly when the Nijenhuis tensor is totally skew-symmetric (the class \( W_1 \oplus W_3 \oplus W_4 \) in the Gray-Hervella classification [31]). Moreover, such a connection is unique [21, 22]. If the almost complex structure is integrable, then this connection is referred as the Bismut connection. It was used by Bismut [2] to derive a local index formula for Hermitian non-Kähler manifolds. When the Bismut connection preserves a given \( SU(n) \)-structure, i.e. it has holonomy contained in \( SU(n) \), then the manifold is called sometimes Calabi-Yau manifold with torsion (CYT) and appears as a possible geometry in heterotic string model due to the work of Ströminger [47] (see e.g. [1, 11, 23, 24, 27, 28, 32, 33, 38] and references therein).

In the following, we consider special almost Hermitian six-manifold, i.e. a six-dimensional smooth manifold endowed with an \( SU(3) \)-structure. We denote the corresponding Lee form by \( \theta^0 \). Let
\[
e_{1C} = e_1 + iJe_1, \quad e_{2C} = e_2 + iJe_2, \quad e_{3C} = e_3 + iJe_3
\]
be a unitary basis such that \( \Psi(e_{1C}, e_{2C}, e_{3C}) = 1 \), i.e. \( \Psi_+(e_1, e_2, e_3) = 1 \), \( \Psi_-(e_1, e_2, e_3) = 0 \). The real orthonormal basis for vectors \( e_1, e_2, e_3, Je_1, Je_2, Je_3 \) is said to be adapted to the \( SU(3) \)-structure. By means of such an adapted basis, the Kähler form \( \omega \) and the three-forms \( \Psi, \Psi_+ \) and \( \Psi_- \) are given by
\[
\omega = -e_1 \wedge Je_1 - e_2 \wedge Je_2 - e_3 \wedge Je_3,
\]
\[
\Psi = e_{1C} \wedge e_{2C} \wedge e_{3C},
\]
\[
\Psi_+ = e_1 \wedge e_2 \wedge e_3 - Je_1 \wedge Je_2 \wedge e_3 - Je_1 \wedge e_2 \wedge Je_3 - e_1 \wedge Je_2 \wedge Je_3,
\]
\[
\Psi_- = -Je_1 \wedge Je_2 \wedge Je_3 + Je_1 \wedge e_2 \wedge e_3 + e_1 \wedge Je_2 \wedge e_3 + e_1 \wedge e_2 \wedge Je_3.
\]
Here and further we freely identify vector field with the dual one-form via the metric.
It is straightforward to check $\omega^3 := \omega \wedge \omega \wedge \omega = 6 e_1 \wedge e_2 \wedge e_3 \wedge J e_1 \wedge J e_2 \wedge J e_3$. If we fix the real volume form $Vol$ such that $6 Vol = \omega^3$, we have the relations \[13, 44\]

\begin{align}
(2.1) & \quad \Psi_+ \wedge \omega = \Psi_- \wedge \omega = 0; \\
& \quad \Psi_+ \wedge \Psi_- = -4 Vol, \quad \Psi_+ \wedge \Psi_+ = \Psi_- \wedge \Psi_- = 0; \\
& \quad x \wedge \Psi_+ = Jx \wedge \Psi_-, \quad x \wedge \Psi_- = Jx \wedge \Psi_+, \quad x \in T_m M,
\end{align}

where $\wedge$ denotes the interior product of vectors and forms.

Note that, defined on $M$, there are two Hodge star operators associated with the volume forms $Vol$ and $\Psi$. Relative to the real Hodge star operator $\ast$, for any one-form $\mu \in \Lambda^1 M$, we have the relations

\begin{align}
(2.2) & \quad \ast ((\mu \wedge \Psi_+) \wedge \Psi_+) = \ast ((\mu \wedge \Psi_-) \wedge \Psi_-) = -2\mu, \\
(2.3) & \quad \ast ((\mu \wedge \Psi_-) \wedge \Psi_-) = -\ast ((\mu \wedge \Psi_+) \wedge \Psi_+) = 2\mu.
\end{align}

For $U(3)$-structures, the minimal $U(3)$-connection is given by $\nabla^{U(3)} = \nabla + \xi^{U(3)}$, where

\begin{equation}
(2.4) \quad \xi^{U(3)}_X Y = -\frac{1}{2} J (\nabla X) J Y
\end{equation}

(see \[18\]). Since $U(3)$ stabilizes the Kähler form $\omega$, it follows that $\nabla^{U(3)} \omega = 0$. Then $\nabla \omega = -\xi^{U(3)} \omega \in T^* M \otimes u(3)^\perp$. Thus, one can identify the $U(3)$-components of $\xi^{U(3)}$ with the $U(3)$-components of $\nabla \omega$.

For $SU(3)$-structures, we have the decomposition $so(6) = su(3) \oplus \mathbb{R} \omega \oplus u(3)^\perp$, i.e. $su(3)^\perp = \mathbb{R} \omega \oplus u(3)^\perp$. Therefore, the intrinsic $SU(3)$-torsion $\eta + \xi^{SU(3)}$ is such that $\eta \in T^* M \otimes \mathbb{R} J \cong T^* M$ and $\xi^{SU(3)}$ is still determined by Equation (2.4). The tensors $\omega$, $\Psi_+$ and $\Psi_-$ are stabilized by the $SU(3)$-action and therefore $\nabla^{SU(3)} \omega = 0$, $\nabla^{SU(3)} \Psi_+ = 0$, $\nabla^{SU(3)} \Psi_- = 0$, where

$$\nabla^{SU(3)} = \nabla + \eta + \xi^{SU(3)}$$

is the minimal $SU(3)$-connection. Since $\nabla^{SU(3)}$ is metric and $\eta \in T^* M \otimes \mathbb{R} J$, we have $\langle Y, \eta X Z \rangle = (J \eta)(X) \omega(Y, Z)$, where $\eta$ on the right hand side is considered to be a one-form. Hence

\begin{equation}
(2.5) \quad \eta X Y = J \eta(X) J Y.
\end{equation}

One can check $\eta \omega = 0$, then from $\nabla^{SU(3)} \omega = 0$ one gets

$$\nabla \omega = -\xi^{SU(3)} \omega \in T^* M \otimes u(3)^\perp = W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$

where the summands $W_i$ are the Gray-Hervella $U(3)$-modules. There is a further splitting of $T^* M \otimes u(3)^\perp$ into six $SU(3)$-modules discovered and first described by Chiossi and Salamon in \[13\] (see also \[44, 45\], for interpretation in physics see \[11, 32, 33\]). We present below the necessary for our considerations part of the description of the $SU(3)$-modules following \[44\].

The spaces $W_3$ and $W_4$ are irreducible also as $SU(3)$-modules. However, $W_1$ and $W_2$ admit the decompositions $W_j = W_j^+ \oplus W_j^-$, $j = 1, 2$, into irreducible $SU(3)$-components, where $W_j^+$ (resp. $W_j^-$) includes those elements $\beta \in W_j \subseteq T^* M \otimes \Lambda^2 T^* M$ such that the bilinear form $r(\beta)$, defined by $2r(\beta)(x, y) = \langle x, J \beta, y \rangle \Psi_+$, is symmetric (resp. skew-symmetric).

On the other hand, we have

$$\nabla \Psi_+ = -\eta \Psi_+ - \xi^{U(3)} \Psi_+, \quad \nabla \Psi_- = -\eta \Psi_- - \xi^{U(3)} \Psi_-,$$
where \( \{e_1, \ldots, e_6\} \) is an orthonormal basis for vectors.

Denote \( W^\pm_5 = T^*M \otimes \Psi_\pm \). It is clear that \(-\eta \Psi_+ \in W^+_5\), \(-\eta \Psi_- \in W^-_5\).

Consider the two \( SU(3) \)-maps \( \Xi_+, \Xi_- : T^*M \otimes \mathfrak{u}(3) \rightarrow T^*M \otimes \Lambda^2 T^*M \) defined by

\[
\nabla_\omega \rightarrow -\frac{1}{2} \sum_{j=1}^{6} ((e_j \nabla \nabla_\omega) \wedge (e_j \nabla \Xi_-)), \quad \nabla_\omega \rightarrow -\frac{1}{2} \sum_{j=1}^{6} ((e_j \nabla \nabla_\omega) \wedge (e_j \nabla \Xi_+)),
\]

respectively. It turns out that the \( SU(3) \)-maps \( \Xi_+ \) and \( \Xi_- \) are injective and

\[
\Xi_+ \left( T^*M \otimes \mathfrak{u}(3) \right) = \Xi_- \left( T^*M \otimes \mathfrak{u}(3) \right) = T^*M \otimes T^*M \wedge \omega.
\]

Denote \((d\Psi_\pm)_{4,5}\) the projections of \( d\Psi_+ \) and \( d\Psi_- \) onto the space \( W^\pm_{4,5} = T^*M \wedge \Psi_\pm = T^*M \wedge \Psi_\pm \subseteq \Lambda^4 T^*M \) respectively defined by the alternating maps \( W^\Xi_4 + W^\Xi^-_5 \rightarrow W^\Xi_{4,5} \) and \( W^\Xi(-5)_{4,5} \rightarrow W^\Xi_{4,5} \), where \( W^\Xi = \Xi_+(W_4) = \Xi_-(W_4) \).

If we compute the \( \Psi_\Xi \)-part \((\nabla_\omega)_4 \) of \( \nabla_\omega \), the images \( \Xi_\pm (\nabla_\omega)_4 \) and then taking the skew-symmetric parts of \( \Xi_\pm (\nabla_\omega)_4 \), we will obtain the \( W^\Xi_{4,5} \)-parts of \( d\Psi_+ \) and \( d\Psi_- \), i.e.

\[
(d\Psi_\pm)_{4,5} = -(3\eta + \frac{1}{2} \theta^0) \wedge \Psi_\pm.
\]

With the help of (2.2), (2.3) and (2.6), one gets that the one-form \( \eta \) satisfies the conditions

\[
* \left( *d\Psi_\pm \wedge \Psi_\pm \right) = 6\eta + \theta^0 = -J * \left( *d\Psi_+ \wedge \Psi_- \right) = J * \left( *d\Psi_- \wedge \Psi_+ \right).
\]

So, we get the \( SU(3) \)-splitting [13]

\[
\eta + \xi U(3) \in T^*M \otimes \mathfrak{su}(3) \perp = W^+_4 \oplus W^-_4 \oplus W^+_2 \oplus W^-_2 \oplus W_3 \oplus W_4 \oplus W_5 \subseteq T^*M \otimes \text{End}(TM).
\]

Moreover, we have also

\[
\nabla_\omega = -\xi U(3) \omega \in T^*M \otimes \mathfrak{u}(3) \perp = W^+_1 \oplus W^-_1 \oplus W^+_2 \oplus W^-_2 \oplus W_3 \oplus W_4 \oplus W_5 \subseteq T^*M \otimes \Lambda^2 T^*M,
\]

\[
d\Psi_+ , d\Psi_- \in W^\Xi_4 \oplus W^\Xi_5 \oplus W^\Xi_{4,5} \subseteq \Lambda^4 T^*M,
\]

where \( W^\Xi_4 = \mathbb{R} \omega \wedge \omega \), and \( W^\Xi_5 = \mathfrak{su}(3) \wedge \omega \). Note that, using the maps \( \xi U(3) \rightarrow -\xi U(3) \omega = \nabla_\omega \) and \( \nabla_\omega \rightarrow (\text{Alt} \circ \Xi_\pm)(\nabla_\omega) \), where \( \text{Alt} \) denotes the alternation map, one has the correspondences

\[
(\xi U(3))_{W^+_j} \leftrightarrow (\nabla_\omega)_{W^-_j} \leftrightarrow (d\Psi_+)_{W^+_j} = \text{Alt} \circ \Xi_+(\nabla_\omega)_{W^-_j},
\]

\[
(\xi U(3))_{W^-_j} \leftrightarrow (\nabla_\omega)_{W^+_j} \leftrightarrow (d\Psi_-)_{W^-_j} = \text{Alt} \circ \Xi_-(\nabla_\omega)_{W^+_j}.
\]
We will also need an alternative approach to describe the summand $\xi^{U(3)}$ of the intrinsic torsion of an $SU(3)$-structure. We can write

\begin{equation}
\nabla \omega = - \xi^{U(3)} \omega = \sum_{j,k=1}^{6} c_{jk} e_j \otimes e_k \omega.
\end{equation}

Consider the $SU(3)$-map $r : T^* M \otimes u(3)^\perp \to \otimes^2 T^* M$ defined by

\begin{equation}
\beta(x, y) = \frac{1}{2} \langle x, y, \omega \rangle.
\end{equation}

It is straightforward to check that, for $\beta = \nabla \omega$ satisfying (2.8), $r(\nabla \omega) = \sum_{j,k=1}^{6} c_{jk} e_j \otimes e_k$ and the coderivative $d^* \omega$ has the form

\begin{equation}
d^* \omega = \sum_{j=1}^{6} \sum_{\{k,l \mid \psi_+(e_j, e_k, e_l) = 1\}} (r(\nabla \omega)(e_k, e_l) - r(\nabla \omega)(e_l, e_k)) e_j.
\end{equation}

A useful explicit description of the $SU(3)$-torsion $\eta + \xi^{U(3)}$ is presented in [45]. Since $\eta$ is given by (2.7), it remains to describe $\xi^{U(3)}$. Write $(d\Psi_{\pm})_{\xi^{U(3)}} = d\Psi_{\pm} + 3 \eta \wedge \Psi_{\pm}$, and $(X \wedge Y) \Omega (d\Psi_{\pm})_{\xi^{U(3)}} = (d\Psi_{\pm})_{\xi^{U(3)}}(X, Y, \cdot, \cdot)$. Then [45]

\begin{align*}
\xi^{U(3)} Y &= -\frac{1}{2} \sum_{j,k=1}^{6} r(\nabla \omega)(X, e_j) \psi_+(e_j, e_k, Y) Je_k, \\
2r(\nabla \omega)(X, Y) &= \langle X, d\omega, Y, \omega \rangle_{\psi_+} + \langle (J X \wedge Y) \Omega (d\Psi_{\pm})_{\xi^{U(3)}} - (X \wedge Y) \Omega (d\Psi_{\pm})_{\xi^{U(3)}}, \omega \rangle,
\end{align*}

for all vectors $X, Y$.

The different classes of $SU(3)$-structures can be characterized in terms $d\omega$, $d\Psi_{\pm}$ and $d\Psi_{-}$, as follows:

- $\mathcal{W}_1 \oplus \mathcal{W}_5 = \mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_5$: The class of nearly Kähler manifolds defined by $d\omega$ to be $(3,0)+(0,3)$-form, i.e. $d\omega \in \mathbb{R}\Psi_+ \oplus \mathbb{R}\Psi_-$, and $d\Psi_{\pm} + 3 \eta \wedge \Psi_{\pm} \in \mathbb{R}\omega \wedge \omega$.
- $\mathcal{W}_2 \oplus \mathcal{W}_5 = \mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_5$: The class of almost Kähler manifolds defined by $d\omega = 0$.
- $\mathcal{W}_3 \oplus \mathcal{W}_5$: The class of balanced Hermitian manifolds determined by $d\Psi_{\pm} = \theta^6 = 0$.
- $\mathcal{W}_4 \oplus \mathcal{W}_5$: The class of locally conformally Kähler spaces defined by $2d\omega = \theta^6 \wedge \omega$.
- $\mathcal{W}_5$: The class of Kähler spaces determined by the one-form $\eta$ given by (2.7).

Note that if all components are zero, then we have a Ricci-flat Kähler manifold. If the complex volume form is closed, $d\Psi = 0$, one gets the observation due to Hitchin [35] that the almost complex structure is integrable.

A new object is the class of half-flat ( or $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$) $SU(3)$-manifolds which can be characterized by the conditions

\begin{equation}
d\Psi_{\pm} = \theta^6 = 0.
\end{equation}

The half-flat $SU(3)$-structures can be lifted to a $G_2$-holonomy metric on the product by the real line solving the Hitchin flow equations [36]. In fact, many new $G_2$-holonomy metrics are obtained in this way [4, 25, 14, 17].
2.2. \textit{Spin}(7)-structures. Now, let us consider \( \mathbb{R}^8 \) endowed with an orientation and its standard inner product. Let \( \{e, e_0, \ldots, e_6\} \) be an oriented orthonormal basis. Consider the four-form \( \Phi \) on \( \mathbb{R}^8 \) given by

\[
\Phi = \sum_{i \in \mathbb{Z}_7} e \wedge e_i \wedge e_{i+1} \wedge e_{i+3} - \sigma \sum_{i \in \mathbb{Z}_7} e_{i+2} \wedge e_{i+4} \wedge e_{i+5} \wedge e_{i+6},
\]

where \( \sigma \) is a fixed constant such that \( \sigma = +1 \) or \( \sigma = -1 \), and + in the subindexes means the sum in \( \mathbb{Z}_7 \). We fix \( e \wedge e_0 \wedge \cdots \wedge e_6 = \frac{\sigma}{14} \Phi \wedge \Phi \) as a volume form.

The subgroup of \( GL(8, \mathbb{R}) \) which fixes \( \Phi \) is isomorphic to the double covering \( Spin(7) \) of \( SO(7) \) [34]. Moreover, \( Spin(7) \) is a compact simply-connected Lie group of dimension 21 [6]. The Lie algebra \( \mathfrak{spin}(7) \) of \( Spin(7) \) is isomorphic to the skew-symmetric two-forms \( \psi \) satisfying the linear equations

\[
\sigma \psi(e_i, e) + \psi(e_{i+1}, e_{i+3}) + \psi(e_{i+4}, e_{i+5}) + \psi(e_{i+2}, e_{i+6}) = 0,
\]

for all \( i \in \mathbb{Z}_7 \). Shorty, \( \mathfrak{spin}(7) \equiv \{ \psi \in \Lambda^2 T^* M | *_8 (\psi \wedge \Phi) = \psi \} \). The orthogonal complement \( \mathfrak{spin}(7) \perp \) of \( \mathfrak{spin}(7) \) in \( \Lambda^2 \mathbb{R}^8^* = so(8) \) is the seven-dimensional space generated by

\[
\beta_i = \sigma e_i \wedge e + e_{i+1} \wedge e_{i+3} + e_{i+4} \wedge e_{i+5} + e_{i+2} \wedge e_{i+6},
\]

where \( i \in \mathbb{Z}_7 \). Equivalently, \( \mathfrak{spin}(7) \perp \) is described as the space consisting of those skew-symmetric two-forms \( \psi \) such that \( *_8 (\psi \wedge \Phi) = -3\psi \).

A \( Spin(7) \)-structure on an eight-manifold \( M^8 \) is by definition a reduction of the structure group of the tangent bundle to \( Spin(7) \); we shall also say that \( M \) is a \( Spin(7) \)-manifold. This can be geometrically described by saying that there exists a nowhere vanishing global differential four-form \( \Phi \) on \( M^8 \) and a local frame \( \{e, e_0, \ldots, e_6\} \) such that the four-form \( \Phi \) can be locally written as in (2.12). The four-form \( \Phi \) is called the fundamental form of the \( Spin(7) \)-manifold \( M \) [3] and the local frame \( \{e, e_0, \ldots, e_6\} \) is called a Cayley frame.

The fundamental form of a \( Spin(7) \)-manifold determines a Riemannian metric \( \langle \cdot, \cdot \rangle \) through \( \langle x, y \rangle = -\frac{1}{8} *_8 ((x \wedge \Phi) \wedge *_8 (y \wedge \Phi)) \) [30]. Thus, \( \langle \cdot, \cdot \rangle \) is referred as the metric induced by \( \Phi \). Any Cayley frame becomes an orthonormal frame with respect to such a metric. We recall that the corresponding three-fold vector cross product \( P \) is defined by

\[
\langle P(X_1, X_2, X_3), X_4 \rangle = \Phi(X_1, X_2, X_3, X_4),
\]

for smooth vector fields \( X_i \) on \( M^8 \).

In general, not every eight-dimensional Riemannian spin manifold \( M^8 \) admits a \( Spin(7) \)-structure. We explain the precise conditions given in [40]. Denote by \( p_1(M) \), \( p_2(M) \), \( \chi(M) \), \( \chi(S_{\pm}) \) the first and the second Pontrjagin classes, the Euler characteristic of \( M \) and the Euler characteristic of the positive and the negative spinor bundles, respectively. It is well known [40] that a spin eight-manifold admits a \( Spin(7) \)-structure if and only if \( \chi(S_+) = 0 \) or \( \chi(S_-) = 0 \). The latter conditions are equivalent to \( p_1^2(M) - 4p_2(M) + 8\chi(M) = 0 \), for an appropriate choice of the orientation.

Let us recall that a \( Spin(7) \)-manifold \( (M, \langle \cdot, \cdot \rangle, \Phi) \) is said to be parallel (torsion-free), if the holonomy of the metric \( Hol((\cdot, \cdot)) \) is a subgroup of \( Spin(7) \). This is equivalent to saying that the fundamental form \( \Phi \) is parallel with respect to the Levi-Civita connection \( \nabla \) of the metric \( \langle \cdot, \cdot \rangle \). Moreover, \( Hol((\cdot, \cdot)) \subseteq Spin(7) \) if and only if \( d\Phi = 0 \) [19, 6] (see also [46]) and any parallel \( Spin(7) \)-manifold is Ricci-flat [3]. The first known explicit example of complete parallel \( Spin(7) \)-manifold with \( Hol((\cdot, \cdot)) = Spin(7) \) was constructed by Bryant and Salamon.
SU(3)-structures on submanifolds of a Spin(7)-manifold

[8, 26]. The first compact examples of parallel Spin(7)-manifolds with \( Hol(\cdot, \cdot) = \text{Spin}(7) \) were constructed by Joyce [39].

There are four classes of Spin(7)-manifolds according to Fernández classification [19] obtained as irreducible Spin(7)-representations of the space \( \mathcal{W} \cong \mathbb{R}^{8*} \otimes \text{spin}(7)^\perp \) of all possible covariant derivatives \( \nabla \Phi \) of the fundamental form with respect to the Levi-Civita connection. The Lee form \( \theta^8 \) is defined by [41]

\[
\theta^8 = -\frac{1}{7} \ast (\ast d\Phi \wedge \Phi) = \frac{1}{7} \ast (\delta \Phi \wedge \Phi).
\]

Fernández classification can be described in terms of the Lee form as follows:

- \( \mathcal{W}_0 : d\Phi = 0 \);
- \( \mathcal{W}_1 : \theta^8 = 0 \);
- \( \mathcal{W}_2 : d\Phi = \theta^8 \wedge \Phi \);
- \( \mathcal{W} : \mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \).

A Spin(7)-structure of the class \( \mathcal{W}_1 \) (i.e. Spin(7)-structure with zero Lee form) is called a balanced Spin(7)-structure. If the Lee form is closed, \( d\theta^8 = 0 \), then the Spin(7)-structure is locally conformal equivalent to a balanced one [37]. It is shown in [41] that the Lee form of a Spin(7)-structure in the class \( \mathcal{W}_2 \) is closed. Therefore, such a manifold is locally conformal equivalent to a parallel Spin(7)-manifold. Compact spaces with closed but not exact Lee form (i.e. the structure is not globally conformal parallel) have very different topology than the parallel ones [37]. Coeffective cohomology and coeffective numbers of Riemannian manifolds with Spin(7)-structure are studied in [48].

3. INTRINSIC TORSION OF Spin(7)-STRUCTURES

In [6], Bryant predicted the existence of a formula expressing the covariant derivative \( \nabla \Phi \) of the fundamental four-form in terms of its exterior derivative \( d\Phi \) (see also [46]). An explicit expression of \( \nabla \Phi \) in terms of \( d\Phi \) has been given in [37]. In this section we use the alternative way of characterizing the different types of Spin(7)-structure proposed in [42, 43]. This help us to describe explicitly the intrinsic torsion of a given Spin(7)-structure and to get a formula for \( \nabla \Phi \) in terms \( d\Phi \). We note that the general properties of the Spin(7)-intrinsic torsion are established in [15].

We consider the Spin(7)-isomorphism \( \pi : \mathcal{W} \to \mathbb{R}^{8*} \otimes \text{spin}(7)^\perp \subset \mathbb{R}^{8*} \otimes \Lambda^2 \mathbb{R}^{8*} \) defined by

\[
\pi(B)(x, y, z) = \frac{1}{8} \langle x \wedge y \wedge z \Phi - z \wedge y \wedge x \Phi \rangle, \quad x, y, z \in \mathbb{R}^8, B \in \mathcal{W}.
\]

It is easy to see that \( \pi \) is a Spin(7)-map. On the other hand, any \( B \in \mathcal{W} \) can be written in the form [41]

\[
B = \sigma \sum_{i \in \mathbb{Z}_8, j \in \mathbb{Z}_7} a_{ij} e_i \otimes (e_j \wedge e \Phi - e \wedge e_j \Phi),
\]

where \( \{e = e_7, e_0, \ldots, e_6\} \) is a Cayley frame. Now one can easily check that

\[
\pi(B) = \sum_{i \in \mathbb{Z}_8, j \in \mathbb{Z}_7} a_{ij} e_i \otimes \beta_j,
\]

where the two-forms \( \beta_j \) are determined in (2.13). Therefore, \( \pi \) is an isomorphism and the four classes of Spin(7)-structures are expressed in terms of \( \pi \) in [42].
Further, we describe the intrinsic \(\text{Spin}(7)\)-torsion in terms of \(d\Phi\). Taking the skew-symmetric part of \(\nabla\Phi\) given by (3.15), we obtain
\[
d\Phi = -\sum_{i \in \mathbb{Z}_7} ((a_{i+2,i+2} + a_{i+4,i+4} + a_{i+5,i+5} + a_{i+6,i+6}) e \wedge e_{i+2} \wedge e_{i+4} \wedge e_{i+5} \wedge e_{i+6} + (\sigma a_{7,i} + a_{i+4,i+5} + a_{i+1,i+3} + a_{i+2,i+6}) e \wedge e_i \wedge e_{i+1} \wedge e_{i+2} \wedge e_{i+4} + (\sigma a_{7,i} - a_{i+5,i+4} - a_{i+3,i+1} + a_{i+2,i+6}) e \wedge e_i \wedge e_{i+2} \wedge e_{i+3} \wedge e_{i+5} + (\sigma a_{7,i} + a_{i+4,i+5} - a_{i+3,i+1} - a_{i+6,i+2}) e \wedge e_i \wedge e_{i+3} \wedge e_{i+4} \wedge e_{i+6} + (\sigma a_{7,i} - a_{i+5,i+4} + a_{i+1,i+3} - a_{i+6,i+2} + \sigma a_{7,i}) e_i + \sigma (a_{i+4,i+5} - a_{i+5,i+4} + a_{i+1,i+3} - a_{i+3,i+1} + a_{i+2,i+6} - a_{i+6,i+2} + \sigma a_{7,i}) e_i + \sigma (a_{i+1,i+3} + a_{i+2,i+6} - a_{i+6,i+2}) e_i \wedge e_{i+6} \wedge e_{i+1} \wedge e_{i+2} \wedge e_{i+3}.
\]

Consequently, for the Lee form \(\theta^8\), (3.17) and (2.14) yield
\[
\theta^8 = -\frac{4}{7} \sum_{i \in \mathbb{Z}_7} (a_{i+4,i+5} - a_{i+5,i+4} + a_{i+1,i+3} - a_{i+3,i+1} + a_{i+2,i+6} - a_{i+6,i+2} + \sigma a_{7,i}) e_i + \frac{4}{7} \sigma \sum_{i \in \mathbb{Z}_7} a_{i,i} e.
\]

The equalities (3.17) and (3.18) imply

**Proposition 3.1.** For a \(\text{Spin}(7)\)-structure, the condition \(d\Phi = \theta^8 \wedge \Phi\) is equivalent to
\[
(3.19) \quad 4\tau(\nabla\Phi) = \sum_{i \in \mathbb{Z}_8} e_i \otimes e_i \wedge \theta^8 + \sigma \theta^8, e\Phi.
\]

Further, we have

**Theorem 3.2.** The minimal \(\text{Spin}(7)\)-connection is given by \(\nabla^{\text{Spin}(7)} = \nabla + \xi^{\text{Spin}(7)}\), where the intrinsic torsion \(\xi^{\text{Spin}(7)}\) is determined by
\[
\langle \xi_X^{\text{Spin}(7)}, Y, Z \rangle = \frac{1}{4} \tau(\nabla\Phi)(X, Y, Z).
\]

Equivalently,
\[
\xi_X^{\text{Spin}(7)} = -\frac{\sigma}{24} \sum_{i,j \in \mathbb{Z}_8} \tau(\nabla\Phi)(X, e_i, e_j) P(e_i, e_j, Y),
\]

where \(\{e = e_7, e_0, \ldots, e_6\}\) is a Cayley frame.

The tensor \(\tau(\nabla\Phi)\) is expressed in terms of \(d\Phi\) due to the next equality
\[
(3.20) \quad 4\tau(\nabla\Phi)(X, Y, Z) = 2(X \wedge d\Phi, Y \wedge Z \wedge \Phi - Z \wedge Y \wedge \Phi) - 7(X \wedge \theta^8)(Y, Z).
\]

**Proof.** Let \(i \in \mathbb{Z}_8\) and \(j \in \mathbb{Z}_7\). Then (3.16) and (2.13) give \(4\tau(\varphi)(e_i, e_j, e) = 4\sigma a_{ij}\). Now, using the expressions (3.17) and (3.18) for \(d\Phi\) and \(\theta^8\), respectively, we check that the right hand side of (3.20) (denote it by \(C\)) gives \(C(e_i, e_j, e) = 4\sigma a_{ij} = 4\tau(\nabla\Phi)(e_i, e_j, e)\). Likewise, using again (3.17) and (3.18), one checks that \(\sigma C(e_i, e_j, e) = C(e_i, e_{j+1}, e_{j+3}) = C(e_i, e_{j+4}, e_{j+5}) = C(e_i, e_{j+2}, e_{j+6})\).

Therefore, \(C \in T^* M \otimes \mathfrak{spin}(7)^\perp\) and \(4\tau(\Phi) = C\). In a similar way, one verifies that \(\xi^{\text{Spin}(7)} \in T^* M^8 \otimes \mathfrak{spin}(7)^\perp\). Finally, it is straightforward to check that \(\nabla^{\text{Spin}(7)} \Phi = 0\). Hence \(\nabla^{\text{Spin}(7)}\) is a \(\text{Spin}(7)\)-connection. \(\square\)
Corollary 3.3. The covariant derivative $\nabla \Phi$ of the fundamental form is expressed in terms of the exterior derivative $d\Phi$ as follows

$$\nabla \Phi = -\xi^{\text{Spin}(7)} \Phi,$$

where $\xi^{\text{Spin}(7)}$ is determined in Theorem 3.2.

4. SU(3)-structures on six-dimensional submanifolds

Let $f : M^6 \to (M^8, \Phi, \langle \cdot, \cdot \rangle)$ be a smooth orientable six-manifold immersed in an eight-dimensional Spin(7)-manifold with fundamental form $\Phi$ and Riemannian metric $\langle \cdot, \cdot \rangle$.

Let $N_1, N_2$ be a local orthonormal frame of the normal bundle $T^1M^6$. The Spin(7)-structure on $M^8$ induces an almost Hermitian structure on $M^6$ defined [30]

$$JX = P(N_1, N_2, X), \quad X \in TM^6,$$

where $P$ is the three-fold vector cross product on $M^8$ determined by the Spin(7)-structure.

It is well known that the almost complex structure $J$ is independent on the particular oriented orthonormal frame and is compatible with the induced Riemannian metric on $M^6$ [30]. Thus, we have a natural global almost Hermitian structure on $M^6$, where the Kähler form $\omega$ and the Hodge star operator $*_6$ are determined by

$$\omega = *_6f^*\Phi, \quad -4\sigma Vol_6 = f^*(N_{1,\cdot}\Phi) \wedge f^*(N_{2,\cdot}\Phi).$$

Also note that $-2\sigma f^*\Phi = \omega \wedge \omega$.

As we have already pointed out, in general, there is not a global SU(3)-structure induced from the Spin(7)-structure on $M^8$. In fact, this assertion is based on the observation, due to Bryant [5], saying that the stabilizer of an oriented two-plane in Spin(7) is the group $U(3)$. In the case $M^8 = \mathbb{R}^8 = \mathbb{R}^1 \oplus \text{Im} \mathbb{O}$, where Im$\mathbb{O}$ is the space of imaginary octonions and $M^6 \subset \text{Im} \mathbb{O}$, there exists a global SU(3)-structure due to the fact that the stabilizer in Spin(7) of two unitary vectors is the group SU(3). This phenomena was discovered and studied by Calabi [10]. More general, any orientable hypersurface of a $G_2$-manifold inherits a global SU(3)-structure [10, 29, 45].

We consider local SU(3)-structures naturally induced from the Spin(7)-structure on $M^8$. Namely, define the real three-forms $\Psi_+, \Psi_-$ by the relations

$$\Psi_+ = \cos \gamma f^*(N_{1,\cdot}\Phi) - \sin \gamma f^*(\sigma N_{2,\cdot}\Phi),$$
$$\Psi_- = \sin \gamma f^*(N_{1,\cdot}\Phi) + \cos \gamma f^*(\sigma N_{2,\cdot}\Phi),$$

where $\gamma$ is a smooth function defined on $M^6$. The complex three-form $\Psi$ with the real part $Re(\Psi) = \Psi_+$ and imaginary part $Im(\Psi) = \Psi_-$ with respect to the induced almost complex structure $J$ defined by (4.21) is clearly a local complex volume form compatible with the induced $U(3)$-structure in the sense that it is a (3,0)-form with respect to $J$, $J(1)\Psi_+ = \Psi_-$. Fixing $-\frac{1}{2}\Psi_+ \wedge \Psi_-$ as real volume form, the metric $\langle \cdot, \cdot \rangle$ and the Kähler form $\omega$ are given by

$$\langle x, y \rangle = \frac{1}{2} *_6((x, y)\Psi_+) \wedge *_6(y, y)\Psi_+), \quad \omega(x, y) = \frac{1}{2} *_6((x, y)\Psi_-) \wedge *_6(y, y)\Psi_+),$$

respectively. The three-forms $\Psi_+$ and $\Psi_-$ clearly depend on the local orthonormal frame on the normal bundle. Therefore, they define a local SU(3)-structure compatible with the global almost Hermitian U(3)-structure $\langle \cdot, \cdot, J \rangle$. 
Remark 4.1. It is clear that all the local $SU(3)$-structures generating the same metric are described by taking all oriented orthonormal frames on the normal bundle and considering the corresponding local $SU(3)$-structures defined above by (4.22). Also note that if we consider the local frame $N'_1, N'_2$ on the normal bundle of $M^6$ given by $N'_1 = \cos \gamma N_1 - \sin \gamma N_2$ and $N'_2 = \sin \gamma N_1 + \cos \gamma N_2$, then the complex volume form $\Psi$ defined in (4.22) satisfy 
$$
\Psi_+ = f^*(N'_1 \Phi) \quad \text{and} \quad \Psi_- = f^*(\sigma N'_2 \Phi).
$$
In this way we recover all local $SU(3)$-structures generating the same almost hermitian structure.

The types of the induced global almost Hermitian $U(3)$-structure depend on the second fundamental form of the immersion and were described by Gray [30] (see also [5]). We show below that the type of the induced local $SU(3)$-structures also depends on the structure of the normal bundle.

We briefly recall some basic notions of the submanifold theory (see e.g. [12]).

Let us fix an oriented orthonormal frame $N_1, N_2$ of the normal bundle. Let $\nabla^8, \nabla^6$ be the Levi-Civita connection on $M^8, M^6$, respectively. The Gauss equations read

$$
\begin{align*}
\nabla^8_X Y &= \nabla^6_X Y + \alpha(X,Y), \\
\nabla^8_X N_j &= -A N_j X + D_X N_j, \quad j = 1, 2, \quad X, Y \in TM^6,
\end{align*}
$$

where

$$
\alpha(X,Y) = \alpha_1(X,Y) N_1 + \alpha_2(X,Y) N_2
$$

is the second fundamental form, $A N_j, j = 1, 2$ is the shape operator and $D$ is the normal connection. Since the normal two-frame is orthonormal, we have

$$
\begin{align*}
\langle A N_j X, Y \rangle &= \alpha_j(X,Y), \quad j = 1, 2, \quad X, Y \in TM^6, \\
D_X N_1 &= a(X) N_2, \\
D_X N_2 &= -a(X) N_1, \quad X \in TM^6,
\end{align*}
$$

where $a(X)$ is a smooth function on $M^6$ depending on $X$.

When the shape operator vanishes, $M^6$ is said to be totally geodesic. The mean curvature $H$ is defined by $H = 1/6 \operatorname{tr} \alpha = h_1 N_1 + h_2 N_2$, where $6h_1 = \operatorname{tr} \alpha_1, \quad 6h_2 = \operatorname{tr} \alpha_2$. The submanifold is said to be minimal, if $H = 0$, and totally umbilic, if $\alpha = \langle \cdot, \cdot \rangle H$.

4.1. Types of local $SU(3)$-structures induced on six-dimensional submanifolds. To investigate special types of local $SU(3)$-structures, we find relations between the local intrinsic $SU(3)$-torsion of $M^6$ and the global intrinsic $Spin(7)$-torsion of the ambient manifold $M^8$. In the next technical result, we get relations involving the intrinsic torsions, the shape operator and the structure of the normal bundle of $M^6$.

**Proposition 4.2.** For the local $SU(3)$-structures on an oriented submanifold $M^6$ of a $Spin(7)$-manifold $M^8$ inherited by the $Spin(7)$-structure of $M^8$ and defined by (4.22), we have the
Proof. On any point of $M^6$, we consider a Cayley frame $\{e = N_1, e_0 = N_2, e_1, \ldots, e_6\}$. Using (3.15) and (3.16), we obtain

$$\sigma\tau(\nabla^8 \Phi)(e_i, e_1, N_1) = a_{i1}(\nabla^8 \Phi)(N_1, N_2, e_1, e_6) = \langle (\nabla^8_{e_i} P)(N_1, N_2, e_1, e_6) \rangle.$$

From these identities it is not hard to show

$$\sigma\tau(\nabla^8 \Phi)(e_i, e_1, N_1) = -\langle \nabla^6_{e_i} \omega, (e_4, e_6) \rangle + \sigma\alpha_1(e_i, e_1) + \sigma_2(e_i, J e_1).$$

Since

$$2(\nabla^6_{e_i} \omega)(e_4, e_6) = \langle \nabla^6_{e_i} \omega, J e_1 \rangle f^*(N_1 \cdot \Phi) = -\langle \nabla^6_{e_i} \omega, (e_1 \cdot f^*(\sigma N_2 \cdot \Phi)) \rangle,$$

we get

$$\sigma\tau(\nabla^8 \Phi)(X, JY, N_1) = \frac{1}{2} \langle \nabla^6_X \omega, J e_1 \cdot f^*(N_1 \cdot \Phi) \rangle = \sigma\alpha_1(X, J Y) - \sigma_2(X, Y),$$

$$\sigma\tau(\nabla^8 \Phi)(X, Y, N_1) = \frac{1}{2} \langle \nabla^6_X \omega, J e_1 \cdot f^*(\sigma N_2 \cdot \Phi) \rangle = \sigma\alpha_1(X, Y) + \sigma_2(X, J Y).$$

Now, (4.27) follows from (4.35) and (4.36), using (4.22) and (2.9).

Next, we derive (4.28) from (3.18), taking (2.10) and (4.27) for $\gamma = 0$ into account. Note that the Lee form $\theta^6$ is independent on the choice of the complex volume form.
From (4.27) we get

\begin{equation}
\sigma \sum_{i=1}^{6} \tau(\nabla \Phi)(e_i, e_i, N_1) - 6\sigma h_1 = -\sin \gamma \text{tr} r(\nabla \omega) + 2 \cos \gamma \langle r(\nabla \omega), \omega \rangle,
\end{equation}

(4.37)

\begin{equation}
\sigma \sum_{i=1}^{6} \tau(\nabla \Phi)(e_i, Je_i, N_1) + 6h_2 = \cos \gamma \text{tr} r(\nabla \omega) + 2 \sin \gamma \langle r(\nabla \omega), \omega \rangle.
\end{equation}

(4.38)

Now (4.29) and (4.30) follow from (3.18), using (4.37) and (4.38).

Take \( \gamma = 0 \). Then \( \Psi_{\pm} = N_{\pm} \Phi \) and \( \Psi_{\mp} = \sigma N_{\pm} \Phi \). Apply (2.7) to get

\begin{equation}
*_{\rho} (*_{\rho}d^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)) = \sigma J *_{\rho} (*_{\rho}d^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)) = \sigma J *_{\rho} (*_{\rho}d^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)) = \sigma J *_{\rho} (*_{\rho}d^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)).
\end{equation}

(4.39)

Use (2.2), (2.3), (4.39) and (2.7) for a generic \( \gamma \) to obtain

\begin{equation}
*_{\rho} (*_{\rho}d^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)) = 6 \eta + \theta^6 = -2Jd \gamma + *_{\rho} (*_{\rho}d^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)),
\end{equation}

(4.40)

\begin{equation}
*_{\rho} (*_{\rho}d^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)) = 6 \eta + \theta^6 = -2Jd \gamma + *_{\rho} (*_{\rho}d^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)).
\end{equation}

(4.41)

From (3.17), (3.18) and (4.28), we obtain

\begin{align*}
*_{\rho} (*_{\rho}f^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)) &= -\theta^6 - 2\tau(\Phi)(N_1, N_1, f_{\pm}) + 2\sigma \tau(\Phi)(f_{\pm}, N_2, N_2), \\
*_{\rho} (*_{\rho}f^*(N_{\pm} \Phi) \wedge f^*(N_{\pm} \Phi)) &= -\theta^6 - 2\sigma \tau(\Phi)(N_2, f_{\pm}, N_1) + 2\sigma \tau(\Phi)(f_{\pm}, N_2, N_2),
\end{align*}

where we used the well known identity

\begin{equation}
d(N_{\pm} \Phi) = L_{N_{\pm}} \Phi - N_{\pm}d \Phi.
\end{equation}

(4.42)

Now, (4.31) and (4.32) follow from (4.40) and (4.41). Finally, (4.33) and (4.34) are consequences of (4.39), (3.17) and (3.18), taking the identity (4.42) into account. \( \square \)

Proposition 4.2 gives us chance to find relations between the Spin(7)-structure on the ambient eight-dimensional manifold and the local SU(3)-structure inherited on the six-dimensional submanifold involving the second fundamental form.

**Theorem 4.3.** Let \( M^6 \) be an eight-dimensional Riemannian manifold with a parallel Spin(7)-structure. Let \( M^6 \) be an oriented six-dimensional submanifold of \( M^8 \) with the local SU(3)-structure defined by (4.21), (4.22). Then \( M^6 \) is of type \( W_1^+ \oplus W_2^+ \oplus W_2^- \oplus W_3 \oplus W_5 \) and the following identities hold

\begin{equation}
*_{\rho} (*_{\rho}f^*(L_{N_1} \Phi) \wedge f^*(N_{\pm} \Phi)) = *_{\rho} (*_{\rho}f^*(L_{N_2} \Phi) \wedge f^*(N_{\pm} \Phi)) = -\sigma J *_{\rho} (*_{\rho}f^*(L_{N_1} \Phi) \wedge f^*(N_{\pm} \Phi)) = \sigma J *_{\rho} (*_{\rho}f^*(L_{N_2} \Phi) \wedge f^*(N_{\pm} \Phi)).
\end{equation}

(4.43)

The precise conditions which characterized the types of local SU(3)-structures on \( M^6 \) are displayed in Table 1.

In particular:

a) \( M^6 \) is a minimal submanifold if and only if the global U(3)-structure belongs to the class \( W_2 \oplus W_3 \) in the Gray-Hervella classification.

b) The global U(3)-structure on \( M^6 \) is nearly Kähler (type \( W_1 \)) if and only if the submanifold is totally umbilical.
c) The global U(3)-structure on $M^6$ is Kähler if and only if the submanifold is totally geodesic.

Proof. The identities (4.43) are direct consequences of (4.39), (4.42) and the condition $d\Phi = 0$. Observe that the latter implies $\tau(\nabla^8 \Phi) = 0$. Now, Table 1 and the remaining part of Theorem 4.3 are consequences of the equations given in Proposition 4.2. □

Remark 4.4. Note that Theorem 4.3 includes the results obtained by Gray in [30].

Theorem 4.5. Let $M^8$ be an eight-dimensional Riemannian manifold with a Spin(7)-structure having zero Lee form, $\theta^8 = 0$. Let $M^6$ be an oriented six-dimensional submanifold of $M^8$ with the local SU(3)-structures defined by (4.21), (4.22). Then:

a) The precise conditions characterizing the types of the local SU(3)-structure are given in Table 2.

b) The following identities hold

$$\theta^6 = -\tau(\nabla^8 \Phi)(N_1, N_1, f_s \cdot) - \sigma \tau(\nabla^8 \Phi)(N_2, f_s J_s, N_1) + \sigma \tau(\nabla^8 \Phi)(f_s J_s, N_2, N_1),$$

$$\text{tr} r(\nabla^6 \omega) = -2 \sin \gamma (\sigma h_1 - \sigma \theta^8(N_1)) + 2 \cos \gamma (h_2 - \tau(\nabla^8 \Phi)(N_1, N_2, N_1)),$$

$$\langle r(\nabla^6 \omega), \omega \rangle = \cos \gamma (\sigma h_1 - \sigma \theta^8(N_1, N_2, N_1)) + \sin \gamma (h_2 - \tau(\nabla^8 \Phi)(N_1, N_2, N_1)).$$

Proof. Using $\theta^8 = 0$, the equalities in Proposition 4.2 imply the assertion. □

Theorem 4.6. Let $M^8$ be an eight-dimensional Riemannian manifold with a locally conformal parallel Spin(7)-structure, i.e. $d\Phi = \theta^8 \wedge \Phi$. Let $M^6$ be an oriented six-dimensional submanifold of $M^8$ with the local SU(3)-structures defined by (4.21), (4.22). Then:

a) The following identities hold

$$*_6 (*_6 f^*(L_{N_1} \Phi) \wedge f^*(N_1 \cdot \Phi)) = *_6 (*_6 f^*(L_{N_2} \Phi) \wedge f^*(N_2 \cdot \Phi)) = -\sigma J *_6 (*_6 f^*(L_{N_1} \Phi) \wedge f^*(N_1 \cdot \Phi)),$$

$$4 r(\nabla^6 \omega) = \cos \gamma (\sigma \theta^8(N_1) + \Phi(\theta^8, N_1, f^*, f^* J_s) + 4 \sigma J(2) \alpha_1 + 4 \alpha_2 - \sin \gamma (\sigma \theta^8(N_1) + \Phi(\theta^8, N_1, f^*, f^* J_s) - 4 \sigma \alpha_1 + 4 J(2) \alpha_2),$$

$$\theta^6 = f^* \theta^8,$$

$$6 \eta = -2 J d \gamma + *_6 (*_6 f^*(L_{N_1} \Phi) \wedge f^*(N_1 \cdot \Phi)) + J d^* \omega,$$

$$\frac{2}{3} \text{tr} r(\nabla^6 \omega) = \sin \gamma (4 \sigma h_1 - \sigma \theta^8(N_1)) + \cos \gamma (4 h_2 - \theta^8(N_2)),$$

$$\frac{4}{3} \langle r(\nabla^6 \omega), \omega \rangle = -\cos \gamma (4 \sigma h_1 - \sigma \theta^8(N_1)) + \sin \gamma (4 h_2 - \theta^8(N_2)).$$

b) The precise conditions characterizing the types of the local SU(3)-structure are given in Table 3. In particular:

i) The global U(3)-structure is locally conformal equivalent to a nearly Kähler structure if and only if $M^6$ is totally umbilic submanifold. If moreover $\theta^8$ is normal to $M^6$, then the structure is nearly Kähler.

ii) The global U(3)-structure is locally conformal Kähler if and only if $M^6$ is totally umbilic submanifold such that $h_1 = \frac{1}{4} \theta^8(N_1)$, $h_2 = \frac{1}{4} \theta^8(N_2)$. If moreover $\theta^8$ is normal to $M^6$, then it is a Kähler structure.
Proof. Since the Spin(7)-structure is locally conformal parallel, the equality (3.19) is valid and \( d\theta^8 = 0 \). Therefore, the equalities in a) as well as the conditions in Table 3 are direct consequences of (3.19) and Proposition 4.2. The totally umbilical conditions are derived in the same way as in the proof of the Theorem 4.3. Now i) follows from the recent result [9] which states that any six-dimensional almost Hermitian manifold in the class \( \mathcal{W}_1 \oplus \mathcal{W}_4 \) is locally conformal to a nearly Kähler space. Finally, if \( \theta^8 \) is normal to \( M^6 \), then (4.44) shows that the Lee form on \( M^6 \) vanishes. □

Corollary 4.7. Let \( M^8 \) be an eight-dimensional Riemannian manifold with a locally conformal parallel Spin(7)-structure, i.e. \( d\Phi = \theta^8 \wedge \Phi \). Let \( M^6 \) be an oriented six-dimensional submanifold of \( M^8 \) with the local SU(3)-structures defined by (4.21), (4.22). If the Lee form \( \theta^8 \) is tangent to \( M^6 \), then the precise conditions characterizing the types of the local SU(3)-structure are given in Table 4. In particular:

i) The global U(3)-structure is Kähler if and only if \( M^6 \) is totally geodesic and the restriction of the Lee form to \( M^6 \) vanishes, i.e. \( f^*\theta^8 = 0 \).

ii) The global U(3)-structure is locally conformal Kähler if and only if \( M^6 \) is totally geodesic.

iii) The global U(3)-structure is of type \( \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \) if and only if \( M^6 \) is minimal.

5. Holomorphic complex volume form

We investigate the case when the induced local complex volume form is closed, which implies, in particular, that the almost complex structure is integrable [35].

We begin with

Proposition 5.1. Let \((M^8, \Phi, g)\) be a Spin(7)-manifold. Let \( M^6 \) be an oriented six-dimensional submanifold and let \( N_1, N_2 \) be any orthonormal frame of the normal bundle. Then the complex volume form

\[
\Psi = \Psi_+ + i\Psi_-,
\]

\[
\Psi_+ = f^*(N_1 \Phi),
\]

\[
\Psi_- = f^*(\sigma N_2 \Phi)
\]

is closed, \( d\Psi = 0 \), if an only if the next two conditions hold simultaneously

\[
\tag{5.45}
L_{N_1} \Phi|_{M^6} = (N_1 \cdot d\Phi)|_{M^6}, \quad L_{N_2} \Phi = (N_2 \cdot d\Phi)|_{M^6}.
\]

In particular, the almost complex structure is integrable.

If the Spin(7)-structure is parallel, \( d\Phi = 0 \), then the complex volume form is closed exactly when

\[
\tag{5.46}
L_{N_1} \Phi|_{M^6} = L_{N_2} \Phi|_{M^6} = 0.
\]

In particular, if the normal bundle is parallel along the submanifold, then there exists a local closed complex volume form compatible with the induced global almost Hermitian U(3)-structure.

Proof. Take the exterior derivative in (4.22) and use (4.42) to get (5.45) and, consequently, (5.46). The integrability of the almost complex structure in the case of closed complex volume form follows from the result of Hitchin [35].
The Lie derivative is expressed in terms of the Levi-Civita connection as follows
\begin{align}
(L_N \Phi)(X, Y, Z, V) &= (\nabla^8_N \Phi)(X, Y, Z, V) + \\
\Phi(\nabla^8_N X, Y, Z, V) + \Phi(X, \nabla^8_N Y, Z, V) + \Phi(X, Y, \nabla^8_N Z, V) + \Phi(X, Y, Z, \nabla^8_N V).
\end{align}

Since the normal bundle is parallel along \( M^6 \), we may choose a parallel oriented normal two-frame. Take the corresponding complex volume form, we see that it is closed due to (5.46) and (5.47).

As a consequence of the proof of Proposition 5.1, we get a result which second part is essentially established in [45].

**Theorem 5.2.** There exist a local half-flat SU(3)-structure induced on a six-dimensional submanifold of a parallel Spin(7)-manifold if and only if there exists a normal vector field which preserves the parallel Spin(7)-form restricted to the submanifold.

In particular, any orientable hypersurface \( M^6 \subset R^7 = Im \mathbb{O} \subset \mathbb{O} \) carries a global half-flat SU(3)-structure.

**Proof.** Since \( M^6 \subset R^7 = Im \mathbb{O} \subset \mathbb{O} \), we may take \( \cos \gamma = 1 \) and \( \nabla^8 N_1 = 0 \). Therefore, \( d\Psi_+ = 0 \) according to the proof of Proposition 5.1. Hence, (2.11) are satisfied since \( \theta^6 = 0 \). \( \square \)

5.1. **Application to Calabi and Bryant examples.** Now we restrict our attention to the case \( M^8 = \mathbb{O} \) studied in detail by Bryant in [5]. In this case (even more general, when the Spin(7)-structure of the ambient manifold is parallel), some of the U(3)-components of the induced almost Hermitian structure are described by Gray [30] (see also [5]). He showed that the Lee form \( \theta^6 \) is always zero and the submanifold \( M^6 \) is necessarily minimal. Therefore, if the almost complex structure is integrable, then it is balanced (type \( W_3 \)). Submanifolds with balanced almost Hermitian structure are investigated by Bryant in [5]. He shows that if \( M^6 \subset \mathbb{O} \) inherits complex and non-Kähler structure, then \( M^6 \) is foliated by four-planes in \( \mathbb{O} \) in a unique way, he calls this foliation asymptotic ruling. He also obtains that if the asymptotic ruling is parallel, then \( M^6 \) is a product of a fixed associative four-plane \( Q^4 \) in \( \mathbb{O} \) with a minimal surface in the orthogonal four-plane. Moreover, Bryant found that the Calabi examples, described in [10], are exactly those complex \( M^6 \) with parallel asymptotic ruling which lie in \( Im \mathbb{O} \subset \mathbb{O} \), i.e. the minimal surface lies in an associative three-plane in \( Im \mathbb{O} \).

We investigate below when the local SU(3)-structures is holomorphic in the case of parallel asymptotic ruling.

To be more precise, we explain the Bryant construction. Let \( R^8 = \mathbb{O} = R^4 \oplus Q^4 \) be an orthogonal sum of Cayley planes and let \( S \subset R^4 \) be a surface. Then \( S \times Q^4 \subset \mathbb{O} \) inherits a complex structure if and only if \( S \) is minimal in \( R^4 \) and non-Kähler provided \( S \) is not a complex curve in \( R^4 \)’s complex structures [5]. We have

**Theorem 5.3.** Let \( S \subset R^4 \) be a minimal surface in \( R^4 \) such that \( M^6 = S \times Q^4 \subset \mathbb{O} \) is a non-Kähler complex manifold with respect to the U(3)-structure induced from \( \mathbb{O} \). There exists a local holomorphic SU(3)-structure compatible with the U(3)-structure if and only if \( S \) is a minimal surface in a three-plane \( R^3 \). In this case the SU(3)-structure is globally defined and the holomorphic volume form is parallel with respect to the Bismut connection. In particular, the SU(3)-structure described by Calabi is holomorphic CYT structure.
Proof. We need information for the Lie derivative of the fundamental four-form in the normal direction due to Proposition 5.1.

Let us fix an oriented orthonormal frame $N_1, N_2$ in the normal bundle $T^\perp S \subset \mathbb{R}^4$ in $\mathbb{R}^4$ and a local frame $X_3, X_4$ of the tangent bundle $TS$. We denote $e_5, e_6, e_7, e_8$ the vectors in $Q^4$. We may write (4.25) and (4.26) in the form

\begin{align*}
A_{N_1} X_3 & = \alpha_1(X_3, X_3)X_3 + \alpha_1(X_3, X_4)X_4, \\
A_{N_1} X_4 & = \alpha_1(X_4, X_3)X_3 + \alpha_1(X_4, X_4)X_4,
\end{align*}

\begin{align*}
A_{N_2} X_3 & = \alpha_2(X_3, X_3)X_3 + \alpha_2(X_3, X_4)X_4, \\
A_{N_2} X_4 & = \alpha_2(X_4, X_3)X_3 + \alpha_2(X_4, X_4)X_4.
\end{align*}

(5.48)

Using (5.48), (5.49), we obtain from (5.47) that

\begin{align*}
D_{X_3} N_1 & = a(X_3) N_2, \\
D_{X_3} N_2 & = -a(X_3) N_1, \\
D_{X_4} N_1 & = a(X_4) N_2, \\
D_{X_4} N_2 & = -a(X_4) N_1.
\end{align*}

(5.49)

The minimality condition implies the equalities

\begin{align*}
\alpha_1(X_3, X_3) + \alpha_1(X_4, X_4) = 0, \\
\alpha_2(X_3, X_3) + \alpha_2(X_4, X_4) = 0.
\end{align*}

(5.50)

Using (5.48), we obtain from (5.47) that $(L_{N_j} \Phi)(X_k, e_l, e_m, e_p) = 0$, for $j = 1, 2$, $k = 3, 4$ and $l, m, p = 5, 6, 7, 8$, since $Q^4$ is a Cayley four-plane. It remains to investigate the case when two of the four vectors are tangent to $S$. We need in addition to take into account the minimality condition (5.50). We obtain

\begin{align*}
(L_{N_1} \Phi)(X_3, X_4, e_l, e_m) & = a(X_3) \Phi(N_2, X_4, e_l, e_m) - a(X_4) \Phi(N_2, X_3, e_l, e_m), \\
(L_{N_2} \Phi)(X_3, X_4, e_l, e_m) & = -a(X_3) \Phi(N_1, X_4, e_l, e_m) + a(X_4) \Phi(N_1, X_3, e_l, e_m).
\end{align*}

(5.51)

Taking into account that $Q^4$ is a Cayley submanifold, we get from (5.51) that $L_{N_1} \Phi|_{M^6} = L_{N_2} \Phi|_{M^6} = 0$ if and only if $a(X_3) = a(X_4) = 0$, i.e. the normal connection is flat. Now, Proposition 5.1 and Remark 4.1 yield that there is a local holomorphic complex volume form compatible with the induced metric exactly when the minimal surface $S$ has flat normal bundle. It is known that a minimal submanifold of an Euclidean space has flat normal connection if and only if it lies in a three-dimensional plane $\mathbb{R}^3$ (see e.g. [12]). In this case, $\theta^6 = d\Psi_+ = d\Psi_- = 0$. Apply Theorem 4.1 of [38] to conclude that the corresponding Bismut connection preserves the complex volume form $\Psi$, i.e. it has holonomy contained in $SU(3)$. Therefore, the structure is Calabi-Yau with torsion which completes the proof. \qed

Applying [38, Theorem 4.1], we obtain in view of Theorem 5.3

**Theorem 5.4.** Let $S \subset \mathbb{R}^4$ be a minimal surface in $\mathbb{R}^4$ such that $M^6 = S \times Q^4 \subset \mathbb{O}$ is a non-Kähler complex manifold with respect to the $U(3)$-structure induced from $\mathbb{O}$. Then the Bismut connection of this $U(3)$-structure has holonomy contained in $SU(3)$ if and only if $S$ is a minimal surface in a three-plane $\mathbb{R}^3$.

In particular, the holonomy of the Bismut connection of the $SU(3)$-structure described by Calabi is contained in $SU(3)$. Consequently, the compact complex non-Kähler six-manifolds with holomorphically trivial canonical bundle constructed by Calabi are balanced CYT-manifolds with respect to the Calabi’s $SU(3)$-structure.
6. Examples

Example 6.1. $S^3 \times S^3$. Let us consider $\mathbb{R}^8$ with its standard parallel Spin(7)-structure. Thus, if $(x, x_0, \ldots, x_6)$ are the global coordinates of $\mathbb{R}^8$, the Spin(7)-structure on $\mathbb{R}^8$ is the one such that $\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^6} \right\}$ is a Cayley frame. For sake of simplicity, we will denote $e = \frac{\partial}{\partial x^6}$ and $e_i = \frac{\partial}{\partial x^i}$, for $i \in \mathbb{Z}_7$.

Let $S^3_1 \times S^3_2$ be the six-submanifold of $\mathbb{R}^8$ consisting of the product of two three-dimensional spheres $S^3_1 \subseteq (\mathbb{R}^4)_1 = \text{span} \left\{ e, e_0, e_1, e_3 \right\}$ and $S^3_2 \subseteq (\mathbb{R}^4)_2 = \text{span} \left\{ e_2, e_4, e_5, e_6 \right\}$. Fixing the oriented normal frame $N_1 = xe + x_0e_0 + x_1e_1 + x_3e_3$, $N_2 = x_2e_2 + x_4e_4 + x_5e_5 + x_6e_6$, we consider the SU(3)-structure on $S^3_1 \times S^3_2$ defined by (4.21) and (4.22). This SU(3)-structure is globally defined on $S^3_1 \times S^3_2$, since the stabilizer of two orthonormal vectors in Spin(7) is the group SU(3) and is compatible with the standard product metric on $S^3 \times S^3$.

The tangent bundle of $S^3 \times S^3$ is decomposed into $T \left( S^3_1 \times S^3_2 \right) = TS^3_1 \oplus TS^3_2$ and, for all $X \in T \left( S^3_1 \times S^3_2 \right)$, we have the corresponding decomposition $X = X_1 + X_2$. The observation

\[
P(N_1, N_2, e), P(N_1, N_2, e_0), P(N_1, N_2, e_1), P(N_1, N_2, e_3) \in T \left( \mathbb{R}^4 \right)_2,
\]

\[
P(N_1, N_2, e_2), P(N_1, N_2, e_4), P(N_1, N_2, e_5), P(N_1, N_2, e_6) \in T \left( \mathbb{R}^4 \right)_1
\]
yields $J \left( T_pS^3_1 \right) = T_pS^3_2$ and $J \left( T_pS^3_2 \right) = T_pS^3_1$, for any point $p \in S^3_1 \times S^3_2$.

The second fundamental form is given by $\alpha_1(X, Y) = -\langle X_1, Y_1 \rangle$, $\alpha_2(X, Y) = -\langle X_2, Y_2 \rangle$. Consequently, $(1 + J)\alpha_1 = \alpha_1 + \alpha_2 = 2h_1(\cdot, \cdot)$, $\alpha_2 = \alpha_2 + \alpha_1 = 2h_2(\cdot, \cdot)$. Using the results in Theorem 4.3 and Table 1, we conclude that the SU(3)-structure on $S^3 \times S^3$ is of type $\mathcal{W}^+_1 \oplus \mathcal{W}^-_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_5$.

We describe the $\mathcal{W}_5$-part $\eta$ of the intrinsic SU(3)-torsion. The Lie derivative $L_{N_1} \Phi$ restricted to $S^3_1 \times S^3_2$ is given by

\[
f^* \left( L_{N_1} \Phi \right) = -\text{Alt} \langle \nabla^8 N_1, P(\cdot, \cdot, \cdot) \rangle = \text{Alt} \left( \alpha_1(\cdot, P(\cdot, \cdot, \cdot)) \right) = -2\sigma \Phi |_{T \left( S^3_1 \times S^3_2 \right)} = \omega \wedge \omega.
\]

This can be checked using a Cayley frame $\left\{ N_1, N_2, u_1, \ldots, u_6 \right\}$, where $u_1, u_2, u_4 \in TS_1^3$ and $u_3, u_5, u_6 \in TS^3_2$. Such a Cayley frame do exist because the almost complex structure $J$ maps the tangent space of one $S^3$ to the tangent space of the another $S^3$. Note also that

\[
*_{\Phi} f^* \left( L_{N_1} \Phi \right) = -2\omega.
\]

Since $f^* \left( L_{N_1} \Phi \right)$ is a linear combination of $\Psi_+$ and $\Psi_-$, we have

\[
*_{\Phi} f^* \left( L_{N_1} \Phi \right) \land f^* \left( N_1 \Phi \right) = -2\omega \land f^* \left( N_1 \Phi \right) = 0.
\]

Now, using Equation (4.31), we get $3\eta = -Jd\gamma$. Hence, the $\mathcal{W}_5$-part, $\eta$, of the intrinsic SU(3)-torsion vanishes exactly when $\gamma$ is a constant.

We compute the exterior derivatives $d\omega$, $d\Psi_+$ and $d\Psi_-$. Consider three orthonormal vector fields $v_1, v_2, v_3$ in $T \left( S^3 \right)_1$ such that $\Phi(v_1, v_2, v_3) = 1$ (or $v_3 = P(N_1, v_1, v_2)$). We know that $Jv_1, Jv_2, Jv_3 \in T \left( S^3 \right)_2$. Taking into account the expression for $\Psi$ given by (4.22), we obtain $\Psi(v_1, v_2, v_3) = e^{i\gamma}$. Therefore, $v_1, v_2, v_3, Jv_1, Jv_2, Jv_3$ is an adapted basis for the $U(3)$-structure but not for the SU(3)-structure considered. However, if we write, for $i = 1, 2, 3$,

\[
u_i = e^{-i\frac{\gamma}{3}} v_i = \cos \frac{\gamma}{3} v_i - \sin \frac{\gamma}{3} Jv_i,
\]

(6.52)
then we have \( \Psi(u_1, u_2, u_3) = 1 \) and hence \( u_1, u_2, u_3, Ju_1, Ju_2, Ju_3 \) is a local frame adapted to the SU(3)-structure. For the second fundamental form we get the expressions

\[
\alpha_1 = -\sum_{i=1}^{3} v_i \otimes v_i = -\cos^2 \frac{\gamma}{3} \sum_{i=1}^{3} u_i \otimes u_i - \sin^2 \frac{\gamma}{3} \sum_{i=1}^{3} Ju_i \otimes Ju_i - \sin \frac{2\gamma}{3} \sum_{i=1}^{3} u_i \vee Ju_i,
\]

\[
\alpha_2 = -\sum_{i=1}^{3} Jv_i \otimes Jv_i = -\sin^2 \frac{\gamma}{3} \sum_{i=1}^{3} u_i \otimes u_i - \cos^2 \frac{\gamma}{3} \sum_{i=1}^{3} Ju_i \otimes Ju_i + \sin \frac{2\gamma}{3} \sum_{i=1}^{3} u_i \vee Ju_i,
\]

where \( \vee \) denotes the symmetric product \( a \vee b = 1/2 (a \otimes b + b \otimes a) \). From equation (4.28), we obtain

\[
\begin{align*}
\rho(\nabla^6 \omega) &= -\frac{1}{2}(\cos \gamma + \sigma \sin \gamma) (\cdot, \cdot) - \frac{1}{2}(\sin \gamma - \sigma \cos \gamma) \omega - (\sin \frac{\gamma}{3} + \sigma \cos \frac{\gamma}{3}) \sum_{i=1}^{3} u_i \vee Ju_i \\
&+ \frac{1}{2}(\cos \frac{\gamma}{3} - \sigma \sin \frac{\gamma}{3}) \sum_{i=1}^{3} (u_i \otimes u_i - Ju_i \otimes Ju_i).
\end{align*}
\]

The first two terms constitute the \( \mathcal{W}_1 \)-part of the tensor \( \rho(\nabla^6 \omega) \), while the \( \mathcal{W}_3 \)-part consists of the last two remaining terms.

We have already deduced at the end of Subsection 2.1 that \( \rho(\nabla^6 \omega) = \sum_{j,k=1}^{6} c_{jk} e_j \otimes e_k \) implies \( \nabla^6 \omega = \sum_{j,k=1}^{6} c_{jk} e_j \otimes e_k - \rho + \Psi_+ \). Then it follows

\[
(6.53) \quad \nabla^6 \omega = -\frac{1}{2}(\cos \gamma + \sigma \sin \gamma) \Psi_+ - \frac{1}{2}(\sin \gamma - \sigma \cos \gamma) \Psi_-
\]

\[
+ \frac{1}{2}(\cos \frac{\gamma}{3} - \sigma \sin \frac{\gamma}{3})(u_1 \wedge u_2 \wedge u_3 + \mathcal{S}_{ijk=123}(u_i \wedge Ju_j \wedge Ju_k - 2u_i \otimes Ju_j \wedge Ju_k))
\]

\[
+ \frac{1}{2}(\sin \frac{\gamma}{3} + \sigma \cos \frac{\gamma}{3})(Ju_1 \wedge Ju_2 \wedge Ju_3 + \mathcal{S}_{ijk=123}(Ju_i \wedge u_j \wedge u_k - 2Ju_i \otimes u_j \otimes u_k)),
\]

where \( \mathcal{S} \) denotes cyclic sum. Thus the exterior derivative \( d\omega \) of the Kähler form is given by

\[
\begin{align*}
d\omega &= -\frac{3}{2}(\cos \gamma + \sigma \sin \gamma) \Psi_+ - \frac{3}{2}(\sin \gamma - \sigma \cos \gamma) \Psi_- \\
&+ \frac{1}{2}(\cos \frac{\gamma}{3} - \sigma \sin \frac{\gamma}{3})(3u_1 \wedge u_2 \wedge u_3 + \mathcal{S}_{ijk=123} u_i \wedge Ju_j \wedge Ju_k)
\]

\[
+ \frac{1}{2}(\sin \frac{\gamma}{3} + \sigma \cos \frac{\gamma}{3})(3Ju_1 \wedge Ju_2 \wedge Ju_3 + \mathcal{S}_{ijk=123} Ju_i \wedge u_j \wedge u_k).
\]

It was shown in [44] that if \( (\nabla^6 \omega)_{\mathcal{W}_1} = \lambda \Psi_+ + \mu \Psi_- \), then \( (d\Psi_+)_F = 2\mu \omega \wedge \omega \) and \( (d\Psi_-)_F = 2\lambda \omega \wedge \omega \). Combining this with (2.6), we get from (6.53) that

\[
(6.54) \quad d\Psi_+ = -(\sin \gamma - \sigma \cos \gamma) \omega \wedge \omega + Jd\gamma \wedge \Psi_+,
\]

\[
(6.55) \quad d\Psi_- = -(\cos \gamma + \sigma \sin \gamma) \omega \wedge \omega + Jd\gamma \wedge \Psi_-.
\]

In particular, one can consider

\[
\begin{align*}
v_1 &= -\sigma x_0 e + \sigma x_0 e + x_3 e_1 - x_1 e_3, \quad Jv_1 = \sigma (x_6 e_2 + x_5 e_4 - x_4 e_5 - x_2 e_6), \\
v_2 &= -\sigma x_1 e - x_3 e_0 + x_1 e_2, \quad Jv_2 = \sigma (x_4 e_2 - x_2 e_4 + x_6 e_5 - x_5 e_6), \\
v_3 &= -x_3 e + \sigma x_1 e_0 + x_0 e_1 + x_3 e_3, \quad Jv_3 = -x_5 e_2 + x_6 e_4 + x_2 e_5 - x_4 e_6.
\end{align*}
\]

It is straightforward to check that \( \Phi(N_1, v_1, v_2, v_3) = 1 \) and

\[
(6.56) \quad dv_1 = -2\sigma v_2 \wedge v_3, \quad dv_2 = -2\sigma v_3 \wedge v_1, \quad dv_3 = -2\sigma v_1 \wedge v_2,
\]

\[
d(Jv_1) = 2Jv_2 \wedge Jv_3, \quad d(Jv_2) = 2Jv_3 \wedge Jv_1, \quad d(Jv_3) = 2Jv_1 \wedge Jv_2.
\]

Now, using (6.52) and (6.56), we can compute \( du_i, d(Ju_i) \). From these, \( d\omega, d\Psi_+ \) and \( d\Psi_- \) can be again obtained by an alternative way.
For the Nijenhuis tensor $N$, we calculate $N = 2\sqrt{2} \Psi \tilde{\tau}$, where $\Psi \tilde{\tau}$ is obtained from (4.22) for $\gamma = \frac{\pi}{4}$. Thus, taking $\sigma = +1$ in (2.12) and $\gamma = \frac{\pi}{4}$ in (4.22), from (6.54) and (6.55) we obtain

$$d\Psi \tilde{\tau} = 0, \quad d\Psi \tilde{\tau} = -\sqrt{2} \omega \wedge \omega - \frac{1}{8}(N, \Psi \tilde{\tau}) \omega \wedge \omega.$$ 

Applying [38, Theorem 4.1], we conclude that the unique $U(3)$-connection $\nabla$ with totally skew-symmetric torsion, defined in [21], preserves the $SU(3)$-structure $(\nabla \Psi \tilde{\tau} = 0)$ on $S^3 \times S^3$ obtained for $\sigma = +1, \gamma = \frac{\pi}{4}$. In particular, the Nijenhuis tensor $N$ is $\nabla$-parallel and nowhere vanishing. Therefore, the structure is strict quasi-integrable $U(3)$-structure in the sense of [7].

More precisely, we have:

- if $\sigma = +1$ and $\gamma = \frac{\pi}{4}, \frac{3\pi}{4}$, then the $SU(3)$-structure on $S^3_1 \times S^3_2$ is compatible with the standard product metric and half-flat of type $\mathcal{W}_1^+ \oplus \mathcal{W}_3$.
- if $\sigma = -1$ and $\gamma = \frac{\pi}{4}, \frac{3\pi}{4}$, then the $SU(3)$-structure on $S^3_1 \times S^3_2$ is compatible with the standard product metric and half-flat of type $\mathcal{W}_1^+ \oplus \mathcal{W}_3$.

Since $S^3_1 \times S^3_2 \subset \mathbb{R}^8$ is neither totally umbilic nor minimal, these structures are neither nearly Kähler nor complex. Moreover, for these cases, we have a global half-flat $SU(3)$-structure on $S^3_1 \times S^3_2$ with totally skew-symmetric $\nabla$-parallel nowhere vanishing Nijenhuis tensor. Therefore, each one of such structures is strict quasi-integrable $U(3)$-structure in the sense of [7] on $S^3 \times S^3$ which is neither nearly Kähler nor complex.

**Remark 6.2.** Consider $S^3 \times S^3 \cong SU(2) \times SU(2)$ as the group manifold $SU(2) \times SU(2)$ and observe that the basis defined by (6.56) is (up to an orientation) the standard left-invariant basis on the group manifold $SU(2) \times SU(2) \cong S^3 \times S^3$. This shows that the $U(3)$-structure defined in Example 6.1 is left-invariant compatible with the bi-invariant Riemannian metric on the group $SU(2) \times SU(2)$. The torsion connection $\nabla$ coincides with the flat canonical connection $\nabla$ on the group manifold $SU(2) \times SU(2)$ defined by making the standard left invariant basis $\nabla$-parallel.

**Example 6.3.** The following examples are already well known, but we pointed out them just to illustrate results here exposed. We consider the product manifold of spheres $S^7 \times S^1$. In [41], it is shown that $S^7 \times S^1$ has a locally conformal parallel $\text{Spin}(7)$-structure such that the Lee form $\theta^8$ is a constant multiple of the Maurer-Cartan one-form on $S^1$. Since $S^5 \times S^1$ is a totally geodesic submanifold of $S^7 \times S^1$ and $\theta^8$ is tangent to $S^5 \times S^1$, by Corollary 4.7, the induced $U(3)$-structure on $S^5 \times S^1$ is locally conformal Kähler. On the other hand, the sphere $S^6$ is totally geodesic in $S^7 \times S^1$, but now $\theta^8$ is normal to $S^6$. Hence, by Theorem 4.6, the induced $U(3)$-structure on $S^6$ is nearly Kähler.

**Example 6.4.** Let $He\mathbb{L}^2$ be the two-dimensional helicoid

$$x^0 = \sinh u \cos v, \quad x^1 = \sinh u \sin v, \quad x^3 = v$$

lying in the Cayley plane $\mathbb{R}^4 = \text{span}\{e, e_0, e_1, e_3\}$. Taking the frame on the normal bundle

$$N_1 \cosh u = -\sin ve_0 + \cos ve_1 - \sinh ue_3, \quad N_2 = e,$$
\[ W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 \quad Jd\gamma = \frac{1}{2} \ast_6 (*_6 f^*(L_N, \Phi) \wedge f^*(N_1, \Phi)) \]
\[ W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 \quad (1 - J)\sigma \alpha_1 = J(1 - J)\alpha_2 \]
\[ W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 \quad \cos \gamma (1 + J)\sigma \alpha_1 - \sin \gamma (1 + J)\alpha_2 = 2 (\sigma h_1 \cos \gamma - h_2 \sin \gamma) \langle \cdot, \cdot \rangle \]
\[ W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 \quad \sin \gamma (1 + J)\sigma \alpha_1 + \cos \gamma (1 + J)\alpha_2 = 2 (\sigma h_1 \sin \gamma + h_2 \cos \gamma) \langle \cdot, \cdot \rangle \]
\[ W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 \quad \sigma h_1 \cos \gamma = h_2 \sin \gamma \]
\[ W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 + W_5 \quad \sigma h_1 \sin \gamma = -h_2 \cos \gamma \]
\[ W_1^+ + W_1^- + W_3 + W_5 \quad (1 + J)\alpha_1 = 2 h_1 \langle \cdot, \cdot \rangle \quad \text{and} \quad (1 + J)\alpha_2 = 2 h_2 \langle \cdot, \cdot \rangle \]
\[ W_2^+ + W_2^- + W_3 + W_5 \quad h_1 = 0 \quad \text{and} \quad h_2 = 0, \text{i.e.} \quad M^6 \text{ is minimal} \]
\[ W_1^+ + W_1^- + W_5 \quad \alpha_1 = h_1 \langle \cdot, \cdot \rangle \quad \text{and} \quad \alpha_2 = h_2 \langle \cdot, \cdot \rangle, \text{i.e.} \quad M^6 \text{ is totally umbilic} \]
\[ W_3 + W_5 \quad J\alpha_1 = -\alpha_1 \quad \text{and} \quad J\alpha_2 = -\alpha_2, \text{ in particular,} \quad M^6 \text{ is minimal} \]
\[ W_5 \quad M^6 \text{ is totally geodesic} \]

**Table 1.** $M^8$ of type parallel ($\mathcal{W}_0$)

the SU(3)-structure on $M^6 = Hel^2 \times Q^4$ induced by the standard Spin(7)-structure (2.12) on $\mathbb{R}^8$, $Q^4 = span\{e_2, e_4, e_5, e_6\}$, is given by the equations
\[
\omega \cosh u = \cos v (e_2 \wedge e_4 + e_5 \wedge e_6) - \sin v (e_2 \wedge e_6 + e_4 \wedge e_5) \\
- \sinh v (e_4 \wedge e_6 - e_2 \wedge e_5) - \cosh^3 u \, du \wedge dv,
\]
\[
\Psi_+ = N_1 \Phi = (\cosh u \sin v du + \cosh u \sin v dv) \wedge (e_2 \wedge e_4 + e_5 \wedge e_6) \\
- (\sinh u \sin v du + \cosh u \cos v dv) \wedge (e_2 \wedge e_6 + e_4 \wedge e_5) \\
- du \wedge (e_2 \wedge e_5 - e_4 \wedge e_6),
\]
\[
\Psi_- = \sigma N_2 \Phi = (\cosh u \cos v du + \sinh u \cos v dv) \wedge (e_2 \wedge e_4 + e_5 \wedge e_6) \\
+ (\cosh u \cos v dv - \sinh u \sin v dv) \wedge (e_2 \wedge e_6 + e_4 \wedge e_5) \\
- dv \wedge (e_2 \wedge e_5 - e_4 \wedge e_6).
\]

Clearly this structure is holomorphic, $d\Psi_\pm = 0$ with zero Lee form, $\theta^6 = 0$. Therefore, the Bismut connection preserves this SU(3)-structure due to [38, Theorem 4.1], i.e. it has holonomy contained in SU(3).

We note that if the helicoid does not lie in a Cayley plane the induced SU(3)-structure could be not closed.

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Remarks on the geometry of almost complex 6-manifolds

\[ W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 + W_4 + W_5 \quad \tau(\nabla^8 \Phi)(N_1, N_1, f, \cdot) = -\sigma \tau(\nabla^8 \Phi)(N_2, N_2, J_*, \cdot) + \sigma \tau(\nabla^8 \Phi)(f, J_*, N_2, N_1) \]

\[ W_1^+ + W_2^+ + W_3 + W_4 + W_5 \quad \sin \gamma \left( \sigma h_1 - \sigma \tau(\nabla^8 \Phi)(N_2, N_2, N_1) \right) = \cos \gamma \left( h_2 - \tau(\nabla^8 \Phi)(N_1, N_2, N_1) \right) \]

\[ W_1^+ + W_2^+ + W_2^- + W_3 + W_4 + W_5 \quad \cos \gamma \left( \sigma h_1 - \sigma \tau(\nabla^8 \Phi)(N_2, N_2, N_1) \right) = -\sin \gamma \left( h_2 - \tau(\nabla^8 \Phi)(N_1, N_2, N_1) \right) \]

\[ W_2^+ + W_2^- + W_3 + W_4 + W_5 \quad h_1 = \tau(\nabla^8 \Phi)(N_2, N_2, N_1) \text{ and } h_2 = \tau(\nabla^8 \Phi)(N_1, N_2, N_1) \]

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\( W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) & \( 2Jd\gamma = *_6 (*_6 f^*(L_{N_1} \Phi) \wedge f^*(N_1, \Phi)) + \theta^6 \) \\
\( W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) & \( \theta^8 \) is normal to \( M^6 \) \\
\( W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) & \( \sigma (1 - J) \alpha_1 = J_{13} (1 - J) \alpha_2 \) \\
\( W_1^+ + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) & \( \cos \gamma (1 + J) \alpha_1 - \sin \gamma (1 + J) \alpha_2 = 2 (\sigma h_1 \cos \gamma - h_2 \sin \gamma) \cdot \cdot \) \\
\( W_1^+ + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) & \( \sin \gamma (1 + J) \sigma h_1 + \cos \gamma (1 + J) \alpha_2 = 2 (\sigma h_1 \sin \gamma + h_2 \cos \gamma) \cdot \cdot \) \\
\( W_1^+ + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) & \( \sigma \theta^6 (N_1) - 4h_1 \cos \gamma = (\theta^6 (N_2) - 4h_2) \sin \gamma \) \\
\( W_1^+ + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) & \( \sigma \theta^6 (N_1) - 4h_1 \sin \gamma = - (\theta^6 (N_2) - 4h_2) \cos \gamma \) \\
\( W_1^+ + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) & \( (1 + J) \alpha_1 = 2h_1 \cdot \cdot \) and \( (1 + J) \alpha_2 = 2h_2 \cdot \cdot \) \\
\( W_1^+ + W_2^- + W_3 + W_4 + W_5 \) & \( 4h_1 = \theta^6 (N_1) \) and \( 4h_2 = \theta^6 (N_2) \) \\
\( W_1^+ + W_2^- + W_3 + W_4 + W_5 \) & \( M^6 \) is totally umbilic \\
\( W_4 + W_5 \) & \( 4\alpha_1 = \theta^6 (N_1) \cdot \cdot \) and \( 4\alpha_2 = \theta^6 (N_2) \cdot \cdot \) \\
\( W_5 \) & \( 4\alpha_1 = \theta^6 (N_1) \cdot \cdot \), \( 4\alpha_2 = \theta^6 (N_2) \cdot \cdot \) and \( \theta^6 \) is normal to \( M^6 \) \\
\hline
\end{tabular}
\caption{\( M^8 \) of type balanced (\( \overline{W}_1 \))}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\( W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) & \( W_1^+ + W_2^+ + W_2^- + W_3 + W_4 + W_5 \) \\
\hline
\hline
\end{tabular}
\caption{\( M^8 \) of type locally conformal parallel (\( \overline{W}_2 \))}
\end{table}

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\begin{align*}
\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4 = 2Jd\gamma = *_6 (*_6 f^*(L_N, \Phi) \wedge f^*(N_1, \Phi)) + \theta^6 \\
\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_5 \quad f^*\theta^8 = 0 \\
\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \sigma(1-J)\alpha_1 = J(1-J)\alpha_2 \\
\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \cos \gamma(1+J)\sigma\alpha_1 - \sin \gamma(1+J)\alpha_2 = 2(\sigma h_1 \cos \gamma - h_2 \sin \gamma) \langle \cdot, \cdot \rangle \\
\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \sin \gamma(1+J)\sigma\alpha_1 + \cos \gamma(1+J)\alpha_2 = 2(\sigma h_1 \sin \gamma + h_2 \cos \gamma) \langle \cdot, \cdot \rangle \\
\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \sigma h_1 \cos \gamma = h_2 \sin \gamma \\
\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \sigma h_1 \sin \gamma = -h_2 \cos \gamma \\
\mathcal{W}_1^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad (1+J)\alpha_1 = 2h_1 \langle \cdot, \cdot \rangle \text{ and } (1+J)\alpha_2 = 2h_2 \langle \cdot, \cdot \rangle \\
\mathcal{W}_1^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad M^6 \text{ is minimal} \\
\mathcal{W}_1^+ + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad M^6 \text{ is totally umbilic} \\
\mathcal{W}_4 + \mathcal{W}_5 \quad M^6 \text{ is totally geodesic and } f^*\theta^8 = 0
\end{align*}

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