FAITHFULNESS OF BI-FREE PRODUCT STATES

CHRISTOPHER RAMSEY

Abstract. Given a non-trivial family of pairs of faces of unital C*-algebras where each pair has a faithful state, it is proved that if the bi-free product state is faithful on the reduced bi-free product of this family of pairs of faces then each pair of faces arises as a minimal tensor product. A partial converse is also obtained.

1. Introduction

The reduced free product was given independently by Avitzour [1] and Voiculescu [7] and it has been foundational in the development of free probability. Dykema proved in [2] that the free product state on the reduced free product of unital C*-algebras with faithful states is faithful. In consequence of this, if \( \{A_i\}_{i \in I} \) is a free family of unital C*-algebras in the non-commutative C*-probability space \((A, \varphi)\) and if \( \varphi \) is faithful on \( C^{*}(\{A_i\}_{i \in I}) \) then

\[
C^{*}(\{A_i\}_{i \in I}) \simeq \ast_{i \in I}(A_i, \varphi|_{A_i}),
\]

the reduced free product of the \( A_i \)'s with respect to the given states. This can be deduced from a paper of Dykema and Rørdam, namely [3, Lemma 1.3].

The present paper is the result of the author’s attempt to prove the same result in the new context of bi-free probability introduced by Voiculescu [8]. To this end, suppose \((A^{(i)}_{l}, A^{(i)}_{r})_{i \in I}\) is a non-trivial family of pairs of faces in the non-commutative C*-probability space \((A, \varphi)\). If \( \varphi_i = \varphi|_{C^{*}(A^{(i)}_{l}, A^{(i)}_{r})} \) is faithful on \( C^{*}(A^{(i)}_{l}, A^{(i)}_{r}) \), for all \( i \in I \), then it will be proven that if the bi-free product state \( **_{i \in I}\varphi_i \) is faithful on the reduced bi-free product \( **_{i \in I}(A^{(i)}_{l}, A^{(i)}_{r}) \) then \( C^{*}(A^{(i)}_{l}, A^{(i)}_{r}) \simeq A^{(i)}_{l} \otimes_{\text{min}} A^{(i)}_{r}, i \in I \). A converse is shown with the added assumption that each \( \varphi_i \) is a product state. Moreover, in this case there is a commensurate result to that which follows from Dykema and Rørdam, mentioned above.

It should be mentioned that the failure in general of the faithfulness of the bi-free product state has been pointed out in [4] and this failure has been the cause of the introduction of weaker versions of faithfulness in the bi-free context [4, 5].
Acknowledgements: The author would like to thank Scott Atkinson for sparking my interest into bi-free independence and for suggesting the reduced bi-free product, Paul Skoufranis for pointing out an error in a previous version of this paper, and the referee for their help in improving several difficult passages.

2. Bi-free independence and the reduced bi-free product

We will first take some time to recall the definition of bi-free independence from [8] and then define the reduced bi-free product of C*-algebras and the bi-free product state.

Fix a non-commutative C*-probability space $(A, \varphi)$, that is a unital C*-algebra and a state. Given a set $I$, suppose that for each $i \in I$ there is a pair of unital C*-subalgebras $A_{l}^{(i)}$ and $A_{r}^{(i)}$ of $A$, a “left” algebra and a “right” algebra. We call the set $(A_{l}^{(i)}, A_{r}^{(i)})_{i \in I}$ a family of pairs of faces in $A$. Such a family will be called non-trivial if $|I| \geq 2$ and $C^{*}(A_{l}^{(i)}, A_{r}^{(i)}) \neq \mathbb{C}$ for all $i \in I$. That is, there are at least two pairs of faces and there are no trivial pairs of faces.

Let $(\pi_{i}, H_{i}, \xi_{i})$ be the GNS construction for $(C^{*}(A_{l}^{(i)}, A_{r}^{(i)}), \varphi_{i})$ where $\varphi_{i} = \varphi|_{C^{*}(A_{l}^{(i)}, A_{r}^{(i)})}$. Voiculescu [8] (and even way back in [7]) observed that there are two natural representations of $B(H_{i})$ on the free product Hilbert space, which we will now introduce. The free product Hilbert space,

$$(H, \xi) = \ast_{i \in I}(H_{i}, \xi_{i}),$$

is given by associating all of the distinguished vectors and then forming a Fock space like structure. Namely, if $H_{j} = H_{j} \oplus \mathbb{C}\xi_{j}$, then

$$H := \mathbb{C}\xi \oplus \bigoplus_{\substack{n \in \mathbb{N} \setminus I \cup \{i_{1}, \ldots, i_{n}\}} \cap (1, \ldots, n) \in I \setminus \{i_{1}, \ldots, i_{n}\}}} \otimes \cdots \otimes H_{i_{n}}.$$ 

To define these representations we need to first build some Hilbert spaces and some unitaries. To this end, define

$$H(l, i) := \mathbb{C}\xi \oplus \bigoplus_{\substack{n \in \mathbb{N} \setminus I \cup \{i_{1}, \ldots, i_{n}\}} \cap (1, \ldots, n) \in I \setminus \{i_{1}, \ldots, i_{n}\}}} \otimes \cdots \otimes H_{i_{n}} \quad \text{and}$$

$$H(r, i) := \mathbb{C}\xi \oplus \bigoplus_{\substack{n \in \mathbb{N} \setminus \{i_{1}, \ldots, i_{n}\}} \cap \{i_{1}, \ldots, i_{n}\}}} \otimes \cdots \otimes H_{i_{n}}.$$ 

Then there are unitaries $V_{l} : H_{i} \otimes H(l, i) \to H$ and $W_{l} : H(r, i) \otimes H_{i}$ given by concatenation (with appropriate handling of $\xi_{i}$ and $\xi$). Finally, the two natural representations are the left representation $\lambda_{l} : B(H_{i}) \to B(H)$ which is defined as

$$\lambda_{l}(T) = V_{l}(T \otimes I_{H(l, i)})V_{l}^{*}.$$
and the right representation \( \rho_i : B(\mathcal{H}_i) \to B(\mathcal{H}) \) which is defined as
\[
\rho_i(T) = W_i(I_{\mathcal{H}(r;i)} \otimes T)W_i^*.
\]

With all of this groundwork established we can finally define bi-free independence. Note that \( * \) below refers to the full (or universal) free product of \( C^* \)-algebras.

**Definition 2.1** (Voiculescu [8]). The family of pairs of faces \( (\mathcal{A}_l^{(i)} , \mathcal{A}_r^{(i)})_{i \in I} \) in the non-commutative probability space \( (\mathcal{A}, \varphi) \) is said to be bi-freely independent with respect to the states \( \varphi \) if the following diagram commutes
\[
\begin{array}{ccc}
*_{i \in I} (\mathcal{A}_l^{(i)} , \mathcal{A}_r^{(i)}) & \overset{\iota}{\longrightarrow} & \mathcal{A} \\
\downarrow_{*_{i \in I}(\pi_i \ast \pi_i)} & & \| \\
*_{i \in I}(B(\mathcal{H}_i) \ast B(\mathcal{H}_i)) & \overset{*_{i \in I}(\lambda_i \ast \rho_i)}{\longrightarrow} & B(\mathcal{H}) \\
\end{array}
\]
where \( \iota \) is the unique \( * \)-homomorphism extending the identity on each \( \mathcal{A}_\chi^{(i)} \), for all \( \chi \in \{l, r\} \) and \( i \in I \).

From this we can now define the main objects of this paper.

**Definition 2.2.** Let \( (\mathcal{A}_l^{(i)} , \mathcal{A}_r^{(i)})_{i \in I} \) be a family of pairs of faces in the non-commutative \( C^* \)-probability space \( (\mathcal{A}, \varphi) \). As before, denote \( \varphi_i \) to be the restriction of \( \varphi \) to \( C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \) and let \( (\pi_i, \mathcal{H}_i, \xi_i) \) be the GNS construction of \( C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i \).

The reduced bi-free product of \( (\mathcal{A}_l^{(i)} , \mathcal{A}_r^{(i)})_{i \in I} \) with respect to the states \( \varphi_i \) is
\[
(**_{i \in I} (\mathcal{A}_l^{(i)} , \mathcal{A}_r^{(i)}), **_{i \in I} \varphi_i) = **_{i \in I}((\mathcal{A}_l^{(i)} , \mathcal{A}_r^{(i)}), \varphi_i)
\]
which is made up of the unital \( C^* \)-subalgebra of \( B(\mathcal{H}) \), called the reduced bi-free product of \( C^* \)-algebras,
\[
**_{i \in I} (\mathcal{A}_l^{(i)} , \mathcal{A}_r^{(i)} ) := C^*((\lambda_i \circ \pi_i(\mathcal{A}_l^{(i)}), \rho_i \circ \pi_i(\mathcal{A}_r^{(i)}))_{i \in I}) \subset B(\mathcal{H})
\]
and the bi-free product state
\[
**_{i \in I} \varphi_i(\cdot) := \langle \xi, \xi \rangle.
\]

It is an immediate fact that the family of pairs of faces \( (\lambda_i \circ \pi_i(\mathcal{A}_l^{(i)}), \rho_i \circ \pi_i(\mathcal{A}_r^{(i)}))_{i \in I} \) is bi-freely independent with respect to the bi-free product state.

It should be noted that we are working within the framework of the original non-commutative \( C^* \)-probability space \( (\mathcal{A}, \varphi) \). This means that the reduced bi-free product is taking into account the behaviour of \( \varphi \) not just on the left and right faces but on the \( C^* \)-algebra they generate, \( C^*(\mathcal{A}_l^{(i)} , \mathcal{A}_r^{(i)})\).

Since bi-free independence is a statement about the behaviour in the original \( C^* \)-probability space this definition makes sense.

That being said, one can create the reduced bi-free product as an external product. Start with pairs of faces in different \( C^* \)-probability spaces and simply create a new \( C^* \)-probability space by taking the full free product of
Theorem 3.1. Let the non-commutative reduced bi-free product family of pairs of faces. This gives that $C^*$ since $\phi$ be using the convention that $H_{\omega}$ suppresses the $\pi$ and thus already a subalgebra of $B_r$ and thus $\phi$ is faithful on $C^*(A^{(i)}_l, A^{(i)}_r)$ for each $i \in I$. If $\phi_{\ast_{i \in I}}$ is faithful on the reduced bi-free product $\phi_{\ast_{i \in I}}(A^{(i)}_l, A^{(i)}_r)$ then

$$C^*(A^{(i)}_l, A^{(i)}_r) \simeq A^{(i)}_l \otimes_{\min} A^{(i)}_r.$$

Proof. First we will establish that $A^{(i)}_l$ and $A^{(i)}_r$ commute in $A$, then we will show that they induce a $C^*$-norm on the algebraic tensor product $A^{(i)}_l \otimes A^{(i)}_r$ and finally that this is in fact the minimal tensor norm.

We will be using the notation from Section 2. To simplify things a little bit, because the $\phi_i$ are assumed to be faithful, consider $C^*(A^{(i)}_l, A^{(i)}_r)$ as already a subalgebra of $B(H_i)$ and so $\phi_i(\cdot) = \langle \xi_i, \xi_i \rangle$. That is, we are suppressing the $\pi_i$ notation from the GNS construction. Moreover, we will be using the convention that $\lambda_i(x), \rho_i(x), \lambda_i \ast \rho_i(x)$ all are living in $B(H)$.

Suppose $a_i \in A^{(i)}_l$ such that $\phi_i(a_i) = 0, \chi \in \{l, r\}$ and $0 \neq b \in A^{(j)}_l \cup A^{(j)}_r$ for $j \neq i$ such that $\phi_j(b) = 0$. Such a $b$ exists by the non-triviality of the family of pairs of faces. This gives that $\langle b^* \xi_j, \xi_j \rangle = \phi_j(b) = 0$ and so $b^* \xi_j \in \mathcal{H}_j$ while $\langle b(b^* \xi_j), \xi_j \rangle = \phi_j(bb^*) \neq 0$ by the faithfulness of $\phi_j$.

Now, [8, Section 1.5] establishes that $[\lambda_i(A^{(i)}_l), \rho_i(A^{(i)}_r)](\mathcal{H}\ominus \mathcal{H}_i) = 0$ which gives that

$$\langle \lambda_i(a_l)\rho_i(a_r)\lambda_j \ast \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j \ast \rho_j(b)\rangle \xi = 0$$

since $b \xi \in \mathcal{H}_j \subset \mathcal{H}$. The faithfulness of $\phi_{\ast_{i \in I}}$ implies that $\xi$ is a separating vector for the reduced bi-free product and thus

$$\lambda_i(a_l)\rho_i(a_r)\lambda_j \ast \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j \ast \rho_j(b) = 0$$

which gives that

$$0 = P_{\mathcal{H}_i} \lambda_i(a_l)\rho_i(a_r)\lambda_j \ast \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j \ast \rho_j(b)b^* \xi_j$$

$$= \lambda_i(a_l)\rho_i(a_r) - \rho_i(a_r)\lambda_i(a_l)\lambda_j \ast \rho_j(b)b^* \xi_j$$

$$= \langle b^* \xi_j, \xi_j \rangle (a_l a_r - a_r a_l) \xi_i.$$
Since $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute, the universal property of $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ gives that there exists a $*$-homomorphism

$$\sum_{k=1}^{m} a_{k,l} \odot a_{k,r} \mapsto \sum_{k=1}^{m} a_{k,l}a_{k,r}.$$ 

We need to establish its injectivity. To this end, consider $h \in \mathcal{H}_j$, $\|h\| = 1$ where $j \neq i$ and the isometric map $V_h : \mathcal{H}_i \otimes \mathcal{H}_i \rightarrow \mathcal{H}_i \otimes h \otimes \mathcal{H}_i$

defined by $V_h(h_l \otimes h_r) = h_l \otimes h \otimes h_r$ for $h_l, h_r \in \mathcal{H}_i$. This map is inspired by Dykema’s proof of the faithfulness of the free product state [2, Theorem 1.1]. Note that in $\mathcal{H}$ we really have that

$$\mathcal{H}_i \otimes h \otimes \mathcal{H}_i = C_h \oplus (\mathcal{H}_i \otimes h) \oplus (h \otimes \mathcal{H}_i) \oplus (\mathcal{H}_i \otimes h \otimes \mathcal{H}_i)$$

but hopefully the reader will pardon the simplified notation.

Now $\mathcal{H}_i \otimes h \otimes \mathcal{H}_i$ is a reducing subspace of $C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))$ since for all $a \in \mathcal{A}_l^{(i)}, b \in \mathcal{A}_r^{(i)}$ and $\eta_1, \eta_2 \in \mathcal{H}_i$ we have that

$$V_h^* \lambda_i(a) \rho_i(b) V_h(\eta_1 \otimes \eta_2) = V_h^* \lambda_i(a) \rho_i(b)(\eta_1 \otimes h \otimes \eta_2)$$

$$= a\eta_1 \otimes b\eta_2.$$ 

Thus, compressing to $\mathcal{H}_i \otimes h \otimes \mathcal{H}_i$ gives

$$V_h^* C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))V_h = \mathcal{A}_l^{(i)} \otimes_{\text{min}} \mathcal{A}_r^{(i)}.$$ 

So, if $\sum_{k=1}^{m} a_{k,l} \otimes a_{k,r} \neq 0 \in \mathcal{A}_l^{(i)} \otimes \mathcal{A}_r^{(i)}$ then $\sum_{k=1}^{m} a_{k,l} \otimes a_{k,r} \neq 0 \in \mathcal{A}_l^{(i)} \otimes_{\text{min}} \mathcal{A}_r^{(i)}$ which implies that

$$0 \neq \sum_{k=1}^{m} a_{k,l} \otimes a_{k,r}(\xi_i \otimes \xi_i)$$

$$= V_h^* \sum_{k=1}^{m} \lambda_i(a_{k,l}) \rho_i(a_{k,r}) V_h(\xi_i \otimes \xi_i)$$

$$= \sum_{k=1}^{m} \lambda_i(a_{k,l}) \rho_i(a_{k,r}) h$$

since the state $(\xi_i \otimes \xi_i, \xi_i \otimes \xi_i)$ is faithful on the min tensor product. But then

$$\sum_{k=1}^{m} \lambda_i(a_{k,l}) \rho_i(a_{k,r}) \neq 0 \in C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))$$ which by the faithfulness
of $\ast_{i\in I} \varphi_i$ gives that $\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\xi \neq 0$. Finally,

$$0 \neq \left(\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\xi, \sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\xi\right)$$

$$= \left(\left(\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\right)^* \left(\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\right)\xi, \xi\right)$$

$$= \varphi_i \left(\left(\sum_{k=1}^m a_{k,l}a_{k,r}\right)^* \left(\sum_{k=1}^m a_{k,l}a_{k,r}\right)\right)$$

which gives by the faithfulness of $\varphi_i$ that $\sum_{k=1}^m a_{k,l}a_{k,r} \neq 0$. Therefore, the claim is verified.

Now, this implies that $C^*(\mathcal{A}_i^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{\alpha} \mathcal{A}_r^{(i)}$ where $\|\cdot\|_\alpha$ is a $C^*$-norm on $\mathcal{A}_l^{(i)} \otimes \mathcal{A}_r^{(i)}$. So by Takesaki’s Theorem [6] we have that there exists a surjective $*$-homomorphism

$$q : C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \to \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}.$$ 

To finish the proof all we need to do is show that $q$ is injective.

To this end, let $a \in C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ such that $q(a) = 0$. Again as in the first part of this proof, find $0 \neq b \in \mathcal{A}_l^{(j)} \cup \mathcal{A}_r^{(j)}$ for $j \neq i$ such that $\varphi_j(b) = 0$ and $h \in \mathcal{H}_j$ such that $\langle bh, \xi_j \rangle \neq 0$. Additionally, assume that $\|b\xi_j\| = 1$.

In the second part of this proof we saw that compressing to $\mathcal{H}_i \otimes b\xi_j \otimes \mathcal{H}_i$ is tantamount to this quotient homomorphism $q$. Namely, suppose

$$\iota_i : \mathcal{A}_l^{(i)} \ast \mathcal{A}_r^{(i)} \to C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \quad (\subseteq B(\mathcal{H}_i) \text{ by assumption})$$

is the unique $*$-homomorphism extending the identity in each component. There then exists $\tilde{\alpha} \in \mathcal{A}_l^{(i)} \ast \mathcal{A}_r^{(i)}$ such that $\iota_i(\tilde{\alpha}) = a$. An important fact to record is that, by uniqueness,

$$\lambda_i \ast \rho_i(\cdot)|_{\mathcal{H}_i} = \iota_i(\cdot),$$

remembering that we have that $\lambda_i \ast \rho_i(\cdot) \in B(\mathcal{H})$. Thus,

$$V_{b\xi_j} \lambda_i \ast \rho_i(\tilde{\alpha})V_{b\xi_j} = q(a) = 0,$$

which implies, by the fact that $V_{b\xi_j}(\mathcal{H}_i \otimes \mathcal{H}_i)$ is reducing for $\lambda_i \ast \rho_i(\mathcal{A}_l^{(i)} \ast \mathcal{A}_r^{(i)})$, that

$$0 = \lambda_i \ast \rho_i(\tilde{\alpha})V_{b\xi_j}(\xi_i \otimes \xi_i)$$

$$= \lambda_i \ast \rho_i(\tilde{\alpha})(b\xi_j)$$

$$= \lambda_i \ast \rho_i(\tilde{\alpha})\lambda_j \ast \rho_j(b)\xi.$$
By the faithfulness of the bi-free product state $\lambda_i \ast \rho_i(\tilde{a}) \lambda_j \ast \rho_j(b) = 0$ and so

$$
0 = P_{H_i} \lambda_i \ast \rho_i(\tilde{a}) \lambda_j \ast \rho_j(b) h
= \lambda_i \ast \rho_i(\langle bh, \xi_j \rangle) \xi
= \langle bh, \xi_j \rangle i_i(\tilde{a}) \xi
= \langle bh, \xi_j \rangle a \xi_i.
$$

Hence, by the faithfulness of $\phi_C$ we have that $a = 0$. Therefore, for all $i \in I$, $C^*(A_l^{(i)}, A_r^{(i)}) \simeq A_l^{(i)} \otimes_{\min} A_r^{(i)}$. \hfill \Box

We turn now to a partial converse of the previous theorem. This is probably known among the experts in bi-free probability but we could not find a published proof. The following proof may be a tad clunky but we find it the clearest from a non-expert perspective.

**Theorem 3.2.** Let $(A_l^{(i)}, A_r^{(i)})_{i \in I}$ be a family of pairs of faces in the non-commutative $C^*$-probability space $(\mathcal{A}, \phi)$. If $C^*(A_l^{(i)}, A_r^{(i)}) \simeq A_l^{(i)} \otimes_{\min} A_r^{(i)}$ and $\phi_i = \phi_{l,i}|A_l^{(i)} \otimes \phi_{r,i}|A_r^{(i)}$ is a faithful product state on $C^*(A_l^{(i)}, A_r^{(i)})$, for all $i \in I$, then $**_{i \in I} \phi_i$ is faithful on the reduced bi-free product and

$$
**_{i \in I} (A_l^{(i)}, A_r^{(i)})_{i \in I} \simeq *_{i \in I} (A_l^{(i)}, \phi) \otimes_{\min} *_{i \in I} (A_r^{(i)}, \phi).
$$

**Proof.** As before, we will be using the notation of Section 2.

For each $i \in I$, since $C^*(A_l^{(i)}, A_r^{(i)}) \simeq A_l^{(i)} \otimes_{\min} A_r^{(i)}$ and $\phi_i$ is a product state we can a priori choose $H_i = H_{i,l} \otimes H_{i,r}$, unit vectors $\xi_{i,l} \in H_{i,l}, \xi_{i,r} \in H_{i,r}$ such that $\xi_i = \xi_{i,l} \otimes \xi_{i,r}$ and $*$-homomorphisms $\pi_{i,\chi} : A_\chi^{(i)} \to B(H_{i,\chi})$ such that $\pi_i = \pi_{i,l} \otimes \pi_{i,r}$. This will give for $a_\chi \in A_\chi^{(i)}, \chi \in \{l, r\}$, that

$$
\phi_i(a_\chi a_\chi) = \langle \pi_{i,l}(a_\chi a_\chi) \xi_{i,l}, \xi_{i,l} \rangle
= \langle \pi_{i,l}(a_\chi) \xi_{i,l}, \xi_{i,l} \rangle \langle \pi_{i,r}(a_\chi) \xi_{i,r}, \xi_{i,r} \rangle.
$$

Along with the free product Hilbert space

$$
(H, \xi) = *_{i \in I}(H_i, \xi_i)
$$

we need to also define, for $\chi \in \{l, r\}$, the free product Hilbert spaces

$$
(H_\chi, \xi_\chi) = *_{i \in I}(H_{i,\chi}, \xi_{i,\chi}).
$$

Since there are multiple free product Hilbert spaces we will use subscripts to denote the different left and right representations, namely,

$$
\lambda_{H_i} : B(H_i) \to B(H) \quad \text{and} \quad \lambda_{H_{i,l}} : B(H_{i,l}) \to B(H_l)
$$

for the left representations and

$$
\rho_{H_i} : B(H_i) \to B(H) \quad \text{and} \quad \rho_{H_{i,r}} : B(H_{i,r}) \to B(H_r)
$$

for the right representations.
Dykema’s original result [2] proves that \( \langle \xi_\chi, \xi_\chi \rangle \) is faithful on \( *_{i \in I}(A_\chi^{(i)}, \varphi) \) for \( \chi \in \{l, r\} \) and it is a folklore result that the minimal tensor product of faithful states is faithful. Thus, \( \langle \cdot \otimes \xi_r, \xi_l \otimes \xi_r \rangle \) is faithful on \( *_{i \in I}(A_\chi^{(i)}, \varphi) \otimes_{\min} *_{i \in I}(A_\chi^{(i)}, \varphi) \).

Fix \( k \geq 1 \) and \( j_1, \cdots, j_k \in I \) such that \( j_i \neq j_{i+1}, 1 \leq i \leq k - 1 \). Now fix a unit vector
\[
\eta_\chi = (\xi_{j_1, l} \otimes h_{j_1, r}) \otimes h_{j_2} \otimes \cdots \otimes h_{j_{k-1}} \otimes (h_{j_{k-1}, l} \otimes \xi_{j_k, r})
\]
\( \in (\xi_{j_1, l} \otimes \hat{\mathcal{H}}_{j_1, r}) \otimes \hat{\mathcal{H}}_{j_2} \otimes \cdots \otimes \hat{\mathcal{H}}_{j_{k-1}} \otimes (\hat{\mathcal{H}}_{j_{k-1}, l} \otimes \xi_{j_k, r}) \).

If \( k = 1 \) the only possible \( h \) is \( \xi = \xi_{j_1, l} \otimes \xi_{j_1, r} \). Call the collection of such \( h \), as \( k \) and the indices vary, \( \mathcal{S} \subset \mathcal{H} \).

As will be shown below, this set of unit vectors \( \mathcal{S} \) plays an important role in decomposing simple tensors in \( \mathcal{H} \), in particular for every simple tensor \( \eta \in \mathcal{H} \) that is also a simple tensor in each component there exists a unique \( h \in \mathcal{S} \) such that \( \eta \in \mathcal{H}_l \otimes h \otimes \mathcal{H}_r \). By abuse of tensor notation this is not very hard to see in one’s mind but the reality of proving this carefully needs plenty of indices.

To this end, for \( m \geq 1 \) suppose \( s_1, \ldots, s_m \in I \) such that \( s_t \neq s_{t+1} \) for \( 1 \leq t \leq m - 1 \), and \( \eta_t \in \mathcal{H}_{s_t, l}, \eta_r \in \mathcal{H}_{s_t, r} \) such that \( \eta_t \otimes \eta_r \in \mathcal{H}_{s_t} \) for \( 1 \leq t \leq m \). This last condition implies that \( \eta_{t, \chi} = \|\eta_{t, \chi}\|\xi_{t, \chi} \) cannot hold for both \( \chi = l \) and \( \chi = r \). In summary,
\[
\eta := (\eta_{1, l} \otimes \eta_{1, r}) \otimes \cdots \otimes (\eta_{m, l} \otimes \eta_{m, r}) \in \mathcal{H}_{s_1} \otimes \cdots \otimes \mathcal{H}_{s_m}.
\]

Note that the conditions imposed on the \( \eta_{t, \chi} \) in the above paragraph imply that the form of \( \eta \) above is as reduced as it can be.

As mentioned above, it will be established that there exists \( h \in \mathcal{S} \) such that
\[
\eta \in \mathcal{H}_l \otimes h \otimes \mathcal{H}_r.
\]

To prove the required decomposition, let
\[
v = \max\{0 \leq t \leq m : \eta_{j, r} = \|\eta_{j, r}\|\xi_{s_j, r}, 1 \leq j \leq t\}
\]
and
\[
w = \min\{1 \leq t \leq m + 1 : \eta_{j, l} = \|\eta_{j, l}\|\xi_{s_j, l}, t \leq j \leq m\}.
\]
This gives that \( v \) is the number of terms in a row from the left with trivial right tensor components and \( m + 1 - w \) is the number of terms in a row from the right with trivial left tensor components.

By the fact that \( \eta_{t, l} \otimes \eta_{t, r} \in \mathcal{H}_{s_t} \), that is \( \eta_{t, \chi} = \|\eta_{t, \chi}\|\xi_{t, \chi} \) cannot hold for both \( \chi = l \) and \( \chi = r \), we have that \( v < w \). If \( v = m \) then \( w = m + 1 \) and \( \eta \in \mathcal{H}_l \), and if \( w = 1 \) then \( v = 0 \) and \( \eta \in \mathcal{H}_r \). Otherwise, when \( 0 \leq v \leq m - 1 \) and \( 2 \leq w \leq m \), define
\[
\eta_l = (\eta_{1, l} \otimes \|\eta_{1, r}\|\xi_{s_1, r}) \otimes \cdots \otimes (\eta_{v, l} \otimes \|\eta_{v, r}\|\xi_{s_v, r}) \otimes (\eta_{v+1, l} \otimes \xi_{s_{v+1}, r}),
\]
\[
\eta_S = (\xi_{v+1, l} \otimes \eta_{v+1, r}) \otimes \cdots \otimes (\eta_{w-1, l} \otimes \xi_{w-1, r}),
\]
\[
\eta_r = (\xi_{w-1, l} \otimes \eta_{w-1, r}) \otimes (\|\eta_{w, l}\|\xi_{w, l} \otimes \eta_{w, r}) \otimes \cdots \otimes (\|\eta_{m, l}\|\xi_{m, l} \otimes \eta_{m, r})
\]
with $\eta_S = \xi$ if $v + 1 = w$. Hence, by the usual slight abuse of the tensor notation, $\eta = \eta_l \otimes \eta_S \otimes \eta_r \in \mathcal{H}_l \otimes \eta_S \otimes \mathcal{H}_r$ with $\|\eta_S\|\in \mathcal{S}$. Therefore,

$$\mathcal{S} \ni \{ \mathcal{H}_l \otimes h \otimes \mathcal{H}_r : h \in \mathcal{S} \} = \mathcal{H}.$$  

For any $h \in \mathcal{S}$, which is a unit vector, there is a natural isometric map $S_h : \mathcal{H}_l \otimes \mathcal{H}_r \to \mathcal{H}$ given by the concatenation $\mathcal{H}_l \otimes \mathcal{H}_r \mapsto \mathcal{H}_l \otimes h \otimes \mathcal{H}_r$, with the appropriate simplification of tensors when needed. In particular, there exist $k \geq 1$ and $j_1, \cdots, j_k \in I$ such that $j_i \neq j_{i+1}, 1 \leq i \leq k - 1$ and then

$$h = (\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{j_k,r})$$

$$\in (\xi_{j_1,l} \otimes \mathcal{H}_{j_1,r}) \otimes \mathcal{H}_{j_2} \cdots \otimes \mathcal{H}_{j_{k-1}} \otimes (\mathcal{H}_{j_k,l} \otimes \xi_{j_k,r}).$$

We can now carefully specify that the isometric map is given by

$$\xi_l \otimes \xi_r \mapsto h,$$

$$\begin{cases} h \otimes (\xi_{i_1,l} \otimes \mathcal{H}_{i_1,r}) \otimes \cdots \otimes (\xi_{i_m,l} \otimes \mathcal{H}_{i_m,r}), & i_1 \neq j_k \\
(\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \mathcal{H}_{j_1,r}) \otimes \\
\cdots \otimes (\xi_{i_m,l} \otimes \mathcal{H}_{i_m,r}), & i_1 = j_k \end{cases}$$

$$((\mathcal{H}_{i_1,l} \otimes \cdots \otimes \mathcal{H}_{i_m,l}) \otimes \xi_r) \mapsto$$

$$\begin{cases} (\mathcal{H}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathcal{H}_{i_m,l} \otimes \xi_{i_m,r}) \otimes h, & i_m \neq j_1 \\
(\mathcal{H}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathcal{H}_{i_m,l} \otimes h_{i_1,r}) \otimes h_{j_2} \otimes \\
\cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{i_1,r}), & i_m = j_1 \end{cases}$$

and

$$((\mathcal{H}_{i_1,l} \otimes \cdots \otimes \mathcal{H}_{i_m,l}) \otimes (\mathcal{H}_{t_1,r} \otimes \cdots \otimes \mathcal{H}_{t_s,r}) \mapsto$$

$$((\mathcal{H}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathcal{H}_{i_m,l} \otimes \xi_{i_m,r}) \otimes h \otimes (\xi_{t_1,l} \otimes \mathcal{H}_{t_1,r}) \otimes \cdots \otimes (\xi_{t_s,l} \otimes \mathcal{H}_{t_s,r})$$

if $i_m \neq j_1$ and $j_k \neq t_1$ with similar statements as the cases above when $i_m = j_1$ or $j_k = t_1$ or both happen. Perhaps the most natural case of $S_h$ is when $h = \xi$. It certainly minimizes, but doesn’t remove, the need for all of the cases above.

A careful examination of the $S_h$ isometric map implies that for $a \in \mathcal{A}^{(1)}_{\chi}$, $b \in \mathcal{A}^{(2)}_{\chi}$ and $\eta_{\chi} \in \mathcal{H}_{\chi}$ for $\chi \in \{l, r\}$ we have that, by abuse of the tensor notation,

$$\lambda_{\mathcal{H}_{i_1}}(\pi_{i_1}(a))\rho_{\mathcal{H}_{i_2}}(\pi_{i_2}(b))S_h(\eta_l \otimes \eta_r)$$

$$= \lambda_{\mathcal{H}_{i_1}}(\pi_{i_1}(a))\rho_{\mathcal{H}_{i_2}}(\pi_{i_2}(b))(\eta_l \otimes h \otimes \eta_r)$$

$$= \lambda_{\mathcal{H}_{i_1,l}}(\pi_{i_1,l}(a))\eta_l \otimes h \otimes \rho_{\mathcal{H}_{i_2,r}}(\pi_{i_2,r}(b))\eta_r$$

$$= S_h(\lambda_{\mathcal{H}_{i_1,l}}(\pi_{i_1,l}(a))\eta_l \otimes \rho_{\mathcal{H}_{i_2,r}}(\pi_{i_2,r}(b))\eta_r)$$
Hence, \( S_h(\mathcal{H}_l \otimes \mathcal{H}_r) \) is a reducing subspace of the reduced bi-free product. Moreover,

\[
S_h^* \lambda_{\mathcal{H}_l} \circ \pi_l(\cdot) S_h = (\lambda_{\mathcal{H}_l} \circ \pi_l(\cdot)) \otimes I_{\mathcal{H}_r} \quad \text{on } \mathcal{A}_{l}^{(i)}
\]

and

\[
S_h^* \rho_{\mathcal{H}_l} \circ \pi_l(\cdot) S_h = I_{\mathcal{H}_l} \otimes (\rho_{\mathcal{H}_r} \circ \pi_r(\cdot)) \quad \text{on } \mathcal{A}_{r}^{(i)}
\]

Therefore, for any \( h \in \mathcal{S} \),

\[
S_h^* (\star_{i \in I}(A_l^{(i)}, A_r^{(i)})) S_h = \star_{i \in I}(A_l^{(i)}, \varphi) \otimes_{\min} \star_{i \in I}(A_r^{(i)}, \varphi)
\]

and furthermore, by the identities involving \( S_h, \lambda \) and \( \rho \), \( S_h^* a S_h = S_{\xi}(a) S_{\xi} \) for all \( h \in \mathcal{S} \) and \( a \in \star_{i \in I}(A_l^{(i)}, A_r^{(i)}) \).

Finally, we want to show that compression to \( S_{\xi}(\mathcal{H}_l \otimes \mathcal{H}_r) \) is a *-isomorphism. Note that this is the same as compression to \( S_h(\mathcal{H}_l \otimes \mathcal{H}_r) \) being injective for any \( h \in \mathcal{S} \). This gives us a way forward. Suppose that \( a \in \star_{i \in I}(A_l^{(i)}, A_r^{(i)}) \) such that \( \star_{i \in I} \varphi_i(a^* a) = 0 \). This implies that

\[
0 = a \xi = S_{\xi}^* a S_{\xi}(\xi_l \otimes \xi_r).
\]

By the faithfulness of \( (\cdot \xi_l \otimes \xi_r, \xi_l \otimes \xi_r) \) this gives that \( S_{\xi}^* a S_{\xi} = 0 \) or rather \( a \) is 0 on the reducing subspace \( S_{\xi}(\mathcal{H}_l \otimes \mathcal{H}_r) \). But then for all \( h \in \mathcal{S} \) we have that

\[
S_h^* a S_h = S_{\xi}^* a S_{\xi} = 0
\]

and \( a \) is 0 on the reducing subspace \( S_h(\mathcal{H}_l \otimes \mathcal{H}_r) \). By what we proved about the set \( \mathcal{S} \), we have that \( a \) is 0 on

\[
\mathfrak{span}\{S_h(\mathcal{H}_l \otimes \mathcal{H}_r) : h \in \mathcal{S}\} = \mathfrak{span}\{\mathcal{H}_l \otimes h \otimes \mathcal{H}_r : h \in \mathcal{S}\} = \mathcal{H}.
\]

Therefore, \( a = 0 \) and thus \( \star_{i \in I} \varphi_i \) is faithful. \( \square \)

There may exist a full converse to Theorem 3.1 but the previous proof highly depends on the state \( \varphi_i \) arising as a tensor product of states. In general, \( \varphi_i \) need not be of this form. We should note here that if \( \varphi_i|_{A_l^{(i)}} \) or \( \varphi_i|_{A_r^{(i)}} \) is a pure state then \( \varphi_i \) will be a tensor product of states.

To end this paper, we summarize with the following corollary.

**Corollary 3.3.** Let \( (A_l^{(i)}, A_r^{(i)})_{i \in I} \) be a non-trivial family of pairs of faces in the non-commutative C*-probability space \( (\mathcal{A}, \varphi) \). If \( \varphi \) is faithful on \( C^*(A_l^{(i)}, A_r^{(i)})_{i \in I} \), \( C^*(A_l^{(i)}, A_r^{(i)}) \simeq \mathcal{A}_{l}^{(i)} \otimes_{\min} \mathcal{A}_{r}^{(i)} \), \( \varphi_i = \varphi_i|_{A_l^{(i)}} \otimes \varphi_i|_{A_r^{(i)}} \) and \( (A_l^{(i)}, A_r^{(i)})_{i \in I} \) is bi-freely independent with respect to \( \varphi \), then

\[
C^*(A_l^{(i)}, A_r^{(i)})_{i \in I} \simeq \star_{i \in I}(A_l^{(i)}, A_r^{(i)})_{i \in I} \simeq \star_{i \in I}(A_l^{(i)}, \varphi) \otimes_{\min} \star_{i \in I}(A_r^{(i)}, \varphi).
\]
Proof. Recall, that by bi-free independence we know that the following diagram commutes
\[
\begin{array}{ccc}
\hat{\pi} \in I & & C^*((A_{\hat{\mathcal{I}}}^{(i)}, \mathcal{A}^{(i)})_{i \in \mathcal{I}}) \\
\mathcal{I} \mathcal{I} \mathcal{I} & & \\
\mathcal{I} \mathcal{I} \mathcal{I} & & C^* (Λ_{\mathcal{I}}^{(i)}, \mathcal{A}^{(i)})_{i \in \mathcal{I}} \\
\end{array}
\]
Because both of the states are faithful on their algebras then for any \( a^*a \in \hat{\pi} \mathcal{I} \mathcal{I} \mathcal{I} (A_{\hat{\mathcal{I}}}^{(i)}, \mathcal{A}^{(i)})_{i \in \mathcal{I}} \), \( a^*a \) is in the kernel of \( \iota \) if and only if \( a^*a \) is in the kernel of \( \mathcal{I} \mathcal{I} \mathcal{I} (Λ_{\mathcal{I}}^{(i)}, \mathcal{A}^{(i)})_{i \in \mathcal{I}} \). Therefore, both quotients are \( \ast \)-isomorphic and Theorem 3.2 gives the final \( \ast \)-isomorphism. \( \square \)

References

[1] D. Avitzour, Free products of \( C^* \)-algebras, Trans. Amer. Math. Soc. 271 (1982), 423–435.
[2] K. Dykema, Faithfulness of free product states, J. Funct. Anal. 154 (1998), 323–329.
[3] K. Dykema and M. Rørdam, Projections in free product \( C^* \)-algebras, Geom. Funct.
Anal., 8 (1998), 1-16; Erratum, idem., 10(4) (2000), 975.
[4] A. Freslon, M. Weber On bi-free de Finetti theorems, Ann. Math. Blaise Pascal 23 (2016), 21–51.
[5] P. Skoufranis, On operator-valued bi-free distributions, Adv. Math. 303 (2016), 638–715.
[6] M. Takesaki, On the cross-norm of the direct product of \( C^* \)-algebras, Tôhoku Math. J. 16 (1964), 111-122.
[7] D. Voiculescu, Symmetries of some reduced free product \( C^* \)-algebras, in Operator algebras and their connection with topology and ergodic theory, Lecture Notes in Math. 1132, Springer, 1985, 556–588.
[8] D. Voiculescu, Free probability for pairs of faces I, Comm. Math. Phys. 332 (2014), 955–980.

Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada
E-mail address: christopher.ramsey@umanitoba.ca