Quartic Balance Theory: Global Minimum With Imbalanced Triangles

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Balance theory proposed by Heider for the first time modeled triplet interaction in a signed network, stating that relationships between two people, friendship or enmity, is dependent on a third person. The Hamiltonian of this model has an implicit assumption that all triads are independent, meaning that state of each triad, being balanced or imbalanced, is ineffective to others. This independence forces the network to have completely balanced final states. However, there exists evidence indicating that real networks are partially balanced raising the question of what is the mechanism preventing the system to be perfectly balanced. Our suggestion is to consider a quartic interaction which dissolves the triad’s independence. We use mean field method to study thermal behavior of such systems where the temperature is a parameter that allows the stochastic behavior of agents. We show that under a certain temperature, the symmetry between balanced and imbalanced triads will spontaneously break and we have a discrete phase transition. As consequence stability arises where either similar balanced or imbalanced triads dominate, hence the system obtains two new imbalanced stable states. In this model, the critical temperature depends on the second power of the number of nodes, which was a linear dependence in thermal balance theory. Our simulations are in good agreement with the results obtained by the mean field method.

I. INTRODUCTION

Thanks to the balance theory introduced by Heider,1 we can now look further than describing society by two people interaction. Heider Balance theory (HBT), brilliantly considers triplet interaction between individuals and proposes that the evolution of interrelationship depends on psychological stress. This model labels all states of triads by balance or imbalance and predicts two final stable balanced states for signed networks: heaven and bipolar [2]. HBT has been applied to many branches of science such as studies of international networks [3–5], sociology [7–10] and ecology [11]. There exists disagreement between researchers on how much the networks based on real data are balanced [12–15]. This disagreement may raise the question that can triplet interaction describe real-world communities as we desire or that we need to consider higher order interactions?

In this paper, bringing a higher order of interaction into account we introduce a new Hamiltonian which favors both balanced and imbalanced triads depending on the status of the neighbors. We think that such a model is needed since the real world environment can have a serious influence on our decisions and cannot be neglected. We focus on the interaction of triads which results in quartic interaction where we call it quartic balance theory (QBT).

Cartwright and Harary [2] expanded the Idea introduced first by Heider, using graph theory. In their model, the community was modeled by a graph in which each node represents an individual. The edge which connects two nodes represents the relationship between the two nodes which it connects and can have the weights of either +1 (friendship), or −1 (enmity). In triplet interaction, there exist four different triads in which balance ones have odd number of positive edges. If a triad is imbalanced, it tends to change to a balanced state. The structural balance is reached when all triads in the network are balanced. The predefined states in structural balance theory, are inspired by the fact that in a three people relationship, if two of them are in some sort of fight but are friend with the third person, then they tend to change their status: whether the third person changing his/her relationship with either of them or that the two in fight, make peace with each other.

We appreciate the early studies after which balance theory was introduced, focusing on the static properties of networks following this model [10]. Discrete-time [8–9] and continues-time [17–18] dynamics of how imbalanced triads turn to be balanced and the concept of jammed states and paths towards it, have been discussed [20]. HBT is tested on the very large online social networks and the result is these networks are extremely balanced [12]. The human relationships can form a multiplex network with a variety of links, the structural balance of this kind of network also has been investigated [10]. The memory effect can affect link flexibility to change and result in a special form of balance theory [21]. Another extension to this model is considered a role for nodes, this can be used to describe the disease-spreading process on the network [22]. In some other work, the temperature has been introduced into balance theory as a parameter of randomness or tension in society [23].

We think this new Hamiltonian is a better demonstration of a community and follow it for a fully connected
network and investigate the results which it brings. We consider temperature as a measure of abnormality which happens randomly in society. The mathematical framework for our analytic approach is statistical physics \textsuperscript{[24]} especially exponential random graphs (ERG) \textsuperscript{[25, 26]}. Park and Newman derive ERG from maximum entropy principle using Boltzmann Gibbs statistical mechanics \textsuperscript{[34]} and discussed two star and Strauss model \textsuperscript{[35, 36]}. We use ERG to find mean values of edges’ variables in our network. These variables result in two self-consistent equations and depending on the temperature, we can have one, three or five simultaneously solutions. The discrete phase transition is seen for our model. The stability of solutions or fixed points, depending on temperature, is being discussed. Finally, we confirm our analytical solutions via simulation.

II. MODEL

Following all of the previous statements, we consider a simple pairwise interaction term between triads with a common edge, as

\begin{equation}
\mathcal{H}(G) = - \sum_{i<j<k<\ell} \Delta_{ijk} \Delta_{ij\ell}
= - \sum_{i<j<k<\ell} \sigma_{ij} \sigma_{jk} \sigma_{ki} \sigma_{ij} \sigma_{ji} \sigma_{\ell i}
= - \sum_{i<j<k<\ell} \sigma_{jk} \sigma_{ki} \sigma_{j\ell} \sigma_{\ell i} = -s(G),
\end{equation}

where, \(\Delta_{ijk}\) represents a triad shaped by \(i, j, k\) nodes and \(\sigma_{ji} = \sigma_{ij}\) is an element of the adjacency matrix which connects node \(i\) to \(j\). The value of edges in adjacency matrix will be \(\pm 1\) which defines friendship or enmity relation between two nodes; so \(\sigma^2_{ij} = 1\) and double appearance of \(\sigma_{ij}\) cancels its effect. In the above equations, the appropriate network observable is the number of squares \(s(G)\) in the specific graph configuration of \(G\).

To understand why our Hamiltonian will result in local states with low energy, look at Fig. 1 which demonstrates a fully connected network with four nodes. The Hamiltonian of this network consists of six terms, as there are six edges that can be counted as a common edge. Consider one of the triads with its neighbor, for example \(\Delta_{qrst}\) with \(\Delta_{qrst}\), the common edge being \(qt\). Consider a sign for each triad which is the product of its edges signs. This triad’s sign would have counted for balanced and imbalanced states in structural balanced theory: negative sign for the imbalanced and positive sign for balanced. There are three structurally different possibilities for the combination of these two triads: \(++\), \(+-\), \(--\), with different probabilities. The first combination, both triads having positive signs, holds a model for four people having the same ideology towards an issue. In structural balance, this community has the lowest energy, being made of two balanced triads. The same holds in our model: the product of four edges will result in the minimum energy. The second combination has higher energy than the previous one, both in structural balanced theory and in our model. However, the difference arises in the third combination: both triads being structurally imbalanced, will result in a state with lower energy in our model.

There exist some similarities between our model and the Ising model \textsuperscript{[37]}. The Ising’s Hamiltonian considers the pairwise interactions between spin sites. This Hamiltonian is in a sense similar to the Hamiltonian of our model \textsuperscript{[1]} if we consider each triad similar to a spin site. However, there exist more degrees of freedom in our model, for each spin site only has two possible configurations, but each triad has eight. We prefer to look at our Hamiltonian in a sense of “square terms”, rather than the triad form.

We consider the temperature in our model as a measure of randomness and use exponential random graph to obtain the probability distribution function \textsuperscript{[38]}. This function is actually the Boltzmann probability in canonical ensemble, \(P(G) \propto e^{-\beta \mathcal{H}(G)}\), where \(\beta = 1/T\).

III. ANALYSIS

A. Mean-field solution

For beginning we want to calculate mean value of edges like \(\langle \sigma_{jk}\rangle\) over all configuration of our network. We rewrite our Hamiltonian as \(\mathcal{H} = \mathcal{H}^\prime + \mathcal{H}_{jk}\) by separating all the terms containing \(\sigma_{jk}\):

\begin{equation}
- \mathcal{H}_{jk} = \sigma_{jk} \sum_{i \neq j,k \neq j,k} \sigma_{ki} \sigma_{ji} \sigma_{\ell i}.
\end{equation}

We can find from statistical mechanics

\begin{equation}
\langle \sigma_{jk}\rangle = \sum_G \sigma_{jk} P(G),
\end{equation}
where \( \mathcal{P}(G) = e^{-\beta \mathcal{H}(G)} / Z \) is Boltzmann probability and 
\( Z = \sum_G e^{-\beta \mathcal{H}(G)} \) is the partition function, we have

\[
\langle \sigma_{jk} \rangle = \frac{1}{Z} \sum_G \langle \sigma_{jk} \rangle e^{-\beta \mathcal{H}(G)}.
\]

By \( \langle \ldots \rangle_G \), we mean the average overall graph configurations that don’t contain \( \sigma_{jk} \). Now we can expand the above fraction and estimate the higher order product using mean field approximation. In this method, we approximate edges’ variables with their averages and the correlation between these variables simply become the product of their averages. For example, if we name some edges’ variable with \( A \), we have \( \langle AA \rangle \approx \langle A \rangle^2 \). Equipped with this method, we can approximate the above quantity, if we name our edge’s variables as: 
\( o \equiv \langle \sigma_{ki} \sigma_{j\ell} \sigma_{\ell i} \rangle \) and \( p \equiv \langle \sigma_{jk} \rangle \), we have

\[
p = \tanh \left( \beta(n-2)(n-3)\sigma \right).
\]

The coefficient in the above equation is the number of all possible squares which contain \( \sigma_{jk} \) and is equal to 2 \times \binom{n}{2}. The factor of 2 comes from the two possible configurations with two selected nodes, Fig. 2.

Now let us calculate the following mean quantities with the same method: 
\( q \equiv \langle \sigma_{jk} \sigma_{ki} \rangle \), which is the mean value of two edges sharing a node or as we call it mean of two stars; 
\( r \equiv \langle \sigma_{ii} \sigma_{jk} \sigma_{ki} \rangle \), mean of open squares; 
\( s \equiv \langle \sigma_{ij} \sigma_{jk} \sigma_{ji} \sigma_{\ell k} \rangle \), mean of triangles and 
\( \mu \equiv \langle \sigma_{jk} \sigma_{ki} \sigma_{j\ell} \sigma_{\ell i} \rangle \), the mean of squares. Rewriting our Hamiltonian for calculating the mean of two stars as \( \mathcal{H} = \mathcal{H}' + \mathcal{H}_v \) with \( \mathcal{H}_v \) being:

\[
-\mathcal{H}_v = \sigma_{jk} \sigma_{ki} \sum_{\ell \neq i,j,k} \sigma_{j\ell} \sigma_{i\ell} + \sigma_{jk} \sum_{\mu \neq i,j,k} \sum_{\ell \neq i,j,k} \sigma_{\mu j} \sigma_{j\ell} \sigma_{i\ell} + \sigma_{ki} \sum_{\mu \neq i,j,k} \sum_{\ell \neq i,j,k} \sigma_{j\ell} \sigma_{\mu j} \sigma_{i\ell}
\]

and \( \mathcal{H}' \) is the remaining terms. Similar to above we have

\[
\langle \sigma_{jk} \sigma_{ki} \rangle = \frac{1}{Z} \sum_G \langle \sigma_{jk} \sigma_{ki} \rangle \mathcal{P}(G)
\]

\[
\approx \frac{\langle e^{-\beta \mathcal{H}_v(\sigma_{jk}=1,\sigma_{ki}=1)} - e^{-\beta \mathcal{H}_v(\sigma_{jk},\sigma_{ki})} - e^{-\beta \mathcal{H}_v(\sigma_{jk=1,\sigma_{ki}=1})} + e^{-\beta \mathcal{H}_v(\sigma_{jk=1,\sigma_{ki}=1})} \rangle_G}{\langle e^{-\beta \mathcal{H}_v(\sigma_{jk}=1,\sigma_{ki}=1)} + e^{-\beta \mathcal{H}_v(\sigma_{jk,\sigma_{ki})}} + e^{-\beta \mathcal{H}_v(\sigma_{jk=1,\sigma_{ki}=1})} + e^{-\beta \mathcal{H}_v(\sigma_{jk=1,\sigma_{ki}=1})} \rangle_G},
\]

where \( G' \) is all graph configurations that doesn’t contain \( \sigma_{jk} \) and \( \sigma_{ki} \). The mean field approximation for above is

\[
q = \frac{e^{2\beta(n-3)(n-4)\sigma + \beta(n-3)q} - 2e^{-\beta(n-3)q} + e^{-2\beta(n-3)(n-4)\sigma + \beta(n-3)q}}{e^{2\beta(n-3)(n-4)\sigma + \beta(n-3)q} + 2e^{-\beta(n-3)q} + e^{-2\beta(n-3)(n-4)\sigma + \beta(n-3)q}}.
\]

Similarly, equations for mean open squares, triangles and squares can be derived. The mean values of two stars and open squares are fundamental which means by knowing these two, all mean quantities like the mean values of triangles and squares, will be calculated. In appendix [A] an equation for mean open value of squares is derived with more details, and equations of mean values of triangles and squares are also shown. By substituting (5) in (A4) and using (7) we can write self consistency equations as

\[
\begin{align*}
q &= f(q, o; \beta, n) \\
o &= g(q, o; \beta, n).
\end{align*}
\]

In Fig. 3 we plot numeric solutions for each equation separately on \( q - o \) plane (in their allowed domains \(-1 \leq q \leq 1, -1 \leq o \leq 1\)). The intersections of curves are our simultaneous solutions for both equations. Right figures show the number of intersections under and above the critical temperature. At \( T = 1050 \) we have five intersections. If temperature is higher than \( T_c \) we have just one intersection which is our trivial solution. Left diagram shows the critical temperature, \( T_c \approx 1062 \), in which we have one intersection and two tangent curves. Depending on temperature, we have one \((T > T_c)\), three \((T = T_c)\) or five solutions \((T < T_c)\). The number of solutions changes abruptly and it is the classical phenomenology of discrete phase transition.

To discuss the stability of solutions or fixed points, we define a two dimensional field with the following compo-
FIG. 3. (Color online) Behavior of simultaneous solutions for $q^*$ in different temperature. We have five different solutions below $T_c$ and one above the critical temperature. The number of nodes is 50.

ments:

\[
\begin{align*}
u_q &\equiv f(q, o; \beta, n) - q \\
u_o &\equiv g(q, o; \beta, n) - o.
\end{align*}
\]  

(10)

In Fig. 4 we show vector field of above quantities with the $q^*$’s solution for different temperatures. This plot shows us the dynamic which $q$ and $o$ evolve to their final state. Below critical temperature $T < T_c$ we have five solutions which three of them are stable or attractive fixed points (blue dots) and other two are unstable or repulsive fixed points (red dots). We can check the stability of these fixed points by considering a point in $q - o$ plane very close to the specified fixed point $(q^*, o^*)$, we have

\[
\begin{align*}
q^* + \delta q' &= f(q^* + \delta q, o^* + \delta o; \beta, n) \\
o^* + \delta o' &= g(q^* + \delta q, o^* + \delta o; \beta, n).
\end{align*}
\]  

(11)

The Taylor expansion of the first equation, will be

\[
q^* + \delta q' \approx f(q^*, o^*; \beta, n) + \frac{\partial f}{\partial q}_{q=q^*} \delta q + \frac{\partial f}{\partial o}_{o=o^*} \delta o.
\]  

(12)

We consider above expansion only in linear regime and the matrix form is

\[
\begin{pmatrix}
\delta q' \\
\delta o'
\end{pmatrix} = J \begin{pmatrix}
\delta q \\
\delta o
\end{pmatrix}
\]  

(13)

Where $J$ is the Jacobian matrix

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial q} & \frac{\partial f}{\partial o} \\
\frac{\partial g}{\partial q} & \frac{\partial g}{\partial o}
\end{pmatrix}_{q=q^*} \\
o=o^*
\]  

We can diagonalize the Jacobian matrix as

\[
\begin{pmatrix}
\delta q' \\
\delta o'
\end{pmatrix} = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} \begin{pmatrix}
\delta q \\
\delta o
\end{pmatrix}
\]  

(15)

The magnitude of eigenvalues of the above equation can distinguish the stability of the fixed points. The only condition for the stable fixed point is $|\lambda_1| < 1$ because right hand side of [15] becomes smaller by each iteration. At least one of the eigenvalues of the red dots in Fig. 4 is bigger than one which makes these fixed points unstable.

We have three stable fixed points in a wide range of temperatures. Trivial solution $(q^* = 0, o^* = 0)$ for $q^*$ corresponds to random or bipolar network which means that the mean value of two stars and open squares are zero. This fixed point is unstable in low temperatures $(T < 50)$ and it becomes stable when the temperature increases. At low temperatures, the unstable fixed point is very close to the trivial solution (Fig. 4c).

We have two completely balanced states corresponding
to heaven \((q^* = +1, o^* = +1)\) and hell \((q^* = +1, o^* = -1)\). In heaven’s fixed point all links are positive which means that there is no hostility in the network. In hell’s fixed point, all links are negative which means that all nodes are enemies with each other. These fixed points move when temperature increases and suddenly disappear when the temperature is larger than the critical temperature.

In Fig. 6, we compare the dependency of the critical temperature on the size of the network, of the presented model with thermal balance theory. In quartic balance theory critical temperature changes as the square of size \(T_c(n) \approx n^2\) and in thermal balance theory it changes linearly \(T_c(n) \approx n\).

### B. Simulations

In our simulations, we work with a fully connected network with \(n\) nodes. We thermalize our system with a given temperature by the Monte Carlo method. In this method, we randomly pick an edge and compute the energy difference with the configuration where it is flipped. We accept this new configuration if this difference is negative. Also, if this difference is positive, we accept the new configuration with the Boltzmann’s probability.

In Fig. 5, we compare our analytic solutions with simulations. Our theory gives us stable and unstable solutions and we can’t see the unstable one (red dashed line) in simulation. We start our simulations with different initial configurations to find how our network changes when the randomness increases.

In random initial configurations, all kinds of balanced and imbalanced squares exist which means that the total energy of our system is near zero \((s \approx 0)\). It is interesting that with this initial condition and in temperature range of \(50 < T < 300\) we expect from our theory that the
method the following results are obtained: the existence of higher order interactions. The quar-

of real data is partially balanced. This can be interpreted

of the triangles is close to one. In Fig. 7 we have plotted

values of two stars and open squares but the mean value

forever, however, our unstable solution is so close to the

trivial solution is stable and system shall remain there

forever, however, our unstable solution is so close to the

stable one which pushes our final state to the bipolar

phases (Fig. 6c). Bipolar networks have small mean

values of two stars and open squares but the mean value

of the triangles is close to one. In Fig. 7 we have plotted

the bipolar networks of our model.

IV. CONCLUSIONS

There exists some evidence indicating that the network

of real data is partially balanced. This can be interpreted

as the existence of higher order interactions. The quartic

interaction is considered and through the mean field

method the following results are obtained:

- We observed that the global minimum of this model

  has four stable phases: heaven/hell and two kinds

  of bipolar. We have imbalance stable phases in two

  of them.

- We observe a phase transition where below a special

  temperature a symmetry between balanced and

  imbalanced triads is broken and similar to the ther-

  mal Heider model this transition is discrete.

- The critical temperature of QBT is proportional to

  the square of the size which is one order higher than

  thermal HBT.

- Our Monte Carlo simulations confirm the prediction

  of the discrete phase transition and the quantita-

  tive value of the critical temperature.

- In simulation, we observe that in low temperatures

  one of the stable points is not stable anymore where

  can be justified by the narrowness of basin of at-

  traction.

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Appendix A: Calculation of The Mean Values of
Open Squares

In this part we want to calculate the mean value of
open squares. As proceeded before, we write the Hamil-
tonian as $\mathcal{H} = \mathcal{H}' + \mathcal{H}_J$; where

$$-\mathcal{H}_J = \sigma_\mu \sum_{\mu \neq i,j,k,\ell} \sum_{\nu \neq i,j,k,\ell} \sigma_\nu \sigma_{ij}\sigma_{\mu\nu}$$

$$+ \sigma_{ki} \sum_{\mu \neq i,j,k,\ell} \sum_{\nu \neq i,j,k,\ell} \sigma_{jk}\sigma_{\mu\nu}\sigma_{vi}$$

$$+ \sigma_{j\ell} \sum_{\mu \neq i,j,k,\ell} \sum_{\nu \neq i,j,k,\ell} \sigma_{ji}\sigma_{\mu\nu}\sigma_{\ell\nu}$$

$$+ \sigma_{jk}\sigma_{ki} \sum_{\mu \neq i,j,k,\ell} \sigma_{\mu j}$$

$$+ \sigma_{jk}\sigma_{j\ell} \sum_{\mu \neq i,j,k,\ell} \sigma_{k\ell}$$

$$+ (\sigma_{jk}\sigma_{ki}\sigma_{j\ell})\sigma_{\ell i}. \quad (A1)$$

The first three terms are all the terms in Hamiltonian
which contain $\sigma_{jk}$, $\sigma_{ki}$ and $\sigma_{\ell i}$ separately, the next two
terms contain $\sigma_{jk}\sigma_{ki}$ and $\sigma_{jk}\sigma_{j\ell}$, the last term is a square
that contains all three edges. We have

$$\langle \sigma_{jk}\sigma_{ki}\sigma_{j\ell} \rangle = \frac{1}{Z} \sum_{\{\sigma_{jk},\sigma_{ki},\sigma_{j\ell} = \pm 1\}} \sigma_{jk}\sigma_{ki}\sigma_{j\ell} P(G). \quad (A2)$$

We can consider all the configurations of three edges in
open square and use mean field approximation. We have

$$-\mathcal{H}_{J}\sigma_{++}^{++} = 3(n - 4)(n - 5) o + 2q(n - 4) + p$$

$$-\mathcal{H}_{J}\sigma_{--}^{--} = -3(n - 4)(n - 5) o + 2q(n - 4) - p$$

$$-\mathcal{H}_{J}\sigma_{--}^{++} = (n - 4)(n - 5) o - 2q(n - 4) - p$$

$$-\mathcal{H}_{J}\sigma_{--}^{--} = (n - 4)(n - 5) o - p$$

$$-\mathcal{H}_{J}\sigma_{++}^{--} = (n - 4)(n - 5) o - p$$

$$-\mathcal{H}_{J}\sigma_{++}^{-+} = -(n - 4)(n - 5) o + p$$

$$-\mathcal{H}_{J}\sigma_{++}^{++} = -(n - 4)(n - 5) o + p$$

$$-\mathcal{H}_{J}\sigma_{++}^{--} = -(n - 4)(n - 5) o - 2q(n - 4) + p.$$ \quad (A3)

Where $\mathcal{H}_{J\sigma}^{abc} = \mathcal{H}_J(\sigma_{jk} = a, \sigma_{ki} = b, \sigma_{j\ell} = c)$.

By defining $\Gamma(o) \equiv (n - 4)(n - 5) o$ and
$\Sigma(o) \equiv (n - 3)(n - 4) o$, we can write:
\[
\sigma = \frac{e^{3\beta \Gamma(o)+2\beta(n-4)q+\beta p} - e^{-3\beta \Gamma(o)+2\beta(n-4)q-\beta p} - e^{3\beta \Gamma(o)-2\beta(n-4)q-\beta p} + 2e^{-\beta \Gamma(o)-\beta p} + 2e^{-\beta \Gamma(o)+\beta p} + e^{-\beta \Gamma(o)-2\beta(n-4)q+\beta p}}{e^{3\beta \Gamma(o)+2\beta(n-4)q+\beta p} + e^{-3\beta \Gamma(o)+2\beta(n-4)q-\beta p} + e^{3\beta \Gamma(o)-2\beta(n-4)q-\beta p} + 2e^{-\beta \Gamma(o)-\beta p} + 2e^{-\beta \Gamma(o)+\beta p} + e^{-\beta \Gamma(o)-2\beta(n-4)q+\beta p}}.
\]

We have calculated \(p\) in equation (5). For the mean value of triangles(5) we used its Hamiltonian which is

\[
-H_{\Delta} = \sum_{\mu \neq i,j,k \mu \neq i,j,k} \sigma_{i\mu} \sigma_{j\nu} \sigma_{\mu\nu} + \sum_{\mu \neq i,j,k} \sigma_{j\mu} \sigma_{j\nu} \sigma_{\mu\nu} + \sum_{\mu \neq i,j,k} \sigma_{i\mu} \sigma_{j\mu} \sigma_{\mu\nu} + \sigma_{i\mu} \sigma_{j\mu} \sigma_{\mu\nu} + \sigma_{i\mu} \sigma_{k\mu} \sigma_{\mu\nu} + \sigma_{j\mu} \sigma_{k\mu} \sigma_{\mu\nu} + \sigma_{i\mu} \sigma_{j\mu} \sigma_{k\mu} \sigma_{\mu\nu}.
\]

Mean value of squares(6) multiplied by minus one, represents the mean field approximation for energy in our network and we derive it by using following Hamiltonian

\[
-H_\Delta = \sum_{\mu \neq i,j,k} \sigma_{i\mu} \sigma_{k\mu} \sigma_{\mu\nu} + \sigma_{j\mu} \sigma_{j\mu} \sigma_{\mu\nu} + \sigma_{i\mu} \sigma_{j\mu} \sigma_{k\mu} \sigma_{\mu\nu} + \sigma_{j\mu} \sigma_{k\mu} \sigma_{\mu\nu} + \sigma_{i\mu} \sigma_{j\mu} \sigma_{k\mu} \sigma_{\mu\nu}.
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