CHEN ROTATIONAL SURFACES OF HYPERBOLIC OR ELLIPTIC TYPE IN THE FOUR-DIMENSIONAL MINKOWSKI SPACE

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ABSTRACT. We study the class of spacelike surfaces in the four-dimensional Minkowski space whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector. These surfaces are determined up to a motion by eight invariant functions satisfying some natural conditions. The subclass of Chen surfaces is characterized by the condition one of these invariants to be zero. In the present paper we describe all Chen spacelike rotational surfaces of hyperbolic or elliptic type.

1. Preliminaries

In [5] we considered the general theory of spacelike surfaces in the four-dimensional Minkowski space \( \mathbb{R}^4 \). The basic feature of our treatment of these surfaces was the introduction of an invariant linear map of Weingarten-type in the tangent plane at any point of the surface, following the approach to the theory of surfaces in \( \mathbb{R}^3 \) [2, 4]. Studying surfaces in the Euclidean space \( \mathbb{R}^4 \), in [2] we introduced a linear map \( \gamma \) of Weingarten-type, which plays a similar role in the theory of surfaces in \( \mathbb{R}^4 \) as the Weingarten map in the theory of surfaces in \( \mathbb{R}^3 \). The map \( \gamma \) generates the corresponding second fundamental form \( II \) at any point \( p \in M^2 \) in the standard way. We gave a geometric interpretation of the second fundamental form and the Weingarten map of the surface in [3].

Let \( M^2 : z = z(u, v), (u, v) \in D (D \subset \mathbb{R}^2) \) be a spacelike surface in \( \mathbb{R}^4 \) with tangent space \( T_p M^2 = \text{span}\{z_u, z_v\} \) at an arbitrary point \( p = z(u, v) \) of \( M^2 \). Since \( M^2 \) is spacelike, \( \langle z_u, z_u \rangle > 0 \), \( \langle z_v, z_v \rangle > 0 \). We use the standard notations \( E(u, v) = \langle z_u, z_u \rangle, F(u, v) = \langle z_u, z_v \rangle, G(u, v) = \langle z_v, z_v \rangle \) for the coefficients of the first fundamental form \( I(\lambda, \mu) := E\lambda^2 + 2F\lambda\mu + G\mu^2, \lambda, \mu \in \mathbb{R} \). Since \( I(\lambda, \mu) \) is positive definite we set \( W = \sqrt{EG - F^2} \). We choose a normal frame field \( \{n_1, n_2\} \) such that \( \langle n_1, n_1 \rangle = 1 \), \( \langle n_2, n_2 \rangle = -1 \), and the quadruple \( \{z_u, z_v, n_1, n_2\} \) is positively oriented in \( \mathbb{R}^4 \).

Considering the tangent space \( T_p M^2 \) at a point \( p \in M^2 \), in [5] we introduced an invariant \( \zeta_{g_1, g_2} \) of a pair of two tangents \( g_1, g_2 \) using the second fundamental tensor \( \sigma \) of \( M^2 \). By means of this invariant we defined conjugate, asymptotic, and principal tangents. The second fundamental form \( II \) of the surface \( M^2 \) at a point \( p \in M^2 \) is introduced on the base of conjugacy of two tangents at the point. The coefficients \( L, M, N \) of the second fundamental form \( II \) are determined as follows:

\[
L = \frac{2}{W} \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}; \quad M = \frac{1}{W} \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}; \quad N = \frac{2}{W} \begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix},
\]

where the functions \( c_{ij}^k \), \( i, j, k = 1, 2 \) are given by

\[
\begin{align*}
c_{11}^1 &= \langle z_{uu}, n_1 \rangle; & c_{12}^1 &= \langle z_{uv}, n_1 \rangle; & c_{12}^2 &= \langle z_{vv}, n_1 \rangle; \\
c_{11}^2 &= \langle z_{uu}, n_2 \rangle; & c_{12}^2 &= \langle z_{uv}, n_2 \rangle; & c_{22}^2 &= \langle z_{vv}, n_2 \rangle.
\end{align*}
\]

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The second fundamental form $II$ determines an invariant linear map $\gamma$ of Weingarten-type at any point of the surface, which generates two invariant functions:

$$k := \det \gamma = \frac{LN - M^2}{EG - F^2}, \quad \varkappa := -\frac{1}{2} \text{tr} \gamma = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$  

The functions $k$ and $\varkappa$ are invariant under changes of the parameters of the surface and changes of the normal frame field. The sign of $k$ is invariant under congruences and the sign of $\varkappa$ is invariant under motions in $\mathbb{R}^4$. However, the sign of $\varkappa$ changes under symmetries with respect to a hyperplane in $\mathbb{R}^4$. We proved that the invariant $\varkappa$ is the curvature of the normal connection of the surface. The number of asymptotic tangents at a point of $M^2$ is determined by the sign of the invariant $k$. In the case $k = 0$ there exists a one-parameter family of asymptotic lines, which are principal.

It is interesting to note that the ”umbilical” points, i.e. points at which the coefficients of the first and the second fundamental forms are proportional, are exactly the points at which the mean curvature vector $H$ is zero. So, the spacelike surfaces consisting of ”umbilical” points in $\mathbb{R}^4$ are exactly the minimal surfaces. Minimal spacelike surfaces are characterized in terms of the invariants $k$ and $\varkappa$ by the equality $\varkappa^2 - k = 0$.

Analogously to $\mathbb{R}^3$ and $\mathbb{R}^4$, the invariants $k$ and $\varkappa$ divide the points of $M^2$ into four types: flat, elliptic, hyperbolic and parabolic points. The surfaces consisting of flat points are characterized by the conditions $k = \varkappa = 0$. We gave a local geometric description of spacelike surfaces consisting of flat points whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector, proving that any such a surface either lies in a hyperplane of $\mathbb{R}^4$ or is part of a developable ruled surface in $\mathbb{R}^4$.

Using the introduced principal lines on a spacelike surface in $\mathbb{R}^4$ whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector, we found a geometrically determined moving frame field on such a surface. Writing the derivative formulas of Frenet-type for this frame field, we obtained eight invariant functions and proved a fundamental theorem of Bonnet-type, stating that these eight invariants under some natural conditions determine the surface up to a motion in $\mathbb{R}^4$.

One of these eight invariants is closely related to the theory of Chen surfaces. We shall recall the notion of Chen submanifolds. Let $M^n$ be an $n$-dimensional submanifold of $(n + m)$-dimensional Riemannian manifold $\tilde{M}^{n+m}$ and $\xi$ be a normal vector field of $M^n$. In [1] B.-Y. Chen defined the allied vector field $a(\xi)$ of $\xi$ by the formula

$$a(\xi) = \frac{||\xi||}{n} \sum_{k=2}^{m} \{\text{tr}(A_1A_k)\} \xi_k,$$

where $\{\xi_1 = \frac{\xi}{||\xi||}, \xi_2, \ldots, \xi_m\}$ is an orthonormal base of the normal space of $M^n$, and $A_i = A_{\xi_i}$, $i = 1, \ldots, m$ is the shape operator with respect to $\xi_i$. In particular, the allied vector field $a(H)$ of the mean curvature vector field $H$ is a well-defined normal vector field which is orthogonal to $H$. It is called the allied mean curvature vector field of $M^n$ in $\tilde{M}^{n+m}$. B.-Y. Chen defined the $A$-submanifolds to be those submanifolds of $\tilde{M}^{n+m}$ for which $a(H)$ vanishes identically [1]. In [3] [7] the $A$-submanifolds are called Chen submanifolds. It is easy to see that minimal submanifolds, pseudo-umbilical submanifolds and hypersurfaces are Chen submanifolds. These Chen submanifolds are said to be trivial $A$-submanifolds.

Now, let $M^2$ be a spacelike surface in $\mathbb{R}^4$ whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector. We denote by $x$ and $y$ the principal unit tangent vector fields of $M^2$, and by $H$ - the mean curvature vector field. One of the invariants in the Frenet-type derivative formulas of $M^2$ (eg. [5]) is $\lambda = \frac{1}{\sqrt{\langle H, H \rangle}} \langle \sigma(x, y), H \rangle$ in the case
\[\langle H, H \rangle > 0, \text{ and } \lambda = -\frac{1}{\sqrt{-\langle H, H \rangle}} \langle \sigma(x, y), H \rangle \] in the case \(\langle H, H \rangle < 0\). Applying the definition of the allied mean curvature vector field and the derivative formulas of \(M^2\) we get

\[a(H) = \frac{\sqrt{x'^2 - k}}{2} \lambda l.\]

Hence, if \(M^2\) is free of minimal points \((\sqrt{x'^2 - k} \neq 0)\), then \(a(H) = 0\) if and only if \(\lambda = 0\). This gives the geometric interpretation of the invariant \(\lambda\). It is clear that \(M^2\) is a non-trivial Chen surface if and only if the invariant \(\lambda\) is zero.

In the present paper we study spacelike rotational surfaces of hyperbolic or elliptic type in the four-dimensional Minkowski space \(\mathbb{R}^4_1\) and we describe the class of Chen rotational surfaces of hyperbolic or elliptic type.

2. Rotational surfaces with two-dimensional axis in \(\mathbb{R}^4_1\)

In [2] we considered the class of the rotational surfaces with two-dimensional axis in the four-dimensional Euclidean space \(\mathbb{R}^4\).

Let \(Oe_1e_2e_3e_4\) be a fixed orthonormal base of \(\mathbb{R}^4\) and \(\mathbb{R}^3\) be the subspace spanned by \(e_1, e_2, e_3\). We consider a smooth curve \(c : \mathbb{R} \rightarrow \mathbb{R}^3\), parameterized by

\[\tilde{z}(u) = (x_1(u), x_2(u), r(u)); \quad u \in J.\]

Without loss of generality we assume that \(c\) is parameterized by the arc-length, i.e. \((x'_1)^2 + (x'_2)^2 + (r')^2 = 1\). We assume also that \(r(u) > 0\), \(u \in J\). Let \(\kappa\) and \(\tau\) be the curvature and the torsion of \(c\). We denote by \(c_1\) the projection of \(c\) into the 2-dimensional plane \(Oe_1e_2\) and by \(\kappa_1\) the curvature of the plane curve \(c_1\), i.e. \(\kappa_1 = x'_1 x''_2 - x''_1 x'_2\).

Let us consider the rotational surface \(M^2\) in \(\mathbb{R}^4\) given by

\[z(u, v) = (x_1(u), x_2(u), r(u) \cos v, r(u) \sin v); \quad u \in J, \quad v \in [0; 2\pi).\]

\(M^2\) is obtained by the rotation of the curve \(c\) about the two-dimensional axis \(Oe_1e_2\) (the rotation of \(c\) that leaves the plane \(Oe_1e_2\) fixed).

In [2] we found that the invariants \(k, \nu\) and the Gauss curvature \(K\) of the rotational surface \(M^2\) are expressed as follows:

\[k = -\frac{(\kappa_1)^2}{r^2}; \quad \nu = 0; \quad K = -\frac{\tau''}{r}.\]

Obviously, the rotational surface \(M^2\) is a surface with flat normal connection since \(\nu = 0\). We described all rotational surfaces, for which the invariant \(k\) is constant [2].

The mean curvature vector field \(H\) of \(M^2\) is given by:

\[H = \frac{\kappa_1 e_2 - \kappa_{12}}{2\kappa r} e_1 - \frac{\kappa_{12}}{2\kappa r} e_2;\]

where \(e_1\) and \(e_2\) are the normal vector fields, defined by

\[e_1 = \frac{1}{\kappa}(x''_1, x''_2, r'' \cos v, r'' \sin v);\]

\[e_2 = \frac{1}{\kappa}(x'_2 r'' - x''_2 r', x''_1 r' - x'_1 r'', (x'_1 x''_2 - x''_1 x'_2) \cos v, (x'_1 x''_2 - x''_1 x'_2) \sin v).\]

Hence, the mean curvature vector field \(H\) vanishes if and only if \(\kappa_1 = 0\) and \(\kappa^2 r - \tau'' = 0\). In this case the rotational surface \(M^2\) is a trivial Chen surface (\(M^2\) is minimal).

In the case \(\kappa_1 = 0\) and \(\kappa^2 r - \tau'' \neq 0\) the rotational surface \(M^2\) lies in a three-dimensional subspace \(\mathbb{R}^3\) of \(\mathbb{R}^4\). Moreover, in this case one can easily get that the allied vector field \(a(H)\) of the mean curvature vector field is zero, and hence \(M^2\) is a Chen surface in \(\mathbb{R}^3\). Hence, \(M^2\) is a trivial Chen surface.
In the case when \( \kappa_1 \neq 0 \) the rotational surface \( M^2 \) is a non-minimal surface in \( \mathbb{R}^4 \) free of flat points. Calculating the invariant \( \lambda \) of \( M^2 \) we get that \( \lambda = 0 \) if and only if
\[
\kappa^4 r^2 - (r'')^2 - (\kappa_1)^2 = 0.
\]
In this case \( M^2 \) is a non-trivial Chen surface in \( \mathbb{R}^4 \).

Now we shall study spacelike rotational surfaces of hyperbolic or elliptic type in Minkowski space \( \mathbb{R}^4_1 \) and we shall describe the class of Chen rotational surfaces.

We consider the Minkowski space \( \mathbb{R}^4_1 \) endowed with the metric \( \langle \cdot, \cdot \rangle \) of signature (3, 1). Let \( Oe_1e_2e_3e_4 \) be a fixed orthonormal coordinate system in \( \mathbb{R}^4_1 \), i.e. \( e_1^2 = e_2^2 = e_3^2 = 1, e_4^2 = -1 \), giving the orientation of \( \mathbb{R}^4_1 \). The standard flat metric is given in local coordinates by
\[
dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2.
\]

A surface \( M^2 : z = z(u, v), (u, v) \in D (D \subset \mathbb{R}^2) \) in \( \mathbb{R}^4_1 \) is said to be spacelike if \( \langle \cdot, \cdot \rangle \) induces a Riemannian metric \( g \) on \( M^2 \). Thus at each point \( p \) of a spacelike surface \( M^2 \) we have the following decomposition \( \mathbb{R}^4_1 = T_pM^2 \oplus N_pM^2 \) with the property that the restriction of the metric \( \langle \cdot, \cdot \rangle \) onto the tangent space \( T_pM^2 \) is of signature \( (2, 0) \), and the restriction of the metric \( \langle \cdot, \cdot \rangle \) onto the normal space \( N_pM^2 \) is of signature \( (1, 1) \).

We consider a smooth spacelike curve \( c : \tilde{z} = \tilde{z}(u), u \in J \), parameterized by
\[
\tilde{z}(u) = (x_1(u), x_2(u), 0, r(u)); \quad u \in J.
\]
The curve \( c \) lies in the three-dimensional subspace \( \mathbb{R}^3_1 = \text{span}\{e_1, e_2, e_4\} \) of \( \mathbb{R}^4_1 \). Without loss of generality we assume that \( c \) is parameterized by the arc-length, i.e. \( (x_1')^2 + (x_2')^2 - (r')^2 = 1 \). We assume also that \( r(u) > 0, u \in J \) and \( \langle t'_c(u), t''_c(u) \rangle \neq 0, u \in J \), where \( t_c(u) = \tilde{z}'(u) \). We have the following possibilities:

1.) \( \langle t'_c, t''_c \rangle > 0 \), i.e. \( (x_1')^2 + (x_2'')^2 - (r'')^2 > 0 \). In this case the Frenet formulas of \( c \) are given by
\[
t_c' = \kappa n_c; \quad n_c' = -\kappa t_c + \tau b_c; \quad b_c' = -\tau n_c,
\]
where \( \{t_c, n_c, b_c\} \) is the Frenet frame field of \( c \); \( \kappa \) and \( \tau \) - the curvature and the torsion of \( c \); \( n_c \) is spacelike, \( b_c \) is timelike.

2.) \( \langle t'_c, t''_c \rangle < 0 \), i.e. \( (x_1')^2 + (x_2'')^2 - (r'')^2 < 0 \). In this case we have the following Frenet formulas of \( c \):
\[
t_c' = \kappa n_c; \quad n_c' = \kappa t_c + \tau b_c; \quad b_c' = \tau n_c,
\]
where \( n_c \) is timelike, \( b_c \) is spacelike.

Let us consider the surface \( M^2 \) in \( \mathbb{R}^4_1 \) given by
\[
(2.1) \quad z(u, v) = (x_1(u), x_2(u), r(u) \sinh v, r(u) \cosh v); \quad u \in J, v \in \mathbb{R}.
\]
The tangent space of \( M^2 \) is spanned by the vector fields
\[
z_u = (x_1', x_2', r' \sinh v, r' \cosh v);
\]
\[
z_v = (0, 0, r \cosh v, r \sinh v).
\]
Hence, the coefficients of the first fundamental form of \( M^2 \) are:
\[
E = \langle z_u, z_u \rangle = 1; \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = r^2(u),
\]
and the induced metric \( g \) on \( M^2 \) is a Riemannian metric:
\[
g = du^2 + r^2(u)dv^2.
\]
So, the surface \( M^2 \) in \( \mathbb{R}^4_1 \), defined by (2.1), is a spacelike surface. It is called a spacelike rotational surface of hyperbolic type in \( \mathbb{R}^4_1 \). It is an orbit of a spacelike regular curve under the action of the orthogonal transformations of \( \mathbb{R}^4_1 \) which leave a spacelike plane point-wise fixed. In our case the two-dimensional plane \( Oe_1e_2 \) is fixed.
A classification of all timelike and spacelike hyperbolic rotational surfaces with non-zero constant mean curvature in the three-dimensional de Sitter space $S^3_1$ is given in [9]. Similarly, a classification of the spacelike and timelike Weingarten rotation surfaces in $S^3_1$ is found in [10].

Here we shall describe the class of Chen spacelike rotational surfaces of hyperbolic type in $\mathbb{R}^4$.

The second partial derivatives of $z(u, v)$ are expressed as follows
\[
\begin{align*}
    z_{uu} &= (x'', x', r'' \sinh v, r'' \cosh v); \\
    z_{uv} &= (0, 0, r' \cosh v, r' \sinh v); \\
    z_{vv} &= (0, 0, r \sinh v, r \cosh v).
\end{align*}
\]

We consider the following orthonormal tangent vector fields
\[
\begin{align*}
    \overline{x} &= (x', x'_2, r' \sinh v, r' \cosh v); \\
    \overline{y} &= (0, 0, \cosh v, \sinh v),
\end{align*}
\]
i.e. $z_u = \overline{x}$, $z_v = r \overline{y}$, and the normal vector fields $n_1$, $n_2$, defined by
\[
\begin{align*}
    n_1 &= \frac{1}{\kappa} (x'', x', r'' \sinh v, r'' \cosh v); \\
    n_2 &= \frac{1}{\kappa} (x'_2 r'' - x'_2 r', r' x'' - x'_1 r'', (x'_2 x'' - x'_1 x'') \sinh v, (x'_2 x'' - x'_1 x'') \cosh v).
\end{align*}
\]

In the case when $n_c$ is spacelike the normal vector field $n_1$ is spacelike and $n_2$ is timelike. If $n_c$ is timelike, then $n_1$ is timelike and $n_2$ is spacelike. We denote $\varepsilon = \langle n_c, n_c \rangle$. Then we have $\langle n_1, n_1 \rangle = \varepsilon, \langle n_2, n_2 \rangle = -\varepsilon$.

We calculate the coefficients of the second fundamental tensor of $M^2$:
\[
\begin{align*}
    c_{11}^1 &= \langle z_{uu}, n_1 \rangle = \varepsilon \kappa; & c_{12}^1 &= \langle z_{uu}, n_2 \rangle = 0; \\
    c_{12}^2 &= \langle z_{uv}, n_1 \rangle = 0; & c_{12}^2 &= \langle z_{uv}, n_2 \rangle = 0; \\
    c_{22}^1 &= \langle z_{vv}, n_1 \rangle = -\frac{r''}{\kappa}; & c_{22}^2 &= \langle z_{vv}, n_2 \rangle = \frac{r}{\kappa} (x'_1 x'' - x'_1 x'').
\end{align*}
\]

Let us denote by $c_1$ the projection of $c$ into the 2-dimensional plane $Oe_1e_2$ and by $\kappa_1$ the curvature of $c_1$, i.e. $\kappa_1 = x'_1 x'' - x'_2 x''$.

Using (2.2) we calculate the coefficients of the second fundamental form of $M^2$:
\[
L = 0; \quad M = \varepsilon \kappa_1; \quad N = 0.
\]

Hence, the invariants $k$ and $\varkappa$ of the rotational surface of hyperbolic type are expressed as
\[
k = -\frac{(\kappa_1)^2}{r^2}; \quad \varkappa = 0.
\]

Consequently, any spacelike rotational surface of hyperbolic type in $\mathbb{R}^4_1$ is a surface with flat normal connection since $\varkappa = 0$.

With respect to the frame field $\{\overline{x}, \overline{y}, n_1, n_2\}$ the derivative formulas of $M^2$ look like:
\[
\begin{align*}
    \nabla'_{\overline{x}} \overline{x} &= \kappa n_1; & \nabla'_{\overline{x}} n_1 &= -\varepsilon \kappa \overline{x} + \tau n_2; \\
    \nabla'_{\overline{x}} \overline{y} &= 0; & \nabla'_{\overline{x}} n_1 &= \frac{r''}{\kappa} \overline{y}; \\
    \nabla'_{\overline{y}} \overline{x} &= \frac{r'}{r} \overline{y}; & \nabla'_{\overline{y}} n_1 &= \tau \kappa n_1; \\
    \nabla'_{\overline{y}} \overline{y} &= -\frac{r'}{r} \overline{x} - \frac{r''}{\kappa} \varepsilon n_1 - \frac{\kappa_1}{\kappa} \varepsilon n_2; & \nabla'_{\overline{y}} n_2 &= -\frac{\kappa_1}{\kappa} \overline{y}.
\end{align*}
\]
So, the Gauss curvature of $M^2$ is:

$$K = -\frac{r''}{r}.$$  

Obviously $M^2$ is not parameterized by the principal lines. The principal tangents of $M^2$ are:

$$x = \frac{\sqrt{2}}{2} \bar{x} + \frac{\sqrt{2}}{2} \bar{y}; \quad y = \frac{\sqrt{2}}{2} \bar{x} - \frac{\sqrt{2}}{2} \bar{y}. $$  

Then formulas (2.3) and (2.4) imply that the normal vector fields $\sigma(x, x)$, $\sigma(x, y)$, and $\sigma(y, y)$ are given by

$$\sigma(x, x) = \frac{r\kappa^2 - \varepsilon r''}{2r\kappa} n_1 - \frac{\varepsilon \kappa_1}{2r\kappa} n_2,$$

$$\sigma(x, y) = \frac{r\kappa^2 + \varepsilon r''}{2r\kappa} n_1 + \frac{\varepsilon \kappa_1}{2r\kappa} n_2,$$

$$\sigma(y, y) = \frac{r\kappa^2 - \varepsilon r''}{2r\kappa} n_1 - \frac{\varepsilon \kappa_1}{2r\kappa} n_2.$$  

The normal mean curvature vector field of $M^2$ is

$$H = \frac{r\kappa^2 - \varepsilon r''}{2r\kappa} n_1 - \frac{\varepsilon \kappa_1}{2r\kappa} n_2.$$  

All Chen spacelike rotational surfaces of hyperbolic type in $\mathbb{R}^4_1$ are described in the following

**Proposition 2.1.** The spacelike rotational surface of hyperbolic type $M^2$, defined by (2.1), is a Chen surface if and only if one of the following cases holds:

(i) $\kappa_1 = 0$ and $r\kappa^2 - \varepsilon r'' = 0$; in such case $M^2$ is a minimal surface (trivial Chen surface);

(ii) $\kappa_1 = 0$ and $r\kappa^2 - \varepsilon r'' \neq 0$; in such case $M^2$ is a surface lying in a three-dimensional subspace of $\mathbb{R}^4_1$ (trivial Chen surface);

(iii) $\kappa_1 \neq 0$ and $r^2\kappa^4 - (r'')^2 + (\kappa_1)^2 = 0$; in such case $M^2$ is a non-trivial Chen surface in $\mathbb{R}^4_1$.

**Proof.** (i) From (2.6) it follows that the mean curvature vector field $H$ vanishes if and only if

$$\kappa_1 = 0; \quad r\kappa^2 - \varepsilon r'' = 0.$$  

In this case the rotational surface $M^2$ is a trivial Chen surface ($M^2$ is minimal).

(ii) In the case $\kappa_1 = 0$ and $r\kappa^2 - \varepsilon r'' \neq 0$ we get that $k = \kappa = 0$, and hence $M^2$ consists of flat points. We shall prove that $M^2$ lies in a hyperplane of $\mathbb{R}^4_1$. The equality $\kappa_1 = 0$ implies that the projection of the curve $c$ into $Oe_1e_2$ lies on a straight line, and hence $c$ lies in a two-dimensional plane orthogonal to $Oe_1e_2$. Thus the torsion of $c$ is $\tau = 0$. Then, using derivative formulas (2.3) we get

$$\nabla'_x n_2 = 0; \quad \nabla'_y n_2 = 0,$$

which imply that the normal vector field $n_2$ is constant. Consequently, $M^2$ lies in the hyperplane $\mathbb{E}^3$ of $\mathbb{R}^4_1$ orthogonal to $n_2$, i.e. $\mathbb{E}^3 = \text{span}\{x, y, n_1\}$. Hence, $M^2$ is a trivial Chen surface.

(iii) In the case when $\kappa_1 \neq 0$ the rotational surface $M^2$ is a non-minimal surface in $\mathbb{R}^4$ free of flat points. According to [5] $M^2$ is a Chen surface if and only if the invariant $\lambda$ is zero. We have that $\lambda = 0$ if and only if $\langle \sigma(x, y), H \rangle = 0$, which in view of (2.5) and (2.6) implies

$$r^2\kappa^4 - (r'')^2 + (\kappa_1)^2 = 0.$$  

In this case $M^2$ is a non-trivial Chen surface in $\mathbb{R}^4_1$. 

$\square$
In a similar way we consider a spacelike surface in $\mathbb{R}^4_1$ which is an orbit of a spacelike regular curve $c$ under the action of the orthogonal transformations of $\mathbb{R}^4_1$ which leave a timelike plane point-wise fixed. Let us consider a spacelike curve $c: \mathbb{R} \to \mathbb{R}^4_1$ which is parameterized by the arc-length, i.e. $(r')^2 + (x'_1)^2 - (x'_2)^2 = 1$. We assume also that $r(u) > 0$, $u \in J$ and $\langle t'_c(u), t'_c(u) \rangle \neq 0$, $u \in J$, where $t_c(u) = z'(u)$.

Now we consider the surface $M^2$ in $\mathbb{R}^4_1$ given by

$$z(u, v) = (r(u) \cos v, r(u) \sin v, x_1(u), x_2(u)); \quad u \in J, \ v \in [0; 2\pi).$$

The tangent space of $M^2$ is spanned by the vector fields

$$z_u = (r' \cos v, r' \sin v, x'_1, x'_2);$$

$$z_v = (-r \sin v, r \cos v, 0, 0).$$

Hence, the coefficients of the first fundamental form of $M^2$ are

$$E = \langle z_u, z_u \rangle = 1; \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = r^2(u),$$

and the induced metric $g$ on $M^2$ is a Rie mannian metric:

$$g = du^2 + r^2(u)dv^2.$$  

The surface $M^2$, defined by (2.7), is a spacelike surface in $\mathbb{R}^4_1$. It is obtained by the rotation of the curve $c$ about the two-dimensional Lorentz plane $Oe_3e_4$. It is called a spacelike rotational surface of elliptic type. A local classification of spacelike surfaces in $\mathbb{R}^4_1$, which are invariant under spacelike rotations, and with mean curvature vector either vanishing or lightlike, is obtained in [8]. Here we shall describe the class of Chen spacelike rotational surfaces of elliptic type in $\mathbb{R}^4_1$.

As in the case of rotational surfaces of hyperbolic type we consider the orthonormal tangent vector fields

$$\overline{x} = z_u; \quad \overline{y} = \frac{z_v}{r},$$

and the normal vector fields $n_1$, $n_2$, defined by

$$n_1 = \frac{1}{\kappa} (r'' \cos v, r'' \sin v, x''_1, x''_2);$$

$$n_2 = \frac{1}{\kappa} ((x'_1 x''_2 - x'_2 x''_1) \cos v, (x'_1 x''_2 - x'_2 x''_1) \sin v, x''_2 r'' - r' x''_2, x'_1 r'' - r' x'_1),$$

where $\kappa$ is the curvature of $c$. The principal tangents of $M^2$ are $x = \frac{\sqrt{2}}{2}(\overline{x} + \overline{y})$; $y = \frac{\sqrt{2}}{2}(\overline{x} - \overline{y})$.

With respect to the principal tangents $x, y$ we get the following formulas:

$$\sigma(x, x) = \frac{r\kappa^2 - \varepsilon r''}{2r\kappa} n_1 + \frac{\varepsilon \kappa_1}{2r\kappa} n_2,$$

$$\sigma(x, y) = \frac{r\kappa^2 + \varepsilon r''}{2r\kappa} n_1 - \frac{\varepsilon \kappa_1}{2r\kappa} n_2,$$

$$\sigma(y, y) = \frac{r\kappa^2 - \varepsilon r''}{2r\kappa} n_1 + \frac{\varepsilon \kappa_1}{2r\kappa} n_2.$$

The invariants $k$, $\varkappa$ and the Gauss curvature $K$ of $M^2$ are expressed as in the hyperbolic case:

$$k = -\frac{(\kappa_1)^2}{r^2}; \quad \varkappa = 0; \quad K = -\frac{r''}{r}. $$
The normal mean curvature vector field of $M^2$ is
\[
H = \frac{r\kappa^2 - \varepsilon r''}{2r\kappa} n_1 + \frac{\varepsilon \kappa_1}{2r\kappa} n_2.
\]

All Chen spacelike rotational surfaces of elliptic type in $\mathbb{R}^4_1$ are described in the following Proposition 2.2. The spacelike rotational surface of elliptic type $M^2$, defined by (2.7), is a Chen surface if and only if one of the following cases holds:

(i) $\kappa_1 = 0$ and $r\kappa^2 - \varepsilon r'' = 0$; in such case $M^2$ is a minimal surface (trivial Chen surface);
(ii) $\kappa_1 = 0$ and $r\kappa^2 - \varepsilon r'' \neq 0$; in such case $M^2$ is a surface lying in a three-dimensional subspace of $\mathbb{R}^4_1$ (trivial Chen surface);
(iii) $\kappa_1 \neq 0$ and $r^2\kappa^4 - (r'')^2 + (\kappa_1)^2 = 0$; in such case $M^2$ is a non-trivial Chen surface in $\mathbb{R}^4_1$.

The proof is similar to the proof of Proposition 2.1.

At the end of the section we shall describe all spacelike rotational surfaces of hyperbolic or elliptic type in $\mathbb{R}^4_1$, for which the invariant $k$ is constant.

1. The invariant $k = 0$ if and only if $\kappa_1 = 0$, i.e. the projection of the curve $c$ into the two-dimensional axis lies on a straight line. In this case $M^2$ lies in a hyperplane $E^3$ of $\mathbb{R}^4_1$.
There are two subcases:

1.1. If $K = 0$, i.e. $r'' = 0$, then $M^2$ is a developable ruled surface in $E^3$.
1.2. If $K \neq 0$, i.e. $r'' \neq 0$, then $M^2$ is a non-flat surface in $E^3$.

2. The invariant $k = \text{const } (k \neq 0)$ if and only if $r(u) = a(x'_1 x''_2 - x'_2 x''_1)$, $a = \text{const}$, $a \neq 0$. Moreover, if $r(u)$ satisfies $r''(u) = c r(u)$, then the Gauss curvature $K$ is also a constant.

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