Fast cooling of trapped ion in strong sideband coupling regime

Shuo Zhang¹, Jian-Qi Zhang², Wei Wu¹,³, Wan-Su Bao¹ and Chu Guo¹∗

¹ Henan Key Laboratory of Quantum Information and Cryptography, Zhengzhou, Henan 450000, People’s Republic of China
² Innovation Academy for Precision Measurement Science and Technology, Wuhan, Hubei 430071, People’s Republic of China
³ Department of Physics, College of Liberal Arts and Sciences, National University of Defense Technology, Changsha 410073, People’s Republic of China

∗ Author to whom any correspondence should be addressed.

Abstract

Trapped ion in the Lamb–Dicke (LD) regime with the LD parameter η ≪ 1 can be cooled down to its motional ground state using sideband cooling. Standard sideband cooling works in the weak sideband coupling (WSC) limit, where the sideband coupling strength is small compared to the natural linewidth γ of the internal excited state, with a cooling rate much less than γ. Here we consider cooling schemes in the strong sideband coupling (SSC) regime, where the sideband coupling strength is comparable or even greater than γ. We derive analytic expressions for the cooling rate and the average occupation of the motional steady state in this regime, based on which we show that one can reach a cooling rate which is proportional to γ, while at the same time the steady state occupation increases by a correction term proportional to η² compared to the WSC limit. We demonstrate with numerical simulations that our analytic expressions faithfully recover the exact dynamics in the SSC regime.

1. Introduction

Trapped ions display rich physical phenomena due to its high degree of controllability and abundant degrees of freedom. In the context of quantum simulations, trapped ions can be used to realize quantum spin systems [1], the Bose–Hubbard chain [2], the Jaynes–Cummings model [3] with tunable interactions, as well as to study energy and particle transport far from equilibrium [4–11]. Trapped ion system is also a promising candidate to build quantum computers [12, 13], where the internal degrees of freedom are used to encode the qubits and the external (motional) degrees of freedom are used to induce effective couplings between those qubits [14]. The motional degrees of freedom of trapped ions play a central role in realizing all the above systems or schemes, being used either directly or indirectly. Particularly, in the context of quantum computing, cooling the motional degrees of freedom down to their ground states is a central step for coherent manipulation of the quantum state [15].

Sideband cooling is one of the first and still widely used cooling scheme [16–18]. The key ideas of sideband cooling are summarized as follows, which are also helpful for understanding other cooling schemes using static lasers. First we assume that an ion with two internal states, say a metastable ground state |g⟩ and an unstable excited state |e⟩ with a lifetime τ = 1/γ (γ is the linewidth and we have set ħ = 1), is trapped in a way that the motional degree of freedom of the ion is described by a harmonic oscillator with equidistant energy levels |n⟩ of energies (n + 1/2)ν, where ν is the trap frequency. A laser with Rabi frequency Ω is then applied onto the ion with detuning Δ which, in the first order of the Lamb–Dicke (LD) parameter η, induces a carrier transition |g⟩ ↔ |e, n⟩ with strength Ω, a red sideband |g⟩ ↔ |e, n + 1⟩ with strengths ηΩ/√n and ηΩ/√(n + 1), respectively. For this first order picture to be valid, the following conditions need to be satisfied: (1) the LD condition η ≪ 1, which requires the oscillations of the trapped ion to be much smaller than the wavelength of the cooling laser, (2)
resolved sideband condition, which requires $\eta \Omega, \gamma \ll \nu$. The laser is often tuned to red sideband resonance, namely $\Delta = -\nu$, such that the blue sideband is far-detuned compared to the red sideband and can often be neglected. The red sideband together with the natural decay $|e\rangle \rightarrow |g\rangle$ form a dissipative cascade [19] which makes the cooling possible. Most existing cooling schemes works in the weak sideband coupling (WSC) regime, which requires additionally $\eta \Omega \ll \gamma$, so that the states $|e, n-1\rangle$ decay to $|g, n-1\rangle$ immediately once pumped up from $|g, n\rangle$ by the red sideband. As a result the states $|e, n\rangle$ can be adiabatically eliminated from the cascade and one gets an effective decay from $|g, n\rangle$ to $|g, n-1\rangle$ with a rate

$$W_{WSC} \propto \frac{\eta^2 \Omega^2}{\gamma}. \tag{1}$$

The WSC condition thus sets a cooling rate which is much less than the natural linewidth $\gamma$.

Subsequent proposals using static lasers mainly aim to improve the quality of the steady state by suppressing the heating effects due to the carrier transition or the blue sideband [20–23], or both of them [24–30], by adding more lasers as well as more internal energy levels. As an outstanding example, cooling by electromagnetically induced transparency (EIT) eliminates the carrier transition [21, 22], which has been demonstrated in various experiments due to its simplicity and effectiveness [31–38]. Quantum control has also been applied in recent years to numerically find an optimal sequence of pulsed lasers which drives the trapped ion towards its motional ground state [39]. An advantage of cooling with pulsed lasers is that the lasers could often be tuned such that fast cooling can be achieved compared to sideband cooling, while the drawbacks are that the time-dependence of the lasers adds more difficulty for the experimental implementation, and that the initial motional state is often required to be known precisely in such schemes.

In this work, we focus on cooling schemes using static lasers. In particular, we aim to improve the cooling rate set by equation (1), which is essential in all applications to reduce the associated dead time [35]. For this goal, we consider cooling schemes in the strong sideband coupling (SSC) regime, where $\eta \Omega$ is comparable with or even larger than $\gamma$, namely $\eta \Omega \gtrsim \gamma$. Equation (1) fails in the SSC regime, which effect has been observed both numerically [26, 27, 29, 30] and experimentally [35]. In particular, we consider both the standing wave sideband cooling and the EIT cooling schemes in the SSC regime such that the heating effects due to the carrier transitions are suppressed. In both cases, we show analytically and numerically that we can reach a cooling rate $W_{SSC} \propto \gamma$ in the SSC regime, independent of the sideband coupling strength $\eta \Omega$. The price to pay is a correction term to the steady state occupation of the motional degree of freedom which is proportional to $\eta^2$. This paper is organized as follows. We first consider the standing wave sideband cooling in the SSC regime in section 2. We derive analytic expressions for the dynamics of the average occupation of the motional state as well as its steady state value, which are verified by comparison to exact numerical results. In section 4, we further generalize those expressions to EIT cooling in the SSC regime. We conclude in section 4.

2. Standing wave sideband cooling in the SSC regime

Standing wave sideband cooling is conceptually the simplest cooling scheme where the carrier transition is suppressed [20]. Here we first consider this scheme both due to its simplicity and that it is also helpful for understanding other cooling schemes that based on a dark state. In standing wave sideband cooling scheme a trapped ion with a mass $m$ and two internal states, a metastable ground state $|g\rangle$ with energy $\omega_g$ and an unstable excited state $|e\rangle$ with linewidth $\gamma$ and energy $\omega_e$, is coupled to a standing wave laser with frequency $\omega_L$ wave number $k$ and Rabi frequency $\Omega$. The ion is assumed to be trapped in a harmonic potential with a trap frequency $\nu$ and at the same time located at the node of the standing wave laser such that the carrier transition vanishes. The equation of motion is described by the Lindblad master equation [40, 41]

$$\frac{d}{dt} \hat{\rho} = -i \left[ \hat{H}_{SW}, \hat{\rho} \right] + D_{SW}(\hat{\rho}), \tag{2}$$

where the Hamiltonian $\hat{H}_{SW}$ takes the form

$$\hat{H}_{SW} = -\Delta |e\rangle \langle e| + \nu \hat{a}^\dagger \hat{a} + \frac{\Omega}{2} (|g\rangle \langle e| + |e\rangle \langle g|) \sin(k\hat{x}). \tag{3}$$

Here $\Delta = \omega_L - (\omega_e - \omega_g)$ is the detuning, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the motional state (the phonons), $\hat{x} = \frac{1}{\sqrt{2m\nu}}(\hat{a}^\dagger + \hat{a})$ is the position operator. The dissipation takes the form
\[ \mathcal{D}_{SW}(\dot{\rho}) = \frac{\gamma}{2} \int_{-1}^{1} d(\cos(\theta)) \left( \frac{3}{4} (1 + \cos^2(\theta)) \right) |g\rangle \langle e| \times e^{ikx \cos(\theta)} e^{-ikx \cos(\theta)} |e\rangle \langle g| - \frac{\gamma}{2} \{ |e\rangle \langle e|, \dot{\rho} \}, \]

In the LD regime, the Hamiltonian \( \hat{H}_{SW} \) can be approximated as
\[ \hat{H}_{SW}^{LD} = -\Delta |e\rangle \langle e| + \nu \hat{a} \hat{a}^\dagger + \frac{\eta \Omega}{2} \left( |g\rangle \langle e| + |e\rangle \langle g| \right) \left( \hat{a} + \hat{a}^\dagger \right) \]
by expanding \( \sin(kx) \) to the first order of the LD parameter \( \eta = k/\sqrt{2\nu} \) in equation (3). The dissipator \( \mathcal{D}_{SW} \) is often kept to the zeroth order of \( \eta \), which is
\[ \mathcal{D}_{SW}^{LD}(\dot{\rho}) = \gamma \left( |g\rangle \langle e| \hat{\rho} |e\rangle \langle g| - \frac{1}{2} \{ \hat{\rho}, |e\rangle \langle e| \} \right), \]
since the higher orders terms only contributes in order \( \eta^4 \) [27]. When the standing wave laser is tuned to red sideband resonance, namely \( \Delta = -\nu \), we further neglect the blue sideband as a first approximation since it is far-detuned compared to the red sideband in the resolved sideband regime. Thus we are left with the following approximate Hamiltonian in the interacting picture
\[ \hat{H}_{SW}^{LDR} = \frac{\eta \Omega}{2} \left( |g\rangle \langle e| \hat{a}^\dagger + |e\rangle \langle g| \hat{a} \right). \]
As a result, equation (2) is approximated by
\[ \frac{d}{dt} \dot{\rho} = -i \left[ \hat{H}_{SW}^{LDR}, \rho \right] + \mathcal{D}_{SW}^{LDR}(\dot{\rho}). \]
\[ \text{In the WSC regime, namely when } \eta \Omega \ll \gamma, \text{ the time required by the transition between the state } |g, n\rangle \text{ and the state } |e, n-1\rangle \text{ is much larger than the life time of } |e, n-1\rangle, \text{ which means that once the state } |e, n-1\rangle \text{ is populated by the red sideband, it immediately decays to the state } |g, n-1\rangle \text{ without being pumped back into } |g, n\rangle, \text{ as shown in the left-hand side box of figure 1(a). The state } |e, n-1\rangle \text{ can thus be adiabatically eliminated and one obtains an effective decay from } |g, n\rangle \text{ to } |g, n-1\rangle \text{ with an effective decay rate as in Equation (1), which is also shown in the right-hand side box of figure 1(a). In contrast, in the SSC regime, the state } |e, n-1\rangle \text{ may oscillate back into the state } |g, n\rangle \text{ before decay into } |g, n-1\rangle \text{ as shown in the left-hand side box of figure 1(b). Therefore it is more convenient to work in the dressed state representation with}
\[ |D_+, n\rangle = \sqrt{2} \left( |g, n\rangle + |e, n-1\rangle \right); \]
\[ |D_-, n\rangle = \sqrt{2} \left( |g, n\rangle - |e, n-1\rangle \right), \]
such that the two states \( |D_+, n\rangle \) and \( |D_-, n\rangle \) are eigenstates of \( \hat{H}_{SW}^{LDR} \). To solve equation (8) in the SSC regime, we further assume that the quantum state \( \dot{\rho} \) can be approximated by the following ansatz
\[ \dot{\rho}(t) = a_0(t) |g, 0\rangle \langle g, 0| + \sum_{n=1}^{\infty} \left( b_{+,n}(t) |D_+, n\rangle \langle D_+, n| + b_{-,n}(t) |D_-, n\rangle \langle D_-, n| \right), \]
that is, only the diagonal terms in the dressed state representation is considered. To this end, we note that the elimination of the carrier transition is important since otherwise \( |g, n\rangle \) will form a dressed state with \( |e, n\rangle \) instead of \( |e, n-1\rangle \) due to the fact that the carrier transition is much stronger than the red sideband, in which case our ansatz in equation (10) will no longer be valid. We can also see from the ansatz in equation (10) that the excited state \( |e, n-1\rangle \) has an equal population to \( |g, n\rangle \), which is very distinct from the WSC limit where \( |e, n-1\rangle \) is rarely excited and thus the population on \( |e, n-1\rangle \) is negligible compared to that on \( |g, n\rangle \). Substituting equation (10) into equation (8), we get the equations for \( b_{+,n}, b_{-,n} \) and \( a_0 \)
\[ \frac{d}{dt} b_{+,n} = -\frac{\gamma}{2} b_{+,n} + \frac{\gamma}{4} \left( b_{+,n+1} + b_{-,n+1} \right); \]
\[ \frac{d}{dt} b_{-,n} = -\frac{\gamma}{2} b_{-,n} + \frac{\gamma}{4} \left( b_{+,n+1} + b_{-,n+1} \right); \]
\[ \frac{d}{dt} a_0 = \frac{\gamma}{2} \left( b_{+,1} + b_{-,1} \right). \]
Now we define $p_n = b_{+,n} + b_{-,n}$ for $n > 0$ and $p_0 = a_0$, and get the equation for $p_n$
\[
\frac{d}{dt} p_0 = \frac{\gamma}{2} p_1; \quad \text{(12a)}
\]
\[
\frac{d}{dt} p_n = \frac{\gamma}{2} p_{n+1} - \frac{\gamma}{2} p_{n-1} \quad (n > 0). \quad \text{(12b)}
\]

The equation of motion for the average phonon number, defined as $\bar{n}(t) = \text{tr}(n \hat{\rho}(t)) = \sum_{n=1}^{\infty} (n - \frac{1}{2}) p_n(t)$, is then
\[
\frac{d}{dt} \bar{n}(t) = -\frac{\gamma}{2} (1 - p_0(t)) + \frac{1}{2} \frac{d}{dt} p_0(t). \quad \text{(13)}
\]

In the following we assume that the initial state of the trapped ion is
\[
\hat{\rho}_0 = |g\rangle \langle g| \otimes \hat{\rho}_b, \quad \text{(14)}
\]

where $\hat{\rho}_b = \sum_{n=0}^{\infty} c_n |n\rangle \langle n|$ is a thermal state for the motional degree of freedom with average phonon number $n_0$, that is, $c_n = n_0^n/(1 + n_0)^{n+1}$ \cite{42}. Due to the SSC, the state $|g\rangle |n\rangle$ will be rapidly mixed with the state $|e\rangle |n-1\rangle$ at the beginning of the cooling dynamics. As a result, after this very short initial dynamics, the system will look as if it starts from another initial state
\[
\hat{\rho}_0 = c_0 |0\rangle \langle g, 0| + \sum_{n=1}^{\infty} \frac{c_n}{2} (|D_{+,n}\rangle \langle D_{+,n}| + |D_{-,n}\rangle \langle D_{-,n}|). \quad \text{(15)}
\]

Compared to our ansatz in equation (10), we can see that $a_0(0) = c_0$, $b_{+,0}(0) = b_{-,0}(0) = c_n/2$. Now we can solve equation (12) with the initial conditions $p_0(0) = c_0$, $p_n(0) = c_n$, for which we can first integrate the equation for $p_1(t)$
\[
p_1(t) = e^{-\frac{\gamma t}{2}} p_1(0) + \frac{\gamma}{2} e^{-\frac{\gamma t}{2}} \int_0^t e^{\frac{\gamma t}{2}} p_2(\tau) d\tau, \quad \text{(16)}
\]
then we can move on to integrate the equation for $p_2(t)$ and substitute it into the above equation. Repeating this process, we will get an infinite series for $p_n(t)$ which sums up to $\frac{n_0}{(1 + n_0)^{n+1}} e^{-\frac{\gamma t}{2(1 + n_0)^{n+1}}}$, plus an additional term containing $p_\infty$ as integrand which is neglected. Similarly, we can obtain the analytic solutions for all $p_n(t)$ with $n > 0$ and then for $p_0(t)$ from the first equation of equation (12) as
\[
p_0(t) = 1 - \frac{n_0}{1 + n_0} e^{-\frac{\gamma t}{2(1 + n_0)^{n+1}}}; \quad \text{(17)}
\]
\[
p_n(t) = \frac{n_0^n}{(1 + n_0)^{n+1}} e^{-\frac{\gamma t}{2(1 + n_0)^{n+1}}}, \quad (n > 0). \quad \text{(18)}
\]
Setting \( t = 0 \) in equations (17) and (18), we can see that they satisfy the initial condition. Substituting equations (17) and (18) into equation (12), it is straightforward to verify that they are indeed the solutions of the latter. Substituting equation (17) into equation (13), we get

\[
\dot{n}(t) = n_0' e^{-\frac{\gamma}{2 n_0}} \tag{19}
\]

with \( n_0' = \left( n_0 - \frac{m}{2(n_0+1)} \right) \). We can identify from equation (19) that the cooling rate in the SSC regime is

\[
W_{SSC}^{SW} = \frac{\gamma}{2} \frac{1}{n_0 + 1}. \tag{20}
\]

There are several important differences between equation (20) derived in the SSC limit and equation (1) derived in the WSC limit. First, \( W_{SSC}^{SW} \) is proportional to the natural linewidth \( \gamma \) and is independent of the sideband coupling \( \eta \Omega \). This is because the effect of \( \eta \Omega \) has already been absorbed into the ansatz in equation (10), where \( |g, n\rangle \) is fully mixed with \( |e, n-1\rangle \). Moreover, the dressed states \( |D_{\pm,n-1}\rangle \) decay to \( |D_{\pm,n}\rangle \) with an effective decay rate of \( \gamma/4 \). As a result each \( p_n \) (with \( n \geq 1 \)) decays with a rate proportional to \( \gamma/2 \), as shown in the right-hand side box of figure 1(b). Second, \( W_{SSC}^{SW} \) is inversely proportional to the initial average photon number \( n_0 \), while in the WSC limit the cooling rate is independent of \( n_0 \). Here the reason is that in the WSC limit, the coefficients of the rate equation (namely the transition rates) are dependent on the energy level \( n \) [20] (see also the box on the right-hand of figure 1(a)), which are then absorbed into the definition of \( \dot{n}(t) \) and the net effect is that \( \dot{n}(t) \) is independent of \( n \). However in the SSC limit, the transition rates as in equation (12) are determined by the linewidth \( \gamma \) (see the box on the right-hand of figure 1(b)), therefore independent of \( n \), as a result \( \dot{n}(t) \) will be dependent on the energy level \( n \) and thus on the initial occupation \( n_0 \).

The steady state occupation predicted by equation (19) is \( \dot{n}_i = \dot{n}(\infty) = 0 \), this is because we have neglected all the heating terms in equation (2). In fact, when \( \dot{n}(t) \) approaches 0, the blue sideband can no longer be neglected since there is no red sideband for the state \( |g, 0\rangle \). To reasonably evaluate \( \dot{n}_i \), we first assume that the trapped ion has already been cooled close to the ground state, namely \( \dot{n}(t) \approx 0 \), such that we can limit ourself to the subspace spanned by \{\{g, 0\}, \{g, 1\}, \{e, 0\}, \{e, 1\}\}. Then we can employ the four-level Bloch equation for this subspace, which is,

\[
\begin{align*}
\frac{d}{dt} \rho_{00,00} &= \frac{\eta \Omega}{2} \sigma^x_{00,11} + \gamma \rho_{00,00} \quad (21a) \\
\frac{d}{dt} \rho_{00,01} &= \frac{\eta \Omega}{2} \sigma^y_{00,01} - \gamma \rho_{01,00} \quad (21b) \\
\frac{d}{dt} \rho_{01,01} &= -\frac{\eta \Omega}{2} \sigma^y_{01,01} + \gamma \rho_{00,01} \quad (21c) \\
\frac{d}{dt} \rho_{01,11} &= \frac{\eta \Omega}{2} \sigma^x_{01,11} + \gamma \rho_{01,01} \quad (21d) \\
\frac{d}{dt} \rho_{11,01} &= -\frac{\eta \Omega}{2} \sigma^x_{11,01} - \gamma \rho_{11,01} \quad (21e) \\
\frac{d}{dt} \rho_{11,11} &= \frac{\eta \Omega}{2} \sigma^y_{11,11} - \gamma \rho_{11,11} \quad (21f)
\end{align*}
\]

Here we have used \( \rho_{i,j} = \text{Tr} \{ \hat{\rho} (|i\rangle \langle j|) \} \), \( \sigma^x_{i,j} = \text{Tr} \{ \hat{\rho} (|i\rangle \langle j| - i |j\rangle \langle i|) \} \), \( \sigma^y_{i,j} = \text{Tr} \{ \hat{\rho} (|i\rangle \langle j| + |j\rangle \langle i|) \} \), with \( |0, 0\rangle, |0, 1\rangle, |e, 0\rangle, |e, 1\rangle \) standing for the states \( |g, 0\rangle, |g, 1\rangle, |e, 0\rangle, |e, 1\rangle \) respectively. By solving equation (21), we get the steady state populations for \( \rho_{00,00}, \rho_{11,11}, \rho_{11,01} \) as

\[
\begin{align*}
\rho_{00,00} &= \frac{(\eta \Omega)^2}{(\eta \Omega)^2 + 16 \nu^2 + \gamma^2} \rho_{00,00} \quad (22a) \\
\rho_{11,11} &= \frac{(\eta \Omega)^2 + \gamma^2}{(\eta \Omega)^2 + 16 \nu^2 + \gamma^2} \rho_{00,00} \quad (22b) \\
\rho_{11,01} &= \frac{(\eta \Omega)^2}{(\eta \Omega)^2 + 16 \nu^2 + \gamma^2} \rho_{00,00} \quad (22c)
\end{align*}
\]

We can see that in the SSC limit, the steady state is mainly occupied in \( |g, 0\rangle \), with excitations which are entangled states between the internal and motional degrees of freedoms. In particular, from equation (22) we have \( \rho_{00,00} \approx \rho_{11,11} \) for the steady state, which is reminiscent of the entanglement between the internal and motional degrees of freedoms from our ansatz in equation (10). This is again in comparison with the
function of the Rabi frequency indeed has a strong dependence on satisfied, and we can see that the dynamics predicted by equation (24) agrees well with the exact numerical sideband condition. In figure 2(c), we have chosen different values of neglected in equation (8). In figure 2(d), we fix solutions, except that equation (24) predicts a slightly faster decay since heating due to the blue sideband is average phonon occupation due to the formation of dressed states by rapidly mixing.

To highlight the sharp differences between the SSC limit and the WSC limit, we compare the dynamics at\( \eta = 0.8, \gamma = 1.5\nu, \gamma = 0.1\nu \). (b) The grey lines from darker to lighter stand for the exact dynamics at\( \Omega = 2.1\nu, 1.6\nu, 0.6\nu \) respectively, while the black dashed line is our analytic prediction. The other parameters used are\( n_0 = 4, \eta = 0.1, \gamma = 0.1\nu \). (c) The grey lines from darker to lighter stand for the exact dynamics at\( \gamma = 0.1\gamma, 0.5\gamma, 0.2\gamma \) respectively with\( \Omega = 9\nu \), while the dashed lines from darker to lighter are the corresponding analytic predictions. The other parameters used are\( n_0 = 4, \eta = 0.1 \). (d) The grey lines from darker to lighter stand for the exact dynamics at\( n_0 = 4, 3, 2, 1 \) respectively, while the dashed lines from darker to lighter are the corresponding analytic predictions. The other parameters used are\( \eta = 0.08, \Omega = 1.5\nu, \gamma = 0.1\nu \).

Figure 2. Average phonon occupation \( \bar{n} \) for the standing wave sideband cooling as a function of time \( t \). In all the panels \( \Delta = -\nu \). (a) The grey lines from darker to lighter stand for the exact dynamics at\( \eta = 0.2, 0.12, 0.04 \) while the black dashed line is our analytic prediction. The other parameters used are\( n_0 = 4, \Omega = 1.5\nu, \gamma = 0.1\nu \). (b) The grey lines from darker to lighter stand for the exact dynamics at\( \Omega = 2.1\nu, 1.25\nu, 0.6\nu \) respectively, while the black dashed line is our analytic prediction. The other parameters used are\( n_0 = 4, \eta = 0.1, \gamma = 0.1\nu \). (c) The grey lines from darker to lighter stand for the exact dynamics at\( \gamma = 0.1\nu, 0.5\nu, 0.2\nu \) respectively with\( \Omega = 9\nu \), while the dashed lines from darker to lighter are the corresponding analytic predictions. The other parameters used are\( n_0 = 4, n = 0.1 \). (d) The grey lines from darker to lighter stand for the exact dynamics at\( n_0 = 4, 3, 2, 1 \) respectively, while the dashed lines from darker to lighter are the corresponding analytic predictions. The other parameters used are\( \eta = 0.08, \Omega = 1.5\nu, \gamma = 0.1\nu \).

WSC limit, where the internal and the motional degrees of freedoms are almost separable during the evolution, with the internal degree of freedom mainly occupied in \( |g\rangle \) and the motional degree of freedom approximately a thermal distribution \( |g\rangle \). Since the system has already been cooled close to its motional average phonon occupation due to the formation of dressed states by rapidly mixing.

To verify our physical picture in the SSC regime, we compare our analytic expression in equation (24) with the numerical solutions of the exact Lindblad equation as in equation (2). Concretely, we plot the dependence of \( \bar{n} \) as a function of time \( t \) in figure 2 for different values of \( \eta \) (panel a), \( \Omega \) (panel b), \( \gamma \) (panel c) and\( n_0 \) (panel d) respectively. From figures 2(a) and (b) we can see that our analytic expression works better for smaller values of\( n_\text{eff}/\nu \), which is as expected since equation (24) is derived based on the resolved sideband condition. In figure 2(c), we have chosen different values of\( \gamma \) such that\( n_\text{eff}/\nu = 0.9\gamma/\nu \ll 1 \) is satisfied, and we can see that the dynamics predicted by equation (24) agrees well with the exact numerical solutions, except that equation (24) predicts a slightly faster decay since heating due to the blue sideband is neglected in equation (8). In figure 2(d), we fix\( n_\text{eff}/\nu = 1.2\gamma/\nu = 0.12 \) and we can see that the cooling rate indeed has a strong dependence on\( n_0 \) as predicted by equation (23). The numerical simulations throughout this work are done using the open source numerical package QuTiP [43].

To highlight the contrast between the SSC limit and the WSC limit, we compare the dynamics as well as the steady state phonon occupation in both regimes. In figure 3(a), we plot\( \bar{n} \) as a function of time \( t \) in the SSC limit (the black line with\( n_\text{eff}/\nu = 1.5\gamma \)) and in the WSC limit (the blue line with\( n_\text{eff}/\nu = 0.1\gamma \)), the black and blue dashed lines are the corresponding analytic predictions. In particular, we can see that the cooling rate in the SSC limit is 10 times larger than in the WSC limit, while the steady state phonon occupation is 5.5 times larger. In the inset of figure 3(a), we plot the short time dynamics in the SSC regime, from which we can see that at the beginning of the cooling dynamics, there is indeed a sudden drop of the average phonon occupation due to the formation of dressed states by rapidly mixing \( |g, n\rangle \) with \( |e, n - 1\rangle \).

In figure 3(b) we plot the cooling rate, which results from an exponential fitting of the exact dynamics, as a function of the Rabi frequency\( \Omega \). The darker black line with star corresponds to\( n_0 = 4 \) while the lighter black line with circle corresponds to\( n_0 = 1 \). The darker and lighter black dashed lines are the
corresponding analytic predictions from equation (20). The blue dot-dashed line is the prediction from equation (1). We can see that equation (1) agrees well with the numerical fitting when $\Omega/\nu < 0.5$, where the WSC condition is satisfied, and then for $1 < \Omega/\nu < 1.5$, the analytic predictions from equation (20) agree well with the numerical fitting. For even larger $\Omega$ such that $\Omega/\nu > 2$, the derivation between equation (20) and the numerical fitting becomes larger since the resolved sideband condition is no longer satisfied.

Finally in figure 4, we plot the steady state phonon occupation $\bar{n}_{ss}$ as a function of $\Omega$. Interestingly, we can see that our analytic prediction in equation (23) agrees well with the exact numerical results in all regimes (the derivations which happen for large $\Omega/\nu$ is because that the resolved sideband condition is no longer satisfied.). This is because that to derive equation (23) we have kept both the red sideband and the blue sideband terms, and only assumed that the final occupation is close to 0. The large derivation of $\bar{n}_{ss,WSC}$ (the blue dot-dashed line) from the exact result (the black line) again signifies that the $\eta^2$ correction term can no longer be neglected when the WSC condition is not satisfied.

3. EIT cooling in the SSC limit

Standing wave sideband cooling, being conceptually simple, may have several drawbacks in experimental implementations: (1) The standing wave laser may not be easy to implement and (2) the natural linewidth
\( \gamma \) is not a tunable parameter and thus the resolved sideband condition \( \gamma \ll \nu \) may not be satisfied. The EIT cooling scheme overcomes both difficulties, while at the same time eliminates the carrier transition.

A standard EIT cooling scheme uses a \( \Lambda \) - type three-level internal structure with an excited state \( |e \rangle \) of linewidth \( \gamma_e \), and two metastable ground states \( |g \rangle \) and \( |r \rangle \). Two lasers are used, which induce transitions \( |g \rangle \leftrightarrow |e \rangle \) and \( |r \rangle \leftrightarrow |e \rangle \), with frequencies \( \omega_g \) and \( \omega_r \), wave numbers \( k_g \) and \( k_r \), Rabi frequencies \( \Omega_g \) and \( \Omega_r \) respectively. The internal level structure of EIT cooling is shown in figure 5(a). The Hamiltonian of EIT cooling can thus be written as

\[
\hat{H}_{\text{EIT}} = -\Delta |e\rangle \langle e| + \nu \hat{a}^\dagger \hat{a} + \frac{\Omega_g}{2} \left( |g\rangle \langle e| e^{-i k_g x} + |e\rangle \langle g| e^{i k_g x} \right) + \frac{\Omega_r}{2} \left( |r\rangle \langle e| e^{-i k_r x} + |e\rangle \langle r| e^{i k_r x} \right),
\]

with \( \eta_g = k_g / \sqrt{2 m \nu} \) and \( \eta_r = k_r / \sqrt{2 m \nu} \), and \( \Delta \) being the detuning for both lasers. Similar to equation (4), the dissipative part of the EIT cooling can be written as

\[
\mathcal{D}_{\text{EIT}}(\hat{\rho}) = \sum_{j=g,r} \frac{\gamma_j}{2} \int_0^1 d(\cos(\theta)) \left( \frac{3}{4} \left( 1 + \cos^2(\theta) \right) \right) |j\rangle \langle j| \times e^{i k_j x \cos(\theta)} \hat{\rho} e^{-i k_j x \cos(\theta)} |j\rangle \langle j| - \frac{\gamma_j}{2} \{ |j\rangle \langle j|, \hat{\rho} \}.
\]

where \( \gamma_g \) and \( \gamma_r \) are the decay rates from \( |e\rangle \) to \( |g\rangle \) and \( |r\rangle \) respectively.

The internal degrees of freedom of \( \hat{H}_{\text{EIT}} \) can be diagonalized with three eigenstates [44]

\[
|+\rangle = \sin \phi |e\rangle - \cos \phi \left( \sin \vartheta |g\rangle + \cos \vartheta |r\rangle \right);
\]

\[
|\rangle = \cos \phi |e\rangle + \sin \phi \left( \sin \vartheta |g\rangle + \cos \vartheta |r\rangle \right);
\]

\[
|d\rangle = \cos \vartheta |g\rangle - \sin \vartheta |r\rangle
\]

with energies

\[
\omega_+ = \frac{1}{2} \left( -\Delta + \sqrt{\Omega_r^2 + \Omega_g^2 + \Delta^2} \right);
\]

\[
\omega_- = \frac{1}{2} \left( -\Delta - \sqrt{\Omega_r^2 + \Omega_g^2 + \Delta^2} \right);
\]

\[
\omega_d = 0.
\]

Here, the angles \( \phi \) and \( \vartheta \) are defined by

\[
\tan 2\phi = -\frac{\sqrt{\Omega_r^2 + \Omega_g^2}}{\Delta};
\]
The EIT cooling condition is chosen as $\omega_+ = \nu$ such that effective red sideband $|d, n \rangle \rightarrow |+, n - 1 \rangle$ is resonant. In the dressed state basis as defined in equation (27), and neglecting the far-detuned state $| - \rangle$ as well as the effective blue sideband $|d, n \rangle \leftrightarrow |+, n + 1 \rangle$, we get an effective Hamiltonian in the interacting picture

$$\hat{H}_{\text{LD}}^{\text{EIT}} = i\gamma \Omega_{\nu} \left( |+\rangle \langle+| \hat{a}^\dagger - |+\rangle \langle+| \hat{a} \right),$$

with $\eta_D = \eta_\nu - \eta_r$, and $\Omega_{\nu} = \frac{\Omega_1 \Omega_2}{\sqrt{\Omega_1^2 + \Omega_2^2}} \sin \phi$. The effective dissipation in the dressed state basis is

$$D_{\text{LD}}^{\text{EIT}}(\hat{\rho}) = \gamma_{\text{eff}} \left( |d\rangle \langle+d| \hat{a}^\dagger \hat{a} - |d\rangle \langle+d| \right) - \frac{1}{2} \left( \hat{\rho}_r \langle+d| + \langle+d| \hat{\rho}_r \right).$$

with

$$\gamma_{\text{eff}} = \frac{\sin^2 \phi}{2} \left( \gamma_\nu \cos^2 \theta + \gamma_r \sin^2 \theta \right).$$

Comparing equations (31) and (32) with equations (7) and (6), we can see that the EIT cooling is equivalent to the standing wave sideband cooling, by making the substitutions $\eta \rightarrow \eta_D$, $\Omega \rightarrow \Omega_\nu$, $\gamma \rightarrow \gamma_{\text{eff}}$, which is shown in figure 5(b). As a result, the cooling rate and the steady state phonon occupation for EIT cooling in the SSC regime $\eta_D \Omega_{\nu} \gg \gamma_{\text{eff}}$ can be read from equations (20) and (23) as

$$W_{\text{EIT}}^{\text{SSC}} = \frac{\eta_{\text{SSC}}}{2} \frac{1}{1 + \eta_0};$$

$$\bar{n}_{\text{SSC}}^{\text{EIT}} = \frac{1}{8} \left( \frac{\eta_D \Omega_{\nu}}{\nu} \right)^2 + \bar{n}_{\text{SSC}}^{\text{EIT}}.$$

with $\bar{n}^{\text{EIT}}_{\text{SSC}} = \left( \frac{\eta_0}{4\Delta} \right)^2$ being steady state average phonon occupation of EIT cooling in WSC regime.

In the experimental implementations of EIT cooling, the coupling strengths are usually chosen such that $\Omega_g \ll \Omega_r$ [21, 31]. As a result the internal dark state $|d\rangle \approx |g\rangle$. Therefore, we have $\gamma_{\text{eff}} \approx \frac{\nu}{\Delta} \gamma_g$ and $\Omega_{\nu} \approx \frac{\Omega_g}{\Delta}$, and the cooling rate in SSC regime becomes

$$W_{\text{EIT}}^{\text{SSC}} \approx \frac{\eta_g \nu}{2\Delta} \frac{1}{1 + \eta_0};$$

Similar to equation (20), we can see that the cooling rate is mainly determined by $\gamma_g$. In contrast, in the WSC regime, the cooling rate is related to $\gamma = \gamma_g + \gamma_r$ as

$$W_{\text{EIT}}^{\text{WSF}} \approx \frac{\eta_{\text{SSC}}^{\text{EIT}}}{\gamma_r}.$$

where in the condition $\Omega_g \ll \Omega_r$ has been used in deriving the above equation.

Similar to figure 3, we compare the sharp difference between EIT cooling in the SSC limit and in the the WSC limit in figure 6. In figure 6(a), we plot the $\bar{n}$ as a function of time in the SSC limit (the black line with
\( \eta_0 \Omega_+ \approx 3.3 \gamma_{\text{eff}} \) and in the WSC limit (the blue line with \( \eta_0 \Omega_+ \approx 0.25 \gamma_{\text{eff}} \)), the black and blue dashed lines are the corresponding analytic predictions, where we can see that our analytic expressions in equations (34) and (35) agree very well with the exact numerical results, and that the cooling rate in the SSC regime is indeed much faster than that in the WSC regime. In figure 6(b) we plot the cooling rate resulting from an exponential fitting of the exact dynamics as a function of the Rabi frequency \( \Omega_x \). We note that in such parameters settings we have \( \Omega_x \approx \frac{2\Omega_+ \Delta}{\Omega_+} \approx 10 \Omega_+ \) and \( \gamma_{\text{eff}} \approx 0.05 \nu \). The black line with circle and the black dashed line correspond to the exact numerical results and the analytic predictions from equation (36) respectively, with \( n_0 = 3 \). The blue dot-dashed line stands for the analytical predictions from equation (37). We can see that equation (37) agrees well with the exact numerical results when \( \Omega_x / \nu < 0.5 \) (corresponding to \( \eta_0 \Omega_+ / \gamma_{\text{eff}} < 0.2 \) where the WSC condition is satisfied). While for \( \Omega_x / \nu > 3 \) (corresponding to \( \eta_0 \Omega_+ / \gamma_{\text{eff}} > 1.2 \) where the SSC condition is satisfied), our analytic predictions from equation (36) agree well with the exact numerical results.

4. Conclusion

In summary, we have studied standing wave sideband cooling and EIT cooling of trapped ion in the SSC regime. We derived analytic expressions for the cooling dynamics as well as for the steady state occupation of the motional state in the SSC regime, showing that in this regime we could reach a cooling rate which is proportional to the linewidth \( \gamma \) of the excited state, and which also depends on the initial occupation \( n_0 \) of the motional state. This is in comparison with current WSC based cooling schemes where the cooling rate is much smaller than \( \gamma \) and is independent of \( n_0 \). Additionally, the steady state occupation of the motional state increases by a term proportional to \( \eta^2 \) compared to the WSC limit. The analytic expressions are verified against the numerical results by solving the exact Lindblad master equation, showing that they could faithfully recover both the short time and long time dynamics for the motional state of the trapped ion. Our results could be experimentally implemented to speed up the cooling of a trapped ion by a factor of 10 compared to current WSC based schemes such as EIT cooling, and can be easily extended to other dark-state based cooling schemes.

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Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

ORCID iDs

Chu Guo https://orcid.org/0000-0002-3411-3076

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