ABSOLUTELY SUMMING OPERATORS ON NON COMMUTATIVE $C^*$-ALGEBRAS AND APPLICATIONS

NARCISSE RANDRIANANTOANINA

Abstract. Let $E$ be a Banach space that does not contain any copy of $\ell^1$ and $A$ be a non commutative $C^*$-algebra. We prove that every absolutely summing operator from $A$ into $E^*$ is compact, thus answering a question of Pełczynski.

As application, we show that if $G$ is a compact metrizable abelian group and $\Lambda$ is a Riesz subset of its dual then every countably additive $A^*$-valued measure with bounded variation and whose Fourier transform is supported by $\Lambda$ has relatively compact range. Extensions of the same result to symmetric spaces of measurable operators are also presented.

1. Introduction

It is a well known result that every absolutely summing operator from a $C(K)$-space into a separable dual space is compact. More generally if $F$ is a Banach space with the complete continuity property (CCP) then every absolutely summing operator from any $C(K)$-spaces into $F$ is compact (see [10]).

It is the intention of the present note to study extensions of the above results in the setting of $C^*$-algebras, i.e., replacing the $C(K)$-spaces above by a general non commutative $C^*$-algebra. Typical examples of Banach spaces with the CCP are dual spaces whose preduals do not contain $\ell^1$. Our main result is that if $E$ is a Banach space that does not contain any copy of $\ell^1$ and $A$ is a $C^*$-algebra then every absolutely summing operator from $A$ into $E^*$ is compact. This answered positively the following question raised by Pełczynski (see [17] Problem 3. P.20): Is every absolutely summing operator from a non commutative $C^*$-algebra into a Hilbert space compact? This result is also used to study

1991 Mathematics Subject Classification. 46E40; Secondary 47D15, 28B05.
Key words and phrases. $C^*$-algebras, vector measures, Riesz sets.
relative compactness of range of countably additive vector measures with values in duals of non commutative $C^\ast$-algebras. In [9], Edgar introduced new types of Radon-Nikodym properties associated with Riesz subsets of countable discrete group (see the definition below) as generalization of the usual Radon-Nikodym property (RNP) and the Analytic Radon-Nikodym property (ARNP). These properties were extensively studied in [7] and [8]. In [7], it was shown that if $\Lambda$ is a Riesz subset of a countable discrete group then $L^1[0,1]$ has the type II-$\Lambda$-RNP. In the other hand, Haagerup and Pisier showed in [14] that non commutative $L^1$-spaces have the ARNP so it is a natural question to ask if non commutative $L^1$-spaces have the type II-$\Lambda$-RNP for any Riesz subset. In this direction, we obtain (as a consequence of our main result) that if a countably additive vector measure of bounded variation is defined on the $\sigma$-field of Borel subsets of a compact metrizable abelian group and takes its values in a dual of a $C^\ast$-algebra then its range is relatively compact provided that its Fourier transform is supported by a Riesz subset of the dual group.

Our terminology and notation are standard. We refer to [8] and [1] for definitions from Banach space theory and [2], [16] and [11] for basic properties from the theory of operator algebras and non-commutative integrations.

2. Preliminary Facts and Notations

We recall some definitions and well known facts which we use in the sequel.

Let $\mathcal{A}$ be a $C^\ast$-algebra, we denote by $\mathcal{A}_h$ the set of Hermitian (self adjoint) elements of $\mathcal{A}$.

**Definition 1.** Let $E$ and $F$ be Banach spaces and $0 < p < \infty$. An operator $T : E \to F$ is said to be absolutely $p$-summing (or simply $p$-summing) if there exists $C$ such that

$$\left( \sum_{i=1}^\ell \| Te_i \|^p \right)^{1/p} \leq C \max \left\{ \sum |\langle e_i, e^* \rangle|^p, \| e^* \| \leq 1 \right\}^{1/p}$$
The following class of operators was introduced by Pisier in \cite{Pisier} as an extension of the $q$-summing operators in the setting of $C^*$-algebras.

**Definition 2.** Let $\mathcal{A}$ be a $C^*$-algebra and $F$ be a Banach space, $0 < q < \infty$. An operator $T : \mathcal{A} \to F$ is said to be $q$-$C^*$-summing if there exists a constant $C$ such that for any finite sequence $(A_1, \ldots, A_n)$ of Hermitian elements of $\mathcal{A}$ one has

\[
\left( \sum_{i=1}^{n} \|T(A_i)\|^q \right)^{1/q} \leq C \left( \sum_{i=1}^{n} |A_i|^q \right)^{1/q}
\|A\|.
\]

The smallest constant $C$ for which the above inequality holds is denoted by $C_q(T)$. It should be noted that if the $C^*$-algebra $\mathcal{A}$ is commutative then every $q$-$C^*$-summing operator from $\mathcal{A}$ into any Banach space is $q$-summing. The following extension of the classical Pietsch's factorization theorem (\cite{Pietsch}) was obtained by Pisier (see Proposition 1.1 of \cite{Pisier}).

**Proposition 1.** If $T : \mathcal{A} \to F$ is a $q$-$C^*$-summing operator then there exists a positive linear form $f$ of norm less than $1$ such that

\[
\|Tx\| \leq C_q(T) \{f(|x|^q)\}^{1/q}, \text{ for every } x \in \mathcal{A}_h.
\]

Let $\mathcal{M}$ be a von-Neumann algebra and $\mathcal{M}_*$ be its predual. We recall that a functional $f$ on $\mathcal{M}$ is called normal if it belongs to $\mathcal{M}_*$. In \cite{Pisier}, it was shown that for the case of von-Neumann algebra and the operator $T$ being weak* to weakly continuous then the positive linear form on the above proposition can be chosen to be normal; namely we have the following lemma (see Lemma 4.1 of \cite{Pisier}).

**Lemma 1.** Let $T : \mathcal{M} \to F$ be a $1$-$C^*$-summing operator. If $T$ is weak* to weakly continuous then there exists a linear form $f \in \mathcal{M}_*$ with $\|f\| \leq 1$ such that

\[
\|Tx\| \leq C_1(T)f(|x|), \text{ for every } x \in \mathcal{M}_h.
\]  \hspace{1cm} (1)

For the next lemma, we recall that for $x \in \mathcal{M}$ and $f \in \mathcal{M}_*$, $xf$ (resp. $fx$) denotes the element of $\mathcal{M}_*$ defined by $xf(y) = f(yx)$ (resp. $fx(y) = f(xy)$) for all $y \in \mathcal{M}$.
Lemma 2. Let \( f \) be a positive linear form on \( M \). For every \( x \in M \),

\[
f \left( \left( \frac{xx^* + x^*x}{2} \right)^{1/2} \right) \leq 2\|xf + fx\|_{M^*}.
\]

(2)

Proof. Assume first that \( x \in M_h \). In this case \( (xx^* + x^*x)/2 = |x|^2 \).

The operator \( x \) can be decomposed as \( x = x_+ - x_- \) where \( x_+, x_- \in M^+ \)

and \( x_+x_- = 0 \). There exists a projection \( p \in M \) such that \( px_- = x_-p = x_- \)

and \( (1-p)x_+ = x_+(1-p) = x_+ \). This yields the following estimates:

\[
f(|x|) = f(x_+ + x_-) = f(x_+) + f(x_-)
\]

\[
= \frac{1}{2}(xf + fx)(1-p) + \frac{1}{2}(xf + fx)(p)
\]

\[
\leq \frac{1}{2} \left( \|xf + fx\| \|1-p\| + \|xf + fx\| \|p\| \right)
\]

\[
\leq \|xf + fx\|.
\]

For the general case, fix \( x \in M \). Let \( a = (x + x^*)/2 \) and \( b = (x - x^*)/2i \).

Clearly \( x = a + ib \) and \( ((xx^* + x^*x)/2)^{1/2} = |a| + |b| \).

Using the Hermitian case, we get:

\[
f(|a| + |b|) \leq \|af + fa\| + \|bf + fb\|
\]

\[
\leq \|xf + fx\| + \|x^f + fx^*\|;
\]

but since \( f \geq 0 \),

\[
\|x^f + fx^*\| = \sup\{|f(sx^* + x^*s)|; \ s \in M, \|s\| \leq 1\}
\]

\[
= \sup\{|f^*(xs^* + s^*x)|; \ s \in M, \|s\| \leq 1\}
\]

\[
= \sup\{|f(xs^* + s^*x)|; \ s \in M, \|s\| \leq 1\}
\]

\[
= \|xf + fx\|,
\]

which completes the proof of the lemma.

\[
\square
\]

3. Main Theorem

Theorem 1. Let \( A \) be a \( C^* \)-algebra, \( E \) be a Banach space that does not contain any copy of \( \ell^1 \) and \( T : A \to E^* = F \) be a 1-summing operator then \( T \) is compact.
We will divide the proof into two steps. First we will assume that the $C^*$-Algebra $A$ is a $\sigma$-finite von-Neumann algebra and the operator $T$ is weak* to weakly continuous; then we will show that the general case can be reduced to this case. We refer to [21] P. 78 for the definition of $\sigma$-finite von-Neumann algebra.

**Proposition 2.** Let $M$ be a $\sigma$-finite von-Neumann algebra. Let $T : M \to E^*$ be a weak* to weakly continuous 1-summing operator then $T$ is compact.

*Proof.* The operator $T$ being weak* to weakly continuous and 1-summing, there exist a constant $C = C_1(T)$ and a normal positive functional $f$ on $M$ such that

$$\|Tx\| \leq Cf(|x|) \text{ for every } x \in M_h.$$  

Since the von-Neumann algebra $M$ is $\sigma$-finite, there exists a faithful normal state $f_0$ in $M_*$ (see [21] Proposition II-3.19). Replacing $f$ by $f + f_0$, we can assume that the functional $f$ on the inequality above is a faithful normal state and using Lemma 2, we get

$$\|Tx\| \leq 2C\|xf + fx\|_{M^*} \text{ for every } x \in M. \quad (3)$$

We may equip $M$ with the scalar product by setting for every $x, y \in M$,

$$\langle x, y \rangle = f \left( \frac{xy^* + y^*x}{2} \right).$$

Since $f$ is faithful, $M$ with $\langle .., \rangle$ is a pre-Hilbertian. we denote the completion of this space by $L^2(M, f)$ (or simply $L^2(f)$).

By construction, the inclusion map $J : M \to L^2(M, f)$ is bounded and is one to one ($f$ is faithful). On the dense subspace $J(M)$ of $L^2(f)$, we define a map $\theta : J(M) \to L^2(f)^*$ by $\theta(Jx) = \langle ., J(x^*) \rangle$. The map $\theta$ is clearly linear and is an isometry; indeed for every $x \in M$,

$$\|\theta(Jx)\|^2 = \sup_{\|u\| \leq 1} \langle u, J(x^*) \rangle^2 = \langle J(x^*), J(x^*) \rangle = f(x^*x + xx^*) = \|Jx\|^2.$$ 

So it can be extended to a bounded map (that we will denote also by $\theta$) from $L^2(f)$ onto $L^2(f)^*$.
Let $S = J^* \circ \theta \circ J$. The operator is defined from $\mathcal{M}$ into $\mathcal{M}_*$ and we claim that for every $x \in \mathcal{M}$, $Sx = xf + fx$. In fact for every $x, y \in \mathcal{M}$, we have:

$$
Sx(y) = J^* \circ \theta \circ Jx(y) = \theta \circ Jx(Jy) = \langle J(y), J(x^*) \rangle = f(xy + yx) = (xf + fx)(y).
$$

Notice also that since $f$ is normal, the functionals $xf$ and $fx$ are both normal for every $x \in \mathcal{M}$; therefore $S(\mathcal{M}) \subset \mathcal{M}_*$. Also since $J$ is one to one, $J^*$ has weak* dense range. The latter with the facts that both $J$ and $\theta$ have dense ranges imply that $S(\mathcal{M})$ is weak* dense in $\mathcal{M}^*$ so $S(\mathcal{M})$ is (norm) dense in $\mathcal{M}^*$.

Let us now define a map $L : S(\mathcal{M}) \rightarrow E^*$ by $L(xf + fx) = Tx$ for every $x \in \mathcal{M}$. The map $L$ is clearly linear and one can deduce from (3) that $L$ is bounded so it can be extended as a bounded operator (that we will denote also by $L$) from $\mathcal{M}_*$ into $E^*$. The above means that $T$ can be factored as follows

$$
\mathcal{M} \xrightarrow{S} \mathcal{M}_* \xrightarrow{L} E^*
$$

Taking the adjoints we get

$$
E^* \xrightarrow{T^*} \mathcal{M}^*_* \xrightarrow{S^*} \mathcal{M}^*_*
$$

To conclude the proof of the proposition, let $(e_n)_n$ be a bounded sequence in $E$. Since $E \nrightarrow \ell^1$, we will assume (by taking a subsequence if necessary) that $(e_n)_n$ is weakly Cauchy. We will show that $(T^*(e_n))_n$ is norm-convergent. For that it is enough to prove that if $(e_n)_n$ is a weakly null sequence in $E$ then $(\|T^*e_n\|)_n$ converges to zero.

Let $(e_n)_n$ be a weakly null sequence in $E$. $(L^*(e_n))_n$ is a weakly null sequence in $\mathcal{M}$. This implies that $((L^*(e_n))^*)_n \geq 1$ (the sequence of the adjoints of the $L^*(e_n)$'s) is weakly null in $\mathcal{M}$. 

Since $T$ is 1-summing, it is a Dunford-Pettis operator (i.e. takes weakly convergent sequence into norm-convergent sequence). Hence

$$\lim_{n \to \infty} \|T((L^* e_n)^*)\|_{E^*} = 0.$$ 

In particular, since $(e_n)_n$ is a bounded sequence in $E$, we have

$$\lim_{n \to \infty} \langle T((L^* e_n)^*), e_n \rangle = 0$$

but

$$\langle T((L^* e_n)^*), e_n \rangle = \langle LS((L^* e_n)^*), e_n \rangle = \langle S((L^* e_n)^*), L^* e_n \rangle = \langle \theta \circ J((L^* e_n)^*), J(L^* e_n) \rangle = \langle J(L^* e_n), J(L^* e_n) \rangle_{L^2(f)} = \|J(L^* e_n)\|_{L^2(f)}.$$ 

So $\|J(L^* e_n)\|_{L^2(f)} \to 0$ as $n \to \infty$ and therefore since $T^* = S^* \circ L^* = J^* \circ \theta \circ J \circ L^*$, we get that $\lim_{n \to \infty} \|T^* e_n\| = 0$.

This shows that $T^*(B_E)$ is compact and since $B_E$ is weak* dense in $B_{E^{**}}$ and $T^*$ is weak* to weakly continuous, $T^*(B_{E^{**}}) \subseteq T^*(B_E)$ so $T^*$ (and hence $T$) is compact. The proposition is proved. \hfill \Box

To complete the proof of the theorem, let $\mathcal{A}$ be a $C^*$-algebra and $T : \mathcal{A} \to E^*$ be a 1-summing operator. The double dual $\mathcal{A}^{**}$ of $\mathcal{A}$ is a von-Neumann and $T^{**} : \mathcal{A}^{**} \to E^*$ is 1-summing. Let $(a_n)_n$ be a bounded sequence in $\mathcal{A}^{**}$. If we denote by $\mathcal{M}$ the von-Neumann algebra generated by $(a_n)_n$ then the predual $\mathcal{M}_*$ of $\mathcal{M}$ is separable and therefore the von-Neumann algebra $\mathcal{M}$ is $\sigma$-finite. Moreover, if we set $I : \mathcal{M} \to \mathcal{A}^{**}$ the inclusion map then $I$ is weak* to weak* continuous. Hence $\mathcal{M}$ and $T^{**} \circ I$ satisfy the conditions of Proposition 2 so $T^{**} \circ I$ is compact and since the sequence $(a_n)_n$ is arbitrary, the operator $T^{**}$ (and hence $T$) is compact. \hfill \Box

Remark. It should be noted that for the proof of Proposition 2, we only require the operator $T$ to be $C^*$-summing and Dunford-Pettis so
the conclusion of Proposition 2 is still valid for \(C^*-\text{summing operators}\) that are Dunford-Pettis.

4. APPLICATIONS TO VECTOR MEASURES

In this section we will provide some applications of the main theorem to study range of countably additive vector measures with values in duals of \(C^*-\text{Algebras}\).

The letter \(G\) will denote a compact metrizable abelian group, \(\hat{G}\) its dual, \(\mathcal{B}(G)\) is the \(\sigma\)-algebra of the Borel subsets of \(G\), and \(\lambda\) the normalized Haar measure on \(G\).

Let \(X\) be a Banach space and \(1 \leq p \leq \infty\), we will denote by \(L^p(G, X)\) the usual Bochner spaces for the measure space \((G, \mathcal{B}(G), \lambda)\); \(M(G, X)\) the space of \(X\)-valued countably additive Borel measures of bounded variation; \(C(G, X)\) the space of \(X\)-valued continuous functions and \(M^\infty(G, X) = \{\mu \in M(G, X), |\mu| \leq C\lambda \text{ for some } C > 0\}\).

If \(\mu \in M(G, X)\), we recall that the Fourier transform of \(\mu\) is a map \(\hat{\mu}\) from \(\hat{G}\) into \(X\) defined by \(\hat{\mu}(\gamma) = \int_G \gamma d\mu\) for \(\gamma \in \hat{G}\).

For \(\Lambda \subset \hat{G}\), we will use the following notation:

\[
\begin{align*}
L^p_\Lambda(G, X) &= \{f \in L^p(G, X), \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\} \\
C_\Lambda(G, X) &= \{f \in C(G, X), \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\} \\
M_\Lambda(G, X) &= \{\mu \in M(G, X), \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\} \\
M^\infty_\Lambda(G, X) &= \{\mu \in M^\infty(G, X), \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}.
\end{align*}
\]

We also recall that \(\Lambda \subset \hat{G}\) is called a Riesz subset if \(M_\Lambda(G) = L^1_\Lambda(G)\). We refer to \[20\] and \[13\] for detailed discussions and examples of Riesz subsets of dual groups.

The following Banach space properties were introduced by Edgar \[9\], and Dowling \[7\].

**Definition 3.** Let \(\Lambda\) be a Riesz subset of \(\hat{G}\). A Banach space \(X\) is said to have type I-\(\Lambda\)-Radon Nikodym Property (resp. type II-\(\Lambda\)-Radon Nikodym property) if \(M^\infty_\Lambda(G, X) = L^\infty_\Lambda(G, X)\) (resp. \(M_\Lambda(G, X) = L^1_\Lambda(G, X)\)).
Our next result deals with property of dual of $C^*$-algebras related to the types of Radon-Nikodym properties defined above.

**Theorem 2.** Let $\Lambda$ be a Riesz subset of $\hat{G}$ and $\mathcal{A}$ be a $C^*$-Algebra. If $F : \mathcal{B}(G) \to \mathcal{A}^*$ is a countably additive measure with bounded variation that satisfies $\hat{F}(\gamma) = 0$ for $\gamma \notin \Lambda$ then the range of $F$ is a relatively compact subset of $\mathcal{A}^*$.

**Proof.** Let $F : \mathcal{B}(G) \to \mathcal{A}^*$ be a measure with bounded variation and $\hat{F}(\gamma) = 0$ for $\gamma \notin \Lambda$. Let $S : C(G) \to \mathcal{A}^*$ be the operator defined by $Sf = \int f \, dF$. Since $F$ is of bounded variation, the operator $S$ is integral (see [4] Theorem IV-3.3 and Theorem IV-3.12) and therefore $S^* : \mathcal{A}^{\ast\ast} \to (C(G))^*$ is also integral. Now since $\hat{F}(\gamma) = 0$ for $\gamma \notin \Lambda$, if we denote by $\Lambda' = \{\gamma \in \hat{G}, \bar{\gamma} \notin \Lambda\}$ then $S(\gamma) = 0$ for all $\gamma \in \Lambda'$ and therefore we have the following factorization

$$
\begin{array}{ccc}
C(G) & \xrightarrow{S} & \mathcal{A}^* \\
\downarrow{q} & & \uparrow{\mathcal{L}} \\
C(G)/C_{\Lambda'}(G) & & \\
\end{array}
$$

where $q$ is the natural quotient map. Taking the adjoints, we get

$$
\begin{array}{ccc}
\mathcal{A}^{\ast\ast} & \xrightarrow{S^*} & (C(G))^* \\
L^* & \searrow & \uparrow{q^*} \\
M_\Lambda(G) & & \\
\end{array}
$$

Since $q^*$ is the formal inclusion and $S^*$ is 1-summing, the operator $L^*$ is 1-summing. The assumption $\Lambda$ being a Riesz subset implies that $M_\Lambda(G) = L_\Lambda^1(G)$ is a separable dual (in particular its predual does not contain $\ell^1$). So by Theorem 1, $L^*$ (and hence $S$) is compact. This proves that the range of the representing measure $F$ of $S$ is relatively compact (see [4] Theorem II-2.18).

Our next result is a generalization of Theorem 2 for the case of symmetric spaces of measurable operators.

Let $(\mathcal{M}, \tau)$ be a semifinite von-Neumann algebra acting on a Hilbert space $H$. Let $\tau$ be a distinguished faithful normal semifinite trace on $\mathcal{M}$. 


Let \( \overline{\mathcal{M}} \) be the space of all measurable operators with respect to \((\mathcal{M}, \tau)\) in the sense of \([14]\); for \( a \in \overline{\mathcal{M}} \) and \( t > 0 \), the \( t^{th} \)-s-number (singular number) of \( a \) is defined by

\[
\mu_t(a) = \inf \{ \|ae\| : e \in \mathcal{M} \text{ projection with } \tau(I - e) \leq t \}.
\]

The function \( t \mapsto \mu_t(a) \) defined on \((0, \tau(I))\) will be denoted by \( \mu(a) \). This is a positive non-increasing function on \((0, \tau(I))\). We refer to \([11]\) for complete detailed study of \( \mu(a) \).

Let \( E \) be a rearrangement invariant Banach function space on \((0, \tau(I))\) (in the sense of \([15]\)). We define the symmetric space \( E(\mathcal{M}, \tau) \) of measurable operators by setting

\[
E(\mathcal{M}, \tau) = \{ a \in \overline{\mathcal{M}} : \mu(a) \in E \}
\]

and \( \|a\|_{E(\mathcal{M}, \tau)} = \|\mu(a)\|_E \).

It is well known that \( E(\mathcal{M}, \tau) \) is a Banach space and if \( E = L^p(0, \tau(I)) \) \((1 \leq p \leq \infty)\) then \( E(\mathcal{M}, \tau) \) coincide with the usual non-commutative \( L^p \)-spaces associated with the von-Neumann algebra \( \mathcal{M} \). The space \( E(\mathcal{M}, \tau) \) is often referred as the non-commutative version of the function space \( E \). Some Banach space properties of these spaces can be found in \([2], [6] \) and \([22]\).

For the case where the trace \( \tau \) is finite, we obtain the following generalization of Theorem 2 for symmetric spaces of measurable operators.

**Corollary 1.** Assume that \( \tau \) is finite. Let \( E \) be a rearrangement invariant function space on \((0, \tau(I))\) that does not contain \( c_0 \) and \( \Lambda \) be a Riesz subset of \( \widehat{G} \). Let \( F : \mathcal{B}(G) \to E(\mathcal{M}, \tau) \) be a countably additive measure with bounded variation and \( \widehat{F}(\gamma) = 0 \) for every \( \gamma \notin \Lambda \) then the range of \( F \) is relatively compact.

**Proof.** We will begin by reducing the general case to the case where \( E(\mathcal{M}, \tau) \) is separable. Since \( \mathcal{B}(G) \) is countably generated, the range of \( F \) is separable. Choose \((A_n)_n \subset \mathcal{B}(G)\) so that \( \{F(A_n), n \geq 1\} \) is dense in \( \{F(A), A \in \mathcal{B}(G)\} \). Let \( \mathcal{M} \) be the von-Neumann algebra generated \( I \) and \( F(A_n) \ (n \geq 1) \) and \( \tau \) the restriction of \( \tau \) in \( \mathcal{M} \). Clearly \( E(\mathcal{M}, \tau) \) is a closed subspace of \( E(\mathcal{M}, \tau) \) and \( F(A) \in E(\mathcal{M}, \tau) \) for all \( A \in \mathcal{B}(G) \).
Moreover the space \( E(\tilde{M}, \tilde{\tau}) \) is separable (see Lemma 5.6 of [22]). So without loss of generalities we will assume that \( E(\mathcal{M}, \tau) \) is separable. It is a well known fact that \( E(\mathcal{M}, \tau) \) is contained in \( L^1(\mathcal{M}, \tau) + \mathcal{M} \) and since \( \tau \) is finite, \( E(\mathcal{M}, \tau) \subset L^1(\mathcal{M}, \tau) \). Let \( J : E(\mathcal{M}, \tau) \to L^1(\mathcal{M}, \tau) \) be the formal inclusion. The measure \( J \circ F \) is of bounded variation and \( \hat{J} \circ \hat{F}(\gamma) = J(\hat{F}(\gamma)) \) for every \( \gamma \in \hat{G} \). One can conclude from Theorem 2 that the range of \( J \circ F \) is relatively compact in \( L^1(\mathcal{M}, \tau) \).

To show that the range of \( F \) is relatively compact, fix \( h : G \to E(\mathcal{M}, \tau)^* \) a weak*-density of \( F \) with respect to the Haar measure \( \lambda \) (see [3]). We have for each \( A \in \mathcal{B}(G) \),

\[
F(A) = \text{weak*} - \int_A h(t) \ d\lambda(t)
\]

and

\[
|F|(A) = \int_A \|h(t)\| \ d\lambda(t).
\]

For each \( N \in \mathbb{N} \), let \( A_N = \{t \in G, \|h(t)\| \leq N\} \) and \( F_N \) the measure defined by \( F_N(A) = F(A \cap A_N) \) for all \( A \in \mathcal{B}(G) \). Clearly \( |F_N| \leq N\lambda \) for every \( N \in \mathbb{N} \).

Define \( T_N : L^1(G) \to E(\mathcal{M}, \tau) \) by \( T_N(f) = \int f(t) \ dF_N(t) \) for every \( f \in L^1(G) \). The operator \( T_N \) is bounded and we claim that \( T_N \) is Dunford-Pettis; for that notice that since the range of \( J \circ F \) is relatively compact so is the range of \( J \circ F_N \) and therefore the operator \( J \circ T_N \) is a Dunford-Pettis operator. The space \( E(\mathcal{M}, \tau) \) is separable and \( J \) is a semi-embedding (see Lemma 5.7 of [22]) so \( J \) is a \( G_\delta \)-embedding (see [4] Proposition 1.8) and one can deduce from Theorem II.6 of [12] that \( T_N \) is a Dunford-Pettis operator. Hence the range of \( F_N \) is relatively compact. Now since

\[
\lim_{N \to \infty} \|F - F_N\| = \lim_{N \to \infty} \int_{G \setminus A_N} \|h(t)\| \ d\lambda(t) = 0,
\]

the range of \( F \) is relatively compact.

Let us finish by asking the following question:

**Question:** Do non-commutative \( L^1 \)-spaces have type II-\( \Lambda \)-RNP for any Riesz set \( \Lambda \)?
In light of Theorem 2, the result of Haagerup and Pisier (14) and so many properties that have been generalized from classical $L^1$-spaces to non-commutative $L^1$-spaces, one tends to conjecture that the answer of the above question is affirmative.

**Acknowledgements:** I would like to thank Professor G. Pisier for many valuable suggestions concerning this work.

**References**

[1] J. Bourgain and H. P. Rosenthal. *Applications of the theory of semi-embeddings to Banach space theory*. J. Funct. Anal., 52:149–188, (1983).

[2] V.I. Chilin and F.A. Sukochev. *Symmetric spaces on semifinite von Neumann algebras*. Soviet Math. Dokl., 42:97–101, (1992).

[3] J. Diestel. *Sequences and Series in Banach Spaces*, volume 92 of Graduate Text in Mathematics. Springer Verlag, New York, first edition, (1984).

[4] J. Diestel and Jr. J.J. Uhl. *Vector Measures*, volume 15 of Math Surveys. AMS, Providence, RI, (1977).

[5] N. Dinculeanu. *Vector Measures*. Pergamon Press, New York, (1967).

[6] P.G. Dodds, T.K. Dodds, and B. Pagter. *Non-commutative Banach function spaces*. Math. Zeit., 201:583–597, (1989).

[7] P. Dowling. *Radon-Nikodym properties associated with subsets of countable discrete abelian groups*. Trans. Amer. Math. Soc., 327:879–890, (1991).

[8] P. Dowling. *Duality in some vector-valued function spaces*. Rocky Mount. Jr. of Math., 22:511–518, (1992).

[9] G. Edgar. *Banach spaces with the analytic Radon-Nikodym property and compact abelian group*. Proc. Int. Conf. on Almost Everywhere convergence in Probability and Ergodic Theory, Academic Press, pages 195–213, (1989).

[10] G. Emmanuele. *Dominated operators on C[0,1] and the (CRP)*. Collect. Math., 41:21–25, (1990).

[11] T. Fack and H. Kosaki. *Generalized s-numbers of $\tau$-measurable operators*. Pac. J. Math., 123:269–300, (1986).

[12] N. Ghoussoub and H. P. Rosenthal. *Martingales, $G_\delta$-Embeddings and quotients of $L_1$*. Math. Ann., 264:321–332, (1983).

[13] G. Godefroy. *On Riesz subsets of Abelian discrete groups*. Isr. J. of Math., 61:301–331, (1988).

[14] U. Haagerup and G. Pisier. *Factorization of Analytic functions with values in non-commutative $L_1$-spaces*. Canad. J. Math., 41:882–906, (1989).

[15] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces II*, volume 97 of Modern Survey In Mathematics. Springer-Verlag, Berlin-Heidelberg-New York, first edition, (1979).

[16] E. Nelson. *Notes on non-commutative integration*. J. Funct. Anal., 15:103–116, (1974).
ABSOLUTELY SUMMING OPERATORS

[17] A. Pelczynski. *Compactness of absolutely summing operators.* In V. Havin and N. Nikolski, editors, *Linear and Complex Analysis*, volume 1573 of Lectures Notes in Mathematics, Berlin Heidlberg New York, (1994). Springer Verlag.

[18] G. Pisier. *Grothendieck’s Theorem for non-commutative C*-algebras with appendix on Grothendieck’s constants.* J. Func. Anal., 29:397–415, (1978).

[19] G. Pisier. *Factorization of operators through L_p∞ or L_1∞ and Non-commutative generalizations.* Math. Ann., 276:105–136, (1986).

[20] W. Rudin. *Fourier analysis on groups,* volume 12 of Interscience Tracts in Pure and Appl. Math. . Interscience, New York, first edition, (1962).

[21] M. Takesaki. *Theory of operator Algebras I.* Springer-Verlag, New-York, Heidelberg, Berlin, (1979).

[22] Q. XU. *Analytic functions with values in Lattices and Symmetric spaces of measurable operators.* Math. Proc. Camb. Phil. Soc., 109:541–563, (1991).

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712-1082

E-mail address: nrandri@math.utexas.edu