HOMOLOGICAL IDEALS AS INTEGER SPECIALIZATIONS OF SOME BRAUER CONFIGURATION ALGEBRAS

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The homological ideals associated with some Nakayama algebras are characterized and enumerated via integer specializations of some suitable Brauer configuration algebras. In addition, it is shown how the number of these homological ideals can be connected with the process of categorification of Fibonacci numbers defined by Ringel and Fahr.

1. Introduction

Homological ideals or strong idempotent ideals are ideals of an algebra introduced byAuslander, Platzeck, and Todorov in [2]. These ideals arise in the research of heredity ideals and quasiherededitary algebras. For these ideals, the corresponding quotient map induces a full and faithful functor between the derived categories. In recent years, homological ideals have been studied in different contexts. Thus, Gatica, Lanzillota, and Platzeck and (independently) Xu and Xi established some relationships between the so-called finitistic dimension conjecture and the Igusa–Todorov functions [6]. Furthermore, De la Peña and Xi in [9] and Armenta in [1] studied the impact of these ideals in the context of Hochschild cohomology and one-point extensions.

The present work deals with the combinatorial properties of homological ideals associated with certain path algebras and their relationships with the novel Brauer configuration algebras recently introduced by Green and Schroll [7]. In particular, we introduce the notion of message of a Brauer configuration. Messages of this kind enable to compute the number of homological ideals associated with some Nakayama algebras. Moreover, this number of ideals enable us to obtain an alternative version of the partition formula for even-index Fibonacci numbers given by Fahr and Ringel in [3] and attain, in this way, a new algebraic interpretation for these numbers. It is worth noting that Fahr and Ringel devoted their works [3–5] to this kind of interpretations, which are also called categorifications.

This paper is organized as follows: In Section 2, we recall the main notation and definitions concerning homological ideals and Brauer configuration algebras. In particular, we introduce the notion of integer specialization of a Brauer configuration and the concept of message of the Brauer configuration. In Section 3, we present combinatorial conditions to determine whether an idempotent ideal corresponding to some Nakayama algebras is homological or not and recall the notion of categorification in a sense of Fahr and Ringel. We also determine the number of these ideals via the integer specialization of a suitable Brauer configuration algebra and its corresponding message. Moreover, we use the number of homological ideals to establish the partition formula for even-index Fibonacci numbers. Some interesting sequences arising from these computations are also described in the Online Encyclopedia of Integer Sequences [11].

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2. Preliminaries

In the present section, we recall the main definitions and notation frequently used throughout the paper [1, 2, 7, 9, 10].

2.1. Homological Ideals. As an algebra $A$, we take a finite-dimensional basic and connected algebra over an algebraically closed field $k$. The category of finite dimensional right $A$-modules is denoted by $\text{mod}(A)$, whereas the bounded derived category of $\text{mod}(A)$ is denoted by $D^b(A)$. We assume that $A$ is a bounded path algebra of the form $kQ/I$, where $Q$ is a finite quiver and $I$ an admissible ideal.

An epimorphism of algebras $\phi: A \to B$ is called homological epimorphism if it induces a full and faithful functor $D^b(\phi^*): D^b(B) \to D^b(A)$.

Let $I$ be a two-sided ideal of $A$. Since the quotient map $\pi: A \to A/I$ is an epimorphism, the induced functor $\pi^*: \text{mod}(A/I) \to \text{mod}(A)$ is full and faithful.

A two-sided ideal $I$ of $A$ is homological if the quotient map $\pi: A \to A/I$ is a homological epimorphism. The following results characterize homological ideals [2, 9]:

Proposition 1. Let $I$ be an ideal of $A$. Then:

(i) $I$ is a homological ideal of $A$ if and only if $\text{Tor}_n^A(I, A/I) = 0$ for all $n \geq 0$; in this case, $I$ is idempotent;

(ii) if $I$ is idempotent and $A$-projective, then $I$ is homological;

(iii) If $I$ is idempotent, then $I$ is homological if and only if $\text{Ext}_n^A(I, A/I) = 0$ for all $n \geq 0$.

We denote the trace of an $A$-module $M$ in an $A$-module $N$ as follows:

$$\text{tr}_M(N) := \sum_{f \in \text{Hom}_A(M,N)} \text{Im}(f) \subset N.$$ 

Remark 1. We recall that, according to Auslander, et al. [2], if $P$ is an $A$-projective module, then $\text{tr}_P(A)$ is an idempotent ideal of $A$ and, in this way, we get all idempotent ideals of $A$.

Remark 2. Note that, since the homological ideals are idempotent ideals and the idempotent ideals are traces of projective modules over $A$, we always have finitely many homological ideals.

Following the assumptions that $A$ is a bounded quiver algebra of the form $kQ/I$ and the number of vertices of $Q$ is finite for every subset $\{a_1, \ldots, a_m\} \subset Q_0$, we use the following notation for any idempotent ideal generated by the trace of $P(a_1) \oplus \ldots \oplus P(a_m)$ in $A$:

$$I_{a_1, \ldots, a_m} = \text{tr}(P(a_1) \oplus \ldots \oplus P(a_m))(A).$$  (1)

In the present paper, we combine the tools developed by Auslander, et al. in [2], Xi and De la Peña in [9] with the integer specializations of some Brauer configuration (see Subsection 2.3) in order to establish an explicit
formula for the number of homological ideals corresponding to some Nakayama algebras. This number makes it possible to establish a partition formula for even-index Fibonacci numbers, as defined by Fahr and Ringel in [3–5].

2.2. Brauer Configuration Algebras. Brauer configuration algebras were introduced by Green and Schroll in [7] as a generalization of Brauer graph algebras, which are biserial algebras of tame-representation type and their representation theory is encoded by some combinatorial data based on graphs. Actually, every Brauer graph algebra has a finite underlying graph with acyclic orientation of the edges at every vertex and a multiplicity function [7]. The construction of a Brauer graph algebra is a special case of construction of a Brauer configuration algebra in a sense that every Brauer graph is a Brauer configuration equipped with the restriction that every polygon is a set with two vertices. In what follows, we give precise definitions of Brauer configurations and Brauer configuration algebras.

A Brauer configuration \( \Gamma \) is a quadruple of the form \( \Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O}) \), where:

(B1) \( \Gamma_0 \) is a finite set whose elements are called vertices;

(B2) \( \Gamma_1 \) is a finite collection of multisets called polygons; in this case, if \( V \in \Gamma_1 \), then the elements of \( V \) are vertices (possibly with repetitions), \( \text{occ}(\alpha, V) \) denotes the frequency of the vertex \( \alpha \) in the polygon \( V \), and the valency of \( \alpha \) is denoted by \( \text{val}(\alpha) \) and defined as follows:

\[
\text{val}(\alpha) = \sum_{V \in \Gamma_1} \text{occ}(\alpha, V);
\]  

(B3) \( \mu \) is an integer-valued function such that \( \mu : \Gamma_0 \to \mathbb{N} \), where \( \mathbb{N} \) denotes the set of positive integers; this function is called the multiplicity;

(B4) \( \mathcal{O} \) denotes an orientation defined on \( \Gamma_1 \) as a choice, for each vertex \( \alpha \in \Gamma_0 \), of a cyclic ordering of polygons in which \( \alpha \) occurs as a vertex, including repetitions; by \( S_\alpha \) we denote the collection of these polygons; more specifically, if \( S_\alpha = \{V_1^{(\alpha_1)}, V_2^{(\alpha_2)}, \ldots, V_t^{(\alpha_t)}\} \) is the collection of polygons where the vertex \( \alpha \) occurs with \( \alpha_i = \text{occ}(\alpha, V_i) \) and \( V_i^{(\alpha)} \) means that \( S_\alpha \) has \( \alpha_i \) copies of \( V_i \); then the orientation \( \mathcal{O} \) is obtained by endowing \( S_\alpha \) with a linear order \( \leq \) and adding the relation \( V_t \leq V_1 \) if \( V_1 = \min S_\alpha \) and \( V_t = \max S_\alpha \);

(B5) every vertex in \( \Gamma_0 \) is a vertex of at least one polygon in \( \Gamma_1 \);

(B6) every polygon has at least two vertices;

(B7) every polygon in \( \Gamma_1 \) has at least one vertex \( \alpha \) such that \( \text{val}(\alpha)\mu(\alpha) > 1 \).

The set \( (S_\alpha, \leq) \) is called the successor sequence at the vertex \( \alpha \).

A vertex \( \alpha \in \Gamma_0 \) is called truncated if \( \text{val}(\alpha)\mu(\alpha) = 1 \), i.e., \( \alpha \) is truncated if it occurs exactly once in exactly one \( V \in \Gamma_1 \) and \( \mu(\alpha) = 1 \). A vertex is called nontruncated if it is not truncated.

The Quiver of a Brauer Configuration Algebra

The quiver \( Q_\Gamma = ((Q_\Gamma)_0, (Q_\Gamma)_1) \) of a Brauer configuration algebra is defined so that the vertex set \( (Q_\Gamma)_0 = \{v_1, v_2, \ldots, v_m\} \) of \( Q_\Gamma \) corresponds to the set of polygons \( \{V_1, V_2, \ldots, V_m\} \) in \( \Gamma_1 \), and we note that there is one vertex in \( (Q_\Gamma)_0 \) for every polygon in \( \Gamma_1 \).

The arrows in \( Q_\Gamma \) are defined by the successor sequences. This means that there is an arrow \( v_i \xrightarrow{\mathcal{O}} v_{i+1} \in (Q_\Gamma)_1 \) provided that \( V_i \leq V_{i+1} \) in \( (S_\alpha, \leq) \cup \{V_i \leq V_1\} \) for some nontruncated vertex \( \alpha \in \Gamma_0 \). In other words, for every nontruncated vertex \( \alpha \in \Gamma_0 \) and each successor \( V' \) of \( V \) at \( \alpha \), there is an arrow from \( v \) to \( v' \) in \( Q_\Gamma \), where \( v \) and \( v' \) are the vertices in \( Q_\Gamma \) associated with the polygons \( V \) and \( V' \) in \( \Gamma_1 \), respectively.
The Ideal of Relations and the Definition of a Brauer Configuration Algebra

We fix a polygon $V \in \Gamma_1$ and suppose that $\text{occ}(\alpha, V) = t \geq 1$. Then there are $t$ indices $i_1, \ldots, i_t$ such that $V = V_{i_j}$. Then the special $\alpha$-cycles at $v$ are the cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_t}$, where $v$ is the vertex of the quiver of $Q_\Gamma$ corresponding to the polygon $V$. If $\alpha$ occurs only once in $V$ and $\mu(\alpha) = 1$, then there is only one special $\alpha$-cycle at $v$.

Let $k$ be a field and let $\Gamma$ be a Brauer configuration. The Brauer configuration algebra associated with $\Gamma$ is defined as the bounded path algebra $\Lambda_\Gamma = kQ_\Gamma/I_\Gamma$, where $Q_\Gamma$ is the quiver corresponding to $\Gamma$ and $I_\Gamma$ is the ideal in $kQ_\Gamma$ generated by the following set of relations $\rho_\Gamma$ of types I, II, and III:

1. **Relations of type I.** For any polygon $V = \{\alpha_1, \ldots, \alpha_m\} \in \Gamma_1$ and each pair of nontruncated vertices $\alpha_i$ and $\alpha_j$ in $V$, the set of relations $\rho_\Gamma$ contains all relations of the form $C_{\mu(\alpha_i)} = C_{\mu(\alpha_j)}$ where $C$ is a special $\alpha_i$-cycle and $C'$ is a special $\alpha_j$-cycle.

2. **Relations of type II.** Relations of types II are all paths of the form $C_{\mu(\alpha)}a$, where $C$ is a special $\alpha$-cycle and $a$ is the first arrow in $C$.

3. **Relations of type III.** These relations are quadratic monomial relations of the form $ab$ in $kQ_\Gamma$, where $ab$ is not a subpath of any special cycle unless $a = b$ and $a$ is a loop corresponding to a vertex of valency 1 and $\mu(\alpha) > 1$.

As an example, for fixed $n \geq 4$, we consider a Brauer configuration $\Gamma_n = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ such that:

(I) $\Gamma_0 = \{n - k - 1 \in \mathbb{N} \mid 2 \leq k \leq n - 1\} \cup \{n - 2\}$,

(II) $\Gamma_1 = \{U_k = \{n - 2, n - k - 1\} \mid 2 \leq k \leq n - 1\}$,

(III) the orientation $\mathcal{O}$ is defined in such a way that

(a) the vertex $n - 2$ is associated with the successor sequence $U_2 < U_3 < \ldots < U_{n-1}$; in this case, $\text{val}(n - 2) = n - 2$,

(b) if $2 \leq k \leq n - 1$, then, at the vertex $n - k - 1$, it is true that the corresponding successor sequence consists only of $U_k$ and, for each $k$, we get $\text{val}(n - k - 1) = 1$,

(IV) $\mu(n - 2) = 1$,

(V) $\mu(n - k - 1) = n - 2$, $2 \leq k \leq n - 1$.

The ideal $I_{\Gamma_n}$ of the corresponding Brauer configuration algebra $\Lambda_{\Gamma_n}$ is generated by the following relations (see Fig. 1) in which we use the following notation for special cycles:

\[
C_{n-2}^{U_k} = \begin{cases} 
  a_1^{n-k-2}a_2^{n-k-2} \ldots a_{k-1}^{n-k-2}, & \text{if } k = 2, \\
  a_1^{n-k-2}a_k^{n-k-2} \ldots a_{k-2}^{n-k-2}, & \text{otherwise},
\end{cases} \tag{3}
\]

1. $a_i^h a_r^s$ if $h \neq s$ for all possible values of $i$ and $r$ unless the loops are associated with the vertices $n-k-1$.

2. $C_{n-2}^{U_k} - \left(C_{n-k-1}^{U_k}\right)^{n-2}$ for all possible values of $k$. 

Fig. 1. The quiver $Q_{\Gamma_n}$ defined by the Brauer configuration $\Gamma_n$.

3. $C_{n-k}^{U_k}a$ with $a$ corresponding to the first arrow of $C_{n-k}^{U_k}$ for all $k$.

4. $(C_{n-k-1}^{U_k})^{n-2}a'$ with $a'$ corresponding to the first arrow of $C_{n-k-1}^{U_k}$ for all $k$.

In Fig. 1, we show the quiver $Q_{\Gamma_n}$ associated with this configuration.

The following results describe the structure of a Brauer configuration algebra [7]:

**Theorem 1.** Let $\Lambda$ be a Brauer configuration algebra with Brauer configuration $\Gamma$.

1. There is a bijective correspondence between the set of projective indecomposable $\Lambda$-modules and the polygons in $\Gamma$.

2. If $P$ is a projective indecomposable $\Lambda$-module corresponding to a polygon $V$ in $\Gamma$, then $\text{rad} P$ is the sum of $r$ indecomposable uniserial modules, where $r$ is the number of (nontruncated) vertices of $V$, and the intersection of any two uniserial modules is a simple $\Lambda$-module.

3. A Brauer configuration algebra is a multiserial algebra.

4. The number of terms in the heart of an indecomposable projective $\Lambda$-module $P$ such that $\text{rad}^2 P \neq 0$ is equal to the number of nontruncated vertices of the polygons in $\Gamma$ corresponding to $P$, counting repetitions.

5. If $\Lambda'$ is a Brauer configuration algebra obtained from $\Lambda$ by removing a truncated vertex of a polygon in $\Gamma_1$ with $d \geq 3$ vertices, then $\Lambda$ is isomorphic to $\Lambda'$. 
**Proposition 2.** Let $\Lambda$ be a Brauer configuration algebra associated with the Brauer configuration $\Lambda$ and let $C = \{C_1, \ldots, C_t\}$ be a full set of equivalence class representatives of special cycles. Assume that, for $i = 1, \ldots, t$, $C_i$ is a special $\alpha_i$-cycle, where $\alpha_i$ is a nontruncated vertex in $\Gamma$. Then

$$\dim_k \Lambda = 2|Q_0| + \sum_{C_i \in C} |C_i| (n_i |C_i| - 1),$$

where $|Q_0|$ denotes the number of vertices of $Q$, $|C_i|$ denotes the number of arrows in the $\alpha_i$-cycle $C_i$, and $n_i = \mu(\alpha_i)$.

**Proposition 3.** Let $\Lambda$ be the Brauer configuration algebra associated with a connected Brauer configuration $\Gamma$. The algebra $\Lambda$ has a length grading induced from the path algebra $kQ$ if and only if there is $N \in \mathbb{Z}_{>0}$ such that $\text{val}(\alpha) \mu(\alpha) = N$ for each nontruncated vertex $\alpha \in \Gamma_0$.

Sierra [10] proved the following result concerning the center of a Brauer configuration algebra:

**Theorem 2.** Let $\Gamma$ be a reduced (i.e., without truncated vertices) and connected Brauer configuration, let $Q$ be its induced quiver, and let $\Lambda$ be the induced Brauer configuration algebra such that $\text{rad}^2 \Lambda \neq 0$. Then the dimension of the center of $\Lambda$ denoted by $\dim_k Z(\Lambda)$ is given by the formula

$$\dim_k Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#(\text{Loops } Q) - |\mathcal{E}_\Gamma|,$$

where $\mathcal{E}_\Gamma = \{\alpha \in \Gamma_0 \mid \text{val}(\alpha) = 1 \text{ and } \mu(\alpha) > 1\}$.

As an example, the following numerology is associated with the algebra $\Lambda_{\Gamma_n} = kQ_{\Gamma_n}/I_{\Gamma_n}$ with $Q_{\Gamma_n}$ as shown in Fig. 1 and special cycles given in (3) ($|r(Q_{\Gamma_n})|$ is the number of indecomposable projective modules and $|C_i| = \text{val}(i)$):

$$|r(Q_{\Gamma_n})| = n - 2,$$

$$|C_{n-2}| = n - 2, \quad |C_{n-k-1}| = 1,$$

$$\sum_{\alpha \in \Gamma_0} \sum_{X \in \Gamma_1} \text{occ}(\alpha, X) = n - 1,$$

the number of special cycles,

$$\dim_k \Lambda_{\Gamma_n} = 2(n - 2) + (n - 2)(n - 3) + (n - 3)(n - 2) = 2(n - 2)^2,$$

$$\dim_k Z(\Lambda_{\Gamma_n}) = 1 + 1 + (n - 2)^2 + (n - 2) - (n - 1) + (n - 2) - (n - 2) = n^2 - 4n + 5.$$

**Remark 3.** $\Lambda_{\Gamma_n}$ is a Brauer graph algebra and, according to Proposition 3, the Brauer configuration algebra $\Lambda_{\Gamma_n}$ with quiver $Q_{\Gamma_n}$ depicted in Fig. 1 has a length grading induced by the path algebra $kQ_{\Gamma_n}$ provided that, for any $\alpha \in \Gamma_0$, it is true that $\mu(\alpha) \text{val}(\alpha) = n - 2$.

2.3. Message of the Brauer Configuration. The concept of message of a Brauer configuration is helpful to categorify some integer sequences in a sense of Fahr and Ringel (see Subsection 3.1 of the present paper and [3, 4]).
Let $\Gamma = \{\Gamma_0, \Gamma_1, \mu, \mathcal{O}\}$ be a Brauer configuration and let $U \in \Gamma_1$ be a polygon such that

$$U = \left\{ \alpha_1^{f_1}, \alpha_2^{f_2}, \ldots, \alpha_n^{f_n} \right\}, \text{ where } f_i = \text{occ}(\alpha_i, U).$$

The term

$$w(U) = \alpha_1^{f_1} \alpha_2^{f_2} \cdots \alpha_n^{f_n}$$

is said to be a word associated with $U$. The sum

$$M(\Gamma) = \sum_{U \in \Gamma_1} w(U)$$

is called a message of the Brauer configuration $\Gamma$.

An integer specialization of a Brauer configuration $\Gamma$ is a Brauer configuration $\Gamma^e = (\Gamma^e_0, \Gamma^e_1, \mu^e, \mathcal{O}^e)$ endowed with an orientation-preserving map $e : \Gamma_0 \to \mathbb{N}$ such that

$$\Gamma^e_0 = \text{Img } e \subset \mathbb{N},$$

$$\Gamma^e_1 = e(\Gamma_1), \quad \text{if } H \in \Gamma_1 \text{ then } e(H) = \{e(\alpha_i) \mid \alpha_i \in H\} \in e(\Gamma_1),$$

$$\mu^e(e(\alpha)) = \mu(\alpha) \quad \text{for any } \alpha \in \Gamma_0.$$

In addition, $e(U) \preceq e(V)$ in $\Gamma^e_1$ provided that $U \preceq V$ in $\Gamma_1$.

Let

$$w^e(U) = (e(\alpha_1))^{f_1} (e(\alpha_2))^{f_2} \cdots (e(\alpha_n))^{f_n}$$

denote the specialization under $e$ of a word $w(U)$. In this case,

$$M(\Gamma^e) = \sum_{U \in \Gamma^e_1} w^e(U)$$

is the specialized message of the Brauer configuration $\Gamma$ with the ordinary integer sum and product (in general, with the sum and product associated with $\text{Img } e$).

**Example 1.** For the Brauer configuration $\Gamma_n$ whose associated quiver is shown in Fig. 1, we define a specialization $e(\alpha) = 2^{\alpha}$, $\alpha \in \Gamma_0$, with concatenation in each word given by the difference of specializations of the vertices that belong to a given polygon; in this case, for fixed $n$, we get

$$w(U_k) = (n - 2)(n - k - 1) \quad \text{for } 2 \leq k \leq n - 1,$$

$$w^e(U_k) = 2^{n-2} - 2^{n-k-1} \quad \text{for } 2 \leq k \leq n - 1,$$

$$M(\Gamma^e_n) = \sum_{U_k \in \Gamma_1} w^e(U_k) = \sum_{k=1}^{n-1} 2^{n-2} - 2^{n-k-1}.$$
3. Homological Ideals Associated with Nakayama Algebras

In this section, we prove some combinatorial conditions, which enable us to establish whether an idempotent ideal in some Nakayama algebras is homological or not. We also present numerous homological ideals associated with these algebras via the integer specialization of the Brauer configuration $\Gamma_n$ defined in Example 1. Moreover, we use the number of homological ideals to deduce a partition formula for even-index Fibonacci numbers.

Let $Q$ be either a linearly oriented quiver with underlying graph $\widetilde{A}_n$ or a cycle $\widetilde{A}_n$ with cyclic orientation. This means that $Q$ is one of the quivers shown in Fig. 2.

A quotient $A$ of $kQ$ by an admissible ideal $I$ is called a Nakayama algebra [8].

In the present work, for fixed $n \geq 3$, we consider the algebras $A_{R(i,j,k)} = kQ/I$, where $Q$ is a linearly oriented Dynkin diagram of type $\widetilde{A}_n$ and $I$ is an admissible ideal generated by a single relation $R_{(i,j,k)}$ of length $k$ with origin at a vertex $i$ and end at a vertex $j$ of the given quiver, $1 \leq i < j \leq n$. The following picture shows the general structure of quivers $Q$ studied in the present paper:

$$\widetilde{A}_n = 1 \rightarrow \ldots \rightarrow i \rightarrow i + 1 \rightarrow \ldots \rightarrow i + k = j \rightarrow j + 1 \rightarrow \ldots \rightarrow n - 1 \rightarrow n.$$

The lemmas presented below enable us to find the idempotent ideals of an algebra $A_{R(i,j,k)}$ that are also homological ideals. In this connection, Lemmas 1 and 2 deal with the case where the idempotent ideal is generated by the trace of exactly one projective module associated with a vertex of the quiver.

**Lemma 1.** Every idempotent ideal $I_r$ of an algebra $A_{R(i,j,k)}$ [see (1)] with $j \leq r$ or $r \leq i$ is homological.

**Proof.** For $r \leq i$, we have the following cases:

(i) $\text{tr}_{P(r)}(P(t)) = 0$ for $t > r$,

(ii) $\text{tr}_{P(r)}(P(t)) = P(r)$ for $t \leq r$, where $P(r)$ denotes the $k$th projective module.

For $r \geq j$, we consider the following cases:

(i) $\text{tr}_{P(r)}(P(t)) = P(r)$ for $i < t \leq r$, where $P(r)$ denotes the $k$th projective module,

(ii) $\text{tr}_{P(r)}(P(t)) = 0$.

In all cases,

$$\text{tr}_{P(r)}(A_{R(i,j,k)}) = P(r)^l$$

for some $l \in \mathbb{N}$. The required result follows from Proposition 1, item 2. We are done.
Lemma 2. Every idempotent ideal $I_t$ of an algebra $A_{R(i,j,k)}$ with $i + 1 \leq t \leq j - 1$ is not homological.

Proof. Consider

$$L_t = \text{tr}_{P(t)} P(i) = P(i)/S(i) \oplus \ldots \oplus S(t - 1),$$

where $S(k)$ denotes the $k$th simple module. We also note that there are nonmorphisms from $P(t)$ to $P(j)$ if $t \neq j$, which means that $\text{Ext}^1_{A_{R(i,j,k)}}(L_t, P(j))$ is a direct summand of $\text{Ext}^1_{A_{R(i,j,k)}}(I_t, A_{R(i,j,k)}/I_t)$ provided that $L_t$ is a direct summand of $I_t$ and $P(j)$ is a direct summand of $A_{R(i,j,k)}/I_t$. Applying the functor $\text{Hom}_{A_{R(i,j,k)}}(-, P(j))$ to the projective resolution of $L_t$ of the form

$$0 \to P(j) \to P(t) \to L_t \to 0,$$

we obtain the exact sequence

$$0 \to \text{Hom}_{A_{R(i,j,k)}}(P(t), P(j)) \to \text{Hom}_{A_{R(i,j,k)}}(P(j), P(j)) \to 0.$$

Thus,

$$\text{Ext}^1_{A_{R(i,j,k)}}(L_t, P(n)) \cong k \quad \text{and} \quad \text{Ext}^1_{A_{R(i,j,k)}}(I_t, A_{R(i,j,k)}/I_t) \neq 0.$$

Then the idempotent ideal $I_t$ is not a homological ideal as follows from Proposition 1, item 3.

Lemma 3. If every idempotent ideal $I_{\alpha_w}$ of an algebra $A_{R(i,j,k)}$ is not homological, then every idempotent ideal of the form $I_{\alpha_1, \ldots, \alpha_l}$ is not homological for $2 \leq l \leq k - 1$.

Proof. For fixed $l$, we start by computing $I_{\alpha_1, \ldots, \alpha_l}$, namely,

$$I_{\alpha_1, \ldots, \alpha_l} = \text{tr}_{P(\alpha_1) \oplus \ldots \oplus P(\alpha_l)}(A_{R(i,j,k)}) = \sum_{w=1}^{l} \text{tr}_{P(\alpha_w)}(A_{R(i,j,k)}).$$

According to the hypothesis, $\alpha_w \in [i + 1, j - 1]$. Thus, by using the fact that

$$\text{tr}_{P(\alpha_w)}(A_{R(i,j,k)}) = \underbrace{L_{\alpha_w} \oplus P(\alpha_w) \oplus 0}_{i \text{ times}} \oplus \underbrace{0 \oplus 0}_{n - \alpha_w \text{ times}},$$

$$\text{tr}_{P(\alpha_1) \oplus \ldots \oplus P(\alpha_l)}(A_{R(i,j,k)}) = \underbrace{L_{\alpha_1} \oplus \bigoplus_{w=1}^{l} P(\alpha_w) \oplus 0}_{i \text{ times}} \oplus \underbrace{0 \oplus 0}_{n - i \text{ times}},$$

we conclude that, in view of identity (10), $P(j)$ is a direct summand of $A_{R(i,j,k)}/I_{\alpha_1 \ldots \alpha_l}$ and $L_{\alpha_1}$ has the following projective resolution:

$$0 \to P(j) \to P(\alpha_1) \to L_{\alpha_1} \to 0.$$

Applying the functor $\text{Hom}_{A_{R(i,j,k)}}(-, P(j))$, we can show that $\text{Ext}^1_{A_{R(i,j,k)}}(L_{\alpha_1}, P(j)) \neq 0$ and, by virtue of Proposition 1, item 3, we conclude that the idempotent ideal $I_{\alpha_1 \ldots \alpha_l}$ is not a homological ideal.
Lemma 4. For fixed \( l \), assume that each idempotent ideal \( I_{\alpha_w} \) of an algebra \( A_{R(i,j,k)} \) with \( 1 \leq w \leq l \) is homological. Then every idempotent ideal of the form \( I_{\alpha_1,...,\alpha_l} \) is also homological.

Proof. It suffices to note that \( \text{tr}_{P(\alpha_w)}(A_{R(i,j,k)}) = P(\alpha_w)^t \) for some \( l \in \mathbb{N} \).

Lemma 5. Every ideal \( I_{t,j} \) or \( I_{t,j} \) of an algebra \( A_{R(i,j,k)} \) is homological.

Proof. By using the previous lemma, it is possible to conclude that if \( I_t \) is homological, then we get the required result. If this is not true, then we consider the following cases:

1. For \( I_t \), which is not homological, we can find

\[
I_{t,j} = \text{tr}_{P(i)\oplus P(j)}(A_{R(i,j,k)})
\]

[see identity (9)]. Since \( r \leq i \), we get \( \text{tr}_{P(i)}P(r) = P(i) \). Therefore, the ideal \( I_{t,j} \) is projective and idempotent. Thus, for Proposition 1, item 2, we conclude that the ideal \( I_{t,j} \) is homological.

2. We start by computing \( I_{t,j} \) as follows:

\[
I_{t,j} = \text{tr}_{P(t)\oplus P(j)}(A_{R(i,j,k)}) = \frac{L_t}{L_t} \oplus \frac{P(t)}{t-i \text{ times}} \oplus \frac{P(j)}{j-t \text{ times}} \oplus \frac{0}{n-j \text{ times}}.
\]

Further, \( A_{R(i,j,k)}/I_{t,j} \) is given by

\[
A_{R(i,j,k)}/I_{t,j} = \frac{P(1) \oplus P(2) \oplus ... \oplus P(i) \oplus ... \oplus P(t) \oplus ... \oplus P(j) \oplus ... \oplus P(n)}{L_t \oplus ... \oplus L_t \oplus P(t) \oplus ... \oplus P(t) \oplus P(j) \oplus ... \oplus P(j) \oplus 0 \oplus ... \oplus 0}.
\]

In order to compute \( \text{Ext}^{k}_{A_{R(i,j,k)}}(I_{t,j}, A_{R(i,j,k)}/I_{t,j}) \) we consider the projective resolution of \( L_t \):

\[
0 \to P(j) \to P(t) \to L_t \to 0.
\]

Applying the functor \( \text{Hom}_{A_{R(i,j,k)}}(-, P(j)) \), we find

\[
0 \to \text{Hom}_{A_{R(i,j,k)}}(P(t), A_{R(i,j,k)}/I_{t,j}) \to \text{Hom}_{A_{R(i,j,k)}}(P(j), A_{R(i,j,k)}/I_{t,j}) \to 0.
\]

In view of the fact that

\[
\begin{align*}
\text{Hom}_{A_{R(i,j,k)}}\left(P(t), \frac{P(z)}{L_t}\right) &= 0 \quad \text{if} \quad 1 \leq z \leq i, \\
\text{Hom}_{A_{R(i,j,k)}}\left(P(t), \frac{P(y)}{P(t)}\right) &= 0 \quad \text{if} \quad i+1 \leq y \leq t-1, \\
\text{Hom}_{A_{R(i,j,k)}}\left(P(t), \frac{P(v)}{P(j)}\right) &= 0 \quad \text{if} \quad t+1 \leq v \leq j-1, \\
\text{Hom}_{A_{R(i,j,k)}}\left(P(t), P(u)\right) &= 0 \quad \text{if} \quad j+1 \leq u \leq n,
\end{align*}
\]
Thus, the ideal \( I_{z_1,\ldots,z_h,t_1,\ldots,t_l,y_1,\ldots,y_m} \) has the following form:

\[
\bigoplus_{a=1}^{h} P(z_a) \oplus \bigoplus_{b=1}^{l} P(t_b) \oplus \bigoplus_{c=1}^{m} P(y_c) \oplus 0,
\]

where

- \( P(z_a) \) appears \( h \) times,
- \( P(t_b) \) appears \( l \) times,
- \( P(y_c) \) appears \( m \) times,
- The \( 0 \) appears \( n-m-j \) times.

These results are used to conclude that if \( I_I \) is a nonhomological ideal, then the previous Lemma 5 is also true.
In view of (13), we conclude that $P(j)/P(y_1)$ is a direct summand of the quotient $A_{R(i,j,k)}/I_{z_1,...,z_l,t_1,...,t_l,...,y_1,...,y_m}$ and $L_{t_1}$ has the following projective resolution:

$$0 \rightarrow P(j) \rightarrow P(t_1) \rightarrow L_{t_1} \rightarrow 0.$$ (14)

Applying the functor $\text{Hom}_{AR(i,j,k)}(-, P(j)/P(y_1))$ to resolution (14), we get the exact sequence

$$0 \rightarrow \text{Hom}_{AR(i,j,k)}(P(t), P(j)/P(y_1)) \rightarrow \text{Hom}_{AR(i,j,k)}(P(j), P(j)/P(y_1)) \rightarrow 0.$$

Thus,

$$\text{Ext}^1_{AR(i,j,k)}(L_t, P(j)/P(y_1)) \cong k$$

and

$$\text{Ext}^1_{AR(i,j,k)}(I_{z_1,...,z_l,t_1,...,t_l,...,y_1,...,y_m}, A_{R(i,j,k)}/I_{z_1,...,z_l,t_1,...,t_l,...,y_1,...,y_m}) \neq 0.$$

Hence, by Proposition 1, item 3, we conclude that the idempotent ideal $I_{z_1,...,z_l,t_1,...,t_l,...,y_1,...,y_m}$ is not a homological ideal.

**Lemma 7.** For fixed $1 \leq h \leq i - 1$, $1 \leq l \leq k - 1$, and $1 \leq m \leq n - j$, the idempotent ideals $I_{z_1,...,z_l,t_1,...,t_l}$ and $I_{t_1,...,t_l,...,y_1,...,y_m}$ of an algebra $A_{R(i,j,k)}$, where $z_a \in [1, i - 1]$, $t_b \in [i + 1, j - 1]$, and $y_c \in [j + 1, n]$, are not homological.

**Proof.** In (11), it is sufficient to consider either the trace

$$\sum_{a=1}^{h} \text{tr}_{P(z_a)}(A_{R(i,j,k)}) = 0$$

or the trace

$$\sum_{c=1}^{m} \text{tr}_{P(y_c)}(A_{R(i,j,k)}) = 0.$$

### 3.1. On the Number of Homological Ideals Associated with Some Nakayama Algebras.

The following results make it possible to compute the number of homological and nonhomological ideals in a bounded algebra $A_{R(i,j,k)}$ by using the integer specialization $e$ of the Brauer configuration $\Gamma_n$ introduced in Example 1.

**Theorem 3.** For fixed $n \geq 4$ and $2 \leq k \leq n - 1$, the number $|\text{NHI}_n^k|$ of nonhomological ideals of an algebra $A_{R(i,j,k)}$ is given by the identity $|\text{NHI}_n^k| = w^e(U_k)$.

**Proof.** Note that, according to Lemmas 2 and 3, there are $2^{k-1} - 1$ nonhomological ideals associated solely with the vertices inside the relation $R(i,j,k)$. By Lemma 6, there are additional $2^{n-k-1}$ nonhomological ideals arising from the combination of vertices located inside and outside the relation. The theorem is proved by taking into account the product rule and Example 1.
Corollary 1. For fixed $n \geq 4$ and $2 \leq k \leq n - 1$, the number of homological ideals $|\mathbb{H}^k_n|$ of an algebra $A_{R(i,j,k)}$ is given by the identity

$$|\mathbb{H}^k_n| = 2^n - w^e(U_k) = 3 \cdot 2^{n-2} + 2^{n-k-1}.$$  

Proof. Since there are $2^n$ idempotent ideals in $A_{R(i,j,k)}$, the required result follows from Theorem 3.

The formula obtained in Theorem 3 induces the following triangle:

**Nonhomological Triangle $\text{NHIT}$**

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|-----|---|---|---|---|---|---|---|-----|
| 3   | 1 | - | - | - | - | - | - |  |
| 4   | 2 | 3 | - | - | - | - | - |  |
| 5   | 4 | 6 | 7 | - | - | - | - |  |
| 6   | 8 | 12| 14| 15| - | - | - |  |
| 7   | 16| 24| 28| 30| 31| - | - |  |

The entries $|\text{NHIT}^k_n|$ of the triangle $\text{NHIT}$ can be inductively computed as follows: we start by defining $|\text{NHIT}^2_n| = 2^{n-3}$ for all $n \geq 3$. We now assume that $|\text{NHIT}^k_n| = 0$ with $k \leq 1$ and, for the sake of clarity, we denote the specialization under $e$ of a word $w(U_k)$ of the polygon $U_k$ in the Brauer configuration $\Gamma_n$ as $w^e(U^n_k)$ (see Example 1). Thus, for $k \geq 3$,

$$w^e(U_k) = w^e(U^n_k) = (w^e(U^n_{k-1}) + w^e(U^{n-1}_{k-1})) - w^e(U^{n-1}_{k-2})$$

or, equivalently,

$$|\text{NHIT}^k_n| = (|\text{NHIT}^{k-1}_n| + |\text{NHIT}^{k-1}_{n-1}|) - |\text{NHIT}^{k-2}_{n-1}|.$$  

These arguments prove the following proposition:

**Proposition 4.** $M(\Gamma_n^e)$ is equal to the sum of elements in the $n$th row of the nonhomological triangle $\text{NHIT}$ (see Example 1).

Remark 5. The integer sequence generated by

$$M(\Gamma_n^e) = \sum_{k=1}^{n-1} 2^{n-2} - 2^{n-k-1} = \{1, 5, 17, 49, 129, 321, 769, 1793, 4097, 9217, \ldots\}$$

is encoded by A000337 in the OEIS. Elements of the sequence A000337 also correspond to the sums of elements in the rows of the Reinhard Zumkeller triangle.

Remark 6. The sum of entries in the diagonals of the nonhomological triangle is given by the sequence A274868 in the OEIS. It is related to the number of partitions of the set $[n]$ into exactly four blocks such that all odd elements belong to the blocks with odd index, whereas all even elements are placed in blocks with even index.
Similarly, for the homological ideals, Corollary 1 induces the following triangle:

\[
\text{Homological Triangle} \quad \text{HIT}
\]

|   | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|-----|
| 3 |  7| - | - | - | - | - | - |     |
| 4 | 14| 13| - | - | - | - | - |     |
| 5 | 28| 26| 25| - | - | - | - |     |
| 6 | 56| 52| 50| 49| - | - | - |     |
| 7 | 112| 104| 100| 98| 97| - | - |     |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |

The elements of the homological triangle are closely related to the research of categorification of the integer sequences. In particular, these numbers deal with the work by Fahr and Ringel regarding the categorification of Fibonacci numbers. In Subsection 3.2, we reconstruct the partition formula for even-index Fibonacci numbers given in [3, 5] by using the number of homological ideals of some Nakayama algebras.

### 3.2. Categorification of Integer Sequences

In this subsection, we present some relationships between the number of homological ideals of an algebra \( A_{R(t,j,k)} \) and the partition formula given by Fahr and Ringel for even-index Fibonacci numbers in [3].

According to Fahr and Ringel [4], the categorification of a sequence of numbers means to consider, instead of these numbers, suitable objects in a category (e.g., representation of quivers) such that the analyzed numbers occur as invariants of the objects, the equality of numbers can be visualized by the isomorphisms of objects by functorial relations by functorial ties. The notion of categorification of this kind arose from the use of suitable arrays of numbers in order to obtain integer partitions of dimensions of the indecomposable preprojective modules over the 3-Kronecker algebra (see Fig. 3, where we show the 3-Kronecker quiver and a piece of the oriented 3-regular tree or universal covering \((T, E, \Omega_i)\), as described by Fahr and Ringel in [3]). First, they indicated that the vector dimension of these kind of modules consists of even-index Fibonacci numbers (denoted by \( f_i \) and such that \( f_i = f_{i-1} + f_{i-2} \) for \( i \geq 2 \), \( f_0 = 0 \), \( f_1 = 1 \)). Then they used the results from the universal covering theory developed by Gabriel and his students to identify these Fibonacci numbers with the dimensions of representations of the corresponding universal covering.

First of all, we note that the road to categorification of the Fibonacci numbers has several stops, some of them deal with the diagonal (bottom) arrays of numbers of the form \( D = (d_{i,j}) \) with \( 0 \leq j \leq i \leq n \) (the columns are...
\begin{align*}
\cdots & \quad a_i[0] \\
\vdots & \\
P_1 & \quad 1 \quad f_2 \quad 0 \\
P_2 & \quad 1 \quad f_4 \quad 1 \\
P_3 & \quad 1 \quad f_6 \quad 2 \\
P_4 & \quad 1 \quad f_8 \quad 3 \\
P_5 & \quad 1 \quad f_{10} \quad 4 \\
P_6 & \quad 1 \quad f_{12} \quad 5 \\
P_7 & \quad 1 \quad f_{14} \quad 6 \\
\vdots & \quad \vdots \quad \vdots \quad \vdots 
\end{align*}

**Fig. 4.** Even-index Fibonacci partition triangle \cite{5}.

numbered from the right to the left; see Fig. 4) for some fixed \( n \geq 0 \) such that

\begin{equation}
\begin{aligned}
d_{i,i} &= 1 \quad \text{for all} \quad i \geq 0, \\
d_{i,j} &= 0 \quad \text{for all} \quad j > i, \\
d_{2k+i,i-1} &= 0 \quad \text{for} \quad i \geq 1, \quad k \geq 0, \\
d_{2k,0} &= 3d_{2k-1,1} - d_{2(k-1),0} \quad \text{for} \quad k \geq 1, \\
d_{i+1,j-1} &= 2d_{i,j} + d_{i,j-2} - d_{i-1,j-1} \quad \text{for} \quad i, j \geq 2.
\end{aligned}
\end{equation}

In addition, if \( i \geq 4 \), then the following identity (hook rule) holds:

\[
\sum_{k=0}^{i-2} d_{i+k,i-k} + d_{2i-2,0} = d_{2i-1,1}.
\]

Note that, each entry \( d_{i,i-j} \) can be associated with a weight \( w_{i,i-j} \) by using the numbers from the homological triangle \( \mathbb{H} \) as follows:

\[
w_{i,i-j} = \begin{cases} 
\frac{[\mathbb{H}]^k_{2s+2} - 2^{2s-k+1}}{[\mathbb{H}]^k_{2s+1}} & \text{if} \quad j \text{ is even,} \quad i \text{ is odd, and} \quad i \neq j + 1, \\
3 & \text{if} \quad i \text{ is odd,} \quad j \text{ is even, and} \quad i = j + 1, \\
1 & \text{if} \quad i = j = 2h \text{ for some} \quad h \geq 0, \\
0 & \text{if} \quad j \text{ is odd and} \quad i \neq j,
\end{cases}
\]

where \( s = \left\lfloor \frac{i-j}{2} \right\rfloor \) and \( \lfloor x \rfloor \) is the greatest integer number smaller than \( x \). If we consider the multiplication
of the entry \(d_{i,i-j}\) with its corresponding weight \(w_{i,i-j}\), then we can define a partition formula for even-index Fibonacci numbers in the following form:

\[
f_{2t+2} = \sum_{j=0}^{2t} (w_{i,i-j})(d_{i,i-j}).
\]

Finally, we recall that Fahr and Ringel interpreted the weights \(w_{i,i-j}\) as distances in a 3-regular tree \((T,E)\) (where \(T\) is a vertex set and \(E\) a set of edges) from a fixed point \(x_0 \in T\) to any point \(y \in T\). They define the sets \(T_r\) whose points are located at a distance \(r\) from \(x_0\). In this case, \(T_0 = \{x_0\}\), \(T_1\) are the neighbors of \(x_0\), and so on (note that \(|T_r| = 3(2^r-1)\) for \(r \geq 1\)). A given vertex \(y\) is said to be even or odd according to this parity [3].

Any vertex \(y \in T\) yields a suitable reflection \(\sigma_y\) on the set of functions \(T \to \mathbb{Z}\) with finite support denoted by \(\mathbb{Z}[T]\). Some reflection products denoted by \(\Phi_0\) and \(\Phi_1\) according to the parity of \(y\) were introduced in [3]. Then some maps \(a_t : \mathbb{N}_0 \to \mathbb{Z} \subseteq \mathbb{Z}[T]\) are defined so that if \(a_0\) is the characteristic function of \(T_0\), then

\[
a_0(x) = 0,
\]

unless \(x = x_0\) (in this case, \(a_0(x_0) = 1\), and

\[
a_t = (\Phi_0\Phi_1)^t a_0 \quad \text{for} \quad t \geq 0
\]

with \(a_t[r] = a_t(x)\) for \(r \in \mathbb{N}_0\) and \(x \in T_r\); these maps \(a_t\) give the values \(d_{i,j}\) of the array (see Fig. 4). The following table is an example of an array of this kind for \(n = 7\). The rows are given by the values of \(t\) and \(P_t\) denotes a 3-Kronecker preprojective module with dimension vector \([f_{2t+2} f_{2t}]\) (see [5]).

According to our discussion, identity (15) takes one of the following forms specified by Fahr and Ringel in [3]:

\[
f_{4t} = \sum_{r \text{ odd}} |T_r|a_t[r] = 3 \sum_{m \geq 1} 2^{2m} \cdot a_t[2m+1],
\]

\[
f_{4t+2} = \sum_{r \text{ even}} |T_r|a_t[r] = a_t[0] + 3 \sum_{m \geq 1} 2^{2m-1} \cdot a_t[2m].
\]

Thus, for \(t = 3\) and \(t = 4\), we compute \(f_8\) and \(f_{10}\) as follows:

\[
21 = f_8 = 0 + 3(3 \cdot 2^0) + 0 + 1(3 \cdot 2^2),
\]

\[
55 = f_{10} = 1 \cdot 7 + 0 + 4(3 \cdot 2^1) + 0 + 1(3 \cdot 2^3).
\]

The sequences \(a_t[0] = d_{2t,0}\) and \(a_t[1] = d_{2t+1,1}\) are encoded as A132262 and A110122 in the OEIS, respectively. Actually, the sequence \(a_t[0]\) was not recorded in the OEIS prior to the publication of Fahr and Ringel.

The following result gives a relationship between the number of homological ideals and Fibonacci numbers. It is obtained as a direct consequence of identities (15) and (16).

\textbf{Theorem 4.}

\[
\sum_{j=0}^{2t} (w_{2t,2t-j})(d_{2t,2t-j}) = \sum_{r \text{ even}} |T_r|a_t[r], \quad t \geq 0,
\]

\[
\sum_{j=0}^{2t-1} (w_{2t-1,2t-1-j})(d_{2t-1,2t-1-j}) = \sum_{r \text{ odd}} |T_r|a_t[r], \quad t \geq 1.
\]
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