Lattice duality for the compact Kardar-Parisi-Zhang equation

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A comprehensive theory of the Kosterlitz-Thouless transition in two-dimensional superfluids in thermal equilibrium can be developed within a dual representation which maps vortices in the superfluid to charges in a Coulomb gas. In this framework, the dissociation of vortex-antivortex pairs at the critical temperature corresponds to the formation of a plasma of free charges. The physics of vortex unbinding in driven-dissipative systems such as fluids of light, on the other hand, is much less understood. Here we make a crucial step to fill this gap by deriving a transformation that maps the compact Kardar-Parisi-Zhang (KPZ) equation, which describes the dynamics of the phase of a driven-dissipative condensate, to a dual electrodynamic theory. The latter is formulated in terms of modified Maxwell equations for the electromagnetic fields and a diffusion equation for the charges representing vortices in the KPZ equation. This mapping utilizes an adaption of the Villain approximation to a generalized Martin-Siggia-Rose functional integral representation of the compact KPZ equation on a lattice.

I. INTRODUCTION

The Kardar-Parisi-Zhang (KPZ) equation [1] represents a paradigm of non-equilibrium statistical mechanics, describing universal scaling behavior in a rich variety of physical systems. To name just a few examples, the range of its applications includes the growth of bacterial colonies [2-4], fluid flow in porous media [5], combustion of paper [6-8], and turbulent liquid crystals [9, 10]. Recently, it has been noted that the KPZ equation emerges also in the context of condensation phenomena out of thermodynamic equilibrium, in systems such as exciton-polaritons [11, 12]. The latter are bosonic quasiparticles, formed via hybridization of photons in a semiconductor microcavity and excitons confined in a two-dimensional quantum well, and have a finite lifetime due to the leakage of the light field out of the cavity. Compensating these losses by continuously injecting energy in the form of laser light into the system drives it into a non-equilibrium steady state that exhibits signatures of Bose-Einstein condensation of exciton-polaritons [11, 12].

Fluctuations of the phase of such a condensate obey the KPZ equation [13-17]. The fundamental difference to the above-mentioned cases of KPZ dynamics is that the phase of the condensate is a compact variable and may thus contain topological defects, i.e., vortices. In fact, driven-dissipative condensates are by far not the only instance of compact KPZ dynamics: further examples include driven vortex lattices in disordered superconductors [18], active smectics [19], and the phase dynamics of other systems obeying the complex Ginzburg-Landau equation (CGLE) with noise [20]; moreover, a KPZ-type non-linearity occurs also in sliding charge-density waves [21, 22] and arrays of coupled limit-cycle oscillators [23]. This raises the question, how topological defects can be incorporated systematically in the compact KPZ equation, and calls for a formulation of the problem that treats these defects explicitly as fundamental degrees of freedom of the system.

In thermal equilibrium, such a description of the physics in terms of topological defects can be obtained by performing a duality transformation which maps the partition function from a functional integral over a compact field to one that is taken over configurations of defects. For systems defined on a lattice, the dual description can be derived systematically using the Villain approximation [24]. A case in point is the duality transformation for the classical XY-model, which can be mapped to a Coulomb gas with charges representing vortices [25-29].

The topological nature of vortices is reflected in the quantization of the charges to integer values. This vortex-charge duality has been extended to a comprehensive theory describing also the dissipative dynamics of superfluid films at finite temperature [30-32], and to a full quantum electrodynamics theory at zero temperature [33]. Moreover, the duality transformation for Abelian (and non-Abelian) lattice systems has been put on a systematic footing by making use of Bianchi identities [34].

In this paper, we derive a systematic lattice duality transformation for compact KPZ dynamics. This derivation complements the heuristic derivation of a dual electrodynamic theory in the continuum, which we introduced in Ref. [35]. There, the dual theory served as the basis for a detailed RG analysis of vortex unbinding in the non-equilibrium steady state.

The rest of the presentation is organized as follows: in Sec. II, we present the derivation of the duality transformation. In particular, we derive the Martin-Siggia-Rose (MSR) action for the compact KPZ equation in Sec. II A, and describe the appropriate form of the Villain approximation in Sec. II B. There we also discuss how charges (vortices) can be introduced as independent degrees of freedom by means of the Poisson summation formula. In Sec. III, we show that the MSR functional integral resulting from the duality transformation can be reduced to the dual electrodynamics theory of Ref. [35], which is
formulated in terms of Langevin equations for the electromagnetic fields and the charges. We finish by giving in Sec. IV an outlook on possible applications of the formalism developed in this paper. Technical details of the Villain transformation are deferred to the Appendix.

II. DUALITY TRANSFORMATION ON A LATTICE

In this section we present our main result, which is the derivation of the lattice duality for the compact KPZ equation. As indicated in the introduction, in thermal equilibrium, the duality transformation can be performed conveniently for systems which are defined on a lattice.

The key step is then to replace the periodic potential of the compact field in the (functional integral representation of the) partition function by a simplified Villain form [24, 27, 28].

Thus, to obtain the appropriate generalization of the Villain approximation to the case of the compact KPZ equation, we first by formulating the latter for a lattice system. Then, in Sec. II.A, we develop a modification of the Martin-Siggia-Rose (MSR) functional integral for the lattice compact KPZ equation. This modification of the usual MSR functional integral [36–38] is necessary in order to properly account for the compactness of the phase field. For a classical system out of thermal equilibrium, the MSR functional integral is the natural counterpart to the partition function. Hence, it provides the appropriate framework for performing a generalized Villain approximation in Sec. II.B.

A. MSR action for the compact KPZ equation

Our starting point is the compact KPZ equation for the phase $\theta$ of a two-dimensional driven-dissipative condensate [13–17]. In spatial continuum, the compact KPZ equation takes the well-known form

$$\partial_t \theta = D\nabla^2 \theta + \frac{\lambda}{2} (\nabla \theta)^2 + \eta,$$  \hspace{1cm} (1)

where — in contrast to the usual KPZ equation [1] — $\theta$ is defined on a circle, i.e., it takes values in the interval $\theta \in [0, 2\pi)$. $\eta$ is a Gaussian noise source with zero mean and white spectrum,

$$\langle \eta(t, x) \eta(t', x') \rangle = 2\Delta \delta(t - t') \delta(x - x').$$  \hspace{1cm} (2)

A systematic way to derive the KPZ equation on a lattice is to start from the CGLE on a lattice (where spatial derivatives are replaced by standard hopping terms) and integrate out density fluctuations as in the continuum case [13–17]. This amounts to replacing spatial derivatives in the KPZ equation (1) with finite differences according to

$$\nabla^2 \theta \rightarrow -\sum_\hat{a} \sin(\theta_x - \theta_{x+\hat{a}}),$$

$$\frac{1}{2} (\nabla \theta)^2 \rightarrow -\sum_\hat{a} \left( \cos(\theta_x - \theta_{x+\hat{a}}) - 1 \right),$$  \hspace{1cm} (3)

where $x + \hat{a}$ are the nearest neighbors of the lattice site $x$, i.e., the sums are over $\hat{a} \in \{\pm x, \pm y\}$ (for convenience we choose the lattice spacing as $a = 1$). Accordingly, the compact KPZ equation reads

$$\partial_t \theta_x = -\sum_\hat{a} \left[ D \sin(\theta_x - \theta_{x+\hat{a}}) + \frac{\lambda}{2} \left( \cos(\theta_x - \theta_{x+\hat{a}}) - 1 \right) \right] + \eta_x,$$  \hspace{1cm} (4)

For $\lambda = 0$, this equation reduces to the form of the two-dimensional classical XY-model with relaxational dynamics. Then, the Langevin equation can be written as

$$\partial_t \theta_x = -\Gamma \delta \mathcal{H}_{XY} / \partial \theta_x + \eta_x,$$  \hspace{1cm} (5)

where $\mathcal{H}_{XY}$ is the XY-Hamiltonian reads (the sum is over pairs of neighboring sites)

Consequently, $D = \Gamma K$ in Eq. (4) is the product of the diffusion constant $\Gamma$ and the spin-stiffness $K$. The stationary state of the Fokker-Planck equation corresponding to relaxational Langevin dynamics is the thermal Gibbs ensemble with distribution function $\mathcal{P}_{\text{Gibbs}} \propto \exp(-\mathcal{H}_{XY} / T)$ at temperature $T = \Delta / \Gamma$. This is the starting point for deriving the dual Coulomb gas representation of the equilibrium XY-model [25–29]. However, in the KPZ problem, the closed form of the stationary distribution is not known in more than one dimension, and therefore we derive the dual representation in terms of the dynamical MSR functional instead.

Hence, the next step is to rewrite the Langevin equation (4) in the form of an equivalent MSR action. We slightly modify the usual approach [36–38] in order to account for the compactness of the phase: as customary, we discretize the stochastic process described by Eq. (4) in time, i.e., we replace the continuous function of time $\theta_x(t)$ by a sequence $\theta_{t,x}$ corresponding to specific points in time $t$ which are multiples of the temporal lattice spacing $\epsilon$. Since each $\theta_{t,x}$ is the phase of a complex number $\psi_{t,x}$, the discrete stochastic process has to be invariant under shifts $\theta_{t,x} \rightarrow \theta_{t,x} + 2\pi n_{t,x}$ for integer-valued $n_{t,x}$ (this is a gauge symmetry in the most general sense of a redundancy of the description that is inherent to the polar representation of the complex number $\psi_{t,x}$). This property is ensured if we write the update of $\theta_{t,x}$ from time $t$ to $t + \epsilon$ in the following way:

$$\theta_{t+\epsilon,x} = \theta_{t,x} + \epsilon (\mathcal{L}[\theta]_{t,x} + \eta_{t,x}) + 2\pi n_{t,x},$$  \hspace{1cm} (6)

where by $\mathcal{L}[\theta]$ we denote the deterministic part of the
compact KPZ equation (4), i.e.,
\[
\mathcal{L}[\theta]_{t,x} = -\sum_{\hat{a}} \left[ D \sin(\theta_{t,x} - \theta_{t,x+\hat{a}}) + \frac{\lambda}{2} (\cos(\theta_{t,x} - \theta_{t,x+\hat{a}}) - 1) \right]. \tag{7}
\]

In Eq. (6), 2\(\pi n_{t,x}\) is the unique multiple of 2\(\pi\) that has to be added to \(\theta_{t,x} + \epsilon (\mathcal{L}[\theta]_{t,x} + \eta_{t,x})\) so that the sum is in the interval from 0 to 2\(\pi\). Thus, Eq. (6) defines a stochastic process for which \(\theta_{t,x}\) remains within this interval at all times. Note that because \(n_{t,x}\) is integer-valued, the straightforward continuum limit \(\epsilon \to 0\) of this stochastic process is ill-defined. However, below we perform a sequence of manipulations leading eventually to a form which allows us to take this limit.

In the following we use the symbol \(\Delta_t\) to denote discrete derivatives with respect to time, i.e., we write \(\Delta_t \theta_{t,x} = \theta_{t+1,x} - \theta_{t,x} \approx \theta_{t+1,x} - \theta_{t-1,x}\); the form after the second equality appears below when we perform summations by parts and we do not make a distinction between the two forms of the discrete derivative as they are equivalent in the limit \(\epsilon \to 0\).

We proceed with the construction of the MSR functional integral for Eq. (6) in the usual way [36–38]. The solution of the stochastic process Eq. (6) for a given realization of the noise is denoted by \(\theta_{t,n}\), and an arbitrary observable, which is a functional of \(\theta_t\), by \(O[\theta]\). Calculating the expectation value of \(O[\theta]\) requires us to take the average of \(O[\theta_{t,n}]\) over different noise realizations, weighted by the Gaussian distribution function
\[
\mathcal{P}[\eta] \propto e^{-\frac{1}{4\pi} \sum_{t,x} \eta_{t,x}^2}.
\]

To be explicit, in the usual derivation of the MSR functional this average is written in the following way:
\[
\langle O[\theta] \rangle = \int \mathcal{D}[\theta] \mathcal{P}[\eta] O[\theta_{t,n}] = \int \mathcal{D}[\theta, \eta] \mathcal{P}[\eta] O[\theta - \eta_{t,n}], \tag{9}
\]
where
\[
\int \mathcal{D}[\eta] = \prod_{t,x} \int_{-\infty}^{\infty} d\eta_{t,x}, \quad \prod_{t,x} \int_{0}^{2\pi} d\theta_{t,x}. \tag{10}
\]

In the second equality in Eq. (9), we introduced an additional integration over \(\theta\), which is fixed to \(\eta_{t,n}\) by the \(\delta\)-functional. The latter can be expressed as the product over \(\delta\)-functions for the values of \(\theta_{t,n}\) at specific points \((t,x)\) on the space-time lattice:
\[
\delta[\theta - \eta_{t,n}] = \prod_{t,x} \delta(\theta_{t,x} - \eta_{t,x}) = \prod_{t,x,\eta_{t,n}} \delta(\Delta_t \theta_{t,x} - \epsilon (\mathcal{L}[\theta]_{t,x} + \eta_{t,x}) + 2\pi n_{t,x}). \tag{11}
\]

In the last equality, we used that the Jacobian of the operator \(\Delta_t \theta_{t,x} - \epsilon \mathcal{L}[\theta]_{t,x}\) for the retarded regularization chosen in Eq. (6) is equal to one. As pointed out above, for a given value of \(\theta_{t,x}\), the value of \(n_{t,x}\) is uniquely specified. In other words, for a given value of \(\theta_{t,x}\) there is a unique combination of \(\theta_{t+\delta \epsilon,x} \in [0, 2\pi)\) and \(n_{t,x} \in \mathbb{Z}\) for which the argument of the \(\delta\)-function becomes zero. Hence, taking the sum over \(n_{t,x} \in \mathbb{Z}\) in Eq. (11) does not correspond to an additional averaging.

Evaluating Eq. (11) further, instead of with a single \(\delta\)-functional as in the usual MSR construction, we have to deal with a train of \(\delta\)-functions, which can in turn be rewritten as a Fourier sum instead of a Fourier integral as in the usual case [36–38]:
\[
\sum_{n_{t,x} = -\infty}^{\infty} \delta(\Delta_t \theta_{t,x} - \epsilon (\mathcal{L}[\theta]_{t,x} + \eta_{t,x}) + 2\pi n_{t,x}) = \frac{1}{2\pi} \sum_{n_{t,x} = -\infty}^{\infty} e^{-i\pi n_{t,x} \lambda}.
\]

(12)

In consequence, the “response field” \(\tilde{n}\) is integer-valued and not continuous as in the non-compact case. The next step in the construction of the MSR functional integral is to insert Eq. (12) in Eq. (11) and perform the integration over the noise field \(\eta\) which leads to
\[
\langle O[\theta] \rangle \propto \sum_{\tilde{n}_{t,x}} \int \mathcal{D}[\theta] \mathcal{P}[\eta] e^{iS}. \tag{13}
\]

The exponent in this expression defines the MSR action,
\[
S = \sum_{t,x} \tilde{n}_{t,x} [(-\Delta_t \theta_{t,x} + \epsilon (\mathcal{L}[\theta]_{t,x} + i\Delta_n_{t,x})], \tag{14}
\]
and the MSR functional integral is thus given by
\[
Z = \sum_{\tilde{n}_{t,x}} \int \mathcal{D}[\theta] e^{iS}. \tag{15}
\]

Note that because \(\tilde{n}_{t,x}\) is integer-valued, the weight \(e^{iS}\) in the MSR functional integral is indeed invariant under the above-mentioned gauge transformation \(\theta_{t,x} \rightarrow \theta_{t,x} + 2\pi n_{t,x}\) as it should be. This is formally similar to the Matsubara functional integral description of Josephson junction array models [33, 39, 40], where \(\tilde{n}\) corresponds to the charge, and the invariance under shifts of the phase by multiples of 2\(\pi\) is guaranteed by the discreteness of the latter.

Finally, we rewrite the action in a form that is more convenient for performing the duality transformation below. It is straightforward to verify the equality
\[
\epsilon \sum_{t,x} \tilde{n}_{t,x} \mathcal{L}[\theta]_{t,x} = -\epsilon \sum_{t,x,i} \left[ D \left( \tilde{n}_{t,x} - \tilde{n}_{t,x+\epsilon_i} \right) \sin(\theta_{t,x} - \theta_{t,x+\epsilon_i}) + \frac{\lambda}{2} \left( \tilde{n}_{t,x} + \tilde{n}_{t,x+\epsilon_i} \right) \cos(\theta_{t,x} - \theta_{t,x+\epsilon_i}) - 1 \right], \tag{16}
\]
where \( i = x, y \), and \( \mathbf{e}_i \) denote the unit vectors in the respective directions. Ultimately, we are interested in the continuum limit both with respect to time and space. Therefore, in the term \((\tilde{n}_{t,x} + \tilde{n}_{t,x+\mathbf{e}_x})\cos(\theta_{t,x} - \theta_{t,x+\mathbf{e}_x})\) we can replace \( \tilde{n}_{t,x+\mathbf{e}_x} \) by \( \tilde{n}_{t,x} \), since the difference vanishes in this limit. Then, to leading order in \( \epsilon \), we have the following equality (to make the notation more compact, in the following we denote the lattice derivative by \( \Delta_i \theta_i \approx \theta_{t,x} + \epsilon \lambda \tilde{n}_{t,x} \)):

\[
\epsilon \sum_{\mathbf{t},x} \tilde{n}_{\mathbf{t},x} \mathcal{L}[\theta]_{\mathbf{t},x} = - \sum_{\mathbf{t},x,i} \sin(\Delta_i \theta_{t,x}) \sin(\epsilon D \Delta_i \tilde{n}_{t,x})
\]

\[
+ \epsilon \lambda \tilde{n}_{t.x} (\cos(\Delta_i \theta_{t,x}) \cos(\epsilon D \Delta_i \tilde{n}_{t,x}) - 1) + O(\epsilon^3),
\]

which can be seen to reduce to Eq. (16) straightforwardly by expanding the trigonometric functions in powers of \( \epsilon \). As a last step, we introduce an additional summation over \( \sigma = \pm 1 \) which allows us to write the RHS of Eq. (17) in the form

\[
\epsilon \sum_{\mathbf{t},x} \tilde{n}_{\mathbf{t},x} \mathcal{L}[\theta]_{\mathbf{t},x} = \frac{1}{2} \sum_{\mathbf{t},x,i} \sum_{\sigma} (\sigma - \epsilon \lambda \tilde{n}_{t,x})
\]

\[
\times (\cos(\Delta_i (\theta_{t,x} + \sigma \epsilon D \tilde{n}_{t,x}))) - 1) + O(\epsilon^3). \tag{18}
\]

Omitting higher order corrections in \( \epsilon \), and using a compact notation in which \( X = (t, x) \), the MSR action (14) can be written as

\[
S = \sum_X \left[ \tilde{n}_X (-\Delta_i \theta_X + i \epsilon \Delta \tilde{n}_X) \right] + \frac{1}{2} \sum_{\sigma} (\sigma - \epsilon \lambda \tilde{n}_X) \right)
\]

\[
= \frac{1}{2} \sum_{\mathbf{t},x} \sum_{\sigma} (\sigma - \epsilon \lambda \tilde{n}_X) \right) - 1) \right]. \tag{19}
\]

Note that the action obeys causality in the sense of Keldysh field theory \[36, 37\], i.e., \( S = 0 \) for \( \tilde{n} = 0 \), which is an essential property of Keldysh and MSR functional integrals \[36-38\]. For the dynamical and noise terms (the first line on the RHS of Eq. (19)), as well as for the term that is proportional to \( \lambda \), this is evident: these terms are of linear or higher order in \( \tilde{n} \); the remaining part of the action resembles a Hamiltonian contribution to a Keldysh action, i.e., a term of the form \( \sum_{\sigma} \sigma \mathcal{H}[\theta] \). In the present case,

\[
\mathcal{H}[\theta] = \frac{1}{2} \sum_{X,i} \cos(\Delta_i (\theta_{X} + \sigma \epsilon D \tilde{n}_X)) - 1, \tag{20}
\]

where \( \theta_{X} = \theta_X + \sigma \epsilon D \tilde{n}_X \) are to some extent analogous to fields on the forward and backward branches of the Keldysh contour. Then, for \( \tilde{n} = 0 \), it follows that \( \theta_{\pm} = \theta_{\pm} \), and the Hamiltonian part of the action vanishes due to the summation over \( \sigma \).

**B. Duality transformation**

The form of the action (19) is the starting point for performing the duality transformation. To this end we write the MSR partition function (15) in the form

\[
Z = \sum_{\{\tilde{n}_X\}} \int D[\theta] e^{\sum_X \tilde{n}_X (-\Delta_0 \theta_X + i \epsilon D \tilde{n}_X)}
\]

\[
\times \prod_{X,i,\sigma} e^{iK_{\sigma,X}(\cos(\Delta_i (\theta_X + \sigma \epsilon D \tilde{n}_X)) - 1)}, \tag{21}
\]

where the prefactor in the exponent is

\[
K_{\sigma,X} = \frac{1}{2} (\sigma - \epsilon \lambda \tilde{n}_X). \tag{22}
\]

The exponential in the second line in Eq. (21) is a periodic function of the variable \( \Delta_i (\theta_X + \sigma \epsilon D \tilde{n}_X) \) which is defined in terms of the values of \( \theta \) and \( \tilde{n} \) on the two sites \( (t, x) \) and \( (t, x + \mathbf{e}_x) \) of the spatio-temporal lattice. Hence, we can expand this exponential as a Fourier series in which the sum is taken over a new variable \( j_{\sigma,X} \) associated with the link connecting these lattice sites,

\[
e^{iK_{\sigma,X}(\cos(\Delta_i (\theta_X + \sigma \epsilon D \tilde{n}_X)) - 1))
\]

\[
= \sum_{j_{\sigma,X}} e^{-i j_{\sigma,X} \Delta_i (\theta_X + \sigma \epsilon D \tilde{n}_X)} + iV_{\sigma,X}(j_{\sigma,X}), \tag{23}
\]

Our choice of the sign \( -\sigma \) in the first term in the exponent on the RHS ensures causality as explained below. The presence of the prefactor \( K_{\sigma,X} \) in the exponent on the LHS implies that also the coefficients in the Fourier series, which we write in the form \( e^{iV_{\sigma,X}(j_{\sigma,X})} \), depend on \( \sigma \) and \( X \), as is indicated by the subscript in the potential \( V_{\sigma,X} \).

In the Villain approximation (more details are provided in App. A below), the latter assumes the form

\[
e^{iV_{\sigma,X}(j_{\sigma,X})} \approx \frac{1}{\sqrt{2 \pi K_{\sigma,X}}} e^{j_{\sigma,X}^2/(2K_{\sigma,X})}. \tag{24}
\]

This leads to the following contribution in the functional integral in Eq. (21) (here we use the shorthand \( j_{\sigma,X}^2 = [j_{\sigma,X}^x]^2 + j_{\sigma,y,X}^2 \)):

\[
\prod_{X,i,\sigma} e^{iK_{\sigma,X}(\cos(\Delta_i (\theta_X + \sigma \epsilon D \tilde{n}_X)) - 1))
\]

\[
= C \sum_{\{j_{\sigma,X}\}} e^{i \sum_X [-\sigma j_{\sigma,X} \nabla(\theta_X + \sigma \epsilon D \tilde{n}_X) + j_{\sigma,X}^2/(2K_{\sigma,X})]}, \tag{25}
\]

where in the second line we denote \( \nabla = (\Delta_x, \Delta_y) \). The prefactor is (in 2D the product over directions gives a square)

\[
C = \prod_{X,i,\sigma} \frac{1}{\sqrt{2 \pi K_{\sigma,X}}} = \prod_{X,\sigma} \frac{1}{2 \pi K_{\sigma,X}}. \tag{26}
\]

\( ^1 \) According to Eq. (12) also \( \tilde{n} \) is defined on links connecting sites \( (t, x) \) and \( (t + \epsilon, x) \) which are separated in the direction of time. This can be seen by noting that both \( \Delta_0 \theta_{X} \) and the increment \( \epsilon (E[\theta]_{t,x} + \eta_{t,x}) \) are defined on such links.
We note that in terms of the newly introduced variables $J_{\sigma X}$ the action is still causal in the above-mentioned sense, i.e., we have $S = 0$ for $\tilde{n} = 0$ and $J_{+} = J_{-}$. To see this, first note that for $\tilde{n} = 0$ in Eq. (22) we have $K_{\sigma X} = \sigma/2$. Then, in the sum in the exponent in Eq. (25) there is an overall prefactor $\sigma$, and therefore the sum vanishes for $\tilde{j}_{+X}^{2} = \tilde{j}_{-X}^{2}$.

As indicated above, eventually we are interested in the continuum limit in time, $\epsilon \to 0$. Hence, in the following we keep only the leading terms in $\epsilon$. This allows us to considerably simplify the factor $C$ defined in Eq. (26). Indeed, reexponentiating $K_{\sigma X}$ and expanding the logarithm to second order we find (in the following we do not keep track of purely numerical, i.e., field-independent, factors; they are inconsequential for our considerations and can be absorbed in the integration measure in the MSR functional integral Eq. (21))

$$C \propto \prod_{X,\sigma} e^{-\ln K_{\sigma X}} = \prod_{X,\sigma} \frac{2\sigma e^{-\sigma \epsilon \tilde{n}_{X}^{2}}}{2 \sigma_{0}} \propto e^{-\epsilon \lambda^{2} \sum_{X} \tilde{n}_{X}^{2} + O(\epsilon^{3})},$$

(27)

The first order term in the exponent vanishes upon taking the product of the exponential over $\sigma$ (or, equivalently, the sum over $\sigma$ in the exponent). As a result, $C$ gives a contribution to the MSR action of second order in $\epsilon$ and can hence be ignored.

Now we examine the last term in the exponent in Eq. (25). Expanding $1/K_{\sigma X}$ to first order in $\epsilon$ we find

$$\frac{i}{2} \sum_{X, \sigma} J_{\sigma X} J_{\sigma X} = i \sum_{X} \left[ J_{X} \cdot \tilde{j}_{X} + \frac{\epsilon \lambda}{2} \tilde{n}_{X} \left( J_{X}^{2} + \tilde{j}_{X}^{2} \right) \right],$$

(28)

where we introduce the current $j = j_{+} + j_{-}$ and the response current $\tilde{j} = j_{+} - j_{-}$. Below in Eq. (43) we replace the integer-valued vector fields $j$ and $\tilde{j}$ by continuous ones by means of the Poisson summation formula. This is a necessary prerequisite for taking the limit $\epsilon \to 0$, for the simple reason that a sequence of integers at times $t, t + \epsilon, t + 2 \epsilon, \ldots$ cannot converge to a continuous function of time for $\epsilon \to 0$. Moreover, taking a sensible $\epsilon \to 0$ limit requires us to rescale the real-valued response current as $\tilde{j} \to \epsilon \tilde{j}$, as becomes clear from Eq. (30) below. Anticipating these steps, we see that the contribution involving $\tilde{j}_{X}^{2}$ in Eq. (28) above can actually be considered to be $O(\epsilon^{2})$, and we discard it already at this point.

Putting the pieces together, the partition function (21) becomes

$$Z \propto \sum_{\tilde{n}, J \cdot j, \tilde{j} \cdot j} \int D[\theta] e^{i \sum_{X} \theta_{X} \left( \Delta_{\xi} \tilde{n}_{X} + \nabla \cdot \tilde{j}_{X} \right)}$$

$$\times e^{i \sum_{X} \left[ \tilde{n}_{X} (D \nabla \cdot j_{X} + i \Delta_{\xi} \tilde{n}_{X}) + j_{X} \cdot \tilde{j}_{X} + \phi_{\xi} \tilde{n}_{X} j_{X} \right]}.$$  

(29)

In the exponent we summed by parts twice. We note that the exponent is linear both in $\theta$ and $\tilde{j}$. Hence, the sum over $J$ can be carried out and gives $\prod_{X} \sum_{m_{X}} \delta(-\nabla \theta_{X} + j_{X} - 2\pi m_{X})$ with an integer-valued vector field $m$. Some intuition can be gained by decomposing this vector field into longitudinal and transverse parts. The longitudinal part can be written as the lattice gradient of an integer field $m_{t}$, and the transverse part as the lattice curl of a vector field $m_{x} \hat{z}$ pointing along the $z$-direction. Absorbing the longitudinal component $m_{t}$ into $\theta$ by extending the integration in Eq. (10) over the whole real axis (in other words, making $\theta$ non-compact) the argument of the $\delta$-function suggests that (up to a prefactor) we can interpret $j$ as the bosonic current. The latter has both a smooth longitudinal contribution $\nabla \theta$ and — in the presence of vortices — a transverse component corresponding to non-zero values of $m_{t}$.

However, instead of summing over $\tilde{j}$ in Eq. (29), we take a different route and integrate out $\theta$. This yields a $\delta$-function corresponding to the constraint that $\tilde{n}_{X}$ and $\tilde{j}_{X}$ should satisfy the continuity equation,

$$\Delta_{\xi} \tilde{n}_{X} + \nabla \cdot \tilde{j}_{X} = 0.$$  

(30)

(Note that as indicated above this equation implies that the continuous fields which replace $\tilde{n}$ and $\tilde{j}$ scale differently in the limit $\epsilon \to 0$.) Formally, the appearance of a continuity equation is again analogous to the duality transformation in a quantum system at $T = 0$ [33, 39, 40]. However, in the latter case, the continuity equation is a consequence of particle number conservation. In driven-dissipative condensates, on the other hand, the number of particles is not conserved. Nevertheless, there is a residual $U(1)$ phase-rotation symmetry [17] which is reflected in the appearance of a Goldstone boson in the (mean-field) condensed phase (indeed, the KPZ equation (1) is the massless equation of motion of the Goldstone boson which is the phase of the condensate), and in the above continuity equation for the response fields $\tilde{n}$ and $\tilde{j}$. From this continuity equation, the Noether charge associated with $U(1)$ symmetry can be seen to be the sum over space of $\tilde{n}_{X}$. However, by construction of the MSR formalism, the response fields have vanishing expectation value, and therefore the Noether charge is always zero and the continuity equation (30) is trivially satisfied on average.

The continuity equation (30) can be interpreted as stating that the three-component vector field $(\tilde{n}, \tilde{j})$ has vanishing divergence. Hence, this vector field can be parametrized as the curl of another vector field,

$$\begin{pmatrix} \tilde{n} \\ \tilde{j} \end{pmatrix} = \begin{pmatrix} \Delta_{\xi} \\ -\phi \end{pmatrix} \times \begin{pmatrix} \hat{z} \\ -\hat{z} \times (\nabla \phi + \Delta_{\xi} \hat{A}) \end{pmatrix}.$$  

(31)

Here and in the following it is understood, that — depending on the context — the gradient operator and vectors such as $\hat{A}$ should be considered as having two components or three components with the third one being zero. The parametrization of $(\tilde{n}, \tilde{j})$ in terms of the potentials $(\phi, \hat{A})$ is not unique. In fact, the “physical” fields...
\((\tilde{n}, \tilde{j})\) are invariant under the gauge transformation
\[
\begin{align*}
\hat{\phi} & \to \hat{\phi} - \Delta \chi, \\
\hat{A} & \to \hat{A} + \nabla \chi,
\end{align*}
\] (32)
with an arbitrary integer field \(\chi\). We can exploit this freedom by choosing a gauge that leads to a simple form of the action. However, for the moment we leave the gauge unspecified. Then, summing over both \(\hat{\phi}\) and \(\hat{A}\) without restriction simply introduces a multiplicative overcounting in the partition function (29).

In the following, we find it instructive to parametrize also the current \(j\) in terms of gauge potentials — although, strictly speaking, this is not necessary. The parametrization is chosen in analogy to Eq. (31) for the response current, however, omitting the time derivative of the vector potential. Hence, we set
\[
j = -\hat{z} \times (\nabla \hat{\phi} + \hat{A}).
\] (33)
As above, this parametrization is not unique. Different, physically equivalent possibilities are related by gauge transformations, which read in the present case
\[
\begin{align*}
\hat{\phi} & \to \hat{\phi} - \chi, \\
\hat{A} & \to \hat{A} + \nabla \chi.
\end{align*}
\] (34)
The quantities \(\hat{\phi}\) and \(\hat{A}\) are the scalar and vector potentials of the dissipative electrodynamics introduced heuristically in Ref. [35] and described below in Sec. III. In terms of these potentials, the gauge-invariant electric and magnetic fields are defined as
\[
\begin{align*}
E &= -\hat{z} \cdot j = -\nabla \hat{\phi} - \hat{A}, \\
B &= D\nabla \times \hat{A}.
\end{align*}
\] (35)
Note that since both \(\nabla\) and \(\hat{A}\) have vanishing components in the \(z\)-direction, the only non-vanishing component of the magnetic field \(B\) is exactly along \(\hat{z}\), i.e., \(B = B\hat{z}\). This implies that the homogeneous Maxwell equation
\[
\nabla \cdot B = 0,
\] (36)
is trivially satisfied. Moreover, due to the absence of the usual time derivative acting on the vector potential \(\hat{A}\) in the definition of the electric field \(E\) in Eq. (35), the magnetic field can be expressed directly in terms of the electric field by means of a modified Faraday’s law [35],
\[
\nabla \times E + \frac{1}{D} B = 0.
\] (37)
According to the expression of the electric field in terms of the current in Eq. (35), we find \(\nabla \times E = - (\nabla \cdot j) \hat{z}\). Inserting this relation in Eq. (37) and keeping in mind that \(B = B\hat{z}\), we see that Faraday’s law is just the continuity equation, where the magnetic field encodes fluctuations of the bosonic density \(\rho\) around the mean value \(\rho_0\), i.e., \(B \propto - (\rho - \rho_0)\). The same identification is made in the dual electrodynamic theory for superfluid films in thermal equilibrium [30, 31].

A crucial difference is that in a driven-dissipative system without particle number conservation the continuity equation includes a source term \(\propto (\rho - \rho_0)\) [13] which dominates over the usual term \(\partial_t \rho\) in the low-frequency limit. It is precisely this limit (in which fluctuations of the phase of a driven-dissipative condensate are described by the KPZ equation) which we are considering here.

Inserting Eqs. (31), (33), and (35) in the partition function in Eq. (29), the latter becomes
\[
Z \propto \sum_{(\hat{\phi}, \hat{A}, \hat{\chi})} e^{iS_{EB}},
\] (38)
where the action is given by
\[
S_{EB} = \sum_X \left[ \hat{\phi}_X \nabla \cdot E_X + \hat{A}_X \cdot \left( \Delta \epsilon E_X - \epsilon \nabla \times B_X \\
+ \frac{\lambda}{2} \hat{z} \times \nabla E_X^2 \right) + i \epsilon \Delta \left( \nabla \times \hat{A}_X \right)^2 \right].
\] (39)
For completeness we mention that in analogy to Eq. (35), we could define response electric and magnetic fields, which are invariant under the gauge transformation given in Eq. (32), as
\[
\begin{align*}
\hat{E} &= -\nabla \hat{\phi} - \Delta \hat{A}, \\
\hat{B} &= \nabla \times \hat{A}.
\end{align*}
\] (40)
Then, the action (39) can be written in a manifestly gauge-invariant form as
\[
S_{EB} = \sum_X \left[ \hat{E}_X \cdot E_X + i \epsilon \hat{B}_X \cdot \left( B_X + \frac{\lambda}{2} \hat{z} E_X^2 \right) + i \epsilon \Delta \hat{B}_X^2 \right].
\] (41)
In the action in Eq. (39), the terms multiplying the response scalar and vector potentials \(\hat{\phi}\) and \(\hat{A}\) are reminiscent of the inhomogeneous Maxwell equations, i.e., of Gauss’ law and Ampère’s law (enriched by the KPZ non-linearity), however, with the source terms missing. To fully establish the equivalence to these Maxwell equations, we introduce charges \(n_v, \tilde{n}_v\) and currents \(J_v, \tilde{J}_v\) by means of the Poisson summation formula. The latter reads for a general function \(g(k)\) (see, e.g., [27]):
\[
\sum_{k=-\infty}^{\infty} g(k) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi g(\phi) e^{-i2\pi n\phi}.
\] (42)
Applying this relation to the summations over \(\phi, \tilde{\phi}, \hat{A}, \) and \(\hat{\chi}\) in Eq. (38), we obtain
\[
Z \propto \sum_{\{n_v, \tilde{n}_v\}, \{J_v, \tilde{J}_v\}} \int D[\phi, \tilde{\phi}, \hat{A}, \hat{\chi}] e^{iS[\phi, \tilde{\phi}, \hat{A}, n_v, \tilde{n}_v, J_v, \tilde{J}_v, \hat{\chi}]}.
\] (43)
Here, we have already included the new summation variables in the action, which reads

\[
S = \sum X \left[ \tilde{\phi}_X (\nabla \cdot \mathbf{E}_X - 2\pi n_{vX}) + \tilde{A}_X \cdot (\Delta_t \mathbf{E}_X - \epsilon \nabla \times \mathbf{B}_X + 2\pi \mathbf{J}_{bX} + \frac{\epsilon A^2}{2} \times \nabla \mathbf{E}_X^2) \right. \\
\left. + i\epsilon \Delta \left( \nabla \times \tilde{A}_X \right)^2 \right] - 2\pi \left( \tilde{n}_{vX} \tilde{\phi}_X - \tilde{J}_{bX} \cdot \tilde{A}_X \right). \tag{44}
\]

The Poisson summation formula allowed us to replace the summations in Eq. (38) over integer-valued fields by integrals over corresponding continuous fields, at the expense of introducing additional summation variables. This, however, is a price we are willing to pay: as indicated above and as we show in the following, the new variables have a clear physical interpretation as (vortex) charge and current densities, acting as sources for the electric and magnetic fields. To establish this identification, we have to make the action Eq. (44) gauge invariant. Indeed, before using the Poisson summation formula in Eq. (43), the action was fully gauge-invariant (cf. Eq. (41)). However, the additional terms appearing in the action (those involving the vortex charge densities and current densities, \(n_{vX}, \tilde{n}_{vX}\) and \(\mathbf{J}_{bX}, \tilde{\mathbf{J}}_{bX}\), respectively, in Eq. (44)) after replacing sums by integrals according to Eq. (42) break gauge invariance. The latter can be restored by means of the following trick: in the partition function (43), we shift the integration variables according to the gauge transformation prescriptions given in Eqs. (32) and (34). Then, in the action the fields \(\chi, \tilde{\chi}\) drop out in all terms with the exception of those which are not gauge-invariant, i.e., the ones involving the charges and currents. To be precise, under the gauge transformation the action becomes

\[
S[\phi - \chi, \tilde{\phi} - \Delta_t \tilde{\chi}, \mathbf{A} + \nabla \chi, \tilde{\mathbf{A}} + \nabla \tilde{\chi}, n_{vX}, \tilde{n}_{vX}, \mathbf{J}_{bX}, \tilde{\mathbf{J}}_{bX}] = S[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}, n_{vX}, \tilde{n}_{vX}, \mathbf{J}_{bX}, \tilde{\mathbf{J}}_{bX}] \\
- 2\pi \sum X \left[ \tilde{\chi}_X \left( \Delta_t n_{vX} + \nabla \cdot \mathbf{J}_{bX} \right) + \chi_X \left( \tilde{n}_{vX} - \nabla \cdot \mathbf{J}_{bX} \right) \right]. \tag{45}
\]

Since we introduced the fields \(\chi, \tilde{\chi}\) in a transformation of integration variables, the value of the functional integral does not depend on them (even though they appear explicitly in the action). Hence, performing an additional integration over these fields leads only to an irrelevant prefactor of the partition function. The benefit of carrying out this integration is that it allows us to arrange the functional integral as

\[
Z \propto \int \mathcal{D}[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}, n_{vX}, \tilde{n}_{vX}, \mathbf{J}_{bX}, \tilde{\mathbf{J}}_{bX}] e^{iS[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}, n_{vX}, \tilde{n}_{vX}, \mathbf{J}_{bX}, \tilde{\mathbf{J}}_{bX}]} \tag{46}
\]

with a gauge-invariant action \(S'\) that is defined as

\[
e^{iS'[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}, n_{vX}, \tilde{n}_{vX}, \mathbf{J}_{bX}, \tilde{\mathbf{J}}_{bX}]} = \int \mathcal{D}[\chi, \tilde{\chi}] e^{iS[\phi - \chi, \tilde{\phi} + \Delta_t \tilde{\chi}, \mathbf{A} + \nabla \chi, \tilde{\mathbf{A}} + \nabla \tilde{\chi}, n_{vX}, \tilde{n}_{vX}, \mathbf{J}_{bX}, \tilde{\mathbf{J}}_{bX}]} = \delta[\Delta_t n_{vX} + \nabla \cdot \mathbf{J}] \delta[\tilde{n}_{vX} - \nabla \cdot \mathbf{J}] e^{iS[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}, n_{vX}, \tilde{n}_{vX}, \mathbf{J}_{bX}, \tilde{\mathbf{J}}_{bX}]} \tag{47}
\]

The representation of \(S'\) after the first equality shows that the action is now manifestly gauge-invariant: any further gauge transformation of the fields appearing in \(S'\) can simply be absorbed in a shift of variables in the integration over \(\chi, \tilde{\chi}\). Since the transformed action in Eq. (45) is linear in \(\chi, \tilde{\chi}\), in the second equality we were able to perform the integrals over these fields explicitly, which yields two \(\delta\)-functions. The first of these expresses conservation of the number of vortices, and in particular, it is not at odds with the discussion below Eq. (30) concerning the absence of conservation of the number of bosons in a driven-dissipative condensate. The second \(\delta\)-functional can be used to evaluate the sum over \(\tilde{n}_{vX}\) in Eq. (46), whereby \(\tilde{n}_{vX}\) is replaced by \(\nabla \cdot \mathbf{J}_{bX}\).

In order to connect the MSR functional integral (46) to the electrodynamics of Ref. [35], which is formulated in terms of Langevin equations for the electromagnetic fields and charges, we follow the usual approach [36–38] and decouple the noise vertex in the action by means of a Hubbard-Stratonovich (HS) transformation,

\[
e^{-\epsilon \Delta \sum X (\nabla \times \mathbf{A}_X)^2} \propto \int \mathcal{D}[\eta] e^{-\epsilon \sum X \left( \frac{1}{2} \eta_X^2 + i\mathbf{A}_X \cdot \hat{z} \times \nabla \eta_X \right)}. \tag{48}
\]

Then, the partition function becomes

\[
Z \propto \sum \left[ \sum_{n_{vX}, \tilde{n}_{vX}, \mathbf{J}_{bX}, \tilde{\mathbf{J}}_{bX}} \right] \left[ \int \mathcal{D}[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}, \eta] \delta[\Delta_t n_{vX} + \nabla \cdot \mathbf{J}_{bX}] e^{iS} \right] \tag{49}
\]

Note that in this form the scalar and vector potentials appear only implicitly in the electric and magnetic fields, i.e., the gauge invariance of the action under the gauge transformation of these fields in Eq. (34) is manifest; it is straightforward to check that the action is also invariant under gauge transformations (32) of the response fields. The action in Eq. (49) now reads

\[
S = \sum X \left\{ \tilde{\phi}_X (\nabla \cdot \mathbf{E}_X - 2\pi n_{vX}) + \tilde{A}_X \cdot \left[ \Delta_t \mathbf{E}_X - \epsilon \nabla \times \mathbf{B}_X + 2\pi \mathbf{J}_{bX} + \frac{\epsilon A^2}{2} \times \nabla \mathbf{E}_X^2 + \eta_X \right] \right\} - 2\pi \tilde{J}_{bX} \cdot \mathbf{E}_X + i\epsilon \frac{\eta_X^2}{4\Delta}. \tag{50}
\]

Due to HS decoupling of the noise vertex, the action is now linear in the response scalar and vector potentials, \(\phi\) and \(\mathbf{A}\), and integration over these fields yields additional
\[ Z \propto \sum_{\{n_v, J_v, \delta J_v\}} \int D[\phi, A, \eta] \delta[\nabla \cdot E - 2\pi n_v] \times \delta[\Delta t n_v + e\nabla \cdot J_v] e^{-\sum x \left( i2\pi J_v(x) + e \hat{x} \cdot \nabla x \right)} \] 

\[ \times \delta \left[ \Delta t B - \varepsilon \nabla \times B + 2\pi J_v + e\hat{x} \times \nabla \left( \frac{\lambda}{2} \varepsilon E^2 + \eta \right) \right] \] 

These \( \delta \)-functionals correspond to the inhomogeneous Maxwell equations of Ref. [35]: the first one is Gauss’ law (note that in 2D electrodynamics the usual factor of \( 4\pi \) on the RHS is replaced by \( 2\pi \)), according to which the charges act as sources for the electric field. Remembering the relation between the electric field and the current, Eq. (35), we see that Gauss’ law is just the differential form of the equation (using continuum notation for clarity)

\[ \oint_{\partial \Omega} d\mathbf{l} \cdot \mathbf{j} = 2\pi \int_{\Omega} d\mathbf{x} n_v, \] 

according to which the circulation of the current along the closed boundary \( \partial \Omega \) of an area \( \Omega \) is determined by the total vortex charge within that area. The second \( \delta \)-functional corresponds to Ampère’s law, which in the present case inherits the non-linearity and the noise source \( \eta \) from the KPZ equation (1). Hence, in Eq. (51), the interpretation of \( n_v \) and \( J_v \) as the vortex and current densities becomes clear. Actually, the \( \delta \)-constraint ensuring the continuity equation for the vortices is redundant: by taking the divergence of Ampère’s law and inserting Gauss’ law, it can be seen that the continuity equation is already contained in the inhomogeneous Maxwell equations.

Equation (51) is already quite close to the dual electrodynamic theory for driven-dissipative condensates of Ref. [35]. However, this equation still assumes discretization in time, and since the vortex and current densities are integer-valued fields, we cannot take the temporal and spatial continuum limits in the present form. In the next section, we resolve this issue by considering a particular representation of \( n_v \) and \( J_v \) (Eqs. (53) and (54) below). Moreover, we give meaning to the term \( \tilde{J}_v \cdot E \) appearing in the exponent in the third line of Eq. (51).

Before proceeding, we note that as mentioned above Eq. (33), we can reach the same result Eq. (51) without ever introducing gauge potentials \( \phi \) and \( A \), but rather using Eqs. (33) and (35) to directly express the current \( j \) in terms of the electric field \( E \). Then, the MSR partition function Eq. (38) contains a summation over \( E \) instead of the double sum over \( \phi \) and \( A \). Applying again the Poisson summation formula as in Eq. (46) leads directly to a term of the form \( \tilde{J}_v \cdot E \) that is also present in Eq. (51). However, it is instructive to see how this term emerges from requiring the action to be gauge invariant.

\[ \text{III. DUAL ELECTRODYNAMIC THEORY} \]

Let us consider a collection of vortices at lattice points \( x_\alpha(t) \) where \( \alpha = 1, 2, \ldots, \) and with vorticity \( n_\alpha \in \mathbb{Z} \). The vortex density \( n_v \) corresponding to such a configuration is given by

\[ n_v, t, x = \sum_\alpha n_\alpha \delta x, x_\alpha(t). \] 

Since the vortex density obeys a continuity equation (expressed by the \( \delta \)-functional in the third line of Eq. (51)), the current density follows immediately from Eq. (53) and is given by

\[ J_v, t, x = \sum_\alpha n_\alpha \Delta t x_\alpha(t) \delta x, x_\alpha(t). \] 

By writing the vortex and current densities in this way, they are fully determined by the vortex trajectories \( x_\alpha(t) \). Quite conveniently, this allows us to take the continuum limit both in space and time: we simply have to replace the sums over \( n_v \) and \( J_v \) in Eq. (51) by integrals over smooth functions \( x_\alpha(t) \),

\[ \sum_{\{n_v, J_v, \delta J_v\}} \rightarrow \int D[\{x_\alpha\}]. \] 

Concomitantly, we replace sums by integrals, \( \sum_\alpha \rightarrow \int dx \) (remember that we set the lattice spacing to 1) and \( \sum_\epsilon \rightarrow \int dt \), discrete derivatives by ordinary ones, \( \Delta t \rightarrow \partial_t \) and \( \Delta t / \epsilon \rightarrow \partial_t \), and finally the Kronecker-\( \delta \)-s in Eqs. (53) and (54) by \( \delta \)-functions, \( \delta x, x_\alpha(t) \rightarrow \delta(x - x_\alpha(t)) \).

Before proceeding we comment on a difference between the summation over \( n_\alpha \) and \( J_\alpha \) and the integral over vortex trajectories \( x_\alpha \) in Eq. (55): while in the former case configurations with different numbers of vortices and antivortices are taken into account, in the latter case these numbers are fixed by the values of the charges \( n_\alpha \). Hence, an additional summation over \( \{n_\alpha\} \) should be included. For simplicity, we consider in the following only a single configuration \( \{n_\alpha\} \).

It remains to specify the dynamics of the vortices. Consistent with the over-damped dynamics of the electric and magnetic fields (cf. Faraday’s law Eq. (37)), we assume that the vortices undergo diffusive motion. In the MSR formalism, diffusive motion corresponds to the following contribution to the action:

\[ S_d = \frac{1}{\mu} \int dt \sum_\alpha p_\alpha \cdot \left( \frac{dx_\alpha}{dt} + iTp_\alpha \right). \] 

This has to be added on phenomenological grounds, as is also the case in the equilibrium treatment of Ref. [30, 31]. There, such a contribution to the action (or, equivalently, to the Langevin equation for the vortex coordinates \( x_\alpha \)) ensures that the stationary distribution is given by a thermal Gibbs ensemble at the vortex “temperature” \( T \). It is
reasonable to assume that the value of the vortex temperature is close to the dimensionless noise strength \( \Delta/D \) in the KPZ equation (1), since the noise acting on the vortices originates from the one acting on phase field \( \theta \). In principle, both the vortex temperature \( T \) and the vortex mobility \( \mu \) could be determined numerically (see Ref. [41] for a related discussion in the context of the complex Ginzburg-Landau equation).

In Eq. (56), \( p_\alpha \) is the momentum that is “conjugate” (in the sense of the MSR formalism) to the position \( x_\alpha \). Exactly the same relation of mutual conjugacy holds between the variables \( J_\alpha \) and \( J_\alpha \), suggesting that similarly to Eq. (54), which expresses \( J_\alpha \) in terms of the vortex coordinates \( x_\alpha \), there should be a representation of \( J_\alpha \) involving the momenta \( p_\alpha \). Indeed, if we replace \( J_\alpha \) in Eq. (51) according to (at the same time taking the continuum limit)

\[
\frac{2\pi}{\epsilon} J_\alpha x - \sum_\alpha n_\alpha p_\alpha (t) \delta (x - x_\alpha (t)),
\]

and combine the resulting contribution to the action with the one in Eq. (56), we obtain the complete vortex or charge action

\[
S_c = \int dt \sum_\alpha p_\alpha \left( \frac{dx_\alpha}{dt} - \mu n_\alpha E(x_\alpha) + i\mu T p_\alpha \right),
\]

where we additionally rescaled \( p_\alpha \) with the vortex mobility \( \mu \). The identification Eq. (57) completely removes \( J_\alpha \) from the action, and correspondingly we replace the summation \( \sum_{\{J_\alpha \}} \) by an integration over momentum trajectories \( \int D\{p_\alpha\} \).

Finally, performing a HS decoupling of the noise vertex in the charge action \( S_c \) (cf. Eq. (48) above), the latter becomes linear in the momenta \( p_\alpha \). Then, the integration over these variables can be performed and yields yet another \( \delta \)-constraint, rendering the functional integral Eq. (51) in the form

\[
Z \propto \int D\{x_\alpha, \xi_\alpha \}, \phi, A, \eta | \delta [\nabla \cdot E - 2\pi n_\alpha] \times \delta \left[ \partial_t E - \nabla \times B + 2\pi J_\alpha + \sum_\alpha \nabla \left( \frac{\lambda}{2} E^2 + \eta \right) \right] \times \prod_\alpha \delta \left[ \frac{dx_\alpha}{dt} - \mu n_\alpha E(x_\alpha) - \xi_\alpha \right] \\
\times e^{-\frac{i}{\hbar} \int dt dx_\alpha n_\alpha^2 - \frac{i}{\hbar} \int dt \sum_\alpha |\xi_\alpha|^2}.
\]

By reverting the logic that leads from a Langevin equation to the corresponding MSR action (cf. the discussion in the paragraph above Eq. (8) in Sec. II A) we can see that this functional integral is equivalent to the electrodynamic theory, which we introduced heuristically in Ref. [35]. It is summarized in the set of modified Maxwell equations, Eqs. (36) and (37) (note that these equations do not appear explicitly in the functional integral Eq. (59) since in the latter the electric and magnetic fields are expressed in terms of the current and the electric field is expressed in terms of the current as in Eq. (35) and the latter is identified with \( j = \nabla \theta \).

The last \( \delta \)-functional in Eq. (59) encodes the equation of motion of the vortices,

\[
\frac{dx_\alpha}{dt} = \mu n_\alpha E(x_\alpha) + \xi_\alpha.
\]

Here, \( \xi_\alpha \) is a Markovian noise source, which is introduced in the course of the HS decoupling of the noise vertex in Eq. (58), with correlations

\[
\langle \xi_{\alpha i}(t) \xi_{\beta j}(t') \rangle = 2\mu T \delta_{\alpha \beta} \delta_{ij} (t - t').
\]

This completes the derivation of the dual electrodynamic theory.

IV. OUTLOOK

Apart from the specific application to the compact KPZ equation in two spatial dimensions, a promising future direction is to generalize the duality transformation developed in this paper to treat other models of stochastic in- or out-of-equilibrium dynamics of compact fields. Even the most straightforward generalization to the one-dimensional compact KPZ equation should make it possible to study the influence of phase slips on the scaling properties of driven-dissipative condensates [14–16]. Moreover, it will be interesting to see whether the same methods can be extended to quantum systems which are described by Keldysh functional integrals [36, 37], and thus allow us to study real-time dynamics of compact fields also in this case.

Finally, we note that the Coulomb gas picture of the static XY-model is the starting point for deriving another representation in terms of a sine-Gordon field theory [32]. The advantage of this form is that it is amenable
to standard field theoretic tools and renormalization procedures. An interesting question is whether a similar mapping exists in the context of the compact KPZ equation.

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Appendix A: Villain approximation

Here we compare the Villain form Eq. (24) to the standard Villain approximation [24, 27] for the static XY-model. Hence, we begin by briefly reviewing the Villain approximation for the latter case.

In view of performing a Villain-type approximation, the main difference between the partition function for the classical XY-model and the MSR partition function for the compact KPZ equation is that in the former case the weight of a specific configuration is the compact KPZ equation is that in the former case the weight of a specific configuration is the real Boltzmann factor, whereas in the latter case we have to deal with a complex weight (the last factor on the RHS in Eq. (21)). For the classical XY-model, the Boltzmann factor can be expanded in a Fourier series as

\[ e^{K(\cos(\theta) - 1)} = \sum_{n=-\infty}^{\infty} e^{i n \theta + V(n)} \],

(A1)

where the potential \( V(n) \) can be expressed in terms of the modified Bessel function of the first kind \( I_n(K) \),

\[ e^{V(n)} = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta e^{-in\theta + K(\cos(\theta) - 1)} = e^{-K I_n(K)} \].

(A2)

This relation holds for any \( K \in \mathbb{C} \). In the XY-model, the prefactor \( K \) is proportional to the inverse temperature, \( K \propto 1/T \). Usually [27] it is argued that the Villain approximation replaces the exact potential \( V(n) \) in Eq. (A2) by an expression that (i) is asymptotically equivalent in the low-temperature limit \( K \rightarrow \infty \) (but see also [42]); moreover, obviously it should be possible to (ii) interpret the approximate potential as a valid free energy functional for \( n \) which (iii) is computationally simpler to handle than the exact expression. The crucial point is that by expanding the Boltzmann factor in a Fourier series, \( e^{V(n)} \) can be expressed in terms of the modified Bessel function as

\[ e^{V(n)} \approx \frac{1}{\sqrt{2\pi K}} e^{-n^2/(2K)} \].

(A3)

Obviously, this satisfies the criteria (ii) and (iii) formulated above. To see in which sense the criterion (i) is satisfied, let us compare the asymptotic expansions of Eqs. (A2) and (A3) for \( K \rightarrow \infty \) [43]: for the modified Bessel function we have

\[ e^{-K I_n(K)} \sim \frac{1}{\sqrt{2\pi K}} \left[ 1 - \frac{4n^2 - 1}{8K} + \frac{(4n^2 - 1)(4n^2 - 9)}{2!(8K)^2} + \cdots \right] \],

(A4)

and the expansion of the Villain potential reads

\[ \frac{1}{\sqrt{2\pi K}} e^{-n^2/(2K)} \sim \frac{1}{\sqrt{2\pi K}} \left[ 1 - \frac{4n^2}{8K} + \frac{(4n^2)^2}{2!(8K)^2} + \cdots \right] \].

(A5)

The pattern (which persists to higher orders) is, that of the polynomials appearing in the numerators in Eq. (A5) is satisfied, let us compare the asymptotic expansions of Eqs. (A2) and (A3) for \( K \rightarrow \infty \) [43]: for the modified Bessel function we have

\[ e^{-K I_n(K)} \sim \frac{1}{\sqrt{2\pi K}} \left[ 1 - \frac{4n^2}{8K} + \frac{(4n^2)^2}{2!(8K)^2} + \cdots \right] \].

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and the expansion of the Villain potential reads

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(A4)
For completeness, let us briefly point out what happens to condition (i) in the dynamical case. Upon replacing \(K\) by \(iK\), the exact Fourier coefficient (A2), becomes

\[
e^{iV(n)} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\theta + iK(\cos(\theta) - 1)} = e^{-iK} I_n(iK) = e^{-i(K-n\pi/2)} J_n(K),
\]

where \(J_n(K)\) is the Bessel function of the first kind. Then, the asymptotic expansions analogous to Eqs. (A4) and (A5) read

\[
e^{-i(K-n\pi/2)} J_n(K) \sim \frac{1}{\sqrt{2\pi K}} \left[ 1 + \frac{i}{8K} \left( \frac{4n^2 - 1}{2! (8K)^2} + \cdots \right) \right]
\]

\[
+ \frac{1}{\sqrt{2\pi K}} \left[ 1 + \frac{i}{8K} \left( \frac{4n^2 - 1}{2! (8K)^2} + \cdots \right) \right],
\]

and

\[
e^{-in^2/(2K)} \sim \frac{1}{\sqrt{2\pi K}} \left[ 1 + \frac{4n^2}{8K} - \frac{(4n^2)^2}{2!(8K)^2} + \cdots \right].
\]

The crucial difference to the static case is that the prefactor of the second term in Eq. (A7) is oscillating and not exponentially decaying, and therefore it gives a contribution in the large-\(K\) limit. Hence, asymptotic equivalence cannot be used as an argument to motivate replacing the exact Fourier coefficient (A7) by the Villain form (A8).

[1] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang, “Dynamic Scaling of Growing Interfaces,” Phys. Rev. Lett. 56, 889–892 (1986).
[2] Tamás Vicsek, Miklós Cserz, and Viktor K. Horváth, “Self-affine growth of bacterial colonies,” Phys. A Stat. Mech. its Appl. 167, 315–321 (1990).
[3] Jun-ichi Wakita, Hiroto Itoh, Tohey Matsuyama, and Mitsugu Matsushita, “Self-Affinity for the Growing Interface of Bacterial Colonies,” J. Phys. Soc. Japan 66, 67–72 (1997).
[4] M A C Huergo, A Dougherty, and JP Gollub, “Self-affine fractal interfaces from immiscible displacement in porous media,” Phys. Rev. Lett. 63, 1685–1688 (1989).
[5] J. Maunuksela, M. Myllys, O.-P. Kähkönen, J. Timonen, M. Prostas, M. J. Alava, and T. Ala-Nissila, “Kinetic Roughening in Slow Combustion of Paper,” Phys. Rev. Lett. 79, 1515–1518 (1997).
[6] M. Myllys, J. Maunuksela, M. Alava, T. Ala-Nissila, J. Merikoski, and J. Timonen, “Kinetic roughening in slow combustion of paper,” Phys. Rev. E 64, 036101 (2001).
[7] L. Miettinen, M. Myllys, J. Merikoski, and J. Timonen, “Experimental determination of KPF height-fluctuation distributions,” Eur. Phys. J. B 46, 55–60 (2005).
[8] Kazumasa A Takeuchi and Masaki Sano, “Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals,” Phys. Rev. Lett. 104, 230601 (2010).
[20] Igor Aranson and Lorenz Kramer, “The world of the complex Ginzburg-Landau equation,” Rev. Mod. Phys. 74, 99–143 (2002).
[21] L Balents and MP Fisher, “Temporal Order in Dirty Driven Periodic Media.” Phys. Rev. Lett. 75, 088701 (2013).
[22] Lee-Wen Chen, Leon Balents, Matthew P. A. Fisher, and M. C. Marchetti, “Dynamical transition in sliding charge-density waves with quenched disorder,” Phys. Rev. B 54, 12798–12806 (1996).
[23] Roland Lauter, Christian Brendel, Steven J M Habraken, and Florian Marquardt, “Pattern phase diagram for two-dimensional arrays of coupled limit-cycle oscillators.” Phys. Rev. E. 92, 012902 (2015).
[24] J. Villain, “Theory of one- and two-dimensional magnets with an easy magnetization plane. II. The planar, classical, two-dimensional magnet,” J. Phys. 36, 581–590 (1975).
[25] Jorge J. José, Leo Kadanoff, Scott Kirkpatrick, and David Nelson, “Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model,” Phys. Rev. B 16, 1217–1241 (1977).
[26] Jorge J. José, Leo P. Kadanoff, Scott Kirkpatrick, and David R. Nelson, “Erratum: Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model,” Phys. Rev. B 17, 1477–1477 (1978).
[27] P. M. Chaikin and T. C. Lubensky, Principles of condensed matter physics (Cambridge University Press, Cambridge, 1995).
[28] Robert Savit, “Duality in field theory and statistical systems.” Rev. Mod. Phys. 52, 453–487 (1980).
[29] Matthew P A Fisher, “Mott insulators, Spin liquids and Quantum Disordered Superconductivity,” in Asp. Topol. la Phys. en basse Dimens. Topol. Asp. low Dimens. Les Houches - Ecole dEte de Physique Theorique, Vol. 69, edited by A Comtet, T Jolicœur, S Ouvry, and F David (Springer Berlin Heidelberg, 1999) pp. 575–641.
[30] V. Ambegaokar, B.I. Halperin, D.R. Nelson, and E.D. Siggia, “Dissipation in Two-Dimensional Superfluids,” Phys. Rev. Lett. 40, 783–786 (1978).
[31] Vinay Ambegaokar, B.I. Halperin, D.R. Nelson, and E.D. Siggia, “Dynamics of superfluid films,” Phys. Rev. B 21, 1806–1826 (1980).
[32] Petter Minnhagen, “The two-dimensional Coulomb gas, vortex unbinding, and superfluid-superconducting films,” Rev. Mod. Phys. 59, 1001–1066 (1987).
[33] Matthew P. A. Fisher and D. H. Lee, “Correspondence between two-dimensional bosons and a bulk superconductor in a magnetic field,” Phys. Rev. B 39, 2756–2759 (1989).
[34] Ghassan G. Batrouni and M. B. Halpern, “Link formulation of lattice spin systems,” Phys. Rev. D 30, 1775–1781 (1984).
[35] G. Wachtel, L. M. Sieberer, S. Diehl, and E. Altman, “Electrodynamic duality and vortex unbinding in driven-dissipative condensates,” Phys. Rev. B 94, 104520 (2016).
[36] Alex Kamenev, Field Theory of Non-Equilibrium Systems (Cambridge University Press, Cambridge, 2011).
[37] Alexander Altland and Ben Simons, Condensed Matter Field Theory, 2nd ed. (Cambridge University Press, Cambridge, 2010).
[38] Uwe C. Täuber, Critical Dynamics: A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior (Cambridge University Press, Cambridge, 2014).
[39] Rosario Fazio and Gerd Schön, “Charge and vortex dynamics in arrays of tunnel junctions,” Phys. Rev. B 43, 1052–1063 (1991).
[40] R Fazio and H van der Zant, “Quantum phase transitions and vortex dynamics in superconducting networks,” Phys. Rep. 355, 235–334 (2001).
[41] Igor S. Aranson, Hugues Chaté, and Lei-Han Tang, “Spiral Motion in a Noisy Complex Ginzburg-Landau Equation,” Phys. Rev. Lett. 80, 2646–2649 (1998).
[42] W. Janke and H. Kleinert, “How good is the villain approximation?” Nucl. Phys. B 270, 135–153 (1986).
[43] Milton Abramowitz and Irene A Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, ninth ed. (Dover, New York, 1964).