Numerical relativity and asymptotic flatness

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Abstract
It is highly plausible that the region of spacetime far from an isolated gravitating body is, in some sense, asymptotically Minkowskian. However theoretical studies of the full nonlinear theory, initiated by Bondi et al (1962 Proc. R. Soc. A 269 21–51), Sachs (1962 Proc. R. Soc. A 270 103–26) and Newman and Unti (1962 J. Math. Phys. 3 891–901), rely on careful, clever, a priori choices of a chart (and tetrad) and so are not readily accessible to the numerical relativist, who chooses her/his chart on the basis of quite different grounds. This paper seeks to close this gap. Starting from data available in a typical numerical evolution, we construct a chart and tetrad which are, asymptotically, sufficiently close to the theoretical ones, so that the key concepts of the Bondi news function, Bondi mass and its rate of decrease can be estimated. In particular, these estimates can be expressed in the numerical relativist’s chart as numerical relativity recipes.

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1. Introduction and motivation

The two threads which underpin this study of asymptotic flatness are theoretical and numerical relativity, respectively. We start by reviewing the former. It is widely believed that the region of spacetime far from an isolated gravitating body is, in some sense, asymptotically Minkowskian. Already in 1962, Bondi and his coworkers (Bondi et al 1962, Sachs 1962) developed asymptotic expansions for the solution of the full nonlinear vacuum field equations, leading to a rigorous concept of gravitational radiation in the far field. Soon afterwards, Newman and Unti (1962) produced an alternative version using the Newman–Penrose (NP) null tetrad formalism (Newman and Penrose 1962). A key ingredient in this and later work was the careful choice of a suitable coordinate chart involving a ‘retarded time’ coordinate $u$. Both groups introduced a $(u, r, \theta, \phi)$ chart where $\theta$ and $\phi$ were spherical polar coordinates. They both developed asymptotic expansions as $r \to \infty$ holding the other coordinates fixed.
Subsequent work by Penrose (1963) showed that by a process of ‘conformal compactification’, infinity could be adjoined to the spacetime manifold and then treated by standard methods. Most modern theoretical treatments use the Penrose conformal approach, and the ‘old-fashioned’ chart-based approach has fallen out of fashion. However it is closer to what most numerical relativists are calculating, and for this reason we shall use it here.

The two groups used different charts and dependent variables. Bondi et al (1962) chose a \((u, r, \theta, \phi)\) chart, where \(r\) was an area coordinate and \(\theta\) and \(\phi\) were standard spherical polar coordinates (see below). Their primary dependent variables were the metric components. Newman and Unti (1962) produced an alternative version using a null ‘retarded time’ coordinate \(u\). Then the null vector \(l^a = g^{ab}u_b\) is geodesic and their coordinate \(r\) was chosen to be an affine parameter for the integral curves of \(l^a\) along which the other three coordinates were fixed. Their primary dependent variables were the tetrad connection components and the tetrad components of the Weyl curvature tensor. These ten independent Weyl tensor components are usually described by five complex scalar functions \(\Psi_n\), where \(n = 0, 1, \ldots, 4\). (For a covariant physical interpretation of the Weyl tensor, see, e.g., Szekeres (1965)).

Both groups considered the limit \(r \to \infty\) with the other coordinates fixed. In the Penrose geometrical picture (Penrose 1963), this region is called future null infinity. Of course, both groups could have considered an ‘advanced time’ coordinate \(v\) where the corresponding limit is past null infinity. \(\Psi_n\) in that case have similar properties and interpretation to \(\Psi_4\) near future null infinity.

If one wants to consider an isolated system with no extrinsic incoming radiation, then, as explained below, the natural place to impose this is past null infinity. However, both groups looked for a condition to be imposed near future null infinity. Bondi et al (1962) introduced an ‘outgoing radiation condition’ which required the vanishing of certain terms in the asymptotic expansion of two of the metric components. This condition will be stated more precisely in section 3. Newman and Unti (1962) made a ‘peeling assumption’: near future null infinity \(\Psi_0 = O(r^{-5})\), and with this assumption they were able to demonstrate a so-called peeling theorem: \(\Psi_4 = O(r^{-4})\) is interpreted as the leading term in the outgoing radiation. In the Bondi et al (1962) picture, the equivalent role is taken by the ‘Bondi news function’ built from the first derivatives of metric components. For a comparison of the conditions in the two schemes, showing that the outgoing radiation condition implies the peeling assumption, see, e.g., Valiente-Kroon (1999), and section 5.

By reversing the direction of time, swapping advanced time for retarded time, one could carry out an almost identical study near past null infinity. There, assuming the analogous peeling condition, \(\Psi_0 = O(r^{-1})\) is to be interpreted as the leading term in the extrinsic incoming radiation, and a natural ‘no incoming radiation condition’ near past null infinity would be \(\Psi_4 = O(r^{-5})\).

It is important to realize that the ‘outgoing radiation condition’ or ‘peeling assumption’ does not preclude the presence of incoming radiation near future null infinity. Even within linearized theory, the ‘peeling theorem’ allows modest amounts of incoming radiation (Deadman and Stewart).

We now turn to numerical relativity where researchers have expended considerable effort on the (numerical) evolution of asymptotically flat spacetimes. A minority of researchers have adopted the Penrose conformal approach, but most have chosen to evolve the spacetime as far out (in both space and time) as is feasible, using the traditional approach. Then, some matching process is required to interpret their numerical data in the Bondi or NP pictures. This, the goal of this paper, turns out to be far from trivial. The choice of a coordinate chart is an intrinsic part of the numerical evolution and the final data are available only in this chosen chart. Each numerical relativity group has its own favoured chart or charts and they usually...
bear little resemblance to the Bondi or NP ones. Furthermore, these data do not contain complete information because the inevitable occurrence of numerical errors will corrupt the values of higher derivatives—from it, one can reliably construct only a few leading terms in asymptotic expansions. The usual approach adopted by numerical relativists is to argue that far from the isolated source the gravitational field is weak, and so linearized theory can be used to match the numerical and the Bondi or NU pictures. Bondi et al. (1962) argued strongly against such an approximation pointing out the fundamental nonlinearity of general relativity. Even if plausible arguments in its favour could be found, linearization carries its own difficulties. The first is that in a non-compactified spacetime, the matching process is a global one. Further, given a spacetime, the choice of a simpler second spacetime of which the first can be considered a linearized perturbation is not unambiguous. Even if such a choice could be justified, the transformation between the charts in the two spacetimes would not, in general, be smooth.

As a concrete example illustrating these points, consider the well-known Schwarzschild metric in the standard $(t, r, \theta, \phi)$ chart:

$$g^S_{ab} = \text{diag}(F, -F^{-1}, -r^2, -r^2 \sin^2 \theta),$$

where $F = 1 - 2M/r$. In the region where $r \gg M$, this might appear to be a small perturbation of the Minkowski spacetime with metric

$$g^M_{ab} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \theta),$$

but this is deceptive. Consider the scalar wave equation $\nabla^a \Psi_{ab} = 0$ on the two spacetimes. We would measure outgoing radiation at future null infinity by taking the limit $r \to \infty$ holding $u$ constant, where $u$ is a retarded time coordinate. Two standard choices for $u$ are

$$u^M = t - r, \quad u^S = t - r^*,$$

where

$$r^* = \int F^{-1} \, dr = r + 2M \log|r/2M - 1| + \text{const.}$$

Thus,

$$u^M = u^S + 2M \log|r/2M - 1| + \text{const.}$$

The Schwarzschild null infinity is given by $r \to \infty$ holding $u^S$ constant, which implies $u^M \to \infty$, known as future timelike infinity for the Minkowski spacetime. Equivalently, the Minkowski null infinity involves taking the limit with $u^M$ constant which corresponds to $r \to \infty$ with $u^S \to -\infty$, known as spacelike infinity for the Schwarzschild spacetime. Thus, the limits in the two charts are different. This happens because of the global nature of the limiting process.

In order to achieve comparable limiting processes, we need to redefine the two charts. Here, both spacetimes are static and so it is simplest to retain the $t$-coordinate. Suppose we invert (for $r > M$) relation (4), $r^* = r^*(r)$, giving $r = r(r^*)$ and introduce a new chart $(t, r^*, \theta, \phi)$. Then the Schwarzschild line element (1) becomes

$$g^S_{ab} = \text{diag}(F, -F, -r^2, -r^2 \sin^2 \theta).$$

1. Consider an asymptotic expansion

$$f(r) = f_0 + f_1 r^{-1} + f_2 r^{-2} + \cdots$$

as $r \to \infty$. If we interpret this as the first few terms in the Taylor series for $f(q)$ about $q = 0$ where $q = r^{-1}$, then $f_n$ are, up to numerical factors, the $q$-derivatives of $f$ evaluated at $q = 0$. 

3
Using the same chart, the Minkowski line element is
\[ g^M_{ab} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \theta). \] (7)

Now the two metrics (6) and (7) are not only small perturbations of each other (for large \( r^* \)), but they share the same causal structure, \( u = t - r^* \), in both cases. (There are of course many other ways of doing this, e.g. retain \( r^* \)'s and change \( i^* \)'s, which is the approach to be adopted in this paper.) Also note the appearance of logarithms, which means that the transformations are not smooth.

The purpose of this paper is to examine in more detail these issues from the point of view of the numerical relativist. In section 2, we state what information we believe is available in a typical numerical evolution and assume that this information is expressed in terms of a given chart \( X^a = (T, R, \Theta, \Phi) \), which is asymptotically Minkowskian. Section 3 addresses the construction of an approximate Bondi-like chart \( x^a = (u, r, \theta, \phi) \) using this information. This circumvents the problem referred to above. We write down here the explicit form of the Bondi et al. outgoing radiation condition. We introduce a NP tetrad (Newman and Penrose 1962) adapted to the problem by Newman and Unti (1962) in section 4. At leading order, this is the usual NP tetrad for the Minkowski spacetime. At each order, \( r^{-1}, r^{-2}, \ldots \), there are 16 real coefficients describing the tetrad. However, from section 3, we know that only ten coefficients are needed to describe the metric. There are six coefficients which describe an infinitesimal Lorentz transformation at each order and, for the moment, we do not make a particular choice for them.

In section 5, we first obtain the asymptotic solution of the full nonlinear vacuum Einstein equations. As we do, so we fine tune our chart and NP tetrad to make them closer to those of Bondi et al. and Newman–Unti (NU). Once we have set the Ricci curvature, to the best of our abilities, to zero we turn to the Weyl curvature described by the Weyl scalars \( \Psi_n \) referred to earlier. We find that \( \Psi_n = O(r^{n-5}) \) for \( n = 4, 3, 2, 1 \), but \( \Psi_0 = O(r^{-3}) \), which would appear to violate the NU peeling assumption. However using the information gleaned from solving the vacuum field equations and the fine-tuning of the chart and tetrad, we can show that the Bondi outgoing radiation condition implies the NU peeling condition so that the peeling theorem then holds.

The bad news is that the leading terms in the Weyl scalars \( \Psi_0 = O(r^{-5}) \) and \( \Psi_1 = O(r^{-4}) \) cannot be estimated using the information we judge to be available from the information extracted in section 2. Although these scalars can be computed in linearized theory, this theory would appear to be an unreliable guide here near future null infinity.

The good news is that we can compute the leading terms in \( \Psi_4 = O(r^{-1}) \), equivalent to the ‘Bondi news function’ (and we can compute this scalar accurately within linearized theory). The same holds for \( \Psi_3 = O(r^{-2}) \), which involves nonlinear terms, but these can be removed by the fine-tuning process. We can also compute \( \Psi_2 = O(r^{-3}) \) which involves nonlinear terms in an essential way. This means that we can offer reliable estimates of the ‘Bondi mass’ \( M_B(u) \) of the isolated system, and its rate of decrease \( dM_B/du \leq 0 \), presumably due to the radiation of energy, both manifestly inaccessible to linearized theory.

Section 6 translates these results back into the \( X^a \) chart of section 2 used by a typical numerical relativist. From her/his standpoint, there is no need to go through the elaborate construction of a theoretical chart and NP tetrad carried out in the intermediate sections. We offer ‘numerical relativity recipes’ so that they can compute the key quantities referred to in the previous paragraph in their own preferred chart.

2 The ‘Bondi mass’ is of course the timelike component of a 4-vector and so frame dependent. But a numerical relativity evolution singles out a well-defined frame, and that is the one in which the mass is computed.
The key ideas in this paper are at least 40 years old, and one might ask why were these results not given before? The nonlinear calculations of Bondi et al (1962), Sachs (1962) and Newman and Unti (1962) were made possible by careful, clever, a priori choices of the chart and tetrad. We have to start from more or less arbitrary choices, and so the resulting expressions are horrendously complicated. In order to handle them accurately, we have utilized a computer algebra system. We used Reduce, and our Reduce 3.8 scripts can be obtained by an email request to the authors. Our choice reflected our experience and knowledge of one particular computer algebra system, but we used no features not available in some other systems.

2. The numerical data

Most numerical relativists would choose a quasi-spherical polar chart \( X^a = (T, R, \Theta, \Phi) \) for the numerical evolution of the spacetime surrounding an isolated gravitational source. We could also define an associated quasi-Cartesian chart \( Y^a = (T, X, Y, Z) \) where

\[
X = R \sin \Theta \cos \Phi, \quad Y = R \sin \Theta \sin \Phi, \quad Z = R \cos \Theta.
\]

We shall be interested in the limit \( R \to \infty \). As stated, this limit is meaningless unless we specify the behaviour of the other three coordinates under the limiting process, and we shall rectify this omission shortly. It proves very convenient to introduce the notation

\[
O_n = O(R^{-n}) \quad \text{as} \quad R \to \infty.
\]

Our fundamental assumption is that the spacetime outside an isolated source is asymptotically Minkowskian, expressed by the idea that, as seen in the \( Y^a \) chart,

\[
g_{ab} = \eta_{ab} + g^{(1)}_{ab} R^{-1} + g^{(2)}_{ab} R^{-2} + O_3,
\]

where \( \eta_{ab} = \text{diag}(1, -1, -1, -1) \) and \( g^{(n)}_{ab} \) are supposed to remain constant during the limiting process. Transforming from the Minkowskian chart to the spherical polar one, we find that the metric components in the \( X^a \) chart look like

\[
\begin{align*}
g_{00} &= 1 + h_{00} R^{-1} + k_{00} R^{-2} + O_1, \\
g_{01} &= h_{01} R^{-1} + k_{01} R^{-2} + O_3, \\
g_{02} &= h_{02} + k_{02} R^{-1} + O_2, \\
g_{03} &= h_{03} + k_{03} R^{-1} + O_2, \\
g_{11} &= -1 + h_{11} R^{-1} + k_{11} R^{-2} + O_3, \\
g_{12} &= h_{12} + k_{12} R^{-1} + O_2, \\
g_{13} &= h_{13} + k_{13} R^{-1} + O_2, \\
g_{22} &= -R^2 + h_{22} R + k_{22} + O_1, \\
g_{23} &= h_{23} R + k_{23} + O_1, \\
g_{33} &= -R^2 \sin^2 \Theta + h_{33} R + k_{33} + O_1.
\end{align*}
\]

Here, the functions \( \{h_{ab}\} \) and \( \{k_{ab}\} \) are required to remain constant during the limiting process.
We shall also need the asymptotic form of the inverse metric \( g^{ab} \) which is readily obtained from the relation \( g^{ac} g_{cb} = \delta^a_b \). We find

\[
\begin{align*}
g_{00} &= 1 + h_{00} R^{-1} + k_{00} R^{-2} + O_3, \\
g_{01} &= h_{01} R^{-1} + k_{01} R^{-2} + O_3, \\
g_{02} &= h_{02} R^{-2} + k_{02} R^{-3} + O_4, \\
g_{03} &= h_{03} R^{-2} + k_{03} R^{-3} + O_4, \\
g_{11} &= -1 + h_{11} R^{-1} + k_{11} R^{-2} + O_3, \\
g_{12} &= h_{12} R^{-2} + k_{12} R^{-3} + O_4, \\
g_{13} &= h_{13} R^{-2} + k_{13} R^{-3} + O_4, \\
g_{22} &= -R^{-2} + h_{22} R^{-3} + k_{22} R^{-4} + O_5, \\
g_{23} &= h_{23} R^{-3} + k_{23} R^{-4} + O_5, \\
g_{33} &= -R^{-2} \csc^2 \Theta_1 + h_{33} R^{-3} + k_{33} R^{-4} + O_5.
\end{align*}
\] (11)

Explicit formulae for \( h^{ab} \) and \( k^{ab} \) are given by equations (A.1) and (A.2) in appendix A. At this level of approximation,

\[
g^{ac} g_{cb} = \delta^a_b + O_3.
\]

A numerical evolution in which the dependent variables include both \( g_{ab} \) and \( g_{ab,c} \), usually called a ‘first-order formulation’, should produce accurate values for \( h_{ab} \) and its first derivatives, and for \( k_{ab} \). Otherwise, we assume that these variables are available for discrete ranges of \( T, \Theta \) and \( \Phi \) so that the corresponding derivatives can be estimated.

3. The Bondi chart

Most of the theoretical work which has been done on outgoing gravitational radiation involves a ‘Bondi chart’ \( (u, r, \theta, \phi) \) in which \( u \) is a retarded time coordinate; see, e.g., Bondi et al (1962), Newman and Penrose (1962), and Newman and Unti (1962).

Here we take the viewpoint that the \( X^a = (T, R, \Theta, \Phi) \) chart introduced in section 2 is the fundamental one in which, ultimately, all numerical calculations will be performed. Starting from this chart, we need to construct an \( x^a = (u, r, \theta, \phi) \) one which has all the essential features of a Bondi chart and we start by studying the function \( u(T, R, \Theta, \Phi) \).

Because \( u \) is a null coordinate, it has to satisfy the relativistic eikonal equation:

\[
g^{ab} u_{,a} u_{,b} = 0. \tag{12}
\]

This is a well-known nonlinear equation with four independent variables which is exceedingly difficult to solve with any generality. (Even the restriction of (12) to the Minkowski spacetime leads to the surprisingly rich structure of light ray caustics.) Note that there is a ‘gauge freedom’—if \( u \) is a solution then so is \( U(u) \) for any differentiable function \( U \).

The standard approach is to specify \( u \) on a spacelike hypersurface in spacetime, and then existence and local uniqueness of \( u \) are guaranteed by standard theorems. The standard approach is of little utility in this context, for no obvious choice of data suggests itself, and so we adopt a different approach.

First, consider the special case of a Minkowski spacetime, where (12) can be rewritten as

\[
(u, r)^2 - (u, \Theta)^2 = R^{-2} [(u, r)^2 + \csc^2 \Theta (u, \phi)^2] = O_2. \tag{13}
\]

6
Suppose we look for spherically symmetric solutions \( u = u(T, R) \). Setting \( \omega = u_R / u_T \) in (13), we find \( \omega^2 = 1 \). Using the gauge freedom mentioned earlier, we may impose \( u_T = 1 \) to find

\[
du = dT \pm dR,
\]

which implies

\[
u = T \pm R + \text{const}.
\]

\( T - R \) is called \textit{retarded time} and \( T + R \) is called \textit{advanced time}.

Although the special case appears trivial, it is the key to the general one. Within this section only, let indices \( i, j \) range over 0, 1 and let indices \( I, J \) range over 2, 3. Perusal of the display (11) shows that \( g_{ij} \) is \( O_0 \) while both \( g_{Ij} \) and \( g_{IJ} \) are \( O_2 \). Thus, the eikonal equation takes the form

\[
g_{ij} u_{,i} u_{,j} = O_2,
\]

which should be compared with (13) above. As boundary conditions (as \( R \to \infty \)), we impose

\[
u_T = 1 + O_1, \quad u_{,I} = O_1.
\]

This means that the eikonal equation takes the form

\[
g_{ij} u_{,i} u_{,j} = O_3,
\]

which we can write as a quadratic equation for \( \omega = u_R / u_T \), and choosing the sign appropriate for retarded time we find the solution

\[
u_R = -(1 + 2m_1 / R + 2m_2 / R^2) u_T + O_3,
\]

where

\[
m_1 = -\frac{1}{4} (h_{00} + 2h_{01} + h_{11})
\]

and

\[
m_2 = -\frac{1}{16} [4k_{00} + 8k_{01} + 4k_{11} + (h_{00} - h_{11})^2
\]

\[
- 4(h_{00} + h_{01})^2 + 4(h_{02} + h_{12})^2 + 4(h_{03} + h_{13})^2 \csc^2 \theta].
\]

Thus,

\[
du = u_T dT - u_T (1 + 2m_1 / R + 2m_2 / R^2) dR + O_3.
\]

We leave some freedom in \( u \) by setting

\[
u_T = 1 + \frac{q_1}{R} + \frac{q_2}{R^2} + O_3,
\]

where \( q_1 \) and \( q_2 \) are \( R \)-independent functions. At the moment, they are arbitrary. The requirement that the vacuum Einstein equations hold then determines inter alia \( q_1 \); see section 6. In our calculation, \( q_2 \) is not used directly.

We next need to specify a radial coordinate \( r = r(T, R, \Theta, \Phi) \), and the simplest choice is \( r = R \). This has the great practical advantage that \( O_2 = O(R^{-n}) = O(r^{-n}) \). It could be argued that our choice of \( r \) is neither the Bondi area coordinate nor an affine parameter along the outgoing null rays as favoured by Newman and Unti (1962). However since both of these approaches are known to be essentially equivalent, it would seem that the discussion is not sensitive to the precise choice of \( r \).

Then, (20) implies

\[
dT = (1 - q_1 / r - (q_2 - q_1^2) / r^2) du + (1 + 2m_1 / r + 2m_2 / r^2) dr + O_3.
\]
and so
\[
\left( \frac{\partial T}{\partial u} \right)_r = 1 - \frac{q_1}{r} - \frac{q_2 - q_1^2}{r^2} + O_3, \quad \left( \frac{\partial R}{\partial u} \right)_r = 0, \quad (23)
\]
\[
\left( \frac{\partial T}{\partial r} \right)_u = 1 + \frac{2m_1}{r} + \frac{2m_2}{r^2} + O_3, \quad \left( \frac{\partial R}{\partial r} \right)_u = 1. \quad (24)
\]

Finally, we consider the choice of angular coordinates \( \theta = \theta(T, R, \Theta, \Phi) \) and \( \phi = \phi(T, R, \Theta, \Phi) \). We shall require \( \theta = \Theta + O_1 \) and \( \phi = \Phi + O_1 \), and so the relations, being close to the identity, are invertible. It is more convenient to posit
\[
\theta = \theta + \frac{y_1}{r} + \frac{z_2}{r^2} + O_3, \quad \Phi = \phi + \frac{y_3}{r} + \frac{z_3}{r^2} + O_3, \quad (25)
\]
where the functions \( y_j \) and \( z_j \) do not depend on \( r \) but are otherwise arbitrary. Equations (25) are certainly consistent with the boundary conditions (15).

We can now specify the limiting process as \( r \to \infty \) holding \( u, \theta \) and \( \phi \) constant. Thus, we regard \( m_\alpha, q_\alpha, y_j, z_j, \{ h_{ab} \} \) and \( \{ k_{ab} \} \) as functions of \( u, \theta \) and \( \phi \).

Because we know the Jacobian \( (\partial X^a/\partial x^b) \), we can write down the metric components in the \( x^a = (u, r, \theta, \phi) \) chart:
\[
\begin{align*}
g_{00} &= 1 + a_{00}/r + b_{00}/r^2 + O_3, \\
g_{01} &= 1 + a_{01}/r + b_{01}/r^2 + O_3, \\
g_{02} &= -ry_{2,u} + a_{02} + b_{02}/r + O_2, \\
g_{03} &= -rz_{2,u} \sin^2 \theta + a_{03} + b_{03}/r + O_2, \\
g_{11} &= a_{11}/r + b_{11}/r^2 + O_3, \\
g_{12} &= a_{12} + b_{12}/r + O_2, \\
g_{13} &= a_{13} + b_{13}/r + O_2, \\
g_{22} &= -r^2 + a_{22}r + b_{22} + O_1, \\
g_{23} &= a_{23}r + b_{23} + O_1, \\
g_{33} &= -r^2 \sin^2 \theta + a_{33}r + b_{33} + O_1.
\end{align*}
\]

Two points should be noted here. First the leading terms in \( g_{02} \) and \( g_{03} \), if non-zero, would violate our notion of an asymptotically Minkowskian spacetime, for they are not present in the standard Minkowski line element. Thus, we must impose the conditions or ‘constraints’
\[
y_{2,u} = z_{2,u} = 0. \quad (27)
\]

Explicit formulae for \( a_{mn} \) in terms of \( h_{mn}, q_1, m_1 \) and \( y_2, z_2 \) (after imposing (27)) are given as \( (A.3) \) in appendix A. We could also give explicit formulae for \( b_{mn} \) in terms of \( h_{mn}, k_{mn}, q_2, m_2, y_3 \) and \( z_3 \) but they are rather lengthy, and are most easily generated using a computer algebra package.

Next, recall that the \( u \)-coordinate was constructed as a solution to the eikonal equation (16). Thus, as seen in the \( (u, r, \theta, \phi) \) chart \( g^{00} = O_1 \) which implies \( g_{11} = O_1 \) and so \( a_{11} = 0 \). One may verify this directly by comparing the explicit expression for \( a_{11} \) given in \( (A.3) \) with (18). We will show later that by making a suitable choice for \( y_2 \) and \( y_3 \), we can achieve \( g_{11} = 0 \) so that \( g_{11} = O_1 \) as expected.

We now have sufficient notation available to write down the ‘outgoing radiation condition’ of Bondi et al. (1962) as
\[
a_{33} = -a_{22} \sin^2 \theta, \quad b_{33} = b_{22} \sin^2 \theta, \quad b_{23} = 0, \quad (28)
\]
which we shall invoke later.
4. The NP tetrad

Since most recent studies of gravitational radiation use a NP null tetrad (Newman and Penrose 1962), we need to introduce one. The basics of tetrad formalisms are due to Schouten (1954). Many textbooks contain more readable, but often succinct, accounts, and Chandrasekhar (1983, chapter 1, section 7) is a good pedagogic compromise. With small, but necessary, changes in notation, this is summarized in appendix B. The specialization of this approach to the original NP formalism has been given by Campbell and Wainwright (1977). It turns out that the calculations that we need to do with it become surprisingly intricate, and so are most conveniently handled using a computer algebra system. One of the authors implemented the appendix B formalism and the other the Campbell and Wainwright (1977) one. Both of course gave the same results, a useful guard against programming errors. Results here are reported for the former.

We use a tetrad of vectors $e_\alpha^a$ and the dual tetrad of covectors $e^\alpha_\beta$. (The tetrad indices are Greek characters and always occur first.) Tetrad indices are lowered and raised using $\epsilon_\alpha^\beta$ and $\epsilon^\alpha_\beta$, where

$$\epsilon_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

In NP notation, we have

$$e_0^a = l_\alpha, \quad e_1^a = n_\alpha, \quad e_2^a = m_\alpha, \quad e_3^a = \overline{m}_\alpha,$$

$$e^a_0 = n^\alpha, \quad e^a_1 = l^\alpha, \quad e^a_2 = -\overline{m}^\alpha, \quad e^a_3 = -m^\alpha. \quad (29)$$

We shall require that, to leading order, $l_\alpha = u_\alpha$.

Setting $s = 2^{-1/2}$, we write the tetrads as

$$e^0_\alpha = (1 + c_{00}/r + d_{00}/r^2 + O_3, c_{01}/r + d_{01}/r^2 + O_3, c_{02} + d_{02}/r + O_2, c_{03} + d_{03}/r + O_2),$$

$$e^1_\alpha = (\frac{1}{2} + c_{10}/r + d_{10}/r^2 + O_3, 1 + c_{11}/r + d_{11}/r^2 + O_3, c_{12} + d_{12}/r + O_2, c_{13} + d_{13}/r + O_2),$$

$$e^2_\alpha = (c_{20}/r + d_{20}/r^2 + O_3, c_{21}/r + d_{21}/r^2 + O_3, -sr + c_{22} + d_{22}/r + O_2, isr \sin \theta + c_{23} + d_{23}/r + O_2),$$

$$e^3_\alpha = (c_{30}/r + d_{30}/r^2 + O_3, 1 + c_{31}/r + d_{31}/r^2 + O_3, -sr + c_{32} + d_{32}/r + O_2, -isr \sin \theta + c_{33} + d_{33}/r + O_2) \quad (30)$$

and

$$e^0_a = (1 + c_{00}/r + d_{00}/r^2 + O_3, -\frac{1}{2} + c_{01}/r + d_{01}/r^2 + O_3, c_{02}/r^2 + d_{02}/r^3 + O_4, c_{03}/r^2 + d_{03}/r^3 + O_4),$$

$$c_{02}/r^2 + d_{02}/r^3 + O_4, c_{03}/r^2 + d_{03}/r^3 + O_4).$$

In our calculations, we actually included one extra term in each of the asymptotic expansions below. For example, the first component of $e^0_\alpha$ was written as

$$e^0_0 = 1 + c_{00}/r + d_{00}/r^2 + j_{00}/r^3 + O_4.$$  

These ‘junk’ terms show up in our expressions for the connection and curvature components. In any expression where a junk term occurs, we regard all terms of that (and any higher order) as being junk, and not computable from the data described in section 2.
\[ e_1^a = (c^{10}/r + d^{10}/r^2 + O_3, 1 + c^{11}/r + d^{11}/r^2 + O_3, \\
\]
\[ c^{12}/r^2 + d^{12}/r^3 + O_4, c^{13}/r^2 + d^{13}/r^3 + O_4), \\
\]
\[ e_2^a = (c^{20}/r + d^{20}/r^2 + O_3, c^{21}/r + d^{21}/r^2 + O_3, \\
\]
\[ -s/r + c^{22}/r^2 + d^{22}/r^3 + O_4, -i s \csc \theta/r + c^{23}/r^2 + d^{23}/r^3 + O_4), \\
\]
\[ e_3^a = (c^{30}/r + d^{30}/r^2 + O_3, 1 + c^{31}/r + d^{31}/r^2 + O_3, \\
\]
\[ -s/r + c^{32}/r^2 + d^{32}/r^3 + O_4, i s \csc \theta/r + c^{33}/r^2 + d^{33}/r^3 + O_4). \]

Each tetrad contains, at each order, 32 real coefficients. For although \(c_{2n}\) and \(c_{3n}\) are complex, \(c_{2n} = c_{3n}\), etc. The relation \(e^a = \tilde{e}^a\) allows one to determine \(e^a\) in terms of \(c_{mn}\) and \(d_{mn}\) in terms of \(c_{mn}\) and \(d_{mn}\), reducing the number of unknowns, at each order, from 32 to 16.

The first set of these is given as equation (A.4) in appendix A. The second set is rather lengthy and best generated using a computer algebra package.

We also have the relation \(\epsilon_{\mu
u
v}e_{\mu}e_{\nu}e_{b} = g_{ab}\), and this enables us to determine \(a_{mn}\) in terms of \(c_{mn}\). Note that there are 10 \(a_{mn}\) and 16 real \(c_{mn}\). Given the tetrad, the metric is uniquely determined. But for a given metric, there is a six-parameter set of tetrad which give rise to it. They are of course Lorentz transformations of each other, and the Lorentz group has six arbitrary parameters. We introduce six arbitrary first-order Lorentz parameters \(a_m(u, \theta, \phi)\) and can determine \(c_{mn}\) in terms of \(a_{mn}\) and \(a_m\). There are many different ways of doing this, and one is written down explicitly as (A.5) in appendix A. We can of course write down \(d_{mn}\) in terms of \(a_{mn}\), \(b_{mn}\), \(a_m\) and extra second-order Lorentz parameters \(\beta_m\), but they are rather lengthy and are best generated by computer algebra.

5. The curvature tensors

We now use the tetrads developed in section 4 to evaluate the Ricci and Weyl curvature tensors using the algorithm outlined in appendix B. At each stage, we convert all instances of \(e^a\) and \(d^a\) to instances of \(e_{\mu}e_{\nu}e_{\beta}\) and its second-order analogue. Then we convert all instances of \(c_{mn}\) and \(d_{mn}\) to instances of the metric coefficients \(e_{\mu}e_{\nu}e_{\beta}\) and \(a_{mn}\) using (A.4) and its second-order analogue. These conversions are implicit and will not be mentioned explicitly again.

We start by looking at the Ricci tensor component \(R_{11} = R_{ab}e_1^ae_1 = R_{ab}e_1^ae_1e_1^b = R_{ab}l^ae^bl^b\). We find

\[ 0 = R_{11} = a_{11, u}/r^2 + O_3. \]

We chose our chart to ensure that \(u\) was approximately a null coordinate or equivalently \(g^{00} = O_3\) or equivalently \(g_{11} = O_1\). This means that we have to enforce \(a_{11} = 0\), and so the leading term in \(R_{11}\) vanishes. We next look at

\[ 0 = R_{01} = R_{ab}e_0^ae_1^b = R_{ab}l^ae^bl^b = -1/4 a_{11, uu}/r + O_2. \]

Again, the leading order term vanishes automatically. Next, consider

\[ 0 = R_{12} + R_{13} = -2a_{12, u}/r^2 + O_3, \quad 0 = R_{12} - R_{13} = -2a_{13, u} \csc \theta/r^2 + O_3, \]

where \(s = 2^{-1/2}\). We deduce that

\[ a_{12, u} = a_{13, u} = 0. \]

Further, we can compute

\[ 0 = R_{02} + R_{03} = sa_{12, uu}/r^2 + O_2, \quad 0 = R_{02} - R_{03} = i s a_{13, uu} \csc \theta/r + O_2, \]

If we inspect \(R_{23}\) and use (35), we find

\[ 0 = R_{23} = 1/2(a_{22, u} + a_{33, uu} \csc^2 \theta)/r^2 + O_3. \]
and we deduce that
\[ a_{22,u} + a_{33,u} \csc^2 \theta = 0. \]  
Further, we may compute
\[ R_{00} = -\frac{1}{2} (a_{22,uu} + a_{33,uu} \csc^2 \theta) / r + O_2, \]  
and we immediately see from (38) that the leading term vanishes and so furnishes no new information. Finally, inspection of the leading \(O_2\) terms in \(R_{22} \pm R_{33}\) reveals that they vanish automatically because of \(a_{11} = 0, (35)\) and (38). Thus, we have
\[ R_{22} + R_{33} = O_3, \quad R_{22} - R_{33} = O_3. \]  
We have found, so far, that the conditions \(a_{11} = 0, (35)\) and (38) imply that \(R_{00}, R_{01}, R_{02}\) and \(R_{03}\) are \(O_2\) while the other components are \(O_3\).

At this point, we need to examine (35) more closely. Using (A.3), we have
\[ (h_{02} + h_{12} + y_2)_u = 0 \quad (h_{03} + h_{13} + z_2 \sin^2 \theta)_u = 0. \]  
Now we know that the functions \(y_2\) and \(z_2\) are arbitrary, apart form the constraints (27), and so (41) implies
\[ (h_{02} + h_{12})_u = 0 \quad (h_{03} + h_{13})_u = 0. \]  
We may therefore choose, consistent with the constraints (27),
\[ y_2 = -(h_{02} + h_{12}), \quad z_2 = -(h_{03} + h_{13}) \csc^2 \theta, \]  
which, using (A.3), sets
\[ a_{12} = a_{13} = 0. \]  

The choice (43) has an added advantage that if we now express \(b_{11}\) in terms of \(h_{ab}, k_{ab}, m_1\) and \(m_2, y_2\) and \(z_2\), we find that \(b_{11} = 0\), so that \(g_{11} = O_1\). At the same time, we can examine \(b_{12}\) and \(b_{13}\) which are linear in \(y_3\) and \(z_3\) respectively. By choosing \(y_3\) and \(z_3\) appropriately, we may arrange \(b_{12} = b_{13} = 0\).

This is a convenient point at which to examine the choice of a specific Lorentz transformation, in our tetrad, exemplified at leading order by the parameters \(\alpha_a\): see, e.g., (A.5). Newman and Penrose (1962) chose \(u\) to be a null coordinate, \(g^{ab}u_au_b = 0\), and the covector \(l_a = u_a\). For a symmetric connection, it easily follows that \(l^a\beta_b^b = 0\). Even if we set \(l_a = f(x^\nu)u_a\) we find that \(l^a\beta_b^b\) is proportional to \(l^a\) so that we still have a null geodesic, albeit not necessarily an affinely parametrized one. The rest of this paragraph relies on some details of the NP formalism which can be checked swiftly using appendix B of Stewart (1993). Within the NP formalism,
\[ l^a,\beta_b^b = (\epsilon + \tau)f^a - \kappa m^a - \kappa m^a, \]  
where \(\kappa\) and \(\epsilon\) are NP spin coefficients defined below. Now our coordinate \(u\) is only approximately null and our covector \(l_a\) is only approximately its gradient. Here, \(\kappa = m^a\beta_b^b l_{a,b} = \gamma_{131}\) (using the notation of appendix B) turns out to be \(O_3\). However if we choose \(\alpha_4 = \alpha_5 = \beta_4 = \beta_5 = 0\,\) and anticipate \(a_{01} = 0\) (see next paragraph), we obtain \(\kappa = O_4\). Also, \(\epsilon + \tilde{\epsilon} = m^a\beta_b^b l_{a,b} = \gamma_{011}\) is \(O_2\) but if we choose \(\alpha_1 = 0\), we find that \(\epsilon + \tilde{\epsilon}\) is \(O_3\). We also found \(\tau = \gamma_{130} = (\alpha_2 + i\alpha_3)/r^2 + O_3\), or \(\tau = O_3\) if we impose \(\alpha_2 = \alpha_3 = 0\). At this stage, we also choose \(\alpha_6 = \beta_6 = 0\) for reasons given below.

Now we need to examine each of the remainder (next order) terms in (32)–(34), (36), (37), (39) and (40). For example, we now find that the \(O_3\) terms in \(R_{11}\) vanish if and only if we set \(a_{01} = 0\), and then \(R_{11} = O_4\). This also implies that the \(O_2\) terms in \(R_{01}\) vanish, so that \(R_{01} = O_3\). We already established that \(R_{12} \pm R_{13} = O_3\). Setting the leading order terms to
zero furnishes expressions for \( a_{02} \) and \( a_{03} \) which we use for subsequent simplifications. Now \( R_{12} \pm R_{13} = O_4 \). We then find that our previous estimate (36) refines to \( R_{02} \pm R_{03} = O_3 \). We also need to refine our estimate (37) to \( R_{23} = O_4 \). We find \( R_{22} \pm R_{23} = O_3 \), where both \( O_3 \) terms deliver the same relation relating the \( u \)-derivatives of \( a_{22}, a_{23}, a_{33} \) and \( b_{22}, b_{23}, b_{33} \) which we save for later use. In deriving this result, we had to choose \( \alpha_6 = \beta_5 = 0 \) and to impose the Bondi outgoing radiation condition (28). Then, \( R_{22} \pm R_{23} = O_4 \). Next, we re-examine (39). The leading \( O_2 \) term gives us an expression for \( a_{00,u} \) which we store for later use. Now \( R_{00} = O_3 \). We have found, so far, that \( R_{00}, R_{01}, R_{02} \) and \( R_{03} \) are \( O_2 \) while the other components are \( O_1 \).

We now try to repeat the procedure of the previous paragraph. However we find that the \( O_4 \) contribution in \( R_{11} \) contains ‘junk’ terms, i.e. terms which involve the third-order metric components which we have not been including; see footnote 3 in section 4. Thus, we can obtain no further information from the vacuum field equation \( R_{11} = 0 \). Similarly, we find that the \( O_3 \) terms in \( R_{01} \) contain junk as do the \( O_4 \) terms in \( R_{12} \pm R_{13} \). The same applies to the \( O_3 \) terms in \( R_{02} \pm R_{03} \), the \( O_4 \) terms in \( R_{23} \) and \( R_{22} \pm R_{33} \) and finally the \( O_3 \) terms in \( R_{00} \). We have therefore exhausted the information available from the vacuum field equations.

Assuming that we have a vacuum we can switch attention to the Weyl tensor, and we first compute

\[
\Psi_4 = R_{0202} = \left( \frac{1}{2} (a_{22,uu} - a_{33,uu} \csc^2 \theta) + \frac{1}{2} ia_{23,uu} \csc \theta \right) / r + O_2.
\]

Using (38), we may rewrite this as

\[
\Psi_4 = N_u/r + O_2,
\]

where

\[
N = \frac{1}{2} (a_{22} + ia_{23} \csc \theta),_{u}
\]

is the Bondi news function (Bondi et al 1962).

This is a highly satisfactory result which, in spite of our rather ad hoc chart and tetrad, mimics the treatment of Bondi et al (1962) and Newman and Unti (1962). Further, we see that it is linear in \( a_{uu} \) and so should appear in linearized theory. Also it does not involve the Lorentz parameters \( \alpha \) and \( \beta \) and so is tetrad invariant (for tetrads which are asymptotically Minkowskian). The remainder term in (47) contains some \( O_2 \) terms and junk \( O_3 \) terms.

Next, consider

\[
\Psi_3 = R_{0120} = \Psi_3^{(2)}/r^2 + O_3,
\]

where

\[
\Psi_3^{(2)} = 2^{-1/2} \left( N_{,\theta} - i N_{,\phi} \csc \theta + N \cot \theta \right) + (a_4 + ia_5) N_{,u}.
\]

First, note that the \( r \)-dependence is precisely what one would have expected from the peeling property. The first term in the coefficient \( \Psi_3^{(2)} \) is linear and would have been predicted within linearized theory. However, the second term is nonlinear for it depends on \( \alpha_6 \) which determine the infinitesimal Lorentz transformation of the NP tetrads (30) and (31). This is to be expected. The NP tetrad used by Newman and Unti (1962) was chosen very specifically, while here we consider a class of tetrads infinitesimally close to the Minkowski one. If we were to restrict attention to the subclass of tetrads where \( \alpha_4 = \alpha_5 = 0 \), then our result would be consistent with linearized theory. On the other hand, another choice of \( \alpha_4 + i \alpha_5 \) would give \( \Psi_3^{(2)} = 0 \). The remainder term in (49) is junk.

Next, we find that

\[
\Psi_2 = R_{1320} = \Psi_2^{(1)}/r^3 + O_4,
\]

where the remainder term is junk. We will return to the leading term shortly.
We next find that
\[ \Psi_1 = R_{0113} = \Psi_1^{(4)}/r^4 + O_5, \]
(52)
The coefficient \( \Psi_1^{(4)} \) contains nonlinear terms, but we are unable to determine it precisely because it also contains junk terms. The peeling property is still holding though.

The peeling property would demand that \( \Psi_0 = R_{113} \) should be \( O_5 \). However, we find
\[ \Psi_0 = \Psi_0^{(4)}/r^4 + O_5, \]
(53)
where
\[ \Psi_0^{(4)} = \frac{1}{8}[(a_{22} + a_{33} \csc^2 \theta)(a_{22} - 2ia_{23} \csc \theta - a_{33} \csc^2 \theta)\]
\[ + 4(b_{22} - b_{33} \csc^2 \theta - 2ib_{23} \csc \theta)], \]
(54)
But we have not made the restrictions that were imposed by Bondi et al (1962) or Newman and Unti (1962) to ensure peeling. The latter restriction was to demand \( \Psi_0^{(4)} = 0 \). The former restriction was that the ‘outgoing radiation condition’ (Bondi et al 1962) held. In our notation, this condition is (see (28))
\[ a_{22} = -a_{33} \csc^2 \theta, \quad b_{22} = b_{33} \csc^2 \theta, \quad b_{23} = 0. \]
(55)
Examining (54), we see that imposing the outgoing radiation condition (55) ensures \( \Psi_0^{(4)} = 0 \), a result first obtained by Valiente-Kroon (1999). With one or other condition, we have
\[ \Psi_0 = \Psi_0^{(5)}/r^5 + O_6, \]
(56)
but the coefficient \( \Psi_0^{(5)} \) contains junk terms and so we cannot evaluate it. (It also contains nonlinear terms not predicted by linearized theory.)

To summarize: if we impose the outgoing radiation condition (55) then we obtain the peeling property, and we can obtain explicitly the leading terms in \( \Psi_1, \Psi_2, \Psi_3 \), but not those for \( \Psi_4 \) and \( \Psi_0 \) because they contain junk terms.

We now return to the discussion of \( \Psi_2 \) given by (51). The leading term coefficient is
\[ \Psi_2^{(3)} = \frac{1}{4}a_{00} - \frac{1}{4}ia_{23} \csc \theta + \frac{1}{4}b_{22,0} + \frac{1}{4}(a_{22} - ia_{23} \csc \theta)(a_{22,0} - ia_{23,0} \csc \theta)\]
\[ + \frac{1}{4}i(a_{23,0} - 2a_{22,0}) \cot \theta \csc \theta - \frac{1}{4}ia_{22,0,\phi} \csc \theta \]
\[ + \frac{1}{4}i(a_{23,0,0} - \csc^2 \theta a_{23,0,\phi}) \csc \theta. \]
(57)
Here we have fixed the Lorentz parameters, as described earlier, and impose the outgoing radiation condition (28).

Now the Bondi mass \( M_B(u) \) can be defined by (Bondi et al 1962, Newman and Unti 1962, Stewart 1989)
\[ 4\pi M_B(u) = -\lim_{r \to \infty} \int_{S(u, r)} r^3 (\Psi_2 + \sigma \lambda) \sin \theta \, d\theta \, d\phi, \]
(58)
where the integral is over the 2-surface \( S(u, r) \) given by \( u = \text{const} \) and \( r = \text{const} \). Here, \( \sigma \) and \( \lambda \) are NP spin coefficients given by
\[ \sigma = m^a \delta l_a = \gamma_{313} = \sigma^{(2)}/r^2 + O_3, \quad \lambda = n^a \delta m_a = \gamma_{022} = \lambda^{(1)}/r + O_2, \]
(59)
where
\[ \sigma^{(2)} = -\frac{1}{2}(a_{22} - ia_{23} \csc \theta), \quad \lambda^{(1)} = -\frac{1}{2}(a_{22,0} + ia_{23,0} \csc \theta). \]
(60)
Taking the limit in (58), we have
\[ 4\pi M_B(u) = -\int_{S(u,1)} (\Psi_2^{(3)} + \sigma^{(2)} \lambda^{(1)}) \sin \theta \, d\theta \, d\phi. \]
(61)
Of course, formula (58) is only valid in a specially chosen Bondi frame. The generalization to an arbitrary NP frame is discussed in Stewart (1989). In the large \( r \) limit, our frame differs from the Bondi one by a Lorentz transformation which is close to the identity. A two-parameter subgroup of the Lorentz group consists of ‘boosts’ and ‘spins’:

\[
\begin{align*}
  l &\rightarrow a^2 l, \\
  n &\rightarrow a^{-2} n, \\
  m &\rightarrow e^{i\psi} m,
\end{align*}
\]

(62)

where \( a \) and \( \psi \) are real. Using the formulae in appendix B of (Stewart 1993), it is easy to verify that the integrand of (61) is invariant under boosts and spins. Next, consider a two-parameter subgroup of ‘null rotations about \( l \)’ given by

\[
\begin{align*}
  l &\rightarrow l, \\
  m &\rightarrow m + \epsilon l, \\
  n &\rightarrow n + cm + \epsilon m + c\epsilon l,
\end{align*}
\]

(63)

where \( c \) is complex. Under such a transformation,

\[
\begin{align*}
  \Psi_2 &\rightarrow \Psi_2 + 2c\Psi_1 + c^2\Psi_0, \\
  \sigma &\rightarrow \sigma + c\epsilon, \\
  \lambda &\rightarrow \lambda + c\pi + 2c\alpha + \epsilon^2(\rho + 2\epsilon) + c^3\kappa + c\epsilon l^a c_{\alpha a} + m^a c_{\alpha a}.
\end{align*}
\]

(64)

We expect \( c = O_1 \), and the NP scalars \( \alpha, \pi, \rho \) and \( \epsilon \) are all \( O_1 \). Thus, the integrand of (61) is not changed. We should also consider null rotations about \( n \) given by

\[
\begin{align*}
  n &\rightarrow n, \\
  m &\rightarrow m + \epsilon n, \\
  l &\rightarrow l + cm + \epsilon m + c\epsilon n,
\end{align*}
\]

(65)

so that

\[
\begin{align*}
  \Psi_2 &\rightarrow \Psi_2 + 2c\Psi_3 + c^2\Psi_4, \\
  \lambda &\rightarrow \lambda + \epsilon \theta, \\
  \sigma &\rightarrow \sigma + c\epsilon + 2c\beta + \epsilon^2(\mu + 2\gamma) + c^3\nu + cn^a c_{\alpha a} + m^a c_{\alpha a}.
\end{align*}
\]

(66)

Now we have taken great care to ensure that \( l \) is almost geodesic (\( \kappa = O_2 \)) and almost affinely parametrized (\( \epsilon + \delta = O_3 \)) and so we should only consider the transformation (65) where \( c = O_1 \). Under this restriction, the integrand of (61) is not changed. Thus, formula (61) evaluated in our frame does indeed give the Bondi mass to leading order. Next, note that

\[
\Im(\Psi_2^{(3)} + \sigma^{(1)} \lambda^{(1)}) = -\frac{1}{2}a_{23} \csc^3 \theta + \frac{1}{3} a_{23,\rho} \csc \theta \cot \theta + \frac{1}{2} a_{23,\rho \theta} \csc \theta
\]

\[
-\frac{1}{2} a_{22,\rho} \csc \theta \cot \theta - \frac{1}{2} a_{22,\rho \theta} \csc \theta - \frac{1}{4} a_{23,\phi \phi} \csc^3 \theta.
\]

(67)

When we integrate this over the unit sphere, the terms in the second line give zero since their contribution to the integrand is \( 2\pi \)-periodic in \( \phi \). Those in the first line contribute

\[
\frac{1}{2} \pi \csc \theta [\sin(\theta a_{23})]_0^\pi.
\]

Now, \( a_{23} \) must scale like \( \sin^2 \theta \) at the end points or else the integrand is singular. It follows that the Bondi mass must be real, and

\[
4\pi M_B(u) = -\frac{1}{2} \int_0^\pi (a_{20} + b_{22,u} + a_{22} a_{22,u} + a_{23} a_{23,u} \csc^2 \theta) \sin \theta \, d\theta \, d\phi.
\]

(68)

Finally, there is a standard result (Bondi et al. 1962, Newman and Unti 1962, Stewart 1989) for the rate of decrease of the Bondi mass:

\[
4\pi \frac{dM_B}{du}(u) = -\int_{S(u,1)} |N|^2 \sin \theta \, d\theta \, d\phi
\]

\[
= -\frac{1}{4} \int_{S(u,1)} ((a_{22,u})^2 + (a_{23,u})^2 \csc^2 \theta) \sin \theta \, d\theta \, d\phi,
\]

(69)

demonstrating the well-known result \( dM_B/du \leq 0 \); the Bondi mass decreases as energy is radiated away, a result not deducible in linearized theory. Although (69) was originally derived in a special Bondi frame, it too holds in our approximate Bondi one, at least to leading order. We should emphasize that although the outgoing radiation condition was used in the derivation of (68), the mass loss formula (69) holds without the need for this restriction.
6. Implications for numerical relativity

At first glance the formalism we set up to carry out this study may seem to be cumbersome, but it has the advantage that the results can be translated back into the $X^a = (T, R, \Theta, \Phi)$ chart, after which the theoretical chart $x^a = (u, r, \theta, \phi)$ can be discarded.

We chose the $x^a$ chart so that the metric coefficients $a_{11}$ and $b_{11}$ vanished as well as $a_{12}$ and $a_{13}$. Now
\[
\left( \frac{\partial}{\partial u} \right)_r = \left( \frac{\partial T}{\partial u} \right)_r, \quad \left( \frac{\partial}{\partial T} \right)_r = \left( \frac{\partial R}{\partial u} \right)_r, \quad \left( \frac{\partial}{\partial R} \right)_T = (1 - q_1 r^{-1}) \left( \frac{\partial}{\partial T} \right)_r + O_2,
\]
(70)
using (23). In principle, the function $q_1$ is arbitrary. But the vacuum field equations implied $a_{01} = 0$, and then equations (A.3) imply
\[
q_1 = \frac{1}{2}(h_{00} - h_{11}).
\]
(71)
The vacuum field equations imply (38)
\[
a_{22,0} + a_{33,0} \csc^2 \theta.
\]
Using (A.3) and (43) we have, to leading order,
\[
[h_{22} + 2h_{02,\theta} + 2h_{12,\phi}, T] + [h_{33} + (2h_{03,\phi} + 2h_{13,\phi}) \csc^2 \Theta]_T \csc^2 \Theta = 0.
\]
(72)
Other vacuum conditions can be handled in a similar way.

In order to discuss the Bondi news function and Bondi mass, it is convenient to introduce some auxiliary functions in the numerical chart:
\[
\mathcal{W} = h_{03} + h_{13},
\]
\[
\mathcal{A} = h_{22} + 2(h_{02,\theta} + h_{12,\phi}),
\]
\[
\mathcal{B} = h_{23} + h_{02,\phi} + h_{12,\phi} + \mathcal{W} \cot \Theta,
\]
\[
\mathcal{C} = k_{22} + 2\mathcal{W} h_{23} \cot \Theta \csc^2 \Theta - 4\mathcal{W}^2 \cot \Theta \csc^2 \Theta + (k_{02} + k_{12}) \phi
\]
\[
- \left( \frac{1}{2} h_{11} + h_{22} \right) h_{02,\theta} - \left( \frac{1}{2} h_{00} + h_{01} \right) h_{02,\theta} + (4\mathcal{W} \cot \Theta - k_{23}) \mathcal{W} \cot \Theta \csc^2 \Theta
\]
\[
- \frac{1}{2} h_{02} h_{11,\phi} - h_{22,12,\phi} + (h_{02} + h_{12}) h_{22,12,\phi} + \mathcal{W} h_{23,\phi} \csc^2 \Theta - (\mathcal{W} \cot \Theta) \csc^2 \Theta,
\]
which should be readily available according to the assumptions in section 2.

Then the leading term in the Bondi news function given by (48) becomes
\[
\mathcal{N} = \mathcal{A}_T + i \mathcal{B}_T \csc \Theta,
\]
(74)
whose calculation might require some sophistication, although there is no reference to the intermediary $x^a = (u, r, \theta, \phi)$ chart. Because the news function is linear in $h_{ab}$ and their derivatives, it could have been calculated within linearized theory.

In particular, formula (68) for the Bondi mass $M_B(u)$ translates into
\[
M_B(T - R) = -\frac{1}{8\pi} \int_{\Phi = 0}^{2\pi} \int_{\Theta = 0}^{\pi} (h_{11} + \mathcal{A}_T + \mathcal{B}_T \csc^2 \Theta + \mathcal{C}_T) \sin \Theta d\Theta d\Phi.
\]
(75)

In a similar way, formula (69) for the rate of change of $M_B$ at fixed large $R$ is
\[
M_B(T - R) = -\frac{1}{16\pi} \int_{\Phi = 0}^{2\pi} \int_{\Theta = 0}^{\pi} ((\mathcal{A}_T)^2 + (\mathcal{B}_T)^2 \csc^2 \Theta) \sin \Theta d\Theta d\Phi.
\]
(76)
Working in linearized theory, numerical relativists often take the leading terms in $\mathcal{A}$ and $\mathcal{B}$ to represent the ‘gravitational waveforms’ $h_\alpha$ and $h_\phi$. Equations (73) suggest that other terms are present, even in linearized theory.
Again, we emphasize that the intermediate $x^a$ chart does not intrude—formulae (74)–(76) apply in the numerical $X^a = (T, R, \Theta, \Phi)$ chart. Only the first can be deduced from linearized theory.

Why are these formulae so complicated, when compared with the original papers (Bondi et al 1962, Newman and Unti 1962) or even the formulae in section 5? Well the coordinates and tetrads of the originals were very carefully chosen to simplify the problem, and much of this paper has been spent building the relationship between the numerical relativist’s $X^a = (T, R, \Theta, \Phi)$ chart and the $x^a = (u, r, \theta, \phi)$ chart and adapted tetrad used in this paper, in which the formulae look almost as simple as in the original approaches. One way to avoid the complexity is to design a numerical approach based on Penrose’s geometrical approach (Penrose 1963), but that brings in different problems and complexities.

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Appendix A. Computational details

$h^{ab}$ and $k^{ab}$ occurring in (11) are given by

\begin{align*}
h^{00} &= -h_{00}, \quad h^{01} = h_{01}, \quad h^{02} = h_{02}, \quad h^{03} = \csc^2 \theta h_{03}, \\
h^{11} &= -h_{11}, \quad h^{12} = -h_{12}, \quad h^{13} = -\csc^2 \theta h_{13}, \\
h^{22} &= -h_{22}, \quad h^{23} = -\csc^2 \theta h_{23}, \quad h^{33} = -\csc^4 \theta h_{33}.
\end{align*}

(A.1)

and

\begin{align*}
k^{00} &= -k_{00} + h_{00}^2 + h_{01}^2 + h_{02}^2 - \csc^2 \theta h_{03}^2, \\
k^{01} &= k_{01} - h_{00} h_{01} + h_{01} h_{11} + h_{02} h_{12} + \csc^2 \theta h_{03} h_{23}, \\
k^{02} &= k_{02} - h_{00} h_{02} + h_{01} h_{12} + h_{02} h_{22} + \csc^2 \theta h_{03} h_{23}, \\
k^{03} &= \csc^2 \theta (k_{03} - h_{00} h_{03} + h_{01} h_{13} + h_{02} h_{23} + \csc^2 \theta h_{03} h_{33}), \\
k^{11} &= -k_{11} + h_{11}^2 - h_{12}^2 - \csc^2 \theta h_{13}^2, \\
k^{12} &= -k_{12} + h_{01} h_{02} - h_{11} h_{12} - h_{12} h_{22} - \csc^2 \theta h_{13} h_{23}, \\
k^{13} &= \csc^2 \theta (-k_{13} + h_{01} h_{03} - h_{11} h_{13} - h_{12} h_{23} - \csc^2 \theta h_{13} h_{33}), \\
k^{22} &= -k_{22} + h_{02}^2 - h_{12}^2 - h_{22}^2 - \csc^2 \theta h_{23}^2, \\
k^{23} &= \csc^2 \theta (-k_{23} + h_{02} h_{03} - h_{12} h_{13} - h_{22} h_{23} - \csc^2 \theta h_{23} h_{33}), \\
k^{33} &= \csc^4 \theta (-k_{33} + h_{03}^2 - h_{13}^2 - h_{23}^2 - \csc^2 \theta h_{33}^2).
\end{align*}

(A.2)

$d_{ab}$ occurring in (26) (after imposing conditions (27)) are given by

\begin{align*}
a_{00} &= h_{00} - 2q_1, \\
a_{01} &= h_{00} + h_{01} + 2m_1 - q_1, \\
a_{02} &= h_{02} - y_3 a, \\
a_{03} &= h_{03} - z_3 a \sin^2 \theta, \\
a_{11} &= h_{00} + 2h_{01} + h_{11} + 4m_1, \\
a_{12} &= h_{02} + h_{12} + y_2, \\
a_{13} &= h_{03} + h_{13} + z_2 \sin^2 \theta, \\
a_{22} &= h_{22} - 2z_3 \sin^2 \theta, \\
a_{23} &= -h_{12} - y_2, \\
a_{33} &= h_{33} + 2z_2 \sin^2 \theta.
\end{align*}
\[ a_{22} = h_{22} - 2y_{2,\theta}, \]
\[ a_{23} = h_{23} - y_{2,\phi} - z_{2,\phi} \sin^2 \theta, \]
\[ a_{33} = h_{33} - 2z_{2,\phi}. \]  \hspace{1cm} (A.3)

The relation between the tetrad components \( c^{\alpha\alpha} \) and \( c_{mn} \) is (here \( s = 2^{-1/2} \))
\[ c^{00} = \frac{1}{2} c_{01} - c_{00}, \]
\[ c^{01} = \frac{1}{2} c_{00} - \frac{1}{2} c_{01} - c_{10} + \frac{1}{2} c_{11}, \]
\[ c^{02} = s(c_{20} + c_{30}) - \frac{1}{2} s(c_{21} + c_{31}), \]
\[ c^{03} = is[(c_{20} - c_{30}) - \frac{1}{2} s(c_{21} - c_{31}) \csc \theta], \]
\[ c^{10} = -c_{01}, \]
\[ c^{11} = \frac{1}{2} c_{01} - c_{11}, \]
\[ c^{12} = s(c_{21} + c_{31}), \]
\[ c^{13} = is(c_{21} - c_{31}) \csc \theta, \]
\[ c^{20} = s(c_{02} + ic_{03} \csc \theta), \]
\[ c^{21} = s(c_{12} - \frac{1}{2} c_{02}) - is \left( \frac{1}{2} c_{03} - c_{13} \right) \csc \theta, \]
\[ c^{22} = -\frac{1}{2}(c_{22} + c_{32}) - \frac{1}{2} i(c_{23} + c_{33}) \csc \theta, \]
\[ c^{23} = -\frac{1}{2} i(c_{22} - c_{32}) \csc \theta + \frac{1}{2} (c_{23} - c_{33}) \csc^2 \theta, \]
\[ c^{30} = s(c_{02} - ic_{03} \csc \theta), \]
\[ c^{31} = s(c_{12} - \frac{1}{2} c_{02}) + is \left( \frac{1}{2} c_{03} - c_{13} \right) \csc \theta, \]
\[ c^{32} = -\frac{1}{2}(c_{22} + c_{32}) + \frac{1}{2} i(c_{23} + c_{33}) \csc \theta, \]
\[ c^{33} = -\frac{1}{2} i(c_{22} - c_{32}) \csc \theta - \frac{1}{2} (c_{23} - c_{33}) \csc^2 \theta. \]  \hspace{1cm} (A.4)

One possible relation between the tetrad coefficients \( c_{mn} \) and the metric coefficients \( a_{mn} \) and Lorentz parameters \( a_m \) is
\[ c_{00} = a_1, \]
\[ c_{01} = \frac{1}{2} a_{11}, \]
\[ c_{02} = a_{12} - 2a_4, \]
\[ c_{03} = a_{13} + 2a_5 \sin \theta, \]
\[ c_{10} = \frac{1}{2} a_{20} - \frac{1}{2} a_1, \]
\[ c_{11} = a_{01} - \frac{1}{2} a_{11} - a_1, \]
\[ c_{12} = a_{02} - \frac{1}{2} a_{12} - s(2a_2 - a_4), \]
\[ c_{13} = a_{03} - \frac{1}{2} a_{13} + s(2a_3 - a_5) \sin \theta, \]
\[ c_{20} = a_2 + ia_3, \]
\[ c_{21} = a_4 + ia_5, \]
\[ c_{22} = \frac{1}{2} s(a_{22} - is(a_{23} - a_6) \csc \theta, \]
\[ c_{23} = sa_6 - \frac{1}{2} ia_{13} \csc \theta, \]
\[ c_{30} = a_2 - ia_3, \]
\[ c_{31} = a_4 - ia_5, \]
\[ c_{32} = \frac{1}{2} s(a_{22} + is(a_{23} - a_6) \csc \theta, \]
\[ c_{33} = sa_6 + \frac{1}{2} ia_{13} \csc \theta, \]  \hspace{1cm} (A.5)

where \( s = 2^{-1/2} \). Other representations are possible.
Appendix B. Tetrad formalism

Here we define the notation and summarize the results. The reader to whom this material is unfamiliar should consult introductory material, e.g., Chandrasekhar (1983) chapter 1, section 7.

In this section, $a, b, c, \ldots$ are coordinate indices while $\alpha, \beta, \gamma, \ldots$ are tetrad indices.

At each spacetime point $P$, we introduce a basis of vectors:

$$ e^a_\alpha, \quad \alpha \in [0, 3], \quad a \in [0, 3]. \quad (B.1) $$

Then the matrix

$$ e^a_\alpha = \begin{pmatrix} e^0_0 & e^0_1 & \ldots \\ e^1_0 & e^1_1 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} $$

is non-singular and we denote its inverse by $e^a_\alpha$. Thus,

$$ e^a_\alpha e^\alpha_b = \delta^a_b, \quad e^a_\alpha e^\alpha_a = \delta^a_a. \quad (B.2) $$

e$ represent the dual basis of covectors. As usual chart, indices are lowered (raised) using $g_{ab}(g^{ab})$.

An additional assumption made here is that

$$ \epsilon^{\alpha\beta} = \epsilon^{\alpha\beta}, \quad \epsilon^{\alpha\beta} = \epsilon^{\alpha\beta} $$

is a constant symmetric matrix with inverse $\epsilon^{\alpha\beta}$. Thus,

$$ \epsilon^{\alpha\beta} \epsilon^{\beta\gamma} = \delta^\gamma_\alpha. \quad (B.3) $$

The choice $\epsilon^{\alpha\beta} = \text{diag}(1, -1, -1, -1)$ gives an orthonormal tetrad, but here we choose

$$ \epsilon^{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (B.5) $$

which gives a NP tetrad (Newman and Penrose 1962).

Then it is easy to see that

$$ \epsilon^{\alpha\beta} \epsilon^{\beta\gamma} = \delta^\gamma_\alpha, \quad \epsilon^{\alpha\beta} \epsilon^{\beta\alpha} = \delta^\alpha_\alpha, \quad \epsilon^{\alpha\beta} \epsilon^{\beta\gamma} = \epsilon^{\alpha\gamma}, \quad (B.6) $$

so that tetrad indices are lowered (raised) using $\epsilon^{\alpha\beta}$ ($\epsilon_{\beta\alpha}$).

The *Ricci rotation coefficients* $\gamma_{\mu\nu}$ are defined via

$$ e_{\mu\beta\gamma} = \gamma_{\mu\nu} e^\gamma_\nu, \quad (B.7) $$

where the metric covariant derivative has been used, and since $\epsilon^{\alpha\beta}$ is constant we must have $\gamma_{\mu\nu} = \gamma_{(\mu|\nu)}$.

The tetrad *structure constants* $C^\gamma_{\alpha\beta}$ are defined via

$$ [e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma, \quad (B.8) $$

and clearly $C^\gamma_{\alpha\beta} = C^\gamma_{[\alpha\beta]}$. If we let (B.8) act on a scalar function $f$, note that the metric connection is symmetric, and use (B.7), then it is easy to see that

$$ C^\gamma_{\alpha\beta} = \gamma^\gamma_{\beta\alpha} - \gamma^\gamma_{\alpha\beta}. \quad (B.9) $$
which implies
\[ \gamma_{\lambda \mu \nu} = \frac{1}{2} (C_{\nu \lambda \mu} - C_{\lambda \mu \nu} - C_{\mu \nu \lambda}). \]  
(B.10)

It is important to realize that \( C_{\lambda \mu \nu} \) do not involve the connection. For (B.9), (B.7) and the fact that the connection is symmetric means that
\[ \gamma_{\lambda \mu \nu} = \epsilon_{\lambda \mu \nu} \gamma. \]  
(B.11)

Next, the Ricci identity applied to \( \epsilon_{\alpha \beta} \) gives
\[ R_{\alpha \beta \gamma \delta} = \gamma_{\alpha \beta \gamma, \delta} \epsilon_{\alpha \beta} + \gamma_{\alpha \beta \delta, \gamma} \epsilon_{\alpha \beta} + \gamma_{\alpha \beta \epsilon} \gamma_{\epsilon \delta \gamma} - \gamma_{\alpha \beta \epsilon} \gamma_{\epsilon \delta \gamma}. \]  
(B.12)

and finally
\[ R_{\alpha \gamma} = \epsilon_{\alpha \beta} R_{\alpha \beta \gamma \delta}. \]  
(B.13)

Note that throughout the paper, we use these forms for the curvature tensors. Thus \( R_{12} \) means \( R_{\alpha \beta} \) with \( \alpha = 1 \) and \( \beta = 2 \), which is not the same as \( R_{ab} \) with \( a = 1 \) and \( b = 2 \).

Our algorithm starts from the sets \( \{ e_{\alpha a} \} \) and \( \{ e_{\alpha a} \} \). We compute \( C_{\alpha \beta \gamma \delta} \) from (B.11) and of course \( \gamma_{\alpha \beta \gamma \delta} = \epsilon_{\alpha \beta \gamma} C_{\alpha \beta \gamma \delta} \). Next, we compute \( \gamma_{\alpha \beta \gamma \delta} \) from (B.12) and finally the curvature tensors from (B.12) and (B.13). Although this looks ponderous, it can easily be automated using a computer algebra system.

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