Riccati Diagonalization of Hermitian Matrices

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Abstract

In this paper a geometric method based on Grassmann manifolds and matrix Riccati equations to make hermitian matrices diagonal is presented. We call it Riccati Diagonalization.

1 Introduction

In this paper we consider a finite dimensional quantum model, so its Hamiltonian is a (finite-dimensional) hermitian matrix. In order to solve the model we want to make the Hamiltonian diagonal.

Although we have a standard method for the purpose, to perform it explicitly is very hard (maybe, almost impossible). Let us make a brief introduction, see [1], [2].

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Let $H$ be a hermitian matrix, namely

$$H \in \{ H \in M(n; \mathbb{C}) \mid H^\dagger = H \}. \quad (1)$$

When we want to make $H$ diagonal Elementary Linear Algebra shows the following diagonalization procedure $[A] \implies [B] \implies [C]:$

[A] First we calculate eigenvalues of $H$,

$$0 = |\lambda E - H| = \lambda^n - \text{tr}H\lambda^{n-1} + \cdots + (-1)^n \det H. \quad (2)$$

There are $n$ real solutions although it is almost impossible to look for exact ones, so let those be $\{ \lambda_1, \lambda_2, \ldots, \lambda_n \}$.

[B] Next we find eigenvectors $\{|\lambda\rangle\}$ corresponding to eigenvalues

$$H|\lambda_j\rangle = \lambda_j|\lambda_j\rangle \quad \text{and} \quad \langle \lambda_i|\lambda_j\rangle = \delta_{ij} \quad (3)$$

for all $1 \leq i, j \leq n$. It is also almost impossible to carry out.

[C] Last by setting

$$U = (|\lambda_1\rangle, |\lambda_2\rangle, \ldots, |\lambda_n\rangle) \quad (4)$$

we finally obtain

$$H = UD_HU^\dagger \quad (5)$$

where $D_H = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal matrix.

The procedure is standard, while to carry out it completely is another problem\footnote{in Japan it is called a pie in the sky}. Even if $n = 3$ it is very hard. In fact, for the hermitian matrix

$$H = \begin{pmatrix} h_1 & \bar{\alpha} & \bar{\beta} \\ \alpha & h_2 & \bar{\gamma} \\ \beta & \gamma & h_3 \end{pmatrix} \in H(3; \mathbb{C})$$

carry out the diagonalization. As far as we know it has not been given in any textbook on linear algebra.
Note that the characteristic equation $f(\lambda) = |\lambda E - H|$ is given by

$$f(\lambda) = \lambda^3 - (h_1 + h_2 + h_2) \lambda^2 + (h_1 h_2 + h_1 h_3 + h_2 h_3 - |\alpha|^2 - |\beta|^2 - |\gamma|^2) \lambda + |\gamma|^2 h_1 + |\beta|^2 h_2 + |\alpha|^2 h_3 - h_1 h_2 h_3 - \alpha \bar{\beta} \gamma - \bar{\alpha} \beta \bar{\gamma}.$$ 

To look for exact solutions by use of Cardano formula is not easy (for example, try it by use of MATHEMATICA or MAPLE).

Therefore we present another diagonalization method based on Grassmann manifolds and matrix Riccati equations.

## 2 Riccati Diagonalization

In this section we show a new diagonalization method. Before it let us explain the idea with simple example for beginners. The target is

$$H = \begin{pmatrix} h_1 & \bar{\alpha} \\ \alpha & h_2 \end{pmatrix} \in H(2; \mathbb{C}). \quad (6)$$

To diagonalize $H$ above we consider a matrix

$$U \equiv U(z) = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \text{ for } z \in \mathbb{C}. \quad (7)$$

It is easy to check $U^\dagger U = UU^\dagger = 1_2$ and $|U| = 1$, so $U$ is special unitary ($U \in SU(2)$). Namely, $U$ is a map

$$U : \mathbb{C} (\subset S^2) \rightarrow SU(2).$$

This map is well–known in Mathematics or Mathematical Physics.

The calculation $U^\dagger HU$ gives

$$U^\dagger HU = \frac{1}{1 + |z|^2} \begin{pmatrix} 1 & z \\ -z & 1 \end{pmatrix} \begin{pmatrix} h_1 & \bar{\alpha} \\ \alpha & h_2 \end{pmatrix} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix}$$

$$= \frac{1}{1 + |z|^2} \begin{pmatrix} h_1 + \bar{\alpha} z + \alpha \bar{z} + h_2 |z|^2 & \bar{\alpha} - (h_1 - h_2) z - \alpha \bar{z}^2 \\ \alpha - (h_1 - h_2) \bar{z} - \bar{\alpha} z^2 & h_2 - \bar{\alpha} z - \alpha \bar{z} + h_1 |z|^2 \end{pmatrix}. \quad (8)$$
so if we assume the equation
\[ \alpha - (h_1 - h_2)z - \alpha z^2 = 0 \iff \bar{\alpha} z^2 + (h_1 - h_2)z - \alpha = 0 \] (9)
we have the diagonal matrix
\[ U^\dagger H U = \begin{pmatrix} \frac{h_1 + \bar{\alpha} z + \alpha \bar{z} + h_2 |z|^2}{1 + |z|^2} & \frac{h_2 - \bar{\alpha} z - \alpha \bar{z} + h_1 |z|^2}{1 + |z|^2} \\ \frac{h_1 + \bar{\alpha} z + \alpha \bar{z} + h_2 |z|^2}{1 + |z|^2} & \frac{h_2 - \bar{\alpha} z - \alpha \bar{z} + h_1 |z|^2}{1 + |z|^2} \end{pmatrix} \] (10)
From this two eigenvalues are obtained by
\[ \lambda_1 = \frac{h_1 + \bar{\alpha} z + \alpha \bar{z} + h_2 |z|^2}{1 + |z|^2}, \quad \lambda_2 = \frac{h_2 - \bar{\alpha} z - \alpha \bar{z} + h_1 |z|^2}{1 + |z|^2} \] (11)
under the equation (9), whose solutions are easily given by
\[ z = \frac{-(h_1 - h_2) \pm \sqrt{(h_1 - h_2)^2 + 4|\alpha|^2}}{2\bar{\alpha}}. \] (12)

A comment is in order. The equation (9) is a special version of (generalized) Riccati equations.

As shown in the example our diagonalization method is different from usual one (a kind of reverse procedure). Let us state our procedure.

**Riccati Diagonalization**

[A] For \( H \in H(n; \mathbb{C}) \) we prepare a unitary matrix \( U = U(Z) \in U(n) \) where \( Z \) is a parameter matrix and calculate
\[ U^\dagger H U \equiv W = (w_{ij}) \].

[B] We set
\[ w_{ij} = 0 \quad \text{for} \quad 1 \leq j < i \leq n \]
and solve these simultaneous equations (a system of Riccati equations) to determine \( Z = (z_{kl}) \).

[C] We finally obtain the diagonal matrix
\[ W = \text{diag}(w_{11}, w_{22}, \cdots, w_{nn}) \]
where each component is an eigenvalue of \( H \) under [B].
3 General Case

In this section we consider a generalization of the example in the preceding section. See \[3\], \[4\] as a general introduction to Grassmann manifolds and also \[5\] and its references as an application.

Namely, we treat a hermitian matrix

\[
H = \begin{pmatrix} H_+ & V^\dagger \\ V & H_- \end{pmatrix} \in H(n; \mathbb{C})
\]  

(13)

where

\[
H_+ \in H(k; \mathbb{C}), \quad H_- \in H(n - k; \mathbb{C}), \quad V \in M(n - k, k; \mathbb{C})
\]

for \(1 \leq k \leq n - 1\). In order to make \(H\) a direct sum form we prepare a unitary matrix

\[
U = U(Z) = \begin{pmatrix} 1_k & -Z^\dagger \\ Z & 1_{n-k} \end{pmatrix} \begin{pmatrix} (1_k + Z^\dagger Z)^{-1/2} \\ (1_{n-k} + ZZ^\dagger)^{-1/2} \end{pmatrix}
\]  

(14)

\[
\equiv U_M U_D
\]

where \(Z \in M(n - k, k; \mathbb{C})\) is a parameter matrix. \(U\) is a map

\[
U : M(n - k, k; \mathbb{C}) \rightarrow SU(n)
\]

and \(Z\) is a local coordinate of the Grassmann manifold \(G_k(\mathbb{C}^n)\) defined by

\[
G_k(\mathbb{C}^n) = \{ P \in M(n; \mathbb{C}) \mid P^2 = P, \; P^\dagger = P, \; \text{tr}P = k \}
\]

\[
= \{ U P_0 U^\dagger \mid U \in U(n) \}
\]

\[
\cong U(n)/U(k) \times U(n - k)
\]

with \(P_0\) given by

\[
P_0 = \begin{pmatrix} 1_k \\ 0_{n-k} \end{pmatrix}.
\]

Note that \(\dim_\mathbb{C} G_k(\mathbb{C}^n) = k(n - k) = \dim_\mathbb{C} M(n - k, k; \mathbb{C})\). The local parametrization of \(G_k(\mathbb{C}^n)\) is more explicitly given by

\[
P(Z) = U(Z) P_0 U(Z)^\dagger = \begin{pmatrix} 1_k & -Z^\dagger \\ Z & 1_{n-k} \end{pmatrix} \begin{pmatrix} 1_k \\ 0_{n-k} \end{pmatrix} \left( \begin{pmatrix} 1_k & -Z^\dagger \\ Z & 1_{n-k} \end{pmatrix} \right)^{-1}
\]  

(15)
where we have used the relation
\[
\begin{pmatrix}
  1_k & Z^\dagger \\
  Z & 1_{n-k}
\end{pmatrix}^{-1} = \begin{pmatrix}
  (1_k + Z^\dagger Z)^{-1} \\
  (1_{n-k} + ZZ^\dagger)^{-1}
\end{pmatrix} \begin{pmatrix}
  1_k & -Z^\dagger \\
  Z & 1_{n-k}
\end{pmatrix}^\dagger.
\]

Let us calculate \( U_M^\dagger H U_M \) :
\[
U_M^\dagger H U_M = \begin{pmatrix}
  1_k & Z^\dagger \\
  -Z & 1_{n-k}
\end{pmatrix} \begin{pmatrix}
  H_+ & V^\dagger \\
  V & H_-
\end{pmatrix} \begin{pmatrix}
  1_k & -Z^\dagger \\
  Z & 1_{n-k}
\end{pmatrix}
= \begin{pmatrix}
  H_+ + Z^\dagger V + V^\dagger Z + Z^\dagger H_- Z & V^\dagger - H_+ Z^\dagger + Z^\dagger H_- - Z^\dagger V Z^\dagger \\
  V - Z H_+ + H_- Z - ZV^\dagger Z & H_- - ZV^\dagger - V Z^\dagger + Z H_+ Z^\dagger
\end{pmatrix}. \tag{16}
\]

From this we set
\[
V - Z H_+ + H_- Z - ZV^\dagger Z = 0 \iff ZV^\dagger Z + Z H_+ - H_- Z - V = 0. \tag{17}
\]

This is just the matrix Riccati equation. Under the condition we obtain the block form
\[
U^\dagger H U = \begin{pmatrix}
  (1_k + Z^\dagger Z)^{-1/2} \tilde{H}_+ (1_k + Z^\dagger Z)^{-1/2} \\
  (1_{n-k} + ZZ^\dagger)^{-1/2} \tilde{H}_- (1_{n-k} + ZZ^\dagger)^{-1/2}
\end{pmatrix}
\tag{18}
\]
where
\[
\tilde{H}_+ = H_+ + Z^\dagger V + V^\dagger Z + Z^\dagger H_- Z, \quad \tilde{H}_- = H_- - ZV^\dagger - V Z^\dagger + Z H_+ Z. \tag{19}
\]

How to solve the Riccati equation (17) is not known as far as we know. In fact, it is very hard, so we must satisfy by finding some approximate solution at the present time.

**Approximation I**

First, by rejecting the quadratic term we have
\[
Z H_+ - H_- Z = V. \tag{20}
\]

This solution is well–known to become
\[
Z = \int_0^\infty e^{tH_-} V e^{-tH_+} dt. \tag{21}
\]

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under some condition on $H_-$ and $H_+$. See for example [6]. In fact,

$$ZH_- - H_+ = \int_0^\infty \{e^{tH_-}V e^{-tH_+} - H_- e^{tH_-}V e^{-tH_+}\}dt$$

$$= -\int_0^\infty \frac{d}{dt}(e^{tH_-}V e^{-tH_+})dt$$

$$= -[e^{tH_-}V e^{-tH_+}]^\infty_0$$

$$= V$$

under the condition

$$\lim_{t \to \infty} e^{tH_-}V e^{-tH_+} = 0.$$  (22)

**Approximation II**

Next, let us consider another approximation. We assume that $n = 2m$, $k = m$ and $V$ is invertible ($V \in GL(m; \mathbb{C})$). Then, by remembering

$$ax^2 + 2bx + c = 0 \implies a(x + \frac{b}{a})^2 = -c + \frac{b^2}{a}$$

we have

$$\{Z - H_-(V^\dagger)^{-1}\}V^\dagger\{Z - (V^\dagger)^{-1}H_+\} = V - H_-(V^\dagger)^{-1}H_+$$  (23)

from (17). Here if we can choose $Z$ as

$$Z + (V^\dagger)^{-1}H_+ \in GL(m; \mathbb{C})$$

then we have a recursive relation

$$Z = H_-(V^\dagger)^{-1} + \{V - H_-(V^\dagger)^{-1}H_+\} \frac{1}{Z + (V^\dagger)^{-1}H_+}(V^\dagger)^{-1}.$$  (24)

Now by inserting an approximate solution (21) into the equation above we obtain the approximate solution

$$Z \approx H_-(V^\dagger)^{-1} + \{V - H_-(V^\dagger)^{-1}H_+\} \frac{1}{\int_0^\infty e^{tH_-}V e^{-tH_+}dt + (V^\dagger)^{-1}H_+}(V^\dagger)^{-1}.$$  (25)

if

$$\int_0^\infty e^{tH_-}V e^{-tH_+}dt + (V^\dagger)^{-1}H_+ \in GL(m; \mathbb{C})$$
or
\[ H_+ + \int_0^\infty V^\dagger e^{tH} - V e^{-tH} \, dt \in GL(m; \mathbb{C}). \]

A comment is in order. We don’t know at the present time whether our approximate solution is convenient enough or not.

## 4 Reduction of Riccati Diagonalization

In this section we give an explicit procedure of Riccati diagonalization. General Hamiltonian is

\[
H = \begin{pmatrix}
  h_1 & \bar{v}_{21} & \bar{v}_{31} & \cdots & \bar{v}_{n-1,1} & \bar{v}_{n1} \\
  v_{21} & h_2 & \bar{v}_{32} & \cdots & \bar{v}_{n-1,2} & \bar{v}_{n2} \\
  v_{31} & v_{32} & h_3 & \cdots & \bar{v}_{n-1,3} & \bar{v}_{n3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  v_{n-1,1} & v_{n-1,2} & v_{n-1,3} & \cdots & h_{n-1} & \bar{v}_{n,n-1} \\
  v_{n1} & v_{n2} & v_{n3} & \cdots & v_{n,n-1} & h_n
\end{pmatrix} \in H(n; \mathbb{C})
\]  \hspace{1cm} (26)

and we write as

\[
H = \begin{pmatrix}
  H_+ & V^\dagger \\
  V & h_n
\end{pmatrix}, \quad V = (v_{n1}, v_{n2}, \cdots, v_{n,n-1})
\]  \hspace{1cm} (27)

for simplicity. We prepare a unitary matrix

\[
U = \begin{pmatrix}
  1_{n-1} & -Z^\dagger \\
  Z & 1
\end{pmatrix} \begin{pmatrix}
  (1_{n-1} + Z^\dagger Z)^{-1/2} \\
  (1 + ZZ^\dagger)^{-1/2}
\end{pmatrix}
\]  \hspace{1cm} (28)

where \( Z = (z_1, z_2, \cdots, z_{n-1}) \). Then the Riccati equation is

\[
ZV^\dagger Z + ZH_+ - h_n Z - V = 0 \iff \left( \sum_{j=1}^{n-1} \bar{v}_{nj} z_j \right) z_k + \sum_{j=1}^{n-1} (H_+)^{jk} z_j - h_n z_k - v_{nk} = 0
\]

for \( 1 \leq k \leq n - 1 \).  \hspace{1cm} (29)

Note that to solve the equation(s) above explicitly is very hard, so in general we must satisfy by constructing some approximate solution.
If we can solve the equation(s) then

\[
U^{\dagger}HU = \begin{pmatrix}
(1_{n-1} + Z^{\dagger}Z)^{-1/2} \tilde{H}_{+}(1_{n-1} + Z^{\dagger}Z)^{-1/2} \\
\tilde{h}_{n} \frac{1}{1 + \sum_{j=1}^{n-1} |z_{j}|^2}
\end{pmatrix}
\]

(30)

\[
\tilde{h}_{n} = h_{n} - \sum_{j=1}^{n-1} (z_{j}\bar{e}_{nj} + c.c.) + \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} z_{j}(H_{+})_{jk}\bar{z}_{k}
\]

and the procedure is reduced to the calculation of

\[
(1_{n-1} + Z^{\dagger}Z)^{-1/2} \tilde{H}_{+}(1_{n-1} + Z^{\dagger}Z)^{-1/2}
\]

so we must calculate the term \((1_{n-1} + Z^{\dagger}Z)^{-1/2}\) exactly.

We write

\[
Z = (z_{1}, z_{2}, \cdots, z_{n-1}) = z_{1}(1, w_{2}, \cdots, w_{n-1}) \equiv z_{1}(1, W), \quad w_{j} = z_{j}/z_{1}
\]

\[
ZZ^{\dagger} = |z_{1}|^2(1 + WW^{\dagger}) = \sum_{j=1}^{n-1} |z_{j}|^2
\]

for simplicity. Then

\[
1_{n-1} + Z^{\dagger}Z = 1_{n-1} + |z_{1}|^2 \begin{pmatrix} 1 & W \\
W^{\dagger} & 1_{n-2} \end{pmatrix}
\]

and a unitary matrix given by

\[
\tilde{U} = \begin{pmatrix} 1 & -W \\
W^{\dagger} & 1_{n-2} \end{pmatrix} \begin{pmatrix} (1 + WW^{\dagger})^{-1/2} \\
(1_{n-2} + W^{\dagger}W)^{-1/2} \end{pmatrix}
\]

(31)

gives

\[
\tilde{U} \begin{pmatrix} 1 \\
0_{n-2} \end{pmatrix} \tilde{U}^{\dagger} = \frac{1}{1 + WW^{\dagger}} \begin{pmatrix} 1 & W \\
W^{\dagger} & W^{\dagger}W \end{pmatrix}.
\]
Therefore

\[
1_{n-1} + Z^\dagger Z = 1_{n-1} + |z_1|^2(1 + WW^\dagger)\tilde{U} \begin{pmatrix} 1 \\ 0_{n-2} \end{pmatrix} \tilde{U}^\dagger
\]

\[
= \tilde{U} \left\{ 1_{n-1} + \sum_{j=1}^{n-1} |z_j|^2 \begin{pmatrix} 1 \\ 0_{n-2} \end{pmatrix} \right\} \tilde{U}^\dagger
\]

\[
= \tilde{U} \begin{pmatrix} 1 + \sum_{j=1}^{n-1} |z_j|^2 \\ 1_{n-2} \end{pmatrix} \tilde{U}^\dagger
\]

and we have

\[
(1_{n-1} + Z^\dagger Z)^{-1/2} = \tilde{U} \begin{pmatrix} 1 \\ \sqrt{1 + \sum_{j=1}^{n-1} |z_j|^2} \\ 1_{n-2} \end{pmatrix} \tilde{U}^\dagger.
\] (32)

As a result the reduced Hamiltonian is

\[
(1_{n-1} + Z^\dagger Z)^{-1/2} \tilde{H}_+ (1_{n-1} + Z^\dagger Z)^{-1/2}
\]

\[
= \tilde{U} \begin{pmatrix} 1 \\ \sqrt{1 + \sum_{j=1}^{n-1} |z_j|^2} \\ 1_{n-2} \end{pmatrix} \tilde{U}^\dagger \tilde{H}_+ \tilde{U} \begin{pmatrix} 1 \\ \sqrt{1 + \sum_{j=1}^{n-1} |z_j|^2} \\ 1_{n-2} \end{pmatrix} \tilde{U}^\dagger
\] (33)

and we obtain

\[
U^\dagger HU
\]

\[
= \tilde{U} \begin{pmatrix} 1 \\ \sqrt{1 + \sum_{j=1}^{n-1} |z_j|^2} \\ 1_{n-2} \end{pmatrix} \tilde{U}^\dagger \tilde{H}_+ \tilde{U} \begin{pmatrix} 1 \\ \sqrt{1 + \sum_{j=1}^{n-1} |z_j|^2} \\ 1_{n-2} \end{pmatrix} \tilde{U}^\dagger
\]

(34)

We have only to continue the reduction process one after another.

A comment is in order. Note that it is not easy to calculate \((1_{n-k} + Z^\dagger Z)^{-1/2}\) for \(Z \in M(k, n-k; \mathbb{C})\) and \(2 \leq k \leq n-2\).
Note There is no need to calculate \((1_{n-2} + W^\dagger W)^{-1/2}\) in \(\tilde{U}\) because

\[
\tilde{U} \left( \frac{1}{\sqrt{1 + \sum_{j=1}^{n-1} |z_j|^2}} \right)_{1_{n-2}} \tilde{U}^\dagger
\]

\[
= \begin{pmatrix} 1 & -W \\ W^\dagger & 1_{n-2} \end{pmatrix} \left( \frac{1}{(1 + WW^\dagger)\sqrt{1 + \sum_{j=1}^{n-1} |z_j|^2}} \right) \begin{pmatrix} (1_{n-2} + W^\dagger W)^{-1} \\ 1_{n-2} \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & -W \\ W^\dagger & 1_{n-2} \end{pmatrix} \left( \frac{1}{(1 + WW^\dagger)\sqrt{1 + \sum_{j=1}^{n-1} |z_j|^2}} \right) \begin{pmatrix} 1_{n-2} - \frac{1}{1 + WW^\dagger} WW^\dagger \\ 1_{n-2} \end{pmatrix}
\]

from the definition of \(\tilde{U}\). This is important.

Last, let us make a comment. If \(n = 3\), namely \(Z = (z_1, z_2)\) then we have a direct method (without using \(W\)). For

\[1_2 + Z^\dagger Z = \begin{pmatrix} 1 + |z_1|^2 & \bar{z}_1 z_2 \\ z_1 \bar{z}_2 & 1 + |z_2|^2 \end{pmatrix}\]

we set

\[U = \frac{1}{\sqrt{|z_1|^2 + |z_2|^2}} \begin{pmatrix} \bar{z}_1 & -z_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \in SU(2).\]

Then we have

\[1_2 + Z^\dagger Z = U \begin{pmatrix} 1 + |z_1|^2 + |z_2|^2 \\ 1 \end{pmatrix} U^\dagger\]

and

\[(1_2 + Z^\dagger Z)^{-1/2} = U \left( \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} \right) U^\dagger\]

\[= \frac{1}{|z_1|^2 + |z_2|^2} \begin{pmatrix} \bar{z}_1 & -z_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \left( \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}} \right) \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}\]

\[= \frac{1}{|z_1|^2 + |z_2|^2} \begin{pmatrix} \sqrt{|z_1|^2} & |z_2|^2 - z_1 \bar{z}_2 \\ |z_2|^2 - z_1 \bar{z}_2 & \sqrt{|z_2|^2} \end{pmatrix} \begin{pmatrix} \bar{z}_1 z_2 \\ \bar{z}_1 \end{pmatrix} + \frac{1}{|z_1|^2 + |z_2|^2} \begin{pmatrix} \sqrt{|z_1|^2} & |z_2|^2 - z_1 \bar{z}_2 \\ |z_2|^2 - z_1 \bar{z}_2 & \sqrt{|z_2|^2} \end{pmatrix} \begin{pmatrix} \bar{z}_1 z_2 \\ \bar{z}_1 \end{pmatrix}. \tag{35}\]

This form looks smart and will be used in the next section.
5 Example

In this section we apply our method to the following important example

\[
H = \begin{pmatrix} h_1 & \bar{\alpha} & \bar{\beta} \\ \alpha & h_2 & \bar{\gamma} \\ \beta & \gamma & h_3 \end{pmatrix}.
\]  

(36)

Since

\[
H^+ = \begin{pmatrix} h_1 & \bar{\alpha} \\ \alpha & h_2 \end{pmatrix}, \quad V = (\beta, \gamma), \quad Z = (z_1, z_2)
\]

the Riccati equation is

\[
\begin{align*}
\bar{\beta} z_1^2 + \bar{\gamma} z_1 z_2 + (h_1 z_1 + \alpha z_2) - h_3 z_1 - \beta &= 0, \\
\bar{\gamma} z_2^2 + \bar{\beta} z_1 z_2 + (\bar{\alpha} z_1 + h_2 z_2) - h_3 z_2 - \gamma &= 0
\end{align*}
\]
or

\[
\begin{align*}
\bar{\beta} z_1^2 + (h_1 - h_3) z_1 - \beta &= -z_2 (\alpha + \bar{\gamma} z_1), \\
\bar{\gamma} z_2^2 + (h_2 - h_3) z_2 - \gamma &= -z_1 (\bar{\alpha} + \bar{\beta} z_2). \tag{37}
\end{align*}
\]

Let us solve the equations. We get \(z_1\) by solving the equation

\[
\begin{align*}
\{ \bar{\beta} \bar{\gamma} (h_1 - h_2) - \alpha \bar{\beta}^2 + \bar{\alpha} \bar{\gamma} \} z_1^3 + \\
[\bar{\gamma} ((h_1 - h_2)(h_1 - h_3) + 2|\alpha|^2 - |\beta|^2 - |\gamma|^2) - \alpha \bar{\beta} (h_1 + h_2 - 2h_3)] z_1^2 + \\
[-\alpha ((h_1 - h_3)(h_2 - h_3) - |\alpha|^2 - |\beta|^2 + 2|\gamma|^2) + \beta \bar{\gamma} (-2h_1 + h_2 + h_3)] z_1 + \\
\beta^2 \bar{\gamma} + \alpha \beta (h_2 - h_3) - \alpha^2 \gamma &= 0 \tag{38}
\end{align*}
\]

by use of Cardano formula and \(z_2\) by solving the equation

\[
\bar{\gamma} z_2^2 + (h_2 - h_3 + \bar{\beta} z_1) z_2 + \bar{\alpha} z_1 - \gamma = 0. \tag{39}
\]
If \((z_1, z_2)\) is given then the reduced Hamiltonian becomes

\[
(1_2 + Z^\dagger Z)^{-1/2} \tilde{H}_+(1_2 + Z^\dagger Z)^{-1/2} = \frac{1}{(\|z_1\|^2 + \|z_2\|^2)^2} \begin{pmatrix} z_1 \bar{z}_2 & \bar{z}_1 z_2 \\ \bar{z}_1 \bar{z}_2 & z_1 z_2 \end{pmatrix} - \frac{\tilde{H}^2}{2} \begin{pmatrix} \bar{z}_1 z_2 & z_1 \bar{z}_2 \\ \bar{z}_1 \bar{z}_2 & z_1 z_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \bar{z}_1 z_2 & z_1 \bar{z}_2 \\ \bar{z}_1 \bar{z}_2 & z_1 z_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \bar{z}_1 \bar{z}_2 & z_1 z_2 \\ \bar{z}_1 \bar{z}_2 & z_1 z_2 \end{pmatrix}
\]

\[
\equiv \begin{pmatrix} k_1 & \bar{\zeta} \\ \zeta & k_2 \end{pmatrix}. \tag{40}
\]

Therefore we have only to solve the equation

\[
\bar{\zeta}z^2 + (k_1 - k_2)z - \zeta = 0 \tag{41}
\]

as shown in the example in section 2.

Finally we obtain three eigenvalues

\[
\lambda_1 = \frac{k_1 + \bar{\zeta}z + \zeta \bar{z} + k_2|z|^2}{1 + |z|^2}, \quad \lambda_2 = \frac{k_2 - \bar{\zeta}z - \zeta \bar{z} + k_1|z|^2}{1 + |z|^2},
\]

\[
\lambda_3 = \frac{h_3 - (\bar{\beta}z_1 + \beta \bar{z}_1) - (\bar{\gamma}z_2 + \gamma \bar{z}_2) + h_1|z_1|^2 + \bar{\alpha}z_1 \bar{z}_2 + \alpha \bar{z}_1 z_2 + h_2|z_2|^2}{1 + |z_1|^2 + |z_2|^2} \tag{42}
\]

under (38), (39) and (41).

In the process of calculation MATHEMATICA or MAPLE is indispensable (calculation by force is very hard).

6 Discussion

In this paper we presented a geometric approach to diagonalization of hermitian matrices. The advantage of our method is quick diagonalization, while to obtain eigenvalues is left in
the final step. Our method is in a certain sense reverse process of the standard one, so they are dual each other.

It is not clear at the present time whether our method is convenient enough or not. Further work will be needed.

Last, let us make a comment on “Riccati structure” of Quantum Mechanics, see for example [7]. We consider the harmonic oscillator given by

\[ H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 \right). \]

In order to solve the model we usually define the annihilation and creation operators like

\[ a = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + x \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + x \right). \]

Then it is well–known that

\[ aa^\dagger = H + \frac{1}{2}, \quad a^\dagger a = H - \frac{1}{2}. \]

Next we define another annihilation and creation operators like

\[ b = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + \beta(x) \right), \quad b^\dagger = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + \beta(x) \right) \]

with unknown \( \beta(x) \).

Then, in order to satisfy the relation \( bb^\dagger = H + \frac{1}{2} \beta(x) \) must satisfy the equation

\[ \beta' + \beta^2 = 1 + x^2. \]

This is a special type of Riccati differential equation.

**Appendix**

In the appendix we consider the important example

\[
H = \begin{pmatrix} h_1 & \bar{\alpha} & \bar{\beta} \\ \bar{\alpha} & h_2 & \bar{\gamma} \\ \beta & \gamma & h_3 \end{pmatrix}
\]

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once more. We want to diagonalize the matrix at a time.

For the purpose we prepare a unitary matrix coming from the flag manifold of the second type (in our terminology, see [8], [9] and also [10], [11]) $SU(3)/U(1) \times U(1)$

$$U = U(x, y, z) = \begin{pmatrix} 1 & -(\bar{x} + \bar{y}z) & \bar{x}z - \bar{y} \\ x & \Delta_1 - x(\bar{x} + \bar{y}z) & -\bar{z} \\ y & z\Delta_1 - y(\bar{x} + \bar{y}z) & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\Delta_1}} \\ \frac{1}{\sqrt{\Delta_1, \Delta_2}} \\ \frac{1}{\sqrt{\Delta_2}} \end{pmatrix} \quad (43)$$

$$\equiv U_MU_D$$

where $\Delta_1$ and $\Delta_2$ are given by

$$\Delta_1 = 1 + |x|^2 + |y|^2, \quad \Delta_1 = 1 + |z|^2 + |xz - y|^2.$$ 

Let us calculate $U^\dagger HU$. Since $U^\dagger HU = U_D U_M^\dagger HU_M U_D$ we have only to calculate $U_M^\dagger HU_M$ because $U_D$ is diagonal. Namely,

$$U_M^\dagger HU_M = \begin{pmatrix} 1 & \bar{x} & \bar{y} \\ -(x + y\bar{z}) & \Delta_1 - \bar{x}(x + y\bar{z}) & \bar{z}\Delta_1 - \bar{y}(x + y\bar{z}) \\ xz - y & -z & 1 \end{pmatrix} \begin{pmatrix} h_1 & \bar{\alpha} & \bar{\beta} \\ \alpha & h_2 & \bar{\gamma} \\ \beta & \gamma & h_3 \end{pmatrix} \times$$

$$\begin{pmatrix} 1 & -(\bar{x} + \bar{y}z) & \bar{x}z - \bar{y} \\ x & \Delta_1 - x(\bar{x} + \bar{y}z) & -\bar{z} \\ y & z\Delta_1 - y(\bar{x} + \bar{y}z) & 1 \end{pmatrix}$$

$$\equiv \begin{pmatrix} w_{11} & \bar{w}_{21} & \bar{w}_{31} \\ w_{21} & w_{22} & \bar{w}_{32} \\ w_{31} & w_{32} & w_{33} \end{pmatrix} \quad (44)$$
where

\[ w_{21} = -(x + y\bar{z})(h_1 + x\alpha + y\bar{\beta}) + \{\Delta_1 - \bar{x}(x + y\bar{z})\}(\alpha + xh_2 + y\gamma) + \\
\{\bar{z}\Delta_1 - \bar{y}(x + y\bar{z})\}(\beta + x\gamma + yh_3), \]

\[ w_{31} = (xz - y)h_1 - z\alpha + \beta + \{(xz - y)\bar{\alpha} - zh_2 + \gamma\}x + \{(xz - y)\bar{\beta} - z\gamma + h_3\}y = (xz - y)(h_1 + x\alpha + y\bar{\beta}) - z(\alpha + xh_2 + y\gamma) + (\beta + x\gamma + yh_3), \quad (45) \]

\[ w_{32} = -(\bar{x} + \bar{y}z)[{(xz - y)h_1 - z\alpha + \beta} + \{(xz - y)\bar{\alpha} - zh_2 + \gamma\}x + \\
{(xz - y)\bar{\beta} - z\gamma + h_3\}y + \Delta_1[(xz - y)\bar{\alpha} - zh_2 + \gamma} + \{(xz - y)\bar{\beta} - z\gamma + h_3\}z] = -(\bar{x} + \bar{y}z)w_{31} + \Delta_1[(xz - y)\bar{\alpha} - zh_2 + \gamma} + \{(xz - y)\bar{\beta} - z\gamma + h_3\}z]. \]

Here by setting \( w_{21} = w_{31} = w_{32} = 0 \), we have

\[ 0 = -(x + y\bar{z})(h_1 + x\alpha + y\bar{\beta}) + \{\Delta_1 - \bar{x}(x + y\bar{z})\}(\alpha + xh_2 + y\gamma) + \\
\{\bar{z}\Delta_1 - \bar{y}(x + y\bar{z})\}(\beta + x\gamma + yh_3), \]

\[ 0 = (xz - y)(h_1 + x\alpha + y\bar{\beta}) - z(\alpha + xh_2 + y\gamma) + (\beta + x\gamma + yh_3), \quad (46) \]

\[ 0 = (xz - y)\bar{\alpha} - zh_2 + \gamma} + \{(xz - y)\bar{\beta} - z\gamma + h_3\}z. \]

Some calculation gives

\[ \bar{z} = \frac{x(h_1 + x\alpha + y\bar{\beta})}{y(h_1 + x\alpha + y\bar{\beta})} - (1 + |y|^2)(\alpha + xh_2 + y\gamma) + \bar{y}x(\beta + x\gamma + yh_3), \]

\[ z = \frac{y(y(h_1 + x\alpha + y\bar{\beta}) - (\beta + x\gamma + yh_3))}{x(h_1 + x\alpha + y\bar{\beta}) - (\alpha + xh_2 + y\gamma)}, \quad (47) \]

\[ (x\bar{\beta} - \bar{\gamma})z^2 + (h_3 - h_2 + x\alpha - y\bar{\beta})z - y\alpha + \gamma = 0 \]

in terms of \( \Delta_1 = 1 + |x|^2 + |y|^2 \).

It is not easy at the present time to solve the equations at a time.

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