Partial regularity for Navier-Stokes and liquid crystals inequalities without maximum principle

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Abstract

In 1985, V. Scheffer discussed partial regularity results for what he called solutions to the “Navier-Stokes inequality”. These maps essentially satisfy the incompressibility condition as well as the local and global energy inequalities and the pressure equation which may be derived formally from the Navier-Stokes system of equations, but they are not required to satisfy the Navier-Stokes system itself.

We extend this notion to a system considered by Fang-Hua Lin and Chun Liu in the mid 1990s related to models of the flow of nematic liquid crystals, which include the Navier-Stokes system when the “director field” \( d \) is taken to be zero. In addition to an extended Navier-Stokes system, the Lin-Liu model includes a further parabolic system which implies an a priori maximum principle for \( d \) which they use to establish partial regularity (specifically, \( P^1(S) = 0 \)) of solutions.

For the analogous “inequality” one loses this maximum principle, but here we nonetheless establish certain partial regularity results (namely \( P^{\frac{9}{2} + \delta}(S) = 0 \), so that in particular the putative singular set \( S \) has space-time Lebesgue measure zero). Under an additional assumption on \( d \) for any fixed value of a certain parameter \( \sigma \in (5, 6) \) (which for \( \sigma = 6 \) reduces precisely to the boundedness of \( d \) used by Lin and Liu), we obtain the same partial regularity (\( P^1(S) = 0 \)) as do Lin and Liu. In particular, we recover the partial regularity result (\( P^1(S) = 0 \)) of Caffarelli-Kohn-Nirenberg (1982) for “suitable weak solutions” of the Navier-Stokes system, and we verify Scheffer’s assertion that the same holds for solutions of the weaker “inequality” as well.

We remark that the proofs of partial regularity both here and in the work of Lin and Liu largely follow the proof in Caffarelli-Kohn-Nirenberg, which in turn used many ideas from an earlier work of Scheffer (1975).

1 Introduction

In [LL95] and [LL96], Fang-Hua Lin and Chun Liu consider the following system, which reduces to the classical Navier-Stokes system in the case \( d \equiv 0 \) (here we have set various parameters equal to one for simplicity):

\[
\begin{align*}
\frac{d}{dt}u - \Delta u + \nabla T \cdot [u \otimes u + \nabla d \odot \nabla d] + \nabla p & = 0 \\
\nabla \cdot u & = 0 \\
\frac{d}{dt}d - \Delta d + (u \cdot \nabla) d + f(d) & = 0
\end{align*}
\]

(1.1)

with \( f = \nabla F \) for a scalar field \( F \) given by

\[
F(x) := (|x|^2 - 1)^2,
\]

so that

\[
f(x) = 4(|x|^2 - 1)x
\]
(and in particular \( f(0) = 0 \)). We take the spatial dimension to be three, so that for some \( \Omega \subseteq \mathbb{R}^3 \) and \( T > 0 \), we are considering maps of the form

\[
u, d : \Omega \times (0, T) \to \mathbb{R}^3, \quad p : \Omega \times (0, T) \to \mathbb{R},
\]

and here

\[
F : \mathbb{R}^3 \to \mathbb{R}, \quad f : \mathbb{R}^3 \to \mathbb{R}^3
\]

are fixed as above. As usual, \( u \) represents the velocity vector field of a fluid, \( p \) is the scalar pressure in the fluid, and, as in nematic liquid crystals models, \( d \) corresponds roughly to the “director field” representing the local orientation of rod-like molecules, with \( u \) also giving the velocities of the centers of mass of those anisotropic molecules.

In (1.1), for vector fields \( v \) and \( w \), the matrix fields \( v \otimes w \) and \( \nabla v \circ \nabla w \) are defined to be the ones with entries

\[
(v \otimes w)_{ij} = v_i w_j \quad \text{and} \quad (\nabla v \circ \nabla w)_{ij} = v_i \cdot w_j := \frac{\partial v_k}{\partial x_i} \frac{\partial w_k}{\partial x_j}
\]

(summing over the repeated index \( k \) as per the Einstein convention), and for a matrix field \( J = (J_{ij}) \), we define\(^2\) the vector field \( \nabla^T \cdot J \) by

\[
(\nabla^T \cdot J)_i := J_{ij,j} := \frac{\partial J_{ij}}{\partial x_j}
\]

(summing again over \( j \)). We think formally of \( \nabla \) (as well as any vector field) as a column vector and \( \nabla^T \) as a row vector, so that each entry of (the column vector) \( \nabla^T \cdot J \) is the divergence of the corresponding row of \( J \). In what follows, for a vector field \( v \) we similarly denote by \( \nabla^T v \) the matrix field with \( i \)-th row given by \( \nabla^T v_i := (\nabla v_i)^T \), i.e.,

\[
(\nabla^T v)_i = v_{i,j} := \frac{\partial v_i}{\partial x_j},
\]

so that for smooth vector fields \( v \) and \( w \) we always have

\[
\nabla^T \cdot (v \otimes w) = (\nabla^T v)w + v(\nabla \cdot w) = (w \cdot \nabla)v + v(\nabla \cdot w).
\]

For a scalar field \( \phi \) we set \( \nabla^2 \phi := \nabla^T(\nabla \phi) \), and for matrix fields \( J = (J_{ij}) \) and \( K = (K_{ij}) \), we let \( J : K := J_{ij,k} \) (summing over repeated indices) denote the (real) Frobenius inner product of the matrices \( (J : K) = \text{tr}(J^T K) \). We set \( |J| := \sqrt{\text{tr}(J^T J)} \) and \( |v| := \sqrt{\nabla \cdot v} \), and to minimize cumbersome notation will often abbreviate by writing \( \nabla v := \nabla^T v \) for a vector field \( v \) where the precise structure of the matrix field \( \nabla^T v \) is not crucial; for example, \( |\nabla v| := |\nabla^T v| \).

We note that by formally taking the divergence \( \nabla \cdot \) of the first line in (1.1) we obtain the usual “pressure equation”

\[
-\Delta p = \nabla \cdot (\nabla^T \cdot (u \otimes u + \nabla d \circ \nabla d)). \tag{1.3}
\]

As in the Navier-Stokes \((d \equiv 0)\) setting, one may formally deduce (see Section 2 for more details) from (1.1) the following global and local energy inequalities which one may expect “sufficiently nice” solutions of (1.1) to satisfy\(^3\)

\[
\frac{d}{dt} \int_\Omega \left[ \frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + F(d) \right] \, dx + \int_\Omega \left[ |\nabla u|^2 + |\Delta d - f(d)|^2 \right] \, dx \leq 0 \tag{1.4}
\]

\(^1\)In principle, for \( d \) to only represent a “direction” one should have \(|d| \equiv 1 \). As proposed in [LL95], \( F(d) \) is used to model a Ginzburg-Landau type of relaxation of the pointwise constraint \(|d| \equiv 1 \). For further discussions on the modeling assumptions leading to systems such as the one above, see e.g. [LW14] or the appendix of [LL95] and the references mentioned therein.

\(^2\)Many authors simply write \( \nabla \cdot J \), which is perhaps more standard.

\(^3\)For sufficiently regular solutions one can show that equality holds.
for each $t \in (0,T)$, as well as a localized version

$$\frac{d}{dt} \int_\Omega \left[ \left( \frac{|u|^2}{2} + \frac{|
abla d|^2}{2} \right) \phi \right] dx + \int_\Omega \left( |\nabla u|^2 + |\nabla^2 d|^2 \right) \phi \, dx$$

$$\leq \int_\Omega \left[ \left( \frac{|u|^2}{2} + \frac{|
abla d|^2}{2} \right) (\phi_t + \Delta \phi) + \left( \frac{|u|^2}{2} + \frac{|
abla d|^2}{2} + p \right) u \cdot \nabla \phi \right]$$

$$+ u \otimes \nabla \phi : \nabla d \otimes \nabla d - \phi \nabla^T \left[ f(d) \right] : \nabla^T d \right] \, dx$$

(1.5)

for $t \in (0,T)$ and each smooth, compactly supported in $\Omega$ and non-negative scalar field $\phi \geq 0$. (For Navier-Stokes, i.e. when $d \equiv 0$, one may omit all terms involving $d$, even though $0 \neq F(0) \notin L^1(\mathbb{R}^3)$.)

In [LL95], for smooth and bounded $\Omega$, the global energy inequality (1.4) is used to construct global weak solutions to (1.1) for initial velocity in $L^2(\Omega)$, along with a similarly appropriate condition on the initial value of $d$ which allows (1.4) to be integrated over $0 < t < T$. This is consistent with the pioneering result of J. Leray [Ler34] for Navier-Stokes (treated later by many other authors using various methods, but always relying on the natural energy as in [Ler34]).

In [LL96], the authors establish a partial regularity result for weak solutions to (1.1) belonging to the natural energy spaces which moreover satisfy the local energy inequality (1.5). The result is of the same type as known partial regularity results for “suitable weak solutions” to the Navier-Stokes equations. The program for such partial regularity results for Navier-Stokes was initiated in a series of papers by V. Scheffer in the 1970s and 1980s (see, e.g., [Sch77, Sch80] and other works mentioned in [CKNS82]), and subsequently improved by various authors (e.g. [CKN82, Lin98, LS99, Vas07]), perhaps most notably by L. Caffarelli, R. Kohn and L. Nirenberg in [CKNS82]. They show (as do [LL96]) that the one-dimensional parabolic Hausdorff measure of the (potentially empty) singular set $S$ is zero ($P^1(S) = 0$, see Definition 1 below), implying that singularities (if they exist) cannot for example form any smooth one-parameter curve in space-time. The method of proof in [LL96] largely follows the method of [CKNS82].

Of course the general system (1.1) is (when $d \neq 0$) substantially more complex than the Navier-Stokes system, and one therefore could not expect a stronger result than the type in [CKNS82]. In fact, it is surprising that one even obtains the same type of result ($P^1(S) = 0$) as in [CKNS82]. The explanation for this seems to be that although (1.1) is more complex than Navier-Stokes in view of the additional $d$ components, one can derive an a priori maximum principle for $d$ because of the third equation in (1.1) which substantially offsets this complexity from the viewpoint of regularity. Therefore, under suitable boundary and initial conditions on $d$, one may assume that $d$ is in fact bounded, a fact which is significantly exploited in [LL96]. More recently, the authors of the preprint [DHW19] establish the same type of result for a related but more complex “Q-tensor” system; however there, as well, one may obtain a maximum principle which is of crucial importance for proving partial regularity. One is therefore led to the following natural question, which we will address below:

**Can one deduce any partial regularity for systems similar in structure to (1.1) but which lack any maximum principle?**

In the Navier-Stokes setting, it was asserted by Scheffer in [Sch85] that in fact the proof of the partial regularity result in [CKNS82] does not require the full set of equations in (1.1). He mentions that the key ingredients are membership of the global energy spaces, the local energy inequality (1.5), the divergence-free condition $\nabla \cdot u = 0$ and the *pressure* equation (1.3) (with $d \equiv 0$ throughout). Scheffer called vector fields satisfying these four requirements solutions to the “Navier-Stokes inequality”.

4Note that in [LL96], the term “$-\mathcal{R}_f(d, \phi)$” in (1.5) actually appears incorrectly as “$+\mathcal{R}_f(d, \phi)$”. See Section 2 for more details.
ity”, equivalent to solutions to the Navier-Stokes equations with a forcing \( f \) which satisfies \( f \cdot u \leq 0 \) everywhere. In contrast, the results in [LL96] do very strongly use the third equation in (1.1) in that it implies a maximum principle for \( d \).

In this paper, we explore what happens if one considers the analog of Scheffer’s “Navier-Stokes inequality” for the system (1.1) when \( d \neq 0 \). That is, we consider triples \((u,d,p)\) with global regularities implied (at least when \( \Omega \) is bounded and under suitable assumptions on the initial data) by (1.4) which satisfy (1.5) and \( \nabla \cdot u = 0 \) weakly as well as (a formal consequence of) (1.5), but are not necessarily weak solutions of the first and third equations (i.e., the two vector equations) in (1.1). In particular, we will not assume that \( d \in L^\infty(\Omega \times (0,T)) \), which would have been reasonable in view of the third equation in (1.1). We see that without further assumptions, the result is substantially weaker than the \( P^1(S) = 0 \) result for Navier-Stokes: following the methods of [LL96] and [CKN82] we obtain (see Theorem 1 below) \( P^2+3(S) = 0 \) for any \( \delta > 0 \). This reinforces our intuition that the situation here is substantially more complex than that of Navier-Stokes. On the other hand, we show that under a suitable uniform local decay condition on \(|\nabla d|^3 + |\nabla d|^3(1-\cdot)\) with \( \sigma \in (5,6) \) (see (1.14) below, which in particular holds when \( d \equiv 0 \) as in [CKN82]), one in fact obtains \( P^1(S) = 0 \) as in [LL96] and [CKN82]. In particular, we verify the above-mentioned assertion made by Scheffer in [Sch85] regarding partial regularity for Navier-Stokes inequalities.

Our key observation which allows us to work without any maximum principle is that, in view of the global energy (1.4) and the particular forms of \( F \) and \( f \), it is reasonable (see Section 2) to assume (1.9); this implies that \( d \in L^\infty(0,T;L^6(\Omega)) \) which is sufficient for our purposes.

As alluded to above, for our purposes we actually do not require all of the information which appears in (1.5) above. In view of the fact that

\[
|R_f(d,\phi)| = |\phi \nabla^T [f(d)] : \nabla^T d| \leq 12|d|^2|\nabla d|^2 \phi + 8\left(\frac{|\nabla d|^2}{2}\right) \tag{1.6}
\]

(see (2.21) below), a consequence of (1.5) is that

\[
\mathcal{A}'(t) + B(t) \leq 8\mathcal{A}(t) + \mathcal{C}(t) \quad \text{for } 0 < t < T, \tag{1.7}
\]

with \( \mathcal{A}, \mathcal{B}, \mathcal{C} \geq 0 \) defined (denoting \( \int_{\Omega \times \{t\}} g := \int_{\Omega} g(\cdot,t) \, dx \)) as

\[
\mathcal{A}(t) := \int_{\Omega \times \{t\}} \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2}\right) \phi, \quad B(t) := \int_{\Omega \times \{t\}} (|\nabla u|^2 + |\nabla d|^2) \phi
\]

and

\[
\mathcal{C}(t) := \int_{\Omega \times \{t\}} \left[ \left(\frac{|u|^2}{2} + \frac{|\nabla d|^2}{2}\right) \phi_t + \Delta \phi + 12|d|^2 |\nabla d|^2 \phi \right]
\]

\[
+ \int_{\Omega \times \{t\}} \left( \frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) u \cdot \nabla \phi + u \cdot \nabla \phi : \nabla d \cdot \nabla d \right].
\]

(1.7) is nearly sufficient, with the appearance of \( \mathcal{A}(t) \) on the right-hand side (in fact, even with \( u \) omitted, which cannot be avoided as “\( R_f(d,\phi) \)” appears on the right-hand side of (1.5) with a minus sign) actually being, for technical reasons, the only troublesome term. We therefore use a Grönwall-type argument to hide this term to the left-hand side of (1.7) so that (if \( \phi|_{t=0} \equiv 0 \))

\[
\mathcal{A}'(t) + B(t) \leq \mathcal{C}(t) + 8e^{8T} \int_0^t \mathcal{C}(\tau) \, d\tau \quad \text{for } 0 < t < T. \tag{1.8}
\]

\(^5\) In fact, one can also show that \( d \in L^\infty_{ic}(0,T;L^\infty(\Omega)) \) for any \( s \in [2,4) \).

\(^6\) See Footnote 4.

\(^7\) In fact, the appearance of \(|d|^2 \) on the right-hand side of (1.6), and hence of (1.7), as well, is handled precisely by the assumption that \( d \in L^\infty(0,T;L^6(\Omega)) \), and is the reason for the slightly weaker results compared to the Navier-Stokes setting (i.e., when \( d \equiv 0 \)).

\(^8\) Note that if \( R_f(d,\phi) \) had appeared with a plus sign in (1.5), one could have simply dropped this troublesome term as a non-positive quantity.
Theorem 1. Fix any open set \( \Omega \subset \mathbb{R}^3 \) and any \( T, C \in (0, \infty) \), set \( \Omega_T := \Omega \times (0, T) \) and suppose \( u, d : \Omega_T \to \mathbb{R}^3 \) and \( p : \Omega_T \to \mathbb{R} \) satisfy the following four assumptions:

1. \( u, d \) and \( p \) belong to the following spaces:
   \[
   u, d, \nabla d \in L^\infty(0, T; L^2(\Omega)), \quad \nabla u, \nabla d, \nabla^2 d \in L^2(\Omega_T)
   \]
   \( p \in L_+^2(\Omega_T); \)

2. \( u \) is weakly divergence-free:
   \[
   \nabla \cdot u = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_T);
   \]

3. the following pressure equation holds weakly:
   \[
   -\Delta p = \nabla \cdot (u \otimes u + \nabla d \otimes \nabla d) \quad \text{in} \quad \mathcal{D}'(\Omega_T);
   \]

4. the following local energy inequality holds:
   \[
   \int_{\Omega_T} \left( |u|^2 + |\nabla d|^2 \right) \phi \, dx + \int_0^T \int_{\Omega_T} |\nabla u|^2 + |\nabla^2 d|^2 \, \phi \, dx \, dt \\
   \leq C \int_0^T \int_{\Omega_T} \left( \left( |u|^2 + |\nabla d|^2 \right) |\phi_e + \Delta \phi| + |d|^2 |\nabla d|^2 \phi \right) \, dx \\
   + \int_{\Omega_T} \left( \left( |u|^2 + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla \phi + u \otimes \nabla \phi : \nabla d \otimes \nabla d \, dx \right) \, dt
   \]
   \[
   \text{for a.e. } t \in (0, T) \quad \text{and} \quad \forall \phi \in C^\infty_0 (\Omega \times (0, \infty)) \, \text{s.t.} \, \phi \geq 0.
   \]

Let \( S \subset \Omega_T \) be the (potentially empty) set of singular points where \( |u| \) and \( |\nabla d| \) are not essentially bounded in any neighborhood of each \( z \in S \), and let \( \mathcal{P}^k \) be the \( k \)-dimensional parabolic Hausdorff outer measure (see Definition 4 below). The following are then true:

1. \( \mathcal{P}^{2+\delta}(S) = 0 \), for any \( \delta > 0 \) arbitrarily small.

2. If
   \[
   g_\sigma := \sup_{z_0 \in \Omega_T} \left( \limsup_{r \searrow 0} \frac{1}{r^{2+\sigma}} \int_{Q_r(z_0)} |d|^\sigma (|u|^3 + |\nabla d|^3)^{1-\frac{\sigma}{3}} \, dz \right) < \infty
   \]
   \[
   \text{for some } \sigma \in (5, 6), \text{ then } \mathcal{P}^1(S) = 0.
   \]

9For a vector field \( f \) or matrix field \( J \) and scalar function space \( X \), by \( f \in X \) or \( J \in X \) we mean that all components or entries of \( f \) or \( J \) belong to \( X \); by \( \nabla^2 f \in X \) we mean all second partial derivatives of all components of \( f \) belong to \( X \); etc.

10Locally integrable functions will always be associated to the standard distribution whose action is integration against a suitable test function so that, e.g., \( [\nabla \cdot u](\psi) = -[u] \nabla \psi \) := \( -\int u \cdot \nabla \psi \) for \( \psi \in \mathcal{D}(\Omega_T) \).

11Note that \( u \otimes u + \nabla d \otimes \nabla d \in L^2(\Omega_T) \subset L^2_{\mathit{loc}}(\Omega_T) \), see [2.18] - [2.19].

12For brevity, for \( \omega \subset \mathbb{R}^3 \), we set \( \int_{\omega \times \{t\}} g \, dx := \int_{\omega} g(x, t) \, dx \).

13In general we set \( z = (x, t) \in \Omega_T, \, dz := dz \, dt \), and recall from Definition 4 that \( Q_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0) \).
Note that in the case $d \equiv 0$, we regain the classical result of $P^1(S) = 0$ for Navier-Stokes as obtained in, for example, [CKN82], and more specifically for the (weaker) Navier-Stokes inequalities mentioned in [Sch85].

We recall that the definition of the outer parabolic Hausdorff measure $P^k$ is given as follows (see [CKN82, pp.783-784]):

**Definition 1** (Parabolic Hausdorff measure). For any $S \subset \mathbb{R}^3 \times \mathbb{R}$ and $k \geq 0$, define

$$P^k(S) := \lim_{\delta \searrow 0} P^k_{\delta}(S),$$

where

$$P^k_{\delta}(S) := \inf \left\{ \sum_{j=1}^{\infty} r_j^k \left| S \subset \bigcup_{j=1}^{\infty} Q_{r_j}, r_j < \delta \forall j \in \mathbb{N} \right\}$$

and $Q_r$ is any parabolic cylinder of radius $r > 0$, i.e.

$$Q_r = Q_r(x, t) := B_r(x) \times (t - r^2, t) \subset \mathbb{R}^3 \times \mathbb{R}$$

for some $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. $P^k$ is an outer measure, and all Borel sets are $P^k$-measurable.

**Remark 1.** In the case $\Omega = \mathbb{R}^3$, the condition (1.10) on the pressure follows (locally, at least) from (1.9) and (1.12) if $p$ is taken to be the potential-theoretic solution to (1.12), since (1.9) implies that $u, \nabla d \in L^{10,3}(\Omega_T)$ by interpolation (see (2.18)) and Sobolev embeddings, and then (1.12) gives $p \in L^5(\Omega_T) \subset L^4_{\text{loc}}(\Omega_T)$ by Calderon-Zygmund estimates. For a more general $\Omega$, the existence of such a $p$ can be derived from the motivating equation (1.1) (e.g. by estimates for the Stokes operator), see [LL96] and the references therein. Here, however, we will not refer to (1.1) at all and simply assume $p$ satisfies (1.10) and address the partial regularity of such a hypothetical set of functions satisfying (1.9) – (1.13).

We note that Theorem 1 does not immediately recover the result of [LL96] (which would correspond to $\sigma = 6$ in (1.14), which holds when $d \in L^\infty$ as assumed in [LL96]). Heuristically, however, one can argue as follows:

If $d$ were bounded, then taking for example $D := 24\|d\|_{L^\infty(\Omega_T)}^2 + 8 < \infty$ one would deduce from (1.6) that

$$|R_f(d, \phi)| \leq D \left( \frac{\|\nabla d\|_{L^2(\Omega_T)}^2}{2} \right) \phi.$$ 

Adjusting the Grönwall-type argument leading to (1.8), one could then deduce from (1.6) that (if $A(0) = 0$)

$$A'(t) + B(t) \leq \bar{C}(t) + De^{DT} \int_0^T \bar{C}(\tau) d\tau \quad \text{for } 0 < t < T,$$

where

$$\bar{C}(t) := \int_{\Omega \times \{t\}} \left( \frac{|u|^2}{2} + \frac{\|\nabla d\|_{L^2(\Omega_T)}^2}{2} \right) |\phi_i + \Delta \phi| + \int_{\Omega \times \{t\}} \left( \frac{|u|^2}{2} + \frac{\|\nabla d\|_{L^2(\Omega_T)}^2}{2} + p \right) u \cdot \nabla \phi + u \otimes \nabla \phi : \nabla d \otimes \nabla d.$$

Using such an energy inequality, one would not need to include the $|d|^6$ term in $E_{d,6}$ (see (3.6)) as one would not need to consider the term coming from $R_f(d, \phi)$ at all in Proposition 2 and (noting that

\footnote{We assume this is roughly the argument in [LL96], although the details are not explicitly given; see, in particular, [LL96] (2.45) which appears without the “remainder” term denoted in [LL96] by $R(f, \phi)$, and here by $R_f(d, \phi)$.}
the $L^\infty$ norm is invariant under the re-scaling on $d$ in (3.25) one could then adjust Lemmas 1 and 2 appropriately to recover the result in [LL96] using the proof of Theorem 1 below.

Finally, we remark that the majority of the arguments in the proofs given below are not new, with many essentially appearing in [LL96] or [CKNS2]. However we feel that our presentation is particularly transparent and may be a helpful addition to the literature, and we include all details so that our results are easily verifiable.

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2 Motivation

We will show in this section that the assumptions in Theorem 1 are at least formally satisfied by smooth solutions to the system (1.1).

2.1 Energy identities

As in [LL96], let us assume that we have smooth solutions to (1.1) which vanish or decay sufficiently at $\partial \Omega$ (assumed smooth, if non-empty) and at spatial infinity as appropriate so that all boundary terms vanish in the following integrations by parts, and proceed to establish smooth versions of (1.4) and (1.5). First, noting the simple identities

$$\nabla^T \cdot (\nabla d \odot \nabla d) = \nabla \left( \frac{|\nabla d|^2}{2} \right) + (\nabla^T d)^T \Delta d$$

and

$$[(\nabla^T d)^T \Delta d] \cdot u = [(\nabla^T d)u] \cdot \Delta d = [(u \cdot \nabla)d] \cdot \Delta d,$$

at a fixed $t$ one may perform various integrations by parts (keeping in mind that $\nabla \cdot u = 0$) to see that

$$0 = \int_{\Omega} \left[ u_t - \Delta u + \nabla^T \cdot (u \odot u) + \nabla p + \nabla^T \cdot (\nabla d \odot \nabla d) \right] \cdot u \, dx$$

$$= \int_{\Omega} \left[ \frac{\partial}{\partial t} \left( \frac{|u|^2}{2} \right) + |\nabla u|^2 + [(u \cdot \nabla)d] \cdot \Delta d \right] \, dx$$

(2.3)

and, recalling that $f = \nabla F$ so that $[d_t + (u \cdot \nabla)d] \cdot f(d) = \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) [F(d)]$, that

$$0 = -\int_{\Omega} [d_t + (u \cdot \nabla)d - (\Delta d - f(d))] \cdot (\Delta d - f(d)) \, dx$$

$$= -\int_{\Omega} \left[ -\frac{\partial}{\partial t} \left( \frac{|\nabla d|^2}{2} + F(d) \right) + [(u \cdot \nabla)d] \cdot \Delta d - |\Delta d - f(d)|^2 \right] \, dx.$$ 

(2.4)

Adding the two gives the

Global energy identity for (1.1):

$$\frac{d}{dt} \int_{\Omega} \left[ \frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + F(d) \right] \, dx + \int_{\Omega} [(|\nabla u|^2 + |\Delta d - f(d)|^2] \, dx = 0$$

(2.5)
in view of the cancelation of the indicated terms in (2.3) and (2.4).

It is not quite straightforward to localize the calculations in (2.3) and (2.4), for example replacing the (global) multiplicative factor \((\Delta d - f(d))\) by \((\Delta d - f(d))\phi\) for a smooth and compactly supported \(\phi\). Arguing as in [LL96], one can deduce a local energy identity by instead replacing \((\Delta d - f(d))\) by only a part of its localized version in divergence-form, namely by \(\nabla \cdot (\phi \nabla T d)\), at the expense of the appearance of \(|\Delta d - f(d)|^2\) anywhere in the local energy.

Recalling (2.1) and (2.2) and noting further that

\[
[u \cdot \nabla d] \cdot [\nabla \cdot (\phi \nabla d)] = [u \cdot \nabla d] \cdot [\phi \Delta d] + [(u \cdot \nabla d) \cdot [(\nabla \phi \cdot \nabla d)]
\]

and that

\[
[\Delta (\nabla d)] : \nabla \cdot (\phi \nabla d) = \Delta \left( \frac{|\nabla d|^2}{2} \right) - |\nabla^2 d|^2,
\]

one may perform various integrations by parts to deduce (as \(\nabla \cdot u = 0\)) that

\[
0 = \int_{\Omega} [u_t - \Delta u + \nabla \cdot (u \otimes u) + \nabla p + \nabla T \cdot (\nabla d \otimes \nabla d)] \cdot u \phi \, dx
\]

and

\[
0 = -\int_{\Omega} [d_t + (u \cdot \nabla d) - (\Delta d - f(d))] \cdot [\nabla \cdot (\phi \nabla d)] \, dx
\]

for smooth and compactly-supported \(\phi\), upon adding which and noting again the cancelation of the indicated terms we obtain the

**Local energy identity for (1.1):**

\[
\frac{d}{dt} \int_{\Omega} \left[ \left( \frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) \phi \right] \, dx + \int_{\Omega} \left( |\nabla u|^2 + |\nabla^2 d|^2 \right) \phi \, dx =
\]

\[
= \int_{\Omega} \left[ \left( \frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} \right) (\phi_t + \Delta \phi) + \left( \frac{|u|^2}{2} + \frac{|\nabla d|^2}{2} + p \right) u \cdot \nabla \phi
\]

\[
+ u \otimes \nabla \phi : \nabla d \otimes \nabla d - \frac{\phi \nabla [f(d)] : \nabla \cdot (\phi \nabla d)}{= R_f(d, \phi)} \right] \, dx.
\]

Note that we have corrected the omission in [LL96] of the “-” preceding \(R_f(d, \phi)\), and the term "\((u \cdot \nabla d \otimes \nabla d) \cdot \nabla \phi\)" which appears in [LL96] has been more accurately written here as \(u \otimes \nabla \phi \cdot \nabla d \otimes \nabla d\), and that \(u \otimes \nabla \phi \cdot \nabla d \otimes \nabla d = [(\nabla d \otimes \nabla d) \nabla \phi] \cdot u = [(u \cdot \nabla d) \cdot [(\nabla \phi \cdot \nabla d)]\)
2.2 Global energy regularity heuristics

Let us first see where the global energy identity (2.5) leads us to expect weak solutions to (1.1) to live (and hence why we assume (1.9) in Theorem 1).

To ease notation, in what follows let’s fix $\Omega \subset \mathbb{R}^3$, and for $T \in (0, \infty)$ let us set $\Omega_T := \Omega \times (0, T)$ and

$$L^i_T L^j_T(\Omega_T) := L^i(0, T; L^j(\Omega)).$$

According to (2.5), we expect, so long as

$$1 \in L^\infty(0, \infty; L^2(\Omega)) \cap L^\infty(0, \infty; L^3(\Omega)),$$

along with (2.8), that

$$\nabla d \in L^\infty L^2(\infty), \quad F(d) \in L^\infty L^1(\infty) \quad \text{and} \quad [\Delta d - f(d)] \in L^2 L^2(\infty).$$

The norms of all quantities in the spaces given in (2.7) and (2.8) are controlled by either $M_0$ (the $F(d)$ term) or $(M_0)^2$ (all other terms), by integrating (2.5) over $t \in (0, \infty)$. Recalling that

$$F(d) := (|d|^2 - 1)^2 \quad \text{and} \quad f(d) := 4(|d|^2 - 1)d,$$

one sees that $|f(d)|^2 = 16F(d)|d|^2$, and one can easily confirm the following simple estimates:

$$\|d\|^2_{L^\infty L^1(\infty)} \leq \|F(d)\|^{1/2}_{L^\infty L^1(\infty)} + \|1\|_{L^\infty L^2(\infty)}, \quad (2.10)$$

$$\|F(d)\|_{L^\infty L^{3/2}(\infty)} \leq \|d\|^2_{L^\infty L^2(\infty)} + \|1\|_{L^\infty L^3(\infty)}, \quad (2.11)$$

$$\|f(d)\|^2_{L^\infty L^2(\infty)} \leq 16\|F(d)\|_{L^\infty L^{3/2}(\infty)}|d|^2_{L^\infty L^2(\infty)} \quad (2.12)$$

and

$$\|\Delta d\|_{L^2(\Omega_T)} \leq \|\Delta d - f(d)\|_{L^2(\Omega_T)} + T^{1/2}\|f(d)\|_{L^\infty L^2(\infty)}. \quad (2.13)$$

Therefore, if we assume that

$$|\Omega| < \infty, \quad (2.14)$$

and hence

$$1 \in L^\infty(0, \infty; L^2(\Omega)) \cap L^\infty(0, \infty; L^3(\Omega)),$$

(2.8) along with (2.10) implies that

$$d \in L^\infty(0, \infty; L^3(\Omega)) \quad (2.14) \quad \text{and} \quad L^\infty(0, \infty; L^2(\Omega)) \quad (2.15)$$

so that (2.8) and (2.15) imply

$$d \in L^\infty(0, \infty; H^1(\Omega)) \to L^\infty(0, \infty; L^3(\Omega)) \quad (2.16)$$

by the Sobolev embedding, from which (2.11) implies that

$$F(d) \in L^\infty L^{3/2}(\infty)$$

which, along with (2.12) and (2.16), implies that

$$f(d) \in L^\infty L^2(\infty).$$
from which, finally, \[ (2.13) \] and the last inclusion in \&(2.8)\ implies that
\[
\Delta d \in L^2(\Omega_T) \quad \text{for any} \quad T < \infty,
\] (2.17)
with the explicit estimate \(2.13\) which can then further be controlled by \(M_0\) via \(2.8\), \(2.10\), \(2.11\) and \(2.12\).

We therefore see that it is reasonable (in view of the usual elliptic regularity theory) to expect that weak solutions to \(1.1\) should have the regularities in \(1.9\) of Theorem 1.

Note further that various interpolations of Lebesgue spaces imply, for example, that for any interval \(I \subset \mathbb{R}\) one has
\[
L^\infty(I; L^2(\Omega)) \cap L^2(I; L^6(\Omega)) \subset L^{\frac{2}{\alpha}}(I; L^{\frac{6}{3-2\alpha}}(\Omega)) \quad \text{for any} \quad \alpha \in [0, 1]
\] (for example, one may take \(\alpha = \frac{3}{5}\) so that \(\frac{2}{\alpha} = \frac{6}{3-2\alpha} = \frac{10}{3}\)). Using this along with the Sobolev embedding we expect (as mentioned in Remark 1) that
\[
(\text{u and } \nabla d) \in L^\frac{2}{\alpha}(0, T; L^{\frac{6}{3-2\alpha}}(\Omega)) \quad \text{for any} \quad \alpha \in [0, 1], \quad T < \infty
\] (2.19)
with the explicit estimate \[15\]
\[
\|\nabla d\|_{L^\frac{2}{\alpha}(0, T; L^{\frac{6}{3-2\alpha}}(\Omega))} \lesssim T\|\nabla d\|_{L^3(\Omega_T)}^{\frac{2}{3}} + \|\nabla d\|_{L^\frac{6}{3-2\alpha}(\Omega)}^{\frac{6}{3-2\alpha}}\|\nabla^2 d\|_{L^2(\Omega_T)}^{\frac{2}{3}}.
\]

### 2.3 Local energy regularity heuristics

Here, we will justify the well-posedness of the terms appearing in the local energy equality \(2.6\), based on the expected global regularity discussed in the previous section. In fact, all but the final term in \(2.6\) (where one can furthermore take the essential supremum over \(t \in (0, T)\)) can be seen to be well-defined by \(2.19\) under the assumptions in \(1.9\) and \(1.10\).

The \(\mathcal{R}_f(d, \phi)\) term of \(2.6\) requires some further consideration: in view of \(2.9\) we see that
\[
\frac{1}{4}\nabla^T [f(d)] = \nabla^T (|d|^2 - 1)d = 2d \odot |d| \cdot (\nabla^T d) + (|d|^2 - 1)\nabla^T d,
\] (2.20)
Recalling that
\[
\mathcal{R}_f(d, \phi) := \phi \nabla^T [f(d)] : \nabla^T d,
\]
we therefore have
\[
\frac{1}{4}\mathcal{R}_f(d, \phi) = \phi \left(2d \odot |d| \cdot (\nabla^T d) + |d|^2 |\nabla d|^2\right) - \phi |\nabla d|^2
\] (2.21)
where we have to be careful how we handle the appearance of, essentially, \(|d|^2\) in the first term (the second term is integrable in view of \(2.8\)). We have, for example, that
\[
\|\phi |d|^2 |\nabla d|^2\|_{L^1(\Omega_T)} \leq \|\phi\|_{L^\infty(\Omega_T)} \|d\|^2_{L^6(\Omega_T)} \|\nabla d\|^2_{L^2(\Omega_T)}
\]
and that
\[
\|d\|_{L^6(\Omega_T)} < \infty \quad \text{for any} \quad T \in (0, \infty)
\] (2.22)
by \(2.16\), and either
\[
\|\phi |\nabla d|^2\|_{L^1(\Omega_T)} \leq \|\phi\|_{L^\infty(\Omega_T)} \|\nabla d\|^2_{L^2(\Omega_T)}
\] or
\[
\|\phi |\nabla d|^2\|_{L^1(\Omega_T)} \leq \|\phi\|_{L^\infty(\Omega_T)} \|\nabla d\|^2_{L^2(\Omega_T)},
\]
(recall that \(\phi\) is assumed to have compact support) and, for example, that
\[
\|\nabla d\|_{L^{10/3}(\Omega_T)} < \infty \quad \text{for any} \quad T \in (0, \infty)
\] (2.23)
by \(2.19\).
3 Proof of Theorem 1

The first part of Theorem 1 is a consequence of the following "$L^3$ $\varepsilon$-regularity" Lemma 1 while the second part is a consequence of the "$H^1$ $\varepsilon$-regularity" Lemma 2 below which is itself a consequence of Lemma 1. In the following, for a given $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$ and $r > 0$, as in \cite{CKNS2} we will adopt the following notation for the standard parabolic cylinder $Q_r(z_0)$ as well as the following time intervals and their "centered" versions:\footnote{These are defined in such a way that $Q^*_r(x_0, t_0) = Q_r(x_0, t_0 + \frac{r^2}{2})$, and subsequently $Q^*_r(z_0) := B_r(x_0) \times (t_0 - \frac{r^2}{2}, t_0 + \frac{r^2}{2})$.} (indicated with a star):

\begin{align}
I_r(t_0) &:= (t_0 - r^2, t_0), \quad I^*_r(t_0) := (t_0 - \frac{r^2}{8}, t_0 + \frac{r^2}{8}) , \\
Q_r(z_0) &:= B_r(x_0) \times I_r(t_0) \quad \text{and} \quad Q^*_r(z_0) := B_r(x_0) \times I^*_r(t_0).
\end{align}

Lemma 1 ($L^3$ $\varepsilon$-regularity, cf. Theorem 2.6 of \cite{LL96} and Proposition 1 of \cite{CKNS2}). Fix any $C \in (0, \infty)$. For each $q \in (5, 6]$, there exists $\varepsilon_q = \varepsilon_q(C) \in (0, 1) \text{ sufficiently small}$\footnote{Roughly speaking, $\varepsilon_q \lesssim (C)^{-\alpha_q} (2^{q_y} - 1)^q$ with $\alpha_q := \frac{2(q-1)}{q-2}$; in particular, $\varepsilon_q \rightarrow 0$ as $q \searrow 5$.} so that for any $\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}$ and $\bar{\rho} \in (0, 1]$, the following holds:

Suppose (see \cite{X}) $u, d : Q_1(\bar{z}) \rightarrow \mathbb{R}^3$ and $p : Q_1(\bar{z}) \rightarrow \mathbb{R}$ with \begin{align}
\nabla \cdot u & = 0 \quad \text{in } D'(Q_1(\bar{z})) ,
\n\Delta p & = \nabla \cdot (\nabla^T \cdot [u \otimes u + \nabla d \odot \nabla d]) \quad \text{in } D'(Q_1(\bar{z})) \quad \text{(3.3)}
\end{align}

and the following local energy inequality holds:\footnote{See Footnote \ref{footnote4} and note that (3.2) implies (3.5) with $\rho = 1$ if $Q_1(\bar{z}) \subseteq \Omega_2$, since \begin{align}
\left| \left( \frac{|u|^2}{2} + \frac{2d^2}{2} \right) u \cdot \nabla \phi + u \odot \nabla \phi \odot \nabla d \odot \nabla d \right| & \leq \left( \frac{3}{2} |u|^3 + \frac{3}{2} |u| |\nabla d|^2 \right) |\nabla \phi| \\
& \leq \left( |u|^3 + |\nabla d|^3 \right) |\nabla \phi|.
\end{align}} \cite{X}\footnote{Note that $E_{3,q} < \varepsilon_q$ by (3.6) and standard embeddings, see Section 3 along with (3.2) with $\sigma = 6$.}

\begin{align}
\int_{B_1(\bar{x}) \times \{t\}} \left( |u|^2 + |\nabla d|^2 \right) \phi \, dx & + \int_{I^*_1} \int_{B_1(\bar{x})} \left( |\nabla u|^2 + |\nabla d|^2 \right) \phi \, dx \, d\tau \\
& \leq C \int_{I^*_1} \int_{B_1(\bar{x}) \times \{t\}} \left[ \left( |u|^2 + |\nabla d|^2 \right) |\phi_x + \Delta \phi| + (|u|^3 + |\nabla d|^3) |\nabla \phi| + \bar{\rho} |d|^2 |\nabla d|^2 |\phi| \right] \, dx \\
& \quad + \int_{I^*_1} \int_{B_1(\bar{x}) \times \{t\}} |p u \cdot \nabla \phi| \, d\tau
\end{align}

for a.e. \( t \in I_1(\bar{t}) \) and \( \forall \phi \in C_0^\infty(B_1(\bar{x}) \times (\bar{t} - 1, \infty)) \) s.t. $\phi \geq 0$.

If $E_{3,q} \leq \varepsilon_q$, then $u, \nabla d \in L^\infty(Q_{1/2}(\bar{z}))$ with

\begin{align}
\|u\|_{L^\infty(Q_{1/2}(\bar{z}))}, \|\nabla d\|_{L^\infty(Q_{1/2}(\bar{z}))} \leq \varepsilon_{q/9}.
\end{align}
In order to prove Lemma 1, we will require the following two technical propositions. In order to state them, let us fix (recalling (3.1)), for a given $z_0 = (x_0, t_0)$ (to be clear by the context), the abbreviated notations

$$r_k := 2^{-k}, \quad B_k := B_{r_k}(x_0), \quad I_k := I_{r_k}(t_0) \quad \text{and} \quad Q_k := B_k \times I_k$$

(3.7)

(so that $Q_k = Q_{2^{-k}}(z_0)$) and, for each $k \in \mathbb{N}$, we define the quantities

$$L_k = L_k(z_0) \quad \text{and} \quad R_k = R_k(z_0)$$

again, the dependence on $z_0 = (x_0, t_0)$ will be clear by context) by\(^{20}\)

$$L_k := \text{ess sup}_{t \in I_k} \int_{B_k} \left( |u(t)|^2 + |\nabla d(t)|^2 \right) \, dx + \int_{I_k} \int_{B_k} \left( |\nabla u|^2 + |\nabla^2 d|^2 \right) \, dx \, dt$$

and

$$R_k := \int_{Q_k} |u|^3 + |\nabla d|^3 \, dz + r_k^{1/3} \int_{Q_k} |u| |p - \bar{p}_k| \, dz$$

where \(\bar{p}_k(t) := \int_{B_k} p(x, t) \, dx\).

$L_k$ and $R_k$ correspond roughly to the left- and right-hand sides of the local energy inequality (3.5). We now state the technical propositions, whose proofs we will give in Section 4.

**Proposition 1** (Cf. Lemma 2.7 of [LL96]). There exists a large universal constant $C_A > 0$ such that the following holds:

Fix any \(\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}\), suppose $u$, $d$ and $p$ satisfy (3.3) and (3.4).

Then for any $z_0 \in Q_{3/2}(\bar{z})$ we have (see (3.7), (3.8), (3.9))

$$R_{n+1}(z_0) \leq C_A \left( \max_{1 \leq k \leq n} L_k^{3/2}(z_0) + \|p\|_{L^{3/2}(Q_{3/2}(z_0))}^{3/2} \right) \quad \forall \ n \geq 2 .$$

(3.10)

The proof of Proposition 1 uses only the Hölder and Poincaré inequalities, Sobolev embedding and Calderon-Zygmund estimates along with a local decomposition of the pressure (see (4.20)) using the pressure equation (3.4).

**Proposition 2** (Cf. Lemma 2.8 of [LL96]). There exists a large universal constant $C_B > 0$ such that the following holds:

Fix any \(\bar{z} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R}\), suppose $u$, $d$ and $p$ satisfy (3.3), (3.5) and (3.8), and set $E_{3,q}$ as in (3.7).

Then for any $z_0 \in Q_{3/2}(\bar{z})$ and any $q \in (5, 6]$, we have (see (3.7), (3.8), (3.9))

$$L_n(z_0) \leq \tilde{C} \cdot C_B \left( \frac{1}{2^q - 1} \cdot \max_{k_0 \leq k \leq n} R_k(z_0) + E_3^{2/3} + (1 + k_0 2^{5k_0}) E_{3,q} \right) \quad \forall \ n \geq 2$$

(3.11)

for any $k_0 \in \{1, \ldots, n - 1\}$, where \(\tilde{C}\) is the constant from (3.4) and

$$\alpha_q := \frac{2(q - 5)}{q - 2} > 0 .$$

\(^{20}\)We use the standard notation for averages, e.g.

$$\int_B f(x) \, dx := \frac{1}{|B|} \int_B f(x) \, dx .$$
The proof of Proposition 2 uses only the local energy inequality (3.5), the divergence-free condition (3.3) on \( u \) and elementary estimates. The quantities on either side of (3.11) do not scale (in the sense of (3.25)) the same way (as do those in (3.10)), which is why the energy inequality is necessary.

Let us now prove Lemma 1 using Propositions 1 and 2.

**Proof of Lemma 1**. Let us fix some \( q \in (5, 6] \) and \( \bar{C} \in (0, \infty) \). We first note that for any \( \phi \geq 0 \) in (3.3) we have\(^{21}\)

\[
\hat{p} \int_{Q^1} |d^2|\nabla d^2 \phi \leq \frac{2}{q} \int_{Q^1} |d^q| |\nabla d^q|^{3(1-\frac{2}{q})} + (1 - \frac{2}{q}) \int_{Q^1} |\nabla d^q|^{\frac{3}{2 q}} \phi^{\frac{5}{2 q} - \alpha_q} \cdot
\]

with \( \alpha_q := \frac{2(q-5)}{q-2} \in (0, \frac{1}{2}] \). Taking \( \phi \) in particular such that \( \phi \equiv 1 \) on \( Q^1 = Q_{1/2}(z_0) \), we see easily from this that

\[
\frac{L_1}{C}(3.5) \lesssim E_{3,q} + E^{2/3}_{3,q} \quad \forall \ z_0 \in Q_{1/2}(\bar{z}) .
\]

(3.12)

It is also easy to see that

\[
L_{n+1} \leq 8L_n \quad \text{for any} \quad n \in \mathbb{N} .
\]

(3.13)

Hence we may pick \( C_0 = C_0(q, \bar{C}) >> 1 \) such that for any \( z_0 \in Q_{\frac{1}{2}}(\bar{z}) \) (and suppressing the dependence on \( z_0 \) in what follows) we have

\[
L_1, L_2, L_3 \leq \frac{3}{13} (C_0)^{2/3} \left( E_{3,q} + E^{2/3}_{3,q} \right) ,
\]

(3.14)

\[
C_A \leq \frac{C_0}{2} \quad \text{and} \quad ((2^{n_a} - 1)^{-1} + 2 + 3 \cdot 2^{15}) \bar{C} \cdot C_B \leq (C_0)^{2/3}
\]

for \( C_A \) and \( C_B \) as in Propositions 1 and 2. Having fixed \( C_0 \) (uniformly over \( z_0 \in Q_{1/2}(\bar{z}) \)), we then choose \( \varepsilon_q \in (0, 1) \) so small that

\[
\varepsilon_q < \frac{1}{(C_0)^6} \quad \iff \quad C_0^2 \varepsilon_q < \varepsilon_q^{2/3} .
\]

Noting first that \( \varepsilon_q \leq (\varepsilon_q)^{2/3} \), under the assumption \( E_{3,q} \leq \varepsilon_q \) we in particular see from (3.12) that

\[
L_1, L_2, L_3 \leq (C_0 \varepsilon_q)^{2/3} .
\]

Then, by Proposition 1 with \( n \in \{2, 3\} \) we have

\[
R_3, R_4 \leq \frac{C_0}{2} (\max \{ L_1^{3/2}, L_2^{3/2}, L_3^{3/2} \} + \varepsilon_q) \leq \frac{C_0 (C_0 + 1)}{2} \varepsilon_q \leq C_0^2 \varepsilon_q < \varepsilon_q^{2/3}
\]

which implies due to Proposition 2 with \( n = 4 \) and \( k_0 = 3 \) that

\[
L_4 \leq C_B \left( (2^{n_a} - 1)^{-1} \max \{ R_3, R_4 \} + E_{3,q}^{2/3} + (1 + 3 \cdot 2^{15}) E_{3,q} \right) \leq (C_0 \varepsilon_q)^{2/3} .
\]

Then in turn, Proposition 1 with \( n = 4 \) gives

\[
L_1, L_2, L_3, L_4 \leq (C_0 \varepsilon_q)^{2/3} \quad \iff \quad R_5 < \varepsilon_q^{2/3} ,
\]

from which Proposition 2 with \( n = 5 \) and, again, \( k_0 = 3 \) gives

\[
R_3, R_4, R_5 < \varepsilon_q^{2/3} \quad \iff \quad L_5 \leq (C_0 \varepsilon_q)^{2/3} ,
\]

\(^{21}\)The inequality in fact holds for any \( q \in (2, 6] \).
and continuing we see by induction that Proposition 1 and Proposition 2 (with \( k_0 = 3 \) fixed throughout) imply that
\[
R_n(z_0) < \epsilon_0^{2/3}, \quad L_n(z_0) \leq (C_0 \epsilon_0)^{2/3} \quad \forall \ n \geq 3.
\]
This, in turn, implies (for example) that (see, e.g., [WZ77, Theorem 7.16])
\[
|u(x_0, t_0)|^3 + |\nabla d(x_0, t_0)|^3 \leq \epsilon_0^{2/3}
\]
for all Lebesgue points \( z_0 \in Q_4(z) \) of \(|u|^3 + |\nabla d|^3\)
which implies the \( L^\infty \) statement, and Lemma 1 is proved. \( \square \)

Lemma 2 will be used to prove the first assertion in Theorem 1 as well as the next lemma, which in turn will be used to prove the second assertion in Theorem 1.

Lemma 2 \((H^1, \epsilon\text{-regularity}, \text{cf. Theorem 3.1 of } [LL96] \text{ and Proposition 2 of } [CKN82])\). Fix any \( \tilde{C} \in (0, \infty) \) and \( \bar{g} \in [1, \infty) \). For each \( \sigma \in (5, 6) \), there exists a small constant \( \epsilon_\sigma = \epsilon_\sigma(\tilde{C}, \bar{g}) > 0 \) such that the following holds. Fix \( \Omega_T := \Omega \times (0, T) \) as in Theorem 1 and suppose \( u, d \) and \( p \) satisfy assumptions (L.1) - (L.5). If (recall (3.1))
\[
\limsup_{r \searrow 0} \frac{1}{r^2+\frac{2}{3}} \int_{Q_r(z_0)} |\nabla|^\sigma (|u|^3 + |\nabla d|^3)^{1-\frac{\sigma}{2}} \, dz \leq \bar{g}
\]
and
\[
\limsup_{r \searrow 0} \frac{1}{r^2} \int_{Q_r(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \, dz \leq \epsilon_\sigma
\]
for some \( z_0 \in \Omega_T \), then \( z_0 \) is a regular point, i.e. \(|u| \) and \(|\nabla d|\) are essentially bounded in some neighborhood of \( z_0 \).

For the proof of Lemma 2 for \( z_0 = (x_0, t_0) \in \Omega_T \) and for \( r > 0 \) sufficiently small, we define \( A_{z_0}, B_{z_0}, C_{z_0}, D_{z_0}, E_{z_0}, F_{z_0} \) (cf. [LL96, (3.3)]) and \( G_{z_0} \) using the cylinders \( Q^*_r(z_0) \) (whose “centers” \( z_0 \) are in the interior, see (3.1)) by
\[
A_{z_0}(r) := \frac{1}{r} \sup_{t \in L_r^p(t_0)} \int_{B_r(x_0)} (|u(t)|^2 + |\nabla d(t)|^2) \, dx,
\]
\[
B_{z_0}(r) := \frac{1}{r} \int_{Q^*_r(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \, dz,
\]
\[
C_{z_0}(r) := \frac{1}{r^2} \int_{Q^*_r(z_0)} (|u|^3 + |\nabla d|^3) \, dz,
\]
\[
D_{z_0}(r) := \frac{1}{r^2} \int_{Q^*_r(z_0)} |p|^{3/2} \, dz,
\]
\[
E_{z_0}(r) := \frac{1}{r^2} \int_{Q^*_r(z_0)} |u| \left( |u|^2 - \frac{3}{2} |u|^2 \right) + |\nabla d|^2 - \frac{3}{2} |\nabla d|^2 \, dz,
\]
\[
F_{z_0}(r) := \frac{1}{r^2} \int_{Q^*_r(z_0)} |u| \, dz
\]
(\text{where } \bar{g}(t) := \int_{B_r(x_0)} g(y, t) \, dy), \quad G_{z_0}(r) := \frac{1}{r^2} \int_{Q^*_r(z_0)} |u| \, dz
\]
and
\[
G_{q, z_0}(r) := \frac{1}{r^2+\frac{2}{3}} \int_{Q^*_r(z_0)} |d|^q (|u|^3 + |\nabla d|^3)^{1-\frac{q}{2}} \, dz
\]
(note that \( G_{0, z_0} \equiv C_{z_0} \) and, for \( q \in [0, 6) \), define
\[
M_{q, z_0}(r) := \frac{1}{2} \left[ C_{z_0}(r) + G_{q, z_0}(r) \right] + D_{z_0}(r) + E_{z_0}(r) + F_{z_0}(r).
\]

The statement in Lemma 2 will follow from Lemma 1 along with the following technical “decay estimate” which will be proved in Section 4.
Proposition 3 (Decay estimate, cf. Lemma 3.1 of [LL96] and Proposition 3 of [CKN82]). Fix any $\tilde{C} \in (0, \infty)$. There exists some constant $\tilde{c} = \tilde{c}(\tilde{C}) > 0$ such that the following holds: fix any $q, \sigma \in \mathbb{R}$ with $2 \leq q < \sigma < 6$, and define
\[
\alpha_{\sigma, q} := \frac{6}{\sigma} \cdot \frac{\sigma - q}{6 - q} \in (0, 1).
\]
If $u, d$ and $p$ satisfy \( (Q.2) \) - \( (Q.5) \) for $\Omega_T$ as in Theorem 1 and $z_0 \in \Omega_T$ and $\rho_0 \in (0, 1]$ are such that $Q_{\rho_0}(z_0) \subseteq \Omega_T$ and furthermore
\[
\sup_{\rho \leq (0, \rho_0]} B_{z_0}(\rho) \leq 1 \quad \text{and} \quad \sup_{\rho \leq (0, \rho_0]} G_{\sigma, z_0}(\rho) \leq \bar{g}
\]
for some finite $\bar{g} \in [1, \infty)$, then for any $\rho \in (0, \rho_0]$ and $\gamma \in (0, \frac{1}{4}]$ we have
\[
M_{q, z_0}(\gamma \rho) \leq \tilde{c} \cdot \bar{g}^{\frac{q}{6} - \frac{\sigma}{\sigma - q}} \left[ \frac{\gamma^{\frac{q}{6} - \frac{\sigma}{\sigma - q}}}{\sigma} \left( \frac{\sigma}{\gamma} \sum_{k=0}^{\infty} B_{z_0}^{\frac{q}{6} - \frac{\sigma}{\sigma - q}} \right) \right] \quad \text{for } q > \frac{1}{2}.
\]
(In fact, in the sum over $k$ in (3.21), one can omit the term with $\alpha_{\sigma, q}$ when $k = 0$.)

The key new element in our statement and proof of Proposition 3 (and hence in achieving Lemma 2) is the fact that, for certain $q > 0$ (so that $G_{q, z_0} \neq C_{z_0}$) and hence $M_{q, z_0}$ is notably different from the quantity found in the standard literature, namely $M_{q, 0}$, we can still derive an estimate for $M_{q, z_0}$ of the form (3.21), with a constant depending only on $C$, $\sigma$ and $\bar{g}$ (and not on $q$). This is made possible (see Claim 4 and its applications in Section 4.4) by the following interpolation-type estimate for the range of the quantities $G_{q, z_0}$ (including $G_{0, z_0} = C_{z_0}$), a simple consequence of Hölder’s inequality:
\[
0 \leq q \leq \sigma \leq 6 \quad \implies \quad G_{q, z_0}(r) \leq G_{z_0}^{\frac{q}{6} - \frac{\sigma}{\sigma - q}}(C_{z_0})^{\frac{1}{6} - \frac{\sigma}{\sigma - q}}(r) \quad \forall \; r > 0.
\]
The estimate (3.22) follows by writing
\[
|d|^q \left( |u|^3 + |\nabla d|^3 \right)^{(1 - \frac{q}{6})} \leq \left[ |d|^\sigma \left( |u|^3 + |\nabla d|^3 \right)^{(1 - \frac{q}{\sigma})} \right]^{\frac{q}{\sigma}} \cdot \left( |u|^3 + |\nabla d|^3 \right)^{\frac{\sigma - q}{\sigma}}
\]
and applying Hölder’s inequality with
\[
1 = \frac{q}{\sigma} + \frac{\sigma - q}{\sigma}
\]
to $G_{q, z_0}$, and noting that $r^{2 + \frac{q}{6}} = [r^{\frac{\sigma}{6} - \frac{\sigma}{\sigma - q}}]^{1 - \frac{q}{\sigma}} \cdot [r^{\frac{\sigma}{\sigma - q} - \frac{\sigma}{\sigma - q}]}^{1 - \frac{q}{\sigma}}$. In particular, if $0 \leq q \leq \sigma < 6$, setting
\[
\alpha_{\sigma, q} := \left( 1 - \frac{q}{\sigma} \right) \cdot \frac{6}{6 - q} \quad \text{and} \quad \beta_{\sigma, q} := \frac{q}{\sigma} \cdot \frac{6}{6 - q}
\]
and noting that
\[
\beta_{\sigma, q} = \frac{6}{6 - \sigma} \cdot (1 - \alpha_{\sigma, q}) \leq \frac{6}{6 - \sigma},
\]
we see that
\[
G_{q, z_0}^{\frac{6}{6 - \sigma}}(r) \leq G_{z_0}^{\frac{q}{6} - \frac{\sigma}{\sigma - q}}(C_{z_0})^{\frac{1}{6} - \frac{\sigma}{\sigma - q}}(r) \leq \bar{g}^{\frac{q}{6} - \frac{\sigma}{\sigma - q}} \cdot \left[ 2M_{q, z_0}(r) \right] \quad \forall \; r > 0
\]
as long as $\bar{g} \geq 1$; this leads to the constants appearing in (3.21).

Let’s now use Proposition 3 and Lemma 1 to prove Lemma 2.

**Proof of Lemma 2** Fix any $\tilde{C} \in (0, \infty), \sigma \in (5, 6)$ and $\bar{g} \in [1, \infty)$, and choose any $q = q(\sigma) \in (5, \min\{\sigma, \frac{11}{2}\})$ which we now also fix, noting that $\frac{q}{\sigma} < 12$ and $2(6 - q) > 1$; for the chosen $q$, let $\tilde{c} = \tilde{c}(\tilde{C}) \in (0, 1)$ be the corresponding small constant from Lemma 1.

---

22In the requirement that $q \in (5, \min\{\sigma, q\})$, the choice of $\tilde{q} := \frac{11}{2}$ is somewhat arbitrary and taken only for concreteness; one could similarly choose any $\tilde{q} \in (5, 6)$ and adjust the subsequent constants accordingly.
Let us first note the following important consequence of Lemma 1: Fix \( \Omega_T \) as in Lemma 1 and \( z_0 := (x_0, t_0) \in \Omega_T \), and suppose that
\[
M_{q,z_0}(r) \leq \frac{1}{2} \left( \frac{\epsilon}{3} \right)^{12}
\]  
for some \( r \in (0, 1] \) such that \( Q^*_r(z_0) \subseteq \Omega_T \). Setting
\[
u_{z_0,r}(x,t) := ru(x_0 + rx + t_0 + r^2 t), \quad p_{z_0,r}(x,t) := r^2 p(x_0 + rx + t_0 + r^2 t)
\]
and \( d_{z_0,r}(x,t) := d(x_0 + rx + t_0 + r^2 t) \), a change of variables from \( z = (x,t) \) to
\[
(y,s) := (x_0 + rx, t_0 + r^2 t)
\]
implies that
\[
\int_{Q^*_1(0,0)} \left( |u_{z_0,r}|^3 + |\nabla d_{z_0,r}|^3 + |p_{z_0,r}|^q + |d_{z_0,r}|^q \left( |u_{z_0,r}|^3 + |\nabla d_{z_0,r}|^3 \right)^{1-q} \right) \, dz
\]
\[
= C_{z_0}(r) + D_{z_0}(r) + G_{q,z_0}(r) \leq \left( \frac{\epsilon}{3} \right)^{12} + \left( \frac{\epsilon}{3} \right)^6 + \left( \frac{\epsilon}{3} \right)^{2(6-q)} < \epsilon_q.
\]
Since \( Q^*_1(0,0) = Q_1(0, \frac{1}{8}) \), it follows from assumptions (1.9) - (1.13) that \( u_{z_0,r}, d_{z_0,r} \) and \( p_{z_0,r} \) satisfy the assumptions of Lemma 1 with \( \tilde{z} = (\tilde{x}, t) := (0, \frac{1}{8}) \) and \( \tilde{\rho} := r^2 \in (0, 1] \), with the same constant \( C \) (see Footnote 18). Since we have just seen that
\[
E_{3,q} = E_{3,q}(u_{z_0,r}, d_{z_0,r}, p_{z_0,r}, \tilde{z}) < \epsilon_q,
\]
we therefore conclude by Lemma 1 that
\[
|u_{z_0,r}(z)|, |\nabla d_{z_0,r}(z)| \leq \frac{\epsilon_q}{2}
\]
for a.e. \( z \in Q_{\frac{1}{8}}(0, \frac{1}{8}) = B_{\frac{1}{8}}(0) \times (-\frac{1}{8}, \frac{1}{8}) \)
and hence
\[
|u(y,s)|, |\nabla d(y,s)| \leq \frac{\epsilon_q}{2r}
\]
for a.e. \( (y,s) \in B_{\frac{1}{8}}(x_0) \times (t_0 - \frac{r^2}{8}, t_0 + \frac{r^2}{8}) \).

In particular, by definition, \( z_0 = (x_0, t_0) \) is a regular point, i.e. \( |u| \) and \( |\nabla d| \) are essentially bounded in a neighborhood of \( z_0 \), so long as (3.24) holds for some sufficiently small \( r > 0 \).

In view of this fact, setting
\[
\delta_\sigma := \frac{1}{2} \left( \frac{\epsilon_q(\sigma)}{3} \right)^{12} \quad \text{and} \quad \bar{c}_\sigma := \tilde{c} \cdot \tilde{g}^{\frac{q}{3-\sigma}},
\]
we choose \( \gamma_\sigma \in (0, \frac{1}{4}] \) so that furthermore
\[
\bar{c}_\sigma(1-\alpha_\sigma,q) \leq \frac{1}{4} \left( \frac{\delta_{\sigma}^{1-\alpha_\sigma,q}}{2} \right),
\]  
\[\]  
\[\text{Footnote 23: For example, if one fixes an arbitrary } \phi \in C_0^\infty(Q^*_1(0,0)) \text{ and sets}
\phi^{z_0,r}(x,\tau) := \phi \left( \frac{x-x_0}{r}, \frac{\tau-t_0}{r^2} \right),
\text{then } \phi^{z_0,r} \in C_0^\infty(Q^*_r(z_0)) \subseteq C_0^\infty(\Omega_T). \text{ One can therefore use the test function } \phi^{z_0,r} \text{ in (1.15), make the change of variables } (\xi, s) := \left( \frac{x-x_0}{r}, \frac{\tau-t_0}{r^2} \right) \text{ (so } (x, \tau) = (x_0 + r\xi, t_0 + r^2s)) \text{ and divide both sides of the result by } r \text{ to obtain the local energy inequality (3.3) for the re-scaled functions with } \tilde{\rho} = r^2 \text{ (as all terms scale the same way except for } |d|^2|\nabla d|^2\phi^{z_0,r} \text{) and } \tilde{z} = (0, \frac{1}{8}). \text{ The other assumptions are straightforward.} \]
where \( \bar{c} = \bar{c}(\bar{C}) \) is the constant from Proposition 3 and \( \alpha_{\sigma,q} \) is defined as in (3.19); finally, we choose \( \epsilon_{\sigma} \in (0, 1) \) so small that

\[
\bar{c}_{\sigma} \gamma_{\sigma}^{-15} \epsilon_{\sigma}^{\frac{3}{2} \alpha_{\sigma,q}} \leq \frac{1}{4} \left( \frac{\delta_{\sigma}^{[1-1/3 \alpha_{\sigma,q}]}}{6} \right).
\]

If \( z_0 \in \Omega_T \) is such that (3.15) and (3.16) hold, it implies in particular that there exists some \( \rho_0 \in (0, 1) \) such that \( Q^*_{\rho_0}(z_0) \subseteq \Omega_T \) and, furthermore,

\[
\sup_{\rho \in (0, \rho_0)} G_{\sigma,z_0}(\rho) \leq \bar{g}
\]

and

\[
\sup_{\rho \in (0, \rho_0)} B_{z_0}(\rho) < \epsilon_{\sigma}.
\]

It then follows from (3.27), (3.28) and (3.30) (and the facts that \( \alpha_{\sigma,q}, \delta_{\sigma} \leq 1 \)) that

\[
\bar{c}_{\sigma} \gamma_{\sigma}^{-15} \epsilon_{\sigma}^{\frac{3}{2} \alpha_{\sigma,q}} \leq \frac{1}{4} \left( \frac{\delta_{\sigma}^{[1-1/3 \alpha_{\sigma,q}]}}{2} \right) = \frac{1}{4} \left( \frac{\min \left\{ 1, \delta_{\sigma}^{[1-1/3 \alpha_{\sigma,q}]}, \delta_{\sigma}^{[1-1/2 \alpha_{\sigma,q}], \delta_{\sigma}^{[1-1/2 \alpha_{\sigma,q}]}} \right\}}{2} \right),
\]

and that

\[
\bar{c}_{\sigma} \gamma_{\sigma}^{-15} B_{z_0}^{\frac{3}{2} \alpha_{\sigma,q}}(\rho) \leq \bar{c}_{\sigma} \gamma_{\sigma}^{-15} \epsilon_{\sigma}^{\frac{3}{2} \alpha_{\sigma,q}} \leq \frac{1}{4} \left( \frac{\delta_{\sigma}^{[1-1/3 \alpha_{\sigma,q}]}}{6} \right) = \frac{1}{4} \left( \frac{\min_{k \in \{0,2\}} \{ \min \left\{ \delta_{\sigma}^{[1-1/4 \alpha_{\sigma,q}]}, \delta_{\sigma}^{[1-1/3 \alpha_{\sigma,q]}}, \delta_{\sigma}^{[1-1/2 \alpha_{\sigma,q}]}, \delta_{\sigma}^{[1-1/2 \alpha_{\sigma,q}]} \right\} }{6} \right)
\]

for all \( \rho \leq \rho_0 \). Suppose now that \( z_0 \) is not a regular point. Then we must have

\[
\delta_{\sigma} < M_{q,z_0}(\rho) \quad \text{for all } \rho \in (0, \rho_0],
\]

or else (3.24) would hold for some \( r \in (0, \rho_0] \) which would imply that \( z_0 \) is a regular point as we established above using Lemma 1.

In view of (3.29) and (3.30) (so that in particular (3.20) holds, as we chose \( \epsilon_{\sigma} \leq 1 \)), we conclude by the estimate (3.21) of Proposition 3 (along with (3.27), (3.28), (3.30), (3.31) and our calculations above) that

\[
M_{q,z_0}(\gamma_{\sigma} \rho) \leq \frac{1}{2} M_{q,z_0}(\rho) \quad \text{for all } \rho \in (0, \rho_0]
\]

for any \( z_0 \) which is not a regular point. However, since \( \gamma_{\sigma}^k \rho_0 \in (0, \rho_0] \) for any \( k \in \mathbb{N} \), by iterating the estimate above we would conclude for such \( z_0 \) that

\[
M_{q,z_0}(\gamma_{\sigma}^n \rho_0) \leq \frac{1}{2} M_{q,z_0}(\gamma_{\sigma}^{n-1} \rho_0) \leq \frac{1}{2^2} M_{q,z_0}(\gamma_{\sigma}^{n-2} \rho_0) \leq \cdots \leq \frac{1}{2^n} M_{q,z_0}(\rho_0) < \delta_{\sigma}
\]

for a sufficiently large \( n \in \mathbb{N} \) which contradicts (3.31) (with \( \rho = \gamma_{\sigma}^n \rho_0 \)), and hence contradicts our assumption that \( z_0 \) is not a regular point. Therefore \( z_0 \) must indeed be regular whenever (3.29) and (3.30) hold for our choice of \( \epsilon_{\sigma} \), which proves Lemma 2.

In order to prove Theorem 1, we now prove the following general lemma, from which Lemma 1 and Lemma 2 will have various consequences (including Theorem 1 as well as various other historical
results, which we point out for the reader’s interest). As a motivation, note first that, for \( r > 0 \) and \( z_1 := (x_1, t_1) \in \mathbb{R}^3 \times \mathbb{R} \), according to the notation in (3.25) a change of variables gives

\[
\int_{Q_r^+(0,0)} |u_{z_1,r}|^q + |p_{z_1,r}|^\frac{q}{2} = \frac{1}{r^{3-q}} \int_{Q_r^+(x_1,t_1)} |u|^q + |p|^\frac{q}{2}, \quad \int_{Q_r^+(0,0)} |\nabla u_{z_1,r}|^q = \frac{1}{r^{5-2q}} \int_{Q_r^+(x_1,t_1)} |\nabla u|^q
\]

and

\[
\int_{Q_r^+(0,0)} |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-\frac{q}{2})} = \frac{1}{r^{2+\frac{q}{2}}} \int_{Q_r^+(x_1,t_1)} |d|^q |\nabla d|^{3(1-\frac{q}{2})} \tag{3.32}
\]

for any \( q \in [1, \infty) \).

**Lemma 3.** Fix any open and bounded \( \Omega \subset \subset \mathbb{R}^3 \), \( T \in (0, \infty) \), \( k \geq 0 \) and \( C_k > 0 \), and suppose \( S \subseteq \Omega_T := \Omega \times (0, T) \) and that \( U : \Omega_T \to [0, \infty] \) is a non-negative Lebesgue-measurable function such that the following property holds in general:

\[
(x_0, t_0) \in S \implies \limsup_{r \searrow 0} \frac{1}{r^T} \int_{Q^+_r(x_0,t_0)} U dz \geq C_k. \tag{3.33}
\]

If, furthermore,

\[
U \in L^1(\Omega_T), \tag{3.34}
\]

then (recall Definition \[\] \( \mathcal{P}^k(S) < \infty \) (and hence the parabolic Hausdorff dimension of \( S \) is at most \( k \)) with the explicit estimate

\[
\mathcal{P}^k(S) \leq \frac{5^3}{C_k} \int_{\Omega_T} U dz; \tag{3.35}
\]

moreover, if \( k = 5 \), then

\[
\mu(S) \leq \frac{4\pi}{3} \mathcal{P}^k(S) \leq \frac{5^5 \cdot 4\pi}{3C^5} \int_{\Omega_T} U dz \tag{3.36}
\]

where \( \mu \) is the Lebesgue outer measure, and if \( k < 5 \), then in fact \( \mathcal{P}^k(S) = \mu(S) = 0 \).

Before proving Lemma 3, let’s first use it along with Lemma 1 and Lemma 2 to give the

**Proof of Theorem 1.**

First note that for any \( r > 0 \) and \( z_1 := (x_1, t_1) \in \mathbb{R}^3 \times \mathbb{R} \) such that \( Q_r(z_1) \subseteq \Omega_T \), it follows (as in the proof of Lemma 2) that the re-scaled triple \((u_{z_1,r}, d_{z_1,r}, p_{z_1,r})\) (see (3.25)) satisfies the conditions of Lemma 1 with \( \tilde{\epsilon} := (0, 0) \) and \( \tilde{\rho} := r^2 \). Therefore if \( q \in (5, 6) \) and

\[
\frac{1}{r^2} \int_{Q_r^+(x_1,t_1)} |u|^3 + |\nabla d|^3 + |p|^\frac{q}{2} + \frac{1}{r^{2+\frac{q}{2}}} \int_{Q_r^+(x_1,t_1)} |d|^q |\nabla d|^{3(1-\frac{q}{2})} = \int_{Q^+(0,0)} |u_{z_1,r}|^3 + |\nabla d_{z_1,r}|^3 + |p_{z_1,r}|^\frac{q}{2} + |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-\frac{q}{2})} < \tilde{\epsilon}_q \tag{3.37}
\]

(with \( \tilde{\epsilon}_q = \epsilon_q(C) \) as in Lemma 1), it follows that \( |u_{z_1,r}|, |\nabla d_{z_1,r}| \leq C \) on \( Q^+_4(0,0) \) for some \( C > 0 \), and hence \( |u|, |\nabla d| \leq \frac{C}{r^2} \) on \( Q^+_2(x_1,t_1) \); in particular, every interior point of \( Q^+_2(x_1,t_1) \) is a regular point, assuming (3.37) holds. Therefore, taking \( z_0 := (x_0, t_0) \) such that

\[
Q^+_2(x_1,t_1) = Q^+_2(x_0,t_0),
\]

(so \( x_0 = x_1 \) and \( t_0 \) is slightly lower than \( t_1 \) so that \((x_0, t_0)\) is in the interior of the cylinder \( Q^+_2(x_1,t_1) \)) and letting \( S \subset \Omega_T \) be the singular set of the solution \((u,d,p)\), we see (in particular) that, since \( r^{2+\frac{q}{2}} < r^2 \) for \( r < 1 \),

\[
(x_0, t_0) \in S \quad \text{and} \quad q \in (5, 6) \quad \implies \quad \limsup_{r \searrow 0} \frac{1}{r^{2+\frac{q}{2}}} \int_{Q^+_r(x_0,t_0)} |u|^3 + |\nabla d|^3 + |p|^\frac{q}{2} + |d|^q |\nabla d|^{3(1-\frac{q}{2})} \geq \tilde{\epsilon}_q \tag{3.38}
\]
and choose \( \gamma \) we may apply Lemma 3 with H"older's inequality implies that so that if \( \sigma \) (Lemma 1), with historical relevance, for the interest of the reader:

Before continuing with the proof of Theorem 1, we describe some intermediate results (using only Lemma 11), with historical relevance, for the interest of the reader:

Suppose that (1.14) holds for some \( \sigma \in (5,6) \) which we now fix. We further fix any \( q \in (5, \sigma) \), and choose \( \gamma_{\sigma,q} > 0 \) small enough that

\[
\gamma_{\sigma,q}^{-1} (\gamma_{\sigma,q} + (g_{\sigma})^{\frac{1}{q}}) < \bar{c}_q.
\]

As in the proof of (3.22), H"older's inequality (along with (3.32)) implies that

\[
\int_{Q_1(0,0)} |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-\frac{2}{p})} \leq (g_{\sigma})^{\frac{1}{q}} \left( \int_{Q_1(0,0)} |\nabla d_{z_1,r}|^{3} \right)^{1-\frac{2}{p}},
\]

so that if

\[
\frac{1}{r^2} \int Q_r(x_0, t_1) |u|^3 + |\nabla d|^3 + |p|^{\frac{3}{2}} = \int_{Q_1(0,0)} |u_{z_1,r}|^3 + |\nabla d_{z_1,r}|^3 + |p_{z_1,r}|^{\frac{3}{2}} < \gamma_{\sigma,q},
\]

it follows that

\[
\int_{Q_1(0,0)} |u_{z_1,r}|^3 + |\nabla d_{z_1,r}|^3 + |p_{z_1,r}|^{\frac{3}{2}} + |d_{z_1,r}|^q |\nabla d_{z_1,r}|^{3(1-\frac{2}{p})} < \bar{c}_q
\]

and hence \((x_0, t_0) \notin S\) for \((x_0, t_0)\) as above.

Therefore under the general assumption (1.14) with \( \sigma \in (5,6) \), there exists \( \gamma_0 > 0 \) (e.g., \( \gamma_0 := \gamma_{\sigma, \frac{5+\epsilon}{2}} \)) such that

\[
(x_0, t_0) \in S \implies \limsup_{r \to 0} \frac{1}{r^2} \int_{Q_r(x_0, t_0)} |u|^3 + |\nabla d|^3 + |p|^{\frac{3}{2}} \geq \gamma_0.
\]

Therefore, as long as \((u, \nabla d, p) \in L^3(\Omega_T) \times L^3(\Omega_T) \times L^2(\Omega_T)\), we may apply Lemma \(\ref{lemma:gamma}\) with \( U := |u|^3 + |\nabla d|^3 + |p|^{\frac{3}{2}} \), \( k = 2 \) and \( C_k := \gamma_0 \) to see (similar to Scheffer’s result in \[\text{Sch77}\]) that

\[
P^2(S) = 0.
\]

On the other hand, we know slightly more than (3.42). The assumptions on \( u \) and \( d \) in (1.9) imply (for example, by (2.18) with \( \alpha = \frac{3}{2} \), along with Sobolev embedding) that \( u, \nabla d \in L^{\frac{10}{3}}(\Omega_T) \). Suppose we also knew (as in the case when \( \Omega = \mathbb{R}^3 \)) that \( p \in L^{\frac{3}{2}}(\Omega_T) \) (which essentially follows from (1.9) and (1.12), see \[\text{LL96}, \text{Theorem 2.5}\]). Then (3.38) holds with \( U := |u|^{\frac{3}{2}} + |\nabla d|^{\frac{3}{2}} + |p|^{\frac{3}{2}} \), and moreover Hölder’s inequality implies that

\[
\left\{ \frac{1}{r^2} \int_{Q_r(z_0)} |u|^3 + |\nabla d|^3 + |p|^{\frac{3}{2}} \right\}^{\frac{2}{3}} \leq 2^{\frac{2}{3}} |Q_1|^{\frac{1}{3}} \left[ \frac{1}{r^2} \int_{Q_r(z_0)} |u|^{\frac{3}{2}} + |\nabla d|^{\frac{3}{2}} + |p|^{\frac{3}{2}} \right]^{\frac{2}{3}}
\]
(\|Q_1\| \text{ is the Lebesgue measure of the unit parabolic cylinder). In view of } (3.41), \text{ one could therefore apply Lemma 3 with }

\[ U := |u|^{10} + |\nabla d|^{10} + |p|^{10}, \quad k = \frac{5}{3} \text{ and } C_k = \frac{\gamma_{a^{10}}}{2^{10} |Q_1|^{10}}, \]

to deduce (similar to Scheffer's result in [Sch80]) that

\[ P^\frac{5}{3}(S) = 0. \]

All of the above follows from Lemma 1 alone. We will now show that Lemma 2 allows one (under assumption (1.14) for some \( \sigma \in (5, 6) \), and even if \( p \notin L^5(\Omega_T) \)) to further decrease the dimension of the parabolic Hausdorff measure, with respect to which the singular set has measure zero, from \( \frac{5}{3} \) to 1. This was essentially the most significant contribution of [CKN82] in the Navier-Stokes setting \( d \equiv 0 \).

Let us now proceed with the proof of the second assertion in Theorem 1. Suppose \( d \) satisfies (1.14) for some \( \sigma \in (5, 6) \). Taking \( \epsilon_\sigma = \epsilon_\sigma (\bar{C}, g_\sigma) > 0 \) as in Lemma 2 with \( \bar{g} := g_\sigma \), we see from (3.16) that

\[ (x_0, t_0) \in S \implies \limsup_{r \to 0} \frac{1}{r} \int_{Q^*_r(x_0, t_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \geq \epsilon_\sigma, \]

so that (3.33) holds with \( U := |\nabla u|^2 + |\nabla^2 d|^2 \) and \( k = 1 \). The second assumption in (1.9) implies that (3.34) holds as well with \( U := |\nabla u|^2 + |\nabla^2 d|^2 \). Therefore Lemma 3 with \( U := |\nabla u|^2 + |\nabla^2 d|^2 \), \( k = 1 \) and \( C_k = \epsilon_\sigma \) implies that

\[ P^1(S) = 0. \]

This completes the proof of Theorem 1 (assuming Lemma 3).

Let us now give the

**Proof of Lemma 3**. Fix any \( \delta > 0 \), and any open set \( V \) such that

\[ S \subseteq V \subseteq \Omega \times (0, T). \]

For each \( z := (x, t) \in S \), according to (3.33) we can choose \( r_z \in (0, \delta) \) sufficiently small so that \( Q_{r_z}^*(z) \subset V \) and

\[ \frac{1}{r_z} \int_{Q_{r_z}^*(z)} U \geq C_k. \]

By a Vitali covering argument (see [CKN82, Lemma 6.1]), there exists a sequence \( (z_j)_{j=1}^\infty \subseteq S \) such that

\[ S \subseteq \bigcup_{j=1}^\infty Q_{r_j}^*(z_j) \]

and such that the set of cylinders \( \{Q_{r_j}^*(z_j)\}_j \) are pair-wise disjoint. We therefore see from (3.44) that

\[ \sum_{j=1}^\infty r_j^k \leq \frac{1}{C_k} \sum_{j=1}^\infty \int_{Q_{r_j}^*(z_j)} U \leq \frac{1}{C_k} \int_V U \leq \frac{1}{C_k} \int_{\Omega_T} U \]

which is finite (and uniformly bounded in \( \delta \)) by (3.34). Note that according to Definition 1 of the parabolic Hausdorff measure \( P^k \), (3.46) implies

\[ P^k(S) \leq \frac{5^k}{C_k} \int_V U \leq \frac{5^k}{C_k} \int_{\Omega_T} U \]

(3.47)
due to (3.46), which establishes (3.35).

Let us now assume that $k \leq 5$. Letting $\mu$ be the Lebesgue (outer) measure, note that

$$\mu(Q^*_{5r_{z_j}}) \leq |B_1|(5r_{z_j})^5$$

so that

$$\mu(S) \leq |B_1| \sum_{j=1}^{\infty} (5r_{z_j})^5 \leq 5^5 |B_1| \sum_{j=1}^{\infty} r_{z_j}^k \leq 5^5 \delta^{-k} \frac{5^5 |B_1|}{C_k} \int_{\Omega_r} U,$$  (3.48)

since we have chosen $r_z < \delta$ for all $z \in S$. If $k = 5$, (3.48) along with Definition 1 gives the explicit estimate (3.36) on $\mu(S)$. If $k < 5$, since $\delta > 0$ was arbitrary, sending $\delta \to 0$ we conclude (by (3.34)) that $\mu(S) = 0$ and hence $S$ is Lebesgue measurable with Lebesgue measure zero. We may therefore take $V$ to be an open set such that $\mu(V)$ is arbitrarily small but so that (3.43) still holds, and deduce that $\mathcal{P}_k(S) = 0$ by (3.34) and (3.47). \qed

4 Proofs of technical propositions

In order to prove Proposition 1 as well as Proposition 3, we will require certain local decompositions of the pressure (cf. [CKN82, (2.15)]) as follows:

4.1 Localization of the pressure

Claim 1. Fix open sets $\Omega_1 \subset \subset \Omega_2 \subset \subset \Omega \subset \mathbb{R}^3$ and $\psi \in C^\infty_0(\Omega_2; \mathbb{R})$ with $\psi \equiv 1$ on $\Omega_1$. Let

$$G^x(y) := \frac{1}{4\pi} \frac{1}{|x-y|}$$  (4.1)

be the fundamental solution of $-\Delta$ in $\mathbb{R}^3$ so that, in particular,

$$\nabla G^x \in L^q(\Omega_2) \quad \text{for any} \quad q \in [1, \frac{3}{2})$$

for any fixed $x \in \mathbb{R}^3$, and set

$$G^x_{\psi,1} := -G^x \nabla \psi$$
$$G^x_{\psi,2} := 2\nabla G^x \cdot \nabla \psi + G^x \Delta \psi$$
$$G^x_{\psi,3} := \nabla G^x \otimes \nabla \psi + \nabla \psi \otimes \nabla G^x + G^x \nabla^2 \psi,$$

so that

$$G^x_{\psi,1}, G^x_{\psi,2}, G^x_{\psi,3} \in C^\infty_0(\Omega_2) \quad \text{for any fixed} \quad x \in \Omega_1.$$

Suppose $\Pi \in C^2(\Omega; \mathbb{R})$, $v \in C^1(\Omega; \mathbb{R}^3)$ and $K \in C^2(\Omega; \mathbb{R}^{3\times3})$.

If

$$-\Delta \Pi = \nabla \cdot v \quad \text{in} \quad \Omega,$$  (4.2)

then for any $x \in \Omega_1$,

$$\Pi(x) = -\int \nabla G^x \cdot \nabla \psi + \int G^x_{\psi,1} \cdot v + \int G^x_{\psi,2} \Pi.$$  (4.3)

Similarly, if

$$-\Delta \Pi = \nabla \cdot (\nabla^T \cdot K) \quad \text{in} \quad \Omega,$$  (4.4)

then for any $x \in \Omega_1$,

$$\Pi(x) = S[\psi K](x) + \int G^x_{\psi,3} \cdot K + \int G^x_{\psi,2} \Pi.$$  (4.5)
Indeed, under the assumptions (1.9), we have due, in particular, to the boundedness of the linear operator passing to limits gives the almost-everywhere convergence (after passing to a suitable subsequence)

\[ t \text{ regularizations one can see that for almost every fixed } x, \]

by Young's convolution inequality (since \( \Omega \) is bounded), so that term is defined for a.e.

\[
\text{Remark 2. We note, therefore, that under the assumptions (1.9), (1.10), and (1.12), by suitable}
\]

Indeed, under the assumptions (1.9), we have \( u, \nabla d \in L^{10}(\Omega_T) \) so that (omitting the \( x \)-dependence)

\[
J(t) \in L^{\frac{5}{2}}(\Omega) \quad \text{for a.e. } t \in (0, T). \tag{4.6}
\]

Moreover, since \( u, \nabla d \in L^{\infty}(0, T; L^3(\Omega)) \cap L^{15}(\Omega_T) \) and \( \nabla u, \nabla^2 d \in L^2(\Omega_T) \), we have

\[
\nabla^T \cdot J \in L^2(0, T; L^1(\Omega)) \cap L^{\frac{5}{2}}(\Omega_T)
\]

so that

\[
\nabla^T \cdot J(t) \in L^1(\Omega) \cap L^{\frac{5}{2}}(\Omega_T) \quad \text{for a.e. } t \in (0, T). \tag{4.7}
\]

Finally, (1.10) implies that

\[
p(t) \in L^{\frac{3}{2}}(\Omega) \quad \text{for a.e. } t \in (0, T). \tag{4.8}
\]

Fix now any \( t \in (0, T) \) such that the inclusions in (4.6), (4.7) and (4.8) hold. Since \( G^x_{\psi, 3} \in C_0^\infty \)

for \( x \in \Omega_1 \), the terms in (4.3) and (4.4) containing \( G^x_{\psi, 3} \) are all well-defined for every \( x \in \Omega_1 \) since

\( J(t), \nabla^T \cdot J(t), p(t) \in L^{10}(\Omega_T) \). The term in (4.3) containing \( \nabla G^x \) is in \( L^r(\Omega_2) \) for any \( r \in [1, \frac{15}{10}] \)

by Young's convolution inequality (since \( \Omega_2 \) is bounded), so that term is well-defined for a.e. \( x \in \Omega_2 \).

Indeed, for \( R > 0 \) such that \( \Omega_2 \subseteq B_R(x_0) \) for some \( x_0 \in \mathbb{R}^3 \), we have \( x - y \in B_R := B_R(0) \) for all \( x, y \in \Omega_2 \). Letting \( G(y) := G^0(y) \) and \( \chi_{B_R} \) the indicator function of \( B_R \), since \( \psi \) is supported in \( \Omega_2 \) we therefore have

\[
- \int \nabla G^x \cdot v \psi = \int [\nabla G \chi_{B_R} \ast (v \psi)](x) \quad \text{for all } x \in \Omega_2.
\]

Therefore

\[
\left\| \int \nabla G^x \cdot v \psi \right\|_{L^s(\Omega_2)} \leq \left\| [\nabla G \chi_{B_R} \ast (v \psi)] \right\|_{L^r(\mathbb{R}^3)}
\]

by Young's inequality for any \( q \in [1, \frac{3}{5}] \), \( s \in [1, \frac{5}{3}] \) and \( r \) such that \( 1 + \frac{1}{s} = \frac{1}{q} + \frac{1}{r} \) (note that \( \frac{2}{3} + \frac{4}{5} - 1 = \frac{7}{15} \)). Finally, \( S[\psi J(t)] \in L^{\frac{5}{2}}(\Omega_2) \) by the Calderon-Zygmund estimates (as \( 1 < \frac{5}{3} < \infty \)), so again that term is defined for a.e. \( x \in \Omega_2 \).

Regularizing the linear equation (1.12) using a standard spatial mollifier at any \( t \in (0, T) \) where (1.12) holds in \( D'(\Omega) \) and where the inclusions in (4.6), (4.7) and (4.8) hold, applying Claim 1 and passing to limits gives the almost-everywhere convergence (after passing to a suitable subsequence) due, in particular, to the boundedness of the linear operator \( S \) on \( L^{\frac{5}{2}}(\Omega_2) \).
Proof of Claim 1. Since (extending $\Pi$ by zero outside of $\Omega$) $\psi \Pi \in C^\infty_0(\mathbb{R}^3)$, by the classical representation formula (see, e.g., [GT01 (2.17)]), for any $x \in \mathbb{R}^3$ we have

$$\psi(x)\Pi(x) = - \int G^x \Delta(\psi\Pi) = - \int G^x (\psi \Delta \Pi + 2 \nabla \psi \cdot \nabla \Pi + \Pi \Delta \psi).$$

(4.9)

In particular, for a fixed $x \in \Omega_1$ where $\psi \equiv 1$, we have $G^x \nabla \psi \in C^\infty_0(\mathbb{R}^3)$ so that integrating by parts in (4.9) we see that

$$\Pi(x) = \int G^x \psi(-\Delta \Pi) + \int G^x_{\psi,2} \Pi.$$  

(4.10)

If (4.2) holds, then by (4.10) we have

$$\Pi(x) = \int G^x \psi \nabla \cdot v + \int G^x_{\psi,2} \Pi$$  

(4.11)

for any $x \in \Omega_1$. One can then carefully integrate by parts once in the first term of (4.11) as follows: for a small $\epsilon > 0$,

$$\int_{|y-x| > \epsilon} G^x \nabla \cdot v dy = - \int_{|y-x| > \epsilon} \left[ \nabla (G^x \psi) \right] \cdot v dy + \frac{1}{4 \pi \epsilon} \int_{|y-x| = \epsilon} \psi \nu_y dS_y = O(\epsilon^2)$$

and since the second term vanishes as $\epsilon \to 0$ due to the fact that $|\partial B_\epsilon(x)| \lesssim \epsilon^2$, we conclude (since $\nabla G^x \in L^1_{loc}$) that

$$\int G^x \psi \nabla \cdot v = - \int \nabla(G^x \psi) \cdot v = - \int \nabla G^x \cdot v \psi + \int G^x_{\psi,1} \cdot v$$

which, along with (4.11), implies (4.3) for any $x \in \Omega_1$.

On the other hand, if (4.4) holds, then by (4.10) we have

$$\Pi(x) = \int G^x \psi \nabla \cdot (\nabla^T \cdot K) + \int G^x_{\psi,2} \Pi$$

(4.12)

and one can write

$$\nabla \cdot (\nabla^T \cdot (\psi K)) = \left[ \nabla^2 \psi \right]^T \cdot K + \nabla^T \psi \cdot \nabla \cdot K + \nabla \psi \cdot [\nabla^T \cdot K] + \psi \nabla \cdot (\nabla^T \cdot K)$$

so that (as $\nabla^2 \psi = \nabla^T (\nabla \psi) = \nabla (\nabla^T \psi) = [\nabla^2 \psi]^T$ since $\psi \in C^2$)

$$\int G^x [\psi \nabla \cdot (\nabla^T \cdot K)] = \int G^x [\nabla \cdot (\nabla^T \cdot (\psi K))] = \int G^x [\nabla^2 \psi \cdot K]$$

$$- \int \left( [G^x \nabla^T \psi \cdot \nabla \cdot K] + [G^x \nabla \psi \cdot [\nabla^T \cdot K]] \right).$$

Since $G^x \nabla \psi \in C^\infty_0$ for $x \in \Omega_1$, one can again integrate by parts in the final term to obtain

$$\Pi(x) = \int G^x [\nabla \cdot (\nabla^T \cdot (\psi K))] + \int G^x_{\psi,3} \cdot K + \int G^x_{\psi,2} \Pi$$

for $x \in \Omega_1$ in view of (4.12). Moreover, since $\psi K \in C^2_0$ and $G^x \in L^1_{loc}$, as usual for convolutions one can change variables to obtain

$$\int G^x \nabla \cdot (\nabla^T \cdot (\psi K)) = \left[ \nabla_x \cdot \left( \nabla^T_x \int G^x \psi K \right) \right] (x) =: S[\psi K](x)$$

which gives us (4.5) for any $x \in \Omega_1$, where (see, e.g., [GT01 Theorem 9.9]) $S$ is a singular integral operator as claimed. (Note that $\nabla^2 G^x \notin L^1_{loc}$ so that one cannot simply integrate by parts twice in this term putting all derivatives on $G^x$, but $\int G^x \psi K$ is the Newtonian potential of $\psi K$ which can be twice differentiated in various senses depending on the regularity of $K$.)

$\Box$
4.2 Proof of Proposition [1]

In what follows, for \( O \subseteq \mathbb{R}^3 \) and \( I \subseteq \mathbb{R} \), we will use the notation
\[
\| \cdot \|_{q;O} := \| \cdot \|_{L^q(O)} , \quad \| \cdot \|_{s;I} := \| \cdot \|_{L^s(I)} , \\
\| \cdot \|_{q,s;O \times I} := \| \cdot \|_{L^q(I;L^s(O))} = \| \| \cdot \|_{L^q(O)} \|_{L^s(I)} \]
and we will abbreviate by writing
\[
\| \cdot \|_{q;O \times I} := \| \cdot \|_{q,s;O \times I} = \| \cdot \|_{L^q(O \times I)} .
\]

We first note some simple inequalities. Letting \( B_r \subseteq \mathbb{R}^3 \) be a ball of radius \( r > 0 \), from the embedding \( W^{1,2}(B_1) \hookrightarrow L^6(B_1) \) applied to functions of the form \( g_r(x) = g(rx) \) (or suitably shifted, if the ball is not centered as zero), we obtain
\[
\| g_r \|_{6,B_1} \lesssim \| g_r \|_{2,B_1} + \| \nabla g_r \|_{2,B_1} = \| g_r \|_{2,B_1} + r \| (\nabla g)_r \|_{2,B_1}
\]
whereupon, noting by a simple change of variables that
\[
\| g_r \|_{q;B_1} = r^{-\frac{d}{q}} \| g \|_{q;B_r}
\]
for any \( q \in [1, \infty) \), we obtain for any ball \( B_r \) of radius \( r > 0 \) and any \( g \) that
\[
\| g \|_{6,B_r} \lesssim \frac{1}{r} \| g \|_{2,B_r} + \| \nabla g \|_{2,B_r} \tag{4.13}
\]
where the constant is independent of \( r \) as well as the center of \( B_r \). Next, for any \( v(x,t) \), using Hölder to interpolate between \( L^2 \) and \( L^6 \) we have
\[
\| v(t) \|_{3,B_r} \leq \| v(t) \|_{\frac{6}{2},B_r} \| v(t) \|_{\frac{3}{2},B_r} \lesssim r^{-\frac{d}{6}} \| v(t) \|_{2,B_r} + \| v(t) \|_{\frac{6}{2},B_r} \| \nabla v(t) \|_{\frac{3}{2},B_r} \tag{4.14}
\]
Then for \( I_r \subseteq \mathbb{R} \) with \( |I_r| = r^2 \) and \( Q_r := B_r \times I_r \), Hölder in the \( t \) variable gives
\[
\| v \|_{3,Q_r} \lesssim r^{-\frac{d}{3}} |I_r|^\frac{1}{2} \| v \|_{2,\infty;Q_r} + \| v \|_{\frac{2}{3},\infty;Q_r} \left[ |I_r|^\frac{1}{2} \| \nabla v \|_{2,Q_r} \right]^{\frac{1}{2}}
\]
so that
\[
\begin{align*}
\| v \|_{3,Q_r} & \lesssim \| v \|_{2,\infty;Q_r} + \| v \|_{\frac{2}{3},\infty;Q_r} \| \nabla v \|_{2,Q_r} \lesssim \| v \|_{2,\infty;Q_r} + \| \nabla v \|_{2,Q_r} \\
(\text{the first of which is sometimes called the "multiplicative inequality"}) \quad \text{with a constant independent of } r.
\end{align*}
\]
From these, noting that \( |B_r| \sim r^d \), \( |Q_r| \sim r^3 \), it follows easily that, for example,
\[
\iint_{Q_{\tilde{r}}} |v|^3 \, dz \lesssim \left( \operatorname{ess sup}_{t \in I_{\tilde{r}}} \int_{B_{\tilde{r}}} |v(t)|^2 \, dx \right)^{\frac{3}{2}} + \left( \int_{I_{\tilde{r}}} \int_{B_{\tilde{r}}} |\nabla v|^2 \, dx \, dt \right)^{\frac{1}{2}} . \tag{4.15}
\]

Note also that a similar scaling argument applied to Poincaré’s inequality gives the estimate
\[
\| g - \overline{g_{B_r}} \|_{q;B_r} \lesssim r \| \nabla g \|_{q;B_r} \sim |B_r|^\frac{1}{q} \| \nabla g \|_{q;B_r} \tag{4.16}
\]
for any \( r > 0 \) and \( q \in [1, \infty) \), where \( \overline{g_{B_r}} \) is the average of \( g \) in \( O \) for any \( O \subset \mathbb{R}^3 \) with \( |O| < \infty \). Note finally that a simple application of Hölder’s inequality gives
\[
\| \overline{g_{O}} \|_{q;O} \leq \| g \|_{q;O} . \tag{4.17}
\]

Proceeding now with the proof, fix some \( \tilde{\phi} \in C_0^\infty(\mathbb{R}^3) \) such that
\[
\tilde{\phi} \equiv 1 \quad \text{in} \quad B_{r_2}(0) = B_{\frac{2}{3}}(0)
\]
and
\[
\operatorname{supp}(\tilde{\phi}) \subseteq B_{r_1}(0) = B_{\frac{1}{3}}(0).
\]
Now fix \( \bar{x} = (\bar{x}, \bar{t}) \in \mathbb{R}^3 \times \mathbb{R} \) and \( z_0 = (x_0, t_0) \in Q_+^4(\bar{x}) \), define \( B^k, I^k \) and \( Q^k \) by (3.7) for this \( z_0 \) and define \( \phi \) by \( \phi(x) := \phi(x - x_0) \). So
\[
\phi \equiv 1 \quad \text{in} \quad B^2 = B^2_+(x_0)
\]
and
\[
\text{supp}(\phi) \subseteq B^1 = B^1_+(x_0) \subset B^*_{1}(\bar{x}),
\]
since \( x_0 \in B^2_+(\bar{x}) \). The following estimates will clearly depend only on \( \tilde{\phi} \), i.e. constants will be uniform for all \( z_0 \in Q_+^4(\bar{x}) \).

First, applying (4.13) to \( v \in \{u, \nabla d\} \) and recalling (3.8) we see that
\[
\frac{1}{r_n} (\|u\|_{3;Q^n}^3 + \|\nabla d\|_{3;Q^n}^3) \lesssim \iint_{Q^n} (|u|^3 + |\nabla d|^3) \, dz \lesssim L_{n}^{3/2}
\]
for any \( n \), with a constant independent of \( n \). In particular,
\[
\|u\|_{3;Q^n} + \|\nabla d\|_{3;Q^n} \lesssim \frac{r_n}{r_n} L_{n}^{1/2}
\]
for any \( n \).

Next, by Claim 11 and Remark 2 with \( \psi := \phi, \Omega_2 := B^1 \) and \( \Omega_1 := B^2 \), at almost every \((x,t) \in Q^2 = Q^4_+(z_0) = B^2_+(x_0) \times (t_0 - (\frac{1}{2})^2, t_0) \) (where \( p = \phi \rho \)), as in (4.1) we have
\[
p(x,t) = S[\phi J(t)](x) + \int_{B^1 \setminus B^2} (2\nabla G^x \otimes_{\sigma} \nabla \phi + G^x \nabla^2 \phi) : J(t) \, dy
\]
\[
+ \int_{B^1 \setminus B^2} (2\nabla G^x \cdot \nabla \phi + G^x \Delta \phi)p(t) \, dy,
\]
where
\[
J := u \otimes u + \nabla d \otimes \nabla d,
\]
\(2a \otimes b := a \otimes b + b \otimes a\) and the operator \( S\) consisting of second derivatives of the Newtonian potential given by
\[
S[\tilde{K}](x) := \nabla_x : \left( \nabla_T \cdot \int_{B^2} G^x \tilde{K} \right)
\]
for \( \tilde{K} \in L^q(B^1) \) is a bounded linear Calderon-Zygmund operator on \( L^q(B^1) \) for \( 1 \leq q \leq \infty \). Hence for any \( n \in \mathbb{N} \), denoting by \( \chi_n \) the indicator function for the set \( B^n = B_{2^{-n}}(x_0) \) and splitting \( \phi = \chi_n \phi + (1 - \chi_n)\phi \) in the first term of (4.20), we can write
\[
p = p^{1,n} + p^{2,n} + p^{3,n} \equiv p^{1,n} + p^{2,n} + p^{3,n},
\]
where, for almost every \((x,t) \in Q^2,\)
\[
p(x,t) = S[\chi_n \phi J(t)](x) + S[(1 - \chi_n) \phi J(t)](x)
\]
\[
+ \int_{B^1 \setminus B^2} (2\nabla G^x \otimes_{\sigma} \nabla \phi + G^x \nabla^2 \phi) : J(t) \, dy + \int_{B^1 \setminus B^2} (2\nabla G^x \cdot \nabla \phi + G^x \Delta \phi)p(t) \, dy
\]
(\( =: p^{3,n}(x,t) \equiv p^3(x,t) \))
Note first that, by the classical Calderon-Zygmund estimates, there is a universal constant $C_{cz} > 0$ such that, for all $n \in \mathbb{N}$, we have
\[
\|p^1,n(t)\|_{\bar{\mathcal{B}}_{n+1}} \leq C_{cz}\|\chi_n \phi J(t)\|_{\frac{3}{2} \mathbb{R}^3} \leq C_{cz}\|\tilde{\phi}\|_{\infty; \mathbb{R}^3}\|J(t)\|_{\frac{3}{2} \mathcal{B}_n} .
\] (4.22)

Next, since the appearance of $\nabla \phi$ in $p^3$ exactly cuts off a neighborhood of the singularity of $G^x$ (see (1.1)) uniformly for all $x \in B_{R_0}^n(x_0)$ (as we integrate over $|x_0 - y| \geq \frac{1}{4}$, hence $|x - y| \geq \frac{1}{4}$), we see that
\[
p^{3,n}(\cdot, t) \in C^\infty(B_{R_0}^n(x_0)) \text{ for } t \in I^n_{R_0}(t_0) \text{ with, in particular,}
\]
\[
\|\nabla_x p^{3,n}(t)\|_{\infty; B_{n+1}^2(x_0)} \leq c(\tilde{\phi}) \left(\|J(t)\|_{1; B^1} + \|p(t)\|_{1; B^1}\right) .
\] (4.23)

In the term $p^{2,n}$, the singularity coming from $G^x$ is also isolated due to the appearance of $\chi_n$, but it is no longer uniform in $n$ so we must be more careful. As we are integrating over a region which avoids a neighborhood of the singularity at $y = x$ of $G^x$, we can pass the derivatives in $S$ under the integral sign to write
\[
\nabla_x p^{2,n}(x, t) = \int_{B_k^2 \setminus B_{k+1}} \nabla_x \left[\nabla_y \left(\frac{G^x}{2}\right)^T \phi J(t)\right] dy = \sum_{k=1}^{n-1} \int_{B_k^2 \setminus B_{k+1}} \nabla_x \left[\nabla_y \left(\frac{G^x}{2}\right)^T \phi J(t)\right] dy
\]
and note, in view of (4.1) that
\[
\left|\nabla_y \left(\frac{G^x}{2}\right)(y)\right| \leq \frac{1}{|x-y|^4} \leq \left(\frac{2^{k+2}}{|B_k|}\right)^4 \leq \frac{2^k}{|B_k|} \quad \forall x \in B^{k+2}, \ y \in (B^{k+1})^c.
\]

Therefore, since
\[
B_{n+1}^k = B^{(n-1)+2}_k \subseteq B^{k+2}_k \quad \text{for } 1 \leq k \leq n-1,
\]
we see that
\[
\|\nabla_x p^{2,n}(\cdot, t)\|_{\infty; B_{n+1}^2} \leq c(\tilde{\phi}) \sum_{k=1}^{n-1} 2^k \int_{B_k^2} |J(y, t)| \ dy
\] (4.24)
for all $t \in I^n_{R_0}(t_0)$.

Now, recalling the notation
\[
\tilde{f}_k(t) := \int_{B_k^2} f(x, t) \ dx
\]
for a function $f(x, t)$ and $k \in \mathbb{N}$, for any $t \in I^2 = (t_0 - \frac{1}{2}, t_0)$ and $n \geq 2$, we estimate
\[
\int_{B_{n+1}} |u(x, t)||p(x, t) - \tilde{p}_{n+1}(t)| \ dx \leq \sum_{j=1}^3 \int_{B_{n+1}} |u(x, t)||p^{j,n}(x, t) - \tilde{p}^{j,n}_{n+1}(t)| \ dx \leq \|u(\cdot, t)||_{3; B_{n+1}} \sum_{j=1}^3 \|p^{j,n}(\cdot, t) - \tilde{p}^{j,n}_{n+1}(t)\|_{\frac{3}{2} B_{n+1}}
\]
\[
\overset{\text{Hölder}}{\leq} \|u(t)||_{3; B_{n+1}} \left(\|p^{1,n}(t)\|_{\frac{3}{2} B_{n+1}} + |B_{n+1}| \sum_{j=2}^3 \|\nabla p^{j,n}(t)\|_{\infty; B_{n+1}}\right)
\]
\[
\overset{\text{Hölder}}{\leq} \|u(t)||_{3; B_{n+1}} \left(\|J(t)\|_{\frac{3}{2} B_{n+1}} + r_{n+1} \left\{\left(\sum_{k=1}^{n-1} 2^k \int_{B_k^2} \|J(t)| dy\right) + \|J(t)\|_{1; B^1} + \|p(t)\|_{1; B^1}\right\}\right).
\]
Note further that, setting
\[ \mathbb{L}_{J,k} := \left\| \int_{B^k} |J(t)| \, dy \right\|_{L^\infty(I^k)}, \] (4.26)
we have
\[ \left\| \sum_{k=1}^{n-1} 2^k \int_{B^k} |J(t)| \, dy \right\|_{L^2(I^{n+1})} \leq |I^{n+1}|^{\frac{1}{2}} \left( \max_{1 \leq k \leq n-1} \mathbb{L}_{J,k} \right) \sum_{k=1}^{n-1} 2^k \leq r_{n+1}^{\frac{n}{2}} \max_{1 \leq k \leq n-1} \mathbb{L}_{J,k}, \]
since \( |I^{n+1}| = r_{n+1}^2 \)
and
\[ \sum_{k=1}^{n-1} 2^k = \frac{2^n - 2}{2 - 1} < 2^n = r_{n+1}^2. \]
Integrating over \( t \in I^{n+1} \) in (4.25), applying Hölder in the variable \( t \) and recalling by (4.19) that
\[ \|u\|_{3;Q^k} \lesssim r_{n+1}^{\frac{1}{2}} L_{n+1}, \]
we obtain
\[ \left\| \int_{Q^{n+1}} |u||p - \bar{p}_{n+1}| \, dz \right\| \lesssim \] (4.27)
\[ \lesssim r_{n+1}^{\frac{3}{2}} L_{n+1} \left\{ \|J\|_{\frac{3}{2},Q^k} + r_{n+1}^{\frac{10}{3}} \max_{1 \leq k \leq n-1} \mathbb{L}_{J,k} + r_{n+1}^{3} \left( \|J\|_{\frac{3}{2},Q^1} + \|p\|_{\frac{3}{2},Q^1} \right) \right\}. \]
It follows now from (4.21) that
\[ \|J\|_{\frac{3}{2},Q^k} \leq \|u\|_{3;Q^k}^{\frac{1}{2}} + \|\nabla d\|_{3;Q^k}^{\frac{1}{2}} \lesssim r_{k}^{\frac{10}{3}} L_{k} \] (4.28)
and
\[ \mathbb{L}_{J,k} \leq \left\| \int_{B^k} (|u(\cdot)|^2 + |\nabla d(\cdot)|^2) \, dy \right\|_{L^\infty(I^k)} \leq L_{k}. \] (4.29)
Now from (4.21), (4.27), (4.28), (4.29) and the simple fact that \( \frac{1}{2} r_{n} = r_{n+1} \leq 1 \) we obtain
\[ r_{n+1}^{\frac{3}{2}} \int_{Q^{n+1}} |u||p - \bar{p}_{n+1}| \, dz \lesssim L_{n+1}^{\frac{1}{2}} \left\{ r_{n+1}^{\frac{1}{2}} L_{n} + r_{n+1}^{\frac{1}{2}} \max_{1 \leq k \leq n-1} L_{k} + r_{n+1}^{\frac{1}{2}} L_{1} + \|p\|_{\frac{3}{2},Q^1} \right\}, \]
\[ \lesssim L_{n+1}^{\frac{1}{2}} \left\{ \max_{1 \leq k \leq n} L_{k} + \|p\|_{\frac{3}{2},Q^1} \right\}. \]
Since
\[ \int_{Q^{n+1}} (|u|^3 + |\nabla d|^3) \, dz \lesssim L_{n+1}^{\frac{3}{2}}, \]
adding the previous estimates and recalling (4.28) and (4.29) we have
\[ R_{n+1} \lesssim L_{n+1}^{\frac{3}{2}} + L_{n+1}^{\frac{1}{2}} \left( \max_{1 \leq k \leq n} L_{k} + \|p\|_{\frac{3}{2},Q^1} \right) \]
(where the constant is universal). This along with (3.13) easily implies (3.10) and proves Proposition \( \Box \)
4.3 Proof of Proposition \[2\]

For simplicity, take $\hat{x} = 0 = (0, 0)$, so that (recall 3.7) $Q^k = Q^k(0, 0)$, etc., as the rest can be obtained by appropriate shifts.

We want to take the test function $\phi$ in 3.5 such that $\hat{\phi} = \phi^2 := \chi^2$, where (recall that here $Q^1 = Q^1(0, 0) = B_{\frac{1}{2}}(0) \times (-\frac{1}{4}, 0)$ so $\chi$ will be zero in a neighborhood of the “parabolic boundary” of $Q^1$)

$$\chi \in C_0^\infty \left( B_{\frac{1}{2}}(0) \times (-\frac{1}{4}, \infty) \right), \quad \chi \equiv 1 \text{ in } Q^2, \quad 0 \leq \chi \leq 1 \quad (4.30)$$

and

$$\psi^n(x, t) := \frac{1}{(r_n^2 - t)^{3/2}} e^{-\frac{|x|^2}{4(r_n^2 - t)}} \text{ for } t \leq 0. \quad (4.31)$$

Note that the singularity of $\psi^n$ would naturally be at $(x, t) = (0, r_n^2) \notin Q^1$, so $\psi^n \in C_0^\infty(Q^1)$ and we may extend $\psi^n$ smoothly to $t > 0$ (where it’s values will actually be irrelevant) for each $n$ so that, in particular, $\phi^n \in C_0^\infty(B(1) \times (-1, \infty))$ as required in 3.7 (with $(\bar{x}, \bar{t}) = (0, 0)$). Furthermore, we have

$$\nabla \psi^n(x, t) = -\frac{x}{2(r_n^2 - t)} \psi^n(x, t) \quad \text{and} \quad \psi^n_t + \Delta \psi^n \equiv 0 \text{ in } Q^1. \quad (4.32)$$

Note first that for $(x, t) \in Q^n (n \geq 2)$, we have

$$0 \leq |x| \leq r_n \quad \text{and} \quad r_n^2 \leq [r_n^2 - t] \leq 2r_n^2$$

so that

$$r_n^3 = (r_n^2)^{\frac{3}{2}} e^{\frac{|x|^2}{r_n^2}} \leq (r^2 - t)^{3/2} e^{\frac{|x|^2}{r_n^2 - t}} \leq (2r_n^2)^{\frac{3}{2}} e^{\frac{r_n^2}{r_n^2}} = 2\frac{3}{4}\pi^3 r_n^3.$$

Hence

$$\frac{1}{2\pi^{\frac{3}{2}}} \cdot \frac{1}{r_n^3} \leq \psi^n(x, t) \leq \frac{1}{r_n^3} \quad \forall (x, t) \in Q^n \quad (4.33)$$

and therefore (as $r_n^2 - t > 0$)

$$|\nabla_x \psi^n(x, t)| = \frac{|x|}{2(r_n^2 - t)} |\psi^n(x, t)| \lesssim \frac{r_n}{r_n^2} \cdot \frac{1}{r_n^3} = \frac{1}{r_n^4} \quad \forall (x, t) \in Q^n. \quad (4.34)$$

Next, note similarly that for $2 \leq k \leq n$ and $(x, t) \in Q^{k-1} \setminus Q^k$, we have

$$r_k \leq |x| \leq r_{k-1} = 2r_k$$

and

$$r_k^2 \leq r_n^2 - t \leq r_k^2 \leq 2r_{k-1}^2 = 2r_k^2,$$

so that

$$e^{\frac{3}{2}} r_k^3 = (r_k^2)^{\frac{3}{2}} e^{\frac{|x|^2}{r_k^2}} \leq (r^2 - t)^{3/2} e^{\frac{|x|^2}{r_k^2 - t}} \leq (2r_k^2)^{\frac{3}{2}} e^{\frac{r_k^2}{2}} = 2\frac{3}{4}\pi^3 e^{r_k^3}.$$

Therefore

$$\frac{1}{2\pi^{\frac{3}{2}}} \cdot \frac{1}{r_k^3} \leq \psi^n(x, t) \leq \frac{1}{e^{\frac{3}{2}}} \cdot \frac{1}{r_k^3} \quad \forall (x, t) \in Q^{k-1} \setminus Q^k \quad (2 \leq k \leq n) \quad (4.35)$$

and hence, as in 4.3.4,24

$$|\nabla_x \psi^n(x, t)| \lesssim \frac{r_k}{r_k^3} \cdot \frac{1}{r_k^3} = \frac{1}{r_k} \quad \forall (x, t) \in Q^{k-1} \setminus Q^k \quad (2 \leq k \leq n). \quad (4.36)$$

\[24\] In 3.5 as well, the values of $\phi$ for $t > \bar{t}$ are actually irrelevant.
We can therefore estimate (for \( n \geq 2 \) where \( \phi^n = \psi^n \) in \( Q^n \)):

\[
\frac{1}{2\pi r_n^2} \cdot \frac{1}{r_n^3} \left[ \text{ess sup}_{I_n} \int_{B^n} (|u|^2 + |\nabla d|^2) + \int_{Q^n} (|\nabla u|^2 + |\nabla^2 d|^2) \right]
\leq \text{ess sup}_{I_n} \int_{B^n} (|u|^2 + |\nabla d|^2) \phi^n + \int_{Q^n} (|\nabla u|^2 + |\nabla^2 d|^2) \phi^n
\leq C \left\{ \int_{Q^1} [(|u|^2 + |\nabla d|^2) \phi_n + \Delta \phi^n] + (|u|^3 + |\nabla d|^3) |\nabla \phi^n| + \rho|d|^2 |\nabla^2 \phi^n| + \int_{B^1} |p u \cdot \nabla \phi^n| \right\}.
\]

Note that

\[
\phi^n + \Delta \phi^n = \psi^n (\chi + \Delta \chi) + 2 \nabla \chi \cdot \nabla \psi^n \equiv 0 \text{ in } Q^2
\]

and hence, taking \( k = 2 \) in (4.35) and (4.36), we see that

\[
|\phi^n + \Delta \phi^n| \lesssim \frac{1}{r_n^2} + \frac{1}{r_2} \lesssim 1 \text{ in } Q^1,
\]

so that

\[
\int_{Q^1} (|u|^2 + |\nabla d|^2) |\phi^n + \Delta \phi^n| \lesssim \int_{Q^1} (|u|^2 + |\nabla d|^2) \lesssim E^{2/3}_{k,q}
\]

by Hölder’s inequality. Note similarly that

\[
|\nabla \phi^n| = |\chi \nabla \psi^n + \psi^n \nabla \chi| \lesssim |\nabla \psi^n| + |\psi^n| \text{ in } Q^1
\]

so that (since \( r_n^4 < r_n^2 \)) (4.33), (4.34) and (4.35), (4.36), respectively, give

\[
|\nabla \phi^n| \lesssim \frac{1}{r_n^2} \text{ in } Q^n, \quad |\nabla \phi^n| \lesssim \frac{1}{r_k^2} \text{ in } Q^{k-1}\setminus Q^k
\]

for any \( n \geq 2 \) and \( 2 \leq k \leq n \). Therefore

\[
\sum_{k=2}^{n} \int_{Q^{k-1}\setminus Q^k} (|u|^3 + |\nabla d|^3) |\nabla \phi^n| \lesssim \max_{1 \leq k \leq n-1} (r_k)^{1-\alpha} \int_{Q^k} (|u|^3 + |\nabla d|^3) \sum_{k=2}^{n} (r_k)^\alpha
\]

and similarly

\[
\int_{Q^n} (|u|^3 + |\nabla d|^3) |\nabla \phi^n| \lesssim (r_n)^{1-\alpha} \int_{Q^n} (|u|^3 + |\nabla d|^3) (r_n)^\alpha
\]

for any \( \alpha \in (0,1] \), and we note that

\[
\sum_{k=1}^{\infty} (r_k)^\alpha = \sum_{k=1}^{\infty} (2^{-\alpha})^k = \frac{1}{2^\alpha - 1} < \infty \quad \forall \quad \alpha > 0.
\]

Hence in view of the disjoint union

\[
Q^1 = \left( \bigcup_{k=2}^{n} Q^{k-1}\setminus Q^k \right) \cup Q^n
\]

(4.40)
we have (taking $\alpha = 1$ in (4.39))
\[
\iint_{Q^n} (|u|^3 + |
abla^2 d|) |\nabla \phi_n| \lesssim \max_{1 \leq k \leq n} \iint_{Q^k} (|u|^3 + |
abla d|^3) .
\]
Similarly, setting
\[\alpha_q := \frac{2(q - 5)}{q - 2}\]
(note $\alpha_q \in (0, \frac{1}{2})$ for $q \in (5, 6]$), we have
\[
\overline{\rho} \iint_{Q^n} |d|^2 |\nabla^2 \phi|^3 \lesssim \frac{2}{3} \iint_{Q^n} |d|^q |\nabla^2 \phi|^{3(1 - \frac{2}{q})} + (1 - \frac{2}{q}) \iint_{Q^n} |\nabla^2 \phi|^3 (\phi_n)^{\frac{3}{2} - (5 - \alpha_q)}
\]
uniformly, of course, over $\overline{\rho} \in (0, 1]$. Since
\[
\iint_{Q^n} |\nabla^2 \phi|^3 (\phi_n)^{\frac{3}{2} - (5 - \alpha_q)} \lesssim (\phi_n)^{\alpha_q - 5} \iint_{Q^n} |\nabla^2 \phi|^3 \lesssim (\phi_n)^{\alpha_q} \iint_{Q^n} |\nabla^2 \phi|^3
\]
for $n \geq 2$ and similarly
\[
\iint_{Q^{k+1} \setminus Q^k} |\nabla^2 \phi|^3 (\phi_n)^{\frac{3}{2} - (5 - \alpha_q)} \lesssim (\phi_n)^{\alpha_q - 5} \iint_{Q^k} |\nabla^2 \phi|^3 \lesssim (\phi_n)^{\alpha_q} \iint_{Q^k} |\nabla^2 \phi|^3
\]
for $1 \leq k \leq n - 1$, we see that (4.39) with $\alpha = \alpha_q$ and (4.40) again give
\[
\iint_{Q^n} |\nabla^2 \phi|^3 (\phi_n)^{\frac{3}{2} - (5 - \alpha_q)} \lesssim (2^{\alpha_q} - 1)^{-1} \max_{1 \leq k \leq n} \iint_{Q^k} |\nabla^2 \phi|^3 .
\]
We therefore see that
\[
\overline{\rho} \iint_{Q^n} |d|^2 |\nabla^2 \phi|^3 \lesssim \frac{2}{3} E_{3,q} + \frac{2}{3} (2^{\alpha_q} - 1)^{-1} \max_{1 \leq k \leq n} \iint_{Q^k} |\nabla^2 \phi|^3 \quad \text{with } \alpha_q := \frac{2(q - 5)}{q - 2},
\]
uniformly for any $\overline{\rho} \in (0, 1]$ and $q \in (5, 6]$.

Putting all of the above together and recalling (3.8), we see that for $n \geq 2$ we have
\[
\frac{L_n}{C} = \frac{1}{C} \left[ \text{ess sup} \frac{1}{I^n} \int_{B^n} (|u|^2 + |\nabla d|^2) + \int_{I^n} \int_{B^n} (|\nabla u|^2 + |\nabla^2 d|^2) \right] \lesssim E_{3,q} + E_{3,q}^{2/3} + (2^{\alpha_q} - 1)^{-1} \max_{1 \leq k \leq n} \iint_{Q^k} (|u|^3 + |\nabla^2 \phi|^3) + \int_{I^n} \int_{\partial B^n} |p - \overline{\rho} \cdot \nabla \phi| .
\]
Furthermore we claim that for $1 \leq k_0 \leq n - 1$ we have
\[
\int_{I^n} \int_{\partial B^n} |p - \overline{\rho} \cdot \nabla \phi| \lesssim \max_{k_0 \leq k \leq n} \left( \frac{1}{r_k} \iint_{Q^k} |p - \overline{\rho}_k||u| \right) + k_0 2^{k_0} \iint_{Q^n} |p||u| .
\]
Assuming this for the moment and continuing, for $n \geq 2$, (4.41), (4.42) and Young's convexity inequality along with the fact that, for any $k_1 \geq 1$, we can estimate
\[
\max_{1 \leq k \leq k_1} \iint_{Q^k} (|u|^3 + |\nabla^2 d|^3) \lesssim k_1 2^{5k_1} \iint_{Q^n} (|u|^3 + |\nabla d|^3)
\]
imply (recalling (3.9)) that
\[
\frac{L_n}{C} \lesssim E_{3,q} + E_{3,q}^{2/3} + (2^{\alpha q} - 1)^{-1} \max_{k_0 \leq k \leq n} R_k + k_0 2^{5k_0} \int_{Q^1} |u|^3 + |\nabla d|^3 + |p|^{3/2} \leq E_{3,q}.
\]
for any \(k_0 \in \{1, \ldots, n - 1\}\), and hence Proposition 2 is proved.

To prove (4.42), we consider additional functions \(\chi_k\) (so that \(\chi_k \phi^n = \chi_k \chi \psi^n\)) satisfying (recall that here \(Q^k = Q^k(0, 0) = B_r(0) \times (-r_k^2, 0)\), so \(\chi_k\) will be zero in a neighborhood of the “parabolic boundary” of \(Q^k\))

\[
\chi_k \in C_0^\infty(\widetilde{Q}_{r_k}) \quad \text{with} \quad \widetilde{Q}_r := B_r(0) \times (-r^2, r^2) \quad \text{for} \ r > 0, \tag{4.43}
\]

\[
\chi_k \equiv 1 \quad \text{in} \quad \widetilde{Q}_r \quad \text{for} \ r > 0, \quad 0 \leq \chi_k \leq 1 \quad \text{and} \quad |\nabla \chi_k| \lesssim \frac{1}{r_k},
\]

\((\chi_k)_{t>0}\) will again actually be irrelevant) so that in particular (as \(\widetilde{Q}_{r_{k+2}} \subset \widetilde{Q}_{r_{k+1}}\) where \(\chi_k \equiv \chi_{k+1} \equiv 1\))

\[
\supp(\chi_k - \chi_{k+1}) \subset \widetilde{Q}_{r_k} \setminus \widetilde{Q}_{r_{k+2}}. \tag{4.44}
\]

Then since \(Q^1 = Q_{1/2}(0, 0) \subset Q_{2r}(0, 0) = Q_{3r}(0, 0), \) we have \(\chi_0 = 1\) on \(Q^1\) and hence for any \(n \geq 2, \)

writing

\[
\chi_0 = \chi_n + \sum_{k=0}^{n-1} (\chi_k - \chi_{k+1}),
\]

for any fixed \(k_0 \in \mathbb{N} \cap \{1, n - 1\}\) and at each fixed \(\tau \in I^1\) we have

\[
\int_{B_1} p u \cdot \nabla \phi^n = \int_{B_1} p u \cdot \nabla [\chi_0 \phi^n] \tag{4.43}
\]

\[
\quad = \int_{B^1} p u \cdot \nabla [\chi_n \phi^n] + \sum_{k=0}^{n-1} \int_{B^1} p u \cdot \nabla [(\chi_k - \chi_{k+1}) \phi^n] \tag{4.43, 4.44}
\]

\[
\quad = \int_{B^n} p u \cdot \nabla [\chi_n \phi^n] + \sum_{k=0}^{n-1} \int_{[B^k \setminus B^{k+2}]} p u \cdot \nabla [(\chi_k - \chi_{k+1}) \phi^n] \tag{3.3}
\]

\[
\quad = \int_{B^n} (p - \bar{p}_n) u \cdot \nabla [\chi_n \phi^n] + \sum_{k=0}^{k_0-1} \int_{[B^k \setminus B^{k+2}]} p u \cdot \nabla [(\chi_k - \chi_{k+1}) \phi^n]
\]

\[
\quad + \sum_{k=k_0}^{n-1} \int_{[B^k \setminus B^{k+2}]} (p - \bar{p}_k) u \cdot \nabla [(\chi_k - \chi_{k+1}) \phi^n], \tag{4.45}
\]

where

\[
\bar{p}_k = \bar{p}_k(\tau) = \int_{B^k} p(x, \tau) \ dx.
\]

Note first that (4.35), (4.36) and (4.44) imply (since \(r_{j+1} = 2r_j\) for any \(j\)) that

\[
|\nabla [(\chi_k - \chi_{k+1}) \phi^n]| \leq |\chi_k - \chi_{k+1}| |\nabla \phi^n| + |\phi^n||\nabla (\chi_k - \chi_{k+1})| \lesssim r_{k-4}^{-1}
\]

on \(Q^k \setminus Q^{k+2} = (Q^k \setminus Q^{k+1}) \cup (Q^{k+1} \setminus Q^{k+2})\)

for any \(k\), and similarly

\[
|\nabla [\chi_n \phi^n]| \leq |\chi_n| |\nabla \phi^n| + |\phi^n||\nabla \chi_n| \lesssim r_n^{-4}
\]

on \(Q^n\).
Therefore we can estimate (recalling again (4.33) and (4.44) when integrating (4.43) over \( \tau \in I^1 \))

\[
\int_{\tau \in I^1} \left( \int_{B^1(\tau)} pu \cdot \nabla \phi^n \right) \lesssim k_0 \omega_{4k_0} \left( \int_{Q^1} |p||u| + \sum_{k=k_0}^{n} r_k \int_{Q^k} |p-p_k||u| \right)
\]

which, along with (4.39) with \( q = \frac{3}{2} \) implies (4.42) for any \( k_0 \in [1, n-1] \) as desired. \( \square \)

### 4.4 Proof of Proposition 3

In this section we prove the technical decay estimate (Proposition 3) used to prove Lemma 2. In all of what follows, recall the definitions in (3.17) and (3.18) of \( A_{z_0}, B_{z_0}, C_{z_0}, D_{z_0}, E_{z_0}, F_{z_0}, G_{q,z_0} \) and \( M_{q,z_0} \). We will require the following three claims which essentially appear in [LL96] and which generalize certain lemmas in [CKN82]; however we include full proofs in order to clarify certain details, and to highlight the role of \( G_{q,z_0} \) (not utilized in [LL96]) in Claim 4 which is therefore a slightly refined version of what appears in [LL96].

**Claim 2** (General estimates (cf. Lemmas 5.1 and 5.2 in [CKN82])). There exist constants \( c_1, c_2 > 0 \) such that for any \( u \) and \( d \) which have the regularities in (3.4) for \( \Omega_T := \Omega \times (0,T) \) as in Theorem 7 the estimates

\[
C_{z_0}(\gamma \rho) \leq c_1 \left[ \gamma^3 A_{z_0} + \gamma^{-3} A_{z_0} B_{z_0} \right] (\rho)
\]

and

\[
E_{z_0}(\gamma \rho) \leq c_2 \left[ C_{z_0}^\frac{1}{2} A_{z_0}^\frac{1}{2} B_{z_0} \right] (\rho)
\]

hold for any \( z_0 \in \mathbb{R}^{3+1} \) and \( \rho > 0 \) such that \( Q_{\rho}(z_0) \subseteq \Omega_T \) and any \( \gamma \in (0,1] \).

**Claim 3** (Estimates requiring the pressure equation (cf. Lemmas 5.3 and 5.4 in [CKN82])). There exist constants \( c_3, c_4 > 0 \) such that for any \( u \), \( d \) and \( p \) which have the regularities in (3.9) and (3.10) for \( \Omega_T := \Omega \times (0,T) \) as in Theorem 7 and which satisfy the pressure equation (3.12), the estimates

\[
D_{z_0}(\gamma \rho) \leq c_3 \left[ \gamma (D_{z_0} + A_{z_0}^\frac{3}{2} B_{z_0} + C_{z_0}^\frac{1}{2}) + \gamma^{-5} A_{z_0}^\frac{3}{2} B_{z_0} \right] (\rho)
\]

and

\[
F_{z_0}(\gamma \rho) \leq c_4 \left[ \gamma (D_{z_0} + A_{z_0}^\frac{3}{2} B_{z_0} + C_{z_0}^\frac{1}{2}) + \gamma^{-10} A_{z_0}^\frac{3}{2} (B_{z_0}^2 + B_{z_0}^2) \right] (\rho)
\]

hold for any \( z_0 \in \mathbb{R}^{3+1} \) and \( \rho > 0 \) such that \( Q_{\rho}(z_0) \subseteq \Omega_T \) and any \( \gamma \in (0,\frac{1}{2}] \).

The crucial aspect of the estimates (4.46), (4.47), (4.48) and (4.49) (which control \( M_{q,z_0}(\gamma \rho) \)) in proving Lemma 2 (through Proposition 3) is that whenever a negative power of \( \gamma \) appears, there is always a factor of \( B_{z_0} \), as well, which will be small when proving Lemma 2. Positive powers of \( \gamma \) will similarly be small; in each term evaluated at \( \rho \) (see also (4.52) below), we must have either \( \gamma^\alpha \) or \( B_{z_0}^\alpha \) for some \( \alpha > 0 \).

To complete the proof of Proposition 3 we require the following:

**Claim 4** (Estimate requiring the local energy inequality (cf. Lemma 5.5 in [CKN82])). There exists a constant \( c_5 > 0 \) such that for any \( u \), \( d \) and \( p \) which have the regularities in (3.9) and (3.10) for \( \Omega_T := \Omega \times (0,T) \) as in Theorem 7 and such that \( u \) satisfies the weak divergence-free property (3.14) and the local energy inequality (3.13) holds for some constant \( C \in (0,\infty) \), the estimate

\[
A_{z_0}(\frac{3}{2}) \leq c_5 \cdot \bar{C} \left[ \bar{C} + (1 + (\cdot)^2) G_{z_0}^\frac{1}{2} + (G_{z_0}^\frac{1}{2} + C \bar{C}) B_{z_0} \right] (\rho)
\]

holds for any \( q \in (2,6) \) and any \( z_0 \in \mathbb{R}^{3+1} \) and \( \rho > 0 \) such that \( Q_{\rho}(z_0) \subseteq \Omega_T \).

\( \text{Note that } G_{z_0}(r) \leq ||d||_\infty \text{ uniformly in } r \text{ (and } z_0) \text{, though in our setting we may have } d \notin L^\infty. \)
Postponing the proof of the claims, let us use them to prove the proposition.

In all of what follows, we note the simple facts that, for any \( \rho > 0 \) and \( \alpha \in (0, 1] \),

\[
K \in \{A_{z_0}, B_{z_0}\} \implies K(\alpha \rho) \leq \alpha^{-1} K(\rho) ,
\]

\[
K \in \{C_{z_0}, D_{z_0}, E_{z_0}, F_{z_0}\} \implies K(\alpha \rho) \leq \alpha^{-2} K(\rho) \tag{4.51}
\]

and \( G_{q, z_0}(\alpha \rho) \leq \alpha^{-2 - \frac{2}{q}} G_{q, z_0}(\rho) \).

**Proof of Proposition** Fixing \( z_0 \) and \( \rho_0 \) as in Proposition 3, under the assumptions in the proposition we see that estimates (4.46), (4.47), (4.48), (4.49) and (4.50) hold for all \( \rho \in (0, \rho_0] \), \( \gamma \in (0, \frac{1}{2}] \) and \( q \in [2, 6] \) by Claims 2, 3 and 4.

Note first that (4.46), (4.47) and (4.51) imply that

\[
E_{z_0}(\gamma \rho) \lesssim [A_{z_0} B_{z_0} + \gamma^{-2} A_{z_0} B_{z_0}^3] (\rho)
\]

and hence, for example, there exists some \( c_6 > 0 \) such that

\[
E_{z_0}(\gamma \rho) \leq c_6 \left[ \gamma^2 A_{z_0} + \gamma^{-2} \left( A_{z_0}^2 B_{z_0} + A_{z_0} B_{z_0} \right) \right] (\rho), \tag{4.52}
\]

for \( \rho \in (0, \rho_0] \) and \( \gamma \in (0, \frac{1}{2}] \) (in fact, for \( \gamma \in (0, 1) \)) and that it follows from (4.50), the assumption (3.20) and the assumption that \( \rho_0 \leq 1 \) that there exists \( c_7 > 0 \) such that

\[
(C^{-1}) A_{z_0}(\gamma \rho/2) \leq c_7 \left[ C_{z_0} + E_{z_0} + F_{z_0} + G_{q, z_0} \left( \gamma^2 A_{z_0}^2 + \gamma^{-3} A_{z_0}^4 B_{z_0}^3 \right) \right] (\rho), \tag{4.53}
\]

and hence, recalling (3.18), we have that, for some \( c_8 > 0 \),

\[
(C^{-1})^\frac{3}{2} A_{z_0}(\rho/2) \leq c_8 \left[ M_{q, z_0}(\rho) + M_{q, z_0}^2(\rho) B_{z_0}^3(\rho) \right] \tag{4.54}
\]

for \( \rho \in (0, \rho_0] \). We note as well that, as in (3.23), if \( \sigma \in \{q, 6\} \) and if \( (3.20) \) holds for some \( \bar{g} \geq 1 \), then

\[
C_{q, z_0}^\frac{q}{q-1} (\gamma \rho) \leq \bar{g}^{\frac{q}{q-1}} C_{q, z_0}^\alpha (\gamma \rho) \leq \bar{g}^{\frac{q}{q-1}} \left[ \gamma^3 A_{z_0}^2 + \gamma^{-3} A_{z_0}^4 B_{z_0}^3 \right] (\rho) \tag{4.55}
\]

for \( \rho \in (0, \rho_0] \). Now, writing \( \gamma \rho = 2 \gamma \cdot \frac{\rho}{2} \) for \( 2 \gamma \leq \frac{1}{2} \) it follows from (4.49), (4.48), (4.49), (4.52), (4.54) and (3.18) followed by an application of (4.51) (with \( \alpha = \frac{2}{q} \)) to all terms except for \( A_{z_0} \), along with the facts that \( \gamma, \rho \), \( z_0, B_{z_0} \) \( \leq 1 \) (so that you can always estimate positive powers by 1) as well as the fact that \( c_{\sigma, q} \in (0, 1) \) that

\[
M_{q, z_0}(\gamma \rho) \leq \left[ C_{z_0} + G_{q, z_0}^{\frac{q}{q-1}} + D_{z_0}^q + E_{z_0}^q + F_{z_0}^1 \right] (\gamma \rho) \leq \left[ \gamma^3 A_{z_0}^2 \left( \frac{\gamma^2}{6} \right) + \gamma^{-3} A_{z_0}^4 \left( \frac{\gamma^2}{6} \right) B_{z_0}^3 \right] (\rho) + \gamma^{\frac{q}{q-1}} \left[ \gamma^3 A_{z_0}^2 \left( \frac{\gamma^2}{6} \right) + \gamma^{-3} A_{z_0}^4 \left( \frac{\gamma^2}{6} \right) B_{z_0}^3 \right] \tag{4.56}
\]

and hence, if \( \gamma \rho \geq 1 \),

\[
M_{q, z_0}(\gamma \rho) \leq \left[ \gamma^3 A_{z_0}^2 \left( \frac{\gamma^2}{6} \right) + \gamma^{-3} A_{z_0}^4 \left( \frac{\gamma^2}{6} \right) B_{z_0}^3 \right] (\rho) + \gamma^{\frac{q}{q-1}} \left[ \gamma^3 A_{z_0}^2 \left( \frac{\gamma^2}{6} \right) + \gamma^{-3} A_{z_0}^4 \left( \frac{\gamma^2}{6} \right) B_{z_0}^3 \right] \tag{4.57}
\]

for all \( \rho \geq 1 \).
so long as $\gamma \in (0, \frac{1}{3}]$. Noting that $1 \leq \frac{6 - \sigma}{2\sigma}$, the estimate (3.21) for such $\gamma$ and for $\rho \in (0, \rho_0]$ now follows from the estimate above along with (4.53) as, in particular, (4.53) implies (as $\gamma, B_{z_0}(\rho) \leq 1$ and $\alpha_{\sigma, q} \in (0, 1)$) that

$$(\bar{C})^{-\frac{3}{2}} A_{\rho_0}^\frac{3}{2} (\frac{\rho}{\rho_0}) \lesssim M_{q, z_0}(\rho) + \gamma^{-15 - \frac{2\sigma}{3\sigma}} M_{q, z_0}^\frac{1}{2}(\rho) B_{z_0}^{\frac{2\alpha_{\sigma, q}}{3\alpha_{\sigma, q}}} (\rho)$$

which we apply to the terms above with the positive power of $\gamma$, and that

$$(\bar{C})^{-\frac{3}{2}} A_{\rho_0}^\frac{3}{2} (\frac{\rho}{\rho_0}) \lesssim M_{q, z_0}(\rho) + \gamma^{-\frac{1}{2}} M_{q, z_0}^\frac{1}{2}(\rho),$$

which we apply to the terms above with the negative power of $\gamma$. This completes the proof of Proposition 3.

Let us now prove the claims:

**Proof of Claim 2.** For simplicity, we will suppress the dependence on $z_0 = (x_0, t_0)$ in what follows.

Let us first prove (4.46). Note that for any $r \leq \rho$, at any fixed $t \in I_r^*$, taking $v \in \{u, \nabla d\}$ we have

$$\int_{B_{r}} |v|^2 \, dx \leq \int_{B_{\rho}} \|v - \bar{v}\|^2 \, dx + |B_{r}| |\bar{v}|^2 \lesssim \rho \int_{B_{\rho}} |\nabla v|^2 \, dx + \frac{(r^3)}{(\rho)} \int_{B_{\rho}} |v|^2 \, dx$$

due to Poincaré’s inequality (4.16). Since $|\nabla v|^2 \leq |v||\nabla v|$ almost everywhere, Hölder’s inequality then implies that

$$\|v\|_{2, B_r}^2 \lesssim \rho \|v\|_{2, B_r} \|\nabla v\|_{2, B_r} + \left(\frac{r}{\rho}\right)^\frac{3}{2} \|v\|_{2, B_r}^2. \quad (4.55)$$

Therefore

$$\|v\|_{3, B_r}^3 \lesssim \frac{1}{r^\frac{2}{3}} \left(\|v\|_{2, B_r}^3 \right)^{\frac{2}{3}} + \|v\|_{2, B_r} \|\nabla v\|_{2, B_r}^\frac{2}{3} \lesssim \left(1 + \left(\frac{r}{\rho}\right)^\frac{2}{3}\right) \|v\|_{2, B_r} \|\nabla v\|_{2, B_r}^\frac{2}{3} + \frac{1}{r^\frac{2}{3}} \left(\frac{r}{\rho}\right)^\frac{2}{3} \|v\|_{2, B_r}^3.$$

Summing over $v \in \{u, \nabla d\}$, we see that

$$\|u\|_{3, B_r}^3 + \|\nabla d\|_{3, B_r}^3 \lesssim \left(1 + \left(\frac{r}{\rho}\right)^\frac{2}{3}\right) \left(\|u\|_{2, B_r}^3 + \|\nabla d\|_{2, B_r}^3\right) \left(\|\nabla u\|_{2, B_r}^2 + \|\nabla^2 d\|_{2, B_r}^2\right)^{\frac{2}{3}} + \frac{r^3}{\rho^{\frac{2}{3}}} \left(\|u\|_{2, B_r}^2 + \|\nabla d\|_{2, B_r}^2\right)^{\frac{2}{3}}.$$

Now integrating over $t \in I_r^*$ (where $|I_r^*| = r^2$), Hölder’s inequality implies that

$$r^2 C(r) \lesssim |I_r^*|^{\frac{3}{2}} \left(1 + \left(\frac{r}{\rho}\right)^\frac{2}{3}\right) \left(\|u\|_{2, B_r}^3 + \|\nabla d\|_{2, B_r}^3\right) \left(\|\nabla u\|_{2, Q_r^*} + \|\nabla^2 d\|_{2, Q_r^*}^2\right)^{\frac{2}{3}}$$

$$+ |I_r^*|^{\frac{3}{2}} \rho^{\frac{2}{3}} \left(\|u\|_{2, B_r}^2 + \|\nabla d\|_{2, B_r}^2\right) \lesssim r^\frac{2}{3} \left(1 + \left(\frac{r}{\rho}\right)^\frac{2}{3}\right) (\rho A(\rho))^{\frac{2}{3}} (\rho B(\rho))^{\frac{2}{3}} + \frac{r^3}{\rho^{\frac{2}{3}}} (\rho A(\rho))^{\frac{2}{3}},$$

which, upon dividing both sides by $r^2$, setting $\gamma := \frac{r}{\rho}$ and noting that $1 \leq \gamma^{-\frac{2}{3}}$, precisely gives (4.46).

Next, to prove (4.47), we use the Poincaré-Sobolev inequality

$$\|g - \bar{g}\|_{q, t; B_r} \leq c_q \|\nabla g\|_{q; B_r}$$
(the constant is independent of \( r \) due to the relationship between \( q \) and \( q^* \)) corresponding to the embedding \( W^{1,q} \to L_q^n \) for \( q < 3 \) (in \( \mathbb{R}^3 \)) and \( q^* = \frac{3q}{q-3} \). Taking \( q = 1 \), at any \( t \in I^*_r \) and for \( v \in \{ u, \nabla d \} \) the H"older and Poincaré-Sobolev inequalities give us

\[
\int_{B_r} |u| \left| \frac{|v|^2 - |v|^2}{|v|^2} \right| dx \leq \|u\|_{3,B_r} \| |v|^2 - |v|^2 \|_{\frac{3}{2};B_r} \lesssim \|u\|_{3,B_r} \| \nabla (|v|^2) \|_{1;B_r} \lesssim \|u\|_{3,B_r} \| v \|_{2;B_r} \| \nabla v \|_{2;B_r} .
\]

Summing this first over \( v \in \{ u, \nabla d \} \) at a fixed \( t \) and then integrating over \( t \in I^*_r \), we see that

\[
\begin{align*}
\int_{I^*_r} r^2 E(r) &\lesssim \int_{I^*_r} \|u\|_{3;B_r} \left( \|u\|_{2;B_r}^2 + \|\nabla d\|_{2;B_r}^2 \right)^{\frac{1}{2}} \left( \|\nabla d\|_{2;B_r}^2 + \|\nabla^2 d\|_{2;B_r}^2 \right)^{\frac{1}{2}} dt \\
&\lesssim \|u\|_{3;\Omega_1} \left( \|u\|_{2;B_r} + \|\nabla d\|_{2;B_r} \right) \left( \|\nabla u\|_{2;\Omega_1}^2 + \|\nabla^2 d\|_{2;\Omega_1}^2 \right)^{\frac{1}{2}} \\
&\lesssim |I^*_r| \frac{1}{2} \left( \|u\|_{3;\Omega_1}^2 + \|\nabla d\|_{2;B_r}^2 \right)^{\frac{1}{2}} \left( \|\nabla u\|_{2;\Omega_1}^2 + \|\nabla^2 d\|_{2;\Omega_1}^2 \right)^{\frac{1}{2}} \\
&\lesssim r^\frac{1}{2} (r^2 C(r) \frac{1}{2} (r A(r)) \frac{1}{2} (r B(r)) \frac{1}{2} = r^2 [C^* A^* B^*]^2(r)
\end{align*}
\]

which proves (4.47) and completes the proof of Claim 2.

\[
\square
\]

**Proof of Claim 3**

As in (4.3) of Claim 1 for any \( t \in I^*_r(z_0) (r \leq \rho) \) we use Remark 2 to decompose \( \Pi := p(\cdot, t) \) for almost every \( x \in B_{\frac{\rho}{4}}(x_0) \) using a smooth cut-off function \( \psi \) equal to one in \( \Omega_1 := B_{\frac{\rho}{4}}(x_0) \) and supported in \( \Omega_2 := B_{\rho}(x_0) \), so that

\[
|\nabla \psi| \lesssim \rho^{-1} \quad \text{and} \quad |\Delta \psi| \lesssim \rho^{-2}, \quad (4.56)
\]

as

\[
p(x,t) = -\int \nabla G^\psi \cdot v(t) \psi \, dy + \int G^\psi \cdot v(t) \, dy + \int G^\psi \cdot p(\cdot, t) \, dy
\]

with

\[
G^\psi_{\psi,1} := -G^\psi \nabla \psi, \quad G^\psi_{\psi,2} := 2 \nabla G^\psi \cdot \nabla \psi + G^\psi \Delta \psi
\]

and

\[
v(t) := |\nabla T \cdot (u \otimes u + \nabla d \otimes \nabla d)](\cdot, t) .
\]

Our goal is to estimate \( p(x,t) \) for \( x \in B_{\frac{\rho}{4}}(x_0) \).

Both \( p_2 \) and \( p_3 \) contain derivatives of \( \psi \) in each term so that the integrand can only be non-zero when \( |y-x_0| > \frac{3\rho}{4} \), and hence for \( x \in B_{\frac{\rho}{2}}(x_0) \) one has

\[
|x-y| \geq \frac{\rho}{4} \quad \Rightarrow \quad |G^\psi(y)| \lesssim \rho^{-1} \quad \text{and} \quad |\nabla G^\psi(y)| \lesssim \rho^{-2} . \quad (4.57)
\]

In view of (4.56) and (4.57) and the fact that \( \psi \) is supported in \( B_{\rho}(x_0) \), we have (omitting the dependence on \( t \), and noting that the constants in the inequalities are independent of \( t \) as they come only from \( G^\psi \) and \( \psi \))
Finally, integrating (4.60) over \( B_r(x_0) \) so that, integrating over \( B_r(x_0) \), we have

\[
\sup_{x \in B_{\frac{r}{2}}(x_0)} |p_2(x)| \lesssim \rho^{-2} \left( \int_{B_r(x_0)} (|u|^2 + |\nabla d|^2) \, dy \right)^{\frac{1}{2}} \left( \int_{B_r(x_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \, dy \right)^{\frac{1}{2}}
\]

and similarly

\[
\sup_{x \in B_{\frac{r}{4}}(x_0)} |p_4(x)| \lesssim \rho^{-3} \int_{B_r(x_0)} |p| \, dy.
\]

For \( p_1 \), Young’s inequality for convolutions (where we set \( R := 2\rho \) as in Remark 2 with \( 2/3 + 1 = 3/4 + 11/12 \) gives

\[
\|p_1\|_{2;B_r(x_0)} \lesssim \frac{1}{\rho} \left\| \int_{B_{2\rho}(0)} (|u| + |\nabla d|)(|\nabla u| + |\nabla^2 d|) \, dy \right\|_{2;B_r(x_0)}
\]

\[
\lesssim \rho^\frac{3}{2} \left\| (|u| + |\nabla d|)(|\nabla u| + |\nabla^2 d|) \right\|_{2;B_r(x_0)}
\]

and then Hölder’s inequality with \( 11/12 = 1/4 + 1/6 + 1/2 \) gives

\[
\|p_1\|_{3;B_r(x_0)} \lesssim \left( \rho^\frac{1}{2} \left\| (|u| + |\nabla d|)^\frac{1}{2} \right\|_{4;B_r(x_0)} \right) \left\| (|u| + |\nabla d|)^\frac{1}{2} \right\|_{6;B_r(x_0)} \left\| |\nabla u| + |\nabla^2 d| \right\|_{2;B_r(x_0)}
\]

\[
\lesssim \rho^\frac{3}{2} \left( \rho A(\rho) \right)^{\frac{3}{2}} \|u| + |\nabla d| \|^3_{4;B_r(x_0)} \left\| |\nabla u| + |\nabla^2 d| \right\|^3_{2;B_r(x_0)}.
\]

For the following, we fix now any \( r \in (0, \frac{3}{2}] \), and omit the dependence on \( x_0, t_0 \) and \( z_0 \) in \( B_r(x_0), B_{\rho}(x_0), I^*(t_0), A_{z_0}, B_{z_0}, C_{z_0} \) and \( D_{z_0} \) (we will retain \( z_0 \) in the notation for \( F_{z_0} \) to distinguish it from \( F = \nabla f \)).

To first prove (4.48), we note that (4.58) implies (since \( r \leq \frac{3}{2} \)) that

\[
\int_{B_r} |p_2|^\frac{3}{2} \, dx \lesssim r^{-3} \rho^{-3} \left( \int_{B_r} (|u|^2 + |\nabla d|^2) \, dy \right)^{\frac{3}{4}} \left( \int_{B_r} (|\nabla u|^2 + |\nabla^2 d|^2) \, dy \right)^{\frac{1}{4}}
\]

\[
\lesssim r^{-3} \rho^{-3} (\rho A(\rho))^{\frac{3}{4}} \left( \int_{B_r} (|\nabla u|^2 + |\nabla^2 d|^2) \, dy \right)^{\frac{1}{4}}
\]

so that, integrating over \( t \in I^*_r \) and using Hölder’s inequality, we have

\[
r^{-2} \int_{Q^*_r} |p_2|^\frac{3}{2} \, dz \lesssim r^{-2} r^3 \rho^{-\frac{3}{2}} A^\frac{1}{2}(\rho) \cdot |I^*_r|^{\frac{3}{4}} (\rho B(\rho))^{\frac{3}{4}} = \frac{r}{\rho} \cdot [(AB)^{\frac{1}{2}}](\rho),
\]

and that (4.59) similarly implies that

\[
r^{-2} \int_{Q^*_r} |p_4|^\frac{3}{2} \, dz \lesssim r \rho^{-\frac{3}{2}} \int_{I^*_r} \left( \int_{B_{\rho}} |p| \, dy \right)^{\frac{3}{4}} \lesssim \frac{r}{\rho} \cdot D(\rho).
\]

Finally, integrating (4.60) over \( t \in I^*_r \), Hölder with \( 1 = 1/4 + 3/4 \) gives

\[
r^{-2} \|p_1\|_{2;Q^*_r} \lesssim r^{-2} r^\frac{3}{2} A^\frac{1}{2}(\rho) \|u| + |\nabla d| \|^3_{2;Q^*_r} \left\| |\nabla u| + |\nabla^2 d| \right\|^3_{2;Q^*_r}
\]

\[
\lesssim r^{-2} \rho^\frac{3}{2} A^\frac{1}{2}(\rho) \left( \rho^2 C(\rho) \right)^{\frac{3}{4}} (\rho B(\rho))^{\frac{3}{4}} = \left( \frac{1}{\rho} \right)^{-2} \left( A^\frac{1}{2}(\rho) B^\frac{1}{2}(\rho) \right).
\]
Multiplying and dividing by \((r/\rho)^{\frac{3}{2}}\) for any \(\alpha \in \mathbb{R}\), Cauchy’s inequality gives

\[
r^{-2} \|p_1\|_{\frac{3}{2};Q_5} \lesssim \left(\frac{r}{\rho}\right)^{\alpha} C^\frac{1}{2}(\rho) + \left(\frac{r}{\rho}\right)^{-\alpha-4} A^\frac{1}{2}(\rho)B^\frac{1}{2}(\rho).
\]  

(4.63)

Since we want a positive power of \(\gamma = r/\rho\) in the first term and a negative one on the second (because it contains \(B\) which will be small), we want to take \(\alpha > 0\). Choosing \(\alpha = 1\) purely to make the following expression simpler, since \(p = p_1 + p_2 + p_1\), we see from (4.51), (4.62) and (4.63) that

\[
D(r) \lesssim \frac{r}{\rho} \cdot [D + (AB)^\frac{1}{2} + C^\frac{1}{2}](\rho) + \left(\frac{r}{\rho}\right)^{-5} \left[A^\frac{1}{2}B^\frac{1}{2}\right](\rho)
\]

which implies (4.68) for \(\gamma := \frac{r}{\rho} \leq \frac{1}{2}\).

To prove (4.49), we note that \(F_{20}(r) \leq F_1(r) + F_2(r) + F_3(r)\), where we set

\[
F_j(r) := \frac{1}{r^2} \int_{Q_r} |p_j| |u| \, dz.
\]

To estimate \(F_1\) we use Hölder and (4.60) to see that (in fact, for \(r \leq \rho\))

\[
\int_{B_r} |p_1| |u| \, dx \leq \|u\|_{3;B_r} \|p_1\|_{\frac{3}{2};B_r}
\]

\[
\lesssim \|u\|_{3;B_r} \cdot \rho^\frac{1}{2} (\rho A(\rho))^\frac{1}{4} \|u\| \|\nabla d\| \|\nabla u\| \|\nabla^2 d\| \|\nabla u\| \|\nabla^2 d\| \|\nabla u\| \|\nabla^2 d\| \|\nabla u\| \|\nabla^2 d\| \|\nabla u\| \|\nabla^2 d\|
\]

and hence Cauchy-Schwarz in time gives

\[
F_1(r) \lesssim r^{-2} \rho^\frac{1}{2} A^\frac{1}{2}(\rho) \|u\| + \|\nabla d\| \|\nabla u\| \|\nabla^2 d\| \|\nabla u\| \|\nabla^2 d\|
\]

\[
\lesssim r^{-2} \rho^\frac{1}{2} A^\frac{1}{2}(\rho) (\rho^2 C(\rho))^\frac{1}{2} (\rho B(\rho))^\frac{1}{2}
\]

\[
= \left(\frac{r}{\rho}\right)^{\alpha} C^\frac{1}{2}(\rho) \cdot \left(\frac{r}{\rho}\right)^{-\alpha-4} [A^\frac{1}{2}B^\frac{1}{2}](\rho)
\]

\[
\lesssim \left(\frac{r}{\rho}\right)^{\alpha} C^\frac{1}{2}(\rho) + \left(\frac{r}{\rho}\right)^{-\alpha-4} [A^\frac{1}{2}B^\frac{1}{2}](\rho)
\]

for any \(\alpha \in \mathbb{R}\). Taking, say, \(\alpha = \frac{1}{2}\), we have

\[
F_1(r) \lesssim \left(\frac{r}{\rho}\right)^{\frac{3}{2}} C^\frac{1}{2}(\rho) + \left(\frac{r}{\rho}\right)^{-10} [AB^2](\rho).
\]

(4.64)

Now for \(F_2\) note that, using (4.58), we have (since \(r \leq \frac{\rho}{2}\))

\[
\int_{B_r} |p_2| |u| \, dx \quad \lesssim \quad \rho^{-2} \int_{B_r} (|u| |\nabla u| + |\nabla d||\nabla^2 d|) \, dy \int_{B_r} |u| \, dx
\]

\[
\lesssim \quad \rho^{-2} \|u\| + \|\nabla d\| \|\nabla u\| \|\nabla^2 d\| \|\nabla u\| \|\nabla^2 d\| \|\nabla u\| \|\nabla^2 d\| \|\nabla u\| \|\nabla^2 d\|
\]

so that integrating over \(t \in I_\rho^*\) and using Hölder in time we have

\[
F_2(r) \lesssim \frac{1}{r^2} \rho^\frac{1}{2} (\rho A(\rho)) (\rho B(\rho))^\frac{1}{2} (r^2)^\frac{1}{2} = \left(\frac{r}{\rho}\right)^{\frac{1}{2}} [AB^\frac{1}{2}](\rho).
\]

(4.65)
For $F_3$, using (4.59) and H"{o}lder, we see that

\[
\frac{1}{r^2} \int_{B_r} |p_3||u| \, dx \leq \frac{1}{r^2 \rho^2} \left( \int_{B_\rho} |p| \, dy \right) \left( \int_{B_r} |u| \, dx \right)
\]

\[
\leq \frac{1}{r^2 \rho^2} \left( \int_{B_\rho} |p|^\frac{2}{3} \, dx \right)^\frac{3}{2} (\rho^3)^\frac{1}{2} \left( \int_{B_r} (|u|^\frac{2}{3})^4 \, dx \right)^\frac{1}{4} \left( \int_{B_r} (|u|^\frac{2}{3})^6 \, dx \right)^\frac{1}{6} (r^3)^\frac{2}{3}
\]

which gives us (setting $\gamma := \frac{r}{\rho}$)

\[
F_3(r) \lesssim \frac{1}{r^2 \rho^2} (rA(r))^{\frac{1}{2}} \left( \int_{Q_3} |p|^\frac{2}{3} \, dx \right)^\frac{3}{2} \left( \int_{Q_3} |u|^3 \, dx \right)^{\frac{3}{2}} (r^2)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{r^2 \rho^2} (rA(r))^{\frac{1}{2}} (\rho^2 D(\rho))^{\frac{1}{2}} (r^2 C(r))^{\frac{1}{2}} (r^2)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{r}{\rho} \right) (\gamma^{-1} A)^{\frac{1}{2}} (\rho) D^\frac{1}{2} (\rho) (\gamma^{-2} C)^{\frac{1}{2}} (\rho) = \left( \frac{r}{\rho} \right) A^\frac{1}{2} (\rho) D^\frac{1}{2} (\rho) C^\frac{1}{2} (\rho)
\]

by (4.51). Hence Young’s inequality implies

\[
F_3(r) \lesssim \left( \frac{r}{\rho} \right)^{\frac{1}{2}} (A(\rho) + D^\frac{1}{2} (\rho) + C^\frac{1}{2} (\rho))
\]

(4.66)

Adding (4.64), (4.65) and (4.66) and passing to the smallest powers of $\gamma = \frac{r}{\rho} (< 1)$ we see that

\[
F_{20}(r) \lesssim \left( \frac{r}{\rho} \right)^{\frac{1}{2}} (A + D^\frac{1}{2} + C^\frac{1}{2}) (\rho) + \left( \frac{r}{\rho} \right)^{-10} [A(B^\frac{1}{2} + B^2)] (\rho)
\]

which implies (4.49), and completes the proof of Claim 3. \qed

**Proof of Claim 4**

We will again omit the dependence on $z_0$ (except in $F_{20}$).

To estimate $A(\frac{1}{2})$, we use the local energy inequality (1.13) with a non-negative cut-off function $\phi \in C_0^\infty (Q_\rho^*)$ which is equal to 1 in $Q_\rho^*$, with

\[
|\nabla \phi| \lesssim \rho^{-1} \quad \text{and} \quad |\phi_t|, |\nabla^2 \phi| \lesssim \rho^{-2}.
\]

We’ll need to estimate terms which control those that appear on the right-hand side of the local energy inequality (1.13), which we’ll call $I - V$ (all of which depend on $\rho$) as follows:

\[
I := \int_{Q_\rho^*} (|u|^2 + |\nabla d|^2) |\phi_t + \Delta \phi| \, dz \lesssim \rho^{-2} \| |u|^2 + |\nabla d|^2 \|_{L^1(Q_\rho^*)} (\rho^5)^\frac{1}{2}
\]

\[
\lesssim \rho^{-2} (\rho^5 C(\rho))^\frac{1}{2} (\rho^5)^\frac{1}{2} = \rho C^\frac{1}{2} (\rho).
\]

Using the assumption (1.11) that $\nabla \cdot u = 0$ weakly and indicating by $\mathbf{v}^\rho$ the average of a function $g$ in $B_\rho$, we have

\[
II := \int_{Q_\rho^*} \left[ \int_{B_\rho} \left( |u|^2 + |\nabla d|^2 \right) u \cdot \nabla \phi \, dx \right] \, dt
\]

\[
= \int_{Q_\rho^*} \left[ \int_{B_\rho} \left[ (|u|^2 - |u|^2) + (|\nabla d|^2 - |\nabla d|^2) \right] u \cdot \nabla \phi \, dx \right] \, dt
\]
Similarly, we have
\[ I I : = \int_{Q_{2}} \int_{B_{2}} |p_{u} \cdot \nabla \phi| \, dz \lesssim \rho_{0}^{-1}(\rho^{2}F_{z_{0}}) = \rho F_{z_{0}}(\rho). \]

Using the weak divergence-free condition \( \nabla \cdot u = 0 \) in (1.11) to write (see (1.2))
\[
(u \cdot \nabla) d = \nabla^{T} \cdot (d \otimes u)
\]
(at almost every \( x \)) and integrating by parts we have
\[
IV: = \int_{I_{p}^{*}} \int_{B_{p}} u \otimes \nabla \phi : \nabla d \otimes \nabla d \, dx \, dt = \int_{I_{p}^{*}} \int_{B_{p}} [(u \cdot \nabla) d] \cdot [(\nabla \phi \cdot \nabla) d] \, dx \, dt
\]
\[
= \int_{I_{p}^{*}} \int_{B_{p}} [\nabla^{T} \cdot (d \otimes u)] \cdot [(\nabla \phi \cdot \nabla) d] \, dx \, dt = \int_{I_{p}^{*}} \int_{B_{p}} d \otimes u : \nabla^{T} [(\nabla \phi \cdot \nabla) d] \, dx \, dt,
\]
and clearly
\[
|\nabla^{T} [(\nabla \phi \cdot \nabla) d]| \lesssim |\nabla^{2} \phi||\nabla d| + |\nabla \phi||\nabla^{2} d|.
\]
Therefore, for \( q \in [2, 6] \) we have\(^{26}\)
\[
IV \lesssim \int_{Q_{2}} \int_{B_{2}} |d| |u| (\rho^{-2}|\nabla d| + \rho^{-1}|\nabla^{2} d|) \, dz
\]
\[
\leq \rho \left( G^{1/2}_{q}(\rho)C^{1/2}(\rho)B^{1/2}(\rho) \right) + G^{1/2}_{q}(\rho)C^{1/2}(\rho)B^{1/2}(\rho).
\]

Similarly, for \( q \in [2, 6] \) we have
\[
V: = \int_{Q_{2}} \int_{B_{2}} |d^{2}| |\nabla d|^{2} \phi \, dz \lesssim \rho G_{2} \leq \rho^{\delta} G_{q}^{\delta}(\rho)C^{1-\delta}(\rho).
\]

Finally, using (4.67) - (4.71), the local energy inequality (1.13) (with constant \( \tilde{C} \)) gives
\[
(\tilde{C})^{-1/2} A(\tilde{C}) \lesssim I + II + III + IV + V
\]
\[
\lesssim \rho \left[ C^{1/2} + E + F_{z_{0}} + G^{1/2}_{q} \left( C^{1/2} + C^{1/2} + B^{1/2} \right) \right] \]
\[
\lesssim \rho \left[ C^{1/2} + E + F_{z_{0}} + (1 + [\cdot]^{2})G^{1/2}_{q} + (G^{1/2}_{q} + C^{1/2})B^{1/2} \right] (\rho)
\]

\(^{26}\)Note that it is only the appearance of \( \nabla^{2} d \) in the estimate of term \( IV \) which forces us to include \( u \) in the definition of \( G_{q, z_{0}} \). Indeed, switching the roles of \( u \) (which appears in \( C_{z_{0}} \) along with \( \nabla d \)) and \( \nabla d \) (which appears in \( G_{q, z_{0}} \) even with \( u \) omitted), one could otherwise control term \( IV \) in precisely the same way. If \( u \) is omitted in \( G_{q, z_{0}} \), one could still obtain the same estimate of \( IV \) if one takes \( q = 6 \), but this would dramatically weaken the statement of Theorem 1. The remainder of the proof of Theorem 1 does not require (but is not harmed by) the inclusion of \( u \) in \( G_{q, z_{0}} \).
as long as \(2 \leq q < 6\), as in that case we have
\[
G_1^q C^{\frac{q}{q} - \frac{1}{q}} = (G_q^q)^{\frac{q}{q} - \frac{1}{q}} (C^2)^{\frac{3q-6}{2q}} \leq \left( \frac{6-q}{4q} \right) G_q^{\frac{q}{q} - \frac{1}{q}} + \left( \frac{5q-6}{4q} \right) C^2 \leq \frac{3}{4} G_q \frac{q}{q} + \frac{3}{4} C^2,
\]
\[
G_1^q C^{\frac{q}{q} - \frac{1}{q}} = (G_q^q)^{\frac{q}{q} - \frac{1}{q}} (C^2)^{\frac{3q-6}{2q}} \leq \left( \frac{6-q}{2q} \right) G_q^{\frac{q}{q} - \frac{1}{q}} + \left( \frac{3q-6}{2q} \right) C^2 \leq \frac{3}{2} G_q \frac{q}{q} + \frac{3}{2} C^2
\]
and
\[
G_1^q C^{1 - \frac{q}{q} - \frac{1}{q}} = (G_q^q)^{1 - \frac{q}{q} - \frac{1}{q}} (C^2)^{\frac{3q-6}{2q}} \leq \left( \frac{6-q}{2q} \right) G_q \frac{q}{q} + \left( \frac{3q-6}{2q} \right) C^2 \leq \frac{3}{2} G_q \frac{q}{q} + \frac{3}{2} C^2
\]
This implies (4.50) and proves Claim 4. □

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