G₂-structure deformations and warped products

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Abstract. We overview the properties of non-infinitesimal deformations of G₂-structures on seven-manifolds, and in particular, focus on deformations that lie in the seven-dimensional representation of G₂ and are thus defined by a vector. We then consider deformations from G₂-structures with the torsion class having one-dimensional and seven-dimensional components (so-called conformally nearly parallel G₂-manifolds) to G₂-structures with just a one-dimensional torsion component (nearly parallel G₂-manifolds). We find that deformations between such structures exist if and only if the metric is a particular warped product metric.

1. Introduction

One of the most general geometric structures that can be constructed on a 7-dimensional manifold is a G₂-structure. A G₂-structure can be considered as a generalization of the vector cross product on R⁷. It is well-known that a 7-manifold admits a G₂-structure if and only if it is orientable and admits a spin structure, or equivalently, if the first two Stiefel-Whitney classes vanish [6, 7]. A very important special case of a G₂-structure is a torsion-free G₂-structure. This implies that the holonomy group lies in G₂. In Section 2 we give a more precise definition and an overview of the properties of G₂-structures. The concept of G₂-structures also has important applications in physics - as shown in [16], the most general backgrounds for M-theory compactifications with fluxes are indeed 7-manifolds with G₂-structures with some particular torsion.

Given a 7-manifold with a G₂-structure defined by the 3-form ϕ with torsion T, a natural question to ask is whether we can modify this 3-form to get a new G₂-structure with torsion that lies a strictly lower torsion class. If ϕ is deformed by a 3-form lying in the 7-dimensional component of Λ³, it is easy to see that such a deformation will always yield a new G₂-structure and in my paper [10], I have explicitly calculated the new torsion in terms of the old one, and the derived the equation that v must satisfy to take a torsion T to torsion ˜T. It was moreover shown that on closed, compact manifolds there are no such deformations from strict torsion classes W₁, W₇, W₁ ⊕ W₇ to the vanishing torsion class W₀, and vice versa.

In this paper we use the general results from [10] and apply them to the situation when we want a transition from the torsion class W₁ ⊕ W₇ to W₁. The torsion class W₁ ⊕ W₇ is known as conformally nearly parallel because a conformal...
transformation takes $W_1 \oplus W_7$ to $W_1$. Here, however, we show that there exists a deformation of $\varphi$ in $\Lambda^3\mathbb{S}$ that takes a torsion in $W_1 \oplus W_7$ to $W_1$ if and only if the metric is a particular warped product. However $G_2$-structures with such torsion have been constructed by Cleyton and Ivanov in [5] as a warped product of an interval over a nearly Kähler manifold, so these examples fit as solutions.

2. $G_2$-structures

The 14-dimensional group $G_2$ is the smallest of the five exceptional Lie groups and is closely related to the octonions. In particular, $G_2$ can be defined as the automorphism group of the octonion algebra. Taking the imaginary part of octonion multiplication of the imaginary octonions defines a vector cross product on $V = \mathbb{R}^7$ and the group that preserves the vector cross product is precisely $G_2$. A more detailed account of the relationship between octonions and $G_2$ can be found in [1, 9]. The structure constants of the vector cross product define a 3-form on $\mathbb{R}^7$, hence $G_2$ can alternatively be defined as the subgroup of $GL(7, \mathbb{R})$ that preserves a particular 3-form [13]. In general, given a $n$-dimensional manifold $M$, a $G$-structure on $M$ for some Lie subgroup $G$ of $GL(n, \mathbb{R})$ is a reduction of the frame bundle $F$ over $M$ to a principal subbundle $P$ with fibre $G$. A $G_2$-structure is then a reduction of the frame bundle on a 7-dimensional manifold $M$ to a $G_2$ principal subbundle. It turns out that there is a 1-1 correspondence between $G_2$-structures on a 7-manifold and smooth 3-forms $\varphi$ for which the 7-form-valued bilinear form $B_\varphi$ as defined by (2.1) is positive definite (for more details, see [2] and the arXiv version of [12]).

\begin{equation}
B_\varphi (u, v) = \frac{1}{6} (u \varphi) \wedge (v \varphi) \wedge \varphi
\end{equation}

Here the symbol $\cdot$ denotes contraction of a vector with the differential form: 

\[(u \varphi)_{mn} = u^a \varphi_{amn}.
\]

Note that we will also use this symbol for contractions of differential forms using the metric.

A smooth 3-form $\varphi$ is said to be positive if $B_\varphi$ is the tensor product of a positive-definite bilinear form and a nowhere-vanishing 7-form. In this case, it defines a unique metric $g_\varphi$ and volume form vol such that for vectors $u$ and $v$, the following holds

\begin{equation}
\begin{aligned}
g_\varphi (u, v) \operatorname{vol} &= \frac{1}{6} (u \varphi) \wedge (v \varphi) \wedge \varphi
\end{aligned}
\end{equation}

In components we can rewrite this as

\begin{equation}
(g_\varphi)_{ab} = (\det s)^{-\frac{1}{2}} s_{ab} \quad \text{where} \quad s_{ab} = \frac{1}{144} \epsilon_{amn} \epsilon_{bpq} \epsilon_{rst} \hat{\epsilon}^{mnpqrst}.
\end{equation}

Here $\hat{\epsilon}^{mnpqrst}$ is the alternating symbol with $\hat{\epsilon}^{12...7} = +1$. Following Joyce ([13]), we will adopt the following definition

**Definition 2.1.** Let $M$ be an oriented 7-manifold. The pair $(\varphi, g)$ for a positive 3-form $\varphi$ and corresponding metric $g$ defined by (2.2) will be referred to as a $G_2$-structure.

Since a $G_2$-structure defines a metric and an orientation, it also defines a Hodge star. Thus we can construct another $G_2$-invariant object - the 4-form $\ast \varphi$. Since the Hodge star is defined by the metric, which in turn is defined by $\varphi$, the 4-form $\ast \varphi$ depends non-linearly on $\varphi$. For convenience we will usually denote $\ast \varphi$ by $\psi$. 

For a general $G$-structure, the spaces of $p$-forms decompose according to irreducible representations of $G$. Given a $G_2$-structure, 2-forms split as $\Lambda^2 = \Lambda^2_2 \oplus \Lambda^2_{14}$, where $\Lambda^2_2 = \{\alpha, \varphi: \text{a vector field } \alpha\}$ and 
\[
\Lambda^2_{14} = \{\omega \in \Lambda^2: (\omega_{ab}) \in g_2\} = \{\omega \in \Lambda^2: \omega, \varphi = 0\}.
\]
The 3-forms split as $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_2 \oplus \Lambda^3_{27}$, where the one-dimensional component consists of forms proportional to $\varphi$, forms in the 7-dimensional component are defined by a vector field $\Lambda^3_7 = \{\alpha, \psi: \text{a vector field } \alpha\}$, and forms in the 27-dimensional component are defined by traceless, symmetric matrices:
\[
(2.4) \quad \Lambda^3_{27} = \{\chi \in \Lambda^3: \chi_{abc} = h^d_{[a} \varphi_{bc]}d \text{ for } h_{ab}, \text{traceless, symmetric}\}.
\]
By Hodge duality, similar decompositions exist for $\Lambda^4$ and $\Lambda^5$. A detailed description of these representations is given in [2, 3]. Also, formulae for projections of differential forms onto the various components are derived in detail in [10, 11, 15].

The *intrinsic torsion* of a $G_2$-structure is defined by $\nabla \varphi$, where $\nabla$ is the Levi-Civita connection for the metric $g$ that is defined by $\varphi$. Following [15], it is easy to see
\[
(2.5) \quad \nabla \varphi \in \Lambda^1_1 \otimes \Lambda^3_2 \cong W.
\]
Here we define $W$ as the space $\Lambda^1_1 \otimes \Lambda^3_2$. Given (2.5), we can write
\[
(2.6) \quad \nabla_a \varphi_{bcd} = T_a^{\ c} \psi_{ebcd}
\]
where $T_{ab}$ is the *full torsion tensor*. From this we can also write
\[
(2.7) \quad T_a^{\ m} = \frac{1}{24} (\nabla_a \varphi_{bcd}) \psi^{mbcd}.
\]
This 2-tensor fully defines $\nabla \varphi$ since pointwise, it has 49 components and the space $W$ is also 49-dimensional (pointwise). In general we can split $T_{ab}$ according to representations of $G_2$ into *torsion components*:
\[
(2.8) \quad T = \tau_1 g + \tau_7 \varphi + \tau_{14} + \tau_{27}
\]
where $\tau_1$ is a function, and gives the 1 component of $T$. We also have $\tau_7$, which is a 1-form and hence gives the 7 component, and, $\tau_{14} \in \Lambda^4_{14}$ gives the 14 component and $\tau_{27}$ is traceless symmetric, giving the 27 component. Hence we can split $W$ as
\[
(2.9) \quad W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}.
\]
As it was originally shown by Fernández and Gray [6], there are in fact a total of 16 torsion classes of $G_2$-structures that arise as the subsets of $W$ to which $\nabla \varphi$ belongs. Moreover, as shown in [15], the torsion components $\tau_i$ relate directly to the expression for $d \varphi$ and $d \psi$. In fact, in our notation,
\[
(2.10a) \quad d \varphi = 4\tau_1 \psi - 3\tau_7 \wedge \varphi - \ast \tau_{27}
\]
\[
(2.10b) \quad d \psi = -4\tau_7 \wedge \psi - 2 \ast \tau_{14}.
\]
Note that in the literature (33, 5, for example) a slightly different convention for torsion components is sometimes used. Our $\tau_1$ then corresponds to $\frac{1}{2}\tau_0$, $\tau_7$ corresponds to $-\tau_1$ in their notation, $\tau_{27}$ corresponds to $-\tau_3$ and $\tau_{14}$ corresponds to $-\frac{1}{2}\tau_{27}$. Similarly, our torsion classes $W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$ correspond to $W_0 \oplus W_1 \oplus W_2 \oplus W_3$.

An important special case is when the $G_2$-structure is said to be torsion-free, that is, $T = 0$. This is equivalent to $\nabla \varphi = 0$ and also equivalent, by Fernández
and Gray, to $d\varphi = d\psi = 0$. Moreover, a $G_2$-structure is torsion-free if and only if the holonomy of the corresponding metric is contained in $G_2$ \cite{13}. The holonomy group is then precisely equal to $G_2$ if and only if the fundamental group $\pi_1$ is finite.

The torsion tensor $T_{ab}$ and hence the individual components $\tau_1, \tau_7, \tau_{14}$ and $\tau_{27}$ must also satisfy certain differential conditions. For the exterior derivative $d$, $d^2 = 0$, so from (2.10), must have

\begin{align}
\text{(2.11a)} & \quad d \left( 4 \tau_1 \psi - 3 \tau_7 \wedge \varphi - \ast \tau_{27} \right) = 0 \\
\text{(2.11b)} & \quad d \left( 4 \tau_7 \wedge \psi + 2 \ast \tau_{14} \right) = 0
\end{align}

These conditions are explored in more detail in \cite{10}. However for some simple torsion classes, the conditions on torsion components simplify. In particular, if the torsion is in the class $W_1$, so that only the $\tau_1$ component is non-vanishing, we just get the condition $d\tau_1 = 0$. Similarly, for the $W_7$ class the condition is $d\tau_7 = 0$. For the class $W_1 \oplus W_7$, both $\tau_1$ and $\tau_7$ are non-vanishing, and the condition is $d\tau_1 = \tau_1 \tau_7$. So if $\tau_1$ is nowhere zero, we have

\begin{equation}
\tau_7 = d(\log \tau_1).
\end{equation}

However if $\tau_1$ does vanish somewhere, it was shown in \cite{5} that it must in fact vanish identically, and so the torsion class reduces to $W_7$.

3. Deformations of $G_2$-structures

Suppose we have a $G_2$-structure on $M$ defined by the 3-form $\varphi$, and we want to obtain a new $G_2$-structure $\tilde{\varphi}$ by adding another 3-form $\chi$

\begin{equation}
\varphi \longrightarrow \tilde{\varphi} = \varphi + \chi
\end{equation}

There are a number of challenges associated with this. Firstly, for a generic 3-form $\chi$, the 3-form $\tilde{\varphi}$ may not even define a $G_2$-structure. In order for $\tilde{\varphi}$ to define a $G_2$-structure it has to be a positive 3-form. In this case, as shown in (\cite{11}), $\tilde{\varphi}$ defines a Riemannian metric $\tilde{g}$ given by

\begin{equation}
\tilde{g}_{ab} = \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} s_{ab}
\end{equation}

for

\begin{equation}
s_{ab} = g_{ab} + \frac{1}{2} \chi_{mn(a} \varphi_{b)}^{mn} + \frac{1}{8} \chi_{amn} \chi_{bpq} \psi^{mnpq} + \frac{1}{24} \chi_{amn} \chi_{bpq} (\ast \chi)^{mnpq}
\end{equation}

In fact, $\tilde{\varphi}$ is positive if and only if $s$ is positive-definite, so in general this gives some algebraic constraints on $\chi$. If we assume that $\tilde{\varphi}$ does in fact define a $G_2$-structure, the next question is the torsion class of the new $G_2$-structure. The metric $\tilde{g}$ defines a Levi-Civita connection $\tilde{\nabla}$, so the new torsion (with lowered indices) is

\begin{equation}
\tilde{T}_{am} = \frac{1}{24} \left( \tilde{\nabla}_a \tilde{\varphi}_{bcd} \right) \tilde{\psi}_m^{bcd}
\end{equation}

Here $\tilde{\psi} = \ast \tilde{\varphi}$, the Hodge dual of $\tilde{\varphi}$ with the Hodge star being defined by the metric $\tilde{g}$. The tilded raised indices on $\tilde{\psi}$ denote indices raised by $\tilde{g}$. In \cite{10}, I derived an explicit expression for $\tilde{T}$ in terms of the old torsion $T$ and the 3-form $\chi$: 
Proposition 3.1 (10). Given a deformation of \( \varphi \) as in (3.7), the full torsion \( \tilde{T} \) of the new \( G_2 \)-structure \( \tilde{\varphi} \) is given by

\[
\tilde{T}_{an} = \frac{1}{24} \left( \frac{\det g}{\det \tilde{g}} \right) \left( 24T_m^a + T_a^{\,\varepsilon e b c d} (\ast \chi)^{mbcd} + \right.
\]

\[
+ \psi^{mbcd} \nabla_a \chi_{bcd} + \nabla_a \chi_{bcd} (\ast \chi)^{mbcd} ) \tilde{s}_{mn} - 3 \left( 4 \varphi_c \varepsilon^{cd} + \varphi_{cpq} \ast \chi^{pqbd} + \chi_{cpq} \psi^{pqbd} + \chi_{cpq} (\ast \chi)^{pqbd} \right) \times
\]

\[
\times \left( \delta^a_n \nabla_b s_{ad} - \frac{1}{9} \delta^a_n g_{bn} \tilde{g}_{bp} \nabla_d \tilde{s}_{pq} \right).
\]

where \( \tilde{s}_{ab} \) is given by (3.3).

Using (3.5), it is then possible to extract the components of \( \tilde{T} \) in \( \tilde{W}_1 \oplus \tilde{W}_7 \oplus \tilde{W}_{14} \oplus \tilde{W}_{27} \) and hence determine the new torsion class. Note that since the \( G_2 \)-structure is different, the decomposition \( \tilde{W}_1 \oplus \tilde{W}_7 \oplus \tilde{W}_{14} \oplus \tilde{W}_{27} \) differs from \( W_1 \oplus W_7 \oplus W_{14} \oplus W_{27} \). An interesting question is whether, given a \( G_2 \)-structure in a specific torsion class, we can find a 3-form \( \chi \) such that the new \( G_2 \)-structure is in a strictly smaller torsion class. From (3.5), this obviously involves solving a non-linear differential equation for \( \chi \), subject to algebraic constraints that (3.3) is positive-definite. One way to simplify the problem is to restrict the choice of \( \chi \). Using the original \( G_2 \)-structure \( \varphi \) we can decompose the 3-form \( \chi \) according to representations of \( G_2 \). So in general it has a \( \Lambda_1^3 \) component that is proportional to \( \varphi \), a \( \Lambda_2^7 \) component that is of the form \( v \ast \psi \) for some vector \( v \) and a more complicated \( \Lambda_3^{27} \) component. For a generic \( \chi \), many of the difficulties come from the \( \Lambda_3^{27} \) component. These of course can be avoided if we only consider deformations by 3-forms that have components only in either \( \Lambda_1^3 \) or \( \Lambda_2^7 \).

A deformation by a 3-form in \( \Lambda_1^3 \) is equivalent to a conformal transformation. So let \( \chi = (f^3 - 1) \varphi \), so that \( \tilde{\varphi} = f^3 \varphi \). Clearly \( \tilde{\varphi} \) still defines a \( G_2 \)-structure. Then from (3.3) and (3.5) we get

\[
s_{ab} = f g_{ab} \quad \text{and thus,} \quad \tilde{g}_{ab} = f^2 g_{ab}.
\]

Substituting into (3.5), we find that

\[
\tilde{T} = f T - df \ast \varphi
\]

In particular, we see that such a transformation only affects the \( W_7 \) component of the torsion. Moreover, if \( T \) has a \( W_7 \) component \( \tau_7 \) that is an exact form, then we can always find a function \( f \) so that a conformal transformation will remove this torsion component.

As an example, suppose \( \varphi \) has torsion in the strict class \( W_1 \oplus W_7 \). Then from (2.12), we know that \( \tau_7 = d (\log \tau_1) \). Hence if we take \( f = \frac{\tau_1}{C} \) for any non-zero constant \( C \), the new torsion will be

\[
\tilde{T} = \frac{\tau_1}{C} \tau_1 g = C \left( \frac{\tau_1}{C} \right)^2 g = C \tilde{g}
\]

and thus in the \( \tilde{W}_1 \) class. Therefore, the conformal transformation \( \tilde{\varphi} = \left( \frac{\tau_1}{C} \right)^3 \varphi \) reduces the class \( W_1 \oplus W_7 \) to \( W_1 \). Conversely, a conformal transformation of the \( W_1 \) class will result in \( W_1 \oplus W_7 \). Since \( G_2 \)-structures in the \( W_1 \) class are sometimes called nearly parallel, the \( G_2 \)-structures in the strict \( W_1 \oplus W_7 \) class are referred to as conformally nearly parallel. If \( W_1 = 0 \), then we just have the \( W_7 \) class. In this
case, we know that $\tau_7$ is closed. So by the Poincaré Lemma, we can at least locally find a function $h$ such that $dh = \tau_7$. By taking a conformal transformation with $f = e^h$, we can thus remove the torsion locally. Hence the $W_7$ class is sometimes called \textit{locally conformally parallel}.

Now suppose we look at deformations where $\chi_{bed} = v^e \psi_{bed} \in A^2$. It was shown by Karigiannis in [14], that in this case, if we let $|v|^2 = M$, with respect to the old metric $g$,

\begin{align}
(3.9a) \quad s_{ab} &= (1 + M) g_{ab} - v_a v_b \\
(3.9b) \quad \left( \frac{\det \tilde{g}}{\det g} \right)^{\frac{1}{2}} &= (1 + M)^{\frac{1}{2}} \\
(3.9c) \quad \tilde{g}_{ab} &= (1 + M)^{\frac{3}{2}} (1 + M) g_{ab} - v_a v_b \\
(3.9d) \quad \tilde{g}^{\tilde{a}\tilde{b}} &= (1 + M)^{-\frac{3}{2}} (g^{mu} + v^m v^u) 
\end{align}

Note that the deformed metric defined above is always positive definite. To see this, suppose $\xi^a$ is some vector, then

\begin{align}
(3.10) \quad \tilde{g}_{ab} \xi^a \xi^b &= \left( 1 + |v|^2 \right)^{-\frac{3}{2}} \left(|\xi|^2 + |v|^2 |\xi|^2 - (v_a \xi^a)^2 \right) \geq 0
\end{align}

since $(v_a \xi^a)^2 \leq |v|^2 |\xi|^2$. Therefore, under such a deformation, the 3-form $\tilde{\varphi}$ is always a positive 3-form, and thus indeed defines a $G_2$-structure.

In [10], the expression for the new torsion was derived using (3.9) and (3.5). This is a very long and messy expression, which we will not reproduce here, but it gives the new torsion in terms of the old torsion components and $\nabla v$, which was also decomposed according to $G_2$-representations as

\begin{align}
(3.11) \quad \nabla v &= v_1 g + v_7 \psi + v_{14} + v_{27}
\end{align}

The expression for $\tilde{T}$ was then inverted to obtain equations for $v_1, v_7, v_{14}$ and $v_{27}$ in terms of the old and new torsion components. By analyzing the equations for $\nabla v$ in the case when the original torsion lies in the class $W_1 \oplus W_7$, it was shown that the new torsion vanishes if and only if the original $G_2$-structure was also torsion free, and moreover $\nabla v = 0$. Similarly it was shown that there are no deformations of this type which preserve the strict $W_1$ torsion class. Here we will attempt something different - what if the torsion is in the class $W_1 \oplus W_2$ and we want to obtain the class $W_1$. From the expression (3.7) we already know that this is possible to do with a conformal transformation. However if it were possible to go from $W_1 \oplus W_7$ to $W_1$ using $\chi \in A^2$, then a composition of the two types of deformation would actually give a much more complicated and interesting deformation that preserves the class $W_1$.

4. Conformally nearly parallel $G_2$-structures

Suppose now we have a $G_2$-structure $(\varphi, g)$ with torsion lying in the strict class $W_1 \oplus W_7$ - that is, both $\tau_1$ and $\tau_7$ are non-zero. We then deform $\varphi$ to $\tilde{\varphi}$ given by

\begin{align}
(4.1) \quad \tilde{\varphi} &= \varphi + v^e \psi_{bed}.
\end{align}

As we know from (3.10), the metric defined by $\tilde{\varphi}$ is positive definite, so $\tilde{\varphi}$ does indeed define a $G_2$-structure. From [10], we can also write down the torsion components of $\tilde{\varphi}$. As before, $M = |v|^2$ with respect to the old metric $g$, and $\nabla v$ is decomposed.
into components as in (3.11). Here we show the expression for \( \tilde{\tau}_1 \) and \( \tilde{\tau}_7 \), the 1- and 7-dimensional components of the new torsion \( \tilde{T}_{ab} \):

\[
\begin{align*}
\tilde{\tau}_1 &= \frac{(1 + \frac{1}{2} M) \tau_1 - v_1 - \frac{6}{7} (\tau_7)^a v_a + \frac{3}{7} (v_7)^a v_a}{(1 + M)^{\frac{4}{7}}} \\
(\tilde{\tau}_7)_c &= (\tau_7)_c - \frac{1}{6} \varepsilon_{ab}^c (\tau_7)_a v_b + \frac{\nu_c (6 \tau_1 - 6 (\tau_7)_a v^a - 8 v_1 + 3 (v_7)_a v^a)}{6 (1 + M)} \\
&\quad - \frac{1}{3} (3 (M + 2) (\tau_7)_c + v^a (v_7)_{ac} + \varphi^b_{ca} v^a (v_7)_{bd} v^d + 3 \varphi^b_{cab} v^a (v_7)^b)}{6 (1 + M)}
\end{align*}
\]

The expressions for \( \tilde{\tau}_{14} \) and \( \tilde{\tau}_{27} \) can similarly be written down in terms of \( \tau_1, \tau_7 \) and the components of \( \nabla v \), using the general results in [10], however they are rather long and not very enlightening. Also, as shown in [10], the linear equations for the torsion components can be solved for the components of \( \nabla v \) in terms of the old torsion and the new torsion. Hence if we require the new torsion to be in a specific torsion class, this would give us a differential equation that \( v \) has to satisfy. We will try to find conditions that will have the \( G_2 \)-structure in \( W_1 \oplus W_7 \) class to a \( G_2 \)-structure that only has a 1-dimensional torsion component. Using the general expression for \( \nabla v \) in [10], and setting \( \tau_{14} = \tau_{27} = 0 \) and \( \tau_7 = \tilde{\tau}_{14} = \tilde{\tau}_{27} = 0 \), we thus have:

**Proposition 4.1.** Suppose \((\varphi, g)\) is \( G_2 \)-structure with the only non-vanishing torsion component \( \tau_1 \) and \( \tau_7 \). The new \( G_2 \)-structure \( \tilde{\varphi} \) obtained via the deformation \((4.1)\) then has torsion in class \( W_1 \) with the non-vanishing component \( \tilde{\tau}_1 \) if and only if \( v \) satisfies:

\[
(4.2) \quad \nabla_a v_b = \left( \tau_1 - (1 + M)^{\frac{1}{2}} \tilde{\tau}_1 - (\tau_7)_c v^c \right) g_{ab} + 4 (1 + M)^{-\frac{1}{2}} \tilde{\tau}_1 v_a v_b + \frac{1}{(M + 9)} (-3 (M - 3) (\tau_7)_c \varphi^c_{ab} \\
- (M + 33) v_a (\tau_7)_b + 3 (1 + M) (\tau_7)_a v_b \\
- \frac{1}{3} \nu^c \varphi_{cab} (9 \tau_1 - 4 \tilde{\tau}_1 (M + 9) (1 + M)^{-\frac{1}{2}} + \tau_1 M - 12 (\tau_7)_d v^d) \\
+ 12 v_a \varphi^c_{d b} (\tau_7)_c v_d - 12 v_b \varphi^d_{a c} (\tau_7)_c v_d + 12 (\tau_7)_c v_d \psi^c_{a b}
\)
\]

Note that \( \tilde{\tau}_1 \) has to be a constant due to the conditions on the torsion (2.11). While it is too difficult to solve equation (4.2) directly, we can obtain conditions under which the equation is at least consistent. In this equation \( v \) has lowered indices, so we can consider this as a 1-form \( v^b \). Then

\[
(dv^b)_{ab} = 2\nabla_{[a} v_{b]}
\]

However \( d^2 = 0 \), and thus the exterior derivative applied to the anti-symmetrization of (4.2) must give zero. From these considerations we get the consistency conditions in Proposition 4.2 below. The extra equations which we get from the consistency conditions are very important to simplify the equation (4.2).

**Proposition 4.2.** The equation (4.2) is consistent with the necessary condition \( d^2 v^b = 0 \) if and only if all of the following conditions are satisfied:
(1) For some smooth function $V$, 
\begin{equation}
\tau^b = V \tau^b
\end{equation}

(2) The 7-dimensional component $\tau_7$ of the original torsion satisfies
\begin{equation}
\nabla \tau_7 = \frac{1}{4} \left( \frac{V^2 |\tau|^2 - 3}{V^2} \right) (V \tau_1 + 1) \tau_7 + \frac{1}{6} (V^2 \tau_1^2 + 6V \tau_1 + 3) \tau_7 \otimes \tau_7
\end{equation}

(3) The 1-dimensional component $\tilde{\tau}_1$ of the new torsion satisfies
\begin{equation}
\tilde{\tau}_1 = \frac{1}{4V} \left( 1 + V^2 |\tau|^2 \right)^{\frac{1}{2}} (V \tau_1 - 3)
\end{equation}

Proof. As outlined above, we apply the condition $\nabla \left[ \nabla_a \nabla_b v_{c_d} \right] = 0$ to (4.2). During the simplification process we apply (4.2) again, and moreover use the conditions $d \tau_7 = 0$, $d \tau_1 = \tau_1 \tau_7$ and $d \tilde{\tau}_1 = 0$. In the end we obtain an expression for the 3-form $d^2 v^b$ in terms of $\tau_1$, $\tilde{\tau}_1$, $\tau_7$ and $\nabla \tau_7$. Since the whole 3-form must vanish, so must the components of the 3-form in $\Lambda^1_3$, $\Lambda^2_3$ and $\Lambda^3_3$. So let $\xi_1$ be the scalar corresponding to the $\Lambda^3_3$ component and let $\xi_7$ and $\xi_{27}$ be the vector and the antisymmetric symmetric tensor corresponding to the $\Lambda^3_7$ and $\Lambda^3_{27}$ components of $d^2 v^b$, respectively.

Then by considering the equations $\xi_1 = 0$, $(\xi^7)^a v_a = 0$ and $(\xi_{27})_{mn} v^m v^n = 0$, we can express $(\nabla_a \tau_7)_b v^a v^b$, $\nabla^a (\tau_7)_a$ and $|\tau_7|^2$ in terms of $M$, $\tau_1$, $\tilde{\tau}_1$ and $\langle \tau_7, v \rangle$. In particular, we find that

\begin{equation}
|\tau_7|^2 = \frac{3 \langle \tau_7, v \rangle^2 (3M^2 - 10M + 51)}{(7M^2 - 66M + 9) M}
\end{equation}

Further, we can consider the vector equations $\xi^d_7 = 0$, $\varphi_{abc} v^b \xi^c_7 = 0$, $(\xi_{27})_{mn} v^m v^n = 0$, and $\varphi_{abc} (\xi_{27})^b_{mn} v^n v^c = 0$. From these, in particular, we find

\begin{equation}
\tau_7 = \frac{\langle \tau_7, v \rangle}{M} v \quad \text{and} \quad |\tau_7|^2 = \frac{(\tau_7, v)^2}{M}.
\end{equation}

Equating (4.0) and (4.7), and solving for $\tilde{\tau}_1^2$, we obtain an expression for $\tilde{\tau}_1$ in terms of $\tau_1$, $\tau_7$ and $v$.

\begin{equation}
\tilde{\tau}_1 = \frac{1}{4} \left( 1 + M \right)^{\frac{1}{2}} \left( \tau_1 + 3 \langle \tau_7, v \rangle \right)
\end{equation}

It can be checked that this expression for $\tilde{\tau}_1$ is in fact consistent with the assumption $d \tilde{\tau}_1 = 0$.

Next, from equations $(\xi_{27})_{ab} = 0$, $\varphi^{cd} (\xi_{27})_{bd} v_e = 0$ and $\varphi_{abc} (\xi_{27})_{cd} v^c v_e (\xi_{27})_{de}$, we finally obtain an expression for $\nabla_a (\tau_7)_b$. Using (4.7) and (4.8) to eliminate $|\tau_7|^2$...
and \( \tilde{\tau}_1 \) from the resulting expression, we overall get:

\[
\nabla \tau_7 = - \langle \tau_7, v \rangle (M - 3) (M \tau_1 + (\tau_7, v)) \frac{g}{4M^2} + \frac{\left(7^4 M^2 + 6 \langle \tau_7, v \rangle M \tau_1 + 3 \langle \tau_7, v \rangle^2 \right)^2}{6 \langle \tau_7, v \rangle^2} \tau_7 \]

Now since \( v \) is proportional to \( \tau_7 \), let us write \( v = V \tau_7 \) for some smooth function \( V \). Then

\[
M = |v|^2 = V^2 |\tau_7|^2 \quad \text{and} \quad \langle \tau_7, v \rangle = V |\tau_7|^2
\]

Thus we get the expressions (4.4) and (4.5) for \( \nabla \tau_7 \) and \( \tilde{\tau}_1 \) in terms of \( V \).

Now it is easy to see that if \( v \) is proportional to \( \tau_7 \) and \( \tilde{\tau}_1 \) satisfies (4.5), then the equation (4.2) for \( v \) is equivalent to the equation (4.4) for \( \tau_7 \). However since these conditions are required for the consistency of (4.2), the equation (4.2) is in fact equivalent to (4.4) together with conditions (4.3) and (4.5). This is now something that we can solve, however for that we will need the following lemma.

**Lemma 4.3** \([4]\). Let \( M \) be a \( n \)-dimensional Riemannian manifold. Then the metric \( g \) satisfies

\[
\nabla_a \nabla_b h = \lambda g_{ab}
\]

for functions \( h \) and \( \lambda \) if and only if the underlying smooth manifold is \((a, b) \times N\), for a \((n - 1)\)-dimensional manifold \( N \), with a warped product metric \( g \) given by

\[
g = \frac{dh^2}{|\nabla h|^2} + |\nabla h|^2 \hat{g}
\]

where \( \hat{g} \) is the induced metric on the \((n - 1)\)-dimensional slices.

**Theorem 4.4.** Consider a deformation of \((\varphi, g)\) with \( T_{ab} \) lying in the strict class \( W_1 \oplus W_7 \) to \((\tilde{\varphi}, \tilde{g})\) with \( \tilde{T}_{ab} \) lying in the class \( \tilde{W}_1 \). Then, such a deformation exists if and only if \( M \) is a warped product manifold \( I \times_f N \) for some interval \( I \) and 6-dimensional manifold \( N \). There are three cases:

1. If \( v^3 = \frac{3}{\tau_7} \tau_7 \), then for a 6-dimensional metric \( \tilde{g} \), the original metric \( g \) and new torsion \( \tilde{\tau}_1 \) must be given by

\[
g = \frac{\tau_7^2}{|\tau_7|^2} + \tau_1^{-10} |\tau_7|^2 \tilde{g}
\]

\[
\tilde{\tau}_1 = 0
\]

2. If \( v^3 = -\frac{3}{\tau_7} \tau_7 \), then

\[
g = \frac{\tau_7^2}{|\tau_7|^2} + \frac{\tau_1^2}{\tau_7} |\tau_7|^2 \tilde{g}
\]

\[
\tilde{\tau}_1^3 = \frac{\tau_7}{8} \left( \frac{\tau_1^2}{\tau_7} + 9 |\tau_7|^2 \right).
\]
(3) If \( v^i = \frac{f}{\tau_1} \tau_7 \) where \( f^2 = \frac{9 \Delta \tau_1^2}{A \tau_1^3 - 1} \) for an arbitrary constant \( A \), then

\begin{align*}
\text{(4.17)} & \quad g = \frac{\tau_1^2}{|\tau_1|^2} + \frac{(f - 3)\Delta \tau_1}{f^3 (f + 3)^2} |\tau_1|^2 \hat{g} \\
\text{(4.18)} & \quad \hat{\tau}_1^3 = \frac{1}{64 A f (f^2 - 9)} \left( 1 + \frac{9 \Delta \tau_1 |\tau_1|^2}{A \tau_1^3 - 1} \right).
\end{align*}

**Proof.** From the expression for \( \nabla v \) (4.2), we find that

\begin{equation}
\text{(4.19)} \quad dM = \frac{3}{2} \left( V^3 |\tau_1|^2 \tau_1 + V^2 |\tau_1|^2 + V \tau_1 + 1 \right) \tau_7
\end{equation}

However, from (4.10),

\begin{equation}
\text{(4.20)} \quad dM = 2 V |\tau_1|^2 dV + 2 V^2 \tau_7 \cdot (\nabla \tau_7)
\end{equation}

So equating (4.19) and (4.20), and using (4.4), we get an expression for \( dV \):

\begin{equation}
\text{(4.21)} \quad dV = \frac{1}{6} V (3 - V^2 \tau_7^2) \tau_7
\end{equation}

Now consider \( d(V \tau_1) \). Using (4.21) and the fact that that \( d \tau_1 = \tau_1 \tau_7 \) we find that

\begin{equation}
\text{(4.22)} \quad d(V \tau_1) = \frac{1}{6} (9 V \tau_1 - V^3 \tau_3) \tau_7
\end{equation}

Let \( f = V \tau_1 \), so we have the equation

\begin{equation}
\text{(4.23)} \quad df = \frac{1}{6} (9 f - f^3) d \log \tau_1.
\end{equation}

Consider the constant solutions first. Since \( \tau_1 \) is non-zero, the solution \( f = 0 \) implies that \( V = 0 \), and hence \( v = 0 \), so this is a degenerate solution. The non-trivial constant solutions are \( f = \pm 3 \). If \( f = V \tau_1 \equiv 3 \), then in (4.5), \( \hat{\tau}_1 = 0 \). In this case, from (4.4), we find

\begin{equation}
\text{(4.24)} \quad \nabla \tau_7 = \left( \frac{\tau_1^2}{3} - |\tau_1|^2 \right) g + 5 \tau_7 \otimes \tau_7
\end{equation}

Now using (4.24) and \( d \tau_1 = \tau_1 \tau_7 \) we can relate the metric to the Hessian of a function via the following expression

\begin{equation}
\text{(4.25)} \quad \nabla_a \nabla_b \left( \tau_1^{-5} \right) = -5 \tau_1^{-5} \left( \frac{\tau_1^2}{3} - |\tau_1|^2 \right) g_{ab}.
\end{equation}

Hence by Lemma 4.3 we get (4.13). Now consider the solution \( f = V \tau_1 \equiv -3 \). In this case, from (4.4), we find

\begin{equation}
\text{(4.26)} \quad V^2 |\tau_1|^2 = -\frac{8 \tau_1^3 V^3}{27} - 1.
\end{equation}

Using the fact that \( V = -\frac{3}{\tau_1} \) and (4.20) in (4.4) we have

\begin{equation}
\text{(4.27)} \quad \nabla \tau_7 = \frac{2}{9 \tau_1} (2 \tau_1^3 - \tau_3) g - \tau_7 \otimes \tau_7
\end{equation}

However, in the \( W_1 \oplus W_7 \) class, \( \tau_7 = d \log \tau_1 \), so we rewrite (4.27) in terms of \( \tau_1 \):

\begin{equation}
\text{(4.28)} \quad \nabla_a \nabla_b \tau_1 = \frac{2}{9} (2 \tau_1^3 - \tau_3) g_{ab}
\end{equation}

Hence, by Lemma 4.3 the metric must be (4.15). Using (4.26), we also find that \( \hat{\tau}_1 \) satisfies (4.16).
Suppose now \( f \) is non-constant. Then whenever the right-hand side of (4.23) is non-zero - that is, \( f \neq \pm 3 \), we can separate variables in (4.23). Integrating, we obtain

\[
f^2 = \frac{9A\tau_1}{A\tau_1^3 - 1}
\]

for some positive constant \( A \). Suppose now \( f = \pm 3 \) at some point, then if \( f \) is non-constant, we must have \( f^2 \to 9 \) in (4.29) for some values of \( A \) and \( \tau_1 \). However, we can see from (4.29) that this happens if and only if \( |\tau_1| \to \infty \), but \( \tau_1 \) is smooth, so in fact, either \( f \equiv \pm 3 \) (so these are singular solutions of (4.23) or \( f \) is nowhere equal to \( \pm 3 \) and is given by (4.29) everywhere. We have already covered the constant cases above, so now we can assume that (4.29) holds everywhere. In particular from (4.29) we also get the relations

\[
V^2 = \frac{9A\tau_1}{A\tau_1^3 - 1} \quad \text{and} \quad V^3 = Af (f^2 - 9)
\]

From (4.29) find can find \( |\tau_1|^2 \) in terms of \( f \) and \( \tilde{\tau}_3 \). Also, note that from (4.23) we get

\[
\nabla_a \nabla_b f = -\frac{1}{6} f (f^2 - 9) \nabla_a (\tau_3) b + \frac{3(f^2 - 3)}{f(f^2 - 9)} \nabla_a f \nabla_b f.
\]

So overall, we can rewrite (4.4) as an equation for \( f \) in the form

\[
\nabla_a \nabla_b f = P(f)g_{ab} + Q(f) \nabla_a f \nabla_b f
\]

for some functions \( P(f) \) and \( Q(f) \). The exact form of \( P(f) \) is not very important, but \( Q(f) \) is given by

\[
Q(f) = \frac{2(f^2 - 3f - 6)}{f(f^2 - 9)}
\]

In order to reduce (4.31) to the form (4.11) we need to find a function \( F(f) \) that satisfies

\[
\frac{d^2 F}{df^2} + \frac{dF}{df}Q(f) = 0.
\]

For such an \( F \), the Hessian would be proportional to the metric. Let \( G = \frac{dF}{df} \), then by separation of variables, we solve (4.32) for \( G \) and get the solution

\[
G = \frac{6(f - 3)\frac{\hat{\tau}}{f}}{f\frac{\hat{\tau}}{(f + 3)\frac{\hat{\tau}}{f}}}
\]

Note that we have chosen a constant factor for convenience. Since \( f \) is nowhere vanishing and is nowhere equal to \(-3\), this can be integrated further to find \( F \). Hence for this \( F \), \( \nabla_a \nabla_b F \) is proportional to \( g_{ab} \). Therefore by Lemma 4.3 the metric must be a warped product of the form

\[
g = \frac{1}{|\nabla F|^2}dF^2 + |\nabla F|^2 \hat{g}
\]
Note that \( \frac{dF}{|\nabla F|} = \frac{df}{|\nabla f|} = \frac{\tau_2}{|\tau_2|} \), and therefore, using (4.23), we get
\[
|\nabla F|^2 = \left( \frac{dF}{df} \right)^2 |\nabla f|^2 = \frac{1}{36} G^2 |\tau_7|^2 f^2 (f^2 - 9)^2
\]
\[
= \frac{(f - 3)^{4/3}}{f^{2/3}} |\tau_7|^2
\]
Thus we obtain the metric (4.17). From (4.5) we get the expression (4.18) for \( \tilde{\tau}_1 \) by substituting (4.30).

It was shown in by Cleyton and Ivanov in [5] that by considering a warped product of an open interval over a 6-dimensional nearly Kähler manifold, it is possible to obtain 7-dimensional manifolds with \( G_2 \)-structures. Moreover, it has been shown that for such a construction the \( \tau_{14} \) torsion component (\( \tau_2 \) in the notation of [5]) will always vanish, and moreover, it is possible to find parameters such that the \( \tau_{27} \) torsion component (\( \tau_3 \) in [5]) will also become zero, leaving only \( \tau_1 \) and \( \tau_7 \) non-zero (\( \tau_0 \) and \( \tau_1 \) in their notation), which are given in terms of the warp factor. In particular, if the metric is \( dt^2 + h(t)^2 \hat{g} \) for warp factor \( h > 0 \), then, using our conventions for the torsion components,
\[
(4.35) \quad \tau_1 = h^{-1} \sigma \sin \theta \quad \text{and} \quad \tau_7 = h^{-1} (\sigma \cos \theta - h') dt
\]
where \( \sigma \) is a constant related to the scalar curvature of the 6-dimensional manifold and \( \theta (t) \) satisfies \( \theta' = h^{-1} \sigma \sin \theta \). These are precisely the kind of manifolds that appear as solutions in Theorem 4.4 - warped product 7-manifolds with torsion in \( W_1 \oplus W_7 \). The warp factors in Theorem 4.4 have to be consistent with the expressions (4.35), and so for each case in Theorem 4.4 we get a system of two first-order ODEs for \( h \) and \( \theta \). Given appropriate initial conditions we can say that solutions exist, however the analysis of these solutions is something to be investigated further. Therefore, we can construct examples of 7-manifolds with a conformally nearly parallel \( G_2 \)-structure such that a non-infinitesimal deformation in \( \Lambda_3^1 \) gives a \( G_2 \)-structure in a strictly smaller torsion class. Moreover, applying a conformal transformation (4.39) we can obtain a nearly parallel \( G_2 \)-structure, for which a combination of a conformal transformation and a \( A_3^7 \) deformation lead to another nearly parallel \( G_2 \)-structure. As we have seen, there are various non-trivial relationships between different \( G_2 \)-structures and it will be a subject of further study whether it is possible find transformations between other \( G_2 \) torsion classes. In particular, so far we only have a grasp on deformations that lie in \( A_3^1 \) and \( A_3^7 \), however it is likely that deformations in \( A_3^27 \) could yield the most interesting results.

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