Renormalizability of Φ-derivable approximations in scalar ϕ⁴ theory

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We discuss the renormalizability of Φ-derivable approximations in scalar ϕ⁴ theory in four dimensions. The formalism leads to self-consistent equations for the 2-point and the 4-point functions which are plagued by ultraviolet divergences. Through a detailed analysis of the one and two-loop self-energy skeletons, we show that both equations can be renormalized simultaneously and determine the corresponding counterterms. These insure the elimination of ultraviolet divergences both at zero and finite temperature.

Self-consistent, “Φ-derivable”, approximations were introduced many years ago in the context of the non-relativistic many body problem \([1, 2]\), and have been extended to field theory \([3, 4]\). They have been found appropriate to treat systems for which the quasiparticle picture is a good starting point and have recently been applied in this spirit to calculate equilibrium thermodynamics of the quark-gluon plasma \([5]\). They are also being used to study the dynamics of quantum fields out of equilibrium \([6]\).

The main difficulty in implementing such approximations in quantum field theory is their renormalization: from the point of view of perturbation theory, the equations that one is led to solve effectively resum infinite sets of Feynman diagrams, and the existence of a procedure for constructing the counterterms needed to eliminate the corresponding divergences is not obvious. This problem becomes particularly acute at finite temperature: While, on general grounds, one expects ultraviolet divergences to be unaffected by the temperature (see e.g. \([7]\)), in self-consistent approximations temperature dependent divergences often do appear, thus casting doubts on the renormalizability (see in particular \([8]\)).

This issue has been addressed recently by van Hees and Knoll in a series of papers \([8, 9]\). The strategy put forward in \([8]\) is based on an expansion of the propagator around the vacuum self-consistent solution, and relies on the real time formalism. The elimination of the divergences proceeds through the BPHZ subtraction scheme. This leads to a systematic and practical renormalization scheme where temperature dependent counterterms never appear. However the dissymmetrical treatment of the vacuum sector and the finite temperature one is unsatisfactory: It hides the fact that the rearrangement of divergences which appears to be necessary at finite temperature is also needed in most renormalization schemes already at zero temperature. And it does not bring out the specific relation between the bare and the renormalized parameters that emerges in Φ-derivable approximations. This makes it difficult, e.g., to compute the β-function, or resolve the apparent discrepancy between the results of Refs. \([5, 6]\) and \([9]\).

We have therefore reconsidered the problem from a more general perspective. Our derivations use the imaginary time formalism, making the connection with conventional equilibrium field theory transparent, and allowing for a simultaneous treatment of the vacuum sector and the finite temperature one: once renormalization is done properly at zero temperature, the extension to finite temperature is straightforward.

The central quantity in Φ-derivable approximations is \(\Phi[D]\), the sum of the 2-particle-irreducible “skeleton” diagrams, a functional of the full propagator \(D\), which enters the expression of the thermodynamical potential. From \(\Phi[D]\) we may calculate the 2-point function (the self-energy) by functional differentiation:

\[
\delta \Phi[D]/\delta D = \frac{1}{2} \Pi.
\]

This relation, together with Dyson’s equation (\(D_0\) denotes the bare propagator):

\[
D^{-1} = D_0^{-1} + \Pi[D],
\]

defines the physical propagator and self-energy in a self-consistent way. We shall refer to Eq. \((2)\), with \(\Pi\) given by \((1)\), as the “gap equation”. A further differentiation of \(\Phi\) with respect to \(D\) yields the 2-particle irreducible kernel

\[
\Lambda(K, P) = \frac{2}{\delta D} \frac{\delta \Pi(K)}{\delta D(P)} = \Lambda(P, K)
\]

of a Bethe-Salpeter (BS) equation

\[
\Gamma(K, P) = \Lambda(K, P) - \frac{1}{2} \int_Q \Gamma(K, Q) D^2(Q) \Lambda(Q, P)
\]

that allows the calculation of the 4-point function \(\Gamma(K, P)\) with a degree of accuracy comparable with that used in the determination of the propagator. Φ-derivable approximations are obtained by selecting a class of skeletons in \(\Phi[D]\) and calculating \(\Pi\) and \(\Gamma\) from the equations above. As we shall see, the renormalizability of such approximations relies on the possibility to simultaneously renormalize \(\Pi\) and \(\Gamma\). In particular, the BS equation is needed to determine coupling constant counterterms which eliminate some divergences of the self-energy.
We consider in this paper a massive scalar field theory with a $\varphi^4$ interaction:

$$\mathcal{L} = \frac{1}{2} (\partial \varphi_0)^2 - \frac{1}{2} m_0^2 \varphi_0^2 - \frac{1}{4!} g_0 \varphi_0^4,$$

and include in $\Phi$ only the 2-loop and 3-loop skeletons (the corresponding self-energy diagrams are displayed in Fig. 1). This allows us to introduce the generic difficulties, deferring the systematic discussion of the general case to a forthcoming publication [12]. In four dimensions, usual power counting indicates that only the 2-point and the 4-point functions are divergent. The divergent parts can be absorbed in local counterterms corresponding to a redefinition of the parameters of the lagrangian. We assume standard relations [7] between the renormalized and bare parameters: $\varphi_0 = \sqrt{Z} \varphi$, $Z m_0^2 = m^2 + \delta m^2$, $Z^2 g_0^2 = g^2 + \delta g^2$, and $\delta Z = Z - 1$.

The gap equation corresponding to the one-loop skeleton (the “tadpole”) reads:

$$\Pi = \frac{g_0^2}{2} \int D(P) + \delta m^2,$$

where $D^{-1}(P) = P^2 + m^2 + \Pi$. The notation $\int_D$ stands for an Euclidean integral over the 4-momentum $P$. At finite temperature, it should be understood as an integral over the 3-momentum together with a sum over Matsubara frequencies. The approximation corresponding to Eq. (6) is a simple self-consistent mean field approximation that has been treated many times before (see for instance [11, 12, 13]). We present it here in a way which will prepare for the more complicated two-loop example that we shall discuss next.

The self-energy $\Pi$ is here a constant, and a single mass counterterm $\delta m^2$ is in principle sufficient to eliminate the ultraviolet divergence. Calculating the integral in Eq. (6) in dimensional regularization we get:

$$\mu^{2\epsilon} \int_D D(P) = -\frac{1}{16\pi^2} (m^2 + \Pi) \left\{ \frac{1}{\epsilon} - \ln \frac{m^2 + \Pi}{\mu^2} + 1 \right\},$$

where $\mu^2 \equiv 4\pi e^{-\gamma} \mu^2$. At this point, one could be tempted to absorb the whole divergence in $\delta m^2$, i.e., set:

$$\delta m^2 = \frac{g_0^2}{32\pi^2} (m^2 + \Pi) \frac{1}{\epsilon}.$$

But this is not a good strategy. If, for instance, the calculation is done at finite temperature, $\Pi$ depends on the temperature, and so does the counterterm $\delta m^2$, which we want to avoid.

In fact, when analyzing the gap equation Eq. (9) in terms of perturbation theory, on finds that its solution effectively resums an infinite set of Feynman diagrams, some of which contribute to the renormalization of the coupling constant. This is best seen by imagining solving this equation by iteration, a procedure which also defines an explicit construction of the counterterms. To do so, we set $D(P) = D_0(P) = (P^2 + m^2)^{-1}$ in the r.h.s. of Eq. (6); one then obtains a first approximation to $\Pi$ on the l.h.s. which can then be used in the r.h.s., and so on. At each iteration, $\delta m^2$ can be adjusted to absorb the overall divergence. But is is easy to see that, starting at the second iteration, a subdivergence appears corresponding to a coupling constant renormalization that needs to be subtracted before adjusting $\delta m^2$ (An illustration of the phenomenon in the less trivial example of the sunset diagram is given in Fig. 2 below). New such subdivergences appear in each iteration, and to take them into account, a term of the form $(\delta g^2/2) \int_P D(P)$ should be added in the r.h.s. of Eq. (6). Equivalently $g^2$ should be replaced by $g_0^2 = g^2 + \delta g^2$ in Eq. (9). As we shall see, $\delta g^2$ is precisely the counterterm that is needed to make finite the BS equation, to which we now turn.

With, here, $\Lambda = g_0^2 = g^2 + \delta g^2$, the BS equation reads:

$$\Gamma = g_0^2 - \frac{g_0^2}{2} \int_P D^2(P),$$

where $\Gamma$ is the renormalized 4-point function, and $\delta g^2$ is chosen so as to absorb the divergence of the integral. Note that this divergence does not depend on the mass (nor therefore on $\Pi$), and for the purpose of determining $\delta g^2$ we could as well use an auxiliary 4-point function $\Gamma_0$ solution of Eq. (10) with $D$ replaced by $D_0$. Clearly, $\Gamma_0$ differs from $\Gamma$ by a finite quantity only.

We now return to the gap equation, Eq. (6) with $g^2$ replaced by $g_0^2$, itself determined in terms of $\Gamma_0$ by the BS equation, and show that its solution, $\Pi$, can be made finite with a counterterm $\delta m^2$ independent of $\Pi$. To this aim, we write $D = D_0 + \delta D$, where

$$\delta D(P) = D_0(P) (-\Pi) D_0(P) + D_\Pi(P),$$

and $D_\Pi(P)$ starts at order $\Pi^2$, so that the integral $\int_P D_\Pi(P)$ is finite. Then, we set

$$\tilde{\Pi}_2 = \frac{g_0^2}{2} \int_P D_0(P) + \delta m^2 \quad \tilde{\Pi}_0 = \frac{g_0^2}{2} \int_P \delta D(P),$$

where only $\tilde{\Pi}_0$ depends on $\Pi$, and when $\Pi$ is solution of the gap equation, $\Pi = \tilde{\Pi}_0 + \tilde{\Pi}_2$. Next, one uses the BS equation to eliminate $g_0^2$ in the defining equation for $\tilde{\Pi}_0$:

$$\tilde{\Pi}_0 = \frac{\Gamma_0}{2} \int_P \delta D(P) + \frac{\Gamma_0}{2} \tilde{\Pi}_0 \int_Q D_0^2(Q).$$
At this point, we have achieved our goal: while both integrals in Eq. (12) are divergent, it is easily verified that no divergence involves \( \Pi \) when \( \Pi \) is solution of the gap equation (so that we can replace \( \tilde{\Pi}_0 \) by \( \Pi - \tilde{\Pi}_2 \) in the r.h.s. of Eq. (12)). The final gap equation reads:

\[
\Pi = \frac{\Gamma}{2} \int_P D_3(P) + \tilde{\Pi}_2 \left( 1 - \frac{\Gamma}{2} \int_Q D_0^2(Q) \right). \tag{13}
\]

The divergence in the last term involves only \( D_0 \) and can be absorbed in \( \delta m^2 \). For instance, in dimensional regularization with minimal subtraction \( \delta m^2 = g_0^2m^2/(32\pi^2\epsilon) \). The factor multiplying \( \tilde{\Pi}_2 \) is \( \Gamma/g_0^2 \) (from the BS equation), so that the resulting expression is indeed finite (but depends on the scheme).

At finite temperature, the one-loop integral in Eq. (12) can be split into a vacuum integral and a 3-dimensional integral involving a statistical factor and giving the following new contribution to \( \Pi \):

\[
\tilde{\Pi}_1 = \frac{g_0^2}{2} \int_P \frac{n(\varepsilon_p)}{\varepsilon_p}, \tag{14}
\]

where \( n(\varepsilon_p) = 1/(e^{\beta \varepsilon_p} - 1) \) and \( \varepsilon_p = \sqrt{p^2 + m^2 + \Pi} \). Eq. (14) involves a temperature dependent counterterm. However the same manipulation as above, with \( \Pi_0 \) replaced by \( \Pi_0 + \tilde{\Pi}_1 \) in Eq. (12), and the use of the \( T = 0 \) counterterms which are calculated entirely from \( D_0 \), eliminate it, leaving a finite gap equation. In the mass-shell subtraction scheme where \( m \) is the physical mass and the vacuum sector is trivial (\( \Pi = 0 \)), this equation is simply:

\[
\Pi = \frac{\Gamma}{2} \int_P D_3(P) + \frac{\Gamma}{2} \int_P \frac{n(\varepsilon_p)}{\varepsilon_p}. \tag{15}
\]

Consider now the 2-loop skeleton (the “sunset”), and the corresponding gap equation:

\[
\Pi(K) = -\frac{g_0^4}{6} \int_P \int_Q D(P)D(Q)D(K + P + Q)
+ \frac{\Delta g^2 + \delta g^2}{2} \int_P D(P) + \Delta m^2 + \delta m^2 + K^2 \delta Z. \tag{16}
\]

That this expression can be made finite with the indicated counterterms follows from a standard analysis: the counterterm \( \Delta g^2 + \delta g^2 \) cancels the subdivergences, while \( \Delta m^2 + \delta m^2 \) and \( \delta Z \) cancel the remaining global divergences (the reason behind the special writing of the counterterms will become clear shortly). The argument assumes, in agreement with Weinberg’s theorem, that the repeated insertions of the self-energy in the propagators, as generated by iterating the gap equation, do not change in an essential way the asymptotic form of these propagators, expected to be typically of the form: \( \Pi(K) \simeq K^2 F(\ln K) \) for \( K \gg m \). Note that the coupling constant counterterm enters only a one-loop diagram: at this order of the sunset expansion, there is no renormalization of the vertices of the sunset diagram. Such renormalizations would involve skeletons whose lowest perturbative order is \( g^6 \).

To proceed to the renormalization of the BS equation we need to take into account the fact that the self-energy \( \Pi(K) \) modifies the asymptotic behavior of the propagator, as indicated above. We then write \( \Pi = \Pi_2 + \Pi_0 \), where \( \Pi_2(K) \) is finite and contains the exact asymptotic behavior of \( \Pi(K) \), and \( \Pi_0(K) \) grows at most logarithmically at large \( K \) (\( \Pi_0(K) \) and \( \delta Z \) can be obtained by solving the gap equation with \( m = 0 \)). We set:

\[
D_{-2}(K) = (K^2 + m^2 + \Pi_2)^{-1}. \tag{17}
\]

\( D_{-2} \) will play here the role of \( D_0 \) in the one-loop example (note that \( D_{-2} \) takes care of field renormalization). Thus, we define an auxiliary 4-point function \( \Gamma_0 \) as the solution to the BS equation (12) with \( D_{-2} \) as propagators. The equation for \( \Gamma_0(K, P) \) contains all the divergences of that for the full 4-point function \( \Gamma(K, P) \), and the renormalizations of \( \Gamma_0 \) and \( \Gamma \) involve therefore the same counterterms. To determine these, we first write the kernel of the BS equation as \( \Lambda_0(K, P) + \delta g^2 \), with:

\[
\Lambda_0(K, P) = \Delta g^2 - g^4 \int_Q D_{-2}(Q)D_{-2}(K + P + Q), \tag{18}
\]

and \( \Delta g^2 \) is chosen so as to make \( \Lambda_0(K, P) \) finite. The counterterm \( \delta g^2 \) is then adjusted, as in Eq. (10), so as to eliminate the divergence of the equation:

\[
\Gamma_0(0, 0) = \frac{\delta g^2 + \Lambda_0(0, 0)}{2} \int_P \Gamma_0(0, P)D_{-2}(P)[\delta g^2 + \Lambda_0(P, 0)], \tag{19}
\]

where \( \Gamma_0(0, 0) \) is fixed by a renormalization condition, and \( \Gamma_0(0, P) \) can be obtained from the following finite equation:

\[
\Gamma_0(0, P) - \Gamma_0(0, 0) = \Lambda_0(0, P) - \Lambda_0(0, 0) - \int_Q \Gamma_0(0, Q)D_{-2}(Q) \{ \Lambda_0(Q, P) - \Lambda_0(Q, 0) \}. \tag{20}
\]

\( \Lambda_0(Q, P) - \Lambda_0(Q, 0) \sim 1/Q^2 \) for \( Q^2 \gg P^2 \), so that the integral over \( Q \) is indeed finite.)
By combining Eqs. (20) and (19) one gets \( \Gamma_0(0, P) \) in terms of \( \Lambda_0(Q, P) \) and \( \delta g^2 \). The result is in fact nothing but Eq. (19) with \( K = 0, D \) replaced by \( D_{-2} \) and \( \Lambda(0, P) \) replaced by \( \Lambda_0(0, P) + \delta g^2 \). We shall now use this expression of \( \Gamma_0(0, P) \) to eliminate the vertex subdivergences from the gap equation. To proceed, we write again \( D = D_{-2} + \delta D \), where \( \delta D = D_{-2}[-\Pi_0 D_{-2} + D_r] \) contains all the dependence on \( \Pi_0 \), and call \( \Pi_2 \) the r.h.s. of Eq. (10) evaluated with \( D \) replaced by \( D_{-2} \). Furthermore, we set \( \Pi_0(K) = (1/2) \int_p [\Lambda_0(K, P) + \delta g^2 \delta D(P)] \). The gap equation is \( \Pi = \Pi_2 + \Pi_0 + \Pi_r \), where \( \Pi_r(K) \) is finite and goes as \( 1/K^2 \) at large \( K \). We then write \( \Pi_0(K) = \Pi_0(K) - \Pi_0(0) + \Pi_0(0) \), where \( \Pi_0(K) - \Pi_0(0) \) is finite, and we express \( \Lambda_0(0, P) + \delta g^2 \) in the defining equation for \( \Pi_0(0) \) in terms of \( \Gamma(0, P) \). We get:

\[
\Pi_0(0) = \frac{1}{2} \int_P \Gamma(0, P) \left[ \delta D(P) + D_{-2}^2(\Pi_0(0)) \right]. \tag{21}
\]

For \( \Pi \) solution of the gap equation, we can set \( \Pi_0 = \Pi - \Pi_2 - \Pi_r \) in the r.h.s., and verify that the divergent terms linear in \( \Pi_0 (= \Pi - \Pi_2) \) cancel, as anticipated. Using the resulting expression of \( \Pi_0 \) we obtain the solution of the gap equation for \( K = 0 \) in the form:

\[
\Pi(0) = \frac{1}{2} \int_P \Gamma(0, P) \left\{ \Pi_2(P) - \Pi_2(0) - \Pi_r(0) \right\} D_{-2}^2(\Pi) + \frac{1}{2} \int_P \Gamma(0, P) D_r(P) + \Pi_r(0) + \Pi_2(0). \tag{22}
\]

To isolate the remaining divergences, we write

\[
\Pi_2(K) = \Pi_2'(K) + \delta m^2 + \frac{\delta g^2}{2} \int_P D_{-2}^2(P) \tag{23}
\]

where \( \Pi_2'(K) \) is finite (owing to the counterterms \( \Delta m^2 \) and \( \Delta g^2 \)). The difference \( \Pi_2(K) - \Pi_2'(K) \) is logarithmic at large \( K \) and contributes to a divergence of the first integral of Eq. (22). But neither this divergence, nor those coming from the counterterms displayed in Eq. (23), depend on the solution \( \Pi_0 \) of the gap equation, and they can be absorbed in the mass counterterm \( \delta m^2 \). This completes the determination of the counterterms which, as we have seen, can all be calculated from \( D_{-2} \).

At this point we emphasize a special feature of \( \Phi \)-derivable approximations: As we have indicated earlier, the renormalization of the two-loop skeleton generates a coupling constant counterterm for the one-loop skeleton, but not for its own vertices. This is a general feature, which persists in higher orders in the loop-expansion of \( \Phi \). Correspondingly, the \( \beta \)-function deviates from that given by perturbation theory beyond the perturbative orders explicitly included in the skeletons \[14\]. For instance, in the present example, the perturbative \( \beta \)-function is correctly reproduced to order \( g^4 \) (when one adds the two contributions of the tadpole and sunset diagrams), but deviates at order \( g^6 \).

The extension of the previous analysis to finite temperature brings no new ultraviolet difficulty. Again, we can separate each loop integral in the sunset into a “vacuum” contribution, and a contribution containing a statistical factor. The final expression for the self-energy takes then a form similar to that at zero temperature, and may be written as \( \Pi = \Pi_2 + \Pi_r + \Pi_0 + \Pi_1 + \Pi_3 \). The first contribution, \( \Pi_3 \), is the same as before and does not depend on the temperature. The last contribution, \( \Pi_3(K) \), is one in which each of the loop integrals contains a statistical factor. It is finite and decreases as \( 1/K^2 \) at large \( K \); thus it is not involved in any divergent term, and it can be regarded as a simple correction to \( \Pi_r \). Finally, \( \Pi_0 \) is defined as at zero temperature, and \[14\]:

\[
\Pi_1(K) = \frac{1}{2} \int_{p_0, p} \left[ \Lambda_0(K, P) + \delta g^2 \right] \rho(p_0, p) n_{|p_0|}^2 \sigma_{p_0} \tag{24}
\]

where \( \rho(p_0, p) \) is the spectral function of the propagator \( D, \sigma_{p_0} \) is the sign of \( p_0 \), and the integral runs over the real \( p_0 \) axis. As in the one-loop example, we can combine \( \Pi_1 \) with \( \Pi_0 \) and show that the zero temperature coupling constant counterterms eliminate the apparent divergence depending on the temperature.

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