ONE-PHASE FREE BOUNDARY PROBLEMS ON RCD METRIC MEASURE SPACES

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Abstract. In this paper, we consider a vector-valued one-phase Bernoulli-type free boundary problem on a metric measure space \((X, d, \mu)\) with Riemannian curvature-dimension condition \(RCD(K, N)\). We first prove the existence and the local Lipschitz regularity of the solutions, provided that the space \(X\) is non-collapsed, i.e. \(\mu\) is the \(N\)-dimensional Hausdorff measure of \(X\). And then we show that the free boundary of the solutions is an \((N-1)\)-dimensional topological manifold away from a relatively closed subset of Hausdorff dimension \(\leq N - 3\).

Contents

1. Introduction 1
   1.1. The Bernoulli-type free boundary problems on Euclidean spaces 3
   1.2. Free boundary problems in RCD-spaces and the main results 4
2. Preliminaries 6
   2.1. RCD\((K, N)\) metric measure spaces and their calculus 7
   2.2. Non-collapsed RCD\((K, N)\) metric measure spaces 12
   2.3. Sets of finite perimeter and the reduced boundary 13
3. Existence of a minimizer 15
4. Hölder continuity of local minimizers 16
5. Lipschitz continuity of local minimizers 19
   5.1. Mean value inequality 20
   5.2. Lipschitz continuity of local minimizers of \(J_Q\) 23
6. Local finiteness of perimeter for the free boundary 26
   6.1. Nondegeneracy 26
   6.2. Density estimates near the free boundary 30
   6.3. Local finiteness of perimeter 32
7. Compactness and the Euler-Lagrange equation 35
8. Regularity of the free boundary 42
Appendix A. Weiss-type monotonicity on cones 52
References 54

1. Introduction

Since the pioneer work of Alt-Caffarelli [AC81], Dirichlet problems with free boundary on Euclidean spaces have been extensively studied. Consider the critical points of the one-phase Bernoulli energy functional:

\[
J(u) := \int_{\Omega} (|\nabla u|^2 + \chi_{\{u>0\}}) \, dx,
\]

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where $\Omega \subset \mathbb{R}^n$ is a bounded open set. The domain $\Omega_u := \{x \in \Omega \mid u(x) > 0\}$ is a priori unspecified and $\partial \{u > 0\} \cap \Omega$ is the free boundary. From [AC81], the fundamental results about existence and regularity of minimizers of $J$ and regularity of free boundaries were established (see also [Caf87, Caf89, Caf88] for a view of viscosity solutions). Recently, the studies on free boundary problems have been extended to the fully nonlinear uniformly elliptic operators [DSFS15] and uniformly elliptic operators with variable coefficients [Tre20a, Tre20b]. In the meantime, the vector-valued Bernoulli-type free boundary problem have been systematically studied by Caffarelli-Shahgholian-Yeressian [CSY18], Mazzoleni-Terracini-Velichkov [MTV17, MTV20] and Kriventsov-Lin [KL18]. We refer the readers to surveys [FS15, F18, DSFS19, CS20] and their references for recent developments of the free boundary problems in the Euclidean settings.

In this paper, we will extend the study on one-phase Bernoulli-type free boundary problems from the Euclidean setting to the setting of non-smooth spaces satisfying a synthetic notion of lower bounds of Ricci curvature. More precisely, letting $(X,d,\mu)$ be a metric measure space (a complete metric space $(X,d)$ equipped with a Radon measure $\mu$ with supp$(\mu) = X$), we assume that it satisfies the Riemannian curvature-dimension condition $RCD(K,N)$ for some $K \in \mathbb{R}$ and $N \in [1, +\infty)$. The main examples in the class of $RCD(K,N)$ spaces include the Ricci limit spaces in the Cheeger-Colding theory [CC96, CC97, CC00] and finite dimensional Alexandrov spaces with curvature bounded from below (see [Pet11] and [ZZ10, Appendix A]). The parameters $K \in \mathbb{R}$ and $N \in [1, +\infty]$ play the role of “Ricci curvature $\geq K$ and dimension $\leq N$” in Riemannian geometry. The theory of $RCD(K,N)$ metric measure spaces and their geometric analysis have fast and remarkable developments, see [Amb18] for a recent survey on this topic.

Let $(X,d,\mu)$ be an $RCD(K,N)$ metric measure space for some $K \in \mathbb{R}$ and $N \in [1, +\infty)$, and let $\Omega \subset X$ be a bounded domain. The Bernoulli-type energy functional is given by

\begin{equation}
J_Q(u) = \int_{\Omega} (|\nabla u|^2 + Q(\{u > 0\})) d\mu,
\end{equation}

where $Q \in L^{\infty}(\Omega) := L^{\infty}(\Omega,\mu)$ with

\begin{equation}
0 < Q_{\min} \leq Q(x) \leq Q_{\max} < +\infty \quad \mu - \text{a.e. } x \in \Omega
\end{equation}

for two positive real numbers $Q_{\min}$ and $Q_{\max}$. According to [Che99], it is now known that the Sobolev space $W^{1,2}(\Omega)$ is well-defined. Given a boundary data $g \in W^{1,2}(\Omega, [0, +\infty)^m)$, we consider the minimization problem:

\begin{equation}
\min_{u \in \mathcal{A}_g} J_Q(u), \quad \mathcal{A}_g := \left\{ u \in W^{1,2}(\Omega, [0, +\infty)^m) \mid u - g \in W^{1,2}_0(\Omega, \mathbb{R}^m) \right\}.
\end{equation}

It is a cooperative vector-valued one-phase Bernoulli-type free boundary problem.

**Definition 1.1.** A map $u \in \mathcal{A}_g$ is called a local minimizer of $J_Q$ in (1.2) if there exists some $\varepsilon_u > 0$ such that $J_Q(u) \leq J_Q(v)$ for every $v \in \mathcal{A}_g$ with $d(u,v) < \varepsilon_u$, where

\begin{equation}
d(u,v) := \|u - v\|_{W^{1,2}(\Omega,\mathbb{R}^m)} + \|\chi_{\{u > 0\}} - \chi_{\{v > 0\}}\|_{L^1(\Omega)}.
\end{equation}

If $\varepsilon_u = +\infty$, we call that it is a minimizer of $J_Q$ in (1.2).

The fundamental problems include the existence and regularity of minimizers (or local minimizers) of $J_Q$ and the regularity of the free boundary $\partial \{\{u \mid u > 0\} \cap \Omega$.}
1.1. The Bernoulli-type free boundary problems on Euclidean spaces.

We first recall some classical results on this problem in the Euclidean setting, i.e., 
\((X, d, \mu) = (\mathbb{R}^n, d_{\text{Eucl}}, \mathcal{L}^n)\). In the seminal work of Alt-Caffarelli [ACS1], for the scalar case where \(m = 1\), they established the following:

- the existence of the minimizer of (1.4),
- Lipschitz continuity of any local minimizer \(u\), and
- when \(Q \in C^\alpha\), the free boundary \(\partial \{u > 0\} \cap \Omega\) is a \(C^{1,\alpha}\)-manifold away from a relatively closed subset \(S_u\) with \(\mathcal{H}^{n-1}(S_u) = 0\).

Nowadays, it is well-known that the singular set \(S_u\) has \(\dim_H(S_u) \leq n - k^*\) for some \(k^* \in \{5, 6, 7\}\) (by Weiss [Wei99], Caffarelli-Jerison-Kenig [CJK04], De Silva-Jerison [DSJ09] and Jerison-Savin [JS15]). Edelein-Engelstein [EE19] explored the rectifiable structure of the singular set \(S_u\).

Recently, the vector-valued case where \(m \geq 2\), of Bernoulli-type free boundary problem for local minimizers of \(J(u)\) have been systematically studied by Caffarelli-Shahgholian-Yeressian [CSY18], Mazzoleni-Terracini-Velichkov [MTV17, MTV20] and Kriventsov-Lin [KL18]. See also [Tre20a, Tre20b] for the uniformly elliptic operators with variable coefficients.

**Theorem 1.2** (Caffarelli-Shahgholian-Yeressian [CSY18]). Let \(m \geq 2\) and let \(\Omega \subset \mathbb{R}^n\) be a bounded domain. Suppose that \(Q \in L^\infty(\Omega)\) satisfies (1.3). For each \(g \in W^{1,2}(\Omega, [0, +\infty)^m)\), there exists a minimizer \(u \in \mathcal{A}_g\) of \(J_g\) in (1.2). Moreover, for any local minimizer \(u = (u_1, \ldots, u_m)\) of \(J_g\), the following properties hold:

1. (Lipschitz regularity) \(u\) is locally Lipschitz continuous on \(\Omega\).
2. (Local finiteness of perimeter and Euler-Lagrange equation) If \(Q \in C(\Omega)\), then the free boundary has locally finite perimeter, and hence the reduced boundary \(\partial_{\text{red}}(\{u > 0\})\) is well-defined. Furthermore, for \(\mathcal{H}^{n-1}\)-a.e. point \(x \in \Omega \cap \partial_{\text{red}}(\{u > 0\})\), \(i = 1, 2, \ldots, m\), and \(\eta > 0\), the non-tangential limit

\[
\lim_{y \to x, y \in \Omega \cap \partial_{\text{red}}(\{u > 0\})} \frac{|u_i(x) - u_i(y)|}{|y - x|} = 0,
\]

(here \(\nu_{\{u > 0\}}(x)\) is the outer normal to \(\{u > 0\}\) at \(x\)) exists, and we have the equations

\[
\Delta u_i = \frac{u_i(y)}{|u(y)|} - \frac{\nu_{\{u > 0\}}(x)}{|y - x|} \quad \text{for} \quad i = 1, 2, \ldots, m.
\]

3. (Regularity of free boundary) If \(Q \in C^\alpha(\Omega)\) and \(u\) is a minimizer of \(J_g\), then the singular part of the free boundary

\[
S_u := (\partial \{u > 0\} \setminus \partial_{\text{red}}(\{u > 0\})) \cap \Omega
\]

is a closed set in the relative topology of \(\Omega\) with \(\dim_H(S_u) \leq n - k^*\) for some \(k^* \in \{5, 6, 7\}\), and the regular part of the free boundary \(\partial_{\text{red}}(\{u > 0\}) \cap \Omega\) is locally \(C^{1,\beta}\) smooth for some \(\beta \in (0, \alpha]\) \((C^{\beta+1,\beta}\) smooth or analytic if \(Q\) is \(C^{k,\alpha}\) smooth or analytic, respectively).

There are many other important developments, for example, two-phase free boundary problems [ACF84, DPSV21a, DPSV21b], free boundary problems for almost minimizer [DT15, DET19], for the fully nonlinear uniformly elliptic operators [DSFS15] and for the fractional \(\alpha\)-Laplace operator [CRS10].

The theory of free boundary problems was used by Caffarelli-Lin [CL08] to the study of the nodal sets of harmonic maps into a singular space with non-positive curvature in the sense of Alexandrov.

In general, the theory of free boundary problems can be divided into two main steps. The first step is to establish the existence and the Lipschitz regularity of the solutions. The second step is to explore the structure of free boundary of these solutions, including the smoothness of its regular part and the size and structure
of its singular part. A blowup argument and an improved flatness property are applied to analyze the structure of the free boundary of solutions. One may notice that some basic ideas in the theory of free boundary problem are similar to the ones in the theory of minimal surfaces [F69, Giu84] and harmonic maps [Sim96, SU82].

1.2. Free boundary problems in $RCD$-spaces and the main results. In this subsection, we state the main results of this paper. Let $(X, d, \mu)$ be an $RCD(K, N)$-space with $K \in \mathbb{R}$ and $N \in (1, +\infty)$, and let $Q$ be a $\mu$-measurable function on $\Omega$ with (1.3). Given a map $g = (g_1, g_2, \ldots, g_m) \in W^{1,2}(\Omega, [0, +\infty)^m)$, we consider the minimization problem (1.4).

The first result is the existence of a minimizer as following.

**Proposition 1.3** (Existence of a minimizer). If $\text{diam}(\Omega) \leq \text{diam}(X)/3$, then for each $g \in W^{1,2}(\Omega, [0, +\infty)^m)$, there exists a $u \in \mathcal{A}_g$ such that

$$J_Q(u) = \inf_{v \in \mathcal{A}_g} J_Q(v).$$

This proposition is somewhat known for experts. For the completeness, we include a proof in Section 3.

We then consider the Lipschitz regularity of a local minimizer $u$ in (1.4). Up to our knowledge, the existing proofs of the Lipschitz regularity of $u$ in the Euclidean setting do not work directly in the setting of $RCD(K, N)$-spaces. In fact, some proofs [AC81, CSY18, Caf87, Caf88, Caf89] make heavy use of the Poisson formula, which is not clear on $RCD(K, N)$-spaces. Other proofs [DT15, DSS20] rely on the fact that gradients of a harmonic function are again harmonic, which fails even on smooth Riemannian manifolds. In this paper, we will overcome this difficulty by using the Cheng-Yau gradient estimates for harmonic functions and a mean value property (see Lemma 5.3), to obtain the following the Lipschitz continuity, provided that the space is non-collapsed.

**Theorem 1.4** (Lipschitz regularity). Let $(X, d, \mu)$ be an $RCD(K, N)$-space with $K \in \mathbb{R}$ and $N \in (1, \infty)$. Assume that $\mu = \mathcal{H}^N$, the $N$-dimensional Hausdorff measure on $X$. (I.e., $X$ is non-collapsed.) Let $\Omega \subset X$ be a bounded domain. Suppose that $u = (u_1, \ldots, u_m)$ is a local minimizer of $J_Q$ in (1.2) and that $Q$ satisfies (1.3), then $u$ is locally Lipschitz continuous on $\Omega$. Precisely, for any ball $B_R(x) \subset \Omega$, there exists a constant $L$ depending only on $N, K, \Omega, R, Q_{\text{max}}, \varepsilon_u$ and $\int_{B_R(x)} |u| d\mu$, such that

$$|u_i(y) - u_i(z)| \leq L \cdot d(y, z), \quad \forall \, y, z \in B_{R/4}(x), \quad i = 1, 2, \ldots, m. \quad (1.9)$$

**Remark 1.5.** The results for the Lipschitz regularity of energy minimizing harmonic maps from/into/between singular spaces were established in [GS92, KS93, ZZ18, GJZ19].

Our next result is about the finiteness of the perimeter of the free boundary of a local minimizer. We will also derive the associated Euler-Lagrange equation.

**Theorem 1.6** (Local finiteness of perimeter and Euler-Lagrange equation). Let $(X, d, \mu)$, $\Omega$ and $u$ be as in the above Theorem 1.4. Suppose $Q \in C(\Omega)$. Then $\Omega_u := \Omega \cap \{|u| > 0\}$ is a set of locally finite perimeter. Moreover, it holds:

1. For all $\Omega' \subseteq \Omega$, $\mathcal{H}^{N-1}(\partial\{|u| > 0\} \cap \Omega') < +\infty$;
2. There exist nonnegative Borel functions $q_i$, $i = 1, 2, \ldots, m$, such that

$$\Delta u_i = q_i \cdot \mathcal{H}^{N-1}\left(\partial\{|u| > 0\} \cap \Omega\right)$$
in the sense of distributions (i.e., $-\int_\Omega (\nabla u_i, \nabla \phi) \, d\mu = \int_{\Omega \setminus \{\{u\} > 0\}} \phi q_i d\mathcal{H}^{N-1}$ for any Lipschitz continuous $\phi$ with compact support in $\Omega$), and

\begin{equation}
(1.10) \quad \sum_{i=1}^{m} q_i^2(x) = Q(x), \quad \text{for } H^{N-1}-\text{a.e. } x \in \partial \{\{u\} > 0\} \cap \Omega.
\end{equation}

Remark 1.7. Recalling in the Euclidean setting, by using the non-tangential limits $w_i, i = 1, \ldots, m$, in (1.6), the densities in the Euler-Lagrange equation (1.7), $q_i = w_i \sqrt{\mathcal{H}^{N-1}}$, fulfill (1.10). The proof of the existence of non-tangential limits $w_i$ relies heavily on a domain variation formula via a $C^1$ vector field (see [CSY18, Lemma 11]). In the $RCD$ setting, the notion of “non-tangential limit” is not well-defined at present. In this paper, we will prove (1.10) by applying a blow-up argument and the theory of sets of finite perimeter in the setting of [Amb01, Mir03, ABS19] (see Corollary 7.2 for the details).

To consider the regularity of free boundary of the local minimizers of $J_Q$, let us recall that on the Euclidean space $\mathbb{R}^n$, the dimension of singular part $\dim_{\mathcal{H}}(S_u) \leq n - k^*$ for some $k^* \in \{5, 6, 7\}$ (see Theorem 1.2(3)). However, in the non-smooth setting, the singularities of the free boundary may arise from the singularities of the space itself, see the following example.

Example 1.8. Let $Y$ be the doubling of an equilateral triangle in $\mathbb{R}^2$ (gluing two same equilateral triangles along their boundaries). This is a two-dimensional Alexandrov space with nonnegative curvature, and thus, $(Y, d_Y, \mathcal{H}^2)$ is an $ncRCD(0, 2)$ metric measure space. Let $X := \mathbb{R} \times Y$. It is clear that $X$ is an $ncRCD(0, 3)$-space. Assume that $\Omega = (-1, 1) \times Y$ and

\[ f(t, y) := \begin{cases} 
  t & \text{if } t \geq 0 \\
  0 & \text{if } t < 0.
\end{cases} \]

It is easy to check that $f$ is a minimizer of $J_{Q=1}$ on $\Omega$. The free boundary $\partial \{f > 0\} = \{0\} \times Y$. It is clear that, assuming that $y \in Y$ is one of vertexes of the equilateral triangle, the point $x := (0, y)$ is a singular point of the free boundary.

This example shows that the best expectation of the bound of the singular part of the free boundary in the $RCD(K, N)$-space without boundary is co-dimension $\geq 3$. We shall prove this bound for non-collapsed $RCD(K, N)$-spaces without boundary.

Two different notions of boundary of $RCD$-spaces have been introduced in [DPG18] and [KM19] respectively. Here we will use the one introduced in [DPG18]. Let $(X, d, \mu := \mathcal{H}^N)$ be a non-collapsed $RCD(K, N)$-space. Recall that [AnBS19, DPG18] the singular part of $X$ has a stratification:

\begin{equation}
(1.11) \quad S^0 \subset S^1 \subset \cdots \subset S^{N-1} = S := X \setminus \mathcal{R},
\end{equation}

where $\mathcal{R}$ is the regular part of $X$ given by

\[ \mathcal{R} := \left\{ x \in X \mid \text{each tangent cone at } x \text{ is } (\mathbb{R}^N, d_{\text{Eucl}}) \right\}, \]

and, for any $0 \leq k \leq N - 1$,

\[ S^k := \left\{ x \in X \mid \text{no tangent cone at } x \text{ splits off } \mathbb{R}^{k+1} \right\}. \]

It holds

\begin{equation}
(1.12) \quad \dim_{\mathcal{H}}(S^k) \leq k, \quad \forall \, k = 1, 2, \ldots, N - 1.
\end{equation}

(This was first given in [CC97] for non-collapsed Ricci limit spaces.) According to [DPG18], the boundary of $X$ is defined by

\[ \partial X := S^{N-1} \setminus S^{N-2}. \]
Our last result is to show that the free boundary has a manifold structure away from a subset having co-dimension 3, which is similar to the one in [MS21] for the boundary of a set minimizing the perimeter in RCD-spaces.

**Theorem 1.9** (Regularity of free boundary). Let $(X,d,\mu)$ and $\Omega$ be as in the above Theorem 1.4. Suppose that $u = (u_1, \ldots, u_m)$ is a minimizer of $J_Q$ in (1.2) and that $Q$ satisfies (1.3). Assume that $\Omega \cap \partial X = \emptyset$ and $Q \in C(\Omega)$. Then for any $\varepsilon > 0$, there exists a relatively open set $O_\varepsilon \subset \partial\{|u| > 0\} \cap \Omega$ satisfying the following properties:

1. ($\varepsilon$-Reifenberg flatness) For any $x \in O_\varepsilon$ there exists a radius $r_x > 0$ such that for any ball $B_r(y)$ with $y \in B_{r_x}(x) \cap \partial\{|u| > 0\}$ and $r \in (0, r_x)$, it holds that $B_r(y)$ is $\varepsilon r$-closed to $B_r(0^N)$ in the pointed measured Gromov-Hausdorff topology and that $B_r(y) \cap \partial\{|u| > 0\}$ is $\varepsilon r$-closed to $B_r(0^{N-1})$ in the Gromov-Hausdorff topology, where $B_r(0^N)$ is the ball in $\mathbb{R}^N$ with centered at $0 \in \mathbb{R}^N$ and radius $r$;

2. (The smallness of the remainder of $O_\varepsilon$)

\[
\dim_H \left( \partial\{|u| > 0\} \cap \Omega \right) \setminus O_\varepsilon \leq N - 3.
\]

Moreover, if $N = 3$, then $(\partial\{|u| > 0\} \cap \Omega) \setminus O_\varepsilon$ is a discrete set of points.

In particular, the relatively open set $O_\varepsilon$ is $C^\alpha$ biHölder homeomorphic to an $(N - 1)$-dimensional manifold, where $\alpha = \alpha(\varepsilon) \in (0, 1)$ with $\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 1$.

**Remark 1.10.** According to the above Example 1.8, the bound (1.13) is sharp.

As a direct consequence, we have the following result.

**Corollary 1.11.** Let $(X,d,\mu), \Omega$ and $u = (u_1, \ldots, u_m)$ be as in the above Theorem 1.9. Assume that $\Omega \cap \partial X = \emptyset$ and $Q \in C(\Omega)$. Let

\[
\partial\{|u| > 0\} \cap \Omega := R^\Omega u \cup S^\Omega u,
\]

where $R^\Omega u$ and $S^\Omega u$ are the regular and singular parts of the free boundary $\partial\{|u| > 0\} \cap \Omega$. (That is, $x \in R^\Omega u$ means that each tangent cone at $x$ is $(\mathbb{R}^N, d_{\text{Eucl}})$ and each blow-up limit of $\partial\{|u| > 0\} \cap \Omega$ at $x$ is an $(N - 1)$-dimensional affine hyperplane in $\mathbb{R}^N$. The singular part $S^\Omega u := (\partial\{|u| > 0\} \cap \Omega) \setminus R^\Omega u$.)

Then we have

\[
\dim_H \left( S^\Omega u \right) \leq N - 3.
\]

**Remark 1.12.** In general, the regular part $R^\Omega u$ might not form a manifold. In fact, it might not be relatively open in the free boundary $\partial\{|u| > 0\} \cap \Omega$. (In the Euclidean case and if $Q \in C^\alpha$, the regular part is relatively open in the free boundary, see Theorem 1.2(3)). This will be seen by the following simple example. Recall that Y. Ostu and T. Shioya in [OS94] constructed a two-dimensional Alexandrov space without boundary, denoted by $Y_{OS}$, such that the singular set $S$ of $Y_{OS}$ is dense in $Y_{OS}$. Recalling the above Example 1.8, we replace the space $Y$ in Example 1.8 by $Y_{OS}$. By using the same construction of $f$, we know that for any singular point $y \in Y_{OS}$, the point $x := (0, y)$ is a singular point of the free boundary. Thus, the singular part $S^\Omega f$ is dense in the free boundary $\partial\{f > 0\} = \{0\} \times Y_{OS}$.

2. Preliminaries

Let $(X,d)$ be a complete metric space and $\mu$ be a Radon measure on $X$ with supp$(\mu) = X$. The triple $(X,d,\mu)$ is called a metric measure space. Given any $p \in X$ and $R > 0$, we denote by $B_R(p)$ the open ball centered at $p$ with radius $R$. 

2.1. \textit{RCD}(K, N) metric measure spaces and their calculus. Let \( K \in \mathbb{R} \) and \( N \in [1, +\infty) \). The curvature-dimension condition \( \text{CD}(K, N) \) for a metric measure space \( (X, d, \mu) \) was introduced by Sturm [Stu06a, Stu06b] and Lott-Villani [LV09, LV07]. The \( \text{RCD}(K, \infty) \)-condition was introduced by Ambrosio-Gigli-Savaré in [AGS14b]. The finitely dimensional case, \( \text{RCD}(K, N) \), was given by Gigli in [Gig13, Gig15]. Erbar-Kuwada-Sturm [EKS15] and Ambrosio-Mondino-Savaré [AMS16] proved that a weak formulation of Bochner inequality is equivalent to the (reduced) Riemannian curvature-dimension condition \( \text{RCD}(K, N) \). In [CM16, Theorem 1.1], Cavalletti-Milman showed that the condition \( \text{RCD}(K, N) \) is equivalent to the condition \( \text{RCD}(K, N) \), if the total measure \( \mu(X) < +\infty \).

We refer the readers to the survey [Amb18] and its references for the basic facts of the theory of \( \text{RCD}(K, N) \) metric measure spaces. Here we only recall some basic properties [LV09, AGS14b, AGMR15, EKS15] as follows:

- \( (X, d) \) is a locally compact length space. In particular, for any \( p, q \in X \), there is a shortest curve connecting them;
- If \( N > 1 \), then the generalized Bishop-Gromov inequality holds. In particular, it implies a local measure doubling property: for all \( 0 < r_1 < r_2 < R \), we have

\[
\frac{\mu(B_{r_2}(p))}{\mu(B_{r_1}(p))} \lesssim C_{N,K,R} \left( \frac{r_2}{r_1} \right)^N
\]

for some constant \( C_{N,K,R} > 0 \) depending only on \( N, K \) and \( R \);

- If \( N > 1 \) and \( \Omega \subset X \) is a bounded set, then there exists a constant \( C_{N,K,\Omega} > 0 \) such that

\[
\frac{\mu(B_{r+\delta}(x) \setminus B_r(x))}{\mu(B_r(x))} \lesssim \frac{\tilde{\mu}(B_{r+\delta} \setminus B_r)}{\tilde{\mu}(B_r)},
\]

where \( \tilde{\mu} \) is the \( N \)-dimensional Hausdorff measure on \( \mathbb{M}^N_{K/(N-1)} \), the simply connected space form with constant sectional curvature \( K/(N-1) \), and \( B_r \) is a geodesic ball of radius \( r \) in \( \mathbb{M}^N_{K/(N-1)} \). It follows

\[
\limsup_{\delta \to 0^+} \frac{\mu(B_{r+\delta} \setminus B_r)}{\delta \cdot \tilde{\mu}(B_r)} \lesssim \frac{C_{N,K}}{r}, \quad \forall r \leq 1.
\]

This gives \( \frac{d^+}{dr} \mu(B_r(x)) \leq \frac{C_{N,K} \cdot \mu(B_r(x))}{r} \) for all \( x \in X \) and \( r \leq 1 \). Thus, by

\[
\mu(B_r(x)) \leq C_{N,K,\Omega} \cdot r, \quad \forall x \in \Omega, \quad \forall r \leq 1,
\]

(see [CC00, Eq.(4.3)] or [KL16, Corollary 5.5]), we conclude (2.2).

Several different notions of Sobolev spaces for metric measure spaces have been given in [Che99, Sha00, AGS13, AGS14a, HK00]. They are equivalent to each other in the setting of \( \text{RCD} \)-metric measure spaces (see, for example, [AGS14a, AGS13]). Given a continuous function \( f \) on \( X \), the pointwise Lipschitz constant ([Che99]) of \( f \) at \( x \) is defined by

\[
\text{Lip}_f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)}
\]

for \( x \) not isolated, and \( \text{Lip}_f(x) = 0 \) if \( x \) is isolated. It is clear that \( \text{Lip}_f \) is \( \mu \)-measurable. Let \( \Omega \subset X \) be an open domain and let \( 1 \leq p \leq +\infty \). The \( W^{1,p} \)-norm of a locally Lipschitz
function \( f \in \text{Lip}_{loc}(\Omega) \) on \( \Omega \) is defined by
\[
\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\text{Lip} f\|_{L^p(\Omega)}.
\]
The Sobolev space \( W^{1,p}(\Omega) \) is defined by the completion of the set of locally Lipschitz functions \( f \) with \( \|f\|_{W^{1,p}(\Omega)} < +\infty \). The space \( W^{1,p}_0(\Omega) \) is defined by the closure of \( \text{Lip}_0(\Omega) \) under the \( W^{1,p}_0 \)-norm, where \( \text{Lip}_0(\Omega) \) is the set of Lipschitz continuous functions on \( \Omega \) with compact support in \( \Omega \). We denote \( f \in \text{Lip}_{loc}^p(\Omega) \) if \( f \in W^{1,p}_0(\Omega') \) for every open subset \( \Omega' \subset \Omega \), where “\( \Omega' \subset \Omega \)” means \( \Omega' \) is compactly contained in \( \Omega \). We itemize some basic properties of Sobolev functions as follows.

**Proposition 2.1.** Let \( 1 < p < \infty \).

1. For each \( f \in W^{1,p}(\Omega) \), there is a function in \( L^p(\Omega) \), denoted by \( |\nabla f| \), (so-called weak upper gradient for \( f \), see [Che99, Sect.2],) such that \( \|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)} \). Moreover, if \( f \in \text{Lip}_0(\Omega) \) then \( |\nabla f| = \text{Lip} f \) holds \( \mu \text{-a.e.} \) in \( \Omega \) (Che99, Theorem 5.1).

2. (Lower semicontinuity of energy.) If \( f_j \in W^{1,p}(\Omega) \) and \( f_j \to f \) in \( L^p(\Omega) \), then \( \liminf_{j \to \infty} \|\nabla f_j\|_{L^p(\Omega)} \geq \|\nabla f\|_{L^p(\Omega)} \).

3. If \( f,g \in W^{1,p}(\Omega) \) and \( f_A = g_A \) for some Borel set \( A \subset \Omega \), then \( |\nabla f|(x) = |\nabla g|(x) \) at \( \mu \text{-a.e.} \) \( x \in A \).

4. The \( W^{1,2}(\Omega) \) is a Hilbert space, and the inner product \( \langle \nabla f, \nabla g \rangle = \|\nabla f\|_{L^2(\Omega)} \) for \( f,g \in W^{1,2}(\Omega) \) can be given by the polarization (see [Gig15]):
\[
\langle \nabla f, \nabla g \rangle = \frac{1}{4}(|\nabla (f + g)|^2 - |\nabla (f - g)|^2).
\]

5. (Poincaré inequality, see [BB11, Eq. (2.6)] or [Raj12]). If \( \Omega \) is bounded, then there exists a constant \( C_\Omega > 0 \) depending only on \( p \), \( K \), and \( \text{diam}(\Omega) \), such that for every ball \( B_R(x) \subset \Omega \) with \( R \leq \text{diam}(\Omega)/3 \), it holds
\[
\int_{B_R(x)} f^p \leq C_\Omega \cdot R^p \int_{B_R(x)} |\nabla f|^p, \quad \forall f \in W^{1,p}_0(B_R(x)).
\]

The following fact is well-known, and a proof is given here, since we are not able to find a reference.

**Proposition 2.2.** Let \( D, \Omega \) be two open sets with \( \overline{D} \subset \subset \Omega \) and \( \mu(\partial D) = 0 \). Let \( 1 < p < \infty \) and \( u \in W^{1,p}(\Omega) \). We denote \( v \in W^{1,p}_0(\Omega) \) whenever \( v - u \in W^{1,p}_0(\Omega) \).

Assume \( g \in W^{1,p}_0(D) \) and \( h \in W^{1,p}_0(\Omega \setminus \overline{D}) \). Then the function
\[
f(x) := \begin{cases} g(x), & x \in D, \\ h(x), & x \in \Omega \setminus \overline{D}, \end{cases}
\]
has a representative in \( W^{1,p}_0(\Omega) \).

In particular, if \( g \in W^{1,p}_0(D) \), then its zero extension \( \tilde{g} \) on \( \Omega \) (namely, \( \tilde{g} = g \) on \( D \) and \( \tilde{g} = 0 \) on \( \Omega \setminus D \)) is in \( W^{1,p}_0(\Omega) \).

**Proof.** Since \( g - u \in W^{1,p}_0(D) \), there are \( \tilde{g}_j \in \text{Lip}_0(D) \) such that \( \tilde{g}_j \to g - u \) in \( W^{1,p}(D) \) as \( j \to \infty \). Similarly, there are \( \tilde{h}_j \in \text{Lip}_0(\Omega \setminus \overline{D}) \) such that \( \tilde{h}_j \to h - u \) in \( W^{1,p}(\Omega \setminus \overline{D}) \). Consider the functions
\[
\tilde{f}_j(x) := \begin{cases} \tilde{g}_j(x), & x \in D, \\ 0, & x \in \partial D, \\ \tilde{h}_j(x), & x \in \Omega \setminus \overline{D}, \end{cases}
\]
for each \( j \in \mathbb{N} \). Then \( \tilde{f}_j \in \text{Lip}_0(\Omega) \) for each \( j = 1, 2, \ldots \). Since \( \mu(\partial D) = 0 \), we have \( \text{Lip}\tilde{f}_j = \text{Lip}\tilde{g}_j \) or \( \text{Lip}\tilde{h}_j \) \( \mu \text{-a.e.} \) in \( \Omega \), and then
\[
\|\tilde{f}_j - \tilde{f}_k\|_{W^{1,p}(\Omega)} = \|\tilde{g}_j - \tilde{g}_k\|_{W^{1,p}(D)} + \|\tilde{h}_j - \tilde{h}_k\|_{W^{1,p}(\Omega \setminus \overline{D})}
\]
for all $j, k = 1, 2, \ldots$. It follows that $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence under $W^{1, p}(\Omega)$-norm. Let $f \in W^{1, p}_0(\Omega)$ be the $W^{1, p}(\Omega)$-limit of $\{f_j\}_{j=1}^\infty$. Meanwhile, since $\mu(\partial D) = 0$, it is clear that $f_j \to (f - u)$ in $L^p(\Omega)$, as $j \to \infty$. Therefore, $\hat{f} = f - u$ in $L^p(\Omega)$. It follows that $\hat{f}$ is a $W^{1, p}_0(\Omega)$-representative of $f - u$.

The second assertion follows from the first one, by taking $u = 0$ and $h = 0$. □

**Definition 2.3** (Distributional Laplacian [Gig15]). For each function $f \in W^{1, 2}_{loc}(\Omega)$, its **Laplacian** $\Delta f$ on $\Omega$ is a linear functional acting on $\mathcal{L}(\Omega)$ given by

\[
\Delta f(\phi) = -\int_\Omega \langle \nabla f, \nabla \phi \rangle \, d\mu
\]

for all $\phi \in \mathcal{L}(\Omega)$. If there is a signed Radon measure $\nu$ such that $\Delta f(\phi) = \int_\Omega \phi \, d\nu$ for all $\phi \in \mathcal{L}(\Omega)$, then we say that $\Delta f = \nu$ in the sense of distribution.

In general, the measure-valued Laplacian $\Delta f$ for a function $f \in W^{1, 2}_{loc}(\Omega)$ may not be absolutely continuous with respect to $\mu$. We consider its Radon-Nikodym decomposition

\[
\Delta f = (\Delta f)^{ac} \cdot \mu + (\Delta f)^{sing}.
\]

This Laplacian on $\Omega$ is linear, and it satisfies the chain rule and the Leibniz rule [Gig15].

**Remark 2.4.** (1) When $f \in W^{1, 2}(\Omega)$, the test function in (2.6) can be taken any $\phi \in W^{1, 2}_{loc}(\Omega)$.

(2) When $\Omega = X$, the inner product (2.5) provides a Dirichlet form $\mathcal{E}(f, g) := \int_X \langle \nabla f, \nabla g \rangle$ on $X$. This Dirichlet form $(\mathcal{E}, W^{1, 2}(X))$ has an infinitesimal generator $\Delta_\mathcal{E}$ with domain $D(\Delta_\mathcal{E}) \subset W^{1, 2}(X)$, i.e., for any $f \in D(\Delta_\mathcal{E})$ and any $g \in W^{1, 2}(X)$, it holds $\langle \Delta_\mathcal{E} f, g \rangle_{L^2(X)} = -\mathcal{E}(f, g)$. In case $f \in D(\Delta_\mathcal{E})$, the measure valued Laplacian $\Delta f$ is absolutely continuous with respect to $\mu$ and $\Delta f = \Delta_\mathcal{E} f \cdot \mu$.

We recall the following Laplacian comparison theorem for distance functions.

**Theorem 2.5** (Laplacian comparison theorem, [Gig15, Corollary 5.15]). Let $(X, d, \mu)$ be an RCD$(K, N)$ space with $K \in \mathbb{R}$ and $N > 1$, and let $p \in X$. Put $\rho(x) := d(x, p)$, then

\[
\Delta \rho \leq (N - 1) \cot_s(\sqrt{\kappa} \rho) \cdot \mu
\]

in the sense of distribution on $X \setminus \{p\}$, where

\[
\kappa = K/(N - 1) \quad \text{and} \quad \cot_s(s) = \begin{cases} 
\sqrt{s} \cot(\sqrt{s}) & \text{if } \kappa > 0, \\
1/s & \text{if } \kappa = 0, \\
\sqrt{-\kappa} \coth(\sqrt{-\kappa}) & \text{if } \kappa < 0.
\end{cases}
\]

In particular, if $K < 0$, $N > 1$, then we have

\[
\Delta \phi_{N, K}(\rho) \leq 0
\]

in the sense of distributions, where

\[
\phi_{N, K}(s) = \int_0^1 \left( \frac{\sinh(\sqrt{-\kappa} t)}{\sqrt{-\kappa}} \right)^{1-N} \, dt, \quad \kappa = K/(N - 1);
\]

and if $\rho \leq 1$ additionally, then we have

\[
\Delta (\rho^2) = 2|\nabla \rho|^2 \cdot \mu + 2 \rho \Delta \rho \leq 2(N + C \rho^2) \cdot \mu
\]

in the sense of distributions on $X$, where $C$ depends only on $N$ and $K$. 

A function $f \in W^{1,2}_{\text{loc}}(\Omega)$ is called subharmonic on $\Omega$ if $\Delta f \geq 0$ in the sense of distributions, that is $\int_\Omega \langle \nabla f, \nabla \phi \rangle \, d\mu \leq 0$ for all $0 \leq \phi \in \text{Lip}_0(\Omega)$. A function $f \in W^{1,2}_{\text{loc}}(\Omega)$ is called harmonic on $\Omega$ if both $f$ and $-f$ are subharmonic on $\Omega$. From [Che99, Theorem 7.17], the maximum principle holds for subharmonic functions. To be precise, if $f \in W^{1,2}(\Omega)$ is a subharmonic function such that $f - g \in W^{1,2}_{\text{loc}}(\Omega)$ for some $g \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$, then $\text{esssup}_\Omega f \leq \text{esssup}_\Omega g$.

According to [Che99, Theorem 7.12], the (relaxed) Dirichlet problem is solvable: for any ball $B_R(z)$ with $B_{2R}(z) \subset \Omega$ and any $f \in W^{1,2}(B_R(z))$, there exists a (unique) solution $f_H$ to $\Delta f_H = 0$ in the sense of distributions with boundary data $f_H - f \in W^{1,2}_{\text{loc}}(B_R(z))$. The classical Cheng-Yau’s estimate [CY75] for harmonic functions has been extended to $RCD$ metric measure spaces [JKY14, ZZ16].

**Theorem 2.6** (see [ZZ16, Theorem 1.6]). Let $(X, d, \mu)$ be an $RCD(K,N)$-space with $K \leq 0$ and $N \in (1, +\infty)$. Then every harmonic function $f$ on a geodesic ball $B_R(x_0) \subset X$ admits a locally Lipschitz continuous representative. Moreover, there exists a constant $C_N$, depending only on $N$, such that every positive harmonic function $f$ on $B_R(x_0)$ satisfies

$$
\sup_{B_{R/2}(x_0)} \frac{|\nabla f|}{f} \leq C_N \frac{1 + \sqrt{-KR}}{R}.
$$

Remark that the Cheng-Yau’s estimate implies the Harnack estimate: if $f$ is a positive harmonic function on $B_R(x_0)$ with $R \leq 1$, then

$$
f(x) \leq C_{K,N} f(y), \quad \forall x, y \in B_{R/2}(x_0)
$$

for some constant $C_{K,N}$ depending only on $K$ and $N$. Indeed, for any two points $x, y \in B_{R/2}(x_0)$, one can connect them by a curve $\gamma(t) \subset B_{R/2}(x_0)$ with length $L(\gamma) \leq R$. By (2.9) and $R \leq 1$, it holds

$$
|\ln f(x) - \ln f(y)| \leq \int_0^{L(\gamma)} |(\ln f \circ \gamma')| \leq \int_0^{L(\gamma)} (|\nabla \ln f| \circ \gamma) \leq \frac{C_N(1 + \sqrt{-KR})}{R} L(\gamma) \leq C_{N,K}.
$$

We recall the notion of the pointed measured Gromov-Hausdorff convergence of a sequence of metric measure spaces (see [Gro07, GMS15]). We focus on $RCD(K,N)$ metric measure spaces.

**Definition 2.7.** (1) Let $(Z, d_Z)$ be a complete metric space. Given $\epsilon > 0$ and subsets $A, B \subset Z$, we say that the Hausdorff distance $d_H^Z(A, B) < \epsilon$ if

$$A \subset B_\epsilon \quad \text{and} \quad B \subset A_\epsilon,$$

where $A_\epsilon$ is the $\epsilon$-neighborhood of $A$ by $A_\epsilon := \{z \in Z \mid d_Z(z, A) < \epsilon\}$. We denote by $A_j \xrightarrow{H} A_\infty$ in $Z$ if $d_H^Z(A_j, A_\infty) \to 0$ as $j \to \infty$.

(2) Let $(X_j, d_j)$ be a sequence of compact metric spaces. We say that $(X_j, d_j)$ converge to a metric space $(X_\infty, d_\infty)$ in the Gromov-Hausdorff topology (GH for short), denoted by $X_j \xrightarrow{GH} X_\infty$ for short, if there exist a complete metric space $(Z, d_Z)$ and a sequence of isometric embedding $\Phi_j : X_j \to Z$ such that $\Phi_j(X_j) \xrightarrow{H} \Phi_\infty(X_\infty)$ in $Z$.

(3) Let $K \in \mathbb{R}$, $N \in [1, \infty]$ and let $\{(X_j, d_j, \mu_j)\}_{j \in \mathbb{N}}$ be a sequence of $RCD(K,N)$ metric measure spaces with based points $p_j \in X_j$ for all $j \in \mathbb{N}$. We say that $(X_j, d_j, \mu_j, p_j)$ converges to a pointed metric measure space $(X_\infty, d_\infty, \mu_\infty, p_\infty)$ in the pointed measured Gromov-Hausdorff topology ($pmGH$ for short), denoted by

$$(X_j, d_j, \mu_j, p_j) \xrightarrow{pmGH} (X_\infty, d_\infty, \mu_\infty, p_\infty),$$
if there exist a metric space \((Z, d_Z)\) and a sequence of isometric embeddings \(\Phi_j : X_j \to Z\), for all \(j \in \mathbb{N}\), and \(\Phi_\infty : X_\infty \to Z\) such that the following hold:

- \(\Phi_j(p_j) \to \Phi_\infty(p_\infty)\) in \(Z\),
- for every \(R > 0\), \(\Phi_j(B_R(p_j)) \xrightarrow{H} \Phi_\infty(B_R(p_\infty))\) in \(Z\), and
- \((\Phi_j)_#\mu_j \rightharpoonup (\Phi_\infty)_#\mu_\infty\) as \(j \to \infty\) (as the duality on \(C_{bs}(Z)\), the space of continuous functions on \(Z\) with bounded support).

This is the so-called extrinsic approach [GMS15, Gro07], and we will fix the choice of \((Z, d_Z)\) and the embeddings \(\Phi_j, \Phi_\infty\) in the rest of this paper. It is well-known that the limit metric measure space \((X_\infty, d_\infty, \mu_\infty)\) is also of \(RCD(K, N)\), and that [GMS15] the ambient space \((Z, d_Z)\) can be chosen to be proper. Hence, the weak convergence of measures \((\Phi_j)_#\mu_j \rightharpoonup (\Phi_\infty)_#\mu_\infty\) can be also understood in the duality on \(C_0(Z)\) (the space of continuous functions on \(Z\) with compact support).

Let \((X_j, d_j, \mu_j, p_j) \xrightarrow{\text{pmGH}} (X_\infty, d_\infty, \mu_\infty, p_\infty)\), and let \(A_j \subset X_j, A_\infty \subset X_\infty\) be Borel subsets. We will denote by \(A_j \xrightarrow{\text{GH}} A\) if \(\Phi_j(A_j) \xrightarrow{H} \Phi_\infty(A_\infty)\) in \(Z\), where the ambient space \(Z\) and the embeddings \(\Phi_j, \Phi_\infty\) are given in Definition 2.7(3).

For any \(x \in X\) and \(r > 0\), we consider the rescaled pointed metric measure space

\[
(X, r^{-1}d, \mu_r^x, x), \quad \text{where} \quad \mu_r^x := c_r^x \cdot \mu := \frac{\mu}{\int_{B_r(x)} (1 - r^{-1}d((\cdot), x))d\mu}.
\]

**Definition 2.8.** A pointed metric measure space \((Y, \rho, \nu, y)\) is called a tangent cone of \((X, d, \mu)\) at \(x\) if there exists \(r_j \to 0\) such that \((X, r_j^{-1}d, \mu_r^x(x)) \xrightarrow{\text{pmGH}} (Y, \rho, \nu, y)\).

A point \(x\) is called a \(k\)-regular if the tangent cone at \(x\) is unique and is isomorphic to

\[
(\mathbb{R}^k, d_E, c_k \mathcal{L}^k, 0^k), \quad \text{where} \quad c_k := \left(\int_{B_1(0^k)} (1 - |x|)dx\right)^{-1}.
\]

Remark that, from [MN19, BS19], now it is known that there exists a unique integer \(k \in [1, N]\) such that \(\mu(X \setminus \mathcal{R}_k) = 0\), where \(\mathcal{R}_k\) is the set of all \(k\)-regular points of \((X, d, \mu)\).

We also consider the convergence of functions defined on varying spaces. Let

\[
(X_j, d_j, \mu_j, p_j) \xrightarrow{\text{pmGH}} (X_\infty, d_\infty, \mu_\infty, p_\infty).
\]

**Definition 2.9.** Let \(R > 0\). Suppose that \(\{f_j\}_{j \in \mathbb{N} \cup \{\infty\}}\) is a sequence of Borel functions on \(B_R(p_j)\). It is said that:

(i) \(f_j \to f_\infty\) over \(B_R(p_j)\), if for any \(x_j \xrightarrow{\text{GH}} x_\infty \in B_R(p_\infty)\) then \(f_j(x_j) \to f_\infty(x_\infty)\) as \(j \to \infty\);

(ii) \(f_j \to f_\infty\) uniformly over \(B_R(p_j)\), if for any \(\varepsilon > 0\) there exist \(N(\varepsilon) \in \mathbb{N}\) and \(\delta := \delta(\varepsilon) > 0\) such that

\[
\sup_{x \in B_R(p_j), \ y \in B_R(p_\infty), \ d_2(\Phi_j(x), \Phi_\infty(y)) < \delta} |f_j(x) - f_\infty(y)| < \varepsilon, \quad \forall \ j \geq N(\varepsilon),
\]

where \(\Phi_j, \Phi_\infty\) and \(Z\) are given in Definition 2.7.

We remark that the Arzela-Ascoli theorem can be generalized to the case where the functions live on varying spaces (see, for example, [LV09]). We recall the following Cheeger’s lifting lemma:

**Lemma 2.10 ([Che99, Lemma 10.7]).** Let \(R > 0\) and let \((X_j, d_j, \mu_j)\) be a sequence of \(RCD(K, N)\) metric measure spaces and

\[
(X_j, d_j, \mu_j, p_j) \xrightarrow{\text{pmGH}} (X_\infty, d_\infty, \mu_\infty, p_\infty).
\]
Then, given any Lipschitz function \( f_\infty \in \text{Lip}(\overline{B_R(p_\infty)}) \) with a Lipschitz constant \( L > 0 \), there exist a sequence of Lipschitz functions \( f_j \in \text{Lip}(\overline{B_R(p_j)}) \) such that \( f_j \to f_\infty \) uniformly over \( B_R(p_j) \), \( \|\nabla f_j\|_{L_\infty(B_R(p_j))} \leq L + 1 \) for all \( j \in \mathbb{N} \), and that

\[
\lim_{j \to \infty} \int_{B_R(p_j)} |\nabla f_j|^2 d\mu_j = \int_{B_R(p_\infty)} |\nabla f_\infty|^2 d\mu_\infty.
\]

We also need a variant of it as follows.

**Lemma 2.11.** Let \( R > 0 \) and let \((X, d, \mu)\) be as above in Lemma 2.10. Let \( F_j \in \text{Lip}(\overline{B_R(p_j)}) \) be a sequence of Lipschitz functions with a uniform Lipschitz constant and satisfy that \( F_j \to F_\infty \) uniformly over \( B_R(p_j) \). Then, given any function \( f_\infty \in \text{Lip}_\infty(\overline{B_R(p_\infty)}) \) with \( f_\infty - F_\infty \in W^{1,2}_0(B_R(p_\infty)) \), for each \( \delta > 0 \), there exist a sequence of functions \( f_j \) on \( B_R(p_j) \) such that \( f_j - F_j \in W^{1,2}_0(B_R(p_j)) \), \( f_j \to f_\infty \) uniformly over \( B_{R-\delta}(p_j) \), and that

\[
\lim_{j \to \infty} \int_{B_R(p_j)} |\nabla f_j|^2 d\mu_j = \int_{B_R(p_\infty)} |\nabla f_\infty|^2 d\mu_\infty.
\]

**Proof.** Fix any \( \delta > 0 \). Since \( f_\infty \in \text{Lip}(\overline{B_{R-\delta}(p_\infty)}) \), we can use the above Cheeger’s lemma, Lemma 2.10, to obtain a sequence of Lipschitz functions \( g_j \in \text{Lip}(\overline{B_{R-\delta}(p_j)}) \) such that \( g_j \to g_\infty := f_\infty\vert_{\overline{B_{R-\delta}(p_\infty)}} \) uniformly over \( B_{R-\delta}(p_j) \), \( \|\nabla g_j\|_{L^2(B_{R-\delta}(p_j))} \to \|\nabla f_\infty\|_{L^2(B_{R-\delta}(p_\infty))} \) as \( j \to \infty \), and \( |\nabla g_j|_{L^\infty(B_{R-\delta}(p_j))} \leq L_\delta \), where \( L_\delta > 0 \) is independent of \( j \) (may depend on \( \delta \)).

Denoted by \( A_\delta(p_j) := B_R(p_j) \setminus B_{R-\delta}(p_j) \) for each \( j \in \mathbb{N} \cup \{ \infty \} \). Let \( G_j \in \text{Lip}(\partial A_\delta(p_j)) \) be defined by \( G_j = F_j \) on \( \partial B_R(p_j) \) and \( G_j = g_j \) on \( \partial B_{R-\delta}(p_j) \). Then, for each \( j \in \mathbb{N} \), we can extend \( G_j \) to a Lipschitz function \( \tilde{G}_j \in \text{Lip}(A_\delta(p_j)) \).

Since \( g_j \) and \( F_j \) have a uniform Lipschitz constant, we can assume that \( \tilde{G}_j \) have a uniform Lipschitz constant \( L'_\delta \). Since \( G_j \to G_\infty \) uniformly on \( \partial A_\delta(p_j) \), we can also assume that \( \tilde{G}_j \to \tilde{G}_\infty \) uniformly (up to a subsequence, by Arzela-Ascoli theorem). Now we have \( f_\infty - \tilde{G}_\infty \in W^{1,2}_0(A_\delta(p_\infty)) \). By [ZZ19, Proposition 3.2(ii)], there exists a sequence \( h_j \) on \( A_\delta(p_j) \) such that \( h_j - \tilde{G}_j \in W^{1,2}_0(A_\delta(p_j)) \) and \( |\nabla h_j|_{L^2(A_\delta(p_j))} \to |\nabla f_\infty|_{L^2(A_\delta(p_\infty))} \) as \( j \to \infty \). At last, we define the function \( f_j \) on \( B_R(p_j) \) by \( f_j := g_j \) on \( \overline{B_{R-\delta}(p_j)} \) and \( f_j := h_j \) on \( A_\delta(p_j) \). From Proposition 2.2, we conclude that \( f_j \in W^{1,2}(B_R(p_j)) \) and \( f_j - F_j \in W^{1,2}_0(B_R(p_j)) \). The functions \( f_j \) satisfy the desired assertions. The proof is finished. \( \square \)

### 2.2. Non-collapsed \( RCD(K,N) \) metric measure spaces

**Definition 2.12.** Let \((X, d, \mu)\) be an \( RCD(K,N) \)-space with \( K \in \mathbb{R} \) and \( N \in [1, \infty) \). It is called a non-collapsed \( RCD(K,N) \)-space, denoted by \( \text{nc}RCD(K,N) \)-space for short, if \( \mu = \mathcal{H}^N \), the \( N \)-dimensional Hausdorff measure on \( X \).

The main examples of \( \text{nc}RCD(K,N) \) metric measure spaces are non-collapsed Ricci limit spaces [CC97, CC00, CJN21] and \( N \)-dimensional Alexandrov space with curvature \( \geq K/(N-1) \). It was shown [DPG18] that if \( \{(X_\iota, d_\iota, \mu_\iota)\} \) is a sequence of \( \text{nc}RCD(K,N) \) metric measure spaces and \((X_\iota, d_\iota, \mu_\iota, p_\iota) \overset{\text{pmGH}}{\rightarrow} (X_\infty, d_\infty, \mu_\infty, p_\infty) \), then \((X_\infty, d_\infty, \mu_\infty) \) is of \( \text{nc}RCD(K,N) \) too.

If \((X, d, \mu)\) is an \( \text{nc}RCD(K,N) \)-space, then \( N \) must be an integer, and there holds (from Corollary 2.14 in [DPG18])

\[
\mu(B_r(x)) \leq \bar{\mu}(B_r) \leq C_{N,K} \cdot r^N, \quad \forall x \in X \text{ and } r \leq 1,
\]
for a constant $C_{N,K} > 0$, where $\bar{\mu}$ is the $N$-dimensional Hausdorff measure on $\mathbb{M}^N_{K/(N-1)}$, the simply connected space form with constant sectional curvature $K/(N-1)$, and $B_r$ is a geodesic ball of radius $r$ in $\mathbb{M}^N_{K/(N-1)}$. Furthermore, if $N > 1$,

\begin{equation}
(2.14) \quad \frac{d}{dr} \mu(B_r(x)) := \limsup_{\delta \to 0^+} \frac{\mu(B_{r+\delta}(x) \setminus B_r(x))}{\delta} \leq C_{N,K} r^{N-1}
\end{equation}

for all $r \leq 1$ and $x \in X$. In fact by (2.3) and $\mu(B_r(x)) \leq \bar{\mu}(B_r)$, we get $\mu(B_{r+\delta}(x) \setminus B_r(x)) \leq \bar{\mu}(B_{r+\delta} \setminus B_r)$. It follows (2.14). Remark that it holds only (2.2) in general $RCD(K,N)$-spaces (without the assumption of non-collapsing).

Let $(X, d, \mu) := \mathcal{MK}$ be an $\mathcal{MK}$RCD($K,N$)-space. For each point $x \in X$, any tangent cone is a metric measure cone (a Euclidean cone with a natural measure). Indeed, the existence of the limit $\lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_{N} r^{N}}$ implies that any tangent cone at $x$ is a volume cone, and hence, by [DPG16], it is a metric cone. It was shown that [DPG18, Corollary 1.7] a point $x \in X$ is regular (i.e. any one tangent cone is isometric to $\mathbb{R}^N$) if and only if

$$\lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_{N} r^{N}} = 1.$$ 

2.3. Sets of finite perimeter and the reduced boundary. The theory of Euclidean sets of finite perimeter of De Giorgi has been extended to $RCD(K,N)$-spaces [Amb01, Mir03, ABS19], and recently [BPS19, BPS21].

**Definition 2.13.** A function $f \in L^1(X, \mu)$ is called a function of bounded variation, denoted by $f \in BV(X)$ for short, if there exists a sequence $f_j \in \text{Lip}_0(X)$ converging to $f$ in $L^1(X)$ such that

$$\limsup_{j \to \infty} \int_X |\nabla f_j| d\mu < +\infty.$$ 

Its total variation is a finite Borel measure and denoted by $|Df|$. Moreover, for any open subset $A \subset X$,

$$|Df|(A) := \inf \left\{ \liminf_{j \to \infty} \int_{\Omega} |\nabla f_j| d\mu \mid f_j \in \text{Lip}_0(A), \ \ f_j \overset{L^1(A)}{\to} f \right\}.$$ 

A function $f \in BV_0(X)$ if $\phi f \in BV(X)$ for any $\phi \in \text{Lip}_0(A)$.

**Definition 2.14.** Let $E \subset X$ be a Borel subset and let $A$ be an open set. The perimeter $\mathcal{P}(E, A)$ is given by

$$\mathcal{P}(E, A) := \inf \left\{ \liminf_{j \to \infty} \int_A |\nabla f_j| d\mu \mid f_j \in \text{Lip}_0(A), \ \ f_j \overset{L^1(A)}{\to} \chi_E \right\}. $$

A Borel set $E$ is called of finite perimeter in $X$ if $\mathcal{P}(E, X) < \infty$. In that case, it is proved [Amb01, Mir03] that the set function $A \mapsto \mathcal{P}(E, A)$ is the restriction to open sets of a finite Borel measure $\mathcal{P}(E, \cdot)$ defined by

$$\mathcal{P}(E, B) := \inf \{ \mathcal{P}(E, A) \mid B \subset A, \ A \subset X \text{ open} \}.$$ 

A subset $E \subset X$ with $\mu(E) < \infty$ is a set of finite perimeter if and only if the characteristic function $\chi_E \in BV(X)$, and its perimeter measure is $\mathcal{P}(E, \cdot) := |D\chi_E|(\cdot)$. A subset $E \subset X$ is called a set of locally finite perimeter if $\chi_E \in BV_0(X)$.

We recall the following basic properties, collected in [Amb01, Mir03, ABS19].

**Proposition 2.15.** Let $(X, d, \mu)$ be an $RCD(K,N)$ metric measure space with $K \in \mathbb{R}$ and $N \in [1, +\infty)$. Then the followings hold:

1. (Lower semicontinuity) $E \mapsto \mathcal{P}(E, X)$ is lower semicontinuous with respect to the $L^1_0(X)$ topology;
(2) (Coarea formula) Let \( v \in BV(X) \). Then \( \{ v > r \} := \{ x \mid v(x) > r \} \) has finite perimeter for \( L^1 \)-a.e. \( r \in \mathbb{R} \). Moreover, if \( v \in BV(X) \) is continuous and nonnegative, then, for any Borel function \( f : X \to [0, +\infty) \), it holds

\[
\int_{s \leq v < t} fd|Dv| = \int_s^t \int_X fd\left(\mathcal{P}\{v > r\}, \cdot\right) dr
\]

for any \( 0 \leq s < t < +\infty \) (see [Mir03, Remark 4.3] or [ABS19, Corollary 1.9]).

By applying to distance functions, we get the following weak convergence of the measures on spheres.

**Lemma 2.16.** Let \( (X_j, d_j, \mu_j) \) be a sequence of \( RCD(K, N) \) measure metric spaces with \( K \in \mathbb{R} \) and \( N \in [1, +\infty) \). Suppose that \( (X_j, d_j, \mu_j, p_j) \xrightarrow{\text{pmGH}} (X_\infty, d_\infty, \mu_\infty, p_\infty) \). Then we have, for \( \text{Lemma 2.16.} \)

\[
|D\chi_{B_r(p_j)}| \rightharpoonup |D\chi_{B_r(p_\infty)}|, \quad \text{as} \quad j \to \infty
\]

in duality with \( C_0(Z) \), where \( Z \) is given in Definition 2.7(3).

**Proof.** By the coarea formula, we know that for \( L^1 \)-a.e. \( r \in \mathbb{R}^+ \), the functions \( \chi_{B_r(p_j)} \in BV(X_j) \). For such \( r > 0 \), from the observation that \( \chi_{B_r(p_j)} \rightharpoonup \chi_{B_r(p_\infty)} \) in \( L^1 \)-strong and Proposition 3.6 [ABS19], we conclude that

\[
\liminf_{j \to \infty} \int_{X_j} g d|D\chi_{B_r(p_j)}|(X_j) \geq \int_{X_\infty} g d|D\chi_{B_r(p_\infty)}|(X_\infty), \quad \forall 0 \leq g \in Lip_0(Z),
\]

where \( Z \) is given in Definition 2.7 and it is proper. On the other hand, for any \( R > 0 \),

\[
\mu(B_R(p_j)) = \int_0^R |D\chi_{B_r(p_j)}|(X_j) dr
\]

\[
\to \mu_\infty(B_R(p_\infty)) = \int_0^R |D\chi_{B_r(p_\infty)}|(X_\infty) dr.
\]

Therefore, we have \( \lim_{j \to \infty} |D\chi_{B_r(p_j)}|(X_j) = |D\chi_{B_r(p_\infty)}|(X_\infty) \) for \( L^1 \)-a.e. \( r \in (0, R) \). At last, the desired assertion (2.16) follows from Corollary 3.7 in [ABS19], and it completes the proof. \( \square \)

**Definition 2.17.** Let \( (X_j, d_j, \mu_j, p_j) \xrightarrow{\text{pmGH}} (X_\infty, d_\infty, \mu_\infty, p_\infty) \), and let \( Z, \Phi_j, \Phi_\infty \) be as in Definition 2.7. A sequence of Borel sets \( E_j \subset X_j \) with \( \mu_j(E_j) < \infty \) for all \( j \in \mathbb{N} \) is called to converge in \( L^1 \)-strong to a Borel set \( F \subset X_\infty \) with \( \mu_\infty(F) < \infty \) if \( \chi_{E_j} \cdot \mu_j \rightharpoonup \chi_F \cdot \mu_\infty \) and \( \mu_j(E_j) \to \mu_\infty(F) \) as \( j \to \infty \).

A sequence of Borel set \( E_j \subset X_j \) is called to converge in \( L^1_{\text{loc}} \) to a Borel set \( F \subset X_\infty \) if \( E_j \cap B_R(p_j) \to F \cap B_R(p_\infty) \) in \( L^1 \)-strong for every \( R > 0 \).

Now recall the notion of reduced boundary of a set of locally finite perimeter in [ABS19].

**Definition 2.18 (Reduced boundary).** Let \( E \) be a set of locally finite perimeter in an \( n cRCD(K, N) \) metric measure space \( (X, d, \mu) \). A point \( x \in X \) is called a reduced boundary point of \( E \), denoted by \( x \in \partial_{\text{red}} E \), if it satisfies the following:

1. it is in \( \text{supp}(\mathcal{P}(E, \cdot)) \) and it is a regular point of \( X \). That is, for each \( \{r_j\} \) with \( r_j \to 0 \) the sequence \( (X, r_j^{-1}d, \mu_r^x, x) \) pointed Gromov-Hausdorff converges to \( \mathbb{R}^N \) with the Euclidean metric; and
2. for each \( \{r_j\} \) with \( r_j \to 0 \), the sequence \( E \subset (X, r_j^{-1}d, \mu_r^x, x) \) converges to the upper half space \( \{x_N > 0\} \subset \mathbb{R}^N \) in \( L^1_{\text{loc}} \).
We need also the following properties for sets of finite perimeter in non-collapsed spaces, see [Amb01, ABS19].

**Proposition 2.19.** Let \((X,d,\mu)\) be an ncRCD\((K,N)\) metric measure space with \(K \in \mathbb{R}\) and \(N \in (1, +\infty)\). Then the followings hold:

1. If \(E\) is a set of finite perimeter in \(X\), then \(\mathcal{H}^{N-1}(\partial^* E) < \infty\), where \(\partial^* E\) is the essential boundary of \(E\), defined by
   \[
   \partial^* E := \left\{ x \in M \mid \limsup_{r \to 0} \frac{\mu(B_r(x) \cap E)}{\mu(B_r(x))} > 0 \wedge \limsup_{r \to 0} \frac{\mu(B_r(x) \setminus E)}{\mu(B_r(x))} > 0 \right\}.
   \]
2. If \(E\) is a set of locally finite perimeter, then \(\mathcal{H}^{N-1}(\partial^* E,\mathcal{F}E) = 0\). Moreover, up to an \(\mathcal{H}^{N-1}\)-null set, it holds
   \[
   \mathcal{F}E = \left\{ x \in E \mid \lim_{r \to 0} \frac{\mathcal{H}^{N}(B_r(x) \cap E)}{\omega_{N} r^N} = \frac{1}{2} \right\},
   \]
   and that \(\mathcal{P}(E,\cdot) = \mathcal{H}^{N-1}_\mathcal{F}E\), (the De Giorgi’s Theorem, see [ABS19, Corollary 4.7].)

3. Existence of a Minimizer

In this section, we will derive the existence of a minimizer of (1.4), where we always assume that \(\Omega\) is a bounded domain in an \(RCD(K,N)\)-space \((X,d,\mu)\) with \(K \in \mathbb{R}\) and \(N \in (1, +\infty)\). Let \(g = (g_1, g_2, \ldots, g_m) \in W^{1,2}(\Omega, [0, +\infty)^m)\) and let \(\mathcal{A}_g\) be given in (1.4).

Now we are ready to prove the existence of a minimizer, which is asserted in Proposition 1.3.

**Proof of Proposition 1.3.** Since \(g \in \mathcal{A}_g\) guarantees that \(\mathcal{A}_g \neq \emptyset\), there exists a minimizing sequence \(\{u^k\}_{k=1}^\infty \subseteq \mathcal{A}_g\) such that

\[
\lim_{k \to \infty} J_Q(u^k) = \inf_{v \in \mathcal{A}_g} J_Q(v) \quad (\leq J_Q(g)).
\]

From the Poincaré inequality (see Proposition 2.1(5)), we get

\[
\int_\Omega |u^k - g|^2 \leq C_1 \int_\Omega |\nabla (u^k - g)|^2,
\]

where the constant \(C_1 > 0\) depends only on \(N, K, \Omega\). Thus,

\[
\begin{align*}
\|u^k\|^2_{W^{1,2}(\Omega, \mathbb{R}^m)} &\leq 2 \|u^k - g\|^2_{W^{1,2}(\Omega, \mathbb{R}^m)} + 2 \|g\|^2_{W^{1,2}(\Omega, \mathbb{R}^m)} \\
&\leq 2(1 + C_1) \int_\Omega |\nabla (u^k - g)|^2 + 2 \|g\|^2_{W^{1,2}(\Omega, \mathbb{R}^m)} \\
&\leq C_2 \left( \int_\Omega |\nabla u^k|^2 + \int_\Omega |\nabla g|^2 \right) + 2 \|g\|^2_{W^{1,2}(\Omega, \mathbb{R}^m)} \\
&\leq C_2 J_Q(u^k) + (C_2 + 2) \|g\|^2_{W^{1,2}(\Omega, \mathbb{R}^m)} \\
&\leq C_2 J_Q(g) + 1 + (C_2 + 2) \|g\|^2_{W^{1,2}(\Omega, \mathbb{R}^m)}
\end{align*}
\]

for all sufficiently large \(k\), where \(C_2 = 4(1 + C_1)\). Then, the fact that \(W^{1,2}(\Omega, \mathbb{R}^m)\) is a Hilbert space implies that there exists a subsequence \(\{u^{k_l}\}_{l=1}^\infty\) of \(\{u^k\}_{k=1}^\infty\) such that \(\{u^{k_l}\}_{l=1}^\infty\) weakly converges to some \(u\) in \(W^{1,2}(\Omega, \mathbb{R}^m)\) and converges to \(u\) almost everywhere on \(\Omega\). Noted that \(\mathcal{A}_g\) is a closed convex subset of \(W^{1,2}(\Omega, \mathbb{R}^m)\), we conclude that \(u \in \mathcal{A}_g\).

By noticing that \(\left\{ X_{\{x \in \Omega \mid |u^k(x)| > 0\}} \right\}_{l=1}^\infty\) converges to \(1\) almost everywhere on \(\{x \in \Omega \mid |u(x)| > 0\}\) and that \(Q \geq 0\), we have

\[
Q\chi_{\{|u| > 0\}} \leq \lim_{l \to \infty} Q\chi_{\{|u^{k_l}| > 0\}} \quad \mu - \text{a.e. in } \Omega.
\]
It follows from the Fatou lemma that
\[
\int_{\Omega} Q\chi_{\{|u| > 0\}} \leq \liminf_{\ell \to \infty} \int_{\Omega} Q\chi_{\{|u^\ell| > 0\}}.
\]
By combining this and the lower semicontinuity of energy in Proposition 2.1(2), we obtain
\[
J_Q(u) = \int_{\Omega} (|\nabla u|^2 + Q\chi_{\{|u| > 0\}})
\leq \liminf_{\ell \to \infty} \int_{\Omega} |\nabla u^\ell|^2 + \liminf_{\ell \to \infty} \int_{\Omega} Q\chi_{\{|u^\ell| > 0\}}
\leq \liminf_{\ell \to \infty} J_Q(u^\ell)
= \inf_{v \in \mathcal{A}_g} J_Q(v),
\]
where we have used (3.1). Therefore, \( u \) is a minimizer of \( J_Q \). The proof is finished.

\[\square\]

Remark 3.1. Here, we only need to assume that \( Q \in L^\infty(\Omega) \) and \( Q \geq 0 \) almost everywhere on \( \Omega \).

4. H"older continuity of local minimizers

In this section, we will derive the locally Hölder regularity for the local minimizers of \( J_Q \) in (1.2). Recall the notations. Let \((X, d, \mu)\) be \( RCD(K,N) \) metric measure space \((X,d,\mu)\) with \( K \in \mathbb{R} \) and \( N \in (1, +\infty) \). Let \( \Omega \subset X \) be a bounded domain and let \( Q \in L^\infty(\Omega) \). Suppose that
\[
u := (u_1, u_2, \cdots, u_m) \in W^{1,2}(\Omega, [0, +\infty)^m)
\]
is a local minimizer of \( J_Q \) in (1.2). Namely, there are a boundary value \( g = (g_1, g_2, \cdots, g_m) \in W^{1,2}(\Omega, [0, +\infty)^m) \) and a number \( \varepsilon_u > 0 \) such that
\[
J_Q(u) \leq J_Q(v), \quad \forall \ v \in \mathcal{A}_g \ \text{with} \ d(u, v) < \varepsilon_u,
\]
where \( \mathcal{A}_g \) and \( d(u, v) \) are given in (1.4) and (1.5) respectively.

To begin, we argue that the components of local minimizers are subharmonic, so that powerful analytic tools can be applied later on.

Lemma 4.1 (Subharmonicity). Let \( u = (u_1, \ldots, u_m) \) be a local minimizer of \( J_Q \) on \( \Omega \) with \( Q \in L^\infty(\Omega) \). Then \( \Delta u_i \geq 0 \) on \( \Omega \) in the sense of distributions, for all \( i = 1, \ldots, m \).

Proof. Fixed any ball \( B_{R(x)}(x) \subset \Omega \) such that \( \mu(B_{R(x)}) \leq \frac{\varepsilon}{2} \), it suffices to show that \( u_i \) is subharmonic on \( B_R(x) \), where \( i = 1, \ldots, m \).

For each \( 0 \leq \phi \in \text{Lip}_0(B_R(x)) \), \( \phi \neq 0 \), let
\[
\nu_{i,\delta} := (u_1, \ldots, u_{i-1}, (u_i - \delta \phi)^+, u_{i+1}, \ldots, u_m),
\]
where \( i \in \{1, \ldots, m\} \) and \( \delta > 0 \), then \( \nu_{i,\delta} \in \mathcal{A}_g \) and, by (1.5),
\[
d(\nu_{i,\delta}, u) \leq \|u - \nu_{i,\delta}\|_{W^{1,2}(\Omega, R^m)} + \|\chi_{\{|u - \nu_{i,\delta}| > 0\}}\|_{L^1(\Omega)}
\leq \delta \|\phi\|_{W^{1,2}(B_R(x))} + \mu(B_R(x)).
\]
This implies \( d(\nu_{i,\delta}, u) < \varepsilon_u \) provided the \( \delta < \delta_0 := \frac{\varepsilon_u}{2\|\phi\|_{W^{1,2}(B_R(x))}} \). Noted that \( \{\nu_{i,\delta} > 0\} \subset \{\|u\| > 0\} \), the local minimality of \( u \) gives
\[
\int_{\Omega} (|\nabla u|^2 + Q\chi_{\{|u| > 0\}}) \leq \int_{\Omega} (|\nabla \nu_{i,\delta}|^2 + Q\chi_{\{|\nu_{i,\delta}| > 0\}})
\leq \int_{\Omega} (|\nabla \nu_{i,\delta}|^2 + Q\chi_{\{|\nu_{i,\delta}| > 0\}})
\]
(4.2)
for all $\delta \in (0, \delta_0)$. Hence, we get $\int_\Omega |\nabla u|^2 \leq \int_\Omega |\nabla v_{i,\delta}|^2$. This implies
\begin{equation}
(4.3) \quad \int_\Omega |\nabla u_i|^2 \leq \int_\Omega |\nabla (u_i - \delta \phi)|^2 = \int_\Omega (|\nabla u_i|^2 - 2\delta \langle \nabla u_i, \nabla \phi \rangle + \delta^2 |\nabla \phi|^2)
\end{equation}
for all sufficiently small $\delta$. Therefore, by letting $\delta \to 0$, we obtain
\begin{equation}
(4.4) \quad \int_\Omega (\nabla u_i, \nabla \phi) \leq 0.
\end{equation}

The arbitrariness of $\phi$ implies $\Delta u_i \geq 0$ on $B_R(x)$ in the sense of distributions. The
proof is finished. \hfill \Box

An immediate consequence is the local boundedness of $u$.

Remark 4.2. Let $u = (u_1, \ldots, u_m)$ be a local minimizer of $J_Q$ on $\Omega$ with $Q \in L^\infty(\Omega)$, then for all $R > 0$, there exists a constant $C = C_{N,K,R} > 0$, depending only on
$N, K, R$, such that
\[
\sup_{B_R(x)} |u| \leq C_m \cdot \int_{B_R(x)} |u|
\]
provided the ball $B_R(x) \subset \Omega$. Indeed, from Lemma 4.1 and [KS01, Theorem 4.2],
we conclude $\sup_{B_R(x)} |u_i| \leq C \cdot \int_{B_R(x)} |u_i|$ for each $i = 1, 2, \ldots, m$.

We will prove the Hölder continuity of $u$ by using Campanato theory, so we need to obtain a decay estimate on $\int_{B_r(x)} |\nabla u|^2 \, d\mu$ (see, for example, [Gó09]).

Lemma 4.3 (Hölder continuity). Let $u = (u_1, \ldots, u_m)$ be a local minimizer of $J_Q$
in (1.2) with $Q \in L^\infty(\Omega)$. Then $u \in C^\alpha_{\text{loc}}(\Omega)$ for some $\alpha \in (0, 1)$ (means that it has
a $C^\alpha_{\text{loc}}$ representative).

Proof. Fixed any ball $B_R(\bar{x}) \subset \subset \Omega$ such that $\bar{R} < \text{diam}(\Omega)/3$, it suffices to show
$u \in C^\alpha(\bar{B}_{\bar{R}/2}(\bar{x}))$. Since $u \in L^\infty(\Omega)$. We denote $M_1 := \sup_{B_R(x)} |u|$.

For each $x_0 \in \bar{B}_{\bar{R}/2}(\bar{x})$ and $R \in (0, \bar{R}/4)$, there exists some $v \in W^{1,2}(B_R(x_0), \mathbb{R}^m)$
that solves the following (relaxed) Dirichlet problem [Che99, Theorem 7.12]:
\begin{equation}
(4.5) \quad \begin{cases}
\Delta v = 0 & \text{on } B_R(x_0), \\
u - v \in W^{1,2}_{\text{loc}}(B_R(x_0), \mathbb{R}^m).
\end{cases}
\end{equation}

After extending $v$ by $u$ on $\Omega \setminus B_R(x_0)$, (that is, $v := u$ in $\Omega \setminus B_R(x_0)$, see Proposition 2.2),
we have $v \in \mathcal{A}_G$ because all components of $v$ are nonnegative on $B_R(x_0)$ by the
maximum principle (see [Che99, Theorem 7.17]).

(i) We first check that $d(u, v) < \varepsilon_u$ whenever $\bar{R} < R_0$ for some $R_0 > 0$ depending
only on $K, N, \bar{R}$ and $u$.

By the Poincaré inequality
\[
\int_{B_R(x_0)} |u - v|^2 \leq C_1 \int_{B_R(x_0)} |\nabla (u - v)|^2
\]
for some constant $C_1$ depending only on $K, N, \bar{R}$ (see, for example, Proposition 2.1(5)), we get
\[
\|v - u\|_{W^{1,2}(\Omega, \mathbb{R}^m)} \leq (\sqrt{C_1} + 1) (\int_{B_R(x_0)} |\nabla (u - v)|^2)^{1/2},
\]
and hence we have
\begin{equation}
(4.6) \quad d(u, v) \leq C_2 \left( \int_{B_R(x_0)} |\nabla (u - v)|^2 \right)^{1/2} + \int_{B_R(x_0)} \left| \chi\{|u| > 0\} - \chi\{|v| > 0\} \right|
\leq C_2 \left( \int_{B_R(x_0)} |\nabla (u - v)|^2 \right)^{1/2} + \mu(B_R(x_0)),
\end{equation}
where $C_2 := \sqrt{\frac{C_1}{2}} + 1$. Note that

$$
\int_{B_R(x_0)} |\nabla (u-v)|^2 \leq 2 \int_{B_R(x_0)} |\nabla u|^2 + 2 \int_{B_R(x_0)} |\nabla v|^2 \\
\leq 4 \int_{B_R(x_0)} |\nabla u|^2,
$$

(4.7)

by the Dirichlet energy minimizing property of $v$. By the combination of (4.6), (4.7) and the facts that $|\nabla u|^2 \in L^1(\Omega)$, we conclude that there is $R_0 \in (0,1)$ (depending only on $K,N,R$ and $u$) such that $d(u,v) < \varepsilon_u$ for all $R \in (0,R_0)$.

(ii) Now by the local minimality of $u$, we have

$$
\int_{B_R(x_0)} (|\nabla u|^2 - |\nabla v|^2) \leq \int_{B_R(x_0)} (\chi_{\{|v|>0\}} - \chi_{\{|u|>0\}})Q \\
\leq \|Q\|_{L^\infty} \cdot \mu(B_R(x_0))
$$

(4.8)

for all $R \in (0,R_0)$. On the other hand, from $u-v \in W^{1,2}(\Omega,\mathbb{R}^m)$ and Remark 2.4(1), it can be taken as test functions for $\Delta u$ and $\Delta v$. Hence, we have (recalling $\Delta v = 0$) that

$$
\int_{B_R(x_0)} |\nabla (u-v)|^2 = -\int_{B_R(x_0)} (u-v)\Delta (u-v) \\
= -\int_{B_R(x_0)} (u-v)\Delta (u+v) \\
= \int_{B_R(x_0)} (|\nabla u|^2 - |\nabla v|^2).
$$

(4.9)

Recall that $M_1 := \sup_{B_R(x)} |u|$. By the maximum principle and the Cheng-Yau’s estimate (Theorem 2.6), we have

$$
\sup_{B_R(x)} |\nabla v| \leq C_3 \sup_{B_R(x)} |v| \leq C_3 \sup_{B_{2R/4}(x)} |u| \leq C_3 M_1,
$$

(4.10)

where we have used $B_R(x_0) \subset B_{2R/4}(x)$. By combining the equations (4.8)-(4.10), we conclude that for all $R \in (0,R_0)$ and $r \leq R/2$,

$$
\int_{B_r(x_0)} |\nabla u|^2 \leq 2 \int_{B_r(x_0)} |\nabla (u-v)|^2 + 2 \int_{B_r(x_0)} |\nabla v|^2 \\
\leq 2 \int_{B_r(x_0)} |\nabla (u-v)|^2 + 2C_3M_1 \cdot \mu(B_r(x_0)) \\
\leq 2\|Q\|_{L^\infty} \cdot \mu(B_R(x_0)) + 2C_3M_1 \cdot \mu(B_r(x_0)).
$$

(4.11)

Let $R_1 := \min\{R_0,2^{-N/2}\}$. Then for any $R < R_1$, by (4.11) and taking $r = R^{1+2/N} (\leq \frac{1}{4} R)$, we have for all $R \in (0,R_1)$ that

$$
r^2 \int_{B_r(x_0)} |\nabla u|^2 \leq 2\|Q\|_{L^\infty} r^2 \frac{\mu(B_R(x_0))}{\mu(B_r(x_0))} + 2C_3M_1 r^2 \\
\leq 2\|Q\|_{L^\infty} r^2 C_4(R/r)^N + 2C_3M_1 r^2 \\
\leq (2\|Q\|_{L^\infty} C_4 + 2C_3M_1)r^{\frac{N}{N-2}}.
$$

(4.12)
where we used $\mu(B_R(x_0)) \leq C_4(R/r)^N$ for some $C_4$ depending only on $K, N$, (by $r < R < 1$, see (2.1)). It follows from the local Poincaré inequality [Raj12] that

$$\int_{B_r(x_0)} |u - \int_{B_r(x_0)} u| \leq C_{N,K,R}(r^2 \int_{B_{2r}(x_0)} |\nabla u|^2)^{\frac{1}{2}} \leq C_{K,N,R}.\|Q\|_{L^\infty,m} \cdot r^{\frac{N}{2}}$$

for all $r$ such that $r^{\frac{N}{2}} \in (0, R_1)$, which guarantees that $u \in C^{1/(N+2)}(B_{R/2}(x))$, due to the Campanato theorem on metric measure spaces [Gör09, Theorem 3.2].

Here the Hölder index $1/(N + 2)$ is not optimal. We will show, in the next section, that $u$ is locally Lipschitz continuous provided that the metric measure space $(X, d, \mu)$ is non-collapsed and $Q$ satisfies (1.3).

In particular, Lemma 4.3 implies that $\{x \in \Omega \mid |u(x)| > 0\}$ is an open set. By combining with the argument in Lemma 4.1, we get the following consequence.

**Lemma 4.4 (Harmonicity).** Let $u = (u_1, \ldots, u_m)$ be a local minimizer of $J_Q$ in (1.2) with $Q \in L^\infty(\Omega)$. Then each component $u_i$ is harmonic on the open set $\Omega_u := \{x \in \Omega \mid |u(x)| > 0\}$ for each $i = 1, \ldots, m$.

**Proof.** We know from Lemma 4.3 that the set $\Omega_u$ is open. It suffices to show that $\Delta u_i = 0$ in the sense of distributions on each small ball $B_R(x_0) \subset \Omega_u$, for each $i = 1, 2, \ldots, m$.

For each $0 \leq \phi \in \text{Lip}_0(B_R(x_0))$ let

$$v_{i,\delta} = (u_1, \ldots, u_{i-1}, u_i + \delta \phi, u_{i+1}, \ldots, u_m),$$

where $i \in \{1, \ldots, m\}$ and $\delta > 0$, then $v_{i,\delta} \in \mathcal{A}_\mathbf{g}$. Noted that $d(v_{i,\delta}, u) \leq \delta \|\phi\|_{W^{1,2}} + Q_{\text{max}}\mu(\text{supp}\phi)$, when both $\delta$ and $\mu(B_R(x_0))$ are sufficiently small, the local minimality of $u$ yields

$$\int_{\Omega} (|\nabla u_i|^2 + Q\chi_{\{|u_i| > 0\}}) \leq \int_{\Omega} (|\nabla v_{i,\delta}|^2 + Q\chi_{\{|v_{i,\delta}| > 0\}}) \leq \int_{\Omega} (|\nabla v_{i,\delta}|^2 + Q\chi_{\{|u_i| > 0\}}),$$

where we have observed $\{|v_{i,\delta}| > 0\} = \{|u_i| > 0\}$, as $\delta$ small enough. Thus, we have

$$\int_{\Omega} |\nabla u_i|^2 \leq \int_{\Omega} (|\nabla u_i + \delta \phi|^2 = \int_{\Omega} (|\nabla u_i|^2 + 2\delta \langle \nabla u_i, \nabla \phi \rangle + \delta^2 |\nabla \phi|^2)$$

for all sufficiently small $\delta$. Therefore, the arbitrariness of $\delta$ gives

$$\int_{\Omega} \langle \nabla u_i, \nabla \phi \rangle \leq 0.$$ (4.16)

This yields $\Delta u_i \leq 0$ on $B_R(x_0)$ in the sense of distributions. Meanwhile, Lemma 4.1 asserts that $\Delta u_i \geq 0$ on $\Omega$ in the sense of distributions. Thus, we conclude that $u_i$ is harmonic on $B_R(x_0) \subset \Omega_u$. The proof is finished.

**Remark 4.5.** Recently, N. Gigli and I. V. Violo [GV21] obtained the locally Hölder continuity of a solution to an obstructed problem on $RCD(K, N)$-spaces.

5. Lipschitz Continuity of Local Minimizers

In this section, we derive the Lipschitz regularity for local minimizers of $J_Q$ in (1.2) on a non-collapsed $RCD$ metric measure space. For this, we will begin with a mean value inequality on general $RCD(K, N)$-spaces.
5.1. Mean value inequality.

Let \((X, d, \mu)\) be an \(RCD(K, N)\) metric measure space with \(K \in \mathbb{R}\) and \(N \in (1, +\infty)\).

**Lemma 5.1** (Stokes formula on balls). Let \(B_R(x_0) \subset X\) and \(\rho(\cdot) = d(\cdot, x_0)\), \(\phi \in C^2([0, R])\) and let \(\psi = \phi \circ \rho\). Suppose that \(u \in C(B_R(x_0)) \cap W^{1,2}(B_R(x_0))\). If \(\Delta \psi\) is a signed Radon measure, then

\[
\int_{B_r(x_0)} u \, d\Delta \psi = - \int_{B_r(x_0)} \langle \nabla u, \nabla \psi \rangle + \phi'(r) \frac{d}{ds} \bigg|_{s=r} \left( \int_{B_r(x_0)} u \right)
\]

holds for almost all \(r \in (0, R)\).

**Proof.** Since \(\Delta \psi\) and \(\mu\) are signed Radon measures, we have for almost all \(r \in (0, R)\) that

\[
\lim_{j \to \infty} |\Delta \psi| (B_{r+1/j}(x_0) \setminus B_r(x_0)) = 0
\]

and

\[
\lim_{j \to \infty} \mu(B_{r+1/j}(x_0) \setminus B_r(x_0)) = 0.
\]

Meanwhile, noted that \(s \mapsto \int_{B_s(x_0)} u\) is locally Lipschitz continuous on \((0, R)\), it is differentiable almost everywhere on \((0, R)\) too. We fix an \(r\) such that both of them hold. For \(j\) sufficiently large, let \(u_j = \eta_j(\rho)u \in W^{1,2}_0(B_R(x_0))\), where

\[
\eta_j(t) = \begin{cases} 
1 & \text{if } t \in [0, r], \\
1 - j(t - r) & \text{if } t \in (r, r + \frac{1}{j}], \\
0 & \text{if } t \in \left[ r + \frac{1}{j}, R \right].
\end{cases}
\]

On the one hand,

\[
\int_{B_R(x_0)} u_j d\Delta \psi = \int_{B_R(x_0) \setminus B_r(x_0)} u_j d\Delta \psi + \int_{B_r(x_0)} u_j d\Delta \psi
\]

\[
= \int_{B_R(x_0) \setminus B_r(x_0)} \eta_j u d\Delta \psi + \int_{B_r(x_0)} u d\Delta \psi,
\]

where

\[
\left| \int_{B_R(x_0) \setminus B_r(x_0)} \eta_j u d\Delta \psi \right| \leq |\Delta \psi|(B_{r+1/j}(x_0) \setminus B_r(x_0)) \|u\|_{C^0(B_R(x_0))}.
\]

By combining this, (5.2) and (5.6), we have

\[
\lim_{j \to \infty} \int_{B_R(x_0)} u_j d\Delta \psi = \int_{B_r(x_0)} u d\Delta \psi.
\]

On the other hand, from Remark 2.4(1) and \(u_j \in W^{1,2}_0(B_R(x_0))\), we have

\[
\int_{B_R(x_0)} u_j d\Delta \psi = - \int_{B_R(x_0)} \langle \nabla u_j, \nabla \psi \rangle
\]

\[
= - \int_{B_R(x_0)} \langle \nabla u, \nabla \psi \rangle \eta_j - \int_{B_R(x_0)} \langle \nabla \eta_j, \nabla \psi \rangle u
\]

\[
= - \int_{B_R(x_0) \setminus B_r(x_0)} \langle \nabla u, \nabla \psi \rangle \eta_j - \int_{B_r(x_0)} \langle \nabla u, \nabla \psi \rangle
\]

\[
- \int_{B_R(x_0)} \langle \nabla \eta_j, \nabla \psi \rangle u.
\]
Remark. The proof is finished.

Therefore, letting \( (5.3) \) and \( \frac{\mu(B_{r_\delta}(x_0))}{\rho} \), we conclude that the first term of right hand side in \((5.8)\) converges to 0 as \( j \to \infty \).

Noted that \( (5.12) \) and that \( \phi \) is a signed Radon measure, then

\[
\int_{B_{r}(x_0)} \langle \nabla \eta_j, \nabla \psi \rangle u = \int_{B_{r_\delta}(x_0) \setminus B_r(x_0)} \langle -j \nabla \rho, \nabla \psi \rangle u
\]

\[
= -j \int_{B_{r_\delta}(x_0) \setminus B_r(x_0)} \phi'(\rho) u
\]

\[
= -j \phi'(r) \int_{B_{r_\delta}(x_0) \setminus B_r(x_0)} u - j \int_{B_{r_\delta}(x_0) \setminus B_r(x_0)} (\phi'(\rho) - \phi'(r))
\]

\[
= -\phi'(r) \int_{B_{r_\delta}(x_0) \setminus B_r(x_0)} u - \int_{B_r(x_0)} u - j \int_{B_{r_\delta}(x_0) \setminus B_r(x_0)} (\phi'(\rho) - \phi'(r)) u,
\]

and that

\[
\left| \int_{B_{r_\delta}(x_0) \setminus B_r(x_0)} (\phi'(\rho) - \phi'(r)) u \right|
\]

\[
\leq j \sup_{[0, R]} \left| \phi''(\rho) \right| u
\]

\[
\leq \sup_{[0, R]} \left| \phi''(\rho) \right| \left( \int_{B_{r_\delta}(x_0) \setminus B_r(x_0)} |u|^2 \frac{\mu(B_{r_\delta}(x_0))}{\rho} \right)^\frac{1}{2},
\]

we conclude, by \((5.3)\) and the fact that \( r \mapsto \int_{B_r(x_0)} u \) is differentiable at \( r \), that the third term of right hand side in \((5.8)\) converges to \( \phi'(r) \frac{d}{dr} \left( \int_{B_r(x_0)} u \right) \), as \( j \to \infty \).

Therefore, letting \( j \to \infty \) in \((5.8)\), we obtain for almost all \( r \in (0, R) \) that

\[
\int_{B_r(x_0)} u d\Delta \phi = -\int_{B_r(x_0)} \langle \nabla u, \nabla \psi \rangle + \phi'(r) \frac{d}{dr} \left( \int_{B_r(x_0)} u \right).
\]

The proof is finished.

A similar argument with a different cut-off function yields the following slight variant.

Remark 5.2 (Stokes formula on annuli). Let \( B_{R_2}(x_0) \setminus B_{R_1}(x_0) \subset X \), \( \rho(\cdot) = d(\cdot, x_0) \), \( \phi \in C^2([R_1, R_2]) \), and let \( \psi = \phi \circ \rho \). Suppose that

\[
u \in C(B_{R_2}(x_0) \setminus B_{R_1}(x_0)) \cap W^{1,2}(B_{R_2}(x_0) \setminus B_{R_1}(x_0)),
\]

and if \( \Delta \psi \) is a signed Radon measure, then

\[
\int_{B_{r_2}(x_0) \setminus B_{r_1}(x_0)} u d\Delta \psi = -\int_{B_{r_2}(x_0) \setminus B_{r_1}(x_0)} \langle \nabla u, \nabla \psi \rangle
\]

\[
+ \phi'(r_2) \frac{d}{ds} \bigg|_{s=r_2} \left( \int_{B_s(x_0)} u \right) - \phi'(r_1) \frac{d}{ds} \bigg|_{s=r_1} \left( \int_{B_s(x_0)} u \right)
\]
Remark

Lemma 5.1 asserts that

\[ r (5.16) \]

for all \( R \). Since \( R \), holds for almost all \( u \).

Proof of Lemma 5.3. Here and in the following of this proof, \( \text{Mean value inequality} \) \( \text{Lemma 5.3} \). By substituting the assumption (5.15) into (5.18) and let \( (5.18) \)

\( \rho \)

\( \hat{\rho} \)

\( \Omega \)

\( (5.15) \)

\( \lim \inf \frac{1}{r^N} \int_{B_r(x_0)} u(x) \mu = 0, \)

then

\[ \int_{B_R(x_0)} u \leq C_1 \int_0^R \frac{e^{-C_2 s^2}}{s^{N+1}} \int_{B_r(x_0)} \langle \nabla u, \nabla \rho^2 \rangle ds \]

for all \( R \), where the constants \( C_1, C_2 \) only depend on \( N, K \) and \( \Omega \).

Remark 5.4. On the Euclidean space \( \mathbb{R}^N \), we have

\[ \int_{B_R(x_0)} u - u(x_0) = \frac{1}{2} \int_0^R \frac{1}{s} \int_{B_s(x_0)} \langle \nabla u, \nabla \rho^2 \rangle ds. \]

Proof of Lemma 5.3. Noted that \( \Delta \rho^2 \) is a signed Radon measure \([\text{Gig15}]\) and \( u \geq 0 \), Lemma 5.1 asserts that

\[ \int_{B_r(x_0)} u d\rho^2 = - \int_{B_r(x_0)} \langle \nabla u, \nabla \rho^2 \rangle + \frac{2r}{dr} \int_{B_r(x_0)} u \]

holds for almost all \( r \). By combining this and the Laplacian comparison theorem (see (2.8) in Theorem 2.5), we get

\[ \frac{d}{dr} \left( \frac{1}{r^N} \int_{B_r(x_0)} u \right) \leq \frac{1}{2r} \int_{B_r(x_0)} u d\Delta (\rho^2) + \frac{1}{2r} \int_{B_r(x_0)} \langle \nabla u, \nabla \rho^2 \rangle 

\]

Here and in the following of this proof, \( C_1, C_2, C_3, \cdots \), will denote positive constants depending only on \( N, K \) and \( \Omega \). This gives

\[ \frac{d}{dr} \left( \frac{1}{r^N} \int_{B_r(x_0)} u \right) \leq C_3 r \cdot \left( \frac{1}{r^N} \int_{B_r(x_0)} u \right) + \frac{1}{2r^{N+1}} \int_{B_r(x_0)} \langle \nabla u, \nabla \rho^2 \rangle \]

for almost all \( r \). Multiplying both sides by \( \exp(-C_3 r^2/2) \), we have

\[ \frac{d}{dr} \left( e^{-\frac{1}{2} C_3 r^2} \frac{1}{r^N} \int_{B_r(x_0)} u \right) \leq e^{-\frac{1}{2} C_3 r^2} \frac{1}{2r^{N+1}} \int_{B_r(x_0)} \langle \nabla u, \nabla \rho^2 \rangle. \]

Since \( r \mapsto \frac{1}{r} e^{-\frac{1}{2} C_3 r^2} \int_{B_r(x_0)} u \) is locally Lipschitz on \((0, R] \), by integrating the above inequality over \((r, R)\) for any \( r < R \), we get

\[ e^{-\frac{1}{2} C_3 R^2} \frac{1}{R^N} \int_{B_R(x_0)} u - e^{-\frac{1}{2} C_3 r^2} \frac{1}{r^N} \int_{B_r(x_0)} u \]

\[ \leq \int_r^R e^{-\frac{1}{2} C_3 s^2} \frac{1}{2s^{N+1}} \int_{B_s(x_0)} \langle \nabla u, \nabla \rho^2 \rangle ds. \]

By substituting the assumption (5.15) into (5.18) and let \( r \to 0^+ \), we get

\[ e^{-\frac{1}{2} C_3 R^2} \frac{1}{R^N} \int_{B_R(x_0)} u \leq \int_0^R e^{-\frac{1}{2} C_3 s^2} \frac{1}{2s^{N+1}} \int_{B_s(x_0)} \langle \nabla u, \nabla \rho^2 \rangle ds. \]
Therefore, by \( \mu(B_R(x_0))/R^N \geq C_5 := \mu(B_{\text{diam}(\Omega)}(x_0))/[\text{diam}(\Omega)]^N \) (this follows from (2.1) and \( R < \text{diam}(\Omega) \)), we conclude
\[
\int_{B_R(x_0)} u \leq e^{\frac{1}{2}C_2R^2} \frac{1}{C_5} \int_0^R e^{-\frac{1}{2}C_2s^2} \int_{B_s(x_0)} \langle \nabla u, \nabla \rho^2 \rangle ds.
\]
This implies (5.16) with \( C_1 := \frac{1}{2}C_2 \) and \( C_2 := \frac{C_5}{2} \). The proof is finished. \( \square \)

### 5.2. Lipschitz continuity of local minimizers of \( J_Q \)

From now on, we shall suppose that \((X, d, \mu)\) is an \( ncRCD(K, N) \) metric measure space with \( K \leq 0 \) and \( N \in (1, +\infty) \). Let \( \Omega \subset X \) be a bounded domain and let \( Q \in L^\infty(\Omega) \) satisfy (1.3) for two positive numbers \( Q_{\min} \) and \( Q_{\max} \). Recall that a map
\[
u = (u_1, u_2, \ldots, u_m) \in W^{1,2}(\Omega, [0, +\infty)^m)
\]
is a local minimizer of \( J_Q \) in (1.2) if there exist a data \( g \in W^{1,2}(\Omega, [0, +\infty)^m) \) and \( \varepsilon_\nu > 0 \) such that \( J_Q(u) \leq J_Q(v) \) for all \( v \in \mathcal{A}_g \) with \( d(u, v) < \varepsilon_\nu \), where the \( \mathcal{A}_g \) and \( d(u, v) \) are given in (1.4) and (1.5), respectively. From Lemma 4.3, we can assume that \( \nu \) is continuous on \( \Omega \). The set \( \Omega_u := \{ x \in \Omega | u(x) > 0 \} \) is open.

The following lemma is inspired by the classical Caccioppoli inequality.

**Lemma 5.5.** Let \( \nu = (u_1, \ldots, u_m) \) be a local minimizer of \( J_Q \) in (1.2) with \( Q \) satisfying (1.3), and let \( \Omega' \subset \Omega \). Then there exists a constant \( R_0 (\leq 1) \) depending only on \( N, K, d(\Omega', \partial \Omega), Q_{\max} \) and \( \varepsilon_\nu \), such that for all balls \( B_r(x) \) with \( r < R_0 \) and \( d(x, \Omega') < R_0 \), it holds
\[
\int_{B_r(x)} \langle \nabla u_i, \nabla \phi \rangle \leq (Q_{\max} \cdot \mu(B_r(x)))^{1/2} \cdot \|\phi\|_{W^{1,2}(B_r(x))}
\]
for all \( i = 1, \ldots, m \) and \( \phi \in W^{1,2}(B_r(x)) \) and \( \phi \geq 0 \).

**Proof.** From (2.13), there is a number \( R_0 \in (0, \frac{1}{2}d(\Omega', \partial \Omega)) \) with \( R_0 < 1 \), (depending only on \( N, K, d(\Omega', \partial \Omega), \Omega, Q_{\max} \) and \( \varepsilon_\nu \)) such that
\[
(Q_{\max} \mu(B_{R_0}(x)))^{1/2} + Q_{\max} \mu(B_{R_0}(x)) < \varepsilon_\nu, \quad \forall \ x \in \Omega.
\]
Fix any ball \( B_r(x) \) with \( r < R_0 \) and \( d(x, \Omega') < R_0 \), where \( R_0 \) is given in the above (5.20). The inequality (5.19) obviously holds if \( \|\phi\|_{W^{1,2}(B_r(x))} = 0 \), so we are assuming that \( \|\phi\|_{W^{1,2}(B_r(x))} > 0 \) in the following. Put
\[
\delta := \left( Q_{\max} \cdot \mu(B_r(x)) \right)^{1/2} \left( \|\phi\|_{W^{1,2}(B_r(x))} \right).
\]
Let \( v = (u_1, \ldots, u_{i-1}, u_i + \delta \phi, u_{i+1}, \ldots, u_m) \in \mathcal{A}_g \). Note that
\[
d(v, u) \leq \delta \cdot \|\phi\|_{W^{1,2}(B_r(x))} + Q_{\max} \mu(\text{supp} \phi)
\]
\[
\leq (Q_{\max} \mu(B_r(x)))^{1/2} + Q_{\max} \mu(B_r(x)) < \varepsilon_\nu
\]
provided \( r < R_0 \), by (5.20). The local minimality of \( \nu \) implies
\[
\int_\Omega |\nabla u|^2 = J_Q(\nu) - \int_\Omega Q\chi_{\{|u| > 0\}} \leq J_Q(v) - \int_\Omega Q\chi_{\{|v| > 0\}}
\]
\[
\leq \int_\Omega |\nabla u|^2 + \int_\Omega (Q\chi_{\{|v| > 0\}} - Q\chi_{\{|u| > 0\}})
\]
\[
\leq \int_\Omega |\nabla v|^2 + Q_{\max} \mu(B_r(x))
\]
\[
\leq \int_\Omega |\nabla u|^2 + 2\delta \int_{B_r(x)} \langle \nabla u_i, \nabla \phi \rangle + \delta^2 \int_{B_r(x)} |\nabla \phi|^2 + Q_{\max} \mu(B_r(x))
\]
for all $r < R_0$. Therefore,
\[
-2 \int_{B_r(x)} \langle \nabla u_i, \nabla \phi \rangle \leq \delta \int_{B_r(x)} |\nabla \phi|^2 + \frac{Q_{\text{max}}}{\delta} \mu(B_r(x))
\]
\[
\leq \delta \cdot \|\phi\|^2_{L^2(B_r(x))} + \frac{Q_{\text{max}}}{\delta} \mu(B_r(x)),
\]
which is equivalent to (5.19) by (5.21), and the proof is finished. \hfill \Box

Combining Lemma 5.5 and Lemma 5.3, we are able to control the growth of local minimizers near the free boundary $\partial \Omega$. Let $\nu$ be the constant given in (5.15) in Lemma 5.3. For each $s < R_0$, by using Lemma 5.5 to $\phi = s^2 - \rho^2(x)$, we get
\[
\int_{B_r(x_0)} (\nabla u_i, \nabla \rho^2) = - \int_{B_r(x_0)} \langle \nabla u_i, \nabla (s^2 - \rho^2) \rangle
\]
\[
\leq \left( Q_{\text{max}} \cdot \mu(B_s(x_0)) \right)^{1/2} \cdot s^2 - \rho^2 \|\nabla s^2_{B_r(x_0)} \|
\leq \left( Q_{\text{max}} \cdot \mu(B_s(x_0)) \right)^{1/2} \left( \int_{B_r(x_0)} (4\rho^2|\nabla \rho|^2 + (s^2 - \rho^2)^2) \right)^{1/2}
\leq 3\sqrt{Q_{\text{max}}} \cdot \mu(B_s(x_0)) \cdot s \quad \text{(by $\rho < s$, $s < R_0 \leq 1$)}
\leq C_{N,K} \sqrt{Q_{\text{max}}} \cdot s^{N+1} \quad \text{(by (2.13)), $s \leq 1$)}.
\]
Since $u$ is continuous on $\overline{B_r(x_0)}$ by Lemma 4.3, it follows from Lemma 5.3 and $u_i(x_0) = 0$ that
\[
(5.23) \quad \int_{B_r(x_0)} u_i \leq C_1 \int_0^r \frac{e^{-C_2 s^2}}{s^{N+1}} \int_{B_s(x_0)} (\nabla u_i, \nabla \rho^2) ds \leq C_3 \sqrt{Q_{\text{max}}} \cdot r,
\]
for all $r < R_0$. Thus, by using the fact that $u_i$ is subharmonic (Lemma 4.1) and $u_i \geq 0$, we get (see, for example, [KS01, Theorem 4.2])
\[
\sup_{B_r(x_0)} u_i \leq C_4 \int_{B_r(x_0)} u_i \leq C_4 C_3 \sqrt{Q_{\text{max}}} \cdot r, \quad \forall \ r < R_0.
\]
The proof is finished. \hfill \Box

As a corollary of the combination of the linear growth and the Cheng-Yau’s gradient estimate for harmonic functions, one can get the following gradient estimate near the free boundary $\partial \{|u| > 0\} \cap \Omega$.

**Lemma 5.7.** Let $u = (u_1, \ldots, u_m)$ be a local minimizer of $J_Q$ in (1.2) with $Q$ satisfying (1.3), and let $\Omega' \Subset \Omega$. There exists a positive constant $C = C_{N,K,Q} > 0$ (depending only on $K, N$ and $\Omega$), such that: if $x_1 \in \Omega'$ and if $d(x_1, \{|u| = 0\} \cap \Omega) < R_0/8$, then it holds
\[
(5.24) \quad \text{Lip } u_i(x_1) \leq C \sqrt{Q_{\text{max}}}, \quad i = 1, 2, \ldots, m,
\]
where $R_0$ is given in Lemma 5.5, and $\text{Lip} \, u_i$ is the pointwise Lipschitz constant defined in (2.4).

**Proof.** We will finish this proof by considering two cases as follows.

(i) In the case where $d(x_1, \{|u| = 0\} \cap \Omega) = 0$. The continuity of $u$ implies the \{|u| = 0\} is relative closed in $\Omega$. This implies $x_1 \in \{|u| = 0\}$ in this case. By Lemma 5.6, we have
\[
\sup_{y \in B_{r/2}(x_1)} \frac{|u_i(y) - u_i(x_1)|}{r} \leq C \sqrt{Q_{\text{max}}}, \quad i = 1, 2, \ldots, m,
\]
for all $r < R_0$. By (2.4) and letting $r \to 0$, this yields $\text{Lip} \, u_i(x_1) \leq C \sqrt{Q_{\text{max}}}$. (ii) In the case where $d(x_1, \{|u| = 0\} \cap \Omega) > 0$. We put
\[
r_1 := d(x_1, \{|u| = 0\} \cap \Omega) \in (0, R_0/8).
\]
Since $B_{r_1/2}(x_1) \subset \{|u| > 0\}$, from Lemma 4.4, we have known that all $u_i$, $i = 1, \ldots, m$, are harmonic on $B_{r_1/2}(x_1)$. By using Cheng-Yau estimate, Theorem 2.6, we obtain
\[
\sup_{y \in B_{r_1/4}(x_1)} u_i(y) \leq C \sup_{y \in B_{r_1/4}(x_1)} u_i(y),
\]
where the constant $C$ depends only on $N, K$ and $\Omega$. Take $x_2 \in B_{2r_1}(x_1) \cap \{|u| = 0\}$. By applying Lemma 5.6 to $B_{2r_1}(x_2)$, (remark that $d(x_2, \Omega') < 2r_1$ and the assumption $8r_1 < R_0$) we have
\[
\sup_{y \in B_{r_1/4}(x_1)} u_i(y) \leq C \sqrt{Q_{\text{max}}} \cdot r_1.
\]
The combination of (5.25) and (5.26) implies the desired estimate (5.24). Now the proof is completed. \hfill \square

Now we are in the position to show the local Lipschitz continuity of $u$.

**Proof of Theorem 1.4.** Let $B_R(x) \subset \Omega$. Let $R_0$ be the constant given in Lemma 5.5 with respect to $\Omega' := B_{R/2}(x)$.

Take any $x_1 \in B_{R/2}(x)$. If $d(x_1, \Omega \cap \{|u| = 0\}) < R_0/8$, then Lemma 5.7 asserts
\[
\text{Lip} \, u_i(x_1) \leq C \sqrt{Q_{\text{max}}}, \quad i = 1, 2, \ldots, m,
\]
If $d(x_1, \Omega \cap \{|u| = 0\}) \geq R_0/8$, that is, $B_{R_0/10}(x_1) \subset \{|u| > 0\}$, then Cheng-Yau’s estimate, Theorem 2.6, asserts
\[
\text{Lip} \, u_i(x_1) \leq \frac{C}{R_0} \sup_{\Omega} |u|, \quad i = 1, 2, \ldots, m.
\]
By summing up the both cases and recalling Remark 4.2, we conclude that there exists a constant $L$ depending only on $N, K, \Omega, R, Q_{\text{max}}, 2\epsilon$ and $\int_{B_{R}(x)} |u|d\mu$, such that
\[
\sup_{B_{R/2}(x)} \text{Lip} \, u_i(x) \leq L, \quad \forall \, i = 1, 2, \ldots, m.
\]
Take any $y, z \in B_{R/4}(x)$. Let $\gamma : [0, d(x, y)] \to \Omega$ be a geodesic from $y$ to $z$. The triangle inequality implies that $\gamma \subset B_{R/2}(x)$. Noted that Lip $u_i$ is one of the upper gradient of $u_i$ (see [Che99]), the estimate (5.27) implies that
\[
|u_i(y) - u_i(z)| \leq \int_0^{d(y, z)} \text{Lip} \, u_i \circ \gamma(s)ds \leq L \cdot d(y, z),
\]
for each $i = 1, 2, \ldots, m$. The proof is finished. \hfill \square
6. LOCAL FINITENESS OF PERIMETER FOR THE FREE BOUNDARY

We continue to assume that \((X, d, \mu)\) is an ncRCD\((K, N)\) metric measure space with \(K \leq 0\) and \(N \in (1, +\infty)\). Let \(\Omega \subset X\) be a bounded domain and let \(Q \in L^\infty(\Omega)\) satisfy (1.3) for two positive numbers \(Q_{\min}\) and \(Q_{\max}\). Let
\[
\mathbf{u} = (u_1, u_2, \cdots, u_m)
\]
be a local minimizer of \(J_Q\) in (1.2) with a boundary data \(g \in W^{1,2}(\Omega, |0, +\infty|^m)\), i.e., there exists \(\varepsilon_u > 0\) such that \(J_Q(u) \leq J_Q(v)\) for all \(v \in \mathcal{A}_g\) with \(d(u, v) < \varepsilon_u\), where the \(\mathcal{A}_g\) and \(d(u, v)\) are given in (1.4) and (1.5), respectively. From Theorem 1.4, we know that \(u\) is locally Lipschitz continuous in the interior of \(\Omega\).

We begin with the nondegeneracy of the local minimizer \(u\) near the free boundary.

6.1. Nondegeneracy.

Theorem 6.1 (Nondegeneracy). Let \(u = (u_1, \ldots, u_m)\) be a local minimizer of \(J_Q\) in (1.2) with \(Q\) satisfying (1.3), and let \(\Omega' \subset \Omega\). Then there is a constant \(R_1 > 0\) (depending only on \(N, K, \Omega', Q_{\max}, \varepsilon_u\) and the Lipschitz constant of \(u\) on \(\Omega'\)) such that for any ball \(B_r(x_0) \subset \Omega\) with \(x_0 \in \partial\{u > 0\} \cap \Omega'\) and \(r < R_1\), it holds
\[
\sup_{B_r(x_0)} |u| \geq c\sqrt{Q_{\min}} \cdot r,
\]
where the positive constant \(c\) depends only on \(N, K, \Omega', \varepsilon_u\) and the Lipschitz constant of \(u\) on \(\Omega'\).

Proof. In the Euclidean setting, this assertion was established in [CSY18, Theorem 3] and [MTV17]. It was extended to smooth Riemannian manifolds in [LS20]. Here we will extend their arguments to nonsmooth setting. Without loss of the generality, we can assume that \(K < 0\).

Fix any \(r \in (0, 1)\) and let \(M = \sup_{B_r(x_0)} |u|\). Since \(u\) is Lipschitz continuous on \(\Omega'\) (see Theorem 1.4) and \(u(x_0) = 0\), we get \(M \leq L\), the Lipschitz constant of \(u\) on \(\Omega'\). Given any \(\theta \in (0, 1)\), as in [CSY18], we consider the map \(v = (v_1, \ldots, v_m)\), where
\[
v_i(y) = \begin{cases} u_i(y), & \text{if } y \in B_r(x_0), \\ \min\{u_i(y), M\psi(\frac{\rho(y)}{r})\}, & \text{if } y \notin B_r(x_0), \end{cases}
\]
for all \(i = 1, \ldots, m\) and \(y \in \Omega\), where \(\rho(\cdot) = d(\cdot, x_0)\),
\[
\psi(\theta) = \frac{(\phi_{N,K}(t) - \phi_{N,K}(\theta))^+}{\phi_{N,K}(1) - \phi_{N,K}(\theta)}
\]
and
\[
\phi_{N,K}(s) = -\int_1^s \left(\frac{\sinh(\sqrt{-K/(N-1)t})}{\sqrt{-K/(N-1)}}\right)^{1-N} dt.
\]
Then it is clear that \(v \in \mathcal{A}_g\) and \(u_i - v_i \in W^{1,2}_0(B_r(x_0))\) for all \(i = 1, 2, \ldots, m\).

(i) We first check that \(d(v, u) < \varepsilon_u\) provided both \(r\) and \(\theta\) are sufficiently small. The co-area formula gives
\[
\int_{B_r(x_0) \setminus B_{r\theta}(x_0)} |\nabla \psi(\frac{\rho}{r})|^2 \, d\mu \leq \int_{\theta r}^r \left[\frac{1}{s} \psi(\frac{s}{r})\right]^2 \cdot d\frac{d\mu}{d\mu(B_s(x_0))} ds.
\]
Since \((X, d, \mu)\) is non-collapsed, substituting (2.14) into the above inequality, we obtain
\[
\int_{B_r(x_0) \setminus B_{r\theta}(x_0)} |\nabla \psi(\frac{\rho}{r})|^2 \, d\mu \leq C_1 r^{N-2} \cdot \left(\int_{\theta r}^r s^{1-N} ds\right)^{-1},
\]
where we have used $|\psi(t)| \leq C_{N,K} t^{1-N} s^{1-N}$ for all $t \in (\theta, 1)$. Here and in the following of this proof, all constants $C_1, C_2, \ldots$ depend only on $N, K, \Omega'$.

Noticed that $|\psi| \leq 1$ and $u_i - v_i = (u_i - M\psi(t))^+ \in W^{1,2}_0(B_r(x_0))$, we get
\begin{equation}
\int_{\Omega} |u_i - v_i|^2 \leq \int_{B_r(x_0)} ((u_i - M\psi(t))^+)^2 \leq M^2 \mu(B_r(x_0))
\end{equation}
and (by the fact that [Che99], for any $w \in W^{1,2}(\Omega)$, $|\nabla w^+| \leq |\nabla w|$ holds almost everywhere in $\Omega$,
\begin{equation}
\int_{\Omega} |\nabla(u_i - v_i)|^2 \leq \int_{B_r(x_0)} |\nabla(u_i - M\psi(t))|^2 \\
\leq 2 \int_{B_r(x_0) \setminus B_{\theta r}(x_0)} |\nabla(M\psi(t))|^2 + 2 \int_{B_r(x_0)} |\nabla u_i|^2 \\
\leq 2M^2 \int_{B_r(x_0) \setminus B_{\theta r}(x_0)} |\nabla \psi(t)|^2 + 2L^2 \mu(B_r(x_0)),
\end{equation}
where for the second inequality we have used $\psi = 0$ on $B_{\theta r}(x_0)$, and for the last inequality we have used $|\nabla u_i| \leq L$. Recall $M \leq L$.

From the combination of (6.4)-(6.6), the fact $M \leq L$, $u - v \in W^{1,2}_0(B_r(x_0), \mathbb{R}^m)$, and taking $\theta$ such that
\begin{equation}
C_1 \left( \int_{\theta}^{1} s^{1-N} \text{d}s \right)^{-1} = \frac{\varepsilon u}{8L^2},
\end{equation}
we conclude that $d(\mathbf{v}, \mathbf{u}) < \varepsilon u$ provided $r < R_1$ for some small number $R_1 > 0$ depending only on $N, K, L, Q_{\max}$ and $\varepsilon u$.

(ii) Fixed any $r \in (0, R_1)$ and taken $\theta$ in (6.7), the local minimality of $\mathbf{u}$ gives
\begin{align}
\int_{B_{\theta r}(x_0)} (|\nabla \mathbf{u}|^2 + Q \chi_{\{|\mathbf{u}| > 0\}}) \\
= \int_{B_r(x_0)} (|\nabla \mathbf{u}|^2 + Q \chi_{\{|\mathbf{u}| > 0\}}) - \int_{B_r(x_0) \setminus B_{\theta r}(x_0)} (|\nabla \mathbf{u}|^2 + Q \chi_{\{|\mathbf{u}| > 0\}}) \\
\leq \int_{B_r(x_0)} (|\nabla \mathbf{v}|^2 + Q \chi_{\{|\mathbf{v}| > 0\}}) - \int_{B_r(x_0) \setminus B_{\theta r}(x_0)} (|\nabla \mathbf{u}|^2 + Q \chi_{\{|\mathbf{u}| > 0\}}) \\
= \int_{B_{\theta r}(x_0) \setminus B_{\theta r}(x_0)} (|\nabla \mathbf{v}|^2 + Q \chi_{\{|\mathbf{v}| > 0\}}) - \int_{B_r(x_0) \setminus B_{\theta r}(x_0)} (|\nabla \mathbf{u}|^2 + Q \chi_{\{|\mathbf{u}| > 0\}}),
\end{align}
where we used that $\mathbf{v}, B_{\theta r}(x_0) \equiv 0$. Noticed that $(B_r(x_0) \setminus B_{\theta r}(x_0)) \cap \{|\mathbf{u}| > 0\} = (B_r(x_0) \setminus B_{\theta r}(x_0)) \cap \{|\mathbf{v}| > 0\}$, this yields
\begin{equation}
\int_{B_{\theta r}(x_0)} (|\nabla \mathbf{u}|^2 + Q \chi_{\{|\mathbf{u}| > 0\}}) \leq \int_{B_r(x_0) \setminus B_{\theta r}(x_0)} (|\nabla \mathbf{v}|^2 - |\nabla \mathbf{u}|^2) \\
= \sum_{i=1}^{m} \int_{B_r(x_0) \setminus B_{\theta r}(x_0)} (|\nabla v_i|^2 - |\nabla u_i|^2).}
\end{equation}
Let \( w_i = (u_i - M\psi_{\theta}(\frac{\rho}{r}))^+ \) for each \( i = 1, 2, \ldots, m \). Then \( u_i = v_i + w_i \) and

\[
\begin{align*}
\int_{B_r(x_0) \setminus B_{sr}(x_0)} \left( |\nabla v_i|^2 - |\nabla u_i|^2 \right) \\
= \int_{B_r(x_0) \setminus B_{sr}(x_0)} \left( -2 \langle \nabla v_i, \nabla w_i \rangle - |\nabla w_i|^2 \right) \\
\leq -2 \int_{B_r(x_0) \setminus B_{sr}(x_0)} \langle \nabla v_i, \nabla w_i \rangle \\
= -2 \int_{B_r(x_0) \setminus B_{sr}(x_0)} \langle \nabla (M\psi_{\theta}(\frac{\rho}{r})), \nabla w_i \rangle,
\end{align*}
\]

(6.9)

for each \( i = 1, \ldots, m \), where for the last equality, we have used that \( |\nabla w_i| = 0 \mu\text{-a.e.} \) on \( \{w_i = 0\} \cap B_r(x_0) \) (see [Che99] or Proposition 2.1(3)) and that \( v_i = M\psi_{\theta}(\frac{\rho}{r}) \) on \( \{w_i \neq 0\} \cap B_r(x_0) \). Remark that \( \{w_i \neq 0\} \) is open by the continuity of \( w_i \).

(iii) Next we want to estimate \( \sum_{i=1}^m I_{\theta r, r}(w_i) \), where we denote

\[
I_{\theta r, r}(w_i) := -\int_{B_{sr}(x_0) \setminus B_{s \theta r}(x_0)} \langle \nabla (M\psi_{\theta}(\frac{\rho}{r})), \nabla w_i \rangle
\]

(6.10)

for any \( r_1, r_2 \in [\theta r, r] \) with \( r_1 < r_2 \).

By the Laplacian comparison theorem (see (2.7) in Theorem 2.5) and that the space \((M, \frac{\rho}{r})\) satisfies \( RCD(K, r^2, N) \), we conclude that

\[
\Delta \psi_{\theta}(\frac{\rho}{r}) \leq 0 \quad \text{on } B_r(x_0) \setminus B_{sr}(x_0)
\]

in the sense of distributions. From Remark 5.2, we have for almost all \( r_1, r_2 \in (\theta r, r) \) with \( r_1 < r_2 \), that

\[
I_{\theta r, r}(w_i) = \int_{B_{sr}(x_0) \setminus B_{s \theta r}(x_0)} w_i d\Delta(M\psi_{\theta}(\frac{\rho}{r})) \\
+ M \frac{\psi'_{\theta}(\frac{\rho}{r})}{r} \frac{d}{ds} \bigg|_{s=r_1} \int_{B_s(x_0)} w_i - M \frac{\psi'_{\theta}(\frac{\rho}{r})}{r} \frac{d}{ds} \bigg|_{s=r_2} \int_{B_s(x_0)} w_i \\
\leq C_{N,K,\theta} \frac{M}{r} \frac{d}{ds} \bigg|_{s=r_1} \int_{B_s(x_0)} w_i,
\]

(6.11)

where we have used that \( 0 \leq \phi_{N,K}'(t) \leq C_{N,K,\theta} \) for all \( t \in (\theta, 1) \) and that \( \frac{d}{ds} \int_{B_s(x_0)} w_i \geq 0 \) for almost all \( s \in (\theta r, r) \).

For almost every \( r_1 \in (\theta r, r) \) such that both \( s \mapsto \int_{B_s(x_0)} w_i \) and \( s \mapsto \int_{B_s(x_0)} u_i \) are differentiable at \( r_1 \), we have

\[
\frac{d}{ds} \bigg|_{s=r_1} \int_{B_s(x_0)} w_i - \frac{d}{ds} \bigg|_{s=r_1} \int_{B_s(x_0)} u_i = \lim_{\delta \to 0^+} \frac{1}{\delta} \int_{A_{r_1-r_1+\delta}} (u_i - w_i) \\
\leq \lim_{\delta \to 0^+} \frac{1}{\delta} \int_{A_{r_1-r_1+\delta}} |u_i - w_i|,
\]

(6.12)

where \( A_{r_1-r_1+\delta} := B_{r_1+\delta}(x_0) \setminus B_{r_1}(x_0) \).

From the definition \( w_i = (u_i - M\psi_{\theta}(\frac{\rho}{r}))^+ \), \( M \leq L \) and \( |\psi_{\theta}'(t)| \leq C_{N,K,\theta} \) in \((\theta, 1)\), it follows that \((u_i - w_i)\) is Lipschitz continuous on \( B_r(x_0) \) with a Lipschitz constant \( C_{L,N,K,\theta} > 0 \). By using \((u_i - w_i)(y) = 0 \) for any \( y \in \partial B_{sr}(x_0) \), we conclude that

\[
\sup_{A_{r_1-r_1+\delta}} |u_i - w_i| \leq C_{L,N,K,\theta} \cdot (r_1 + \delta - \theta r).
\]

(6.13)
Substituting this into (6.12), we have

\[
\frac{d}{ds}igg|_{s=r_1} \int_{B_s(x_0)} u_i = \frac{1}{2r} \left( \int_{B_r(x_0)} u_i d\Delta(\rho^2) + \int_{B_{r_1}(x_0)} \langle \nabla u_i, \nabla \rho^2 \rangle \right) \\
\leq C_{N,K,r} \frac{1}{2r} \int_{B_{r_1}(x_0)} u_i + \int_{B_{r_1}(x_0)} |\nabla u_i| \\
\leq C_{N,K,r} \frac{1}{r} \int_{B_{r_1}(x_0)} |u| + \int_{B_{r_1}(x_0)} |\nabla u|,
\]

(6.14)

where for the last inequality we have used (2.2).

On the other hand, by using Lemma 5.1 and Theorem 2.5 (the Laplacian comparison theorem), we have for almost all \( r_1 \in (\theta r, 3\theta r/2) \)

\[
\frac{d}{ds}igg|_{s=r_1} \int_{B_s(x_0)} u_i = \frac{1}{2r_1} \left( \int_{B_{r_1}(x_0)} u_i d\Delta(\rho^2) + \int_{B_{r_1}(x_0)} \langle \nabla u_i, \nabla \rho^2 \rangle \right) \\
\leq C_{N,K,r} \frac{1}{2r_1} \int_{B_{r_1}(x_0)} u_i + \int_{B_{r_1}(x_0)} |\nabla u_i| \\
\leq C_{N,K,r} \frac{1}{r} \int_{B_{r_1}(x_0)} |u| + \int_{B_{r_1}(x_0)} |\nabla u|,
\]

where we have used \( u_i \geq 0 \) and \( r_1 \geq \theta r \). This implies for almost all \( r_1 \in (\theta r, 3\theta r/2) \)

\[
\sum_{i=1}^{m} \frac{d}{ds}igg|_{s=r_1} \int_{B_s(x_0)} u_i \\
\leq C_{N,K,m,r} \int_{B_{r_1}(x_0)} \left( \frac{1}{r} |u| + |\nabla u| \right) \\
\leq C_{N,K,m,r} \int_{B_{r_1}(x_0)} \left( \frac{1}{r} |u| \chi_{\{|u|>0\}} + |\nabla u| \chi_{\{|u|>0\}} \right) \\
\leq C_{N,K,m,r} \int_{B_{r_1}(x_0)} \left( \frac{M}{r} \chi_{\{|u|>0\}} + \frac{1}{2\sqrt{Q_{\min}}} (Q\chi_{\{|u|>0\}} + |\nabla u|^2) \right)
\]

(6.15)

where for the second inequality we have used \(|\nabla u| = 0\) \( \mu \)-a.e. in \(|u| = 0\) ([Che99]), and for the third inequality we have used \( \sup_{B_{r_1}(x_0)} |u| \leq M \). From the combination of (6.11), (6.14), (6.15) and the fact

\[
\lim_{r_1 \to \theta r} I_{\theta r, r_1}(w_i) = 0, \quad \lim_{r_2 \to r} I_{r_2, r}(w_i) = 0,
\]

by letting \( r_1 \to \theta r \) and \( r_2 \to r \), we conclude

\[
\sum_{i=1}^{m} I_{\theta r, r_1}(w_i) = \sum_{i=1}^{m} \left( I_{\theta r, r_1}(w_i) + I_{r_1, r_2}(w_i) + I_{r_2, r}(w_i) \right) \\
\leq \lim_{r_1 \to \theta r} \sum_{i=1}^{m} C_{N,K,m} \frac{M}{r} \frac{d}{ds}igg|_{s=r_1} \int_{B_s(x_0)} u_i \\
\leq C_{N,K,m} \frac{M}{r} \left( \frac{M}{Q_{\min}} + \frac{1}{2\sqrt{Q_{\min}}} (Q\chi_{\{|u|>0\}} + |\nabla u|^2) \right).
\]

(6.16)
(iv) At last, the assumption $x_0 \in \partial \{|u| > 0\}$ and the continuity of $u$ implies
\[
\int_{B_{\theta r}(x_0)} \left(|\nabla u|^2 + Q \chi_{\{|u|>0\}}\right) \geq Q_{\min} \cdot \mu(B_{\theta r}(x_0) \cap \{|u| > 0\}) > 0.
\]
Thus, by combining this with (6.8), (6.9), (6.10) and (6.16), we obtain
\[
1 \leq C'_{N,K,m,\theta} \frac{2M}{r} \left(\frac{M}{\sqrt[3]{Q_{\min} \cdot r}} + \frac{1}{2\sqrt{Q_{\min}}} \right).
\]
This implies
\[
\frac{M}{\sqrt[3]{Q_{\min} \cdot r}} \geq \min \left\{ \frac{1}{2} \cdot \frac{1}{C'_{N,K,m,\theta}}, \frac{1}{2\sqrt{Q_{\min}}} \right\}.
\]
This is the desired estimate (6.1), since $M = \sup_{B_r(x)} |u|$ and $\theta$ is given in (6.7). The proof is finished. \hfill \square

Remark 6.2. If $u$ is an absolute minimizer of $J_Q$, the previous proof (step (ii)–(iv)) still works for general $\text{RCD}(K,N)$-spaces. That is:

Let $(X,d,\mu)$ be an $\text{RCD}(K,N)$-space with some $K < 0$ and $N \in (1,+)\infty$, and let $u$ be an absolute minimizer of $J_Q$ in (1.2). Let $\Omega' \subseteq \Omega$. If $u$ is Lipschitz continuous on $\Omega'$, then for any ball $B_r(x_0) \subseteq \Omega$ with $x_0 \in \partial \{|u| > 0\} \cap \Omega'$, it holds (6.1) for a positive constant $c$ depending only on $m, N$ and $K$.

6.2. Density estimates near the free boundary. In the subsection, we will show that both $\{|u| > 0\}$ and $\{|u| = 0\}$ have positive density along the free boundary.

Lemma 6.3. Let $u = (u_1, \ldots, u_m)$ be a local minimizer of $J_Q$ in (1.2) with $Q$ satisfying (1.3), and let $\Omega' \subseteq \Omega$. Then for any ball $B_r(x_0) \subseteq \Omega'$ with $x_0 \in \partial \{|u| > 0\}$ and $r < 2R_1$, where $R_1$ is given in Theorem 6.1, we have
\[
\mu(B_r(x_0) \cap \{|u| > 0\}) \geq c_1 \cdot \mu(B_r(x_0))
\]
for some constant $c_1$ depending on $m, N, K, \Omega', Q_{\min}, \varepsilon_u$ and $L$, the Lipschitz constant of $u$ on $\Omega'$.

Proof. From Theorem 6.1, there exists some $i_0 \in \{1,2,\cdots,m\}$ such that
\[
\sup_{B_{\varepsilon_u}/2(x_0)} u_{i_0} \geq \frac{c\sqrt{Q_{\min}}}{m} \cdot r/2
\]
for all $r/2 < R_1$, where $R_1$ and $c$ are given in Theorem 6.1. Choose $y_0 \in B_{r/2}(x_0)$ such that $u_{i_0}(y_0) \geq c_2 r/2$, where $c_2 := \frac{c\sqrt{Q_{\min}}}{m}$. Since $u_{i_0}$ is Lipschitz continuous on $B_{r/2}(y_0) \subseteq \Omega'$ with a Lipschitz constant $L$, we have
\[
\inf_{B_{c_3r}(y_0)} u_{i_0} \geq u_{i_0}(y_0) - c_3 L r \geq c_2 r/4, \quad c_3 := \min \left\{ \frac{1}{8}, \frac{c_2}{8L} \right\}.
\]
In particular, this yields $B_{c_3r}(y_0) \subseteq \{|u| > 0\} \cap B_r(x_0)$. It follows
\[
\mu(B_r(x_0) \cap \{|u| > 0\}) \geq \mu(B_{c_3r}(y_0)).
\]
By combining this and the Bishop-Gromov inequality
\[
\mu(B_{c_3r}(y_0)) \geq c_{N,K}(c_3/2)^N \mu(B_{2r}(y_0)) \geq c_{N,K}(c_3/2)^N \mu(B_r(x_0)),
\]
we get the desired estimate (6.17). \hfill \square

Lemma 6.4. Let $u = (u_1, \ldots, u_m)$ be a local minimizer of $J_Q$ in (1.2) with $Q$ satisfying (1.3), and let $\Omega' \subseteq \Omega$. Then there exists a number $R_0 \in (0,1)$ depending only on $N,K,\Omega',Q_{\max}$ and $u$, such that for any ball $B_r(x_0) \subseteq \Omega'$ with $x_0 \in \partial \{|u| > 0\} \cap \Omega'$ and $r < R_0$, we have
\[
\mu(B_r(x_0) \cap \{|u| = 0\}) \geq c_4 \cdot \mu(B_r(x_0))
\]
for some constant $c_4$ depending on $m, N, K, \Omega', Q_{\min}, Q_{\max}, \epsilon_u$ and the Lipschitz constant of $u$ on $\Omega'$.

Proof. Let $v = (v_1, v_2, \cdots, v_m) \in W^{1,2}(B_r(x_0), \mathbb{R}^m)$ be the map such that each component $v_i$ is the (unique) solution of the following (relaxed) Dirichlet problem [Che99, Theorem 7.12]:

\begin{equation}
\begin{cases}
\Delta v_i = 0 & \text{on } B_r(x_0), \\
v_i - u_i \in W^{1,2}_0(B_r(x_0)).
\end{cases}
\end{equation}

After extending $v$ by $u$ on $\Omega \setminus B_r(x_0)$, we have $v \in \mathcal{A}_g$ because all $v_i \geq 0$ on $B_R(x_0)$ by the maximum principle (see [Che99, Theorem 7.17]). In the proof of Lemma 4.3, it was showed that there is $R_0 \in (0,1)$ (depending only on $N, K, \Omega'$ and $u$) such that $d(u, v) < \epsilon_u$ for all $r \in (0, R_0)$.

The local minimality of $u$ implies

\begin{equation}
\int_{B_{r/2}(x_0)} (|\nabla u|^2 - |\nabla v|^2) \leq \int_{B_r(x_0)} Q(\chi_{\{|v|>0\}} - \chi_{\{|u|>0\}})
\end{equation}

for all $r \in (0, R_0)$. From the harmonicity of $v$, the same argument in (4.9) gives

\begin{equation}
\int_{B_{r/2}(x_0)} (|\nabla u|^2 - |\nabla v|^2) = \int_{B_r(x_0)} |\nabla (u - v)|^2.
\end{equation}

Note that $|v|$ is not identical zero on $B_r(x_0)$ by Theorem 6.1. We get from the strong maximum principle (see, for example, [KS01, Corollary 6.4]) that $|v| > 0$ on $B_r(x_0)$. Hence,

\begin{equation}
\int_{B_r(x_0)} Q(\chi_{\{|v|>0\}} - \chi_{\{|u|>0\}} = \int_{B_r(x_0)} Q(\chi_{\{|u|=0\}} \leq Q_{\max} \cdot \mu(B_r(x_0) \cap \{|u| = 0\})
\end{equation}

By combining (6.20)–(6.22) and the Poincaré inequality (see Proposition 2.1(5)), we get

\begin{equation}
\int_{B_{r/2}(x_0)} |u - v|^2 \leq C_P \cdot r^2 \cdot Q_{\max} \cdot \mu(B_r(x_0) \cap \{|u| = 0\}),
\end{equation}

where the Poincaré constant $C_P$ depends only on $N, K$ and $\Omega'$.

From the nondegeneracy Theorem 6.1, there exists some $i_0 \in \{1, 2, \cdots, m\}$ such that $\sup_{B_{r/2}(x_0)} u_{i_0} \geq c_{\sqrt{Q_{\min}r}}$. Recalling $\Delta (u_{i_0} - u_{i_0}) \geq 0$ on $B_r(x_0)$ in the sense of distributions, the maximum principle implies that $u_{i_0} \geq u_{i_0}^\prime$ on $B_r(x_0)$. Hence, from this and the Harnack inequality, we have

\begin{equation}
\inf_{B_{r/2}(x_0)} v_{i_0} \geq C_1 \sup_{B_{r/2}(x_0)} u_{i_0} \geq C_1 \sup_{B_{r/2}(x_0)} u_{i_0} \geq C_1 c_{\sqrt{Q_{\min}r}}/2,
\end{equation}

where the constant $C_1$ depends only on $N, K, \Omega'$. Since $u_{i_0}(x_0) = 0$ and that $u_{i_0}$ is Lipschitz continuous on $B_r(x_0) \subset \Omega'$ with a Lipschitz constant $L$, we have

\begin{equation}
\sup_{B_r(x_0)} u_{i_0} \leq L c_1 r, \quad \text{with } c_1 := \min\left\{\frac{C_1 c_{\sqrt{Q_{\min}r}}}{4L}, \frac{1}{4}\right\}.
\end{equation}

Combining this with (6.24), we conclude that

\begin{equation}
\inf_{B_{r/2}(x_0)} (v_{i_0} - u_{i_0}) \geq \frac{C_1 c_{\sqrt{Q_{\min}r}}}{4} := C_2 \sqrt{Q_{\min}r},
\end{equation}

where the constant $C_2$ depends on $m, N, K, \Omega', \epsilon_u$ and $L$. This yields

\begin{equation}
\int_{B_{r/2}(x_0)} |u - v|^2 \geq \int_{B_{r/2}(x_0)} (v_{i_0} - u_{i_0})^2 \geq C_2^2 Q_{\min} \cdot r^2 \cdot \mu(B_{r/2}(x_0)).
\end{equation}
From (6.23), (6.25) and that \( \mu(B_{\epsilon_1 r}(x_0)) \geq C_{N,K} c_N^r \cdot \mu(B_r(x_0)) \), it follows
\[
C_P Q_{\max} \cdot \mu(B_r(x_0) \cap \{ |u| = 0 \}) \geq C_P^2 Q_{\min} C_{N,K} c_N^r \cdot \mu(B_r(x_0)).
\]
This is the desired (6.18) with the constant \( c_4 := C_P^{-1} C_P^2 C_{N,K} c_N^r \cdot Q_{\min} / Q_{\max} \). The proof is finished. \( \square \)

6.3. Local finiteness of perimeter. In this subsection, we will derive the local finiteness of perimeter of the free boundary in the sense of [Amb01, Mir03, ABS19], via the estimates of density, Lemma 6.3 and Lemma 6.4, and a similar argument in [MTV20]. We need an estimate on perimeter which is similar to the one in [MTV20, Lemma 2.4].

**Lemma 6.5.** Let \( (X,d,\mu) \) be an RCD\((K,N)\) space with \( K \in \mathbb{R} \) and \( N \in (1, +\infty) \), and let \( D \subset X \) be an open subset. Suppose that \( 0 \leq \psi \in \text{Lip}_{\text{loc}}(D) \cap W^{1,1}(D) \). If there are positive constants \( \varepsilon \) and \( C \) such that
\[
(6.26) \quad \int_{\{0 < \psi < \varepsilon\} \cap D} |\nabla \psi| \leq C \varepsilon
\]
for all \( \varepsilon \in (0, \varepsilon) \), then
\[
(6.27) \quad \mathcal{P}(\{ \psi > 0 \}, D) \leq 2C.
\]

**Proof.** Given \( \varepsilon \in (0, \varepsilon/2) \), then, by the coarea formula [Amb01, Theorem 3.3] (see also Proposition 2.15(2)) to \( \chi_D \), we have
\[
(6.28) \quad \int_{\varepsilon}^{2\varepsilon} \mathcal{P}(\{ \psi > t \}, D) dt = \int_{\varepsilon \leq \psi < 2\varepsilon} \chi_D d|D\psi| = \int_{\{\varepsilon \leq \psi < 2\varepsilon\} \cap D} |\nabla \psi| \leq \int_{\{0 < \psi < \varepsilon\} \cap D} |\nabla \psi| \leq 2C \varepsilon.
\]
For each integer \( j \) so large that \( 2^{-j} \leq \varepsilon/2 \), by setting \( \varepsilon_j = 2^{-j} \), there exists some \( \varepsilon_j' \in (\varepsilon_j, 2\varepsilon_j) \) such that
\[
(6.29) \quad \mathcal{P}(\{ \psi > \varepsilon_j' \}, D) \leq \varepsilon_j^{-1} \int_{\varepsilon_j}^{2\varepsilon_j} \mathcal{P}(\{ \psi > t \}, D) dt \leq \varepsilon_j^{-1} 2C \varepsilon_j \leq 2C.
\]
It is clear that \( \chi_{\{\psi > \varepsilon_j'\}} \xrightarrow{\text{Lip}_{\text{loc}}(D)} \chi_{\{\psi > 0\}} \), by Lebesgue’s dominated convergence theorem. From (6.29) and the lower semicontinuity of perimeter [Amb01], it follows that
\[
(6.30) \quad \mathcal{P}(\{ \psi > 0 \}, D) \leq \liminf_{j \to \infty} \mathcal{P}(\{ \psi > \varepsilon_j' \}, D) \leq 2C.
\]
This finishes the proof. \( \square \)

Now let us check the condition (6.26) of the above lemma for the local minimizers of \( J_Q \).

**Lemma 6.6.** Let \( u = (u_1, \ldots, u_m) \) be a local minimizer of \( J_Q \) in (1.2) with \( Q \) satisfying (1.3), and let \( \Omega' \subset \Omega \). Then there exist \( \bar{r} > 0, \bar{\varepsilon} > 0 \) and \( C > 0 \) such that
\[
(6.31) \quad \int_{\{ |u| \leq \varepsilon \} \cap B_{\bar{r}/2}(x_0)} |\nabla u|^2 + \mu(\{ 0 < |u| \leq \varepsilon \} \cap B_{\bar{r}/2}(x_0)) \leq C \mu(B_{\bar{r}}(x_0)) \cdot \frac{\bar{\varepsilon}}{\bar{r}^2}
\]
for all \( \varepsilon \in (0, \bar{\varepsilon}) \), \( r \in (0, \bar{r}) \) and \( B_r(x_0) \subset \Omega' \). Here the constants \( \bar{r}, \bar{\varepsilon}, C \) depend only on \( N, K, \Omega', \text{diam}(\Omega), Q_{\min}, \varepsilon_u \) and \( L \), the Lipschitz constant of \( u \) on \( \Omega' \).
Proof. Fix any ball $B_r(x_0) \subset \Omega'$ such that $r \in (0, \bar{r})$, where $\bar{r} < 1$ will be determined later. Let $\phi : M \to [0, 1]$ be a cutoff function with $\phi \equiv 1$ on $B_{r/2}(x_0)$, supp$(\phi) \subset B_r(x_0)$, and $r |\nabla \phi| \leq C_N$. For each $\varepsilon > 0$, we set

$$v := \begin{cases} (1 - \phi)u & \text{if } |u| \leq \varepsilon, \\ (1 - \varepsilon/|u|)u & \text{if } |u| > \varepsilon. \end{cases}$$

(6.32)

It is easy to check $v \in \mathcal{E}_d$ and $v - u \in W^{1,2}_0(B_r(x_0), \mathbb{R}^m)$.

(i) We will first show that $d(u, v) < \varepsilon u$ provided both $r$ and $\varepsilon$ are sufficiently small. By the Poincaré inequality (Proposition 2.1(5)) and $v - u \in W^{1,2}_0(B_r(x_0), \mathbb{R}^m)$, we have

$$|\nabla (v - u)| = |\nabla \phi| \leq L + \varepsilon \cdot C_N r^{1/2}, \quad \mu\text{-a.e. on } \{|u| \leq \varepsilon\} \cap B_r(x_0),$$

where $L$ is a Lipschitz constant of $u$ on $\Omega'$, and we have used $|\nabla \phi| \leq C_N r^{1/2}$. Thus,

$$d(u, v) \leq C_P \left( \int_{B_r(x_0)} |\nabla (u - v)|^2 \right)^{1/2} + \mu(B_r(x_0)),$$

where $C_P$ depends on $N, K$ and diam$(\Omega)$. From $u - v = \phi u$ on $\{|u| \leq \varepsilon\} \cap B_r(x_0)$ and [Che99, Corollary 2.25], we have

$$|\nabla (u - v)| = |\nabla (\phi u)| \leq L + \varepsilon \cdot C_N r^{1/2}, \quad \mu\text{-a.e. on } \{|u| \leq \varepsilon\} \cap B_r(x_0),$$

(6.33)

where $L$ is a Lipschitz constant of $u$ on $\Omega'$, and we have used $|\nabla \phi| \leq C_N r^{1/2}$. Thus,

$$\int_{B_r(x_0) \cap \{|u| \leq \varepsilon\}} |\nabla (u - v)|^2 \leq (L + C_N \cdot \varepsilon r^{1/2})^2 \cdot \mu(B_r(x_0)).$$

(6.34)

On $\{|u| > \varepsilon\} \cap B_r(x_0)$, we denote by $h := \varepsilon \phi / |u|$. Then, from the Chain rule (see [Gig15]) it follows

$$|\nabla h| \leq \frac{\varepsilon}{|u|} \cdot \frac{C_N}{r} + \frac{\varepsilon\phi|\nabla |u|}{|u|^2} \leq \frac{\varepsilon}{|u|} \cdot \frac{C_N}{r} + \frac{|\nabla \phi|}{|u|} \leq \frac{C_N}{r},$$

$\mu$-a.e. on $\{|u| > \varepsilon\} \cap B_r(x_0)$, where we have used $|\nabla |u|| \leq |\nabla u| \leq L$, $\varepsilon \phi \leq \varepsilon |u|$ and $|\nabla \phi| \leq C_N r^{-1}$. From this and $u - v = hu$ on $\{|u| > \varepsilon\}$, we have by [Che99, Corollary 2.25], that

$$|\nabla (u - v)| = |\nabla (hu)| \leq L + \varepsilon \cdot \frac{C_N}{r} + |h| |\nabla u| \leq 2L + \varepsilon \cdot \frac{C_N}{r},$$

$\mu$-a.e. on $\{|u| > \varepsilon\} \cap B_r(x_0)$. Thus, we obtain

$$\int_{B_r(x_0) \cap \{|u| > \varepsilon\}} |\nabla (u - v)|^2 \leq (2L + C_N \cdot \varepsilon r^{-1})^2 \cdot \mu(B_r(x_0)).$$

(6.35)

From (6.33), (6.34) and (6.35), it follows that

$$d(u, v) \leq C_1 \left( L + \frac{\varepsilon}{r} \right) \cdot \sqrt{\mu(B_r(x_0))} + \mu(B_r(x_0)),$$

(6.36)

where $C_1$ depending only on $N, K$ and diam$(\Omega)$. By the non-collapsing property of $X$, we have $N \geq 2$ (since $N$ is an integer and $N > 1$), and then $\mu(B_r(x_0)) \leq C_2 r^N \leq C_2 r^2$ for $r < 1$, where $C_2$ depends only on $N, K$ (see (2.13)). Therefore, we conclude $d(u, v) < \varepsilon u$ provided

$$\varepsilon \leq \varepsilon := \frac{\varepsilon u}{4C_1 C_2^{1/2}} \quad \text{and} \quad r \leq \bar{r} := \min \left\{ 1, \frac{\varepsilon u}{4(C_1 LC_2^{1/2} + C_2)} \right\}.$$

(6.37)

(ii) Fix any $r \in (0, \bar{r})$ and $\varepsilon \in (0, \bar{\varepsilon})$ given in (6.37). From the local minimality of $u$, we obtain
\[
\int_{B_r(x_0)} (|\nabla u|^2 - |\nabla v|^2) \leq \int_{B_{r/2}(x_0)} Q(\chi_{\{|v|>0\}} - \chi_{\{|u|>0\}})
\]

\[
\leq \int_{B_{r/2}(x_0) \cap \{|u| \leq \varepsilon\}} Q(\chi_{\{|v|>0\}} - \chi_{\{|u|>0\}})
\]

\[
\leq -\int_{B_{r/2}(x_0) \cap \{|u| \leq \varepsilon\}} Q\chi_{\{|u|>0\}}
\]

\[
\leq -Q_{\min} \cdot \mu(B_{r/2}(x_0) \cap \{0 < |u| \leq \varepsilon\}),
\]

where for the second inequality we have used \(\chi_{\{|v|>0\}} - \chi_{\{|u|>0\}} \leq 0\) on \(B_r(x_0)\), and for the third inequality we have used \(v = 0\) on \(B_{r/2}(x_0) \cap \{|u| \leq \varepsilon\}\).

On the other hand, from \(v = (1 - \phi)u\) on \(\{|u| \leq \varepsilon\} \cap B_r(x_0)\) and [Che99, Corollary 2.25], it follows \(|\nabla v| = |\nabla ((1 - \phi)u)| \mu\text{-a.e. in } \{|u| \leq \varepsilon\} \cap B_r(x_0)\). Thus, we have

\[
\int_{\{|u| \leq \varepsilon\} \cap B_r(x_0)} (|\nabla u|^2 - |\nabla v|^2)
\]

\[
= \int_{\{|u| \leq \varepsilon\} \cap B_r(x_0)} (|\nabla u|^2 - |\nabla((1 - \phi)u)|^2)
\]

\[
\geq \int_{\{|u| \leq \varepsilon\} \cap B_r(x_0)} (\chi_{B_{r/2}(x_0)} |\nabla u|^2) - \int_{\{|u| \leq \varepsilon\} \cap B_r(x_0)} (2C_NLr^2|u| + C_N^2r^2|u|^2)
\]

\[
\geq \int_{\{|u| \leq \varepsilon\} \cap B_r(x_0)} |\nabla u|^2 - C_NLr^2 \mu(B_r(x_0)),
\]

where for the first inequality we have used \((2\phi - \phi^2) = 1\) on \(B_{r/2}(x_0)\), \(|\nabla u| \leq L\) and \(|\nabla \phi| \leq C_N/r\) on \(B_r(x_0)\).

On the other hand, we denote \(h = \frac{\phi}{|u|}\). Then \(v = (1 - h)u\) on \(\{|u| > \varepsilon\} \cap B_r(x_0)\), and by [Che99, Corollary 2.25], we have that, for \(\mu\text{-a.e. in } \{|u| > \varepsilon\} \cap B_r(x_0)\),

\[
|\nabla u|^2 - |\nabla v|^2 = |\nabla u|^2 - |\nabla((1 - h)u)|^2
\]

\[
= (2h - h^2)|\nabla u|^2 - |\nabla h|^2|u|^2 + 2(1 - h) \sum_{i=1}^m u_i \langle \nabla h, \nabla u_i \rangle
\]

\[
= (2h - h^2)|\nabla u|^2 - |\nabla h|^2|u|^2 + (1 - h) \langle \nabla h, \nabla (|u|^2) \rangle
\]

\[
= (2h - h^2)|\nabla u|^2 - |\nabla h|^2|u|^2 + 2(1 - h)|u| \langle \nabla h, \nabla |u| \rangle
\]

\[
= (2h - h^2)|\nabla u|^2 - |\nabla h|(|u||u|) + 2 \langle \nabla |u|, \nabla (h|u|) \rangle + (h - 2h)|\nabla u|^2
\]

\[
= (2h - h^2)(|\nabla u|^2 - |\nabla |u||^2) - |\nabla (\varepsilon \phi)|^2 + 2 \langle \nabla |u|, \nabla (\varepsilon \phi) \rangle.
\]

Thus, by \(|\nabla u| \geq |\nabla u| \mu\text{-a.e. everywhere and } h \leq 1\), we have

\[
\int_{\{|u| > \varepsilon\} \cap B_r(x_0)} (|\nabla u|^2 - |\nabla v|^2)
\]

\[
\geq \int_{\{|u| > \varepsilon\} \cap B_r(x_0)} (- \varepsilon^2 |\nabla \phi|^2 + 2 \varepsilon \langle \nabla |u|, \nabla \phi \rangle)
\]

\[
\geq (- \varepsilon^2 \frac{C_N^2}{r^2} - 2 \varepsilon \frac{CNL}{r}) \mu(B_r(x_0)) \geq -C_NL \varepsilon \frac{r}{2} \mu(B_r(x_0)),
\]

where we have used \(|\nabla u| \leq L \) and \(|\nabla \phi| \leq C_N/r\). The inequality (6.31) follows from the combination of (6.38)–(6.40). The proof is finished. \qed

Now we are in the position to show the free boundary of \(u\) is a set of locally finite perimeter in the sense of [Amb01, Mir03, ABS19].
Proposition 6.7 (Local finiteness of perimeter). Let $u = (u_1, \ldots, u_m)$ be a local minimizer of $J_Q$ in (1.2) and let $Q$ satisfy (1.3). Then $\Omega_u = \Omega \cap \{|u| > 0\}$ is of locally finite perimeter. Moreover, the followings hold:

1. For all $\Omega' \subset \Omega$, $\mathcal{H}^{N-1}(\partial |\{u| > 0\} \cap \Omega') < +\infty$;
2. There exist nonnegative Borel functions $q_i, i = 1, 2, \cdots, m$, such that
   \[ \Delta u_i = q_i \cdot \mathcal{H}^{N-1}(\partial |\{u| > 0\} \cap \Omega). \]

Proof. Fix any $\Omega' \subset \Omega$. By using Lemma 6.6, we get
\[
\int_{0 < |u| \leq \varepsilon} |\nabla u| \leq \int_{0 < |u| \leq \varepsilon} \frac{|\nabla u|^2 + 1}{2} \leq C_{N,K,L,r,\mu(B_r(x_0))} \cdot \varepsilon
\]
for all $\varepsilon \in (0, \varepsilon)$, which holds for any ball $B_r(x_0) \subset \Omega'$ with radius $r < \bar{r}$. Then, by using Lemma 6.5, we conclude that $\{|u| > 0\} \cap B_{r/2}(x_0)$ has finite perimeter. Hence, $\{|u| > 0\} \cap \Omega'$ has finite perimeter.

From Proposition 2.19(1), we have $\mathcal{H}^{N-1}(\partial^* |\{u| > 0\} \cap \Omega') < +\infty$. The density estimates in Lemma 6.3 and Lemma 6.4 imply
\[
(6.41) \quad \partial^* |\{u| > 0\} \cap \Omega = \partial |\{u| > 0\} \cap \Omega.
\]
Now the assertion (1) follows.

For the assertion (2), by noticing that Proposition 2.19(2) and that $\Delta u_i$ is a Radon measure supported in $\partial |\{u| > 0\} \cap \Omega$ (see Lemma 4.1 and Lemma 4.4), we need only to show that $\Delta u_i$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$, for each $i = 1, 2, \cdots, m$.

Let $B_r(x) \subset \Omega$ with $x \in \partial |\{u| > 0\} \cap \Omega$. Taking a cut-off function $\phi : \Omega \to [0,1]$ with $\phi = 1$ on $B_{r/2}(x)$, $\text{supp}(\phi) \subset B_r(x)$ and $|\nabla \phi| \leq C_{N,r}$, we have
\[
\Delta u_i(B_{r/2}(x)) \leq \Delta u_i(\phi) = -\int_{B_r(x)} \langle \nabla u_i, \nabla \phi \rangle d\mu \leq \frac{C_{N,L}}{r} \cdot \mu(B_r(x)),
\]
where we have used $\Delta u_i \geq 0$, $|\nabla u_i| \leq L$. Thus, by (2.13), we get $\Delta u_i(B_{r/2}(x)) \leq C_{N,K,L} \cdot r^{N-1}$. This shows, that the Radon measure $\Delta u_i$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$, and then shows the assertion (2). \hfill \Box

7. Compactness and the Euler-Lagrange equation

In this section, we consider the compactness of local minimizers of $J_Q$ living in a sequence of $\text{pmGH}$-converging $\text{ncRCD}$-spaces, under some uniformity assumptions. Let $K \leq 0, N \in (1, +\infty)$ and let $(X_j, d_j, \mu_j)$ be a sequence of $\text{ncRCD}(K, N)$ metric measure spaces. Fix $p_j \in X_j$ for each $j \in \mathbb{N}$. Suppose that
\[
(X_j, d_j, \mu_j, p_j) \xrightarrow{\text{pmGH}} (X_\infty, d_\infty, \mu_\infty, p_\infty).
\]
According to [DPG16], the limit $(X_\infty, d_\infty, \mu_\infty)$ is still an $\text{ncRCD}(K, N)$ metric measure space.

Fix $R > 0$. For each $j \in \mathbb{N}$, let $Q_j \in C(B_R(p_j))$ and let $u_j := (u_{j,1}, u_{j,2}, \cdots, u_{j,m}) \in W^{1,2}(B_R(p_j), \mathbb{R}^m)$ be a local minimizer of $J_{Q_j}$ on $B_R(p_j)$ with size $\varepsilon_{u_j} > 0$. That is, for each $j \in \mathbb{N}$, there exists a data $g_j \in W^{1,2}(B_R(p_j), [0, \infty)^m)$ such that $J_{Q_j}(u_j) \leq J_{Q_j}(v_j)$ for all $v_j \in \mathcal{A}_{g_j}$ with $d(u_j, v_j) < \varepsilon_{u_j}$, where the $\mathcal{A}_{g_j}$ and $d(u_j, v_j)$ are given in (1.4) and (1.5), respectively.
Theorem 7.1 (Compactness). Let \( R, Q_j, u_j \) be as above. Let \( Q_\infty \in C(B_R(p_\infty)) \) such that
\[
\lim_{j \to \infty} Q_j(x_j) = Q_\infty(x_\infty) \quad \text{whenever} \quad x_j \xrightarrow{GH} x_\infty \in B_R(p_\infty).
\]
(Recall that \( x_j \xrightarrow{GH} x_\infty \) means \( \Phi_j(x_j) \to \Phi_\infty(x_\infty) \) in \( Z \), where \( \Phi_j, \Phi_\infty \) and \( Z \) are given in the Definition 2.7(3).) Assume that \( \{u_j\} \) are uniformly bounded on \( B_R(p_j) \),
\[
\lim_{j \to +\infty} \varepsilon_{u_j} = +\infty,
\]
and there exist positive constants \( Q_{\min}, Q_{\max} \) and \( L \) such that
\[
0 < Q_{\min} \leq Q_j \leq Q_{\max} < \infty \quad \text{on} \quad B_R(p_j), \quad \forall j \in \mathbb{N},
\]
and
\[
|\nabla u_j| \leq L \quad \text{on} \quad B_R(p_j), \quad \forall j \in \mathbb{N}.
\]
Then there exist a subsequence, denoted by \( \{u_j\}_j \) again, and a map \( u_\infty \in \text{Lip}(B_R(p_\infty)) \) such that \( u_j \to u_\infty \) uniformly over \( B_R(p_\infty) \) as \( j \to \infty \) and that for any \( R' \leq R \), the limit map \( u_\infty \) is a minimizer of \( J_{Q_\infty} \) on \( B_{R'}(p_\infty) \). Moreover, for any \( R' \leq R \), the followings hold:
\[
\lim_{j \to \infty} \int_{B_{R'}(p_j)} |\nabla u_j|^2 \, d\mu_j = \int_{B_{R'}(p_\infty)} |\nabla u_\infty|^2 \, d\mu_\infty,
\]
\[
\lim_{j \to \infty} \int_{B_{R'}(p_j)} Q_j \chi_{\{|u_j| > 0\}} \, d\mu_j = \int_{B_{R'}(p_\infty)} Q_\infty \chi_{\{|u_\infty| > 0\}} \, d\mu_\infty,
\]
\[
\partial \{|u_j| > 0\} \cap B_{R'}(p_j) \xrightarrow{GH} \partial \{|u_\infty| > 0\} \cap B_{R'}(p_\infty),
\]
\[
\mu_j(B_{R'}(p_j) \cap \{|u_j| > 0\}) \to \mu_\infty(B_{R'}(p_\infty) \cap \{|u_\infty| > 0\}).
\]

Proof. From (7.3) and the Arzela-Ascoli theorem, there exist a subsequence of \( \{u_j\}_j \)
converging uniformly to some \( u_\infty \in \text{Lip}(B_R(p_\infty)) \) with the same Lipschitz constant \( L \).

(i) Fix any \( R' \leq R \). We first show the minimality of \( u_\infty \) on \( B_{R'}(p_\infty) \). The lower
semicontinuity of the Cheeger energy (see, for example, [ZZ19, Lemma 2.12]) gives
\[
\liminf_{j \to \infty} \int_{B_{R'}(p_j)} |\nabla u_j|^2 \, d\mu_j \geq \int_{B_{R'}(p_\infty)} |\nabla u_\infty|^2 \, d\mu_\infty.
\]
Since \( \{|u_\infty| > 0\} \) is an open set, it is easy to check that
\[
x_j \xrightarrow{GH} x_\infty \implies \liminf_{j \to \infty} Q_j \chi_{\{|u_j| > 0\}}(x_j) \geq Q_\infty \chi_{\{|u_\infty| > 0\}}(x_\infty).
\]
By the Fatou’s lemma (see [DPG18, Lemma 2.5] or [ZZ19, Appendix A], for the
Fatou’s lemma for functions defined on varying spaces), we have
\[
\liminf_{j \to \infty} \int_{B_{R'}(p_j)} Q_j \chi_{\{|u_j| > 0\}} \, d\mu_j \geq \int_{B_{R'}(p_\infty)} Q_\infty \chi_{\{|u_\infty| > 0\}} \, d\mu_\infty.
\]
Let \( v_\infty : B_{R'}(x_\infty) \to \mathbb{R}^m \) be a minimizer of \( J_{Q_\infty} \) with \( v_\infty - u_\infty \in W^{1,2}_0(B_{R'}(p_\infty), \mathbb{R}^m) \).
We first claim that
\[
J_{Q_\infty}(v_\infty) \geq \limsup_{j \to \infty} J_{Q_j}(u_j).
\]
If this claim holds, by combining (7.8), (7.9), the minimality of \( v_\infty \) and (7.10), then
we have
\[
\liminf_{j \to \infty} J_{Q_j}(u_j) \geq J_{Q_\infty}(u_\infty) \geq J_{Q_\infty}(v_\infty) \geq \limsup_{j \to \infty} J_{Q_j}(u_j).
\]
This yields \( \lim_{j \to \infty} J_{Q_j}(u_j) = J_{Q_{\infty}}(u_{\infty}) = J_{Q_{\infty}}(v_{\infty}) \). Thus, we conclude that \( u_{\infty} \) is also a minimizer of \( J_{Q_{\infty}} \) on \( B_R(p_{\infty}) \), and that the inequalities in (7.8) and (7.9) must be equalities, i.e., the both assertions of (7.4) and (7.5) hold.

Now let us prove the claim (7.10) by a contradiction argument. Suppose not, there exists some \( \delta_0 > 0 \) and a subsequence of \( \{ u_j \} \), denoted by \( \{ u_j \} \) again, such that

\[
J_{Q_{\infty}}(v_{\infty}) \leq \lim_{j \to \infty} J_{Q_j}(u_j) - \delta_0.
\]

Let \( \delta \in (0, \delta_0) \) be a constant so small that

\[
4m \sqrt{\Lambda_{\infty}} \cdot \mu_{\infty}^{1/2}(A_{R_{\infty}} - \delta R_{\infty}(p_{\infty})) + (4m + Q_{\max}) \cdot \mu_{\infty}(A_{R_{\infty}} - \delta R_{\infty}(p_{\infty})) \leq \frac{\delta_0}{2},
\]

where \( \Lambda_{\infty} := \int_{B_{R_{\infty}}(p_{\infty})} |\nabla v_{\infty}|^2 d\mu_{\infty} \), and \( A_{r_1, r_2}(p) := B_{r_2}(p) \setminus B_{r_1}(p) \).

From Theorem 1.4, we have \( v_{\infty} \in L_{\text{loc}}(B_R(p_{\infty})) \). By combining this and the facts that \( u_j \) converges uniformly to \( u_{\infty} \) over \( B_R(p_j) \), \( |\nabla u_j| \leq L \) for all \( j \in \mathbb{N} \), we conclude, by Lemma 2.11, that there exist a sequence of maps \( v_j : B_{R_{\infty}R_{\infty}}(p_{\infty}) \to \mathbb{R}^m \) such that \( v_j - u_j \in W^{1,2}_0(B_{R_{\infty}}(p_j), \mathbb{R}^m) \), \( v_j \to v_{\infty} \) uniformly over \( B_{R_{\infty}}(p_j) \) as \( j \to \infty \), and

\[
\lim_{j \to \infty} \int_{B_{R_{\infty}}(p_j)} |\nabla v_j|^2 d\mu_j = \int_{B_{R_{\infty}}(p_{\infty})} |\nabla v_{\infty}|^2 d\mu_{\infty} =: \Lambda_{\infty}.
\]

For each \( j \in \mathbb{N} \), we put \( w_j = (w_{j,1}, \ldots, w_{j,m}) \), where

\[
w_{j, \alpha} := \max \{ 0, v_{j, \alpha} - \delta \phi_j \} \quad \forall \alpha = 1, 2, \ldots, m,
\]

and \( \phi_j \) is a cut-off function on \( B_{R_{\infty}}(p_j) \) such that \( \text{supp}(\phi_j) \subset B_{R_{\infty}}(p_j) \), \( \phi_j(x) = 1 \) if \( x \in B_{R_{\infty}}(p_j) \), and \( |\nabla \phi_j| \leq 2/\delta \). It is clear that \( w_j \) is an admissible map for \( u_j \) in (1.4). From the definition of \( d(u_j, v_j) \) in (1.5), by using the Poincaré inequality to \( v_j - u_j \in W^{1,2}_0(B_{R_{\infty}}(p_j), \mathbb{R}^m) \), and then \( |\nabla u_j| \leq L \) (7.13) and (2.13), we get

\[
d(v_j, u_j) \leq C'
\]

for some constant \( C' > 0 \) independent of \( j \) (may depend on \( N, K, R', L, \Lambda_{\infty} \) and the Poincaré constant \( C_P \) in Proposition 2.1(5)). Thus, by combining this and (7.14), we obtain

\[
d(w_j, u_j) \leq d(w_j, v_j) + d(v_j, u_j)
\]

\[
\leq \delta \cdot \| \phi_j \|_{W^{1,2}(B_{R_{\infty}}(p_j))} + \mu(B_{R_{\infty}}(p_j)) + C' \leq C''
\]

for some constant \( C'' > 0 \) independent of \( j \). Thus, for any \( j \) sufficiently large (such that \( \varepsilon_{u_j} > C'' \) since the assumption (7.1)), the local minimality of \( u_j \) implies

\[
J_{Q_j}(u_j) \leq J_{Q_j}(w_j) = \int_{B_{R_{\infty}}(p_j)} |\nabla w_j|^2 + Q_j x_1 |\nabla w_j| > 0 d\mu_j
\]

\[
\leq \int_{B_{R_{\infty}}(p_j)} |\nabla (v_j - \delta \phi_j)|^2 + \int_{B_{R_{\infty}}(p_j)} Q_j x_1 |\nabla w_j| > 0 d\mu_j,
\]

where we have used \( |\nabla w_j| \leq |\nabla (v_j - \delta \phi_j)| \mu_j \text{-a.e. in } B_{R_{\infty}}(p_j) \) (see, for instance, Proposition 2.1(3)). Noticing that \( \phi_j = 1 \) on \( B_{R_{\infty}}(p_j) \) and \( |\nabla \phi_j| \leq 2/\delta \) on
\[ A_{R^\delta, R'}(p_j) \], we have

\[
\int_{B_{R^\delta}(p_j)} |\nabla (v_j - \delta \phi_j)|^2 \\
\leq \int_{B_{R^\delta}(p_j)} |\nabla v_j|^2
\]

\[+ \int_{A_{R^\delta, R'}(p_j)} \left( |\nabla v_j|^2 + 2m \cdot \delta |\nabla v_j| \cdot |\nabla \phi_j| + m \cdot \delta^2 |\nabla \phi_j|^2 \right)\]

\[
\leq \int_{B_{R^\delta}(p_j)} |\nabla v_j|^2 + 4m \int_{A_{R^\delta, R'}(p_j)} |\nabla v_j| + 4m \cdot \mu_j \left( A_{R^\delta, R'}(p_j) \right)
\]

\[
\leq \int_{B_{R^\delta}(p_j)} |\nabla v_j|^2 + 4m \left( \int_{B_{R^\delta}(p_j)} |\nabla v_j|^2 \right)^{1/2} \cdot \mu_j \left( A_{R^\delta, R'}(p_j) \right)^{1/2}
\]

\[+ 4m \cdot \mu_j \left( A_{R^\delta, R'}(p_j) \right).\]

By the definition of \( w_j \), (7.14), we have

\[\{ |w_j| > 0 \} \cap B_{R^\delta}(p_j) \subset \{ |v_j| > \delta \} \cap B_{R^\delta}(p_j),\]

and then

\[\int_{B_{R^\delta}(p_j)} Q_j \chi_{\{w_j > 0\}} \leq \int_{B_{R^\delta}(p_j)} Q_j \chi_{\{|v_j| > \delta \}} + Q_{\max} \cdot \mu_j \left( A_{R^\delta, R'}(p_j) \right).\]

Substituting this and (7.16) into (7.15), we obtain

\[
J_{Q_j}(u_j) \leq \int_{B_{R^\delta}(p_j)} |\nabla v_j|^2 + \int_{B_{R^\delta}(p_j)} Q_j \chi_{\{|v_j| > \delta \}}
\]

\[+ 4m \left( \int_{B_{R^\delta}(p_j)} |\nabla v_j|^2 \right)^{1/2} \cdot \mu_j \left( A_{R^\delta, R'}(p_j) \right)^{1/2}
\]

\[+ (4m + Q_{\max}) \cdot \mu_j \left( A_{R^\delta, R'}(p_j) \right).
\]

Recall that \( v_j \to v_\infty \) uniformly over \( B_{R^\delta}(p_j) \) as \( j \to \infty \). Hence,

\[
\lim_{j \to \infty} \sup \chi_{\{|v_j| > \delta \}}(x_j) \leq \chi_{\{|v_\infty| > 0 \}}(x_\infty), \quad \forall \, x_j \xrightarrow{GH} x_\infty.
\]

By using this and letting \( j \to \infty \) in (7.17), we obtain

\[
\lim_{j \to \infty} \sup J_{Q_j}(u_j) \leq A_\infty + \int_{B_{R^\delta}(p_\infty)} Q_\infty \chi_{\{|v_\infty| > 0 \}} + 4m \sqrt{A_\infty} \cdot \mu_{\infty}^{1/2} \left( A_{R^\delta, R'}(p_\infty) \right)
\]

\[+ (4m + Q_{\max}) \cdot \mu_{\infty} \left( A_{R^\delta, R'}(p_\infty) \right)
\]

\[\leq J_{Q_\infty}(v_\infty) + \frac{\delta_0}{2},\]

where we have used (7.13), (7.12) and \( \mu_j \left( A_{R^\delta, R'}(p_j) \right) \to \mu_{\infty} \left( A_{R^\delta, R'}(p_\infty) \right) \). This contradicts with (7.11), and then proves the claim (7.10). Therefore, we have proved the minimality of \( u_\infty \) and the equalities (7.4), (7.5).

(ii) Next we will prove (7.6). On the one hand, let \( x_j \in \partial \{|u_j| > 0 \} \cap B_R(p_j) \) such that \( x_j \xrightarrow{GH} x_\infty \in B_R(p_\infty) \). Fixing any \( r > 0 \), we have \( B_r(x_j) \xrightarrow{GH} B_r(x_\infty) \).

By Theorem 6.1, we get \( \sup_{B_r(x_\infty)} \|u_j\| \geq C_r \) for a constant \( C > 0 \) independent of \( j \). Since \( u_j \to u_\infty \) over \( B_r(p_j) \) uniformly, we have \( |u_\infty(x_\infty)| = 0 \) and \( \sup_{B_r(x_\infty)} \|u_\infty\| \geq C_r \). By the arbitrariness of \( r > 0 \), we conclude \( x_\infty \in \partial \{|u_\infty| > 0 \} \cap B_R(p_\infty) \). On the other hand, for each \( y_\infty \in \partial \{|u_\infty| > 0 \} \cap B_R(p_\infty) \), we can find a sequence \( y_j \in B_{R}(p_j) \) such that \( y_j \xrightarrow{GH} y_\infty \) and \( |u_j(y_j)| \to 0 \). By the nondegeneracy,
we get
\[ d(y_j, \partial \{ |u_j| > 0 \}) \to 0 \quad \text{as} \quad j \to \infty. \]

Therefore, there exists a sequence \( \{z_j\} \) such that \( z_j \in \partial \{ |u_j| > 0 \} \cap B_{R^j}(p_j) \) and
\[ d(y_j, z_j) \to 0 \quad \text{as} \quad j \to \infty. \]
So we have \( z_j \xrightarrow{GH} y_\infty \). This proves the assertion (7.6).

(iii) The assertion (7.7) follows from Lemma 6.6. Since all sets \( B_{R^j}(p_j) \cap \{ |u_j| > 0 \} \) are open, and \( \mu_j \to \mu_\infty \), we have
\[ \liminf_{k \to \infty} \mu_j \left( B_{R^j}(p_j) \cap \{ |u_j| > 0 \} \right) \geq \mu_\infty \left( B_{R^\infty}(p_\infty) \cap \{ |u_\infty| > 0 \} \right). \]

Similarly, given any \( \varepsilon > 0 \), the fact that all sets \( B_{R^j}(p_j) \cap \{ |u_j| \geq \varepsilon \} \) are closed implies
\[ \limsup_{k \to \infty} \mu_j \left( B_{R^j}(p_j) \cap \{ |u_j| \geq \varepsilon \} \right) \leq \mu_\infty \left( B_{R^\infty}(p_\infty) \cap \{ |u_\infty| \geq \varepsilon/2 \} \right) \leq \mu_\infty \left( B_{R^\infty}(p_\infty) \cap \{ |u_\infty| > 0 \} \right). \]

From Lemma 6.6, there exists a constant \( C > 0 \) (independent of \( j \)) such that for all \( \varepsilon \in (0, \varepsilon) \), we have
\[ \mu_j \left( B_{R^j}(p_j) \cap \{ |u_j| > 0 \} \right) - \mu_j \left( B_{R^j}(p_j) \cap \{ |u_j| \geq \varepsilon \} \right) \leq C \varepsilon, \quad \forall \ j \in \mathbb{N}. \]

By combining these three inequality and the arbitrariness of \( \varepsilon \in (0, \varepsilon) \), the assertion (7.7) follows. Now the proof is finished. \( \square \)

We now apply Theorem 7.1 to the special case of blow-up limits, to get the Euler-Lagrange equation of local minimizers.

**Corollary 7.2.** Let \( u = (u_1, u_2, \cdots, u_m) \) be a local minimizer of \( J_Q \) on \( \Omega \) with \( Q \in C(\Omega) \). Recall that \( \Delta u_i = q_i \cdot \mathcal{H}^{N-1}_\nu(\partial \{ |u| > 0 \} \cap \Omega) \) for some nonnegative Borel functions \( q_i, \ i = 1, 2, \cdots, m \) (see Proposition 6.7). Then
\[ \sum_{i=1}^m q_i^2(x_0) = Q(x_0), \quad \mathcal{H}^{N-1} \text{-a.e. } x_0 \in \partial \{ |u| > 0 \} \cap \Omega. \]

**Proof.** Since \( \Omega_u = \Omega \cap \{ |u| > 0 \} \) is of locally finite perimeter (by Proposition 6.7), we know from Proposition 2.19(2) and (6.41) that the reduced boundary \( \mathcal{F}(\{ |u| > 0 \} \cap \Omega) \) has full \( \mathcal{H}^{N-1} \) measure in \( \partial \{ |u| > 0 \} \cap \Omega \). Suppose that \( x_0 \in \mathcal{F}(\{ |u| > 0 \} \cap \Omega) \) and that it is a Lebesgue’s point of \( q_i \), for all \( i = 1, 2, \cdots, m \), with respect to \( \mathcal{H}^{N-1} \). It suffices to show (7.18) at such \( x_0 \).

Let \( \{r_j\}_{j=1,2,\cdots} \) be a sequence of real numbers such that \( r_j \to 0^+ \) as \( j \to \infty \), and consider the blow-up sequence of spaces
\[ X_j := (X, d_j := r_j^{-1}d, \mu_j, x_0), \quad \mu_j := \mu_{r_j}^{x_0} = c_j \cdot \mu, \]
where \( c_j^{-1} = \int_{B_{r_j}(x_0)} (1 - r_j^{-1}d(x, x_0))d\mu(x) \) (is given in (2.11)). We denote by \( B_{R}^{(j)}(x_0) \) the ball in \( X_j \) with radius \( R \) with respect to the metric \( d_j \).

Given \( \mathbf{v} \in W^{1,2}(B_{R}(x_0), \mathbb{R}^m) \), it is clear that the blow-up sequence of maps \( \mathbf{v}_j := r_j^{-1}\mathbf{v} \in W^{1,2}(B_{r_j}(x_0), \mathbb{R}^m) \). Moreover, if \( \mathbf{v} \in C(B_{R}(x_0)) \), then for each \( j \in \mathbb{N} \), we have
\[ \int_{B_{R}^{(j)}(x_0)} |\mathbf{v}_j|^2 d\mu_j = r_j^{-2}c_j \int_{B_{R}^{(j)}(x_0)} |\mathbf{v}|^2 d\mu, \]
\[ \int_{B_{R}^{(j)}(x_0)} |\nabla^{(j)}\mathbf{v}_j|^2 d\mu_j = c_j \int_{B_{R}^{(j)}(x_0)} |\nabla\mathbf{v}|^2 d\mu, \]
\[ \int_{B_{R}^{(j)}(x_0)} |\nabla^{(j)}\mathbf{v}_j|^2 d\mu_j = c_j \int_{B_{R}^{(j)}(x_0)} |\nabla\mathbf{v}|^2 d\mu, \]
where $|\nabla u| v$ is the minimal weak upper gradient for $v$ with respect to $d_j$, and

$$
\int_{B_R^{(j)}(x_0)} Q(x)\chi_{\{|v_j|>0\}} d\mu_j = c_j \int_{B_{Rr_j}(x_0)} Q(x)\chi_{\{|v|>0\}} d\mu,
$$

since $\{|v_j| > 0 \} \cap B_R^{(j)}(x_0) = \{|v| > 0 \} \cap B_{Rr_j}(x_0)$. Denoting by $B_R^{(j)} := B_R^{(j)}(x_0)$ and $B_{Rr_j} := B_{Rr_j}(x_0)$, the combination of (7.20)-(7.21) gives, for each $j \in \mathbb{N}$, that

$$
J_Q(v_j, B_R^{(j)}): = \int_{B_R^{(j)}} |\nabla u_j|^2 d\mu_j + \int_{B_R^{(j)}} Q(x)\chi_{\{|v_j|>0\}} d\mu_j = c_j \cdot J_Q(v, B_{Rr_j})
$$

and that, by the definition (1.5),

$$
d_{B_R^{(j)}}(v_j, w_j): = \|v_j - w_j\|_{W^{1,2}(B_R^{(j)}, \mathbb{R}^m)} + \|\chi_{\{|v_j|>0\}} - \chi_{\{|w|>0\}}\|_{L^1(B_R^{(j)})}
$$

$$
= c_j \left( d_{B_{Rr_j}}(v, w) - \|v - w\|_{L^2(B_{Rr_j}, \mathbb{R}^m)} \right) + c_j r_j^{-1} \|v - w\|_{L^2(B_{Rr_j}, \mathbb{R}^m)} \geq c_j d_{B_{Rr_j}}(v, w) \quad (\text{by } r_j \leq 1).
$$

Noticing that $x_0$ is a regular point, we have $X_j \xrightarrow{pmGH} (\mathbb{R}^N, d_E, c_N, \mathcal{H}^N, 0)$ as $j \to +\infty$. Since $u$ is a local minimizer of $J_Q$ on $B_R^{(j)}(x_0) \subset \Omega$ with size $\varepsilon_u > 0$, we conclude, for each $j \in \mathbb{N}$, that the blow-up map $u_j := r_j^{-1}u$ is a local minimizer of $J_Q$ on $B_R^{(j)}(x_0)$ with size $c_j \cdot \varepsilon_u$ (from (7.23)). Since $c_j \to +\infty$ as $j \to +\infty$, by using Theorem 7.1 and a diagonal argument, there exist a subsequence of $r_j$ such that $u_j$ converges to a limit map $u_0 = (u_{0,1}, u_{0,2}, \cdots, u_{0,m})$ on the tangent cone $(\mathbb{R}^N, d_E, c_N, \mathcal{H}^N, 0)$, and that for each $R > 0$, $u_0$ is a minimizer of $J_{Q_0}$ on each Euclidean ball $B_R^{(0)}(0) \subset \mathbb{R}^N$, where $Q_0 = Q(x_0)$. Moreover, by applying (7.4), (7.5), (7.20) and (7.21), we obtain

$$
\int_{B_R^{(0)}} |\nabla u_0|^2 dx = R^N \cdot \omega_N \cdot \lim_{j \to \infty} \int_{B_{Rr_j}(x_0)} |\nabla u|^2 d\mu,
$$

and

$$
\int_{B_R^{(0)}} Q_0\chi_{\{|u_0|>0\}} dx = R^N \cdot \omega_N \cdot \lim_{j \to \infty} \int_{B_{Rr_j}(x_0)} Q(x)\chi_{\{|u|>0\}} d\mu,
$$

where we have used $c_j \mu(B_{r_j}(x_0)) = \mu_j(B_1^{(j)}(x_0))$, $\lim_{j \to \infty} \mu_j(B_1^{(j)}(x_0)) = c_N \omega_N$ and $\lim_{j \to \infty} \frac{\mu(B_{r_j}(x_0))}{\mu(B_{r_j}(x_0))} = R^N$ (see [DPG18, Corollary 1.7]).

Remark that $u_0$ is also a blow-up limit of itself on $\mathbb{R}^N$ (Indeed, by taking a subsequence of $\{r_j\}$, say $\{r_j'\}$, such that $r_j' / r_j := \varepsilon_u \to 0$ as $j \to +\infty$, we get that $u_0$ is the blow-up limit of $c_j^{-1}u_0(\cdot \varepsilon_u)$ on $(\mathbb{R}^N, d_E, c_N, \mathcal{H}^N, 0)$.) According to the classification of blow-up limits on Euclidean space $\mathbb{R}^N$ (see [MTV17, Proposition 4.2] or [CSY18, Lemma 23]), we conclude that there is a 1-homogeneous nonnegative global minimizer $u : \mathbb{R}^N \to [0, \infty)$ of the one-phase Alt–Caffarelli functional

$$
J(u) := \int (|\nabla u|^2 + Q_0 \cdot \chi_{\{|u|>0\}}) dx
$$

such that $u_0(x) = \xi \cdot u(x)$, where $\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{R}^N$ with $|\xi| = 1$. On the other hand, since $x_0 \in \mathcal{F}\{|u| > 0\}$, we have $\lim_{j \to \infty} \int_{B_{Rr_j}(x_0) \subset X} \chi_{\{|u|>0\}} d\mu = 1/2$, and
hence, by (7.25) and that $Q$ is continuous at $x_0$, we get
\[
\int_{B^c_R(0)} \chi_{\{u > 0\}} \, dx = \int_{B_R^c(0)} \chi_{\{|u| > 0\}} \, dx = 1/2.
\]
This yields, by Theorem 5.5 in [AC81], that $\partial\{u > 0\}$ is a $(N-1)$-dimensional hyperplane in $\mathbb{R}^N$ and $\Delta u = \sqrt{Q_0} \cdot \mathcal{H}^{N-1} \partial\{u > 0\}$ in the sense of measures. Thus, we obtain
\[
(7.26) \quad \Delta u(B^c_R(0)) = \sqrt{Q_0} \cdot \mathcal{H}^{N-1}(B^c_R(0) \cap \partial\{u > 0\}) = \sqrt{Q_0} \cdot \omega_{N-1} R^{N-1}.
\]
For each $i_0 \in \{1, 2, \cdots, m\}$, recalling that $\Delta u_{i_0}$ is a Radon measure supported on $\partial\{|u| > 0\} \cap \Omega$, we will calculate the density of $\Delta u_{i_0}$ at $x_0$. Fix any $\delta \in (0, 1/8)$. For each $j \in \mathbb{N} \cup \{+\infty\}$, we take the Lipschitz cut-off $\phi_j : X_j \to [0, 1]$ as
\[
\phi_j(x) := \begin{cases} 
\min \left\{1, \frac{r_j + \delta}{\delta} - \frac{d(x, x_0)}{\delta}\right\}, & x \in B^{(j)}_{1+\delta}(x_0) \\
0, & x \notin B^{(j)}_{1+\delta}(x_0)
\end{cases}
\]
where $X_\infty = (\mathbb{R}^N, d_E, \mu_\infty = c_N \mathcal{H}^N, 0), x_\infty = 0$ and $B^{(\infty)}_R(0) = B^c_R(0)$. Then
\[
(7.27) \quad \Delta u_{i_0}(B^c_R(x_0)) \leq \Delta u_{i_0}(\phi_j) = -\int_{B^{(j)}_{1+\delta}(x_0)} \langle \nabla u_{i_0} , \nabla \phi_j \rangle \, d\mu
\]
\[
= -r_j^{-1} \int_{B^{(j)}_{1+\delta}(x_0)} \langle \nabla \phi_j(r_j^{-1} u_{i_0}), \nabla \phi_j(r_j^{-1} u_{i_0}) \rangle \, d\mu_j,
\]
where we have used $|\nabla \phi_j| = |\nabla \phi_j(r_j^{-1} \phi_j)|$ and $\Delta u_{i_0} \geq 0$. Letting $j \to \infty$, by using (7.24) and
\[
\lim_{j \to +\infty} \int_{B^{(j)}_{1+\delta}(x_0)} |\nabla \phi_j|^2 \, d\mu_j = \int_{B^c_{1+\delta}(0)} |\nabla \phi_\infty|^2 \, d\mu_\infty,
\]
we get
\[
\limsup_j r_j c_j \Delta u_{i_0}(B^c_R(x_0)) \leq -\int_{B^{(j)}_{1+\delta}(0)} \langle \nabla u_{i_0}, \nabla \phi_j \rangle \, d\mu_\infty
\]
\[
= c_N \cdot \Delta u_{i_0}(\phi_\infty) \leq c_N \cdot \Delta u_{i_0}(B^{c}_{1+\delta}(0)).
\]
By combining with the fact $c_j \cdot \mu(B^c_R(x_0)) \to c_N \omega_N$, we obtain
\[
(7.28) \quad \limsup_{j \to +\infty} \frac{r_j \Delta u_{i_0}(B^c_R(x_0))}{\mu(B^c_R(x_0))} \leq \frac{1}{\omega_N} \cdot \Delta u_{i_0}(B^{c}_{1+\delta}(0)).
\]
By replacing the Lipschitz cut-off $\phi_j$ by another $\psi_j : X_j \to [0, 1]$, defined by
\[
\psi_j(x) := \begin{cases} 
\min \left\{1, \frac{1 + \delta}{\delta} - \frac{d(x, x_0)}{\delta}\right\}, & x \in B^{(j)}_{1+\delta}(x_0) \\
0, & x \notin B^{(j)}_{1+\delta}(x_0)
\end{cases}
\]
the same argument implies that $\Delta u_{i_0}(B^c_R(x_0)) \geq \Delta u_{i_0}(\psi_j)$ and then
\[
(7.29) \quad \liminf_{j \to +\infty} \frac{r_j \Delta u_{i_0}(B^c_R(x_0))}{\mu(B^c_R(x_0))} \geq \frac{1}{\omega_N} \Delta u_{i_0}(B^{c}_{1-\delta}(0)).
\]
By combining the two inequalities, $u_{i_0} = \xi_{i_0} \cdot u$, (7.26) and $\lim_{r \to 0} \frac{\mu(B_r(x_0))}{\omega_N r^{N-1}} = 1$ (see Corollary 1.9 of [DPG18]), we get
\[
\xi_{i_0} \sqrt{Q_0}(1 + \delta)^{N-1} \geq \limsup_{j \to +\infty} \frac{\Delta u_{i_0}(B^c_R(0))}{\omega_N r_j^{N-1}} \geq \liminf_{j \to +\infty} \frac{\Delta u_{i_0}(B^{c}_{j}(0))}{\omega_N r_j^{N-1}} \geq \xi_{i_0} \sqrt{Q_0}(1 - \delta)^{N-1}.
\]
Since $x_0$ is a Lebesgue’s point of $q_{i_0}$, letting $\delta \to 0$, we get $q_{i_0}(x_0) = \xi_{i_0} \sqrt{Q_{i_0}}$. This completes the proof.

Proof of Theorem 1.6. It follows from the combination of Proposition 6.7 and Corollary 7.2.

8. Regularity of the free boundary

Suppose that $(X,d,\mu := \mathcal{H}^N)$ is a non-collapsed $RCD(K,N)$ metric measure space with some $K \leq 0$, $N \in (1, +\infty)$, and that $u$ is a minimizer of $J_Q$ on a bounded domain $\Omega \subset X$ and that $Q \in C(\Omega)$ and satisfies (1.3). In this section, we consider the regularity of free boundary $\partial\{\{u\} > 0\} \cap \Omega$.

Let $x_0 \in \partial\{\{u\} > 0\} \cap \Omega$ and $R > 0$ with $B_R(x_0) \subset \Omega$. We have known that for almost all $s \in (0, R)$, the ball $B_u(x_0)$ has finite perimeter. We define the Weiss’ density by

$$W_u(x_0,s,Q) := \frac{1}{s^N} \int_{B_r(x_0)} \left( |\nabla u|^2 + Q\chi_{\{u > 0\}} \right) d\mu$$

$$- \frac{1}{s^{N+1}} \int_X |u|^2 d|\text{D}\chi_{B_r(x_0)}|,$$

for almost all $s \in (0, R)$.

Lemma 8.1. For every $x_0 \in \partial\{\{u\} > 0\} \cap \Omega$, the function $r \mapsto W_u(x_0,r,Q)$ is in $L^\infty(0, R)$ provided $R \leq 1$.

Proof. From the Lipschitz continuity of $u$, we know that $|\nabla u|(x) \leq L$ and $|u|(x) \leq Lr$ for all $x \in B_r(x_0)$, since $u(x_0) = 0$. By combining (2.13) and (2.14), we have

$$-C_{N,K} L^2 \leq W_u(x_0,r,Q) \leq C_{N,K} (L^2 + Q_{\text{max}})$$

provided $R \leq 1$. This finishes the proof.

Theorem 7.1 implies the following continuity of $W$ under pointed-measured Gromov-Hausdorff topology.

Lemma 8.2. Let $K \leq 0, N \in (1, +\infty)$ and let $(X_j,d_j,\mu_j)$ be a sequence of $n$-RCD$(K,N)$ metric measure spaces such that $(X_j,d_j,\mu_j, p_j) \xrightarrow{\text{pmGH}} (X, \infty, d, \infty, p)$. Let $R > 0$. Suppose that $Q_j, u_j$ are given in Theorem 7.1 satisfying (7.2) and (7.3) with uniform constants $Q_{\text{min}}, Q_{\text{max}}$ and $L$. Assume that $u_j$, for all $j \in \mathbb{N} \cap \{+\infty\}$, satisfy the conclusions in Theorem 7.1. Then for almost every $s \in (0, R)$, we have

$$\lim_{j \to \infty} W_{u_j}(p_j, s, Q_j) = W_{u_\infty}(p_\infty, s, Q_\infty).$$

Proof. This follows from the combination of (7.4), (7.5), Lemma 2.16 and the fact that $u_j$ converge uniformly to $u_\infty$ on $B_R(p_j)$.

Let us recall some properties of Weiss’ density in the special case where $X = \mathbb{R}^N$ (with Euclidean metric and Lebesgue measure $\mathcal{H}^N$). It is well-known [Wei99, MTV17, CSY18] that for $x_0 \in \partial\{\{u\} > 0\} \cap \Omega$, $W_u(x_0,r,Q)$ is absolutely continuous and almost monotonicity in $r$:

$$W_u(x_0,r,Q) \geq W_u(x_0,s,Q) - C_{N,Q_{\text{max}}} \cdot \int_s^r \frac{\text{osc}_{B_r(x_0)} Q}{t} dt$$

for all $r > s > 0$. In particular, when $Q$ is a constant then $r \mapsto W_u(x_0,r,Q)$ is non-decreasing in $r$, and strictly increasing unless $u$ is homogeneous of degree one. Moreover, we have [Wei99, MTV17, CSY18] that

$$\lim_{r \to 0} W_u(x_0,r,Q) = \lim_{r \to 0} \frac{\{\{u\} > 0\} \cap B_r(x_0)}{|B_r(x_0)|} \geq \frac{1}{2} Q(x_0) \omega_N,$$
Definition 8.3. Let \( \varepsilon > 0 \) and a point \( x_0 \in \partial\{|u| > 0\} \cap \Omega \). The set \( \partial\{|u| > 0\} \) is called \( \varepsilon \)-regular at \( x_0 \), if \( B_1(x_0) \subset \Omega \) and if the followings hold:

1. \( x_0 \in \mathcal{R}_{\varepsilon,1} \), that is, \( d_{pGH}(B_1(x_0), B_1(0^N)) < \varepsilon \), where \( B_1(x_0) \subset X_1 := (X, d, \mu_1^{x_0}, x_0) \) given in (2.11) and \( B_1(0^N) \) is the unit ball in \( \mathbb{R}^N \) centered at 0, with measure \( c_N \mathcal{H}^N \) given in (2.12),

2. we have

\[
\int_0^1 \overline{W}_u(x_0, s, Q)ds < \frac{c_N \omega_N}{2} Q(x_0) (1 + \varepsilon),
\]

where \( \overline{W}_u(x_0, s, Q) \) is the Weiss’ density of \( u \) with respect to the rescaled metric measure space \( (X, d, \mu_1^{x_0}, x_0) \).

The notion that \( \partial\{|u| > 0\} \) is \( \varepsilon \)-regular at \( x_0 \) in the scalar \( r \) can be introduced by scaling. That is, the set \( \partial\{|u| > 0\} \) is \( \varepsilon \)-regular at \( x_0 \) in the scalar \( r \), if on the rescaling space \((X, r^{-1}d, \mu_1^{x_0}, x_0)\) and putting \( u_r := r^{-1}u \), the set \( \partial\{|u_r| > 0\} \) is \( \varepsilon \)-regular at \( x_0 \) in the ball \( B_1^r(x_0) \), where \( B_1^r(x_0) \) is the unit ball centered at \( x_0 \) in \( (X, r^{-1}d, \mu_1^{x_0}, x_0) \).

We introduce some notations for the quantitative estimates for singular sets of the free boundary of \( u \). Given \( \varepsilon > 0 \) and \( r > 0 \), we put

\[
\mathcal{R}_{\varepsilon, r}^{\Omega u} := \text{all points where } \partial\{|u| > 0\} \text{ is } \varepsilon\text{-regular in the scalar } r,
\]

\[
\mathcal{S}_{\varepsilon}^{\Omega u} := \cup_{r>0}\mathcal{R}_{\varepsilon, r}^{\Omega u} = \{x \in \partial\{|u| > 0\} \cap \Omega | \exists r > 0 \text{ such that } x \in \mathcal{R}_{\varepsilon, r}^{\Omega u}\},
\]

and finally,

\[
\mathcal{R}_{\varepsilon}^{\Omega u} := \cap_{\varepsilon > 0}\mathcal{R}_{\varepsilon}^{\Omega u} \quad \text{ and } \quad \mathcal{S}_{\varepsilon}^{\Omega u} := (\partial\{|u| > 0\} \cap \Omega) \setminus \mathcal{R}_{\varepsilon}^{\Omega u}.
\]

Clearly, by Lemma 8.2 and \( Q \in C(\Omega) \), \( \mathcal{R}_{\varepsilon}^{\Omega u} \) is relatively open in \( \partial\{|u| > 0\} \cap \Omega \) for all \( \varepsilon > 0 \) and \( r > 0 \), and then \( \mathcal{R}_{\varepsilon}^{\Omega u} \) is also relatively open. It is easy to check \( \mathcal{R}_{\varepsilon}^{\Omega u} \subset \mathcal{R}_{\varepsilon}^{\Omega u} \) for any \( 0 < \varepsilon_1 < \varepsilon_2 \).

We need the following two simple facts for rescaling metric measure spaces.

Lemma 8.4. Let \( a, b > 0 \) and let \( u_a := a^{-1}u \). Suppose that \( \overline{W}_{u_a}(x_0, s, Q) \) is the Weiss’ density of \( u_a \) with respect to the rescaled space \( X_{a,b} := (X, a^{-1}d, b \cdot \mu, x_0) \). Then

\[
\overline{W}_{u_a}(x_0, s, Q) = b \cdot a^N \cdot \overline{W}_u(x_0, as, Q)
\]

for almost all \( s \in (0, R/a) \). In particular, \( \overline{W}_{u_a}(x_0, s, Q) = (c_{2a}^{x_0}/c_1^{x_0}) \cdot r^N \cdot \overline{W}_u(x_0, rs, Q) \) for almost all \( s \in R/r \), where \( c_{2a}^{x_0} \) is given in (2.11).

Proof. We denote \( |\nabla^{(a)}v| \) be the minimal weak upper gradient of \( v \) with respect to \( X_{a,b} \). Then \( |\nabla^{(a)}u_a| = |\nabla u| \). Therefore, from this and the definition of perimeter measure (see Definition 2.14), we have \( |DX_{B_r^{(a)}(x_0)}| = ba|DX_{B_{ar}(x_0)}| \) for almost all \( s \in (0, R/a) \). According to (8.1), it is easy to check that \( \overline{W}_{u_a}(x_0, s, Q) = b \cdot a^N \cdot \overline{W}_u(x_0, sa, Q) \) for almost all \( s \in (0, R/a) \).

\[\text{(1)}\] Here the reason for the factor \( c_N \) is that the measure on \( \mathbb{R}^N \) is chosen by \( c_N \mathcal{H}^N \).
Notice that \( X_r := (X, r^{-1}d, \mu_r, x_0) \) in \((2.11)\) and \( \mu_r x_0 = c_f^0 \cdot r = \frac{c_f^0}{c_1} \mu_1 x_0 \). The second assertion follows from \((8.6)\), by taking \( a = r \) and \( b = \frac{c_f^0}{c_1} \).

**Lemma 8.5.** For any \( \varepsilon > 0 \), there exists a constant \( \delta := \delta(\varepsilon; K, N) > 0 \) such that it holds: Let \((X, d, \mu)\) be an \( RCD(K, N)\)-space. If \( d_{\text{pmGH}}(B_1(x), B_1(0^N)) < \delta \), then \( d_{\text{pmGH}}(B_r(x), B_r(0^N)) < \varepsilon \cdot r \) for all \( r \in (0, 1) \).

**Proof.** This fact is a well-known consequence of the standard compactness of \( RCD(K, N) \)-spaces. For the completeness we include a proof here.

Suppose that the statement is not true, then there are a sequence of \( RCD(K, N) \)-spaces \((X_j, d_j, \mu_j, x_j)\) and a sequence \( r_j \in [0, 1]\) such that
\[
(8.7) \quad d_{\text{pmGH}}(B_j(x_j), B_j(0^N)) < j^{-1},
\]
but the rescaling balls
\[
(8.8) \quad d_{\text{pmGH}}(B_1^{(j)}(x_j), B_1(0^N)) \geq \varepsilon_0, \quad \forall j \in \mathbb{N},
\]
for some \( \varepsilon_0 > 0 \), where \( B_1^{(j)}(x_j) := r_j^{-1} B_r(x_j) \) are the unit ball in the rescaling spaces \((X_j, r_j^{-1}d_j, \mu_j)\) given in \((2.11)\). It is obvious that \( r_j \to 0 \), indeed, \( \varepsilon_0 \cdot r_j \leq d_{\text{pmGH}}(B_{r_j}(x_j), B_{r_j}(0^N)) \leq j^{-1} \).

Let \((X_\infty, d_\infty, \mu_\infty, x_\infty)\) be one of the limit space of the sequence \((X_j, d_j, \mu_j, x_j)\) under the pmGH-converging. From \((8.7)\), we know that \( B_1(x_\infty) \subset X_\infty \) is isometric to \( B_1(0^N) \). In particular, \( x_\infty \) is a regular point in \( X_\infty \).

On the other hand, there is a subsequence of \( \{r_j\} \) such that the rescaling spaces \((X_\infty, r_j^{-1}d_\infty, \mu_\infty, x_\infty)\) converges to one of tangent cone at \( x_\infty \), denoted by \( Y \) with vertex \( \nu \). By a diagonal argument, there exists a further subsequence of \( \{r_j\} \) such that \( B_1^{(j)}(x_j) \xrightarrow{\text{pmGH}} B_1(\nu) \subset Y \). Hence, by \((8.8)\), we have \( d_{\text{pmGH}}(B_1(\nu), B_1(0^N)) \geq \varepsilon_0 \). Thus, \( x_\infty \) is singular point in \( X_\infty \). We get a contradiction. The proof is finished. \( \square \)

**Theorem 8.6** (\( \varepsilon \)-regularity). For any \( \varepsilon > 0 \), there exists a positive constant \( \delta := \delta(\varepsilon; K, N, R, Q_{\text{min}}, Q_{\text{max}}, L) > 0 \) such that the following holds:

Let \((X, d, \mu)\) be an \( ncRCD(K, N)\)-space and let \( u \) be a minimizer of \( J_Q \) on \( B_R(x_0) \subset X \) with \( R \geq 2 \) and \( Q, u \) satisfying \((7.2), (7.3)\). If \( \text{osc}_{B_2(x_0)} Q < \delta \) and if \( x \in \mathcal{R}^{\alpha_1}_{x, r} \) for all \( x \in B_1(x_0) \cap \partial\{ |u| > 0 \} \), then \( y \in \mathcal{R}^{\alpha_1}_{x, r} \) for all \( y \in B_{1/4}(x_0) \cap \partial\{ |u| > 0 \} \) and all \( r \in (0, 1/4) \).

**Proof.** We argue by contradiction. Suppose not, then there exists \( \varepsilon_0 > 0 \) such that for each \( j \in \mathbb{N} \), there is an \( ncRCD(K, N)\)-spaces \((X_j, d_j, \mu_j)\) and a minimizer \( u_j \) of \( J_{Q_j} \) on \( B_{R}(x_j) \subset X_j \) with \( Q_j \) satisfying the uniform estimates \((7.2), (7.3)\) such that followings hold:

(i) \( \text{osc}_{B_{2}(x_j)} Q_j \leq j^{-1} \),
(ii) \( x' \in \mathcal{R}^{\alpha_1}_{x_j, r} \) for all \( x' \in B_{1}(x_j) \cap \partial\{ |u_j| > 0 \} \),
(iii) there exist \( y_j \in B_{1/4}(x_j) \cap \partial\{ |u_j| > 0 \} \) and \( r_j \in (0, 1/4) \) such that \( y_j \notin \mathcal{R}^{\alpha_1}_{x_j, r_j} \).

By applying the standard compactness of \( RCD \)-spaces and Theorem 7.1 to \( X_j, Q_j \) and \( u_j \), there exists subsequences of \( \{y_j\}, X_j \) and \( \{u_j\} \) such that:
\[
(8.9) \quad (X_j, d_j, \mu_j^{y_j}, y_j) \xrightarrow{\text{pmGH}} X_\infty := (X_\infty, d_\infty, \mu_\infty, y_\infty),
\]
and \( u_j \to u_\infty \) uniformly in any \( B_s(y_j) \) for all \( s < 1 \), where \( \mu_1^{y_j} \) is given in \((2.11)\). The limit map \( u_\infty \) is a minimizer of \( J_{Q_\infty} \) on \( B_1(y_\infty) \). By \( y_j \in \mathcal{R}^{\alpha_1}_{x_j, r_j} \) and Definition 8.3(1), we conclude that the limit ball \( B_1(y_\infty) \subset X_\infty \) is isometric.
to \(B_t(0^N) \subset \mathbb{R}^N\) with the Euclidean metric \(d_e\) and the measure \(c_N \mathcal{H}^N\). The limit \(Q_\infty := \lim_{j \to +\infty} Q_j\) is a constant (because \(\text{osc}_{B_t(y_j)} Q_j \leq \text{osc}_{B_t(x_j)} Q_j \leq 1/j\)). By Lemma 8.2, we get \(\overline{W}_{u_j}(y_j, s, Q_j) \to \overline{W}_{u_\infty}(y_\infty, s, Q_\infty)\) for almost all \(s \in (0, 1)\). By integrating on \((0, 1)\) and using Lemma 8.1, we obtain

\[
\int_0^1 \overline{W}_{u_\infty}(y_\infty, s, Q_\infty) ds = \lim_{j \to +\infty} \int_0^1 \overline{W}_{u_j}(y_j, s, Q_j) ds \leq \frac{1}{2} \lim_{j \to +\infty} Q_j(y_j) c_N \cdot \omega_N (1 + j^{-1}) = \frac{c_N \omega_N}{2} Q_\infty,
\]

where we have used \(y_j \in R_{\omega_j}^{\rho_j}\) and Definition 8.3(2). From the fact that the limit \(B_t(y_\infty)\) is the Euclidean ball and the monotonicity (8.3), we get that \(y_\infty\) is a regular point in the free boundary of \(u_\infty\) and that

\[
(8.10) \quad \overline{W}_{u_\infty}(y_\infty, s, Q_\infty) = \frac{c_N}{2} Q_\infty \omega_N, \quad \forall \ s \in (0, 1).
\]

Now let us consider the rescaled spaces \(X_j := (X_j, d_j := r_j^{-1} d_j, \mu_j^{\rho_j}, y_j)\) and maps \(\overline{u}_j := r_j^{-1} u_j\), where \(\mu_j^{\rho_j}\) is given in (2.11). Remark that the Lipschitz constant of \(\overline{u}_j\) is the same as the one of \(u_j\) for each \(j \in \mathbb{N}\). By applying Theorem 7.1 to \(X_j, Q_j := Q, \overline{u}_j\), there exists subsequences of \(\{r_j\}, \{y_j\}\) such that:

\[
 r_j \to r_0 \in [0, 1/4], \quad X_j \overset{pmGH}{\to} X_\infty := (X_\infty, d_\infty, \mu_\infty, y_\infty),
\]

and \(\overline{u}_j \to \overline{u}_\infty\) uniformly in any \(B_t^{(j)}(y_j)\) for all \(s < 1\), where \(B_t^{(j)}(y_j)\) denotes the ball in \(X_j\).

From \(y_j \in R_{\omega_j}^{\rho_j}\) (see Definition 8.3(1)) and Lemma 8.5, we know that

\[
d_{pmGH}(B_{\rho_j}(y_j), B_t(0^N)) < \frac{\epsilon_0}{2} \cdot r_j
\]

for all sufficiently large \(j\). That is, \(y_j \in R_{\epsilon_0/r_j}\) for all \(j\) large enough. By combining with the condition \(y_j \not\in R_{\epsilon_0}^{\rho_j}\), and by using Lemma 8.2, we obtain

\[
(8.11) \quad \int_0^1 \overline{W}_{u_\infty}(y_\infty, s, Q_\infty) ds \geq \frac{c_N}{2} Q_\infty \omega_N (1 + \epsilon_0).
\]

Clearly, by a diagonal argument, and up to a subsequence, \(X_\infty\) is a \(pmGH\)-limit of \((X_\infty, r_j^{-1} d_\infty, \mu_\infty^{\rho_j}, y_\infty)\), and that \(\overline{u}_\infty\) is the limit of \(r_j^{-1} u_\infty\). Thus, from Lemma 8.4, \(\mu_\infty^{\rho_j} = \frac{\epsilon_1}{c_1} \mu_\infty^{\rho_j}\) and \(\mu_\infty = \mu_\infty = c_N \mathcal{H}^N\), we get, for all \(s \in (0, 1)\) that

\[
\overline{W}_{r_j^{-1} u_\infty}(y_\infty, s, Q_\infty) = \frac{c_N}{c_1} \cdot r_j \cdot \overline{W}_{u_\infty}(y_\infty, r_j s, Q_\infty).
\]

By using (8.10), \(r_j \leq 1\) and

\[
\frac{c_N}{c_1} = \frac{c_1}{\epsilon_1} \frac{1}{\int_{B_{\rho_j}(y_\infty=0^N)} \left(1 - r_j^{-1} d_e(z, y_\infty)\right) d\mathcal{H}^N(z)} = \frac{1}{\int_{B_{\rho_j}(0^N)} \left(1 - r_j^{-1} |z|\right) d\mathcal{H}^N(z)} = \frac{1}{r_j},
\]

we conclude that \(\overline{W}_{r_j^{-1} u_\infty}(y_\infty, s, Q_\infty) = \frac{c_N}{c_1} Q_\infty \omega_N\) for all \(s \in (0, 1)\), and hence \(\overline{W}_{u_\infty}(y_\infty, s, Q_\infty) = \frac{c_N}{c_1} Q_\infty \omega_N\) for all \(s \in (0, 1)\). By integrating on \((0, 1)\), it contradicts with (8.11), and hence, the proof is completed.

This \(\varepsilon\)-regularity is the reason for us to define the almost regular part of the free boundary \(R_\omega = \bigcup \limits_{r > 0} R_{\rho_j}^{\rho_j}\). A simple but important corollary of this definition is that singular points do not disappear under \(pmGH\)-converging as follows.
Lemma 8.7. Let \((X_j, d_j, \mu_j)\) be a sequence of \(ncRCD(K, N)\)-spaces and \((X_j, d_j, \mu_j, p_j) \xrightarrow{pmGH} (X_\infty, d_\infty, \mu_\infty, p_\infty)\). Let \(Q_j \in C(B(x_j))\) and \(u_j\) be a minimizer of \(J_{Q_j}\) on \(B(x_j) \subset X_j\). Suppose that \(Q_j\) and \(u_j\) satisfy the uniformly estimates (7.2), (7.3) and that \(u_j\) converges uniformly to \(u_\infty\) on \(B(p_j)\). Then, for any \(\varepsilon > 0\), if \(x_j \in S_{\varepsilon}^{\Omega_u} \cap B_\varepsilon(p_j)\) and \(x_j \xrightarrow{GH} x_\infty\), we have \(x_\infty \in S_{\varepsilon}^{\Omega_u}\).

Proof. From (7.6) in Theorem 7.1, we know that \(x_\infty \in \partial\{|u_\infty| > 0\} \cap B_\varepsilon(p_\infty)\).

Fix arbitrarily \(r > 0\). We know that \(x_j \notin R_{\varepsilon, r}\), from \(R_{\varepsilon, r} = \mathbb{U}_{r>0} R_{\varepsilon, r}\). By the definition of \(R_{\varepsilon, r}\), we know either
\[
d_{pmGH}\left(B_1^{r^{-1}d_j}(x_j), B_1(0^{N-1})\right) \geq \varepsilon,
\]
where \(B_1^{r^{-1}d_j}(x_j)\) is the unit ball centered at \(x_j\) on the rescaling space \((X_j, r^{-1}d_j, \mu_j^{p_j}, p_j)\), or
\[
\int_0^1 W_{u_j}(x_j, s, Q)ds \geq \frac{C_N \omega_N}{2} Q(x_j) (1 + \varepsilon),
\]
where \(u_j := r^{-1}u_j\). From (7.5), Lemma 8.2 and \(Q(x_j) \rightarrow Q(x_0)\), we get \(x_\infty \notin R_{\varepsilon, r, 1}\), and then \(x_\infty \notin R_{\varepsilon}^{\Omega_u}\). The proof is finished. \(\square\)

A similar argument in the proof of the \(\varepsilon\)-regularity gives also the following Reifenberg’s property.

Lemma 8.8. For any \(\varepsilon > 0\), there exists a constant \(\delta = \delta(\varepsilon|N, K, R, L, Q_{\max}, Q_{\min}) > 0\) such that the following holds: Let \((X, d, \mu)\) be an \(ncRCD(K, N)\)-space and let \(u\) be a minimizer of \(J_Q\) on \(B_R(x) \subset X\) with \(R \geq 1\), \(0 < Q_{\min} \leq Q \leq Q_{\max} < +\infty\) and \(|\nabla u| \leq L\). If \(x \in R_{\varepsilon, 1}^{\Omega_u}\) and if \(osc_{B_{\varepsilon/2}(x)} Q \leq \delta\), then
\[
d_{GH}\left(B_1(x) \cap \partial\{|u| > 0\}, B_1(0^{N-1})\right) \leq \varepsilon,
\]
where \(B_1(0^{N-1})\) is the unit ball in \(\mathbb{R}^{N-1}\) centered at 0.

Proof. Suppose that this assertion is not true. There exists some \(\varepsilon_0 > 0\) such that for each \(j \in \mathbb{N}\), there is an \(ncRCD(K, N)\)-spaces \((X_j, d_j, \mu_j)\), \(Q_j \in C(B(x_j))\) and a minimizer \(u_j\) of \(J_{Q_j}\) on \(B_R(x_j) \subset X_j\) with the uniform estimates (7.2), (7.3) such that:

(i) \(x_j \in R_{\varepsilon_0, 1}^{\Omega_u}\), \(osc_{B_{\varepsilon_0/2}(x_j)} Q_j \leq \varepsilon_0\) and
(ii) \(d_{GH}\left(B_1(x_j) \cap \partial\{|u_j| > 0\}, B_1(0^{N-1})\right) \geq \varepsilon_0\).

By applying the compactness of \(RCD\)-spaces and Theorem 7.1 to \(X_j, Q_j\) and \(u_j\), there exist subsequences of \(\{x_j\}, X_j\) and \(\{u_j\}\) such that:
\[
(X_j, d_j, \mu_j^{p_j}, x_j) \xrightarrow{pmGH} (X_\infty, d_\infty, \mu_\infty, x_\infty),
\]
and \(u_j \rightarrow u_\infty\) uniformly in any \(B_r(x_j)\) for all \(r < R\), where \(\mu_j^{p_j}\) is given in (2.11). The limit map \(u_\infty\) is a minimizer of \(J_{Q_\infty}\) on \(B_1(x_\infty)\), and from (7.6) that
\[
d_{GH}\left(B_1(x_\infty) \cap \partial\{|u_\infty| > 0\}, B_1(0^{N-1})\right) \geq \varepsilon_0.
\]

On the other hand, since \(x_j \in R_{\varepsilon_0, 1}^{\Omega_u}\), similar to the proof of (8.10), by Definition 8.3 and Lemma 8.2, we conclude that the limit ball \(B_1(y_\infty) \subset X_\infty\) is isometric to \(B_1(0^{N}) \subset \mathbb{R}^N\) with the Euclidean metric \(d_\varepsilon\) and the measure \(\varepsilon_N \mathcal{H}^N\), and that
\[
\int_0^1 W_{u_\infty}(x_\infty, s, Q_\infty)ds = \frac{C_N \omega_N}{2} Q_\infty.
\]
where the limit \(Q_\infty := \lim_{j \rightarrow +\infty} Q_j\) is a constant (because \(\lim_{j \rightarrow +\infty} osc_{B_1(x_j)} Q_j = 0\)). From this and the monotonicity (8.5), we get that \(W_{u_\infty}(x_\infty, s, Q_\infty) \leq \frac{C_N \omega_N}{2} Q_\infty\omega_N\).
for all \( s \in (0, 1) \). Thus, we obtain that \( x_\infty \) is a regular point and that \( u_\infty \) is homogeneous of degree one. This implies

\[
d_{GH} \left( B_1(x_\infty) \cap \partial \{|u_\infty| > 0\}, B_1(0^{N-1}) \right) = 0.
\]

This contradicts with (8.12). The proof is finished. \( \square \)

Consequently, we have the following the topological regularity for the almost regular part of the free boundary \( \partial \{|u| > 0\} \cap \Omega \).

**Corollary 8.9.** Suppose that \( (X, d, \mu := \mathcal{H}^N) \) is a non-collapsed \( \text{RCD}(K, N) \) metric measure space, and that \( u \) is a minimizer of \( J_Q \) on a bounded domain \( \Omega \subset X \) and that \( Q \in C(\Omega) \) satisfies (1.3). Then for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that the set \( \mathcal{R}^{\Omega u}_\delta \) satisfies the following property: for any \( x \in \mathcal{R}^{\Omega u}_\delta \), there exists \( r_x > 0 \) such that it holds for all \( y \in B_{r_x}(x) \cap \partial \{|u| > 0\} \) and all \( r \in (0, r_x) \) that

1. \( y \in \mathcal{R}^{\Omega u}_\delta \), and
2. \( d_{GH} \left( B_r(y) \cap \partial \{|u| > 0\}, B_r(0^{N-1}) \right) < \varepsilon r \).

Consequently, for any \( \alpha \in (0, 1) \), there exists \( \delta_0 > 0 \) such that the almost regular set \( \mathcal{R}^{\Omega u}_\delta \) is a \( C^\alpha \)-biHölder homeomorphic to an \( (N - 1) \)-dimensional topological manifold.

**Proof.** For any \( \delta > 0 \) sufficiently small, for each \( x \in \mathcal{R}^{\Omega u}_\delta \), there exists \( r_x > 0 \) such that \( x \in \mathcal{R}^{\Omega u}_{\delta, r_x} \). Notice that \( \mathcal{R}^{\Omega u}_{\delta, r} \) is relatively open in \( \partial \{|u| > 0\} \cap \Omega \) for any \( r > 0 \). Hence, there exists a neighborhood \( B_{r'}(x) \) such that \( x' \in \mathcal{R}^{\Omega u}_{\delta', r'} \) for all \( x' \in \partial \{|u| > 0\} \cap B_{r'}(x) \) and \( \text{osc}_{B_{r'}(x)} Q \leq \delta \). We can assume that \( r'' \leq r' \). Thus, by \( \varepsilon \)-regularity Theorem 8.6, we conclude that

\[
y \in \mathcal{R}^{\Omega u}_{\delta', r'} \quad \text{for all} \quad y \in B_{r'/4}(x) \cap \partial \{|u| > 0\} \quad \text{and all} \quad r \in (0, r''/4),
\]

where \( \delta' = \delta' (\delta) > 0 \) with \( \lim_{\delta \to 0} \delta'(\delta) = 0 \). According to Lemma 8.8, we have

\[
d_{GH} \left( B_r(y) \cap \partial \{|u| > 0\}, B_r(0^{N-1}) \right) < \delta''(\delta, r).
\]

for all \( r \in (0, r''/4) \) and all \( y \in B_{r'/4}(x) \), where \( \delta'' = \delta''(\delta') > 0 \) with \( \lim_{\delta' \to 0} \delta''(\delta') = 0 \). We put \( r_x := r''/4 \) and take \( \delta \) sufficiently small that \( \max \{ \delta''(\delta), \delta''(\delta') \} \leq \varepsilon \). Now the first assertion follows.

The second assertion comes from the first one and the Reifenberg’s disk theorem for metric spaces, see [CC97, Appendix A]. In fact, for any \( \alpha \in (0, 1) \), we know that \( B_{r_x}(x) \cap \partial \{|u| > 0\} \cap \Omega \) is \( C^\alpha \)-homeomorphic to the ball \( B_{r_x}(0^{N-1}) \) provided that \( \varepsilon < \varepsilon(\alpha) \), where \( \varepsilon(\alpha) > 0 \) such that \( \varepsilon(\alpha) \to 0 \) as \( \alpha \to 1^- \). The proof is finished. \( \square \)

In the rest of this section, we want to estimate the size of the singular part of \( \partial \{|u| > 0\} \cap \Omega \). Firstly, we need to deal with the minimizers of \( J_Q \) on metric measure cones. Let \( N \geq 2 \) and let \( (\Sigma, d_\Sigma, \mu_\Sigma) \) be a metric measure space with \( \text{diam}(\Sigma) \leq \pi \). The metric measure cone over \( \Sigma \) is the metric measure space \( (C(\Sigma), d_C, \mu_C) \) given by

\[
C_N(\Sigma) = [0, \infty) \times \Sigma / (\{0\} \times \Sigma)
\]

with the distance

\[
d_C((r_1, \xi_1), (r_2, \xi_2)) = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos d_\Sigma(\xi_1, \xi_2)},
\]

and the measure

\[
d\mu_C(r, \xi) = r^{N-1} dr \times d\mu_\Sigma(x).
\]

In the following, we always assume that \( (C(\Sigma), d_C, \mu_C) \) is an \( \text{ncRCD}(0, N) \)-space. Remark that, for any point \( x_0 \) of an \( \text{ncRCD}(K, N) \)-space, any tangent cone at \( x_0 \) must be a metric measure cone satisfying \( \text{ncRCD}(0, N) \) (see [DPG18]).
Lemma 8.10. Let $u$ be a global minimizer of $J_{Q_0}$ on a cone $(C(\Sigma), d_C, \mu_C)$ with the vertex $p$, where $Q_0 > 0$ is a constant. Then the Weiss’ density $W_u(p, r, Q_0)$ is non-decreasing in $r$; moreover, if $W_u(p, r, Q_0)$ is a constant then $u$ is homogeneous of degree one, i.e. $u(\xi, t) = t \cdot u(\xi, 1)$ for any $t > 0$ and $\xi \in \Sigma$.

In particular, if $u_0$ is one of blow up limits of a minimizer $u$ at a point $x_0$ in an $ncRCD(K, N)$-space $(X, d, \mu)$. Then $u_0$ must be homogeneous of degree one.

Proof. The first assertion is similar to the case of Euclidean space. For the completeness, we give the details in the Appendix A (see Lemma A.3).

Since $u_0$ is one of blow up limits of $u$, there exists a sequence $r_j \rightarrow 0$ such that $(Y, d_Y, \mu_Y, o_Y)$ is the pmGH-limit of $(X, r^{-1}_j d, \mu^{r_j}_x, x_0)$ and that $u_0$ is the limit of $r^{-1}_j u$. For the second assertion, we only need to check that the Weiss’ density $W_{u_0}(o_Y, r, Q(x_0))$ is a constant.

By taking a subsequence of $\{r_j\}$, says $\{r_j' := \epsilon_j \cdot r_j\} \subset \{r_j\}$ such that $\epsilon_j \rightarrow 0$, then it is clear that $u_0$ is the limit of $u_{0, \epsilon_j}(\xi, s) := \epsilon_j^{-1} u_{0}(\xi, \epsilon_j \cdot s)$. That is, $u_0$ is also one of blow up limits of $u_0$ under the rescaling space $(Y, \epsilon_j^{-1} d_Y, \mu^{\epsilon_j}_{\epsilon_j} Y, o_Y)$ as $\epsilon_j \rightarrow 0^+$. By Lemma 8.4, for such sequence $\epsilon_j$, we obtain for any $s > 0$ that

\[
W_{u_0}(o_Y, s, Q(x_0)) = \lim_{\epsilon_j \rightarrow 0} e^{\epsilon_j^N} / c_1^{\epsilon_j^N} \cdot \epsilon_j^N \cdot W_{u_0}(o_Y, \epsilon_j \cdot s, Q(x_0)) = \lim_{s \rightarrow 0} W_{u_0}(o_Y, s, Q(x_0)),
\]

where we have used $e^{\epsilon_j^N} / c_1^{\epsilon_j^N} \cdot \epsilon_j^N = 1$ for all $r > 0$ and the existence of the limit $\lim_{s \rightarrow 0} W_{u_0}(o_Y, s, Q(x_0))$, by the monotonicity in the first assertion. $\square$

Lemma 8.11. Let $Q_0$ be a positive constant and $u = (u_1, u_2, \cdots, u_m)$ be a minimizer of $J_{Q_0}$ on a two-dimensional cone $C(S_a)$, where $S_a$ is a circle with length $a \in (0, 2\pi)$. Assume that $u$ is homogeneous of degree one. Then the vertex $o \notin \partial \{|u| > 0\}$.

Consequently, the singular set $S^{\partial u}$ is empty for any minimizer $u$ of $J_{Q_0}$ on a bounded domain $\Omega$ of an $ncRCD(K, 2)$-space without boundary.

Proof. When $m = 1$, this assertion is the main result in [ACL15].

For $m \geq 1$, we shall reduce it to the case $m = 1$, by an argument in [CL08]. Since $u$ is homogeneous of degree one, the set $\{|u| > 0\}$ is a cone over an interval $(b_1, b_2) \subset (0, a)$. By Lemma 4.4, we know that $u_i$ is harmonic on $\{|u| > 0\}$ for each $i = 1, 2, \cdots, m$. From the fact that $u(\xi, r)$ is homogeneous on $r$ of degree one and that any locally Lipschitz continuous function $v$ on $C(S_a)$ satisfies

\[
\Delta_{C(S_a)} v = r^{-2} \Delta_{S_a} v + r^{-1} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2}
\]

in the sense of distributions, we know that $u_i(\xi, 1)$ is a Dirichlet eigen-function of $\Delta_{S_a}$ on $(b_1, b_2)$ with respect to the eigenvalue $\lambda = 1$, for all $i = 1, 2, \cdots, m$. Notice that $u_i(\xi, 1) \geq 0$ and the fact that the first Dirichlet eigenvalue of $\Delta_{S_a}$ is single. Thus, $u_i(\xi, 1)/u_1(\xi, 1) = c_i$ for some constant $c_i > 0$, for all $i = 2, 3, \cdots, m$. Combining with the fact that $u$ is homogeneous, we get

\[
u = (u_1, c_2 u_1, \cdots, c_m u_1).
\]

Thus, from the minimality of $u$, it is clear that $u_1$ is a minimizer of $J_{Q_0}$ with the constant

\[
Q_0 := \frac{Q_0}{1 + \sum_{i=2}^m c_i^2}.
\]

According to [ACL15], we get the vertex $o \notin \partial \{|u| > 0\} = \partial \{|u| > 0\}$.

For the second assertion. Suppose not, if there exists a minimizer $u$ on an 2-dimensional $ncRCD(K, 2)$-space without boundary, such that it has a singular
point \( x_0 \). Then we blow up the spaces \((X, 2\langle d, \mu^{x_0} \rangle, x_0)\) and the maps \( u_j := 2^j u \).

By Theorem 7.1 and Lemma 8.7, up to a subsequence, we can obtain a blow up limit map \( u_0 \) on one of tangent cone at \( x_0 \) such that \( u_0 \) is minimizer and has a singular point at the vertex \( o \). This contradicts the first assertion. The proof is finished. \( \square \)

We shall estimate the size of singular part of \( \partial \{ |u| > 0 \} \cap \Omega \) by a variant of the classical dimension reduction argument. See [F69] and [Giu84] for the case of perimeter minimizers in the Euclidean setting, [Wei99] for the case of free boundary problems in the Euclidean setting, [DPG18] for the dimension bounds for the singular strata on non-collapsed RCD spaces, and [MS21] for the dimension bounds of the singular part of perimeter minimizers in the non-collapsed RCD spaces.

**Theorem 8.12.** Suppose that \((X, d, \mu := \mathcal{H}^N)\) is a non-collapsed RCD\((K, N)\) metric measure space with \( N \geq 3 \), and that \( u \) is a minimizer of \( J_\Omega \) on a bounded domain \( \Omega \subset X \) and that \( Q \in C(\Omega) \) satisfies (1.3). Assume that \( \Omega \cap \partial X = \emptyset \). Then for any \( \varepsilon > 0 \),

\[
\dim_{\mathcal{H}} (S^\Omega_{\varepsilon} u) \leq N - 3.
\]

Moreover, if \( N = 3 \), then \( S^\Omega_{\varepsilon} u \) contains at most isolated points.

**Proof.** Fix any \( \varepsilon > 0 \). Assume that \( \mathcal{H}^{N-3+\eta}(S^\Omega_{\varepsilon} u) > 0 \) for some \( \eta > 0 \). Then there exists a point \( x_0 \in S^\Omega_{\varepsilon} u \) such that (see for example Lemma 3.6 in [DPG18]):

\[
\limsup_{r \to 0} \frac{\mathcal{H}^{N-3+\eta}(S^\Omega_{\varepsilon} u \cap B_r(x_0))}{r^{N-3+\eta}} \geq C_0 := 2^{-N+3-\eta} \omega_{N-3+\eta},
\]

where \( \mathcal{H}^{N-3} \) is the \( \infty \)-Hausdorff premeasure. Let \( r_j \) be a sequence such that \( r_j \to 0 \) and

\[
\frac{\mathcal{H}^{N-3+\eta}(S^\Omega_{\varepsilon} u \cap B_{r_j}(x_0))}{r_j^{N-3+\eta}} \geq C_0/2 > 0.
\]

Now we consider the blow-up sequence of pointed metric measure spaces \( X_j := (X, \frac{1}{r_j} d, \mu^{x_0}_{r_j}, x_0) \) and let \( u_j := r_j^{-1} u \). From (8.14), we have

\[
\mathcal{H}^{N-3+\eta}(S^{\Omega_{\varepsilon} u_j} \cap B^j_1(x_0)) \geq C_0/2,
\]

where \( B^j_1(x_0) \) is the unit ball in \( X_j \). By Theorem 7.1 and Lemma 8.10, up to a subsequence of \( \{r_j\} \), the \( X_j \) converges to a tangent cone at \( x_0 \) in the \( \text{pGH} \)-topology, denoted by \( (Y, d_C, \mu_C, \alpha_Y) \), and \( u_j \) converges to a blow up limit \( u_0 \) defined on \( C(Y) \), which is a global minimizer of \( J_{Q_{\Omega_0=Q(x_0)}} \) on \( C(Y) \) and homogeneous of degree one. By using the upper semicontinuity of the \( \infty \)-Hausdorff premeasure \( \mathcal{H}^{N-3+\eta} \) under \( \text{GH} \)-convergence (see [DPG18]) and Lemma 8.7, we get

\[
\mathcal{H}^{N-3+\eta}(S^{\Omega_{\varepsilon} u_0} \cap B_1(\alpha_Y)) \geq C_0/2.
\]

This implies

\[
\mathcal{H}^{N-3+\eta}(S^{\Omega_{\varepsilon} u_0} \cap B_1(\alpha_Y)) > 0.
\]

Since \( N - 3 + \eta > 0 \), it follows that there exists a point \( x_1 \neq \alpha_Y \) such that \( x_1 \in S^{\Omega_{\varepsilon} u_0} \cap B_1(\alpha_Y) \) and \( \limsup_{r \to 0} \frac{\mathcal{H}^{N-3+\eta}(S^{\Omega_{\varepsilon} u_0} \cap B_r(x_1) \subset C(Y))}{r^{N-3+\eta}} \geq C_0 \). By the same argument, we shall blow up again \( u_0 \) at \( x_1 \) along a sequence \( s_j \to 0^+ \) such that

\[
\frac{\mathcal{H}^{N-3+\eta}(S^{\Omega_{\varepsilon} u_0} \cap B_{s_j}(x_1) \subset C(Y))}{s_j^{N-3+\eta}} \geq C_0/2.
\]
We consider the blow up sequence of metric measure spaces $C_{s_j} := (C(Y), d_j := s_j^{-1}d_C, \mu_j := \mu^{x_j, x_1}_C)$, and the blow up sequence of maps $u_{0,j} := s_j^{-1}u_0$. Letting $j \to +\infty$, up to a subsequence, the metric measure spaces $C_{s_j}$ converge to a limit space $C_{\infty}$ in the $\text{pmGH}$-topology, which is isometric to a product space $C_{\infty} = Z \times \mathbb{R}$ with the natural product metric and product measure (by the splitting theorem in [Gig13]), and the maps $u_{0,j}$ converge to a limit map $u_{00}$, which a global minimizer on $Z \times \mathbb{R}$ of homogeneous of degree one, and

\begin{equation}
H^{N-3+\eta}\left(S^{\text{conf}}_0 \cap B_1((z_0, 0))\right) > 0,
\end{equation}

where $(z_0, 0)$ is the limit of the points $x_1 \in C_{s_j}$ as $j \to +\infty$. To continue the proof, we need the following lemma.

**Lemma 8.13.** The map $u_{00}|_{Z \times \{0\}}$ is a global minimizer of $J_{\Phi}$ on $Z$.

**Proof.** We first claim that $u_{00}(z, t) = u_{00}(z, 0)$ for any $z \in Z$ and $t \in \mathbb{R}$. It can be intuitively observed from homogeneity of $u_0$ on each $C_{s_j}$, and the converging of $C_{s_j} \xrightarrow{\text{pmGH}} Z \times \mathbb{R}$. Here, for clarity, we include the details for the realization of this observation as follows.

Let $\gamma : [0, +\infty) \to C(Y)$ is the ray with $\gamma(0) = o_Y$ and $\gamma(L) = x_1$, where $L = d_C(x_1, o_Y) > 0$. On each $C_{s_j}$, the curve $\gamma_j(t) := (s_j, o_Y)$ is one of the shortest on every sub-interval $[a, b] \subset [-L/s_j, +\infty)$. We first consider the functions $f_j$ on $C_{s_j}$ given by

$f_j(x) := d_j(x_1, o_Y) - d_j(x, o_Y) = s_j^{-1}(d_C(x_1, o_Y) - d_C(x, o_Y)).$

Since $u_0 = (u_0^1, u_0^2, \cdots, u_0^m)$ is homogeneous of degree one on $C(Y)$, we have for all $s_j$, that

\begin{equation}
\langle \nabla u_0^\alpha_{s_j}, \nabla f_j \rangle(x) = \langle \nabla(s_j^{-1}u_0^\alpha), \nabla d_j(x, o_Y) \rangle = 0 \quad \text{a.e. in } C_{s_j},
\end{equation}

for all $\alpha = 1, 2, \cdots, m$.

Letting $s_j \to 0^+$, up to a subsequence, the curves $\gamma_j : [-L/s_j, +\infty) \subset C_{s_j}$ converge to a line $\gamma_{\infty}$ on the limit space $C_{\infty}$. According to the splitting theorem in [Gig13], we know that $C_{\infty}$ splits isometrically to a product space $Z \times \mathbb{R}$. Moreover, letting $b$ be the Busemann function with respect to the ray $\gamma_{\infty}|_{[-L, 0]}$ on $C_{\infty}$, then $Z = b^{-1}(0)$ and that the gradient flow of $b$ exists, denoted by $\Phi_t$, and furthermore $(z, t) = \Phi_t(z, 0)$ for any $z \in Z$ and $t \in \mathbb{R}$. On the other hand, from the definition of $f_j(x) = d_j(x_1, o_Y) - d_j(x, o_Y) = d_j(\gamma_j(0), \gamma_j(-L/s_j)) - d_j(x, \gamma_j(-L/s_j))$ and the fact that $f_j$ is 1-Lipschitz on $C_{s_j}$, it is clear that, up to a subsequence, the functions $f_j$ converge uniformly on each compact set to the Busemann function $b(x)$. Notice that $|\nabla f_j|(x) = 1$ a.e. $x \in C_{s_j}$ and $|\nabla b| = 1$ a.e. on $C_{\infty}$. In particular, we get that $\langle \nabla f_j \rangle = |\nabla b|$ in the $L^2_{\text{loc}}$ as $j \to \infty$. By combining this and (7.4), we conclude that $\langle \nabla u_0^\alpha_{s_j}, \nabla f_j \rangle = \langle \nabla u_0^\alpha_{d_0}, \nabla b \rangle$ in $L^2_{\text{loc}}$, for all $\alpha = 1, 2, \cdots, m$. From (8.18), we get that

$\langle \nabla u_0^\alpha_{d_0}, \nabla b \rangle = 0 \quad \text{a.e. in } C_{\infty} = Z \times \mathbb{R}, \quad \forall \alpha = 1, 2, \cdots, m.$

This implies for almost all $z \in Z$, $u_{00} \circ \Phi_t$ is a constant map, where $\Phi_t$ is the gradient flow of $b$. Recalling $\Phi_t(z, 0) = (z, t)$ for all $z \in Z$, this gives $u_{00}(z, t) = u_{00}(z, 0)$ for almost all $z \in Z$. Finally, from the fact that $u_{00}$ is Lipschitz continuous, we know that $u_{00}(z, t) = u_{00}(z, 0)$ for all $z \in Z$. This claim is proved.

With the help of the fact that $u_{00}(z, t) = u_{00}(z, 0)$ for any $z \in Z$ and $t \in \mathbb{R}$, we will use the argument in [Wei99] to prove this Lemma 8.13. Suppose that $u_{00}$ is not a
minimizer of $J_{Q_0}$ on a ball $B \subset Z$. Then there exists a map $v \in W^{1,2}(B, [0, +\infty)^m)$ such that $v - u_{00}|_{Z \times \{0\}} \in W^{1,2}_0(B, \mathbb{R}^m)$ and

$$\int_B (|\nabla v|^2 + Q_0 \chi_{|v|>0}) d\mu_Z \leq \int_B (|\nabla u_{00}|^2 + Q_0 \chi_{|u_{00}|>0}) d\mu_Z - \epsilon_0$$

for some $\epsilon_0 > 0$. We define a map $v_T$ on $B \times (-T, T)$, for any $T > 1$, by

$$v_T(z, t) := \begin{cases} v(z), & |t| < T - 1 \\ (T - |t|) v(z) + (|t| - T + 1) u_{00}(z, 0), & T - 1 \leq |t| \leq T \\ u_{00}(z, 0), & |t| \geq T. \end{cases}$$

It is clear that $v_T - u_{00} \in W^{1,2}_0(B \times (-T, T), [0, +\infty)^m)$, by using the fact $u_{00}(z, t) = u_{00}(z, 0)$ for all $z \in Z$. Note that

$$\int_{B \times (-T, T)} (|\nabla v_T|^2 + Q_0 \chi_{|v_T|>0}) d\mu_Z dt - \int_{B \times (-T, T)} (|\nabla u_{00}|^2 + Q_0 \chi_{|u_{00}|>0}) d\mu_Z dt \leq 2(T - 1) \left( \int_B (|\nabla v|^2 + Q_0 \chi_{|v|>0}) d\mu_Z - \int_B (|\nabla u_{00}|^2 + Q_0 \chi_{|u_{00}|>0}) d\mu_Z \right) + \int_{B \times ((-T, -T+1) \cup (T-1, T))} (|\nabla v_T|^2 + Q_0 \chi_{|v_T|>0}) d\mu_Z dt \leq -2(T - 1) \epsilon_0 + 4 \left( \int_B (|\nabla v|^2 + |v|^2 + |\nabla u_{00}|^2 + |u_{00}|^2) d\mu_Z + 2Q_0 \mu_Z(B) \right),$$

which contradicts the fact that $u_{00}$ is a minimizer on $B \times (-T, T)$ when $T$ is large enough. Therefore, the Lemma 8.13 is proved.

We now come back to the proof of Theorem 8.12. From the assumption $\partial X \cap \Omega = \emptyset$, we know that both $C(Y)$ and $Z \times \mathbb{R}$ have no boundary. Thus, $Z$ has no boundary.

If $N - 1 \geq 3$, by the combination of the above Lemma 8.13, (8.17) and the fact that $u_{00}(z, t) = u_{00}(z, 0)$ for all $z \in Z$ and $t \in \mathbb{R}$, we obtain that there exists an $(N - 1)$-dimensional $ncRCD(0, N - 1)$-space without boundary, $Z$, and a minimizer of $J_{Q_0}$ on $Z$, $\hat{u} := u_{00}|_{Z \times \{0\}}$, such that $\mathcal{H}^{N-4+\eta}(S_\epsilon^{\Omega_\alpha} \cap B_1(z_0) \subset Z) > 0$.

Iterating this procedure we conclude that there exists a 3-dimensional $ncRCD(0, 3)$-space without boundary, denoted by $\hat{X}$, and a minimizer of $J_{Q_0}$, denoted by $\hat{u}$, on $\hat{X}$, such that $\mathcal{H}^{\eta}(S_\epsilon^{\Omega_\alpha} \cap B_1(\hat{x})) > 0$.

We claim that the singular set of $\hat{u}$ must contain only isolated points. Suppose that a sequence $\hat{x}_j \in S_\epsilon^{\Omega_\alpha}$ and $\hat{x}_j \to \hat{x}_0$. Let $\hat{s}_j = d(\hat{x}_0, \hat{x}_j) \to 0^+$. We consider the blow up sequence of spaces $(\hat{X}, \hat{s}_j^{-1}d, \mu_{\hat{x}_j}^{\hat{s}_j}, \hat{x}_0)$ and maps $\hat{u}_j = \hat{s}_j^{-1} \hat{u}$. Letting $j \to +\infty$, we get a blow up limit map $\hat{u}_0$ on a tangent cone $C(Y)$ at $\hat{x}_0$. From Lemma 8.7, we know that $\hat{u}_0$ has at least two singular points $\hat{y}_\vee$ and $\hat{y}_\infty$, the limit of $\hat{x}_j$. By $\hat{s}_j^{-1}d(\hat{x}_0, \hat{x}_j) = 1$, we have $\hat{y}_\infty \neq \hat{y}_\vee$. Now, we blow up again at $\hat{y}_\infty$ as above, from Lemma 8.13, we get a minimizer of $J_{Q_0}$, $u_{00}|_Z$, on some two-dimensional $ncRCD(0, 2)$-space $\hat{Z}$. Moreover, it has at least one singular point at $\hat{z}_0$, where the point $(\hat{z}_0, 0)$ is the limit of points $\hat{y}_\infty$ under this blow up procedure. This contradicts with Lemma 8.11. The claim is proved, and hence the proof of Theorem 8.12 is finished.

Proof of Theorem 1.9. It follows from the combination of Corollary 8.9 (by putting $O_\epsilon := \mathcal{R}_{\epsilon}^{\Omega_\alpha}$ in Corollary 8.9), Theorem 8.12 and Lemma 8.11.

Proof of Corollary 1.11. It follows from Theorem 8.12 and $S_\epsilon^{\Omega_\alpha} = \cup_{\epsilon > 0} S_{\epsilon}^{\Omega_\alpha}$. \qed
Appendix A. Weiss-type monotonicity on cones

A Weiss-type monotonicity for minimizers $u$ of $J_Q$ defined on $\mathbb{R}^N$ has been obtained in [CSY18] and [MTV17]. The same argument can be extended to the case where $u$ is defined on a metric measure cone. We will provide the details as follows.

Let $N \geq 2$ and let $(C(\Sigma), d_C, \mu_C)$ be a metric measure cone, with the vertex $p$, over $(\Sigma, d_\Sigma, \mu_\Sigma)$, and assume that $(C(\Sigma), d_C, \mu_C)$ satisfies RCD$(0,N)$.

Lemma A.1. For $\mathcal{L}^1$-a.e. $r \in \mathbb{R}^+$, it holds
\begin{equation}
(\text{A.1}) \quad \int_{C(\Sigma)} g \cdot d\chi_{B_r(p)} = r^{N-1} \int_{\Sigma} g_r \cdot d\mu_S,
\end{equation}
for any Borel function $g(t, \xi)$ on $C(\Sigma)$, where $g_r(\xi) := g(r, \xi)$.

Proof. For $\mathcal{L}^1$-a.e. $r \in \mathbb{R}^+$, the set $B_r(p)$ has finite perimeter and that, by coarea formula,
\begin{equation}
\int_s^t \int_{C(\Sigma)} g \cdot d\chi_{B_r(p)} = \int_{B_r(p)} g \cdot d\mu_C = \int_{(s,t) \times \Sigma} g_r^{N-1} |\mathrm{d}r| \cdot d\mu_\Sigma
\end{equation}
for all $0 \leq s < t < \infty$ and all Borel function $g$, where we have used $d\mu_C(t, \xi) = t^{N-1} |\mathrm{d}r| \cdot d\mu_\Sigma(x)$. It follows that the function $t \mapsto \int_s^t \int_{C(\Sigma)} g \cdot d\chi_{B_r(p)}$ is absolutely continuous, and then the desired assertion (A.1) holds. \hfill \Box

Let $Q_0 > 0$ be a constant and let $u = (u_1, \ldots, u_m)$ be a global minimizer of $J_{Q_0}$ on $(C(\Sigma), d_C, \mu_C)$. I.e., for each $R > 0$, $u$ is minimizer of $J_{Q_0}$ on $B_R(p)$. For any $r \in (0, +\infty)$, we denote by $u_r(\xi) = u(r, \xi), \forall \xi \in \Sigma$.

Lemma A.2. For each $r \in (0, \infty)$, we have
\begin{equation}
\frac{N}{r^{N-2}} \int_{B_r(p)} (|\nabla u|^2 + Q_0 \chi_{\{|u| > 0\}}) \leq \int_{\Sigma} (|\nabla u|^2 + |u_r|^2) \cdot d\mu_\Sigma, \tag{A.2}
\end{equation}
\begin{equation}
+ r^2 \int_{\Sigma} (Q_0 \chi_{\{|u| > 0\}}) \cdot d\mu_\Sigma,
\end{equation}
where $\nabla_S v$ is the weak upper gradient of $v \in W^{1,2}(\Sigma)$.

Proof. Fix any $r \in (0, \infty)$. We set the function $v := (v_1, \ldots, v_m) : B_r(p) \rightarrow \mathbb{R}^m$ by
\begin{equation}
v_i(t, \xi) := \frac{t}{r} u_i(r, \xi), \quad \forall t \in (0, r), \quad \xi \in \Sigma, \quad \forall i \in \{1, 2, \ldots, m\}.
\end{equation}
Then we first have $v \in Lip(B_r(p))$ and $v = u$ on $\partial B_r(p)$. By the minimizer of $u$, we have
\begin{equation}
\int_{B_r(p)} (|\nabla u|^2 + Q_0 \chi_{\{|u| > 0\}}) \leq \int_{B_r(p)} (|\nabla v|^2 + Q_0 \chi_{\{|v| > 0\}}).
\end{equation}
Following Proposition 3.4 of [Ket15], we know that for any $v \in W^{1,2}(B_r(p))$, it holds, for almost all $(t, x) \in C(\Sigma)$,
\begin{equation}
|\nabla v|^2(t, x) = |\nabla_S u + v_\xi|^2(t) + t^{-2} |\nabla_S v_\xi|^2(\xi),
\end{equation}
where $v_1(\cdot) := v(t, \cdot)$ and $v_\xi(\cdot) := v(\cdot, \xi)$. By applying this to each component of $v$, we get
\begin{equation}
\int_{B_r(p)} |\nabla v|^2 = \int_{B_r(p)} \left(\frac{1}{r^2} |\nabla_S v_1|^2(\xi) + |v_1(v_\xi)|^2(t)\right) \cdot d\mu_C
\end{equation}
\begin{equation}
= \int_0^r \int_{\Sigma} \left(\frac{1}{r^2} |\nabla_S u_r|^2(\xi) + \frac{1}{r^2} |u_r|^2(\xi)\right) t^{N-1} |\mathrm{d}t| \cdot d\mu_\Sigma
\end{equation}
\begin{equation}
= \frac{r^{N-2}}{N} \int_{\Sigma} \left(\frac{1}{r^2} |\nabla_S u_r|^2(\xi) + |u_r|^2(\xi)\right) \cdot d\mu_\Sigma.
\end{equation}
Noticing that $|\psi(t, \xi)| > 0 \iff |u_r|(|\xi|) > 0$, we have

$$
\int_{B_r(p)} Q_0 \chi_{\{\psi > 0\}} = \int_0^r \int_{\Sigma} \left( Q_0 \chi_{\{|u_r| > 0\}} \right) t^{N-1} \, dt \, d\mu_{\Sigma} = \frac{r^N}{N} \int_{\Sigma} \left( Q_0 \chi_{\{|u_r| > 0\}} \right) \, d\mu_{\Sigma}.
$$

(A.5)

Now the desired estimate (A.2) follows from the combination of (A.3)-(A.5), and the proof is finished. \(\square\)

Now we give the monotonicity of $W_u(p, r, Q_0)$.

**Lemma A.3.** Suppose the cone $C(\Sigma)$ is non-collapsed. Then the function $r \mapsto W_u(p, r, Q_0)$ is non-decreasing. Moreover, if $W_u(p, r, Q_0)$ is a constant then $u$ is homogeneous of degree one.

**Proof.** Since $u$ is locally Lipschitz continuous in $C(\Sigma)$, it is clear that $W_u(p, r, Q_0)$ is locally Lipschitz continuous in $(0, \infty)$, and then it is differentiable at $\mathcal{L}^1$-a.e. $r \in (0, \infty)$. At a such $r$, we have

$$(A.6) \quad \frac{d}{dr} W_u(p, r, Q_0) = -\frac{N}{r^{N+1}} \int_{B_r(p)} (|\nabla u|^2 + Q_0 \chi_{\{|u| > 0\}}) \, d\mu + \frac{1}{N} \int_{C(\Sigma)} (|\nabla u|^2 + Q_0 \chi_{\{|u| > 0\}}) \, d|D\chi_{u_r}(p)| + \frac{2}{r^3} \int_{\Sigma} |u_r|^2 \, d\mu_{\Sigma} \geq \frac{1}{r} \int_{\Sigma} |\nabla_{\Sigma} u_r|^2 (r) \, d\mu_{\Sigma} + \frac{1}{r^3} \int_{\Sigma} |u_r|^2 \, d\mu_{\Sigma} - \frac{1}{r^2} \frac{d}{dr} \int_{\Sigma} |u_r|^2 \, d\mu_{\Sigma},$$

where we have used (A.2), (A.1) and

$$(\nabla u)^2(r, x) = |\nabla_{\Sigma} u_r|^2(r) + r^{-2} |\nabla_{\Sigma} u_r|^2(\xi)$$

(by Proposition 3.4 of [Ket15]). Since $C(\Sigma)$ is assumed to be non-collapsed, we know that $u$ is locally Lipschitz on $C(\Sigma)$. Thus, we get

$$\frac{d}{dr} \int_{\Sigma} |u_r|^2 \, d\mu_{\Sigma} = \sum_{i=1}^m \frac{d}{dr} \int_{\Sigma} u_i^2(r, \xi) \, d\mu_{\Sigma} = \sum_{i=1}^m \int_{\Sigma} 2 u_i(r, \xi) \frac{\partial u_i}{\partial r} (r, \xi) \, d\mu_{\Sigma}.$$ 

Putting this into (A.6), we get

$$r \cdot \frac{d}{dr} W_u(p, r, Q_0) \geq \sum_{i=1}^m \int_{\Sigma} \left( |\nabla_{\Sigma} u_i|_d|^2 + \frac{u_i^2}{r^2} - 2 \frac{u_i}{r} |\nabla_{\Sigma} u_i| \right) \, d\mu_{\Sigma} = \sum_{i=1}^m \int_{\Sigma} \left( |\nabla_{\Sigma} u_i| - \frac{u_i}{r} \right)^2 \, d\mu_{\Sigma} \geq 0,$$

where $u_i(\cdot, \xi) := u_i(\cdot, \xi)$ and $u_i(r, \cdot) := u_i(r, \cdot)$. It follows that $W_u(p, r, Q_0)$ is non-decreasing. Moreover, if $W_u(p, r, Q_0)$ is a constant, then one have

$$\frac{\partial u_i}{\partial r} (r, \xi) = \frac{u_i(r, \xi)}{r} = \frac{u_i(r, \xi)}{r}$$

for almost all $(r, \xi)$ in $C(\Sigma)$. This implies for almost $\xi \in \Sigma$ that $u_{i, \xi}(r) = ru_{i, \xi}(1)$. Therefore, in this case $u$ is homogeneous of degree one. The proof is finished. \(\square\)
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