On quasi subordination for analytic and biunivalent function class

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Abstract
For two analytic functions $f$ and $F$, if there exists analytic functions $F$ and $w$, with $|F(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that the equality $f(z) = F(z)\varphi(\omega(z))$ holds, then the function $f$ is said to be quasi subordinate to $\varphi$, written as follows $f(z) \prec f \varphi(z), z \in E$. In this work, the subclass $M^{q,\phi}_{\Sigma}(\gamma, \lambda, \delta)$ of the function class $S$ of bi-univalent functions associated with the quasi-subordination is defined and studied. Also some relevant classes are recognized and connections to previous results are made.

Keywords: Analytic function, Bi-univalent function; Quasi-subordination; Subordination; Univalent function
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1 Introduction and Definitions

Thoroughly this study, we assume that the open unit disc $\{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$ is shown by $E$, all normalized analytic functions of the form

$$ f(z) = z + \sum_{n=2}^{\infty} a_n z^n. $$

satisfying the conditions

$$ f(0) = 0 \text{ and } f'(0) = 0 $$

in $E$ is demonstrated by the symbol $A$, and the subclass of all functions in $A$ which are univalent in $E$ is shown by $S$. Due to the fact that the univalent functions are one to one, these functions are possessed of inverse. While the inverse of univalent functions are invertible they don’t need to be defined on the entire unit disc $E$. Absolutely, according the Koebe one-quarter theorem, a
and Clunie [3] conjectured that the bound 1.51 for the modulus of the second coefficient of the study of Srivastava et al. One can see the basic definitions of the analytic and bi-univalent function class and their properties and interesting of functions in the class Σ, in the study of Srivastava et al. [19].

Lewin ([11]) is the first mathematician working this subject. ([11]) obtained the bound 1.51 for the modulus of the second coefficient |a_2|. Later, Brannan and Clunie [3] conjectured that |a_2| ≤ √2 for f ∈ Σ. Later, if f(z) ∈ Σ then max |a_2| = 1 is proven by Netanyahu [14]. Then, a certain subclasses of class Σ analogous subclasses S^∗ (β) of starlike functions and Κ (β) convex functions of order β (0 ≤ β < 1) in U was expressed by Brannan and Taha [2], in turn (see [14]). The classes S^*_Σ (β) and Κ_Σ (β) of bi-starlike functions of order β in U and bi-convex functions of order β in U, corresponding to the function classes S^∗ (β) and Κ (β), were also introduced congruently. For each of the function classes S^*_Σ (β) and Κ_Σ (β), these mathematicians obtained some estimates for the initial coefficients but these estimates were not sharp. Recently, motivated substantially by the following work on this area Srivastava et al. [19], many authors searched the coefficient bounds for diversified subclasses of bi-univalent functions (see, for instance, [5], [20]). Dealing with the bounds on the general coefficient |a_n| for n ≥ 4, there isn’t enough knowledge. In the literature, only a few works has been made to identify the general coefficient bounds for |a_n| for the analytic bi-univalent functions (see, for instance, [7], [8]). Today, the problem of identifying the coefficient for each of the coefficients |a_n| (n ∈ N \ {1, 2} : N = {1, 2, 3, ···}) is an unsoluble problem.

For two analytic functions f and F, as long as there is an analytic function w defined on E by ω(0) = 0, |ω(z)| < 1 satisfying f(z) = ϕ(ω(z)) , then the function f is named to be subordinate to F and demonstrated by f ≺ ϕ. The class of Ma-Minda starlike as well as convex functions [12] are expressed as follows:

\[ S^*_Σ (ϕ) = \left\{ f \in Σ : \frac{zf'(z)}{f(z)} ≺ ϕ(z) \right\} \]

and

\[ KΣ (ϕ) = \left\{ f \in Σ : \left(1 + \frac{zf'(z)}{f(z)} \right) ≺ ϕ(z) \right\} \]

where ϕ be analytic function having positive real part in E, ϕ(0) = 1 and ϕ'(0) > 0 also ϕ(E) is a region which is starlike with respect to 1 and symmetric with respect to the real axis. In 2013, the coefficient bounds for biunivalent Ma-Minda starlike and convex functions were described in the study of Ali at all.
The classes $S^*(\varphi)$ and $K(\varphi)$ includes several famous subclasses starlike functions like special case.

In 1970, the concept of quasi subordination was first defined by [18]. For the functions $f$ and $F$, if there exists analytic functions $F$ and $w$, with $|F(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that the equality

$$f(z) = F(z)\varphi(\omega(z))$$

holds, then the function $f$ is said to be quasi subordinate to $\varphi$, demonstrated by

$$f(z) \prec_q \varphi(z), \ z \in \mathbb{E}. \quad (3)$$

Prefering $F(z) \equiv 1$, the quasi subordination given in (3) turns into the subordination $f(z) \prec \varphi(z)$. Thus, the quasi subordination is a universality of the well known subordination and majorization (see [18]).

The work on quasi subordination is very wide and it includes some recent investigations ([9], [17], [18], [13]).

From beginning to the end this study, it is assumed that

$$F(z) = A_0 + A_1z + A_2z^2 + \cdots, \quad (|F(z)| \leq 1, z \in \mathbb{E}) \quad (4)$$

and

$$\varphi(z) = 1 + B_1z + B_2z^2 + \cdots \quad (B_1 > 0) \quad (5)$$

where $\varphi(z)$ is analytic function in $\mathbb{E}$.

Guided by the above mentioned studies, we define a subclass of function class $S$ in such a way.

**Definition 1** The class $M^q_{\Sigma}(\gamma, \lambda, \delta) \ (0 \neq \gamma \in \mathbb{C}, \lambda \geq 1, \delta \geq 0$ and $(\forall z, w \in \mathbb{E})$ consists of the function $f \in \Sigma$, expressed in the relation (1), if the quasi subordination conditions

$$\frac{1}{\gamma} \left[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta zf''(z) - 1 \right] \prec_q (\varphi(z) - 1), \quad (6)$$

$$\frac{1}{\gamma} \left[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) + \delta wg''(w) - 1 \right] \prec_q (\varphi(w) - 1) \quad (7)$$

are satisfied, where the function $f^{-1}$ is the restriction of $g$ described in the relation (2), $\varphi$ is the function given in (5).

By choosing the special values for $\delta, \gamma, \lambda$ and the class $M^q_{\Sigma}(\gamma, \lambda, \delta)$ reduces to several earlier known classes of analytic and biunivalent functions studied in the literature.

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Remark 2 Taking $\gamma = 1$ and $\delta = 0$ in the above class, so we obtain

\[ M_{q,\varphi}^{q,\varphi}(1, \lambda, 0) = R_{\Sigma}^{q}(\lambda, \varphi) \]

This study was firstly introduced by Patil and Naik (15). The class $R_{\Sigma}^{q}(\lambda, \varphi)$ contains functions $f \in \mathcal{S}$ satisfying

\[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) - 1 \prec q (\phi(z) - 1) \]

and

\[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) - 1 \prec q (\phi(w) - 1). \]

where $z, w \in \mathbb{E}$ and $\lambda \geq 1$.

Remark 3 Taking $F(z) \equiv 1$, so the quasi subordination reduces the subordination and we get

\[ M_{\varphi}^{\varphi}(\gamma, \lambda, \delta) = M_{\Sigma}^{\varphi}(\gamma, \lambda, \delta) \]

This new class containes functions $f \in \Sigma$ satisfying

\[ \frac{1}{\gamma} \left[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) \right] \prec \varphi(z) \]

and

\[ \frac{1}{\gamma} \left[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) \right] \prec \varphi(w). \]

The class $M_{\Sigma}^{\varphi}(\gamma, \lambda, \delta)$ involves many well known classes, which are given following:

1. Taking $\gamma = 1$ and $\delta = 0$, so we obtain

\[ M_{\Sigma}^{\varphi}(1, \lambda, 0) = M_{\Sigma,1,\lambda}^{0}(\phi) = R_{\Sigma}(\lambda, \phi). \]

The class $R_{\Sigma}(\lambda, \phi)$ was first introduced by Kumar et al. (10). This class involves functions $f \in \Sigma$ satisfying

\[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) \prec \varphi(z) \]

and

\[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) \prec \varphi(w) \]

for the function $\phi$ analytic and $\lambda \geq 1$.

The class $M_{\Sigma}^{\varphi}(1, \lambda, 0)$ involves many earlier classes. These classes are given following:
(i) Taking $\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, 0 \leq \beta < 1$, we obtain the class $B_\Sigma(\beta, \lambda)$ defined by Frasin and Aouf ([5]), see Definition 3.1.

(ii) Taking $\varphi(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha, 0 < \alpha \leq 1, \lambda \geq 1$, then we have the class $B_\Sigma(\alpha, \lambda)$ defined by Frasin and Aouf ([5]), see Definition 2.1.

(iii) Taking $\lambda = 1$ then the class reduces $H_\Sigma(\varphi)$ investigated and defined by Ali et all. [1].

(iv) Taking $\lambda = 1$ and $\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, 0 \leq \beta < 1$, then the class reduces $H_\Sigma(\beta)$ defined and investigated by Srivastava et al. [19] (see Definition 2).

(v) Taking $\lambda = 1$ and $\varphi(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha, 0 < \alpha \leq 1$, then the class reduces $H_\Sigma(\alpha)$, defined and investigated by Srivastava et al. [19] (see Definition 1).

2. Taking $\gamma = 1, \lambda = 1$ and $\delta = 0$ then we get

$$M_\Sigma^\varphi(1, 1, 0) = R_\Sigma(\varphi).$$

The class $R_\Sigma(\varphi)$ was introduced by Ali et al. [1]. This class involves of functions $f \in \Sigma$ satisfying

$$f'(z) \prec \varphi(z)$$

and

$$g'(w) \prec \varphi(w).$$

3. Taking $\lambda = 1$ and $0 \leq \delta < 1$, so we have

$$M_\Sigma^\varphi(\gamma, 1, \delta) = R_\Sigma(\eta, \lambda, \varphi)$$

The class $R_\Sigma(\eta, \lambda, \varphi)$ was studied by Deniz ([6]). This class involves of the functions $f \in \Sigma$ satisfying

$$1 + \frac{1}{\gamma} \left[ f'(z) + \eta f'(z) - 1 \right] \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma} \left[ g'(w) + \eta g'(w) - 1 \right] \prec \varphi(w).$$

4. Taking $\gamma = 1, \lambda = 1$ and $\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, 0 \leq \beta < 1$, then we have
\[ M^{q,\varphi}_{\Sigma}(\gamma, \lambda, 1) = M^{q}_{\Sigma}(\lambda, \delta). \]

The class \( M^{q}_{\Sigma}(\lambda, \delta) \) was defined and investigated by Bulut [4]. This class involves the functions \( f \in \Sigma \) satisfying

\[
\text{Re}\left( (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta zf''(z) \right) > \alpha
\]

and

\[
\text{Re}\left( (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) + \delta wg''(w) \right) > \alpha
\]

where \( 0 \leq \alpha < 1 \).

2 Coefficient Estimates for the Function Class \( M^{q,\varphi}_{\Sigma}(\gamma, \lambda, \delta) \)

Firstly, we will state the Lemma 4 to obtain our result.

**Lemma 4** ([10]) If \( p \in \mathcal{P} \), then \( |p_i| \leq 1 \) for each \( i \), where \( \mathcal{P} \) is the family all functions \( p \), analytic in \( E \), for which

\[
\text{Re}\{p(z)\} > 0 \quad (z \in E)
\]

where

\[
p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in E).
\]

We begin this section by finding the estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in the class \( M^{q,\varphi}_{\Sigma}(\gamma, \lambda, \delta) \) proposed by Definition 1.

**Theorem 5** If \( f \in \Sigma \) expressed by (1) belongs the class \( M^{q,\varphi}_{\Sigma}(\gamma, \lambda, \delta) \), \( 0 \neq \gamma \in \mathbb{C}, \lambda \geq 1, \delta \geq 0 \) and \( z, w \in E \), then

\[
|a_2| \leq \min \left\{ \frac{|A_0||B_1|}{1 + \lambda + 2\delta}, \sqrt{\frac{|A_0||B_1| + |B_2 - B_1|}{1 + 2\lambda + 6\delta}} \gamma \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{A^2_0 B^2_1}{(1 + \lambda + 2\delta)^2} |\gamma| + \frac{|(A_0||B_1| + (\lambda_0 + \lambda_1)|B_1|}{1 + 2\lambda + 6\delta}} |\gamma|, \right\}
\]

\[
\frac{|A_0||B_1| + |B_2 - B_1| + |A_0| + |A_1||B_1|}{1 + 2\lambda + 6\delta} |\gamma|
\]

\[ (8) \]

\[ (9) \]
Proof. If $f \in \mathcal{M}_{\gamma, \lambda}^{s, \delta} (\varphi)$ then, there are analytic functions $u, v : E \rightarrow E$ with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$ and a function $F$ given by (3), such that

$$
\frac{1}{\gamma} \left[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta zf'(z) - 1 \right] = F(z) [\varphi (u(z)) - 1] \tag{10}
$$

and

$$
\frac{1}{\gamma} \left[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) + \delta wg'(w) - 1 \right] = F(w) [\varphi (u(w)) - 1] \tag{11}
$$

Determine the functions $p_1$ and $p_2$ in $P$ given by

$$
p_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \cdots
$$

and

$$
p_2(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + d_1 w + d_2 w^2 + \cdots.
$$

Thus,

$$
u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left[ c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \right] \tag{12}
$$

and

$$
v(w) = \frac{p_2(w) - 1}{p_2(w) + 1} = \frac{1}{2} \left[ d_1 w + \left( d_2 - \frac{d_1^2}{2} \right) w^2 + \cdots \right]. \tag{13}
$$

The fact that $p_1$ and $p_2$ are analytic in $E$ with $p_1(0) = p_2(0) = 1$ and have their real part in $E$ is obvious. Due to the fact that all of the functions $u, v : E \rightarrow E$ and $p_1, p_2$ have their real part in $E$, the relations $|c_1| \leq 2$ and $|d_1| \leq 2$ are true (16). Using (10) and (14) together with (4) and (5) in the right hands of the relations (10) and (11), we obtain

$$
F(z) [\varphi (u(z)) - 1] = \frac{1}{2} A_0 B_1 c_1 z + \left\{ \frac{1}{2} A_1 B_1 c_1 + \frac{1}{2} A_0 B_1 \left( c_2 - \frac{c_1^2}{2} \right) \right\} z^2 + \cdots \tag{14}
$$

and

$$
F(w) [\varphi (v(w)) - 1] = \frac{1}{2} A_0 B_1 d_1 w + \left\{ \frac{1}{2} A_1 B_1 d_1 + \frac{1}{2} A_0 B_1 \left( d_2 - \frac{d_1^2}{2} \right) \right\} w^2 + \cdots \tag{15}
$$

By using the form of the functions $f$ and $g$, which are given by (1) and (2), $g = f^{-1}$, we have
\[
\frac{1}{\gamma} \left[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) \right] = \frac{1}{\gamma} \left[ \sum_{n=2}^{\infty} [1 + (n - 1)\lambda + n(n - 1)\delta] a_n z^{n-1} \right]
\]

and

\[
\frac{1}{\gamma} \left[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) \right] = \frac{1}{\gamma} \left[ \sum_{n=2}^{\infty} [1 + (n - 1)\lambda + n(n - 1)\delta] a_n w^{n-1} \right].
\]

Comparing the coefficients of (14) with (16) and (15) with (17), then we have

\[
\frac{1}{\gamma} (1 + \lambda + 2\delta) a_2 = \frac{1}{2} A_0 B_1 c_1
\]

(18)

\[
\frac{1}{\gamma} (1 + 2\lambda + 6\delta) a_3 = \frac{1}{2} A_1 B_1 c_1 + \frac{1}{2} A_0 B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{A_0 B_2}{4} c_1^2
\]

(19)

\[
- \frac{1}{\gamma} (1 + \lambda + 2\delta) a_2 = \frac{1}{2} A_0 B_1 d_1
\]

(20)

\[
\frac{1}{\gamma} (1 + 2\lambda + 6\delta) \left( 2a_2^2 - a_3 \right) = \frac{1}{2} A_1 B_1 d_1 + \frac{1}{2} A_0 B_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{A_0 B_2}{4} d_1^2
\]

(21)

From (18) and (20), we have

\[
c_1 = -d_1
\]

(22)

and

\[
\frac{8}{\gamma^2} (1 + \lambda + 2\delta)^2 a_2^2 = A_0 B_1^2 \left( c_1^2 + d_1^2 \right)
\]

(23)

Adding (19) and (21), we get

\[
\frac{1}{\gamma} (1 + 2\lambda + 6\delta) a_2^2 = \frac{2A_0 B_1 (c_2 + d_2) + A_0 (B_2 - B_1) (c_1^2 + d_1^2)}{8}
\]

(24)

Using (22) end Lemma 4 in equalities (23) and (24), we obtain desired result given by the inequality (8).

Now, to find the bound on \(|a_3|\), by using the relations (21) and (19), then we have

\[
\frac{2}{\gamma} (1 + 2\lambda + 6\delta) \left( a_3 - a_2^2 \right) = \frac{2A_1 B_1 c_1 + A_0 B_1 (c_2 - d_2)}{2 (1 + 2\lambda + 6\delta)} \gamma
\]

(25)
Using the Lemma in (26) and (27), we complete the proof of theorem.

3 Corollaries and Consequences

Choosing $\gamma = 1$ and $\delta = 0$ in Theorem 5, we get the consequences below

**Corollary 6** ([15]). Let the function $f \in \mathcal{R}_q^\lambda (\lambda, \phi)$. Then,

$$|a_2| \leq \min \left\{ \frac{|A_0| B_1}{1+\lambda}, \sqrt{\frac{|A_0| (B_1 + |B_2 - B_1|)}{1+2\lambda}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{A_0^2 B_1^2}{(1+\lambda)^2}, \frac{(|A_0| + |A_1|) B_1}{1+2\lambda}, \frac{|A_0| (B_1 + |B_2 - B_1|) + (|A_0| + |A_1|) B_1}{1+2\lambda} \right\}.$$

**Remark 7** The estimates for $|a_2|$ and $|a_3|$ of Corollary 6 show that Theorem 5 ([?]) is an improvement of the estimates obtained by Patil nd Naik ([7], (15)), Theorem 2.2).

Choosing $F(z) \equiv 1$ in Theorem 5, we obtain the following

**Corollary 8** Let the function $f \in \mathcal{M}_q^\gamma (\gamma, \lambda, \delta)$. Then,

$$|a_2| \leq \sqrt{\left( \frac{B_1 + |B_2 - B_1|}{1+2\lambda+2\delta} \right)} \gamma$$

and

$$|a_3| \leq \left( \frac{B_1 + 2|B_2 - B_1|}{1+2\lambda+2\delta} \right) \gamma.$$

If we let $F(z) \equiv 1$, $\gamma = 1$ and $\delta = 0$ in Theorem 5, we get the consequences below:
**Corollary 9**  

Let the function \( f \in \mathcal{R}_\Sigma(\lambda, \phi) \). Then,

\[
|a_2| \leq \min \left\{ \frac{B_1}{1 + \lambda}, \sqrt{\frac{B_1 + |B_2 - B_1|}{1 + 2\lambda}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{B_1^2}{1 + 2\lambda} + \frac{B_1^2}{(1 + \lambda)^2}, \frac{B_1 + |B_2 - B_1|}{1 + 2\lambda} \right\}.
\]

**Remark 10**  

In light of the Corollary 9, we can state the following remarks:

1. The estimates for \( |a_2| \) and \( |a_3| \) of Corollary 9 (??) show that Theorem 5 (??) is an improvement of the estimates obtained by Kumar et all. (??).

2. Further, if we let \( \varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \) (0 \( \leq \beta < 1 \)) and \( \lambda = 1 \), then Theorem 5 gives the bounds on \( |a_2| \leq \sqrt{\frac{2(1 - \beta)}{3}} \) for functions \( f \in \mathcal{R}_\sigma(\beta) \) which coincides with the consequence of Xu et all (??). Also if we let \( \beta = 1 \), then estimates of Xu et all (??) becomes \( |a_2| \leq \sqrt{\frac{2}{3}} \) for functions in the class \( \mathcal{R}_\sigma(0) \). Since the estimate on \( |a_2| \) for \( f \in \mathcal{R}_\sigma(0) \) is improved over the assumed estimate \( |a_2| \leq \sqrt{2} \) for \( f \in \sigma \), the functions in \( \mathcal{R}_\sigma(0) \) can not be the nominee for the sharpness of the estimate in the class \( \sigma \).

3. Taking \( \lambda = 1 \), then we get the consequences below

**Corollary 11**  

Let the function \( f \in \mathcal{R}_\Sigma(\phi) \). Then,

\[
|a_2| \leq \min \left\{ \frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2 - B_1|}{3}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{B_1}{3} + \frac{B_1^2}{4}, \frac{B_1 + |B_2 - B_1|}{3} \right\}.
\]

**Remark 12**  

Corollary 11 is the improvement of the estimates given by Ali et all. (Theorem2.1., (??))

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