A NEW CHARACTERIZATION OF FLAT AFFINE MANIFOLDS AND THE ASSOCIATIVE ENVELOPE OF A LEFT SYMMETRIC ALGEBRA

MEDINA A.*, SALDARRIAGA, O.**, AND VILLABON, A.**

Abstract. This paper deals with affine connections on real manifolds. We give a new characterization of flat affine connections on real manifolds by means of certain affine representations of the Lie group of automorphisms preserving the connection. Then we specialize the characterization to the case of a left invariant connection on a Lie group. In the last case, we show the existence of a Lie group endowed with a flat affine bi-invariant connection whose Lie algebra contains the Lie algebra of complete infinitesimal affine transformations of the given Lie group. We also prove some results about flat affine manifolds whose group of diffeomorphisms admit a flat affine bi-invariant structure. The paper is illustrated with several examples.

Keywords: Flat affine manifolds, Infinitesimal affine transformations, Sheaves of Lie algebras, Associative Envelope.

* Université Montpellier, Institute A. Grothendieck, UMR 5149 du CNRS, France and Universidad de Antioquia, Colombia
  e-mail: alberto.medina@univ-montp2.fr

** Instituto de Matemáticas, Universidad de Antioquia, Medellín-Colombia
  e-mails: omar.saldarriaga@udea.edu.co, andresvillabon2000@gmail.com

1. INTRODUCTION

The objects of study of this paper are flat affine manifolds and their affine transformations. A well understanding of the category of Lagrangian manifolds assumes a good knowledge of the category of flat affine manifolds (Theorem 7.8 in [We], see also [Fu]). Recall that flat affine manifolds with holonomy reduced to $GL_n(\mathbb{Z})$ appear naturally in integrable systems and Mirror Symmetry (see [KoSo]).

In what follows $M$ is a connected real n-dimensional manifold, $P = L(M)$ its bundle of linear frames, $\theta$ the fundamental 1-form, $\Gamma$ a linear connection on $P$ of connection form $\omega$, and $\nabla$ the covariant derivative on $M$ associated to $\Gamma$. The manifold is called flat affine if its curvature and torsion tensors are both null. In the case where $M$ is a Lie group and the connection is left invariant, we call it a flat affine Lie group. Recall that a Lie group is flat affine if and only if its Lie algebra is endowed with a left symmetric product compatible with the bracket.

An affine transformation of $M$ is a diffeomorphism $f$ of $M$ such that its derivative map $f_* : TM \to TM$ sends parallel vector fields into parallel vector fields, and therefore geodesics into geodesics (together with its affine parameter). An infinitesimal affine transformation is a smooth vector field $X$ on $M$ whose local 1-parameter groups $\phi_t$ are local affine transformations. We will denote by $\mathfrak{a}(M, \nabla)$ the vector space of infinitesimal affine transformations and for $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields. An element $X$ of $\mathfrak{X}(M)$ belongs to $\mathfrak{a}(M, \nabla)$ if and only if
if it verifies
\[ \mathcal{L}_X \circ \nabla_Y - \nabla_Y \circ \mathcal{L}_X = \nabla_{[X,Y]}, \quad \text{for all} \quad Y \in \mathfrak{X}(M), \]
where \( \mathcal{L}_X \) denotes the Lie derivative. For a vector field \( X \) on \( M \), we will denote by \( \tilde{X} \) its natural lift on \( P \). It is also well known that
\[ X \in \mathfrak{a}(M, \nabla) \quad \text{if and only if} \quad \mathcal{L}_{\tilde{X}} \omega = 0 \]
this is also equivalent to say that \( \tilde{X} \) commutes with every standard horizontal vector field. We will denote by \( \mathfrak{a}(P, \omega) \) the set of vector fields \( Z \) on \( P \) satisfying
\[ Z \text{ is invariant by } R_a \text{ for every } a \in GL_n(\mathbb{R}) \]
\[ \mathcal{L}_Z \theta = 0 \]
\[ \mathcal{L}_Z \omega = 0. \]
The map \( X \mapsto \tilde{X} \) is an isomorphism of Lie algebras from \( \mathfrak{a}(M, \nabla) \) to \( \mathfrak{a}(P, \omega) \). The vector subspace \( \text{aff}(M, \nabla) \) of \( \mathfrak{a}(M, \nabla) \) whose elements are complete, with the usual bracket of vector fields, is the Lie algebra of the group \( \text{Aff}(M, \nabla) \) of affine transformations of \( (M, \nabla) \), (see \[KoNo\]).

The image of \( \text{aff}(M, \nabla) \) under the isomorphism \( X \mapsto \tilde{X} \) will be denoted by \( \mathfrak{a}_c(P) \). For the purposes of this work, it is useful to recall that \( P \) admits an absolute parallelism, that is, its tangent bundle admits \( n^2 + n \) sections independent at every point.

This paper is organized as follows. In Section 2 we state a new characterization of flat affine manifolds and its consequences. In the theory of flat affine manifolds appears, in a natural way, a finite dimensional associative algebra (see Lemma 3.1). Section 3 is devoted to the study of this algebra. Section 4 is used to study the following question. Does the group of affine transformations, \( \text{Aff}(M, \nabla) \), of a flat affine manifold \( (M, \nabla) \) admit a left invariant flat affine or flat projective structure? This question was answered positively in some particular cases in \[M-S-G\] in the special case where \( M = G \) is a Lie group and \( \nabla = \nabla^+ \) is left invariant.

2. Characterization of flat affine manifolds

Before stating the main result of the section, let us denote by \( A^* \) the fundamental vector field associated to \( A \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}) \) and by \( R_a \) the right action of \( a \in G = GL_n(\mathbb{R}) \) on \( P \) and by \( B(\xi) \) the standard horizontal vector field corresponding to \( \xi \). Recall that if \( \Gamma \) is a linear connection over \( M \) with connection form \( \omega \) and \( \theta \) is the canonical form, they satisfy the following properties:

1. \( \omega(A^*) = A \), for all \( A \in \mathfrak{gl}_n(\mathbb{R}) \),
2. \( \theta(B(\xi)) = \xi \), for all \( \xi \in \mathbb{R}^n \),
3. \( R_a^* \omega = \text{Ad}_{a^{-1}} \omega \) and \( (R_a^* \theta)(Z) = a^{-1}(\theta(Z)) \), for all \( a \in GL_n(\mathbb{R}) \) and \( Z \in \mathfrak{X}(P) \).

Consider the trivial vector bundle \( P \times \text{aff}(\mathbb{R}^n) \) over \( P \) and its corresponding sheaf of sections \( \text{Sec}(P \times \text{aff}(\mathbb{R}^n)) \). This sheaf is endowed with a Lie bracket given by
\[ [s_1, s_2](u) := [s_1(u), s_2(u)] \]
with \([s_1(u), s_2(u)]\) the Lie bracket of the Lie algebra \( \text{aff}(\mathbb{R}^n) \). Notice that \( \text{Sec}(TP) \) is also a sheaf of Lie algebras with the natural bracket of vector fields over \( P \).

The following fact can be easily verified.

Lemma 2.1. The sheaf of Lie algebras \( \text{Sec}(P \times \text{aff}(\mathbb{R}^n)) \) acts infinitesimally on \( \mathbb{R}^n \) by
\[ (u, s(u)) \cdot w = (u, (v_u, f_u)) \cdot w := v_u + f_u(w). \]
Moreover, if \( \eta : \text{Sec}(TP) \rightarrow \text{Sec}(P \times \text{aff}(\mathbb{R}^n)) \) is a homomorphism of sheaves of Lie algebras, then \( \text{Sect}(TP) \) acts infinitesimally on \( \mathbb{R}^n \) via \( \eta \).

In these terms we have the following result.

**Theorem 2.1.** Having a flat affine connection \( \Gamma \) on \( P \) is equivalent to the existence of a homomorphism of sheaves of Lie algebras

\[
\eta : \text{Sec}(TP) \rightarrow \text{Sec}(P \times \text{aff}(\mathbb{R}^n)),
\]

such that conditions (1), (2) and (3) above are verified and there exists \( v \in \mathbb{R}^n \) such that the map \( Z \mapsto \theta(Z) + \omega(Z)v \) is surjective.

**Proof.** Suppose that \( \Gamma \) is flat and torsion free. Then the structure equations (E. Cartan) for \( \Gamma \) reduce to

\[
d\theta(Z_1, Z_2) &= -\frac{1}{2}(\omega(Z_1) \cdot \theta(Z_2) - \omega(Z_2) \cdot \theta(Z_1)), \\
\omega(Z_1, Z_2) &= \frac{1}{2}[\omega(Z_1), \omega(Z_2)].
\]

It is clear that the map

\[
\eta : \text{Sec}(TP) \rightarrow \text{Sec}(P \times \text{aff}(\mathbb{R}^n)),
\]

is \( \mathbb{R} \)-linear. That \( \eta \) is a homomorphism of sheaves of Lie algebras means that the map \( Z \mapsto \omega(Z) \) is a linear representation of \( \text{Sec}(TP) \) and that \( \theta \) is 1-cocycle relative to this representation. It is sufficient to verify these conditions in the following cases. When \( Z_1 \) and \( Z_2 \) are vertical vector fields, when \( Z_1 \) is a vertical vector field and \( Z_2 \) is a horizontal vector field, and in the case where both \( Z_1 \) and \( Z_2 \) are horizontal vector fields. The first two cases are readily checked. In the third case we can choose \( \xi_1, \xi_2 \in \mathbb{R}^n \) such that \( Z_{1,u} = B(\xi_1)_u \) and \( Z_{2,u} = B(\xi_2)_u \), then we have

\[
[\eta(Z_1), \eta(Z_2)](u) = [(\theta(B(\xi_1)), \omega(B(\xi_1))), (\theta(B(\xi_2)), \omega(B(\xi_2)))](u) \\
= [(\xi_1, 0), (\xi_2, 0)] \\
= 0.
\]

On the other hand, since the torsion and curvature forms, \( \Theta \) and \( \Omega \), vanish everywhere, the structure equations give \( \omega([B(\xi_1), B(\xi_2)]) = 0 \) and \( \theta([B(\xi_1), B(\xi_2)]) = 0 \). That is, \( \eta([Z_1, Z_2]) = 0 \). The existence of \( v \) so that the map \( Z \mapsto \theta(Z) + \omega(Z)v \) is surjective follows from the fact that \( \eta(Z)(0) = \theta(B(\xi)) = \xi \).

For the converse, assume that \( \eta \) is a homomorphism of sheaves of Lie algebras verifying (1), (2), and (3). Taking \( \omega \) as the connection form of a connection \( \Gamma \), as \( \eta \) is a homomorphism of sheaves of Lie algebras it follows that \( \omega([Z_1, Z_2]) = [\omega(Z_1), \omega(Z_2)] \) and \( \theta([Z_1, Z_2]) = \omega(Z_1) \cdot \theta(Z_2) - \omega(Z_2) \cdot \theta(Z_1) \) for \( Z_1, Z_2 \in \text{Sec}(TP) \). The first equality means that the horizontal distribution on \( P \) determined by \( \omega \) is completely integrable, that is, its curvature \( \Omega \) vanishes.
As the torsion $\Theta$ is a tensorial 2-form on $P$ of type $(GL_n(\mathbb{R}), \mathbb{R}^n)$, if $Z_1 = B(\xi_1)$ and $Z_2 = B(\xi_2)$ are horizontal vector fields for some $\xi_1, \xi_2 \in \mathbb{R}^n$, we get

$$\Theta(Z_1, Z_2) = d\theta(Z_1, Z_2)$$

$$= \frac{1}{2} (Z_1(\theta(Z_2)) - Z_2(\theta(Z_1)) - \theta([Z_1, Z_2]))$$

$$= \frac{1}{2} (Z_1(\xi_2) - Z_2(\xi_1) - \omega(Z_1) \cdot \theta(Z_2) + \omega(Z_2) \cdot \theta(Z_1))$$

$$= 0.$$

This means that $\Theta$ vanishes locally.

Finally, since the connection is flat, i.e., $\omega([Z_1, Z_2]) = [\omega(Z_1), \omega(Z_2)]$ for all $Z_1, Z_2 \in \text{Sec}(TP)$, the bundle $P \to M$ has discrete structure group, that is, the transition functions are locally constant. Consequently $\Theta = 0$ globally. \hfill \Box

For any linear connection $\Gamma$ we have the following.

Remark 2.1. (a) The connection $\Gamma$ on $P$ is flat, i.e., the horizontal distribution is completely integrable, if and only if for any vector fields $Z_1, Z_2 \in \mathfrak{X}(P)$ it holds

$$d\omega(Z_1, Z_2) = -\frac{1}{2} [\omega(Z_1), \omega(Z_2)].$$

(b) The connection $\Gamma$ on $P$ is torsion free if and only if for any horizontal vector fields $Z_1$ and $Z_2$ we have

$$\theta([Z_1, Z_2]) = Z_1 \cdot \theta(Z_2) - Z_2 \cdot \theta(Z_1).$$

Recall that if $\Gamma$ is any linear connection on $P$, the fiber bundle $P = L(M)$ is a parallelizable manifold. More precisely, the set of vector fields $\{E_{ij}^*, B(e_k) \mid 1 \leq i, j, k \leq n\}$ defines an absolute parallelism of $P$, i.e., these vector fields form a basis of $T_uP$ for every $u \in P$. In other words, the linear connection $\Gamma$ determines a $\{1\}$-structure on $P$. Consequently the group $K$ of transformations of this parallelism is a Lie group of dimension at most $\dim P = n^2 + n$. More precisely, given $\sigma \in K$ and for any $u \in P$, the map $\sigma \mapsto \sigma(u)$ is injective and its image $\{\sigma(u) \mid \sigma \in K\}$ is a closed submanifold of $P$. The submanifold structure of $\{\sigma(u) \mid \sigma \in K\}$ turns $K$ into a Lie transformation group (see KoNo and Kob). The group $\text{Aut}(P, \omega)$ is a closed subgroup of $K$, so it is a Lie group of transformations of $P$. The Lie algebra of $K$ will be denoted by $\mathfrak{K}$ and it can be viewed as a submodule of $\text{Sec}(TP)$.

Theorem 2.2. Let $M$ be a connected real $n$-dimensional manifold, $P = L(M)$ its principal bundle of linear frames, $\theta$ the fundamental form of $P$ and $\Gamma$ a linear connection on $P$ of connection form $\omega$. We have then:

The connection is flat affine if and only if $(\theta, \omega)$ determines a (unique) homomorphism of Lie groups

$$\rho : \text{Aut}(P, \omega, \theta) \longrightarrow \text{Aut}(L(\mathbb{R}^n), \omega^0, \theta^0)$$

where $\omega^0$ is the connection form of the usual connection on $\mathbb{R}^n$, such that there exists $v \in \mathbb{R}^n$ with open orbit relative to the action $\rho$.

Proof. Suppose that $\Gamma$ is a flat affine connection. According to Theorem 2.1 the map $\eta$ is a homomorphism of sheaves of Lie algebras, $\eta(Z)(0) = (\omega(Z), \theta(Z))(0) = \theta(Z)$, for any $Z \in \text{Sec}(TP)$ with $\theta$ surjective. It is clear that the restriction $\eta_1$ of $\eta$ to $a_c(P)$ is a homomorphism of Lie algebras, $\eta_1(Z)(0) = \theta(Z)$, for any $Z \in a_c(P)$ with $\theta$ surjective. Now $a_c(P)$ is the Lie algebra
of the Lie group Aut(P, ω, θ). Since the category of Lie algebras is equivalent to the category of (analytic) Lie groups, η₁ can be integrated on a homomorphism of Lie groups ρ (See [Bo]). Finally the existence of an open orbit for ρ follows from the fact that \( \{ θ(Z)|Z ∈ a_c(P)\} = R^n \).

Now suppose given ρ with derivative \( ρ_∗,ε = (θ, ω) \) and \( \{ θ(Z)|Z ∈ a_c(P)\} = R^n \). Clearly \( ρ_∗,ε \) can be extended to \( Sec(TP) \) as a homomorphism of Lie algebras η verifying the hypothesis of Theorem 2.1. Consequently the linear connection ω is flat and torsion free.

□

Remark 2.2. Notice that the homomorphism ρ can be also obtained applying Lie’s third theorem.

Let \( O(M) \) denote the fiber bundle of orthogonal frames, given a metric (or pseudo-metric) connection \( Γ \) we have

Corollary 2.1. (1) The Levi-Civita connection \( Γ \) associated to a pseudo-Riemannian metric is flat if and only if the homomorphism \( ρ \) of the theorem, takes values in the pseudo Euclidean group \( E_{(p,q)}(n) = R^n ⋊ O(p,q) \), where \( (p,q) \) is the signature of the pseudo-metric.

(2) If the connection \( Γ \) preserves a volume form, then \( Γ \) is flat affine if and only if the homomorphism \( ρ \) of the theorem takes values in the group \( R^n ⋊ SL_n(R) \).

Recall that there exists topological obstructions to the existence of a linear flat connection (see [Mi]) and therefore to the existence of a flat affine connection (see for instance [Sm]). The following fact is well known (see [Mi])

Lemma 2.2. The bundle \( L(M) \) possesses a flat linear connection if and only if it is induced from the universal covering bundle by a homomorphism \( h : \pi_1(M) → GL_n(R) \).

As the image \( h(\pi_1(M)) \) of the map \( h \) is the holonomy group of the connection \( Γ \), the holonomy of a flat connection is the linear part of the holonomy of the foliation along the leaf through the point (see for instance [La]).

In particular a closed manifold \( M \) with finite fundamental group \( π_1 \), does not admit a flat affine connection.

The following question arises.

Question What conditions must be satisfied by \( h \) so that the linear flat connection be torsion free? (See [KaTo])

The existence of a left invariant flat affine connection on a Lie group \( G \), is equivalent to the existence of an étale affine representation of \( G \). This means, we have a homomorphism of Lie groups \( τ : G → Aff(R^n) \) with an open orbit and discrete isotropy (see for instance [Mc]). In these terms we get the following consequence of Theorem 2.2

Theorem 2.3. Let \( Γ \) be a left invariant linear connection on \( G \). The following assertions are equivalent

(1) The connection \( Γ \) is flat affine.

(2) There exists a homomorphism of Lie groups \( ρ : Aut(G, ω^+) → Aut(L(R^n), ∇^0) \) with an open orbit relative to \( ρ \).

(3) There exists a point in \( R^n \) with open orbit and discrete isotropy relative to restriction of \( ρ \) to \( G \).

Proof. The assertion (1) implies (2) was proved in Theorem 2.2.
Now suppose that (2) is true. Since $\Gamma$ is left invariant, the group $\text{Aut}(G, \omega^+)$ contains the group $G$. Moreover, right invariant vector fields are infinitesimal affine transformations of $(G, \nabla^+)$. Let $\eta = \rho_{*\epsilon}$ be the infinitesimal version of $\rho$ and $\eta'$ the restriction of $\eta$ to $\text{Lie}(G)$. It is easy to see that $\eta'(0)(Z) = \theta_u(Z_u)$, with $u \in P = L(G)$, hence $\eta'(0)$ is a linear isomorphism from $\text{Lie}(G)$ to $\mathbb{R}^n$. That is $v = 0$ is a point with open orbit and discrete isotropy.

The statement (3) implies (1) is proved in [Mc] (see also [Kos]). □

3. THE ASSOCIATIVE ENVELOPE OF A FINITE LEFT SYMMETRIC ALGEBRA

Given a flat affine Lie group $G$, in this section we show the existence of a simply connected flat affine Lie group whose connection is also right invariant and its Lie algebra contains the Lie algebra of $G$ (see Theorem 3.1).

For this purpose we recall some known facts and introduce some notation.

Let $M$ be an $n$ dimensional manifold with a linear connection $\Gamma$ and corresponding covariant derivative $\nabla$, consider the product

$$XY := \nabla_X Y, \quad \text{for } X, Y \in \mathfrak{X}(M).$$

If the curvature tensor relative to $\nabla$ is identically zero, this product verifies the condition

$$[X, Y]Z = X(YZ) - Y(XZ), \quad \text{for all } X, Y, Z \in \mathfrak{X}(M).$$

If both torsion and curvature vanish identically we have

$$(XY)Z - X(YZ) = (YX)Z - Y(XZ), \quad \text{for all } X, Y, Z \in \mathfrak{X}(M).$$

A vector space endowed with a bilinear product satisfying Equation (5) is called a left symmetric algebra. In this case the product given by $[X, Y]_1 := XY - YX$ defines a Lie bracket on the space. If $\nabla$ is torsion free this Lie bracket agrees with the usual Lie bracket of $\mathfrak{X}(M)$, that is

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

Recall that product (4) satisfies $(fX)Y = f(XY)$ and $X(gY) = X(gY) + g(XY)$, for all $f, g \in C^\infty(M, \mathbb{R})$, i.e., the product defined above is $\mathbb{R}$-bilinear and $C^\infty(M, \mathbb{R})$-linear in the first component. Also, if $(M, \nabla)$ is a flat affine manifold then $M$ and $\nabla$ are analytic with respect to the atlas of normal coordinates (see [KoNo] Theorem 7.7 page 263).

If $M = G$ is a Lie group, in what follows we identify its Lie algebra $\mathfrak{g}$ with the real vector space of left invariant vector fields and also with the tangent space at the unit $e \in G$. For any $x \in T_eG$, we denote by $x^+$ (respectively by $x^-$) the left (right) invariant vector field determined by $x$. The group $G$ acts on itself on the left (respectively right). The left (respectively right) action of $\sigma \in G$ will be denoted by $L_{\sigma}$ (respectively $R_{\sigma}$). These actions can be lifted to actions of $G$ on $P = L(G)$ by $\psi_{1,\sigma}(u) = \sigma \cdot u = (\sigma \tau, (L_{\sigma})_{*\tau}(X_{ij}))$ (respectively $\psi_{2,\sigma}(u) = u \cdot \sigma = (\tau, (X_{ij}))$). A linear connection $\Gamma$ on $G$ is said left invariant (respectively right invariant) if the horizontal distribution is preserved by $\psi_1$ (respectively by $\psi_2$). The connection is flat and bi-invariant if the corresponding horizontal distribution is preserved by $\psi_1$ and $\psi_2$.

If $\nabla = \nabla^+$ is left invariant, it is well known that $(G, \nabla^+)$ is flat affine if and only if Product (1) turns $\mathfrak{g}$ into a left symmetric algebra (see [Mc]). In what follows, given an associative or a left symmetric algebra $(\mathcal{A}, \cdot)$, we will denote by $\mathcal{A}_-$ the Lie algebra of commutators of $\mathcal{A}$, i.e., the Lie algebra with bracket given by

$$[a, b] = a \cdot b - b \cdot a.$$
Lemma 3.1. Let \((M, \nabla)\) be a flat affine manifold. Then Product (4) induces a structure of associative algebra on the vector space \(\mathfrak{a}(M, \nabla)\) whose commutator is the Lie bracket of vector fields on \(M\). In particular if \(M = G\) is a Lie group and \(\nabla = \nabla^+\) is left invariant, \(\mathfrak{a}(G, \nabla^+)\) contains \(\mathfrak{g}^\text{op}\) as a subalgebra, where \(\mathfrak{g}^\text{op}\) is the opposite Lie algebra of \(\mathfrak{g} = \text{Lie}(G)\).

Proof. Since \(\nabla\) is flat affine, it follows from (1) that a smooth vector field \(X\) is an infinitesimal affine transformation if and only if
\[
\nabla_{\nabla_Y Z}X = \nabla_Y \nabla_Z X,
\]
for all \(Y, Z \in \mathfrak{X}(M)\). This equality implies that \(\nabla_X Y \in \mathfrak{a}(M, \nabla)\), whenever \(X, Y \in \mathfrak{a}(M, \nabla)\). It follows from (7) that the product \(X \cdot Y = \nabla_X Y\) is an associative product on \(\mathfrak{a}(M, \nabla)\). Moreover, from (6) the commutator of \(\mathfrak{a}(M, \nabla)\) is the Lie bracket of vector fields on \(M\).

When \(M = G\) is a Lie group, as \(\nabla^+\) is left invariant, the real vector space \(\mathfrak{g}\) of right invariant vector fields on \(G\) is a subspace of \(\text{aff}(G, \nabla^+)\). Hence from (6) we get that \(\mathfrak{g}^\text{op}\) is a Lie subalgebra of \(\mathfrak{a}(G, \nabla^+)\).

Let \(\mathcal{A}^\text{op}\) be the opposite algebra of a finite dimensional associative algebra \(\mathcal{A}\). We have

Theorem 3.1. If \((\mathfrak{g}, \cdot)\) is an \(n\)-dimensional left symmetric algebra over \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\), then there exists:

1. A unique connected and simply connected flat affine Lie group \((G, \nabla^+)\) whose Lie algebra \(\text{Lie}(G) = \mathfrak{g}_-\), the Lie algebra of commutators of \(\mathfrak{g}\).
2. An associative algebra \(\mathcal{A}\) so that \(\mathfrak{g}_-\) is a Lie subalgebra \(\mathcal{A}_-\), the Lie algebra of commutators of \(\mathcal{A}\). Moreover \(\mathcal{A}^\text{op}\) is the subalgebra of the associative algebra \(\mathfrak{a}(G, \nabla^+)\) generated by \(\text{aff}(G, \nabla^+)\).

Proof. Let \(G\) be the connected and simply connected Lie group of Lie algebra \(\mathfrak{g}_-\). The product of the left symmetric algebra \(\mathfrak{g}\) determines a flat affine left invariant connection \(\nabla^+\) on \(G\). From Lemma 3.1 the connection \(\nabla^+\) induces an associative product over \(\mathfrak{a}(G, \nabla^+)\) compatible with the Lie bracket of vector fields on \(G\), that is
\[
[X, Y] = XY - YX.
\]
Let \(\mathcal{B}\) the subalgebra of the associative algebra \(\mathfrak{a}(G, \nabla^+)\) generated by \(\text{aff}(G, \nabla^+)\). Then \(\mathcal{A}\) is precisely the opposite associative algebra of \(\mathcal{B}\). As the right invariant vector fields on \(G\) are complete infinitesimal affine transformations, it follows that \(\mathfrak{g}_-\) is a Lie subalgebra of \(\mathcal{A}_-\).

This theorem motivates the following

Definition 3.1. Under the terms of the previous theorem we define the following.

1. the associative envelope \(\text{Env}(\mathfrak{g})\) of a left symmetric algebra \((\mathfrak{g}, \cdot)\) is the opposite of the associative subalgebra of \(\mathfrak{a}(G, \nabla^+)\) generated by \(\text{aff}(G, \nabla^+)\). That is, \(\text{Env}(\mathfrak{g})\) is the opposite of the least associative subalgebra of \(\mathfrak{a}(G, \nabla^+)\) containing \(\text{aff}(G, \nabla^+)\).
2. Any Lie group of Lie algebra \(\text{Env}(\mathfrak{g})_-\) will be called an enveloping Lie group of \((G, \nabla^+)\).

The following result is useful in what follows (see for instance [Mc] or [BoMc]).

Theorem 3.2. Let \(G\) be a Lie group and \(\mathfrak{g}\) its Lie algebra. The group \(G\) admits a flat affine bi-invariant structure if and only if \(\mathfrak{g}\) is the underlying Lie algebra of an associative algebra so that \([a, b] = ab - ba\), for all \(a, b \in \mathfrak{g}\).
Remark 3.1.

i. Any enveloping Lie group of a flat affine Lie group $G$ is endowed with a flat affine bi-
invariant connection.

ii. Although the elements of the associative envelope $\text{Env}(\mathfrak{g})$ of the left symmetric algebra
$\mathfrak{g} = \text{Lie}(G)$ are differential operators of order less than or equal to 1 on $G$, the associative
envelope is not a subalgebra of the universal enveloping algebra of the Lie algebra.

Example 3.1. Let $\nabla^+$ be the flat affine left invariant connection on $G = \text{Aff}(\mathbb{R})$ defined by

$$
\begin{align*}
\nabla^+_{e_1} e_1 &= 2e_1, & \nabla^+_{e_2} e_2 &= e_2, & \nabla^+_{e_2} e_1 &= 0, & \nabla^+_{e_2} e_2 &= e_1.
\end{align*}
$$

(8)

A direct calculation shows that a linear basis of $\mathfrak{a}(G, \nabla^+)$ is given by the following vector fields

$$
e_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial y}, \quad C_3 = \frac{1}{x} \frac{\partial}{\partial x}, \quad C_4 = \frac{y}{x} \frac{\partial}{\partial x}, \quad C_5 = \left(x + \frac{y^2}{x}\right) \frac{\partial}{\partial x}
$$

and

$$
C_6 = \left(-xy - \frac{y^3}{x}\right) \frac{\partial}{\partial x} + (x^2 + y^2) \frac{\partial}{\partial y},
$$

where $e_1$ and $e_2$ denote the right invariant vector fields. As the connection is left invariant, the
vector fields $e_1$ and $e_2$ are complete. Moreover, it can be checked that no real linear combination
of the fields $C_3$, $C_4$, $C_5$, and $C_6$ is complete. Consequently $(e_1, e_2)$ is a linear basis of $\text{aff}(G, \nabla^+)$. On the other hand, the multiplication table of the product defined by $\nabla^+$, i.e., the product $X \ast Y = \nabla^+_X Y$, on the basis of $\mathfrak{a}(G, \nabla^+)$ displayed above is given by

$$
\begin{array}{c|c|c|c|c|c}
\ast & e_1^- & e_2^- & C_3 & C_4 & C_5 & C_6 \\
\hline
e_1^- & e_1 + C_5 & C_4 & 0 & C_4 & 2C_5 & 2C_6 \\
e_2^- & e_2 + C_5 & C_4 & 0 & C_3 & 2C_4 & 2e_1 - 2C_5 \\
C_3 & 2C_3 & 0 & 0 & 2C_3 & 2e_2 - 2C_4 & \\
C_4 & 2C_4 & 0 & 0 & 2C_4 & 2e_1 - 2C_5 & \\
C_5 & 2C_5 & 0 & 0 & 2C_5 & 2C_6 & \\
C_6 & C_6 & C_5 & 0 & C_5 & 0 & 0
\end{array}
$$

(9)

It follows from Table 9 that the real associative subalgebra of $\mathfrak{a}(G, \nabla^+)$ generated by $\{e_1^-, e_2^-\}$
has linear basis $(e_1, e_2, C_3, C_4, C_5)$. That is, $\text{Env}(\mathfrak{g})$ is the real 5-dimensional associative algebra
with linear basis $(e_1, e_2, C_3, C_4, C_5)$ and the opposite product of Table 9.

The animation below shows the lines of flow of each of the vector fields $e_1^-, e_2, C_3, C_4, C_5$, and
$C_6$ with the initial condition $(1.5, -1)$. The fact that the flows of the vector fields $C_3, C_4, C_5$
and $C_6$ cross the boundary to the outside of the orbit of $(0, 0)$ determined by the affine étale
representation relative to $\nabla$, corresponds to the fact that the vector fields are not complete in $G_0$. (To play the animation click on the image).
Remark 3.2. The reader can verify that the simply connected Lie group of Lie algebra $Env(g)$ is isomorphic to the group $E := (\mathbb{R}^2 \rtimes_{\theta_1} \mathbb{R}) \rtimes_{\theta_2} G^\text{op}$, where $G = \text{Aff}(\mathbb{R})_0$ is the connected component of the unit of $\text{Aff}(\mathbb{R})$ and the actions $\theta_1$ and $\theta_2$ are given by

$$\theta_1(t)(y, z) = (e^{-2t}x, e^{-2t}y) \quad \text{and} \quad \theta_2(x, y)(z_1, z_2, z_3) = \left( \frac{z_1 + yz_2 + y^2}{x^2}, \frac{z_2 + 2y}{x}, z_3 \right)$$

The Lie group $E$ is also isomorphic to the semidirect product $\mathcal{H}_3 \rtimes_\rho \mathbb{R}^2$ of the additive group $\mathbb{R}^2$ acting on the 3-dimensional Heisenberg group where $\rho$ is given by

$$\rho(a, b)(x, y, z) = (e^{-a-2b}x, e^{-2a-2b}y, e^{-a-2b}z).$$

In fact we have the following result more general than Theorem 3.1 whose proof is left to the reader.

**Proposition 3.1.** Given a flat affine manifold $(M, \nabla)$, there exists a unique connected and simply connected Lie group endowed with a flat affine bi-invariant structure induced by $\nabla$ whose Lie algebra contains the Lie algebra $\text{aff}(M, \nabla)$ of complete infinitesimal affine transformations of $(M, \nabla)$.

At this point we have some questions for which we do not have an answer.

**Questions.**

1. To determine a transformation group $E$ of diffeomorphisms of $P = L(M)$ so that $\text{Lie}(E) = Env(a_c(P))$ in the case when $\dim(Env(a_c(P))) < n^2 + n$.

2. Is it possible to realize an enveloping Lie group of a flat affine Lie group $(G, \nabla^+)$ as a group of transformations of $L(G)$ (eventually as a subgroup of the Lie group $K$)?

If $a_c(P) = a(P, \omega)$ the answer to the first question is positive. When $\dim(Env(a_c(P))) < n^2 + n$, the method described on Proposition 3.2 could give an answer to these questions.
Example 3.2. Let us consider the family of left symmetric products on \( g = \text{aff}(\mathbb{R}) \) given by

\[
\begin{array}{c|cccc}
\cdot & e_1 & e_2 & e_1 & e_2 \\
\hline
e_1 & a e_1 & e_2 & e_2 & 0 \\
e_2 & 0 & 0 & 0 & C_2 \\
\end{array}
\]

Let \( \alpha \neq 0 \) be fixed and \( \nabla^+ = \nabla(\alpha) \) be the affine connection determined by this product. An easy calculation shows that a linear basis of the space of complete infinitesimal affine transformations of \( \text{Aff}(\mathbb{R}) \) relative to \( \nabla^+ \) is

\[
C_1 = \frac{1}{\alpha} x \frac{\partial}{\partial x}, \quad C_2 = \frac{1}{\alpha} x^\alpha \frac{\partial}{\partial y}, \quad C_3 = y \frac{\partial}{\partial y}, \quad \text{and} \quad C_4 = \frac{\partial}{\partial y}.
\]

It can be verified that the product determined by \( \nabla^+ \) on these vector fields is given by

\[
\begin{array}{c|cccc}
\ast & C_1 & C_2 & C_3 & C_4 \\
\hline
C_1 & C_1 & C_2 & 0 & 0 \\
C_2 & 0 & 0 & C_2 & 0 \\
C_3 & 0 & 0 & C_3 & 0 \\
C_4 & 0 & 0 & C_4 & 0 \\
\end{array}
\]

By Lemma 3.1, the vector space of infinitesimal affine transformations relative to \( \nabla^+ \) is an associative 6-dimensional algebra under the product defined by \( \nabla^+ \). Therefore we get that the vector space with linear basis \( C_1, C_2, C_3, C_4 \) and the product on the table above is an associative 4-dimensional subalgebra. This 4-dimensional algebra is the associative envelope of \((g, \cdot)\).

The following Proposition describes a natural method to construct Lie groups whose Lie algebra is the underlying Lie algebra \( A^- \) of a real finite dimensional associative algebra \( A \). The proof is outlined in [BoMe].

**Proposition 3.2.** Given a real (or complex) finite dimensional associative algebra \( A \), set \( A' = A \oplus \mathbb{R} e \) the associative algebra obtained from \( A \) by the adjunction of a neutral element \( e \). The set of units on the set \( \{ x = a + e \mid a \in A \} \) is a Lie group of Lie algebra \( A^- \).

**Example 3.3.** Using Proposition 3.2, one can easily verify that

\[
G' = \left\{ \begin{pmatrix} 1 + \beta_1 & \beta_2 & 0 \\ 0 & 1 + \beta_3 & 0 \\ \beta_1/\alpha & \beta_2/\alpha + \beta_4 & 1 \end{pmatrix} \mid \beta_1 \neq -1 \text{ and } \beta_3 \neq -1 \right\}
\]

is an enveloping Lie group of the flat affine Lie group \((G, \nabla^+)\) of the previous example.

We finish the section with a more general example.

**Example 3.4.** Let \( G = GL_n(\mathbb{R}) \) endowed with the flat affine bi-invariant connection \( D \) determined by composition of linear endomorphisms. Given the local coordinates \([x_{ij}]\) with \( i, j = 1, \ldots, n \), it is easy to check that linear bases of left and right invariant vector fields are given by

\[
E^+_r = \sum_{i=1}^n x_{i r} \frac{\partial}{\partial x_{is}} \quad \text{and} \quad E^-_r = \sum_{i=1}^n x_{si} \frac{\partial}{\partial x_{ri}},
\]

with \( r, s = 1, \ldots, n \). The group \( \text{Aff}(G, D) \) is of dimension \( 2n^2 - 1 \) ([BoMe]) and the Lie bracket of its Lie algebra is the bracket of vector fields on \( G \). Using the product determined by \( D \) on
left and right invariant vector fields one gets

\[ D_{E_{pq}E_{rs}} = x_{sp} \frac{\partial}{\partial x_{rq}}, \text{ for all } p, q, r, s = 1, \ldots, n. \]

It follows that an enveloping Lie group of \( G \) is \( GL_n^2(\mathbb{R}) \).

The following remark describes the algorithm to compute the associative envelope of a finite dimensional left symmetric algebra.

**Remark 3.3.** Given a finite dimensional real or complex left symmetric algebra \( A \), do as follows.

Start by applying Lie’s third theorem to the Lie algebra \( A_\pm \) of commutators of \( A \). In this way we find the connected and simply connected Lie group \( G(A) \) of Lie algebra \( A_\pm \). This group is endowed with a left invariant connection \( \nabla^+ \) determined by the product on \( A \).

Then compute the associative subalgebra \( B \) of \( \mathfrak{a}(G(A), \nabla^+) \) generated by the vector space \( \text{aff}(G(A), \nabla^+) \).

Finally, the associative envelope \( Env(A) \) of \( A \) is \( B^{op} \).

To finish the section, let us pose the following interesting problem.

**Problem.** To find an algebraic method to determine the associative envelope, in the sense of the Definition 3.1, of a real or complex finite dimensional left symmetric algebra.

4. **Affine Transformation Groups endowed with a flat Affine or Projective bi-invariant structure**

We start with the following result that completes and generalizes Theorem 13. in [M-S-G].

**Theorem 4.1.** Let \( (M, \nabla) \) be flat affine and connected. If \( \nabla \) is geodesically complete, then \( \text{Aff}(M, \nabla) \) admits a flat affine bi-invariant structure.

Moreover \( \text{Aff}(M, \nabla) \) contains a normal Lie subgroup \( T(M, \nabla) \) so that

\[ \text{Id} \rightarrow T(M, \nabla) \rightarrow \text{Aff}(M, \nabla) \rightarrow T(M, \nabla) \backslash \text{Aff}(M, \nabla) \rightarrow \text{Id} \]

is a split exact sequence of Lie groups endowed with flat affine bi-invariant structures.

**Proof.** Let \( \hat{M} \) be the universal covering of \( M \) with covering map \( p : \hat{M} \rightarrow M \) and \( \hat{\nabla} \) the connection on \( \hat{M} \) pullback of \( \nabla \) by \( p \). From Ehresmann’s Developing theorem (see [Eh], for more details see [Shil]), there exists an affine immersion \( D : \hat{M} \rightarrow \mathbb{R}^n \) and a Lie group homomorphism \( A : \text{Aff}(\hat{M}, \hat{\nabla}) \rightarrow \text{Aff}(\mathbb{R}^n, \nabla^0) \) so that the following diagram commutes

\[
\begin{array}{ccc}
\hat{M} & \xrightarrow{D} & \mathbb{R}^n \\
\downarrow{F} & & \downarrow{A(F)} \\
\hat{M} & \xrightarrow{D} & \mathbb{R}^n \\
\end{array}
\]

for all \( F \in \text{Aff}(\hat{M}, \hat{\nabla}) \).

Now, if \( F \in \text{Ker}A \), that is, \( A(F) = \text{Id}_{\mathbb{R}^n} \), as the diagram is commutative and \( \nabla \) is complete, we get that \( F = \text{Id}_{\hat{M}} \). Moreover, the completeness of \( \nabla \) also implies that \( \dim(\text{Aff}(M, \nabla)) = \dim(\text{Aff}(\hat{M}, \hat{\nabla})) = \dim(\text{Aff}(\mathbb{R}^n, \nabla^0)) \). Consequently \( A \) is an isomorphism of Lie groups.
On the other hand, from Theorem 13. in [M-S-G], the group \( \text{Aff}(\mathbb{R}^n) \) admits a flat affine bi-invariant structure \( \nabla' \), inducing the usual connection \( \nabla^0 \) of \( \mathbb{R}^n \) and the connection on \( GL_n(\mathbb{R}) \), given by composition of linear endomorphisms, so that the canonical sequence

\[
0 \to \mathbb{R}^n \to \text{Aff}(\mathbb{R}^n) \to GL_n(\mathbb{R}) \to \text{Id}
\]

is a split sequence of Lie groups admitting flat affine bi-invariant structures.

Hence transporting \( \nabla' \) by means of \( A \), we get that \( \text{Aff}(\hat{M}, \hat{n}) \) admits a flat affine bi-invariant structure. As \( \text{aff}(\hat{M}, \hat{n}) \cong \text{aff}(M, n) \), the Lie group \( \text{Aff}(M, \nabla) \) admits a flat affine bi-invariant structure \( \tilde{\nabla} \) with the suitable properties. \( \square \)

If \( \text{Aff}_x(M, \nabla) \) denotes the affine transformation of \((M, \nabla)\) fixing \( x \in M \), we have

**Corollary 4.1.** Let \((M, \nabla)\) be a flat affine manifold. If \( \dim \text{Aff}_x(M, \nabla) \geq n^2 \) for each \( x \in M \). Then \( \text{Aff}(M, \nabla) \) is endowed with a flat affine bi-invariant structure as that of Theorem 4.1.

**Proof.** The result follows by recalling that the only obstruction to the completeness of \( \nabla \) is the existence of a point \( x \in M \) so that \( \dim \text{Aff}_x(M, \nabla) < n^2 \) (see [BoMe] and [Ise]). \( \square \)

The following proposition is a consequence of the results presented in the previous section.

**Proposition 4.1.** Let \( \nabla \) be a flat affine connection on \( M \). If \( M \) is a compact manifold then the Lie group \( \text{Aff}(M, \nabla) \) is naturally endowed with a flat affine bi-invariant connection \( \tilde{\nabla} \).

**Proof.** Since \( \nabla \) is flat affine, it follows from (7) that Product (4) induces an associative product on \( \mathfrak{a}(M, \nabla) \). As \( M \) is a compact manifold every vector field on \( M \) is complete, in particular every infinitesimal affine transformation of \((M, \nabla)\) is complete. Hence the Lie algebra \( \mathfrak{a}_c(M, \nabla) = \mathfrak{a}(M, \nabla) \) of the Lie group \( \text{Aff}(M, \nabla) \) has an associative product given by \( X \ast Y = \nabla_X Y \) with \( X, Y \in \mathfrak{a}(M, \nabla) \). Therefore \( \text{Aff}(M, \nabla) \) admits a flat affine bi-invariant structure (see for instance [Me] inherited from \( \nabla \). \( \square \)

**Corollary 4.2.** Let \( G \) be a Lie group endowed with a flat affine left invariant connection and let \( D \) be a discrete cocompact subgroup of \( G \). The group of affine transformations of the flat affine compact manifold \( M = G/D \) has a flat affine bi-invariant structure.

The following result follows from a theorem by Yano (see [Ya], see also Corollary 3.9, page 244 in [KoNo]).

**Corollary 4.3.** The group of isometries \( \mathfrak{I}(M, g) \) of a compact flat Riemannian manifold \((M, g)\), admits a flat affine bi-invariant structure as that of Theorem 4.1.

Let us recall that another interesting case where there is a positive answer to Medina’s question is the following (see Theorem 18 in [M-S-G]).

**Theorem 4.2.** Let \( G = GL_n(\mathbb{K}) \) with \( \mathbb{K} = \mathbb{R} \) or the quaternions \( \mathbb{H} \) and \( \nabla^+ \) be the flat affine bi-invariant connection on \( G \) determined by composition of linear endomorphisms of \( \mathbb{K}^n \). Then the group \( \text{Aff}(G, \nabla^+) \) admits a flat projective bi-invariant structure.

5. Appendix

5.1. Miscellaneous Results. In this subsection we state some results, without proof, consequence of the previous work. The reader can easily verify them.
Proposition 5.1. Let \((G, \nabla^+)\) be a flat affine Lie group of dimension \(n\) and \(a(G, \nabla^+)\) (respectively \(\text{aff}(G, \nabla^+)\)) be the real vector space of infinitesimal affine transformations of \((G, \nabla^+)\) (respectively complete). Then we have the following:

1. \(a(G, \nabla^+)\) with the product defined by \(\nabla^+\) is an associative algebra of dimension at most \(n^2 + n\).
2. Let \(G'\) be an enveloping Lie group of \((G, \nabla^+)\), then \(G\) is a Lie subgroup of \(G'\) and \(\text{Aff}(G, \nabla^+)\) is a Lie subgroup of \(G'_{\text{rop}}\). Moreover \(G'\) is equipped with a flat bi-invariant affine structure \(\nabla'\). The Lie bracket of the Lie algebra of \(G'\) is given by the commutator of the product determined by \(\nabla^+\).

Lemma 5.1. Consider the flat affine Lie group \(\mathbb{R}^n\) with the usual connection \(\nabla^0\).

1. Any set \(S\) of infinitesimal affine transformations of \(\mathbb{R}^n\) relative to \(\nabla^0\) determines a unique real finite dimensional associative algebra \(A(S)\) with a product defined by \(\nabla^0\).
2. The Lie algebra of commutators \(A(S)_{-}\) is of dimension less than or equal to \(n^2 + n\).

Proposition 5.2. The Lie group \(\text{Aff}(G, \nabla^+)\) is locally isomorphic to a Lie subgroup of an enveloping Lie group \(\text{Env}(G, \nabla^+)\) endowed with a flat affine bi-invariant structure induced by \(\nabla^+\).

Theorem 4.1 implies the following.

Corollary 5.1. Let \((G, \nabla^+)\) be a flat affine Lie group and let \((\mathfrak{g}, \cdot)\) be the left symmetric algebra determined by \(\nabla^+\) on \(\mathfrak{g} = \text{Lie}(G)\). If \(\nabla^+\) is geodesically complete then the associative envelope of \((\mathfrak{g}, \cdot)\) is isomorphic to \(a(\mathbb{R}^n, \nabla^0)\) with the product given by composition of affine endomorphisms. In fact, the vector space \(a(G, \nabla^+)\) with the product \(XY \coloneqq \nabla^+_X Y\) is an associative algebra isomorphic to \(a(\mathbb{R}^n, \nabla^0)\).

Proposition 5.3. Let \((G, \nabla^+)\) be a flat affine Lie group and \(H\) a Lie group endowed with a flat bi-invariant structure. If \(\rho : G \rightarrow \text{Int}(H)\) is a representation of Lie groups then the Lie group \(H \rtimes_\rho G\) is endowed with a natural flat affine left invariant structure so that the sequence

\[ \{e\} \rightarrow H \rightarrow H \rtimes_\rho G \rightarrow G \rightarrow \{e\} \]

is an exact sequence of flat affine Lie groups.

5.2. Case \(\text{Aff}(\mathbb{R})_0\). This subsection is devoted to the study of enveloping Lie groups of the affine group \((\text{Aff}(\mathbb{R})_0, \nabla^+)\) where \(\nabla^+\) is any of the flat affine left invariant linear connections listed in Section 3 of [M-S-G].

Recall that the associative envelope \(\text{Env}(\mathfrak{g})\) of a left symmetric algebra contains the Lie algebra of right invariant vector fields. Hence, in view of Lemma 5.1, the associative envelope of \((\mathfrak{g}, \cdot)\) agrees with the associative envelope of \((\mathfrak{g}^{op}, \star)\).

Example 5.1. Consider the product on \(a(\mathbb{R}) = \text{span}\{e_1, e_2\}\) determined by the connection defined by Equation \((\ref{eq:connection})\). The associative envelope \(\text{Env}(a(\mathbb{R}), \cdot)\) of the left symmetric algebra \(a(\mathbb{R})\) under this product is the algebra with linear basis

\[ e_\overline{1}, e_\overline{2}, C_3, C_4, C_5 \]

with product given by the opposite of the product in Table \(\ref{table:products}\).

The corresponding connected and simply connected enveloping Lie group of \((\text{Aff}(\mathbb{R}), \nabla^+)\), with \(\nabla^+\) defined by Equation \((\ref{eq:connection})\), is given by \(\mathcal{H}_3 \times \mathbb{R}^2\), where \(\mathcal{H}_3\) is the 3-dimensional Heisenberg Lie group.
Remark 5.1. Given a flat affine Lie group \((G, \nabla^+)\) then the opposite Lie group \(G^{\text{op}}\) inherits a flat left invariant structure defined by the product
\[
x \star y := -xy,
\]
where \(xy\) is the left symmetric product corresponding to \(\nabla^+\).

Let \(\alpha\) be a real parameter and \(\nabla_1(\alpha), \, \nabla_2(\alpha), \, \nabla_1, \) or \(\nabla_2\) be the flat affine left invariant connections on \(\text{Aff}(\mathbb{R})_0\) described in [M-S-G] (page 198). In the following proposition, whose proof is left to the reader, \(\nabla^+\) denotes anyone of these connections.

Proposition 5.4. Let \(G = \text{Aff}(\mathbb{R})_0\) endowed with the connection \(\nabla^+\) and \((\mathfrak{g}, \cdot)\) the corresponding left symmetric algebra, then we have
1. The algebra \(\text{aff}(G, \nabla^+)\) is an associative algebra with the product determined by \(\nabla^+\). Therefore \(\text{aff}(G, \nabla^+)\) is the associative envelope of \((\mathfrak{g}, \cdot)\).
2. The associative product on \(\text{aff}(G, \nabla^+)\) given in 2., naturally determines a flat affine bi-invariant structure \(\nabla\) on \(\text{Aff}(G, \nabla^+)\).
3. The connection \(\nabla_1\) is bi-invariant and it induces a flat affine bi-invariant structure \(\nabla'\) on the opposite Lie group \(G^{\text{op}}\).
4. The flat affine Lie group \(\text{Aff}(G, \nabla^+)\) contains \(G\) as a normal Lie subgroup which is totally geodesic relative to \(\nabla\) and \(\text{Aff}(G, \nabla^+)/G\) is isomorphic to \(G^{\text{op}}\) and the natural sequence
\[
1 \longrightarrow (G, \nabla_1) \longrightarrow (\text{Aff}(G, \nabla^+)\!, \nabla) \longrightarrow (G^{\text{op}}, \nabla') \longrightarrow 1
\]
is an exact sequence of affine Lie groups where the affine structures are bi-invariant, where \(\nabla'\) is determined by the product \(\star\) defined in the previous remark.

5.3. Enveloping group of \((\text{Aff}(\mathbb{R}), \nabla_3)\) as a group of transformations of the bundle of linear frames \(L(\text{Aff}(\mathbb{R}))\) of \(\text{Aff}(\mathbb{R})\). A linear basis for the Lie algebra of infinitesimal affine transformations of \(M = \text{Aff}(\mathbb{R})\) relative to \(\nabla_3\) is given in Example 3.1 The associative envelope of the corresponding left symmetric algebra is generated by the first five elements in this basis. The corresponding affine transformations of \(L(M)\) determined by these vector fields, in local coordinates \((x, y, X_{11}, X_{12}, X_{21}, X_{22})\), are given by
\[
\phi_{1,t} = (xe^t, ye^t, x_{11}e^t, x_{12}e^t, x_{21}e^t, x_{22}e^t),
\phi_{2,t} = (x, y + t, X_{11}, X_{12}, X_{21}, X_{22})
\phi_{3,t} = \left( \frac{x X_{11}}{\sqrt{2t + x^2}}, \frac{x X_{12}}{\sqrt{2t + x^2}}, \sqrt{2t + x^2}, X_{21}, X_{22} \right)
\phi_{4,t} = \left( \frac{x X_{11} + t X_{21}}{\sqrt{2yt + x^2}}, \frac{x X_{12} + t X_{22}}{\sqrt{2yt + x^2}}, \sqrt{2yt + x^2}, X_{21}, X_{22} \right)
\phi_{5,t} = \left( \sqrt{(x^2 + y^2)e^{2t} - y^2}, y, \frac{(x X_{11} + y X_{21})e^{2t} - y X_{21}}{\sqrt{(x^2 + y^2)e^{2t} - y^2}}, \frac{(x X_{12} + y X_{22})e^{2t} - y X_{22}}{\sqrt{(x^2 + y^2)e^{2t} - y^2}}, X_{21}, X_{22} \right)
\]
Therefore, the enveloping Lie group of $\text{Aff}(\mathbb{R})$ is the group generated by $\{ \phi_{1,t}, \phi_{2,t}, \phi_{3,t}, \phi_{4,t}, \phi_{5,t} \}$. These generators are subject to relations, we exhibit some of them

\[
\begin{align*}
[\phi_{1,t}, \phi_{2,t}] &= \phi_{2,s(t^2-1)}, \\
[\phi_{1,t}, \phi_{3,t}] &= \phi_{2,s(t^2-1)}, \\
[\phi_{1,t}, \phi_{4,t}] &= \phi_{2,s(t^2-1)}, \\
[\phi_{2,t}, \phi_{4,t}] &= \phi_{3,s(t^2-1)}, \\
[\phi_{3,t}, \phi_{5,t}] &= \phi_{2,s(t^2-1)}
\end{align*}
\]

where $[\ , \ ]$ denotes the commutator.

Acknowledgment

We are very grateful to Martin Bordemann for his suggestions and comments which were helpful to improve this manuscript.

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