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Localized modes in mini-gaps opened by periodically modulated intersite coupling in two-dimensional nonlinear lattices

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Spatially periodic modulation of the intersite coupling in two-dimensional (2D) nonlinear lattices modifies the eigenvalue spectrum by opening mini-gaps in it. This work aims to build stable localized modes in the new bandgaps. Numerical analysis shows that single-peak and composite two- and four-peak discrete static solitons and breathers emerge as such modes in certain parameter areas inside the mini-gaps of the 2D superlattice induced by the periodic modulation of the intersite coupling along both directions. The single-peak solitons and four-peak discrete solitons are stable in a part of their existence domain, while unstable stationary states (in particular, two-soliton complexes) may readily transform into robust localized breathers.

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Nonlinear lattices offer a possibility to create a vast variety of self-trapped (i.e., spontaneously localized) discrete wave packets, alias discrete solitons. They are supported by the stable balance between the onsite nonlinearity and discrete diffraction in the lattice. In optics, discrete solitons have been observed in one- and two-dimensional (1D and 2D) arrays of nonlinear waveguides. In recent years, such photonic lattices have been implemented as permanent structures, or optically induced as virtual ones, using various materials, including those with cubic, quadratic, photorefractive, and liquid-crystal nonlinearities. In these systems, discrete solitary modes (fundamental and multi-peak solitons, 2D vortex solitons, etc.) are observed under both the self-focusing (in-phase states) and defocusing nonlinearity. In the former case, the discrete solitons have an in-phase structure (in particular, the fundamental single-peak solitons are represented by real positive solutions), while in the latter case solitons exist in the staggered form, with alternating signs of the discrete field at adjacent sites. Discrete solitons are also known in many other fields, such as Bose-Einstein condensates (BEC), electric transmission lines, solid-state lattice media, polymer molecules, etc.

In this work, we address the formation and ensuing dynamics of localized structures in 2D photonic superlattices, induced by a relatively long-wave periodic modulation of the intersite coupling imposed on the underlying lattice with the onsite cubic nonlinearity. In addition to photonics, similar BEC-trapping settings can be built in the form of modulated optical lattices, by means of a superposition of laser beams illuminating the condensate. We demonstrate that the periodic modulation of the intersite coupling opens narrow gaps in the linear spectrum of the 2D superlattices, and thus enables the creation of new solitary modes (gap solitons) in these mini-gaps. Some of these modes are robust (not only static ones, but also periodically oscillating breathers), therefore they can be experimentally created in the photonic and matter-wave (BEC) settings.
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I. INTRODUCTION

Discrete solitons represent self-trapped states in nonlinear lattice systems. They result from the interplay between the lattice diffraction and material nonlinearity. In optics, these states have been experimentally observed in both one- and two-dimensional (1D and 2D) nonlinear waveguiding arrays. Such photonic lattices have been built as permanent structures, or induced as virtual ones, in a variety of optical media, including those with cubic, quadratic, photorefractive (saturable), and liquid-crystal nonlinearities\(^1-8,10-14\). Similar spatially periodic trapping structures for quasi-discrete states of matter waves, based on optical lattices (induced by the interference of laser beams illuminating the Bose-Einstein condensate, BEC), have been created too\(^15-20\). In addition, discrete solitons are known in chains of micromechanical oscillators\(^21\), liquid crystals\(^22\), biological macromolecules\(^23,24\), and in other settings.

Discrete fundamental\(^4,5\) and vortical\(^7-10\) solitons in 2D nonlinear waveguide arrays were first observed in biased photorefractive crystals. Properties of fundamental discrete solitons were studied in detail theoretically too\(^25-28\). In these systems, solitons exist in unstaggered (i.e., in-phase) and staggered (i.e., with phase shift $\pi$ between adjacent lattice sites) forms, under the self-focusing (SF) and self-defocusing (SDF) onsite nonlinearity, respectively. Discrete solitary vortices were also studied in detail by means of numerical methods\(^25,29-31\). In particular, attention has been drawn to the 2D band structure associated with photonics lattices and respective aspects of the stability of 2D discrete solitons, see, e.g., Ref.\(^32\). Specific features of lattice solitons in BEC models were addressed in Refs.\(^33-35\).

Discrete localized modes were also studied in inhomogeneous 1D lattices, subject to a quasiperiodic spatial modulation of the intersite coupling constant\(^36\), as well as with an inhomogeneous onsite nonlinearity\(^37\). The modulation of the lattice coupling constant merely implies a varying spacing between the sites, as the coupling constant depends on it exponentially. An interesting finding is that the strength of the onsite SDF nonlinearity, which grows from the center to periphery of the 1D lattice at any rate faster than the distance, $|n|$, supports solitons of the unstaggered type, which are impossible in the uniform lattice with the SDF sign of the nonlinearity\(^37\). Equivalently, solitons of the staggered type are supported by the growing onsite SF nonlinearity, which is not possible either in the uniform lattice. It was demonstrated too that unstaggered solitons exist in the lattice with homogeneous onsite
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SDF nonlinearity, if the coupling constant decays fast enough at \( |n| \rightarrow \infty \). In this paper, we aim to extend the study of discrete solitons to 2D inhomogeneous lattices, namely, to those with periodic modulation of the intersite coupling. These settings may be considered as superlattices, which, in the general case, are defined as structured created by imposing relatively long-wave periodic spatial modulations onto the underlying lattice. The model, based on the corresponding discrete nonlinear Schrödinger (DNLS) equations with the SF onsite nonlinearity, is introduced in Section II. First, we study linear properties of the modulated 2D lattices, and demonstrate opening of mini-gaps in the corresponding linear spectrum, where self-trapping of new types of discrete solitons may be expected. For the sake of the completeness, in subsection III.A we briefly present results of the study of self-trapped modes (fundamental and vortex solitons) residing in the semi-infinite spectral gap in the superlattice, and compare them to the corresponding results in uniform lattices. The main topic of this work, which is presented in subsection III.B, is the creation of stable discrete solitons in the mini-gaps. We find families of fundamental solitons in mini-gaps, as well as two- and four-soliton complexes. While the fundamental solitons are stable, numerical simulations demonstrate that the complexes evolve into localized breathing structures. The paper is concluded by Section IV.

II. THE MODEL

A. Basic equations

The 2D discrete model is based on the following DNLS equation, with the cubic onsite SF nonlinearity, for complex field amplitudes \( \psi_{m,n} \):

\[
i \frac{d\psi_{m,n}}{dz} + C_{m,n} (\psi_{m+1,n} + \psi_{m-1,n}) + K_{m,n} (\psi_{m,n+1} + \psi_{m,n-1}) + \gamma |\psi_{m,n}|^2 \psi_{m,n} = 0, \tag{1}
\]

where the horizontal and vertical coupling constants are modulated, respectively, along the horizontal and vertical directions, as shown in Fig. 1.

\[
C_{m,n} = C_0 [1 + \Delta_1 \cos(Q_am)] \equiv C_m
\]

\[
K_{m,n} = C_0 [1 + \Delta_2 \cos(Q_bn)] \equiv K_n. \tag{2}
\]

This setting can be easily implemented in photonic lattices by properly selecting distances between the constituent waveguides, as well as for the BEC loaded into deep optical lattices,
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shaped as shown in Fig. 1. In the former case, evolution variable $z$ is the propagation distance along individual waveguides, while in the matter-wave (BEC) realization, $z$ is replaced by time $t$.

![Schematic plot of the lattice with corresponding modulation pattern](image)

**FIG. 1.** (a) The schematic plot of the lattice with corresponding to modulation pattern (2) with $Q_a = Q_b = \pi$, see also Eq. (3). (b) A fragment of the superlattice corresponding to $Q_a = Q_b = \pi/3$. Different values of the coupling constants are designated in the plots.

In addition, we also considered the 2D lattice with another modulation pattern, corresponding to the horizontal and vertical couplings periodically modulated along the vertical and horizontal directions, respectively:

$$C_{m,n} = C_0 [1 + \Delta_1 \cos(Q_a n)]$$
$$K_{m,n} = C_0 [1 + \Delta_2 \cos(Q_b m)].$$

Unlike the model based on Eq. (1), the analysis has not revealed any dynamically stable localized structures in Eq. (3), except for the usual onsite solitons in the semi-infinite spectral gap, and it is not straightforward to implement the latter model experimentally. Therefore, we here focus on Eq. (2).

The results are presented below for $\gamma = +1$ (which corresponds to the SF nonlinearity), $C_0 = 1$, and $\Delta_1 = \Delta_2 = 0.5$ in Eqs. (1) and (2). Furthermore, $Q_a = Q_b = \pi/3$ is fixed in Eq. (2), which corresponds to the superlattice displayed in Fig. 1(b), unless it is specified otherwise. Comparison with many results obtained at other values of the parameters, including those with $Q_a \neq Q_b$, demonstrates that this case adequately represents the generic situation.
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Stationary solutions to Eqs. (1) with real propagation constant $-\mu$ (or chemical potential for the BEC) are looked for as

$$\psi_{m,n}(z) = e^{-i\mu z} U_{m,n},$$

where stationary discrete function $U_{m,n}$ obeys the following equation:

$$\mu U_{m,n} + C_m (U_{m+1,n} + U_{m-1,n}) + K_n (U_{m,n+1} + U_{m,n-1}) + |U_{m,n}|^2 U_{m,n} = 0,$$

with the coupling constants defined as per Eq. (2). The power of the discrete soliton is defined as usual,

$$P = \sum_{m,n} |U_{m,n}|^2.$$  

Stationary equation (5) was solved by means of a numerical algorithm based on the modified Powell minimization method\textsuperscript{28}. Stability of the so found discrete solitons was checked, in the framework of the linear stability analysis, by numerically solving the corresponding eigenvalue equation for modes of small perturbations. Finally, the evolution equation (1) was directly simulated by dint of the Runge-Kutta procedure of the sixth order, cf. Ref.\textsuperscript{28}. The simulations were used to verify the stability properties predicted by the linear analysis.

B. The linear spectrum

The linearized version of Eq. (5) is

$$\mu U_{m,n} + C_m (U_{m+1,n} + U_{m-1,n}) + K_n (U_{m,n+1} + U_{m,n-1}) = 0.$$  

Its eigenvalue (EV) spectrum can be derived analytically for binary lattices, which are characterized by alternating values of the coupling constants ($Q_a = Q_b = \pi$), see Fig. II(a):

$$C_m = C_0 [1 + (-1)^m \Delta_1] = 1 \pm 0.5 \equiv C_{1(2)},$$

$$K_m = C_0 [1 + (-1)^n \Delta_2] = 1 \pm 0.5 \equiv K_{1(2)},$$

with the top and bottom signs corresponding, severally, to even and odd values of $m$ or $n$. The corresponding solution for amplitudes in this case can be looked for as

$$(a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n}) = (A, B, C, D) \exp [i(\kappa_a m + \kappa_b n)].$$
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where \( \kappa_a, \kappa_b \) are the Bloch wavenumbers in the \( m \) and \( n \) directions, while \( a, b, c, d \) pertain to four sets of sites distinguished by the four different values of the coupling constants in Eq. (8). The substitution of ansatz (9) into Eq. (7) leads to the following eigenvalue equation:

\[
16C_1^2C_2^2 \cos^4 \kappa_a - 4 \left( \cos^2 \kappa_a \right) \left[ (C_1^2 + C_2^2)\mu^2 + 8C_1C_2K_1K_2 \cos^2 \kappa_b \right]
+ \left( \mu^2 - 4K_1^2 \cos^2 \kappa_b \right) \left( \mu^2 - 4K_2^2 \cos^2 \kappa_b \right) = 0.
\]

(10)

For \( \kappa_{a,b} = \pi/2 \), Eq. (10) degenerates to \( \mu^4 = 0 \), at which point the gaps get closed. In this situation, we have found a few different families of nonstationary localized solutions, which radiate due to coupling to linear lattice modes in the absence of the gap.

In the general case, the eigenvalue problem was solved numerically. It has been found that, in addition to the semi-infinite gap, the spectrum of the superlattice contains new narrow (mini-) gaps, in which the linear Bloch waves do not exist, while their nonlinear counterparts are modulationally unstable, thus opening the way to create localized structures (discrete gap solitons) with the propagation constant falling into the mini-gaps, via the interplay of the nonlinearity, discreteness and modulated intersite coupling. Our main objective here is to demonstrate that some of such gap solitons are stable. They can be observed experimentally, and may be used to control the light propagation in photonic structures. It is relevant to mention that mini-gaps, and gap solitons existing in them, are known in continual nonlinear models of Bragg supergratings, i.e., gratings subject to a long-wave modulation.

As an illustration, the linear spectrum for the lattice with \( Q_a = Q_b = \pi/3 \) is presented in Fig. 2. Clearly visible are mini-gaps around \(|\mu| = 4\), which are placed symmetrically with respect to \( \mu = 0 \). Extensive numerical calculations, performed for lattices with different ratios between \( Q_a \) and \( Q_b \), produce similar spectra.

III. LOCALIZED MODES

It is known that 2D nonlinear lattices with the uniform intersite coupling give rise to fundamental bright discrete solitons of the unstaggered type, and different types of vortices in the parameter region corresponding to the semi-infinite gap in the corresponding linear spectrum. In the case of the 2D lattices with the SDF onsite nonlinearity, bright solitons are produced by the staggering transformation.
FIG. 2. The linear spectrum for the lattice with $Q_a = Q_b = \pi/3$ and $\Delta_1 = \Delta_2 = 0.5$. The spectrum features semi-infinite gaps at (approximately) $|\mu| > 4.73$, and mini-gaps around $|\mu| = 4$ (both presented by gray areas). Similar spectra, with slightly different positions and widths of the gaps, are obtained for other values of $Q_a, Q_b$.

In comparison with the uniform lattice, the periodic modulation of the intersite coupling opens the mini-gaps, in which new discrete solitons are expected, as said above. We demonstrate below that discrete solitons residing in the semi-infinite gap of the modulated lattices are not significantly altered by the spatially periodic inhomogeneity, while completely novel species of staggered discrete solitons are found in the mini-gaps.

A. Soliton families in the semi-infinite gap

Three types of fundamental unstaggered solitons have been found in the semi-infinite gap ($\mu < -4.73$ in Fig. 2): onsite, hybrid, and intersite ones, see Fig. 3. They feature dynamical properties similar to those of their counterparts in the 2D uniform lattice. In particular,
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solely the onsite family is stable, in almost the entire existence region, see Fig. 4. The modulation of the lattice only slightly extends the area where the stable onsite modes are found.

Vortex solitons with topological charge $S = 1$ and $S = 2$ are formed too in the semi-infinite gap. In terms of their stability and dynamics, they are also similar to their counterparts in the 2D uniform lattice. In particular, vortices with topological charges $S = 1$ and $S = 2^{25,29,30}$ are stable in certain parts of their existence region. The anisotropic intersite coupling in the modulated 2D lattice affects the shape of the solitons, as illustrated in Fig. 3 for the intersite and hybrid fundamental solitons.

FIG. 3. (Color online) Amplitude profiles of the 2D solitons belonging to the semi-infinite gap in the superlattice created by periodic modulation (3) with $Q_a = Q_b = \pi/3$: (a) onsite-centered, (b) inter-site-centered, and (c) hybrid fundamental solitons. Plot (d) shows the dependence of the soliton’s norm $P$ vs. $\mu$ for the fundamental modes (solid line - onsite, dashed line - hybrid, dotted line - inter-site); the norm is defined as per Eq. (6).
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![Graph](image)

**FIG. 4.** (Color online) The \( P(\mu) \) dependence for 2D onsite solitons belonging to the semi-infinite gap: black solid and red dashed lines correspond to the uniform and periodically modulated \((Q_a = Q_b = \pi/3)\) lattices, respectively. The stability region for the uniform and modulated lattices are located, severally, on the left of the dotted black and red vertical lines.

### B. Soliton families in the mini-gaps

#### 1. Fundamental solitons

Several different single-soliton families have been found in the mini-gaps. We here consider the single one, which is stable in a part of its existence region, see Fig. 5(a), while all the other families are completely unstable. The \( P(\mu) \) dependence for the soliton family is shown in Fig. 5(b). By approaching both edges of the mini-gap the single soliton families disappear in the sense that corresponding localized patterns cannot be created. In the lower bound \( P \) vanishes, while in the upper grows. In both cases background is characterized by highly irregular amplitudes with corresponding magnitude (small and high, respectively) without clearly distinguishing localized structure. It is found that, except for the light-gray area, \( 3.98 < \mu < 4.1 \) [the rectangle in Fig. 5(b)], where the calculation of the EVs for small perturbations shows that the solitons are stable, in other domains they are unstable. These predictions are corroborated by direct simulations, as shown in Fig. 5(c,d). In particular, the strong instability observed in Fig. 5(c) is accounted for by purely real EVs.
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FIG. 5. (Color online) Solitons in the mini-gap opened around \( \mu = 4 \), cf. Fig. 3. (a) A stable soliton found for \( \mu = 4.05 \) [this value of \( \mu \) is designated by the right green vertical dashed line in panel (b)]. (b) The solitons’s norm \( P \) vs. \( \mu \). The gray rectangle designates the stability region of the soliton’s type presented in plot (a). Plots (c) and (d) show the amplitude at the central site vs. time, produced by direct simulations of Eq. (1) at \( \mu = 3.54 \) [a strongly unstable soliton, designated by the left green vertical dashed line in panel (b)], and \( \mu = 4.05 \) [the stable soliton displayed in (a)].

2. Solitons complexes

In addition to dynamically stable single-soliton gap modes, we have found different bound states of solitons in the minigap. First, two-soliton complexes are built of two identical single-soliton constituents, see Fig. 6. It is found that the two-soliton complex keeps its compactness and the staggered structure in the course of perturbed evolution, which gives rise to regular amplitude oscillations, as seen in (c). In direct simulations, the two-soliton
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complex is actually found to be more robust than predicted by the linear-stability analysis, which demonstrate the presence of pure real EVs for small perturbations, i.e., weak instability of such complexes.

![Figure 6](image)

FIG. 6. (Color online) Dynamics of two-soliton complexes in the mini-gap opened around $\mu = 4$. (a) The profile of the two-peak complex ($\mu = 4$). (b) The norm of the complex, $P$, vs. $\mu$ (c) The amplitudes of the constituent solitons vs. time, in the complex designated by the vertical line in (b).

On the other hand, four-soliton complexes, which are built of four identical in-phase individual solitons (with zero phase shifts between them), see Fig. 7, feature stability properties similar to those of the constituent solitons, as shown in Fig. 7(b). In particular, the instability of the four-soliton complex is accounted for by purely real EVs, when they are present, and the complex is stable in the absence of such eigenvalues, which is corroborated by direct simulations. Thus, the in-phase four-soliton complexes are essentially more stable than their two-soliton counterparts. On the other hand, out-of-phase composites were always found to be unstable, independently on the number of constituent solitons. Note that both multi-soliton families cannot be formed in the neighborhood of the mini-gap edges. The corresponding soliton branches are changed by solution branches corresponding to more or less irregular background similarly to the case of fundamental solitons.

In the case of the SDF nonlinearity, single-soliton modes and their two- and four-soliton bound states can be generated by the staggering transformation from their unstaggered counterparts found for the SF nonlinearity in the mini-gap opened around $\mu = -4$, see Fig. 2. Naturally, these staggered modes feature the same stability and dynamics as their counterparts which were considered above.
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FIG. 7. (Color online) Four-soliton complexes in the mini-gap. (a) The profile of the stable complex at $\mu = 4.05$. (b) The norm, $P$, vs. $\mu$, for the family of the four-soliton complexes. The gray rectangle denotes the stability region for solitons of the type presented in panel (a). (c) Amplitudes of the constituent solitons vs. time, for the stable complex designated by the vertical line in (b), in the course of its perturbed evolution.

C. Other solutions

Direct simulations clearly demonstrate that all the modes observed in the mini-gaps are immobile in the 2D lattice: in direct simulations, the application of a kick to stable modes, in any direction with respect to the underlying superlattice, causes oscillations of the kicked solitons, but not their progressive motion (not shown here in detail). In fact, such simulations, although they fail to produce motile discrete solitons, corroborate their stability against positional perturbations.

The lack of the mobility of the discrete solitons in the present model is not surprising, taking into regard the general property of immobility of 2D discrete solitons supported by the cubic nonlinearity\cite{42}, which is, in turn, explained by the vulnerability of the continuum-limit counterparts of such solitons to the collapse. As a result, a soliton will be compressed by the (quasi-) collapse until it will become strongly localized, on few lattice sites, i.e., strongly pinned to the lattice, which provides for its stabilization but prevents it from being mobile.

It is known that the mobility of discrete solitons may be enhanced by special modes of “management”, i.e., by the application of time-periodic [or $z$-periodic, in terms of Eq. (1)] modulation of the local nonlinearity (at least, in 1D settings)\cite{43}. The consideration of such mechanisms may be a subject for a separate work, being beyond the scope of the present one.
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Lastly, in the superlattice with the spatially periodic modulation of the intersite coupling defined as per Eq. (3), new gaps in the linear spectrum open too. In those gaps, different stationary localized solutions can be found. The linear-stability analysis predicts that all these states are subject to an oscillatory instability, while in direct simulations some of them may seem as robust breathers, which is explained by the saturation of the instability with the increase of the oscillation amplitude.

IV. CONCLUSION

We have demonstrated that periodic modulation of the intersite coupling opens narrow gaps in the linear spectrum of the 2D square-shaped lattices, offering the possibility to create new species of discrete solitons in these mini-gaps, if the onsite cubic nonlinearity acts in the system. A part of the family of fundamental solitons and the respective four-soliton complexes are dynamically stable. Some other modes, which are unstable, develop into robust localized breathers. These modes can be experimentally created in arrays of nonlinear optical waveguides and in BEC trapped in a deep optical lattice, be means of currently available techniques4–10. It may be interesting to extend the analysis to 2D lattices with different geometries, such as triangular, honeycomb, and quasi-periodic.

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