Generation of Six-Qubit Cluster State in Ion-Trap System

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Based on resonant sideband excitation, we present a scheme for the generation of six-qubit cluster state in ion-trap system. One can realize experimentally this scheme with presently available techniques.

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I. INTRODUCTION

Recently Briegel et al. introduced cluster states as a fundamental resource aimed at the linear optics one way quantum computation [1, 2]. Walther et al. [3] and Tame et al. [4] demonstrated independently the experimental feasibility of one way computation by four-photon cluster states.

Many schemes for the generation of four-qubit cluster states have been proposed so far [3–8]. In this paper, based on resonant sideband excitation we generalize Zheng’s scheme for the generation of four-qubit cluster state [3] to the case of six-qubit to present a scheme for the generation of six-qubit cluster state in ion-trap system. It can be realized with presently available experimental techniques.

II. GENERATION OF SIX-QUBIT CLUSTER STATE IN ION-TRAP SYSTEM

The $N$-qubit cluster states can be written in the form

$$|\Psi_N\rangle = \frac{1}{2^{N/2}} \otimes_{\alpha=1}^{N} (|0\rangle_\alpha \sigma_z^{(\alpha+1)} + |1\rangle_\alpha),$$

(1)

with the convention $\sigma_z^{(N+1)} \equiv 1$. Here $\sigma_z = |0\rangle \langle 0| - |1\rangle \langle 1|$ is Pauli operator.

Suppose that six ions are confined in the ion-trap system and each ion has two excited metastable states $|e\rangle$ and $|e'\rangle$ and one ground state $|g\rangle$. The state of the qubit is $\alpha|e\rangle + \beta|g\rangle$, where $\alpha$ and $\beta$ are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. We prepare initially the first ion in the state $|e_1\rangle$ and the center-of-mass vibrational mode in the vacuum state $|0\rangle$. When a laser tuned to the first lower vibrational sideband was applied to the first ion, it induces the transition between $|g_1\rangle|1\rangle$ and $|e_1\rangle|0\rangle$. Assume the laser is off resonant with the transition $|g_1\rangle|1\rangle \rightarrow |e'_1\rangle|0\rangle$ and hence this laser will leave the state $|e'_1\rangle$ unaffected during the interaction. When the Lamb-Dicke criterion is satisfied, i.e. the Lamb-Dicke parameter $\eta \ll 1$, and in the weak-excitation regime, where Rabi frequency $\Omega$ is much smaller than vibrational frequency, the Hamiltonian in an interaction picture reads [3, 8]

$$H = i\frac{\eta}{2} \Omega e^{-i\phi} a |e_1\rangle \langle g_1| + \text{h.c.},$$

(2)

where $a^+$ and $a$ are the creation and annihilation operators of the center-of-mass vibrational mode of the trapped ions, $\phi$ is the phase of this laser field. As the Hamiltonian for the quantum system is not a time-varying, hence if the laser beam is on for a certain time $t$, the evolution of the system will be described by the unitary operator

$$U(t) = e^{-iHt} = \exp\{-i\left(\frac{\eta}{2} \Omega e^{-i\phi} a |e_1\rangle \langle g_1| + \text{h.c.}\right)t\}$$

$$= \cos\left(\frac{\eta t}{2} \Omega\right) a |e_1\rangle \langle g_1| + \text{h.c.}$$

(3)

After an interaction time $\tau_1$, the state of the system consisted of this ion and the center-of-mass mode evolves into

$$U(\tau_1)|e_1\rangle|0\rangle = \cos\left(\frac{\eta \tau_1}{2} \Omega\right)|e_1\rangle|0\rangle - e^{i\phi} \sin\left(\frac{\eta \tau_1}{2} \Omega\right)|g_1\rangle|1\rangle.$$ 

(4)

By choosing $\phi = \pi$ and $\eta \Omega \tau_1 = \pi/2$, one has [5]

$$\frac{1}{\sqrt{2}}(|e_1\rangle|0\rangle + |g_1\rangle|1\rangle).$$

(5)

Following the ideas introduced in Ref. [5, 8], we now drive the third ion with a laser tuned to the first lower vibrational sideband with respect to the transition $|g_3\rangle|1\rangle \rightarrow |e'_3\rangle|0\rangle$. In this case, the interaction Hamiltonian in an interaction picture is

$$H' = i\frac{\eta}{2} \Omega' e^{-i\phi'} a |e'_3\rangle \langle g_3| + \text{h.c.},$$

(6)
where parameters Ω', φ' have the similar meaning with Ω, φ respectively.

If the laser beam is on for a certain time t, the evolution operator of the system combined by the third ion and the center-of-mass mode reads

\[ U'(t) = e^{-iH't} = \exp\left\{-i \left( \frac{\Omega'}{2} e^{-i\phi'} a + \text{h.c.} \right) t \right\} \]

\[ \begin{align*}
&\cos\left(\frac{\Omega'}{2} t \sqrt{1 + a^+ a} \right| e_3 \rangle \langle e_3 | = e^{-i\phi'} \sin\left(\frac{\Omega'}{2} t \sqrt{\frac{1}{a+a^+}} \right| e_3 \rangle \langle e_3 | \\
&- e^{i\phi'} \sin\left(\frac{\Omega'}{2} t \sqrt{\frac{1}{a+a^+}} \right| e_3 \rangle \langle e_3 | + \cos\left(\frac{\Omega'}{2} t \sqrt{\frac{1}{a+a^+}} \right| e_3 \rangle \langle e_3 |. \\
\end{align*} \]

(7)

After an interaction time τ₂, the state |g₃⟩|1⟩ evolves to

\[ U'(τ₂)|g₃⟩|1⟩ = \cos\left(\frac{\Omega'}{2} τ₂ \right| g₃⟩|1⟩\rangle + e^{-i\phi'} \sin\left(\frac{\Omega'}{2} τ₂ \right| e_3 \rangle \langle e_3 |. \]

(8)

By choosing ηΩ'τ₂ = 2π, we have |1⟩|g₃⟩ → −|1⟩|g₃⟩, but other states do not affect during the interaction. This is just a phase gate between the third ion and the center-of-mass vibrational mode, proposed by Cirac and Zoller [9].

If the third ion is prepared initially in the state \( \frac{1}{\sqrt{2}}(|g_3⟩ - |e_3⟩) \), the system consisting of the first ion and the center-of-mass mode is in state given by Eq.(5), then after the phase gate operation between the third ion and the center-of-mass mode has been implemented, the state of the system combined by the first ion, the third ion and the center-of-mass mode is in [10]

\[ \frac{1}{2} (|e_1⟩(|g₃⟩ - |e_3⟩)|0⟩ - |g₃⟩|g₃⟩ + |e₃⟩)|1⟩. \]

(9)

Suppose the second ion is prepared initially in the state |g₂⟩. Now one drives the second ion with a laser tuned to the first lower vibrational sideband with respect to the transformation |g₄⟩|1⟩ → |e₂⟩|0⟩, |g₂⟩|0⟩ → |g₂⟩|0⟩. After that the state of the system consisting of the first ion, the second ion, the third ion and the center-of-mass mode becomes [11]

\[ \frac{1}{2} (|e₁⟩|g₂⟩(|g₃⟩ - |e₃⟩) - |g₁⟩|e₂⟩(|e₃⟩ + |g₃⟩)|0⟩. \]

(10)

It is easy to obtain the state [12]

\[ \frac{1}{\sqrt{2}} (|g₁⟩ + |e₁⟩)|g₂⟩(|g₃⟩ - |e₃⟩) - |g₁⟩ - |e₁⟩)|e₂⟩(|e₃⟩ + |g₃⟩)|0⟩. \]

(11)

by making the following unitary operation

\[ |e₁⟩ → \frac{1}{\sqrt{2}} (|e₁⟩ + |g₁⟩), |g₁⟩ → \frac{1}{\sqrt{2}} (|g₁⟩ - |e₁⟩). \]

(12)

Assume that the fourth ion is prepared initially in the state \( \frac{1}{\sqrt{2}}(|e₄⟩ - |g₄⟩) \). One maps this state to the vibrational mode by applying a laser tuned to the first lower vibrational sideband to the fourth ion, that induces the transformation |e₄⟩|0⟩ → |g₄⟩|1⟩, |g₄⟩|0⟩ → |g₄⟩|0⟩, one obtains the state

\[ \frac{1}{\sqrt{2}} (|g₁⟩ + |e₁⟩)|g₂⟩(|g₃⟩ - |e₃⟩) - |g₁⟩ - |e₁⟩)|e₂⟩(|e₃⟩ + |g₃⟩)|0⟩. \]

(13)

By completing the phase gate operation between the third ion and the vibrational mode, the state given by Eq.(13) becomes

\[ \frac{1}{\sqrt{2}} (|g₁⟩ + |e₁⟩)|g₂⟩(|g₃⟩ - |e₃⟩) - |g₁⟩ - |e₁⟩)|e₂⟩(|e₃⟩ + |g₃⟩)|0⟩. \]

(14)

Assume that the fifth ion is initially in the state \( \frac{1}{\sqrt{2}} (|e₅⟩ - |g₅⟩) \). Now we apply a laser tuned to the first lower vibrational sideband, that induces a phase gate between the fifth ion and the center-of-mass vibrational mode, |g₅⟩|1⟩ → −|g₅⟩|1⟩. Then we have

\[ \frac{1}{\sqrt{2}} (|g₁⟩ + |e₁⟩)|g₂⟩(|g₃⟩ - |e₃⟩) - |g₁⟩ - |e₁⟩)|e₂⟩(|e₃⟩ + |g₃⟩)|0⟩. \]

(15)

Now we map the state of the vibrational mode to the fourth ion. By applying a laser tuned to the first lower vibrational sideband to the fourth ion, we obtain the transformation \(|g₄⟩|1⟩ → |e₄⟩|0⟩, |g₄⟩|0⟩ → |g₄⟩|0⟩\). Then we have

\[ \frac{1}{\sqrt{2}} (|g₁⟩ + |e₁⟩)|g₂⟩(|g₃⟩ - |e₃⟩) - |g₁⟩ - |e₁⟩)|e₂⟩(|e₃⟩ + |g₃⟩)|0⟩. \]

(16)

Suppose that the sixth ion is initially in the state \( \frac{1}{\sqrt{2}} (|e₆⟩ + |g₆⟩) \). By applying a laser tuned to the first lower vibrational sideband to this ion, one can map the state of the sixth ion to the vibrational mode. So one has the state

\[ \frac{1}{\sqrt{2}} (|g₁⟩ + |e₁⟩)|g₂⟩(|g₃⟩ - |e₃⟩) - |g₁⟩ - |e₁⟩)|e₂⟩(|e₃⟩ + |g₃⟩)|0⟩. \]

(17)

We now perform a phase gate operation between the fifth ion and the vibrational mode, leading to

\[ \frac{1}{\sqrt{2}} (|g₁⟩ + |e₁⟩)|g₂⟩(|g₃⟩ - |e₃⟩) - |g₁⟩ - |e₁⟩)|e₂⟩(|e₃⟩ + |g₃⟩)|0⟩. \]

(18)
Mapping the state of the vibrational mode to the sixth ion, we have

\[
\frac{1}{2}(\{(|g_1\rangle - |e_1\rangle)|g_2\rangle(|g_3\rangle - |e_3\rangle)
-\{(|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle + |g_3\rangle)|g_4\rangle(|g_5\rangle - |e_5\rangle)
+\{(|g_1\rangle + |e_1\rangle)|g_2\rangle(-|g_3\rangle - |e_3\rangle)
-\{(|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle - |g_3\rangle)|e_4\rangle(|g_5\rangle + |e_5\rangle))\}|g_6\rangle
+\{(|g_1\rangle + |e_1\rangle)|g_2\rangle(|g_3\rangle - |e_3\rangle)
-\{(|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle + |g_3\rangle)|e_4\rangle(-|g_5\rangle - |e_5\rangle))\}|g_6\rangle)
+\{(|g_1\rangle + |e_1\rangle)|g_2\rangle(-|g_3\rangle - |e_3\rangle)
-\{(|g_1\rangle - |e_1\rangle)|e_2\rangle(|e_3\rangle - |g_3\rangle)|e_4\rangle(-|g_5\rangle + |e_5\rangle))\}|e_6\rangle)|0\rangle.
\]

We can rewrite the state of the six ions stated in Eq. (19) as

\[
\frac{1}{2}(\{|g_1\rangle\sigma_z^g + |e_1\rangle\rangle)|g_2\rangle\sigma_z^e + |e_2\rangle\rangle(|g_3\rangle\sigma_z^g + |e_3\rangle\rangle)
(|g_4\rangle\sigma_z^g + |e_4\rangle\rangle(|g_5\rangle\sigma_z^g + |e_5\rangle\rangle)|g_6\rangle\rangle + |e_6\rangle\rangle),
\]

where \(\sigma_z^i = |g_i\rangle\langle g_i| - |e_i\rangle\langle e_i|\). The state of Eq. (20) is just a six-qubit cluster state.

Now we estimate the fidelity of our scheme. Obviously, our scheme consists of eight sideband excitations which couple the internal and external degrees of freedom and trivial single-qubit operations. The fidelity of each sideband excitation is about 0.93 [10, 11], so the fidelity of our whole procedure is about 0.56. We expect that the fidelity of each sideband excitation can be improved. In that case, the fidelity of our scheme can be enhanced.

### III. SUMMARY

We have proposed a scheme for the generation of six-qubit cluster state with trapped ions. The required experimental techniques are presently available. We hope the scheme will be useful in quantum information proceeding.

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