FIBERWISE VOLUME DECREASING DIFFEOMORPHISMS ON PRODUCT MANIFOLDS

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ABSTRACT. Given a closed connected Riemannian manifold $M$ and a connected Riemannian manifold $N$, we study fiberwise, i.e. $M \times \{z\}, \ z \in N$, volume decreasing diffeomorphisms on the product $M \times N$. Our main theorem shows that in the presence of certain cohomological condition on $M$ and $N$ such diffeomorphisms must map a fiber diffeomorphically onto another fiber and are therefore fiberwise volume preserving. As a first corollary, we show that the isometries of $M \times N$ split. We also study properly discontinuous actions of a discrete group on $M \times N$. In this case, we generalize the first Bieberbach theorem and prove a special case of an extension of Talelli’s conjecture.

1. INTRODUCTION

When one studies a problem on a product manifold $M \times N$, then it is convenient if this problem “reduces” to two separate problems concerning $M$ and $N$. We have noticed this in an attempt to generalize the first Bieberbach theorem to a case related to product manifolds. Here, we needed the fact that under certain conditions the isometries of $M \times N$ split. An isometry is said to split if its $M$-component $M \times N \to M$ is independent of the $N$-coordinates and its $N$-component $M \times N \to N$ is independent of the $M$-coordinates. The component mappings can then be seen as isometries of $M$ and $N$, respectively. In this article, we find conditions on $M$ and $N$ that admit such a splitting of isometries and thus allow for the reduction of geometric symmetries of $M \times N$ to the components, $M$ and $N$. At least one intermediate result is useful on its own and will be referred to as our main theorem.

The structure of the article is as follows. In section 2, we recall definitions and state the preliminary results. Our main theorem is proven in section 3. Section 4 is concerned with applications of this theorem to properly discontinuous actions. Throughout the article, $M$ and $N$ are Hausdorff and second countable Riemannian manifolds. Also, we will only deal with smooth maps, so differentiable means $C^\infty$. We denote $n = \dim(M)$. Let us formulate our results.

If $z \in N$, then $M \times \{z\} \subset M \times N$ is a manifold isometric to $M$ and $\text{Vol}(M \times \{z\}) = \text{Vol}(M)$ (see Definition 2.4). Let $f : M \times N \to M \times N$ be a diffeomorphism. We say that $f$ is fiberwise volume decreasing at $z$ if

$$\text{Vol}(f(M \times \{z\})) \leq \text{Vol}(M),$$

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where $f(M \times \{z\})$ has the induced metric from $M \times N$. A diffeomorphism is fiberwise volume decreasing (fvd) if it is fiberwise volume decreasing at each $z \in N$. Note that isometries are fvd.

We say that a pair $(M, N)$ satisfies the $*$-condition if
\[\pi^* : H^n(M; \mathbb{Z}_2) \to H^n(M \times N; \mathbb{Z}_2)\]
is an isomorphism. Here, $\pi : M \times N \to M$ is the natural projection map. We obtain the following result.

**Theorem 1.1 ((Main Theorem)).** Let $M$ be a closed connected Riemannian manifold and let $N$ be a Riemannian manifold such that $(M, N)$ satisfies the $*$-condition. If $f : M \times N \to M \times N$ is a diffeomorphism which is fvd at $z \in N$, then there exists $w \in N$ such that $f(M \times \{z\}) = M \times \{w\}$.

Under the assumptions of the theorem, one can show that the collection of fiberwise volume decreasing self-maps of $M \times N$, denoted by $\text{FVD}(M \times N)$, is a group. To describe its structure, we define the following map $\psi$ whose definition does not depend on the chosen $y_0 \in M$.

\[\psi : \text{FVD}(M \times N) \to \text{Diffeo}(N), \quad g \mapsto \tilde{p} \circ g,\]

where $p : M \times N \to N$ is the natural projection and
\[\tilde{p} \circ g : N \to N, \quad z \mapsto p \circ g(y_0, z).\]

For every element of $K := \{f : N \to \text{Diffeo}(M) \mid f \text{ is Fréchet differentiable}\}$, we can associate the map
\[\overline{f} : M \times N \to M, \quad (y, z) \mapsto f(z)(y)\]
and so $(\overline{f}, p) \in \text{FVD}(M \times N)$. We obtain the following exact sequence.

**Theorem 1.2.** The sequence:
\[1 \to K \hookrightarrow \text{FVD}(M \times N) \xrightarrow{\psi} \text{Diffeo}(N) \to 1\]
is short exact.

An important application of our main theorem is related to the splitting of isometries of a product manifold. Cheeger and Gromoll (see [6]) have shown that the isometries of $M \times \mathbb{R}^k$ split for any $k \in \mathbb{N}$. They use the fact that any point of $\mathbb{R}^k$ lies on a line through a given point to eventually show that isometries map fibers of the form $M \times \{z\}$ to fibers of the same form. A line in a complete Riemannian manifold $N$ is a geodesic $\gamma : (-\infty, \infty) \to N$ that minimizes the arc length between any two of its points. The 3-dimensional Heisenberg group shows that not even contractible Lie groups with a left-invariant metric need to satisfy this property (see [12]). Using Theorem 1.1, we are able to avoid these complications.
Corollary 1.3 (Splitting Theorem). If $M$ is a closed connected Riemannian manifold and if $N$ is a connected Riemannian manifold such that $(M, N)$ satisfies the $*$-condition, then the isometries of $M \times N$ split, i.e. $\text{Iso}(M \times N) = \text{Iso}(M) \times \text{Iso}(N)$.

It is worth noting that for a complete $N$ the theorem follows by the de Rham decomposition (see [8]). Throughout the article we make no completeness assumption on $N$, except in theorem 1.3.

An interesting application of our splitting theorem is related to groups that can act properly discontinuously, cocompactly and isometrically on products $M \times N$. When $M$ is a singleton and $N$ is a simply connected, connected, nilpotent Lie group, equipped with a left-invariant metric, such groups are called \textit{almost-crystallographic groups}. All of the three famous Bieberbach theorems (see [4], [5] and [9]) have been generalized to almost-crystallographic groups. The following generalization of the first Bieberbach theorem was given by L. Auslander.

\textbf{Bieberbach 1.4 (Generalized first Bieberbach theorem, Auslander, [2])}. Every almost-crystallographic group contains a finite index subgroup isomorphic to a uniform lattice, i.e. a discrete and cocompact subgroup of $N$.

We obtain the following generalization.

\textbf{Theorem 1.5}. Let $M$ be a closed connected Riemannian manifold and let $N$ be a simply connected, connected, nilpotent Lie group equipped with a left-invariant metric. If $\Gamma$ is a group acting properly discontinuously, cocompactly and isometrically on $M \times N$, then $\Gamma$ contains a finite index subgroup isomorphic to a uniform lattice of $N$.

Since $N$ is a Lie group with a left-invariant metric, it must be complete. Then, the de Rham decomposition gives an alternate proof of this result (see Remark 4.1.7, Section 4). We note that one can easily find examples showing that the other Bieberbach-theorems do not generalize to the $M \times N$ case.

Another interesting setting for applying Theorem 1.4 is Talelli’s conjecture (Conjecture III of [15]). Let us denote the cohomological dimension of a group $\Gamma$ by $\text{cd}(\Gamma)$. We study the following, slightly different version of the conjecture (see [15]).

\textbf{Conjecture 1.6 (Talelli conjecture reformulated, 2005)). If $\Gamma$ is a torsion-free group that acts smoothly and properly discontinuously on $S^n \times \mathbb{R}^k$, then $\text{cd}(\Gamma) \leq k$.

By a result of Mislin and Talelli ([16]), we know that the conjecture holds for the large class of $LHF$-groups (see [10]).

In the context of this article it feels natural to replace $S^n$ by any closed, connected Riemannian manifold $M$, and to replace $\mathbb{R}^k$ by any contractible Riemannian manifold $N$. By doing this, we obtain the following

\textbf{Conjecture 1.7 (Petrosyan, 2007)). If $\Gamma$ is a torsion-free group acting smoothly and properly discontinuously on $M \times N$, then $\text{cd}(\Gamma) \leq \dim(N)$.

Petrosyan has proven this conjecture in the case of $HF$-groups and when $N$ is 1-dimensional (see [14]). We prove the following
Theorem 1.8. Let $M$ be a closed and connected Riemannian manifold and let $N$ be a contractible Riemannian manifold. If $\Gamma$ is a torsion-free group acting properly discontinuously and fiberwise volume decreasingly on $M \times N$, then $\Gamma$ acts freely and properly discontinuously on $N$. In particular, we have that $\text{cd}(\Gamma) \leq \dim(N)$.

2. Background and preliminary results

Let $M$ be a Riemannian manifold of dimension $n$ and let $x : U \to M, U \subset \mathbb{R}^n$ be a parametrization of $M$. For $i, j \in \{1, 2, \ldots, n\}$, vector fields $X_i$ and functions $g_{ij}$ on $x(U)$ are defined as follows: let $p \in M$ and $q = (q_1, q_2, \ldots, q_n) \in U$ such that $x(q) = p$. For each $i \in \{1, 2, \ldots, n\}$, consider the curve $x_i(t) := x(q_1, q_2, \ldots, q_i+t, q_{i+1}, \ldots, q_n)$ in $M$. We define $X_i(p) = \frac{d}{dt}x_i(t)|_{t=0}$ and $g_{ij}(p) = \langle X_i(p), X_j(p) \rangle_p$. The $g_{ij}$ are called the components of the metric tensor relative to the parametrization $x$. To simplify notation, we will sometimes denote $g_{ij}(x(q))$ by $g_{ij}(q)$.

In section 3, we will need the notions of measure 0 and of volume of subsets of $M$. We define these here.

Definition 2.1. A subset $A$ of a manifold $M$ has measure 0 if $x^{-1}(A)$ has Lebesgue measure 0 in $\mathbb{R}^n$ for every parametrization $x$ of $M$.

Observe that the notion of measure 0 is invariant under diffeomorphisms.

Definition 2.2. Assume that $x : U \to M$ is a parametrization. If $C$ is an open connected set such that $C \subset x(U)$ is compact and such that the boundary $\partial(C)$ of $C$ has measure 0, then we call $C$ a nice open of $M$.

The volume of a nice open $C$ of $M$ is defined by

$$\text{Vol}(C) = \int_{x^{-1}(C)} \sqrt{\det(g_{ij})} \, d\mu,$$

where the $g_{ij}$ are the components of the metric tensor relative to $x$ and where $\mu$ is the Lebesgue measure on $\mathbb{R}^n$. The definition is independent of the parametrization used.

Definition 2.3. A diffeomorphism $f : M \to M$ is volume preserving if it preserves the volume of all nice opens of $M$.

Definition 2.4. Take a countable number of nice opens, say $(C_i)_{i \in I}$ where $I$ is some index set, such that the $C_i$ are pairwise disjoint and such that $M \setminus \bigcup_{i \in I} C_i$ has measure 0. We call such a family a nice family for $M$. We define the volume of $M$ as

$$\text{Vol}(M) = \sum_{i \in I} \text{Vol}(C_i).$$

Let us elaborate on this definition. First of all, note that this definition of volume is independent of the nice family chosen.
Secondly, there is a standard way of finding a nice family \((C_i)_{i \in I}\) for \(M\). Start with a countable number of parameterizations \(x_1, x_2, \ldots\) whose images contain the closures of nice open sets \(B_1, B_2, \ldots\) respectively. Make sure that \(\bigcup_{i=1}^{\infty} B_i \supset M\). Consider the sets

\[
B'_1 := B_1, B'_2 := B_2 \setminus \overline{B_1}, B'_3 := B_3 \setminus (\overline{B_1} \cup B_2), \ldots, B'_{n} := B_n \setminus \bigcup_{i=1}^{n-1} B_i, \ldots
\]

You can take the family \((C_i)_{i \in I}\) as the family of connected components of the sets \(B'_i\).

Finally, note that volume preserving diffeomorphisms preserve \(\text{Vol}(M)\).

An important class of volume preserving diffeomorphisms is the class of isometries of a Riemannian manifold. We will be primarily interested in isometries of \(\prod_{1}^{k} M_i\), which is independent of its coordinates. In this case, the component mappings \(f_1 \times f_2 \times \ldots \times f_k\) can be seen as isometries of \(M_i\) for \(i = 1, \ldots, k\) respectively.

Note that all isometries of \(M \times N\) split if and only if \(\text{Iso}(M \times N) = \text{Iso}(M) \times \text{Iso}(N)\).

The following theorem is a standard result from algebraic topology.

**Theorem 2.6** (Poincaré-Lefschetz Duality, [3]). Let \(M\) be a compact orientable \(n\)-manifold and let \(L\) be a closed subset of \(M\). Denoting Čech cohomology by \(\check{H}\), we have the following commutative diagram where the rows are exact and all the vertical arrows (cap products with the orientation class) are isomorphisms:

\[
\begin{array}{ccccccccc}
\cdots & \to & \check{H}^p(M,L) & \to & \check{H}^p(M) & \to & \check{H}^p(L) & \to & \check{H}^{p+1}(M,L) & \to & \cdots \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \cdots \\
\cdots & \to & H_{n-p}(M\setminus L) & \to & H_{n-p}(M) & \to & H_{n-p}(M, M\setminus L) & \to & H_{n-p-1}(M\setminus L) & \to & \cdots
\end{array}
\]

For non-orientable \(M\) the theorem holds with \(\mathbb{Z}_2\)-coefficients.

One obtains the following interesting

**Corollary 2.7.** If \(L\) is a proper closed subset of a compact \(n\)-manifold \(M\), then

\[
| \check{H}^n(L; \mathbb{Z}_2) | \leq | \check{H}^n(M; \mathbb{Z}_2) |.
\]

**Proof.** The corollary follows from the fact that \(\check{H}^n(L; \mathbb{Z}_2)\) is isomorphic to \(H_0(M, M\setminus L; \mathbb{Z}_2)\) and this group contains less elements than \(H_0(M; \mathbb{Z}_2) \cong \check{H}^n(M; \mathbb{Z}_2)\), by Theorem 2.6.

We end this section by a purely algebraic lemma. Recall the following definitions.
**Definition 2.8.** A symmetric matrix $G$ in $\mathcal{M}_n(\mathbb{R})$ is positive definite if $x^T G x > 0$ for every non-zero vector $x \in \mathbb{R}^k$. A symmetric matrix $H \in \mathcal{M}_n(\mathbb{R})$ is positive semi-definite if $x^T H x \geq 0$ for every vector $x \in \mathbb{R}^k$.

**Lemma 2.9.** If $G \in \mathcal{M}_n(\mathbb{R})$ is positive definite and $H \in \mathcal{M}_n(\mathbb{R})$ is positive semi-definite, then $\det(G + H) \geq \det(G)$. The inequality is strict when $H \neq 0$.

**Proof.** We start by proving the special case where $H = E = (\mu, 0, 0, \ldots, 0)$ with $\mu \geq 0$. Here, the notation $(e_{11}, e_{22}, \ldots, e_{nn})$ stands for a diagonal matrix whose $(i, i)$th entry is $e_{ii}$. Denote by $\tilde{G}$ the matrix obtained from $G$ by removing the first row and column, i.e. $\tilde{G}_{ij} = G_{(i+1)(j+1)}$ for $i, j \in \{1, 2, \ldots, n-1\}$. Expanding $\det(G + E)$ by the first row gives

$$\det(G + E) = \det(G) + \mu \det(\tilde{G}).$$

Since

$$(x_1, x_2, \ldots, x_{n-1}) \tilde{G}(x_1, x_2, \ldots, x_{n-1})^T = (0, x_1, x_2, \ldots, x_{n-1}) G(0, x_1, x_2, \ldots, x_{n-1}),$$

for all $(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, we have that $\tilde{G}$ is positive definite. This implies that $\det(\tilde{G}) > 0$ and thus $\det(G + E) \geq \det(G)$. Strict inequality holds if and only if $\mu > 0$. Notice that a similar proof exists when $H$ equals a diagonal matrix of the form $(0, 0, \mu, 0, 0, \ldots, 0)$.

In general, take an orthogonal matrix $O$ such that $D = OHO^T = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Clearly, $\lambda_i \geq 0$ for all $i$. We have

$$\det(G + H) = \det(OHO^T + D) = \det(OHO^T + E_1 + E_2 + \ldots + E_n),$$

where $E_i$ is the matrix that has $\lambda_i$ as its $(i, i)$th entry and zeros everywhere else. By positive definiteness of $OHO^T$ we have that $OHO^T + E_1 + E_2 + \ldots + E_k$ is positive definite for each $k \in \{1, 2, \ldots, n\}$. The proof now follows from the special case proven above. \qed

### 3. Main theorem

#### 3.1. Proof of the main theorem and splitting of isometries.

From now on, assume that $M$ is an $n$-dimensional closed Riemannian manifold. Apart from being Riemannian, we put no conditions on $N$. Consider the product manifold $M \times N$ and a point $z \in N$. Define the inclusion

$$i : M \rightarrow M \times N$$

$$y \mapsto (y, z)$$

and the projection

$$\pi : M \times N \rightarrow M$$

$$(y, w) \mapsto y.$$ 

Clearly, the composition $\pi \circ i$ is the identity mapping of $M$ and so the mapping

$$i^* \circ \pi^* : H^n(M; \mathbb{Z}_2) \mapsto H^n(M; \mathbb{Z}_2)$$

is an isomorphism. Therefore, $\pi^*$ must be injective.
Definition 3.1.1. If $M$ and $N$ are such that
\[
\pi^* : H^n(M; \mathbb{Z}_2) \to H^n(M \times N; \mathbb{Z}_2)
\]
is an isomorphism or equivalently that
\[
i^* : H^n(M \times N; \mathbb{Z}_2) \to H^n(M; \mathbb{Z}_2)
\]
is an isomorphism, then we say that $(M, N)$ satisfies the ∗-condition. Note that this definition does not depend on the choice of $z \in N$.

The following propositions will be useful in the proof of our main theorem.

Proposition 3.1.2. If $(M, N)$ satisfies the ∗-condition, then we have that
\[
\phi := \pi \circ f \circ i : M \to M
\]
is surjective for any homeomorphism $f : M \times N \to M \times N$.

Proof. Since $f$ is a homeomorphism and since $(M, N)$ satisfies the ∗-condition, we know that
\[
\phi^* : H^n(M; \mathbb{Z}_2) \xrightarrow{\pi^*} H^n(M \times N; \mathbb{Z}_2) \xrightarrow{f^*} H^n(M \times N; \mathbb{Z}_2) \xrightarrow{i^*} H^n(M; \mathbb{Z}_2)
\]
is an isomorphism. Assume now that $\phi$ is not surjective. The image of $\phi$ is compact and thus closed. Since it misses a point, say $p$, it has to miss an open subset of $M$, say $U$. Take a CW-complex structure on $M$ containing an open $n$-cell $\sigma$ with $p \in \sigma \subset U$. Now, the forgetful map $\phi_1 : M \to M \setminus \sigma$ of $\phi$ induces the mapping $\phi_1^* : H^n(M \setminus \sigma; \mathbb{Z}_2) \to H^n(M; \mathbb{Z}_2)$. Let $j$ be the inclusion mapping of $M \setminus \sigma$ into $M$. On the cohomology level we obtain
\[
\phi_1^* \circ j^* : H^n(M; \mathbb{Z}_2) \to H^n(M; \mathbb{Z}_2),
\]
and this mapping equals $\phi^*$. Since $\phi^*$ is surjective, we conclude that $\phi_1^*$ must be surjective, which is a contradiction to Corollary 2.7 because Čech cohomology and singular cohomology are isomorphic for CW-complexes. □

Proposition 3.1.3. Let $f : M \times N \to M \times N$ be a diffeomorphism and let $z \in N$. On $f(M \times \{z\})$, we consider the induced metric from $M \times N$. Suppose that $C$ is a nice open in $M$ such that the natural projection map $\pi : f(M \times \{z\}) \to M$ restricts to a diffeomorphism $\phi$ onto an open set containing $\overline{C}$. Then, Vol($\phi^{-1}(C)$) ≥ Vol($C$). Moreover, the equality is strict if and only if the projection $p : M \times N \to N$ is not constant on $\phi^{-1}(C)$.

Proof. Let $x : U \to M$ be a parametrization for $M$ such that $\overline{C} \subset x(U)$. Let $V = x^{-1}(C)$ and consider the parametrization
\[
\psi := \phi^{-1} \circ x : V \to \phi^{-1}(C).
\]
Write $\psi = (x, \eta)$ where $x : V \to M$ is the $M$-component map and where $\eta : V \to N$ is the $N$-component map of $\psi$. Denote the components of the metric tensor relative to $x$ and $\psi$ by $g_{ij}$ and $\tilde{g}_{ij}$ respectively. By definition we have that
\[
\text{Vol}(C) = \int_V \sqrt{\det(g_{ij})(q)} \, d\mu
\]
and
\[ \text{Vol}(\phi^{-1}(C)) = \int_V \sqrt{\det(\widetilde{g}_{ij})(q)} \, d\mu. \]

To prove that \( \text{Vol}(\phi^{-1}(C)) \geq \text{Vol}(C) \) it thus suffices to show that \( \det(g_{ij}(q)) \leq \det(\widetilde{g}_{ij}(q)) \) for all \( q \in V \). Let us investigate the functions \( \widetilde{g}_{ij} \).

For each \( i \in \{1,2,\ldots,n\} \) and \( q = (q_1,q_2,\ldots,q_n) \in V \), denote the curve
\[ x(q_1,q_2,\ldots,q_i + t,q_{i+1},\ldots,q_n) \]
by \( x^q_i(t) \) and
\[ \psi(q_1,q_2,\ldots,q_i + t,q_{i+1},\ldots,q_n) \]
by \( \psi^q_i(t) = (x^q_i(t),\eta^q_i(t)) \in M \times N \). For simplicity, we drop the upper index \( q \) in the following calculation.

\[ \widetilde{g}_{ij}(q) = \langle (\psi^j_i(0),\psi^j_j(0)) \psi(q) \rangle = \langle ((x_j(t),\eta_j(t))'(0), (x_j(t),\eta_j(t))'(0)) \psi(q) \rangle = \langle x_i'(0),x_j'(0) \rangle x(q) + \langle \eta_i'(0),\eta_j'(0) \rangle \eta(q) = g_{ij}(q) + h_{ij}(q), \]

where
\[ h_{ij}(q) = \langle (\eta_i^j)'(0), (\eta_j^j)'(0) \rangle \eta(q) \]

This shows that \( \widetilde{g}_{ij}(q) = g_{ij}(q) + h_{ij}(q) \) for all \( q \in V \). The first part of the proposition now follows from Lemma 3.1.9.

If \( p \circ \phi^{-1} \) is not constant on \( C \), then \( \text{Vol}(\phi^{-1}(C)) > \text{Vol}(C) \). Indeed, in this case there exists an open set \( O \subset C \) such that the linear map \( D(p \circ \phi^{-1})_y \neq 0 \) for each \( y \in O \). Let \( W = x^{-1}(O) \). We have that for each \( q \in W \) there exists \( i_q \in \{1,2,\ldots,n\} \) such that \( \langle \eta_{i_q}^q'(0) \rangle D(p \circ \phi^{-1})_y((x^q_{i_q})'(0)) \neq 0 \). The matrices \( h_{ij}(q) \) are thus non-zero. Our claim now follows from Lemma 3.1.9.

We give one more definition before proceeding with our main result.

**Definition 3.1.4.** Let \( f : M \times N \to M \times N \) be a diffeomorphism and let \( z \in N \). Equip both \( M \times \{z\} \) and \( f(M \times \{z\}) \) with the Riemannian metric induced from \( M \times N \) and note that \( \text{Vol}(M \times \{z\}) = \text{Vol}(M) \). We say that \( f \) is fiberwise volume decreasing at \( z \) if
\[ \text{Vol}(f(M \times \{z\})) \leq \text{Vol}(M). \]

A diffeomorphism is fiberwise volume decreasing (fvd) if it is fiberwise volume decreasing at every point of \( N \). We denote the set of all fiberwise volume decreasing maps of \( M \times N \) by \( \text{FVD}(M \times N) \).

**Theorem 3.1.5.** Let \( M \) be a closed connected Riemannian manifold and let \( N \) be a Riemannian manifold such that \((M,N)\) satisfies the \( s \)-condition. If \( f : M \times N \to M \times N \) is fvd at \( z \in N \), then there exists \( w \in N \) such that \( f(M \times \{z\}) = M \times \{w\} \).
Proof. Assume that $f$ is fiberwise volume decreasing at $z$. We prove the theorem by showing that
\[ \text{Vol}(f(M \times \{z\})) > \text{Vol}(M), \]
if $f(M \times \{z\})$ is not of the form $M \times \{w\}$ for some $w \in N$. For the remainder of the proof we will denote $f(M \times \{z\})$ by $f(M)$.

Let $\pi$ be the natural projection map of $f(M)$ onto $M$. From Proposition 3.1.2 it follows that $\pi \circ f|_{M \times \{z\}}$ is surjective. Let’s look at the set $A$ of critical values of $\pi$. This set is closed and we know by Sard’s theorem that it is of measure 0 in $M$. Take a family of nice opens $(C_i)_{i \geq 1}$ of $M$ that are pairwise disjoint, and such that their union equals $M \setminus \tilde{A}$ where $\tilde{A} \supset A$ has measure 0. We can assume this family to be such that the $C_i$ satisfy the hypotheses of proposition 3.1.3. We conclude that $\text{Vol}(f(M)) \geq \text{Vol}(M)$.

Assume there exists a nice open $C \subset M$ such that

1. there are open subsets $V \subset f(M)$ and $O \subset M$ with $\phi := \pi|_V : V \to O$ a diffeomorphism and $C \subset O$,
2. $\text{Vol}(\phi^{-1}(C)) > \text{Vol}(C)$.

We can then look at a nice family of $M$ containing $C$ to conclude that $\text{Vol}(f(M)) > \text{Vol}(M)$, obtaining the desired contradiction. It remains thus to prove the existence of a nice open $C$, satisfying the two conditions above, in case $f(M)$ is not a fiber.

Denote $p : f(M) \to N$ the projection map. Assume by contradiction that for all $x \in f(M)$ the differential $(Dp)_x = 0$ whenever $(D\pi)_x$ is an isomorphism, then
\[ \mathcal{A}_1 = \{x \in f(M) \mid (Dp)_x \neq 0\}, \]
and
\[ \mathcal{A}_2 = \{x \in f(M) \mid (D\pi)_x \text{ is an isomorphism}\}. \]
are disjoint, open, nonempty sets. Since $f$ is a diffeomorphism, we have that $\mathcal{A}_1 \cup \mathcal{A}_2 = f(M)$. Since $M$ is connected, this is a contradiction. Hence, there exists an element $y \in f(M)$ such that $(Dp)_y \neq 0$ and $(D\pi)_y$ is an isomorphism. Take a nice open $U \subset f(M)$ consisting of such points $y$. Let $u \in U$ with $\pi(u) \notin \tilde{A}$. We can find a nice open $C \subset M \setminus \tilde{A}$ containing $\pi(u)$ that satisfies the hypotheses of Proposition 3.1.3. Now, $\text{Vol}(\phi^{-1}(C)) > \text{Vol}(C)$, as desired.

Remark 3.1.6. For $(M, N)$ satisfying the *-condition, the proof can be generalized to the case that $M$ is not connected. If $M_1, M_2, \ldots, M_k$ are the connected components of $M$, and $(M, N)$ satisfies the *-condition, then there exist $z_1, z_2, \ldots, z_k \in N$ such that
\[ f(M \times \{z\}) = (M_1 \times \{z_1\}) \cup (M_2 \times \{z_2\}) \cup \ldots \cup (M_k \times \{z_k\}). \]

Proof. Proposition 3.1.2 does not use the fact that $M$ is connected and so we know that $\pi \circ f|_{M \times \{z\}}$ is surjective. This implies that $\pi \circ f$ maps each $M_i \times \{z\}$ surjectively onto an $M_j$. The same reasoning as in the proof of Theorem 3.1.5 then shows that $\text{Vol}(f(M_i \times \{z\})) \geq \text{Vol}(M_j)$. Since $f$ is fvd we can conclude that
\[ \sum_{l=1}^k \text{Vol}(f(M_l \times \{z\})) = \sum_{l=1}^k \text{Vol}(M_l), \]
and so \( \text{Vol}(f(M_i \times \{z\})) = \text{Vol}(M_j) \). If we suppose that \( f(M_i \times \{z\}) \) is not of the form \( M_j \times \{z\} \) for some \( z_j \in N \), then, as in the proof of Theorem 3.1.5, using the connectedness of \( f(M_i \times \{z\}) \), we can find a point \( y \in f(M_i \times \{z\}) \) such that \( Dp_y \neq 0 \) and \( D\pi_y \) is an isomorphism. We can thus find a nice open \( C \) of \( M_j \setminus A \) containing \( \pi(y) \) that satisfies the hypotheses of Proposition 3.1.3. Therefore, \( \text{Vol}(\phi^{-1}(C)) > \text{Vol}(C) \), implying \( \text{Vol}(f(M_i \times \{z\})) > \text{Vol}(M_j) \) and giving us a contradiction. 

We obtain the following interesting corollary.

**Corollary 3.1.7 ((Splitting Theorem)).** If \( M \) is a closed connected Riemannian manifold and if \( N \) is a connected Riemannian manifold such that \((M, N)\) satisfies the \(*\)-condition, then the isometries of \( M \times N \) split, i.e. \( \text{Iso}(M \times N) = \text{Iso}(M) \times \text{Iso}(N) \).

**Proof.** Let \( f = (f_1, f_2) \) be an isometry of \( M \times N \). Then, \( f \) satisfies the hypothesis of Theorem 3.1.5 and therefore \( f_2 \) is independent of its \( M \)-coordinates. Notice that \( f_2 \) can thus be seen as a map from \( N \) to \( N \).

Let \((y, z) \in M \times N \) and denote \( f_1(y, z) = x \). A path \( \gamma \) in \( \{y\} \times N \), containing \((y, z)\), is orthogonal to every fiber \( M \times \{w\} \). Since \( f \) is an isometry which maps each fiber to another fiber, we have that \( f \circ \gamma \) is orthogonal to each fiber \( M \times \{w\} \). It is therefore a path in \( \{x\} \times N \) and connectedness of \( N \) implies that \( f_1(\{y\} \times N) = \{x\} \). Since \( y \in M \) is arbitrary, we conclude that \( f_1 \) does not depend on its \( N \)-coordinates. It can thus be seen as a map from \( M \) to \( M \).

Since \( f \) is an isometry, we obtain that \( f_1 \) and \( f_2 \) are isometries of \( M \) and \( N \) respectively. \( \square \)

### 3.2. Fiberwise volume decreasing maps.

It is interesting to investigate which maps exactly are fvd. First of all, we note that there is no immediate connection with volume preserving maps. For example, on the cylinder \( S^1 \times \mathbb{R} \subset \mathbb{R}^2 \), one can consider the diffeomorphism mapping \((\cos(x), \sin(x), y) \in S^1 \times \mathbb{R} \to (\cos(x), \sin(x), \frac{y}{2}) \). This map is clearly not volume preserving, but it is fvd. Conversely, the diffeomorphism

\[
f : \quad S^1 \times \mathbb{R} \quad \to \quad S^1 \times \mathbb{R} \\
(\cos(x), \sin(x), y) \quad \mapsto \quad (\cos(x), \sin(x), y + \sin(x))
\]

is volume preserving, since the Jacobian of the map \( f \) has determinant one at each point of \( \mathbb{R}^2 \). Yet, \( f \) is not fvd.

Note further that \( \text{FVD}(M \times N) \) has a natural group structure, because in our setting “fiberwise volume decreasing” and “fiber preserving” are equivalent notions. Our main theorem implies the following

**Corollary 3.2.1.** Given a point \( y_0 \in M \), consider

\[
\psi : \quad \text{FVD}(M \times N) \quad \to \quad \text{Diffeo}(N) \\
(\alpha, \beta) \quad \mapsto \quad \tilde{\beta},
\]

where

\[
\tilde{\beta} : \quad N \quad \to \quad N \\
z \quad \mapsto \quad \beta(y_0, z).
\]
This definition is independent of the chosen \( y_0 \). Furthermore, the map \( \psi \) is a group homomorphism with kernel

\[ K = \{ f : N \to \text{Diffeo}(M) \mid \overline{f} : M \times N \to M, (y, z) \mapsto f(z)(y) \text{ is differentiable} \}. \]

Additionally, there is a short exact sequence

\[ 1 \to K \cong \text{kernel}(\psi) \hookrightarrow \text{FVD}(M \times N) \xrightarrow{\psi} \text{Diffeo}(N) \to 1. \]

**Proof.** Theorem 3.1.5 implies that the definition of \( \psi \) is independent of the chosen \( y_0 \in M \).

To show that \( \psi \) is a group homomorphism, let \((y, z) \in M \times N\) and \((\alpha, \beta), (\alpha', \beta') \in \text{FVD}(M \times N)\). Then,

\[ (\alpha, \beta) \circ (\alpha', \beta')(y, z) = (\alpha(\alpha_2(y, z), \beta_2(y, z)), \beta_1(\alpha_2(y, z), \beta_2(y, z))) \]

and thus

\[ \psi((\alpha, \beta) \circ (\alpha', \beta'))(z) = \beta_1(\alpha_2(y_0, z), \beta_2(y_0, z)). \]

On the other hand,

\[ \psi(\alpha, \beta) \circ \psi(\alpha', \beta')(z) = \beta_1(\alpha_2(y_0, z), \beta_2(y_0, z)). \]

Both expressions are equal since \( \beta_1 \) doesn’t depend on its first argument.

Observe that \( \psi \) maps each \((\alpha, \beta) \in \text{FVD}(M \times N)\) to a diffeomorphism of \( N \). This follows from the fact that \((\alpha, \beta) \in \text{FVD}(M \times N)\) has an inverse \((\alpha', \beta') \in \text{FVD}(M \times N)\) and so \( \psi(\alpha', \beta') \) is an inverse for \( \psi(\alpha, \beta) \). We conclude that \( \psi \) is a well-defined group homomorphism.

Given a diffeomorphism \( \gamma \) of \( N \), define

\[ \hat{\gamma} : M \times N \to N \]

\[ (y, z) \mapsto \gamma(z). \]

Let \( \pi : M \times N \to M \) be the natural projection onto \( M \). Then, \((\pi, \hat{\gamma}) \in \text{FVD}(M \times N)\) and \( \psi(\pi, \hat{\gamma}) = \gamma \). Hence, \( \psi \) is surjective.

If \( f \) is an element of \( K \) and \( p : M \times N \to N \) is the natural projection map, then \((\overline{f}, p)\) is clearly an element of \( \text{kernel}(\psi) \). Conversely, if \((\alpha, \beta) \in \text{kernel}(\psi) \), then \( \beta = p \) and \( \alpha = \overline{g} \) for some \( g \in K \). There is thus a bijective correspondence between \( K \) and \( \text{kernel}(\psi) \). We define the group law on \( K \) such that this bijection is an isomorphism. \( \square \)

It would be desirable to have an “easier” description of \( K \). For this, let us look at the set

\[ \mathcal{D} = \{ f : N \to \text{Diffeo}(M) \}, \]

equipped with the following group law:

\[ f \ast g : N \to \text{Diffeo}(M), z \mapsto f(z) \circ g(z) \ \forall f, g \in \mathcal{D}. \]

It is clear that \( K \subset (\mathcal{D}, \ast) \) and that \( K \) contains those elements of \( \mathcal{D} \) that satisfy a certain differentiability condition: for a given \( f \in K \), the diffeomorphisms \( f(z) \) should change “smoothly in \( z \)” in order for the corresponding map \( \overline{f} \) to be differentiable. Recall that \( \text{Diffeo}(M) \) need not be a differentiable manifold, but that it does have the structure of a
Fréchet manifold. In fact, it is an open subset of the Fréchet manifold $C^\infty(M)$ of smooth self-maps of $M$ (see [11], [13]). We will show that

$$K = \{ f \in D \mid f \text{ is Fréchet } C^\infty \}.$$  

Let us start by fixing some notation. Take $g \in C^\infty(M)$. Consider the tangent bundle $\pi : TM \to M$ and denote its pullback under $g$ by $g^*(TM)$:

$$g^*(TM) = \{(y, \epsilon) \mid y \in M, \epsilon \in TM \text{ with } \pi(\epsilon) = g(y)\}.$$  

Given an open, relatively compact set $U \subset M$, we say that $TM|_U$ is trivial if $\overline{U}$ is contained in the image of a coordinate chart $x$. Then, there is local trivialization mapping $v \in T_yM, y \in \text{Im}(x)$ to $(y, b_1, b_2, \ldots, b_n) \in \text{Im}(x) \times \mathbb{R}^n$ where the real numbers $b_i$ are the coordinates of $v$ relative to the basis of $T_yM$ induced by $x$. Furthermore, we shall say that $g^*(TM)|_U$ is trivial if $x$ can be chosen such that $g(\text{Im}(x)) \subset \text{Im}(\overline{x})$ for some chart $\overline{x}$. Again, there is a local trivialization mapping $(y, \epsilon) \in g^*(TM)$ with $y \in \text{Im}(x)$ to $(y, c_1, c_2, \ldots, c_n) \in \text{Im}(x) \times \mathbb{R}^n$ where the $c_i$ are the coordinates of $\epsilon$ relative to the basis of $T_{g(y)}M$ induced by $\overline{x}$.

Cover $M$ by finitely many open sets $U_\alpha$, such that each $g^*(TM)|_{U_\alpha}$ is trivial. Denote the corresponding charts, analogously to $x$ and $\overline{x}$ above, by $x_\alpha$ and $\overline{x}_\alpha$. We call the finite set of triples

$$(U_\alpha, x_\alpha, \overline{x}_\alpha)$$  

a trivializing family for $g : M \to M$. By definition of trivializing family, the restriction to one of the $U_\alpha$ of a section $s : M \to g^*(TM)$, can be seen as a map $s_\alpha : U_\alpha \to U_\alpha \times \mathbb{R}^n$. The first component $U_\alpha \to U_\alpha$ is just the identity. Using $x_\alpha$, we denote the second component map $s_\alpha^\top : x_\alpha^{-1}(U_\alpha) \to \mathbb{R}^n$. By definition, we say that a sequence $(s_n)_{n \in \mathbb{N}}$ converges to $s$ in the Fréchet space of smooth sections of $g^*(TM)$ if

$$\lim_{n \to \infty} \frac{\partial s_n^\top}{\partial k_1 \partial k_2 \ldots \partial k_l} = \frac{\partial s^\top}{\partial k_1 \partial k_2 \ldots \partial k_l}$$  

uniformly over $x_\alpha^{-1}(U_\alpha)$ for all $(U_\alpha, x_\alpha, \overline{x}_\alpha)$, all $l \in \mathbb{N}$ and all $k_1, k_2, \ldots, k_l \in \mathbb{R}^n$.

**Proposition 3.2.2.** A map $f : N \to C^\infty(M)$ is Fréchet $C^\infty$ if and only if the corresponding map $\overline{f} : M \times N \to M, (m, n) \mapsto f(n)(m)$ is $C^\infty$.

**Proof.** Assume first that $f$ is Fréchet $C^\infty$. Then,

$$j : M \times N \to M \times C^\infty(M)$$  

$$(y, z) \mapsto (y, f(z))$$  

is Fréchet $C^\infty$. So, differentiability of $\overline{f}$ is implied by Fréchet differentiability of

$$i : M \times C^\infty(M) \to M$$  

$$(y, g) \mapsto g(y).$$  

Choose $(y, g) \in M \times C^\infty(M)$, fix a trivializing family for $g : M \to M$ and denote $S$ the Fréchet space of smooth sections of the pullback bundle $g^*(TM)$. Take an open neighbourhood $U = U_1 \times U_2 \ni (y, g)$ such that $(U_1, x_1, \overline{x}_1)$ is inside the chosen compactifying
family. Denote \( x_1^{-1}(U_1) = O, \) \( \text{Im}(\tilde{x}_1) = W, \) \( C^\infty(M) = \{ f : M \to M \mid f \) is \( C^\infty \} \) and let \( x_2 : O \to C^\infty(M), O \subset S \) be a chart with image \( U_2. \) Define
\[
x : O \times O \to M \times C^\infty(M) \quad (o, v) \mapsto (x_1(o), x_2(v)).
\]
Using the structure of \( C^\infty(M) \) as a Fréchet manifold, we can assume that \( i \circ x : (o, v) \mapsto \exp(\pi(v(x_1(o)))) \) with \( \pi \) the natural projection of \( g^*(TM) \) onto \( TM. \) Now, Fréchet differentiability of \( i \) on \( U_1 \times U_2 \) is equivalent with Fréchet differentiability of
\[
\tilde{i} := \tilde{x}_1^{-1} \circ i \circ x : O \times O \to \tilde{x}_1^{-1}(W) \quad (o, v) \to \tilde{x}_1^{-1}(\exp(\pi(v(x_1(o))))),
\]
on \( O \times O \) (where we can assume without loss of generality that \( U_2 \) is small enough for \( \tilde{i} \) to be defined). Since \( \exp : TM \to M \) is smooth, it suffices to prove that
\[
\gamma : O \times O \to TM|\text{Im}(\tilde{x}_1) \cong \text{Im}(\tilde{x}_1) \times \mathbb{R}^n \quad (o, v) \mapsto \pi \circ v(x_1(o))
\]
is Fréchet differentiable on \( O \times O. \) By differentiability of \( g, \) we only need to prove Fréchet differentiability for the second component map \( \gamma_2 : (o, v) \mapsto v^T(o). \) It is an easy exercise to prove by induction on \( l \) that the \( l^{th} \) differential \( D^l \gamma_2 \) exists and that it is given by
\[
D^l \gamma_2 : (O \times O) \times (\mathbb{R}^n \times S)^l \to \mathbb{R}^n \quad (o, s, k_1, h_1, k_2, h_2, \ldots, k_l, h_l) \mapsto \frac{\partial^l \gamma_2}{\partial k_1 \partial k_2 \ldots \partial k_l}(o) + \sum_{j=1}^l \frac{\partial^j \gamma_2}{\partial k_j \partial h_2 \ldots \partial k_l \partial h_l}(o).
\]
Continuity of the differentials of \( \gamma_2 \) then follows automatically and so we have proven the forward claim of the proposition.

To prove the converse, choose \( z \in N^k, \) denote \( f(z) = g \) and for some chart \( \bar{x} \) of \( N, \) let \( V \subset \text{Im}(\bar{x}) \) be a neighbourhood of \( z \) in \( N. \) Since \( M \) is compact, we can choose \( V \) such that the map
\[
v : M \times \bar{x}^{-1}(V) \to g^*(TM) \quad (y, w) \mapsto (y, (\exp_{g(y)})^{-1}(\bar{f}(y, \bar{x}(w))))
\]
is well-defined in the sense that for all \( (y, w) \in M \times \bar{x}^{-1}(V) \) there is a totally normal neighbourhood containing \( g(y) \) and \( \bar{f}(y, \bar{x}(w)). \) The differentiability of \( \bar{f} \) clearly implies that of \( v. \) It suffices to prove Fréchet differentiability of
\[
\bar{f} : \bar{x}^{-1}(V) \to S \quad w \mapsto v(z, w).
\]
Fix a compactifying family \((U_{\alpha}, x_{\alpha}, \bar{x}_\alpha)_{\alpha \in A} \) for \( g : M \to M \) where \( A \) is some index set. We claim that the \( l^{th} \) differential \( D^l(\bar{f}) \) maps \((w', h_1, h_2, \ldots, h_l) \in \bar{x}^{-1}(V) \times (\mathbb{R}^k)^l \) to the section \( s \) such that
\[
s_{\alpha} : x_{\alpha}^{-1}(U_{\alpha}) \to \mathbb{R}^n \quad o' \mapsto \frac{\partial x_{\alpha}(w')}{\partial h_1 \partial h_2 \ldots \partial h_l}(o', w'), \quad \forall \alpha \in A.
\]
The claim would imply continuity of $D f$. Further, in order for the $\overline{\sigma^\alpha}$ to determine a section, we need to show that for $\alpha, \beta \in \mathcal{A}, y \in U_\alpha \cap U_\beta$, the vector in $T_{g(y)}M$ with coordinates $\partial^n_{\partial h_1 \partial h_2 \ldots \partial h_l} (x_\alpha^{-1}(y), w')$ relative to the basis induced by $\tilde{x}_\alpha$ is the same as the vector with coordinates $\partial^n_{\partial h_1 \partial h_2 \ldots \partial h_l} (x_\beta^{-1}(y), w')$ relative to the basis induced by $\tilde{x}_\beta$. To this end, let $A$ be the change of base matrix from the basis of $T_{g(y)}M$ induced by $\tilde{x}_\alpha$ to the one induced by $\tilde{x}_\beta$. It is clear by definition that

$$A(v(\cdot, w^\alpha(x_\alpha^{-1}(y)))) = v(\cdot, w^\beta(x_\beta^{-1}(y))), \forall w \in \overline{x}^{-1}(V).$$

We obtain the desired equality since $D f$ only involves partial derivatives in the second coordinates of $v$.

It remains to prove the claim. By induction, assume the hypothesis is true for some natural number $l$, let us prove it for $l + 1$. We choose $(U_\alpha, x_\alpha, \tilde{x}_\alpha)$ inside our trivializing family. Let $j \in \mathbb{N}$ and $u_1, u_2, \ldots, u_j \in \mathbb{R}^n$. We need to prove that

$$\lim_{t \to 0} \frac{1}{t} \partial^l D^l(f)(w' + th_{l+1}, h_1, h_2, \ldots, h_{l})^\alpha - \partial^l D^l(f)(w', h_1, h_2, \ldots, h_{l})^\alpha$$

converges uniformly for $t \to 0$ over $x_\alpha^{-1}(U_\alpha)$ to

$$\partial^{l+1} v(\cdot, w)^\alpha(y)$$

uniformly over $x \in K$.

**Lemma 3.2.3.** Given $n, k, d \in \mathbb{N}, h \in \mathbb{R}^k$, a $C^1$-map $v : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^d$, and a compact subset $K \subset \mathbb{R}^n$, then

$$\lim_{t \to 0} \frac{1}{t} (v(x, y + th) - v(x, y)) = \frac{\partial v}{\partial h}(x, y)$$

uniformly over $x \in K$.

**Proof.** Without loss of generality we can assume that $d = 1$. Assume, by contradiction, that the convergence is not uniform over $K$. Then,

$$\exists \epsilon > 0 \forall N \in \mathbb{N} \exists t_N < \frac{1}{N} \exists x_N \in K \text{ such that } \left| \frac{1}{t_N}(v(x_N, y + t_N h) - v(x_N, y)) - \frac{\partial v}{\partial h}(x_N, y) \right| \geq \epsilon.$$

Consequently,

$$\exists \epsilon > 0 \forall N \in \mathbb{N} \exists t'_N < \frac{1}{N} \exists x_N \in K \text{ such that } \left| \frac{\partial v}{\partial h}(x_N, y + t'_N h) - \frac{\partial v}{\partial h}(x_N, y) \right| \geq \epsilon.$$

Since $K$ is compact, continuity of $\frac{\partial v}{\partial h}(x, y)$ gives us a contradiction. \qed

We obtain the following
Theorem 3.2.4. We have the following short exact sequence:
\[ 1 \to K \hookrightarrow \text{FVD}(M \times N) \xrightarrow{\psi} \text{Diffeo}(N) \to 1 \]
with \( \psi \) as in corollary 3.2.1, \( K \cong \{ f : N \to \text{Diffeo}(M) \mid f \text{ is Fréchet differentiable} \} \).

4. Properly discontinuous actions

4.1. The Bieberbach theorems. A group \( \Gamma \) acts properly discontinuously on a space \( X \) if the set
\[ \{ \gamma \in \Gamma \mid \gamma K \cap K \neq \phi \} \]
is finite for any compact \( K \subset X \). A \( k \)-dimensional crystallographic group is a group acting isometrically, properly discontinuously and cocompactly on \( \mathbb{R}^k \). Its structure and some of its properties are described by the three famous Bieberbach theorems (see [4], [5], [9]). Let us recall what they are.

Bieberbach 4.1.1. Let \( \Gamma \subset \mathbb{R}^k \rtimes O(k) = \text{Iso}(\mathbb{R}^k) \) be a \( k \)-dimensional crystallographic group. Then \( \Gamma \) contains a finite index subgroup \( \Gamma^* = \Gamma \cap \mathbb{R}^k \) which is a uniform lattice, i.e. a discrete cocompact subgroup of \( \mathbb{R}^k \).

Bieberbach 4.1.2. Let \( \Gamma_1, \Gamma_2 \subset \mathbb{R}^k \rtimes O(k) \) be two \( k \)-dimensional crystallographic groups. If \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic, then they are conjugated by an element of \( \text{Aff}(\mathbb{R}^k) = \mathbb{R}^k \rtimes \text{GL}(k, \mathbb{R}) \).

Bieberbach 4.1.3. Up to isomorphism, there are only finitely many \( k \)-dimensional crystallographic groups.

All three Bieberbach theorems have been generalized to the case of almost-crystallographic groups.

Definition 4.1.4. An almost-crystallographic group is a group that acts properly discontinuously, cocompactly and isometrically on a simply connected, connected, nilpotent Lie group \( N \) that is equipped with a left-invariant metric.

The left-invariant metric on \( N \) is determined by the choice of an inner product on the Lie algebra \( \eta \) of \( N \). Then, \( \text{Iso}(N) = N \rtimes C \) where \( C \) is the group of automorphisms of \( N \) whose differential at the identity preserves the chosen inner product on \( \eta \) (see [17]).

In 1960, Auslander generalized the first Bieberbach theorem to almost-crystallographic groups.

Bieberbach 4.1.5 ((Generalization first Bieberbach theorem, Auslander, [2])). Let \( \Gamma \subset N \rtimes C \) be an almost-crystallographic group. Then \( \Gamma \) contains a finite index subgroup \( \Gamma^* = \Gamma \cap N \) which is a uniform lattice of \( N \).

It turns out that the first Bieberbach theorem can be generalized in our setting.

Theorem 4.1.6. Let \( M \) be a closed connected Riemannian manifold and let \( N \) be a simply connected, connected, nilpotent Lie group equipped with a left invariant metric. If \( \Gamma \) is a group acting properly discontinuously, cocompactly and isometrically on \( M \times N \), then \( \Gamma \) contains a finite index subgroup isomorphic to a uniform lattice of \( N \).
Proof. Since $N$ is contractible, we have that $(M,N)$ satisfies the $*$-condition. Corollary 3.1.7 thus implies that $\text{Iso}(M \times N) = \text{Iso}(M) \times \text{Iso}(N)$. Denote $$\psi : \text{Iso}(M \times N) \to \text{Iso}(N)$$ the canonical projection. Let $\overline{\Gamma} = \psi(\Gamma)$ and let $\Gamma_1$ be the kernel of $\psi|_\Gamma$. We obtain the following short exact sequence:

$$1 \to \Gamma_1 \to \Gamma \to \overline{\Gamma} \to 1.$$  

Since $\Gamma$ acts properly discontinuously and since $\Gamma_1 \subset \Gamma$ maps $M \times \{1\}$ to itself, we have that $\Gamma_1$ is finite. Clearly, $\Gamma$ is an almost-crystallographic group. Theorem 4.1.5 then shows that $\Gamma$ contains a finite index subgroup isomorphic to a uniform lattice of $N$. It is thus virtually-(finitely generated and nilpotent). Hence, it is poly-(cyclic or finite). In total, we have that $\Gamma$ is poly-(cyclic or finite) and therefore poly-$\mathbb{Z}$-by-finite. We obtain the following short exact sequence:

$$1 \to P\mathbb{Z} \to \Gamma \to F \to 1,$$

where $P\mathbb{Z}$ is a poly-$\mathbb{Z}$ group and $F$ is a finite group.

The restriction of $\psi$ to the $P\mathbb{Z}$-subgroup is injective since poly-$\mathbb{Z}$-groups are torsion-free. Then, $P\mathbb{Z}$ is isomorphic to a finite index subgroup of the almost-crystallographic group $\overline{\Gamma}$. Thus, it is itself an almost-crystallographic group with a finite index subgroup isomorphic to a uniform lattice of $N$. We conclude that $\Gamma$ contains a finite index subgroup isomorphic to a uniform lattice of $N$. □

Remark 4.1.7. The main tool in proving Theorem 4.1.6 is Corollary 3.1.7. Since $N$ is locally compact with a left-invariant metric, it is complete. Then, de Rham decomposition implies that $\text{Iso}(M \times N) = \text{Iso}(M) \times \text{Iso}(N)$ and therefore gives an alternate proof.

We recall that two isomorphic groups of isometries, acting freely, properly discontinuously and cocompactly on $\mathbb{R}$, are conjugated by an element of $\text{Aff}(\mathbb{R}) = \mathbb{R} \rtimes \text{GL}_1(\mathbb{R})$. It is also true that two finite isomorphic groups acting freely and isometrically on $S^1$ are equal. The following example implies that there is no similar rigidity for $S^1 \times \mathbb{R}$. More concretely, we find two isomorphic groups acting properly discontinuously, cocompactly and isometrically on $S^1 \times \mathbb{R}$ such that the induced actions on $S^1$ and $\mathbb{R}$ are free, but these groups cannot be conjugated by an element of $\text{Diffeo}(S^1) \times \text{Diffeo}(\mathbb{R})$.

Example 4.1.8. Consider $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. Choose $\theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q}$ such that $\theta_1 \pm \theta_2 \notin \mathbb{Z}$. Let $\Gamma \subset \text{Iso}(M \times N)$ be the group generated by $(\alpha_1, \alpha_2)$ where $\alpha_1 : S^1 \to S^1$ is multiplication by $e^{2\pi i \theta_1}$ and $\alpha_2 : \mathbb{R} \to \mathbb{R}, x \mapsto x + 1$. Analogously, let $\overline{\Gamma}$ be the group generated by $(\beta_1, \beta_2)$ where $\beta_1 : S^1 \to S^1$ is multiplication by $e^{2\pi i \theta_2}$ and where $\beta_2 = \alpha_2$. Clearly, both groups are infinite cyclic and they act isometrically, properly discontinuously and cocompactly on $S^1 \times \mathbb{R}$. Also, the induced actions on $S^1$ and $\mathbb{R}$ are free. However, with little effort one can show that $\langle \alpha_1 \rangle$ and $\langle \beta_1 \rangle$ are not conjugated by a diffeomorphism of $S^1$.

The third Bieberbach theorem does not generalize either. There are infinitely many non-isomorphic groups acting isometrically, properly discontinuously and cocompactly on $S^1 \times \{1\}$. 
4.2. Talelli’s Conjecture. Let us begin by recalling the definition of cohomological dimension.

**Definition 4.2.1.** The cohomological dimension of a group $\Gamma$ is defined by

$$cd(\Gamma) = \sup\{n \mid H^n(\Gamma; M) \neq 0 \text{ for some } \mathbb{Z}\Gamma\text{-module } M\}.$$ 

There are two definitions in literature for periodic cohomology of a group. We use the following

**Definition 4.2.2.** A group $\Gamma$ has periodic cohomology after $k$ steps if there exists an integer $q > 0$ such that $H^i(\Gamma, -)$ and $H^{i+q}(\Gamma, -)$ are naturally isomorphic functors for all $i > k$.

In 2005, Talelli stated the following (Conjecture III of [15]):

**Conjecture 4.2.3 ((Talelli, 2005)).** A torsion-free group $\Gamma$ that has periodic cohomology after some steps has finite cohomological dimension.

By a result of Mislin and Talelli ([16]) we know that this conjecture holds for the large class of $LHF$-groups (see [10]). Among others, this class contains all linear and all elementary amenable groups.

In 2001, Adem and Smith have proven that a countable group acts freely, properly discontinuously and smoothly on some $S^n \times \mathbb{R}^k$ if and only if it has periodic cohomology. Actually, they use the other definition of periodic cohomology which states that the isomorphisms of cohomological functors are induced by a cup product map (see [1] for more details). For the large class of $HF$-groups it is known that these definitions are equivalent. Furthermore, it has been conjectured by Talelli that they are equivalent for all groups. The Adem-Smith Theorem suggests the following slightly weaker reformulation of the Talelli conjecture.

**Conjecture 4.2.4 ((Talelli reformulated, 2005)).** If $\Gamma$ is a torsion-free group that acts smoothly and properly discontinuously on $S^n \times \mathbb{R}^k$, then $cd(\Gamma) \leq k$.

Now, let us replace $S^n$ by any closed, connected Riemannian manifold $M$ and replace $\mathbb{R}^k$ by any $k$-dimensional contractible Riemannian manifold $N$. We obtain the following generalization.

**Conjecture 4.2.5 ((Petrosyan, 2007)).** If $\Gamma$ is a torsion-free group acting smoothly and properly discontinuously on $M \times N$, then $cd(\Gamma) \leq \dim(N)$.

In [14], Petrosyan has verified this conjecture in the case of $HF$-group and when $N$ is 1-dimensional. We prove the following

**Theorem 4.2.6.** Let $\Gamma$ be a torsion-free group that acts properly discontinuously on $M \times N$ where $M$ is closed and connected and where $N$ is contractible. If each $\gamma \in \Gamma$ acts as a fiberwise volume decreasing map, then $\Gamma$ acts freely and properly discontinuously on $N$. In particular, $cd(\Gamma) \leq \dim(N)$. 

Proof. Let \( y_0 \in M \) and consider the map
\[
\psi : \text{FVD}(M \times N) \rightarrow \text{Diffeo}(N)
\]
where
\[
\tilde{\beta} : N \rightarrow N
z \mapsto \beta(y_0, z).
\]
By Corollary 3.2.1, we have that \( \psi \) is a well-defined epimorphism.

This gives us the following short exact sequence
\[
1 \rightarrow \Gamma_1 \rightarrow \Gamma \rightarrow \overline{\Gamma} \rightarrow 1,
\]
where \( \overline{\Gamma} = \psi(\Gamma) \) and \( \Gamma_1 \) is the kernel of \( \psi|_\Gamma \). Let \( z \in N \) and observe that every element of \( \Gamma_1 \) maps \( M \times \{z\} \) onto itself. Since \( \Gamma \) acts properly discontinuously on \( M \times N \) we have that \( \Gamma_1 \) is finite. Since \( \Gamma \) is torsion-free, \( \Gamma_1 \) must be trivial and therefore, \( \Gamma \cong \overline{\Gamma} \).

Now, \( \overline{\Gamma} \) acts freely, smoothly and properly discontinuously on \( N \). Since \( N \) is contractible, we have \( \text{cd}(\Gamma) \leq \dim(N) \).

\[\square\]

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