A Neumann problem for a diffusion equation with n-dimensional fractional Laplacian

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Abstract
We study an initial-boundary value problem for a n-dimensional stochastic diffusion equation with fractional Laplacian on \( \mathbb{R}^n \). In order to prove existence and uniqueness, we generalize the Fokas method to construct the Green function for the associated linear problem and then we apply a fixed point argument. Also, we present an example where the explicit solutions are given.

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1 Introduction
The classical diffusion phenomenon is governed by a second order linear partial differential equation, whose Green function is given by a Gaussian probability density function and which describes the movement of energy through a medium in response to a gradient of energy. On the other hand, the diffusion processes in various systems with complex structure, such as liquid crystals, glasses, polymers, biopolymers, and proteins, usually do not follow a Gaussian density, as a consequence the phenomenon is described by a fractional partial differential equation [7]. Dipierro et al., [4] have studied the asymptotic behavior of the solutions of the time-fractional diffusion equation.

There is some previous work for the initial-boundary value problem on the first quadrant \( \mathbb{R}^2_+ \) for fractional diffusion equations, where the Green function has been constructed and an integral representation of the solution was found [3, 6]. In this note, we consider the equation

\[
    u_t = \Delta^\alpha u, \tag{1}
\]

where the operator \( \Delta^\alpha \) is defined via the Riesz fractional derivative, for each coordinate. Let us notice that the generalization of the Laplacian most commonly used [1, 9] is different from the one we use in this work.

However, Eq. (1) is an idealized version because many aspects are missing in the modeling; such as the inhomogeneity of the medium, external sources, and measurement errors.
Then a more realistic version is obtained by considering a stochastic version with additive noise. For example, Balanzario and Kaikina [2] studied the stochastic nonlinear Landau–Ginzburg equations on the half-line with Dirichlet white-noise boundary conditions, Shi and Wang [11] studied the solution for a stochastic fractional partial differential equation driven by an additive fractional space–time white noise. In Sanchez et al. [10], studied the stochastic version of (1) for the 2-dimensional case; however, the \( n \)-dimensional case on \( \mathbb{R}^n_+ := \{ x = (x_1, \ldots, x_n) : x_j \geq 0, j = 1, \ldots, n \} \) has not been studied. In the present work we tackle this problem via the main ideas of the Fokas method (unified transform) [5], this method is a technique for solving initial-boundary value problems for partial differential equations. Moreover, it generates integral representation formulas for solutions, where the integrals converge uniformly on the boundary.

2 Preliminaries

Let us give some known definitions and results.

**Definition 1** The \( n \)-dimensional Fourier–Laplacian transform is defined as follows:

\[
\hat{u}(k, t) = \int_{\mathbb{R}^n} e^{-ik \cdot x} u(x, t) \, dx,
\]

where \( x \in \mathbb{R}^n, k \in \mathbb{C}^n = \{ k = (k_1, \ldots, k_n) : k_j \in \mathbb{C}, j = 1, \ldots, n \} \) and \( \Im(k_j) \leq 0 \), \( k \cdot x \) is the usual inner product, and its inverse is defined by

\[
u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ik \cdot \hat{u}(k, t)} \, dk.
\]

**Definition 2** The Riesz fractional operator is defined by

\[
D^\alpha_{x_j} u(x, t) = -\frac{1}{2\Gamma(3-\alpha) \cos(\frac{\pi}{2} \alpha)} \int_{0}^{\infty} \frac{\text{sgn}(x_j - y_j)}{|x_j - y_j|^\alpha} \partial_j^2 u(x_j, t) \, dy_j.
\]

Here, \( \alpha \in (2, 3), x_j \in \mathbb{R}^n_+ \) is the vector \( x \), where the \( j \)th coordinate is \( y_j, j = 1, \ldots, n \).

Note that the operator, using integration by parts, \( D^\alpha_{x_j} \) can be represented in the following form [8]:

\[
(-\Delta)^\alpha_{x_j} u(x, t) = \frac{\alpha}{2\Gamma(1-\alpha) \cos(\frac{\pi}{2} \alpha)} \int_{0}^{\infty} \frac{u(x_j, t) - u(x, t)}{|x_j - y_j|^{1+\alpha}} \, dy_j.
\]

**Lemma 1** If \( \Delta^\alpha, \alpha \in (2, 3), \) is the fractional \( n \)-dimensional Laplace operator

\[
\Delta^\alpha = D^\alpha_{x_1} + D^\alpha_{x_2} + \cdots + D^\alpha_{x_n},
\]

then, for \( \Im(k_j) \leq 0 \),

\[
\widehat{\Delta^\alpha u}(k) = |k|^\alpha \hat{u}(k, t) = \sum_{l=1}^{n} \sum_{j=0}^{2} \frac{|k|^\alpha}{|ik_j|^{1+\alpha}} \partial^\alpha_{x_j} \hat{u}(k_{[-l]}, t).
\]

Here, \( |k|^\alpha := \sum_{l=1}^{n} |k_l|^\alpha \) and \( k_{[-l]} \in \mathbb{C}^n \) is the \( k \) vector, where its \( l \)th coordinate is zero.
Proof. The theorem follows from the linearity of the operator $\Delta^\alpha$ and the well-known equation

$$\widehat{D^\alpha_t}u(k) = |k|^\alpha \widehat{u}(k,t) - \sum_{j=0}^{3} \frac{|k|^\alpha}{(ik)^{j+1}} \frac{\partial^j_x \widehat{u}(0,t)}{i^j}.$$ 

\square

3 Green function

We consider a linear problem for an evolution equation with initial condition $u_0$ and boundary conditions $h_j$, $j = 1, \ldots, n$,

$$\begin{cases}
u_t = \Delta^\alpha u, \\
u(x,0) = u_0(x), \\
u_x(x_{[-j]}, t) = h_j(x_{[-j]}, t),
\end{cases}$$ (2)

where $\alpha \in (2,3)$, $t > 0$, $x_{[-j]} \in \mathbb{R}_n^+$ means that the $j$th coordinate of $x$ is zero, with the compatibility conditions $h_j(x_{[-j]}, t) = h_l(x_{[-l]}, t)$ where $x_{[-j, l]} \in \mathbb{R}_n^+$ is such that $j$th and $l$th coordinates, $x_l$ and $x_j$, are equal to zero for $j \neq l$.

Theorem 1. Let the initial data $u_0(x) \in L^1(\mathbb{R}_n^+)$ and the boundary data $h_j(x_{[-j]}, t) \in C(\mathbb{R}_+; L^1(\mathbb{R}_n^+))$. Suppose that there exists some function $u(x, t)$, which satisfies (2). Then $u(x, t)$ has the following integral representation:

$$u(x, t) = G^l(t)u_0 - \sum_{l=1}^{n} \int_0^t G^l(t-s) h_l ds,$$

where the Green operators are given by

$$G^l(t)u_0 = \int_{\mathbb{R}_n^+} G^l(x, y, t) u_0(y) dy,$$

$$G^l(t)h_l = \int_{\mathbb{R}_n^+} G^l(x, y_{[-l]}, t) h_l(y_{[-l]}, s) dy_{[-l]},$$ (3)

and the Green functions are

$$G^l(x, y, t) = \frac{2^n}{\pi^n} \int_{\mathbb{R}_n^+} e^{-k^\alpha \tau} \prod_{l=1}^{n} \cos[k_i x_l] \cos[k_j y_l] \, dk,$$

$$G^l(x, y, t) = \frac{2^n}{\pi^n} \int_{\mathbb{R}_n^+} e^{-k^\alpha \tau} k_i^{\alpha - 2} \cos[k_i x_l] \prod_{m=1}^{n} \cos[k_m x_m] \cos[k_m y_m] \, dk.$$

Here, $k^\alpha = \sum_{l=1}^{n} k_l^\alpha$.

Proof. Applying Theorem 1 to Eq. (2), we obtain

$$\widehat{\nu}_l(k, t) + |k|^\alpha \widehat{u}(k, t) = \sum_{l=1}^{n} \sum_{j=0}^{3} \frac{|k|^\alpha}{(ik)^{j+1}} \hat{\partial}^j_x \widehat{u}(l_{[-j]}, t).$$
Now, we multiply the above equation by $e^{ik|x|t}$ and integrate from 0 to $t$,

$$e^{ik|x|t} \tilde{u}(k, t) - \tilde{u}_0(k) = \sum_{l=1}^{n} \sum_{j=0}^{2} \frac{|k_j|^2}{(ik_j)^{n+1}} g_f^l ([k]^a, [k_{-l}], t)$$

(4)

for $\Im m(k_j) \leq 0$, where

$$g_f^l (\sigma, [k_{-l}], t) = \int_0^t e^{is\sigma} \partial_s g_f^l (\sigma, [k_{-l}], s) \, ds.$$

Now, we initially consider 2-dimensional case. Thus, Eq. (4) is expressed as

$$e^{ik|x|t} \tilde{u}(k, t) - \tilde{u}_0(k) = \sum_{j=0}^{2} \frac{|k_j|^a}{(ik_j)^{n+1}} g_f^1 ([k]^a, [k_{-j}], t)$$

$$e^{ik|x|t} \tilde{u}(k, t) - \tilde{u}_0(k) = \sum_{j=0}^{2} \frac{|k_j|^a}{(ik_j)^{n+1}} g_f^1 ([k]^a, [k_{-j}], t)$$

(5)

Applying the inverse transform in (5) with respect to $k_1$ and moving the contour of integration for the terms with $g_f^1$ in the integrand, we obtain

$$\tilde{u}(x_1, x_2, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik_1 x_1 - |k|^2 t} \left[ \tilde{u}_0(k) + \sum_{j=0}^{2} \frac{|k_j|^a}{(ik_j)^{n+1}} g_f^1 ([k]^a, [k_{-j}], t) \right] \, dk_1$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik_1 x_1 - |k|^2 t} \sum_{j=0}^{2} \frac{|k_j|^a}{(ik_j)^{n+1}} g_f^1 ([k]^a, [k_{-j}], t) \, dk_1,$$

(6)

where $D_1 = \{ k_1 \in \mathbb{C} : 0 \leq \Im m(k_1) \leq \frac{1}{2\pi} \theta \}$. Let us note the following: if we substitute $k_1$ by $-k_1$, the functions $g_f^1$ from Eq. (5) are invariant. Then, making this change of variables in (5), we get

$$e^{ik|x|t} \tilde{u}(-k_1, x_2, t) - \tilde{u}_0(-k_1, x_2) = \sum_{j=0}^{2} \frac{|k_j|^a}{(-ik_j)^{n+1}} g_f^1 ([k]^a, [k_{-j}], t)$$

$$+ \sum_{j=0}^{2} \frac{|k_j|^a}{(ik_j)^{n+1}} g_f^1 ([k]^a, [-k_{-j}], t),$$

(7)

for $\Im m(-k_1), \Im m(k_2) \leq 0$. Substituting $g_f^1$ from Eq. (7) in (6) and using the fact that

$$\int_{\mathbb{R}} e^{ik_1} \tilde{u}(-k_1, x_2, t) \, dk_1 = 0,$$

by the Cauchy theorem, we obtain the following integral representation:

$$\tilde{u}(x_1, x_2, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik_1 x_1 - |k|^2 t} \left[ \tilde{u}_0(k) + \tilde{u}_0(-k_1, x_2) - \frac{2|k_1|^a}{k_1^2} g_f^1 ([k]^a, [k_{-1}], t) \right.$$

$$+ \sum_{j=0}^{2} \frac{|k_j|^a}{(ik_j)^{n+1}} \left[ g_f^1 ([k]^a, [k_{-j}], t) + g_f^2 ([k]^a, [-k_{-j}], t) \right] \, dk_1.$$  

(8)
Applying the inverse transform in (8) with respect to \( k_2 \) and moving the contour of integration for the terms with \( g_2^2 \) in the integrand, we obtain

\[
u(x, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ik \cdot x} [\tilde{u}_0(k) + \tilde{u}_0(-k_1, k_2)]
- \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ik \cdot x} \frac{|k_1|^\alpha}{k_1^2} \mathcal{F}\{(|k|^\alpha, k_{[-1]}), t\} dk
+ \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{D^2_2} e^{ik \cdot x} \frac{2}{k_1^2} \mathcal{F}\{(|k|^\alpha, k_{[-1]}), t\}
\times \left[ \mathcal{F}\{(|k|^\alpha, k_{[-2]}), t\} + g_2^2 (|k|^\alpha, -k_{[-2]}, t) \right] dk,
\]

(9)

where \( D^2_2 = \{k_2 \in \mathbb{C} : 0 \leq \Im(k_2) \leq \frac{\pi}{2}\} \). Let us note the following: if we substitute \( k_2 \) by \(-k_2\), the functions \( g_2^2 \) from Eq. (8) are invariant. Then, making this change of variables in (7), we get

\[
\hat{u}(x_1, -k_2, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx_1 \cdot |k|^\alpha \tau} [\tilde{u}_0(k_1, -k_2) + \tilde{u}_0(-k) - \frac{2|k_1|^\alpha}{k_1^2} \mathcal{F}\{(|k|^\alpha, -k_{[-1]}), t\}
+ \sum_{j=0}^2 \frac{|k_2|^\alpha}{(-ik_{[-2]})^j+1} \mathcal{F}\{(|k|^\alpha, k_{[-2]}), t\} + g_2^2 (|k|^\alpha, -k_{[-2]}, t) \}] dk_1,
\]

(10)

for \( \Im(k_1), \Im(k_2) \geq 0 \). Substituting \( g_2^2 (|k|^\alpha, \pm k_{[-2]}, t) \) from Eq. (10) in (9) and using the fact that

\[
\int_{D^2_2} e^{ikx_2} \hat{u}(x_1, -k_2, t) dk_2 = 0,
\]

by the Cauchy theorem, we obtain the following integral representation:

\[
u = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ik \cdot x} \left[ \sum_{r \in S_2} \tilde{u}_0(r) - 2 \sum_{l=1}^2 \sum_{r_{[-l]} \in S_2} \frac{|k_l|^\alpha}{k_l^2} \mathcal{F}\{(|k|^\alpha, r_{[-l]}), t\} \right] dk,
\]

(11)

where \( r \in S_2 = \{(\pm k_1, \pm k_2)\} \) and \( r_{[-l]} \) is such that the \( l \)th coordinate is equal to zero. In Eq. (11) we have, after interchanging the integration order, integrals of the form

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{ik_1x_1 + y_{21} x_2 + y_{22} x_3} u_0(y) dy dk,
\]

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{ik_1x_1 + y_{21} x_2 + y_{22} x_3} \frac{|k_1|^\alpha}{k_1^2} u_{x_1}(0, \pm y_{21}, s) dy_{21} ds dk,
\]

and

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{ik_1x_1 + y_{21} x_2 + y_{22} x_3} \frac{|k_2|^\alpha}{k_2^2} u_{x_2}(\pm y_{12}, 0, s) dy_{12} ds dk.
\]
We notice that all the integrals above are absolutely integrable, then using the Fubini theorem, after some simplifications, we arrive from Eq. (11) at the following equation:

\[ u(x, t) = G^I(t)u_0 - \sum_{l=1}^{2} \int_0^t G^{Bi}(t - s)h_l ds, \]

where the Green operators are given by

\[ G^I(t)u_0 = \int_{\mathbb{R}_2^+} e^{-k^\alpha \cdot x} \prod_{l=1}^{2} \cos[k_l y_l] d\mathbf{k}, \]

\[ G^{Bl}(t)h_l = \int_{\mathbb{R}_2^+} e^{-k^\alpha \cdot x} \cos[k_l y_l]k_0^{\alpha - 2} \prod_{m=1, m \neq l}^{2} \cos[k_m x_m] d\mathbf{k}, \]

and the Green functions are

\[ G^I(x, y, \tau) = \left( \frac{2}{\pi} \right)^2 \int_{\mathbb{R}_2^+} e^{-k^\alpha \cdot x} \prod_{l=1}^{2} \cos[k_l y_l] d\mathbf{k}, \]

\[ G^{Bl}(x, y_{[-l]}, \tau) = \left( \frac{2}{\pi} \right)^2 \int_{\mathbb{R}_2^+} e^{-k^\alpha \cdot x} \cos[k_l y_l]k_0^{\alpha - 2} \prod_{m=1, m \neq l}^{2} \cos[k_m x_m] d\mathbf{k}, \]

where \( k^\alpha = k_1^\alpha + k_2^\alpha \). Now, following the previous arguments we can tackle the \( n \)-dimensional case. This can be achieved, via mathematical induction over \( n \), passing from Eq. (4) to Eq. (12), through the steps that we describe in the 2-dimensional case. Analogous to Eq. (11), we obtain an integral representation for \( u \),

\[ u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_n^+} e^{i\mathbf{k} \cdot x - |k|^2t} \left[ \sum_{r \in S_n} \check{u}_0(r) + 2 \sum_{l=1}^{n} \sum_{r_{[-l]} \in S_n} \frac{|k_l|^2}{k_l^2} g^l(|k|^2, r_{[-l]}, t) \right] d\mathbf{k}, \]

(12)

where \( r \in S_n = \{ (\pm k_1, \pm k_2, \ldots, \pm k_n) \} \) and \( r_{[-l]} \) is such that the \( l \)th coordinate is equal to zero. Interchanging the integrals in the above equation, by Fubini’s theorem, we obtain the desired result.

4 Stochastic nonlinear problem

In order to state the problem, we define the Brownian sheet \( \dot{B} \) on \( \mathbb{R}_n^+ \times [0, T] \) on a complete probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \), where \( \mathcal{F} \) is a \( \sigma \)-algebra, \( \{ \mathcal{F}_t \}_{t \geq 0} \) is a right-continuous filtration on \( (\Omega, \mathcal{F}) \) such that \( \mathcal{F}_0 \) contains all \( P \)-negligible subsets and \( P \) is a probability measure. We consider a center Gaussian field \( B = \{ B(x, t) \mid x \geq 0, t \geq 0 \} \) with covariance function given by

\[ K((x, t), (y, s)) = \min(t, s) \text{diag}(\min(x, y), \ldots, \min(x_n, y_n)). \]

We suppose that \( B \) generates a \( (\mathcal{F}_t, t \geq 0) \)-martingale measure in the sense of Walsh [12]. Let the initial condition \( u_0 \) be \( \mathcal{F}_0 \times B(\mathbb{R}_n^+) \) measurable, where \( B(\mathbb{R}_n^+) \) is the Borelian \( \sigma \)-algebra over \( \mathbb{R}_n^+ \).
Now, we consider the following initial-boundary value problem for a nonlinear equation:

\[
\begin{aligned}
    u_t - \Delta u &= Nu + B, \\
    u(x, 0) &= u_0(x), \\
    u_x(x_{[-j]}, t) &= h_j(x_{[-j]}, t),
\end{aligned}
\]  

where \( x \in \mathbb{R}_+^n, t > 0, \alpha \in (2, 3), N \) is a Lipschitzian operator; i.e., \(|Nu - Nv| \leq C|u - v|, C > 0\), and the compatibility conditions \( h_j(x_{[-j]}, t) = h_j(x_{[-j]}, t) \) are satisfied. We understand the solutions for the problem (13) in the following sense: \( u \) is a solution if, for all \( x \in \mathbb{R}_+^n \) and \( t > 0 \), the following equation is fulfilled:

\[
\begin{aligned}
u(x, t) &= G^l(t)u_0 + \sum_{l=1}^{n} \int_0^t G^{B_l}(t - s)h_l ds \\
&+ \int_0^t \int_{\mathbb{R}_+^n} G(x - y, t - s)Nv(y, s) dy ds \\
&+ \int_0^t \int_{\mathbb{R}_+^n} G(x - y, t - s)dB(y, s),
\end{aligned}
\]  

where the Green operators \( G^l(t), G^{B_l}(t) \) are given in Eq. (3) and the Green function is

\[
G(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_+^n} e^{k \cdot x - |k|^\alpha t} dk.
\]  

**Theorem 2** Let the initial data \( u_0(x) \in L^1(\mathbb{R}_+^n) \) and the boundary data \( h_j(x_{[-j]}, t) \in C(\mathbb{R}_+; L^1(\mathbb{R}_+^n)) \). Suppose that, for each \( T > 0 \), there exists a constant \( C > 0 \) such that, for each \( x \in \mathbb{R}_+^n, t \in [0, T] \) and \( u, v \in \mathbb{R}_+^n, |Nu - Nv| \leq C|u - v|, \) and for some \( p \geq 1 \),

\[
\sup_{x \geq 0} \mathbb{E}(|u_0(x)|^p) < \infty.
\]  

Then, there exists a unique solution \( u(x, t) \) to Eq. (13). Moreover, for all \( T > 0 \) and \( p \geq 1 \),

\[
\sup_{x \geq 0, t \in [0, T]} \mathbb{E}(|u(x, t)|^p) < \infty.
\]

**Proof** First, we define a Picard succession:

\[
\begin{aligned}
u^{n+1}(x, t) &= u^0(x, t) + \sum_{l=1}^{n} \int_0^t \int_{\mathbb{R}_+^n} G^{B_l}(x, y_{[-j]}, t - s)h_l(y_{[-j]}, s) dy_{[-j]} ds \\
&+ \int_0^t \int_{\mathbb{R}_+^n} G(x - y, t - s)Nu^n(y, s) dy ds \\
&+ \int_0^t \int_{\mathbb{R}_+^n} G(x - y, t - s)dB(y, s)
\end{aligned}
\]  

where

\[
u^0(x, t) = \int_{\mathbb{R}_+^n} G^l(x, y, t)u_0(y) dy.
\]
Now, let us prove that \( \{u^n(x,t)\}_{n \geq 0} \) converges in \( L^p(\Omega) \). Using the fact that, for all \( t \geq 0 \), \( G(x,t) \) from Eq. (15) is a probability density function with respect to \( x \), we obtain, for \( n \geq 2 \),

\[
\mathbb{E}(\|u^{n+1}(x,t) - u^n(x,t)\|^p) \\
= \mathbb{E}\left( \left\| \int_0^t \int_{\mathbb{R}^n_*} G(x-y,t-s)[\mathcal{N}u^n(y,s) - \mathcal{N}u^{n-1}(y,s)] \, dy \, ds \right\|^p \right) \\
\leq C(p) \int_0^t \int_{\mathbb{R}^n_*} G(x-y,t-s) \mathbb{E}(\|u^n(y,s) - u^{n-1}(y,s)\|^p) \, dy \, ds \\
\leq C(p) \int_0^t \sup_{y \geq 0} \mathbb{E}(\|u^n(y,s) - u^{n-1}(y,s)\|^p) \, ds
\]

and by (16) and Burkholder’s inequality we have

\[
\sup_{x \geq 0} \mathbb{E}(\|u^1(x,t) - u^0(x,t)\|^p) \\
\leq C(p) \left( \sup_{x \geq 0} \mathbb{E}(\|u^1(x,t)\|^p) + \sup_{x \geq 0} \mathbb{E}(\|u^0(x,t)\|^p) \right) < \infty.
\]

Then, by Gronwall’s lemma we obtain

\[
\sum_{n \geq 0} \sup_{t \in [0,T]} \mathbb{E}(\|u^n(x,t) - u^{n-1}(x,t)\|^p) < \infty.
\]

Hence, \( \{u^n(x,t)\}_{n \geq 0} \) is a Cauchy sequence in \( L^p(\Omega) \). Let

\[
u(x,t) = \lim_{n \to \infty} u^n(x,t).
\]

Thus,

\[
\sup_{x \geq 0} \mathbb{E}(\|\nu(x,t)\|^p) < \infty.
\]

Taking \( n \to \infty \) in \( L^p(\Omega) \) at both sides of (17) shows that \( \nu(x,t) \) satisfies the problem (2).

Finally, we have to prove the uniqueness of the solution. Let \( u \) and \( v \) be the two solutions of problem (2), then

\[
\mathbb{E}(\|u(x,t) - v(x,t)\|^p) \\
= \mathbb{E}\left( \left\| \int_0^t \int_{\mathbb{R}^n_*} G(x-y,t-s)[\mathcal{N}u(y,s) - \mathcal{N}v(y,s)] \, dy \, ds \right\|^p \right) \\
\leq C(p) \int_0^t \int_{\mathbb{R}^n_*} G(x-y,t-s) \mathbb{E}(\|u(y,s) - v(y,s)\|^p) \, dy \, ds \\
\leq C(p) \int_0^t \sup_{y \geq 0} \mathbb{E}(\|u(y,s) - v(y,s)\|^p) \, ds.
\]

Therefore, Gronwall’s lemma yields

\[
\mathbb{E}(\|u(x,t) - v(x,t)\|^p) = 0.
\]
Figure 1: Anomalous diffusion for $\alpha = 2.5$
5 Example

In this section, we consider an example for the case $n = 2$, with the initial condition

$$u_0(x_1, x_2) = \begin{cases} 
1, & 1 \leq x_1, x_2 \leq 2, \\
0, & \text{in the other case,}
\end{cases}$$

and the boundary conditions, for $l = 1, 2$,

$$h_l(x_{|l|}, t) = \begin{cases} 
(-1)^{l+1}, & 3/4 \leq x_{|l|} \leq 5/4, \\
0, & \text{in the other case.}
\end{cases}$$

In Fig. 1, we present the plot of the solution $u(x, t)$ for $t = 0.02, 0.1, 0.5, 1$, and $\alpha = 2.5$.

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