Distance Matrix of Weighted Cactoid-type Digraphs

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Abstract
A strongly connected digraph is called a cactoid-type, if each of its blocks is a digraph consisting of finitely many oriented cycles, sharing a common directed path. In this article, we find the formula for the determinant of the distance matrix for weighted cactoid-type digraphs and find its inverse, whenever it exists. We also compute the determinant of the distance matrix for a class of unweighted and undirected graphs consisting of finitely many cycles, sharing a common path.

Keywords: weighted digraphs, cycles, distance matrix, determinant, inverse.
MSC: 05C12, 05C50

1 Introduction and Motivation
A graph $G = (V, E)$ is said to be a weighted graph if each edge $e \in E$ is associated with a real number $W(e)$, called the weight of $e$. In the case that all of the weights are equal to 1, we refer to $G$ as an unweighted graph. Similarly, $G$ is said to be a directed graph (or digraph) if edge $e \in E$ is assigned with an orientation; otherwise, $G$ is called an undirected graph. We write, $x \rightarrow y$ to indicate the directed edge from $x$ to $y$.

A vertex $v$ of a graph $G$ is a cut-vertex of $G$, if $G - v$ is disconnected. A block of the graph $G$ is a maximal connected subgraph of $G$ that has no cut-vertex. Recall that, a digraph $G$ is strongly connected if a directed path connects each pair of vertices.

We define the total weight of a path $P$ in a graph $G$, the sum of the weights on edges of $P$ and it is easy to see that if $G$ is unweighted, then the total weight represents the number of edges on $P$. Given a graph $G$, the distance between vertices $x, y \in V$ is denoted by $d(x, y)$, is the minimum total weight of a path from $x$ to $y$ and we set $d(x, x) = 0$ for all $x \in V$. The distance matrix $D(G)$, is the $|V| \times |V|$ matrix with $D(G) = [d_{xy}]$, where $d_{xy} = d(x, y)$.

We now introduce some notations. Let $I_n$ and $\mathbf{1}$ denote the identity matrix and the column vector of all ones respectively. Further, $J_{m \times n}$ denotes the $m \times n$ matrix of all ones and if $m = n$, we use the notation $J_n$. We will use $\mathbf{0}$ to represent zero matrix if there is no scope of confusion with the order of the matrix. Given a matrix $A$, we use $A^T$ to denote the transpose of the matrix.

Let $G = (V, E)$ be an undirected and unweighted graph. For $x, y \in V$, the adjacency matrix of the graph $G$ is, $A(G) = [a_{xy}]$, where

$$a_{xy} = \begin{cases} 1 & \text{if } x \text{ is adjacent to } y, \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian matrix of $G$, is defined as $L(G) = D(G) - A(G)$, where $D = \text{Diag}(\mathbf{1}) = \text{Diag}(\delta_x)$, where $\delta_x$ denotes the degree of the vertex $x$. It is well known that $L(G)$ is a symmetric and positive
semi-definite matrix. The constant vector \( \mathbf{1} \) is the eigenvector of \( L(G) \) corresponding to the smallest eigenvalue 0 and hence satisfies \( L(G) \mathbf{1} = 0 \) and \( \mathbf{1}^T L(G) = 0 \) (for details see [1]).

Let \( T \) be a (unweighted and undirected) tree with \( n \) vertices. In [11], the authors proved that the determinant of the distance matrix \( D(T) \) of \( T \) is given by \( \det D(T) = (-1)^{n-1}(n-1)2^{n-2} \). Thus the determinant does not depend on the structure of the tree but only on the number of vertices. In [13], it was shown that the inverse of the distance matrix of a tree is given by 

\[
D(T)^{-1} = -\mathbf{L} + \frac{1}{\lambda} \beta \alpha^T
\]

where \( \alpha \) and \( \beta \) are column vectors and \( \lambda \) is a suitable constant. Except for the weighted cactoid digraph in [19], \( \alpha = \beta \) for all other cases mentioned above.

Before proceeding further, we first recall the definition of cactoid digraph. A graph is said to be a cactoid digraph if each of its blocks is an oriented cycle. In literature, an undirected cycle with \( n \) vertices is denoted by \( C_n \), and for the oriented case it is denoted as \( dC_n \).

Let \( G \) be a graph consisting of finitely many cycles sharing a common path \( P \) such that the intersection between any two cycles is also precisely the path \( P \). If the path \( P \) is of length \( n \) and \( G \) consists of \( r \) cycles, then we denote \( G \) by \( C(n, m_1, \ldots, m_r) \). If \( G \) consists of \( r \) oriented cycles and the common path \( P \) is of length \( n \) such that \( G \) is strongly connected (i.e. the orientation on path \( P \) agrees with all the \( r \) cycles), then we denote \( G \) by \( dC(n, m_1, \ldots, m_r) \). The choice of parameters involved in the above notation is explained in Section 2. We define cactoid-type digraphs as follows: A strongly connected digraph is said to be cactoid-type, if each of its blocks is a digraph \( dC(n, m_1, \ldots, m_r) \), whenever \( n, r \geq 1 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

**Remark 1.1.** 1. It is easy to see, if \( r = 1 \), then \( C(n, m_1, \ldots, m_r) \) (or \( dC(n, m_1, \ldots, m_r) \)) represent a cycle (or oriented cycle) of certain length. So the notion of cactoid-type digraphs is a extension to the cactoid digraphs.

2. If \( n = 1 \), the \( r \) cycles in a block shares a common edge or common cut edge. So, we are interested in graphs with a common cut edge (if \( n > 1 \), it is a common path).

In this article, we are interested in computing the determinant of the distance matrix for weighted cactoid-type digraphs and find its inverse, whenever it exists. Due to results in [12, 19],
it is sufficient to find the determinant and the inverse (in the requisite form, see Eqn. (1.1)) for individual blocks, the weighted $dC(n, m_1, \cdots, m_r)$ and these sufficient conditions are discussed in Section 1.1.

In this search, we found that the Laplacian-like matrix $L$ involved in the inverse of the distance matrix for the weighted $dC(n, m_1, \cdots, m_r)$, is a “perturbed weighted Laplacian” for the same. Interestingly, the adjacency matrix involved in the weighted Laplacian for the digraph is not dependent on the weights on the individual edge, but only on the total weight of each of the $r$ cycles involved. It allows us to choose the weight on edge to be 0, without disturbing the structure.

This article is organized as follows. In Section 1.1, we summarized few existing results useful for this article. In Section 2, we find the determinant and cofactor of the distance matrix for weighted $dC(n, m_1, \cdots, m_r)$ and compute its inverse, whenever it exists. Consequently, we extend these results to cactoid-type digraphs. Finally, in Section 3, we compute the determinant of the distance matrix for a class of unweighted and undirected graph $C(n, m_1, \cdots, m_r)$.

1.1 Some Preliminary Results

In this section, we recall some existing literature on distance matrix, that allow us to extend the results for determinant and inverse from individual blocks to the full graph.

Given a matrix $A$, we use the notation $A(i \mid j)$ to denote the submatrix obtained by deleting the $i^{th}$ row and the $j^{th}$ column. Let $A$ be an $n \times n$ matrix. For $1 \leq i, j \leq n$, the cofactor $c_{i,j}$ is defined as $(-1)^{i+j} \det A(i \mid j)$. We use the notation cof $A$ to denote the sum of all cofactors of $A$.

Lemma 1.2. [1] Let $A$ be an $n \times n$ matrix. Let $M$ be the matrix obtained from $A$ by subtracting the first row from all other rows and then subtracting the first column from all other columns. Then

$$\text{cof } A = \det M(1|1).$$

The result below gives the determinant of a strongly connected graph based the determinant and cofactor of its blocks.

Theorem 1.3. [12] Let $G$ be strongly connected digraph with blocks $G_1, G_2, \cdots, G_b$. Then

$$\text{cof } D(G) = \prod_{i=1}^{b} \text{cof } D(G_i),$$

$$\det D(G) = \sum_{i=1}^{b} \det D(G_i) \prod_{j \neq i} \text{cof } D(G_j).$$

Now using our notations, we summarize few definitions and results in [18] and [19, Section 3], that gives the sufficient condition to compute the inverse of the distance matrix for weighted cactoid-type digraph.

Lemma 1.4. [19] Let $D$ and $L$ be two $n \times n$ matrices, $\alpha$ and $\beta$ be two $n \times 1$ column vectors and $\lambda$ be a nonzero number. If one of the following two conditions holds:

1. $\alpha^T D = \lambda \mathbf{1}^T$ and $LD + I = \beta \mathbf{1}^T$,
2. $D\beta = \lambda \mathbf{1}$ and $DL + I = \mathbf{1}\alpha^T$,

then $D$ is invertible and $D^{-1} = -L + \frac{1}{\lambda} \beta \alpha^T$. 

Definition 1.5. Let $D$ and $L$ be two $n \times n$ matrices, $\alpha$ and $\beta$ be two $n \times 1$ column vectors and $\lambda$ be a number. Then

1. $D$ is a left LapExp($\lambda, \alpha, \beta, L$) matrix, if $\alpha^T \mathbf{1}, \ L \mathbf{1} = 0, \ \alpha^T D = \lambda \mathbf{1}^T$ and $LD + \beta = \mathbf{1}^T$,

2. $D$ is a right LapExp($\lambda, \alpha, \beta, L$) matrix, if $\beta^T \mathbf{1} = 1, \ \mathbf{1}^T L = 0, \ D \beta = \lambda \mathbf{1}$ and $D L + I = \mathbf{1} \alpha^T$.

Assumption 1.6. Let $G = (V, E)$ be a strongly connected digraph on $|V|$ vertices with blocks $G_t = (V_t, E_t), 1 \leq t \leq b$. Let $D$ and $D_t$ be the distance matrix of $G$ and $G_t$ respectively. For each $1 \leq t \leq b$, the $D_t$ is a left LapExp($\lambda_t, \alpha_t, \beta_t, L_t$) matrix (or right LapExp($\lambda_t, \alpha_t, \beta_t, L_t$) matrix).

For each block $G_t$ is also considered as a graph on vertex set $V$ with perhaps isolated vertices, and let its edge set be $E_t$ (i.e, $G_t$ is a graph on $|V|$ vertices, consider it as a graph on vertex set $V$). Let $L_t$ be the $|V| \times |V|$ matrix defined as above for the vertices of $G_t$ and 0 for others. Define

1. $L = \sum_{t=1}^{b} L_t$.

2. $\lambda = \sum_{t=1}^{b} \lambda_t$.

3. $\alpha(v) = \sum_{t=1}^{b} \alpha_t(v) - (k - 1), \ \text{where} \ v \ \text{is a vertex belongs to} \ k \ \text{blocks of} \ G$.

4. $\beta(v) = \sum_{t=1}^{b} \beta_t(v) - (k - 1), \ \text{where} \ v \ \text{is a vertex belongs to} \ k \ \text{blocks of} \ G$.

The next result gives the inverse of the distance matrix of strongly connected digraphs.

Theorem 1.7. [19, Theorem 3.8] Let $G = (V, E)$ be a graph with blocks $G_t = (V_t, E_t), 1 \leq t \leq b$. Let $D$ and $D_t$ be the distance matrix of $G$ and $G_t$ respectively. For each $1 \leq t \leq b$, if $D_t$ is a left LapExp($\lambda_t, \alpha_t, \beta_t, L_t$) matrix (or right LapExp($\lambda_t, \alpha_t, \beta_t, L_t$) matrix) then using Assumption 1.6, $D$ is a left LapExp($\lambda, \alpha, \beta, L$) matrix (or right LapExp($\lambda, \alpha, \beta, L$) matrix). Furthermore, if $\lambda \neq 0$, then

$$D^{-1} = -L + \frac{1}{\lambda} \beta \alpha^T.$$  

We conclude this section with the result to determine the determinant using the Schur complement technique. Let $B$ be an $n \times n$ matrix partitioned as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$  

(1.2)

where $B_{11}$ and $B_{22}$ are square matrices. If $B_{11}$ is nonsingular, then the Schur complement of $B_{22}$ in $B$ is defined to be the matrix $B/B_{11} = B_{22} - B_{21}B_{11}^{-1}B_{12}$. Similarly, if $B_{22}$ is nonsingular, then the Schur complement of $B_{22}$ in $B$ is defined to be the matrix $B/B_{22} = B_{11} - B_{12}B_{22}^{-1}B_{21}$.

Proposition 1.8. [17] Let $B$ be an $n \times n$ matrix partitioned as in Eqn. (1.2). If $B_{11}$ is nonsingular, then $\det B = \det B_{11} \times \det(B_{22} - B_{21}B_{11}^{-1}B_{12})$. Similarly, if $B_{22}$ is nonsingular, then $\det B = \det B_{22} \times \det(B_{11} - B_{12}B_{22}^{-1}B_{21})$.

In the next section, we compute the determinant of the distance matrix for weighted cactoid-type digraphs and find its inverse whenever it exists.

2 Distance Matrix on Cactoid-type Digraphs

We begin this section, by recalling the definition of $dC(n, m_1, \cdots, m_r)$ and the reasoning behind the choice of the notation for the blocks of cactoid-type digraphs. Let $G$ be a strongly connected digraph consisting of finitely many oriented cycles sharing a common path $P^{(s)}(\text{say})$ such that the
intersection between any two cycles is also precisely the path $P^{(c)}$. For notational convenience, we give the following representation to the graph $G$.

Let $P^{(c)}$ be a directed path $u_0 \to u_1 \to \cdots \to u_n$ and for $j = 1, 2, \cdots, r$, let $P^{(j)}$ be a directed path $v^{(j)}_0 \to v^{(j)}_1 \to \cdots v^{(j)}_{m_j} \to v^{(j)}_{m_j+1}$. For all $j = 1, 2, \cdots, r$, we identify the vertex $u_0$ with $v^{(j)}_{m_j+1}$, denote it as $u_0$ and similarly for all $j = 1, 2, \cdots, r$ we identify the vertex $u_n$ with $v^{(j)}_1$ and denote it as $u_n$. Then the resulting graph $G = (V, E)$ with $|V| = (n + 1) + \sum_{j=1}^r m_j$ vertices, is a strongly connected digraph consisting of $r$ oriented cycles sharing the common path $P^{(c)}$ as defined above and we denote such digraphs as $dC(n, m_1, \ldots, m_r)$. Note that whenever $r = 1$, the $dC(n, m_1)$ is an oriented cycle on $(n + 1) + m_1$ vertices, i.e. $dC(n, m_1) = dC(n+1)+m_1$. In this above choice if we consider undirected $r$ cycles, we denote the resulting graph as $C(n, m_1, \ldots, m_r)$.

We will use the following convention to index the weights on the directed edges. If $e : u \to v$ represent a directed edge from $u$ to $v$, then we index the weight “$W(e)$” on the edge $e$ as $W_v$. Now with the identification $u_0 = v^{(j)}_{m_j+1}$ and $u_n = v^{(j)}_1$, for all $j = 1, 2, \ldots, r$, the weights on the edges (see Figure 2) of $dC(n, m_1, \ldots, m_r)$ are denoted as follows:

$$W(e) = \begin{cases} W_{i+1}, & \text{if } e : u_i \to u_{i+1}, \ i = 1, \cdots, n-1, \\ W_{i}^{(j)}, & \text{if } e : v^{(j)}_{i-1} \to v^{(j)}_i, \ j = 1, \cdots, r, \ i = 1, \cdots, m_j, \\ W_{i}^{(0)}, & \text{if } e : v^{(0)}_{m_j} \to u_0, \ j = 1, \cdots, r. \end{cases}$$

![Figure 2: $dC(n, m_1, \cdots, m_r)$](image)

Before proceeding further, we first introduce a few notations useful for the subsequent results. For $j = 1, \ldots, r$, let us denote

$$\begin{align*}
 w_j &= w_c + \hat{w}_j, \text{ where} \\
 w_c &= \sum_{i=1}^n W_i \text{ and } \hat{w}_j = \sum_{i=0}^{m_j} W_i^{(j)}. \tag{2.1}
\end{align*}$$

Notice that, $dC(n, m_1, \cdots, m_r)$ consists of $r$ cycles with total weights $w_1, w_2, \ldots, w_r$. For $1 \leq j \leq r$, let $dC^{(j)}$ denote the oriented cycle due to the path $P^{(c)}$ and $P^{(j)}$ with total weight $w_j$. Without
Observation 2.1. Let \( \theta_0, \theta_1, \ldots, \theta_{n-1} \) be \( n \) weights. Let \( w^{(2)} = \sum_{0 \leq s < t < n-1} \theta_s \theta_t \) be the sum of these weights taken two at a time (without repetition). Now we will look into different rearrangement to obtained the sum \( w^{(2)} \) which are useful in our subsequent calculations.

Let us consider oriented cycle \( dC_n \) with vertex \( \{0, 1, \ldots, n-1\} \) with \( \theta_i \) is the weight on the edge \( i-1 \to i \) (see Figure 1(a)). Observe that, the sum \( w^{(2)} \) can be oriented by starting with any weight \( \theta_s, \) when \( 0 \leq s \leq n-1, \) considering the sum in clockwise direction in \( dC_n \) and using the notation \( [i] \equiv i(mod \ n) \), we have

\[
w^{(2)} = \sum_{i=s}^{s+n-2} \left( \theta_{[i]} \sum_{j=i+1}^{s+n-1} \theta_{[j]} \right) = \sum_{i=s}^{s+n-2} \theta_{[i]} \ d([i], [s-1]).
\]

Similarly, considering the sum in anticlockwise direction in \( dC_n \), we have

\[
w^{(2)} = \sum_{i=0}^{n-2} \left( \theta_{[s-i]} \sum_{j=s+1}^{s-i-1} \theta_{[j]} \right) = \sum_{i=0}^{n-2} \theta_{[s-i]} \ d(s, [s-i-1]).
\]

For example: let us consider the sum \( w^{(2)} \), for the weights \( \theta_0, \theta_1, \theta_2, \theta_3, \theta_4 \). Now starting with \( \theta_2 \), the sum in clockwise direction is:

\[
w^{(2)} = \theta_2(\theta_3 + \theta_4 + \theta_0 + \theta_1) + \theta_3(\theta_4 + \theta_0 + \theta_1) + \theta_4(\theta_0 + \theta_1) + \theta_0\theta_1,
\]

and the sum in anticlockwise direction is:

\[
w^{(2)} = \theta_2(\theta_3 + \theta_4 + \theta_0 + \theta_1) + \theta_1(\theta_3 + \theta_4 + \theta_0) + \theta_0(\theta_3 + \theta_4) + \theta_4\theta_3.
\]

The lemmas below give rearrangements of the \( w^{(2)} \)-sum discussed above for the weights on oriented cycle \( dC^{(j)} \) (due to the path \( P^{(c)} \) and \( P^{(j)} \) in \( dC(n, m_1, \ldots, m_r) \)) and on path \( P^{(j)} \) respectively.

Lemma 2.2. For \( 1 \leq j \leq r \), let \( w^{(2)}(dC^{(j)}) \) denote the sum of weights on \( dC^{(j)} \) taken two at a time without repetition. Then

\[
w^{(2)}(dC^{(j)}) = w_c\widehat{w}_j + w_c^{(2)} + w_j^{(2)}.
\]

Proof. Using Observation 2.1, computing the sum in clockwise direction starting with the weight \( W_1 \) yields

\[
w^{(2)}(dC^{(j)}) = \sum_{i=1}^{n-1} W_i \left( \sum_{s=i+1}^{n} W_s + \sum_{s=0}^{m_j} W_s^{(j)} \right) + W_n \sum_{s=0}^{m_j} W_s^{(j)} + \sum_{i=1}^{m_j-1} W_i \left( W_0^{(j)} + \sum_{s=i+1}^{m_j} W_s^{(j)} \right) + W^{(j)} W_0^{(j)}
\]

\[
= \sum_{i=1}^{n-1} W_i \left( \sum_{s=i+1}^{n} W_s \right) + \left( W_n + \sum_{i=1}^{n-1} W_i \right) \left( \sum_{s=i+1}^{m_j} W_s^{(j)} \right) + \sum_{i=0}^{m_j-1} W_i^{(j)} \left( \sum_{s=i+1}^{m_j} W_s^{(j)} \right)
\]

and the result follows. \( \square \)
Lemma 2.3. For $1 \leq j \leq r$, let $v_s^{(j)}$ be a vertex on weighted graph $dC(n, m_1, \cdots, m_r)$. Then

$$w_j^{(2)} = w_j \ d(u_n, v_s^{(j)}) + \left[ \sum_{i=1}^{m_j} W_i^{(j)} d(v_i^{(j)}, v_s^{(j)}) \right] - d(u_n, v_m^{(j)}) d(u_0, v_s^{(j)})$$.

Proof. In view of $d(u_0, v_m^{(j)}) = \sum_{i=1}^{m_j} W_i^{(j)}$ and $d(v_s^{(j)}, v_s^{(j)}) = 0$, we have

$$\left( \sum_{i=1}^{m_j} W_i^{(j)} d(v_i^{(j)}, v_s^{(j)}) \right) - d(u_n, v_m^{(j)}) d(u_0, v_s^{(j)})$$

$$= \sum_{i=1}^{m_j} W_i^{(j)} d(v_i^{(j)}, v_s^{(j)}) - d(u_0, v_s^{(j)}) \sum_{i=1}^{m_j} W_i^{(j)}$$

$$= \sum_{i=1}^{m_j} W_i^{(j)} \left( d(v_i^{(j)}, u_0) - d(v_s^{(j)}, u_0) \right) + \sum_{i=1}^{m_j} W_i^{(j)} \left( d(v_i^{(j)}, u_0) + d(u_0, v_s^{(j)}) \right) - \sum_{i=1}^{m_j} W_i^{(j)} d(u_0, v_s^{(j)})$$

$$= \sum_{i=1}^{m_j} W_i^{(j)} d(v_i^{(j)}, u_0) - \sum_{i=1}^{s} W_i^{(j)} \left( d(v_s^{(j)}, u_0) + d(u_0, v_s^{(j)}) \right)$$

$$= w_j^{(2)} - w_j \sum_{i=1}^{s} W_i^{(j)} = w_j^{(2)} - w_j \ d(u_n, v_s^{(j)})$$

and hence the result follows. \qed

Now we compute the determinant and the cofactor of the distance matrix for $dC(n, m_1, \cdots, m_r)$.

2.1 Determinant and Cofactor of Distance Matrix for weighted $dC(n, m_1, \cdots, m_r)$

In this section, we will calculate the determinant and cofactor of the distance matrix $D$ for weighted graph $dC(n, m_1, \cdots, m_r)$. Let $V$ be the vertex set of $dC(n, m_1, \cdots, m_r)$. Henceforth, we will use

$$V = \{ u_0, u_1, \ldots, u_n, v_1^{(1)}, \ldots, v_m^{(1)}, v_1^{(2)}, \ldots, v_m^{(2)}, \ldots, v_1^{(r)}, \ldots, v_m^{(r)} \}$$

as the vertex ordering for the distance matrix $D$. We begin with the following lemma.

Lemma 2.4. Let $D$ be the distance matrix of a weighted graph $dC(n, m_1, \cdots, m_r)$ with $w_j = 0$, for some $1 \leq j \leq r$. Then $\det D = 0$.

Proof. Assume that the oriented cycle $dC^{(j)}$ in $dC(n, m_1, \cdots, m_r)$ with total weight $w_j = 0$. Let $x, y, z$ be three vertices in $dC^{(j)}$ such that $x \to y \to z$. Let $C_u$ denote the column corresponding to the vertex $u$ in the distance matrix $D$ and let $\tilde{D}$ be the resulting matrix after elementary column operations $C_z \leftarrow C_z - C_y$ followed by $C_y \leftarrow C_y - C_z$ on $D$. For any vertex $v$ in $dC(n, m_1, \cdots, m_k)$, we have

$$(C_z - C_y)(v) = d(v, z) - d(v, y) = \begin{cases} d(y, z) & \text{if } v \neq z, \\ -d(z, y) & \text{if } v = z. \end{cases}$$

Note that $d(z, y) = w_j - d(y, z)$. Since $w_j = 0$, so $(C_z - C_y)(v) = W_z$. Thus, the column corresponding to $z$ in $\tilde{D}$, that is $C_z - C_y = W_z 1$ and similar argument yields that the column corresponding to $y$ in $\tilde{D}$, $C_y - C_z = W_y 1$. Hence $\det D = \det \tilde{D} = 0$. \qed

Now we compute the determinant of distance matrix for weighted digraph $dC(n, m_1, \cdots, m_r)$.  

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Theorem 2.5. Let $D$ be the distance matrix of the weighted digraph $dC(n, m_1, \ldots, m_r)$ with $|V| = (n + 1) + \sum_{j=1}^{r} m_j$ vertices. Then the determinant of the distance matrix is given by

$$
\det D = \begin{cases} 
(-1)^{|V|-1} \left( w_1^n \prod_{j=1}^{r} w_j^{m_j} \right) \left( \frac{w_1}{w_1} + \frac{w_2}{w_1} + \sum_{j=1}^{r} \frac{w_j}{w_1} \right), & \text{if } w_j \neq 0, 1 \leq j \leq r, \\
0, & \text{otherwise.}
\end{cases}
$$

Proof. Let $R_u/C_u$ denote the row/column of $D$ corresponding to the vertex $u_i$, $0 \leq i \leq n$ and for $1 \leq j \leq r$, let $R_{v_i}^{(j)}/C_{v_i}^{(j)}$ denote the row/column of $D$ corresponding to the vertex $v_i^{(j)}$, $1 \leq i \leq m_j$. Now we begin with few elementary matrix operations on the distance matrix $D$, listed below in the following steps:

1. For each $j = 1, 2, \ldots, r$, first compute $C_{u_i}^{(j)} \leftarrow C_{v_i}^{(j)} - C_{v_{i-1}}^{(j)}$ recursively, for $i = m_j, m_j-1, \ldots, 2$, followed by $C_{v_i}^{(j)} \leftarrow C_{v_i}^{(j)} - C_{u_n}$.

2. Next, compute $C_{u_i} \leftarrow C_{u_i} - C_{u_{i-1}}$ recursively, for $i = n, n-1, \ldots, 1$.

3. Further, first compute $R_{u_i} \leftarrow R_{u_i} - R_{u_{i+1}}$ recursively, for $i = 1, 2, \ldots, n-1$, followed by $R_{u_n} \leftarrow R_{u_n} - R^{(1)}$.

4. Next, for $j = 1, 2, \ldots, r$, first compute $R_{v_i}^{(j)} \leftarrow R_{v_i}^{(j)} - R_{v_{i+1}}^{(j)}$ recursively, for $i = 1, 2, \ldots, m_j-1$, followed by $R_{v_{m_j}}^{(j)} \leftarrow R_{v_{m_j}}^{(j)} - R_{u_0}$.

5. Now compute $C_{u_{i+1}} \leftarrow C_{u_{i+1}} + C_{u_i}$ recursively for $i = 1, 2, \ldots, n$, followed by computing $C_{v_i}^{(1)} \leftarrow C_{v_i}^{(1)} - C_{u_n}$. Next, for each $j = 1, 2, \ldots, r$, compute $C_{v_i}^{(j)} \leftarrow C_{v_i}^{(j)} - C_{v_{i-1}}^{(j)}$ recursively, for $i = 2, 3, \ldots, m_j$.

6. Finally, for each $j = 2, 3, \ldots, k$, compute $C_{v_i}^{(j)} \leftarrow C_{v_i}^{(j)} - C_{v_{i-1}}^{(j)}$ recursively, for $i = 2, 3, \ldots, m_j$.

Then, the resulting matrix is of the following block form $\begin{bmatrix} 0 & X^T \\ Y & Z \end{bmatrix}$, where

$$
Z = \begin{bmatrix}
-w_1 I_{n+m_1} & 0 & \cdots & 0 \\
0 & -w_2 I_{m_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -w_k I_{m_k}
\end{bmatrix},
$$

$$
X^T = [X_1^T X_2^T \cdots X_r^T] \text{ and } Y^T = [Y_1^T Y_2^T \cdots Y_r^T] \text{ with } X_1^T = \begin{bmatrix} W_1, \sum_{i=1}^{2} W_i, \cdots, \sum_{i=1}^{n} W_i, n W_i + W_1^{(1)}, \cdots, n W_i + \sum_{i=1}^{m_1} W_i^{(1)} \end{bmatrix},
$$

$$
Y_1^T = \begin{bmatrix} W_2, W_3, \cdots, W_n, W_1^{(1)}, \cdots, W_0^{(1)} \end{bmatrix},
$$

$$
X_j^T = \begin{bmatrix} W_1^{(j)}, \sum_{i=1}^{2} W_i^{(j)}, \cdots, \sum_{i=1}^{m_j} W_i^{(j)} \end{bmatrix}, \quad Y_j^T = \begin{bmatrix} W_2^{(j)}, W_3^{(j)}, \cdots, W_0^{(j)} \end{bmatrix}, \quad 2 \leq j \leq r.
$$

If $w_j = 0$ for some $j$, then result follows from Lemma 2.4. Next, assume that $w_j \neq 0$, $1 \leq j \leq r$. Then $Z$ is invertible, so using Proposition 1.8, we have

$$
\det D = (\det Z) \times (\det(-X^T Z^{-1} Y)),
$$

(2.4)
Further, using Eqn. (2.3), we have
\[
\det(-X^T Z^{-1} Y) = \sum_{j=1}^{r} \frac{\langle X_j, Y_j \rangle}{w_j} = \frac{\langle X_1, Y_1 \rangle}{w_1} + \sum_{j=2}^{r} \frac{\langle X_j, Y_j \rangle}{w_j},
\]  
(2.5)
where \( \langle X_j, Y_j \rangle \) represent the inner product of the column vectors \( X_j \) and \( Y_j \). By Observation 2.1, starting the sum with \( W_0^{(1)} \), in anticlockwise direction with the weights on \( dC^{(1)} \) and using Lemma 2.2, we have
\[
\langle X_1, Y_1 \rangle = W_2 W_1 + W_3 \left( \sum_{i=1}^{2} W_i \right) + \cdots + W_n \left( \sum_{i=1}^{n-1} W_i \right) + W_1^{(1)} \left( \sum_{i=1}^{n} W_i \right) + W_2^{(1)} \left( W_1^{(1)} + \sum_{i=1}^{n} W_i \right)
+ \cdots + W_{m_1}^{(1)} \left( \sum_{i=1}^{m_1-1} W_i + \sum_{i=1}^{n} W_i \right) + W_0^{(1)} \left( \sum_{i=1}^{m_1} W_i + \sum_{i=1}^{n} W_i \right)
= w^{(2)}(dC^{(1)}) = w_c \tilde{w}_1 + w_c^{(2)} + w^{(1)}_1.
\]
Similarly, for \( 1 \leq j \leq r \), considering the sum starting with \( W_0^{(j)} \) in anticlockwise direction with the weights on \( P^{(j)} \), we have
\[
\langle X_j, Y_j \rangle = W_2^{(j)} W_1^{(j)} + W_3^{(j)} \left( \sum_{i=1}^{2} W_i^{(j)} \right) + \cdots + W_n^{(j)} \left( \sum_{i=1}^{n-1} W_i^{(j)} \right) + W_1^{(j)} \left( \sum_{i=1}^{n} W_i^{(j)} \right) + W_2^{(j)} \left( W_1^{(j)} + \sum_{i=1}^{n} W_i^{(j)} \right)
+ \cdots + W_{m_j}^{(j)} \left( \sum_{i=1}^{m_j-1} W_i^{(j)} + \sum_{i=1}^{n} W_i^{(j)} \right) + W_0^{(j)} \left( \sum_{i=1}^{m_j} W_i^{(j)} + \sum_{i=1}^{n} W_i^{(j)} \right)
= w^{(2)}(dC^{(j)}) = w_c \tilde{w}_1 + w_c^{(2)} + w^{(1)}_1.
\]
Using the above calculations in Eqns.(2.4) and (2.5), yields the result. 

The next result gives the cofactor of the distance matrix for weighted graph \( dC(n, m_1, \ldots, m_r) \).

**Theorem 2.6.** Let \( D \) be the distance matrix of the weighted graph \( dC(n, m_1, \ldots, m_r) \) with \( |V| = (n + 1) + \sum_{j=1}^{r} m_j \) vertices. Then the cofactor of the distance matrix \( D \) is given by
\[
\text{cof } D = (-1)^{|V|-1} w_1^n \prod_{j=1}^{r} w_j^{m_j}.
\]

**Proof.** Let \( M \) be the matrix obtained from \( D \) by subtracting the first row from all other rows and then subtracting the first column from all other columns. Then the matrix \( M(1|1) \) in block form is given by
\[
M(1|1) = \begin{bmatrix}
-w_1 U_n & -w_1 J_{n \times m_1} & -w_1 J_{n \times m_2} & \cdots & -w_1 J_{n \times m_r} \\
0 & -w_1 U_{m_1} & 0 & \cdots & 0 \\
0 & 0 & -w_2 U_{m_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -w_r U_{m_r}
\end{bmatrix},
\]
where \( U_s \) is a upper triangular matrix of order \( s \), with all its non-zero entries 1. Note that, \( M(1|1) \) is obtained after deleting row and column of \( M \) corresponding to the vertex \( u_0 \). Using similar notations for rows and columns as in Theorem 2.5, we use the following elementary column operations on \( M(1|1) \):

1. For each \( j = 1, 2, \ldots, r \), first compute \( C_{v_1}^{(j)} \leftarrow C_{v_1}^{(j)} - C_{v_{i-1}}^{(j)} \), recursively for \( i = m_j, m_{j-1}, \ldots, 2 \) followed by \( C_{v_1}^{(j)} \leftarrow C_{v_1}^{(j)} - C_{uv} \).
2. Next, compute \( C_{u_i} \leftarrow C_{u_i} - C_{u_{i-1}} \), recursively for \( i = n, n-1, \ldots, 2 \).
and the resulting matrix is given by
\[
\begin{bmatrix}
-w_1 I_n & 0 & 0 & \cdots & 0 \\
0 & -w_1 I_{m_1} & 0 & \cdots & 0 \\
0 & 0 & -w_2 I_{m_2} & \cdots & 0 \\
& & & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -w_r I_{m_r}
\end{bmatrix}.
\]
Hence the result follows from Lemma 1.2.

By Theorem 2.5, the det \( D \neq 0 \) iff \( w_j \neq 0, \ 1 \leq j \leq r \) and \( \frac{w_i w_j}{w_1} + \frac{w_i^{(2)}}{w_1} + \sum_{j=1}^{r} \frac{w_i^{(2)}}{w_j} \neq 0 \). Thus, Theorem 2.6, gives det \( D \neq 0 \) implies cof \( D \neq 0 \). The next section deals with the inverse of the distance matrix \( D \) of weighted digraph \( dC(n, m_1, \ldots, m_r) \), so henceforth we only consider the case whenever det \( D \neq 0 \). It is easy to see, for unweighted digraph \( dC(n, m_1, \ldots, m_r) \), the det \( D \neq 0 \).

### 2.2 Inverse of Distance Matrix for weighted \( dC(n, m_1, \ldots, m_r) \)

In this section, we find the inverse of the distance matrix \( D \) for weighted digraph \( dC(n, m_1, \ldots, m_r) \), by proving that \( D \) is a left LapExp(\( \lambda, \alpha, \beta, \mathcal{L} \)) matrix for weighted \( dC(n, m_1, \ldots, m_r) \). We begin with defining the parameters \( \lambda, \alpha, \beta \) and subsequently show that these parameters satisfying the requisite conditions. We define

\[
\begin{align*}
\lambda &= \frac{\det D}{\text{cof } D} = \frac{w_i \hat{w}_1}{w_1} + \frac{w_i^{(2)}}{w_1} + \sum_{j=1}^{r} \frac{w_i^{(2)}}{w_j}, \\
\alpha &= \begin{bmatrix}
\sum_{j=1}^{r} \frac{W_0^{(j)}}{w_j} - \sum_{j=2}^{r} \frac{\hat{w}_j}{w_j}, & W_1, & \ldots, & W_n, & W_1^{(1)} / w_1, & \ldots, & W_1^{(r)} / w_1, & \ldots, & W_n^{(1)} / w_1, & \ldots, & W_n^{(r)} / w_r \\
W_1 / w_1, & \ldots, & W_n / w_1, & \sum_{j=1}^{r} \frac{W_1^{(j)}}{w_j} - \sum_{j=2}^{r} \frac{\hat{w}_j}{w_j}, & \ldots, & W_2 / w_1, & \ldots, & W_2^{(r)} / w_r, & \ldots, & W_n / w_1, & \ldots, & W_n^{(r)} / w_r
\end{bmatrix}, \\
\beta &= \begin{bmatrix}
W_1, & \ldots, & W_n, & \sum_{j=1}^{r} \frac{W_1^{(j)}}{w_j} - \sum_{j=2}^{r} \frac{\hat{w}_j}{w_j}, & \ldots, & W_2, & \ldots, & W_2^{(r)}, & \ldots, & W_n, & \ldots, & W_n^{(r)}
\end{bmatrix}.
\end{align*}
\]

#### Lemma 2.7.
Let \( \alpha, \beta \) be the vectors as defined in Eqn. (2.6). Then \( \alpha^T \mathbb{1} = \beta^T \mathbb{1} = 1 \).

**Proof.** Since

\[
\sum_{j=1}^{r} \frac{W_0^{(j)}}{w_j} - \sum_{j=2}^{r} \frac{\hat{w}_j}{w_j} = \frac{W_0^{(1)}}{w_1} + \sum_{j=2}^{r} \left( \frac{W_0^{(j)} - \hat{w}_j}{w_j} \right) = \frac{W_0^{(1)}}{w_1} - \sum_{j=2}^{r} \sum_{i=1}^{m_j} \frac{W_i^{(j)}}{w_j},
\]

so using the definition of \( \alpha \), we have

\[
\alpha^T \mathbb{1} = \frac{W_0^{(1)}}{w_1} - \sum_{j=2}^{r} \sum_{i=1}^{m_j} \frac{W_i^{(j)}}{w_j} + \sum_{i=1}^{n} \frac{W_i}{w_1} + \sum_{j=1}^{r} \sum_{i=1}^{m_j} \frac{W_i^{(j)}}{w_j} = \frac{W_0^{(1)}}{w_1} + \sum_{i=1}^{n} \frac{W_i}{w_1} + \sum_{i=1}^{m_1} \frac{W_i^{(1)}}{w_1} = 1,
\]

and similar calculations leads to \( \beta^T \mathbb{1} = 1 \).

#### Lemma 2.8.
Let \( D \) be the distance matrix of the weighted graph \( dC(n, m_1, \ldots, m_r) \). Then \( \alpha^T D = \lambda \mathbb{1}^T \), where \( \lambda \) and \( \alpha \) are defined as in Eqn. (2.6).
Let $\eta^T = \alpha^T D$ and $V$ be the vertex set of the weighted graph $dC(n, m_1, \ldots, m_r)$. We will prove this result by repeated application of Observation 2.1, to show $\lambda = \eta(v) = \sum_{z \in V} \alpha(z)d(z, v)$, for all $v \in V$.

**Case 1:** For $v = u_0$, \[
\eta(v) = \sum_{i=1}^{n} \frac{W_i}{w_1} d(u_i, u_0) + \sum_{j=1}^{r} \sum_{i=1}^{m_j} \frac{W_i^{(j)}}{w_j} d(v_i^{(j)}, u_0) \]

By Observation 2.1, $\Sigma_{11}$ corresponds to the clockwise sum of weights on $dC^{(1)}$ starting with $W_1$. Next, for $2 \leq j \leq r$ each term in the sum $\Sigma_{12}$, corresponds to the clockwise sum of weights on path $P^{(j)}$ starting with $W_1^{(j)}$. Therefore, using the Lemma 2.2, we have \[
\eta(v) = \frac{w^{(2)}(dC^{(1)})}{w_1} + \sum_{j=2}^{r} \frac{w^{(2)}_j}{w_j} = \lambda.
\]

**Case 2:** For $v \in \{u_1, u_2, \ldots, u_n, v_1^{(1)}, \ldots, v_r^{(1)}\}$.

Note that, \[
\sum_{j=1}^{r} \frac{W_0^{(j)}}{w_j} - \sum_{j=2}^{r} \frac{W_0^{(1)}}{w_j} = \frac{W_0^{(1)}}{w_1} - \sum_{j=2}^{r} \frac{d(u_n, v_{m_j}^{(j)})}{w_j}. \]

Thus, \[
\eta(v) = \left( \frac{W_0^{(1)}}{w_1} - \sum_{j=2}^{r} \frac{d(u_n, v_{m_j}^{(j)})}{w_j} \right) d(u_0, v) + \sum_{i=1}^{n} \frac{W_i}{w_1} d(u_i, v) + \sum_{j=1}^{r} \sum_{i=1}^{m_j} \frac{W_i^{(j)}}{w_j} d(v_i^{(j)}, v) \]

\[
= \frac{1}{w_1} \left[ W_0^{(1)} d(u_0, v) + \sum_{i=1}^{n} W_i d(u_i, v) + \sum_{i=1}^{m_1} W_i^{(1)} d(v_i^{(1)}, v) \right] \]

\[
+ \sum_{j=2}^{r} \frac{1}{w_j} \left[ \sum_{i=1}^{m_j} W_i^{(j)} d(v_i^{(j)}, v) - d(u_n, v_{m_j}^{(j)})d(u_0, v) \right] \]

\[
= \Sigma_{21} + \Sigma_{22}.
\]

For $2 \leq j \leq r$, $1 \leq i \leq m_j$, we have $d(v_i^{(j)}, v) = d(v_i^{(j)}, u_0) + d(u_0, v)$. So

\[
\Sigma_{22} = \sum_{j=2}^{r} \frac{1}{w_j} \left[ \sum_{i=1}^{m_j} W_i^{(j)} d(v_i^{(j)}, u_0) + \left( \sum_{i=1}^{m_j} W_i^{(j)} \right) d(u_0, v) - d(u_n, v_{m_j}^{(j)})d(u_0, v) \right] \]

\[
= \sum_{j=2}^{r} \frac{1}{w_j} \left( \sum_{i=1}^{m_j} W_i^{(j)} d(v_i^{(j)}, u_0) \right) = \Sigma_{12} = \sum_{j=2}^{r} \frac{w_j^{(2)}}{w_j},
\]

and the sum $\Sigma_{21}$ corresponds to the clockwise sum of weights on $dC^{(1)}$ starting with $W_z$, where $v \rightarrow z$ and $z \in dC^{(1)}$. Hence, by similar argument as in the Case 1, yields $\eta(v) = \lambda$.  

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\textbf{Case 3}: For \( v \in \{ v_i^{(j)} : 2 \leq j \leq r; 1 \leq i \leq m_j \} \).

Let \( v = v_a^{(j_0)} \). Similar to Case 2, using \( \sum_{j=1}^{r} \frac{W_0^{(j)}}{w_j} - \sum_{j=2}^{r} \frac{w_j}{w_j} = \frac{W_0^{(1)}}{w_1} - \sum_{j=2}^{r} \frac{d(u_n, v_{m_j}^{(j)})}{w_j} \) and along with the fact that \( d(y, v) = d(y, u_n) + d(u_n, v) \) for every vertex \( y \) in \( dC(1) \), we have

\[
\eta(v) = \frac{1}{w_1} \left[ W_0^{(1)} d(u_0, u_n) + \sum_{i=1}^{n} W_i d(u_i, u_n) + \sum_{i=1}^{m_1} W_i^{(1)} d(v_i^{(1)}, u_n) \right] + \frac{1}{w_1} \left[ \sum_{i=1}^{n} W_i + \sum_{i=0}^{m_1} W_i^{(1)} \right] \left( d(u_n, v) - \sum_{j=2}^{r} \frac{1}{w_j} \left( \sum_{i=1}^{m_j} W_i^{(j)} d(v_i^{(j)}, v) - d(u_n, v_{m_j}^{(j)}) d(u_0, v) \right) \right)
\]

\[
= \frac{1}{w_1} \left[ W_0^{(1)} d(u_0, u_n) + \sum_{i=1}^{n} W_i d(u_i, u_n) + \sum_{i=1}^{m_1} W_i^{(1)} d(v_i^{(1)}, u_n) \right] + \left[ d(u_n, v) - \sum_{j=2}^{r} \frac{1}{w_j} \left( \sum_{i=1}^{m_j} W_i^{(j)} d(v_i^{(j)}, v) - d(u_n, v_{m_j}^{(j)}) d(u_0, v) \right) \right]
\]

\[
= \Sigma_{31} + \Sigma_{32} + \Sigma_{33}.
\]

Note that, the sum \( \Sigma_{31} \) corresponds to the clockwise sum of weights on \( dC(1) \), starting with \( W_0^{(1)} \) and for \( \Sigma_{33} \), proceeding similar to Case 2, whenever \( 2 \leq j \leq r; 1 \leq i \leq m_j \), with \( j \neq j_0 \), we have \( d(v_i^{(j)}, v) = d(v_i^{(j)}, u_0) + d(u_0, v) \) and hence

\[
\Sigma_{31} + \Sigma_{33} = \frac{w^{(2)}(dC(1))}{w_1} + \sum_{j=2}^{r} \frac{w_j^{(2)} w_{j_0}}{w_j}.
\]

By Lemma 2.3, \( \Sigma_{32} = \frac{w^{(2)}(dC(1))}{w_1} \) and this completes the proof. \( \square \)

Consider an adjacency matrix \( A^{(in)} = [A_{xy}] \) of weighted graph \( dC(n, m_1, \cdots, m_r) \), where

\[
A_{xy} = \begin{cases} 
\frac{1}{w_1}, & \text{if } x \rightarrow y \text{ and } x \in \{ u_0, u_1, \ldots, u_{n-1} \}, \\
\frac{1}{w_j}, & \text{if } x = u_n \text{ and } y = v_i^{(j)}; 1 \leq j \leq r, \\
\frac{1}{w_j}, & \text{if } x \rightarrow y \text{ and } x = v_i^{(j)}; 1 \leq i \leq m_j, \ 1 \leq j \leq r, \\
0, & \text{otherwise},
\end{cases}
\]

and let \( D^{(out)} = \text{diag}(\partial(x)) \) be the diagonal matrix, where \( \partial(x) = \sum_y A_{x,y} \), the out degree of \( x \).

Let \( L = D^{(out)} - A^{(in)} \) denote the Laplacian matrix of the weighted graph \( dC(n, m_1, \cdots, m_r) \), which satisfies \( L1 = 0 \) and similar matrices for directed graphs were studied in [5]. Now we define the Laplacian-like matrix \( \mathcal{L} \) of weighted graph \( dC(n, m_1, \cdots, m_r) \), as the Laplacian matrix \( L \) perturbed by a matrix \( \tilde{L} \) such that \( \mathcal{L}1 = 1^T \tilde{L} = 0 \) and is given by

\[
\mathcal{L} = L + \tilde{L},
\]

(2.7)
with \( \hat{L} = [\hat{L}_{xy}] \) where
\[
\hat{L}_{xy} = \begin{cases}  
\sum_{j=2}^{r} \frac{1}{w_j}, & \text{if } x = u_n \text{ and } y = u_0, \\
-\sum_{j=2}^{r} \frac{1}{w_j}, & \text{if } x = y = u_n, \\
0, & \text{otherwise.}
\end{cases}
\]

The lemma below gives one of the requisite condition for the distance matrix \( D \) to be a Laplacian-like expressible matrix.

**Lemma 2.9.** Let \( D \) be the distance matrix of the weighted graph \( dC(n, m_1, \cdots, m_r) \) and \( L \) be the Laplacian-like expressible matrix.

**Proof.** Let \( \hat{L}D = [(\hat{L}D)_{xy}] \) and it is easy to see
\[
(\hat{L}D)_{xy} = \begin{cases}  
-\sum_{j=2}^{r} \frac{\hat{w}_1}{w_j}, & \text{if } x = u_n, \ y = u_0, u_1, \ldots, u_{n-1}, \\
\sum_{j=2}^{r} \frac{w_j}{w_j}, & \text{if } x = u_n, \ y = u_n, v^{(j)}_i; 1 \leq i \leq m_j, 1 \leq j \leq r, \\
0, & \text{otherwise.}
\end{cases}
\]

Now we will compute \( LD + I = [(LD + I)_{xy}] \) and using the definition of Laplacian matrix \( L = D^{(out)} - A^{(in)} \), we have
\[
(LD + I)_{xy} = \begin{cases}  
\sum_{x \rightarrow z} \frac{d(x, y) - d(z, y)}{w_z} & \text{if } x \neq y, \\
1 - \sum_{x \rightarrow z} \frac{d(z, x)}{w_z} & \text{if } x = y,
\end{cases}
\]
where
\[
w_z = \begin{cases}  
w_1, & \text{if } z \in \{u_0, u_1, \ldots, u_n\}, \\
w_j, & \text{if } z \in \{v^{(j)}_i; 1 \leq i \leq m_j, 1 \leq j \leq r\}.
\end{cases}
\]

**Case 1:** For \( x \neq u_n \).

Using Eqn. (2.9) for \( x \neq u_n \), yields
\[
(LD + I)_{xy} = \begin{cases}  
\frac{W_z}{w_z} & \text{if } x \rightarrow z \text{ and } x \neq y, \\
1 - \frac{w_z - W_z}{w_z} & \text{if } x \rightarrow z \text{ and } x = y,
\end{cases}
\]
and hence \( (LD + I)_{xy} = \frac{W_z}{w_z} \), where \( x \rightarrow z \).

**Case 2:** For \( x = u_n \).

For \( x = u_n \) and \( y \in \{u_0, u_1, \ldots, u_{n-1}\} \), we get \( d(u_n, y) = \hat{w}_1 + d(u_0, y) \) and \( d(v^{(j)}_i, y) = (\hat{w}_j - W^{(j)}_1) + d(u_0, y); 1 \leq j \leq r \). Then, Eqn. (2.9) yields
\[
(LD + I)_{xy} = \sum_{j=1}^{r} \frac{d(u_n, y) - d(v^{(j)}_i, y)}{w_j} = \sum_{j=1}^{r} \frac{\hat{w}_1 - (\hat{w}_j - W^{(j)}_1)}{w_j} = \left[ \sum_{j=1}^{r} \frac{W^{(j)}_1}{w_j} - \sum_{j=2}^{r} \frac{\hat{w}_j}{w_j} \right] + \sum_{j=2}^{r} \frac{\hat{w}_1}{w_j}.
\]
For \( x = u_n = y \), using Eqn. (2.9) and \( d(v^{(j)}_1, u_n) = w_j - W^{(j)}_1; 1 \leq j \leq r \), we have

\[
(LD + I)_{xy} = 1 - \sum_{j=1}^{r} \frac{d(v^{(j)}_1, u_n)}{w_j} = 1 - \sum_{j=1}^{r} \frac{w_j - W^{(j)}_1}{w_j} = \sum_{j=1}^{r} \frac{W^{(j)}_1}{w_j} - (r - 1)
\]

\[
= \sum_{j=1}^{r} \frac{W^{(j)}_1}{w_j} - \sum_{j=2}^{r} w_c + \hat{w}_j = \left[ \sum_{j=1}^{r} \frac{W^{(j)}_1}{w_j} - \sum_{j=2}^{r} \frac{\hat{w}_j}{w_j} \right] - \sum_{j=2}^{r} \frac{w_c}{w_j}.
\]

For \( x = u_n \) and \( y = v^{(j_0)}_i \), for some \( 1 \leq j_0 \leq r \) and \( 1 \leq i \leq m_{j_0} \). Note that, if \( j \neq j_0 \), then \( d(v^{(j)}_1, v^{(j_0)}_i) = (w_j - W^{(j)}_1) + d(u_n, v^{(j_0)}_i) \). Again, using Eqn. (2.9), we get

\[
(LD + I)_{xy} = \frac{d(u_n, v^{(j_0)}_i) - d(v^{(j_0)}_1, v^{(j_0)}_i)}{w_j} + \sum_{j=1}^{r} \frac{d(u_n, v^{(j_0)}_i) - d(v^{(j)}_1, v^{(j_0)}_i)}{w_j}
\]

\[
= \frac{W^{(j_0)}_1}{w_{j_0}} + \sum_{j=1}^{r} \frac{W^{(j)}_1}{w_j} - w_j = \sum_{j=1}^{r} \frac{W^{(j)}_1}{w_j} - (r - 1)
\]

\[
= \left[ \sum_{j=1}^{r} \frac{W^{(j)}_1}{w_j} - \sum_{j=2}^{r} \frac{\hat{w}_j}{w_j} \right] - \sum_{j=2}^{r} \frac{w_c}{w_j}.
\]

By Eqn. (2.7), \( LD + I = (LD + I) + \hat{LD} \). So using Eqn. (2.8) and the above cases, we have shown \( (LD + I)_{xy} = \beta(x) \), for every \( y \). Hence the result follows. \( \square \)

The next result gives the inverse of the distance matrix of \( dC(n, m_1, \cdots, m_r) \), whenever it exists.

**Theorem 2.10.** Let \( D \) be the distance matrix of the weighted digraph \( dC(n, m_1, \cdots, m_r) \). Let \( L \) be the Laplacian-like matrix as defined in Eqn. (2.7) and \( \lambda, \alpha, \beta \) be as defined in Eqn. (2.6). Then \( D \) is a left LapExp\( (\lambda, \alpha, \beta, L) \) matrix. Moreover, if \( \det D \neq 0 \), then

\[
D^{-1} = -L + \frac{1}{\lambda} \beta \alpha^T.
\]

**Proof.** Using Lemmas 2.7, 2.8 and 2.9, it is clear that \( D \) is a left LapExp\( (\lambda, \alpha, \beta, L) \) matrix. Further, \( \det D \neq 0 \) implies that \( \lambda \neq 0 \) and so by Lemma 1.4, the result follows. \( \square \)

**Remark 2.11.** Calculations similar to Lemmas 2.8 and 2.9, also leads to \( D\beta = \lambda I \) and \( DL + I = \alpha \alpha^T \) and hence the distance matrix \( D \) of the weighted \( dC(n, m_1, \cdots, m_r) \), is also a right LapExp\( (\lambda, \alpha, \beta, L) \) matrix.

The next section deals with the determinant and inverse of the distance matrix for weighted cactoid-type digraphs.

### 2.3 Distance Matrix of Weighted Cactoid-type Digraphs: Determinant and Inverse

The result below gives the determinant of weighted cactoid-type digraph which follows directly from Theorem 1.3. But in view of Theorems 2.5 and 2.6 it is possible to find a exact form for the determinant.
Theorem 2.12. Let \( G \) be a weighted cactoid-type digraph with blocks \( G_1, G_2, \cdots, G_b \). For \( 1 \leq t \leq b \), let \( D_t \) and \( D_t \) be the distance matrix of \( G \) and \( G_t \) respectively. If \( \det D_t \neq 0 \), for all \( 1 \leq t \leq b \), then the determinant of the distance matrix \( D \) is given by
\[
\det D = \lambda \prod_{t=1}^b \det D_t = \lambda \det D,
\]
where \( \lambda = \sum_{t=1}^b \lambda_t \) with \( \lambda_t = \frac{\det D_t}{\det D} \).

Now we state and prove a result which gives the inverse of a weighted cactoid-type digraph.

Theorem 2.13. Let \( D \) be the distance matrix of a weighted cactoid-type digraph \( G \) with blocks \( G_t \); \( 1 \leq t \leq b \). If \( \det D \neq 0 \) and \( \det D_t \neq 0 \), for all \( 1 \leq t \leq b \), then there exists a Laplacian-like matrix \( L \), column vectors \( \alpha, \beta \) and a real number \( \lambda \), such that
\[
D^{-1} = -L + \frac{1}{\lambda} \beta \alpha^T.
\]

Proof. For \( 1 \leq t \leq b \), let \( D_t \) be the distance matrix of the block \( G_t \). Since a block \( G_t \) is a weighted digraph \( dC(n, m_1, \cdots, m_r) \), so by Theorem 2.10, \( D_t \) is a left LapExp(\( \lambda_t, \alpha_t, \beta_t, L_t \)) matrix. Therefore, using the Assumption 1.6 on graph \( G \) and Theorem 1.7, \( D \) is a left LapExp(\( \lambda, \alpha, \beta, L \)) matrix. Further, by Theorem 2.12 the \( \det D \neq 0 \) implies that \( \lambda \neq 0 \) and so by Lemma 1.4, the result follows.

The next section deal with the determinant of the distance matrix for undirected and unweighted graph \( C(n, m_1, m_2, \cdots, m_r) \).

3 Determinant of the Distance Matrix for \( C(n, m_1, m_2, \cdots, m_r) \)

We first recall \( C(n, m_1, m_2, \cdots, m_r) \) is a undirected graph consisting of \( r \) cycles sharing a common path of length \( n \). In this section, we compute the determinant of the distance matrix for a class of \( C(n, m_1, m_2, \cdots, m_r) \). In literature, similar problems has been studied for the cases \( r = 1, 2 \) (for details see [3, 8, 9, 10, 15]) and we have shown that some of their results can be extended for \( r \geq 3 \). We recall some results for the case \( r = 1 \), i.e. cycles \( C_n \); \( n \geq 1 \), useful for the subsequent results. For cycles of even length \( C_{2k} \), by [3, Theorem 3.1] we get \( \det D(C_{2k}) = 0 \). Next, for the odd cycle \( C_{2k+1} \) the vertices are labeled so that the vertex \( i \) is adjacent to vertices \( i-1 \) and \( i+1 \) (the indices are taken modulo \( 2k+1 \)). Let \( C \) be the cyclic permutation matrix of order \( 2k+1 \) such that \( C_{i,i+1} = 1, 1 \leq i \leq 2k+1 \) (again taking indices modulo \( 2k+1 \)). The next result gives the determinant and the inverse of the distance matrix for odd cycles.

Theorem 3.1. ([3, Theorem 3.1] and [15, Lemma 2.1]) Let \( D \) be the distance matrix of the odd cycle \( C_{2k+1} \) on \( 2k+1 \) vertices. Then, \( \det D = k(k+1) \) and the inverse is given by
\[
D^{-1} = -2I - C^k - C^{k+1} + \frac{2k+1}{k(k+1)}J.
\]

In Section 2, we found that \( C(n, m_1, m_2, \cdots, m_r) \) can be identified with \( r+1 \) paths. Due to this identification, we will be able to show for a large class the determinant of distance matrix is zero. We will follow the same indexing of vertices and conventions, as in the Section 2, i.e., the vertex set of \( C(n, m_1, m_2, \cdots, m_r) \) is \( \{u_i : 1 \leq i \leq n\} \cup \{u_i^{(j)} : 1 \leq i \leq m_j \ ; 1 \leq j \leq r\} \) with \( m_1 \leq m_2 \leq \cdots \leq m_r \).
Given a unweighed path $P$, if we draw a line through the middle of the path, then the following are true: (a) if $P$ has even number of vertices, then vertex set is partitioned into two equal halves, (b) if $P$ has odd number of vertices, then vertex set is partitioned into two equal halves and a middle vertex. Since $C(n, m_1, m_2, \cdots, m_r)$ is a planar graph and also can be identified with $r + 1$ paths, so if we draw a line “$L$” passing through the middle of all the $r + 1$ paths, then the vertex set is partitioned in the following:

1. The set of vertices in the upper half of the line $L$ is denoted by $UH_L$.
2. The set of vertices in the lower half of the line $L$ is denoted by $LH_L$.
3. Vertices on the line $L$ (non-empty only when the path contains an odd number of vertices).

![Figure 3](a) $C(2k, 1, 1, \ldots, 1)$ (b) $C(2k, 1, 1, \ldots, 1, 2l)$ (c) $C(2k - 1, 1, 1, \ldots, 1)$

We begin with the result which gives the determinant of the distance matrix, whenever the common path is of even length $\geq 4$.

**Theorem 3.2.** Let $r \geq 2$ and $n \geq 4$ with $n$ is even. Let $D$ be the distance matrix of unweighted $C(n, m_1, m_2, \cdots, m_r)$. Then $\det D = 0$.

**Proof.** We will prove this result case by case, by dividing the problem in the following cases:

Case 1. $m_j = 1$, for all $1 \leq j \leq r$.

Case 2. For $1 \leq j \leq r$, exactly one of the $m_j$’s is even and all other $m_j = 1$.

Case 3. For $1 \leq j \leq r$, at least two of $m_j$’s are even.

Case 4. For $1 \leq j \leq r$, at least one of $m_j$’s (say $m_s$), is odd with $m_s \geq 3$.

For $n \geq 4$ and $n$ is even, suppose $n = 2k$ whenever $k \geq 2$. Now we provide the proofs.

**Case 1.** Since $m_j = 1; 1 \leq j \leq r$, so $d(u_0, u_n) = 2$, (see Figure 3(a)). For any vertex $y$ on the line $L$, we have $d(y, u_0) - d(y, u_{k-1}) = d(y, u_0) - d(y, u_{k+1})$. Next, if $y$ is in UH$L$, then $d(y, u_0) = d(y, u_n) + 2$ and $d(y, u_{k-1}) = d(y, u_{k+1}) + 2$. Similarly, if $y$ is in LH$L$, then $d(y, u_0) = d(y, u_0) + 2$ and $d(y, u_{k+1}) = d(y, u_{k-1}) + 2$. Thus $d(y, u_0) - d(y, u_{k-1}) + d(y, u_{k+1}) = d(y, u_n) = 0$ for every vertex $y$ in $C(n, m_1, m_2, \cdots, m_r)$. It implies that, the column in $D$ corresponding to vertices \{u_0, u_{k-1}, u_{k+1}, u_n\} are linearly dependent and hence $\det D = 0$.

**Case 2.** Following the conventions, assume $m_j = 1$, $1 \leq j \leq r - 1$ and $m_r = 2l$; where $l \geq 1$. Since $r \geq 2$, so $d(u_0, u_n) = 2$ (see Figure 3(b)).

For any vertex $y$ on the line $L$, we have $d(y, u_0) = d(y, u_n)$ and $d(y, u_{k-1}) = d(y, u_{k+1})$. Next, for $y$ in UH$L$, we have the following sub cases.
• Whenever \( y \neq v_1^{(r)} \), we have \( d(y, u_0) = d(y, u_n) + d(u_n, u_0) = d(y, u_n) + 2 \) and similar arguments also leads to \( d(y, u_{k+1}) = d(y, u_{k-1}) + 2 \).

• Whenever \( y = v_1^{(r)} \), we have \( d(y, u_0) = d(y, u_n) + 1 \) and \( d(y, u_{k-1}) = d(y, u_{k+1}) + 1 \).

Similarly, for \( y \) in LHC:

• If \( y \neq v_{i+1}^{(r)} \), then \( d(y, u_n) = d(y, u_0) + 2 \) and \( d(y, u_{k+1}) = d(y, u_{k-1}) + 2 \).

• If \( y = v_{i+1}^{(r)} \), then \( d(y, u_n) = d(y, u_0) + 1 \) and \( d(y, u_{k+1}) = d(y, u_{k-1}) + 1 \).

Thus, \( d(y, u_0) - d(y, u_{k-1}) + d(y, u_{k+1}) - d(y, u_n) = 0 \), for every vertex \( y \). Hence, the column in \( D \) corresponding to vertices \( \{u_0, u_{k-1}, u_{k+1}, u_n\} \) are linearly dependent and the results holds good for this case.

**Case 3.** For \( 1 \leq s, t \leq r \), let us assume \( m_s = 2l \) and \( m_t = 2p \), where \( l, p \geq 1 \). If a vertex \( y \) is on the line \( \mathfrak{L} \), then \( d(y, v_{i+1}^{(s)}) = d(y, v_1^{(s)}) + d(y, v_{p+1}^{(t)}) = d(y, v_{p+1}^{(t)}) \). For \( y \) in UH\( \mathfrak{L} \), we have \( d(y, v_{i+1}^{(s)}) = d(y, v_1^{(s)}) + 1 \) and \( d(y, v_{p+1}^{(t)}) = d(y, v_p^{(t)}) + 1 \). Similarly, for \( y \) in LHC, we get \( d(y, v_{i+1}^{(s)}) = d(y, v_1^{(s)}) + 1 \) and \( d(y, v_{p+1}^{(t)}) = d(y, v_p^{(t)}) + 1 \). Thus, \( d(y, v_1^{(s)}) - d(y, v_{i+1}^{(s)}) - d(y, v_p^{(t)}) + d(y, v_{p+1}^{(t)}) = 0 \), for every vertex \( y \). Hence, the columns in \( D \) corresponding to vertices \( \{v_1^{(s)}, v_{i+1}^{(s)}, v_p^{(t)}, v_{p+1}^{(t)}\} \) are linearly dependent and the result follows.

**Case 4.** Assume that, \( m_s = 2l + 1 \), where \( l \geq 1 \). If a vertex \( y \) is on the line \( \mathfrak{L} \), then \( d(y, u_{k-1}) = d(y, u_{k+1}) + 1 \) and \( d(y, v_1^{(s)}) = d(y, v_{i+2}^{(s)}) \). Next, for \( y \) in UH\( \mathfrak{L} \), we have \( d(y, v_{i+2}^{(s)}) = d(y, v_1^{(s)}) + 2 \) and \( d(y, u_{k-1}) = d(y, u_{k+1}) + 2 \). Finally, for \( y \) in LHC, we get \( d(y, v_1^{(s)}) = d(y, v_{i+2}^{(s)}) + 2 \) and \( d(y, u_{k-1}) = d(y, u_{k+1}) + 2 \). Thus, for all vertices \( y \), we have \( d(y, u_{k-1}) - d(y, u_{k+1}) + d(y, v_1^{(s)}) - d(y, v_{i+2}^{(s)}) = 0 \), which implies that, the columns of \( D \) corresponding to vertices \( \{u_{k-1}, u_{k+1}, v_1^{(s)}, v_{i+2}^{(s)}\} \) are linearly dependent. Hence \( \det D = 0 \). This completes the proof.

Now we consider the cases whenever the common path is of odd length \( \geq 3 \), and we begin with the lemma below.

**Lemma 3.3.** Let \( r \geq 2 \), \( n \geq 3 \) and \( n \) is odd. Let \( D \) be the distance matrix of \( C(n, m_1, m_2, \ldots, m_r) \). If \( n = 2k - 1 \), where \( k \geq 2 \) and \( m_j = 1 \), where \( 1 \leq j \leq r \), then the determinant of \( D \) given by \( \det D = (-2)^{r-1} \left[ k(k+1) - (r-1)(3k^2 - k - 2) \right] \).

**Proof.** Let \( D_{11} \) be the distance matrix of the odd cycle of length \( 2k+1 \), the induced subgraph generated by the vertices \( \{u_0, u_1, \ldots, u_n, v_1^{(1)}\} \) in \( C(n, m_1, m_2, \ldots, m_r) \), see Figure 3(c). We partitioned the distance matrix of \( C(n, m_1, m_2, \ldots, m_r) \) as below:

\[
D = \begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}.
\]

For \( 1 \leq j \leq r \), let \( R_{v_1^{(j)}}^{(j)}/C_{v_1^{(j)}}^{(j)} \) denote the row/column of \( D \) corresponding to the vertex \( v_1^{(j)} \). With this observation, all the rows and columns of \( D \) corresponding the vertices \( \{v_1^{(j)} : 1 \leq j \leq r\} \) are equal, except at the diagonal entries. So, we begin with the elementary column operations on \( D \), \( C_{v_1^{(j)}}^{(j)} \leftarrow C_{v_1^{(j)}}^{(j)} - C_{v_1^{(1)}}^{(1)} \), for all \( 2 \leq j \leq r \), followed by the row operations \( R_{v_1^{(j)}}^{(j)} \leftarrow R_{v_1^{(j)}}^{(j)} - R_{v_1^{(1)}}^{(1)} \), for all \( 2 \leq j \leq r \). Then the block form of the resulting matrix is given by,

\[
\tilde{D} = \begin{bmatrix}
\tilde{D}_{11} & \tilde{D}_{12} \\
\tilde{D}_{21} & \tilde{D}_{22}
\end{bmatrix},
\]
where $\tilde{D}_{12} = \begin{bmatrix} 0_{2k \times (r-1)} \\ 2I_{1 \times (r-1)} \end{bmatrix}$ with $\tilde{D}_{21} = 2(I_{r-1} + J_{r-1})$.

Using Theorem 3.1, $D_{11}$ is invertible and the inverse is given by

$$D_{11}^{-1} = -2I - C^k - C^{k+1} + \frac{2k+1}{k(k+1)}J,$$

where $C_{ij}^k = \begin{cases} 1 & \text{if } j = [i + k] \pmod{2k+1}, \\ 0 & \text{otherwise.} \end{cases}$

It is easy to see, the only non zero row of $C^k \tilde{D}_{12}$ is the $(k+1)$th row, equals to $2I$. Similarly, for $C^{k+1}\tilde{D}_{12}$ the $k$th row is $2I^T$ and remaining are zero. Thus $\tilde{D}_{21} D_{11}^{-1} \tilde{D}_{12} = 4 \left( \frac{2k+1}{k(k+1)} - 2 \right) J_{r-1}$.

Using Proposition 1.8, we have

$$\det \tilde{D} = \det D_{11} \times \det(D_{22} - \tilde{D}_{21} D_{11}^{-1} \tilde{D}_{12})$$

$$= k(k+1) \det \left( -2(I_{r-1} + J_{r-1}) - 4 \left( \frac{2k+1}{k(k+1)} - 2 \right) J_{r-1} \right)$$

$$= k(k+1) \det \left( -2I_{r-1} + \left( 6 - \frac{4(2k+1)}{k(k+1)} \right) J_{r-1} \right)$$

$$= (-2)^{r-1} \left[ k(k+1) - (r-1)(3k^2 - k - 2) \right]$$

and hence the result follows. \(\square\)

The next theorem gives the determinant of the distance matrix, whenever the common path is of odd length $\geq 3$.

**Theorem 3.4.** Let $r \geq 2$, $n \geq 3$ and $n$ is odd. Let $D$ be the distance matrix of the $C(n, m_1, m_2, \ldots, m_r)$. Then,

$$\det D = \begin{cases} (-2)^{r-1} \left[ k(k+1) - (r-1)(3k^2 - k - 2) \right], & \text{if } n = 2k-1; \ m_j = 1, 1 \leq j \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** In view of Lemma 3.3, it is enough to establish the result for the following cases:

Case 1. For $1 \leq j \leq r$, exactly one of the $m_j$’s is odd (say $m_s$), with $m_s \geq 3$ and all other $m_j = 1$.

Case 2. For $1 \leq j \leq r$, at least two of $m_j$’s are odd ($\geq 3$).

Case 3. For $1 \leq j \leq r$, at least one of $m_j$’s is even.

**Case 1.** Suppose $m_s = 2l + 1$, where $l \geq 3$. Note that $d(u_0, u_n) = 2$. If a vertex $y$ is on the line $\mathcal{L}$, then $d(y, u_0) = d(y, u_n)$ and $d(y, v_{i+2}^{(s)}) = d(y, v_{i}^{(s)})$. For $y$ in $\mathcal{UH}_s$, we have $d(y, u_0) = d(y, u_n) + 2$ and $d(y, v_{i+2}^{(s)}) = d(y, v_{i}^{(s)}) + 2$. Finally, for $y$ in $\mathcal{LU}_s$, we get $d(y, u_0) = d(y, u_n) + 2$ and $d(y, v_{i}^{(s)}) = d(y, v_{i+2}^{(s)})$. Thus, $d(y, u_0) - d(y, u_n) + d(y, v_{i}^{(s)}) - d(y, v_{i+2}^{(s)}) = 0$ for all vertex $y$. Hence $\det D = 0$.

**Case 2.** For $1 \leq s, t \leq r$, let us assume $m_s = 2l$ and $m_t = 2p$ where $l, p \geq 1$. If a vertex $y$ is on the line $\mathcal{L}$, then $d(y, v_{i}^{(s)}) = d(y, v_{i+2}^{(s)})$ and $d(y, v_{p}^{(t)}) = d(y, v_{p+2}^{(t)})$. Next for $y$ in $\mathcal{UH}_s$, we have $d(y, v_{i+2}^{(s)}) = d(y, v_{i}^{(s)}) + 2$ and $d(y, v_{p+2}^{(t)}) = d(y, v_{p}^{(t)}) + 2$. Similarly, for $y$ in $\mathcal{LU}_s$, we get $d(y, v_{i}^{(s)}) = d(y, v_{i+2}^{(s)}) + 2$ and $d(y, v_{p}^{(t)}) = d(y, v_{p+2}^{(t)}) + 2$. Thus, $d(y, v_{i}^{(s)}) - d(y, v_{i+2}^{(s)}) - d(y, v_{p}^{(t)}) + d(y, v_{p+2}^{(t)}) = 0$ for every vertex $y$ and the result holds true for this case.

**Case 3.** Since $n \geq 3$ and $n$ is odd, so let $n = 2k + 1$ where $k \geq 1$. Also assume $m_s = 2l$, where $l \geq 1$. For this case we will show column corresponding to vertices $\{u_k, u_{k+1}, v_{i}^{(s)}, v_{i+1}^{(s)}\}$ are linearly
dependent. If a vertex $y$ is on the line $2$, then $d(y, u_k) = d(y, u_{k+1})$ and $d(y, v^{(s)}_l) = d(y, v^{(s)}_{l+1})$. For $y$ in $UH\Sigma$, we have $d(y, u_k) = d(y, u_{k+1}) + 1$ and $d(y, v^{(s)}_l) = d(y, v^{(s)}_{l+1}) + 1$. Finally, for $y$ in $LH\Sigma$, we have $d(y, u_{k+1}) = d(y, u_k) + 1$ and $d(y, v^{(s)}_l) = d(y, v^{(s)}_{l+1}) + 1$. Thus $d(y, u_k) - d(y, u_{k+1}) + d(y, v^{(s)}_l) - d(y, v^{(s)}_{l+1}) = 0$ for every vertex $v$ and hence the result follows.

Now we consider the distance matrix for few cases whenever the common path is of length $n = 1$ and $n = 2$, respectively.

**Theorem 3.5.** Let $n = 1$ and $D$ be the distance matrix of the $C(n, m_1, m_2, \ldots, m_r)$. Then

(i) if $m_j = 1, 1 \leq j \leq r$, then $\det D = (-1)^{r-1} 2^{r-2}$ and

(ii) if one of the $m_j$'s is even, $1 \leq j \leq r$, then $\det D = 0$.

**Proof.** In literature, the graph $n = 1$ with $m_j = 1, 1 \leq j \leq r$, is denoted by $T_r$ and in [6], it was shown that $\det D(T_r) = (-1)^{r-1} 2^{r-2}$. To prove (ii), we assume $m_s = 2l$, where $l \geq 1$. If a vertex $y$ is on the line $2$, then $d(y, u_0) = d(y, u_1)$ and $d(y, v^{(s)}_l) = d(y, v^{(s)}_{l+1})$. Next, for $y$ in $UH\Sigma$, we have $d(y, u_0) = d(y, u_1) + 1$ and $d(y, v^{(s)}_l) = d(y, v^{(s)}_{l+1}) + 1$. Similarly, for $y$ in $LH\Sigma$, we have $d(y, u_1) = d(y, u_0) + 1$ and $d(y, v^{(s)}_l) = d(y, v^{(s)}_{l+1}) + 1$. Hence $d(y, u_1) - d(y, u_0) - d(y, v^{(s)}_l) + d(y, v^{(s)}_{l+1}) = 0$, for every vertex $y$ and the result follows.

**Theorem 3.6.** Let $r \geq 2, n = 2$ and $D$ be the distance matrix of the $C(n, m_1, m_2, \ldots, m_r)$. Then

(i) if $m_j = 1, 1 \leq j \leq r$, then $\det D = (-1)^{r+1} 2^{r+2}(r - 1)$ and

(ii) if one of the $m_j$’s (say, $m_s$) is odd, with $m_s \geq 3$, $1 \leq j \leq r$, then $\det D = 0$.

**Proof.** Observe that, for $n = 2$ with $m_j = 1, 1 \leq j \leq r$, the graph is a complete bipartite graph $K_{2, r+1}$, with the following vertex partition $\{u_0, u_2\}$ and $\{u_1, v^{(s)}_l; 1 \leq j \leq r\}$. Thus, using [6, 16], the result follows. Next to prove (ii), we assume $m_s = 2l + 1$, where $l \geq 1$. If a vertex $y$ is on the line $2$, then $d(y, u_0) = d(y, u_2)$ and $d(y, v^{(s)}_l) = d(y, v^{(s)}_{l+2})$. For $y$ in $UH\Sigma$, we have $d(y, u_0) = d(y, u_2) + 2$ and $d(y, v^{(s)}_l) = d(y, v^{(s)}_{l+2}) + 2$. Finally, for $y$ in $LH\Sigma$, we have $d(y, u_2) = d(y, u_0) + 2$ and $d(y, v^{(s)}_l) = d(y, v^{(s)}_{l+2}) + 2$. Hence $d(y, u_0) - d(y, u_2) + d(y, v^{(s)}_l) - d(y, v^{(s)}_{l+2}) = 0$, for every vertex $y$ and the result follows.

**Remark 3.7.** The cases when $n = 1$ with $m_i \geq 3$ is odd and when $n = 2$ with $m_i$ even has been computationally seen to have non-zero determinant but there is no constructive proof yet.

4 Conclusion

In this article, we first compute the determinant and cofactor of the distance matrix for weighted digraphs consisting of finitely many oriented cycles, that shares a common directed path. If the distance matrix is invertible, then we have shown the inverse as a rank one perturbation of a multiple of the Laplacian-like matrix similar to trees. In this search, we found the Laplacian-like matrix is a perturbed weighted Laplacian for the digraph. Interestingly, the adjacency matrix involved in the weighted Laplacian for the digraph is not dependent on the weights on the individual edge, but only on the total weight of each of the cycles involved. It allows us to choose the weight on edge to be 0, without disturbing the structure.

Further, we compute the determinant of the distance matrix for weighted cactoid-type digraphs, if the cofactor of each block is non zero and find its inverse, whenever it exists. We also compute the
determinant of the distance matrix for a class of graphs consisting of finitely many cycles, sharing a common path.

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