Scale factor duality in string Bianchi cosmologies

Elisa Di Pietro and Jacques Demaret

Institute of Astrophysics and Geophysics
Group of Theoretical Cosmology
University of Liège
B-4000 LIEGE-BELGIUM

Abstract

We apply the scale factor duality transformations introduced in the context of the effective string theory to the anisotropic Bianchi-type models. We find dual models for all the Bianchi-types [except for types VIII and IX] and construct for each of them its explicit form starting from the exact original solution of the field equations. It is emphasized that the dual Bianchi class B models require the loss of the initial homogeneity symmetry of the dilatonic scalar field.

PACS numbers: 9890H, 9530S, 0450H.

1E-mail: dipietro@astro.ulg.ac.be
1 Introduction

String theory has recently motivated the study of cosmological models because its application to cosmology provides an alternative solution to the inflationary paradigm, the pre-Big Bang scenario\[1\]. Indeed, the low-energy string action possesses a symmetry property, called scale factor duality, which lets us expect that the present phase of the Universe evolution is preceded in time by a "naturally" inflationary pre-Big Bang phase. In such a scenario, the Big Bang represents only the peculiar instant of the Universe evolution in which its curvature and its density are maximal. Unfortunately, as far as we know, it is not already possible in this context to avoid this maximum to be infinite and so, to remove the initial singularity present in the standard model.

Several reasons lead us to think that the standard model characterized by an isotropic and spatially homogeneous spacetime cannot be extended until the first phases of the Universe\[2\]. Indeed, most cosmologists think that the primordial Universe was not necessarily isotropic: for them, it seems more natural and more general to say that the Universe began in a less symmetric state and became isotropic after some time. So, more general spacetimes than FLRW models are often invoked to describe the real dynamical behavior of the very early Universe. The cosmological spatially homogeneous and anisotropic spacetimes are the most symmetric models after those of FLRW type. They are called the Bianchi-type models.

Since the pre-Big Bang scenario concerns principally the early Universe that has to be described by an anisotropic spacetime and since the scale factor duality is at the root of this scenario, it is important to demonstrate that it remains also valid for any anisotropic spacetime. The aim of this paper is to inquire about the possibility of building explicit exact dual solutions – in the sense of scale factor duality – for different Bianchi models in order to check the validity of the pre-Big Bang scenario in this more general cosmological framework.

The problem of the scale factor duality has already been considered within the context of spatially homogeneous vacuum Brans-Dicke cosmologies by Clancy et al.\[3\]. This study is based on the Lagrangian formulation of the field equations and is restricted to Bianchi type A cosmological models, since it is well known that such formulation for class B models is ambiguous\[4\]. Moreover, the dilaton is supposed constant on the surfaces of homogeneity.
and the dual and the original equations are assumed to have the same isometry group, so that, in [3], duality symmetries are found to exist for type I, II, VI$_{-1}$ and VII$_0$.

In the present paper, we address this problem in the framework of effective string theory$^{2}$, in a more direct way. We rely on the construction of the dual metric from the original one, using the explicit scale factor duality transformations as deduced from the study of O(d,d)-invariance of the effective action of string theory ([1], [5]). In particular, we give explicit expressions of the dual counterparts of the exact solutions known for different Bianchi models in presence of a dilatonic field [3].

With the exception of Bianchi class $A$ types $VIII$ and $IX$ models, we are able to construct explicit dual solutions for each Bianchi spacetime. However, for class $B$ models ($III$, $IV$, $V$, $VI_{h\neq-1}$ and $VII_{h\neq0}$), we have to get rid of the hypothesis of spatial homogeneity of the dilatonic field and permit its effective inhomogeneity character.

2 The SFD symmetry of the effective action of string theory

In the four dimensional spacetime context, the low energy string effective action (in the string frame) can be written as$^{3}$

$$S_{eff} = \frac{1}{2\kappa^2} \int e^{-\phi} \left[ R + \nabla_\alpha \phi \nabla^\alpha \phi - \frac{1}{12} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \right] \sqrt{-g} \, d^4x \quad (1)$$

Our application of the scale factor duality (SFD) to the Bianchi-type models will be made assuming a vanishing antisymmetric field strength $H_{\alpha\beta\gamma}$. Within this assumption and by varying action (1) with respect to the metric $g_{\alpha\beta}$ and to the dilatonic scalar field $\phi$, we find respectively the following field equations [4]:

$$R^\alpha_{\ \beta} + g^{\alpha\delta} \nabla_\delta \nabla_\beta \phi = 0 \quad (2)$$

and

$$R + 2 \Box \phi - [\nabla \phi]^2 = 0 \quad (3)$$

$^{2}$Effective string theory with $H = 0$ corresponds to a special case of Brans-Dicke cosmologies with $\omega = -1$, where $\omega$ is the Brans-Dicke parameter.

$^{3}$Greek indices always run from 0 to 3.
where □ stands for the dalembertian operator.

It is well known that action (1) is invariant under a SFD transformation (1). We have showed recently [5] that SFD has the same form for any kind of metric, i.e.

$$G \rightarrow \bar{G} = G^{-1}$$  \hspace{1cm} (4)

$$\phi \rightarrow \bar{\phi} = \phi - \ln(\det G)$$  \hspace{1cm} (5)

The only difference between a FLRW metric, an anisotropic metric or an inhomogeneous metric appears in the building of the matrix $G$: it always contains the metric components relative to the coordinates the metric does not depend on, the other components remaining unchanged after duality transformations. We can express the only condition necessary for the application of these transformations as follows: "If the metric depends on a particular $x^\alpha$ coordinate, then we must have

$$g_{\alpha\beta} = 0$$  \hspace{1cm} (6)

for all $\beta \neq \alpha$" [5]. For example, if the metric depends explicitly on $x^0$ and $x^1$ and does not depend on $x^2$ and $x^3$, then we can use the transformations (4) and (5) only if the spacetime metric can take the following form:

$$g_{\alpha\beta} = \begin{pmatrix}
g_{00} & 0 & 0 \\
0 & g_{11} & 0 \\
0 & 0 & 0
\end{pmatrix} G$$

where $G$ is the following $2 \times 2$ matrix:

$$G = \begin{pmatrix}
g_{22} & g_{23} \\
g_{32} & g_{33}
\end{pmatrix}$$

3 SFD in Bianchi-type models

Bianchi-type models are anisotropic and spatially-homogeneous models in which a three-dimensional Lie group of isometries acts simply-transitively on the hypersurfaces of homogeneity (for an introduction to the anisotropic cosmologies, see e.g. [8]). They are nine in number but their classification permits to split them into two classes: there are six models in the class A (I, 4

4In what follows, we shall use the Bianchi models classification presented by Ryan and Shepley in [9].
As we shall see, for some of these models, the dual dilatonic field loses the spatial homogeneity symmetry of the initial field. This is due to the fact that, in some cases, the determinant of the matrix $G$ becomes dependent on a spacelike coordinate and so, in view of (5), the initial homogeneous dilaton is transformed in an inhomogeneous field after SFD. As will be shown later, this loss of symmetry does not affect the spacetime metric which still keeps its initial symmetry after duality.

The nine Bianchi-type metrics can all be written as follows if we adopt the hypothesis of diagonality of the spatial metric

$$ds^2 = - (\tilde{\omega}^0)^2 + a^2 (\tilde{\omega}^1)^2 + b^2 (\tilde{\omega}^2)^2 + c^2 (\tilde{\omega}^3)^2$$

(7)

where $a$, $b$ et $c$ are the scale factors, functions of the timelike coordinate only and where the $\tilde{\omega}^\alpha$ (with $\alpha = 0, 1, 2, 3$) are the Cartan 1-forms characterizing the different Bianchi metrics. But, as we want to apply relations introduced previously in [5], we will have to write each metric in its natural basis and so to develop Cartan 1-forms in terms of natural 1-forms.

Most of the string cosmological exact solutions given in the literature are not written in terms of the proper time $t$ but rather in terms of the logarithmic time $\tau$ related to proper time by the following differential expression:

$$dt(\tau) = a(\tau) b(\tau) c(\tau) e^{-\phi(\tau)} d\tau$$

(8)

So, unless explicit mention, we shall write the metric and perform the duality transformations in terms of the logarithmic time and use the following coordinate system: for any metric, we shall take $(x^0, x^1, x^2, x^3) = (\tau, x, y, z)$. Every initial solution will be noted $a$, $b$, $c$ and $\phi$ whereas the barred variables, $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{\phi}$, will always refer to a dual solution.

For some of the class $A$ spacetimes, similar results have been found by Clancy et al. [3] in the framework of scalar-tensor theories. In this paper, the authors impose to the dilatonic scalar field to remain homogeneous after duality and so they are not able to find a dual solution for each Bianchi-type model. Our method to find dual solutions appears however simpler and more direct than theirs.

5The Bianchi-type model called $VI_h$ (resp. $VII_h$) corresponds to a class B model only for $h \neq -1$ (resp. $h \neq 0$); for $h = -1$ (resp. $h = 0$), it becomes a class $A$ model.

6In fact, in all the class B models.
We shall present in the next section a table with explicit exact solutions and their corresponding dual solutions for Bianchi classes A and B models.

3.1 Bianchi I model

The Bianchi I metric describes the simplest spatially homogeneous anisotropic model:

\[ ds^2 = -(abc)^2 e^{-2\phi} d\tau^2 + a^2 dx^2 + b^2 dy^2 + c^2 dz^2 \]  

(9)

As this metric depends on the timelike coordinate only, the matrix \( G \) needed for the SFD transformation is the following \( 3 \times 3 \) matrix:

\[
G = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}
\]  

(10)

with determinant \((abc)^2\). From (4), we see that the dual metric can be obtained by simply inverting the matrix \( G \):

\[ ds^2 = -(\bar{a}\bar{b}\bar{c})^2 e^{-2\bar{\phi}} d\tau^2 + \bar{a}^2 dx^2 + \bar{b}^2 dy^2 + \bar{c}^2 dz^2 \]  

(11)

and using (5), the dual dilatonic scalar field can be written as

\[ \bar{\phi} = \phi - 2 \log(abc) = \phi + 2 \log(\bar{a}\bar{b}\bar{c}) \]  

(12)

with, for the dual scale factors,

\[ \bar{a} = a^{-1}, \quad \bar{b} = b^{-1} \quad \text{and} \quad \bar{c} = c^{-1} \]  

(13)

3.2 Bianchi II model

As mentioned above, the SFD transformations given by (4) and (5) have to be applied to a metric written in its natural basis. The Bianchi II metric developed in terms of the natural basis 1-forms can be written as:

\[
\begin{align*}
   ds^2 &= -(abc)^2 e^{-2\phi} d\tau^2 + a^2 (dy - xdz)^2 + b^2 dz^2 + c^2 dx^2 \\
   &= -(abc)^2 e^{-2\phi} d\tau^2 + c^2 dx^2 + a^2 dy^2 - 2x a^2 dy dz + (a^2 x^2 + b^2) dz^2
\end{align*}
\]  

(14)
A spacelike coordinate appears now explicitly in the metric despite its spatial homogeneity, so it is necessary to take it into account in the construction of the matrix $G$. As the metric depends on the two coordinates, $\tau$ and $x$, $G$ is the following $2 \times 2$ matrix:

$$G = \begin{pmatrix} a^2 & -xa^2 \\ -xa^2 & x^2a^2 + b^2 \end{pmatrix}$$

(15)

with $(ab)^2$ as determinant. We note that despite the presence of the coordinate $x$ in $G$, its determinant remains time-dependent only. In view of (5), the dual dilatonic field remains spatially homogeneous as the dual metric.

In order to retrieve the initial Cartan 1-forms

$$\omega^0 = abc e^{-\phi} d\tau$$
$$\omega^1 = dy - xdz$$
$$\omega^2 = dz$$
$$\omega^3 = dx$$

in the dual metric, we have to add to the inversion of the matrix $G$ the following variable change:

$$x \rightarrow -x$$
$$y \rightarrow z$$
$$z \rightarrow y$$

and so, the dual metric takes the following form:

$$ds^2 = -(\bar{a}\bar{b}\bar{c})^2 e^{-2\bar{\phi}} d\tau^2 + \bar{a}^2 (dy - xdz)^2 + \bar{b}^2 dz^2 + \bar{c}^2 dx^2$$

(16)

with the dual scale factors defined by

$$\bar{a} = b^{-1}, \quad \bar{b} = a^{-1} \text{ and } \bar{c} = c$$

(17)

The relation (5) and the definitions (17) enable one to write the dual dilaton in terms of the dual scale factors as follows:

$$\bar{\phi} = \phi - 2 \log(ab) = \phi + 2 \log(\bar{a}\bar{b})$$

(18)
3.3 Bianchi III model

The Bianchi III metric is a class B spacetime with the following metric:

$$\begin{align*}
ds^2 &= -(abc)^2 e^{-2\phi} d\tau^2 + a^2 dx^2 + b^2 dy^2 + c^2 e^{2x} dz^2 
\end{align*}$$

(19)

As it depends explicitly on two coordinates, $\tau$ and $x$, the matrix $G$ is again a $2 \times 2$ matrix:

$$G = \begin{pmatrix}
b^2 & 0 \\
0 & c^2 e^{2x}
\end{pmatrix}$$

(20)

with determinant $(bc)^2 e^{2x}$. To retrieve the initial Cartan 1-forms, it is again necessary to add the transformation given by (4) and (5) the following transformation on $x$: $x \rightarrow -x$, with the following result for the dual metric:

$$\begin{align*}
ds^2 &= -(ab\bar{c})^2 e^{-2\phi} d\tau^2 + \bar{a}^2 dx^2 + \bar{b}^2 dy^2 + \bar{c}^2 e^{2x} dz^2 
\end{align*}$$

(21)

with

$$\begin{align*}
\bar{a} &= a, \quad \bar{b} = b^{-1} \quad \text{and} \quad \bar{c} = c^{-1}
\end{align*}$$

(22)

and for the dual dilaton

$$\bar{\phi} = \phi - 2 \log(bc) + 2x$$

(23)

This is an example of a dilatonic field becoming inhomogeneous after SFD transformations due to the dependence of the determinant of $G$ with respect to $x$.

Using (22) and (23), we can write

$$a b c e^{-\phi} = \bar{a} \bar{b} \bar{c} e^{-\bar{\phi}+2x}$$

(24)

The expressions of the dual metric and the dual dilaton in terms of the dual scale factors are finally given by:

$$\begin{align*}
ds^2 &= -\bar{a} \bar{b} \bar{c} e^{-2(\bar{\phi} - 2x)} d\tau^2 + \bar{a}^2 dx^2 + \bar{b}^2 dy^2 + \bar{c}^2 e^{2x} dz^2 
\end{align*}$$

(25)

$$\bar{\phi} = \phi + 2 \log(\bar{b}\bar{c}) + 2x$$

(26)

The inhomogeneity of the metric (23) is only apparent, since from (23), we can see that the exponential term $e^{-2(\phi - 2x)}$ present in the $g_{00}$ metric component is only $\tau$-dependent, so that the dual metric remains spatially homogeneous after SFD transformations.
3.4 Bianchi IV model

The Bianchi IV metric belongs to class B models. As the corresponding exact solution we are going to present later is known in terms of the proper time and is non-diagonal in Cartan’s basis (cf. the table in the next section), we shall use the following form for its spacetime metric [10]:

\[ ds^2 = -\sigma_0^0 + a^2 \sigma_1^1 + b^2 \left\{ c^2 \sigma_2^2 + d^2 (\sigma_3^3 + \sigma_2^3 + \sigma_3^3) \right\} \] (27)

with \( a, b, c \) and \( d \), functions of \( t \) and where the 1-forms \( \sigma^i \) \((i = 0, 1, 2, 3)\) are defined by

\[
\begin{align*}
\sigma^0 &= dt \\
\sigma^1 &= dx \\
\sigma^2 &= e^{-x} \, dz \\
\sigma^3 &= e^{-x} (dy - xdz)
\end{align*}
\] (28)

Developing this metric in terms of the natural basis 1-forms, we obtain:

\[
\begin{align*}
ds^2 &= -dt^2 + a^2 dx^2 + b^2 e^{-2x} \left\{ dy^2 + \left[ c^2 - 2x \, d^2 + x^2 \right] dz^2 + 2 (d^2 - x) dydz \right\}
\end{align*}
\] (29)

Again we can see that the metric depends on two coordinates, \( t \) and \( x \), so that we take for \( G \) the following \( 2 \times 2 \) matrix:

\[
G = b^2 e^{-2x} \left( \begin{array}{cc}
1 & d^2 - x \\
0 & c^2 - 2xd^2 + x^2 \end{array} \right)
\] (30)

Its determinant depends again on the coordinate \( x \): \( \det G = b^4 e^{-4x} (c^2 - d^4) \).

Inverting \( G \) and making the variable change:

\[
\begin{align*}
x &\to -x \\
y &\to z \\
z &\to y
\end{align*}
\]

we build the following dual metric:

\[
\begin{align*}
ds^2 &= -\sigma_0^0 + \bar{a}^2 \sigma_1^1 + \bar{b}^2 \left\{ \bar{c}^2 \sigma_2^2 - \bar{d}^2 (\sigma_3^3 + \bar{c}^3 \sigma_2^3) + \bar{c}^3 \bar{c}^3 \right\}
\end{align*}
\] (31)

with

\[
\bar{a} = a, \quad \bar{b} = b^{-1} (c^2 - d^4)^{-1/4}, \quad \bar{c} = c \quad \text{and} \quad \bar{d} = d
\] (32)

and with the same 1-forms \( \sigma^i \) \((i = 0, 1, 2, 3)\) as defined in (28). Using (30) and (32), we can also write the dual dilatonic field as

\[
\bar{\phi} = \phi - 4x - 4 \log \left[ b^4 (c^2 - d^4) \right] = \phi - 4x + 4 \log (\bar{b})
\] (33)
3.5 Bianchi V model

The Bianchi V metric developed in terms of its natural basis 1-forms can be written as:

\[ ds^2 = -(abc)^2 e^{-2\phi} d\tau^2 + a^2 dx^2 + b^2 e^{2x} dy^2 + c^2 e^{2x} dz^2 \] (34)

The corresponding matrix \( G \) is seen to be the following two-dimensional squared matrix:

\[ G = e^{2x} \begin{pmatrix} b^2 & 0 \\ 0 & c^2 \end{pmatrix} \] (35)

with determinant \( e^{4x} (bc)^2 \). This determinant being \( x \)-dependent, it will lead to the inhomogeneity of the dual dilatonic field.

The building of the dual metric needs both the inversion of the matrix \( G \) and the change: \( x \to -x \). We can then write the dual metric and the dual dilaton respectively as follows:

\[ ds^2 = -(\bar{a}\bar{b}\bar{c})^2 e^{-2(\bar{\phi}-4x)} d\tau^2 + \bar{a}^2 dy^2 + \bar{b}^2(dz - xdy)^2 + \bar{c}^2 dx^2 \] (36)

\[ \bar{\phi} = \phi + 4x + 2 \log(\bar{b}\bar{c}) \] (37)

with

\[ \bar{a} = a, \quad \bar{b} = b^{-1} \quad and \quad \bar{c} = c^{-1} \] (38)

Again the inhomogeneity of the metric above is only apparent.

3.6 Bianchi VI\(_h\) model

Developed in its natural basis, the Bianchi VI\(_h\) metric can be written as

\[ ds^2 = -(abc)^2 e^{-2\phi} d\tau^2 + a^2 dx^2 + b^2 e^{2hx} dy^2 + c^2 e^{2x} dz^2 \] (39)

For all \( h \neq -1 \), this metric is a Bianchi class B spacetime. The particular case of \( h = -1 \) transforms this metric in a Bianchi class A model. For the purpose of this paper, it is not necessary to consider those cases separately.

The metric being independent of \( y \) and \( z \), \( G \) is the following 2 × 2 matrix:

\[ G = \begin{pmatrix} b^2 e^{2hx} & 0 \\ 0 & c^2 e^{2x} \end{pmatrix} \] (40)
with $e^{2x(h+1)}(bc)^2$ as determinant.

The dual metric can be obtained by performing both the inversion of $G$ and a transformation on $x$: $x \rightarrow -x$ and can thus be written as

$$ds^2 = -(\bar{a}\bar{b}\bar{c})^2 e^{-2\tilde{\phi}} e^{4x(h+1)} d\tau^2 + \bar{a}^2 dx^2 + \bar{b}^2 e^{2hx} dy^2 + \bar{c}^2 e^{2z} dz^2$$

with

$$\bar{a} = a, \quad \bar{b} = b^{-1} \quad \text{and} \quad \bar{c} = c^{-1}$$

Using (43), the dual dilatonic scalar field takes the following form

$$\tilde{\phi} = \phi + 2x(h+1) + 2 \log(\bar{b}\bar{c})$$

Note that the dual dilatonic field remains spatially homogeneous only for the special case $h = -1$, which corresponds to a class A spacetime.

### 3.7 Bianchi $VII_h$ model

We can write the Bianchi $VII_h$ metric in its natural basis as follows:

$$ds^2 = -(abc)^2 e^{-2\phi} e^{4x(h+1)} d\tau^2 + a^2 [(X-kY) dy - Ydz]^2$$

$$+ b^2 [Y dy + (X+kY) dz]^2 + c^2 dx^2$$

$$= -(abc)^2 e^{-2\phi} d\tau^2 + c^2 dx^2 + [a^2 (X-kY)^2 + b^2 Y^2] dy^2$$

$$+ 2 [b^2 Y (X+kY) - a^2 Y (X-kY)] dydz$$

$$+ [a^2 Y^2 + b^2 (X+kY)^2] dz^2$$

where

$$X(x) = e^{-kx} \cos(qx)$$

$$Y(x) = \frac{1}{q} e^{-kx} \sin(qx)$$

$$q = \sqrt{1-k^2}$$

$$k = h/2$$

(45)

For all $h \neq 0$, this metric belongs to Bianchi class $B$ but for the special case $h = 0$, it becomes a Bianchi class $A$ metric. Again the purpose of this paper does not require to consider this particular case separately from the others.

The explicit presence of the coordinate $x$ in the metric implies $G$ to be the following $2 \times 2$ matrix:

$$G = \begin{pmatrix}
    a^2 (X-kY)^2 + b^2 Y^2 & b^2 Y (X+kY) - a^2 Y (X-kY) \\
    b^2 Y (X+kY) - a^2 Y (X-kY) & b^2 (X+kY)^2 + a^2 Y^2
\end{pmatrix}$$

(46)
Using the definitions (45), we can write the determinant of $G$ as: $det(G) = a^2 b^2 e^{-4kx}$.

The dual metric can be obtained making both the inversion of $G$ and a transformation on the constant $k$: $k \rightarrow -k$, i.e. $h \rightarrow -h$. We can thus write

$$ds^2 = -(\bar{a}\bar{b}\bar{c})^2 e^{-2\phi} e^{-8kx} d\tau^2 + \bar{a}^2 [(X - kY) dy - Y dz]^2$$
$$+ \bar{b}^2 [Y dy + (X + kY) dz]^2 + \bar{c}^2 dx^2$$

with

$$\bar{a} = a^{-1}, \quad \bar{b} = b^{-1} \quad and \quad \bar{c} = c$$

(47)

Using the relation (5) and the definitions (48), we can build the dual dilatonic field as follows:

$$\bar{\phi} = \phi - 4kx - 2 \log(ab) = \phi - 4kx + 2 \log(\bar{a}\bar{b})$$

(49)

Again the inhomogeneity of the $g_{00}$ component in the metric (47) is only apparent. On the other hand, the dual dilaton remains spatially homogeneous only for $k = 0$, i.e. for $h = 0$ (class A model). Indeed, for all $h \neq 0$ (class B model), we lose the initial spatial homogeneity symmetry of the dilatonic field.

### 3.8 Bianchi VIII and Bianchi IX models

These are both class A models respectively given (in terms of the natural 1-forms) by (5):

- **Bianchi VIII** metric:

$$ds^2 = -(abc)^2 e^{-2\phi} d\tau^2 + a^2 [2x \; dz + (1 - 2xz) \; dy]^2$$
$$+ b^2 [dx + (x^2 - 1) \; dz + (x + z - z \; x^2) \; dy]^2$$
$$+ c^2 [dx + (x^2 + 1) \; dz + (x - z - z \; x^2) \; dy]^2$$

(50)

7This transformation on $h$ is possible because the constant $h$, in our conventions, has to be comprised between $-2$ and 2.
Bianchi IX metric:

\[ ds^2 = -(abc)^2 e^{-2\phi} d\tau^2 + a^2 \left[ \cos(x)dy + dz \right]^2 + b^2 \left[ -\sin(z)dx + \sin(x)\cos(z)dy \right]^2 + c^2 \left[ \cos(z)dx + \sin(x)\sin(z)dy \right]^2 \quad (51) \]

In these cases, we cannot apply SFD transformations as given by (4) and (5) in the building of the dual solutions because the conditions (6) are not realized. So the method presented in [8] does not enable one to build dual solutions for these two models.

Nevertheless, it is possible to determine a dual solution for these models using Busher’s relations given in [11] and which can be written, for \( B = 0 \), as follows:

\[ \bar{g}_{yy} = \frac{1}{g_{yy}} \quad \bar{B}_{yy} = 0 \]

\[ \bar{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{y\alpha}g_{y\beta}}{g_{yy}} \quad \bar{B}_{\alpha\beta} = 0 \quad (52) \]

\[ \bar{g}_{y\alpha} = 0 \quad \bar{B}_{y\alpha} = \frac{g_{y\alpha}}{g_{yy}} \]

where \( y \) is, for both models, the only coordinate the metric does not depend on and where \( \alpha \) and \( \beta \) stand for the coordinates the metric depends on, i.e. \( \tau, x \) and \( z \).

It is not necessary to write explicitly the two dual solutions to see clearly that these SFD transformations introduce a torsion field \( B \) which was absent initially but, above all, cancel the non-diagonal components of the metric so that we lose the initial symmetry of the metric after this SFD. Thus, for these cases, the relevance of the SFD transformations is less evident than for the others.

4 Exact dual solutions for Bianchi models

We shall present in the following table exact Bianchi-type solutions of field equations (2) and (3) with their dual expressions. In the first column, we shall note the Bianchi-type. In the second and in the third columns,
the initial exact solution and its dual will be respectively displayed. All the explicit initial solutions presented in the second column come from Batakis and Kehagias paper’s [6] except for the Bianchi IV solution which has been obtained by Harvey and Tsoubelis [10]. After any explicit solution, we also give the constraint on the constants present therein. Note that the quantities $N$, $p_i$ and $q_j$ (with $i=1,2,3$ and $j=1,2$) appearing in the table are constants. Introducing both solutions and dual solutions in the field equations (2) and (3), we have checked that they satisfy exactly these equations.

The Bianchi VII$_h$ as well as VII$_0$ general exact solutions of field equations (2) and (3) being not known, we have not been able to build their dual counterparts in explicit form.
| Type | Solution | Dual solution |
|------|----------|---------------|
| I    | \(a(\tau)^2 = e^{(p_1+N)\tau}\)  
\(b(\tau)^2 = e^{(p_2+N)\tau}\)  
\(c(\tau)^2 = e^{(p_3+N)\tau}\)  
\(\phi(\tau) = N\tau\)  
with \(\sum_{i<j} p_i p_j = N^2\) | \(\bar{a}(\tau)^2 = e^{-(p_1+N)\tau}\)  
\(\bar{b}(\tau)^2 = e^{-(p_2+N)\tau}\)  
\(\bar{c}(\tau)^2 = e^{-(p_3+N)\tau}\)  
\(\bar{\phi}(\tau) = -(2N + p_1 + p_2 + p_3)\tau\) |
| II   | \(a(\tau)^2 = X^{-1} e^{N\tau}\)  
\(b(\tau)^2 = X e^{(2p_1+N)\tau}\)  
\(c(\tau)^2 = X e^{(2p_2+N)\tau}\)  
\(\phi(\tau) = N\tau\)  
with \(X(\tau) = 1/p_3 \cosh(p_3 \tau)\) | \(\bar{a}(\tau)^2 = X^{-1} e^{-(2p_1+N)\tau}\)  
\(\bar{b}(\tau)^2 = X e^{-N\tau}\)  
\(\bar{c}(\tau)^2 = X e^{(2p_2+N)\tau}\)  
\(\bar{\phi}(\tau) = -(N + 2p_1)\tau\)  
\(4p_1 p_2 - p_3^2 = N^2\) |
| III  | \(a(\tau)^2 = p_1 e^{(p_2+N)\tau} \sinh^{-2}(p_1 \tau)\)  
\(b(\tau)^2 = p_1 e^{(N-p_2)\tau}\)  
\(c(\tau)^2 = p_1 e^{(p_2+N)\tau} \sinh^{-2}(p_1 \tau)\)  
\(\phi(\tau) = N\tau\)  
with \(4p_1^2 - p_2^2 = N^2\) | \(\bar{a}(\tau)^2 = p_1 e^{(p_2+N)\tau} \sinh^{-2}(p_1 \tau)\)  
\(\bar{b}(\tau)^2 = p_1^{-1} e^{(p_2-N)\tau}\)  
\(\bar{c}(\tau)^2 = p_1^{-1} \sinh^2(p_1 \tau) e^{-(p_2+N)\tau}\)  
\(\bar{\phi}(\tau) = -N\tau + 2x - 2 \log(p_1)\)  
\(+2 \log(\sinh(p_1 \tau))\) |
| IV   | \(a(t)^2 = (a_0^{-1} t)^2\)  
\(b(t)^2 = (a_0^{-1} t)^2 a_0\)  
\(c(t)^2 = 1 + \left[a_0 \log \left(a_0^{-1} t\right)\right]^2\)  
\(d(t)^2 = a_0 \log \left(a_0^{-1} t\right)\)  
\(\phi(t) = 0\)  
with \(a_0 = 4/5\) | \(\bar{a}(t)^2 = (a_0^{-1} t)^2\)  
\(\bar{b}(t)^2 = (a_0^{-1} t)^{-2 a_0}\)  
\(\bar{c}(t)^2 = 1 + \left[a_0 \log \left(a_0^{-1} t\right)\right]^2\)  
\(\bar{d}(t)^2 = a_0 \log \left(a_0^{-1} t\right)\)  
\(\bar{\phi}(t) = -4x - 4 a_0 \log \left(a_0^{-1} t\right)\) |
| Type | Solution | Duale |
|------|----------|-------|
| $V$  | $a(\tau)^2 = \frac{q_1}{2} X e^{N\tau}$ | $\bar{a}(\tau)^2 = \frac{q_1}{2} X e^{N\tau}$ |
|      | $b(\tau)^2 = \frac{q_2}{2} X e^{(p_2+N)\tau}$ | $\bar{b}(\tau)^2 = \frac{2}{q_2 X} e^{-(p_2+N)\tau}$ |
|      | $c(\tau)^2 = \frac{1}{2 q_2} X e^{(-p_2+N)\tau}$ | $\bar{c}(\tau)^2 = \frac{2 q_2}{X} e^{(p_2-N)\tau}$ |
|      | $\phi(\tau) = N\tau$ | $\bar{\phi}(\tau) = -2 \log(X(\tau)) - N\tau + 4 + 2 \log(2)$ |
|      | $X(\tau) = p_1 / \sinh(p_1 \tau)$ | $3 p_1^2 - p_2^2 = N^2$ |
| $VI_{-1}$ | $a(\tau)^2 = q_1^2 p_1 \exp[p_3^2 e^{2p_2\tau}] e^{(p_1+N)\tau}$ | $\bar{a}(\tau)^2 = q_1^2 p_1 \exp[p_3^2 e^{2p_2\tau}] e^{(p_1+N)\tau}$ |
|      | $b(\tau)^2 = p_3^2 p_2 e^{(p_2+N)\tau}$ | $\bar{b}(\tau)^2 = (p_3^2 p_2)^{-1} e^{-(p_2+N)\tau}$ |
|      | $c(\tau)^2 = p_3^2 p_2 e^{(p_2+N)\tau}$ | $\bar{c}(\tau)^2 = (p_3^2 p_2)^{-1} e^{-(p_2+N)\tau}$ |
|      | $\phi(\tau) = N\tau$ | $\bar{\phi}(\tau) = -(N + 2p_2)\tau - 2 \log(p_3^2 p_2)$ |
|      | $p_2^2 + 2p_1 p_2 = N^2$ | |
| $VI_{h}$ | $a(\tau)^2 = q_1^2 X^{(h^2+1)} e^{(\frac{h+1}{h+1} p_2+N)\tau}$ | $\bar{a}(\tau)^2 = q_1^2 X^{(h^2+1)} e^{(\frac{h+1}{h+1} p_2+N)\tau}$ |
|      | $b(\tau)^2 = q_2^2 X^{-(h+1)} e^{(p_2-N)\tau}$ | $\bar{b}(\tau)^2 = q_2^2 X^{-(h+1)} e^{-(p_2-N)\tau}$ |
|      | $c(\tau)^2 = q_2^2 X^{-(h+1)} e^{(p_2-N)\tau}$ | $\bar{c}(\tau)^2 = q_2^2 X^{-(h+1)} e^{(p_2-N)\tau}$ |
|      | $\phi(\tau) = N\tau + 2x (h+1)$ | $\bar{\phi}(\tau) = -(h+1)^2 \log[X(\tau)]$ |
|      | $X(\tau) = \left[ \frac{h+1}{p_1} \sinh(p_1 \tau) \right]^{\frac{2}{(h+1)^2}}$ | $4 \frac{(h^2 + h + 1)}{(h+1)^2} p_1^2 - p_2^2 = N^2$ |
Acknowledgments

This work was supported in part by Belgian Interuniversity Attraction Pole P4/05 as well as by a grant from “Fonds National de la Recherche Scientifique”.

To the memory of my professor Jacques Demaret. It was a pleasure to work with him. He will be most deeply missed.

References

[1] K. A. Meissner and G. Veneziano, Phys. Lett. B 267, 33 (1991);
    K. A. Meissner and G. Veneziano, Mod. Phys. Lett. A 6, 3397 (1991);
    M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993).
    See also the references given in "http://carmen.to.infn.it/gasperin".

[2] C. W. Misner, K. S. Thorne, J. A. Wheeler, Gravitation, Freeman, San Francisco (1973) chapter 30, 800.

[3] D. Clancy, J. E. Lidsey and R. Tavakol, Class. Quantum Grav. 15, 257 (1998).

[4] M. MacCallum and A. Taub, Commun. Math. Phys. 25, 173 (1972);
    M. P. Ryan Jr. and S. M. Waller, gr-qc/9709012.

[5] J. Demaret and E. Di Pietro, "O(d,d)-invariance in inhomogeneous string cosmologies with perfect fluid", Gen. Rel. Grav. 31, 323 (1999).

[6] N. A. Batakis and A. A. Kehagias, Nucl. Phys. B 449, 248 (1995).

[7] E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. D 50, 4868 (1994).

[8] M. H. H. MacCallum, Anisotropic and inhomogeneous relativistic cosmologies, in General relativity - An Einstein centenary survey, ed. S. W. Hawking and W. Israel (Cambridge University Press, 1979) Chapter 11, 533.

[9] M. P. Ryan and L. C. Shepley, Homogeneous Relativistic Cosmologies, Princeton (Princeton University Press, 1975).
[10] A. Harvey and D. Tsoubelis, *Phys. Rev. D* **15**, 2734 (1977).

[11] T. Busher, *Phys. Lett. B* **194**, 59 (1987);
    T. Busher, *Phys. Lett. B* **201**, 466 (1988);
    G. T. Horowitz, *The Black Side of String Theory: Black Holes and Black Strings in String Theory and Quantum Gravity*, ed. J. Harvey et al. (World Scientific, 1993), [hep-th/9210119](http://arxiv.org/abs/hep-th/9210119).