Simple and Faster algorithm for Reachability in a Decremental Directed Graph

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Abstract

We consider the problem of maintaining source sink reachability ($st$-Reachability), single source reachability (SSR) and strongly connected component (SCC) in an edge decremental directed graph. In particular, we design a randomized algorithm that maintains with high probability $1$:

1. $st$-Reachability in $\tilde{O}(mn^{4/5})$ total update time.
2. $st$-Reachability in a total update time of $\tilde{O}(n^{8/3})$ in a dense graph.
3. SSR in a total update time of $\tilde{O}(mn^{9/10})$.
4. SCC in a total update time of $\tilde{O}(mn^{9/10})$.

For all the above problems, we improve upon the previous best algorithm (by Henzinger et. al. (STOC 2014)).

Our main focus is maintaining $st$-Reachability in an edge decremental directed graph (other problems can be reduced to $st$-Reachability). The classical algorithm of Even and Shiloach (JACM 81) solved this problem in $O(1)$ query time and $O(mn)$ total update time. Recently, Henzinger, Krinninger and Nanongkai (STOC 2014) designed a randomized algorithm which achieves an update time of $O(mn^{0.98})$ with high probability and broke the long-standing $O(mn)$ bound of Even and Shiloach.

In their breakthrough work, Henzinger et. al. introduced a randomized strategy that breaks the $st$-reachability problem into many reachability problems in smaller graphs. However, they designed four algorithms $A_i (1 \leq i \leq 4)$ such that for graphs having total number of edges between $m_i$ and $m_{i+1}$ ($m_{i+1} > m_i$), $A_i$ outperforms other three algorithms. That is, one of the four algorithms may be faster for a particular density range of edges, but it may be too slow asymptotically for the other ranges. Our main contribution is that we design a single algorithm which works for all types of graphs. Not only is our algorithm faster, it is much simpler than the algorithm designed by Henzinger et. al. (STOC 2014).

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$1$ with probability $\geq 1 - 1/n^{O(1)}$  
$2$ we hide $\text{polylog}(n)$ factor under the $\tilde{O}$ notation
1 Introduction

In a dynamic graph, at each update step an edge is added or deleted from the graph. If only insertions are allowed, then the graph is said to be an incremental dynamic graph. If only deletions are allowed, then the graph is said to be a decremental dynamic graph. If both additions and deletions are allowed, then the graph is said to be a fully dynamic graph. We consider the problem of maintaining a source $s$-sink $t$ reachability in a decremental directed graph. In this problem, we are given a sequence of edge deletions on a fixed set of vertices that includes two special vertices, $s$ and $t$. Let $D = d_1, d_2, \ldots, d_m$ be the deletion sequence where $d_i \leftarrow \text{delete}(u, v)$, i.e., at the $i$th update step, an edge $(u, v)$ is deleted from the graph. After each edge deletion we may have to answer the following question:

Is $t$ reachable from $s$?

Note that the above problem is nearly solved for an undirected graph even in a fully dynamic setting i.e., the classical problem of undirected connectivity. After many years of persistent research work spanning 20 years [Fre85, EGIN97, HK99, HdLT01], a fully dynamic algorithm with a worst-case polylogarithmic time was discovered recently [KKM13]. However, if we look at the case of directed graph, even for decremental graphs the results are few. Even and Shiloach [ES81] designed a simple and elegant algorithm that maintains a BFS tree —also called an Even-Shiloach tree—from the source $s$ in $O(mn)$ total update time. An Even-Shiloach tree can be used to answer $st$-Reachability queries as well as other complex queries: (1) Is any vertex $v$ reachable from $s$? (2) What is the length of the shortest path from $s$ to any other vertex $v$? It can be shown that all the above queries can be satisfied in $O(1)$ time and same total update time. However, these queries are much more complex than answering $st$-Reachability query and intuitively one feels that there ought to be a faster algorithm for maintaining $st$-Reachability. Note that the algorithm developed in [ES81] has frequently been used as a subroutine in many decremental dynamic graph problems like all pair shortest path [Ber13, BHS07, RZ04], strongly connected component [RZ11, BHS07, HK95]. So improving the $st$-Reachability problem may lead to better algorithms for above problems.

For more than 30 years, the algorithm in [ES81] was the best known for $st$-Reachability problem. Recently, Henzinger, Krinninger and Nanongkai [HKN14] designed an algorithm for $st$-Reachability problem with a total update time $\tilde{O}(mn^{0.98})$. Our work refines the framework built in [HKN14] and greatly simplifies it.

Using their framework, one of the sub-problem we face is to maintain an Even-Shiloach tree in a partially dynamic graph, i.e., the graph is not strictly decremental, and at some time steps edges might be added to the graph. We now ask a critical question: Can we maintain an Even-Shiloach tree under addition and deletion of edges? In general, the answer to this question is no as the analysis of an Even-Shiloach tree breaks down if insertions are allowed. However, the above assertion is true only if the insertions are arbitrary. We will show that in the framework built in [HKN14], the insertions are not completely arbitrary. By tweaking the Even-Shiloach tree algorithm [ES81], we are able to maintain an Even-Shiloach tree even in this partially dynamic graph. This unique feature of our algorithm might be of independent interest and may find application in other problems.

Owing to the above observation and the consequent improvement in the total update time, we prove the following theorem:

**Theorem 1.1.** There exists a randomized algorithm that maintains $st$-Reachability in a decremental directed graph in a total update time of $\tilde{O}(mn^{4/5})$ with high probability

We now move on to some extensions of the $st$-Reachability problem. For dense graphs, the previous best total update time for $st$-Reachability is $\tilde{O}(n^{11/4})$ [HKN14]. We improve the above result and show the following:
Theorem 1.2. There exists a randomized algorithm that maintains $st$-Reachability in a decremental directed graph in a total update time of $\tilde{O}(n^{8/3})$ with high probability if the initial size of the graph is $\Omega(n^2)$.

Consider the problem of maintaining single source reachability (SSR): Given a decremental directed graph where after each edge deletion, we may have to answer the following query.

Is $v$ reachable from the source $s$?

where $v$ is any vertex in the graph. The previous best total update time for this problem was $\tilde{O}(mn^{0.982})$ [HKN14]. We can use our improved algorithm for $st$-Reachability to prove the following:

Theorem 1.3. There exists a randomized algorithm that maintains SSR in a decremental directed graph in a total update time of $\tilde{O}(mn^{9/10})$ with high probability.

Consider the problem of maintaining strongly connected component (SCC): Given a decremental directed graph where after each edge deletion, we may have to answer the following query.

Are $u$ and $v$ in the same component?

There are few algorithms that solve this problem in $O(mn)$ total update time [RZ02, Lac13, Rod13]. Henzinger et. al. [HKN14] improved the above bound to $\tilde{O}(mn^{0.982})$. We can use our improved algorithm for SSR to prove the following:

Theorem 1.4. There exists a randomized algorithm that maintains SCC in a decremental directed graph in a total update time of $\tilde{O}(mn^{9/10})$ with high probability.

2 Preliminaries

A directed graph is represented by $G = (V, E)$, where $V$ represents the set of vertices and $E$ represents the set of edges in the graph. The vertex set contains two special vertices — the source $s$ and the sink $t$. We will use $n$ to denote the number of vertices $|V|$, and $m$ to denote the number of edges $|E|$ in the initial graph. In the ensuing discussion, we focus on the $st$-Reachability problem. The Even-Shiloach tree takes two parameters: a source $s$ and a range $R$. It is a BFS tree from the vertex $s$ and all the other nodes up to distance $R$. The Even-Shiloach tree for $s(ES(s))$ is initially constructed as follows: Construct a BFS tree from the source $s$ till a distance $R$ from $s$. For each vertex $v$ in $ES(s)$, we maintain the following information:

1. $l(v)$ — the distance from $s$ to $v$ in $ES(s)$. It represents the current level of the vertex $v$ in $ES(s)$.
2. For each edge $(v, w)$, the current level of $w$ is stored in $CL_v(w)$.

Since an Even-Shiloach tree is a BFS tree, it implicitly maintains the following invariant:

**Invariant 2.1.** If $L$ is the shortest path from $s$ to $v$ in the current graph such that $|L| \leq R$, then the distance of $s$ to $v$ in $ES(s)$, $l(v) = |L|$.

We now follow the deletion algorithm in Figure 1. Assume that a tree edge $(u, v)$ is deleted. If the level of $v$ is greater than $R$, we do not process it. Else, observe that the deletion of $(u, v)$ has no effect on $u$ as the path between $s$ and $u$ in $ES(s)$ remains intact. However, we need to find a replacement edge for $v$. We add $v$ to the dirty vertex list $D$. We try to find if there exists any neighbor $w$ of $v$ at level $l(v) - 1$ (Note that $v$ does not have any neighbor at level $l(v) - 1$). Finding such a neighbor takes $O(1)$ time as we already
have the correct information of the level of all the neighbors of $v$. If we find such a neighbor $w$, then we add edge $(w, v)$ in $\mathcal{ES}(s)$ and we have found the replacement path from $s$ to $v$. Else, $v$ moves to level $l(v) + 1$ and we set $l(v) \leftarrow l(v) + 1$. At this point, $v$ updates all its neighbors about its increase in level. This step takes $O(d(v))$ time where $d(v)$ is the maximum degree of $v$ at any point of time. We then add $v$ and all its immediate children in the dirty vertex list and repeat the process. For the analysis, note that when the level of a vertex $v$ increases, we incur a cost of $d(v)$. Since the level of a vertex can increase at most $R$ times, the total cost is $\sum_v d(v)R = mR$, where $m$ is the total number of edges in the initial graph.

1. if $l(v) > R$ then
2. return;
3. $D \leftarrow \{v\}$;
4. while $D \neq \emptyset$ do
5. Let $v$ be a vertex in $D$ having least $l(v)$ (Break ties arbitrarily);
6. Remove $v$ from $D$;
7. if $\exists$ an edge $(w, v)$ such that $CL_w(v) = l(v) - 1$ then
8. Add edge $(w, v)$ in $\mathcal{ES}(s)$;
9. else
10. $l(v) \leftarrow l(v) + 1$; //increase the level of $v$;
11. foreach neighbor $w$ of $v$ do
12. $CL_w(v) \leftarrow l(v)$;
13. Add $v$ to $D$;
14. for each child $w$ of $v$ in $\mathcal{ES}(s)$ do
15. Add $w$ to $D$;

Figure 1: After the deletion of edge $(u, v)$ from the $\mathcal{ES}(s)$

In Section 3, we review the approach taken by Henzinger et. al. [HKN14]. In section 4, we give an overview of our technique and in Section 5 we give the full detail of our algorithm.

3 Review of the approach taken in [HKN14]

In this section, we will sketch the approach taken by Henzinger et. al [HKN14] to solve the st-Reachability problem. At the end of this section, we will not the possible opportunities to improve upon this approach. The basic algorithm is as follows:

1. Randomly choose $\tilde{O}(c)$ vertices called centers. For each pair of centers $(c_1, c_2)$, we will maintain the following information: Is there a path from $c_1$ to $c_2$ of length $\leq n/c$?

2. Randomly choose $\tilde{O}(b)$ hubs in the following way:
   a. Choose each vertex with probability $\frac{b \text{ polylog } n}{n}$
   b. Choose each edge with probability $\frac{b \text{ polylog } n}{m}$. Add both the endpoints of all randomly chosen edges to the set of hub.

Maintain two Even-Shiloach tree from each hub as follows:

   a. An Even-Shiloach tree in $G$ till a distance $n/c$ from the hub $h$, say $\mathcal{ES}(G, h)$. 


(b) Consider the graph $G'$ obtained by reversing each edge in $G$. Maintain another Even-Shiloach tree in $G'$ from the hub $h$ till a distance of $n/c$. Let us denote this tree as $\mathcal{ES}(G', h)$.

Instead of maintaining the reachability information between $s$ and $t$, we maintain reachability information between all pair of centers. Let $\mathcal{C}(G)$ denote the directed graph on all centers plus the source $s$ and the sink $t$. There is a directed edge from $c_1$ to $c_2$ in $\mathcal{C}(G)$ if there exists a path from $c_1$ to $c_2$ of length $\leq n/c$ in $G$. In [HKN14], the authors show that if at any update step, $t$ is reachable from $s$ in $G$, then $t$ is also reachable from $s$ in $\mathcal{C}(G)$ with a very high probability.

**Lemma 3.1.** [HKN14] With high probability, $s$ can reach $t$ in $G$ iff $s$ can reach $t$ in $\mathcal{C}(G)$.

For any two vertices $u, c_1 \in \mathcal{C}(G)$, we say that $c_1$ is reachable from $u$ via a hub if there exists a hub $h$ such that the distance of $u$ to $h$ plus the distance of $h$ to $c_1$ is $\leq n/c$. We use $\mathcal{ES}(G, h)$ and $\mathcal{ES}(G', h)$ to check the above constraint. Maintaining $\mathcal{ES}(G, h)$ and $\mathcal{ES}(G', h)$ for a hub $h$ takes at most $O(mn/c)$ total update time as we are interested in maintaining an Even-Shiloach tree till a distance of $n/c$ only. Additionally, for each hub $h$, and each pair of center $u, c_1$, we need to maintain the distance of $u$ to $h(d'(h, u)$ in $\mathcal{ES}(G', h))$ and distance of $h$ to $c_1(d(h, c_1)$ in $\mathcal{ES}(G, h))$. If $d'(h, u) + d(h, c_1) \leq n/c$, then $c_1$ is reachable from $u$ via a hub. The distance $d'(h, u)$ (or $d(h, c_1)$) increases only when the level of $u$ (or $c_1$) increases in $\mathcal{ES}(G', h)$ (or $\mathcal{ES}(G, h)$). Whenever the distance of $u$ increases in $\mathcal{ES}(G', h)$, we have to check at most $\tilde{O}(c)$ such distances. So the time taken to check all the constraints associated with $u$ is $\tilde{O}(c)$. Since the level of a vertex can increase only $n/c$ times, the total time for checking constraints of $u$ can be bounded by $\tilde{O}(c.n/c) = \tilde{O}(n)$. Since there are $\tilde{O}(b)$ and $\tilde{O}(c)$ centers, the total time taken for the above process is $\tilde{O}(bnc)$. The total time taken to maintain Even-Shiloach trees from all hubs can be bounded as follows:

**Lemma 3.2.** Even-Shiloach trees ($\mathcal{ES}(G, h)$ and $\mathcal{ES}(G', h)$ for each hub $h$) can be maintained from all hubs in $\tilde{O}(\frac{mn}{c} + bnc)$ total update time.

However, if $c_1$ is not reachable from $u$ via a hub, then consider the path union graph $\mathcal{P}(u, c_1)$ which is defined as follows: if a vertex $v$ lies on a path from $u$ to $c_1$ of length $\leq n/c$, then $v \in \mathcal{P}(u, c_1)$. Similarly, if an edge $(v, v')$ lies on a path from $u$ to $c_1$ of length $\leq n/c$, then $(v, v') \in \mathcal{P}(u, c_1)$. In [HKN14], the authors proved the following lemma which bounds the number of vertices and edges in a path union graph.

**Lemma 3.3.** [HKN14] With high probability, if $c_1$ is not reachable from $u$ via a hub, then $\mathcal{P}(u, c_1)$ has at most $n/b$ vertices and $m/b$ edges.

Note that we can extend the above lemma for any number of centers.

**Lemma 3.4.** With a high probability, if vertices in $\{c_1, c_2, \ldots, c_j\}$ are not reachable from $u$ via a hub, $\cup_{i=1}^{j} \mathcal{P}(u, c_i)$ has at most $n/b$ vertices and $m/b$ edges.

**Proof.** We give a sketch of the proof that uses the standard hitting set argument (see [UY90]). Assume that $\cup_{i=1}^{j} \mathcal{P}(u, c_i)$ contains greater than $n/b$ vertices. Initially, hubs were selected by sampling each vertex with a probability $\frac{1}{n \log n}$. So, with a very high probability, at least one of these $n/b$ vertices, say $x$, will be a hub. Without loss of generality, assume that $x$ first appeared in the path union graph $\mathcal{P}(u, c_i)$. This implies that there existed a path from $u$ to $c_i$ of length less than $n/c$ such that $x$ lies on it. So, $c_i$ was reachable from $u$ via the hub $x$. This contradicts our assumption that when we constructed $\mathcal{P}(u, c_i)$, $c_i$ is not reachable to $u$ via any hub.

Using a similar argument, we can show that the number of edges in $\cup_{i=1}^{j} \mathcal{P}(u, c_i)$ is $\leq m/b$. \qed
In order to find $P(u, c_1)$, we perform a BFS from $u$ in $G$ and a BFS from $c_1$ in $G'$. For each vertex $v$, we denote $d(u, v)$ as the distance from $u$ to $v$ in $G$ and $d'(c_1, v)$ as the distance from $c_1$ to $v$ in $G'$. If $d(u, v) + d'(c_1, v) \leq n/c$, then the vertex $v$ is added to the path union graph $P(u, c_1)$. Similarly, edges can be added in the path union graph. All the above operations take $O(m)$ time, so we have proved the following lemma:

**Lemma 3.5.** A path union graph can be constructed in $O(m)$ time.

In [HKN14], once $c_1$ is not reachable from $u$ via a hub, a path union graph $P(u, c_1)$ is constructed. After this construction, an Even-Shiloach tree is maintained on the graph $P(u, c_1)$ till a distance of $n/c$ from $u$. Let us denote this Even-Shiloach tree as $ES(u, c_1)$. The sole purpose of $ES(u, c_1)$ is to find if $c_1$ is at a distance $\leq n/c$ from $u$. By Lemma 3.3, there are $m/b$ edges in $P(u, c_1)$ and we maintain $ES(u, c_1)$ till a height of $n/c$ only. So, the total time to maintain $ES(u, c_1)$ is $\frac{mn}{bc}$.

In the worst case, we have to build $O(c^2)$ path union graphs as there are $O(c)$ vertices in $C(G)$. So the total time to build all path union graphs is $O(mc^2)$. Also, the total time to maintain Even-Shiloach trees on all path union graphs is $\tilde{O}(\frac{mn}{bc}c^2) = \tilde{O}(\frac{mnc}{b})$.

Lastly, we analyze the time taken to maintain connectivity information between $s$ and $t$ in $C(G)$. Even $C(G)$ is a decremental graph, where a directed edge from $c_1$ to $c_2$ is deleted from $C(G)$ if there is no path of length less than $n/c$ from $c_1$ to $c_2$ in $G$. We maintain an Even-Shiloach tree on $C(G)$ from the vertex $s$. Since there are $\tilde{O}(c^2)$ edges in $C(G)$ and we need to maintain a tree of height $\tilde{O}(c)$ (since there are $\tilde{O}(c)$ vertices in $C(G)$), the total update time to maintain an Even-Shiloach tree in $C(G)$ is $\tilde{O}(c^3)$.

The total time taken by the basic algorithm is calculated in Figure 2. If we set $b = c = 1$, then we obtain a total running time of $\tilde{O}(mn)$ which is $\text{polylog}(n)$ factor away from the classical algorithm of Even and Shiloach [ES81]. In [HKN14], the authors introduce the above approach and perform some non-trivial additions in the basic algorithm to obtain a running time of $\tilde{O}(mn^{0.98})$.

In this paper, we also perform some non-trivial additions in the above basic algorithm. Specifically, we concentrate on the third step of the algorithm, i.e., maintaining Even-Shiloach trees on path union graphs. When a vertex $c_1$ is not reachable from $u$ via a hub, an Even-Shiloach tree $ES(u, c_1)$ is constructed on the path union graph $P(u, c_1)$. In the worst case, we have to maintain $\tilde{O}(c)$ such Even-Shiloach trees for $u$. We will show that we can maintain just one Even-Shiloach tree for the union of all path union graphs of $u$, i.e., $\cup_iP(u, c_i)$. Using our improved algorithm, we will show that we can maintain this Even-Shiloach tree for $u$ in $\tilde{O}(\frac{mn}{bc} + \frac{m}{b})$ time. Since there are $\tilde{O}(c)$ centers, the total running time of Step 3 in Figure 2 is reduced to $\tilde{O}(\frac{mn}{bc} + \frac{mc^2}{b})$. Note that this is an improvement over the running time of $\tilde{O}(\frac{mnc}{b})$ of the basic algorithm. If we set $c = n^{2/5}$ and $b = n^{1/5}$, we can show that our algorithm has a total running time $\tilde{O}(mn^{4/5})$. 

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**Figure 2:** Total running time calculation of the algorithm in [HKN14]
4 Overview of our Techniques

Let us see the potential problem in maintaining just one Even-Shiloach tree on two path union graphs \( \mathcal{P}(u, c_1) \) and \( \mathcal{P}(u, c_2) \) of \( u \). Let \( t_1 \) and \( t_2 (t_1 \leq t_2) \) respectively be the update step at which \( c_1 \) and \( c_2 \) becomes unreachable from \( u \) via a hub. At update step \( t_1 \), we build an Even-Shiloach tree \( \mathcal{ES}(u, c_1) \). In order to maintain just one Even-Shiloach tree, at update step \( t_2 \) we will try to construct a single Even-Shiloach tree for \( \mathcal{P}(u, c_1) \) and \( \mathcal{P}(u, c_2) \). The analysis of an Even-Shiloach tree crucially relies on the fact the level of a vertex may only increase in a decremental setting—which we will refer to as the decremental property of an Even-Shiloach tree. In our case, we add edges in \( \mathcal{P}(u, c_2) \) at time step \( t_2 \). It is known that addition of edges spoils the decremental property of an Even-Shiloach tree. In the ensuing discussion, we will show that the above case does not arise in our algorithm. In particular, we will show the following:

We tweak the Even-Shiloach algorithm such that even when edges in \( \mathcal{P}(u, c_2) \) are added, we need not decrease the level of any vertex in the current Even-Shiloach tree. This is an important property of an Even-Shiloach tree that is used crucially in its analysis. Since we don’t break this property even in our modified Even-Shiloach tree, we can show that we can maintain just one Even-Shiloach tree for all path union graphs of \( u \).

We will use the following notations in the ensuing discussion:

1. edge \((u, v)\) ← a directed edge from \( u \) to \( v \).
2. \( G(t) \) ← the graph at update step \( t \). We will abuse notation and use \( G(t) \) to represent the vertices and edges in the graph \( G(t) \).
3. \( G^r(t) \) ← the graph at update step \( t \) obtained by reversing edges in \( G(t) \).
4. At time steps \( t_1, t_2 \), vertices \( c_1, c_2 \in \mathcal{C}(G) \) respectively becomes unreachable from \( u \) via a hub. At update step \( t_i (i \in \{1, 2\}) \), we find the the path union graph \( \mathcal{P}(u, c_i) \). Between update steps \( t_1 \) and \( t_2 \) edges might be deleted from \( \mathcal{P}(u, c_1) \). For any \( t \geq t_1 \), let \( \mathcal{P}(u, c_1, t) \) denote the edges in \( \mathcal{P}(u, c_1) \) that are not deleted till update step \( t \).
5. At update step \( t_1 \leq t \leq t_2 \), we will maintain an Even-Shiloach tree for the graph \( \mathcal{P}(u, c_1, t) \). We will use \( \mathcal{ES}(u) \) to denote the current Even-Shiloach tree. Also, \( \mathcal{ES}(u, t) \) will denote the Even-Shiloach tree of \( u \) at the end of update step \( t \).
6. \( l(v, t) \) denote the level of a vertex in \( \mathcal{ES}(u, t) \), i.e., the distance of \( v \) from \( u \) in \( \mathcal{ES}(u, t) \). Also, we will use \( l(v) \) to denote the current level of vertex \( v \) in \( \mathcal{ES}(u) \).

For each vertex \( v \) at update step \( t_1 \), we denote \( d(u, v, t_1) \) as the distance from \( u \) to \( v \) in \( G(t_1) \) and \( d''(c_1, v, t_1) \) as the distance from \( c_1 \) to \( v \) in \( G''(t_1) \). If \( d(u, v, t_1) + d''(c_1, v, t_1) \leq n/c \), then the vertex \( v \) is added to \( \mathcal{P}(u, c_1) \). The level of \( v \), \( l(v, t_1) = d(u, v, t_1) \) in \( \mathcal{ES}(u, t_1) \). Define \( M(v, t_1) \leftarrow d''(c_1, v, t_1) \). We maintain the following important invariant:

\[
\text{For any vertex } v \in \mathcal{ES}(u, c_1) \text{ and for any update step } t_1 \leq t \leq t_2, l(v, t) \leq n/c - M(v, t_1).
\]

That is, we never let a vertex \( v \) reach a level greater than \( n/c - M(v, t_1) \). If \( v \) tries to move to a level greater than \( n/c - M(v, t_1) \), then we stop processing it and it remains at level \( n/c - M(v, t_1) \). We now justify the above invariant. Note that we are maintaining \( \mathcal{ES}(u) \) in order to find if there exists a path of length \( n/c \) from \( u \) to \( c_1 \). We now make the following important claim:

**Claim 4.1.** At an update step \( t (t_1 \leq t \leq t_2) \), if a vertex \( v \) tries to move to a level greater than \( n/c - M(v, t_1) \), then there is no path of length less than \( n/c \) from \( u \) to \( c_1 \) that passes through \( v \) in \( \mathcal{P}(u, c_1, t) \).
That is, if a vertex $v$ tries to move to a level greater than $n/c - M(v, t_1)$, then it has become uninteresting as there is no path from $u$ to $c_1$ of length $\leq n/c$ that passes through $v$. We can thus stop processing this vertex and it remains at level $n/c - M(v, t_1)$. We prove that the above claim holds (Lemma 5.1). We give a short justification here.

**Proof.** Since, at update step $t$, $v$ tries to move to a level greater than $n/c - M(v, t_1)$, we conclude that there is no path from $u$ to $v$ of length $n/c - M(v, t_1)$ in $P(u, c_1, t)$. The shortest path from $v$ to $c_1$ in $G^r(t_1)$ is $d^r(c_1, v, t_1)$ (and in decremental graph the length of the shortest path can only increase). So at update step $t$, the length of the shortest path from $u$ to $c_1$ that passes through $v$ is $> n/c - M(v, t_1) + d^r(c_1, v, t_1) = n/c$.

At update step $t_2$, edges in $P(u, c_2)$ are added to $P(u, c_1, t_2 - 1)$. We now show the following:

**Claim 4.2.** Let $L_w$ be the shortest path from $u$ to $w$ in $P(u, c_1, t_2 - 1) \cup P(u, c_2)$ such that $|L_w| \leq n/c - M(w, t_1)$, then $l(w, t_2 - 1) = |L_w|$.

If the above claim is true, then even after the addition of edge in $P(u, c_2)$, $w$ is at a correct level for the graph $P(u, c_1, t_2 - 1) \cup P(u, c_2)$. So, we need not decrease the level of $w$ thus preserving the decremental property of an Even-Shiloach tree. In Lemma 5.3, we show that the above claim holds. We now give a short justification by showing an intermediate result. If $|L_w| \leq n/c - M(w, t_1)$, then we make the following important claim (see Lemma 5.3):

**Claim 4.3.** If $|L_w| \leq n/c - M(w, t_1)$, then $L_w \in P(u, c_1)$.

The above claim implies that the path $L_w$ was already the part of the path union graph $P(u, c_1)$. Since no edge of this path is deleted till the update step $t_2$ and we are maintaining an Even-Shiloach tree, we can show that $l(w, t - 1) = |L_w|$. We now give a short justification for the above claim.

**Proof.** Since $w \in P(u, c_1)$, let $d^r(c_1, w, t_1)$ denote the shortest path from $c_1$ to $w$ in $G^r(t_1)$. Let $L'_w$ denote this path. Also, we set $M(w, t_1) \leftarrow d^r(c_1, w, t_1)$. So the concatenation of $L_w$ and $L'_w$ gives us a path of length $|L_w| + |L'_w| \leq n/c - M(w, t_1) + d^r(c_1, w, t_1) = n/c - M(w, t_1) + M(w, t_1) \leq n/c$. So each vertex (and each edge) of $L_w$ (and $L'_w$) is a part of $P(u, c_1)$.

Now, we fill in the last missing link, i.e., the case when $|L_w| > n/c - M(w, t_1)$. However, note that we mandate that $w$ never moves to a level greater than $n/c - M(w, t_1)$ before update step $t_2$. So we can conveniently move $w$ to level $|L_w|$ at update step $t_2$. Note that we are only increasing the level of $w$. Thus we still preserve the decremental property of Even-Shiloach tree. In the above argument, we crucially use the fact that there is no need to move the vertex $w$ to a level greater than $n/c - M(w, t_1)$ between update steps $t_1$ and $t_2$.

Thus, addition of edges $P(u, c_2)$ does not violate the decremental property of the Even-Shiloach tree $ES(u)$. We will extend the above technique to maintain one Even-Shiloach tree on many path union graphs associated with $u$, i.e., $\cup_i P(u, c_i)$. This will allow us to maintain just one Even-Shiloach tree for $u$ instead of $O(c)$ (as done in [HKNL4]). This completes the short overview of our main technique. In the next section, we give full details of our technique.

## 5 Improved Algorithm for st-Reachability

We will use the notation as defined in the previous section. In addition, we extend the following notations:
1. At time steps $t_1, t_2, \ldots, t_l$, vertices $c_1, c_2, \ldots, c_l \in C(G)$ respectively becomes unreachable from $u$ via a hub. At update step $t_i$, we find the path union graph $\mathcal{P}(u, c_i)$. Also, for any $t \geq t_i$, let $\mathcal{P}(u, c_i, t)$ denote the edges in $\mathcal{P}(u, c_i)$ that are not deleted till update step $t$.

2. The union graph $\mathcal{U}(u, \cdot)$ denotes the union of all path union graphs of $u$. It is formally defined as follows: for $t > t_i$, $\mathcal{U}(u, t) \leftarrow \bigcup_{t_i \leq t} \mathcal{P}(u, c_i, t)$. For notational convenience, we will use $\mathcal{U}(u)$ to denote the current union graph.

3. At update step $t$, we will maintain an Even-Shiloach tree for the union graph $\mathcal{U}(u, t)$. We will use $\mathcal{E}S(u)$ to denote the current Even-Shiloach tree. Also, $\mathcal{E}S(u, t)$ will denote the Even-Shiloach tree of $u$ at the end of update step $t$.

4. Also for book-keeping, we use the following data-structure: for each edge $(v, w) \in \mathcal{E}S(u)$, the current level of $w$ in $\mathcal{E}S(u, l(w))$, is stored in $CL_u(w)$.

At update step $t_i$, edges in $\mathcal{P}(u, c_i)$ are added to $\mathcal{U}(u, t_i - 1)$. Let $d(u, v, t_i)$ denote the distance between $u$ and $v$ in $G(t_i)$. Also, let $d^r(c_i, v, t_i)$ denote the distance between $c_i$ and $v$ in $G^r(t_i)$. We maintain the following invariant:

**Invariant 5.1.** For $1 < i < l$, let $t$ be an update step such that $t_{i-1} \leq t < t_i$. Let $M(v, t) \leftarrow \min_{t_{i-1} \leq t} d^r(c_j, v, t_j)$ for each vertex $v$. For any vertex $v \in \mathcal{U}(u, t)$, the level $l(v, t) \leq n/c - M(v, t)$.

We now describe this invariant in detail. Let $t \geq t_{i-1} \geq \cdots \geq t_2 \geq t_1$. At update step $t_j$ ($1 \leq j \leq i - 1$), we find the distance between $c_j$ and $v$ in $G^r(t_j)$. We define $M(v, t)$ as the minimum of all such distances calculated at update step $\{t_1, t_2, \ldots, t_{i-1}\}$, i.e., $M(v, t) = \min_{t_{i-1} \leq t} d^r(c_j, v, t_j)$. Our invariant mandates that we never let the level of $v$ increase over $n/c - M(v, t)$ where $t_{i-1} \leq t < t_i$. At update step $t$, if $l(v, t)$ tries to increase to level $n/c - M(v, t) + 1$ and violate Invariant 1, we add $v$ to a spurious set $S$. We then remove all the edges emanating out of $v$ from $\mathcal{E}S(u, t - 1)$. Note that these edges are not deleted from $\mathcal{U}(u, t - 1)$, but we forcefully remove them from $\mathcal{E}S(u, t - 1)$. The invariant also indicates that the value $M(v, \cdot)$ does not change from update step $t_{i-1}$ to update step $t_i - 1$. In the ensuing discussion, we will assume that $M(v, t_i) \leftarrow M(v, t - 1)$ for $t_{i-1} \leq t < t_i$. Even at update step $t_i$, we will assume that $M(v, t_i) \leftarrow M(v, t_i - 1)$ unless our algorithm explicitly sets a new value of $M(v, t_i)$ at update step $t_i$.

We now describe an important property of a vertex in $S$, henceforth called a spurious vertex. If $v \in \mathcal{U}(u, t)$, it should imply that there exists a path of length less than $n/c$ from $u$ to some other vertex in $C(G)$ that passes through $v$. However, we will show that the above implication does not hold for spurious vertices. Let $v$ be a spurious vertex at update step $t$ ($t_{i-1} \leq t < t_i$). Since $v$ tries to move to a level $> n/c - M(v, t)$, there is no path from $u$ to $v$ of distance less than $n/c - M(v, t)$ in $\mathcal{U}(u, t)$. Let $t_k \leftarrow \arg\min_{t_{i-1} \leq t} d^r(c_j, v, t_j)$. That is, the distance between $c_k$ and $v$ is $d^r(c_k, v, t_k)$ at update step $t_k$. In a decremental graph, the distance between vertices can only increase. So, even at update step $t$, the distance between $c_k$ and $v$ is $\geq d^r(c_k, v, t_k)$. In $G(t)$, the length of the shortest path from $u$ to $c_k$ that passes through $v$ should be $> n/c - M(v, t) + d^r(c_k, v, t_k)$. Since $M(v, t) = \min_{t_{i-1} \leq t} d^r(c_j, v, t_j) = d^r(c_k, v, t_k)$, there is no path of length less than $n/c$ from $u$ to $v$ that passes through $v$. Similarly, we can also prove that there is no path of length less than $n/c$ from $u$ to vertices in $\{c_1, c_2, \ldots, c_{i-1}\}$ that passes through $v$. So we have proved the following lemma:

**Lemma 5.1.** For all $t_{i-1} \leq t < t_i$, if $v$ is spurious at update step $t$, then there is no path of length less than $n/c$ from $u$ to vertices in $\{c_1, c_2, \ldots, c_{i-1}\}$ that passes through $v$ in $\mathcal{U}(u, t)$.  

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We revisit the basics of an Even-Shiloach tree. An important invariant of an Even-Shiloach tree is defined in Invariant 5.2. for a vertex \( v \), \( l(v, t) \) = the length of the shortest path from \( u \) to \( v \) in the current graph. We tweak the above property to take into account spurious vertices and Invariant 5.1. Our algorithm will maintain \( ES(u) \) with the following invariant:

**Invariant 5.2.** If \( L \) is the shortest path from \( u \) to \( v \) in \( U(u, t) \) such that

1. \( |L| \leq n/c - M(v, t) \)
2. for any vertex \( w \in L \), \( w \notin S \)

Then \( l(v, t) = |L| \).

We need to show that at any update step, Invariant 5.2 is satisfied. Since \( U(u, t) \) does not contain any edge before update step \( t < t_1 \), Invariant 5.2 is trivially satisfied. At update step \( t_1 \), \( U(u, t_1) \) is initialized by adding all the edges in \( P(u, c_1) \) and \( ES(u) \) is found out. At this update step, no vertex \( v \in U(u, t_1) \) is spurious. So, Invariant 5.2 holds at update step \( t_1 \). Between update steps \( t_1 \) and \( t_2 \), edges may be deleted from \( U(u) \). In general, let \( t_{i-1} \leq t < t_i \) be the update step at which an edge \( (v, w) \) is deleted from \( U(u, t - 1) \).

We now describe the Procedure DELETION that handles the deletion of this edge. This procedure is similar to the classical Even-Shiloach algorithm. If \( (v, w) \notin U(u, t - 1) \), then there is nothing to be done. So assume that \( (v, w) \in U(u, t - 1) \). We now follow the Procedure DELETION in Figure 3. If \( l(w) > n/c \), then there is nothing to be done as we maintain a tree till a distance \( n/c \). Else we add \( w \) to the dirty list \( D \).

The procedure is similar to the Procedure in Figure 1 except that we need to take care of vertices that may become spurious. In Step 10 (see Figure 3), if a vertex \( w \) tries to move to a level \( > n/c - M(w, t) \), then we add \( w \) to \( S \). For each child \( w' \) of \( w \) in \( ES(u) \), we then remove the edge \( (w, w') \) from \( ES(u) \) and add \( w' \) to the dirty list \( D \). Also, with each vertex \( w \), we will maintain a set \( S_w \) that contains all the spurious neighbors of \( w \). When \( w \) is added to \( S \), then for each neighbor \( w' \) of \( w \), \( w \) is added to \( S_{w'} \). The other difference in procedure DELETION is in Step 7. When \( w \) searches for a replacement edge, we make sure that the other endpoint of that edge is not spurious – the set \( S_w \) is used for this purpose. The last difference in procedure DELETION is in Step 1. We do not process a vertex \( w \) once it becomes spurious.

Assume that Invariant 5.2 holds at update step \( t_{i-1} \). We will crucially use two observations of the algorithm in Figure 3:

1. If a vertex \( w \) is added to \( S \), then all its outgoing edges in \( ES(u) \) are removed. Due to this, we claim that no vertex in \( ES(u) \) is reachable from \( u \) via a spurious vertex.
2. A vertex added to \( S \) is never removed from \( S \) between update steps \( t_{i-1} \) and \( t_i \).

Apart from the above changes, our algorithm is same as the classical Even-Shiloach algorithm. So we can claim the following lemma:

**Lemma 5.2.** Assuming that Invariant 5.2 holds at update step \( t_{i-1} \), it holds for all update steps between \( t_{i-1} \) and \( t_i \) where \( 1 < i < l \).

We now prove a basic lemma which will be helpful in proving the main result of the paper.

**Lemma 5.3.** Assuming that Invariant 5.2 holds at update step \( t_{i-1} \). At an update step \( t (t_{i-1} \leq t < t_i) \), for any vertex \( v \in U(u, t) \), if there exists a path \( L \) from \( u \) to \( v \) in \( G(t) \) such that \( |L| \leq n/c - M(v, t) \), then

1. \( \forall w \in L, w \in U(u, t) \)
2. \( \forall w \in L, w \notin S \)

The important part of the lemma is that \( L \) can be any path in \( G(t) \). This implies that the path \( L \) was a part of at least one of the path union graph in \( \{P(u,c_1),P(u,c_2),\ldots,P(u,c_{i-1})\} \).

**Proof.** Let \( L = \{u,v_1,v_2,\ldots,v_l = v\} \) be a path from \( u \) to \( v \) such that \( |L| = l \leq n/c - M(v,t) \)

1. We have to show that each vertex in \( L \) lies in \( \mathcal{U}(u,t) \). By definition, \( M(v,t) = \min_{t_j < t} d^r(c_j,v,t_j) \). Let \( t_k \leftarrow \arg\min_{t_j < t} d^r(c_j,v,t_j) \). So \( M(v,t_k) = M(v,t) = d^r(c_k,v,t_k) \). Let \( L' \) be the path of length \( d^r(c_k,v,t_k) \) from \( v \) to \( c_k \) at update step \( t_k \). Also, note that since the path \( L \) exists in \( G(t) \), it also existed in \( G(t_k) \). So the concatenation of path \( L \) and \( L' \) gives us a path from \( u \) to \( v \) having length \( \leq n/c - M(v,t) + d^r(c_k,v,t_k) \leq n/c \). This implies that all the vertices on the path \( L \) were part of the graph \( \mathcal{P}(u,c_k) \).

2. Again consider the the update step \( t_k \). Consider a vertex \( v_i \in L \). Since there is a path of length \( l-i \) from \( v \) to \( v_i \) in \( G(t_k) \), \( M(v_i,t_k) \leq M(v,t_k) + (l-i) \). By definition, the value of \( M(v_i,\cdot) \) can only decrease. So, even at update step \( t \), \( M(v_i,t) \leq M(v_i,t_k) \leq M(v,t_k) + (l-i) \). Since, \( M(v,t) \) is same as \( M(v,t_k) \), \( M(v_i,t) \leq M(v,t) + (l-i) \). By Lemma 5.2, Invariant 5.2 holds at update step \( t \). Since

```latex
\begin{figure}
\begin{verbatim}
if \( l(w) > n/c \) or \( w \in S \) then
  return;
D \leftarrow \{w\};
while \( D \neq \emptyset \) do
  Let \( w \) be a vertex in \( D \) having least \( l(w) \) (Break ties arbitrarily);
  Remove \( w \) from \( D \);
  if \( \exists (w',w) \) such that \( w \) is not spurious and \( l(w') = l(w) - 1 \) then
    Add edge \((w',w)\) in \( ES(u) \);
  else
    if \( l(w) = n/c - M(w,t) \) then
      Add \( w \) to \( S \);
      foreach child \( w' \) of \( w \) in \( ES(u) \) do
        Remove \((w,w')\) from \( ES(u) \);
        Add \( w' \) to \( D \);
      foreach neighbor \( w' \) of \( w \) do
        \( S_{w'} \leftarrow S_{w'} \cup \{w\} \);
    else
      \( l(w) \leftarrow l(w) + 1 \); //increase the level of \( w \);
      foreach neighbor \( w' \) of \( w \) do
        \( CL_{w'}(w) \leftarrow l(w) \);
      Add \( w \) to \( D \);
      for each child \( w' \) of \( w \) in \( ES(u) \) do
        Add \( w' \) to \( D \);
end while
\end{verbatim}
\end{figure}
```
there is a path from $u$ to $v_i$ in $G(t)$ of length $i$ (the path from $u$ to $v_i$ in $L$), the level of $v_i$, $l(v_i, t) \leq i$. For $v$ not to be spurious, $i$ should be less than equal to $n/c - M(v, t) \geq n/c - M(v, t) - (l - i)$. That is, $l$ should be less than equal to $n/c - M(v, t)$, which is indeed true.

\[ \square \]

At update step $t_i$, edges in $P(u, c_i)$ are added to $U(u, t_i - 1)$. We now describe the procedure $Add(P(u, c_i))$ that adds edges from $P(u, c_i)$ to $U(u)$ at update step $t_i$ (see Figure 3). Let $F$ denote the set of vertices in $P(u, c_i)$. We process vertices from $F$ in the following order: at each iteration, we process that vertex $v$ from $F$ which is closest to $u$ in $P(u, c_i)$. For each $(v, w)$ such that $w \in P(u, c_i)$, we check if distance of $c_i$ to $w$ in $G^r(t_i)$ is less than $M(w, t_i - 1)$. If yes, then we set $M(w, t_i) \leftarrow d^r(c_i, w, t_i)$ (else $M(w, t_i)$ remains same as $M(w, t_i - 1)$). If $w$ was spurious, then it is now free to move to a higher level (as $M(w, t_i) < M(w, t_i - 1)$). So we remove $w$ from $S$ and increase the level of $w$ to $l(w) + 1$. If $w$ is not an element of $U(u, t_i - 1)$, then we add $w$ along with the edge $(v, w)$ to $E\mathcal{S}(u, t_i - 1)$ and initialize its level. If not, then edge $(v, w)$ is added to $U(u, t_i - 1)$ (if it is not already present in $U(u, t_i - 1)$) and the other associated data-structure is initialized.

\[
\begin{align*}
1 & \text{ Let } F \text{ be the set of vertices in } P(u, c_i); \\
2 & \text{ while } F \text{ is not empty do } \\
3 & \quad \text{ Let } v \in F \text{ be a vertex closest to } u \text{ in } P(u, c_i) \text{ (Break ties arbitrarily);} \\
4 & \quad \text{ Remove } v \text{ from } F; \\
5 & \quad \text{ foreach } (v, w) \text{ such that } w \in P(u, c_i) \text{ do } \\
6 & \quad \quad \text{ if } d^r(c_i, w, t_i) < M(w, t_i - 1) \text{ then } \\
7 & \quad \quad \quad M(w, t_i) \leftarrow d^r(c_i, w, t_i); \\
8 & \quad \quad \quad \text{ if } w \in S \text{ then } \\
9 & \quad \quad \quad \quad \text{ Remove } w \text{ from } S; \\
10 & \quad \quad \quad \quad \quad l(w) \leftarrow l(v) + 1; \\
11 & \quad \quad \quad \quad \text{ foreach edge } (w, w') \in U(u, t_i - 1) \text{ do } \\
12 & \quad \quad \quad \quad \quad \text{ Remove } w \text{ from } S_{w'}; \\
13 & \quad \quad \quad \quad \quad CL_{w'}(w) \leftarrow l(w); \\
14 & \quad \quad \text{ if } w \notin U(u, t - 1) \text{ then } \\
15 & \quad \quad \quad \text{ Add } (v, w) \text{ to } E\mathcal{S}(u); \\
16 & \quad \quad \quad \quad l(w) \leftarrow l(v) + 1; \\
17 & \quad \quad \quad \text{ U}(u) \leftarrow U(u) \cup \{(v, w)\}; \\
18 & \quad \quad \quad CL_{w}(v) \leftarrow l(v);
\end{align*}
\]

Figure 4: $Add(P(u, c_i))$ : Procedure used to add edges from $P(u, c_i)$ to $U(u)$ at update step $t_i$

We first present an intermediate lemma which will help us in proving our main result.

**Lemma 5.4.** If $w$ is processed in the procedure $Add(P(u, c_i))$, then either $w$ is not spurious or is removed from the spurious set list during its processing in this procedure.

**Proof.** If $w \notin S$ at update step $t_i - 1$, then there is nothing to prove. So let $w \in S$ at update step $t_i - 1$. Since $w$ is processed in the procedure $Add(P(u, c_i))$, $w$ is an element of $F$. This implies that there is a path form $u$ to $w$ in $P(u, c_i)$ in $G(t_i)$ and a path from $c_i$ to $w$ in $G^r(t_i)$. If $w$ was spurious before this update step, then using Lemma [5.3] there is no path from $u$ to $w$ in $G(t_i - 1)$ of length $\leq n/c-$
Let \( M(w, t_i - 1) \). Since \( w \in \mathcal{P}(u, c_i) \), the length of a path from \( u \) to \( w \) in \( \mathcal{P}(u, c_i) \), \( d(u, w, t_i) \) should be greater than \( n/c - M(w, t_i - 1) \). So, \( d^k(c_i, w, t_i) \) must be less than \( n/c - (n/c - M(w, t_i - 1)) \). That is, \( d^k(c_i, w, t_i) < M(w, t_i - 1) \). In that case, in Step 2 of \( \text{Add}(\mathcal{P}(u, c_i)) \), we remove \( w \) from \( S \).

We have to show that Invariant 5.2 holds even after the completion of Procedure \( \text{Add}(\mathcal{P}(u, c_i)) \). We will assume that Invariant 5.2 holds at update step \( t_i - 1 \). Using Lemma 5.2, we claim that Invariant 5.2 holds at update step \( t_i - 1 \). Let us first show that Invariant 5.2 holds for a vertex \( w \) such that \( w \in U(u, t - 1) \). The bad case arises at Step 17 of Procedure \( \text{Add}(\mathcal{P}(u, c_i)) \). While we were processing a vertex \( v \in \mathcal{P}(u, c_i) \), we may add an edge \((v, w)\) to \( U(u) \). Addition of an edge may violate the decremental property of the Even-Shiloach tree. At this point in the algorithm, we are not even checking if the addition of this edge violates Invariant 5.2. We need to show that Invariant 5.2 holds even after the addition of edge \((v, w)\) at Step 17.

Another bad case might be the following: by Lemma 5.4, \( v \) may be removed from \( S \) at update step \( t_i \). Let \( L_v \) be the shortest path from \( u \) to \( v \) in \( \mathcal{P}(u, c_i) \). After \( v \) is processed, for a neighbor \( w \) of \( v \), there may now be a path of length \( |L_v| + 1 \) from \( u \) to \( w \) that does not contain any spurious vertex. So, we need to show that the level of \( w \) at update step \( t_i - 1 \), \( l(w, t_i - 1) \leq |L_v| + 1 \).

In general, let \( v \) be a vertex processed at update step \( t_i \). Let \( L_v \) be the shortest path from \( u \) to \( v \) in \( \mathcal{P}(u, c_i) \). For any neighbor \( w \) of \( v \) in \( U(u) \), there now exists a path \( L_w \) from \( u \) to \( w \) of length \( |L_v| + 1 \). We have to show that after the completion of the procedure \( \text{Add}(\mathcal{P}(u, c_i)) \), \( w \) satisfies Invariant 5.2 and its level in \( U(u, t_i) \leq |L_w| \).

There are two cases to be considered:

1. \( |L_w| \leq n/c - M(w, t_i - 1) \)

   If \( |L_w| \leq n/c - M(w, t_i - 1) \), then the condition of Lemma 5.2 is satisfied. That is, at update step \( t_i - 1 \), there exists a path of length \( \leq n/c - M(w, t_i - 1) \) in \( G(t_i - 1) \). So, none of the vertex in \( L_w \) is spurious. By Lemma 5.2, Invariant 5.2 is satisfied at update step \( t_i - 1 \). So, \( l(w, t_i - 1) \leq |L_w| \).

   Procedure \( \text{Add}(\mathcal{P}(u, c_i)) \) sets a level of a vertex only if it is spurious (Step 10) or a newly added vertex in \( U(u) \) (Step 16). Since \( w \) is neither spurious (since \( w \in L_w \)) nor a newly added vertex at update step \( t_i \), we do not set a new level for \( w \) even at update step \( t_i \). So, we can conclude that after the completion of procedure \( \text{Add}(\mathcal{P}(u, c_i)) \), \( l(w, t_i) \leq |L_w| \) (where the inequality becomes an equality only if \( L_w \) represents the shortest path from \( u \) to \( w \)).

2. \( |L_w| > n/c - M(w, t_i - 1) \)

   There are two cases:

   (a) If \( w \in S \) and \( w \in \mathcal{P}(u, c_i) \)

      Since \( w \in \mathcal{P}(u, c_i) \), by Lemma 5.4, it is removed from \( S \) at update step \( t_i \). In this case, we set a new value of \( M(w, t_i) \). Also, \( |L_w| \leq n/c - M(w, t_i) \) as the edge \((v, w)\) cannot be a part of \( \mathcal{P}(u, c_i) \) if the above condition is not true. The procedure \( \text{Add}(\mathcal{P}(u, c_i)) \) removes \( w \) from \( S \) (Step 9) and correctly sets the level of \( w \) to \( |L_w| \) to maintain Invariant 5.2.

   (b) Otherwise

      In this case, procedure \( \text{Add}(\mathcal{P}(u, c_i)) \) does not change the value of \( M(w) \). So, \( M(w, t_i) \) is same as \( M(w, t_i - 1) \). In that case, the condition of Invariant 5.2 is not satisfied, since \( |L_w| > n/c - M(w, t_i) \). So, Invariant 5.2 is trivially satisfied here.

For a vertex \( w \) such that \( w \notin U(u, t_i - 1) \) but \( w \in \mathcal{P}(u, c_i) \), \( w \) is a newly inserted vertex to \( U(u) \) at update step \( t_i \). Since we process each vertex based on its distance from \( u \) to \( w \) in \( \mathcal{P}(u, c_i) \), when we process \( w \) at Step 14 all the vertices on the shortest path from \( u \) to \( w \) (except \( w \)) in \( \mathcal{P}(u, c_i) \) are processed. By Lemma 5.4, none of these vertices are spurious. So, to satisfy Invariant 5.2 we should set \( l(w) \) to be the
length of its shortest path from $u$. Indeed, at Step 16 we set $l(w)$ to the length of its shortest path (by setting $l(w) ← l(v) + 1$). So, we have proved the following lemma.

Lemma 5.5. Assume that Invariant 5.2 holds at update step $t_{i−1}$. After the completion of procedure Add($P(u, c_i)$) at update step $t_i$, Invariant 5.2 holds.

Invariant 5.2 holds at update step $t_1$ (when $P(u, c_1)$ is added to $U(u)$). We can use Lemma 5.5 and Lemma 5.3 and prove by induction that Invariant 5.2 holds at every update steps from $t_1$.

We now show the correctness of the algorithm. To this end, we have to show that our algorithm correctly maintains the connectivity information between $u$ and $c_1, c_2, \ldots, c_i$. To achieve this, we will use the following simple approach: Fix a vertex $c_i$. We check the update step $t$ at which $c_i$ becomes spurious. This can be easily tracked by our algorithm. From update step $t_i ≤ t' < t$, we claim that there exists a path of length less than $n/c$ from $u$ to $c_i$ in $U(u, t')$. And from update step $t' ≥ t$, we claim that no such path exists.

We now show that the above assertion is true. Consider an update step $t' ≥ t_i$. Note that $M(c_i, t_i) = 0$, so $M(c_i, t'_i) = 0$ as well. If there exists a path of length less than $n/c$ from $u$ to $c_i$ at update step $t'$, then by Lemma 5.3 all the vertices on this path are in $U(u, t')$ and none of them are spurious. So $c_i$ is also not spurious. If there is no path of length less than $n/c$ from $u$ to $c_i$ at update step $t'$, then there is no such path even in $U(u, t')$. Since, $n/c ≥ n/c - M(c_i, t'$), either $c_i$ is added to $S$ at update step $t'$ or it was in $S$ before the start of update step $t'$. So our algorithm correctly maintains the connectivity information between $u$ and $c_i$.

Now we bound the running time of the algorithm. For each $c_i ∈ \{c_1, c_2, \ldots, c_i\}$, we have to find $P(u, c_i)$. We perform two BFS to find $P(u, c_i)$: one in $G(t_i)$ and other in $G^c(t_i)$. This takes $O(m)$ time. In the Procedure Add($P(u, c_i)$), for each vertex $v$ in $F$, all its adjacent edges are processed in $O(1)$ time. So the total running time of the procedure Add($P(u, c_i)$) is $O(m)$ which is asymptotically same as the running time of BFS performed at update step $t_i$.

Now we bound the time DELETION procedure in Figure 3. Its total processing time can be divided into the following two categories:

1. Processing done when a vertex is added to $S$.

Suppose that we add a vertex $v$ to $S$ at update step $t$. This implies that $l(v, t)$ tries to increase beyond $n/c - M(v, t)$. To this end, we remove the edge between $v$ and its child $w$ from $ES(u)$. The time taken to remove edges of $v$ is $d_{max}(v, U(u))$ – the maximum degree of $v$ in $U(u)$ at any point during the algorithm. The child of $v$, $w$ may either find a replacement edge in $O(1)$ time or increase its level. In the first case, the processing time is same as the time taken to remove $(v, w)$ from $ES(u)$. In the second case, $w$ increases its level and notifies its neighbors about its new level. Since, even in our algorithm, we never decrease the level of any vertex, the processing time for this step can be accounted by the analysis done in the classical Even-Shiloach algorithm.

Since $v$ is removed from $S$ only during the procedure Add($P(u, c_i)$) and the number of centers in $C(G)$ is $\tilde{O}(c)$, the number of times a vertex can move out of set $S$ is $\tilde{O}(c)$. So, $v$ can be added to $S$ atmost $\tilde{O}(c)$ times. The total processing time required when $v$ is added to $S$ can then be bounded by $\sum_{v \in U(u)} \tilde{O}(cd_{max}(v, U(u)))$. Using Lemma 3.4, the total number of edges in $U(u)$ at any point of time can be bounded by $O(m/b)$ with high probability. So the total time is $\sum_{v \in U(u)} \tilde{O}(cd_{max}(v, U(u))) = \tilde{O}(\frac{mn^2}{b^2})$. Adding up the computation time for all $u$ in $C(G)$, the total time processing done when a vertex is added to $S$ is bounded by $\tilde{O}(\frac{mn^2}{b^2})$.

2. Total update time to maintain $ES(u)$ in $U(u)$.

Using Lemma 3.4 the total number of edges in $U(u)$ is $O(m/b)$. The DELETION procedure for maintaining Even-Shiloach tree is same as the classical Even-Shiloach algorithm since even in our tree
we never decrease the level of a vertex. So the time taken to maintain $\mathcal{E}S(u)$ is $\tilde{O}(\frac{mRu}{b}) = \tilde{O}(\frac{mn}{b^c})$.

The total update time to maintain Even-Shiloach tree for all $u \in \mathcal{C}(G)$ is bounded by $\tilde{O}(\frac{mn}{b^c})$.

So, the total update time of DELETION algorithm is $\tilde{O}(\frac{mn}{b^c} + \frac{mc^2}{b^c})$. Thus, we have improved the running time of Step 3 of the Figure 2 from $\tilde{O}(\frac{mn}{b^c})$ to $\tilde{O}(\frac{mn}{b^c} + \frac{mc^2}{b^c})$. If we set $c = n^{2/5}$ and $b = n^{1/5}$, we see that the total running time of all steps in Figure 2 is $\tilde{O}(mn^{4/5} + n^{8/5})$. The term $n^{8/5}$ is greater than the term $mn^{4/5}$ only when $m \leq n^{4/5}$. For a graph with $m \leq n^{4/5}$, we can use the trivial Even-Shiloach algorithm [ES81]. Since the total number edges is $n^{4/5}$, the maximum distance between $s$ and any other vertex is $n^{4/5}$. So we have to maintain an Even-Shiloach tree till a distance $R = n^{4/5}$. The total update time of such an Even-Shiloach tree is $O(mR) = O(mn^{4/5})$. So, we use the classical Even-Shiloach algorithm when $m < n^{4/5}$.

We can now claim the following theorem:

**Theorem 5.6.** There exists a randomized algorithm that maintains st-Reachability in a decremental directed graph in a total update time of $\tilde{O}(mn^{4/5})$ with high probability.

### 6 Extentions

We now move on to some extension of st-Reachability problem. For dense graphs, we redo the calculation performed in the previous section. The only change we will make is that instead of bounding the size of $\mathcal{U}(u)$ by $m/b$, we will bound it by $n^2/b^2$. Note that by Lemma 3.4, the total number of vertices in $\mathcal{U}(u)$ is $n/b$ with high probability, so the total number of edges in $\mathcal{U}(u)$ is $n^2/b^2$. So replacing $m/b$ by $n^2/b^2$ in the above calculation, we see that the time taken to maintain $\mathcal{E}S(u)$ for all $u$ is $\tilde{O}(\frac{n^3}{b^c} + \frac{n^3c^2}{b^c})$.

We now use one result presented in [HKN14]. For dense graph, they build an approximate reachability data structure that builds all the path union graph (step 2 in Figure 2) in $\tilde{O}(n^2c + n^2c^2/b^2)$. So the time bound in Figure 2 changes as follows:

1. $\tilde{O}(\frac{mn}{b^c} + bnc)$ – time taken to maintain an Even-Shiloach tree from the hubs
2. $\tilde{O}(n^2c + \frac{n^3c^2}{b^c})$ – time taken to construct the path union graphs ([HKN14])
3. $\tilde{O}(\frac{n^3}{b^c} + \frac{n^3c^2}{b^c})$ – time taken to maintain Even-Shiloach trees on all the path union graphs (using our algorithm)
4. $\tilde{O}(c^3)$ – time taken to maintain an Even-Shiloach tree on $\mathcal{C}(G)$

If we set $c = n^{2/3}$ and $b = n^{1/3}$ and $m = n^2$, we see that the total running time of our algorithm is $O(n^{8/3})$.

**Theorem 6.1.** There exists a randomized algorithm that maintains st-Reachability in a decremental directed graph in a total update time of $\tilde{O}(n^{8/3})$ if the initial size of the graph is $\Omega(n^2)$ with high probability.

Consider the problem of maintaining single source reachability (SSR): Given a decremental directed graph where after every edge deletion, we may have to answer the following query.

Is $v$ reachable from the source $s$?

where $v$ is any vertex in the graph. The previous best total update time for this problem was $\tilde{O}(mn^{0.982})$ [HKN14]. In [HKN14], the authors prove that if we can maintain st-Reachability in $\tilde{O}(mn^{1 - \epsilon})$ update time, then we can maintain SSR in $O(mn^{1 - \epsilon/2})$ update time. In our algorithm $1 - \epsilon = 4/5$, so $\epsilon = 1/5$. So, we can prove the following:
Theorem 6.2. There exists a randomized algorithm that maintains SSR in a decremental directed graph in a total update time of $\tilde{O}(mn^{9/10})$ with high probability.

Consider the problem of maintaining strongly connected component (SCC): Given a decremental directed graph where after every edge deletion, we may have to answer the following query.

Are $u$ and $v$ in the same component?

There are few algorithms that solve this problem in $O(mn)$ total update time \cite{RZ02, Lac13, Rod13}. Henzinger et. al. \cite{HKN14} improved the above bound to $\tilde{O}(mn^{0.982})$. In \cite{HKN14}, the authors showed that if we can maintain SSR in $\tilde{O}(mn^\alpha)$ update time, then we can maintain SCC in $O(mn^\alpha)$ update time. We can use our improved algorithm for SSR to prove the following:

Theorem 6.3. There exists a randomized algorithm that maintains SCC in a decremental directed graph in a total update time of $\tilde{O}(mn^{9/10})$ with high probability.

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