Uncertainty principle for the Riemann-Liouville operator

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ABSTRACT

A Beurling-Hörmander theorem’s is proved for the Fourier transform connected with the Riemann-Liouville operator. Nextly, Gelfand-Shilov and Cowling-Price type theorems are established.

RESUMEN

Se demuestra el teorema de Beurling-Hörmander por la transformada de Fourier conectada con el operador de Riemann-Liouville. Además, se establecen teoremas tipo de Gelfand-Shilov y Cowling-Price.
Keywords: Beurling-Hörmander theorem, Gelfand-Shilov theorem, Cowling-Price theorem, Fourier transform, Riemann-Liouville operator.

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1 Introduction

The uncertainty principles play an important role in harmonic analysis and have been studied by many authors, and from many points of view [12, 15]. These principles state that a function $f$ and its Fourier transform $\hat{f}$ cannot be simultaneously sharply localized. Theorems of Hardy, Morgan, Gelfand-Shilov, or Cowling-Price,... are established for several Fourier transforms [8, 11, 13, 20, 21], the most recent being the well known Beurling-Hörmander theorem’s which has been proved by Hörmander [16], who took an idea of Beurling [4]. This theorem states that if $f$ is an integrable function on $\mathbb{R}$ with respect to the Lebesgue measure, and if

$$\int_{\mathbb{R}^2} |f(x)||\hat{f}(y)||e^{\langle xy\rangle} \, dx \, dy < +\infty,$$

then $f = 0$ almost everywhere.

Later, Bonami, Demange and Jaming [5] have generalized the above theorem and have established a strong multidimensional version of this uncertainty principle [15], by showing the following result

If $f$ is a square integrable function on $\mathbb{R}^n$ with respect to the Lebesgue measure, then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)||\hat{f}(y)||}{(1 + |x| + |y|)^d} e^{\langle x/y \rangle} \, dx \, dy < +\infty,$$

if and only if $f$ may be written as

$$f(x) = P(x)e^{-(A x / x)},$$

where $A$ is a real positive definite symmetric matrix and $P$ is a polynomial with degree($P$) < $d - n$. In particular for $d \leq n$, $f$ is identically zero.

The Beurling-Hörmander uncertainty principle in its weak and strong forms has been studied by many authors, and for various Fourier transforms. In particular, Bouattour and Trimeche [6] have showed this theorem for the hypergroup of Chébli-Trimèche, Kamoun and Trimeche [17] have proved an analogue of the Beurling-Hörmander theorem for some singular partial differential operators, Trimeche [22] has showed this uncertainty principle for the Dunkl transform, we cite also Yakubovich [26], who has established the same result for the Kontorovich-Lebedev transform.

The Beurling-Hörmander uncertainty principle implies many other known quantitative uncertainty principles as those of Gelfand-Shilov [13], Cowling-Price [8], Morgan [3, 19] or also the one of Hardy [14].

In [2], the third author with the others have considered the singular partial differential oper-
ators defined by
\[
\begin{aligned}
\Delta_1 &= \frac{\partial}{\partial x}, \\
\Delta_2 &= \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2} ; (r, x) \in ]0, +\infty[ \times \mathbb{R} ; \alpha \geq 0,
\end{aligned}
\]
and they associated to \(\Delta_1\) and \(\Delta_2\) the following integral transform, called the Riemann-Liouville operator which is defined on \(\mathcal{C}_c^\prime(\mathbb{R}^2)\) (The space of continuous functions on \(\mathbb{R}^2\), even with respect to he first variable) by
\[
\mathcal{R}_\alpha(f)(r, x) = \begin{cases}
\frac{\alpha}{\pi} \int_{-1}^{1} \int_{-1}^{1} f(rs\sqrt{1-t^2}, x + rt)(1-t^2)^{-\frac{\alpha}{2}}(1-s^2)^{-\frac{\alpha}{2}-1} \frac{dt}{\sqrt{1-t^2}} \frac{ds}{\sqrt{1-s^2}}, & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{-1}^{1} f(r\sqrt{1-t^2}, x + rt) \frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0.
\end{cases}
\]
The Fourier transform connected with the operator \(\mathcal{R}_\alpha\) is defined by
\[
\mathcal{F}_\alpha(f)(\mu, \lambda) = \int_{\mathbb{R}} f(r, x) \varphi_{\mu, \lambda}(r, x) d\nu_\alpha(r, x),
\]
where
\[
\varphi_{\mu, \lambda}(r, x) = \mathcal{R}_\alpha(\cos(\mu_x)e^{-i\lambda_x})(r, x).
\]
\(d\nu_\alpha\) is the measure defined on \([0, +\infty[ \times \mathbb{R}\) by,
\[
d\nu_\alpha(r, x) = \frac{\pi^{\alpha+1}}{2^{\alpha+1} \Gamma(\alpha + 1)\sqrt{2\pi}} dr \otimes dx.
\]
Many harmonic analysis results are established for the Fourier transform \(\mathcal{F}_\alpha\) (Inversion formula, Plancherel’s formula, Paley-Winer and Plancherel’s theorems...).

The aim of this work is to establish the Beurling-Hörmander theorem for the fourier transform \(\mathcal{F}_\alpha\) and to deduce the analogues of the Gelfand-Shilov and the Cowling-Price theorems for this transform.

More precisely, in the second section, we give some basic harmonic analysis results related to the Fourier transform \(\mathcal{F}_\alpha\). The third section is devoted to establish the main result of this paper, that is the the Beurling-Hörmander theorem.
Let $f$ be a square integrable function on $[0, +\infty \times \mathbb{R}$ with respect to the measure $d\nu_{\alpha}$. Let $d$ be a real number, $d \geq 0$. If

$$\int_{\Gamma_+} \int_{\mathbb{R}} |f(r,x)||\mathcal{F}_{\alpha}(f)(\mu,\lambda)| e^{d(r,x)(\theta(\mu,\lambda))} d\nu_{\alpha}(r,x) \, d\tilde{\gamma}_{\alpha}(\mu,\lambda) < +\infty.$$ 

Then

i) For $d \leq 2$, $f = 0$.

ii) For $d > 2$, there exist a positive constant $a$ and a polynomial $P$ on $\mathbb{R}^2$ even with respect to the first variable, such that

$$f(r,x) = P(r,x) e^{-a(r^2 + x^2)},$$

with degree($P$) $< \frac{d}{2} - 1$,

where

$$\Gamma_+ = [0, +\infty \times \mathbb{R} \cup \{(it,x) \mid (t,x) \in [0, +\infty \times \mathbb{R}, t \leq |x|\}.$$ 

$\theta$ is the function defined on the set $\Gamma_+$ by

$$\theta(\mu,\lambda) = (\sqrt{\mu^2 + \lambda^2}, \lambda).$$

d$\tilde{\gamma}_{\alpha}$ the measure defined on the set $\Gamma_+$ by

$$\int_{\Gamma_+} g(\mu,\lambda) \, d\tilde{\gamma}_{\alpha}(\mu,\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} \int_{[0, +\infty)} g(\mu,\lambda)(\mu^2 + \lambda^2)^{-\frac{1}{2}} \mu \, d\mu \, d\lambda$$

$$+ \int_{\mathbb{R}} \int_{[0, +\infty]} g(\mu,\lambda)(\lambda^2 - \mu^2)^{-\frac{1}{2}} \mu \, d\mu \, d\lambda.$$ 

The last section of this paper contains the following results that are respectively the Gelfand-Shilov and the Cowling-Price theorems for $\mathcal{F}_{\alpha}$

Let $p, q$ be two conjugate exponents, $p, q \in ]1, +\infty[$. Let $d, \xi, \eta$ be non negative real numbers such that $\xi \eta \geq 1$. Let $f$ be a measurable function on $\mathbb{R}^2$, even with respect to the first variable, such that $f \in L^2(d\nu_{\alpha})$. If

$$\int_{0}^{+\infty} \int_{\mathbb{R}} |f(r,x)| e^{\frac{p(\xi(r,x)) \mu}{\nu}} \frac{1}{(1 + (r,x))^{d}} \, d\nu_{\alpha}(r,x) < +\infty,$$

and

$$\int_{\Gamma_+} |\mathcal{F}_{\alpha}(f)(\mu,\lambda)| e^{\frac{q(\eta(\mu,\lambda)) \lambda}{\nu}} \frac{1}{(1 + \theta(\mu,\lambda))^{d}} \, d\tilde{\gamma}_{\alpha}(\mu,\lambda) < +\infty,$$

then

i) For $d \leq 1$, $f = 0$.

ii) For $d > 1$, we have
a) \( f = 0 \) for \( \xi \eta > 1 \).
b) \( f = 0 \) for \( \xi \eta = 1 \), and \( p \neq 2 \).
c) \( f(r, x) = P(r, x)e^{-a(r^2 + x^2)} \), for \( \xi \eta = 1 \), and \( p = q = 2 \).

where \( a > 0 \), and \( P \) is a polynomial on \( \mathbb{R}^2 \) even with respect to the first variable, with degree\( |P| < d - 1 \).

Let \( \xi, \eta, \omega_1, \omega_2 \) be non-negative real numbers such that \( \xi \eta \geq 1/4 \). Let \( p, q \) be two exponents, \( p, q \in [1, +\infty) \), and let \( f \) be a measurable function on \( \mathbb{R}^2 \), even with respect to the first variable such that \( f \in L^2(d\nu_\alpha) \). If

\[
\left\| \frac{e^{\xi \sqrt{(\cdot)^2}}}{(1 + |(\cdot,\cdot)|)^{d/2}} f \right\|_{p,\nu_\alpha} < +\infty,
\]

and

\[
\left\| \frac{e^{\eta \sqrt{(\cdot)^2}}}{(1 + |(\cdot,\cdot)|)^{d/2}} \mathcal{F}_\alpha(f) \right\|_{a,\gamma_\alpha} < +\infty,
\]

then

i) For \( \xi \eta > 1/4 \), \( f = 0 \).

ii) For \( \xi \eta = 1/4 \), there exist a positive constant \( a \) and a polynomial \( P \) on \( \mathbb{R}^2 \), even with respect to the first variable, such that

\[
f(r, x) = P(r, x)e^{-a(r^2 + x^2)}.
\]

2 The Fourier transform associated with the Riemann-Liouville operator

It’s well known \cite{2} that for all \( (\mu, \lambda) \in \mathbb{C}^2 \), the system

\[
\begin{align*}
\Delta_1 u(r, x) &= -i \mu u(r, x), \\
\Delta_2 u(r, x) &= -\mu^2 u(r, x), \\
\end{align*}
\]

admits a unique solution \( \varphi_{\mu, \lambda} \), given by

\[
\forall (r, x) \in \mathbb{R}^2; \quad \varphi_{\mu, \lambda}(r, x) = j_\alpha(r \sqrt{\mu^2 + \lambda^2})e^{-i\lambda x},
\]

where

\[
j_\alpha(z) = \frac{2^\alpha \Gamma(\alpha + 1)}{z^\alpha} J_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\alpha + n + 1)} \left( \frac{z^2}{2} \right)^{2n}, \quad z \in \mathbb{C}, \quad (2.1)
\]
and $J_\alpha$ is the Bessel function of the first kind and index $\alpha$ \cite{9, 10, 18, 25}.

The modified Bessel function $j_\alpha$ has the following integral representation \cite{18, 25}, for all $z \in \mathbb{C}$, we have

$$j_\alpha(z) = \begin{cases} \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}(\alpha + \frac{1}{2})} \int_0^1 (1 - t^2)^{\alpha - \frac{1}{2}} \cos(zt) dt, & \text{if } \alpha > -\frac{1}{2}; \\ \cos(z), & \text{if } \alpha = -\frac{1}{2}. \end{cases} \quad (2.2)$$

From the relation (2.2), we deduce that for all $z \in \mathbb{C}$, we have

$$|j_\alpha(z)| \leq e^{\left|\text{Im}(z)\right|}. \quad (2.3)$$

From the properties of the modified Bessel function $j_\alpha$, we deduce that the eigenfunction $\varphi_{\mu, \lambda}$ satisfies the following properties

$$\sup_{(r, x) \in \mathbb{R}^2} |\varphi_{\mu, \lambda}(r, x)| = 1, \quad (2.4)$$

if and only if $(\mu, \lambda)$ belongs to the set

$$\Gamma = \mathbb{R}^2 \cup \{(it, x) | (t, x) \in \mathbb{R}^2, |t| \leq |x|\}.$$ 

The eigenfunction $\varphi_{\mu, \lambda}$ has the following Mehler integral representation

$$\varphi_{\mu, \lambda}(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \cos(\mu rs \sqrt{1 - t^2}) e^{i\lambda(x + rt)} (1 - t^2)^{\frac{\alpha}{2}} (1 - s^2)^{\frac{\alpha - 1}{2}} dt ds; & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 \cos(\mu r \sqrt{1 - t^2}) e^{i\lambda(x + rt)} \frac{dt}{\sqrt{1 - t^2}}; & \text{if } \alpha = 0. \end{cases}$$

This integral representation allows to define the so-called Riemann-Liouville operator associated with $\Delta_1, \Delta_2$ by

$$\mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 f(rs \sqrt{1 - t^2}, x + rt)(1 - t^2)^{\frac{\alpha}{2}} (1 - s^2)^{\frac{\alpha - 1}{2}} dt ds; & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 f(r \sqrt{1 - t^2}, x + rt) \frac{dt}{\sqrt{1 - t^2}}; & \text{if } \alpha = 0. \end{cases}$$

where $f$ is a continuous function on $\mathbb{R}^2$, even with respect to the first variable.

The transform $\mathcal{R}_\alpha$ generalizes the ”mean operator” defined by

$$\mathcal{R}_0(f)(r, x) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \theta, x + r \cos \theta) d\theta.$$

In the following, we denote by
\(dm_{n+1}\) the measure defined on \([0, +\infty \times \mathbb{R}^n]\) by,
\[
dm_{n+1}(r, x) = \sqrt{\frac{2}{\pi}} \frac{1}{(2\pi)^{n/2}} dr \otimes dx.
\]

\(L^p(dm_{n+1})\) the space of measurable functions \(f\) on \([0, +\infty \times \mathbb{R}^n]\), such that
\[
\|f\|_{p, m_{n+1}} = \left( \int_0^{+\infty} \left( \int_{\mathbb{R}^n} |f(r, x)|^p dm_{n+1}(r, x) \right)^{\frac{1}{p}} dr \right)^{\frac{1}{p}} < +\infty, \quad \text{if } p \in [1, +\infty[, \\
\|f\|_{\infty, m_{n+1}} = \text{ess sup}_{(r, x) \in [0, +\infty \times \mathbb{R}^n]} |f(r, x)| < +\infty, \quad \text{if } p = +\infty.
\]

\(d\nu_\alpha\) the measure defined on \([0, +\infty \times \mathbb{R}\), by
\[
d\nu_\alpha(r, x) = r^{2\alpha + 1} 2^{\alpha+1} \Gamma(\alpha + 1)^{1/2} \sqrt{\pi} d\gamma_\alpha(r, x).
\]

\(L^p(d\nu_\alpha)\) the space of measurable functions \(f\) on \([0, +\infty \times \mathbb{R}\) such that \(\|f\|_{p, \nu_\alpha} < +\infty\).

\(\Gamma_+ = [0, +\infty \times \mathbb{R} \cup \{(it, x) \mid (t, x) \in [0, +\infty \times \mathbb{R}, t \leq |x|\}.
\]

\(B_{\Gamma_+}\) the \(\sigma\)-algebra defined on \(\Gamma_+\) by
\[
B_{\Gamma_+} = \{\theta^{-1}(B) \mid B \in B([0, +\infty \times \mathbb{R}]),\}
\]
where \(\theta\) is the bijective function defined on the set \(\Gamma_+\) by
\[
\theta(\mu, \lambda) = (\sqrt{\mu^2 + \lambda^2}, \lambda).
\]

\(d\gamma_\alpha\) the measure defined on \(B_{\Gamma_+}\) by
\[
\forall A \in B_{\Gamma_+}; \gamma_\alpha(A) = \nu_\alpha(\theta(A)).
\]

\(L^p(d\gamma_\alpha)\) the space of measurable functions \(f\) on \(\Gamma_+\), such that \(\|f\|_{p, \gamma_\alpha} < +\infty\).

\(d\tilde{\gamma}_\alpha\) the measure defined on \(B_{\Gamma_+}\) by
\[
d\tilde{\gamma}_\alpha(\mu, \lambda) = \frac{2^{\alpha+\frac{1}{2}} \Gamma(\alpha + 1)}{\sqrt{\pi} (\mu^2 + \lambda^2)^{\alpha+\frac{1}{2}}} d\gamma_\alpha(\mu, \lambda).
\]

\(S_*(\mathbb{R}^2)\) the Schwartz's space formed by the infinitely differentiable functions on \(\mathbb{R}^2\), rapidly decreasing together with all their derivatives, and even with respect to the first variable.

Then we have the following properties.
Proposition 2.1. i) For all non negative measurable function $g$ on $\Gamma_+$, we have
\[
\int \int_{\Gamma_+} g(\mu, \lambda) \, d\gamma_{\alpha}(\mu, \lambda) = \frac{1}{2^{\alpha+1} \sqrt{2\pi}} \left( \int_0^{\infty} g(\mu, \lambda) (\mu^2 + \lambda^2)^{\alpha} \, d\mu \, d\lambda + \int_0^{\frac{\lambda}{\mu}} g(i\mu, \lambda) (\lambda^2 - \mu^2)^{\alpha} \, d\mu \, d\lambda \right).
\]

ii) For all measurable function $f$ on $[0, +\infty \times \mathbb{R}$, the function $f \circ \theta$ is measurable on $\Gamma_+$. Furthermore if $f$ is non negative or integrable function on $[0, +\infty \times \mathbb{R}$ with respect to the measure $d\nu_\alpha$, then we have
\[
\int \int_{\Gamma_+} (f \circ \theta)(\mu, \lambda) \, d\gamma_{\alpha}(\mu, \lambda) = \int_0^{+\infty} \int_\mathbb{R} f(r, x) \, d\nu_\alpha(r, x).
\]

iii) For all non negative measurable function $f$, respectively integrable on $[0, +\infty \times \mathbb{R}$ with respect to the measure $dm_2$, we have
\[
\int \int_{\Gamma_+} (f \circ \theta)(\mu, \lambda) \, d\tilde{\gamma}_{\alpha}(\mu, \lambda) = \int_0^{+\infty} \int_\mathbb{R} f(r, x) \, dm_2(r, x).
\] (2.5)

In the following we shall define the Fourier transform $T_\alpha$ associated with the operator $R_\alpha$, and we shall give some properties that we use in the sequel.

Definition 2.1. The Fourier transform $T_\alpha$ associated with the Riemann-Liouville operator $R_\alpha$ is defined on $L^1(d\nu_\alpha)$ by
\[
\forall (\mu, \lambda) \in \Gamma; \quad T_\alpha(f)(\mu, \lambda) = \int_0^{+\infty} \int_\mathbb{R} f(r, x) \psi_{\mu, \lambda}(r, x) \, d\nu_\alpha(r, x).
\]

Then, for all $(\mu, \lambda) \in \Gamma$,
\[
T_\alpha(f)(\mu, \lambda) = \tilde{T}_\alpha(f) \circ \theta(\mu, \lambda),
\] (2.6)

where for all $(\mu, \lambda) \in [0, +\infty \times \mathbb{R}$,
\[
\tilde{T}_\alpha(f)(\mu, \lambda) = \int_0^{+\infty} \int_\mathbb{R} f(r, x) j_\alpha(r\mu) e^{-i\lambda x} \, d\nu_\alpha(r, x).
\] (2.7)

Moreover, the relation (2.4) implies that the Fourier transform $T_\alpha$ is a bounded linear operator from $L^1(d\nu_\alpha)$ into $L^\infty(d\gamma_\alpha)$, and that for all $f \in L^1(dm_2)$, we have
\[
\|T_\alpha(f)\|_{L^\infty, \gamma_\alpha} \leq \|f\|_{1, \nu_\alpha}.
\] (2.8)

Theorem 2.1 (Inversion formula). Let $f \in L^1(dm_2)$ such that $T_\alpha(f) \in L^1(dm_\alpha)$, then for almost every $(r, x) \in [0, +\infty \times \mathbb{R}$, we have
\[
f(r, x) = \int \int_{\Gamma_+} T_\alpha(f)(\mu, \lambda) \psi_{\mu, \lambda}(r, x) \, d\gamma_{\alpha}(\mu, \lambda)
= \int_0^{+\infty} \int_\mathbb{R} \tilde{T}_\alpha(f)(\mu, \lambda) j_\alpha(r\mu) e^{i\lambda x} \, d\nu_\alpha(\mu, \lambda).
\]
Lemma 2.2. Let $\mathcal{R}_\alpha$ be the mapping defined for all non negative measurable function $g$ on $[0, +\infty \times \mathbb{R}$ by
\[
\mathcal{R}_\alpha(g)(r, x) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_0^1 (1 - s^2)^{\alpha - \frac{1}{2}} g(rs, x) \, ds
\]
\[
= \frac{2\Gamma(\alpha + 1)r^{-2\alpha}}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_0^r (r^2 - s^2)^{\alpha - \frac{1}{2}} f(s, x) \, ds, \quad r > 0.
\]
Then for all non negative measurable functions $f, g$ on $[0, +\infty \times \mathbb{R}$, we have
\[
\int_0^{+\infty} \int_\mathbb{R} f(r, x) \mathcal{R}_\alpha(g)(r, x) \, d\nu_\alpha(r, x) = \int_0^{+\infty} \int_\mathbb{R} \mathcal{W}_\alpha(f)(r, x) g(r, x) \, d\mu_2(r, x),
\]
where $\mathcal{W}_\alpha$ is the classical Weyl transform defined for all non negative measurable function on $[0, +\infty \times \mathbb{R}$ by
\[
\mathcal{W}_\alpha(f)(r, x) = \frac{1}{2^{\alpha + \frac{1}{2}} \Gamma(\alpha + \frac{1}{2})} \int_0^{+\infty} (t^2 - r^2)^{\alpha - \frac{1}{2} f(t, x) 2t \, dt.
\]
Proposition 2.2. For all $f \in L^1(d\nu_\alpha)$, the function $\mathcal{W}_\alpha(f)$ belongs to $L^1(d\mu_2)$, and we have
\[
\|\mathcal{W}_\alpha(f)\|_{1, \mu_2} \leq \|f\|_{1, \nu_\alpha}.
\]
Moreover, for all $(\mu, \lambda) \in [0, +\infty \times \mathbb{R}$, we have
\[
\mathcal{F}_\alpha(f)(\mu, \lambda) = (\Lambda_2 \circ \mathcal{W}_\alpha)(f)(\mu, \lambda),
\]
where $\Lambda_2$ is the usual Fourier transform defined on $L^1(d\mu_2)$ by
\[
\Lambda_2(g)(\mu, \lambda) = \int_0^{+\infty} \int_\mathbb{R} g(r, x) \cos(r\mu) e^{-i\lambda x} \, d\mu_2(r, x).
\]
Remark 2.1. It’s well known [23, 24] that the transforms $\mathcal{F}_\alpha$ and $\Lambda_2$ are topological isomorphisms from $S_\alpha(\mathbb{R}^2)$ onto itself. Then by the relation (2.13), we deduce that the classical Weyl transform $\mathcal{W}_\alpha$ is also a topological isomorphism from $S_\alpha(\mathbb{R}^2)$ onto itself.

Proposition 2.3. For all $f \in S_\alpha(\mathbb{R}^2)$, we have
\[
\mathcal{W}_\alpha^{-1}(f) = (-1)^{1+[\alpha+\frac{1}{2}]} \mathcal{W}_{[\alpha+\frac{1}{2}]}^{-1+\alpha+\frac{1}{2}} \left( \left( \frac{\partial}{\partial t^2} \right)^{1+[\alpha+\frac{1}{2}]} f(t, x) \right),
\]
where
\[
\left( \frac{\partial}{\partial t^2} \right)(f)(t, x) = \frac{1}{t} \frac{df}{dt}(t, x).
\]
Proof. For $\sigma \in \mathbb{R}, \sigma > 0$, let us define the so-called fractional transform $\mathcal{H}_\sigma$, defined on $S_\alpha(\mathbb{R}^2)$ by
\[
\mathcal{H}_\sigma(f)(r, x) = \frac{1}{2^{\alpha + 1} \sigma} \int_r^{+\infty} (t^2 - r^2)^{\sigma - \frac{1}{2} f(t, x) 2t \, dt = \mathcal{W}_{[\sigma+\frac{1}{2}]}(f)(r, x).
From the remark [2.1] it follows that for all real number $\sigma > 0$, the mapping $\mathcal{H}_\sigma$ is a topological isomorphism from $S_+(\mathbb{R}^2)$ onto itself.

Moreover, we have the following properties

For all $\sigma, \delta \in \mathbb{R}; \quad \sigma, \delta > 0$ and for every $f \in S_+(\mathbb{R}^2)$, we have

$$(\mathcal{H}_\sigma \circ \mathcal{H}_\delta)(f) = \mathcal{H}_{\sigma + \delta}(f).$$

For all $\sigma \in \mathbb{R}, \quad \sigma > 0$, and for every integer $k$, we have

$$\mathcal{H}_\sigma(f) = (-1)^k \mathcal{H}_{\sigma + k} \left( \left( \frac{\partial}{\partial t^2} \right)^k (f) \right). \tag{2.15}$$

where $\frac{\partial}{\partial t^2}$ is the linear continuous operator defined on $S_+(\mathbb{R}^2)$ by

$$\frac{\partial}{\partial t^2}(f)(t, x) = \frac{1}{t} \frac{\partial f}{\partial t}(t, x).$$

The relation (2.15) allows us to extend the mapping $\mathcal{H}_\sigma$ on $\mathbb{R}$, by setting

$$\mathcal{H}_\sigma(f)(r, x) = (-1)^k \mathcal{H}_{\sigma + k} \left( \left( \frac{\partial}{\partial t^2} \right)^k (f) \right),$$

where $k$ is any integer such that $\sigma + k > 0, \quad \sigma \in \mathbb{R}$.

The extension $\mathcal{H}_\sigma, \quad \sigma \in \mathbb{R}$ satisfies

$$(\mathcal{H}_\sigma \circ \mathcal{H}_\delta)(f) = \mathcal{H}_{\sigma + \delta}(f), \quad \sigma, \delta \in \mathbb{R}, \quad f \in S_+(\mathbb{R}^2),$$

and $\mathcal{H}_0(f) = f$, for all $f \in S_+(\mathbb{R}^2)$.

In particular, for all $\sigma \in \mathbb{R}$, the transform $\mathcal{H}_\sigma$ is a topological isomorphism from $S_+(\mathbb{R}^2)$ onto itself, and the isomorphism inverse is given by

$$\mathcal{H}_\sigma^{-1} = \mathcal{H}_{-\sigma}.$$

Thus, for all real number $\sigma$, we have

$$\mathcal{H}_\sigma^{-1}(f) = (-1)^{1 + |\sigma|} \mathcal{H}_{1 + |\sigma| - \sigma} \left( \left( \frac{\partial}{\partial t^2} \right)^{1 + |\sigma|} (f) \right).$$

In particular

$$\mathcal{H}_\sigma^{-1}(f) = \mathcal{H}_{\sigma + \frac{1}{2}}^{-1}(f) = (-1)^{1 + |\sigma + \frac{1}{2}|} \mathcal{H}_{1 + |\sigma + \frac{1}{2}| - \sigma + \frac{1}{2}} \left( \left( \frac{\partial}{\partial t^2} \right)^{1 + |\sigma + \frac{1}{2}|} (f) \right).$$

$\square$

3 The Beurling-Hörmander theorem for the Riemann-Liouville operator

In this section, we shall establish the main result of this paper, that is the Beurling-Hörmander theorem for the Fourier transform $\mathcal{F}_\alpha$.

We recall firstly the following result that has been established by Bonami, Demange and Jaming [5].
Theorem 3.1. Let \( f \) be a measurable function on \( \mathbb{R} \times \mathbb{R}^n \), even with respect to the first variable such that \( f \in L^2(dm_{n+1}) \), and let \( d \) be a real number, \( d \geq 0 \). If
\[
\int_0^{+\infty} \int_{\mathbb{R}^n} \int_0^{+\infty} \int_{\mathbb{R}^n} |f(r, x)||a_{n+1}(f)(s, y)| e^{\alpha(r, x)\theta(s, y)} dm_{n+1}(r, x) dm_{n+1}(s, y) < +\infty,
\]
then there exist a positive constant \( a \) and a polynomial \( P \) on \( \mathbb{R} \times \mathbb{R}^n \), even with respect to the first variable, such that
\[
f(r, x) = P(r, x)e^{-a(r^2+|x|^2)},
\]
with degree\( (P) < \frac{d - (n + 1)}{2} \).

In the following, we will establish some intermediary results that we use nextly.

Lemma 3.2. Let \( f \in L^2(d\nu_\alpha) \) such that
\[
\int_0^{+\infty} \int_{\mathbb{R}^n} \int_0^{+\infty} \int_{\mathbb{R}^n} |f(r, x)||\mathcal{F}_\alpha(f)(\mu, \lambda)| \frac{e^{\alpha(r, x)|\theta(\mu, \lambda)|}}{1 + ||r, x|| + |\Theta(\mu, \lambda)|} d\nu_\alpha(r, x) d\bar{\nu}_\alpha(\mu, \lambda) < +\infty,
\]
then the function \( f \) belongs to the space \( L^1(d\nu_\alpha) \).

Proof. From the hypothesis, and the relations (2.5) and (2.6), we have
\[
\int_0^{+\infty} \int_{\mathbb{R}^n} \int_0^{+\infty} \int_{\mathbb{R}^n} |f(r, x)||\mathcal{F}_\alpha(f)(\mu, \lambda)| \frac{e^{\alpha(r, x)|\theta(\mu, \lambda)|}}{1 + ||r, x|| + |\Theta(\mu, \lambda)|} d\nu_\alpha(r, x) d\bar{\nu}_\alpha(\mu, \lambda)
= \int_0^{+\infty} \int_{\mathbb{R}^n} \int_0^{+\infty} \int_{\mathbb{R}^n} |f(r, x)||\mathcal{F}_\alpha(f)(\mu, \lambda)| \frac{e^{\alpha(r, x)|\theta(\mu, \lambda)|}}{1 + ||r, x|| + |\Theta(\mu, \lambda)|} d\nu_\alpha(r, x) dm_2(\mu, \lambda) < +\infty.
\]
We assume of course that \( f \neq 0 \). Then, there exists \( (\mu_0, \lambda_0) \in [0, +\infty[ \times \mathbb{R} \), such that \( (\mu_0, \lambda_0) \neq (0, 0) \), \( \mathcal{F}_\alpha(f)(\mu_0, \lambda_0) \neq 0 \), and
\[
|\mathcal{F}_\alpha(f)(\mu_0, \lambda_0)| \int_0^{+\infty} \int_{\mathbb{R}^n} |f(r, x)| \frac{e^{\alpha(r, x)|\mu_0, \lambda_0|}}{1 + ||r, x|| + |\mu_0, \lambda_0|} d\nu_\alpha(r, x) < +\infty,
\]
hence
\[
\int_0^{+\infty} \int_{\mathbb{R}^n} |f(r, x)| \frac{e^{\alpha(r, x)|\mu_0, \lambda_0|}}{1 + ||r, x|| + |\mu_0, \lambda_0|} d\nu_\alpha(r, x) < +\infty.
\]
Let \( h \) be the function defined on \( [0, +\infty[ \) by
\[
h(s) = \frac{e^{\alpha|\mu_0, \lambda_0|}}{1 + s + |\mu_0, \lambda_0|},
\]
then the function \( h \) admits a minimum attained at
\[
s_0 = \begin{cases} \frac{d}{||\mu_0, \lambda_0||} - 1 - |\mu_0, \lambda_0|, & \text{if } \frac{d}{||\mu_0, \lambda_0||} > 1 + ||\mu_0, \lambda_0||; \\ 0, & \text{if } \frac{d}{||\mu_0, \lambda_0||} \leq 1 + ||\mu_0, \lambda_0||. \end{cases}
\]
Consequently,
\[
\int_0^{+\infty} \int_{\mathbb{R}} |f(r,x)| \, d\nu_{\alpha}(r,x) \leq \frac{1}{h(s_0)} \int_0^{+\infty} \int_{\mathbb{R}} |f(r,x)||e^{[(r,x)||\mu_0, \lambda_0]}| \, d\nu_{\alpha}(r,x) < +\infty.
\]

Lemma 3.3. Let \( f \in L^2(d\nu_{\alpha}) \) such that
\[
\int \int_{\mathbb{R}^2} \frac{|f(r,x)||\mathcal{F}_{\alpha}(f)(\mu, \lambda)|}{(1 + |r| + |\mu_0, \lambda_0|)^{\alpha}} \, d\nu_{\alpha}(r,x) \, d\nu_{\alpha}(\mu, \lambda) < +\infty.
\]
Then, there exists \( a > 0 \) such that the function \( \mathcal{F}_{\alpha}(f) \) is analytic on the set
\[
B_a = \{ (\mu, \lambda) \in \mathbb{C}^2 \mid |\text{Im}(\mu)| < a, \ |\text{Im}(\lambda)| < a \}.
\]

Proof. From the proof of the lemma 3.2, there exists \( (\mu_0, \lambda_0) \neq (0, 0) \), such that
\[
\int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r,x)|e^{[(r,x)||\mu_0, \lambda_0]}|}{(1 + |r| + |\mu_0, \lambda_0|)^{\alpha}} \, d\nu_{\alpha}(r,x) < +\infty.
\]

Let \( a > 0 \), such that \( 0 < 2a < |(\mu_0, \lambda_0)| \). Then we have
\[
\int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r,x)|e^{[(r,x)||\mu_0, \lambda_0]}|}{(1 + |r| + |\mu_0, \lambda_0|)^{\alpha}} \, d\nu_{\alpha}(r,x)
= \int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r,x)|e^{2a[(r,x)]}}{(1 + |r| + |\mu_0, \lambda_0|)^{\alpha}} \, d\nu_{\alpha}(r,x) < +\infty.
\]

Let \( g \) be the function defined on \( [0, +\infty[ \) by
\[
g(s) = \frac{e^{s(\mu_0, \lambda_0)^{-2a}}}{(1 + s + |(\mu_0, \lambda_0)|)^{d}},
\]
then \( g \) admits a minimum attained at
\[
s_0 = \begin{cases} 
\frac{d}{|\mu_0, \lambda_0| - 2a} - 1 - |\mu_0, \lambda_0|, & \text{if } \frac{d}{|\mu_0, \lambda_0| - 2a} > 1 + |\mu_0, \lambda_0|; \\
0, & \text{if } \frac{d}{|\mu_0, \lambda_0| - 2a} \leq 1 + |\mu_0, \lambda_0|. 
\end{cases}
\]

Consequently,
\[
\int_0^{+\infty} \int_{\mathbb{R}} |f(r,x)|e^{2a[(r,x)]} \, d\nu_{\alpha}(r,x)
\leq \frac{1}{g(s_0)} \int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r,x)||e^{[(r,x)||\mu_0, \lambda_0]}|}{(1 + |r| + |\mu_0, \lambda_0|)^{\alpha}} \, d\nu_{\alpha}(r,x) < +\infty.
\]
On the other hand, from the relation (2.1) we deduce that for all \((r, x) \in [0, +\infty] \times \mathbb{R}\), the function
\[
(\mu, \lambda) \mapsto j_\alpha(r \mu)e^{-ix\lambda}
\]
is analytic on \(\mathbb{C}^2\) [7], even with respect to the first variable, and by the relation (2.3) we have
\[
|j_\alpha(r \mu)e^{-ix\lambda}| \leq e^{|r \mu|(|\text{Im}\mu|+|\text{Im}\lambda|)}.
\] (3.3)
From the relations (2.7), (3.2), and (3.3), it follows that the function \(\widetilde{F}_\alpha(f)\) is analytic on \(B_\alpha\), even with respect to the first variable. \(\Box\)

**Corollary 3.1.** Let \(f \in L^2(d\nu_\alpha)\); \(f \neq 0\); and let \(d\) be a real number, \(d \geq 0\). If
\[
\int_{\mathbb{R}^+} \int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r, x)||\mathcal{F}_\alpha(f)(\mu, \lambda)|}{1 + ||(r, x)| + ||(\mu, \lambda)||} e^{-\frac{1}{2}(r \mu)(|\text{Im}\mu| + |\text{Im}\lambda|)} d\nu_\alpha(r, x) d\gamma_\alpha(\mu, \lambda) < +\infty,
\]
then for all real number \(\alpha > 0\), we have
\[
\nu_\alpha\left( \left\{ (r, x) \in \mathbb{R}^2 \mid \widetilde{F}_\alpha(f)(r, x) \neq 0 \text{ and } ||(r, x)|| > a \right\} \right) > 0.
\]

**Proof.** From lemma 3.2, the function \(f\) belongs to \(L^1(d\nu_\alpha)\), and consequently the function \(\widetilde{F}_\alpha(f)\) is continuous on \(\mathbb{R}^2\), even with respect to the first variable.

Then for all \(\alpha > 0\), the set
\[
\left\{ (r, x) \in \mathbb{R}^2 \mid \widetilde{F}_\alpha(f)(r, x) \neq 0 \text{ and } ||(r, x)|| > a \right\},
\]
is on open subset of \(\mathbb{R}^2\).

Assume that
\[
\nu_\alpha\left( \left\{ (r, x) \in \mathbb{R}^2 \mid \widetilde{F}_\alpha(f)(r, x) \neq 0 \text{ and } ||(r, x)|| > a \right\} \right) = 0,
\]
then for all \((r, x) \in \mathbb{R}^2; ||(r, x)|| > a\), we have \(\widetilde{F}_\alpha(f)(r, x) = 0\).

Applying lemma 3.3 and analytic continuation, we deduce that \(\widetilde{F}_\alpha(f)\) vanishes on \(\mathbb{R}^2\), and by theorem 2.1 it follows that \(f = 0\). \(\Box\)

**Lemma 3.4.** Let \(f \in L^2(d\nu_\alpha)\) and let \(d\) be a real number \(d \geq 0\). If
\[
\int_{\mathbb{R}^+} \int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r, x)||\mathcal{F}_\alpha(f)(\mu, \lambda)|}{1 + ||(r, x)| + ||(\mu, \lambda)||} e^{-\frac{1}{2}(r \mu)(|\text{Im}\mu| + |\text{Im}\lambda|)} d\nu_\alpha(r, x) d\gamma_\alpha(\mu, \lambda) < +\infty,
\]
then the function \(\mathcal{W}_\alpha(f)\), belongs to \(L^2(d\mu_2)\), where \(\mathcal{W}_\alpha\) is the mapping defined by the relation (2.11).

**Proof.** From the hypothesis and the relations (2.5) and (2.6), we have
\[
\int_{\mathbb{R}^+} \int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r, x)||\mathcal{F}_\alpha(f)(\mu, \lambda)|}{1 + ||(r, x)| + ||(\mu, \lambda)||} e^{-\frac{1}{2}(r \mu)(|\text{Im}\mu| + |\text{Im}\lambda|)} d\nu_\alpha(r, x) d\gamma_\alpha(\mu, \lambda)
= \int_0^{+\infty} \int_{\mathbb{R}^+} \int_0^{+\infty} \frac{|f(r, x)||\mathcal{F}_\alpha(f)(\mu, \lambda)|}{1 + ||(r, x)| + ||(\mu, \lambda)||} e^{-\frac{1}{2}(r \mu)(|\text{Im}\mu| + |\text{Im}\lambda|)} d\nu_\alpha(r, x) d\mu_2(\mu, \lambda) < +\infty.
\]
By the same way as inequality (3.2) of the lemma 3.3 there exists \( b \in \mathbb{R}, b > 0 \), such that
\[
\int_0^{+\infty} \int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(\mu, \lambda)| e^{b(|\mu, \lambda|)} dm_2(\mu, \lambda) < +\infty. \tag{3.4}
\]
Consequently, the function \( \mathcal{F}_\alpha(f) \) lies in \( L^1(d\nu_\alpha) \) and by theorem 2.1, we get
\[
f(r, x) = \int_0^{+\infty} \int_{\mathbb{R}} \mathcal{F}_\alpha(f)(\mu, \lambda) e^{i\lambda x} d\nu_\alpha(\mu, \lambda); \quad \text{a.e.}
\]
In particular the function \( f \) is bounded and
\[
\|f\|_{\infty, \nu_\alpha} \leq \|\mathcal{F}_\alpha(f)\|_{1, \nu_\alpha}. \tag{3.5}
\]
Now, we have
\[
|\mathcal{W}_\alpha(f)(r, x)| \leq \frac{1}{2^{2s+\frac{1}{2}} \Gamma(\alpha + 1)} \int_r^{+\infty} \left( \int_0^{+\infty} r^{2\alpha+1}(u^2 - 1)^{\alpha-\frac{1}{2}} |f(ru, x)|^2 du \right)^{\frac{1}{2}} dm_2(r, x) \tag{3.3}
\]
\[
\leq \frac{1}{2^{2s+\frac{1}{2}} \Gamma(\alpha + 1)} \int_{1}^{+\infty} \left( \int_0^{+\infty} r^{2\alpha+1}(u^2 - 1)^{\alpha-\frac{1}{2}} |f(ru, x)|^2 du \right)^{\frac{1}{2}} dm_2(r, x) \tag{3.4}
\]
Using Minkowski's inequality for integrals [11], we get
\[
\left( \int_0^{+\infty} \int_{\mathbb{R}} |\mathcal{W}_\alpha(f)(r, x)|^2 dm_2(r, x) \right)^{\frac{1}{2}} \leq M_\alpha \|f\|_{\infty, \nu_\alpha} \left( \int_0^{+\infty} \int_{\mathbb{R}} |f(r, x)| e^{2\alpha|t, x|} d\nu_\alpha(t, x) \right)^{\frac{1}{2}} < +\infty.
\]

Remark 3.1. Let \( f \) be a function satisfying the hypothesis (3.1), then from the relations (3.3) and (3.4), we can prove that the function \( f \) belongs to the Schwartz's space \( S_\alpha(\mathbb{R}^2) \). Since the Weyl transform \( \mathcal{W}_\alpha \) is an isomorphism from \( S_\alpha(\mathbb{R}^2) \) onto itself, then the function \( \mathcal{W}_\alpha(f) \) belongs to \( S_\alpha(\mathbb{R}^2) \), in particular \( \mathcal{W}_\alpha(f) \in L^2(dm_2). \)
Remark 3.2. Let \( \sigma \) be a positive real number such that 
\( \sigma + \sigma^2 > d \geq 0 \). Then, the function
\[
|t| \mapsto \frac{e^{\sigma t}}{|1 + t + \sigma|^d},
\]
is increasing on \([0, +\infty)\).

Theorem 3.5. Let \( f \in L^2(d\nu_\alpha) \), and let \( d \) be a real number, \( d \geq 0 \). If
\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |f(|r,x|)||F_\alpha(f)(\mu,\lambda)|^2 e|\mu,\lambda| \text{d} \nu_\alpha(r,x) \text{d} \gamma_\alpha(\mu,\lambda) < +\infty,
\]
then
\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |\mathcal{W}_\alpha(f)(r,x)||\mathcal{F}_\alpha(f)(\mu,\lambda)|^2 e|\mu,\lambda| \text{d} m_2(r,x) \text{d} m_2(\mu,\lambda) < +\infty.
\]

Proof. From the hypothesis and the relations (2.5) and (2.6), we have
\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |f(|r,x|)||\mathcal{F}_\alpha(f)(\mu,\lambda)|^2 e|\mu,\lambda| \text{d} \nu_\alpha(r,x) \text{d} m_2(\mu,\lambda) < +\infty. \tag{3.6}
\]

(i) If \( d = 0 \), then by Fubini’s theorem we have
\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |\mathcal{W}_\alpha(f)(r,x)||\mathcal{F}_\alpha(f)(\mu,\lambda)|^2 e|\mu,\lambda| \text{d} m_2(r,x) \text{d} m_2(\mu,\lambda)
\leq \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |\mathcal{F}_\alpha(f)(\mu,\lambda)|^2 \left( \int_0^\infty \int_0^\infty |\mathcal{W}_\alpha(f)(r,x)|^2 \text{d} m_2(r,x) \right) \text{d} m_2(\mu,\lambda)
\leq \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |\mathcal{F}_\alpha(f)(\mu,\lambda)|^2 \left( \int_0^\infty \int_0^\infty |\mathcal{W}_\alpha(f)(r,x)|^2 \text{d} m_2(r,x) \right) \text{d} m_2(\mu,\lambda). \tag{3.7}
\]

Using the relation (2.10), we deduce that
\[
\int_0^\infty \int_0^\infty |\mathcal{W}_\alpha(f)(r,x)|^2 \text{d} m_2(r,x) = \int_0^\infty \int_0^\infty |f(r,x)|^2 \mathcal{R}_\alpha(e^{\sigma t}) \text{d} \nu_\alpha(r,x) \text{d} \gamma_\alpha(\mu,\lambda), \tag{3.8}
\]
but for all \( (r,x) \in [0, +\infty) \times \mathbb{R} \)
\[
\mathcal{R}_\alpha(e^{\sigma t}) \leq e^{\sigma t} \tag{3.9}
\]
Combining the relations (3.6), (3.7), (3.8), and (3.9), we get
\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |\mathcal{W}_\alpha(f)(r,x)||\mathcal{F}_\alpha(f)(\mu,\lambda)|^2 e|\mu,\lambda| \text{d} m_2(r,x) \text{d} m_2(\mu,\lambda)
\leq \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |f(r,x)|^2 \text{d} \nu_\alpha(r,x) \text{d} \gamma_\alpha(\mu,\lambda)
< +\infty.
\]
ii) If $d > 0$, let

$$B_d = \{ (u, v) \in [0, +\infty \times \mathbb{R} \mid ||u, v|| \leq d \}.$$ 

By Fubini’s theorem, we have

$$\int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)||f(r, x)| e^{(r, x)||\mu, \lambda||} \, dm_2(r, x) \, dm_2(\mu, \lambda) \leq \int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)| \left( \int_{\mathbb{R}} |f(r, x)| \, \mathcal{R}_{\alpha} \left( \frac{e^{(r, x)||\mu, \lambda||}}{1 + ||r, x|| + ||\mu, \lambda||} \right) \right) \, dm_2(\mu, \lambda),$$

and by the relation (2.10), we get

$$\int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)||f(r, x)| e^{(r, x)||\mu, \lambda||} \, dm_2(r, x) \, dm_2(\mu, \lambda) \leq \int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)| \left( \int_{\mathbb{R}} |f(r, x)| \mathcal{R}_{\alpha} \left( \frac{e^{(r, x)||\mu, \lambda||}}{1 + ||r, x|| + ||\mu, \lambda||} \right) \right) \, dm_2(\mu, \lambda) \, \text{d} \nu_{\alpha}(r, x).$$

However, by the relation (2.9) and remark 3.2 we have for all $(\mu, \lambda) \in B_d^\circ$

$$\mathcal{R}_{\alpha} \left( \frac{e^{(r, x)||\mu, \lambda||}}{1 + ||r, x|| + ||\mu, \lambda||} \right)(r, x) \leq e^{(r, x)||\mu, \lambda||} \left( \frac{1}{1 + ||r, x|| + ||\mu, \lambda||} \right).$$

Combining the relations (3.10) and (3.11), we obtain

$$\int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)||f(r, x)| e^{(r, x)||\mu, \lambda||} \, dm_2(r, x) \, dm_2(\mu, \lambda) \leq \int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)| \left( \int_{\mathbb{R}} |f(r, x)| \mathcal{R}_{\alpha} \left( \frac{e^{(r, x)||\mu, \lambda||}}{1 + ||r, x|| + ||\mu, \lambda||} \right) \right) \, dm_2(\mu, \lambda) \, \text{d} \nu_{\alpha}(r, x) \, \text{d} \nu_{\alpha}(r, x),$$

and

$$\int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)||f(r, x)| e^{(r, x)||\mu, \lambda||} \, dm_2(r, x) \, dm_2(\mu, \lambda) \leq \int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)| \left( \int_{\mathbb{R}} |f(r, x)| \mathcal{R}_{\alpha} \left( \frac{e^{(r, x)||\mu, \lambda||}}{1 + ||r, x|| + ||\mu, \lambda||} \right) \right) \, dm_2(\mu, \lambda) \, \text{d} \nu_{\alpha}(r, x) \, \text{d} \nu_{\alpha}(r, x).$$

Therefore, the following inequalities hold:

$$\int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)||f(r, x)| e^{(r, x)||\mu, \lambda||} \, dm_2(r, x) \, dm_2(\mu, \lambda) \leq \int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)| \left( \int_{\mathbb{R}} |f(r, x)| \mathcal{R}_{\alpha} \left( \frac{e^{(r, x)||\mu, \lambda||}}{1 + ||r, x|| + ||\mu, \lambda||} \right) \right) \, dm_2(\mu, \lambda) \, \text{d} \nu_{\alpha}(r, x) \, \text{d} \nu_{\alpha}(r, x).$$

and

$$\int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)||f(r, x)| e^{(r, x)||\mu, \lambda||} \, dm_2(r, x) \, dm_2(\mu, \lambda) \leq \int_{B_d} \int_{\mathbb{R}} |\tilde{\mathcal{F}}_{\alpha}(f)(\mu, \lambda)| \left( \int_{\mathbb{R}} |f(r, x)| \mathcal{R}_{\alpha} \left( \frac{e^{(r, x)||\mu, \lambda||}}{1 + ||r, x|| + ||\mu, \lambda||} \right) \right) \, dm_2(\mu, \lambda) \, \text{d} \nu_{\alpha}(r, x) \, \text{d} \nu_{\alpha}(r, x).$$
The relations (3.12), (3.13), and (3.14) imply that Hence,

\begin{align*}
\mathcal{B}_d \int_{\mathcal{B}_d} \mathcal{W}_\alpha(f)(r, x) \frac{e^{(r,x)||[\mu, \lambda]|}}{1 + (|r,x| + |[\mu, \lambda]|)^d} \, dm_2(r, x)
= \int_0^{+\infty} \int_{\mathbb{R}} |f(r, x)| \mathcal{R}_\alpha \left( \frac{e^{(-r)||[\mu, \lambda]|}}{(1 + (|r,x| + |[\mu, \lambda]|)^d} \mathbf{1}_{\mathcal{B}_d}(r, x) \right) \, d\nu_\alpha(r, x)
\leq \int_{\mathcal{B}_d} |f(r, x)| \frac{e^{d||r,x||}}{(1 + (|r,x| + d)^d} \, d\nu_\alpha(r, x).
\end{align*}

In virtue of the relation (2.8), we have

\begin{align*}
\mathcal{B}_d \int_{\mathcal{B}_d} \mathcal{W}_\alpha(f)(r, x) \frac{|\mathcal{F}_\alpha(f)(\mu, \lambda)|}{(1 + (|r,x|)^d} \, dm_2(r, x) \, dm_2(\mu, \lambda)
\leq \left( \int_{\mathcal{B}_d} |\mathcal{F}_\alpha(f)(\mu, \lambda)| \, dm_2(\mu, \lambda) \right) \left( \int_{\mathcal{B}_d} |f(r, x)| \frac{e^{d||r,x||}}{(1 + (|r,x| + d)^d} \, d\nu_\alpha(r, x) \right). \quad (3.12)
\end{align*}

On the other hand, from corollary (3.1) and the relation (3.6), there exists \((\mu_0, \lambda_0) \in [0, +\infty \times \mathbb{R}, ||[\mu_0, \lambda_0]| > d, \mathcal{F}_\alpha(f)(\mu_0, \lambda_0) \neq 0, and

\begin{align*}
\mathcal{B}_d \int_{\mathcal{B}_d} |f(r, x)| \frac{e^{(\mu_0, \lambda_0)||r,x||}}{(1 + (|r,x| + ||[\mu_0, \lambda_0]|)^d} \, d\nu_\alpha(r, x) < +\infty, \quad (3.13)
\end{align*}

so, by remark 3.2

\begin{align*}
\mathcal{B}_d \int_{\mathcal{B}_d} |f(r, x)| \frac{e^{d||r,x||}}{(1 + (|r,x| + d)^d} \, d\nu_\alpha(r, x)
\leq \int_{\mathcal{B}_d} |f(r, x)| \frac{e^{(\mu_0, \lambda_0)||r,x||}}{(1 + (|r,x| + ||[\mu_0, \lambda_0]|)^d} \, d\nu_\alpha(r, x)
< +\infty. \quad (3.14)
\end{align*}

The relations (3.12), (3.13), and (3.14) imply that

\begin{align*}
\mathcal{B}_d \int_{\mathcal{B}_d} \mathcal{W}_\alpha(f)(r, x) \frac{|\mathcal{F}_\alpha(f)(\mu, \lambda)|}{(1 + (|r,x| + |[\mu, \lambda]|)^d} \, e^{(r,x)||[\mu, \lambda]|} \, dm_2(r, x) \, dm_2(\mu, \lambda) < +\infty.
\end{align*}
Finally
\[
\int_{B_d} \int_{B_d} |\mathcal{W}_\alpha(f)(r,x)||\tilde{\mathcal{F}}_\alpha(f)(\mu,\lambda)| \frac{e^{r|x|}||\mu,\lambda||}{(1 + ||r,x|| + ||\mu,\lambda||)^d} \, dm_2(r,x) \, dm_2(\mu,\lambda) 
\leq e^{d^2} \left( \int_{B_d} |\tilde{\mathcal{F}}_\alpha(f)(\mu,\lambda)| \, dm_2(\mu,\lambda) \right) \left( \int_{B_d} |\mathcal{W}_\alpha(f)(r,x)| \, dm_2(r,x) \right) 
\leq e^{d^2} m_2(B_d) \|\tilde{\mathcal{F}}_\alpha(f)\|_{\infty,\gamma_\alpha} \|\mathcal{W}_\alpha(f)\|_{1,m_2},
\]
and therefore by the relations \(2.28\) and \(2.42\), we deduce that
\[
\int_{B_d} \int_{B_d} |\mathcal{W}_\alpha(f)(r,x)||\tilde{\mathcal{F}}_\alpha(f)(\mu,\lambda)| \frac{e^{r|x|}||\mu,\lambda||}{(1 + ||r,x|| + ||\mu,\lambda||)^d} \, dm_2(r,x) \, dm_2(\mu,\lambda) 
\leq e^{d^2} m_2(B_d) \|f\|_{1,\gamma_\alpha}^2 
< +\infty,
\]
and the proof of theorem 3.5 is complete.

\[\square\]

**Theorem 3.6** (Beurling-Hörmander for \(\mathcal{A}_2\)). Let \(f \in L^2(\mathcal{d}v_\alpha)\), and let \(d\) be a real number, \(d \geq 0\). If
\[
\int_{\mathbb{R}^n} \int_{0}^{+\infty} \int_{\mathbb{R}} |f(r,x)||\tilde{\mathcal{F}}_\alpha(f)(\mu,\lambda)| \frac{e^{r|x|}||\mu,\lambda||}{(1 + ||r,x|| + ||\mu,\lambda||)^d} \, d\mathcal{v}_\alpha(r,x) \, d\tilde{\mathcal{F}}_\alpha(\mu,\lambda) < +\infty.
\]
Then
i) For \(d \leq 2\), \(f = 0\).
ii) For \(d > 2\), there exist a positive constant \(\alpha\) and a polynomial \(P\), even with respect to the first variable, such that
\[
f(r,x) = P(r,x)e^{-\alpha(r^2 + x^2)},
\]
with degree \(P < \frac{d}{2} - 1\).

**Proof.** Let \(f \in L^2(\mathcal{d}v_\alpha)\), satisfying the hypothesis. From proposition 2.7, lemma 3.2 and lemma 3.4, we deduce that the function \(\mathcal{W}_\alpha(f)\) belongs to the space \(L^1(\mathcal{d}m_2) \cap L^2(\mathcal{d}m_2)\) and that
\[
\tilde{\mathcal{F}}_\alpha(f) = \Lambda_2 \circ \mathcal{W}_\alpha(f).
\]
Thus from theorem 3.5 we get
\[
\int_{0}^{+\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}} |\mathcal{W}_\alpha(f)(r,x)||\Lambda_2(\mathcal{W}_\alpha(f))(\mu,\lambda)| \frac{e^{r|x|}||\mu,\lambda||}{(1 + ||r,x|| + ||\mu,\lambda||)^d} \, dm_2(r,x) \, dm_2(\mu,\lambda) < +\infty.
\]
Applying theorem 3.1, when \(f\) is replaced by \(\mathcal{W}_\alpha(f)\), we deduce that

If \(d \leq 2\), \(\mathcal{W}_\alpha(f) = 0\), and by remark 2.1, \(f = 0\).
If \( d > 2 \), then there exist \( a > 0 \) and a polynomial \( Q \) even with respect to the first variable such that
\[
H_\alpha(f)(r, x) = Q(r, x)e^{-a(r^2 + x^2)} = \sum_{2p+q \leq m} a_{p, q} r^{2p} x^q e^{-a(r^2 + x^2)}.
\]

In particular, the function \( H_\alpha(f) \) belongs to the space \( S_a(\mathbb{R}^2) \). From remark \( 2.1 \) the function \( f \) belongs to \( S_a(\mathbb{R}^2) \) and from the relation \( 2.14 \), we get
\[
f(r, x) = H_{-d - \frac{1}{2}} (Q(t, y)e^{-a(t^2 + y^2)}) (r, x)
\]
\[
= (-1)^{[\alpha + \frac{1}{2}]} H_{\alpha + \frac{1}{2} - \alpha - \frac{1}{2}} \left( \left( \frac{\partial}{\partial t^2} \right)^{[\alpha + \frac{1}{2}]} \right) \left( P(t, y)e^{-a(t^2 + y^2)} \right) (r, x)
\]
\[
= \sum_{2p+q \leq m} a_{p, q} (-1)^{[\alpha + \frac{1}{2}]} H_{\alpha + \frac{1}{2} - \alpha - \frac{1}{2}} \left( \left( \frac{\partial}{\partial t^2} \right)^{[\alpha + \frac{1}{2}]} \right) \left( t^{2p} y^q e^{-a(t^2 + y^2)} \right) (r, x).
\]

However, for all \( k \in \mathbb{N} \),
\[
(\frac{\partial}{\partial t^2})^k (t^{2p} y^q e^{-a(t^2 + y^2)}) = \left( \sum_{j=0}^{\min(p, k)} C_j^k \frac{2^j p!}{(p-j)!} (-2a)^{k-j} t^{2(p-j)} \right) y^q e^{-a(t^2 + y^2)},
\]
and for all \( \sigma \in \mathbb{R}, \sigma > 0 \),
\[
H_\alpha(t^{2p} y^q e^{-a(t^2 + y^2)}) (r, x) = \frac{1}{2\pi \Gamma(\sigma)} \left( \sum_{j=0}^{p} C_j^p \frac{\Gamma(\sigma + p - j)}{a^{\sigma + p - j}} r^{2j} \right) x^q e^{-a(r^2 + x^2)}.
\]

Combining the relations \( 3.15, 3.16 \) and \( 3.17 \), we deduce that
\[
f(r, x) = P(r, x)e^{-a(r^2 + x^2)}.
\]
Where \( P \) is a polynomial, even with respect to the first variable and \( \text{degree}(P) = \text{degree}(Q) \).

4 Applications of Beurling-Hörmander theorem

In this section, we shall deduce from the precedent Beurling-Hörmander theorem two most important uncertainty principles for the Fourier transform \( \mathcal{F}_\alpha \), that are the Gelfand-Shilov and the Cowling-Price theorems.

**Lemma 4.1.** Let \( P \) be a polynomial on \( \mathbb{R}^2 \), \( P \neq 0 \), with \( \text{degree}(P) = m \). Then there exist two positive constants \( A \) and \( C \) such that
\[
\forall t \geq A, \quad p(t) = \int_{\theta}^{2\pi} |P(t \cos(\theta), t \sin(\theta))| d\theta \geq Ct^m.
\]
Proof. Let $P$ be a polynomial on $\mathbb{R}^2$, $P \neq 0$ and with degree$(P) = m$. We have

$$p(t) = \int_0^{2\pi} \left| \sum_{j=0}^m a_j(\theta) t^j \right| d\theta,$$

where the functions $a_j$, $0 \leq j \leq m$, are continuous on $[0, 2\pi]$. It’s clear that the function $p$ is continuous on $[0, +\infty[$, and by dominate convergence theorem’s, we have

$$p(t) \sim C_m t^m \quad (t \rightarrow +\infty), \quad (4.1)$$

where $C_m = \int_0^{2\pi} |a_m(\theta)| d\theta > 0$.

Now the relation $(4.1)$ involves that there exists $A > 0$ such that

$$\forall t \geq A, \; p(t) \geq \frac{C_m}{2} t^m.$$

Theorem 4.2 (Gelfand-Shilov for $\mathcal{R}_\alpha$). Let $p, q$ be two conjugate exponents, $p, q \in ]1, +\infty[$. Let $\xi, \eta$ be non negative real numbers such that $\xi \eta \geq 1$. Let $f$ be a measurable function on $\mathbb{R}^2$, even with respect to the first variable, such that $f \in L^2(d\nu_\alpha)$.

If

$$\int_0^{+\infty} \int_{\mathbb{R}^2} |f(r, x)| e^{\frac{\xi \eta}{\theta} (r^2 + x^2)} \frac{\nu_\alpha(r, x)}{1 + |(r, x)|^d} d\nu_\alpha(r, x) < +\infty,$$

and

$$\int_{\mathbb{R}^2} \frac{|\mathcal{F}_\alpha(f)(\mu, \lambda)| e^{\frac{\xi \eta}{\theta} (\theta(\mu, \lambda))}}{1 + |(\mu, \lambda)|} d\tilde{\nu}_\alpha(\mu, \lambda) < +\infty, \quad d \geq 0;$$

Then

i) For $d \leq 1$, $f = 0$.

ii) For $d > 1$, we have

a) $f = 0$ for $\xi \eta > 1$.

b) $f = 0$ for $\xi \eta = 1$, and $p \neq 2$.

c) $f(r, x) = P(r, x) e^{-a(r^2 + x^2)}$ for $\xi \eta = 1$ and $p = q = 2$,

where $a > 0$ and $P$ is a polynomial on $\mathbb{R}^2$ even with respect to the first variable, with degree$(P) < d - 1$.

Proof. Let $f$ be a function satisfying the hypothesis. Since $\xi \eta \geq 1$, and by a convexity argument,
we have
\[
\left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(r,x)||F_{\alpha}(f)(\mu,\lambda)|}{(1 + ||(r,x)|| + |\theta(\mu,\lambda)||)^2} e^{\xi \eta ||(r,x)|| |\theta(\mu,\lambda)||} d\nu_{\alpha}(r,x) d\tilde{\nu}_{\alpha}(\mu,\lambda) \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(r,x)||F_{\alpha}(f)(\mu,\lambda)|}{(1 + ||(r,x)|| + |\theta(\mu,\lambda)||)^{a}} e^{\xi \eta ||(r,x)|| |\theta(\mu,\lambda)||} d\nu_{\alpha}(r,x) d\tilde{\nu}_{\alpha}(\mu,\lambda) \right)^{\frac{1}{2}}
\]
\[
\times \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(r,x)|}{(1 + ||(r,x)||)^{a}} e^{\frac{1}{p}(\mu^2 + \lambda^2)} d\nu_{\alpha}(r,x) \right)^{\frac{1}{2}}
\] < +\infty. \tag{4.2}
\]

Then from the Beurling-Hörmander theorem, we deduce that
i) For $d \leq 1$, $f = 0$.
ii) For $d > 1$, there exist a positive constant $a$, and a polynomial $P$ on $\mathbb{R}^2$, even with respect to the first variable such that
\[
f(r,x) = P(r,x)e^{-a(r^2 + x^2)}, \tag{4.3}
\]
with degree($P$) < $d - 1$, and by a standard calculus, we obtain
\[
\tilde{F}_{\alpha}(f)(\mu,\lambda) = Q(\mu,\lambda)e^{-\frac{1}{\mu}(\mu^2 + \lambda^2)}, \tag{4.4}
\]
where $Q$ is a polynomial on $\mathbb{R}^2$, even with respect to the first variable, with degree($P$) = degree($Q$).

On the other hand, from the relations (2.5), (2.7), (4.8), (4.9) and (4.10), we get
\[
\int_0^{\infty} \int_0^{\infty} \int_{\mathbb{R}^n} \frac{|P(r,x)||Q(\mu,\lambda)|}{(1 + ||(r,x)||)^a(1 + ||(r,x)||)^{a}} e^{\xi \eta ||(r,x)|| |\theta(\mu,\lambda)|| - a(r^2 + x^2)}
\]
\[
\times e^{-\frac{1}{\mu}(\mu^2 + \lambda^2)} d\nu_{\alpha}(r,x) d\mu d\lambda < +\infty,
\]
so
\[
\int_0^{\infty} \int_0^{\infty} \frac{\varphi(t)}{(1 + t)^{d} (1 + \rho)^{d}} e^{\xi \eta t} e^{-\alpha t^2} e^{-\frac{1}{\rho^2} \rho^2 t^2 \gamma + \epsilon} d\mu d\rho < +\infty, \tag{4.5}
\]
where
\[
\varphi(t) = \int_0^{2\pi} |P(t \cos(\theta), t \sin(\theta))| |\cos(\theta)|^{2\alpha + 1} d\theta,
\]
and
\[
\psi(\rho) = \int_0^{2\pi} |Q(\rho \cos(\theta), \rho \sin(\theta))| d\theta.
\]
Suppose that $\xi \eta > 1$. If $f \neq 0$, then each of the polynomials $P$ and $Q$ is not identically zero, let $m = \text{degree}(P) = \text{degree}(Q)$.

From lemma 4.1, there exist two positive constants $A$ and $C$ such that
\[
\forall t \geq A, \quad \varphi(t) \geq C t^m,
\]
and
\[ \forall \rho \geq A, \quad \psi(\rho) \geq C\rho^m. \]

Then, the inequality (4.5) leads to
\[ \int_{A}^{+\infty} \int_{A}^{+\infty} \frac{e^{\xi \eta t \rho}}{(1 + t)^d(1 + \rho)^d} e^{-\frac{1}{4} \rho^2} dt \, d\rho < +\infty. \]

(4.6)

Let \( \varepsilon > 0 \), such that \( \xi \eta - \varepsilon = \sigma > 1 \). The relation (4.6) implies that
\[ \int_{A}^{+\infty} \int_{A}^{+\infty} e^{\varepsilon t \rho} e^{-\frac{1}{4} \rho^2} dt \, d\rho < +\infty. \]

(4.7)

However, for all \( t \geq A \geq \frac{d}{\varepsilon} \) and \( \rho \geq A \), we have
\[ e^{\varepsilon \rho t} \frac{1}{(1 + t)^d(1 + \rho)^d} \geq \frac{e^{\varepsilon A^2}}{(1 + A)^{2d}}, \]
and by the relation (4.7) it follows that
\[ \int_{A}^{+\infty} \int_{A}^{+\infty} e^{\varepsilon t \rho} e^{-\frac{1}{4} \rho^2} dt \, d\rho < +\infty. \]

(4.8)

Let \( F(t) = \int_{A}^{+\infty} e^{\sigma t \rho - \frac{1}{2} \rho^2} d\rho \), then \( F \) can be written
\[ F(t) = e^{\sigma t^2} \left( \int_{A}^{+\infty} e^{-\frac{1}{2} \rho^2} d\rho + 2a \sigma e^{-\frac{A^2}{4}} \int_{0}^{t} e^{A \sigma s - a \sigma s^2} ds \right), \]
in particular
\[ F(t) \geq e^{\sigma t^2} \int_{A}^{+\infty} e^{-\frac{1}{2} \rho^2} d\rho. \]

Thus
\[ \int_{A}^{+\infty} \int_{A}^{+\infty} e^{\sigma t \rho} e^{-\frac{1}{4} \rho^2} dt \, d\rho = \int_{A}^{+\infty} e^{-\frac{1}{2} \rho^2} \int_{A}^{+\infty} e^{\sigma t^2} F(t) \, dt \]
\[ \geq \int_{A}^{+\infty} e^{-\frac{1}{2} \rho^2} \int_{A}^{+\infty} e^{a(\sigma^2 - 1) t^2} \, dt = +\infty, \]

because \( \sigma > 1 \). This contradicts the relation (4.8) and shows that \( f = 0 \).

Suppose that \( \xi \eta = 1 \) and \( p \neq 2 \). In this case we have \( p > 2 \) or \( q > 2 \). Suppose that \( q > 2 \), then from the second hypothesis and the relation (4.4), we have
\[ \int_{0}^{+\infty} \psi(\rho) e^{-\frac{\tau^2}{2}} e^{\frac{a \rho^p}{p}} \frac{1}{(1 + \rho)^d} d\rho < +\infty. \]

(4.9)

If \( f \neq 0 \), then the polynomial \( Q \) is not identically zero, and by lemma (4.1) and the relation (4.9), it follows that
\[ \int_{0}^{+\infty} e^{-\frac{\tau^2}{2}} e^{\frac{a \rho^p}{p}} \frac{1}{(1 + \rho)^d} d\rho < +\infty. \]
which is impossible because $q > 2$.

The proof of theorem 4.2 is complete.

\[ \tag{4.10} \]

\[ \tag{4.11} \]

**Theorem 4.3** (Cowling-Price for $R_\alpha$). Let $\xi, \eta, \omega_1, \omega_2$ be non negative real numbers such that $\xi \eta \geq \frac{1}{4}$. Let $p, q$ be two exponents, $p, q \in [1, +\infty]$, and let $f$ be a measurable function on $\mathbb{R}^2$, even with respect to the first variable such that $f \in L^2(d\nu_\alpha)$.

If

\[ \left\| \frac{e^{\xi |r| \theta(x, \cdot)}}{(1 + |r| \theta(x, \cdot))^{\omega_1}} f \right\|_{p, \nu_\alpha} < +\infty, \]

and

\[ \left\| \frac{e^{\eta |\theta(x, \cdot)| \theta(x, \cdot)}}{(1 + |\theta(x, \cdot)|)^{\omega_2}} F_\alpha(f) \right\|_{q, \tilde{\gamma}_\alpha} < +\infty, \]

then

i) For $\xi \eta > \frac{1}{4}$, $f = 0$.

ii) For $\xi \eta = \frac{1}{4}$, there exist a positive constant $a$ and a polynomial $P$ on $\mathbb{R}^2$, even with respect to the first variable, such that

\[ f(r, x) = P(r, x)e^{-a(r^2 + x^2)}. \]

**Proof.** Let $p'$ and $q'$ be the conjugate exponents of $p$ respectively $q$. Let us pick $d_1, d_2 \in \mathbb{R}$, such that $d_1 > 2\alpha + 3$ and $d_2 > 2$. Finally, let $d$ be a positive real number such that $d > \max(\omega_1 + \frac{d_1}{p}, \omega_2 + \frac{d_2}{q}, 1)$.

From Hölder’s inequality and the relations (4.10) and (4.11), we deduce that

\[ \int_{\mathbb{R}} \int_{\mathbb{R}} |f(r, x)| e^{\xi |r, x| \theta(x, \cdot)} \frac{d\nu_\alpha(r, x)}{(1 + |r| \theta(x, \cdot))^{\omega_1 + \frac{d_1}{p}}} < +\infty, \]

and

\[ \int_{\Gamma} \int_{\Gamma} |F_\alpha(f)(\mu, \lambda)| e^{\eta |\theta(\mu, \lambda)| \theta(\mu, \lambda)} \frac{d\tilde{\gamma}_\alpha(\mu, \lambda)}{(1 + |\theta(\mu, \lambda)|)^{\omega_2 + \frac{d_2}{q}}} < +\infty. \]

Consequently we have

\[ \int_{\mathbb{R}} \int_{\mathbb{R}} |f(r, x)| e^{\xi |r, x| \theta(x, \cdot)} \frac{d\nu_\alpha(r, x)}{(1 + |r| \theta(x, \cdot))^{d}} < +\infty, \]

and

\[ \int_{\Gamma} \int_{\Gamma} |F_\alpha(f)(\mu, \lambda)| e^{\eta |\theta(\mu, \lambda)| \theta(\mu, \lambda)} \frac{d\tilde{\gamma}_\alpha(\mu, \lambda)}{(1 + |\theta(\mu, \lambda)|)^{d}} < +\infty. \]

Then, the desired result follows from theorem 4.2.

**Remark 4.1.** The Hardy’s theorem is a special case of theorem 4.3 when $p = q = +\infty$.

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