Prediction Intervals based on Doubly Type-II Censored Data from Gompertz Distribution in the Presence of Outliers

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Abstract—The study aims at getting the Bayesian predication intervals for some order statistics of future observations from the distribution of Gompertz \((\alpha, \beta)\). Doubly Type-II censored data has assisted obtaining in the presence of single outlier that arose from the different same family members of distribution. Single outlier of type \(\beta_0\) and \(\beta + \beta_0\) are considered and bivariate independent prior density for \(\alpha\) and \(\beta\) are used. The problem of solving the Double integral to obtain the closed form for \(\alpha\) and \(\beta\), leads us to use MCMC for calculating the Bayesian Predication Intervals. The use of numerical examples and statistical data has enabled to properly present and describe the procedure. We conclude that the Bayesian predication intervals are shorter for \(y_1\), when we are increasing the \(\beta_0\) value.

Keywords—Bayesian prediction; Gompertz distribution; predictive distribution; doubly Type-II censored data; Markov Chain Monte Carlo; single outliers

I. INTRODUCTION

The adult death patterns can be effectively described through the use of the Gompertz distribution ([17]; [6]). Moreover, the Gompertz mortality force for the decreased infant and young adult levels of mortality extends to the whole life population span without any observed deceleration of mortality ([16]). A continuous probability density function (pdf) and a cumulative distribution function (cdf) are the constituents of the Gompertz distribution.

The pdf as follows:

\[
f(x) = \alpha \beta e^{\alpha x} - \beta (e^{\alpha x} - 1), \quad x > 0, \quad \alpha > 0, \quad \beta > 0,
\]

and The cdf as follows:

\[
F(x) = 1 - e^{-\beta (e^{\alpha x} - 1)}.
\]

This distribution should be denoted with two Gomp \(\alpha\) and \(\beta\) parameters. The research conducted by [1] indicated that a simple transformation relates the Gompertz distribution to a certain distribution in the family of distributions. A further research conducted by [7] showed that it is possible to get the maximum likelihood parameter estimates the Gompertz model. The study by [3] suggests the ways to apply it and provides a more recent survey that enables to better understand the model. At the same time, [19] made an attempt to reformulate the Gompertz mortality force and get an insight into the new formation relationship.

The analysis of the research by [18] enabled to trace the connections between the Weibull, the Gompertz, and other Type I extreme value distributions. Later, [9] managed to obtain a Bayesian prediction, mixing two-component lifetime model of Gompertz. In another study [10] derived a Bayesian record statistics analysis from the Gompertz model. A negative Gompertz distribution was presented by later, [11] who focused on the discussion of the negative aging parameter rate. A generalized three-parameter Gompertz distribution was presented by [8]), who provided a deep insight into the topic under investigation. Furthermore, [2] worked on the Gompertz model, and attempted to introduce a more generalized four-parameter version of the model that was referred to as a beta-Gompertz distribution. Also, the paper provides some commonly used distributions, including generalized and beta-exponential Gompertz distributions as sub-models. [15] proposed a distribution of an exponentiated Weibull extension; however, it was modified. It was further generalized and discussed in the study by [8]. Author in [13] focused on the investigation and discussion of the obtained prediction intervals that are based on Gompertz doubly censored data. There are some cases make Progressive Hybrid Censored schemes (PHCS) difficult to apply when the failures may occur before time [21]. Some researchers estimated and predicted the Generalized Progressive Hybrid Censored Data for Gompertz Distribution [20]. Whoever Gompertz distribution was studied by many researchers such as [22].

The main objective of this paper, we assume that \(X_1, X_2, \ldots, X_n\), is an ordered random sample of size \(n\) drawn from a population whose pdf, is Gomp\((\alpha, \beta)\), which is defined by equation 1, and that \(Y_1, Y_2, \ldots, Y_m\), is a second independent random sample (of size \(m\)) of future observations from the same distributions. Bayesian prediction bounds for the future observations \(Y_1, Y_2, \ldots, Y_m\) in the presence of a single outlier of type \(\beta_0\) and \(\beta + \beta_0\) are obtained.
Observation is an outlier in the data set that is inconsistent with the data set remainder ([5]). Hence, a single \( \beta \beta_0 \) and \( \beta + \beta_0 \) type outliers are present in the future Gompertz population sample. Gomp\((\alpha, \beta \beta_0)\) is taken for a single type \( \beta + \beta_0 \) outlier of the pdf, while in the case of single type \( \beta + \beta_0 \) outlier the pdf is taken Gomp\((\alpha, \beta + \beta_0)\).

In the study, the bounds of the Bayesian prediction are received for the future Gomp\((\alpha, \beta)\) distribution observations in the presence of a single outlier of type. It is considered that both parameters \( \alpha \) and \( \beta \) are unknown. The true value \( (\beta, \alpha) \) uncertainty is measured through the function of the bivariate prior density that was discussed and applied with the same model in the research conducted by [10]. Furthermore, the current research presupposes the construction of the predictive interval that will be used for the future observation with the presence of a single outlier of type with MCMC. The use of statistics will assist in illustrating and presenting the procedure.

In this article, Section II explains the Likelihood Function. After that Section III discuss the Posterior distribution. Moreover, Section IV clarify the Bayesian predication in the presence of outliers for future observations with two schemes \( \beta \beta_0 \) and \( \beta + \beta_0 \). Section V shows numerical example, which are considered the previous two schemes. In the final Section VI, we give the conclusion and opens future direction.

II. LIKELIHOOD FUNCTION

In this section, we assume \( x_1, x_2, \ldots, x_n \) is an ordered random size \( n \) sample from the Gomp\((\alpha, \beta)\). The pdf and cdf are given by (1) and (2), respectively. Also, let \( x_1 \leq x_2 \leq \cdots \leq x_k \) be the \( k \) smallest ordered observation, while \( x_{r+1} \leq x_{r+2} \leq \cdots \leq x_n \) the \( n-r \) largest ordered observations in the sample. The statistical analysis contains the application of only the remaining ordered observations, that is, \( \bar{x} = (x_{k+1}, x_{s+2}, \cdots, x_r) \). Moreover, it is evident that when \( k = 1 \), the sample will be a Type-II right censored sample. A doubly censored sample pulled from population with pdf and cdf as given in (1) and (2) that likelihood function is given as follows:

\[
L(\alpha, \beta; \bar{x}) \propto [F_X(x_{k+1}; \alpha, \beta)]^k [1 - F_X(x_s; \alpha, \beta)]^{n-r} \times \prod_{i=k+1}^{r} [f_X(x_i; \alpha, \beta)]_x_{s+1} \geq 0
\]

\[
= (\alpha \beta)^{r-k} [1 - \exp(-\beta T_1(\alpha; x_{k+1}))]^k \times \exp\left\{ \alpha \sum_{i=s+1}^{r} x_i - \beta T_2(\alpha; \bar{x}) \right\}.
\]

where

\[
T_1(\alpha; x_{k+1}) = e^{\alpha x_{k+1}} - 1,
\]

\[
T_2(\alpha; \bar{x}) = (n-r) e^{\alpha x_r} + \sum_{i=k+1}^{r} e^{\alpha x_i} - n + s.
\]

The Bayesian prediction tends to bound the future observations in the presence of a single outlier of type Gomp\((\alpha, \beta)\) distribution when two parameters types \( \alpha \) and \( \beta \) are both dependent and unknown.

III. THE POSTERIOR DISTRIBUTION

To obtain the joint posterior density of \( \alpha \) and \( \beta \), we use a bivariate prior density of the form:

\[
\pi(\alpha, \beta) = \pi_1(\alpha) \pi_2(\beta),
\]

where

\[
\pi_1(\alpha) = \frac{\gamma_1}{\Gamma(\gamma_1)} \alpha^{\gamma_1 - 1} e^{-\alpha \gamma_1}, \ (\eta_1, \gamma_1 > 0)
\]

and

\[
\pi_2(\beta) = \frac{\gamma_2}{\Gamma(\gamma_2)} \beta^{\gamma_2 - 1} e^{-\beta \gamma_2}, \ (\eta_2, \gamma_2 > 0).
\]

The paper assumes that the joint prior density for the parameter \( \alpha \) and \( \beta \) is the form (5) and presented by Jaheen [10] for the progressive censored data prediction from the Gompertz model and applied by [13] for the prediction Gompertz doubly censored data intervals.

The likelihood of the function presented by (3) and the function of the joint posterior density of parameters \( \alpha \) and \( \beta \) is

\[
\pi^*(\alpha, \beta | \bar{x}) = \frac{L(\alpha, \beta; \bar{x})\pi_1(\alpha) \pi_2(\beta)}{\int_0^\infty \int_0^\infty L(\alpha, \beta; \bar{x})\pi_1(\alpha) \pi_2(\beta) d\alpha d\beta}.
\]

The joint posterior density function of \( \alpha \) and \( \beta \) given data can be written as

\[
\pi^*(\beta, \alpha | \bar{x}) \propto h_1(\beta | \alpha, \text{data})h_2(\alpha | \text{data})h_3(\alpha, \beta | \text{data})
\]

where \( h_1(\beta | \alpha, \text{data}) \) is a gamma density where the shape parameter \( m = r - k + \eta_1 \) and the scale parameter is \( \eta_1 + T_2(\alpha; \bar{x}) \). At the same time, \( h_2(\alpha | \text{data}) \) is a proper density function of the form

\[
h_2(\alpha | \text{data}) = \frac{1}{[\gamma_1 + T_2(\alpha; \bar{x})]^{\gamma_1}} e^{-\alpha \gamma_1 + T_2(\alpha; \bar{x})},
\]

and \( h_3(\alpha, \beta | \text{data}) \) is given by

\[
h_3(\alpha, \beta | \text{data}) = [1 - e^{-\beta T_1(\alpha; x_{k+1})}]^s.
\]

From equation (8) and it enables to see that a simple closed form cannot express the equation. Therefore, the Bayes estimators of the parameter \( \alpha \) and \( \beta \) cannot be received in simple closed forms. Hence, the paper suggests the approximation (9) by applying the importance sampling technique that is also presented by [14]. The importance sampling details are presented below.

In this paper, we used the importance sampling procedure to calculate the Bayes estimates for \( \alpha \) and \( \beta \) as well as any function of the parameters \( g(\alpha, \beta) \). Moreover, the Algorithm 1 (presented below) is used to generate \( \alpha \) and \( \beta \) from the posterior density function (7).

Algorithm 1:
Step 1: Start with an \((\alpha^0; \beta^0)\).

Step 2: set \(t = 1\).

Step 3: Generate \(\alpha^t\) from \(h_2(\alpha|\text{data})\) using the method developed by [12] with the \(N(\alpha^{t-1}; \sigma)\) proposal distribution, where \(\sigma^2\) is the variance of the parameter \(\alpha\).

Step 4: Generate \(\beta^t\) from gamma distribution with pdf \(h_2(\beta | \alpha, \text{data})\).

Step 5: Put \(t = t+1\).

Step 6: Repeat steps 3-5 \(M\) times to obtain \(\{(\alpha^t, \beta^t), t = 1, 2, \cdots, M\}\).

The approximate Bayes are applied to estimate any function of the parameters say \(g(\alpha, \beta)\) under the squared functions of error loss using the procedure of importance sampling, as shown below:

\[
\hat{g}_{BS}(\alpha, \beta) = \frac{\sum_{i=1}^{M} g(\alpha_i, \beta_i) g_3(\alpha_i, \beta_i|\text{data})}{\sum_{i=M_0}^{M} g_3(\alpha_i, \beta_i|\text{data})},
\]

(12)

IV. BAYESIAN PREDICTION IN THE PRESENCE OF A SINGLE OUTLIER FOR FUTURE OBSERVATIONS

The section introduces the prediction of the future observations in the presence of a single outlier. Also, it is assumed that \(X_1, X_2, \cdots, X_n\) is a random size \(n\) sample drawn from the Gomp(\(\alpha, \beta\)) population, where the pdf is presented by (1). Let us assume that \(Y_1, Y_2, \cdots, Y_m\) is a second, independent, unobserved size \(m\) sample received from the same population. This sample is the future sample, and the aim of the study is to get Bayesian prediction bounds for the \(s^{th}\) oncoming observation \(Y_s\), \(s = 1, 2, \cdots, m\) in the presence of a single outlier.

In the case of the size \(m\) sample, let \(Y_s\) be the \(s^{th}\) ordered lifetime, \(1 \leq s \leq m\). Then the \(Y_s\) density function for a given \(\theta\) in the presence of a single outlier is of the form \(f = f(y|\theta)\) and \(F = F(y|\theta)\) are the distribution and density functions of all \(y_s\) which are not referred to be outliers as \(f^* = f^*(y|\theta)\) and \(F^* = F^*(y|\theta)\) are those of an outlier (41). The \(f^*\) and \(F^*\) functions are received for the Gomp(\(\alpha, \beta\)) model through the replacement of parameter \(\beta\) by \(\beta_0\) or \(\beta + \beta_0\) depending on the outlier type.

\[
f(y|\theta) = D(s)[(s-1)F^* - (1-F)(m-s)F^*f + (m-s)F^{s-1} - (1-F)(m-s-1)F^{s-1}(1-F^*)f + F^{s-1}(1-F)(m-s-1)F^*],
\]

(13)

where

\[
D(s) = \binom{m-1}{s-1},
\]

(14)

A. Outliers of type \(\beta_0\)

The \(Y_s\) density function, in the presence of a single outlier of type \(\beta_0\), in the Gomp(\(\alpha, \beta\)) case may be received through the substituting of (1) and (2) for \(f\) and \(F\) in (13). The \(f^*\) and \(F^*\) values presented by (1) and (2), after the replacement of \(\beta\) by \(\beta_0\). It is possible to simplify the density function implementing the pdf \(g_1(y_2|\alpha, \beta)\), where the cdf \(G_1(y_s|\alpha, \beta)\) is given as follows:

\[
g_1(y_s|\alpha, \beta) = D(s) \alpha \beta \exp\{\alpha y_s^2[(m + \beta_0 - s) \sum_{j=0}^{s-1} A_{1j}(y_s) + (s-1) \sum_{j=0}^{s-2} A_{2j}(y_s)]\}, \quad y_s > 0, \quad (15)
\]

where

\[
A_{1j}(y_s) = a_{1j}(s) \exp\{-\beta \omega_j(s)\phi(y_s; \alpha)\},
\]

\[
A_{2j}(y_s) = a_{2j}(s) \left[\exp\{-\beta \omega_j(s)\phi(y_s; \alpha)\} - \exp\{-\beta \omega_{j+1}(s)\phi(y_s; \alpha)\}\right],
\]

\[
\phi(y_s; \alpha) = (e^{\alpha y_s} - 1), \omega_j(s) = m - s + \beta_0 + j, \omega_{1j}(s) = m - s + j + 1
\]

(16)

and for \(\ell = 1, 2\),

\[
a_{\ell j}(s) = (-1)^\ell \binom{s-\ell}{j}.
\]

(17)

and the pdf \(g_1(y_s|\alpha, \beta)\) the cdf \(G_1(y_s|\alpha, \beta)\) is given by

\[
G_1(y_s|\alpha, \beta) = D(s) \left[(m + \beta_0 - s) \sum_{j=0}^{s-1} A_{1j}^*(y_s) + (s-1) \sum_{j=0}^{s-2} A_{2j}^*(y_s)\right], \quad y_s > 0
\]

(18)

where

\[
A_{1j}^*(y_s) = a_{1j}(s) \omega_j(y_s) F(y_s; \alpha, \beta \omega_j(s)),
\]

\[
A_{2j}^*(y_s) = a_{2j}(s) \omega_{1j}(y_s) F(y_s; \alpha, \beta \omega_{1j}(s)) - a_{2j}(s) \omega_{1j+1}(y_s) F(y_s; \alpha, \beta \omega_{1j+1}(s)).
\]

(19)

The Bayesian predictive density of \(y_s, s = 1, 2, \cdots, m\) given \(x\) is represented by

\[
g_1^*(y_s|x) = \int_0^{\infty} \int_0^{\infty} g_1(y_s|\alpha, \beta) \pi^*(\alpha, \beta|x) \, d\alpha \, d\beta.
\]

(20)

The Bayesian predictive distribution function of \(y_s, s = 1, 2, \cdots, m\) given \(x\), \(\alpha\) and \(\beta\) is given by

\[
G_1^*(y_s|x) = \int_0^{\infty} \int_0^{\infty} G_1(y_s|\alpha, \beta) \pi^*(\alpha, \beta|x) \, d\alpha \, d\beta.
\]

(21)

Supposing that \(\{(\alpha_i, \beta_i); i = 1, 2, \cdots, M\}\) are MCMC samples received from \(\pi^*(\alpha, \beta|x)\), it is possible to get the
simulation consistent estimators of $g^*_1(y_s|x)$ and $G^*_1(y_s|x)$ can be obtained as

$$g^*_1(y_s|x) = \sum_{i=1}^{M} g_1(y_s|x_i, \alpha_i, \beta_i) h_i$$

(22) 

and

$$G^*_1(y_s|x) = \sum_{i=1}^{M} G_1(y_s|x_i, \alpha_i, \beta_i) h_i$$

(23) 

where

$$h_i = \frac{h_3(\alpha_i, \beta_i)}{\sum_{i=1}^{M} h_3(\alpha_i, \beta_i)}; \quad i = 1, 2, \ldots, M.$$ 

(24)

A $(1 - \tau)\%$ Bayesian prediction interval for $Y_s$ is as follows: $P[L(x) \leq Y_s \leq U(x)] = 1 - \tau$, where $L(x)$ and $U(x)$ are the lower and the upper bounds for $y_s$, $s = 1, 2, \ldots, m$. Thus, equating of (23) $1 - \frac{\tau}{2}$ and $\frac{\tau}{2}$ enables to get the following:

$$P[Y_s \geq L(x)|x] = 1 - \frac{\tau}{2} \Rightarrow \hat{G}_1^*(L(x)|x) = \frac{\tau}{2}$$

(25)

and

$$P[Y_s \leq U(x)|x] = \frac{\tau}{2} \Rightarrow \hat{G}_1^*(U(x)|x) = 1 - \frac{\tau}{2}.$$ 

(26)

B. Type $\beta + \beta_0$ Outliers

The $y_s$ density function, in the presence of a single outlier of type $\beta + \beta_0$, in the Gompf($\alpha$, $\beta$) case, can be received through the substituting of (1) and (2) for $F$ and $f$ in (3). The $F^*$ and $f^*$ are presented by (1) and (2) after the replacement of $\beta$ by $\beta + \beta_0$. Consequently, the density begins to form:

$$g_2(y_s|\alpha, \beta) = D(s) e^{\alpha y_s} \left[ (\beta (m-s+1) + \beta_0) \sum_{j=0}^{s-1} B_{1j}(y_s) + \beta (s-1) \sum_{j=0}^{s-2} B_{2j}(y_s) \right], \quad y_s > 0,$$  

(27)

where

$$B_{1j}(y_s) = a_{1j}(s) \exp \left\{ -[\beta \omega_{1j}(s) + \beta_0] \phi(y_s; \alpha) \right\}$$

$$B_{2j}(y_s) = a_{2j}(s) \left[ \exp \left\{ -[\beta \omega_{1j}(s) \phi(y_s; \alpha) \right\} - \exp \left\{ -[\beta \omega_{1(j+1)}(s) + \beta_0] \phi(y_s; \alpha) \right\}, \right.$$ 

(28)

$\phi(y_s; \alpha) \omega_{1j}(s)$ are given in (16) and $a_{1j}(s), a_{2j}(s)$ is given for $\ell = 1, 2$, respectively, by (17).

The cdf corresponding to the pdf $g_2(y_s|\alpha, \beta)$ is presented by

$$G_2(y_s|\alpha, \beta) = D(s) \left[ (\beta (m-s+1) + \beta_0) \sum_{j=0}^{s-1} B_{1j}^*(y_s) + \beta (s-1) \sum_{j=0}^{s-2} B_{2j}^*(y_s) \right], \quad y_s > 0,$$ 

(29)

where

$$B_{1j}^*(y_s) = \frac{a_{1j}(s)}{\beta \omega_{1j}(s) + \beta_0} F(y_s; \alpha, \beta \omega_{1j}(s) + \beta_0),$$

$$B_{2j}^*(y_s) = \frac{a_{2j}(s)}{\beta \omega_{1j}(s)} F(y_s; \alpha, \beta \omega_{1j}(s)) - \frac{a_{2j}(s)}{\beta \omega_{1(j+1)}(s) + \beta_0} F(y_s; \alpha, \beta \omega_{1(j+1)}(s) + \beta_0),$$

(30)

where $F(y_s; \alpha, \beta m + \beta_0)$ is given by (2). The Bayesian predictive distribution function of $y_s$, $s = 1, 2, \ldots, m$ given $x$, $\alpha$ and $\beta$ is given by

$$g_2^*(y_s|x) = \int_0^{\infty} \int_0^{\infty} g_2(y_s|\alpha, \beta) \pi^*(\alpha, \beta|x) \; d\alpha d\beta,$$  

(31)

and the predictive cdf of $y_s$, $G_2^*(y_s|x)$ is given by

$$G_2^*(y_s|x) = \int_0^{\infty} \int_0^{\infty} G_2(y_s|\alpha, \beta) \pi^*(\alpha, \beta|x) \; d\alpha d\beta,$$  

(32)

where $G_2(y_s|\alpha, \beta)$ is given by (29) and $\pi^*(\alpha, \beta|x)$ is given by (9). It is evident that it is impossible to express (31) and (32) in closed form. Therefore, they cannot be analytically evaluated. The use of MCMC samples \{(\alpha_i, \beta_i), i = 1, 2, \ldots, M\}, enable the obtaining of $\hat{g}_2^*(y_s|x)$ and $\hat{G}_2^*(y_s|x)$ simulation consistent estimator, as follows:

$$\hat{g}_2^*(y_s|x) = \sum_{i=1}^{M} g_2(y_s|x_i, \alpha_i, \beta_i) h_i,$$ 

(33)

and

$$\hat{G}_2^*(y_s|x) = \sum_{i=1}^{M} G_2(y_s|x_i, \alpha_i, \beta_i) h_i,$$ 

(34)

Where $h_i$ is given by (24). It is essential to highlight that it is possible to use the same MCMC samples \{(\alpha_i, \beta_i), i = 1, 2, \ldots, M\}, to compute $\hat{g}_2^*(y_s|x)$ and $\hat{G}_2^*(y_s|x)$ simulation consistent estimators for $\ell = 1, 2$, respectively, by (17).
V. NUMERICAL EXAMPLE

Example 1. This example shows a doubly Type-II censored sample, \( x_{(s+1)}, x_{(s+2)}, \ldots, x_{(r)} \), that is received through the application of the following steps:

1. For the hyperparameters given values \( \eta_1 = 1.2 \) and \( \gamma_1 = 1.8 \) a generated value of \( \alpha = 0.860986 \) is received from the prior distribution with pdf (6).

2. For the hyperparameters given values \( \eta_2 = 1.4 \) and \( \gamma_2 = 1.7 \) a generated value of \( \beta = 0.409442 \) is received from the prior distribution with pdf (7).

3. The use of the generated values of \( \alpha \) and \( \beta \) from two prior steps, enables to generate a sample of size \( n = 30 \) from the Gomp(\( \alpha, \beta \)) distribution with pdf, that is represented by (2).

4. The application of some sorting routine, assists in obtaining a doubly Type-II censored different value sample of size \( r = 20, 25, 30 \) and \( k = 0, 5, 10 \) from the Gomp(\( \alpha, \beta \)) distribution, where the deferent value of \( r \) and \( k \) is presented in Tables I, II and III.

5. Generate \( (\alpha_i, \beta_i), i = 1, 2, \ldots, M \), through the use of MCMC shown in Algorithm 1.

6. The above generated doubly Type-II censored size \((r-s)\) sample, the 95\% Bayesian prediction links to the future ordered values, \( y(1), y(2), \ldots, y(m), m = 5 \) in the single types \( \beta \) \( \beta \) outliers, enable a numerical calculation through solving the equations (25) and (26).

Let us assume that we have one more size \( m = 5 \) sample in the presence of a single outlier of type \( \beta \) \( \beta \). Hence, for the given \( \beta_0 \) values we seek to receive 95% Bayesian prediction bounds for \( y_1 \) to \( y_5 \) of the failure future sample times. Tables I, II and III represents these bounds with the corresponding \( \beta_0 \) values.

Example 2. The 95% Bayesian prediction interval for a future unobserved \( y_1 \) to \( y_5 \), which are the failure times in the future size 5 sample in the presence of a single outlier of type \( \beta + \beta_0 \) can be obtained on the basis of a generated doubly Type-II censored sample of size \( m \) from the Gomp(\( \alpha, \beta \)) distribution. Same different \( \eta_1, \gamma_1, \eta_2, \gamma_2 \) hyper-parameter values and the same data set is presented in Example 1. Hence, these bounds with the corresponding \( n = 30, r = 20, 25, 20 \) and \( k = n-r \) and \( \beta_0 \) values are shown in Tables IV, V and VI.

VI. CONCLUSION

The study investigated and discussed the single \( \beta_0 \) and \( \beta + \beta_0 \) type outliers through the application of the predictive distribution function. Hence, the Bayesian prediction intervals in the case of future homogeneous case observations can be received by \( \beta_0 = 1 \) in (18) or \( \beta_0 = 0 \) in (29).

However, it is impossible in the no outlier case. The Gibbs sampling technique was applied to generate MCMC samples. Afterwards, the importance sampling methodology was used to compute the Bayesian prediction problems in the presence of a single outlier of both types. It is essential to highlight that the Bayesian prediction intervals are shorter for \( y_1 \) and larger for the Bayesian prediction intervals for \( y_5 \) due to the increase of \( \beta_0 \) value.

### Table I.

| \( y_0 \) | \( \eta_1 \) | \( \gamma_1 \) | \( \alpha \) | \( \beta \) | \( \eta_2 \) | \( \gamma_2 \) |
|----------|----------|----------|----------|----------|----------|----------|
| PP       | 0.492271 | 0.393387 | 1.35569 | 1.80314 | 3.7285   |
| LB       | 0.017663 | 0.170507 | 0.65082 | 0.842849| 1.29551  |
| UB       | 1.39910 | 1.84811 | 2.30933 | 2.76994 | 3.36565  |
| CP       | 0.945075 | 0.96924 | 0.95131 | 0.94792 | 0.9386   |

### Table II.

| \( y_0 \) | \( \eta_1 \) | \( \gamma_1 \) | \( \alpha \) | \( \beta \) | \( \eta_2 \) | \( \gamma_2 \) |
|----------|----------|----------|----------|----------|----------|----------|
| PP       | 0.492271 | 0.393387 | 1.35569 | 1.80314 | 3.7285   |
| LB       | 0.017663 | 0.170507 | 0.65082 | 0.842849| 1.29551  |
| UB       | 1.39910 | 1.84811 | 2.30933 | 2.76994 | 3.36565  |
| CP       | 0.945075 | 0.96924 | 0.95131 | 0.94792 | 0.9386   |

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TABLE III. 95% Bayesian prediction intervals for $y_1, \ldots, y_5$ in the presence of a single outlier of type $\beta_3$, where $n = 30$, $r = 30$, $k = 30$. Note: Obs. is observations PP is point predictors, LB is lower bound, UB is upper bound, CP is coverage percentages.

| $\beta_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|-----------|-------|-------|-------|-------|-------|
| PP        | 0.475422 | 0.902867 | 1.31418 | 1.74964 | 2.30966 |
| LB        | 0.016929 | 0.170929 | 0.442908 | 0.796926 | 1.25459 |
| UB        | 1.35561 | 1.83021 | 2.24682 | 2.69093 | 3.35434 |
| CP        | 95.56% | 95.69% | 95.74% | 95.87% | 95.97% |
| P         | 0.413396 | 0.090015 | 1.20817 | 1.64613 | 2.22353 |
| LB        | 0.014124 | 0.145269 | 0.388644 | 0.717329 | 1.16018 |
| UB        | 1.21088 | 1.68822 | 2.16118 | 2.53056 | 2.90356 |
| CP        | 94.93% | 95.14% | 95.24% | 95.38% | 95.46% |
| 3 PP      | 0.366148 | 0.7499 | 1.15607 | 1.61019 | 2.20664 |
| LB        | 0.021216 | 0.129143 | 0.336338 | 0.677551 | 1.11961 |
| UB        | 1.09588 | 1.61323 | 2.90543 | 2.60433 | 3.03526 |
| CP        | 98.69% | 98.83% | 98.98% | 99.13% | 99.33% |
| 54 PP     | 0.299642 | 0.681802 | 1.10578 | 1.87207 | 2.19956 |
| LB        | 0.009434 | 0.108767 | 0.317832 | 0.630847 | 1.08604 |
| UB        | 0.92483 | 1.55811 | 2.08696 | 2.60376 | 3.03526 |
| CP        | 96.94% | 97.49% | 98.04% | 98.49% | 98.73% |

TABLE IV. 95% Bayesian prediction intervals for $y_1, \ldots, y_5$ in the presence of a single outlier of type $\beta_4 + \beta_5$, where $n = 30$, $r = 30$, $k = 10$. Note: Obs. is observations PP is point predictors, LB is lower bound, UB is upper bound, CP is coverage percentages.

| $\beta_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|-----------|-------|-------|-------|-------|-------|
| PP        | 0.314436 | 0.624192 | 0.947281 | 1.31177 | 1.80927 |
| LB        | 0.001001 | 0.103158 | 0.235667 | 0.56839 | 0.98417 |
| UB        | 0.96839 | 1.36921 | 1.7411 | 2.15974 | 2.78612 |
| CP        | 95.88% | 96.47% | 97.12% | 97.78% | 98.33% |
| 1 PP      | 0.249322 | 0.524206 | 0.833376 | 1.20209 | 1.62088 |
| LB        | 0.00759 | 0.081351 | 0.230135 | 0.45227 | 0.78826 |
| UB        | 0.792035 | 1.20258 | 1.61103 | 2.07199 | 2.73866 |
| CP        | 94.87% | 95.47% | 96.17% | 96.87% | 97.47% |
| 2 PP      | 0.206843 | 0.478438 | 0.794469 | 1.17926 | 1.71243 |
| LB        | 0.006111 | 0.06950 | 0.205725 | 0.42648 | 0.757922 |
| UB        | 0.67229 | 1.14618 | 1.59745 | 2.07048 | 2.73858 |
| CP        | 91.36% | 91.74% | 92.15% | 92.56% | 92.97% |
| 3 PP      | 0.176862 | 0.446821 | 0.777449 | 1.17201 | 1.71076 |
| LB        | 0.005115 | 0.086181 | 0.209861 | 0.429094 | 0.740202 |
| UB        | 0.584882 | 1.29271 | 1.56839 | 2.07045 | 2.73858 |
| CP        | 87.96% | 89.34% | 90.76% | 92.17% | 92.67% |
| 4 PP      | 0.154538 | 0.429348 | 0.76897 | 1.16911 | 1.71014 |
| LB        | 0.004398 | 0.050299 | 0.181013 | 0.398183 | 0.740829 |
| UB        | 0.517254 | 1.12552 | 1.59631 | 2.07045 | 2.73858 |
| CP        | 84.05% | 84.95% | 90.15% | 91.65% | 92.28% |

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