Non-Gaussianity as a signature of thermal initial condition of inflation

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Abstract

We study non-Gaussianities in the primordial perturbations in single field inflation where there is radiation era prior to inflation. Inflation takes place when the energy density of radiation drops below the value of the potential of a coherent scalar field. We compute the thermal average of the two, three and four point correlation functions of inflaton fluctuations. The three point function is proportional to the slow roll parameters and there is an amplification in $f_{NL}$ by a factor of 65 to 90 due to the contribution of the thermal bath, and we conclude that the bispectrum is in the range of detectability with the 21-cm anisotropy measurements. The four point function on the other hand appears in this case due to the thermal averaging and the fact that thermal averaging of four-point correlation is not the same as the square of the thermal averaging of the two-point function. Due to this fact $\tau_{NL}$ is not proportional to the slow-roll parameters and can be as large as $-42$. The non-Gaussianities in the four point correlation of the order 10 can also be detected by 21-cm background observations. We conclude that a signature of thermal inflatons is a large trispectrum non-Gaussianity compared to the bispectrum non-Gaussianity.

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1. INTRODUCTION

The experimental determination of signature of primordial non-Gaussianity in the CMB spectrum is of great interest as non-Gaussianity can give us insights into the dynamics of inflation models [1, 2]. Interacting fields have non-Gaussian correlations proportional to the coupling strength, which for the inflaton is unmeasurably small [3]. In quasi-de-Sitter space, single field inflation models predict a level of non-Gaussianity proportional to the slow roll parameters [4, 5, 6, 7]. The current constraint on CMB bispectrum from the WMAP 5yr data is $-151 < f_{NL}^{eq} < 253$ (95% CL), $f_{NL}$ being the bispectrum non-linear parameter and a measure of non-Gaussianity. The single field inflation models prediction of $f_{NL} \sim 10^{-2}$ is too small to be detectable in WMAP or the upcoming PLANCK mission where non-Gaussianities at the level of $f_{NL} \sim 5$ can be probed [8]. The most sensitive probe of primordial non-Gaussianities can come from the measurement of anisotropies of the 21-cm background whose bispectrum can probe $f_{NL} < 0.1$ [9]. WMAP constrains the non-Gaussianity from trispectrum at $|\tau_{NL}| \lesssim 10^8$ [11] while PLANCK is expected to reach the sensitivity upto $|\tau_{NL}| \sim 560$ [12]. The anisotropies of the 21-cm background can constrain the trispectrum of primordial perturbations to the level of $\sim 10^8$ [10], which is still too large compared to the predictions of the single field models of $\tau_{NL} \approx (6/5)^2 f_{NL}$ [13, 14].

It was shown by Gangui et al. [15] and more recently in [6, 16] that if the initial state of the inflatons is not the Bunch-Davies vacuum but some excited state then there is an enhancement of the non-Gaussianity from such initial state effects. A natural example of a non-Bunch Davies initial state arises if there is a pre-inflation radiation era prior to inflation [17]. Inflation takes place when the energy density of radiation $\rho_r$ drops below the value of the potential of a coherent scalar field. In such models it is seen that the power-spectrum is enhanced at low $k$ which can be used to put constraint on the comoving temperature at the time of inflation [17]. Inflation scenario with a pre-radiation era have an interesting prediction that the B-mode polarization spectrum is enhanced at low $l$ due the contribution of thermal gravitons [18, 19].

The scenario of thermal initial condition is very general and would be applicable for any model of inflation if there was a pre-inflationary radiation dominated era. The effects of the initial thermal era to be observable either in the CMB anisotropy spectrum or in the non-gaussianities the perturbations entering the horizon today should have left the de-Sitter
horizon at a temperature $T$ not too small compared to $H$ (the Hubble parameter at the time of inflation). If there were a large number of e-foldings prior to the present perturbation modes leaving the inflation horizon then the effect of the pre-inflationary thermal era would be unobservable. In models where the total number of e-foldings are just enough to solve the flatness and horizon problems, there can be a imprint of the spatial curvature at the time of inflation on the power spectrum. This has been studied in [20]. A natural model where inflation commences just as the temperature falls below a critical temperature and is of limited duration is where a fermion pair forms a scalar condensate which acts as the inflaton. Such models have been studied in [21, 22].

In this paper we study non-Gaussianities in the primordial perturbation in single field inflation where there is radiation era prior to inflation. The thermal background of inflaton, gravitons and other fields is decoupled from the actual dynamical evolution of the inflaton unlike in the warm inflation models [23], where there can be large non-Gaussianities [24, 25] due to dissipative coupling between the inflaton and the radiation bath. In this model the temperature of the decoupled radiation bath goes down as $T_{ph} = T/a$ where $T$ is the constant comoving temperature. The thermal distribution functions which depend on the ratio $k_{ph} = k T$ (where $k$ is the comoving wavenumber of the perturbations) retain the same form during inflation.

We calculate the thermal average of the three point correlation function, otherwise known as the bispectrum, of the comoving density perturbations. The non-linear parameter $f_{NL}$ being a measure of non-Gaussianity due to bispectrum is turned out to be a function of the magnitude of the three momenta of comoving perturbations. In order to quantify $f_{NL}$ three distinct configurations of those momenta are analyzed namely the “squeezed” triangle, the “equilateral” triangle and the “folded” triangle and it is observed that the maximum contribution for $f_{NL}$ comes from the “equilateral” configuration. We find that the thermal contribution can result in the enhancement of $f_{NL}$ by factors ranging from $65 − 90$.

We show that due to the presence of the initial temperature the contribution to four point correlation function of the density perturbations comes from different factors other than the mere disconnected diagrams. We evaluate the thermal average of the four point function and calculate the contribution of the thermal initial states to $\tau_{NL}$. We find that in the leading order $\tau_{NL}$ is independent of the slow roll parameters and we find that it can be as large as $\tau_{NL} \sim −42$. 

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We organize the rest of the paper as follows. In Section 2, we compute the thermal average of the two point correlation of the density perturbations. We see that the two point function is enhanced at low $k$ due to the contribution of the thermal inflatons as observed in [17]. In Section 3, we compute the thermal averaged three-point correlations and calculate $f_{NL}$ for various momentum configurations. In Section 4, we compute the thermal contribution to $\tau_{NL}$. We discuss the feasibility of measuring the non-Gaussianity due to thermal inflatons in the Concluding section. We have given a outline of the calculation of the non-Gaussianity parameter $f_{NL}$ at zero temperatures for the single field slow-roll inflation models in the Appendix.

2. THERMAL AVERAGE OF INFLATON POWER SPECTRA

If there was a radiation era prior to inflation one expects a thermal distribution of inflatons to be present which might have decoupled from other fields prior to inflation. It has been shown in [17] that this thermal distribution of inflaton modifies the power spectrum of inflaton fluctuations and the curvature power spectrum will have an additional temperature depended term. In this section we compute the two point correlation of inflaton perturbations taking this thermal distribution of inflatons into consideration.

The Fourier expansion of inflaton fluctuations in de-Sitter space is

$$\delta \phi(x, t) = \int \frac{d^3k}{(2\pi)^3} \left( b_k \varphi_k(t) + b_k^\dagger \varphi_k^*(t) \right) e^{ik \cdot x},$$

where $\varphi_k(t)$ are the mode functions which satisfy the Klein-Gordon equation in Fourier space and $b_k$ and $b_k^\dagger$ are the annihilation and creation operators respectively. In Fourier space the inflaton fluctuations can be written as

$$\delta \phi(k, t) = b_k \varphi_k(t) + b_k^\dagger \varphi_k^*(t).$$

The canonical commutation relation satisfied by these creation and annihilation operators is

$$\left[ b_k, b_{k'}^\dagger \right] = \delta^3(k_1 - k_2),$$

with the vacuum satisfying $b_k |0\rangle = 0$ at zero temperature, which ensures that the vacuum has zero occupation $N_k |0\rangle = 0$ where $N_k \equiv b_k^\dagger b_k$ is the number operator. The power
spectrum of inflaton $P_{\delta\phi}(k)$ is the two-point correlation function of the inflaton fluctuations in momentum space which is defined as

$$P_{\delta\phi}(k) \equiv \frac{k^3}{2\pi^2} \langle \delta\phi(k,t)\delta\phi(k,t) \rangle ,$$  

(4)

where $k \equiv |k|$.

In this case, where there is a radiation era prior to inflation, the inflaton will have a thermal distribution during inflation. Due to this distribution the thermal vacuum $|\Omega\rangle \equiv |n_{k_1}, n_{k_2}, \cdots \rangle$ will now contain real particles yielding

$$N_k|\Omega\rangle = n_k|\Omega\rangle ,$$  

(5)

where $n_k$ is the number of particles with momentum $k$ present in the thermal vacuum. In general, for creation-annihilation operators with different momenta one gets

$$b_{k_1}^\dagger b_{k_2}|\Omega\rangle = \delta^3(k_1 - k_2)n_{k_1}|\Omega\rangle .$$  

(6)

Throughout this paper we will consider non-interacting real scalar field for which the chemical potential $\mu = 0$. For a single inflaton with momentum $k$ the partition function will be

$$z = \sum_{n_k=0}^{\infty} e^{-\beta n_k k} = \frac{1}{1 - e^{-\beta k} } ,$$  

(7)

where $\beta$ is the inverse of the comoving temperature $T$. Due to this thermal distribution of the inflaton fluctuation a thermal statistical average of the two-point correlation function will determine the power spectrum

$$P_{\delta\phi}^{\text{th}}(k) = \frac{k^3}{2\pi^2} \langle \Omega|\delta\phi(k,t)\delta\phi(k,t)|\Omega\rangle_\beta$$

$$= \frac{k^3}{2\pi^2} \sum_{\varepsilon_k} p(\varepsilon_k) \langle \Omega|\delta\phi(k,t)\delta\phi(k,t)|\Omega\rangle .$$  

(8)

Here $p(\varepsilon_k)$ is the probability of the system to be in the state $\varepsilon_k \equiv n_k k$ which is defined as

$$p(\varepsilon_k) \equiv \frac{e^{-\beta n_k k}}{\sum_{n_k} e^{-\beta n_k k}} = \frac{e^{-\beta n_k k}}{z} ,$$  

(9)

where $z$ is given in Eq. (7). However, due to the thermal distribution of the inflaton field the inflaton fluctuations will follow the relations given in Eq. (3) and Eq. (5) which yield

$$\langle \Omega|\delta\phi(k,t)\delta\phi(k,t)|\Omega\rangle = |\varphi_k(t)|^2 \langle \Omega| (1 + 2N_k) |\Omega\rangle$$

$$= |\varphi_k(t)|^2 (1 + 2n_k) .$$  

(10)
Hence the power spectrum given in Eq. (8) will be

\begin{equation}
\mathcal{P}_{\delta \phi}^{th}(k) = \frac{k^3}{2\pi^2} |\varphi_k(t)|^2 \frac{1}{z} \sum_{n_k} e^{-\beta n_k} (1 + 2n_k),
\end{equation}

where $f_B(k) \equiv \frac{1}{e^{\beta k} - 1}$ is the Bose-Einstein distribution. To get the last equality in the above equation the following relation is used [27]

\begin{equation}
\sum_{n=0}^{\infty} n q^n = \frac{q}{(1-q)^2}.
\end{equation}

Now for a light inflaton ($m_\phi \ll H$, $m_\phi$ being the mass of the inflaton and $H$ being the Hubble parameter during inflation) the mode function has the solution [28]

\begin{equation}
|\varphi_k| \simeq \frac{H}{\sqrt{2k^3}} \left( \frac{k}{aH} \right)^{\frac{3}{2} - \nu_\phi},
\end{equation}

where $a$ is the cosmic scale factor and $\nu_\phi \simeq \frac{3}{2} - \frac{m_\phi^2}{H^2}$. In a generic single field inflationary model this mode function solution along with the $k^3$ factor in the power spectrum gives a nearly scale invariant spectra for inflaton fluctuations. But due to the thermal distribution of the inflaton fluctuations, expression for power spectrum in Eq. (11) contains an additional temperature dependent factor of $(1 + 2f_B(k)) = \coth(\beta k/2)$. Thus the thermal power spectrum of inflaton fluctuations is given by

\begin{equation}
\mathcal{P}_{\delta \phi}^{th}(k) = \coth(\beta k/2) \mathcal{P}_{\delta \phi}(k),
\end{equation}

and hence the thermal average of the power spectrum for comoving curvature perturbations defined in Eq. (A5) will be

\begin{equation}
\mathcal{P}_{R}^{th}(k) = \coth(\beta k/2) \mathcal{P}_{R}(k),
\end{equation}

as has been already stated in [17].

In [17] the CMB power spectrum generated using the thermal comoving curvature power spectrum is compared with WMAP data and a constraint on comoving temperature is given as $T < 1.0 \times 10^{-3}$ Mpc$^{-1}$ (with the convention that the present scale factor $a_0 \equiv 1$). Such a bound is also found in [19] from thermal primordial gravitational waves. Since $T = a_i T_{ph}$ where $T_{ph}$ and $a_i$ are the physical temperature and the scale factor when our
current horizon scale crossed the de-Sitter horizon during inflation, this constraint can be rewritten as \( T_0 < 4.2H \). As the comoving wavenumber \( k = a_i H \) one can put a lower bound on \( \beta k \) as

\[
\beta k = \frac{a_i H}{a_i T_{ph}} > 0.238.
\]

This lower bound on \( \beta k \) will be used in following sections to quantify the maximum value of non-Gaussianity in thermal bispectrum and thermal trispectrum.

3. NON-GAUSSIANITY IN BISPECTRUM FROM THERMAL DISTRIBUTION OF INFLATON

The three point correlation function of comoving curvature perturbations \( \mathcal{R} \) or the bispectrum is defined in Eq. (A10) and the non-linear parameter for bispectrum in the case of single field slow-roll model is given in Eq. (A14). In presence of a pre-inflationary radiation era the bispectrum will also receive a modification as in the case of the thermal power spectrum. Hence in this case the three point correlation function of the non-linear curvature perturbation will be

\[
\langle \mathcal{R}_{NL}(k_1) \mathcal{R}_{NL}(k_2) \mathcal{R}_{NL}(k_3) \rangle_\beta = \frac{1}{2} \left( \frac{H}{\dot{\phi}} \right)^2 \frac{\partial}{\partial \phi} \left( \frac{H}{\dot{\phi}} \right) \int \frac{d^3 p}{(2\pi)^3} \times
\]

\[
\left[ \langle \delta \phi_L(p) \delta \phi_L(k_1 - p) \delta \phi_L(k_2) \delta \phi_L(k_3) \rangle_\beta + 2 \text{ perms} \right],
\]

where R.H.S. of the above equation contains the thermal average of four-point correlation functions of the inflaton perturbations.

We will first generalise the case of thermal average of the two-point correlation function to derive the thermal average of the four-point correlation function of scalar perturbations with any four momenta. The thermal average of higher order correlation functions is of the form

\[
\langle \phi_{k_1} \phi_{k_2} \phi_{k_3} \cdots \rangle_\beta = \sum_{\{n_k\}} p(k_1, k_2, k_3, \cdots) \langle \Omega | \phi_{k_1} \phi_{k_2} \phi_{k_3} \cdots | \Omega \rangle,
\]

where the thermal probability of the occupancy of different momenta \( k_i \) and \( \varepsilon \equiv \sum_{n_k} n_k, k_r \) is

\[
p(k_1, k_2, k_3, \cdots) = \frac{\Pi_r e^{-\beta n_{k_r} k_r}}{\Pi_r \sum_{n_k} e^{-\beta n_{k_r} k_r}} = \frac{\Pi_r e^{-\beta n_{k_r} k_r}}{Z}.
\]
Here $Z$ is the grand partition function of massless inflatons with energies $E_{k_r} = \sqrt{k_r^2} = k_r$ which is given as

$$Z = \prod_r \sum_{n_{k_r} = 0}^{\infty} e^{-\beta n_{k_r} k_r} = \prod_r \frac{1}{1 - e^{-\beta k_r}},$$

(20)

where $r$ is the index for different energy levels.

The four-point correlation function of inflaton fluctuations with four different momenta contains six different combinations of two creation and two annihilation operators and thermal average of one of these combinations can be derived as follows:

Let us consider the thermal average of \( \langle b_{-k_1}^\dagger b_{k_2} b_{-k_3}^\dagger b_{k_4} \rangle \), which yields

$$\langle b_{-k_1}^\dagger b_{k_2} b_{-k_3}^\dagger b_{k_4} \rangle_\beta = \sum_{\epsilon} p(k_1, k_2, k_3, k_4) \langle \Omega | b_{-k_1}^\dagger b_{k_2} b_{-k_3}^\dagger b_{k_4} | \Omega \rangle \delta^3(k_1 + k_2) \delta^3(k_3 + k_4) \frac{1}{Z} \sum_{n_{k_1}} \sum_{n_{k_2}} e^{-\beta(n_{k_1} k_1 + n_{k_3} k_3)} [n_{k_1} n_{k_3}],$$

(21)

where $Z = \left( \frac{1}{1 - e^{-\beta k_1}} \right) \left( \frac{1}{1 - e^{-\beta k_3}} \right)$. The summations in the above equation yields

$$\langle b_{-k_1}^\dagger b_{k_2} b_{-k_3}^\dagger b_{k_4} \rangle_\beta = \delta^3(k_1 + k_2) \delta^3(k_3 + k_4) [f_B(k_1) f_B(k_3)],$$

(22)

where the identity stated in Eq. (12) is used. Similarly the thermal average of other combinations of the two creation and two annihilation operators will be

$$\langle b_{k_1} b_{k_2} b_{-k_3}^\dagger b_{k_4} \rangle_\beta = \delta^3(k_1 + k_4) \delta^3(k_2 + k_3) [1 + f_B(k_1)] + \delta^3(k_1 + k_3) \delta^3(k_2 + k_4) [1 + f_B(k_1) + f_B(k_2) + f_B(k_1) f_B(k_2)],$$

(23)

$$\langle b_{k_1} b_{-k_2}^\dagger b_{k_3} b_{k_4} \rangle_\beta = \delta^3(k_1 + k_2) \delta^3(k_3 + k_4) [1 + f_B(k_1) + f_B(k_3) + f_B(k_1) f_B(k_3)],$$

(24)

$$\langle b_{k_1} b_{-k_2}^\dagger b_{-k_3}^\dagger b_{k_4} \rangle_\beta = \delta^3(k_1 + k_2) \delta^3(k_3 + k_4) [f_B(k_3) + f_B(k_1) f_B(k_3)],$$

(25)

$$\langle b_{-k_1}^\dagger b_{k_2} b_{k_3} b_{k_4} \rangle_\beta = \delta^3(k_1 + k_2) \delta^3(k_3 + k_4) [f_B(k_1) + f_B(k_1) f_B(k_3)],$$

(26)

$$\langle b_{-k_1}^\dagger b_{-k_2}^\dagger b_{k_3} b_{k_4} \rangle_\beta = -\delta^3(k_1 + k_4) \delta^3(k_2 + k_3) f_B(k_1) + \delta^3(k_1 + k_3) \delta^3(k_2 + k_4) [f_B(k_1) f_B(k_3)].$$

(27)

Hence the thermal average of a general four-point correlation function with four different momenta will be

$$\langle \delta \phi(k_1, t) \delta \phi(k_2, t) \delta \phi(k_3, t) \delta \phi(k_4, t) \rangle_\beta = |\varphi_{k_1}(t)|^2 |\varphi_{k_2}(t)|^2 \left[ \delta^3(k_1 + k_4) \delta^3(k_2 + k_3) + \delta^3(k_1 + k_3) \delta^3(k_2 + k_4) \right] \{ 1 + f_B(k_1) + f_B(k_2) + 2f_B(k_1) f_B(k_2) \} + |\varphi_{k_1}(t)|^2 |\varphi_{k_3}(t)|^2 \times \left[ \delta^3(k_1 + k_2) \delta^3(k_3 + k_4) \{ 1 + 2f_B(k_1) + 2f_B(k_3) + 4f_B(k_1) f_B(k_3) \} \right].$$

(28)
With this general result one can calculate the thermal average of the three-point correlation function of the comoving curvature perturbations using Eq. (17) as

\[
\langle R_{NL}(k_1)R_{NL}(k_2)R_{NL}(k_3) \rangle_\beta \simeq (2\pi)^{-\frac{3}{2}} \delta^3(k_1 + k_2 + k_3)(2m_{Pl}^2\epsilon) \frac{\partial}{\partial \phi} \left( \frac{H}{\phi} \right) \times \left[ \frac{P_R(k_1)}{k_1^3} \frac{P_R(k_2)}{k_2^3} \left( 1 + \frac{1}{2} f_B(k_1) + \frac{1}{2} f_B(k_2) + f_B(k_1)f_B(k_2) \right) \right. \\
+ \frac{P_R(k_2)}{k_2^3} \frac{P_R(k_3)}{k_3^3} \left( 1 + \frac{1}{2} f_B(k_2) + \frac{1}{2} f_B(k_3) + f_B(k_2)f_B(k_3) \right) \\
+ \frac{P_R(k_3)}{k_3^3} \frac{P_R(k_1)}{k_1^3} \left( 1 + \frac{3}{2} f_B(k_3) + \frac{3}{2} f_B(k_1) + 3f_B(k_3)f_B(k_1) \right) \right],
\]

where \( P_R(k) \) is defined in Eq. (A11). The three momenta form a triangle due to the presence of the delta function. In general three different triangle configurations are considered to determine the non-Gaussian effect. The non-linear parameter \( f_{NL} \) for these three momenta configurations are discussed below:

- **Squeezed triangle case**: For a “squeezed” triangle the configuration suggests \( |k_1| \approx |k_2| \approx k \gg |k_3| \). In this configuration the \( f_{NL} \) will be

\[
f_{NL}^{th} = \frac{5}{6}(\delta - \epsilon) \left( 2 + 2f_B(k_3) \coth \left( \frac{\beta k_3}{2} \right) \right).
\]

At low temperature \( \beta \rightarrow \infty \) and \( f_B(k_3) \rightarrow 0 \), yielding the same contribution to the \( f_{NL} \) for super-cool inflation. The minimum value \( k_3 \) can obtain when the corresponding wavelength is of Hubble size while crossing the horizon such that \( \lambda_3 = \frac{1}{k_3} \sim H^{-1} \) which implies \( \beta k_3 \sim 0.238 \). Hence it yields

\[
f_{NL}^{th} = \frac{5}{6}(\delta - \epsilon) \times 2 \left( 1 + 3.72 \coth \left( \frac{\beta k_3}{2} \right) \right) \\
= f_{NL} \times 2 \left( 1 + 3.72 \coth \left( \frac{\beta k_3}{2} \right) \right).
\]

A lower bound on \( \beta k \) can be given from thermal power spectrum which is given in Eq. (16) and for this constraint \( f_{NL} \) will be maximum yielding \( f_{NL}^{th} = 64.82f_{NL} \sim 0.65 \).

- **Equilateral triangle case**: For a “equilateral” triangle we have \( |k_1| = |k_2| = |k_3| = k \)
and in this case the $f_{NL}$ will be

$$f_{NL}^{th} = \frac{5}{6} (\delta - \epsilon) \left( 3 + \frac{5}{4 \sinh^2 \left( \frac{\beta k}{2} \right)} \right)$$

$$= f_{NL} \left( 3 + \frac{5}{4 \sinh^2 \left( \frac{\beta k}{2} \right)} \right).$$  \hspace{1cm} (32)

This implies that for the modes corresponding to our present horizon $\beta k > 0.238$ and the $f_{NL}^{th} = 90.85 f_{NL} \sim 0.9$.

- **Folded triangle case**: For “flattened” isosceles triangle or the “folded” triangle case we have $|k_1| = |k_3| = \frac{1}{2} |k_2| = k$ and in this case the $f_{NL}$ will be

$$f_{NL}^{th} = \frac{5}{6} (\delta - \epsilon) \left( 3 + \frac{1}{\sinh^2 \left( \frac{\beta k}{2} \right)} \right)$$

$$= f_{NL} \left( 3 + \frac{1}{\sinh^2 \left( \frac{\beta k}{2} \right)} \right).$$  \hspace{1cm} (33)

In this configuration the non-linearity will be $f_{NL}^{th} = 73.28 f_{NL} \sim 0.73$ at horizon crossing for the modes corresponding to our current horizon.

![FIG. 1: $f_{NL}^{th} / f_{NL}$ as a function of $\beta k$ for different triangle configurations of the three momenta.](image)

In Fig. 11 the thermal enhancement factor $f_{NL}^{th} / f_{NL}$ is plotted as a function of $\beta k$ for three different triangle configurations of the three momenta. From the above discussion it is seen
that the maximum contribution for $f_{NL}$ comes from the “equilateral” configuration, though the contribution from the other two configurations are of the same order. Non-Gaussianity in all these three cases may be measurable by the 21-cm background radiation observations.

4. NON-GAUSSIANITY IN TRISPECTRUM DUE TO THERMAL DISTRIBUTION OF INFLATON

In a generic slow-roll single field model the trispectrum $T(k_1, k_2, k_3, k_4)$ is defined as the Fourier counterpart of the connected part of four point correlation function of comoving curvature perturbation

$$\langle R(k_1)R(k_2)R(k_3)R(k_4) \rangle_c = \tau_{NL} \delta^3(k_1 + k_2 + k_3 + k_4) T(k_1, k_2, k_3, k_4), \quad (34)$$

where $\tau_{NL}$ is the non-linear parameter for trispectrum and is a measure of non-Gaussianity. The connected part in the above equation is defined as

$$\langle R(k_1)R(k_2)R(k_3)R(k_4) \rangle_c = \langle R(k_1)R(k_2)R(k_3)R(k_4) \rangle - \langle (R_L(k_1)R_L(k_2)) (R_L(k_3)R_L(k_4)) + 2 \text{ perm} \rangle. \quad (35)$$

The comoving curvature perturbation $R$ has been expanded non-linearly up to $O(\delta \phi^2_L)$. Hence the term $\langle R_L(k_1)R_L(k_2)R_L(k_3)R_{NL}(k_4) \rangle$ vanishes as it turns out to be an expectation value of odd number of Gaussian variables $\delta \phi_L$. If one expands $R$ up to $O(\delta \phi^3_L)$ then a term like $\langle R_L(k_1)R_L(k_2)R_L(k_3)R_{NL}(k_4) \rangle$ will survive and the non-linear parameter $\tau_{NL}$ will be $\tau_{NL} = \left(\frac{4}{5} f_{NL}\right)^2$ i.e. $O(\epsilon^2) [14]$. The trispectrum $T(k_1, k_2, k_3, k_4)$ in this case will be proportional to product of three power spectrums and the four momenta form quadrilateral configuration due to the delta function in Eq. (34).

But in this model of slow-roll inflation with a radiation era prior to inflation, the analysis for trispectrum turns out to be quite different. In presence of a pre-inflationary radiation era the four point correlation function which contributes to the non-Gaussianity will be thermal averaged as in the case of power spectrum and bispectrum. It is worth to point out that due to thermal averaging the four point function is not just the square of the two-point function as that would have been the case at zero temperature. So in this case, by connected part of the four-point function defined in Eq. (35) we will simply mean the excess of the thermal
averaged of four-point function than the square of its two-point Gaussian part and will now define the non-linear parameter $\tau_{NL}$ in the following way

$$
\langle R_L(k_1)R_L(k_2)R_L(k_3)R_L(k_4)\rangle_c \equiv \langle R_L(k_1)R_L(k_2)R_L(k_3)R_L(k_4)\rangle_{\beta} - \left(\langle R_L(k_1)R_L(k_2)\rangle_{\beta} \langle R_L(k_3)R_L(k_4)\rangle_{\beta} + 2 \text{ perm.}\right)
$$

$$
= \tau_{NL} \left[ \frac{P_R(k_1)P_R(k_2)}{k_1^3} \frac{P_R(k_3)P_R(k_4)}{k_2^3} \delta^3(k_1 + k_3)\delta^3(k_2 + k_4)
\right. + 2 \text{ perm.} \right].
$$

(36)

Hence in this case $\tau_{NL}$ will not depend upon the slow-roll parameters. The thermal average of the four-point correlation function of inflaton fluctuation has been calculated in the last section in Eq. (28). Using this equation the thermal average of the four-point correlation of curvature perturbation can be derived as

$$
\langle R_L(k_1)R_L(k_2)R_L(k_3)R_L(k_4)\rangle_{\beta} = \frac{P_R(k_1)P_R(k_2)}{k_1^3} \frac{P_R(k_3)P_R(k_4)}{k_2^3} \left[ \delta^3(k_1 + k_3)\delta^3(k_2 + k_4)
\right.
$$

$$
+ \delta^3(k_1 + k_3)\delta^3(k_2 + k_4) \left\{ 1 + f_B(k_1) + f_B(k_2) + 2f_B(k_1)f_B(k_2) \right\} + \frac{P_R(k_1)P_R(k_3)}{k_1^3} \frac{P_R(k_4)}{k_2^3}
$$

$$
\times \left[ \delta^3(k_1 + k_2)\delta^3(k_3 + k_4) \left\{ 1 + 2f_B(k_1) + 2f_B(k_2) + 4f_B(k_1)f_B(k_2) \right\} \right],
$$

(37)

and the thermal average of two-point function can be given in terms of the power spectrum as

$$
\langle R_L(k_1)R_L(k_2)\rangle_{\beta} = \frac{P_R(k_1)}{k_1^3} \left( 1 + 2f_B(k_1) \right) \delta^3(k_1 + k_2).
$$

(38)

Hence the connected part will be

$$
\langle R_L(k_1)R_L(k_2)R_L(k_3)R_L(k_4)\rangle_c = -\frac{P_R(k_1)P_R(k_2)}{k_1^3} \frac{P_R(k_3)P_R(k_4)}{k_2^3} \left[ \delta^3(k_1 + k_3)\delta^3(k_2 + k_4) \left\{ f_B(k_1) + f_B(k_2) + 2f_B(k_1)f_B(k_2) \right\} + 2\delta^3(k_1 + k_4)\delta^3(k_2 + k_3) \left\{ f_B(k_1) + f_B(k_2) + f_B(k_1)f_B(k_2) \right\} \right].
$$

(39)

The four momenta in this case will not form a quadrilateral as in other trispectrum cases. But due to the presence of two delta functions on the R.H.S. of the above equation the non-linear parameter $\tau_{NL}$ can be calculated in the following two cases:

1. $k_1 = -k_3$, $k_2 = -k_4$ and $k_i = k$ ($i = 1, 2, 3, 4$):

$$
\tau_{NL}^{th} = -\frac{1}{\cosh(\beta k) - 1}.
$$

(40)
The maximum observable value of $|\tau_{NL}^{th}|$ can be obtained using the constraint on the comoving temperature as $\beta k > 0.238$. Hence for $\beta k \sim 0.238$ one finds that $\tau_{NL}^{th} \sim -35.14$.

2. $k_1 = -k_4$, $k_2 = -k_3$ and $k_i = k$ ($i = 1, 2, 3, 4$):

$$\tau_{NL}^{th} = -2 \frac{1 - 2 e^{\beta k}}{(e^{\beta k} - 1)^2}.$$  \hspace{1cm} (41)

The maximum value of $\tau_{NL}^{th}$ for this case will be $\tau_{NL}^{th} \sim -42.58$.

![Figure 2](image.png)

**FIG. 2:** Plot of $\tau_{NL}^{th}$ for two different momenta configurations as a function of $\beta k$

In Fig. (2) we have plotted $\tau_{NL}^{th}$ as a function of $\beta k$. We find that the maximum value of non-Gaussianity comes from the configuration when $k_1 = -k_4$, $k_2 = -k_3$ and $k_i = k$ ($i = 1, 2, 3, 4$) which is $\sim -42$. We do not compare this contribution to non-Gaussianity due to thermal initial states with the zero temperature case as there is no contribution from the later at this order and the leading order $\tau_{NL}$ in zero temperature is $O(\epsilon^2)$.

5. **CONCLUSION**

We studied the effect of a decoupled thermal spectrum of inflatons (which exist in the scenario where the inflation is preceded by a prior thermal era) on the non-Gaussianity of the
primordial perturbation. We found that thermal inflatons can enhance the bispectrum non-Gaussianity parameter \( f_{NL} \) by a factors of \((65-90)\) depending upon the momentum configuration. The zero temperature non-Gaussianity parameter \( f_{NL} \) in single field inflation models is proportional to the slow roll parameters and is expected to be of order \( \sim 10^{-2} \). Therefore the observed value of \( f_{NL} \) in thermal history models will be of \( \sim 1 \). This is too small to be measured by WMAP or even the forthcoming PLANCK experiment. Measurements of anisotropies in the Hydrogen 21-cm radiation background can detect non-Gaussianities as low as \( f_{NL} \sim 0.1 \) \[9\], and this may be the ideal experiment in which non-Gaussianities with a thermal origin can be observed. The 21-cm observations may also be able to measure the non-Gaussianity in the trispectrum \( \tau_{NL} \sim \mathcal{O}(10) \) and for which the prediction from thermal history inflation scenarios is \( 0 > \tau_{NL} > -43 \). We conclude that a signature of thermal inflaton background at the time of inflation is a large trispectrum non-Gaussianity compared to the bispectrum non-Gaussianity.

**APPENDIX A: NON-GAUSSIANITY IN A SINGLE-FIELD SLOW-ROLL INFLATION MODEL**

In a single field slow-roll model Non-Gaussianity appears generically once the inflaton field has self-interactions such as \( V(\phi) \sim \phi^3 \) or \( V(\phi) \sim \phi^4 \). But the Non-Gaussianity due to these self-interactions is very small where the bispectrum Non-Gaussian parameter \( f_{NL} \sim \mathcal{O}(\epsilon^2) \). Larger contribution to Non-Gaussianity in such models come from non-linear curvature perturbations and in this cases \( f_{NL} \sim \mathcal{O}(\epsilon, \eta) \) \[29\]. Here we will briefly discuss how non-linearity in comoving curvature perturbation gives rise to Non-Gaussianity of the order of slow-roll parameters \( \epsilon \) and \( \eta \).

The quantum fluctuations in inflaton field generates fluctuation in the metric which is coupled to it through Einstein’s equation. The perturbed FRW metric has the form (considering only scalar perturbations)

\[
\tilde{g}_{\mu\nu} = a^2(\eta) \begin{pmatrix}
1 + 2A & 0 \\
0 & -(1 - 2\psi)\delta_{ij}
\end{pmatrix},
\]

(A1)

where the quantity \( \psi \) is known as the spatial curvature perturbation, \( \eta \) is the conformal time and \( a(\eta) \) is the cosmic scale factor. The gauge invariant quantity formed out of this spatial
curvature perturbation is known as the comoving curvature perturbation and defined as

\[ \mathcal{R}(t, \mathbf{x}) = \psi(t, \mathbf{x}) + \frac{H}{\dot{\phi}} \delta \phi(t, \mathbf{x}). \]  \hspace{1cm} (A2)

These perturbations are conserved on super-horizon scales throughout the evolution. Hence during inflation in the spatially flat gauge this reduces to

\[ \mathcal{R}(t, \mathbf{x}) = \frac{H}{\dot{\phi}} \delta \phi(t, \mathbf{x}), \]  \hspace{1cm} (A3)

and after inflation when \( \delta \phi \sim 0 \) this represents the gravitational potential on comoving hypersurfaces

\[ \mathcal{R}(t, \mathbf{x}) = \psi(t, \mathbf{x}). \]  \hspace{1cm} (A4)

The CMB anisotropy spectrum is determined by the power spectrum of this comoving curvature perturbation which is related to the power spectrum of scalar perturbation as

\[ \mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2} \langle \mathcal{R}(k) \mathcal{R}(k) \rangle = \frac{1}{2m_{Pl}^2 \epsilon} P_\delta(k), \]  \hspace{1cm} (A5)

where the slow-roll parameter \( \epsilon \equiv 4\pi G \frac{\dot{\phi}^2}{H^2} \), \( m_{Pl} \equiv \frac{1}{\sqrt{8\pi G}} \) is the reduced Planck mass and \( G \) being the Newton’s Gravitational constant.

Presuming that the inflaton fluctuations \( \delta \phi \) are initially Gaussian, the comoving curvature perturbations \( \mathcal{R} \) given in Eq. (A3) also obeys Gaussian statistics in the linear order

\[ \mathcal{R}_L(t, \mathbf{x}) = \frac{H}{\dot{\phi}} \delta \phi_L(t, \mathbf{x}), \]  \hspace{1cm} (A6)

where \( \mathcal{R}_L(t, \mathbf{x}) \) and can be expanded in Fourier space as

\[ \mathcal{R}_L(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^\frac{3}{2}} e^{i\mathbf{k} \cdot \mathbf{x}} \mathcal{R}_L(t, \mathbf{k}). \]  \hspace{1cm} (A7)

Being constant in time outside the horizon these comoving curvature fluctuations after entering the horizon in later times produces curvature perturbations which are Gaussian in nature.

In the non-linear limit one observes that \( \frac{H}{\dot{\phi}} \equiv -\frac{1}{m_{Pl}^2} \frac{V(\phi)}{V'(\phi)} \) is a function of \( \phi \) and hence

\[ \mathcal{R}_{NL}(t, \mathbf{x}) = \frac{H}{\dot{\phi}} \delta \phi_L(t, \mathbf{x}) + \frac{1}{2 \partial \phi} \left( \frac{H}{\dot{\phi}} \right) \delta \phi^2_L(t, \mathbf{x}) + \mathcal{O}(\delta \phi^3). \]  \hspace{1cm} (A8)
Therefore in the Fourier space one gets

$$\mathcal{R}_{NL}(t, k) = \frac{H}{\dot{\phi}} \delta \phi_L(t, k) + \frac{1}{2} \frac{\partial}{\partial \phi} \left( \frac{H}{\dot{\phi}} \right) \int \frac{d^3p}{(2\pi)^3} \delta \phi_L(t, p) \delta \phi_L(t, k - p),$$  \hspace{1cm} (A9)

where $\delta \phi_L(t, k)$ has the same form as given in Eq. (2).

The non-linear parameter $f_{NL}$ for bispectrum $B(k_1, k_2, k_3)$ or the three-point correlation function of the comoving curvature perturbation is defined as

$$\langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle = (2\pi)^{-\frac{3}{2}} \delta^3(k_1 + k_2 + k_3) \frac{6}{5} f_{NL} \left( \frac{P_R(k_1)}{k_1^3} \frac{P_R(k_2)}{k_2^3} + 2 \text{ perms.} \right)$$

$$= (2\pi)^{-\frac{3}{2}} \delta^3(k_1 + k_2 + k_3) \frac{6}{5} f_{NL} B(k_1, k_2, k_3),$$  \hspace{1cm} (A10)

where we have used the definition

$$P_R(k) = (2\pi)^2 \mathcal{P}_R(k),$$  \hspace{1cm} (A11)

and the normalization $(2\pi)^{-\frac{3}{2}}$ has been chosen accordingly. Hence using Eq. (A9) one can compute the bispectrum in this case as follows

$$\langle \mathcal{R}_{NL}(k_1) \mathcal{R}_{NL}(k_2) \mathcal{R}_{NL}(k_3) \rangle \simeq (2\pi)^{-\frac{3}{2}} \delta^3(k_1 + k_2 + k_3) (2m_{Pl}^2 \epsilon) \frac{\partial}{\partial \phi} \left( \frac{H}{\dot{\phi}} \right)$$

$$\times \left( \frac{P_R(k_1)}{k_1^3} \frac{P_R(k_2)}{k_2^3} + 2 \text{ perms.} \right).$$  \hspace{1cm} (A12)

Comparing the above two equations the non-linearity parameters in this case will be

$$f_{NL} = \frac{5}{3} \frac{m_{Pl}^2 \epsilon}{m_{Pl}^2} \frac{\partial}{\partial \phi} \left( \frac{H}{\dot{\phi}} \right)$$

$$= - \frac{5 \epsilon}{3} \frac{\partial}{\partial \phi} \left( \frac{V(\phi)}{V'(\phi)} \right).$$  \hspace{1cm} (A13)

The non-linear parameter $f_{NL}$ given in Eq. (A13) can be fully expressed in terms of the slow-roll parameters $\epsilon \equiv \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2$, $\eta \equiv \frac{1}{8\pi G} \left( \frac{V''}{V} \right)$ and $\delta \equiv \eta - \epsilon$ as

$$f_{NL} = \frac{5}{6} (\delta - \epsilon).$$  \hspace{1cm} (A14)

The form of the $f_{NL}$ derived above is same as in [14]. Here $f_{NL}$ being proportional to the slow-roll parameters is too small to be detected by the ongoing experiments.

The non-linearity parameter can also be expressed in terms of the potential $V(\phi)$ using Eq. (A13) as

$$f_{NL} = - \frac{5}{6} m_{Pl}^2 \left( \frac{V'}{V} \right) \frac{\partial}{\partial \phi} \left( \frac{V(\phi)}{V'(\phi)} \right).$$  \hspace{1cm} (A15)
This equation is useful in the case where the form of the potential is known. Such as, for a power-law potential where \( V(\phi) \sim \phi^n \) the non-linearity parameter will be

\[
f_{NL} = - \left( \frac{5}{6} \right) n \frac{m_{Pl}^2}{\phi^2}, \tag{A16}
\]

which is same as the one given in [29].

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