A Uniqueness Theorem for Constraint Quantization

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Abstract

This work addresses certain ambiguities in the Dirac approach to constrained systems. Specifically, we investigate the space of so-called “rigging maps” associated with Refined Algebraic Quantization, a particular realization of the Dirac scheme. Our main result is to provide a condition under which the rigging map is unique, in which case we also show that it is given by group averaging techniques. Our results comprise all cases where the gauge group is a finite-dimensional Lie group.

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I. INTRODUCTION

Our goal here is to develop and cement the mathematical structure of the Dirac Quantization procedure [1] for constrained systems. This ‘procedure’ involves introducing the constraints as operators on some space and then taking only those states which are annihilated by the constraints to be ‘physical’. These physical states should then be made into a (physical) Hilbert space. Before discussing the details, we remind the reader that this procedure is expected to be an essential part of any attempt to quantize gravity using canonical methods [2–4], and in particular has been of great interest in the proposed ‘loop representation’ (see, e.g. [5]) of quantum gravity. Dirac style methods are also commonly used in quantum cosmology, whether inspired by Einstein-Hilbert gravity (e.g., [6]–[9]) or string/M-theory (e.g., [10,11]).

The Dirac proposal involves a large number of ambiguities. As a result, one must make certain choices in implementing the proposal, and those may be roughly divided into two categories: Category I has to do with setting the stage for solving the constraints. It includes, for example, both the factor ordering of the constraints and the choice of the space on which they should act. Category II has to do with implementing the rest of the scheme. This includes the questions of in just what sense the constraints should be ‘solved’, how to make the solutions into a Hilbert space, and how observables should then act on this space.

Choices in category I are arguably shared with familiar non-relativistic quantum mechanics, while category II choices are particular to Dirac’s scheme for constrained systems. Thus, while one suspects that category I questions can be answered only in the context of particular physical systems, category II questions could in principle have quite general solutions. The discovery of such solutions could be of great help in the study of quantum gravity and/or cosmology. Some efforts to address or at least to structure the discussion of category II issues were made under the names of geometric quantization [12–15] (although this approach addresses certain category I issues as well), Klein-Gordon methods [16], path integral methods [17,18], coherent state quantization [19,20], C*-algebra methods [21,22], algebraic quantization [2], and refined algebraic quantization (see, for example [23,24] as well as the closely related work of [25,26,27]).

Each of these uses a different strategy to implement the Dirac procedure, and, unfortunately, uses a different language to discuss the details. However, the Refined Algebraic Quantization (RAQ) scheme has been shown to have a certain generality [24] and, in this sense, to include the other approaches. This provides some motivation for our study of RAQ in this work. However, it should be noted that RAQ is not at this point compatible with systems of constraints involving structure functions instead of structure constants. This means that it cannot even in principle be applied to full 3+1 gravity. As will be discussed

\footnote{Note that in this paper all elements of the algebra of observables will be called observables, and not just its self-adjoint elements.}
further in section IV, we hope that our treatment of non-unimodular groups represents a step toward removing this restriction.

Like the other approaches, RAQ codifies the Dirac scheme and formalizes the choices that must be made for its completion. RAQ first requires that the system be quantized without imposing the gauge symmetry; that is, that a Hilbert space representation of the gauge-dependent operator algebra be chosen. In particular, the constraints are to be represented as Hermitian operators. This first Hilbert space is called the auxiliary Hilbert space ($H_{aux}$) to distinguish it from the physical Hilbert space to be constructed. Choosing the constraint operators and $H_{aux}$ clearly falls under category I.

The next choice made by RAQ is that of a dense subspace $\Phi \subset H_{aux}$ which is mapped into itself by the constraints. It is this choice which sets the final arena in which the constraints are to be solved, as RAQ seeks solutions of the constraints in the algebraic dual space $\Phi^*$ of linear functionals on $\Phi$. To complete RAQ, one must then find a map $\eta$ from $\Phi$ to $\Phi^*$ with certain special properties. This map is called the ‘rigging’ map in vague analogy to the theory of rigged Hilbert spaces. In RAQ, it simultaneously solves the constraints (i.e., it’s image contains only solutions) and defines the physical Hilbert space $H_{phys}$. The choice of $\Phi$ and $\eta$ fall into Category II. In the work below, we will address the choice of the rigging map $\eta$.

One popular idea for constructing $\eta$ is to use the group averaging procedure (discussed below and in [26,23,28]). We find that, when the group averaging integral converges sufficiently rapidly, the rigging map is unique (and given by group averaging). We hope to address the choice of $\Phi$ in future work.

The uniqueness of the group averaging map is our main result. The surprising feature is that this result holds subject only to the restriction that the gauge group be a locally compact Lie group. In particular, it applies to arbitrary discrete groups (which we consider as zero-dimensional Lie groups) and it includes the non-unimodular groups as well as non-amenable groups (see [29] for a definition) and groups that are ‘wild’ in the sense of [29] (non-type I, i.e., type-II and type-III groups in the terminology of [30] and [31]).

Arriving at a general result of this nature entails a significant change of perspective from certain earlier works [23,27,28,32] on RAQ, which emphasized the connections of RAQ with spectral theory and the decomposition of group representations as direct integrals of Hilbert spaces. Roughly speaking, the idea there was to decompose the unitary representation of the gauge group on the auxiliary Hilbert space into a direct integral, and then select as the physical Hilbert space the integrand which carries the trivial representation. This analogy is a powerful tool for dealing with Abelian groups (and one dimensional groups in particular), but leads one to expect certain (nonexistent) difficulties with non-amenable and wild groups.

First of all, it is clear that mere ‘inspection’ of a concretely given direct integral repre-

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2Thus, the group must be finite dimensional and our results do not apply directly to field theories. Extensions to the infinite dimensional case may be possible using path integrals but, at the level of rigor used here, the infinite dimensional case is beyond the scope of this paper.
sentation reveals the constituents only up to sets of measure zero. Hence a proper definition of ‘support’ or ‘containment’ of a representation needs to be given. This is achieved by the notion of *weak containment*, which, in brief, works as follows: Consider the space $C(G)$ of continuous complex-valued functions on $G$ in the topology of uniform convergence on compact sets. Matrix elements of representations of $G$ are now considered as elements in $C(G)$. Now, a representation $\rho$ is weakly contained in a family $\mathcal{S}$ of representations if the matrix elements of $\rho$ can be approximated by those of representations in $\mathcal{S}$. This defines a topology on the space of representations by defining the closure of a set as the set of weakly contained ones. For more information we refer to [34] and also chapter 18.1 of [31].

In the case of wild (non-type I) groups the direct integral decomposition into irreducibles of a representation is far more ambiguous over and above those trivial measure-theoretic ambiguities mentioned above. In these cases it is therefore unclear how to interpret the appearance or absence of certain irreducible representations, and in particular the trivial representation, in the various direct integral decompositions.

The non-amenable case is characterized by the fact that the trivial one-dimensional representation is not weakly contained in the regular representation. Recall that in the regular representation the gauge group acts on the space $L^2(G,dLg)$ of square integrable functions with respect to the left-invariant Haar measure $dLg$ on $G$. Now, non-amenable gauge groups suggest problems of the following kind: Consider a mechanical system with configuration space $G$ and left-translations of $G$ as gauge group. All momenta are then constrained to vanish and the classical reduced phase-space consists of a single point. Accordingly, one expects a single gauge invariant quantum state to survive the quantization procedure. However, for a non-amenable group the trivial representation is not weakly contained, which suggests that one ends up with no states at all. Similar difficulties have been encountered for the action of the large diffeomorphisms on the state space of 2+1 gravity [35,36] on $R \times T^2$.

However, it turns out that refined algebraic quantization is not in fact directly related to spectral analysis and weak containment. Instead, it is more closely tied to another (coarser) topology on the space of group representations. This new topology is related to a new notion of containment, which we call “ultraweak containment,” in the same way as the old topology was related to the notion of weak containment. As will be discussed in section IV below, in order for group averaging to produce a nontrivial result it is necessary that a particular representation be ultraweakly contained in the representation of the gauge group on the

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3If $G$ is locally compact and Hausdorff, as it is here since we consider only finite-dimensional Lie groups, this topology is complete and equivalent to the compact-open topology (see [33], chapter 7).

4A group is amenable iff all its irreducible representations are weakly contained in the regular representation. It turns out that this is equivalent to the weak containment of just the trivial representation.

5The trivial one, for unimodular groups.
auxiliary Hilbert space. Under a natural restriction, the same is true of RAQ. It would, of course, be of interest to show (perhaps, along the lines of [24]), that ultraweak containment of the trivial representation is a sufficient condition for the success of RAQ. However, no results of that sort are yet available. It turns out that, even for non-amenable groups, the trivial representation is always ultraweakly contained in the regular representation. In addition, there is in general no direct integral decomposition of representations in terms of the irreducibles that they contain ultraweakly, so that there is no distinction between wild and ‘tame’ groups in this context.

Below, we briefly review Refined Algebraic Quantization and the method of group averaging in section II. We then proceed in section III to show that, when group averaging converges in a sufficiently strong sense, it gives the unique implementation of Refined Algebraic Quantization (and therefore the unique implementation of the Dirac procedure in the sense of [24]). A key step in deriving this result for the non-unimodular case is to realize (following [13] and [15]) that, for such groups, the sense in which the constraints should be solved is somewhat different from what one might naively expect. However, as this represents a diversion from the main thrust of the paper and as it will be of less interest to some readers, our discussion of this point is relegated to two appendices (A and B). In the context of geometric quantization, this result was argued in [13,15] by comparison with reduced phase space methods and in [4] by using the fact that a gauge-system with non-unimodular gauge-group may always be suitably enlarged to a system with unimodular gauge group but isomorphic reduced phase space. This system can then be treated by known techniques including in particular the Dirac condition in its familiar form. But if projected to the original, non-unimodular, gauge system this then turns out to be equivalent to an alteration of the Dirac condition by an additional term. References [13] and [14] also present some simple models of gauge-systems with non-unimodular gauge groups where the failure of the naive Dirac condition is studied explicitly. In Appendix B we summarize the unimodularisation technique just mentioned and also discuss its group-averaging version. Section IV contains a discussion of the results and introduces the topology of ultraweak containment.

II. PRELIMINARIES

In this section we briefly review both refined algebraic quantization and group averaging. More detailed treatments can be found in [23,24,28] and other references. In particular we refer the reader to [24] for a thorough motivation of RAQ and some results on its generality.

Refined Algebraic Quantization is a framework for the implementation of the Dirac constraint quantization procedure which begins by first considering an ‘unconstrained’ quantum system in which even gauge dependent quantum operators act on an auxiliary Hilbert space $H_{aux}$. On this auxiliary space are defined self-adjoint constraint operators $C_i$. The com-
mutator algebra of these quantum constraints is assumed to close and form a Lie algebra \[.\] Exponentiation of the operators will then yield a unitary representation of the corresponding Lie group. We will choose to formulate refined algebraic quantization entirely in terms of this unitary representation \(U\) in order to avoid dealing with unbounded operators.

As with any version of the Dirac procedure, physical states \(|\psi\rangle_{\text{phys}}\) must be identified which in some sense solve the quantum constraints \(C_i\). Physically the same requirement is given\(^7\) by the statement that the unitary operators \(U(g)\) should act trivially on the physical states for any \(g\) in the gauge group: \(U(g)|\psi\rangle_{\text{phys}} = |\psi\rangle_{\text{phys}}\). Now, as the discrete spectrum of \(U(g)\) need not contain one, the Hilbert space \(\mathcal{H}_{\text{aux}}\) will in general not contain any such solutions. However, by choosing some subspace \(\Phi \subset \mathcal{H}_{\text{aux}}\) of ‘test states’, we can seek solutions in the algebraic dual \(\Phi^*\) of \(\Phi\). This means that we take \(\Phi^*\) to be the space of all complex-valued linear functions on \(\Phi\) and topologize this space by the topology of pointwise convergence, i.e., \(f_n \to f\) in \(\Phi^*\) iff \(f_n(\phi) \to f(\phi)\) in the complex numbers for all \(\phi \in \Phi\). The space \(\Phi\) should be chosen so that the operators \(U(g)\) map \(\Phi\) into itself. In this case, there is a well-defined dual action of \(U(g)\) on \(f \in \Phi^*\) given by \(U(g)f[\phi] = f[U(g^{-1})\phi]\) for all \(\phi \in \Phi\). Solutions of the constraints are then elements \(f \in \Phi^*\) for which \(U(g)f = f\) for all \(g\).

Of at least equal importance is a discussion of observables: gauge invariant operators on \(\mathcal{H}_{\text{aux}}\) that will become operators on the physical Hilbert space. In RAQ, observables together with their adjoints are required to include \(\Phi\) in their domain and to map \(\Phi\) to itself\(^2\). The star algebra of observables consists of all such gauge invariant operators. ‘Gauge invariance’ of such an operator \(O\) then means that \(O\) commutes with the \(G\)-action on the domain \(\Phi\): \(OU(g)|\phi\rangle = U(g)O|\phi\rangle\) for all \(g \in G, \phi \in \Phi\).

Refined Algebraic Quantization then asks that a ‘rigging map’ be chosen. This is an anti-linear map from \(\Phi\) into \(\Phi^*\) satisfying the following properties (here and in the rest of this work, an overline denotes complex-conjugation):

i) The image of \(\eta\) solves the constraints. For unimodular \(G\), this means that for all \(\phi_1, \phi_2 \in \Phi\) and all \(g \in G\), \(\eta(\phi_1)[U(g)\phi_2] = \eta(\phi_1)[\phi_2]\). In the non-unimodular case, we will see that the right hand side needs to be multiplied by the square-root of \(G\)’s modular function (compare with equation (4.2)).

ii) The map is real: \(\eta(\phi_1)[\phi_2] = \eta(\phi_2)[\phi_1]\).

iii) The map is positive: \(\eta(\phi_1)[\phi_1] \geq 0\).

iv) The map should commute with the observables or, in other words, intertwine the representations of the observables on \(\Phi\) and \(\Phi^*\):

\[
O(\eta \phi) = \eta(O \phi). \quad (2.1)
\]

\(^6\)In particular, this excludes systems such as General Relativity, which have structure functions instead of structure constants.

\(^7\)At least for unimodular groups. See Appendices A and B for a discussion of the non-unimodular case.
RAQ then constructs the physical Hilbert space by introducing a Hermitian inner product on the image of $\eta$ and completing this into a Hilbert space. The inner product is just given by the rigging map:

$$(\eta(\phi_1), \eta(\phi_2))_{\text{phys}} = \eta(\phi_1)[\phi_2].$$

This is equivalent to defining a new inner product on $\Phi$, taking the quotient by the zero norm vectors, and completing to a physical Hilbert space.

As stated in the introduction, the goal of this work is to study the possible rigging maps. A priori, it is not at all clear how large the space of rigging maps might be. See, for example, [28] for a discussion of some of the freedoms. However, a natural idea is given by ‘group averaging’ [26]. This is the idea that one could construct such a rigging map by integrating the operators $U(g)$ with respect to the Haar measure of the group. Suppose for the moment that the group is unimodular so that the left and right Haar measures agree. We will call this (unique) Haar measure $dg$. In this case, the group averaging proposal is

$$\eta|\phi\rangle := \langle \phi | \int dg \, U(g).$$

Let us also, for the moment, ignore convergence issues and note that the expression (2.3) formally qualifies as a rigging map (except perhaps for the positivity condition). The invariance of the Haar measure guarantees that any state in the image of the rigging map is invariant under the action of $U(g_0)$ and so ‘solves the constraints.’ The reality and symmetry properties follow from the fact that $dg = d(g^{-1})$.

When the group is non-unimodular, the left- and right- Haar measures ($d_Lg$ and $d_Rg$) do not agree, and are related by $d_Lg = [\Delta(g)]^{-1}d_Rg$, where $\Delta(g)$ is the so-called modular function, a homomorphism from the group $G$ to the positive real numbers. For finite dimensional Lie groups one has $\Delta(g) = \det\{\text{Ad}_g\}$, where $\text{Ad}$ denotes the adjoint representation. (See Appendix A for details and an explanation of our notation.) Neither the right- nor the left- Haar measure is invariant under the map $g \to g^{-1}$. However, there is another measure, halfway between the left- and right- Haar measure, which is invariant under $g \to g^{-1}$. We will call this measure $d_0g$, the symmetric measure. It is given by $d_0g = [\Delta(g)]^{1/2}d_Lg = [\Delta(g)]^{-1/2}d_Rg$ (see Appendix A).

We will work below with the entire class of measures

$$d_ng = \Delta^{n/2}(g)d_0g, \text{ for } n \in \mathbb{Z}. \quad (2.4)$$

In particular, $d_Lg = d_{-1}g$ and $d_Rg = d_1g$. Note that these measures satisfy the following properties:

$$d_n^{-1} = d_{-n} = \Delta^{-n}(g)d_ng$$

$$d_n(\alpha g) = \Delta^{-\frac{n}{2}}(\alpha)d_ng$$

$$d_n(\alpha g) = \Delta^{-\frac{n-1}{2}}(\alpha)d_ng. \quad (2.5)$$

Thus, the following expression, replacing (2.3),
\[ \eta|\phi\rangle := \langle \phi| \int d_0 g \ U(g) \] (2.6)

has the right properties for \( \eta \) to be real in the sense of RAQ. However, this expression is not invariant under multiplication on the right by \( U(g_0) \). Instead, \( (\eta|\phi\rangle)U(g_0) = (\eta|\phi\rangle)\Delta^{1/2}(g_0) \). As argued in Appendix B (based on [15,14,13]), this is in fact the correct result in the non-unimodular case. If we consider RAQ in terms of defining a new inner product on \( \Phi \), then the group averaging inner product is just

\[ \langle \phi_1|\phi_2\rangle_{ga} = \int d_0 g \langle \phi_2|U(g)|\phi_1\rangle. \] (2.7)

In order for the group averaging procedure to be well defined, the integral (2.7) must converge absolutely for all \( \phi_1 \) and \( \phi_2 \) in \( \Phi \). However, to complete our uniqueness proof, we will need to impose the somewhat stronger condition that the integrals

\[ \int d_n g \langle \phi_2|U(g)|\phi_1\rangle. \] (2.8)

converge absolutely for all \( n \in \mathbb{Z} \). In this case the function \( \langle \phi_2|U(g)|\phi_1\rangle \) defines an element of \( L^1(G,d_n g) \) for all \( n \) and we will refer to \( \Phi \) as an \( L^1 \) space. Similarly, a state \( \phi \) will be called \( L^1 \) when \( \langle \phi|U(g)|\phi\rangle \) is in \( L^1(G,d_n g) \) for all \( n \). In this case, we denote the function \( \phi \) by \( U_\phi(g) \).

III. A UNIQUE RIGGING MAP

This section contains the proof of our uniqueness theorem. A key feature of our derivation will be the use of a certain group algebra for \( G \), which we introduce and study in subsection A below. The main result then follows in subsection B.

A. The group algebra

We now construct a group algebra \( \mathcal{A}_G \) based on the functions on \( G \) that are \( L^1 \) with respect to all of the measures \( d_n g \) for \( n \in \mathbb{Z} \). We will denote the \( L^1(G,d_n g) \) norm of \( f \) by \( ||f||_n \). The group algebra multiplication of two functions \( f_1, f_2 \in \mathcal{A}_G \) will be defined to be convolution with respect to the right Haar measure:

\[ f_1 \cdot f_2(g) = \int_G (d_R g_1) \ f_1(g g_1^{-1}) f_2(g_1). \] (3.1)

To see that \( f_1 \cdot f_2 \) is in \( L^1(G,d_n g) \), note that

\[ |f_1 \cdot f_2(g)| \leq \int_G d_R g_1 |f_1(g g_1^{-1})| \ |f_2(g_1)|. \] (3.2)

Thus, it is sufficient to check the result for positive \( f_1 \) and \( f_2 \). Consider such positive real \( f_1, f_2 \) and consider some compact subset \( K \subset G \). We have
\[
\int_K d_ng \cdot \langle f_1 \cdot f_2 \rangle = \int_K d_ng \cdot \int_G dRg_1 \cdot f_1(gg_1^{-1}) \cdot f_2(g_1) = \int_G d_ng_1 \left( \int_{Kg_1^{-1}} d_ng_2 \cdot f_1(g_2) \right) f_2(g_1).
\] (3.3)

Since the integral in parentheses converges absolutely to \(|f_1|_n\) in the limit as \(K \to G\), it follows that the entire expression converges to \(|f_1|_n \cdot |f_2|_n\). Thus, for any \(f_1, f_2 \in \mathcal{A}_G\), we have \(|f_1 \cdot f_2|_n \leq |f_1|_n \cdot |f_2|_n\), with equality if and only if \(f_1\) and \(f_2\) are real and positive. It follows that the operation \(f_1 \cdot f_2\) defines a product on \(\mathcal{A}_G\).

The group algebra \(\mathcal{A}_G\) is in fact a star algebra. The star operation is defined by

\[
f^*(g) = \overline{f(g^{-1})}.
\] (3.4)

From (2.5), we have that \(|f|_n = |f|_{-n}\), so \(\mathcal{A}_G\) is closed under the action of \(*\). From (3.4) it follows that \((f_1 \cdot f_2)^* = f_2^* \cdot f_1^*\). Another (linear) involution on \(\mathcal{A}_G\) of which we will make use is \(*\), given by

\[
f^*(g) = f(g^{-1}) \Delta^{-\frac{1}{2}}(g).
\] (3.5)

Now, the important part of this construction is that, to each \(f \in \mathcal{A}_G\), we may associate an operator \(\hat{f}\) on \(H_{aux}\). This operator is given by

\[
\hat{f} = \int_G d_0g \cdot f(g)U(g).
\] (3.6)

Note that the operator norm of \(\hat{f}\) is bounded by \(|f|_0\), so that \(\hat{f}\) is defined on all of \(H_{aux}\).

As may be readily verified, the map \(f \mapsto \hat{f}\) satisfies \(\hat{f}^* = (\hat{f})^\dagger\) and \(f_1 \cdot f_2 = \hat{f}_1 \hat{f}_2\). Hence it defines a \(*\)-homomorphism from \(\mathcal{A}_G\) to \(B(H_{aux})\), the bounded operators on \(H_{aux}\). Thus, the group algebra \(\mathcal{A}_G\) acts on \(H_{aux}\) via its image \(\hat{A}_G \subseteq B(H_{aux})\).

An important property of this action is that it preserves the \(L^1\) property of a state and of a subspace. To see this, we first prove the result:

**Lemma 1.** If \(\phi\) is an \(L^1\) state, then \(\langle \phi | \hat{f}_1U(g)\hat{f}_2| \phi \rangle\) is in \(L^1(G, d_ng)\).

**Proof.** For a compact subset \(K \subseteq G\), consider the expression

\[
\int_K d_ng \cdot |\langle \phi | \hat{f}_1U(g)\hat{f}_2| \phi \rangle| = \int_K d_ng \cdot \int_{G \times G} d_0g_1 \cdot d_0g_2 \cdot |\mathcal{F}_1(g_1)f_2(g_2)U_{\phi}(g_1^{-1}g_2)|.
\] (3.7)

Setting \(t = g_1^{-1}g_2\) and replacing \(g\) by \(g = g_1tg_2^{-1}\) leads to the expression

\[
\int_{G \times G} d_0g_1 \cdot d_0g_2 \cdot \int_{g_1^{-1}Kg_2} d_n t \cdot |\mathcal{F}_1(g_1)f_2(g_2)U_{\phi}(t)\Delta^{\frac{1}{2n}}(g_1)\Delta^{\frac{1}{2n}}(g_2)|.
\] (3.8)

In the limit as \(K \to G\) (and hence \(g_1^{-1}Kg_2 \to G\)), the \(t\)-integral converges absolutely and the factors of \(\Delta^{1/2}\) combine with the symmetric measures to make \(d_{n+1}g_1\) and \(d_{1-n}g_2\). Thus, \(\langle \phi | \hat{f}_1U(g)\hat{f}_2| \phi \rangle\) is in \(L^1(G, d_ng)\) and \(\int_G d_ng |\langle \phi | \hat{f}_1U(g)\hat{f}_2| \phi \rangle| = ||f_1||_{1+n} \cdot ||f_2||_{1-n} \cdot ||U_{\phi}||_n\). In particular, if \(\phi\) is an \(L^1\) state then so is \(\hat{f}_\phi\). Thus, we arrive at the conclusion that the space of \(L^1\) states carries a \(*\)-representation of the group algebra \(\mathcal{A}_G\).
Two immediate extensions of the above Lemma should be noted. The first is that the result continues to hold if either \( \hat{f}_1 \) or \( \hat{f}_2 \) is replaced by \( U(g_0) \) for any \( g_0 \in G \). Thus, we may extend Lemma 1 to include any \( \hat{f}_1, \hat{f}_2 \) in the algebra \( \hat{A}_G \) generated by \( \hat{f} \) for \( f \in A_G \) and \( U(g_0) \) for \( g_0 \in G \). To state the second, let us say that \( \phi_1, \phi_2 \) are \( L^1 \) with respect to each other when \( \langle \phi_1 | U(g) | \phi_2 \rangle \) lies in \( L^1(G, d_n g) \). It then follows that, if \( \phi_1 \) and \( \phi_2 \) are \( L^1 \) with respect to each other, then so are \( \hat{f}_1 \phi_1 \) and \( \hat{f}_2 \phi_2 \) for any \( \hat{f}_1, \hat{f}_2 \in \hat{A}_G \). To see this, simply let \( M(\phi; g) = \langle \phi | \hat{f}_1 \hat{f}_2 | \phi \rangle \) and note the usual polarization identity:

\[
\langle \phi_1 | \hat{f}_1 U(g) \hat{f}_2 | \phi_2 \rangle = \frac{1}{4} \left( M(\phi_1 + \phi_2; g) - M(\phi_1 - \phi_2; g) - iM(\phi_1 + i\phi_2; g) + iM(\phi_1 - i\phi_2; g) \right).
\]

(3.9)

Taken together, these results give the following:

**Lemma 2.** If \( \Phi \) is an \( L^1 \) subspace of \( \mathcal{H}_{aux} \), then so is any space generated by the action of \( \hat{A}_G \) on \( \Phi \).

**B. The Main Result**

We will now show that, given an \( L^1 \) subspace \( \Phi \) that is invariant under the action of \( \hat{A}_G \) and on which the group averaging inner product (2.7) is not identically zero, any rigging map is some constant times the group averaging map. Note that, given any \( L^1 \) space \( \Phi_0 \), the results of subsection III A show that an invariant \( L^1 \) space can be constructed by taking the closure of \( \Phi_0 \) under the action of \( \hat{A}_G \).

It is important to remark that we do not know whether the group averaging map is in general positive, and that we will not need to assume positivity of the group averaging map in this section. In particular, this means that if in some case the group averaging map fails to be positive semi-definite, then there cannot exist a positive rigging map. No cases of this sort are yet known, but neither are they ruled out by general theorems known to the authors.

The most restrictive property of the rigging map is that it must intertwine representations of the observable algebra \( A_{obs} \) acting on \( \Phi \) and \( \Phi^* \). Our uniqueness theorem will follow by making use of a certain class of observables. These observables are of the sort constructed in [27] by averaging gauge-dependent operators over the gauge group. To this end, we will need the following Lemma:

**Lemma 3.** Let \( \phi_1, \phi_2 \) be states in the invariant \( L^1 \) space \( \Phi \). The expression

\[
O_{\phi_1 \phi_2} = \int dLg U(g) |\phi_1 \rangle \langle \phi_2 | U(g^{-1})
\]

(3.10)
defines an observable.

**Proof.** Recall that, in the context of RAQ, an observable is an operator whose domain includes our chosen dense subspace \( \Phi \), maps \( \Phi \) to itself, and which commutes with the
action of the gauge group on the domain $\Phi$. Let us therefore consider the action of $O_{\phi_1,\phi_2}$ on some state $\phi_0 \in \Phi$. We introduce the notation $U_{\phi_2\phi_0}(g) = \langle \phi_2 | U(g) | \phi_0 \rangle$. Note that since $\phi_2$ and $\phi_0$ are in $\Phi$, we know that $U_{\phi_2\phi_0}$ is $L^1$ with respect to each of the measures $d_n g$, and so defines an element of $A_G$. Recalling the definition of the * involution from (3.5), we may write

$$O_{\phi_1,\phi_2} | \phi_0 \rangle = \int d_n g U_{\phi_2\phi_0} (g^{-1}) U(g) | \phi_1 \rangle = U^*_{\phi_2\phi_0} | \phi_1 \rangle. \quad (3.11)$$

As an element of $\hat{A}_G$, the operator $U^*_{\phi_2\phi_0}$ is bounded, so that the integral in (3.11) converges in the Hilbert space norm. The domain of $O_{\phi_1,\phi_2}$ thus contains all of $\Phi$. Furthermore, since $\Phi$ is invariant under the action of $\hat{A}_G$, equation (3.11) shows that $O_{\phi_1,\phi_2}$ maps $\Phi$ into itself. Since the measure in (3.10) is invariant under left translations, it follows immediately that $O_{\phi_1,\phi_2} U(g) | \phi_0 \rangle = U(g) O_{\phi_1,\phi_2} | \phi_0 \rangle$ for any $\phi_0 \in \Phi$, and that $O$ is an observable. \hfill $\blacksquare$

Note that the operators (3.11) in fact define a complete set of operators on the physical Hilbert space in the sense that they generate the entire algebra of bounded operators.

We are now ready to prove our main result.

**Theorem.** Suppose that $\Phi$ is an $L^1$ subspace of $\mathcal{H}_{\text{aux}}$ which is invariant under the action of $\hat{A}_G$ and on which the group averaging bilinear form (2.7) is not identically zero. Then the rigging map is unique up to an overall scale. Furthermore, it is given by the group averaging expression (2.6).

**Proof.** We begin by choosing any two states $\phi_1$ and $\phi_2$ in $\Phi$ and supposing that $\eta$ is a rigging map. It suffices to consider the case where $\eta$ is not the zero map. Associated with these states are the observables $O_{\phi_1,\phi_2}$ and $O_{\phi_2,\phi_1}$ defined above. As observables, $O_{\phi_1,\phi_2}$ and $O_{\phi_2,\phi_1}$ act in the physical Hilbert space and $\eta$ must intertwine this action with the action of these operators on $\Phi$. We now consider their action on the physical states $\eta \phi_1$ and $\eta \phi_2$. For this we note that $\eta \hat{f} = (\int_G d_1 g f(g)) \eta$ for any $f \in A_G$. Then, for $i,j$ ranging over 1,2, we have

$$O_{\phi_1,\phi_2} \eta \phi_j = \eta O_{\phi_1,\phi_2} \phi_j = \eta \hat{U}_{\phi_2,\phi_1} \phi_j = \left[ \int_G d_0 g \langle \phi_j | U(g) | \phi_i \rangle \right] \eta \phi_i. \quad (3.12)$$

Finally, recall (e.g. [23,24]) that any rigging map preserves *-relations in the sense that, since $\langle \psi_2 | O_{\phi_1,\phi_2} | \psi_1 \rangle = \langle \psi_1 | O_{\phi_2,\phi_1} | \psi_2 \rangle$ for all $\psi_1, \psi_2 \in \Phi$, we also have

$$O_{\phi_1,\phi_2} \eta (\psi_2) | \psi_1 \rangle = \eta (\psi_2) | O^*_{\phi_1,\phi_2} \eta (\psi_1) | \psi_2 \rangle. \quad (3.13)$$

Taking $\psi_1 = \phi_1$, $\psi_2 = \phi_2$, we apply (3.12) with $i = 1, j = 2$ to the left hand side and with $i = 2, j = 1$ to the right hand side. Taking into account the hermiticity of $\int_G d_0 g \langle \phi_i | U(g) | \phi_i \rangle$, and the hermiticity of the physical inner product, we arrive at

---

8Proof: $O \eta (\psi_2) | \psi_1 \rangle = \eta (\psi_2) | O^* \psi_1 \rangle$ (by definition of dual action) = $\eta (O^* \psi_1) | \psi_2 \rangle$ (by reality) = $\overline{O^* \eta (\psi_1)} | \psi_2 \rangle$ (by intertwining property).
\[
\left( \int_G d_0g \langle \phi_1 | U(g) | \phi_1 \rangle \right) \eta(\phi_2)[\phi_2] = \left( \int_G d_0g \langle \phi_2 | U(g) | \phi_2 \rangle \right) \eta(\phi_1)[\phi_1]
\] (3.14)

which must hold for all \( \phi_1, \phi_2 \in \Phi \).

Now, both Hermitian forms, the one defined by group averaging and the one defined by \( \eta \), are non-zero. Suppose \( \phi_1 \) is in the kernel of the first form. Choosing \( \phi_2 \) in the complement of this kernel, (3.14) implies that \( \phi_1 \) is also in the kernel of the second form. Reversing the roles of the two forms in this argument shows that their kernels coincide. Equation (3.14) also implies that in the complement of this kernel the ratio \( k := \frac{\eta(\phi_1)[\phi_1]}{\int_G d_0g \langle \phi_1 | U(g) | \phi_1 \rangle} \) is a constant, independent of \( \phi_1 \). Furthermore, the rigging map \( \eta \) is just this constant \( k \) times the group averaging map defined by the symmetric inner product \( d_0g \).

IV. DISCUSSION

In this work we have shown that, when group averaging converges sufficiently quickly (and gives a nontrivial result), it defines the unique rigging map and thus the unique physical inner product. Convergence of group averaging is not uncommon for finite dimensional gauge groups. A particular case of interest is the group \( \mathbb{R} \), which has a single generator. This case includes the quantization of homogeneous cosmological models, in which the single generator is the Hamiltonian constraint. The convergence of group averaging is most easily analyzed by considering a complete set of (generalized) eigenstates of the self-adjoint constraint. So long as such states can be parameterized by points on a smooth manifold, it is usually not difficult to construct an \( L^1 \) domain \( \Phi \) from smooth functions on that manifold and to see that group averaging defines a positive rigging map. A dense \( L^1 \) space is also easy to find in the (say left-) regular representation of any group \( G \), where \( G \) acts unitarily on \( L^2(G, d_Lg) \) via \( (U(h)\phi)(g) := \phi(h^{-1}g) \). One need only take the space \( \Phi \) to be the closure under the action of \( \hat{A}_G \) on the space of functions of compact support on \( G \), and all integrals are guaranteed to converge. Moreover, for the regular (and hence any sub-) representation the group averaging map is positive. This follows from

\[
\int_G d_0h \langle \phi, U(h)\phi \rangle_{L^2} = \int_{G \times G} d_0h d_Lg \overline{\phi(g)} \phi(h^{-1}g) = \int_G d_0g \phi(g)^2 \geq 0.
\] (4.1)

Further remarks on positivity may be found in [37].

Our result is quite strong, as it applies to an arbitrary locally compact group, even if non-unimodular, non-amenable, or wild. The discussion of non-unimodular groups is especially interesting and may hold important insights for future work. Following [13, 14], our physical states ‘solve’ the equations

\[
U(g) | \phi \rangle_{\text{phys}} = \Delta^{1/2} | \phi \rangle_{\text{phys}},
\] (4.2)

where \( U(g) \) is a unitary representation of the gauge group \( G \). However, this is equivalent to introducing the non-unitary representation \( \rho(g) = \Delta^{-\frac{1}{2}}(g)U(g) \) and taking physical states
to be invariant under this action. This may represent the first step in moving away from self-adjoint constraints and unitary group actions which, as argued in \[38\], may be necessary for a treatment of gravity-like systems where the presence of structure functions means that the constraints do not form a Lie algebra.

Even staying within the context of Lie groups and Lie algebras, there are clearly a number of important issues remaining, such as understanding what happens when group averaging does not converge sufficiently rapidly (or at all!) or when the group averaging inner product is zero. In fact, based on our result, one expects this to be a common occurrence. To see this, suppose that there is some superselection rule (in the sense of \[23\]) on \(H_{aux}\). By this we mean that the set of observables defined as above for some choice of \(\Phi\) leaves invariant some nontrivial decomposition \(H_1 \oplus H_2\) of \(H_{aux}\). Let \(\eta_1, \eta_2\) be the restrictions of any rigging map \(\eta\) to \(H_1\) and \(H_2\) respectively. Then, it is clear that any map of the form \(a_1 \eta_1 \oplus a_2 \eta_2\) is also a rigging map when \(a_1, a_2\) are real and positive. So long as both \(\eta_1\) and \(\eta_2\) are nontrivial, this will give a real ambiguity in the choice of rigging map. Thus, by our result above, group averaging cannot converge rapidly on any subspace in the common domain of \(\eta_1\) and \(\eta_2\) where both rigging maps act nontrivially. In such cases, one might hope that some procedure related to group averaging can be applied (perhaps along the lines of the “renormalization” of group averaging used in \[23\]). Some exploration in this direction will appear in \[39\].

As remarked above, our uniqueness theorem holds regardless of non-amenability or the possible ‘wild’ character of a group. In the case of a non-amenable group, the regular representation does not weakly contain the trivial representation, so the trivial representation does not appear in a direct integral decomposition. Nonetheless, as we have seen, group averaging does succeed in producing a trivial representation from the regular representation. Thus, it is clear that refined algebraic quantization in general (and group averaging in particular) is not tied to the concept of weak containment. However, if we are to move beyond our work here and study in detail the space of rigging maps in the case where group averaging does not converge, it would be useful to find a similar concept which might be used in place of weak containment. Our suggestion is the notion of ‘ultraweak containment,’ which we now introduce.

Ultraweak containment is a topology on the space of all representations of a group \(G\). It is explicitly our intention to include non-unitary representations (so that we may discuss the representation \(\Delta^{1/2}\) associated with non-unimodular groups). We specify this topology by stating when a representation \(\rho_0\) lies in the closure of a set \(R\) of representations.

**Definition.** A representation \(\rho\) of a group \(G\) on a Hilbert space \(H\) lies in the ultraweak closure of a set \(R\) of Hilbert space representations of \(G\) when, for every pair of states \(\phi_1, \phi_2 \in H\) there exists a sequence of triples \(\{\rho_n, \phi_1^n, \phi_2^n\}\), of representations \(\rho_n \in R\) and states \(\phi_1^n\) and \(\phi_2^n\) which lie in the Hilbert space carrying \(\rho_n\), which satisfy

\[
\lim_{n \to \infty} \langle \phi_1^n | \rho_n(g) | \phi_2^n \rangle = \langle \phi_1 | \rho(g) | \phi_2 \rangle.
\]

(4.3)
as functionals on \(C_0^\infty(G)\); i.e., that
\[
\lim_{n \to \infty} \int_G d_0 g \ f(g) \langle \phi^n_1 | \rho_n(g) | \phi^n_2 \rangle = \int d_0 g \ f(g) \langle \phi_1 | \rho(g) | \phi_2 \rangle,
\]
(4.4)
for all smooth functions \( f \) of compact support on \( G \). Note that this definition differs from weak containment just in the choice of topology in which limit (4.3) is to converge. Instead of the compact open topology on \( C(G) \) we now use the topology of \( C^\infty_0 \)'s dual.

Note that the sequence is not allowed to depend on the group element \( g \) (or, on the test function \( f \)), but that as the sequence depends on the pair \( \phi_1, \phi_2 \), taking \( \phi_1 = \phi_2 \) does not require taking \( \phi^n_1 = \phi^n_2 \).

The following result shows the relevance of this topology to group averaging, and also illustrates the fact that this topology is distinctly non-Hausdorff.

**Proposition.** Consider any unitary representation \( \rho \) of \( G \) on a Hilbert space \( \mathcal{H} \) having an \( L^1 \) subspace \( \Phi \) on which group averaging yields a nontrivial physical Hilbert space. The representation \( \Delta^{1/2} \) lies in the ultraweak closure of the single element set \( \{ \rho \} \).

**Proof.** To see this, simply choose any state \( \phi_0 \in \Phi \) and a sequence of compact sets \( K_n \subset G \) which expand to all of \( G \). We let \( \phi^n_2 = k \phi_0 \), where \( k \) is some number to be determined, and \( \phi^n_1 = \int_{K_n} d_0 g \ \rho(g) | \phi_0 \rangle \). It follows that \( \lim_{n \to \infty} \int_G f(g) \langle \phi^n_1 | \rho(g) | \phi^n_2 \rangle d_0 g = k \lim_{n \to \infty} \int_G d_0 g f(g) \int_{K_n} d_0 g_1 (\phi_0 | \rho(g_1) \rho(g) | \phi_0) = \left[ \int_G d_0 g \Delta^{1/2}(g) f(g) \right] k \left( \int_G d_0 g_2 (\phi_0 | \rho(g_2) | \phi_0) \right) \). Choosing \( k^{-1} = \int_G d_0 g_2 (\phi_0 | \rho(g_2) | \phi_0) \) shows the result.

A similar result holds whenever RAQ can be implemented on a space \( \Phi \) which is dense in its dual \( \Phi^* \). The restriction of RAQ to this case may be natural on the grounds that \( \Phi^* \) should provide a ‘completion’ of \( \Phi \). With this understanding, ultraweak containment of the \( \Delta^{1/2} \) representation is essential to the success of RAQ.

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**APPENDIX A: MEASURES ON LIE GROUPS**

Let \( G \) be a finite dimensional, connected Lie group. The maps of left and right multiplication with \( g \in G \) are denoted by \( L_g \) and \( R_g \) respectively; push-forwards and pull-backs of maps are denoted by a lower and upper \( * \). By \( \text{Ad} \) and \( \text{Ad}^* \) we denote the adjoint representation of \( G \) on its Lie algebra \( \mathcal{L} \) and its transpose, the anti-representation on the dual \( \mathcal{L}^* \) of the Lie algebra. Note that \( g \mapsto \text{Ad}_g^{\mathcal{L}^*} \) is a proper representation of \( G \) on \( \mathcal{L}^* \).

If \( n = \dim(G) \), let \( \mu \) be an \( n \)-form over the tangent-space at the identity \( e \in G \). We extend it to left- and right-invariant \( n \)-forms over \( G \) by defining \( \mu_L(g) := L_{g^{-1}}^* \mu \) and \( \mu_R(g) := \mu \\

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transformation formulae for the measures in the main text. Of odd-dimensional $G$ \cite{41}, and differential geometry.

In the third part we show that this procedure fits naturally into the framework of group averaging. In the following we assume familiarity with the basic concepts of symplectic mechanics, including the notion of momentum maps (see Chapt. 4 of \cite{40} or Appendix 5 of \cite{11}).

Our discussion is divided into three parts: in the first we explain the classical aspect and allows one to reduce the cases of non-unimodular gauge groups to the unimodular ones. In the second its realization in geometric quantization. Here we basically follow ref. \cite{15}.

In this appendix we explain in some detail the process of unimodularisation, which allows one to reduce the cases of non-unimodular gauge groups to the unimodular ones. Our discussion is divided into three parts: in the first we explain the classical aspect and in the second its realization in geometric quantization. Here we basically follow ref. \cite{15}. In the third part we show that this procedure fits naturally into the framework of group averaging. In the following we assume familiarity with the basic concepts of symplectic mechanics, including the notion of momentum maps (see Chapt. 4 of \cite{40} or Appendix 5 of \cite{11}), and differential geometry.

1. Unimodularisation Classically

We consider a finite dimensional first-class gauge system $(M, \omega, G, \Phi, J)$, consisting of an even dimensional manifold $M$, a symplectic structure $\omega$, a gauge group $G$, a symplectic left action\footnote{Here we follow standard notational conventions of geometric quantization. This left-action $\Phi$ should not be confused with the dense subspace $\Phi$ of RAQ (which will make no appearance in this appendix).} $\Phi : G \times M \to M$ and an $L^*$-valued, $\text{Ad}^*$-equivariant (i.e. $J \circ \Phi_g = \text{Ad}^*_g \circ J$)

\begin{equation}
R^*_{g^{-1}} \mu \text{ respectively. } G \text{ is unimodular iff } \mu_L = \mu_R. \text{ But we have } R^*_h \mu_L(g) = R^*_h L^*_{(gh)^{-1}} \mu = L^*_{g^{-1}} \text{Ad}^*_h \mu = \det \{ \text{Ad}_{h^{-1}} \} \mu_L(g), \text{ so that for Lie groups unimodularity of } G \text{ is equivalent to (hence its name) } \Delta(g) := \det \{ \text{Ad}_g \} = 1 \forall g \in G. \text{ Note that } \Delta : G \to R_+ \text{ is a homomorphism into the multiplicative group of the positive real numbers. Let } I : g \mapsto g^{-1} \text{ denote the inversion map, then clearly } L_h \circ I = I \circ R_{h^{-1}} \text{ and } I^*_\mu = (-1)^n \mu, \text{ implying } I^* \mu_L = (-1)^n \mu_R.
\end{equation}

We define the ‘inversion-symmetric’ $n$-form $\mu_0 := \Delta^{-\frac{1}{2}} \mu_L = \Delta^{-\frac{1}{2}} \mu_R$ which satisfies $I^* \mu_0 = (-1)^n \mu_0$.

Suppose some $\mu$ has been fixed, we then write $d_{L\mu} := \mu_L(g), d_{R\mu} := \mu_R(g)$ and $d_0\mu = \mu_0(g)$. From now on we agree that the integration variable is always $g$. We then write $(L^*_h \tilde{\mu})(g) =: \tilde{d}(hg), (R^*_h \tilde{\mu})(g) =: \tilde{d}(gh)$ and $(I^* \tilde{\mu})(g) = \tilde{d}(g^{-1})$, where $\tilde{\mu}$ stands for $\mu_L, \mu_R, \mu_0$ and $\tilde{d}$ for $d_{L\mu}, d_{R\mu}, d_0\mu$ respectively. From the above one easily shows that:

\begin{equation}
\begin{aligned}
d_{L\mu}(gh) &= \Delta^{-1}(h) d_{L\mu}(g), & d_{L\mu}(g^{-1}) &= (-1)^n \Delta(g) d_{L\mu}(g), \\
d_{R\mu}(hg) &= \Delta(h) d_{R\mu}(g), & d_{R\mu}(g^{-1}) &= (-1)^n \Delta^{-1}(g) d_{R\mu}(g), \\
d_0\mu(hg) &= \Delta^\frac{1}{2}(h) d_0\mu(g), & d_0\mu(g^{-1}) &= \Delta^{-\frac{1}{2}}(h) d_0\mu(g).
\end{aligned}
\end{equation}

Note that the factor $(-1)^n$, which results from the orientation reversing nature of $I$ in case of odd-dimensional $G$, does not appear in the integrals. Hence we also suppressed it in the transformation formulae for the measures in the main text.

**APPENDIX B: NON-UNIMODULAR GAUGE GROUPS AND THEIR UNIMODULARISATION**

In this appendix we explain in some detail the process of unimodularisation, which allows one to reduce the cases of non-unimodular gauge groups to the unimodular ones. Our discussion is divided into three parts: in the first we explain the classical aspect and in the second its realization in geometric quantization. Here we basically follow ref. \cite{15}.

In the third part we show that this procedure fits naturally into the framework of group averaging. In the following we assume familiarity with the basic concepts of symplectic mechanics, including the notion of momentum maps (see Chapt. 4 of \cite{40} or Appendix 5 of \cite{11}), and differential geometry.
momentum map $J : M \to \mathcal{L}^\ast$. That the gauge system is first-class means that the constraint submanifold, given by $J^{-1}(0) \subset M$, is coisotropic, i.e. $T^\perp(J^{-1}(0)) \subset T(J^{-1}(0))$, where $\perp$ refers to the $\omega$-orthogonal complement. In order for the reduced phase space to become a manifold we tacitly assume here $\Phi$ to be proper and free\footnote{Freeness is not really necessary. It may be relaxed to the condition that the stabilizer subgroups of the action are all conjugate. See exercise 4.1.M. on p. 276 of \cite{10}.}. The constraint set still supports a left action of $G$, and the reduced phase-space is the orbit space of this action: $M_{\text{red}} := G \backslash J^{-1}(0)$, with its symplectic structure $\omega_{\text{red}}$ induced by $\omega$. Now the point is that if $G$ is not unimodular, we can always replace $(M,\omega,G,\Phi,J)$ in a canonical fashion by some ‘larger’ $(\tilde{M},\tilde{\omega},\tilde{G},\tilde{\Phi},\tilde{J})$, such that (i) $\tilde{G}$ is unimodular and (ii) $(\tilde{M}_{\text{red}},\tilde{\omega}_{\text{red}})$ is symplectomorphic to $(M_{\text{red}},\omega_{\text{red}})$. This process, which may be understood as a special case of symplectic induction (\cite{12}), is called \textit{unimodularisation}. In many aspects it closely resembles the BRST-method, albeit in a purely bosonic setting.

More concretely, unimodularisation starts with the choice $\tilde{G} = T^*G$. We globally trivialize $T^*G$ by left translations, i.e. we map $\alpha_g \in T_g G$ to $(g, L_g^* \alpha_g) \in G \times \mathcal{L}^\ast$, and identify (as manifolds) $T^*G$ with $G \times \mathcal{L}^\ast$. The group structure is that of a semi-direct product, with $G$ acting on $\mathcal{L}^\ast$ via $(\text{Ad}^\ast)^{-1}$, so that we can write $\tilde{G} = G \ltimes \mathcal{L}^\ast$ and have $(g,\alpha)(h,\beta) = (gh, \alpha + \text{Ad}^\ast_{g^{-1}}(\beta))$ and $(g,\alpha)^{-1} = (g^{-1}, -\text{Ad}^\ast_g(\alpha))$. Using these formulae it is straightforward to show that the matrix of the adjoint action of $(g,\alpha) \in \tilde{G}$ on $\tilde{\mathcal{L}} \cong \mathcal{L} \oplus \mathcal{L}^\ast$ has the block from

$$
\begin{pmatrix}
\text{Ad}_g & 0 \\
0 & \text{Ad}_g^{-1}
\end{pmatrix}.
$$

Hence it has unit determinant so that $\tilde{G}$ is unimodular.

Further, one chooses $\tilde{M} = M \times \mathcal{L}^\ast \oplus \mathcal{L}$ with $\mathcal{L}^\ast \oplus \mathcal{L}$ understood as cotangent bundle $T^*\mathcal{L}^\ast$, carrying its canonical symplectic structure \footnote{In the standard Darboux-coordinates $(q,p)$ for (base space,fibre), this is just the familiar $\wtilde{\omega} = dp \wedge dq = d\sigma$ with $\sigma = pdq$.}$\tilde{\omega} = d\sigma$, where $\sigma_{(a,b)}(\beta) := \langle a, \beta \rangle$ and where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathcal{L}$ and $\mathcal{L}^\ast$. The symplectic structure is then given by $\wtilde{\omega} = \omega + \tilde{\omega}$. The left action $\tilde{\Phi}$ of $T^*G$ on $\mathcal{L}^\ast \oplus \mathcal{L}$ is the canonical lift to the cotangent bundle of the affine action of $T^*G$ on the base $\mathcal{L}^\ast$: $((g,\alpha),\beta) \mapsto \text{Ad}^\ast_{g^{-1}}(\beta) + \alpha$, so that for $\tilde{\Phi}$ we thus obtain

$$
\tilde{\Phi}(g,\alpha)(m,\beta,b) := \left(\Phi_g(m), \text{Ad}^\ast_{g^{-1}}(\beta) + \alpha, \text{Ad}_g(b)\right).
$$

Since $\tilde{\Phi}$ is a cotangent lift, it automatically preserves $\wtilde{\omega}$ and hence $\tilde{\Phi}$ preserves $\wtilde{\omega}$. Also, by the same token, the momentum map $\tilde{J}$ for $\tilde{\Phi}$ is simply given by

$$
\langle (a,\alpha), \tilde{J}(\beta,b) \rangle = \sigma \left( \frac{d}{dt} \bigg|_{t=0} \tilde{\Phi}_{(g(t),\alpha(t))}(\beta,b) \right) = \langle b, -\text{ad}_a^\ast(\beta) + \alpha \rangle,
$$

\cite{10}
where \((g(0), \alpha(0)) = (e, 0)\) and \((a, \alpha) := \frac{d}{dt}|_{t=0}(g(t), \alpha(t))\). Hence the total momentum map for \(\Phi\) is defined by

\[
\langle (a, \alpha), \bar{J}(m, \beta, b) \rangle = \langle a, J(m) \rangle - \langle b, \text{ad}^\ast_b(\beta) \rangle + \langle a, \alpha \rangle.
\]  
\[(B4)\]

The extended constraint set \(\bar{J}^{-1}(0) \subset M \times \mathcal{L}^* \oplus \mathcal{L}\) is now seen to be given by \(J^{-1}(0) \times \mathcal{L}^* \oplus \{0\}\), which contains the extra dimensions parameterized by \(\mathcal{L}^*\). However, \((B2)\) shows that the enlarged gauge group just cancels these additional (ghost-) degrees of freedom and that \(T^*G \setminus \bar{J}^{-1}(0) \cong G \setminus J^{-1}(0)\), i.e., \(\tilde{M}_{\text{red}} \cong M_{\text{red}}\).

2. Unimodularisation in Geometric Quantization

In geometric quantization, the Hilbert space is built by completion from certain smooth sections in a line bundle over the symplectic manifold \((M, \omega)\) of dimension, say, \(n\). A central structural element is a foliation \(P\) of \(M\) by \(n\)-dimensional Lagrangian submanifolds such that \(M/P\) is smooth. For simplicity we shall here restrict attention to the subclass of so-called ‘half-density quantizations’, where the line bundles are of the form \(L = \hat{L} \otimes |\Lambda^n P|^{1 \over 2}\), where \(\hat{L}\) is a line bundle whose curvature is proportional to \(\omega\), and where \(|\Lambda^n P|^{1 \over 2}\) is the line bundle of half-densities associated to \(P\). The Hilbert space is obtained by completion from smooth sections, which are covariantly constant along \(P\) (they are ‘polarized’) and compactly supported on \(M/P\). For this we clearly also need to choose a Hermitian structure that is compatible with the connection, which is always possible. The quantizable observables are those whose Hamiltonian flows preserve \(P\) and there is a quantization map that maps smooth functions on \(M\) to differential operators on \(L\).

If \((M, \omega, G, \Phi, J)\) is a gauge system as above, then, provided that \(P\) is compatible with the action \(\Phi\) of \(G\) in the sense that \(P\) is invariant under \(\Phi\) and satisfies \(TP \cap T^\perp(P^{-1}) = \{0\}\), there is an induced polarization \(P_{\text{red}}\) in \((M_{\text{red}}, \omega_{\text{red}})\). Furthermore, if \(\Phi\) lifts to a connection preserving action on \(L\), we can perform a “\(G\)-invariant” quantization based on \((M, \omega, P)\). The natural question then is whether this quantization is unitarily equivalent to the already reduced quantization based on \((M_{\text{red}}, \omega_{\text{red}}, P_{\text{red}})\). Here the main result is (see [L3]), that this is indeed the case if \(G\) is unimodular and the condition of \(G\)-invariance on sections in \(L\) is just the Dirac condition (where \(J_a := \langle a, J \rangle\) and \(J_a\) is its quantization)

\[
(J_a)\psi = 0 \quad \forall a \in \mathcal{L}.
\]  
\[(B5)\]

Hence in this sense the Dirac condition may be thought of as being derived from the requirement of equivalence with the quantization based on the reduced phase space. However, in the proof unimodularity is needed to (naturally) relate half densities on \((M, P)\) to those on \((M_{\text{red}}, P_{\text{red}})\) so that this result is not directly applicable to the non-unimodular case. But the trick is of course to apply it to the unimodularized setting. Then the Dirac condition for the \(\hat{G}\) invariant quantization of the extended system can be shown to the equivalent to the following modified Dirac condition for a \(G\)-invariant quantization of the original system:
\[ (\mathcal{J}_a)\psi = \frac{i}{2} \text{tr}(\text{ad}_a)\psi \quad \forall a \in \mathcal{L}. \]  

which is just the infinitesimal version of (4.2) and which reduces to (B5) for unimodular \( G \), since then the \( \text{ad}_a \) are traceless. Hence we conclude that in the non-unimodular case the physical Hilbert space is to be built from solutions of (B6) (as a differential equation, not as an eigenvalue equation in Hilbert space) rather than (B5). The failure of the Dirac condition has also been explicitly studied in simple examples with non-unimodular gauge group, like the pseudo-rigid body [14] and other systems [13] [15].

3. Unimodularisation and Group Averaging

Let \( G \) and \( \tilde{G} = T^*G \cong G \ltimes \mathcal{L}^* \) be as above and recall the multiplication law: 
\[(g_1, \alpha_1)(g_2, \alpha_2) = (g_1 g_2, \alpha_1 + \text{Ad}_{g_1^{-1}}(\alpha_2)). \]
We shall identify \( G \) and \( \mathcal{L}^* \) with subgroups of \( \tilde{G} \) as follows: \( G \ni g \leftrightarrow (g, 0) \in \tilde{G} \) and \( \mathcal{L}^* \ni \alpha \leftrightarrow (e, \alpha) \in \tilde{G} \), where \( e \) denotes the identity in \( G \). Since \( \tilde{g} = (g, \alpha) = (e, \alpha)(g, 0) \), this allows us to simply write \( \tilde{g} = \alpha g \) and hence to uniquely decompose each element of \( \tilde{G} \) into the product of elements from \( \mathcal{L}^* \) and \( g \) respectively. The multiplication of \( \tilde{g}_1 \) with \( \tilde{g}_2 \) can then simply be written as follows:
\[ \tilde{g}_1 \tilde{g}_2 = \alpha_1 g_1 \alpha_2 g_2 = \alpha_1 (g_1 \alpha_2 g_1^{-1}) g_1 g_2, \quad \text{(B7)} \]
where \( g_1 \alpha_2 g_1^{-1} = \text{Ad}_{g_1^{-1}}^*(\alpha_2) \), so that in \( \tilde{G} \), the automorphism \( \text{Ad}_{g_1}^* \) of \( \mathcal{L}^* \) is written as simple conjugation.

Since \( \tilde{G} \) and \( \mathcal{L}^* \) are unimodular, their invariant measures are necessarily bi-invariant. We now construct the bi-invariant measure \( d\tilde{g} \) on \( \tilde{G} \) as a product measure of the bi-invariant measure \( d\alpha \) on \( \mathcal{L}^* \) and the right-invariant measure \( dRg \) on \( G \). More precisely, if \( \tilde{g} = \alpha g \), we claim:
\[ d\tilde{g} = d\alpha \, dRg, \quad \text{(B8)} \]
where this equality should be understood as saying that there exists equality if the undetermined constant scale-factors in each measure are chosen appropriately. For the proof it suffices to show left- and right-invariance under \( \tilde{G} \)-multiplication of the right hand side of (B8). Left-Invariance follows from
\[
d(\tilde{g}_1 \tilde{g}) = d(\alpha_1 + \text{Ad}_{g_1^{-1}}^*(\alpha)) \, d(g_1 g) \\
= \det\{\text{Ad}_{g_1^{-1}}^*\} \, d\alpha \, \Delta(g_1) \, dg \\
= d\alpha \, dg = d\tilde{g}, \quad \text{(B9)}
\]
where we used (A1) and \( \Delta(g) = \det\{\text{Ad}_g\} \). Right-Invariance is even simpler:
\[
d(\tilde{g} g_1) = d(\alpha + \text{Ad}_{g^{-1}_1}^*(\alpha_1)) \, d_R(g g_1) \\
= d\alpha \, dg = d\tilde{g}. \quad \text{(B10)}
\]
Suppose now that the unimodular group $\tilde{G}$ acts on a Hilbert space $\tilde{H}_{\text{aux}}$ via a unitary representation $\tilde{U}$. Suppose further that we have an associated rigging map $\tilde{\eta}$, given by averaging over $\tilde{G}$. We can then write (in the following all integrals are to be understood in the weak sense):

\[ \tilde{\eta} := \int_{\tilde{G}} d\tilde{g} \tilde{U}(\tilde{g}) = \int_{\mathcal{L}^*} d\alpha \tilde{U}(\alpha) \int_{G} dRg \tilde{U}(g) \]
\[ = \int_{G} dLg \tilde{U}(g) \int_{\mathcal{L}^*} d\alpha \tilde{U}(\alpha) \]
\[ =: \eta \circ \hat{\eta}, \quad \text{(B13)} \]

where we denoted the averaging maps over $G$ and $\mathcal{L}^*$ in (B12) by $\eta$ and $\hat{\eta}$. The equality of the first with the second line follows from $dLg = \Delta^{-1}(g)dRg$ and

\[ \int_{\mathcal{L}^*} d\alpha \tilde{U}(\alpha) \tilde{U}(g) = \tilde{U}(g) \int_{\mathcal{L}^*} d\alpha \tilde{U}(g^{-1}\alpha g) \]
\[ = \tilde{U}(g) \int_{\mathcal{L}^*} d\alpha \tilde{U}(\text{Ad}^*_g(\alpha)) \]
\[ = \Delta^{-1}(g)\tilde{U}(g) \int_{\mathcal{L}^*} d\alpha \tilde{U}(\alpha), \quad \text{(B14)} \]

which we may write in the form

\[ \tilde{U}(g) \circ \hat{\eta} = \Delta(g) \hat{\eta} \circ \tilde{U}(g). \quad \text{(B15)} \]

The point of the reordering in (B12) is that we wish to first reduce the normal subgroup $\mathcal{L}^*$. Since in our conventions the rigging-maps act to the right, we need to commute the $\mathcal{L}^*$-integration to the right. At this intermediate step, i.e. after applying the rigging map connected with $\mathcal{L}^*$, we will then obtain an effective prescription for the rigging map $\eta$ connected with the remaining part $G$ of $\tilde{G}$. The task is to show that this prescription coincides with (2.6).

To verify this, we simply note that (B15) leads to the statement that with respect to the ‘intermediate-inner-product’, given by $\langle \phi_1, \phi_2 \rangle \text{int} := \hat{\eta}(\phi_1)[\phi_2]$, the maps $\tilde{U}(g)$ differ from being unitary by a factor of $\Delta^{\frac{1}{2}}(g)$. In fact, applying (B15) to the second slot, we have

\[ \langle \tilde{U}(g)\hat{\eta}(\phi_1), \tilde{U}(g)\hat{\eta}(\phi_2) \rangle \text{int} = \Delta(g)\langle \tilde{U}(g)\hat{\eta}(\phi_1), \hat{\eta}(\tilde{U}(g)\phi_2) \rangle \text{int} \]
\[ = \Delta(g)\langle \tilde{U}(g)\hat{\eta}(\phi_1)\rangle[\tilde{U}(g)\phi_2] \]
\[ = \Delta(g)\hat{\eta}(\phi_1)[\phi_2] \]
\[ = \Delta(g)\langle \hat{\eta}(\phi_1), \hat{\eta}(\phi_2) \rangle \text{int}, \quad \text{(B16)} \]

where the third equality simply uses the fact that $\tilde{U}$ acts on the image of $\hat{\eta}$ via the dual action. Hence we see that the restriction of $\tilde{U}$ to $G \subset \tilde{G}$ can be written as a product of representations of $G$

\[ \tilde{U} \bigg|_G =: \Delta^{\frac{1}{2}} \otimes U, \quad \text{(B17)} \]
where $U$ is unitary. Hence we can write the remaining integration over $G$, i.e. the $\eta$-map, as

$$
\eta = \int_G d_\mathcal{L} g \, \bar{U}(g) = \int_G d_0 g \, U(g) ,
$$

(B18)

which indeed coincides with (2.6).
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