Soliton Solution of the Integrable Coupled Nonlinear Schrödinger Equation of Manakov Type

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Abstract

The soliton solution of the integrable coupled nonlinear Schrödinger equation (NLS) of Manakov type is investigated by using Zakharov-Shabat (ZS) scheme. We get the bright N-solitons solution by solving the integrable uncoupled NLS of Manakov type. We also find that there is an elastic collision of the bright N-solitons.

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I. Introduction

The integrable coupled nonlinear Schrödinger equation of Manakov type is widely used in recent developments in the field of optical solitons in fibers. The use and applications of the equation is to explain how the solitons waves transmit in optical fiber, what happens when the interaction among optical solitons influences directly the capacity and quality of communication and so on.\cite{1,2,3} On the other hand, many interesting physical phenomena can be modeled by discrete nonlinear equations. Examples include vibration of particles in a one-dimensional lattice, ladder type electric circuits, collapse of Langmuir waves in plasma physics, growth of conflicting populations in biological science, different simulations of differential equation, etc. Hence it is undoubtedly significant that the ideas of the inverse scattering transform apply to certain types of discrete evolution equations.\cite{4}

Some of exact solutions have been derived for the system related to the Manakov type equation:

\begin{align*}
  iq_{1x} + q_{1tt} + 2\mu \left(|q_1|^2 + |q_2|^2\right) q_1 &= 0 \\
  iq_{2x} + q_{2tt} + 2\mu \left(|q_1|^2 + |q_2|^2\right) q_2 &= 0
\end{align*}

(1.1)

with different procedures \cite{5,6,7} including the use of Hirota method\cite{8}. Based on ref.\cite{8}, Radhakrishnan, et. al. got the one and two soliton solution. However promising their methods, the solution of N-solitons of the equation has not been derived in an exact result yet. In this paper, we investigate the N-soliton solution. However, we show that it is possible when the Zakharov-Shabat (ZS) scheme (in its final form) is expressed solely in terms of certain operators. Based on the choice of the operator, we find that there is a constraint $q_2 = \sqrt{2m} q_1^*$ (where $m$ is an arbitrary positive real parameter and $q_1^*$ is a complex conjugate of $q_1$) when $\mu$ is an arbitrary positive real value.

This paper is organized as follows. In section II, we will perform the ZS scheme and will obtain the constraint $q_2 = \sqrt{2m} q_1^*$, as well as the positive real parameter $\mu$. In section III, we will solve the bright N-solitons solution. We will compare one and two soliton solution with the result in ref.\cite{8}. Section IV is devoted for discussions and conclusions.
II. ZS Scheme for the Integrable Uncoupled NLS of Manakov Type

We start by choosing the following two operators related to Zakharov-Shabat (ZS) scheme
\[
\Delta_0^{(1)} = I \left( \frac{i}{(2m+1)} \frac{\partial}{\partial x} - \frac{\partial^2}{\partial t^2} \right),
\]
and
\[
\Delta_0^{(2)} = \begin{pmatrix} m+1 & 0 \\ 0 & -m \end{pmatrix} \frac{\partial}{\partial t},
\]
where \(m\) is an arbitrary positive real, and \(I\) is the 2x2 unit matrix. We can then define the following operators by using this scheme related to inverse scattering techniques\[^2\]
\[
\Delta^{(1)} = \Delta_0^{(1)} + U(t,x),
\]
and
\[
\Delta^{(2)} = \Delta_0^{(2)} + V(t,x).
\]
Here operators \(\Delta^{(i)}, (i = 1, 2)\) satisfy the following equation
\[
\Delta^{(i)} (I + \Phi_\pm) = (I + \Phi_\pm) \Delta_0^{(i)},
\]
where the integral operator \(\Phi_\pm(\psi)\) are defined according to equation
\[
\Phi_\pm(\psi) = \int_{-\infty}^{\infty} k_\pm(t,z) \psi(z) \, dz.
\]

We now suppose that operators \(\Phi_F(\psi)\) and \(\Phi_\pm(\psi)\) are related to the following operator identity
\[
(I + \Phi_+) (I + \Phi_F) = (I + \Phi_-),
\]
where the integral operator \(\Phi_F(\psi)\) is
\[
\Phi_F(\psi) = \int_{-\infty}^{\infty} F(t,z) \psi(z) \, dz.
\]
Both $k_+$ and $F$ in eq.(2.4) and (2.6) are the 2x2 matrix chosen as follows

$$k_+ = \begin{pmatrix} a(t, z; x) & q_1(t, z; x) \\ \pm q_1^*(t, z; x) & d(t, z; x) \end{pmatrix}, \quad (2.7a)$$

and

$$F = \begin{pmatrix} 0 & \alpha_n'(t, z; x) \\ \beta_n'(t, z; x) & 0 \end{pmatrix}. \quad (2.7b)$$

Here $a$, $q_1$, $\pm q_1^*$, $d$, $\alpha_n'$, and $\beta_n'$ are parameters which will be calculated in section III.

In eq.(2.5), we have assumed that $(I + \Phi_+)$ is invertible, then

$$(I + \Phi_F) = (I + \Phi_+)^{-1}(I + \Phi_-), \quad (2.8)$$

so that operator $(I + \Phi_F)$ is factorisable. From eq.(2.5), we can derive Marchenko matrix equation,

$$k_+(t, z) + F(t, z) + \int_t^\infty k_+(t, t') F(t', z; x) \, dt' = 0, \quad \text{for } z > t \quad (2.9a)$$

and

$$k_-(t, z) = F(t, z) + \int_t^\infty k_+(t, t') F(t', z; x) \, dt', \quad \text{for } z < t. \quad (2.9b)$$

In eq.(2.9b), $k_-$ is obviously defined in terms of $k_+$ and $F$. Both eq.(2.9a) and (2.9b) require $F$, and $F$ is supplied by the solution of equations:

$$\left[ \Delta_{(1)}^0, \Phi_F \right] = 0, \quad (2.10a)$$

and

$$\left[ \Delta_{(2)}^0, \Phi_F \right] = 0. \quad (2.10b)$$

After a little algebraic manipulation, we get

$$\begin{pmatrix} m+1 & 0 \\ 0 & -m \end{pmatrix} F_t + F_z \begin{pmatrix} m+1 & 0 \\ 0 & -m \end{pmatrix} = 0, \quad (2.11a)$$

and

$$\frac{i}{(2m+1)} F_x + F_{zz} - F_{tt} = 0, \quad (2.11b)$$
where $F_t \equiv \frac{\partial F}{\partial t}$, $F_z \equiv \frac{\partial F}{\partial z}$, etc.

$U(t, x)$ and $V(t, x)$ can be found by solving eq.(2.3) in which we have substituted eq.(2.7a) and (2.4) (for $k_+)$ to that equation:

\[ V(t, x) = (m + 1) \begin{pmatrix} 0 & q_1 \\ \mp q_1^* & 0 \end{pmatrix}, \]  

(2.12a)

and

\[ U(t, x) = -2k_+ = -2 \begin{pmatrix} a_t & q_{1t} \\ \pm q_{1t}^* & d_t \end{pmatrix}. \]  

(2.12b)

Based on the solution of equation $\Delta^{(2)} (I + \Phi_+ = (I + \Phi_+) \Delta_0^{(2)}$, we get that $k_+ (t, z; x)$ must obey the following equation:

\[ \left( \begin{array}{cc} m + 1 & 0 \\ 0 & -m \end{array} \right) k_+ + k_+ \left( \begin{array}{cc} m + 1 & 0 \\ 0 & -m \end{array} \right) + (2m + 1) \left( \begin{array}{cc} 0 & q_1 \\ \mp q_1^* & 0 \end{array} \right) k_+ = 0, \]  

(2.13)

and if we evaluate this eq.(2.13) on $z = t$, we find

\[ a_t = \mp \frac{(2m + 1)}{(m + 1)} |q_1|^2, \quad d_t = \mp \frac{(2m + 1)}{(m)} |q_1|^2, \]  

(2.14a)

\[ \pm q_{1t}^* = \mp \frac{(2m + 1)}{(m)} a q_{1t}^*, \quad q_{1t} = -\frac{(2m + 1)}{(m + 1)} q_1 d. \]  

(2.14b)

Plugging the above equations into eq.(2.12b) yields

\[ U(t, x) = -2 \begin{pmatrix} \mp \frac{(2m + 1)}{(m + 1)} |q_1|^2 & \frac{q_{1t}}{\pm q_{1t}^*} \\ \pm q_{1t}^* & \mp \frac{(2m + 1)}{(m)} |q_1|^2 \end{pmatrix}. \]  

(2.15)

Now we substitute equations (2.12a) and (2.15) into eq.(2.2), we get

\[ \Delta^{(1)} = I \left( \frac{i}{(2m + 1)} \partial - \frac{\partial^2}{\partial t^2} \right) - 2 \left( \frac{(2m + 1)}{(m + 1)} |q_1|^2 \right) \begin{pmatrix} q_{1t} \\ \pm q_{1t}^* \end{pmatrix}, \]  

(2.16a)

and

\[ \Delta^{(2)} = \left( \begin{array}{cc} m + 1 & 0 \\ 0 & -m \end{array} \right) \partial + (2m + 1) \left( \begin{array}{cc} 0 & q_1 \\ \mp q_1^* & 0 \end{array} \right). \]  

(2.16b)

5
Since $\Delta^{(1)}$ commutes with $\Delta^{(2)}$, we find

$$
\begin{align*}
i q_{1x} + q_{1tt} & \pm 2 \left( \frac{(2m+1)^2}{-m} + \frac{(2m+1)^2}{(m+1)} \right) | q_1 |^2 q_1 = 0 \\
i q_{1x}^* + q_{1tt}^* & \pm 2 \left( \frac{(2m+1)^2}{-m} + \frac{(2m+1)^2}{(m+1)} \right) | q_1 |^2 q_1^* = 0. \quad (2.17)
\end{align*}
$$

Eq.(2.17) can then be manipulated as

$$
\begin{align*}
i q_{1x} + q_{1tt} + 2\mu \left( | q_1 |^2 + | q_2 |^2 \right) q_1 & = 0 \\
i q_{1x}^* + q_{1tt}^* + 2\mu \left( | q_1 |^2 + | q_2 |^2 \right) q_1^* & = 0, \quad (2.18)
\end{align*}
$$

where we have used the relationship between $q_1$ and $q_2$, namely

$$
q_2 = \sqrt{2m} q_1^*, \quad (2.19)
$$

and parameter $\mu$ is an arbitrary positive real value

$$
\mu = \frac{2m+1}{m^2 + m}. \quad (2.20)
$$

It is obvious that eq.(2.18) is one part of the integrable coupled nonlinear Schrödinger equation of Manakov type (eq.(1.1)).

III. The Bright N-Solitons Solution

We consider a general matrix function $F$ in eq.(2.7b) and substitute it into eq.(2.11b), we find two differential equations

$$
\begin{align*}
\frac{i}{2m+1} \alpha'_{nx} + \alpha'_{nzz} - \alpha'_{ntt} & = 0, \quad (3.1a) \\
\frac{i}{2m+1} \beta'_{nx} + \beta'_{nzz} - \beta'_{ntt} & = 0. \quad (3.1b)
\end{align*}
$$

The solution of the above equations can be derived by using separable variable method. We then find

$$
\begin{align*}
\alpha'_n (t, z; x) & = \sum_{n=1}^N \alpha'_{n_0} e^{-\gamma_n (m+1)z} \left[ e^{-\gamma_n (mt-i\gamma_n (2m+1)^2 x)} \right], \quad (3.2a)
\end{align*}
$$
and

\[ \beta'_n(t, z; x) = \sum_{n=1}^{N} \beta'_n e^{\theta_n(z)} \left[ e^{\theta_n((m+1)t-i\theta_n(2m+1)^2x)} \right], \quad (3.2b) \]

where \( \alpha'_n \) and \( \beta'_n \) are arbitrary constant parameters.

To get the final solution of the integrable coupled NLS equation of Manakov type, we have to substitute eq.(2.7b), eq.(3.2a) and eq.(3.2b) into Marchenko matrix equation (eq.(2.9a)). We get (for \( a \) and \( q_1 \))

\[ a(t, z; x) = -\sum_{n=1}^{N} e^{m\theta_n z} \int_{t}^{\infty} q_1 \beta'_n e^{\theta_n(m+1)t'} e^{-i\theta_n^2(2m+1)^2 x} dt', \quad (3.3a) \]

and

\[ q_1 = -\sum_{n=1}^{N} e^{-(m+1)\gamma_n x} \alpha'_n e^{-m\gamma_n t} e^{i(2m+1)^2 \gamma_n^2 x} \]
\[ -\sum_{n=1}^{N} e^{-(m+1)\gamma_n x} \int_{t}^{\infty} a(t, z; x) \alpha'_n e^{-m\gamma_n t'} e^{i\gamma_n^2(2m+1)^2 x} dt'. \quad (3.3b) \]

Finally, by substituting eq.(3.3a) to eq.(3.3b) we find the solution :

\[ q_1 = \sum_{n=1}^{N} \frac{-\alpha'_n e^{-(2m+1)\gamma_n [t-i(2m+1)\gamma_n x)}}{1 + \left( \frac{-\mu(2m+1)\alpha'_n e^{\gamma_n t} e^{i(2m+1)\gamma_n x}}{(-2m+1)\gamma_n + (2m+1)\gamma_n x} \right) e^{-(2m+1)\gamma_n [t-i(2m+1)\gamma_n x]} e^{(2m+1)\theta_n [t-i(2m+1)\theta_n x]}} \quad (3.4) \]

We define

\[ \eta_n = k_n (t + ik_n x), \quad (3.5a) \]

and

\[ \eta_n^* = k_n^* (t - ik_n^* x), \quad (3.5b) \]

where

\[ k_n = -(2m+1) \gamma_n, \quad (3.5c) \]

and

\[ k_n^* = (2m+1) \theta_n. \quad (3.5d) \]

Here \( \gamma_n \) and \( \theta_n \) are arbitrary complex parameters and \(-\gamma_n^* = \theta_n\).
Now $q_1$ (and $q_2$) can be rewritten as

$$q_1 = \sum_{n=1}^{N} \frac{-\alpha'_n e^{\eta_n}}{1 + e^{R_n + \eta_n + \eta_n}}; \quad (3.6)$$

and

$$q_2 = \sum_{n=1}^{N} \frac{-\sqrt{2m} (\alpha'_{n_0})^* e^{\eta_n}}{1 + e^{R_n + \eta_n + \eta_n}}; \quad (3.7)$$

where

$$e^{R_n} = -\frac{\mu (1 + 2m) \alpha'_{n_0} \beta'_{n_0}}{(k_n + k'_n)^2}. \quad (3.8)$$

Based on our solution in eq.(3.6) and (3.7), we can see that our results are the bright N-solitons since $\mu > 0$. In the equations, there are only two arbitrary complex parameters ($\gamma_n$ and $\theta_n$), and two arbitrary parameters $\alpha'_{n_0}$ and $\beta'_{n_0}$ which can directly influence the phase of the solitons interaction. The results of $q_1$ (eq.(3.6)) and $q_2$ (eq.(3.7)) can be rewritten in the more conventional form by introducing $-\gamma_n^* = -l_n + i\lambda_n$ (where $l_n$ and $\lambda_n$ are arbitrary real parameter),

$$q_1 = \sum_{n=1}^{N} (2m + 1) l_n \left( \frac{\alpha'_{n_0}}{\sqrt{\mu(2m+1) \alpha'_{n_0} \beta'_{n_0}}} \right) e^{i \left[ (2m+1) \lambda_n t + (2m+1)^2 \left( \lambda_n^2 - l_n^2 \right) x \right]} \cosh \left[ -(2m + 1) l_n t - 2 (2m + 1)^2 l_n \lambda_n x + \varphi_n \right], \quad (3.9)$$

and

$$q_2 = \sum_{n=1}^{N} (2m + 1) \sqrt{2m} l_n \left( \frac{(\alpha'_{n_0})^*}{\sqrt{\mu(2m+1) (\alpha'_{n_0} \beta'_{n_0})^*}} \right) e^{i \left[ (2m+1) \lambda_n t + (2m+1)^2 \left( \lambda_n^2 - l_n^2 \right) x \right]} \cosh \left[ -(2m + 1) l_n t - 2 (2m + 1)^2 l_n \lambda_n x + \varphi_n \right], \quad (3.10)$$

where $\varphi_n$ is a real N-solitons phase,

$$\varphi_n = \frac{1}{2} R_n. \quad (3.12)$$

In our results, we have put a real phase $\varphi_n$ by replacing the arbitrary negative parameters $\alpha'_{n_0}$, the arbitrary positive parameter $\beta'_{n_0}$, and an arbitrary complex parameter $k_n$ which is related to arbitrary complex parameters $\gamma_n$ and $\theta_n$. The results derived above have contributed on the obtaining a
general form of an exact positive real parameter \( \mu \) and have also shown that there is a special relationship (eq. (2.19)) between \( q_1 \) and \( q_2 \). Hence, by finding \( q_1 \), we can directly get \( q_2 \).

Based on our results in eq.(3.6) and (3.7), the bright N-solitons solution can be derived into the bright one and two soliton solutions related to that in the works that have been done before by Radhakrishnan, et. al. using Hirota method in ref.[8]. According to the comparison of both methods, their results of the bright one soliton solution is equal to our results when

\[
\alpha e^{\eta(0)} = -\alpha'_1, \quad \beta e^{\eta(0)} = -\sqrt{2m} (\alpha'_1)^* e^{\eta} e^{-\eta_1}, \text{and}
\]

\[
(|\alpha|^2 + |\beta|^2) = -(1 + 2m) \alpha'_1 \beta'_1.
\]

On the other hand, their results of the inelastic collision of the bright two soliton can be reduced to our elastic collision of the solution if we put \( \alpha_1 : \alpha_2 = \beta_1 : \beta_2 \) in their result. So, we can generalize that our result is the solution of an elastic collision of the bright N-solitons.

IV. Discussions and Conclusions

We have presented the bright N-solitons solution of the integrable coupled NLS equation of Manakov type using the N-solitons solution of the integrable uncoupled NLS equation based on Zakharov-Shabat scheme. We can conclude that the solution of the Manakov type (where \( \mu \) is exactly a positive real parameter) can be solved by using an expanded inverse scattering Zakharov-Shabat scheme in which we have chosen a certain operator in our solution (eq.(2.1)). The constraint or the relationship between \( q_1 \) and \( q_2 \) happens absolutely when parameter \( \mu \) is equal to an exact positive real value shown in eq.(2.20).

Finally, we find the novel result that the solution in eq.(3.6) and (3.7) corresponds to an elastic collision of the bright N-solitons, as long as there is a constraint \( q_2 = \sqrt{2mq_1^*} \). From the eq.(3.9) and eq.(3.10), we can also conclude that although the collision between the bright N-solitons their velocities and amplitudes or intensities do not change, their phases do change.

According to the comparison between our one and two soliton solution of the integrable coupled NLS equation of Manakov type and the solution provided by Radhakrishnan, R., et.al., we get that their inelastic collision of the bright two-soliton solution can be our elastic collision solution, if some certain parameters in the results are omitted by putting \( \alpha_1 : \alpha_2 = \)
$\beta_1 : \beta_2$. In our results, the parameter $\exp(\eta_j^{(0)})$ appeared in ref.[8] has been absorbed into $\alpha'_{no}$. The explanations mentioned above mean that there is an $N$-solitons solution of eq.(1.1) in an inelastic collision of the bright $N$-solitons if there is a constraint $q_2 = \sqrt{2mq_1^*}$, where $\mu$ is an arbitrary positive real parameter. Investigations concerning this problem are now in progress.

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