Quasi-optimal convergence rate for adaptive mixed finite element methods *

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Abstract. For adaptive mixed finite element methods (AMFEM), we first introduce the data oscillation to analyze, without the restriction that the inverse of the coefficient matrix of the partial differential equations (PDEs) is a piecewise polynomial matrix, efficiency of the a posteriori error estimator presented by Carstensen [Math. Comput., 1997, 66: 465-476] for Raviart-Thomas, Brezzi-Douglas-Morini, Brezzi-Douglas-Fortin-Marini elements. Second, we prove that the sum of the stress variable error in a weighted norm and the scaled error estimator is of geometric decay, namely, it reduces with a fixed factor between two successive adaptive loops, up to an oscillation of the right-hand side term of the PDEs. Finally, with the help of this geometric decay, we show that the stress variable error in a weighted norm plus the oscillation of data yields a decay rate in terms of the number of degrees of freedom as dictated by the best approximation for this combined nonlinear quantity.

Key words. mixed finite element, error reduction, convergence, optimal cardinality, adaptive algorithm

AMS subject classifications. 65N30, 65N50, 65N15, 65N12, 41A25

1 Introduction and main results

Adaptive methods for the numerical solution of the PDEs are now standard tools in science and engineering to achieve better accuracy with minimum degrees of freedom. The adaptive procedure consists of loops of the form

\[
SOLVE \rightarrow ESTIMATE \rightarrow MARK \rightarrow REFINE. \tag{1.1}
\]

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A posteriori error estimation (ESTIMATE) is an essential ingredient of adaptivity. We refer to [5, 6, 2, 7, 11, 26, 42, 61, 40] for related work on this topic. The analysis of convergence and optimality of the above whole algorithm is still in its infancy. In recent years, there have been some results for the standard adaptive finite element method [39, 49, 50, 51, 29]. In [27, 33, 12, 28], convergence analysis has been carried out for the AMFEM.

Let \( \Omega \) be a bounded polygonal in \( \mathbb{R}^2 \). We consider the following homogeneous Dirichlet boundary value problem for a second order elliptic PDE:

\[
\begin{align*}
-\text{div}(A \nabla u) &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

(1.2)

where \( A \in L^\infty(\Omega; \mathbb{R}^{2 \times 2}) \) is a symmetric and uniformly positive definite matrix, and \( f \in L^2(\Omega) \). The choice of boundary conditions is made for ease of presentation, since similar results are valid for other boundary conditions. In [23], Carstensen presented an a posteriori error estimate for the mixed finite element method of (1.2), and analyzed its efficiency under some restriction of the coefficient matrix. In this paper, we shall prove its efficiency without the restriction, but at the expense of introducing data oscillation, which is right a component of the new concept of error (the total error) (see [29]).

To summarize the first main result, let \( T_h, M_h \times L_h, (p_h, u_h), \eta_{h, \kappa}, \text{osc}_h \) denote the meshes, a pair of finite element spaces, a pair of corresponding discrete solutions, the estimators and oscillations in turn. We avoid the assumption that \( A^{-1}p_h \) is a polynomial on each element, which is required in [23] for the proof of efficiency of the a posteriori error estimator, and obtain the following efficient estimates for the Raviart-Thomas, the Brezzi-Douglas-Marini, or the Brezzi-Douglas-Fortin-Marini elements

\[
\eta_{h, \kappa}^2 \lesssim ||A^{-1/2}(p - p_h)||_{L^2(\Omega)}^2 + ||h^\kappa \text{div}(p - p_h)||_{L^2(\Omega)}^2 + ||u - u_h||_{L^2(\Omega)}^2 + \text{osc}_h^2.
\]

Secondly, we shall analyze convergence and optimality of the AMFEM of the form (1.1). Here we only concern the stress variable error, which is of interest in many applications.

The convergence analysis of the adaptive finite element method (AFEM) is very recent, it started with Döfler [39], who introduced a crucial marking, and proved the strict energy error reduction of the standard AFEM for the Laplacian under the condition that the initial mesh \( T_0 \) satisfies a fineness assumption. Morin, Nochetto and siebert [49, 51] showed that such strict energy error reduction can not be expected in general. Introducing the concept of data oscillation and the interior node property, they proved convergence of the standard AFEM without fineness restriction on \( T_0 \) which is valid only for \( A \) in (2) being piecewise constant on \( T_0 \). Inspired by the work by Chen and Feng [32], Mekchay and Nochetto [51]
extended this result to general second elliptic operators and proved that the standard AFEM is a contraction for the total error, namely the sum of the energy error and oscillation. Recently, Cascon, Kreuzer, Nochetto and Seibert [29] presented a new error notion, the so-called quasi-error, namely the sum of the energy error and the scaled estimator, and showed without the interior node property for the self-adjoint second elliptic problem that the quasi-error is strictly reduced by the standard AFEM even though each term may not be.

However, for convergence of the AMFEM, present works are done only for the Laplacian for the lowest order Raviart-Thomas elements [27, 12] and any order Raviart-Thomas and Brezzi-Douglas-Marini elements [33]. Since the approximation of mixed finite element methods is a saddle point of the corresponding energy, there is no orthogonality available, as is one of main difficulties for convergence of the AMFEM. In this paper, motivated by the two new notions of the error, by establishing a quasi-orthogonality result (see Theorem 4.1) we proved that the AMFEM is a contraction with respect to the sum of the stress variable error in a weighted norm and the scaled error estimator, which is also called the quasi-error.

To summarize the second main result, let \( \{T_k, (M_k, L_k), (p_k, u_k), \eta_k, \text{osc}_k\}_{k \geq 0} \) be the sequence of the meshes, a pair of finite element spaces, a pair of corresponding discrete solutions, the estimators and oscillations produced by the AMFEM in the \( k \)-th step. We prove in Section 5 that the quasi-error uniformly reduces with a fixed rate between two successive meshes, up to an oscillation of data \( f \), namely

\[
E_{k+1}^2 + \gamma \eta_{k+1}^2 \leq \alpha^2 (E_k^2 + \gamma \eta_k^2) + C \text{osc}^2(f, T_k),
\]

where \( \alpha \in (0, 1), \gamma > 0, \)

\[
E_k^2 := ||A^{-1/2}(p - p_{k})||_{L^2(\Omega)}^2 + ||h_k \text{div}(p - p_{k})||_{L^2(\Omega)}^2,
\]

and \( \text{osc}(f, T_k) \) is the oscillation of \( f \) over \( T_k \) (see Section 2.5). We point out here that in some cases, even though the stress variable error is monotone, strict error reduction may fail. For instance, when \( p_k = p_{k+1} \) and \( f \in L_k \), from the second equation of (2.3) it follows \( \text{div} p_k = -f \). Then it holds \( E_k^2 = E_{k+1}^2 \). On the other hand, the residual estimator \( \eta_k := \eta_k(p_k, T_k) \) displays strict reduction when \( p_k = p_{k+1} \) but no monotone behavior in general.

Besides convergence, optimality is another important issue in AFEM which was first addressed by Binev, Dahmen, DeVore [13] and further studied by Stevenson [57], who showed optimality without additional coarsening required in [13]. Both papers [13, 57] are restricted to Laplace operator and rely on suitable marking by data oscillation and the interior node property. Cascon, Kreuzer, Nochetto and Seibert [29] succeeded in establishing quasi-optimality of the AFEM without
both the assumption of the interior node property and marking by data oscillation for the self-adjoint second elliptic operator.

Since all decisions of the AMFEM in MARK are based on the estimator $\eta_k$, a decay rate for the true error is closely related to the quality of the estimator, which is described by the global lower bound

$$\eta_k^2 \lesssim \mathcal{E}_k^2 + \text{osc}_k^2.$$  

Hereafter, following the idea in [29], we refer to the square root of right-hand side above as the total error [46]. The lower bound demonstrates that the estimator is controlled by the error with weights except up to an oscillation term and one can observe the difference between $\mathcal{E}_k$ and $\eta_k$ only when oscillation is large. Furthermore, from the upper bound $\mathcal{E}_k^2 \lesssim \eta_k^2$ and $\text{osc}_k^2 \leq \eta_k^2$ it follows $\mathcal{E}_k^2 + \text{osc}_k^2 \lesssim \eta_k^2$. This implies that the total error, which is the quantity reduced by the AMFEM, is controlled by the estimator. Since the estimator itself is an upper bound for the quasi-error, in view of the global lower bound it holds

$$\mathcal{E}_k^2 + \text{osc}_k^2 \approx \eta_k^2 \approx \mathcal{E}_k^2 + \gamma \eta_k^2. \quad (1.3)$$

In short, the behavior of the AMFEM is intrinsically bonded to the total error, which measures the approximability of both the flux $p = A \nabla u$ and data encoded in the oscillation term. Note that when $A^{-1}p_h$ is a piecewise polynomial vector, oscillation will reduce to approximation of the right-hand side term $f$ of (1.2) (see Section 2.5). In general cases, approximation of data $A$ appeared in $\text{osc}_k^2$ couples in nonlinear fashion with the discrete solutions $p_k$.

In Section 6, we shall introduce two approximation classes $\mathcal{A}_s$ and $\mathcal{A}_0^s$ based on the total error and oscillation of $f$, respectively. Using a quasi-monotonicity property of oscillation and a localized discrete upper bound, we prove the following quasi-optimal convergence rate for the AMFEM in terms of DOFs by assuming the marking parameter $\theta \in (0, \theta_*)$ with $0 < \theta_* < 1$ (see Theorem 6.3):

$$(\mathcal{E}_N^2 + \text{osc}_N^2)^{1/2} \leq C^s \Theta^s \langle s, \theta \rangle \langle (p, f, A) \rangle_s + ||f||_{\mathcal{A}_0^s} (\# \mathcal{T}_N - \# \mathcal{T}_0)^{-s}.$$  

The rest of this paper is organized as follows. In Section 2, we shall give some preliminaries and details on notations. Some auxiliary results are included in Section 3 for later usage. Section 4 is devoted to the analysis of efficiency of the a posteriori error estimator. The proof of convergence for the AMFEM is placed in Section 5. Finally, we shall prove the quasi-optimal convergence rate for the AMFEM in Section 6 and give conclusions in Section 7.
2 Preliminaries and notations

2.1 Weak formulation

By splitting (1.2) into two equations, the mixed formulation is given as

\[
\begin{cases}
- \text{div } p = f \quad \text{and} \quad p = A \nabla u \\
u = 0
\end{cases}
\text{in } \Omega \quad \text{on } \partial \Omega.
\tag{2.1}
\]

Since the coefficient matrix \( A \) is symmetric and uniformly positive definite, by the Lax-Milgram theorem, there exists a unique solution \( u \in H^1_0(\Omega) \) to the problem (2.1). Moreover, the weak formulation of (2.1) reads as: Find \((p, u) \in H(\text{div}, \Omega) \times L^2(\Omega)\) such that

\[
(A^{-1} p, q)_{0, \Omega} + (\text{div } q, u)_{0, \Omega} = 0 \quad \text{for all } q \in H(\text{div}, \Omega),
\]

\[
(\text{div } p, v)_{0, \Omega} = -(f, v)_{0, \Omega} \quad \text{for all } v \in L^2(\Omega),
\tag{2.2}
\]

where \( H(\text{div}, \Omega) := \{ q \in L^2(\Omega)^2 : \text{div } q \in L^2(\Omega) \} \) is endowed with the norm given by \( ||q||_{H(\text{div}, \Omega)}^2 := ||q||_{L^2(\Omega)}^2 + ||\text{div } q||_{L^2(\Omega)}^2 \), and \((\cdot, \cdot)_{0, \Omega}\) denotes \( L^2 \) inner product on \( \Omega \).

For a given shape-regular triangulation \( \mathcal{T}_h \) of \( \Omega \) into triangles, let \( M_h \) and \( L_h \) denote finite dimensional subspaces of \( H(\text{div}, \Omega) \) and \( L^2(\Omega) \), respectively. In the step \textit{SOLVE} a mixed finite element method reads as: Find \((p_h, u_h) \in M_h \times L_h\) such that

\[
(A^{-1} p_h, q_h)_{0, \Omega} + (\text{div } q_h, u_h)_{0, \Omega} = 0 \quad \text{for all } q_h \in M_h,
\]

\[
(\text{div } p_h, v_h)_{0, \Omega} = -(f_h, v_h)_{0, \Omega} \quad \text{for all } v_h \in L_h,
\tag{2.3}
\]

where \( f_h \) is the \( L^2 \) projection of \( f \) over \( L_h \).

It is well-known that existence and uniqueness of the solution of (2.2) hold true, and that the discrete problem (2.3) has a unique solution when a discrete inf-sup-condition is satisfied by the discrete spaces \( M_h \) and \( L_h \) (cf. \cite{22}). So we are interested in controlling stress variable error \( \epsilon := p - p_h \in H(\text{div}, \Omega) \) and displacement error \( e := u - u_h \in L^2(\Omega) \), and suppose that the module \textit{SOLVE} outputs a pair of discrete solutions over \( \mathcal{T}_h \), namely, \((p_h, u_h) = \text{SOLVE}(\mathcal{T}_h)\).

2.2 Mixed finite elements

We consider some well-known mixed finite elements for the discretization problem (2.3), such as Raviart-Thomas (RT) elements, Brezzi-Douglas-Marini (BDM) elements and Brezzi-Douglas-Fortin-Marini (BDFM) elements \cite{52,21,22}, which are briefly described for all triangle \( T \in \mathcal{T}_h \) by some \( D_l(T) \subset C(T) \) and \( M_l(T) \subset C(T)^2 \) given in the following table.
Examples for mixed finite elements

| Element | $M_l(T)$ | $L_l(T)$ |
|---------|----------|----------|
| R | $P^l \oplus P^l(x)$ | $P^l$ |
| BDM | $P^l_{0,0}$ | $P^l$ |
| BDFM | $\{q \in P^l_{0,0} : (q \cdot v)|_{\partial T} \in R_l(\partial T)\}$ | $P^l$ |

Here $P^l$ denotes the set of polynomials of total degree $\leq l$ and $R_l(\partial T)$ denotes the set of polynomials of degree at most $l$ on each edge of $T$ (not necessary continuous).

By using the above sets $M_l(T)$ and $D_l(T)$, the discrete spaces $M_h$ and $L_h$ are given by

$$
M_h := \{ q_h \in H(\text{div}, \Omega) : q_h|_T \in M_l(T) \quad \text{for} \ T \in \mathcal{T}_h\},
$$

$$
L_h := \{ v_h \in L^2(\Omega) : v_h|_T \in D_l(T) \quad \text{for} \ T \in \mathcal{T}_h\}.
$$

### 2.3 Assumption on $\mathcal{T}_h$

Let $\mathcal{T}_h$ be a shape regular triangulation in the sense of [34] which satisfies the angle condition, namely, there exists a constant $c_1$ such that for all $T \in \mathcal{T}_h$

$$
c_1^{-1}h_T^2 \leq |T| \leq c_1h_T^2,
$$

where $h_T := \text{diam}(T)$, and $|T|$ is the area of $T$.

Let $\varepsilon_h$ denote the set of element edges in $\mathcal{T}_h$, $J(v)|_E := (v|_{T_+})|_E - (v|_{T_-})|_E$ denote the jump of $v \in H^1(\bigcup \mathcal{T}_h)$ over an interior edge $E := T_+ \cap T_-$ of length $h_E := \text{diam}(E)$, shared by the two neighboring (closed) triangles $T_\pm \in \mathcal{T}_h$, specifically, $J(v)|_E := (v|_E)$ if $E = \partial T \cap \partial \Omega$. Furthermore, for $T \in \mathcal{T}_h$, we denote by $\omega_T$ the union of all elements in $\mathcal{T}_h$ sharing one edge with $T$, and define the patch of $E \in \varepsilon_h$ by

$$
\omega_E := \bigcup\{ T \in \mathcal{T}_h : E \subset T \}.
$$

Denote $\Gamma_h := \bigcup \varepsilon_h$, and let $J : H^1(\bigcup \mathcal{T}_h) \rightarrow L^2(\Gamma_h)$ be an operator with $H^1(\bigcup \mathcal{T}_h) := \{ v \in L^2(\Omega) : \forall T \in \mathcal{T}_h, v|_T \in H^1(T)\}$.

Throughout the paper, the local versions of the differential operators div, $\nabla$, curl are understood in the distribution sense, i.e., in $D'(\Omega)$, namely, div$_h$, curl$_h : H^1(\bigcup \mathcal{T}_h)^2 \rightarrow L^2(\Omega)$ and $\nabla_h : H^1(\bigcup \mathcal{T}_h) \rightarrow L^2(\Omega)^2$ are defined such that, e.g., div$_h v|_T := \text{div}(v|_T)$ in $D'(T)$, for all $T \in \mathcal{T}_h$.

### 2.4 A posteriori error estimators

For all $E \in \varepsilon_h$, let $\tau$ be the unit tangential vector along $E$, and $(p_h, u_h) \in M_h \times L_h$ be the solution of (2.3) with respect to the triangulation $\mathcal{T}_h$. Then the local
estimator is defined by (see [24])
\[
\eta_{T,\kappa}^2 := ||h^\kappa(f + \text{div } p_h)||^2_{L^2(T)} + h_T^2 ||\text{curl}(A^{-1}p_h)||^2_{L^2(T)}
+ ||h^{1/2}J(A^{-1}p_h \cdot \tau)||^2_{L^2(\partial T)} + ||h(A^{-1}p_h - \nabla_h u_h)||^2_{L^2(T)}
\]
with \(0 \leq \kappa \leq 1\) and the global estimator is given as
\[
\eta_{h,\kappa}^2 := \sum_{T \in T_h} \eta_{T,\kappa}^2.
\]
Here \(\text{curl} \psi := \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2}\) for \(\psi = (\psi_1, \psi_2)^T\). For convenience we also define the stress variable error in weighted norm
\[
\mathcal{E}_h^2 := ||A^{-1/2}(p - p_h)||^2_{L^2(\Omega)} + ||h \text{div}(p - p_h)||^2_{L^2(\Omega)}.
\]
Note that in this paper, the Curls of a scalar function \(\phi\) are involved as
\[
\text{Curl} \phi := (-\frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_1})^T.
\]
In [23], reliability of the a posteriori error estimator \(\eta_{h,\kappa}\) with estimates of the stress and displacement variables in weighted norm was obtained under a weak regularity assumption on \(A\) (see Section 4.2 in [23]), whereas efficiency of \(\eta_{h,\kappa}\) was derived by assuming additionally that \(A^{-1}p_h\) is a piecewise polynomial vector. In Section 3, we shall prove the efficiency without this additional assumption on the coefficient matrix. But we pay the price to introduce oscillation of data.

In many applications the stress variable is of interest. We define the local estimator for the stress variable error as
\[
\eta_{T,h}^2(p_h, T) := h_T^2 ||f - f_h||^2_{L^2(T)} + h_T^2 ||\text{curl}(A^{-1}p_h)||^2_{L^2(T)}
+ h_T ||J(A^{-1}p_h \cdot \tau)||^2_{L^2(\partial T)},
\]
and define the global error estimator as
\[
\eta_{h,h}^2(p_h, T_h) := \sum_{T \in T_h} \eta_{T,h}^2(p_h, T).
\]
We assume that, for a given triangulation \(T_h\) and a pair of corresponding discrete solutions \((p_h, u_h) \in M_h \times L_h\), the module \(ESTIMATE\) for the stress variable outputs the indicators
\[
\{\eta_{T,h}^2(p_h, T)\}_{T \in T_h} = ESTIMATE(p_h, T_h).
\]
Then the estimates of the stress variable error \(\epsilon\) in a weighted norm are reduced to (see [23])
\[
||A^{-1/2}\epsilon||^2_{L^2(\Omega)} + ||h \text{div } \epsilon||^2_{L^2(\Omega)} \leq C_1 \eta_{T,h}^2(p_h, T_h), \tag{2.4}
\]
where \(C_1\) is a constant independent of the mesh size. In Section 3, we shall show efficiency of the estimator \(\eta_{T,h}(p_h, T_h)\) for the stress variable error in a weighted norm.
2.5 Oscillation of data

For an integer $n \geq l+1$, we denote by $\Pi_n^2$ the $L^2$—best approximation operator onto the set of piecewise polynomials of degree $\leq n$ over $T \in T_h$ or $E \in E_h$, denote by $\text{id}$ the identity operator, and set $P_n^2 := \text{id} - \Pi_n^2$. We define oscillation $\widetilde{\text{osc}}_h$ of data as:

$$\widetilde{\text{osc}}_h^2 := ||hP_n^2\text{curl}(A^{-1}p_h)||^2_{L^2(\Omega)} + ||h^{1/2}P_{n+1}^2J(A^{-1}p_h \cdot \tau)||^2_{L^2(T_h)}$$

$$+ ||hP_n^2(A^{-1}p_h - \nabla_h u_h)||^2_{L^2(\Omega)}.$$

For the stress variable error in a weighted norm, convergence and quasi optimality of the AMFEM are involved in the oscillations of the data including the right-hand side term $f$. Then we define the oscillation of data as

$$\text{osc}^2_{T_h}(p_h, T) := h_T^2||P_n^2\text{curl}(A^{-1}p_h)||^2_{L^2(T)} + h_T||P_{n+1}^2J(A^{-1}p_h \cdot \tau)||^2_{L^2(\partial T)}$$

$$+ ||h(f - f_h)||^2_{L^2(T)} \quad \text{for all } T \in T_h.$$

Finally, for any subset $T'_h \subset T_h$, we set

$$\text{osc}^2_{T_h}(p_h, T'_h) := \sum_{T \in T'_h} \text{osc}^2_{T_h}(p_h, T) \quad \text{and} \quad \text{osc}^2 := \text{osc}^2_{T_h}(p_h, T_h).$$

We also define oscillation of $f$ as

$$\text{osc}^2(f, T_h) := ||h(f - f_h)||^2_{L^2(\Omega)}.$$

**Remark 2.1.** For the estimator of the stress variables, let $T_h$ be a triangulation, $q_h \in M_h$ be given. By substituting $p_h$ with $q_h$ in the definitions of $\eta_{T_h}(p_h, T)$ and $\text{osc}_{T_h}(p_h, T)$, we can see that the indicator $\eta_{T_h}(q_h, T)$ controls oscillation $\text{osc}_{T_h}(q_h, T)$, i.e., $\text{osc}_{T_h}(q_h, T) \leq \eta_{T_h}(q_h, T)$ for all $T \in T_h$. In addition, for the stress variables, the definitions of the error indicator and oscillation are fully localized to $T$, which means there holds $\eta_{T_h}(q_h, T) = \eta_{T_h}(q_h, T)$ and $\text{osc}_{T_h}(q_h, T) = \text{osc}_{T_h}(q_h, T)$ for any refinement $T_h$ of $T_H$ with $T \in T_h \cap T_H$ and $q_h \in M_H$. Moreover, a combination of the monotonicity of local mesh sizes and properties of the local $L^2$—projection yields

$$\eta_{T_h}(q_h, T_h) \leq \eta_{T_h}(q_h, T_H) \quad \text{and} \quad \text{osc}_{T_h}(q_h, T_h) \leq \text{osc}_{T_h}(q_h, T_H) \quad \forall q_h \in M_H.$$

We note that in this paper, the triangulation $T_h$ means a refinement of $T_H$, all notations with respect to the mesh $T_H$ are defined similarly. Throughout the rest of the paper we use the notation $A \lesssim B$ to represent $A \leq CB$ with a mesh-size independent, generic constant $C > 0$. Moreover, $A \approx B$ abbreviates $A \lesssim B \lesssim A$. 
2.6 The module MARK

By relying on Dörfler marking, while only concerning the stress variable error in a weighted norm, we select the elements to mark according to the indicators for the stress variables, namely, given a grid $T_H$ with the set of indicators $\{\eta_{TH}(p_H, T)\}_{T \in T_H}$ and marking parameter $\theta \in (0, 1]$, the module $MARK$ outputs a subset of making elements $M_H \subset T_H$, i.e.,

$$M_H = MARK(\{\eta_{TH}(p_H, T)\}_{T \in T_H}, T_H, \theta),$$

such that $M_H$ satisfies Dörfler property

$$\eta_{TH}(p_H, M_H) \geq \theta \eta_{TH}(p_H, T_H). \quad (2.5)$$

2.7 The module REFINE

In the $REFINE$ step, we suppose that the refinement rule, such as the longest edge bisection [53, 54] and newest vertex bisection [56, 47, 48], is guaranteed to produce conforming and shape regular mesh. Given a fixed integer $b \geq 1$, a mesh $T_H$, and a subset $M_H \subset T_H$ of marked elements, a conforming triangulation $T_h$ is output by

$$T_h = REFINE(T_H, M_H),$$

where all elements of $M_H$ are at least bisected $b$ times. Note that not only marked elements get refined but also additional elements are refined to recovery the conformity of triangulations. Let $R := R_{T_H \rightarrow T_h} := T_H / (T_H \cap T_h)$ denote the set of refined elements, which means $M_H \subset R_{T_H \rightarrow T_h}$. In general, the number of these additionally refined elements is not controlled by $\#M_H$, that is to say, $\#T_h - \#T_H$ cannot be bounded by $CM_H$ with a positive constant $C$, which is independent of $T_H$ and may depend on the refinement level. On the other hand, by arguing with the entire sequence $\{T_k\}_{k \geq 0}$ of refinement, Binev, Dahman, and DeVore showed in two dimensions that the cumulative number of elements added by insuring conformity does not inflate the total number of marked elements [13]. Stevenson generalized this result to higher dimensions [58].

**Lemma 2.1.** ([58]) (Complexity of $REFINE$). Assume that $T_0$ verifies condition (b) of section 4 in [58]. Let $\{T_k\}_{k \geq 0}$ be any conforming triangulation sequence refined from a shape regular triangulation $T_0$, where $T_{k+1}$ is generated from $T_k$ by $T_{k+1} = REFINE(T_k, M_k)$ with a subset $M_k \subset T_k$. Then there exists a constant $C_0$ solely depending on $T_0$ and $b$ such that

$$\#T_k - \#T_0 \leq C_0 \sum_{j=0}^{k-1} \#M_j \quad \text{for all } k \geq 1.$$
2.8 Adaptive algorithm

We now collect the modules described in the previous sections to obtain the AM-FEM of the stress variables. In doing this, we replace the subscript \( H \) (or \( h \)) by an iteration counter called \( k \geq 0 \). Let \( \mathcal{T}_0 \) be a shape regular triangulation, \( \eta_0 := \eta_{\mathcal{T}_0}(p_0, T_0) \) denote the error indicator onto the initial mesh \( \mathcal{T}_0 \), with a right hand side \( f \in L^2(\Omega) \), a tolerance \( \varepsilon \), and a parameter \( \theta \in (0, 1] \). The basic loop of the AMFEM is then given by the following iterations:

\[
\text{Algorithm for AMFEM} \\
[T_N, (p_N, u_N)] = \text{AMFEM}(T_0, f, \varepsilon, \theta) \\
\text{set } k = 0, \eta_k = \eta_0 \text{ and iterate} \\
\text{WHILE } \eta_k \geq \varepsilon \text{ DO} \\
(1) \quad (p_k, u_k) = \text{SOLVE}(T_k); \\
(2) \quad \{\eta_k(p_k, T)\}_{T \in \mathcal{T}_k} = \text{ESTIMATE}(p_k, T_k); \\
(3) \quad M_k = \text{MARK}(\{\eta_k(p_k, T)\}_{T \in \mathcal{T}_k}, T_k, \theta); \\
(4) \quad T_{k+1} = \text{REFINE}(T_k, M_k); k = k + 1. \\
\text{END WHILE} \\
T_N = T_k. \\
\text{END AMFEM}
\]

We note that the AMFEM for the stress variables is a standard algorithm in which it employs only the error estimator \( \{\eta_{\mathcal{T}_k}(p_k, T)\}_{T \in \mathcal{T}_k} \), does not use the oscillation indicators \( \{\text{osc}_{\mathcal{T}_k}(p_k, T)\}_{T \in \mathcal{T}_k} \), and does not need the interior node property for marked elements.

3 Analysis of efficiency for estimators

We devote this section to the analysis of efficiency of the a posteriori error estimator \( \eta_{h, \kappa} \) for the stress and displacement variables in a weighted norm. Herein, We avoid the additional assumption that \( A^{-1} p_h \) is a polynomial vector on each element, which is necessary for the proof of the efficiency of \( \eta_{h, \kappa} \) in [23] for the RT, BDM, and BDFM elements.

Lemma 3.1. Let \( (p_h, u_h) \in M_h \times L_h \) be a pair of discrete solutions of (2.3), \( P_n^2 \) denote the operator defined in Section 2.5. Then, for all \( T \in \mathcal{T}_h \), it holds

\[
h_T \| \text{curl}(A^{-1} p_h) \|_{L^2(T)} \lesssim \| A^{-1/2} \xi \|_{L^2(T)} + h_T \| P_n^2 \text{curl}(A^{-1} p_h) \|_{L^2(T)}. \tag{3.1}
\]

Proof. From the triangle inequality, we have

\[
\| \text{curl}(A^{-1} p_h) \|_{L^2(T)} \leq 2(\| P_n^2 \text{curl}(A^{-1} p_h) \|_{L^2(T)} + \| \Pi_0^2 \text{curl}(A^{-1} p_h) \|_{L^2(T)}). \tag{3.2}
\]
For all $T \in \mathcal{T}_h$, let $\psi_T$ denote the bubble function on $T$ with zero boundary values on $T$ and $0 \leq \psi_T \leq 1$, the equivalence of norms $\|\psi_T^{1/2} \cdot \|_{L^2(T)}$ and $\| \cdot \|_{L^2(T)}$ for polynomials implies

$$\|\Pi_n^2 \text{curl}(A^{-1} p_h)\|_{L^2(T)}^2 \approx \|\psi_T^{1/2} \Pi_n^2 \text{curl}(A^{-1} p_h)\|_{L^2(T)}^2 = (\psi_T \Pi_n^2 \text{curl}(A^{-1} p_h), \text{curl}(A^{-1} p_h))_{0,T} \tag{3.3}$$

$$- (\psi_T \Pi_n^2 \text{curl}(A^{-1} p_h), P^2 \text{curl}(A^{-1} p_h))_{0,T}.$$ 

From $\epsilon := p - p_h$, $P := A \nabla u$, Stokes theory, and integration by parts, we have

$$\int_T \psi_T \Pi_n^2 \text{curl}(A^{-1} p_h) \cdot \text{curl}(A^{-1} p_h) = \int_T \psi_T \Pi_n^2 \text{curl}(A^{-1} p_h) \cdot \text{curl}(\nabla u - A^{-1} \epsilon)$$

$$= - \int_T \psi_T \Pi_n^2 \text{curl}(A^{-1} p_h) \cdot \text{curl}(A^{-1} \epsilon) \tag{3.4}$$

$$= \int_T \text{Curl}(\psi_T \Pi_n^2 \text{curl}(A^{-1} p_h)) \cdot (A^{-1} \epsilon) \leq |\psi_T \Pi_n^2 \text{curl}(A^{-1} p_h)|_{H^1(T)} \|A^{-1} \epsilon\|_{L^2(T)}.$$ 

A combination of (3.2)-(3.4), together with an inverse estimation and the property of $L^2$—projection, yields

$$\|\text{curl}(A^{-1} p_h)\|_{L^2(T)}^2 \lesssim (\|P_n^2 \text{curl}(A^{-1} p_h)\|_{L^2(T)} + h_T^{-1} \|A^{-1} \epsilon\|_{L^2(T)}) \times \|\text{curl}(A^{-1} p_h)\|_{L^2(T)}.$$ 

This implies the desired result. \qed

**Lemma 3.2.** Let $(p_h, u_h) \in M_h \times L_h$ be the discrete solutions of (2.3), $P_{n+1}^2$ denote the operator defined in Section 2.5. Then, for all $E \in \varepsilon_h$, it holds

$$h_E^{1/2} \|J(A^{-1} p_h \cdot \tau)\|_{L^2(E)} \lesssim h_E^{1/2} \|P_{n+1}^2 J(A^{-1} p_h \cdot \tau)\|_{L^2(E)} + h_E \|P_n^2 \text{curl}_h(A^{-1} p_h)\|_{L^2(\omega_E)} + \|A^{-1/2} \epsilon\|_{L^2(\omega_E)}. \tag{3.5}$$

**Proof.** For all $E \in \varepsilon_h$, let $\psi_E$ denote the bubble function on $E$ with the support set $\omega_E$ and $0 \leq \psi_E \leq 1$. Put $\sigma := J(A^{-1} p_h \cdot \tau)$. since $\Pi_{n+1}^2 : L^2(E) \rightarrow P_{n+1}(E)$ is an $L^2$—projection operator, where $P_{n+1}(E)$ is a set of polynomials of total degree $\leq n + 1$ over $E$, there exists an extension operator $P : C(E) \rightarrow C(\omega_E)$ \cite{59,60} such that

$$P^2_{n+1} \sigma|_E = \Pi_{n+1}^2 \sigma \quad \text{and} \quad \|\psi_E P^2_{n+1} \sigma\|_{L^2(\omega_E)} \approx h_E^{1/2} \|\Pi_{n+1}^2 \sigma\|_{L(E)}. \tag{3.6}$$
From the triangle inequality, we obtain
\[ \|\sigma\|^2_{L^2(E)} \leq 2(\|P^2_n\sigma\|^2_{L^2(E)} + \|\Pi^2_{n+1}\sigma\|^2_{L^2(E)}). \] (3.7)

The equivalence of norms \( \|\psi^{1/2} E\|_{L^2(E)} \) and \( \|\cdot\|_{L^2(E)} \) for polynomials implies
\[ \|\Pi^2_{n+1}\sigma\|^2_{L^2(E)} \approx \|\psi^{1/2} E\Pi^2_{n+1}\sigma\|^2_{L^2(E)} = (\psi E \Pi^2_{n+1}\sigma, \Pi^2_{n+1}\sigma)_{0,E} \]
\[ = (\psi E \Pi^2_{n+1}\sigma, \sigma)_{0,E} - (\psi E \Pi^2_{n+1}\sigma, P^2_{n+1}\sigma)_{0,E}. \] (3.8)

From integration by parts, we get
\[ (\psi E \Pi^2_{n+1}\sigma, \sigma)_{0,E} = \int_E \psi E P\Pi^2_{n+1}\sigma \cdot \sigma = \int_{\omega_E} A^{-1} p_h \cdot \text{curl}(\psi E P\Pi^2_{n+1}\sigma) \]
\[ + \int_{\omega_E} \text{curl}_h(A^{-1} p_h) \psi E P\Pi^2_{n+1}\sigma. \] (3.9)

From Stokes theory, and noticing that \( A^{-1} p_h = \nabla u - A^{-1} \epsilon \), we have
\[ \int_{\omega_E} A^{-1} p_h \cdot \text{curl}(\psi E P\Pi^2_{n+1}\sigma) = - \int_{\omega_E} A^{-1} \epsilon \cdot \text{curl}(\psi E P\Pi^2_{n+1}\sigma). \] (3.10)

A combination of (3.9) and (3.10) yields
\[ (\psi E \Pi^2_{n+1}\sigma, \sigma)_{0,E} \leq \|A^{-1} \epsilon\|^2_{L^2(\omega_E)} \|\psi E P\Pi^2_{n+1}\sigma\|^2_{H^1(\omega_E)} \]
\[ + \|\text{curl}_h(A^{-1} p_h)\|_{L^2(\omega_E)} \|\psi E P\Pi^2_{n+1}\sigma\|_{L^2(\omega_E)}. \] (3.11)

This inequality, together with an inverse estimate and (3.6), yields
\[ (\psi E \Pi^2_{n+1}\sigma, \sigma)_{0,E} \lesssim \left( \|A^{-1} \epsilon\|^2_{L^2(\omega_E)} + h_{E} \|\text{curl}_h(A^{-1} p_h)\|_{L^2(\omega_E)} \right) \]
\[ \times h_{E}^{-1/2} \|\Pi^2_{n+1}\sigma\|_{L^2(E)}. \] (3.12)

We apply Lemma 3.1 to the above inequality (3.12) and use the property of \( L^2 \)–projection to get
\[ (\psi E \Pi^2_{n+1}\sigma, \sigma)_{0,E} \lesssim (h_{E} \|P^2_n\text{curl}_h(A^{-1} p_h)\|_{L^2(\omega_E)} + \|A^{-1/2} \epsilon\|^2_{L^2(\omega_E)}) h_{E}^{-1/2} \|\sigma\|_{L^2(E)}. \] (3.13)

From (3.7), (3.8), the property of \( L^2 \)–projection, equivalence of norms \( \|\psi^{1/2} E\|_{L^2(E)} \) and \( \|\cdot\|_{L^2(E)} \) for polynomials, and (3.13), we obtain
\[ h_{E}^{1/2} \|\sigma\|^2_{L^2(E)} \lesssim (h_{E}^{1/2} \|P^2_n\sigma\|_{L^2(E)} + h_{E} \|P^2_n\text{curl}_h(A^{-1} p_h)\|_{L^2(\omega_E)} \]
\[ + \|A^{-1/2} \epsilon\|^2_{L^2(\omega_E)}) \|\sigma\|_{L^2(E)}. \] (3.14)

The above inequality (3.14) implies the desired result (3.5) by canceling one \( \|\sigma\|_{L^2(E)} \).
Lemma 3.3. Let \((p_h, u_h) \in M_h \times L_h\) be a pair of discrete solutions of (2.3). Let \(P_n^2\) denote the operator defined in Section 2.5. Then, for all \(T \in \mathcal{T}_h\), it holds

\[
\eta \left| h_T |A^{-1} p_h - \nabla_h u_h| L^2(T) \right| \lesssim h_T |P_n^2(A^{-1} p_h - \nabla_h u_h)| L^2(T) + \|e\|_{L^2(T)} + \|A^{-1/2} e\|_{L^2(T)}. \tag{3.15}
\]

Proof. Denote \(\mu := A^{-1} p_h - \nabla_h u_h\), and let \(\psi_T\) be the bubble function defined in the proof of Lemma eflem 3.1. Since \(\Pi_n^2\) is the \(L^2\)-best approximation operator onto the set of polynomials of degree \(\leq n\) over \(T \in \mathcal{T}_h\), we have

\[
\|e\|_{L^2(T)} \leq 2(\|P_n^2(\mu)\|_{L^2(T)} \|\mu\|_{L^2(T)} + \|\Pi_n^2(\mu)\|_{L^2(T)}). \tag{3.16}
\]

The property of the bubble function \(\psi_T\) indicates

\[
\|\Pi_n^2(\mu)\|_{L^2(T)}^2 \approx \|\psi_T^{1/2}\Pi_n^2(\mu)\|_{L^2(T)}^2 = (\psi_T^{1/2}\Pi_n^2(\mu), \Pi_n^2(\mu))_{0,T} = (\psi_T(\Pi_n^2(\mu), \mu))_{0,T} - (\psi_T(\Pi_n^2(\mu), P_n^2(\mu)))_{0,T}. \tag{3.17}
\]

Since \(\epsilon = p - p_h\), integration by parts and an inverse estimate lead to

\[
\int_T \psi_T \Pi_n^2(\epsilon) \cdot \mu = \int_T (\nabla_h e - A^{-1} \epsilon) \cdot \psi_T \Pi_n^2(\mu) = - \int_T \text{div}(\psi_T \Pi_n^2(\mu)) e - \int_T A^{-1} \epsilon \cdot \psi_T \Pi_n^2(\mu) \lesssim \|\mu\|_{L^2(T)} + \|A^{-1/2} \epsilon\|_{L^2(T)} \|\mu\|_{L^2(T)}.
\]

A combination of (3.16)-(3.18) yields

\[
h_T \|\mu\|_{L^2(T)}^2 \lesssim (h_T \|P_n^2(\mu)\|_{L^2(T)} + \|\epsilon\|_{L^2(T)} + \|A^{-1/2} \epsilon\|_{L^2(T)}) \|\mu\|_{L^2(T)}. \tag{3.19}
\]

The assertion (3.15) then follows from the above inequality (3.19).

\[\square\]

We now prove efficiency of the estimator \(\eta_{h,\kappa}\) by using the above three lemmas.

Theorem 3.1. Let \((p, u)\) and \((p_h, u_h)\) \(\in M_h \times L_h\) be the solutions of (2.7) and (2.3), respectively, and \(\eta_{h,\kappa}\) and \(\widetilde{osc}_h\) be defined as in Section 2.4 and 2.5. Then, for the estimator of the stress and displacement variables for the RT, BDM, and BDFM elements, there exists a constant hidden in \(\lesssim\), independent of mesh-size, such that

\[
\eta_{h,\kappa}^2 \lesssim \|A^{-1/2}(p - p_h)\|_{L^2(\Omega)}^2 + \|h^\kappa \text{div}(p - p_h)\|_{L^2(\Omega)}^2 + \|u - u_h\|_{L^2(\Omega)}^2 + \widetilde{osc}_h^2. \tag{3.20}
\]
Proof. Notice that for all \( T \in \mathcal{T}_h \), it holds
\[
\|h^\kappa (f + \text{div } p_h)\|_{L^2(T)} = \|h^\kappa \text{div } \epsilon\|_{L^2(T)} \quad \text{for all } 0 \leq \kappa \leq 1.
\]
A combination of Lemmas 3.1-3.3 yields the assertion (3.20) by summing over all \( T \in \mathcal{T}_h \) and \( E \in \varepsilon_h \).

Theorem 3.2. Under the assumptions of Theorem 3.1, let \( \eta_{T_h}(p_h, T_h) \), \( \mathcal{E}_h \), and \( \text{osci}_h \) be defined as in Section 2.4 and 2.5. Then, for the estimator of the stress variables for the RT, BDM, and BDFM elements, there exists a constant \( C_2 \) independent of mesh-size, such that
\[
C_2 \eta_{T_h}(p_h, T_h)^2 \leq \mathcal{E}_h^2 + \text{osci}_h^2.
\]
(3.21)

Proof. Since \(-\text{div } \epsilon = f + \text{div } p_h\), combining Lemmas 3.1-3.2 and summing over all \( T \in \mathcal{T}_h \) and \( E \in \varepsilon_h \), we obtain the desired result (3.21).

4 Auxiliary results

In this section, we will give some auxiliary results for convergence and quasi-optimality of the AMFEM for the stress variable.

4.1 Quasi-orthogonality

Lemma 4.1. Given a function \( f \in L^2_0(\Omega) \), there exists a function \( q \in H^1_0(\Omega)^2 \) such that
\[
\text{div } q = f \quad \text{and} \quad ||q||_{H^1(\Omega)^2} \leq ||f||_{L^2(\Omega)}.
\]

We refer to [20, 4, 41] for detailed proofs of this lemma respectively on smooth or convex, non-convex and general Lipschitz domains.

Lemma 4.2. Let \( L_h, M_h \) be respectively the discrete displacement and stress spaces given in Section 2.2. Set \( W := H(\text{div}, \Omega) \cap L^2(\Omega)^2 \) for some \( q > 2 \), and let \( \Pi_{L_h}, \text{id} \) and \( \perp \) be respectively the \( L^2(\Omega) \)--projection onto \( L_h \), the identity operator and \( L^2(\Omega) \)--orthogonality. Then there exists an operator \( \Pi_h : W \to M_h \) with the following commuting diagram
\[
\begin{array}{c}
W \xrightarrow{\text{div}} L^2(\Omega) \\
\downarrow \Pi_h \downarrow \Pi_{L_h} \\
M_h \xrightarrow{\text{div}} L_h
\end{array}
\]
(4.1)
such that
(I) it holds a local estimate (note that $H^1(\bigcup T_h)^2 \cap H(\text{div}, \Omega) \subset W$)

\[ ||h^{-1}(id - \Pi_h)q||_{L^2(\Omega)} \lesssim |q|_{H^1(\bigcup T_h)} \quad \text{for all } q \in H^1(\bigcup T_h)^2 \cap H(\text{div}, \Omega); \quad (4.2) \]

(II) $\Pi_h$ approximates the normal components on element edges with

\[ \int_E \nu_h(id - \Pi_h)q \cdot \nu_E ds = 0 \quad \text{for all } E \in \epsilon_h, \nu_h \in L_h, q \in W; \]

where $\nu_E$ is the unit normal vector along $E$.

For the detailed construction of such interpolation operator $\Pi_h$ and proof of these properties, we refer to [44, 3, 22]. Note that the above commuting diagram means

\[ \text{div}(id - \Pi_h)W \perp L_h. \]

**Lemma 4.3.** Let $T_h$ and $T_H$ be two nested triangulations, $\Pi_{L_H}$ be the $L^2(\Omega)$ projection onto $L_H$, and $(p_h, u_h) \in M_h \times L_h$ be the solutions of \(2.3\). Then for any $T \in T_H$, there exists a positive constant $C_0$ depending only on the shape regularity of $T_h$, such that

\[ ||u_h - \Pi_{L_H}u_h||_{L^2(T)} \leq \sqrt{C_0} ||H^1(T)|| \cdot A^{-1/2} p_h ||L^2(T). \quad (4.3) \]

**Proof.** Let $\Pi_{L_h}$ denote the $L^2$—projection operator over $L_h$. For any $T \in T_H$, by the definition of $L^2$—projection operator $\Pi_{L_H}$, we have $\int_T (\Pi_{L_h} - \Pi_{L_H}) u_h = 0$, i.e., $(\Pi_{L_h} - \Pi_{L_H}) u_h \in L_0^2(T)$. We thus can apply Lemma 4.1 to find a function $q \in H^1_0(T)^2$ such that

\[ \text{div} q = (\Pi_{L_h} - \Pi_{L_H}) u_h \text{ in } T \text{ and } ||q||_{H^1(T)} \lesssim ||(\Pi_{L_h} - \Pi_{L_H}) u_h||_{L^2(T)}. \quad (4.4) \]

We extend $q$ to $H^1_0(\Omega)^2$ by zero, since $\Pi_h$ (or $\Pi_H$) approximates the normal components on elements edge, the second result (II) of Lemma 4.2 implies that $(\Pi_h - \Pi_H) q \in M_h$ and supp$(\Pi_h - \Pi_H) q \subseteq T$. Noticing that div $q \in L_h$, we have

\[ (\text{div} q - (\Pi_{L_h} - \Pi_{L_H}) \text{div} q, (\Pi_{L_h} - \Pi_{L_H}) u_h)_{0,T} = 0 \quad (4.5) \]

and

\[ (u_h - (\Pi_{L_h} - \Pi_{L_H}) u_h, (\Pi_{L_h} - \Pi_{L_H}) \text{div} q)_{0,T} = 0. \quad (4.6) \]

Since $\Pi_{L_h} u_h = u_h$, a combination of the first equality of (4.4), and (4.5)-(4.6) yields

\[ ||u_h - \Pi_{L_H}u_h||_{L^2(T)}^2 = ||(\Pi_{L_h} - \Pi_{L_H}) u_h||_{L^2(T)}^2 = (u_h, (\Pi_{L_h} - \Pi_{L_H}) \text{div} q)_{0,T}. \quad (4.7) \]
Using the locality of \(q\), the commuting property (4.1) and (2.3), we obtain

\[
(u_h, (\Pi_{Lh} - \Pi_{LH}) \text{div } q)_{0,T} = (u_h, (\Pi_{Lh} - \Pi_{LH}) \text{div } q)_{0,\Omega} \\
= (u_h, \text{div}(\Pi_h - \Pi_H)q)_{0,\Omega} \\
= - (A^{-1} p_h, (\Pi_h - \Pi_H)q)_{0,\Omega} \\
= - (A^{-1} p_h, (\Pi_h - \Pi_H)q)_{0,T}.
\] (4.8)

The local approximation (4.2) of Lemma 4.2 indicates

\[
| (A^{-1} p_h, (\Pi_h - \Pi_H)q)_{0,T} | \leq |A^{-1} p_h|_{L^2(T)} (|q - \Pi_H q|_{L^2(T)} + |q - \Pi_H q|_{L^2(T)}) \\
\leq C_0 H_T |A^{-1/2} p_h|_{L^2(T)} |q|_{H^1(T)}. \tag{4.9}
\]

Finally, the desired result (4.3) follows from the second inequality of (4.4) and (4.7) - (4.9).

In order to prove the quasi-orthogonality, we need to introduce a pair of auxiliary solutions. Let \(f_H := \Pi_{LH} f\) denote the \(L^2\)-projection of \(f\) over \(L_H\), and consider the following problem: Find \((\tilde{p}_h, \tilde{u}_h) \in M_h \times L_h\) such that

\[
(A^{-1} \tilde{p}_h, q_h)_{0,\Omega} + (\text{div } q_h, \tilde{u}_h)_{0,\Omega} = 0 \quad \text{for all } q_h \in M_h, \\
(\text{div } \tilde{p}_h, v_h)_{0,\Omega} = -(f_H, v_h)_{0,\Omega} \quad \text{for all } v_h \in L_h.
\] (4.10)

In fact, the solution \((\tilde{p}_h, \tilde{u}_h)\) of this auxiliary problem may be regarded as another approximation to the flux and displacement \((p, u)\).

**Lemma 4.4.** Let \(T_H\) and \(T_H\) be two nested triangulations, \(\text{osc}(f_H, T_H)\) denote the oscillation of \(f_H := \Pi_{LH} f\) over \(T_H\), \((p_h, u_h)\) and \((\tilde{p}_h, \tilde{u}_h)\) be the solutions of (2.3) and (4.10), respectively. Then there exists a constant \(C_0\) depending only on the shape regularity of \(T_H\) such that

\[
|A^{-1/2}(p_h - \tilde{p}_h)|_{L^2(\Omega)} \leq \sqrt{C_0 \text{osc}(f_H, T_H)}.
\] (4.11)

**Proof.** Recall that \((p_h - \tilde{p}_h, u_h - \tilde{u}_h) \in M_h \times L_h\) satisfies the equations

\[
(A^{-1}(p_h - \tilde{p}_h), q_h)_{0,\Omega} + (\text{div } q_h, u_h - \tilde{u}_h)_{0,\Omega} = 0 \quad \text{for all } q_h \in M_h, \\
(\text{div}(p_h - \tilde{p}_h), v_h)_{0,\Omega} = -(f_h - f_H, v_h)_{0,\Omega} \quad \text{for all } v_h \in L_h.
\] (4.12)

According to the above equations (4.12), we choose \(q_h = p_h - \tilde{p}_h\) and \(v_h = u_h - \tilde{u}_h\) to obtain

\[
|A^{-1/2}(p_h - \tilde{p}_h)|_{L^2(\Omega)}^2 = -(\text{div}(p_h - \tilde{p}_h), u_h - \tilde{u}_h)_{0,\Omega} \\
= (f_h - \Pi_{LH} f, u_h - \tilde{u}_h)_{0,\Omega} = (f_h - \Pi_{LH} f, v_h)_{0,\Omega}.
\] (4.13)
Since $L_H \subset L_h$, it holds
\[(\Pi_{L_h} f, v_H)_{0, \Omega} = (f, v_H)_{0,\Omega} = (\Pi_{L_H} \Pi_{L_h} f, v_H)_{0,\Omega} = (\Pi_{L_H} f, v_H)\]
for all $v_H \in L_H$. This implies
\[\Pi_{L_H} \Pi_{L_h} f = \Pi_{L_H} f \quad \text{and} \quad (\Pi_{L_h} f - \Pi_{L_H} f, \Pi_{L_H} v_h)_{0,\Omega} = 0.\]

From (4.13), we have
\[||A^{-1/2}(p_h - \tilde{p}_h)||^2_{L^2(\Omega)} = ((\Pi_{L_h} - \Pi_{L_H}) f, v_h - \Pi_{L_H} v_h)_{0,\Omega} = \sum_{T \in \mathcal{T}_h} ((\Pi_{L_h} - \Pi_{L_H}) f, v_h - \Pi_{L_H} v_h)_{0,T}. \tag{4.14}\]

We apply Lemma [4.3] to $v_h - \Pi_{L_H} v_h$ in (4.14) and obtain
\[||A^{-1/2}(p_h - \tilde{p}_h)||^2_{L^2(\Omega)} \leq C_0 \sum_{T \in \mathcal{T}_h} H_T ||f_h - f_H||_{L^2(T)} ||A^{-1/2}(p_h - \tilde{p}_h)||_{L^2(T)} \leq \sqrt{C_0} \text{osc}(f_h, \mathcal{T}_H) ||A^{-1/2}(p_h - \tilde{p}_h)||_{L^2(\Omega)},\]
which leads to the desired result (4.11). \qed

We state the property of quasi-orthogonality as follows.

**Theorem 4.1.** (Quasi-orthogonality) Given $f \in L^2(\Omega)$ and two nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, let $(p_h, u_h)$ and $(p_H, u_H)$ be the solutions of (2.3) with respect to $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively. Then it holds
\[(A^{-1}(p - p_h), p_h - p_H)_{0,\Omega} \leq C_0^{1/2} ||A^{-1/2}(p - p_h)||_{L^2(\Omega)} \text{osc}(f_h, \mathcal{T}_H). \tag{4.15}\]

Furthermore, for any $\delta_1 > 0$, it holds
\[(1 - \delta_1)||A^{-1/2}(p - p_h)||^2_{L^2(\Omega)} \leq ||A^{-1/2}(p - p_h)||^2_{L^2(\Omega)} - ||A^{-1/2}(p_h - p_H)||^2_{L^2(\Omega)} + \frac{C_0}{\delta_1} \text{osc}^2(f_h, \mathcal{T}_H). \tag{4.16}\]

In particular, if $\text{osc}(f_h, \mathcal{T}_H) = 0$, then it holds
\[||A^{-1/2}(p - p_h)||^2_{L^2(\Omega)} = ||A^{-1/2}(p - p_H)||^2_{L^2(\Omega)} - ||A^{-1/2}(p_h - p_H)||^2_{L^2(\Omega)}. \tag{4.17}\]

**Proof.** Let $(\tilde{p}_h, \tilde{u}_h)$ solve the problem (4.10), then we have
\[(A^{-1}(p - p_h), \tilde{p}_h - p_H)_{0,\Omega} = -(\text{div}(\tilde{p}_h - p_H), u - \tilde{u}_h)_{0,\Omega} \]
\[= (f_H - f_H, u - \tilde{u}_h)_{0,\Omega} = 0. \tag{4.18}\]
From the above identity (4.18) and Lemma 4.4, we obtain

\[
(A^{-1}(p - p_h), p_h - p_H)_{0, \Omega} = (A^{-1}(p - p_h), p_h - \tilde{p}_h)_{0, \Omega}
\]
\[
\leq ||A^{-1/2}(p - p_h)||_{L^2(\Omega)}||A^{-1/2}(p_h - \tilde{p}_h)||_{L^2(\Omega)}
\]
\[
\leq C_0^{1/2}||A^{-1/2}(p - p_h)||_{L^2(\Omega)} \Omega \leq \frac{1}{2} ||A^{-1/2}(p - p_h)||_{L^2(\Omega)} + \frac{1}{2} ||A^{-1/2}(p_h - \tilde{p}_h)||_{L^2(\Omega)} + \frac{1}{2} \Omega \osc(f_h, \mathcal{T}_H),
\]
which implies the first result (4.15).

Furthermore, notice that

\[
||A^{-1/2}(p - p_h)||_{L^2(\Omega)}^2 = ||A^{-1/2}(p - p_h)||_{L^2(\Omega)}^2 - 2(A^{-1}(p - p_h), p_h - p_H)_{0, \Omega},
\]
then for any \(\delta_1 > 0\), from (4.19) and Young’s inequality we have

\[
||A^{-1/2}(p - p_h)||_{L^2(\Omega)}^2 \leq ||A^{-1/2}(p - p_h)||_{L^2(\Omega)}^2 - ||A^{-1/2}(p_h - p_H)||_{L^2(\Omega)}^2 + \delta_1 ||A^{-1/2}(p - p_h)||_{L^2(\Omega)}^2 + C_0 \delta_1 \Omega \osc(f_h, \mathcal{T}_H),
\]
which implies the estimate (4.16).

In particular, if \(\osc(f_h, \mathcal{T}_H) = 0\), then from (4.19) it follows \((A^{-1}(p - p_h), p_h - p_H)_{0, \Omega} = 0\). This, together with (4.20), yields the relation (4.17).

Although the oscillation of \(f_h\) over the triangulation \(\mathcal{T}_H\) appears in the estimate of quasi-orthogonality, it is dominated by \(\osc(f, \mathcal{T}_H)\). We refer to [33] for the proof of the following observation.

**Lemma 4.5.** Let \(f_h\) denote the \(L^2\)–projection of \(f\) over \(L_h\), then it holds

\[
\osc(f_h, \mathcal{T}_H) \leq \osc(f, \mathcal{T}_H).
\]

**4.2 Estimator and oscillation reduction**

In this subsection, we aim at reduction of estimator and oscillation. To this end, we relate the error indicators and oscillation of two nested triangulations to each other. The link involves weighted maximum-norms of the inverse matrix, \(A^{-1}\), of coefficient matrix \(A\) and its oscillation.

For a nonnegative integer \(m = n - l\), any given triangulation \(\mathcal{T}_H\), and \(v \in L^\infty(\Omega)\), we denote by \(\Pi_m^\infty v\) the best \(L^\infty(\Omega)\)–approximation of \(v\) in the space of piecewise polynomials of degree \(\leq m\), and denote by \(\omega_T\) the union of elements in \(\mathcal{T}_H\) sharing a edge with \(T\). We further set

\[
\Pi_m^\infty v := 0, \quad P_m^\infty v := (id - \Pi_m^\infty)v,
\]
\[ \eta_{TH}^2(A^{-1}, T) := H_T^2(||\text{Curl } A^{-1}||^2_{L^\infty(T)} + H_T^{-2}||A^{-1}||^2_{L^\infty(\omega_T)}) \text{ for all } T \in \mathcal{T}_H, \]

\[ \text{osc}_{TH}^2(A^{-1}, T) := H_T^2(||P_{m-1} \text{Curl } A^{-1}||^2_{L^\infty(T)} + H_T^{-2}||P_m A^{-1}||^2_{L^\infty(\omega_T)}). \]

Noticing that \( P_m^{\infty} \) is defined elementwise, for any subset \( \mathcal{T}'_H \subset \mathcal{T}_H \) we finally set

\[ \eta_{TH}(A^{-1}, T'), \text{ osc}_{TH}(A^{-1}, T') := \max_{T \in \mathcal{T}'_H} \eta_{TH}(A^{-1}, T), \text{ osc}_{TH}(A^{-1}, T) := \max_{T \in \mathcal{T}'_H} \text{ osc}_{TH}(A^{-1}, T). \]

**Remark 4.1.** (Monotonicity) The use of best approximation in \( L^\infty \) in the definition of \( \eta_{TH}(A^{-1}, \mathcal{T}_H) \) and \( \text{osc}_{TH}(A^{-1}, \mathcal{T}_H) \) implies the following monotonicity: for any refinement \( \mathcal{T}'_H \) of \( \mathcal{T}_H \), it holds

\[ \eta_{T'H}(A^{-1}, T) \leq \eta_{TH}(A^{-1}, T) \text{ and } \text{osc}_{T'H}(A^{-1}, T) \leq \text{osc}_{TH}(A^{-1}, T). \]

To avoid any smoothness assumptions on the coefficient matrix of PDEs, we need to quote a result about implicit interpolation, whose proof can be found in [29].

**Lemma 4.6.** (Implicit interpolation) Let \( \bar{m} \) and \( \bar{n} \) be two nonnegative integer, and \( \omega \) be either one or two dimension simplex. For a positive integer \( i \) we denote by \( \Pi_{\bar{m}}^2 : L^2(\omega, \mathbb{R}^i) \rightarrow P_{\bar{m}}(\omega, \mathbb{R}^i) \) the operator of best \( L^2 \)–approximation in \( \omega \), and \( P_{\bar{m}}^2 := \text{id} - \Pi_{\bar{m}}^2 \). Then for all \( v \in L^\infty(\omega, \mathbb{R}^i), V \in P_{\bar{n}}(\omega, \mathbb{R}^i) \) and \( \bar{m} \geq \bar{n} \), it holds

\[ ||P_{\bar{m}}^2(vV)||_{L^2(\omega)} \leq ||P_{\bar{m}-\bar{n}}^\infty v||_{L^\infty(\omega)}||V||_{L^2(\omega)}. \tag{4.21} \]

**Lemma 4.7.** Let \( \mathcal{T}_H \) be a triangulation. For all \( T \in \mathcal{T}_H \) and any pair of discrete functions \( \sigma_H, \tau_H \in M_H \), there exists a constant \( \bar{\Lambda}_1 > 0 \) depending only on the shape regularity of \( T_0 \), the polynomial degree \( l + 1 \), and the eigenvalues of \( A^{-1} \), such that

\[ \eta_{TH}(\sigma_H, T) \leq \eta_{TH}(\tau_H, T) + \bar{\Lambda}_1 \eta_{TH}(A^{-1}, T)||A^{-1/2}(\sigma_H - \tau_H)||_{L^2(\omega_T)}, \tag{4.22} \]

\[ \text{osc}_{TH}(\sigma_H, T) \leq \text{osc}_{TH}(\tau_H, T) + \bar{\Lambda}_1 \text{osc}_{TH}(A^{-1}, T)||A^{-1/2}(\sigma_H - \tau_H)||_{L^2(\omega_T)}. \tag{4.23} \]

**Proof.** We only prove the second estimate (4.23), since the first one (4.22) is somewhat simpler and can be derived similarly. We denote by \( L^2(\Gamma_H) \) the square integrable function spaces on \( \Gamma_H := \bigcup \mathcal{E}_H \). The jump of the tangential component defines a linear mapping \( J : M_H \rightarrow L^2(\mathbb{H}_H) \) by \( J(q_H) = J(A^{-1}q_H \cdot \tau) \) for all \( q_H \in M_H \) from \( M_H \) into \( L^2(\mathbb{H}_H) \). Recalling \( P_{\bar{n}}^2 = \text{id} - \Pi_{\bar{n}}^2 \) with \( \Pi_{\bar{n}}^2 \) being the \( L^2 \)–projection, denoting \( q_H := \sigma_H - \tau_H \) and using the triangle inequality, we have

\[ \text{osc}_{TH}(\sigma_H, T) \leq \text{osc}_{TH}(\tau_H, T) + H_T||P_{\bar{n}}^2 \text{curl}(A^{-1}q_H)||_{L^2(T)} \]

\[ + H_T^{1/2}||P_{n+1}^2 J(A^{-1}q_H \cdot \tau)||_{L^2(\partial T)}. \tag{4.24} \]

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We split the curl term as
\[
\text{curl}(A^{-1}q_H) = \text{Curl } A^{-1} \cdot q_H + A^{-1} : \tilde{\text{curl }} q_H,
\]
where \( \text{Curl } A^{-1} \) is a vector whose every component is the curl of the corresponding column vector of \( A^{-1} \), and \( \tilde{\text{curl }} q_H \) is a matrix whose column vector is the Curl of the corresponding vector of \( q_H \). Invoking Lemma 4.6 with \( \omega = T \) and noticing that polynomial degree of \( q_H \) is \( l + 1 \), we infer for the first term that
\[
\| P_n^2(\text{curl } A^{-1} \cdot q_H) \|_{L^2(T)} \lesssim \| P_{n-l-1}^\infty \text{curl } A^{-1} \|_{L^\infty(T)} \| A^{-1/2} q_H \|_{L^2(T)}. \tag{4.25}
\]

Since \( \tilde{\text{curl }} q_H \) is a polynomial of degree \( \leq l \), applying \( (4.21) \) again in conjunction with an inverse inequality, we obtain for the second term that
\[
\| P_n^2(A^{-1} : \tilde{\text{curl }} q_H) \|_{L^2(T)} \leq \| P_{n-l-1}^\infty A^{-1} \|_{L^\infty(T)} \| \tilde{\text{curl }} q_H \|_{L^2(T)}
\leq \| P_{n-l-1}^\infty A^{-1} \|_{L^\infty(T)} \| q_h \|_{H^1(T)}
\lesssim H^{-1} \| P_{n-l-1}^\infty A^{-1} \|_{L^\infty(T)} \| A^{-1/2} q_H \|_{L^2(T)}. \tag{4.26}
\]

We now deal with the jump residual. Let \( T' \in T_H \) share an interior edge \( E \) with \( T \). We write \( J(A^{-1} q_H \cdot \tau) = ((A^{-1} q_H)_T - (A^{-1} q_H)_{T'}) \cdot \tau \) and use the linearity of \( \Pi_{n+1}^2 \), Lemma 4.6 with \( \omega = E \), and the inverse inequality \( \| q_H \|_{L^2(E)} \lesssim H^{-1/2} \| q_H \|_{L^2(T)} \) to deduce that
\[
\| P_{n+1}^2((A^{-1} q_H)_T \cdot \tau) \|_{L^2(E)} = \| (P_{n+1}^2(A^{-1} q_H)_T) \cdot \tau \|_{L^2(E)} \leq \| P_{n+1}^2(A^{-1} q_H)_T \|_{L^2(E)}
\leq \| P_{n-l-1}^\infty A^{-1} \|_{L^\infty(T)} \| q_h \|_{L^2(E)} \lesssim H^{-1/2} \| P_{n-l-1}^\infty A^{-1} \|_{L^\infty(T)} \| q_H \|_{L^2(T)}. \tag{4.27}
\]

Since \( T_H \) is shape-regular, we can replace \( H_T' \) by \( H_T \), a similar argument leads to
\[
\| P_{n+1}^2((A^{-1} q_H)_{T'} \cdot \tau) \|_{L^2(E)} \lesssim H^{-1/2} \| P_{n-l-1}^\infty A^{-1} \|_{L^\infty(T')} \| q_H \|_{L^2(T')}. \tag{4.28}
\]

A combination of \( (4.27) \) and \( (4.28) \) yields
\[
\| P_{n+1}^2 J(A^{-1} q_H \cdot \tau) \|_{L^2(E)} = \| P_{n+1}^2(((A^{-1} q_H)_T - (A^{-1} q_H)_{T'}) \cdot \tau) \|_{L^2(E)}
\leq \| P_{n+1}^2((A^{-1} q_H)_T \cdot \tau) \|_{L^2(E)} + \| P_{n+1}^2((A^{-1} q_H)_{T'} \cdot \tau) \|_{L^2(E)}
\lesssim H^{-1/2} \| P_{n-l-1}^\infty A^{-1} \|_{L^\infty(E)} \| A^{-1/2} q_H \|_{L^2(E)}. \tag{4.29}
\]
By summing over all edges of element \( T \), from the above inequality (4.29), we get

\[
\| P_{n+1}^{2} J(A^{-1} q_{H} \cdot \tau) \|_{L^{2}(\partial T)} \lesssim H_{T}^{-1/2} \| P_{n+1}^{\infty} A^{-1} \|_{L^{\infty}(\omega_{T})} \| A^{-1/2} q_{H} \|_{L^{2}(\omega_{T})}. \tag{4.30}
\]

Finally, the desired result (4.23) follows from (4.24)-(4.26) and (4.30).

The following two corollaries are global forms of the above lemma.

**Corollary 4.1.** (Estimator reduction) **For a triangulation** \( \mathcal{T}_{H} \) **with** \( \mathcal{M}_{H} \subset \mathcal{T}_{H} \), **let** \( \mathcal{T}_{h} \) **be a refinement of** \( \mathcal{T}_{H} \) **obtained by** \( \mathcal{T}_{h} := \text{REFINE}(\mathcal{T}_{H}, \mathcal{M}_{H}) \). **Denote** \( \Lambda_{1} := 3 \Lambda_{1}^{2} \) **with** \( \Lambda_{1} \) **given in Lemma 4.7** **and** \( \lambda := 1 - 2^{-b/2} > 0 \) **with** \( b \) **given in Section 2.7**. **Then it holds**

\[
\eta_{\mathcal{T}_{h}}^{2}(\sigma_{h}, \mathcal{T}_{h}) \leq (1 + \delta_{3}) \{ \eta_{\mathcal{T}_{h}}^{2}(\sigma_{H}, \mathcal{T}_{H}) - \lambda \eta_{\mathcal{T}_{h}}^{2}(\sigma_{H}, \mathcal{M}_{H}) \} + (1 + \delta_{3}^{-1}) \Lambda_{1} \eta_{\mathcal{T}_{h}}^{2}(A^{-1}, \mathcal{T}_{h}) \| A^{-1/2}(\sigma_{H} - \sigma_{h}) \|_{L^{2}(\Omega)}^{2} \tag{4.31}
\]

**for all** \( \sigma_{H} \in \mathcal{M}_{H}, \sigma_{h} \in \mathcal{M}_{h} \) **and any** \( \delta_{3} > 0 \).

**Proof.** For \( T \in \mathcal{T}_{h} \), applying the first estimate (4.22) of Lemma 4.7 with \( \sigma_{H}, \sigma_{h} \in \mathcal{M}_{h} \) and using Young’s inequality with parameter \( \delta_{3} > 0 \), we derive

\[
\eta_{\mathcal{T}_{h}}^{2}(\sigma_{h}, T) \leq (1 + \delta_{3}) \eta_{\mathcal{T}_{h}}^{2}(\sigma_{H}, T) + (1 + \delta_{3}^{-1}) \Lambda_{1} \eta_{\mathcal{T}_{h}}^{2}(A^{-1}, T) \| A^{-1/2}(\sigma_{H} - \sigma_{h}) \|_{L^{2}(\Omega)}^{2}. \tag{4.32}
\]

By summing over all elements \( T \in \mathcal{T}_{h} \) and using the finite overlap of patches \( \omega_{T} \), the above inequality (4.32) indicates

\[
\eta_{\mathcal{T}_{h}}^{2}(\sigma_{h}, \mathcal{T}_{h}) \leq (1 + \delta_{3}) \eta_{\mathcal{T}_{h}}^{2}(\sigma_{H}, \mathcal{T}_{h}) + (1 + \delta_{3}^{-1}) 3 \Lambda_{1}^{2} \eta_{\mathcal{T}_{h}}^{2}(A^{-1}, \mathcal{T}_{h}) \| A^{-1/2}(\sigma_{H} - \sigma_{h}) \|_{L^{2}(\Omega)}^{2}. \tag{4.33}
\]

For a marked element \( T \in \mathcal{M}_{H} \), we set \( \mathcal{T}_{h,T} := \{ T' \in \mathcal{T}_{h} | T' \subset T \} \). Since \( \sigma_{H} \in \mathcal{M}_{H} \) and \( A^{-1} \) jumps only across edges of \( \mathcal{T}_{0} \), we have \( J(A^{-1} \sigma_{H} \cdot \tau) = 0 \) on edges of \( \mathcal{T}_{h,T} \) in the interior of \( T \). Notice that \( \| f - f_{h} \|_{L^{2}(\omega_{T})} \leq \| f - f_{H} \|_{L^{2}(\omega_{T})} \), we then obtain

\[
\sum_{T \in \mathcal{T}_{h,T}} \eta_{\mathcal{T}_{h}}^{2}(\sigma_{H}, T') \leq 2^{-b/2} \eta_{\mathcal{T}_{h}}^{2}(\sigma_{H}, T), \tag{4.34}
\]

since refinement by bisection implies

\[
h_{T'} = \| T' \|^{1/2} \leq (2^{-b} | T |)^{1/2} \leq 2^{-b/2} H_{T} \quad \text{for all} \ T' \in \mathcal{T}_{h,T}.
\]

On the other hand, for an element \( T \in \mathcal{T}_{H} \setminus \mathcal{M}_{H} \), Remark efrem 2.1 yields

\[
\eta_{\mathcal{T}_{h}}(\sigma_{H}, T) \leq \eta_{\mathcal{T}_{h}}(\sigma_{H}, T).
\]
Hence, from (4.34) and the above inequality, by summing overall $T \in T_h$ we arrive at

$$\eta_{T_h}^2(\sigma_{T_h}, T_h) \leq \sum_{T \in T_h} \left( 2^{b/2} \eta_{T_h}^2(\sigma_{T_h}, \mathcal{M}_T) + \eta_{T_h}^2(\sigma_{T_h}, T_h \setminus \mathcal{M}_T) \right) = \eta_{T_h}^2(\sigma_{T_h}, T_h) - \lambda \eta_{T_h}^2(\sigma_{T_h}, \mathcal{M}_T).$$

(4.35)

From (4.33), (4.35), and the monotonicity $\eta_{T_h}(A^{-1}, T_h) \leq \eta_{T_0}(A^{-1}, T_0)$ stated in Remark 4.1, we get the desired result (4.31).

**Corollary 4.2.** (Perturbation of oscillation) Let $T_h$ be a refinement of $T_H$, and let $\Lambda_1$ be the same as in Corollary 4.1. Then for all $\sigma_{T_h} \in M_h, \sigma_{T_0} \in M_h$, it holds

$$\text{osc}_{T_h}(\sigma_{T_h}, T_h) \leq 2 \text{osc}_{T_0}(\sigma_{T_0}, T_0) + 2 \Lambda_1 \text{osc}_{T_0}(A^{-1}, T_0)|||A^{-1/2}(\sigma_{T_0} - \sigma_{T_h})|||_{L^2(\Omega)}^2.$$

(4.36)

Proof. Remark 2.1 yields $\text{osc}_{T_h}(\sigma_{T_h}, T) = \text{osc}_{T_0}(\sigma_{T_0}, T)$ for all $T \in T_h \cap T$. Hence, by the estimate (4.23) and Young’s inequality, we get

$$\text{osc}_{T_h}(\sigma_{T_h}, T) \leq 2 \text{osc}_{T_0}(\sigma_{T_0}, T) + 2 \Lambda_1 \text{osc}_{T_0}(A^{-1}, T_0)|||A^{-1/2}(\sigma_{T_0} - \sigma_{T_h})|||_{L^2(\Omega)}^2.$$

(4.36)

By summing over $T \in T_h \cap T$ and using the monotonicity property $\text{osc}_{T_h}(A^{-1}, T_h) \leq \text{osc}_{T_0}(A^{-1}, T_0)$ stated in Remark 4.1 the inequality (4.36) indicates the desired assertion.

**5 Convergence for the AMFEM**

We shall prove in this section that the so-called quasi-error, i.e., the sum of the stress variable error plus the scaled estimator, uniformly reduces with a fixed rate on two successive meshes, up to an oscillation term of $f$. This means the AMFEM is a contraction with respect to the quasi-error. To this end, subsequently we replace the subscripts $H, h$ respectively with iteration counters $k, k+1$, and denote by

$$\eta_k := \eta_{T_k}(p_k, T_k)$$

the scaled estimator over the whole mesh $T_k$.

**Theorem 5.1.** (Contraction property) Given $\theta \in (0, 1]$, let $\{T_k; (M_k, L_k); (p_k, u_k)\}_{k \geq 0}$ be the sequence of meshes, a pair of finite element spaces, and discrete solutions produced by the AMFEM. Then there exits constants $\gamma > 0, 0 < \alpha < 1$, and $C > 0$ depending solely on the shape-regularity of $T_0$, $b, \eta_{T_0}(A^{-1}, T_0)$, and the marking parameter $\theta$, such that

$$\varepsilon_{k+1}^2 + \gamma \eta_{k+1}^2 \leq \alpha^2 (\varepsilon_k^2 + \gamma \eta_k^2) + C \text{osc}^2(f, T_k).$$

(5.1)
Proof. For convenience, we use the notations \( \epsilon_k := p - p_k, E_k := p_k - p_{k+1}, \eta_k(\mathcal{M}_k) := \eta_{\Gamma_k}(p_k, \mathcal{M}_k), \eta_0(A^{-1}) := \eta_{\Gamma_0}(A^{-1}, T_0). \)

For any \( \delta_2 > 0 \), by Young’s inequality and the mesh-size functions \( h_{k+1} \leq h_k \), we have

\[
\| h_{k+1} \text{div} \epsilon_{k+1} \|_{L^2(\Omega)}^2 = \| h_{k+1} \text{div} \epsilon_k \|_{L^2(\Omega)}^2 - \| h_{k+1} \text{div} E_k \|_{L^2(\Omega)}^2 \\
\leq \| h_k \text{div} \epsilon_k \|_{L^2(\Omega)}^2 - \| h_{k+1} \text{div} E_k \|_{L^2(\Omega)}^2 \\
+ \delta_2 \| h_{k+1} \text{div} \epsilon_{k+1} \|_{L^2(\Omega)}^2 + \delta_2^{-1} \text{osc}^2(f_{k+1}, T_k),
\]

which implies the inequality

\[
(1 - \delta_2)\| h_{k+1} \text{div} \epsilon_{k+1} \|_{L^2(\Omega)}^2 \leq \| h_k \text{div} \epsilon_k \|_{L^2(\Omega)}^2 - \| h_{k+1} \text{div} E_k \|_{L^2(\Omega)}^2 \\
+ \delta_2^{-1} \text{osc}^2(f_{k+1}, T_k),
\]

where \( f_{k+1} := \Pi_{L_{k+1}} f \) is the \( L^2 \)-projection of \( f \) over \( L_{k+1} \).

We combine the quasi-orthogonality (Theorem 4.1) and (5.2), and take \( \delta_2 = \delta_1 \) to obtain

\[
(1 - \delta_1)\mathcal{E}_{k+1}^2 \leq \mathcal{E}_k^2 - \| A^{-1/2} E_k \|_{L^2(\Omega)}^2 + \frac{C_0 + 1}{\delta_1} \text{osc}^2(f_{k+1}, T_k).
\]

Applying the estimator reduction (Corollary 4.1) to (5.3), we get for any \( \bar{\gamma} \geq 0, \)

\[
(1 - \delta_1)\mathcal{E}_{k+1}^2 + \bar{\gamma}\eta_{k+1}^2 \leq \mathcal{E}_k^2 - \| A^{-1/2} E_k \|_{L^2(\Omega)}^2 \\
+ \bar{\gamma}(1 + \delta_3)\eta_k^2 - \lambda \eta_k^2(\mathcal{M}_k) \\
+ \bar{\gamma}(1 + \delta_3^-)\Lambda_1 \eta_0^2(\Lambda^{-1}) \| A^{-1/2} E_k \|_{L^2(\Omega)}^2 \\
+ \frac{C_0 + 1}{\delta_1} \text{osc}^2(f_{k+1}, T_k).
\]

In what follows we choose

\[
\bar{\gamma} := 1 / \left( (1 + \delta_3^-)\Lambda_1 \eta_0^2(\Lambda^{-1}) \right)
\]

so as to obtain

\[
(1 - \delta_1)\mathcal{E}_{k+1}^2 + \bar{\gamma}\eta_{k+1}^2 \leq (1 - \delta_1)\mathcal{E}_k^2 + \bar{\gamma}(1 + \delta_3)\eta_k^2 - \Lambda \eta_k^2(\mathcal{M}_k) + \frac{C_0 + 1}{\delta_1} \text{osc}^2(f_{k+1}, T_k).
\]

By using the reliable estimation (2.4) of the stress variable error, and invoking
Dörfler marking property (2.5), the above inequality yields for any constant \( \alpha \),

\[
(1 - \delta_1) \mathcal{E}_{k+1}^2 + \gamma \eta_{k+1}^2 \leq \alpha^2 (1 - \delta_1) \mathcal{E}_k^2 + (1 - \alpha^2(1 - \delta_1)) \mathcal{E}_k^2
\]

\[
+ \bar{\gamma}(1 + \delta_3) \{ \eta_k^2 - \lambda \eta_k^2(\mathcal{M}_k) \} + \frac{C_0 + 1}{\delta_1} \text{osc}^2(f_{k+1}, T_k)
\]

\[
\leq \alpha^2 (1 - \delta_1) \mathcal{E}_k^2 + (1 - \alpha^2(1 - \delta_1)) \mathcal{E}_k^2
\]

\[
+ \bar{\gamma}(1 + \delta_3) (1 - \lambda \theta^2) \eta_k^2 + \frac{C_0 + 1}{\delta_1} \text{osc}^2(f_{k+1}, T_k)
\]

\[
\leq \alpha^2 \{ (1 - \delta_1) \mathcal{E}_k^2 + (1 - \alpha^2(1 - \delta_1)) \mathcal{E}_k^2 + \bar{\gamma}(1 + \delta_3)(1 - \lambda \theta^2) \eta_k^2 \}
\]

\[
+ \frac{C_0 + 1}{\delta_1} \text{osc}^2(f_{k+1}, T_k).
\] (5.6)

We choose \( \alpha \) such that

\[
(1 - \alpha^2(1 - \delta_1)) C_1 + \bar{\gamma}(1 + \delta_3)(1 - \lambda \theta^2) = \alpha^2 \bar{\gamma},
\]

which indicates

\[
\alpha^2 = \frac{(1 - \delta_1) C_1 + \bar{\gamma}(1 + \delta_3)(1 - \lambda \theta^2)}{(1 - \delta_1) C_1 + \bar{\gamma}}.
\]

We now choose \( \delta_3 \) and \( \delta_1 \) such that

\[
\delta_3 \leq \lambda \theta^2 / (1 - \lambda \theta^2) \quad \text{and} \quad \delta_1 < \min\{1, \frac{(1 - \lambda \theta^2) \delta_3 \bar{\gamma}}{C_1} \}.
\]

Then it follows

\[
\delta_1 C_1 / \bar{\gamma} + (1 + \delta_3)(1 - \lambda \theta^2) < (1 - \lambda \theta^2) \delta_3 + (1 + \delta_3)(1 - \lambda \theta^2)
\]

\[
\leq (1 - \lambda \theta^2)(1 + \frac{\lambda \theta^2}{1 - \lambda \theta^2}) = 1,
\]

which leads to \( \alpha^2 < 1 \). Finally we set \( \gamma = \bar{\gamma} / (1 - \delta_1) \). Then the desired result (5.1) follows from (5.6).

We note that the oscillation \( \text{osc}(f, T_k) \) of the right-hand side term \( f \) measures intrinsic information missing in the average process associated with finite elements, but fails to detect fine structures of \( f \). When the oscillation term \( \text{osc}(f, T_k) \) is marked, it is easy to show the following convergence result.

**Corollary 5.1.** (Convergence result) Under the assumptions of Theorem 5.1, there exist constants \( \rho \in (0, 1) \), \( \gamma > 0 \), and \( C > 0 \) depending solely on the shape-regularity of \( T_0 \), \( b \), \( \eta_{T_0(A^{-1}, T_0)} \), and the marking parameter \( \theta \), such that

\[
\mathcal{E}_k^2 + \gamma \eta_k^2 \leq C \rho^{2k}.
\]
6 Quasi-optimal convergence rate for the AMFEM

6.1 Auxiliary results

In this subsection, we aim at the discrete upper bound, which is one key for the proof for the quasi-optimal convergence rate. Simultaneously, we shall prove and quote some preliminary results.

Theorem 6.1. (Discrete upper bound) Let $\mathcal{T}_h$ and $\mathcal{T}_H$ be two nested conforming triangulations, $(p_h, u_h) \in M_h \times L_h$ and $(p_H, u_H) \in M_H \times L_H$ be the discrete solutions with respect to the meshes $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively, and $\mathcal{F}_H := \{ T \in \mathcal{T}_H : T$ is not included in $\mathcal{T}_h \}$. Then there exist constants $C_1$ and $C_0$ depending only on the shape regularity of $\mathcal{T}_H$ such that

$$||A^{-1/2}(p_h - p_H)||^2_{L^2(\Omega)} \leq C_1 \eta_H^2(p_H, \mathcal{F}_H) + C_0 \text{osc}^2(f_h, \mathcal{T}_H), \quad (6.1)$$

and

$$\# \mathcal{F}_H \leq \# \mathcal{T}_h - \# \mathcal{T}_H. \quad (6.2)$$

Proof. The second inequality, i.e., (6.2), follows from the definition of $\mathcal{F}_H$. To prove the first one, we introduce the solution $(\tilde{p}_h, \tilde{u}_h) \in M_h \times L_h$ to the problem (4.10). From (2.3) and (4.10), we obtain $\text{div}(\tilde{p}_h - p_H) = 0$, which implies

$$\int_{\partial\Omega} (\tilde{p}_h - p_H) \cdot \nu ds = 0, \quad (6.3)$$

where $\nu$ is the outward unit normal vector along $\partial\Omega$. Thus $\tilde{p}_h - p_H$ satisfies the conditions of Theorem 3.1 in [43] on the polygonal domain $\Omega$, namely it is divergence-free and fulfills (6.3). As a result, there exists $\psi_h \in H^1(\Omega)$ with $\tilde{p}_h - p_H = \text{Curl} \psi_h$. Since $\tilde{p}_h - p_H \in M_h$, this leads to

$$\psi_h \in S^{l+2}_H := \{ \psi_h \in C(\Omega) : \text{Curl} \psi_h \in M_h, \psi_h|_T \in P_{l+2}(T) \text{ for all } T \in \mathcal{T}_h \}$$

(the definition of $S^{l+2}_H$ is analogous). From (4.10) with $q_h = \tilde{p}_h - p_H$, we get

$$||A^{-1/2}(\tilde{p}_h - p_H)||^2_{L^2(\Omega)} = (A^{-1}(\tilde{p}_h - p_H), \tilde{p}_h - p_H)_{0,\Omega}$$

$$= (A^{-1}\tilde{p}_h, \tilde{p}_h - p_H)_{0,\Omega} - (A^{-1}p_H, \tilde{p}_h - p_H)_{0,\Omega}$$

$$= -(A^{-1}p_H, \text{Curl} \psi_h)_{0,\Omega} = -(A^{-1}p_H, \text{Curl} \psi_h)_{0,\Omega} \quad (6.4)$$

Since $\text{div}(\text{Curl} \psi_H) = 0$ for any $\psi_H \in S^{l+2}_H$, from (2.3) with $q_H = \text{Curl} \psi_H$, we have

$$(A^{-1}p_H, \text{Curl} \psi_H)_{0,\Omega} = -(\text{div Curl} \psi_H, u_H)_{0,\Omega} = 0 \text{ for all } \psi_H \in S^{l+2}_H. \quad (6.5)$$
To connect $S_{h}^{l+2}$ with $S_{H}^{l+2}$, we need to use some local quasi-interpolation, e.g. the Scott-Zhang interpolation, \[ \mathcal{I}_{H} : S_{h}^{l+2} \rightarrow S_{H}^{l+2} \] with

\[ \| \psi_{h} - \mathcal{I}_{H} \psi_{h} \|_{L^{2}(E)} \leq H_{E}^{1/2} \| \psi_{h} \|_{H^{1}(\omega_{E})} \quad \text{for all } E \in \varepsilon_{H} \quad (6.6) \]

and

\[ \| \psi_{h} - \mathcal{I}_{H} \psi_{h} \|_{L^{2}(T)} \leq H_{T} \| \psi_{h} \|_{H^{1}(\omega_{T})} \quad \text{for all } T \in \mathcal{T}_{H}. \quad (6.7) \]

We note the quasi-interpolation $\mathcal{I}_{H}$ is local in the sense that if $T \in \mathcal{T}_{h} \cap \mathcal{T}_{H}$ or $E \in \varepsilon_{h} \cap \varepsilon_{H}$ (i.e., $T$ or $E$ is not refined), then $(\psi_{h} - \mathcal{I}_{H} \psi_{h})|_{T} = 0$ or $\{\psi_{h} - \mathcal{I}_{H} \psi_{h}\}|_{E} = 0$.

With $\mathcal{F}_{H}$ defined in Theorem 6.1 and $\psi_{H} = \mathcal{I}_{H} \psi_{h}$, by using integration by parts, a combination of (6.4) and (6.5) yields

\[ \| A^{-1/2}(\tilde{p}_{h} - p_{H}) \|_{L^{2}(\Omega)}^{2} = \sum_{T \in \mathcal{T}_{H}} - \int_{T} A^{-1} p_{H} \cdot \nabla(\psi_{h} - \psi_{H}) \]

\[ = \sum_{T \in \mathcal{T}_{H}} \int_{T} \nabla( A^{-1} p_{H})(\psi_{h} - \psi_{H}) \]

\[ - \sum_{E \in \varepsilon_{H}} \int_{E} J(A^{-1} p_{H} \cdot \tau)(\psi_{h} - \psi_{H}). \quad (6.8) \]

Applying (6.6) and (6.7) to the identity (6.8), we arrive at

\[ \| A^{-1/2}(\tilde{p}_{h} - p_{H}) \|_{L^{2}(\Omega)}^{2} \leq \eta_{T}^{H}(p_{H}, \mathcal{F}_{H}) \| \psi_{h} \|_{H^{1}(\Omega)} \]

\[ \leq C_{1}^{1/2} \eta_{T}^{H}(p_{H}, \mathcal{F}_{H}) \| A^{-1/2}(\tilde{p}_{h} - p_{H}) \|_{L^{2}(\Omega)}, \]

which yields

\[ \| A^{-1/2}(\tilde{p}_{h} - p_{H}) \|_{L^{2}(\Omega)} \leq C_{1}^{1/2} \eta_{T}^{H}(p_{H}, \mathcal{F}_{H}). \quad (6.9) \]

On the other hand, from $\tilde{p}_{h} - p_{H} \in M_{h}$ we have

\[ (A^{-1}(p_{h} - \tilde{p}_{h}), \tilde{p}_{h} - p_{H})_{0,\Omega} = - (\text{div}(\tilde{p}_{h} - p_{H}), u_{h} - \bar{u}_{h})_{0,\Omega} = 0. \quad (6.10) \]

Then a combination of (6.10), (6.9) and Lemma 4.4 yields

\[ \| A^{-1/2}(p_{h} - p_{H}) \|_{L^{2}(\Omega)}^{2} = (A^{-1}(p_{h} - p_{H}), p_{h} - p_{H})_{0,\Omega} \]

\[ = (A^{-1}(p_{h} - \tilde{p}_{h} + \tilde{p}_{h} - p_{H}), p_{h} - \tilde{p}_{h} + \tilde{p}_{h} - p_{H})_{0,\Omega} \]

\[ = \| A^{-1/2}(p_{h} - \tilde{p}_{h}) \|_{L^{2}(\Omega)}^{2} + \| A^{-1/2}(\tilde{p}_{h} - p_{H}) \|_{L^{2}(\Omega)}^{2} \]

\[ \leq C_{1} \eta_{T}^{2}(p_{H}, \mathcal{F}_{H}) + C_{0}\text{osc}^{2}(f_{h}, \mathcal{T}_{H}), \]

namely the result (6.1) holds. \( \square \)
Remark 6.1. One can also include the second term, \( \text{osc}^2(f_h, \mathcal{T}_H) \), of the right-hand side in (6.1) in the first term \( \eta_{T_h}^2(p_H, \mathcal{F}_H) \). In doing so, however, one cannot expect any relaxation of complexity of analysis, because the oscillation of \( f_h \) over \( \mathcal{T}_H \) still appears in the contraction property (see Theorem 5.1).

Corollary 6.1. (Discrete upper bound) Under the assumption of Theorem 6.1, there exist constants \( C_1 \) and \( C_0 \) depending only on the shape regularity of \( \mathcal{T}_H \) such that

\[
||A^{-1/2}(p_h-p_H)||_{L^2(\Omega)}^2 + ||\text{div}(p_h-p_H)||_{L^2(\Omega)}^2 \leq C_1 \eta_{T_h}^2(p_H, \mathcal{F}_H) + C_0 \text{osc}^2(f, \mathcal{T}_H). 
\]  

(6.11)

Proof. Since

\[
||\text{div}(p_h-p_H)||_{L^2(\Omega)}^2 \leq ||H\text{div}(p_h-p_H)||_{L^2(\Omega)}^2 = \text{osc}^2(f_h, \mathcal{T}_H),
\]

in view of (6.1) and Lemma 4.5 we obtain the desired result (6.11).

Note that the constant \( C_0 \) in Corollary 6.1 is actually the constant \( C_0 \) appeared in Theorem 6.1 plus 1. Here, for simplicity we still denote it by \( C_0 \).

In what follows we shall prove a stable result in the continuous level. Let \( f_H \) denote the \( L^2 \)-projection of \( f \) over \( L_H \), we consider the following problem:

\[
\begin{align*}
-\text{div}(A\nabla \tilde{u}) &= f_H \quad \text{in } \Omega \\
\tilde{u} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(6.13)

By the Lax-Milgram lemma, there exists unique solution \( \tilde{u} \in H^1_0(\Omega) \) to the problem (6.13).

Lemma 6.1. (Stable result) Given a shape regular triangulation \( \mathcal{T}_H \) of \( \Omega \), let \( u \in H^1_0(\Omega) \) and \( \tilde{u} \in H^1_0(\Omega) \) are respectively the weak solutions to the problems (1.2) and (6.13), and let \( p = A\nabla u \) and \( \tilde{p} = A\nabla \tilde{u} \) denote the continuous flux. Then there exists a positive constant \( C_0 \) depending only on the shape regularity of \( \mathcal{T}_H \) and the eigenvalues of \( A \) such that

\[
||A^{-1/2}(p - \tilde{p})||_{L^2(\Omega)} \leq C_0^{1/2} \text{osc}(f, \mathcal{T}_H). 
\]  

(6.14)

Proof. From Green’s formula we have

\[
||A^{-1/2}(p - \tilde{p})||_{L^2(\Omega)}^2 = ||A^{1/2}\nabla (u - \tilde{u})||_{L^2(\Omega)}^2 \\
= - \int_{\Omega} (u - \tilde{u}) \nabla \cdot (A \nabla (u - \tilde{u})) \\
= \langle f - f_H, u - \tilde{u} \rangle_{0, \Omega}.
\]

(6.15)
Let $V^c_H \subset L^2_H$ be a conforming finite element space, and set $\bar{\omega}_T := \bigcup\{T' \in T_H : T' \cap \overline{T} \neq \emptyset\}$ for $T \in T_H$. We consider the Clément or Scott-Zhang interpolation operator or other regularized conforming finite element approximation operator $J : H^1_0(\Omega) \rightarrow V^c_H$ which satisfies
\[
H^{-1}_T ||v - J v||_{L^2(T)} \lesssim ||\nabla v||_{L^2(\bar{\omega}_T)} \quad \text{for all } T \in T_H, v \in H^1_0(\Omega).
\] (6.16)

Existence of such an operator is guaranteed (cf. [31, 55, 24, 25]). Recall that $f_{H} := \Pi_{L^2_H} f$ is $L^2$-projection of $f$ over $L^2_H$. This means $(f - f_{H}, v_{H})_{0,\Omega} = 0$ for all $v_{H} \in L^2_H$. Denoting $v := u - \tilde{u}$, from (6.15) and (6.16) we arrive at
\[
||A^{-1/2}(p - \tilde{p})||^2_{L^2(\Omega)} = (f - f_{H}, v - J v)_{0,\Omega} \\
= \sum_{T \in T_H} (f - f_{H}, v - J v)_{0,T} \\
\lesssim \sum_{T \in T_H} ||H(f - f_{H})||_{L^2(T)} ||\nabla v||_{L^2(T)} \\
\leq C^1_0 2^{\text{osc}}(f, T_H) ||A^{-1/2}(p - \tilde{p})||_{L^2(\Omega)},
\]
which implies the desired result (6.14).

We finish this subsection by quoting a counting conclusion from [57] for the overlay $T := T_1 \oplus T_2$ of two conforming triangulations $T_1$ and $T_2$, which shows $T$ is the smallest conforming triangulation for the triangulations $T_1$ and $T_2$.

**Lemma 6.2.** (Overlay of meshes) For two conforming triangulations $T_1$ and $T_2$ the overlay $T := T_1 \oplus T_2$ is conforming, and satisfies
\[
\#T \leq \#T_1 + \#T_2 - \#T_0.
\]

### 6.2 Quasi-optimal convergence rate

In this subsection, we shall prove the quasi-optimal convergence rate of the AM-FEM for the stress variable error in a weighted norm. To this end, we need to introduce two nonlinear approximation classes. Let $\mathcal{P}_N$ be the set of all triangulations $T$ which is refined from $T_0$ and $\#T \leq N$. For a given triangulation $T$, let $M_T$ and $L_T$ denote respectively the approximation spaces to the flux and displacement, $(p_T, u_T) \in M_T \times L_T$ be the approximation to $(p, u)$, and $h_T$ be mesh-size functions with respect to the triangulation $T$. The quantity of the best approximation to the total error in $\mathcal{P}_N$ is given by
\[
\sigma(N; p, f, A) := \inf_{T \in \mathcal{P}_N} \{||A^{-1/2}(p - p_T)||^2_{L^2(\Omega)} \\
+ ||h_T \text{div}(p - p_T)||^2_{L^2(\Omega)} + \text{osc}_T^2(p_T, T)\}^{1/2},
\]

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and for $s > 0$ we define the nonlinear approximation class $A_s$ as

$$A_s := \{(p, f, A) | |(p, f, A)|_s := \sup_{N > N_0 = \#T_0} N^s \sigma(N; p, f, A) < \infty\}.$$  

Moreover, the quantity of the best approximation to the right-hand side term $f$ in $P_N$ is described by

$$||f||_{A^*_0} := \sup_{N > N_0 = \#T_0} N^s \inf_{T \in P_N} \text{osc}(f, T).$$

By the nonlinear approximation theory \[37, 38\], we know that if $f \in L^2(\Omega)$ then $||f||_{A^*_0} < \infty$. Here, we recall a result of Binev, Dahmen, and DeVore \[13\] which shows that the approximation of data $f$ can be done in an optimal way. The proof of this result can be found in \[13, 14\].

**Lemma 6.3.** (Approximation of data $f$) Given an $f \in L^2(\Omega)$, a tolerance $\varepsilon$, and a shape regular triangulation $T_0$, there exists an algorithm

$$T_H = \text{APPROX}(f, T_0, \varepsilon)$$

such that

$$\text{osc}(f, T_H) \leq \varepsilon, \quad \#T_H - \#T_0 \leq C ||f||_{A^*_0}^{1/s} \varepsilon^{-1/s}.$$  

We now prove that the approximation $p_k$ generated by the AMFEM concerning the stress variable converges to $p$ in a weighted norm with the same rate $(\#T_k - \#T_0)^{-s}$ as the best approximation described by $A_s$ up to a multiplicative constant. We need to count elements added by handling hanging nodes to keep mesh conformity (see Lemma \[2.1\]), as well as those marked by the estimator (the cardinality of $\mathcal{M}_k$). To this end, we impose more stringent requirements than for convergence of the AMFEM.

**Assumption 6.1** (Optimality). we assume the following properties of the AMFEM:

(a) The marking parameter $\theta$ satisfies $\theta \in (0, \theta_*]$ with

$$\theta_*^2 = \frac{C_2}{1 + C_1(1 + 2A_1 \text{osc}_{T_0}^2 (A^{-1}, T_0)};$$

(b) Procedure MARK selects a set $\mathcal{M}_k$ of marked elements with minimal cardinality;

(c) The distribution of refinement edges on $T_0$ satisfies condition (b) of section 4 in \[58\].

The limit value $\theta_*$ depends on the ratio $(C_2/C_1)^{1/2} \leq 1$, which quantifies the quality of approximation to the stress variable of estimator $\eta_{T_k}(p_k, T_k)$, as well as the oscillation $\text{osc}_{T_0}(A^{-1}, T_0)$ of coefficient matrix of the PDEs over $T_0$. \[29\]
The following lemma establishes a link between nonlinear approximation theory and the AMFEM through the Dörfler marking strategy. Roughly speaking, we prove that, if an approximation satisfies a suitable total error reduction from \( \Omega \) and the AMFEM through the Dörfler marking strategy. Roughly speaking, we prove that, if an approximation satisfies a suitable total error reduction from \( T_h \) to \( T_h \) (\( T_h \) is a refinement of \( T_H \)), the error indicators of the coarser solutions must satisfy a Dörfler property on the set \( \mathcal{R} \) of refined elements. In other words, the total error reduction and Dörfler marking are intimately connected.

**Lemma 6.4.** (Optimality marking) Assume that the marking parameter \( \theta \) verifies (a) of Assumption 6.1, and that \( f \) is a piecewise polynomial of degree \( \leq l \) on \( T_H \). Let \( T_H \) be a shape regular triangulation of \( \Omega \), \( (p_H, u_H) \in M_H \times L_H \) be a pair of discrete solutions of (2.3). Set \( \mu := \frac{1}{2}(1 - \frac{a^2}{c^2}) > 0 \), and let \( T_{h_*} \) be any refinement of \( T_H \) such that a pair of discrete solutions \( (p_{h_*}, u_{h_*}) \in M_{h_*} \times L_{h_*} \) satisfies

\[
\mathcal{E}^2_{h_*} + \text{osc}^2_{T_{h_*}}(p_{h_*}, T_{h_*}) \leq \mu \{ \mathcal{E}^2_H + \text{osc}^2_{T_H}(p_H, T_H) \}.
\]  

Then the set \( \mathcal{R} := \mathcal{R}_{T_H \rightarrow T_{h_*}} \) satisfies the Dörfler property

\[
\eta_{T_{h_*}}(p_{h_*}, \mathcal{R}) \geq \theta \eta_{T_H}(p_H, T_H).
\]

**Proof.** Since \( f \) is a piecewise polynomial of degree \( \leq l \) on \( T_H \), it holds

\[
\text{osc}(f, T_H) = 0 \quad \text{and} \quad \text{osc}(f_{h_*}, T_{h_*}) = 0,
\]

which imply

\[
||H \text{div}(p - p_H)||_{L^2(\Omega)} = 0 \quad \text{and} \quad ||h_\ast \text{div}(p - p_{h_*})||_{L^2(\Omega)} = 0.
\]

These two relations, together with the lower bound (3.21), the condition (6.17), and the quasi-orthogonality (4.17), yield

\[
(1 - 2\mu)C_2 \eta^2_{T_H}(p_H, T_H) \leq (1 - 2\mu)\{ \mathcal{E}^2_H + \text{osc}^2_{T_H}(p_H, T_H) \}
\]

\[
\leq ||A^{-1/2}(p - p_H)||^2_{L^2(\Omega)} - ||A^{-1/2}(p - p_{h_*})||^2_{L^2(\Omega)}
\]

\[
+ ||H \text{div}(p - p_H)||^2_{L^2(\Omega)} - ||h_\ast \text{div}(p - p_{h_*})||^2_{L^2(\Omega)}
\]

\[
+ \text{osc}^2_{T_{h_*}}(p_{h_*}, T_{h_*}) - 2\text{osc}^2_{T_{h_*}}(p_{h_*}, T_{h_*}).
\]

We estimate separately the error and oscillation terms. By the quasi-orthogonality (4.17) and discrete upper bound (6.1), we get

\[
||A^{-1/2}(p - p_H)||^2_{L^2(\Omega)} - ||A^{-1/2}(p - p_{h_*})||^2_{L^2(\Omega)} + ||H \text{div}(p - p_H)||^2_{L^2(\Omega)}
\]

\[
- ||h_\ast \text{div}(p - p_{h_*})||^2_{L^2(\Omega)} = ||A^{-1/2}(p_{h_*} - p_H)||^2_{L^2(\Omega)}
\]

\[
\leq C_1 \eta^2_{T_H}(p_H, \mathcal{R})
\]

(6.19)
For the oscillation term we argue according to whether an element $T \in T_H$ belongs to the set of refined elements $\mathcal{R}$ or not. For $T \in \mathcal{R}$ we use the dominance $\text{osc}^2_{T_H}(p_H, T) \leq \eta^2_{T_H}(p_H, T)$ (see Remark 2.1). For $T \in T_H \cap \mathcal{T}_h$, Corollary 4.2

(Perturbation of oscillation) together with $\sigma^\prime_H = p_H$ and $\sigma_h = p_h$, yields

$$\text{osc}^2_{T_H}(p_H, T \cap \mathcal{T}_h) - 2\text{osc}^2_{T_h}(p_h, T \cap \mathcal{T}_h) \leq 2\Lambda_{1} \text{osc}^2_{T_0}(A^{-1}, T_0) ||A^{-1/2}(p_h - p_H)||^2_{L^2(\Omega)}.$$  \hspace{1cm} (6.20)

Combining (6.19) and (6.20) we infer that

$$\text{osc}^2_{T_H}(p_H, T_H) - 2\text{osc}^2_{T_h}(p_h, T_h) = \text{osc}^2_{T_H}(p_H, \mathcal{R}) + \text{osc}^2_{T_H}(p_H, T_H \cap \mathcal{T}_h) - 2\text{osc}^2_{T_h}(p_h, T_H \cap \mathcal{T}_h) - 2\text{osc}^2_{T_h}(p_h, T_h \setminus T_H) \leq \text{osc}^2_{T_0}(p_H, \mathcal{R}) + 2\Lambda_{1} \text{osc}^2_{T_0}(A^{-1}, T_0) ||A^{-1/2}(p_h - p_H)||^2_{L^2(\Omega)} \leq (1 + 2\Lambda_{1} \text{osc}^2_{T_0}(A^{-1}, T_0)) \eta^2_{T_H}(p_H, \mathcal{R}).$$ \hspace{1cm} (6.21)

From (6.18), (6.19), and (6.21), we finally deduce that

$$\eta^2_{T_H}(p_H, \mathcal{R}) \geq \frac{(1 - 2\mu)C_2}{1 + C_1(1 + 2\Lambda_{1} \text{osc}^2_{T_0}(A^{-1}, T_0))} \eta^2_{T_H}(p_H, T_H) = \theta^2 \eta^2_{T_H}(p_H, T_H).$$

In light of the definitions of $\theta_s$, $\theta$, and $\mu$, this concludes the proof. \hfill \Box

The fact that procedure MARK selects the set of marked elements $\mathcal{M}_k$ with minimal cardinality, establishes a link between the best mesh and triangulations generated by AMFEM, and forms crucial idea of AFEM (see [57]). In what follows we shall use this fact.

**Lemma 6.5.** (Cardinality of $\mathcal{M}_k$) Assume that the marking parameter $\theta$ verifies (a) of Assumption 6.1, and procedure MARK satisfies (b) of Assumption 6.1, and that $f$ is the piecewise polynomial of degree $\leq l$ onto $\mathcal{T}_0$. Let $(p, u)$ solve the problem (2.7), and let $\{\mathcal{T}_k; (M_k, L_k); (p_k, u_k); \mathcal{E}_k\}_{k \geq 0}$ be the sequence of meshes, finite element spaces, the discrete solution produced by the AMFEM, and the stress variable error in weighted norm $\mathcal{H}_{s}$, then the following estimate is valid:

$$\#\mathcal{M}_k \lesssim (1 - \frac{\theta_s^2}{\theta^2})^{-1/2s} ||(p, f, A)||^{1/s}_{L^1(\Omega)} C_{A}^{1/2s} \{\mathcal{E}_k^2 + \text{osc}^2_{T_0}(p_k, \mathcal{T}_k)\}^{-1/2s}. \hspace{1cm} (6.22)$$

Furthermore, if $\mathcal{T}_{k+1}$ is a refinement of $\mathcal{T}_k$ obtained by the algorithm REFINE with $\theta \in (0, \theta_s)$, then

$$\#\mathcal{T}_{k+1} - \#\mathcal{T}_k \lesssim (1 - \frac{\theta_s^2}{\theta^2})^{-1/2s} ||(p, f, A)||^{1/s}_{L^1(\Omega)} C_{A}^{1/2s} \{\mathcal{E}_k^2 + \text{osc}^2_{T_0}(p_k, \mathcal{T}_k)\}^{-1/2s}. \hspace{1cm} (6.23)$$

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Proof. Let $[\varepsilon^{-1/s}((p, f, A)]_{s/8}^{1/s}$ denote the integer component of $\varepsilon^{-1/s}((p, f, A)]_{s/8}^{1/s}$. We set $\varepsilon^2 := 4^{-1}C_A^2 \mu (\mathcal{E}^2_k + \text{osc}^2 (p_k, \mathcal{T}_k))$, where $\mu = \frac{1}{2} (1 - \frac{\rho_2}{\rho_1^2}) > 0$ and $C_A = \max \{1 + 2\Lambda^2 \text{osc}^2 (\mathcal{T}_0, 2)\}$, and set $N_\varepsilon := [\varepsilon^{-1/s}((p, f, A)]_{s/8}^{1/s}] + 1$. Recall that

$$\sigma(N_\varepsilon + \# \mathcal{T}_0 - 1; p, f, A) := \inf_{\mathcal{T} \in \mathcal{P}_{N_\varepsilon + \# \mathcal{T}_0 - 1}} \{\mathcal{E}^2_\mathcal{T} + \text{osc}^2 (p_\mathcal{T}, \mathcal{T})\}^{1/2},$$

where $\mathcal{E}^2_\mathcal{T} := ||A^{-1/2}(p - p_\mathcal{T})||^2_{L^2(\Omega)} + ||h_\mathcal{T} \text{div}(p - p_\mathcal{T})||^2_{L^2(\Omega)}$, and

$$\mathcal{E}^2_{h_\mathcal{T}} := ||A^{-1/2}(p - p_\mathcal{T})||^2_{L^2(\Omega)} + ||h_\mathcal{T} \text{div}(p - p_\mathcal{T})||^2_{L^2(\Omega)},$$

where $p_\mathcal{T}$ is the discrete flux approximation to $p := A \nabla u$ with respect to the mesh $\mathcal{T}_\varepsilon$, and $h_\mathcal{T}$ is the mesh-size function with respect to $\mathcal{T}_\varepsilon$.

Since there exists $\mathcal{T}_\varepsilon \in \mathcal{P}_{N_\varepsilon + \# \mathcal{T}_0 - 1}$ with $\# \mathcal{T}_\varepsilon \leq N_\varepsilon + \# \mathcal{T}_0 - 1$ such that

$$\{\mathcal{E}^2_{h_\mathcal{T}} + \text{osc}^2 (p_\mathcal{T}, \mathcal{T}_\varepsilon)\}^{1/2} \leq (1 + \bar{\varepsilon}) \sigma(N_\varepsilon + \# \mathcal{T}_0 - 1; p, f, A),$$

where $\bar{\varepsilon} := \min\{1, \varepsilon\}$. This inequality leads to

$$N_\varepsilon \{\mathcal{E}^2_{h_\mathcal{T}} + \text{osc}^2 (p_\mathcal{T}, \mathcal{T}_\varepsilon)\}^{1/2} \leq (N_\varepsilon + \# \mathcal{T}_0 - 1) \{\mathcal{E}^2_{h_\mathcal{T}} + \text{osc}^2 (p_\mathcal{T}, \mathcal{T}_\varepsilon)\}^{1/2} \leq (1 + \bar{\varepsilon}) \sigma(N_\varepsilon + \# \mathcal{T}_0 - 1; p, f, A) \leq (1 + \bar{\varepsilon}) \sup_{\mathcal{T} \in \mathcal{P}_{N_\varepsilon + \# \mathcal{T}_0 - 1}} \{\mathcal{E}^2_{h_\mathcal{T}} + \text{osc}^2 (p_\mathcal{T}, \mathcal{T})\}^{1/2}.$$

From the above inequality (6.24), we obtain

$$\{\mathcal{E}^2_{h_\mathcal{T}} + \text{osc}^2 (p_\mathcal{T}, \mathcal{T}_\varepsilon)\}^{1/2} \leq \frac{(1 + \bar{\varepsilon}) \sigma(N_\varepsilon + \# \mathcal{T}_0 - 1; p, f, A)}{N_\varepsilon} \leq 2\varepsilon$$

(6.25)

and

$$\# \mathcal{T}_\varepsilon - \# \mathcal{T}_0 \leq N_\varepsilon - 1 \leq \varepsilon^{-1/s}((p, f, A)]_{s/8}^{1/s}.$$

(6.26)

Let $\mathcal{T}_* := \mathcal{T}_\varepsilon \oplus \mathcal{T}_k$ be the overlay of $\mathcal{T}_\varepsilon$ and $\mathcal{T}_k$, $h_*$ denote the mesh-size function with respect to $\mathcal{T}_*$, and $(p_*, u_*)$ be a pair of discrete solutions onto $\mathcal{T}_*$. We shall show that there is a reduction with a factor $\mu$ of the total error between $p_*$ and $p_k$. Notice that $\mathcal{T}_*$ is a refinement of $\mathcal{T}_\varepsilon$, and since $f$ is the piecewise polynomial of degree $\leq l$ onto $\mathcal{T}_0$. Recall that $\mathcal{E}^2_{h_*} := ||A^{-1/2}(p - p_*)||^2_{L^2(\Omega)} + ||h_* \text{div}(p - p_*)||^2_{L^2(\Omega)}$, by the quasi-orthogonality (4.17) and $\text{div}(p_* - p_k) = 0$, we get

$$\mathcal{E}^2_{h_*} + \text{osc}^2 (p_*, \mathcal{T}_*) = ||A^{-1/2}(p - p_*)||^2_{L^2(\Omega)} - ||A^{-1/2}(p_* - p_k)||^2_{L^2(\Omega)} + ||h_* \text{div}(p_* - p_k)||^2_{L^2(\Omega)} + \text{osc}^2 (p_*, \mathcal{T}_*) \leq \mathcal{E}^2_{h_*} + \text{osc}^2 (p_*, \mathcal{T}_*).$$

(6.27)
By the second inequality (4.23) of Lemma 4.7 with \( p_\varepsilon, p_* \in M_{h_*} \), for all \( T \in T_* \), we have

\[
\text{osc}_{T_*}^2(p_\varepsilon, T) \leq 2\text{osc}_{T_*}^2(p_\varepsilon, T) + 2\Lambda_2^2\text{osc}_{T_*}^2(A^{-1}, T)||A^{-1/2}(p_* - p_\varepsilon)||^2_{L^2(\omega_T)}.
\]

Summing on \( T_* \), the monotonicity of the data oscillation (see Remarks 2.1 and 4.1), we get

\[
\text{osc}_{T_*}^2(p_\varepsilon, T_* \) \leq 2\text{osc}_{T_*}^2(p_\varepsilon, T) + 2\Lambda_2^2\text{osc}_{T_*}^2(A^{-1}, T)||A^{-1/2}(p_* - p_\varepsilon)||^2_{L^2(\Omega)}
\]

\[
\leq 2\text{osc}_{T_*}^2(p_\varepsilon, T) + 2\Lambda_2^2\text{osc}_{T_*}^2(A^{-1}, T_0)||A^{-1/2}(p_* - p_\varepsilon)||^2_{L^2(\Omega)}
\]

\[
\leq 2\text{osc}_{T_*}^2(p_\varepsilon, T) + 2\Lambda_2^2\text{osc}_{T_*}^2(A^{-1}, T_0)
\]

\[
\times ((||A^{-1/2}(p_* - p_\varepsilon)||^2_{L^2(\Omega)} - ||A^{-1/2}(p - p_*)||^2_{L^2(\Omega)})).
\]

(6.28)

A combination of (6.25), (6.27), and (6.28) yields

\[
E_{h_*}^2 + \text{osc}_{T_*}^2(p_\varepsilon, T_* \) \leq C_A(E_{h_*}^2 + \text{osc}_{T_*}^2(p_\varepsilon, T_*))
\]

\[
\leq 4\varepsilon^2 C_A = \mu\{E_{k}^2 + \text{osc}_{T_*}^2(p_k, T_k)\}.
\]

Hence, we deduce from optimality marking (Lemma 6.4) that the subset \( R := R_{T_k \rightarrow T_*} \subset T_k \) verifies the Dörfler property (2.5) for \( \theta < \theta_* \). The fact that procedure MARK selects a subset \( M_k \subset T_k \) with minimal cardinality satisfying the same property (2.5), and (6.26) leads to

\[
\#M_k \leq \#R \leq \#T_* - \#T_k \leq \#T_* - \#T_0 - \#T_k
\]

\[
= \#T_* - \#T_0 \leq |(p, f, A)|_{1/s}^{1/s} \varepsilon^{-1/s}
\]

\[
\leq (8C_A)^{1/2s}(1 - \frac{\theta^2}{\theta_*^2})^{-1/2s}|(p, f, A)|_{s}^{1/s}\{SE_{k}^2 + \text{osc}_{T_*}^2(p_k, T_k)\}^{-1/2s},
\]

which implies the desired result (6.22). In the third step above, we have used the overlay of two meshes (Lemma 6.2).

The second assertion (6.23) follows from \( \#T_{k+1} - \#T_k \lesssim \#M_k \) and the first result (6.22).

\[ \Box \]

**Theorem 6.2.** Assume that \( f \) is a piecewise polynomial of degree \( \leq l \) onto \( T_0 \), then the algorithm AMFEM will terminate in finite steps for a given tolerance \( \varepsilon \). Furthermore, set the algorithm AMFEM terminating in the \( N \)-th step, and denote by \( T_N \) the triangulation obtained in the \( N \)-th step. Let \( (p, f, A) \in A_s \), and \( \Theta(s, \theta) := (1 - \frac{\theta^2}{\theta_*^2})^{-1/2s} \frac{1}{1 - \alpha^{1/s}} \) describes the asymptotics of the AMFEM as \( \theta \to \theta_* \) or \( s \to 0 \). Then there exists a constant \( C \), depending on data, the refinement depth \( b \), and \( T_0 \), but independent of \( s \), such that

\[
\#T_N - \#T_0 \lesssim C\Theta(s, \theta)|(p, f, A)|_{s}^{1/s} \varepsilon^{-1/s}.
\]

(6.29)
Furthermore, it holds
\[ \{ E_N^2 + \text{osc}_{T_N}^2(p_N, T_N) \}^{1/2} \lesssim C^s \Theta^s(s, \theta) \|(p, f, A)\|_s (\#T_N - \#T_0)^{-s}. \]  
\( \text{(6.30)} \)

**Proof.** By the contraction property (Theorem 5.1), and the fact that \( f \) is a piecewise polynomial of degree \( \leq l \) onto \( T_0 \), there exists \( \alpha \in (0, 1) \) such that
\[ E_{k+1}^2 + \gamma \eta_{T_{k+1}}^2 (p_{k+1}, T_{k+1}) \leq \alpha^2 (E_k^2 + \gamma \eta_{T_k}^2 (p_k, T_k)). \]  
\( \text{(6.31)} \)

The first assertion is a direct consequence of the above inequality (6.31).

Since
\[ E_k^2 + \text{osc}_{T_k}^2(p_k, T_k) \approx E_k^2 + \gamma \eta_{T_k}^2 (p_k, T_k) \approx \eta_{T_k}^2 (p_k, T_k), \]
Combining the complexity of \( \text{REFINE} \) (Lemma 2.1) and the cardinality (6.22) of \( \mathcal{M}_k \), we deduce that
\[ \#T_N - \#T_0 \lesssim \sum_{k=0}^{N-1} \#M_k \lesssim \beta \sum_{k=0}^{N-1} \{ E_k^2 + \text{osc}_{T_k}^2(p_k, T_k) \}^{-1/2}, \]
\( \text{(6.32)} \)

where \( \beta := (1 - \frac{\theta^2}{\eta^2})^{-1/2s} C_A^{1/2s} \|(p, f, A)\|_s^{1/s} \).

From the lower bound (5.21), we infer that
\[ E_k^2 + \gamma \text{osc}_{T_k}^2(p_k, T_k) \leq E_k^2 + \gamma \eta_{T_k}^2 (p_k, T_k) \leq (1 + \frac{\gamma}{C_2}) \{ E_k^2 + \text{osc}_{T_k}^2(p_k, T_k) \}. \]
\( \text{(6.33)} \)

On the other hand, the linear rate \( \alpha = \alpha(\theta) < 1 \) of convergence for the quasi error implies that for \( 0 \leq k \leq N - 1 \)
\[ E_{N-1}^2 + \gamma \eta_{T_{N-1}}^2 (p_{N-1}, T_{N-1}) \leq \alpha^{2(N-1-k)} \{ E_k^2 + \gamma \eta_{T_k}^2 (p_k, T_k) \}. \]  
\( \text{(6.34)} \)

We combine the above three inequalities (6.32)-(6.34) to obtain
\[ \#T_N - \#T_0 \lesssim \beta (1 + \frac{\gamma}{C_2})^{1/2s} \{ E_{N-1}^2 + \gamma \eta_{T_{N-1}}^2 (p_{N-1}, T_{N-1}) \}^{-1/2s} \sum_{k=0}^{N-1} \alpha^{k/s}. \]

Since \( \alpha < 1 \), the geometric series is bounded by the constant \( s_0 = 1/(1 - \alpha^{1/s}) \).
By recalling \( \eta_{T_{N-1}}^2 (p_{N-1}, T_{N-1}) \approx E_{N-1}^2 + \gamma \eta_{T_{N-1}}^2 (p_{N-1}, T_{N-1}) \), we end up with
\[ \#T_N - \#T_0 \lesssim s_0 \beta (1 + \frac{\gamma}{C_2})^{1/2s} \{ \eta_{T_{N-1}}^2 (p_{N-1}, T_{N-1}) \}^{-1/2s}. \]  
\( \text{(6.35)} \)

A combination of the above inequality (6.35) and the stopping criteria
\[ \eta_{T_N}^2 (p_N, T_N) \leq \varepsilon^2 \]  
\( \text{(6.36)} \)
yields the second assertion (6.29).

By raising the second result (6.29) to the $s-$th power and recording, we obtain

$$
\varepsilon \lesssim C^s \Theta^s(s, \theta) \left| (p, f, A) \right|_s \left( \# \mathcal{T}_N - \# \mathcal{T}_0 \right)^{-s}.
$$

Since \( \left\{ E^2_N + \text{osc}^2_{\mathcal{T}_N}(p_N, \mathcal{T}_N) \right\}^{1/2} \approx \eta_{\mathcal{T}_N}(p_N, \mathcal{T}_N) \), the above inequality (6.37) and the stopping criteria (6.36) imply the third assertion (6.30).

Since some results concerning quasi-optimal convergence rate are obtained under the assumption that \( f \) is a piecewise polynomial of degree \( \leq l \) onto \( \mathcal{T}_0 \). By inspiration of these results, we consider the following algorithm which separates the oscillation reduction of data \( f \) and the quasi error:

**Step 1** \([T_H, f_H] = \text{APPROX}(f, \mathcal{T}_0, \varepsilon/2)\),

**Step 2** \([T_N, (p_N, u_N)] = \text{AMFEM}(T_H, f_H, \varepsilon/2, \theta)\).

The advantage of separating data error and discretization error is that in second Step 2, oscillation of data \( f \) is always zero since the input data \( f_H \) is piecewise polynomial of degree \( \leq l \) over the initial mesh \( \mathcal{T}_H \) for AMFEM.

We are now at the position to show the quasi-optimal convergence rate.

**Theorem 6.3.** (Quasi-optimal convergence rate) For any \( f \in L^2(\Omega) \), a shape regular triangulation \( \mathcal{T}_0 \) and a tolerance \( \varepsilon > 0 \). Let \( \theta \in (0, \theta^*) \), \([T_N, (p_N, u_N)] = \text{AMFEM}(\mathcal{T}_0, f, \varepsilon, \theta)\), and \( \text{osc}^2_{\mathcal{T}_N}(p_N, \mathcal{T}_N) \) then there exist a positive constant \( C \) depending only on data \( (A, f) \), refinement depth \( b \), and \( \mathcal{T}_0 \), but independent of \( s \), such that

$$
(\mathcal{E}^2_N + \text{osc}^2_{\mathcal{T}_N}(p_N, \mathcal{T}_N))^{1/2} \leq C^s \Theta^s(s, \theta) \left| (p, f, A) \right|_s + \| f \|_{A^s_0} \left( \# \mathcal{T}_N - \# \mathcal{T}_0 \right)^{-s}.
$$

**Proof.** By Lemma 6.3 there exists a triangulation \( \mathcal{T}_k \) such that

$$
\text{osc}(f, \mathcal{T}_k) \leq \varepsilon, \text{ and } \# \mathcal{T}_k - \# \mathcal{T}_0 \leq C \left\| f \right\|_{A^0_0}^{1/s} \varepsilon^{-1/s}.
$$

Let \( f_k := \Pi_{L_k} f \) denote the \( L^2 \)-projection of \( f \) over \( L_k \), and \((\hat{p}, \hat{u})\) be a pair of solutions of (2.1) with respect to the right-hand side term \( f_k \). By Lemma 6.1 (stable result), we obtain

$$
\left\| A^{-1/2}(p - \hat{p}) \right\|_{L^2(\Omega)} \leq C_0^{1/2} \text{osc}(f, \mathcal{T}_k) \leq C_0^{1/2} \varepsilon.
$$

By the definition of \( A^s \), and the assumption that \((p, f, A) \in A^s \), it follows \((\hat{p}, f_k, A) \in A^s \) and

$$
\left| (\hat{p}, f_k, A) \right|_s \lesssim \left| (p, f, A) \right|_s + \left\| f \right\|_{A^s_0}.
$$
We then use the stopping criteria (6.36) and apply Theorem 6.2 to \((\tilde{p}, \tilde{u})\) to obtain
\[
\|A^{-1/2}(\tilde{p} - p_N)\|^2_{L^2(\Omega)} + \|h_N \text{div}(\tilde{p} - p_N)\|^2_{L^2(\Omega)} + \text{osc}^2_{T_N}(p_N, T_N) \lesssim \eta^2_{T_N}(p_N, T_N) \leq \varepsilon^2
\] (6.42)
and
\[
\#T_N - \#T_k \lesssim C\Theta(s, \theta)|\langle \tilde{p}, f, A \rangle|^{1/s} \varepsilon^{-1/s}.
\] (6.43)

Notice that \(\|h_N \text{div}(p - \tilde{p})\|_{L^2(\Omega)} \leq \|h_k \text{div}(p - \tilde{p})\|_{L^2(\Omega)} = \text{osc}(f, T_k)\), and \(\Theta(s, \theta) > 1\). A combination of (6.39)-(6.43) yields
\[
\mathcal{E}^2_N + \text{osc}^2_{T_N}(p_N, T_N) \leq 2\|A^{-1/2}(p - \tilde{p})\|^2_{L^2(\Omega)} + 2\|h_N \text{div}(p - \tilde{p})\|^2_{L^2(\Omega)} + 2\|A^{-1/2}(\tilde{p} - p_N)\|^2_{L^2(\Omega)} + \text{osc}^2_{T_N}(p_N, T_N) \lesssim 2C_0\varepsilon^2 + 2\text{osc}^2(f, T_k) + 2\varepsilon^2 \lesssim \varepsilon^2
\] (6.44)
and
\[
\#T_N - \#T_0 = \#T_N - \#T_k + \#T_k - \#T_0 \\
\lesssim C\Theta(s, \theta)|\langle \tilde{p}, f, A \rangle|^{1/s} \varepsilon^{-1/s} + C||f||_{A^s_0}^{1/s} \varepsilon^{-1/s} \\
\lesssim C\Theta(s, \theta)|\langle (p, f, A) \rangle|_s + ||f||_{A^s_0}^{1/s} \varepsilon^{-1/s}.
\] (6.45)
The above inequality (6.45) implies
\[
\varepsilon \lesssim C^s\Theta^s(s, \theta)|\langle (p, f, A) \rangle|_s + ||f||_{A^s_0}^{1/s} (#T_N - #T_0)^{-s}.
\] (6.46)
The desired result (6.38) then follows from the above two inequalities (6.44) and (6.46).

7 Conclusions

It is time to recall the main results in this paper. Firstly, we have removed the restriction of the coefficient matrix of the PDEs which is required in Carstensen’ work, and have analyzed the efficiency of the a posteriori error estimator obtained by Carstensen, for RT, BDM, and BDFM elements. Secondly, For the AMFEM, as is customary in practice, the AMFEM marks exclusively according to the error estimator and performs a minimal element refinement without the interior node property. We have proved that the sum of the stress variable error in a weighted norm and the scaled error estimator, reduces with a fixed factor between two successive adaptive loops, up to an oscillation of data \(f\). This geometric decay is instrumental to obtain the optimal cardinality of the AMFEM. Finally, we have
shown that the stress variable error in a weighted norm plus the oscillation of data yields a decay rate in terms of the number of degrees of freedom as dictated by the best approximation for this combined nonlinear quantity, namely we have obtain the quasi-optimal convergence rate for the AMFEM.

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