Haga’s theorems in paper folding and related theorems in Wasan geometry Part 2

HIROSHI OKUMURA

Abstract. We generalize problems in Wasan geometry which involve no folded figures but are related to Haga’s fold in origami. Using the tangent circles appeared in those problems we give a parametric representation of the generalized Haga’s fold given in the first part of this two-part paper.

Keywords. Haga’s fold, parametric representation of generalized Haga’s fold

Mathematics Subject Classification (2010). 01A27, 51M04

1. INTRODUCTION

This is the second part this two-part paper. In the first part we considered the generalized Haga’s fold. There are several problems in Wasan geometry, which do not involve folded figures but are closely related Haga’s fold. In this second part we consider those problems in a general way. Using tangent circles appeared in those problems, we give a parametric representation of the generalized Haga’s fold.

2. RELATED PROBLEMS IN WASAN GEOMETRY

In this section we consider several problems in Wasan geometry closely related to Haga’s fold, though they are not involving folded figures. A general solution of the problems is given in the next section. We start with two similar problems. The following problem can be found in [1], [13, 17], [23] and [25].

Problem 2.1. Let $\delta$ be a circle of radius $d$ and let $ABCD$ be a rectangle sharing its center with $\delta$, where the side $AB$ touches $\delta$ and the side $BC$ intersect $\delta$ in two points (see Figure 1). The inradius of the curvilinear triangle made by $AB$, $BC$ and $\gamma$ is $r$ and the circle touching $BC$ at its midpoint and touching the minor arc of $\delta$ cut by $BC$ has radius $r$. Find $d$ in terms of $r$ (or find $r$ in terms of $d$).

Figure 1

Figure 2
Problem 2.2. Let $\delta$ be a circle of radius $d$ and let $ABC$ be a right triangle with right angle at $A$. The side $CA$ touches $\delta$, and each of the sides $AB$ and $BC$ intersects $\delta$ in two points. The inradius of the curvilinear triangle made by $CA$, $AB$ and $\delta$ equals $r$. The maximal circle touching $AB$ from the side opposite to $C$ and touching $\delta$ internally, and the maximal circle touching $BC$ from the side opposite to $A$ and touching $\delta$ internally have radius $r$. Find $r$ in terms of $d$.

![Figure 3](image1.png)  ![Figure 4](image2.png)  ![Figure 5](image3.png)

We show that the two problems are essentially the same. Let $\gamma$ be the incircle of the curvilinear triangle made by $AB$, $BC$ and $\delta$ in Problem 2.1 (see Figure 3). If we draw the line parallel to $BC$ touching $\delta$ and the reflection of $\gamma$ in the line $BC$ and extend the side $AB$, we get Figure 5. We can also get the same figure from Figure 2 in a similar way (see Figure 4). Therefore the two problems are essentially the same. Problems with Figure 5 can also be found in [3], [4], [13], [17], [20], [21], [22], [24] and [26]. A generalization of Problem 2.1 can be found in [12].

We state Problems 2.1 and 2.2 so that the body text gives enough information without the figures. However the most informations of the problems in Wasan geometry are given by the figures, thereby the body texts play only subsidiary roles. The next sangaku problem is stated in such a way [2]:

![Figure 6](image4.png)  ![Figure 7](image5.png)

Problem 2.3. There are a large circle of radius $d$ and two small circles of radius $r$ in a square as in Figure 6. Show $r$ in terms of $d$.

We show the problem is incorrect using the next proposition.

Proposition 2.1. If an external common tangent of externally touching two circles of radii $r$ and $s$ touches the circles at points $P$ and $Q$, then $|PQ| = 2\sqrt{rs}$. 
The answer says \( r = d/9 \). But we have \( d = 2\sqrt{rd} + \sqrt{2}r + r \) by the proposition. Hence we get \( d = \left(3 + \sqrt{2} + 2\sqrt{2} + \sqrt{2}\right)r \approx 8.11r \). Therefore the assertion of the problem is incorrect. It seem that the two small circles were described as in Figure 5 in the original problem, however Figure 6 was used by transcription error. A general case was considered by Toyoyoshi (see Figure 7):

**Problem 2.4** ([14]). Let \( \delta \) be a circle of radius \( d \) with center \( C \) passing through \( B \) for a square \( ABCD \). Let \( \gamma_1, \gamma_2, \cdots, \gamma_n \) be congruent circles of radius \( r \) touching \( DA \) from the same side such that \( \gamma_1 \) and \( \gamma_2 \) touch and \( \gamma_i \) \((i = 3, 4, \cdots n)\) touches \( \gamma_{i-1} \) from the side opposite to \( \gamma_1 \), also \( \gamma_1 \) touches \( \delta \) externally and \( \gamma_n \) touches \( AB \) from the same side as \( \delta \). Show \( r \) in terms of \( s \) and \( n \).

3. Generalized figure

We generalize the figures of Problems 2.4. Let \( k \) and \( l \) be perpendicular lines intersecting in a point \( A \) and let \( \gamma \) be a circle of radius \( r \) touching \( k \) at a point \( K \). Let \( \delta_1 \) and \( \delta_2 \) be circles of radii \( d_1 \) and \( d_2 \) \((d_2 \leq d_1)\), respectively, touching \( l \) from the same side, such that they touch \( k \) from the same side as \( \gamma \), and also touch \( \gamma \) externally, where if \( \gamma \) touches \( k \) at \( A \), we consider \( \delta_1 \) is the circle touching \( k, l \) and \( \gamma \) externally, \( \delta_2 = A \) and \( d_2 = 0 \). We denote the figure of \( k, l, \gamma, \delta_1 \) and \( \delta_2 \) by \( T(n) \), where

\[
(1) \quad n = \tau|AK|/2r + 1/2
\]

and \( \tau = 1 \) if \( \delta_1 \) and \( K \) lies on the same side of \( l \) otherwise \( \tau = -1 \) (see Figures 8 and 9). We also consider the case in which \( \gamma \) degenerates to a point \( K \neq A \) on \( k \). We consider that \( \delta_1 \) and \( \delta_2 \) coincide and touch \( k \) at \( K \) in this case, and figure is denoted by \( T(0) \) (see Figure 10). We define the value of \( 0 \) equals 0.
Notice that $K$ coincides with $A$ if and only if $n = 1/2$ (see Figure 11), also $1/2 < n$ or $0 < n < 1/2$ according as $K$ and $\delta_1$ lie on the same side of $l$ or not. If $n = 0$, $\delta_1$ coincides with $\delta_2$ and is the reflection of $\gamma$ in $l$ (see Figure 12). Our definition of $\mathcal{T}(n)$ implies $n \geq 0$ for any $n$. If $n$ is a natural number, there are circles $\gamma_1$, $\gamma_2$, $\cdots$, $\gamma_n$ of radius $r$ touching $k$ from the same side such that $\gamma = \gamma_1$, $\gamma_1$ and $\gamma_2$ touch, $\gamma_i$ ($i = 3, 4, \cdots n$) touches $\gamma_{i-1}$ from the side opposite to $\gamma_1$, and $l$ is the external common tangent of $\gamma_n$ and $\delta_1$ (see Figure 13). This is the case considered by Toyoyoshi stated as Problem 2.4. If $n = 2$, the circles $\gamma_2$ and $\delta_2$ coincide (see Figure 25 where regard $\gamma_1 = \gamma(2)$ in the figure). If we add the reflection of $\delta_1$ in $l$ and remove $\delta_2$ and $l$ for $\mathcal{T}(n/2)$ in the case $n$ being a natural number, the resulting figure is a part of the figure $B(n)$ in [9], i.e., $\mathcal{T}(n)$ is a generalization of $B(n)$ in this sense. The first half of Theorem 3.1(i) gives a solution of Problems 2.1, 2.2 and 2.4.

**Theorem 3.1.** The following statements are true for $\mathcal{T}(n)$.

(i) If $n \neq 0$, $d_1 = (\sqrt{2n} + 1)^2 r$ and $d_2 = (\sqrt{2n} - 1)^2 r$.

(ii) $|AK| = \sqrt{d_1d_2}$.

(iii) $\sqrt{\mathcal{T}} = \begin{cases} \frac{\sqrt{d_1} + \sqrt{d_2}}{2} & \text{if } 0 \leq n \leq 1/2, \\ \frac{\sqrt{d_1} - \sqrt{d_2}}{2} & \text{if } 1/2 < n \text{ or } n = \bar{0}. \end{cases}$

**Proof.** If $n \neq \bar{0}$, by Proposition 2.1 $d_2 = \tau |AK| + 2\sqrt{\tau d_2} = (2n - 1)r + 2\sqrt{\tau d_2}$ if $\tau = 1$, and $d_2 = \tau |AK| - 2\sqrt{\tau d_2} = (2n - 1)r - 2\sqrt{\tau d_2}$ if $\tau = -1$, which yield $d_2 = (\sqrt{2n} \pm 1)^2 r$ in both the cases. Also we have $d_1 = \tau |AK| + 2\sqrt{\tau d_1} = (2n - 1)r + 2\sqrt{\tau d_1}$, which also yields $d_1 = (\sqrt{2n} \pm 1)^2 r$. This proves (i). The part (ii) is obvious if $n = \bar{0}$. If $n \neq \bar{0}$, (ii) follows from $|AK| = |2n - 1|r$ and (i). The part (iii) is trivial if $n = \bar{0}$. If $n \neq \bar{0}$, eliminating $n$ from the two equations in (i) we get (iii). 

If $n = 4$, $\delta_1$ and $\delta_2$ intersect and the maximal circle touching $\delta_1$ and $\delta_2$ from inside of them has radius $r$, which is obtained by translating $\gamma_2$ parallel to $l$, through distance $4r$ (see Figure 13). Let $D_1$ be the point of contact of $\delta_1$ and $k$. If $n = 9/2$, then $d_1 = 4d_2 = 16r$ and $K$ is the midpoint of $D_1A$ (see Figure 14). Problems considering this case with the circle $\delta_2$ can be found in [15, 16], [18, 19] and [20]. However the circle $\delta_2$ seems to have been ignored for the figure $\mathcal{T}(n)$ in most cases.
except this case. Let $E_i$ be the point of intersection of $k$ and the internal common tangent of $\delta_i$ and $\gamma$, if $\delta_i$ and $\gamma$ are proper circles (see Figure 15).

**Theorem 3.2.** If $n \neq 0$, the following statements hold for $T(n)$.

(i) The point $E_1$ divides $D_1A$ internally in the ratio $1 : \sqrt{2}n$.

(ii) If $n \neq 1/2$, the point $E_2$ divides $D_2A$ externally in the ratio $1 : \sqrt{2}n$.

**Proof.** Let $n \neq 0$. We prove (ii). Let $n \neq 1/2$. Since $E_2$ is the midpoint of the segment $D_2K$, $|D_2E_2| = \sqrt{d_2r} = |\sqrt{2}n - 1| r$ by Proposition 2.1 and Theorem 6.1(i). If $1/2 < n$, $D_2$ lies between $A$ and $E_2$. Hence $|AE_2| = |D_2E_2| + d_2 = (\sqrt{2n} - 1) r + (\sqrt{2n} - 1)^2 r = \sqrt{2n} |\sqrt{2n} - 1| r$. If $0 < n < 1/2$, $A$ lies between $D_2$ and $E_2$ (see Figure 15). Therefore $|AE_2| = |D_2E_2| - d_2 = (1 - \sqrt{2n}) r - (1 - \sqrt{2n})^2 r = \sqrt{2n} |\sqrt{2n} - 1| r$. Hence we get $|D_2E_2| : |AE_2| = 1 : \sqrt{2n}$ in both the cases. The part (i) is proved in a similar way.

\[ \text{Figure 15: } T(n) \text{ and } T(\pi) \text{ in the case } 0 < n < 1/2 \]

Let $\gamma = \gamma$ if $n = 1/2$, $\gamma = D_1$ if $n = 0$, $\gamma$ be the reflection of $\delta_1$ in $\ell$ if $n = 0$, $\gamma$ be the remaining circle touching $k$ and $\delta_1$, $\delta_2$ externally if $n \neq 1/2, 0, \overline{0}$ for the figure $T(n)$. We denote the figure consisting of the circles $\delta_1$, $\delta_2$, $\gamma$ and the lines $k$ and $\ell$ by $T(\pi)$ (see Figure 15). Notice that $\overline{\gamma} = \gamma$, $\overline{T(\pi)} = T(n)$, $T(1/2) = T(1/2)$. (see Figures 13 to 11). Let $K$ be the point of contact of $\gamma$ and $k$, and let $\pi$ be the radius of $\gamma$. We now assume the definition of the division by zero [7], i.e., $n/0 = 0$ for any real number $n$. Hence we also get $n/0 = 0$.

**Theorem 3.3.** The following statements hold for $T(n)$ and $T(\pi)$.

(i) $|AK| = |AK|$. 

(ii) $2n = \frac{1}{2\pi}$.

(iii) \(2(r + \pi) = d_1 + d_2\).

**Proof.** The part (i) follows from Theorem 3.1(ii). We prove (ii). Since $n = 0$ and $\pi = 0$ are equivalent, and $n = \overline{0}$ and $\pi = 0$ are also equivalent, both side of (ii) equal 0 if $n = 0$ or $n = \overline{0}$ by the definition of the division by zero. Hence (ii) holds in this case. Let $n \neq 0, \overline{0}$. The case $n = 1/2$ is trivial. If $1/2 < n$, the points $K$, $D_2$, $A$, $\overline{K}$ lie in this order. Hence $2\sqrt{d_2r} = |D_2K| < |D_2\overline{K}| = 2\sqrt{d_2\pi}$ by (i), i.e., $r < \pi$. Therefore $r = (\sqrt{d_1} - \sqrt{d_2})^2/4$ and $\pi = (\sqrt{d_1} + \sqrt{d_2})^2/4$ by Theorem 3.1(iii). Hence

\[ n\pi = \frac{|AK| + r}{2r} \cdot \frac{|AK| + \pi}{2\pi} = \frac{(\sqrt{d_1} + \sqrt{d_2})^2}{2 (\sqrt{d_1} - \sqrt{d_2})^2} \cdot \frac{(\sqrt{d_1} - \sqrt{d_2})^2}{2 (\sqrt{d_1} + \sqrt{d_2})^2} = \frac{1}{4}. \]

The rest of (ii) is proved similarly. The part (iii) follows from Thereom 3.1(iii).
Remark. One may think that the equation in (ii) should be expressed as \(4n\overline{n} = 1\). But this does not hold in the case \(n = 0\) or \(n = \overline{0}\). However, the equation \(2n = \overline{2n}\) holds even in this case by the definition of the value \(\overline{0}\) and the definition of the division by zero.

4. Parametric representation of the generalized Haga’s fold

Let \(ABCD\) be a square with a point \(E\) on the line \(DA\). We assume that \(m\) is the perpendicular bisector of the segment \(CE\), \(G\) is the reflection of \(B\) in \(m\), and \(F\) is the point of intersection of the lines \(AB\) and \(EG\) if they meet, where we define \(F = B\) in the case \(E = A\). The figure consisting of \(ABCD\) and the points \(E, F\) (if exists) and \(G\) is called the figure made by the generalized Haga’s fold and denoted by \(H\) [8]. Ordinary Haga’s fold is obtained if \(E\) lies between \(D\) and \(A\). Let \(\delta\) be the circle of radius \(d = |AB|\) with center \(C\). In this section we give a parametric representation of \(H\) using the circle touching the line \(DA\) and the circle \(\delta\) externally.

Let \(\gamma\) be the circle touching the line \(DA\) and \(\delta\) at the point of contact of \(\delta\) and the remaining tangent of \(\delta\) from \(E\). Then \(\gamma \mapsto H\) is one-to-one correspondence between the set of the circles touching the line \(DA\) and \(\delta\) externally and the set of the figures made by the generalized Haga’s fold, where we consider that the point \(D\) is a member of the former set and \(D\) corresponds to the figure made by the generalized Haga’s fold with \(D = E\). If \(\gamma\) is a circle of radius \(r\) touching \(\delta\) externally and the line \(DA\) at a point \(K\), we define \(n\) by (1). Let \(T\) be the point of contact of \(\gamma\) and \(\delta\). We explicitly denote the circle \(\gamma\) by \(\gamma(n)\) or \(\gamma(-n)\) according as \(T\) lies inside of \(ABCD\) or outside of \(ABCD\). If \(T = B\), \(\gamma\) is denoted by \(\gamma(0)\). The point \(D\) is denoted by \(\gamma(\overline{0})\). Now any proper circle touching the line \(DA\) and \(\delta\) externally can be expressed by \(\gamma(n)\) for a real number \(n\), and we also explicitly denote the figure \(H\) by \(H(n)\). If \(D = E\), the figure is denoted by \(H(\overline{0})\) [8].

There are seven cases to be considered for \(H\):

- \((h1)\) \(D\) lies between \(E\) and \(A\) and \(b > d\),
- \((h2)\) \(D\) is the midpoint of \(EA\), i.e., \(b = d\),
- \((h3)\) \(D\) lies between \(E\) and \(A\) and \(b < d\),
- \((h4)\) \(D = E\),
- \((h5)\) \(E\) lies between \(D\) and \(A\),
- \((h6)\) \(E = A\),
- \((h7)\) \(A\) lies between \(D\) and \(E\),

\[^{1}\text{\(H(\overline{0})\) is denoted by \(H(\infty)\) in [8].}]}
Figure 16: $\mathcal{H}(n), (-2 < n < -1/2)$

Figure 17: $\mathcal{H}(-2)$

Figure 18: $\mathcal{H}(n), (n < -2)$

Figure 19: $\mathcal{H}(0)$

Figure 20: $\mathcal{H}(n), (0 < n)$

Figure 21: $\mathcal{H}(0)$

Figure 22: $\mathcal{H}(n), (-1/2 < n < 0)$

Figure 23: $\mathcal{H}(-\rho^{-1})$
We consider the value of $n$ for $H(n)$ as a function of the point $E$ when $E$ moves on the line $DA$. We also specify the case in which the crease line $m$ passes through the inside of $ABCD$. In the cases (h1), (h2), (h3), the point $T$ lies outside of $ABCD$. Therefore $n < 0$, and by Proposition 2.1 we have

$$
(2) \quad n = - \left( \frac{|AK|}{2r} + \frac{1}{2} \right) = -\frac{d + 2\sqrt{dr}}{2r} - \frac{1}{2} = -\frac{d}{r} \left( \frac{1}{2} \sqrt{\frac{d}{r}} + 1 \right) - \frac{1}{2}.
$$

Therefore $n$ is a monotonically increasing function of $r$, i.e., $n$ decreases when $E$ moves with moving direction same as to $\overrightarrow{DA}$. Hence $n$ approaches $-1/2$ when $D$ lies between $A$ and $E$ and $E$ moves away from $D$, i.e., $n < -1/2$. Therefore we get $-2 < n < -1/2$, $n = -2$, $n < -2$ in the cases h(1), h(2), h(3), respectively. The crease line $m$ does not pass through the inside of $ABCD$ (see Figure 16) in the case h(1). In the case (h2), $m$ passes through $D$ but does not pass through the inside of $ABCD$ (see Figure 17). In the cases (h3), (h4), (h5), (h6), $m$ passes through the inside of $ABCD$ (see Figures 18 to 21), and we get $n = 0$, $0 < n$, $n = 0$ in the cases (h4), h(5), (h6), respectively. The values of $n$ also decreases when $E$ moves from $D$ to $A$ in the case (h5).

Let us consider the case (h7) (see Figure 22). Since $T$ lies outside of $ABCD$, we get

$$
(3) \quad n = - \left( \frac{|AK|}{2r} + \frac{1}{2} \right) = \frac{2\sqrt{dr} - d}{2r} - \frac{1}{2} = -\frac{d}{r} \left( \frac{1}{2} \sqrt{\frac{d}{r}} - 1 \right) - \frac{1}{2}.
$$

Hence $n$ is a monotonically decreasing function of $r$. Therefore $n$ approaches $-1/2$ when $r$ increases to $+\infty$, i.e., $n$ decreases when $E$ moves away from $A$, and we have $-1/2 < n < 0$.

Let $\rho = \left(1 + \sqrt{2} \right)^2$. We consider the special case in which $m$ passes through the point $A$. This happens if and only if $|AE| = \sqrt{2}d$, or $AEF$ is an isosceles right triangle (see Figure 23). In this event $|DE| = |DK|/2 = \sqrt{dr} = d + \sqrt{2}d$ holds by Proposition 2.1. The equation implies $d/r = \rho^{-1}$, and we get $n = -\rho^{-1}$ by (3). Therefore $m$ passes through the inside of $ABCD$ if and only if $-\rho^{-1} < n < 0$ in the case (h7).

We summarize the results (see Table 1 also). The arrows in the table show that $n$ is a monotonically decreasing function of $E$ when $E$ moves on the line $DA$ with moving direction same as to $\overrightarrow{DA}$. The crease line $m$ passes through the inside of $ABCD$ in the cases (h3), (h4), (h5), (h6), and (h7) if $-\rho^{-1} < n < 0$.

| case | (h1) | (h2) | (h3) | (h4) | (h5) | (h6) | (h7) |
|------|------|------|------|------|------|------|------|
| $n$  | $-2 < n < -1/2$ | $-2$ | $n < -2$ | $0$ | $0 < n$ | $0$ | $-1/2 < n < 0$ |

Table 1.

The above observation shows that $n \neq -1/2$, while the remaining tangent of the circle $\delta$ parallel to $DA$ is not a member of the set of circles touching the line $DA$ and $\delta$ externally. The fact suggests to describe the tangent by $\gamma(-1/2)$. 
5. Special cases

In this section we consider special cases for the figures $\mathcal{H}(n)$. At first we consider two special cases of Haga’s fold in the case $(h5)$. If $E$ is the midpoint of $DA$, $F$ divides $AB$ in the ratio $2:1$ internally [8, Theorem 3.1]. The fact is called Haga’s first theorem [6]. While Theorem 3.2(i) shows that this happens if and only if $n = 1/2$. Therefore the figure of Haga’s first theorem coincides with $\mathcal{H}(1/2)$ (see Figure 24). A problem considering $\mathcal{H}(1/2)$ giving the relation $d = 4r$ can be found in [5].

Similarly if $F$ is the midpoint of $AB$, $E$ divides $DA$ in the ratio $1:2$ internally. The fact is called Haga’s third theorem [6]. While Theorem 3.2(i) shows that this happens if and only if $n = 2$. Hence the figure of Haga’s third theorem coincides with $\mathcal{H}(2)$ (see Figure 25). Since $E$ is the midpoint of the segment $DK$, $E$ and $K$ are the points of trisection of the side $DA$. The remaining circle touching $DA$ and $\delta_1$ and $\delta_2$ externally is $\gamma(1/8)$ by Theorem 3.3(ii). Theorem 3.2(ii) shows that $K$ coincides with the point of intersection of $DA$ and the internal common tangent of $\gamma(1/8)$ and $\delta$. Notice that $\mathcal{H}(2)$ seems to be most frequently considered among $\mathcal{H}(n)$ in Wasan geometry as we have shown in section 2.

6. Conclusion

We argued the merit of considering circles in the geometry of origami in [10] [11]. In this two-part paper we have shown several examples to verify the validity of our assertion. The circles we have considered are tangent circles except the circumcircle of a triangle considered in the first part of the paper. In this sense we can say that the geometry of origami is a geometry of tangent circles. In particular, the incircle and the excircles of a right triangle play important roles in the geometry of origami using a square piece of paper as shown in the first part of this paper.

References

[1] Furuya ed., Sampō Tsūsho, 1854, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005506
[2] Gunmaken Wasan Kenkyūkai ed., The Sangaku in Gunma, Gunmaken Wasan Kenkyūkai, 1987.
[3] Ishida et al. ed., Mishō Sampō Vol. 10, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100007110
[4] Itô ed., Sampo Zatsushō Semmonkai, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005302

[5] Kimura ed., Sampō, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005210

[6] Koshiro, How to divide the side of square paper, http://www.origami.gr.jp/Archives/People/CAGE_/divide/02-e.html

[7] M. Kuroda, H. Michiwaki, S. Saitoh and M. Yamane, New meanings of the division by zero and interpretations on 100/0 = 0 and on 0/0 = 0, Int. J. Appl. Math. 27(2) (2014), 191-198.

[8] H. Okumura, Haga’s theorems in paper folding and related theorems in Wasan geometry Part 1, Sangaku J. Math., (2017) 42–56.

[9] H. Okumura, Configurations of congruent circles on a line, Sangaku J. Math., (2017) 24–34.

[10] H. Okumura, Origamis with Wasan geometry, Symmetry: Culture and Science, 28(3) (2016) 312–320.

[11] H. Okumura, Origamis involving circles, RIMS Kōkyūroku, volume 1982 (2016) 37–42.

[12] H. Okumura, Concentric figures with tangent circles, Mathematics and Informatics Quarterly, 4(2) (1994) 87–91.

[13] Toyoyoshi ed., Santei Yōdai Katsuyō, Digital Library Department of Mathematics, Kyoto University, http://edb.math.kyoto-u.ac.jp/vasan/068

[14] Toyoyoshi ed., Chikusaku, Digital Library Department of Mathematics, Kyoto University, http://edb.math.kyoto-u.ac.jp/vasan/154

[15] Toyoyoshi, Tenzan, Digital Library Department of Mathematics, Kyoto University, http://edb.math.kyoto-u.ac.jp/vasan/159

[16] Toyoyoshi ed., Tenzan Zatsunmon, Digital Library Department of Mathematics, Kyoto University, http://edb.math.kyoto-u.ac.jp/vasan/161

[17] Toyoyoshi ed., Tenzan Sanyō, Digital Library Department of Mathematics, Kyoto University, http://edb.math.kyoto-u.ac.jp/vasan/162

[18] Yamamoto ed., Sampo Tenzan Tebiki Gusa, 1833, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005576

[19] Yamamoto ed., Taizen Jinkōki, 1832, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100000422

[20] Yasuda et al. ed., Žozoku Kiōshū, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100004196

[21] Enrui Tekitōshū, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100003918

[22] Kansui Heishin Sanron, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100004185

[23] Sandai Kenmonki, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005039

[24] Sansoku, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005018

[25] Tenzan Kaitei, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100006710

[26] Wasan Henshū, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100007321

Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.