New Class of Generalized Extensive Entropies for Studying Dynamical Systems in Highly Anisotropic Phase Space.

Giorgio SONNINO$^{1,2}$

$^1$Université Libre de Bruxelles (ULB), Department of Physics
Bvd du Triomphe, Campus de la Plaine CP 231
1050 Brussels, Belgium. Email: gsonnino@ulb.ac.be

$^2$Royal Military School (RMS), Av. de la Renaissance 30,
1000 Brussels, Belgium

and

György STEINBRECHER$^3$

$^3$Association EURATOM-MEdC, University of Craiova,
A. I. Cuza 13, 200585 Craiova, Romania
Email: gyorgy.steinbrecher@gmail.com

November 20, 2013

Abstract

Starting from the geometrical interpretation of the Rényi entropy, we introduce further extensive generalizations and study their properties. In particular, we found the probability distribution function obtained by the MaxEnt principle with generalized entropies. We prove that for a large class of dynamical systems subject to random perturbations, including particle transport in random media, these entropies play the role of Liapunov functionals. Some physical examples, which can be treated by the generalized Rényi entropies are also illustrated.

Generally, to characterize some unknown events with a statistical model, it is chosen the one that has maximum entropy (MaxEnt principle) i.e., the one that has maximum uncertainty. The corresponding probability distribution functions (PDF) are obtained by maximizing entropy, under a set of constraints. When the system is close to the thermodynamic equilibrium, the PDF may be obtained by extremizing the entropy production subject to restrictions. For this reason, the entropy production for plasmas with electromagnetic turbulence was analyzed and carefully calculated in many paper, by using the standard definition of the Boltzmann-Gibbs theory [1]-[3]. In other terms, entropy is regarded as a measure of information, and it is a quantity able to quantify the uncertainty,
or the randomness of a system. Information theory was initially developed by C. E. Shannon to quantify the expected value of the information contained in a message, usually in units such as bits [4]. However, the Shannon entropy, used in standard top-down decision trees, does not guarantee the best generalization (see, for example, Ref. [5]). The Shannon entropy has then been generalized, successively. A. Rényi introduced the most general definition of information measures that preserve the additivity for independent events and are compatible with the axioms of probability [6]. C. Tsallis, on the contrary, introduced a nonadditive entropy, such as nonextensive statistical mechanics, generalizing then the Boltzmann-Gibbs theory [7]. Rényi’s and Tsallis’s entropies are algebraically related, and both definitions include the Shannon entropy as a limit case. A wide spectrum of natural, artificial and social complex systems are now analyzed by means of these two entropies. At present, information theory finds applications in broad areas of science such as, in neurobiology [8], the evolution of molecular codes [9], model selection in ecology [10], thermal physics [11], plagiarism detection [12] or quantum information [13].

The purpose of this work is to introduce and study a class of anisotropic generalization of Rényi’s entropy (RE). An obvious reason is that, in some problems of complex systems, the phase space may be highly anisotropic. Let us illustrate this concept by the following simple example. We consider two small identical particles of mass $m$, immersed in a fluid governed by the Stokes law, with friction coefficient $\tilde{\gamma}$, and subject to central two body force $-\nabla V(r)$. In this case, it is easy to check that the problem is equivalent to the one of a single particle, of reduced mass $\mu = m/2$, immersed in a bath with friction coefficients $\gamma = \tilde{\gamma}/2$. After little calculations, we see that in this case the statistical properties of the center of mass $R = (r_1 + r_2)/2$ are completely decoupled by those of the relative motion $r \equiv r_1 - r_2$. It is also easily checked that the velocity of the center of mass i.e., $\dot{R} = (\dot{r}_1 + \dot{r}_2)/2$, obeys to the Ornstein-Uhlenbeck-Fokker-Planck equation, whereas the statistical properties of the relative motion are described by the following Kramer equation

$$\frac{\partial \rho(r, v, t)}{\partial t} = \left[ -\frac{\partial}{\partial r} v + \frac{1}{\mu} \frac{\partial}{\partial v} \left( \gamma v + \nabla V(r) \right) + \frac{\gamma K_B T}{\mu^2} \frac{\partial^2}{\partial v^2} \right] \rho(r, v, t)$$

with $K_B$ denoting Boltzmann’s constant. Moreover, $T$ and $v$ stand for temperature and $\dot{r}$, respectively. The steady state solution is of the Boltzmann distribution type [14]

$$\rho_{stat.}(r, v) = \rho_0 \exp[-V(r)/(K_B T)] \exp[-\mu v^2/(2K_B T)]$$

We are interested in diffusion of particles in the presence of a logarithmically growing potential. This potential has attracted much interest since it serves as a model of several physical systems. For example, charged particles near a long and uniformly charged polymers, are subject to a logarithmic potential. Other examples of systems having potentials showing a logarithm and power-law potentials are: in condensation in polyelectrolyte solutions [14], nanoparticles with arbitrary two-dimensional force field [16], and vortex dynamics in the two-dimensional model [17]. For a logarithmically growing potential
\[ V(x) \approx V_0 \log(x/a), \text{ for } x \gg a \text{ and } V_0 > 0, \] we get an equilibrium distribution with power-law tail \( \rho_{\text{stat}}(x) \sim x^{-V_0/(K_B T)} \). Ref. [18] reports on the study of an over damped motion of a Brownian particle in the logarithmic-harmonic potential. In Ref. [20] it can be found the study of the trajectories of a Brownian particle moving in a confining asymptotically logarithmic potential, obeying the over damped Langevin equation with potential

\[ V(r) = g \log(1 + r^2) \]  

where the parameter \( g > 0 \) specifies the strength of the attractive potential. Notice that this situation may be realized in experiments [21]-[22]. In this case, the stationary solution reads

\[ \rho_{\text{stat}}(r, v) = \frac{1}{(1 + r^2)^{g/(K_B T)}} \exp[-\mu v^2/(2 K_B T)] \]  

Eq. (3) clearly shows that the steady-state solution has a short tail in velocity and long tail in distance \( r \). We shall show that this anisotropy of the PDF in the \((r, v)\) phase-space can be retrieved from the MaxEnt principle applied to the new class of entropy here introduced, subject to natural scale-invariant restrictions (see below). This anisotropy of the phase space is manifest in magnetically confined plasmas and, in general, in the case of integrable systems subject or not to perturbations. Another example is provided by the evolution of dynamical systems under the effect of noise where we consider the extended phase space of the system plus source of noise. Also in this quite general example, the extended phase space is anisotropic. Our main task is to propose a generalization of the Rényi entropy (GRE) that, still preserving the additivity, is able to treat these anisotropic situations. We prove that the GRE provides a new set of Liapunov functionals (more exactly H theorems) for a large class of Fokker-Planck equations describing particles transport in a random physical environment. In this case the GRE is always monotonic. We encounter this situation when we study the dynamics of charged particles in random electric and magnetic fields [23].

Our generalization of RE results from the reinterpretation of the RE and MaxEnt principles in the terms geometrical concepts in the functional spaces of all PDF in a given phase space. Let us now proceed by introducing a rigorous definition of the GRE and illustrating its main properties. The definition of GRE starts from the reformulation of RE in the geometric term of norm (or pseudo norm). Starting from the initial axiomatic definition in the case of discrete probability field, Rényi proved that, for a fixed value of parameter \( q \), appearing in his set of axioms, in particular the ”axiom 5'” (see Ref. [6]),

\[ S_{R,q}(p_i) = \frac{q}{q-1} \log \left( \sum_{i=1}^{N} p_i^q \right)^{1/q} \]  

is unique, up to a multiplicative constant. In general, i.e. including the continuum case, when the probability is defined in the terms of some (possibly preferred invariant) measure \( dm(x) \) with PDF \( \rho(x) \) in the space \( \Omega \), the previous definition of RE can be extended, as
\[ S_{R,q}[\rho] = \frac{1}{1-q} \log \left[ \int_{\Omega} [\rho(x)]^q \, dm(x) \right]^{\frac{1}{q}}. \] In the particular case when \( \Omega \) is a set with \( N \) elements and \( m \) is the counting measure the original form is obtained, the permutation symmetry from the original Axiom 1 (see Ref. [6]) became the invariance under transformations that preserve the measure \( m \). Observe that \( S_{R,q}[\rho] \) for \( q > 1 \) can geometrically be reinterpreted in terms of the norm \( \| \rho \|_q \):

\[ S_{R,q}[\rho] = \frac{1}{1-q} \log \| \rho \|_q ; \quad \| \rho \|_q = \left[ \int_{\Omega} [\rho(x)]^q \, dm(x) \right]^{\frac{1}{q}} ; \quad q > 1 \quad (4) \]

with the norm, \( \| \rho \|_q \), defining the distance in the standard Lebesgue \( L_q(\Omega, dm) \) spaces [24], [25]. However, in the case \( 0 < q < 1 \) we are no longer able to interpret \( \| \rho \|_q \) in Eq.(4), as a distance. This problem may easily be overcome by observing that the functional

\[ N_q[\rho] = \int_{\Omega} [\rho(x)]^q \, dm(x) \]

may be interpreted as a distance [24], [26] and [27] so, for \( 0 < q < 1 \), the definition of entropy reads

\[ S_{R,q}[\rho] = \frac{1}{1-q} \log N_q[\rho] \]

with \( N_q[\rho] \) playing the role of "distance", but in the more complicated space \( L_{q<1}(\Omega, dm) \) [24]. We remark that \( N_q[\rho] \) and spaces \( L_{q<1} \) were used for studying the steady state distributions of linear stochastic differential equations [27] or of the stable distributions with heavy tail [26]. However, it should be mentioned that, due to the complexity of the formalism, mathematicians prefer to transfer \( f(x) \in L_{q<1} \) in the standard \( L_1 \) space by \( f(x) \rightarrow |f(x)|^q \in L_1 \) [24]. The physical counterpart of this transformation is the Tsallis averaging rule [7]: if \( \rho(x) \) is a PDF then for averaging it use \( [\rho(x)]^q / \langle [\rho(x)]^q \rangle \), not \( \rho(x) \). Under a natural set of restrictions (consisting in normalization, fixing the expectation values, positivity) on \( \rho(x) \), the MaxEnt principle, applied to \( S_{R,q}(\rho) \) with \( q < 1 \), generates distribution functions with heavy tail. The set of the PDFs satisfying the restrictions (this set of PDFs will be denoted with \( \mathcal{K} \)) is always convex. According to this interpretation, geometrically we have that, for \( 0 < q < 1 \), the MaxEnt PDF is the PDF \( \in \mathcal{K} \) corresponding to the maximal distance from the origin \( \rho \equiv 0 \), and reversely, for \( q > 1 \), it is the PDF \( \in \mathcal{K} \), closest to the origin \( \rho \equiv 0 \). Despite the corresponding equations for the Lagrange multipliers could be quite complicate, from general arguments on convex analysis we have that the convexity of the functional \( \| \rho \|_q \) with respect to the variable \( \rho \) (or, similarly, the concavity of the functional \( N_q[\rho] \) with respect to \( \rho \)), ensures the uniqueness of the solution of the MaxEnt problem [28]. These properties, as well as the
The functional \( \parallel f \parallel_{p_y,p_z} \) is convex with respect to \( f \). The corresponding new entropy is defined as

\[
S_{p_y,p_z}^{(1)}[\rho] = \frac{p_y}{1 - p_z} \log \parallel \rho \parallel_{p_y,p_z}; \ p_y, p_z > 1
\]  

(6)

Similarly, for \( 0 < q_y < 1, 0 < q_z < 1 \) we have a concave functional \( N_{q_y,q_z}(f) \) that, in analogy of the standard Rényi case, may also be interpreted as the distance in the corresponding functional space and the corresponding entropy:

\[
N_{q_y,q_z}(f) = \int_{\Omega_y} dm_y(y) \left[ \int_{\Omega_z} dm_z(z) |f(y,z)|^{q_z} \right]^{q_y}
\]  

(7)

\[
S_{q_y,q_z}^{(2)}[\rho] = \frac{1}{1 - q_z} \log N_{q_y,q_z}(\rho); 0 < q_y, q_z < 1
\]  

(8)

The distance between PDF’s \( \rho_1, \rho_2 \) is \( d(\rho_1, \rho_2) := \parallel \rho_1 - \rho_2 \parallel_{p_y,p_z} \) for \( p_y, p_z > 1 \), and \( d(\rho_1, \rho_2) := N_{q_y,q_z}(\rho_1 - \rho_2) \) for \( 0 < q_y, q_z < 1 \). Notice that, as shown in Ref. [30], for \( p_y, p_z > 1 \), the function \( d(\rho_1, \rho_2) \) preserves the triangle inequality:

\[
d(\rho_1, \rho_3) \leq d(\rho_1, \rho_2) + d(\rho_2, \rho_3)
\]

\(^1\)A similar method illustrated by Rudin in his textbook [24] may be adopted for showing the validity of the triangle inequality also for \( 0 < q_y, q_z < 1 \).
The norm $\|\rho\|_{p_y,p_z}$ is a convex functional whereas $N_{q_y,q_z}[\rho]$ is concave functional. These properties give the intuitive geometric interpretation of MaxEnt problem subject to linear constraints, in the framework of convex analysis. It follows that GRE is related to the geometry of generalized Lebesgue space $L_{p_y,p_z}$ consisting in the set of functions $f(y,z)$ such that $N_{q_y,q_z}[f]$ or $\|f\|_{p_y,p_z}$ are finite (like the Rényi entropy in the Lebesgue space $L_p$). Convexity properties imply the uniqueness of the solution of MaxEnt problem with restrictions, despite the equations for Lagrange multipliers could be very complex. Instead of working with two different definitions of entropy (for two separate cases $0 < q_y, q_z < 1$ and $q_y, q_z > 1$) we prefer to compact the definitions in only one expression. To this end, we observe that, for fixed $\rho \geq 0$, both functions $\|\rho\|_{p_y,p_z}$ and $N_{q_y,q_z}[\rho]$ are analytic in the variables $p_y, p_z, q_y, q_z$ (at least near the positive real axis) so we can do a unique analytic continuation in their formula outside their initial domains in the following manner

\begin{align}
N_{p_y/p_z,p_y}[\rho] &= \|\rho\|_{p_y,p_z}^p_y \quad (9) \\
S^{(2)}_{p_y/p_z,p_z}[\rho] &= S^{(1)}_{p_y,p_z}[\rho] \quad (10)
\end{align}

For compactness of the formulae, we use for all $q_y, q_z > 0$, and $q_z \neq 1$

\begin{align}
N_{q_y,q_z}[\rho] &= \int_{\Omega_y} dm_y(y) \left[ \int_{\Omega_z} dm_z(z) |f(y,z)|^{q_y} \right]^{q_z} \quad (11) \\
S^{(2)}_{q_y,q_z}[\rho] &= \frac{1}{1 - q_z} \log N_{q_y,q_z}[\rho] \quad (12)
\end{align}

Remark that the Axiom 1, the symmetry, invariance under permutations for Rényi entropy appears in a more general form: the invariance under transformations that acts independently in the spaces $\Omega_y$ and $\Omega_z$ that preserves the measures $m_y$ respectively $m_z$. In the example given below, passive advection-diffusion of a tracer in turbulent field, the variables $y$ and measure $m_y$ are related to the statistical properties of a macroscopic, external, given turbulent velocity field, while $z$ and the measure $m_z$ give a statistical description of the effects of molecular diffusion. In this case is no symmetry transformation that mixes these very different type of variables, rather it is meaningful to relate this asymmetry of the GRE to the hierarchical relation between multiple scales, or causality effects, between spaces $\Omega_y, \Omega_z$.

**Properties of the GRE**

\[2\text{.i.e., for } 0 \leq \alpha \leq 1 \text{ we have for } \rho(y,z) = \alpha \rho_1(y,z) + (1 - \alpha) \rho_2(y,z)\]

\[\|\rho\|_{p_y,p_z} \leq \alpha \|\rho_1\|_{p_y,p_z} + (1 - \alpha) \|\rho_2\|_{p_y,p_z}\]

\[N_{q_y,q_z}[\rho] \geq \alpha N_{q_y,q_z}[\rho] + (1 - \alpha) N_{q_y,q_z}[\rho]\]
Notice that in the limit case $q_y \to 1$ we obtain the standard Rényi entropy

$$
S^{(2)}_{1,q_y} [\rho] = \frac{1}{1-q_y} \log \int_{\Omega_y} dm_y(y) \int_{\Omega_z} dm_z(z) |\rho(y, z)|^{q_z}
$$

and for $p_z \to 1$ the Shannon entropy

$$
\lim_{p_y \to 1} \lim_{p_z \to 1} S^{(2)}_{p_y,p_z} [\rho] = -\int_{\Omega_y} dm_y(y) \int_{\Omega_z} dm_z(z) \rho(y, z) \log \rho(y, z)
$$

We would like to underline that, if we perform the following scaling of the variables $y \to \alpha y$, $z \to \beta z$, and the measures transform like as $dm_y(y) \to \alpha^\mu dm_y(y)$, $dm_z(z) \to \beta^\nu dm_z(z)$, then the previously defined entropies changes by constant. In this context, the variation of the GRE is invariant under scaling, exactly like as in the case of the Shannon entropy. In addition, notice that the GRE is extensive, like the Rényi entropy, because the norm $\|\rho\|_{p_y,p_z}$ and the functional $N_{q_y,q_z} [\rho]$ are multiplicative, in analogy with properties of the norm in the $L_p$ space.

**The MaxEnt principle**

The probability distribution functions may be obtained by the MaxEnt principle. Here, we shall determine the PDF by generalizing the calculations made for the case of the Shannon entropy, subject to the most general scale-invariant restrictions 32. To this end, we maximize the GRE, $S^{(1,2)}_{p_y,p_z} [\rho]$, subject to the constraints

$$
\int_{\Omega_y} dm_y(y) \int_{\Omega_z} dm_z(z) \rho(y, z) f_k(y, z) = c_k; \ 1 \leq k \leq M
$$

$$
\rho(y, z) \geq 0; \ f_0(y, z) = 1; \ c_0 = 1
$$

This means to find the extrema of $N_{q_y,q_z} [\rho]$. From Kuhn-Tucker theorem for maximization 33, we get

$$
\frac{\delta}{\delta \rho(y, z)} \left\{ N_{p_y,p_z} [\rho] + \int_{\Omega_y \times \Omega_z} dm_y(y) dm_z(z) \rho(y, z) \left[ \mu(y, z) - \sum_{k=0}^{N} \lambda_k f_k(y, z) \right] \right\} = 0
$$

$$
\mu(y, z) \geq 0; \ \mu(y, z) \rho(y, z) = 0
$$

where $\mu(y, z)$ and $\lambda_\kappa$ and are the multipliers corresponding to the positivity inequality and the linear restrictions, respectively. Here, we consider only the case $0 < p_y, p_z < 1$. We introduce the notations

$$
g(\lambda, y, z) := \frac{1}{p_y p_z} \sum_{k=0}^{N} \lambda_k f_k(y, z)
$$

$$
a := \frac{1-p_y}{1-p_y p_z} \quad \text{;} \quad b := \frac{p_z}{1-p_z} \quad \text{;} \quad h(\lambda, y) := \int_{\Omega_z} dm_z(z') |g(\lambda, y, z')|^{-b}
$$

$$
\Omega_z
$$
By straightforward calculations, we get
\[ \rho(\lambda, y, z) = g(\lambda, y, z)^{1/(p_z-1)} |h(\lambda, y)|^{-a} \] (16)

Consider the particular case:
\[ \int_{\mathbb{R}^2} dydz \rho(y, z) y^2 = c_1 \quad \int_{\mathbb{R}^2} dydz \rho(y, z) z^2 = c_2 \] (17)

From Eq. (16), we obtain (up to a multiplicative constant)
\[ \rho(\lambda, y, z) = \frac{(1 + (\lambda_1 y)^2)^m}{(1 + (\lambda_1 y)^2 + (\lambda_2 z)^2)^{1-p_z}} \quad m = \frac{1 - 3p_z}{2(1-p_z)} \frac{1 - p_y}{1 - p_y p_z} \] (18)

which corresponds to a PDF with different tail-exponents in the variables \( y, z \).

If to Eqs. (17), we add the supplementary restriction
\[ \int_{\mathbb{R}^2} dydz \rho(y, z) y^2 z^2 = c_3 \] (19)

it is possible to find a combination of the Lagrange multipliers such that, up to a multiplicative constant, we get
\[ g(\lambda, y, z) = (1 + a_1 y^2)(1 + a_2 z^2) \]

So that \( c_{1,2} > a_{1,2} \)
\[ \rho(y, z) = K \frac{1}{(1 + a_1 y^2)^{\kappa_y}} \frac{1}{(1 + a_2 z^2)^{\kappa_z}} \quad \kappa_y = 1/(1 - p_y p_z) ; \quad \kappa_z = 1/(1 - p_z) \] (20)

Hence, by putting \( c_2 = (1 - p_z)/\sigma^2 \), and in the limit case \( p_z \to 1 \) (with the rest of the parameters kept constant), we get
\[ \rho(y, z) = K \frac{1}{(1 + c_1 y^2)^{\kappa_y}} \exp(-z^2/\sigma^2) \] (21)

which is exactly the form of the stationary solution analyzed in our example Eq. (13). Let us now consider the problem of variation of GRE in a dynamical system, whose microscopic statistical features are described by the Fokker Planck equation, when additional random effects, due to turbulence at macroscopic scale, are taken into account by a random variable \( \omega \). In general, the diffusion term describes the effect of the interactions at the atomic scale. The space of the additional random variable \( \omega \), will be denoted simply by \( \Omega \). \( \Omega \) corresponds to the previous \( \Omega_y \), but now it describes the effect of turbulent environment. The previous space \( \Omega_z \) is the usual phase space of the dynamical system, with coordinates \( z = \{z_1, \ldots, z_m\} \). The typical example is the passive advection-diffusion of tracer by a velocity field with turbulent components and molecular diffusion. Consider the case when the evolution is modeled by the
more general advection-diffusion stochastic differential equation (SDE) driven by the white noise \( \zeta_i(t) \)

\[
dz_i/dt = V_i(z, \omega) + \zeta_i(t) \quad ; \quad 1 \leq i \leq m
\]  

where \( V_i(z, \omega) \) and \( D_{i,j}(z, \omega) \) satisfy the conditions

\[
\frac{\partial V_i(z, \omega)}{\partial z_i} = 0 \quad ; \quad \frac{\partial D_{i,j}(z, \omega)}{\partial z_i} = 0
\]  

where the convention of summation on repeated indices is adopted. The corresponding Fokker-Planck equation, for a fixed \( \omega \)

\[
\frac{\partial \rho(t, \omega, z)}{\partial t} = -\frac{\partial}{\partial z_i}(V_i \rho) + \frac{\partial^2}{\partial z_i \partial z_j}(D_{i,j} \rho)
\]  

Notice that the first condition of Eqs (23) is satisfied e.g., by the most general Hamiltonian system, with \( m/2 \) degrees of freedom (Liouville theorem). This general model contains some important particular cases. When \( m = 3 \) this corresponds to the passive tracer transport by advection and molecular diffusion, in a turbulent flow, whose statistical properties are encoded in the probability measure \( dP(\omega) \). It also describes the stochastic magnetic field line dynamics in tokamak [23]. For \( m = 2 \) it may describe the transversal motion (transversal to a constant magnetic field \( B \)) of the charged particles in the drift approximation, and subject to a random electric field \(-\nabla \phi(z, t, \omega)\) and collisions modelled by white noise \( \zeta_i(t) \) [34].

Here \( e_{i,j} \) is the Levi-Civita symbol and \( \sigma \) describe the effects of the collisions. We prove now the following important theorem, which is a sort of \( H \)-theorem describing the tendency of the GRE to increase in time. In general, we define the Liapunov function \( L(t) := \int dP(\omega) \left[ \int_\Gamma d^m x (\rho(t, \omega, x))^p \right]^{\frac{1}{p}} \) where \( \rho(t, \omega, z) \) is the solution of the Fokker-Planck equation Eq. (24) for a fixed \( \omega \). Then we have the following

**Proposition**

Under the conditions Eqs (23), \( \frac{d}{dt} L(t)(p_z - 1) > 0 \) and the corresponding GRE \( S_{q, q_z}^{(2)}[\rho] = \frac{1}{1-q_z} \log L(t) \) is non decreasing in time.

**Proof:** We start from the definition of the GRE and differentiate the expression with respect to time. Then we use the Fokker-Planck equation for the time derivative of \( \rho \), and after integration by part in the \( z \) coordinate, and by taking
into account Eqs (23), we obtain

$$\frac{d}{dt} L(t) = \int_{\Omega} dP(\omega) M(\omega, t) \left( \int_{\Gamma} d\mathbf{z} \left( \rho(t, \omega, \mathbf{z}) \right)^{p_{z} - 2} \frac{\partial \rho}{\partial z_{i}} \frac{\partial \rho}{\partial z_{j}} D_{i,j} \right)$$

$$M(\omega, t) = -p_{y}p_{z}(p_{z} - 1) \left[ \left( \int_{\Gamma} d\mathbf{z}' \left( \rho(t, \omega, \mathbf{z}') \right)^{p_{z}} \right) \right]^{p_{y} - 1}$$

with $$\frac{\partial \rho}{\partial z_{i}}, \frac{\partial \rho}{\partial z_{j}}, D_{i,j} \geq 0$$ from the second law of thermodynamics.

In conclusion, we have introduced a generalization of the Rényi entropy (GRE) that, still preserving the additivity, is able to treat dynamical systems in a highly anisotropic phase space. This is the case of magnetically confined plasmas or of integrable systems subject to perturbations. The anisotropy of the PDF in the phase-space can be retrieved from the MaxEnt principle applied to GRE, subject to natural scale-invariant restrictions. In these situations, the PDF may show different tail-exponents in the variables; this property belongs only to GRE and not to the standard Rényi entropy. We have also seen that the Rényi and Shannon entropies are re-obtained by GRE as limit cases. Even though the extensivity of the RE is preserved in the GRE, the symmetry of the RE (axiom 1 of the RE), is not completely preserved in our generalized version. The symmetry group of Rényi entropy, i.e. the measure preserving transformations of $$\Omega_{y} \times \Omega_{z}$$ splits into direct product of measure preserving transformations of $$\Omega_{y}, \Omega_{z}$$ respectively. This is the ”price” we have to pay to treat dynamical systems in highly anisotropic phase-space. This point deserves attention. Strictly speaking, we can have the possibility to have a PDF with variables having different tails also in the Shannon theory (see the example illustrated at the beginning of the manuscript). However, in this case, it turns out that the PDF may be expressed in terms of a product of two PDFs, each of which depends on only one variable. The GRE may analyze examples ”less trivial”. We mean by this that the corresponding PDFs, obtained by the MaxEnt principle, may show different tail behaviours of the variables without necessarily being expressed in products of PDFs. However, as seen, the GRE includes the possibility to analyze simpler cases where the PDF is expressed as products of PDFs, each of which depends only on one variable, or on subgroups of independent variables [see Eq. (21)].

The functionals that appears in the definition of the GRE may be interpreted as the distance in the corresponding functional space and in a wide range of the parameters $$p_{y}, p_{z}$$ have useful concavity, respectively convexity, properties. Again, when the evolution of the system is modeled by the general advection-diffusion SDE driven by the white noise, we proved the validity of a sort of H-theorem, which results to be satisfied by the GRE when the velocity flows and the diffusion coefficients are divergenceless in the phase-space of the dynamical system [see Eq. (21)].

This work gives several perspectives. Through the thermodynamical field theory (TFT) [35], it is possible to estimate the PDF when the nonlinear contributions cannot be neglected [36]. The next task should be to establish the relation
between the reference, stationary PDF, derived by the MaxEnt principle applied to GRE, subject to scale-invariant restrictions, with the ones found by the TFT.

G. Sonnino is very grateful to M. Malek Mansour, of the Université Libre de Bruxelles, for his scientific suggestions and for his help in the development of this work. G. Steinbrecher acknowledges J. Misquich for useful discussions. We also acknowledge M. Van Schoor of the Royal Military School.

References

[1] X. Garbet et al., Phys. of Plasmas, 12 082511 (2005).
[2] N. Pometescu, Plasma Phys. and Control. Fusion, 41, 1453–1468 (1999).
[3] H. Sugama and W. Horton, Phys. of Plasmas, 5, 2560 (1998).
[4] C.E. Shannon, A Mathematical Theory of Communication, Bell System Technical Journal, 27, 379423 & 623656 (1948).
[5] T. Maszczyk and X. Duch, Comparison of Shannon, Rényi and Tsallis Entropy used in Decision Trees, Lecture Notes in Computer Science, 5097, 643 (2008).
[6] A. Rényi (1961), On measures of information and entropy, Proceedings of the fourth Berkeley Symposium on Mathematics, Statistics and Probability 1960, pp. 547–561.
[7] C. Tsallis, Journal of Statistical Physics, 52, 479–487 (1988).
[8] F. Rieke, D. Warland, R Ruyter van Steveninck and W Bialek (1997), Spikes: Exploring the Neural Code, The MIT press. ISBN 978-0262681087.
[9] J.P. Huelsenbeck, F. Ronquist, R. Nielsen and J. P. Bollback, Bayesian inference of phylogeny and its impact on evolutionary biology, Science, 29, 2310 (2001).
[10] K. P. Burnham and D.R. Anderson (2002), Model Selection and Multimodel Inference: A Practical Information-Theoretic Approach, (Springer Science) Second Edition, New York ISBN 978-0-387-95364-9.
[11] E.T. Jaynes, Phys. Rev., 106, 620 (1957).
[12] C.H. Bennett, M. Li and B. Ma, Scientific American 288, 76 (2003).
[13] F. Franchini, F., A.R. Its and V.E. Korepin, Journal of Physics A: Mathematical and Theoretical, 4, 25302 (2008). A.R. Its, V.E. Korepin (2010), Generalized entropy of the Heisenberg spin chain, Theoretical and Mathematical Physics (Springer), 164, 1136.
[14] H. Risken (1996), *The Fokker-Planck Equation. Methods of Solution and Applications*, Berlin, Heidelberg, New York, Paris, Tokio, (Springer, 3rd printing).

[15] G. S. Manning, *J. Chem. Phys.*, 51, 924 (1969).

[16] A. E. Cohen, *Phys. Rev. Lett.*, 94, 118102 (2005).

[17] A. J. Bray, *Phys. Rev. E*, 62, 103 (2000).

[18] S. Marksteiner, K. Ellinger and P. Zoller, *Phys. Rev. A*, 53, 3409 (1996).

[19] A. Ryabov, M. Dierl, P. Chvosta, M. Einax and P. Maass, *Work distribution in time-dependent logarithmic-harmonic potential: exact results and asymptotic analysis*, [http://arxiv.org/pdf/1302.0976v1.pdf](http://arxiv.org/pdf/1302.0976v1.pdf), 5 February (2013).

[20] A. Dechant, E. Lutz, D.A. Kessler and E. Barkai, *Super-aging correlation function and ergodicity breaking for Brownian motion in logarithmic potentials*, [http://arxiv.org/pdf/1202.6250.pdf](http://arxiv.org/pdf/1202.6250.pdf), 18 May (2012).

[21] D. A. Kessler and E. Barkai, *Phys. Rev. Lett.*, 105, 120602 (2010).

[22] V. Bickler and C. Beckinger, *Nat. Phys.*, 8, 103 (2012).

[23] M. Vlad, F. Spineanu, J. H. Misguich, and R. Balescu, *Phys. Rev. E*, 58, 7359 (1998). E. Vanden Eijnden and R. Balescu, *Phys. Plasmas*, 3, 815 (1996). M. Negrea, I. Petrisor and R. Balescu, *Phys. Rev. E*, 70, 046409 (2004).

[24] W. Rudin (1987), *Real and Complex Analysis*, McGraw Hill Inc. 3rd Ed. page 74.

[25] M. Reed and B. Simon (1981), *Functional Analysis (Methods of Modern Mathematical Physics)*. Vol. 1, Academic Press.

[26] H. Luschgy and G. Pagès, *Moment estimates for Lévy processes*. Electronic Communication in Probability, 13, 422-434 (2008).

[27] G. Steinbrecher, X. Garbet and B. Weyssow (2010), *Large time behavior in random multiplicative processes*, [arXiv:1007.0952v1](http://arxiv.org/abs/1007.0952v1).

[28] S. Boyd and L. Vandenberghe (2004), *Convex Optimization*, Cambridge University Press, Cambridge.

[29] G. Sonnino and G. Steinbrecher, *Study of dynamical systems in highly anisotropic phase space by generalized Rényi entropies defined through N-iterated norms*, to be submitted to *J. of Stat. Phys.* (2013).

[30] O.V. Besov, IV.P. l’in and S.M. Nikol’skii (1975), *Integral representations of functions and embedding theorems*, Ed. Nauka, pp. 9-40, Moscow (in Russian).
[31] G. Comte (2006), *Entropy and Quantitative Transversality*, Encyclopedia of Mathematical Physics, Pages 237–242, Academic Press.

[32] G. Sonnino, G. Steinbrecher, A. Cardinali, A. Sonnino and M. Tlidi, *Phys. Rev. E*, **87**, 014104 (2013).

[33] S. Stephen and L. Vandenberghe (2004), *Convex Optimization*, Cambridge University Press. Cambridge, p. 244.

[34] M. Vlad, F. Spineanu, J. H. Misguich, and R. Balescu, *Phys.Rev. E*, **63**, 066304 (2001).

[35] G. Sonnino, *Phys. Rev. E*, **79**, 051126 (2009).

[36] G. Sonnino, *Eur. Phys. J. D*, **62**, 81 (2011).