SUSY partners of the complex oscillator and Painlevé IV equation

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Dedicated to Professor Bogdan Mielnik for his 50 years of scientific career.

Abstract. Supersymmetry transformations will be used to obtain new exactly solvable potentials from the complex oscillator. The corresponding Hamiltonians are ruled by polynomial Heisenberg algebras. By applying a mechanism to reduce to second the order of these algebras, the connection with the Painlevé IV equation is achieved, supplying us with an algorithm to generate solutions to such equation.

1. Introduction
Supersymmetric quantum mechanics (SUSY QM) is a simple method to generate exactly solvable potentials from a given initial one such that the spectra of the original and built Hamiltonians differ by a finite number of levels [1–7]. This procedure has been applied mainly to hermitian Hamiltonians, in order to generate either real or complex new potentials [8–13], with an associated spectrum which in the last case could contain a finite number of additional complex energy levels. The method has been used also for non-hermitian Hamiltonians with the so-called PT-symmetry [14–16]. Indeed, in the context of optics it is possible to find families of complex potentials with entirely real spectra with a missing single level, the one associated to an arbitrary initial mode [17]. However, up to our knowledge there are not many works employing Hamiltonians with purely complex eigenvalues.

In this work we will address this subject by taking as initial system the complex oscillator [18–21], whose potential has the harmonic oscillator form but with a complex frequency \( \omega \). We shall implement interesting first-order SUSY transformations which employ excited state eigenfunctions as transformation functions, leading to solvable potentials whose spectra are completely determined.

Let us note that, in general, the set of SUSY partners of the harmonic oscillator Hamiltonian realize the even-order polynomial Heisenberg algebras (PHA) [22–27]. In addition, some subsets are ruled simultaneously by second-order PHA, which are linked with the Painlevé IV (PIV) equation; hence, a simple method for generating solutions of such equation can be designed [28–31]. Since the complex oscillator is mapped in a simple way into the harmonic oscillator, thus complex solutions of the PIV equation will be found by applying supersymmetry transformations to the complex oscillator.

This paper is organized as follows. In Section 2 we will sketch the SUSY QM. Then, we will discuss the PHA in Section 3, paying special attention to those of second-order, which are linked with the PIV equation. In Section 4 we will generate the SUSY partners of the complex oscillator, while in Section 5 we will analyze the corresponding solutions to the PIV equation. Section 6 contains our conclusions and some discussion about the perspectives of our work.
2. Supersymmetric quantum mechanics

Let us consider a chain of intertwined Hamiltonians \( \hat{H}_j \) such that \([4,5,32–34]\):

\[
\hat{H}_j \hat{A}_j^+ = \hat{A}_j^+ \hat{H}_{j-1}, \quad \hat{H}_{j-1} \hat{A}_j^- = \hat{A}_j^- \hat{H}_j, \quad j = 1, \ldots, k, \tag{1a}
\]

\[
\hat{A}_j^+ = \frac{1}{\sqrt{2}} \left[ \mp \frac{d}{dx} + \beta_j (x, \varepsilon_j) \right], \tag{1b}
\]

\[
\hat{H}_j = -\frac{1}{2} \frac{d^2}{dx^2} + V_j (x), \tag{1c}
\]

where \( \beta_j (x, \varepsilon_j) \) and \( V_j (x) \) are complex functions, i.e., \( \hat{A}_j^+ \) and \( \hat{A}_j^- \) are not adjoint to each other. Thus, the following equations must be satisfied:

\[
\beta'_j (x, \varepsilon_j) + \beta^2_j (x, \varepsilon_j) = 2 [V_{j-1} (x) - \varepsilon_j], \tag{2}
\]

\[
V_j (x) = V_{j-1} (x) - \beta'_j (x, \varepsilon_j). \tag{3}
\]

The solution \( \beta_j (x, \varepsilon_j) \) of the \( j \)-th Riccati equation (2) is expressed in terms of two solutions of the \((j-1)\)-th one:

\[
\beta_j (x, \varepsilon_j) = -\beta_{j-1} (x, \varepsilon_{j-1}) - \frac{2 (\varepsilon_j - \varepsilon_{j-1})}{\beta_{j-1} (x, \varepsilon_{j-1}) - \beta_{j-1} (x, \varepsilon_{j-1})}, \quad j = 1, \ldots, k, \tag{4}
\]

which implies that the method depends either of \( k \) solutions \( \beta_1 (x, \varepsilon_j) \) of the initial Riccati equation, or of the associated Schrödinger solutions \( u_j (x) \) such that \( \beta_1 (x, \varepsilon_j) = u'_j (x) / u_j (x) \), i.e.,

\[
\frac{1}{2} u''_j (x) + V_0 (x) u_j (x) = \varepsilon_j u_j (x). \tag{5}
\]

The spectrum of the Hamiltonian \( \hat{H}_k \) consists of the initial one plus \( k \) new energy levels \([4,34–36]\):

\[
\text{Sp}(\hat{H}_k) = \text{Sp}(\hat{H}_0) \cup \{ \varepsilon_1, \ldots, \varepsilon_k \}. \tag{6}
\]

Moreover, the Hamiltonians \( \hat{H}_0 \) and \( \hat{H}_k \) are intertwined as follows:

\[
\hat{H}_k \hat{B}_k^+ = \hat{B}_k^+ \hat{H}_0, \quad \hat{H}_0 \hat{B}_k^- = \hat{B}_k^- \hat{H}_k, \tag{7}
\]

with

\[
\hat{B}_k^+ = \hat{A}_k^+ \cdots \hat{A}_1^+, \quad \hat{B}_k^- = \hat{A}_1^- \cdots \hat{A}_k^-, \tag{8}
\]

being \( k \)-th order differential intertwining operators. The initial and final potentials, \( V_0 (x) \) and \( V_k (x) \) respectively, are related by

\[
V_k (x) = V_0 (x) - [\ln W (u_1, \ldots, u_k)]'' \tag{9}
\]

where \( W (u_1, \ldots, u_k) \) is the Wronskian of the \( k \) seed solutions \( \{u_1, \ldots, u_k\} \). Let us note that this technique has been employed successfully to generate new solvable potentials \( V_k (x) \) departing from several interesting initial Hamiltonians (mainly Hermitian \([5,6,34]\)).

3. Polynomial Heisenberg algebras

A \((m-1)\)-th order PHA is a deformation of the Heisenberg-Weyl algebra with three generators \( \{\hat{H}, \hat{L}_m^+, \hat{L}_m^-\} \) which obey:

\[
[\hat{H}, \hat{L}_m^+] = \pm \hat{L}_m^+, \tag{10a}
\]

\[
[\hat{L}_m^-, \hat{L}_m^+] = \hat{N}_m (\hat{H} + \hat{1}) - \hat{N}_m (\hat{H}) = P_m (\hat{H}), \tag{10b}
\]

\[
\hat{N}_m (\hat{H}) = \prod_{j=1}^{m} (\hat{H} - \varepsilon_j \hat{1}), \tag{10c}
\]
where $\hat{H}$ has the form given in Eq. (1c), $\hat{L}_m^\pm$ are $m$-th order differential ladder operators, $\hat{N}_m(\hat{H}) = \hat{L}_m^+ \hat{L}_m^-$ is a generalization of the number operator, $P_{m-1}(\hat{H})$ is a polynomial in $\hat{H}$ of degree $m - 1$. Note that the zeros $\mathcal{E}_j$ of $\hat{N}_m(\hat{H})$ correspond to the associated extremal state energies.

The PHA generated by $\{\hat{H}, \hat{L}_m^+, \hat{L}_m^-\}$ gives information about $\text{Sp}(\hat{H})$ as follows. First the extremal states of the system, such that

$$\hat{L}_m^- \psi = 0,$$

have to be chosen. From Eqs. (10c) and (11) we obtain

$$\hat{L}_m^+ \hat{L}_m^- \psi = \prod_{j=1}^m (\hat{H} - \mathcal{E}_j \hat{1}) \psi = 0,$$

which suggests that, in the diagonalizable case, we can choose the $\psi_{\mathcal{E}_j}$ satisfying the stationary Schrödinger equation with complex eigenvalues $\mathcal{E}_j$:

$$\hat{H} \psi_{\mathcal{E}_j} = \mathcal{E}_j \psi_{\mathcal{E}_j}.$$  

(13)

Departing now from these extremal states, by acting iteratively with $\hat{L}_m^+$ onto them, we can build $m$ mathematical energy ladders with the same spacing $\Delta \mathcal{E} = 1$ starting from each $\mathcal{E}_j$.

A particularly interesting case appears for $m = 3$, when the ladder operators are of third order. Next, it is analyzed the corresponding second-order PHA.

### 3.1. Second order PHA

The connection between PIV equation and PHA appears for $m = 3$ [27]. Hence, let $\hat{L}_3^+$ be a third-order differential ladder operator factorized as the product of a first and a second-order differential one $\hat{L}_3^+ = \hat{L}_1^+ \hat{L}_2^+$, where

$$\hat{L}_1^+ = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + f(x) \right],$$

(14a)

$$\hat{L}_2^+ = \frac{1}{2} \left[ \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right].$$

(14b)

The complex functions $f, g, h$ and the potential $V$ are unknown. For simplicity, we suppose the existence of an auxiliary Hamiltonian $\hat{H}_a$ which is intertwined with $\hat{H}$ in the way

$$\hat{H} \hat{L}_1^+ = \hat{L}_1^+ (\hat{H}_a + 1), \quad \hat{H}_a \hat{L}_2^+ = \hat{L}_2^+ \hat{H}.$$  

(15)

These relationships lead to a coupled system of equations for $f, g, h, V$ and its derivatives. Its solution supplies an expression for the potential:

$$V = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \mathcal{E}_3 - \frac{1}{2},$$

(16)

where $g$ satisfies the PIV equation with parameters $a, b \in \mathbb{C}$ [37], namely

$$g'' = \frac{g'^2}{2g} + \frac{3}{2} g^3 + 4xg^2 + 2 (x^2 - a) g + \frac{b}{g},$$

(17a)

$$a := \mathcal{E}_1 + \mathcal{E}_2 - 2 \mathcal{E}_3 - 1, \quad b := -2 (\mathcal{E}_1 - \mathcal{E}_2)^2.$$  

(17b)
We need to know the function $g$ for a pair of parameters $a$, $b$; then, the extremal energies $E_j$ and the potential $V$ in (16) can be found. Moreover, the three extremal states are expressed in terms of $g$ as:

$$\psi_{E_1} \propto \left(\frac{g'}{2g} - \frac{g}{2} - \frac{b}{g} \sqrt{-\frac{b}{2} - x}\right) e^{\int \left(\frac{g'}{2g} + \frac{g}{2} + \frac{i}{\omega} \sqrt{-\frac{b}{2} - x}\right) dx}, \quad (18a)$$

$$\psi_{E_2} \propto \left(\frac{g'}{2g} - \frac{g}{2} + \frac{b}{g} \sqrt{-\frac{b}{2} - x}\right) e^{\int \left(\frac{g'}{2g} + \frac{g}{2} + \frac{i}{\omega} \sqrt{-\frac{b}{2} - x}\right) dx}, \quad (18b)$$

$$\psi_{E_3} \propto e^{-\frac{1}{4}x^2} g \sqrt{g} \, dx. \quad (18c)$$

Conversely, if we are able to identify Hamiltonians having third-order differential ladder operators, it is possible to design a simple mechanism to obtain solutions of the PIV equation (see [27] and Section 5 in this paper).

4. SUSY partners of the complex oscillator

In order to implement the SUSY technique, we need to find the general solution $u(x)$ of the stationary Schrödinger equation for an arbitrary complex factorization energy $\varepsilon$. The potential associated to the complex oscillator reads:

$$V_0(x) = \frac{1}{2} \omega^2 x^2, \quad \omega = e^{i\theta}, \quad -\frac{\pi}{2} \leq \theta < \frac{3\pi}{2}, \quad (19)$$

$\omega$ being a dimensionless complex frequency and $\theta$ its phase. A direct calculation leads to [38]

$$u(x) = e^{-\frac{1}{4}\omega x^2} \left[1 F_1 \left(\frac{1}{4} - \frac{\varepsilon}{2\omega} - \frac{1}{2}; \omega x^2\right) + 2\nu x^2 \frac{\Gamma\left(\frac{3}{4} - \frac{\varepsilon}{2\omega}\right)}{\Gamma\left(\frac{1}{4} - \frac{\varepsilon}{2\omega}\right)} 1 F_1 \left(\frac{3}{4} - \frac{\varepsilon}{2\omega}; 3\omega x^2\right)\right], \quad (20)$$

where we will take $|\nu| < 1$ and $1 F_1$ is the confluent hypergeometric function (Kummer). In general, $\lim_{|x| \to \infty} |u(x)| = 0$ is not satisfied. In order to ensure the square-integrability of $u$, any of the two series $1 F_1$ in Eq. (20) must reduce to a polynomial. Thus, three different cases arise, depending on the value of $\theta$: (i) for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ the energy levels of the complex oscillator are given by

$$E_n(\theta) = \left(n + \frac{1}{2}\right) e^{i\theta}, \quad n = 0, 1, 2, \ldots, \quad (21a)$$

and the corresponding eigenfunctions read

$$\phi_n(x) = C_n H_n(\sqrt{\omega}x) e^{-\frac{1}{4}\omega x^2}, \quad \sqrt{\omega} = e^{i\frac{\theta}{2}}, \quad (21b)$$

where $C_n$ are normalization factors and $H_n(\sqrt{\omega}x)$ are the Hermite polynomials. (ii) For $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ the eigenfunctions and eigenvalues are

$$\phi_n(x) = D_n H_n(\sqrt{-\omega}x) e^{\frac{1}{4}\omega x^2}, \quad \sqrt{-\omega} = e^{i\frac{\theta - \pi}{2}}, \quad (22a)$$

$$E_n(\theta) = \left(n + \frac{1}{2}\right) e^{i(\theta - \pi)}, \quad n = 0, 1, 2, \ldots \quad (22b)$$

(iii) For $\theta = \pm \frac{\pi}{2}$ there do not exist square-integrable eigenfunctions of $\hat{H}$, since $V_0(x)$ corresponds to the repulsive oscillator [39].

The eigenvalues of the complex oscillator lie either in the first or in the fourth quadrant of the complex energy plane, since $V_0(x)$ is invariant under the change $\omega \to -\omega$, thus the eigenfunctions and eigenvalues for both $\omega$ and $-\omega$ are the same. Moreover, since

$$E_n(-\theta) = [E_n(\theta)]^*, \quad (23)$$
we will work from now in with the domain \( \theta \in [0, \frac{\pi}{2}) \).

Let us define now the annihilation and creation operators \( \hat{a}_\omega^\pm \) in analogy to the harmonic oscillator:

\[
\hat{a}_\omega^\pm := \frac{1}{\sqrt{2}} \left( \mp \frac{d}{dx} + \omega x \right).
\]

(24)

Note that they define a fixed direction on the complex energy plane (see Figure 1). In addition, \( \hat{a}_\omega^+ \neq (\hat{a}_\omega^-)^\dagger \), since \( \omega \in \mathbb{C} \). The algebra of these operators is similar to the harmonic oscillator one [40]:

\[
\begin{align*}
[\hat{a}_\omega^-, \hat{a}_\omega^+] &= \omega \hat{1}, \\
\{\hat{a}_\omega^-, \hat{a}_\omega^+\} &= 2\hat{H}_0, \\
[\hat{H}_0, \hat{a}_\omega^\pm] &= \pm \omega \hat{a}_\omega^\pm.
\end{align*}
\]

(25a)

(25b)

(25c)

By using this algebra, it is possible to find the square-integrable eigenfunctions of \( \hat{H}_0 \) and their corresponding eigenvalues in a very elegant way. They turn out to be given by:

\[
\phi_n(x) = \frac{1}{\sqrt{n!}} (\hat{a}_\omega^+)^n \phi_0(x) = C_n H_n(\omega^2 x) e^{-\frac{1}{2} \omega x^2},
\]

(26)

where \( \phi_0 \propto e^{-\frac{1}{2} \omega x^2} \) is the ground state. Note that when \( \varepsilon = E_n \), the general solution in Eq. (20) can be reduced to the expression in Eq. (26).

A diagram of the complex energy plane, the eigenvalues \( E_n \) of the complex oscillator for a fixed \( \theta \) and the action of the annihilation and creation operators \( \hat{a}_\omega^\pm \) can be seen in Figure 1.

**Figure 1.** Diagram of the complex energy plane. Some eigenvalues for a fixed \( \theta \) are shown at their right positions.

Let us apply now a non-singular \( k \)-th order supersymmetry transformation which creates \( k \) new levels, additional to \( E_n \), for the new Hamiltonian \( \hat{H}_k = -\frac{1}{2} \frac{d^2}{dx^2} + V_k \), i.e.,

\[
\text{Sp}(\hat{H}_k) = \text{Sp}(\hat{H}_0) \cup \{\varepsilon_1, \ldots, \varepsilon_k\}.
\]

In order to do that, we take \( k \) solutions \( \{u_j; j = 1, \ldots, k\} \) of the initial Schrödinger equation. Thus, the expression for the new potential \( V_k \) is given by Eq. (9) with \( V_0(x) = \frac{1}{2} \omega^2 x^2 \).

The algebra for systems described by \( \hat{H}_k \) is generated by its natural ladder operators

\[
\hat{L}^\pm_{2k+1} = \hat{B}^+_k \hat{a}_\omega^\pm \hat{B}^-_k,
\]

(27)
where $\hat{B}_k^\pm$ are the $k$-th order differential intertwining operators of Eq. (8). The standard commutation relations of the polynomial Heisenberg algebra are satisfied:

$$[\hat{H}_k, \hat{L}_{2k+1}^\pm] = \pm \omega \hat{L}_{2k+1}^\pm,$$  

(28)

which, up to the factor $\omega$, are the same as in Eq. (10a). Thus, it is natural to redefine the Hamiltonian as $\hat{H}_k = \hat{H}_k / \omega$, so that Eq. (28) is transformed into

$$[\hat{H}_k, \hat{L}_{2k+1}^\pm] = \pm \hat{L}_{2k+1}^\pm. $$  

(29)

The Hamiltonians $\hat{H}_k$ associated with this algebra read:

$$\hat{H}_k = -\frac{1}{2\omega} \frac{d^2}{dx^2} + \frac{1}{\omega} V_k(x) = -\frac{1}{2\omega} \frac{d^2}{dx^2} + \frac{1}{2} \omega x^2 - \frac{1}{\omega} \frac{d^2}{dx^2} \ln W(u_1, \cdots, u_k), $$  

(30)

which take the standard Schrödinger form through the substitution $y = \omega^{1/2} x$.

In conclusion, the $(2k+1)$-th order operators $\hat{L}_{2k+1}^\pm$ and the Hamiltonian $\hat{H}_k$ (in fact a family) generate a $2k$-th order PHA. As we saw in Section 3, systems which are connected with the PIV equation must have third-order ladder operators and satisfy a second-order PHA. Thus, we need to identify the subfamily of $\hat{H}_k$ (if any) having as well third-order ladder operators. In fact, there is a theorem which enables us to reduce to second the initial order $(2k)$ of the PHA, producing then solutions of the PIV equation [28]. Some results of this formalism, for first and second-order SUSY, are next illustrated.

4.1. First-order transformation

For a transformation function as given in Eq. (20) with $\epsilon \in \mathbb{C}$, the first-order SUSY partner potential of the complex oscillator turns out to be:

$$V_1(x) = \frac{1}{2} \omega^2 x^2 - \frac{u''}{u} + \left( \frac{u'}{u} \right)^2. $$  

(31)

Figure 2 shows some of these potentials for several values of the factorization energy $\epsilon$. The probability densities associated to some eigenfunctions for fixed values of $\epsilon$ and $\nu$, are plotted in Figure 3.
Figure 3. Normalized probability densities for: (a) the eigenstates of $\hat{H}_0$ with $n = 0$ (blue), $n = 1$ (red) and $n = 2$ (green); (b) their corresponding densities for $\hat{H}_1$. We have taken $\varepsilon = 0.01 + 0.2i$, $\nu = 0.7 + 0.3i$ and $\theta = \frac{\pi}{5}$.

4.2. Higher-order transformation

As it was said previously, a $k$-th order SUSY transformation requires, in general, $k$ independent seed solutions of the Schrödinger equation. In order to induce the reduction theorem, however, we have to consider just connected seed solutions satisfying:

$$u_j = (\hat{a}_\omega)^{-1}u_1, \quad \varepsilon_j = \varepsilon_1 - (j - 1)\omega, \quad j = 1, \ldots, k,$$

(32a)

(32b)

i.e., the only free function is a nodeless solution $u_1$ of the stationary Schrödinger equation for the complex factorization energy $\varepsilon_1$, with form given in Eq. (20). For instance, the second-order SUSY partner potential of the complex oscillator (for $k = 2$) is given by:

$$V_2(x) = \frac{1}{2}\omega^2 x^2 - \frac{W''}{W} + \left(\frac{W'}{W}\right)^2,$$

(33)

where

$$W := W(u_1, \hat{a}_\omega u_1) \propto (\omega^2 x^2 + \omega - 2\varepsilon_1) u_1^2 - (u_1')^2.$$

(34)

In Figure 4 we have plotted the real and imaginary parts of $V_0(x)$ and $V_2(x)$ for different values of $\varepsilon_1$, with the other parameters $\nu_1$ and $\theta$ remaining fixed.

5. Solutions of the Painlevé IV equation

Systems having third-order ladder operators, ruled then by second-order PHA, are connected with the PIV equation (see Section 3). Thus, the first-order SUSY partners of the complex oscillator are linked directly with the PIV equation. Moreover, from Eq. (18c) we can obtain an expression to generate solutions to such equation in terms of the extremal states $\psi_{\varepsilon_j}$ by making carefully the change from the harmonic to the complex oscillator:

$$g_1^{(j)}(x) = -\omega^{\frac{1}{2}} x - \omega^{-\frac{1}{2}} \left[\ln \psi_{\varepsilon_j}(x)\right]', \quad j = 1, 2, 3,$$

(35)

where the subscripting indicates the order of the SUSY transformation. This expression already includes the fact that, by making cyclic permutations of the indices of the extremal states, we can generate three different solutions to the PIV equation. For a fixed value of $\varepsilon_1$ the solutions (35) of the PIV equation constitute a 3-parametric family.

In Figure 5 we show some complex solutions to the PIV equation for several values of the factorization energy $\varepsilon_1$ and a fixed $\nu_1$, with $|\nu_1| < 1$. 

Figure 4. Real (a) and imaginary parts (b) of the potential $V_2$ of Eq. (33) for $\varepsilon_1 = 0.01 + 0.1i$ (blue), $\varepsilon_1 = 0.4 + 0.1i$ (red) and $\varepsilon_1 = 0.95 + 0.1i$ (green). The discontinuous line represents the complex oscillator potential $V_0 = \frac{1}{2} \omega^2 x^2$. We have taken $\nu_1 = 0.7 - 0.3i$ and $\theta = \frac{\pi}{5}$.

Figure 5. Real (a) and imaginary (b) parts of $g_1^{(2)}$ for $\varepsilon_1 = 0.04 + 0.8i$ (blue), $\varepsilon_1 = 0.9 + 0.8i$ (red) and $\varepsilon_1 = 3 + 0.8i$ (green). We have taken $\theta = \frac{\pi}{5}$ and $\nu_1 = 0.9 - 0.1i$.

Note that the solution $g_1^{(j)}$ is non-singular if the extremal state $\psi_{\varepsilon_j}$ does not have zeros. In addition, these solutions have a null asymptotic behavior when $|x| \to \infty$.

In a more general context, for $k$-th order SUSY an explicit expression for a solution of the PIV equation appears after the reduction theorem is applied, which is given by:

$$g_k(x) = -\omega \frac{1}{2} x - \omega^{-\frac{1}{2}} \ln \left[ \frac{W(u_1, \cdots, u_{k-1})}{W(u_1, \cdots, u_k)} \right], \quad k \geq 2,$$

where the $u_j$, $j = 1, \cdots, k$ satisfy Eqs. (32). In Figure 6 we illustrate the typical behavior of $g_2$ for several values of $\varepsilon_1$ and $\nu_1 = 0.8 + 0.5i$.

6. Conclusions
In this paper we have introduced a complex generalization of the standard factorization method to analyze the SUSY partners of the non-Hermitian complex oscillator Hamiltonian. The corresponding potentials were generated through SUSY transformations for complex factorization energies. An algorithm to generate complex solutions of the Painlevé IV equation, from the extremal state eigenfunctions of these Hamiltonians, was as well explored.

Let us note that the physical interpretation of complex energies remains open, although possible applications related with absorbive (dissipative) systems have been noticed recently [13].
Figure 6. Real (a) and imaginary (b) parts of $g_2$ for $\varepsilon_1 = 0.03 - 0.4i$ (blue), $\varepsilon_1 = 1 - 0.4i$ (red) and $\varepsilon_1 = 3 - 0.4i$ (green). We have taken $\theta = \frac{\pi}{5}$ and $\nu_1 = 0.4 - 0.3i$.

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