The Wilson renormalization group for low $x$ physics: towards the high density regime

Jamal Jalilian-Marian$^1$, Alex Kovner$^{1,2}$*, Andrei Leonidov$^3$, Heribert Weigert$^4$

$^1$ Physics Department, University of Minnesota, Minneapolis, MN 55455, USA
$^2$ Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, UK
$^3$ Theoretical Physics Department, P.N. Lebedev Physics Institute, Leninsky pr. 53 Moscow, Russia
$^4$ University of Cambridge, Cavendish Laboratory, HEP, Madingley Road, CB3 0HE UK

Abstract

We continue the study of the effective action for low $x$ physics based on a Wilson renormalization group approach. We express the full nonlinear renormalization group equation in terms of the average value and the average fluctuation of extra color charge density generated by integrating out gluons with intermediate values of $x$. This form clearly exhibits the nature of the phenomena driving the evolution and should serve as the basis of the analysis of saturation effects at high gluon density at small $x$.

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Understanding of hadronic systems at large energies is one of the most exciting open problems in QCD today (see, e.g., a recent review [1]). The recent wave of interest was sparked by experimental data on deep inelastic scattering [2] showing a steep rise in the cross section at small values of Bjorken $x$. Theoretically it is believed that this rise is mainly triggered by fast growth of gluon density in a proton and indeed most fits to the HERA data use gluonic distributions that grow as a power at small $x$ [3]. It argued a long time ago [4] that eventually the rate of growth of the gluon density should slow down and eventually saturate, thus curing a potential conflict with unitarity of the underlying scattering. Physically this should happen as the gluonic system becomes dense and gluons start overlapping in space causing self interactions, which are normally neglected in a linear QCD evolution, to become important. The most exciting problem of low $x$ physics therefore is the question of how to deal with finite density partonic systems.

A formulation which is naturally directed towards this type of problems was suggested by McLerran and Venugopalan [5] in the context of large nuclei. Their approach was considerably modified and generalized later [6, 7] and evolved into an effective action approach to the low $x$ hadronic structure. In this paper we continue to study this effective action with the help of Wilson renormalization group. The general Wilsonian renormalization group procedure for study of the low $x$ physics was discussed first in [6], and then in more detail in [7]. It was also shown in the latter paper that in the linear regime, i.e. for small partonic densities, this renormalization group equation reduces to the celebrated BFKL equation for the evolution of unintegrated gluon density [8]. This is a very important test for the consistency of the whole approach, however our goal is to be able to deal with the nonlinear regime where the color charge densities are not small and therefore have to be treated nonperturbatively.

In this note we take one further step in this direction. The renormalization group procedure involves functional integration over two sets of variables: the fluctuation modes
of the gluon field with longitudinal momenta above the cutoff and the surface color charge density $\rho$ at fixed value of $\rho'$ which then becomes the charge density in the effective theory that describes only low longitudinal momentum glue modes. The goal of this note is to show how to perform one part of this program, namely the integration over the color charge density distribution. This will result in a gratifyingly compact and simple representation of the full nonlinear RG equation in terms of the average value and the average fluctuation of extra color charge density generated by integrating out gluons with intermediate values of $x$. The evolution towards smaller $x$ is understood therefore in a simple way in terms of these two physical quantities. The simple form of the result is ideally suited to discuss the validity and the limitations of the BFKL equation from a new vantage point, as the low density limit of the full nonlinear RG equations.

First, let us recall the framework as set up in [7]. The starting point is the following action given in the Light Cone gauge $A^+ = 0$

$$S = i \int d^2 x_t F[\rho^a(x_t)]$$

$$- \int d^4x \frac{1}{4}G^2 + \frac{i}{N_c} \int d^2 x_t dx^- \delta(x^-)\rho^a(x_t) tr W \int_\infty^{-\infty} [A^-](x^-, x_t)$$

Here $G^{\mu\nu}$ is the gluon field strength tensor

$$G^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + gf_{abc} A^\mu_b A^\nu_c$$

$T_a$ are the $SU(N)$ color matrices in the adjoint representation and $W$ is the path ordered exponential along the $x^+$ direction in the adjoint representation of the $SU(N_c)$ group

$$W_{-\infty,\infty}[A^-](x^-, x_t) = P \exp \left[ -ig \int dx^+ A^-_a(x^-, x_t) T_a \right]$$

The average of any gluonic operator $O(A)$ in the hadron is calculated as

$$< O > = \frac{\int D[\rho^a, A^\mu_a] O(A) \exp\{iS[\rho, A]\}}{\int D[\rho^a, A^\mu_a] \exp\{iS[\rho, A]\}}$$
The exponential of the imaginary part of the action

$$\text{Im } S = \int d^2 x_t F[\rho^a]$$

(5)

can be thought of as a kind of “free energy” for the ensemble of the color charge density \(\rho(x_t)\). The “Boltzmann factor”

$$\exp \left( -\int d^2 x_t F[\rho^a] \right)$$

(6)

appearing in Eq.(5) controls the statistical weight of a particular configuration of the two dimensional color charge density \(\rho^a(x_t)\) inside the hadron.

In the classical approximation to the path integral in Eq.(4) one finds the classical solution for the equations of motion at fixed \(\rho\), and then averages over the charge density distribution with the “Boltzmann weight” Eq.(6). The classical solution for any fixed \(\rho(x_t)\) has the structure\(\text{\footnote{To be more precise, there is an infinite number of solutions to the classical equations of motion at fixed } \rho. \text{ This is a consequence of the residual gauge invariance of the action. In the following we will be working in the gauge } \partial_i A_i(x^- \to -\infty) = 0. \text{ This is a complete gauge fixing and it therefore picks a unique solution of the equations of motion.}}\] \(\text{\footnote{\text{\text{\footnote{To be more precise, there is an infinite number of solutions to the classical equations of motion at fixed } \rho. \text{ This is a consequence of the residual gauge invariance of the action. In the following we will be working in the gauge } \partial_i A_i(x^- \to -\infty) = 0. \text{ This is a complete gauge fixing and it therefore picks a unique solution of the equations of motion.}}}}}}\)

\[
A_{cl}^- = 0
\]

\[
A_{cl}^i \equiv b^i = \theta(x^-)\alpha^i(x_t)
\]

(7)

where \(\alpha^i\) is related to \(\rho\) through

\[
\alpha^i(x_t) = \frac{i}{g} U(x_t) \partial_i U^\dagger(x_t)
\]

\[
\partial_i \alpha^i = -g\rho
\]

and \(U(x_t)\) is a unitary matrix.

The quantum corrections to this classical approximation are large at small longitudinal momenta \(\text{\footnote{To be more precise, there is an infinite number of solutions to the classical equations of motion at fixed } \rho. \text{ This is a consequence of the residual gauge invariance of the action. In the following we will be working in the gauge } \partial_i A_i(x^- \to -\infty) = 0. \text{ This is a complete gauge fixing and it therefore picks a unique solution of the equations of motion.}}\]

To resum these large corrections we follow the renormalization group procedure as developed in \(\text{\footnote{To be more precise, there is an infinite number of solutions to the classical equations of motion at fixed } \rho. \text{ This is a consequence of the residual gauge invariance of the action. In the following we will be working in the gauge } \partial_i A_i(x^- \to -\infty) = 0. \text{ This is a complete gauge fixing and it therefore picks a unique solution of the equations of motion.}}\]
Let us introduce the following decomposition of the gauge field:

\[ A^a_\mu(x) = b^a_\mu(x) + \delta A^a_\mu(x) + a^a_\mu(x) \]  \hspace{1cm} (9)

where \( b^a_\mu(x) \) is the solution of the classical equations of motion, \( \delta A^a_\mu(x) \) is the fluctuation field containing longitudinal momentum modes \( q^+ \) such that \( P^+_n < q^+ < P^+_{n-1} \) while \( a_\mu \) is a soft field with momenta \( k^+ < P^+_n \), with respect to which the effective action is computed. The initial path integral is formulated with the longitudinal momentum cutoff on the field \( \delta A, q^+ < P^+_{n-1} \). The effective action for \( a_\mu \) is calculated by integrating over the fluctuations \( \delta A \). This integration is performed with the assumption that the fluctuations are small as compared to the classical fields \( b^a_\mu \). More quantitatively, at each step of this RG procedure the scale \( P^+_n \) is chosen such that \( \ln \frac{P^+_n}{P^+_{n-1}} > 1 \), but \( \alpha_s \ln \frac{P^+_n}{P^+_{n-1}} \ll 1 \).

Expanding the action around the classical solution \( b^a_\mu(x) \) and keeping terms of the first and second order in \( \delta A \) we get

\[ S = -\frac{1}{4} G(a)^2 - \frac{1}{2} \delta A_\mu[D^{-1}(\rho)]^{\mu\nu} \delta A_\nu + ga^-\rho' + O((a^-)^2) + iF[\rho] \]  \hspace{1cm} (10)

where

\[ \rho' = \rho + \delta \rho_1 + \delta \rho_2 \]  \hspace{1cm} (11)

with

\[ \delta \rho_1^a(x, t, x^+) = -2f^{abc}a_i^b \delta A_c^i(x^- = 0) \]  \hspace{1cm} (12)

\[ -\frac{g}{2} f^{abc} \rho^b(x_t) \int dy^+ \left[ \theta(y^+ - x^+) - \theta(x^+ - y^+) \right] \delta A^{-c}(y^+, x_t, x^- = 0) \]

and

\[ \delta \rho_2^a(x) = -f^{abc}[\partial^+ \delta A^b_\mu(x)] \delta A^c_\mu(x) \]

\[ -\frac{g^2}{N_c} \rho^b(x_t) \int dy^+ \delta A^{-c}(y^+, x_t, x^- = 0) \int dz^+ \delta A^{-d}(z^+, x_t, x^- = 0) \]

\[ \times \left[ \theta(z^+ - y^+) \theta(y^+ - x^+) \text{tr} T^a T^c T^d T^b + \theta(x^+ - z^+) \theta(z^+ - y^+) \text{tr} T^a T^b T^c T^d \right. \]

\[ \left. + \theta(z^+ - x^+) \theta(x^+ - y^+) \text{tr} T^a T^d T^b T^c \right] \]  \hspace{1cm} (13)
The first term in both $\delta \rho_1^a$ and $\delta \rho_2^a$ is coming from expansion of $G^2$ in the action while the rest of the terms proportional to $\rho$ are from the expansion of the Wilson line term. The three terms correspond to different time ordering of the fields. Since the longitudinal momentum of $a^-$ is much lower than of $\delta A$, we have only kept the eikonal coupling (the coupling to $a^-$ component of the soft field), which gives the leading contribution in this kinematics. The inverse propagator $[D^{-1}]^{\mu \nu}$ is given by

\[
[D^{-1}]^{ij}_{ab}(x, y) = \left[ D^2(b) \delta^{ij} + D^i(b) D^j(b) \right]_{ab} \delta^{(4)}(x, y)
\] (14)

\[
[D^{-1}]^{i+}_{ab}(x, y) = -[\partial^+ x] D^i_{ab}(b) \delta^{(4)}(x, y) + 2 f_{abc} \alpha^c_\gamma(x_t) \delta(x^-) \delta(y^-) \delta^{(2)}(x_t, y_t) \delta(x^+, y^+)]
\]

\[
[D^{-1}]^{++}_{ab}(x, y) = (\partial^+)^2 \delta_{ab} \delta^{(4)}(x, y) + f_{abc} \rho^c(x_t) \delta(x^-) \delta(y^-) \theta(x^+ - y^+) \delta^{(2)}(x_t, y_t)
\]

The procedure now is the following:

1. Integrate over $\delta A^\mu$ at fixed $\rho$ and fixed $\delta \rho$.

2. Integrate over $\rho$ at fixed $\rho' = \rho + \delta \rho$.

This generates the new effective action which formally can be written as

\[
\exp \left( i S[\rho', a^\mu] \right) = \exp \left( -F'[\rho'] - \frac{i}{4} G^2(a) + i g \alpha \rho \right)
\] (15)

with

\[
\exp \left( -F'[\rho'] \right) = \int D[\rho, \delta A] \delta(\rho' - \rho - \delta \rho(\delta A)) \exp \left( -F[\rho] - \frac{i}{2} \delta A D^{-1}[\rho] \delta A \right)
\] (16)

Of course, to leading order in $\ln 1/x$ only terms linear in $\alpha_s \ln 1/x$ should be kept in $F'$. Defining

\[
\alpha_s \ln \frac{1}{x} \Delta[\rho] \equiv F'[\rho] - F[\rho]
\] (17)

gives the RG equation

\[
\frac{d}{dy} F[\rho] = \alpha \Delta[\rho]
\] (18)
In order for Eq.(18) to be a bona fide renormalization group equation, we should be able to find the functional $\Delta$ for arbitrary Boltzmann weight $F$. That requires being able to integrate over the charge density $\rho(x_t)$ at arbitrary $F[\rho]$. The aim of this paper is to show how this is done. Let us start by considering the functional integral over $\delta A$ at fixed $\rho$ and $\rho'$. Although this integration is complicated [10], the structure of the result is simple and can be understood by simple counting of powers of the coupling constant. From the explicit expression of $\delta \rho$ in terms of $\delta A$, Eqs.(12), (13), it is readily seen that

$$< \delta \rho >_{\delta A} = O(\alpha_s)$$
$$< \delta \rho \delta \rho >_{\delta A} = O(\alpha_s)$$

while all other (connected) correlation functions of $\delta \rho$ are higher order in $\alpha_s$. Since we are working to the lowest order in $\alpha_s$ we can neglect all these other terms. It therefore follows that to lowest order, after integrating over $\delta A$ we are left with the weight function for $\delta \rho$, which generates only connected one- and two-point functions. Such weight is obviously a Gaussian. Introducing the following notations

$$\eta := \alpha_s \ln(1/x)$$
$$< \delta \rho^a(x_t) >_{\delta A} =: \eta \sigma^a(x_t)$$
$$< \delta \rho^a(x_t, x^+) \delta \rho^b(y_t, x^+) >_{\delta A} =: \eta \chi^{ab}(x_t, y_t)$$

we can therefore write the result of the $\delta A_{\mu}$ integration in the form

$$\int D[\rho, \rho'][\text{Det}(\chi)]^{-1/2} \exp \left( -F[\rho] \right) \times \exp \left( -\frac{1}{2\eta} [\delta_x - \rho_x - \eta \sigma_x] [\chi_x^{-1}] [\delta_x - \rho_y - \eta \sigma_y] \right)$$
$$=: \int D[\rho, \rho'] \exp \{-U[\rho, \rho']\}$$

In the above equation we adopted condensed notations: the indices $x$ stand for the set of indices and coordinates $\{x_t, a\}$, and repeated indices are understood to be summed (integrated) over. We will use these notations in the rest of the paper.
We note here that this result can be derived formally by introducing the variable \( \rho' \) with the help of Lagrange multiplier

\[
\delta (\rho' - \rho - \delta \rho [\delta A]) = \int D[\lambda] \ e^{i\lambda (\rho' - \rho - \delta \rho [\delta A])}
\]

and subsequently integrating out \( \lambda \) in perturbation theory to order \( \alpha_s \).

The calculation of \( \sigma \) and \( \chi \) is technically nontrivial \cite{10} and is beyond the scope of this note. Here we want to show, that assuming those quantities are known functionals of \( \rho \), the integral over \( \rho \) in Eq.(21) can be performed explicitly.

Consider again Eq.(21). We will assume that \( F[\rho] \) is a smooth functional of \( \rho \) and that the scale of its variation is independent of the strong coupling constant \( \alpha_s \). The last factor in Eq.(21) however is a very steep function of \( \rho \) which is sharply peaked around \( \rho' + O(\alpha_s) \). This is due to the factor \( 1/\alpha_s \) in the Gaussian weight for \( \delta \rho \). It is therefore clear that the integral can be calculated straightforwardly in the steepest descent approximation. We should also remember that we need to retain the terms up to first order in the coupling constant. Simple counting of the powers of \( \alpha_s \) shows that to reach this accuracy one needs to take into account up to the third order contributions in the steepest descent integration. In other words, the expansion of the exponential factor in Eq.(21) is needed up to the fourth order in fluctuations around the stationary point.

The steepest descent equation to first order in \( \alpha_s \) reads

\[
\rho'_x - \rho_x - \eta \sigma_x = \eta \chi_{xx} \left[ \frac{\delta F}{\delta \rho_u} + \frac{1}{2} \text{tr} (\chi^{-1} \frac{\delta \chi}{\delta \rho_u}) \right]
\]

(23)

Substituting this into Eq.(21) we find that the logarithm of the integrand in Eq.(21) becomes

\[
U = F + \frac{1}{2} \text{tr} \ln(\chi) + \eta \left[ \frac{\delta F}{\delta \rho_u} + \frac{1}{2} \text{tr} (\chi^{-1} \frac{\delta \chi}{\delta \rho_u}) \right] \chi_{uv} \left[ \frac{\delta F}{\delta \rho_v} + \frac{1}{2} \text{tr} (\chi^{-1} \frac{\delta \chi}{\delta \rho_v}) \right]
\]

(24)
In the above expression all the functionals are taken at $\rho^0$ - the solution of the steepest descent equation Eq. (23).

Next we have to evaluate the correction due to Gaussian fluctuations of $\rho$ around $\rho^0$.

After some algebra we find

$$\frac{\delta^2 U}{\delta \rho_x \delta \rho_y} = \frac{1}{\eta} \chi_{xy}^{-1} + \frac{\delta^2}{\delta \rho_x \delta \rho_y} \left[ F + \frac{1}{2} \text{tr} \ln(\chi) \right]$$

(25)

$$+ \chi^{-1}_{xy} \frac{\delta \sigma_u}{\delta \rho_y} + \delta \frac{\sigma_u}{\delta \rho_x} \chi^{-1}_{uy}$$

$$+ \left[ \chi^{-1}_{xy} \frac{\delta \chi_{uv}}{\delta \rho_y} + \chi^{-1}_{yu} \frac{\delta \chi_{uv}}{\delta \rho_x} \right] \left[ \frac{\delta F}{\delta \rho_v} + \frac{1}{2} \text{tr} \left( \frac{\delta \chi}{\delta \rho_v} \right) \right]$$

Here the argument of all the functionals is again $\rho^0$.

As noted earlier to order $\alpha_s$ we have to retain two more corrections to the steepest descent result. This is because the third as well as all higher derivatives of $U$ in Eq. (21) is of the order $1/\alpha_s$ while the fluctuation is of order $\alpha_s^{1/2}$. Schematically, we can therefore write all order $\alpha_s$ contributions as

$$\int D[\rho] \exp \left( -U[\rho] \right) \quad (26)$$

$$= \exp \left( -U[\rho^0] \right) \int D[\rho] \exp \left( -\frac{1}{2} \frac{U''(\rho - \rho^0)^2}{4!} - \frac{1}{3} \frac{U'''(\rho - \rho^0)^3}{3!} - \frac{1}{4} \frac{U''''(\rho - \rho^0)^4}{4!} \right)$$

$$= \exp \left( -U[\rho^0] \right) \int D[\rho] \exp \left( -\frac{1}{2} \frac{U''(\rho - \rho^0)^2}{4!} \right)$$

$$\times \left[ 1 - \frac{1}{4!} \frac{U'''(\rho - \rho^0)^4}{3!} + \frac{1}{2} \left( \frac{1}{3!} \frac{U''''(\rho - \rho^0)^3}{3!} \right)^2 \right]$$

$$= \exp \left( -U[\rho^0] - \frac{1}{2} \text{tr} \ln \eta U'' - \frac{1}{4!} \left( 3 \text{ - point vertex } + \frac{1}{2(3!)^2} \left( 9 \text{ - point vertex } + 6 \text{ - point vertex } \right) \right) \right) \quad (27)$$

Here we have used the obvious simplified notations and graphical representation in terms of the “propagator” $[U'']^{-1}$ and three- and four-point vertices $U'''$ and $U''''$. We have also re-exponentiated the result of the integration which is valid to leading order in $\alpha_s$.

Our task is simplified, however by the fact that we need to retain only the leading contribution to this term. That for $U'''$ and $U''''$ we only need to keep the pieces of order
$1/\alpha_s$, and for $< (\rho - \rho_0)^n >$ only pieces of order $\alpha_s^n/2$. Some straightforward algebra yields

$$
F_4 := \frac{3}{4!} \delta^4 U \frac{\delta^2 U}{\delta \rho_0 \delta \rho_u \delta \rho_v \delta \rho_z} \left[ \frac{\delta^2 U}{\delta \rho \delta \rho} \right]^{-1}_{wz} \left[ \frac{\delta^2 U}{\delta \rho \delta \rho} \right]^{-1}_{wz} \left[ \frac{\delta^2 U}{\delta \rho \delta \rho} \right]^{-1}_{yz} \left[ \frac{\delta^2 U}{\delta \rho \delta \rho} \right]^{-1}_{xz}
$$

(28)

and

$$
F_3 := -\frac{1}{2(3!)} \left( \frac{3}{9} \right) + 6
$$

(29)

Putting Eqs. (25), (26), (28) and (29) together and eliminating the stationary value of $\rho$ in favor of $\rho'$ through Eq. (23), we find

$$
F' = U + \frac{1}{2} \left( \frac{\alpha_s}{1/x} \right) + F_3 + F_4
$$

(30)

Equation (31) gives the Wilson renormalization group equation.

$$
\frac{d}{d \ln(1/x)} F = \frac{\alpha_s}{2} \left[ \frac{\delta^2 F}{\delta \rho_0 \delta \rho_v} - \frac{\delta^2 \chi_{uv}}{\delta \rho_0 \delta \rho_v} \right] + 2 \frac{\delta F}{\delta \sigma_u}
$$

(31)
This equation is extremely simple when written for the weight function $Z \equiv \exp\{-F\}$

\[
\frac{d}{d\ln(1/x)} Z = \alpha_s \left\{ \frac{1}{2} \frac{\delta^2}{\delta \rho_u \delta \rho_v} [ Z \chi_{uv} ] - \frac{\delta}{\delta \rho_u} [ Z \sigma_u ] \right\}
\]  

Equations (31) and (32) are the central result of this paper. They provide the closed form of the renormalization group equation in terms of the functionals $\sigma[\rho]$ and $\chi[\rho]$.

We want to make now several remarks. First, the two terms in Eqs.(32) have a natural interpretation as the real and the virtual contributions to the evolution. The average fluctuation $\chi$ is generated by the real ladder-like diagrams of the gluon fluctuation $\delta A$ while the average color density comes from the virtual one loop diagrams [10].

Second, Eq.(32) can be written directly as evolution equation for the correlators of the charge density. Multiplying Eq.(32) by $\rho_1 \cdots \rho_n$ and integrating over $\rho$ yields

\[
\frac{d}{d\ln(1/x)} < \rho_1 \cdots \rho_n > = \alpha_s \left[ \sum_{0<m<k<n+1} < \rho_1 \cdots \rho_{m-1} \rho_{m+1} \cdots \rho_{k-1} \rho_{k+1} \cdots \rho_n \chi_{m,k} > - \sum_{0<l<n+1} < \rho_1 \cdots \rho_{l-1} \rho_{l+1} \cdots \rho_n \sigma_{l} > \right]
\] (33)

In particular, taking $n = 2$ we obtain the evolution equation for the two point function

\[
\frac{d}{d\ln(1/x)} < \rho_x \rho_y > = \alpha_s \left\{ < \chi_{xy} + \rho_x \sigma_y + \rho_y \sigma_x > \right\}
\] (34)

In general this is not a closed equation since $\sigma$ and $\chi$ depend on $\rho$ in a complicated nonlinear way. However in the weak field limit, $\sigma$ and $\chi$ are linear and quadratic functionals of $\rho$ respectively. In this case Eq.(34) becomes a closed equation for the two point correlation function of $\rho$. The explicit expressions for $\sigma$ and $\chi$ in this limit have been calculated in [7] where it is also shown that Eq.(34) is precisely the famous BFKL equation.

Since the correlator of the colour charge density is directly related to the unintegrated gluon density in a hadron (nucleus) [7], Eq. (34) can be straightforwardly rewritten as
a nonlinear evolution equation for the gluon density. It would be very interesting to compare this equation to the generalized evolution equation derived by Levin and Laenen [11]. In the second order in the gluon density (and neglecting the virtual contributions in the double logarithmic approximation) Eq. (34) should reduce to the famous GLR equation [4] with the coefficient calculated by Mueller and Qiu [12].

Another remark we want to make is that a general Gaussian form of the weight function

$$ F = \int dx_t dy_t \rho(x_t)\mu^{-1}(x_t, y_t)\rho(y_t) $$

(35)
is not an eigenfunction of equation (31). This form of $F$ was argued in [3] to be a good approximation for valence partons in a large enough nucleus, due to incoherence of color charges of partons coming from different nucleons. In general it should also be a good approximation in perturbative regime where $F$ can be expanded in powers of $\rho$. We see however, that at low enough $x$ non quadratic terms are generated through the evolution Eq.(31). In particular, even if we take the weak field expressions for $\sigma$ and $\chi$, the term quadratic in $F$ on the r.h.s. of Eq.(31) generates a correction quartic in $\rho$. Obviously, at weak coupling and small $\rho$ this term can be neglected (or rather approximated by a quadratic term in a mean field like manner) so that a Gaussian $Z$ is recovered in the perturbative regime. This suggests that a study of the RG flow Eq.(31) should indeed use Eq.(35) as an initial condition for the evolution starting at not too small values of $x$. The natural choice for the initial value of $\mu(x_t, y_t)$ is

$$ \mu(x_t, y_t) = S(b_t) \int \frac{d^2 k_t}{(2\pi)^2} e^{ik_t(x_t-y_t)} \phi(k_t, x_0) $$

(36)

Here $b_t = \frac{x_t+y_t}{2}$ is the impact parameter, $S(b_t)$ is a nucleon shape function, $x_0$ is the value of $x$ from which we start evolving $F$ according to Eq.31 and $\phi(k_t, x)$ is the unintegrated gluon density.

Finally, we note that $\sigma$ and $\chi$ are indeed calculable in a closed form. The result of this calculation will be presented elsewhere [10].
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