Optimal eavesdropping on noisy states in quantum key distribution

Z. Shadman, H. Kampermann, T. Meyer, and D. Bruß
Institute für Theoretische Physik III, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany
(Dated: February 2, 2022)

We study eavesdropping in quantum key distribution with the six state protocol, when the signal states are mixed with white noise. This situation may arise either when Alice deliberately adds noise to the signal states before they leave her lab, or in a realistic scenario where Eve cannot replace the noisy quantum channel by a noiseless one. We find Eve’s optimal mutual information with Alice, for individual attacks, as a function of the qubit error rate. Our result is that added quantum noise can make quantum key distribution more robust against eavesdropping.

PACS numbers: 03.67.-a,03.67.Dd,42.50.EX

I. INTRODUCTION

In quantum cryptography – or, more precisely, quantum key distribution – a secret key is established between two trusted parties (Alice and Bob), by employing certain quantum states as signals, and suitable measurements [1]. In a typical implementation, polarized photons are sent from Alice to Bob along an optical fiber [2]. An eavesdropper (Eve) is usually assumed to have every possible power that is compatible with the laws of quantum mechanics. This power is not necessarily realistic, with respect to existing tools and technology. In particular, Eve is supposed to be able to replace the quantum channel (i.e. the optical fiber in our example given above), which in reality always introduces some noise, by a noiseless fiber. Even though this assumption is compatible with quantum mechanics, it is too restrictive from a realist’s point of view.

In this paper, we will assume that Eve does not possess a noiseless fiber. Thus, the quantum states will experience noise in transit from Alice to Bob. We will consider white noise, where the noise parameter $p$ describes the best existing fiber that Eve can possibly get hold of. Our results also hold for the scenario where Alice deliberately adds noise to the state before sending it out of her laboratory, and the quantum channel is noiseless.

Note that we are considering additional noise at the quantum level. A related question has been studied in [6]: there, it has been shown that if one of the parties (Alice or Bob) adds some noise to their classical measurement data before error correction, then the BB84 [1], B92 [1] and six state protocol [3,4] are more robust with respect to quantum noise, i.e. the secret key rate is non-zero up to higher values of the quantum bit error rate. Adding noise at the quantum level, i.e. before the measurement, will also lead to additional noise at the classical level. In this sense our scenario should lead, at least qualitatively, to a similar result as shown in [6].

The aim of our paper is to present an intuitive understanding for the counter-intuitive fact (shown in [6]) that noise may help the trusted parties to improve the performance of a quantum cryptographic protocol. We will derive the optimal mutual information that Eve can obtain, when using individual attacks on noisy quantum signals, and compare it to the mutual information achievable by eavesdropping on pure states. One expects that Alice and Bob, but also Eve, will lose some information, due to the additional noise. It is not evident, however, how the relation between the two mutual information curves (Alice and Bob versus Alice and Eve) changes, when the noise increases.

In this paper we will discuss the six state protocol with additional (equal) noise on all signal states. The six signal states of the protocol with pure states are \{\ket{0_x}, \ket{1_x}, \ket{0_y}, \ket{1_y}, \ket{0_z}, \ket{1_z}\}, where \ket{0_\alpha} and \ket{1_\alpha} with $\alpha = x,y,z$ denote the eigenstates of Pauli operator $\sigma_\alpha$. Here, the states \ket{0_\alpha} symbolize the classical bit value 0, and \ket{1_\alpha} represents the classical bit value 1.

II. EAVESDROPPING ON MIXED STATES

In the six state protocol with mixed states Alice sends instead of pure states one of the following six mixed states (either deliberately, or due to unavoidable noise in the transmission channel):

$$\rho' = (1-p)|i\rangle\langle i| + \frac{p}{2} \mathbb{I}, \quad i \in \{0_\alpha,1_\alpha\}$$

with $\alpha = x,y,z$. The parameter $p$ describes the amount of noise, with $0 \leq p \leq 1$. Here, we assume the noise to be equal in all bases, i.e. we study the depolarising channel. (In a more general model, polarization dependent noise could be treated in an analogous way, by letting $p_{\alpha}$ depend on $\alpha = x,y,z$.)

For the eavesdropping strategy we assume that Eve is restricted to interfering separately with each of the single systems sent by Alice (i.e. individual attack). In this class of attacks she attaches to each qubit an independent probe which is initially in the state $|X\rangle$ and applies some unitary transformation. The dimension of the probes and the interaction are in principle arbitrary, but in [1] it has been shown that the most general unitary eavesdropping

*Electronic address: shadman@thphy.uni-duesseldorf.de*
attack on a $d$-dimensional signal state needs only $d^2$ linearly independent ancilla states of Eve. (This argument also holds when the signal states are mixed, as the unitary transformation of the basis states already uniquely defines the transformation of any superposition, due to linearity, and thus also of a mixture of projectors onto superpositions of basis states.) Thus, it is enough for Eve to use two qubits for her probe states.

The most general unitary transformation $U$ that Eve can design is defined via its action on the basis states (where we use for the computational basis the notation $|0\rangle = |0_z\rangle$ and $|1\rangle = |1_z\rangle$),

\[
U|0\rangle|X\rangle = \sqrt{1 - D}|0\rangle|A\rangle + \sqrt{D}|1\rangle|B\rangle, \quad (2a)
\]

\[
U|1\rangle|X\rangle = \sqrt{1 - D}|1\rangle|C\rangle + \sqrt{D}|0\rangle|D\rangle, \quad (2b)
\]

where $D$ is called the disturbance, with $0 \leq D \leq \frac{1}{4}$. Eve’s normalized probes after interaction are $|A\rangle$, $|B\rangle$, $|C\rangle$, and $|D\rangle$. They have to be chosen such that $U$ is a unitary operator.

The quantum bit error rate in the $z$-basis is denoted as $Q_z$, and given as the fraction of original signals $|0\rangle(|1\rangle)$ sent by Alice, but interpreted as $|1\rangle(|0\rangle)$ by Bob, namely

\[
Q_z = \frac{1}{2}\langle 0 | \rho^B_0 | 0 \rangle + \frac{1}{2}\langle 1 | \rho^B_1 | 1 \rangle, \quad (3)
\]

where $\rho^B_0$ and $\rho^B_1$ are the states that Bob receives when Alice sends $|0\rangle$ and $|1\rangle$, respectively. We define $Q_{x,y}$ in an analogous way for the $x$ and $y$-basis.

As we assume the noise to be uniform, a quantum bit error rate that is basis-dependent indicates the presence of an eavesdropper. We therefore suppose that Eve uses a strategy that produces the same quantum bit error rate in the three different bases, i.e.

\[
Q = Q_z = Q_x = Q_y. \quad (4)
\]

It can be easily verified that the relationship between the quantum bit error rate $Q$ and $D$ is

\[
Q = D(1 - p) + \frac{p}{2}. \quad (5)
\]

Additionally, we restrict Eve to attack in such a way that the two terms of $Q$ are identical, i.e.

\[
\langle 0 | \rho^B_0 | 0 \rangle = \langle 1 | \rho^B_1 | 1 \rangle, \quad (6)
\]

which can be tested by Alice and Bob, by comparing a part of their bit string for the $z$-basis. Again, an analogous requirement has to hold in the other two bases, too.

Equations (3) and (6), together with the unitarity of $U$ lead to the four following conditions for Eve’s states:

\[
\langle B | D \rangle = 0, \quad (7a)
\]

\[
\text{Re}\langle A | C \rangle = \frac{2(1 - 2D)}{2 - p - 2D}, \quad (7b)
\]

\[
\langle A | B \rangle + \langle D | C \rangle = 0, \quad (7c)
\]

\[
\langle A | D \rangle + \langle B | C \rangle = 0. \quad (7d)
\]

Note that the quantum bit error rate $Q$ only depends on the scalar product between $|A\rangle$ and $|C\rangle$. Eve’s two-qubit states can be written as an expansion of four basis vectors with complex coefficients. As explained above, Eve’s states only need to be four-dimensional. We have the freedom to choose $|B\rangle = |0\rangle$. Equation (7a) allows to assign $|D\rangle$ one of the other three basis vectors, e.g., $|D\rangle = |1\rangle$. The general expansion for the normalized vectors $|A\rangle$ and $|C\rangle$ is

\[
|A\rangle = \alpha_A|00\rangle + \beta_A|10\rangle + \gamma_A|01\rangle + \delta_A|11\rangle, \quad (8a)
\]

with $|\alpha_A|^2 + |\beta_A|^2 + |\gamma_A|^2 + |\delta_A|^2 = 1, \quad (8b)
\]

and

\[
|C\rangle = \alpha_C|00\rangle + \beta_C|10\rangle + \gamma_C|01\rangle + \delta_C|11\rangle, \quad (9a)
\]

with $|\alpha_C|^2 + |\beta_C|^2 + |\gamma_C|^2 + |\delta_C|^2 = 1. \quad (9b)
\]

We have to determine the free parameters $\alpha_A, ..., \delta_A$ and $\alpha_C, ..., \delta_C$ such that Eve’s transformation is optimized. As a figure of merit we will calculate the mutual information between Eve and Alice, and optimize Eve’s transformation such that she acquires the maximal mutual information.

### III. RESULTS

The mutual information measures the information that two parties share. Here the parties have variables $X,Y$ that can take values $x,y$, respectively. The mutual information is defined as

\[
I_{XY} = \sum_{x,y} p(x,y) \log p(y|x) - \sum_y p(y) \log p(y), \quad (10)
\]

where $p(x,y)$ is the joint probability to find $x$ and $y$, and $p(y|x)$ is the conditional probability of $y$, given $x$. All logarithms are taken to base 2.

Eve wishes to retrieve the maximal information, i.e. she has to choose the optimal coefficients $\alpha_{A,C}, \beta_{A,C}, \gamma_{A,C}, \delta_{A,C}$, for fixed $p$ and $Q$. The full problem can be simplified with the following argument: As mentioned above, for a fixed noise parameter $p$ the quantum bit error rate $Q$ only depends on the real part of the overlap between $|A\rangle$ and $|C\rangle$, see equation (7b). Therefore, Eve is free to choose those states on which $Q$ does not depend in such a way that her information is maximal, as long as the constraints given in equations (7a) and (7b) are fulfilled. Thus, Eve will choose her ancilla states orthogonal (whenever possible), i.e. $\langle A | B \rangle = \langle B | C \rangle = \langle A | D \rangle = \langle D | C \rangle = 0$, which corresponds to $\alpha_A = \delta_A = \alpha_C = \delta_C = 0$. One realizes by looking at equation (2a) and (2b) that in this way Eve’s probe states are made as distinguishable as possible (for given $\langle A | C \rangle \neq 0$). The best measurement for the two remaining non-orthogonal states $|A\rangle$ and $|C\rangle$ is rank one and orthogonal.

\[\]
With the above definition (10) for the mutual information besides the explicit expansions of Eve’s states as in [20], [30], the information that Eve acquires is

\[
I_{AE} = 1 + \frac{1}{2} \left( 1 - \frac{Q}{1 - p} \right),
\]

\[
\cdot \left\{ \tau \left[ (1 - p) |\beta_A|^2 + \frac{p}{2} |\gamma_C|^2, \frac{p}{2} |\beta_A|^2 + (1 - \frac{p}{2}) |\beta_C|^2 \right] 
+ \tau \left[ (1 - \frac{p}{2}) |\gamma_A|^2 + \frac{p}{2} |\gamma_C|^2, \frac{p}{2} |\gamma_A|^2 + (1 - \frac{p}{2}) |\gamma_C|^2 \right] \right\}
+ (Q - \frac{p}{1 - p}) \tau \left[ 1 - \frac{p}{2}, \frac{p}{2} \right],
\]

(11)

where we used the definition

\[
\tau[x, y] = x \log x + y \log y - (x + y) \log(x + y).
\]

We used the method of Lagrange multipliers for the optimization problem. Some details and the equations to be solved are given in the Appendix.

As sketched in the Appendix, we arrive at \(|\beta_C|^2 = 1 - |\beta_A|^2|\). Thus, we find that the maximal mutual information between Alice & Eve is

\[
I_{AE} = 1 + \left( 1 - \frac{Q}{1 - p} \right).
\]

\[
\cdot \left\{ \left( (1 - p) |\beta_A|^2 + \frac{p}{2} \right) \log \left( (1 - p) |\beta_A|^2 + \frac{p}{2} \right) 
+ (1 - \frac{p}{2} - (1 - p) |\beta_A|^2) \log \left( 1 - \frac{p}{2} - (1 - p) |\beta_A|^2 \right) \right\}
+ (Q - \frac{p}{1 - p}) \left\{ \frac{p}{2} \log \frac{p}{2} + (1 - \frac{p}{2}) \log (1 - \frac{p}{2}) \right\},
\]

(13)

where

\[
|\beta_A|^2 = \frac{1}{2} \left( 1 + \frac{1}{1 - 2Q} \sqrt{(Q - \frac{p}{2})(2 - 3Q - \frac{p}{2})} \right).\]

Obtaining the mutual information between Alice and Bob is an easy task. From the definition of the mutual information we find

\[
I_{AB} = 1 + Q \log Q + (1 - Q) \log(1 - Q).
\]

(15)

One easily confirms that in the absence of white noise, i.e. for \(p = 0\), the mutual information functions reduce to the noiseless case described in [3].

Let us turn our attention to the crossing point between the two mutual information curves. A secret key can be established if \(I_{AB} - I_{AE} \geq 0\) [10], i.e. for values of \(Q\) which are smaller than the value for the crossing point. In Fig. 1 we observed that for one example of non-zero \(p\) the crossing point moves towards a larger \(Q\). We therefore studied the \(Q\)-value for the crossing-point as a function of \(p\). The result is shown in Fig. 2. Remember the relation between qubit error rate \(Q\), disturbance \(D\) and noise \(p\), given in equation (15). One might expect that the crossing point of the two information curves obeys this linear dependence, i.e. \(Q_{cross} = D_{cross}(1 - p) + p/2\); this is the dashed line in Fig. 2. However, the true value for the crossing point lies above that straight line. Thus, as \(p\) increases, the crossing point moves to a higher quantum bit error rate than expected from the additional noise. So, by adding noise to the quantum data the secure parameter range of \(Q\) will increase. In other words, the six state protocol with mixed states is more robust against eavesdropping than the six state protocol with pure states.

In summary, we have studied individual eavesdropping attacks on the six state protocol with added white noise. We found the optimal mutual information between Eve and Alice, for a fixed amount of noise, as a function of the qubit error rate. We showed that the crossing point between the mutual information curves for Alice/Bob and Alice/Eve (Csiszár-Körner threshold) moves to a qubit error rate that is higher than expected from the noise alone. Thus, we gave a simple explanation for the counter-intuitive fact that added noise may improve the robustness of a quantum key distribution protocol. Our results are of importance in a realistic scenario, where the eavesdropper does not have a noiseless fiber available. Alternatively, Alice can deliberately add noise to the signal states. In this paper, we have focused on a particular QKD protocol and a particular type of noise. As an outlook, it would be interesting to study other protocols, and/or other types of noise.

Acknowledgements: We acknowledge stimulating discussions with Matthias Kleinmann and Chiara Macchiavello. This work was partially supported by the EU Integrated Projects SECOQC and SCALA.
Eve for six pure and six mixed state cases, as a function of qubit error rate (Q) when noise parameter is $\beta$.

FIG. 1. Mutual information between Alice & Bob and Alice & Eve for six pure and six mixed state cases, as a function of the noise parameter $p = 0.05$.

Here we briefly explain our method for obtaining the optimized $I^{AE}$. First, we redefine the complex coefficients $\beta_{A,C}, \gamma_{A,C}$ in polar coordinates:

$$
\beta_A = r_{\beta_A},
$$

$$
\beta_C = r_{\beta_C},
$$

$$
\gamma_A = r_{\gamma_A} \exp(i\Phi_{\gamma_A}),
$$

$$
\gamma_C = r_{\gamma_C} \exp(i\Phi_{\gamma_C}).
$$

Note that because of the unphysical global phase we have the freedom to choose $\beta_A$ and $\beta_C$ real. Using the Lagrange multiplier method we then write the Lagrangian $L$ as

$$
L = I^{AE} + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3,
$$

where $g_1, g_2$ and $g_3$ are the constraints (7b), (8b) and (9b). The derivative $dL = 0$ yields the following system of equations:

$$
g_1 = r_{\beta_A} r_{\beta_C} + r_{\gamma_A} r_{\gamma_C} \cos(\Phi_{\gamma_A} - \Phi_{\gamma_C}) - \frac{2(1 - 2Q)}{2 - p - 2Q} = 0,
$$

$$
g_2 = r_{\beta_A}^2 + r_{\gamma_A}^2 - 1 = 0,
$$

$$
g_3 = r_{\beta_C}^2 + r_{\gamma_C}^2 - 1 = 0,
$$

$$
r_{\gamma_A} r_{\gamma_C} \sin(\Phi_{\gamma_A} - \Phi_{\gamma_C}) = 0,
$$

$$
r_{\beta_A} \{\frac{1 - \frac{Q}{2}}{1 - p}\} \{(1 - p) \log M_2 + \frac{p}{2} \log M_6
$$

$$
- \log(M_2 + M_6) + 2 \lambda_2\} + \lambda_1 r_{\beta_A} = 0,
$$

$$
r_{\beta_C} \{\frac{1 - \frac{Q}{2}}{1 - p}\} \{(1 - p) \log M_2 + (1 - \frac{p}{2}) \log M_6
$$

$$
- \log(M_2 + M_6) + 2 \lambda_3\} + \lambda_1 r_{\beta_C} = 0,
$$

$$
r_{\gamma_A} \{\frac{1 - \frac{Q}{2}}{1 - p}\} \{(1 - p) \log M_3
$$

$$
+ \frac{p}{2} \log M_7 - \log(M_3 + M_7)\} + 2 \lambda_2\}
$$

$$
+ \lambda_1 r_{\gamma_A} \cos(\Phi_{\gamma_A} - \Phi_{\gamma_C}) = 0,
$$

$$
r_{\gamma_C} \{\frac{1 - \frac{Q}{2}}{1 - p}\} \{(1 - p) \log M_3
$$

$$
+ (1 - \frac{p}{2}) \log M_7 - \log(M_3 + M_7)\} + 2 \lambda_3\}
$$

$$
+ \lambda_1 r_{\gamma_C} \cos(\Phi_{\gamma_A} - \Phi_{\gamma_C}) = 0.
$$

IV. APPENDIX

Here we briefly explain our method for obtaining the optimized $I^{AE}$. First, we redefine the complex coefficients $\beta_{A,C}, \gamma_{A,C}$ in polar coordinates:

$$
\beta_A = r_{\beta_A},
$$

$$
\beta_C = r_{\beta_C},
$$

$$
\gamma_A = r_{\gamma_A} \exp(i\Phi_{\gamma_A}),
$$

$$
\gamma_C = r_{\gamma_C} \exp(i\Phi_{\gamma_C}).
$$

It is not straightforward to extract the solution from this set of equations. We will follow a strategy based...
on analytical and numerical methods. Due to equation (24) there are two possible solutions, which are \( \cos(\Phi_{\gamma_A} - \Phi_{\gamma_C}) = 1 \) and \( \cos(\Phi_{\gamma_A} - \Phi_{\gamma_C}) = -1 \) (as \( r_{\gamma_A} \) and \( r_{\gamma_C} \) cannot be zero).

Let us first assume the option \( \cos(\Phi_{\gamma_A} - \Phi_{\gamma_C}) = 1 \). Note that in this case the set of equations (21) - (28) is invariant under the simultaneous exchange \( r_{\gamma_A} \leftrightarrow r_{\gamma_C} \) and \( r_{\beta_A} \leftrightarrow r_{\beta_C} \). As we can see from equation (11), the mutual information function is also symmetric under this exchange. Now, we combine the set of the equations (21)-(28) to one joint equation in terms of \( r, p, Q \) and \( r_{\beta_A} \). The task is to find all roots for \( r_{\beta_A}^2 \). From equations (22) and (23) we have \( r_{\gamma_A}^2 = 1 - r_{\beta_A}^2 \) and \( r_{\gamma_C}^2 = 1 - r_{\beta_C}^2 \). This fact, together with the symmetry mentioned above, means that there has to be an even number of roots for \( r_{\beta_A}^2 \) (if one finds a solution for \( r_{\beta_A}^2 \), then \( 1 - r_{\beta_A}^2 \) is also a solution). Numerically (by plotting the joint equation in terms of \( r_{\beta_A}^2 \)) we show that for different \( p \) and \( Q \) there are always exactly two roots. Analytically, \( r_{\beta_C}^2 = 1 - r_{\beta_A}^2 \) is a possible solution for the equations (22)-(28). By inserting this expression for \( r_{\beta_C}^2 \) as well as \( r_{\gamma_A}^2 \) and \( r_{\gamma_C}^2 \) (see above) into (21) we find two solutions for \( r_{\beta_A}^2 \) which are parametrized in terms of \( p \) and \( Q \). One of them is the equation (13), already given in the text, and the other one is

\[
\begin{align*}
\gamma_{\beta_A}^2 &= |\beta_A|^2 = \frac{1}{2} \left( 1 + \frac{1}{2 - Q} \sqrt{(Q - \frac{5}{4})(2 - 3Q - \frac{5}{4})} \right) \\
(30)
\end{align*}
\]

Both of them lead to the same mutual information (this is clear from the symmetry, as explained above). Hence, we arbitrarily chose the one given in equation (14). Comparing analytical and numerical results made us sure that \( r_{\beta_C}^2 = 1 - r_{\beta_A}^2 \) is the unique relation between \( r_{\beta_A}^2 \) and \( r_{\beta_C}^2 \).

For the case \( \cos(\Phi_{\gamma_A} - \Phi_{\gamma_C}) = -1 \), we repeat the above process. However, equation (21) is now not symmetric under the exchange \( r_{\beta_A} \leftrightarrow r_{\gamma_A} \) and \( r_{\beta_C} \leftrightarrow r_{\gamma_C} \). If we plot the joint function for equations (21)-(28) in terms of \( r_{\beta_A}^2 \), we just find one root, and thus just expect one solution of the set of equations. Analytically we obtain \( r_{\beta_C}^2 = r_{\beta_A}^2 \) as a possible solution. This leads to the following mutual information between Alice and Eve:

\[
I^{AE} = \left( \frac{Q - \frac{5}{4}}{1 - p} \right) \left\{ 1 + \frac{p}{2} \log \frac{p}{2} + \left( 1 - \frac{p}{2} \right) \log \left( 1 - \frac{p}{2} \right) \right\}
\]

Comparing the equations (13) and (31) analytically, we see that for all \( p \) and \( Q \) the function in (13) is bigger than (31). Therefore, equation (13) is the optimal mutual information.

[1] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing*, Bangalore, India (IEEE, New York, 1984), pp. 175-179.
[2] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).
[3] R. Renner, N. Gisin and B. Kraus, Phys. Rev. A 72, 012332 (2005).
[4] C. H. Bennett, Phys. Rev. Lett. 68, 3121 (1992).
[5] D. Bruß, Phys. Rev. Lett. 81, 3018 (1998).
[6] H. Bechmann-Pasquinucci and N. Gisin, Phys. Rev. A 59, 4238 (1999).
[7] C. A. Fuchs and A. Peres, Phys. Rev. A 53, 2038 (1996).
[8] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, United Kingdom, 2000).
[9] E. B. Davies, IEEE Inf. Theory, IT-24, 596 (1978).
[10] I. Csiszár and J. Körner, IEEE Trans. Inf. Theory IT-24, 339 (1978).