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The ground state of a two-dimensional electron gas (2DEG) in the presence of a high perpendicular magnetic field is well known to appear as a multitude of highly correlated incompressible fractional quantum Hall effect (FQHE) states at a few special values of the Landau level filling factors [1, 2]. It is also well established that the strongest effect appears at the lowest Landau level filling factor (\( \nu = \frac{1}{3} \)) that has the largest quasiparticle-quasihole gap [1, 2]. The gaps are much smaller for \( \nu < \frac{1}{3} \), and as a consequence, nature of the electron states in that regime has remained a challenge until now. This is due to the fact that in the low density regime, the incompressible liquid is expected to undergo a phase transition to a crystalline state [3]. However, a definitive conclusion about the onset of this quantum phase transition has remained elusive because experimentally one observes weak effects for filling factors \( \frac{1}{7}, \frac{1}{9}, \) etc., and theoretically, one compares two very small energies (ground state) in order to determine which phase is energetically favored. For more than two decades, investigations of the FQHE have focused largely on 2DEGs that are embedded in GaAs heterostructures, where spin-related effects are small (though important [4]) compared to other effects, because of the small value of the \( g \)-factor of electrons in GaAs [5]. However, studies of the spin-orbit (SO) coupling in a 2DEG within an InAs (or InSb) quantum well with very large \( g \) values, are at the cusp of a rapid advance, due largely to their relevance to spin transport in low-dimensional electron channels [6]. In order to investigate the influence of SO interaction on the incompressible Laughlin states, we have carried out the well established finite-size studies in a periodic rectangular geometry, but for the first time with the SO coupling included in the Hamiltonian. We find that as the SO coupling strength is increased, there is an increase in the quasiparticle-quasihole gap, indicating that the incompressible state can be rendered more stable by appropriate tuning of the SO coupling. This might prove to be particularly useful in the low-electron density regime where, as described above, the FQHE state is usually very weak for conventional systems.

For a 2DEG in the \( xy \) plane with a magnetic field \( \mathbf{B} \) along the \( z \) direction, in the Landau gauge [with the choice of vector potential \( \mathbf{A} = (0, Bx, 0) \)], the single-particle states and the corresponding energies are obtained by solving the one-electron Hamiltonian

\[
H = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m^*} + \frac{\alpha}{\hbar}[\sigma \times (\mathbf{p} + e\mathbf{A})] + g\mu_B B\sigma_z
\]

that includes the Bychkov-Rashba term [7] and the Zeeman term. Here \( \mathbf{p} \) is the momentum operator, \( \alpha \) is the SO coupling strength and \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli spin matrices. Experimental values of the SO coupling constant lie in the range of 5–45 meV-nm [8]. The high values of \( \alpha \) are deduced from magnetotransport experiments where the SO interaction is tuned for a fixed carrier density in a 2DEG by using gate electrodes in a square asymmetric InAs quantum well [8]. We will therefore focus our investigation on InAs, as it represents the most promising material, as yet, for achieving large SO coupling.

We solve the Schrödinger equation

\[
H \psi = E \psi
\]

in a rectangular geometry with supercell sides \( L_x \) and \( L_y \) (i.e., aspect ratio \( \lambda = L_x/L_y \)) and expand the single-particle wavefunctions \( \psi_{kn}(\mathbf{r}) \) as a superposition of solutions of the Hamiltonian in the absence of SO interaction [9]

\[
\psi_{kn}(\mathbf{r}) = e^{ik_y y} \sum_{n, \sigma} \phi_n(x - x_0) C_n^\sigma |\sigma|/\sqrt{L_y}
\]

where

\[
\phi_n(x - x_0) = \beta_n e^{- (x-x_0)^2/2\sigma_x^2} H_n[(x - x_0)/l_0] / \sqrt{\sigma_x l_0}
\]

is the usual solution to the harmonic oscillator problem, with \( H_n(x) \) the Hermite polynomial, \( l_0 = (\hbar/m^* \omega_c)^{1/2} \) the radius of the cyclotron orbit with frequency \( \omega_c = eB/m^* \) and center \( x_0 = k_y l_0^2 \), \( n \) is the Landau level index,
\[ \beta_n = 1/\sqrt{2^n n!}, \] and
\[ \sigma = up, dn = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
is the electron spinor.

Substituting Eq. (3) into the Schrödinger equation (2), multiplying both sides by \( \phi_i(x - x_0) \) and integrating over \( x \), we obtain a system of equations [9]
\[ i(\alpha / l_0)\sqrt{2H_{\text{up}}^{\alpha}} + [(l + 1/2)\hbar \omega_c + E_d]C_{\text{dn}}^\alpha = 0 \]
\[ [(l + 1/2)\hbar \omega_c + E_u]C_{\text{up}}^\alpha - i(\alpha / l_0)\sqrt{2(l + 1)}C_{\text{dn}}^{\alpha + 1} = 0 \]
for \( l = 0, 1, 2, \ldots \) (4)
whose solution yields [9]
\[ (1/2\hbar \omega_c + E_d)C_{\text{dn}}^\alpha = 0, \quad s = 0, \]
\[ \left[ (s - 1/2)\hbar \omega_c + E_u - i(\alpha / l_0)\sqrt{2s} \right] \left( C_{\text{up}}^{s-1} / C_{\text{dn}}^s \right) = 0, \]
s = 1, 2, 3, \ldots , where \( E_u = g_\mu_B B - E \) and \( E_d = g_\mu_B B - E \). Corresponding to \( s = 0 \) there is only one level, the same as the lowest Landau level without SO interaction, with energy
\[ E_0 = 1/2\hbar \omega_c - g_\mu_B B \]
and wavefunction
\[ \psi_{0,0} = e^{ik_y y} \phi_0(x - x_0)/\sqrt{L_y} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]. \]
For all other values of \( s \neq 0 \) there are two branches of levels [9]
\[ \psi_{s,0}^+ = \frac{e^{ik_y y}}{\sqrt{L_y}A_s} \left( \begin{pmatrix} -iD_s \phi_{s-1}(x - x_0) \phi_s(x - x_0) \end{pmatrix} \right) \]
and
\[ \psi_{s,0}^- = e^{ik_y y} \left( \begin{pmatrix} \phi_{s-1}(x - x_0) -iD_s \phi_s(x - x_0) \end{pmatrix} \right) \]
with energies
\[ E_s^+ = s\hbar \omega_c \pm \sqrt{E_0^2 + 2s\alpha^2 / l_0^2}. \]
Here
\[ D_s = \frac{\sqrt{2s\alpha / l_0}}{E_0 + \sqrt{E_0^2 + 2s\alpha^2 / l_0^2}} \]
and \( A_s = 1 + D_s^2 \). From Eq. (5) and (6) we see that SO interaction couples two Landau levels. While previous works [9] were restricted to the study of the single-particle states, we use these equations as a starting point for our exact many-body treatment. Applying periodic boundary conditions, we obtain \( (\beta_y = x_0/l_0) \)
\[ x_0 = X_j = 2\pi l_0^2 / L_y, \quad L_x = 2\pi l_0^2 m / L_y, \]
and consequently,
\[ \psi_{s,j}^+(r) = \frac{1}{\sqrt{\pi l_0^2 A_s}} \sum_n \exp \left[ i(X_j + nL_x) \frac{y}{l_0^2} \right] \]
\[ - \frac{X_j + nL_x - x_0}{2l_0^2} \]
\[ \left( \begin{pmatrix} \phi_{s-1}(x - x_0) -iD_s \phi_s(x - x_0) \end{pmatrix} \right) \]
\[ \psi_{s,j}^-(r) = \frac{1}{\sqrt{\pi l_0^2 A_s}} \sum_n \exp \left[ i(X_j + nL_x) \frac{y}{l_0^2} \right] \]
\[ - \frac{X_j + nL_x - x_0}{2l_0^2} \]
\[ \left( \begin{pmatrix} \phi_{s-1}(x - x_0) -iD_s \phi_s(x - x_0) \end{pmatrix} \right). \]
We then build the antisymmetrized products (Slater determinants) using \( \psi^+ \) and \( \psi^- \) as a complete basis for the many-body wavefunction expansion
\[ \Psi = \sum_{\{i_n\}} \mathcal{P}(i_1, i_2, \ldots , i_n) a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_n}^\dagger |0\rangle \]
where \( i_k = (s_k, j_k, \sigma_k) \), \( \sigma_k = \pm \) and \( \mathcal{P}(i_1, i_2, \ldots , i_n) \) is the antisymmetrization operator.

The many-body Schrödinger equation was then solved by performing an exact diagonalization of the many-body Hamiltonian
\[ H = \sum_j W_j a_j^\dagger a_j + \sum_{j_1} \sum_{j_2} \sum_{j_3} \sum_{j_4} A_{j_1 j_2 j_3 j_4} a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3} a_{j_4}. \]
The kinetic energy term

$$W_j = S + E_j$$

includes the effects of a neutralizing background. Here $E_j = E_j^\pm$, 

$$S = -\frac{e^2}{\epsilon_0 \sqrt{2\pi m}} \left[ 2 - \sum_{k_1, k_2} \sqrt{\frac{\pi}{2}} (1 - \text{erf}(\sqrt{z})) \right]$$

is the Madelung energy, $e$ the electron charge, $\epsilon$ the dielectric constant, $z = \pi(\lambda k_i^2 + \lambda k_j^2)/\lambda$, and the prime in the summation means that the term $k_1 = k_2 = 0$ is excluded. The expression for the scattering matrix element $A_{ij|i2i3i4}$ depends on the quantum numbers $i_1i_2i_3i_4$, where, again, $i_k = (s_k,j_k,\sigma_k)$. For the case of positively polarized “spins” (i.e., $\sigma_k = +$ for all $k$) we have:

$$A_{i_1i_2i_3i_4}|_{\sigma_i=+} = \frac{1}{2} \frac{e^2}{\epsilon_0} \sqrt{\frac{\lambda}{2\pi m}} \prod_{i=1}^{4} \left( \frac{D_{s_i}}{A_{s_i}} \right) (-1)^{s_2+s_4} s_1 s_2 s_3 s_4 \
\times \sum_{k_1=1}^{\infty} \sum_{k_2=-\infty}^{0} \frac{\delta_{j_1,j_4,k_2}}{k_1^2 + \lambda^2 k_2^2} e^{-\pi(k_1^2 + \lambda^2 k_2^2)/\lambda m} \left( k_2 \sqrt{\frac{2\pi \lambda}{m}} \right)^{\sum_{i=1}^{4} s_i} \cdot \left[ \cos \left( \frac{2\pi}{m} k_1 (j_1 - j_3) \right) \right]$$

$$\times \left\{ \sum_{s_1} \sum_{s_3} \sum_{s_3} \sum_{s_3} \left( s_1 \left( s_1 - t_1 \right) \left( s_3 - t_2 \right) \right) \left( s_4 \left( s_4 + t_1 + 2t_2 + p_1 + p_2 \right) \right) \left( s_2 \left( s_2 + t_2 + 2p_2 \right) \right) \left( -1 \right)^{t_1 + t_2 + p_1 + p_2} \right\}$$

where

$$L_n^\alpha(x) = \sum_{m=0}^{n} (-1)^m \frac{(n + \alpha)^m x^m}{m!}$$

are the Laguerre polynomials and

$$u_{p_2} = \left\{ \begin{array}{ll} (s_4 - t_1) & \text{if } (s_4 - t_1) < 0 \\ \text{Int}\{ (s_4 - t_1)/2 \} & \text{otherwise} \end{array} \right.$$ 

and the same yields for $u_{p_2}$, with $s_4 \rightarrow s_2$ and $t_1 \rightarrow t_2$, $\text{Int}\{x\}$ is the integer part of $x$.

Due to the presence of the spinors $\sigma$, the two branches $\psi^\pm$ and the coupling of the Landau levels in pairs, the derivation of the complete expression for $A_{i_1i_2i_3i_4}$ is highly nontrivial. Moreover, the Hamiltonian matrix can be very large and its diagonalization computationally very intensive. We calculated the ground state energy per particle $E_0$ for a system with four electrons in the lowest two Landau levels with different filling factors $\nu = 4/N_s$, where $N_s = 8, 9, \ldots, 20$. The results are shown in Fig. 1 for $B = 10$ Tesla. We see that the presence of SO coupling lowers considerably the value of $E_0$, compared to the result for $\alpha = 0$, which coincides with the usual results obtained for the FQHE with no SO, with both the Zeeman and the kinetic energies included.

The most intriguing effect of SO coupling in 2DEG, however, is found in connection with the magnitude of the quasiparticle-quasihole energy gap $E_g$, derived from the positive discontinuity of the chemical potential at the filling factor $\nu$ [2].

As shown in Fig. 2 for four electrons and a filling factor of $\nu = \frac{1}{3}$ and in Fig. 3 for even smaller values of $\nu = \frac{1}{5}$ and $\frac{1}{7}$ [10], large values of $\alpha$ cause the enhancement of $E_g$. This enhancement is larger for small magnetic fields ($B \sim 1$ Tesla, i.e., fields for which the Rashba term is still comparable to the Zeeman term), and can be of the order of 25% for $\nu = \frac{1}{32}$ and mid-range values of the coupling constant ($\alpha = 20$). This is seen in Fig. 2, where $E_g$ is plotted.

![FIG. 2: Quasiparticle-quasihole energy gap as a function of magnetic field B for a filling factor $\nu = 1/3$ and three different values of the SO coupling constant $\alpha = 0, 20, 40$ calculated for four electrons in the lowest two Landau levels in InAs.](image-url)
FIG. 3: Quasiparticle-quasihole energy gap as a function of magnetic field $B$ for a filling factor $\nu = 1/5$ and $1/7$ and two different values of the SO coupling constant $\alpha = 0, 40$ calculated for three electrons in the lowest two Landau levels in InAs.

Incompressible states (Laughlin at $\nu = \frac{1}{3}, \frac{2}{5}$) are particularly advantageous for filling factors $\nu < \frac{1}{5}, \frac{1}{7}$ where a larger gap would signify a more stable liquid state that will push the liquid-solid transition further down in the density.

In summary, we have investigated the influence of the SO coupling (Bychkov-Rashba) on the incompressible state proposed by Laughlin at $\nu = \frac{1}{3}, \frac{2}{5}$ using an exact diagonalization scheme for finite-size systems in a periodic rectangular geometry. We found that, as the SO coupling strength is increased, there is an increase of the quasiparticle-quasihole gap. This is particularly advantageous for filling factors $\nu < \frac{1}{5}, \frac{1}{7}$ where a larger gap would signify a more stable liquid state that will push the liquid-solid transition further down in the density.

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