Pseudo-Hermitian version of the charged harmonic oscillator and its “forgotten” exact solutions

Miloslav Znojil

Nuclear Physics Institute of Academy of Sciences of the Czech Republic,
250 68 Řež, Czech Republic
e-mail: znojil@ujf.cas.cz

Abstract

An unusual type of the exact solvability is reported. It is exemplified by the Coulomb plus harmonic oscillator in $D$ dimensions after a complexification of its Hamiltonian which keeps the energies real. Infinitely many bound states are found in closed form which generalizes the popular harmonic-oscillator states at zero charge and even parity. Apparently, the model is halfway between exact and quasi-exact.

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1 Introduction

Schrödinger equation for the shifted and charged harmonic oscillator in $D$ dimensions reads

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + \frac{f}{r} + 2g r + r^2\right] \varphi(r) = E \varphi(r), \quad r \in (0, \infty).$$

(1)

Thirty years ago, André Hautot [1] noticed that at certain non-vanishing couplings $f$ and/or $g$ it may possess elementary solutions for the equidistant set of the energy levels

$$E = E_{n,\ell}^{(\text{Hautot})} = 2n + 2 + \mathcal{L} - g^2,$$

(2)

$$n = 0, 1, \ldots, \quad \mathcal{L} = \mathcal{L}(\ell) = 2\ell + 1 = D - 2, D, D + 2, \ldots$$

which do not depend on the charge $f$. The charge itself is not arbitrary (for this reason, the models of this type are called quasi-exactly solvable, QES). One has to evaluate the admissible values of $f = f_n$ as real roots of certain polynomials of the $(n - 1)$–st degree (see below).

The undeniable mathematical as well as physical appeal of QES solutions has been revealed by many independent authors whose work was summarized by Alex Ushveridze [2]. Very recently, the QES models helped to clarify some counterintuitive formal features of the so called $\mathcal{PT}$ symmetric quantum mechanics of Bender et al. [3] who replaced the usual Hermiticity $H = H^\dagger$ by the mere commutativity of the Hamiltonian with the product of parity $\mathcal{P}$ and time reversal $\mathcal{T}$. In the early stages of development, the studies of this formalism were strongly motivated by its relevance in field theory [4]. In such a setting, it was very impressive when Bender and Boettcher [5] demonstrated that in contrast to the current Hermitian case, quartic polynomial oscillators belong to the QES class after their appropriate $\mathcal{PT}$ symmetrization (cf. also ref. [6] for more details).

The charged harmonic oscillator [7] does not possess similar appeal in field theory but it was still amusing to reveal in paper [7] that its Schrödinger equation does not lose its partial elementary solvability even after the weakening of the Hermiticity to the mere $\mathcal{PT}$ symmetry of its Hamiltonian. In accord with the Buslaev’s and Grecchi’s recipe [8] we used the shifted coordinates $x \in (-\infty, \infty)$ on complex line
\[ r(x) = x - i \varepsilon \text{ at a constant distance } \varepsilon > 0 \text{ from the real axis,} \]

\[
\left[ -\frac{d^2}{dx^2} + \frac{\ell(\ell + 1)}{r^2(x)} + i \frac{F_n}{r(x)} + 2ibr(x) + r^2(x) \right] \psi_n(x) = E_n \psi_n(x). \quad (3)
\]

Unfortunately, we only analyzed the \( \ell = 0 \) solutions in the quasi-odd regime (cf. a more detailed explanation below).

Since the publication of paper [7] a significant progress has been achieved in the interpretation of the non-Hermitian equations. Several authors [9] emphasized that within the domain of quantum mechanics, the \( \mathcal{P} \mathcal{T} \) symmetry of eq. (3) should be replaced by the (formally equivalent but mathematically more natural) requirement of the pseudo-Hermiticity of the Hamiltonian,

\[ H = \eta^{-1} H^\dagger \eta, \quad \eta = \eta^\dagger. \quad (4) \]

One of the oldest illustrations of the efficiency of the use of the pseudo-Hermitian Hamiltonians [11] with an auxiliary indefinite metric \( \eta \) appears in the relativistic quantum mechanics where it very naturally originates from the Feshbach’s and Villars’ Hamiltonian formulation of the Klein Gordon equation for the particles with zero spin [10]. The same choice of a suitable invertible \( \eta \) helps to clarify some contemporary problems in the cosmological models based on the equations of Bryce de Witt [11].

In the light of eq. (4), physical interpretation of the non-Hermitian bound states is more transparent and does not depend too much on the specification of the operator \( \eta \) itself. This operator only plays the role of a certain auxiliary transformation of the dual Hilbert space. For a detailed explanation of this idea we recommend the older review [12] where the physical meaning of the nontrivial “metric” \( \eta \) was illustrated by its emergence in the many-fermion models where \( \eta \neq I \) characterizes the so called Dyson’s mapping of the “physical” (and Hermitian) fermionic Hamiltonians onto their “more easily solvable” \( \eta \)–Hermitian bosonic equivalents [13].

One must keep in mind the non-uniqueness of the metric \( \eta \) which belongs to the given \( H \). According to A. Mostafazadeh [14] one can replace the initial indefinite (and, in particular, \( \mathcal{P} \mathcal{T} \)–symmetric) metric \( \eta_1 \) by an alternative \( \eta_2 \) which is positive definite. In the other words, the puzzling quasi-unitary evolution generated by
the indefinite \( \eta_1 \) may be declared an artifact of our constructions. *Vice versa*, all the phenomenological considerations should necessarily be related to the positive definite version \( \eta_2 > 0 \) of the metric (one can show that it exists for all the diago-
nalizable Hamiltonians \[9, 14\]). Then, the time evolution remains compatible with
the probabilistic interpretation of the norm in the Hilbert space of states.

In the light of the possible peaceful coexistence of \( \eta_1 = \mathcal{P} \) with some \( \eta_2 > 0 \) in
eq (4), our attention has been re-directed to the solutions of eq. (3) which diverge in
the simple-minded Hermitian limit \( \varepsilon \to 0 \) and which were omitted from the scope of
our preceding study \[7\]. We are going to correct the omission now. Our expository
section 2 will outline an improvement of the method which generates the old QES
solutions of ref. \[7\]. Section 3 will then modify the basic ansatz which opens the
way towards the new (or, in the wording of our title, “forgotten”) QES solutions of
eq (3). Some of their properties and possible applications will be finally discussed
in section 4.

## 2 The standard quasi-exact solutions

As long as the differential eq. (3) is of the second order, its general solution is a
superposition of some two linearly independent solutions. This independence may
be deduced from their available leading-order form near the origin,

\[ \psi_n(r) = c_- \psi_n^-(r) + c_+ \psi_n^+(r), \quad \psi_n^{(\pm)}(r) = \mathcal{O} \left( r^{1/2\mp(\ell+1/2)} \right). \]  

(5)

In the spirit of ref. \[7\] one usually searches for the polynomial solutions compatible
with the correct physical boundary conditions in the origin (i.e., \( c_+ = 0 \) \[10\]) as well
as with their asymptotic normalizability. In the spirit of the general QES philosophy
one may meet both these requirements by employing the special elementary ansatz
of the very common harmonic-oscillator-like form

\[ \psi(x) = \psi_n^-(x) = e^{-r^2/2-i b r} \sum_{n=0}^N h_n^{(-)}(i r)^n \ell + 1, \quad r = r(x) = x - i \varepsilon. \]  

(6)

The construction of the solutions of this type degenerates to the insertion of eq. (6)
into the differential eq. (3) which gives the homogeneous set of \( N+2 \) linear algebraic
equations for the \( N + 1 \) coefficients \( h_j^{(-)} \). We may drop the superscript and turn to the explicit non-square matrix form of the latter equations,

\[
\begin{pmatrix}
B_0 & C_0 \\
A_1 & B_1 & C_1 \\
& \ddots & \ddots \\
A_{N-1} & B_{N-1} & C_{N-1} \\
A_N & B_N \\
A_{N+1}
\end{pmatrix}
\begin{pmatrix}
h_0 \\
h_2 \\
\vdots \\
h_N
\end{pmatrix} = 0 \quad (7)
\]

with elements

\[
A_n = A_n^{(-)} = b^2 + 2n + L - E, \quad B_n = B_n^{(-)} = -(2n + 1 + L)b - F, \\
C_n = C_n^{(-)} = (n + 1)(n + 1 + L), \quad L = 2\ell + 1, \quad n = 0, 1, \ldots .
\]

This is a finite-dimensional and over-determined linear algebraic re-incarnation of the original differential equation \((\mathbb{R})\). Its matrix structure enables us to define the wave function (i.e., all its energy-dependent coefficients) as determinants,

\[
h_{j-1} = \frac{h_N}{(-A_j)(-A_{j+1}) \ldots (-A_N)} \cdot \det
\begin{pmatrix}
B_j & C_j \\
A_{j+1} & B_{j+1} & C_{j+1} \\
& \ddots & \ddots & \ddots \\
A_{N-1} & B_{N-1} & C_{N-1} \\
A_N & B_N
\end{pmatrix} \quad (8)
\]

with \( j = N, N - 1, \ldots, 1 \) and under any choice of the normalization \( h_N \neq 0 \).

The latter normalization convention converts the last row \( A_{N+1}h_N = 0 \) of eq. \((7)\) into the constraint

\[
E = E^{(-)} = 2N + 2 + L + b^2. \quad (9)
\]

In the other words, the condition of the mutual compatibility of the original over-determined linear system \((\mathbb{R})\) fixes the energy which coincides with the old Hautot’s formula \((\mathbb{R})\). At any \( N = 0, 1, \ldots \) the energy is an increasing function of the angular momentum \( \ell \) or \( L \) and of the size of the shift \( b \). The QES construction is complete and

- simplifies the prescription of ref. \([7]\) (where the special cases of eqs. \((\mathbb{R})\) and \((\mathbb{R})\) contained the less compact matrix with four diagonals),
• leads to the polynomial wave functions (with the closed form (8) of the coefficients),
• preserves the Hautot’s explicit form (14) of the energies,
• reduces the differential Schrödinger equation to its \( n \)-dimensional square-matrix form.

The first observation (simplification) is a marginal technical merit due to our transition to a better ansatz. In contrast, the last feature of the QES solutions remains highly unpleasant as it forces us to guarantee that the related secular determinant vanishes,

\[
\begin{vmatrix}
B_0^{(-)} & C_0^{(-)} \\
A_1^{(-)} & B_1^{(-)} & C_1^{(-)} \\
& \ddots & \ddots & \ddots \\
& & A_{N-1}^{(+)} & B_{N-1}^{(-)} & C_{N-1}^{(-)} \\
& & & A_N^{(-)} & B_N^{(-)}
\end{vmatrix} = 0 .
\]

Such a constraint determines the set of the \( N + 1 \) admissible couplings \( F = F_k^{(-)}(N) \) \[1\] and its purely numerical nature is an example of the most serious practical shortcoming of the majority of the QES models \[2\]. We are now going to describe a remarkable exception from this discouraging rule.

### 3 Nonstandard, quasi-even QES states

#### 3.1 The concept of quasi-parity

Above we emphasized that at a fixed, non-vanishing shift \( \varepsilon > 0 \) the ambiguity of the metric \( \eta \) opens the possibility of using the “simpler” conjugation \( \eta \) with \( \eta = \eta_1 = \mathcal{P} \) during the explicit constructions of the solutions while switching to their “physical” re-interpretation based on an alternative scalar product with the positive definite metric \( \eta_2 > 0 \). In such an approach both components \( c_- \neq 0 \) and \( c_+ \neq 0 \) of wave functions in eq. \( (3) \) may be equally useful.

Once we relax the redundant boundary conditions in the origin we get more solutions of course. One of the most transparent illustrations of the emergence of
the additional “quasi-even” \( c_+ \neq 0 \) solutions was described in ref. [17] where eq. (9) has been solved at the vanishing \( b = F_n = 0 \). The superscripts in the resulting states \( \psi_n^{(+)}(r) \) and \( \psi_n^{(-)}(r) \) have been interpreted as the so called quasi-parity. The introduction of this concept was motivated by the observation that the quasi-parity degenerated to the current parity at \( b = F_n = \ell = 0 \) (cf. eq. (5)).

Its independent additional support appeared in ref. [18] revealing its connection with the \( \mathcal{PT} \) parity and with the pseudo-norm using \( \eta = \eta_1 = \mathcal{P} \). It even plays its role in the supersymmetric quantum mechanics (cf. ref. [19] for more details) but is missing from our ansatz (6) inherited from ref. [7]. As we understand it now, ansatz (6) is unnecessarily restrictive as it represents merely quasi-odd solutions. From its generalization

\[
\psi_n(r) = e^{-r^2/2-i b r} \sum_{n=0}^{N} h_n(i r)^{n-\ell}. \tag{11}
\]

(where we dropped all the superscripts for the time being) one can always return to the old quasi-odd option via the additional \((-)\)–superscripted constraint

\[
h_0^{(-)} = h_1^{(-)} = \ldots = h_{L-1}^{(-)} = 0. \tag{12}
\]

Whenever necessary, the \((+)\)–superscripted “quasi-even” QES solutions may be characterized by the alternative criterion

\[
|h_0^{(+)}| + |h_1^{(+)}| + \ldots + |h_{L-1}^{(+)}| > 0. \tag{13}
\]

We are now close to our key claim that the structure of the quasi-even QES solutions is exceptionally simple.

### 3.2 QES states having the even quasi-parity

The source of the latter claim lies in the improved ansatz (11) which leads to the same equation (11) with the new matrix elements

\[
A_n = A_n^{(+)} = b^2 + 2n - \mathcal{L} - E, \quad B_n = B_n^{(+)} = -(2n + 1 - \mathcal{L})b - F, \quad C_n = C_n^{(+)} = (n + 1)(n + 1 - \mathcal{L}), \quad \mathcal{L} = 2\ell + 1, \quad n = 0, 1, \ldots .
\]

The energy formula (13) is only marginally modified,

\[
E = E^{(+)} = 2N + 2 - \mathcal{L} + b^2. \tag{14}
\]
Still, we immediately notice the much more important difference connected with the presence of the vanishing matrix element $C_{L-1}^{(+)} = 0$ in the upper diagonal of our new form of the QES secular equation. This means that the $(+)$—superscripted secular determinant may be re-written as the product of a “small”, $L-$dimensional

$$S^{(S)} = \det \begin{pmatrix} B_0^{(+)} & C_0^{(+)} & 0 & \cdots & 0 \\ A_1^{(+)} & B_1^{(+)} & C_1^{(+)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{L-2}^{(+)} & B_{L-2}^{(+)} & C_{L-2}^{(+)} & \cdots & 0 \\ A_{L-1}^{(+)} & B_{L-1}^{(+)} & C_{L-1}^{(+)} & \cdots & 0 \end{pmatrix}$$

with another determinant $S^{(L)}$ of a “large”, $(N-L)-$dimensional matrix. The latter factor

$$S^{(L)} = \det \begin{pmatrix} B_L^{(+)} & C_L^{(+)} & 0 & \cdots & 0 \\ A_{L+1}^{(+)} & B_{L+1}^{(+)} & C_{L+1}^{(+)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{N-1}^{(+)} & B_{N-1}^{(+)} & C_{N-1}^{(+)} & \cdots & 0 \\ A_N^{(+)} & B_N^{(+)} & C_N^{(+)} & \cdots & 0 \end{pmatrix}$$

precisely coincides with the left-hand side expression in eq. (10) which guarantees, in its turn, the existence of the QES solutions with the property (12). The use of the condition $S^{(L)} = 0$ would return us back to the old quasi-odd ansatzs of section 2. In what follows we shall ignore these solutions as standard and assume that $S^{(L)} \neq 0.$

### 3.3 Facilitated QES constructions

After one concentrates attention solely to the quasi-even states, the QES construction degenerates, basically, to the secular equation $S^{(S)} = 0$. One verifies easily that the acceptance of this condition is consistent with the quasi-parity (13). The wave function coefficients themselves remain formally determined by the $(+)$—superscripted version of the determinants (8) whose dimension grows with $N$. For this reason we recommend a re-interpretation of these coefficients as quantities evaluated by the
recurrences initiated at an initial $h_N^{(+)} \neq 0$ and defining $h_{N-1}^{(+)}$, $h_{N-2}^{(+)} \ldots$ step by step,

\[
\begin{pmatrix}
-2N & \beta_1 - F & 4 - 2\mathcal{L} \\
\vdots & \ddots & \vdots \\
-4 & \beta_{N-1} - F & N^2 - N\mathcal{L} \\
-2 & \beta_N - F & \\
\end{pmatrix}
\begin{pmatrix}
h_0^{(+)} \\
\vdots \\
h_{N-1}^{(+)} \\
h_N^{(+)} \\
\end{pmatrix} = 0 .
\]

We abbreviated here $\beta_n \equiv -(2n + 1 - \mathcal{L})b$ and note that in the last step of these recurrences, the secular equation $\mathcal{S}^{(S)} = 0$ replaces the redundant definition of a ghost coefficient $h_{-1}^{(+)} = 0$. Finally, the explicit form of our secular equation

\[
\begin{pmatrix}
\beta_0 - F & 1 - \mathcal{L} \\
-2N & \beta_1 - F & 4 - 2\mathcal{L} \\
\vdots & \ddots & \vdots \\
-2(N + 3 - \mathcal{L}) & \beta_{\mathcal{L}-2} - F & 1 - \mathcal{L} \\
-2(N + 2 - \mathcal{L}) & \beta_{\mathcal{L}-1} - F \\
\end{pmatrix}
\begin{pmatrix}
h_0^{(+)} \\
\vdots \\
h_{\mathcal{L}-2}^{(+)} \\
h_{\mathcal{L}-1}^{(+)} \\
\end{pmatrix} = 0
\]  

(16)

specifies the family of the admissible charges $F_N = F^{(+)}_k(N)$ in a way which remains virtually purely non-numerical for the first few dimensions $\mathcal{L}$.

### 3.3.1 $\mathcal{L} = 2$

One of the key reasons why both the formal appeal and practical importance of the quasi-even spectrum \([14]\) remained unnoticed in ref. \([7]\) was purely psychological. Indeed, at the simplest choice of $\mathcal{L} = 1$ in eq. \((16)\) (which may mean both the $s$–wave in three dimensions and an even state at $D = 0$) one does not obtain anything new. Equation \((16)\) provides the single root $F_0^{(+)}(N) = 0$ and one just reveals the well known fact that at the vanishing eigencharge our model degenerates to the linear harmonic oscillator defined on the whole line.

Let us move, therefore, to the first nontrivial choice of $\mathcal{L} = 2$ corresponding to the $p$–wave in two dimensions or to the $s$–wave in four dimensions. This gives the following two series of the fully non-numerical eigencharges,

\[
F_{[1,2]}(N) = \pm \sqrt{(b^2 + 2N)} .
\]  

(17)
The wave functions retain the even quasi-parity in a way compatible with eq. (13),

\[ h_{0[1,2]}^{(+)}(N) = -\frac{1}{2N} \left[ b + F_{[1,2]}(N) \right] h_{1[1,2]}^{(+)}(N). \]

Both the eigencharges grow with the increasing size of the shift \( b \) and with the number \( N \) of nodes (i.e., with the energy \( E_N \)). In the coupling-energy plane our QES states may be visualized as families located along certain curved lines which, in a way, lie somewhere in between the harmonic-oscillator \( F = 0 \) straight line and the numerous Hautot’s sets of roots \( F_k(N) \) each of which is defined at a fixed energy or integer \( N \). Sometimes, the similar families of the bound states with the “energy = constant” property are being called Sturmians. In this sense one could speak here about a certain further generalization of the latter concept.

### 3.3.2 \( \mathcal{L} = 3 \)

A mild formal shortcoming of the present Coulomb + harmonic oscillators lies in a quick growth of complexity of eq. (11) for the larger \( \mathcal{L} \). At any \( \mathcal{L} \geq 3 \) one should not be tempted to generate the formulae verifying, say, the smooth \( N- \) and \( b- \) dependence of the eigencharges. In practice, the other approaches may definitely prove preferable.

Even at the very first \( \mathcal{L} = 3 \) the comparatively compact form of our eq. (11),

\[ \det \begin{pmatrix} 2b - F & -2 & 0 \\ -2N & -F & -2 \\ 0 & -2N + 2 & -2b - F \end{pmatrix} = 0, \tag{18} \]

should not inspire a search for the triplet of charges \( \{F_0, F_1, F_2\} \) via the closed (i.e., Cardano) formulae (the reader is recommended to try to generate them using the computer symbolic manipulations in order to see that they really are enormously clumsy). A significantly better strategy consists in an elimination of the (unique) value of \( N \) from the above secular determinant (18) giving

\[ N = N(F, b) = -\frac{1}{8F} \left( 4Fb^2 + 8b - F^3 - 4F \right). \]

After we fix any left-hand-side integer \( N \) we may pick up the two eigenshifts \( b_{[1,2]} \) as elementary functions of the indeterminate variable \( F \) (we skip the details which are trivial).
3.3.3 \( \mathcal{L} > 3 \)

The main advantage of the semi-implicit techniques of the solution of eq. (16) is that they may work at a few larger integers \( \mathcal{L} > 3 \). For illustration, the choice of \( \mathcal{L} = 5 \) (which corresponds to the \( d \)-wave in three dimensions) leads already to the purely numerical determination of eigencharges \( F \). In an alternative approach we eliminate

\[
N^\pm = \frac{1}{512F} \left[ -768b - 256Fb^2 + 768F + 40F^3 \pm 24\sqrt{(1024b^2 + 192bF^3 + 512F^2 + F^6)} \right]
\]

and recommend the graphical determination of the eigenvalues \( F = F(b) \) and/or \( b = b(F) \) afterwards.

The most practical possibility consists in a direct selection of a suitable shift \( b \) followed by the subsequent diagonalization of the purely numerical matrix. In an illustration using \( \mathcal{L} = 3 \) and \( b = 5 \) one gets the three eigencharges

\[
F = \{10.757, -10.400, -0.35755\}
\]

at the smallest possible \( N = 2 \). In an opposite-extreme using \( N = 1000 \) the computed values

\[
F = \{89.98, -89.975, -0.004907\}
\]

already lie very close to their large-\( N \) estimate obtainable in closed form,

\[
F \approx \{\sqrt{8N}, -\sqrt{8N}, -b/N\} \approx \{89.44, -89.44, -0.005\}.
\]

This simplifies the verification of the reality of the eigencharges and confirms the smoothness of their \( N \)-dependence. Such a type of calculation is very quick and gives results sampled at \( \mathcal{L} = 4 \) and \( b = 5 \) in Table 1.

4 Discussion

The main merit of our key eq. (16) (which defines the eigencharges) is that its dimension is independent of the quantum number \( N \). Equation (17) is the best illustration of the related new form of the solvability which we intended to describe here. Still, the principle of the whole construction is more general and one might try to apply the similar recipe to the quartic oscillator of refs. [5, 6], to the sextic
oscillators studied by many authors [20], to the decadic oscillator of ref. [21] and to the numerous existing modifications [2] of these most popular or “canonical” models.

4.1 Non-orthogonal QES states as a basis?

In our particular example, the strength of the Coulombic interaction appears to be an energy- or \( N \)-dependent quantity. One deals with an \( \mathcal{L} \geq 2 \) generalization of the common \( \mathcal{L} = 1 \) harmonic oscillator. Its most important features are an apparent completeness “in a relevant subspace” (a guess inspired by their infinite number) and a compact form (reflecting the \( N \)-independent evaluation of the eligible charges, each of which is selected as a function of the main quantum number \( N = 0, 1, \ldots \)). Both these features make our infinite set(s) of the quasi-even states very similar to the ordinary harmonic oscillator basis. In the context of concluding remarks, let us pay some attention to the possible analogies of the latter type.

Firstly, due to the non-Hermiticity of the Hamiltonian \( H(F) = H(0) + F W \) we have to distinguish between the left (= double-ket) and right (= single-ket) QES eigenstates,

\[
\begin{align*}
[H(0) + F_N W] \, \langle N \rangle &= E_N \, \langle N \rangle, \\
\langle \langle N | \, [H(0) + F_N W] &= E_N \, \langle \langle N |.
\end{align*}
\]

The integer \( N \) numbers the energies \( E_N \) as well as the selected charges \( F_N = F[k]^{(+)}(N) \) so that the left and right eigenstates exist at the common energies (14) and charges [say, (17)]. The wave functions are defined in closed form, as polynomials in the coordinates (cf. (11)) and in the couplings and quantum numbers (cf. (8)). This is of paramount importance, making our quasi-even QES solutions extremely similar to the even bound states of the exactly solvable chargeless oscillators which form one of the most popular complete bases in \( L_2(0, \infty) \).

In a tentative parallel one could search for the appropriately weakened biorthogonality relations. This is a real mathematical challenge since our QES solutions are only defined at the exceptional and \( \ell \)- and \( N \)-dependent charges. Still, one can easily verify the manifest non-orthogonality of the pairs of many randomly selected QES states. Of course, the closed form as well as the infinite number of these states inspires their use in a perturbative or variational context.
For the similar purposes one has to truncate their set to a finite subset \((N < \infty)\), assuming that the matrix of their overlaps

\[ Q_{m,n} = \langle m|n \rangle, \quad m, n = 0, 1, \ldots N \]

is invertible. We may then proceed, say, in the variational spirit and reduce our Hilbert space and its dual to the finite-dimensional subspaces spanned by our subset of the selected QES eigenvectors. The approximate identity operator becomes defined by the usual series

\[ I = \sum_{m,n=0}^{N} |m\rangle R_{m,n} \langle n| \]

where \(R = Q^{-1}\) is, in general, fully non-diagonal. Also the Hamiltonian \(H(F)\) itself becomes approximated by a non-diagonal matrix. At almost all \(F\), the search for the energies \(E = E(F)\) becomes, therefore, a numerical task.

### 4.2 Speculations about applicability

In spite of the unpleasant character of the latter conclusion, one should still feel the difference between a fully general matrix diagonalization and our “next-to-solvable” Coulomb + harmonic problem considered at any charge \(F\),

\[ [H(0) + FW] |\Psi\rangle = E(F) |\Psi\rangle. \tag{21} \]

In particular, our Schrödinger equation may be (e.g., perturbatively) connected to its special cases with QES character. In an attempted step towards making such a connection explicit, let us imagine that equations (19) and (20) share the energy and charge (though not the eigenvectors) at every given \(N\). This means that, respectively, we have the relations

\[ \langle\langle N| [H(0) + F_M W] |M\rangle = E_M \langle\langle N|M\rangle, \tag{22} \]

\[ \langle\langle N| [H(0) + F_N W] |M\rangle = E_N \langle\langle N|M\rangle \tag{23} \]

the subtraction of which gives the strongest constraint

\[ (F_M - F_N) \langle\langle N|W|M\rangle = (E_M - E_N) Q_{N,M}. \tag{24} \]
This relation is an immediate generalization of the bi-orthogonality of the states which would result from it in the hypothetical case of the subscript-independent charges $F_M = F_N$.

Let us now return to the relations (22) and (23) and deduce the matrix form of the Coulomb + harmonic Hamiltonian $H(F)$ at any value of the charge,

$$\langle\langle \langle N | H(F) | M \rangle \rangle = (F - F_M) \langle\langle \langle N | W | K \rangle \rangle R_{K,J} p_J = (E - F_N) p_N. \tag{26}$$

Next, the necessary input information is further reduced by the generalized biorthogonality relations (24) which express all the off-diagonal elements of the (in our example, Coulombic) operator $W$ in terms of the known overlap matrix $Q$. In this way the numerical or perturbative diagonalization of the matrix Schrödinger eq. (26),

$$\langle\langle \langle N | W | N \rangle \rangle \sum_{J} \left[ T_N R_{N,J} + \sum_{K \neq N} \frac{E_N - E_K}{F_N - F_K} Q_{N,K} R_{K,J} \right] p_J = (E - E_N) p_N, \tag{27}$$

will require the independent input evaluation of the mere diagonal matrix elements $T_N = \langle\langle \langle N | W | N \rangle \rangle$ assuming of course that we always have $F_M \neq F_N$ for $M \neq N$. We may summarize that the main merit of the use of the QES states lies in the compact and easily generated matrix form of our Coulomb + harmonic matrix Schrödinger equation at any non-QES charge $F$. 

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4.3 Outlook

We have seen that our system (3) may quite efficiently be treated by the non-numerical as well as almost purely numerical means. In the former sense it has been shown to lie somewhere in between the QES and exactly solvable category (cf. Table 2). In comparison with its completely solvable neighbor (i.e., with its own harmonic oscillator special case of ref. [17]), its nontrivial $b \neq 0$ and $F \neq 0$ versions do not generate all their bound states in the elementary form. Still, in contrast to the quasi-odd QES model [4] in Table 2, its present quasi-even partner supports an infinite number of bound states in the closed, elementary form. Thus, our example (as well as any one of its many analogs) could be assigned a “midway” status in some of its applications and interpretations.

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Table 1. $N$–dependence of eigencharges at $\mathcal{L} = 4$ and $b = 5$.

| $N$ | $F_N$          |
|-----|---------------|
| 3   | -15.611 -5.9279 4.888 7 16.651 |  |
| 30  | -27.149 -9.2909 8.9294 27.511 |  |
| 100 | -44.732 -15 14.865 44.867 |  |
| 200 | -61.665 -20.602 20.531 61.736 |  |
| 300 | -74.856 -24.984 24.936 74.904 |  |
| 1000| -134.93 -44.985 44.970 134.94 |  |
| 3000| -232.82 -77.610 77.605 232.83 |  |
| 30000| -734.99 -245.00 245.00 734.99 |  |

Table 2. Solvable pseudo-Hermitian potentials: Tentative classification.

| class | quasi – exact | intermediate | exact |
|-------|---------------|--------------|-------|
| solutions available | at a finite set of $N$ | at infinitely many $N$ | at all $N$ |
| range of couplings | restricted | restricted | any |
| illustrative example | [7] | here | [17] |
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