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A new approach to study nonlinear space-time fractional sine-Gordon and Burgers equations

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Abstract

In this study, we investigate a couple of nonlinear fractional differential equations namely, the sine-Gordon and Burgers equations in the sense of Riemann-Liouville fractional derivative. In order to examine exact solutions effectively applicable in relaxation and diffusion problems, crystal defects, solid-state physics, plasma physics, vibration theory, astrophysical fusion plasmas, scalar electrodynamics, etc. we introduce the new generalized $(G'/G)$-expansion method. The method is highly effective and a functional mathematical scheme to examine solitary wave solutions to diverse fractional physical models.

1. Introduction

The fractional derivatives and integrals is not a new thinking. It has been found that the roots of it were educated over three hundred years ago. In recent decades, considerable efforts have been paid in the fields of NFDEs and drawn attention for their frequent appearance in different engineering and scientific arenas, like plasma physics, controlled thermonuclear fusion, solid-state physics, acoustics, diffusive transport, stochastic dynamical system, electrical network, astrophysics, electromagnetic theory, etc. Fractional calculus is used to formulate and interpret different physical models for a continuous transition from relaxation to oscillation phenomena. Therefore, diverse methods have been presented to attain solutions of NFDEs, such as the local fractional Riccati differential equation [1], modified auxiliary equation [2], FRDTM [3], sine-Gordon expansion [4], IBSEF [5], $(G'/G)$-expansion [6, 7], modified simple equation [8], Lie symmetry group [9–11], Exp-function [12], exp$(\psi(x))$-expansion [13], modified Kudryashov [14] method etc.

Chen and Lin [15] established new exact solutions to the $(2 + 1)$-dimensional sine-Gordon equation by utilizing generalized tanh-function expansion method. Later, applying the Jacobi elliptic function method, Zhou et al [16] investigated the soliton solutions to the coupled sine-Gordon equation in nonlinear optics. Recently, Hosseini et al [14] determined the new exact solutions to the coupled sine-Gordon equations by applying the modified Kudryashov method. Rawashdeh [3], Yang et al [1] and Cenesiz et al [17] established exact wave solutions to the Burgers-type equation utilizing fractional reduced differential transform, local fractional Riccati differential equation, and first integral methods respectively.

In this article, we investigate the nonlinear space-time fractional sine-Gordon equation (STFSGE) and space-time fractional Burgers equation (STFBE) which have not been studied by using new generalized $(G'/G)$-expansion method. The purpose of this analysis is to achieve wide-ranging and typical wave solutions via hyperbolic, trigonometric, rational functions and establish wave profiles.

2. The preliminaries of fractional derivative

In this section, we discuss some definitions of fractional order $\alpha > 0$, namely the Riemann-Liouville, Caputo and conformable fractional derivatives.
**Definition 2.1:** The Riemann-Liouville fractional derivative of order $\alpha$ is defined as [18]:

$$D^\alpha f(x) = \frac{d^p}{dx^p} \left[ \frac{1}{\Gamma(p-\alpha)} \int_0^x (x-y)^{\alpha-1} f(y) dy \right],$$

where $\alpha > 0$ and $p - 1 < \alpha < p$.

**Definition 2.2:** The Caputo fractional derivative of order $\alpha$ is defined as [19]:

$$D^\alpha_0 f(x) = 1^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-y)^{m-\alpha-1} \left( \frac{d}{dy} \right)^m f(y) dy,$$

where $\alpha > 0$ for $m \in \mathbb{N}$, $m - 1 < \alpha < m$, $D^\alpha_0(\cdot)$ and $1^{m-\alpha}(\cdot)$ denote the Caputo fractional derivative and Caputo fractional integral operator respectively.

**Definition 2.3:** Recently, Khalil *et al* [20] proposed a new definition of derivative named conformable fractional derivative defined as:

$$T_\alpha(f)(y) = \lim_{\epsilon \to 0} \frac{f(y + \epsilon y^{1-\alpha}) - f(y)}{\epsilon},$$

For $f: [0, \infty) \to \mathbb{R}$ and for all $y > 0$, $\alpha \in (0, 1)$. The characteristics of this definition are given underneath as $g$ and $h$ are differentiable function at $y > 0$, then

- $T_\alpha(ag + bh) = aT_\alpha(h) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.
- $T_\alpha(y^q) = py^{q-\alpha}$ for all $q \in \mathbb{R}$.
- $T_\alpha(g, h) = gT_\alpha(h) + hT_\alpha(g)$.
- $T_\alpha \left( \frac{g}{h} \right) = \frac{hT_\alpha(g) - gT_\alpha(h)}{h^2}$.
- $T_\alpha(k) = 0$, for all constant functions $f(y) = k$.
- If $f$ is differentiable, then $T_\alpha(f)(y) = y^{1-\alpha} \frac{df(y)}{dy}$.

These are the important and useful properties of fractional order derivative. In this study, we will use these properties during establish general and broad-ranging wave solutions.

### 3. Algorithm of the method

Suppose a general NFDE as:

$$\mathcal{H}(u, D_0^{\alpha} u, D_0^{\beta} u, D_0^{\gamma} D_0^\tau u, D_0^{\nu} D_0^\mu u, D_0^\delta D_0^\lambda u, \ldots) = 0, \ 0 < \alpha, \beta, \gamma, \nu, \delta \leq 1,$$

where $\mathcal{H}$ is a polynomial in $u(x, t)$, $u = u(x, t)$ is unspecified function. It consists of the nonlinear terms and fractional-order derivatives. The subscripts represent the partial derivatives.

**Step 1:** Associating the real variables $(x, t)$ by a compound variable $\xi$ yields

$$u(x, t) = u(\xi), \quad \xi = \frac{\alpha x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)},$$

where $\nu$ is the wave speed, and $\alpha$ is the fractional order. Utilizing $u = u(\xi)$ by (5), the equation (4) becomes

$$\mathcal{R}(u, u', u'', u''', u'''', \ldots) = 0,$$

where $\mathcal{R}$ is the polynomial of $u$ having its derivatives.

**Step 2:** As per feasibility, integrate (6) several times give arbitrary constants supposed to be zero.

**Step 3:** We consider the solution structure of equation (6) as:

$$u(\xi) = \sum_{i=0}^{i=n} a_j (d + \mathcal{F})^j + \sum_{i=1}^{i=n} b_j (d + \mathcal{F})^{-j},$$

where $a_n$ and $b_n$ may separately be zero, but both cannot be zero together, $a_j$ for $j = 0, 1, 2, \ldots, n$ and $b_j$ for $j = 1, 2, \ldots, n$ and $d$ are arbitrary constants. Also, $\mathcal{F}(\xi)$ is defined as:
\[ F(\xi) = \frac{G'}{G} \]  

where \( G = G(\xi) \) satisfies the nonlinear ODE:

\[ DGG'' - EGG' - TG^2 - S(G')^2 = 0, \]  

where \( D, E, S \) and \( T \) be the indeterminate.

Step 4: To calculate the balance number \( n \), we check the homogeneous balance between the nonlinear highest order exponents and highest order derivatives arising in (6).

Step 5: Substituting (7) and (9) along with (8) into equation (6) having the value of \( n \) gained from Step 4, we get a polynomial in \((d + F)n\) and \((d + F)n(n = 0, 1, 2, \ldots)\). Afterward, we assimilate each coefficient of the attained polynomial to zero delivers a set of algebraic equations for \( a_i \) and \( b_j \), \( j = 0, 1, 2, \ldots, n, d, \delta \) and \( \nu \).

Step 6: The constants \( a_i \) for \( j = 0, 1, 2, \ldots, n \) and \( b_j \) for \( j = 1, 2, \ldots, n, d, \delta \) be estimated by calculating the algebraic equations gained in Step 5. As the solution of equation (10) is familiar, substituting \( a_i \) and \( b_j \), \( j = 0, 1, 2, \ldots, n, d, \omega \) and \( \nu \) into equation (8), we achieve standard and further-reaching wave solutions to the FPDE (4).

Using equation (10) and combining (9), it is reported:

Family 1: If \( E \neq 0, \psi = D - S \) and \( \tau = E^2 + 4T(D - S) > 0 \), then

\[ F(\xi) = \frac{G'}{G} = \frac{E}{2\psi} + \frac{K_1}{2\psi} \sinh\left(\frac{\sqrt{\tau}}{2\psi} \xi\right) + K_2 \cosh\left(\frac{\sqrt{\tau}}{2\psi} \xi\right), \]  

Family 2: If \( E = 0, \psi = D - S \) and \( \tau = E^2 + 4T(D - S) < 0 \), then

\[ F(\xi) = \frac{G'}{G} = \frac{E}{2\psi} - \frac{K_1}{2\psi} \sin\left(\frac{\sqrt{-\tau}}{2\psi} \xi\right) + K_2 \cos\left(\frac{\sqrt{-\tau}}{2\psi} \xi\right), \]  

Family 3: If \( E = 0, \psi = D - S \) and \( \tau = E^2 + 4T(D - S) = 0 \), then

\[ F(\xi) = \frac{G'}{G} = \frac{E}{2\psi} + \frac{K_2}{K_1 + K_2 \xi}, \]  

Family 4: If \( E = 0, \psi = D - S \) and \( \sigma = \psi T > 0 \), then

\[ F(\xi) = \frac{G'}{G} = \frac{\sqrt{\sigma}}{\psi} + \frac{K_1}{2\psi} \sin\left(\frac{\sqrt{\sigma}}{\psi} \xi\right) + K_2 \cosh\left(\frac{\sqrt{\sigma}}{\psi} \xi\right), \]  

Family 5: If \( E = 0, \psi = D - S \) and \( \sigma = \psi T < 0 \), then

\[ F(\xi) = \frac{G'}{G} = \frac{-\sqrt{\sigma}}{\psi} + \frac{K_1}{2\psi} \sin\left(\frac{-\sqrt{\sigma}}{\psi} \xi\right) + K_2 \cos\left(\frac{-\sqrt{\sigma}}{\psi} \xi\right), \]  

4. Formulation of solutions

In this paragraph, we discuss the applications of the two important models, such as the nonlinear STFSGE and STFBE.

4.1. The space-time fractional sine-Gordon equation

The STFSGE involves sine function and widely used in classical lattice dynamics, the propagation of waves, the extension of biological membrane, the spread of crystal defects, relativistic field theory, etc. Besides, the fractional order \( \alpha \) has notable impact in a bridge, a nonlinear beam and other vibration theories. We consider the STFSGE [12]:
where $\alpha$ is fractional order. If $\alpha = 1$, the equation (15) reduces to the integer-order sine-Gordon equation as:

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t) - \frac{1}{6}u^3(x, t) = 0; \quad t > 0, \quad 0 < \alpha < 1,$$

(16)

Applying $\xi = \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)}$, equation (15) transforms into the ODE as:

$$(\nu^2 - \delta^2)u'' + u - \frac{1}{6}u^3 = 0,$$

(17)

We achieve the balance number $n = 1$ through homogeneous balance technique. Thus

$$u(\xi) = a_0 + a_1(d + \mathcal{F}) + b_1(d + \mathcal{F})^{-1},$$

(18)

where $a_0, a_1, b_1, d$ are constants.

Substituting (18) together with (8) and (9) into (17), the left-hand side is modified to the polynomial in $(d + \mathcal{F})^n$ for $n = 0, 1, 2, 3, \ldots$ and $(d + \mathcal{F})^{-n}$ for $n = 0, 1, 2, 3, \ldots$ Picking up the particular coefficient of the reported polynomial and setting them to zero yields a set of equations for $a_0, a_1, b_1, d, \delta,$ and $\nu$. Set-1:

$$\delta = \pm \sqrt{-4E^2 \psi(4\nu^2E^2 - 4\nu^2T^2\psi)} - \frac{E}{4\psi}, \quad d = \pm \frac{6\psi}{\sqrt{-3E^2 - 12T^2\psi}},$$

$$b_1 = \pm \frac{\sqrt{-3E^2 - 12T^2\psi}}{2\psi},$$

(19)

where $\psi = D - S, \quad d, \quad D, \quad E, \quad S, \quad T$ are free parameters, $\nu$ is the wave celerity.

Set-2: $\delta = \pm \sqrt{4(E^2 + 4T\psi)(2D^2 + \nu^2E^2 + 4\nu^2T^2\psi)} - \frac{E}{4\psi}, \quad a_0 = \pm \frac{(E + 2d\psi)\sqrt{6E^2 + 24T^2\psi}}{E^2 + 4T^2\psi},$

$$a_1 = \pm \frac{12\psi}{\sqrt{6E^2 + 24T^2\psi}}, \quad b_1 = 0,$$

(20)

Set-3: $\delta = \pm \sqrt{2(E^2 + 4T^2\psi)(2D^2 + 2\nu^2E^2 + 8\nu^2T^2\psi)} - \frac{E}{2}\psi, \quad a_0 = 0, \quad a_1 = \pm \frac{6\psi}{\sqrt{6E^2 + 24T^2\psi}},$

$$b_1 = \pm \frac{\sqrt{6E^2 + 24T^2\psi}}{4\psi},$$

(21)

Set-4: $\delta = \pm \sqrt{(E^2 + 4T^2\psi)(2D^2 + \nu^2E^2 + 4\nu^2T^2\psi)} - \frac{E}{4\psi}, \quad a_0 = \pm \frac{(E + 2d\psi)\sqrt{6E^2 + 24T^2\psi}}{E^2 + 4T^2\psi},$

$$a_1 = 0, \quad b_1 = \pm \frac{12(Ed - T + d^2\psi)}{\sqrt{6E^2 + 24T^2\psi}},$$

(22)

Since $E \neq 0, \quad \psi = D - S$ and $\tau = E^2 + 4T(D - S) > 0$, we obtain the traveling wave solutions after inserting the values in (19) into (18) along with (10) and simplify as $(K_1 \neq 0, \quad K_2 = 0$ and $K_3 \neq 0, \quad K_4 = 0)$:

$$u_1(\xi) = \pm \frac{3\sqrt{\nu}}{\sqrt{-3E^2 - 12T^2\psi}} \tanh \left(\frac{\sqrt{\nu} \xi}{2\psi}\right) \pm \frac{\sqrt{-3E^2 - 12T^2\psi}}{\rho} \left\{ \tanh \left(\frac{\sqrt{\nu} \xi}{2\psi}\right) \right\}^{-1},$$

$$u_2(\xi) = \pm \frac{3\sqrt{\nu}}{\sqrt{-3E^2 - 12T^2\psi}} \coth \left(\frac{\sqrt{\nu} \xi}{2\psi}\right) \pm \frac{\sqrt{-3E^2 - 12T^2\psi}}{\rho} \left\{ \coth \left(\frac{\sqrt{\nu} \xi}{2\psi}\right) \right\}^{-1},$$

where $\xi = \pm \sqrt{-(E^2 + 4T\psi)(D^2 - \nu^2E^2 - 4\nu^2T^2\psi)} \frac{x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)}$ and $\nu$ is the wave celerity.
The above solutions can be re-written in terms of \((x, t)\)-variables as:

\[
\begin{align*}
    u_1(x, t) &= \pm \frac{3 \sqrt{\psi}}{\sqrt{-3 E^2 - 12 T \psi}} \tanh \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right) \\
    \pm \sqrt{-3 E^2 - 12 T \psi} \frac{\tanh \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right)}{\tau}^{-1}, \\
    \times u_2(x, t) &= \pm \frac{3 \sqrt{\psi}}{\sqrt{-3 E^2 - 12 T \psi}} \coth \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right) \\
    \pm \sqrt{-3 E^2 - 12 T \psi} \frac{\coth \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right)}{\tau}^{-1}.
\end{align*}
\]

Since \(E \neq 0\), \(\psi = D - S\) and \(\tau = E^2 + 4 T (D - S) < 0\), employing (18) accompanying with (11) and calculating from (19), we establish the desired wave solutions \((K_1 \neq 0, K_2 = 0 \text{ and } K_2 \neq 0, K_1 = 0)\) in terms of spatial and temporal variables in the next:

\[
\begin{align*}
    u_3(x, t) &= \pm \frac{3 \tau}{\sqrt{E^2 + 4 T \psi}} \tan \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right) \\
    \pm \sqrt{E^2 + 4 T \psi} \frac{\tan \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right)}{\tau}^{-1}, \\
    \times u_4(x, t) &= \pm \frac{3 \tau}{\sqrt{E^2 + 4 T \psi}} \cot \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right) \\
    \pm \sqrt{E^2 + 4 T \psi} \frac{\cot \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right)}{\tau}^{-1}.
\end{align*}
\]

Moreover, since \(E \neq 0\), \(\psi = D - S\) and \(\pi = E^2 + 4 T (D - S) = 0\), we establish further exact solutions by placing (19) into (18) together with (12) and using the primary wave variable \((x, t)\) and after simplification \((K_1 \neq 0, K_2 = 0 \text{ and } K_2 \neq 0, K_1 = 0)\), we derive

\[
\begin{align*}
    u_5(x, t) &= \pm \frac{6 \psi}{\sqrt{-3 E^2 - 12 T \psi}} \left\{ \frac{K_3}{K_1 + K_2 \left( \frac{\omega x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)} \right\} \\
    \pm \sqrt{-3 E^2 - 12 T \psi} \frac{K_3}{2 \psi} \left\{ \frac{K_1 + K_2 \left( \frac{\omega x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{\tau} \right\}^{-1}.
\end{align*}
\]

Furthermore, since \(E \neq 0\), \(\psi = D - S\) and \(\sigma = \psi T > 0\), we achieve considerable wave solutions in terms of elementary wave variables by setting (19) into (18) along with (13) \((K_1 \neq 0, K_2 = 0 \text{ and } K_2 \neq 0, K_1 = 0)\) in the underneath:

\[
\begin{align*}
    u_6(x, t) &= \pm \frac{3}{\sqrt{-E^2 - 4 \sigma}} \left[ E - 2 \sqrt{\sigma} - \sqrt{\psi} \tan \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right) \right] \\
    \pm \sqrt{-E^2 - 4 \sigma} \left[ E - 2 \sqrt{\sigma} - \sqrt{\psi} \tan \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right) \right]^{-1}, \\
    \times u_7(x, t) &= \pm \frac{3}{\sqrt{-E^2 - 4 \sigma}} \left[ E - 2 \sqrt{\sigma} - \sqrt{\psi} \coth \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right) \right] \\
    \pm \sqrt{-E^2 - 4 \sigma} \left[ E - 2 \sqrt{\sigma} - \sqrt{\psi} \coth \left( \frac{\sqrt{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right)}{2 \psi} \right) \right]^{-1}.
\end{align*}
\]

Finally, since \(E \neq 0\), \(\psi = D - S\) and \(\sigma = \psi T < 0\), we obtain the trigonometric function solutions by inserting (19) into (18) together with (14) and using the introductory wave variables \((K_1 \neq 0, K_2 = 0 \text{ and } K_2 \neq 0, K_1 = 0)\) in the following:
\[ u_8(x, t) = \pm \sqrt{-\frac{3}{E^2 + 4\sigma}} \left[ E - 2 \sqrt{-\sigma} + \sqrt{\tau} \tan \left( \frac{\sqrt{-\sigma}}{\psi} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu a^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right] \]
\[ \mp \sqrt{-3E^2 + 12\sigma} \left[ E - 2 \sqrt{-\sigma} - \sqrt{\tau} \cot \left( \frac{\sqrt{-\sigma}}{\psi} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu a^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right]^{-1}, \]
\[ u_9(x, t) = \pm \sqrt{-\frac{3}{E^2 + 4\sigma}} \left[ E - 2 \sqrt{-\sigma} - \sqrt{\tau} \cot \left( \frac{\sqrt{-\sigma}}{\psi} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu a^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right]^{-1}. \]

We emphasize that the traveling wave solutions \( u_8(x, t) \) to \( u_9(x, t) \) be meaningful and resourceful which are applicable in analyzing architectural points of view, vibration theory, applied mathematics, etc.

4.2. The space-time fractional Burgers equation

We consider the STFBE [21]:
\[ \frac{\partial^\alpha w(x, t)}{\partial t^\alpha} - 2w(x, t) \frac{\partial^\beta w(x, t)}{\partial x^\beta} - a \frac{\partial^{2\beta} w(x, t)}{\partial x^{2\beta}} = 0; \quad 0 \leq \alpha, \beta \leq 1, t, x, 0, \]

where \( \sigma \) is physical parameter; \( \alpha \) and \( \beta \) are fractional orders. The Burgers equation generally appears in the fields of applied sciences, namely, acoustic waves, ocean engineering, dynamics etc. It interprets the propagation of nonlinear acoustic waves, decay of non-planar shock waves.

If \( \alpha = \beta = 1 \), the equation (23) reduces to the integer-order Burgers equation as:
\[ \frac{\partial w(x, t)}{\partial t} - 2w(x, t) \frac{\partial w(x, t)}{\partial x} - a \frac{\partial^2 w(x, t)}{\partial x^2} = 0. \]

Applying \( \xi = \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu a^\alpha}{\Gamma(\alpha + 1)} \), equation (23) converts into the ODE for \( w(x, t) = w(\xi) \):
\[ \delta w' - 2\nu w w' - aw^2 w'' = 0, \]

where \( w' = \frac{dw}{d\xi} \). We integrate it one time, yields
\[ \delta w - \nu w^2 - aw^4 = 0. \]

We find the balance number \( n = 1 \). Therefore, the solution shape is
\[ w(\xi) = a_0 + a_1(d + \mathcal{F}) + b_1(d + \mathcal{F})^{-1}, \]

where \( a_0, a_1, b_1, d \) are constants.

In the same procedure, we obtain the new and resourceful sets of solutions as:

Set \(-1\): \( \delta = \pm \frac{2a\psi \sqrt{E^2 + 4T\psi}}{D} \), \( a_0 = \pm \frac{a\psi \sqrt{E^2 + 4T\psi}}{D} \), \( a_1 = \frac{a\psi}{D} \), \( b_1 = \pm \frac{a\psi (E^2 + 4T\psi)}{4D\psi} \).

It is expected to ascertain further advanced solution to the fractional Burgers equation for the early mentioned values accumulated in (28)–(31).
For $E \neq 0$, $\psi = D - S$ and $\tau = E^2 + 4T(D - S) > 0$, placing (28) into solution (27) along with (10) relation to spatial-temporal variables and after simplification, we obtain the wave solutions as ($K_1 = 0$, $K_2 = 0$ and $K_2 = 0$, $K_1 = 0$):

$$w_1(x, t) = \pm \frac{a \nu \sqrt{E^2 + 4T\psi}}{D} + \frac{a \nu \sqrt{T}}{2D} \coth \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right),$$

$$+ \frac{a \nu (E^2 + 4T\psi)}{2D} \left[ \frac{\sqrt{T}}{2} \coth \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right]^{-1},$$

$$\times w_2(x, t) = \pm \frac{a \nu \sqrt{E^2 + 4T\psi}}{D} + \frac{a \nu \sqrt{T}}{2D} \coth \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right),$$

$$+ \frac{a \nu (E^2 + 4T\psi)}{2D} \left[ \frac{\sqrt{T}}{2} \coth \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right]^{-1},$$

where

$$\xi = \pm \frac{2 a \nu \sqrt{E^2 + 4T\psi}}{D} \frac{x^\alpha}{\Gamma(\beta + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \quad \text{and} \quad \delta = \pm \frac{2 a \nu \sqrt{E^2 + 4T\psi}}{D}.$$

In addition, since $E \neq 0$, $\psi = D - S$ and $\tau = E^2 + 4T(D - S) < 0$, by means of (28) associating with (11) and from solution (27) subject to the initial variables, we establish the wave solutions in the underneath ($K_1 = 0$, $K_2 = 0$ and $K_2 = 0$, $K_1 = 0$):

$$w_3(x, t) = \pm \frac{a \nu \sqrt{E^2 + 4T\psi}}{D} - \frac{a \nu \sqrt{T}}{2D} \tan \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right),$$

$$- \frac{a \nu (E^2 + 4T\psi)}{2D} \left[ \frac{\sqrt{T}}{2} \tan \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right]^{-1},$$

$$\times w_4(x, t) = \pm \frac{a \nu \sqrt{E^2 + 4T\psi}}{D} + \frac{a \nu \sqrt{T}}{2D} \cot \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right),$$

$$+ \frac{a \nu (E^2 + 4T\psi)}{2D} \left[ \frac{\sqrt{T}}{2} \cot \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right]^{-1}.$$

Moreover, insasmuch as $E \neq 0$, $\psi = D - S$ and $\tau = E^2 + 4T(D - S) = 0$, we bring out more advanced wave solutions about the primary variable $(x, t)$ by introducing (28) into solution (27) together with (12) given in the underneath ($K_1 = 0$, $K_2 = 0$ and $K_2 = 0$, $K_1 = 0$):

$$w_5(x, t) = \pm \frac{a \nu \sqrt{E^2 + 4T\psi}}{D} + \frac{a \nu \psi}{D} \frac{K_2}{\Gamma(\alpha + 1)} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right).$$

On the other hand, forasmuch as $E = 0$, $\psi = D - S$ and $\sigma = \psi T > 0$, we formulate more useful wave solutions by setting (28) into (27) along with (13) concerning basic variable waves as ($K_1 = 0$, $K_2 = 0$ and $K_2 = 0$, $K_1 = 0$):

$$w_6(x, t) = \pm \frac{2 a \nu \sqrt{T}}{D} + \frac{a \nu \sqrt{\tau}}{2D} \left[ 2 \sqrt{\tau} + \frac{\sqrt{T}}{2} \tan \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right],$$

$$\times w_7(x, t) = \pm \frac{2 a \nu \sqrt{T}}{D} + \frac{a \nu \sqrt{\tau}}{2D} \tan \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right),$$

$$+ \frac{2 a \nu \sqrt{T}}{D} + \frac{a \nu \sqrt{\tau}}{2D} \cot \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right),$$

$$+ \frac{2 a \nu \sqrt{T}}{D} + \frac{a \nu \sqrt{\tau}}{2D} \cot \left( \frac{\sqrt{T}}{2} \left( \frac{\delta x^\alpha}{\Gamma(\alpha + 1)} + \frac{\nu t^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right]^{-1}.$$
Insofar as $E = 0$, $\psi = D - S$ and $\sigma = \psi T < 0$, we obtain the functional exact wave solutions by embedding \((28)\) into \((27)\) together with \((14)\) as $(K_0 = 0, K_2 = 0)$ and $(K_0 = 0, K_1 = 0)$:
Figure 4. Periodic wave profile of solution $u_4(x, t)$ for $\tau = -4$.

Figure 5. Periodic wave profile of solution $u_5(x, t)$ for $\tau = 0$.

Figure 6. Periodic multi-wave profile of solution $u_6(x, t)$ for $\sigma = 9$.

\[
w_6(x, t) = \pm \frac{2au}{D} \sqrt{-\sigma} + \frac{au}{2D} \left[ 2\sqrt{-\sigma} - \sqrt{\tau} \tan \left\{ \frac{-\sqrt{-\sigma}}{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right) \right\} \right] \\
- \frac{2au\Delta}{D} \left[ 2\sqrt{-\sigma} - \sqrt{\tau} \tan \left\{ \frac{-\sqrt{-\sigma}}{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right) \right\} \right]^{-1}, \\
\]

\[
w_9(x, t) = \pm \frac{2au}{D} \sqrt{-\sigma} + \frac{au}{2D} \left[ 2\sqrt{-\sigma} + \sqrt{\tau} \cot \left\{ \frac{-\sqrt{-\sigma}}{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right) \right\} \right] \\
- \frac{2au\Delta}{D} \left[ 2\sqrt{-\sigma} + \sqrt{\tau} \cot \left\{ \frac{-\sqrt{-\sigma}}{\psi} \left( \frac{\delta x^a}{\Gamma(\alpha + 1)} + \frac{\nu t^a}{\Gamma(\alpha + 1)} \right) \right\} \right]^{-1}. 
\]
The above reported wave solutions from $w_1(x, t)$ to $w_9(x, t)$ be fresh, novel and newly finding resources. The solutions are highly effective and efficient for describing physical systems, such as: modeling, acoustic waves, heat conduction, etc.

5. Graphical representation and physical interpretations

A graph is a crucial tool to depict the physical structures of the phenomena in the sense of real-world applications. The wave solutions are established in terms of trigonometric, hyperbolic and rational function for $\tau < 0, \tau > 0$ and $\tau = 0$ respectively. For the values $D = 5, E = 3, S = 3, T = 3, a = 3, \nu = 2$ and for the fractional order $\alpha = 1.0$, the results $u_1(x, t)$ and $u_2(x, t)$ reflect the kink and singular kink shape wave profile in the interval $-10 \leq x, t \leq 10$ sketched in figures 1 and 2.

The solutions $u_3(x, t)$ and $u_4(x, t)$ provide the singular periodic wave structured maintaining the boundary $-1 \leq x, t \leq 20$ and $-1 \leq x, t \leq 30$ exhibited in figures 3 and 4 for $D = 4, E = 3, S = 2, T = 3, a = 3, \nu = 2$ and fractional constant $\alpha = 0.04$.

The solution $u_5(x, t)$ for $D = 4, E = 3, S = 2, T = 5, \nu = 3, a = 3, K_3 = 2, K_4 = 2$ and $\alpha = 1.0$ within the range $-5 \leq x, t \leq 5$ and the solutions $u_6(x, t)$ for $D = 4, E = 3, S = 2, T = 3, a = 3, \nu = 3$ and $\alpha = 0.025$ within the range $-1 \leq x, t \leq 50$ provide periodic wave profile shown in figures 5 and 6.

For $D = 4, E = 3, S = 2, T = 3, a = 3, \nu = 3$ and $\alpha = 0.015$ within the range $-1 \leq x, t \leq 30$, the solution $u_7(x, t)$ is traced in figure 7. To end, for $\sigma < 0$, the solutions $u_8(x, t)$ represent exact periodic wave profiles and displayed in figure 8 for $D = 4, E = 3, S = 2, T = 3, a = 3, \nu = 3$ and $\alpha = 0.003$, within $-1 \leq x, t \leq 30$.

Remarkably, the fractional-order derivative $\alpha$ leads a significant contribution in modulating the amplitude of the soliton solutions.
6. Conclusion

In this work, we have investigated advanced and functional exact solitary wave solutions of FDEs, namely the sine-Gordon and the Burgers equations. The obtained solutions are kink, singular kink, and exact periodic type wave profiles. The Burgers equation models the nuclear fusion reactor, traffic flow, and steady rainfall on layered field soils. Besides, the sine-Gordon equation is applied for crystal dislocations, the propagation, and creation of ultra-short optical pulses, the motion of rigid pendulum attached to a stretched wire, etc. It also arises in nonlinear optics. The introduced method is effectively applicable for further studies for other nonlinear fractional models.

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References

[1] Yang X J, Gao F and Srivastava H M 2017 Exact travelling wave solutions for the local fractional two-dimensional Burgers-type equations Comput. Math. Appl. 73 203–10
[2] Akbar M A, Ali N H M and Tanjim T 2020 Adequate soliton solutions to the perturbed Boussinesq equation and the KdV-caudrey-dodd-gibbon equation J. King Saud Univ.-Sci. 32 2777–85
[3] Rawashdeh M S 2017 A reliable method for the space-time fractional Burgers and time-fractional cahn-allen equations via the FRDTM Adv. Differ. Equ. 2017 99
[4] Kadkhoda N and Jafari H 2019 An analytical approach to obtain exact solutions of some space-time conformable fractional differential equations Adv. Differ. Equ. 2019 428
[5] Islam M E and Akbar M A 2020 Stable wave solutions to the Landau–Ginzburg–Higgs equation and the modified equal width have equation using the IBSEF method. Arab J. Basic Appl. Sci. 27 270–8
[6] Roy R, Akbar M A and Wazwaz A M 2018 Exact wave solutions for the time fractional sharma-tasso-olver equation and the fractional Klein–Gordon equation in mathematical physics Opt. Quant. Electron. 50 25
[7] Akbar M A, Ali N H M and Roy R 2018 Closed form solutions of two nonlinear time fractional wave equations Results Phys. 9 1031–9
[8] Roy R et al 2020 Study on nonlinear partial differential equation by implementing MSE method. Global Scientific J. 8 1651–65 ISSN 2320–9186
[9] Jafari H et al 2017 Group classification of the time-fractional Kaup–Kupershmidt equation Scientia Iranica. 24 302
[10] Jafari H, Kadkhoda N and Baleanu D 2015 Fractional Lie group method of the time-fractional Boussinesq equation Nonlinear Dyn. 81 1569–74
[11] Jafari H, Kadkhoda N and Baleanu D 2020 Lie group theory for nonlinear fractional K(m,n) type equation with variable coefficients arXiv:2006.08014 [math.AP]
[12] Guner O and Bekir A 2015 Exact solutions of some fractional differential equations arising in mathematical biology Int. J. Biomath. 8 1550003
[13] Kadkhoda N and Jafari H 2017 Analytical solutions of the Gerdjikov–Ivanov equation by using exp (φ(ξ))-expansion method Optik 139 72–6
[14] Hosseini K, Mayeli P and Kumar D 2018 New exact solutions of the coupled sine-Gordon equations in nonlinear optics using the modified Kudryashov method J. Mod. Opt. 65 361–4
[15] Chen W-X and Lin J 2014 Some new exact solutions of (1+2)-dimensional sine-Gordon equation Abstr. Appl. Anal. 2014 645456
[16] Zhou Q et al 2017 The investigation of soliton solutions of the coupled sine-Gordon equation in nonlinear optics J. Mod. Opt. 64 1677–82
[17] Genesio Y et al 2017 New exact solutions of Burgers’ type equations with conformable derivative Waves Random Complex Media 27 103–16
[18] Das S 2011 Functional fractional calculus (Heidelberg, Germany: Springer Nature)
[19] Caputo M 1967 Linear models of dissipation whose q is almost frequency independent–II Geophysical J. Int. 13 529–39
[20] Khalil R et al 2014 A new definition of fractional derivative J. Comput. Appl. Math. 264 65–70
[21] Bekir A and Guner O 2014 The (G'/G)-expansion method using modified Riemann–Liouville derivative for some space-time fractional differential equations Ain Shams Engg. J. 5 959–65