Quantum Theories of Dilaton Gravity

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Abstract

Quantization of two-dimensional dilaton gravity coupled to conformal matter is investigated. Working in conformal gauge about a fixed background metric, the theory may be viewed as a sigma model whose target space is parameterized by the dilaton φ and conformal factor ρ. A precise connection is given between the constraint that the theory be independent of the background metric and conformal invariance of the resulting sigma model. Although the action is renormalizable, new coupling constants must be specified at each order in perturbation theory in order to determine the quantum theory. These constants may be viewed as initial data for the beta function equations. It is argued that not all choices of this data correspond to physically sensible theories of gravity, and physically motivated constraints on the data are discussed. In particular a recently constructed subclass of initial data which reduces the full quantum theory to a soluble Liouville-like theory has energies unbounded from below and thus is unphysical. Possibilities for modifying this construction so as to avoid this difficulty are briefly discussed.

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1. Introduction

Two-dimensional dilaton gravity is a useful model for developing understanding of the quantum properties of higher-dimensional gravity. This theory captures several of the essential features of its higher-dimensional cousins, and in particular has black hole solutions\cite{1-3} and Hawking radiation. Thus we might expect it to help us unravel some of the mysteries of real black holes. In \cite{3} a program to investigate the collapse and evaporation of two-dimensional black holes was initiated. It was shown that effects due to Hawking radiation of $N$ massless scalar fields are incorporated by including the Polyakov-Liouville term, arising from the one-loop matter functional integral, in the action. A semiclassical $1/N$ treatment of the resulting equations indeed produces an evaporating black hole. However, as was shown in \cite{1-5}, the large-$N$ semiclassical equations become singular inside the black hole and predictability fails. The resulting breakdown of the semiclassical limit necessitates consideration of higher order quantum corrections to the theory. In \cite{1} the one-loop semiclassical equations (including effects of the ghost measure) were analyzed for finite $N$, and such singularities were not found. However, the interesting physics occurs in a region where the one-loop approximation can not be justified by a small expansion parameter. Clearly, it is important to understand the quantum theory beyond the one-loop semiclassical level.

In this note we will discuss some aspects of the quantization of dilaton gravity. Much of the discussion is not new: previous work on this subject can be found in \cite{7-17}. An important point is that there are infinitely many theories of quantum dilaton gravity; although the theory is power counting renormalizable, it is non-predictable in that an infinite number of coupling constants must be specified to determine the theory, a finite number of which arise at each order in perturbation theory. Such theories can be characterized in terms of initial value data for the renormalization group equations. However not all choices of initial data correspond to physically sensible theories. For example, recently it was shown that a subset of the space of dilaton gravity theories are equivalent to soluble Liouville-like conformal field theories\cite{11,13-15,18}. We show that this particular subset does not have physical behavior: the ADM mass is unbounded from below\cite{10}. It would be of great interest to find modifications of this construction which do not have this difficulty. Constraints on the initial data required by physical sensibility are discussed.

\footnote{This objection is not relevant to the recent work of Russo, Susskind and Thorlacius\cite{19}, who use the transformation to Liouville theory as a trick to solve semiclassical equations, and are not trying to define the full quantum theory.}
2. Semiclassical Dilaton Gravity Coupled to Conformal Matter

Dilaton gravity coupled to $N$ conformal matter fields is given by the action $S_D + S_M$ with

\[ S_D = \frac{1}{2\pi} \int d^2x \sqrt{-g} \ e^{-2\phi} \left[ R + 4(\nabla \phi)^2 + 4\lambda^2 \right] \]

\[ S_M = -\frac{1}{4\pi} \int d^2x \sqrt{-g} \sum_{i=1}^{N} (\nabla f_i)^2 . \]

At the classical level the theory is exactly soluble [3]. This is easily seen by passing to conformal gauge,

\[ ds^2 = -e^{2\rho} d\sigma^+ d\sigma^- , \]

with $\sigma^\pm = \sigma^0 \pm \sigma^1$. The action then becomes

\[ S = \frac{1}{\pi} \int d^2\sigma \left[ -2\partial_+ e^{-2\phi} \partial_- (\rho - \phi) + \lambda^2 e^{2(\rho - \phi)} + \frac{1}{2} N \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right] . \]

Given an arbitrary solution $f = f_+ (\sigma^+) + f_- (\sigma^-)$ of the matter equations, the general solution for $\rho$ and $\phi$ is

\[ e^{-2\phi} = \frac{M}{\lambda} - \lambda^2 \int d\sigma^+ e^{w_+} \int d\sigma^- e^{w_-} \]

\[ -\frac{1}{2} \int d\sigma^+ e^{w_+} \int d\sigma^+ e^{-w_+} (\partial_+ f)^2 - \frac{1}{2} \int d\sigma^- e^{w_-} \int d\sigma^- e^{-w_-} (\partial_- f)^2 \]

\[ \rho - \phi = \frac{w_+ + w_-}{2} \]

where $w^+ (\sigma^+)$ and $w^- (\sigma^-)$ are arbitrary gauge functions. This solution includes the general black hole formed by collapse of $f$-matter.

Quantum effects of matter can be understood from the functional integral,

\[ Z = \int \mathcal{D}(g, \phi, f) e^{iS_D + iS_M} . \]

The measure for the matter fields is induced from the natural metric for matter fluctuations:

\[ < \delta f, \delta f > = \int d^2x \sqrt{-g} \delta f^2 \]

and depends on $g$. The matter functional integral can be performed and yields

\[ \int \mathcal{D}f \ e^{iS_M} = e^{iNS_{PL}} \]
with
\[ S_{PL} = -\frac{1}{96\pi} \int d^2x \sqrt{-g(x)} \int d^2x' \sqrt{-g(x')} R(x) \Box^{-1}(x,x') R(x') \] (2.9)
the Polyakov-Liouville action. This action incorporates both the Hawking radiation and its backreaction on the geometry\[20,3,21].

The semiclassical equations resulting from the large-$N$ action $S_D + S_{PL}$ have been analyzed in\[3-5,22-24] and have evaporating black holes as solutions. However, these equations become singular at a critical value of $\phi_c$, $e^{2\phi_c} = \frac{12}{N}$, signalling the need to go beyond a semiclassical analysis.

3. Quantum Dilaton Gravity

Quantization of dilaton gravity proceeds from the functional integral
\[ \int \mathcal{D}(g,\phi) e^{iS_D+iNS_{PL}}. \] (3.1)
from which the matter fields have been integrated out. A convenient approach to gauge fixing and quantization is to express the metric as a conformal rescaling of a fixed background metric $\hat{g}$,
\[ g = e^{2\rho} \hat{g}. \] (3.2)
The measure in (3.1) is defined from the metric on the tangent space to the space of fields\[.\] It is natural to derive this metric from the metric appearing in the kinetic term in the action (2.4). In the gauge (3.2) this leads to
\[ <\delta\phi,\delta\phi> = -\int d^2x \sqrt{-\hat{g}} e^{2\rho-2\phi} \delta\phi^2, \]
\[ <\delta\phi,\delta\rho> = \int d^2x \sqrt{-\hat{g}} e^{2\rho-2\phi} \delta\phi\delta\rho, \] (3.3)
\[ <\delta\rho,\delta\rho> = 0. \]
with a similar metric for the ghosts. With the gauge and measure choice (3.2) and (3.3) the functional integral becomes\[.\]
\[ \int \mathcal{D} e^{2\rho-2\phi} \hat{g} \det e^{2\rho-2\phi} \hat{g} P e^{iS_D+iNS_{PL}} \] (3.4)

\[^2\text{In this topologically trivial context there are no moduli.}\]
\[^3\text{We are grateful to Herman Verlinde for discussions on this point.}\]
where \( \det_{e^{2\rho - 2\phi} \hat{g}} P \) is the Fadeev-Popov determinant, and the subscripts on \( D \) and \( \det \) indicate the metric used to define them. The naive Ansatz for the transformation of the measures is

\[
D_{e^{2\rho - 2\phi} \hat{g}}(\rho, \phi) \det_{e^{2\rho - 2\phi} \hat{g}} P = e^{-24iS_L[\hat{g}, \rho - \phi]} D_{\hat{g}}(\rho, \phi) \det_{\hat{g}} P \tag{3.5}
\]

with the Liouville action

\[
S_L[\hat{g}, \rho - \phi] = \frac{1}{24\pi} \int d^2 x \sqrt{-\hat{g}} \left( (\nabla (\rho - \phi))^2 + \hat{R}(\rho - \phi) \right). \tag{3.6}
\]

Note that the \( \rho, \phi \) dependence of the ghost-gravity measure differs from that of the matter measure. We have motivated that difference here by choosing the measure naturally associated with the action (2.1). This choice is somewhat arbitrary in that other choices of measure (e.g. obtained by shifting \( \rho \) by different multiples of \( \phi \)) would also lead to covariant theories, corresponding to the freedom of adding finite, covariant, local counterterms at the one-loop level. However the physically sensibility of the choice we have made is confirmed by the observation [6] that it implies, at one-loop order, that black holes do not Hawking radiate ghosts or (non-dynamical) \( \rho \) or \( \phi \) modes.

The ambiguity associated with finite counterterms does not end at the one-loop level. Although dilaton gravity is renormalizable, this doesn’t buy as much in two dimensions as in higher dimensions since the fields \( \rho \) and \( \phi \) have dimension zero. In general, quantization will therefore introduce renormalizable counterterms of the form

\[
S = -\frac{1}{2\pi} \int d^2 x \sqrt{-\hat{g}} \left( G_{\mu\nu}(X^\lambda) \nabla X^\mu \cdot \nabla X^\nu + \frac{1}{2} \Phi(X^\lambda) \hat{R} + T(X^\lambda) \right), \tag{3.7}
\]

with \( X^\lambda = (\rho, \phi) \) (or more general coordinates on \( \rho, \phi \) space). In this sense, dilaton gravity is closer to its non-renormalizable cousins in higher dimensions than we might have hoped. Indeed, to define the theory we must specify an infinite number of coupling constants (giving the complete functions in (3.7)). Therefore the theory is non-predictable: a two-dimensional observer must perform an infinite number of experiments to determine the lagrangian.

In quantum dilaton gravity, one must therefore consider \( \sigma \)-model actions of the form (3.7). However, as discussed in several papers [23], not every action of the form (3.7) corresponds to a theory of gravity. One one way of stating the reason for this is two-dimensional

\[\text{4} \] One might also consider the addition of (1,1) operators with more than two derivatives, corresponding to massive string modes. We will ignore this possibility in the following.
general covariance, or equivalently background independence: given the decomposition (3.2), the theory should be invariant under the transformation

\[ \hat{g} \to e^{2\delta \omega} \hat{g}, \ \rho \to \rho - \delta \omega \] (3.8)

(quantum corrections may modify the form of the latter transformation). This is equivalent to the requirement that (3.7) be a \( c = 26 - N \) conformal field theory. To see the reason for this connection, note that quantization of the theory by fixing the gauge as in (3.2) leaves unfixed a group of residual diffeomorphisms which is isomorphic to the conformal group. These residual symmetries are generated by a set of operators in the theory which obey (two copies of) the Virasoro algebra (with \( c = 0 \) when ghosts and matter are included). The correlation functions will then exhibit the corresponding Ward identities, which may be taken as the defining characteristic of a conformal field theory.

Thus the couplings \( G, \Phi \) and \( T \) are restricted by conformal invariance to satisfy \( \beta \)-function equations. Except in very special cases it is not possible to solve these equations exactly; one must find a perturbation expansion. One such expansion is the loop expansion\(^5\) in \( \hbar \). From (2.1) it is evident that an equivalent loop expansion parameter for dilaton gravity is \( e^{2\phi} \), so this will be useful when \( e^{2\phi} \) is small. Also by working in the loop expansion, we can provide a more explicit description of the connection between background independence and conformal invariance.

At the classical level, the conditions on \( G, \Phi \) and \( T \) that follow from invariance under the transformation (3.8) should be precisely those which enable one to eliminate \( \hat{g} \) in (3.7) and rewrite it in a covariant form as a theory of dilaton gravity. We will now verify that this is indeed the case, and find the connection to conformal invariance. Let us consider the generalization of the transformation (3.8) incorporating some (as-yet-unspecified) variation \( \delta X^\mu \) of the \( X^\mu \)'s. Invariance of the action requires

\[
0 = \frac{2\pi}{\sqrt{-\hat{g}}} \left( \frac{\delta S}{\delta \hat{g}} \delta \hat{g} + \frac{\delta S}{\delta X^\mu} \delta X^\mu \right) \\
= (\partial_\mu \partial_\nu \Phi \nabla X^\mu \cdot \nabla X^\nu + \partial_\mu \Phi \Box X^\mu + 2T) \delta \omega \\
+ 2\pi \frac{\delta S}{\delta X^\mu} \delta X^\mu. \tag{3.9}
\]

\(^5\) This is related to the \( \alpha' \) expansion in string theory, but it is not identical because in string theory different powers of \( \alpha' \) conventionally appear in front of the three terms in (3.7). That reordering of the loop expansion is convenient in string theory but not in the present context.
where
\[
\frac{2\pi}{\sqrt{-g}} \frac{\delta S}{\delta X^\mu} = 2G_{\mu\nu} \Box X^\lambda + 2\Gamma_{\mu,\nu} \nabla X^\nu \cdot \nabla X^\lambda - \frac{1}{2} \partial_\mu \Phi \tilde{R} - \partial_\mu T
\] (3.10)
and spacetime indices are suppressed. Eq. (3.9) may be rewritten in a target-space covariant form:
\[
0 = \left( \beta^G_{0\mu\nu} \nabla X^\mu \cdot \nabla X^\nu + \frac{1}{2} \beta^\Phi_0 \tilde{R} + \beta^T_0 \right) \delta \omega
+ \frac{2\pi}{\sqrt{-g}} \frac{\delta S}{\delta X^\mu} (\delta X^\mu + \frac{1}{2} \nabla^\mu \Phi \delta \omega),
\] (3.11)
where the zeroth-order beta functions are
\[
\beta^G_{0\mu\nu} = \nabla_\mu \nabla_\nu \Phi,
\beta^\Phi_0 = \frac{1}{2} (\nabla \Phi)^2,
\beta^T_0 = \frac{1}{2} \nabla_\mu \Phi \nabla^\mu T - 2T.
\] (3.12)
One sees that (to leading order) background independence and conformal invariance are equivalent on-shell. More generally, (3.11) shows that the theory is background independent even off-shell if the \(\beta\)-functions vanish,
\[
\beta^G_0 = \beta^\Phi_0 = \beta^T_0 = 0,
\] (3.13)
and if the variation \(\delta X^\mu\) is given by
\[
\delta X^\mu = -\frac{1}{2} \nabla^\mu \Phi \delta \omega.
\] (3.14)

The vanishing of the \(\beta\)'s can be interpreted geometrically: \(\beta^G_0 = 0\) implies that \(k_\nu = \nabla_\nu \Phi\) is a Killing vector, while \(\beta^\Phi = 0\) implies \(k\) is null. Every two-dimensional geometry with a null Killing vector is flat. If the Killing vector is the gradient of a globally defined function, the space is flat two-dimensional Minkowski space up to possible periodic identifications in the direction transverse to \(k\). It also follows that \(\Phi\) runs from \(+\infty\) to \(-\infty\) as \(X\) runs over the target space.

In order to check that the general background-independent lagrangian (3.7) can be rewritten as a theory of dilaton gravity it is necessary to specify \(\rho\) and \(\phi\) in terms of the general fields \(X^\lambda\). This identification is determined from (3.14) which we would like to identify with the Weyl transformation as in (3.8). Under Weyl transformations one should find \(\delta \phi = 0\) while \(\delta \rho = -\delta \omega\). The correct transformation law for \(\phi\) is obtained by defining
\[
\phi = \phi(\Phi(X))
\] (3.15)
for a smooth single valued function $\phi(\Phi)$ (for example $\phi = \Phi$). There is no preferred function, corresponding to the inherent ambiguity in field redefinitions of $\phi$. However we shall see below that regularity of $G$ in the $(\rho, \phi)$ coordinates requires that $\phi$ is a monotonically increasing function of $\Phi$. This then implies that the range of $\phi$ is unrestricted.\footnote{Note however that in the action (2.1), $\Phi = -2e^{-2\phi}$. If only finite real values of $\phi$ are allowed, then the sigma-model target space has a boundary at $\Phi = 0$.} One finds using (3.14) and $\beta_0^\Phi = 0$ that

$$\delta\phi = \frac{\partial \phi}{\partial \Phi} \nabla_\mu \Phi \delta X^\mu = -\frac{1}{2} \frac{\partial \phi}{\partial \Phi} (\nabla \Phi)^2 \delta \omega = 0, \quad (3.16)$$

as desired.

To define $\rho$ as a function of the $X^\mu$'s, note that the flatness of $G$ implies the existence of a second Killing vector $\ell_\mu$ obeying

$$\nabla_\mu \ell_\nu = 0 \quad (3.17)$$

and

$$\ell^\mu k_\mu = 1. \quad (3.18)$$

The conformal factor $\rho$ may then be defined by

$$\rho(X) = 2 \int^X \ell_\mu dX^\mu. \quad (3.19)$$

The variation of $\rho$ is then

$$\delta \rho = 2 \ell_\mu \delta X^\mu = -\ell_\mu \nabla^\mu \Phi \delta \omega = -\delta \omega. \quad (3.20)$$

in agreement with (3.8). Note that single-valuedness of $\rho$ requires that there are not periodic identifications of $X$ as discussed below equation (3.14). The range of $\rho$ is then unrestricted. There is also an ambiguity in $\rho$ corresponding to shifts by arbitrary functions of $\phi$.

Constraints may now be obtained on the components of $G$ in the $\rho, \phi$ coordinates. Firstly, using $(\ell_\rho, \ell_\phi) = (\frac{1}{2}, 0)$ and $(k_\rho, k_\phi) = (0, \frac{\partial \phi}{\partial \Phi})$, (3.18) becomes

$$G^{\rho\phi} = 2 \frac{\partial \phi}{\partial \Phi}. \quad (3.21)$$
Secondly $\beta^T_0 = 0$ implies
\[ 0 = \left( \frac{\partial \Phi}{\partial \phi} \right)^2 G_{\phi\phi} \]  
(3.22)
or equivalently
\[ G_{\rho\rho} = 0. \]  
(3.23)
Finally, given (3.23) and (3.21) flatness of $G$ requires that
\[ G_{\phi\phi} = K(\phi) \]  
(3.24)
and is independent of $\rho$.

The equation $\beta^T_0 = 0$ for $T$ is, in $\rho, \phi$ coordinates simply
\[ \frac{\partial T}{\partial \rho} = 2T. \]  
(3.25)
The general solution to this is
\[ T = e^{2\rho U(\phi)}, \]  
(3.26)
for some $\rho$-independent function $U$.

Using (3.23), (3.21), (3.24) and (3.25), (3.7) may be rewritten in $\rho, \phi$ coordinates as
\[ S = -\frac{1}{2\pi} \int d^2x \sqrt{-\hat{g}} \left( K \nabla \hat{\phi} \cdot \nabla \hat{\phi} + \frac{\partial \Phi}{\partial \phi} \nabla \hat{\phi} \cdot \nabla \rho + \frac{1}{2} \Phi \hat{R} + e^{2\rho U} \right). \]  
(3.27)
Defining
\[ g = e^{2\rho} \hat{g}, \]  
(3.28)
and integrating by parts one finds
\[ S = -\frac{1}{2\pi} \int d^2x \sqrt{-g} \left( K \nabla \phi \cdot \nabla \phi + \frac{1}{2} \Phi \hat{R} + U \right). \]  
(3.29)

Eq. (3.29) is the most general power-counting renormalizable theory of two-dimensional gravity coupled to a single scalar field $\phi$. Thus we see explicitly that classical background independence or conformal invariance of the sigma model of the form (3.7) implies equivalence to a theory of gravity. Conversely, by repeating the preceding steps in reverse, it can be seen that every theory of gravity coupled to a scalar field is equivalent to a conformally invariant sigma model of the form (3.7).
As discussed above, this equivalence persists at the quantum level. Quantum conformal invariance requires that the quantum corrected beta functions vanish. To first order in the $\hbar$ expansion these take the form (temporarily reinstating $\hbar$)

$$
\beta^T = \frac{1}{2} \nabla_{\mu} \Phi \nabla^{\mu} T - 2 T - \frac{\hbar}{4} \Box T + \cdots = 0,
$$

$$
\beta^G_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} \Phi + \frac{\hbar}{2} R_{\mu\nu} + \cdots = 0,
$$

$$
\beta^\Phi = \frac{1}{2} (\nabla \Phi)^2 - \frac{\hbar}{4} \Box \Phi + 2 \hbar \gamma + \cdots = 0,
$$

where $\mathcal{R}$ is the curvature of $G$ and we define

$$
\gamma \equiv \frac{N - 24}{12}. \tag{3.31}
$$

There are many solutions of the $\beta$-function equations. This may be viewed as a target space initial value problem. For example we could specify $G$, $\Phi$ and $T$ as functions of $\phi$ at fixed $\rho$, and then use the $\beta$ function equations to determine them at other values of $\rho$.\footnote{The target space initial data is subject to the usual constraints of a covariant theory. These can be used to solve for $G$ and $\Phi$ (up to a finite number of integration constants), leaving $T$ as the only freely specifiable function. There are an infinite number of further functions (corresponding to the massive string modes) in the most general setting.}

In general, explicitly writing down these equations, let alone solving them, is a difficult task. Nonetheless, we can hope to make progress by considering special cases where they simplify.

In particular, it is worth pointing out that the loop expansion is not the only way to investigate solutions of the $\beta$ function equations. An alternative procedure is to begin with a solution that is known to be an exact conformal field theory on other grounds, and then perturb it in the direction defined by some marginal operator. In this case the small expansion parameter is the coefficient of the perturbation. For the dilaton gravity theory (2.1), this expansion is good when the cosmological constant term, and hence $e^{-2\phi}$, is small. This is in contrast to the usual loop expansion about the linear dilaton vacuum, which is good when $e^{-2\phi}$ is large. There are thus in general two inequivalent (and possibly overlapping) expansion schemes that may be helpful in finding particular solutions. We will consider one such special case where the marginal operator expansion can be used in the next section.

However, even if we could find all solutions to the \( \beta \)-function equations, not all of these covariant theories are useful for studying black hole physics. We are interested in those theories which reduce to dilaton gravity in the classical \((e^{2\phi} \to 0)\) limit. The subleading (one-loop) \( e^{2\phi} \) dependence is also constrained by the demand that Hawking radiation should proceed only by matter emission, and to leading non-trivial order should take the form described in [3,6]. These conditions are enforced by requiring the full action (3.7) to duplicate

\[
S_D + NS_L \left[ \hat{g}, \rho \right] - 24S_L \left[ \hat{g}, \rho - \phi \right]
\]

to one-loop level at weak coupling. Rewriting (3.32) in the form (3.7), this gives

\[
G_{\mu\nu} \overset{\phi \to -\infty}{\to} \left( -4e^{-2\phi} + 2 \frac{2e^{-2\phi} - 2}{-\gamma} \right) + O(e^{2\phi}),
\]

\[
\Phi \overset{\phi \to -\infty}{\to} -2e^{-2\phi} - 4\phi - 2\gamma\rho + O(e^{2\phi}),
\]

and

\[
T \overset{\phi \to -\infty}{\to} -4\lambda^2 e^{2\rho - 2\phi} + O(1).
\]

There are still further conditions which must be met if the theory is to serve as a useful model for studying black hole dynamics. One such physical condition is that the higher order corrections preserve the existence of a stable ground state.

In summary, we wish to study quantum theories of dilaton gravity described by actions of the form (3.7), where \( G, \Phi \) and \( T \) are constrained by the requirements that

1. They must agree with the couplings of classical dilaton gravity to leading order in \( e^{2\phi} \).
2. Hawking radiation should proceed only by matter emission, and to leading non-trivial order the action should take the form described in [3,6].
3. They should define a \( c = 26 - N \) conformal field theory.
4. There must be a ground state of the full quantum theory.

Further investigations of the physical sensibility of these models may lead to yet further restrictions.
4. Soluble Models

One approach to solving (3.30), pursued by deAlwis [13,15] and Bilal and Callan[14], is to search for appropriate exact conformal theories. In particular, the asymptotic target space metric (3.33) is easily seen to be lorentzian and flat. They therefore consider the special case of exactly flat lorentzian metrics (in this section we take $\hat{g}$ to be flat and use coordinates $\sigma^\pm$)

$$G_{\mu\nu}\partial_+ X^\mu \partial_- X^\nu = -\gamma \partial_+ X \partial_- X + \frac{1}{\gamma} \partial_+ Y \partial_- Y.$$  (4.1)

The relationship of the natural flat coordinates $X$ and $Y$ with $\rho$ and $\phi$ will be described momentarily. To leading order, eqs. (3.30) then imply

$$\Phi = -2\gamma X$$  (4.2)

and that the operator corresponding to the tachyon background,

$$T = -4\lambda^2 e^{2(X-Y/\gamma)}$$  (4.3)

is (1,1). In fact, the resulting action

$$S = \frac{1}{\pi} \int d^2 \sigma \left[ -\gamma \partial_+ X \partial_- X + \frac{1}{\gamma} \partial_+ Y \partial_- Y + \lambda^2 e^{2(X-Y/\gamma)} \right]$$  (4.4)

defines an exact CFT, hence an exact solution of (3.30).

The lagrangian (4.4) is also classically soluble. The change of coordinates

$$V = X - \frac{Y}{\gamma}$$  \hspace{1cm} $$U = \frac{1}{2}(\gamma X + Y)$$  (4.5)

puts it in the form

$$S = \frac{1}{\pi} \int d^2 \sigma \left( -2\partial_+ U \partial_- V + \lambda^2 e^{2V} \right)$$  (4.6)

which is the same as the gravitational part of (2.4) if one identifies

$$V \leftrightarrow \rho - \phi$$

$$U \leftrightarrow e^{-2\phi}.$$  \hspace{1cm} (4.7)

This has the general classical solution

$$U = u_+ + u_- - \lambda^2 \int d\sigma^+ e^{w_+} \int d\sigma^- e^{w_-}$$

$$V = \frac{w_+ + w_-}{2}$$  (4.8)
where \( u_+ \) and \( u_- \) are determined by the classical matter distribution as in (2.3).

\( X \) and \( Y \) are related to the usual variables \( \rho \) and \( \phi \) for dilaton gravity via a target space coordinate transformation, which restores the action to the original form (3.33)-(3.35) (up to order \( e^{2\phi} \)). Let

\[
Y = - \int d\phi \sqrt{4e^{-4\phi} - 4(\gamma + 2)e^{-2\phi} + 2(\gamma + 2)} + \delta Y
\]
\[
X = \rho + \frac{e^{-2\phi} + 2\phi}{\gamma} + \delta X
\]

where

\[
\delta Y, \delta X \sim O(e^{2\phi})
\]

and

\[
\delta \left( X - \frac{Y}{\gamma} \right) \sim O(e^{4\phi})
\]

It is straightforward to verify that the target space fields \( G, \Phi \) and \( T \) take the form (3.33)-(3.35) in terms of the new variables.

This relation can be illustrated by considering the static solutions. Taking \( w_+ = \lambda \sigma^+, \ w_- = -\lambda \sigma^- \), and \( u_+ + u_- = \mu + F \sigma \), (with \( F \) a constant and \( \sigma^+ - \sigma^- = 2 \sigma \)) in (4.8), one finds

\[
U = \mu + F \sigma + e^{2\lambda \sigma}
\]

and

\[
V = \lambda \sigma .
\]

One can solve for the \( \sigma \to \infty \) behavior of \( \rho \) and \( \phi \) by expanding (4.9) in \( e^{2\phi} \) and equating the expansions to (4.12), (4.13). One finds for large negative \( \phi \)

\[
U = \frac{\gamma}{2} \rho + e^{-2\phi} + \left( \frac{\gamma}{2} + 2 \right) \phi + \frac{\gamma(\gamma + 2)}{16} e^{2\phi} + \ldots
\]
\[
V = \rho - \phi - \frac{\gamma + 2}{8} e^{2\phi} + \ldots
\]

which then implies

\[
\phi = -\lambda \sigma - \frac{1}{2} \left[ \mu + \left( F + \frac{\gamma \lambda}{2} + 2 \lambda \right) \sigma \right] e^{-2\lambda \sigma} + O(e^{-4\lambda \sigma})
\]
\[
\rho = -\frac{1}{2} \left[ \mu - \frac{\gamma + 2}{4} + \left( F + \frac{\gamma \lambda}{2} + 2 \lambda \right) \sigma \right] e^{-2\lambda \sigma} + O(e^{-4\lambda \sigma})
\]
For \( F + \frac{\gamma^2}{2} + 2\lambda = 0 \) these are asymptotic solutions similar to the usual black holes, modulo the extra term in \( \rho \). The ADM mass may be computed in the usual fashion, by linearizing the constraints about a given solution, e.g. that with \( \mu = 0 \). One finds

\[
M = 2e^{\lambda(\sigma^+ - \sigma^-)}(\lambda \delta \rho + \partial_+ \delta \phi - \partial_- \delta \phi)
\]

where \( \delta \rho \) and \( \delta \phi \) are the asymptotically vanishing deviations of \( \rho \) and \( \phi \) from their \( \mu = 0 \) values. The resulting mass is the expected

\[
M = \lambda \mu.
\]

The solutions with \( F + \frac{\gamma^2}{2} + 2\lambda \neq 0 \) have infinite mass \(^8\), like the black holes in equilibrium with incoming radiation described in \(^9\). At first sight it appears that one has made great progress by finding an exact conformal field theory describing black holes together with Hawking radiation. However, further examination reveals a difficulty with the identification of the Liouville-like model, \((4.4)\), as a physical theory of dilaton gravity. In the conformal field theory variables \( U \) and \( V \), the solution \((4.12), (4.13)\) is non-singular independent of the value of \( \mu \), and for all \( \sigma \): \( U = 0 \) is a regular value for the field. Thus, in particular, there is nothing wrong with taking \( \mu \) large and negative\(^9\). This means that the mass \( M \) is unbounded from below, and the theory doesn’t have a ground state, in contradiction with our requirement \((4.10)\). Interpretation of the theory \((4.4)\) as a physical model for black hole physics is therefore very problematic. One might attempt to evade this problem by constraining the range of \( U \) and \( V \) in such a way that the mass is bounded. However this is throwing out the baby with the bath water: the resulting constrained theory is very complicated and is unlikely to correspond to a soluble conformal field theory.

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\(^8\) Although a finite mass \( M' \) can be defined for solutions with any value of \( F \) by linearizing around a reference solution with that fixed value of \( F \). \( M' \) is then easily seen to be invariant under asymptotically trivial field redefinitions, which explains why \( M \) in \((4.17)\) agrees with the mass defined directly in the \( U - V \) variables from \((4.12) \) and \((4.13)\).

\(^9\) In fact the theory we wish to describe presumably does not have static solutions with a continuously varying mass of any sign: black holes should evaporate.

\(^10\) This is in agreement with the observations of \([14, 15]\) that the Hawking radiation rate asymptotes to a constant value in the future, rather than shutting off when the mass reaches zero. The conclusion here is not affected by quantum corrections to the mass formula discussed in \([26]\).
5. Conclusions

We have argued that there are an infinite number of quantum theories of dilaton gravity. These are parameterized by initial value data for the renormalization group equations (3.30) that produce solutions with the asymptotic behavior (3.33), (3.34). One would like to find further criteria to narrow the class of solutions, and a more explicit description of the allowed solutions. One approach is to specify initial data corresponding to a known soluble conformal field theory, but we have found that the simplest such choice leads to a theory with unphysical behavior.

Having said this, we would like to add that we nevertheless find the basic idea of deAlwis and Bilal and Callan of fixing the higher order corrections in order to obtain a soluble conformal field theory very appealing. If this were indeed possible, it would certainly greatly clarify the quantum properties of two-dimensional black holes. While their attempt was not completely successful, perhaps some variant on this idea may yet work. For example there are, in addition to (4.3), a one parameter family of solutions of the tachyon equation of motion which correspond to (1, 1) operators. Some of these vanish at weak coupling, and could be used to perturb the lagrangian. Alternately, the target space metric need not be flat beyond leading order. Thus one might consider a conformal theory with an asymptotically flat but curved target space, such as the $SU(1,1)/U(1)$ black hole (perturbed by a tachyon), as a candidate quantum extension of classical dilaton gravity\textsuperscript{11}.

On the other hand there is no obvious a priori reason to believe that a physical theory describing black hole evaporation should correspond to a simple conformal field theory. (Indeed the existence of a unique static solution suggests the opposite.) In general, one must confront the difficult problems of finding physically sensible initial data and solving the renormalization group equations (3.30). One approach is to try to generally characterize the space of physically sensible theories, together with the behavior of black holes in such theories. Another approach is to try find and analyze special examples of such theories. Such special examples may be obtained from low-energy limit of two-dimensional string theory, or extremal black holes in higher-dimensional string theory. Alternately,

\textsuperscript{11} Much more generally, studies of (tree level) $c = 1$ matrix models have produced (albeit in an unusual and indirect manner) infinite families of exact conformal field theories corresponding to a metric, dilaton and tachyon in a two-dimensional target space. Perhaps this spectacular technological development has applications in the present context.
extended supersymmetry might be used to constrain the space of theories. All of these paths deserve further exploration.

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