Symmetries of Vector Perturbations during the de Sitter Epoch

M. Biagetti\textsuperscript{a}, A. Kehagias\textsuperscript{b}, E. Morgante\textsuperscript{a}, H. Perrier\textsuperscript{a} and A. Riotto\textsuperscript{a}

\textsuperscript{a} Department of Theoretical Physics and Center for Astroparticle Physics (CAP)  
24 quai E. Ansermet, CH-1211 Geneva 4, Switzerland

\textsuperscript{b} Physics Division, National Technical University of Athens,  
15780 Zografou Campus, Athens, Greece

Abstract

We analyze the class of models where a suitable coupling between the inflaton field and the vector field gives rise to scale-invariant vector perturbations. We exploit the fact that the de Sitter isometry group acts as conformal group on the three-dimensional Euclidean space for the super-Hubble fluctuations in order to characterize the correlators involving the inflaton and the vector fields.
1 Introduction

It has recently become clear that symmetries play a crucial role in characterizing the properties of the cosmological perturbations generated by an inflationary stage \[1\]. During inflation the de Sitter isometry group acts as conformal group on \( \mathbb{R}^3 \) when the fluctuations are on super-Hubble scales. During such a stage, correlators are constrained by conformal invariance as the SO(1,4) isometry of the de Sitter background is realized as conformal symmetry of the flat \( \mathbb{R}^3 \) sections \[2–6\]. This happens when the cosmological perturbations are sourced by light scalar fields other than the inflaton (the field that drives inflation). In the opposite case in which the inflationary perturbations originate from only one degree of freedom, conformal consistency relations among the inflationary correlators have also been recently investigated \[7–12\]. Similarly, one may study the symmetries enjoyed by the Newtonian equations of motion of the non-relativistic dark matter fluid coupled to gravity which give rise to the phenomenon of gravitational instability and reveal consistency relations involving the soft limit of the \((n+1)\)-correlator functions of matter and galaxy overdensities \[13, 14\].

On the other hand, there has been recently a lot of interest in models which can produce vector field perturbations during inflation. There are mainly two reasons. On one side, one might hope to generate large-scale magnetic fields if vector field perturbations are excited during a de Sitter stage \[15\]; on the other side claims of broken statistical invariance of the CMB modes, as hinted also by the recent Planck satellite data \[16\], have put forward the proposal that such a breaking might be due to vector fields \[17–26\].

In this paper we shall investigate the symmetry properties of the vector field models with a kinetic term given by

\[
\mathcal{L} = -\frac{1}{4} I^2(\phi) F_{\mu\nu}^2, \tag{1.1}
\]

where \( \phi \) indicates the inflaton field. Vector perturbations can be generated if the function \( I(\phi) \) has the appropriate time dependence \[27, 28\]. In particular, if \( I \sim a^n \), being \( a \) the scale factor, magnetic modes are generated during inflation with a scale-invariant spectrum for \( n = 2 \) and \( n = -3 \). In the first case, however, a too large electromagnetic coupling constant is generated during inflation \[29, 30\], while the second case implies a too large energy density in the electric modes. Nevertheless, some recent work have investigated the cross-correlations between primordial perturbations and large-scale magnetic fields induced by the coupling (1.1) \[31–35\] as well as the (possibly too large) contribution from the vector modes to the anisotropic power spectrum of the curvature perturbation \[36–40\]. For these reasons, the reader should be aware that it might not be healthy to identify the vector field with the electromagnetic field. At any rate, the goal of this paper is to analyze the conformal symmetries enjoyed by the vector perturbations on super-Hubble scales. We will see that the action associated to the Lagrangian (1.1) respects the conformal group on \( \mathbb{R}^3 \) when the fluctuations are on super-Hubble scales and therefore the correlators involving the inflaton and the vector fields must be invariant under conformal transformations of Euclidean three-space on the future boundary. This will allow us to write down the appropriate Ward
identities as well as the two- and three-point correlator between the inflaton field and the vector fields, thus explaining some features found recently in the literature.

The paper is organized as follows. In section 2, as a warm-up, we describe the conformal symmetries of de Sitter for the action of a massive scalar field; in section 3 we analyze the conformal symmetries of de Sitter in the presence of a vector field, and describe the correlators in section 4. Finally, section 5 contains our conclusions.

2 Conformal symmetries of de Sitter and the scalar field

Let us start by recalling some of the properties of the conformal symmetry in de Sitter. Conformal invariance in three-dimensional space $\mathbb{R}^3$ is connected to the symmetry under the group $SO(1, 4)$ in the same way conformal invariance in a four-dimensional Minkowski spacetime is connected to the $SO(2, 4)$ group. As $SO(1, 4)$ is the isometry group of de Sitter spacetime, a conformal phase during which fluctuations were generated could be a de Sitter stage. In such a case, the kinematics is specified by the embedding of $\mathbb{R}^3$ as flat sections in de Sitter spacetime. The de Sitter isometry group acts as conformal symmetry of the flat $\mathbb{R}^3$ sections. Correlators are expected to be constrained by conformal invariance. All these reasonings apply in the case in which the cosmological perturbations are generated by light scalar fields other than the inflaton, in particular vector perturbations. Indeed, it is only in such a case that correlators inherit all the isometries of de Sitter.

Let us first describe the case of the scalar field. The de Sitter space in conformally flat coordinates is described by the metric

$$ds^2 = \frac{1}{H^2 \tau^2} (-d\tau^2 + d\vec{x}^2). \quad (2.1)$$

It can easily be checked that the transformations

$$x \rightarrow x' = a_i + M^j_i x_j, \quad (2.2)$$

$$x_i \rightarrow x'_i = \lambda x_i, \quad \tau \rightarrow \tau' = \lambda \tau, \quad (2.3)$$

$$x_i \rightarrow x'_i = \frac{x_i + b_i (-\tau^2 + \vec{x}^2)}{1 + 2\vec{b} \cdot \vec{x} + b^2 (-\tau^2 + \vec{x}^2)}, \quad \tau \rightarrow \tau' = \frac{\tau}{1 + 2\vec{b} \cdot \vec{x} + b^2 (-\tau^2 + \vec{x}^2)}, \quad (2.4)$$

are isometries of the de Sitter metric. They correspond to translations (by a vector $\vec{a}$), rotations ($M^j_i$), dilations (by a real parameter $\lambda$) and special conformal transformations (parametrized by a real vector $\vec{b}$), respectively. In particular, for infinitesimal parameters and for super-Hubble scales ($\tau \rightarrow 0$), the special conformal transformations read

$$\vec{x} \rightarrow \vec{x}' = \vec{x} + \delta \vec{x} = \vec{x}(1 - 2\vec{b} \cdot \vec{x}) + \vec{b} \vec{x}^2, \quad \tau \rightarrow \tau' = \tau + \delta \tau = \tau(1 - 2\vec{b} \cdot \vec{x}). \quad (2.5)$$
and we recognize the 3D special conformal transformations in flat \( \mathbb{R}^3 \). We also note that special conformal transformations can be written as

\[
\begin{align*}
\tau' &= \frac{\tau}{-\tau^2 + x'^2}, \\
x'_i &= \frac{x_i}{-\tau^2 + x^2} + b_i
\end{align*}
\]

(2.6)

(2.7)

and therefore they can be generated by

\[
\text{(inversion)} \times \text{(translation)} \times \text{(inversion)},
\]

(2.8)

where by inversion we mean

\[
\tau \rightarrow \tau' = \frac{\tau}{-\tau^2 + x^2}, \quad x_i \rightarrow x'_i = \frac{x_i}{-\tau^2 + x^2}.
\]

(2.9)

In other words, the special conformal transformations are generated by the transformation chain

\[
\begin{align*}
\tau \rightarrow \tau' &= \frac{\tau}{-\tau^2 + x^2}, \quad x_i \rightarrow x'_i = \frac{x_i}{-\tau^2 + x^2}, \quad \text{(inversion)} \tag{2.10} \\
\tau' \rightarrow \tau'' &= \tau', \quad x'_i \rightarrow x''_i = x'_i + b_i, \quad \text{(translation)} \tag{2.11} \\
\tau'' \rightarrow \tau''' &= \frac{\tau''}{-\tau''^2 + x''^2}, \quad x''_i \rightarrow x'''_i = \frac{x''_i}{-\tau''^2 + x''^2}, \quad \text{(inversion)}. \tag{2.12}
\end{align*}
\]

Now, the action for a scalar field

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \left( -\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right)
\]

(2.13)

and on a de Sitter background it is written as

\[
S = \frac{1}{2} \int d^3x d\tau \left( \eta^\mu_\nu \partial_\mu \phi \partial_\nu \phi - \frac{m^2}{H^2 \tau^2} \phi^2 \right).
\]

(2.14)

It is easy to verify that (2.14) is invariant under the transformations (2.4) if \( \phi(\bar{x}, \tau) \) transforms as

\[
\phi(\bar{x}, \tau) \rightarrow \phi'(\bar{x}, \tau),
\]

(2.15)

satisfying \( \phi'(\bar{x}', \tau') = \phi(\bar{x}, \tau) \). Indeed, for an inversion

\[
x^\mu \rightarrow x'^\mu = \frac{x^\mu}{x^2}
\]

(2.16)

we have that

\[
\partial_\mu = (x^2) J^\mu_\nu \partial'_\nu,
\]

(2.17)

where

\[
J^\mu_\nu = \delta^\mu_\nu - 2 \frac{x^\mu x_\nu}{x^2}.
\]

(2.18)
Then invariance of the action (2.14) follows from (2.15) and the relation
\[ J_\mu, J_\nu = \delta^\mu_\nu. \] (2.19)

The field equation for the scalar field is
\[ \partial^2 \tau \phi - \frac{2}{\tau} \partial_\tau \phi - \nabla^2 \phi + \frac{m^2}{H^2 \tau^2} \phi = 0. \] (2.20)

On super-Hubble scales we can isolate the time-dependent behavior as
\[ \phi(\vec{x}, \tau) \sim \tau^{\Delta_\pm} \chi(\vec{x}), \quad \Delta_\pm = \frac{3}{2} \left( 1 \pm \sqrt{1 - \frac{4m^2}{9H^2}} \right). \] (2.21)

Let us therefore write for \( H \tau \ll 1 \)
\[ \phi(\vec{x}, \tau) = (H \tau)^\Delta \chi(\vec{x}), \quad \partial_\tau \phi(\vec{x}, \tau) = \frac{\Delta}{\tau} \phi(\vec{x}, \tau), \] (2.22)
where we have kept the dominant solution at large times, that is that with \( \Delta = \Delta_- \). In this case we get
\[ S = -\frac{1}{2} \int d^3x d\tau \left( (\nabla \phi)^2 - \frac{H^2 \Delta^2 - m^2}{H^2 \tau^2} \phi^2 \right). \] (2.23)

Note that for \( \tau^2 \ll \vec{x}^2 \), the inversion (2.9) is written as
\[ \tau \rightarrow \tau' = \frac{\tau}{|\vec{x}|^2}, \quad x_i \rightarrow x'_i = \frac{x_i}{|\vec{x}|^2}, \] (2.24)
and leaves invariant the action (2.23) as can easily be checked for a \( \phi(\vec{x}, \tau) \) satisfying (2.15). In addition, in this case, (2.23) is also invariant under rescalings (2.4). When the inversion (2.24) and the dilatation (2.4) are combined with translations and rotations, they generate the 3D conformal group acting on \( \mathbb{R}^3 \) and (2.20) therefore possesses full 3D conformal invariance. Note that the transformation of \( \phi(\vec{x}, \tau) \), see Eq. (2.15), is written in terms of \( \chi \) as
\[ \phi'(\vec{x}', \tau') = (H \tau')^\Delta \chi'(\vec{x}') = \phi(\vec{x}, \tau) = (H \tau)^\Delta \chi(\vec{x}), \] (2.25)
so that \( \chi \) transforms under dilations (2.4) and inversions (2.24) as
\[ \chi'(\lambda \vec{x}) = \lambda^{-\Delta} \chi(\vec{x}), \quad \chi' \left( \frac{\vec{x}}{|\vec{x}|^2} \right) = |\vec{x}|^{2 \Delta} \chi(\vec{x}), \] (2.26)
respectively. These are exactly the transformations we are expecting of a primary field of dimension \( \Delta \).

Therefore, on super-Hubble scales, the theory possesses 3D conformal invariance if the time-independent field \( \chi(\vec{x}) \) has scaling dimension \( \Delta \).

Let us rephrase in other terms and look at the infinitesimal version of the special conformal transformations on super-Hubble scales, that is the transformation (2.5). Under such an infinitesimal (passive) transformation, the scalar field variation reads
\[ \delta \phi(\vec{x}, \tau) \simeq \delta x^\mu \partial_\mu \phi(\vec{x}, \tau) \]
\[ = -\delta \tau \partial_\tau \phi(\vec{x}, \tau) + \delta \vec{x} \cdot \nabla \phi(\vec{x}, \tau) = (2\vec{b} \cdot \vec{x}) \tau \partial_\tau \phi(\vec{x}, \tau) - (2\vec{b} \cdot \vec{x}) \vec{x} \cdot \nabla \phi(\vec{x}, \tau) + x^2 \vec{b} \cdot \nabla \phi(\vec{x}, \tau) \]
\[ = \Delta (2\vec{b} \cdot \vec{x}) \phi(\vec{x}, \tau) - (2\vec{b} \cdot \vec{x}) \vec{x} \cdot \nabla \phi(\vec{x}, \tau) + x^2 \vec{b} \cdot \nabla \phi(\vec{x}, \tau), \quad (2.27) \]

which gives

\[ \delta \chi(\vec{x}) = \Delta (2\vec{b} \cdot \vec{x}) \chi(\vec{x}) - (2\vec{b} \cdot \vec{x}) \vec{x} \cdot \nabla \chi(\vec{x}) + x^2 \vec{b} \cdot \nabla \chi(\vec{x}). \quad (2.28) \]

This is precisely the transformation of a primary field with weight \( \Delta \) under 3D special conformal transformation. Therefore, correlation functions of scalar fields on de Sitter must be invariant under conformal transformations of Euclidean three-space on the future boundary \( \tau = 0 \). Similarly, an interaction of the form

\[ S_3 = \int d^4x \sqrt{-g} \mu \phi^3(\vec{x}, \tau) = \int \frac{d^3x d\tau}{H^4 \tau^4} \mu (H\tau)^{2\Delta} \chi^3(\vec{x}, \tau) \quad (2.29) \]

is invariant under 3D conformal symmetry. In fact, all non-derivative interactions of the form

\[ S_N = \int d^4x \sqrt{-g} \mu \phi^N = \int \frac{d^3x d\tau}{H^4 \tau^4} \mu (H\tau)^{N\Delta} \chi^N(\vec{x}, \tau), \quad (2.30) \]

are invariant under 3D conformal symmetry, although subleading with respect to (2.29) on super-Hubble scales for \( N > 3 \).

## 3 Conformal symmetries of de Sitter and the vector field

Let us consider now the case of a massless vector field \( A_\mu(\vec{x}, \tau) \). In particular we consider the action

\[ S_A = -\frac{1}{4} \int d^4x \sqrt{-g} I^2(\phi) F_{\mu\nu} F_{\kappa\rho} g^{\mu\kappa} g^{\nu\rho}, \quad (3.1) \]

with gauge coupling \( g = 1/I(\phi) \). In the de Sitter background, this action is written as

\[ S_A = -\frac{1}{4} \int d^3x d\tau I^2(\phi) F_{\mu\nu} F_{\kappa\rho} \eta^{\mu\kappa} \eta^{\nu\rho}, \quad (3.2) \]

where \( \eta_{\mu\nu} \) is standard Minkowski metric. In the \( A_0 = 0 \) gauge, the action (3.2) turns out to be

\[ S_A = -\frac{1}{4} \int d^3x d\tau I^2(\phi) \left( A_i^2 + (\partial_i A_j - \partial_j A_i)^2 \right). \quad (3.3) \]

We will assume that

\[ I(\phi) = I(\tau) = (H\tau)^{-n} \quad (3.4) \]
and we would like to see for which values of $n$ the action (3.3) is invariant under the 3D conformal group. It is clear that the most stringent constraint will arise from the invariance under the inversion (2.24).

We should recall that a vector $X_i$ with scaling dimension $\Delta_X$ in $D$-dimensions transforms under the conformal group as $X_i(x) \to X'_i(x')$ where

$$X'_i(x') = \left| \det \left( \frac{\partial x'^j}{\partial x^i} \right) \right|^{(1-\Delta_X)/D} \frac{\partial x^j}{\partial x'^i} X_j(x).$$

In general, a tensor $T_{i_1i_2...i_n}$ transforms as

$$T'_{i_1i_2...i_n}(x') = \left| \det \left( \frac{\partial x'^j}{\partial x^i} \right) \right|^{(n-\Delta_T)/D} \left( \frac{\partial x'^j_1}{\partial x'^i_1} \frac{\partial x'^j_2}{\partial x'^i_2} \cdots \frac{\partial x'^j_n}{\partial x'^i_n} \right) T_{j_1j_2...j_n}(x),$$

where $\Delta_T$ is its conformal dimension. Thus, for example, under rescalings we have

$$X'_i(x') = \lambda^{-\Delta} X_i(x),$$

whereas for the inversion (2.24)

$$X'_i(x') = |\vec{x}|^{2(\Delta-1)} J_i^j(x) X_j(x).$$

The action (3.3) is

$$S_A = -\frac{1}{4} \int d^3x d\tau (H\tau)^{-2n} \left\{ (\partial_\tau A_i)^2 - (\partial_i A_j - \partial_j A_i)^2 \right\}.$$

Expressing $A_i$ in terms of the field $a_i$ as

$$A_i(\vec{x}, \tau) = (H\tau)^n a_i(\vec{x}, \tau),$$

we get that

$$S_A = -\frac{1}{4} \int d^3x d\tau \left\{ (\partial_\tau a_i)^2 + \frac{n(n+1)}{\tau^2} a_i^2 - (\partial_i a_j - \partial_j a_i)^2 \right\}.$$

It can be checked that the action (3.3) can only be invariant under inversion (2.24) if the dimension of the vector $a_i$ is $\Delta_a = 1$. Indeed, under inversions, the vector $a_i$ transforms as

$$a'_i(x') = \left| \det \left( \frac{\partial x'^j}{\partial x^i} \right) \right|^{(1-\Delta_a)/D} \frac{\partial x^j}{\partial x'^i} a_j(x).$$

This is a coordinate transformation augmented by the factor $J^{(1-\Delta_a)/D}$ where $J = \left| \det(\partial x'^j/\partial x^i) \right|$. Then, when transforming $f_{ij} = (\partial_i a_j - \partial_j a_i)$, there will appear cross terms of the form $a_j \partial_i J$ which cannot be canceled and will spoil conformal invariance. In fact, conformal invariance can be maintained if the $J$ factor in the transformation (3.12) is missing. This is possible for

$$\Delta_a = 1.$$
Then, by using that
\[ f'_{ij} = |\vec{x}|^4 J_{ik} J_{jl} f_{kl}, \quad \partial_{\tau} a'_i = |\vec{x}|^4 J_{ij} \partial_{\tau} a_j \] (3.14)
and the orthogonality relation (2.19), the action (3.11) turns out to be under the inversion (2.24)
\[
S_A' = -\frac{1}{4} \int d^3 x' d\tau' \left\{ (\partial_{\tau'} a'_i)^2 + \frac{n(n+1)}{\tau'^2} a'_i^2 - (\partial'_a a'_j - \partial'_a a'_i)^2 \right\} \\
= -\frac{1}{4} \int d^3 x d\tau |\vec{x}|^8 \left\{ (\partial_{\tau} a_i)^2 + \frac{n(n+1)}{\tau^2} a_i^2 - (\partial_i a_j - \partial_j a_i)^2 \right\} \\
= S_A. \quad (3.15)
\]
As we did for the scalar, the field equations for \( a_i \) at super-Hubble scales is
\[
\partial_{\tau}^2 a_i - \frac{n(n+1)}{\tau^2} a_i = 0. \quad (3.16)
\]
The general solution is written as
\[
a_i = (H\tau)^{-n} V_i(\vec{x}) + (H\tau)^{1+n} U_i(\vec{x}). \quad (3.17)
\]
Depending on the value of \( n \), the leading term is different. So, there will be two cases, which we will call “magnetic” and “electric” respectively.

### 3.1 The magnetic case

The excitations of the vector field during a de Sitter epoch generate magnetic-like fluctuations if \( n > -1/2 \). In this case, the solution is
\[
a_i = (H\tau)^{-n} V_i(\vec{x}) \quad (3.18)
\]
and the conformal dimension of \( V_i \) can be determined by the conformal dimension of \( a_i \) (which is \( \Delta_a = 1 \)) to be
\[
\Delta_V = -n + 1. \quad (3.19)
\]
Indeed, under rescaling \( \tau \rightarrow \tau' = \lambda \tau \) and \( \vec{x} \rightarrow \vec{x}' = \lambda \vec{x} \) we get
\[
a'_i(\vec{x}', \tau') = (H\tau')^{-n} V'_i(\vec{x}') = \lambda^{-n}(H\tau)^{-n} V'_i(\vec{x}) = \lambda^{-1} a_i(\vec{x}, \tau) = \lambda^{-1}(H\tau)^{-n} V_i(\vec{x}) \quad (3.20)
\]
and therefore,
\[
V'_i(\vec{x}') = \lambda^{n-1} V_i(\vec{x}). \quad (3.21)
\]
Then, a simple comparison with (3.7) reveals that \( V_i \) transforms as a vector of conformal dimension \((−n + 1)\), that is Eq. (3.19). Let us note that the electric and magnetic fields are given by
\[
E_i = -\frac{I}{a^2} A'_i, \quad B_i = \frac{I}{a^2} \epsilon_{ijk} \partial_j A'_k. \quad (3.22)
\]
Then, by using (3.10) and (3.18) we get

\[ E_i = -\frac{I}{a^2} V'_i = 0, \quad B_i = \frac{I}{a^2} \epsilon_{ijk} \partial_j V_k. \]  

(3.23)

Therefore, in the magnetic case, the electric field vanish and the magnetic field is constant on super-Hubble scales only for

\[ I = a^2, \]  

(3.24)

that is only for \( n = 2 \).

### 3.2 The electric case

The excitations of the vector field during a de Sitter epoch generate electric-like fluctuations if \( n < -1/2 \). In such a case, the solution is

\[ a_i = (H\tau)^{n+1} U_i(\vec{x}) \]  

(3.25)

and the conformal dimension of \( U_i \) is

\[ \Delta U = n + 2. \]  

(3.26)

Indeed, under rescalings

\[ a'_i(\vec{x}', \tau') = (H\tau')^{n+1} U'_i(\vec{x}') = \lambda^{n+1} a_i(\vec{x}, \tau) = \lambda^{-1} a_i(\vec{x}, \tau) = \lambda^{-1}(H\tau)^{n+1} U_i(\vec{x}) \]  

(3.27)

and (3.26) easily follows. Note also that in this case, by using (3.10) and (3.25) the electric and magnetic fields are

\[ E_i = -\frac{I}{a^2} (2n + 1) H(H\tau)^{2n} U_i = -(2n + 1) H(H\tau)^{n+2} U_i, \]
\[ B_i = \frac{I}{a^2} (H\tau)^{2n+1} \epsilon_{ijk} \partial_j U_k = (H\tau)^{n+3} \epsilon_{ijk} \partial_j U_k. \]  

(3.28)

Therefore, in the electric case, the magnetic field vanish on super-Hubble scales whereas the electric field is constant only for

\[ I = a^{-2}, \]  

(3.29)

i.e., for \( n = -2 \). Of course, this is related to the electric-magnetic duality in this case since under \( I \rightarrow 1/I \), electric and magnetic fields exchange their role [39].
4 The correlators of the vector field

In order to describe the higher-order correlator of the vector fields we need to specify the dependence of the gauge coupling to the inflaton field, that is the function $I(\phi)$. We consider $I(\tau) = I(\phi_0(\tau))$ as $I(\phi)$ at the background value of $\phi = \phi_0(\tau)$ during inflation. Fluctuations of the latter produce interactions of the form

$$S_{\text{int}} = \int d^3x d\tau \left. \frac{\partial I^2}{\partial \phi} \right|_{\phi_0} \delta \phi \left(A_i' A_i' - (\nabla A_i)^2\right). \quad (4.1)$$

Following Ref. [28] for $I(\phi) = a^n(\tau)$, we should have

$$I^2 = \exp \left( -2n \int^\phi \frac{V(\phi')}{V'(\phi')} d\phi' \right). \quad (4.2)$$

Then, we easily find that

$$\left. \frac{\partial I^2}{\partial \phi} \right|_{\phi_0} = -2n I^2(\phi_0) \frac{V(\phi_0)}{V'(\phi_0)}. \quad (4.3)$$

By combining Friedmann and inflaton equations we find that

$$\frac{V(\phi_0)}{V'(\phi_0)} = -\frac{d \ln a}{d \phi} \quad (4.4)$$

and since

$$a = \left( \frac{\tau}{\tau_0} \right)^{1+b}, \quad \phi_0 = v + \sqrt{2\epsilon}(1 + b) \ln \tau, \quad (4.5)$$

where $\epsilon$ is the slow roll parameter, we get that

$$\frac{d \ln a}{d \phi} = \frac{1}{\sqrt{2\epsilon}}. \quad (4.6)$$

Therefore, we find

$$\left. \frac{\partial I^2}{\partial \phi} \right|_{\phi_0} = \sqrt{\frac{2}{\epsilon}} n I^2(\phi_0). \quad (4.7)$$

The above equation (4.7) can be integrated to give

$$I^2(\phi) = I^2_0 e^{\sqrt{\frac{2}{\epsilon}} n (\phi - \phi_0)} \quad (4.8)$$

Therefore the action (3.3) turns out to be

$$S_A = -\frac{1}{4} \int d^3x d\tau I^2_0 e^{\sqrt{\frac{2}{\epsilon}} n (\phi - \phi_0)} \left(A_i'^2 - (\partial_i A_j - \partial_j A_i)^2\right), \quad (4.9)$$
Expanding around $\phi_0$ given by the relation (4.5), the action turns out to be

$$S_A = -\frac{1}{4} \int d^3x d\tau (H\tau)^{-2n} \left( 1 + \sqrt{\frac{2}{n}} n \delta \phi + \frac{n^2}{\epsilon} \delta \phi^2 + \cdots \right) \left( A_i^2 - (\partial_i A_j - \partial_j A_i)^2 \right).$$

(4.10)

Clearly, the interactions in the action (4.10) are conformal invariant if $\phi(\vec{x}, \tau)$ has scaling dimension zero $\Delta = 0$ which we will assume from now on (therefore neglecting the small deviations from scale-invariance). In other words, the interaction of the vector field with the inflaton field are conformal invariant at any order in perturbation theory. This will become important in the following. We are now in the position to characterize the correlators involving the vector field.

Let us first recall that in momentum space that the Ward identities associated to dilations and special conformal transformations in general $D$-dimensions of a symmetric, two-tensor $N$-point amplitude $A'_{lm}$ (primes indicate they are computed without Dirac delta functions), using (3.6), are given by

$$\delta \lambda A'_{lm} = \left\{ -d(N - 1) + \sum_{a=1}^{N} \left( \Delta_a - \vec{k}_a \cdot \vec{\partial}_{k_a} \right) \right\} A'_{lm},$$

(4.11)

$$\delta \mu A'_{lm} = i \sum_{a=1}^{N} \left\{ 2(\Delta_a - d) \partial_{k_a^i} + k_a^i \nabla^2_{k_a} - 2\vec{k}_a \cdot \vec{\partial}_{k_a} \right\} A'_{lm}$$

$$- 2i \sum_{a=1}^{N} \left\{ \left( \delta_{ni} \partial_{k_a^i} - \delta_{i} \partial_{k_a} \right) A'_{nm} + \left( \delta_{ni} \partial_{k_a^m} - \delta_{i} \partial_{k_a} \right) A'_{ln} \right\}. \quad (4.12)$$

We will see that the above transformations, together with the fact that $A'_{lm}$ transforms as a symmetric two-tensor under SO(3) rotations is enough to determine their form.

### 4.1 The magnetic case

In the temporal $A_0 = 0$ gauge, the field $A_i$ as well as $a_i$ and $V_i$ are divergenceless $\nabla^i A_i = \nabla^i a_i = \nabla^i V_i = 0$. Then, by SO(3) covariance and momentum conservation, we get

$$\left\langle A_i(\vec{k}) A_j(-\vec{k}) \right\rangle' = P(k) \left( \delta_{ij} - \alpha(k) k_i k_j \right)$$

(4.13)

for the Fourier modes $A_i(\vec{k})$ of the vector $A_i(\vec{x})$. The divergenceless condition for $A_i(\vec{x})$ is written as $k_i \left\langle A_i(\vec{k}) A_j(-\vec{k}) \right\rangle' = 0$, which specifies $\alpha = 1/k^2$ so that

$$\left\langle A_i(\vec{k}) A_j(-\vec{k}) \right\rangle' = P(k) \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right).$$

(4.14)

Finally, by using that

$$\vec{k} \cdot \vec{\partial}_{\vec{k}} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) = 0,$$

(4.15)
$P(k)$ is specified by the invariance under dilations and special conformal transformations (4.11) and (4.12) to satisfy

$$\left\{-3 + 2(-n + 1) - k \partial_k \right\} P(k) = 0. \quad (4.16)$$

The SO(3) symmetric solution to the above equation is

$$P(k) = \frac{C_M}{k^{2n+1}} , \quad (4.17)$$

where $C_M$ is a constant.

We can similarly determine the three-point function $\langle \delta \phi A_i A_j \rangle$ by rotational and 3D conformal symmetry. SO(3) covariance imposes for the Fourier modes $A_i(\vec{k})$ of $A_i(\vec{x})$ the form

$$\langle \delta \phi(\vec{k}_1) A_i(\vec{k}_2) A_j(\vec{k}_3) \rangle' = c_{ij} \delta_{ij} + c_2 (k_2)_i(k_3)_j + c_3 (k_2)_j(k_3)_i + c_4 (k_2)_i(k_2)_j + c_5 (k_3)_j(k_3)_i , \quad (4.18)$$

where $c_i = c_i(\vec{k}_1, \vec{k}_2, \vec{k}_3)$. By multiplying by $(k_2)_i$ and by $(k_3)_j$, and using that $A_i$ is divergenceless ($\vec{k}^i \cdot A_i(\vec{k}) = 0$), we get the conditions

$$0 = c_1 (k_2)_j + c_2 k_2^2(k_3)_j + c_3 (\vec{k}_2 \cdot \vec{k}_3)(k_2)_j + c_4 k_2^2(k_2)_j + c_5 (\vec{k}_2 \cdot \vec{k}_3)(k_3)_j$$

$$0 = c_1 (k_3)_i + c_2 k_3^2(k_2)_i + c_3 (\vec{k}_2 \cdot \vec{k}_3)(k_3)_i + c_4 (\vec{k}_2 \cdot \vec{k}_3)(k_2)_i + c_5 k_3^2(k_3)_i . \quad (4.19)$$

The above equations specify the constants as

$$c_3 = \frac{c_2 k_2^2 k_3^2}{(k_2 \cdot k_3)^2} - \frac{c_1}{(k_2 \cdot k_3)} , \quad c_4 = - \frac{c_2 k_3^2}{(k_2 \cdot k_3)} , \quad c_5 = - \frac{c_2 k_2^2}{(k_2 \cdot k_3)} , \quad (4.20)$$

and therefore, by appropriate parametrization, the three-point correlator can be written as

$$\langle \delta \phi(\vec{k}_1) A_i(\vec{k}_2) A_j(\vec{k}_3) \rangle' = I_1 D_{ij} + I_2 \Delta_{ij} , \quad (4.21)$$

where $I_{1,2} = I_{1,2}(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ and

$$\Delta_{ij} = \vec{k}_2 \cdot \vec{k}_3 \delta_{ij} - (k_2)_j(k_3)_i , \quad D_{ij} = \delta_{ij} - \frac{(k_2)_i(k_2)_j}{k_2^2} - \frac{(k_3)_i(k_3)_j}{k_3^2} + \frac{(k_2)_i(k_3)_j}{k_2^2 k_3^2} \vec{k}_2 \cdot \vec{k}_3 . \quad (4.22)$$

The final step is to implement conformal invariance. Since the two terms in (4.20) containing $I_1$ and $I_2$ respectively are independent, they should be conformal invariant independently. On super-Hubble scales we may assume that $I_{1,2}$ have a series expansion in terms of $k_1, k_2, k_3$ as

$$I_1 = \alpha_M k_1^{a_1}(k_2 k_3)^{b_1} + \ldots , \quad I_2 = \beta_M k_1^{a_2}(k_2 k_3)^{b_2} + \ldots , \quad (4.23)$$
where $\alpha_1, \alpha_2$ are constants and we have also used the symmetry of the correlator (4.21) under simultaneous interchange $i \leftrightarrow j$ and $\vec{k}_2 \leftrightarrow \vec{k}_3$. By using the relations
\[
\vec{k}_2 \cdot \delta \vec{k}_2 \Delta_{ij} = \vec{k}_3 \cdot \delta \vec{k}_3 \Delta_{ij} = \Delta_{ij}, \\
\vec{k}_2 \cdot \delta \vec{k}_2 D_{ij} = \vec{k}_3 \cdot \delta \vec{k}_3 D_{ij} = 0, 
\]
and the fact the scaling dimensions of $\delta \phi$ and $a_i$ are $\Delta_{\delta \phi} = 0$ and $\Delta_A = \Delta_V = -n + 1$, respectively, we get from scaling invariance (4.11) that
\[
b_1 = -n - 2 - a_1/2, \quad b_2 = -n - 3 - a_2/2, \quad i.e., 
\]
\[
I_1 = \frac{\alpha_M k_1^{\alpha_1}}{(k_2k_3)^{n+2+\frac{\alpha_1}{2}}}, \quad I_2 = \frac{\beta_M k_1^{\alpha_2}}{(k_2k_3)^{n+3+\frac{\alpha_2}{2}}}. 
\]
In addition, the implementation of special conformal invariance (4.12) gives that $a_2 = 0$ and $\alpha_M = 0$ since it turns out that only $I_2$ is invariant (up to $O((k_2k_3)^{-n-2})$ terms). Thus, conformal invariance with constant magnetic field at super Hubble scales (i.e. $n = 2$), specify the three-point function to be
\[
\langle \delta \phi(\vec{k}_1)A_i(\vec{k}_2)A_j(\vec{k}_3) \rangle' = \frac{\beta_M}{k_2k_3} \left( \vec{k}_2 \cdot \vec{k}_3 \delta_{ij} - (k_2)_j(k_3)_i \right). 
\]
We should mention that we could have consider instead of (4.23), the most general form
\[
I_1 = \alpha_M k_1^{\alpha_1}(k_2k_3)^{b_1}(\vec{k}_2 \cdot \vec{k}_3)^{q_1} + \ldots, \quad I_2 = \beta_M k_1^{\alpha_2}(k_2k_3)^{b_2}(\vec{k}_2 \cdot \vec{k}_3)^{q_2} + \ldots, 
\]
However, although scaling symmetry (4.11) can be satisfied with
\[
I_1 = \frac{\alpha_M k_1^{\alpha_1}(\vec{k}_2 \cdot \vec{k}_3)^{q_1}}{(k_2k_3)^{n+2+q_1+\frac{\alpha_1}{2}}}, \quad I_2 = \frac{\beta_M k_1^{\alpha_2}(\vec{k}_2 \cdot \vec{k}_3)^{q_2}}{(k_2k_3)^{n+3+q_2+\frac{\alpha_2}{2}}}, 
\]
special conformal symmetry leads again to $q_2 = 0, a_2 = 0$ and $\alpha_M = 0$, that is again to the solution (4.26).

### 4.2 The electric case

For the case of electric-like excitations, again by $SO(3)$ covariance and momentum conservation, we get
\[
\langle U_i(\vec{k})U_j(-\vec{k}) \rangle' = P(k) \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) 
\]
and $P(k)$ is similarly specified by the invariance under dilations and special conformal transformations (4.11) and (4.12) to satisfy
\[
\left\{ -3 + 2(n + 2) - k \partial_k \right\} P(k) = 0. 
\]
The $SO(3)$ symmetric solution to the above equation is
\[
P(k) = C_E k^{2n+1}, 
\]
where $C_E$ is a constant. Then, since $A_i = (H\tau)^{2n+1}U_i$, we get

$$\langle A_i(\vec{k})A_j(-\vec{k})\rangle' = (H\tau)^{4n+2}C_E k^{2n+1} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right).$$ (4.32)

In particular, for a constant electric field at super-Hubble scales, $n = -2$ and we get

$$\langle A_i(\vec{k})A_j(-\vec{k})\rangle' = (H\tau)^{-6}C_E k^3 \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right),$$ (4.33)

whereas, the two-point correlator of the Fourier modes of the electric field is

$$\langle E_i(\vec{k})E_j(-\vec{k})\rangle' = 9H^2 C_E k^3 \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right).$$ (4.34)

Similarly, it is straightforward to specify the three-point correlator $\langle \delta\phi A_i A_j \rangle$ by rotational and 3D conformal symmetry. In this case we get

$$\langle \delta\phi(\vec{k}_1)U_i(\vec{k}_2)U_j(\vec{k}_3)\rangle' = I_1 D_{ij} + I_2 \Delta_{ij},$$ (4.35)

where $I_{1,2} = I_{1,2}(\vec{k}_1,\vec{k}_2,\vec{k}_3)$. Again, we may assume that $I_{1,2}$ have a series expansion in terms of $k_1, k_2, k_3$ at super-Hubble scales as

$$I_1 = \alpha_E k_1^{a_1}(k_2 k_3)^{b_1} + \cdots, \quad I_2 = \beta_E k_1^{a_2}(k_2 k_3)^{b_2} + \cdots.$$ (4.36)

Then, by recalling that the scaling dimensions of $\delta\phi$ and $a_i$ are $\Delta_{\delta\phi} = 0$ and $\Delta_U = n + 2$, respectively, scaling invariance (4.11) of the three-point correlator gives that $b_1 = n - 1 - a_1/2$, $b_2 = n - 2 - a_2/2$, that is

$$I_1 = \frac{\alpha_E k_1^{a_1}}{(k_2 k_3)^{1-n+a_1/2}}, \quad I_2 = \frac{\beta_E k_1^{a_2}}{(k_2 k_3)^{2-n+a_2/2}}.$$ (4.37)

In addition, by employing special conformal invariance we get now $a_1 = 0$ and $\beta_E = 0$ since in this case only $I_1$ is invariant (up to order $O((k_2 k_3)^{n-1})$ terms) and hence

$$\langle \delta\phi(\vec{k}_1)U_i(\vec{k}_2)U_j(\vec{k}_3)\rangle' = \alpha_E \frac{k_1^{a_1}}{(k_2 k_3)^{1-n}} \left( \delta_{ij} - \frac{(k_2)_i (k_2)_j}{k_2^2} - \frac{(k_3)_i (k_3)_j}{k_3^2} + \frac{(k_2)_i (k_3)_j \vec{k}_2 \cdot \vec{k}_3}{k_2^2 k_3^2} \right).$$ (4.38)

For a constant electric field at super Hubble scales $n = -2$, we therefore get

$$\langle \delta\phi(\vec{k}_1)A_i(\vec{k}_2)A_j(\vec{k}_3)\rangle' = (H\tau)^{-6} \frac{\alpha_E}{(k_2 k_3)^3} \left( \delta_{ij} - \frac{(k_2)_i (k_2)_j}{k_2^2} - \frac{(k_3)_i (k_3)_j}{k_3^2} + \frac{(k_2)_i (k_3)_j \vec{k}_2 \cdot \vec{k}_3}{k_2^2 k_3^2} \right),$$ (4.39)

and

$$\langle \delta\phi(\vec{k}_1)E_i(\vec{k}_2)E_j(\vec{k}_3)\rangle' = 9H^2 \frac{\alpha_E}{(k_2 k_3)^3} \left( \delta_{ij} - \frac{(k_2)_i (k_2)_j}{k_2^2} - \frac{(k_3)_i (k_3)_j}{k_3^2} + \frac{(k_2)_i (k_3)_j \vec{k}_2 \cdot \vec{k}_3}{k_2^2 k_3^2} \right).$$ (4.40)
Note that again, introducing powers $(\vec{k}_2 \cdot \vec{k}_3)^q$ in $I_1, I_2$ as we did in the magnetic case, do not change the (4.39) as special conformal invariance (4.12) leads again to $q = 0$.

The results (4.26) and (4.40) are fully dictated by dilation and special conformal invariance. Indeed, since the de Sitter isometry group acts as conformal group on $\mathbb{R}^3$ when the fluctuations are super-Hubble, the correlators must be of the form we found when perturbations are on super-Hubble scales. In other words, correlators may have a more complicated form in momentum space. Nevertheless, as soon as they are evaluated with all the corresponding wavelengths outside the Hubble radius, the symmetries of the problem dictate their form. This simple argument imposes that, while the amplitude of the correlators is not fixed by the symmetries, for sure it has to depend on a parameter signaling how long modes live outside the Hubble radius till the end of inflation. An educated guess for such a parameter is the number of e-folds $N(k_t) = \ln(-k_t \tau)$ the given mode $k_t = (k_1 + k_2 + k_3)$ spends outside the Hubble radius.

This expectation is nicely confirmed by comparing our results with those of Refs. [32, 33, 39] where the three-point correlator among the inflaton field and two vector fields have been computed. A simple inspection of their results, e.g. Eq. (4.16) of Ref. [33] for the case $n = 2$ or Eqs. (46) and (47) of Ref. [32] for the case $n = -2$ (once some errors on the tensor parts are properly corrected), reveal that, despite the complexity of the full result, on super-Hubble scales there is only one dominant piece which is indeed proportional to $N(k_t) = \ln(-k_t \tau)$. The reason is simple: as soon as $N(k_t)$ gets larger than unity, the symmetry arguments apply and the correlator has to acquire the shape dictated by the symmetries at hand.

The symmetry arguments are in fact even more powerful. In the case in which there exists a background electric field (possibly sustained by IR super-Hubble modes), but whose associated energy density is negligible, the 3D conformal symmetry is not spontaneously broken. This is as a consequence of the fact that the electric field $\vec{E}$, or equivalently the field $\vec{U}$ in Eq. (3.27) (corresponding to $n = -2$), has zero conformal weight. This implies that the shape of correlators are the same at any order in perturbation theory. This is indeed what has been found in Ref. [39] (once the super-Hubble modes contributing to the electric field renormalize the background value), where the one-loop corrections for the curvature power spectrum and bispectrum have been shown to be the same as the tree-level ones.

5 Conclusions

In this paper we have investigated the properties of the vector perturbations generated during a primordial period of inflation on the basis of the symmetries present during such a stage. The key point of all our logic is that the de Sitter isometry group acts as conformal group on the three-dimensional Euclidean space for the super-Hubble fluctuations. This allows to characterize the correlators involving the inflaton and the vector fields, determining their shapes and shedding some light on some results found in the recent literature. Our results may be relevant for the following reason: when analyzing the broken statistical invariances of the CMB modes, one needs some sort of guidance to parametrize such deviations from the standard set-up. Symmetries may provide such a guidance in the very same way the do when writing effective field theories in the infrared because the ultraviolet completion is missing. We will investigate
this issue in a separate publication.

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