Cardinal B-spline dictionaries on a compact interval

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Abstract

A prescription for constructing dictionaries for cardinal spline spaces on a compact interval is provided. It is proved that such spaces can be spanned by dictionaries which are built by translating a prototype B-spline function of fixed support into the knots of the required cardinal spline space. This implies that cardinal spline spaces on a compact interval can be spanned by dictionaries of cardinal B-spline functions of broader support that the corresponding basis function.

Keywords: cardinal spline spaces, B-spline dictionaries, sparse representation, nonlinear approximation.

1 Introduction

The problem of signal approximation outside the orthogonal basis setting is a non-linear problem, per se, in the following sense: Let us consider that a signal $f$ in a separable Hilbert space, equipped with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|f\| = (\langle f, f \rangle)^{1/2}$, is to be approximated as the $M$-term superposition

$$f^M = \sum_{n=1}^{M} c_n^M \alpha_n,$$

where $\{\alpha_n\}_{n=1}^{M}$ are fixed elements of a non-orthogonal basis, which if well localized are often refereed to as atoms. In order to construct the approximation $f^M$ of $f$ minimizing the distance $\|f - f^M\|$ the coefficients $c_n^M$ in (1) should be computed as $c_n = \langle \alpha_n^M, f \rangle$, where the dual sequence $\{\alpha_n^M\}_{n=1}^{M}$ is biorthogonal to $\{\alpha_n\}_{n=1}^{M}$, and, in addition, the superscript is meant to indicate that $\text{span}\{\alpha_n\}_{n=1}^{M} \equiv \text{span}\{\alpha_n^M\}_{n=1}^{M}$. Hence, in order to account for the inclusion (or respective elimination) of one term in (1), all the elements of the dual sequence need to be modified for the coefficients of the new approximation to minimize the distance to the signal $f$. Such modifications can be performed in an effective manner by means of adaptive biorthogonalization techniques [14,15]. However, the problem of choosing the $M$ elements of a non-orthogonal basis best representing the signal is a highly nonlinear one [8,9,16]. Since the general problem has not feasible solution in polynomial time, in some practical situations it is addressed by algorithms which evolve by fixing atoms at each iteration step. These approaches are known as adaptive pursuit strategies [2,8,10,11,13]. They operate by selecting atoms from a redundant set, called a dictionary [10].

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It should be stressed that, as far as complexity is concerned, there is not much to lose by applying the above mentioned practical selection strategies on a redundant dictionary, rather than on a non-orthogonal basis. Nevertheless, as has been shown in the context of several applications, there is potentially much to win in relation to sparseness of the signal representation. Hence the motivation of this Communication: *We prove here that cardinal spline spaces on a compact interval are amenable to be spanned by dictionaries of cardinal B-spline functions of broader support than the corresponding basis functions.*

It goes without saying that splines have been used with success in wavelet theory and applications to signal processing [4, 5, 7, 17]. In particular, the construction of multiresolution based spline wavelets on a bounded interval is explained in great detail in [5, 6, 12]. Here we focus on B-spline functions and discuss the way of going from B-spline basis on the interval to B-spline dictionaries which are endowed with a very interesting property. Cardinal spline spaces on a bounded interval are finite dimensional linear spaces. The dimension is actually given by the number $d = m + N$, where $m$ is the order of the splines being considered and $N$ the number of knots in the interval. By fixing the order of splines, the usual way of increasing the subspace dimension is to increase the number of knots, decreasing thereby the distance, $b$, say, between two adjacent knots. Let us recall that the basis functions for cardinal spline spaces (B-spline basis) have compact support which, except for the boundary functions, is of length $mb$. Thus, by decreasing the distance $b$ between knots, the support of the functions is reduced. With the exception of the boundary functions [5, 6] a B-spline basis for the subspace arises by successive translations of a prototype function. The translation parameter is the distance between knots. It is clear then that if the translation operation is carried out with a translation parameter $b'$, such that $b/b'$ is an integer, some redundancy will be introduced. However, the main result of this contribution is to prove that, by such a procedure, one can generate the span for the cardinal spline space associated with the distance $b'$ between knots. We believe this to be a remarkable feature of splines, which is potentially useful in relation to signal representation. Such a possibility will be illustrated using matching pursuit strategies for representing a signal by selecting atoms from our cardinal spline dictionaries.

The paper is organized as follows: Section 2 provides some definitions and background on splines relevant for our purpose. Section 3 gives the proof of the main theorem establishing the above mentioned property of the proposed dictionaries. The potential suitability of such dictionaries for recursive signal approximation is illustrated in Section 4. The conclusions are drawn in Section 5.

2 Splines on a compact interval

We introduce here some notation and basic definitions which are relevant for our purpose. For an in depth treatment on splines we refer to [3, 18].

**Definition 1.** Given a compact interval $[c, d]$ we define a partition of $[c, d]$ as the finite set of points

$$\Delta = \{x_i\}_{i=0}^{N+1}, N \in \mathbb{N}, \text{ such that } c = x_0 < x_1 < \cdots < x_N < x_{N+1} = d. \quad (2)$$

We further define $N$ subintervals $I_i, i = 0, \ldots, N$ as: $I_i = [x_i, x_{i+1}), i = 0, \ldots, N - 1$ and $I_N = [x_N, x_{N+1}]$. 


**Definition 2.** Let $\Pi_m$ be the space of polynomials of degree smaller or equal to $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $m$ be a positive integer and define

$$S_m(\Delta) = \{ f \in C^{m-2}[c,d] ; \ f|_{I_i} \in \Pi_{m-1}, i = 0, \ldots, N \},$$

where $f|_{I_i}$ indicates the restriction of the function $f$ on the interval $I_i$.

We call $S_m(\Delta)$ the space of polynomial splines (or splines) of order $m$ with simple knots at the points $x_1, \ldots, x_N$.

Let us recall two well known properties of $S_m(\Delta)$.

**Property 1.** $S_m(\Delta)$ is a linear space of dimension $m + N$ [18, Theorem 4.4].

Moreover, it readily follows from Definition 2 that

**Property 2.** If $\Delta$ and $\Delta'$ are two partitions of the interval $[c,d]$ such that $\Delta \subset \Delta'$, then $S_m(\Delta) \subset S_m(\Delta')$, $m \in \mathbb{N}$.

In order to construct a particular basis for $S_m(\Delta)$ it is necessary to introduce the so-called extended partition.

**Definition 3.** Let $\Delta$ be a partition of $[c,d]$ and let us consider

$$y_1 \leq y_2 \leq \cdots \leq y_{2m+N}$$

such that

$$y_1 \leq \cdots \leq y_m \leq c, \ d \leq y_{m+N+1} \leq \cdots \leq y_{2m+N}$$

and

$$y_{m+1} < \cdots < y_{m+N} = x_1, \ldots, x_N.$$  

We call $\tilde{\Delta} = \{y_i\}_{i=1}^{2m+N}$ an extended partition with single inner knots associated with $S_m(\Delta)$.

The points $\{y_i\}_{i=m+1}^{m+N}$ in an extended partition $\tilde{\Delta}$ associated with $S_m(\Delta)$ are uniquely determined, however the first and last $m$ points in $\tilde{\Delta}$ can be chosen arbitrarily.

With each fixed extended partition $\tilde{\Delta}$ there is associated a unique B-spline basis for $S_m(\Delta)$, that we denote as $\{B_i\}_{i=1}^{m+N}$. Full details on how to construct such basis are given in [18, Theorem 4.9] and [5]. The basis functions $B_i$ satisfy for $x \in [c,d]$

$$B_i(x) = 0 \quad \text{if } x \notin [y_i, y_{i+m}],$$

$$B_i(x) > 0 \quad \text{if } x \in (y_i, y_{i+m}).$$

In the case of equally spaced knots the corresponding splines are called cardinal. Moreover all the cardinal B-splines of order $m$ can be obtained from one cardinal B-spline $B(x)$ associated with the uniform simple knot sequence $0, 1, \ldots, m$. Such a function is given as

$$B(x) = \frac{1}{m!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x - i)^{m-1},$$

where $(x - i)^{m-1} +$ is equal to $(x - i)^{m-1}$ if $x - i > 0$ and 0 otherwise. If $y_i, \ldots, y_{i+m}$ are equally spaced, then

$$B_i(x) = \frac{1}{b} B \left( \frac{x - y_i}{b} \right),$$
where $b$ is the distance between two adjacent knots.

Let as recall that, given a partition $\Delta$ and a real number $r$ by the operation $\Delta + r$ we obtain:

$$\Delta + r = \{x_i + r\}_{i=0}^{N+1} = \{x_0 + r, x_1 + r, \ldots, x_{N+1} + r\}.$$ 

In order to retain the boundary points $x_0$ and $x_{N+1}$ we define a new operation $\Delta \uplus r$ as follows:

**Definition 4.** Given a partition $\Delta$ and a real number $r$, $0 < r < \min_i(x_{i+1} - x_i)$, $i = 0, \ldots, N$ by the operation $\Delta \uplus r$ we obtain:

$$\Delta \uplus r = \{x_i + r\}_{i=0}^{N} \cup \{x_0, x_{N+1}\} = \{x_0, x_0 + r, \ldots, x_N + r, x_{N+1}\}.$$ 

Since we are interested only in cardinal spline spaces hereafter we will consider an equidistant partition of $[c, d]$. Such a partition is thereby uniquely determined by the interval $[c, d]$ and the distance, $b$ say, between two adjacent points. It is assumed that the interval $[c, d]$ contains at least one complete B-spline function, i.e., $d - c \geq mb$. The definition of equidistant partition is extended to open/semi-open intervals as indicated below.

**Definition 5.** We construct an equidistant partition of $[c, d], (c, d), (c, d), \text{ and } [c, d]$ with distance $b$ between adjacent points (such that $(d - c)/b = \text{integer}$) as follows:

$$\mathcal{P}_b[c, d] = \{c, c + b, \ldots, d - b, d\}, \quad \mathcal{P}_b(c, d) = \{c + b, \ldots, d - b\},$$

$$\mathcal{P}_b[c, d] = \{c + b, \ldots, d - b\}, \quad \mathcal{P}_b[c, d] = \{c, c + b, \ldots, d - b\}.$$ 

As already mentioned, the selection of a particular extended partition $\tilde{\Delta}$ yields a particular B-spline basis for $S_m(\Delta)$. Two possible choices for the first and last $m$ points determining the extended partition are the following:

i) The points $\{y_i\}_{i=1}^{m}$ and $\{y_i\}_{i=N+m+1}^{2m+N}$ are determined in order for the whole sequence $\tilde{\Delta} = \{y_i\}_{i=1}^{2m+N}$ to be endowed with the equidistant property, i.e., $\tilde{\Delta} = \mathcal{P}_b(c - mb, d + mb)$. Such a sequence is called an equally spaced extended partition (ESEP).

ii) The points lying outside $[c, d]$ are given the value of the closest point in the interval, i.e., $y_1 = \cdots = y_m = c$ and $y_{m+N+1} = \cdots = y_{2m+N} = d$. This is called an extended partition with $m$-tuple knots on the border (EPKB).

## 3 Building B-spline dictionaries

For the sake of a simpler notation we prefer to build dictionaries by considering the extended partition ESEP, i.e., $\tilde{\Delta} = \mathcal{P}_b(c - mb, d + mb)$. The reason is that, since the whole partition is then equidistant, we can construct the spanning functions by translation of one prototype B-spline of support of length $mb$ into the points $\{y_i\}_{i=1}^{m+N} = \mathcal{P}_b(c - mb, d)$ and the restriction of the functions to the interval $[c, d]$. This process is equivalent to constructing the boundary functions by truncation (see Figure 1), which allows us to use a simple notation to label the dictionary functions. Shifting indices, for later convenience, we indicate the basis for $S_m(\Delta)$ associated to the ESEP $\tilde{\Delta}$ as

$$\{\phi_k(x)\}_{k \in \mathcal{P}_b(c - mb, d)} = \{\phi(x - k)\}_{k \in \mathcal{P}_b(c - mb, d)}, \quad (9)$$

where $\phi(x) = \frac{1}{b}B\left(\frac{x}{b}\right)$. Note that $\text{supp}(\phi) = [0, mb]$. 


Since $S_m(\Delta)$ is a space of dimension $m + N$, it is clear that the dimension can be increased by decreasing the distance between knots. The main contribution of the present effort is to propose an alternative way of increasing dimension, namely: maintaining the same distance between knots and including functions arising by simple translations of the basis function for $S_m(\Delta)$. This is established by the following theorem.

**Theorem 1.** Let $\Delta = \mathcal{P}_b[c,d]$ and $\Delta' = \mathcal{P}_{b'}[c,d]$ be such that $\Delta \subset \Delta'$. Let us denote as $\{\phi_k\}_{k \in \mathcal{P}_b[c-mb,d)}$ and $\{\phi'_k\}_{k \in \mathcal{P}_{b'}[c-mb',d)}$ the corresponding ESEP B-spline basis for $S_m(\Delta)$ and $S_m(\Delta')$, respectively.

We construct a dictionary, $\mathcal{D}_m(\Delta, b')$, of B-spline functions on $[c, d]$ as

$$\mathcal{D}_m(\Delta, b') = \{\phi_k(x)\}_{k \in \mathcal{P}_{b'}(c-mb,d)},$$

for which it holds that

$$\text{span}\{\mathcal{D}_m(\Delta, b')\} = S_m(\Delta').$$

Note that the number of functions in the above defined dictionary is equal to the cardinality of $\mathcal{P}_{b'}(c-mb, d)$, which happens to be $K = (d - c + mb)/b' - 1$. Before advancing the proof of this theorem let us assert the following remark:

**Remark 1.** Setting $p = b/b' - 1$, for $0 < i \leq p$, we have

$$S_m(\Delta \uplus ib') = \text{span}\{\phi_k(x)\}_{k \in \mathcal{P}_b[c-mb,d] + ib'} = \text{span}\{\phi(x - (k + ib'))\}_{k \in \mathcal{P}_b[c-mb,d]},$$

$$\mathcal{P}_{b'}(c - mb, d) = \mathcal{P}_b(c - mb, d) \cup \bigcup_{i=1}^{p}\{\mathcal{P}_b[c - mb, d] + ib\}.$$ 

Hence, an alternative definition for the dictionary $\mathcal{D}_m(\Delta, b')$, which is equivalent to (10), is the following:

$$\mathcal{D}_m(\Delta, b') = \{\phi(x - k)\}_{k \in \mathcal{P}_b[c-mb,d]} \cup \bigcup_{i=1}^{p}\{\phi(x - (k + ib'))\}_{k \in \mathcal{P}_b[c-mb,d]}.$$ 

Moreover, from (12) and (14), it follows that

$$\text{span}\{\mathcal{D}_m(\Delta, b')\} = \bigcup_{i=0}^{p}S_m(\Delta \uplus ib').$$

We are now in a position to start the proof of Theorem 1.

**Proof of Theorem 1.**

We will use (15) to prove that $S_m(\Delta') = \text{span}\{\mathcal{D}_m(\Delta, b')\}$.

The inclusion $S_m(\Delta') \supset \text{span}\{\mathcal{D}_m(\Delta, b')\}$ follows from (15) and the fact that, from Property 2,

$S_m(\Delta), S_m(\Delta \uplus b'), \ldots, S_m(\Delta \uplus pb') \subset S_m(\Delta')$. We can then construct scaling-like equations by expressing each function $\phi_k \in \mathcal{D}_m(\Delta, b'), k \in \mathcal{P}_{b'}(c-mb, d)$ as a linear combination of functions $\phi'_n \in S_m(\Delta'), n \in \mathcal{P}_{b'}(c-mb', d)$. Considering the support of functions $\phi_k$ we introduce the scaling equations by grouping them into three classes:

$$\phi_k(x) = \sum_{n \in J_L} h_{n,k} \phi'_n(x) \quad \text{for } k \in \mathcal{P}_{b'}(c - mb, c), \quad J_L = \mathcal{P}_{b'}(c - mb', k + m(b - b')), \quad (16a)$$

$$\phi_k(x) = \sum_{n \in J_I} h_{n,k} \phi'_n(x) \quad \text{for } k \in \mathcal{P}_{b'}[c, d - mb], \quad J_I = \mathcal{P}_{b'}[k, k + m(b - b')], \quad (16b)$$

$$\phi_k(x) = \sum_{n \in J_R} h_{n,k} \phi'_n(x) \quad \text{for } k \in \mathcal{P}_{b'}(d - mb, d), \quad J_R = \mathcal{P}_{b'}[k, d]. \quad (16c)$$
Equation (16a) describes the \textit{left boundary} functions, i.e., all the functions which have support of length smaller than \( mb \) and contain the point \( c \) in their support. Equation (16b) describes the \textit{inner} functions, which are functions of support of length \( mb \) arising just as translations of a fixed prototype B-spline function. Finally, equation (16c) corresponds to the \textit{right boundary} functions, which have support of length smaller than \( mb \) and contain the point \( d \) in their support.

The proof of the inclusion \( S_m(\Delta') \subset \text{span}\{D_m(\Delta, b')\} \) will be achieved by showing that every \( \phi'_n(x) \in S_m(\Delta') \), \( n \in \mathcal{P}_{b'}(c - mb', d) \) can be expressed as a linear combination of functions \( \phi_k \in \mathcal{P}_m(\Delta, b'), \, k \in \mathcal{P}_b'(c - mb, d) \). For such an end we find it convenient to merge the equations (16b) and (16c). Notice that (16b) and (16c) can be recast as:

\[
\phi_k(x) = \sum_{n \in \mathcal{J}_R} h_{n,k} \phi_n'(x) \quad \text{for} \quad k \in \mathcal{P}_{b'}[c, d], \quad \mathcal{J}_R = \mathcal{P}_{b'}[k, d].
\]

Now, since \( \mathcal{P}_b'(c - mb', d) = \mathcal{P}_b'(c - mb', c) \cup \mathcal{P}_{b'}'(c, d) \), in order to decompose every function \( \phi'_n, \, n \in \mathcal{P}_b'(c - mb', d) \) in terms of the broader functions \( \phi_k, k \in \mathcal{P}_b'(c - mb, d) \) we make use of (17) if \( n \in \mathcal{P}_b'(c, d) \) and (16a) if \( n \in \mathcal{P}_b'(c - mb', c) \). The steps leading to the required decompositions are spelled out below by focusing on the case \( n \in \mathcal{P}_{b'}[c, d] \).

Let us recall that each function \( \phi_k, k \in \mathcal{P}_{b'}[c, d] \) is supported in [\( k, k + mb \) \( \cap [c, d] \)] and that \( \phi_k|_{[k, k+b']} = h_{k,k} \phi_k'|_{[k, k+b']} \). Hence one can be certain that the coefficient \( h_{k,k} \) in (17) is non-zero and use this equation to express \( \phi'_k(x) \) for \( k \in \mathcal{P}_{b'}[c, d] \), as:

\[
\phi'_k(x) = \frac{1}{h_{k,k}} \phi_k(x) - \sum_{n \in \mathcal{P}_{b'}[k+b', d]} \frac{h_{n,k}}{h_{k,k}} \phi'_n(x).
\]

The proof that for all \( l \in \mathcal{P}_{b'}[c, d] \) each function \( \phi'_l(x) \) is expressible as a linear combination of functions \( \phi_k(x), k \in \mathcal{P}_{b'}[c, d] \) follows by subsequent evaluations of (18) at the explicit values \( k = l, l + b', l + 2b', \ldots, d - b' \). Indeed, for \( k = l \) we have:

\[
\phi'_l(x) = \frac{1}{h_{l,l}} \phi_l(x) - \sum_{n \in \mathcal{P}_{b'}[l+b', d]} \frac{h_{n,l}}{h_{l,l}} \phi'_n(x) = \frac{1}{h_{l,l}} \phi_l(x) - \frac{h_{l+b',l}}{h_{l,l}} \phi'_{l+b'}(x) - \sum_{n \in \mathcal{P}_{b'}[l+2b', d]} \frac{h_{n,l}}{h_{l,l}} \phi'_n(x),
\]

and the recursive process evolves as follows: The first step entails to evaluate (18) at \( k = l + b' \) and introduce the corresponding expression for \( \phi'_{l+b'}(x) \) in (19). Thus we have:

\[
\phi'_l(x) = \frac{1}{h_{l,l}} \phi_l(x) - \frac{h_{l+b',l}}{h_{l,l} h_{l+b',l+b'}} \phi_{l+b'}(x) + \sum_{n \in \mathcal{P}_{b'}[l+2b', d]} \left( \frac{h_{l+b',l} h_{n,l+b'}}{h_{l,l} h_{l+b',l+b'}} - \frac{h_{n,l}}{h_{l,l}} \right) \phi'_n(x).
\]

The next step consists of evaluating (18) at \( k = l + 2b' \) and introducing the corresponding expression for \( \phi'_{l+2b'}(x) \) in (20). The process continues repeating equivalents steps, i.e., at step \( s \) say, we evaluate (18) at \( k = l + sb' \) and introduce the equation in (20). Finally, at step \( (d - b' - l)/b' \), for \( k = d - b' \) we obtain:

\[
\phi'_{d-b'}(x) = \frac{1}{h_{d-b',d-b'}} \phi_{d-b'}(x).
\]

When \( \phi'_{d-b'}(x) \) as given in (21) is introduced in (20), the right hand side of this equation turns out to be a linear combinations of functions \( \phi_l(x), \phi_{l+b'}(x), \phi_{l+2b'}(x), \ldots, \phi_{d-b'} \). We have thereby proved that \( \phi'_l(x) \in \text{span}\{\mathcal{D}_m(\Delta, b')\}, l \in \mathcal{P}_{b'}[c, d] \).
The proof concerning functions $φ'_n(x)$, $n ∈ P_b(c - mb', c)$ parallels the proof given above, but the following considerations are in order: To obtain the corresponding equation for $φ'_n(x)$ we use (16a) and the fact that $h_{n,k} ≠ 0$ for $k = n - m(b - b')$. The required decomposition arises by evaluating such an equation at values of $k$ in decreasing order ranging from $k = n - m(b - b')$ to $k = c - mb + b'$.

**Corollary 1.** The dictionary $D_m(∆, b') = \{φ_k(x)\}_{k∈P_b(c - mb, d)}$ is a frame for $S_m(∆')$, i.e., for all $f ∈ S_m(∆')$ there exist two constants $0 < A ≤ B$ such that

$$A∥f∥^2 ≤ \sum_{k∈P_b(c - mb, d)} |⟨f, φ_k⟩|^2 ≤ B∥f∥^2.$$  \hfill (22)

**Proof.** The proof is a direct consequence of Theorem 1 since the dictionary $D_m(∆, b')$ is a finite dimension set of functions of finite norm and, as such, a frame for its span. The upper bound is a consequence of Schwartz inequality and the non-zero lower bound is ensured by the fact that for $f ∈ S_m(∆')$ it is true that $⟨f, φ_k⟩ ≠ 0$ at least for one $k$ in $P_b(c - mb, d)$.

**Remark 2.** It is appropriate to point out that, although the proof of Theorem 1 was given for dictionaries constructed within an ESEP setting, the result can be extended to more general extended partition settings. The corresponding proof is equivalent to the one given here, but it involves a less handy notation.

Figure 1 shows some examples of B-spline dictionaries. The top graph on the left depicts a B-spline basis of order $m = 1$. These are piecewise constant functions and, since there are not boundary functions, the ESEP and EPKB basis are equivalent. The corresponding dictionary of functions of double support is depicted in the top right graph of the same figure. As emphasized by the thicker lines, the dictionary contains two boundary functions of support equal to the basis function of the left graph. The middle left graph of Figure 1 shows the B-spline basis of order $m = 4$ corresponding to the ESEP extended partition. The middle right graph shows the dictionary, for the same space, consisting of functions of twice as much support as the corresponding basis functions. The graphs at the bottom have the same description as the middle graphs, but correspond to the EPKB extended partition.

### 4 Application to recursive signal approximation

We present here two examples to illustrate the relevance of the proposed dictionaries to a typical problem of signal representation: the problem of achieving a sparse atomic decomposition approximating a given signal.

As a first example we consider the randomly generated blocky signal on the interval $[0, 4]$ depicted in the left graph of Figure 2. Such a signal has an acceptable approximation in the subspace $S_1(∆')$, with distance between knots $b' = 2^{-8}$, spanned by a B-spline basis of order one. The graph indicating the signal approximation coincides with the one of Figure 2. For such an approximation we need to use almost all the basis functions (956 out of 1024). This is of course an expected result, since the support of the basis functions for $S_1(∆')$ is considerably small ($|supp| = 2^{-8}$) in comparison to the length of the blocks composing the signal. It is then convenient to construct the identical approximation using dictionary functions of much larger support ($|supp| = 1$) spanning the same space $S_1(∆')$. In this case the approximation is obtained by means of only $M = 23$ functions, chosen through a forward and backward
Figure 1: Examples of bases (graphs on the left) and the corresponding dictionaries consisting of functions of double support. The top graphs correspond to B-splines of order $m = 1$. The middle and bottom graphs involve B-splines of order $m = 4$ and ESEP and EPKB settings, respectively.

optimized orthogonal matching pursuit strategy [2,13], from a dictionary of 1279 atoms. For further comparison we have used the Haar wavelet representation for the same space, which contains wavelets of varied support and scaling functions of the same support as the dictionary functions. Since Haar wavelets are orthonormal, to obtain the desired signal approximation we need just to retain $M = 50$ basis functions corresponding to the 50 coefficients of largest absolute value.

The second example involves the piece of modulated chirp signal plotted in right graph of Figure 2. This signal has an acceptable representation in the subspace $S_4(\Delta')$, with $\delta' = 2^{-5}$, spanned by the B-splines basis of order four. Since in this case the basis is non-orthogonal, in order to approximate the signal by a subset of basis functions we applied the same pursuit strategy as in the previous example. An approximation coinciding with the graph of Figure 2...
is obtained with $M = 101$ basis functions. Using a dictionary of functions having four times larger support (i.e., distance $2^{-3}$ between knots), and the same matching pursuit strategy for selecting functions, we need $M = 58$ dictionary functions. In this case, the selection from the wavelet representation of the subspace needs to be carried out also through the matching pursuit approach. The required number of wavelet functions is 61.

It is interesting to notice that in both examples we have achieved representations which in terms of sparseness are comparable (superior in the first case) to wavelet basis representation. This is a surprising result, since wavelet basis are composed by functions of different support and our dictionaries by functions of fixed support. These outcomes are certainly worth to be investigated further.

MATLAB codes for generating the proposed dictionaries and implementing pursuit strategies are available at [1].

5 Conclusions

An interesting feature of B-spline functions has been discussed. We have shown that a dictionary for a cardinal spline space on a compact interval can be constructed by translating a prototype B-spline function into the knots of the corresponding space. This property allows to span a cardinal spline space by using B-spline functions of larger support than the one corresponding to the basis functions for the same space.

As an example of application of the proposed dictionaries, two signals of different nature have been represented by selecting atoms from dictionaries of B-splines. The results illustrate the possible relevance of B-spline dictionaries to problems requiring sparse representation.

Finally, we would like to point out some lines of follow up work that we believe to be interesting: The possibility of spanning a fixed space by B-spline dictionaries, each of which consists of functions of different support, arises the question as to how to choose in an effective manner the dictionary of B-splines of ‘optimal support’ for representing a given signal. Another matter that appears definitely worth looking at is the possibility of extending the proposed construction in order to generate suitable subspaces by translating wavelets generated by B-splines.
References

[1] M. Andrle and L. Rebollo-Neira, “Biorthogonal techniques for optimal signal representation,” Web page [http://www.ncrg.aston.ac.uk/Projects/BiOrthog](http://www.ncrg.aston.ac.uk/Projects/BiOrthog).

[2] M. Andrle, L. Rebollo-Neira, and E. Sagianos, “Backward-Optimized Orthogonal Matching Pursuit Approach”, IEEE Signal Processing Letters, Vol (11,9), 705–708 (2004).

[3] C. de Boor, A Practical Guide to Splines, Applied Mathematical Sciences, Vol 27, Springer-Verlag, New York, 1978.

[4] C. K. Chui, An Introduction to Wavelets, Academic Press, 1992.

[5] C. K. Chui, Wavelets: A Mathematical Tool for Signal Processing, SIAM, Philadelphia, 1997.

[6] C. K. Chui and E. Quak, “Wavelets on a bounded interval”, in Numerical Methods of Approximation Theory, Vol 9, Eds. D. Braess and L. L. Schumaker, pp. 53–75, Birkhäuser, Basel, 1992.

[7] C. K. Chui and J. Z. Wang, “A general framework of compactly supported splines and wavelets”, J. Approx. Theory, Vol 71(3), 263–304 (1992).

[8] G. M. Davis, S. Mallat, and M. Avellaneda, “Adaptive greedy approximations”, Conts. Approx., Vol 13, 57–98 (1997).

[9] A. DeVore, “Nonlinear approximation”, Acta Numer., 51–150 (1998).

[10] S. Mallat and Z. Zhang, “Matching Pursuit with time-frequency dictionary”, IEEE Transactions on Signal Processing, Vol 41, 3397–3415 (1993).

[11] Y. C. Pati, R. Rezaifar, and P. S. Krishnaprasad, “Orthogonal matching pursuits: recursive function approximation with applications to wavelet decomposition”, in Proceedings of the 27th Asilomar Conference on Signals, Systems and Computers, 1993.

[12] E. Quak and N. Weyrich, “Decomposition and Reconstruction Algorithms for Spline Wavelets on a Bounded Interval”, Applied and Comp. Harm. Analysis, Vol 1, 217–231 (1994).

[13] L. Rebollo-Neira and D. Lowe, “Optimized Orthogonal Matching Pursuit Approach”, IEEE Signal Processing Letters, Vol (9,4), 137–140 (2002).

[14] L. Rebollo-Neira, “Backward Adaptive Biorthogonalization”, Int. Jour. of Math. and Math. Sciences, Vol (2004,35), 1843–1853 (2004).

[15] L. Rebollo-Neira, “On nonorthogonal signal representation”, chapter in Progress in Mathematical Physics, Nova Science Publishers, New York, to be published in 2005; “Recursive bi-orthogonalisation approach and orthogonal projector”, math-ph /0209026 (2002).

[16] V. N. Temlyakov, “Greedy algorithms and M-term approximation with regard to redundant dictionaries”, Journal of Approximation Theory, Vol (98,1), 117–145 (1999).
[17] M. Unser, “Splines. A Perfect Fit for Signal and Image Processing”, *IEEE Signal Processing Magazine*, 22–38 (1999).

[18] L. L. Schumaker, *Spline Functions: Basic Theory*, Wiley, New-York, 1981.