The Torus Universe in the Polygon Approach to 2+1-Dimensional Gravity

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Abstract

In this paper we describe the matter-free toroidal spacetime in ’t Hooft’s polygon approach to 2+1-dimensional gravity (i.e. we consider the case without any particles present). Contrary to earlier results in the literature we find that it is not possible to describe the torus by just one polygon but we need at least two polygons. We also show that the constraint algebra of the polygons closes.

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Introduction

As is well known nowadays, gravity in 2+1 dimensions is flat everywhere outside sources \([1, 2]\). This means that the gravitational field itself has no local degrees of freedom. One can make the theory non-trivial by adding sources (e.g. point-particles) or considering a non-trivial topology of a closed universe. For \(N\) point-particles that live on a genus-\(g\) surface for instance, the phase space is \(12g - 12 + 4N\) dimensional \([3, 6, 8]\). This formula is however wrong in the case of a torus in the absence of particle sources. This is due to the fact that the torus has some symmetries because of which the counting argument breaks down. The toroidal universe, with or without cosmological constant, has been extensively studied in the past. Its classical solutions \([11, 12]\) as well as the quantum theory \([8, 9, 10, 13, 16, 17]\) are well understood in both the ADM formalism and in the Chern-Simons formalism. From this work we know that the dimension of the space of ADM solutions is four (i.e. there are only four independent degrees of freedom) The torus is therefore a particularly simple model and a convenient starting point for a quantization program. As we are interested in the polygon description of 2+1-D gravity invented by ‘t Hooft \([3, 4]\) we decided to study the torus in this approach. Guadagnini and Franzosi had already worked on this problem \([15]\). But their counting of degrees of freedom was a bit puzzling to us. We found that they described a subset of all possible solutions for a torus universe. This is due to the fact that they use only one polygon for their slicing of spacetime. This is not enough to cover all possible tori. The simple solution to this problem is to add another polygon. This unfortunately implies that the description loses its simplicity due to the fact that polygon transitions may take place during evolution. This fact also considerably complicates the quantization. The temporary conclusion is that the polygon approach is not the most convenient description for the matter-free torus universe as compared with other approaches.

In section 1 we recapitulate the way Carlip describes a toroidal spacetime and stress the fact that the phase space is 4 dimensional.

Section 2 contains an introduction to the polygon approach. We compute the constraint algebra of the polygons and conclude that the algebra closes but is highly nonlinear. We propose to define a new constraint for which the constraint algebra closes linearly. In this section we also reproduce the one-polygon solution for the toroidal universe of Guadagnini and Franzosi and it is shown that it contains only part of phase space.

In section 3 we propose a 2 polygon representation for the torus and show that

\[ \mathcal{M} = \Sigma(g) \times R \]

where \(\Sigma(g)\) is a genus-\(g\) spacelike surface and \(R\) is in the time direction.

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\(^2\)The topology of the universe under consideration is:

\[ \mathcal{M} = \Sigma(g) \times R \]

where \(\Sigma(g)\) is a genus-\(g\) spacelike surface and \(R\) is in the time direction.
Figure 1: The torus as constructed by Carlip.

the phase space is now 4 dimensional.
In the discussion we comment on possible roads to quantization.
Appendix A gives some details of the calculation of the constraint algebra.
In appendix B we carefully count the number of degrees of freedom for the two-polygon torus.

1 The Torus Universe

In this section we recapitulate the work of Carlip [8] and Louko and Marolf [10].
The construction of the torus starts by studying its first homotopy group (or fundamental group):

\[ \pi_1(T^2 \times \mathbb{R}) = \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \]  

(1)

The fact that the fundamental group is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \) implies that there are 2 closed paths \( \alpha_1 \) and \( \alpha_2 \) that cannot be deformed into one another or in the trivial path. They fulfill the relation:

\[ \alpha_1 \circ \alpha_2 \circ \alpha_1^{-1} \circ \alpha_2^{-1} = I \]  

(2)

where \( \circ \) stands for path composition. Next we must construct the representation of this fundamental group in the three dimensional Poincaré group ISO(2,1).
From Witten’s gauge formulation of 2+1-D gravity [6] we know that 2 holonomies around a loop are equivalent if they can be conjugated into each other (i.e. \( \Lambda \sim SAS^{-1} \)). Louko and Marolf have shown that the possible holonomies (up to conjugation) split into 4 sectors; a spacelike sector, a timelike sector, a null sector
and a static (spacelike) sector. We will not consider the timelike and null sector in the following as they suffer from causality problems (closed timelike curves). The spacelike sector can always be conjugated into:

\[
\Lambda(\alpha_i) = B_y(\eta_i)T_x(a_i)
\]

where \(B_y(\eta_i)\) are boost matrices in the y-direction with rapidity \(\eta_i\), and \(T_x(a_i)\) are translations in the x-direction over a distance \(a_i\). The transformations \(\Lambda(\alpha_i)\) are in the spacelike sector because the projection to \(\text{SO}(2,1)\) (i.e. \(B_y(\eta_i)\)) stabilizes a spacelike vector.

The static sector is given by:

\[
\Lambda(\beta_i) = T_x(b_i)
\]

Only these physical sectors are expected to be contained in the set of “ADM-solutions” \[11, 12\]. The construction of the torus proceeds by finding a non-degenerate triad \(e^a\) that reproduces these holonomies. Another way is to find a fundamental region of Minkowski space \(\mathcal{J}\) upon which the holonomies act properly discontinuously. The quotient space \(\mathcal{J}/\{\Lambda(\alpha_i)\}\) will then be our torus.

In the spacelike case this construction is as follows: first introduce the coordinates:

\[
\begin{align*}
t &= \tau \cosh u \\
y &= \tau \sinh u \\
x &= x
\end{align*}
\]

The Minkowski metric in these coordinates is:

\[
ds^2 = -d\tau^2 + \tau^2 du^2 + dx^2
\]

The transformations \(\Lambda(\alpha_i)\) act on these coordinates as:

\[
\Lambda(\alpha_i) : (\tau, u, x) \rightarrow (\tau, u + \eta_i, x + a_i)
\]

The torus is thus constructed by identifying these points on a \(\tau =\)constant surface (see figure (1)).

In the case of the static torus the holonomies act on the Minkowski coordinates as:

\[
\begin{align*}
\Lambda(\beta_1) : (t, x, y) &\rightarrow (t, x + b_1, y + b_2) \\
\Lambda(\beta_2) : (t, x, y) &\rightarrow (t, x + b_3, y)
\end{align*}
\]

In this case the torus is constructed by making these identifications on a flat 2-dimensional \(t =\)constant slice (see figure (2)). It is important to notice that in the spacelike sector it is possible that one of the \(\eta_i\) is zero. Actually the construction works only if \(\varepsilon^{\alpha\beta} \eta_\alpha a_\beta \neq 0\). This implies that the static torus is not in the set considered by Carlip. As we will see, the static torus is included in the polygon construction.

\[3\] Our definition of time is different from the one used in Carlip’s paper \[8\]. We use \(\tau\) where he uses \(\frac{1}{\tau}\).
2 The Torus in the Polygon Approach

In this section we give an introduction to the polygon approach and calculate the constraint algebra of the polygons. We also construct a one-polygon torus \[15\]. The polygon approach to 2+1-D gravity was invented by ’t Hooft \[3, 4\] as a Hamiltonian description of 2+1-D gravity. The main idea behind this approach is to construct a Cauchy surface by gluing together piecewise flat patches of spacetime. Let’s define coordinates $x^\mu_I$ on polygon I. These can be transformed to a new frame (to be used in polygon II) by the following Poincaré transformation (see figure (3)):

$$x^\mu_{II} = [T_y(-l_3)B_y(2\eta)T_x(-l_2)T_y(-l_1)]^\mu^\nu x^\nu_I$$  \hspace{1cm} (8)

where $B$ is a boost and $T$ is a translation. To construct a Cauchy surface we consider the condition $t_I = t_{II}$. If we put this in (8) we find an equation for the boundary between the 2 frames:

$$y_s^I(t) = -vt + l_1 \quad v = \tanh \eta$$

$$y_s^{II}(t) = vt - l_3$$  \hspace{1cm} (9)
where $y_s$ denotes the y-position of the boundary. Next we define coordinates $x_{I,II}^\mu$ on a third frame and repeat the same procedure. The boundary between II and III can now be calculated. The three edges meet in a vertex (see figure (4)). The fact that the three dimensional curvature must vanish at the vertex implies a relation between the angles $\alpha_i$ and $\eta_j$.\footnote{The $\eta_j$ should however always obey the triangle relation: $|\eta_i| + |\eta_j| \geq |\eta_k|$ $i, j, k = 1, 2, 3.$} If we perform the three Lorentz transformations in sequel we should end up in the original frame:

$$x^\mu = [R(\alpha_k)B(2\eta_k)R(\alpha_j)B(2\eta_j)R(\alpha_i)B(2\eta_i)]^\mu_{\quad \nu} x^\nu$$ \hfill (10)

The equations generated from this constraint are the vertex relations:

$$s_i : s_j : s_k = \sigma_i : \sigma_j : \sigma_k$$ \hfill (11)

$$\gamma_j s_k + s_i c_j + c_i s_j \gamma_k = 0$$

$$c_i = c_j c_k - \gamma_i s_j s_k$$

$$\gamma_i = \gamma_j \gamma_k + c_i \sigma_j \sigma_k$$

$$c_j = -\frac{c_i \gamma_k}{s_i} - \frac{\gamma_j \sigma_k}{s_i \sigma_j}$$ \hfill (12)

where we defined:

$$\sigma_i = \sinh(2\eta_i)$$ \hfill (13)

$$\gamma_i = \cosh(2\eta_i)$$

$$c_i = \cos \alpha_i$$

$$s_i = \sin \alpha_i$$ \hfill (14)

They allow us to calculate for instance the three angles $\alpha_i$ from the three rapidities $\eta_j$. If we continue to introduce new frames (and thus new edges) we end up with a $t=$constant surface made out of polygons. (see figure (5)).

Figure 4: The three-vertex.
Particles can also be included by putting them at the end of a boundary. The transformation over that line is however a bit more complicated:

\[ \tilde{x}^\mu = [T(\tilde{a})BRB^{-1}T(-\tilde{a})]^\mu_\nu x^\nu \] (15)

where \( T(\tilde{a}) \) is a translation to the position of the particle \( \tilde{a} \), \( B \) is a boost in the direction of the velocity of the particle and \( R \) is a rotation over an angle proportional to the particle’s mass. From the above we may also notice that the coordinates are multivalued in the presence of the particles (see [7]). We will not go into this further as we will not need particles in the following. The interested reader is referred to [3, 4, 7]. As was shown by ’t Hooft we can turn this description into a Hamiltonian formulation. The edge lengths \( L_i \) are the canonically conjugate variables to the rapidities \( \eta_j \):

\[ \{2\eta_j, L_i\} = \delta_{ij} \] (16)

The Hamiltonian is then given by the sum of all the deficit angles at the particles and the vertices:

\[ H = \sum_P \beta_P + \sum_V (2\pi - \alpha_1^V - \alpha_2^V - \alpha_3^V) \] (17)

Remember that the angles \( \alpha_i \) can be expressed in terms of the rapidities \( \eta_j \) using the vertex relations. The same is true for the angles \( \beta_P \). In the above expression \( V \) labels the vertices and \( P \) the particles. One can now for instance check that the following relation hold:

\[ \frac{d}{dt}L_i = \{H, L_i\} \quad \frac{d}{dt}\eta_j = \{H, \eta_j\} \quad (= 0) \] (18)

Another important aspect is the constraints of the theory. First of all the angles within one polygon should fulfill the relation:

\[ C_1 = \sum_{k=1}^N \alpha_k - (N - 2)\pi = 0 \] (19)

Figure 5: Cauchy surface made out of polygons.
where $N$ is the total number of corners inside the polygon. Also, if temporarily we view the edge lengths $L_i$ as vectors, the sum of the edges that enclose a polygon should add up to zero. In complex notation:

$$C_2 = \sum_{k=1}^{N} L_k e^{-i\theta_k} = 0 \quad \theta_k = \sum_{l=2}^{k} \alpha_l - (k-1)\pi$$

(20)

These are first class constraints and they generate “gauge-transformations” of the system. As can be verified $C_1$ pushes the particular polygon forward in time. After this time evolution of one polygon, the surface must still be a Cauchy surface. This can only be the case in general if the boundaries rearrange themselves. So the phase space variables change as follows:

$$\delta L_i = \{C_1, L_i\} \quad \delta \eta_j = \{C_1, \eta_j\}$$

(21)

The complex constraint $C_2$ generates boost transformations of the polygon (in 2 independent directions). If we boost the polygon to a new Lorentz frame, the $L_i$ and $\eta_j$ change in order to keep the surface a $t=$constant surface. This change is generated in the same way as in (21). For consistency we must check that the constraint algebra closes. The result of this rather lengthy calculation is:

$$\{C_1^A, C_1^B\} = 0$$

$$\{C_1^A, C_2^B\} = \frac{1}{2} \left( \frac{\sigma_1 \gamma_N + \sigma_N \sigma_1 \gamma_1}{s_1 \sigma_N \sigma_1} + i \frac{\gamma_1}{\sigma_1} \right) (1 - e^{-iC_1^A}) \delta_{A,B}$$

$$\{\bar{C}_2^A, C_2^B\} = \frac{1}{2} \frac{\gamma_1}{\sigma_1} (C_2^A + \bar{C}_2^A) \delta_{A,B}$$

The index $A$ ($B$) labels the different polygons. For the definition of the $\eta_i$, $L_i$ and $\alpha_i$ see figure (A.1) in the appendix A where one can also find some details of the calculation. Although this algebra closes on-shell the first bracket is very nonlinear. To remedy this situation we change $C_1$ to a new constraint:

$$D_1 = e^{iC_1} - 1$$

(23)

The first bracket is now replaced by:

$$\{D_1^A, C_2^B\} = \frac{i}{2} \left( \frac{\sigma_1 \gamma_N + \sigma_N \sigma_1 \gamma_1}{s_1 \sigma_N \sigma_1} + i \frac{\gamma_1}{\sigma_1} \right) D_1^A \delta_{A,B}$$

(24)

Using this new constraint we find that the algebra closes linearly. If we quantize the theory we should replace poisson-brackets by commutators. Moreover, the constraint $D_1$ acts as the generator of time translations of one unit of time. The constraint algebra of infinitesimal Lorentz transformations and time steps of one unit now closes linearly. This is consistent with the fact that time should be quantized in this way in the quantum theory as was noted by ’t Hooft [4, 5].
Finally we like to comment on the transitions that may occur during evolution as one of the lengths $L_i$ shrinks to zero. Two out of nine possible transitions are shown in figure (6). The rapidities $\eta_j$ of the newly opened edges can be calculated from the rapidities of the pre-transition diagram. It is in fact these transitions that render the theory non-trivial.

Next we will construct a toroidal spacetime using only one polygon. In the next section we will give an alternative method to construct the torus. We must cut the torus open according to figure (7). To do the cutting one could use the double line representation of \[15\]. Next we impose the constraint $C_1 = 0$:

$$C_1 = 2(\alpha_1 + \alpha_2 + \alpha_3 - 2\pi) = 0 \tag{25}$$

This implies that both vertices $A$ and $B$ have no angle deficit: $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$. Moreover, if we use this constraint in $C_2$ we see that it is automatically fulfilled. It means that $C_2$ is no longer an independent constraint. The only possible vertices that have no angle deficit are vertices where all adjacent $\eta_j$ vanish or vertices where 2 rapidities are equal and one vanishes (see figure (8)).

![Figure 6: 2 possible transitions.](image)

![Figure 7: Cutting open a torus using only one polygon.](image)
Consider first the case where all rapidities are vanishing. As the rapidities are zero, the $\alpha_i$ are free and we may choose them as simple as possible: $\alpha_3 = \pi$ and $\alpha_1 = \alpha_2 = \frac{1}{2} \pi$. (Note that the free angles are not degrees of freedom.) The identification rules we have found are thus (see figure (7) and take all $\eta_j = 0$):

$$x^\mu = [T_x(L_2)T_y(L_3)]^\mu_\nu x^\nu$$

$$x^\mu = [T_x(L_1 + L_2)]^\mu_\nu x^\nu$$

Comparison with (8) gives that this is precisely the static torus as constructed using the methods of section 1.

Next consider the case where both vertices are of the type of figure (8). In this case only $\alpha_1 = \alpha$ is free and we choose it to be equal to $\frac{1}{2} \pi$ for convenience. Again we stress that it cannot be considered as a degree of freedom as it "separates" two Lorentz frames that are really the same ($\eta_3 = 0$). Different choices of $\alpha$ will therefore not give different tori but just another way of describing the same torus.\(^5\) The torus that we constructed is thus an expanding or shrinking torus in one direction and is therefore in the spacelike sector (see figure (7) and take $\eta_1 = \eta_2 = \eta$ and $\eta_3 = 0$). Counting degrees of freedom we find $L_1, L_2, L_3, \eta$. As we have a closed universe, time cannot be defined at infinity, but must be defined in terms of internal degrees of freedom. $L_3$ is the only degree of freedom that changes (and it does so at a constant rate: $\frac{d}{dt} L_3 = 2 \tanh \eta$), so we choose it as our time variable:

$$T = L_3$$

Another way of seeing this is that in generally covariant systems time evolution is a gauge transformation connected with the freedom of reparametrizing time. This implies that we should fix this gauge by choosing explicitly time in terms of the phase space variables.

So we end up with an odd, three dimensional phase space which hints at the fact that we missed part of the possible configurations by our choice of slicing.\(^6\) To see this more clearly we must analyze the identification rules and compare them

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\(^5\)In this aspect we clearly disagree with the authors of [5].

\(^6\)I would like to thank T. Jacobson for pointing this out to me.
with (3). We find:

\[ x^\mu = [T_x(L_1 + L_2)]^\mu _\nu x^\nu \]

\[ x^\mu = [T_x(L_2)T_y(L_3)B_y(\eta)]^\mu _\nu x^\nu \]  

(28)

Changing variables from \((L_1 + L_2; L_2; \eta)\) to \((a_1; a_2; \eta_2)\) and conjugating both transformations by:

\[ \Lambda_i \rightarrow T_t(L_3^2)v T_y(L_3^2) \]  

\[ v = \tanh \eta \]  

(29)

we arrive at:

\[ x^\mu = [T_x(a_1)]^\mu _\nu x^\nu \]  

(30)

\[ x^\mu = [T_x(a_2)B_y(\eta_2)]^\mu _\nu x^\nu \]

The fact that we can transform \(T_x(L_3)\) away is consistent with the fact that it is simply time which is not an independent degree of freedom. The transformation (29) is precisely a time translation to \(t = 0\) where the identification is indeed given by (31). It is now clear that we can only access the phase space with \(\eta_1 = 0\) using a one polygon slicing. The next thing to try is then obviously to add another polygon in the construction of the torus.

3 The Two-Polygon Torus

In this section we discuss the general method for constructing a particle-free torus. Using this method due to ’t Hooft we prove once more that one polygon is certainly not enough to describe whole of phase space. We then discuss a two polygon torus. In appendix B we show that its phase space is four dimensional.

Let us define a Lorentz frame with an observer on it. The observer has coordinates \(x^\mu = (t, 0, 0)\). There are also Poincaré transformed copies of the observer floating around. The Poincaré transformations of these copies are given by combinations of the \(\Lambda(\alpha_i)\) of (3) i.e. \(\Lambda(\alpha_1)^n\Lambda(\alpha_2)^m\) with \(n\) and \(m\) integers. At \(t = 0\) all the copies are situated at the x-axis with different velocities in the y-direction. As we let the system evolve for a while the copies move up and are situated on a lattice (see figure (9)).

Using the method described in section 2 we can define a boundary between any two copies in such a way that the coordinates of the copies are indeed transformed according to the transformations \(\Lambda_i\). If we choose to slice spacetime with only one polygon we construct one boundary between each copy. Where the boundaries meet, vertices are formed as described in section one. The edges now enclose a region with precisely one copy on it. We may take one of the regions as our

\footnote{Remember that \(\Lambda(\alpha_1)\) and \(\Lambda(\alpha_2)\) commute.}
fundamental region and identify points on the boundary. In the case of the torus however all the boundaries are horizontal lines. Because on the total lattice of copies one can always find a copy as close to the y-axis as one pleases, the torus will become an infinitely small, infinitely long cylinder. This is of course a singular situation and we will try to remedy it by adding another polygon. In this case we may define two boundaries between certain copies. For the construction of such a two-polygon torus we add two extra vertex points on the torus and cut it open along the lines drawn in figure (10). Of course there are many possibilities of drawing the lines between the vertices corresponding with different kinds of two-polygon tori. Not all cuttings are successful however in improving the singular situation of the one-polygon torus. We also added arrows on the boundaries. We will define a boost to be positive as the arrows are directed towards the center (like in $L_1$) and negative if they are directed towards the vertices (like in $L_5$). The arrows are turned into vectors by giving them lengths equal to $|\eta_i|$. For small $\eta_i$ the vectors should add up to 0 at a vertex due to the vertex relations. For large $\eta_i$ this is not exactly true anymore but still a good indication for how to draw the diagram approximately. The boundary between 2 polygons that represent the same piece of (transformed) spacetime should be horizontal because observers see their copies move in the y-direction (so $L_2$ and $L_4$ are horizontal). Moreover we cannot change the values $\eta_2$ and $\eta_4$ by defining new Lorentz frames since both frames (I and I') change by the same amount. In both figures we have included an observer (X). Observer X sees his copy X’ move in the y direction with velocity $v = \tanh(2\eta_2)$ and translated in the x-direction and y-direction. As we have seen before, the translation in the y-direction is due to the time-evolution and is immaterial. To see what the other independent Poincaré transformation is we consider the copy X”. To see that this copy is moving in the y-direction we consider the path $\gamma$. To get to the copy we must move over the boundaries $L_5$ and $L_6$. But because the path $\gamma$ is contractable to a point (this is precisely what the vertex relations tell us) we see that the Lorentz transformation must
be in the \( y \)-direction with rapidity \( 2\eta_4 - 2\eta_2 \). This proves that the observer \( X \) sees his copy \( X'' \) move in the \( y \)-direction with velocity \( \tanh(2\eta_4 - 2\eta_2) \) and translated in the \( x \)- and \( y \)-direction. The movement in the \( y \)-direction is once again due to time evolution. If we would take for instance \( \eta_2 = \eta_4 \) it is easy to see that the translation in the \( y \)-direction vanishes. In the appendix we give a detailed proof that the number of degrees of freedom is indeed four. There are two configuration variables \( L_2 \) and \( L_4 \) and two independent momentum variables \( 2\eta_2 \) and \( 2\eta_4 \). Using gauge transformations we can choose the magnitude and orientation of \( \eta_6 \). Furthermore, the length \( L_6 \) will be taken as the physical time. Using this input we can uniquely calculate the two-polygon torus of figure (10). It is however hard to prove that this particular two-polygon construction works for all possible tori corresponding every Poincaré transformations of (3). In figure (10) we took the translations \( a_i \) positive. For negative \( a_i \) we might for instance consider the mirror-image of figure (10). It might even be necessary to add another polygon in certain regimes of phase space. During time evolution ’t Hooft-transitions may occur. This complicates the description considerably.

4 Discussion

In this paper we studied the torus in the polygon representation. We found that we need at least two polygons to describe the torus universe. This implies that the description of [15] covers only part of phase space. We also gave an explicit construction of such a two-polygon representation. As soon as particles
are present, generic single polygon representations do exist. Originally we had hoped that the torus was so simple that we could try to quantize the system. It first seemed that we did not have to include transitions in this model. As we try to quantize the two-polygon torus we can follow two routes. The first is to try to find reduced phase space variables and substitute commutators for poisson-brackets. However finding the effective Hamiltonian will be a difficult job due to the non-linearity of the constraints. Another possibility is to keep all classical variables and reduce the Hilbert space by promoting the classical constraints to quantum operator constraints. Of course, the non-linearity of the constraints will be hard to handle also in this case. But even if one has overcome these problems one still has to add the transitions as boundary conditions on the wave functions. Moreover the torus should be invariant under the group of modular transformations. There is a possibility however that we can simplify things a bit. This was already suggested by ’t Hooft in [4]. Say we start with a two-polygon torus. As the system evolves transitions will take place to for instance to a three polygon representation. Using the gauge transformations (generated by the constraints) we might be able to transform back to the two-polygon representation. For instance, we could evolve one polygon forward (or backward) in time until it disappears. Alternatively we could Lorentz transform one polygon in such a way that it has the same Lorentz frame as a neighboring polygon. The rapidity $\eta$ between the two-polygons disappears and one is really left with one polygon less.

As all modular transformed universes must be considered equal it might also happen that these modular transformations can “move us back” to the original diagram. We did however not investigate what the action of the modular group on the two-polygon torus is.

A last issue concerning the quantization is the question if time is quantized. For open systems where the coordinate $t$ can be taken as the physical time, ’t Hooft argued that time is indeed quantized [4, 5]. We found that the constraint algebra supports this idea. For closed systems however, the parameter $t$ cannot be considered as the physical time but merely as a gauge parameter connected with reparametrization invariance. Whether the physical time is also quantized in a closed system needs to be investigated further. Waelbroeck seems to have found evidence that this is not the case [14].

To summarize we must conclude that the polygon approach is not the most convenient way of treating the particle-free torus.

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Appendix A: Calculation of the Constraint Algebra

In this appendix we will give some details of the calculation of the algebra (22). First we define the variables $L_i, \eta_i, \alpha_i$ for polygon I and $K_j, \xi_j, \beta_j$ for polygon II in figure (A.1). The result will not depend on the precise definition as one can check (i.e. where $L_1$ is situated as compared to polygon II). Next we establish some basic results that we will need in the following. Consider figure (4). From the vertex relations (11) we have the identity:

$$c_i = \frac{\gamma_i - \gamma_j \gamma_k}{\sigma_j \sigma_k} \quad \text{and cyclic permutations} \quad (A.1)$$

From this we calculate:

$$\frac{\partial \alpha_i}{\partial \eta_i} = -\frac{\sigma_i}{s_i \sigma_j \sigma_k} \quad (A.2)$$

$$\frac{\partial \alpha_i}{\partial \eta_j} = -\frac{\sigma_i c_k}{s_i \sigma_j \sigma_k} = \frac{\sigma_j \gamma_k + c_i \sigma_k \gamma_j}{s_i \sigma_j \sigma_k} \quad (A.3)$$

If we take for instance the partial derivative $\frac{\partial}{\partial \eta_1}$ in the above formula, we keep the other rapidities ($\eta_2$ and $\eta_3$) fixed. Another useful identity that is derived using the last two identities is (see figure (A.2)):

$$\frac{\partial \alpha_{k+1}}{\partial \eta_k} - \frac{\gamma_k}{\sigma_k} = e^{-i \alpha_{k+1}} \left( \frac{\partial \alpha_{k+1}}{\partial \eta_{k+1}} + i \frac{\gamma_{k+1}}{\sigma_{k+1}} \right) \quad (A.4)$$

A last (vertex) relation that we will need in the following is:

$$s_i : s_j : s_k = \sigma_i : \sigma_j : \sigma_k \quad (A.5)$$

Figure A.1: Definition of rapidities and angles for the calculation of the constraint algebra.
The bracket $\{C_1^I, C_1^I\}$ must vanish because the angles only depend on the momenta $\eta_i$. Next we calculate the following bracket between two different polygons:

$$\{C_1^I, C_2^I\} = \{\alpha_{N-1} + \alpha_N, e^{-i\varphi_{M-2}}(K_{M-2} - K_{M-1}e^{-i\beta_{M-1}} + K_Me^{-i(b_{M-1}+\beta_M)}) = \frac{1}{2}e^{-i\varphi_{M-2}}\left(\frac{\partial \alpha_N}{\partial \xi_{M-2}} - \frac{\partial (\alpha_{N-1} + \alpha_N)}{\partial \xi_{M-1}}e^{-i\beta_{M-1}} + \frac{\partial \alpha_{N-1}}{\partial \xi_{M}}e^{-i(\beta_{M-1}+\beta_M)}\right)$$

Note that only $\alpha_{N-1}$ and $\alpha_N$ contribute. Analogous to (20) we used here the definition: $\varphi_i = \sum_{l=1}^{i} \alpha_l - (i - 1)\pi$. Using (A.2), (A.3) and (A.5) we find that it vanishes. Next we calculate:

$$\{C_1^I, C_2^I\} = e^{-i(\theta_{N-2}+\varphi_{M-2})}\{(L_{N-2} - L_{N-1}e^{-i\alpha_{N-1}} + L_Ne^{-i(\alpha_{N-1}+\alpha_N)}), (K_{M-2} - K_{M-1}e^{-i\beta_{M-1}} + K_Me^{-i(\beta_{M-1}+\beta_M)}) = -ie^{-i(\theta_{N-2}+\varphi_{M-2})}(L_{N}e^{-i(\alpha_{N-1}+\alpha_N)}\{\alpha_N, K_{M-2}\} + L_{N-1}e^{-i(\beta_{M-1}+\alpha_{N-1})}\{\alpha_{N-1}, K_{M-1}\} - e^{-i(\beta_{M-1}+\alpha_{N-1})}\{\alpha_{N-1}, K_{M}\} + L_{N}e^{-i(\beta_{M-1}+\beta_M+\alpha_{N-1})}\{\alpha_{N-1}, K_{M}\} - (\beta \leftrightarrow \alpha, L \leftrightarrow K)$$

All terms proportional to $L_i$ (or $K_i$) should vanish independently. The calculation of the term proportional to $L_N$ (and $K_M$) turns out to be proportional to the bracket (A.7) and therefore vanishes. For the calculation of the remaining terms we notice that $L_{N-1} = K_{M-1}$. Using again (A.2), (A.3) and (A.5) we find that it vanishes. The calculation of $\{C_1^I, C_2^I\}$ follows the same lines. Next we calculate the brackets between constraints of the same polygon:

$$\{C_1^I, C_2^I\} = \sum_{i=1}^{N} \alpha_i, \sum_{j=1}^{N} L_j e^{-i\theta_j}$$

$$\frac{1}{2}\left(\frac{\partial \alpha_1}{\partial \eta_1} + \frac{\partial \alpha_2}{\partial \eta_2} - e^{-i\alpha_2}\left(\frac{\partial \alpha_2}{\partial \eta_2} + \frac{\partial \alpha_3}{\partial \eta_2}\right) + ... + e^{-i\alpha_1}\left(\frac{\partial \alpha_N}{\partial \eta_N} + \frac{\partial \alpha_1}{\partial \eta_N}\right)\right)$$

In the first term we add and subtract $\frac{i\sigma_1}{\sigma_2}$. Idem for the second term, where we add and subtract $\frac{i\sigma_2}{\sigma_2}$ etc. Using relation (A.4) we see that pairs of terms cancel.\footnote{for the definitions of $\sigma, \gamma, c, s$ see (13).}
The last term and the first term must be calculated explicitly using (A.3) to find the desired result (22). Finally we calculate:

\[ \{ C^I_2, \bar{C}^I_2 \} = \{ \sum_{i=1}^{N} L_i e^{-i\theta_i}, \sum_{j=1}^{N} L_j e^{i\theta_j} \} \] (A.9)

This can be simplified to:

\[ -i \sum_{i,j=1}^{N} L_j \cos(\theta_i - \theta_j) \frac{\partial \theta_j}{\partial \eta_i} \] (A.10)

Again using (A.3) the desired result will follow.

**Appendix B: Degrees of Freedom for the Two-Polygon Torus**

In this appendix we will carefully count the number of degrees of freedom present in the two-polygon representation of the torus given in figure (B.1). First we notice that all the vertices and angles are already present in and around polygon II (see figure (B.4)). If we count the momentum degrees of freedom we only have to consider this part of the diagram. Once all rapidities and angles are determined here, they are also fixed in the rest of the diagram. We claim that the momentum degrees of freedom are \( \eta_2 \) and \( \eta_4 \). They can not be changed by gauge transformations as explained in section 3. Different choices of \( \eta_2 \) and \( \eta_4 \) imply different tori. By using Lorentz transformations we can choose the magnitude and orientation of \( \eta_6 \). This determines the relative Lorentz frames of I and II. So we have now \( \eta_2, \eta_4, \eta_6, A_1 \) and \( C_3 \). (Remember that \( L_2 \) and \( L_4 \) must be horizontal.) Given two \( \eta_i \) and one angle around a vertex we can determine all angles and rapidities at that vertex using the vertex relations. It implies that we can calculate now \( A_6, A_4, \eta_1, C_3, C_6 \) and \( \eta_3 \). Using the constraint that \( A_6 + B_5 = 2\pi \) we can determine \( B_5 \). But now we have two rapidities \( (\eta_2 \) and \( \eta_1 ) \) and one angle \( (B_5) \) at the vertex B. So we can calculate \( B_1, B_2 \) and \( \eta_5 \). Because the angles inside polygon II must add up to \( 2\pi \) we know \( D_4 \). Again we have two rapidities and one angle at vertex D which determines \( \xi, D_5 \) and \( D_3 \). Of course we would like to prove now that \( \xi = \eta_4 \) and its orientation to be horizontal. Then the constraints \( B_1 + D_3 = \pi \) and \( D_5 + C_6 = 2\pi \) are also obeyed. We calculate \( \xi \) and its orientation by noting that the holonomy around the vertex D must be trivial (see [10]). From that we derive the vertex relations. As long as the vertex relations are obeyed one can move the loop of the holonomy over a vertex without changing its holonomy. Let’s change the loop around vertex D to the loop \( \gamma' \). We

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9 The notation for the angles is defined in figure (B.1)
find for the holonomy:

\[ B(2\eta_5)B(2\eta_6)B(-2\eta_2)B(2\xi) = I \] \hspace{1cm} (B.1)

Consider also the holonomy around loop \( \gamma \) if we traverse it in the direction as is indicated in the figure:

\[ B(2\eta_5)B(2\eta_6)B(2\eta_4)B(-2\eta_4) = I \] \hspace{1cm} (B.2)

Because \( B(2\eta_2) \) and \( B(2\eta_4) \) commute we derive that \( B(2\xi) = B(2\eta_4) \) which proves the statement. As mentioned earlier we can finish the rest of the diagram (except for the fact that we have to choose the lengths \( L_2 \) and \( L_4 \)) because all angles are known now. We are also assured that the angles inside polygon I will add up to \( 6\pi \). To see this we notice that the constraint splits actually in four constraints:

\[ C_3 + A_1 = \pi \] \hspace{1cm} (B.3)
\[ D_5 + C_6 = 2\pi \]
\[ B_1 + D_3 = \pi \]
\[ B_5 + A_6 = 2\pi \]

But these constraints were used or verified in the previous proof so they are automatically obeyed.

Next we count the independent length variables. It is clear that given a diagram we can change the lengths \( L_2 \) and \( L_4 \) independently and still have a valid diagram. So to make life easy we choose them to be zero. We have now two polygons with four of the same sides but in general different internal angles. Given the size of one of these sides (say \( L_6 \)) and the internal angles there is generally only one solution for the remaining three edge lengths for which we can draw these two
“four-gons”. The only independent variable (the scale or $L_6$) will be used to define the physical time. This concludes the proof that $L_2$ and $L_4$ are the only two degrees of freedom in configuration space.

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