Appendix. Properties of multiplier matrices

In the stability analysis of random dynamical processes we have used the properties of multiplier matrices following from the conditions 1 and 2 of sec.2. Here we shall consider some additional properties of these matrices related to other conditions. For simplicity, we put \( t_0 = 0 \) dealing with \( M(\xi, t) \equiv M(\xi, t, 0) \). Then eq.(9) takes the form

\[
\frac{\partial}{\partial t} M(\xi, t) = L(y, \xi, t) M(\xi, t),
\]

\[
M(\xi, 0) = 1. \tag{A.1}
\]

Suppose that the eigenvalue problem for the multiplier matrix has a solution with the eigenfunctions forming a complete orthonormal basis:

\[
M(\xi, t) \varphi_n(\xi, t) = \mu_n(\xi, t) \varphi_n(\xi, t),
\]

\[
\varphi_m^+(\xi, t) \varphi_n(\xi, t) = \delta_{mn}, \tag{A.2}
\]

\[
\sum_n \varphi_n(\xi, t) \varphi_n^+(\xi, t) = \hat{1}.
\]

**Property 1.** If a multiplier matrix satisfying conditions (A.1) and (A.2) is Hermitian, then its eigenvalues are

\[
\mu_n(\xi, t) = \exp \left\{ \int_0^t L_n(y, \xi, t') dt' \right\}
\]

where

\[
L_n(y, \xi, t) \equiv \varphi_n^+(\xi, t) L(y, \xi, t) \varphi_n(\xi, t).
\]

**Proof.** Differentiating the normalization equality in (A.2), we get

\[
\frac{\partial \varphi_m^+}{\partial t} \varphi_n + \varphi_m^+ \frac{\partial \varphi_n}{\partial t} = 0.
\]

Using this, we find

\[
\frac{\partial}{\partial t} \left( \varphi_m^+ M \varphi_n \right) = \varphi_m^+ \frac{\partial M}{\partial t} \varphi_n + (\mu_m^* - \mu_n) \varphi_m^+ \frac{\partial \varphi_n}{\partial t}.
\]

Since, by assumption, the multiplier matrix is Hermitian, \( M^+ = M \), its eigenvalues are real, \( \mu_n^* = \mu_n \). Because of this, we have the equality

\[
\varphi_n^+ \frac{\partial M}{\partial t} \varphi_n = \frac{\partial \mu_n}{\partial t}.
\]
The latter, together with (A.1) and (A.2), gives

$$\frac{\partial \mu_n}{\partial t} = L_n \mu_n, \quad \mu_n(\xi, 0) = 1,$$

from where we obtain Property 1.

Note that $L_n$ here is not necessarily an eigenvalue of the Lyapunov matrix, but just a matrix element of the latter with respect to the eigenfunctions of the multiplier matrix. These eigenfunctions of $M$ are not the eigenfunctions of $L$.

Another condition which yields an explicit relation between the multiplier and Lyapunov matrices is the commutation relation

$$\left[ L(y, \xi, t), \int_0^t L(y, \xi, t') dt' \right] = 0,$$

which sometimes called the Lappo–Danilevsky condition [26].

**Property 2.** Let the multiplier matrix enjoy conditions (A.1), (A.2), (A.3) and be Hermitian, then it possesses a common set of eigenfunctions with the Lyapunov matrix, their eigenvalues being related as

$$\mu_n(\xi, t) = \exp \left\{ \int_0^t \lambda_n(\xi, t') dt' \right\},$$

where $\lambda_n$ is an eigenvalue of the Lyapunov matrix,

$$L(y, \xi, t) \varphi_n(\xi, t) = \lambda_n(\xi, t) \varphi_n(\xi, t).$$

**Proof.** From (A.1) and (A.3) it follows that

$$M(\xi, t) = \exp \left\{ \int_0^t L(y, \xi, t') dt' \right\},$$

so that $[M, L] = 0$. Commuting matrices have a common set of eigenfunctions. Because of the self–adjointness of the multiplier matrix, Property 1 holds true. Since now $\varphi_n$ are the eigenfunctions of the Lyapunov matrix, we have $L_n = \lambda_n$, which accomplishes the proof.

The properties considered can simplify the consideration. However, a principal difficulty remains, related to the fact that the eigenfunctions $\varphi_n(\xi, t)$ depend, generally, on the random variable $\xi$ as well as on time $t$. 
Property 3. Assume that the multiplier matrix satisfies condition (A.2), then the variation (11) yields
\[ \delta c_n(\xi, t) = \mu_n(\xi, t) \delta f_n(\xi, t), \]
where
\[ c_n(\xi, t) = \varphi_n^+(\xi, t)y(\xi, t), \]
\[ f_n(\xi, t) = \varphi_n^+(\xi, t)f. \]

Proof. The proof is straightforward.

The latter property shows that, because of the dependence of \( f_n(\xi, t) \) on the random variable, one cannot, in general, define the simple relation (26) with the averaged local multiplier (25). This multiplier can be defined if the eigenfunctions \( \varphi_n(\xi, t) \) are stochastically invariant, that is \( \varphi_n(\xi, t) = \varphi_n(t) \), or if random fields are weak and the stochastic invariance of \( \varphi_n(t) \) is an acceptable approximation. In such a case the equation from Property 3 reduces to
\[ \delta c_n(t) = \mu_n(t) \delta f_n(t), \]
where \( c_n(t) \) and \( \mu_n(t) \) are the same as in (23) and (25) and \( f_n(t) = \varphi_n^+(t)f \).

However, in the obtained variation the multiplier \( \mu_n(t) \) does not play the role of a quantity characterizing stability with respect to the variation of initial conditions, since \( f_n(t) \) depends on time. Only if the eigenfunctions \( \varphi_n(t) \) are stationary, i.e. \( \varphi_n(t) = \varphi_n \), then we return to (26), and the local multiplier (25) becomes a genuine characteristic of local stability. This is why the condition of stationarity (condition 1 of sec.2) has been essential for the stability analysis.
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Local Stability of Dynamical Processes in Random Media

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Abstract

A particular type of random dynamical processes is considered, in which the stochasticity is introduced through randomly fluctuating parameters. A method of local multipliers is developed for treating the local stability of such dynamical processes corresponding to infinite-dimensional dynamical systems. The method is illustrated by several examples, by the random diffusion equation, random wave equation, and random Schrödinger equation. The evolution equation for the density matrix of a quasiopen statistical system subject to the action of random surrounding is considered. The stationary solutions to this equation are found to be unstable against arbitrary small finite random perturbations. The notion of random structural stability is introduced.
1 Introduction

The stability of dynamical systems described by ordinary differential equations can be characterized by Lyapunov exponents. There are several kinds of the latter, for example, the classical Lyapunov exponents characterizing asymptotic stability [1,2], the finite–interval exponents describing stability on finite intervals [3,4], the pointwise local–time Lyapunov exponents giving information on the local stretching and contraction rates of trajectories [5,6], and generalized Lyapunov exponents taking into account correlations [7].

More difficult is the situation with the so–called infinite–dimensional dynamical systems modelled by partial differential equations. There is no general stability theory for such equations, although in many cases one can analyse the asymptotic stability by linearizing these equations about a stationary solution [8].

Even more complicated is the case when partial differential equations contain random fields. The present paper addresses just this type of equations describing random dynamical processes. More precisely, only one particular type of these processes will be considered here, when the stochasticity is introduced into differential equations by means of randomly distributed parameters. Such a kind of equations may be called parametrically random differential equations. This is a specific case of stochastic differential equations. The latter are usually treated as containing Gaussian stochastic fields related to the Wiener or Ornstein–Uhlenbeck processes [9-12]. These will not be touched on in the paper – we shall deal only with parametrically random equations. And, although the words random and stochastic mean the same, we shall rather use the former instead of the latter in order to distinguish our case from what one usually implies under stochastic processes.

Parametrically random equations form an important class of equations that can be often met in applications, such as quantum electronics. Thus, the influence of interference on propagating signals may be modelled by randomly fluctuating amplitude and phase. The random shift of frequency describes various local defects leading to the so–called nonuniform broadening. Some local interactions can be treated as random forces whose
existence may become incomparably more essential for a nonequilibrium system than the presence of the Nyquist noise. For example, such random local fields, due to nonsecular dipole interactions, trigger pure spin superradiance in nonequilibrium nuclear magnets [13,14].

The approach suggested below may be generalized in several aspects. However, for the sake of clarity, here I prefer to stick myself to the parametrically random dynamical processes. To treat the stability of the latter, the method of multipliers is developed. For ordinary differential equations without random fields the method of multipliers is completely equivalent to that of the Lyapunov exponents. However, for partial differential equations, especially with random fields, the method of multipliers is more natural and general: one may define multipliers when it is difficult or even impossible to introduce Lyapunov exponents. Multipliers characterize both the local as well as asymptotic stability. They can describe not only the exponential stability, as the Lyapunov exponents do, but also other types of stability. The effectiveness of using multipliers for analysing the local stability of dynamical processes has been demonstrated earlier for approximation cascades and approximation flows [15-17].

2 Method of local multipliers

Let $x \in \mathbb{D}$ be a set of continuous variables pertaining to a domain $\mathbb{D}$ which can be bounded or not; let $t \in \mathbb{R}_+$ denote time; and let $\xi$ be a set of random variables with a probability distribution $p(\xi)$. Consider a set, enumerated by the index $i = 1, 2, \ldots, d$, of real or complex functions $y_i(x, \xi, t)$ of these variables. Define the column

$$y(\xi, t) = [y_i(x, \xi, t)]$$

with respect to $i$ and $x$. This column is assumed to be a solution of the evolution equation

$$\frac{\partial}{\partial t} y(\xi, t) = v(y, \xi, t)$$

(1)

with the velocity field

$$v(y, \xi, t) = [v_i(x, y, \xi, t)]$$
which may contain differential and integral operators acting on \( y(\xi, t) \). The evolution equation (1) is to be supplemented by the initial condition

\[
y(\xi, 0) = f = [f_i(x)],
\]

(2)
in which \( f_i(x) \) are given functions, and by boundary conditions. The quantity of final interest is the solution

\[
y(t) \equiv \int y(\xi, t)p(\xi)d\xi
\]

(3)
averaged over random variables.

Equation (1) is a compact form of writing down a wide class of evolution equations. In particular, it includes the finite \( d \)–dimensional dynamical systems described by ordinary differential equations. But in general, this class embraces infinite–dimensional dynamical systems with various partial differential equations, such as can be met in considering hydrodynamic and plasma turbulence [18,19], dynamics of structures in liquid flows [20], soliton dynamics in condensed matter [21], electromagnetic radiation in large systems [22-24], and phase–ordering kinetics in quenched systems [25].

The stability of motion is related to the variation of the solution \( y(\xi, t) \) at time \( t \),

\[
\delta y(\xi, t) = M(\xi, t, t_0)\delta y(\xi, t_0),
\]

(4)
with respect to its variation at time \( t_0 \). Here the \textit{multiplier matrix}

\[
M(\xi, t, t_0) = [M_{ij}(x, x', \xi, t, t_0)]
\]
is introduced with the elements

\[
M_{ij}(x, x', \xi, t, t_0) \equiv \frac{\delta y_i(x, \xi, t)}{\delta y_j(x', \xi, t_0)}.
\]

(5)
From this definition it follows that

\[
M_{ij}(x, x', \xi, t, t) = \delta_{ij}\delta(x - x').
\]

Therefore, at the coinciding times, \( t_0 = t \), the multiplier matrix is the unity matrix

\[
M(\xi, t, t) = \hat{1} \equiv [\delta_{ij}\delta(x - x')]
\]

(6)
in the space of the indices $i$ and variables $x$. Remind that in (4) the matrix notation is used according to which the right-hand side of (4) is the column

$$M(\xi, t, t_0) \delta y(\xi, t) = \left[ \sum_k \int M_{ik}(x, x', \xi, t, t_0) \delta y_k(x', \xi, t_0) dx' \right].$$

From the variational-derivative property

$$\frac{\delta y_i(x, \xi, t)}{\delta y_j(x', \xi, t_0)} = \sum_k \int \frac{\delta y_k(x, \xi, t)}{\delta y_k(x_1, \xi, t_1)} \frac{\delta y_k(x_1, \xi, t_1)}{\delta y_j(x', \xi, t_0)} dx_1$$

we find that

$$M(\xi, t, t_1)M(\xi, t_1, t_0) = M(\xi, t, t_0). \quad (7)$$

Putting in (7) $t_0 = t$ and using (6), we obtain the definition of the inverse multiplier matrix

$$M^{-1}(\xi, t, t_0) = M(\xi, t_0, t). \quad (8)$$

The properties (6)–(8) show that the multiplier matrices form a group

$$\mathcal{M}(\xi) = \{M(\xi, t, t_0) | t, t_0 \in \mathbb{R}_+ \},$$

which may be called the multiplier group. It is also evident that the multiplier matrices are the evolution operators for the variation $\delta y(\xi, t)$.

If the random variable $\xi$ were fixed, then the transformation (4) would be contracting provided that $||M(\xi, t, t_0)|| < 1$. However, this does not yield, in general, the stability of the averaged solution (3).

Variating the evolution equation (1), we get the equation

$$\frac{\partial}{\partial t} M(\xi, t, t_0) = L(y, \xi, t)M(\xi, t, t_0) \quad (9)$$

for the multiplier matrix, where the matrix

$$L(y, \xi, t) = [L_{ij}(x, x', y, \xi, t)]$$

consists of elements

$$L_{ij}(x, x', y, \xi, t) \equiv \frac{\delta v_i(x, y, \xi, t)}{\delta y_j(x', \xi, t)}. \quad (10)$$
In particular cases, the real parts of the eigenvalues of the matrix $L(y, \xi, t)$ define the Lyapunov exponents, because of which we shall call it the Lyapunov matrix. The property (6) plays the role of the initial condition for eq.(9). The boundary conditions for eq. (9) are to be obtained by variating those of eq.(1). For example, if the functions $y_i(x, \xi, t)$ are given at the boundary of $D$, denoted by $\partial D$, then

$$M_{ij}(x, x', \xi, t, t_0) = 0 \quad (x \in \partial D),$$

and if $\partial y_i/\partial x$ is given at $\partial D$, then

$$\frac{\partial}{\partial x}M_{ij}(x, x', \xi, t, t_0) = 0 \quad (x \in \partial D).$$

Generally, averaging the variation (4) over the random variables does not lead a simple relation between the variation of (3) and an averaged multiplier matrix, except for the case $t_0 = 0$, when $y(\xi, 0)$ is fixed by the initial condition (2) not containing $\xi$. In such a case the variation of (3) is

$$\delta y(t) = M(t)\delta f$$

with the averaged multiplier matrix

$$M(t) \equiv \int M(\xi, t, 0)p(\xi)d\xi.$$  \hspace{1cm} (12)

The transformation (11) is contracting provided that $||M(t)|| < 1$, from where $||\delta y(t)|| < ||\delta f||$. Then we may say that the motion is locally stable on the time interval $[0, t]$.

As we see, to characterize the stability of motion we need to know the multiplier matrix. If we are interested only in sufficient conditions of stability, we can evaluate the norm $||M(t)||$ by employing various inequalities [26] valid for the solutions of the matrix equations of the type (9). Since the multiplier matrix satisfies eq.(9) involving the Lyapunov matrix, the properties of these matrices are closely interrelated. Some properties of the multiplier matrices are considered in Appendix. Throughout the paper we shall mainly deal with the case when the Lyapunov matrix meets the following two conditions greatly simplifying the analysis.

**Condition 1.** The Lyapunov matrix is stationary:

$$L(y, \xi, t) = L(\xi) \equiv [L_{ij}(x, x', \xi)],$$

\hspace{1cm} (13)
that is, does not depend on time.

**Condition 2.** The Lyapunov matrix possesses a set of eigenfunctions forming a complete stochastically invariant basis:

\[
L(\xi)\varphi_n = \lambda_n(\xi)\varphi_n, \quad \varphi_n = [\varphi_n(x)],
\]  

which means that the eigenvalue problem (14) has solutions with the eigenfunctions forming a complete basis independent of the random variable \(\xi\).

If these conditions are not fulfilled for eq. (9), it may happen, nevertheless, that there exists a transformation reducing (9) to an equivalent equation with an effective Lyapunov matrix enjoying conditions (13) and (14). Another important case is when these conditions are fulfilled approximately, thus, defining an initial approximation for perturbation theory.

Assuming condition 1, one immediately gets the solution

\[
M(\xi,t,t_0) = \exp\{L(\xi)(t-t_0)\}
\]  

(15)
to the matrix equation (9). From here, under condition (2), it follows that the multiplier and Lyapunov matrices possess a common set of eigenfunctions,

\[
M(\xi,t,t_0)\varphi_n = \mu_n(\xi,t,t_0)\varphi_n,
\]  

(16)
with their eigenvalues related as

\[
\mu_n(\xi,t,t_0) = \exp\{\lambda_n(\xi)(t-t_0)\}.
\]  

(17)
The eigenvalues of the multiplier matrix will be called the *local multipliers*. As is obvious from (17),

\[
\mu_n(\xi,t,t) = 1.
\]  

(18)

With a complete basis \(\{\varphi_n\}\), we may define the expansions for the solution

\[
y(\xi,t) = \sum_n c_n(\xi,t)\varphi_n
\]  

(19)
and for its initial value

\[
f = \sum_n f_n\varphi_n,
\]  

(20)
where, according to the initial condition (2),

\[ c_n(\xi, 0) = f_n. \]

The coefficient functions \( c_n(\xi, t) \) in the expansion (19) have the meaning of natural variables, or natural components, of the solution \( y(\xi, t) \). For these components, from eq.(4) we find the variation

\[ \delta c_n(\xi, t) = \mu_n(\xi, t, t_0) \delta c_n(\xi, t_0). \] (21)

The expansion for the averaged solution (3) reads

\[ y(t) = \sum_n c_n(t) \varphi_n \] (22)

with the averaged components

\[ c_n(t) \equiv \int c_n(\xi, t) p(\xi) d\xi. \] (23)

Averaging eq.(16), we come to the eigenproblem

\[ M(t) \varphi_n = \mu_n(t) \varphi_n \] (24)

showing that the averaged multiplier matrix (12) has the eigenvalues

\[ \mu_n(t) = \int \mu_n(\xi, t, 0) p(\xi) d\xi. \] (25)

According to (18), we have \( \mu_n(0) = 1 \).

Using (22)–(25), we come to the conclusion that the variation (11) is equivalent to the set of variations

\[ \delta c_n(t) = \mu_n(t) \delta f_n \] (26)

for the natural components (23). The apparent form of (26), in which the averaged multiplier (25) is just a function, makes it possible to classify the local stability properties at time \( t \).

We shall say that the \( n \)-component at time \( t \) is locally stable, locally neutral, or locally unstable against the variation of initial conditions if, respectively,

\[ |\mu_n(t)| < 1 \quad (locally \ stable), \]

\[ |\mu_n(t)| = 1 \quad (locally \ neutral), \]

\[ |\mu_n(t)| > 1 \quad (locally \ unstable). \]
\[ |\mu_n(t)| = 1 \quad (locally \ neutral), \]  
\[ |\mu_n(t)| > 1 \quad (locally \ unstable). \]  

The motion as a whole, or the process, will be called locally stable, locally neutral or locally unstable according to whether \( \sup_n |\mu_n(t)| \) is less than one, equal to or more than one.

If the property of local stability, neutrality or instability of an \( n \) –component holds at each point \( t \) of an interval \([t_1, t_2]\), then we say that the \( n \) –component is, respectively, \textit{uniformly stable}, \textit{uniformly neutral} or \textit{uniformly unstable} on this interval. Similarly, the process is uniformly stable, uniformly neutral or uniformly unstable on an interval \([t_1, t_2]\), if the corresponding property for \( \sup_n |\mu_n(t)| \) holds at each point \( t \) of that interval.

In the case of local neutrality of an \( n \) –component the latter may be found to be unstable with respect to higher–order multipliers defined as second, third or higher variational derivatives.

As time \( t \to \infty \), the \( n \) –component can be asymptotically stable, neutral or unstable if, accordingly,

\[ \mu_n(t) \to 0 \quad (asymptotically \ stable), \]
\[ |\mu_n(t)| \in (0, \infty) \quad (asymptotically \ neutral), \]
\[ |\mu_n(t)| \to \infty \quad (asymptotically \ unstable). \]

The process is asymptotically stable, asymptotically neutral or asymptotically unstable if \( \sup_n |\mu_n(t)| \) tends to zero, remains finite or tends to infinity as \( t \to \infty \). Note that \( \lim_{t \to \infty} |\mu_n(t)| \) may not exist in the case of asymptotic neutrality.

Asymptotic stability can be of different types depending on the law by which \( \mu_n(t) \) tends to zero as \( t \to \infty \). For instance, this can be exponential decrease, power–law decay or another tendency. Recall the possibility of the polynomial asymptotic stability [27] of solutions to stochastic differential equations. Therefore, the multipliers allow a more refined description of stability than the Lyapunov exponents, since the latter describe only the exponential asymptotic stability.

Moreover, in the case of random processes the averaged multiplier (25) is, actually, the sole natural characteristic for defining stability. This follows from the fact that the
stability properties (27) and (28) are directly related to the multiplier $\mu_n(t)$, but not to the eigenvalues of the Lyapunov matrix $\lambda_n(\xi)$, these two quantities being connected through the integral

$$\mu_n(t) = \int \exp\{\lambda_n(\xi)t\} p(\xi) d\xi.$$ (29)

One can, of course, introduce effective Lyapunov exponents, such as effective finite–interval exponents

$$\lambda_n^{eff}(t, t_0) \equiv \frac{1}{t - t_0} \ln |\mu_n(t, t_0)|,$$

where

$$\mu_n(t, t_0) \equiv \int \mu_n(\xi, t, t_0) p(\xi) d\xi,$$

effective pointwise exponents

$$\lambda_n^{eff}(t, t) \equiv \lim_{\Delta t \to 0} \frac{1}{\Delta t} \ln |\mu_n(t + \Delta t, t)|$$

and effective asymptotic exponents

$$\lambda_n^{eff} \equiv \lim_{t \to \infty} \lambda_n^{eff}(t, t_0).$$

However, these would be excessive unnecessary definitions.

What useful we can get from defining the effective Lyapunov exponents is the following. Assume that $\{\lambda_n\}$ is a set of Lyapunov exponents ordered in the nonincreasing law, that is $\lambda_n \geq \lambda_{n+1}$. Let $N$ be the largest integer for which $\sum_{n=1}^{N} \lambda_n \geq 0$. Then the Lyapunov dimension [28] is

$$D_L \equiv N + \frac{1}{|\lambda_{N+1}|} \sum_{n=1}^{N} \lambda_n.$$

If we denote by $\lambda_n^+$ positive Lyapunov exponents, then the metric entropy is $h \equiv \sum_n \lambda_n^+$. These definitions can be transformed to the language of multipliers as follows.

Let the local multipliers (29), at each fixed $t$, be ordered so that

$$|\mu_n(t)| \geq |\mu_{n+1}(t)|.$$

And let $N = N(t)$ be the largest integer for which

$$\prod_{n=1}^{N(t)} |\mu_n(t)| \geq 1.$$
Then we define the *local Lyapunov dimension* as

\[ D_L(t) \equiv N(t) + \frac{\ln \prod_{n=1}^{N(t)} |\mu_n(t)|}{\ln |\mu_{N+1}(t)||}. \]  

The asymptotic Lyapunov dimension is

\[ D_L \equiv \lim_{t \to \infty} D_L(t), \]

if this limit exists. Similarly, the *local metric entropy* is

\[ h(t) = \frac{1}{t} \ln \prod_n |\mu_n^+(t)| \quad (|\mu_n^+(t)| > 1). \]

Concluding this section, it is worth making several remarks. First, the method of local multipliers seems to be a natural and convenient tool for analysing the local stability of motion. The appearance of local instability, in the case of quantum–mechanical models, may be connected with the quantum chaos [29] which is a temporal phenomenon. Note, however, that there are models of quantum systems [30,31] exhibiting the same asymptotic chaos as that in the models of classical mechanics.

Second, instead of emphasizing the time dependence, one could separate out one of the space coordinate considering the stability properties along the chosen space direction [32,33]. Then we could introduce the local, with respect to the separated variable, multipliers. Also, one could consider the motion treating a coupling parameter as a variable [34,35]. Then the stability properties with respect to the change of the coupling parameter could be analysed.

Third, we could be interested in the stability of stochastic processes against the variation, not of initial conditions, but of boundary conditions or of model parameters. Then a similar scheme of analysing the local stability could be developed.

### 3 Examples of random processes

The method described in the previous section will be illustrated here by several simple examples of random processes. We shall consider two types of the probability distribution
$p(\xi)$ for the stochastic variable $\xi$ : the *uniform distribution*

$$p(\xi) = \frac{1}{2\xi_0} [\Theta(\xi + \xi_0) - \Theta(\xi - \xi_0)],$$

in which $\Theta(\xi)$ is the unit–step function; and the *Gaussian distribution*

$$p_\gamma(\xi) = \frac{1}{\sqrt{2\pi\gamma}} \exp \left\{ -\frac{1}{2} \left( \frac{\xi}{\gamma} \right)^2 \right\}.$$

The average value of $\xi$ in both cases is zero, and the average of $\xi^2$, respectively, is

$$\int_{-\infty}^{+\infty} \xi^2 p(\xi) d\xi = \frac{1}{3}\xi_0^2, \quad \int_{-\infty}^{+\infty} \xi^2 p_\gamma(\xi) d\xi = \gamma^2.$$

### 3.1 Random diffusion equation

Consider the diffusion equation

$$\frac{\partial y}{\partial t} = (D + \xi) \frac{\partial^2 y}{\partial x^2}, \quad (31)$$

in which $x \in [0, L]$, $t \geq 0$, the diffusion coefficient $D > 0$, and $\xi$ describes random fluctuations of the diffusion coefficient. Such a situation can occur, for example, in a diffusion process through a nonhomogeneous medium consisting of randomly distributed regions with different diffusion coefficients. Eq.(31) is supplemented by an initial condition

$$y(x, \xi, 0) = f(x)$$

and boundary conditions

$$y(0, \xi, t) = c_1, \quad y(L, \xi, t) = c_2.$$

Comparing (1) with (31), we see that the velocity field is

$$v(x, y, \xi, t) = (D + \xi) \frac{\partial^2}{\partial x^2} y(x, \xi, t). \quad (32)$$

For the Lyapunov matrix (10) we get

$$L(x, x', \xi) = (D + \xi) \frac{\partial^2}{\partial x^2} \delta(x - x'). \quad (33)$$
The equation (9) for the multiplier matrix is
\[ \frac{\partial M}{\partial t} = (D + \xi) \frac{\partial^2 M}{\partial x^2}, \tag{34} \]
where \( M = M(x, x', \xi, t) \). Variating initial conditions and boundary conditions gives
\[ M(x, x', \xi, 0) = \delta(x - x'), \]
\[ M(0, x', \xi, t) = M(L, x', \xi, t) = 0. \]

Since the Lyapunov matrix (33) is stationary, it has common eigenfunctions with the multiplier matrix. These eigenfunctions, satisfying the boundary conditions
\[ \varphi_n(0) = \varphi_n(L) = 0 \]
and the normalization
\[ \int_0^L |\varphi_n(x)|^2 dx = 1, \]
are
\[ \varphi_n(x) = \sqrt{\frac{2}{L}} \sin k_n x \quad (k_n \equiv \frac{\pi n}{L}), \]
where \( n = 1, 2, \ldots \). The basis \( \{\varphi_n(x)\} \) is stochastically invariant, that is, does not depend on the random variable \( \xi \).

The eigenvalues of (33) are
\[ \lambda_n(\xi) = -(D + \xi)k_n^2. \tag{35} \]

Therefore, the local multipliers (17), with \( t_0 = 0 \), become
\[ \mu_n(\xi, t) = \exp\{-(D + \xi)k_n^2 t\}. \tag{36} \]

The same answer can be obtained by solving the equation (34) yielding the multiplier matrix
\[ M(x, x', \xi, t) = \sum_{n=1}^{\infty} \mu_n(\xi, t) \varphi_n(x) \varphi_n(x'). \tag{37} \]
The eigenvalues of (37) are, evidently, given by (36).
The multiplier matrix \((37)\) can also be found from the direct variation of the solution
\[
y(x, \xi, t) = \sum_{n=1}^{\infty} b_n \mu_n(\xi, t) \varphi_n(x) + y_\infty(x)
\]
to eq.\((31)\), where
\[
b_n = \frac{1}{L} \int_0^L \left[ f(x) - y_\infty(x) \right] \varphi_n(x) dx,
\]
\[
y_\infty(x) = c_1 + \frac{c_2 - c_1}{L} x.
\]

Now, let us find the averaged multiplier \((25)\). For the case of the uniform noise, the latter is
\[
\mu_n(t) = \frac{\sinh(\xi_0 k_n^2 t)}{\xi_0 k_n^2 t} \exp \left( -Dk_n^2 t \right).
\]

The asymptotic behaviour of \((38)\) at small and large times is
\[
\mu_n(t) \simeq 1 - Dk_n^2 t \quad (t \to 0),
\]
\[
\mu_n(t) \simeq \exp \left\{ \frac{\left( \xi_0 - D \right) k_n^2 t}{2 \xi_0 k_n^2 t} \right\} \quad (t \to \infty).
\]

According to \((27)\) and \((28)\), the behaviour of \((38)\) defines the local stability at time \(t\) of an \(n\)-component. Thus, the \(n\)-component for \(\xi_0 < D\) is uniformly stable for all \(t > 0\). As \(t \to \infty\), it is exponentially stable. Since \(\sup_n |\mu_n(t)| < 1\) for all \(n \geq 1\), the process is uniformly stable for all \(t > 0\).

When \(\xi_0 = D\), the \(n\)-component is also uniformly stable for all \(t > 0\). But as \(t \to \infty\), it displays now, not exponential, but power-law stability,
\[
\mu_n(t) \simeq \left( 2 \xi_0 k_n^2 t \right)^{-1} \quad (t \to \infty).
\]
The process is uniformly stable for \(t > 0\).

If \(\xi_0 > D\), then the \(n\)-component is locally stable on the interval \((0, t_n)\), where \(t_n\) is given by the condition \(|\mu_n(t_n)| = 1\). After this, it becomes unstable for all \(t > t_n\). Because of the limit \(\lim_{n \to \infty} |\mu_n(t)| = \infty\), the process is uniformly unstable for any \(t > 0\).

In the case of the noise with Gaussian distribution, the averaged multiplier \((25)\) becomes
\[
\mu_n(t) = \exp \left\{ \frac{1}{2} \left( \gamma k_n^2 t \right)^2 - Dk_n^2 t \right\}.
\]

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In this case, the \( n \)-component for any \( \gamma > 0 \) is locally stable till \( t_n = 2D/(\gamma k_n)^2 \), when \( |\mu_n(t_n)| = 1 \) and it is uniformly unstable for \( t > t_n \). As far as \( \sup_n |\mu_n(t)| = \infty \), the process is uniformly unstable for all \( t > 0 \).

### 3.2 Random wave equation

In the wave equation

\[
\frac{\partial^2 u}{\partial t^2} = (c + \xi)^2 \frac{\partial^2 u}{\partial x^2},
\]

(41)

where \( x \in [0, L] \), \( t \geq 0 \), the inclusion of the random variable \( \xi \) models fluctuations of the sound, or light, velocity \( c \) in a randomly inhomogeneous medium. Write the initial conditions to (41),

\[
u(x, \xi, 0) = f_1(x), \quad \nu_t(x, \xi, 0) = f_2(x),
\]

where \( \nu_t \) means the derivative with respect to \( t \). Take the boundary conditions as

\[
u(0, \xi, t) = 0, \quad \nu(L, \xi, t) = 0.
\]

To reduce (41) to the standard form (1), define

\[
y_1 \equiv u, \quad y_2 \equiv \frac{\partial u}{\partial t}.
\]

Then (41) is equivalent to the system

\[
\frac{\partial y_1}{\partial t} = y_2, \quad \frac{\partial y_2}{\partial t} = (c + \xi)^2 \frac{\partial^2 y_1}{\partial x^2}.
\]

(42)

The corresponding initial and boundary conditions are

\[
y_1(x, \xi, 0) = f_1(x), \quad y_2(x, \xi, 0) = f_2(x),
\]

\[
y_1(0, \xi, t) = 0, \quad y_1(L, \xi, t) = 0.
\]

For the velocity field we have

\[
v_1(x, y, \xi, t) = y_2(x, \xi, t), \quad v_2(x, y, \xi, t) = (c + \xi)^2 \frac{\partial^2}{\partial x^2} y_1(x, \xi, t).
\]

(43)
Thence, for the Lyapunov matrix (10) we get

\[ L(x, x', \xi) = \begin{bmatrix} 0 & 1 \\ (c + \xi)^2 \frac{\partial^2}{\partial x^2} & 0 \end{bmatrix} \delta(x - x'). \]  

(44)

Variating (41), we obtain the equations

\[ \frac{\partial M_{11}}{\partial t} = M_{21}, \quad \frac{\partial M_{22}}{\partial t} = (c + \xi)^2 \frac{\partial^2 M_{12}}{\partial x^2}, \]

\[ \frac{\partial M_{12}}{\partial t} = M_{22}, \quad \frac{\partial M_{21}}{\partial t} = (c + \xi)^2 \frac{\partial^2 M_{11}}{\partial x^2} \]  

(45)

for the multiplier matrix \( M_{ij} = M_{ij}(x, x', \xi, t) \) with the initial and boundary conditions

\[ M_{ij}(x, x', \xi, 0) = \delta_{ij} \delta(x - x'), \]

\[ M_{ij}(0, x', \xi, t) = 0, \quad M_{ij}(L, x', \xi, t) = 0. \]

Eigenfunctions of the Lyapunov matrix (44), satisfying the boundary and normalization conditions

\[ \varphi_n(0) = 0, \quad \varphi_n(L) = 0, \]

\[ \int_0^L |\varphi_n(x)|^2 dx = 1 \quad (i = 1, 2) \]

are

\[ \varphi_{n1}(x) = \sqrt{\frac{2}{L}} \frac{\sin k_n x}{\sqrt{1 + \omega_n^2}} \begin{bmatrix} i \\ \omega_n \end{bmatrix}, \quad \varphi_{n2}(x) = \varphi_{n1}^*(x), \]

where

\[ \omega_n = \omega_n(\xi) = (c + \xi) k_n, \quad k_n \equiv \frac{\pi n}{L} \quad (n = 1, 2, \ldots). \]

Note that these eigenfunctions are not orthogonal to each other, since

\[ \int_0^L \varphi_{n1}^*(x) \varphi_{n2}(x) dx = \frac{\omega_n^2 - 1}{\omega_n^2 + 1}. \]

The eigenvalues of (44) are

\[ \lambda_{n1}(\xi) = i\omega_n(\xi), \quad \lambda_{n2}(\xi) = -i\omega_n(\xi). \]  

(46)
Due to the stationarity of the Lyapunov matrix (44), the multiplier matrix has the same eigenfunctions with the eigenvalues

$$
\mu_{n1}(\xi, t) = \exp\{i\omega_n(\xi)t\}, \quad \mu_{n2}(\xi, t) = \mu^*_{n1}(\xi, t).
$$

(47)

Equations in (45) can be solved explicitly giving

$$
M(x, x', \xi, t) = 2 \sum_{n=1}^{\infty} \sin(k_n x') \sin(k_n x) \begin{pmatrix}
\cos \omega_n t & \sin \omega_n t \\
\sin \omega_n t & \cos \omega_n t
\end{pmatrix}.
$$

(48)

It can be checked that (48) has, really, the eigenfunctions $\varphi_{ni}(x)$ and the eigenvalues (47). The multiplier matrix (48) could be also obtained from the direct variation, with respect to initial functions $f_1(x)$ and $f_2(x)$, of the solution

$$
u(x, \xi, t) = \sum_{n=1}^{\infty} \left[ A_n \cos \omega_n(\xi)t + B_n \sin \omega_n(\xi)t \right] \sin k_n x,
$$

in which

$$
A_n = \frac{2}{L} \int_0^L f_1(x) \sin(k_n x) \, dx, \\
B_n = \frac{2}{\omega_n(\xi)L} \int_0^L f_2(x) \sin(k_n x) \, dx.
$$

In the considered case, the eigenfunctions $\varphi_{ni}(x)$ depend on $\xi$ through $\omega_n(\xi)$. Therefore, eq.(26) is not valid. However, if the noise is weak, that is if $\xi_0 \ll c$ or $\gamma \ll c$, then, we can resort to perturbation theory with the zero–order basis $\{\varphi_{ni}(x)\}$ in which $\omega_n = \omega_n(0)$. Then the spectrum $\omega_n(\xi)$ is the first–order approximation. Respectively, the averaged multiplier (25) can be defined as a first–order approximation.

Averaging (47) with the uniform distribution, we get

$$
\mu_{n1}(t) = \frac{\sin(\xi_0 k_n t)}{\xi_0 k_n t} \exp(ick_n t), \\
\mu_{n2}(t) = \mu^*_{n1}(t).
$$

(49)

Since $|\mu_{n1}(t)| < 1$ for all $n \geq 1$ and $t > 0$, the process is uniformly stable, its asymptotic stability being of power–law type.
For the Gaussian noise we find
\[ \mu_{n1}(t) = \exp \left\{ i c k_n t - \frac{1}{2} (\gamma k_n t)^2 \right\}, \]
\[ \mu_{n2}(t) = \mu_{n1}^*(t). \quad (50) \]
The process is uniformly stable for all \( t > 0 \), and as \( t \to \infty \), it is exponentially stable.

Note that perturbation theory can be employed for defining the spectrum of the Lyapunov matrix. But after averaging over stochastic fields, the local multipliers (49) or (50) cannot be expanded in powers of \( \xi_0 \) or \( \gamma \) because now these parameters enter being factored by the time \( t \).

### 3.3 Random Schrödinger equation

Take the time–dependent Schrödinger equation
\[ i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \frac{\partial^2 \psi}{\partial x^2} + (U + \xi) \psi \quad (51) \]
for a complex wave function \( \psi = \psi(x, \xi, t) \), where \( x \in (-\infty, +\infty) \), \( t \geq 0 \); \( m \) and \( U \) are real constants, and the random variable \( \xi \) imitates random fluctuations of the potential \( U \). An initial conditions to (51) is
\[ \psi(x, \xi, 0) = f(x). \]

A special case of the initial condition will be considered here, when the initial function is periodic,
\[ f(x + L) = f(x). \quad (52) \]
We opt for a periodic initial condition in order to compare the influence of random fields on the Schrödinger equation with the influence of nonlinearity. When periodic boundary conditions are enforced for the cubic Schrödinger equation, then its periodic solutions are well known to be subject to modulational long–wavelength instability [36,37]. The Cauchy problem for the linear Schrödinger equation with periodic initial data, but without random fields, has also been intensively studied [38,39]. The behaviour of the solution,
being closely related to incomplete Gaussian sums [39,40], has been found to look so chaotically as it would display the property of quantum chaos [41].

In the case of eq.(51), for the Lyapunov matrix we have

$$L(x, x', \xi) = \left[ \frac{i}{2m} \frac{\partial^2}{\partial x^2} - i(U + \xi) \right] \delta(x - x'). \quad (53)$$

Its eigenfunctions, satisfying the periodicity and normalization conditions

$$\varphi_n(x + L) = \varphi_n(x), \quad \int_0^L |\varphi_n(x)|^2 dx = 1,$$

are the plane waves

$$\varphi_n(x) = \frac{1}{\sqrt{L}} e^{-ik_n x}, \quad k_n = \frac{2\pi}{L} n, \quad (54)$$

where $n$ is any integer. The eigenvalues of (53) are

$$\lambda_n(\xi) = -i \left( \frac{k_n^2}{2m} + U + \xi \right). \quad (55)$$

The Lyapunov matrix (53) is stationary, therefore the local multiplier is

$$\mu_n(\xi, t) = \exp \left\{ -i \left( \frac{k_n^2}{2m} + U + \xi \right) t \right\}. \quad (56)$$

The multiplier matrix satisfies the equation

$$\frac{\partial M}{\partial t} = i \frac{\partial^2 M}{\partial x^2} - i(U + \xi) M \quad (57)$$

with the initial and periodicity conditions

$$M(x, x', \xi, 0) = \delta(x - x'),$$

$$M(x + L, x', \xi, t) = M(x, x', \xi, t).$$

The solution to (57) is

$$M(x, x', \xi, t) = \sum_{n=-\infty}^{+\infty} \mu_n(\xi, t) \varphi_n(x) \varphi_n^*(x'), \quad (58)$$

from where it is evident that (58) has eigenfunctions (54) and eigenvalues (56). The same matrix (58) could be obtained by a direct variation of the solution

$$\psi(x, \xi, t) = \sum_{n=-\infty}^{+\infty} c_n \mu_n(\xi, t) \varphi_n(x)$$
to eq. (51), where
\[ c_n = \int_0^L f(x) \varphi_n^*(x) dx. \]

The eigenfunctions (54) are stochastically invariant, which permits to define the averaged local multiplier (25). For the uniform noise one gets
\[ \mu_n(t) = \sin \frac{\xi_0 t}{2 \xi_0 t} \exp \left\{ -i \left( \frac{k_n^2}{2m} + U \right) t \right\}. \tag{59} \]
The process is uniformly stable for all \( t > 0 \) with a power–law asymptotic stability.

For the Gaussian noise, the local multipliers are
\[ \mu_n(t) = \exp \left\{ -i \left( \frac{k_n^2}{2m} + U \right) t - \frac{1}{2} (\gamma t)^2 \right\}. \tag{60} \]
So, the process is uniformly stable for any \( t > 0 \) with the exponential asymptotic stability.

If random fields are absent, then \( |\mu_n(0, t)| = 1 \), that is the motion is neutral. The solution \( \psi(x, 0, t) \) is quasiperiodic with a countable number of frequencies
\[ \omega_n = \frac{1}{n} \left( \frac{k_n^2}{2m} + U \right) \quad (n \neq 0). \]
As is known, the behaviour of many–frequency quasiperiodic solutions can be quite complicated, often reminding chaotic one. But, of course, there is no chaos here. The neutral process, after the inclusion of random fluctuations, becomes stable.

4 Stability of quasiopen systems

From the simple models of the previous section we now pass to the consideration of a realistic statistical system. Take the the standard Hamiltonian
\[ H = \int \psi^\dagger(\vec{r}) \left( -\frac{\nabla^2}{2m} - \mu \right) \psi(\vec{r}) d \vec{r} + \]
\[ + \frac{1}{2} \int \psi^\dagger(\vec{r}) \psi^\dagger(\vec{r}') \Phi(\vec{r} - \vec{r}') \psi(\vec{r}') \psi(\vec{r}) d \vec{r} d \vec{r}' \tag{61} \]
of spinless particles with a chemical potential \( \mu \), a symmetric interaction potential \( \Phi(\vec{r}) = \Phi(-\vec{r}) \) and with field operators \( \psi(\vec{r}) \equiv \psi(\vec{r}, t) \). Consider the density matrix
\[ \rho(\vec{r}, \vec{r}', t) = \langle \psi^\dagger(\vec{r}', t) \psi(\vec{r}, t) \rangle, \tag{62} \]
in which the brackets $\langle \ldots \rangle$ mean the statistical averaging with a statistical operator $\hat{\rho}(0)$. The diagonal element of (62) is the density of particles

$$\rho(\vec{r}, t) \equiv \rho(\vec{r}, \vec{r}, t) \geq 0. \quad (63)$$

The average density of particles in a system of volume $V$ is

$$\frac{1}{V} \int_V \rho(\vec{r}, t) \, d\vec{r} = \rho. \quad (64)$$

In the thermodynamic limit, $V \to \infty$, the right-hand side of (64) remains constant.

Another property of the density matrix (62) following from its definition is

$$\rho^*(\vec{r}, \vec{r}', t) = \rho(\vec{r}', \vec{r}, t). \quad (65)$$

Differentiating (62) with respect to time and invoking the Heisenberg equations for the field operators, one gets the evolution equation relating (62) with the two-particle density matrix

$$\rho_2(\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2', t) = \langle \psi^i(\vec{r}_1') \psi(\vec{r}_2') \psi(\vec{r}_2) \psi(\vec{r}_1) \rangle, \quad (66)$$

in which all field operators contain the same time variable $t$, that is, $\psi(\vec{r}_i) \equiv \psi(\vec{r}_i, t)$.

Suppose that the considered system is randomly open in the sense that it is subject to the action of random forces from surrounding whose influence can be taken into account by supplementing the evolution equation with a random field. Thus, the evolution equation for the density matrix (62) takes the form

$$i \frac{\partial}{\partial t} \rho(\vec{r}_1, \vec{r}_2, \xi, t) = \left[ -\frac{1}{2m} \left( \nabla_1^2 - \nabla_2^2 \right) + \xi \right] \rho(\vec{r}_1, \vec{r}_2, \xi, t) +$$

$$+ \int \left[ \Phi(\vec{r}_1 - \vec{r}_3) - \Phi(\vec{r}_2 - \vec{r}_3) \right] \rho_2(\vec{r}_1, \vec{r}_3, \vec{r}_2, \vec{r}_3, \xi, t) \, d\vec{r}_3, \quad (67)$$

where the random variable $\xi$ can, in general, be complex. If we put $\xi = 0$ in (67), then we return to the usual evolution equation for an isolated system.

Emphasize the difference between a random isolated system and a random open system. In the former case, one should add random fields into the Hamiltonian (61), so such fields are to be Hermitian. In the latter case, random fields are to be inserted into the evolution equation, and they are not necessarily self-adjoint.
To make the evolution equation (67) closed, we need to resort to an approximation for the two–particle density matrix (66). Let us use the Hartree–Fock approximation
\[
\rho_2(\vec{r}_1, \vec{r}_2, \vec{r}'_1, \vec{r}'_2, t) = \rho(\vec{r}_1, \vec{r}_1', t)\rho(\vec{r}_2, \vec{r}_2', t) \pm \rho(\vec{r}_1, \vec{r}_2, t)\rho(\vec{r}_2, \vec{r}'_1, t). 
\]  
(68)

If the interaction potential \( \Phi(\vec{r}) \) is strongly singular, then the Hartree–Fock approximation leads to divergences. In that case, one must use the correlated Hartree–Fock approximation [42], in which the interaction potential is smoothed by a correlation function. Keeping in mind these both possibilities, we imply in what follows that \( \Phi(\vec{r}) \) is integrable. The evolution equation (67) becomes
\[
\left[ i\frac{\partial}{\partial t} + \frac{1}{2m} \left( \nabla_1^2 - \nabla_2^2 \right) - \xi \right] \rho(\vec{r}_1, \vec{r}_2, \xi, t) =
\]
\[
= \int \left[ \Phi(\vec{r}_1 - \vec{r}_3) - \Phi(\vec{r}_2 - \vec{r}_3) \right] \left[ \rho(\vec{r}_1, \vec{r}_2, \xi, t)\rho(\vec{r}_3, \vec{r}_3, \xi, t) \right. 
\]
\[
\left. \pm \rho(\vec{r}_1, \vec{r}_3, \xi, t)\rho(\vec{r}_3, \vec{r}_2, \xi, t) \right] d^3 \vec{r}_3. 
\]  
(69)

The solution to eq.(69) has to satisfy conditions (63)–(65) and an initial condition
\[
\rho(\vec{r}_1, \vec{r}_2, \xi, 0) = f(\vec{r}_1, \vec{r}_2). 
\]  
(70)

To analyse the stability of processes described by eq.(69), we need to consider the multiplier matrix
\[
M(\vec{r}_1, \vec{r}_2, \vec{r}'_1, \vec{r}'_2, \xi, t) = \frac{\delta \rho(\vec{r}_1, \vec{r}_2, \xi, t)}{\delta \rho(\vec{r}_1, \vec{r}_2, \xi, t)} 
\]  
(71)

with an initial condition
\[
M(\vec{r}_1, \vec{r}_2, \vec{r}'_1, \vec{r}'_2, \xi, 0) = \delta(\vec{r}_1 - \vec{r}_1')\delta(\vec{r}_2 - \vec{r}_2'). 
\]  
(72)

For the multiplier matrix (71) we may write the evolution equation (9) with the Lyapunov matrix
\[
L(\vec{r}_1, \vec{r}_2, \vec{r}'_1, \vec{r}'_2, \rho, \xi, t) = \frac{\delta \nu(\vec{r}_1, \vec{r}_2, \rho, \xi, t)}{\delta \rho(\vec{r}_1, \vec{r}_2, \xi, t)}, 
\]  
(73)

where the velocity field is
\[
\nu(\vec{r}_1, \vec{r}_2, \rho, \xi, t) = \left[ \frac{i}{2m} \left( \nabla_1^2 - \nabla_2^2 \right) - i\xi \right] \rho(\vec{r}_1, \vec{r}_2, \xi, t)
\]
From (73) and (74) we find that the average of \(| \xi |^2 \) is the limiting situation when the average of randomly quasiopen equation (67) subject to the action of an asymptotically small random field will be called the random field is arbitrary small although finite. A statistical system with the evolution \( \rho(r_1, r_2, r_3, \xi, t) \) where \( \rho \) is a stationary solution. Then the density of particles that an isolated statistical system with any initial condition will finally, as \( t \to \infty \), tend

\[
-i \int \left[ \Phi(\vec{r}_1 - \vec{r}_3) - \Phi(\vec{r}_2 - \vec{r}_3) \right] \left[ \rho(\vec{r}_1, \vec{r}_2, \xi, t) \rho(\vec{r}_3, \vec{r}_3, \xi, t) \pm \rho(\vec{r}_1, \vec{r}_3, \xi, t) \rho(\vec{r}_3, \vec{r}_2, \xi, t) \right] d \vec{r}_3.
\]

(74)

From (73) and (74) we find

\[
\begin{align*}
L(\vec{r}_1, \vec{r}_2, \vec{r}_3, \rho, \xi, t) &= \left[ \frac{i}{2m} \left( \nabla^2 + \frac{\partial^2}{\partial \xi^2} \right) - i\xi \right] \delta(\vec{r}_1 - \vec{r}_1') \delta(\vec{r}_2 - \vec{r}_2') - \\
- i &\delta(\vec{r}_1 - \vec{r}_1') \delta(\vec{r}_2 - \vec{r}_2') \int \left[ \Phi(\vec{r}_1 - \vec{r}_3) - \Phi(\vec{r}_2 - \vec{r}_3) \right] \rho(\vec{r}_3, \vec{r}_3, \xi, t) d \vec{r}_3 - \\
& - i \left[ \Phi(\vec{r}_1 - \vec{r}_2') - \Phi(\vec{r}_2 - \vec{r}_1') \right] \rho(\vec{r}_1, \vec{r}_2, \xi, t) \delta(\vec{r}_1 - \vec{r}_2') \pm \\
& \mp i \left[ \Phi(\vec{r}_1 - \vec{r}_2') - \Phi(\vec{r}_2 - \vec{r}_1') \right] \rho(\vec{r}_1, \vec{r}_1', \xi, t) \delta(\vec{r}_2 - \vec{r}_2') \pm \\
& \mp i \left[ \Phi(\vec{r}_1 - \vec{r}_2') - \Phi(\vec{r}_2 - \vec{r}_1') \right] \rho(\vec{r}_2, \vec{r}_1', \xi, t) \delta(\vec{r}_1 - \vec{r}_1').
\end{align*}
\]

(75)

Consider the case when the random field \( \xi \) is weak. More precisely, this means that the average of \( |\xi|^2 \) is much less than the average energy of the system. Of great importance is the limiting situation when the average of \( |\xi|^2 \) tends to +0 , that is, when the random field is arbitrary small although finite. A statistical system with the evolution equation (67) subject to the action of an asymptotically small random filed will be called randomly quiasiopen.

If in eq.(69) we put \( \xi \equiv 0 \), then it is easy to check that the density matrix

\[
\rho(\vec{r}_1, \vec{r}_2, 0, t) = \rho_0(\vec{r}_1 - \vec{r}_2),
\]

(76)

where \( \rho_0(\vec{r}) \) is an arbitrary function satisfying just two conditions

\[
\rho_0(\vec{r}) = \rho_0(- \vec{r}), \quad \rho_0(0) = \rho > 0,
\]

(77)

is a stationary solution. Then the density of particles

\[
\rho(\vec{r}, \vec{r}, 0, t) = \rho_0(0) = \rho
\]

is uniform in real space. It is worth stressing that there is infinite number, actually, a functional continuum, of stationary solutions (76), for which conditions (77) are fulfilled.

A common convention, concluded from the existence of stationary solutions (76), is that an isolated statistical system with any initial condition will finally, as \( t \to \infty \), tend
to a stationary state called the state of absolute equilibrium. The fact that such a state is infinitely degenerate, in the sense that there exist infinitely many density matrices with the same diagonal part but different nondiagonal parts, is interpreted as follows. All these density matrices are treated as statistically equivalent provided that they define the same set of statistical averages for local observables. As is obvious, the convention about the existence of absolute equilibrium is based on the assumption of its asymptotic stability.

Using the method of multipliers we can check the stability of a stationary solution given by (76). The Lyapunov matrix with the solution (76) is stationary, i.e. according to the notation (13),

$$L(\xi) = \left[ L(\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2', \rho_0, \xi) \right]$$

does not depend on time. When there are no random fields, that is $\xi \equiv 0$, then the eigenfunctions of $L(0)$ are arbitrary functions of the type $\varphi(\vec{r}_1 - \vec{r}_2)$ with the eigenvalue $\lambda(0) = 0$. The corresponding motion is neutral, as it should be expected for a mean-field approximation. In the presence of weak random fields, the eigenvalue problem for $L(\xi)$ can be solved using perturbation theory. Then in the first-order approximation

$$\lambda(\xi) = -i\xi.$$

And for the local multiplier we have

$$\mu(\xi, t) = \exp(-i\xi t). \quad (78)$$

Consider the averaged local multiplier (25) assuming that the random variable $\xi$ pertains to the complex plane. For the uniform distribution, let $Re\xi \in [-\xi_1, +\xi_1]$ and $Im\xi \in [-\xi_2, +\xi_2]$. Then

$$\mu(t) = \frac{\sin(\xi_1 t) \sinh(\xi_2 t)}{\xi_1 \xi_2 t^2}.$$  \quad (79)

The asymptotic behaviour of (80) is

$$\mu(t) \simeq 1 - \frac{1}{6} (\xi_1^2 - \xi_2^2) t^2 \quad (t \to 0),$$

$$\mu(t) \simeq \frac{\sin(\xi_1 t)}{2\xi_1 \xi_2 t^2} e^{\xi_2 t} \quad (t \to \infty).$$
If $\xi_1 \geq \xi_2$, then the motion is locally stable at small time. With increasing time the stability is lost, but it is recovered around the recurrence time $t_{rec} = \frac{\pi}{\xi_1} n \ (n = 1, 2, \ldots)$, where $\mu(t_{rec}) = 0$. So, there occurs a peculiar stability echo. When $\xi_1 = \xi_2$, then the first region of local stability is inside the interval $0 < \xi_1 t < 6.8$; the second, inside the interval $6.1 < \xi_1 t < 6.4$, and so on. There is no asymptotic stability, but the regions of local stability and instability change one another. Such a kind of behaviour is usually called intermittent. If $\xi_1 < \xi_2$, then the motion at small time is unstable, but with increasing time it is again intermittent, when instability regions are interrupted by the stability echo. Here the intermittency happens with respect to time, but it may occur in space as well [43].

If the noise is Gaussian, we denote by $\gamma_1$ the dispersion of $\text{Re} \xi$; and by $\gamma_2$, the dispersion of $\text{Im} \xi$. Then the averaged local multiplier is

$$\mu(t) = \exp \left\{ -\frac{1}{2} \left( \gamma_1^2 - \gamma_2^2 \right) t^2 \right\}. \quad (80)$$

For $\gamma_1 > \gamma_2$, the motion is uniformly stable for all $t > 0$. If $\gamma_1 = \gamma_2$, the motion is neutral. And it is uniformly unstable for any $t > 0$, if $\gamma_1 < \gamma_2$.

In this way, for both uniform as well as Gaussian noise there exist such small random fields that make the stationary solutions for the density matrix unstable. In the presence of such fields, the solution to (69) will wander between infinite many of the stationary solutions, always remaining unstable and nonstationary. Therefore, the density of particles (63) will overlastingly depend on time fluctuating in real space. Accepting that no realistic statistical system can be ideally isolated, but should be treated rather as randomly quasiopen, we come to the conclusion that such a system never reaches the state of absolute equilibrium. A randomly quasiopen system can become not more than quasiequilibrium, with perpetually persisting mesoscopic fluctuations [44].

5 Random structural stability

An important question is how the qualitative behaviour of a dynamical system without random fields changes when the latter are switched on. Define by $\delta = \{\delta_i\}$ a set of
parameters characterizing the distribution of random variables. For example, in the case of real random fields, \( \delta = \xi_0 \) for the uniform distribution, and \( \delta = \gamma \) for the Gaussian distribution. In the case of complex random fields, \( \delta = \{\xi_1, \xi_2\} \) for the uniform noise, and \( \delta = \{\gamma_1, \gamma_2\} \) for the Gaussian noise. A dynamical system without random fields corresponds to \( \delta = 0 \), which implies that all \( \delta_i = 0 \).

Define the parametric neighborhood of \( \delta = 0 \) as a manifold \( \{\delta_i \in [0, \varepsilon_i] | \varepsilon_i > 0\} \) of stochastic parameters \( \delta_i \) pertaining to arbitrary small finite intervals \([0, \varepsilon_i]\) with \( \varepsilon_i \) independent of each other.

We shall say that the \( n \)-component is random-structurally stable with respect to a given random field, if there exists a parametric neighborhood of \( \delta = 0 \) such that

\[
\lim_{t \to \infty} \lim_{\delta \to 0} |\mu_n(t)| = \lim_{\delta \to 0} \lim_{t \to \infty} |\mu_n(t)|
\]

for any sequence of \( \delta \) from this parametric neighborhood. The limit \( \delta \to 0 \) means the all \( \delta_i \to 0 \). The commutativity of the limits in the above equality can be expressed shorter as

\[
\left[ \lim_{t \to \infty}, \lim_{\delta \to 0} \right] |\mu_n(t)| = 0.
\]

When this commutativity holds true for all \( n \)-components, we shall say that the dynamical process is random-structurally stable.

Let us illustrate the notion of this stability for the examples of section 3. Thus, for the random diffusion equation we have

\[
\lim_{t \to \infty} \lim_{\delta \to 0} |\mu_n(t)| = 0
\]

for all \( n = 1, 2, \ldots \). And the opposite order of the limits gives

\[
\lim_{\delta \to 0} \lim_{t \to \infty} |\mu_n(t)| = \begin{cases} 0 & (\text{uniform noise}) \\ \infty & (\text{Gaussian noise}) \end{cases}
\]

again for any \( n \geq 1 \). Therefore, the diffusion process is random-structurally stable with respect to the uniform noise and unstable with respect to the Gaussian noise.

For the random wave equation we get

\[
\lim_{t \to \infty} \lim_{\delta \to 0} |\mu_{ni}(t)| = 1
\]
for all $n = 1, 2, \ldots$ and $i = 1, 2$. While

$$\lim_{\delta \to 0} \lim_{t \to \infty} |\mu_{ni}(t)| = 0$$

for both uniform and Gaussian noise, and any $n \geq 1$ and $i = 1, 2$. Consequently, the wave process is random–structurally unstable with respect to the uniform as well as to the Gaussian noise.

Similarly to the previous case, for the random Schrödinger equation we find

$$\lim_{\delta \to 0} \lim_{t \to \infty} |\mu_n(t)| = 1,$$

$$\lim_{\delta \to 0} \lim_{t \to \infty} |\mu_n(t)| = 0$$

for all $n \in \mathbb{N}$ and for both uniform and Gaussian noise. Thus, the process is also random–structurally unstable with respect to these noises.

For the randomly quasiopen system of sec.4, we obtain

$$\lim_{t \to \infty} \lim_{\delta \to 0} |\mu(t)| = 1.$$

But interchanging the limits makes the result undefined. For the uniform noise we get the intermittent behaviour, and for the Gaussian noise we can come to $-\infty$, $1$ or $+\infty$ depending on the relation between $\gamma_1$ and $\gamma_2$. Thence, the dynamical process for a randomly quasiopen system is random–structurally unstable.

The notion of random–structural stability is analogous to that of structural stability [28,45]. Neutral dynamical processes are known to be structurally unstable. As we have seen, such processes are also random–structurally unstable.