A REALISATION OF THE QUANTUM SUPERGROUP $U(\mathfrak{gl}_{m|n})$

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Abstract. We reconstruct the quantum enveloping superalgebra $U(\mathfrak{gl}_{m|n})$ over $\mathbb{Q}(\nu)$ via (finite dimensional) quantum Schur superalgebras. In particular, we obtain a new basis containing the standard generators for $U(\mathfrak{gl}_{m|n})$ and explicit multiplication formulas between the generators and an arbitrary basis element.

When an algebra is defined by generators and relations, the realisation problem of the algebra is to reconstruct the algebra via a certain vector space—the realisation space—together with some explicit multiplication rules on a basis of the space. For example, for Kac–Moody algebras, the affine case is constructed in [12] via loop algebras and central extensions while, for symmetrisable Kac–Moody algebras, the problem was solved by Peng and Xiao [19] via root categories and Hall multiplication. If the realisation space happens to be the Grothendieck group of a certain representation category, such a realisation is also known as categorification.

Since the introduction of quantum groups in late eighties, their realisation and applications have achieved significant progress. For example, Ringel [20] gave a realisation for the $\pm$-parts of quantum enveloping algebras of finite type. This work has also been generalised by Lusztig [13, 14], Green [10] to the symmetrisable case. The realisation problem for the entire quantum enveloping algebras has also attracted much attention. For example, Beilinson, Lusztig and MacPherson solved the problem for quantum $\mathfrak{gl}_n$ via endomorphism algebras of certain representations, namely, the quantum Schur algebras. This approach has also been investigated in the affine case by Deng, Fu and first author; see [8] and [3]. Recently, Bridgeland [2] has solved the problem for all quantum enveloping algebras of finite type, generalising the construction in [19]. In this paper, we will tackle the problem for the quantum groups associated with the Lie superalgebra $\mathfrak{gl}_{m|n}$.

The work was motivated by several recent developments about the quantum Schur or $\nu$-Schur (if the parameter needs to be specified) superalgebras $\mathcal{S}(m|n, r)$. Firstly, Rui and the first author [9] introduced these algebras and established several properties in analogy to those of quantum Schur algebras. For example, the construction of standard bases via super double cosets gives the base change property, the
Kazhdan–Lusztig combinatorics via super cells and super Robinson–Schensted-Knuth correspondence gives a construction of simple modules in terms of cells modules. Moreover, these algebras have also alternative characterisations as a homomorphic image of the quantum supergroup $U(\mathfrak{gl}_{m|n})$ and as the dual algebras of the homogeneous components of the super matrix coordinate bialgebra. Secondly, a presentation for these algebras have been given by Turkey and Kujawa \cite{Tur21} via the epimorphism from $U(\mathfrak{gl}_{m|n})$ to $\mathcal{S}(m|n,r)$. Thirdly, by applying the relative norm method of Hoefsmit and Scott developed in 1977 (see \cite{Hoef77}) and its application to quantum Schur algebras \cite{GuW07}, the authors together with Wang \cite{GuW07} determined a classification of their irreducible representations at a root of unity when $m + n \geq r$.

All these works suggest that the realisation of $U(\mathfrak{gl}_{m|n})$ via quantum Schur superalgebras should exist. However, since the BLM realisation for quantum $\mathfrak{gl}_n$ relies on a geometric setting for quantum Schur algebras, it is inevitable that one has to develop a new and algebraic approach in the super case. Motivated from a large amount of computation with relative norms in \cite{GuW07}, we discover in this paper that the relative norm method of Hoefsmit and Scott can replace the geometric method to establish two key multiplication formulas and, thus, to solve the realisation problem for the quantum supergroup $U(\mathfrak{gl}_{m|n})$.

We organise the paper as follows. The basics of symmetric groups, Hecke algebras and quantum Schur superalgebras are briefly discussed in §1. In particular, we use the Hecke algebra action on the tensor superspace and display the relative norm basis for the $q$-Schur superalgebra. In §2, we develop some algorithms for computing a reduced expression of a distinguished double coset representative associated with a matrix and computing the corresponding Hecke algebra action on various tensors. With these algorithms, we are able to derive in §3 the super version of two key multiplication formulas which in the non super or affine case were obtained by analysis on (partial) flag varieties.

Now, like \cite{GuW07}, we are in position to get more multiplication formulas first with respect to the parameter $q = \nu^2$ in §4 and then their normalised version with respect to parameter $\nu$ in §5. Uniform spanning sets for $\nu$-Schur superalgebras $\mathcal{S}(m|n,r)$ for all $r \geq 0$ are constructed in §6. The uniformness allows to define a $\mathbb{Q}(\nu)$-subspace $\mathfrak{A}(m|n)$ inside the direct product $\mathcal{S}(m|n)$ of $\mathcal{S}(m|n,r)$ over all $r \geq 0$. We then prove that, by deriving explicit multiplication formulas between (candidates of) generators and basis elements of $\mathfrak{A}(m|n)$, $\mathfrak{A}(m|n)$ is closed under multiplication by these generators. By producing the defining relations for $U(\mathfrak{gl}_{m|n})$ in $\mathcal{S}(m|n)$, we find in §7 an algebra homomorphism $\eta$ from $U(\mathfrak{gl}_{m|n})$ to $\mathcal{S}(m|n)$. Finally, as a further application of the multiplication formulas, we establish a super triangular relation in §8. This relation gives us a monomial basis which is crucial to prove that $\mathfrak{A}(m|n)$ is the image of $\eta$ and $\eta$ is injective.

Like the non super case, this realisation will have many applications such as a super version of the quantum Schur–Weyl duality at a root of unity and realisation
of the finite dimensional quantum supergroups. We plan to complete these tasks in forthcoming papers.

Throughout the paper, let $\nu$ be an indeterminate and let $q = \nu^2$. Let $Z = \mathbb{Z}[\nu, \nu^{-1}]$ and let $A = \mathbb{Z}[q, q^{-1}]$. For any integers $0 \leq t \leq s$, define Gaussian polynomials in $A$ by

$$\left[ \begin{array}{c} t \\ s \end{array} \right] = \frac{[s]!}{[t]![s-t]!}$$

where $[r]^! := [1][2] \cdots [r]$ with $[i] = 1 + q + \cdots + q^{i-1}$. By introducing $[i] = \frac{\nu^i - \nu^{-i}}{\nu - \nu^{-1}}$, we define the symmetric Gaussian polynomials $[r]^!$ and $[\cdot]^!$ in $Z$ similarly. Clearly, $\left[ \begin{array}{c} t \\ s \end{array} \right] = \nu^{s(t-s)} \left[ \begin{array}{c} t \\ s \end{array} \right]$.

The letters $m, n$ denote two arbitrary fixed nonnegative integers (not both zero) and, for $h \in [1, m+n] := \{1, 2, \ldots, m + n\}$, define the parity function

$$\widehat{h} = \begin{cases} 0, & \text{if } 1 \leq h \leq m; \\ 1, & \text{if } m + 1 \leq h \leq m + n. \end{cases} \tag{0.0.1}$$

Let $q_h = q^{\widehat{h}}$ and $\nu_h = \nu^{\widehat{h}}$.

1. Preliminaries

Let $(W, S)$ be the symmetric group on $r$ letters where $W = \mathfrak{S}_r = \mathfrak{S}_{\{1, 2, \ldots, r\}}$ and $S = \{s_k \mid 1 \leq k < r\}$ is the set of basic transpositions $s_k = (k, k + 1)$, and let $\ell : W \to \mathbb{N}$ be the length function with respect to $S$.

An $N$-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ of non-negative integers is called a composition of $r$ into $N$ parts if $|\lambda| := \sum_i \lambda_i = r$. Let $\Lambda(N, r)$ denote the set of compositions of $r$ into $N$-parts.

The parabolic (or standard Young) subgroup $W_\lambda$ of $W$ associated with a composition $\lambda$ consists of the permutations of $\{1, 2, \ldots, r\}$ which leave invariant the following sets of integers

$$R_i^\lambda = \{\tilde{\lambda}_{i-1} + 1, \tilde{\lambda}_{i-1} + 2, \ldots, \tilde{\lambda}_{i-1} + \lambda_i \} \quad (1 \leq i \leq N).$$

Here $\tilde{\lambda}_0 = 0$ and $\tilde{\lambda}_i = \sum_{j=1}^{i} \lambda_j$ so that $\tilde{\lambda}$ is the partial sum sequence associated with $\lambda$.

We will also denote by $D_\lambda := D_{W_\lambda}$ the set of all distinguished (or shortest) coset representatives of the right cosets of $W_\lambda$ in $W$. Let $D_{\rho\lambda} = D_{\rho} \cap D_{\lambda}^{-1}$, where $\rho \in \Lambda(N, r)$. Then $D_{\rho\lambda}$ is the set of distinguished $W_\rho W_\lambda$ double coset representatives. For $d \in D_{\rho\lambda}$, the subgroup $W_d^\rho \cap W_\lambda = d^{-1} W_d^\rho \cap W_\lambda$ is a parabolic subgroup associated with a composition which is denoted by $\rho d \cap \lambda$. In other words, we define

$$W_{\rho d \cap \lambda} = W_d^\rho \cap W_\lambda. \tag{1.0.2}$$

The composition $\rho d \cap \lambda$ can be easily described in terms of the following matrix. For $\rho, \lambda \in \Lambda(N, r), d \in D_{\rho\lambda}$, let

$$A = (a_{i,j}) = j(\rho, d, \lambda), \quad \text{where } a_{i,j} = |R_i^\rho \cap d(R_j^\lambda)|, \tag{1.0.3}$$
be the $N \times N$ matrix associated to the double coset $W_{\rho}dW_{\lambda}$. Then
\begin{equation}
\rho d \cap \lambda = (\nu^1, \nu^2, \ldots, \nu^N), \tag{1.0.4}
\end{equation}
where $\nu^j = (a_{1,j}, a_{2,j}, \ldots, a_{N,j})$ is the $j$th column of $A$. In this way, the matrix set
\[ M(N, r) = \{j(\rho, d, \lambda) \mid \rho, \lambda \in \Lambda(N, r), d \in D_{\rho \lambda}\} \]
is the set of all $N \times N$ matrices over $\mathbb{N}$ whose entries sum to $r$. For $A \in M(N, r)$, let
\[ \text{ro}(A) := (\sum_{j=1}^{N} a_{1,j}, \ldots, \sum_{j=1}^{N} a_{N,j}) = \rho \quad \text{and} \quad \text{co}(A) := (\sum_{i=1}^{N} a_{i,1}, \ldots, \sum_{i=1}^{N} a_{i,N}) = \lambda. \]

We also let row$_j(A)$ and col$_j(A)$ denote the $j$th row and $j$th column of $A$, respectively.

For a composition $\lambda = (\lambda_1, \ldots, \lambda_{m+n}) \in \Lambda(m+n, r)$, we often rewrite
\[ \lambda = (\lambda(0)|\lambda(1)) = (\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_m^{(0)}|\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_n^{(1)}) \]
to indicate the “even” and “odd” parts of $\lambda$ and identify $\Lambda(m+n, r)$ with the set
\[ \Lambda(m|n, r) = \{\lambda = (\lambda(0)|\lambda(1)) \mid \lambda \in \Lambda(m+n, r)\} \]
\[ = \bigcup_{r_1+r_2=r} (\Lambda(m, r_1) \times \Lambda(n, r_2)) \]
and identify the set $\Lambda(m+n, 1)$ with the set of standard basis
\[ \{e_1, e_2, \ldots, e_{m+n}\}. \tag{1.0.5} \]

Thus, a parabolic subgroup $W_{\lambda}$ associated with $\lambda = (\lambda(0)|\lambda(1))$ has even part $W_{\lambda(0)}$ and odd part $W_{\lambda(1)}$.

For $d \in D_{\rho \lambda}$ with $\rho, \lambda \in \Lambda(m|n, r)$, the parabolic subgroup $W_{\rho d \cap \lambda}$ decomposes into four parabolic subgroups
\[ W_{\rho d \cap \lambda} = W_{\rho d \cap \lambda}^{(0)} \times W_{\rho d \cap \lambda}^{(1)} \times W_{\rho d \cap \lambda}^{(0)} \times W_{\rho d \cap \lambda}^{(1)}, \tag{1.0.6} \]
where $W_{\rho d \cap \lambda}^{(ij)} = W_{\rho d}^{(i)} \cap W_{\lambda(j)}$. In this case, the composition $\nu = \rho d \cap \lambda$ has the form $\nu = (\nu^1, \ldots, \nu^{m+n})$ with $\nu^j \in \Lambda(m+n, \lambda_i)$.

We say that $d$ satisfies the even-odd trivial intersection property if $W_{\rho d \cap \lambda}^{(0)} = 1 = W_{\rho d \cap \lambda}^{(1)}$. For $\rho, \lambda \in \Lambda(m|n, r)$, let
\[ D_{\rho \lambda}^{o} = \{d \in D_{\rho \lambda} \mid W_{\rho d}^{(0)} \cap W_{\lambda(1)} = 1, W_{\rho d}^{(1)} \cap W_{\lambda(0)} = 1\}. \tag{1.0.7} \]
This set is the super version of the usual $D_{\rho \lambda}$. Let
\[ M(m|n, r) = \{j(\rho, d, \lambda) \mid \rho, \lambda \in \Lambda(m|n, r), d \in D_{\rho \lambda}^{o}\} \]
and
\[ M(m|n) = \bigcup_{r \geq 0} M(m|n, r). \tag{1.0.8} \]

Write a matrix $A \in M(m|n)$ in the form $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{11}$ is $m \times m$, and write $|A| = r$ if $A \in M(m|n, r)$. Then the even-odd trivial intersection property implies that the entries in $A_{12}$ and $A_{21}$ are either 0 or 1. In fact, $M(m|n)$ is the set of all
and, for basis

\[ V \]

and denote the transpose of \( A \) by \( A' \). Clearly, \( \hat{A} = A' \).

For \( d \in D_{\rho, \lambda}^w \), if we put \( W_{\nu(0)} = W_{\rho(0)}^d \cap W_{\lambda(0)}, W_{\nu(1)} = W_{\rho(1)}^d \cap W_{\lambda(1)} \), \( \nu = (\nu(0)|\nu(1)) \), then \( W_v = W_{\nu(0)} \times W_{\nu(1)} \). We have, in general, \( \nu \in \Lambda(m'|n', r) \) where \( m' = m(m+n) \) and \( n' = n(m+n) \).

The Hecke algebra \( H_q(r) = H_q(W) \) corresponding to \( W = S_r \) is a free \( A \)-module with basis \( \{T_w; w \in W\} \) and the multiplication is defined by the rules: for \( s \in S \),

\[
T_wT_s = \begin{cases} 
T_{ws}, & \text{if } \ell(w) > \ell(s); \\
(q-1)T_w + qT_{ws}, & \text{otherwise.}
\end{cases}
\]

Let \( H_v(r) = H_q(r) \otimes_A Z \).

Let \( V(m|n) \) be a free \( A \)-module of rank \( m+n \) with basis \( v_1, v_2, \ldots, v_{m+n} \). Its tensor product \( V(m|n)^{\otimes r} \) has basis \( \{v_i\}_{i \in I(m|n, r)} \) where

\[
I(m|n, r) = \{i = (i_1, i_2, \ldots, i_r) \mid 1 \leq i_j \leq m+n, \forall j\}
\]

and, for \( i = (i_1, i_2, \ldots, i_r) \in I(m|n, r) \), let

\[
v_i = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} = v_{i_1}v_{i_2}\cdots v_{i_r}.
\]

The symmetric group \( W \) acts on \( I(m|n, r) \) by place permutation: for \( w \in W, i \in I(m|n, r) \)

\[
iw = (i_{w(1)}, i_{w(2)}, \ldots, i_{w(r)}).
\]

Thus, following \( [\text{10}] \), \( V(m|n)^{\otimes r} \) is a right \( H_q(r) \)-module with the following action:

\[
v_iT_{s_k} = \begin{cases} 
(-1)^{\hat{k}k+1}v_{i_{s_k}}, & \text{if } i_k < i_{k+1}; \\
q^{\ell_i}v_i, & \text{if } 1 \leq i_k = i_{k+1} \leq m; \\
-v_i, & \text{if } m+1 \leq i_k = i_{k+1}; \\
(-1)^{\hat{k}k+1}q^{\ell_i}v_{i_{s_k}} + (q-1)v_i, & \text{if } i_k > i_{k+1}\end{cases}
\]

Note that when \( n = 0 \), this action coincides with the action on the usual tensor space as given in \( [\text{4}] (9.1.1) \). Clearly, we have the following decomposition into \( H_q(r) \)-modules:

\[
V(m|n)^{\otimes r} = \bigoplus_{\lambda \in \Lambda(m|n, r)} v_\lambda H_q(r).
\]

**Definition 1.1.** The algebra

\[
S_q(m|n, r) := \text{End}_{H_q(r)}(V(m|n)^{\otimes r})
\]

\(^1\)Replacing the basis \( \{v_i\}_{i \in I(m|n, r)} \) by the basis \( \{v^{-\ell_i}v_i\}_{i \in I(m|n, r)} \), where \( \ell_i \) is the number of inversions in \( i \), the action of \( v^{-1}T_{s_k} \) on the new basis coincides with the action given in \( [\text{7}] (2.6.1) \).
is called a $q$-Schur superalgebra over $A$ with a $\mathbb{Z}_2$-grading
\[ S_q(m|n,r) = \bigoplus_{|\lambda^{(1)}|+|\mu^{(1)}|\equiv i (\text{mod } 2)} \text{Hom}_{H_q(r)}(v_{\lambda}H_q(r), v_{\mu}H_q(r)). \]

Let $S_v(m|n,r) = S_q(m|n,r) \otimes \mathbb{Z}$ and $S_R(m|n,r) = S_q(m|n,r) \otimes R$ for any commutative ring $R$ which is an $A$-module.

Note that $E = \text{End}_{A}(V(m|n)^{\otimes r})$ is an $H_q(r)$-$H_q(r)$-bimodule and $S_q(m|n,r)$ is the set of all $H_q(r)$-fixed points:
\[ Z_E(H_q(r)) = \{ x \in E \mid hx = xh, \forall h \in H_q(r) \}. \]

We will use a right-handed function notation for the elements $e$ in $E$: $v \mapsto (v)e$ for all $v \in V(m|n)^{\otimes r}$. This notation is convenient for the $H_q(r)$-$H_q(r)$-bimodule structure on $E$.

For $i, j \in I(m|n,r)$, we define $e_{i,j} \in \text{End}_{A}(V(m|n)^{\otimes r})$ to be the $A$-linear map
\[ (v_{i'})e_{i,j} = \begin{cases} v_{j}, & \text{if } i' = i, \\ 0, & \text{otherwise}. \end{cases} \tag{1.1.1} \]

For each $i \in I(m|n,r)$, define $\lambda \in \Lambda(m|n,r)$ to be the weight $wt(i)$ of $v_i$ by setting $\lambda_k = \# \{ j \mid i_j = k, 1 \leq j \leq r \}$. For each $\lambda \in \Lambda(m|n,r)$, define $i_{\lambda} \in I(m+n,r)$ by
\[ i_{\lambda} = (1, \ldots, 1, 2, \ldots, 2, \ldots, m+n, \ldots, m+n) = (1^{\lambda_1}, 2^{\lambda_2}, \ldots, (m+n)^{\lambda_{m+n}}). \]

Thus, for every $i$, there is a unique $d \in D_{\lambda}$ such that $i = i_{\lambda}d$ ($\lambda = wt(i)$).

For $\rho, \lambda \in \Lambda(m|n,r)$ and $d \in D_{\rho\lambda}$, let $A = j(\rho,d,\lambda)$ be as defined in (1.0.3) and define the relative norm
\[ N_{A'} = N_{A'}^{d} = N_{W_{\rho},W_{\rho}^{\prime}}(e_{\lambda,\rho,d}) := \sum_{w \in D_{\rho \lambda}} q^{-\ell(w)}T_{w^{-1}}e_{\lambda,\rho,d}T_{w} \in E, \tag{1.1.2} \]

where $e_{\lambda,\rho,d} = e_{i_\lambda,i_\rho,d}$ and $A' = j(\lambda,d^{-1},\rho)$ is the transpose of $A$. Clearly, the grading degree of $N_{A'}$ is $\tilde{A}$. The first assertion of the following is given in [\textbf{1} T 5.1].

**Lemma 2.1.** (1) The set \{ $N_{A'}^{d} \mid \rho, \lambda \in \Lambda(m|n,r), d \in D_{\rho\lambda}$ \} forms an $A$-basis of homogeneous elements for $S_q(m|n,r)$. Moreover, $N_{A'}N_{\lambda} = \delta_{\lambda,\text{col}(A)}N_{A'}$ and $N_{\lambda}N_{A'} = \delta_{\lambda,\text{col}(A)}N_{A'}$, where $N_{\lambda} = N_{\lambda}^{A}$. 

(2) Matrix transposing induces a superalgebra anti-automorphism
\[ \tau : S_q(m|n,r) \longrightarrow S_q(m|n,r), \quad N_A \longmapsto N_{A'}. \]

Sign manipulating is an important issue in the super theory. The number of “odd inversions” in $i_{\lambda}d$ will be frequently encountered later on.

\[ \text{However, we will turn it to a left action from } \S 6 \text{ onwards via the anti-involution } \tau \text{ below.} \]
Definition 1.3. For $\lambda \in \Lambda(m|n)$, $d \in D_\lambda$, if $i_\lambda d = (i_1, i_2, \ldots, i_r)$, define a number in $\mathbb{N}$ (not in $\mathbb{Z}_2$)
\[(\lambda, d)^\wedge = \sum_{k=1}^{r-1} \sum_{k < l, j_k > i_k} \hat{i}_k \hat{i}_l.\]
When $\lambda$ is clear from the context, we write $\hat{d} = (\lambda, d)^\wedge$.

Note that $(-1)\hat{d}(-1)^{\hat{i}_k \hat{i}_k+1} = (-1)^{d_{s_k}}$ for all $d \in D_\lambda$, $s_k \in S$ with $d_{s_k} \in D_\lambda$.

2. Some Algorithms

Recall from [1.0.3] the map $j$ taking a double coset $W_\rho d W_\lambda$ to a matrix $j(\rho, d, \lambda)$. We now look at the inverse map. Given a matrix $N$, the element satisfying
\[\text{Algorithm 2.1 (Computing } \rho, d, \lambda \text{ from } N)\]

Set $\rho := \text{ro}(A)$, $\lambda := \text{co}(A)$, $d = 1$, $\text{col}_0(A) = 0$, $\theta = 0$, $\lambda_0 = 0$;

for $j$ from 1 to $N-1$, do $w_j := 1$, $\rho := \rho - \text{col}_1(A)$, $\theta := \theta + (\lambda_j - 1, \ldots, \lambda_j - 1)$;

for $i$ from 2 to $N$, do
\[w_{ij} := s_{\theta + (\hat{\rho}_1, \hat{\rho}_{i-1}, \ldots, \hat{a}_{i-1,j} + 1)} s_{\theta + (\hat{\rho}_1, \hat{\rho}_{i-1}, \ldots, \hat{a}_{i-1,j} + 2)} \ldots s_{\theta + (\hat{\rho}_1, \hat{\rho}_{i-1} + a_{i,j} - 2, \ldots, \hat{a}_{i,j})};\]
\[w_j := w_j \ast w_{ij};\]
end do $d := d \ast w_j$; end do; OUTPUT($d$);

Here, the inner loops build the subsequence $1^{a_{1,j}}, 2^{a_{2,j}}, \ldots, N^{a_{N,j}}$, while the outer loops indicate the construction is done from column 1 to column $N-1$ of $A$.

The element $w_{ij}$ moves the block $i^{a_{i,j}}$ to the left of the sequence $1^{\rho_1 - \sum_{i=1}^l a_{1,i}}, 2^{\rho_2 - \sum_{i=1}^l a_{2,i}}, \ldots, (i-1)^{\rho_{i-1} - \sum_{i=1}^l a_{i-1,i}}$.

By the Algorithm, we see that $w_j$ is a product of $N-1$ permutations $w_{2,j}, \ldots, w_{N,j}$ which move $2^{a_{2,j}}$ leftwards by $w_{2,j}$ to form $1^{a_{1,j}}, 2^{a_{2,j}}, \ldots$, and move $N^{a_{N,j}}$ leftwards.

\[\text{Once the construction is done for column } N-1, \text{ that for column } N \text{ is automatic.}\]
by $w_{N,j}$ to form $1^{a_{1,j}}, 2^{a_{2,j}}, \ldots, N^{a_{N,j}}$. For example,

$$w_{2,1} = s(\tilde{\rho}_1, \tilde{\rho}_1 - 1, \tilde{\rho}_1 + 1, \tilde{\rho}_1 + 2) \cdots s(\tilde{\rho}_1 + a_{2,1} - 1, \tilde{\rho}_1 + a_{2,1} - 2, \ldots, a_{2,1})$$

$$w_{3,1} = s(\tilde{\rho}_2, \tilde{\rho}_2 - 1, \tilde{\rho}_2 + 1, \tilde{\rho}_2 + 2) \cdots s(\tilde{\rho}_2 + a_{3,1} - 1, \tilde{\rho}_2 + a_{3,1} - 2, \ldots, a_{3,1})$$

$$\ldots$$

$$w_{N,1} = s(\tilde{\rho}_{N-1}, \tilde{\rho}_{N-1} - 1, \tilde{\rho}_{N-1} + 1, \ldots, \tilde{\rho}_{N-1}) \cdots s(\tilde{\rho}_{N-1} + a_{N,1} - 1, \tilde{\rho}_{N-1} + a_{N,1} - 2, \ldots, a_{N,1}).$$

Moreover, we have

$$\ell(w_1) = (\tilde{\rho}_1 - \tilde{a}_1, 1)a_{2,1} + (\tilde{\rho}_2 - \tilde{a}_2, 1)a_{3,1} + \cdots + (\tilde{\rho}_{N-1} - \tilde{a}_{N-1}, 1)a_{N,1}$$

$$\ell(w_2) = (\tilde{\rho}_1 - \tilde{a}_1, 1 - \tilde{a}_2, 2)a_{2,2} + (\tilde{\rho}_2 - a_{2,2}, 2)a_{3,2} + \cdots + (\tilde{\rho}_{N-1} - \tilde{a}_{N-2} - \tilde{a}_{N-1}, 2)a_{N,2}$$

$$\ldots$$

$$\ell(w_{N-1}) = a_{1,N}a_{2,N} - 1 + (a_{1,N} + a_{2,N})a_{3,N-1} + \cdots + (a_{1,N} + a_{2,N} + \cdots + a_{N-1,N})a_{N,N-1}.$$}

Hence, we obtain from the algorithm

$$d = w_1w_2 \cdots w_{N-1} \quad \text{and} \quad \ell(d) = \sum_{i < k, j > l} a_{i,j}a_{k,l}.$$}

We now use the Hecke algebra action on $V(m|n)^{\otimes r}$ defined in (1.0.11) to compute $v^\rho \cdot T_d$ by modifying Algorithm 2.1.

For $\lambda, \rho \in \Lambda(m|n, r)$, $d \in D^\rho_\alpha$, set $A = (a_{i,j}) \in M(m|n, r)$ to be the matrix $j(\rho, d, \lambda)$ associated to the double coset $W_\rho d W_\lambda$ and let $\nu = \rho d \cap \Lambda$.

Since only the $i_k < i_{k+1}$ case occurs when computing $v^\rho \cdot T_d$, by applying the first formula in (1.0.11) repeatedly, the algorithm above continues to hold with the sign recorded. Hence, we have $v^\rho \cdot T_d = (-1)^d \tilde{v}A$, where

$$v^A = v_1^{a_{1,1}}v_2^{a_{1,2}} \cdots v_N^{a_{N,1}}v_1^{a_{1,2}}v_2^{a_{2,2}} \cdots v_N^{a_{N,2}}v_1^{a_{1,N}}v_2^{a_{2,N}} \cdots v_N^{a_{N,N}} \quad (N = m + n).$$

More precisely, putting $d_i = w_i + 1 \cdots w_{N-1}$,

$$v^\rho T_d = (-1)^d v_1^{a_{1,1}}v_2^{a_{2,1}} \cdots v_N^{a_{N,1}}v_1^{a_{1,1}}v_2^{a_{1,2}} \cdots v_N^{a_{N,2}}v_1^{a_{1,N}}v_2^{a_{2,N}} \cdots v_N^{a_{N,N}} \quad (N = m + n).$$

Algorithm 2.2 (Computing $v_1^{\rho_1}v_2^{\rho_2} \cdots v_m^{\rho_m} T_d$). Let $A = (a_{i,j})$ be the matrix associated with $(\rho, d, \lambda)$.

for $j$ from 1 to $m + n$, do

for $i$ from 2 to $m + n$, move the factor $v_1^{a_{i,j}}$ to the left of the block $v_1^{\rho_1 - \sum_{l=1}^i a_{l,1}} \cdots v_i^{\rho_{i-1} - \sum_{l=1}^{i-1} a_{l-1,1}}$ to form $v_1^{a_{i,1}} \cdots v_{i-1}^{a_{i-1,1}} v_i^{a_{i,j}}$ end move; end do;
Lemma 2.3. In the algorithm of computing $v_\rho \cdot T_d$, moving $v^{a_{i,j}}_i$ for $i > 1$ to the left of the block $v_1^{\rho-\sum_{l=1}^{i-1} a_{i,l}} \cdots v_1^{\rho - \sum_{l=1}^{i-1} a_{1,l}}$ contributes the sign

$(-1)^{\sum_{k=1}^{i-1} a_{j,k}(\rho_k - \sum_{l=1}^{j} a_{k,l})i\bar{k}}$

Hence, $v_\rho \cdot T_d = (-1)^{\hat{d}} v_\rho$ and

$$\hat{d} = \sum_{j=1}^{m+n-1} \sum_{i=2}^{m+n} (\sum_{k=1}^{i-1} a_{i,j}(\rho_k - \sum_{l=1}^{j} a_{k,l})i\bar{k}).$$

For $1 \leq k \leq m+n-1$ and $0 \leq t \leq a_{h+1,k} - 1$, let

$$v_{h+1}^{k,t} = v_{h+1}^{\hat{a}_{h+1,k-1}+t} v_{h+1}^{\hat{q} a_{h+1,k-1} - \hat{a}_{h+1,k-1} - t - 1}$$

$$= v_{h+1}^{\hat{a}_{h+1,k-1} \cdots v_{h+1}^{\hat{q} a_{h+1,k+1} - \hat{a}_{h+1,k+1} - \hat{a}_{h+1,k-1} - t - 1}}$$

$$= v_{h+1}^{\hat{a}_{h+1,k-1} \cdots v_{h+1}^{b_{h+1,k}^{(t)}}} v_{h+1}^{\hat{q} a_{h+1,k+1} - \hat{a}_{h+1,k+1} - \hat{a}_{h+1,k-1} - t - 1}$$

where $b_{h+1,k}^{(t)} = v_{h+1}^{t} v_{h+1}^{a_{h+1,k} - \hat{a}_{h+1,k} - t - 1}$ and $\hat{a}_{i,j} = a_{i,1} + \cdots + a_{i,j}$. Applying Algorithm 2.2 to the element

$$v_\rho(v_{h+1}^{k,t}) = v_1^{\rho_1} \cdots v_{h+1}^{\rho_{h}v_{h+1}^{k,t}} v_{h+1}^{\rho_{h+2}} \cdots v_{m+n}^{\rho_{m+n}}$$

yields the following.

Lemma 2.4. Maintain the notation above and assume $a_{h+1,k} \geq 1$. We have

$$v_\rho(v_{h+1}^{k,t}) \cdot T_d = \begin{cases} (-1)^{\hat{d}} q^{\sum_{j=1}^{h} a_{h,j}} v^A(b_{h+1,k}^{(t)}), & h < m; \\ (-1)^{\hat{d}}(-1)^{\sum_{j=m+1}^{m+n-1} a_{j,k} q^{\sum_{j=1}^{h} a_{j,k}} v^A(b_{h+1,k}^{(t)})}, & h = m; \\ (-1)^{\hat{d}} q^{\sum_{j=1}^{h} a_{h,j}} v^A(b_{h+1,k}^{(t)}), & h > m. \\ \end{cases}$$

where $v^A(b_{h+1,k}^{(t)})$ is the element obtained from $v^A$ by replacing the factor $v_{h+1}^{\rho_{h+1}}$ by $b_{h+1,k}^{(t)}$ for $0 \leq t \leq a_{h+1,k} - 1$.

Proof. We first observe that the sign contribution occurs only when moving $v^{a_{i,j}}_i$ to the left of $v^{a_{i,j}}_i - \sum_{j=1}^{i-1} a_{i,j}$, where $m < t' < i$. We now have three cases to consider.

If $h < m$, then it is clear that the sign is the same as the sign $(-1)^{\hat{d}}$ when computing $v_\rho \cdot T_d$. In this case, moving the factor $b_{h+1,k}^{(t)}$ to the left of the block $v_1^{\rho_1 - \sum_{l=1}^{k} a_{h,l}} \cdots v_1^{\rho_{h} - \sum_{l=1}^{k} a_{h,l}}$ requires applying the second formula in ($1.0.11$) $\sum_{j=k}^{h} a_{h,j} = \rho_h - \sum_{l=1}^{k} a_{h,l}$ times. This gives the power $q^{\sum_{j=k}^{h} a_{h,j}}$.

If $h = m$, then one factor of $v_{m+1}^{\rho_{m+1}}$ is replaced by $v_{m}$. Thus, for $i > m + 1$ and $j \leq k - 1$, moving $v_i^{a_{i,j}}$ to the left of $v_{m+1}^{\rho_{m+1} - \sum_{l=1}^{k} a_{h,l}}$ loses the sign contribution $(-1)^{a_{i,j}}$. The power of $q$ is produced similarly by taking $h = m$ in the argument above.

Finally, assume $h > m$. In this case, moving $v_i^{a_{i,j}}$ ($i > h + 1$ and $j < k$) to the left of $v_{h+1}^{a_{h+1,k-1} - \hat{a}_{h+1,k-1} - t - 1}$ contributes the same sign as in computing $v_\rho \cdot T_d$, while moving $b_{h+1,k}^{(t)}$ to the left of the block $v_1^{\rho_1 - \sum_{l=1}^{k} a_{h,l}} \cdots v_1^{\rho_{h} - \sum_{l=1}^{k} a_{h,l}}$ requires applying the third formula in ($1.0.11$) $\rho_h - \sum_{l=1}^{k} a_{h,l} = \sum_{j=k}^{h} a_{h,j}$ times, resulting again in the same sign. □
3. Super version of two key multiplication formulas

The realisation of quantum $\mathfrak{gl}_n$ requires several important multiplication formulas between certain standard basis elements and between certain elements of a spanning set—the BLM spanning set—for the $\nu$-Schur algebra. These formulas were built on two key formulas, see [1] Lem. 3.2, whose proof used the geometry of (partial) flag varieties. In this section, we will use an algebraic method to derive the two corresponding formulas in the super case. The algorithm discussed in §2 is helpful to the success of the method.

Let $\rho, \lambda \in \Lambda(m|n, r)$, $d \in \mathcal{D}^o_{\rho, \lambda}$, and $A = j(\rho, d, \lambda)$. Fix $1 \leq h < m + n$, and let

\[ \mu = (\rho_1, \rho_2, \cdots, \rho_h + 1, \rho_{h+1} - 1, \cdots, \rho_{m+n}) = \rho + e_h - e_{h+1}. \]

It is easy to show $\mathcal{D}_{\rho \mu} = \{1\}$. Set $B = j(\rho, 1, \mu)$ to be the matrix associated to the double coset $W_{\rho}1W_{\mu}$.

Notice also that, if $W_\nu = W_\rho^d \cap W_\lambda$, then $\nu = (\nu^1, \nu^2, \cdots, \nu^{m+n})$ where $\nu^i = \mathrm{col}_1(A) = (a_{1,i}, a_{2,i}, \cdots, a_{m+n,i}) \in \Lambda(m|n, \lambda_i)$. If we set $W_{\nu'} = W_{\mu} \cap W_\rho$, then $\nu' = (\rho_1, \cdots, \rho_h, 1, \rho_{h+1} - 1, \rho_{h+2}, \cdots, \rho_{m+n})$. Observing the compositions $\rho$ and $\nu'$, we have

\[ \mathcal{D}_{\nu'} \cap W_\rho = \{1, s_{\rho_1 + \cdots + \rho_h + 1} s_{\rho_1 + \cdots + \rho_h + 2} \cdots s_{\rho_1 + \cdots + \rho_h + 1} \cdots s_{\rho_1 + \cdots + \rho_h + 1} - 1\}. \]

For $A = (a_{i,j}) \in M(m|n, r)$ and $h, k \in [1, m + n]$ with $h < m + n$, define

\[ f_k(q; A, h) = \begin{cases} 
q^{\sum_{j>k} a_{h,j}}, & \text{if } h < m; \\
(-1)^{\sum_{j<m,k} a_{i,j}} q^{-\sum_{j<k} a_{m+1,j}} q^{\sum_{j>k} a_{m,j}}, & \text{if } h = m; \\
(q^{-1})^{\sum_{j<k} a_{h+1,j}}, & \text{if } h > m,
\end{cases} \]

and

\[ g_k(q; A, h + 1) = \begin{cases} 
q^{\sum_{j<k} a_{h+1,j}}, & \text{if } h < m; \\
(-1)^{\sum_{j>m,k} a_{i,j}} q^{\sum_{j<k} a_{m,j}}, & \text{if } h = m; \\
(q^{-1})^{\sum_{j>k} a_{h,j}}, & \text{if } h > m.
\end{cases} \]

Here is the key lemma of the paper.

**Lemma 3.1.** Maintain the notation above. For $A = (a_{i,j}) \in M(m|n, r)$ and $h \in [1, m+n)$, let $D = \mathrm{diag}(\mathrm{ro}(A))$, $B = D + E_{h,h+1} - E_{h+1,h+1}$ and $C = D - E_{h,h} + E_{h+1,h}$. Then, in the $q$-Schur superalgebra $S_q(m|n, r)$ over $A$, the following multiplication formulas hold (and are homogeneous):

\[ \begin{align*}
(1) & \quad N_A N_{B'} = \sum_{k \in [1, m+n], a_{h+1,k} \geq 1} f_k(q; A, h)[a_{h,k} + 1] q_{h} N(\lambda + E_{h,h} + E_{h+1,k})'; \\
(2) & \quad N_A N_{C'} = \sum_{k \in [1, m+n], a_{h,k} \geq 1} g_k(q; A, h + 1)[a_{h+1,k} + 1] q_{h+1} N(\lambda - E_{h,h} + E_{h+1,k})'.
\end{align*} \]

where $N(A \pm E_{h,h} \mp E_{h+1,k})' = 0$ whenever $A \pm E_{h,h} \mp E_{h+1,k} \not\in M(m|n, r)$. \[\quad \]

---

\footnote{The affine analogue of the two key formulas has also been discovered by Lusztig (see [15, Lem. 3.5]) and the proof is also geometric in nature.}
Proof. We first prove (1). Since \( N_{A'} = N_{d'\rho} N_{B'} = N_{\rho\mu} \) and \( N_{(A+E_h,k-E_{h+1,k})'} = N_{d'\mu} \) for some \( d' \in D'_{\mu\lambda} \), it suffices to prove that the actions of both sides on \( v_\lambda \) is the same. By \((1.1.2)\),

\[
(v_\lambda) N_{d'\rho} N_{\rho\mu} = \big((v_\lambda) N_{W_\rho \cap W_\lambda} (e_{\lambda,\rho}) N_{W_\rho \cap W_\mu} (e_{\rho,\mu}) \big) (v_\lambda) \sum_{x \in D'_{\rho} \cap W_\lambda} q^{-\ell(x)} T_{x-1} e_{\lambda,\rho} T_x \sum_{w \in D'_{\mu}} q^{-\ell(w)} T_{w-1} e_{\rho,\mu} T_w.
\]

Write \( x = x_0 x_1 \) with \( x_1 \in W_\lambda(i) \). By \((1.0.11)\), \( v_\lambda \cdot T_x = x^{\ell(x_0)} (-1)^{\ell(x_1)} v_\lambda \). Thus,

\[
(v_\lambda) N_{d'\rho} N_{\rho\mu} = \big( \sum_{x \in D'_{\rho} \cap W_\lambda} (-q^{-1})^{\ell(x_1)} v_{\rho \cdot T_x} \big) \sum_{w \in D'_{\mu}} q^{-\ell(w)} (v_{\rho \cdot T_x} T_{w-1}) e_{\rho,\mu} T_w
\]

\[
= \sum_{x \in D'_{\rho} \cap W_\lambda, w \in D'_{\mu}} (-q^{-1})^{\ell(x_1)} q^{-\ell(w)} (v_{\rho \cdot T_x} T_{w-1}) e_{\rho,\mu} T_w, \text{ by \((1.0.11)\)}.
\]

Now, \( (v_{\rho \cdot T_x} T_{w-1}) e_{\rho,\mu} \neq 0 \) if and only if \( w = w' dx \) for some \( w' \in D'_{\rho} \cap W_\rho \). Write \( w' = w_0' \cdot w' \) with \( w_0' \in W_\rho(i) \). Then, in this case, \( q^{-\ell(w')} (v_{\rho \cdot T_x} T_{w-1}) e_{\rho,\mu} = q^{-\ell(w') (-1)^{d'}} (v_{\rho \cdot T_x} T_{w-1}) e_{\rho,\mu} \). Hence,

\[
(v_\lambda) N_{d'\rho} N_{\rho\mu} = \sum_{x \in D'_{\rho} \cap W_\lambda, \ w' \in D'_{\rho} \cap W_\rho} (-q^{-1})^{\ell(x_1)} (-1)^{d'} q^{-\ell(w')} (v_\lambda) (v_{\rho \cdot T_x} T_{w-1}) e_{\rho,\mu} T_{w' dx}
\]

\[
= \sum_{x \in D'_{\rho} \cap W_\lambda, \ w' \in D'_{\rho} \cap W_\rho} (-1)^{d'} (-q^{-1})^{\ell(x_1)} (-q^{-1})^{\ell(w')} (v_\lambda) T_{w' dx} \tag{3.1.1}
\]

\[
= (-1)^{d'} \sum_{x \in D'_{\rho} \cap W_\lambda} (-q^{-1})^{\ell(x_1)} (T \cdot T_d) T_x,
\]

where \( T := \sum_{w' \in D'_{\rho} \cap W_\rho} (-q^{-1})^{\ell(w')} v_\mu T_{w'} \text{ and } \hat{d} = (\rho, d) = \sum_{i>k>m,j<l} a_{i,j} a_{k,l} \). Thus, by observing

\[
D'_{\rho} \cap W_\rho = \begin{cases} D_{\rho} \cap W_{\rho} \cap W_{\rho}, & \text{if } h < m; \\ D_{\rho} \cap W_{\rho}, & \text{if } h \geq m. \end{cases}
\]

we see that

\[
\mathcal{Y} = \begin{cases} \sum_{w' \in D'_{\rho} \cap W_\rho} v_\mu T_{w'}, & \text{if } h < m; \\ \sum_{w' \in D'_{\rho} \cap W_\rho} (-q^{-1})^{\ell(w')} v_\mu T_{w'}, & \text{if } h \geq m. \end{cases}
\]

Since \( D'_{\rho} \cap W_\rho = \{1, s_{\rho h+1}, s_{\rho h+1}s_{\rho h+2}, \ldots, s_{\rho h+1} \cdots s_{\rho h+\rho h+1-1}\} \), it follows from \((1.0.11)\) that,

\[
\mathcal{Y} = \begin{cases} v_1^{\rho_1} \cdots v_h^{\rho_h} (\sum_{i=1}^{\rho_{h+1}} v_{h+1}^{i-1} v_{h+2}^{i} v_{h+3}^{1-i}) v_{h+2}^{\rho_{h+2}} \cdots v_{m+n}^{\rho_{m+n}}, & \text{if } h < m; \\ v_1^{\rho_1} \cdots v_h^{\rho_h} (\sum_{i=1}^{\rho_{h+1}} (-q^{-1})^{i-1} v_{h+1}^{i-1} v_{h+2}^{1-i}) v_{h+2}^{\rho_{h+2}} \cdots v_{m+n}^{\rho_{m+n}}, & \text{if } h = m; \\ v_1^{\rho_1} \cdots v_h^{\rho_h} (\sum_{i=1}^{\rho_{h+1}} q^{i-1} v_{h+1}^{i-1} v_{h+2}^{1-i}) v_{h+2}^{\rho_{h+2}} \cdots v_{m+n}^{\rho_{m+n}}, & \text{if } h > m. \end{cases}
\]

Let

\[
v^A(b_{h+1,k}^{\bullet}) = \begin{cases} \sum_{t=0}^{a_{h+1,k}-1} v^A(b_{h+1,k}^{\bullet}), & \text{if } h < m; \\ \sum_{t=0}^{a_{m+1,k}-1} (-q^{-1})^{t} v^A(b_{h+1,k}^{\bullet}), & \text{if } h = m; \\ \sum_{t=0}^{a_{h+1,k}-1} q^{-t} v^A(b_{h+1,k}^{\bullet}), & \text{if } h > m; \tag{3.1.2} \end{cases}
\]
If \( h < m \), then 
\[
\sum_{i=1}^{\rho_{h+1}} v_{h+1}^{i-1} v_h^{\rho_{h+1} - i} = \sum_{k \in [1, m+n]} \sum_{a_{h+1,k} \geq 1} a_{h+1,k}^{\rho_{h+1} - 1} v_{h+1}^{k,t},
\]
and so
\[
\Upsilon \cdot T_d = \sum_{k \in [1, m+n]} \sum_{a_{h+1,k} \geq 1} \sum_{t=0}^{a_{h+1,k}^{\rho_{h+1} - 1}} v_1^\rho_1 \cdots v_h^\rho_h v_{h+1}^{k,t} v_{h+2}^{\rho_{h+2}} \cdots v_{m+n}^{\rho_{m+n}} \cdot \Upsilon \cdot T_d.
\]

If \( h \geq m \) and \( \star := \sum_{i>m+1,j<k} a_{i,j} \), then
\[
\Upsilon \cdot T_d = \sum_{k \in [1, m+n]} \sum_{a_{h+1,k} \geq 1} \sum_{t=0}^{a_{h+1,k}^{\rho_{h+1} - 1}} v_1^\rho_1 \cdots v_h^\rho_h ((-1)^a_{h,m} q^{-1})^{\delta_{h,m}} q_{a_{h+1,k} - 1}^{\rho_{h+1} - 1} v_{h+1}^{k,t} v_{h+2}^{\rho_{h+2}} \cdots v_{m+n}^{\rho_{m+n}} \cdot \Upsilon \cdot T_d
\]
\[
= \begin{cases} 
(-1)^{a_{h,m}} \sum_{k \in [1, m+n]} (-1)^{\star} (-q^{-1})^{\sum_{j \rightarrow h} a_{m+1,j}^q} q_{\sum_{j \rightarrow h} a_{m+1,j}^q} v^A(b_{h+1,k}^\star), & \text{if } h = m \\
(-1)^{\star} \sum_{k \in [1, m+n]} q_{\sum_{j \rightarrow h} a_{h+1,j}^q} v^A(b_{h+1,k}^\star), & \text{if } h > m.
\end{cases}
\]

Thus, with the notation given in (3.0.1), the RHS of (3.1.1) becomes
\[
\sum_{k \in [1, m+n]} a_{h+1,k} \geq 1 \quad f_k(q; A, h) \sum_{x \in D_{\nu} \cap W_\lambda} (-q^{-1})^{\ell(x)} v^A(b_{h+1,k}^\star) \cdot T_x = \sum_{k \in [1, m+n]} a_{h+1,k} \geq 1 \quad f_k(q; A, h) \Upsilon'(k; A, h),
\]
where
\[
\Upsilon' = \Upsilon'(k; A, h) := \sum_{x \in D_{\nu} \cap W_\lambda} (-q^{-1})^{\ell(x)} v^A(b_{h+1,k}^\star) \cdot T_x.
\]

By comparing this with the formulas in the theorem, it remains to prove that, for all \( k \in [1, m+n] \) with \( a_{h+1,k} \geq 1 \),
\[
\Upsilon' = \begin{cases} 
[a_{h,k} + 1] q_h(v_\lambda) N(A + E_{h,k} - E_{h+1,k}), & \text{if } A + E_{h,k} - E_{h+1,k} \in M(m|n, r), \\
0, & \text{otherwise.}
\end{cases}
\]

We now compute the left hand side of (3.1.3). Recall the composition \( \nu = \nu(A) \) defined by the columns of \( A \). Take a note of the right hand side of (3.1.3): for \( a_{h+1,k} \geq 1 \),
\[
(v_\lambda) N(A + E_{h,k} - E_{h+1,k})' = \sum_{x \in D_{\nu} \cap W_\lambda} (-q^{-1})^{\ell(x)} v_\sigma \cdot T_x,
\]
where
\[
\sigma = \sigma(h, k) = \nu(A + E_{h,k} - E_{h+1,k}) = (\sigma^1, \sigma^2, \ldots, \sigma^{m+n}).
\]

Clearly, \( \sigma \) is obtained from \( \nu \) by replacing \( \nu^k = (a_{1k}, a_{2k}, \ldots, a, b, \ldots, a_{m+n,k}) \), where \( a = a_{h,k}, b = a_{h+1,k} \), by
\[
\sigma^k = (a_{1k}, a_{2k}, \ldots, a, b + 1, b - 1, \ldots, a_{m+n,k}).
\]
We also define $\tau = \tau(h, k)$ to be the composition obtained from $\nu$ by replacing $\nu^k$ by

$$\tau^k = (a_{1,k}, a_{2,k}, \ldots, a, 1, b - 1, \ldots, a_{m+n,k}).$$

Note that the relationship between $\nu^k$, $\sigma^k$, and $\tau^k$ are similar to the relationship between $\rho$, $\mu$, and $\rho'$. Note also that $W_\nu \cap W_\sigma = W_\tau$ and so $W_\tau \leq W_\sigma$ and $W_\tau \leq W_\nu$. In particular, we have

$$\begin{cases} (1) & b = 1 \implies W_\tau = W_\nu; \\ (2) & a = 0 \implies W_\tau = W_\sigma; \\ (3) & a = 1 \implies W_\sigma = W_\tau \cup W_\tau \sigma_\tilde{a}, \end{cases} \quad (3.1.4)$$

where

$$\tilde{a} = |\nu^1| + \cdots + |\nu^{k-1}| + a_{1,k} + \cdots + a_{h-1,k} + a_{h,k} = a' + a (a' = a_{h-1,k}).$$

Since $\sigma^{k'} = \nu^{k'} = \tau^{k'}$ for all $k' \neq k$, it follows that, for each $\iota \in \{\nu, \sigma, \tau\}$, there is a commuting set decomposition

$$\mathcal{D}_\iota \cap W_\lambda = \mathcal{D}(\mathcal{D}_\iota \cap W_{\lambda_k}) \mathcal{D}', \quad (3.1.5)$$

where $\mathcal{D} = \mathcal{D}_{\nu^1} \cap W_{\lambda_1} \cdots \mathcal{D}_{\nu^k} \cap W_{\lambda_k-1}$ and $\mathcal{D}' = \mathcal{D}_{\nu^{k+1}} \cap W_{\lambda_{k+1}} \cdots \mathcal{D}_{\nu^{m+n}} \cap W_{\lambda_{m+n}}$. Moreover,

$$\mathcal{D}_\tau \cap W_{\lambda_k} = (\mathcal{D}_\tau \cap W_{\nu^k}) \times (\mathcal{D}_\nu \cap W_{\lambda_k}) = (\mathcal{D}_\tau \cap W_{\sigma^k}) \times (\mathcal{D}_\sigma \cap W_{\lambda_k}), \quad (3.1.6)$$

where

$$\mathcal{D}_\tau \cap W_{\nu^k} = \{1, s_{\tilde{a}+1}, s_{\tilde{a}+1}s_{\tilde{a}+2}, \ldots, s_{\tilde{a}+2} \cdots s_{\tilde{a}+b-1}\} \text{ and}$$

$$\mathcal{D}_\tau \cap W_{\sigma^k} = \{1, s_{\tilde{a}}, s_{\tilde{a}}s_{\tilde{a}-1}, \ldots, s_{\tilde{a}} \cdots s_{\tilde{a}'+1}\}.$$

We now complete our computation in two cases.

**Case 1.** $a_{h+1,k} \geq 1$ and $k = 0$. In this case, $\mathcal{D}_\tau \cap W_{\lambda_k} \subseteq W_{\lambda(0)}$. So $x = x_0$ for all $x \in \mathcal{D}_\tau \cap W_{\lambda_k}$ (or $x_1 = 1$).

If $h < m$, by the decomposition $[3.1.5]$ for $\iota = \nu$, we have

$$\mathcal{Y}' = \sum_{w \in \mathcal{D}_\nu \cap W_\lambda} (-q^{-1})^{l_\nu(w)} \sum_{x \in \mathcal{D}_\nu \cap W_{\lambda_k}} v^A (v_h v_{h+1}^{a_{h+1,k}-1} + \cdots + v_{h+1}^{a_{h+1,k}-1} v_h) \cdot T_w$$

$$= \sum_{d \in \mathcal{D}_\nu, d' \in \mathcal{D}'} (-q^{-1})^{l_\nu(d_1 d')} \sum_{x \in \mathcal{D}_\nu \cap W_{\lambda_k}} v^A (v_h v_{h+1}^{a_{h+1,k}-1} + \cdots + v_{h+1}^{a_{h+1,k}-1} v_h) \cdot T_x T_{dd'}$$
However,
\[ \sum_{x \in D_{\nu} \cap W_{\lambda}} v^A (v_h v_{h+1}^{a_{h+1,k} - 1} + \cdots + v_{h+1}^{a_{h+1,k} - 1} - v_h) \cdot T_x \]
\[ = \sum_{y \in D_{\nu} \cap W_{\lambda} \cap W_{\lambda}'} (\cdots v_1^{a_{1,k}} \cdots v_h^{a_{h-1,k}} (v_h v_{h+1}^{a_{h+1,k} - 1} v_{h+2}^{a_{h+2,k}} \cdots v_{m+n}^{a_{m+n,k}}) \cdots) \cdot T_y T'_y \]
\[ = \sum_{z \in D_{\nu} \cap W_{\lambda} \cap W_{\lambda}'} (\cdots v_1^{a_{1,k}} \cdots v_h^{a_{h-1,k}} v_{h+1} v_{h+2}^{a_{h+2,k}} \cdots v_{m+n}^{a_{m+n,k}} \cdots) \cdot T_z' \]
\[ = [a + 1] \sum_{z' \in D_{\nu} \cap W_{\lambda}'} (\cdots v_1^{a_{1,k}} \cdots v_h^{a_{h-1,k}} v_{h+1} v_{h+2}^{a_{h+2,k}} \cdots v_{m+n}^{a_{m+n,k}} \cdots) \cdot T_z' \]

(Here, and in the sequel, only the part associated with column $k$ is displayed. Other parts unchanged from $v^A$ are omitted.) By (3.1.6), substituting gives

\[ \Upsilon' = [a_{h,k} + 1] \sum_{z \in D_{\nu} \cap W_{\lambda}} (-q^{-1})^{0} v_0^A \cdot T_x = [a_{h,k} + 1] (N_{A+E_{h,k} = E_{h+1,k}}) \cdot T_x' \]

Assume now $h \geq m$, then $b = a_{h+1,k} = 1$ and so $W_{\nu} = W_{\sigma} = W_{\tau}$ as seen in (3.1.4). If $h = m$, by a similar argument as above. we have

\[ \Upsilon' = \sum_{x \in D_{\nu} \cap W_{\lambda}} (-q^{-1})^{0} (v_1^{a_{m+1,m}} (v_1 v_{m+1}^{a_{m+1,k} - 1} \cdots) \cdot T_x \]
\[ = \sum_{x \in D_{\nu} \cap W_{\lambda}} (-q^{-1})^{0} (v_1^{a_{m+1,m}} \cdots v_{m+n}^{a_{m+n,k} - 1} \cdots) \cdot T_x \]
\[ = [a_{m,k} + 1] \sum_{z \in D_{\nu} \cap W_{\lambda}} (-q^{-1})^{0} v_0^A \cdot T_z. \]

If $h > m$, then $a = a_{h,k} = 0, 1$. For $a = 0$, by (3.1.4), $W_{\nu} = W_{\sigma} = W_{\tau}$ and so

\[ \Upsilon' = \sum_{x \in D_{\nu} \cap W_{\lambda}} (-q^{-1})^{0} (v_1^{a_{1,k}} \cdots v_h v_{h+2}^{a_{h+2,k}} \cdots v_{m+n}^{a_{m+n,k}} \cdots) \cdot T_x \]
\[ = [a_{h,k} + 1] \sum_{x \in D_{\nu} \cap W_{\lambda}} (-q^{-1})^{0} v_0^A \cdot T_x. \]

For $a = 1$, $W_{\nu} = W_{\tau}$, $D_{\tau} \cap W_{\lambda} = \{1, s_\beta\}$ and $A + E_{h,k} = E_{h+1,k} \notin M(m|n, r)$. But

\[ \Upsilon' = \sum_{x \in D_{\nu} \cap W_{\lambda}} (-q^{-1})^{0} (v_1^{a_{1,k}} \cdots v_h v_{h+2}^{a_{h+2,k}} \cdots v_{m+n}^{a_{m+n,k}} \cdots) \cdot T_x \]
\[ = \sum_{z' \in D_{\nu} \cap W_{\lambda}} (-q^{-1})^{0} (v_1^{a_{1,k}} \cdots v_h v_{h+2}^{a_{h+2,k}} \cdots v_{m+n}^{a_{m+n,k}} \cdots) \cdot T_z T_{z'}. \]

Case 2. $a_{h+1,k} \geq 1$ and $\hat{k} = 1$. In this case, $D_{\nu} \cap W_{\lambda} \subseteq W_{\lambda}(1).$ So $x = x_1$ for $x \in D_{\nu} \cap W_{\lambda}$. 

\[- \]
If \( h < m \) then \( b = a_{h+1,k} = 1 \) and \( a = a_{h,k} \in \{0, 1\} \). Repeatedly applying (3.1.4) yields

\[
\Upsilon' = \sum_{x \in D_x \cap W_\lambda} (-q^{-1})^\ell(x_1)(\ldots v_1^{a_{1,k}} \ldots v_h^{a_{h+1,k-1}} v_{h+1}^{a_{h+1,k}} v_{h+2}^{a_{h+2,k}} \ldots v_{m+n}^{a_{m+n,k}}) \cdot T_x
\]

or

\[
\begin{cases}
0, & \text{if } a = 1; \\
\sum_{x \in D_x \cap W_\lambda} (-q^{-1})^\ell(x_1) v_\sigma \cdot T_x, & \text{if } a = 0.
\end{cases}
\]

Assume now \( h = m \). Then \( a = a_{h,k} \in \{0, 1\} \) (but \( a_{h+1,k} \geq 1 \)). By the first formula of (1.0.11),

\[
\Upsilon' = \sum_{x \in D_x \cap W_\lambda} (-q^{-1})^\ell(x_1)(\ldots v_1^{a_{1,k}} \ldots v_h^{a_{h+1,k}} v_{h+1}^{a_{h+1,k}} v_{h+2}^{a_{h+2,k}} \ldots v_{m+n}^{a_{m+n,k}}) \cdot T_x.
\]

By (3.1.6) and (3.1.4)(3), a similar argument shows that

\[
\Upsilon' = \begin{cases}
0, & \text{if } a = 1; \\
\sum_{x \in D_x \cap W_\lambda} (-q^{-1})^\ell(x_1) v_\sigma \cdot T_x, & \text{if } a = 0.
\end{cases}
\]

Finally, assume \( h > m \). Then, by the decomposition (3.1.5),

\[
\Upsilon' = \sum_{w \in D_x \cap W_\lambda} (-q^{-1})^\ell(w_1)(\ldots v_1^{a_{1,k}} \ldots v_h^{a_{h,k}} (v_h v_{h+1}^{a_{h+1,k-1}} + q^{-1} v_{h+1} v_h v_{h+1}^{a_{h+1,k-2}} + \ldots + (q^{-1})^{a_{h+1,k-1}} v_{h+1}^{a_{h+1,k-1}} v_{h+2}^{a_{h+2,k}} \ldots v_{m+n}^{a_{m+n,k}}) \cdot T_w
\]

or

\[
\begin{cases}
0, & \text{if } a = 1; \\
\sum_{x \in D_x \cap W_\lambda} (-q^{-1})^\ell(x) \sum_{d \in D_d \cap D'} (-q^{-1})^\ell(d_1 d_1') (\ldots v_1^{a_{1,k}} \ldots v_h^{a_{h,k}} (v_h v_{h+1}^{a_{h+1,k-1}} + q^{-1} v_{h+1} v_h v_{h+1}^{a_{h+1,k-2}} + \ldots + (q^{-1})^{a_{h+1,k-1}} v_{h+1}^{a_{h+1,k-1}} v_{h+2}^{a_{h+2,k}} \ldots v_{m+n}^{a_{m+n,k}}) \cdot T_x T_{d d'}.
\end{cases}
\]

By (3.1.3) and repeatedly applying the third formula in (1.0.11), the inner summation becomes

\[
\sum_{y_{\in D_{x} \cap W_{\lambda}}} (-q^{-1})^\ell(y_1)(\ldots v_1^{a_{1,k}} \ldots v_h^{a_{h,k}} v_{h+1}^{a_{h+1,k-1}} v_{h+2}^{a_{h+2,k}} \ldots v_{m+n}^{a_{m+n,k}}) T_{y_1 x}
\]

or

\[
\begin{cases}
0, & \text{if } a = 1; \\
\sum_{x \in D_x \cap W_\lambda} (-q^{-1})^\ell(x') \sum_{z' \in D_x \cap W_{\lambda}} (-q^{-1})^\ell(z') (\ldots v_1^{a_{1,k}} \ldots v_h^{a_{h,k}} v_{h+1}^{a_{h+1,k-1}} v_{h+2}^{a_{h+2,k}} \ldots v_{m+n}^{a_{m+n,k}}) T_{z' x'}.
\end{cases}
\]

Hence, substituting gives

\[
\Upsilon' = [a_{h,k} + 1] q^{-1} \sum_{z' \in D_x \cap W_{\lambda}} (-q^{-1})^\ell(z') v_\sigma \cdot T_{z' x'}.
\]

So we have proved all cases for the first formula.
Thus, \( \Upsilon \) formulas which will be used to establish the realisation of the supergroup \( \mathbf{U} \).

Lemma 4.1. Maintain the notation above and assume we also have \( \text{RHS} = 0 \) (Note that, for \( h \)).

Remark 3.2. As seen in Lemma 3.1, we will make the convention \( N_{\mathcal{A}'} = 0 \) for any matrix \( A \in M_{m+n}(\mathbb{Z}) \) but \( A \notin M(m|n,r) \).

4. Some multiplication formulas in \( S_q(m|n,r) \)

We now use the key formulas given in Lemma 3.1 to derive more multiplication formulas which will be used to establish the realisation of the supergroup \( \mathbf{U}(\mathfrak{gl}_{m|n}) \).

For \( \lambda = (\lambda_1, \ldots, \lambda_{m+n}) \in \Lambda(m|n,r) \) and fixed \( 1 \leq h < m+n \), let

\[
B_p = B_p(h, \lambda) = \text{diag}(\lambda + pe_h - (p+1)e_{h+1}) + E_{h,h+1}, \quad \text{resp.,}
\]

\[
C_p = C_p(h, \lambda) = \text{diag}(\lambda - (p+1)e_h + pe_{h+1}) + E_{h,h+1},
\]

for all \( 0 \leq p < \lambda_{h+1}, \) resp., \( 0 \leq p < \lambda_h, \) and let

\[
U_p = U_p(h, \lambda) = \text{diag}(\lambda - pe_{h+1}) + pE_{h,h+1}, \quad \text{resp.,}
\]

\[
L_p = L_p(h, \lambda) = \text{diag}(\lambda - pe_h) + pE_{h+1,h},
\]

for all \( 0 \leq p \leq \lambda_{h+1}, \) resp., \( 0 \leq p \leq \lambda_h, \) Clearly, \( \text{co}(U_p) = \text{co}(L_p) = \lambda, \) \( ro(U_p) = \text{co}(B_p) \) and \( ro(L_p) = \text{co}(C_p) \). We need the following relations for a proof by induction.

Lemma 4.1. Maintain the notation above and assume \( 1 \leq h < m+n \). We have

\[
\begin{align*}
(1) \quad N_{U_p}N_{B_p} &= [p+1]_{q_h}N_{U_{p+1}} \quad \text{for all } p \geq 1; \\
(2) \quad N_{U_p}N_{C_p} &= [p+1]_{q_h}N_{U_{p+1}} \quad \text{for all } p \geq 1.
\end{align*}
\]

(Not that, for \( h = m \), the RHS in both cases is 0 and, for \( p \geq \lambda_{h+1}, \) resp., \( p \geq \lambda_h, \) we also have \( \text{RHS} = 0 \) in (1), resp., in (2).)

Proof. Take \( A \) to be \( U_p \) in Lemma 3.1(1). If \( a_{h+1,k} \geq 1 \), then \( k = h+1 \) and \( a_{h,h+1} = p \). Thus, \( f(q; A,h) = 1 \) for \( h \neq m \) and, hence, \( N_{U_p}N_{B_p} = [p+1]_{q_h}N_{U_{p+1}} \). If \( h = m \), then \( U_{p+1} \notin M(m|n,r) \) and, hence, \( N_{U_{p+1}} = 0 \). The second formula can be proved similarly. \( \square \)
For \( A \in M(m|n,r) \), set \( \text{row}_h(A) = (a_{h,1}, a_{h,2}, \ldots, a_{h,m+n}) \) to be the \( h \)-th row of \( A \). Then \( \text{row}_h(A) \in \mathbb{N}^{m+n} \). For \( \nu, \nu' \in \mathbb{N}^{m+n} \), set
\[
\nu \leq \nu' \iff \nu_j \leq \nu'_j \text{ for all } j \in [1,m+n].
\]

Like the polynomials defined in (3.0.1) and (3.0.2), we define, for \( A = (a_{i,j}) \in M(m|n,r) \), \( h \in [1,m+n] \), and \( \nu \in \mathbb{N}^{m+n} \) with \( |\nu| = p \geq 2 \),
\[
f_\nu(q; A, h) = \begin{cases} 
q^{\sum_{j \geq t} a_{h,j} \nu_t}, & \text{if } h < m; \\
0, & \text{if } h = m; \\
q^{\sum_{j < t} \nu_{t-1}(q^{-1}) \sum_{k \geq l} a_{h,k} \nu_l}, & \text{if } h > m,
\end{cases}
\]
and
\[
g_\nu(q; A, h+1) = \begin{cases} 
q^{\sum_{j < t} a_{h+1,k} \nu_t}, & \text{if } h < m; \\
0, & \text{if } h = m; \\
q^{\sum_{j < t} \nu_{t-1}(q^{-1}) \sum_{k \geq l} a_{h,k} \nu_l}, & \text{if } h > m.
\end{cases}
\]

Note that, if \( |\nu| = 1 \), we may extend the definition by setting \( f_\nu(q; A, h) = f_k(q; A, h) \) and \( g_\nu(q; A, h+1) = g_k(q; A, h + 1) \) as in (3.0.1) and (3.0.2), if \( \nu_k = 1 \).

Now the multiplication formulas in Lemma 3.1 can be generalised from \( U_1 = B \) and \( L_1 = C \) to \( U_p \) and \( L_p \), respectively.

**Theorem 4.2.** Let \( A = (a_{i,j}) \in M(m|n,r) \). Assume \( 1 \leq h < m + n \). Define \( U_p = U_p(h, \text{ro}(A)), L_p = L_p(h, \text{ro}(A)) \in M(m|n,r) \) as in (4.0.1). The following multiplication formulas hold in the \( q \)-Schur superalgebra \( S_q(m|n,r) \) over \( A \):

1. \[
N_A N_{U_p} = \sum_{\nu \in \Lambda(m|n,p)} \prod_{\nu \leq \text{row}_{h+1}(A)} f_\nu(q; A, h) \prod_{k=1}^{m+n} \left[ a_{h,k} + \nu_k \right] q_h \left( N(A+\sum_i \nu_i(E_{h,l} - E_{h+1,l}))' \right);
\]
2. \[
N_A N_{L_p} = \sum_{\nu \in \Lambda(m|n,p)} \prod_{\nu \leq \text{row}_{h+1}(A)} g_\nu(q; A, h+1) \prod_{k=1}^{m+n} \left[ a_{h+1,k} + \nu_k \right] q_{h+1} \left( N(A-\sum_i \nu_i(E_{h,l} - E_{h+1,l}))' \right).
\]

Here, as before, \( N_X = 0 \) for a matrix \( X \notin M(m|n,r) \).

**Proof.** We only prove (1). The proof of (2) is similar.

By the definition of the coefficients \( f_\nu(q; A, h) \), (1) is true for \( p = 1 \). Assume now \( p > 1 \) and \( h = m \), then \( f_\nu(q; A, h) = 0 \) for all \( \nu \) and \( U_p \notin M(m|n,r) \). So \( N_{U_p} = 0 \) by the convention in Remark 3.2. Hence, \( N_A N_{U_p} = 0 \), proving the formula in this case. We now assume \( h \neq m \) and let \( e_k = (0, \ldots, 0, 1, 0, \ldots) \); see (1.0.5). The proof is parallel to that in [1] by applying induction on \( p \).
By Corollary 4.1, $N_{U_p'}N_{B_p'} = [p + 1]q_hN_{U_p'}$. We assume $U_{p+1} \in M(m|n, r)$ (so $p + 1 \leq \sum_{k=1}^{m+n} a_{h+1,k}$). Then

$$N_{A'}N_{U_p'} = \frac{1}{[p + 1]q_h}N_{A'}N_{U_p'}N_{B_p'}$$

$$= \frac{1}{[p + 1]q_h} \left( \sum_{\nu \in \Lambda(m+n,p)} f_\nu(q; A, h) \prod_{k=1}^{m+n} \left[ a_{h,k} + \nu_k \right] q_h N_{A' + \sum_\nu \left( \nu(E_{h,1} - E_{h+1,1}) \right)} \right) N_{B_p'}.$$ (4.2.1)

Putting $A_\nu = A + \sum_\nu \nu(E_{h,l} - E_{h+1,l})$, we have, by Lemma 3.1(1),

$$N_{A_\nu}N_{B_p'} = \sum_{s \in \Lambda(m+n)} f_s(q; A, h)[a_{h,s} + \nu_s + 1]q_hN_{(A_\nu + E_{h,s} - E_{h+1,s})'}. \tag{4.2.2}$$

For fixed $\eta \in \Lambda(m+n, p+1)$, if $A + \sum_\eta \eta(E_{h,l} - E_{h+1,l}) \in M(m|n, r)$, then $\eta \leq \text{row}_{h+1}(A)$. By inserting (4.2.2) into (4.2.1), we see that the coefficient $f_\eta' (q; A, h)$ of $N_{A_\eta} = N(A + \sum_\eta \eta(E_{h,l} - E_{h+1,l}))'$ is a sum over pairs $(\eta, \nu)$ with $\eta = \nu + e_s$. Hence, $f_\eta' (q; A, h)$ is equal to

$$\frac{1}{[p + 1]q_h} \sum_{s \in \Lambda(m+n)} f_{\eta - e_s} (q; A, h) \prod_{k=1}^{m+n} \left[ a_{h,k} + (\eta - e_s)_k \right] \left( \eta - e_s \right)_k q_h f_s(q; A_{\eta - e_s}, h)[a_{h,s} + \eta_s].$$

If $h < m$, then $q_h = q$, $f_s(q; A_{\eta - e_s}, h) = q^{\sum_{j>s} a_{h,j} + (\eta - e_s)_j} e_s$, and $f_{\eta - e_s} (q; A, h) = q^{\sum_{j>t} a_{h,j} (\eta - e_s)_j} e_s$. Since

$$\prod_{k=1}^{m+n} \left[ a_{h,k} + (\eta - e_s)_k \right] \cdot [a_{h,s} + \eta_s] = \prod_{k=1}^{m+n} \left[ a_{h,k} + \eta_k \right] \cdot [\eta_s],$$

$$\sum_{j>t} a_{h,j} (\eta - e_s)_t + \sum_{j>s} (a_{h,j} + (\eta - e_s)_j) = \sum_{j>t} a_{h,j} \eta_t + \sum_{j>s} a_{h,j} + \sum_{j>s} a_{h,j} + \sum_{j>s} \eta_j$$

$$= \sum_{j>t} a_{h,j} \eta_t + \sum_{j>s} \eta_j,$$

and

$$[p + 1] = \sum_{s \in \Lambda(m+n,p)} q^{\sum_{j>s} \eta_j} [\eta_s],$$

it follows that

$$f_\eta' (q; A, h) = \frac{1}{[p + 1]q_h} q^{\sum_{j>t} a_{h,j} \eta} \prod_{k=1}^{m+n} \left[ a_{h,k} + \eta_k \right] \sum_{s \in \Lambda(m+n,p)} q^{\sum_{j>s} \eta_j} [\eta_s]$$

$$= q^{\sum_{j>t} a_{h,j} \eta} \prod_{k=1}^{m+n} \left[ a_{h,k} + \eta_k \right] = f_\eta (q; A, h) \prod_{k=1}^{m+n} \left[ a_{h,k} + \eta_k \right],$$

as desired in this case.
If $h > m$, then $q_h = q^{-1}$. We first observe that $q^{2p(p-1)} = \frac{[p]!}{[p]_q^{-1}}$ and $\frac{1}{2}p(p - 1) - \sum_{k=1}^{m+n} \frac{1}{2} \nu_k (\nu_k - 1) = \sum_{l < t} \nu_l \nu_t$. It follows that

\[
\frac{[p]!}{[p]_q^{-1}} (q^{-1})^{\sum_{j < t} a_{h+1,j} t} \prod_{k=1}^{m+n} \frac{[a_{h,k} + \nu_k]_q^{m-1}}{[a_{h,k}]_q^{m-1}} = f_{q'}(q; A, h) \prod_{k=1}^{m+n} \frac{[a_{h,k} + \nu_k]_q^{m-1}}{[a_{h,k}]_q^{m-1}}.
\]

In this case, $f_{q'}(q; A_{-e_s}, h) = (q^{-1})^{\sum_{j < s} a_{h+1,j} h} a_{h,s} + \eta_s q^{-1}$.

Since

\[
\prod_{k=1}^{m+n} \frac{[a_{h,k} + (\eta - e_s)_k]_q^{m-1}}{[a_{h,k}]_q^{m-1}} (q^{-1})^{\sum_{j < t} (a_{h+1,j} - (\eta - e_s) j)} \prod_{k=1}^{m+n} \frac{[a_{h,k} + \eta_k]_q^{m-1}}{[a_{h,k}]_q^{m-1}} = \frac{[p]!}{[p + 1]_{q^{-1}}^m} (q^{-1})^{\sum_{j < t} a_{h+1,j} j} a_{h,s} + \eta_s q^{-1},
\]

and since

\[
\sum_{j < t} a_{h+1,j} (\eta - e_s) j + \sum_{j < s} (a_{h+1,j} - (\eta - e_s) j)
\]

\[
= \sum_{j < t} a_{h+1,j} \eta_j - \sum_{j < s} a_{h+1,j} + \sum_{j < s} a_{h+1,j} - \sum_{j < s} \eta_j
\]

\[
= \sum_{j < t} a_{h+1,j} \eta_j - \sum_{j < s} \eta_j;
\]

it follows that

\[
f_{q'}(q; A, h) = \frac{[p]!}{[p + 1]_{q^{-1}}^m} (q^{-1})^{\sum_{j < t} a_{h+1,j} j} \prod_{k=1}^{m+n} \frac{[a_{h,k} + \eta_k]_q^{m-1}}{[a_{h,k}]_q^{m-1}} \sum_{s \in \{1, m+n\}, \eta - e_s \in \Lambda(m+n, p)} q^{\sum_{j < s} \eta j} \frac{[q]!}{[q]_{q^{-1}}^m} (q^{-1})^{\sum_{j < t} a_{h+1,j} j} \prod_{k=1}^{m+n} \frac{[a_{h,k} + \eta_k]_q^{m-1}}{[a_{h,k}]_q^{m-1}}
\]

\[
= f_{q'}(q; A, h) \prod_{k=1}^{m+n} \frac{[a_{h,k} + \eta_k]_q^{m-1}}{[a_{h,k}]_q^{m-1}} q^{-1},
\]

as desired. \qed

5. Normalised multiplication formulas in $S_\nu(m|n, r)$

The multiplication formulas discovered in the last section are derived through a $T$-basis action on the tensor superspace. It is somewhat symmetric relative to $\nu^2$ and $-1$, the eigenvalues of $T_s$. For example, the $h = m$ case in (3.0.1) and (3.0.2) reflects such symmetry. In order to obtain a realisation for quantum super $\mathfrak{gl}_{m|n}$, we need to recover a symmetry relative to $\nu$ and $-\nu^{-1}$, the eigenvalues of the normalised generators $\nu^{-1}T_s$. Thus, we need to normalise the basis $\{N_A\}_A$ and twist the right action to a left action via the anti-involution $\tau$ given in Lemma 1.2(2).
For $A = (a_{ij}) \in M(m|n, r)$, define
\[
d(A) = \sum_{i > k, j < l} a_{i,j} a_{k,l} + \sum_{j < l} (-1)^i a_{i,j} a_{i,l}, \quad \xi_A = q^{-d(A)} \tau(N_A).
\] (5.0.3)

Remark 5.1. Actually, if $A$ is corresponding to the triple $(\lambda | \mu, d, \xi | \eta)$ where $\lambda | \mu, \xi | \eta \in \Lambda(m|n, r)$ and $d \in D_{\lambda | \mu, \xi | \eta}$, using the notation in [9, §6], $d(A) = \ell(d^*) - \ell(d^*) + \ell(d) - \ell(w_0, \xi) + \ell(w_0, \eta)$.

Recall $q = v^2$ and $v_h = v^{-1} h$. The multiplication formulas in Theorem 4.2 turn out to be quite symmetric if we use the normalised basis $\{\xi_A\}_{\lambda \in M(m|n, r)}$ for $S_v(m|n, r)$. We first look at the even generator case.

**Proposition 5.2 (The $h \neq m$ case).** Let $A = (a_{ij}) \in M(m|n, r)$. Assume $1 \leq h < m + n$ and $h \neq m$. Define $U_p = U_p(h, \text{ro}(A))$, $L_p = L_p(h, \text{ro}(A)) \in M(m|n, r)$ as in (4.0.1). The following multiplication formulas hold in the $v$-Schur superalgebra $S_v(m|n, r)$ over $\mathbb{Z}$.

1. $\xi U_p \xi_A = \sum_{\nu \in \lambda(m|n, p)} v^{f_h (\nu, A)} \prod_{k=1}^{m+n} \left[ \begin{array}{c} a_{h,k} + \nu_k \\ \nu_k \end{array} \right] v^2 \xi_A + \sum_{\ell \in \ell(E_{h,l} - E_{h+1,l})} v^{\mu (E_{h,l} - E_{h+1,l})}$, where

\[
f_h (\nu, A) = \sum_{j > t} a_{h,j} \nu_t - \sum_{j < t} a_{h+1,j} \nu_t + \sum_{t < t'} \nu_t \nu_{t'}.
\] (5.2.1)

2. $\xi L_p \xi_A = \sum_{\nu \in \lambda(m|n, p)} v^{f_{h+1} (\nu, A)} \prod_{k=1}^{m+n} \left[ \begin{array}{c} a_{h+1,k} + \nu_k \\ \nu_k \end{array} \right] v^2 \xi_A - \sum_{\ell \in \ell(E_{h,l} - E_{h+1,l})} v^{\mu (E_{h,l} - E_{h+1,l})}$, where

\[
f_{h+1} (\nu, A) = \sum_{j < t} a_{h+1,j} \nu_t - \sum_{j < t} a_{h,j} \nu_t + \sum_{t < t'} \nu_t \nu_{t'}.
\] (5.2.2)

Note that $v_h = v_{h+1}$ if $h \neq m$. Note also that, in this $h \neq m$ case, these formulas are the super version of the formulas (a2) and (b2) in [11, Lem. 3.4]. Formerly, they can be obtained by replacing $v$ there by $v_h$.

**Proof.** We just give the proof of (1). The proof of (2) is similar.

For $\nu \in \Lambda(m|n, p)$ with $\nu \leq \text{ro}(h+1)(A)$, set $X = A + \sum_{\ell} \nu(\ell(E_{h,l} - E_{h+1,l})$. Since
\[
\left[ \begin{array}{c} N \\ s \end{array} \right] = v^{2s(N-s)} \left[ \begin{array}{c} N \\ -s \end{array} \right], \quad \text{for all } N \geq s \geq 0,
\]
it follows that
\[
\prod_{k=1}^{m+n} \left[ \begin{array}{c} a_{h,k} + \nu_k \\ \nu_k \end{array} \right] v^2 = v^{2 \sum_{k=1}^{m+n} a_{h,k} \nu_k} \prod_{k=1}^{m+n} \left[ \begin{array}{c} a_{h,k} + \nu_k \\ \nu_k \end{array} \right].
\]

By a comparison with Theorem 4.2(1), the proof of (1) is equivalent to the identity.
\[
v^{d(X) - d(A) - d(U_p)} f_\nu(q; h) v^{2 \sum_{k=1}^{m+n} a_{h,k} \nu_k} = v^{f_h (A, \nu)}.
\] (5.2.3)
Let
\[ J = \{(i, j, k, l) \mid 1 \leq i, j, k, l \leq m + n, i > k, j < l\}, \]
\[ I = \{(i, j, k, l) \in J \mid i, k \notin \{h, h + 1\}\}, \]
\[ J' = \{(i, j, i, l) \mid 1 \leq i, j, l \leq m + n, j < l\}, \] and
\[ I' = \{(i, j, i, l) \in J' \mid i \notin \{h, h + 1\}\}. \]

By definition, \(x_{i,j} = a_{i,j}\) for all \(i, j\) with \(i \notin \{h, h + 1\}\). Hence,
\[
d(X) = \sum_{i} a_{i,j}a_{k,l} + \sum_{J \setminus I} x_{i,j}x_{k,l} + \sum_{I'} (-1)^{\hat{h}} a_{i,j}a_{i,l} + \sum_{J' \setminus I'} (-1)^{\hat{h}+1} x_{i,j}x_{i,l},
\]
where
\[
\sum_{J \setminus I} x_{i,j}x_{k,l} = \sum_{h > k, j < l} (a_{h,j} + \nu_j) a_{k,l} + \sum_{i = h, h + 1, k = h, j < l} (a_{h+1,j} - \nu_j) a_{k,l} + \sum_{i = h + 1, k = h, j < l} a_{i,j}(a_{h+1,l} - \nu_l)
\]
\[
+ \sum_{i > h + 1, k = h, j < l} a_{i,j}(a_{h+1,l} + \nu_l) + \sum_{i > h + 1, k = h, j < l} a_{i,j}(a_{h+1,l} - \nu_l) - \sum_{j < l} \nu_j \nu_l + \sum_{i > h + 1, j < l} a_{i,j} \nu_l - \sum_{i > h + 1, j < l} a_{i,j} \nu_l
\]
\[
= \sum_{J \setminus I} a_{i,j}a_{k,l} - \sum_{h > k, j < l} a_{h,l} \nu_j - \sum_{h > k, j < l} a_{k,l} \nu_j - \sum_{j < l} a_{h,l} \nu_j - \sum_{j < l} a_{h+1,l} \nu_l
\]
\[
+ \sum_{j < l} (a_{h,j} + \nu_j) a_{h+1,l} + \sum_{j < l} (a_{h+1,j} - \nu_j) a_{h+1,l} + \sum_{j < l} (a_{h+1,j} + \nu_j) a_{h+1,l} + \sum_{j < l} (a_{h+1,j} - \nu_j) a_{h+1,l}
\]

and
\[
\sum_{J' \setminus I'} (-1)^{\hat{h}} x_{i,j}x_{i,l} = \sum_{j < l} (-1)^{\hat{h}}(a_{h,j} + \nu_j)(a_{h+1,l} + \nu_l) + \sum_{j < l} (-1)^{\hat{h}+1}(a_{h+1,j} - \nu_j)(a_{h+1,l} - \nu_l)
\]
\[
= \sum_{J' \setminus I'} (-1)^{\hat{h}} a_{i,j}a_{i,l} + \sum_{j < l} (-1)^{\hat{h}} a_{h,j} \nu_l + \sum_{j < l} (-1)^{\hat{h}} a_{h,l} \nu_j - \sum_{j < l} (-1)^{\hat{h}+1} a_{h+1,j} \nu_l
\]
\[
- \sum_{j < l} (-1)^{\hat{h}+1} a_{h+1,j} \nu_j + \sum_{j < l} ((-1)^{\hat{h}} + (-1)^{\hat{h}+1}) \nu_j \nu_l.
\]
Thus, since $d(U_p) = (-1)^{h} p \sum_{j=1}^{m+n} a_{h,j} = (-1)^{h} \sum_{1 \leq j,k \leq m+n} a_{h,j} \nu_k$ by (4.0.1), it follows that
\[
d(X) = d(A) - \sum_{j<l} a_{h,j} \nu_j + \sum_{j<l} a_{h+1,j} \nu_l - \sum_{j<l} \nu_j \nu_l + \left( d(U_p) - (-1)^{h} \sum_{j} a_{h,j} \nu_j \right)
- \sum_{j<l} (-1)^{h+1} a_{h+1,j} \nu_j - \sum_{j<l} (-1)^{h+1} a_{h+1,l} \nu_j + \sum_{j<l} ((-1)^{h} + (-1)^{h+1}) \nu_j \nu_l.
\]

In other words,
\[
d(X) - d(A) - d(U_p)
= \sum_{j<l} a_{h,j} \nu_j + \sum_{j<l} a_{h+1,j} \nu_l - (-1)^{h} \sum_{j} a_{h,j} \nu_j - \sum_{j<l} (-1)^{h+1} a_{h+1,j} \nu_l +
- \sum_{j<l} (-1)^{h+1} a_{h+1,l} \nu_j + \sum_{j<l} ((-1)^{h} + (-1)^{h+1} - 1) \nu_j \nu_l
\]
\[
= \begin{cases}
- \sum_{j<l} a_{h,j} \nu_j - \sum_{j<l} a_{h+1,j} \nu_j + \sum_{j<l} \nu_j \nu_l, & \text{if } h < m; \\
- \sum_{j<l} a_{h,j} \nu_j + 2 \sum_{j<l} a_{h+1,j} \nu_l + \sum_{j} a_{h,j} \nu_j + \sum_{j<l} \nu_j \nu_l, & \text{if } h > m; \\
- \sum_{j>k} a_{m,j} + 2 \sum_{j<k} a_{m+1,j} + \sum_{j>k} a_{m+1,j} - a_{m,k}, & \text{if } h = m, \nu = e_k.
\end{cases}
\]

Thus, for $h < m$, the exponent of the LHS of (5.2.3) becomes
\[
d(X) - d(A) - d(U_p) + \sum_{j \geq l} 2a_{h,j} \nu_l + 2 \sum_{k=1}^{m+n} a_{h,j} \nu_j
= \sum_{j \geq l} a_{h,j} \nu_l + \sum_{k=1}^{m+n} a_{h,j} \nu_j - \sum_{j \leq k} a_{h+1,j} \nu_j + \sum_{j \leq k} \nu_j \nu_l
= \sum_{j \geq l} a_{h,j} \nu_l - \sum_{j \leq k} a_{h+1,j} \nu_j + \sum_{j \leq k} \nu_j \nu_l = f_h(\nu, A),
\]

while, for $h > m$, it has the form
\[
d(X) - d(A) - d(U_p) + 2 \sum_{j \leq l} \nu_j \nu_l - 2 \sum_{j \leq l} a_{h+1,j} \nu_l - 2 \sum_{k=1}^{m+n} a_{h,k} \nu_k
= - \sum_{j \leq l} a_{h,j} \nu_j + 2 \sum_{j \leq l} a_{h+1,j} \nu_l + \sum_{j \leq l} a_{h,j} \nu_j + \sum_{j \leq l} a_{h+1,j} \nu_j - 3 \sum_{j \leq l} \nu_j \nu_l
- 2 \sum_{j \leq l} a_{h+1,j} \nu_l - 2 \sum_{k=1}^{m+n} a_{h,k} \nu_k + 2 \sum_{j \leq l} \nu_j \nu_l
= - \sum_{j \leq l} a_{h,j} \nu_j + \sum_{j \leq l} a_{h+1,j} \nu_j - \sum_{j \leq l} \nu_j \nu_l = -f_h(\nu, A),
\]
as desired.

Let

\[ f_m(e_k, A) = \sum_{j \geq k} a_{m,j} + \sum_{j > k} a_{m+1,j}, \]

\[ f'_m(e_k, A) = \sum_{j \leq k} a_{m+1,j} + \sum_{j < k} a_{m,j}. \]  

We now look at the odd generator case.

**Proposition 5.3 (The \( h = m \) case).** Let \( A = (a_{i,j}) \in M(m|n, r) \). Assume \( h = m \). Define \( U_1 = U_1(h, \text{ro}(A)), L_1 = L_1(h, \text{ro}(A)) \in M(m|n, r) \) as in (4.0.1). The following multiplication formulas hold in the \( v \)-Schur superalgebra \( S_{v}(m|n, r) \) over \( \mathbb{Z} \):

1. \( \xi_U, \xi_A = \sum_{k \in [1, m+n]} \sum_{a_{m+1,k} \geq 1} (-1)^{\sum_{i > m-j < k} a_{i,j}} v^{f_m(e_k, A)} a_{m,k}^{m+1,k} \xi_A + v^{m-1,k} \xi_A - E_{m+1,k} \).

2. \( \xi_L, \xi_A = \sum_{k \in [1, m+n]} \sum_{a_{m+1,k} \geq 1} (-1)^{\sum_{i > m-j < k} a_{i,j}} v^{f'_m(e_k, A)} a_{m+1,k}^{m+1,k} \xi_A - E_{m+1,k} \).

**Proof.** We only prove (1); the proof of (2) is similar. By Lemma 3.1 and noting \( U_1 = B \), we have

\[ N_A, N'_U = \sum_{k \in [1, m+n]} \sum_{a_{m+1,k} \geq 1} (-1)^{\sum_{i > m-j < k} a_{i,j}} q^{-\sum_{j < k} a_{m+1,j}} q^{\sum_{j > k} a_{m,j}} [a_{m,k} + 1] N_{(A + E_{h,k} - E_{h+1,k})}. \]

Recalling \( q = v^2 \) and \( [a_{m,k} + 1] = v^{2a_{m,k}} [a_{m,k} + 1] \), assertion (1) is equivalent to

\[ v^{d(X) - d(A) - d(U_1)} v^{-2 \sum_{j < k} a_{m+1,j}} v^{2 \sum_{j > k} a_{m,j}} v^{2a_{m,k}} = v^{\sum_{j > k} a_{m,j} + \sum_{j < k} a_{m+1,j}}. \]

By (5.2.4),

\[ d(X) - d(A) - d(U_1) = -\sum_{j > k} a_{m,j} + 2 \sum_{j < k} a_{m+1,j} + \sum_{j < k} a_{m+1,j} - a_{m,k}. \]

Hence,

\[ d(X) - d(A) - d(U_1) - 2 \sum_{j < k} a_{m+1,j} + 2 \sum_{j > k} a_{m,j} + 2a_{m,k} \]

\[ = \sum_{j > k} a_{m,j} + \sum_{j < k} a_{m+1,j} + a_{m,k}, \]

as desired. \( \square \)

6. **Uniform spanning sets over \( \mathbb{Q}(v) \)**

We now use the multiplication formulas to show the structure of quantum group type for \( v \)-Schur superalgebras. First, we describe a spanning set for \( S_{v}(m|n, r) \) whose members are linear combination of \( \xi_A \)'s with the same off-diagonal entries. To achieve
Lemma 6.1. We will see in §8 that this number occurs naturally in a triangular relation.

By the definition, we have the following.

Lemma 6.1. Let $\mu \in \mathbb{N}^{m+n}$ and $B = (b_{i,j}) \in M(m|n)$.

- (1) If $A = \text{diag}(\mu)$ or $A = E_{h,h+1} + \mu$ or $A = E_{h+1,h} + \mu$, where $X + \mu := X + \text{diag}(\mu)$, then $\overline{A} = 0$;
- (2) If $h \neq m$, the $B + E_{h,k} - E_{h+1,k} = B$;
- (3) If $h = m$, then

$$
\overline{B + E_{m,k} - E_{m+1,k}} = \begin{cases} 
\overline{B}, & \text{if } k \leq m; \\
\overline{B} - \sum_{i \leq m,j > m+1} b_{i,j}, & \text{if } m+1 = k; \\
\overline{B + \sum_{m,m < j < k} b_{i,j} - \sum_{i \leq m,j > k} b_{i,j}}, & \text{if } m+1 < k.
\end{cases}
$$

Proof. Write a matrix $A \in M(m|n)$ in the form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{11}$ is $m \times m$.

Then $\overline{A}$ is a sum of $a_{i,j}a_{k,l}$ with $a_{i,j} \in A_{22}$ and $a_{k,l} \in A_{12}$ and $j < l$. So (1) is clear from the definition and (2) for $h < m$ and (3) for $k \leq m$ are clear. For the case $h > m$ of (2), $(a_{h+1,k} - 1)a_{i,j}$ and $(a_{h,k} - 1)a_{i,j}$ occur in the sum. The remaining cases can be seen similarly. \[\square\]

Consider now the set of matrices with zero diagonal:

$$
M(m|n)^\pm = \{ A = (a_{i,j}) \in M(m|n) \mid a_{i,i} = 0, 1 \leq i \leq m+n \}.
$$

For $A \in M(m|n)^\pm$ and $\mathbf{j} = (j_1, j_2, \ldots, j_{m+n}) \in \mathbb{Z}^{m+n}$, define

$$
A(\mathbf{j}, r) = \begin{cases} 
\sum_{\lambda \in \Lambda(m,n,r-|A|)} (-1)^{A+\lambda} \nu^{\lambda \cdot \mathbf{j}} \xi_{A+\lambda}, & \text{if } |A| \leq r; \\
0, & \text{otherwise},
\end{cases}
$$

(6.1.1)

where $\cdot = \cdot^\pm$ denotes the super (or signed) “dot product”:

$$
\lambda \cdot \mathbf{j} = \sum_{i=1}^{m+n} (-1)^i \lambda_i j_i = \lambda_1 j_1 + \cdots + \lambda_m j_m - \lambda_{m+1} j_{m+1} - \cdots - \lambda_{m+n} j_{m+n}.
$$

(6.1.2)

Also, for notational simplicity in the multiplication formulas below, we define $A(\mathbf{j}, r)$ to be 0 if $A \notin M(m|n, r)$. In particular, this is the case when $A$ has a negative entry or an entry $> 1$ in the $m \times n$ or $n \times m$ block. Note that, since $\xi_{A+\lambda}$ has the same $\mathbb{Z}_2$-grading degree $\widehat{A}$ (see (1.0.9)), $A(\mathbf{j}, r)$ is homogeneous of degree $\widehat{A}$.

Let

$$
\mathcal{S}(m|n, r) = \mathcal{S}_\nu(m|n, r) \otimes \mathbb{Q}(\nu).
$$
We want to prove that \( \mathcal{S}(m|n, r) \) is the span of \( \{ A(\mathbf{j}, r) \mid A \in M(m|n)^\pm, \mathbf{j} \in \mathbb{Z}^n \} \). The following result shows that this span is closed under the multiplication by \( O(\mathbf{j}, r) \).

**Proposition 6.2.** For \( \mathbf{j}, \mathbf{j}' \in \mathbb{Z}^{m+n} \) and \( A = (a_{k,l}) \in M(m|n)^\pm \), we have in \( \mathcal{S}(m|n, r) \)

1. \( O(\mathbf{j}, r)A(\mathbf{j}', r) = \mathbf{v}^{\text{ro}(A)}\mathbf{j} A(\mathbf{j} + \mathbf{j}', r) \);  
2. \( A(\mathbf{j}', r)O(\mathbf{j}, r) = \mathbf{v}^{\text{co}(A)}\mathbf{j} A(\mathbf{j} + \mathbf{j}', r) \).

**Proof.** If \( |A| > r \), then both sides are zero. Assume now \( |A| \leq r \). The proof of (1) follows from definition:

\[
\text{LHS} = \sum_{\lambda \in \Lambda(m|n, r)} \mathbf{v}^{\lambda} \xi_{\lambda} \sum_{\mu \in \Lambda(m|n, r - |A|)} (-1)^{\lambda + \mu} \mathbf{v}^{\mu} \xi_{\lambda + \mu} = \sum_{\mu \in \Lambda(m|n, r - |A|)} (-1)^{\mu} \mathbf{v}^{\mu} \xi_{\mu} \xi_{\lambda + \mu} \text{ (by Lem 1.2)} \]

The proof of (2) is similar. \( \square \)

Let \( \mathcal{S}(m|n, 0) = \mathbb{Q}(\mathbf{v}) \) and \( O(\mathbf{j}, 0) = 1 \) for all \( \mathbf{j} \in \mathbb{Z}^{m+n} \). By Proposition 6.2 similar to [4, 13.29], we can get the following result.

**Corollary 6.3.** For all \( r \geq 0 \), the set \( \mathcal{L}_r = \{ A(\mathbf{j}, r) \mid A \in M(m|n)^\pm, \mathbf{j} \in \mathbb{Z}^{m+n} \} \) spans the \( \mathbf{v} \)-Schur superalgebra \( \mathcal{S}(m|n, r) \) over \( \mathbb{Q}(\mathbf{v}) \).

**Proof.** Let \( \mathcal{T} \) be the \( \mathbb{Q}(\mathbf{v}) \)-span of \( \mathcal{L}_r \). Then, by Proposition 6.2, \( \mathcal{T} \) is closed under multiplication by \( O(\mathbf{j}, r) \) for all \( \mathbf{j} \in \mathbb{Z}^{m+n} \). The first thing we need to prove is that \( \xi_{\lambda} = \xi_{\text{diag}(\lambda)} \in \mathcal{T} \) for all \( \lambda \in \Lambda(m|n, r) \). This is because

\[
\xi_{\lambda} = \prod_{i=1}^{m+n} \left[ O(e_i, r); \lambda_i \right] \in \mathcal{T},
\]

where

\[
\left[ O(e_i, r); a \right] := \frac{\lambda_i}{\lambda_{ij}} \frac{O(e_i, r)v_i^{a-j+1} - O(-e_i, r)v_i^{a+j-1}}{v_i^j - v_i^{-j}} \in \mathcal{T}.
\]

Next, for any \( A \in M(m|n, r) \), let \( A^\pm \) be the matrix obtained by replacing all the diagonal entries of \( A \) by 0. Then, putting \( \lambda = \text{co}(A) \) and \( \mu = \text{ro}(A) \), we have \( \xi_A = \xi_{\mu} A^\pm(0, r) = A^\pm(0, r) \xi_{\lambda} \in \mathcal{T} \), by again Proposition 6.2. Hence, \( \mathcal{T} = \mathcal{S}(m|n, r) \). For a more detailed argument, see the proof of [4, Lem. 3.29]. \( \square \)

By these uniform spanning sets \( \mathcal{L}_r \), we define

\[
\mathcal{S}(m|n) := \prod_{r \geq 0} \mathcal{S}(m|n, r) \quad \text{and} \quad \mathfrak{A}(m|n) := \text{span}\{ A(\mathbf{j}) \mid A \in M(m|n)^\pm, \mathbf{j} \in \mathbb{Z}^{m+n} \},
\]

where

\[
A(\mathbf{j}) := \sum_{r \geq 0} A(\mathbf{j}, r) \in \mathcal{S}(m|n).
\]

(6.3.1)
Note that, if we assign to $A(j)$ the grading degree $\hat{A}$, then $\mathfrak{A}(m|n)$ is a superspace.

**Lemma 6.4.** The set $\{A(j) \mid A \in M(m|n)^\pm, j \in \mathbb{Z}^{m+n}\}$ is linearly independent and forms a basis of homogeneous elements for the superspace $\mathfrak{A}(m|n)$.

**Proof.** The proof is almost identical to the proof of [8, Prop. 4.1(2)] if we replace the vector $1 = (1, 1, \ldots, 1)$ used in the last paragraph by $1^\pm = (1, 1, \ldots, -1, \ldots, -1)$. □

We will eventually prove that the subspace $\mathfrak{A}(m|n)$ is a subalgebra isomorphic to the quantum supergroup $U(\mathfrak{gl}_{m|n})$. We now show that it is closed under the multiplication by “generators”.

First, the formulas given in Proposition 6.2 gives, by taking sum over all $r \geq 0$ (i.e. removing $r$ throughout), the corresponding formulas in $\mathcal{S}(m|n)$.

**Proposition 6.5.** For $j, j' \in \mathbb{Z}^{m+n}$ and $A = (a_{k,l}) \in M(m|n)^\pm$, we have in $\mathcal{S}(m|n)$

1. \(O(j)A(j') = \nu_{O(A)j} A(j + j')\);
2. \(A(j')O(j) = \nu_{O(A)j} A(j + j')\).

In particular, $\mathfrak{A}(m|n)$ is closed under the multiplication by $O(j)$.

Note that the sign \((-1)^{\hat{A}+\hat{A}}\) in (6.1.1) disappears in the following cases:

\[
E_{h,h+1}(j, r) = \sum_{\lambda \in \Lambda(m|n,r-1)} \nu^{\lambda} j \xi_{E_{h,h+1},\lambda} \quad \text{and} \quad E_{h+1,h}(j, r) = \sum_{\lambda \in \Lambda(m|n,r-1)} \nu^{\lambda} j \xi_{E_{h+1,h},\lambda}.
\]

For given $1 \leq h, k < m + n$ and $A \in M(m|n)$, let

\[
\alpha_h = e_h - e_{h+1},
\]

\[
\beta_h = -e_h - e_{h+1},
\]

\[
f(k) = f_h(e_k, A) = \sum_{j \geq k} a_{h,j} - (-1)^{\varepsilon} \sum_{j > k} a_{h+1,j}, \quad \text{and} \quad (6.5.1)
\]

\[
f'(k) = f'_h(e_k, A) = \sum_{j \leq k} a_{h+1,j} - (-1)^{\varepsilon} \sum_{j < k} a_{h,j},
\]

\[
\varepsilon = \varepsilon_{h,h+1} = \hat{h} + \hat{h} + 1 \in \{0, 1\}
\]

(see (5.2.1), (5.2.2), and (5.2.3)). For $A \in M(m|n)$ and $k \in [1, m + n]$, define

\[
\sigma(k) = \sigma(k, A) = \begin{cases} 
\sum_{i>m,j<k} a_{i,j} & \text{if } k \leq m; \\
\sum_{i\leq m,j>k} a_{i,j} + \sum_{i>m,j\leq m} a_{i,j} & \text{if } k \geq m + 1.
\end{cases} (6.5.2)
\]

So, for $k \leq m$, $\sigma(k)$ is the entry sum of the submatrix with the upper right corner $(m + 1, k - 1)$, while, for $k \geq m + 1$, $\sigma(k)$ is the entry sum of the left bottom $n \times m$ submatrix and the submatrix with the lower left corner $(m, k + 1)$.

Propositions 5.2 for $p = 1$ and 5.3 can now be combined and extended as follows.
Proposition 6.6. Maintain the notation introduced above. For given \( h \in [1, m + n) \) and \( A = (a_{k,l}) \in M(m|n) \), the following multiplication formulas of homogeneous elements hold in \( \mathcal{S}(m|n,r) \) for all \( r \geq 0 \):

\[
E_{h,h+1}(0, r)A(j, r) 
= \sum_{k<h,a_{k,h+1,k} \geq 1} (-1)^{\varepsilon(k)} v^{(k)}_h \left[ a_{h,k} + 1 \right] v^2_h (A + E_{h,k} - E_{h+1,k})(j + \alpha_h, r) 
+ \sum_{k=h+1,a_{k,h+1,k} \geq 1} (-1)^{\varepsilon(k)} v^{(k)}_h \left[ a_{h,k} + 1 \right] v^2_h (A + E_{h,k} - E_{h+1,k})(j, r) 
+ (-1)^{\varepsilon(h)} v^{(h+1)}_h (A - E_{h+1,h})(j - \alpha_h, r) 
+ (-1)^{\varepsilon(h)} v^{(h+1)}_h (A - E_{h+1,h})(j + \beta_h, r),
\]

(1)

\[
E_{h+1,h}(0, r)A(j, r) 
= \sum_{k<h,a_{k,h,k} \geq 1} (-1)^{\varepsilon(k)} v^{(k)}_{h+1} \left[ a_{h+1,k} + 1 \right] v^2_{h+1} (A - E_{h,k} + E_{h+1,k})(j, r) 
+ \sum_{i=h+1,a_{i,h,k} \geq 1} (-1)^{\varepsilon(k)} v^{(k)}_{h+1} \left[ a_{h+1,k} + 1 \right] v^2_{h+1} (A - E_{h,k} + E_{h+1,k})(j - \alpha_h, r) 
+ (-1)^{\varepsilon(h)} v^{(h+1)}_{h+1} (A + E_{h+1,h})(j, r) 
+ (-1)^{\varepsilon(h)} v^{(h+1)}_{h+1} (A + E_{h+1,h})(j + \beta_h, r),
\]

(2)

Moreover, summing over all \( r \geq 0 \) (i.e. removing \( r \) throughout) gives the corresponding formulas in \( \mathcal{S}(m|n) \). In particular, \( \mathcal{A}(m|n) \) is closed under multiplication by \( E_{h,h+1}(0) \) and \( E_{h+1,h}(0) \).

Proof. If \( |A| > r \), then there is nothing to prove.\footnote{If \( |A| = r + 1 \), then \( A - E_{h+1,h} = r \) and so \( (A - E_{h+1,h})(j) = \xi_{A-E_{h+1,h}}. \) Thus, the last summand in the RHS of (1) is also zero.}

We now only prove (1); the proof of (2) is symmetric. First, we prove the \( h \neq m \) case. In this case, the sign \((-1)^{\varepsilon(h)}\) and \((-1)^{\varepsilon}\) become 1.

\[
E_{h,h+1}(0, r)A(j, r) = \sum_{\lambda \in \Lambda(m|n,r-1)} \sum_{\mu \in \Lambda(m|n,r-|A|)} (-1)^{\lambda+\mu} v^{(\lambda)}_{h} \xi_{E_{h,h+1} + \lambda} \xi_{A+\mu}
\]

(1)

\[
E_{h+1,h}(0, r)A(j, r) = \sum_{\mu \in \Lambda(m|n,r-|A|)} \sum_{\lambda \in \Lambda(m|n,r-1)} (-1)^{\lambda+\mu} v^{(\lambda)}_{h} \xi_{E_{h,h+1} + \lambda} \xi_{A+\mu} \text{ (by Lem. 1.21)}
\]

(2)

\[
E_{h,h+1}(0, r)A(j, r) = \sum_{\mu \in \Lambda(m|n,r-|A|)} \sum_{k \in [1,m+n]} \sum_{a_{h,k+1,k} \geq 1} (-1)^{\lambda+\mu} v^{(\lambda)}_{h} \xi_{E_{h,h+1} + \lambda} \xi_{A+\mu}
\]

(by Proposition 5.2, where \( A + \mu = (a_{i,j}) \)).
By Lemma 6.1 for $h \neq m$, $A + E_{h,k} - E_{h+1,k} + \mu = A + \mu$. Note also that $\nu_{h}^{\mu_{h} - \mu_{h+1}} = \nu_{h}^{\mu_{h}}$, $\nu_{h}^{-\mu_{h} - \mu_{h+1}} = \nu_{h}^{\mu_{h}}$ and $\alpha_{j,k}^{\mu_{h}} = a_{j,k}$ whenever $j \neq k$. Moreover, by definition, $f_{h}(e_{k}, A + \mu) = \sum_{j \geq k} a_{h,j}^{\mu} - \sum_{j > k} \alpha_{j,h}^{\mu}$ for $h \neq m$. Thus,

$$f_{h}(e_{k}, A + \mu) = \begin{cases} f_{h}(e_{k}, A), & \text{if } k > h; \\ f_{h}(e_{k}, A) + \mu_{h} - \mu_{h+1}, & \text{if } k \leq h. \end{cases} \quad (6.6.1)$$

Hence,

$$\text{LHS} = \sum_{k > h+1, a_{h+1,k} \geq 1} \nu_{h}^{f(k)}[a_{h,k} + 1] \nu_{h}^{2}(A + E_{h,k} - E_{h+1,k})(j, r) + \sum_{k < h, a_{h+1,k} \geq 1} \nu_{h}^{f(k)}[a_{h,k} + 1] \nu_{h}^{2}(A + E_{h,k} - E_{h+1,k})(j + \alpha_{h}, r) + Y_{h} + Y_{h+1},$$

where, noting $a_{i,i} = 0$,

$$Y_{h+1} = \sum_{\mu \in \Lambda(m, n, r - |A|)} (-1)^{A+\mu} \nu_{h}^{\mu \cdot j} \xi_{A+E_{h,h+1}-E_{h+1,h+1}+\mu} \nu_{h}^{2} \sum_{\mu' \in \Lambda(m, n, r - |A| - 1)} (-1)^{A+\mu} \nu_{h}^{\mu' \cdot j} \xi_{A+E_{h,h+1}+\mu'}$$

and, if $a_{h+1,h} \geq 1$,

$$Y_{h} = \sum_{\mu \in \Lambda(m, n, r - |A|)} (-1)^{A+\mu} \nu_{h}^{\mu \cdot j} \xi_{A+E_{h,h}+\mu} \nu_{h}^{2} \xi_{A+E_{h,h}+\mu}$$

$$= \sum_{\mu \in \Lambda(m, n, r - |A|)} (-1)^{A+\mu} \nu_{h}^{\mu \cdot j} \xi_{A+E_{h,h}+\mu} \nu_{h}^{2} \xi_{A+E_{h,h}+\mu}$$

$$= \nu_{h}^{f(h)} \sum_{\mu \in \Lambda(m, n, r - |A|)} (-1)^{A+\mu} \nu_{h}^{\mu \cdot j} \frac{\nu_{h}^{\mu_{h} - \mu_{h+1} - \mu_{h+1} - 2}}{1 - \nu_{h}^{-2}} \xi_{A-E_{h+1,h}+\mu+\epsilon_{h}}$$

$$= \nu_{h}^{f(h)-j_{h}-1} \sum_{\mu \in \Lambda(m, n, r - |A|)} (-1)^{A+\mu} \frac{\nu_{h}^{(j+\alpha_{h})+\mu_{h}+\epsilon_{h}} - \nu_{h}^{(j+\beta_{h})+\mu_{h}+\epsilon_{h}}}{1 - \nu_{h}^{-2}} \xi_{A-E_{h+1,h}+\mu+\epsilon_{h}}$$

$$= \nu_{h}^{f(h)-j_{h}-1} \frac{A - E_{h+1,h}(j + \alpha_{h}, r) - (A - E_{h+1,h})(j + \beta_{h}, r)}{1 - \nu_{h}^{-2}},$$

proving (1) for $h \neq m$. 
We now prove the $h = m$ case. By definition,

$$E_{m,m+1}(0, r) A(j, r) = \sum_{\lambda \in \Lambda(m,n,r)} \xi_{E_{m,m+1} + \lambda} \sum_{\mu \in \Lambda(m,n,r-|A|)} (-1)^{\lambda+\mu} a^{\lambda+j} \xi_{A+\mu}$$

$$= \sum_{\mu \in \Lambda(m,n,r-|A|)} (-1)^{\lambda+\mu} a^{\lambda+j} \xi_{E_{m,m+1} + \mu - \text{co}(A) + \text{co}(E_{m,m+1})} \xi_{A+\mu}$$

$$= \sum_{\mu \in \Lambda(m,n,r-|A|)} \sum_{k \in [1,m+n], a_{m+1,k} \geq 1} (-1)^{\lambda+\mu} a^{\lambda+j} (-1)^{m_j} \sum_{i > m, j < k} a_{i,j}^{\mu} f_m(e_k, A+\mu$$

$$\sum_{i > m, j < k} a_{i,j}^{\mu} f_m(e_k, A+\mu)$$

Now, by (5.2.5), (6.6.1) becomes, for $h$

$$= \sum_{\mu \in \Lambda(m,n,r-|A|)} \sum_{k \in [1,m+n], a_{m+1,k} \geq 1} (-1)^{\lambda+\mu} a^{\lambda+j} (-1)^{m_j} \sum_{i > m, j < k} a_{i,j}^{\mu} f_m(e_k, A+\mu)$$

$$\sum_{i > m, j < k} a_{i,j}^{\mu} f_m(e_k, A+\mu)$$

Thus, in this case, the term

$$\sum_{i > m, j < k} a_{i,j}^{\mu} f_m(e_k, A+\mu)$$

Substituting gives the first sum in the formula.

If $k < m$ and $a_{m+1,k} = a_{m+1,k}^{\mu} \geq 1$, then $A + \mu = A + E_{m,k} - E_{m+1,k} + \mu$ by Lemma 6.1 and, by (6.1.2), $\mu \cdot \alpha_m = \mu_m + \mu_{m+1}$. Thus, in this case, the term

$$\sum_{i > m, j < k} a_{i,j}^{\mu} f_m(e_k, A+\mu)$$

Substituting gives the first sum in the formula.

If $k > m$ and $a_{m+1,m} \geq 1$, then we have

$$\sum_{i > m, j < k} a_{i,j}^{\mu} f_m(e_k, A+\mu)$$

Applying Lemma 6.1 and substituting give the last term in the formula.

If $k > m + 1$ and $a_{m+1,k} \geq 1$ then, by Lemma 6.1 $A + E_{m,k} - E_{m+1,k} + \mu = A + \mu + \sum_{i > m, m < k} a_{i,j}^{\mu} + \sum_{i \leq m, j > k} a_{i,j}^{\mu}$, and

$$\sum_{i > m, j < k} a_{i,j}^{\mu} f_m(e_k, A+\mu)$$

So substituting gives the second sum in the formula.
Finally, if \( k = m + 1 \) and \( \mu_{m+1} = a_{m+1,m+1}^\mu \geq 1 \), by Lemma 6.11 \( A + \mu = A + E_{m,m+1} + \mu - e_{m+1} + \sum_{i \leq m,j > m+1} a_{i,j} \) and the corresponding term becomes
\[
\nu^{\mu,j}(-1)^{\mu+k+1}(-1)^{\sum_{i > m,j < m+1} a_{m+1,i,j}^\mu} v^{f_{m+1,A}[a_{m+1,m+1}^\mu + 1]} \nu^2 \xi_{A+E_{m,m+1}-E_{m+1,m+1}+\mu} \\
= (-1)^{\sigma(m+1)} v^{f(m+1) - j_{m+1} + 1} [a_{m,m+1} + 1] \nu^2 (-1)^{A+E_{m,m+1}+\mu-e_{m+1}} v^{(\mu-e_{m+1})} \xi_{\epsilon + E_{m,m+1}+\mu-e_{m+1}+1}.
\]
Now substituting gives the second last term.

7. Generators and Relations for the Quantum Supergroup \( U(\mathfrak{gl}_{m|n}) \)

We now use the formulas in \( S(m|n) \) given in Propositions 6.5 and 6.6 to derive the defining relations for the quantum supergroup \( U(\mathfrak{gl}_{m|n}) \) inside \( S(m|n) \). First, we recall the definition from [23]. Recall also the definition of the super commutator on homogeneous elements of a superngebra with parity function \( \epsilon \):
\[
[X,Y] = XY - (-1)^{\hat{X}\hat{Y}} YX.
\]

Definition 7.1. The quantum enveloping superalgebra \( U(\mathfrak{gl}_{m|n}) \) is the algebra over \( \mathbb{Q}(\nu) \) generated by
\[
K_a, K_a^{-1}, E_{h,h+1}, E_{h+1,h}, \quad (1 \leq a \leq m+n, 1 \leq h < m+n)
\]
which satisfy the following relations:

(QS1) \( K_a K_a^{-1} = 1, K_a K_b = K_b K_a \); 

(QS2) \( K_a E_{h,h+1} = \nu^{\delta_{a,h}} - \delta_{a,h+1} E_{h,h+1} K_a, K_a E_{h+1,h} = \nu^{\delta_{a,h}} - \delta_{a,h+1} E_{h+1,h} K_a \); 

(QS3) \( [E_{h,h+1}, E_{k+1,k}] = \delta_{h,k} \frac{K_a K_a^{-1} - K_a^{-1} K_a}{\nu_a - \nu^{-1}_a} \); 

(QS4) \( E_{h+1,k} E_{k+1,h} = E_{k+1,k} E_{k+1,h}, E_{h+1,h} E_{k+1,k} = E_{k+1,k} E_{h+1,h} \), where \( |k-h| > 1 \); 

(QS5) For \( h \neq m \),
\[
E_{h,h+1}^2 E_{h+1,h+2} - (\nu_h + \nu^{-1}_h) E_{h,h+1} E_{h+1,h+2} E_{h+1,h} + E_{h+1,h+2} E_{h,h+1} = 0, \\
E_{h+1,h}^2 E_{h,h+1} - (\nu_h + \nu^{-1}_h) E_{h+1,h} E_{h,h+1} E_{h,h+1} + E_{h,h+1} E_{h+1,h} = 0, \\
E_{h+1,h}^2 E_{h+2,h+1} - (\nu_h + \nu^{-1}_h) E_{h+1,h} E_{h+2,h+1} + E_{h+2,h+1} E_{h+1,h} + E_{h+2,h+1} E_{h+1,h} = 0, \\
E_{h+1,h}^2 E_{h,h-1} - (\nu_h + \nu^{-1}_h) E_{h+1,h} E_{h-1,h} + E_{h-1,h} E_{h+1,h} = 0;
\]

(QS6) \( E_{m,m+1}^2 = E_{m+1,m}^2 = 0 \) and \( [E_{m,m+1}, E_{m-1,m+2}] = [E_{m+1,m}, E_{m+2,m-1}] = 0 \).

Here, only \( E_{m,m+1}, E_{m+1,m} \) are odd generators and others are even generators. Moreover, the quantum root vectors \( E_{a,b} \), for \( a, b \in [1, m+n] \) with \( |a-b| > 1 \), is defined recursively as follows
\[
E_{a,b} = \begin{cases} 
E_{a,c} E_{c,b} - \nu c E_{c,b} E_{a,c}, & \text{if } a > b; \\
E_{a,c} E_{c,b} - \nu^{-1} c E_{c,b} E_{a,c}, & \text{if } a < b,
\end{cases} \quad (7.1.1)
\]
where \( c \) can be taken to be an arbitrary index strictly between \( a \) and \( b \), and \( E_{a,b} \) is homogeneous of degree \( \hat{a} + \hat{b} \).
By the quantum Schur-Weyl duality ([17, Thm. 4.4, [21 §8, [21 §4]]), it is known that there are algebra homomorphisms from $U(gl_{m|n})$ to $S(m|n, r)$ for all $r \geq 0$. These homomorphisms induce an algebra homomorphism from $U(gl_{m|n})$ to the direct product $S(m|n)$. We now use the multiplication formulas given in Propositions 6.5 and 6.6 to explicitly define a $\mathbb{Q}(\mathfrak{v})$-algebra homomorphism $\eta : U(gl_{m|n}) \to S(m|n)$.

Let $M(m|n)^+$ (resp. $M(m|n)^-$) be the subset of $M(m|n)$ consisting of those matrices $(a_{i,j})$ with $a_{i,j} = 0$, for all $i \geq j$ (resp. $i \leq j$). Recall also the symmetric Gaussian numbers $[k]_{\mathfrak{v}_h} = \frac{\mathfrak{v}_h^k - \mathfrak{v}_h^{-k}}{\mathfrak{v}_h - \mathfrak{v}_h^{-1}}$.

For the convenience of computation below, we list some special cases of Proposition 6.6.

**Lemma 7.2.** For $1 \leq h < m + n$, let $\varepsilon = \hat{h} + \hat{h} + 1$ and, for $A^+ = (a_{i,j}) \in M(m|n)^+$ and $A^- = (a_{i,j}) \in M(m|n)^-$, let (see (6.5.2))

\[ \sigma^+(k) := \sigma(k, A^+) = \sum_{i \leq m, j > k} a_{i,j} \quad \text{and} \quad \sigma^-(k) := \sigma(k, A^-) = \sum_{i \geq m+1, j < k} a_{i,j}. \]

Then the following multiplication formulas hold in $S(m|n)$:

1. $E_{h,h+1}(0)A^+(0) = (-1)^{\varepsilon(h+1)}\mathfrak{v}_h^{f(h+1)}\prod_{k=h,h+1}^+ [a_{h,k} + 1]_{\mathfrak{v}_h^k} (A^+ + E_{h,h+1})(0)
   \quad + \sum_{k > h+1, a_{h+1,k} \geq 1} (-1)^{\varepsilon(h+1)}\mathfrak{v}_h^{f(k)}\prod_{k=h+1}^+ [a_{h,k} + 1]_{\mathfrak{v}_h^k} (A^+ + E_{h+1,k})(0)$;
2. $E_{h+1,h}(0)A^-(0) = (-1)^{\varepsilon(h)}\mathfrak{v}_h^{f(h)}\prod_{k=h+1}^+ [a_{h,k} + 1]_{\mathfrak{v}_h^k} (A^- + E_{h+1,h})(0)
   \quad - \sum_{k < h, a_{h,k} \geq 1} (-1)^{\varepsilon(h)}\mathfrak{v}_h^{f(k)}\prod_{k=h+1}^+ [a_{h,k} + 1]_{\mathfrak{v}_h^k} (A^- + E_{h,k} + E_{h+1,k})(0)$.

In particular, for $k \geq 0$, we have

3. $E_{h,h+1}(0)^k = [k]_{\mathfrak{v}_h}^l (kE_{h,h+1})(0)$ and $E_{h+1,h}(0)^k = [k]_{\mathfrak{v}_h}^l (kE_{h+1,h})(0)$, if $h \neq m$;
4. $E_{m,m+1}(0)^k = 0$ and $E_{m+1,m}(0)^k = 0$, if $k > 1$.

**Proof.** Formulas (1) and (2) follows from Proposition 6.6 and (4) was seen from Theorem 4.2. We prove (3) by induction on $k$. By (1),

\[ E_{h,h+1}(0)E_{h,h+1}(0) = \mathfrak{v}_h [2]_{\mathfrak{v}_h^2} (2E_{h,h+1})(0) = [2]_{\mathfrak{v}_h} (2E_{h,h+1})(0). \]

By induction and (1),

\[ (E_{h,h+1}(0))^{k+1} = [k]_{\mathfrak{v}_h}^l E_{h,h+1}(0)(kE_{h,h+1})(0)
   = [k]_{\mathfrak{v}_h}^l \mathfrak{v}_h^{k+1} \prod_{k=h}^k [(k+1)E_{h+1,h})(0) \]

The second formula in (3) can be proved similarly by applying (2).

Let

\[ K_i = O(e_i), \quad K_i^{-1} = O(-e_i), \quad E_h = E_{h,h+1}(0), \quad \text{and} \quad F_h = E_{h+1,h}(0). \]
**Theorem 7.3.** There is a $\mathbb{Q}(\mathfrak{v})$-algebra homomorphism
\[ \eta : \mathfrak{u}(\mathfrak{gl}_{m|n}) \longrightarrow \mathcal{S}(m|n) \]
sending $E_{h,h+1}, E_{h+1,h}$ and $K^{\pm 1}_a$ to $E_h, F_h$ and $K^{\pm 1}_a$, respectively.

**Proof.** It is enough to show that $\eta$ preserves all relations (QS1)–(QS6).

(QS1) By Proposition 6.3, we have
\[ K_a K_a^{-1} = O(e_a) O(-e_a) = O(0) = 1, \]
and
\[ K_a K_b = O(e_a) O(e_b) = O(e_{a+b}) = O(e_b) O(e_a) = K_b K_a. \]

(QS2) For $a \in [1, m+n]$, by Proposition 6.3, we have
\[ K_a E_h = O(e_a) E_{h,h+1}(0) = \mathfrak{v}^\text{ro}(E_{h,h+1}) e_a E_{h,h+1}(e_a) \]
\[ E_h K_a = E_{h,h+1}(0) O(e_a) = \mathfrak{v}^\text{co}(E_{h,h+1}) e_a E_{h,h+1}(e_a). \]
Since $\text{co}(E_{h+1,h}) = e_{h+1}$ and $\text{ro}(E_{h,h+1}) = e_h$, by (6.1.2),
\[ K_a E_h = \mathfrak{v}(e_h - e_{h+1}) e_a E_h K_a = \mathfrak{v}_a h^{-1} e_a h \hat{K}_a. \]
Similarly, one proves the second formula.

(QS3) Let $\hat{O} = 0, \hat{E}_{i,j} = \hat{i} + \hat{j}$. By definition, for $1 \leq h, k \leq m+n-1$,
\[ [E_h, F_k] = E_h F_k - (-1)^{\hat{E}_{h,h+1} F_{k} F_{h}}. \]
(i) If $h = k \neq m$, by Proposition 6.6, then
\[ E_h F_h = (E_{h,h+1} + E_{h+1,h})(0) + \frac{O(\alpha_h) - O(\beta_h)}{\mathfrak{v}_h - \mathfrak{v}_h^{-1}} \]
and
\[ F_h E_h = (E_{h,h+1} + E_{h+1,h})(0) + \frac{O(-\alpha_h) - O(\beta_h)}{\mathfrak{v}_h - \mathfrak{v}_h^{-1}}. \]
Since $\hat{E}_{h,h+1} = 0$,
\[ [E_h, F_h] = \frac{O(\alpha_h) - O(-\alpha_h)}{\mathfrak{v}_h - \mathfrak{v}_h^{-1}} = \frac{K_h K_h^{-1} - K_h^{-1} K_h}{\mathfrak{v}_h - \mathfrak{v}_h^{-1}}. \]
(ii) If $h = k = m$, then $\hat{E}_{m,m+1} = \hat{E}_{m+1,m} = 1$. By Proposition 6.6,
\[ E_m F_m = -(E_{m,m+1} + E_{m+1,m})(0) + \frac{O(\alpha_m) - O(\beta_m)}{\mathfrak{v} - \mathfrak{v}^{-1}} \]
and
\[ F_m E_m = (E_{m,m+1} + E_{m+1,m})(0) - \frac{O(-\alpha_m) - O(\beta_m)}{\mathfrak{v} - \mathfrak{v}^{-1}}. \]
Hence,
\[ [E_m, F_m] = E_m F_m + F_m E_m = \frac{O(\alpha_m) - O(-\alpha_m)}{\mathfrak{v} - \mathfrak{v}^{-1}} = \frac{K_m K_{m+1}^{-1} - K_{m+1}^{-1} K_m}{\mathfrak{v} - \mathfrak{v}^{-1}}. \]
(iii) If \( h \neq k \), then \( \hat{E}_{h,h+1}\hat{E}_{k+1,k} = 0 \). By Proposition 6.6, \[ E_h F_k = (E_{h,h+1} + E_{k+1,k})(0) = F_k E_h. \]

Thus
\[ [E_h, F_k] = E_h F_k - F_k E_h = 0. \]

(QS4) Similarly, for \(|h - k| > 1\), we have \( \hat{E}_{h+1,h}\hat{E}_{k+1,k} = 0 \) and \[ F_h F_k = (E_{h+1,h} + E_{k+1,k})(0) = F_k E_h \]
and
\[ E_h E_k = (E_{h+1,h} + E_{k,k+1})(0) = E_k E_h. \]

(QS5) These are the quantum Serre relations when \( h \neq m \). By applying Lemma 7.2, the argument in the proof of [11, Lem. 5.6] carries over; see also the proof of [4, Th. 13.33]. Note that, when \( h = m - 1 \), there is no sign affecting the computation of \( E_{m-1} E_m E_{m-1} \) or \( E_m^2 E_{m-1} \).

(QS6) The relation \( 0 = F_m^2 \) is given in Lemma 7.2(4). Finally, we prove
\[ [\eta(E_{m+2,m-1}), F_m] = \eta(E_{m+2,m-1})F_m + F_m \eta(E_{m+2,m-1}) = 0, \]
where, by Lemma 7.1.1,
\[ \eta(E_{m+2,m-1}) = F_m + F_m E_{m-1} - \nu^{-1} F_m F_{m+1} F_{m-1} - \nu F_{m-1} F_{m+1} F_m + F_{m-1} F_m F_{m+1}. \]

Applying Lemma 7.2(2) repeatedly yields
\[
(1) \quad F_{m+1} F_m F_{m-1} F_m = F_{m+1} F_m = F_{m+1} F_m (E_{m,m-1} + E_{m+1,m})(0) \\
= F_{m+1} (E_{m+1,m-1} + E_{m+1,m})(0) \\
= (E_{m+2,m-1} + E_{m+1,m})(0) + \nu (E_{m+1,m-1} + E_{m+2,m})(0) \\
+ \nu^2 (E_{m+1,m-1} + E_{m+1,m} + E_{m+2,m+1})(0),
\]
\[
(2) \quad F_m F_{m+1} F_{m-1} F_m \\
= F_m F_{m+1} (E_{m,m-1} + E_{m+1,m})(0) \\
= F_m ((E_{m,m-1} + E_{m+2,m})(0) + \nu ((E_{m,m-1} + E_{m+1,m} + E_{m+2,m+1})(0))) \\
= (E_{m+1,m-1} + E_{m+2,m})(0) + \nu^{-1} (E_{m,m-1} + E_{m+1,m} + E_{m+2,m})(0) \\
+ \nu (E_{m+1,m-1} + E_{m+1,m} + E_{m+2,m+1})(0),
\]
\[
(3) \quad F_{m-1} F_{m+1} F_m F_m = 0, \text{ and}
\]
\[
(4) \quad F_{m-1} F_{m+1} F_m F_m \\
= F_{m-1} F_m ((E_{m+2,m})(0) + \nu (E_{m+1,m} + E_{m+2,m+1})(0)) \\
= F_{m-1} (E_{m+2,m} + E_{m+1,m})(0) = (E_{m,m-1} + E_{m+1,m} + E_{m+2,m})(0).
\]

Thus,
\[
\eta(E_{m+2,m-1}) F_m = (1) - \nu^{-1}(2) - \nu(3) + (4) \\
= (E_{m+2,m-1} + E_{m+1,m})(0) + (\nu - \nu^{-1})(E_{m,m-1} + E_{m+2,m})(0) \\
+ (\nu^2 - 1)(E_{m+1,m-1} + E_{m+1,m} + E_{m+2,m+1})(0) \\
+ (1 - \nu^{-2})(E_{m,m-1} + E_{m+1,m} + E_{m+2,m})(0).
\]
We now compute the summands of $F_m\eta(E_{m+2,m-1})$. By applying Lemma 7.2 repeatedly, we have

$$\begin{align*}
(1)' & F_mF_{m+1}F_mF_{m-1} = F_mF_{m+1}((E_{m+1,m-1})(0) + \nu^{-1}(E_{m+1,m} + E_{m,m-1})(0)) \\
& = F_m((E_{m+2,m-1})(0) + \nu(E_{m+1,m-1} + E_{m,m-1})(0)) + \\
& \quad \nu^{-1}(E_{m+2,m} + E_{m,m-1})(0) + (E_{m+1,m} + E_{m+2,m+1} + E_{m,m-1})(0)) \\
& = -(E_{m+2,m-1} + E_{m+1,m})(0) + \nu^{-1}(E_{m+1,m-1} + E_{m+2,m})(0) + \\
& \quad \nu^{-2}(E_{m+1,m-1} + E_{m+1,m} + E_{m+2,m})(0),
\end{align*}$$

and

$$\begin{align*}
(2)' & \nu^{-1}F_mF_{m+1}F_mF_{m-1} = 0, \\
(3)' & F_mF_{m+1}F_mF_{m-1} = F_mF_{m-1}((E_{m+2,m})(0) + \nu(E_{m+1,m} + E_{m+2,m+1})(0)) \\
& = (E_{m+1,m-1} + E_{m+2,m})(0) + \nu^{-1}(E_{m+2,m} + E_{m+1,m} + E_{m,m-1})(0) + \\
& \quad \nu(E_{m+1,m-1} + E_{m+1,m} + E_{m+2,m+1})(0),
\end{align*}$$

and

$$\begin{align*}
(4)' & F_mF_{m-1}F_mF_{m+1} = F_mF_{m-1}((E_{m+2,m+1} + E_{m+1,m})(0)) \\
& = F_m((E_{m+2,m+1} + E_{m+1,m} + E_{m,m-1})(0)) \\
& = (E_{m+1,m-1} + E_{m+1,m} + E_{m+2,m+1})(0).
\end{align*}$$

Thus,

$$\begin{align*}
F_m\eta(E_{m+2,m-1}) &= (1)' - \nu^{-1}(2)' - \nu(3)' + (4)' \\
& = -(E_{m+2,m-1} + E_{m+1,m})(0) + \nu^{-1}(E_{m+1,m-1} + E_{m+2,m})(0) \\
& \quad + (1 - \nu^2)(E_{m+1,m-1} + E_{m+1,m} + E_{m+2,m+1})(0) \\
& \quad + (\nu^{-2} - 1)(E_{m+1,m-1} + E_{m+1,m} + E_{m+2,m})(0).
\end{align*}$$

Hence, $[\eta(E_{m+2,m-1}), F_m] = 0$. Similarly, we can prove $[E_m, \eta(E_{m-1,m+2})] = 0$. This completes the proof of the theorem. \hfill \square

**Corollary 7.4.** There is a $\mathbb{Q}(\nu)$-superalgebra homomorphism

$$\eta_r : U(\mathfrak{gl}_m|n) \to \mathcal{S}(m|n)$$

sending $E_{h,h+1}, E_{h+1,h}$ and $K_a^\pm$ to $E_{h,h+1}(0,r), F_{h+1,h}(0,r)$ and $O(\pm e_a, r)$, respectively.

In the last two section, we will prove that $\eta$ induces an algebra isomorphism from $U(\mathfrak{gl}_m|n)$ to $\mathfrak{A}(m|n)$. In other words, we want to show that $\eta$ is injective and its image is $\mathfrak{A}(m|n)$. For the latter, we prove that $\mathfrak{A}(m|n)$ is generated by $K_a^\pm$, $E_h$, and $F_h$; while, for the former, we show that $\eta$ sends a basis to a linearly independent set. We will see a monomial basis plays a key role in the proof.

### 8. A Super Triangular Relation

We now aim to construct a monomial basis and a triangular relation from the monomial basis to the basis $\{A(j)\}_{A,j}$. In this section, we first make the construction at the $\nu$-Schur superalgebra level.
Lemma 8.1. and \( B \) be defined as in \([9, (5.3.2)]\), where \( \xi \) continue to hold. For a double coset \( D = W_\lambda d W_\mu \) \((d \in D_\mu\), let
\[
T_D = \sum_{z \in D_\lambda \cap W_\mu} (-q)^{-\ell(z)} x_{\lambda(0)} y_{\lambda(1)} T_d T_z
\]
be defined as in \([9, (5.3.2)]\), where \( x_{\lambda(0)} y_{\lambda(1)} = \sum_{w \in W_\lambda} (-q)^{\ell(w)} T_w \).

Lemma 8.2. For \( d \in D_\lambda\) and \( d' \in D_\mu\), let \( D = W_\lambda d W_\mu\), \( D' = W_\mu d W_\nu\), \( A = j(\lambda, d, \mu) \) and \( A' = j(\mu, d', \nu) \).

1. If \( T_D T_{D'} = \sum_{D'' \in W_\lambda D_\mu W_\nu} f_{D,D',D''} T_{D''} \), then there exists a \( D_0 \) with \( f_{D,D',D''} \neq 0 \)
\( \) and \( D'' \leq D_0 \) whenever \( f_{D,D',D''} \neq 0 \).

2. If \( \xi_A \xi_{A'} = \sum_{A'' \in M(m|n,r)} g_{A,A',A''} \xi_{A''} \), then there exists a \( A_0 \) with \( g_{A,A',A''} \neq 0 \)
\( \) and \( A'' \leq A_0 \) whenever \( g_{A,A',A''} \neq 0 \).

Proof. For (1), the proof of \([4, Prop. 7.38]\) carries over to the super case. A key observation is that the leading term in \( T_D T_{D'} \) is \( T_{w_D w_{D'}} \) which must occur in some \( T_{D''} \) of the RHS. By \([7, 8.3.8.4]\), \( \xi_A = \pm \nu^* \phi_A \) where \( \phi_A \) is defined in \([9, (5.7.1)]\). Thus, \( f_{D,D',D''} = d_\mu g_{A,A',A''} \) for some \( d_\mu \in Z \), (2) follows from (1).

The following lemma is the super version of \([1, Lem. 3.8]\) (see also \([4, Lem. 13.22]\)). The “lower terms” in an expression of the form “\( \xi_A + \) lower terms” means a \( \mathbb{Q}(\nu) \)-linear combination of elements \( \xi_B \) with \( B \in M(m|n,r) \) and \( B < A \) (the Bruhat ordering on \( M(m+n, r) \) defined above).

Lemma 8.3. Assume that \( 1 \leq k \leq m+n \), \( 1 \leq h \leq m+n \), and \( p \in \mathbb{N} \). Let \( A \in M(m|n,r) \), \( U_p = U_p(h, \rho(A)) \), and \( L_p = L_p(h, \rho(A)) \) as in \([4,0.1]\).

1. If \( A \in M(m|n,r) \) has the form
\[
A = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots \\
\ldots & a_{h,k-1} & 0 & 0 & \ldots \\
\ldots & a_{h+1,k-1} & a_{h+1,k} & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix},
\]
with \( a_{h,j} = 0 \), for \( k \leq j \leq m+n \), \( a_{h+1,j} = 0 \), for \( k+1 \leq j \leq m+n \), and \( a_{h+1,k} \geq p \), then
(a) \( \xi_{U_p} \xi_A = \xi_{A_p} + \) lower terms, for \( h \neq m \), and
(b) \( \xi_{U_1} \xi_A = (-1)^{\sum_{i>h,j<k} a_{i,j}} \xi_{A_1} + \) lower terms, for \( h = m \).

where \( A_p = A + pE_{h,k} - pE_{h+1,k} \).
(2) If \( A \in M(m|n,r) \) has the form

\[
A = \begin{pmatrix}
0 & \cdots & 0 & a_{h,k} & a_{h,k+1} & \cdots \\
0 & \cdots & 0 & a_{h+1,k+1} & \cdots \\
\vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]

with \( a_{h,j} = 0, \) for \( 1 \leq j \leq k - 1, \) \( a_{h+1,j} = 0, \) for \( 1 \leq j \leq k, \) and \( a_{h,k} \geq p, \) then

(a) \( \xi_{L_p} \xi_A = \xi_{A_p} + \) lower terms, for \( h \neq m, \) and

(b) \( \xi_{L_1} \xi_A = (-1)^{i > m, j < k} a_{i,j} \xi_{A_1} + \) lower terms, for \( h = m, \)

where \( A_p = A - pE_{h,k} + pE_{h+1,k}. \)

**Proof.** When \( h \neq m, \) comparing Proposition 5.2 with [4, theorem 13.18](1') and (2'), we find that if we replace \( v \) in [4, theorem 13.18](1') and (2') with \( v_h, \) we get Proposition 5.2. Similar to the proof of [4, 13.22], we have \( \xi_{U_p} \xi_A = \xi_{A_p} + \) lower terms and \( \xi_{L_p} \xi_A = \xi_{A_p} + \) lower terms respectively.

When \( h = m, \) by Lemma 3.1, if \( \xi_{U_p} \xi_A \neq 0 \) or \( \xi_{L_p} \xi_A \neq 0, \) then \( p = 1. \) Applying Proposition 5.3 to the matrix \( A \) in (1) or (2) gives the required formulas. \( \square \)

Let

\( \mathcal{T} = \mathcal{T}(m + n) = \{(i, h, j) \mid 1 \leq i \leq h < j \leq m + n\}. \)

There are two total ordering \( \leq_1 \) (\( i = 1, 2 \)) on \( \mathcal{T} \) which are defined as follows (see [4 p.562]):

\[
\begin{align*}
(i, h, j) &\leq_1 (i', h', j') \\
\iff \text{one of the following three conditions is satisfied:} \\
&\quad (i) j > j', (ii) j = j', i > i', \text{ and (iii) } j = j', i = i', h \leq h'; \\
(i, h, j) &\leq_2 (i', h', j') \\
\iff \text{one of the following three conditions is satisfied:} \\
&\quad (i) i < i', (ii) i = i', j < j', \text{ and (iii) } j = j', i = i', h \geq h'.
\end{align*}
\]

For example, \( \mathcal{T}(3) = \{(2, 2, 3), (1, 1, 3), (1, 2, 3), (1, 1, 2)\} \) under \( \leq_1. \) In general,

\[
(\mathcal{T}, \leq_1) = (\mathcal{T}_{m+n}, \ldots, \mathcal{T}_3, \mathcal{T}_2) \quad \text{and} \quad (\mathcal{T}, \leq_2) = (\mathcal{T}_1', \mathcal{T}_2', \ldots, \mathcal{T}_{m+n-1}')
\]

(8.2.2)

where, for every \( m + n \geq j \geq 2 \) and \( 1 \leq l < m + n, \) \( \mathcal{T}_j \) is the sequence (formed by inserting top row entries into the * entry below):

\[
j - 1 \quad j - 2 \quad j - 1 \quad \cdots \quad i, i + 1, \ldots, j - 1 \quad 1, 2, \ldots, j - 1
\]

(8.2.3)

while \( \mathcal{T}_j' \) is the sequence:

\[
l \quad \cdots \quad l + k - 1, \ldots, l + 1, l \quad \cdots \quad m + n - 1, \ldots, l + 1, l
\]

(8.2.4)

The following definition is taken from [4 Def. 13.23] which modifies the definition given in [1].
Definition 8.3. Let $A = (a_{i,j}) \in M(m|n)$ and define recursively almost diagonal matrices $E^{(A)}_{i,h,j}$, for $(i,h,j) \in (\mathcal{T}, \leq_1)$ as follows:

1. $E^{(A)}_{1,1,2}$ is the matrix defined by the conditions that $\text{co}(E^{(A)}_{1,1,2}) = \text{co}(A)$ and $E^{(A)}_{1,1,2} - a_{1,2} E_{1,2}$ is diagonal;
2. If $(i,h,j)$ is the immediate predecessor of $(i',h',j')$, then $E^{(A)}_{i,h,j}$ is defined by the conditions that $\text{co}(E^{(A)}_{i,h,j}) = \text{ro}(E^{(A)}_{i',h',j'})$ and $E^{(A)}_{i,h,j} - a_{i,j} E_{h+1,h+1}$ is diagonal.

Similarly, define recursively almost diagonal matrices $F^{(A)}_{i,h,j}$, for $(i,h,j) \in (\mathcal{T}, \leq_2)$, with respect to $\leq_2$ as follows:

1. $F^{(A)}_{N-1,N-1,N} (N = m+n)$ is the matrix defined by the conditions that $F^{(A)}_{N-1,N-1,N} - a_{N,N-1} E_{N,N-1}$ is diagonal and $\text{co}(F^{(A)}_{N-1,N-1,N}) = \text{ro}(E^{(A)}_{N-1,N-1,N})$;
2. If $(i,h,j)$ is the immediate predecessor of $(i',h',j')$, then $F^{(A)}_{i,h,j}$ is defined by the conditions that $\text{co}(F^{(A)}_{i,h,j}) = \text{ro}(F^{(A)}_{i',h',j'})$ and $F^{(A)}_{i,h,j} - a_{i,j} E_{h+1,h}$ is diagonal.

It is easy to see from the definition that, if $A \in M(m|n,r)$, then all $E^{(A)}_{i,h,j}$ and $F^{(A)}_{i,h,j}$ are in $M(m|n,r)$.

For $A \in M(m|n)$, define $\bar{A}$ as in (6.0.1).

Theorem 8.4. Maintain the notation introduced above and let $A = (a_{i,j}) \in M(m|n,r)$. The following triangular relation holds in $\mathcal{S}_v(m|n,r)$:

$$
\Psi_A := \prod_{(i,h,j) \in (\mathcal{T}, \leq_2)} \xi_{E^{(A)}_{i,h,j}} \cdot \prod_{(i,h,j) \in (\mathcal{T}, \leq_1)} \xi_{E^{(A)}_{i,h,j}} = (-1)^{T_A} \xi_A + \text{lower terms}, \quad (8.4.1)
$$

where the products are taken over the total orderings listed in (8.2.2). In particular, the set $\{\Psi_A\}_{A \in M(m|n,r)}$ forms a basis for $\mathcal{S}_v(m|n,r)$.

Proof. We follow the proof of [4, 13.24] by repeatedly applying Lemma 8.2 and carefully manipulating the sign occurred in Lemma 8.2(1b) & (2b).

We start with the largest element $(1,1,2)$ in $(\mathcal{T}, \leq_1)$ and let $A_{1,1,2} = E^{(A)}_{1,1,2}$. Repeatedly applying Lemma 8.2(1) (and noting Lemma 6.4) yields, for $(i,h,j) \in (\mathcal{T}, \leq_1)$

$$
\prod_{(i',h',j') \in (\mathcal{T}, \leq_1) \setminus \{(i,h,j) \leq_1 (i',h',j')\}} \xi_{E^{(A)}_{i',h',j'}} = (-1)^{N_{i,h,j}} \xi_{A_{i,h,j}} + \text{lower terms},
$$
where $A_{i,h,j}$ is the upper triangular matrix
\[
\begin{pmatrix}
\lambda_1 & a_{1,2} & \cdots & a_{1,j-1} & a_{1,j} & 0 & \cdots \\
0 & \lambda_2 - a_{1,2} & \cdots & a_{2,j-1} & a_{2,j} & 0 & \cdots \\
& \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & a_{i-1,j-1} & a_{i-1,j} & 0 & \cdots \\
0 & 0 & \cdots & a_{i,j-1} & 0 & 0 & \cdots \\
& \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \lambda_{j-1} - \sum_{i=1}^{j-2} a_{i,j-i} & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & \lambda_j - \sum_{i=1}^{j} a_{i,j} & 0 & \cdots \\
& \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \lambda_{j+1} & \cdots
\end{pmatrix}
\] \hspace{1cm} (8.4.2)

Note that, if $j \leq m$ and $(i', h', j') \geq 1 (i, h, j)$, then $h' < m$. So only Lemma 8.2(1a) applies. Hence, all $N_{i,h,j} = 0$ unless $j \geq m + 1$. Similarly, if $j \geq m + 1$ and $(i, h, j)$ is an immediate predecessor or an immediate successor of $(i', h', j')$ with $h \neq m$, then $N_{i,h,j} = N_{i',h',j'}$. Thus, if $j > m + 1$, then (see (8.2.3))

1. $N_{1,m+1,j} = N_{1,m+2,j} = \cdots = N_{1,j-1,j} = N_{j-2,j-2,j-1} = \cdots = N_{m,m,j-1}$,
2. $N_{i+1,m+1,j} = \cdots = N_{i+1,j-1,j} = N_{i,i,j} = \cdots = N_{i,m-1,j} = N_{i,m,j}$, for $i \leq m$,
3. $N_{j-1,j-1,j} = \cdots = N_{i+1,i+1,j} = \cdots = N_{i+1,j-1,j} = N_{i,i,j} = \cdots = N_{i,j-1,j}$, for $i > m$.

(8.4.3)

Hence, all $N$-values are determined by $N_{i,m,j}$ with $1 \leq i \leq m$ and $m + 1 \leq j \leq m + n$.

We first claim the following recursive formula:

\[
N_{m,m,j} = \begin{cases} 
0, & \text{if } j = m + 1; \\
N_{m,m,j-1} + \sum_{m+n \geq i', m \geq j', j \geq 1, m < j' < j} a_{i',j'} a_{i,j}, & \text{if } j \geq m + 2.
\end{cases}
\]

Indeed, $N_{m,m,m+1} = 0$ is clear since the $(k, l)$ entry $(A_{i,h,m+1})_{k,l} = 0$ whenever $k > m$ and $l < m + 1$. Assume now $j \geq m + 2$. For $i = 1$, by (8.4.3)(1), $N_{1,m+1,j} = N_{m,m,j-1}$.

Then, by (8.4.3)(2) and noting $a_{i,j} = 0$ or 1 for all $i \leq m$,

\[
\begin{align*}
N_{1,m,j} &= N_{1,m+1,j} + \sum_{i' > m, m < j' < j} a_{i',j'} a_{1,j} = N_{m,m,j-1} + \sum_{i' > m, m < j' < j} a_{i',j'} a_{1,j}, \\
N_{2,m,j} &= N_{1,m,j} + \sum_{i' > m, m < j' < j} a_{i',j'} a_{2,j} = N_{m,m,j-1} + \sum_{i' > m, m < j' < j} a_{i',j'} (a_{1,j} + a_{2,j}), \\
& \quad \cdots \quad \cdots, \\
N_{m,m,j} &= N_{m-1,m,j} + \sum_{i' > m, m < j' < j} a_{i',j'} a_{m,j} = N_{m,m,j-1} + \sum_{i' > m, m < j' < j} a_{i',j'} a_{i,j},
\end{align*}
\]

proving the claim.
By the claim, we obtain a close formula:

\[ N_{m,m,m+n} = \sum_{m+n \geq i > m \geq k \geq 1, \ m < j < l \leq m+n} a_{i,j}a_{k,l} = \bar{A}. \]

Thus, by (8.4.3)(3), \( N_{m,m+1} = N_{m+1,m,m+1,m+n} \) and so

\[ \xi(E) := \prod_{(i,h,j) \in (\mathcal{T}, \leq 1)} \xi_{E_{i,h,j}}^{(A)} = (-1)^{\bar{A}}\xi_{A_{m+n-1,m+n,m+n} + \text{lower terms}}. \]

Now, for \((i, h, j) \in (\mathcal{T}, \leq 2)\),

\[ \prod_{(i', h', j') \in (\mathcal{T}, \leq 2)} \xi_{E_{i', h', j'}}^{(A)} \cdot \xi(E) = (-1)^{N'_{i,h,j} + \bar{A}}\xi_{A'_{i,h,j}} + \text{lower terms}, \]

where \( A'_{i,h,j} \) is a matrix of the form

\[
\begin{pmatrix}
\lambda_1 & a_{1,2} & \cdots & a_{1,i-1} & a_{1,i} & a_{1,i+1} & \cdots \\
0 & \lambda_2 - a_{1,2} & \cdots & a_{2,i-1} & a_{2,i} & a_{2,i+1} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & \lambda_{i-1} - \sum_{l=1}^{i-2} a_{l,i-1} & a_{i-1,i} & a_{i-1,i+1} & \cdots \\
0 & 0 & \cdots & 0 & \lambda_i - \sum_{l<i} a_{l,i} - \sum_{l \geq j} a_{l,i} & a_{i+1,i} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & 0 & 0 & a_{h,i+1} & \cdots \\
0 & 0 & \cdots & 0 & a_{j,i} & a_{h+1,i+1} & \cdots \\
0 & 0 & \cdots & 0 & 0 & a_{h+2,i+1} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & 0 & a_{j+1,i} & a_{j+1,i+1} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & 0 & a_{m+n,i} & a_{m+n,i+1} & \cdots
\end{pmatrix}
\]

(In particular, \( A'_{1,1,2} = A \).) From the matrix above, we see that \((A'_{i,m+1,j'})_{j' < 0} = 0 \) for \( j' > m, i' < i \). Hence, by Lemma (8.2), all \( N'_{i,h,j} = 0 \) and

\[ \prod_{(i,h,j) \in (\mathcal{T}, \leq 2)} \xi_{E_{i,h,j}}^{(A)} \cdot \xi(E) = (-1)^{\bar{A}}\xi_A + \text{lower terms}, \]

as required.

\[ \square \]

9. Realisation of the quantum general linear supergroups

We are now in position to solve the realisation problem for \( U(gl_{m|n}) \) by first determining the image of the homomorphism \( \eta \) and then proving that \( \eta \) is injective. This requires another triangular relation with respect the preorder \( \preceq \) on \( M(m|n) \) (or more precisely on \( M(m|n)^{\pm} \)) which we define now. This order has already been implicitly used in the proof of Lemma (8.2).
For $A = (a_{i,j}), A' = (a'_{i,j}) \in M(m|n)$, define

$$A' \preceq A \iff \begin{cases} 
(1) & \sum_{i \leq s,j \geq t} a'_{i,j} \leq \sum_{i \leq s,j \geq t} a_{i,j}, \quad \text{for all } s \leq t; \\
(2) & \sum_{i \geq s,j \leq t} a'_{i,j} \leq \sum_{i \geq s,j \leq t} a_{i,j}, \quad \text{for all } s \geq t
\end{cases} \tag{9.0.4}$$

and

$$\|A\| = \sum_{i < j} \left( j - i + 1 \right) (a_{i,j} + a_{j,i}).$$

These definitions are independent of the diagonal entries of a matrix. Moreover, the following is taken from [4] Lem. 3.6(1) (see also [4] Lem. 13.20,13.21).

$$A < B \text{ (the Bruhat order)} \implies A^\pm < B^\pm \implies \|A^\pm\| < \|B^\pm\|, \tag{9.0.5}$$

where $A, B \in M(m|n, r)$ and $X^\pm$ is the matrix obtained by replacing the diagonal entries by 0.

**Theorem 9.1.** The (super) subspace $\mathfrak{A}(m|n)$ of the algebra $\mathcal{S}(m|n)$ defined as in (9.3.1) is the (super) subalgebra generated by $E_h = E_{h,h+1}(0), F_h = E_{h+1,h}(0), K_i^{\pm 1} = O(\pm e_i)$, for all $1 \leq h < m + n$ and $1 \leq i \leq m + n$.

**Proof.** Let $\mathfrak{A}(m|n)'$ be the subalgebra of $\mathcal{S}(m|n)$ generated by $E_h, F_h, K_i^{\pm 1}$. By Proposition 6.6, we have $\mathfrak{A}(m|n)' \subseteq \mathfrak{A}(m|n)$. We now prove $\mathfrak{A}(m|n) \subseteq \mathfrak{A}(m|n)'$ by induction on $\|A\|$. Clearly, if $\|A\| = 0$, $A(j) = O(j) = \prod_{i=1}^{m+n} K_i^{j_i} \in \mathfrak{A}(m|n)'$ for all $j \in \mathbb{Z}^{m+n}$. Assume now $\|A\| > 0$ and $B(j) \in \mathfrak{A}(m|n)'$ for all $j \in \mathbb{Z}^{m+n}$, whenever $\|B\| < \|A\|$. We need a triangular relation to complete the proof.

By Lemma 7.2, we have

$$(a_{i,j} E_{h,h+1})(0) = \frac{E_{h,h+1}^{a_{i,j}}(0)}{[a_{i,j}]^h_b} \quad \text{and} \quad (a_{i,j} E_{h+1,h})(0) = \frac{E_{h+1,h}^{a_{i,j}}(0)}{[a_{i,j}]^{h+1}_{b+1}}.$$

Thus, repeatedly applying Proposition 6.6 yields

$$\prod_{(i,h,j) \in \mathcal{T}} (a_{i,j} E_{h,h+1})(0) \prod_{(i,h,j) \in \mathcal{T}} (a_{i,j} E_{h+1,h})(0) = \sum_{B \in M(m|n)^\pm} \sum_{j \in \mathbb{Z}^{m+n}} g_{A,B,j} B(j).$$

We now prove that $g_{A,A,0} = 1$ and $B < A$ whenever $g_{A,B,j} \neq 0$ and $B \neq A$. In other words, we show that the triangular relation established in Theorem 8.4 induces a triangular relation of the form

$$\prod_{(i,h,j) \in \mathcal{T}} (a_{i,j} E_{h+1,h})(0) \prod_{(i,h,j) \in \mathcal{T}} (a_{i,j} E_{h,h+1})(0) = A(0) + \sum_{B \in M(m|n)^\pm} \sum_{j \in \mathbb{Z}^{m+n}} g_{B,A,j} B(j), \tag{9.1.1}$$
or equivalently, for all \( r \geq 0 \),

\[
\prod_{(i,h,j) \in \mathcal{T}} (a_{j,i} E_{h+1,h})(0,r) \prod_{(i,h,j) \in \mathcal{T}} (a_{i,j} E_{h,h+1})(0,r) = A(0,r) + \sum_{B \in M(m|n)^\pm} g_{B,A,j} B(j,r).
\]

Observe, for \( \lambda, \mu \in \Lambda(m|n, r - |A|) \) and \( \lambda \neq \mu \), the orthogonality relations

\[
\xi_{E_i}^{(A+\lambda)} \xi_{E_j}^{(A+\mu)} = 0, \quad \xi_{F_i}^{(A+\lambda)} \xi_{F_j}^{(A+\mu)} = 0, \text{ and } \xi_{F_i}^{(A+\lambda)} \xi_{F_j}^{(A+\mu)} = 0,
\]

where \( N = m + n \). Thus,

\[
\prod_{(i,h,j) \in \mathcal{T}} (a_{j,i} E_{h+1,h})(0,r) \prod_{(i,h,j) \in \mathcal{T}} (a_{i,j} E_{h,h+1})(0,r)
= \sum_{\lambda \in \Lambda(m|n, r - |A|)} \prod_{(i,h,j) \in \mathcal{T}} (-1)^{\xi_{E_i}^{(A+\lambda)}} \prod_{(i,h,j) \in \mathcal{T}} (-1)^{\xi_{E_j}^{(A+\lambda)}}
\]

Since \( E_i^{(A+\lambda)} = 0 \) and \( F_i^{(A+\lambda)} = 0 \), by Theorem 8.4,

\[
\prod_{(i,h,j) \in \mathcal{T}} (a_{j,i} E_{h+1,h})(0,r) \prod_{(i,h,j) \in \mathcal{T}} (a_{i,j} E_{h,h+1})(0,r)
= \sum_{\lambda \in \Lambda(m|n, r - |A|)} \prod_{(i,h,j) \in \mathcal{T}} \xi_{E_i}^{(A+\lambda)} \prod_{(i,h,j) \in \mathcal{T}} \xi_{E_j}^{(A+\lambda)}
= \sum_{\lambda \in \Lambda(m|n, r - |A|)} ((-1)^{A+\lambda} \xi_{A+\lambda} + \text{lower terms})
= A(0,r) + \sum_{B \in M(m|n)^\pm, B < A} g_{B,A,j} B(j,r), \text{ by (9.0.4),}
\]

where “lower terms” is a linear combination of \( \xi_B \) with \( B \in M(m|n) \) and \( B < A \) under Chevalley-Bruhat ordering in \( M(m|n) \). Hence, (9.1.1) follows.

To complete the proof, by (9.0.5) and induction, (9.1.1) implies that \( A(0) \in \mathfrak{A}(m|n)' \). Finally, by Proposition 6.8 for any \( j = (j_1, j_2, \cdots, j_{m+n}) \in \mathbb{Z}^{m+n} \), \( A(j) = \nu^{-\co(A)} j A(0) O(j) \in \mathfrak{A}(m|n)' \). \( \square \)

For any \( A = (a_{i,j}) \in M(m|n) \) and \( j \in \mathbb{Z}^{m+n} \), let

\[
M_{A,j} := \prod_{(i,h,j) \in \mathcal{T}} (a_{j,i} E_{h+1,h})(0) \cdot O(j) \cdot \prod_{(i,h,j) \in \mathcal{T}} (a_{i,j} E_{h,h+1})(0).
\]

Now, Lemma 6.3 and (9.1.1) implies immediately the first part of the following.

**Corollary 9.2.** (1) The set \( \overline{M} = \{ M_{A,j} | A \in M(m|n)^\pm, j \in \mathbb{Z}^{m+n} \} \) forms a basis of homogeneous elements for \( \mathfrak{A}(m|n) \).
(2) The image of the algebra homomorphism $\eta$ established in Theorem 7.3 is the subalgebra $\mathfrak{A}(m|n)$.

We now use this basis to derive a basis, a monomial basis, for the quantum supergroup $U = U(\mathfrak{gl}_{m|n})$.

For any $A \in M(m|n)^\pm$, $j \in \mathbb{Z}^{m+n}$, define

$$M^A_j = M'_1 M'_2 \cdots M'_{m+n-1} K^{j_1}_1 \cdots K^{j_{m+n}}_{m+n} M_m \cdots M_1 M_2,$$

where, for $2 \leq j \leq m + n$ and $1 \leq k \leq m + n - 1$,

$$M_j = E^{(a_{j-1,j})}_{j-1,j} (E^{(a_{j-2,j})}_{j-2,j-1} E^{(a_{j-1,j})}_{j-1,j}) \cdots (E^{(a_{1,j})}_{1,2} E^{(a_{j,j})}_{j-1,j}),$$

$$M'_k = E_{k+1,k}^{(a_{k+1,k})} (E_{k+2,k+1}^{(a_{k+2,k})} E_{k+3,k+2}^{(a_{k+3,k+2})}) \cdots (E_{m+n,m+n-1}^{(a_{m+n,k})} E_{m+n+1,m+n}^{(a_{m+n+1,k})}),$$

following the orders in (8.2.3) and (8.2.4).

**Corollary 9.3.** The set $\mathfrak{M} = \{M^A_j \mid A \in M(m|n)^\pm, j \in \mathbb{Z}^{m+n}\}$ forms a basis of homogeneous elements for $U(\mathfrak{gl}_{m|n})$.

**Proof.** Let $Q^+ = \sum_{i=1}^{m+n} N \alpha_i$ be the $+$-part of the root lattice, where $\alpha_i = \alpha_{i,i+1}$ are simple roots and let $U^\pm$ be the $\pm$-part of $U$. Then $U^+ = \bigoplus_{\alpha \in Q^+} U^\alpha$ is $Q^+$-graded. By inspecting a PBW type basis\(^6\) we see that

$$\dim U^\alpha = \# \{A \in M(m|n)^+ \mid \alpha = \sum_{i < j} a_{i,j} \alpha_{i,j}\},$$

where, for $i < j$, $\alpha_{i,j} = \alpha_i + \cdots + \alpha_{j-1}$. Since $\eta(\mathfrak{M}) = \mathfrak{M}$ is linearly independent, $\mathfrak{M}$ is linearly independent. In particular, $\mathfrak{M}^+ := \{M^A_0 \mid A \in M(m|n)^+\}$ is linearly independent. A dimensional comparison shows that $\mathfrak{M}^+$ forms a basis for $U^+$. Hence, $\mathfrak{M}$ forms a basis for $U$ since $U \cong U^- \otimes U^0 \otimes U^+$. \(\square\)

Now, $\eta$ sends $\mathfrak{M}$ to a basis. Hence $\eta$ is injective and so we have established the following realisation.

**Theorem 9.4.** The quantum enveloping superalgebra $U(\mathfrak{gl}_{m|n})$ is isomorphic to the superalgebra $\mathfrak{A}(m|n)$. In particular, $U(\mathfrak{gl}_{m|n})$ can be regarded as the $\mathbb{Q}(v)$-superalgebra whose underlying superspace is spanned by

$$\{A(j) \mid A \in M(m|n)^\pm, j \in \mathbb{Z}^{m+n}\}$$

and whose multiplication is given by the formulas in Propositions 6.5 and 6.6.

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\(^6\)For example, replacing $M_j$ (resp., $M'_k$) by $E^{a_{j-1,j}}_{j-1,j} E^{a_{j-2,j}}_{j-2,j} \cdots E^{a_{1,j}}_{1,1}$ (resp., $E^{a_{k+1,k}}_{k+1,k} E^{a_{k+2,k}}_{k+2,k} \cdots E^{a_{m+n,k}}_{m+n,k}$) yields such a basis.
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