Abstract. In this paper we establish the persistence property for solutions of the quartic generalized Korteweg-de Vries equation with initial data in weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx)$ for $s = 1/12 + \varepsilon$ and any $r \in (0, R)$, for some $0 < \varepsilon < 1/4$ and $0 < R < s/2$.

1. Introduction

In this work we are interested in the initial value problem for the k-generalized Korteweg-de Vries equation (k-gKdV)

\[ \begin{cases} 
\partial_t u + \partial_x^3 u + \partial_x (u^{k+1}) = 0, & x \in \mathbb{R}, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R},
\end{cases} \]

which was originally proposed to model the unidirectional propagation of nonlinear dispersive water waves ([20]), but has also been considered in connection with other physical systems, such as elastic rods ([26]), gravity waves ([33]) and plasma physics ([2]).

Next, we briefly review well-posedness results (i.e. existence, uniqueness and stability of solutions) obtained for the k-gKdV equation with initial data in Sobolev spaces $H^s(\mathbb{R})$, with emphasis on the minimal regularity $s \in \mathbb{R}$ possible.

When $k = 1$, (1) is simply referred as the Korteweg-de Vries equation (KdV). Z. Guo [14] and N. Kishimoto [19] showed independently that the KdV is locally and globally (in time) well-posed for initial data in $H^{-3/4}(\mathbb{R})$.

For $k = 2$, (1) is known as the modified Korteweg-de Vries equation (mKdV). Its local well-posedness, provided that $s \geq 1/4$, was shown by C. Kenig, G. Ponce and L. Vega [18] and extended globally by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao [9] (see also [14, 19] for the endpoint $s = 1/4$).

If $k = 3$, the best results for $s \geq -1/6$ are due to A. Grünrock [13] and T. Tao [32].

Finally, for $k \geq 4$ and $s > (k-4)/(2k)$ the local well-posedness was established in [18]. However, even for $k = 4$, blow up in finite time has been shown by Y. Martel and F. Merle [21].
Building upon the seminal work of T. Kato [15], the solvability of various dispersive nonlinear equations has been explored within the framework of weighted Sobolev spaces \( H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx) \), aiming to achieve enhanced control over the decay at infinity of solutions (as evidenced in studies such as [6, 7, 8, 11, 25] and their associated references). Notably, for equations like the modified Korteweg-de Vries (mKdV) and the generalized Korteweg-de Vries (k-gKdV) with \( k \geq 4 \), optimal results within this context were established by J. Nahas [24]. Moreover, A. Muñoz-García [23] established the local well-posedness of the Korteweg-de Vries equation for \( s > 3/4 \) in weighted Sobolev spaces. In the same spirit, our objective is to extend these investigations to encompass the quartic generalized Korteweg-de Vries equation (in the case of \( k = 3 \)), as elaborated upon in the subsequent sections.

As is standard, we are going to apply the contraction mapping principle to the integral equation version of the IVP (1) with \( k = 3 \), i.e.

\[
(2) \quad u = e^{-t\partial_x^3} u_0 - \int_0^t e^{-(t-t')\partial_x^3} \partial_x (u^4)(\cdot, t') \, dt',
\]

where \( e^{-t\partial_x^3} \) is the Airy semigroup introduced in Subsection 2.3.

We now state our main result.

**Theorem 1.1.** Let \( u_0 \in H^{1/12+\varepsilon}(\mathbb{R}) \cap L^2(|x|^{2r}dx) \) and \( r \in (0, R) \), for some \( 0 < \varepsilon < 1/4 \) and certain \( 0 < R < 1/24 + \varepsilon/2 \). Then, there exist \( T > 0 \) and a unique solution \( u \) of the integral equation (2) such that

\[
u(\cdot, t) \in H^{1/12+\varepsilon}(\mathbb{R}) \cap L^2(|x|^{2r}dx), \quad t \in (0, T].\]

In light of the Sobolev optimal results mentioned earlier for \( k = 3 \), it is natural to expect an improvement of Theorem 1.1 on the Sobolev regularity \( s \), at least for some \( 0 < s < 1/12 \). Similar observation applies in the situation \( k = 1 \), if one compares [14, 19] with [23]. We believe that a treatment of the factor \( |x|^r u \) (see Section 4.8 below) in the realm of Bourgain-type spaces might be advantageous, as has been demonstrated in the low regularity unweighted cases ([4, 13, 14, 17, 19, 32] among others). We are currently investigating this approach.

2. Definitions and Preliminaries

In this article we write \( A \lesssim B \), if there exists a constant \( C > 0 \) such that \( A \leq CB \). Moreover, \( A \sim B \) represents \( A \lesssim B \) and \( B \lesssim A \).

2.1. Function spaces. For a function \( f \in L^2(\mathbb{R}) \), consider its Fourier transform

\[
\hat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx, \quad \xi \in \mathbb{R}
\]

and its inverse Fourier transform by

\[
f(y(x)) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} f(\xi) \, d\xi, \quad x \in \mathbb{R}.
\]

In this work we use the inhomogeneous Sobolev space \( H^s(\mathbb{R}) \), of order \( s \in \mathbb{R} \), defined via the norm

\[
\|f\|_{H^s} := \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2},
\]
which satisfy the inclusion $H^s(\mathbb{R}) \subset H^{s'}(\mathbb{R})$ for $s \leq s'$, that is,
\begin{equation}
\|f\|_{H^s} \lesssim \|f\|_{H^{s'}}.
\end{equation}

In order to measure the regularity of functions defined in the space–time domain $\mathbb{R} \times [0, T]$ we introduce the mixed–norm Lebesgue spaces $L^p_t L^q_x$ or $L^p_t L^q_x$, $1 \leq p, q \leq \infty$, given respectively by the norms
\[
\|f\|_{L^p_t L^q_x} := \left\{ \int_0^T \left( \int_{\mathbb{R}} |f(x,t)|^q \, dx \right)^{p/q} \, dt \right\}^{1/p}
\]
and
\[
\|f\|_{L^p_t L^q_x} := \left\{ \int_0^T \left( \int_{\mathbb{R}} |f(x,t)|^q \, dx \right)^{p/q} \, dt \right\}^{1/q}
\]
with the standard modifications involving the essential supremum when $p$ or $q$ are equal to infinity.

2.2. Fractional derivatives. For $\alpha \in \mathbb{C}$, we define the fractional derivative $D^\alpha_x$ as the Fourier multiplier given by
\[
(D^\alpha_x f)(\xi) := |\xi|^\alpha \hat{f}(\xi).
\]
Analogously, we introduce the operator $(1 + D^2_x)^\alpha$ via
\[
((1 + D^2_x)^\alpha f)(\xi) := (1 + |\xi|^2)^\alpha \hat{f}(\xi).
\]
Hence, the Plancherel identity allows us to write
\[
\|f\|_{H^s} \sim \|(1 + D^2_x)^{s/2} f\|_{L^2} \lesssim \|f\|_{L^2} + \|D^2_x f\|_{L^2}.
\]
Also, if we invoke the Hilbert transform $H$ determined by
\[
(Hf)(\xi) := -i \text{sgn}(\xi) \hat{f}(\xi),
\]
we can relate $D_x$ with the standard derivative $\partial_x$ as $D_x = H \partial_x$ or $\partial_x = H D_x$.

A highly useful property in Section [18] will be the following fractional Leibniz rule-type inequality (see [18] Theorem A.8)
\begin{equation}
\|D^\alpha_x (fg) - f D^\alpha_x g - g D^\alpha_x f\|_{L^p_t L^q_x} \lesssim \|D^\alpha_x f\|_{L^p_t L^q_x} \|D^\alpha_x g\|_{L^p_t L^q_x},
\end{equation}
where $\alpha \in (0, 1)$, $\alpha_1, \alpha_2 \in [0, \alpha]$ verify $\alpha = \alpha_1 + \alpha_2$ and the exponents $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$ are given by
\begin{equation}
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.
\end{equation}
Moreover, for $\alpha_1 = 0$ the value $q_1 = \infty$ is allowed.

We also recall the chain rule for fractional derivatives ([18] Theorem A.6]
\begin{equation}
\|D^\alpha_x F(f)\|_{L^p_t L^q_x} \lesssim \|F'(f)\|_{L^p_t L^q_x} \|D^\alpha_x f\|_{L^p_t L^q_x}
\end{equation}
for $\alpha \in (0, 1)$, $p, p_1, p_2, q, q_2 \in (1, \infty)$ and $q_1 \in (1, \infty]$ satisfying [5].

Sometimes it is more convenient to use this other definition of fractional derivative,
\[
D^\alpha_x f(x) := \left( \int_{\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x - y|^{1 + 2\alpha}} \, dy \right)^{1/2}, \quad \alpha \in (0, 1),
\]
which is related to $D^\alpha_x$ in the following sense (see for example [25, eq. (1.23)])

(7) \[ \|f\|_{L^p} + \|D^\alpha_x f\|_{L^p} \sim \|f\|_{L^p} + \|D^\alpha_x f\|_{L^p}, \]

and also satisfies ([25, eq. (2.2)])

(8) \[ \|D^\alpha_x (fg)\|_{L^p} \lesssim \|f\|_{L^\infty} \|D^\alpha_x g\|_{L^p} + \|g D^\alpha_x f\|_{L^p}. \]

2.3. Airy semigroup. The solution of the Airy equation

\[ \begin{align*}
\partial_t u + \partial_x^3 u &= 0, \quad x \in \mathbb{R}, \ t > 0, \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R}
\end{align*} \]

can be written as $u(x, t) = e^{-t\partial_x^3} u_0(x)$, where $e^{-t\partial_x^3}$ denotes the Fourier multiplier given by

\[(e^{-t\partial_x^3} u_0)(\xi) := e^{it\xi^3} \hat{u}_0(\xi).\]

The Plancherel theorem easily provides

(9) \[ \|e^{-t\partial_x^3} u_0\|_{L^2_x} \sim \|u_0\|_{L^2_x}. \]

Next, we recall additional boundedness properties of the family of operators \(\{e^{-t\partial_x^3}\}_{t>0}\), which will be very helpful in proving our main result. In [18, Theorem 3.5] it was established that

(10) \[ \|\partial_t e^{-t\partial_x^3} u_0\|_{L^p_t L^2_x} \lesssim \|u_0\|_{L^2_x}. \]

In addition, for any $\theta \in [0, 1]$, $\alpha \in [0, 1/2]$, $p = 2/(1 - \theta)$ and $q = 6/(\theta \alpha + \theta)$ it was shown in [16, Lemma 2.4] that

(11) \[ \|D^{\theta \alpha/2}_x e^{-t\partial_x^3} u_0\|_{L^p_t L^q_x} \lesssim \|u_0\|_{L^2_x}. \]

and

\[ \left\| \int_0^t D^{\theta \alpha}_x e^{-(t-t')\partial_x^3} f(\cdot, t') dt' \right\|_{L^p_t L^q_x} \lesssim \|f\|_{L^{p', q'}_{t'} L^q_x}, \]

provided $1/p + 1/p' = 1 = 1/q + 1/q'$.

We continue with some other crucial estimates involving integrals of the semigroup. In [18, Theorem 3.5(ii)] it was proved that

(12) \[ \|\partial_x \int_0^t e^{-(t-t')\partial_x^3} f(\cdot, t') dt'\|_{L^p_t L^2_x} \lesssim \|f\|_{L^p_t L^2_x} \]

and

(13) \[ \|\partial_x^2 \int_0^t e^{-(t-t')\partial_x^3} f(\cdot, t') dt'\|_{L^p_t L^2_x} \lesssim \|f\|_{L^p_t L^2_x}, \]

and in [12, eq. (2.13)]

(14) \[ \left\| \int_0^t e^{-(t-t')\partial_x^3} f(\cdot, t') dt' \right\|_{L^p_t L^q_x} \lesssim \|f\|_{L^{p', q'}_{t'} L^q_x}, \]

provided $1/p_2 + 1/p'_2 = 1 = 1/q_2 + 1/q'_2$ and

\[ \frac{1}{q_j} = \frac{1}{6} - \frac{1}{3p_j}, \quad 2 \leq p_j \leq \infty, \quad j = 1, 2. \]

Next, we deduce other related estimates that will be used in the proof of Theorem [11].
Lemma 2.1. For $0 < \varepsilon < 1/4$, one has that
\begin{align}
(15) \quad \| e^{-i \theta^3} u_0 \|_{L_x^{120}/(26+51 \varepsilon)} \lesssim \| D_x^{1/12+\varepsilon} u_0 \|_{L^2}
\end{align}
and
\begin{align}
(16) \quad \| e^{-i \theta^3} u_0 \|_{L_x^{60}/(13-12 \varepsilon)} \lesssim \| u_0 \|_{H^{1/12+\varepsilon}}.
\end{align}

Proof. It follows from Stein’s theorem of analytic interpolation (see [11] Theorem 1, p. 313) to the family of operators
\[ T^{(1)}_z := D_x^{-\frac{2\varepsilon}{12}} e^{-i \theta^3} \]
for (15) and
\[ T^{(2)}_z := (1 + D_x^2)^{-\frac{1}{2}} e^{-i \theta^3} \]
for (16), where $z \in \mathbb{C}$, with $0 \leq \Re z \leq 1$. More precisely, for any $\gamma \in \mathbb{R}$, observe from [18] Lemma 3.2.6] that
\[ \| T^{(1)}_\gamma u_0 \|_{L_x^{60}/13 L_T^{15}} = \| D_x^{-1/12} e^{-i \theta^3} D_x^{-2 \gamma/3} u_0 \|_{L_x^{60}/13 L_T^{15}} \lesssim \| D_x^{-2 \gamma/3} u_0 \|_{L^2} \sim \| u_0 \|_{L^2}.
\]
Moreover, we obtain from [18] eq. (3.39.a)] that
\[ \| T^{(1)}_\gamma u_0 \|_{L_x^{60}/13 L_T^{15}} \lesssim \| u_0 \|_{L^2}.
\]
Similarly, [18] Lemma 3.2.6 implies that
\[ \| T^{(2)}_\gamma u_0 \|_{L_x^{60}/13 L_T^{15}} \lesssim \| u_0 \|_{L^2}
\]
and [29] Theorem A.] gives
\[ \| T^{(2)}_\gamma u_0 \|_{L_x^{60}/13 L_T^{15}} \lesssim \| u_0 \|_{L^2}.
\]
Therefore, we have for any $0 < \theta_1, \theta_2 < 1$ and $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, satisfying
\[ \frac{1}{p_1} = \frac{1 - \theta_1}{60} + \frac{\theta_1}{2}, \quad \frac{1}{q_1} = \frac{1 - \theta_1}{15} + \frac{\theta_1}{\infty},
\]
and
\[ \frac{1}{p_2} = \frac{1 - \theta_2}{60} + \frac{\theta_2}{6}, \quad \frac{1}{q_2} = \frac{1 - \theta_2}{15} + \frac{\theta_2}{\infty},
\]
that
\[ \| T^{(1)}_{\theta_1} u_0 \|_{L_p^p L_T^{q_1}} \lesssim \| u_0 \|_{L^2}
\]
and
\[ \| T^{(2)}_{\theta_2} u_0 \|_{L_p^p L_T^{q_2}} \lesssim \| u_0 \|_{L^2}.
\]
Thus, (15) and (16) are derived by choosing $\theta_1 = 3 \varepsilon/2$ and $\theta_2 = 4 \varepsilon$, respectively. \qed

Lemma 2.2. Let $0 < \varepsilon < 1/4$. Then, there hold that
\[ \| \int_0^t e^{-(t-t') \theta^3} f(\cdot, t') \, dt \|_{L_x^{120}/(26+51 \varepsilon)} \lesssim T^{\delta} \| D_x^{1/12+\varepsilon} f \|_{L_x^{10}(7-3 \varepsilon) L_T^{30}/(23-12 \varepsilon)}
\]
and
\[ \| \int_0^t e^{-(t-t') \theta^3} f(\cdot, t') \, dt \|_{L_x^{60}/(13-12 \varepsilon)} \lesssim T^{\delta} \left( \| f \|_{L_x^{10}(7-8 \varepsilon) L_T^{30}/(23-32 \varepsilon)} + \| D_x^{1/12+\varepsilon} f \|_{L_x^{10}(7-8 \varepsilon) L_T^{30}/(23-32 \varepsilon)} \right)
\]
for some $\delta > 0$. 

We can proceed similarly to the proof of Lemma 2.1 using (see [18, eq. (3.55)] and [29, Theorem A.])

$$
\|T_{\tau}^{(1)} f\|_{L^{60/13}_x L^1_T} = \left\| D^{-1/12}_x \int_0^t e^{i(t-t')\partial_x^3} D^{-2i\gamma/3}_x f(\cdot, t') dt' \right\|_{L^{60/13}_x L^1_T} \lesssim T^{1/6} \|f\|_{L^{10/7} L^{30/23}_T},
$$

$$
\|T_{\tau}^{(1)} f\|_{L^2_x L^\infty_T} = \left\| D^{-3/4}_x \int_0^t e^{i(t-t')\partial_x^3} D^{-2i\gamma/3}_x f(\cdot, t') dt' \right\|_{L^2_x L^\infty_T} \lesssim T^{1/2} \|f\|_{L^2_x L^2_T},
$$

and the estimate (see [18, eq. (3.55)] and [29, Theorem A.])

$$
\|T_{\tau}^{(2)} f\|_{L^{60/13}_x L^1_T} = \left\| (1 + D^2_x)^{-1/2} \int_0^t e^{i(t-t')\partial_x^3} (1 + D^2_x)^{-i\gamma/3} f(\cdot, t') dt' \right\|_{L^{60/13}_x L^1_T} \lesssim T^{1/6} \|f\|_{L^{10/7} L^{30/23}_T},
$$

and

$$
\|T_{\tau}^{(2)} f\|_{L^2_x L^\infty_T} = \left\| (1 + D^2_x)^{-1/2} \int_0^t e^{i(t-t')\partial_x^3} (1 + D^2_x)^{-i\gamma/3} f(\cdot, t') dt' \right\|_{L^2_x L^\infty_T} \lesssim T^{1/2} \|f\|_{L^2_x L^2_T}. \tag{16}
$$

2.4. Fonseca-Linares-Ponce pointwise formula. In this section, we present a pointwise formula derived in [10] that enables the commutation of fractional powers \(|x|\nu\) and the Airy semigroup \(e^{-t\partial_x^3}\) with the necessary adjustments. Specifically, for \(u_0 \in H^s(\mathbb{R}) \cap L^2(|x|^{2r} dx)\), where \(0 < s < 2\) and \(0 < r < s/2\), the following identity holds:

$$
|x|\nu e^{-t\partial_x^3} u_0(x) = e^{-t\partial_x^3} (|x|\nu u_0)(x) + e^{-t\partial_x^3} \{\Psi_t, r, (\tilde{u}_0)(\xi)\}^\nu(x), \tag{17}
$$

for all \(t > 0\) and almost any \(x \in \mathbb{R}\). Moreover, the \(L^2\)–norm of the last term above can be controlled as follows:

$$
\|\{\Psi_t, r, (\tilde{u}_0)(\xi)\}^\nu\|_{L^2_x} \lesssim (1 + t) \left( \|u_0\|_{L^2} + \|D_x^{2r} u_0\|_{L^2} \right). \tag{18}
$$

It is important to note that in [10], they specifically considered the case of \(s = 2\alpha\) and \(r = \alpha\) for \(0 < \alpha < 1\). However, a careful analysis of their proof reveals that this result can be extended to the slightly more general situation described here.

3. Main estimates

In this section, we gather several interpolation formulas that will prove useful in the subsequent discussions.
3.1. Interpolation formulas. We commence by revisiting a well-known estimate, the proof of which we include here for the sake of completeness.

Lemma 3.1. Let $p, p_1, p_2, p_3, q, q_1, q_2, q_3 \in [1, \infty]$ and $\theta_1, \theta_2 \in [0, 1]$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1 - \theta_1 - \theta_2}{p_3}$$

and

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1 - \theta_1 - \theta_2}{q_3},$$

that is, $(1/p, 1/q)$ is a point in the convex hull generated by $(1/p_1, 1/q_1)$, $(1/p_2, 1/q_2)$ and $(1/p_3, 1/q_3)$. Then,

$$\|f\|_{L^p L^q_T} \leq \|f\|_{L^{p_1} L^{q_1}_T} \|f\|_{L^{p_2} L^{q_2}_T} \|f\|_{L^{p_3} L^{q_3}_T}^{1 - \theta_1 - \theta_2}.$$

Proof. Since

$$1 = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = \frac{q_1}{q_1(1 - \theta_1 - \theta_2)} + \frac{q_2}{q_2(1 - \theta_1 - \theta_2)} + \frac{q_3}{q_3(1 - \theta_1 - \theta_2)},$$

the Hölder inequality allows for the following representation:

$$\|f\|_{L^p L^q_T}^q = \int_0^T |f|^{q_1} |f|^{q_2} |f|^{q(1 - \theta_1 - \theta_2)} dt$$

$$= \int_0^T \left( |f|^{q_1} \frac{q_1}{q_1(1 - \theta_1 - \theta_2)} \left( |f|^{q_2} \frac{q_2}{q_2(1 - \theta_1 - \theta_2)} \right) |f|^{q(1 - \theta_1 - \theta_2)} \right) \frac{q}{q} dt$$

$$\leq \left( \|f\|_{L^p L^q_T}^{q_1} \frac{q_1}{q_1(1 - \theta_1 - \theta_2)} \|f\|_{L^p L^q_T}^{q_2} \frac{q_2}{q_2(1 - \theta_1 - \theta_2)} \|f\|_{L^p L^q_T}^{q(1 - \theta_1 - \theta_2)} \right) \frac{q}{q}$$

$$= \|f\|_{L^{p_1} L^{q_1}_T}^{q_1} \|f\|_{L^{p_2} L^{q_2}_T}^{q_2} \|f\|_{L^{p_3} L^{q_3}_T}^{q(1 - \theta_1 - \theta_2)}.$$

Hence,

$$(19) \quad \|f\|_{L^p L^q_T} \leq \|f\|_{L^{p_1} L^{q_1}_T} \|f\|_{L^{p_2} L^{q_2}_T} \|f\|_{L^{p_3} L^{q_3}_T}^{1 - \theta_1 - \theta_2}.$$
provided that
\[
\frac{1}{p} = \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{2}.
\]

**Proof.** Firstly, recall from [22, Lemma 3.3] the interpolation formula for fractional derivatives
\[
\|D^s_x f\|_{L^p_t L^{q_2}_x} \lesssim \|D^{s_1}_x f\|_{L^{p_1}_t L^{q_1}_x} \|D^{s_2}_x f\|_{L^{p_2}_t L^{q_2}_x}^{1-\theta}
\]
for \(p, q_i \in (1, \infty), s_i \in \mathbb{R}\) and \(\theta \in (0, 1)\) related as
\[
\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad s = \theta s_1 + (1-\theta)s_2.
\]
Furthermore, it is allowed to take \(p_2 = \infty\) in (22) if one uses the interpolation result of O. Blasco for vector valued functions of bounded mean oscillations ([3, Corollary 1]). Finally, to derive (20) and (21) we also need to take into account the result of O. Blasco for vector valued functions of bounded mean oscillations ([3, Theorem 2] and [23, Theorem 1.3])
\[
\|\mathcal{H} f\|_{L^p_t L^q_x} \lesssim \|f\|_{L^p_t L^q_x}, \quad p, q \in (1, \infty),
\]
and
\[
\|\mathcal{H} f\|_{\text{BMO}, L^q_t} \lesssim \|f\|_{L^p_t L^q_x}, \quad q \in (1, \infty).
\]

The following lemma is a consequence of the fractional Caffarelli-Kohn-Nirenberg inequality established in [27].

**Lemma 3.3.** Let \(r, \tau, \gamma > 0, \sigma \leq 0, 0 < a < 1, 0 \leq \theta \leq 1\) and \(0 \leq \alpha \leq 1/2\) satisfy
\[
\gamma = a\sigma + (1-a)r \quad \text{and} \quad \frac{1}{r} + \gamma = a\left(\frac{1-\theta}{2} - \theta\alpha\right) + (1-a)\left(\frac{1}{2} + r\right).
\]
Then, there exists \(\delta > 0\) such that
\[
\|\|x\|^{\gamma} f\|_{L^p_t L^{q_2}_x} \lesssim \|f\|_{W^{\theta a, 2/(1-\theta)} L^2_x} \|\|x\|^{\gamma} f\|_{L^2_x}^{1-a},
\]
where
\[
|f|_{W^{s, p}} := \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^p}{|x-y|^{1+sp}} \, dx \, dy \right)^{1/p}.
\]

Since (see for example [31, Theorem 5, p.155])
\[
|f|_{W^{s, p}} + \|f\|_{L^p_t} \sim \|D^s_x f\|_{L^p_t} + \|f\|_{L^p_t}, \quad p \geq 2,
\]
we can write
\[
\|\|x\|^{\gamma} f\|_{L^p_t} \lesssim \left( \|D^s_x f\|_{L^p_t} + \|f\|_{L^2_x} \right) \|\|x\|^{\gamma} f\|_{L^2_x}^{1-a}.
\]
Next, the Hölder inequality leads us to
\[
\|\|x\|^{\gamma} f\|_{L^p_t L^{q_2}_x} \lesssim \left( \|D^s_x f\|_{L^p_t} \|\|x\|^{\gamma} f\|_{L^{q_2}_x} \right) \|\|x\|^{\gamma} f\|_{L^{q_2}_x}^{1-a}
\]
\[
= \left( \|D^s_x f\|_{L^p_t} \|\|x\|^{\gamma} f\|_{L^{q_2}_x} \right) \|\|x\|^{\gamma} f\|_{L^{q_2}_x}^{1-a}
\]
\[
\lesssim T^\delta \left( \|D^s_x f\|_{L^p_t} \|\|x\|^{\gamma} f\|_{L^{q_2}_x} \right) \|\|x\|^{\gamma} f\|_{L^{q_2}_x}^{1-a}
\]
for some \(\delta > 0\).
3.2. Commutators. The following is our first commutator result.

**Lemma 3.4.** Let $0 < r, \rho < 1$ such that $r + \rho < 1$, $1 \leq \tilde{p}_1 < 1/(1 - r - \rho)$ and $1 \leq q, p \leq \infty$ satisfying $1/q + 1 = 1/\tilde{p}_1 + 1/p$. Then,

$$\| \{\langle \xi \rangle^{-\rho}, D_\xi^p \} \hat{f} \|_{L^2_\rho} \lesssim \| \hat{f} \|_{L^2_\rho}.$$

**Proof.** Define

$$K_\rho(x) := \int_\mathbb{R} e^{-ix\xi} \langle \xi \rangle^{-\rho} \, d\xi.$$

It is known that (see for example [30, Lemma 0.3.9])

$$\| K_\rho(x) \| \lesssim \frac{1}{|x|^{1-\rho(1+|x|)^2}}.$$

By using the estimate

$$\| \{\langle \xi \rangle^{-\rho}, D_\xi^p \} \hat{f} \|^\wedge (x) \| = \| \{\langle \xi \rangle^{-\rho} D_\xi^p f - D_\xi^p (\langle \xi \rangle^{-\rho} f) \} \hat{f} \|^\wedge (x),$$

$$\sim |K_\rho \ast (| \cdot |^r \hat{f}) - |x|^r K_\rho \ast \hat{f}(x)| = \left| \int_{\mathbb{R}} \left( |y|^r - |x|^r \right) K_\rho(x - y) \hat{f}(y) \, dy \right|$$

$$\leq \int_{\mathbb{R}} |x - y|^r |K_\rho(x - y)| \| \hat{f}(y) \| \, dy = (| \cdot |^r K_\rho) \ast | \hat{f}(x),$$

it follows from the Young inequality that

$$\| \{\langle \xi \rangle^{-\rho}, D_\xi^p \} \hat{f} \|^\wedge \| \lesssim \| \cdot |^r K_\rho \|_{L^p_{\tilde{p}_1}} \| \hat{f} \|_{L^q_\rho}.$$  

Now, inequality (23) implies

$$\| \cdot |^r K_\rho \|_{L^p_{\tilde{p}_1}} \lesssim \int_0^1 dx \frac{1}{|x|^{(1-r-\rho)\tilde{p}_1}} + \int_1^\infty dx \frac{1}{(1+|x|)^{2\tilde{p}_1}} < \infty,$$

provided

$$1 \leq \tilde{p}_1 < \frac{1}{1 - r - \rho}. \quad \Box$$

Next, we establish a consequence of the Leibniz rule for fractional derivatives.

**Lemma 3.5.** If $0 < \alpha < 1$, $1 \leq p \leq 2 \leq q < \infty$ and $1/2 = 1/p - 1/q$ we have that

$$\| [g, D_\alpha^p] f \|_{L^2_\rho} \lesssim \| D_\alpha^p g \|_{L^2_\rho} \| \hat{f} \|_{L^q_\rho}.$$

**Proof.** For $1/2 = 1/q + 1/p'$, it is known from [18] that

$$\| D_\alpha^p (fg) - g D_\alpha^p f - f D_\alpha^p g \|_{L^2} \lesssim \| D_\alpha^p g \|_{L^2_\rho} \| f \|_{L^q_\rho}.$$  

So, the Hölder and Hausdorff-Young inequalities imply that for $1/p + 1/p' = 1$ and $1 \leq p \leq 2$,

$$\| [g, D_\alpha^p] f \|_{L^2_\rho} \lesssim \| D_\alpha^p (fg) - g D_\alpha^p f - f D_\alpha^p g \|_{L^2} + \| f D_\alpha^p g \|_{L^2_\rho}$$

$$\lesssim \| D_\alpha^p g \|_{L^2_\rho} \| f \|_{L^q_\rho} \lesssim \| D_\alpha^p g \|_{L^2_\rho} \| \hat{f} \|_{L^q_\rho}. \quad \Box$$

In Section 4.8.1 we will apply the previous result to the function $g(\xi) := e^{it\xi^\alpha} \langle \xi \rangle^{-\rho}$, therefore Lemma 3.6 below will be very helpful.
Lemma 3.6. Let $0 < r, \rho < 1$ such that $r+\rho < 1$ and $2r < \rho$. Then, for any $t > 0$ and $q \in \{\max\{2,1/(\rho-2r)\}, \infty\}$ one has

$$
\left\| D_\xi^{(\rho)}(e^{it\xi^3}) \right\|_{L_q^T} \lesssim 1 + t^\delta
$$

for certain $\delta > 0$.

Proof. Using properties (7) and (8), we have

$$
\left\| D_\xi^{(\rho)}(e^{it\xi^3}(\xi)^{-\rho}) \right\|_{L_q^T} \leq \left\| D_\xi^{(\rho)}(e^{it\xi^3}(\xi)^{-\rho}) \right\|_{L_q^T} + \left| \xi \right| \left\| \xi \right\|_{L_q^T}
$$

$$
\sim \left\| D_\xi^{(\rho)}(e^{it\xi^3}(\xi)^{-\rho}) \right\|_{L_q^T} + \left\| \xi \right\|_{L_q^T}
$$

$$
\leq \left\| \xi \right\|^{-\rho} D_\xi^{(\rho)} e^{it\xi^3} \right\|_{L_q^T} + \left\| D_\xi^{(\rho)}(\xi)^{-\rho} \right\|_{L_q^T} + \left\| \xi \right\|^{-\rho} \right\|_{L_q^T}
$$

$$
\sim \left\| \xi \right\|^{-\rho} D_\xi^{(\rho)} e^{it\xi^3} \right|_{L_q^T} + \left\| D_\xi^{(\rho)}(\xi)^{-\rho} \right\|_{L_q^T} + \left\| \xi \right\|^{-\rho} \right\|_{L_q^T}.
$$

The third term on the right-hand side of the above inequality is finite if $q > 1/\rho$.

As for the second term, it is sufficient to choose $q > \max\{2,1/(r+\rho)\}$ and utilize the Hausdorff-Young inequality in combination with (24). Finally, by applying [6, Lemma 2.2], the first term is estimated as follows for some $\delta > 0$:

$$
\left\| \xi \right\|^{-\rho} D_\xi^{(\rho)} e^{it\xi^3} \right\|_{L_q^T} \lesssim t^\delta \left\| \xi \right\|^{-(\rho-2r)} \right\|_{L_q^T} \lesssim t^\delta,
$$

provided $q > 1/(\rho-2r)$. \qed

4. Proof of Theorem 1.1

As is standard, our aim is to apply the Banach fixed-point theorem to the mapping

$$
\Phi(u) := e^{-t03^2}u_0 - \int_0^t e^{-(t-t')03^2} \partial_x(u^4)(\cdot, t') \, dt'.
$$

Therefore, it suffices to demonstrate that this mapping is a contraction on a suitable subspace of $L_T^q H_x^{1/12+\varepsilon} \cap L_T^q L_x^2(|x|^{2r} \, dx)\), as detailed below.

For some $\rho$ and $T > 0$, which we will fix later, let us consider the complete metric space

$$
X_T^\rho := \{u : \|u\|_{X_T} \leq \rho\},
$$

equipped with the norm

$$
\|u\|_{X_T} := \|u\|_{L_T^q H_x^{1/12+\varepsilon}} + \|u\|_{L_T^q L_x^2} + \sum_{j=2}^{7} \nu_j^T(u) + \sum_{j=1}^{6} \mu_j^T(u),
$$

where

$$
\nu_j^T(u) := \left(1 + T\right)^{-\rho} \|u\|_{L_x^{10/3} L_T^4}, \quad \nu_j^T(u) := T^{-1/6} \|D_x^{1/12} u\|_{L_x^{10/3} L_T^{30/7}},
$$

$$
\nu_j^T(u) := \|u\|_{L_x^{60/13} L_T^5}, \quad \nu_j^T(u) := \|\partial_x u\|_{L_x L_T^2},
$$

$$
\nu_j^T(u) := T^{-1/6} \|u\|_{L_x^{10/3} L_T^{30/7}}, \quad \nu_j^T(u) := \|D_x^{1/12} \partial_x u\|_{L_T^2 L_x^2},
$$

and

$$
\mu_j^T(u) := T^{-1/2} \|u\|_{L_T^2 L_x^2}, \quad \mu_j^T(u) := \|D_x^{\theta(\alpha)} u\|_{L_T^{2/1-(\theta-1)}}
$$

$$
\mu_j^T(u) := \|D_x^{\theta(\alpha)} u\|_{L_T^{2/1-(\theta-1)}}, \quad \mu_j^T(u) := \|D_x^{1/12+\varepsilon} \partial_x u\|_{L_T^\infty L_x^2}.
$$


\[ \mu_3^T(u) := \|u\|_{L_x^p(0+\theta) L_t^{2/(1-\theta)}} , \quad \mu_6^T(u) := \|u\|_{L_x^{10/(13-12\varepsilon)} L_t^{15/(1-4\varepsilon)}} . \]

Throughout this proof, we fix
\[ \varepsilon := \frac{633}{5000} , \quad R := \frac{51}{1000} , \quad \theta := \frac{23}{25} , \quad \alpha := \frac{41}{100} . \]

**Remark 4.1.** To provide a better view of the proof, numerology, and technical constraints, it is essential to note that controlling the norms of the nonlinear contribution of \( \Phi(u) \) involves several considerations. Throughout our analysis, we make frequent use of Lemma 3.1 to interpolate various \( L_p L_q \)-norms of \( u \). This interpolation involves terms such as \( \nu_j^T(u) \), \( \mu_3^T(u) \), \( \mu_2^T(u) \), or \( \mu_0^T(u) \). As a consequence, we cannot arbitrarily reduce the parameter \( \varepsilon \) to a very small value. On the other hand, in the treatment of \( \text{NL}_1 \) (see Section 4.8.1 below), we rely on Lemma 3.6, which imposes the condition that \( q > 1/(1/12 + \varepsilon - 2r) \). However, for the application of Lemma 3.7, we must also consider smaller values of \( q \). Therefore, we need to restrict the range of \( r \) to be within the interval \( 0 < r < R \lesssim 1/24 + \varepsilon/2 \). It is also crucial to mention that the parameters \( \theta \) and \( \alpha \) play a vital role in estimating \( \text{NL}_{3,2} \). See Section 4.8.3 for more details.

**Remark 4.2.** It is worth noting that the \( \nu_j^T \) terms were previously considered in [18, p. 585] for establishing the well-posedness in \( H^{1/12}(\mathbb{R}) \) in the unweighted case \( (r = 0) \). Therefore, we already know that
\[ \sum_{j=2}^7 \nu_j^T(\Phi(u)) \lesssim \|u_0\|_{H^{1/12}} + T^2 \left( \max_{j=2, \ldots, 7} \nu_j^T(u) \right)^4 \lesssim \|u_0\|_{H^{1/12+\varepsilon}} + T^2 \|u\|_{X_T}^4 \]
for certain \( \delta > 0 \).

### 4.1. Analysis of the \( L_T^\infty H_x^{1/12+\varepsilon} \)-norm of \( \Phi(u) \)

The Plancherel formula readily yields
\[ \|e^{-\partial_x^2} u_0\|_{L_T^\infty H_x^{1/12+\varepsilon}} \lesssim \|u_0\|_{H^{1/12+\varepsilon}} . \]

To control the nonlinear term, observe that (12) and the chain rule (6) guarantee
\[ \left\| \int_0^t e^{-(t-t')\partial_x^3} \partial_x(u^4) \ dt' \right\|_{L_T^\infty H_x^{1/12+\varepsilon}} \]
\[ \lesssim \left\| \partial_x \int_0^t e^{-(t-t')\partial_x^3} D_x^{1/12+\varepsilon}(u^4) \ dt' \right\|_{L_x^\infty L_t^2} + \left\| \partial_x \int_0^t e^{-(t-t')\partial_x^3} u^4 \ dt' \right\|_{L_x^\infty L_t^2} \]
\[ \lesssim \|D_x^{1/12+\varepsilon}(u^4)\|_{L_x^1 L_t^2} + \|u_4\|_{L_x^1 L_t^2} \]
\[ \lesssim \|u_4\|_{L_x^3 L_t^3} \|D_x^{1/12+\varepsilon} u\|_{L_x^1 L_t^1} + T^{1/10} \|u_4\|_{L_x^4 L_t^{10}}^4 , \]
provided that
\[ \frac{1}{r_1} + \frac{1}{p} + \frac{1}{q} = \frac{1}{s_1} + \frac{1}{q} . \]

On the other hand, estimate (21) implies
\[ \|D_x^{1/12+\varepsilon} u\|_{L_x^p L_t^q} \lesssim \|u\|_{L_x^{p_1} L_t^{q_1}} \|D_x^{1/12+\varepsilon} \partial_x u\|_{L_x^{p_0} L_t^{q_0}} \]
for
\[ \theta_0 := \frac{12}{13 + 12\varepsilon} , \quad \frac{1}{p} = \frac{\theta_0}{p_1} , \quad \frac{1}{q} = \frac{\theta_0}{q_1} + \frac{1 - \theta_0}{2} . \]
Now, if we set \( r_1 := 127/100, s_1 := 287/100 \) and take \( p, q, p_1, q_1 \) satisfying (28) and (29), it is possible to find by Lemma 3.1 the numbers \( \theta_1, \theta_1', \theta_2, \theta_2', \theta_2'' \in (0, 1) \) such that

\[
\|u\|_{L_{2}^{\infty}L_{2}^{2+1}} \lesssim T^{\theta_2/2}(\nu^T_s u(0))^{\theta_1} (\mu^T_s u(0))^{\theta_2} (\mu^T_{2s} u(0))^{1-\theta_1-\theta_2}, \\
\|u\|_{L_{2}^{0}L_{2}^{1+1}} \lesssim T^{\theta_2/2}(\nu^T_s u(0))^{\theta_1} (\mu^T_s u(0))^{\theta_2} (\mu^T_{2s} u(0))^{1-\theta_1-\theta_2}, \\
\|u\|_{L_{2}^{0}L_{2}^{1+0}} \lesssim T^{\theta_2/2}(\nu^T_s u(0))^{\theta_1} (\mu^T_s u(0))^{\theta_2} (\mu^T_{2s} u(0))^{1-\theta_1-\theta_2}.
\]

Consequently, all the above inequalities lead us to the following bound:

\[
\|\Phi(u)\|_{L_{T}^{p}H_{x}^{1/12+\varepsilon}} \lesssim \|u_0\|_{H_{x}^{1/12+\varepsilon}} + T^\delta \|u\|_{X_T}^4
\]

for some \( \delta > 0 \).

4.2. Analysis of \( \mu^T_s(\Phi(u)) \). Inequalities 9 and 3 yield

\[
\|e^{-t\partial_x^3}u_0\|_{L_{x}^{2}L_{T}^{2+2}} \lesssim T^{1/2}\|u_0\|_{H_{x}^{1/12+\varepsilon}}.
\]

Meanwhile, utilizing estimate 14 and the Hölder inequality, we obtain

\[
\left\| \int_0^t e^{-\left(t-t'\right)\partial_x^3} \partial_x(u^4)\, dt' \right\|_{L_{x}^{2}L_{T}^{2}} \lesssim T^{1/2}\left\| \int_0^t e^{-\left(t-t'\right)\partial_x^3} \partial_x(u^4)\, dt' \right\|_{L_{T}^{2}L_{x}^{2}} \\
\lesssim T^{1/2}\|u^3\|_{L_{x}^{6/7}L_{T}^{6/7}} \lesssim T^{1/2}\|u^3\|_{L_{x}^{8/7}L_{T}^{8/7}}\|\partial_x u\|_{L_{x}^{2}L_{T}^{2}}
\]

(31)

Moreover, an application Lemma 3.1 reveals that

\[
\|u\|_{L_{x}^{24/7}L_{T}^{8}} \lesssim T^{\theta_2/2}(\nu^T_s u(0))^{\theta_1} (\mu^T_s u(0))^{\theta_2} (\mu^T_{2s} u(0))^{1-\theta_1-\theta_2}
\]

for suitable \( \theta_1, \theta_2 \in (0, 1) \). Hence, we can find \( \delta > 0 \) that satisfies

\[
\mu^T_s(\Phi(u)) \lesssim \|u_0\|_{H_{x}^{1/12+\varepsilon}} + T^\delta \|u\|_{X_T}^4.
\]

4.3. Analysis of \( \mu^T_{2s}(\Phi(u)) \). Lemma 2.1 immediately leads to

\[
\|e^{-t\partial_x^3}u_0\|_{L_{x}^{120/(26+51\varepsilon)}L_{T}^{30/(2-3\varepsilon)}} \lesssim \|u_0\|_{H_{x}^{1/12+\varepsilon}}
\]

Applying Lemma 2.2 and utilizing the Leibniz rule for fractional derivatives, along with the inequality 6, we conclude that

\[
\left\| \int_0^t e^{-\left(t-t'\right)\partial_x^3} \partial_x(u^4)\, dt' \right\|_{L_{x}^{120/(26+51\varepsilon)}L_{T}^{30/(2-3\varepsilon)}} \\
\lesssim T^\delta\|D_{x}^{1/12+\varepsilon}(u^3\partial_x u)\|_{L_{x}^{10/(7-3\varepsilon)}L_{T}^{20/(23-12\varepsilon)}} \\
\lesssim T^\delta\left(\|u^3\|_{L_{x}^{12/7+\varepsilon}}\|\partial_x u\|_{L_{x}^{10/(7-3\varepsilon)}L_{T}^{20/(23-12\varepsilon)}} + \|\partial_x u\|_{L_{x}^{2}L_{T}^{2}}\|D_{x}^{1/12+\varepsilon}(u^3)\|_{L_{x}^{1}L_{T}^{1}}\right) \\
\lesssim T^\delta\left(\|u^3\|_{L_{x}^{10/(7-3\varepsilon)}L_{T}^{45/(4-6\varepsilon)}}\|D_{x}^{1/12+\varepsilon}(\partial_x u)\|_{L_{x}^{2}L_{T}^{2}} \\
\quad + \|\partial_x^2 u\|_{L_{x}^{2}L_{T}^{2}}\|u\|_{L_{T}^{1}L_{x}^{2}}\|D_{x}^{1/12+\varepsilon} u\|_{L_{x}^{2}L_{T}^{2}}\right),
\]

where

\[
7 - 3\varepsilon = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{23 - 12\varepsilon}{30} = \frac{1}{q_1} + \frac{1}{q_2}
\]

and

\[
\frac{1}{p_1} = \frac{1}{r_1} + \frac{1}{p}, \quad \frac{1}{q_1} = \frac{1}{s_1} + \frac{1}{q}.
\]
On the other hand, we obtain from (20) that
\[
\|\partial_x u\|_{L^p_x L^q_t} \lesssim \|u\|_{L^{p_3}_x L^{q_3}_t}^{\theta_1} \|D_x^{1/12+\varepsilon} \partial_x u\|_{L^2_x L^2_t}^{1-\theta_0},
\]
where
\[
(35) \quad \theta_0 := \frac{1 + 12\varepsilon}{13 + 12\varepsilon}, \quad \frac{1}{p_2} = \frac{\theta_0}{p_3}, \quad \frac{1}{q_2} = \frac{\theta_0}{q_3} + \frac{1 - \theta_0}{2}.
\]
We also have from (21) and Hölder’s inequality
\[
\|D_x^{1/12+\varepsilon} u\|_{L^p_x L^q_t} \lesssim \|u\|_{L^{p_4}_x L^{q_4}_t} \|D_x^{1/12+\varepsilon} \partial_x u\|_{L^2_x L^2_t} \leq T^3 \|u\|_{L^{p_1}_x L^{q_1}_t}^{\theta_1} \|\mu_T^T(u)\|_{L^{q_2}_t}^{1-\theta_0},
\]
where
\[
(36) \quad \theta_1' := \frac{12}{13 + 12\varepsilon}, \quad \frac{1}{p} = \frac{\theta_0'}{p_4}, \quad \frac{1}{q} = \frac{\theta_0'}{q_4} + \frac{1 - \theta_0'}{2}.
\]
Taking \( r_1 := 23/10, s_1 := 15/2, p_2 := 24, q_2 := 233/100 \) and \( p, q, p_1, q_1, p_3, q_3, p_4, q_4 \), satisfying \( (33), (34), (35) \) and \( (36) \), by Lemma 3.1 we find \( \theta_1, \theta_1', \theta_2, \theta_2', \theta_2'' \) such that
\[
\|u\|_{L^{30/(7-3\varepsilon)}_x L^{45/(4-6\varepsilon)}_t} \leq T^{\theta_2/2}(\nu_2^T(u))^{\theta_1} (\mu_2^T(u))^{\theta_2} (\mu_T^T(u))^{1-\theta_0-\theta_2},
\]
\[
\|u\|_{L^{2r_1}_x L^{2s_1}_t} \leq T^{\theta_2/2}(\nu_2^T(u))^{\theta_1} (\mu_2^T(u))^{\theta_2} (\mu_T^T(u))^{1-\theta_0-\theta_2},
\]
\[
\|u\|_{L^{p_3}_x L^{q_3}_t} \leq T^{\theta_2'/2}(\nu_3^T(u))^{\theta_1'} (\mu_3^T(u))^{\theta_2'} (\mu_T^T(u))^{1-\theta_0-\theta_2'},
\]
\[
\|u\|_{L^{p_4}_x L^{q_4}_t} \leq T^{\theta_2''/2}(\nu_3^T(u))^{\theta_1''} (\mu_3^T(u))^{\theta_2''} (\mu_T^T(u))^{1-\theta_0-\theta_2''}.
\]
Combining all the above, we conclude that
\[
(37) \quad \mu_T^T(\Phi(u)) \lesssim \|u_0\|_{H^{1/12+\varepsilon}} + T^{\delta} \|u\|_{X_T}^4,
\]
for some \( \delta > 0 \).

4.4. Analysis of \( \mu_T^T(\Phi(u)) \). By Hölder’s inequality and (11) (taken with \( \alpha = 0 \)) we obtain
\[
\|e^{-t\partial_x^3} u_0\|_{L^{6/\theta_0+\theta_0}_x L^{2/(1-\theta)_t}} \leq T^{\delta} \|e^{-t\partial_x^3} u_0\|_{L^{6/\theta_0}_x L^{2/(1-\theta)_t}} \lesssim T^\delta \|u_0\|_{L^2} \lesssim T^\delta \|u_0\|_{H^{1/12+\varepsilon}}.
\]
Moreover, inequality (14) with \( p_2 = q_2 = 8 \) gives
\[
\|\int_0^t e^{-((t-t')\partial_x^3)} \partial_x(u^4) dt'\|_{L^{6/\theta_0}_x L^{2/(1-\theta)_t}} \lesssim T^\delta \|\int_0^t e^{-((t-t')\partial_x^3)} \partial_x(u^4) dt'\|_{L^{6/\theta_0}_x L^{2/(1-\theta)_t}} \lesssim T^\delta \|u^3 \partial_x u\|_{L^{8/7}_x L^{8/7}_t},
\]
and using the same argument as in (31) we can conclude
\[
(38) \quad \mu_T^T(\Phi(u)) \lesssim \|u_0\|_{H^{1/12+\varepsilon}} + T^{\delta} \|u\|_{X_T}^4
\]
for some \( \delta > 0 \).
4.5. **Analysis of** $\mu_4^T(\Phi(u))$. By (11) and taking $\theta\alpha/2 < 1/12 + \varepsilon$ we have

$$
\left\| D_{t}^{\alpha} e^{-t\partial_0^2} u \right\|_{L_T^{6/(\theta\alpha+\theta)} L_x^{2/(1-\theta)}} \lesssim \left\| D_{t}^{\alpha/2} e^{-t\partial_0^2} D_{x}^{\alpha/2} u \right\|_{L_T^{6/(\theta\alpha+\theta)} L_x^{2/(1-\theta)}}
\lesssim \left\| D_{x}^{\alpha/2} u \right\|_{L_x^{2}} \lesssim \left\| u \right\|_{H^{1/12+\varepsilon}}.
$$

Inequality (11), the Minkowski inequality (since $6/(6-\theta\alpha-\theta) > 2/(1 + \theta)$ for the particular choice $\theta := 23/25$ and $\alpha := 41/100$), and the Hölder inequality imply

$$
\left\| D_{x}^{\alpha} \int_{0}^{t} e^{-(t-t')^{\alpha/2} \partial_0^2} D_{x}^{\alpha} u(t') \, dt' \right\|_{L_T^{6/(\theta\alpha+\theta)} L_x^{2/(1-\theta)}} \lesssim \left\| u^{3/2} \partial_0^3 u \right\|_{L_T^{6/(6-\theta\alpha-\theta)} L_x^{2/(1+\theta)}} \lesssim \left\| u \right\|_{L_{x}^{3/2}, L_x^{3}} \left\| \partial_0^3 u \right\|_{L_T^{1/2}, L_x^{1}}
$$

for

$$
\frac{1 + \theta}{2} = \frac{1}{p_1} + \frac{1}{q}, \quad \frac{6-\theta\alpha-\theta}{6} = \frac{1}{s_1} + \frac{1}{q}.
$$

Next, estimate (20) produces

$$
\left\| \partial_0^3 u \right\|_{L_T^{1/2}, L_x^{1}} \lesssim \left\| u \right\|_{L_{x}^{3/2}, L_x^{3}} \left\| D_{x}^{1/12+\varepsilon} \partial_0^3 u \right\|_{L_T^{1/2}, L_x^{1/2}}
$$

for

$$
\theta_0 := \frac{1 + 12\varepsilon}{13 + 12\varepsilon}, \quad \frac{1}{p} = \frac{\theta_0}{p_1}, \quad \frac{1}{q} = \frac{\theta_0}{q_1} + \frac{1 - \theta_0}{2}.
$$

Finally, if we set $r_1 := 1111/1000$, $s_1 := 299/100$ and take $p, q, p_1, q_1$ satisfying (39) and (40), by Lemma 3.1 it is possible to find $\theta_1', \theta_2, \theta_2' \in (0, 1)$ such that

$$
\left\| u \right\|_{L_{x}^{3/2}, L_x^{3}} \lesssim T^{\theta_2'/2} \left\| \mu_2^T(u) \right\|_{L_{x}^{3/2}, L_x^{3}} \lesssim T^{\theta_2'/2} \left\| \mu_2^T(u) \right\|_{L_{x}^{3/2}, L_x^{3}}
$$

Hence, there exists $\delta > 0$ such that

$$
\mu_4^T(\Phi(u)) \lesssim \left\| u \right\|_{H^{1/12+\varepsilon}} + T^{\delta} \left\| u \right\|_{H^{4/7}}.
$$

4.6. **Analysis of** $\mu_5^T(\Phi(u))$. Inequality (10) leads to

$$
\left\| \partial_0^3 D_{x}^{1/12+\varepsilon} u \right\|_{L_T^{1/2}, L_x^{1/2}} \lesssim \left\| D_{x}^{1/12+\varepsilon} u \right\|_{L_T^{1/2}, L_x^{1/2}} \lesssim \left\| u \right\|_{H^{1/12+\varepsilon}}.
$$

Moreover, by (13)

$$
\left\| \partial_0^2 \int_{0}^{t} e^{-(t-t')^{\alpha/2} \partial_0^2} D_{x}^{1/12+\varepsilon}(u^4) \, dt' \right\|_{L_T^{1/2}, L_x^{1/2}} \lesssim \left\| D_{x}^{1/12+\varepsilon}(u^4) \right\|_{L_T^{1/2}, L_x^{1/2}},
$$

and proceeding as in (27) above we can get

$$
\mu_5^T(\Phi(u)) \lesssim \left\| u \right\|_{H^{1/12+\varepsilon}} + T^{\delta} \left\| u \right\|_{H^{4/7}}
$$

for certain $\delta > 0$. 
4.7. Analysis of \( \mu_6^T(\Phi(u)) \). Lemma 2.1 immediately leads to
\[
\|e^{-t\partial^2_x}u_0\|_{L_{x,t}^{60/(13-12\epsilon)}L_{x,t}^{15/(1-4\epsilon)}} \lesssim \|u_0\|_{H^{1/12+\epsilon}}.
\]
Applying Lemma 2.2, we have
\[
\left\| \int_0^t e^{-\sigma(t-t')} \partial_x^3 u(t') \, dt' \right\|_{L_{x,t}^{60/(13-12\epsilon)}L_{x,t}^{15/(1-4\epsilon)}} \\
\lesssim T^5 \left( \|u\|_{L_{x,t}^{10/(7-8\epsilon)}L_{x,t}^{30/(23-32\epsilon)}} + \|D_x^{1/12+\epsilon} u\|_{L_{x,t}^{10/(7-8\epsilon)}L_{x,t}^{30/(23-32\epsilon)}} \right).
\]
Next, the Hölder inequality yields to
\[
\|u^3 \partial_x u\|_{L_{x,t}^{10/(7-8\epsilon)}L_{x,t}^{30/(23-32\epsilon)}} \\
\lesssim \|u\|_{L_{x,t}^{10/(7-8\epsilon)}L_{x,t}^{15/(4-16\epsilon)}} \|\partial_x u\|_{L_{x,t}^{\infty}L_{x,t}^1} \\
\lesssim \|u\|_{L_{x,t}^{10/(7-8\epsilon)}L_{x,t}^{45/(4-16\epsilon)}} \|D_x^{1/12+\epsilon} u\|_{L_{x,t}^{15/(4-16\epsilon)}L_{x,t}^1} \|\partial_x u\|_{L_{x,t}^{15/(4-16\epsilon)}L_{x,t}^1} \|D_x^{1/12+\epsilon} u\|_{L_{x,t}^{45/(4-16\epsilon)}L_{x,t}^1},
\]
Furthermore, utilizing the Leibniz rule for fractional derivatives, along with the inequality [6], we conclude that
\[
\|D_x^{1/12+\epsilon} (u^3 \partial_x u)\|_{L_{x,t}^{10/(7-8\epsilon)}L_{x,t}^{30/(23-32\epsilon)}} \\
\lesssim \|u^3 D_x^{1/12+\epsilon} \partial_x u\|_{L_{x,t}^{10/(7-8\epsilon)}L_{x,t}^{30/(23-32\epsilon)}} + \|\partial_x u\|_{L_{x,t}^{p_2}L_{x,t}^{q_2}} \|D_x^{1/12+\epsilon} (u^3)\|_{L_{x,t}^{p_1}L_{x,t}^{q_1}} \\
\lesssim \|u\|_{L_{x,t}^{30/(7-8\epsilon)}L_{x,t}^{45/(4-16\epsilon)}} \|D_x^{1/12+\epsilon} \partial_x u\|_{L_{x,t}^{15/(4-16\epsilon)}L_{x,t}^1} + \|\partial_x u\|_{L_{x,t}^{p_2}L_{x,t}^{q_2}} \|u\|_{L_{x,t}^{15/(4-16\epsilon)}L_{x,t}^1} \|D_x^{1/12+\epsilon} u\|_{L_{x,t}^{45/(4-16\epsilon)}L_{x,t}^1},
\]
provided that
\[
7 - 8\epsilon = \frac{1}{p_1} + \frac{1}{p_2} \quad \frac{23 - 32\epsilon}{30} = \frac{1}{q_1} + \frac{1}{q_2}.
\]
and
\[
1 = \frac{1}{p_1} + \frac{1}{r_1} = \frac{1}{s_1} = \frac{1}{q}.
\]
On the other hand, we obtain from [20] and the Hölder inequality that
\[
\|\partial_x u\|_{L_{x,t}^{p_2}L_{x,t}^{q_2}} \lesssim \|u\|_{L_{x,t}^{p_3}L_{x,t}^{q_3}} \|D_x^{1/12+\epsilon} \partial_x u\|_{L_{x,t}^{15/(4-16\epsilon)}L_{x,t}^1} \lesssim T^5 \|u\|_{L_{x,t}^{10/p_3}L_{x,t}^{10/q_3}} \|D_x^{1/12+\epsilon} \partial_x u\|_{L_{x,t}^{15/(4-16\epsilon)}L_{x,t}^1},
\]
where
\[
\theta_0 := 1 + \frac{12\epsilon}{13 + 12\epsilon}, \quad \frac{1}{p_2} = \frac{\theta_0}{q_3}, \quad \frac{1}{q_2} = \frac{\theta_0}{q_3} + \frac{1}{q_2}.
\]
We also have from [21] that
\[
\|D_x^{1/12+\epsilon} u\|_{L_{x,t}^{p_2}L_{x,t}^{q_2}} \lesssim \|u\|_{L_{x,t}^{p_3}L_{x,t}^{q_3}} \|D_x^{1/12+\epsilon} \partial_x u\|_{L_{x,t}^{15/(4-16\epsilon)}L_{x,t}^1},
\]
for
\[
\theta_0' := \frac{12}{13 + 12\epsilon}, \quad \frac{1}{p} = \frac{\theta_0'}{p_2}, \quad \frac{1}{q} = \frac{\theta_0'}{q_2} + \frac{1}{q_2}.
\]
Taking \( r_1 := 63/25, s_1 := 27/2, p_2 := 27, q_2 := 23/10 \) and \( p, q, p_1, q_1, p_3, q_3, p_4, q_4 \), satisfying [43], [44], [45] and [46], by Lemma 3.1 we find \( \theta_1, \theta_1', \theta_1'', \theta_2, \theta_2', \theta_2'', \theta''_2 \) in (0, 1) such that
\[
\|u\|_{L_{x,t}^{45/(4-16\epsilon)}L_{x,t}^{15/(4-16\epsilon)}} \lesssim \left( \nu_3^T (u) \right)^{\theta_1} \left( \mu_2^T (u) \right)^{\theta_2} \left( \mu_6^T (u) \right)^{1-\theta_1-\theta_2},
\]
\[
\|u\|_{L_{x,t}^{15/(4-16\epsilon)}L_{x,t}^{15/(4-16\epsilon)}} \lesssim \left( \nu_3^T (u) \right)^{\theta_1'} \left( \mu_2^T (u) \right)^{\theta_2'} \left( \mu_6^T (u) \right)^{1-\theta_1'-\theta_2'},
\]
\[
\|u\|_{L_{x,t}^{21/(4-16\epsilon)}L_{x,t}^{21/(4-16\epsilon)}} \lesssim \left( \nu_3^T (u) \right)^{\theta_1''} \left( \mu_2^T (u) \right)^{\theta_2''} \left( \mu_6^T (u) \right)^{1-\theta_1''-\theta_2''}.
\]
Combining all the above, we conclude that
\begin{equation}
\mu^T_6(\Phi(u)) \lesssim \|u_0\|_{H^{1/12+\varepsilon}} + T^\delta \|u\|_{\tilde{X}}^4.
\end{equation}

4.8. Analysis of the $L^\infty_T L^2_x(|x|^{2r})$-norm of $\Phi(u)$. First, we note that
\[
\|x|^r \Phi(u)\|_{L^\infty_T L^2_x} \leq ||x|^r e^{-t\partial_x u} u_0\|_{L^\infty_T L^2_x} + \|x|^r \int_0^t e^{-(t-t')\partial_x u} u(t') \, dt'\|_{L^\infty_T L^2_x} =: L + NL.
\]
The linear term $L$ can be controlled by (17), (18) and the Plancherel formula (9)
\[
L \lesssim \|x|^r u_0\|_{L^2} + \|\partial_t u_0\|_{L^2} + \|D_x^{2r} u_0\|_{L^2} \lesssim \|x|^r u_0\|_{L^2} + \|u_0\|_{H^{1/12+\varepsilon}},
\]
where we used (3) and the fact $T \lesssim 1$.

To treat the nonlinear factor $NL$, we decompose it as follows:
\[
NL \sim \left\| \int_0^t e^{i(t-t')\xi^3} (\partial_x (u^4)^\wedge (\xi, t')) \, dt' \right\|_{L^\infty_T L^2_x}.
\]

\[
= \left\| \int_0^t D_x^t \left\{ e^{i(t-t')\xi^3} \langle \xi \rangle^{1/12+\varepsilon} (\partial_x (u^4)^\wedge (\xi, t')) \right\} \, dt' \right\|_{L^\infty_T L^2_x}.
\]
\[
\leq \left\| \int_0^t e^{i(t-t')\xi^3} \langle \xi \rangle^{1/12+\varepsilon} (\partial_x (u^4)^\wedge (\xi, t')) \, dt' \right\|_{L^\infty_T L^2_x}
+ \left\| \int_0^t e^{i(t-t')\xi^3} \langle \xi \rangle^{1/12-\varepsilon} (\partial_x (u^4)^\wedge (\xi, t')) \, dt' \right\|_{L^\infty_T L^2_x}
+ \left\| \int_0^t e^{i(t-t')\xi^3} (\partial_x (u^4)^\wedge (\xi, t')) \, dt' \right\|_{L^\infty_T L^2_x}
=: NL_1 + NL_2 + NL_3.
\]

We investigate each of the above terms in the next subsections.

4.8.1. Commutator estimate for $NL_1$. Let $r \in (0, R)$, for $R := 51/1000$. By Lemma 3.5, Lemma 3.6 and the Minkowski inequality we deduce
\[
NL_1 \leq \int_0^T \left\| \int e^{i(t-t')\xi^3} \left\{ \langle \xi \rangle^{1/12+\varepsilon} (\partial_x (u^4)^\wedge (\xi, t')) \right\} \, dt' \right\|_{L^\infty_T L^2_x} \, dt'
\lesssim \int_0^T \left\| D_x^t \left\{ e^{i(t-t')\xi^3} \langle \xi \rangle^{1/12+\varepsilon} (\partial_x (u^4)^\wedge (\xi, t')) \right\} \, dt' \right\|_{L^\infty_T L^2_x} \, dt'
\lesssim (1 + T^\delta) \left( \|\partial_t (u^4)\|_{L^1_T L^p_x} + \|D_x^{1/12+\varepsilon} \partial_x (u^4)\|_{L^1_T L^p_x} \right)
\lesssim T^\delta \left( \|u^3 \partial_x u\|_{L^p_T L^r_x} + \|D_x^{1/12+\varepsilon} (u^3 \partial_x u)\|_{L^p_T L^r_x} \right)
\]

for some $\delta > 0$ and $1 \leq p \leq 2$ such that
\begin{equation}
\frac{1}{2} = \frac{1}{p} - \frac{1}{q} \quad \text{and} \quad q > \frac{1}{12 + \varepsilon - 2r}.
\end{equation}

The Hölder inequality yields to
\[
\|u^3 \partial_x u\|_{L^p_T L^r_x} \lesssim \|u^3\|_{L^{p_0}_x L^{p_0}_T} \|\partial_x u\|_{L^\infty_T L^2_x} \lesssim \|u\|_{L^{3\cdot p_0}_x L^{3\cdot p_0}_T}^\delta \mu^T_6(u)
\]

provided that
\begin{equation}
\frac{1}{p} = \frac{1}{p_1}, \quad \frac{1}{p} = \frac{1}{q_1} + \frac{1}{2}.
\end{equation}
Furthermore, inequalities \[4\] and \[6\] give
\begin{align*}
\|D_x^{1/12+\varepsilon}(u^3 \partial_x u)\|_{L^p_t L^q_x} & \lesssim \|u\|_{L_t^p L^q_x}^3 \|D_x^{1/12+\varepsilon} \partial_x u\|_{L_t^p L^q_x} + \|\partial_x u\|_{L_t^p L^q_x} \|D_x^{1/12+\varepsilon}(u^3)\|_{L_t^p L^q_x} \\
& \lesssim \|u\|_{L_t^{p_1} L_x^{q_1}}^3 \mu_5^T(u) + \|\partial_x u\|_{L_t^{p_2} L_x^{q_2}} \|u^2\|_{L_t^{p_4} L_x^{q_4}} \|D_x^{1/12+\varepsilon} u\|_{L_t^{q_5} L_x^{p_5}} \\
& \lesssim \|u\|_{L_t^{p_1} L_x^{q_1}}^3 \mu_5^T(u) + \|\partial_x u\|_{L_t^{p_2} L_x^{q_2}} \|u\|_{L_t^{p_4} L_x^{q_4}}^2 \|D_x^{1/12+\varepsilon} u\|_{L_t^{q_5} L_x^{p_5}},
\end{align*}
provided that
\begin{equation}
\frac{1}{p} = \frac{1}{p_2} + \frac{1}{p_3}, \quad \frac{1}{p} = \frac{1}{q_2} + \frac{1}{q_3}, \quad \frac{1}{p_3} = \frac{1}{p_4} + \frac{1}{p_5}, \quad \frac{1}{q_3} = \frac{1}{q_4} + \frac{1}{q_5}.
\end{equation}
Next, we employ Lemma \[3.2\] to get
\begin{equation}
\|\partial_x u\|_{L_t^{p_2} L_x^{q_2}} \lesssim \|u\|_{L_t^{p_1} L_x^{q_1}}^{\theta_0} \|D_x^{1/12+\varepsilon} \partial_x u\|_{L_t^{p_4} L_x^{q_4}} \lesssim \|u\|_{L_t^{p_1} L_x^{q_1}}^{\theta_0} (\mu_5^T(u))^{1-\theta_0}
\end{equation}
for
\begin{equation}
\theta_0 := \frac{1 + 12\varepsilon}{13 + 12\varepsilon}, \quad \frac{1}{p_2} = \frac{\theta_0}{r_1}, \quad \frac{1}{q_2} = \frac{\theta_0}{s_1} + \frac{1}{2} - \theta_0
\end{equation}
and
\begin{equation}
\|D_x^{1/12+\varepsilon} u\|_{L_t^{q_5} L_x^{p_5}} \lesssim \|u\|_{L_t^{p_2} L_x^{q_2}}^{\theta_0} \|D_x^{1/12+\varepsilon} \partial_x u\|_{L_t^{p_4} L_x^{q_4}} \lesssim T^\delta \|u\|_{L_t^{p_2} L_x^{q_2}}^{\theta_0} (\mu_5^T(u))^{1-\theta_0},
\end{equation}
provided that
\begin{equation}
\theta_0' := \frac{12}{13 + 12\varepsilon}, \quad \frac{1}{p_5} = \frac{\theta_0'}{r_2}, \quad \frac{1}{q_5} = \frac{\theta_0'}{s_2} + \frac{1}{2} - \theta_0'.
\end{equation}
Take \(q := 15000/1619\), \(p_1 := 3p_1/2\), \(q_1 := 3q_1/2\), \(r_2 := 49/10\), \(s_2 := 55/2\) and \(p_1, q_1, p_3, q_3, p_5, q_5, r_1, s_1\), satisfying \[49\]-\[53\]. Then, we can apply Lemma \[3.1\] to obtain \(\theta_1, \theta_1', \theta_2, \theta_2', \theta_2'' \in (0, 1)\) such that
\begin{align*}
\|u\|_{L_t^{p_1} L_x^{q_1}} & \lesssim (\nu_3^T(u))^\theta_1 (\mu_2^T(u))^{\theta_3} (\mu_4^T(u))^{1-\theta_1-\theta_2}, \\
\|u\|_{L_t^{p_1} L_x^{q_1}} & \lesssim (\nu_3^T(u))^\theta_1 (\mu_2^T(u))^{\theta_3} (\mu_4^T(u))^{1-\theta_1-\theta_2}, \\
\|u\|_{L_t^{p_2} L_x^{q_2}} & \lesssim (\nu_3^T(u))^{\theta_2'} (\mu_2^T(u))^{\theta_2''} (\mu_4^T(u))^{1-\theta_1-\theta_2}. 
\end{align*}
Summarizing, we get
\begin{equation}
NL_1 \lesssim T^\delta \|u\|_{L_t^{p_1} L_x^{q_1}}^3
\end{equation}
for some \(\delta > 0\).

4.8.2. Commutator estimate for \(NL_2\). Let \(r \in (0, 1/24 + \varepsilon/2]\) and for simplicity, denote
\[f(\xi, t') := (\xi)^{1/12+\varepsilon} (\partial_x(u^4))^{\langle \xi, t' \rangle}.
\]
By using property \[14\], Lemma \[3.4\] and the Hölder and Minkowski inequalities we obtain
\begin{align*}
NL_2 & \sim \left\| \int_0^t e^{i(t-t')\partial_x^2} \left( (\langle \xi \rangle^{-1/12-\varepsilon}, D_x^2[f])^{\langle \xi, t' \rangle} (x, t') dt' \right) \right\|_{L_t^{p_1} L_x^{q_1}} \\
& \lesssim \| \{(\langle \xi \rangle^{-1/12-\varepsilon}, D_x^2[f])^{\langle \xi, t' \rangle} \|_{L_t^{p_1} L_x^{q_1}} \lesssim \| \{f\}_{L_t^{p_1} L_x^{q_1}} \|_{L_t^{p_1} L_x^{q_1}} 
\end{align*}
\[
\begin{align*}
\lesssim \| \partial_x (u^4) \|_{L^{q_2'}_x L^p_y} & + \| D_x^{1/12 + \varepsilon} \partial_x (u^4) \|_{L^{q_2'}_x L^p_y} \\
& \lesssim T^8 \left( \| u^3 \partial_x u \|_{L^{p_2}_x L^{q_2}_y} + \| D_x^{1/12 + \varepsilon} (u^3 \partial_x u) \|_{L^{p_2}_x L^{q_2}_y} \right),
\end{align*}
\]

(55)

provided that

\[
\begin{align*}
p_2 & \geq 2, \quad \frac{1}{q_2} = \frac{1}{6} - \frac{3}{3p_2}, \quad \frac{1}{p_2} + \frac{1}{p_2'} = 1 = \frac{1}{q_2} + \frac{1}{q_2'}, \\
\frac{1}{p_2'} + 1 & = \frac{1}{p_1} + \frac{1}{p'}, \quad 1 \leq p_1 \leq \frac{1}{12} - r - \varepsilon
\end{align*}
\]

and \(q_2' + d \geq p\).

The first term in (55) can be controlled via the Hölder inequality and (20) by

\[
\begin{align*}
\| u^3 \partial_x u \|_{L^{p_2}_x L^{q_2}_y} & \lesssim \| u^3 \|_{L^{p_0}_x L^{q_0}_y} \| \partial_x u \|_{L^{p_2}_x L^{q_2}_y} \\
& \lesssim T^8 \| u \|_{L^{p_0}_x L^{q_0}_y}^3 \| \partial_x u \|_{L^{p_2}_x L^{q_2}_y} \cdot \| D_x^{1/12 + \varepsilon} \partial_x u \|_{L^{p_0}_x L^{q_0}_y}^{1 - \theta_0'},
\end{align*}
\]

for

\[
\begin{align*}
p & = \frac{1}{p_0} + \frac{1}{r_0}, \quad \frac{1}{p_0} = \frac{1}{s_0} + \frac{1}{s_1}
\end{align*}
\]

and

\[
\theta_0' = \frac{1 + 12\varepsilon}{13 + 12\varepsilon}, \quad \frac{1}{r_0} = \frac{\theta_0'}{r_2}, \quad \frac{1}{s_1} = \frac{\theta_0'}{s_2} + \frac{1 - \theta_0'}{2}.
\]

As for the second term in (55) we invoke (4) and (6),

\[
\begin{align*}
\| D_x^{1/12 + \varepsilon} (u^3 \partial_x u) \|_{L^{p_2}_x L^{q_2}_y} & \lesssim \| u^3 \|_{L^{p_0}_x L^{q_0}_y} \| \partial_x u \|_{L^{p_2}_x L^{q_2}_y} \cdot \| D_x^{1/12 + \varepsilon} (u^3) \|_{L^{p_2}_x L^{q_2}_y} \\
& \lesssim \| u \|^3_{L^{p_0}_x L^{q_0}_y} \mu_T(u) + \| \partial_x u \|_{L^{p_2}_x L^{q_2}_y} \cdot \| D_x^{1/12 + \varepsilon} (u^3) \|_{L^{p_0}_x L^{q_0}_y} \\
& \lesssim \| u \|^3_{L^{p_0}_x L^{q_0}_y} \mu_T(u) + \| \partial_x u \|_{L^{p_2}_x L^{q_2}_y} \cdot \| u \|_{L^{p_0}_x L^{q_0}_y}^2 \cdot \| D_x^{1/12 + \varepsilon} u \|_{L^{p_0}_x L^{q_0}_y} \cdot \| D_x^{1/12 + \varepsilon} u \|_{L^{p_0}_x L^{q_0}_y},
\end{align*}
\]

(59)

where

\[
\begin{align*}
\frac{1}{q_2'} + d & = \frac{1}{q_2} + \frac{1}{2}, \quad \frac{1}{p} = \frac{1}{p_4} + \frac{1}{p_5}, \quad \frac{1}{p_2'} + 1 & = \frac{1}{p_4} + \frac{1}{q_5},
\end{align*}
\]

and

\[
\begin{align*}
\frac{1}{q_5} & = \frac{1}{r_3} + \frac{1}{p_7}, \quad \frac{1}{p_5} = \frac{1}{s_3} + \frac{1}{q_7}.
\end{align*}
\]

Furthermore, we can bound some of the factors in (59) by means of (20) and (21) as follow

\[
\begin{align*}
\| \partial_x u \|_{L^{p_0}_x L^{q_0}_y} & \lesssim \| u \|^{	heta_0'}_{L^{p_0}_x L^{q_0}_y} \cdot \| D_x^{1/12 + \varepsilon} \partial_x u \|_{L^{p_0}_x L^{q_0}_y}^{1 - \theta_0'},
\end{align*}
\]

and

\[
\begin{align*}
\| D_x^{1/12 + \varepsilon} u \|_{L^{p_0}_x L^{q_0}_y} & \lesssim \| u \|^\theta_0_{L^{p_0}_x L^{q_0}_y} \cdot \| D_x^{1/12 + \varepsilon} \partial_x u \|_{L^{p_0}_x L^{q_0}_y}^{1 - \theta_0'},
\end{align*}
\]

where

\[
\begin{align*}
\frac{1}{p_4} = \frac{\theta_0'}{p_0}, \quad \frac{1}{q_4} = \frac{\theta_0'}{q_0} + \frac{1}{\theta_0'} - \frac{2}{2}.
\end{align*}
\]

(62)
and

\begin{equation}
\theta_0 := \frac{12}{13 + 12\varepsilon}, \quad \frac{1}{p_T} = \frac{\theta_0}{p_8}, \quad \frac{1}{q_T} = \frac{\theta_0}{q_8} + \frac{1 - \theta_0}{2}.
\end{equation}

If we set \( p_2 := 233/50, p_1 := 1 \), then by (56) one can easily compute \( p, q_2 \). And we choose \( d := 17/100 \) such that \( q_2 + d \geq p \). Also taking \( r_0 := 7/5, s_0 := 16/5, r_3 := 19/10, s_3 := 26/5, p_4 := 17, q_4 := 11/5 \) and \( r_i, s_i = 1 \) for \( i = 1, 2, 3 \), \( j = 5, \ldots, 8 \), satisfying \((57), (58), (60), (61), (62)\) and \((63)\), by Lemma 3.1 we find \( \theta_1, \theta'_1, \theta''_1, \theta_2, \theta'_2, \theta_3, \theta_4, \theta''_3, \theta''_4 \in (0, 1) \) such that

\begin{align*}
\|u\|_{L_x^8 L_t^6} & \lesssim T^{\theta_1/2}(\nu_3^T(u))^{\theta_1} (\mu_1^T(u))^{\theta_2} (\mu_2^T(u))^{1-\theta_1-\theta_2}, \\
\|u\|_{L_x^8 L_t^6} & \lesssim T^{\theta_2/2}(\nu_3^T(u))^{\theta_2} (\mu_1^T(u))^{\theta_3} (\mu_2^T(u))^{1-\theta_1-\theta_2}, \\
\|u\|_{L_x^8 L_t^6} & \lesssim T^{\theta_3/2}(\nu_3^T(u))^{\theta_3} (\mu_1^T(u))^{\theta_4} (\mu_2^T(u))^{1-\theta_1-\theta_2}, \\
\|u\|_{L_x^8 L_t^6} & \lesssim T^{\theta_4/2}(\nu_3^T(u))^{\theta_4} (\mu_1^T(u))^{\theta_5} (\mu_2^T(u))^{1-\theta_1-\theta_2}, \end{align*}

Combining all the above, we conclude that

\begin{equation}
NL_2 \lesssim T^\delta \|u\|_{X_T}^4
\end{equation}

for some \( \delta > 0 \).

4.8.3. Boundedness of \( NL_3 \). First we observe that by the Plancherel theorem that

\begin{equation}
NL_3 \sim \| \int_0^t e^{-\langle t-t' \rangle \partial_x^3 \{ \langle x \rangle^2 \partial_x (u^4)(x, t') \} } dt' \|_{L_x^\infty L_t^2}.
\end{equation}

By choosing \( \varphi \in C_0^\infty(\mathbb{R}) \) such that \( \varphi = 1 \) when \( |x| < 1/2 \), while \( \varphi = 0 \) when \( |x| \geq 1 \), we have

\[ |x|^2 \partial_x (u^4) = \varphi(x) |x|^2 \partial_x (u^4) + \partial_x ((1 - \varphi(x))|x|^2 u^4) - \partial_x ((1 - \varphi(x))|x|^4) u^4. \]

Hence,

\begin{align*}
NL_3 & \lesssim \left\| \int_0^t e^{-\langle t-t' \rangle \partial_x^3 \{ \varphi(x) |x|^2 \partial_x (u^4) \} } dt' \right\|_{L_x^\infty L_t^2} \\
& \quad + \left\| \int_0^t e^{-\langle t-t' \rangle \partial_x^3 \{ (1 - \varphi(x)) |x|^2 u^4 \} } dt' \right\|_{L_x^\infty L_t^2} \\
& \quad + \left\| \int_0^t e^{-\langle t-t' \rangle \partial_x^3 \{ \partial_x ((1 - \varphi(x)) |x|^4) u^4 \} } dt' \right\|_{L_x^\infty L_t^2} \\
& =: NL_{3,1} + NL_{3,2} + NL_{3,3}.
\end{align*}

Analysis of \( NL_{3,1} \). Inequality (14) with \( p_2 = q_2 = 8 \) leads to the same estimate as in (31) above:

\begin{equation}
NL_{3,1} \lesssim \| \varphi(x) |x|^2 \partial_x (u^4) \|_{L_x^8 L_t^{5/7}} \lesssim \| u^3 \partial_x u \|_{L_x^{5/7} L_t^{5/7}} \lesssim \| u^3 \partial_x u \|_{L_x^{5/7} L_t^{5/7}}.
\end{equation}

Thus, repeating the argument for the nonlinear part of \( \mu_1^T(\Phi(u)) \), we conclude

\begin{equation}
NL_{3,1} \lesssim T^\delta \|u\|_{X_T}^4
\end{equation}

for some \( \delta > 0 \).
Analysis of NL3.3. By using (14) with $q_2 = 90/11$, $p_2 = 15/2$ and $q'_2 = 90/79$, $p'_2 = 15/13$ and then the Minkowski inequality, we obtain

$$\text{NL}(66) \quad \text{NL3.3} \lesssim \|u^4||_{L_T^{90/79}L_x^{15/13}} \lesssim T^\delta \|u^4||_{L_T^{15/4}L_x^{15/13}} \lesssim T^\delta \|u||_{L_T^{15}L_x^{60/13}}^4 \leq T^\delta (v_3^T(u))^4,$$

since

$$\|\partial_x (|x|^r (1 - \varphi(x)))\|_{L_x^\infty} \lesssim 1, \quad 0 < r < 1.$$ It follows that

$$\text{NL}(66) \quad \text{NL3.3} \lesssim T^\delta \|u\|_{X_T}^4$$

for some $\delta > 0$.

Analysis of NL3.2. An application of (12), the Hölder inequality and the Minkowski inequality produce

$$\text{NL3.2} = \|\partial_x \int_0^t e^{-(t-t')\sigma_3^2} (1 - \varphi(x)|x|^r u^4) \, dt'\|_{L_T^\infty L_x^2} \lesssim \||x|^r u^4||_{L_T^\infty L_x^2} \lesssim \|||x|^r u^4||_{L_T^\infty L_x^2}$$

$$\lesssim \||x|^r u^c||_{L_T^{p_1}L_x^{q_1}} \|u^b||_{L_T^{p_2}L_x^{q_2}} \lesssim \|||x|^r u^c||_{L_T^{p_1}L_x^{q_1}} \|u||_{L_T^{p_2}L_x^{q_2}}^b$$

with $p_1 \geq q_1$ and

$$\text{NL}(67) \quad \text{NL3.2} \lesssim T^\delta \|u\|_{X_T}^4$$

Next, we apply Lemma 3.3

$$\|\|x|^r u^c||_{L_T^{p_1}L_x^{q_1}} \lesssim T^\delta \left(\||D_x^\theta a^\alpha||_{L_T^{p_1}L_x^{q_1}} + \||u^a||_{L_T^{p_1}L_x^{q_1}}\right) \|\|x|^r u^c||_{L_T^{p_1}L_x^{q_1}} \lesssim T^\delta \left(\|\mu_3^T(u)^a + (\mu_1^T(u))^a\| \|\|x|^r u^c||_{L_T^{p_1}L_x^{q_1}} \right)^{1-a}$$

for

$$\text{NL}(68) \quad \text{NL3.2} \lesssim T^\delta \|u\|_{X_T}^4$$

Recall the choices of $\varepsilon$, $\theta$ and $\alpha$ given by (25). For any $r \in (0,1/24 + \varepsilon/2]$, by taking $a := 69/200$ and $c := 153/100$ one can easily compute $p_1, q_1, \sigma$ from (68) satisfying $p_1 \geq q_1$ and $\sigma < 0$. Furthermore, for $b, p_2, q_2$ verifying (67), Lemma 3.1 gives

$$\|\|u\|_{L_T^{p_2}L_x^{q_2}} \lesssim T^{\theta_2/2}(\nu_3^T(u)^{\theta_1}(\mu_3^T(u)^{\theta_2}(\mu_1^T(u)^{1-\theta_1-\theta_2})$$

for certain $\theta_1, \theta_2 \in (0, 1)$. Hence

$$\text{NL}(69) \quad \text{NL3.2} \lesssim T^\delta \|u\|_{X_T}^4$$

for some $\delta > 0$.

4.9. Contraction mapping. Let $u \in X_T^p$. Collecting (26), (30), (32), (37), (38), (41), (42), (47), (48), (54), (64), (65), (66) and (69) we get

$$\|\Phi(u)\|_{X_T} \leq C \left(\|u_0\|_{H^{1/12+\varepsilon}} + \|||x|^r u_0||_{L_x^2}\right) + CT^\delta \rho^4,$$

for some positive constants $C$ and $\delta$. Thus, by taking

$$\rho := 2(\|u_0\|_{H^{1/12+\varepsilon}} + \|||x|^r u_0||_{L_x^2})$$
and choosing $T > 0$ sufficiently small such that
\begin{equation}
\frac{\rho}{2} + C T^\delta \rho^4 \leq \rho, \tag{70}
\end{equation}
we can justify that $\Phi(u) \in X^\rho_T$, that is $\Phi : X^\rho_T \rightarrow X^\rho_T$ is a well-defined mapping.

Furthermore, for $u, v \in X^\rho_T$ we observe that
\[ \Phi(u) - \Phi(v) = \int_0^t e^{-(t-t')\partial_x^3} \partial_x (u^4 - v^4) \, dt', \]
and since
\begin{align*}
\partial_x (u^4 - v^4) &= \partial_x ((u - v)((u^3 + u^2 v + uv^2 + v^3)) \\
&= \partial_x(u - v)(u^3 + u^2 v + uv^2 + v^3) \\
&+ (u - v)(3u^2 \partial_x u + 2uv \partial_x v + u^2 \partial_x v + 2uv \partial_x v + 3v^2 \partial_x v),
\end{align*}
on one can essentially proceed as above to deduce that
\[ \|\Phi(u) - \Phi(v)\|_{X^\rho_T} \leq C' T^\delta \rho^3 \|u - v\|_{X^\rho_T}, \]
for certain $C' > 0$. Therefore, if we pick $T > 0$ satisfying simultaneously $C' T^\delta \rho^3 < 1$ and (70) we can guarantee that $\Phi$ is a contraction.

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