Loewner driving functions for off-critical percolation clusters

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We numerically study the Loewner driving function $U_t$ of a site percolation cluster boundary on the triangular lattice for $p < p_c$. It is found that $U_t$ shows a drifted random walk with a finite crossover time. Within this crossover time, the averaged driving function $(U_t)$ shows a scaling behavior $-(p_c - p) t^{(v+1)/2v}$ with a superdiffusive fluctuation whereas, beyond the crossover time, the driving function $U_t$ undergoes a normal diffusion with Hurst exponent $1/2$ but with the drift velocity proportional to $(p_c - p)^{\nu}$, where $\nu = 4/3$ is the critical exponent for two-dimensional percolation correlation length. The crossover time diverges as $(p_c - p)^{-2v}$ as $p \to p_c$.

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Introduction. Loewner evolution has recently drawn much attention in physics community because of the development of Schramm-Loewner evolution (SLE) [1], which has provided us a new tool for the study of two-dimensional (2-d) continuous phase transition. The basic device of SLE is a conformal mapping that transforms a motion along a non-intersecting 2-d curve into another motion along the real axis; Loewner driving function is a real function that represents this transformed motion. It turns out that, for a certain class of stochastic and conformally invariant curves in 2-d, the driving function shows Brownian motion in one dimension. What makes SLE especially remarkable is that it gives us a method that describes all the geometrical properties of the curves through a single parameter of the Brownian motion, namely, the diffusion constant. The class of curves includes the self-avoiding walk, the uniform-spanning trees, the loop-erased random walk, and boundaries of critical clusters in various 2-d lattice models in physics such as percolation, Ising model, $O(n)$ loop models, and Potts models (see [2] for review).

Being inspired by mathematically oriented development, people start using SLE formalism to test the conformal invariance by calculating Loewner driving functions obtained from 2-d curves in a number of physical systems, such as vorticity clusters and temperature iso-lines in turbulence [2], domain walls in 2-d spin glass [3], isohight lines on growing solid surface [2], nodal domains of chaotic maps [4], etc.

In study of physical systems, it is important to ask how driving functions may look like when the system departs from the critical point, because in real life there are a number of sources that may drive a system away from it. There are some mathematical approaches to study the effects of off-criticality on SLE based upon probability theory and conformal field theory [2,5], which mainly pursue mathematical consistency in the continuum limit, but general feature of the off-critical driving function has not been known yet. In this Letter, we report our results of numerical simulations to study how the Loewner driving function deviates from the ideal Brownian motion when the system departs from the criticality in the case of percolation clusters.

Loewner evolution. Let us start by reviewing basic elements of Loewner evolution briefly. Consider a non-intersecting continuous curve $\gamma$ which starts from the origin and extends toward infinity in the upper half plane $\mathbb{H}$. We parametrize $\gamma$ by $t \geq 0$, and denote a point on $\gamma$ as $\gamma_t = 0$. A part of $\gamma$ between $\gamma_t_1$ and $\gamma_t_2$ is represented by $[\gamma_t_1, \gamma_t_2]$. It is known that, for given $[\gamma_{t_3}, \gamma_{t_4}]$, there exists a unique conformal map $g_t : \mathbb{H} \setminus [\gamma_{t_3}, \gamma_{t_4}] \to \mathbb{H}$ that satisfies the condition

$$g_t(z) = z + a_t/z + O(|z|^{-2}) \quad \text{as} \quad |z| \to \infty$$

with $a_t \geq 0$ (Fig. 1). The parameter $t$, which we will call time, is now defined by $t := a_t/2$. The driving function $U_t$ is defined by the image of $\gamma_t$ by the map $g_t$: $U_t := \lim_{z \to \infty} g_t(z)$. Then, it can be shown that $g_t(z)$ satisfies the Loewner evolution, $\partial_z g_t(z) = 2/(g_t(z) - U_t)$.

Note that $g_t(z)$ represents the complex electrostatic potential for the equipotential boundary of $[\gamma_{t_3}, \gamma_{t_4}]$ and the real axis $[0,1]$, then one can see that the time $a_t = 2t$ is equal to $p/2\pi$, where $p$ is the 2-d dipole moment induced by $[\gamma_{t_3}, \gamma_{t_4}]$ on a flat electrode. In this electro-static picture, $U_t$ is just given by the charge induced by $[\gamma_{t_3}, \gamma_{t_4}]$ along the right side of $[\gamma_{t_3}, \gamma_{t_4}]$ and the positive part of the real axis in the unit system with $\varepsilon_0 = 1$ (Fig. 1).

Schramm has shown that $U_t$ becomes a Brownian motion if the curve $\gamma$ is a conformally invariant random curve with the domain Markov property [1]. This means that any properties of the curve such as the fractal dimension are determined solely by the diffusion constant of the Brownian motion.

Simulations. We performed numerical simulations to obtain $U_t$ for cluster boundaries of the site percolation on the triangular lattice with the occupation probability $p \leq p_c = 0.5$. Percolation clusters are generated in the unit system with $\varepsilon_0 = 1$.

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For each boundary curve, we numerically generate the conformal map $g_t$ on $\mathbb{D}$ with $\gamma_t$ to $U_t$ on the real axis. The deformed grid in the left panel can be regarded as equipotential and electric force lines, and is mapped to the straight grid in the right panel.

Fig. 2(a–c) shows typical examples of boundaries $\gamma$ up to 20000 steps at $p = 0.5, 0.49, \text{and} 0.48$, and Fig. 3(d) shows corresponding driving functions $U_t$ for $p = 0.5$. The boundary tends to extend toward left in larger scale, but it can hardly be distinguished from that at $p = p_c$ within the scale of correlation length $\xi$ with a finite lattice constant. Accordingly, $U_t$ wander around the origin at $p = p_c$, whereas it drifts toward $-\infty$ for large $t$ for $p < p_c$. Note that the time does not increases uniformly with the step along a boundary because the time is proportional to the dipole moment induced by $\gamma_{[0,t]}$.

The averaged behavior of $U_t$ is shown for several values of $p$ in the inset of Fig. 4 where one can see $(U_t)$ appears to drift at a constant velocity $v_d$. In Fig. 4 the drift velocity $v_d$ averaged up to $t = 21000$ over 4000 samples is plotted against $p_c - p$ in the logarithmic scale; $v_d$ is shown to be proportional to $(p_c - p)$ for $(p_c - p) \lesssim 0.01$,

$$v_d \sim (p_c - p),$$

whereas it behaves as

$$v_d \sim (p_c - p)^\nu$$

with $\nu \approx 4/3$ for $p_c - p \gtrsim 0.01$.

The time dependence of the variance $\text{Var}[U_t]$ is shown in the logarithmic scale in Fig. 5. As one can see in the plot for $p = 0.42$ in Fig. 5(a), there are two time regimes: the superdiffusion regime and the normal diffusion regime,

$$\text{Var}[U_t] \sim \begin{cases} t^\alpha & \text{for} \ t \ll t_c, \\ t & \text{for} \ t \gg t_c, \end{cases}$$

with the exponent $\alpha > 1$ and the crossover time $t_c$. The plots for several values of $p$ in Fig. 5(b) shows the tendency that the exponent $\alpha$ decreases to 1 and the crossover time $t_c$ increases as $p \rightarrow p_c$. The estimated values for $t_c$ are plotted against $(p_c - p)$ in the logarithmic scale by filled circles in Fig. 5(a).

**Discussions.** Our results can be understood as in the following. First, we consider the curve of $p = p_c$. In this case, $\gamma_t$ explores along a critical boundary, which extends indefinitely without bias. Consider a part of the curve $\gamma_{[0,t]}$. This may look like a blob, whose size we denote by $l_t$. The time $t$ that corresponds to the blob can be estimated as

$$t \sim l_t^2,$$
FIG. 4: (Color online) Drift velocity $v_d$ vs. $(p_c - p)$. The drift velocities of the driving function averaged up to $t = 21000$ are plotted in logarithmic scale as a function of $(p_c - p)$. There is a crossover around $(p_c - p) \sim 0.01$ between the two regimes. The solid (dashed) line with the slope 1 (4) using the electro-static analogy; The time $t$ is proportional to the dipole moment $p$ induced by $\gamma(0, t)$ attached to a flat electrode, and both the induced charge and the charge displacement are of order of $l_t$. On the other hand, $U_t$ may be approximated to be $x_t := \Re \gamma_t$ because $U_t$ is determined from the number of electric force lines landed on the right side of $\gamma(0, t)$ and the positive part of real axis. Since $x_t$ wanders along the blob of size $l_t$, one can see that $U_t$ undergoes normal diffusion without drift.

\begin{equation}
U_t \sim l_t \sim \sqrt{t}.
\end{equation}

For $p < p_c$, the curve $\gamma$ is the boundary of finite off-critical clusters that are connected by the bottom. Since typical size of each cluster is the percolation correlation length $\xi$, the curve $\gamma$ looks like a chain of the blobs of size $\xi$ (see for example Fig. 3(c)). Within each blob, $\gamma$ is almost like a critical boundary but with a bias toward the left; the strength of the bias is proportional to $p_c - p$.

There should be two time regimes: the short time regime where $\gamma_t$ is still in the first blob, and the long time regime where $\gamma_t$ is traveling over blobs. The crossover time $t_c$ between the two regimes is the time that $\gamma_t$ goes over the first blob of size $\xi$, therefore, it is estimated as $t_c \sim \xi^2$ as in the case of critical blob. Within the short time regime, $\gamma_t$ explores in the first blob that looks almost like a critical blob in the smaller length scale than $\xi$ but with a small bias toward left. We assume the scaling behavior $\langle x_t \rangle \sim -\xi(t/t_c)^{\beta}$ with an exponent $\beta$. If we determine the exponent so that the effect of bias should be proportional to $(p_c - p)$ for a fixed $t$, we obtain $\beta = (\nu+1)/2\nu = 7/8$, using the critical exponent $\nu = 4/3$ for the correlation length of the 2-d percolation. On the other hand, the behavior of $\langle x_t \rangle$ in the long time regime may be obtained by considering the situation where $\gamma_t$ is at $n$-th blob, i.e., $\langle x_t \rangle \sim n^c$. The time corresponding to this is $t \sim n^c \xi^2$ because the dipole moment induced at each blob is of order of $\xi^2$, with which we can express

\begin{align*}
\langle x_t \rangle &\sim n^c \\
\text{when } n &\leq \xi/t_c, \\
\text{and } \langle x_t \rangle &\sim \xi^2 t_c \text{ when } n \gg \xi/t_c.
\end{align*}
\( \langle x_t \rangle \) as \( t/\xi \). With all these argument, we finally obtain

\[
\langle x_t \rangle \sim \begin{cases} 
-(p_c - p) t^{(\nu+1)/2\nu} & (t \lesssim t_c) \\
-(p_c - p)^\nu t & (t \gtrsim t_c)
\end{cases}
\] (7)

with \( t_c \sim (p_c - p)^{-2\nu} \).

Within the approximation \( U_t \sim x_t \), this is consistent with our results for the drift velocity \( v_d \) in Fig. 4. The slope we obtained for \( p_c - p \gtrsim 0.01 \) is very close to \( \nu = 4/3 \). The linear dependence for \( p_c - p \lesssim 0.01 \) should correspond to the short time behavior. The crossover around \( p_c - p \approx 0.01 \) is due to the fact that our simulation time length is not long enough in comparison with \( t_c \). If we can simulate longer time, the crossover value of \( p_c - p \) will be smaller. The expected weak non-linear \( t \)-dependence in \( x_t \) is difficult to distinguish from the behavior numerically; Actually, one might notice slight convexity of the plots in the inset of Fig. 4.

In order to check further consistency of the data with our interpretation, we plot \( t_c \) for \( \text{Var}[U_t] \), \( \xi \), and \( \xi^2 \) against \( p_c - p \) in Fig. [a]. The horizontal solid line indicates the time range of our simulations. The plotted range of the crossover time (filled circles) in \( \text{Var}[U_t] \) is too narrow to determine its behavior in a reliable way, but it seems consistent with that of \( \xi^2 \) (open circles). The value of \( p_c - p \) where the extrapolated \( t_c \) of \( \text{Var}[U_t] \) reaches the simulation time is about 0.04 (arrow), which is larger than the crossover point in \( v_d \) by the factor 4; This discrepancy may come from the difference between \( v_d \) and \( \text{Var}[U_t] \), or simply due to the uncertainty in the estimates. In Fig. [b], \( \langle x_t \rangle \) for several values of \( t \) are plotted against \( (p_c - p) \). From Eq. (7), we expect linear dependence on \( (p_c - p) \) for \( (p_c - p) \lesssim t^{-3/8} \) and \((p_c - p)^{4/3}\)-dependence beyond that; The data seem to be consistent.

With these observations, we interpret our results as follows. For an off-critical percolation cluster boundary, the driving function \( U_t \) undergoes a random walk with drift. The drift velocity \( v_d \) appears to be proportional to \( (p_c - p) \) when \( (p_c - p) \lesssim 0.01 \), but it is because time lengths of our simulations are finite; With the approximation \( U_t \sim x_t \), the averaged driving function should be given by Eq. (7). Within the crossover time, the random walk is superdiffusive with exponent \( \alpha > 1 \), which decreases toward 1 as \( p \to p_c \). Beyond the crossover time, the fluctuation around the drift motion is normal diffusion with larger diffusion constant than that at \( p = p_c \).

Finally, let us discuss the scaling limit where the lattice constant \( a \to 0 \) with keeping the correlation length \( \xi \) constant. Our argument to derive Eq. (7) holds for any \( \xi \) and \( a \) as long as \( \xi \gg a \), thus if we scale the variables as \( x_t \equiv x_t/\xi, U_t \equiv U_t/\xi \), and \( t \equiv t/\xi^2 \), then we obtain the corresponding equation for the scaled variables, i.e. the same as Eq. (7) but without the factor of \((p_c - p)\).

[1] O. Schramm, Isr. J. Math. 118, 221 (2000).
[2] G. Lawler, Conformally Invariant Processes in the Plane (American Mathematical Society, Providence, 2005; W. Kager and B. Nienhuis, J. Stat. Phys. 115, 1149 (2004); J. Cardy, Ann. Phys. (N.Y.) 318, 81 (2005); I. A. Gruzberg, J. Phys. A: Math. Gen. 39, 12601 (2006); M. Bauer and D. Bernard, Phys. Rep. 432, 115 (2006).
[3] D. Bernard, G. Boffetta, A. Celani, and G. Falkovich, Nature Phys. 2, 124 (2006); Phys. Rev. Lett. 98, 024501 (2007).
[4] C. Amoruso, A. K. Hartmann, M. B. Hastings, and M. A. Moore, Phys. Rev. Lett. 97, 267202 (2006); D. Bernard, P. Le Doussal, and A. A. Middleton, Phys. Rev. B 76, 020403(R) (2007).
[5] A. A. Saberi, M. A. Rajabpour, and S. Rouhani, Phys. Rev. Lett. 100, 044504 (2008); A. A. Saberi, M. D. Niry, S. M. Fazeli, M. R. Rahimi Tabar, and S. Rouhani, Phys. Rev. E 77, 051607 (2008).
[6] J. P. Keating, J. Marklof, and I. G. Williams, Phys. Rev. Lett. 97, 034101 (2006); New J. Phys. 10, 083023 (2008).
[7] P. Nolin and W. Werner, J. Amer. Math. Soc. 22, 797 (2009).
[8] M. Bauer, D. Bernard, and K. Kytölä, J. Stat. Phys. 132 (2008); M. Bauer, D. Bernard, and L. Cantini, arXiv:0903.1023v1 [math-ph].
[9] L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media (Pergamon Press, 1960).
[10] T. Kennedy, J. Stat. Phys. 131, 803 (2008).
[11] D. E. Marshall and S. Rohde, SIAM J. Numer. Anal. 45, 2577 (2007).
[12] Note that, under the employed boundary condition, the finite system size effect comes into the problem as the limitation that the available length of \( \gamma \) is finite.