L-S CATEGORIES OF SIMPLY-CONNECTED COMPACT
SIMPLE LIE GROUPS OF LOW RANK

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Abstract. We determine the L-S category of $Sp(3)$ by showing that the 5-fold reduced diagonal $\Sigma_5 X$ is given by $\nu^2$, using a Toda bracket and a generalised cohomology theory $h^*$ given by $h^*(X, A) = \{X/A, S[0, 2]\}$, where $S[0, 2]$ is the 3-stage Postnikov piece of the sphere spectrum $S$. This method also yields a general result that $\text{cat}(Sp(n)) \geq n + 2$ for $n \geq 3$, which improves the result of Singhof [17].

1. Introduction

In this paper, each space is assumed to have the homotopy type of a CW complex. The (normalised) L-S category of $X$ is the least number $m$ such that there is a covering of $X$ by $(m + 1)$ open subsets each of which is contractible in $X$. Hence $\text{cat}\{\} = 0$. By Lusternik and Schnirelmann [10], the number of critical points of a smooth function on a manifold $M$ is bounded below by $\text{cat} M + 1$.

G. Whitehead showed that $\text{cat}(X)$ coincides with the least number $m$ such that the diagonal map $\Delta_{m+1} : X \to \prod^{m+1} X$ can be compressed into the ‘fat wedge’ $T^{m+1}(X)$ (see Chapter X of [20]). Since $\prod^{m+1} X/T^{m+1}(X)$ is the $(m + 1)$-fold smash product $\wedge^{m+1} X$, we have a weaker invariant $\text{wcat} X$, the weak L-S category of $X$, given by the least number $m$ such that the reduced diagonal map $\Sigma_{m+1} : X \to \wedge^{m+1} X$ is trivial. Hence $\text{wcat} X \leq \text{cat} X$.

T. Ganea has also introduced a stronger invariant $\text{Cat} X$, the strong L-S category of $X$, by the least number $m$ such that there is a covering of $X$ by $(m + 1)$ open subsets each of which is contractible in itself. Thus $\text{wcat} X \leq \text{cat} X \leq \text{Cat} X$.

The weak and strong L-S categories usually give nice estimates of L-S category especially for manifolds. Actually, we do not know any example of a closed manifold whose strong L-S, L-S and weak L-S categories are not the same. The following problems are posed by Ganea [4]:

i) (Problem 1) Determine the L-S category of a manifold.

ii) (Problem 4) Describe the L-S category of a sphere-bundle over a sphere in terms of homotopy invariants of the characteristic map of the bundle.

Problem 1 has been studied by many authors, such as Singhof [16, 17, 18], Montejano [12], Schweizer [17], Gomez-Larrañaga and Gonzalez-Acuña [3], James and Singhof [3] and Rudyak [15, 19]. In particular for compact simply-connected simple Lie groups, $\text{cat}(SU(n+1)) = n$ for $n \geq 1$ by [16], $\text{cat}(Sp(2)) = 3$ by [3] and
cat(\text{Sp}(n)) \geq n + 1 for n \geq 2 by \cite{17}. It was also announced recently that Problem 4 was solved by the first author \cite{7}.

The method in the present paper also provides a result for \( G_2 \), and thus we have the following result.

**Theorem 1.1.** The following is the complete list of L-S categories of a simply-connected compact simple Lie group of rank \( \leq 2 \):

| Lie groups | \text{Sp}(1) = SU(2) = Spin(3) | SU(3) | \text{Sp}(2) = Spin(5) | \text{G}_2 |
|------------|--------------------------------|-------|------------------------|----------|
| wcat       | 1                              | 2     | 3                      | 4        |
| cat        | 1                              | 2     | 3                      | 4        |
| Cat        | 1                              | 2     | 3                      | 4        |

Although the above result is known for experts, we give a short proof for \( G_2 \). In fact, the result for \( G_2 \) has never been published and is obtained in a similar but easier manner than the following result for \( \text{Sp}(3) \):

**Theorem 1.2.** \( \text{wcat} (\text{Sp}(3)) = \text{cat} (\text{Sp}(3)) = \text{Cat} (\text{Sp}(3)) = 5 \).

**Remark 1.3.** The argument given to prove Theorem 1.2 provides an alternative proof of Schweizer’s result

\[ \text{wcat} (\text{Sp}(2)) = \text{cat} (\text{Sp}(2)) = \text{Cat} (\text{Sp}(2)) = 3. \]

The authors know that a similar result to Theorem 1.2 is obtained by Lucía Fernández-Suárez, Antonio Gómez-Tato, Jeffrey Strom and Daniel Tanré \cite{3}. Our method is, however, much simpler and providing the following general result:

**Theorem 1.4.** \( n + 2 \leq \text{wcat} (\text{Sp}(n)) \leq \text{cat} (\text{Sp}(n)) \leq \text{Cat} (\text{Sp}(n)) \) for \( n \geq 3 \).

This improves Singhof’s result: \( \text{cat} (\text{Sp}(n)) \geq n + 1 \) for \( n \geq 2 \). We propose the following conjecture.

**Conjecture 1.5.** Let \( G \) be a simply-connected compact Lie group with \( G = \prod_{i=1}^{n} H_i \) where \( H_i \) is a simple Lie group. Then \( \text{wcat} (G) = \text{cat} (G) = \text{Cat} (G) \) and \( \text{cat} (G) = \sum_{i=1}^{n} \text{cat} (H_i) \).

It might be difficult to say something about \( \text{cat} \text{Sp}(n) \), but an old conjecture says the following.

**Conjecture 1.6.** \( \text{cat} \text{Sp}(n) = 2n - 1 \) for all \( n \geq 1 \).

The authors thank John Harper for many helpful conversations.

2. **Proof of Theorem 1.2**

Let us recall a CW decomposition of \( G_2 \) from \cite{11}:

\[ G_2 = e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}. \]

On the other hand, we have the following cone-decomposition.

**Theorem 2.1.** There is a cone-decomposition of \( G_2 \) as follows:

\[ G_2^{(5)} = \Sigma \mathbb{CP}^2, \quad S^5 \cup e^7 \rightarrow G_2^{(5)} \hookrightarrow G_2^{(8)}, \]
\[ S^8 \cup e^{10} \rightarrow G_2^{(8)} \hookrightarrow G_2^{(11)}, \quad S^{13} \rightarrow G_2^{(11)} \hookrightarrow G_2. \]
Proof. The first and the last formulae are obvious. So we show the 2nd and 3rd formulae: By taking the homotopy fibre $F_1$ of $G_2 \leftrightarrow G_2$, we can easily observe using the Serre spectral sequence that the fibre has a CW structure given by $S^5 \cup e^7 \cup (\text{cells in dimensions } \geq 7)$, where the cohomology generators corresponding to $S^5$ and $e^7$ are transgressive. Thus the mapping cone of $S^5 \cup e^7 \subset F_1 \to G_2$ has the homotopy type of $G_2^{(8)}$. Similarly, the homotopy fibre $F_2$ of $G_2^{(8)} \leftrightarrow G_2$ has a CW structure given by $S^8 \cup e^{10} \cup (\text{cells in dimensions } \geq 10)$, where the cohomology generators corresponding to $S^8$ and $e^{10}$ are transgressive. Thus the mapping cone of $S^8 \cup e^{10} \subset F_2 \to G_2^{(8)}$ has the homotopy type of $G_2^{(11)}$. QED.

Corollary 2.1.1. $1 \geq \text{Cat}(G_2^{(5)}) \geq \text{Cat}(G_2^{(3)})$, $2 \geq \text{Cat}(G_2^{(8)}) \geq \text{Cat}(G_2^{(6)})$, $3 \geq \text{Cat}(G_2^{(11)}) \geq \text{Cat}(G_2^{(9)})$ and $4 \geq \text{Cat}(G_2)$.

Let us recall the following well-known fact due to Borel.

Fact 2.2. $H^*(G_2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x, x_3]/(x^3, x_3^2)$.

Corollary 2.2.1. $\text{wcat}(G_2^{(5)}) \geq \text{wcat}(G_2^{(3)}) \geq 1$, $\text{wcat}(G_2^{(8)}) \geq \text{wcat}(G_2^{(6)}) \geq 2$, $\text{wcat}(G_2^{(11)}) \geq \text{wcat}(G_2^{(9)}) \geq 3$ and $\text{wcat}(G_2) \geq 4$.

Corollaries 2.1.1 and 2.2.1 yield the following.

Theorem 2.3.

| Skeleta | $G_2^{(5)}$ | $G_2^{(3)}$ | $G_2^{(6)}$ | $G_2^{(8)}$ | $G_2^{(9)}$ | $G_2^{(11)}$ |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|
| wcat    | 1           | 1           | 1           | 2           | 2           | 2           |
| cat     | 1           | 1           | 1           | 2           | 2           | 2           |

This completes the proof of Theorem 2.1.

3. The Ring Structure of $h^*(Sp(3))$

To show Theorem 2.2, we introduce a cohomology theory $h^*(-)$ such that $h^*(X, A) = \{X/A, S[0, 2]\}$, where $S[0, 2]$ is the spectrum obtained from $S$ by killing all homotopy groups of dimensions bigger than 2. Then $S[0, 2]$ is a ring spectrum with $\pi^S_\ast(S[0, 2]) \cong \mathbb{Z}[[\eta]]/(\eta^3, 2\eta)$, where $\eta$ is the Hopf element in $\pi^S_1(S) = \pi^S_1(S[0, 2])$. Thus $h^*$ is an additive and multiplicative cohomology theory with $h^* = h^*(pt) \cong \mathbb{Z}[\varepsilon]/(\varepsilon^3, 2\varepsilon)$, deg $\varepsilon = -1$, where $\varepsilon \in h^{-1} = \pi^S_0(\Sigma^{-1}S) \cong \pi^S_1(S)$ corresponds to $\eta$.

The characteristic map of the principal $Sp(1)$-bundle

$$Sp(1) \hookrightarrow Sp(2) \to S^7$$

is given by $\omega = (t_3, t_3) : S^6 \to Sp(1) \cong S^3$ the Samelson product of two copies of the identity $t_3 : S^3 \to S^3$, which is a generator of $\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$. We state the following well-known fact (see Whitehead [P]).

Fact 3.1. Let $\mu : S^3 \times S^3 \to S^3$ be the multiplication of $Sp(1) \cong S^3$. Then we have

$$Sp(2) \simeq S^3 \cup_{\mu \circ (1 \times \omega)} S^3 \times C(S^6) \cup S^3 \cup_\omega C(S^6) \cup_{\mu \circ [3, \omega]} C(S^9),$$

where $\mu : S^3 \times S^3 \cup_{\times \omega} \{\ast\} \times C(S^6) \to S^3 \cup_\omega C(S^6)$ is given by $\mu|_{S^3 \times S^3} = \mu$ and $\mu|_{S^3 \cup_\omega C(S^6)} = 1$ the identity and $[3, \omega] : S^9 \to S^3 \times S^3 \cup_{\times \omega} \{\ast\} \times C(S^6)$ is the relative Whitehead product of the identity $t_3 : S^3 \to S^3$ and the characteristic map $\chi_\omega : C(S^9, S^6) \to (S^3 \cup e^7, S^3)$ of the 7-cell. Thus we have $1 \geq \text{Cat}(Sp(2)^{(3)}), 2 \geq \text{Cat}(Sp(2)^{(7)})$ and $3 \geq \text{Cat}(Sp(2))$.  


Let $\nu : S^7 \to S^4$ be the Hopf element whose suspension $\nu_n = \Sigma^{n-4}\nu$ ($n \geq 4$) gives a generator of $\pi_{n+3}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 5$. Then we remark that $\omega_n = \Sigma^{n-3}\omega$ ($n \geq 3$) satisfies the formula $\omega_n = 2\nu_n \in \pi_{n+3}(S^n)$ for $n \geq 5$. By Zabrodsky [21], there is a natural splitting

$$\Sigma(S^3 \times S^3 \cup \{\ast\} \times (S^3 \cup e^7)) \cong \Sigma S^3 \vee \Sigma(S^3 \cup e^7) \vee \Sigma S^3 \wedge S^3.$$  

Then by the definition of a relative Whitehead product, the composition of $[\iota_3, \omega]^r$ with the projections to $S^3$ and $S^3 \cup e^7$ are trivial and the composition with the projection to $S^3 \wedge S^3$ is given by $\iota_3 \wedge \omega$. Thus we have

$$\Sigma(\mu[\iota_3, \omega]^r) = H(\mu) \circ \Sigma(\iota_3 \wedge \omega) = \pm \nu \omega = 2\nu \omega \neq 0$$

in $\pi_{10}(S^4) \cong \mathbb{Z}/2\mathbb{Z}(\nu \cdot \nu) \oplus \mathbb{Z}/2\mathbb{Z}(\omega \cdot \nu \gamma)$, and hence we have

$$\Sigma^2(\mu[\iota_3, \omega]^r) = \nu \omega \omega = 2\nu^2 = 0 \in \pi_{11}(S^4) \cong \mathbb{Z}/2\mathbb{Z}$$

by Proposition 5.11 of Toda [19]. The following two facts are also well-known.

**Fact 3.2.** We have the following homotopy equivalences:

$$Sp(2)/S^3 \cong (S^3 \times C(S^6))/\langle S^3 \times S^6 \rangle = S^1 \wedge \Sigma(S^6) = S^7 \vee S^{10},$$

$$\Sigma^2 Sp(2) \cong \Sigma^2(S^3 \cup C(S^6)) \vee \Sigma^2 S^{10} = S^5 \cup \Sigma^2 C(S^8) \vee S^{12}.$$

**Fact 3.3.** The 11-skeleton $X_{3,2}^{(11)}$ of $X_{3,2} = Sp(3)/Sp(1)$ has the homotopy type of $S^7 \uplus e^{11}$.

Restricting the principal $Sp(1)$-bundle $Sp(1) \to Sp(3) \to X_{3,2}$ to the subspace $X_{3,2}^{(11)} = S^7 \uplus e^{11}$ of $X_{3,2}$, we obtain the subspace $q^{-1}(X_{3,2}^{(11)}) = Sp(3)^{(11)}$ of $Sp(3)$ as the total space of the principal $Sp(1)$-bundle $Sp(1) \to Sp(3)^{(14)} \to \Sigma(S^6 \cup e^{10})$ with a characteristic map $\phi : S^6 \cup e^{10} \to Sp(1) \approx S^1$, which is an extension of $\omega : S^6 \to S^3$.

**Proposition 3.4.** We have the following homotopy equivalences:

$$Sp(3)^{(14)} \cong S^3 \cup_{\phi \circ (1 \times \phi)} S^3 \times C(S^6 \cup e^{10})$$

$$\cong S^3 \cup_{\phi} C(S^6 \cup e^{10}) \cup C(S^9 \cup e^{13}),$$

$$Sp(3)^{(14)}/S^3 \cong (S^3 \times C(S^6 \cup e^{10}))/\langle S^3 \times (S^6 \cup e^{10}) \rangle \cong S^3 \wedge \Sigma(S^6 \cup e^{10}) = (S^7 \cup e^{11}) \vee (S^{10} \cup e^{14}),$$

$$Sp(n) \cong Sp(n-1) \cup Sp(n-1) \times C(S^{4n-2}),$$

where $Sp(n-1) \subset Sp(n)^{(2n+1)n-11})$ for $n \geq 3$, and hence

$$Sp(n)/Sp(n)^{(2n+1)n-11})$$

$$\cong (Sp(n-1) \times C(S^{4n-2}))/\langle Sp(n-1) \times S^{4n-2} \cup Sp(n-1)^{(2n-1)(n-1)-11)} \times C(S^{4n-2}) \rangle$$

$$= (Sp(n-1)/Sp(n-1)^{(2n-1)(n-1)-11}) \wedge S^{4n-2}$$

$$= \cdots = (Sp(2)/\emptyset) \wedge \Sigma S^{10} \wedge \cdots \wedge \Sigma S^{4n-2} = (Sp(2)) \wedge S^{(2n+1)n-10},$$

$$= S^{(2n+1)n-10} \vee S^{(2n+1)n-10} \wedge Sp(2)$$

$$= S^{(2n+1)n-10} \vee (S^{(2n+1)n-7} \cup e^{(2n+1)n-3}) \vee S^{(2n+1)n}, \quad \text{for } n \geq 3.$$ 

This yields the following result.
Proposition 3.5. Let $\tilde{\mu}: S^3 \times S^3 \times \times \{*\} \times (S^3 \cup \varnothing C(S^6 \cup \varnothing e^{10})) \rightarrow S^3 \cup \varnothing C(S^6 \cup \varnothing e^{10})$ be the map given by $\tilde{\mu}|_{S^3 \times S^3} = \mu$ and $\tilde{\mu}|_{S^3 \cup \varnothing C(S^6 \cup \varnothing e^{10})} = 1$ the identity. Then we have the following cone decomposition of $Sp(3)$:

$$Sp(3) \simeq S^3 \cup \varnothing C(S^6 \cup \varnothing e^{10}) \cup \mu_{\varnothing} \circ C(S^6 \cup \varnothing e^{13}) \cup C(S^7) \cup C(S^{20}).$$

Corollary 3.5.1. $1 \geq \text{Cat}(Sp(3)^{(3)})$, $2 \geq \text{Cat}(Sp(3)^{(7)})$, $3 \geq \text{Cat}(Sp(3)^{(14)}) \geq \text{Cat}(Sp(3)^{(11)}) \geq \text{Cat}(Sp(3)^{(10)})$, $4 \geq \text{Cat}(Sp(3)^{(18)})$ and $5 \geq \text{Cat}(Sp(3))$.

To determine the ring structures of $h^*(Sp(2))$ and $h^*(Sp(3))$, we show the following lemma.

Lemma 3.6. Let $h^*$ be any multiplicative generalised cohomology theory and let $Q = S^r \cup_i e^q$ for a given map $f: S^{r-1} \rightarrow S^r$ with $h^*(Q) \cong h^*(1, x, y)$, where $x$ and $y$ correspond to the generators of $h^*(S^r) \cong h^*(x_0)$ and $h^*(S^3) \cong h^*(y_0)$. Then

$$x^2 = \pm H^2(\lambda_2(f)) \cdot y \quad \text{in} \quad h^*(Q),$$

where $\lambda_2(f) \in \pi^q q(S^{2r})$ is the Boardman-Steer Hopf invariant equal to $\Sigma \lambda_2(f)$ the suspension of the James-Hopf invariant $\lambda_2(f)$ (see $(1)$) and $H^2: \pi^q q(S^{2r}) \rightarrow h^2(S^{2r}) \cong h^{2r-q}$ is the Hurewicz homomorphism given by $H^2(y) = \Sigma^r \nu^q(x_0 \otimes x_0)$.

Proof. By Boardman and Steer $(1)$, $Q: Q_2 = S^r \cup_i e^q \rightarrow Q_2 \wedge Q_2$ equals the composition $(i_2 \wedge i_2) \circ \lambda_2(f) \circ q_2$, where $q_2: Q_2 \rightarrow Q_2 / S^q = S^q$ is the collapsing map and $i_2: S^r \hookrightarrow Q_2$ is the bottom-cell inclusion. Thus we have

$$x^2 = \Delta (x \otimes x) = ((i_2 \wedge i_2) \circ \lambda_2(f) \circ q_2)^*(x \otimes x)$$

$$= q_2^*(\lambda_2(f)^* (i_2^* (x \otimes i_2^*(x)))) = q_2^*(\lambda_2(f)^*(x_0 \otimes x_0)) = q_2^*(\Sigma^q H^2(\lambda_2(f))).$$

Since $\Sigma^q H^2(\lambda_2(f))$ is $H^2(\lambda_2(f)) \cdot y_0 \in h^{2r}(S^q)$ up to sign, we proceed as

$$x^2 = q_2^*(\pm H^2(\lambda_2(f)) \cdot y_0) = \pm H^2(\lambda_2(f)) \cdot q_2^*(y_0) = \pm H^2(\lambda_2(f)) \cdot y.$$ 

This completes the proof of the lemma. \hfill QED

Using cohomology long exact sequences derived from the cell structure of $Sp(3)$ and a direct calculation using Proposition 3.4 and Lemma 3.6, we deduce the following result for the cohomology theory $h^*$ considered at the beginning of this section.

Theorem 3.7. The ring structures of $h^*(Sp(2))$ and $h^*(Sp(3))$ are as follows:

$$h^*(Sp(2)) \cong h^*[1, x_3, x_7, y_{10}],$$

$$h^*(Sp(3)) \cong h^*[1, x_3, x_7, x_{11}, y_{10}, y_{14}, y_{18}, z_{21}]$$

with ring structures given by $x_3^2 = \varepsilon \cdot x_7, x_7^2 = 0, x_{11}^2 = 0, x_3 x_7 = y_{10}, x_3 x_{11} = y_{14}, x_7 x_{11} = y_{18}$ and $x_3 x_7 x_{11} = z_{21}$, where $\varepsilon$ is the non-zero element in $h^{-1}$.

Remark 3.8. The two possible attaching maps: $S^{10} \rightarrow S^3 \cup \varnothing e^7$ of $e^{11}$ discovered by Lucía Fernández-Suárez, Antonio Gómez-Tato and Daniel Tanré $(3)$ are homotopic in $Sp(2)$. So, we can not find any effective difference in the ring structure of $h^*(Sp(3))$ by altering, as is performed in $(3)$, the attaching map of $e^{11}$.

Corollary 3.8.1. $\text{wcat}(Sp(3)^{(3)}) \geq 1$, $\text{wcat}(Sp(3)^{(7)}) \geq 2$, $\text{wcat}(Sp(3)^{(14)}) \geq \text{wcat}(Sp(3)^{(11)}) \geq \text{wcat}(Sp(3)^{(10)}) \geq 3$ and $\text{wcat}(Sp(3)) \geq 4$, together with $\text{wcat}(Sp(2)^{(3)}) \geq 1$, $\text{wcat}(Sp(2)^{(7)}) \geq 2$ and $\text{wcat}(Sp(2)) \geq 3$. 

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Corollary 3.8.2.

| Skeleta | $\text{Sp}(2)^{(3)}$ | $\text{Sp}(2)^{(14)}$ | $\text{Sp}(2)_{\text{cat}}$ |
|---------|---------------------|---------------------|---------------------|
| $\text{ucat}$ | 1 | 2 | 3 |
| $\text{cat}$ | 1 | 2 | 3 |
| $\text{Cat}$ | 1 | 2 | 3 |

4. PROOF OF THEOREM

By Facts 3.3 and 3.2, the smash products $\wedge^4 \text{Sp}(3)$ and $\wedge^5 \text{Sp}(3)$ satisfy

$(\wedge^4 \text{Sp}(3))^{(19)} \simeq S^{12} \cup_{\omega_{12}} e^{16} \lor (S^{16} \lor S^{16} \lor S^{16}) \lor (S^{19} \lor S^{19} \lor S^{19} \lor S^{19})$,

$(\wedge^5 \text{Sp}(3))^{(22)} \simeq S^{15} \cup_{\omega_{15}} e^{19} \lor (S^{19} \lor S^{19} \lor S^{19}) \lor (S^{22} \lor S^{22} \lor S^{22} \lor S^{22})$.

Then we have the following two propositions.

**Proposition 4.1.** The bottom-cell inclusions $i : S^{12} \hookrightarrow \wedge^4 \text{Sp}(3)$ and $i' : S^{15} \hookrightarrow \wedge^5 \text{Sp}(3)$ induce injective homomorphisms

$i_* : \pi_{18}(S^{12}) \to \pi_{18}(\wedge^4 \text{Sp}(3)^{(18)})$ and $i'_* : \pi_{21}(S^{15}) \to \pi_{21}(\wedge^5 \text{Sp}(3))$

respectively.

**Proof.** We have the following two exact sequences

$$\pi_{18}(S^{15}) \xrightarrow{i'} \pi_{18}(S^{12}) \xrightarrow{i} \pi_{18}(\wedge^4 \text{Sp}(3)^{(18)}) \to \pi_{18}(S^{16} \lor S^{16} \lor S^{16} \lor S^{16})$$

$$\pi_{21}(S^{18}) \xrightarrow{i'} \pi_{21}(S^{15}) \xrightarrow{i} \pi_{21}(\wedge^5 \text{Sp}(3)) \to \pi_{21}(S^{19} \lor S^{19} \lor S^{19} \lor S^{19})$$

where $\pi_{18}(S^{12}) \cong \pi_{21}(S^{15}) \cong \mathbb{Z}/2\mathbb{Z}^2$ and $\psi$ and $\psi'$ are induced from $\omega_{12} = 2\nu_{12}$ and $\omega_{15} = 2\nu_{15}$. Thus $\psi$ and $\psi'$ are trivial, and hence $i_*$ and $i'_*$ are injective. QED.

**Proposition 4.2.** The collapsing maps $q : \text{Sp}(3)^{(18)} \to \text{Sp}(3)^{(18)}/\text{Sp}(3)^{(14)} = S^{18}$ and $q' : \text{Sp}(3) \to \text{Sp}(3)/\text{Sp}(3)^{(18)} = S^{21}$ induce injective homomorphisms

$q^* : \pi_{18}(\wedge^4 \text{Sp}(3)^{(18)}) \to [\text{Sp}(3)^{(18)}, \wedge^4 \text{Sp}(3)^{(18)}]$ and

$q'^* : \pi_{21}(\wedge^5 \text{Sp}(3)) \to [\text{Sp}(3), \wedge^5 \text{Sp}(3)]$

respectively.

**Proof.** Firstly, we show that $q'^*$ is injective: Since we have $[\text{Sp}(3), \wedge^5 \text{Sp}(3)] = [S^{14} \lor_{\omega_{14}} e^{18} \lor S^{21}, \wedge^5 \text{Sp}(3)] = [S^{14} \lor_{\omega_{14}} e^{18}, \wedge^5 \text{Sp}(3)] \oplus \pi_{21}(\wedge^5 \text{Sp}(3))$ by Proposition 4.3, $q'^*$ is clearly injective.

Secondly, we show that $q^*$ is injective: Similarly we have $[\text{Sp}(3)^{(18)}, \wedge^4 \text{Sp}(3)^{(18)}] = [S^{14} \lor_{\omega_{14}} e^{18}, \wedge^4 \text{Sp}(3)^{(18)}]$ by Proposition 4.2. Thus it is sufficient to show that $q^* : \pi_{18}(\wedge^4 \text{Sp}(3)^{(18)}) \to [S^{14} \lor_{\omega_{14}} e^{18}, \wedge^4 \text{Sp}(3)^{(18)}]$ is injective, where $q : S^{14} \lor_{\omega_{14}} e^{18} \to S^{18} = \text{the collapsing map}$. In the exact sequence

$$\pi_{15}(\wedge^4 \text{Sp}(3)^{(18)}) \xrightarrow{\omega_{15}^*} \pi_{18}(\wedge^4 \text{Sp}(3)^{(18)}) \xrightarrow{\omega_{15}} [S^{14} \lor_{\omega_{14}} e^{18}, \wedge^4 \text{Sp}(3)^{(18)}]$$

we know that $\pi_{15}(\wedge^4 \text{Sp}(3)^{(18)}) \cong \pi_{15}(S^{12} \lor_{\omega_{12}} e^{16}) = \mathbb{Z}/2\mathbb{Z}$ is generated by the composition of $\nu_{12}$ and the bottom-cell inclusion. Since $\nu_{12} \omega_{15} = 0 \in \pi_{18}(S^{12})$, the homomorphism $\omega_{15}^*$ is trivial, and hence $q^*$ is injective. QED.

Then the following lemma implies that $\Sigma_4$ and $\Sigma_5$ are non-trivial by Propositions 3.1 and 4.2.
Lemma 4.3. We obtain that $\overline{\Delta}_4 = i'\nu^2_{12}q : \text{Sp}(3)^{(18)} \to \Lambda^4\text{Sp}(3)^{(18)}$ and that $\overline{\Delta}_5 = i''\nu^2_{15}q' : \text{Sp}(3) \to \Lambda^5\text{Sp}(3)$. 

Proof. Firstly, we show that $\overline{\Delta}_4 = i'\nu^2_{12}q$ implies $\overline{\Delta}_5 = i''\nu^2_{15}q'$. For dimensional reasons, the image of $\overline{\Delta} : \text{Sp}(3) \to \text{Sp}(3)\wedge\text{Sp}(3)$ is in $\text{Sp}(3)^{(14)} \wedge \text{Sp}(3)^{(14)} \cup \text{S}^3\wedge\text{Sp}(3)^{(18)}$. Since $\text{Sp}(3)^{(14)}$ is of cone-length 3 by Corollary 3.5.1, the restriction of the map $(1\wedge\overline{\Delta}_4)\circ\overline{\Delta} = \overline{\Delta}_5$ to $\text{Sp}(3)^{(18)}\wedge\text{Sp}(3)^{(14)}$ is trivial. Thus $\overline{\Delta}_5$ equals the composition 

$$\overline{\Delta}_5 : \text{Sp}(3) \to S^3\wedge\text{Sp}(3)^{(18)} 1\wedge\overline{\Delta}_4 \Lambda^5\text{Sp}(3)^{(18)} \subset \Lambda^5\text{Sp}(3).$$

Then by $\overline{\Delta}_4 = i'\nu^2_{12}q$, we observe that $\overline{\Delta}_5 = i''\nu^2_{15}q'$. For dimensional reasons, the image of $\overline{\Delta} : \text{Sp}(3)^{(18)} \to \text{Sp}(3)^{(18)}\wedge\text{Sp}(3)^{(18)}$ is in $\text{Sp}(3)^{(14)} \wedge S^3 \cup \text{Sp}(3)^{(11)} \wedge \text{Sp}(3)^{(7)} \cup \text{Sp}(3)^{(7)} \wedge \text{Sp}(3)^{(11)} \cup \text{S}^3\wedge\text{Sp}(3)^{(14)}$. Since $S^3\cup S^6_C(U_{10})$ is of cone-length 2 by Corollary 3.5.1, the restriction of $\overline{\Delta}_3 : \text{Sp}(3)^{(18)} \to \wedge^3\text{Sp}(3)^{(18)}$ to $S^3\cup S^6_C(U_{10})$ is trivial. Hence $1\wedge\overline{\Delta}_3 : \text{Sp}(3)^{(14)} \wedge S^3 \cup \text{Sp}(3)^{(11)} \wedge \text{Sp}(3)^{(7)} \cup \text{Sp}(3)^{(7)} \wedge \text{Sp}(3)^{(11)} \cup \text{S}^3\wedge\text{Sp}(3)^{(14)} \to \wedge^3\text{Sp}(3)^{(18)}$ equals the composition 

$$1\wedge\overline{\Delta}_3 : (\text{Sp}(3)\wedge\text{Sp}(3)^{(18)}) \to (S^3 \cup e^7) \wedge S^10 \cup S^3 \wedge (S^{10} \cup U_{14} e^{14}) \to (S^3 \cup e^7).$$

The map $\alpha \overline{\Delta} : \text{Sp}(3)^{(18)} \to (S^3 \cup e^7) \wedge S^10 \cup S^3 \wedge (S^{10} \cup U_{14} e^{14})$ equals the composition 

$$\alpha \overline{\Delta} : \text{Sp}(3)^{(18)} \to S^{14} \cup U_{14} e^{14} \to (S^3 \cup e^7) \wedge S^10 \cup S^3 \wedge (S^{10} \cup U_{14} e^{14}).$$

Collapsing the subspace $S^3\wedge(S^{10} \cup U_{14} e^{14})$ of $(S^3 \cup e^7) \wedge S^10 \cup S^3 \wedge (S^{10} \cup U_{14} e^{14})$, we obtain a map 

$$q' \circ \alpha \overline{\Delta} : \text{Sp}(3)^{(18)} \to S^7 \wedge S^{10},$$

where $q' : (S^3 \cup e^7) \wedge S^10 \cup S^3 \wedge (S^{10} \cup U_{14} e^{14}) \to S^3 \wedge S^{10}$ is the collapsing map. For dimensional reasons, $q' \circ \alpha \overline{\Delta}$ equals the composition:

$$q' \circ \alpha \overline{\Delta} : \text{Sp}(3)^{(18)} \to \text{Sp}(3)^{(18)} / \text{Sp}(3)^{(14)} = S^{18} \to S^7 \wedge S^{10}.$$ 

If $\gamma$ were non-trivial, then $\gamma$ would be $\eta_{17} : S^{18} \to S^{17}$, and hence we should have $x_{7}y_{15} = \varepsilon y_{18} \neq 0$. However, from the ring structure of $h^*(\text{Sp}(3))$ given in Theorem 3.4 we know $x_{7}y_{15} = 0$, and hence we obtain $\gamma = 0$. Then the image of $\alpha \overline{\Delta}$ is in the subspace $S^3\wedge(S^{10} \cup U_{14} e^{14})$ of $(S^3 \cup e^7) \wedge S^10 \cup S^3 \wedge (S^{10} \cup U_{14} e^{14})$, since they are 12-connected. Hence $\overline{\Delta}_4 = (1\wedge\overline{\Delta}_3)\circ\overline{\Delta}_5$ equals the composition 

$$\overline{\Delta}_4 : \text{Sp}(3)^{(18)} \to S^3 \wedge (S^{10} \cup U_{14} e^{14}) 1\wedge\overline{\Delta}_4 \Lambda^4\text{Sp}(3)^{(18)},$$

where $(\Lambda^3(S^3 \cup e^7))^{(15)}$ is given by $(S^3 \cup e^7) \wedge S^3 \wedge (S^3 \cup e^7) \wedge S^3 \wedge S^3 \wedge S^3 \wedge (S^3 \cup e^7)$. Collapsing the subspace $\Lambda^3S^3$ of $(\Lambda^3(S^3 \cup e^7))^{(15)}$, we obtain a map 

$$q' \circ \overline{\Delta}_4 : S^{10} \cup U_{14} e^{14} \to S^7 \wedge S^3 \wedge S^3 \cup S^3 \wedge S^7 \wedge S^3 \cup S^3 \wedge S^3 \wedge S^3 \wedge S^7,$$

where $q' : (\Lambda^3(S^3 \cup e^7))^{(15)} \to S^7 \wedge S^3 \wedge S^3 \cup S^3 \wedge S^7 \wedge S^3 \cup S^3 \wedge S^3 \wedge S^3 \wedge S^7$ is the collapsing map. For dimensional reasons, $q' \circ \overline{\Delta}_4$ is the composition 

$$q' \circ \overline{\Delta}_4 : S^{10} \cup U_{14} e^{14} \to S^{14} \to S^7 \wedge S^3 \wedge S^3 \cup S^3 \wedge S^7 \wedge S^3 \cup S^3 \wedge S^3 \wedge S^7.$$ 

If $\gamma'$ were non-trivial, then its projection to $S^{13}$ would be $\eta_{13} : S^{14} \to S^{13}$, and hence we should have $x_{3}x_{7} = \varepsilon y_{14} \neq 0$. However, from the ring structure of
contains a single element $\nu$ where the restriction $\alpha\gamma$ equals the composition
\begin{align*}
\alpha\Delta : Sp(3)^{(18)} &\to S^{14} \cup e^{18} \to S^{3} \land (S^{10} \cup e_{10}^{18}) \to e^{14},
\end{align*}
where the restriction $\alpha'|_{S^{14}}$ equals the composition
\begin{align*}
\alpha'|_{S^{14}} : S^{14} \to S^{3} \land (S^{10} \cup e_{10}^{18} e^{14}).
\end{align*}
If it were non-trivial, then $\gamma''$ would be $\eta_{13} : S^{14} \to S^{13}$, and hence we should have $x_{3}y_{10} = e_{10}y_{14} \neq 0$. However, from the ring structure of $h^{*}(Sp(3))$ given in Theorem 3.7 we know $x_{3}y_{10} = x_{3}^2 x_{7} = e_{10}y_{4} = 0$, and hence we obtain $\gamma'' = 0$. Hence $\alpha\Delta$ equals the composition
\begin{align*}
\alpha\Delta : Sp(3)^{(18)} &\to S^{18} \to S^{3} \land (S^{10} \cup e_{10}^{14} e^{14}),
\end{align*}
and hence $\Delta_{4}$ equals the composition
\begin{align*}
\Delta_{4} : Sp(3)^{(18)} &\to S^{18} \to S^{3} \land (S^{10} \cup e_{10}^{14} e^{14}) \to S^{3} \land S^{3} \land S^{3} \to \lambda^{4} S^{3}.\end{align*}
Now, we are ready to determine $\Delta_{4}$. By Theorem 3.7 we know $x_{3}^{2} x_{11} = e_{10}y_{18}$ and $x_{4}^{2} = e_{7} x_{7}$, hence $\alpha'' : S^{18} \to S^{13} \cup e_{13}^{17}$ is a co-extension of $\eta_{16} : S^{17} \to S^{16}$ on $S^{13} \cup e_{13}^{17}$ and $\lambda\beta : S^{13} \cup e_{13}^{17} \to S^{12}$ is an extension of $\eta_{12} : S^{13} \to S^{12}$. Thus the composition $(1\lambda\beta) \alpha''$ is an element of the Toda bracket $\{\eta_{12}, \eta_{13}, \eta_{16}\}$ which contains a single element $\nu_{12}^{12}$ by Lemma 5.12 of [19] and hence $\Delta_{4} = \nu_{12}^{12} \Delta$. QED.

**Corollary 4.3.1.** $\text{wcat}(Sp(3)^{(18)}) \geq 4$ and $\text{wcat}(Sp(3)) \geq 5$.

This yields the following result.

**Theorem 4.4.**

| Skeleta | $Sp(3)^{(10)}$ | $Sp(3)^{(11)}$ | $Sp(3)^{(10)}$ | $Sp(3)^{(11)}$ | $Sp(3)^{(14)}$ | $Sp(3)^{(15)}$ | $Sp(3)^{(18)}$ | $Sp(3)$ |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|-------|
| wcat    | 1              | 2              | 3              | 3              | 3              | 4              | 4              | 5     |
| cat     | 1              | 2              | 3              | 3              | 3              | 4              | 4              | 5     |
| Cat     | 1              | 2              | 3              | 3              | 3              | 4              | 4              | 5     |

This completes the proof of Theorem 1.2.

5. PROOF OF THEOREM 1.3

We know that for $n \geq 4$,
\begin{align*}
Sp(n)^{(16)} &= Sp(4)^{(15)} = Sp(3)^{(14)} \cup e_{15}, \\
Sp(n)^{(19)} &= \begin{cases} Sp(4)^{(15)} \cup (e_{18} \lor e_{18}) & n = 4, \\ Sp(4)^{(15)} \cup (e_{18} \lor e_{18}) \lor e_{19} & n \geq 5, \end{cases} \\
Sp(n)^{(21)} &= Sp(n)^{(19)} \lor e_{21}
\end{align*}
and that $\text{wcat}(Sp(3)^{(14)}) = \text{cat}(Sp(3)^{(14)}) = \text{Cat}(Sp(3)^{(14)}) = 3$. Firstly, we show the following.

**Proposition 5.1.** $\text{wcat}(Sp(4)^{(15)}) = 3$. 


Proof. Since $\text{cat}(Sp(2)) = 3$, it follows that $\text{wcat}(Sp(4)^{(15)}) \geq 3$ by Theorem 5.1 of [1]. Hence we are left to show $\text{wcat}(Sp(4)^{(15)}) \leq 3$: For dimensional reasons, $\Delta_4 = (\Delta \wedge \Delta) \circ \Delta : Sp(4)^{(15)} \rightarrow \wedge^4 Sp(4)^{(15)}$ equals the composition

$$\Delta_4 : Sp(4)^{(15)} \xrightarrow{\alpha_0} Sp(4)^{(11)} \wedge Sp(4)^{(11)} \xrightarrow{\Delta^3} \wedge^4 Sp(4)^{(11)} \hookrightarrow \wedge^4 Sp(4)^{(15)},$$

for some $\alpha_0$. By Fact 5.5 of [19] $\Delta : Sp(4)^{(11)} \rightarrow \wedge^2 Sp(4)^{(11)}$ equals the composition

$$\Delta : Sp(4)^{(11)} \xrightarrow{\beta_0} (S^7 \vee S^{10}) \cup e^{11} \xrightarrow{\Delta^2} \wedge^2 (S^3 \cup e^7) \hookrightarrow \wedge^2 Sp(4)^{(11)},$$

for some $\beta_0$ and $\gamma_0$. Then for dimensional reasons, $(\beta_0 \wedge \beta_0) \circ \alpha_0 : Sp(4)^{(15)} \rightarrow ((S^7 \vee S^{10}) \cup e^{11}) \wedge ((S^7 \vee S^{10}) \cup e^{11})$ and $(\gamma_0 \wedge \gamma_0) |_{ST \wedge ST} : S^7 \wedge S^7 \rightarrow \wedge^4 (S^3 \cup e^7)$ are respectively equal to the compositions

$$(\beta_0 \wedge \beta_0) \circ \alpha_0 : Sp(4)^{(15)} \xrightarrow{\beta_0^*} S^7 \wedge S^7 \hookrightarrow ((S^7 \vee S^{10}) \cup e^{11}) \wedge ((S^7 \vee S^{10}) \cup e^{11}),$$

$$(\gamma_0 \wedge \gamma_0) |_{ST \wedge ST} : S^7 \wedge S^7 \xrightarrow{\gamma_0^*} \wedge^4 (S^3 \cup e^7),$$

for some $\beta_0^*$ and $\gamma_0^*$. Hence $\Delta_4 : Sp(4)^{(15)} \rightarrow \wedge^4 Sp(4)^{(15)}$ equals the composition

$$\Delta_4 : Sp(4)^{(15)} \xrightarrow{\alpha_0^*} S^7 \wedge S^7 \xrightarrow{\gamma_0^*} \wedge^4 S^3 \rightarrow \wedge^4 Sp(4)^{(15)},$$

where $Sp(4)^{(15)} = Sp(3)^{(14)} \cup e^{15}$. By Theorem 3.7, $x_7^2 = 0$ in $h^*(Sp(3))$, and hence $\alpha_0$ annihilates $Sp(3)^{(14)}$. Thus $\Delta_4 : Sp(4)^{(15)} \rightarrow \wedge^4 Sp(4)^{(15)}$ equals the composition

$$\Delta_4 : Sp(4)^{(15)} \xrightarrow{i' \circ \eta_1} S^15 \xrightarrow{\beta_0^*} S^7 \xrightarrow{\gamma_0^*} S^{12} \xrightarrow{i''} \wedge^4 Sp(4)^{(15)}$$

for some $\beta_0^*$, where $q'' : Sp(4)^{(15)} \rightarrow Sp(4)^{(15)} / Sp(4)^{(14)} = S^{15}$ is the projection and $i'' : S^{12} = S^3 \wedge S^3 \wedge S^3 \wedge S^3 \rightarrow \wedge^4 Sp(4)^{(15)}$ is the inclusion. Hence the non-triviality of $\Delta_4$ implies the non-triviality of $\beta_0^*$ and $\gamma_0^*$. Therefore $\Delta_4$ should be $i'' \circ \eta_1 \circ q''$, if it were non-trivial. However, we also know from (5.5) of [14] that $\eta_1^3 = 12 \nu_{12} = 6 \omega_{12}$ and that $i'' \circ \omega_{12}$ is trivial by Fact 3.7. Therefore, $\Delta_4 : Sp(4)^{(15)} \rightarrow \wedge^4 Sp(4)^{(15)}$ is trivial, and hence $\text{wcat}(Sp(4)^{(15)}) \leq 3$. This implies that $\text{wcat}(Sp(4)^{(15)}) = 3$. QED.

Secondly, we show the following.

Proposition 5.2. $\text{wcat}(Sp(n)^{(19)}) = 4$ for $n \geq 4$.

Proof. Since $\Delta_5 = ((1\circ Sp(n)) \wedge \Delta_4) \circ \Delta : Sp(n)^{(19)} \rightarrow \wedge^5 Sp(n)^{(19)}$, it equals the composition

$$\Delta_5 : Sp(n)^{(19)} \xrightarrow{\alpha} Sp(n)^{(16)} \wedge Sp(n)^{(16)} = Sp(4)^{(15)} \wedge Sp(4)^{(15)} \xrightarrow{(1\circ Sp(n)) \circ \Delta_4} \wedge^5 Sp(4)^{(15)} \rightarrow \wedge^5 Sp(n)^{(19)},$$

which is trivial, since $\Delta_4 : Sp(4)^{(15)} \rightarrow \wedge^4 Sp(4)^{(15)}$ is trivial by Proposition 5.1. Thus $\text{wcat}(Sp(n)^{(19)}) \leq 4$, and hence $\text{wcat}(Sp(n)^{(19)}) = 4$. QED.

Let $p_j : Sp(n) \rightarrow X_{n,j} = Sp(n) / Sp(n - j)$ be the projection for $j \geq 1$. Then we have the following.

Proposition 5.3. Let $q'' : Sp(n) \rightarrow Sp(n) / Sp(n)^{(2n+1)n-3} = S^{(2n+1)n}$ be the collapsing map and $i'' : S^{(2n+1)n-6} \hookrightarrow (\wedge^5 Sp(n)) \cap X_{n,n-3} \cap \cdots \cap X_{n,1}$ the inclusion. Then

$$q'' \circ i'' : \pi_{2n+1}((S^{(2n+1)n-6}) \rightarrow [Sp(n), (\wedge^5 Sp(n)) \cap X_{n,n-3} \cap \cdots \cap X_{n,1}]$$
is injective.

Proof. Firstly, we have the following exact sequence
\[
\pi((2n+1)_n(S(2n+1)^{n-3}) \xrightarrow{\psi''} \pi((2n+1)_n S(2n+1)^{n-6})
\]

\[
i'''' \pi((2n+1)_n ((\wedge^5 Sp(n)) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}) \to \pi((2n+1)_n (S(2n+1)^{n-2}),
\]

where \(\pi((2n+1)_n S(2n+1)^{n-6}) \cong \mathbb{Z}/2\mathbb{Z}^2\) and \(\psi''\) is induced from \(\omega((2n+1)_n \wedge \cdots \wedge X_{n,1}) = 2\nu((2n+1)_n - 6\). Thus \(\psi''\) is trivial, and hence \(i''''\) is injective.

Secondly, since \((\wedge^5 Sp(n)) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}\) is \((n(2n+1) - 11)\)-connected, we have
\[
[Sp(n), (\wedge^5 Sp(n)) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}]
\]

\[
= [(S(2n+1)^{n-7} \cup \omega((2n+1)_n \wedge (\wedge^5 Sp(n)) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1})
\]

\[
= [S(2n+1)^{n-7} \cup \omega((2n+1)_n \wedge (\wedge^5 Sp(n)) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}]
\]

\[
\oplus \pi((2n+1)_n ((\wedge^5 Sp(n)) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1})
\]

by Proposition 4, and hence \(q''''\) is injective. Thus \(q'''' \circ i''''\) is injective. QED.

Then the following lemma implies that \(((1_{w^5 Sp(n)}) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}) \otimes \Delta_{n+2}\) is non-trivial by Proposition 3, and hence we obtain Theorem 4.

Lemma 5.4. \(((1_{w^5 Sp(n)}) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}) \otimes \Delta_{n+2}\) is non-trivial by Proposition 3, and hence we obtain Theorem 4.

Proof. We have
\[
((1_{w^5 Sp(n)}) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}) \otimes \Delta_{n+2}\]

\[
= (\Delta_5 \wedge (1_{w^5 Sp(n)}) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}) \otimes \Delta_{n+2}.
\]

For dimensional reasons, the image of \(((1_{Sp(n)}) \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}) \otimes \Delta_{n+2}\) lies in
\[
Sp(n)^{(21)} \wedge S^{15} \wedge \cdots \wedge S^{4n-1} \cup Sp(n)^{(19)} \wedge X_{n-3} \wedge \cdots \wedge X_{n,1}.
\]

From Proposition 5.2, it follows that \(\Delta_5\) annihilates \(Sp(n)^{(19)}\), and hence it equals the composition
\[
\Delta_5 : Sp(n)^{(21)} \to S^{21} \xrightarrow{\delta} \wedge^5 Sp(n)^{(21)}
\]

for some \(\delta \in \pi_{21}(\wedge^5 Sp(3))\). Then we obtain the following diagram except for the dotted arrow using Lemma 4, which is commutative up to homotopy:
Since the pair $(\wedge^5 Sp(n), \wedge^5 Sp(3))$ is 26-connected for $n \geq 4$, we can compress $\delta$ into $\wedge^5 Sp(3)$ as $\delta \sim j \circ \delta_0$. Thus we have

$$j \circ \delta_0 \circ q' \sim j \circ i' \circ \nu^2_{15} \circ q'.$$

Now we know that $\dim Sp(3) = 21 < 26 - 1$, and hence we can drop $j$ from the above homotopy relation and obtain

$$\delta_0 \circ q' \sim i' \circ \nu^2_{15} \circ q'.$$

By Proposition 4.2, $q'^* : \pi_{21} (\wedge^5 Sp(3)) \to [Sp(3), \wedge^5 Sp(3)]$ is injective, and hence we obtain

$$\delta_0 \sim i' \circ \nu^2_{15}.$$

Thus $\Delta_5$ equals the composition

$$\Delta_5 : Sp(n)^{(21)} \to S^{21} \overset{\nu^2_{15}}{\to} S^{15} \overset{\wedge^5 Sp(n)^{(21)}}{\hookrightarrow}.$$

Thus $((1 \wedge^5 Sp(n)) \wedge_{n-3} \cdots \wedge p_1) \circ \Delta_{n+2}$ equals the composition

$$((1 \wedge^5 Sp(n)) \wedge_{n-3} \cdots \wedge p_1) \circ \Delta_{n+2} : Sp(n) \to S^{21} \wedge S^{(2n+7)(n-3)} \overset{\nu^2_{15}}{\to} S^{15} \wedge S^{(2n+7)(n-3)} \hookrightarrow (\wedge^5 Sp(n)) \wedge X_{n,n-3} \cdots \wedge X_{n,1}.$$

This completes the proof of the lemma. \textit{QED.}

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