Witness for initial system-environment correlations in open-system dynamics

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Abstract – We study the evolution of a general open quantum system when the system and its environment are initially correlated. We show that the trace distance between two states of the open system can increase above its initial value, and derive tight upper bounds for the growth of the distinguishability of open-system states. This represents a generalization of the contraction property of quantum dynamical maps. The obtained inequalities can be interpreted in terms of the exchange of information between the system and the environment, and lead to a witness for system-environment correlations which can be determined through measurements on the open system alone.

When the initial state of an open quantum system is statistically independent of its environment, the evolution of the reduced system can be described by a family of completely positive dynamical maps between the reduced system states. The most simple evolution for initially uncorrelated states is described by a Markov process for which this family of maps forms a dynamical semigroup [1,2]. However, in many systems the Markov description gives an overly simplified picture of the dynamics and a more rigorous treatment is needed [3]. Many methods for treating non-Markovian dynamics have been developed in recent years [4], but the effect of initial correlations is often ignored. The assumption of initially uncorrelated states is well justified whenever the system and the environment are weakly interacting, but it has been argued [5] that the assumption of initially uncorrelated states generally is too restrictive. Therefore, the influence of initial correlations on the open-system dynamics has been recently under intensive study [5–10].

If one considers initial states of the total system with different system-environment correlations, the open-system dynamics can in general no longer be described by a dynamical map acting on the reduced state space [5,6,9]. This is a common physical situation which occurs, for example, when the initial correlations are created by an earlier interaction between the system and its environment. The question thus arises, whether in this situation one can still find general quantitative features that characterize the reduced system dynamics. The answer to this question is, in fact, yes: We demonstrate below that the distinguishability between any two states of the open system can increase above its initial value. This increase has a tight upper bound which can be interpreted as the initial information in the total system which is inaccessible for the open system, \textit{i.e.}, which cannot be obtained through measurements on the open system at the initial time. The existence of this upper bound can be seen as a generalization of the contraction property of dynamical maps and is shown to lead to a measurable witness for correlations in the initial system-environment states.

In the following we shall use the trace distance as a distance measure for quantum states. The trace distance of two quantum states represented by trace class operators $\rho^1$ and $\rho^2$ is defined as $D(\rho^1, \rho^2) = \frac{1}{2} \text{Tr}[\rho^1 - \rho^2]$. It is a metric on the space of physical states, satisfying $0 \leq D \leq 1$, and represents the achievable upper bound for the distinguishability between the probability distributions arising from measurements performed on the quantum states [11].

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Thus, the trace distance can be interpreted as distinguishability between quantum states, or in other words, as the “which state” information that can be gained through measurements on the system.

Consider now a bipartite quantum system consisting of a system $S$ coupled to an environment $E$, such that together they form an isolated system described by the initial state $\rho_{SE}$. The state of the system $S$ at time $t$ can then be written as

$$\rho_S(t) = \text{Tr}_E[U_t \rho_{SE} U_t^\dagger],$$

where $U_t = \exp[-i H t/\hbar]$ represents the unitary time evolution operator of the composite system with total Hamiltonian $H$, and $\text{Tr}_E$ denotes the partial trace over the environment. Given two initial states $\rho_{SE}^1$ and $\rho_{SE}^2$ of the composite system the trace distance between the corresponding reduced system states $\rho_S^1(t)$ and $\rho_S^2(t)$ at time $t$ is given by $D(t) \equiv D(\rho_S^1(t), \rho_S^2(t))$. For $dD/dt < 0$ the trace distance and, hence, the distinguishability between the reduced states decrease. We interpret this as a flow of information from the system to the environment. Correspondingly, whenever we have $dD/dt > 0$ the distinguishability of the pair of reduced states increases, and we interpret this increase as a reversed flow of information from the environment to the system [12].

If one assumes that the system and the environment are initially uncorrelated with a fixed environmental state $\rho_E$, i.e., $\rho_{SE} = \rho_S \otimes \rho_E$, one can describe the time evolution of the reduced system given by eq. (1) through a family of completely positive dynamical maps $\Phi_t$,

$$\rho_S \mapsto \rho_S(t) = \Phi_t \rho_S = \text{Tr}_E[U_t \rho_S \otimes \rho_E U_t^\dagger],$$

which map the state space of the reduced system into itself. It is a well known fact [11,13] that such dynamical maps are contractions for the trace distance, i.e., for any initial pair of states $\rho_S^1, \rho_S^2$ and for any time $t \geq 0$ we have

$$D(\rho_S^1, \Phi_t \rho_S^2) \leq D(\rho_S^1, \rho_S^2).$$

However, if initial correlations are present eq. (2) does not apply. This allows the situation where the trace distance of the reduced system states grows to values which are larger than the initial trace distance as is illustrated by the upmost curve in fig. 1(a).

Our aim is to construct upper bounds for the growth of the trace distance in the presence of initial correlations. To this end, we consider an arbitrary pair of initial states $\rho_{SE}^{1,2}$ of the total system with corresponding reduced system states $\rho_S^{1,2} = \text{Tr}_E[\rho_{SE}^{1,2}]$ and environment states $\rho_E^{1,2} = \text{Tr}_S[\rho_{SE}^{1,2}]$. One then finds the inequality

$$D \left( \text{Tr}_E[U_t \rho_{SE}^1 U_t^\dagger], \text{Tr}_E[U_t \rho_{SE}^2 U_t^\dagger] \right) - D(\rho_S^1, \rho_S^2) \leq D(\rho_{SE}^{1,2}) - D(\rho_S^1, \rho_S^2) \equiv I(\rho_{SE}^{1,2}),$$

which states that the increase of the trace distance of the states $\rho_S^1$ and $\rho_S^2$ during the time evolution given by eq. (1) is bounded from above by the quantity $I(\rho_{SE}^{1,2})$. The inequality can easily be derived by employing the invariance of the trace distance under unitary transformations and by using the fact that the trace distance is non-increasing under the partial trace operation. The upper bound $I(\rho_{SE}^{1,2})$ represents the distinguishability of the initial states $\rho_{SE}^{1,2}$ of the total system minus the distinguishability of the corresponding reduced system states $\rho_S^{1,2}$. Hence, this quantity represents the loss of distinguishability of the initial total states which results when measurements on the reduced system only can be performed. One can thus interpret $I(\rho_{SE}^{1,2})$ as the information which lies initially outside the open system and is inaccessible for it. The inequality (3) therefore leads to the following physical interpretation: The maximal amount of information the open system can gain from the environment is the amount of information flowed out,
earlier from the system since the initial time, plus the information which is initially outside the open system.

As we shall demonstrate in several examples below, the inequality (3) can in fact become an equality, showing that the upper bound of this inequality is tight. When the bound of eq. (3) is actually reached at a certain time $t$, the distinguishability of the reduced system states at time $t$ is equal to the distinguishability of the total system states at time zero. This means that the information in the total system at time zero has been transferred completely to the reduced system at time $t$.

Obviously, the contraction property eq. (2) for completely positive maps is a special case of the inequality (3) which occurs if both total initial states are taken to be uncorrelated with the same environmental state, i.e. $\rho_{SE}^{1,2} = \rho_S^{1,2} \otimes \rho_E$. This follows by using the invariance of the trace distance under the tensor product which yields

$$I(\rho_S \otimes \rho_E, \rho_S^2 \otimes \rho_E) = 0,$$

implying that initially there is no information outside the reduced system.

A further important special case of the inequality (3), which reveals most clearly the role of initial correlations, is obtained if we choose $\rho_{SE}^{1,2}$ to be the fully uncorrelated state constructed from the marginals of $\rho_{SE}^{1,2}$, i.e., $\rho_{SE}^{1,2} = \rho_S^{1,2} \otimes \rho_E$. In this case inequality (3) simplifies to

$$D\left(\mathrm{Tr}_E[U_1 \rho_{SE}^1 U_1^†], \mathrm{Tr}_E[U_1 \rho_{SE}^2 \otimes \rho_E^2 U_1^†]\right) \leq D(\rho_{SE}^1, \rho_S^1 \otimes \rho_E).$$

This inequality shows how far from each other two initially indistinguishable reduced states can evolve when only one of the two total initial states is correlated. Here, the initial information $I(\rho_{SE}^1, \rho_{SE}^2)$ in the total system which is inaccessible for the open system is equal to $D(\rho_{SE}^2, \rho_S^1 \otimes \rho_E^2)$. This quantity describes how well the state $\rho_{SE}^2$ can be distinguished from the corresponding fully uncorrelated state $\rho_S^1 \otimes \rho_E^2$ and, therefore, provides a measure for the amount of correlations in the state $\rho_{SE}^2$. Thus, the increase of the trace distance is bounded from above by the correlations in the initial state.

Returning to the general case, we use the subadditivity of the trace distance with respect to tensor products to conclude from inequality (3)

$$D\left(\mathrm{Tr}_E[U_1 \rho_{SE}^1 U_1^†], \mathrm{Tr}_E[U_1 \rho_{SE}^2 \otimes \rho_E^2 U_1^†]\right) - D(\rho_{SE}^1, \rho_{SE}^2) \leq D(\rho_{SE}^2, \rho_S^1 \otimes \rho_E) = D(\rho_{SE}^1, \rho_S^1 \otimes \rho_E) \leq D(\rho_{SE}^2, \rho_S^1 \otimes \rho_E) + D(\rho_{SE}^1, \rho_S^1 \otimes \rho_E),$$

where the second inequality follows by using twice the triangle inequality for the trace distance. This inequality clearly shows that in the most general case an increase of the trace distance of the reduced states implies that there are initial correlations in $\rho_{SE}^1$ or $\rho_{SE}^2$, or that the initial environmental states are different. For identical environmental states any increase of the trace distance is a witness for the presence of initial correlations.

We discuss some examples to illustrate the above inequalities. First, we consider two qubits under the controlled-NOT gate, i.e., under an interaction given by the unitary operator $U_C: |00\rangle \rightarrow |00\rangle, |01\rangle \rightarrow |01\rangle, |10\rangle \rightarrow |11\rangle, |11\rangle \rightarrow |10\rangle$. We regard the first qubit as the system $S$ (control qubit), and the second qubit as the environment (target qubit). Let us study the states $|9\rangle$

$$\rho_{SE}^1 = (|\alpha| |00\rangle + |\beta| |11\rangle)(|\alpha^*| |00\rangle + |\beta^*| |11\rangle),$$

with $\alpha, \beta \neq 0$. The state $\rho_{SE}^1$ is a pure entangled state, while $\rho_{SE}^2$ is a mixed, classically correlated state with nonzero quantum discord [14]. For these two total states the system states and the environmental states coincide:

$$\rho_S = \rho_E = |\alpha|^2 |00\rangle \langle 00| + |\beta|^2 |11\rangle \langle 11| = \rho_{SE}^1 = \rho_{SE}^2.$$

Under the action of the controlled-NOT gate the increase of the trace distance is found to be

$$D(\mathrm{Tr}_E[U_C \rho_{SE}^1 U_C^†], \mathrm{Tr}_E[U_C \rho_{SE}^2 U_C^†]) = |\alpha, \beta|,$$

witnessing that at least one of the initial states must have been correlated. The right-hand side of the inequality (3) is given by $D(\rho_{SE}^1, \rho_{SE}^2) = |\alpha, \beta|$ which shows that the upper bound of this inequality is indeed reached here and that, therefore, the initial information in the total system is transferred completely to the reduced system by the controlled-NOT gate.

Let us also study a situation described by the inequality (4) for a pair consisting of a correlated and the corresponding uncorrelated state. For the state $\rho_{SE}^1$ in eq. (6) we obtain

$$D\left(\mathrm{Tr}_E[U_C \rho_{SE}^1 U_C^†], \mathrm{Tr}_E[U_C \rho_S \otimes \rho_E^2 U_C^†]\right) = |\alpha, \beta|.$$
in which the initial states are purely classically correlated. Moreover, we observe from eq. (7) and eq. (8) that the equality sign in eq. (4) holds, demonstrating again the tightness of the bound provided by this inequality.

Finally, we consider a central spin with Pauli operator $\sigma$ (the open system) which interacts with a bath of $N$ spins with Pauli operators $\sigma^{(k)}$ through the Hamiltonian $H = A_0 \sum_{k=1}^{N} (\sigma^z + \sigma^x)$. An example for a pair of initial states which leads to an increase of the trace distance over its initial is given by

$$\rho_{SE}^1 = |\Psi\rangle\langle \Psi|, \quad |\Psi\rangle = \alpha |+\rangle \otimes |\chi_+\rangle + \beta |+\rangle \otimes |\chi_-\rangle,$$

$$\rho_{SE}^2 = |\alpha|^2 |\Psi\rangle\langle \Psi| + |\beta|^2 |\Psi\rangle\langle \Psi| + 1 \otimes |\chi_-\rangle \langle \chi_-|,$$

where $|\pm\rangle$ are central spin states, and $|\chi_+\rangle = |++\ldots+\rangle$ and $|\chi_-\rangle = \frac{1}{\sqrt{N}} \sum_k |k\rangle$ are environment states. The state $|k\rangle$ is obtained from $|\chi_+\rangle$ by flipping the $k$-th bath spin. The states $\rho_{SE}^1$ and $\rho_{SE}^2$ have the same marginals and, thus, differ from one another only by the correlations. We find that the increase of the trace distance is given by

$$D\left( \text{Tr}_E[U_{\rho_{SE}^1} U_{\rho_{SE}^2} U_{\rho_{SE}^1}^\dagger], \text{Tr}_E[U_{\rho_{SE}^1} U_{\rho_{SE}^2} U_{\rho_{SE}^1}^\dagger]\right) = \mathbb{E}(\alpha \beta) \sin(2At),$$

where $A = \sqrt{N}A_0$. The trace distance thus oscillates periodically between the initial value zero and the maximal value $|\mathbb{E}(\alpha \beta)|$. We conclude that for almost all values of the amplitudes $\alpha$ and $\beta$ there is an increase of the trace distance witnessing the initial correlations. Moreover, we have $D(\rho_{SE}^1, \rho_{SE}^2) = |\alpha \beta|$. Hence, if $\alpha \beta$ is real the upper bound of inequality (3) is reached periodically whenever $|\sin(2At)| = 1$, as is shown in fig. 1(b) for the case $\alpha = \beta = 1/\sqrt{2}$. A further example for the increase of the trace distance as a correlation witness has recently been discussed in ref. [15].

How can one use the above results to develop experimental methods for the detection of correlations in an unknown initial state $\rho_{SE}^1$? To this end, one has to perform a state tomography on the open system at the initial time zero and at some later time $t$, to determine the reduced states $\rho_{SE}^1(0) = \text{Tr}_E[\rho_{SE}^1]$ and $\rho_{SE}^2(t) = \text{Tr}_E[U_{\rho_{SE}^1} U_{\rho_{SE}^2} U_{\rho_{SE}^1}^\dagger]$. In order to apply inequality (5) we need a second reference state $\rho_{SE}^2$ for the evolution of which is compared with that of the state $\rho_{SE}^1$, and which has the same environmental state as $\rho_{SE}^1$, i.e., $\rho_{SE}^2 = \rho_{SE}^1$. This can be achieved by performing a local trace-preserving quantum operation on $\rho_{SE}^1$ to obtain the state $\rho_{SE}^2 = (S \otimes I) \rho_{SE}^1$. The operation $S$ acts locally on the variables of the open system ($I$ denotes the identity), and may be realized, for instance, by the measurement of an observable of the open system, or by a unitary transformation induced, e.g., through an external control field. For example, the state $\rho_{SE}^1$ in eq. (9) is obtained from $\rho_{SE}^1$ through a non-selective measurement of the $z$-component of the central spin. The state tomography carried out on the open system will then yield the states $\rho_{SE}^2(0) = \text{Tr}_E[\rho_{SE}^2]$ and $\rho_{SE}^2(t) = \text{Tr}_E[U_{\rho_{SE}^1} U_{\rho_{SE}^2} U_{\rho_{SE}^1}^\dagger]$. With these results one can check whether the trace distance increased over its initial value. If one finds that $D(\rho_{SE}^1(t), \rho_{SE}^2(t)) > D(\rho_{SE}^1(0), \rho_{SE}^2(0))$, the inequality (5) implies that the original system-environment state $\rho_{SE}^1$ was correlated. In fact, if $\rho_{SE}^1$ was uncorrelated then also $\rho_{SE}^2$ was uncorrelated since it has been obtained from $\rho_{SE}^1$ through a local operation. Therefore, for an uncorrelated state $\rho_{SE}^1$ the trace distance cannot increase according to inequality (5). Thus, any such increase represents a witness for correlations in the initial state $\rho_{SE}^1$.

We note that this strategy for the observation of initial correlations presupposes only local control and measurements of the open quantum system. It neither requires a knowledge of the structure of the environment or of the system-environment interaction, nor a full knowledge of the initial system-environment state $\rho_{SE}^1$. Moreover, there is no principal restriction on the operation $S$ used to generate the reference state $\rho_{SE}^2$, which opens a large number of possible experimental realizations. For example, an experimental implementation can be done using single photons from a quantum dot travelling in a PM fiber [16].

In summary, we have studied the dynamics of open systems with initial system-environment correlations. It has been shown that the growth of the distinguishability of the reduced states is bounded from above by the initial information lying outside the open system. The obtained inequalities can be interpreted in terms of the exchange of information between the system and its environment: If the trace distance increases over its initial value, information which is locally inaccessible at the initial time is transferred to the open system. This transfer of information enlarges the distinguishability of the open-system states which suggests various ways for the experimental detection of initial correlations.

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