Higher Dimensional Particle Model Construction in Third-Order Lovelock Gravity

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By using the formalism of thin-shells, we construct a geometrical model of particle in third-order Lovelock gravity. This particular theory which is valid at least in 7 dimensions, provides enough degrees of freedom and grounds towards such a construction. The particle consists of a flat interior and a non-black hole exterior spacetimes whose mass, charge and radius are determined from the junction conditions, in terms of the parameters of the theory.

I. INTRODUCTION

Geometrical description of elementary particles attracted interest of physicists at different stages of physics history [1, 2]. With the advent of general relativity all attempts in that direction focussed on the curvatures and singularities of spacetimes as potential sites to represent particles. In his description Geometrodynamics [3], John Wheeler advocated the view of concentrated field points - the Geons - as particle-like structures in spacetimes. More recently, the trend of constructing particle models from geometry oriented towards non-singular spacetimes such as the de Sitter core with the cosmological constant. Since a particle can also have charge besides mass, the electromagnetic spacetimes such as the non-singular Bertotti-Robinson geometry [4, 5] also attracted attentions [6], in this regard. Having no interior singularity and possessing both mass and electric charge became basic criteria in search for a geometrical model of particles. The method has been to consider a spherical shell as the representative surface of the particle with inner/outer regions satisfying certain boundary junction conditions. Among those conditions we cite continuity of the first fundamental form (the metric) and possible surface energy-momentum from the discontinuity conditions of the second fundamental form (the extrinsic curvature).

In this letter, we show that geometrical model of a particle is possible within the context of third-order Lovelock gravity combined with the thin-shell formalism. Lovelock gravity [7], is known to have the most general combination of curvature invariants that still maintains the second-order field equations. Our model will cover up to the third-order terms, supplemented by the Maxwell Lagrangian, apt for the dimension of \( n+1 \)-dimensional spacetimes with \( n \geq 6 \), in which the theory admits non-trivial solutions. We made choice amongst available solutions of the theory that suits our purpose. A spherical thin-shell is assumed as surface of the particle, whose inside is the flat Minkowski spacetime with a suitable Lovelock solution to represent the outer region. Deliberate choice of the Lovelock’s coupling constants in the action, renders physical boundary conditions possible for a surface energy-momentum on the shell. As the final step, we set the pressure and surface energy density to zero and search for viable geometrical criteria. Those conditions determine the mass and charge of the particle constructed, entirely from the geometrical parameters of the third-order Lovelock gravity in 6 + 1 dimensions, as an example.

Overall, the letter is organized in the following sections. In section II, rather than the thin-shell formalism in general, the particular choices of inner/outer solutions within the third-order Lovelock gravity framework are introduced. In section III, we briefly review the proper junction conditions for a thin-shell in the third-order Lovelock gravity. Section IV is devoted to conclusion. All over the letter, the unit convention \( 4\pi \varepsilon_0(n+1) = 8\pi G(n+1) = c = 1 \) is applied.

II. THIN-SHELL FORMALISM

Consider a spherically symmetric Riemannian manifold in \( n+1 \) dimensions, with two distinct regions, say \( (\Sigma, g) = \{ x_\pm \mid r_+ \geq a > r_e, r_- \leq a \} \) (where \( r_e \) is a (probable) event horizon), distinguished by their common timelike hypersurface \( \partial \Sigma = \{ \xi^\mu | r = a \} \). The hypersurface \( \partial \Sigma \) is therefore a thin-shell separating the two regions with different line elements and probably different coordinates \( x_\pm^\mu \). The inner spacetime (marked with (−)) is (preferably) non-singular and the outer spacetime (marked with (+)) has its event horizon behind the thin–shell (if there is any).
For the purpose of this study, we would like to have for both inner and outer spacetimes, the solutions to the third-order Lovelock gravity. Expectedly, the two spacetimes cannot be matched randomly at the thin-shell location, and it takes some proper junction conditions which will be discussed below.

In the $n + 1$-dimensional third-order Lovelock gravity ($n \geq 6$), the action is given by

$$ I = \int d^{n+1}x \sqrt{-g} \left( \mathcal{L}_1 + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 - F_{\mu\nu} F_{\mu\nu} \right), \quad (1) $$

in the presence of an electromagnetic field with the anti-symmetric field tensor $F_{\mu\nu}$. Here,

$$ \mathcal{L}_1 = R, \quad \mathcal{L}_2 = R^\alpha_{\lambda\mu\nu} R_{\alpha\lambda\mu\nu} - 4R_{\mu\nu} R^{\mu\nu} + R^2, \quad (2) $$

and

$$ \mathcal{L}_3 = 2R^\alpha_{\lambda\rho\sigma} R_{\rho\sigma\mu\nu} R^\mu_{\alpha\lambda} + 8R^\mu_{\alpha\lambda} R^{\rho\sigma}_{\nu\rho} R^\lambda_{\mu\nu} + 24R^\mu_{\alpha\lambda\nu} R_{\nu\lambda\rho} R^\rho_{\mu} + 3RR^\alpha_{\lambda\mu\nu} R_{\alpha\lambda\mu\nu} $$

$$ + 24R^\alpha_{\lambda\rho\sigma} R_{\mu\alpha\nu} R_{\nu\lambda\rho} + 16R_{\mu\nu} R_{\nu\sigma} R^\sigma_{\mu} - 12RR_{\mu\nu} R^{\mu\nu} + R^3 \quad (4) $$

are the first (Einstein-Hilbert), the second (Gauss-Bonnet), and the third-order Lovelock Lagrangians, respectively, accompanied with their respective coefficients $\alpha_2$, and $\alpha_3$. The spherically symmetric line element suitable for this action is given by

$$ ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\Omega^2_{n-1}, \quad (5) $$

where $d\Omega^2_{n-1}$ is the line element of the $n - 1$-dimensional unit sphere. Although there are three general solutions for $f(r)$, the one we consider here is (see [8, 9] and references therein)

$$ f(r) = 1 - r^2 \left( -\frac{\tilde{\alpha}_2}{3\tilde{\alpha}_3} + \delta + u\delta^{-1} \right), \quad (6) $$

where

$$ \left\{ \begin{array}{l}
\tilde{\alpha}_2 = (n - 3)(n - 2)\alpha_2 \\
\tilde{\alpha}_3 = (n - 5)(n - 4)(n - 3)(n - 2)\alpha_3 ,
\end{array} \right. \quad \delta = \left( v + \sqrt{v^2 - u^2} \right)^{1/3} \quad (7) $$

and

$$ \left\{ \begin{array}{l}
u = \frac{9\tilde{\alpha}_2 - 3\tilde{\alpha}_3}{54\tilde{\alpha}_3} \\
v = \frac{9\tilde{\alpha}_2 - 2\tilde{\alpha}_3}{54\tilde{\alpha}_3} + \frac{1}{2\tilde{\alpha}_3} \left[ m \frac{r^n}{r^{2(n-1)}} - \frac{q^2}{r^{2(n-1)}} \right] \end{array} \right. \quad (8) $$

In the special case where $\tilde{\alpha}_2 = 3\tilde{\alpha}_3 = 3^2$, one finds $u = 0$ and $\delta = (2v)^{1/3}$, with which the solution in Eq. (6) reduces to

$$ f(r) = 1 + \frac{r^2}{\beta} \left\{ 1 + 3\beta \left( m \frac{r^n}{r^{2(n-1)}} - \frac{q^2}{r^{2(n-1)}} \right)^{1/2} \right\}. \quad (9) $$

Depending on the values of the mass $m$, the charge $q$, and the number of spatial dimensions $n$, this particular solution could represent a black hole with two horizons, an extremal black hole with a single horizon, or a non-black hole solution with a naked singularity. In this study, we assign to the inner spacetime an $n + 1$-dimensional Minkowski geometry, which amounts to choosing $m_- = q_- = 0$, hence, $f_- (r_-) = 1$. Furthermore, we consider $m_+ = m$, $q_+ = q$, and that $a > r_+$ for the outer spacetime, where $r_+$ is the event horizon of $f_+ (r_+)$ (if there is any), and $f_+ (r_+)$ has the general form in Eq. (6). Note that, although we exploited the special choice of $\tilde{\alpha}_2 = 3\tilde{\alpha}_3 = 3^2$ to obtain Eq. (9), we distinguish $\beta_+$ and $\beta_-$ for the outer and inner spacetimes, respectively. With the choice $f_- (r_-) = 1$, however, the parameter $\beta_-$ remains arbitrary, which will play role in determining the mass and the charge of the particle. In the next section, this will arise (see Eq. (19) below) explicitly.
As it is already mentioned, the matching at \( r_\pm = a \) follows certain junction conditions. In general relativity these are known as Darmois-Israel junction conditions which are, however, inapplicable when it comes to modified theories of gravity. In the case of the third-order Lovelock gravity the proper junction conditions are given by Dehghani et al. in [10, 11] (Also see [9, 12]). These junction conditions firstly demand the continuity of the first fundamental form at the thin-shell’s surface, so one has a smooth transition while passing across the shell. The line element of the thin-shell can be stated as

\[
    ds_{ab}^2 = \gamma_{ab} d\xi^a d\xi^b = -d\tau^2 + a^2 d\Omega_{n-1},
\]

where \( \tau \) is the proper time on the shell and

\[
    \gamma_{ab} = \frac{\partial x^a}{\partial \xi^\pm} \frac{\partial x^b}{\partial \xi^\mp} g_{\mu
u}.
\]

The first junction condition, then, demands that

\[
    f_+ (a) = f_- (a),
\]

for a static shell, which for our choice of outer and inner metric functions immediately leads to

\[
    a = \left( \frac{q^2}{m} \right)^{1/(n-2)}.
\]

This static radius, is therefore, where the two spacetimes could join smoothly. Note that, in three spatial dimensions \( (n = 3) \) this radius is interestingly in full agreement with the classical radius of a charged particle, remarking the unit convention used here. Consequently, this expression for the radius of the particle in this model can be regarded as the classical radius of a charged particle in higher dimensional third-order Lovelock gravity.

Secondly, there exists a non-zero surface energy-momentum tensor on the shell \( S_{ab} \), with their components given in orthonormal coordinates for a static thin-shell by

\[
    - S_b^a = [K_b^\alpha - K_0^\alpha + 2\alpha_2 (3J_b^\alpha - J_0^\alpha) + 3\alpha_3 (5P_b^\alpha - P_0^\alpha + L_2 (K_b^\alpha + K_0^\alpha))],
\]

where \([\ ]^+\) denotes a jump in the expression inside the brackets, i.e.

\[
    [\Psi]^+ = \Psi_+ - \Psi_-.
\]

Herein, \( K_b^\alpha \) are the mixed tensor components of the extrinsic curvature tensor of the shell given by

\[
    K_{ab}^\pm = -n_a^\pm \left( \frac{\partial^2 x^\pm}{\partial \xi^a \partial \xi^b} - \Gamma^{\pm}_{\alpha\beta} \frac{\partial x^a}{\partial \xi^\alpha} \frac{\partial x^b}{\partial \xi^\beta} \right),
\]

where \( n_\lambda^\pm \) are the unit spacelike normals to the surface identified by \( n^\pm_\mu \frac{\partial x^\pm}{\partial \xi^\mu} = 0, n^\pm_\mu n^\pm_\mu = 1 \), and \( \Gamma^\pm_{\alpha\beta} \) are the Christoffel symbols of the outer and inner spacetimes, compatible with \( g^{\pm}_{\mu\nu} \). Note that, since the metric of our bulks are diagonal, for our radially symmetric static timelike shell the extrinsic curvature tensor will be diagonal, as well. Also, \( J_0^\alpha \) are the Kronecker symbol and \( J \) and \( P \) are the respective traces of \( J_{ab} \) and \( P_{ab} \), with their corresponding mixed tensors for a diagonal metric, such as the one in Eq. (9), are specified by

\[
    \begin{align*}
    J_0^a &= \text{diag} \left( -\frac{2}{3} \left\{ \sum_{s=0}^{\pm} \frac{(-1)^s}{n} \left[ sK_0^\tau + (n-s) K_0^\alpha \right] (K_0^\alpha)^{s-1} (K_0^\beta)^{n-s} \right\} \right), \\
    P_0^a &= \text{diag} \left( \frac{4}{3} \left\{ \sum_{s=0}^{\pm} \frac{(-1)^s}{n} \left[ sK_0^\tau + (n-s) K_0^\alpha \right] (K_0^\alpha)^{s-1} (K_0^\beta)^{n-s} \right\} \right).
    \end{align*}
\]

Here, \( K_0^\tau \) and \( K_0^\beta \) are the components associated with the time and angular coordinates of the thin-shell, since \( K_0^\theta = K_0^{\theta_1} = K_0^{\theta_2} = \ldots = K_0^{\theta_{n-1}} \). In what follows we take \( n = 6 \), noting that the same argument can be applied to higher dimensions, at equal ease.

Having everything done on the second junction condition, we arrive at

\[
    \begin{align*}
    \sigma &= -\frac{5}{3a^2} \sum_{i=+,-} i\sqrt{f_i} \left\{ \left[ 3a^2 + \beta_i (3 - f_i) \right]^2 + \frac{4}{3a} \beta_i^2 f_i^2 \right\}, \\
    p &= \frac{1}{2a^2} \sum_{i=+,-} \frac{1}{\sqrt{f_i}} \left\{ f_i^4 \left[ a^2 + \beta_i (1 - f_i) \right]^2 + 8a f_i \left[ a^2 + \beta_i (1 - f_i/3) \right] \right\},
    \end{align*}
\]

which are the surface energy density and angular pressure of the fluid on the shell, in accordance with the energy-momentum tensor \( S_b^a = \text{diag} (-\sigma, p, \ldots, p) \), respectively. By applying the radius of the particle acquired from the
first junction condition (Eq. (12)), we set both $\sigma$ and $p$ to zero [2, 6] to obtain the one and only solution for $m$ and $q$ as

$$\{m, q\} = \left\{ \frac{8 (\beta^2 - \beta_+^2)}{15}; \pm \frac{8 (\beta_- + \beta_+) \sqrt{\beta_-^2 - \beta_+^2}}{5\sqrt{30}} \right\}. \tag{17}$$

This in turn yields

$$a = \sqrt{\frac{2}{5} |\beta_- + \beta_+|}, \tag{18}$$

for the radius of the particle, according to Eq. (12). To have physically meaningful quantities, then, we must impose $|\beta_-| > |\beta_+|$. This condition is specially surprising in the sense that the inner spacetime is flat, and one may naively think that the value of $\beta_-$ would not affect the results in a great deal; say, it could be set to zero from the beginning. However, as can be perceived from the condition $|\beta_-| > |\beta_+|$, setting $\beta_- = 0$ (which directly assigns a usual Minkowski geometry to the inner spacetime), will not lead to a physically sensible mass for the particle. Moreover, note that the steps leading to Eq. (17) could have been looked at in a different way. Theoretically, having $\beta_-$ and $\beta_+$ as the constants of the theory, one can find the mass, the charge and the radius of the particle using Eqs. (17) and (18).

However, an experimentalist would rather find $\beta_-$ and $\beta_+$ by fine-tuning them such that the exact values of the mass and the charge of the fundamental particles come out of the theory. In other words, we could have solved the system of equations $\sigma = 0$ and $p = 0$ for $\beta_-$ and $\beta_+$ instead of $m$ and $q$. This would result in

$$\beta_\pm = \pm \frac{(3m^2 + 10q^2)}{8\sqrt{mq^2}}. \tag{19}$$

Consequently, taking into account the positivity of the mass, $\beta_-$ will always be a negative constant. On the contrary, depending on the mass and the charge of the particle, $\beta_+$ can be either positive or negative as long as it satisfies $|\beta_-| > |\beta_+|$.\n
Fig. 1 is plotted for $f(a)$ against $a$ for different values of $\beta_+$, where we have applied $\beta_- = -1$ and hence $-1 < \beta_+ < 1$. Inside the particle we have $f_-(a) = 1$ and outside the particle we have $f_+(a)$. Under these conditions the solution is real and non-black hole for all the values of $\beta_+$, post-particle’s radius, and therefore, can be considered as a consistent particle model.

Here are some remarks. From Eq. (17) it is evident that the charge $q$ is zero either when $\beta_- = \beta_+$ or $\beta_- = -\beta_+$. However, these two choices are banned since they also lead to a null mass (and a null equilibrium radius in the latter case). Consequently, at least for $n = 6$ and the particular solution that we considered here (Eq. (9)), the existence of the Maxwell Lagrangian in the action is essential. In other words, for $n = 6$ and the metric function in Eq. (9) as the exterior region, the charge must be non-zero. Let us also note that we narrowed down our degrees of freedom by choosing the special case $\tilde{\alpha}_2^2 = 3\tilde{\alpha}_3 = \beta^2$ and $n = 6$.

**IV. CONCLUSION**

In a satisfactory geometric description of a particle the physical properties are expressible in terms of the parameters of the theory. For a charged spherical model, for instance in $3 + 1$-dimensional, the mass $m$ and the charge $q$ are constrained to satisfy the condition $a =$radius$= q^2/m$. In search for an analogous model in higher dimensions, we employ the third-order Lovelock gravity as a useful model. The reason for this choice relies on the existence of enough free parameters to define the particle properties. The existence of exact solutions, suitable to define thin, spherical shells, provides enough motivating factors towards construction of a particle model. For $n + 1$ dimensions ($n \geq 6$), we consider a spherical shell as representative of a particle whose inside is a flat vacuum to be connected through the junction conditions to a curved, asymptotically flat outside region. In the process, the emergent fluid’s energy-momentum components on the shell are required to vanish. This yields the mass and the charge of the shell (a particle) in terms of the parameters of the theory, which are to be tuned-finely. The dimensionality naturally reflects in the radius of the particle, i.e. $a = (q^2/m)^{1/(n-2)}$. Finally, let us add that further tuning of the parameters (without the Maxwell Lagrangian) renders the construction of a chargeless massive particle also possible.
FIG. 1: The graphs illustrate the metric function $f(a)$ ($f_+(a)$ of the exterior and $f_-(a) = 1$ of the interior spacetimes) versus the equilibrium radius $a$, for five different values of $\beta_+$ where $\beta_- = -1$. The results suggest that for the admissible domain of $\beta_+$ the particle model is feasible. The vertical line is the location of the particle’s radius.
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