Reorientational solitons in nematic liquid crystals with modulated alignment

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This paper is dedicated to one of its coauthors, Professor Antonmaria (Tim) A. Minzoni, who prematurely passed away during its preparation. N.F.S. and G.A. remember Tim as a generous person of vast culture, a dear friend and an outstanding colleague.

I. INTRODUCTION

Nematic liquid crystals (NLCs) are anisotropic, typically uniaxial, soft matter with several peculiar properties. As suggested by the name, derived from the Greek, they consist of thread-like molecules and exhibit orientational but no spatial order [1]. The anisotropic molecules are in a fluid state, linked by elastic forces, and exhibit two refractive index eigenvalues, ordinary and extraordinary, for light polarized perpendicular or parallel to the optic axis, termed the molecular director and usually denoted by the unit vector \( \hat{n} \). The refractive index of extraordinary polarized light has a nonlinear optical dependence through the reorientational response: the electric field of the light beam induces dipoles and the wavevector are not mutually orthogonal since the wave vector of the light beam and the molecular director and wavevector. Nematicon walk-off can be exploited in optical devices, for instance, signal demultiplexers or routers [20–26].

In uniform NLCs, nematicons propagate along rectilinear trajectories along their Pointing vector. The corresponding graded index waveguides associated with these spatial solitons are therefore straight. Curved light induced waveguides have been investigated in NLCs by means of graded interfaces [4, 24, 25, 27, 28], localized refractive index perturbations [21, 29, 30], interactions with boundaries [11, 31, 32] as well as other nematicons [6, 12, 33, 34]. At variance with previous approaches, in this article we introduce and study curved reorientational spatial solitons as they propagate in nematic liquid crystals with a linearly varying orientation of the optic axis across the transverse coordinate in the principal plane (defined by director and wavevector). We consider nematicons excited in a planar cell of fixed (uniform) thickness, with upper and lower interfaces treated to ensure planar anchoring of the NLC molecules. This geometry is radically different from those entailing spin-orbit interactions of light with matter [35, 38], as the optic axis and the wavevector are not mutually orthogonal since the light beam is an extraordinary wave. As the molecular
alignment varies across the sample, both the extraordinary refractive index and the birefringent walk-off vary as well. These two variations determine the resulting trajectory of extraordinarily polarized beams in the cell, including the path of self-confined nematicons. To investigate nematicon paths in a transversely modulated uniaxial nematic liquid crystal, we use two different approaches in the weakly nonlinear regime (i.e. power independent walk-off): (i) numerical solutions of the full governing Maxwell’s equations employing a fully vectorial beam propagation method for the beam and the Frank-Oseen elastic theory for the NLC response [2]; (ii) an adiabatic (slowly varying) approximation to yield simplified forms of these equations, invoking momentum conservation [2, 39]. The adiabatic approximation is based on the high nonlocality of the NLCs, which implies that the nonlinear response extends far beyond the transverse size of the optical wavepacket [2, 40, 41] and decouples the amplitude/width evolution of the beam from its trajectory [42, 43]. In this study the background director angle is slowly varying, typically 0.002 rad/μm in a cell of width 200 μm, so that the nematicon trajectory can be determined by “momentum conservation”, in the sense of invariances of the Lagrangian for the NLC equations. The latter approach yields simple equations which have an exact solution and provides excellent agreement with the full numerical solutions, proving more than adequate to model beam evolution in non-uniform birefringent media.

II. GEOMETRY AND GOVERNING EQUATIONS

We consider the propagation of a linearly polarized, coherent light beam in a cell filled with an undoped positive uniaxial nematic liquid crystal. The extraordinary polarized beam is taken to initially propagate forward in the z direction, with electric field \( \vec{E} \) oscillating in the \( y \) transverse direction and \( x \) completing the coordinate triad. To eliminate the Fréedericksz threshold [1] and maximize the nonlinear optical response [2], we use two different approaches in the weakly nonlinear regime (i.e. power independent walk-off): (i) numerical solutions of the full governing Maxwell’s equations employing a fully vectorial beam propagation method (FVBPM) [46] in conjunction with elastic theory based on the Frank-Oseen model for the NLC response [1, 47, 48]. The FVBPM can be derived directly from Maxwell’s equations [49, 50], considering harmonically oscillating electric and magnetic fields in an anisotropic dielectric

\[
\begin{align*}
\frac{\partial \mathbf{H}_x}{\partial y} - \frac{\partial \mathbf{H}_y}{\partial z} &= i\omega\varepsilon_0 (\varepsilon_{11}\mathbf{E}_x + \varepsilon_{12}\mathbf{E}_y + \varepsilon_{13}\mathbf{E}_z), \\
\frac{\partial \mathbf{H}_x}{\partial z} - \frac{\partial \mathbf{H}_z}{\partial x} &= i\omega\varepsilon_0 (\varepsilon_{21}\mathbf{E}_x + \varepsilon_{22}\mathbf{E}_y + \varepsilon_{23}\mathbf{E}_z), \\
\frac{\partial \mathbf{H}_y}{\partial x} - \frac{\partial \mathbf{H}_x}{\partial y} &= i\omega\varepsilon_0 (\varepsilon_{31}\mathbf{E}_x + \varepsilon_{32}\mathbf{E}_y + \varepsilon_{33}\mathbf{E}_z), \\
\mathbf{H}_x &= -\frac{1}{i\mu_0\omega} \left( \frac{\partial \mathbf{E}_z}{\partial y} - \frac{\partial \mathbf{E}_y}{\partial z} \right), \\
\mathbf{H}_y &= -\frac{1}{i\mu_0\omega} \left( \frac{\partial \mathbf{E}_x}{\partial z} - \frac{\partial \mathbf{E}_z}{\partial x} \right), \\
\mathbf{H}_z &= -\frac{1}{i\mu_0\omega} \left( \frac{\partial \mathbf{E}_y}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial y} \right).
\end{align*}
\]

Here the complex amplitudes \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic fields, respectively, \( \varepsilon \) is the electric permittivity tensor, \( \omega \) is the angular frequency and \( \mu_0 \) is the magnetic permeability.
Numerical solution of the electromagnetic model (1), so-
parameter Ω = 1 using successive over-relaxations (SOR)
Numerical solutions of this elliptic equation (5) are found
tromagnetic equations are coupled to the NLC response,
that in the examined configuration the molecular direc-
energy. However, doing so results in a large system of
with ∆ε = n∥ − n⊥, the optical anisotropy. These elec-
tors corresponding to the standard nematic liquid crys-
tal 6CHBT, with Frank elastic constants K11 = 8.57 pN,
K12 = 3.7 pN and K33 = 9.51 pN and indices n∥ = 1.6335
and n⊥ = 1.4967 at temperature T=20°C and wave-
length λ = 2π/k0 = 1.064 μm [12 53]. The input beam
Gaussian and y-polarized, with a full width half max-
maximum FWHM = 7 μm and power 1 mW.

B. Momentum conservation

The full system [11] and [14] governing the propaga-
tion of a light beam in a non-uniform NLC cell is exten-
sive and amenable to numerical solutions only. However,
these equations can be simplified to yield a reduced sys-
tem for which an adiabatic approximation applies based
on the slow variation of the director orientation. This
adiabatic approximation shows that the beam trajectory
is determined by an overall “momentum conservation”
(MC) equation. This is not physical momentum, but mo-
den in the sense of the invariances of the Lagrangian
in the reduced system. Such reduction of the full system
and the resulting momentum conservation equation will
now be derived.

The first approximation is that the imposed linear
modulation θb in the director orientation is much smaller
than the constant background θ0, |θb| ≪ θ0. For the ex-
amples considered here, typical values are θ0 = 45° and
maximum |θb| ranging from 5° to 20°. While the largest
|θb| is not strictly much smaller than θ0, nevertheless the
asymptotic results are found to be in good agreement
with the numerical ones even at this upper limit. As dis-
cussed in the previous section, we denote the additional
nonlinear reorientation by θ, so that the total pointwise
orientation is ψ = θ0 + θb + θ. In the paraxial, slowly
varying envelope approximation, the equations [11] and
[14] governing the propagation of the light beam through
the NLC can be reduced to \[2, 3, 40\]
\[
\begin{align*}
&i k_0 n_e \frac{\partial E_y}{\partial z} + 2 i k_0 n_e \Delta(\psi) \frac{\partial E_y}{\partial y} + \nabla^2 E_y \\
&+ k_0^2 \left( n_e^2 \cos^2 \psi + n_e^2 \sin^2 \psi \right) \\
&- n_e^2 \cos^2 \theta_0 - n_e^2 \sin^2 \theta_0 \right) E_y = 0, \\
&K \nabla^2 \psi + \frac{1}{4} \epsilon_0 \Delta \epsilon |E|^2 \sin 2 \psi = 0.
\end{align*}
\]
(6)
\(7\)
As for the full equations of Section II A, \(E_y\) is the complex valued envelope of the electric field of the beam, since in the paraxial approximation the components \(E_x\) and \(E_z\) are neglected. The Laplacian \(\nabla^2\) is in the transverse \((x, y)\) plane. In the single constant approximation, the parameter \(K\) is a scalar on the assumption that bend, splay, and twist in the full director equation (5) have comparable strengths. The wavenumber \(k_0\) of the input light beam is intended in vacuum and \(n_e\) is the background extraordinary refractive index of the NLC \[2, 3\]
\[
n_e^2(\psi) = \frac{n_e^2 n_{\parallel}^2}{n_e^2 \cos^2 \psi + n_e^2 \sin^2 \psi},
\]
(8)
in the linear limit \(\theta = 0\). The coefficient \(\Delta\) is related to the birefringent walk-off angle \(\delta\) of the extraordinary-wave beam, with \(\tan \delta = \Delta\) in the \((y, z)\) plane, and is given by
\[
\Delta \psi = \frac{\Delta \epsilon \sin 2\psi}{\Delta \epsilon + 2 n_e^2 + \Delta \epsilon \cos 2\psi}.
\]
(9)
Throughout this work, despite the nonlinear dependence of \(\Delta\) on the beam power through the reorientation \(\theta\) \[20, 23, 53\], we assume \(\Delta = \Delta(\theta_0 + \theta_b)\) in the low power limit. In the single elastic constant approximation, the director equations \[53, 71\] differ by a factor of 1/2 in the dipole term involving \(\epsilon_0 \Delta \epsilon\), owing to definitions of the electric field based on either the maximum amplitude or the RMS (Root Mean Square) value. In this context, this difference is equivalent to a rescaling of \(K\), with the latter constant \(K\) cancelling out in the adiabatic momentum conservation approximation.

The reduced equations \[6\] and \[7\] can be set in non-dimensional form via the variable and coordinate transformations
\[
x = W X, \quad y = W Y, \quad z = B Z, \quad E_y = A u,
\]
(10)
where
\[
W = \frac{\lambda}{\pi \sqrt{\Delta \epsilon \sin 2\theta_0}}, \quad B = \frac{2 n_e \lambda}{\pi \Delta \epsilon \sin 2\theta_0},
\]
\[
A^2 = \frac{2 P_0}{\pi \Gamma W^2}, \quad \Gamma = \frac{1}{2} \epsilon_0 c n_e
\]
(11)
for a Gaussian input beam power of \(P_0\) and wavelength \(\lambda\) \[53\]. With these non-dimensional variables, Eqs. \[6\] and \[7\] become
\[
\begin{align*}
i \frac{\partial u}{\partial z} + i \gamma \Delta(\theta_0 + \theta_b) \frac{\partial u}{\partial y} + &\frac{1}{2} \nabla^2 u \\
&+ 2 (\theta_0 + \theta_b + \theta) u = 0,
\end{align*}
\]
(12)
\[
\nu \nabla^2 \theta = -2|u|^2.
\]
(13)
In deriving these equations we assumed that the NLC director rotation from \(\theta_0\) is small, i.e., \(|\theta_b| \ll \theta_0\), as discussed above. We further assumed that the nonlinear response is small, with \(|\theta| \ll \theta_0\). The trigonometric functions in the dimensional equations \[6\] and \[7\] have been expanded in Taylor series \[32\]. The scaled parameters in these non-dimensional equations are
\[
\gamma = \frac{2 n_e}{\sqrt{\Delta \epsilon \sin 2\theta_0}} \quad \text{and} \quad \nu = \frac{8 K}{\epsilon_0 \Delta \epsilon A^2 W^2 \sin 2\theta_0}.
\]
(14)
The equations \[12\] and \[13\] have the Lagrangian formulation
\[
L = i \left( u^* \frac{\partial u}{\partial Y} u - u^* \frac{\partial u}{\partial X} u \right) - 4 (\theta_0 + \theta_b + \theta)|u|^2 - \nu|\nabla \theta|^2,
\]
(15)
where the * superscript denotes the complex conjugate. Equations \[12\] and \[13\] have no general exact solitary wave, or nematicon, solution; the only known exact solutions are for specific, related values of the parameters \[50\]. For this reason, variational and conservation law methods have proved to be useful to study nematicon evolution \[50, 57\], as they give solutions in good agreement with numerical and experimental results \[53, 58\]. In particular, they provide accurate results for the refraction of nematics due to variations in the dielectric constant \[39, 58, 61\]. Conservation laws based on the Lagrangian \[15\] are used below to determine the nematicon trajectory in a cell with an imposed linear modulation of the orientation angle \(\theta_0 + \theta_b\).

The easiest way to obtain the approximate momentum conservation equations for Eqs. \[12\] and \[13\] is from the Lagrangian \[15\] \[42, 13\]. We assume the general functional forms
\[
u a g(\rho) e^{i \sigma + i \nu(Y - \xi)} \quad \text{and} \quad \theta = a g^2(\mu),
\]
(16)
where
\[
\rho = \frac{\sqrt{X^2 + (Y - \xi)^2}}{w}, \quad \mu = \frac{\sqrt{X^2 + (Y - \xi)^2}}{\beta},
\]
(17)
for the nematicon and the director responses, respectively \[42, 13\]. The actual beam profile \(g\) is not specified, as its trajectory is found to be independent of this functional form \[42\]. In response to the change in the NLC refractive index, the extraordinary wave beam undergoes refraction, as well as amplitude and width oscillations. If the length scale of the refractive index change is larger than the beam width, the beam refraction decouples from the amplitude/width oscillations \[53, 58, 64\]. Consistent with this decoupling, the electric field amplitude \(a\) and
the width \( w \) of the beam, the amplitude \( \alpha \) and width \( \beta \) of the director response can be taken as constant if just the beam trajectory is required. Only the beam center position \( \xi \) and (transverse) “velocity” \( V \) are then taken to depend on \( Z \), as well as the phase \( \sigma \). This approximation is equivalent to momentum conservation for the Lagrangian \((15)\) \((62)\).

Substituting the profile forms \((16)\) into the Lagrangian \((19)\) and averaging by integrating in \( X \) and \( Y \) from \( -\infty \) to \( \infty \) gives the averaged Lagrangian \((57)\)

\[
\mathcal{L}_m = -2S_2 (\sigma' - V \xi') a^2 w^2 - S_{22} a^2 w^2 - S_2 (V^2 + 2VF_1 - 4F) a^2 w^2 + \frac{2A_2^2 B_2^2 \alpha \beta^3 a^2 w^2}{A_2^2 \beta^3 + B_2^2 w^2} - 4\nu S_{42} \alpha^2 - 2qS_{42} \alpha^2 \beta^2,
\]

where primes denote differentiation with respect to \( Z \). Here \( F \) and \( F_1 \), which determine the beam trajectory, are expressed by

\[
F(\xi) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \theta_0 + \theta_b \right) g^2 \, dx \, dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^2 \, dx \, dy}, \quad F_1(\xi) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma \Delta \left( \theta_0 + \theta_b \right) g^2 \, dx \, dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^2 \, dx \, dy}.
\]

The integrals \( S_2, S_4 \) and \( S_{22} \) and \( S_{42} \) appearing in this averaged Lagrangian are

\[
S_2 = \int_0^\infty \zeta g^2(\zeta) \, d\zeta, \quad S_{22} = \int_0^\infty \zeta g''(\zeta) \, d\zeta,
\]

\[
S_4 = \int_0^\infty \zeta g^4(\zeta) \, d\zeta, \quad S_{42} = \frac{1}{4} \int_0^\infty \zeta \left[ \frac{d}{d\zeta} g^2(\zeta) \right]^2 \, d\zeta.
\]

Taking variations of this averaged Lagrangian with respect to \( \xi \) and \( V \) yields the modulation equations

\[
\frac{dV}{dZ} = 2 \frac{dF}{d\xi} - V \frac{dF_1}{d\xi}, \quad \frac{d\xi}{dZ} = V + F_1,
\]

which determine the beam trajectory. Eq. \((22)\) is the momentum equation.

A simple reduction of the trajectory Eqs. \((22)\) and \((23)\) can be carried out when the beam width is much less than the length scale for the variation of the refractive index, that is the length scale of the variation of \( \theta_0 \) \((22)\). For the examples in this work, \( \theta'_b \sim 0.002 \text{ rad/\mu m} \). Hence, a length scale for the variation of \( \theta_0 \) is \( 500 \mu \text{m} \), while the typical beam width is \( 7 \mu \text{m} \). The linear variation of the angle \( \theta_0 \) from the background angle \( \theta_0 \) starts at \( \theta_0 = 0 \) at \( Y = 0 \). Since the beam is launched at the mid-section of the cell \( Y = L/2 \), where the total angle in the absence of light is \( \theta_0 + \theta_b(L/2) = \theta_m \), it is more accurate to expand the walk-off \( \Delta \) in a Taylor series about \( \theta_m \) rather than \( \theta_0 \). If we set \( \theta_b = \theta_b - \theta_b(L/2) \), the integrals \( F \) \((19)\) and

\[
\begin{align*}
\theta_b(Y) &= \theta_r Y/L, \\
\end{align*}
\]

sketched in Fig.\((22)\)b). For this linear case, \( \theta_b \) goes from 0 at \( Y = 0 \) to \( \theta_r \) at \( Y = L \). This variation of \( \theta_b \) enables the momentum equations \((25)\) and \((26)\) to be solved exactly and gives the position of the beam center \( \xi \) as

\[
\xi = \left[ \xi_0 + \frac{1 + \gamma^2 \Delta' \theta_m \Delta' \theta_m}{\gamma^2 \Delta' \theta_m^2} e^{\gamma \Delta' \theta_m} \theta_b \right] \frac{1 + \gamma^2 \Delta' \theta_m \Delta' \theta_m}{\gamma^2 \Delta' \theta_m^2} e^{-\gamma \Delta' \theta_m} \theta_b Z
\]

\[
+ \left[ \frac{2 + \gamma^2 \Delta' \theta_m \Delta' \theta_m}{\gamma^2 \Delta' \theta_m^2} e^{-\gamma \Delta' \theta_m} \theta_b \right] \frac{1}{\gamma^2 \Delta' \theta_m^2} e^{-\gamma \Delta' \theta_m} \theta_b Z
\]

as \( \theta_b = \theta_r \) is a constant. We assumed that the beam is launched at \( \xi = \xi_0 \) with \( V = 0 \) at \( Z = 0 \).
Since \( \theta_0 \) is slowly varying, the trajectory solution given by Eq. (28) can be expanded in a Taylor series to yield

\[
\xi \sim \left[ \xi_0 + \gamma \Delta(\theta_m)Z \right] \\
\quad + \left[ \xi_0 \left( \gamma \Delta'(\theta_m)\theta'_0 Z + \frac{1}{2} \gamma^2 \Delta^2(\theta_m)\theta'_0 Z^2 \right) \right] + \\
\quad \left( 1 + \frac{1}{2} \gamma^2 \Delta(\theta_m)\Delta'(\theta_m) \theta'_0 Z^2 \right) \ldots \quad (29)
\]

The first term in square brackets is the trajectory in a uniform NLC and the terms in the second set of square brackets are the correction due to a changing orientation. For the examples hereby, \( \theta'_0 \sim 0.002 \text{ rad}/\mu \text{m} \) and \( \Delta' \sim 0.05/\mu \text{m} \). So, to first order in small quantities

\[
\xi \sim \left[ \xi_0 + \gamma \Delta(\theta_m)Z \right] + \theta'_0 Z^2 \quad (30)
\]

as \( \Delta'(\theta_m) \) is small. Hence, the trajectory is described by the term for a uniform medium and a quadratic correction; the walk-off change due to the varying background director orientation dominates the change in the nematicon trajectory.

To convert the non-dimensional solution (28) back to dimensional variables, the scalings (11) are used. In particular, for the \( z \) scaling factor \( B \), the angle for the extraordinary index (8) needs to be calculated. The obvious choice is to use the uniform background angle \( \theta_0 \). However, while this leads to good agreement with the numerical solutions, near exact agreement is obtained by using the total director angle \( \theta_0 + \theta_b \) in the absence of light. The imposed component \( \theta_b \) is not constant, but a slowly varying (linear) function of \( Y \), as discussed above, so its local value can be used to transform back to dimensional variables, consistent with a multiple scales analysis [24]. This local variation in the scaling factor for \( z \) gives a metric change in this coordinate, with a small, slowly varying alteration of the trajectory. Nevertheless, the overall effect of this small local change is significant over propagation distances of 500 \( \mu \text{m} \) and larger.

### III. RESULTS AND DISCUSSION

Figure 3 shows a comparison of nematicon trajectories in the modulated NLC as given by the adiabatic momentum approximation (28) and by the FVBPM solution of the full system (11) and (5). The considered cell has a range of linear variations in the background director angle \( \theta_0 \) of the form \( 2\theta \). Each individual case, \( \theta_0 + \theta_b \), is indicated in the figure. A Gaussian beam is launched at the center of the cell, with its trajectory becoming curved due to the non-uniform director alignment. In a uniform medium the (straight) nematicon trajectory is determined solely by the walk-off, which leads to a rectilinear path in the \( (y, z) \) plane. For the modulated uniaxial medium, not only the walk-off changes due to the varying anchoring, but the phasefront of the wavepacket is also distorted as the dielectric properties are modified and the NLC behaves like a lens with an index distribution \( n_e \) given by (8). Clearly, the momentum conservation approximation gives trajectories in close agreement with the numerical results. This validates the approximations made to arrive at the momentum conservation equations (25) and (26), in particular the assumption that the beam trajectory is not influenced by its amplitude-width oscillations. Furthermore, it shows how powerful such adiabatic approximations can be. Nonetheless, the momentum result is a kinematic approximation and so does not give all the information for the evolving beam, whereas the full system (11) and (5) can also provide the amplitude-width evolution. A final point regarding Figure 3 is that if the background angle for the extraordinary refractive index (8) in the \( z \) scaling (11) was chosen as \( \theta_0 \) rather than \( \theta_0 + \theta_b \), there would have been a noticeable difference between the momentum conservation and numerical results. The local variation of the propagation metric \( z \) due to the modulated director angle in the absence of light, in fact, has a significant effect on beam propagation.

These results are further analyzed in Fig. 4(a). The data is plotted to a logarithmic scale with an exponen-
tial regression fitted through the numerical trajectories. As \( z \) increases the trajectories are well approximated by an exponential evolution, in agreement with the momentum conservation solution (28) as for large \( z \) the decaying exponential is negligible and the growing exponential dominates. Furthermore, when the rectilinear nematicon path in a uniform NLC is subtracted from the trajectory in the modulated case, the resulting beam position has a quadratic evolution in \( z \), as shown in Fig. 4 (b). These exponential and quadratic fittings of the trajectories are consistent with \( z \theta \) and \( z \Delta \) being small, as demonstrated by reducing the full trajectory (28) to the quadratic approximation (30) via (29).

For a positive change of the anchoring conditions, i.e. \( \theta_r > \theta_0 \), walk-off and phase distortion both increase the beam deviation. In the opposite case for which \( \theta_r < \theta_0 \) these two phenomena counteract. The influence of walk-off and phase change on the nematicon path was analyzed for the case of the director orientation changing by \( 30^\circ / 200 \mu m \), as shown in Fig. 4. When \( \theta_r > \theta_0 \) the beam bends strongly due to both the walk-off and phase distortions acting in the same direction, as illustrated in Fig. 4 (a). The phase change is strongest at the launch position as the molecules are oriented at approximately \( 45^\circ \) there, so walk-off (given by (9) with \( \psi = \theta_0 + \theta_b \)) is close to its maximum. All the trajectories are monotonic and the beam transverse deviation increases with propagation distance. As for the comparisons in Figure 3 the agreement between the momentum conservation and numerical trajectories is near perfect, except for the lowest angle variation from \( 5^\circ \) to \( 35^\circ \), for which the agreement is still satisfactory. In the latter case the initial director angle at the input is far from the walk-off maximum at \( 45^\circ \), so the trajectory bending is weak. Small errors in the momentum approximation then become relevant.

In the opposite case \( \theta_0 > \theta_r \), the walk-off and the phase change along the cell counteract, resulting in the solitary beam reversing its transverse velocity, as illustrated in the comparison of Fig. 4 (b). The agreement between the momentum conservation and numerical trajectories is nearly perfect, except for two noticeable cases. The first is for the modulation from \( 35^\circ \) to \( 5^\circ \), opposite to what noted in the previous paragraph. The reason for the disagreement is again the weak bending of the beam and the enhanced role of small errors in the momentum approximation. The other case is the \( 75^\circ \) to \( 45^\circ \) modulation. It can be seen from Fig. 4 (b) that as the range of \( \theta_0 \) varies the beam reaches a maximum deviation in \( y \). The \( 75^\circ \) to \( 45^\circ \) variation is just after this turning point. As for the \( 35^\circ \) to \( 5^\circ \) case, small errors in the momentum conservation approximation can then result in large trajectory deviations, in particular errors in the \( \theta_0 \) changes required for the maximum displacement in \( y \).

Finally, we note that comparable beam powers are needed to obtain nematicons in uniform and linearly modulated NLCs, as a 1 mW input beam is sufficient to excite them in both cases, i.e. the rate of change in anchoring does not significantly modify the threshold power for reorientational solitons.

**IV. CONCLUSIONS**

We have studied the optical propagation of reorientational spatial solitons in nematic liquid crystals encompassing a transverse modulation of their optic axis (director) orientation. Even in the simplest limit of a linear change in anchoring angle, as considered here, non-uniform walk-off and wavefront distortion determine a bending of the resulting nematicon trajectory, leading to curved paths and curved optical waveguides induced by light through reorientation. Based on comparisons with numerical solutions obtained by FVBPM and elastic theory for self-localized light beam propagation in non-uniform nematic liquid crystals, we found that “momentum conservation” is an excellent approximation for modelling soliton paths in highly nonlocal media. It provides simple results for these trajectories and a highly intuitive explanation for their evolution, at variance with the highly coupled form of the full governing equations. While full numerical solutions can well describe nemati-
con evolution under generic conditions, the simplicity of the momentum conservation theory and its analytical solution speak in its favour for specific limits within the adiabatic category. Due to the slow variation of the anchoring conditions, both models show that the nematic trajectory can be described as propagation in a uniform medium with a quadratic correction. Additionally, the power needed to excite reorientational solitons in either uniform or linearly non-uniform NLCs is comparable. Further studies will investigate the role of longitudinal director modulations, as well as combinations of transverse and longitudinal changes, unveiling scenarios for the design of arbitrary nematicon paths and corresponding all-optical waveguides.

**ACKNOWLEDGEMENTS**

F. A. Sala thanks the Faculty of Physics, Warsaw University of Technology, for a grant. U. A. L. thanks the National Centre for Research and Development in Poland under the grant agreement LiD/ER/018/309/L-5/13/NCBR/2014. G. Assanto thanks the Academy of Finland for support through the Finland Distinguished Professor grant no. 282858.
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