Magnetic Field Control of Optical Transmission and Narrow Linewidth Superradiant Lasing by Strontium Atoms

Yuan Zhang\(^1\) and Klaus Mølmer\(^2\)

\(^1\)Donostia International Physics Center, Paseo Manuel de Lardizabal 4, 20018 Donostia-San Sebastian (Gipuzkoa), Spain
\(^2\)Department of Physics and Astronomy, Aarhus University, Ny Munkegade 120, DK-8000 Aarhus C, Denmark

We study the magnetic field controlled optical transmission observed in [Phys. Rev. Lett. 118, 263601 (2017)] with ultra-cold \(^{88}\)Sr atoms trapped in a one-dimensional optical lattice inside an optical cavity. We show that this phenomenon can be understood with a Jaynes-Cumming-like model for the atoms with two Zeeman-split excited states. This model yields three peaks in the transmission spectrum associated with three singly excited dressed states. The transmission is controlled by adjusting the photonic components in the dressed states via the Zeeman splitting of the atomic levels. If the atomic ensemble is incoherently pumped, lasing can be achieved in a superradiant crossover regime. Without the magnetic field the lasing relies on bright dressed states featuring a spectrum with line-width down to 5 kHz. However, in the presence of the magnetic field, the lasing relies on the dressed states, which are originally uncoupled but acquire photonic component due to the Zeeman splitting, and can yield a spectrum with line-width down to 2 Hz.

I. INTRODUCTION

Conventional lasers rely on the optical coherence established by stimulated photon emission from a population-inverted medium and have a spectrum line-width set by the Schawlow-Townes limit \(^[1]\). In contrast, superradiant lasers rely on coherence in the medium, established by collective atom-light interaction, and they can reach a line-width given by the Purcell enhanced atomic decay rate \(^[2,5]\). Recent theoretical \(^[6,7]\) and experimental \(^[8]\) studies showed that the two lasing mechanisms may co-exist in a so-called superradiant crossover regime with optical lattice clock systems, e.g., alkaline earth-metal \((^{88}\)Sr\)) atoms trapped in a one-dimensional optical lattice inside an optical cavity, see Fig. 1(a). The theoretical studies \(^[6,7]\) revealed that lasing in this regime benefits from the optical and atomic coherence and achieves a linewidth which is even smaller than the Purcell enhanced decay rate.

The recent experiment by Winchester, et. al. \(^[9]\) demonstrated that transmission of light by the same system can be controlled with a magnetic field. A magnetic field splits the triplet excited state \(3P_1\) into three states denoted as \(|e_\pm\rangle, |e_0\rangle\), see Fig. 1(b), by \(\sigma_\pm, \pi\) transitions of frequency \(\omega_\pm = \omega_0 \pm \Delta/2\) and \(\omega_\pi = \omega_0\) and spontaneous decay rate \(\gamma = 2\pi \times 2.5\) kHz. Here, \(\omega_0 = 2\pi c/\lambda\) is the frequency of the \(1S_0 \rightarrow 3P_1\) transition with the wavelength \(\lambda = 689\) nm and \(\Delta = 2\pi \times 2.1B\) MHz with the static magnetic field \(B\) given in unit of Gauss. A fundamental cavity mode, which has a frequency \(\omega_c\) close to \(\omega_0\) and a cavity loss rate \(\kappa = 2\pi \times 150\) kHz, couples to the three transitions with a strength \(g = 2\pi \times 7.5\) kHz. If the cavity supports mode with \(\pi\)- and \(\sigma_\pm\)-polarization, the excited state \(|e_0\rangle\) would couple only to the mode with the \(\pi\)-polarization and the states \(|e_\pm\rangle\) to those with \(\sigma_\pm\)-polarization.

For a system with up to \(N \approx 1.3 \times 10^6\) atoms \(^[9]\), the atomic ensemble behaves like a giant atom with collectively enhanced coupling \(\sqrt{N}g\) \(^[10]\), which is stronger than the cavity loss rate \(\kappa\) and the atomic decay rate \(\gamma\). In this case, by measuring the transmission of the probe field through the cavity, we can probe the system dressed states as identified in the Jaynes-Cumming model. However, for the atoms excited through the \(\sigma_\pm\)-transitions, our extended model identifies three dressed eigen-states for each photon number and their photonic component can be controlled by the Zeeman splitting, which explains the transmission spectra observed in \(^[9]\) (see below).

We also explore the radiation from the system subject to incoherent pumping of the atoms from \(|g\rangle\) to \(|e_\pm\rangle\) via higher excited states as in \(^[5]\). We study its variation with the applied magnetic field and hence the Zeeman splitting of the atomic excited state. We find that the system displays narrow linewidth lasing due to excitation of the dressed states, which are originally uncoupled but acquire...
photonic components due to the Zeeman splitting.

To describe the system shown in Fig. 1(a), in the following section, we develop a quantum master equation and obtain its approximate solution with second-order mean-field theory. In Sec. III we utilize our extended Jaynes-Cummings model, which goes beyond the simple bosonic model developed in [9][10], to understand our results for magnetic field controlled transmission. In Sec. IV we study the magnetic field controlled lasing in details. In the end, we summarize our work and discuss prospects for further theoretical and experimental studies.

II. QUANTUM MASTER EQUATION

The atomic dynamics restricted to the ground and two extreme excited states are described by the quantum master equation

$$\frac{\partial}{\partial t} \rho = -\frac{i}{\hbar} [H_a + H_c + H_{a-c} + H_d, \rho] - \kappa D (a) \rho$$

$$- \gamma_{+} \sum_k D (e_{+,k}) \langle g_k | \rho \rangle - \gamma_{-} \sum_k D (e_{-,k}) \langle g_k | \rho \rangle$$

$$- \eta_{+} \sum_k D (g_k) \langle e_{+,k} | \rho \rangle - \eta_{-} \sum_k D (g_k) \langle e_{-,k} | \rho \rangle.$$

(1)

Here, we have introduced the atomic Hamiltonian

$$H_a = \sum_{k=1}^N H_k = \sum_{k=1}^N \sum_{s=e_{+,e,-}g} \hbar \omega_{s} | s_k \rangle \langle s_k |,$$

the cavity-mode Hamiltonian

$$H_c = \hbar \omega_c a^+ a,$$

and the atom-cavity mode coupling

$$H_{a-c} = \hbar \langle g_{+} \sum_k | e_{+,k} \rangle \langle g_k | + g_{-} \sum_k | e_{-,k} \rangle \langle g_k | \rangle + \text{h.c.},$$

together with the coherent driving of the cavity mode

$$H_d = \sqrt{\kappa_{1}} \hbar \Omega \omega_d^\ast a + \text{h.c.},$$

$$H_k$$ has two excited eigenstates $| e_{+,k} \rangle$, $| e_{-,k} \rangle$, and one ground state $| g_k \rangle$ with frequencies $\omega_{e_{+,k}}$, $\omega_{e_{-,k}}$, $\omega_g$, and $\omega_d^\ast$ for $k$th atom. $H_k$ is expressed by the creation $a^\ast$ and annihilation $a$ operator of photons with the frequency $\omega_d$. $H_{a-c}$ couples the linearly polarized cavity mode to the atomic $\sigma_{\pm}$-transitions with equal strength $g_{\pm} = g$. $H_d$ drives the cavity mode through the left mirror of transmission rate $\kappa_1$ with a strength $\Omega$ and a frequency $\omega_d$. The three Lindblad terms in Eq. (1) describe cavity loss with the rate $\kappa = \kappa_1 + \kappa_2$ due to the left and right cavity mirror, atomic decay with rates $\gamma_+ = \gamma_- = \gamma$, and incoherent atomic excitation with rates $\eta_+ = \eta_- = \eta$. The superoperators are defined as $D (a) \rho = \{ a^\ast a, \rho \} / 2 - \rho a a^\ast$. Here, we ignore atomic dephasing since it is very small.

In our model, we assume single cavity mode, identical atom-cavity mode coupling and identical atomic decay and pumping. These assumptions are fulfilled by the optical lattice clock system. The first assumption is ensured because the atoms are trapped along the cavity axis and couple only with one fundamental transverse mode (near resonant to the atomic transitions). The lattice confinement of the atomic motion provides a nearly environment and hence coupling and dissipation for all the atoms.

We might solve Eq. (1) exactly with the density matrix in the collective number basis [11], which however requires a demanding computational effort and is thus limited to tens of atoms. To simulate the system with tens of thousands of atoms as encountered in the experiment, we rely on second-order mean-field theory. In this theory, we derive the equation $\partial \langle o \rangle / \partial t = \text{tr} \{ o d \rho / \partial t \}$ with Eq. (1) for the expectation value of any observable $o$, see Appendix A. We start from the equation for the mean photon number $\langle a^\ast a \rangle$ in the cavity and find that it couples to the atom-photon correlations $\langle a_{s,t} \rangle$ $(s,t = g_k, e_{+,k}, e_{-,k})$ through the atom-cavity mode coupling. Then, the equation for these correlations depend on the atom-atom correlation $\langle A_{s,t} A_{s',t'} \rangle$ and third-order correlations, e.g. $\langle a^\ast a A_{s,t} \rangle$, through the same coupling. Similarly, the equations for the atom-atom correlations depend on both second and third-order correlations which, in turn, depend on a hierarchy of higher order correlations. To close the equations, we apply the third-order cumulant expansion to truncate this hierarchy by approximating the third-order correlations with products of lower-order correlations, e.g. $\langle a^\ast a A_{s,t} \rangle = \langle a^\ast a \rangle \langle a_{s,t} \rangle + \langle a^\ast \rangle \langle a_{s,t} \rangle^2 - 2 \langle a \rangle^2 \langle a_{s,t} \rangle$. By doing so, we obtain a set of non-linear equations involving the aforementioned quantities but also the photon-photon correlation $\langle aa \rangle$, the cavity field amplitude $\langle a \rangle$, the atomic state populations $\langle A_{s,t} \rangle$ and the atomic polarization $\langle A_{s,t} \rangle (s \neq t)$. For details, see Appendix A.

Since the atoms are assumed to be identical, their permutation symmetry admits the same atom-photon correlation for all the atoms and the same atom-atom correlation for any atom pair, which reduces the computation complexity dramatically. We consider up to second-order correlations because they account for properly the collective atom-cavity mode coupling and the equations are also sufficiently simple to allow numerical solution. Notice that the equations discussed above can be generalized straightforwardly to atoms with multi-level ground and excited states, such as $^{87}$Sr atoms, by allowing $s, t$ to take more values.

III. MAGNETIC FIELD CONTROLLED TRANSMISSION

In this section, we present solutions of the set of second-order mean field equations given in Appendix A to calculate the transmission efficiently, we expose the system to a laser pulse, which leads to a Gaussian coupling strength $\Omega = \Omega_0 \exp \left\{ - (t - \tau)^2 / [2 \sigma^2] \right\}$ characterized by a maximum $\Omega_0$, a center $\tau$ and a duration $\sigma$, and we define the transmission as the ratio between the Fourier transform of the time-dependent input strength $\sqrt{\kappa_{1}} \Omega$ and output amplitude $\sqrt{\kappa_{2}} \langle \langle a \rangle \rangle$ (here and in the following, we assume $\kappa_1 = \kappa_2$).

Fig. 2 shows the transmission spectrum for a system with $6.25 \times 10^4$ $^{88}$Sr atoms under a weak probe field, see
Without the magnetic field, we observe
\begin{align}
\langle g_e | e_{\pm} \rangle = 1 / \sqrt{N} \sum_k (g_e g_{e \pm} \cdots g_e) \text{ and then approximate the collective atomic ground state } |g\rangle = |g_e g_{e \pm} \cdots g_e\rangle.
\end{align}

In this approximation, the atomic ensemble works like a giant atom with a collectively enhanced coupling $g_c = \sqrt{N} g$ to the cavity mode. Here, we consider only singly excited atomic states. Multiple atomic excitations are, however, included in our numerical second-order mean-field equations. As the Jaynes-Cummings model for two-level atoms [12], we can introduce the product states $|\psi_{1n}\rangle = |n\rangle |e_+\rangle$, $|\psi_{2n}\rangle = |n\rangle |e_-\rangle$ and $|\psi_{3n}\rangle = |n+1\rangle |g\rangle$ with the photon number states $|n\rangle$ to decompose the system Hamiltonian as a direct sum $H_s = \bigoplus_n H^{(n)}$ with
\begin{align}
H^{(n)} = h \begin{pmatrix}
\omega_+ - \omega_c & 0 & g_c \sqrt{n+1} \\
0 & \omega_- - \omega_c & g_c \sqrt{n+1} \\
g_c \sqrt{n+1} & g_c \sqrt{n+1} & 0
\end{pmatrix} + h (n+1) \omega_c.
\end{align}

In the special case with $\omega_+ - \omega_c = \pm \Delta/2$ as considered in the experiment, we can diagonalize the above matrix and get eigen-frequencies $\omega_{1n} = (n+1) \omega_c$ and $\omega_{2n} = (n+1) \omega_c + \delta_{n+1}$ and $\omega_{3n} = (n+1) \omega_c - \delta_{n+1}$ with the splitting $\delta_{n+1} = (\Delta/2)^2 + (\sqrt{2} g_c)^2 (n+1)$, and the corresponding dressed-eigen states
\begin{align}
|\alpha_{1n}\rangle = N_{1n} \begin{pmatrix}
- g_c \sqrt{n+1} / \Delta/2 \\
g_c \sqrt{n+1} / \Delta/2 \\
1
\end{pmatrix},
\end{align}
\begin{align}
|\alpha_{2n}\rangle = N_{2n} \begin{pmatrix}
- g_c \sqrt{n+1} / \Delta/2 + \delta_{n+1} \\
- g_c \sqrt{n+1} / \Delta/2 - \delta_{n+1} \\
1
\end{pmatrix},
\end{align}
\begin{align}
|\alpha_{3n}\rangle = N_{3n} \begin{pmatrix}
- g_c \sqrt{1+n} / \Delta/2 - \delta_{n+1} \\
- g_c \sqrt{1+n} / \Delta/2 + \delta_{n+1} \\
1
\end{pmatrix}.
\end{align}

To understand the transmission spectrum, we might consider the input field as a perturbation to the atom-cavity system. In this case, through the spectrum we actually probe the eigen-states of the system Hamiltonian $H_s = H_a + H_c + H_{a-c}$. In the weak atomic excitation limit, we can introduce the collective atomic ground state $|g\rangle = |g_e g_{e \pm} \cdots g_e\rangle$ and singly excited states $|e_{\pm}\rangle = 1 / \sqrt{N} \sum_k (g_e g_{e \pm} \cdots g_e)$ and then approximate the Hamiltonians as $H_a \approx h \sum_{s=e,g,+,-} \omega_s \langle s | \langle s |$ and $H_{a-c} \approx h g_c \langle e_+ | (g_e | e_- \rangle (g_e) + h.c.$.

Figure 2. Magnetic field controlled transmission (relative to $\omega_c$) of systems with $6.25 \times 10^4$ $^{88}$Sr atoms probed with weak pulsed field ($\Omega_0 = 10 \sqrt{kHz}$) (a,b) and strong pulsed field ($\Omega_0 = 400 \sqrt{kHz}$, c,d). Panel (a) shows transmission for different Zeeman splitting $\Delta/2\pi$. Panel (b) shows the transmission (solid black curve) and phase (dashed blue curve) of the output field (relative to the input field) for a given Zeeman splitting $\Delta/2\pi = 2 \text{ MHz}$. Panel (c) and (d) show the results with and without Zeeman splitting $\Delta/2\pi = 7.5 \text{ MHz}$, respectively. The laser pulses have a duration $\tau = 26.4 \text{ ns}$ and a center $\tau = 264.1 \text{ ns}$. Other parameters are specified in the Introduction.
peaks about double distances from the cavity mode frequency, corresponding to the transitions of frequencies \( \omega_c \pm (\delta_{n+1} + \delta_n) \) in the third group. In Fig. 2 (d) without the Zeeman splitting \( (\Delta = 0) \), the center peak disappears since the states \(|\alpha_{1n}\rangle\) have no photonic component.

### IV. MAGNETIC FIELD CONTROL OF SUPERRADIANT LASING

In the previous section, we studied the magnetic field controlled transmission of a coherent optical probe field. In this section, we study the possibility to utilize a magnetic field induced Zeeman splitting to induce a small photonic component of the otherwise uncoupled dressed states, and observe lasing with this component. To characterize the lasing properties, we calculate the intracavity photon number and the emission spectrum in steady-state. These calculations are done in a second-order mean-field theory derived in the Appendix [A] and we mimic the spectral measurement by calculating the photon number in a filter cavity as a function of its frequency, see Appendix [B]. This way of calculating the spectrum is equivalent to the use of the quantum regression theorem and it also lends itself to calculations for transient signals and light pulses.

For systems with no initial atomic or field coherence, the atomic polarization \( \langle A_{eg} \rangle \) \((r = e_+ \text{ or } e_-)\) and the field amplitudes \( \langle a \rangle \), \( \langle b \rangle \) in the system and filter cavity will be zero at all later times. We can therefore eliminate these quantities from the full equations and obtain the coupled differential equations only for few quantities, such as the photon number \( \langle a^+a\rangle \), the atom-photons correlations \( \langle aA_{eg}\rangle \), and the atom-atom correlations \( \langle A_{gr}A_{rg}\rangle \), see Appendix [C]. The truncation of our system at second order leads to products of lower order terms, and hence a set of non-linear differential equations, which we solve numerically in steady-state and check among different possible solutions the physically correct one, e.g. by yielding a real value for \( \langle a^+a\rangle \).

Fig. 4 (a) shows the mean photon number and emission spectrum for a system with \( 2.5 \times 10^5 \) atoms in the absence of the magnetic field. Panel (a) shows the intracavity photon number versus the incoherent excitation rate \( \eta \) for increasing Zeeman splitting \( \Delta \). Panel (b) and (c) show the normalized steady-state spectra without \( (\Delta = 0) \) and with the splitting \( \Delta = 2\pi \times 0.1 \text{ MHz} \) for increasing pumping \( \eta \) (from bottom to top curve). Panel (d) shows the spectra with increasing Zeeman splitting \( \Delta \) for weak pumping \( \eta = 0.1\gamma \) (lower part) and strong pumping \( \eta = 5\gamma \) (upper part). Other parameters are specified in the Introduction.

Fig. 4 (b) shows the corresponding steady-state spectra in the three regimes with different \( \eta \) in the absence of the magnetic field \( (\Delta = 0) \). In this case the system behaves like many two-level atoms as studied in detail in [7]. For the weakest pumping with \( \eta = 0.01\gamma \), we observe two emission peaks at the frequency \( 5.2 \text{ MHz} \) (relative to the
cavity mode frequency), which agrees with the collective Rabi-frequency $\sqrt{2} \times \sqrt{N} g$. Interestingly, the two peaks approach the cavity frequency with increasing pumping. With further increase of the pumping, the peaks start merging, see the middle plot. Here, even a small variation of the pumping has strong impact on the spectrum since the system undergoes a transition from superradiance to lasing, where the stimulated photon emission starts contributing. In the strong pumping regime, where the absorption balances the emission, we observe only single peaks whose linewidths decrease slightly with increasing pumping. The minimal linewidth is about 5 kHz, which agrees with earlier calculations \[\text{[7]}\] and the experimental measurements \[\text{[8]}\].

Fig. 4 (c) shows the spectra for different $\eta$ in the presence of a Zeeman splitting of $2\pi \times 0.1$ MHz. The lower plot shows two side peaks and a new peak around the cavity mode frequency for the weak and intermediate pumping. These peaks correspond to the first and second group of transitions shown in Fig. 3, which were used to explain the transmission spectra. With increasing pumping, the two side peaks approach the center peak while getting weaker and ultimately vanishing. The center peak, on the other hand, gets stronger and becomes spectrally more narrow under further increase of the pumping. Surprisingly, the center peak approaches a linewidth around only $2\pi \times 5$ Hz (note the difference in scale in the different panels).

Fig. 4 (d) shows the spectrum in the superradiance and lasing regime as function of the Zeeman splitting $\Delta$. In the former regime with weak pumping $\eta = 0.1 \gamma$ the two side peaks become weaker while the center peak gets stronger with increasing $\Delta$ due to the increased photonic component of the transitions around the cavity mode frequency. In the latter regime with strong pumping $\eta = 5\gamma$, the single peaks become narrower when $\Delta$ increases from 0.1 MHz to 0.3 MHz, but it widens when $\Delta$ increases further.

The minimal linewidth achieved here is about $2\pi \times 2$ Hz, which is about three orders of magnitude smaller than that without the magnetic field induced Zeeman splitting. It is also orders of magnitude smaller than the atomic decay rate of $2\pi \times 2.5$ kHz, the pumping rate $2\pi \times 37.5$ kHz and the cavity loss rate of $2\pi \times 150$ kHz.

We note that using transitions between the ground state and a single excited state in the same atoms, lasing in the superradiant crossover regime can be also achieved with a linewidth in the Hz regime \[\text{[4]}\], but this requires a three orders of magnitude larger pumping rate than found with Zeeman split excited states.

We have solved the equation for the mean photon number in the filter cavity analytically and obtained a quadratic equation for the linewidth, see Appendix \[\text{D}\]. This yields an approximate equation for the linewidth that is valid for strong pumping [see the upper parts of Fig. 5(b,c,d)].

$$\Gamma / 2 = \frac{\kappa / 2 + \text{Im} Z'_1 - (\gamma / 2 + \eta) Z'_1}{1 + |Z'_1|^2}. \quad (6)$$

where we have introduced

$$Z'_1 = \frac{N g^2 \sum_{r, r'} (\langle A_{r r} \rangle - \delta_{r, r'} \langle A_{gg} \rangle)}{(\Delta / 2)^2 + (\gamma / 2 + \eta)^2}, \quad (7)$$

$$Z'_2 = \frac{N g^2 \sum_{r} \sum_{r'} (\langle A_{r r'} \rangle - \delta_{r, r'} \langle A_{gg} \rangle)}{(\Delta / 2)^2 + (\gamma / 2 + \eta)^2}. \quad (8)$$

$\Delta_c$ is defined as $\Delta_{c_+} = \Delta / 2$ and $\Delta_{c_-} = -\Delta / 2$.

Eq. (6) indicates that the spectrum linewidth is determined by the population inversion $\langle A_{r r} \rangle - \langle A_{gg} \rangle$ ($r = e_{\pm k}$) and the polarization of the two excited states $\langle A_{r r'} \rangle$ ($r \neq r'$) through the parameter $Z'_1$ and $Z'_2$. In Fig. 5 (a), we present these quantities for the systems without Zeeman splitting ($\Delta = 0$). Notice that $Z'_2$ is zero in this case. We see that the population of the ground state $\langle A_{gg} \rangle$ (black solid curve) reduces while that of excited states $\langle A_{r r} \rangle$ (black dashed curve) increases with increasing pumping $\eta$, which leads to negative and positive population inversion for $\eta < \gamma$ and $\eta > \gamma$, respectively. The polarization $\langle A_{r r'} \rangle$ (black dotted curve) is non-zero only for $\eta > \gamma$ and increases with increasing
pumping. Since the polarization is negative, it compensates the population inversion, see Eq. (7). Their interplay leads to the increase of \((\gamma/2 + \eta)Z_1^2\) (blue solid curve) and this quantity changes sign at \(\eta \approx 2.2\gamma\), which indicates that the polarization dominates for \(\eta < 2.2\gamma\) while the population inversion dominates for \(\eta > 2.2\gamma\). The behavior of \((\gamma/2 + \eta)Z_1^2\) causes the spectrum linewidth to decrease dramatically when \(\eta\) overcomes \(\gamma\), see the solid curve in Fig. 5 (b), and to approach a constant for strong pumping (large \(\eta\)). Here, the linewidth is computed with the expression derived in Appendix D which is valid for the system under lasing (large pumping). The minimal linewidth is estimated around \(2\pi \times 3.74\text{ kHz}\) for \(\eta = 5\gamma\), which is close to the \(2\pi \times 5\text{ kHz}\) obtained from the fully numerical simulations. The simplified expression \((\gamma/2 + \eta)Z_1^2\) yields the dashed curve following the same trend, which indicates that lasing here is mainly because the positive \((\gamma/2 + \eta)Z_1^2\) dominates by the population inversion, compensates the cavity loss, see Eq. (6).

In the following, we consider the systems with the Zeeman splitting \((\Delta = 2\pi \times 0.3\text{ MHz})\), see Fig. 5 (c,d). In this case, \(Z_1^2\) is non-zero and its real part \(\text{Re}Z_2^2\) contributes to the lasing frequency and its imaginary part \(\text{Im}Z_2^2\) to the spectrum linewidth, see Eq. (6). We observe that the population and the polarization change similar as before except that the population becomes complex and its real part is much larger than the population inversion. As a result, \((\gamma/2 + \eta)Z_1^2\) is negative and its absolute value increases with the pumping, see the dotted curve in Fig. 5 (c), which increases the line-width according to Eq. (6). In contrast, \(\text{Im}Z_2^2\) is negative and reduces with the pumping, which reduces the line-width. In total, \(\text{Im}Z_2^2 - Z_1^2\) is negative and reduces with pumping, which compensates the cavity loss according to Eq. (6) and leads to the reduced linewidth, see the dashed curve in Fig. 5 (d). The expression derived in Appendix D leads to the solid curve and predicts a linewidth of \(2\pi \times 1\text{ Hz}\), which is close to the \(2\pi \times 2\text{ Hz}\) from the fully numerical simulations.

The above analysis indicates that for the systems without Zeeman splitting the lasing is related to the population inversion, established through the compensation of the atomic decay and the cavity loss with the atomic pumping. As revealed in [7], this brings the atomic ensembles to superradiant states, which maintain coherence longer than any dissipative process and lead to the narrowing emission. In contrast, for the systems with Zeeman splitting, the lasing is sustained by the polarization of the two excited states, which results in a constant exchange of coherence between two atomic excited states via the cavity model.

V. CONCLUSION

In summary we have studied the magnetic field controlled transmission and lasing of ultra-cold \(^{88}\text{Sr}\) atoms trapped in a one-dimensional optical lattice inside an optical cavity. A Jaynes-Cummings-like model for three-level atoms yields dressed eigenstates and energies that explain the main features of the calculated transmission. Our calculations show that the transmission of a strong driving field can be utilized to produce higher excited dressed states. We have also shown that an incoherently excited atomic ensemble may yield lasing using the dressed states, which become weakly coupled by the introduction of a magnetic field. By tuning the corresponding atomic Zeeman splitting, we can achieve lasing with a line-width down to 2 Hz. This is about three orders of magnitude smaller than the lasing using the bright dressed states. We expect similar results for other atoms like calcium and ytterbium, and we imagine that studies of other level schemes, e.g., with more levels and with asymmetric energy splittings may yield further insight and proposals for applications, e.g., for photon storage and precise measurement.

ACKNOWLEDGMENTS

This work was supported by the Villum Foundation (K. Mølmer) and the European Union’s Horizon 2020 research and innovation program (No. 712721, NanoQtech, K. Mølmer).

Appendix A: Second-order Mean Field Theory

In the main text, we outlined the procedure to obtain closed equations in the second-order mean field theory. In this appendix, we present the resulting equations. We start from the equation for the mean photon number:

\[
\frac{\partial}{\partial t} \langle a^+ a \rangle = -N2\text{Im} \sum_r g_r (aA_{rg})
- \sqrt{\kappa}2\text{Im}\Omega e^{i\omega _t t} \langle a \rangle - \kappa \langle a^+ a \rangle.
\]

Here and in the following, the label \(r = e_{+k}, e_{-k}\) counts only the excited states. The above equation indicates that the photon number is coupled to the atom-photon correlation \( \langle aA_{st} \rangle \) \( s,t = g_k, e_{+k}, e_{-k} \) through the atom-cavity mode coupling, which is identical for all the atoms and satisfies the equation

\[
\frac{\partial}{\partial t} \langle aA_{st} \rangle = \left[ i(\omega_s - \omega_t - \omega_e) - (\kappa/2) \right] \langle aA_{st} \rangle
+ i\sum_r \left[ g_r (\delta_{s,g}\langle aA_{rt} \rangle - \delta_{t,r}\langle aaA_{sg} \rangle) + g^*_r (\delta_{s,r}\langle a^+ aA_{gt} \rangle - \delta_{t,g}\langle (a^+ aA_{sr}) + \langle A_{sr} \rangle \rangle) \right]
- i(N-1) \sum_r g^*_r \langle A_{gr} A_{st} \rangle - i\sqrt{\kappa}\Omega^* e^{-i\omega _t} \langle A_{st} \rangle
- \sum_r \frac{\gamma_r}{2} [\delta_{s,r} + \delta_{t,r}] \langle aA_{st} \rangle - 2\delta_{s,g}\delta_{t,g} \langle aA_{rt} \rangle
- \sum_r \frac{\eta_r}{2} [\delta_{s,g} + \delta_{t,g}] \langle aA_{st} \rangle - 2\delta_{s,r}\delta_{t,r} \langle aA_{gg} \rangle.
\]
The above equation indicates that this correlation couples to third-order correlations \( \langle aa_{st} \rangle \), \( \langle a^+ a_{st} \rangle \) through the same coupling, see the second and third line, and we apply the third-order cumulant expansion to approximate them with the lower-order correlations. In addition, Eq. (A2) depends on the atom-atom correlations \( \langle A_{s't'} A_{st} \rangle (s,t,s',t' = g_k, e_{+k}, e_{-k}) \), which satisfy the equation

\[
\begin{align*}
\frac{\partial}{\partial t} \langle A_{s't'} A_{st} \rangle &= i (\omega_{s'} - \omega_{t'} + \omega_s - \omega_t) \langle A_{s't'} A_{st} \rangle \\
&+ i \sum_r \left[ g_r \left( \delta_{s,g} \langle a A_{s't'} A_{rt} \rangle - \delta_{t,r} \langle a A_{s't'} A_{sg} \rangle \right) \\
&+ g^*_r \left( \delta_{s,r} \langle a^+ A_{s'r} A_{gt} \rangle - \delta_{t,g} \langle a^+ A_{s'r} A_{sr} \rangle \right) \right] \\
&+ \sum_r \left[ g_r \left( \delta_{s',g} \langle a A_{s't'} A_{st} \rangle - \delta_{t',r} \langle a A_{s't'} A_{sg} \rangle \right) \\
&+ g^*_r \left( \delta_{s',r} \langle a^+ A_{s't'} A_{st} \rangle - \delta_{t',g} \langle a^+ A_{s't'} A_{sg} \rangle \right) \right] \\
&- i \sum_r \left[ \frac{\gamma_r}{2} \left[ (\delta_{s,r} + \delta_{t,r}) \langle A_{s't'} A_{st} \rangle - 2 \delta_{s,g} \delta_{t,r} \langle A_{s't'} A_{rr} \rangle \right] \\
&- \frac{\gamma_r}{2} \left[ \delta_{s',r} + \delta_{t,r'} \langle A_{s't'} A_{st} \rangle - 2 \delta_{s',g} \delta_{t,r} \langle A_{s't'} A_{rg} \rangle \right] \\
&- \frac{\eta_r}{2} \left[ \delta_{t,g} + \delta_{s,g} \langle A_{s't'} A_{st} \rangle - 2 \delta_{t,g} \delta_{s',r} \langle A_{s't'} A_{gg} \rangle \right] \\
&- \frac{\eta_r}{2} \left[ \delta_{t',g} + \delta_{s',g} \langle A_{s't'} A_{st} \rangle - 2 \delta_{t',g} \delta_{s,r} \delta_{t,r'} \langle A_{gg} A_{gg} \rangle \right] \right]
\end{align*}
\]

(A3)
the spectrum that we want to measure. Under these assumptions, we can first determine the steady-state expectation values of the system observables, and then obtain the filter cavity correlations and mean values for different filter-cavity frequencies \(\omega_f\).

Appendix C: Systems without Initial Atomic and Field Coherence

In the previous sections, we outlined the equations for rather general systems. However, if there is no initial coherence in the field or the atoms, the cavity field amplitude \(\langle a \rangle, \langle b \rangle\) and the polarization \(\langle A_{gr} \rangle\) vanish at all times. In this case, we can neglect all these quantities in all the equations to get the following sufficient set of simplified equations

\[
\frac{\partial}{\partial t} \langle a^+ a \rangle = -N 2 \text{Im} \sum_r g_r \langle a_{Ar} \rangle - \kappa \langle a^+ a \rangle, \quad (C1)
\]

\[
\frac{\partial}{\partial t} \langle a_{Ar} \rangle = i (\tilde{\omega}_r - \tilde{\omega}_c) \langle a_{Ar} \rangle + i g_{r^r} \langle a^+ a \rangle \langle A_{Ar} \rangle - i \left( \langle a^+ a \rangle + 1 \right) \sum_{r'} g_{r'^r} \langle A_{rr} \rangle, \quad (C2)
\]

\[
\frac{\partial}{\partial t} \langle A_{gr} \rangle = -i \tilde{\omega}_{gr,rg} \langle A_{gr} \rangle + i \langle a_{Ar}^r \rangle^* \left[ g_{r^r} \langle A_{Ar} \rangle - \sum_{r''} g_{r''r} \langle A_{rr'} \rangle \right] - i \left[ g_{r'} \langle A_{Ar} \rangle - \sum_{r''} g_{r''r} \langle A_{rr'} \rangle \right] \langle a_{Ar} \rangle, \quad (C3)
\]

\[
\frac{\partial}{\partial t} \langle A_{gg} \rangle = -\sum_r 2 \text{Im} g_r \langle a_{Ar} \rangle + \sum_r \gamma_r \langle A_{rr} \rangle - \sum_r \eta_r \langle A_{gg} \rangle, \quad (C4)
\]

\[
\frac{\partial}{\partial t} \langle A_{rr} \rangle = -i \tilde{\omega}_{rr} \langle A_{rr} \rangle - i g_{r^r} \langle a_{Ar} \rangle + i g_{r^r}^* \langle a_{Ar} \rangle^* + \delta_{r',r} \eta_r \langle A_{gg} \rangle, \quad (C5)
\]

where we have introduced \(\tilde{\omega}_r = \omega_r - \omega_c + i(\gamma_r + \sum_{r''} \eta_{r''})/2\), \(\tilde{\omega}_c = \omega_c - i\kappa/2\), \(\tilde{\omega}_{gr,rg} = \omega_{gr,rg} - i \sum_{r''} \eta_{r''}\) and \(\tilde{\omega}_{rr} = \omega_{rr} - \omega_r - i(\gamma_r + \gamma_r)/2\).

To calculate the spectrum, we first solve the above equations in the steady-state and then utilize the results as the input parameters for the following simplified equations for the filter cavity-related quantities:

\[
\frac{\partial}{\partial t} \langle b^+ b \rangle = -\chi \langle b^+ b \rangle + \beta 2 \text{Im} \langle b^+ a \rangle, \quad (C6)
\]

\[
\frac{\partial}{\partial t} \langle b^+ a \rangle = [i(\omega_g - \omega_r - (\chi + \kappa)/2) \langle b^+ a \rangle - i\beta (\langle b^+ b \rangle - \langle a^+ a \rangle) - i N \sum_r g_{r'} \langle b^+ A_{gr} \rangle, \quad (C7)
\]

\[
\frac{\partial}{\partial t} \langle b^+ A_{gr} \rangle = [i(\omega_g - \omega_r + \omega_f) - (\gamma_r + \sum_r \eta_r + \chi)/2] \langle b^+ a \rangle + i \beta \langle a^+ A_{gr} \rangle + i \langle b^+ a \rangle \sum_{r'} g_{r'} \left( \langle A_{rr'} \rangle - \delta_{r,r'} \langle A_{gg} \rangle \right). \quad (C8)
\]

In the last equation, we have utilized \(\langle b^+ a_{Ar} \rangle = \langle b^+ a \rangle \langle A_{Ar} \rangle\) and \(\langle b^+ a_{Ar} A_{gg} \rangle = \langle b^+ a \rangle \langle A_{Ar} \rangle \langle A_{gg} \rangle\). We obtain the steady-state solutions for the above equations by setting the left hand sides of the differential equations equal to zero and then solve the non-linear coupled equations numerically.

Appendix D: Analytical Expression of Spectrum Linewidth

In the main text, we have identified lasing mechanisms relying on the bright dressed states and the dark dressed states, respectively. Our numerical studies show that the second mechanism leads to lasing with a much narrower spectrum. To understand this lasing mechanism, it is necessary to analyze how the spectrum is determined by the atomic observables, i.e., to consider the steady-state version of Eqs. \(C6\) and \(C8\):

\[
\langle b^+ b \rangle = -i \left( \beta/\chi \right) \left( \langle b^+ a \rangle - \langle a^+ b \rangle \right), \quad (D1)
\]

\[
\langle b^+ A_{gr} \rangle = -\xi^{-1}_{r} \beta \langle a^+ A_{gr} \rangle - \xi^{-1}_{r} \sum_{r'} g_{r'} \left( \langle A_{rr'} \rangle - \delta_{r,r'} \langle A_{gg} \rangle \right) \langle b^+ a \rangle, \quad (D2)
\]

\[
\langle b_{Ar} \rangle = -\xi^{-1}_{r} \beta \langle A_{Ar} \rangle - \xi^{-1}_{r} \sum_{r'} g_{r'} \left( \langle A_{rr'} \rangle - \delta_{r,r'} \langle A_{gg} \rangle \right) \langle a^+ b \rangle. \quad (D3)
\]

Here, we have introduced \(\xi_{r} = \omega_g - \omega_r + \omega_f - i(\gamma_r + \sum_{r'} \eta_{r'} + \chi)/2\). Inserting the above expressions in the steady-state version of Eq. \(C7\), we get the following closed equations

\[
\begin{bmatrix}
\tau^* & -\beta^2/\chi \\
-\beta^2/\chi & \tau
\end{bmatrix}
\begin{bmatrix}
\langle b^+ a \rangle \\
\langle a^+ b \rangle
\end{bmatrix}
= i \beta
\begin{bmatrix}
\psi^* \\
-\psi
\end{bmatrix}. \quad (D4)
\]

Here, we have introduced the abbreviations

\[
\tau = i(\omega_f - \omega_r) + (\chi + \kappa)/2 + \beta^2/\chi + i\zeta, \quad (D5)
\]

\[
\zeta = N \sum_{r,r'} \xi^{-1}_{r} g_{r'}^* \langle A_{rr'} \rangle - \delta_{r,r'} \langle A_{gg} \rangle, \quad (D6)
\]

\[
\psi = \langle a^+ a \rangle + N \sum_{r} \xi^{-1}_{r} g_{r} \langle A_{rg} \rangle. \quad (D7)
\]
We solve the above equations \((b^+a) = iβ (v^*τ - v\beta^2/\chi)/((ττ^* - \beta^4/\chi^2)),\) \((a^+b) = \langle b^+a \rangle^*\) and then insert the solutions back to Eq. (D1) to get the analytical expression for the mean photon number in the filter cavity
\[
\langle b^+b \rangle = \frac{\beta^2 v^*τ + vτ^* - (v + v^*) β^2/\chi}{ττ^* - \beta^4/\chi^2} . \tag{D8}
\]

To obtain the spectrum, we calculate \(\langle b^+b \rangle\) for different values of the frequency \(ω_f\). The linewidth is determined by the denominator in Eq. (D8) and also in the definition of \(τ, ζ, ν\). In the systems as considered in the main text, we have the parameters \(g_+ = g_- = g\), \(γ_+ = γ_- = γ\), \(η_+ = η_- = η\), \(ω_c - (ω_r - ω_0) = −Δ_r (Δ_± = ±Δ/2)\) and thus we can approximate the parameter \(τ\) as
\[
τ (ω_f, Γ) ≈ i (ω_f - ω_c - ReZ_2) + κ/2 + ImZ_2 - (γ/2 + ν ± Γ/2) Z_1 . \tag{D9}
\]

with the abbreviations
\[
Z_1 = \frac{N g^2 \sum_{r,r'} (⟨A_{rr'}⟩ - δ_{r,r'} ⟨A_{gg}⟩)}{(Δ/2)^2 + (γ/2 + ν ± Γ/2) Z_2} , \tag{D10}
\]
\[
Z_2 = \frac{N g^2 \sum_{r} Δ_r \sum_{r'} (⟨A_{rr'}⟩ - δ_{r,r'} ⟨A_{gg}⟩)}{(Δ/2)^2 + (γ/2 + ν ± Γ/2) Z_1} . \tag{D11}
\]

To achieve Eq. (D9), we have replaced \(ω_f\) with \(ω_c ± iΓ/2\) in \(ζ, ν\), where the spectrum linewidth \(Γ\) should be determined in a self-consistent way, see below.

We expect that the mean photon number \(\langle b^+b \rangle\) at \(ω_f = ω_c\), which is inversely proportional to the square of \(τ(ω_c, Γ)\) according to Eq. (D9), is about two times of that at \(ω_f = ω_c - iΓ\), which is inversely proportional to the square of \(τ(ω_c, Γ)\) according to Eq. (D9). This leads to the relation between the real parts
\[
\sqrt{2}Re [τ (ω_c, Γ = 0)] = Re [τ (ω_c, Γ)] . \tag{D12}
\]

This relation leads to a cubic equation for \(Γ\), whose analytical solution can be obtained but is very complex. However, considering that \(Γ\) is very small, we can ignore \((Γ/2)^2\) in the denominator in Eqs. (D10) and (D11) and then Eq. (D12) leads to a quadratic equation \((Γ/2)^2 + B (±Γ/2) + C = 0\) with the coefficients
\[
B = \left[ (1 - \sqrt{2}) κ/2 - \sqrt{2} (ImZ'_1 - (γ/2 + ν) Z'_1) \right] + (1 - Z_1') (Δ/2)^2 + (γ/2 + ν)^2 ,
\]
\[
C = \frac{(Δ/2)^2 + (γ/2 + ν)^2}{2 (γ/2 + ν)} × (1 - \sqrt{2}) (κ/2 + ImZ'_2 - (γ/2 + ν) Z'_2) . \tag{D14}
\]

The parameters \(Z'_1\) and \(Z'_2\) are achieved from \(Z_1\) and \(Z_2\) by setting \(Γ = 0\), see Eq. (D10) and (D11) in the main text. Finally, the quadratic equation can be simply solved and we get
\[
± Γ/2 = -B ± \sqrt{B^2 - 4C} . \tag{D15}
\]

There are two solutions and we should choose the one with positive value for \(Γ\). Notice that \(Z'_1\) is a real number while \(Z'_2\) can be complex.

Eq. (D12) yields the spectrum linewidth \(Γ/2 = κ/2 + ImZ'_2 - (γ/2 + ν ± Γ/2) Z'_1\), where we assume \(Γ\) is much smaller than \(γ/2 + ν\) in order to replace \(Z_1\) and \(Z_2\) by \(Z'_1\) and \(Z'_2\). This relation, valid for the system under lasing conditions, leads to Eq. (6) in the main text and approximates the numerical solution well.

[1] A. L. Schawlow and C. H. Townes, Phys. Rev. 112, 1940 (1958).
[2] D. Meiser, J. Ye, D. R. Carlson, and M. J. Holland, Phys. Rev. Lett. 102, 163601 (2009).
[3] D. Meiser and M. J. Holland, Phys. Rev. A 81, 033847 (2010).
[4] M. A. Norcia, M. N. Winchester, J. R. K. Cline, and J. K. Thompson, Sci. Adv. 2, 31601231 (2016).
[5] J. G. Bohnet, Z. Chen, J. M. Weiner, D. Meiser, M. J. Holland, and J. K. Thompson, Nature 484, 78 (2012).
[6] D. A. Tieri, M. Xu, D. Meiser, J. Cooper, and M. J. Holland, arXiv preprint: 1702.04830 (2017).
[7] K. Deb Nath, Y. Zhang and K. Mølmer, Lasing in the Superradiant Crossover Regime, arXiv:1809.01620 (2018).