The nature of three dimensions: Non-local behavior in the three-dimensional (3D) Ising model

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Abstract. In this work, we first reveal the nature of three dimensions: the existence of non-local behavior. Taking the three-dimensional (3D) Ising model as an example, we analyze the transfer matrices as well as the partition function to discuss in details the origin of non-local behavior in three dimensions. We then point out that fail to deal with non-local behavior is one of the main disadvantages of several theoretical approaches (such as conventional low-temperature expansions, conventional high-temperature expansions, perturbative renormalization group and Monte Carlo methods, etc), all of which take local environments into account only. This work provides a deep understanding on the natural difference between 3D bulk materials and 2D materials, and shows that how to treat the non-local behavior and the interaction between spins along the third dimension is a key point for solving exactly the problems of the 3D systems with the nearest neighboring interactions. Finally, we will briefly review recent advance in the 3D Ising model and related topics.

1. Introduction

The trend of study on materials science has flocked to nanostructured materials (such as 1D, 2D materials and their composites) since 1980s. However, the study on the physical properties of 3D bulk materials is still important for application in various fields, such as electronic, energy, etc. Furthermore, the understanding of the 3D materials provides a chance to understand well the 3D world we are living. For a full understanding on a physical system, it is important to solve exactly the model as representation of the system. Unfortunately, even for the simplest model, it has been a great challenge for solving exactly the Ising model in three dimensions.

The study on the Ising models has a long history in fields of both physics and mathematics. After the pioneer work of Onsager [1] for the exact solution of two-dimensional (2D) Ising model, there have been many developments for 2D Ising models and 3D Ising models. In the case of 2D Ising models, it is easy to find out the exact solution following the Onsager’s approach [1], when no non-trivial topological structures exist. However, almost all the results for 3D Ising models had been
obtained by approximation approaches, such as low-temperature expansions, high-temperature expansions, perturbative renormalization group and Monte Carlo methods, etc. The author conjectured that the non-trivial topological structures of the 3D Ising model can be trivialized in a higher dimensional space and the 3D Ising model can be realized as the free statistic model on them with topological/geometrical phases on eigenvectors [2, 3].

In this work, on the observation of the formula of the partition functions, we first reveal the non-local behavior of the 3D Ising model. We then point out that fail to deal with non-local behavior is one of the main disadvantages of several theoretical approaches (such as conventional low-temperature expansions, conventional high-temperature expansions, renormalization group and Monte Carlo methods, etc), all of which take local environments into account only. It is clear now that these approximation approaches cannot serve as the standard for judging the putative exact solution of the 3D Ising model. Finally, we will briefly review recent advance in the 3D Ising model and related topics.

2. Non-local behavior of 3D Ising model

As in [2,3], we consider the Ising Hamiltonian on an orthorhombic lattice in 3D Euclidean space, with up-spin or down-spin at each lattice point:

\[ H = -J \sum_{(r,s)} S^x_{r} S^x_{s} - J' \sum_{(r,s)} S^z_{r} S^z_{s} - J'' \sum_{(r,s)} S^y_{r} S^y_{s} \]

Here only the nearest neighboring interaction between spins at each lattice point is considered. We follow the notation of Onsager-Kaufman-Zhang for Pauli matrices [1-4]:

\[ s_i^x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s_i^y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_i^z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

and we have the relation \( \tanh K^* = e^{2K} \). We introduce \( K = \beta J \), \( K' = \beta J' \) and \( K'' = \beta J'' \) with \( \beta = 1/k_BT \). The partition function of the 3D Ising model can be given as follows [3]:

\[ Z = (2 \sinh 2K)^{-3/2} \cdot \text{trace}(V_3 V_2 V_1) = (2 \sinh 2K)^{-3/2} \cdot \sum_{i=1}^{n} \lambda_i^m \]

\[ V_3 = \prod_{j=1}^{n} \exp(-iK'' \Gamma_{j \rightarrow 2n}^{\prime \prime}) \prod_{j=1}^{n} \Gamma_{2j-2n+1}^{\prime \prime} = \prod_{j=1}^{n} \exp(K'' s_{j} s_{j+n}) \]

\[ V_2 = \prod_{j=1}^{n} \exp(-iK' \Gamma_{j \rightarrow 2j}^{\prime}) = \prod_{j=1}^{n} \exp(K' s_{j} s_{j+1}) \]

\[ V_1 = \prod_{j=1}^{n} \exp(iK \Gamma_{j-1 \rightarrow 2j}) = \prod_{j=1}^{n} \exp(K \Gamma_{j \rightarrow 2j}) \]

Here we introduce the following generators of Clifford algebra of 3D Ising model:

\[ \Gamma_{j \rightarrow 1} = C \otimes C \otimes \cdots \otimes C \otimes s_{j} \otimes \cdots \otimes 1 \] (j times C) \hspace{1cm} (6a)

\[ \Gamma_{j \rightarrow 2} = C \otimes C \otimes \cdots \otimes C \otimes (i s_{j}^x) \otimes 1 \otimes \cdots \otimes 1 \] (j times C) \hspace{1cm} (6b)

The Clifford algebra is generated by the Pauli matrices. Clearly, we have the following formulas:
The problem becomes to how to deal with the trace of the transfer matrices $V = V_3 V_2 V_1$, specially, $V_3$ with non-linear terms of $\Gamma$ matrices. There are direct products of $(n-1)$ terms of unit matrices in between two Pauli matrices $s'$ in Eq. (9).

It is noticed that the transfer matrix $V_1$ or $V_2$ with the product of two $\Gamma$ matrices has trivial topologic structure, which can be represented as Lie group for a rotation. The transfer matrix $V_3$ is the root of difficulties to hinder the direct application of Onsager-Kaufmann approach to 3D Ising model, due to the high-order terms of $\Gamma$-matrices. The non-trivial topological effect as well as the non-local behavior can be seen also in the language of the spin ($\sigma$) variables although all expressions seems local. Although the formulas for the transfer matrices $V_2$ and $V_3$, i.e., Eqs. (3) and (4), look very similar, they are quite different. The difference in the subscripts of $s'$ in Eqs. (3) and (4) represents different meanings, as shown in Eqs. (8) and (9). $s'_j s'_{j+1}$ represents the interaction between the most neighboring spins along the second dimension in a plane, while $s'_j s'_{j+n}$ represents the interaction between the most neighboring spins along the third dimension. The latter term looks like an interaction between two spins located far from each other, via a chain of $n$ spins in the plane (actually, via $n$ spins in the plane if one considers the periodic condition already used along the first dimension, as shown in Eq. (2)). The reason is that the third transfer matrix $V_3$ must follow the sequence of the spin ($\sigma$) variables arranged and fixed already in the first two transfer matrices $V_1$ and $V_2$. According to this fixed order, although the interaction between a spin and one of its neighboring spins along the third dimension is the nearest neighboring one, its effect is correlated with the states of $n$ other spins in the plane. It is equivalent effectively to a long-range and many-body interaction in which $(n+1)$ spins are involved. The same effect happens for every interaction along the third dimension in the 3D Ising model. Therefore, the non-local correlation and the global effect indeed exist in the 3D Ising system. From another angle of points of view, all the spins in the 3D Ising model are entangled. With the language of the formionic ($\gamma$) variables (see Eq. (3)), it is easy to obtain the same conclusion.

It was understood that the different topological states (e.g. knots/links) formed by up or down spins also contribute to the partition function and hence the free energy and other physical properties of the system [3]. Therefore, the partition function of the 3D Ising model can be written in the following two parts as:

$$ Z = Z_{\text{Local}} + Z_{\text{Non-local}} $$  \hfill (10)

Here $Z_{\text{Local}}$ and $Z_{\text{Non-local}}$ represent the contributions from the local environment and the non-local behavior to the partition function of the 3D Ising system. Correspondingly, the free energy of the 3D Ising model can be written as the sum of the local and non-local parts:

$$ F = F_{\text{Local}} + F_{\text{Non-local}} $$  \hfill (11)

Here $F_{\text{Local}}$ and $F_{\text{Non-local}}$ represent the contributions from the local environment and the non-local behavior to the free energy of the 3D Ising system. Consequently, each of the physical properties, such as specific heat, magnetization, susceptibility, correlation function, etc., can be treated as the sum of local and non-local parts.

3. Disadvantages of several theoretical approaches

Attempts to apply the algebraic method used for solving the 2D model to the 3D problem are seriously hindered at an early stage, because the operators of interest generate a much large Lie algebra being so large that it would seem to be of little value [5]. The combinatorial method developed by Kac and...
Ward [6] introduces some problems in topology. This combinatorial method of counting the closed graph cannot be generated in any obvious way to the 3D problem, since the peculiar topological property is that a polygon in three dimensions does not divide the space into an "inside and outside" [5]. These problems are what an approach for the 3D Ising model must face. It is clear that any approaches based on only local environments, such as conventional low-temperature expansions, conventional high-temperature expansions, perturbative renormalization group, Monte Carlo simulations, etc, cannot be exact for the 3D Ising model, even though they may be exact for the 2D cases [2,3]. We will discuss the disadvantages of these theoretical techniques as follows (for details, the readers refer to [2,3]):

3.1 Low-temperature expansions

All systematic methods for the determination of series coefficients are at some level graphical or diagrammatic. With each coefficient is associated a set of graphs of some given topological type. To each graph corresponds a numerical contribution according to a well-defined rule. To calculate the required coefficient, one simply sums all contributions. For conventional low-temperature expansions, the appropriate choice is to define the partition function such that the fully aligned spin-up state is taken as having zero energy, since the low-temperature series is a perturbation expansion about this state [5,7-10]. The conventional low-temperature expansions, evaluated by systematically overturning spins from the ground state with all spins "up", consider only the local environment of spins and their interactions, neglecting the non-local behavior of the 3D Ising model (i.e., the interaction via the chain of n spins in the plane). Therefore, the conventional low-temperature expansions take into account only the local part of the partition function and the free energy of the system. It gives the same fundamental leading term for the 2D triangular Ising lattice and the 3D simple cubic lattice, because it relates directly with the Ising model lattice of coordination number q, regardless of what the dimension it is. This is also the origin of the divergence of the low-temperature expansions for the 3D Ising model. The reason for the zero radius of the circle of convergence of the conventional low-temperature series for the 3D Ising model is the direct result that the non-local effect lacks in the series. It is noticed that the conjectured solution for the 3D Ising model agrees well with C.N. Yang's magnetization for 2D. If one accepted the conjectured exact solution in [2], which consists of both local and non-local parts of the physical properties of the 3D Ising system, one would derive the non-local contribution by extracting the local part from the total property.

3.2 High-temperature expansions

It is known that the conventional high-temperature series expansion is an exact expansion for the 2D Ising model and it is easy to write down every terms of the high-temperature series expansion, by accounting the loops (polygons) in the 2D lattice. The loops are the boundary between the areas of spin up and spin down [5,7,11]. However, for the 3D Ising model, one first meets the challenge to write down every terms of the high-temperature series expansion. A basic difficulty is that in 3D, the boundaries between domains of spin up and spin down are not polygons, but polyhedrons. The polygons cannot separate the domains of spin up and spin down. Furthermore, the existence of the non-trivial topological objects (such as Mobius band, Klein bottle, crosses and knots/links, etc.) in the 3D Ising lattice makes the well-known topological troubles. Actually, any polygons are local, which cannot take into account the non-local behavior of the 3D Ising system, which involves all the states of nm spins in a plane (for an interaction between the neighboring spins along the third dimension). Because of the duality between the high-temperature and low-temperature expansions, the divergence of the conventional low-temperature expansions corresponds to the zero radius of the convergence of the conventional high-temperature series. It was the observation of the author [2,3] that there exists a topological phase transition at/near infinite temperature for the 3D Ising model. In the 3D Ising system, the trivial topological structure at infinite temperature is changed to be the non-trivial topological structure below infinite. The existence of such a topological phase transition makes the possibilities of
the existence of several different high-temperature expansions. Namely, the high-temperature expansions are not unique, and the correct one should be that including the non-local behavior.

3.3 Renormalization group

Although the renormalization group theory is described in mathematical terms, it is not rigorous. Although the renormalization group theory has been widely applied to study the critical phenomena, and although it shows the results with high precision for 2D Ising model (in comparison with the Onsager's exact solution) [12-17], it possesses the disadvantages similar to those of the conventional series expansions. During the renormalization group procedures for 3D Ising model, various approximations, such as expansions, perturbations, linearizations, normalizations, etc. are performed. In all the ways, the disadvantages, similar to those of conventional low- and high-temperature expansions, have not been removed. The real-space renormalization group techniques are concerned with the construction of new models from old by averaging dynamical variables of the old model to form the block variable of the new one. The final results depend sensitively on how to divide Kadanoff blocks, define the effective Hamiltonian, determine the details of the block variables, and calculate approximately the partial trace. The larger the Kadanoff blocks, the more accurate the calculations, but the more complicated the procedure. The final results would approach the exact solution if and only if the size of the Kadanoff blocks were chose to be infinite and the infinite terms of the expansion were remained. This is impossible to be done, because much more variables would emerge with increasing the size of the Kadanoff blocks and taking into account more terms of the expansion, which make the calculations become extremely difficult. A serious trouble is that the non-local behavior in 3D involves in the interaction of long-range-like type and all the possible states of nm spins in a plane (n \to \infty and m \to \infty) even for only one interaction between two spins along the third dimension, which make any procedure with finite blocks cut the plane of nm spins with the non-local effect. The field theoretical or k-space renormalization group techniques meets the same difficulty that cannot be overcome, since at the deeper level there is a close connection, because the basic idea of them is the same and because there are connections between \phi and a set of block variables. It is clear now that the perturbative renormalization group with finite blocks cannot take into account the non-local behavior of the 3D Ising model. Therefore, one has to develop the nonperturbative renormalization group for studying the critical phenomena of the 3D Ising model.

3.4 Monte Carlo methods

It is known that Monte Carlo simulations are powerful techniques for numerical calculations [16,18-21], which numerically evaluate canonical thermal averages of some observable A by an approximate one, where M states \{x_\mu\} are selected by importance-sampling process. Any computer simulations including Monte Carlo method for the 3D Ising model are limited by the size effects since the number of the configurations of the 3D Ising model increases in a fashion of 2^N, with the number of the spins or atoms, N \to \infty. Usually, the period boundary condition is used as that the simulation is performed on a small system with finite size. This technique breaks down the long-range many-body interaction between the two nearest neighboring spins along the third dimension in the 3D Ising model, which is due to the non-local behavior. As mentioned above, the interaction between the two spins along the third dimension depends on all the possible configurations 2^{nm} of nm spins in a plane. The number nm of the spins in the plane can be infinite, which makes the simulation on any finite subsystems plus the periodic boundary condition meaningless. This results in that for the finite system Monte Carlo simulation there is not a sharp transition between zero and non-zero magnetization or a sharp peak in the specific heat. Therefore the critical point cannot be located precisely, which hinders to understand its singularity analytically at the critical point. The limitations of the Monte Carlo simulations were summarized in [2]. For the 2D Ising model, the Monte Carlo simulations give much better results, not only because much larger lattice sizes can be dealt with and but also because there is no non-local problem in 2D. Although the combination of the Monte Carlo simulations and the renormalization group techniques reduces the calculation time and give better results by allowing us to
study much larger than those possibly by the direct summation methods, the difficulties above of the Monte Carlo simulations are not removed essentially.

4. Some recent advances in 3D Ising model

In this section, we will review some recent advances in the 3D Ising model and related topic.

4.1 Fractals and chaos in 3D Ising lattices

Inspired by the Zhang’s generation of Onsager’s approach to the 3D Ising model [2], Lawrynowicz et al. discussed the 3D Ising lattices in the context of possible applications of the Jordan-von Neumann-Wigner approach [22]. They reformulated the algebraic part of the theory in terms of the quaternionic sequence of Jordan algebras and looked at some of the geometrical aspects of the 3D Ising lattices [22,23]. They discussed the relationship with Bethe-type fractals, Kikuchi-type fractals and fractals of the algebraic structure, and also the duality for fractal sets and lattice models on fractal sets. It was found that it is possible to have a simpler description in terms of fractals corresponding to algebraic structure involving the quaternionic sequence of Jordan algebras [22]. It was suggested that the commutativity of the 3D Ising model can be realized by application of Jordan algebras in relation with the Clifford algebras. Moreover, the quaternionic sequence of Jordan algebras has the mathematical basis of the quantum mechanics. In [24-26], Lawrynowicz et al. continued to modify and simplify the Ising-Onsager-Zhang procedure for analyzing the 3D Ising lattices by considering some fractal structures in connection with Jordan and Clifford algebras and by following the Jordan-von Neumann-Wigner approach. They concentrated on duality of complete and perfect Jordan-von Neumann-Wigner systems, particular ternary systems, and analyzed algebras of complete Jordan-von Neumann-Wigner systems. It was proven that in the case of a composition algebra, there is a self-dual perfect Jordan-von Neumann-Wigner system related to quaternion or octonion algebras [24]. It was pointed out [22,23,25] that the Onsager-Zhang approach reformatted with the use of the quaternionic sequence of Jordan algebra, is closely related to Heisenberg’s approach to quantum theories. The Jordan structures are closely related to some types of fractals, in particular, fractals of the algebraic structures [25]. The study in [25] included fractal renormalization, the renormalized Dirac operator, meromorphic Schauder basis and hyperfunctions on fractal boundaries.

Scaling properties near the critical point indicates the existence of self-similarity behavior for the critical phenomena. Although the 3D Ising system considered is not a truly dynamic one, Zhang and March proposed a specific set of relations between fractal dimensions and critical exponents in the Ising model of statistical mechanics [27]. In particular, we put forward, corresponding to six critical exponents for the Ising model, six fractal dimensions. Assuming the latter proposals, we then derived relationships between such fractal dimensions.

4.2 Critical exponents in 3D Ising models and related systems

The 3D Ising universality for critical indices in magnets and at fluid-fluid phase transition was reviewed in [28]. Experimental data for critical exponents in magnetic materials were compared with theoretical results on the 3D Ising model, as derived based on two conjectures [2]. It was found that critical exponents in some bulk magnetic materials indeed form a 3D Ising universality. The attention was then focused on the critical indices at fluid-fluid phase transition. It is commonly accepted that pure fluids at gas-liquid critical points, and binary liquid mixtures at liquid-liquid critical points belong to the same universality class as the Ising model. On the other hand, the analogy between magnetic behaviour near criticality and the corresponding liquid – vapour behaviour was used to discuss the 2D – 3D crossover in the latter case [29]. The experimentally observed magnetic behaviour
near criticality was considered for the ferromagnet CrBr$_3$ to allow fingerprints of the 3D Ising Hamiltonian to be anticipated. The magnetic equation of state for the 3D Ising model near criticality was proposed for CrBr$_3$ [29,30]. Moreover, the dimensionality and the critical exponents of the models for polymer growth in solution and the dynamic epidemic model were discussed in [31]. We also predicted the critical exponents for the Potts model in three dimensions and the percolation exponents ($q = 1$) for $d = 1, 2, 3, 4, 5,$ and $d \geq 6$.

A relationship between the exponents $\gamma$ and $\eta$ at liquid–vapour critical point was proposed via the dimensionality $d$ [32]. Similarities and contrasts between critical point behavior of heavy fluid alkalis and d-dimensional Ising model were discussed in [33]. Crucial combinations of critical exponents were given for liquids-vapour and ferromagnetic second-order phase transitions [34]. A theory of critical exponents was developed in terms of dimensionality $d$ plus universality class $n$ [35]. March also presented a unified theory of critical exponents generated by the Ising Hamiltonian embracing solely the discrete values $d = 2, 3$ and $4$, in terms of the critical exponent $\eta$ [36]. The formula contains the exact values for $d = 2$ and $4$, while for $d = 3$, it yields the value obtained eerily by Zhang [2]. Furthermore, statistical - mechanical models with separable many-body interactions, especially partition functions and thermodynamic consequences, for a variety of classical and quantum phase transition, were reviewed in [37,38].

Power law singularities and critical exponents in $n$-vector models were considered within a theoretical approach called grouping of Feynman diagrams (GFD theory [39]). Predictions for corrections to scaling of the perturbative renormalization group and GFD approaches are different. A nonperturbative proof was provided, supporting corrections to scaling of the GFD theory. Highly accurate experimental data very close to the $\lambda$-transition point in liquid helium, as well as the Goldstone mode singularities in n-vector spin models, evaluated from Monte Carlo simulation results, are discussed. The analysis showed that in both cases the data can be well interpreted within GFD theory. The singularity of specific heat $C_V$ of the 3D Ising model was studied based on Monte Carlo data for lattice sizes $L \leq 1536$ [40]. A direct estimation from $C_V^{\text{MC}}$ data suggests that $\alpha/\nu$ has a small value (e.g., $\alpha/\nu = 0.113$). The conventional power-law scaling ansatz can be questioned and the data are well described by certain logarithmic ansatz. Corrections to scaling in the 3D Ising model were studied based on non-perturbative analytical arguments and Monte Carlo simulation data for different lattice sizes $L$ [41].

There is a new development in the 3D Ising model, using convex optimization of $c$-parameter within the conformal bootstrap approach to the four-point correlation functions [42]. In particular the estimates of $\eta = 0.036$ and $\nu = 0.629$ were reported in [42]. However, it was pointed out in [41] that these estimates in [42] were obtained based on certain hypotheses (e.g. the existence of a sharp kink) and that if these hypotheses are not used, then the conformal bootstrap analyses appear to be consistent with the GFD values $\eta = 1/8$ and $\nu = 2/3$. It is noticed that these GFD values are consistent with the solutions obtained in [2].

One- and two-phase isochoric heat capacities $C_V$ and saturated liquid and vapour densities of sec-butanol near the critical point were measured with a high-temperature and high-pressure nearly constant-volume adiabatic calorimeter [43]. The measured thermodynamic properties of sec-butanol near the critical point were interpreted in terms of the complete scaling theory of critical phenomena and the theory of logarithmic singularity of $C_V$.

4.3 Temperature-time duality and time

Zhang and March presented a detailed analysis of temperature-time duality in the 3D Ising model, by inspecting the resemblance between the density operator in quantum statistical mechanics and the evolution operator in quantum field theory, with the mapping $\beta = (k_BT)^{-1} \rightarrow \text{it}$ [44]. We pointed out that in systems like the 3D Ising model, for the nontrivial topological contributions, the time necessary for the time averaging must be infinite, and being comparable with or even much larger than the time...
of measurement of the physical quantity of interest. The time averaging is equivalent to the
temperature averaging. The phase transitions in the parametric plane \((\beta, t)\) are discussed, and a
singularity (a second-order phase transition) was found to occur at the critical time \(t_c\), corresponding to
the critical point \(\beta_c\) (i.e. \(T_c\)). It is necessary to use the 4-fold integral form for the partition function for
the 3D Ising model. The time is needed to construct the \((3+1)D\) framework for the quaternionic
sequence of Jordan algebras, in order to employ the Jordan-von Neumann-Wigner procedure. We then
turned to discuss quite briefly temperature-time duality in quantum-chemical many-electron theory.
We found that one can use the known one-dimensional differential equation for the Slater sum \(S(x, \beta)\)
to write a corresponding form for the diagonal element of the Feynman propagator, again with the
mapping \(\beta \rightarrow it\).

Based on the quaternionic approach developed in [2] for the three-dimensional (3D) Ising model,
we study conformal invariance in three dimensions. The 2D conformal field theory was generalized to
be appropriate for three dimensions, within the framework of the quaternionic coordinates with
complex weights [45]. The Virasoro algebra still works, but for each complex plane of quaternionic
coordinates. The quaternionic geometric phase appears in quaternionic Hilbert space as a result of
diagonalization procedure which involves the smoothing of knots/crossings in the 3D many-body
interacting spin Ising system. Possibility for application of conformal invariance in three dimensions
on studying the behaviour of the world volume of the brane, or the world sheet of the string in 3D or
\((3+1)D\), was discussed.

Zeng discussed the possibility of emergence of time as the holographic dimension of gauge
systems in Euclidean space [46], which take statistic, e.g. Ising model as concrete implementations.
By identifying the renormalization group flow of statistic models with the time flow of dual gravities,
he got a universe whose evolution history is qualitatively the same as our real world (also for 3D Ising
model). He also commented highlights projected by this idea on the cosmological constant problem
and developed preliminary evidences for the validity of this idea. It is interesting to note that the
partition function obtained in [2] for 3D Ising model was used for calculation of the entropy.

5. Summary

In summary, we have investigated the non-local behavior of the 3D Ising model. The interaction
between the nearest neighboring spins along the third dimension is effectively equivalent to the long-
range many-body interaction in which \((n+1)\) spins are involved. This nonlocal property is an intrinsic
property of the 3D Ising model, which could cause that any approximation techniques (including the
conventional low-temperature expansions, the conventional high-temperature expansions, the
perturbative renormalization group and Monte Carlo method) taking into account only the local
property cannot be exact for the 3D Ising model, though these approaches work well for other 2D or
1D models. Any approaches taking into account only the local environments (such as local alignments
of spins in low-temperature expansions, polygons in high-temperature expansions, finite blocks in
renormalization group, or finite unit cells plus the periodic boundary condition in Monte Carlo
method) are inconsistency with the global effect induced by the chain of \(n\) spins. The exact solution of
the 3D Ising model must be calculated by evaluating the partition function of the whole system
involving all the possible states of all the spins of the whole 3D system. Therefore, the systematical
errors exist seriously in the approximation techniques, which are caused by their disadvantages. These
systematical errors of these approximation techniques are related directly to the physical
conceptions/pictures at the first beginning and the neglects of important non-local factors during
procedures. The systematical errors are intrinsic, which cannot be removed by the efforts of improving
technically the precision. Finally, we reviewed briefly some recent advances in the 3D Ising model.

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