Curvaton Dynamics

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Abstract

In contrast to the inflaton’s case, the curvature perturbations due to the curvaton field depend strongly on the evolution of the curvaton before its decay. We study in detail the dynamics of the curvaton evolution during and after inflation. We consider that the flatness of the curvaton potential may be affected by supergravity corrections, which introduce an effective mass proportional to the Hubble parameter. We also consider that the curvaton potential may be dominated by a quartic or by a non-renormalizable term. We find analytic solutions for the curvaton’s evolution for all these possibilities. In particular, we show that, in all the above cases, the curvaton’s density ratio with respect to the background density of the Universe decreases. Therefore, it is necessary that the curvaton decays only after its potential becomes dominated by the quadratic term, which results in (Hubble damped) sinusoidal oscillations. Finally, we study the effects of thermal corrections to the curvaton’s potential and show that, if they ever dominate the effective mass, they lead to premature thermalization of the curvaton condensate. To avoid this danger, a stringent bound has to be imposed on the coupling of the curvaton to the thermal bath.

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I. INTRODUCTION

Observation of the Cosmic Microwave Background Radiation (CMBR) anisotropy has now confirmed that structure in the Universe originates as density perturbation, already present a few Hubble times before cosmological scales enter the horizon, with an almost flat spectrum [1–4]. The only known explanation for this ‘primordial’ density perturbation is that it originates, during an era of almost exponential inflation, from the vacuum fluctuation of some ‘curvaton’ field [5]. The primordial density perturbation is predominantly adiabatic in character [2, 14], and hence characterized by a single quantity which may be taken to be the spatial curvature perturbation ζ. According to observation, the spectrum of the curvature perturbation (roughly its mean-square) on cosmological scales is [1]

\[ P_\zeta = (2 \times 10^{-5})^2. \] (1)

According to the usual hypothesis, the primordial curvature perturbation is generated solely by the inflaton field, defined as the one whose value determines the end of inflation. No other field is involved, and the curvature perturbation is conserved after inflation ends. As a result, the spectrum of the primordial curvature perturbation in this ‘inflaton scenario’ is essentially determined by the inflation model alone [15, 16]. The only influence of the subsequent cosmology is to determine precisely the number \( N \gtrsim 60 \) of e-folds of inflation after our Universe leaves the horizon.

It has been pointed out in Ref. [17] (see also Refs. [18, 19]) that the inflaton scenario is not the only possible way of generating large scale structure from inflation. The primordial density perturbation may instead originate from the vacuum fluctuation of some ‘curvaton’ field \( \sigma \), different from the inflaton field. The curvaton scenario has received a lot of attention [6, 20–47] because it opens up new possibilities both for model-building and for observation [48].

In the curvaton scenario, the curvaton energy density is supposed to be initially negligible so that spacetime is practically unperturbed. It becomes significant only during some era of radiation domination, when the curvaton field is undergoing a (Hubble damped) sinusoidal oscillation causing its energy density to grow relative to the radiation background. (In this paper, we take \( \sigma \) to be a real scalar field.) While this happens, the curvaton energy density \( \rho_\sigma \) is proportional to the square of the amplitude \( \bar{\sigma} \) of the oscillation, and the fractional perturbation in both of these quantities is time-independent. Finally, the curvaton decays before nucleosynthesis, leaving behind a conserved curvature perturbation which is proportional to the fractional energy density perturbation that existed before the curvaton decayed,

\[ \zeta = 1 - \frac{\delta \rho_\sigma}{\rho_\sigma} = 2 - \frac{6 \delta \bar{\sigma}}{3 \bar{\sigma}}. \] (2)

In the (quite good [32, 47]) approximation of sudden curvaton decay, the constant \( r \) is the energy density fraction
of the curvaton at the decay epoch,

$$r \simeq \left. \frac{\rho_{\sigma}}{\rho} \right|_{\text{dec}} .$$  \hfill (3)

The fractional perturbation in $\delta$ will be related to the fractional perturbation of the curvaton field during inflation by some factor $q$,

$$\frac{\delta \sigma}{\bar{\sigma}} = q \frac{\delta \sigma^*}{\sigma^*} ,$$  \hfill (4)

where the subscript $*$ denotes the epoch when the cosmological scales exit the horizon during inflation. Using the well-known expression $P_{\sigma^*} = H_* / 2\pi$, valid for any light scalar field, with $H$ the Hubble parameter, one obtains the prediction of the curvaton scenario [17, 21]

$$P_{\xi}^{2} = \frac{2}{6\pi} r q \frac{H_*}{\sigma^*} .$$  \hfill (5)

(We are ignoring, in this paper, the possible scale-dependence of the curvature perturbation.)

This prediction depends on the numbers $q$ and $r$, which encode the evolution of the curvaton field between horizon exit during inflation and the epoch of curvaton decay. The evolution is given in terms of the effective potential $V(\sigma, H)$ by the well-known equation

$$\ddot{\sigma} + 3H \dot{\sigma} + V' = 0 ,$$  \hfill (6)

and its perturbation

$$\left( \ddot{\delta \sigma} \right) + 3H(\dot{\delta \sigma}) + V''\delta \sigma = 0 ,$$  \hfill (7)

where the prime and the dot denote differentiation with respect to $\sigma$ and the cosmic time $t$ respectively. The explicit dependence of the potential on $H$ comes from the time-dependent fields, which must be present in the early Universe in order to generate the energy density.

The simplest case is the one where the effective curvaton potential is so flat that the curvaton field actually has negligible evolution until the onset of (Hubble damped) sinusoidal oscillation. Then $q = 1$, and $r$ may be calculated in terms of the curvaton decay rate as described in Ref. [41]. This case may be expected to occur [17, 41] if the curvaton is a pseudo-Nambu-Goldstone boson (PNGB), so that it has a potential of the form

$$V \simeq (vm)^2 \left( 1 - \cos \left( \frac{\sigma}{v} \right) \right) ,$$  \hfill (8)

whose flatness is protected by a global symmetry.

In this paper, we consider the case that the curvaton is not a PNGB. Instead of Eq. (8), we adopt a form for the potential which allows a range of possibilities,

$$V(\sigma, H) = \frac{1}{2} \left( m^2 \pm cH^2 + g^2T^2 \right) \sigma^2 + \frac{\lambda_n}{(n+4)!} \frac{\sigma^{n+4}}{M^n} .$$  \hfill (9)

For the curvaton model to work $m$ must be far smaller than $H_*$. The term $\pm cH^2$ represents the effect of those fields to which the curvaton has gravitational-strength couplings, on the assumption that supergravity is valid. It may be positive or negative, and its value will change at the end of inflation and again at reheating [52–54]. Generically $c \sim 1$, but during inflation one must have $c_* \ll 1$ so that the curvaton qualifies as a ‘light’ field. The term $g^2T^2$ represents the effective coupling $g$ of the curvaton to those fields which are in equilibrium at temperature $T$ after inflation. Finally, we have kept the leading term $n \geq 0$ in the presumably infinite sum of quartic and higher terms. In the case of non-renormalizable terms ($n > 0$), we take $M$ to be the ultra-violet cutoff of the effective field theory, say the Planck scale, and assume that $\lambda_n \lesssim 1$.

Using this form for the potential, we study the evolution of the curvaton field in a very general way. We verify that the curvaton contribution to the energy density, if initially negligible, remains so until the (Hubble damped) sinusoidal oscillation sets in, and we also show how to calculate the numbers $r$ and $q$ which enter into the prediction for the curvature perturbation.

The plan of the paper is as follows. In Section II, we lay out the stage for our study. In particular, we provide the basic equations for the background evolution of the Universe and briefly introduce the curvaton model with emphasis on how the curvaton imposes its perturbations onto the Universe and on what the requirements for a successful curvaton are. Also, in this section, we discuss and justify the choice of the form of the curvaton’s effective potential. In Section III, we study the evolution of the curvaton during inflation and estimate the value of the field at the end of inflation as well as the amplitude of its perturbations at that time. In Section IV, we investigate the evolution of the curvaton field and its density ratio with respect to the background density after the end of inflation. We find analytic solutions in all cases considered, which describe both the curvaton’s fractional perturbation and the background density ratio at any time. In Section V, we employ the results of the previous section to calculate, in all cases considered, the time when the curvaton comes to dominate the Universe or, if it decays earlier than this, the corresponding value of the density ratio $r$ at the time of decay. We also calculate in detail the value of $q$, relating the curvaton’s fractional perturbation at decay with its initial fractional perturbation, obtained in inflation. In Section VI, we study the effects of temperature corrections to the effective potential on the evolution of the curvaton and find upper bounds on the curvaton’s coupling $g$ to the thermal bath, which prevent these effects from being destructive. In Section VII, we present a concrete model realization of our findings. Assuming a realistic set of values for the inflationary parameters, we study the model in detail and, by enforcing all the constraints and requirements, we identify the parameter space for a successful curvaton. Finally, in Section VIII, we discuss our results and present our conclusions.

In this paper, we use units such that $c = \hbar = 1$ so that Newton’s gravitational constant is $G^{-1} = 8\pi G m_P$, where $m_P = 2.4 \times 10^{18}$GeV is the reduced Planck mass.
II. THE SET UP

A. The background evolution of the Universe

The dynamics of the Universe expansion is determined by the Friedmann equation, which, for a spatially flat Universe, reads

$$\rho = 3(m_P H)^2, \quad (10)$$

where $\rho$ is the density of the Universe and $H \equiv \dot{a}/a$ with $a$ being the scale factor of the Universe.

The dynamics of a real, homogeneous scalar field $\sigma$ in the expanding Universe is determined by the Klein-Gordon equation of motion given in Eq. (6). It is well-known that a homogeneous scalar field can be treated as a perfect fluid with density and pressure given, respectively, by

$$\rho_\sigma = \rho_{\text{kin}} + V \quad \text{and} \quad p_\sigma = \rho_{\text{kin}} - V \quad (11)$$

with $\rho_{\text{kin}} \equiv \frac{1}{2} \dot{\sigma}^2$ being its kinetic energy density. In view of this fact, we will model the content of the Universe as a collection of perfect fluids with equations of state of the form: $\rho_i = w_i \rho_i$, where $\rho_i$ and $p_i$ is the density and pressure of each fluid respectively and $w_i$ is the corresponding barotropic parameter. In particular, for matter (radiation), we have $w_m = 0 \{w_\gamma = \frac{1}{3}\}$. In most of the history of the Universe, the total density $\rho$ is dominated by one of the above components.

The energy conservation (continuity) equation for the Universe, reads

$$\dot{\rho} = -3H(\rho + p). \quad (12)$$

The above can be applicable to each of the component fluids individually, if only they are independent so that their densities $\rho_i$ are conserved, diluted only by the expansion of the Universe according to Eq. (12).

If $w$ is the barotropic parameter of the dominant fluid then, for a constant $w$, Eq. (12) gives [55]

$$\rho \propto a^{-3(1+w)}. \quad (13)$$

Using this and Eq. (10), it is easy to find that, for $w \neq -1$, the Hubble parameter and the density of the Universe evolve in time as

$$H(t) = \frac{2t^{-1}}{3(1 + w)} \quad \text{and} \quad \rho = \frac{4}{3(1+w)^2} \left(\frac{m_P}{t}\right)^2, \quad (14)$$

from which it is evident that $a \propto t^{2/3(1+w)}$.

During inflation the Universe is dominated by the potential density of another scalar field, the inflaton. According to this, Eq. (11) suggests that the equation of state of the Universe during inflation is $w \simeq -1$, which, in view of Eq. (13), suggests that $\rho \simeq \text{const.}$. Hence (c.f. Eq. (10)) the Hubble parameter remains roughly constant.

B. The curvaton model

The curvaton is a scalar field $\sigma$ other than the inflaton that may be responsible for the curvature perturbation in the Universe. One of the merits of considering such a field is the liberation of inflation model-building from the stringent requirements imposed by the observations of the Cosmic Background Explorer (COBE). Indeed, the curvaton model changes the COBE constraint on the energy scale of inflation $V_{\text{inf}}^{1/4}$ into an upper bound [20]. Furthermore, in the context of the curvaton model, one can achieve a remarkably flat superhorizon spectrum of curvature perturbations, which is in agreement with the recent data from the Wilkinson Microwave Anisotropy Probe (WMAP) [1, 2]. Finally, since the curvaton (in contrast to the inflaton) is not related to the Universe dynamics during inflation it can be associated with much lower energy scales than the ones corresponding to inflation and, therefore, may be easily linked with lower energy (e.g. TeV) physics. This is why physics beyond the standard model provides many candidates for the curvaton, such as sfermions, string axions, or even the radion of large extra dimensions.

According to the curvaton scenario, the curvature perturbations, generated during inflation due to the quantum fluctuations of the inflaton field, are not the ones which cause the observed CMBR anisotropy and seed the formation of large scale structure. This is because their contribution is rendered negligible compared to the curvature perturbations introduced by another scalar field, the curvaton. In a similar manner as with the inflaton, the curvaton field receives an almost scale invariant, superhorizon spectrum of perturbations during inflation. However, at that time, the density of the curvaton field is subdominant, and the field lies frozen at some value displaced from the minimum of its potential. After the end of inflation, the field eventually unfreezes and begins oscillating. While doing so, the density fraction of the curvaton to the overall energy density increases. As a result, before decaying (or being thermalized) the curvaton may come to dominate (or nearly dominate) the Universe, thereby imposing its own curvature perturbation spectrum. After this, it decays into standard model particles, creating the thermal bath of the standard hot big bang.

1. The curvature perturbation

On a foliage of spacetime corresponding to spatially flat hypersurfaces, the curvature perturbation attributed to each of the Universe components is given by [21]

$$\zeta_i = -H \frac{\delta \rho_i}{\rho_i}. \quad (15)$$

Then the total curvature perturbation $\zeta(t)$, which also satisfies Eq. (15), may be calculated as follows. Using
that \( \delta \rho = \sum_i \delta \rho_i \) and Eq. (12), it is easy to find

\[
(\rho + \rho) \zeta = \sum_i (\rho_i + p_i) \zeta_i .
\]  

(16)

Now, since in the curvaton scenario all contributions to the curvature perturbation other than the curvaton’s are negligible, we find that

\[
\zeta_0 = \zeta_0 \left( 1 + w_\sigma \right)_{\text{dec}} \frac{\rho_\sigma}{\rho} \bigg|_{\text{dec}},
\]

(17)

where \( \zeta_0 \equiv P^-_{\zeta} = 2 \times 10^{-5} \) is the curvature perturbation observed by COBE, and \( \zeta_\sigma \) is the curvature perturbation of the curvaton when the latter decays (or thermalizes). From the above, it is evident that (c.f. Eq. (3))

\[
r \equiv \frac{\zeta_0}{\zeta_\sigma} \simeq \frac{\rho_\sigma}{\rho} \bigg|_{\text{dec}} .
\]

(18)

The curvaton decays when \( H \sim \Gamma \), where

\[
\Gamma = \max \{ \Gamma_\sigma, \Gamma_T \}
\]

(19)

with \( \Gamma_\sigma \) being the decay rate of the curvaton and \( \Gamma_T \) the thermalization rate, corresponding to the thermal evaporation of the curvaton condensate. Note that, after the curvaton decay, the total curvature perturbation remains constant, i.e. \( \zeta_0 = \zeta_{\text{dec}} \).

Suppose that the curvaton, just before decaying (or being thermalized), is oscillating in a potential of the form \( V \propto \sigma^\alpha \), where \( \alpha \) is an even, positive number. Then, according to Ref. [56], the density of the oscillating field scales as

\[
\rho_\sigma \propto a^{-\frac{6\alpha}{\alpha + 2}} ,
\]

(20)

which means that \( \dot{\rho} = -\frac{6\alpha}{\alpha + 2} \rho H \). Using this, Eq. (15) gives

\[
\zeta_\sigma = \frac{\alpha + 2}{6} \delta \sigma \sigma \bigg|_{\text{dec}} ,
\]

(21)

where we used that \( \delta \rho_\sigma / \rho_\sigma = \alpha \delta \sigma / \sigma \).

Note also that, using Eq. (12) for the oscillating curvaton, it is easy to show that

\[
w_\sigma = \frac{\alpha - 2}{\alpha + 2} = \text{const} .
\]

(22)

Thus, in the case of a quadratic potential, the oscillating scalar field behaves like pressureless matter, whereas for a quartic potential it behaves like radiation. For a (Hubble damped) sinusoidal oscillation (\( \alpha = 2 \)), the above give \( \zeta_\sigma = \frac{\delta \sigma}{\sigma} \delta \sigma \bigg|_{\text{dec}} \) (c.f. Eq. (2)).

2. The curvaton requirements

Apart from providing the correct curvature perturbation, a successful curvaton needs to satisfy a number of additional requirements. These can be outlined as follows:

- **Masslessness:** In order for the curvaton field to obtain a superhorizon spectrum of perturbations during inflation, at least on cosmological scales, the field needs to be effectively massless, when these scales exit the horizon. Thus, at that time, we require that

\[
V'' \ll H_s^2 .
\]

(23)

- **Misalignment:** For the curvature perturbations due to the curvaton to be Gaussian, it is necessary that, when the cosmological scales exit the horizon during inflation, the curvaton field is significantly displaced from the minimum of its potential. Thus, we require that

\[
(\sigma - \sigma_{\text{min}})_s > H_s .
\]

(24)

- **WMAP:** According to the recent wmap observations, to avoid excessive non-Gaussianity in the density perturbation spectrum, the curvaton needs to decay (or thermalize) when the density ratio \( r \) of Eq. (18) is sufficiently large and satisfies the bound [3]

\[
r > 9 \times 10^{-3} .
\]

(25)

Obviously, in order for the field not to be thermalized too soon after the end of inflation, we require that its coupling \( g \) to the thermal bath, generated by the inflaton’s decay, is small enough.

- **Nucleosynthesis:** Due to the above wmap bound on \( r \), one has to demand that the curvaton should have decayed (and not just be thermalized) by the time when big bang nucleosynthesis (BBN) takes place (at temperature \( T_{\text{BBN}} \sim 1 \text{ MeV} \)), which imposes the constraint

\[
\Gamma_\sigma > H_{\text{BBN}} \sim 10^{-24} \text{GeV} .
\]

(26)

C. The effective potential

The form of the perturbative scalar potential is [57]

\[
V(\sigma) = \frac{1}{2} m^2 \sigma^2 + \sum_{n \geq 0} \frac{\lambda_n}{(n + 4)!} \sigma^{n + 4} ,
\]

(27)

where \( 0 < \lambda_n \leq 1 \) and \( M \) is some large mass-scale (e.g. \( m_p \)) determining when the non-renormalizable terms become important. We will consider that the sum is dominated by the term of the \( (n + 4)\)-th order. We will also take into account temperature corrections due to the possible coupling \( g \) of the field with an existing thermal bath of temperature \( T \). Finally, we will consider corrections to the scalar potential due to supergravity effects. These corrections introduce an effective mass, whose magnitude
is determined by the Hubble parameter $H(t)$ [53]. All in all, the form of our scalar potential is

$$V(\sigma) = \frac{1}{2} (m^2 \pm c H^2 + g^2 T^2) \sigma^2 + \frac{\lambda_n}{(n+4)!} \frac{\sigma^{n+4}}{M^n}, \quad (28)$$

where $0 < c, g \leq 1$ [58]. The masslessness requirement demands that, at least during inflation, $c_* \ll 1$. This may be due to some (approximate) symmetry or due to accidental cancellations of specific Kähler corrections, which means that it depends on the particular curvaton model [59]. Obviously, the masslessness requirement also demands $m \ll H_*$ during inflation.

Depending on the sign in front of $c$, the above potential may have a global minimum or a local maximum at the origin. In the latter case and when $\sqrt{c} H \gg m, g T$, the potential is characterized by global minima at $\pm \sigma_{\text{min}}$ such that

$$\sigma_{\text{min}}(H) = \left[ (n+4)! \frac{c H^2 M^n}{\lambda_n} \right]^{\frac{1}{n+2}} \sim (c H^2 M^n)^{\frac{1}{n+2}}, \quad (29)$$

where, in the last equation, we have absorbed $\lambda_n/(n+4)!$ into $M$ (for $n \to 0$, we can set $M^n \to 4!/\lambda_n$). In this case, in order to ensure that the potential remains positive we may add a constant $V_0 \gtrsim c H^2 \sigma_{\text{min}}^2$ [60].

If the local minimum is displaced from the origin then one can express the potential in terms of

$$\hat{\sigma} \equiv \sigma - \sigma_{\text{min}}, \quad (30)$$

in which case, for $\sqrt{c} H \gg m, g T$, the potential of Eq. (28) is recast as

$$V(\hat{\sigma}) = \frac{1}{2} (n + 2) c H^2 \hat{\sigma}^2 + \frac{\lambda_n}{M^n} \sum_{k=0}^{n+1} \alpha_k \sigma_{\text{min}}^{n+1-k} \hat{\sigma}^{3+k}, \quad (31)$$

where $\alpha_k^{-1} \equiv (n + 1 - k)! (3 + k)!$. Since the series is finite, the above is an exact expression and not an approximation. It is evident that all the terms of the sum in Eq. (31) are comparable to each other when $\hat{\sigma} \simeq \sigma_{\text{min}}$ and the sum reduces to $\sigma_{\text{min}}^{n+1} \sum_{k=0}^{n+1} \alpha_k$. For $\hat{\sigma} \gg \sigma_{\text{min}}$, however, the sum is dominated by the term of the largest order which is $\hat{\sigma}^{n+4}/(n+4)!$. Obviously, when $\hat{\sigma} \ll \sigma_{\text{min}}$, only the quasi-quadratic (‘quasi’ in the sense that $H = H(t)$) term is important. Thus, in all cases, the potential can be roughly approximated by

$$V(\sigma) \sim c H^2 \sigma^2 + \frac{\sigma^{n+4}}{M^n}, \quad (32)$$

where we considered that, when $\hat{\sigma} \gg \sigma_{\text{min}}$, then $\hat{\sigma} \simeq \sigma$.

In the case of $+c$ in Eq. (28), $\sigma_{\text{min}} = 0$ and $\sigma = \sigma_*$. In the $-c$ case, $\sigma_{\text{min}} \neq 0$ but the complicated structure near the local maximum is felt only when the field is near $\hat{\sigma} \sim \sigma_{\text{min}}$. Since we attempt a generic study of the behaviour of the field’s dynamics, Eq. (32) will suffice for $\sqrt{c} H \gg m, g T$. It is true, however, that a different higher-order term may dominate for different values of the field. This is one of the reasons for treating $n$ as a free parameter here. In general, we expect that the order of the dominant higher–order term increases with $c$. This can be understood as follows.

Reinstating the $\lambda_n$ coefficients, it is easy to see that the ratio of two higher–order terms of different orders is

$$\frac{V_n}{V_{n+k}} \sim \frac{\lambda_n}{\lambda_{n+k}} \left( \frac{M}{\sigma} \right)^k. \quad (33)$$

This means that, if all the $\lambda_n$’s are of the same magnitude then, for $\sigma \ll M$, the sum of the higher–order terms in Eq.(27) will be dominated by the lowest order (quartic term), while, for $\sigma \gg M$, it will be the highest order that dominates. Since the sum is infinite, this means that $V(\sigma \gg M)$ blows up and the perturbative approximation is no longer valid. However, if the $\lambda_n$ coefficients are not of the same magnitude, then it is possible that the sum of the higher–order terms is dominated by a single term, higher than the quartic, before blowing up. Increasing $\sigma$ may change the order of this term to an even higher one for given $\lambda_n$’s as shown by Eq. (33).

### III. DURING INFLATION

#### A. Regimes of evolution

During inflation, our scalar field is expected to roll down its potential toward the minimum. This roll may be of three distinct types, depending on the initial conditions of the field or, more precisely, on where exactly the field lies originally in its potential. Below, we briefly discuss these regimes assuming that, for all practical purposes, the Hubble parameter during inflation is almost constant. We also assume, for simplicity, that the minimum of the potential lies at the origin (if not one should simply set $\sigma \to \hat{\sigma}$) and consider the potential of Eq. (32). Without loss of generality we take $\sigma > 0$.

- **The fast-roll regime:** The field is in this regime when it lies at a region of its potential such that $V'' \gg H_*^2$. Due to the masslessness requirement during inflation, we expect $c_* \ll 1$. Thus, the field may be in the fast-roll regime only if it lies into the higher–order part of its potential. Then it is easy to see that the border of the fast-roll regime corresponds to $V''(\sigma_{fl}) \sim H_*^2$, where

$$\sigma_{fl} \sim (H_*^2 M^n)^{\frac{1}{n+2}} \sim c_*^{-\frac{1}{n+2}} \sigma_{\text{min}}^* \quad (34)$$

with $\sigma_{\text{min}}^* \equiv \sigma_{\text{min}}(H_*)$.

Once in the fast-roll regime (i.e. for $\sigma > \sigma_{fl}$), the field rapidly rolls until it reaches $\sigma_{fr}$. After this, the motion of the field is no longer underdamped but feels instead the drag of the Universe expansion. Any kinetic energy assumed during the fast-roll will be depleted very fast after crossing $\sigma_{fr}$ as can be easily seen from Eq. (6) [61].
• The slow-roll regime: This regime corresponds to \( V'' \ll H^2 \). The field’s motion is overdamped by the Universe expansion and Eq. (6) can be approximated as

\[
3H\dot{\sigma} \simeq -V'(\sigma),
\]

which is the familiar form of the inflaton’s equation of motion in slow-roll inflation.

• The quantum regime: Classically, the field will continue to slow-roll until it reaches the minimum of its potential. However, an effectively massless scalar field during inflation undergoes particle production by generating superhorizon perturbations due to its quantum fluctuations. In that respect, the quantum fluctuations may affect the classical motion of the field if the generation of the perturbations becomes comparable to the slow-roll motion of the field. Let us elaborate a bit on this issue.

In a region of fixed size \( H^2_r \) (horizon), the effect of the quantum fluctuations can be represented [62] as a random walk with step given by the Hawking temperature \( \delta \sigma = T_H = H_s / 2\pi \) per Hubble time. The slow-roll motion is comparable to this “quantum kick” when \( \delta \sigma \sim \dot{\sigma} / H_s \), or equivalently when \( \sigma \simeq \sigma_Q \), where \( \delta \) is given by Eq. (35) and \( \sigma_Q \) is implicitly defined by the condition

\[
V'(\sigma_Q) \equiv H_s^3.
\]

(Here \( \sigma \) corresponds to region of fixed size \( H^{-1}_r \)). One can understand the above also as follows. The energy density corresponding to the quantum fluctuations, which exit the horizon, is \( \rho_{\text{kin}} \sim T_H^4 \sim H_s^4 \) [63]. The quantum regime begins when this energy density is comparable to the kinetic density of the slow-roll \( \rho_{\text{kin}} \equiv \frac{1}{2}\dot{\sigma}^2 \). Comparing the two, one finds the same condition, Eq. (36), for the border of the quantum regime \( \sigma_Q \) [64].

After the onset of the quantum regime, the coherent motion of \( \sigma \) ceases. Instead, the field configuration spreads toward the origin by random walk. As a result, the mean value at horizon exit of the curvaton field in our Universe (denoted by \( \sigma \)) may lie anywhere within the range \( |\sigma| \leq \sigma_Q \), which means that, typically, its value would be \( \sigma \sim \sigma_Q \). The same is true if the field finds itself already into the quantum regime at the onset of inflation.

Thus, from the above, we can sketch the typical evolution of the field during inflation. Since the curvaton is expected to have negligible energy density when the cosmological scales exit the horizon during inflation, if we demand that initially \( \rho_\sigma \sim V_{\text{int}} \), this means that the field must originally lie in the steep (and curled) part of its potential (where it cannot act as a curvaton) and, therefore, it will engage into fast-roll. Very soon, however, fast-roll sends the field below \( \sigma_{fr} \) and the curvaton enters its slow-roll regime. Whether it remains there until the end of inflation or not depends on the duration of the inflationary period. If inflation lasts long enough, the field will slow-roll down to \( \sigma_Q \), where it will be “stabilized” by the action of its quantum fluctuations. The quantum drift may further reduce \( \sigma \), somewhat but, since in the quantum regime the field is oblivious of the potential \( V \), there is no real motivation to drag it towards the origin. Still, \( \sigma \ll \sigma_Q \) can be selected anthropically, i.e. by assuming that we happen to live at a special place in the Universe.

B. Initial conditions at the end of inflation

The above enable us to speculate on the initial conditions of the curvaton after the end of inflation. Indeed, given sufficiently long inflation, we expect \( \sigma_{fr} \sim \sigma_Q \) and \( \dot{\sigma}_{fr} \sim 0 \) because the coherent motion of the field ceases once it enters into the quantum regime. However, if inflation ends before the field reaches the quantum regime, we have \( \sigma_{fr} > \sigma_Q \), where the subscript ‘end’ denotes the end of inflation. Thus, we see that

\[
\sigma_Q \lesssim \sigma_{fr} < \sigma_r.
\]

Using Eq. (36) and also that \( V' \sim \sigma V'' \) for a perturbative potential, it is easy to see that

\[
\sigma_Q \sim \left( \frac{H_s^2}{V''} \right) \sigma_r \gg H_s.
\]

Since, in all cases, \( \sigma \geq \sigma_{fr} \gtrsim \sigma_Q \), we see that the misalignment requirement for the curvaton is naturally guaranteed by masslessness.

Let us estimate \( \sigma_Q \) using Eq. (32). Whether \( \sigma_Q \) lies in the quasi-quadratic part of the potential or not depends on how large is the value of \( c_* \). Indeed it is easy to see that

\[
\sigma_Q \sim \begin{cases} \frac{H_s}{c_*} & c_0 < c_* < 1 \\ \frac{H_s}{c_0} \sim \left( \frac{H_s^4}{M^n} \right)^{\frac{1}{n+2}} & c_* \leq c_0 \end{cases},
\]

where

\[
c_0 \equiv \left( \frac{H_s}{M} \right)^{\frac{1}{n+2}}.
\]

From the above, since \( c_0, c_* \ll 1 \) during inflation, it is evident that the misalignment requirement is, quite generally, satisfied. Using Eqs. (34), (39) and (40), it is easy to show that, when \( c_* \leq c_0 \),

\[
\frac{\sigma_Q}{\sigma_{fr}} \sim c_0^{(n+2)/(n+3)} < 1
\]

and, therefore, fast-roll always ends before reaching the quantum regime.

We should point out here that \( c_* \) during inflation cannot be arbitrarily small. Indeed, due to the coupling \( g \) between the curvaton and other fields, one expects a
contribution of the form \( g^2 T_H^2 \) to the effective mass during inflation, which introduces the bound \( c_\ast \geq g^2 \). Thus, during inflation,
\[
g^2 \leq c_\ast \ll 1. \tag{42}
\]

Note, also, that, if \( c_\ast < (m/H_*)^2 \), the soft mass \( m \) dominates the quasi-quadratic term of the potential.

In the \( -c \) case, the above are valid with the substitution \( \sigma \rightarrow \delta \sigma \). If \( c_\ast \leq c_0 \) then \( \sigma Q \approx \delta Q \gg \sigma_{\text{min}} \). However, if \( c_0 < c_\ast \ll 1 \) then \( \sigma Q \ll \sigma_{\text{min}} \), which means that \( \sigma Q \sim \sigma_{\text{min}} \) in this case.

**C. The amplitude of perturbations**

We can now estimate the amplitude of the curvaton's almost scale invariant superhorizon spectrum of perturbations. The evolution of \( \delta \sigma \) after horizon crossing is determined by Eq. (7), which, for \( V'' \ll H_*^2 \) and \( 0 \leq (\delta \sigma)_0 < H_*^2 \), gives
\[
\delta \sigma \simeq \delta \sigma_0 \exp \left[ -\frac{1}{3} \left( \frac{V''}{H_*^2} \right) H_* \Delta t \right] \sim H_* e^{-\bar{\eta} \Delta N}, \tag{43}
\]
where \( \delta \sigma_0 = H_* / 2\pi \) at horizon exit, \( \bar{\eta} \equiv V''/3H_*^2 \) and \( \Delta N = H_* \Delta t \) is the elapsed e-foldings of inflation. For the cosmological scales, \( \Delta N \lesssim 60 \) by the end of inflation. Thus, for \( V''/H_*^2 < 0.05 \), we see that \( \delta \sigma \approx \delta \sigma_0 \sim H_* \), i.e. the perturbation remains frozen. Thus, in view also of Eq. (39), for the amplitude of the perturbation spectrum we find
\[
\left. \frac{\delta \sigma}{\sigma} \right|_{\text{end}} \sim \frac{H_*}{\sigma Q} \lesssim \frac{H_*}{\sigma Q} \sim \max \{ c_\ast, c_0 \}. \tag{44}
\]

Here we should mention the fact that, while \( \bar{\eta} \neq 0 \) and \( \bar{H}_* \neq 0 \) during inflation leave the amplitude of the perturbation spectrum largely unaffected, they do affect the slope of the spectrum, causing departure from scale invariance. Indeed, in Ref. [21] it is shown that the spectral index of the curvature perturbation spectrum is given by
\[
n_s = 1 + 2(\bar{\eta} - \epsilon), \tag{45}
\]
where \( \epsilon \equiv -\bar{H}/H_*^2 \ll 1 \).

**IV. AFTER THE END OF INFLATION**

After the end of inflation, the inflaton begins oscillating around its vacuum expectation value. These coherent oscillations correspond to massive particles (inflatons), whose energy density dominates the Universe. Thus, after inflation ends the Universe enters a matter dominated period. The inflatons eventually decay creating a thermal bath of temperature \( T \). The Universe becomes radiation dominated only after the decay of the inflaton field (reheating), which takes place at \( H_{\text{reh}} \sim \Gamma_{\text{inf}} \), where \( \Gamma_{\text{inf}} \) is the decay rate of the inflaton field and the subscript ‘reh’ corresponds to reheating. The temperature at this moment is the so-called reheat temperature \( T_{\text{reh}} \). After reheating, the Universe enters a radiation dominated period.

In the following, we will use Eq. (6) to follow the evolution of the curvaton field after the end of inflation. At the first stage, we will assume that \( cH \gg m, gT \) so that the potential may be approximated by Eq. (32). Because, after the end of inflation, there are no perturbations generated due to quantum fluctuations to inhibit the field’s motion, the curvaton evolves entirely classically, following Eq. (6). Its evolution depends on which of the two terms in Eq. (32) is the dominant. Below we study both cases individually.

**A. The quasi-quadratic case**

Suppose that it is the quasi-quadratic term that dominates the potential in Eq. (32). Then we can write
\[
V(\sigma) \simeq \frac{1}{2} cH^2(t)\sigma^2, \tag{46}
\]
where, for simplicity, we have dropped the hat on \( \sigma \). Inserting the above into Eq. (6) and substituting \( dt = (1 + w)td\tau \), we obtain
\[
\frac{\partial^2 \sigma}{\partial \tau^2} + (1 - w) \frac{d\sigma}{d\tau} + \frac{4c}{9} \sigma = 0. \tag{47}
\]

The above admits oscillating solutions if \( c \) is larger that \( c_\ast \), where
\[
\sqrt{c_\ast} \equiv \frac{3}{4} (1 - w). \tag{48}
\]

1. Case \( c > c_\ast \)

Assuming negligible initial kinetic density and after some algebra, it can be shown that, in this case, the solution for \( \sigma(t) \) is
\[
\sigma(t) = \sigma_0 \left( \frac{t}{t_0} \right)^{-\frac{c}{2-3w}} \left\{ \cos \left[ \frac{2\sqrt{c-c_\ast}}{3(1+w)} \ln \left( \frac{t}{t_0} \right) \right] + \frac{3(1-w)}{4\sqrt{c-c_\ast}} \sin \left[ \frac{2\sqrt{c-c_\ast}}{3(1+w)} \ln \left( \frac{t}{t_0} \right) \right] \right\}. \tag{49}
\]

It is evident from the above that, when \( c \rightarrow c_\ast \), the oscillation frequency reduces to zero.

Since \( V \propto H^2 \sigma^2 \propto (\sigma/t)^2 \) and \( \rho_{\text{kin}} \propto \dot{\sigma}^2 \propto (\sigma/t)^2 \) in view of Eq. (14), we find that
\[
\rho_{\sigma} \propto a^{-\frac{4}{3} + w}. \tag{50}
\]

Comparing the above with Eq. (13), we obtain
\[
\frac{\rho_{\sigma}}{\rho} \propto a^{-\frac{4}{3} + w}. \tag{51}
\]
which means that the density ratio of the curvaton is decreasing and cannot lead to curvaton domination.

Note here that, although during inflation masslessness requires $c_\ast \ll 1$, this is not necessarily so after inflation, when $c$ may assume a different value. In fact, unless the (approximate) symmetry that enables masslessness persists after the end of inflation, we would expect $c \sim 1$ and, therefore, $c > c_\ast$ is quite plausible.

2. Case $c \leq c_\ast$

In this case, Eq. (47) does not admit oscillating solutions. Instead, for $c < c_\ast$ and negligible initial kinetic density, one finds the scaling solution (see Fig. 1)

$$
\sigma(t) = \sigma_0 \left( \frac{t}{t_0} \right)^{-\frac{1}{2} \left( \frac{3(1-w)}{2c_\ast - c} \right)} \times
$$

$$
\times \left[ 1 + \frac{3(1-w)}{4\sqrt{c_\ast - c}} \left( \frac{t}{t_0} \right)^{2\sqrt{c_\ast - c}} + \left( 1 - \frac{3(1-w)}{4\sqrt{c_\ast - c}} \right) \left( \frac{t}{t_0} \right)^{-2\sqrt{c_\ast - c}} \right].
$$

The second term in the square brackets soon becomes negligible. Again $V \propto \rho_{\text{kin}} \propto (\sigma/t)^2$, which means that [65]

$$
\rho_{\sigma} \propto a^{-\frac{3(1+w)}{2} + 2\sqrt{c_\ast - c}}.
$$

From the above, it is evident that, when $c \to c_\ast$ the scaling of $\rho$ reduces to the one given by Eq. (50). Using Eq. (13), we now find for the curvaton’s density ratio

$$
\frac{\rho_{\sigma}}{\rho} \propto a^{-\frac{3(1-w)}{2} + 2\sqrt{c_\ast - c}}. \tag{54}
$$

The above again shows that the curvaton’s density ratio is decreasing and cannot lead to curvaton domination. To make this clearer, consider that, for $c \to 0$, Eq. (48) suggests that $\rho_{\sigma}/\rho \to \text{const.}$, which means that, for all $c > 0$, the exponent in Eq. (54) is negative.

The solution for $c = c_\ast$, with negligible initial kinetic density, is easily found to be

$$
\sigma(t) = \sigma_0 \left[ 1 + \frac{1}{2} \left( \frac{1-w}{1+w} \right) \frac{3}{4\sqrt{c_\ast - c}} \left( \frac{t}{t_0} \right)^{2\sqrt{c_\ast - c}} \right]^{-\frac{1}{2} \left( \frac{3(1-w)}{2c_\ast - c} \right)}.
$$

Ignoring the logarithmic time dependence, the above results in the same scaling for $\rho_{\sigma}$ as in Eq. (50).

From Eqs. (49) and (52), we see that $\delta \sigma/\sigma = \text{const.}$, regardless of whether the field is oscillating or not. In that sense was Eq. (52) called a ‘scaling’ solution. Thus, in view also of Eq. (21), in the quasi-quadratic case we find

$$
\left. \frac{\delta \sigma}{\sigma} \right|_{\text{end}} \sim \left. \frac{\delta \sigma}{\sigma} \right|_{\text{dec}} \sim \zeta_\sigma. \tag{56}
$$

B. The higher–order case

This case corresponds to $\delta \approx \sigma \gg \sigma_{\text{min}}$, when the scalar potential in Eq. (32) can be approximated as

$$
V(\sigma) \approx \frac{\sigma^{n+4}}{M^n}. \tag{57}
$$

In contrast to the previous case, where the effective mass was determined by the Hubble parameter, in this case the roll of the field does not necessarily commence immediately after the end of inflation. Indeed, since at least during the latest stage of inflation (when the cosmological scales exited the horizon and afterwards) the masslessness requirement demands $V'' \ll H$, just after the end of inflation, the field originally lies in the potential at a point where its motion is largely overdamped [66]. Consequently, the field remains frozen at the value $\sigma_{\text{end}}$. One can easily show this as follows. Considering a negligible initial kinetic density, the motion of the field is determined by Eq. (35), which gives

$$
\left( \frac{\sigma_0}{\sigma} \right)^{n+2} \simeq 1 + \frac{1}{9(1+w)} \frac{n+2}{n+3} \left( 1 - \frac{H^2}{H_0^2} \right) \frac{V''}{V_0}. \tag{58}
$$

Thus, as long as $V'' \ll H^2$, we see that $\sigma \approx \sigma_0$, i.e. the field remains frozen. However, while $H(t)$ decreases there will be a moment when $V''(\sigma_{\text{end}}) \sim H$ after which the field unfreezes and begins to roll again down its potential. Note that, as expected, a similar result applies to the perturbation of the field $\delta \sigma$. For example, solving Eq. (7) for the matter era following the end of inflation, one finds

$$
\delta \sigma = \delta \sigma_0 \left( \frac{H}{H_0} \right) \left\{ \cos \left[ \frac{2}{3} \frac{\sqrt{V_0}}{H} \left( 1 - \frac{H}{H_0} \right) \right] + \frac{3}{2} \frac{H_0}{\sqrt{V_0}} \sin \left[ \frac{2}{3} \frac{\sqrt{V_0}}{H} \left( 1 - \frac{H}{H_0} \right) \right] \right\}, \tag{59}
$$
which, in the regime \( \sqrt{V_0} \ll H(t) \ll H_0 \), becomes \( \delta \sigma \approx \delta \sigma_0 [1 + (\frac{H}{\sqrt{V_0}})] \approx \delta \sigma_0 \). Thus, for the amplitude of the perturbation spectrum we expect

\[
\frac{\delta \sigma}{\sigma}|_{\text{osc}} \simeq \frac{\delta \sigma}{\sigma}|_{\text{end}},
\]

(60)

where the subscript ‘osc’ denotes the onset of the \( \sigma \) oscillations.

As demonstrated in Ref. [56], the energy density of an oscillating scalar field in an expanding Universe with potential \( V \propto \sigma^n + 4 \) scales as

\[
\rho_\sigma \propto a^{-6(n+4)}. \quad (61)
\]

However, when the scalar field unfreezes, it does not always engage into oscillations. Indeed, it is easy to show that Eq. (6) with a scalar potential of the form of Eq. (57) has the following exact solution:

\[
\sigma_{\text{solo}}(t) = \left[ 2M^n \frac{(1-w)(n+4) - 4}{(n+4)(n+2)^2(1+w)} \right]^{\frac{1}{n+2}} t^{-\frac{2}{n+2}}. \quad (62)
\]

Of course, in order for the system to follow this solution, it has to have the correct initial conditions. However, in Ref. [67], a stability analysis has shown that the above solution is an attractor to the system for any \( n > n_c \), where

\[
n_c = 2 \left( 1 + \frac{3w}{1-w} \right). \quad (63)
\]

Attractor solutions are lethal for the curvaton scenario. Indeed, one of the principal characteristics of attractor solutions is that they are insensitive to initial conditions. In other words, if the system is to follow an attractor solution, all the memory of the initial conditions will be eventually lost. This is a disaster for the curvaton because it means that the superhorizon spectrum of perturbations will be erased. To understand this better, consider the above solution \( \sigma_{\text{solo}}(t) \). If this is an attractor solution then, after the field unfreezes, it will tend to approach this solution, i.e. \( \sigma(t) \to \sigma_{\text{solo}}(t) \). However, if we consider a perturbed value of the field \( \delta \sigma = \sigma + \delta \sigma \) then again \( \delta \sigma(t) \to \sigma_{\text{solo}}(t) \), which means that \( \delta \sigma \to 0 \). Thus, the perturbation spectrum \( (\delta \sigma/\sigma)_{\text{end}} \), present at the end of inflation, will rapidly diminish so that \( (\delta \sigma/\sigma)_{\text{dec}} \to 0 \) and, therefore, from Eq. (21), we see that \( \zeta_{\sigma} \to 0 \).

Consequently, in order to avoid the above catastrophe, we have to enforce the attractor constraint:

\[
n < n_c, \quad (64)
\]

i.e. we forbid the curvaton field to lie at the end of inflation deep enough in the higher–order part of its potential so that the attractor constraint is violated.

The attractor constraint is more stringent in the matter dominated era (before reheating) because then \( n_c = 2 \). Thus, if the curvaton unfreezes before reheating is completed then it should not lie into the higher–order part of \( V \) beyond quartic terms. Similarly, for the radiation era, \( n_c = 6 \) so the constraint is relaxed up to terms of order 8, if the field is to unfreeze after reheating.

Of course, one can imagine that, even if the initial value after the end of inflation does lie in the forbidden regime, after unfreezing there is a chance that, while approaching the attractor, the field manages to roll down to parts of its potential where the attractor constraint is not violated before its perturbation spectrum becomes entirely erased. Although this is, in principle, marginally possible it does not result in anything more than effectively making the borders of the forbidden zone somewhat “fuzzy”. Thus, in the following, we will disregard this possibility and consider the attractor constraint as strict.

The attractor constraint has a number of consequences on the evolution of the field. For example, from Eqs. (57) and (61), one finds for the amplitude \( \tilde{\sigma} \) of the oscillations

\[
\tilde{\sigma} \propto a^{-\frac{n}{n+2}} \Rightarrow \frac{\tilde{\sigma}}{\sigma_{\text{min}}} \propto H \left( \frac{\tilde{\sigma}}{\sigma_{\text{min}}} \right)^{\frac{n+2}{n+4}}, \quad (65)
\]

where we used also Eqs. (29) and (63). The above shows that the attractor constraint in Eq. (64) guarantees that the ratio \( \tilde{\sigma}/\sigma_{\text{min}} \) is increasing and, therefore, the oscillation amplitude will never drop as low as \( \sigma_{\text{min}} \). This means that once the higher–order term is dominant it will continue to be so without ever the quasi-quadratic term in Eq. (32) taking over. One can also show this as follows. The effective mass for the potential of Eq. (57) is

\[
m_{\text{eff}}^2 \equiv V''(\sigma) \sim \sigma^{n+4}/M^n. \quad (66)
\]

from which it is evident that, due to the attractor constraint, \( m_{\text{eff}}^2 \) decreases slower than \( cH^2 \) and, therefore, the quasi-quadratic term in Eq. (32) will never take over once the field finds itself in the higher–order part of its potential at the end of inflation.

From Eqs. (13) and (61), we find that the density ratio of the oscillating curvaton scales as

\[
\frac{\rho_\sigma}{\rho} \propto a^{-6\left(\frac{n+2}{n+4}\right) + 3(1+w)}. \quad (67)
\]

It is easy to see that the exponent in the above is non-positive for \( n \geq 2\left(\frac{3w}{1-w}\right) \), which is valid both in the matter and the radiation eras for all \( n \geq 0 \). Thus we conclude that the oscillating curvaton in the higher–order case is unable to dominate the Universe, as in the quasi-quadratic case. In fact, we have found that the scaling of the curvaton’s density is such that, for \( 0 \leq n < n_c \), it is decreasing slower than the quasi-quadratic case but not as slow as the background density of the Universe.

It is easy to show that the solution in Eq. (62) scales as

\[
\rho_\sigma(\sigma_{\text{solo}}) \propto a^{-3(1+w)}\left(\frac{n+2}{n+4}\right). \quad (68)
\]

Comparing this with Eq. (61), one can see that the scaling coincides in the border case, when \( n = n_c \). As shown
in Ref. [67], in this border case, even though the energy density of the scalar field does scale according to the attractor solution, the field engages into oscillations. However, as we show in Sec. V D, its perturbation spectrum is not preserved and so \( n = n_c \) cannot be allowed.

For the oscillating curvaton, we have \( \delta \sigma / \sigma = \text{const.} \), since both \( \sigma \) and \( \sigma + \delta \sigma \) undergo the same scaling given by Eq. (65). This fact, in view also of Eqs. (21) and (60), suggests that

\[
\frac{\delta \sigma}{\sigma}_{\text{end}} \sim \zeta_{\sigma}, \quad (69)
\]

which is the same result as in the quasi-quadratic case (c.f. Eq. (56)). We elaborate more on this point in Sec. V D.

Therefore, in both the quasi-quadratic and in the higher–order cases, we have shown that there is no significant damping of the curvaton’s curvature perturbation after the end of inflation [68]. Consequently, using Eq. (44), we find the range for the amplitude of the curvature perturbations of \( \sigma \):

\[
\zeta_0 \leq \zeta_{\sigma} \lesssim \max\{c_s, c_0\}. \quad (70)
\]

We have also shown that, in both the quasi-quadratic and in the higher–order cases, the curvaton is unable to dominate or even to increase its density fraction with respect to the background density of the Universe. This means that for the curvaton scenario to work, the soft mass \( m \) should take over before the curvaton decays (or thermalizes). In Sec. V, we investigate this further.

### C. The \(-c\) case revisited

Before concluding this section some reference to the \(-c\) case in Eq. (28) is called upon. Due to Eq. (65), a negative sign in front of \( c \) has no effect on the dynamics of the higher–order case, because the quasi-quadrantic term remains always subdominant. In contrast, it may have an important effect in the quasi-quadratic case.

The minus in front of \( c \) may exist in inflation and turn into a plus after inflation ends. This has no other effect than setting the value of \( \sigma_{\text{end}} \) to

\[
\sigma_{\text{end}} = \hat{\sigma}_{\text{end}} + \sigma_{\text{end}}^\text{par} \approx \sigma_{\text{end}}^\text{min}, \quad (71)
\]

where \( \sigma_{\text{end}}^\text{min} \equiv \sigma_{\text{end}}(H_{\text{end}}) \). The above is so because, in the quasi-quadrantic case, \( \sigma_{\text{end}} < \sigma_{\text{min}} \). Another possibility is that the minus existed there both before and after the end of inflation, or that it appeared after the end of inflation. Again Eq. (71) applies [69].

However, now that the quasi-quadrantic term, after the end of inflation, has a negative sign, the evolution of the field may be affected. Let us study this issue.

In this case, Eq. (46) becomes

\[
V(\sigma) \simeq \frac{1}{2} cH^2(t)(\sigma - \sigma_{\text{min}})^2. \quad (72)
\]

Substituting this into Eq. (6), we can recast it as

\[
\ddot{\sigma} + 3H(t)\dot{\sigma} + cH^2(t)\sigma = cH^2(t)\sigma_{\text{min}}(t), \quad (73)
\]

where we see that there is now a source term on the right-hand-side (RHS). The solution to the above equation, therefore, is of the form

\[
\sigma(t) = \sigma_{\text{nom}}(t) + \sigma_{\text{par}}(t), \quad (74)
\]

where the homogeneous solution \( \sigma_{\text{nom}}(t) \) is given in Eq. (49) or Eq. (52) depending on the value of \( c \). It is easy to see that the particular solution \( \sigma_{\text{par}}(t) \) is of the form

\[
\sigma_{\text{par}}(t) = A \sigma_{\text{min}}(t), \quad (75)
\]

where, from Eq. (29), \( \sigma_{\text{min}}(t) \propto t^{-\frac{w}{w+1}} \) and \( A \) is the constant

\[
A = \left[ 1 + \frac{9}{c} \left( \frac{1 - w}{n + 2} \right) \left( \frac{1 + w}{n + 2} - 1 \right) \right]^{-1}. \quad (76)
\]

According to Eq. (71), the initial conditions, in this case, suggest that \( \sigma_{\text{end}} \approx \sigma_{\text{min}}^\end \). Thus, \( \sigma_{\text{end}} \) in Eq. (49) or Eq. (52) for the \( \sigma_{\text{hom}}(t) \) is \( \sigma_{\end} \approx (1 - A)\sigma_{\text{min}}^\end \), because initially \( \sigma(t_0) = \sigma_{\text{end}} \). Therefore, for \( t \gg t_{\text{end}} \), the solution is

\[
\sigma(t) \sim (1 - A)\sigma_{\text{min}}^\end \left( \frac{t}{t_{\text{end}}} \right)^{-\frac{1}{2}} \left( \frac{1 - w}{1 + w} \right)^{\frac{1}{2}} \left( 1 - \sqrt{1 - \frac{c}{c_s}} \right) + A \sigma_{\text{min}}^\end \left( \frac{t}{t_{\text{end}}} \right)^{\frac{1}{2}} \left( \frac{1 - w}{1 + w} \right)^{\frac{1}{2}} \left( 1 - \sqrt{1 - \frac{\dot{c}}{c_s}} \right), \quad (77)
\]

where \( \dot{c} \equiv \min\{c, c_s\} \) and, in the oscillating case (i.e. when \( c > c_s \)), \( \sigma(t) \) corresponds to the amplitude of the oscillations.

It is interesting to note that the attractor constraint in Eq. (64) results in

\[
n < n_c \Rightarrow -\frac{2}{n + 2} < -\frac{1}{2} \left( \frac{1 - w}{1 + w} \right)^{\frac{1}{2}} \left( 1 - \sqrt{1 - \frac{\dot{c}}{c_s}} \right). \quad (78)
\]

Therefore, we see that, if the attractor constraint is satisfied, the particular solution decreases faster than the homogeneous and, hence, it soon becomes negligible. If this is the case then there is no modification to the dynamics described in Sec. IV A other than the initial condition, which is the one given in Eq. (71).

However, since the field is not in the higher–order part of its potential, there is no compelling reason for the attractor constraint to be satisfied. Indeed, for sufficiently large \( n \), the particular solution decreases less fast than the homogeneous. When this is so, the domination of the particular solution results in \( \sigma(t) \propto \sigma_{\text{min}} \). In other words, for sufficiently large \( n \), \( \sigma_{\text{par}} \) takes over and keeps the field at a constant ratio \( A > 1 \) to \( \sigma_{\text{min}} \). The problem is that, in contrast to the homogeneous solution, the particular solution does not carry the memory of the initial conditions and, therefore, does not preserve the perturbation spectrum of the field. This means
that the domination of the particular solution should be avoided, which imposes the constraint

\[ n + 2 < (n_c + 2) \left( 1 - \sqrt{1 - \min \left\{ 1, \frac{c}{c_x} \right\}} \right)^{-1}, \]  

(79)

The above constraint is more relaxed than the attractor constraint of Eq. (64) and becomes even more so the smaller \( c \) is.

One final point to note before we conclude this section is that, from Eqs. (56) and (71), in the quasi-quadratic case with negative sign, after the end of inflation the curvature perturbation of the field is

\[ \zeta \sim \frac{H}{\sigma_{\text{min}}^{\text{end}}} \ll c_\ast \ll 1. \]  

(80)

We elaborate more on this point in Sec. V D 3.

V. CURVATON DOMINATION

We now turn our attention to the evolution of the curvaton’s density ratio over the background density \( \rho_\sigma / \rho \), which will determine when the curvaton will dominate the Universe, or, if it decays (or thermalizes) earlier, what will determine when the curvaton will dominate

We implement them in a minimalistic scenario, where there is one, long period of inflation and one curvaton field. In this minimalistic scenario, the inflationary period is followed by a matter era, dominated by the oscillating inflaton, and a radiation era, commencing after reheating. We still ignore temperature corrections by assuming a negligible \( g \) postponing their study to Sec. VI.

As mentioned in the previous section, the curvaton has a chance to dominate only if it decays (or thermalizes) after the soft mass \( m \) dominates the effective mass \( m_{\text{eff}} \equiv \sqrt{\sigma m} \). Thus, we require

\[ H_m > \Gamma, \]  

(81)

where \( H_m \) is the Hubble parameter when \( m \sim m_{\text{eff}} \), which is different in the quasi-quadratic case and the higher–order case. Below we treat both cases individually again, as in the previous section.

A. The quasi-quadratic case

From Eqs. (50) and (53) and in view also of Eq. (14), we find

\[ \frac{\rho_\sigma}{\rho} \propto H^K, \]  

(82)

where

\[ K \equiv \left( \frac{1 - w}{1 + w} \right) \left( 1 - \sqrt{1 - \min \left\{ 1, \frac{c}{c_x} \right\}} \right). \]  

(83)

From the above and Eq. (48), it is evident that the exponent \( K \) depends on whether we are in the matter or the radiation era. Indeed, for the matter and radiation eras respectively, we have

\[ K_{\text{MD}} \equiv 1 - \sqrt{1 - \min \{1, 16c/9\}}, \]  

(84)

and

\[ K_{\text{RD}} \equiv \frac{1}{2} \left( 1 - \sqrt{1 - \min \{1, 4c\}} \right), \]  

(85)

which results in

\[ \frac{\rho_\sigma}{\rho} \bigg|_{\text{MD}} \sim \left( \frac{H}{H_{\text{end}}} \right)^{K_{\text{MD}}} \frac{\rho_\sigma}{\rho} \bigg|_{\text{end}}, \]  

(86)

and

\[ \frac{\rho_\sigma}{\rho} \bigg|_{\text{RD}} \sim \left( \frac{H}{H_{\text{inf}}} \right)^{K_{\text{RD}}} \frac{\rho_\sigma}{\rho} \bigg|_{\text{reh}}. \]  

(87)

Note that, for \( c \ll 1 \), we have

\[ K_{\text{MD}} \xrightarrow{c \ll 1} \frac{8}{9} c \quad \text{and} \quad K_{\text{RD}} \xrightarrow{c \ll 1} c. \]  

(88)

Thus, for \( c \to 0 \), the ratio \( \rho_\sigma / \rho \) remains constant as expected according to the discussion after Eq. (54).

Using Eqs. (10), (39) and (46), we find

\[ \frac{\rho_\sigma}{\rho} \bigg|_{\text{end}} \sim c \left( \frac{\sigma_{\text{end}}}{m_p} \right)^2 \approx \frac{1}{c} \left( \frac{H_{\text{end}}}{m_p} \right)^2 \]  

(89)

and

\[ \frac{\rho_\sigma}{\rho} \bigg|_{\text{reh}} \sim \left( \frac{H_{\text{inf}}}{H_{\text{end}}} \right)^{K_{\text{MD}}} \frac{\rho_\sigma}{\rho} \bigg|_{\text{end}}. \]  

(90)

Now, in the quasi-quadratic case, it is easy to see that

\[ H_m \sim m/\sqrt{c}, \]  

(91)
Using this and Eqs. (86) and (87), we obtain

\[
\frac{\rho_\sigma}{\rho} \bigg|_{H > H_m} \sim c \left( \frac{\sigma_{\text{end}}}{m_p} \right)^2 \left( \frac{m/\sqrt{c}}{H_{\text{end}}} \right)^{K_{\text{MD}}} \times \\
\times \min \left\{ 1, \frac{m/\sqrt{c}}{\Gamma_{\text{inf}}} \right\}^{K_{\text{RD}} - K_{\text{MD}}}.
\]  

(92)

When \( H(t) \) falls below \( H_m \), the quasi-quadratic term becomes dominated by the quadratic term corresponding to the soft mass \( m \) in Eq. (28). If \( c < 1 \) then there is a regime where something curious happens. Indeed, for \( m < H(t) < m/\sqrt{c} \), the scalar potential is dominated by the quadratic term, for which \( V'' = m^2 \ll H^2 \), i.e. the field becomes overdamped and, consequently, freezes [70]. This does not happen in the case where the field was oscillating before \( H_m \) was reached, because in this case \( c > c_\ast \sim 1 \). However, it happens when the field is following the scaling solution in Eq. (52), according to which, until \( H_m \) was reached, the field was rolling steadily towards the origin. After \( H_m \), this roll is halted. The important point to be stressed here is that, roughly when \( H_m \) is reached, the field stops regardless of its value. That is the whole configuration of \( \sigma \) freezes at the same moment and, therefore, the perturbation spectrum is unaffected by the freezing. The reason is evident: the possible freezing has to do with the comparison of \( m \) and \( m_{\text{eff}}(t) \) with each-other and with \( H(t) \). None of these quantities, in the quasi-quadratic case, is related to the actual value of \( \sigma \) and, therefore, the freezing of the field configuration is \( \sigma \)-independent.

If the field configuration does freeze then, in the interval \( m < H < H_m \), the scaling of the curvaton’s density ratio is only due to the decrease of \( \rho \). Thus, from Eq. (13), in this interval, we have

\[
\frac{\rho_\sigma}{\rho} \propto a^{3(1+w)},
\]  

(93)

which is, for the first time, increasing. Is it possible for the curvaton to dominate (or nearly dominate) during this interval? Well, since \( \rho_\sigma/\rho \propto \rho^{-1} \times H^{-2} \), we find

\[
\frac{\rho_\sigma}{\rho} \bigg|_{H \sim m} \sim \frac{1}{c} \frac{\rho_\sigma}{\rho} \bigg|_{H \sim H_m}.
\]  

(94)

Thus, after the whole interval of freezing has passed the amplification of the density ratio is only a factor of \( 1/c \). Hence, at the end of the freezing interval, the curvaton’s density ratio is

\[
\rho_\sigma \bigg|_{m} \sim \left( \frac{\sigma_{\text{end}}}{m_p} \right)^2 \left( \frac{m/\sqrt{c}}{H_{\text{end}}} \right)^{K_{\text{MD}}} \min \left\{ 1, \frac{m/\sqrt{c}}{\Gamma_{\text{inf}}} \right\}^{K_{\text{RD}} - K_{\text{MD}}},
\]  

(95)

where we have set \( (\rho_\sigma/\rho)_m \equiv (\rho_\sigma/\rho)_{H \sim m} \). Notice now that, in the RHS of the above, all the factors are smaller than (or at most equal to) unity, while all the exponents are positive. This means that the density ratio of Eq. (95), even though somewhat increased compared to Eq. (92), is still expected to be much smaller than unity. It is hardly possible, therefore, that this ratio is able to satisfy the constraint of Eq. (25), let alone lead to domination. Therefore, curvaton domination will have to wait for the quadratic oscillations to begin. The initial density ratio for the curvaton at the onset of these final quadratic oscillations is given by Eq. (95).

An important comment to be made here, regarding Eq. (95), is that the curvaton’s density ratio (with respect to the background density at a given time) strongly depends on the value of \( c \). Indeed, when \( c \ll 1 \), Eq. (88) suggests that the two last factors on the RHS of Eq. (95) are negligible. On the other hand, if \( c \sim 1 \) then these factors are not negligible but, instead, they may strongly suppress the density ratio. Indeed, it is evident that the curvaton’s density ratio, in this case, is suppressed at least by a factor of \( (m/H_{\text{end}}) \), which can be very small. The reason is that, for \( c \ll 1 \), the effective mass is small and the field is overdamped. On the other hand, if \( c \) is comparable to unity both the quasi-quadratic oscillations and the scaling solution substantially diminish the density of the curvaton. In fact, it is possible that this suppression is so efficient that the final quadratic regime is unable to counteract it. Hence, we have shown that the quasi-quadratic evolution disfavours curvatons which admit substantial supergravity corrections to their effective potential. This strengthens the case of the curvaton being a PNG as discussed in Ref. [41]. The above are more clearly demonstrated in the example analysed in Sec. VII.

**B. The higher-order case**

As discussed in Sec. IVB, in this case, the field unfreezes and begins oscillating when \( m_{\text{eff}} \sim H \). Suppose, at first, that, between the onset of the field’s oscillation and the time when the soft mass \( m \) dominates \( m_{\text{eff}} \), the dominant fluid component of the Universe remains the same so that \( w \) is unchanged. Then, using Eq. (66), it is easy to find that

\[
H_m \sim \left( \frac{m}{H_{\text{osc}}} \right)^{\left( \frac{n+6}{n+2} - \frac{n+2}{n+6} \right)},
\]  

(96)

where we used that \( m_{\text{eff}}(H_{\text{osc}}) \sim H_{\text{osc}} \). Due to the attractor constraint in Eq. (64) we see that the exponent in the above is always positive. This means that

\[
H_{\text{osc}} \geq m \implies H_m \leq m.
\]  

(97)

The above shows that, the oscillation of the field will continue uninterrupted when the quadratic term takes over. This is because, for a pure quadratic potential, the field oscillations begin when \( H(t) \sim \sqrt{V''} = m \). Also, since \( H_{\text{osc}} \geq H_m \) (for the higher–order oscillations to take place), the condition \( H_m \leq m \) is guaranteed if \( H_{\text{osc}} \leq m \). Therefore, in all cases \( H_m \leq m \) and there is no intermediate freezing of the field once it unfreezes after the end of inflation.
Now, employing Eqs. (63), (67) and (96), we obtain
\[
\rho_\sigma / \rho_m \sim \left( \frac{m}{H_{\text{osc}}} \right)^{\frac{(1-w)(n+6)-4}{n+2}} \left( \frac{\sigma_{\text{end}}}{\sigma_P} \right)^2 ,
\]
where we also used
\[
\rho_\sigma / \rho_{\text{osc}} \sim \left( \frac{\sigma_{\text{end}}}{\sigma_P} \right)^2 ,
\]
which is evident in view of Eq. (10) and considering that \( \rho_\sigma \sim m_{\text{eff}}^2 \sigma^2 \) with \( m_{\text{eff}}(H_{\text{osc}}) \sim H_{\text{osc}} \) and \( \rho_{\text{osc}} = \rho_{\text{end}}^\text{\text{MD}} \).

From the above, it is straightforward to find the curvaton’s density ratio at the onset of the quadratic oscillations in the case where this occurs before reheating (i.e., \( H_m > \Gamma_{\text{inf}} \)). Indeed, from Eq. (98), one gets
\[
\rho_\sigma / \rho_m \bigg|_{\text{MD}} \sim \left( \frac{m}{H_{\text{osc}}} \right) \left( \frac{\sigma_{\text{end}}}{\sigma_P} \right)^2 ,
\]
(corresponding to (c.f. Eq. (96))
\[
H_m \sim \left( \frac{m}{H_{\text{osc}}} \right)^{\frac{1}{2} \frac{3(n+2)}{n+6}} m .
\]
Note that, for unfreezing before reheating, the attractor constraint demands \( n < 2 \). Similarly, we obtain the curvaton’s density ratio at the onset of the quadratic oscillations in the case when unfreezing occurs after reheating (i.e., \( H_{\text{osc}} < \Gamma_{\text{inf}} \)). This time, Eq. (98) gives
\[
\rho_\sigma / \rho_m \bigg|_{\text{RD}} \sim \left( \frac{m}{H_{\text{osc}}} \right)^{\frac{2n}{3(n+2)}} \left( \frac{\sigma_{\text{end}}}{\sigma_P} \right)^2 ,
\]
(corresponding to (c.f. Eq. (96))
\[
H_m \sim \left( \frac{m}{H_{\text{osc}}} \right)^{\frac{6-n}{3(n+2)}} m .
\]
Note that, for unfreezing after reheating, the attractor constraint is relaxed to \( n < 6 \).

What happens, however, if \( H_m < \Gamma_{\text{inf}} < H_{\text{osc}} \)? In this case, during the oscillations and before the quadratic term dominates, reheating is completed and the Universe switches from being matter dominated to being radiation dominated. We may obtain \( H_m \) and \( (\rho_\sigma / \rho)_m \) working in the same manner as above.

Employing Eq. (66) both for the matter and radiation eras, we get
\[
m \sim \left[ \frac{H_m}{\Gamma_{\text{inf}}} \right]^{3/2} \left( \frac{\Gamma_{\text{inf}}}{H_{\text{osc}}} \right)^{\frac{4+n+2}{3(n+2)}} H_{\text{osc}} .
\]
Solving for \( H_m \) we obtain
\[
H_m \sim \left( \frac{m}{H_{\text{osc}}} \right)^{\frac{4}{3} \frac{n+2}{3(n+2)}} \left( \frac{m}{\Gamma_{\text{inf}}} \right)^{1/3} .
\]
Since unfreezing occurs before reheating, the attractor constraint demands \( n < 2 \), which suggests that \( H_m \leq m \) if \( H_{\text{osc}} \geq m \) and \( m \leq \Gamma_{\text{inf}} \). Since \( H_{\text{osc}} \geq H_m \), the condition \( H_m \leq m \) is automatically satisfied if \( H_{\text{osc}} \leq m \). Also, if \( m \geq \Gamma_{\text{inf}} \) then again \( H_m \leq m \) is satisfied because \( H_m < \Gamma_{\text{inf}} \) in the first place. Therefore \( H_m \leq m \) in all cases, which means that the oscillations continue uninterrupted after the quadratic term dominates.

Now, using Eq. (67), we get
\[
\rho_\sigma / \rho_m \bigg|_{\text{MD}} \sim \left( \frac{m}{H_{\text{osc}}} \right)^{\frac{2n}{3(n+2)}} \left( \frac{\Gamma_{\text{inf}}}{H_{\text{osc}}} \right)^{2/3} \left( \frac{\sigma_{\text{end}}}{\sigma_P} \right)^2 ,
\]
Putting these together and also using Eqs. (99) and (105), we find
\[
\rho_\sigma / \rho_m \bigg|_{\text{MD}\rightarrow\text{RD}} \sim \left( \frac{m}{H_{\text{osc}}} \right)^{\frac{2n}{3(n+2)}} \left( \frac{\Gamma_{\text{inf}}}{H_{\text{osc}}} \right)^{2/3} \left( \frac{\sigma_{\text{end}}}{\sigma_P} \right)^2 .
\]

Summing up, in the above we have obtained the value of \( H_m \) [Eqs. (101), (103) and (105)] and of the ratio \( (\rho_\sigma / \rho)_m \) [Eqs. (100), (102) and (108)] in the cases \( \Gamma_{\text{inf}} \leq H_m \leq H_{\text{osc}}, \ H_m \geq H_{\text{osc}} \geq \Gamma_{\text{inf}} \) and \( H_m \leq \Gamma_{\text{inf}} \leq H_{\text{osc}} \) respectively. We have found that the attractor constraint ensures that \( H_m \leq m \) in all cases and, therefore, guarantees that the oscillations of the field continue uninterrupted after the quadratic term due to the soft mass m dominates the effective mass.

Before moving on to the final quadratic oscillations, we provide an estimate for \( H_{\text{osc}} \). Indeed, since \( H_{\text{osc}}^2 \sim m_{\text{end}}^2 (\sigma_{\text{end}}) \sim \sigma_{\text{end}} / M^2 \), Eqs. (37) and (39) suggest that
\[
H_{\text{osc}} \gtrsim \sqrt{c_0} \sigma_{\text{end}} .
\]

C. The final quadratic oscillations

When \( H(t) < H_m \), the scalar potential becomes dominated by the quadratic term due to the soft mass
\[
V(\sigma) \sim \frac{1}{2} m^2 \sigma^2 .
\]
According to Eq. (22), a scalar field coherently oscillating in a quadratic potential corresponds to a collection of massive particles at rest (curvatons in our case) and behaves like pressureless matter, i.e. \( \rho_\sigma \propto \alpha^{-3} \). Therefore, from Eq. (13), one finds
\[
\rho_\sigma / \rho \propto \alpha^3 \propto H^{-3/2} .
\]
The above shows that, in the quadratic regime the curvaton’s density ratio is constant in the matter era and begins growing only after reheating. Thus, the curvaton
will have a chance to dominate (or nearly dominate) the Universe only if
\[ \Gamma < \Gamma_{\text{inf}}. \]  
(112)

Using Eq. (111), one obtains
\[ \frac{\rho_\sigma}{\rho} = \left( \frac{H_{\bar{m}}}{H} \right)^{1/2} \frac{\rho_\sigma}{\rho} \bigg|_m, \]
(113)
where
\[ H_{\bar{m}} \equiv \min\{m, H_m, \Gamma_{\text{inf}}\}. \]
(114)

The above suggests that, if the curvaton decays (or thermalizes) before domination, then, from Eq. (18), we have
\[ r \sim \sqrt{H_{\bar{m}}/\Gamma} \times \frac{\rho_\sigma}{\rho} \bigg|_m, \]
(115)
If, on the other hand, the curvaton does manage to dominate, then Eq. (113) gives
\[ H_{\text{dom}} \sim H_{\bar{m}} \left( \frac{\rho_\sigma}{\rho} \bigg|_m \right)^2. \]
(116)

Hence, in the quasi-quadratic case, considering that \( H_{\bar{m}} = \min\{m, \Gamma_{\text{inf}}\} \) and in view of Eq. (95), the above gives
\[ H_{\text{dom}} \sim \min\{m, \Gamma_{\text{inf}}\} \left( \frac{\sigma_{\text{end}}}{m_P} \right)^4 \left( \frac{m/\sqrt{c}}{H_{\text{end}}} \right)^{2K_{\text{MD}}} \times \]
\[ \times \min\left\{1, \frac{m/\sqrt{c}}{\Gamma_{\text{inf}}} \right\}^{2(K_{\text{RD}}-K_{\text{MD}})}. \]
(117)

In the higher-order case there are three possibilities, as follows. The first possibility is that all the higher-order oscillations occur before reheating. Then, \( H_{\bar{m}} = \Gamma_{\text{inf}} \) and using Eqs. (100) and (116) we obtain
\[ H_{\text{dom}} \sim \left( \frac{m}{H_{\text{osc}}} \right)^2 \left( \frac{\sigma_{\text{end}}}{m_P} \right)^4 \Gamma_{\text{inf}}. \]
(118)

The next possibility is that all the higher-order oscillations occur after reheating, in which case \( H_{\bar{m}} = H_m \). Then, substituting Eqs. (102) and (103) into Eq. (116), we find
\[ H_{\text{dom}} \sim \left( \frac{m}{H_{\text{osc}}} \right)^2 \left( \frac{\sigma_{\text{end}}}{m_P} \right)^4 m. \]
(119)
Finally, the last possibility is that the higher-order oscillations begin before reheating but they continue afterwards into the radiation era. Again we have \( H_{\bar{m}} = H_m \). Substituting Eqs. (108) and (105) into Eq. (116), we get
\[ H_{\text{dom}} \sim \left( \frac{m}{H_{\text{osc}}} \right)^2 \left( \frac{\sigma_{\text{end}}}{m_P} \right)^4 \Gamma_{\text{inf}}. \]
(120)

which is identical to the first possibility (c.f. Eq. (118)). Thus, in general, for the higher-order oscillations, we have found that
\[ H_{\text{dom}} \sim \left( \frac{m}{H_{\text{osc}}} \right)^2 \left( \frac{\sigma_{\text{end}}}{m_P} \right)^4 \min\{m, \Gamma_{\text{osc}}, \Gamma_{\text{inf}}\}. \]
(121)

From the above, we see that, remarkably, in all cases \( H_{\text{dom}} \) is independent of \( n \).

What if the quadratic term was dominant right from the start? This is indeed possible if
\[ c_0 < c < \left( \frac{m}{H_{\text{end}}} \right)^2 \ll 1. \]
(122)
Then, \( (\rho_\sigma/\rho)_m = (\rho_\sigma/\rho)_{\text{osc}} \), which is given by Eq. (99). Thus, in this case, we find
\[ H_{\text{dom}} \sim \left( \frac{\sigma_{\text{end}}}{m} \right)^4 \min\{m, \Gamma_{\text{inf}}\}. \]
(123)

Note that the above can be obtained by setting \( c \rightarrow (m/H_{\text{end}})^2 \) in Eq. (117) and \( H_{\text{osc}} \rightarrow m \) in Eq. (121).

**D. The value of \( q \)**

In this section, we take a closer look to the value of \( q \), which relates the fractional perturbation of the curvaton at horizon crossing and at curvaton decay as defined in Eq. (4). We have already seen that, in all cases (c.f. Eqs. (56) and (69)),
\[ \frac{\delta \sigma}{\sigma} \bigg|_{\text{end}} \sim \frac{\delta \sigma}{\sigma} \bigg|_{\text{dec}} \sim \zeta_\sigma. \]
(124)
Below, however, we will study this relation in more detail. For simplicity, we drop the bar from the oscillation amplitude of the field.

During inflation and after the fast-roll regime, the curvaton is overdamped. As shown by Eq. (43), this means that the value of the curvaton perturbation is frozen. It is trivial to show, using Eq. (35), that a very similar relation is true for the value of \( \sigma_\sigma \) itself. This is especially true if the curvaton has reached the quantum regime. Thus, if \( c \) does not change significantly at the end of inflation, it is safe to consider that
\[ \frac{\delta \sigma}{\sigma} \bigg|_{\star} \approx \frac{\delta \sigma}{\sigma} \bigg|_{\text{end}}. \]
(125)
Also, since we have shown that, before decaying, the curvaton needs to be in the final quadratic oscillation regime, Eq. (21) suggests that
\[ \zeta_\sigma = \frac{2}{3} \frac{\delta \sigma}{\sigma} \bigg|_{\text{dec}} = \frac{2}{3} \frac{\delta \sigma}{\sigma} \bigg|_{\star}. \]
(126)
1. The quasi-quadratic case

In the quasi-quadratic case, Eqs. (49) and (52) show clearly that

$$\sigma(t) = \sigma_0 \times f(t/t_0),$$

(127)

where \( f \) is some function with \( f(1) = 1 \), which is independent of \( \sigma_0 \). It is clear from the above that, if we consider a perturbed value of the field, we will have

$$\frac{\delta \sigma}{\sigma}(t) = \frac{\delta \sigma_0}{\sigma_0} = \text{const}.$$  

(128)

This is hardly surprising since, for potentials of the form \( V \propto \sigma^2 \), the equations of motion for the field and its perturbation (Eqs. (6) and (7) respectively) are identical. Now it is crucial to note that the onset of the final quadratic oscillations occurs at the same time regardless of the value of the field. This is due to Eq. (91), which shows that \( H_m \) is \( \sigma \)-independent. Thus, choosing \( t_0 = t_{\text{end}} \), Eq. (128) gives

$$\frac{\delta \sigma}{\sigma}(H_m) = \frac{\delta \sigma}{\sigma}(H_{\text{end}}).$$

(129)

Eq. (128) is obviously valid during the final quadratic oscillations as well, carrying the perturbations intact until the time of curvaton decay, which again is \( \sigma \)-independent. Thus,

$$\frac{\delta \sigma}{\sigma}|_{\text{dec}} = \frac{\delta \sigma}{\sigma}(H_m).$$

(130)

As explained in Sec. V A, if \( c \ll 1 \), it is possible that the field briefly freezes again before the onset of the final quadratic oscillations, but this is not expected to affect the amplitude of the perturbation spectrum. All in all, in view also of Eq. (125), we have found that, in the quasi-quadratic case

$$q = 1,$$

(131)

which means that

$$\zeta_{\sigma} = \frac{2}{3} \frac{\delta \sigma}{\sigma}|_{*}.$$  

(132)

2. The higher-order case

The case of higher-order term domination is a bit more complicated. Due to Eq. (65), one easily sees that, in this case too, Eqs. (127) and (128) are valid. However, because \( m_{\text{eff}} \sim \sigma^{n+2}/M^n \) and in contrast to the quasi-quadratic case, the onset of both the higher-order and the final quadratic oscillations does depend on the value of \( \sigma \). Therefore, perturbing this value somewhat affects also the scaling of the field. Below we calculate what this means for \( q \).

Suppose we consider two causally disconnected regions of the Universe, characterized by two different values for the curvaton field: \( \sigma^{(1)} \) and \( \sigma^{(2)} \). Suppose also that, in both these regions, the field lies in the higher-order regime. Then, in these two regions, the onset of the higher-order oscillations and the final quadratic oscillations will occur at different times. Without loss of generality, we assume that it is in region–(1) that the field begins its (Hubble damped) sinusoidal oscillations first. Using Eq. (65), we obtain the following for the amplitudes of the field at this moment:

$$\sigma^{(1)}_{m:1} = \sigma^{(1)}_{\text{end}} \left( a^{(1)}_{\text{osc}}/a^{(1)}_{m} \right)^{q/3},$$

$$\sigma^{(2)}_{m:1} = \sigma^{(2)}_{\text{end}} \left( a^{(2)}_{\text{osc}}/a^{(1)}_{m} \right)^{q/3}.$$  

(133)

Considering the fact that \( a \propto H^{-2/3(1+w)} \) and also that \( H^2_{\text{osc}} \sim m^2_{\text{eff}}(\sigma_{\text{end}}) \propto \sigma^{n+2}_{\text{end}} \), the above gives

$$\frac{\sigma^{(2)}_{m:1}}{\sigma^{(1)}_{m:1}} = \left( \frac{\sigma^{(2)}_{\text{end}}}{\sigma^{(1)}_{\text{end}}} \right)^{1 - (n+6)/(n+2)}. \quad (134)$$

Now, after the onset of the final quadratic oscillations for \( \sigma^{(1)} \), its density decreases as \( \rho_\sigma \propto \sigma^3 \propto a^{-3} \). On the other hand, \( \sigma^{(2)} \) still oscillates in the higher-order regime, until the final quadratic oscillations begin for \( \sigma^{(2)} \) too. Therefore, at this moment, we have

$$\sigma^{(1)}_{m:2} = \sigma^{(1)}_{\text{end}} \left( a^{(1)}_{m}/a^{(2)}_{m} \right)^{q/3},$$

$$\sigma^{(2)}_{m:2} = \sigma^{(2)}_{\text{end}} \left( a^{(2)}_{m}/a^{(2)}_{m} \right)^{q/3}.$$  

(135)

Let us assume now that, during the period of the higher-order oscillations, the equation of state of the Universe remains unmodified. Then we can use Eq. (96) to relate \( H_m \) with \( H_{\text{osc}} \). Using again that \( a \propto H^{-2/3(1+w)} \) and also that \( H^2_{\text{osc}} \sim m^2_{\text{eff}}(\sigma_{\text{end}}) \propto \sigma^{n+2}_{\text{end}} \), we find

$$\frac{\sigma^{(2)}_{m:2}}{\sigma^{(1)}_{m:1}} = \frac{\sigma^{(2)}_{\text{end}}}{\sigma^{(1)}_{\text{end}}} \left( \frac{\sigma^{(2)}_{\text{end}}}{\sigma^{(1)}_{\text{end}}} \right)^{1 - (n+6)/(n+2)}.$$  

(136)

Now, after \( \sigma^{(2)} \) begins its final quadratic oscillations both fields oscillate sinusoidally (with Hubble damping) and, therefore, Eq. (128) is valid. Since the decay of the curvaton occurs independently of the value of \( \sigma \), we see that \( (\sigma^{(2)}/\sigma^{(1)})_{\text{dec}} = (\sigma^{(2)}/\sigma^{(1)})_{m:2} \). Using this and combining Eqs. (134) and (136), we find that

$$\frac{\sigma^{(2)}_{m:2}}{\sigma^{(1)}_{m:1}} = \left( \frac{\sigma^{(2)}_{\text{end}}}{\sigma^{(1)}_{\text{end}}} \right)^{1 - (n+6)/(n+2)}.$$  

(137)
Therefore, if we take that $\sigma = \sigma^{(1)}$ and $\sigma + \delta \sigma = \sigma^{(2)}$ and also $\delta \sigma \ll \sigma$, we obtain

$$\frac{\delta \sigma}{\sigma} \bigg|_{\text{dec}} \approx \frac{n + 6}{4} \left[ 1 - \left( \frac{n_c + 6}{n_c + 2} \right) \frac{n + 6}{n + 2} \right] \frac{\delta \sigma}{\sigma} \bigg|_{\text{end}} + \cdots ,$$

where the ellipsis denotes terms of higher power in $(\delta \sigma/\sigma)_{\text{end}}$. Hence, in view also of Eq. (125), we have found that

$$q = \frac{1}{4}(n + 6) \left[ 1 - \left( \frac{n_c + 6}{n_c + 2} \right) \frac{n + 6}{n + 2} \right] ,$$

which, for matter and radiation eras respectively, becomes

$$q_{\text{MD}} = \frac{2 - n}{4} \quad \text{and} \quad q_{\text{RD}} = \frac{6 - n}{8} .$$

From Eq. (139), it is evident that, for $n = n_c$, we have $q = 0$ and all the perturbation spectrum is erased. From Eq. (137), it is clear that this is so to all orders in $(\delta \sigma/\sigma)_{\text{end}}$. Hence, we see that, the fact that the onset of the higher-order and the final quadratic oscillations is $\sigma$-dependent suppresses strongly the perturbation spectrum if $n = n_c$. This is why the border case $n = n_c$ for the attractor constraint in Eq. (64) is not acceptable, even though the field oscillates and does not converge towards the attractor solution. For $n < n_c$, however, we see that $q \approx 1$ and so Eq. (124) is justified.

In the case where the higher-order oscillations begin during matter domination but end after reheating, working as above and using Eq. (105) instead of Eq. (96), one can show that

$$q_{\text{MD}} = \frac{(n + 5)(2 - n)}{3(n + 6)} .$$

Since, when unfreezing occurs during matter domination, only the case of the quartic term is allowed by the attractor constraint in Eq. (64), we see that, in this case, $q = 1/2 \{ q = 5/9 \}$ if the curvaton decays before {after} reheating.

### 3. Effective $q$ for significant change in $c$

Before concluding this section, we should point out the possibility that $c$ does change significantly at the end of inflation. This will not have an effect if the field remains in the higher-order regime. However, if $c \sim 1$ after the end of inflation, this will probably relocate the field into the quasi-quadrical part of the potential. In this case, as discussed also in Sec. IV C, $(\delta \sigma/\sigma)_{\text{end}}$ may be substantially reduced. We will demonstrate this assuming that $\sigma_*$ has reached the quantum regime.

In view of Eq. (71), one finds that Eq. (125) becomes

$$\frac{\delta \sigma}{\sigma} \bigg|_{\text{end}} \sim \left( \frac{\sigma_{\text{end}}(c_*)}{\sigma_{\text{min}}(c)} \right) \frac{\delta \sigma}{\sigma} \bigg|_{\ast} ,$$

where $c \sim 1$ corresponds to the value after the end of inflation. Thus, since $\sigma_{\text{end}}(c_*) \sim \sigma_0 \ll \sigma_{\text{min}}(c) < \sigma_{\text{end}}(c)$, we see that the effective value of $q$ may be substantially reduced if $c$ changes significantly at the end of inflation in agreement with Eq. (80). Considering that $\sigma_{\text{end}}(c_*) \approx \sigma_*$ and also that $\delta \sigma_* \sim H_*$, the above becomes

$$\frac{\delta \sigma}{\sigma} \bigg|_{\text{end}} \sim \left( \frac{c_0}{c} \right) \frac{1}{\sqrt{\sigma_*}} \sim \left( \frac{c_0}{c} \right) \frac{1}{\sigma_*} .$$

From Eq. (44), we have $(\delta \sigma/\sigma)_* \approx \max \{ c_*, c_0 \}$, which, in view of the above, gives

$$q_{\text{eff}} \sim \left( \frac{c_0}{c} \right) \frac{1}{\sqrt{\sigma_*}} \min \left\{ 1, \frac{c_0}{c_*} \right\} .$$

### VI. THERMAL CORRECTIONS

In this section, we introduce also the temperature correction to the potential, as it appears in Eq. (28). We will investigate the effect of this correction on the dynamics of the inflating and oscillating curvaton. The mass of the curvaton is taken now to include all the contributions

$$V'' \sim m^2 + cH^2(t) + m^2_{\text{eff}} + g^2T^2 ,$$

where, in this section, with $m_{\text{eff}}$ we refer to the contribution due to the higher-order terms: $m_{\text{eff}}^2 \sim \sigma^{n+2}/M^2$. During the matter era of the inflaton’s oscillations, there exists a subdominant thermal bath due to the inflaton’s decay products. One would naively expect that any thermal bath during a matter era is to be rapidly diluted away since radiation density redshifts faster than matter, according to Eq. (13). However, due to the continuous decay of the inflaton field, the thermal bath receives an ever-growing contribution to its density, so that the continuity equation, Eq. (12), does not apply and, therefore, neither does Eq. (13). The temperature of this thermal bath is [71]

$$T \sim \left( m_p^2 \Gamma_{\text{inf}} H \right)^{1/4} .$$

According to the above, the radiation density during the matter era scales as $\rho_r \propto T^4 \propto H$. Since this density is negligible compared to the one of the oscillating inflaton field, the loss of energy from the latter is also negligible so that, for the inflaton, the continuity equation does apply. Now, from Eq. (10), we have $\rho \propto H^2$, which means that there is a moment when the radiation density becomes comparable to the one of the inflaton. This is the moment of reheating and it occurs when $H_{\text{reh}} \sim \Gamma_{\text{inf}}$. In view of Eq. (146), the reheat temperature is

$$T_{\text{reh}} \sim \sqrt{m_p^2 \Gamma_{\text{inf}}} .$$

After reheating, the Universe becomes radiation dominated with $\rho \sim T^4$. In view of Eq. (10), therefore, we can write for the temperature of the Universe

$$T \sim \left( m_p^2 H \min \{ H, \Gamma_{\text{inf}} \} \right)^{1/4} \propto H^{1+2/3} .$$
Due to its interaction with the thermal bath, the scalar condensate is in danger of evaporating, that is of becoming thermalized, if $\Gamma_T > \Gamma_m$, i.e. if the decay rate of the field is smaller than the thermalization rate. If thermalization does occur, even if, strictly speaking, the curvaton have not decayed yet into other particles, they constitute a component of the thermal bath and their equation of state is that of radiation. Thus, after thermalization the density ratio of the curvaton to the radiation background remains constant. As a result, $r$ is decided not at the time of the curvaton decay but at the time of thermalization and this is why, so far, we have used $\Gamma$ as given by Eq. (19) to determine the amplitude of the total curvature perturbation.

Let us estimate $\Gamma_T$. The scattering cross-section for scalar particles is $\sigma \sim g^2/E_{cm}^2$, where $E_{cm} \sim \sqrt{Tm_\sigma}$ is the centre-of-mass energy. Thus, for the scattering rate $\Gamma_{sc} \sim \sigma T^3$, we find $\Gamma_{sc} \sim g^2 T^2/m_\sigma$. When the particles thermalize, $m_\sigma \sim gT$ and so, for thermalization, we find

$$\Gamma_T \sim gT.$$  \hfill (149)

Suppose, now, that the temperature corrections were indeed dominating the effective mass of Eq. (145). Then $V'' \sim g^2 T^2 \sim \Gamma_T^2$. As usual, the field is able to oscillate only when $H(t)$ drops enough so that $V'' \sim H^2$. However, in this case, Eq. (149) suggests that, when the field is about to begin oscillating, $H \sim \Gamma_T$ and, therefore, the field thermalizes and the condensate evaporates. Hence, we see that the field thermalizes before engaging into oscillations, when the effective mass is dominated by the temperature corrections.

As discussed in the previous section, the curvaton has a chance to dominate (or nearly dominate) the Universe only if it decays or thermalizes after the onset of the final quadratic oscillations. Thus, in order to protect the curvaton from thermalizing too early, we have to demand that

$$gT_m < m,$$  \hfill (150)

where $T_m = T(H_m)$ is the temperature when $\sqrt{V''}$ becomes dominated by the soft mass $m$. Note that, for later times, $gT < m$ and the thermalization rate is suppressed.

The above condition, Eq. (150), suffices in order to avoid thermalization, only if, before $H \sim H_m$, the temperature corrections have been always subdominant. This is ensured provided both $cH^2$ and $m_{\text{eff}}^2$ decrease with time faster than $(gT)^2$. This is indeed the case as we show below.

Consider first the quasi-quadratic case. Then, in view of Eq. (148), we find

$$\frac{cH^2}{(gT)^2} \propto H^{2(1-w)},$$  \hfill (151)

which is indeed decreasing with the Universe expansion. Now, in the higher–order case, we have the following. Eqs. (66) and (148), in view also of Eq. (63), give

$$\frac{m_{\text{eff}}^2}{(gT)^2} \propto H^{(n+2)/n} \left(\frac{\Gamma}{\Gamma_{\text{inf}}}\right)^{\frac{n+3}{2}}.$$  \hfill (152)

The exponent in the rhs of the above is positive when $n > -2 \left(\frac{5-w}{7+5w}\right)$, which is always true for $n \geq 0$. Thus, the effective mass $m_{\text{eff}}$ also decreases faster than the $gT$ and, hence, the condition in Eq. (150) ensures that the temperature corrections to the effective mass are always subdominant. This is illustrated in Fig. 2.

The condition in Eq. (150) can be translated into an upper bound on the coupling $g$ of the curvaton to the thermal bath. Indeed, from Eqs. (148) and (150), we find

$$g^2 < \frac{m^2}{m_{\text{eff}} H_m} \max \left\{1, \frac{H_m}{\Gamma_{\text{inf}}}\right\}^{1/2}.$$  \hfill (153)

In the quasi–quadratic case, using Eq. (91), the above becomes

$$g^2 < \sqrt{c} \frac{m}{m_{\text{eff}} H_m} \max \left\{1, \frac{m/\sqrt{c}}{\Gamma_{\text{inf}}}\right\}^{1/2}.$$  \hfill (154)

Now, using Eqs. (101), (103) and (105), in the higher–order case, we find

$$g^2 < \frac{m}{m_{\text{eff}}} \left(\frac{H_m}{m_{\text{eff}}}\right)^{\frac{1-n}{2}} \left(\frac{m}{\Gamma_{\text{inf}}}\right)^{1/2}.$$  \hfill (155)
for $\Gamma_{\text{inf}} \leq H_m < H_{\text{osc}}$,}

$$g^2 < \frac{m}{m_p} \left( \frac{H_{\text{osc}}}{m} \right)^{\frac{2}{1 - \frac{1}{n}}} \left( \frac{m}{\Gamma_{\text{inf}}} \right)^{-1/3},$$

(156)

for $H_m < \Gamma_{\text{inf}} \leq H_{\text{osc}}$, and

$$g^2 < \frac{m}{m_p} \left( \frac{H_{\text{osc}}}{m} \right)^{\frac{4}{1 - \frac{1}{n}}}$$

(157)

for $H_m < H_{\text{osc}} < \Gamma_{\text{inf}}$.

As discussed above, the constraint in Eq. (150) guarantees that the temperature correction to the potential has negligible effect to the dynamics of the curvaton. In this case, $\Gamma_{\text{T}}$ becomes irrelevant and, therefore, $\Gamma = \Gamma_{\sigma}$. For strong coupling $g$, we expect $\Gamma_{\sigma} \sim g^2 m$. However, if $g$ is very small then the curvaton may have to decay gravitationally, in which case its decay will be suppressed by $m_p$. Thus, we can write

$$\Gamma_{\sigma} \sim m \text{ max } \left( g^2, \left( \frac{m}{m_p} \right)^2 \right).$$

(158)

VII. A CONCRETE EXAMPLE

In this section, we will demonstrate our findings by employing them in a specific model, under our minimalistic assumption of a single inflationary period and a single curvaton field. By assuming certain quantities, such as the inflationary energy scale, the decay rate of the inflaton, the mass scale $M$ and so on, we aim to obtain bounds on the curvaton model and its parameters, such as $m$ or $g$.

In particular, for our example, we choose the following values:

$$M = m_p, \quad \lambda_n \simeq 1$$

(159)

and also

$$V_{\text{inf}}^{1/4} \sim 10^{14}\text{GeV}, \quad H_s \sim 10^{11}\text{GeV},$$

$$m_{\text{inf}} \sim 10^{10}\text{GeV}, \quad \Gamma_{\text{inf}} \sim 10^{-6}\text{GeV}.$$  

(160)

In the above choice, we took into account the COBE upper bound on the inflationary scale $V_{\text{inf}}^{1/4} < 10^{18}\text{GeV}$ necessary to render the inflaton’s perturbations negligible [20]. Also, we considered that the mass of the inflaton $m_{\text{inf}}$ should be small enough for slow roll, $m_{\text{inf}}^2 \ll H_s^2$, but not too small since, for most inflation models, $\eta \sim (m_{\text{inf}}/H_s)^2 \simeq 0.01$. We chose $m_{\text{inf}}$ to be of the so-called intermediate scale $\sim \sqrt{m_pm_{\text{ew}}}$, where $m_{\text{ew}} \sim 1\text{ TeV}$ is the electroweak scale. Finally, we assumed that the inflaton is a gauge singlet and, therefore, decays only gravitationally so that $\Gamma_{\text{inf}} \sim m_{\text{inf}}^3/m_p^2$. The reheat temperature, for the above parameters, is $T_{\text{reh}} \sim 10^6\text{GeV}$ (c.f. Eq. (147)), which is well above $T_{\text{BBN}}$ but satisfies also comfortably the gravitino bound: $T_{\text{reh}} < 10^9\text{GeV}$ [72].

We assume that $H_{\text{end}} \approx H_\ast$ and also that inflation lasts long enough for the curvaton to reach the quantum regime. Then Eq. (39) suggests

$$\sigma_{\text{end}} \sim \sigma_Q \sim H_\ast/\max\{c_s, c_0\}.$$  

(161)

This means that, in view of Eqs. (4), (18), (44), (56) and (69), one finds

$$\zeta_0 \sim r q \max\{c_s, c_0\},$$  

(162)

where $\zeta_0 = 2 \times 10^{-5}$. Using this, one can readily exclude the quartic case $n = 0$ regardless of the choice of parameters in Eq. (160). Indeed, reinstating the $\lambda_n$ in Eq. (40), one finds

$$c_0 \sim \frac{1}{2}(\lambda_0/3)^{1/3} \sim 0.1.$$  

(163)

This means that, since $c_s \ll 1$, we have $c_s < c_0$. Thus, using Eq. (162) and employing also the WMAP constraint in Eq. (25), it is easy to find that we require

$$q \leq 10^{-2}.$$  

(164)

which is possible to realize only if $q = q_{\text{eff}}$ since, otherwise, $q \sim 1$. Thus, we need to have a change of sign for the quasi-quadratic term after the end of inflation. Now, using Eq. (144) in this case, we obtain

$$q_{\text{eff}} \sim \sqrt{c_0/c},$$  

(165)

which can never be as small as $10^{-2}$. Hence, the quartic case is excluded, unless $\lambda_0$ is extremely small or inflation ends well before the curvaton reaches the quantum regime. The remaining two possibilities are $n = 2, 4$ (we assume that symmetries in the potential allow only even powers). In the following, we choose $n = 4$, because it offers the largest parameter space. Using this, Eqs. (40) and (160) give

$$c_0 \sim 10^{-4}. $$  

(166)

We consider two cases. In the first case, we assume that the (approximate) symmetry which keeps $c$ small during inflation persists after inflation ends, so that $c = c_\ast \ll 1$. In the second case, we assume that $c \simeq 1$ after the end of inflation, considering both the $\pm c$ cases. In all cases, we assume that $c$ does not change significantly at the transition between the matter and radiation dominated epochs.

A. Case $c = c_\ast \ll 1$

In this case, there is no change of sign for the quasi-quadratic term so we will drop the hat on $\sigma$, since, when both the constraints of Eqs. (64) and (79) are satisfied, the curvaton’s evolution does not depend on the sign in front of $c$. 

Suppose, at first, that $c \leq c_0$. Then the field lies into the higher-order part of the potential, where the attractor constraint in Eq. (64) is applicable. For $n > 0$, we see that the curvaton must unfreeze after reheating, that is

$$\Gamma_{\text{inf}} > H_{\text{osc}}.$$  \hfill (167)

Now, in view of Eq. (109), we find

$$H_{\text{osc}} \sim \sqrt{c_0} H_* \sim 10^9 \text{GeV},$$  \hfill (168)

where we used Eqs. (160) and (166). From Eqs. (160) and (168), it is clear that Eq. (167) is not satisfied and, therefore, the possibility of higher-order evolution (i.e. $c \leq c_0$) is excluded.

Let us assume now that $c_0 < c \ll 1$, which results in quasi-quadratic evolution. Since $q = 1$, in this case, Eq. (162) suggests that

$$r \sim 10^{-5}/c.$$  \hfill (169)

Using the WMAP constraint of Eq. (25), we find that $c \leq 10^{-3}$. However, from Eq. (166), we have $c > c_0 \sim 10^{-4}$, which allows only one possibility

$$c \sim 10^{-3}$$  \hfill (170)

and, therefore,

$$r \sim 10^{-2},$$  \hfill (171)

which marginally satisfies Eq. (25).

Now, since $c \ll 1$, Eqs. (84) and (85) suggest that $K_{\text{MD}} \simeq K_{\text{RD}} \simeq 0$ (c.f. Eq. (88)). Then, using Eq. (95) in Eq. (115), we obtain

$$r \sim \left(\frac{\sigma_{\text{end}}}{m_P}\right)^2 \left(\frac{H_\ast}{\Gamma_{\sigma}}\right)^{1/2}.$$  \hfill (172)

Eq. (161) gives $\sigma_{\text{end}} \sim 10^{14} \text{GeV}$. Substituting this into Eq. (172) and using also Eq. (160), we find the condition

$$\min\{m, \Gamma_{\text{inf}}\} \sim 10^{12} \Gamma_{\sigma},$$  \hfill (173)

where we have also employed Eqs. (91) and (114). To study this, we need to use Eq. (158). Thus, we consider the following two cases:

- **Small $g$:** This case corresponds to

$$g < \frac{m}{m_P} \Rightarrow \Gamma_{\sigma} \sim \frac{m^3}{m_P},$$  \hfill (174)

which means that the curvaton decays gravitationally. Using the above, the BBN constraint in Eq. (26) results in the bound

$$m > 10^4 \text{GeV}.$$  \hfill (175)

Hence, in view also of Eq. (160), we see that $m \gg \Gamma_{\text{inf}}$. Consequently, Eq. (173) gives

$$m \sim 10^6 \text{GeV},$$  \hfill (176)

which satisfies the bound of Eq. (175). Inserting the above into Eq. (174), we obtain

$$g < 10^{-12},$$  \hfill (177)

which, in view of Eqs. (170) and (176), can be easily shown to satisfy the bound of Eq. (154).

- **Large $g$:** This case corresponds to

$$g \geq \frac{m}{m_P} \Rightarrow \Gamma_{\sigma} \sim g^2 m.$$  \hfill (178)

Using this, Eq. (173) is recast as

$$\min\left\{1, \frac{\Gamma_{\text{inf}}}{m}\right\} \sim 10^{12} g^2.$$  \hfill (179)

Taking $m \geq \Gamma_{\text{inf}}$, one finds $g \sim 10^{-6}$, which can be easily shown, however, to violate the constraint of Eq. (154). Thus, we have

$$m \geq \Gamma_{\text{inf}} \sim 10^{-6} \text{GeV}.$$  \hfill (180)

In view of the above, Eq. (179) is now recast as

$$g \sim 10^{-9} (m/\text{GeV})^{-1/2}.$$  \hfill (181)

Using this and Eq. (178), one can show that the BBN constraint in Eq. (26) is trivially satisfied. Further, Eq. (178) demands that

$$m \leq 10^6 \text{GeV}.$$  \hfill (182)

Finally, employing the constraint of Eq. (154), we find the bound

$$m \geq 0.1 \text{ GeV}.$$  \hfill (183)

Summing up, in this case, we have ended up with the following parameter space. For large $g$, we have found

$$10^{-7} \text{GeV} \leq m \leq 10^6 \text{GeV},$$

$$10^{-9} \geq g \geq 10^{-12},$$  \hfill (184)

where the relation between $m$ and $g$ is given in Eq. (181). For small $g$, we have found

$$m \sim 10^6 \text{GeV},$$

$$g < 10^{-12}.$$  \hfill (185)

The above parameter space is shown in Fig. 3. Note that, for all the parameter space, the masslessness requirement $m \ll H_* \sim 10^{11} \text{GeV}$ is well satisfied.

**B. Case $c \simeq 1 \gg c_\ast$.**

In this case, Eqs. (162) and (166) yield

$$r q \leq 0.1.$$  \hfill (186)

The results depend on the sign of $c$. We treat the two cases separately.
Eq. (25). The above and Eq. (162) suggest that \( w_{\text{map}} \) where we also took into account the \( g_{\text{T}} \) we find, from Eq. (115), that \( c = \frac{5}{3} \) for a successful curvaton in the case when \( c = c_{x} \sim 10^{-3} \), with model parameters given by Eq. (160). The upper bound \( q \leq 10^{-9} \) is determined by the requirement of Eq. (154), which ensures that \((gT)^2\) never dominates the effective mass. For \( q < 10^{-12} \), the curvaton decays predominantly through gravitational couplings.

1. The case of \( +c \)

In this case, \( q \sim 1 \) and, therefore, Eq. (186) suggests that

\[
10^{-2} \leq r \leq 10^{-1},
\]

where we also took into account the WMAP constraint in Eq. (25). The above and Eq. (162) suggest that

\[
1 \leq \gamma \leq 10,
\]

where we defined

\[
\gamma \equiv \max\{1, c_{x}/c_{0}\} .
\]

Using this and Eqs. (160) and (161), we obtain

\[
\sigma_{\text{end}} \sim \gamma^{-1}10^{15}\text{GeV} .
\]

Now, when \( c \approx 1 \), it can be shown that, after inflation, the field finds itself in the quasi-quadratic regime. Eq. (187) shows that the curvaton has to decay before it dominates. For \( c \geq c_{x} \), Eqs. (84) and (85) show that \( K_{\text{MD}} = 1 \) and \( K_{\text{RD}} = 1/2 \). Substituting this in Eq. (95), we find, from Eq. (115), that

\[
r \sim \left( \frac{\sigma_{\text{end}}}{m_{p}} \right)^{2} \frac{m}{H_{*}} \left( \frac{\Gamma_{\text{inf}}}{\Gamma_{\sigma}} \right)^{1/2} .
\]

By comparing the above with Eq. (172), we see that \( r \) is now smaller by a factor \((m/H_{*})\). This is because, in contrast to the case \( c \ll 1 \) when the curvaton’s density ratio \( \rho_{\sigma}/\rho \) remains almost constant after the end of inflation, in the case \( c \approx 1 \) this ratio is substantially reduced until the onset of the final quadratic oscillations. This makes things much more difficult and, as a result, the parameter space is diminished.

Now, employing Eqs. (160) and (190), one finds from Eq. (191) the condition

\[
m^{2} \sim \Gamma_{\sigma} \times \gamma^{2}10^{38}\text{GeV} .
\]

In view of the BBN constraint in Eq. (26), the above gives the bound

\[
m > \gamma 10^{7}\text{GeV} .
\]

Now suppose that \( g \) is small enough for Eq. (174) to be valid. Then, Eq. (192) gives \( m \sim \gamma^{-2}10^{-2}\text{GeV} \), which strongly violates the bound in Eq. (193) for the whole range of \( \gamma \). Thus, we have to consider the case when \( g \) is large and Eq. (178) is valid. Using this, Eq. (192) can be recast as

\[
g \sim \gamma^{-1}10^{-19}(m/\text{GeV})^{1/2} .
\]

In view of the above, the lower bound on \( g \) from Eq. (178) gives \( m \sim \gamma^{-1}10^{-2}\text{GeV} \) which again violates Eq. (193) for the whole range of \( \gamma \). Hence, in this case there is no parameter space for a successful curvaton.

2. The case of \( -c \)

In this case, \( q = q_{\text{eff}} \) as given in Eq. (144). Using this and Eq. (162), one, generically, obtains

\[
\zeta_{0} \sim r c_{0} \left( \frac{c_{0}}{c} \right)^{\frac{1}{n+2}} .
\]

In view of Eq. (166) in our example, the above gives

\[
r \sim 1
\]

and, therefore, it is possible for the curvaton to dominate the Universe before its decay. Using Eqs. (144), (161) and (195), it is easy to show that, generically,

\[
\sigma_{\text{min}}(c, H_{*)} \gg \bar{\sigma}_{\text{end}} \Leftrightarrow q_{\text{eff}} < 1 .
\]

Thus, if \( q_{\text{eff}} < 1 \), compared to the minimum after the end of inflation, the field appears to be at the origin. Therefore, the initial amplitude of the quasi-quadratic oscillations is

\[
\bar{\sigma}_{\text{end}} \sim \sigma_{\text{min}}(c, H_{*}) .
\]

Using this in our example, Eqs. (29) and (160) suggest

\[
\bar{\sigma}_{\text{end}} \sim 10^{15}\text{GeV} .
\]

As before, for \( c \geq c_{x} \), Eqs. (84) and (85) show that \( K_{\text{MD}} = 1 \) and \( K_{\text{RD}} = 1/2 \). Substituting this in Eq. (117), we obtain

\[
H_{\text{dom}} \sim \left( \frac{\bar{\sigma}_{\text{end}}}{m_{p}} \right)^{4} \left( \frac{m}{H_{*}} \right)^{2} \Gamma_{\text{inf}} .
\]
Inserting Eqs. (160) and (199) in the above, we find
\[ H_{\text{dom}} \sim 10^{-40}(m/\text{GeV})^2\text{GeV}. \] (201)
Since the curvaton decays after domination, we require that
\[ \Gamma_{\sigma} \leq H_{\text{dom}}. \] (202)
Combining Eqs. (201) and (202) with the BBN constraint in Eq. (26), we find the bound
\[ m > 10^8\text{GeV}. \] (203)

Now suppose that \( g \) is small so that Eq. (174) is valid. Then, Eqs. (201) and (202) give \( m \leq 10^{-4}\text{GeV} \), which strongly violates the bound in Eq. (203). Thus, we have to consider the case when \( g \) is large and Eq. (178) is valid instead. Using this, Eqs. (201) and (202) result in
\[ g \leq 10^{-20}(m/\text{GeV})^{1/2}. \] (204)
It can be easily shown that, when the above is true the constraint of Eq. (154) is also satisfied.

Now the lower bound on \( g \) coming from Eq. (178) is
\[ g \geq 10^{-18}(m/\text{GeV}). \] (205)

One can be easily check that this bound, in view of Eq. (178), suffices to satisfy also the BBN constraint in Eq. (26).

The available parameter space is spanned by Eqs. (204) and (205) and it is a surface on the \( m - g \) plane, in contrast to being a line, as shown in Fig. 3 for the previous case. This is because, when the curvaton dominates, the necessary condition is the inequality in Eq. (202), whereas if it decays before domination the necessary condition is the equality in Eq. (115) (i.e. \( r \) has to have exactly the right value).

Unfortunately, from Eqs. (204) and (205), we find that the parameter space exists only if \( m < 10^{-4}\text{GeV} \), which strongly violates the bound in Eq. (203). Therefore, we have shown that there is no parameter space for a successful curvaton in the case when \( c \approx 1 \) regardless of the sign of \( c. \) This is due to the fact that the quasi-quadratic oscillations drastically reduce the energy density of the field. This reduction is impossible to be counteracted during the final quadratic oscillations.

**VIII. DISCUSSION AND CONCLUSIONS**

In this paper, we have investigated and analysed the dynamics of the curvaton field. Apart from its soft mass \( m \), we have assumed that the field receives a contribution to its effective mass from supergravity corrections. This contribution is determined by the Hubble parameter and, hence, the potential acquires what we called the quasi-quadratic term. Another contribution to the effective mass may arise due to the presence of a quartic term or of non-renormalizable terms. Finally, we also considered possible thermal corrections to the potential, which arise due to the coupling of the curvaton to the thermal bath present after the end of inflation.

Firstly, we discussed the behaviour of the field during inflation. We explained that the field is expected to engage, initially, in fast-roll, which soon, however, is terminated to be followed by slow-roll evolution. Eventually, slow roll is also halted due to the action of field perturbations generated by quantum fluctuations. These perturbations effectively stabilize the coherent motion of the field for the remaining part of inflation.

After inflation, we have studied the evolution of the curvaton’s energy density in the cases when the effective mass is dominated by the quasi-quadratic term or a higher-order one. In the case of the quasi-quadratic domination, we have shown that there are two types of evolution depending on how suppressed is the effective mass. For effective mass of order the Hubble parameter, we found that the field begins to oscillate, whereas, for suppressed mass, we have found a scaling solution, according to which the field rolls gently towards the minimum of its potential. Both solutions preserve the amplitude of the perturbation spectrum of the curvaton, obtained during inflation.

In the case of the higher-order term domination, we have shown again that the field, even though frozen initially, will, eventually, begin to oscillate (preserving also the amplitude of the perturbation spectrum) provided the order of the higher-order term is not too high. This is because, in the opposite case, after unfreezing, the field was found to follow an attractor solution, which loses all memory of initial conditions. Consequently, were the field to follow such an attractor, all the superhorizon spectrum of its perturbations would have been erased and the field could not act as a curvaton.

A similar situation was also found in the quasi-quadratic case, when the effective mass was negative after the end of inflation. In this case, if the order of the higher-order term (which decides the temporal position of the minimum of the potential) is too high, instead of rolling, the field was found to follow a particular solution and remain at fixed, constant ratio with respect to the temporal minimum, losing thereby all memory of initial conditions. The constraints obtained on the order of the principle higher-order term in both the above cases were comparable and their implementation was decisive for the evolution of the field.

Generically, we have found the following behaviour for the curvaton after the end of inflation. Both in the quasi-quadratic case and the higher-order case, the energy density of the field was found to be decreasing faster than the background density. Consequently, curvaton domination (or near domination) is possible only after the effective mass of the field becomes dominated by the soft mass. In all cases, when the attractor was avoided, we have found that the damping on the amplitude of the perturbation spectrum is negligible. Finally, we have shown that the
temperature corrections should not be allowed to dominate the effective mass of the field because if they do the field thermalizes before having a chance to dominate (or nearly dominate). This requirement introduces a stringent constraint on the coupling of the curvaton to the thermal bath.

In all cases, we have calculated the density ratio $r$ of the curvaton to the background density of the Universe at the time of the curvaton’s decay. If the curvaton is to decay before dominating, this density ratio is necessary to determine the total curvature perturbation imposed by the curvaton field onto the Universe. We have also calculated when the curvaton dominates, if it does indeed so, in all cases considered. Furthermore, we calculated in detail the factor $q$ relating the curvaton perturbation at horizon crossing with the one at its decay. Finally, we have employed our findings in a concrete model realization, for which we have shown that there is hardly any parameter space available for a successful curvaton if $c \sim 1$ after the end of inflation, regardless of the sign in front of $c$. This is due to the fact that, for large values of $c$, the density fraction of the curvaton is drastically reduced by either the quasi-quadratic oscillations or the scaling solution. As explained also in Sec. VA, we expect this problem to be generic and, if so, this would favour models where $c$ after inflation remains strongly suppressed, such as the PNGB curvaton models discussed in Ref. [41].

We should mention here that, due to the fact that our attempt to describing the curvaton evolution was aimed at the bulk of the parameter space, there are possibly special situations, not studied here, where the evolution may be significantly modified. For example, if the quasi-quadratic term is dominant but negative after the end of inflation and the field is undergoing oscillations, there will be a moment, just before the soft mass taking over, when the quadratic and quasi-quadratic terms cancel each other. For a brief period, therefore, the field will be oscillating into a potential of higher order and, hence, will have an equation of state with a different (larger) effective barotropic parameter as shown by Eq. (22). The outcome of all this will be a “glitch” to the scaling of the density ratio that may affect both the time of curvaton domination and/or the total curvature perturbation if the curvaton decays before domination. The importance of such an effect is possible to be studied only numerically.

Another example, in the case of negative quasi-quadratic mass, is the effect of the local maximum at the origin, which may be felt if the initial condition of the field is $\sigma_{\text{end}} \sim \sigma_{\text{min}}$, which is exactly what we expect in this case, if the sign of $c$ changes at the end of inflation. Indeed, if the field engages into oscillations, there is a chance that, at some point, the oscillation amplitude marginally sends the field on top of the local maximum. The effect of such an event is the possible tachyonic amplification of the perturbation spectrum. In an analogous situation, this tachyonic amplification (when the field, as it oscillates, is temporarily stabilized near a local maximum of the potential) was thoroughly investigated by us in an earlier paper [6], where it was found that the amplification of the perturbations could be substantial. One could even imagine that such an amplification mechanism may counteract the suppression of the amplitude of the perturbation spectrum due to attractor evolution.

Furthermore, we can imagine a number of modifications to our considerations arising due to a Universe history more complicated than the one considered here. For example, one can insert a brief period of thermal inflation before or after curvaton domination. Another example would be to consider a period of ‘kination’, for which the background density has a stiff equation of state $w \approx 1$, which arises naturally in models of quintessential inflation [36, 37] or non-oscillatory inflationary models [34]. Indeed, the curvaton is a convenient mechanism to increase the effective reheating efficiency of such models [38]. We have not considered such possible complications in this paper mainly because there is an infinity of them. However, we tried to express our results in as much a model independent way as possible, so that they will be easy to implement in particular, more complicated scenarios of the Universe history.

All in all, we have investigated the evolution of the curvaton field during and after inflation. We have shown that a successful curvaton either oscillates or scales down to its minimum but always preserves the amplitude of the spectrum of its perturbations. We also showed that, in order to dominate (or nearly dominate), the curvaton has to decay after its effective mass becomes dominated by the soft mass term and the oscillation becomes (Hubble damped) sinusoidal. Furthermore, to avoid premature thermalization, the temperature corrections to the curvaton’s potential should remain negligible throughout its evolution. Finally, there is a part of the parameter space which results in destructive attractor evolution, which erases the perturbation spectrum and should be avoided.

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[4] http://map.gsfc.nasa.gov/
[5] In special cases, such as the one studied in Ref. [6], two or more scalar fields might be involved. As an alternative to the inflation hypothesis, a "pre-big-bang" [7–9] or "ekpyrotic" [10–12] era of collapse has been proposed, but there is so far no accepted theory of a bounce and therefore no firm prediction from collapsing cosmologies. In particular, there is so far no accepted string-theoretic description of a bounce [13].

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[44] Recently, a different alternative to the inflaton scenario has been proposed in Refs. [49–51], in which some ‘modulating’ field perturbs the inflaton decay rate without ever contributing significantly to the energy density. The properties of the modulating field have to be much more special than those of the curvaton field.

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[50] Note that, in view of Eq. (11), $w_\sigma$ may be in principle time-dependent.

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[52] We ignore cubic $A$-terms.

[53] Here, $g$ may also include loop factors or number of degrees of freedom etc.

[54] For example, in the case of no-scale supergravity, an (approximate) Heisenberg symmetry suggests that $c \sim 10^{-2} g^2$.

[55] Note that the full scalar potential includes also the contribution of the inflaton field, which, at least until reheating, is much larger than $V_0$.

[56] Indeed, by retaining only the kinetic terms in Eq. (6), one easily finds that $\sigma = \sigma_0 + (\sigma_0/3H)[1 - e^{-3H(t_0 - t)}]$, which means that the rapid roll due to the kinetic energy will cease in less than a Hubble time.

[57] A. A. Starobinsky, Phys. Lett. B 117, 175 (1982); A. D. Linde, Particle Physics And Inflationary Cosmology, (Harwood, Chur Switzerland, 1990).

[58] This is easy to understand using the uncertainty principle (which governs quantum fluctuations) $\Delta E \cdot \Delta t = 1$, if one considers the energy within a Hubble volume $\Delta E \simeq \rho_{\text{eq}} \times H_*^{-1}$ and the time necessary for horizon exit $\Delta t \simeq H_*^{-1}$.

[59] This is the source of our disagreement with Ref. [33], where it is claimed that the border of the quantum regime is determined by comparing the energy density of the quantum fluctuations $\rho_{\text{eq}}$ with the potential $V$ and not the kinetic energy density $\rho_{\text{kin}}$ of the rolling scalar field. If one adopted such a view then there would be no quantum regime on top of local maxima (e.g. in the case of eternal inflation) or at inflection points, where $V' = 0$, as long as $V$ is large enough. Moreover, the fact that one can always
add a cosmological constant would have really caused confusion since a change of $V$ would have affected the location of the borders of the quantum regime.

[65] Note that, since $V(\sigma)/p_{\text{kin}} = \text{const.}$, the equation of state for the curvaton has a constant $w_\sigma$ both in the oscillating case and in the case of the scaling solution.

[66] This is true if $c$ does not change dramatically at the end of inflation. If, however, $c \sim 1$, at the end of inflation the field may find itself in the quasi-quadratic regime instead.

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[68] This is our main difference with Ref. [51] where it is claimed that there is substantial damping of the field’s perturbation between the end of inflation and the onset of the oscillations. The damping found in Ref. [51] is due to the incorrect assumption that the field is “critically damped”, i.e. that $V'' \sim H^2$, during this time interval, which would lead to substantial roll that reduces the perturbation.

[69] Note that, if $c$ is positive during inflation, then $\sigma_{\text{end}} \ll \sigma_{\text{end}}^{\text{min}}$. This, however, means that the oscillations will still have initial amplitude $\sim \sigma_{\text{end}}^{\text{min}}$ and so, for the amplitude of the oscillating field, Eq. (71) is valid.

[70] For example, in the matter era the value of the field is described by the Eq. (59) with the substitutions $\delta \sigma \rightarrow \sigma$ and $\sqrt{V_0''} \rightarrow m$.

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[72] Note that the gravitino bound on $T_{\text{reh}}$ may be substantially relaxed by the extra entropy production if the curvaton decays after it dominates the Universe.