Spin flip of electron in static electric fields

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The effects on the spin state of an electron in a time independent electric field are examined. The probability of spin flipping is calculated, and other effects are studied using the minimally coupled Dirac equation.

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INTRODUCTION

There is a vast yet growing interest in spin states and their control. This attention comes from a number of sectors, from the quantum information world and the need to spin flip and control the spin, to a myriad number of applications of spintronics. It is well-known how to flip a spin, and basic physics tells us the most obvious way is through the use of a magnetic field. This is because the interaction between a charged particle with spin and an external electromagnetic field is \( \mu \cdot \mathbf{B} \). But this is the non-relativistic limit. We know moving through a \( \mathbf{B} \)-field with velocity \( v \) produces an electric field, so we expect electric field effects to go like \( v/c \) times the magnetic effect.

In fact, it is well known spin can be changed without using a magnetic field. Nuclear spin flips induced by an electric field have recently been observed, and electric field effects with a laser were demonstrated a while ago. Electric fields are also known to cause spin flipping through the Rashba effect, and dynamic spin process such as the Kapitza-Dirac effect are well known. Spins can also be flipped using non-electromagnetic means.

Although, at first glance, one expects the magnetic field to be used in the control of spin, there are other reasons for considering electric effects. In small devices, electric fields are usually easier to make, both from size considerations of power usage, so it is worthwhile to investigate a little more carefully the effect of the electric field on spin. In addition, if there is a current present, then there is an electric field present as well.

The main objective of this paper is to examine and assess the effect of a known external electric field on the spin state of an electron. In the applications given above, it is impossible to isolate the effects of an electric field. This is because many of those effects are in various shaped bulk materials with an electric field of unknown functional form, some of which have current density or accompanying magnetic fields. Under these conditions it is impossible to really know the effect of the electric field alone. In addition, many times the properties of the material is modeled by a Hamiltonian that is assumed to hold, at least at some approximation. In this paper, we treat the problem of an electron in an electric field exactly (although we stop at first order perturbation), the interaction of which comes directly from the Dirac equation. Other than using first order theory, no approximations are made, and we can be confident we are seeing the effects of the electric alone.

Before the effect of the electric field is considered, there is an important conceptual issue that should be addressed. Let us consider an electron in the non-relativistic regime. Then we say it has either spin up or spin down (or spin right or spin left, etc.), reflecting the two allowed states of spin for the electron. Spins up or down are meaningless concepts without a fiduciary field by which to measure them. So we place the electron in an external magnetic field and now the concept of spin up and down makes sense.

In particular, we consider flipping the spin. We should not think of flipping the spin of a free particle for the reasons discussed, so the initial and final states are not those of a free particle, they are those of an electron in a magnetic field, which is assumed to be constant. It has been
shown that, assuming the final and initial states are those of the eigentstates of an electron in a magnetic field, then the scattering matrix is

\[ S_{fi} = -i \frac{e}{\hbar c} \int d^4x \psi_f A_x \gamma^\sigma \psi_i \]  

(1)

which, of course, looks the same as the free particle form, but it is important to note that in (1) the initial and final states are those of a particle in a magnetic field. The above formula is given in cgs units, but below natural units \((\hbar = 1 = c)\) are used in general, although some formulas are given in cgs for clarity, including the final results.

**KNOWN SOLUTIONS**

For convenience, this brief section will summarize known properties of an electron in a constant magnetic field \(B\). The energy is quantized in Landau levels with quantum number \(n\)

\[ E_n^2 = m^2 + p_z^2 + 2neB \]  

(2)

where \(p_z\) is the \(z\) component of the momentum, \(e\) is the magnitude of the charge of the electron, and \(n = 0, 1, 2, \ldots\), and the wavefunctions are given by

\[ \psi_n = C_f e^{-i(E_i t - p_i^x x - p_i^z z)} e^{-\xi^2/2} u_n \]  

(3)

where the momentum terms are eigenvalues, and the dimensionless coordinate is

\[ \xi = \sqrt{eB} y - \frac{p_x}{\sqrt{eB}}. \]  

(4)

In terms of the energy quantum number \(n\) the solutions are given by for spin up,

\[ u_n = \begin{pmatrix} h_n \hbar^{-1} \\ 0 \\ p_n^z h_{n-1}/(E_n + m) \\ -\sqrt{2neB}h_n/(E_n + m) \end{pmatrix} \]  

(5)

and for spin down,

\[ u_n = \begin{pmatrix} 0 \\ h_n \\ -\sqrt{2neB}h_{n-1}/(E_n + m) \\ -p_n^z h_n/(E_n + m) \end{pmatrix} \]  

(6)

where \(u_n = u_n(\xi)\) and \(h_n = h_n(\xi)\) are the orthonormal Hermite polynomials, \(h_n = N_n H_n\), and \(H_n\) are the Hermite polynomials, \(N_n = 1/\sqrt{2^nn!\sqrt{\pi}}\), and by definition the Hermite polynomial with a negative subscript is zero.

The solution is normalized to a two-dimensional box in the \(x\) and \(z\) directions, and the normalization constants are

\[ C = \sqrt{\frac{(E_n + m)}{2L_x L_z E_n}}, \quad L_\xi = \sqrt{\frac{\hbar c}{eB}} \]  

(7)

where \(L_\xi\) is given in cgs. It is useful to note the solutions have a two-fold degeneracy.

**SCATTERING AMPLITUDE**

We are now in a position to describe all the effects of an electromagnetic perturbation, \(A_x\), on an electron trapped in a magnetic field. Using the above in (1) we have

\[ S_{fi} = -\frac{iC_f C_i}{L} \int d^4x e^{i(E_f t - p_f^x x - p_f^z z)} e^{-\xi^2} \]  

(8)

(to avoid double subscripting, here and below, when \(f\) or \(i\) appears as a subscript or superscript, it really stands for \(n_f\) or \(n_i\), the final or initial quantum number) where \(L \equiv \hbar \omega/eE\) and the matrix element is

\[ M_{fi} = u_f^\dagger \gamma^0 \gamma^\sigma A_\sigma u_i, \]  

(9)

which becomes,
This last term has a wealth of information. For example, for a static electric field the only contributions come from the $A_0$ term, whereas magnetic effects are described by the other terms. If one compares the $A_0$ term to the $A_3$ term we find the ratio is like $p c / m c^2 \sim v/c$. This is expected, and we also note the $A_0$ term vanishes unless the electron is moving, (we know an observer moving through and electric field sees a magnetic field $\mathbf{v} \times \mathbf{B}/c$).

Another interesting situation arises from the fact the energy levels are determined by two parameters, the momentum and the magnetic interaction energy. Thus, the electron can suffer a spin flip even though the total energy of the system remains constant. This can be seen by looking at the energy levels and noting, from \[ E \Delta E = p \Delta p + \Delta n e B, \] so that the total energy can remain constant while the electron gains (loses) magnetic energy as the electron loses (gains) kinetic energy.

\[ S_{fi} = \frac{L_c C_f}{2} \int d^4 x e^{-i(E_f - E_i) t} e^{i(p^f_z - p^i_z) z} e^{i\left(p^f_x - p^i_x\right)x} e^{-\xi^2} \left( e^{i(k z - \omega t)} - e^{-i(k z - \omega t)} \right) M_{fi}. \]

The spin flip probability is the integral of $|S_{fi}|^2$ times the density of states. Details were given elsewhere[6] and the result is, in the low velocity limit

\[ S_{fi} = \kappa \left( \frac{\sin(\omega - \omega_0)t/2}{(\omega - \omega_0)/2} \right)^2 \]

where $\kappa = E^2 \mu^2 / h^2$ and $\mu = e h / 2 m c$. This is the well-known Rabi formula.
TIME INDEPENDENT ELECTRIC FIELD

Now we would like to consider the effect of an external electric field. Since a time dependent electric field creates a magnetic field, only “pure electric” effects can be seen from a static field, which we assume is imposed externally. The only other requirement is that the field obey Maxwell’s equations, for otherwise we would end up with unrealistic effects. A relatively simple electric field is one in which the potential is given by \( A_0 = (E/k) \sin kze^{-ky} \) where \( E \) is the electric field strength and \( k \) is a parameter describing the spatial variation of the field. With this we may perform the integrations is \( \text{[3]} \) and obtain the transition matrix. A few details will be discussed.

This time the time integral, \( L_t \), is of the form \( \int e^{-i(E_i-E_f)t} dt \). Integrating from minus to plus infinity gives a delta function, showing \( E_i = E_f \) (recall from above that this can induce a spin flip through a loss or gain in momentum). In order to perform the integral over some finite time, we assume \( E_i - E_f = \epsilon \) and take the limit as \( \epsilon \rightarrow 0 \). The square of this gives \( 4t^2 \). The \( x, z \) integrations give \( L_x, L_z \), which are conservation of momentum and in particular \( \delta(p^x_f - p^x_i \pm \hbar k) \). That leaves the \( y \) integral, which may be accomplished using the orthogonality of the Hermite polynomials. The result is

\[
S_{fi} = -i \frac{C_i C_f}{L} L_t L_x L_z L_\xi \Gamma \sum_{n=0}^{n_f} \left( \begin{array}{c} n_f \\ n \end{array} \right) \left( \begin{array}{c} n_i \\ n \end{array} \right) \sqrt{n_f(n_f + n_i + n_x)} p^i z - R
\]

where the terms with parentheses are the binomial coefficients,

\[
\Gamma = \frac{\sqrt{2eB}}{(E_f + m)(E_i + m)} e^{kL\xi \phi} e^{(kL\xi/2)^2}
\]

and \( R \) is the same as the term preceding it with \( n_f \) replaced by \( n_f-1 \), \( n_i \) replaced by \( n_i-1 \), and \( p^i_z \) replaced by \( p^i_z \).

In order to find the transition probability \( |S_{fi}|^2 \) is integrated over the density of states as before. Let us simplify things a bit by assuming the momentum in the \( x \) direction is small. This means we can set \( \phi \) to zero (the momentum would have to be relativistic for this term to be important). Now let us consider \( kL\xi \). For a \( B \) field of 100 G, this would be on the order of \( 2 \times 10^{-5}k \) (\( k \) in cm) and will assumed to be small. This is valid unless the electric field varies appreciably over the scale of a tenth of a micron. However if such fields are imposed, then this term, being exponential, will have a dramatic effect on the spin flip probability.

Let us also assume \( n_i = 0 \). The spin flip probability \( W \) becomes

\[
W = \frac{B(p^f_1)^2 E^2 e^3}{m^4 k^2} t^2
\]

but we must impose the conditions of the delta functions: From the momentum delta function we have \( p_f x = p_i x - k \) and from energy we have \( p_f^2 = p_i^2 - 2eB \).

**Electric Cooling**

As discussed above, it is possible for the scattering amplitude to be non-vanishing with the total energy remaining constant. In this case, if the spin flips from a higher to lower state, the energy is given to kinetic energy of the electron, as the delta functions dictate. On the other hand, if the spin flips to a higher energy state, the electron and loses that much kinetic energy, cooling the sample. The amount of cooling can be controlled by the amplitude of the applied electric and magnetic fields. Laser cooling with spin has been discussed for condensed matter systems. [8]

As an example, for the \( z \) component of the momentum, setting \( p^f_1 \) to zero we find \( p^f = \mu B/2mc^2 \), which we expect on simple energy considerations.

**Summary**

The main goal has been achieved, to show that an electron moving through a static electric field can suffer spin flips. The transition amplitude was calculated using the Dirac equation, and the spin flip probability for an electron was calculated from that. The Rabi formula was derived from the formalism, although a large number of relativistic effects may be gleaned from [10]. In the present case, we examined the effect of a static electric field.
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