January 4, 2017

LIMIT THEOREMS FOR HILBERT SPACE-VALUED LINEAR PROCESSES UNDER LONG RANGE DEPENDENCE

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Abstract. Let \((X_k)_{k \in \mathbb{Z}}\) be a linear process with values in a separable Hilbert space \(\mathbb{H}\) given by \(X_k = \sum_{j=0}^{\infty} (j+1)^{-N} \varepsilon_{k-j} \) for each \(k \in \mathbb{Z}\), where \(N : \mathbb{H} \rightarrow \mathbb{H}\) is a bounded, linear normal operator and \((\varepsilon_k)_{k \in \mathbb{Z}}\) is a sequence of independent, identically distributed \(\mathbb{H}\)-valued random variables with \(E \varepsilon_0 = 0\) and \(E \| \varepsilon_0 \|^2 < \infty\). We investigate the central and the functional central limit theorem for \((X_k)_{k \in \mathbb{Z}}\) when the series of operator norms \(\sum_{j=0}^{\infty} \| (j+1)^{-N} \|_{op}^2 \) diverges. Furthermore we show that the limit process in case of the functional central limit theorem generates an operator self-similar process.

1. Introduction

In this paper, we study long-range dependent linear processes with values in a separable Hilbert space \(\mathbb{H}\). Given a sequence of bounded linear operators \(u_j : \mathbb{H} \rightarrow \mathbb{H}\), \(j \geq 0\), and a sequence of independent, identically distributed \(\mathbb{H}\)-valued random variables \((\varepsilon_k)_{k \in \mathbb{Z}}\) with \(E \varepsilon_0 = 0\) and \(E \| \varepsilon_0 \|^2 < \infty\), we define the linear process

\[
X_k = \sum_{j=0}^{\infty} u_j(\varepsilon_{k-j}), \quad k \in \mathbb{Z}.
\]

We investigate the asymptotic distribution of the partial sums \(S_n = \sum_{k=1}^{n} X_k\) and of the partial sums process \(\zeta_n(t) = S_{\lfloor nt \rfloor} + \{nt\}X_{\lfloor nt \rfloor + 1}\) with \(t \in [0, 1]\), where \(\lfloor \cdot \rfloor\) denotes the floor function and \(\{x\} = x - \lfloor x \rfloor\).

The behaviour of the linear process \((X_k)_{k \in \mathbb{Z}}\) depends crucially on the convergence respectively divergence of the series \(\sum_{j=0}^{\infty} \| u_j \|_{op}\), where \(\| \cdot \|_{op}\) denotes the operator norm. If \(\sum_{j=0}^{\infty} \| u_j \|_{op} < \infty\), the process \((X_k)_{k \in \mathbb{Z}}\) is short range dependent. In this case, the central limit theorem holds with the usual normalizing sequence \(n^{-\frac{1}{2}}\) and the normalized partial sums converge in distribution to an \(\mathbb{H}\)-valued Gaussian random element (see Račkauskas and Suquet (2010) and Merlevde et al. (1997)). We are interested in the situation when the series diverges.

Račkauskas and Suquet (2011) investigate a functional central limit theorem for \((X_k)_{k \in \mathbb{Z}}\) as in (1) with values in a Hilbert space \(\mathbb{H}\) when \(\sum_{j=0}^{\infty} \| u_j \|_{op}\) diverges with \(u_0 = I\) and \(u_j = j^{-T}\) for \(j \geq 1\), where \(T \in L(\mathbb{H})\) satisfies \(\frac{1}{2} I < T < I\) and is self-adjoint. Additionally they assume that the operator \(T\) commutes with the covariance operator of \(\varepsilon_0\).

Characiejus and Račkauskas (2014, 2013) consider \((X_k)_{k \in \mathbb{Z}}\) with values in the Hilbert space \(L_2(\mu) = L_2(S, \mathcal{S}, \mu)\) of square-integrable real-valued functions, where \((S, \mathcal{S}, \mu)\) is a \(\sigma\)-finite measure space. They choose \(u_j = (j+1)^{-D}\) without requiring that the operator commutes with the covariance operator of \(\varepsilon_0\). In their case \(D\) is a multiplication operator given by

\[Df(x) = \sum_{j=0}^{\infty} j^{-D} f(j+1) \delta_j(x)\]

Key words and phrases. Linear processes, Long memory, functional central limit theorem, self-similarity, Hilbert space.

Research supported by the Research Training Group 2131 - High-dimensional Phenomena in Probability - Fluctuations and Discontinuity.
We combine both results, constructing a process with values in a complex Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$, choosing

$$u_j = (j + 1)^{-N} \tag{2}$$

for each $j \geq 0$, where $N \in L(\mathbb{H})$ is a normal operator, i.e. $N$ commutes with its hermitian adjoint denoted by $N^*$, that is $NN^* = N^*N$.

To be more precise we give some details about operators. Let $A \in L(\mathbb{H})$, then it is called non-negative if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{H}$. For an additional operator $B \in L(\mathbb{H})$ the inequality $A \geq B$ means $A - B \geq 0$. We set $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ and $a^A = \exp(A \log a)$ for $a > 0$.

As in Characiejus and Račkauskas (2014) we get an operator self-similar process. Such processes were first introduced by Lamperti (1962) and play an important role in the context of long memory. Later operator self-similar processes were studied by Laha and Rohatgi (1981). In our case we get a self-similar process with values in a complex Hilbert space $\mathbb{H}$. With this in mind, we repeat the definition of self-similarity of Hilbert space-valued random sequences referring to Matache and Matache (2006).

**Definition 1.1.**

A stochastic process $\{Y(t)\} \mid t \geq 0\}$ on a Hilbert space $\mathbb{H}$ is called operator self-similar, if there exists a family $\{T(a)\} \mid a > 0\} \subset L(\mathbb{H})$, such that

$$\{Y(at)\} \mid t \geq 0\} \overset{\text{f.d.d.}}{=} \{T(a)Y(t)\} \mid t \geq 0\},$$

for each $a > 0$, where $\overset{\text{f.d.d.}}{=} \text{ denotes the equality of the finite-dimensional distributions.}$

The set $\{T(a)\} \mid a > 0\} \subset L(\mathbb{H})$ is also called scaling family of operators. If $T(a) = a^G I$, where $G$ is a fixed scalar and $I$ is the identity operator, the process is called self-similar.

In the following section we first present our main results with sufficient conditions for the central and the functional central limit theorem for the process $(X_k)_{k \in \mathbb{Z}}$ with values in a general Hilbert space $\mathbb{H}$ constructed as in (1) with $(u_j)_{j \geq 0}$ given by (2). In section 3 we give an application to a convolution operator. Next, we present an extension of the results given in Characiejus and Račkauskas (2014, 2013), which are needed to proof our main results. Especially we consider a process $(X_k)_{k \in \mathbb{Z}}$ with values in the Hilbert space $L_2(\mu, \mathbb{C})$ of square-integrable complex-valued functions.

In section 5 we present the proofs of our main results, including the proof of the existence of an operator self-similar process with values in a Hilbert space $\mathbb{H}$.

The appendix consists of the proofs of the preliminary results given in section 4.
2. Main results

Before presenting our main results, we need some introducing definitions and preliminary results.

The spectral theorem for normal operators (see Conway (1994, chapter 9, theorem 4.6)) states that it is possible to decompose each normal operator \( N \in L(\mathbb{H}) \) into a unitary operator \( U : \mathbb{H} \to L_2(\mu, \mathbb{C}) \) and a multiplication operator \( D : L_2(\mu, \mathbb{C}) \to L_2(\mu, \mathbb{C}) \). More precisely there exist a \( \sigma \)-finite measure space \((\mathcal{S}, \mathcal{S}, \mu)\) and a unitary operator \( U : \mathbb{H} \to L_2(\mu, \mathbb{C}) \) together with a bounded function \( d : \mathcal{S} \to \mathbb{C} \), such that

\[
UNU^* = D,
\]

where \( D \) is a multiplication operator given by \( Df = \{d(s)f(s) : s \in \mathcal{S}\} \) for each \( f \in L_2(\mu, \mathbb{C}) \) with \( d : \mathcal{S} \to \mathbb{C} \). We denote by

\[
h(s) = \frac{1}{2}(d(s) + \overline{d(s)})
\]

the real part of \( d(s) \) and write \( d(r, s) = d(r) + d(s) \). It is well known that the so called beta function is a function of two complex numbers \( a, b \) with positive real part defined by

\[
\text{Beta}(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} dx.
\]

It may be also written as \( \text{Beta}(a, b) = \int_0^\infty x^{a-1}(x + 1)^{-a-b} dx \). We define the function \( c : \mathcal{S} \times \mathcal{S} \to \mathbb{C} \) by

\[
c(r, s) = \text{Beta}(1 - d(r), d(r, s) - 1) = \int_0^\infty x^{-d(r)}(x + 1)^{-d(s)} dx.
\]

and introduce the further notations

\[
\sigma_U(r, s) := E((U\varepsilon_0)(r)(U\varepsilon_0)(s)), \quad \sigma_U^2(s) := E|(U\varepsilon_0)(s)|^2 \quad r, s \in \mathcal{S}.
\]

If \( h(s) > \frac{1}{2} \mu \)-almost surely, the series

\[
X_k = \sum_{j=0}^\infty (j + 1)^{-N}\varepsilon_{k-j},
\]

which defines the process \((X_k)_{k \in \mathbb{Z}}\), converges. We postpone the proof to section 5.

Now, we are ready to present the central limit theorem.

**Theorem 2.1.** Suppose that \( h(s) \in (\frac{1}{2}, 1) \) for each \( s \in \mathcal{S} \) and that the integrals

\[
\int_{\mathcal{S}} \frac{\sigma_U^2(s)}{(1 - h(s))^2} \mu(ds) \quad \text{and} \quad \int_{\mathcal{S}} \frac{\sigma_U^2(s)}{(1 - h(s))(2h(s) - 1)} \mu(ds)
\]

are finite. Then

\[
n^{-H}S_n \xrightarrow{D} G \quad \text{as} \quad n \to \infty,
\]

in \( \mathbb{H} \) with \( H = \frac{3}{2}I - N \). The Hilbert space-valued random variable \( G \) is Gaussian with covariance operator \( C_G : \mathbb{H} \to \mathbb{H} \) defined by

\[
C_G(x) = U^* \int_{\mathcal{S}} \left( \frac{c(r, s) + c(s, r)}{(2 - d(r, s))(3 - d(r, s))} \sigma_U(r, s) \right) (Ux)(s) \mu(ds).
\]
Before we continue with the functional central limit theorem we need some further preliminaries. We define the Gaussian stochastic process \( \mathcal{G} = \{ \mathcal{G}(t) | t \in \mathbb{R}_{\geq 0} \} \) with the help of its covariance operator \( C_{\mathcal{G}} : \mathbb{H} \to \mathbb{H} \) given by

\[
C_{\mathcal{G}}(x) = U^* \int_\mathbb{S} V_U((r, t)(s, u))(Ux)(s) \mu(ds),
\]

where

\[
V_U((r, t)(s, u)) = \frac{\sigma_U(r, s)}{(2 - d(r, s))(3 - d(r, s))} \left[ c(s, r) t^{3 - d(r, s)} + c(r, s) u^{3 - d(r, s)} - C(r, s; t - u) |t - u|^{3 - d(r, s)} \right]
\]

and

\[
C(r, s; t) = \begin{cases} 
  c(r, s), & \text{if } t < 0 \\
  c(s, r), & \text{if } t > 0
\end{cases}
\]

**Theorem 2.2.** Suppose that \( h(s) \in (\frac{1}{2}, 1) \) for each \( s \in \mathbb{S} \), the integrals

\[
E \left( \int_\mathbb{S} \frac{|U_{\leq 0}(s)|^2}{(1 - h(s))^2} \mu(ds) \right)^{\frac{\pi}{2}} \quad \text{and} \quad \int_\mathbb{S} \frac{\sigma_U^2(v)}{(1 - h(s))(2h(s) - 1)} \mu(ds)
\]

are finite and \( p = 2 \) and \( \bar{h} = \text{ess sup}_{s \in \mathbb{S}} h(s) < 1 \) or \( p > 2 \). Then

\[
n^{-H} \zeta_n \xrightarrow{D} G \quad \text{as} \quad n \to \infty
\]

in \( C([0, 1], \mathbb{H}) \) with \( H = \frac{3}{2} I - N \). The process \( G = \{ G(t) | t \in [0, 1] \} \) is the restriction of \( \mathcal{G} \) to the unit interval.

### 3. Application: Convolution operator

An example of a normal operator is the so called convolution operator \( F : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \) defined by

\[
(7) \quad F(f) = \int_\mathbb{R} K(x - y) f(y) dy =: K \ast f(x)
\]

with \( K \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \), where \( L_p(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} | f \text{ measurable and } \int_\mathbb{R} |f(x)|^p dx < \infty \} \).

It should be noted that the operator is self-adjoint, if the kernel function \( K \) is hermitian, i.e. if \( K(-x) = \overline{K(x)} \).

The Fourier transform of \( (\mathcal{F}g)(s) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} g(t) e^{-ist} dt \) with \( s \in \mathbb{R} \) factorizes the convolution, i.e. \( \mathcal{F}(K \ast f) = \sqrt{2\pi} \mathcal{F}(K) \cdot \mathcal{F}(f) \).

Defining the multiplication operator \( D : L_2(\mathbb{R}) \to L_2(\mathbb{R}), f \mapsto d \cdot f \) by \( d(s) = \sqrt{2\pi} \mathcal{F}(K)(s) \), we obtain the spectral decomposition in (3) of a convolution operator \( F \), given by \( \mathcal{F} \mathcal{F}^{-1} = D \).

Choosing \( u_j = (j + 1)^{-F} \) in (1) the process \( (X_k)_{k \in \mathbb{Z}} \) is defined by

\[
X_k = \sum_{j=0}^{\infty} (j + 1)^{-F^{-1}DF} \varepsilon_{k-j}.
\]

To formulate the central limit theorem for the above process, we maintain the notations introduced in the previous section

\[
\sigma_F(r, s) := E((\mathcal{F} \varepsilon_0)(r)(\mathcal{F} \varepsilon_0)(s)), \quad \sigma_F^2(s) := E|\mathcal{F}(\varepsilon_0)(s)|^2 \quad r, s \in \mathbb{S}
\]
and get the following corollary.

**Corollary 3.1.** Suppose that the real part of $\sqrt{2\pi}F(K(s))$ takes values in the interval $(\frac{1}{2}, 1)$ for each $s \in \mathbb{S}$ and that the integrals

$$\int_{\mathbb{R}} \frac{\sigma^2_F(s)}{(1 - h(s))^2} ds \quad \text{and} \quad \int_{\mathbb{R}} \frac{\sigma^2_F(s)}{(1 - h(s))(2h(s) - 1)} ds$$

are finite. Then

$$n^{-H}S_n \xrightarrow{D} G \quad \text{as} \quad n \to \infty,$$

in $L_2(\mathbb{R})$ with $H = \frac{3}{2}I - F$. The $L_2(\mathbb{R})$-valued random variable $G$ is Gaussian with covariance operator $C_G : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ defined by

$$C_G(x) = F^{-1} \left( \int_{\mathbb{R}} \frac{c(r, s) + c(s, r)}{(2 - d(r, s))(3 - d(r, s))} \sigma_F(r, s) \right) (Fx)(s)ds$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{0}^{\infty} \int_{0}^{\infty} x - \sqrt{2\pi}FK(t_1)(x + 1) - \sqrt{2\pi}F^{-1}K(s)(x + 1) - \sqrt{2\pi}FK(t_1)_s ds \right) dx$$

$$\sigma_F(t_1, s) e^{-it_{2 - tr_1}}x(t_2) dt_2 dt_1 ds.$$ 

For an explicit example we take the kernel function $K(x) = e^{-a|x|} + \frac{\delta(x)}{2}$ with $a > 0$, where $\delta(x)$ denotes the Dirac delta function. So the convolution operator $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is defined by

$$F(f)(x) = \int_{\mathbb{R}} (e^{-a|x-y|} + \delta(x - y)) f(y) dy = \int_{\mathbb{R}} e^{-a|x-y|} f(y) dy + f(x).$$

The Fourier transform equals $FK(s) = \frac{1}{\sqrt{2\pi}} \left( \frac{-2a}{a^2 + s^2} + \frac{1}{2} \right)$ and takes values in the interval $(\frac{1}{2\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}}(\frac{a}{2} + \frac{1}{2}))$. So $h(s) = \sqrt{2\pi}FK(s) \in (\frac{1}{2}, 1)$, if $a > 4$, and the first assumption of the above corollary is fulfilled.

### 4. Preliminary results

Characiejus and Račkauskas (2013, 2014) investigate a central and a functional central limit theorem for a linear process ($X_k$)$_{k \in \mathbb{Z}}$ in form of (1) with values in the real Hilbert space $L_2(\mu)$ of square-integrable real-valued functions. We extend their result to the complex Hilbert space of square-integrable complex-valued functions denoted by $L_2(\mu, \mathbb{C})$ with inner product

$$\langle f, g \rangle = \int_{\mathbb{S}} f(s) \overline{g(s)} \mu(ds), \quad f, g \in L_2(\mu, \mathbb{C}).$$

With this in mind, we choose a complex-valued multiplication operator defined by $Df = \{ d(s)f(s) | s \in \mathbb{S} \}$ for each $f \in L_2(\mu, \mathbb{C})$ with $d : \mathbb{S} \to \mathbb{C}$ and consider the process (1) with

$$(8) \quad u_j = (j + 1)^{-D}. \quad \text{We denote} \quad h(s) = \frac{1}{2} \left( d(s) + \overline{d(s)} \right) \quad \text{and introduce the notations}$$

$$\sigma(r, s) := E(\varepsilon_0(r)\varepsilon_0(s)), \quad \sigma^2(s) := E|\varepsilon_0(s)|^2, \quad r, s \in \mathbb{S}.$$ 

We start with the central limit theorem.
Theorem 4.1. Suppose that \( h(s) \in (\frac{1}{2}, 1) \) for each \( s \in \mathbb{S} \) and that the integrals
\[
\int_{\mathbb{S}} \frac{\sigma^2(s)}{(1-h(s))^2} \mu(ds) \quad \text{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(s)}{(1-h(s))(2h(s)-1)} \mu(ds)
\]
are finite. Then
\[
n^{-H} S_n \xrightarrow{D} G \quad \text{as} \quad n \to \infty,
\]
in \( L_2(\mu, \mathbb{C}) \) with \( H = \frac{3}{2} I - D \). The \( L_2(\mu, \mathbb{C}) \)-valued random variable \( G \) is Gaussian with zero mean and covariance
\[
E(G(r)G(s)) = \frac{c(r, s) + \bar{c}(s, r)}{(3-d(r, s))(2-d(r, s))} \sigma(r, s),
\]
with \( c(r, s) = \text{Beta}(1-d(r), d(r, s) - 1) \), see (5).

The proof of this theorem may be found in the appendix.

Before continuing with the functional central limit theorem, we introduce the function \( V : \mathbb{T} \times \mathbb{T} \to \mathbb{C} \) with \( \mathbb{T} = \mathbb{S} \times \mathbb{R}_{\geq 0} \) and

\[
(9) \quad V((r, t), (s, u)) = \frac{\sigma(r, s)}{(3-d(r, s))(2-d(r, s))} \left[ c(s, r) t^{3-d(r, s)} + c(r, s) u^{3-d(r, s)} - C(r, s; t-u) |t-u|^{3-d(r, s)} \right],
\]
where
\[
C(r, s; t) = \begin{cases} c(r, s), & \text{if } t < 0 \\ c(s, r), & \text{if } t > 0. \end{cases}
\]

Especially there exists a Gaussian stochastic process \( \mathcal{G} = \{ \mathcal{G}(s, t) | (s, t) \in \mathbb{T} \} \) with zero mean and covariance function \( V \). For more details see Lemma A.10.

Theorem 4.2. Suppose that \( h(s) \in (\frac{1}{2}, 1) \) for each \( s \in \mathbb{S} \), the integrals
\[
E \left[ \int_{\mathbb{S}} \frac{|\epsilon_0(v)|^2}{(1-h(v))^2} \mu(dv) \right]^\frac{p}{2} \quad \text{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(v)}{(1-h(v))(2h(v)-1)} \mu(dv)
\]
are finite and either \( p = 2 \) and \( \bar{h} = \text{ess sup}_{s \in \mathbb{S}} h(s) < 1 \) or \( p > 2 \). Then
\[
n^{-H} \zeta_n \xrightarrow{D} G \quad \text{as} \quad n \to \infty
\]
in \( C([0, 1], L_2(\mu, \mathbb{C})) \) with \( H = \frac{3}{2} I - D \). The process \( G = \{ G(s, t) | (s, t) \in \mathbb{S} \times [0, 1] \} \) is a restriction to \( \mathbb{S} \times [0, 1] \) of the process \( \mathcal{G} \).

Again we refer to the appendix for the proof details.

5. PROOFS OF THE MAIN RESULTS

We start with some general results about the process \( (X_k)_{k \in \mathbb{Z}} \) given by (1) with \( u_j = (j+1)^{-N} \). First, we need to show the convergence of the series. Therefore we rewrite the process \( (X_k)_{k \in \mathbb{Z}} \) with the help of (3) and obtain

\[
X_k = \sum_{j=0}^{\infty} (j+1)^{-N} \epsilon_{k-j} = \sum_{j=0}^{\infty} (j+1)^{-U^*DU} \epsilon_{k-j} = \sum_{j=0}^{\infty} U^* (j+1)^{-D}(U \epsilon_{k-j})
\]
To avoid confusion we denote the inner product of $L^2(\mu, \mathbb{C})$-space as $\langle \cdot, \cdot \rangle_2$ and the corresponding norm as $\| \cdot \|_2$.

The series of operator norms $\sum_{j=0}^{\infty} \| (j+1)^{-N} \|^\text{op}$ diverges if and only if $\text{ess inf}_{s \in S} h(s) \leq 1$, because

$$
\| u_j \|_{\text{op}} = \sup \left\{ \frac{\| (j+1)^{-N} f \|_H}{\| f \|_H} \mid f \in H \text{ with } f \neq 0 \right\} = \sup \left\{ \frac{\| (j+1)^{-D}(Uf) \|_2}{\| Uf \|_2} \mid Uf \in L_2(\mu, \mathbb{C}) \text{ with } Uf \neq 0 \right\} = (j+1)^{-\text{ess inf}_{s \in S} h(s)},
$$

where the last step follows, since the operator norm of the multiplication operator $(j+1)^{-D}$ is known. Therefore we refer to the appendix, especially to (21).

We define the process $(Z_k)_{k \in \mathbb{Z}}$ by

$$
Z_k = \sum_{j=0}^{\infty} (j+1)^{-D}(U\epsilon_{k-j}). \tag{11}
$$

Our aim is to apply the results of Theorems 4.1 and 4.2 to the new process $(Z_k)_{k \in \mathbb{Z}}$. With this in mind, we prove that the series $(U\epsilon_k)_{k \in \mathbb{Z}}$ fulfills the assumptions. Since $(\epsilon_k)_{k \in \mathbb{Z}}$ is a sequence of independent, identically distributed random variables with values in $\mathbb{H}$ and $U : \mathbb{H} \to L_2(\mu, \mathbb{C})$ is a unitary operator, $(U\epsilon_k)_{k \in \mathbb{Z}}$ is a sequence of $L_2(\mu, \mathbb{C})$-valued random variables.

Moreover, the expected value is zero since $E(U\epsilon_0) = U(E\epsilon_0) = 0$ holds.

We still need to verify the interchangeability of the expected value and the unitary operator. Referring to properties of expectation in Bosq (2000) about the interchange of an operator and the expectation of a Hilbert space-valued random variable, it suffices to prove if $\epsilon_0 \in L_H^1(P) := \{ X \|\|X\|_1 = E\|X\| < \infty \}$. This easily follows from

$$
E\|\epsilon_0\| = E \left( \|\epsilon_0\|^2 \right)^{\frac{1}{2}} \leq (E\|\epsilon_0\|^2)^{\frac{1}{2}} < \infty.
$$

Using unitarity of $U$, we get the finite second moments

$$
E\|U\epsilon_0\|^2 = E\|\epsilon_0\|^2 < \infty.
$$

So, referring to the assumptions in Theorems 2.1 and 2.2, the process $(Z_k)_{k \in \mathbb{Z}}$ fulfills the assumptions of Theorems 4.1 and 4.2, and we are able to apply Lemma A.1. Therefore, the process converges in $L_2(\mu, \mathbb{C})$ if and only if $h(s) > \frac{1}{2}$ for $\mu$-almost all $s \in S$ and if the integral

$$
\int_S \frac{\sigma^2 U(s)}{(1-h(s))^2} \mu(ds).
$$

is finite. The almost sure convergence and boundedness of the unitary operator enables the interchangeability of the series and the hermitian adjoint of $U$, so we obtain

$$
X_k = U^* \left( \sum_{j=0}^{\infty} (j+1)^{-D}(U\epsilon_{k-j}) \right) \tag{12}
$$

and its convergence.

The rest of the section is divided into two parts. First we prove the central limit theorem, secondly the functional central limit theorem. Additionally we investigate a self-similar Gaussian process with values in $\mathbb{H}$.

Before we start let us recall some properties of random variables with values in abstract
With reference to Ledoux and Talagrand (1991) a random variable $Y$ with values in $\mathbb{H}$ is called Gaussian, if for any continuous linear function $f : \mathbb{H} \to \mathbb{C}$, $f(Y)$ is a Gaussian random variable. A stochastic process $(Y_t)_{t \in T}$ with values in $\mathbb{H}$ is Gaussian, if each linear combination $\sum_{i=1}^n a_i Y_{t_i}$ is a Gaussian random element, i.e. for each $n \geq 1$, $a_1, ..., a_n$ in $\mathbb{C}$ and $t_1, ..., t_n$ in $T$.  

Proof of Theorem 2.1. As announced above we want to apply Theorem 4.1 to the process $(Z_k)_{k \in \mathbb{Z}}$ defined by (11). Under our assumptions we get the convergence in distribution of the partial sums $S_n = \sum_{k=1}^n Z_k$ normalized by $n^{-\frac{3}{2}(1-D)}$ to a Gaussian zero mean process $\tilde{G} = \{\tilde{G}(s)|s \in S\}$ with covariance function

$$E(\tilde{G}(r)\tilde{G}(s)) = \frac{c(r, s) + c(s, r)}{(2 - d(r, s))(3 - d(r, s))} \sigma_U(r, s).$$

In other words

$$n^{-\frac{3}{2}(1-D)} \tilde{S}_n \xrightarrow{D} \tilde{G} \quad \text{as } n \to \infty \text{ in } L_2(\mu, \mathbb{C}).$$

Rewriting the expression $n^{-\frac{3}{2}(1-N)} S_n$ with $S_n = \sum_{k=1}^n X_k$ and help of the spectral theorem, we get

$$n^{-\frac{3}{2}(1-N)} S_n = U^* (n^{D-\frac{3}{2}}(US_n)) = U^* \left( n^{D-\frac{3}{2}} \left( \sum_{k=1}^n (US_k) \right) \right)$$

$$= U^* \left( n^{-\frac{3}{2}(1-D)} \sum_{k=1}^n Z_k \right) = U^* \left( n^{-\frac{3}{2}(1-D)} \tilde{S}_n \right).$$

Denote $z_n := n^{\frac{3}{2}-D}$. We use the continuous mapping theorem and the fact that $U^* : L_2(\mu, \mathbb{C}) \to \mathbb{H}$ is a unitary operator, i.e. it is bounded and so particularly continuous. Applying $U^* : L_2(\mu, \mathbb{C}) \to \mathbb{H}$ to the convergence (14), we get

$$U^* \left( n^{-\frac{3}{2}(1-D)} \tilde{S}_n \right) \xrightarrow{D} U^* \tilde{G} \quad \text{as } n \to \infty \text{ in } \mathbb{H}.$$  

So the limit process is of the form $G = \{U^* \tilde{G}\}$ with values in $\mathbb{H}$. Referring to Theorem 4.1 $\{\tilde{G}(s)|s \in S\}$ is Gaussian with values in $L_2(\mu, \mathbb{C})$, i.e. $\tilde{f}(\tilde{G})$ is a complex-valued Gaussian random element for each continuous, linear function $\tilde{f} : L_2(\mu, \mathbb{C}) \to \mathbb{C}$. So

$$f(G) = f(U^* \tilde{G}) = (f \circ U^*)(\tilde{G})$$

is Gaussian for each continuous, linear function $f : \mathbb{H} \to \mathbb{C}$, since $f \circ U^* : L_2(\mu, \mathbb{C}) \to \mathbb{C}$.

In the final step of the proof we calculate the covariance operator of the limit process. In general the covariance operator $C_G : \mathbb{H} \to \mathbb{H}$ is given by $C_G(x) = E((G,x)\overline{G})$ (see Bosq (2000)). An alternative definition says that $C_G$ is the covariance operator of $G$ if and only
if \( E(\langle G, x \rangle, y, G) = \langle x, C_G y \rangle \) for each \( x, y \in \mathbb{H} \). Therefor we use equality (13) and obtain

\[
E(\langle G, x \rangle, y, G) = E(\langle Ux, \tilde{G} \rangle, y, G) = E \left( \int_\mathcal{S} (Ux)(r)G(r)\mu(dr) \int_\mathcal{S} \tilde{G}(s)Uy(s)\mu(ds) \right)
\]

\[
= \int_\mathcal{S} \int_\mathcal{S} (Ux)(r)(Uy)(s)E(\tilde{G}(r)\tilde{G}(s))\mu(dr)\mu(ds)
\]

\[
= \langle Ux, \int_\mathcal{S} E(\tilde{G}(\cdot)\tilde{G}(s))(Uy)(s)\mu(ds) \rangle_2
\]

\[
= \langle x, U^* \int_\mathcal{S} \left( \frac{c(r, s) + c(s, r)}{(2 - d(r, s))(3 - d(r, s))} \sigma_U(r, s) \right) (Uy)(s)\mu(ds) \rangle.
\]

The assumed interchangeability of the expected value and the integrals easily follows from Fubini. To prove this, we first apply Lemma A.7 to \( n^{-(\frac{d}{2} - D)}\tilde{S}_n \), then the inequality also holds for the limit process \( \tilde{G} \), i.e.

\[
E|\tilde{G}(s)|^2 \leq \left( \frac{\sigma_U^2(s)}{(1 - h(s))^2} + \frac{\sigma_U^2(s)}{(1 - h(s))(2h(s) - 1)} \right) =: \tilde{g}(s).
\]

Using Hölder’s inequality yields

\[
\int_\mathcal{S} \int_\mathcal{S} E[(Ux)(r)(Uy)(s)\tilde{G}(r)\tilde{G}(s)]\mu(dr)\mu(ds)
\]

\[
\leq \int_\mathcal{S} \int_\mathcal{S} |(Ux)(r)(Uy)(s)|\tilde{g}(r)^{\frac{1}{2}}\tilde{g}(s)^{\frac{1}{2}}\mu(dr)\mu(ds)
\]

\[
\leq \left( \int_\mathcal{S} |(Ux)(r)|^2\mu(dr) \right)^{\frac{1}{2}} \left( \int_\mathcal{S} |(Uy)(s)|^2\mu(ds) \right)^{\frac{1}{2}} \left( \int_\mathcal{S} |\tilde{g}(s)|\mu(ds) \right)^{\frac{1}{2}}
\]

\[
\leq \|x\| \|y\| \int_\mathcal{S} |\tilde{g}(s)|\mu(ds) < \infty.
\]

Under our assumptions in Theorem 4.1 the function \( \tilde{g} \) is integrable and it follows the assertion. \( \square \)

**Proof of Theorem 2.2.** First we rewrite the piecewise linear function using (12)

\[
S_{[nt]} + \{nt\}X_{[nt]+1} = \sum_{k=1}^{[nt]} U^*Z_k + \{nt\}U^*Z_{[nt]+1}
\]

\[
= U^* \left( \sum_{j=-\infty}^{[nt]+1} a_{nj}(t)U\xi_j \right) := U^*\tilde{\zeta}_n(t)
\]

with \( a_{nj}(t) = \sum_{k=1}^{[nt]} v_{k-j} + \{nt\}v_{[nt]+1-j} \) and

\[
v_j = \begin{cases} 
(j + 1)^{-D} & \text{if } j \geq 0 \\
0 & \text{if } j < 0
\end{cases}
\]

We consider the sequence of piecewise linear functions \( \zeta_n \) and the stochastic Process \( G \) as random elements in the separable Banach space \( C([0, 1], \mathbb{H}) \). To prove the theorem, we have
to show the convergence of the finite-dimensional distributions and tightness.

Analogously to the proof of the central limit theorem we first use the fact that the process $(Z_k)_{k \in \mathbb{Z}}$ fulfills the assumptions of Theorem 4.2. So the process $(\tilde{\zeta}_n(t))_{n \in \mathbb{N}}$ normalized by $z_n = n^{3-D}$ converges to a Gaussian process $\tilde{G} = \{\tilde{G}(s,t) | (s,t) \in \mathbb{S} \times [0,1]\}$ with zero mean and covariance function $V_U$ in $C([0,1],L_2(\mu,\mathcal{C}))$. As stated in section 4 the process $\tilde{G}$ is defined as a restriction of a Gaussian stochastic process $\tilde{G} = \{\tilde{G}(s,t) | (s,t) \in \mathbb{S} \times [0,\infty)\}$.

5.0.1. **Convergence of the finite-dimensional distributions.** The statements we have shown for the proof of Theorem 4.2 are applicable to the process $(\tilde{\zeta}_n(t))_{n \in \mathbb{N}}$ defined in (16) since $(Z_k)_{k \in \mathbb{Z}}$ fulfills the conditions in Theorem 4.2. So using the convergence (33) yields

$$z_n^{-1} \tilde{\zeta}_n^q := (z_n^{-1} \tilde{\zeta}_n(t_1), ..., z_n^{-1} \tilde{\zeta}_n(t_q)) \xrightarrow{D} (\tilde{G}(t_1), ..., \tilde{G}(t_q)) =: \tilde{G}^q(n) \quad \text{as} \quad n \to \infty$$

in $L_2^q(\mu,\mathcal{C})$ for each $q \in \mathbb{N}$ and $t_1, ..., t_q \in [0,1]$. In other words

$$z_n^{-1} \tilde{\zeta}_n^q \xrightarrow{D} \tilde{G}^q(n) \quad \text{as} \quad n \to \infty.$$

To show the convergence of the finite-dimensional distributions we have to prove

$$n^{-H} \zeta_n^q := (n^{-H} \zeta_n(t_1), ..., n^{-H} \zeta_n(t_q)) \xrightarrow{D} (G(t_1), ..., G(t_q)) =: G^q(n) \quad \text{as} \quad n \to \infty$$

in $\mathbb{H}^q$ for each $q \in \mathbb{N}$ and $t_1, ..., t_q \in [0,1]$. Using the known notations and (16) this is equivalent to

$$(U^* (z_n^{-1} \zeta_n(t_1)), ..., U^* (z_n^{-1} \zeta_n(t_q))) \xrightarrow{D} (G(t_1), ..., G(t_q)) \quad \text{as} \quad n \to \infty$$

in $\mathbb{H}^q$, since $n^{-H} = U^* z_n^{-1} U$. Defining the mapping

$$\hat{U}^*: \begin{cases} L_2^q(\mu,\mathcal{C}) & \to \mathbb{H}^q \\ (g_1, ..., g_q) & \mapsto (U^* g_1, ..., U^* g_q) \end{cases},$$

it is possible to rewrite the convergence condition to

$$\hat{U}^* (z_n^{-1} \zeta_n(t_1), ..., z_n^{-1} \zeta_n(t_q)) \xrightarrow{D} (G(t_1), ..., G(t_q)) \quad \text{as} \quad n \to \infty,$$

in other terms

$$\hat{U}^* \zeta_n^q \xrightarrow{D} G^q(n) \quad \text{as} \quad n \to \infty.$$ Analogously to the proof of the central limit theorem, we get the convergence statement, using the continuous mapping theorem.

$$\hat{U}^* \zeta_n^q \xrightarrow{D} \hat{U}^* \tilde{G}^q(n) \quad \text{as} \quad n \to \infty.$$ So, the limit process $G$ is of the form $U^* \hat{G}$.

5.0.2. **Tightness.** In 1968, Billingsley establishes sufficient conditions for tightness for a sequence of random elements with values in $C([0,1],\mathbb{R})$. Referring to Račkauskas and Suquet (2011), we use the following extension to the space $C([0,1],\mathbb{H})$.

**Proposition 5.1.** Let $\mathbb{H}$ be a separable Hilbert space. A sequence of random elements $(Y_n)_{n \geq 1}$ with values in $C([0,1],\mathbb{H})$ is tight if

(i) for every $t \in [0,1]$, $(Z_n(t))_{n \geq 1}$ is tight in $\mathbb{H}$.

(ii) there exist constants $\gamma \geq 0$, $\alpha > 1$ and a continuous increasing function $F : [0, 1] \to \mathbb{R}$, such that

$$P(||Y_n(t) - Y_n(u)|| > \lambda) \leq \lambda^{-\gamma}|F(t) - F(u)|^\alpha.$$  

The first point easily follows since the central limit theorem is still proved. In detail, since the process $n^{-H}S_n$ converges in distribution in $\mathbb{H}$, the sequence $\{n^{-H}\zeta_n(t)\}$ converges in distribution in $\mathbb{H}$ and using Prohorov, it is tight on $\mathbb{H}$ for each $t \in [0, 1]$

The second point is an implication of Lemma A.16. Using additionally the linearity of $U^*$, we obtain

$$E\|n^{-H}\zeta_n(t) - n^{-H}\zeta_n(u)\|^p = E\|U^*(z_n^{-1}\zeta_n(t)) - U^*(z_n^{-1}\zeta_n(u))\|^p$$

$$= E\|U^*(z_n^{-1}\zeta_n(t) - z_n^{-1}\zeta_n(u))\|^p$$

$$= E\|z_n^{-1}\zeta_n(t) - z_n^{-1}\zeta_n(u)\|_2^p$$

$$\leq C|t - u|^{(3-2\gamma)p},$$

where the last inequality follows with the help of the mentioned lemma and under the assumptions of Theorem 4.2. \hfill \Box

Properties of the process $\mathcal{G}$. Finally we show some properties of the process $\mathcal{G} = \{\mathcal{G}(t)|t \in \mathbb{R}_{\geq 0}\}$, especially that it is Gaussian, calculate the cross-covariances and show self-similarity.

The process is Gaussian, if

$$f\left(\sum_{i=1}^{n} a_i \mathcal{G}(t_i)\right)$$

is a Gaussian random element in $\mathbb{C}$ for each $n \geq 1$, $a_1, \ldots, a_n$ in $\mathbb{C}$ and $t_1, \ldots, t_n$ in $\mathbb{R}_{\geq 0}$ and also for all continuous, linear functions $f : \mathbb{H} \to \mathbb{C}$.

Using the linearity of the unitary operator $U^*$, we get

$$f\left(\sum_{i=1}^{n} a_i \mathcal{G}(t_i)\right) = f\left(\sum_{i=1}^{n} a_i U^* \tilde{\mathcal{G}}(t_i)\right) = (f \circ U^*)\left(\sum_{i=1}^{n} a_i \tilde{\mathcal{G}}(t_i)\right).$$

The composition $f \circ U^*$ is a mapping from $L_2(\mu, \mathbb{C})$ into the complex numbers. Since the process $\tilde{\mathcal{G}} = \{\tilde{\mathcal{G}}(s, t)|s, t \in \mathbb{S} \times \mathbb{R}_{\geq 0}\}$ is Gaussian, we get the assertion.

The following calculation yields the cross-covariance of the process $\mathcal{G}$.

$$E(\langle x, \mathcal{G} \rangle \langle y, \mathcal{G} \rangle) = E(\langle U x, \tilde{\mathcal{G}} \rangle_2 \langle \tilde{\mathcal{G}}, U y \rangle_2)$$

$$= \int_{\mathbb{S}} \int_{\mathbb{S}} \langle U x(r) \tilde{\mathcal{G}}(r, t) \tilde{\mathcal{G}}(s, u)\rangle \mu(dr) \mu(ds)$$

$$= \langle U x, \int_{\mathbb{S}} \tilde{V}_U((\cdot, t), (s, u)) \tilde{\mathcal{G}}(s, u) \mu(ds) \rangle_2$$

$$= \langle x, U^* \int_{\mathbb{S}} \tilde{V}_U((r, t), (s, u)) \tilde{\mathcal{G}}(s, u) \mu(ds) \rangle$$
It remains to show the interchangeability of the expected value and the integrals. Applying inequality (32) to the process \( \tilde{\mathcal{G}} \), we obtain

\[
(17) \quad E \left( \int_\Sigma |\tilde{\mathcal{G}}(r,t)\|^2 \mu(dr) \right) \leq \max\{t, t^2\} \left( \int_\Sigma \frac{\sigma^2(r)}{(1-h(r))^2} \mu(dr) + \int_\Sigma \frac{\sigma^2(r)}{1-h(r)(2h(r)-1)} \mu(dr) \right).
\]

With the help of Fubini’s theorem, the unitarity of \( U \) and Hölder’s inequality we get

\[
\begin{align*}
&\int_\Sigma \int_\Sigma E[(Ux)(r)(Uy)(s)|\tilde{\mathcal{G}}(r,t)|\tilde{\mathcal{G}}(s,u)]\mu(dr)\mu(ds) \\
&\leq \int_\Sigma \int_\Sigma |(Ux)(r)(Uy)(s)|E[|\tilde{\mathcal{G}}(r,t)|^2]^\frac{1}{2} (E[|\tilde{\mathcal{G}}(s,u)|^2]^\frac{1}{2} \mu(dr)\mu(ds) \\
&\leq ||x|| ||y|| \left( \int_\Sigma |E[\tilde{\mathcal{G}}(r,t)|^2| \mu(ds) \right)^{\frac{1}{2}} \left( \int_\Sigma |E[\tilde{\mathcal{G}}(s,u)|^2| \mu(ds) \right)^{\frac{1}{2}} < \infty.
\end{align*}
\]

The finiteness follows by using inequality (17) and the assumptions in theorem 2.2.

As a last step we show the existence of an operator self-similar process with values in \( \mathbb{H} \).

**Lemma 5.2.** The stochastic process \( \{\mathcal{G}(t)|t \in [0, \infty)\} \) is operator self-similar with scaling family \( \{a^H|a > 0\} \), where \( H \) is equal to \( \frac{3}{2}I - N \) and \( N \) is a normal operator.

**Proof.** It is necessary to prove

\[
\{\mathcal{G}(at)|t \in [0, \infty)\} \overset{f.d.d.}{=} a^H \{\mathcal{G}(t)|t \in [0, \infty)\}.
\]

Since both sides are Gaussian processes with zero mean it suffices to show that the covariance operators are equal.

\[
\langle E[(\mathcal{G}(at), f)\mathcal{G}(aw)], g \rangle = \langle E[(a^H\mathcal{G}(t), f)a^H\mathcal{G}(u)], g \rangle \text{ for each } f, g \in \mathbb{H}
\]

Using the decomposition into a unitary operator \( U^* \) and the process \( \tilde{\mathcal{G}} \) of which we still know the operator self-similarity, we get

\[
\begin{align*}
\langle E[(a^H\mathcal{G}(t), f)a^H\mathcal{G}(u)], g \rangle &= \langle E[(a^H(U^*\tilde{\mathcal{G}}(x,t)), f)a^H(U^*\tilde{\mathcal{G}}(y,u))], g \rangle \\
&= \langle E[(U^*(a^{\frac{3}{2}d(x)}\tilde{\mathcal{G}}(x,t)), f)U^*(a^{\frac{3}{2}d(y)}\tilde{\mathcal{G}}(y,u))], g \rangle \\
&= \langle E[(a^{\frac{3}{2}d}(\cdot,t), Uf)a^{\frac{3}{2}d}(\cdot,u)], g \rangle.
\end{align*}
\]

At this point we interchange the operator \( U^* \) and the expected value. The regularity will checked out later.

\[
\begin{align*}
&= \langle U^*E[(a^{\frac{3}{2}d}(\cdot,t), Uf)a^{\frac{3}{2}d}(\cdot,u)], g \rangle \\
&= \langle E[(a^{\frac{3}{2}d}(\cdot,t), Uf)a^{\frac{3}{2}d}(\cdot,u)], U^*g \rangle \\
&= \langle E[(a^{\frac{3}{2}d}(\cdot,t), \tilde{f})a^{\frac{3}{2}d}(\cdot,u)], g \rangle
\end{align*}
\]
Since $\tilde{f} := Uf$ and $\tilde{g} := \overline{Ug}$ in $L_2(\mu, \mathbb{C})$, the operator self-similarity of $\hat{G}$ (see Lemma A.12) is applicable and therefore
\[
\langle E[\langle \hat{G}(\cdot, at), \tilde{f} \rangle | \hat{G}(s, au)] \rangle, \tilde{g} \rangle.
\]
As announced we prove the interchangeability of the operator $U^*$ and the expected value. Define $b(t) := \langle a^{-\frac{d}{2}-(\cdot)} \hat{G}(\cdot, t), Uf \rangle_2$. We have to prove $b(t) a^{-\frac{d}{2}-(y)} \hat{G}(y, u) \in L^1_{L_2(\mu, C)}(P)$.
\[
E\|b(t) a^{-\left(\frac{d}{2}+(\cdot)\right)} \hat{G}(\cdot, u)\|_2 = E\left( \int_S |b(t) a^{\frac{d}{2}-(s)} \hat{G}(s, u)|^2 \mu(ds) \right)^{\frac{1}{2}}
\]
\[
\leq \left( E \int_s |a^{\frac{d}{2}-(r)} \hat{G}(r, t)|^2 \mu(dr) \int_S |(Uf)(r)|^2 \mu(dr) \right)^{\frac{1}{2}} \left( E \int_s |a^{\frac{d}{2}-(s)} \hat{G}(s, u)|^2 \mu(ds) \right)^{\frac{1}{2}}
\]
\[
= \left( \int_s |(Uf)(r)|^2 \mu(dr) \right)^{\frac{1}{2}} \left( E \int_s |a^{\frac{d}{2}-(r)} \hat{G}(r, t)|^2 \mu(dr) \right)^{\frac{1}{2}} \left( E \int_s |a^{\frac{d}{2}-(s)} \hat{G}(s, u)|^2 \mu(ds) \right)^{\frac{1}{2}}
\]
\[
\leq \|f\| \max\{a^{\frac{d}{2}}, a\} \left( \int_s E|\hat{G}(r, t)|^2 \mu(dr) \right)^{\frac{1}{2}} \left( \int_s E|\hat{G}(s, u)|^2 \mu(ds) \right)^{\frac{1}{2}} < \infty
\]
We used Hölder’s inequality and (17).

\[\square\]

**APPENDIX A. PROOFS OF THE PRELIMINARY RESULTS**

The proofs of theorems 4.1 and 4.2 are closely related to the proofs in Characiejus and Račkauskas (2013, 2014). So we will focus on the passages which differ.

We start with some preliminaries. It is well known that the Beta function can be expressed as
\[
\text{Beta}(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}.
\]
Using this identity and $d(r, s) = d(r) + \overline{d(s)}$, we obtain
\[
\text{Beta}(1 - d(r), d(r, s) - 1) = \frac{\Gamma(d(r, s) - 1) \text{Beta}(d(r), 1 - d(r))}{\Gamma(d(r)) \Gamma(d(s))}
\]
For simplicity we denote $h(r, s) = h(r) + h(s)$. Recall that we write $h(s)$ for the real part of $d(s)$. Now, let $\tilde{h}(s)$ denote the imaginary part of $d(s)$. We define $c(s) := c(s, s)$, for the definition of $c(s, s)$ see (18). Using $a^{-ih(s)} = \exp(-i\tilde{h}(s) \log(a))$ and Euler’s formula yields
\[
\frac{1}{2} \left( c(s) + \overline{c(s)} \right) = \int_0^\infty (x(x + 1))^{-h(s)} \cos \left( \log \left( \frac{x}{x + 1} \right)^{-\tilde{h}(s)} \right) dx
\]
\[
\leq \int_0^\infty (x(x + 1))^{-h(s)} dx =: c_h(s).
\]
Continuing estimation of the right hand side gives
\[
c_h(s) \leq \frac{1}{1 - h(s)} + \frac{1}{2h(s) - 1}.
\]
Since \((u_j)_{j \in \mathbb{Z}}\) given by (8) are multiplication operators in \(L(L_2(\mu, \mathbb{C}))\), we have the operator norm
\[
\|(j + 1)^{-D}\|_{\text{op}} = (j + 1)^{-\text{ess inf}_{s \in \mathbb{S}} h(s)}
\]
referring to Comway (1994). So the series of operator norms \(\sum_{j=0}^{\infty} \|(j + 1)^{-D}\|_{\text{op}}\) diverges if and only if \(\text{ess inf}_{s \in \mathbb{S}} h(s) \leq 1\).

**Lemma A.1.** The series (1) with \(u_j\) as in (8) converges in mean square if and only if \(h(s) > \frac{1}{2}\) for \(\mu\)-almost all \(s \in \mathbb{S}\) and if the integral
\[
\int_{\mathbb{S}} \frac{\sigma^2(s)}{2h(s) - 1} \mu(ds)
\]
is finite. Then the series converges also almost surely.

**Proof.** We want to apply Cauchy’s criterion. Defining \(X_{nk} = \sum_{j=0}^{n} (j + 1)^{-D} \varepsilon_{k-j}\), we have to prove
\[
\lim_{M, N \to \infty} E\|X_{nk} - X_{Mk}\|^2 = 0.
\]
Since \((j + 1)^{-D} f = \{(j + 1)^{-d(s)} f(s) : s \in \mathbb{S}\}\) for each \(j \geq 1\) and \(f \in L_2(\mu, \mathbb{C})\)
\[
E\|X_{nk} - X_{Mk}\|^2 = \sum_{j=M+1}^{N} \int_{\mathbb{S}} (j + 1)^{-2h(s)} \sigma^2(s) \mu(ds).
\]
Since
\[
\frac{1}{2h(s) - 1} \leq \sum_{j=1}^{\infty} j^{-2h(s)} \leq 1 + \frac{1}{2h(s) - 1}
\]
we get
\[
\int_{\mathbb{S}} \frac{\sigma^2(s)}{2h(s) - 1} \mu(ds) \leq \int_{\mathbb{S}} \sum_{j=0}^{\infty} (j + 1)^{-2h(s)} \sigma^2(s) \mu(ds) \leq E\|\varepsilon_0\|^2 + \int_{\mathbb{S}} \frac{\sigma^2(s)}{2h(s) - 1} \mu(ds).
\]
Mean square convergence implies convergence in probability and applying the Lévy-Itô-Nisio theorem (Ledoux and Talagrand (1991)) almost sure convergence follows. \(\square\)

We need a generalization of the geometric sum
\[
\sum_{j=m_2}^{m_1} (m_3 - j) e^{-jx} = \frac{(m_3 - a_1)e^{(2-m_1)x} - (m_3 - m_2 - 1)e^{(1-m_2)x} - (m_3 - m_1 + 1)e^{(1-m_1)x} + (m_3 - m_2)e^{-m_2x}}{(e^x - 1)^2}
\]
with \(m_i \in \mathbb{N}\) for each \(i \in \{1, 2, 3\}\) (see Rakauskas and Suquet (2011)).

We calculate the autocovariance function of \((X_k(r))_{k \geq 1}\) and \((X_k(s))_{k \geq 1}\), which are stationary for fixed \(r, s\). Using \((j + 1)^{-D} f = \{(j + 1)^{-d(s)} f(s) : s \in \mathbb{S}\}\) we get
\[
\gamma_h(r, s) := E(X_0(r)X_h(s)) = \sigma(r, s) \sum_{j=0}^{\infty} (j + 1)^{-d(r)} (j + 1 + h)^{-d(s)}.
\]
Before we present the proofs we cite some helpful theorems. First we refer to Cremers and Kadelka (1986) for a theorem, which gives sufficient conditions for the weak convergence of random sequences with paths in $L_p(\mu, \mathbb{B})$.

**Theorem A.2.** H. Cremers and D. Kadelka Let $(Y_n)_{n \in \mathbb{N}}$ be a sequence of stochastic processes with paths in $L_p(\mu, \mathbb{B})$. Then $Y_n \xrightarrow{D} Y$ as $n \to \infty$, if

1. the finite-dimensional distributions of $Y_n$ converge weakly to those of $Y$ almost everywhere, i.e.
   $$(Y_n(s_1), \ldots, Y_n(s_k)) \xrightarrow{D} (Y(s_1), \ldots, Y(s_k))$$
   for each $k \in \mathbb{N}$ and almost all $s_1, \ldots, s_k$ in $\mathbb{S}$,
2. $\lim_{n \to \infty} E\|Y_n(s)\|^p = E\|Y(s)\|^p$ for each $s \in \mathbb{S}$ and $n \in \mathbb{N}$,
3. there exists a square $\mu$-integrable function $f : \mathbb{S} \to \mathbb{R}$ such that $E\|Y_n(s)\|^p \leq f(s)$ for each $s \in \mathbb{S}$.

Next we refer to Račkauskas and Suquet (2011) for a theorem, which gives sufficient conditions for the weak convergence of linear processes with values in a Hilbert space. Let $\mathbb{H}$ and $\mathbb{E}$ be two Hilbert spaces and $(\varepsilon_j)_{j \in \mathbb{Z}}$ a sequence of independent, identically distributed random variables with values in $\mathbb{E}$. Define $(X_n)_{n \in \mathbb{Z}}$ with

$$X_n = \sum_{j \in \mathbb{Z}} A_{nj} \varepsilon_j$$

and $A_{nj} \in L(\mathbb{H}, \mathbb{E})$. Now, we define a second process $(Y_n)_{n \in \mathbb{Z}}$ with

$$Y_n = \sum_{j \in \mathbb{Z}} A_{nj} \tilde{\varepsilon}_j,$$

where $A_{nj}$ is the same operator as above and $\tilde{\varepsilon}_j$ is a sequence of Gaussian random elements with values in $\mathbb{E}$, zero-mean and the same covariance operator as $\varepsilon_j$.

Before we state Račkauskas and Suquet’s lemma we need a definition of a metric on the space of probability measures on Hilbert spaces.

**Definition A.3.** Let $X, Y$ be $\mathbb{H}$-valued random variables, then the metric $\varrho_k$ is defined by

$$\varrho_k(X, Y) = \sup_{f \in F_k} \left| Ef(X) - Ef(Y) \right|,$$

where $F_k$ is the set of all $k$ times Fréchet differentiable functions $f : \mathbb{H} \to \mathbb{R}$ such that $\sup_{x \in \mathbb{H}} |f^{(i)}(x)| \leq 1$ for $i = 0, \ldots, k$.

No we are able to present the lemma.

**Lemma A.4.** If the conditions

$$\lim_{n \to \infty} \sup_{j \in \mathbb{Z}} \|A_{nj}\|_{op} = 0 \quad \text{and} \quad \limsup_{n \to \infty} \sum_{j \in \mathbb{Z}} \|A_{nj}\|^2_{op} < \infty$$

are fulfilled, then

$$\lim_{n \to \infty} \varrho_3(X_n, Y_n) = 0.$$

Referring to Ginè and León (1980), the processes have the same convergence behaviour if $\lim_{n \to \infty} \varrho_3(X_n, Y_n) = 0$, since the metric induces the weak topology on the set of probability measures on $\mathbb{H}$.
Proof of theorem 4.1. We start with the main part of the proof. At the same time this is the part, which completely differs from the real-valued case. We calculate the limit behaviour of the cross-covariances of the partial sums $S_n$.

Lemma A.5. If $h(r) \in (\frac{1}{2}, 1)$ and $h(s) \in (\frac{1}{2}, 1)$, then

$$ \lim_{n \to \infty} n^{d(r,s)-3} E(S_n(r)S_n(s)) = \frac{c(r, s) + c(s, r)}{(3 - d(r, s))(2 - d(r, s))} \sigma(r, s). $$

Proof. Changing the order of summation, we have

$$ E(S_n(r)S_n(s)) = n^3 + \sum_{k=1}^n (n - k) \gamma_k(r, s) + \sum_{k=1}^n (n - k) \gamma_k(s, r). $$

Adding the normalization, we calculate the limit for each summand. First

$$ \lim_{n \to \infty} n^{d(r,s)-3} n^3 \gamma_0(r, s) = 0, $$

since $|n^{d(r,s)-2}| \leq n^{h(r,s)-2}$ and $h(r, s) - 2 \in (-1, 0)$.

To prove convergence of the second summand, we calculate the Laplace transform $\mathcal{L}$ of the power function $f : [0, \infty) \to \mathbb{C}$, $t \mapsto t^a$.

$$ \mathcal{L}(t^a)(s) = \int_0^\infty t^{a-1} e^{-st} dt = \frac{\Gamma(a)}{s^a} \quad \text{for} \ Re(s) > 0 \text{ and } Re(a) > 0. $$

Rearranging the terms, we get

$$ s^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-st} dt \quad \text{for} \ Re(s) > 0 \text{ and } Re(a) > 0. \tag{26} $$

Combining (23) and (26), and applying (22) we have

$$ n^{d(r,s)-3} \sum_{k=1}^n (n - k) \gamma_k(r, s) = \frac{n^{d(r,s)-3} \sigma(r, s)}{\Gamma(d(r)) \Gamma(d(s))} \sum_{k=1}^n (n - k) \int_0^\infty \int_0^\infty e^{(j+1)(x_1+x_2)} e^{-kx_2 x_1} x_1^{d(r)-1} x_2^{d(s)-1} dx_1 dx_2 $$

$$ = \frac{n^{d(r,s)-3} \sigma(r, s)}{\Gamma(d(r)) \Gamma(d(s))} \int_0^\infty \int_0^\infty \frac{1}{e^{x_1+x_2} - 1} \sum_{k=1}^n (n - k) e^{-kx_2} x_1^{d(r)-1} x_2^{d(s)-1} dx_1 dx_2 $$

$$ = \frac{n^{d(r,s)-3} \sigma(r, s)}{\Gamma(d(r)) \Gamma(d(s))} \int_0^\infty \int_0^\infty \frac{1}{e^{x_1+x_2} - 1} \left( (n - 1) e^{x_2} + e^{(1-n)x_2} - n x_1^{d(r)-1} x_2^{d(s)-1} \right) dx_1 dx_2. $$
Next, we substitute \( nx_1 = t_1, nx_2 = t_2 \) and take the limit by the dominated convergence theorem.

\[
\lim_{n \to \infty} \frac{\sigma(r, s)}{\Gamma(d(r))\Gamma(d(s))} \int_0^\infty \int_0^\infty \frac{(n-1)e^{\frac{t_2}{n}} + e^{(1-n)\frac{t_2}{n}} - n}{n^3(e^{\frac{t_2}{n}} - 1)^2} \frac{1}{t_1^{d(r)-1}t_2^{d(s)-1}} \, dt_1 dt_2
\]

\[
\sigma(r, s) \int_0^\infty \int_0^\infty \frac{t_2 + e^{-t_2} - 1}{t_2(t_1 + t_2)} \, dt_1 dt_2
\]

\[
= \frac{\sigma(r, s) B(d(r), 1 - d(r))}{\Gamma(d(r))\Gamma(d(s))} \frac{1}{3 - d(r, s)} \frac{1}{2 - d(r, s)} \Gamma(d(r, s) - 1) = \frac{\sigma(r, s) c(r, s)}{(3 - d(r, s))(2 - d(r, s))}
\]

After a repeated substitution, in this case with \( t_1 = xt_2 \) and integration by parts twice of the second integral we applied (26) again. The last step then follows from (18).

Analogously we obtain the limit of the third summand by interchanging \( r \) and \( s \) and complex conjunction.

Finally, we verify that the dominated convergence theorem is applicable. We have to prove the existence of an integrable function \( g(t_1, t_2) \), which fulfills

\[
\left| \frac{(n-1)e^{\frac{t_2}{n}} + e^{(1-n)\frac{t_2}{n}} - n}{n^3(e^{\frac{t_2}{n}} - 1)^2} \frac{1}{t_1^{d(r)-1}t_2^{d(s)-1}} \right| \leq g(t_1, t_2)
\]

for each \( n \geq 1 \). The numerator is obviously non-negative. The same holds for the denominator, since the function \( n(1 - e^{-\frac{t_2}{n}}) \) monotone in \( n \)

\[1 - e^{-t_2} \leq n(1 - e^{-\frac{t_2}{n}})\]

holds, it follows

\[
(n-1)e^{\frac{t_2}{n}} + e^{(1-n)\frac{t_2}{n}} - n \geq 0.
\]

Using the inequalities \( e^{-x} \geq 1 - x \) and \( x \leq e^x - 1 \) for each value \( x \in \mathbb{R} \), we obtain

\[
\frac{(n-1)e^{\frac{t_2}{n}} + e^{(1-n)\frac{t_2}{n}} - n}{n^3(e^{\frac{t_2}{n}} - 1)^2} \leq \frac{\frac{e^{\frac{t_2}{n}}}{n^2(e^{\frac{t_2}{n}} - 1)^2}}{n^3(e^{\frac{t_2}{n}} - 1)^2} \leq \frac{t_2 - 1 + e^{-t_2}}{t_1 + t_2} \leq \frac{t_2 - 1 + e^{-t_2}}{t_2(t_2 + t_1)}.
\]

In the final step, we have applied the inequality

\[
\frac{e^{x}}{n^2(e^{\frac{x}{n}} - 1)^2} \leq \frac{1}{x^2}.
\]

Using the power series representation we prove the inequality \( e^{\frac{x}{2n}} - e^{\frac{x}{n}} \leq e^{\frac{x}{n}} - 1 \) by

\[
e^{\frac{x}{2n}} = \sum_{k=0}^{\infty} \frac{(\frac{x}{2n})^k}{k!} = \sum_{j=1}^{\infty} \frac{\binom{\frac{x}{n}}{j}}{j!} \leq \sum_{j=1}^{\infty} \frac{(\frac{x}{n})^j}{j!} = e^{\frac{x}{n}} - 1.
\]
All in all we obtain
\[
\left| \frac{(n - 1)e^{t_2/n} + e^{(1-n)t_2/n} - n^{-1}(t_2)}{n^{3/(e^{t_2/n} - 1)^2}} \right| \leq \frac{t_2 - 1 + e^{-t_2}}{t_2^2(t_2 + t_1)} =: g(t_1, t_2)
\]
for each \( n \geq 1 \). The integrability follows from the rewritten limit above. \( \square \)

To prove the weak convergence we use theorem A.2. For the first point, we rewrite the process in form of (24). Denote
\[
v_j(s) = \begin{cases} (j + 1)^{-d(s)}, & \text{if } j \geq 0 \\ 0, & \text{if } j < 0 \end{cases}
\]
and rewrite the partial sums as
\[
S_n(s) = \sum_{k=1}^{n} \sum_{j=0}^{\infty} v_j(s) \varepsilon_{k-j}(s) = \sum_{j=-\infty}^{n+1} a_{nj}(s) \varepsilon_j(s)
\]
with \( a_{nj}(s) = \sum_{k=1}^{n} v_{k-j}(s) \). We define \( z_n(s) = n^{d(s)} - \frac{3}{2} \) and we consider for \( s_1, \ldots, s_p \in \mathbb{S} \) the sequence of random vectors
\[
(n^{-H}s_n(s_1), \ldots, n^{-H}s_n(s_p))^t = \sum_{j=-\infty}^{\infty} A_{nj} \varepsilon_j
\]
with \( A_{nj} = \text{diag}(z_n^{-1}(s_1)a_{nj}(s_1), \ldots, z_n^{-1}(s_p)a_{nj}(s_p)) \) and \( \varepsilon_j = (\varepsilon_j(s_1), \ldots, \varepsilon_j(s_p))^t \).

**Lemma A.6.** If \( h(s) \in (\frac{1}{2}, 1) \), the sequence of operators \( A_{nj} \) fulfills the conditions (25).

**Proof.** Since \( \|A_{nj}\|_{op} = \max_{1 \leq i \leq q} |n^{d(s_i)} - \frac{3}{2} a_{nj}(s_i)| \), first using the triangle inequality
\[
\sup_{j \in \mathbb{Z}} |n^{d(s_i)} - \frac{3}{2} a_{nj}(s)| \leq \sup_{j \in \mathbb{Z}} n^{h(s_i)} - \frac{3}{2} \sum_{k=1}^{n} |v_{k-j}(s)| \sim n^{h(s_i)} - \frac{3}{2} \frac{n^{1-h(s_i)}}{1 - h(s)}.
\]
Second using the relationship between the variances of the partial sums and the sequence \( a_{nj}(s) \)
\[
E|S_n(s)|^2 = \sigma^2(s) \sum_{j=-\infty}^{n+1} |a_{nj}(s)|^2,
\]
we obtain
\[
\sum_{j \in \mathbb{Z}} \|A_{nj}\|^2 = \sum_{j \in \mathbb{Z}} |n^{d(s_i)} - \frac{3}{2} a_{nj}(s)|^2 = n^{2h(s_i) - 3} \sum_{j \in \mathbb{Z}} |a_{nj}(s)|^2 = n^{2h(s_i) - 3} \frac{1}{\sigma^2(s)} E|S_n(s)|^2
\]
and using Lemma A.5
\[
\lim_{n \to \infty} n^{2h(s) - 3} \frac{1}{\sigma^2(s)} E|S_n(s)|^2 = \frac{c(s) + c(s)}{(3 - 2h(s))(2 - 2h(s))}.
\] \( \square \)
It remains to show the convergence of the process
\[(n^{-H}S_n(s_1), \ldots, n^{-H}S_n(s_p)) = \sum_{j=-\infty}^{\infty} A_{nj}\tilde{\epsilon}_j,\]
which follows from Lemma A.5.

Setting \(r = s\) in Lemma A.5, we get the proof of point (II) in theorem A.2.

The following lemma provides the proof of point (III).

**Lemma A.7.** There exists a \(\mu\)-integrable function \(g\) with \(|E|n^{-H}S_n(s)|^2| \leq g(s)\) for each \(s\) in \(S\) and \(n \in \mathbb{N}\), if the integrals
\[
\int_S \frac{\sigma^2(s)}{(1 - h(s))^2} \mu(ds) \quad \text{and} \quad \int_S \frac{\sigma^2(s)}{(1 - h(s))(2h(s) - 1)} \mu(ds)
\]
are finite.

**Proof.** Rewriting the variances and using the triangular inequality, we get
\[
|E|n^{-H}S_n(s)|^2| \leq n^{2h(s) - 3} \left(n|\gamma_0(s)| + \sum_{k=1}^{n} (n - k)|\gamma_k(s)| + \sum_{k=1}^{n} (n - k)|\gamma_k(s)|\right).
\]
The first two summands are approximable as
\[
n^{2h(s) - 3}n|\gamma_0(s)| \leq \sigma^2(s) \left(1 + \frac{1}{2h(s) - 1}\right)
\]
and
\[
n^{2h(s) - 3}\sum_{k=1}^{n} (n - k)|\gamma_k(s)| \leq n^{2h(s) - 3}\sigma^2(s)\sum_{k=1}^{n} (n - k)k^{1 - 2h(s)}c_h(s)
\]
\[
\leq \sigma^2(s)\frac{1}{2}\left(\frac{1}{(1 - h(s))^2} + \frac{1}{(1 - h(s))(2h(s) - 1)}\right),
\]
where the last inequality follows from (20). The approximation of the third one works analogously and altogether we obtain
\[
|E|n^{-H}S_n(s)|^2| \leq 2\left(\frac{\sigma^2(s)}{(1 - h(s))^2} + \frac{\sigma^2(s)}{1 - h(s))(2h(s) - 1)}\right) =: g(s).
\]

\(\square\)

The proof of the central limit theorem is completed. \(\square\)

**Proof of theorem 4.2.** We are interested in the convergence behaviour of the piecewise linear process \((\zeta_n(t))_{n \in \mathbb{N}}\) with
\[
\zeta_n(t) = S_{[nt]} + \{nt\}X_{[nt]+1} \quad \text{for} \quad t \in [0, 1].
\]
With reference to Characiejus and Račkauskas (2014) the expression may be represented as a linear combination of a sequence of operators and the random process \((\varepsilon_k)_{k \in \mathbb{Z}}\). Let
\[
v_j(s) = \begin{cases} (j + 1)^{-d(s)} & \text{if } j \geq 0 \\ 0 & \text{if } j < 0 \end{cases},
\]
then
\[ \zeta_n(s,t) = S_{\lfloor nt \rfloor}(s) + \{nt\}X_{\lfloor nt \rfloor+1}(s) = \sum_{j=-\infty}^{\lfloor nt \rfloor+1} a_{nj}(s,t) \varepsilon_j(s) \]

with
\[ a_{nj}(s,t) = \sum_{k=1}^{\lfloor nt \rfloor} v_{k-j}(s) + \{nt\}v_{\lfloor nt \rfloor+1-j}(s). \]

Next we show the relation between the limit of the normalised cross-covariances of the piecewise linear process \( \zeta_n \) and of the normalised cross-covariances of the partial sums \( S_n \).

**Lemma A.8.** If \( h(s) \in (\frac{1}{2}, 1) \) and \( h(r) \in (\frac{1}{2}, 1) \), then
\[ \lim_{n \to \infty} n^{d(r,s)-3} E(\zeta_n(r)\zeta_n(s,u)) = \lim_{n \to \infty} n^{d(r,s)-3} E(S_{\lfloor nt \rfloor}(r)S_{\lfloor nu \rfloor}(s)). \]

**Lemma A.9.** If \( h(s) \in (\frac{1}{2}, 1) \) and \( h(r) \in (\frac{1}{2}, 1) \), then
\[ \lim_{n \to \infty} n^{d(r,s)-3} E(S_{\lfloor nt \rfloor}(r)S_{\lfloor nu \rfloor}(s)) = V((r,t),(s,u)) \]
for \((r,t),(s,u) \in \mathbb{S} \times [0,1] \), where \( V \) is given in (9).

**Proof.** We assume \( t < u \) and supplement \( S_{\lfloor nt \rfloor}(s) \) in the expression of the cross-covariances. \( E(S_{\lfloor nt \rfloor}(r)S_{\lfloor nu \rfloor}(s)) = E(S_{\lfloor nt \rfloor}(r)S_{\lfloor nt \rfloor}(s)) + E(S_{\lfloor nt \rfloor}(r)(S_{\lfloor nu \rfloor}(s) - S_{\lfloor nt \rfloor}(s)) \)

Using Lemma A.5, we are able to calculate the limit of the first summand as
\[ \lim_{n \to \infty} n^{d(r,s)-3} E(S_{\lfloor nt \rfloor}(r)S_{\lfloor nt \rfloor}(s)) = \frac{c(r,s) + c(s,r)}{(3 - d(r,s))(2 - d(r,s))} \sigma(r,s)t^{3-d(r,s)}. \]

The second term may be separated into three sums, where \( m_n = \min(\lfloor nt \rfloor, \lfloor nu \rfloor - \lfloor nt \rfloor) \).
\[ = \sum_{k=1}^{m_n-1} k \gamma_k(r,s) + m_n \sum_{k=m_n}^{\lfloor nu \rfloor - m_n} \gamma_k(r,s) + \sum_{k=\lfloor nu \rfloor - m_n+1}^{\lfloor nu \rfloor-1} (\lfloor nu \rfloor - k) \gamma_k(r,s) \]

We remind of the representation of the autocovariance function, we already used in the proof of Lemma A.5 by using the Laplace transform.
\[ \gamma_k(r,s) = \int_0^\infty \int_0^\infty \frac{1}{e^{x_1+x_2}-1} e^{-kx_2} x_1^{d(r)-1} x_2^{d(r)-1} dx_1 dx_2 \]

With the help of (22) we get three expressions:
\[ G_n^1(x_2) := \sum_{k=1}^{m_n-1} k e^{-kx_2} = \frac{e^{x_2} - m_n e^{(2-m_n)x_2} - (1-m_n) e^{(1-m_n)x_2}}{(e^{x_2} - 1)^2} \]
\[ G_n^2(x_2) := m_n \sum_{k=m_n}^{\lfloor nu \rfloor - m_n} e^{-kx_2} = m_n \frac{e^{(2-m_n)x_2} - e^{(1-(\lfloor nu \rfloor-m_n))x_2} - e^{(1-m_n)x_2} + e^{(\lfloor nu \rfloor-m_n)x_2}}{(e^{x_2} - 1)^2} \]
\[ G_n^3(x_2) := \sum_{k=\lfloor nu \rfloor - m_n+1}^{\lfloor nu \rfloor-1} (\lfloor nu \rfloor - k) e^{-kx_2} \]
Combining our results, adding the normalization sequence and substituting $x_1 = \frac{1}{n}$ and $x_2 = \frac{t_2}{n}$, we obtain

\[
\int_0^\infty \int_0^\infty \frac{1}{c_{x_1, x_2} - 1} \left( G_n(x_2) + G_n^2(x_2) + G_n^3(x_2) \right) x_1 (d(r) - 1) x_2 (d(s) - 1) \, dx_1 \, dx_2
\]

\[
= n^{d(r,s) - 3} \frac{\sigma(r,s)}{\Gamma(d(r)) \Gamma(d(s))} \int_0^\infty \int_0^\infty \frac{1}{c_{x_1, x_2} - 1} \left( G_n^1(x_2) + G_n^2(x_2) + G_n^3(x_2) \right) x_1 (d(r) - 1) x_2 (d(s) - 1) \, dx_1 \, dx_2
\]

Let $m = \min(t, u - t)$, then we use the dominated convergence theorem and substitute with $t_1 = xt_2$.

\[
\int_0^\infty \int_0^\infty \frac{1}{c_{x_1, x_2} - 1} \left( G_n^1(x_2) + G_n^2(x_2) + G_n^3(x_2) \right) x_1 (d(r) - 1) x_2 (d(s) - 1) \, dx_1 \, dx_2
\]

In the last step we used integration twice by parts. Combining the limits and (30), we get

\[
\lim_{n \to \infty} n^{3-d(r,s)} E(S_{[nt]}(r)S_{[nu]}(s))
\]

\[
= \frac{\sigma(r,s)}{(3 - d(r,s))(2 - d(r,s))} \left[ (c(r,s) + c(s,r)) t^{3-d(r,s)} + c(r,s)(-m^{3-d(r,s)} + u^{3-d(r,s)} - (u - m)^{3-d(r,s)}) \right]
\]

in case of $t < u$. Analogously for $t > u$

\[
\lim_{n \to \infty} n^{3-d(r,s)} E(S_{[nt]}(r)S_{[nu]}(s))
\]

\[
= \frac{\sigma(r,s)}{(3 - d(r,s))(2 - d(r,s))} \left[ c(r,s) u^{3-d(r,s)} + c(s,r)(t^{3-d(r,s)} - (t - u)^{3-d(r,s)}) \right]
\]

follows. Both cases together provide

\[
\lim_{n \to \infty} n^{3-d(r,s)} E(S_{[nt]}(r)S_{[nu]}(s))
\]

\[
= \frac{\sigma(r,s)}{(3 - d(r,s))(2 - d(r,s))} \left[ c(r,s) t^{3-d(r,s)} + c(r,s) u^{3-d(r,s)} + C(r,s; t - u)|t - u|^{3-d(r,s)} \right].
\]
We still need to prove the interchangeability of the integrals and the limit, so, using the dominated convergence theorem, we have to show the existence of an integrable function $g : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that

$$\left| \frac{E_n^{t_2} - e^{(1-m_n)t_2/n} - e^{(m_n-[nu]+1)t_2/n} + e^{(1-[nu])t_2/n} t_1^{d(r)-1} t_2^{d(s)-1}}{n^2 (e^{t_2/n} - 1)^2} \right| \leq g(t_1, t_2)$$

for each $n \geq 1$. Under the assumption $t < u$ the numerator is obviously non-negative. The denominator is positive, since by case analysis in dependence of $m_n$

$$e^{t_2/n} - e^{(1-m_n)t_2/n} - e^{(m_n-[nu]+1)t_2/n} + e^{(1-[nu])t_2/n} \geq 0$$

$$\Leftrightarrow 0 \geq m_n - [nu]$$

$$\Leftrightarrow 0 \geq \begin{cases} \lfloor nt \rfloor - \lfloor nu \rfloor, & \text{for } m_n = \lfloor nt \rfloor \\ -\lfloor nt \rfloor, & \text{for } m_n = \lfloor nu \rfloor - \lfloor nt \rfloor, \end{cases}$$

which is fulfilled under the assumption $t < u$. So using (27), we obtain

$$\left| \frac{E_n^{t_2} - e^{(1-m_n)t_2/n} - e^{(m_n-[nu]+1)t_2/n} + e^{(1-[nu])t_2/n} t_1^{d(r)-1} t_2^{d(s)-1}}{n^2 (e^{t_2/n} - 1)^2} \right|$$

$$= \frac{e^{t_2/n}}{n^2 (e^{t_2/n} - 1)^2} \frac{1}{n} \left( 1 - e^{-m_n t_2/n} - e^{(m_n-[nu])t_2/n} + e^{-[nu]t_2/n} \right) h(r)-1 t_1^{h(s)-1}$$

$$\leq \frac{1 - e^{-mt_2} - e^{(m-u)t_2} + e^{-ut_2}}{t_2^2(t_2 + t_1)} t_1^{h(r)-1} t_2^{h(s)-1} =: g(t_1, t_2)$$

for each $n \geq 1$. In the last step, we applied the inequality

$$1 - e^{-m_n t_2/n} - e^{(m_n-[nu])t_2/n} + e^{-[nu]t_2/n} \leq 1 - e^{-mt_2} - e^{(m-u)t_2} + e^{-ut_2},$$

which holds, since

$$\frac{e^{m_n t_2/n}}{e^{mt_2/n} - 1} \left( 1 - e^{(m_n-[nu]+mn)t_2/n} \right) \leq \frac{e^{m_n t_2/n} - mt_2}{1 - e^{(m_n-[nu]+mn)t_2/n}} = e^{m_n t_2/n} - e^{(2m_n-[nu])t_2/n} \leq 1 - e^{(m_n-[nu]+mn)t_2/n}$$

$$\Rightarrow 1 - e^{-m_n t_2/n} - e^{(m_n-[nu])t_2/n} + e^{-[nu]t_2/n} \leq 1 - e^{-mt_2} - e^{(m-u)t_2} + e^{-ut_2}$$

$$\Leftrightarrow (e^{m_n t_2/n} - 1) (1 - e^{(m_n-[nu]+mn)t_2/n}) \leq (e^{mt_2} - 1) (1 - e^{(m_n-un+mn)t_2/n}).$$

where the last inequality follows from $m_n t_2/n - mt_2 \leq 0$ and $2m_n - [nu] \geq m_n - un + mn$. The function $g$ is integrable, because the calculated limit above is finite. \hfill \Box

**Proof of Lemma A.8.** It holds

$$E(\zeta_n(r, t)\zeta_n(s, u)) = E(S_{[nt]}(r)S_{[nu]}(s)) + \{nu\} E(S_{[nt]}(r)X_{[nu]+1}(s))$$

$$\{nt\} E(S_{[nu]}(s)X_{[nt]+1}(r)) + \{nu\}\{nt\} E(X_{[nu]+1}(r)X_{[nt]+1}(s)).$$
We have to show that
\[
\lim_{n \to \infty} n^{d(r,s)-3} \left( \{nu\} E(S_{nt}(r)X_{nu+1}(s)) + \{nt\} E(S_{nu}(s)X_{nt+1}(r)) + \{nu\} \{nt\} E(X_{nu+1}(r)X_{nt+1}(s)) \right) = 0.
\]
The first summand could be estimated as
\[
|n^{d(r,s)-3}\{nu\} E(S_{nt}(r)X_{nu+1}(s))| \leq \frac{1}{n^{2-h(r,s)}} \gamma_0(r,s),
\]
the convergence to zero of the other summands follows analogously. \(\square\)

Combining the Lemmas A.8 and A.9 yields
\[
\lim_{n \to \infty} n^{3-d(r,s)} E(\zeta_n(r,t)\zeta_n(s,u)) = V((r,t), (s,u)).
\]
Next we show the existence of a Gaussian process \(\mathcal{G}\) with zero-mean and cross-covariance function \(V\). Additionally we will prove that the process is operator self-similar.

**Lemma A.10.** The function \(V : \mathbb{T} \times \mathbb{T} \to \mathbb{C}\) defined in (9) is hermitian and non-negative definite.

**Proof.** Clearly \(V\) is hermitian, since \(\sigma(r,s)\) is hermitian. Let \(N \in \mathbb{N}, \tau_1, \ldots, \tau_N \in \mathbb{T}, w_1, \ldots, w_N \in \mathbb{C}\) and \(M := \max\{t_1, \ldots, t_N\}\) then
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} w_i \overline{w_j} V(\tau_i, \tau_j) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i M^{\frac{3}{2}-d(s_i)} \overline{w_j} M^{\frac{3}{2}-d(s_j)} V\left(\left(s_i, \left(\frac{t_i}{M}\right)\right), \left(s_j, \left(\frac{t_j}{M}\right)\right)\right).
\]
We denote \(\bar{w}_i := w_i M^{\frac{3}{2}-d(s_i)}\) and use Lemma A.9
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \bar{w}_i \overline{\bar{w}_j} \lim_{n \to \infty} \frac{1}{n^{3-d(s_i,s_j)}} E\left(\zeta_n\left(s_i, \left(\frac{t_i}{M}\right)\right) \overline{\zeta_n\left(s_j, \left(\frac{t_j}{M}\right)\right)}\right) \geq 0.
\]
Since \(\frac{h}{M}\) is in the unit interval \([0, 1]\) and \(\frac{1}{n^{3-d(s_i,s_j)}} E(\zeta_n(r,s)\overline{\zeta_n(s,u)})\) is a covariance function for each \((r,t), (s,u) \in \mathbb{S} \times [0, 1]\). \(\square\)

So there exists a Gaussian process \(\mathcal{G}\) with covariance function \(V\).

**Lemma A.11.** If \(h(s) \in (\frac{1}{2}, 1)\) and the integrals
\[
\int_{\mathbb{S}} \frac{\sigma^2(v)}{(1-h(v))^2} \mu(dv) \quad \text{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(v)}{(1-h(v))(2h(v)-1)} \mu(dv)
\]
are finite, then for each \(t \in \mathbb{R}_{\geq 0}\) the stochastic process \(\{\mathcal{G}(s,t)\mid s \in \mathbb{S}\}\) has sample paths in \(L_2(\mu, \mathbb{C})\). Furthermore \(\mathcal{G}(\cdot, t)\) are Gaussian random elements with values in \(L_2(\mu, \mathbb{C})\) and the process \(\{\mathcal{G}(\cdot, t)\mid t \in \mathbb{R}_{\geq 0}\}\) is Gaussian.
Proof. Using the inequalities (19) and (20), we get the approximation

\begin{align}
E \left( \int_{S} |G(v, t)|^2 \mu(dv) \right) &= \int_{S} \frac{\sigma^2(v)}{2(1-h(v))(3-2h(v))} (c(v) + c(v)) t^{3-2h(v)} \mu(dv) \\
&\leq \max\{t, t^2\} \left( \int_{S} \frac{\sigma^2(v)}{(1-h(v))^2} \mu(dv) + \int_{S} \frac{\sigma^2(v)}{(1-h(v))(2h(v)-1)} \mu(dv) \right) < \infty.
\end{align}

So the paths of the stochastic process \( \{G(s, t) : s \in S\} \) are \( P \)-almost surely in \( L_2(\mu, \mathbb{C}) \) for each \( t \in \mathbb{R}_{\geq 0} \) and so \( G(\cdot, t) \) is a Gaussian random element in \( L_2(\mu, \mathbb{C}) \). \( \square \)

Lemma A.12. The stochastic process \( \{G(\cdot, t) : t \in [0, \infty)\} \) is operator self-similar with scaling family \( \{a^H : a > 0\} \), where \( a^H \) with \( a > 0 \) is given by \( a^H f = \{a^{\frac{3}{2}-d(s)} f(s) | s \in S\} \) for \( f \in L_2(\mu, \mathbb{C}) \).

Proof. We have to prove

\[ \langle E[a^H G(\cdot, t), f], a^H G(\cdot, u) \rangle = \langle E[G(\cdot, t), f], a^H G(\cdot, u) \rangle \]

for each \( f, g \in L_2(\mu, \mathbb{C}) \).

Since

\[ E[G(r, at)G(s, au)] = E(a^{\frac{3}{2}-d(r)} G(r, t) a^{\frac{3}{2}-d(s)} G(s, u)), \]

we obtain

\begin{align}
\langle E[a^H G(\cdot, t), f], a^H G(\cdot, u) \rangle &= \int_{S} E[\int_{S} a^{\frac{3}{2}-d(r)} G(r, t) f(r) \mu(dr) a^{\frac{3}{2}-d(s)} G(s, u)] g(s) \mu(ds) \\
&= \int_{S} \int_{S} E[a^{\frac{3}{2}-d(r)} G(r, t) a^{\frac{3}{2}-d(s)} G(s, u)] f(r) \mu(dr) g(s) \mu(ds) \\
&= \langle E[G(\cdot, t), f], G(s, u) \rangle
\end{align}

for all \( f, g \in L_2(\mu, \mathbb{C}) \). \( \square \)

We are now able to define the limit process, which is a restriction to the unit interval of the process \( \{G(s, t) : (s, t) \in S \times \mathbb{R}_{\geq 0}\} \) given by

\[ G = \{G(s, t) : (s, t) \in S \times [0, 1]\}. \]

The following lemma establishes sufficient conditions for the existence of a continuous version of this process.

Lemma A.13. If the integrals

\[ \int_{S} \frac{\sigma^2(v)}{(1-h(v))^2} \mu(dv) \quad \text{and} \quad \int_{S} \frac{\sigma^2(v)}{(1-h(v))(2h(v)-1)} \mu(dv) \]

are finite, there exists a continuous version of the \( L_2(\mu, \mathbb{C}) \)-valued stochastic process \( G = \{G(\cdot, t) : t \in [0, 1]\} \).

Proof. We use Kolmogorov’s continuity theorem (see Kallenberg (2014)) and an inequality for the moments of a Hilbert space-valued Gaussian random element (see Ledoux and Talagrand (1991)).

\[ E\|G(\cdot, t) - G(\cdot, u)\|^4 \leq K_{4,2}^4 E\|G(\cdot, t) - G(\cdot, u)\|^2 \]

Proof. \(A.0.3.\) tightness. We used the inequalities (19) and (20).

The first condition holds, since
\[ \| \cdot \| \]

where
\[ C \]

Lemma \(A.14.\) If \( \zeta_n, G : \Omega \to C([0, 1], L_2(\mu, \mathbb{C})) \), i.e.
\[ \zeta_n, G : \Omega \to C([0, 1], L_2(\mu, \mathbb{C})). \]

We get as a necessary condition for the convergence of the finite-dimensional distributions
\[ (n^{-H} \zeta_n(t_1), \ldots, n^{-H} \zeta_n(t_q)) \xrightarrow{D} (G(t_1), \ldots, G(t_q)) \]
in \( L^2_2(\mu, \mathbb{C}) \) for each \( q \in \mathbb{N} \) and all \( t_1, \ldots, t_q \in [0, 1] \).

The space \( L^2_2(\mu, \mathbb{C}) \) is isomorph to the space of all \( \mu \)-integrable functions \( f : \mathcal{S} \to \mathbb{C}^q \) denoted by \( L_2(\mu, \mathbb{C}^q) \) and endowed with the norm
\[ \| f \| = \left( \int_\mathcal{S} \| f(s) \|^2 \mu(ds) \right)^{\frac{1}{2}} \text{ for } f \in L_2(\mu, \mathbb{C}^q), \]

where \( \| \cdot \|_{\mathbb{C}^q} \) is the euclidean norm in \( \mathbb{C}^q \). We define
\[ \zeta^{(q)}_n(s) = (\zeta_n(s, t_1), \ldots, \zeta_n(s, t_q))^T \text{ and } G^{(q)}(s) = (G(s, t_1), \ldots, G(s, t_q))^T \]
for \( s \in \mathcal{S} \) and fixed \( t_1, \ldots, t_q \in [0, 1] \) and the processes \( \zeta^{(q)}_n = \{ \zeta^{(q)}_n(s) | s \in \mathcal{S} \} \) and \( G^{(q)} = \{ G^{(q)}(s) | s \in \mathcal{S} \} \). Using the isomorphy of \( L^2_2(\mu, \mathbb{C}) \) and \( L_2(\mu, \mathbb{C}^q) \) we are able to prove equivalently
\[ (33) \]
\[ n^{-H} \zeta^{(q)}_n \xrightarrow{D} G^{(q)} \]
in \( L_2(\mu, \mathbb{C}^q) \). We use theorem \(A.2.\) To prove part (I) we consider the sequence of random vectors
\[ (n^{-H} \zeta^{(q)}_n(s_1), \ldots, n^{-H} \zeta^{(q)}_n(s_p)) = \sum_{j=-\infty}^{\infty} A_{nj} \varepsilon_j \]
for \( s_1, \ldots, s_p \in \mathcal{S} \) with \( A_{nj} = (n^{-\left(\frac{q}{2} - d(s_a)\right)}a_{nj}(s_a, t_b))_{a=1 \ldots p, b=1 \ldots q} \), \( a_{nj}(s, t) \) like in (29) and \( \varepsilon_j = \text{diag}(\varepsilon_j(s_1), \ldots, \varepsilon_j(s_p)) \).

Lemma \(A.14.\) If \( h(s) \in (\frac{1}{2}, 1) \), the sequence of operators \( A_{nj} \) fulfills the conditions (25).

Proof. The operator norm may be calculated as \( \|A_{nj}\|_{op} = \max_{1 \leq i \leq q} \sum_{t=1}^{p} |n^{d(s_i) - \frac{q}{2}}a_{nj}(s_i, t_i)| \).

The first condition holds, since
\[ \sup_{j \in \mathbb{Z}} |n^{d(s) - \frac{q}{2}}a_{nj}(s, t)| \leq n^{h(s) - \frac{q}{2}} \sup_{j \in \mathbb{Z}} \left\{ \sum_{k=1}^{nt} |v_{k-j}(s)| + \{nt\}|v_{nt+1-j}(s)| \right\} \sim \frac{t^{1-h(s)}}{1 - h(s)} n^{-\frac{q}{2}}. \]
The second one may be proved by using the Jensen inequality and the relationship between the variance of the piecewise linear process and \( a_{nj}(s, t) \) given by

\[
E|\zeta_n(s, t)|^2 = \sigma^2(s) \sum_{j=-\infty}^{\lfloor nt \rfloor+1} |a_{nj}(s, t)|^2,
\]
then

\[
\sum_{j \in \mathbb{Z}} n^{d(s)-\frac{2}{3}} a_{nj}(s, t)^2 = n^{2h(s)-3} \sum_{j \in \mathbb{Z}} |a_{nj}(s, t)|^2 = n^{2h(s)-3} \frac{1}{\sigma^2(s)} E|\zeta_n(s, t)|^2
\]
and with the help of (31) and the inequalities (19) and (20)

\[
\lim_{n \to \infty} \frac{1}{\sigma^2(s)} n^{2h(s)-3} E|\zeta_n(s, t)|^2 = \frac{1}{2} \frac{c(s) + c(s)}{(3 - 2h(s))(1 - h(s))} t^{3-2h(s)} \leq \max\{t, t^2\} \left( \frac{1}{(1 - h(s))^2} + \frac{1}{(1 - h(s))(2h(s) - 1)} \right).
\]

We construct the process

\[
(n^{-H} \tilde{\zeta}^{(q)}_n(s_1), ..., n^{-H} \tilde{\zeta}^{(q)}_n(s_p)) = \sum_{j=-\infty}^{\infty} A_{nj} \tilde{\varepsilon}_j.
\]

Since \( \{\tilde{\varepsilon}_j\} \) are zero-mean Gaussian stochastic processes with the same covariance operator as \( \{\varepsilon_j\} \) and

\[
\lim_{n \to \infty} n^{-(\frac{2}{3}-d(s_1))} n^{-(\frac{2}{3}-d(s_i))} E(\tilde{\zeta}_n(r_i, t_j) \tilde{\zeta}_n(s_i, u_j)) = E(G(r_i, t_j)G(s_i, u_j)),
\]
concerning to (31), the process converges in distribution to \( G \). Calculating

\[
E\|n^{-H} \zeta_n^{(q)}(s)\|_C^2 = \sum_{i=1}^{q} E[|n^{-\left(\frac{2}{3}-d(s)\right)} \zeta_n(s, t_i)|^2] \quad \text{and} \quad E\|G^q(s)\|_C^2 = \sum_{i=1}^{q} E[|G(s, t_i)|^2],
\]
the second point in theorem A.2 follows by setting \( r = s \) and \( t = u \) in (31). We prove point (III) in the next lemma.

**Lemma A.15.** There exists a \( \mu \)-integrable function \( g \) such that \( |E|n^{-\left(\frac{2}{3}-d(s)\right)} \zeta_n(s, t)|^2| \leq g(s) \) for each \( s \) in \( S \) and all \( n \in \mathbb{N} \), if the integrals

\[
\int_S \frac{\sigma^2(s)}{(1 - h(s))^2} \mu(ds) \quad \text{and} \quad \int_S \frac{\sigma^2(s)}{(1 - h(s))(2h(s) - 1)} \mu(ds)
\]
are finite.

**Proof.** Using the triangular inequality

\[
|E|\zeta_n(s, t)|^2| \leq 2 \sum_{k=1}^{\lfloor nt \rfloor} (|nt| - k) |\gamma_k(s)| + 2\{nt\} \sum_{k=1}^{\lfloor nt \rfloor} |\gamma_k(s)| + (|nt| + \{nt\}) |\gamma_0(s)|.
\]

Each particular summand may be restricted as follows

\[
\sum_{k=1}^{\lfloor nt \rfloor} (|nt| - k) |\gamma_k(s)| \leq |nt|^{3-2h(s)} \frac{1}{2} \sigma^2(s) \left( \frac{1}{(1 - h(s))^2} + \frac{1}{(1 - h(s))(2h(s) - 1)} \right),
\]
\[ \sum_{k=1}^{[nt]} |\gamma_k(s)| \leq [nt]^{2-2h(s)} \sigma^2(s) \frac{1}{2} \left( \frac{1}{(1-h(s))^2} + \frac{1}{(1-h(s))(2h(s) - 1)} \right) \]

and

\[ |\gamma_0(s)| \leq \sigma^2(s) \sum_{j=0}^{\infty} (j + 1)^{-2h(s)} \]

\[ \leq \sigma^2(s) \left( 1 + \int_{1}^{\infty} x^{-2h(s)} dx \right) = \sigma^2(s) \left( 1 + \frac{1}{2h(s) - 1} \right). \]

\[ \square \]

Altogether \( E\|n^{-H} \zeta_n^g(s)\|_{\mathbb{C}^n}^2 \leq q \ast g(s) =: f(s) \), where \( f \) is integrable.

A.0.4. **Tightness.** To investigate tightness of the process \( \{n^{-H} \zeta_n(t) : n \in \mathbb{N} \mid t \in [0, 1]\} \) we use proposition 5.1. The first condition follows since \( (n^{-H} S_n)_{n \in \mathbb{N}} \) converges in distribution in \( L_2(\mu, \mathbb{C}) \) referring to theorem 4.1. Then \( (n^{-H} \zeta_n(t))_{n \in \mathbb{N}} \) converges in distribution in \( L_2(\mu, \mathbb{C}) \) for each \( t \in [0, 1] \).

The next lemma proves the second condition in proposition 5.1.

**Lemma A.16.** If \( h(s) \in (\frac{1}{2}, 1) \) and the integrals

\[ E \left[ \int_{\mathcal{S}} \frac{|\varepsilon_0(v)|^2}{(1-h(v))^2} \mu(dv) \right]^\frac{p}{2} \text{, for } p \geq 2 \text{ and } \int_{\mathcal{S}} \frac{\sigma^2(v)}{(1-h(v))(2h(v) - 1)} \mu(dv) \]

are finite, then there exists a constant \( C \), such that

\[ E\|n^{-H} [\zeta_n(t) - \zeta_n(u)]\|_p \leq C|t - u|^{(2-2h)p} \text{, for } n \geq 1 \]

with \( h = \text{ess sup}_{s \in \mathcal{S}} h(s) \).

The following lemma is part of the proof of the previous one.

**Lemma A.17.** Let \( 0 \leq u < t \leq 1 \) and \( \{nt\} = \{nu\} = 0 \). If \( h(s) \in (\frac{1}{2}, 1) \), then

\[ n^{-(3-2h(s))} \sum_{j=-\infty}^{nt} \sum_{k=nu+1}^{nt} v_{k-j}(s) \left| v_k(s) - v_j(s) \right|^2 \leq \frac{2}{(1-h(s))^2} + \frac{1}{2h(s) - 1} \]

\[ |t - u|^{3-2h(s)}. \]

We refer to Characiejus and Račkauskas (2014)) for the proofs. They may be easily extended using the triangular inequality and the inequalities (19) and (20) as we did so far. \( \square \)

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