Quantum approximate optimization is computationally universal

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Abstract: The quantum approximate optimization algorithm (QAOA) applies two Hamiltonians to a quantum system in alternation. The original goal of the algorithm was to drive the system close to the ground state of one of the Hamiltonians. This paper shows that the same alternating procedure can be used to perform universal quantum computation: the times for which the Hamiltonians are applied can be programmed to give a computationally universal dynamics. The Hamiltonians required can be as simple as homogeneous sums of single-qubit Pauli X’s and two-local ZZ Hamiltonians on a one-dimensional line of qubits.

Quantum information processing supports a broad range of platforms ranging from universal quantum computers capable of performing quantum algorithms such as factoring, to quantum annealers, which are not computationally universal but which can be used to try to find answers to hard optimization problems. The quantum approximate optimization algorithm (QAOA) is a dynamic optimization method that is related to quantum annealing [1-2]. As originally proposed, QAOA is not obviously computationally universal. This paper shows that QAOA is capable of universal quantum computation in a simple and natural way.

The quantum approximate optimization algorithm operates by applying two different Hamiltonians to a quantum system in alternation to try to drive the system to the ground state of one of the Hamiltonians [1-2]. In its original form, the first Hamiltonian $H_Z = \text{poly}(Z_j)$ is a low-order polynomial in the Pauli Z operator over the qubits, and the second
Hamiltonian \( H_X = \sum_j X_j \) is a uniform sum of Pauli \( X \) operators. Starting in the uniform superposition of logical qubits, \( |I\rangle = 2^{-n/2} \sum_{j=0}^{2^n-1} |j\rangle \), one first applies \( H_Z \) for time \( t_1 \), then \( H_X \) for time \( \tau_1 \), then \( H_Z \) for time \( t_2 \), then \( H_X \) for time \( \tau_2 \), and so on, \( p \) times in alteration, yielding the state

\[
U(\vec{t}, \vec{\tau})|I\rangle = e^{-i\tau_p H_X} e^{-it_p H_Z} \ldots e^{-i\tau_2 H_X} e^{-it_2 H_Z} e^{-i\tau_1 H_X} e^{-it_1 H_Z} |I\rangle.
\]  

(1)

Because of its alternating form, the procedure for performing the QAOA is sometimes referred to as the quantum alternating operator ansatz.

The original purpose of QAOA was to vary the \( t \)'s and \( \tau \)'s for fixed \( p \) to try to make \( U(\vec{t}, \vec{\tau})|I\rangle \) approximate the ground state of \( H_Z \). This procedure works rather well, even for small \( p \) [1-2]. The form of the QAOA dynamics (1) exhibits a variety of features. The alternating form of the application of operators in QAOA makes it an application of ‘bang-bang’ quantum control [3-4], which like its classical cousin, is known to be time-optimal via the Pontryagin minimum principle [5]. Its simplicity and flexibility means that QAOA can be repurposed for investigations of quantum supremacy/advantage [6], and for quantum search [7].

In this paper I show that QAOA represents a natural framework for performing universal quantum computation. In particular, when \( H_Z \) is a simple, homogeneous two-qubit Hamiltonian on a one-dimensional lattice, the times \( \vec{t}, \vec{\tau} \) can be selected to program the system to implement any desired sequence of quantum logic gates. The method is to apply QAOA to implement computationally universal broadcast quantum cellular automaton architectures [8-11].

The Hamiltonians required to implement universal quantum computation via QAOA are particularly simple. Let \( H_X = \sum_j X_j \) as before, and let

\[
H_Z = \sum_j \omega_A Z_{2j} + \omega_B Z_{2j+1} + \gamma_{AB} Z_{2j} Z_{2j+1} + \gamma_{BA} Z_{2j+1} Z_{2j+2}
\]

\[
\equiv \omega H_A + \omega_B H_B + \gamma_{AB} H_{AB} + \gamma_{BA} H_{BA},
\]

(2)

where \( \omega_A, \omega_B, \gamma_{AB}, \gamma_{BA} \) are not rationally related. We can then choose \( t \) to effectively ‘turn on’ one of the Hamiltonians \( H_A, H_B, H_{AB}, \) or \( H_{BA} \), while ‘turning off’ the others. For example, choose \( t \) so that

\[
|\omega_A t - (2\pi n_A + \phi)| < \epsilon/4, |\omega_B t - 2\pi n_B| < \epsilon/4, |\gamma_{AB} t - 2\pi n_{AB}| < \epsilon/4, |\gamma_{BA} t - 2\pi n_{BA}| < \epsilon/4,
\]

(3)
where \( n_A, n_B, n_{AB}, n_{BA} \) are integers. The amount of time it takes to attain this accuracy is \( O(1/\epsilon^4) \): with four incommensurate Hamiltonians one must ‘wrap around’ \( O(\epsilon^4) \) times to line up the appropriate phases to accuracy \( \epsilon \). The resulting transformation obeys
\[
\| e^{-itH_Z} - e^{-i\phi_A H_A} \|_1 < \epsilon. \tag{4}
\]

In a similar fashion, we can implement the transformations \( e^{-i\phi_B H_B}, e^{-i\phi_{AB} H_{AB}}, e^{-i\phi_{BA} H_{BA}} \) to any desired degree of accuracy.

Implementing \( e^{-i\phi_A H_A}, e^{-i\phi_B H_B}, \) and \( e^{-i\phi_{AB} H_{AB}} \) transformations allows us to construct transformations of the form
\[
U_{AB} = e^{-i\phi_A H_A}e^{-i\phi_B H_B}e^{-i\phi_{AB} H_{AB}} = U_{01} \otimes U_{23} \otimes \ldots \otimes U_{2j,2j+1} \otimes \ldots \tag{5}
\]
where \( U_{2j,2j+1} \) can be any desired two-qubit unitary with determinant equal to 1, diagonal in the \( Z \) basis, that acts on pairs of spins \( 2j, 2j + 1 \). Similarly, we can construct transformations of the form
\[
U_{BA} = e^{-i\phi_A H_A}e^{-i\phi_B H_B}e^{-i\phi_{BA} H_{BA}} = U_{12} \otimes U_{34} \otimes \ldots \otimes U_{2j+1,2j+2} \otimes \ldots \tag{6}
\]
where again \( U_{2j+1,2j+2} \) can be any desired unitary diagonal in the \( Z \) basis.

By adjusting the \( \tau \) terms in the QAOA formula (1), we see that we can also implement transformations of the form
\[
e^{-i3\pi X/4}e^{-itH_Z}e^{-5\pi X/4} = e^{-itH_Y}, \tag{7}
\]
where \( H_Y \) has the same form as \( H_Z \) in equation (2), but all the \( Z \) Pauli matrices have been transformed into \( Y \) Pauli matrices. Consequently, we can implement transformations of the form (5) and (6), but where the unitaries \( U_{2j,2j+1}, U_{2j+1,2j+2} \) are now diagonal in the \( Y \) basis. But the ability to perform transformations of the form (5) and (6) with the \( U \)‘s diagonal in \( Z \) or in \( Y \) basis implies that one can perform transformations of the form (5) and (6) for any desired \( U_{2j,2j+1}, U_{2j+1,2j+2} \).

To summarize, the QAOA procedure of equation (1) allows us to implement a broadcast quantum cellular automaton dynamics [8-11] of the form
\[
U_{BA}^p U_{AB}^p \ldots U_{BA}^1 U_{AB}^1, \tag{8}
\]
where the \( U_{AB}^k, U_{BA}^k \) can be varied at will from step to step. If, in addition to being able to apply the broadcast quantum CA dynamics of equation (8), one can measure and
prepare the first qubit of the one-dimensional array, one can perform universal quantum computation using well-established methods developed in [8-11]. These methods operate by first implementing pulse sequences that load data onto the array, next, by applying pulses that implement a parallel quantum computation, and finally by moving the results of the computation to the first qubit of the array where they can be measured out sequentially.

**Discussion:**

This paper showed that the dynamics of quantum approximate optimization algorithm can be programmed to perform any desired quantum computation. Note that the results of the paper imply that the even simpler dynamics are quantum computationally universal. In particular, since the Hadamard operation transforms the $Z$ Hamiltonian into the corresponding operator with $Z$ Pauli matrices transformed into $X$’s, the programmable dynamics

$$U(\vec{t}) = H e^{-i t_1 H_Z} \ldots H e^{-i t_2 H_Z} H e^{-i t_1 H_Z},$$

where $H = H_1 \otimes \ldots \otimes H_j \ldots$ is the tensor product of single qubit Hadamard operations, is also computationally universal. Equation (9) represents the repeated application of an instantaneous quantum polynomial time dynamics (IQP), using the same $Z$ Hamiltonian each time [12]. Our results show that such IQP dynamics with fixed $Z$ Hamiltonian is computationally universal.

The architecture used to prove universality here was a simple one-dimensional array with nearest neighbor interactions: such one-dimensional architectures do not support thresholds for scalable fault-tolerant quantum computation. The Hamiltonian averaging trick introduced here can easily be extended to higher-dimensional arrays, and long-range interactions could be added to implement long-range ‘wires’ to transmit quantum information between distant parts of the network. Such architectures might be able to support fault-tolerant quantum computation via QAOA. For each additional type of interaction Hamiltonian added, the time required for each ‘active’ parallel quantum logic operation is multiplied by an additional factor of $1/\epsilon$ so that all the ‘non-active’ interactions can wrap around. How many different types of wires/interactions are required to give a network with a desired degree of long-range connectivity is an open question.

Although the QAOA dynamics is phrased in terms of turning on and turning off interactions, a physical implementation of the computationally universal dynamics exhibited here could be implemented with an always-on $Z$ Hamiltonian, with the $X$ rotations or Hadamards implemented by a single, strong, globally applied pulse. The simplicity of the
addressing required to perform universal quantum computation in such systems suggests their implementation via superconducting systems, arrays of atoms in optical lattices, spin, or quantum dot systems. Even when such systems are not capable of scalable quantum computation, their ability to exhibit quantum supremacy/advantage suggests that they could be used to construct near term quantum information processing devices for problems such as deep quantum learning [13], where the weights of the deep quantum network are given by the adjustable times $\vec{t}$. For example, the goal of the deep quantum learning procedure could be to train the computationally universal QAOA system to implement a unitary transformation that maps inputs to outputs, given a training set of such input-output pairs. Or the system could be trained to try to create quantum states on which measurement results match the statistics of a classical data set.

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