UNIFORM BOUNDS ON SYMBOLIC POWERS IN REGULAR RINGS

TAKUMI MURAYAMA

Abstract. We prove a uniform bound on the growth of symbolic powers of arbitrary (not necessarily radical) ideals in arbitrary (not necessarily excellent) regular rings of all characteristics. This gives a complete answer to a question of Hochster and Huneke. In equal characteristic, this result was proved by Ein, Lazarsfeld, and Smith and by Hochster and Huneke. For radical ideals in excellent regular rings of mixed characteristic, this result was proved by Ma and Schwede. We also prove a generalization of these results involving products of various symbolic powers and a uniform bound for regular local rings related to a conjecture of Eisenbud and Mazur, which are completely new in mixed characteristic. In equal characteristic, these results are due to Johnson and to Hochster–Huneke, Takagi–Yoshida, and Johnson, respectively.

Contents

1. Introduction 2
   1.1. Main theorem 2
   1.2. A generalization and a containment for regular local rings 3
   1.3. Discussion of proof strategies 4
   1.4. Outline 11

2. Preliminaries 11
   2.1. Symbolic powers 11
   2.2. Reductions for main theorems 12
   2.3. Absolute integral closures 13
   2.4. Big Cohen–Macaulay algebras 13
   2.5. Weakly functorial big Cohen–Macaulay \( R^+ \)-algebras in equal characteristic zero 14

Part I. Proof via closure theory 15

   3. Closure operations 15
   3.1. Closure operations in mixed characteristic 15
   3.2. Axioms for closure operations 16
   3.3. Algebra closures 17
   4. Proof of main theorems via closure theory 18

Part II. Proof via multiplier/test ideals 21

   5. Big Cohen–Macaulay test ideals, \( + \)-test ideals, and multiplier ideals 21
   5.1. Big Cohen–Macaulay test ideals of divisor pairs 21
   5.2. \( + \)-test ideals 21
   5.3. Multiplier ideals for excellent schemes of equal characteristic zero 23
   6. Big Cohen–Macaulay test ideals of triples 24
   6.1. Definition and preliminaries 24
   6.2. Unambiguity of exponents 28

2020 Mathematics Subject Classification. Primary 13A15, 13H05; Secondary 14G45, 13C14, 13A35, 14F18.
Key words and phrases. uniform bounds, symbolic powers, regular rings, big Cohen–Macaulay algebras, closure operations, test ideals.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1902616.
1. Introduction

Let $R$ be a Noetherian ring, and let $I \subseteq R$ be an ideal. For all integers $n \geq 1$, the $n$-th symbolic power of $I$ is the ideal

$$I^{(n)} := \bigcap_{p \in \text{Ass}(R)} I^n p \cap R.$$  

Hartshorne asked when the $p$-adic and $p$-symbolic topologies are equivalent, where $p \subseteq R$ is a prime ideal and the $p$-symbolic topology is the topology defined by the symbolic powers $p^{(n)}$ [Har70, p. 160]. One motivation for Hartshorne’s question was a conjecture of Grothendieck asking whether the modules $\text{Hom}_R(R/J, H^i_J(M))$ are finitely generated for finitely generated $M$ [SGA2, Exposé XIII, Conjecture 1.1]. Hartshorne answered this question when $\dim(R/p) = 1$ [Har70, Proposition 7.1], and Schenzel gave a complete characterization for when the $I$-adic and $I$-symbolic topologies are equivalent on a Noetherian ring $R$ for arbitrary ideals $I \subseteq R$ [Sch85, Theorem 1; Sch86, Theorem 3.2] (see also [Sch98, §2.1]). Schenzel’s result implies for example that if $R$ is a domain whose local rings are analytically irreducible, then the $p$-adic and $p$-symbolic topologies on $R$ are equivalent for every prime ideal $p \subseteq R$ (see [HKV15, p. 545]). This latter result is originally due to Zariski [Zar51, Lemma 3 on p. 33] (see also [Ver88, Proposition 3.11]).

In [Swa00, Main Theorem 3.3], Swanson proved that in fact, when the $I$-adic and $I$-symbolic topologies on $R$ are equivalent, there exists an integer $k$ (depending on $I$) such that

$$I^{(kn)} \subseteq I^n$$

for all $n \geq 1$. When $R$ is regular of equal characteristic, Ein, Lazarsfeld, and Smith [ELS01, Theorem 2.2 and Variant on p. 251] and Hochster and Huneke [HH02, Theorems 2.6 and 4.4(a)] proved that (1.2) holds for all ideals $I \subseteq R$ and all $n \geq 1$, where $k$ can be taken to be the largest analytic spread of the ideals $IR_p \subseteq R_p$ as $p$ ranges over the associated primes of $R/I$. In particular, when $R$ is a finite-dimensional regular ring of equal characteristic, one can take $k = \dim(R)$ by [SH06, Proposition 5.1.6], and hence there is a uniform $k$ that does not depend on $I$.

In [HH02, p. 368], Hochster and Huneke asked whether the results of Ein–Lazarsfeld–Smith and Hochster–Huneke hold in mixed characteristic. Ma and Schwede answered their question when $R$ is excellent (or, more generally, has geometrically reduced formal fibers) and $I$ is radical [MS18, Theorem 7.4]. However, the cases for (I) arbitrary ideals (i.e. not necessarily radical), and (II) non-excellent rings remained open.

1.1. Main theorem. Our main result gives a complete answer to Hochster and Huneke’s question by proving the following uniform bound on the growth of symbolic powers of arbitrary ideals in arbitrary regular rings of all characteristics. This resolves the remaining cases of Hochster and Huneke’s question, namely when $I$ is an arbitrary ideal (i.e. not necessarily radical) and $R$ is not necessarily excellent.

**Theorem A.** Let $R$ be a regular ring, and let $I \subseteq R$ be an ideal. Let $h$ be the largest analytic spread of the ideals $IR_p \subseteq R_p$, where $p$ ranges over all associated primes $p$ of $R/I$. Then, for all
$n \geq 1$, we have

\[ I^{(hn)} \subseteq I^n. \]

See [SH06, Definition 5.1.5] for the definition of analytic spread. Note that

\[ h \leq \text{bight}(I) := \max_{p \in \text{Ass}(R/I)} \{ \text{ht}(p) \} \leq \dim(R) \]

by [SH06, Proposition 5.1.6]. Thus, Theorem A implies all finite-dimensional regular rings satisfy the uniform symbolic topology property of Huneke, Katz, and Validashti [HKV09, Question on p. 337; HKV15, p. 543].

We provide two proofs of Theorem A in this paper. In the mixed characteristic setting, where the full statement of Theorem A was not known, both proof strategies use Scholze’s theory of perfectoid spaces [Sch12]. André and Bhatt applied perfectoid techniques to commutative algebra in mixed characteristic to resolve Hochster’s direct summand conjecture [And18a; And18b] (conjectured in [Hoc73, p. 25]) and de Jong’s derived variant of the direct summand conjecture [Bha18], respectively.

Our first proof of Theorem A uses axiomatic closure operations in the sense of [Die10; RG18] analogous to the theory of tight closure [HH90] in positive characteristic. Tight closure was used by Hochster and Huneke [HH02] to prove Theorem A in equal characteristic. Our closure-theoretic proof of Theorem A provides a short, axiomatic proof of Theorem A that is almost characteristic free and unifies all known cases of Theorem A. See §1.3.1 for more discussion.

Our second proof of Theorem A uses perfectoid/big Cohen–Macaulay test ideals. Perfectoid test ideals were introduced by Ma and Schwede [MS18] to prove Theorem A for radical ideals in excellent regular rings in mixed characteristic. The strategy in [MS18] adapts the strategy of Ein, Lazarsfeld, and Smith [ELS01] using multiplier ideals on smooth complex varieties. A crucial difference between the approach in [MS18] and in this paper is that we use a different definition for perfectoid/big Cohen–Macaulay test ideals due to Robinson [Rob22]. Robinson’s definition has the advantage of working in arbitrary characteristic and not using almost mathematics. Moreover, we show that Robinson’s test ideals can be compared to other notions of test ideals [MS21; PRG21; ST; HLS], in particular to the +-test ideals introduced by Hacon, Lamarche, and Schwede [HLS]. This flexibility is a key component of our test ideal-theoretic proof of Theorem A, since in mixed characteristic, only the +-test ideals from [HLS] are known to be compatible with open embeddings (see (4a′) below) and to satisfy a Skoda-type theorem (see (4b′) below). See §1.3.2 for more discussion.

1.2. A generalization and a containment for regular local rings. We in fact show the following more general version of Theorem A, again for arbitrary ideals in arbitrary regular rings of all characteristics. When the $s_i$ are not all zero, this result is completely new in mixed characteristic, even for radical ideals in excellent regular rings. In equal characteristic, the special case of the result below when all the $s_i$ are equal is due to Hochster and Huneke [HH02, Theorems 2.6 and 4.4(a)]. The full statement in equal characteristic is due to Johnson [Joh14, Theorem 4.4].

**Theorem B.** Let $R$ be a regular ring, and let $I \subseteq R$ be an ideal. Let $h$ be the largest analytic spread of $IR_p$, where $p$ ranges over all associated primes $p$ of $R/I$. Then, for all $n \geq 1$ and for all non-negative integers $s_1, s_2, \ldots, s_n$ with $s = \sum_{i=1}^{n} s_i$, we have

\[ I^{(s+nh)} \subseteq \prod_{i=1}^{n} I^{(s_i+1)}. \]

(1.3)

As a consequence, many known results on symbolic powers in regular rings of equal characteristic can now be extended to regular rings of arbitrary characteristic. These include results on Grifo’s stable version of Harbourne’s conjecture [Gri20, Conjecture 2.1] and resurgence. For example, the results in [Gri20, Theorem 2.5], [GHM20, Theorems 3.2 and 3.3], and [DPD21, Theorem 4.5, Corollary 4.8, Theorem 4.12, and Corollary 4.13] (see [DPD21, Remark 4.23]) now hold for arbitrary regular rings. Theorem A follows from Theorem B by setting $s_i = 0$ for all $i$. 
We also show the following uniform bound on the growth of symbolic powers of arbitrary ideals in arbitrary regular local rings of all characteristics. This result is related to a conjecture of Eisenbud and Mazur [EM97, p. 190], which asks whether for a regular local ring $(R, \mathfrak{m})$ of equal characteristic zero, one has $I^{(2)} \subseteq \mathfrak{m}I$ for every unmixed ideal $I$. This result is completely new in mixed characteristic, even for radical ideals in excellent regular rings. In equal characteristic, special cases of the result below are due to Hochster and Huneke [HH07, Theorems 3.5 and 4.2(1)] and to Takagi and Yoshida [TY08, Theorems 3.1 and 4.1]. The full statement in equal characteristic is due to Johnson [Joh14, Theorem 4.3(2)].

**Theorem C.** Let $(R, \mathfrak{m})$ be a regular local ring, and let $I \subseteq R$ be an ideal. Let $h$ be the largest analytic spread of $IR_p$, where $p$ ranges over all associated primes $\mathfrak{p}$ of $R/I$. Then, for all $n \geq 1$ and for all non-negative integers $s_1, s_2, \ldots, s_n$ with $s = \sum_{i=1}^{n} s_i$, we have

$$I^{(s+nh+1)} \subseteq \mathfrak{m} \cdot \prod_{i=1}^{n} I^{(s_i+1)}.$$  \hspace{2cm} (1.4)

While the conjecture of Eisenbud and Mazur has counterexamples in positive characteristic [EM97, Example on p. 200] and mixed characteristic [KR00, Remark 3.3(b)], Theorem C holds for all regular local rings. In particular, Theorem C implies that for ideals $I$ in regular local rings $(R, \mathfrak{m})$, we have $I^{(h+1)} \subseteq \mathfrak{m}I$. See [HH07, p. 172; TY08, p. 719] for more discussion.

1.3. **Discussion of proof strategies.** We now describe our strategy for proving Theorems A, B, and C. We give two logically independent strategies for proving these results, corresponding to the two parts of this paper. Both strategies work in all characteristics and are of independent interest.

(I) A closure-theoretic proof using axiomatic closure operations in the sense of [Die10; RG18] to replace the theory of tight closure [HH90; HH] used in [HH02]. In mixed characteristic, this proof uses Heitmann’s full extended plus (epf) closure [Hei01], Jiang’s weak epf (wepf) closure [Jia21], and R.G.’s results on closure operations that induce big Cohen–Macaulay algebras [RG18].

(II) A proof using various versions of perfectoid/big Cohen–Macaulay test ideals from [MS18; MS21; PRG21; MSTWW22; Rob22; ST; HLS] to replace the theory of multiplier ideals used in [ELS01]. Note that Ma and Schwede introduced their theory of perfectoid test ideals to prove their special case of Theorem A in [MS18].

As mentioned above, in mixed characteristic, both proof strategies use Scholze’s theory of perfectoid spaces [Sch12].

Removing the condition that $I$ is radical is the key obstacle to proving Theorem A in the most general setting. In mixed characteristic, the closure operations and versions of test ideals available to us require working over a complete local ring. We therefore need to reduce to the case where we can work over a complete local ring. Even if $I$ is radical, it may no longer be radical after this reduction. Thus, even if one’s main interest is the special case of Theorem A when $I$ is radical or even prime, one must consider non-radical ideals.

While our proofs of Theorems A, B, and C using test ideals predate our closure-theoretic proofs, we have presented the closure-theoretic proofs first because the proofs are shorter and require fewer technical prerequisites. After reading the preliminaries in §2, the reader can proceed directly to reading Part I or Part II depending on their interests. In the rest of this introduction, we summarize some key aspects of each of our proofs and how they relate to previous work in the literature.

1.3.1. **Proof via closure theory.** The proof of Theorem A in equal characteristic due to Hochster and Huneke [HH02] uses the theory of tight closure in equal characteristic $p > 0$ [HH90] or equal characteristic zero [HH]. This proof is remarkably short, especially in equal characteristic $p > 0$ when $I$ is radical [HH02, pp. 350–351].
In Part I of this paper, we prove Theorems A, B, and C in mixed characteristic using an appropriate replacement for tight closure. In fact, our proofs apply to any context where a sufficiently well-behaved closure operation exists, and hence gives new proofs of Theorems A, B, and C in all characteristics. We found these proofs after reconstructing an unpublished proof of Theorem A in equal characteristic $p > 0$ due to Hochster and Huneke [HH02], which uses plus closure [HH92; Smi94]. Hochster and Huneke sketch their strategy in [HH02, p. 353]. According to Hochser and Huneke, their strategy used the following two ingredients:

1. Plus closure localizes [HH92, Lemma 6.5(b); Smi94, p. 45].
2. A Briançon–Skoda-type theorem holds for plus closure [HH95, Theorem 7.1].

Our closure-theoretic proofs of Theorems A, B, and C in mixed characteristic are possible because sufficiently powerful replacements for tight closure and plus closure in mixed characteristic are now available. Heitmann introduced full extended plus (epf) closure [Hei01] as one such possible replacement for tight closure and plus closure in mixed characteristic. Following [Hei01; HM21], for a domain $R$ such that the image of $p$ lies in the Jacobson radical of $R$, the full extended plus (epf) closure of an ideal $I \subseteq R$ is

$$I^{\text{epf}} := \left\{ x \in R \mid \begin{array}{l}
\text{there exists } c \in R - \{0\} \text{ such that } \\
ocline
\epsilon^cx \in (I,p^N)R^+ \\
\text{for every } \epsilon \in \mathbb{Q}_{>0} \text{ and every } N \in \mathbb{Z}_{>0}
\end{array}\right\}. $$

Here, $R^+$ denotes the absolute integral closure of $R$, i.e., the integral closure of a domain $R$ in an algebraic closure of its fraction field (see §2.3). We note that epf closure coincides with tight closure for complete Noetherian rings of equal characteristic $p > 0$ by [HH91, Theorem 3.1] (see [HM21, Proposition 2.11]).

Heitmann proved a Briançon–Skoda-type theorem for epf closure [Hei01, Theorem 4.2], and used epf closure to prove Hochster’s direct summand conjecture [Hoc73, p. 25] in mixed characteristic for rings of dimension three [Hei02]. More recently, using perfectoid techniques, Heitmann and Ma showed that Heitmann’s Briançon–Skoda-type theorem for epf closure implies the Briançon–Skoda theorem in mixed characteristic [HM21, Theorem 3.20] and gave a closure-theoretic proof of Hochster’s direct summand conjecture in mixed characteristic [HM21, p. 135]. The Briançon–Skoda theorem was originally proved by Briançon and Skoda [SB74] for $C\{z_1, z_2, \ldots, z_n\}$ and by Lipman and Sathaye [LS81] in general. The direct summand conjecture was originally proved by Hochster [Hoc73] in equal characteristic and by Heitmann [Hei02] (in dimension three) and André [And18a; And18b] (in general) in mixed characteristic.

While epf closure is powerful enough to prove the direct summand conjecture and the Briançon–Skoda theorem in mixed characteristic, our proofs of Theorems A, B, and C require working with other closure operations. One reason for this is that epf closure does not localize [Hei01, p. 817], and hence Hochster and Huneke’s unpublished proof of Theorem A using plus closure in equal characteristic $p > 0$ [HH02, p. 353] cannot readily be adapted to the mixed characteristic case.

Instead, our strategy in Part I utilizes epf closure in conjunction with Jiang’s weak epf (wepf) closure [Jia21] and R.G.’s results on closure operations that induce big Cohen–Macaulay algebras [RG18]. We use wepf closure and R.G.’s results to work around the fact that epf closure does not localize. Following [Jia21], for a domain $R$ such that the image of $p$ lies in the Jacobson radical of $R$, the weak epf (wepf) closure of an ideal $I \subseteq R$ is

$$I^{\text{wepf}} := \bigcap_{N=1}^{\infty} (I,p^N)^{\text{epf}}. $$

The advantage of using wepf closure is that it satisfies key additional properties not known for epf closure, namely Dietz’s axioms [Die10, Axioms 1.1] and R.G.’s algebra axiom [RG18, Axiom 3.1]. Dietz formulated the axioms in [Die10] to characterize when a closure operation can be
used to construct big Cohen–Macaulay modules. R.G. formulated the algebra axiom in [RG18] to characterize when a closure operation can be used to construct big Cohen–Macaulay algebras. Using perfectoid techniques, Jiang [Jia21, Theorem 4.8] showed that \text{wepf} closure satisfies Dietz’s axioms and R.G.’s algebra axiom when \( R \) is a complete local domain of mixed characteristic with \( F \)-finite residue field. As a consequence, we have the inclusions

\[ I^{\text{epf}} \subseteq I^{\text{wepf}} \subseteq IB \cap R \] (1.5)

for some big Cohen–Macaulay algebra \( B \) over \( R \) by a result of R.G. [RG18, Proposition 4.1].

To illustrate our proof strategy for Theorems A, B, and C, we sketch the proof of Theorem A when \( R \) is a complete regular local ring of mixed characteristic with an \( F \)-finite residue field and \( I \) is generated by \( h \) elements. If \( u \in I^{(hn)} \), there exists an element \( c \) avoiding the associated primes of \( R/I \) such that \( cu \in I^h \). We then consider a module-finite extension \( R' \) of \( R \) that is complete local and contains an \( n \)-th root \( u^{1/n} \) of \( u \). We have

\[ (cu^{1/n})^n = c^nu \in (I^h R')^n, \]

and hence \( cu^{1/n} \in \overline{I^h R'} \). By Heitmann’s Briançon–Skoda-type theorem for \( \text{epf} \) closure [Hei01, Theorem 4.2] and using (1.5), we know that

\[ cu^{1/n} \in (IR')^{\text{epf}} \subseteq (IR')^{\text{wepf}} \subseteq IB \cap R', \]

where \( B \) is a big Cohen–Macaulay algebra over \( R' \). Then, \( B \) is also a big Cohen–Macaulay algebra over \( R \) and \( R \to B \) is faithfully flat. We therefore see that \( c \) is a nonzerodivisor on \( B/IB \), and hence \( u^{1/n} \in IB \cap R' \). Taking \( n \)-th powers, we have \( u \in I^n B \cap R = I^n \).

We highlight three key aspects of our proof of Theorems A, B, and C in Part I that already appear in the proof sketch above.

1. Instead of passing to a large, possibly non-Noetherian extension of \( R \) (such as \( R^+ \) or a big Cohen–Macaulay algebra), we pass to the smallest integral extension \( R' \) of \( R \) where \( u^{1/n} \) makes sense. Since \( R' \) is module-finite over \( R \), it is Noetherian, and hence we can use closure operations on \( R' \) as usual.

2. After applying the Briançon–Skoda theorem for \( \text{epf} \) closure on \( R' \), we can apply R.G.’s result on \( R' \) as in (1.5) to pass to a big Cohen–Macaulay algebra \( B \) that captures \( \text{wepf} \) closure. We pass to \( IB \cap R' \) instead of working with \( (IR')^{\text{epf}} \) or \( (IR')^{\text{wepf}} \) because there are no elements \( c^e \) or \( p \)-powers involved in the definition.

3. Since \( B \) is flat over \( R \), we can detect when nonzerodivisors in \( R/I \) map to nonzerodivisors in \( B/IB \). Thus, we can get rid of the element \( c \) multiplying \( u^{1/n} \) into \( IB \).

Note that (2) and (3) require the extension ring \( R' \) in (1) to be Noetherian. Working with the big Cohen–Macaulay algebra \( B \) in (2) and (3) allows us to avoid localizations.

Our proof strategy using closure operations also applies to any closure operation satisfying Dietz’s axioms, R.G.’s algebra axiom, and a Briançon–Skoda-type theorem. We therefore obtain new proofs of Theorems A, B, and C in all characteristics, since tight closure [HH90] and plus closure [HH92; Smi94] satisfy these properties in equal characteristic \( p > 0 \), and \( \mathfrak{B} \)-closure [AS07] satisfies these properties in equal characteristic zero. In mixed characteristic, one can alternatively use Heitmann’s full rank 1 (\( \text{r1f} \)) closure [Hei01] instead of \( \text{wepf} \) closure (still using results from [RG18; Jia21]). See Table 1 for references to proofs of these axioms for these closure operations. We can also adapt our proofs to use small or big equational tight closure in equal characteristic zero [HH]. See Remark 4.2.
1.3.2. Proof via multiplier/test ideals. The proof of Theorem A for smooth complex varieties due to Ein, Lazarsfeld, and Smith [ELSO01] uses the theory of multiplier ideals \( I(X, \mathfrak{a}^t) \) (see [Laz04, Part Three]). Here, \( X \) denotes a smooth complex variety, \( \mathfrak{a} \subseteq \mathcal{O}_X \) is a coherent ideal sheaf, and \( t \) is a non-negative real number. One version of their proof (see [ST12, Theorem 6.23; CRS20, p. 27]) proceeds by showing the following sequence of inclusions and equalities:

\[
I^{(hn)}(1) \subseteq \mathcal{J}
(2) \subseteq \mathcal{J}(3) \subseteq \mathcal{J}(4) \subseteq I^n.
\]

These inclusions and equalities follow from the following formal properties of multiplier ideals (using terminology from [MS18, p. 915]):

1. (Not too small) \( \mathfrak{a} \subseteq \mathcal{J}(X, \mathfrak{a}^1) \).
2. (Unambiguity of exponent) For every integer \( n > 0 \), we have \( \mathcal{J}(X, \mathfrak{a}^{tn}) = \mathcal{J}(X, (\mathfrak{a}^t)^n) \).
3. (Subadditivity [DEL00, Variant 2.5]) \( \mathcal{J}(X, \mathfrak{a}^n) \subseteq \mathcal{J}(X, \mathfrak{a}^{tn}) \).
4. (Not too big [ELS01, Proof of Variant on p. 251; TY08, Proof of Theorem 4.1]) If \( h \) is the largest analytic spread of \( \mathfrak{a}R_p \) where \( p \) ranges over all associated primes of \( R/\mathfrak{a} \), then

\[
\mathcal{J}(X, (\mathfrak{a}^{tn})^{1/n}) \subseteq \mathfrak{a}.
\]

Hara [Har05, Theorem 2.12] adapted this approach to give an alternative proof of Theorem A in equal characteristic \( p > 0 \) using the theory of test ideals \( \tau(R, \mathfrak{a}^t) \) introduced in [HH90, §8; HY03].

In mixed characteristic, Ma and Schwede [MS18] used perfectoid techniques to define and develop the theory for a mixed characteristic analogue of multiplier/test ideals, the perfectoid test ideals

\[
\tau(R, [f]^t) \quad \text{and} \quad \tau(R, \mathfrak{a}^t).
\]

Here, \( R \) is a complete regular local ring of mixed characteristic, \([f] \) represents a choice of elements \( f_1, f_2, \ldots, f_r \in R \), and \( \mathfrak{a} \) is the ideal \( (f_1, f_2, \ldots, f_r) \). These two ideals are related in the following manner:

\[
\tau(R, [f]^t) \subseteq \tau(R, \mathfrak{a}^t).
\]

To prove the special case of Theorem A when \( R \) is a regular ring of mixed characteristic with geometrically reduced formal fibers and \( I \) is radical, Ma and Schwede first reduce to the complete local case (the assumption on the formal fibers of \( R \) is needed to preserve the radicalness of \( I \)). Ma and Schwede then prove and apply analogues of properties (1)–(3) for the ideal \( \tau(R, [f]^t) \) [MS18, Proposition 3.8, Proposition 3.9, and Theorem 4.4]. The assumption that \( I \) is radical is used to prove the special case of (4) for the ideal \( \tau(R, \mathfrak{a}^t) \) when \( \mathfrak{a} \) is radical [MS18, Theorem 5.11].

As mentioned above, removing the condition that \( I \) is radical is the key obstacle to proving Theorem A in the most general setting. To apply [MS18, Theorem 5.11], it is necessary in [MS18] to reduce to the case when \( I \) is a radical ideal in a complete regular local ring \( R \). Even if one starts with a radical ideal \( I \), after reducing to the complete local case, it may no longer be radical. In addition, our strategy for Theorems B and C in Part II requires stronger forms of Ma and Schwede’s unambiguity statement for exponents (2) [MS18, Proposition 3.8] and subadditivity (3) [MS18, Theorem 4.4].

In Part II of this paper, we instead use the test ideals

\[
\tau_B(R, \Delta, [f]^t) \quad \text{and} \quad \tau_B(R, \Delta, \mathfrak{a}^t)
\]

which were defined by Robinson [Rob22] for normal complete Noetherian local rings of arbitrary characteristic such that \( K_R + \Delta \) is \( \mathbb{Q} \)-Cartier (see Definition 6.1). As with Ma and Schwede’s definition, \([f] \) represents a choice of generators for the ideal \( \mathfrak{a} \), and the two ideals are related in the following manner:

\[
\tau_B(R, \Delta, [f]^t) \subseteq \tau_B(R, \Delta, \mathfrak{a}^t).
\]
Robinson’s definition combines features of the perfectoid test ideals of [MS18] and of the big Cohen–Macaulay test ideals \( \tau_B(R, \Delta) \). Ma and Schwede [MS21] and Pérez and R.G. [PRG21] introduced big Cohen–Macaulay test ideals \( \tau_B(R, \Delta) \) of pairs \((R, \Delta)\). Robinson [Rob22] extended their definition to triples of the form \((R, \Delta, \bigcup \alpha^t)\) and \((R, \Delta, a^t)\) (see also [ST]). These definitions rely on the existence of \( R^+\)-algebras \( B \) that are big Cohen–Macaulay over \( R \), where \( R^+ \) is the absolute integral closure of \( R \). Big Cohen–Macaulay algebras were introduced by Hochster [Hoc75] and were shown to exist in [HH92; And18b] (see also [Hoc94; HH95; Shi18; DRG19; And20] and Remark 2.7).

Compared to the perfectoid test ideals of [MS18], Robinson’s definition works in arbitrary characteristic. It also does not use almost mathematics, and there is no perturbation present in the definition in contrast to [MS18, Definition 3.5]. These differences in the definition allow us in this paper to prove versions of the unambiguity statement for exponents \((2)\) (Proposition 6.13) and subadditivity \((3)\) (Theorem 6.18) that are stronger than those proved in [MS18] for perfectoid test ideals. These results are essential for our proofs of Theorems B and C.

We are able to prove Theorems A, B, and C for arbitrary (not necessarily radical) ideals in arbitrary (not necessarily excellent) regular rings by proving the following version of \((4)\) that holds for arbitrary (not necessarily radical) ideals \( I \).

**Theorem D.** Let \( R \) be a normal complete Noetherian local ring such that \( K_R \) is \( \mathbb{Q} \)-Cartier. Consider an ideal \( I \subseteq R \), and assume that the localizations of \( R \) at the associated primes of \( R/I \) have infinite residue fields.

Let \( h \) be the largest analytic spread of \( IR_p \), where \( p \) ranges over all associated primes of \( R/I \). For every integer \( M > 0 \) and every finite set of non-negative integers \( s_1, s_2, \ldots, s_n \), there exists an \( R^+\)-algebra \( B \) that is big Cohen–Macaulay over \( R \) such that

\[
\tau_B(R, (I^{(M)})^{s_i + h}) \subseteq I^{(s_i + 1)}
\]

for every \( i \). If \( R \) is of residue characteristic \( p > 0 \), then setting \( B = \widehat{R^+} \) suffices.

Here, \( \widehat{R^+} \) denotes the \( p \)-adic completion of the absolute integral closure of \( R \), which is a big Cohen–Macaulay algebra for complete Noetherian local domains of residue characteristic \( p > 0 \). This is due to Hochster and Huneke [HH92, Main Theorem 5.15] in equal characteristic \( p > 0 \) and to Bhatt [Bha, Corollary 5.17] in mixed characteristic.

Theorem D can be viewed as a version of the equal characteristic statements in [ELS01, Proof of Variant on p. 251], [Har05, Proof of Theorem 2.12], and [TY08, Proposition 2.3 and Proof of Theorem 4.1] that applies in all characteristics. Using Robinson’s test ideals and Theorem D, we then prove Theorems A, B, and C for arbitrary ideals in arbitrary regular rings of all characteristics simultaneously. The idea is to use our results for the test ideals \( \tau_B(R, \Delta, \bigcup \alpha^t) \) and \( \tau_B(R, \Delta, a^t) \) mentioned above to construct a sequence of inclusions as in \((1.6)\), instead of those for multiplier ideals or for existing versions of test ideals used in [ELS01; Har05; TY08; ST12; MS18] to prove previously known cases of Theorem A.

A key difficulty in proving Theorem D is that no single version of perfectoid/big Cohen–Macaulay test ideals satisfies all the properties necessary to prove Theorem D. To prove the analogous result \((4)\) for rings essentially of finite type over a field of characteristic zero, Takagi and Yoshida [TY08, Proof of Theorem 4.1] proceed by localizing at each associated prime \( p \) of \( R/I \) and show the following sequence of inclusions and equalities (see also [ELS01, Proof of Variant on p. 251] and Proposition...
These inclusions and equalities follow from the following formal properties of multiplier ideals which hold whenever $X$ is a normal scheme essentially of finite type over the complex numbers such that $K_X$ is $\mathbb{Q}$-Cartier:

(4a) (Localizing) The formation of $\mathcal{J}(X, a^t)$ is compatible with arbitrary localization.

(4b) (Skoda-type theorem [Laz04, Theorem 9.6.36]) Suppose that $X = \text{Spec}(R)$ for a local ring $R$ and that $a$ has analytic spread $h$. For all $t \geq h$, we have $\mathcal{J}(X, a^t) = a \cdot \mathcal{J}(X, a^{t-1})$.

Hara [Har05, Proof of Theorem 2.12] and Takagi–Yoshida [TY08, Proposition 2.3] used this approach for test ideals in equal characteristic $p > 0$.

In mixed characteristic, no version of perfectoid/big Cohen–Macaulay test ideals is currently known to be compatible with arbitrary localizations. However, the $+t$-test ideals of Hacon, Lamarche, and Schwede [HLS] satisfy a weaker version of (4a) and also satisfy (4b). The definition of $+t$-test ideals relies on the result that the $p$-adic completion $\hat{R}^+$ of the absolute integral closure of $R$ is a big Cohen–Macaulay algebra for complete Noetherian local domains of residue characteristic $p > 0$ mentioned above [HH92, Main Theorem 5.15; Bha, Corollary 5.17]. Building on these results and subsequent developments due to Takamatsu and Yoshikawa [TY23] and to Bhatt, Ma, Patakfalvi, Schwede, Tucker, Waldron, and Witaszek [BMPSTWW], Hacon, Lamarche, and Schwede [HLS] introduced the $+t$-test ideals $\tau_+(\mathcal{O}_X, \Delta)$ and $\tau_+(\mathcal{O}_X, a^t)$ for divisor pairs $(X, \Delta)$ and ideal pairs $(X, a^t)$, respectively, where $X$ is a normal integral quasi-projective scheme over a complete Noetherian local ring $(R, \mathfrak{m})$ of residue characteristic $p > 0$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

These $+t$-test ideals satisfy the following versions of (4a) and (4b), where $X$ and $R$ are as in the previous paragraph:

(4a') (Compatibility with open embeddings [HLS, Corollary 5.8]) If $U \hookrightarrow X$ is an open embedding, then $\tau_+(\mathcal{O}_X, \Delta)|_U = \tau_+(\mathcal{O}_U, \Delta|_U)$.

(4b') (Skoda-type theorem [HLS, Theorem 6.6]) Suppose that $X = \text{Spec}(R)$ and that $a$ has analytic spread $h$. For all $t \geq h$, we have $\tau_+(\mathcal{O}_X, a^t) = a \cdot \tau_+(\mathcal{O}_X, a^{t-1})$.

We highlight six key aspects of our proof of Theorems A, B, and C in Part II.

(1) Because Robinson’s big Cohen–Macaulay test ideals $\tau_B(R, \Delta, [f]^t)$ do not use almost mathematics and do not incorporate perturbations, we are able to prove stronger versions of the unambiguity statement for exponents (2) (Proposition 6.13) and subadditivity (3) (Theorem 6.18) compared to [MS18]. This extra flexibility is essential in our proofs of Theorems B and C. Our subadditivity theorem (Theorem 6.18) requires working with a fixed set of generators $f_1, f_2, \ldots, f_r$ as was also the case in [MS18].

To prove Theorem D in residue characteristic $p > 0$, we proceed as follows.

(2) To make properties (4a') and (4b') satisfied by Hacon–Lamarche–Schwede’s $+t$-test ideals available to us, we prove the following comparisons between the various versions of test ideals in [MS21; PRG21; Rob22; ST; HLS]. For normal complete Noetherian local rings $(R, \mathfrak{m})$ of residue characteristic $p > 0$ whose canonical divisors $K_R$ are $\mathbb{Q}$-Cartier, and for all
ideals \( a \subseteq R \) generated by elements \( f_1, f_2, \ldots, f_r \), the special instances of Robinson’s test ideals when \( B = \hat{R}^+ \) and \( \Delta = 0 \) are related to the definitions in [MS21; PRG21; ST; HLS] in the following manner for rational numbers \( t \geq 0 \):

\[
\tau_{\hat{R}^+}(R, [f]^t) \subseteq \tau_{\hat{R}^+}(R, a^t) = \sum_{m=1}^{\infty} \sum_{g \in a^m} \tau_{\hat{R}^+}\left( R, \frac{t}{m} \text{div}_R(g) \right) = \sum_{m=1}^{\infty} \sum_{g \in a^m} \tau_+ \left( \mathcal{O}_{\text{Spec}(R)}, \frac{t}{m} \text{div}_R(g) \right) \subseteq \tau_+ \left( \mathcal{O}_{\text{Spec}(R)}, a^t \right).
\]

See Proposition 6.4, Lemma 6.6, Remark 5.5, and Lemma 5.8.

(3) Instead of localizing at prime ideals as in (1.8), it is enough to find a suitable principal localization where we can show that the sequence of inclusions and equalities in (1.8) hold. Since we need to apply (4a’), which holds for +-test ideals of divisor pairs \((X, \Delta)\), we use the penultimate ideal in (1.9) together with properties of reductions of ideals (see [SH06, Chapter 8]).

(4) We use the last inclusion in (1.9) to pass to a +-test ideal of an ideal pair \((X, a^t)\) and apply the Skoda-type theorem (4b’). This step requires using the fact that +-test ideals are compatible with integral closures of ideals [HLS, Remark 6.4].

Finally, to prove Theorem D in equal characteristic zero, we proceed as follows.

(5) We prove a comparison between Robinson’s big Cohen–Macaulay test ideal and multiplier ideals (Theorem 6.22) that strengthens [MS18, Theorem 6.3; MS21, Proposition 3.7 and Theorem 6.21; MSTWW22, Theorem 5.1; Rob22, Theorem 3.9]. This requires proving that weakly functorial big Cohen–Macaulay \( \hat{R}^+ \)-algebras exist in equal characteristic zero (2.8). This existence result uses ultraproducts, Lefschetz hulls, and the theory of seed algebras in equal characteristic zero [AS07; Sch10; DRG19] and is of independent interest.

(6) We then apply the theory of multiplier ideals on excellent schemes of equal characteristic zero, which was developed in [dFM09; JM12; ST]. An essential new ingredient is our recent generalization of the Kawamata–Viehweg vanishing theorem for proper morphisms of schemes of equal characteristic zero [Mur, Theorem A], which is used to verify the Skoda-type theorem (4b) when \( X \) is no longer essentially of finite type over a field.

As a result, our proof strategy using multiplier/test ideals applies to all characteristics. We also obtain new proofs of Theorems B and C in equal characteristic that use multiplier/test ideals. As far as we are aware, the only proofs of Theorems B and C in full generality in equal characteristic that exist in the literature [Joh14] use tight closure (see [TY08, Theorems 3.1 and 4.1] for a proof of a special case of Theorem C using multiplier/test ideals).

Lastly, in equal characteristic zero, we also give an independent proof of Theorems A, B, and C using only multiplier ideals. This new proof of Theorem A in equal characteristic zero does not rely on the Néron-type desingularization theorem due to Artin and Rotthaus [AR88, Theorem 1]. This proof therefore answers a question of Schoutens [Sch03, p. 179 and p. 187], who asked whether one could show Theorem A in equal characteristic zero without using [AR88, Theorem 1]. Note that the proof described above using Robinson’s version of test ideals and Theorem D still depends on the theorem of Artin and Rotthaus [AR88, Theorem 1] because as far as we are aware, all proofs for the existence of big Cohen–Macaulay \( \hat{R}^+ \)-algebras in equal characteristic zero rely on [AR88, Theorem 1] or stronger results.

\textit{Remark 1.10.} After writing this paper, we were informed by Wenliang Zhang that Zhang had independently shown a version of Theorem B when \( I \) is radical and \( R \) has geometrically reduced formal fibers using the methods in [MS18].
1.4. Outline. This paper is structured as follows.

We review some preliminaries in §2. We show that the definition of symbolic powers in (1.1) matches the definition used in [HH02] (Lemma 2.1). We also prove a reduction step for our main theorems (Lemma 2.4) that will be used in both parts of this paper. We then review the notions of absolute integral closures originating from [Art71] and of (balanced) big Cohen–Macaulay algebras from [Hoc75; Sha81]. We then prove that weakly functorial big Cohen–Macaulay \( R^+ \)-algebras exist in equal characteristic zero (Theorem 2.8) using the methods developed in [AS07; DRG19]. This last result is of independent interest.

The remainder of the paper is divided into two parts, corresponding to the two proof strategies for our main theorems using closure theory and multiplier/test ideals, respectively.

In Part I, we prove Theorems A, B, and C via closure theory. In §3, we review the necessary background material we need from closure theory. The closure operations we use in mixed characteristic, namely full extended plus (epf) closure [Hei01] and weak epf (wepf) closure [Jia21], are defined in §3.1. We then state Dietz’s axioms for closure operations [Die10] and R.G.’s algebra axiom [RG18], and collect references for proofs of these axioms and for Briançon–Skoda-type theorems for various closure operations in Table 1. We also state a result of R.G. from [RG18] stating that Dietz closures satisfying R.G.’s algebra axiom are contained in big Cohen–Macaulay algebra closures (Proposition 3.10). Finally, we prove Theorems A, B, and C in §4.

In Part II, we prove Theorems A, B, C, and D via multiplier/test ideals. In §5, we review some definitions and preliminaries on multiplier/test ideals. In particular, we define the big Cohen–Macaulay test ideals from [MS21], the + -test ideals from [HLS], and the multiplier ideals from [dFM09; ST]. In §6, we state the definition of Robinson’s version of perfectoid/big Cohen–Macaulay test ideals from [Rob22]. We then prove that they are related to the test ideals from §5 (Lemma 6.6). In the remainder of §6, we prove foundational results on Robinson’s test ideals necessary for our proofs of our main theorems, in particular an unambiguity statement for exponents (Proposition 6.13), the subadditivity theorem (Theorem 6.18), and a comparison with multiplier ideals (Theorem 6.22). Finally, in §7, we prove Theorems A, B, C, and D using the various versions of test ideals in [MS21; PRG21; Rob22; ST; HLS]. In equal characteristic zero, we also use multiplier ideals.

Conventions. All rings are commutative with identity, and all ring maps are unital.

Acknowledgments. We are grateful to Hailong Dao, Rankeya Datta, Neil Epstein, Eloïsa Grifo, Jack Jeffries, Zhan Jiang, János Kollár, Zhenqian Li, Mircea Mustaţă, Rebecca R.G., Daniel Smolkin, Kevin Tucker, and Farrah Yhee for helpful conversations. We would especially like to thank Melvin Hochster, Linquan Ma, Karl Schwede, Irena Swanson, Jugal Verma, and Wenliang Zhang for discussing their results with us. We are also grateful to Karl Schwede for pointing out the reference [Rob22]. Finally, we would like to thank Farrah Yhee for helpful edits on multiple drafts of the introduction to this manuscript.

2. Preliminaries

2.1. Symbolic powers. For completeness, we show that the definition of symbolic powers in (1.1) matches the definition used in [HH02]. The former definition is the one used in the survey [DDSGHNBJB18, p. 388]. A similar argument will appear in the proof of Proposition 4.1(iii).

Lemma 2.1. Let \( R \) be a Noetherian ring, and let \( I \subseteq R \) be an ideal. Denote by \( R_W \) the localization of \( R \) with respect to the multiplicative set \( W = \bigcup_{p \in \text{Ass}_R(R/I)} p \). Then, we have

\[
I^n R_W \cap R = \bigcap_{p \in \text{Ass}_R(R/I)} I^n R_p \cap R. \tag{2.2}
\]
Proof. The natural maps $R \to R_p$ factor through $R_W$ by the universal property of localization. We therefore have inclusions $I^n R_W \cap R \subseteq I^n R_p \cap R$ for every $p \in \text{Ass}_R(R/I)$, which yields the inclusion “$\subseteq$” in (2.2). Thus, it suffices to show the inclusion “$\supseteq$” in (2.2).

Let $u \in \bigcap_{p \in \text{Ass}_R(R/I)} I^n R_p \cap R$. Denote by $\{p_\ell\}$ the subset of $\text{Ass}_R(R/I)$ consisting of the associated primes that are maximal in $\text{Ass}_R(R/I)$ with respect to inclusion. For each $\ell$, there exists an element $c_\ell \notin p_\ell$ such that $c_\ell u \in I^n$. For each $\ell$, we can also choose $d_\ell \in p_\ell - \bigcup_{j \neq \ell} p_j$ by prime avoidance. Now consider the element
\[
c = \sum_{\ell} \left( c_\ell \prod_{j \neq \ell} d_j \right).
\]
Note that $c \notin \bigcup_{\ell} p_\ell$, for otherwise if $c \in p_{\ell_0}$ for some $\ell_0$, we have
\[
c_{\ell_0} \prod_{j \neq \ell_0} d_j = c - \sum_{\ell \neq \ell_0} \left( c \prod_{j \neq \ell} d_j \right) \in p_{\ell_0},
\]
which contradicts the fact that $c_{\ell_0} \notin p_{\ell_0}$ and $d_j \notin p_{\ell_0}$ for every $j \neq \ell_0$. Since $c_\ell u \in I^n$ by assumption, we see that
\[
c u = \sum_{\ell} \left( c_\ell u \prod_{j \neq \ell} d_j \right) \in I^n.
\]
Finally, since $c \in W = R - \bigcup_{\ell} p_\ell$, we have $u \in I^n R_W \cap R$. \qed

Remark 2.3. There is a different notion of symbolic powers in the literature due to Verma [Ver87, p. 205] obtained by intersecting over minimal primes instead of associated primes on the right-hand side of (2.2). Using notation from [HJKN23, p. 691], this version of the $n$-th symbolic power is defined as follows:
\[
m I^{(n)} := \bigcap_{p \in \text{Min}_R(R/I)} I^n R_p \cap R.
\]
The analogue of Theorem A using this notion of symbolic powers is not true. For example, let $K$ be a field and let $R = K[x, y]$. For every integer $k > 0$, the ideal
\[
I_k = (x)(x, y)^k = (x^{k+1}, x^k y, \ldots, x y^k) = (x) \cap (x, y)^{k+1} \subseteq R
\]
satisfies
\[
m(I_k)^{(kn)} = (x)^{kn} \nsubseteq (x)^n(x, y)^{kn} = (I_k)^n.
\]

2.2. Reductions for main theorems. We prove some initial reductions for Theorems B and C that will be used in both Parts I and II.

Lemma 2.4 (cf. [HH02, Lemma 2.4(b) and Theorem 4.4; TY08, Theorem 3.1]). It suffices to show Theorems B and C under the additional assumptions that $R$ is complete local ring $(R, m)$, and that the localizations of $R$ at the associated primes of $R/I$ have infinite residue fields. Moreover, after this reduction, we may further replace the residue field $k$ of $(R, m)$ with any field $k'$ containing it.

Proof. We proceed in four steps.

Step 1. It suffices to consider the case when the localizations of $R$ at the associated primes of $R/I$ have infinite residue fields.

Let $X$ be an indeterminate. Since $R \to R[X]$ is faithfully flat, it suffices to show (1.3) and (1.4) after extending scalars to $R[X]$. By [HH02, Discussion 2.3(b)], the associated primes of $IR[X]$ are of the form $p R[X]$ where $p \in \text{Ass}_R(R/I)$, the analytic spreads of $IR[X]_{p R[X]}$ are the same as those for $IR_p$, and the symbolic powers of $IR[X]$ are the extensions of the symbolic powers of $I$. We may therefore replace $R$ by $R[X]$ and $I$ by $I[X]$ to assume that the localizations of $R$ at the associated primes of $R/I$ have infinite residue fields. For Theorem C, we furthermore replace $R[X]$ by $R[X]_{mR[X]}$ to remain in the local case.
Step 2. It suffices to consider the case when $R$ is local and the localizations of $R$ at the associated primes of $R/I$ have infinite residue fields.

For Theorem B, the inclusion (1.3) can be checked after localizing at every prime ideal of $R$, and hence we may assume that $R$ is local. For Theorem C, we are already in the local case.

Step 3. For $R$ as in Step 2, it suffices to show Theorems B and C after passing to a faithfully flat local extension $R \to S$.

By [HH02, Discussion 2.3(c)], the analytic spread of $IS_q$ for every $q \in \text{Ass}_S(S/IS)$ is at most the analytic spread of $IR_{q\cap R}$. Moreover, we have

$$(IS)^{(n)} \cap R = \bigcap_{q \in \text{Ass}_S(S/IS)} I^nS_q \cap R = \bigcap_{p \in \text{Ass}_R(R/I)} I^nR_p \cap R = I^{(n)}$$

for all $n \geq 1$, where the middle equality follows from [Bou72, Chapter IV, §2, no. 6, Theorem 2 and Corollary 1] and the fact that $R_p \to S_q$ is faithfully flat when $p = q \cap R$. We may therefore replace $R$ by $S$. Note that for every $q \in \text{Ass}_S(S/IS)$, the residue field of $S_q$ contains the residue field of $R_{q \cap R}$, and hence is still infinite.

Step 4. For $R$ as in Step 2, we can find a faithfully flat local extension $(R, m, k) \to (S, n, l)$ such that $l = k'$ and $S$ is complete.

By [Bou06, Chapitre IX, Appendice, n° 2, Corollaire au Théorème 1], there exists a gonflement $(R, m) \to (R', m')$ where $R'/m' = k'$. This map is a flat local map such that $R'$ is regular [Bou06, Chapitre IX, Appendice, n° 2, Proposition 2]. We now consider the composition $R \to R' \to \widehat{R'}$, where the second map is the $m'$-adic completion map. This map is faithfully flat, and Step 3 shows that we may replace $R$ by $S = \widehat{R'}$.

2.3. Absolute integral closures. We now define the absolute integral closure of an integral scheme following [HLS]. This definition is modeled after Artin’s notion of the absolute integral closure of an integral domain [Art71, p. 283].

Definition 2.5 (see [HLS, §2.2]). Let $X$ be an integral scheme. Fix an algebraic closure $\overline{K(X)}$ of the function field $K(X)$ of $X$, and consider the collection $\{f: Y \to X\}$ of all finite surjective morphisms from integral schemes fitting into the commutative diagram

$$\text{Spec}(\overline{K(X)}) \longrightarrow Y \quad \downarrow f \quad \longrightarrow X.$$ 

The absolute integral closure of $X$ is the inverse limit

$$X^+ := \lim_{\longrightarrow} Y$$

in the category of schemes, which exists by [EGAIV3, Proposition 8.2.3]. We denote the canonical projection morphism by $\nu: X^+ \to X$.

Now suppose that $X = \text{Spec}(R)$ for an integral domain $R$. The absolute integral closure of $R$ is the ring $R^+$ of global sections of $X^+$, which is an affine scheme by [EGAIV3, (8.2.2)]. The absolute integral closure $R^+$ can also be described as the integral closure of $R$ in an algebraic closure $\text{Frac}(R)$ of its fraction field.

2.4. Big Cohen–Macaulay algebras. Next, we define (balanced) big Cohen–Macaulay algebras.

Definition 2.6 [Hoc75, pp. 110–111; Sha81, Definition 1.4]. Let $(R, m)$ be a Noetherian local ring of dimension $d$, and let $x_1, x_2, \ldots, x_d$ be a system of parameters of $R$. An $R$-algebra $B$ is big Cohen–Macaulay over $R$ with respect to $x_1, x_2, \ldots, x_d$ if $x_1, x_2, \ldots, x_d$ is a regular sequence on $B$,
and is a (balanced) big Cohen–Macaulay algebra over $R$ if it is big Cohen–Macaulay over $R$ with respect to every system of parameters $x_1, x_2, \ldots, x_d$.

Remark 2.7. Let $(R, \mathfrak{m})$ be a Noetherian local ring.

(i) If $R$ is of equal characteristic, then there is a big Cohen–Macaulay algebra over $R$ [HH92, Theorem 8.1] (see also [Hoc94, Theorems 11.1 and 11.4]). This big Cohen–Macaulay algebra can be taken to be an $R^+$-algebra if $R$ is a domain. In equal characteristic $p > 0$, this follows from the construction in [HH92, Theorem 8.1] and applying [HH95, Proposition 1.2] (see also [Hoc94, Theorem 11.1; Die07, Theorem 6.9]). In equal characteristic zero, this was shown in [DRG19, Corollary 5.2], and can also be shown using the methods in [HH95; HH], as pointed out to us by Hochster (cf. [Hoc94, Theorem 11.4]).

(ii) If $R$ is an excellent biequidimensional domain of equal characteristic $p > 0$ or a domain of equal characteristic $p > 0$ that is a homomorphic image of a Gorenstein local ring, then the absolute integral closure $R^+$ of $R$ is a big Cohen–Macaulay algebra over $R$ [HH92, Main Theorem 5.15; HL07, Corollary 2.3(b)].

(iii) If $R$ is of mixed characteristic, then $R$ has a big Cohen–Macaulay algebra [And18b, Théorème 0.7.1]. This big Cohen–Macaulay algebra can be taken to be an $R^+$-algebra if $R$ is a domain [Shi18, Corollary 6.5 and Remark 6.6; And20, Theorem 3.1.1].

(iv) If $R$ is an excellent domain of mixed characteristic, then the $p$-adic completion $\hat{R}^+$ of the absolute integral closure $R^+$ of $R$ is a big Cohen–Macaulay algebra over $R$ [Bha, Corollary 5.17; BMPSTWW, Corollary 2.10].

2.5. Weakly functorial big Cohen–Macaulay $R^+$-algebras in equal characteristic zero.

We prove the following result showing the existence of weakly functorial big Cohen–Macaulay $R^+$-algebras in equal characteristic zero using the work of Aschenbrenner and Schoutens [AS07] and Dietz and R.G. [DRG19]. This result is of independent interest and will only be used in Part II of this paper when proving Theorem D in equal characteristic zero.

The formulation of this existence result is based on the mixed characteristic statements in [And20, Theorems 1.2.1 and 4.1.1; MSTWW22, Theorem A.5]. See [Sch10, §2.1] for definitions of ultraproducts and ultrarings, see [AS07, §4] for the definition of a Lefschetz hull and [AS07, p. 261] for the definition of an absolutely normalizing Lefschetz hull, and see [DRG19, Definition 2.3 and Lemma 2.5] for the definition of a rational seed.

**Theorem 2.8.** Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local map of Noetherian local domains of equal characteristic zero. Fix faithfully flat, absolutely normalizing Lefschetz hulls

\[ \mathcal{D}(R) = \operatorname{ulim}_{w \in W} R_w \quad \text{and} \quad \mathcal{D}(S) = \operatorname{ulim}_{w \in W} S_w \]

for $R$ and $S$ with characteristic $p > 0$ approximations $R_w$ and $S_w$, where $W$ is an infinite set with a non-principal ultrafilter $W$, such that the diagram

\[ \begin{array}{ccc} \mathcal{D}(R) & \to & \mathcal{D}(S) \\ \uparrow & & \uparrow \\ R & \to & S \end{array} \]

commutes.

(i) There exists an ultraring $B_2$ with respect to $W$ that is an $R^+$-algebra and is a big Cohen–Macaulay algebra over $R$. 
(ii) For every choice of $B_\natural$ as in (i), there exists a commutative diagram

\[
\begin{array}{ccc}
B_\natural & \longrightarrow & C_\natural \\
\uparrow & & \uparrow \\
R^+ & \xrightarrow{f^+} & S^+ \\
\uparrow & & \uparrow \\
R & \xrightarrow{f} & S \\
\end{array}
\] (2.9)

where $C_\natural$ is a big Cohen–Macaulay algebra over $S$ that is an ultraring with respect to $W$. Moreover, $f^+$ can be given in advance.

Proof. The first diagram exists by [AS07, Remark 4.26], where we note that the choice of Lefschetz hull depends on the choice of a Lefschetz field of sufficiently large cardinality (e.g. larger than $2^{\lvert R \rvert}$ and $2^{\lvert S \rvert}$). We use the construction in [AS07, Remark 4.26] to moreover choose $\mathcal{D}(R)$ and $\mathcal{D}(S)$ to be absolutely normalizing.

We next show (i). Since $\mathcal{D}(R)$ was chosen to be absolutely normalizing, the construction in [AS07, (7.7)] shows that $R$ is a rational seed in the sense of [DRG19, Definition 2.3] (see also [DRG19, Theorem 2.4]). By [DRG19, Theorem 5.1], the absolute integral closure $R^+$ is also a rational seed over $R$ (see [DRG19, Corollary 5.2]), and hence $B_\natural$ exists.

For (ii), if $f^+$ is not given in advance, then we can construct the bottom square in the diagram by [HH95, Proposition 1.2]. Now given such a square, since $B_\natural \otimes_R S$ and $S^+$ are rational seeds over $S$ by [DRG19, Theorem 3.5 and Corollary 5.2], the tensor product $B_\natural \otimes_R S^+ \cong B_\natural \otimes_R S \otimes_S S^+$ is a rational seed over $S$ by [DRG19, Theorem 3.3]. Thus, there is a map $B_\natural \otimes_R S^+ \to C_\natural$ to a big Cohen–Macaulay algebra over $S$ that is an ultraring with respect to $W$ and makes the diagram commute. \qed

Part I. Proof via closure theory

3. Closure operations

In this section, we review some preliminaries on closure operations. We first define full extended plus (epf) closure [Hei01] and weak epf (wepf) closure [Jia21]. Next, we state Dietz’s axioms for closure operations [Die10] and R.G.’s algebra axiom [RG18], and collect references for proofs of these axioms and for Briançon–Skoda-type theorems for various closure operations in Table 1. We also state a result of R.G. from [RG18] stating that Dietz closures satisfying R.G.’s algebra axiom are contained in big Cohen–Macaulay algebra closures (Proposition 3.10), which is a key ingredient in our closure-theoretic proof of Theorems A, B, and C.

3.1. Closure operations in mixed characteristic. We recall the definition of Heitmann’s full extended plus (epf) closure.

Definition 3.1 [Hei01, Definition on pp. 804–805; RG16, Definition 7.1; HM21, Definition 2.3; Jia21, Definition 2.3]. Let $p > 0$ be a prime number. Let $R$ be a domain such that the image of $p$ lies in the Jacobson radical of $R$. For an inclusion $Q \subseteq M$ of finitely generated $R$-modules, the full extended plus (epf) closure of $Q$ in $M$ is

\[
Q_M^{\text{epf}} := \left\{ u \in M \mid \exists c \in R - \{0\} \text{ such that } c^e \otimes u \in \text{im}(R^+ \otimes_R Q \to R^+ \otimes_R M) + p^N(R^+ \otimes_R M) \right\},
\]

for every $e \in Q_{>0}$ and every $N \in \mathbb{Z}_{>0}$.

We also recall the definition of Jiang’s weak epf (wepf) closure.
We consider the following axioms:

We denote $I$ by $I_{\text{axiom}}$.

Notation 3.3. We denote by $R$ a ring, and by $Q, M$, and $W$ arbitrary finitely generated $R$-modules such that $Q \subseteq M$. We consider an operation $cl$ sending submodules $Q \subseteq M$ to an $R$-module $Q_{cl}$. We denote $I_{cl} := I_{R}^{cl}$ for ideals $I \subseteq R$.

We state Dietz’s axioms for closure operations from [Die10] with some conventions from [Eps12, Definition 2.1.1; Die18, Definition 1.1].

Axioms 3.4 [Die10, Axioms 1.1]. Fix notation as in Notation 3.3. We say that the operation $cl$ is a closure operation if it satisfies the following three axioms:

1. (Extension) $Q_{cl}^{cl}$ is a submodule of $M$ containing $Q$.
2. (Idempotence) $(Q_{cl}^{cl})^{cl} = Q_{cl}^{cl}$.
3. (Order-preservation) If $Q \subseteq M \subseteq W$, then $Q_{cl}^{cl} \subseteq M_{cl}^{cl}$.

We also consider the following axioms:

4. (Functoriality) Let $f : M \rightarrow W$ be a map of $R$-modules. Then, $f(Q_{cl}^{cl}) \subseteq f(Q)^{cl}$.
5. (Semi-residuality) If $Q_{cl}^{cl} = Q$, then $0_{cl}^{cl} = 0$.

Now suppose that $R$ is a Noetherian local domain $(R, \mathfrak{m})$. We say that $cl$ is a Dietz closure if it satisfies axioms (1)–(5) above, and the following two additional axioms:

6. The maximal ideal $\mathfrak{m}$ and the zero ideal $0$ are $cl$-closed in $R$, i.e., $\mathfrak{m}^{cl} = \mathfrak{m}$ and $0^{cl} = 0$.
7. (Generalized colon-capturing) Let $x_1, x_2, \ldots, x_{k+1}$ be a partial system of parameters for $R$ and let $J = (x_1, x_2, \ldots, x_k)$. Suppose there exists a surjective map $f : M \rightarrow R/J$ of $R$-modules, and let $v \in M$ be an arbitrary element such that $f(v) = x_{k+1} + J$. Then,

\[ (Rv)^{cl}_{M} \cap \ker(f) \subseteq (Jv)^{cl}_{M}. \]

To state R.G.’s algebra axiom, we first define the notion of a $cl$-phantom extension.

Definition 3.5 [Die10, Definition 2.2]. Fix notation as in Notation 3.3. Suppose $cl$ satisfies axioms (1)–(5) above. Let $M$ be a finitely generated $R$-module and let $\alpha : R \rightarrow M$ be an injective map of $R$-modules. Consider the short exact sequence

\[ 0 \rightarrow R \xrightarrow{\alpha} M \rightarrow Q \rightarrow 0, \]

and let $\epsilon \in \text{Ext}_{R}^{1}(Q, R)$ be the element corresponding to this short exact sequence via the Yoneda correspondence.

Fix a projective resolution $P_{\bullet}$ of $Q$ consisting of finitely generated projective $R$-modules $P_{i}$. We say that $\epsilon$ is $cl$-phantom if

\[ \epsilon \in \left( \text{im} \left( \text{Hom}_{R}(P_{0}, R) \rightarrow \text{Hom}_{R}(P_{1}, R) \right) \right)^{cl}_{\text{Hom}_{R}(P_{1}, R)}. \]

This definition does not depend on the choice of $P_{\bullet}$ by [Die10, Discussion 2.3].

We say that $\alpha$ is a $cl$-phantom extension if $\epsilon$ is $cl$-phantom.

We now state R.G.’s algebra axiom.
### Axiom 3.6 [RG18, Axiom 3.1]
Fix notation as in Notation 3.3. If cl satisfies axioms (1)-(5) above, we consider the following axiom:

(8) (Algebra axiom) Let $\alpha : R \to M$ be a map of $R$-modules. If $\alpha$ is an $\text{cl}$-phantom extension, then the map $\alpha' : R \to \text{Sym}_R^2(M)$ where $1 \mapsto \alpha(1) \otimes \alpha(1)$ is a $\text{cl}$-phantom extension.

We also introduce an axiom asserting that a closure-theoretic version of the Briançon–Skoda theorem holds.

### Axiom 3.7
Fix notation as in Notation 3.3. If cl satisfies axiom (1) above, we consider the following axiom:

(9) (Briançon–Skoda-type theorem) Let $I \subseteq R$ be an ideal generated by at most $h$ elements. Then, $I^{h+k} \subseteq (I^{k+1})^{\text{cl}}$ for every integer $k \geq 0$.

### 3.3. Algebra closures
The key result we need about Dietz closures satisfying R.G.’s algebra axiom (8) is that they are related to algebra closures, which are defined as follows.

#### Definition 3.8 [RG16, Definition 2.3 and Remark 2.4]
Let $R$ be a ring, and let $S$ be an $R$-algebra. Let $Q \subseteq M$ be an inclusion of finitely generated $R$-modules. We then set

$$Q^{\text{cl}_S} := \{ u \in M \mid 1 \otimes u \in \text{im}(S \otimes_R Q \to S \otimes_R M) \}.$$  

We call the operation $\text{cl}_S$ an algebra closure.

#### Examples 3.9
If $B$ is a big Cohen–Macaulay algebra over a Noetherian local domain, then the algebra closure $\text{cl}_B$ is a Dietz closure [Die10, Theorem 4.2] and satisfies R.G.’s algebra axiom (8) [RG18, Proposition 3.11]. We list two examples of such algebra closures appearing in Table 1 for which a Briançon–Skoda-type theorem (9) holds.

(i) Let $R$ be an excellent biequidimensional local domain of equal characteristic $p > 0$, or a Noetherian local domain of equal characteristic $p > 0$ that is a homomorphic image of a Gorenstein local ring. Then, $R^+$ is a big Cohen–Macaulay algebra over $R$ by [HH92, Main Theorem 5.15; HL07, Corollary 2.3(b)]. The associated algebra closure $\text{cl}_{R^+}$ is called plus closure [Smi94, Definition 2.13], and is denoted by $+$.  

(ii) Let $R$ be a Noetherian local ring of equal characteristic zero. Using ultraproducts, Aschenbrenner and Schoutens construct a big Cohen–Macaulay algebra $\mathcal{B}(R)$ over $R$ [AS07, (7.7)]. The associated algebra closure $\text{cl}_{\mathcal{B}(R)}$ is called $\mathcal{B}$-closure [AS07, (7.13)], and is also denoted by $+$.  

---

**Table 1.** Some closure operations on Noetherian complete local domains.

| characteristic | closure (cl) | Dietz closure | R.G.’s algebra axiom | Briançon–Skoda |
|----------------|-------------|---------------|----------------------|----------------|
| char. $p > 0$  | tight (+)  | [Die10, Ex. 5.4] | [RG18, Prop. 3.6]   | [HH94, Thm. 8.1] |
|                | plus (+)   | [Die10, Ex. 5.1] | [RG18, Prop. 3.11]  | [HH95, Thm. 7.1] |
| equal char. 0   | small equational tight (+eq) | [Die10, Ex. 5.7] | open$^*$             | [HH, Thm. 4.1.5] |
|                | big equational tight (+EQ)   | [Die10, Ex. 5.7] | open$^*$             | [HH, Thm. 4.1.5] |
|                | $\mathcal{B}_-$ (+)         | [Die10, Thm. 4.2] | [RG18, Prop. 3.11]  | [AS07, Thm. 7.14.3] |
| mixed char.     | full extended plus (epf)     | open$^*$         | [Hei01, Thm. 4.2]    |
|                | weak epf (wepf)             | [Jia18, Thm. 4.8] | $F$-finite           | [Hei01, Thm. 4.2] |
|                | full rank 1 (rf)            | [Jia18, Thm. 4.8 and Rem. 4.7] | $F$-finite          | [Hei01, Thm. 4.2] |

* As far as we are aware, it is open whether R.G.’s algebra axiom holds for these closure operations. However, a suitable replacement that works for our proofs does hold. See Remark 4.2.

$F$-finite Jiang’s results hold when $R/m$ is $F$-finite.
We now state the following result due to R.G., which says that Dietz closures satisfying R.G.’s algebra axiom (8) are contained in an algebra closure defined by a big Cohen–Macaulay algebra.

**Proposition 3.10 [RG18, Proposition 4.1].** Let $R$ be a Noetherian local domain and let $\text{cl}$ be a Dietz closure on $R$ that satisfies R.G.’s algebra axiom (8). Then, there exists a big Cohen–Macaulay algebra $B$ over $R$ such that $Q^\text{cl}_M \subseteq Q^B_M$ for every inclusion $Q \subseteq M$ of finitely generated $R$-modules.

### 4. Proof of Main Theorems via Closure Theory

We are now ready to prove Theorems A, B, and C using closure operations. Theorem A follows from Theorem B by setting $s_i = 0$ for all $i$. It therefore suffices to show Theorems B and C.

We start with the following result.

**Proposition 4.1.** Let $(R, \mathfrak{m})$ be a complete local domain. Consider an ideal $I \subseteq R$. Let $u \in I^{(M)}$ for an integer $M > 0$, and fix non-negative integers $s_1, s_2, \ldots, s_n$.

(i) Denote by $R'$ the integral closure of 
\[ R[u^{1/M}] \subseteq \text{Frac}(R)[u^{1/M}] \]

in $\text{Frac}(R)[u^{1/M}]$, where $u^{1/M}$ is a choice of $M$-th root of $u$ in an algebraic closure of $\text{Frac}(R)$. Then, $R'$ is a complete normal local domain $(R', \mathfrak{m}')$. Moreover, if $R/\mathfrak{m}$ is $F$-finite and of characteristic $p > 0$, then $R'/\mathfrak{m}'$ is $F$-finite and of characteristic $p > 0$.

Now assume that the localizations of $R$ at the associated primes of $R/I$ have infinite residue fields. Let $h$ be the largest analytic spread of $IR_\mathfrak{p}$, where $\mathfrak{p}$ ranges over all associated primes $\mathfrak{p}$ of $R/I$.

(ii) Fix a closure operation $\text{cl}$ on $R$ for which the Briançon–Skoda-type theorem (9) holds. Let $\mathfrak{p}_\ell \in \text{Ass}_R(R/I)$. Then, there exists $c_\ell \in R - \mathfrak{p}_\ell$ such that for every $i$, we have
\[ c_\ell u^{s_\ell + h} \in (I^{s_\ell + 1}R')^{\text{cl}}. \]

(iii) Suppose that if $R$ is of mixed characteristic, then $R/\mathfrak{m}$ is $F$-finite and of characteristic $p > 0$. Then, there exists a big Cohen–Macaulay algebra $B$ over $R'$ such that the following holds: For every $\mathfrak{p}_\ell \in \text{Ass}_R(R/I)$, there exists $c_\ell \in R - \mathfrak{p}_\ell$ such that for every $i$, we have
\[ c_\ell u^{s_\ell + h} \in I^{s_\ell + 1}B. \]

Moreover, if $R$ is regular, then for every $i$, we have
\[ u^{s_\ell + h} \in I^{(s_\ell + 1)}B. \]

**Proof.** We first prove (i). The ring $R'$ is module-finite over $R$ and is complete local by [EGAIV₁, Chapitre 0, Théorème 23.1.5 and Corollaire 23.1.6]. If $R/\mathfrak{m}$ is $F$-finite and of characteristic $p > 0$, then $R'/\mathfrak{m}'$ is $F$-finite and of characteristic $p > 0$ because the field extension $R/\mathfrak{m} \hookrightarrow R'/\mathfrak{m}'$ is a finite field extension.

Next, we prove (ii). Fix $\mathfrak{p}_\ell \in \text{Ass}_R(R/I)$. Since the residue field of $R_{\mathfrak{p}_\ell}$ is infinite, there exists an ideal $J_\ell \subseteq I$ with at most $h$ generators such that $J_\ell R_{\mathfrak{p}_\ell}$ is a reduction of $IR_{\mathfrak{p}_\ell}$ by [SH06, Proposition 8.3.7]. By [SH06, Lemma 8.1.3(1)], we also know that $J_\ell R_{\mathfrak{p}_\ell}'$ is a reduction of $I_\ell R_{\mathfrak{p}_\ell}'$, and by [SH06, Proposition 8.1.5], we know that $J_\ell^{s_\ell + h}R_{\mathfrak{p}_\ell}'$ is a reduction of $I^{s_\ell + h}R_{\mathfrak{p}_\ell}'$ for all $i$.

Since $u \in I^{(M)}$, there exists an element $x_\ell \in R - \mathfrak{p}_\ell$ such that $x_\ell u \in I^M$. We then have
\[ (x_\ell^{s_\ell + h} u^{s_\ell + h})^M = x_\ell^M (s_\ell + h) u^{s_\ell + h} = (x_\ell^M u)^{s_\ell + h} \in (I^{s_\ell + h}R')^M, \]
and hence
\[ x_\ell^{s_\ell + h} u^{s_\ell + h} \in I^{s_\ell + h}R'. \]
Since \( J^{s_i+h}_{\ell} R'_{p_{\ell}} = I^{s_i+h} R'_{p_{\ell}} \), there exists an element \( y_{\ell,i} \in R - p_{\ell} \) such that 
\[
y_{\ell,i} x^{s_i+h}_{\ell} u^\frac{a_i+h}{m} \in J^{s_i+h}_{\ell} R'.
\]
By the Briançon–Skoda-type theorem (9), we have 
\[
y_{\ell,i} x^{s_i+h}_{\ell} u^\frac{a_i+h}{m} \in \left( J^{s_i+1}_{\ell} R' \right)^{cl} \subseteq (I^{s_i+1} R')^{cl},
\]
where the inclusion on the right holds by the order-preservation axiom (3). Setting 
\[
y = y_{\ell,1} y_{\ell,2} \cdots y_{\ell,n},
\]
we therefore have 
\[
y x^{s_i+h}_{\ell} u^\frac{a_i+h}{m} \in (I^{s_i+1} R')^{cl}
\]
for all \( i \). Setting \( c_{\ell} = y x^{s_i+h}_{\ell} \), we obtain (ii).

Finally, we show (iii). Fix a Dietz closure \( cl \) on \( R \) satisfying R.G.’s algebra axiom and the Briançon–Skoda-type theorem (9). Note that such a closure operation exists in all characteristics by Table 1. By (ii), there exist \( c_{\ell} \in R - p_{\ell} \) such that 
\[
c_{\ell} u^\frac{a+i+h}{m} \in (I^{s_i+1} R')^{cl}
\]
for all \( i \). By R.G.’s result stating that Dietz closures satisfying R.G.’s algebra axiom are contained in a big Cohen–Macaulay algebra \( B \) over \( R' \) such that
\[
c_{\ell} u^\frac{a+i+h}{m} \in I^{s_i+1} B.
\]
This proves the first statement in (iii).

For the second statement in (iii), for each \( \ell \) such that \( p_{\ell} \) is maximal in \( \text{Ass}_R(R/I) \) with respect to inclusion, choose \( d_{\ell} \in p_{\ell} - (\bigcup_{j \neq \ell} p_j) \), which is possible by prime avoidance. As in the proof of Lemma 2.1, we have 
\[
c = \sum_{\ell} \left( c_{\ell} \prod_{j \neq \ell} d_j \right) \notin \bigcup_{\ell} p_{\ell}.
\]
We also have 
\[
c u^\frac{a+i+h}{m} \in I^{s_i+1} B \subseteq I^{(s_i+1)} B.
\]
We now note that \( B \) is a big Cohen–Macaulay algebra over \( R \), since every system of parameters in \( R \) maps to a system of parameters in \( R' \). Since \( R \) is regular, \( R \to B \) is therefore faithfully flat by [Hoc75, Lemma 5.5] (see also [HH92, (6.7); HH95, Lemma 2.1(d)]). Thus, since \( c \) is a nonzerodivisor on \( R/I^{(s_i+1)} \) by [Mat89, Theorem 6.1(ii)], it is also a nonzerodivisor on \( B/I^{(s_i+1)} B \). We therefore see that for every \( i \), we have
\[
u^\frac{a+i+h}{m} \in I^{(s_i+1)} B.
\]
\( \square \)

Remark 4.2. The proof of Proposition 4.1(iii) in fact applies to any closure operation satisfying a Briançon–Skoda-type theorem (9) and for which the following property holds for all inclusions \( Q \subseteq M \) of finitely generated \( R \)-modules:

\( (*) \) If \( u \in Q^n_M \), then there exists a big Cohen–Macaulay algebra \( B \) over \( R \) such that \( 1 \otimes u \in \text{im}(B \otimes_R Q \to B \otimes_R M) \).

The property (\( * \)) holds for all Dietz closures satisfying R.G.’s algebra axiom (8) by R.G.’s result stating that Dietz closures satisfying R.G.’s algebra axiom are contained in a big Cohen–Macaulay algebra closure (Proposition 3.10). However, (\( * \)) holds for the other closure operations listed in Table 1 as well.
(i) The property (\(\star\)) holds for big (and hence also small) equational tight closure on all Noetherian rings of equal characteristic zero [Hoc94, Theorem 11.4]. See [HH, Definition 3.4.3(b) and (4.6.1)] for definitions of these closure operations.

(ii) The property (\(\star\)) holds for epf closure for complete local domains of mixed characteristic with \(F\)-finite residue field. This follows from the fact that wepf closure is a Dietz closure satisfying R.G.’s algebra axiom (8) [Jia21, Theorem 4.8] since \(Q_M^{epf} \subseteq Q_M^{wepf}\).

(iii) A stronger version of (\(\star\)) holds for tight closure on all analytically irreducible excellent local domains of equal characteristic \(p > 0\) [Hoc94, Theorem 11.1]. We can therefore use these closure operations and property (\(\star\)) instead of Proposition 3.10 to prove Theorems B and C below.

Finally, we can prove Theorems B and C using closure operations.

**Proof of Theorems B and C via closure theory.** By Lemma 2.4, we may assume that \(R\) is a complete regular local ring \((R, \mathfrak{m})\) with a perfect residue field, and that the localizations of \(R\) at the associated primes of \(R/I\) have infinite residue fields.

We start with Theorem B. Setting \(M = s + nh\) in Proposition 4.1(iii), for \(u \in I^{(s+nh)}\), we have

\[
u^{s+h}_{s+nh} \in I^{(s+1)}B
\]

for some big Cohen–Macaulay algebra \(B\) over \(R\). Multiplying together these inclusions for every \(i\), we have

\[
u \in \prod_{i=1}^{n} I^{(s_i+1)}B.
\]

Since \(R\) is regular, \(R \rightarrow B\) is faithfully flat by [Hoc75, Lemma 5.5] (see also [HH92, (6.7); HH95, Lemma 2.1(d)]), and hence we have

\[
u \in \prod_{i=1}^{n} I^{(s_i+1)}.
\]

We now prove Theorem C. Setting \(M = s + nh + 1\) in Proposition 4.1(iii), for \(u \in I^{(s+nh+1)}\), we have

\[
u^{s+h}_{s+nh+1} \in I^{(s+1)}B
\]

for some big Cohen–Macaulay algebra \(B\) over \(R\). Multiplying together these inclusions for every \(i\), we have

\[
u^{s+h}_{s+nh+1} \in \prod_{i=1}^{n} I^{(s_i+1)}B.
\]

Let \(l\) be the largest integer for which the composition

\[
R \rightarrow B \xrightarrow{\nu^{s+h}_{s+nh+1}} B
\]

splits as a map of \(R\)-modules. We claim such an \(l\) exists. When \(l = 0\), the map \(R \rightarrow B\) is faithfully flat, hence pure [Mat89, Theorem 7.5(i)]. Thus, the map \(R \rightarrow B\) splits by an argument of Auslander [HH90, p. 59].

We now have

\[
u^{l}_{s+nh+1} u = u^{l+1}_{s+nh+1} u^{s+nh}_{s+nh+1} \in u^{l+1}_{s+nh+1} \cdot \prod_{i=1}^{n} I^{(s_i+1)}B.
\]

Applying \(\varphi \in \text{Hom}_R(B, R)\), we have

\[
\varphi\left(u^{l}_{s+nh+1}\right) \cdot u = \varphi\left(u^{l+1}_{s+nh+1} u^{s+nh}_{s+nh+1}\right) \in \varphi\left(u^{l+1}_{s+nh+1}\right) \cdot \prod_{i=1}^{n} I^{(s_i+1)}.
\]
Setting $\varphi$ to be a splitting of the map (4.3), we obtain
\[ u \in \mathfrak{m} \cdot \prod_{i=1}^{n} I^{(s_{i}+1)} \]
as claimed, since $\varphi(u^{\frac{l+1}{s_{i}+1}}) \in \mathfrak{m}$ for all $\varphi \in \text{Hom}_{R}(B, R)$ by the assumption on $l$. \hfill $\square$

Part II. Proof via multiplier/test ideals

5. Big Cohen–Macaulay test ideals, +-test ideals, and multiplier ideals

In this section, we review the theory of big Cohen–Macaulay test ideals developed by Ma and Schwede [MS21] and the theory of +-test ideals developed by Hacon, Lamarche, and Schwede [HLS]. We also define multiplier ideals for excellent schemes of equal characteristic zero, following [dFM09; ST].

When working with big Cohen–Macaulay test ideals throughout Part II of this paper, we will often use the following notation.

**Notation 5.1** (see [MS21, Setting 6.1]). Let $(R, \mathfrak{m})$ be a normal complete Noetherian local ring of dimension $d$. Fix a dualizing complex $\omega_{\bullet} R$ on $R$ with associated canonical module $\omega_{R}$, and fix an embedding $R \subseteq \omega_{R} \subseteq \mathbb{K}(R)$. This yields a choice of an effective canonical divisor $K_{R}$ on $\text{Spec}(R)$.

Let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $\text{Spec}(R)$ such that $K_{R} + \Delta$ is $\mathbb{Q}$-Cartier. Since $K_{R} + \Delta$ is effective, there is an integer $N > 0$ and an element $h \in R$ such that $K_{R} + \Delta = \frac{1}{N} \text{div}_{R}(h)$.

5.1. Big Cohen–Macaulay test ideals of divisor pairs. We define the big Cohen–Macaulay test ideals from [MS21]. See also [PRG21, Definition 3.1] for a related definition, which applies when $\Delta = 0$ but does not assume that $K_{R}$ is $\mathbb{Q}$-Cartier.

**Definition 5.2** [MS21, Definitions 6.2 and 6.9]. With notation as in Notation 5.1, let $B$ be an $R^{+}$-algebra that is big Cohen–Macaulay over $R$. We set
\[ 0^{B, K_{R} + \Delta}_{H^{d}_{m}(R)} := \ker \left( H^{d}_{m}(R) \xrightarrow{h^{1/N}} H^{d}_{m}(B) \right), \]
which does not depend on the choice of $h^{1/N}$. The **big Cohen–Macaulay test ideal** of $(R, \Delta)$ with respect to $B$ is
\[ \tau_{B}(R, \Delta) := \text{Ann}_{R} \left( 0^{B, K_{R} + \Delta}_{H^{d}_{m}(R)} \right). \]
This is an $R$-submodule of $\omega_{R}$, and is in fact an ideal in $R$ by [MS21, Lemma 6.8].

5.2. +-test ideals. Next, we define Hacon, Lamarche, and Schwede’s notion of +-test ideals [HLS].

5.2.1. +-stable global sections. To define +-test ideals, we first need to define Hacon, Lamarche, and Schwede’s version of the space of +-stable global sections first introduced in [TY23, Definition 3.10] and [BMPSTWW, Definition 4.2].

**Definition 5.3** [HLS, Notation 3.1 and Definition 3.2]. Let $(R, \mathfrak{m})$ be a complete Noetherian local ring of residue characteristic $p > 0$, and consider a proper morphism $\pi: X \to \text{Spec}(R)$ from a normal integral scheme of dimension $d$. Fixing a dualizing complex $\omega_{X}^{\bullet} = \pi^{!} \omega_{R}^{\bullet}$ on $X$ with associated canonical sheaf $\omega_{X}$. We can then fix a canonical divisor $K_{X}$ on $X$, which satisfies $\omega_{X} = \mathcal{O}_{X}(K_{X})$. 

Fix an algebraic closure \( \overline{K(X)} \) of the function field \( K(X) \) of \( X \), and consider the absolute integral closure \( \nu: X^+ \to X \). For a \( \mathbb{Q} \)-Weil divisor \( B \) on \( X \), we set
\[
\nu_* \mathcal{O}_{X^+}(\nu^* B) := \varprojlim_{f: Y^+ \to X} f_* \mathcal{O}_Y([f^* B]),
\]
where the colimit is taken over finite surjective morphisms \( f: Y \to X \) as in Definition 2.5. This yields a sheaf \( \mathcal{O}_{X^+}(\nu^* B) \).

Now consider a Weil divisor \( M \) and an effective \( \mathbb{Q} \)-Weil divisor \( B \) on \( X \). The Matlis dual of the \( R \)-module
\[
im \left( H^d_m \left( R\Gamma \left( X, \mathcal{O}_X(-M) \right) \right) \to H^d_m \left( R\Gamma \left( X^+, \mathcal{O}_{X^+}(-\nu^* M + \nu^* B) \right) \right) \right)
\]
yields an \( R \)-submodule
\[
\mathcal{B}^0(X, B, \mathcal{O}_X(K_X + M)) \subseteq H^0(X, \mathcal{O}_X(K_X + M))
\]
by Lipman’s local-global duality [Lip78, Theorem on p. 188] (see [BMPSTWW, Lemma 2.2]). This submodule \( \mathcal{B}^0(X, B, \mathcal{O}_X(K_X + M)) \) is called the module of \( + \)-stable global sections.

5.2.2. \( + \)-test ideals for divisor pairs. We can now define \( + \)-test ideals for pairs \((X, \Delta)\), where \( \Delta \) is a \( \mathbb{Q} \)-Weil divisor, following [HLS].

**Definition 5.4** [HLS, Definition 4.3, Notation 4.6, Definition 4.14, and Definition 4.15]. With notation as in Definition 5.3, fix an effective \( \mathbb{Q} \)-Weil divisor \( \Delta \) on \( X \) and a Cartier divisor \( H \) on \( X \) such that \( K_X + \Delta + H \geq 0 \). The **\( + \)**-test ideal of the pair \((X, \Delta)\) is the coherent subsheaf
\[
\tau_+(\mathcal{O}_X, \Delta) \subseteq \omega_X \otimes \mathcal{O}_X(H)
\]
such that \( \tau_+(\mathcal{O}_X, \Delta) \otimes \mathcal{O}_X(-H) \otimes \mathcal{L} \subseteq \omega_X \otimes \mathcal{L} \) is globally generated by
\[
\mathcal{B}^0(X, K_X + \Delta + H, \omega_X \otimes \mathcal{L}) \subseteq H^0(X, \omega_X \otimes \mathcal{L}),
\]
where \( \mathcal{L} = \mathcal{O}_X(L) \) is a sufficiently ample invertible sheaf. The subsheaf \( \tau_+(\mathcal{O}_X, \Delta) \) is independent of the choice of \( \mathcal{L} \) and \( H \) by [HLS, Proposition 4.5 and Lemma 4.8(b)], and is an ideal sheaf by [HLS, Lemma 4.18].

Big Cohen–Macaulay test ideals and \( + \)-test ideals are related in the following manner.

**Remark 5.5.** With notation as in Notation 5.1, assume that \( R \) is of residue characteristic \( p > 0 \), and set \( X = \text{Spec}(R) \). By [HLS, Remark 5.12], we have
\[
\tau_{\hat{R}^+}(R, \Delta) = \tau_+(\mathcal{O}_X, \Delta),
\]
where we recall from Remark 2.7(iv) that \( \hat{R}^+ \) is a big Cohen–Macaulay \( \hat{R}^+ \)-algebra.

This definition also extends to quasi-projective morphisms \( U \to \text{Spec}(R) \) under \( \mathbb{Q} \)-Cartier assumptions.

**Definition 5.6** [HLS, Definition 5.10]. Let \((\hat{R}, \mathfrak{m})\) be a complete Noetherian local ring of residue characteristic \( p > 0 \), and consider a quasi-projective morphism \( U \to \text{Spec}(R) \) from a normal integral scheme. As in Definition 5.3, by fixing a choice of dualizing complex \( \omega_R^* \) on \( R \), we have an induced canonical sheaf \( \omega_U \) on \( U \), and we can fix a canonical divisor \( K_U \) on \( U \) that satisfies \( \omega_U = \mathcal{O}_U(K_U) \).

Fix an effective \( \mathbb{Q} \)-Weil divisor \( \Delta_U \) on \( U \) such that \( K_U + \Delta_U \) is \( \mathbb{Q} \)-Cartier, and fix a Cartier divisor \( G_U \) on \( U \) such that \( K_U + \Delta_U + G_U \geq 0 \). The **\( + \)**-test ideal of the pair \((U, \Delta_U)\) is the coherent ideal sheaf
\[
\tau_+(\mathcal{O}_U, \Delta_U) := \tau_+(\mathcal{O}_X, \Delta)|_U \subseteq \mathcal{O}_U,
\]
where \( \tau_+(\mathcal{O}_X, \Delta) \) is defined by finding a normal integral projective compactification \( X \to \text{Spec}(R) \) of \( U \to \text{Spec}(R) \), together with compactifications \( G \) for \( G_U \) and \( K_X + \Delta + G \) for \( K_U + \Delta_U + G_U \) such
that $G \geq 0$ and $K_X + \Delta + G \geq 0$. This definition does not depend on the choice of compactifications by [HLS, Proposition 5.7].

5.2.3. $+\text{-test ideals for ideal pairs.}$ We also define $+\text{-test ideals}$ for ideal pairs, following [HLS].

**Definition 5.7** [HLS, p. 24]. With notation as in Definition 5.3, suppose that $K_X$ is $\mathbb{Q}$-Cartier. Let $a \subseteq \mathcal{O}_X$ be a coherent ideal sheaf, and let $\mu : X' \to X$ be the normalized blowup of $a$ in which case $\mu^{-1}a \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ for an effective Cartier divisor $F$ on $X'$. For every rational number $t \geq 0$, the $+\text{-test ideal of the pair $(X, a^t)$}$ is the coherent ideal sheaf

$$
\tau_+(\mathcal{O}_X, a^t) \subseteq \mathcal{O}_X
$$

such that $\tau_+(\mathcal{O}_X, a^t) \otimes \mathcal{L} \subseteq \mathcal{L}$ is globally generated by

$$
\mathcal{B}^0 \left( X', tF, \mathcal{O}_{X'}(K_{X'} - \mu^*K_X + L) \right)
$$

$$
\subseteq \mathcal{B}^0 \left( X', \{tf\}, \mathcal{O}_{X'}(K_{X'} - \mu^*K_X - \lfloor tf \rfloor + L) \right)
$$

$$
\subseteq H^0 \left( X', \mathcal{O}_{X'}(K_{X'} - \mu^*K_X + L) \right)
$$

$$
\subseteq H^0 \left( X, \mathcal{O}_X(L) \right),
$$

where $\mathcal{L} = \mathcal{O}_X(L)$ is a sufficiently ample invertible sheaf and $L' = \mu^*L$. The ideal sheaf $\tau_+(\mathcal{O}_X, a^t)$ is independent of the choice of $\mathcal{L}$ by [HLS, p. 24].

We compare the definitions of $+\text{-test ideals}$ for divisor pairs and for ideal pairs. We note that the ideal on the left-hand side of (5.9) below is related to Robinson’s notion of the big Cohen–Macaulay test ideal for a triple $(R, \Delta, a^t)$ [Rob22] defined in Definition 6.1. See Lemma 6.6(iii).

**Lemma 5.8** (cf. [HLS, Remark 6.4]). With notation as in Definition 5.3, if $K_X$ is $\mathbb{Q}$-Cartier, we have

$$
\sum_{m=1}^{\infty} \sum_{\Gamma \in |\mathcal{L} \otimes m \mathcal{O}_X|} \tau_+ \left( \mathcal{O}_X, \frac{t}{m} \Gamma \right) \subseteq \tau_+ \left( \mathcal{O}_X, a^t \right).
$$

(5.9)

In particular, if $X = \text{Spec}(R)$, we have

$$
\sum_{m=1}^{\infty} \sum_{g \in a^m} \tau_+ \left( \mathcal{O}_X, \frac{t}{m} \text{div}_R(g) \right) \subseteq \tau_+ \left( \mathcal{O}_X, a^t \right).
$$

**Proof.** We adapt the proof of [HLS, Remark 6.4], which is a version of our lemma for test modules $\tau_+ (\omega_X, \Delta)$ and $\tau_+ (\omega_X, a^t)$.

Let $\mathcal{L} = \mathcal{O}_X(L)$ be a sufficiently ample invertible sheaf on $X$ such that $a \otimes \mathcal{L}$ is globally generated. Let $\mu : X' \to X$ be the normalized blowup of $a$ in which case $\mu^{-1}a \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ for an effective Cartier divisor $F$ on $X'$. Set $L' = \mu^*L$. First, we note that for every integer $m \geq 1$, we have $\mu^* \frac{n}{m} \Gamma \geq F$ for all $\Gamma \in |\mathcal{L} \otimes m \mathcal{O}_X|$. Thus, by [HLS, Lemma 3.3(a) and Proposition 3.9], we see that $\mathcal{B}^0 \left( X', tF, \mathcal{O}_{X'}(K_{X'} - \mu^*K_X + L) \right)$ contains

$$
\mathcal{B}^0 \left( X', t \cdot \mu^* \frac{1}{m} \Gamma, \mathcal{O}_{X'}(K_{X'} - \mu^*K_X + L) \right) \subseteq \mathcal{B}^0 \left( X, \frac{t}{m} \Gamma, \mathcal{O}_X(L) \right).
$$

By the definitions of $\tau_+ (\mathcal{O}_X, \frac{1}{m} \Gamma)$ and of $\tau_+ (\mathcal{O}_X, a^t)$, we therefore have (5.9). □

5.3. Multiplier ideals for excellent schemes of equal characteristic zero. Finally, we define multiplier ideals for excellent schemes of equal characteristic zero, following [dFM09, §2; ST, §2.1].

**Definition 5.10** (cf. [Laz04, Generalization 9.2.8; dFM09, p. 495; ST, Definition 2.4]). Let $X$ be an excellent normal integral scheme of equal characteristic zero with a dualizing complex $\omega_X^+$ and associated choice of canonical divisor $K_X$. 


Let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $a_1, a_2, \ldots, a_n \subseteq \mathcal{O}_X$ be coherent ideal sheaves, and let $\mu : X' \to X$ be a log resolution for the triple $(X, \Delta, a_1 a_2 \cdots a_n)$, which exists by [Tem08, Theorem 1.1]. For each $i$, write $\mu^{-1}a_i \cdot \mathcal{O}_X = \mathcal{O}_{X'}(-F_i)$ for effective Cartier divisors $F_i$ on $X'$. For real numbers $t_1, t_2, \ldots, t_n \geq 0$, the multiplier ideal of the triple $(X, \Delta, a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n})$ is the coherent ideal sheaf

$$
\mathcal{J}(X, \Delta, a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}) := \mu_* \left( \mathcal{O}_{X'} \left( K_{X'} - \left[ \mu^*(K_X + \Delta) + \sum_{i=1}^n t_i F_i \right] \right) \right) \subseteq \mathcal{O}_X.
$$

This definition does not depend on the choice of $\mu$ by [dFM09, Proposition 2.2; ST, Remark 2.5(ii)].

## 6. Big Cohen–Macaulay test ideals of triples

In this section, we define a slight variation of Robinson’s version of test ideals from [Rob22], which combines aspects of the big Cohen–Macaulay test ideals defined in [MS21] and [PRG21] together with aspects of the perfectoid test ideals defined in [MS18]. We then prove fundamental results on Robinson’s test ideals necessary to prove Theorems A, B, C, and D. Throughout this section, we will use the notation from Notation 5.1.

### 6.1. Definition and preliminaries

We start with slight variations of Robinson’s definitions of big Cohen–Macaulay test ideals for triples from [Rob22] (see Remark 6.2). There are two versions of test ideals in [Rob22]: the ideals $\tau_B(R, \Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n})$ that fix sets of generators for ideals, and the ideals $\tau_B(R, \Delta, a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n})$ that do not fix sets of generators. While our main results in this paper can be shown using only the ideals $\tau_B(R, \Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n})$, we will prove many of our results for both versions of Robinson’s test ideals. We note that it is necessary to consider the ideals $\tau_B(R, \Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n})$ since we do not know whether the subadditivity theorem (Theorem 6.18) and the strongest version of our unambiguity statement (Proposition 6.13) hold for $\tau_B(R, \Delta, a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n})$.

The definition with a fixed set of generators combines the definitions in [MS18] and in [MS21]. Compared to [MS18], this definition works in arbitrary characteristic, and does not include the factor $p^{1/p\infty}$ coming from almost mathematics. One also ranges over all integers $m_1, m_2, \ldots, m_n$ instead of just $p$-th powers, and there is no perturbation present in contrast to [MS18, Definition 3.5]. See [Rob22, Remark 3.7].

A similar definition that does not fix a set of generators also appeared in the work of Sato and Takagi [ST, Definition 3.2(i)]. See Lemma 6.6(ii).

**Definition 6.1** (see [Rob22, Definition 3.6]). With notation as in Notation 5.1, fix sets

$$
\{f_{i,1}, f_{i,2}, \ldots, f_{i,r_i}\}
$$

of elements in $R$ for $i \in \{1, 2, \ldots, n\}$. For fixed $i$, we denote these data by $[f_i]$, and we denote by $a_i$ the ideal $(f_{i,1}, f_{i,2}, \ldots, f_{i,r_i})$. Fix real numbers $t_1, t_2, \ldots, t_n \geq 0$. For an $R^+$-algebra $B$ that is big Cohen–Macaulay over $R$, we set

$$
0^B_{H^d_\Delta(R)} := \left\{ \eta \in H^d_m(R) : \begin{aligned}
\text{for all integers } m_1, m_2, \ldots, m_n > 0, & \text{ we have } \\
h^{1/N} g_1^{1/m_1} g_2^{1/m_2} \cdots g_n^{1/m_n} \eta = 0 \text{ in } H^d_m(B) & \text{ for all } g_i \in a_i^{[m_i,t_i]} \end{aligned} \right\},
$$

$$
0^B_{H^d_\Delta(R)} := \left\{ \eta \in H^d_m(R) : \begin{aligned}
\text{for all integers } m_1, m_2, \ldots, m_n > 0, & \text{ we have } \\
h^{1/N} g_1 g_2 \cdots g_n \eta = 0 \text{ in } H^d_m(B) & \text{ for all } g_i = \prod_{k=1}^{a_i} f_{i,j_k}^{1/m_i} \text{ where } a_i \geq m_i t_i \end{aligned} \right\}.
$$
where \( \eta \) in the equations \( h^{1/N}g_1^{1/m_1}g_2^{1/m_2} \cdots g_n^{1/m_n} \eta = 0 \) and \( f^{1/N}g_1g_2 \cdots g_n \eta = 0 \) is interpreted as the image of \( \eta \) under the map \( H^d_m(R) \to H^d_m(B) \). We then set

\[
\tau_B(R, \Delta, a_1^i, a_2^i \cdots a_n^i) := \text{Ann}_R \left( 0^{B,K_R+\Delta,a_1^i,a_2^i \cdots a_n^i}_{H^d_m(R)} \right)
\]

\[
\tau_B(R, \Delta, [f_1]^{t_1}, [f_2]^{t_2} \cdots [f_n]^{t_n}) := \text{Ann}_R \left( 0^{B,K_R+\Delta,[f_1]^{t_1}[f_2]^{t_2} \cdots [f_n]^{t_n}}_{H^d_m(R)} \right)
\]

which are ideals in \( R \) since they are \( R \)-submodules of \( \tau_B(R, \Delta) \). For both \( \tau^B \) and \( \tau_B \), we omit the divisor \( \Delta \) (resp. \( a_i \) or \([f_i]\)) from our notation if \( \Delta = 0 \) (resp. if \( n = 0 \)).

**Remark 6.2.** In addition to allowing for multiple tuples of generators \([f_i]\) or ideals \( a_i \), Definition 6.1 differs slightly from Robinson’s definition in [Rob22, Definition 3.6] in that Robinson ranges over all sufficiently large \( m_i \) instead of all \( m_i \). However, the ideal defined in Definition 6.1 and Robinson’s original definition from [Rob22, Definition 3.6] coincide by Lemma 6.5 below.

**Remark 6.3.** Robinson’s definitions in Definition 6.1 allow the exponents \( t_1, t_2, \ldots, t_n \) to be real numbers. Some of our results below (Lemma 6.6(iii), Proposition 6.13, and Proposition 6.17) require rational exponents \( t_1, t_2, \ldots, t_n \), at least in their strongest forms. In [MS18], Ma and Schwede are able to prove the analogues of these results for their perfectoid test ideals with real exponents \( t_1, t_2, \ldots, t_n \) because their definitions include perturbations. According to [Rob22, Remark 3.7], it is expected that these perturbations are unnecessary for sufficiently large choices of the big Cohen–Macaulay algebra \( B \).

The two definitions in Definition 6.1 are related in the following manner. This comparison appears in [MS18, Proposition 3.3(a)] for perfectoid test ideals when \( \Delta = 0 \) and in [Rob22, Proposition 3.8] for big Cohen–Macaulay test ideals of triples when \( n = 1 \).

**Proposition 6.4** (cf. [MS18, Proposition 3.3(a); Rob22, Proposition 3.8]). Fix notation as in Notation 5.1 and Definition 6.1. We have

\[
\tau_B(R, \Delta, [f_1]^{t_1}, [f_2]^{t_2} \cdots [f_n]^{t_n}) \subseteq \tau_B(R, \Delta, a_1^i, a_2^i \cdots a_n^i).
\]

**Proof.** It suffices to show that

\[
0^{B,K_R+\Delta,[f_1]^{t_1}[f_2]^{t_2} \cdots [f_n]^{t_n}}_{H^d_m(R)} \geq 0^{B,K_R+\Delta,a_1^i,a_2^i \cdots a_n^i}_{H^d_m(R)}.
\]

Let \( \eta \) be an element in the module on the right-hand side, and consider elements

\[
g_i = \prod_{k=1}^{a_i} f_{i,j_k}^{1/m_i} = \left( \prod_{k=1}^{a_i} f_{i,j_k} \right)^{1/m_i}
\]

for each \( a_i \geq m_i t_i \). Since each \( f_{i,j_k} \) is a generator of \( a_i \), we have

\[
\prod_{k=1}^{a_i} f_{i,j_k} \in a_i^{[m_i t_i]} \subseteq a_i^{[m_i t_i]}.
\]

Thus, each \( g_i \) can be written as an \( m_i \)-th root of an element in \( a_i^{[m_i t_i]} \), and \( h^{1/N}g_1g_2 \cdots g_n \eta = 0 \). □

We also note that in Definition 6.1, we may restrict to \( m_i \) sufficiently large and divisible.

**Lemma 6.5** (cf. [MS18, Lemma 3.6]). Fix notation as in Notation 5.1 and Definition 6.1. In the definitions of \( 0^{B,K_R+\Delta,a_1^i,a_2^i \cdots a_n^i}_{H^d_m(R)} \) and \( 0^{B,K_R+\Delta,[f_1]^{t_1}[f_2]^{t_2} \cdots [f_n]^{t_n}}_{H^d_m(R)} \), we may restrict to \( m_1, m_2, \ldots, m_n > 0 \) sufficiently large and divisible.
Proof. Fix integers \(m_{i,0} > 0\) for each \(i\). We want to show that

\[
0_{H_m^d(R)}^{B,K_R + \Delta} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n} = \bigcap_{m_1,m_2,\ldots,m_n \in \mathbb{Z}_{>0}} \bigcap_{g_i \in a_i^{[m_i t_i]}} \ker \left( H_m^d(R) \xrightarrow{h^{1/N} g_1^{1/m_1} g_2^{1/m_2} \cdots g_n^{1/m_n}} H_m^d(B) \right),
\]

\[
0_{H_m^d(R)}^{B,K_R + \Delta} [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n} = \bigcap_{m_1,m_2,\ldots,m_n \in \mathbb{Z}_{>0}} \bigcap_{g_i = \prod_{k=1}^{i'} f_{j,k}^{1/m_i}} \ker \left( H_m^d(R) \xrightarrow{h^{1/N} g_1 g_2 \cdots g_n} H_m^d(B) \right).
\]

The inclusion \(\subseteq\) holds in both cases since restricting to \(m_i\) divisible by \(m_{i,0}\) results in fewer conditions.

For the inclusion \(\supseteq\), we first consider the case for the \(a_i\). Let \(\eta\) be an element in the module on the right-hand side, and consider elements \(g_i \in a_i^{[m_i t_i]}\) for each \(i\). We then have

\[
g_i^{m_{i,0}} \in a_i^{m_{i,0}[m_i t_i]} \subset a_i^{[m_{i,0} m_i t_i]}
\]

for each \(i\), and hence

\[
h^{1/N} g_1^{1/m_1} g_2^{1/m_2} \cdots g_n^{1/m_n} \eta = h^{1/N} (g_1^{m_{i,0}^{1/(m_1,0)m_1)} a_i^{g_2^{m_{i,0}^{1/(m_2,0)m_2)}} \cdots (g_n^{m_{i,0}^{1/(m_n,0)m_n)} \eta = 0.
\]

It remains to show the inclusion \(\supseteq\) for the \([f_i]\). Let \(\eta\) be an element in the module on the right-hand side, and consider elements \(g_i = \prod_{k=1}^{i'} f_{j,k}^{1/m_i}\) for each \(i\) where \(a_i \geq m_i t_i\). We can then write

\[
g_i = \prod_{k=1}^{i'} (f_{j,k}^{m_{i,0}^{1/m_i}})^{1/(m_i,0)m_i} = \prod_{k'=1}^{i'} f_{j,k'}^{1/(m_i,0)m_i}
\]

for some sequence \((j'_{k'})_{k'=1}^{i'}\), and hence \(h^{1/N} g_1 g_2 \cdots g_n \eta = 0\).

We connect Definition 6.1 to the big Cohen–Macaulay test ideals reviewed in §5. In (ii) below, (6.7) connects Robinson’s definition (Definition 6.1) without a fixed choice of generators to Sato and Takagi’s definition in [ST, Definition 3.2(i)]. A version of (6.8) when \(n = 1\) appears in the proof of [Rob22, Theorem 3.11].

**Lemma 6.6.** Fix notation as in Notation 5.1 and Definition 6.1.

(i) We have

\[
0_{H_m^d(R)}^{B,K_R + \Delta} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n} = \bigcap_{m_1,m_2,\ldots,m_n \in \mathbb{Z}_{>0}} \bigcap_{g_i \in a_i^{[m_i t_i]}} \ker \left( H_m^d(R) \xrightarrow{h^{1/N} g_1^{1/m_1} g_2^{1/m_2} \cdots g_n^{1/m_n}} H_m^d(B) \right),
\]

\[
0_{H_m^d(R)}^{B,K_R + \Delta} [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n} = \bigcap_{m_1,m_2,\ldots,m_n \in \mathbb{Z}_{>0}} \bigcap_{g_i = \prod_{k=1}^{i'} f_{j,k}^{1/m_i}} \ker \left( H_m^d(R) \xrightarrow{h^{1/N} g_1 g_2 \cdots g_n} H_m^d(B) \right).
\]

Moreover, on the right-hand sides we may restrict to \(m_1,m_2,\ldots,m_n > 0\) sufficiently large and divisible.

(ii) We have

\[
\tau_B (R,\Delta, a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}) = \sum_{m_1,m_2,\ldots,m_n \in \mathbb{Z}_{>0}} \sum_{g_i \in a_i^{[m_i t_i]}} \tau_B \left( R,\Delta + \sum_{i=1}^{n} \frac{1}{m_i} \text{div}_R (g_i) \right),
\]

\[
\tau_B (R,\Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n}) = \sum_{m_1,m_2,\ldots,m_n \in \mathbb{Z}_{>0}} \sum_{g_i = \prod_{k=1}^{i'} f_{j,k}^{1/m_i}} \tau_B \left( R,\Delta + \sum_{i=1}^{n} \text{div}_R (g_i) \right),
\]

where \(a_i \geq m_i t_i\).
where $\text{div}_R(g_i) := \frac{1}{m_i} \sum_{k=1}^{a_i} \text{div}_R(f_{i,j_k})$. Moreover, on the right-hand sides we may restrict to $m_1, m_2, \ldots, m_n > 0$ sufficiently large and divisible.

(iii) If the $t_i$ are rational numbers, then we have

$$\tau_B(R, \Delta, a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}) = \sum_{m_1, m_2, \ldots, m_n \in \mathbb{Z}} \sum_{g_i \in a_i^{m_i}} \tau_B \left( R, \Delta + \sum_{i=1}^{n} \frac{t_i}{m_i} \text{div}_R(g_i) \right). \quad (6.9)$$

If, moreover, $r_i = 1$ for all $i$, then we have

$$\tau_B(R, \Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n}) = \tau_B(R, \Delta, a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}) = \tau_B \left( R, \Delta + \sum_{i=1}^{n} t_i \text{div}_R(f_{i,1}) \right). \quad (6.10)$$

Proof. (i) follows from Definition 6.1. The last statement about taking $m_i > 0$ sufficiently large and divisible follows from Lemma 6.5.

For (ii), we note that the sums on the right-hand side of (6.7) and (6.8) are finite sums since $R$ is Noetherian, and that the intersection in (i) is a finite intersection since $H^d_m(R)$ is Artinian [Gro67, Proposition 6.4(2); Har68, §2, Example 1; Har70, Proposition 1.1]. Applying $\text{Ann}_{\omega_R}(-)$ to both sides of (i), we therefore obtain (6.8). The penultimate statement about taking $m_i > 0$ sufficiently large and divisible follows from Lemma 6.5 by taking $m_i > 0$ sufficiently large and divisible in (i).

For (iii), we first show the inclusion $\subseteq$ in (6.9). By (ii), it suffices to show that for $m_i$ sufficiently large and divisible, the ideal

$$\tau_B \left( R, \Delta + \sum_{i=1}^{n} \frac{1}{m_i} \text{div}_R(g_i) \right) \quad (6.10)$$

for $g_i \in a_i^{[m_i t_i]}$ appears on the right-hand side of (6.9). Since $m_i$ is sufficiently large and divisible, we have $m_i t_i \in \mathbb{Z}$, and hence we can write $m_i t_i = s_i$ for some integer $s_i \geq 0$. We then have

$$\frac{1}{m_i} \text{div}_R(g_i) = \frac{t_i}{m_i t_i} \text{div}_R(g_i) = \frac{t_i}{s_i} \text{div}_R(g_i),$$

where $g_i \in a_i^{[m_i t_i]} = a_i^{s_i}$. Thus, the ideal (6.10) appears on the right-hand side of (6.9).

We now show the inclusion $\supseteq$ in (6.9). It suffices to show that the ideal

$$\tau_B \left( R, \Delta + \sum_{i=1}^{n} \frac{t_i}{m_i} \text{div}_R(g_i) \right) \quad (6.11)$$

for $g_i = \prod_{k=1}^{a_i} f_{i,j_k}^{1/m_i}$ where $a_i \geq m_i t_i$ appears on the right-hand side of (6.7). Write $t_i = p_i/q_i$ for integers $p_i, q_i \geq 0$. We then have

$$\tau_B \left( R, \Delta + \sum_{i=1}^{n} \frac{t_i}{m_i} \text{div}_R(g_i) \right) = \tau_B \left( R, \Delta + \sum_{i=1}^{n} \frac{p_i}{m_i q_i} \text{div}_R(g_i) \right) \quad (6.10)$$

$$= \tau_B \left( R, \Delta + \sum_{i=1}^{n} \frac{1}{m_i q_i} \text{div}_R(g_i^{p_i}) \right).$$

Since $g_i^{p_i} \in a_i^{[m_i p_i]} = a_i^{[m_i q_i t_i]} = a_i^{[m_i q_i t_i]}$, we see that the ideal (6.11) appears as an ideal on the right-hand side of (6.9).

The last statement in (iii) follows by looking at the monomials appearing in the sum on the right-hand sides of (6.8) and (6.9). \qed

We use Lemma 6.6 to show the following result.
Proposition 6.12 (cf. [MS18, Proposition 3.9; MS21, Lemma 6.6]). Fix notation as in Notation 5.1 and Definition 6.1. Fix another set of elements \( \{f_1, f_2, \ldots, f_r\} \), which we abbreviate by \([f]_r\), and denote by \( a \) the ideal \((f_1, f_2, \ldots, f_r)\). We have
\[
\tau_B(R, \Delta, a_1^a a_2^{a^2} \cdots a_n^{a^n}) \cdot (f_1, f_2, \ldots, f_r) \subseteq \tau_B(R, \Delta, a_1^a a_2^{a^2} \cdots a_n^{a^n}) \cdot (f_1, f_2, \ldots, f_r) \subseteq \tau_B(R, \Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n} \cdot (f_1, f_2, \ldots, f_r)
\]
In particular, if \((R, \Delta)\) is big Cohen–Macaulay-regular with respect to \( B \) (which holds for example when \( R \) is regular and \( \Delta = 0 \)), then
\[
(f_1, f_2, \ldots, f_r) \subseteq \tau_B(R, \Delta, [f]_r) \subseteq \tau_B(R, \Delta, a^1).
\]

We recall from [MS21, Definition 8.9] that a pair \((R, \Delta)\) as in Notation 5.1 is big Cohen–Macaulay-regular with respect to a big Cohen–Macaulay \( R^+\)-algebra \( B \) if \( \tau_B(R, \Delta) = R \).

Proof. By Lemma 6.6(ii), we have
\[
\tau_B(R, \Delta, a_1^a a_2^{a^2} \cdots a_n^{a^n}) \cdot (f_1, f_2, \ldots, f_r)
\]
\[
= \sum_{j=1}^r \tau_B(R, \Delta, a_1^a a_2^{a^2} \cdots a_n^{a^n}) \cdot f_j
\]
\[
= \sum_{j=1}^r \sum_{m_1, m_2, \ldots, m_n \in \mathbb{Z}_{>0} \, g_i \in a_i^{m_i t_i}} \tau_B(R, \Delta + \sum_{i=1}^n \frac{1}{m_i} \text{div}_R(g_i)) \cdot f_j
\]
\[
= \sum_{j=1}^r \sum_{m_1, m_2, \ldots, m_n \in \mathbb{Z}_{>0} \, g_i \in a_i^{m_i t_i}} \tau_B(R, \Delta + \sum_{i=1}^n \frac{1}{m_i} \text{div}_R(g_i) + \text{div}_R(f_j))
\]
\[
\tau_B(R, \Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n}) \cdot (f_1, f_2, \ldots, f_r)
\]
\[
= \sum_{j=1}^r \tau_B(R, \Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n}) \cdot f_j
\]
\[
= \sum_{j=1}^r \sum_{m_1, m_2, \ldots, m_n \in \mathbb{Z}_{>0} \, g_i \in [a_i^{m_i t_i}]} \tau_B(R, \Delta + \sum_{i=1}^n \text{div}_R(g_i) + \text{div}_R(f_j))
\]
where the last equality in each case holds by the Skoda-type result in [MS21, Lemma 6.6]. Applying Lemma 6.6(ii) again, we see that these ideals are contained in \( \tau_B(R, \Delta, a_1^a a_2^{a^2} \cdots a_n^{a^n}) \) and \( \tau_B(R, \Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n} \cdot (f_1, f_2, \ldots, f_r) \), respectively.

The assertion for big Cohen–Macaulay-regular pairs \((R, \Delta)\) now follows because \( \tau_B(R, \Delta) = R \) by definition in [MS21, Definition 8.9]. Pairs \((R, \Delta)\) where \( R \) is regular and \( \Delta = 0 \) are big Cohen–Macaulay-regular with respect to all big Cohen–Macaulay \( R^+\)-algebras \( B \) since \( R \to B \) is faithfully flat [Hoc75, Lemma 5.5] (see also [HH92, (6.7); HH95, Lemma 2.1(d)]), and hence \( H^m_{\text{in}}(R) \to H^m_{\text{in}}(B) \) is injective. \( \square \)

6.2. Unambiguity of exponents. We next prove the following unambiguity-type statement for exponents for the test ideals \( \tau_B(R, \Delta, [f_1]^{t_1} [f_2]^{t_2} \cdots [f_n]^{t_n}) \), which is stronger than that proved for the perfectoid test ideals in [MS18]. This statement does not need the perturbations present in [MS18, Definition 3.5 and Proposition 3.8] because we do not have the extra factor \( p^{1/p^n} \) coming.
from almost mathematics in [MS18, Definitions 3.1 and 4.1], and because we can clear the denominators in $s_1, s_2$ after passing to sufficiently large and divisible $m_1, m_2$ using Lemma 6.5, at least when $s_1, s_2$ are rational.

**Proposition 6.13** (cf. [MS18, Proposition 3.8]). Fix notation as in Notation 5.1 and Definition 6.1. Fix another set of elements $\{f_1, f_2, \ldots, f_r\}$, which we abbreviate by $[f]$, and denote by $a$ the ideal $(f_1, f_2, \ldots, f_r)$. For all real numbers $s_1, s_2 \geq 0$, we have

$$\tau_B(R, \Delta, a^{s_1}a^{s_2}) \subseteq \tau_B(R, \Delta, a^{s_1}a^{s_2})$$

$$\tau_B(R, \Delta, f_1^{s_1}f_2^{s_2} \ldots f_n^{s_1}f_1^{s_2}) \subseteq \tau_B(R, \Delta, f_1^{s_1}f_2^{s_2} \ldots f_n^{s_1}f_1^{s_2})$$

(6.14)

with equality in (6.14) if $s_1$ and $s_2$ are rational.

**Proof.** Fix $m_1, m_2, \ldots, m_n$ and $g_1, g_2, \ldots, g_n$ as in Lemma 6.6(ii), and set

$$\Delta' = \sum_{i=1}^n \text{div}_R(g_i).$$

By Lemma 6.6(ii), it suffices to show that

$$\tau_B(R, \Delta + \Delta', a^{s_1}a^{s_2}) \subseteq \tau_B(R, \Delta + \Delta', a^{s_1}a^{s_2})$$

$$\tau_B(R, \Delta + \Delta', f_1^{s_1}f_2^{s_2} \ldots f_n^{s_1}f_1^{s_2}) \subseteq \tau_B(R, \Delta + \Delta', f_1^{s_1}f_2^{s_2} \ldots f_n^{s_1}f_1^{s_2})$$

with equality if $s_1$ and $s_2$ are rational. Replacing $\Delta$ with $\Delta + \Delta'$, it therefore suffices to prove the case when there are no $[f]$. By Definition 6.1, it moreover suffices to show that

$$0^{B, K_R + \Delta, a^{s_1}a^{s_2}} \subseteq 0^{B, K_R + \Delta, a^{s_1}a^{s_2}}$$

(6.15)

$$0^{B, K_R + \Delta, f_1^{s_1}f_2^{s_2}} \subseteq 0^{B, K_R + \Delta, f_1^{s_1}f_2^{s_2}}$$

(6.16)

with equality in (6.16) if $s_1$ and $s_2$ are rational.

For the inclusion $\supseteq$ in (6.15), suppose that $\eta \in 0^{B, K_R + \Delta, a^{s_1}a^{s_2}}$. It suffices to show that for all $m_1, m_2 > 0$, we have $h^{1/N} g_1^{1/m_1} g_2^{1/m_2} \eta = 0$ in $H^d_m(B)$ for all choices of elements $g_i \in a_i^{[m_i, s_i]}$. Writing

$$g_1^{1/m_1} g_2^{1/m_2} = g_1^{m_2/(m_1 m_2)} g_2^{m_1/(m_1 m_2)} = (g_1^{m_2} g_2^{m_1})^{1/(m_1 m_2)},$$

we see that $g_1^{1/m_1} g_2^{1/m_2}$ can be written as the $(m_1 m_2)$-th root of the element

$$g = g_1^{m_2} g_2^{m_1} \in a^{m_2 [m_1 s_1] + m_1 [m_2 s_2]} = a^{[m_1 m_2 s_1 + m_1 m_2 s_2]} \subseteq a^{[m_1 m_2 (s_1 + s_2)]}.$$
we see that $h^{1/N}g_1g_2\eta = h^{1/N}g\eta = 0$.

We now show the inclusion $\subseteq$ in (6.16) assuming that $s_1$ and $s_2$ are rational. Suppose that $\eta \in \mathcal{O}_{H^i_{\Delta}(R)}$ with equality when $h$ is rational. The inclusion $\subseteq$ holds since $\left\lfloor \frac{a}{s} \right\rfloor + \left\lfloor \frac{b}{s} \right\rfloor = \max\left\lfloor \frac{a}{s} \right\rfloor, \left\lfloor \frac{b}{s} \right\rfloor$. By Lemma 6.5, it suffices to show that for all sufficiently large and divisible $m > 0$, we have $h^{1/N}g\eta = 0$ in $H^{1/N}_g(B)$ for all choices of elements $g \in R$ which are expressed as a product of $a$ choices of $m$-th roots of elements corresponding to $\lfloor f \rfloor$, where $a \geq m(s_1 + s_2)$. By ranging over $m$ divisible enough, we may assume that $s_m$ and $s_2$ are integers, in which case we may write $a = a_1 + a_2$ where $a_1 \geq ms_1$ and $a_2 \geq ms_2$. Write

$$g = \left( \prod_{k_1=1}^{a_1} f_{j_k}^{1/m} \right) \left( \prod_{k_2=1}^{a_2} f_{j_k}^{1/m} \right).$$

Thus, we have

$$h^{1/N}g\eta = h^{1/N}\left( \prod_{k_1=1}^{a_1} f_{j_k}^{1/m} \right) \left( \prod_{k_2=1}^{a_2} f_{j_k}^{1/m} \right) \eta = 0. \quad \Box$$

We note that the unambiguity statement in [MS18, Proposition 3.7] for perfectoid test ideals holds for the test ideal $\tau_B(R, \Delta, a_1^i a_2^j \cdots a_n^k)$ as well.

**Proposition 6.17** (cf. [MS18, Proposition 3.7]). Fix notation as in Notation 5.1 and Definition 6.1. Fix another set of elements $\{f_1, f_2, \ldots, f_r\}$ and a real number $s \geq 0$, and denote by $a$ the ideal $(f_1, f_2, \ldots, f_r)$. We have

$$\tau_B(R, \Delta, a_1^i a_2^j \cdots a_n^k (a^k)^s) \subseteq \tau_B(R, \Delta, a_1^i a_2^j \cdots a_n^k (a^k)^s)$$

for every integer $k \geq 0$, with equality if $s$ is rational.

**Proof.** It suffices to show

$$0^{B, K+\Delta, a_1^i a_2^j \cdots a_n^k (a^k)^s} \supseteq 0^{B, K+\Delta, a_1^i a_2^j \cdots a_n^k (a^k)^s}$$

with equality when $s$ is rational. The inclusion $\supseteq$ holds since $\left\lfloor \frac{a^k}{s} \right\rfloor = \left\lfloor \frac{a^k}{s} \right\rfloor \subseteq \left\lfloor \frac{a^k}{s} \right\rfloor$ for every $m$. When $s$ is rational, by Lemma 6.5, we may restrict to all $m$ sufficiently large and divisible such that $ms$ is an integer. In this case, the inclusion $\left\lfloor \frac{a^k}{s} \right\rfloor \subseteq \left\lfloor \frac{a^k}{s} \right\rfloor$ is an equality. $\Box$

### 6.3. Subadditivity

We now use the strategy in [MS18, Theorem 4.4] (which is in turn based on the strategy in [Tak06, Theorem 2.4] and [BMS08, Proposition 2.11(iv)]) to prove a stronger version of the subadditivity theorem for the big Cohen–Macaulay test ideals $\tau_B(R, \{f\})$ than is proved for perfectoid test ideals in [MS18].

**Theorem 6.18** (cf. [MS18, Theorem 4.4]). Let $(R, m)$ be a complete regular local ring. Fix sets

$$\{f_{i,1}, f_{i,2}, \ldots, f_{i,r_i}\}$$

of elements in $R$ for $i \in \{1,2,\ldots,n\}$, and fix real numbers $t_1, t_2, \ldots, t_n \geq 0$. For every big Cohen–Macaulay $R^+$-algebra $B$, we have

$$\tau_B(R, [f_{i_1}^{t_1} f_{i_2}^{t_2} \cdots f_{i_n}^{t_n}]) \subseteq \tau_B(R, [f_{i_1}^{t_1} f_{i_2}^{t_2} \cdots f_{i_n}^{t_n}]) \tau_B(R, [f_{i_1}^{t_1} f_{i_2}^{t_2} \cdots f_{i_n}^{t_n}])$$

(6.19)

for every $i_0 \in \{1,2,\ldots,n-1\}$, where we define $\tau_B$ using the choice of canonical divisor $K_R = 0$. In particular, if $\Delta_1$ and $\Delta_2$ are effective $Q$-divisors on $X = \text{Spec}(R)$, we have

$$\tau_B(R, \Delta_1 + \Delta_2) \subseteq \tau_B(R, \Delta_1) \cdot \tau_B(R, \Delta_2)$$

$$\tau_+(O_X, \Delta_1 + \Delta_2) \subseteq \tau_+(O_X, \Delta_1) \cdot \tau_+(O_X, \Delta_2).$$
We note that the proof of the special case \( \tau_+(O_X, \Delta_1 + \Delta_2) \subseteq \tau_+(O_X, \Delta_1) \cdot \tau_+(O_X, \Delta_2) \) (still assuming \( X = \text{Spec}(R) \)) was suggested in [HLS, Bottom of p. 32].

**Proof.** Let \( d = \text{dim}(R) \). The two "in particular" statements follow from (6.19) by Lemma 6.6(iii) and Remark 5.5, respectively. It therefore suffices to show (6.19).

For (6.19), we want to show the containment

\[
\begin{align*}
0^{B,K,R}[f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k} & \\
\subseteq \text{Ann}_{H_m^d(R)}(\tau_B(R, [f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k}) \cdot \tau_B(R, [f_{i_0+1}]^{t_{i_0+1}}[f_{i_0+2}]^{t_{i_0+2}} \cdots [f_k]^{t_k}))
\end{align*}
\]

(6.20)

by [Smi95, Lemma 2.1(iii)]. We claim it suffices to show that

\[
\begin{align*}
0^{B,K,R}[f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k} & \\
\subseteq \left\{ \eta \in H_m^d(R) \mid \tau_B(R, [f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k}) \cdot \eta \subseteq 0^{B,K,R}[f_{i_0+1}]^{t_{i_0+1}}[f_{i_0+2}]^{t_{i_0+2}} \cdots [f_k]^{t_k} \right\}.
\end{align*}
\]

(6.21)

If \( \eta \) is in the module on the right-hand side of (6.20), then

\[
\tau_B(R, [f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k}) \cdot \eta
\]

\[
\subseteq \text{Ann}_{H_m^d(R)}(\tau_B(R, [f_{i_0+1}]^{t_{i_0+1}}[f_{i_0+2}]^{t_{i_0+2}} \cdots [f_k]^{t_k}))
\]

\[
= 0^{B,K,R}[f_{i_0+1}]^{t_{i_0+1}}[f_{i_0+2}]^{t_{i_0+2}} \cdots [f_k]^{t_k}
\]

again by [Smi95, Lemma 2.1(iii)], and hence \( \eta \) lies in the module on the right-hand side of (6.21). By (6.21), we see that \( \eta \in 0^{B,K,R}[f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k} \), showing (6.20) as claimed.

It remains to show (6.21). Suppose \( \eta \) is in the module on the right-hand side of (6.21). By definition, we know that

\[
g_{i_0+1}g_{i_0+2} \cdots g_{n} \cdot \tau_B(R, [f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k}) = 0 \subseteq H_m^d(B)
\]

for all \( g_i = \prod_{j=1}^{a_i} f_1^{1/m_i} \) where \( a_i \geq m_i t_i \) and \( i > i_0 \). We therefore have

\[
g_{i_0+1}g_{i_0+2} \cdots g_{n} \in \text{Ann}_{H_m^d(B)}(\tau_B(R, [f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k}) \cdot B).
\]

Since \( R \) is regular and \( B \) is a big Cohen–Macaulay algebra over \( R \), we know that \( R \to B \) is faithfully flat [Hoc75, Lemma 5.5] (see also [HH92, (6.7); HH95, Lemma 2.1(d)]), and hence taking annihilators commutes with extending scalars to \( B \) [Bou72, Chapter I, §2, no. 10, Proposition 12 and Remark on p. 24]. Thus, we have

\[
g_{i_0+1}g_{i_0+2} \cdots g_{n} \in B \otimes_R \text{Ann}_{H_m^d(R)}(\tau_B(R, [f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k}))
\]

\[
= B \otimes_R 0^{B,K,R}[f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k}
\]

again by [Smi95, Lemma 2.1(iii)], and we can write

\[
g_{i_0+1}g_{i_0+2} \cdots g_{n} = b_1 \eta_1 + b_2 \eta_2 + \cdots + b_\ell \eta_\ell \in H_m^d(B)
\]

where \( b_1, b_2, \ldots, b_\ell \in B \) and \( \eta_1, \eta_2, \ldots, \eta_\ell \in 0^{B,K,R}[f_1]^{t_1}[f_2]^{t_2} \cdots [f_k]^{t_k} \). Finally, multiplying this sum by any product of elements \( g_1g_2 \cdots g_i \) where \( g_i = \prod_{k=1}^{a_i} f_1^{1/m_i} \) where \( a_i \geq m_i t_i \) and \( i \leq i_0 \), we see
that
\[
g_1g_2 \cdots g_{i_0}g_{i_0+1}g_{i_0+2} \cdots g_n \eta = b_1(g_1g_2 \cdots g_{i_0} \eta_1) + b_2(g_1g_2 \cdots g_{i_0} \eta_2) + \cdots + b_l(g_1g_2 \cdots g_{i_0} \eta_l) = 0
\]
in \(H^i_m(B)\), and hence \(\eta \in H^i_m(R)\) as claimed in (6.21).

6.4. Comparison with multiplier ideals. Finally, we show that Robinson’s big Cohen–Macaulay test ideals are contained in multiplier ideals. While we will only use this comparison in the proof of Theorem D in equal characteristic zero, the proof yields a statement valid in arbitrary characteristic. Ma and Schwede showed a version of this statement when \(\Delta = 0\) for perfectoid test ideals in mixed characteristic \(p > 0\) in [MS18, Theorem 6.3]. The statement for big Cohen–Macaulay test ideals of pairs \((R, \Delta)\) in residue characteristic \(p > 0\) is also due to Ma and Schwede [MS21, Proposition 3.7 and Theorem 6.21] (see also [MSTWW22, Theorem 5.1]). The statement below is due to Robinson in residue characteristic \(p > 0\) when \(n = 1\) [Rob22, Theorem 3.9]. A version of this statement for pairs \((R, \Delta)\) appears in the arXiv version of [MSTWW22] (see footnote 1 on page 33 of this paper), although in equal characteristic zero the big Cohen–Macaulay algebra constructed therein is not necessarily an \(R^+\)-algebra.

We note that recently, Yamaguchi showed that various versions of big Cohen–Macaulay test ideals for normal local rings \((R, m)\) essentially of finite type over \(\mathbb{C}\) coincide with the multiplier ideal [Yam, Theorem 6.4 and Proposition 7.7] when \(B\) is chosen to be the \(m\)-adic completion of the big Cohen–Macaulay algebra constructed by Schoutens for this class of rings in [Sch04, Theorem A].

**Theorem 6.22** (cf. [MS18, Theorem 6.3; MS21, Proposition 3.7 and Theorem 6.21; MSTWW22, Theorem 5.1; Rob22, Theorem 3.9]). Fix notation as in Notation 5.1 and Definition 6.1. Let \(\mu: Y \to \text{Spec}(R)\) be a proper birational morphism with \(Y\) normal such that for every \(i\), there exists a Weil divisor \(F_i\) on \(Y\) for which
\[
\frac{1}{m_i} \mu^* \text{div}_R(g_i) \geq t_i F_i
\]
for every \(g_i \in a_i^{[m_it_i]}\) as \(m_i\) ranges over all positive integers. Then, there exists an \(R^+\)-algebra \(B\) that is big Cohen–Macaulay over \(R\) such that
\[
\tau_B(R, \Delta, a_1^{t_1}a_2^{t_2} \cdots a_n^{t_n}) \subseteq \mu_* \mathcal{O}_Y\left(K_Y - \left[\mu^*(K_R + \Delta) + \sum_{i=1}^n t_i F_i\right]\right).
\]

In particular, if \(R\) is of equal characteristic zero, there exists an \(R^+\)-algebra \(B\) that is big Cohen–Macaulay over \(R\) such that for all choices of \(t_1, t_2, \ldots, t_n \geq 0\), we have
\[
\tau_B(R, \Delta, a_1^{t_1}a_2^{t_2} \cdots a_n^{t_n}) \subseteq \mathcal{J}(R, \Delta, a_1^{t_1}a_2^{t_2} \cdots a_n^{t_n})
\]

We note that the hypothesis that the \(F_i\) exist holds in particular when \(\mu^{-1} a_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_i)\) for some Cartier divisors \(F_i\) on \(Y\).

**Proof.** First, we claim that we may assume that \(\mu\) is projective, and that if we write \(Y = \text{Proj}_R(R[Jt])\) for an ideal \(J \subseteq R\), we may assume that \(R[Jt]\) is normal. We adapt the proof of [MSTWW22, Theorem 5.1]. By Chow’s lemma, there exists a projective birational morphism \(\mu': Y' \to Y\) such that \(\mu \circ \mu'\) is projective. Now writing \(Y' = \text{Proj}_R(R[J't])\), we may replace \(R[J't]\) by its normalization to assume that \(R[J't]\) is normal, in which case \(\text{Proj}_R(R[J't])\) is also normal.
Finally, we note that
\[
(\mu \circ \mu')_* \mathcal{O}_{Y'} \left( K_{Y'} - \left( (\mu \circ \mu')^* (K_R + \Delta) + \sum_{i=1}^n t_i \mu^* F_i \right) \right) 
\subseteq \mu_* \mathcal{O}_Y \left( K_Y - \left[ \mu^* (K_R + \Delta) + \sum_{i=1}^n t_i F_i \right] \right)
\]
by the projection formula since \( \mu'_* \mathcal{O}_{Y'} (K_{Y'}) \subseteq \mathcal{O}_{Y'} (K_Y) \). We may therefore replace \( Y \) by \( Y' \) to assume that \( \mu \) is projective, and that writing \( Y = \text{Proj}_{R}[R[Jt]] \), the Rees algebra \( R[Jt] \) is normal.

We now construct the big Cohen–Macaulay algebra \( B \), following the proof of [MS21, Proposition 3.7]. Let \( S = R[Jt] \) and let \( n \) denote the maximal ideal \( m + Jt \), and consider the surjective ring map \( \hat{S}_n \rightarrow R \). Since \( R \) is a domain, this map factors through \( \hat{S}_n / \mathfrak{p} \), where \( \mathfrak{p} \) is a minimal prime of \( \hat{S}_n \), and \( \dim(\hat{S}_n / \mathfrak{p}) = \dim(R) + 1 \). We now note that \( \hat{S}_n / \mathfrak{p} \) and \( R \) have the same characteristic. Thus, by [HH92, Theorem 1.1; HH95, Proposition 1.2] in equal characteristic \( p > 0 \), [And20, Theorem 1.2.1] in mixed characteristic, and Theorem 2.8 in equal characteristic zero, there exists a commutative diagram

\[
\begin{array}{ccc}
B' & \longrightarrow & B \\
\uparrow & & \uparrow \\
(\hat{S}_n / \mathfrak{p})^+ & \longrightarrow & R^+ \\
\uparrow & & \uparrow \\
\hat{S}_n / \mathfrak{p} & \longrightarrow & R
\end{array}
\]

where \( B' \) is big Cohen–Macaulay over \( \hat{S}_n / \mathfrak{p} \) and \( B \) is big Cohen–Macaulay over \( R \).

We now prove the theorem for the choice of \( B \) constructed in the previous paragraph. By Lemma 6.6(ii), we have
\[
\tau_B \left( R, \Delta, a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} \right) = \sum_{m_1, m_2, \ldots, m_n \in \mathbb{Z}_{>0}} \sum_{g_i \in a_i^{[m_i]}} \tau_B \left( R, \Delta + \sum_{i=1}^n \frac{1}{m_i} \text{div}_{R}(g_i) \right).
\]

By the proof of [MSTWW22, Theorem 5.1]\(^1\), we see that
\[
\tau_B \left( R, \Delta + \sum_{i=1}^n \frac{1}{m_i} \text{div}_{R}(g_i) \right) \subseteq \mu_* \mathcal{O}_Y \left( K_Y - \left[ \mu^* (K_R + \Delta) + \sum_{i=1}^n \frac{1}{m_i} \mu^* \text{div}_{R}(g_i) \right] \right)
\]
for every \( m_i \) and \( g_i \). Since \( \frac{1}{m_i} \mu^* \text{div}_{R}(g_i) \geq t_i F_i \), we see that
\[
\tau_B \left( R, \Delta + \sum_{i=1}^n \frac{1}{m_i} \text{div}_{R}(g_i) \right) \subseteq \mu_* \mathcal{O}_Y \left( K_Y - \left[ \mu^* (K_R + \Delta) + \sum_{i=1}^n t_i F_i \right] \right).
\]

Finally, the last statement when \( R \) is of equal characteristic zero follows by choosing \( \mu \) to be a log resolution for the triple \( (R, \Delta, a_1 a_2 \cdots a_n) \), since in this case
\[
\frac{1}{m_i} \mu^* \text{div}_{R}(g_i) \geq \frac{[m_i t_i]}{m_i} F_i \geq t_i F_i
\]
for all choices of \( t_1, t_2, \ldots, t_n \geq 0 \). \( \square \)

---

\(^1\)See also Remark 5.1.1 in the arXiv version of [MSTWW22] available at [https://arxiv.org/abs/1910.14665v5](https://arxiv.org/abs/1910.14665v5) in the equal characteristic zero case.
7. Proof of main theorems via multiplier/test ideals

Our goal in this section is to prove Theorems A, B, and C using Robinson’s version of big Cohen–Macaulay test ideals and our results for these test ideals from §6. In equal characteristic zero, we also use multiplier ideals. The idea is to use our results for the test ideals \( \tau_B(R, \Delta, [\mathcal{J}]^i) \) and \( \tau_B(R, \Delta, \mathfrak{a}^i) \) proved above, instead of those for multiplier ideals or for existing versions of test ideals used in [ELS01, Theorem 2.2 and Variant on p. 251], [Har05, Theorem 2.12], [TY08, Theorem 3.1], and [MS18, Theorem 7.4] (see also [ST12, Theorem 6.23]) to prove previously known cases of Theorem A.

To prove Theorems A, B, and C, we prove Theorem D using the test ideals \( \tau_B(R, \Delta, [\mathcal{J}]^i) \) and \( \tau_B(R, \Delta, \mathfrak{a}^i) \) and the big Cohen–Macaulay test ideals from [MS21; PRG21; ST], together with the +-test ideals from [HLS] in residue characteristic \( p > 0 \) and with multiplier ideals in equal characteristic zero as developed in [dFM09; JM12; ST]. As mentioned in §1.3.2, we also give a proof of Theorems B and C in equal characteristic zero using only multiplier ideals. We present these proofs in equal characteristic zero using multiplier ideals in §7.1, followed by the proofs in arbitrary characteristic in §7.2.

Since Theorem A follows from Theorem B by setting \( s_i = 0 \) for all \( i \), it suffices to show Theorems B, C, and D.

7.1. Proof via multiplier ideals in equal characteristic zero. We prove Theorems B and C in equal characteristic zero using multiplier ideals. The proof below uses multiplier ideals (see Definition 5.10) and illustrates the strategy we will want to use for Theorems B, C, and D in arbitrary characteristic. Our proof also yields the first proof of Theorem A for all regular rings of equal characteristic zero that does not rely on the Néron-type desingularization theorem due to Artin and Rotthaus [AR88, Theorem 1] or stronger results. This proof therefore answers a question of Schoutens [Sch03, p. 179 and p. 187], who asked whether one could show Theorem A in equal characteristic zero without using [AR88, Theorem 1].

We first prove the analogue of Theorem D for multiplier ideals.

**Proposition 7.1** (cf. [ELS01, Proof of Variant on p. 251; TY08, Proof of Theorem 4.1]). Let \( R \) be an excellent normal domain of equal characteristic zero with a dualizing complex \( \omega_X^* \) and associated choice of canonical divisor \( K_R \). Suppose that \( K_R \) is \( \mathbb{Q} \)-Cartier, and set \( X = \text{Spec}(R) \).

Consider an ideal \( I \subseteq R \), and let \( h \) be the largest analytic spread of \( IR_p \), where \( p \) ranges over all associated primes of \( R/I \). Then, for every integer \( M > 0 \), we have
\[
\mathcal{J}\left(X, \left(I^{(M)}\right)^{s_i+h/M}\right) \subseteq I^{(s_i+1)}
\]
(7.2)
for every integer \( s_i \geq 0 \).

**Proof.** Let \( \{p_\ell\} \) denote the associated primes of \( R/I \). It suffices to show that (7.2) holds after localizing at every \( p_\ell \). Since the residue field of \( R_{p_\ell} \) contains \( \mathbb{Q} \), it is infinite, and hence for every \( \ell \) there exists an ideal \( J_\ell \subseteq I \) with at most \( h \) generators such that \( J_\ell R_{p_\ell} \) is a reduction for \( IR_{p_\ell} \) by [SH06, Proposition 8.3.7 and Corollary 1.2.5]. Note that after localizing at \( p_\ell \), the ordinary and symbolic powers of \( I \) coincide. Setting \( X_{p_\ell} = \text{Spec}(R_{p_\ell}) \), which is excellent by [EGAIV2, Scholie 7.8.3(ii)], we have
\[
\mathcal{J}\left(X, \left(I^{(M)}\right)^{s_i+h/M}\right) \cdot R_{p_\ell} = \mathcal{J}\left(X_{p_\ell}, \left(I^{(M)}R_{p_\ell}\right)^{s_i+h/M}\right)
\]
\[
= \mathcal{J}\left(X_{p_\ell}, \left(I^M R_{p_\ell}\right)^{s_i+h/M}\right)
\]
\[
= \mathcal{J}(X_{p_\ell}, I^{s_i+h}R_{p_\ell})
\]
since the formation of multiplier ideals is compatible with localization by [dFM09, Proposition 2.2; ST, Remark 2.5(ii)]. Finally, by the Skoda-type result in [Laz04, Theorem 9.6.36] (whose proof holds in this generality using [Mur, Theorem A] instead of the local vanishing theorem [Laz04, Variant 9.4.4]), we have
\[ J(X_{p_\ell}, I_{s+h} R_{p_\ell}) = J_{\ell+1} R_{p_\ell}, \]
where the last inclusion holds since \( J_\ell \subseteq I \) and \( J(X_{p_\ell}, I_{h-1} R_{p_\ell}) \subseteq I_{s+1} R_{p_\ell}. \) □

We now show Theorems B and C in equal characteristic zero.

Proof of Theorems B and C in equal characteristic zero via multiplier ideals. By Lemma 2.4, we may assume that \( R \) is a complete local ring, which is excellent by [EGAIV_2, Scholie 7.8.3(iii)]. Note that Step 1 of Lemma 2.4 is unnecessary in this case, since the complete local ring produced in Step 3 contains \( Q. \)

Set \( X = \text{Spec}(R). \) We start with Theorem B. We have
\[ I^{(s+nh)} = I^{(s+nh)} \cdot J(X, R^1) \subseteq J(X, (I^{(s+nh)})^1), \]
where the first equality holds by [dFM09, Proposition 2.3(4)], and the second inclusion holds by Definition 5.10. Next, by Definition 5.10, we have
\[ J(X, (I^{(s+nh)})^1) = J(X, (I^{(s+nh)})^{\frac{1}{s+nh}}) \]
\[ = J(X, (I^{(s+nh)})^{\frac{s+1}{s+nh}} (I^{(s+nh)})^{\frac{s+2}{s+nh}} \cdots (I^{(s+nh)})^{\frac{s+n}{s+nh}}) \]
\[ \subseteq \prod_{i=1}^{n} J(X, (I^{(s+nh)})^{\frac{s+i}{s+nh}}), \]
where the last inclusion holds by applying the subadditivity theorem [JM12, Theorem A.2] \((n-1)\) times. We can therefore apply Proposition 7.1 when \( M = s + nh \) and combine these inclusions to show (1.3).

We now prove Theorem C. Let \( l \) be the largest integer such that
\[ J(X, (I^{(s+nh+1)})^{\frac{l}{s+nh+1}}) = R. \]
Note that such an \( l \) exists since \( l = 0 \) satisfies this equality by [dFM09, Proposition 2.3(4)]. We then have
\[ I^{(s+nh+1)} = I^{(s+nh+1)} \cdot J(X, (I^{(s+nh+1)})^{\frac{1}{s+nh+1}}) \]
\[ \subseteq J(X, (I^{(s+nh+1)})^{\frac{s+1}{s+nh+1}} (I^{(s+nh+1)})^{\frac{s+2}{s+nh+1}}) \]
\[ = J(X, (I^{(s+nh+1)})^{\frac{s+1}{s+nh+1}}) \]
\[ = J(X, (I^{(s+nh+1)})^{\frac{s+1}{s+nh+1}} (I^{(s+nh+1)})^{\frac{s+2}{s+nh+1}} \cdots (I^{(s+nh+1)})^{\frac{s+n}{s+nh+1}}) \]
by Definition 5.10. Applying the subadditivity theorem [JM12, Theorem A.2] \( n \) times, we have
\[ J(X, (I^{(s+nh+1)})^{\frac{l+1}{s+nh+1}} (I^{(s+nh+1)})^{\frac{s+1}{s+nh+1}} \cdots (I^{(s+nh+1)})^{\frac{s+n}{s+nh+1}}) \]
\[ \subseteq J(X, (I^{(s+nh+1)})^{\frac{l+1}{s+nh+1}}) \cdot \prod_{i=1}^{n} J(X, (I^{(s+nh+1)})^{\frac{s+i}{s+nh+1}}) \]
\[ \subseteq m \cdot \prod_{i=1}^{n} I^{(s_i+1)} \]
where the last inclusion holds since $J(X, (I^{(s+nh+1)})^{s+nh+1}) \subseteq \mathfrak{m}$ by our assumption on $l$ and by applying Proposition 7.1 when $M = s + nh + 1$. We can therefore combine these inclusions to show (1.4).

\section{Proof via big Cohen–Macaulay test ideals in all characteristics.}

We now prove Theorems B, C, and D in all characteristics using Robinson’s version of big Cohen–Macaulay test ideals from [Rob22] and the results we proved for them in §6.

We start by proving Theorem D. To prove Theorem D, we use Robinson’s test ideals from [Rob22] and the big Cohen–Macaulay test ideals from [MS21; PRG21; ST], together with the +-test ideals from [HLS] in residue characteristic $p > 0$ and multiplier ideals in equal characteristic zero. In the proof below, $\widehat{R^+}$ denotes the $p$-adic completion of the absolute integral closure $\widehat{R^+}$ of a complete local ring $R$ of residue characteristic $p > 0$. As in Remark 2.7(iv), $\widehat{R^+}$ is a big Cohen–Macaulay $R^+$-algebra by [HH92, Theorem 8.1; Bha, Corollary 5.17].

\textit{Proof of Theorem D.} We first consider the case when $R$ is of residue characteristic $p > 0$, in which case we will show that setting $B = \widehat{R^+}$ suffices. Let $\{p_\ell\}$ denote the associated primes of $R/I$. It suffices to show that (1.7) holds after localizing at every $p_\ell$. Since the residue field of $R_{p_\ell}$ is infinite, for every $\ell$ there exists an ideal $J_\ell \subseteq I$ with at most $h$ generators such that $J_\ell R_{p_\ell}$ is a reduction of $IR_{p_\ell}$ by [SH06, Proposition 8.3.7 and Corollary 1.2.5]. Note that after localizing at $p_\ell$, the ordinary and symbolic powers of $I$ coincide, and the integral closures of $I$ and $J_\ell$ coincide. Since $R$ is Noetherian, for every $\ell$, we can then choose an element $x_\ell \notin p_\ell$ such that

\begin{equation}
I\langle M \rangle R_{x_\ell} = I^M R_{x_\ell} \quad \text{and} \quad \overline{TR_{x_\ell}} = \overline{J_\ell R_{x_\ell}}.
\end{equation}

Since $x_\ell \notin p_\ell$, it suffices to show that (1.7) holds after inverting each $x_\ell$.

By Lemma 6.6(iii), we have

\[ \tau_{\widehat{R^+}}\left(R, (I^{(M)})^{s_i + h/M}\right) \subseteq \sum_{m=1}^{\infty} \sum_{g \in (I^{(M)})^m} \tau_{\widehat{R^+}}\left(R, s_i + h/mM \ divisor R(g)\right). \]

We now invert $x_\ell$ and use the test ideal theory from [HLS] (see §5). Setting $X = \text{Spec}(R)$ and $U = \text{Spec}(R_{x_\ell})$, we have $\tau_{\widehat{R}}(R, -) = \tau_{\widehat{R}}(O_X, -)$ and $\tau_{\widehat{R}}(R, -) \cdot R_{x_\ell} = \tau_{\widehat{R}}(O_U, -|U)$ by Remark 5.5 and Definition 5.6. For fixed $m$, we have

\[ \sum_{g \in (I^{(M)})^m} \tau_{\widehat{R^+}}\left(R, s_i + h/mM \ divisor R(g)\right) \cdot R_{x_\ell} \]

\[ \quad = \sum_{g \in (I^{(M)})^m} \tau_{\widehat{R}}\left(O_U, s_i + h/mM \ divisor U(g)\right) \]

\[ \quad = \sum_{g \in (I^{(M)})^m} \tau_{\widehat{R}}\left(O_U, s_i + h/mM \ divisor U(g)\right) \]

\[ \subseteq \sum_{g \in (I^{(M)})^m} \tau_{\widehat{R}}\left(O_U, s_i + h/mM \ divisor U(g)\right). \]
by [HLS, Corollary 5.8], since the ideals \((I^{(M)})^m\) and \((I^{M})^m\) coincide after inverting \(x_\ell\) by (7.3), as is also the case for \((J^{(M)})^m\) and \((T^{M})^m\). We therefore have

\[
\tau_{R^+(R, (I^{(M)})^{\frac{s_i+h}{M}})} \cdot R_{x_\ell} \subseteq \sum_{m=1}^{\infty} \sum_{g \in (J^{M})^m} \tau_+ \left( O_X, \frac{s_i + h}{mM} \text{div}_R(g) \right) \cdot R_{x_\ell} \\
\subseteq \sum_{m=1}^{\infty} \sum_{g \in J^{m}} \tau_+ \left( O_X, \frac{s_i + h}{m} \text{div}_R(g) \right) \cdot R_{x_\ell}.
\]

Next, by Lemma 5.8, we have

\[
\sum_{m=1}^{\infty} \sum_{g \in J^{m}} \tau_+ \left( O_X, \frac{s_i + h}{m} \text{div}_R(g) \right) \subseteq \tau_+ \left( O_X, \mathcal{J}^{s_i+h} \right)
\]

where the middle equality is by definition of \(\tau_+\) (see the last paragraph in [HLS, Remark 6.4]), and the last equality is by the Skoda-type result in [HLS, Theorem 6.6] (see also [HLS, Footnote on p. 27]). Here, we use the fact that the analytic spread of \(J_\ell\) is bounded above by the number of generators of \(J_\ell\) [SH06, Corollary 8.2.5], which is at most \(h\). Inverting \(x_\ell\) in this last inclusion and combining the containments so far, we obtain

\[
\tau_{R^+(R, (I^{(M)})^{\frac{s_i+h}{M}})} \cdot R_{x_\ell} \subseteq J_\ell^{s_i+1} \cdot (X, J_\ell^{h-1}) \cdot R_{x_\ell} \subseteq I^{s_i+1} R_{x_\ell}
\]

where the last inclusion holds since \(J_\ell \subseteq I\) and \((X, J_\ell^{h-1}) \subseteq R\).

We now consider the case when \(R\) is of equal characteristic zero. Set \(X = \text{Spec}(R)\), which is excellent by [EGAIV_2, Scholie 7.8.3(iii)]. Let \(B_\ell\) be the \(R^+\)-algebra that is big Cohen–Macaulay over \(R\) as constructed in Theorem 6.22. By Lemma 6.6(iii) and Theorem 6.22, we then have

\[
\tau_{B_\ell} \left( R, (I^{(M)})^{\frac{s_i+h}{M}} \right) \subseteq \mathcal{J} \left( X, (I^{(M)})^{\frac{s_i+h}{M}} \right) \subseteq I^{(s_i+1)}
\]

where the last inclusion holds by Proposition 7.1.

Finally, we prove Theorems B and C.

**Proof of Theorems B and C via big Cohen–Macaulay test ideals.** By Lemma 2.4, we may assume that \(R\) is a complete local ring and that the localizations of \(R\) at the associated primes of \(R/I\) have infinite residue fields. Throughout the proof, we define \(\tau_B\) using the choice of canonical divisor \(K_R = 0\) and the big Cohen–Macaulay algebra \(B\) that satisfies the conclusion of Theorem D.

We start with Theorem B. Set \(M = s + nh\), and choose generators \(f_1, f_2, \ldots, f_r\) for \(I^{(s+nh)}\). We have

\[
I^{(s+nh)} = (f_1, f_2, \ldots, f_r) \subseteq \tau_B (R, [f])
\]

by Proposition 6.12 (with notation as in Definition 6.1). We then have

\[
\tau_B (R, [f]) = \tau_B \left( R, \left[ \frac{f + h}{nh} \right] \right)
\]

where

\[
\tau_B (R, \left[ \frac{f + h}{nh} \right]) \subseteq \prod_{\ell = 1}^{\infty} \tau_B \left( R, \left[ \frac{f^{s_i+h}}{s_i+n} \right] \right).
\]
where the second equality holds by the unambiguity-type statement in Proposition 6.13, and the last inclusion holds by applying our version of the subadditivity theorem (Theorem 6.18) \((n - 1)\) times. We can therefore apply Proposition 6.4 and Theorem D when \(M = s + nh\) and combine these inclusions to show \((1.3)\).

We now prove Theorem C. Set \(M = s + nh + 1\), and choose generators \(f_1, f_2, \ldots, f_r\) for \(I^{(s+nh+1)}\). Let \(l\) be the largest integer such that

\[
\tau_B\left(R, \left[ f_{\frac{s+nh+1}{n}} \right] \right) = R.
\]

Note that such an \(l\) exists since \(l = 0\) satisfies this equality by the faithful flatness of \(R \to B\) [Hoc75, Lemma 5.5] (see also [HH92, (6.7); HH95, Lemma 2.1(d)]), and hence \(H^d_m(R) \to H^d_m(B)\) is injective. We then have

\[
I^{(s+nh+1)} = (f_1, f_2, \ldots, f_r) \cdot \tau_B\left(R, \left[ f_{\frac{s+nh+1}{n}} \right] \right)
\]

\[
\subseteq \tau_B\left(R, \left[ f_{\frac{s+nh+1}{n}} \right] \right) = \tau_B\left(R, \left[ f_{\frac{s+nh+1}{n}} \right] \right)
\]

\[
\tau_B\left(R, \left[ f_{\frac{s+nh+1}{n}} \right] \right) = \tau_B\left(R, \left[ f_{\frac{s+nh+1}{n}} \right] \right)
\]

where the inclusion holds by Proposition 6.12 (with notation as in Definition 6.1), and the last two equalities hold by the unambiguity-type statement in Proposition 6.13. By applying our version of the subadditivity theorem (Theorem 6.18) \(n\) times, we have

\[
\tau_B\left(R, \left[ f_{\frac{s+nh+1}{n}} \right] \right) \subseteq \tau_B\left(R, \left[ f_{\frac{s+nh+1}{n}} \right] \right) \subseteq \tau_B\left(R, \left[ f_{\frac{s+nh+1}{n}} \right] \right)
\]

where the last inclusion holds since \(\tau_B(R, \left[ f_{\frac{s+nh+1}{n}} \right] ) \subseteq m\) by our assumption on \(l\) and by applying Proposition 6.4 and Theorem D when \(M = s + nh + 1\). We can therefore combine these inclusions to show \((1.4)\). \(\square\)

References

[And18a] Y. André. “Le lemme d’Abhyankar perfectoïde.” *Publ. Math. Inst. Hautes Études Sci.* 127 (2018), pp. 1–70. DOI: 10.1007/s10240-017-0096-x. MR: 3814650. 3, 5

[And18b] Y. André. “La conjecture du facteur direct.” *Publ. Math. Inst. Hautes Études Sci.* 127 (2018), pp. 71–93. DOI: 10.1007/s10240-017-0097-9. MR: 3814651. 3, 5, 8, 14

[And20] Y. André. “Weak functoriality of Cohen-Macaulay algebras.” J. Amer. Math. Soc. 33.2 (2020), pp. 363–380. DOI: 10.1090/jams/937. MR: 4073864. 8, 14, 33

[AR88] M. Artin and C. Rotthaus. “A structure theorem for power series rings.” *Algebraic geometry and commutative algebra, Vol. I.* Tokyo: Kinokuniya, 1988, pp. 35–44. DOI: 10.1016/B978-0-12-348031-6.50009-7. MR: 977751. 10, 34

[Art71] M. Artin. “On the joins of Hensel rings.” *Advances in Math.* 7 (1971), pp. 282–296. DOI: 10.1016/S0001-8708(71)80076-1. MR: 289501. 11, 13

[AS07] M. Aschenbrenner and H. Schoutens. “Lefschetz extensions, tight closure and big Cohen-Macaulay algebras.” *Israel J. Math.* 161 (2007), pp. 221–310. DOI: 10.1007/s11856-007-0080-0. MR: 2350164. 6, 10, 11, 14, 15, 17

[Bha18] B. Bhatt. “On the direct summand conjecture and its derived variant.” *Invent. Math.* 212.2 (2018), pp. 297–317. DOI: 10.1007/s00222-017-0768-7. MR: 3787829. 3
G. D. Dietz. “A characterization of closure operations that induce big Cohen-Macaulay modules.” Proc. Amer. Math. Soc. 138.11 (2010), pp. 3849–3862. DOI: 10.1090/S0002-9939-2010-10417-3. MR: 2679608. 3, 4, 5, 11, 15, 16, 17.

G. D. Dietz. “Axiomatic closure operations, phantom extensions, and solidity.” J. Algebra 502 (2018), pp. 123–145. DOI: 10.1016/j.jalgebra.2018.01.023. MR: 3774887. 16.

M. DiPasquale and M. Drabkin. “On resurgence via asymptotic resurgence.” J. Algebra 587 (2021), pp. 64–84. DOI: 10.1016/j.jalgebra.2021.07.021. MR: 430520. 3.

G. D. Dietz and R. R.G. “Big Cohen-Macaulay and seed algebras in equal characteristic zero via ultraproducts.” J. Commut. Algebra 11.4 (2019), pp. 511–533. DOI: 10.1216/jca-2019-11-4-511. MR: 4039980. 8, 10, 11, 14, 15.

A. Grothendieck and J. Dieudonné. “Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I.” Inst. Hautes Études Sci. Publ. Math. 20 (1964), pp. 1–259. DOI: 10.1007/BF02684747. MR: 173675. 18.

A. Grothendieck and J. Dieudonné. “Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II.” Inst. Hautes Études Sci. Publ. Math. 24 (1965), pp. 1–231. DOI: 10.1007/BF02684322. MR: 199181. 34, 35, 37.

A. Grothendieck and J. Dieudonné. “Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III.” Inst. Hautes Études Sci. Publ. Math. 28 (1966), pp. 1–255. DOI: 10.1007/BF02684343. MR: 217086. 13.

L. Ein, R. Lazarsfeld, and K. E. Smith. “Uniform bounds and symbolic powers on smooth varieties.” Invent. Math. 144.2 (2001), pp. 241–252. DOI: 10.1007/s002220100121. MR: 1826369. 2, 3, 4, 7, 8, 34.

D. Eisenbud and B. Mazur. “Evolutions, symbolic squares, and Fitting ideals.” J. Reine Angew. Math. 488 (1997), pp. 189–201. DOI: 10.1515/crll.1997.488.189. MR: 1465370. 4.

N. Epstein. “A guide to closure operations in commutative algebra.” Progress in commutative algebra 2. Berlin: Walter de Gruyter, 2012, pp. 1–37. DOI: 10.1515/9783110287806. 1. MR: 2932590. 16.

E. Grifo, C. Huneke, and V. Mukundan. “Expected resurgences and symbolic powers of ideals.” J. Lond. Math. Soc. (2) 102.2 (2020), pp. 453–469. DOI: 10.1112/jlms.12324. 3.

E. Grifo. “A stable version of Harbourne’s conjecture and the containment problem for space monomial curves.” J. Pure Appl. Algebra 224.12 (2020), 106435, 23 pp. DOI: 10.1016/j.jalgebra.2020.106435. MR: 410479. 3.
[Gro67] A. Grothendieck. Local cohomology. Lecture notes by R. Hartshorne from a seminar given at Harvard, Fall, 1961. Lecture Notes in Math., Vol. 41. Berlin-New York: Springer-Verlag, 1967. DOI: 10.1007/BFb0073971. MR: 224620.

[Har68] R. Hartshorne. “Cohomological dimension of algebraic varieties.” Ann. of Math. (2) 88 (1968), pp. 403–450. DOI: 10.2307/1970720. MR: 232780.

[Har70] R. Hartshorne. “Affine duality and cofiniteness.” Invent. Math. 9 (1969/70), pp. 145–164. DOI: 10.1007/BF01404554. MR: 257096.

[Har05] N. Haras. “A characteristic p analog of multiplier ideals and applications.” Comm. Algebra 33.10 (2005), pp. 3375–3388. DOI: 10.1080/00927870500357386. MR: 2175438.

[Hei01] R. C. Heitmann. “Extensions of plus closure.” J. Algebra 238.2 (2001), pp. 801–826. DOI: 10.1006/jabr.2000.8661. MR: 1823784.

[Hei02] R. C. Heitmann. “The direct summand conjecture in dimension three.” Ann. of Math. (2) 156.2 (2002), pp. 695–712. DOI: 10.2307/3597204. MR: 1933722.

[HH90] M. Hochster and C. Huneke. “Tight closure, invariant theory, and the Briançon-Skoda theorem.” J. Amer. Math. Soc. 3.1 (1990), pp. 31–116. DOI: 10.2307/1990984. MR: 1017784.

[HH91] M. Hochster and C. Huneke. “Tight closure and elements of small order in integral extensions.” J. Pure Appl. Algebra 71.2-3 (1991), pp. 233–247. DOI: 10.1016/0022-4049(91)90149-V. MR: 1117636.

[HH92] M. Hochster and C. Huneke. “Infinite integral extensions and big Cohen-Macaulay algebras.” Ann. of Math. (2) 135.1 (1992), pp. 53–89. DOI: 10.2307/2946653. MR: 1147957.

[HH94] M. Hochster and C. Huneke. “Tight closure of parameter ideals and splitting in module-finite extensions.” J. Algebraic Geom. 3.4 (1994), pp. 599–670. DOI: 1297848.

[HH95] M. Hochster and C. Huneke. “Applications of the existence of big Cohen-Macaulay algebras.” Adv. Math. 113.1 (1995), pp. 45–117. DOI: 10.1006/aima.1995.1035. MR: 1332808.

[HH02] M. Hochster and C. Huneke. “Comparison of symbolic and ordinary powers of ideals.” Invent. Math. 147.2 (2002), pp. 349–369. DOI: 10.1007/s002220010176. MR: 1881923.

[HH07] M. Hochster and C. Huneke. “Fine behavior of symbolic powers of ideals.” Illinois J. Math. 51.1 (2007), pp. 171–183. DOI: 10.1215/ijm/1258735331. MR: 2346193.

[HH] M. Hochster and C. Huneke. “Tight closure in equal characteristic zero.” Version of Oct. 8, 2020. URL: http://www.math.ias.umich.edu/~hochster/tcz.pdf.

[HJKN23] H.T. Hà, A.V. Jayanthan, A. Kumar, and H.D. Nguyen. “Binomial expansion for saturated and symbolic powers of ideals.” J. Algebra 620 (2023), pp. 690–710. DOI: 10.1016/j.jalgebra.2022.12.037. MR: 4539106.

[HKV09] C. Huneke, D. Katz, and J. Validashti. “Uniform equivalence of symbolic and adic topologies.” Illinois J. Math. 53.1 (2009), pp. 325–338. DOI: 10.1215/ijm/1264170853. MR: 2584949.

[HKV15] C. Huneke, D. Katz, and J. Validashti. “Uniform symbolic topologies and finite extensions.” J. Pure Appl. Algebra 219.3 (2015), pp. 543–550. See also [HKV21]. DOI: 10.1016/j.jpaa.2014.05.012. MR: 3279373.

[HKV21] C. Huneke, D. Katz, and J. Validashti. “Corrigendum to “Uniform symbolic topologies and finite extensions”.” J. Pure Appl. Algebra 225.6 (2021), Paper No. 106587, 2 pp. DOI: 10.1016/j.jpaa.2020.106587. MR: 4197929.

[HL07] C. Huneke and G. Lyubeznik. “Absolute integral closure in positive characteristic.” Adv. Math. 210.2 (2007), pp. 498–504. DOI: 10.1016/j.aim.2006.07.001. MR: 2303230.

[HLS] C. Hacon, A. Lamarche, and K. Schwede. “Global generation of test ideals in characteristic and applications.” Dec. 25, 2022. arXiv:2106.14329v4 [math.AG]. 3, 4, 9, 10, 11, 13, 21, 22, 23, 31, 34, 36, 37.

[HM21] R. C. Heitmann and L. Ma. “Extended plus closure in complete local rings.” J. Algebra 571 (2021), pp. 134–150. DOI: 10.1016/j.jalgebra.2018.10.006. MR: 4200713.

[Hoc75] M. Hochster. “Contracted ideals from integral extensions of regular rings.” Nagoya Math. J. 51 (1973), pp. 25–43. DOI: 10.1017/s0027763000015701. MR: 349656.

[Hoc75] M. Hochster. “Big Cohen-Macaulay modules and algebras and embeddability in rings of Witt vectors.” Conference on Commutative Algebra–1975 (Queen’s Univ., Kingston, Ont., 1975). Queen’s Papers on Pure and Applied Math., Vol. 42. Kingston, Ont.: Queen’s Univ., 1975, pp. 106–195. MR: 396544.

[Hoc91] M. Hochster. “Solid closure.” Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992). Contemp. Math., Vol. 159. Providence, RI: Amer. Math. Soc., 1994, pp. 103–172. DOI: 10.1090/conm/159/01508. MR: 1266182.
[Sch12] P. Scholze. “Perfectoid spaces.” *Publ. Math. Inst. Hautes Études Sci.* 116 (2012), pp. 245–313. DOI: 10.1007/s10240-012-0042-x; MR: 3090258. 3, 4

[SGA2] A. Grothendieck. *Séminaire de géométrie algébrique du Bois Marie, 1962. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2).* With an exposé by Mme M. Raynaud. With a preface and edited by Y. Laszlo. Revised reprint of the 1968 French original. Doc. Math. (Paris), Vol. 4. Paris: Soc. Math. France, 2005. MR: 2171939. 2

[SH06] I. Swanson and C. Huneke. *Integral closure of ideals, rings, and modules.* London Math. Soc. Lecture Note Ser., Vol. 336. Cambridge: Cambridge Univ. Press, 2006. Online corrected version available at https://www.math.purdue.edu/~iswanso/book; DOI: 2266432. 2, 3, 10, 18, 34, 36, 37

[Sha81] R. Y. Sharp. “Cohen-Macaulay properties for balanced big Cohen-Macaulay modules.” *Math. Proc. Cambridge Philos. Soc.* 90.2 (1981), pp. 229–238. DOI: 10.1017/s0305004100058680; MR: 620732. 11, 13

[Shi18] K. Shimomoto. “Integral perfectoid big Cohen-Macaulay algebras via André’s theorem.” *Math. Ann.* 372.3-4 (2018), pp. 1167–1188. DOI: 10.1007/s00208-018-1704-x; MR: 3880296. 8, 14

[Smi94] K. E. Smith. “Tight closure of parameter ideals.” *Invent. Math.* 115.1 (1994), pp. 41–60. DOI: 10.1007/BF01231753; MR: 1248078. 5, 6, 17

[Smi95] K. E. Smith. “Test ideals in local rings.” *Trans. Amer. Math. Soc.* 347.9 (1995), pp. 3453–3472. DOI: 10.1007/BF01231753; MR: 1248078. 31

[ST] K. Schwede and K. Tucker. “A survey of test ideals.” *Progress in commutative algebra 2.* Berlin: Walter de Gruyter, 2012, pp. 39–99. DOI: 10.1515/9783110278606. 7, 8, 34

[ST12] K. Schwede and K. Tucker. “Artithmetic and geometric deformations of F-pure and F-regular singularities.” Mar. 18, 2021. arXiv:2103.03721v2 [math.AC]. 3, 4, 8, 9, 10, 11, 21, 23, 24, 26, 34, 35, 36

[Swa00] I. Swanson. “Linear equivalence of ideal topologies.” *Math. Z.* 234.4 (2000), pp. 755–775. DOI: 10.1007/s002080050007; MR: 1778408. 2

[Tak06] S. Takagi. “Formulas for multiplier ideals on singular varieties.” *Amer. J. Math.* 128.6 (2006), pp. 1345–1362. DOI: 10.1353/ajm.2006.0049; MR: 2275023. 30

[Tem08] M. Temkin. “Desingularization of quasi-excellent schemes in characteristic zero.” *Adv. Math.* 219.2 (2008), pp. 488–522. DOI: 10.1016/j.aim.2008.05.006; MR: 2436474. 24

[TY08] S. Takagi and K. Yoshida. “Generalized test ideals and symbolic powers.” *Michigan Math. J.* 57 (2008): *Special volume in honor of Melvin Hochster*, pp. 711–724. DOI: 10.1307/mmj/1220879433; MR: 2492477. 4, 7, 8, 9, 10, 12, 34

[TY23] T. Takamatsu and S. Yoshikawa. “Minimal model program for semi-stable threefolds in mixed characteristic.” *J. Algebraic Geom.* 32.3 (2023), pp. 429–476. DOI: 10.1090/jag/813; MR: 462225? 9, 21

[Ver87] J. K. Verma. “On ideals whose adic and symbolic topologies are linearly equivalent.” *J. Pure Appl. Algebra* 47.2 (1987), pp. 205–212. DOI: 10.1016/0022-4049(87)90062-4; MR: 906971. 12

[Ver88] J. K. Verma. “On the symbolic topology of an ideal.” *J. Algebra* 112.2 (1988), pp. 416–429. DOI: 10.1016/0021-8693(88)90099-3; MR: 926614. 2

[Yam] T. Yamaguchi. “Big Cohen-Macaulay test ideals in equal characteristic zero via ultraproducts.” To appear in *Nagoya Math. J.* DOI: 10.1017/mmj.2022.41. 32

[Zar51] O. Zariski. “Theory and applications of holomorphic functions on algebraic varieties over arbitrary ground fields.” *Mem. Amer. Math. Soc.* 5 (1951), 90 pp. DOI: 10.1090/memo/0005; MR: 41487. 2

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907-2067, USA

Email address: murayama@purdue.edu

URL: https://www.math.purdue.edu/~murayama/