Moyal multiplier algebras of the test function spaces of type $S$

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Abstract

The Gel’fand-Shilov spaces of type $S$ are considered as topological algebras with respect to the Moyal star product and their corresponding algebras of multipliers are defined and investigated. In contrast to the well-studied case of Schwartz’s space $S$, these multipliers are allowed to have nonpolynomial growth or infinite order singularities. The Moyal multiplication is thereby extended to certain classes of ultradistributions, hyperfunctions, and analytic functionals. The main theorem of the paper characterizes those elements of the dual of a given test function space that are the Moyal multipliers of this space. The smallest nontrivial Fourier-invariant space in the scale of $S$-type spaces is shown to play a special role, because its corresponding Moyal multiplier algebra contains the largest algebra of functions for which the power series defining their star products are absolutely convergent. Furthermore, it contains analogous algebras associated with cone-shaped regions, which can be used to formulate a causality condition in quantum field theory on noncommutative space-time.

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I. Introduction and motivation

This paper continues the work started in [1], where it was shown that, under the condition $\beta \geq \alpha$, the test function spaces $S^{\alpha}_\beta$ introduced by Gel’fand-Shilov [2] are algebras with respect to the Weyl-Groenewold-Moyal star product, or Moyal product as it is more commonly called now. The spaces $S^{\alpha}_\beta$ are smaller than the Schwartz space $S$ of all smooth functions of fast decrease and are specified by additional restrictions on smoothness and behavior at infinity. Accordingly, their dual spaces $(S^{\alpha}_\beta)'$ are larger than the space $S'$ of tempered distributions, and this scale of spaces provides a wider framework for various applications. The Moyal multiplication is a basic notion of the Wigner-Weyl phase-space representation of quantum mechanics because it is just the composition rule for functions on the classical phase space that corresponds, via the Weyl transformation, to the product of operators acting on a Hilbert space. (And then the functions are treated as symbols of their corresponding operators [3].)

This notion is also central to noncommutative quantum field theory (see, e.g, [4] for a review), whose intensive development over the last years has been stimulated by the in-depth analysis [5] of the quantum limitations on localization of events in space-time and by the study [6] of the low energy limit of string theory. The Moyal product is a noncommutative deformation of the usual pointwise product, generated by a Poisson structure. In the simplest case of functions on the space $\mathbb{R}^d$ endowed with a linear Poisson structure, it can be written as

$$
(f \ast_\theta g)(x) = f(x)\exp\left\{ \frac{i}{2} \frac{1}{\theta} \frac{\partial^j g}{\partial x^j} \frac{\partial^j f}{\partial x^j} \right\} g(x) = \sum_{n=0}^{\infty} \left( \frac{i}{2} \frac{1}{\theta} \frac{\partial^j g}{\partial x^j} \frac{\partial^j f}{\partial x^j} \right)^n f(x)g(y) \bigg|_{x=y},
$$

where $\theta^{\mu\nu}$ is a real constant antisymmetric $d \times d$ matrix. This formula goes back to the celebrated works of Groenewold [7] and Moyal [8]. The right-hand side of (1) is usually understood as a formal power series in the noncommutativity parameter $\theta$ and the validity of this formula is rather restrictive, though it is ideally suited for polynomial symbols. There is another representation for the Moyal star product, which came into use later and is easily derivable from (1) under favorable assumptions about $f$ and $g$. Namely, if these functions decrease faster than any inverse power of $|x|$ and their Fourier transforms $\hat{f}$ and $\hat{g}$ decrease faster than the Gaussian function, then the standard theorems of analysis justify the formal application of the operator $\exp\left\{ \frac{1}{2} \frac{1}{\theta} \frac{\partial^j g}{\partial x^j} \frac{\partial^j f}{\partial x^j} \right\}$ to the Fourier representation of $f(x)g(y)$ and we can write

$$
(f \ast_\theta g)(x) = \frac{1}{(2\pi)^d} \int \hat{f}(p)\hat{g}(q) e^{ipx-\frac{1}{2}i\theta pq} dp dq =
$$

$$
\frac{1}{(2\pi)^d} \int f(x-\frac{1}{2}i\theta q)\hat{g}(q) e^{ipx} dq = \frac{1}{(2\pi)^d} \int \hat{f}(p)g(x+\frac{1}{2}i\theta p) e^{ipx} dp =
$$

$$
\frac{1}{(2\pi)^d} \int f(x-\frac{1}{2}i\theta q)g(x+y) e^{-i\theta qy} dy dq = \frac{1}{(2\pi)^d} \int \hat{f}(p)g(x+\frac{1}{2}i\theta p) e^{-ip\theta y} dp dy.
$$

where $px = \sum_{j=1}^{d} p_j x_j$ and $\theta p = \sum_{j=1}^{d} \theta^{ij} p_j$. Clearly, the last two integrals are well defined for any integrable $f$ and $g$ if the matrix $\theta$ is invertible, and we obtain

$$
(f \ast_\theta g)(x) = \frac{1}{\pi^d \det \theta} \int f(x+y)g(x+z) e^{-2i\theta^{-1}yz} dy dz.
$$

(Note that the determinant of each invertible antisymmetric matrix is positive.) Neither of definitions (1) and (3) is universal, but the second one can be naturally extended to a
larger class of symbols by using the methods of distribution theory, which opens the way to unification. It is readily seen that the Schwartz space $S$ is an associative topological algebra with respect to product (3). Antonets [9, 10] was the first to propose the extension of (3) by duality to those elements of $S'$ that are Moyal multipliers of $S$. Such an extension has been investigated by Kammerer [11], Maillard [12] and, most thoroughly, by Gracia-Bondia and Várilly [13, 14]. Here we generalize this approach to the spaces $S^α_β$. From the above it is clear that different forms of Moyal star product can be obtained by extension (depending on the problem under consideration) from an appropriate function space, whereon (1) and (3) are equally well defined and interconvertible. Because of this we use the same notation for them and we will sometimes write $⋆$ instead of $⋆_θ$ when this cannot cause confusion. The relation between formulas (1) and (3) was also discussed at length in [15], where (1) was systematically treated as an asymptotic expansion of (3).

For the role of Moyal analysis in quantum field theory on noncommutative space-time, we refer the reader to [16] and [17]. The question of causality is crucial for the physical interpretation of noncommutative field theory and, when analyzing its causal structure, it should be taken into account that the Moyal star product is inherently nonlocal. As stressed in [18], the framework of tempered distributions is apparently too restrictive for the nonperturbative study of general properties of noncommutative field theories. In [1], an enlarged framework with the use of test function spaces $S^α_β$ was proposed for this purpose and series (1) was shown to converge for all elements of $S^α_β$ if and only if $β < 1/2$. A similar proposal was made in [19]. An examination [20, 21] of locality violations in some noncommutative models shows that they fail to obey the microcausality axiom of the standard quantum field theory [22, 23]. Nevertheless, they obey a weaker causality condition [24, 25] formulated in terms of analytic test functions. As argued elsewhere [26], this condition is sufficient to ensure such fundamental physical properties as the spin-statistics relation and the existence of CPT-symmetry. The Fourier invariant spaces $S^β_β$ were used by Fischer and Szabo [27, 28] in their analysis of the renormalization properties of scalar field theories on noncommutative Minkowski space, but as shown by Zahn [29], this issue requires a more careful study. Because of all the above, it is desirable to construct and investigate the Moyal multiplier algebras of the spaces of type $S$. We will show that for $α = β$, these algebras contain all polynomials and all distributions of compact support as does the Moyal multiplier algebra of $S$. But in contrast to the latter, they also contain a large class of functions with non-polynomial growth and with non-tempered singularities. In this paper, we focus on the extension of the Weyl symbolic calculus of itself. The results have direct applications to deformation quantization and to field theory on noncommutative spaces, but this applications will be considered in a subsequent work.

The paper is organized as follows. In Sec. II, we recall the definition and main properties of the spaces $S^α_β$ and their associated algebras of pointwise multipliers and of convolution multipliers. The aim of this paper is to investigate a noncommutative deformation of these algebras, but the basic construction is presented in a more general setting, for an arbitrary test function space $E ⊂ S$ allowing such deformation. In Sec. III, we define by duality the algebras $ℳ_{θ,L}(E)$ and $ℳ_{θ,R}(E)$ of left and right Moyal multipliers of $E$. In Sec. IV, we show that in the case of spaces $S^β_β$, $β ≥ α$, every Moyal multiplier can be approximated by test functions in the operator topology. This result gives an alternative way of defining the algebras $ℳ_{θ,L}(S^α_β)$ and $ℳ_{θ,R}(S^α_β)$, via a completion procedure applied to the sets of operators of the left and right
star multiplication by elements of $S^\beta_\alpha$. Moreover, the approximation theorem lets us prove that the intersection $M_\theta(S^\beta_\alpha)$ of these two algebras is also an algebra. In the same section, we show that $M_\theta(S^\beta_\alpha)$ equipped with a natural topology acts continuously on the dual space $(S^\beta_\alpha)'$, i.e., this space has the structure of an $M_\theta(S^\beta_\alpha)$-bimodule. In Sec. V, we characterize the smoothness properties and the behavior at infinity of the Fourier transforms of the star products $f \star_\theta u$ and $u \star_\theta f$, where $f \in S^\beta_\alpha$ and $u \in (S^\beta_\alpha)'$. In particular, we prove that for any matrix $\theta$, the Fourier transforms of these products are pointwise multipliers of $S^\beta_\alpha = S^\beta_\alpha$. Making use of this result, we prove in Sec. VI that the Moyal multiplier algebra of the Fourier-invariant space $S^\beta_\alpha$ contains all elements of $(S^\beta_\alpha)'$ decreasing sufficiently fast at infinity and, moreover, contains their Fourier transforms. In Sec. VII, we discuss a special role of the space $S^{1/2}$ and show that $M_\theta(S^{1/2})$ contains the largest algebra with the property that the Moyal power series converges absolutely for all its elements. Furthermore, we demonstrate that $M_\theta(S^{1/2})$ has a family of subalgebras associated naturally with cone-shaped regions, which can be used for formulating causality in a rigorous development of quantum field theory on noncommutative space-time. Sec. VIII is devoted to concluding remarks. Appendix presents the proof of a lemma on the properties of the Fourier transforms of functions in $S^\beta_\alpha$, which is used in deriving the main results.

II. The pointwise multipliers of the spaces of type S

We recall that the space $S^\beta_\alpha(\mathbb{R}^d)$ with indices $\alpha \geq 0$, $\beta \geq 0$ consists of all infinitely differentiable functions on $\mathbb{R}^d$ satisfying the inequalities

$$|\partial^n f(x)| \leq C B |n|^\beta e^{-|x|/A}, \quad (4)$$

where $C$, $A$, and $B$ are constants depending on $f$ and $n = (n_1, \ldots, n_d)$ is an arbitrary $d$-tuple of nonnegative integers. From here on we use the standard multiindex notation: $|n| = n_1 + \cdots + n_d$, $n^\beta = n_1^{\beta_1} \cdots n_d^{\beta_d}$, and $\partial^n = \partial_{x_1}^{n_1} \cdots \partial_{x_d}^{n_d}$. The above definition is independent of the choice of the norm in $\mathbb{R}^d$ because all these norms are equivalent, but the uniform norm $|x| = \max_{1 \leq j \leq d} |x^j|$ is most convenient for use below. We write $S^\beta_\alpha$ instead of $S^\beta_\alpha(\mathbb{R}^d)$ when there is no risk of confusion. By $S^\beta_\alpha,B$ we denote the set of functions satisfying (4) with fixed $A$ and $B$ and equip it with the norm

$$\|f\|_{A,B} = \sup_{x,n} \frac{|\partial^n f(x)|}{B |n|^\beta e^{x/A}}.$$

which turns $S^\beta_{\alpha,A}$ into a Banach space. Accordingly, $S^\beta_\alpha$ is equipped with the inductive limit topology determined by the canonical embeddings $S^\beta_{\alpha,A} \to S^\beta_{\alpha}$, hence it is a barrelled space. The space $S^\beta_{\alpha}$ is nontrivial if and only if $\alpha + \beta > 1$ or $\alpha + \beta = 1$ but $\alpha$ and $\beta$ are nonzero. In what follows, we assume that this condition is fulfilled. Every nontrivial space $S^\beta_{\alpha}$ is dense in $S$. As shown in [2], the natural maps $S^\beta_{\alpha,A} \to S^\beta_{\alpha,A'}$, where $A' > A$ and $B' > B$, are compact. It follows that $S^\beta_{\alpha}$ are DFS spaces (dual Fréchet-Schwartz spaces). We refer the reader to [30] for the definition and properties of FS and DFS spaces. Furthermore, they are nuclear [31] and an analog of the Schwartz kernel theorem holds for them. The pointwise multiplication $(f, g) \to f \cdot g$ is well defined for elements of $S^\beta_{\alpha}$ and is separately continuous [2]. Since $S^\beta_{\alpha}$ is a DFS space, the separate continuity implies continuity, see, e.g., [32], Sec. 44.2, where the
The corresponding theorem is proved even for a larger class of spaces. Therefore every space $S^\beta_\alpha$ is a topological algebra with respect to the pointwise product and with respect to the (ordinary) convolution product. The Fourier transform is a topological isomorphism between $S^\beta_\alpha$ and $S^\beta_\alpha$ and converts one of these products to the other.

The space of pointwise multipliers of $S^\beta_\alpha$ has been completely characterized by Palamodov [33]. Let $E$ be a topological vector space contained in the Schwartz space $S$ and let $\mathcal{L}(E)$ be the algebra of continuous linear operators on $E$, equipped with the topology of uniform convergence on bounded subsets of $E$. By Palamodov’s definition, the space $M(E)$ of multipliers of $E$ is the closure in $\mathcal{L}(E)$ of the set of all operators of multiplication by elements of $S$. Accordingly, $M(E)$ is equipped with the topology induced by that of $\mathcal{L}(E)$. The space $C(E)$ of convolution multipliers is defined in a similar manner. Clearly, these spaces are subalgebras of the operator algebra $\mathcal{L}(E)$. Let $E^\beta,B_\alpha,A(R^d)$ be the space of all smooth functions $u$ on $R^d$ with the property that

$$\|u\|_{-A,B} = \sup_{x,n} \left| \frac{\partial^n u(x)}{B^n |n^n \beta} e^{-|x/A|^{1/\alpha}} \right| < \infty.$$  

(6)

**Theorem 1** (Palamodov): The spaces of pointwise multipliers and convolution multipliers of $S^\beta_\alpha$ can be presented as follows:

$$M(S^\beta_\alpha) = \text{proj lim } \text{inj lim } E^{\beta,B}_{\alpha,A},$$  

(7)

$$C(S^\beta_\alpha) = \text{proj lim } \text{inj lim } (E^{\beta,B}_{\alpha,A})',$$  

(8)

where $(E^{\beta,B}_{\alpha,A})'$ is the dual of $E^{\beta,B}_{\alpha,A}$, equipped with the strong topology. Both these spaces are complete, nuclear, and semireflexive. The Fourier transform is an isomorphism between $M(S^\beta_\alpha)$ and $C(S^\beta_\alpha)$.

The above-listed topological properties of the spaces (7) and (8) follows from the corresponding properties of $S^\beta_\alpha$. In [33], it was noted that these spaces are quasicomplete. But if $E$ is a complete DF space, then $\mathcal{L}(E)$ is complete [32] and so is every closed subspace of $\mathcal{L}(E)$. If in addition $E$ is nuclear, than $\mathcal{L}(E)$ is also nuclear [31] and hence semireflexive, and these properties are also inherited by closed subspaces.

Since spaces (7) and (8) are embedded into $(S^\beta_\alpha)'$, the Fourier transforms of their elements are defined in the ordinary way: $\langle \hat{u}, \hat{f} \rangle = (2\pi)^d \langle u, f \rangle$, where $f \in S^\beta_\alpha$ and $\hat{f}(x) = f(-x)$. The space $S^\beta_\alpha$ is obviously contained in either of these two spaces and is dense in both of them by Theorem 1. Hence their duals $M'(S^\beta_\alpha)$ and $C'(S^\beta_\alpha)$ can also be identified with linear subspaces of $(S^\beta_\alpha)'$, which are described by the next theorem.

**Theorem 2**: The duals of $M(S^\beta_\alpha)$ and $C(S^\beta_\alpha)$ have, respectively, the form

$$M'(S^\beta_\alpha) = \bigcup_A \bigcap_B (E^{\beta,B}_{\alpha,A})',$$  

(9)

and

$$C'(S^\beta_\alpha) = \bigcup_B \bigcap_A E^{\beta,B}_{\alpha,A}.$$  

(10)

**Proof.** These formulas follow from the well known duality relations [34] between projective and inductive limits. The only subtlety is that the projective limit is assumed to be in reduced
form in these relations. Let \( \mathcal{E}_{\alpha,A}^{\beta,B} := \text{proj lim}_{\epsilon \to 0} E_{\alpha,A - \epsilon}^{\beta,B + \epsilon} \). A reasoning similar to that used by Gel’fand and Shilov for \( \text{proj lim}_{\epsilon \to 0} S_{\alpha,A + \epsilon}^{\beta,B + \epsilon} \) shows that the natural maps \( E_{\alpha,A}^{\beta,B} \to E_{\alpha,A - \epsilon}^{\beta,B + \epsilon} \) are compact. Hence the spaces \( \mathcal{E}_{\alpha,A}^{\beta,B} \) are perfect, i.e., are FS spaces (Fréchet-Schwartz spaces) in the modern terminology. In particular, they are Montel spaces. For any \( A > A' > A'' \) and \( B < B' \), we have the commutative diagram

\[
\begin{array}{ccc}
E_{\alpha,A}^{\beta,B} & \xrightarrow{\mathcal{E}_{\alpha,A}^{\beta,B}} & E_{\alpha,A'}^{\beta,B'} \\
\downarrow & & \downarrow \\
E_{\alpha,A'}^{\beta,B'} & & E_{\alpha,A''}^{\beta,B''}
\end{array}
\]

where all arrows are natural embeddings. It follows that in definition \( \mathcal{A} \), the spaces \( E_{\alpha,A}^{\beta,B} \) can be replaced with \( \mathcal{E}_{\alpha,A}^{\beta,B} \), leaving the limit space unchanged. We claim that \( S_{\alpha,A}^{\beta} \) is dense in \( \text{proj lim}_B \mathcal{E}_{\alpha,A}^{\beta,B} \) for each \( A > 0 \). Let \( f \in \mathcal{E}_{\alpha,A}^{\beta,B} \) and \( 2A_1 < A \). Let \( B_1 \) be large enough for \( \mathcal{E}_{\alpha,A_1}^{\beta,B_1}(\mathbb{R}^d) \) to be nontrivial. We choose a function \( e \in S_{\alpha,A_1}(\mathbb{R}^d) \) with the property \( \int e(\xi)d\xi = 1 \) and set

\[ e_{\nu}(x) = \int_{|\xi|<\nu} e(x-\xi)d\xi, \quad \nu = 1, 2, \ldots. \]

Then \( e_{\nu}(x) \to 1 \) at every point \( x \). Using Leibniz’s formula and the inequality

\[-|x-\xi|^{1/\alpha} \leq -|x/2|^{1/\alpha} + |\xi|^{1/\alpha}, \quad (12)\]

we obtain

\[ |\partial^n(f e_{\nu})(x)| \leq C_\epsilon \sum_m \left( \frac{n}{m} \right) (B + \epsilon)^{|m|} B_1^{n-m} m^{\beta(n-m)} (n-m)^{\beta(n-m)} c |x/(A-\epsilon)|^{1/\alpha} \int_{|\xi|<\nu} e^{-|x-\xi|/A_1^{1/\alpha}} d\xi \leq C_\epsilon'(B + B_1 + \epsilon) |n|^{\alpha} c |x/(A-\epsilon)|^{1/\alpha} \int_{\mathbb{R}^d} e^{-|\xi|/A_1^{1/\alpha}} d\xi, \quad (13)\]

where \( \epsilon > 0 \) can be taken arbitrarily small. Therefore, \( f e_{\nu} \in S_{\alpha,A}^{\beta} \). On the other hand,

\[ |\partial^n(f e_{\nu})(x)| \leq C_\epsilon (B + B_1 + \epsilon) |n|^{\alpha} c |x/(A-\epsilon)|^{1/\alpha} \int_{\mathbb{R}^d} e^{-|\xi|/A_1^{1/\alpha}} d\xi \quad (14)\]

and the sequence \( f e_{\nu} \) is hence bounded in \( \mathcal{E}_{\alpha,A}^{\beta,B+B_1} \). Because it is a Montel space and its topology is stronger than that of pointwise convergence, we conclude that \( f e_{\nu} \to f \) in \( \mathcal{E}_{\alpha,A}^{\beta,B+B_1} \). This proves our claim. As a consequence, \( M(S_{\alpha}^{\beta}) \) is dense in \( \text{proj lim}_B \mathcal{E}_{\alpha,A}^{\beta,B} \) for any \( A \), so the projective limit \( \text{proj lim}_A \{ \text{proj lim}_B \mathcal{E}_{\alpha,A}^{\beta,B} \} \) is reduced. Therefore, for every \( v \in M'(S_{\alpha}^{\beta}) \), there is an \( A \) such that \( v \) has a unique continuous extension to \( \text{proj lim}_B \mathcal{E}_{\alpha,A}^{\beta,B} \) and thereby to each space \( \mathcal{E}_{\alpha,A}^{\beta,B} \), \( B > 0 \). This is equivalent to saying that for some \( A \), the functional \( v \) extends continuously to each space \( E_{\alpha,A}^{\beta,B} \), \( B > 0 \). Thus, formula \( \mathcal{B} \) is proved.

From the commutativity of diagram \( \mathcal{A} \), it follows that the corresponding diagram for dual spaces is also commutative, which shows that in \( \mathcal{A} \) the spaces \( (E_{\alpha,A}^{\beta,B})' \) can be replaced by the DFS spaces \( (\mathcal{E}_{\alpha,A}^{\beta,B})' = \text{proj lim}_{\epsilon \to 0} (E_{\alpha,A - \epsilon}^{\beta,B + \epsilon})' \). By the Hahn-Banach theorem, the functionals \( \delta(x - \xi) \), where \( \xi \) ranges over \( \mathbb{R}^d \), form a total set in every \( (\mathcal{E}_{\alpha,A}^{\beta,B})' \) because any FS space is reflexive and \( (\mathcal{E}_{\alpha,A}^{\beta,B})'' = \mathcal{E}_{\alpha,A}^{\beta,B} \). The space \( C(S_{\alpha}^{\beta}) \) contains all these functionals and
hence is dense in every $E_{A,B}^{\beta,A}$, i.e., the projective limit $\text{proj lim}_{B}(\text{inj lim}_{A}(E_{A,B}^{\beta,A}))$ is reduced. Therefore, for every $u \in C'(S_{A}^{\beta})$, there is $B$ such that $u$ has a unique continuous extension to $\text{inj lim}_{A}(E_{A,B}^{\beta,A})$, and so belongs to each of the spaces $E_{A,B}^{\beta,A}$, $A > 0$. This amounts to saying that for some $B$, the functional $u$ belongs to each of the spaces $E_{A,B}^{\beta,A}$, $A > 0$.

### III. Extension of the Moyal product by duality

Let $E$ be a locally convex function space embedded densely and continuously into the Schwartz space $S$. Then we have the sequence of natural continuous injections

$$E \to S \to S' \to E'.$$

The third map in (15), being the transpose of the first one, is continuous for the strong as well as for the weak topology on $E'$ (see [34], Sec. IV.7.4) and has a weakly dense image. For $u \in E'$, we write $\langle u, f \rangle$ for the value of the functional $u$ evaluated at $f \in E$. If $E$ is a topological algebra under the Moyal multiplication, then the products $u \star f$ and $f \star u$ can be defined by

$$\langle u \star f, g \rangle = \langle u, f \star g \rangle, \quad \langle f \star u, g \rangle = \langle u, g \star f \rangle, \quad g \in E,$n

in complete analogy with the case $E = S$ studied in [7]–[13]. Since the expressions on the right-hand side are linear and continuous in $g$, these products are well defined as elements of $E'$. From (2), it follows that

$$\int (f \star g)(x) \, dx = \int f(x)g(x) \, dx, \quad \text{for any } f, g \in E.$$

This simple but important relation called the tracial property implies that the products (16) are extensions of the initial $\star$-multiplication on $E$. Indeed, using (17) and the associativity of the algebra $(E, \star)$, we obtain that

$$\langle h \star f, g \rangle = \langle h, f \star g \rangle = \langle f, g \star h \rangle = \int (h \star g \star f)(x) \, dx, \quad \text{for all } h, f, g \in E.$$n

For every fixed $f$, the maps $u \to u \star f$ and $u \to f \star u$ of $E'$ into itself are continuous because they are the transposes of the continuous maps $g \to f \star g$ and $g \to g \star f$. Since $E$ is dense in $E'$, there are no other continuous extensions of the $\star$-multiplication to the case where one of factors belongs to $E'$. For every fixed $u \in E'$, the maps $f \to f \star u$ and $f \to u \star f$ from $E$ into $E'$ are also continuous. Consider for instance the first of them. For any $\epsilon > 0$, we can find a neighborhood $W$ of the origin in $E$ such that $u$ is bounded by $\epsilon$ on $W$. Because the map $(f,g) \to f \star g$ is jointly continuous, there are neighborhoods $U$ and $V$ in $E$ such that $f \star g \in W$ for all $f \in U$ and all $g \in V$. For any bounded set $Q \subset E$, there is $\delta > 0$ such that $\delta Q \subset V$, hence $\sup_{g \in Q} |(u, f \star g)| \leq \epsilon$ for any $f \in \delta U$, which proves the statement. From (16) and the associativity of the $\star$-multiplication in $E$, it immediately follows that

$$(u \star f) \star h = u \star (f \star h), \quad h \star (f \star u) = (h \star f) \star u \quad \text{for all } u \in E', f, h \in E.$$

This means that $E'$ has the structure of a (nonunital) bimodule over the ring $(E, \star)$. 

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Now we introduce the spaces of left and right \( \star \)-multipliers of \( E \):

\[
\mathcal{M}_{\theta,L}(E) \overset{\text{def}}{=} \{ u \in E' : u \star f \in E, \quad \text{for all } f \in E \},
\]

\[
\mathcal{M}_{\theta,R}(E) \overset{\text{def}}{=} \{ u \in E' : f \star u \in E, \quad \text{for all } f \in E \}.
\]

The linear maps \( f \rightarrow u \star f \) and \( f \rightarrow f \star u \) of \( E \) into itself have closed graphs. Indeed, if \( f_\nu \rightarrow f \) and \( u \star f_\nu \rightarrow h \), then for any \( g \in E \), we have

\[
\langle h, g \rangle = \lim_\nu \langle u \star f_\nu, g \rangle = \lim_\nu \langle u, f_\nu \star g \rangle = \langle u, f \star g \rangle = \langle u \star f, g \rangle
\]

and hence \( h = u \star f \). If one or other version of the closed graph theorem \([32,33]\) is applicable to \( E \), then these maps are continuous and so belong to \( \mathcal{L}(E) \). This allows us to define the star products of the multipliers with elements of \( E' \) by the formulas

\[
\langle w \star u, f \rangle = \langle w, u \star f \rangle, \quad \langle v \star w, f \rangle = \langle w, f \star v \rangle,
\]

(20)

where \( w \in E' \), \( u \in \mathcal{M}_{\theta,L}(E) \), \( v \in \mathcal{M}_{\theta,R}(E) \), and \( f \) ranges over \( E \). So, the left (right) Moyal multipliers of \( E \) serve as right (left) multipliers for \( E' \). Clearly, \( E \) is contained in \( \mathcal{M}_{\theta,L}(E) \) as well as in \( \mathcal{M}_{\theta,R}(E) \), and operations (20) extend operations (16). For fixed \( u \) and \( v \), the maps \( w \rightarrow w \star u \) and \( w \rightarrow v \star w \) of \( E' \) into itself are continuous because they are the transposes of the maps \( f \rightarrow v \star f \) and \( f \rightarrow f \star u \) from \( E \) into \( E \).

It is readily seen that \( \mathcal{M}_{\theta,L}(E) \) and \( \mathcal{M}_{\theta,R}(E) \) are unital associative algebras with respect to the star product. First we show that \( u_1, u_2 \in \mathcal{M}_{\theta,L}(E) \) implies \( u_1 \star u_2 \in \mathcal{M}_{\theta,L}(E) \). Let \( f, g \in E \). From (16), it follows that

\[
\langle (u_1 \star u_2) \star f, g \rangle = \langle u_1 \star u_2, f \star g \rangle,
\]

and by (20) we have

\[
\langle u_1 \star u_2, f \star g \rangle = \langle u_1, u_2 \star (f \star g) \rangle.
\]

Using (19) and again (16), we obtain

\[
\langle u_1, u_2 \star (f \star g) \rangle = \langle u_1, (u_2 \star f) \star g \rangle = \langle u_1 \star (u_2 \star f), g \rangle, \quad \text{for all } g \in E.
\]

Therefore, \( (u_1 \star u_2) \star f = u_1 \star (u_2 \star f) \in E \) for all \( f \in E \) and hence \( u_1 \star u_2 \in \mathcal{M}_{\theta,L}(E) \). In a similar way, \( v_1, v_2 \in \mathcal{M}_{\theta,R}(E) \Rightarrow v_1 \star v_2 \in \mathcal{M}_{\theta,R}(E) \). Furthermore,

\[
\langle (u_1 \star u_2) \star u_3, f \rangle = \langle u_1 \star u_2, u_3 \star f \rangle = \langle u_1, u_2 \star (u_3 \star f) \rangle = \langle u_1, (u_2 \star u_3) \star f \rangle = \langle u_1 \star (u_2 \star u_3), f \rangle,
\]

which proves the associativity of the algebra \( \mathcal{M}_{\theta,L}(E) \). We write \( \mathbb{1} \) for the functional \( f \rightarrow \int f(x)dx \). From (17),

\[
\langle \mathbb{1} \star f, g \rangle = \langle f \star \mathbb{1}, g \rangle = \int f(x)g(x)dx, \quad \text{for all } f, g \in E.
\]

Hence \( \mathbb{1} \) belongs to both \( \mathcal{M}_{\theta,L}(E) \) and \( \mathcal{M}_{\theta,R}(E) \) and is the identity of these algebras.

If \( E \) is invariant under the complex conjugation \( f \rightarrow f^\star \) and hence is an involutive algebra, then \( E' \) also has an involution \( u \rightarrow u^\star \), where \( u^\star \) is defined by

\[
\langle u^\star, f \rangle = \overline{\langle u, f^\star \rangle}.
\]

(21)
The involution (21) is an antilinear isomorphism of \( \mathcal{M}_{\theta,L}(E) \) onto \( \mathcal{M}_{\theta,R}(E) \). Indeed, let \( u \) be a left \(*\)-multiplier of \( E \) and let \( f, g \in E \). Then

\[
\langle f \ast u^*, g \rangle = \langle u^*, g \ast f \rangle = \langle u, (g \ast f)^* \rangle = \langle u, f^* \ast g^* \rangle = \langle u \ast f^*, g^* \rangle = (\langle u \ast f^* \rangle)^*, g),
\]

and we see that the functional \( f \ast u^* \) is generated by the test function \( (u \ast f^*)^* \in E \), hence \( u^* \in \mathcal{M}_{\theta,R}(E) \).

We let \( \hat{E} \) denote the Fourier transform of \( E \) and equip it with the topology induced by the map \( E \to \hat{E} \). It follows from definition (2), that

\[
(\hat{f} \ast \hat{g})(q) = (2\pi)^{-d} \int \hat{f}(p)\hat{g}(q-p) e^{i\theta qp} dp
\]

for all \( f, g \in S \). The integral expression on the right-hand side of (22) is called the twisted convolution product of \( \hat{f} \) and \( \hat{g} \). We denote \( \hat{E} \) this product by \( \hat{f} \ast_\theta \hat{g} \) and, as before, omit the explicit reference to \( \theta \) whenever this cannot cause confusion. Then (22) takes the form

\[
(\hat{f} \ast \hat{g}) = (2\pi)^{-d} \hat{f} \ast_\theta \hat{g},
\]

which is analogous to the familiar relation between the pointwise multiplication and the ordinary convolution and turns into it at \( \theta = 0 \). If \( E \) is a topological algebra under the Moyal multiplication, then \( \hat{E} \) is a topological algebra under the twisted convolution. From the foregoing it is clear that the twisted convolution has a unique extension by continuity to the case where one of factors is in \( \hat{E}' \). For \( v \in \hat{E}' \) and \( g \in \hat{E} \), this extension is defined by

\[
v \ast g = (2\pi)^d \mathcal{F}(\mathcal{F}^{-1}v \ast \mathcal{F}^{-1}g), \quad g \ast v = (2\pi)^d \mathcal{F}(\mathcal{F}^{-1}g \ast \mathcal{F}^{-1}v).
\]

The Fourier transform is an isomorphism of the Moyal multiplier algebras \( \mathcal{M}_{\theta,L}(E) \) and \( \mathcal{M}_{\theta,R}(E) \) onto the twisted convolution multiplier algebras

\[
\mathcal{C}_{\theta,L}(\hat{E}) \overset{\text{def}}{=} \{ v \in \hat{E}': \ v \ast_\theta g \in \hat{E}, \ \text{for all } g \in \hat{E} \}
\]

and

\[
\mathcal{C}_{\theta,R}(\hat{E}) \overset{\text{def}}{=} \{ v \in \hat{E}': \ g \ast_\theta v \in \hat{E}, \ \text{for all } g \in \hat{E} \}.
\]

Clearly, the Dirac \( \delta \)-function is the identity of the algebras \( \mathcal{C}_{\theta,L}(\hat{E}) \) and \( \mathcal{C}_{\theta,R}(\hat{E}) \).

**IV. Moyal multipliers of the spaces of type \( S \)**

The spaces \( S_\alpha^\beta \) are barrelled and fully complete. Therefore, Ptak’s version (see [34], Sec. IV.8.5) of the closed graph theorem is applicable to their linear endomorphisms. Each of them is dense in \( S \) and, by Theorem 1 of [1], the spaces \( S_\alpha^\beta \) with \( \alpha \geq \beta \) are topological algebras under the Moyal multiplication. Therefore, it follows from the above general consideration that the algebras \( \mathcal{M}_{\theta,L}(S_\alpha^\beta), \mathcal{M}_{\theta,R}(S_\alpha^\beta), \mathcal{C}_{\theta,L}(S_\alpha^\beta), \) and \( \mathcal{C}_{\theta,R}(S_\alpha^\beta) \) are well defined for \( \alpha \geq \beta \). To describe their properties we need the following lemma.

\footnote{This notation follows Kammerer [11], whereas in [12] the twisted convolution operation was denoted by \( *_\theta \) and in [13] [14] by \( \diamond \).}
Lemma 1: Let $\alpha \geq \beta$ and let $e$ be a function in $S_0^\beta(\mathbb{R}^d)$ such that $e(0) = 1$. Let $e_\nu(x) = e(x/\nu), \nu = 1, 2, \ldots$. Then either of the operator sequences $f \to e_\nu * f$ and $f \to f * e_\nu$ converges to the identity map of $S_0^\beta$ uniformly on the bounded subsets of $S_0^\beta$.

Proof. From (24), we have

\[ (f * e_\nu)(x) = \int \omega_\nu(q)f(x - \frac{1}{2}q)e^{iqx}dq, \]  

where $\omega_\nu(q) \overset{\text{def}}{=} (\nu/2\pi)^d \hat{\nu}(\nu q)$. The function $\hat{\nu}$ belongs to $S_0^\alpha$, hence $\omega_\nu$ satisfies the inequality

\[ |\omega_\nu(q)| \leq C_0|\nu|e^{-|\nu q|/B_0}1/\beta \]

with some positive constants $C_0, B_0$. Since $\int \omega_\nu(q)dq = e(0) = 1$, it follows from (25) that

\[ \partial^n(f * e_\nu - f)(x) = \int \omega_\nu(q)\left( e^{iqx}\partial^n f(x - \frac{1}{2}q) - \partial^n f(x) \right)dq + \int \omega_\nu(q)\sum_{m \neq 0} \left( \frac{n}{m} \right)^{m} e^{iqx}\partial^n-m f(x - \frac{1}{2}q)dq. \]  

(27)

We define $F_x(q) = e^{iqx}\partial^n f(x - \frac{1}{2}q)$. By the mean value theorem,

\[ |F_x(q) - F_x(0)| \leq |q| \sup_{|p| \leq |q|} \sum_{j=1}^d \left| \frac{\partial F_x(p)}{\partial p_j} \right| \leq \frac{1}{2} \sum_{i,j} |\theta^{ij} \partial_{xi}\partial^n f(x - \frac{1}{2}q)| \right\}. \]  

(28)

Using definition (5) and inequality (12), we obtain

\[ |x| |\partial^n f(x - \frac{1}{2}q)| \leq \left\| f \right\|_A, B^{|n|}n^{\beta n}|x|e^{-|x| - \theta|q|/A1/\alpha} \leq C_A' \left\| f \right\|_A, B^{|n|}n^{\beta n}e^{-|x|/A1/\alpha + |(\theta|q|/2A)|1/\alpha}, \]

where $|\theta| = \sum |\theta^{ij}|$ and $A'$ is an arbitrary constant greater than $2A$. The sum in the braces on the right-hand side of (28) is estimated in a similar manner, using the inequality $(n+1)^{\beta(n+1)} \leq C_e(1 + e)^n n^{\beta n}$, and this yields

\[ |F_x(q) - F_x(0)| \leq C \left\| f \right\|_A, B^{|n|}n^{\beta n}e^{-|x|/A'1/\alpha + |(\theta|q|/2A)|1/\alpha}, \]  

(29)

where $B' > B$ and can be taken arbitrarily close to $B$. We let $I^{(1)}_\nu(x)$ and $I^{(2)}_\nu(x)$ denote the integrals on the right-hand side of (27). Since $\alpha \geq \beta$, from (26) and (29) it follows that for $\nu > B_0|\theta|/A$, the first integral satisfies the estimate

\[ |I^{(1)}_\nu(x)| \leq \frac{1}{\nu} C' \left\| f \right\|_A, B^{|n|}n^{\beta n}e^{-|x|/A'1/\alpha}, \]  

(30)

where $C' = C C_0 \int |q|e^{-|q|/B_01/\beta + |q|/2B_01/\beta} dq$. To estimate the second integral we take into account that if $m \neq 0$, then

\[ |q^m| \leq |q| B_1^{|m|-1}m^{\beta m} \prod_{j=1}^d e^{(\beta/e)|q|/B_11/\beta} \text{ for each } B_1 > 0. \]  

(31)
This is obtained by writing \( q^m = |q|q^{m'} \), where \( |m'| = |m| - 1 \), and evaluating \( \sup_m |q^m|/m^{\beta m} \).

Using (31) together with (12) and (26), we find that

\[
|I^{(2)}_\nu(x)| \leq \frac{C_0}{B_1} \|f\|_{A,B} e^{-|x/2A|^{1/\alpha}} \sum_m \left( \frac{n}{m} \right) B_1^m B_1^{n-m} m^{\beta m}(n-m)^{\beta(n-m)} \times \int |q|e^{-\nu q/B_0|1/\beta + (d/\nu)|q/B_1|1/\beta + (|\theta|q/2A)^{1/\beta}} \nu^d dq
\]

Hence, if \( \nu \) is so large that \( \nu^{1/\beta} > (d/\nu)(2B_0/B_1)^{1/\beta} + (|\theta|B_0/A)^{1/\beta} \), we have

\[
|I^{(2)}_\nu(x)| \leq \frac{1}{\nu} C' \|f\|_{A,B} (B + B_1)^{|n|} n^{\beta n} e^{-|x/2A|^{1/\alpha}}.
\]  

(32)

Now we recall that the inductive limit \( \text{inj lim}_{A,B \to \infty} S_{\alpha,A}^{\beta,B} \) is regular. In other words for any bounded subset \( Q \) of \( S_{\alpha}^{\beta} \), there are \( A \) and \( B \) such that \( Q \) is contained in \( S_{\alpha,A}^{\beta,B} \) and is bounded in its norm. Let \( A' > 2A \) and \( B' > B \) as before. Using (31) and (32) with \( B_1 = B' - B \), we conclude that if \( f \) ranges over \( Q \) and \( \nu \) is large enough, then the functions \( f \ast e_\nu - f \) belong to \( S_{\alpha,A'}^{\beta,B'} \) and \( \sup_{f \in Q} \|f \ast e_\nu - f\|_{A',B'} \to 0 \) as \( \nu \to 0 \). Thus, the sequence of the operators of right \( \ast \)-multiplication by \( e_\nu \) converges to the unit operator uniformly on every bounded subset of \( S_{\alpha}^{\beta} \). The argument for the case of left \( \ast \)-multiplication is analogous.

Theorem 3: The algebra \( \mathcal{M}_{\theta,L}(S_{\alpha}^{\beta}) \), where \( \alpha \geq \beta \), can be canonically identified with the closure in \( \mathcal{L}(S_{\alpha}^{\beta}) \) of the set of operators of left \( \ast \)-multiplication by elements of \( S_{\alpha}^{\beta} \). An analogous statement is true for \( \mathcal{M}_{\theta,R}(S_{\alpha}^{\beta}) \).

Proof. Let \( u \in \mathcal{M}_{\theta,L}(S_{\alpha}^{\beta}) \) and let \( U \in \mathcal{L}(S_{\alpha}^{\beta}) \) be the operator that takes each function \( f \in S_{\alpha}^{\beta} \) to \( u \ast f \), so that

\[
\langle u \ast f, g \rangle = \int U(f)g \, dx, \quad \text{for all } g \in S_{\alpha}^{\beta}.
\]

(33)

If \( u_1 \) and \( u_2 \) determine the same operator, then \( \langle u_1, f \ast g \rangle = \langle u_2, f \ast g \rangle \) for all \( f, g \in S_{\alpha}^{\beta} \). Lemma 1 shows in particular that the set \( \{f \ast g; f, g \in S_{\alpha}^{\beta}\} \) is dense in \( S_{\alpha}^{\beta} \). Therefore the map

\[
\mathcal{M}_{\theta,L}(S_{\alpha}^{\beta}) \to \mathcal{L}(S_{\alpha}^{\beta}): u \to U
\]

(34)

is one-to-one. Every operator belonging to its image has the property

\[
U(h \ast f) = U(h) \ast f,
\]

(35)

where \( f \) and \( h \) are arbitrary elements of \( S_{\alpha}^{\beta} \). Indeed, from (18) and (19) it follows that

\[
\int U(h \ast f)g \, dx = \langle u \ast (h \ast f), g \rangle = \langle u \ast h, f \ast g \rangle = \int U(h)(f \ast g) \, dx = \int (U(h) \ast f)g \, dx,
\]

for all \( g \in S_{\alpha}^{\beta} \). Let \( e_\nu \) be the sequence defined in Lemma 1. From (33), we see that the operator \( U_\nu: f \to U(e_\nu \ast f) \) consists in the left \( \ast \)-multiplication by \( U(e_\nu) \) and by Lemma 1, the operator sequence \( U_\nu \) converges to \( U \) in \( \mathcal{L}(S_{\alpha}^{\beta}) \) as \( \nu \to \infty \).

On the other hand, to each \( T \in \mathcal{L}(S_{\alpha}^{\beta}) \) we can assign a functional \( t \in (S_{\alpha}^{\beta})' \) by setting

\[
\langle t, f \rangle = \int T(f) \, dx, \quad f \in S_{\alpha}^{\beta}.
\]

(36)
The map \( T \to t \) from \( \mathcal{L}(S^3_\alpha) \) into \((S^3_\alpha)'\) is continuous by the definition of topologies of these spaces. Using (35) again, we see that the composition of (34) and this map is the identity map of \( \mathcal{M}_{\theta,L}(S^3_\alpha) \). Let \( T = \lim_\nu T_\nu \), where \( T_\nu \) is the operator of left \(*\)-multiplication by \( g_\nu \in S^3_\alpha \). Then we have

\[
T(h \ast f) = \lim_\nu g_\nu \ast (h \ast f) = \lim_\nu (g_\nu \ast h) \ast f = T(h) \ast f, \quad \text{for all } h \in S^3_\alpha.
\]

This implies that the functional \( t \) corresponding to \( T \) by (36) satisfies \( t \ast h = T(h) \) and hence belongs to \( \mathcal{M}_{\theta,L}(S^3_\alpha) \). The algebra \( \mathcal{M}_{\theta,R}(S^3_\alpha) \) can be considered in a similar way, and this completes the proof.

Theorem 3 shows that the natural topology on the left and right multiplier algebras of \( S^3_\alpha \) is the topology induced by that of \( \mathcal{L}(S^3_\alpha) \). Now we define analogs of the involutive algebra of \(*\)-multipliers introduced by Antonets [9, 10] for \( S \). Namely, we consider the intersection

\[
\mathcal{M}_\theta(S^3_\alpha) = \mathcal{M}_{\theta,L}(S^3_\alpha) \cap \mathcal{M}_{\theta,R}(S^3_\alpha).
\]

The Moyal product of two its elements \( u \) and \( v \) can be defined by

\[
\langle u \ast v, f \rangle = \langle u, v \ast f \rangle,
\]

or, with the same right, by

\[
\langle u \ast v, f \rangle = \langle v, f \ast u \rangle.
\]

To prove that these definitions are equivalent, we again use the sequence \( e_\nu \) of Lemma 1. By (16) and (19), we have

\[
\langle e_\nu \ast u, v \ast f \rangle = \langle u, (v \ast f) \ast e_\nu \rangle \to \langle u, v \ast f \rangle \quad (\nu \to \infty),
\]

\[
\langle v, f \ast (e_\nu \ast u) \rangle = \langle v, (f \ast e_\nu) \ast u \rangle \to \langle v, f \ast u \rangle \quad (\nu \to \infty).
\]

It remains to note that \( \langle g, v \ast f \rangle = \langle v, f \ast g \rangle \) for any \( g \in S^3_\alpha \) and, in particular, for \( g = e_\nu \ast u \). This is obtained by passing to the limit at \( \nu \to \infty \) in the equality \( \langle g, (v \ast e_\nu) \ast f \rangle = \langle v \ast e_\nu, f \ast g \rangle \) which holds by (18). The space \( \mathcal{M}_\theta(S^3_\alpha) \) can be made into a locally convex space by giving it the least upper bound of the topologies induced by those of \( \mathcal{M}_{\theta,L}(S^3_\alpha) \) and \( \mathcal{M}_{\theta,R}(S^3_\alpha) \).

Before formulating the next theorem we recall that in the theory of bilinear maps of locally convex spaces, a large role is played by the notion of \((\mathfrak{B}_1, \mathfrak{B}_2)\)-hypocontinuity [34] which takes an intermediate position between continuity and separate continuity. In the case where \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) are families of all bounded subsets of the spaces \( E_1 \) and \( E_2 \) on whose direct product a bilinear map is defined, this property is often termed hypocontinuity for short.

**Theorem 4:** Under the condition \( \alpha \geq \beta \), \( \mathcal{M}_\theta(S^3_\alpha) \) is a complete nuclear semireflexive unital \(*\)-algebra with hypocontinuous multiplication and continuous involution. The space \((S^3_\alpha)'\) is an \( \mathcal{M}_\theta(S^3_\alpha) \)-bimodule with hypocontinuous operations \( (v, w) \to w \ast v \) and \( (v, w) \to v \ast w \), where \( w \in (S^3_\alpha)' \), and \( v \in \mathcal{M}_\theta(S^3_\alpha) \).

**Proof.** We noticed already that \( \mathcal{L}(S^3_\alpha) \) has the topological properties listed above. Its closed subspaces \( \mathcal{M}_{\theta,L}(S^3_\alpha) \) and \( \mathcal{M}_{\theta,R}(S^3_\alpha) \) as well as their intersection also are complete, nuclear, and semireflexive by the well known hereditary properties [34].

From the definition of the topology on \( \mathcal{M}_\theta(S^3_\alpha) \), it immediately follows that the multiplication in this algebra is separately continuous. Indeed, a base of neighborhoods of 0 in \( \mathcal{M}_\theta(S^3_\alpha) \)
is formed by the sets of the form \( \mathcal{V}_{Q, \mathcal{U}} = \{ v : (v \star Q) \cup (Q \star v) \subset \mathcal{U} \} \), with \( Q \) a bounded subset of \( S^\beta_\alpha \) and \( \mathcal{U} \) a neighborhood of 0 in \( S^\beta_\alpha \). For any fixed \( u \in \mathcal{M}_\theta(S^\beta_\alpha) \) and for each neighborhood \( \mathcal{U} \), there is a neighborhood \( \mathcal{U}_1 \) such that \( u \star \mathcal{U}_1 \subset \mathcal{U} \). Taking \( \mathcal{U}' = \mathcal{U} \cap \mathcal{U}_1 \), we have the implication

\[
v \in \mathcal{V}_{Q', \mathcal{U}'} \implies u \star v \in \mathcal{V}_{Q, \mathcal{U}},
\]

which shows that the map \( v \to u \star v \) of \( \mathcal{M}_\theta(S^\beta_\alpha) \) into itself is continuous. Similarly, the map \( u \to u \star v \) is continuous for every fixed \( v \). Now let \( u \) range over a bounded subset \( \mathcal{Q} \) of \( \mathcal{M}_\theta(S^\beta_\alpha) \). Then the set \( \mathcal{Q} \times \mathcal{Q} \) is bounded in \( S^\beta_\alpha \). Since the space \( S^\beta_\alpha \) is barrelled, we may apply the general principle of uniform convergence \([34]\) and conclude that there is a neighborhood \( \mathcal{U}_2 \) such that \( \mathcal{Q} \star \mathcal{U}_2 \subset \mathcal{U} \). Taking this time \( \mathcal{Q}'' = \mathcal{Q} \cup (\mathcal{Q} \star \mathcal{Q}) \) and \( \mathcal{U}' = \mathcal{U} \cap \mathcal{U}_2 \), we again have implication \([40]\), hence the bilinear map \( (u, v) \to u \star v \) is \( \mathcal{B}_1 \)-hypocontinuous. Analogously, it is \( \mathcal{B}_2 \)-hypocontinuous.

A base of neighborhoods for \( S^\beta_\alpha \) can obviously be formed of sets invariant under the involution \( f \to f^* \), and every bounded subset of \( S^\beta_\alpha \) is contained in an invariant bounded subset. Therefore, the family of sets of the form \( \{ u : f \star u \in \mathcal{U}, u \in \mathcal{U}, \forall f \in \mathcal{Q} \} \), with \( \mathcal{U} \) an invariant neighborhood and \( \mathcal{Q} \) an invariant bounded set in \( S^\beta_\alpha \), forms a base of neighborhoods for \( \mathcal{M}_\theta(S^\beta_\alpha) \) which is invariant under the map \( u \to u^* \).

To prove that \((S^\beta_\alpha)'\) is a bimodule over the algebra \( \mathcal{M}_\theta(S^\beta_\alpha) \), it suffices to show that

\[
(u \star v) \star w = u \star (v \star w), \quad (u \star w) \star v = u \star (w \star v), \quad (w \star u) \star v = w \star (u \star v),
\]

for all \( w \in (S^\beta_\alpha)' \) and for any \( u, v \in \mathcal{M}_\theta(S^\beta_\alpha) \). These associativity relations follow immediately from analogous relations for the action of the algebra \( \mathcal{M}_\theta(S^\beta_\alpha) \) on test functions. In particular, for any \( f \in S^\beta_\alpha \), we have the chain of equalities

\[
\langle (u \star v) \star w, f \rangle = \langle w, f \star (u \star v) \rangle = \langle w, (f \star u) \star v \rangle = \langle v \star w, f \star u \rangle = \langle u \star (v \star w), f \rangle,
\]

which proves the first of relations \([41]\). As was already noted in Sec. III, the maps \( w \to w \star v \) and \( w \to v \star w \), being transposes of continuous maps, are continuous. Now we fix \( w \in (S^\beta_\alpha)' \) and show that the maps \( v \to v \star w \) and \( v \to v \star w \) of \( \mathcal{M}_\theta(S^\beta_\alpha) \) into \((S^\beta_\alpha)'\) are also continuous. By the definition of the strong topology, every neighborhood of 0 in \((S^\beta_\alpha)'\) contains a set of the form \( Q^\circ = \{ t : \sup_{f \in \mathcal{Q}} |\langle t, f \rangle| \leq 1 \} \), where \( Q \) is a bounded subset of \( S^\beta_\alpha \). Because the functional \( w \) is continuous, there is a neighborhood \( \mathcal{U} \) in \( S^\beta_\alpha \) such that \( \sup_{f \in \mathcal{U}} |\langle w, f \rangle| \leq 1 \). Clearly, \( v \in \mathcal{V}_{Q, \mathcal{U}} \) implies \( w \star v \in Q^\circ \) and \( v \star w \in Q^\circ \). Hence the bilinear maps \( (v, w) \to w \star v \) and \( (v, w) \to v \star w \) are separately continuous. Moreover, since the space \((S^\beta_\alpha)'\) is barrelled, they are \( \mathcal{B}_1 \)-hypocontinuous. Now let \( u \) range over a bounded set \( B \subset (S^\beta_\alpha)' \). Because \( S^\beta_\alpha \) is barrelled, this set of functionals is equicontinuous, i.e., there is a neighborhood \( \mathcal{U} \) of 0 in \( S^\beta_\alpha \) such that \( \sup_{f \in \mathcal{U}} |\langle w, f \rangle| \leq 1 \) for all \( w \in B \). Therefore, the bilinear maps under consideration are \( \mathcal{B}_2 \)-hypocontinuous, and this completes the proof of Theorem 4.

In concluding this section, we note that the Fourier transform is an isomorphism of the Moyal multiplier algebra \( \mathcal{M}_\theta(S^\beta_\alpha) \), where \( \alpha \geq \beta \), onto the algebra

\[
\mathcal{C}_\theta(S^\beta_\alpha) = \mathcal{C}_{\theta,L}(S^\beta_\alpha) \cap \mathcal{C}_{\theta,R}(S^\beta_\alpha)
\]

and \((S^\beta_\alpha)'\) is a unital \( \mathcal{C}_\theta(S^\beta_\alpha) \)-bimodule with hypocontinuous operations.
V. Smoothness and growth properties of the twisted convolution product

We now focus our attention on the smoothness and growth properties of the twisted convolution products $g \ast v$ and $g \ast g$, where $g \in S^0_\beta$, $v \in (S^0_\beta)'$, and $\alpha \geq \beta$. We will show that these properties are not too much different from those of the undeformed convolution product $g \ast v$. For this purpose, we use the following lemma.

**Lemma 2:** If $f \in S^{\alpha \cdot B}_{\alpha,A}(\mathbb{R}^d)$, then its Fourier transform $\hat{f}$ belongs to $S^{\alpha \cdot r^A \cdot B}_{\beta \cdot r B}(\mathbb{R}^d)$, where the number $r$ depends only on $\alpha$, $\beta$, and $d$. The map $S^{\alpha \cdot B}_{\alpha,A} \rightarrow S^{\alpha \cdot r^A \cdot B}_{\beta \cdot r B}$: $f \rightarrow \hat{f}$ is bounded, i.e., there is a constant $C$ such that

$$
\| \hat{f} \|_{r^B,r^A} \leq C \| f \|_{A,B}, \quad \text{for all } f \in S^{\beta \cdot B}_{\alpha,A}.
$$

(43)

The proof of Lemma 2 is presented in Appendix. It is based on an inequality proposed in [35], which allows us to simplify considerably the theory [2] of Fourier transform on the spaces of type $S$.

**Theorem 5:** Suppose that $\alpha \geq \beta$, $g \in S^0_\beta$ and $v \in (S^0_\beta)'$. Then the twisted convolution products $v \ast g$ and $g \ast v$ belong to the space $M(S^0_\beta)$ and can be written as

$$
(v \ast g)(q) = \left< v, g(k - \cdot) e^{i q \theta(\cdot)} \right>, \quad (g \ast v)(q) = \left< v, g(q - \cdot) e^{i q \theta(\cdot)} \right>.
$$

(44)

The maps $(g,v) \rightarrow v \ast g$ and $(g,v) \rightarrow g \ast v$ from $S^0_\beta \times (S^0_\beta)'$ into $M(S^0_\beta)$ are hypocontinuous.

**Proof.** If $v$ is a regular functional generated by a function in $S$, then (44) is obviously consistent with the above definition of the twisted convolution product for elements of the Schwartz space. The functions on the right-hand sides of (44) are well defined because $S^0_\beta$ is invariant under the reflection and translations of $R^d$, and under the multiplication by $e^{\pm \frac{i}{2} q \theta(\cdot)}$ which is equivalent to a translation of the Fourier transforms in $S^0_\beta$. It suffices to show that these functions belong to $M(S^0_\beta)$ and depend continuously on $v$. Then we may state that the products defined by (44) extend continuously the twisted convolution multiplication to the case where one of factors belongs to $(S^0_\beta)'$, because $M(S^0_\beta)$ is contained in $(S^0_\beta)'$ and its topology is stronger than that induced from $(S^0_\beta)'$. Such an extension is unique because $S^0_\beta$ is dense in $(S^0_\beta)'$, hence functions (44) coincide with the products defined by (24).

Now we introduce the notation

$$
g_q^\pm(p) = g(q - p) e^{\pm \frac{i}{2} q \theta}.\quad (45)
$$

The functions $\langle v, g_q^\pm \rangle$ are infinitely differentiable and

$$
\partial_\theta^n (v, g_q^\pm) = (v, \partial_\theta^n g_q^\pm)
$$

(46)

for any $d$-tuple $n \in \mathbb{Z}_+^d$. To prove formula (46), we first observe that the operators $\partial_\theta^n$ are defined and continuous in $S^0_\beta$ because by definition (5) and the inequality $(n + m)^{(n+m)} \leq 2^{(n+m)} n^m m^n$ we have

$$
\| \partial_\theta^n g \|_{B,2^\alpha A} \leq (2^\alpha A)^{|n|} n^m \| g \|_{B,A}.
$$

(47)

for all $g \in S^{\alpha \cdot A}_{\beta,B}$. The translation operators $T_q: g(p) \rightarrow g(p - q)$ are also defined and continuous in $S^0_\beta$. Using the inequality $-|p - q|^{1/\beta} \leq -|p/2|^{1/\beta} + |q|^{1/\beta}$, we obtain

$$
\| T_q g \|_{2B,A} \leq e^{-|q|/B^{1/\beta}} \| g \|_{B,A}
$$

(48)
and conclude that the set of functions $T_g g$, where $|q| \leq c < \infty$, is bounded in $S^\alpha_\overline{\beta}$ for any fixed $g$. Since $S^\alpha_\overline{\beta}$ is a Montel space, it follows that $T_g$ is continuous in $q$. As shown in Sec. III.3.3 of [2], these properties of differentiation and translation in $S^\alpha_\overline{\beta}$ imply that the difference quotient $(T_{-q}f - f)/q_j$, where $T_{q_j}$ is the operator of translation in the $j$th coordinate, converges to $\partial_j f$ in $S^\alpha_\overline{\beta}$. The operator of multiplication by $e^{\pm \frac{i}{2} \theta q}$ is also strongly differentiable in $q$, because the (inverse) Fourier transform converts it into the translation by $\pm \frac{1}{2} \theta q$ in $S^\alpha_\overline{\beta}$. As a result, we arrive at (50).

From (46), it follows that

$$|\partial^n (v \ast g)(q)| \leq \|v\|_{B,A} \|\partial^n g_q^+\|_{B,A} \quad \text{and} \quad |\partial^n (g \ast v)(q)| \leq \|v\|_{B,A} \|\partial^n g_q^-\|_{B,A},$$

(49)

where by the Leibniz rule,

$$\|\partial^n g_q^\pm\|_{B,A} \leq \sum_m \left( \frac{n}{m} \right) \frac{1}{m!} \left( (\theta p)^m e^{\pm \frac{i}{2} \theta q} \partial^n g_p \right),$$

(50)

Let $g \in S^\alpha_\overline{\beta}$ and let $A_1 = 2^\alpha A_0$, $B_1 = 2B_0$. Using (47) and (48), we obtain

$$\|\partial^n-m g(q-p)\|_{B_1,A_1} \leq \|g\|_{B_0,A_0(2^\alpha A_0)^{n-m}}(n-m)^\alpha \|e^{\frac{1}{2} \theta q/B_0}\|^{1/\beta}.$$  

(51)

For any $h \in S^\alpha_\overline{\beta}$, we have the inequality

$$\|\|\theta p\|^m e^{\pm \frac{i}{2} \theta q} h(p)\|_{B,A} \leq \|\theta\|^m \max_{|k| \leq |m|} \|p^k e^{\pm \frac{i}{2} \theta q} h(p)\|_{B,A},$$

(52)

where the norms are finite if $A$ and $B$ are large enough. Furthermore,

$$\mathcal{F}^{-1} \left( \sum_{k} p^k e^{\pm \frac{i}{2} \theta q} h(p) \right) = \frac{1}{(2\pi)^d} \int p^k e^{i p \cdot (x + \frac{1}{2} \theta q)} h(p) dp = \left( -i \right)^{|k|} \hat{f}(x + \frac{1}{2} \theta q),$$

where $\hat{f} = h$. If $h \in S^\alpha_\overline{\beta}$, then $\|f\|_{rA_1,rB_1} \leq C\|h\|_{B_1,A_1}$ by Lemma 2. (More exactly, by an analog of this lemma for $\mathcal{F}^{-1}$.) Next we use analogs of inequalities (47) and (48) for functions in $S^\beta_\alpha$. This gives the estimate

$$\|\partial^k f(x + \frac{1}{2} \theta q)\|_{rA_1,2^\beta rB_1} \leq \|f(x + \frac{1}{2} \theta q)\|_{2rA_1,2^\beta rB_1}(2^\beta rB_1)^{k} \|k\hat{f}\|_{S^\alpha_\overline{\beta}} \leq C\|h\|_{B_1,A_1}(2^\beta rB_1)^{k} \|k\hat{f}\|_{S^\alpha_\overline{\beta}} e^{\frac{1}{2} \theta q/(2rA_1)^{1/\alpha}}.$$  

(53)

By Lemma 2, we have

$$\|p^k e^{\pm \frac{i}{2} \theta q} h(p)\|_{B,A} \leq C'\|\partial^k f(x + \frac{1}{2} \theta q)\|_{2rA_1,2^\beta rB_1}, \quad \text{where} \quad A = 2rr' A_1, B = 2^\beta rr' B_1.$$  

(54)

Combining (52) and (53) with (51) and taking into account that $\max_{|k| \leq |m|} k^k = |m|^{|m|} \leq d^{|m|} m^m$, we get

$$\|\|\theta p\|^m e^{\pm \frac{i}{2} \theta q} h(p)\|_{B,A} \leq C''\|h\|_{B_1,A_1}(2^\beta d^\beta |\theta| rB_1)^{m} \|m^m e^{\frac{1}{2} \theta q/(2rA_1)^{1/\alpha}}.$$  

Substituting here $h(p) = \partial^{n-m} g(q-p)$ and using (50) and (51), we conclude that

$$\|\partial^n g_q^\pm\|_{B,A} \leq C'''\|g\|_{B_0,A_0} \sum_m \left( \frac{n}{m} \right) (2^\beta d^\beta |\theta| rB_0)^{m} (2^\alpha A_0)^{n-m} \|m^m (n-m)^\alpha (n-m) \times e^{\frac{1}{2} \theta q/B_0)^{1/\alpha} + \frac{1}{2} |\theta q/(2rA_0)^{1/\alpha}}.$$  

15
where $A = 2^{\alpha+1}rr'A_0$ and $B = 2^{\beta+1}rr'B_0$. If $\alpha \geq \beta$, then $m^{\beta m}(n-m)^{(n-m)} \leq n^{\alpha n}$, and we arrive at the inequality

$$\|\partial^\alpha_q g^\pm\|_{B,A} \leq C''\|g\|_{B_0,A_0} \left(2^\beta d^3|\theta| r B_0 + 2^\alpha A_0\right)^{\alpha n} e^{|q/B_0|^{1/\beta} + \frac{1}{2}|\theta q|(2r A_0)^{1/\beta}}. \quad (55)$$

A similar inequality obviously holds for any $A_0 > A_0$, $B_0' > B_0$ and their corresponding $A' = 2^{\alpha+1}rr'A_0'$, $B' = 2^{\beta+1}rr'B_0'$, because $S_{\beta,B_0}^\alpha \subset S_{\beta,B_0'}^{\alpha A_0}$. Let $A_0' = A_0 + |\theta| r/2 r$ and $B_0' = B_0 + 2^\beta R$, where $R$ is a positive number. Then

$$|q/B_0'|^{1/\beta} + \frac{1}{2}|\theta q|(2r A_0')^{1/\beta} \leq |q/R|^{1/\beta},$$

and we obtain

$$\|\partial^\alpha_q g^\pm\|_{B',A'} \leq C''\|g\|_{B_0,A_0} A_R^{\alpha n} e^{|q/R|^{1/\beta}}. \quad (56)$$

where $A_R = 2^\alpha A_0 + s|\theta|(B_0 + R)$ with a positive coefficient $s$ depending only on $\alpha$, $\beta$ and $d$, whose explicit form is irrelevant. From (49) and (56), it follows that

$$\|v \ast g\|_{R,A_R} \sup_{q,n} \left|\frac{\partial^\alpha v \ast g}{A_R^{\alpha n} e^{-|q/R|^{1/\beta}}}\right| \leq C''\|v\|_{B',A'} \|g\|_{B_0,A_0}. \quad (57)$$

An analogous inequality holds for $g \ast v$. We conclude that the functions $v \ast g$ and $g \ast v$ belong to any space $E_{\alpha,A}^\beta, R$ with $R > 0$, and $a fortiori$ to $M(S^\beta_\alpha)$. From the form of the right-hand side of (57), it is clear that the maps $(g,v) \mapsto g \ast v$ and $(g,v) \mapsto v \ast g$ from $S^\alpha_\beta \times (S^\beta_\beta)'$ into $M(S^\beta_\alpha)$ are separately continuous. Since the spaces $S^\alpha_\beta$ and $(S^\beta_\alpha)'$ are barrelled, this amounts to saying that these maps are hypocontinuous, and thus we have proved Theorem 5.

**Corollary:** If $\alpha \geq \beta$, $f \in S^\alpha_\beta$ and $u \in (S^\beta_\alpha)'$, then the functionals $u \ast f$ and $f \ast u$ belong to $C(S^\beta_\alpha)$. The maps $(f,u) \mapsto u \ast f$ and $(f,u) \mapsto f \ast u$ from $S^\alpha_\beta \times (S^\beta_\alpha)'$ into $C(S^\beta_\alpha)$ are hypocontinuous.

We notice that for $\theta = 0$, the constant $A_R$ equals $2^\alpha A_0$ and becomes independent of $R$. In this case, formula (57) shows that for all $g \in S^\alpha_\beta$ and $v \in (S^\beta_\alpha)'$, the usual convolution $v \ast g$ belongs to the space $C'(S^\beta_\alpha)$ defined by (10), which is smaller than $M(S^\beta_\alpha)$. The deformed convolution need not belong to $C'(S^\beta_\alpha)$ but is contained in $M(S^\beta_\alpha)$ for any $\theta$. This is analogous to the result obtained previously [10] in the framework of tempered distributions, where the roles of $C'(S^\beta_\alpha)$ and $M(S^\beta_\alpha)$ are played respectively by the spaces $O_C = C'(S)$ and $O_M = M(S)$. The analogy is even more complete. The linear dependence of $A_R$ on $R$, established by Theorem 5 for $\theta \neq 0$, is an analog of the fact that for $g \in S$ and $v \in S'$, the twisted convolutions $v \ast g$ and $g \ast v$ belong to the space denoted by $O_T$ by $O_T$. This space is smaller than the Schwartz multiplier space $O_M$ and consists of all smooth functions with polynomially bounded derivatives for which the degree of the polynomial bound increases linearly with the order of the derivative.

VI. How large are the extended Moyal algebras?

The next theorem establishes inclusion relations between the duals of the Palamodov spaces (7) and (8) and the algebras $M_\theta(S^\beta_\alpha)$ and $C_\theta(S^\beta_\alpha)$.
Theorem 6: Let $\alpha \geq \beta$. For any $\theta$, the space $C'(S^\beta_\alpha) = \bigcup_{B \to \infty} \bigcap_{A \to \infty} E_{\alpha,B}^C$ is contained in the algebra $M_{\theta}(S^\beta_\alpha)$ and the space $M'(S^\beta_\alpha) = \bigcup_{B \to \infty} \bigcap_{A \to \infty} (E_{\alpha,A}^\beta)'$ is contained in the algebra $C_{\theta}(S^\beta_\alpha)$.

Proof. In the theorem’s formulation, we use the presentation of $C'(S^\beta_\alpha)$ and $M'(S^\beta_\alpha)$ proved in Theorem 2. Let $u \in C'(S^\beta_\alpha)$. We need show that for any $f \in S^\beta_\alpha$, the product $u * f$ also belongs to $S^\beta_\alpha$. By definition, (60),

$$\langle u * f, h \rangle = \langle u, f * h \rangle$$

for all $h \in S^\beta_\alpha$. We let $L_f$ denote the linear map $w \to f * w$ from $(S^\beta_\alpha)'$ into $C(S^\beta_\alpha)$. By Corollary of Theorem 5, this map is continuous. Consequently, its transpose $L'_f$ is well defined as a map from $C'(S^\beta_\alpha)$ into $(S^\beta_\alpha)'$. The second dual coincides with $S^\beta_\alpha$ because this space is reflexive, hence $L'_f u \in S^\beta_\alpha$. For all $w \in (S^\beta_\alpha)'$, we have the equality

$$\langle w, L'_f u \rangle = \langle u, f * w \rangle.$$}

If $w = h \in S^\beta_\alpha$, then the right-hand side of (59) becomes equal to that of (58) and the left-hand side of (59) takes the form $\int (L'_f u)(x)h(x)dx$. Thus, the function $L'_f u$ considered as an element of $(S^\beta_\alpha)'$ coincides with the functional $u * f$ and we conclude that $u \in M_{\theta,L}(S^\beta_\alpha)$. Analogously, $u \in M_{\theta,R}(S^\beta_\alpha)$ and hence $u \in M_{\theta}(S^\beta_\alpha)$. Now let $v \in M'(S^\beta_\alpha)$. Then $F^{-1}v \in C'(S^\beta_\alpha)$ by Theorem 1 and $F^{-1}v \in M_{\theta}(S^\beta_\alpha)$ by what has just been said. Therefore, $v$ belongs to $M_{\theta}(S^\beta_\alpha) = C_{\theta}(S^\beta_\alpha)$, which completes the proof.

Of special interest are the Moyal multiplier algebras of the Fourier-invariant spaces $S^\beta_\alpha$.

Theorem 7: Suppose that $\beta \geq 1/2$ and the matrix $\theta$ is invertible. In this case, the Moyal multiplier algebra $M_{\theta}(S^\beta_\alpha)$ contains both the spaces $C'(S^\beta_\alpha)$ and $M'(S^\beta_\alpha)$. The algebra $C_{\theta}(S^\beta_\alpha)$ also contains these spaces and consists of the same elements as $M_{-4\theta^{-1}}(S^\beta_\alpha)$.

Proof. Changing variables in one of integrals (2), we obtain

$$(f *_{\theta} g)(x) = \frac{1}{\pi^d \det \theta} \int f(x - \xi) \hat{g}(2\theta^{-1}\xi) e^{2ix\theta^{-1}\xi} d\xi = \frac{1}{\pi^d \det \theta} (f *_{-4\theta^{-1}} F_{\theta} g)(x),$$

where $(F_{\theta} g)(\xi) = \int f(x) e^{-2ix\theta^{-1}\xi} dx$ is the symplectic Fourier transform of $g$, which obviously belongs to $S^\beta_\alpha$ if $g \in S^\beta_\alpha$. An analogous manipulation with another integral in (2) gives

$$(f *_{\theta} g)(x) = \frac{1}{\pi^d \det \theta} (F_{\theta} f *_{-4\theta^{-1}} g)(x),$$

where $(F_{\theta} f)(\xi) = \int f(x) e^{2ix\theta^{-1}\xi} dx$. Because $S^\beta_\alpha$ is dense in $(S^\beta_\alpha)'$, the Moyal product and the twisted convolution product have unique continuous extensions to the case where one of factors is in $(S^\beta_\alpha)'$, and we conclude that

$$u *_{\theta} g = \frac{1}{\pi^d \det \theta} u *_{-4\theta^{-1}} F_{\theta} g \quad \text{and} \quad f *_{\theta} u = \frac{1}{\pi^d \det \theta} F_{\theta} f *_{-4\theta^{-1}} u$$

for each $u \in (S^\beta_\alpha)'$ and for all $f, g \in S^\beta_\alpha$. Therefore, an element of $(S^\beta_\alpha)'$ belongs to $M_{\theta}(S^\beta_\alpha)$ if and only if it belongs to $C_{-4\theta^{-1}}(S^\beta_\alpha)$. 17
Combining Theorem 5 with (61), we see that the products $u \ast_\theta g$ and $f \ast_\theta u$ are contained in $M(S^\beta_\theta)$. Moreover, the maps $(g, u) \mapsto u \ast_\theta g$ and $(f, u) \mapsto f \ast_\theta u$ are hypocontinuous from $S^\beta_\theta \times (S^\beta_\theta)'$ into $M(S^\beta_\theta)$. The rest of proof is similar to the proof of Theorem 6. Let $v \in M'(S^\beta_\theta)$. We show that for any $g \in S^\beta_\theta$, the product $g \ast_\theta v$ also belongs to $S^\beta_\theta$. By (16), we have
\[
\langle g \ast_\theta v, h \rangle = \langle v, h \ast_\theta g \rangle
\]  
for all $h \in S^\beta_\theta$. Let $R_g$ be the continuous linear map $u \mapsto u \ast_\theta g$ from $(S^\beta_\theta)'$ into $M(S^\beta_\theta)$. Then
\[
\langle u, R_g' v \rangle = \langle v, u \ast_\theta g \rangle
\]  
for all $u \in (S^\beta_\theta)'$. If $u = h \in S^\beta_\theta$, then the right hand sides of (62) and (63) coincide and \( \langle h, R_g' v \rangle = \int (R_g' v)(x) h(x) dx \). Since $h$ is an arbitrary element of $S^\beta_\theta$, we conclude that $g \ast_\theta v = R_g' v$ and hence $v \in M_{\theta,R}(S^\beta_\theta)$. Analogously, $v \in M_{\theta,L}(S^\beta_\theta)$ and consequently $v \in M_{\theta}(S^\beta_\theta)$. This result together with the equalities $\widehat{M(S^\beta_\theta)} = C(S^\beta_\theta)$ and $\widehat{M_{\theta}(S^\beta_\theta)} = C_{\theta}(S^\beta_\theta)$ implies that $C'(S^\beta_\theta) \subset C_{\theta}(S^\beta_\theta)$. Thus, Theorem 7 is proved.

Remark: Theorem 7 shows that the algebra $M_{\theta}(S^\beta_\theta)$ is invariant under the Fourier transform if (and really only if) $\theta^2/4 = -I$. In the phase space representation of quantum mechanics, one usually uses the symplectic Fourier transform, which is natural when dealing with the Weyl correspondence. In this connection, it should be noted that either of the two operators $F_{\theta}$ or $F_{\theta}$ maps $M_{\theta}(S^\beta_\theta)$ isomorphically onto $C_{\theta-1}(S^\beta_\theta)$, and these algebras consist of the same elements.

Like the Moyal multiplier algebra of the Schwartz space, any algebra $M_{\theta}(S^\beta_\theta)$ with $\beta \geq 1/2$ contains all polynomials and all distributions of compact support. Moreover, Theorem 6 shows that for $\beta > 1$, this algebra contains all smooth functions that belong to the Gervey class $[36]$ of order $\beta$ and grow at infinity not faster than exponentially of order $1/\beta$, type 0. By Theorem 7, it also contains all ultradistributions of the Roumieu $[37, 38]$ class $\{n^\beta\}$ that decrease at infinity not slower than exponentially of order $1/\beta$, finite type. The elements of $(S^1_\theta)'$ are called Fourier-hyperfunctions $[39]$, and the algebra $M_{\theta}(S^1_\theta)$ contains all real-analytic functions growing not faster than exponentially of order 1, type 0 and also all Fourier-hyperfunctions decreasing not slower than order 1, finite type. Analogous statements hold for the analytic functionals defined on the spaces $S^\beta_\theta$ with $\beta < 1$.

VII. A special role of the algebra $M_{\theta}(S^{1/2})$

As noted in Sec. II, the space $S^\beta_\theta$ is nontrivial if and only if $\beta \geq 1/2$. The algebra $M_{\theta}(S^{1/2})$ plays a special role in the theory. By Theorem 6, it contains the space
\[
E^{1/2}_{1/2} = \lim_{A \to \infty,B \to 0} E^{1/2,B}_{1/2,A},
\]  
which is a linear subspace of $C'(S^{1/2})$. By the arguments used in the proof of Theorem 2, $E^{1/2}_{1/2}$ is an FS space. Using Taylor’s formula and Cauchy’s inequality, it is easy to verify that $E^{1/2}_{1/2}(\mathbb{R}^d)$ coincides with the space of restrictions to $\mathbb{R}^d$ of all entire functions on $\mathbb{C}^d$ that are of order at most two, type zero. In other words, these entire functions satisfy the inequalities
If \( |f(z)| \leq C_{f_\epsilon} e^{\epsilon |z|^2} \), where \( \epsilon > 0 \) and can be taken arbitrarily small. It is readily seen that series \( (1) \) converges pointwise for such functions. Moreover, the following theorem holds.

**Theorem 8:** The space \( E_{1/2}^{1/2} \) is a topological algebra with respect to the Moyal star product. If \( f, g \in E_{1/2}^{1/2} \), then series \( (1) \) representing \( f \ast_\theta g \) converges absolutely in the topology of this space.

**Proof.** Let \( G(s) = \sum_n c_n s^n \) be an entire function of order \(< 2\) or of order \(2\) and finite type. We will show that then the differential operator \( G(\partial) \) is a continuous endomorphism of \( E_{1/2}^{1/2} \) and the series \( \sum_n c_n \partial^n f \), where \( f \in E_{1/2}^{1/2} \), converges absolutely in every norm of this space. According to the theory of entire functions (see, e.g., Sec. IV.5.2 of [2]) the above bound on the growth of \( G(s) \) is equivalent to the condition

\[
|c_n| \leq C (b/n)^{n/2},
\]

where \( C \) and \( b \) are positive constants. The space \( E_{1/2}^{1/2} \) consists of all functions such that

\[
|\partial^n f(x)| \leq \|f\|_{-A,B} B^n |x/A|^n, e^{A |x/A|^2},
\]

for any \( A, B > 0 \). Let \( B' \geq B \sqrt{2} \). Using \( (65), (66) \), and the inequality \((n + m)^{n+m} \leq 2^{n+m} n^m m^m \), we obtain

\[
\|c_n \partial^m f\|_{-A,B'} = \sup_{x, m} \left| \frac{c_n \partial^{n+m} f(x)}{B^n m^m} e^{-|x/A|^2} \right| \leq C \|f\|_{-A,B}(B \sqrt{2B})^{n/2}.
\]

Because \( B \) can be taken arbitrarily small, we conclude that the series \( \sum_n c_n \partial^n f \) converges absolutely in every norm of \( E_{1/2}^{1/2} \). Moreover, the factor \( \|f\|_{-A,B} \) on the right-hand side of \( (67) \) shows that the map \( E_{1/2}^{1/2} \rightarrow E_{1/2}^{1/2} \): \( f \rightarrow G(\partial) f \) is continuous. In particular, the operator \( e^{\frac{i}{2} |\partial_x |^2} \) is well defined and acts continuously on \( E_{1/2}^{1/2}(\mathbb{R}^{2d}) \). The Moyal product \( f \ast_\theta g \) is obtained by applying this operator to the function \( (f \otimes g)(x, y) \) in \( E_{1/2}^{1/2}(\mathbb{R}^{2d}) \) and then identifying \( x \) with \( y \). It is easily verified that the restriction to the diagonal \( x = y \) is a continuous map from \( E_{1/2}^{1/2}(\mathbb{R}^{2d}) \) into \( E_{1/2}^{1/2}(\mathbb{R}^d) \), which completes the proof.

It should be emphasized that \( E_{1/2}^{1/2} \) is the largest star product algebra with the property of absolute convergence of the series determining this product. Indeed, the bound \( |\partial^n f(0)| \leq C_{B} B^n |n|^{n/2}, \forall B > 0 \), on the derivatives of \( f \) at the origin implies that \( f \) cannot grow faster than with order \( 2 \) and minimum type. This subalgebra of \( \mathcal{M}_\theta(S^{1/2}) \) in turn has various subalgebras which are specified by additional restrictions on the behavior of their elements at the infinity of real space. In particular, \( E_{1/2}^{1/2} \) contains the space \( \mathcal{S}^{1/2} = \text{proj lim}_{B \rightarrow 0, N \rightarrow \infty} S_{N}^{1/2,B} \), where \( S_{N}^{1/2,B} \) is the Banach space of analytic functions such that

\[
\|f\|_{B,N} = \sup_{x,n} (1 + |x|)^N \left| \frac{\partial^n f(x)}{B^n n^n} \right| < \infty.
\]

As shown in [21], the space \( \mathcal{S}^{1/2} \) is a topological algebra with respect to the Moyal star product and, being adequate to the nonlocal nature of this product, is suitable for using as a test function space in a general formulation of quantum field theory on noncommutative space-time. The definitions \( (1) \) and \( (2) \) of Moyal multiplication are equivalent for functions in this space and the algebras \( (S, \ast) \) and \( (E_{1/2}^{1/2}, \ast) \) can be regarded as different extensions of
Moreover, to each closed cone $V \subset \mathbb{R}^d$ we can assign an algebra $\mathcal{S}^{1/2}(V)$ which also consists of entire functions and is defined similarly but with supremum over $x \in V$ in an analog of (68). All the algebras $\mathcal{S}^{1/2}(V)$ are also contained in $E^{1/2}_{1/2}$. If a functional $u \in (\mathcal{S}^{1/2})'$ has a continuous extension to $\mathcal{S}^{1/2}(V)$, then the cone $V$ can be thought of as a carrier of $u$. As argued in [24, 25], the spaces $\mathcal{S}^{1/2,B}_N(V)$ can be used as a tool for formulating causality in noncommutative quantum field theory. Another family of subalgebras of $E^{1/2}_{1/2}$ is formed by the spaces $\mathcal{S}^{1/2,\alpha}_B = \text{proj } \lim_{A \to \infty, B \to 0} \mathcal{S}^{1/2,B}_{\alpha,A}$, $\alpha > 1/2$, and by their siblings associated with cones in $\mathbb{R}^d$. The basic reason for considering the spaces over cones is explained in [40]. It lies in the fact that the continuous functionals defined on the $S$-type spaces with superscript $\beta < 1$ retain the angular localizability property, in spite of the failure of the notion of support in the case of entire analytic test functions.

VIII. Concluding remarks

In this paper, we content ourselves with considering the Moyal multiplier algebras of the spaces $S_\alpha^\beta$. However the general construction of Sec. III is applicable to any test function space on which the Weyl-Heisenberg group acts continuously and whose topological properties are more or less like those of $S$. In particular, it immediately extends to the Gel’fand-Shilov spaces $S_\alpha^0$ and $W_\Omega^0$ specified by more flexible restrictions on the smoothness and behavior at infinity of their elements. (See Supplements 1 and 2 in [2] for the definition of these spaces.)

The Schwartz space $S$ can formally be considered as a limit of the spaces $S_\beta^\beta$ as $\beta \to \infty$, and an analog of Lemma 1 shows the existence of an approximation of the identity for $S$. Namely, if $f \in S$, $e \in S$, and $e(0) = 1$, then the sequences of operators of left and right Moyal multiplication by $e(x/\nu)$ converges to the unit operator in the topology of $L(S)$ as $\nu \to \infty$. Because of this, an analog of Theorem 1 also holds for $S$, which gives an alternative definition of the algebras $\mathcal{M}_{\theta,L}(S)$ and $\mathcal{M}_{\theta,R}(S)$, different from the original definition [9, 10, 13, 14]. It should be mentioned that the existence of an approximation of the identity with the weaker property of pointwise convergence on elements of $S$ was previously indicated in [12].

It is worth noting that the Fréchet space $E^{1/2}_{1/2}$ is a topological algebra not only with respect to the Moyal product but also with respect to the Wick star product. (We refer the reader to [3] for the definition and main properties of this product which also is often called the Wick-Voros product.) The proof of this fact is similar to that of Theorem 8, with the replacement of $e^{x^\theta_1 y_1 \theta_2 y_2}$ by the bi-differential operator corresponding to the Wick product. Thereby we obtain a simple and explicit solution to the problem [11] of constructing the largest Fréchet space of analytic functions for which the Wick star product converges and depends continuously on the deformation parameter. As shown in [25] the space $\mathcal{S}^{1/2}$ is also a topological algebra with respect to the Wick product. Finally we note that the Weyl transformation can be naturally extended to the Moyal multipliers discussed here and their definition can be expressed in terms of the corresponding operators on a Hilbert space, but this is beyond the scope of this paper.
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Appendix: Proof of Lemma 2

For each function $f \in S(\mathbb{R}^d)$ and for any multiindices $k, n$, the following inequality holds:

$$\int_{\mathbb{R}^d} |\partial^k x^n| |f(x)| \, dx \leq \sqrt{2} \int_{\mathbb{R}^d} |x^n| |\partial^k f(x)| \, dx, \quad (A1)$$

It suffices to prove (A1) for functions of one variable and for $k = 1$, because the general case can be easily reduced to this one. We suppose first that $f(x)$ is a real-valued function on $\mathbb{R}$ and divide the semiaxis $x \geq 0$ into the three parts $M^+, M^-$, and $M^0$, where $f(x)$ takes positive, negative, and zero values respectively. After numbering the connected components of $M^+$ and $M^-$, we write

$$\int_0^\infty (x^n)^j |f(x)| \, dx = \sum_j \int_{M^+} f(x) dx^n - \sum_j \int_{M^-} f(x) dx^n \leq \int_0^\infty x^n |f'(x)| \, dx.$$ 

Thus, (A1) holds for all real-valued functions in $S(\mathbb{R})$ even without the coefficient $\sqrt{2}$. This coefficient is relevant to the case of complex-valued functions because $|u| + |v| \leq \sqrt{2}|u + iv|$.

Let $f \in S_{\alpha,A}^d$ and $A' > d^n A$. Using (A1) and the inequality $\sum_{j=1}^N |x_j|^{1/\alpha} \leq d |x|^{1/\alpha}$, we get

$$|p^m \partial^n \hat{f}(p)| = \left| \int e^{-ipx} \partial^m [x^n f(x)] \, dx \right| \leq \int \sum_k \binom{m}{k} |\partial^k x^n| |\partial^{m-k} f(x)| \, dx \leq \sqrt{2} \int 2^m |x^n \partial^m f(x)| \, dx \leq C A' \|f\|_{A,B}(2B)^{|m|} \sup_x |x^n| \prod_j e^{-|x_j/A'|^{1/\alpha}}.$$ 

The supremum over $x$ on the right-hand side equals $A'^{|n|}(\alpha/e)^{|n|} n^{|n|}$ and hence

$$|\partial^n \hat{f}(p)| \leq C A' \|f\|_{A,B} A'^{|n|}(\alpha/e)^{|n|} n^{|n|} \inf_m (2B)^{|m|} \frac{|m^{|m|}}{|p^m|} \leq C \|f\|_{A,B} A'^{|n|}(\alpha/e)^{|n|} n^{|n|} e^{-(\beta/e)|p/2B|^{1/\beta}}.$$ 

A subtlety in the calculation of infimum over $m$ is that the components of $m$ range over integers, but this affects only the magnitude of the coefficient $C$. We conclude that (13) holds for any $r > \max\{(\alpha d/e)^{\alpha}, 2(e/\beta)^{\beta}\}$. Lemma 2 is proved.

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