INVERSE GROUP 1-MEDIAN PROBLEM ON TREES

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ABSTRACT. In location theory, group median generalizes the concepts of both median and center. We address in this paper the problem of modifying vertex weights of a tree at minimum total cost so that a prespecified vertex becomes a group 1-median with respect to the new weights. We call this problem the inverse group 1-median on trees. To solve the problem, we first reformulate the optimality criterion for a vertex being a group 1-median of the tree. Based on this result, we prove that the problem is \(NP\)-hard. Particularly, the corresponding problem with exactly two groups is however solvable in \(O(n^2 \log n)\) time, where \(n\) is the number of vertices in the tree.

1. Introduction. Location theory has been playing an important role in operational research due to its theoretical and practical contributions. Here, we want to find optimal locations for new facilities such as schools, warehouses, hospitals, garbage dumps, mega airports, railway stations, etc. According to the specific applications, we often study the planar locations, locations on networks, or the facility location problems with various types of objective functions; see Hamacher [17], Kariv and Hakimi [18, 19], Drezner and Hamacher [9], Puerto et al. [31], and references therein.

If the locations of facilities are already known and they cannot be relocated due to cost or security reasons, we aim to modify parameters, e.g., edge lengths or facility weights, at minimum total cost to make the predetermined facilities optimal w.r.t. the perturbed parameters. This problem is the so-called inverse location problem, which has recently become an interesting topic in location theory. For the inverse median location problems, Burkard et al. [7] developed polynomial time algorithm for the inverse 1-median location problems on the plane with rectilinear norm and on trees. Since then, there were intensive investigations concerning the inverse 1-median problems; for example, the inverse Fermat-Weber problem [6], the problem on trees under Chebyshev norm and Hamming distance [14], the problem on block graphs [23], etc. Additionally, Bonab et al. [5] showed that the inverse \(p\)-median problem is \(NP\)-hard on general graphs and considered the underlying problem on

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trees with polynomial time solution approach. The inverse center location problem is \( NP \)-hard; see Cai et al. [8]. However, the corresponding problem on trees is solvable in polynomial time; see the problem on unweighted trees [2, 3], the problem on weighted trees [30], the problem under Chebyshev norm and Hamming distance [27]. Moreover, inverse obnoxious location problems were also under study with efficient solution methods; see Alizadeh et al. [4, 1]. Recently, Keshtkar and Ghiyasvand [20] solved the inverse quickest 1-center problem on trees with augmentations or reductions of capacities in linear time based on the optimality condition of a quickest 1-center.

According to the best of our knowledge, there are recently two extensions regarding the objective of a location problem, say the ordered median function and the group median/center function. In one hand, one uses the multipliers and the sorting of distances to define the ordered median function. For properties of the ordered median functions and algorithmic approaches concerning ordered median problems, we can refer to the book of Nickel and Puerto [22]. On the other hand, the group median/center function is defined based on the partition of facilities into groups. Gupta and Punnen [15] pioneered in investigating the group median and group center problems which generalize both of the classical median and center problems. Then, the same authors [16] considered the group 1-median and group 1-center problem on trees with improved algorithms.

For the inverse version of the unified location theory, the inverse ordered median location problem was first investigated by Gassner [12]. She proved that the inverse convex ordered 1-median problem on trees is \( NP \)-hard. Moreover, she developed a polynomial time algorithm to solve the inverse \( k \)-centrum problem on trees. Since then, many papers concerned the inverse ordered median location problem with complexity and algorithmic results; see [24, 26, 29]. Although group median and group center problems showed their importance to real life model, e.g., the customers can be grouped according to the demand, analogy, or relation, the inverse version of these problems with promising applications has not been under study so far. In this paper, we focus on the inverse group 1-median problem on tree graphs and mainly study on the complexity of the problem and special cases with efficient solution approaches.

We organize the paper as follows. Sect. 2 recalls the group 1-median problem on trees. We also modify the optimality criterion, which was first constructed by Gupta and Punnen [16], based on the directional derivative. For the complexity result, we show in Sect. 3 that the problem is \( NP \)-hard if the number of groups are not a part of the input. Moreover, a special case of the problem with exactly two groups can be solved in \( O(n^2 \log n) \) time, where \( n \) is the number of vertices in the tree.

2. Preliminaries. Let a graph \( G = (V, E) \) be given with \( n \) vertices. Each vertex \( v \in V \) is associated with a non-negative weight \( w_v \) and each edge \( e \in E \) has a positive length \( \ell_e \). The length of the shortest path \( P(u, v) \) connecting two vertices \( u \) and \( v \) is, by definition, the distance \( d(u, v) \) between them. A point on \( G \) is either a vertex or lies on an edge of the graph. For the later situation, a point \( \rho \) on an edge \( e = (u, v) \) is identified by a parameter \( \mu \in (0, 1) \) with \( d(u, \rho) = \mu \ell_e \) and \( d(v, \rho) = (1 - \mu) \ell_e \). The distance between two points on \( G \) is defined analogously to the distance between two vertices. In the rest of this paper, we focus on a special type of graphs, say the tree graph \( T = (V, E) \), that is a connected graph without cycle. In a location problem on \( T \), one wants to find optimal locations of
new facilities under the assumption that the customers are located at the vertex set. Concerning single facility location problem on trees with median and center functions, we can refer to Goldman [13] and Megiddo [21].

In what follows, we recall the group 1-median problem on trees that was investigated by Gupta [16]. For convenience, we always associate the number of group $S_i$, indexed by $j$, for $i = 1, \ldots, m$. Concerning single facility location problem on trees with median and center functions are also convex, we conclude that the $m$-group 1-median function is convex along each simple path of the tree. Moreover, as the max and the sum functions are also convex, we conclude that the $m$-group 1-median function is convex along each simple path of $T$. Taking a vertex $v'$ in a subtree $T' \subseteq T(v)$ that is

A point $\rho^0$ that minimizes $F(\rho)$ is called a $m$-group 1-median of $G$.

Now we reformulate the optimality criterion for a vertex that is a $m$-group 1-median on the tree $T$. For a predetermined vertex $v$, we denote by $T(v)$ the set of all subtrees induced by deleting $v$ and its incident edges. Moreover, let

$$J_i(v) := \left\{ v_j \in S_i \mid w_{i,j}d(v, v_j) = \max_{j=1}^{k_i} \left\{ w_{i,j}d(v, v_j) \right\} \right\}$$

the set of vertices corresponding to the maximum weighted distance to $v$ in the group $S_i$. Considering a branch $T^\text{sub}$ of $T(v)$, we denote by

- $w_{i,t(i)}(T^\text{sub}) := \min \left\{ w_{i,j} \mid v_j \in J_i(v) \cap T^\text{sub} \right\}$ the minimum weight in $T^\text{sub}$ that corresponds to the largest weighted distance to $v$ in the group $S_i$.
- $w_{i,t'(i)}(T^\text{sub}) := \max \left\{ w_{i,j} \mid v_j \in J_i(v) \setminus T^\text{sub} \right\}$ the maximum weight in $T \setminus T^\text{sub}$ that corresponds to the largest weighted distance to $v$ in the group $S_i$.

Based on these denotations, we can choose an index $j$ ($j'$) such that $I(i) := j$ ($I'(i) := j'$) if $w_{i,j} = w_{i,t(i)}(w_{i,j} = w_{i,t'(i)})$. For a group $S_i$ and a subtree $T^\text{sub}$ in $T(v)$, we define

$$\lambda(i, T^\text{sub}) := \begin{cases} 1, & \text{if } J_i(v) \cap (T \setminus T^\text{sub}) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

The denotation $\lambda(i, T^\text{sub})$ helps to know that whether there exists a vertex in $T^\text{sub}$ with maximum weighted distance between $v$ and the group $S_i$.

We derive the optimality criterion, that is similar to the formulation of Goldman [13] for 1-median on tree graphs, for a predetermined vertex $v$ that is a $m$-group 1-median of $T$ as follows.

**Theorem 2.1.** [Optimality criterion - a reformulated version; see Gupta and Punnen [16]] The vertex $v$ is a group 1-median of $T$ iff

$$\sum_{i=1}^{m} (1 - \lambda(i, T^\text{sub}))w_{i,t(i)}(T^\text{sub}) \leq \sum_{i=1}^{m} \lambda(i, T^\text{sub})w_{i,t'(i)}(T^\text{sub}), \quad \forall T^\text{sub} \in T(v) \quad (2)$$

**Proof.** We know that the distance function from a point to any vertex $v$ in $T$ is a convex function in each simple path of the tree. Moreover, as the max and the sum functions are also convex, we conclude that the $m$-group 1-median function is convex along each simple path of $T$. Taking a vertex $v'$ in a subtree $T^\text{sub} \subseteq T(v)$ that is
adjacent to \( v \) and a point \( \rho \) on edge \((v, v')\) such that \( d(v, \rho) = t \), for sufficiently small \( t \). Elementary computation yields
\[
F(\rho) - F(v) = \left( \sum_{i=1}^{m} \lambda(i, T_{sub}) w_{i(T')} - \sum_{i=1}^{m} (1 - \lambda(i, T_{sub})) w_{i(T')} \right) t.
\]
By the convexity of the objective, the vertex \( v \) is a \( m \)-group 1-median if and only if it is a local minimizer of \( F(\cdot) \), i.e., \( F(\rho) \geq F(v) \). As the inequality holds for all subtrees \( T_{sub} \) in \( T(v) \), the result follows.

If a vertex \( v \) is not a \( m \)-group 1-median of \( T \), there exists exactly one subtree \( T' \in T(v) \) such that
\[
\sum_{i=1}^{m} (1 - \lambda(i, T')) w_{i(T')} > \sum_{i=1}^{m} \lambda(i, T') w_{i(T')}.
\]

Let a tree \( T = (V, E) \) and a prespecified vertex \( v^* \) be given. The weight of each vertex \( v_j \) in \( V \) can be increased or reduced by \( p_j \) or \( q_j \) respectively and this yields the new weight \( \bar{w}_j := w_j + p_j - q_j \) for \( j = 1, \ldots, k_i \) and \( i = 1, \ldots, m \). Furthermore, increasing or reducing one unit weight of vertex \( v_j \) incurs the corresponding cost \( c_j \). The inverse \( m \)-group 1-median on \( T \) is stated as follows.

1. The vertex \( v^* \) becomes a \( m \)-group 1-median with respect to the new weights \( \bar{w} \), i.e.,
\[
\sum_{i=1}^{m} \max_{j=1}^{k_i} \left\{ \bar{w}_j d(v^*, v_j) \right\} \leq \sum_{i=1}^{m} \max_{j=1}^{k_i} \left\{ w_j d(\rho, v_j) \right\}
\]
for all points \( \rho \).
2. The cost \( \sum_{i=1, \ldots, m} \sum_{j=1, \ldots, k_i} c_j (p_j + q_j) \) is minimized.
3. Modifications are in certain bounds, i.e., \( 0 \leq p_j \leq \bar{p}_j \) and \( 0 \leq q_j \leq \bar{q}_j \).

As the new weights are always assumed to be non-negative, we can further assume that \( q_j \leq \bar{w}_j \) for all \( v_j \in V \).

Note that, we also call \( p_v \) (\( q_v \)) the augmentation (reduction) and \( \bar{p}_v \) (\( \bar{q}_v \)) the corresponding upper bounds for \( v \in V \) if the indices are not specified. This convention is also applied for the cost coefficients \( c_v \) for \( v \in V \).

3. Complexity results.

3.1. General problem. We now investigate the complexity of the inverse \( m \)-group 1-median problem on trees, where \( m \) is not a part of the input as in the following.

**Theorem 3.1.** The inverse \( m \)-group 1-median problem on trees is \( NP \)-hard.

*Proof.* Given an instance \((I)\) of the partition problem: A set \( S := \{a_1, \ldots, a_n\} \subset \mathbb{N} \) such that \( \sum_{i=1}^{n} a_i = 2B \). Does there exist a subset \( S' \) of \( S \) such that \( \sum_{a_i \in S'} a_i = B \)? This problem is \( NP \)-complete; see, e.g., Garey and Johnson [11].

Let us state the decision version of the inverse \( m \)-group 1-median problem on a tree \( T \) \((Inv)\). Given an instance of the corresponding problem, does there exist a modification such that the prespecified vertex becomes a \( m \)-group 1-median of the perturbed tree and the total cost is at most \( C \)?

Given an instance \((I)\), we can derive an instance \((Inv)\) in polynomial time as follows.

- A tree \( T = (V, E) \) where the vertex set \( V := \{v_j^1, v_j^2, \ldots, v_j^{n+1}\} \) and the edge set \( E := \{(v_j^{n+1}, v_j^1)\}_{j=1,2} \cup \{(v_j^{n+1}, v_j^2)\}_{i=1,2} \cup \{(v_j^{n+1}, v_j^{n+1})\} \).
The lengths of edges \((v_1^{n+1}, v_i^1)\) are all equal to 1 for \(i = 1, \ldots, n\). The lengths of \((v_1^{n+1}, v_i^2)\) and \((v_1^{n+1}, v_2^{n+1})\) are \(\frac{1}{2}\) for \(i = 1, \ldots, n\). We set \(w_1^i := a_i\) and 
\[w_2^i := 2a_i \quad \text{for} \quad i = 1, \ldots, n; \quad w_1^{n+1} := B, \quad w_2^{n+1} := 0.\]

- Set \(\bar{q}_2^i := a_i, \quad \bar{p}_2^i := 0, \quad \forall i = 1, \ldots, n; \quad \bar{q}_2^{n+1} := 0 \quad \text{and} \quad \bar{p}_1^i := \bar{q}_1^i := 0 \quad \forall i = 1, \ldots, n + 1.

- Cost coefficients are all equal to 1 and \(C := B.\)
- There are \(n + 1\) groups, say \(S_i := \{v_1^i, v_2^i\}\) for \(i = 1, \ldots, n + 1.\)
- The prespecified vertex is \(v^* = v_1^{n+1}.\)

Considering a subtree \(T'\) in \(T(v_1^{n+1})\) induced by \(v_i^2\) for \(i = 1, \ldots, n + 1.\) As \(w_2^i d(v_2^i, v_1^{n+1}) > w_1^i d(v_1^i, v_1^{n+1})\) for all \(i = 1, \ldots, n\) and 
\[w_2^{n+1} d(v_2^{n+1}, v_1^{n+1}) = w_1^{n+1} d(v_1^{n+1}, v_1^{n+1}) = 0,\]
we get \(\lambda(i, T') = 0\) for \(i = 1, \ldots, n\) and \(\lambda(n + 1, T') = 1.\) The subtree \(T'\) violates the optimality criterion as
\[\sum_{i=1}^m (1 - \lambda(i, T')) w_{1(i)}^i (T') = 4B\]
and
\[\sum_{i=1}^m \lambda(i, T') w_{l(i)}^i (T') = B.\]

Therefore, we reduce the weights of vertices in \(T'\) to obtain the optimality criterion. In the following we prove that answer to \((I)\) is ‘yes’ if and only if the answer to \((\text{Inv})\) is ‘yes’. 

\(" \Rightarrow \)\] If the answer to \((I)\) is ‘yes’, there exists a subset \(S' \subset S\) such that \(\sum_{a_i \in S'} a_i = B.\) Then, we reduce the weights of \(v_2^i\) by \(a_i\), i.e., \(\bar{w}_2^i := \bar{w}_2^i - a_i = a_i,\) for all indices \(i\) with \(a_i \in S'.\) The total cost is \(\sum_{a_i \in S'} a_i = B\) and 
\[\sum_{i=1}^m (1 - \lambda(i, T')) \bar{w}_{1(i)}^i (T') = \sum_{i=1}^m \lambda(i, T') \bar{w}_{l(i)}^i (T') = 2B.\]

Hence, the answer to \((\text{Inv})\) is ‘yes’.

\(" \Leftarrow \)\] If the answer to \((\text{Inv})\) is ‘yes’, we prove that the answer to \((I)\) is also ‘yes’. We claim that there is at most one vertex whose weight is reduced but the reduction does not obtain the upper bound. Indeed, assuming that \(v_2^i\) and \(v_2^k\) are reduced by \(q_2^i\) and \(q_2^k\) with \(0 < q_2^i < a_i\) and \(0 < q_2^k < a_k\). In the corresponding modified weights \(\bar{w}\), we know \(\bar{w}_2^i d(v_2^i, v_1^{n+1}) > \bar{w}_1^i d(v_2^i, v_1^{n+1})\) and \(\bar{w}_2^k d(v_2^k, v_1^{n+1}) > \bar{w}_1^k d(v_2^k, v_1^{n+1})\). The weight modifications of \(v_2^i\) and \(v_2^k\) do not lead to the reduction of the optimality gap \(\sum_{i=1}^m (1 - \lambda(i, T')) w_{1(i)}^i (T') = \sum_{i=1}^m \lambda(i, T') w_{l(i)}^i (T').\) Hence, the following cases.

- If \(q_2^i + q_2^k < \min\{a_i, a_k\}\), we set \(q_2^k := q_2^i + q_2^k\) and \(q_2^i := 0.\)
- If \(q_2^i + q_2^k \geq \min\{a_i, a_k\}\), assuming without loss of generality that \(a_i \leq a_k\), then we set \(q_2^k := q_2^i + q_2^k - a_i\) and \(q_2^i := a_i.\)

The cost is indeed non-increasing but the gap is possibly reduced w.r.t. new modifications. For any pair of modified vertices, we can induce a zero or maximum modification and this proves the claim.

Let \(P := \{i \in \{1, \ldots, n\} : q_2^i = a_i\}\) and \(i_0\) (if exists) be the index such that \(0 < q_2^{i_0} < a_{i_0}.\) As the optimality holds for \(T',\)
\[\sum_{i=1}^m (1 - \lambda(i, T')) \bar{w}_{1(i)}^i (T') \leq \sum_{i=1}^m \lambda(i, T') \bar{w}_{l(i)}^i (T'),\]
or equivalently saying, \( \sum_{i \in P} 2a_i \leq B + \sum_{i \in P} a_i \), we get \( \sum_{i \in P} a_i \geq B \). Moreover, due to the total cost \( \sum_{i \in P} a_i + B = B \), it yields \( \sum_{i \in P} a_i = B \). Hence, the answer to (I) is ‘yes’.

By the polynomial time reduction of the partition problem to (Inv), the inverse \( m \)-group 1-median problem is indeed \( NP \)-hard.

As the inverse \( m \)-group 1-median is \( NP \)-hard, we can hardly find a polynomial time algorithm for the problem unless \( P = NP \). Thus, special cases with polynomial time algorithm are interesting. If there are \( n \) groups in the tree \( (m = n) \), we obtain the classical inverse 1-median problem on the underlying tree which can be solved in linear time by Galavii [10]. Another special case is \( m = 1 \), i.e., there is exactly one group in the tree. This is indeed the inverse 1-center problem on trees and thus can be solved in quadratic time; see Nguyen and Anh [25]. In what follows we develop an efficient algorithm that solves the inverse 2-group 1-median problem on trees in polynomial time.

3.2. Problem with 2 groups. For the 2-group 1-median problem on trees, the objective function can be written as

\[
F(\rho) := \max_{j=1}^{k_2} \{ w_j \cdot d(v_j, \rho) \} + \max_{j=1}^{k_2} \{ w_j \cdot d(v_j, \rho) \},
\]

for a point \( \rho \) on \( T \).

By the optimality criterion in Theorem 2.1, the prespecified vertex \( v^* \) is not a 2-group 1-median of \( T \) if there exists a subtree \( T' \in T(v^*) \) and a group \( S_i \) for \( i \in \{1, 2\} \) such that \( \mathcal{J}_i(v^*) \cap T' \neq \emptyset \), \( \mathcal{J}_i(v^*) \cap (T \setminus T') = \emptyset \) and either one of the following conditions hold:

1. If any vertex \( v \in (T \setminus T') \cap S_{i'} \) with \( w_{i'} d(v, v^*) = \max_{u \in S_{i'}} w_{i'} d(u, v^*) \) for \( i' \in \{1, 2\} \setminus \{i\} \), then \( w_{i'} I(i) (T') > w_{i'} I(i') (T') \).
2. There does not exist any vertex \( v \in (T \setminus T') \cap S_{i'} \) such that \( w_{i'} d(v, v^*) = \max_{u \in S_{i'}} w_{i'} d(u, v^*) \) for \( i' \in \{1, 2\} \setminus \{i\} \).

Assuming that a subtree \( T' \) violates the optimality criterion of \( v^* \), we get the following result.

**Proposition 1.** In an optimal solution of the inverse 2-group 1-median problem on \( T \), we may increase either one vertex weight in \( T \setminus T' \) or the weights of two vertices with one vertex in \( (T \setminus T') \cap S_1 \) and another in \( (T \setminus T') \cap S_2 \).

**Proof.** Assuming that, in the modified weights \( \tilde{w} \) regarding the optimal solution, we consider the two cases. In the first case, there exists a vertex \( v \in S_i \cap (T \setminus T') \) for \( i \in \{1, 2\} \) such that \( \tilde{w}_{i'} d(v, v^*) = \max_{v' \in S_{i'}} \tilde{w}_{i'} d(v', v^*) \). Furthermore, the modified weight \( \tilde{w}_v \) is also larger than or equal to \( \tilde{w}_{i'} I(i) (T') \) with \( i' \in \{1, 2\} \setminus \{i\} \) and \( v' I(i') (T') \) in \( T' \). Then, we can just augment the weight of only one vertex, say \( v \), in order to reduce the cost. In the second case, there are two vertices \( u \in S_1 \cap (T \setminus T') \) and \( v \in S_2 \cap (T \setminus T') \) such that \( \tilde{w}_u d(u, v^*) = \max_{v' \in S_1} \tilde{w}_{i'} d(v', v^*) \) and \( \tilde{w}_v d(v, v^*) = \max_{v' \in S_2} \tilde{w}_{i'} d(v', v^*) \). Then, the optimality criterion also holds. We modify just two vertices \( u \) and \( v \) for the minimum cost in this case.

To illustrate the second case described in Proposition 1, where the weights of two vertices with one vertex in \( (T \setminus T') \cap S_1 \) and another in \( (T \setminus T') \cap S_2 \) are augmented, let us investigate the following example.

**Example 1.** Let a tree \( T \) be given as in Figure 1, where all edge lengths are 1 and weights are labeled on the vertices. We consider the inverse 2-group 1-median
problem with two groups $S_1 := \{v_1^1, v_1^2, v_1^3, v_1^5\}$ and $S_2 := \{v_2^1, v_2^2\}$, and the prespecified vertex $v_1^1$. A subtree $T'$ induced by $\{v_1^2, v_2^1, v_2^2\}$ violates the optimality criterion as there exists a group $S_2$ such that $w_i^2(2) = 9$ ($I(2) = 2$) and there does not exist any vertex in $S_1 \cap (T\setminus T')$ with maximum weighted distance between $v_1^1$ and vertices in $S_1$. We can also take the group $S_1$ to convince that $T'$ violates the optimality criterion.

![Figure 1. An instance of the inverse 2-group 1-median problem on trees](image)

| $j$ | 1 | 2 | 3 | 5 |
|-----|---|---|---|---|
| $q^1_j$ | 0 | 0 | 0 | 0 |
| $q^2_j$ | 0 | 0 | 0 | - |
| $p^1_j$ | 0 | 0 | 3 | - |
| $p^2_j$ | 0 | 16 | - | - |

Table 1. An instance of the inverse 2-group 1-median problem

For the instance of modifications in Table 1, we have to augment the weight of $v_3^3$ by 3 and the weight of $v_2^3$ by 16 to make $v_1^1$ a 2-group 1-median of $T$. 

By Proposition 1, we can solve the inverse 2-group 1-median problem on $T$ by taking into account the two situations. For the first situation, we augment the weight of one vertex in $T\setminus T'$ and denote this problem by $(P^1)$. The second situation is to solve the problem $(P^2)$ with two vertices, one in $(T\setminus T') \cap S_1$ and another in $(T\setminus T') \cap S_2$, whose weights are augmented. In the following, let us analyze the two problems in detail.

**Problem $(P^1)$**

We can choose a vertex $v \in T\setminus T'$, which is either in $S_1$ or $S_2$, to augment its weight. Furthermore, we reduce the weights of other vertices in order to obtain the optimality criterion at minimum cost. We further assume that $v \in S_1$, as the case $v \in S_2$ can be solved by the similar argument. From here on, we denote by $d(u) := d(u, v^*)$, $\forall u \in V$, for simplicity. The corresponding problem, where the weight of $v$ is increased, is abbreviated by $(P^1_v)$.

We next solve $(P^1_v)$ by a parameterization approach. Let $t := \bar{w}_v d(u)$ be the modified weighted distance from $v$ to $v^*$, we take into account the following types of modifications in order to obtain the optimality criterion.
1. Augmenting the weighted distance from $v$ to $v^*$, to obtain the value $t$, i.e., $(w_v + \hat{p}_v)d(v) = t$. This modification costs

$$c_u\left(\frac{t}{d(u)} - w_v\right).$$

2. Reducing the weights of vertices in $S_1$ such that the modified weighted distance from $v$ to $v^*$ becomes the largest one in $S_1$, i.e.,

$$t := \max\{\hat{w}_u d(u) : u \in S_1\}.$$

Let us denote by $B := \{w_u d(u) : u \in S_1 \text{ and } w_u d(u) > w_v d(v)\}$. Furthermore, let $t := \max\{w_v d(v), \min_{u \in B}(w_u - \hat{q}_u) d(u)\}$ and $\hat{t} := (w_v + \hat{p}_v) d(v)$, one can always assume $t \leq \hat{t}$, since $(P^1_v)$ is infeasible otherwise. As $t$ and $\hat{t}$ are the possible smallest and largest value of $t$, we can set $B := (B \cup \{t, \hat{t}\}) \cap [\bar{t}, \bar{f}]$. After sorting all elements in $B$ nondecreasingly, we get $B := \{t_1, t_2, t_3, \ldots, t_r\}$, where $t_j < t_{j+1}$ for $j = 1, \ldots, r-1$. For a fixed parameter $t$, we also define $B(t) := \{u \in S_1 : w_u d(u) > t\}$. One can observe that $B(t) = B(t')$ for $t$ and $t'$ in a half open interval $(t_j, t_{j+1})$, $j = 1, \ldots, r$.

As $(w_u - \hat{q}_u) d(u) = t$ leads to $t := w_u - \frac{t}{d(u)}$ for $u \in B(t)$, the cost for reducing the maximum weighted distance between $v^*$ and vertices in $S_1$ to $t \in [\bar{t}, \bar{f}]$ is

$$\sum_{u \in B(t)} c_u \left(w_u - \frac{t}{d(u)}\right).$$

3. Reducing the weights of vertices in $S_2 \cap T'$ such that the minimum weight of vertices in $J_2(v^*) \cap T'$ is smaller than or equal to the weight of $v$. Let $z := \max_{v' \in S_2} \hat{w}_v d(v')$ the maximum weighted distance between $v^*$ and vertices in $S_2$, then $z$ is reduced as much as possible to

$$\hat{z} := \max\{\max_{v' \in S_2 \cap T'}\{(w_v - \hat{q}_v) d(v')\}, \max_{u \in S_2 \cap (T \setminus T')} w_u d(u)\}$$

in order to obtain the optimal solution. Let

$$D := \{w_v d(v') : v' \in S_2 \cap T' \text{ and } w_v d(v') \geq \hat{z}\} \cup \{\hat{z}\}$$

and after sorting elements in $D$ nondecreasingly, we get $D := \{z_1, z_2, \ldots, z_r\}$ with $z_j < z_{j+1}$ for $j = 1, \ldots, r-1$. For a vertex $v'$ in $S_2 \cap T'$ such that $w_v d(v') = z_j$, let $\hat{d}_j := d(v')$ and $u_j$ be the vertex in $S_2$ such that $w_{u_j} \hat{d}_j = z_j$ for $j = 1, \ldots, r$. As we are interested in the minimum value in $J_2(v^*) \cap T'$, the possible corresponding modified weights are given in the following set

$$L := \{w_{u_j} : \exists i \text{ such that } z_j < z_i \text{ and } \hat{d}_j < \hat{d}_i\}.$$ 

Indeed, if $z_j < z_i$ and $\hat{d}_j < \hat{d}_i$, the weight of $u_i$ is always smaller than the weight of $u_j$ if $u_i = u_j = \arg \max_{v' \in S_2} \{\hat{w}_v d(v')\}$. The set $L$ can be found in linear time by checking the corresponding condition for each element in $D$.

Now we translate the weights in $L$ in a relation to parameter $t$. For the modified weights $\hat{w}$, let

$$\hat{w} := \min\{\hat{w}_v : v' \in S_2 \cap T' \text{ and } \hat{w}_v d(v') = \hat{z}\},$$

then we can reset $t := \max\{L, \hat{w} d(v)\}$ the minimum value of $t$ such that the modified weight of $v$ is possible to obtain the minimum modified weight in $J_2(v^*) \cap T'$. Let $B' := \{w_u d(v) : u \in L\} \cup \{\{t, \hat{t}\}\}$ be the set regarding the multiplication of weights in $L$ and the distance $d(v)$. After sorting the set $B'$ nondecreasingly, we get $B' := \{t_1, t_2, \ldots, t_p\}$ with $t_j < t_{j+1}$ for $j = 1, \ldots, p-1$. Moreover, let $B'(t) := \{u \in S_2 \cap T' : w_u d(u) > t\}$ for a fixed parameter $t$. Note that, if $t, t' \in (t_j, t_{j+1})$, then
\( B'(t) = B'(t') \). Let \( \alpha(t) := d(u_{j+1}) \) if \( t \in (t'_j, t'_{j+1}] \) be a function on \( t \). The function \( \alpha(t) \) is a nonincreasing function.

To assure that \( \bar{w}_v \) is larger than or equal to \( \bar{w}^2_{l(2)} \), we have to reduce the weight of vertices in \( B'(t) \) such that their weighted distance to \( v^* \) is \( \frac{t}{d(v)} \alpha(t) \), i.e., the modifications are \( w_u - \frac{\alpha(t)}{d(v)d(u)} \cdot t \) for all \( u \in B'(t) \).

Hence, the cost for reducing the weights of vertices in \( J_2(v^*) \cap T' \) to \( \bar{w}_v \) is:

\[
\sum_{v' \in B'(t)} c_{v'} \left( w_{v'} - \frac{\alpha(t)}{d(v)d(v')} \cdot t \right).
\]

By summing up (4), (5), (6), we write the cost function as

\[
C(t) := c_v \left( \frac{t}{d(v)} - w_v \right) + \sum_{u \in B(t)} c_u \left( w_u - \frac{t}{d(u)} \right) + \sum_{v' \in B'(t)} c_{v'} \left( w_{v'} - \frac{\alpha(t)}{d(v)d(v')} \cdot t \right),
\]

\[
= \left( \frac{c_v}{d(v)} - \sum_{u \in B(t)} \frac{c_u}{d(u)} - \sum_{v' \in B'(t)} \frac{c_{v'}}{d(v') \alpha(t)} \right) \cdot t + \text{constant}, \text{ for } t \in ([t, \bar{t}].
\]

The cost function is indeed a piecewise linear function with breakpoints in \( (B \cup B') \cap [t, \bar{t}] \). Furthermore, considering \( t' \geq t \), then \( B(t') \subset B(t) \). Hence, \( \alpha(t') \leq \alpha(t) \). Hence, the slope of \( C(t) \) is increasing or \( C(t) \) is a convex function. By the convexity of the cost function, we can find the optimal value of \( C(t) \) by pruning a half of elements in \( B \cup B' \) and it yields a linear time algorithm; see e.g., [4].

Hence, the sorting of elements in \( B \), \( B' \) costs \( O(n \log n) \) time, we can solve the problem \( (P_{v^1}^1) \) in \( O(n \log n) \) time.

After solving all subproblems \( (P_{v^1}^1) \) for all \( v \) in \( T \setminus T' \), we obtain the minimum cost and the corresponding solution is also the optimal solution for \( (P^1) \). As the subproblem \( (P_{v^1}^1) \) is solvable in \( O(n \log n) \) time for all \( v \) in \( T \setminus T' \), the complexity of \( (P^1) \) is \( O(n^2 \log n) \). Let us illustrate the algorithm for solving a subproblem \( (P_{v^1}^1) \) in the following.

**Example 2. An enumerical example for a subproblem \( (P_{v^1}^1) \)**

We consider subproblem \( (P_{v^1}^1) \) for an instance of a tree \( T \) in Figure 1 (see Example 1). Here, the prespecified vertex \( v^1 \) is not a 2-group 1-median of \( T \). The cost coefficients and modification bounds are given in Table 2. Furthermore, \( \bar{t}_1 := 8 \).

\[
\begin{array}{c|ccccc}
(j) & 1 & 2 & 3 & 4 & 5 \\
q_j & 1 & 5 & 3 & 2 & - \\
q^*_j & 10 & 5 & 1 & - & - \\
\end{array}
\]

**Table 2. An instance of the subproblem \( (P_{v^1}^1) \)**

Let \( t := \bar{w}_1 d(v^1) \), then \( t := 4 \) and \( \bar{t} := 20 \). We can also compute the set \( B := \{8, 10, 20\} \), \( D := \{15, 18\} \). Furthermore, we get the corresponding distance and weights \( \bar{d}_1 := 1, \bar{d}_2 := 2; \bar{w}_1 := 15, \bar{w}_2 := 9; \bar{L} := \{9\}, \bar{B} := \{18\} \). The set of breakpoints is \{8, 10, 18, 20\}. 

By elementary computation, we get the corresponding costs at the breakpoints in Table 3. Hence, in the optimal solution to \((P_1^1)\) is \(p_1^4 = 3, q_1^1 = 5, q_2^2 = 4,\) and other vertices are unchanged.

**Problem \((P^2)\)**
We first induce instances of trees \(T^{S_j}\), for \(j = 1, 2,\) where \(w_v = \bar{p}_v = \bar{q}_v := 0\) for all \(v \in S_k, k \in \{1,2\}\setminus\{j\}\) and all vertices in \(S_j\) get the same weights, costs, and modification bounds from the ones in \(T\). We solve the inverse 1-center problem on \(T^{S_j}\), denoted by \(P^2(S_j)\), for \(j = 1, 2\) with the prespecified vertex \(v^*\). It is trivial that the optimal objective value of \((P^2)\) equals the sum of optimal values of \(P^2(S_1)\) and \(P^2(S_2)\). Moreover, it requires quadratic time to solve the inverse 1-center problem on a tree with variable vertex weights; see Nguyen and Anh [25]. Hence, the complexity of \((P^2)\) is quadratic.

**Table 3.** Cost values at breakpoints

| \(t\) | 8   | 10  | 18  | 20  |
|-------|-----|-----|-----|-----|
| \(C(t)\) | 46  | 44  | 49  | 56  |

To summarize, we solve the two problem \((P_1)\) and \((P_2)\) and get the minimum objective value among them which is also the optimal objective of the inverse 2-group 1-median problem on \(T\); see Algorithm 1. As the complexity of \((P_1)\) and \((P_2)\) are respectively \(O(n^2 \log n)\) and \(O(n^2)\), we get the main result of this section as follows.

**Theorem 3.2.** The inverse 2-group 1-median on trees with variable vertex weights can be solved in \(O(n^2 \log n)\) time.

4. **Conclusions.** We addressed the inverse \(m\)-group 1-median problem on trees with variable vertex weights. Despite of the polynomial solvability of its two special cases, say the inverse 1-median and the inverse 1-center problems, the problem is \(NP\)-hard in general. We also developed an \(O(n^2 \log n)\) algorithm for the corresponding problem with exactly two groups. For future research topic, one may concern some other special the cases of the inverse group median problem on trees with polynomial algorithms or consider the problem under other norms. Also, the inverse group center location problem is a promising topic.
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