Forces at the Sea Bed using a Finite Element Solution of the Mild Slope Wave Equation

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Abstract

An algorithm to compute forces at the sea bed from a finite element solution to the mild slope wave equation is devised in this work. The algorithm is best considered as consisting of two logical parts: The first is concerned with the computation of the derivatives to a finite element solution, given the associated mesh; the second is a bi–quadratic least squares fit which serves to model the sea bed locally in the vicinity of a node. The force at the sea bed can be quantified in terms of either lift and drag, the likes of Stokes’ formula or traction. While the latter quantity is the most desirable, the direct computation of tractions at the sea bed is controversial in the context of the mild slope wave equation as a result of the irrotationality implied by the use of potentials. This work ultimately envisages a “Monte Carlo” approach using wave induced forces to elucidate presently known heavy mineral placer deposits and, consequently, to predict the existence of other deposits which remain as yet undiscovered.

Keywords: waves; sediment; Berkhoff equation; mild slope wave equation; lift; drag; Stokes’ formula; traction; placer deposits; heavy minerals; waves; refraction; diffraction; reflection; interference; standing waves; resonance

1 Introduction

The mild slope wave equation is a model for break water diffraction, reflection and refraction which has been used with considerable success for the quantitative prediction of ocean dynamics in a great variety of circumstances (see Booij [3] for limitations). The model is linearised, assumes the sea bed to be locally flat, uses potential theory and there
is no turbulence (see Berkhoff [1], Bettess and Zienkiewicz [2], Gonsalves [10]). Despite this, a remarkable resemblance between the geometries of some heavy mineral placer deposits and those of computer-generated wave height envelopes (predicted using the mild slope wave equation for waves moving over fairly simple, idealised bathymetries) is documented in Childs and Shillington [8]. Wave reflection, refraction, diffraction and resonance would appear to have played a major concentrating role in the formation of these deposits.

An algorithm to compute forces at the sea bed from a finite element solution to the mild slope wave equation and the associated mesh is devised in this work. Two main components are fundamental to the logic of the algorithm. One is concerned with the computation of the derivatives to a finite element solution, given the associated mesh; the other is a bi-quadratic least squares fit which serves to model the sea bed locally in the vicinity of a node. There is a considerable advantage in developing a routine to compute the derivatives separate from the existing code (adapting the code to an alternative wave model would be one example). The computation of the wave number using a Newton–Raphson scheme and other components essential to the algorithm are also discussed.

This work ultimately envisages a “Monte Carlo” approach using wave induced forces to elucidate presently known heavy mineral placer deposits and, consequently, to predict the existence of other deposits which remain as yet undiscovered. The intention is therefore to use the results in an empirical or qualitative (as opposed to quantitative) manner.

1.1 Traction and the Boundary Layer Controversy

The flow forces at the sea bed can be quantified in terms of either lift and drag, Stokes’ formula or traction. While the latter is most desirable in physical terms, the direct computation of traction at the sea bed is controversial in the context of the mild slope wave equation as a result of the irrotationality implied by the use of potentials and the consequent lack of a thorough treatment of the boundary layer. Computing the traction indirectly (by using the solution to the mild slope wave equation as a boundary condition in a model more suited to boundary layer application eg. Childs [4], [5], [7] and [6]), though not impossible, is computationally exhorbitant. The aforementioned controversy, practicality and the observed negligible effect of the pressure gradient on the mechanical character of fluid motion in the vicinity of the bed (Yalin [17]) suggest that velocity¹ might be the more attractive option. Stokes’ formula is probably the most conventional option advocated by classical texts such as Landau and Lifshitz [12]. A comparative study involving all four approaches is ultimately what is required.

The traction formulae are by far the most complicated and they incorporate all the elements necessary for the calculation of the other quantities mentioned. Lift, drag and the quantities necessary to evaluate Stokes’ formula are all incidental to the traction calculation and it is for this reason that the traction algorithm is supplied as the central theme to this work.

¹to which lift and drag are squarely proportional
This work is also concerned with the stability of fairly small, sediment grains, grains whose threshold is presently reached at deep to intermediate wave depths where the orbitals are relatively small. Scaling arguments suggest that an oscillatory flow in which oscillations are relatively small in comparison to the wave length is a potential flow to first approximation. The lateral extent of the sediment deposits of interest, taken in conjunction with observations that the convective term is negligible (Yalin [17]), suggests a fairly uniform boundary layer may be assumed. It may therefore be possible to ignore the exact physics of the boundary layer at the scale on which the sediments of interest occur, leaving the way open for the qualitative use of a traction calculated directly from the solution of the mild slope wave equation. Under these circumstances the tracional flow driving, what is assumed to be a relatively thin and uniform boundary layer is what is being considered. The modelled motion for a linear sea bed would be that of a number of layers of fluid slapping up and down, a kind of pumping action on the sea bed.

2 Stress in Terms of a Solution to the Mild Slope Wave Equation

The approximated velocity potential based on the solution to the mild slope wave equation is

\[ \Phi(x_1, x_2, x_3, t) = \text{Re}\{f^h(x_1, x_2)e^{-i\omega t}\} Z(x_3, h) \]  

(1)

where \( \Phi \) is the velocity potential, \( \text{Re}\{ \} \) indicates the real part of a complex number, \( f^h \) is the finite element solution to the mild slope wave equation, \( Z \) is a function which describes attenuation with depth, \( x_3 \) is the vertical coordinate measured from mean water level, \( h \) is the depth below mean water level and \( \omega \) is a frequency. The stress tensor is given by the constitutive relation

\[ \sigma = -pI + \mu (\nabla v + (\nabla v)^\dagger) \]

where, in terms of the approximation [11],

\[
\begin{align*}
\nu_{1,1} &= \text{Re}\left\{ \left( \frac{\partial^2 f^h}{\partial x_1^2} Z + 2 \frac{\partial f^h}{\partial x_1} \frac{\partial Z}{\partial x_1} + f^h \frac{\partial^2 Z}{\partial x_1^2} \right) e^{-i\omega t} \right\} \\
\nu_{2,2} &= \text{Re}\left\{ \left( \frac{\partial^2 f^h}{\partial x_2^2} Z + 2 \frac{\partial f^h}{\partial x_2} \frac{\partial Z}{\partial x_2} + f^h \frac{\partial^2 Z}{\partial x_2^2} \right) e^{-i\omega t} \right\} \\
\nu_{3,3} &= \text{Re}\left\{ \left( f^h \frac{\partial^2 Z}{\partial x_3^2} \right) e^{-i\omega t} \right\} \\
\nu_{1,2} &= \text{Re}\left\{ \left( \frac{\partial^2 f^h}{\partial x_2 \partial x_1} Z + \frac{\partial f^h}{\partial x_2} \frac{\partial Z}{\partial x_1} + \frac{\partial f^h}{\partial x_1} \frac{\partial Z}{\partial x_2} + f^h \frac{\partial^2 Z}{\partial x_2 \partial x_1} \right) e^{-i\omega t} \right\} \\
\nu_{1,3} &= \text{Re}\left\{ \left( \frac{\partial f^h}{\partial x_3} \frac{\partial Z}{\partial x_1} + f^h \frac{\partial^2 Z}{\partial x_3 \partial x_1} \right) e^{-i\omega t} \right\} \\
\nu_{2,3} &= \text{Re}\left\{ \left( \frac{\partial f^h}{\partial x_3} \frac{\partial Z}{\partial x_2} + f^h \frac{\partial^2 Z}{\partial x_3 \partial x_2} \right) e^{-i\omega t} \right\}.
\end{align*}
\]
Forthcoming sections are devoted to the modelling and computation of these values.

3 The Analytic Derivatives of a Finite Element Solution

The finite element method approximates a solution to a problem in a finite dimensional subspace $\bar{F}^h$. Thus for $f^h \in \bar{F}^h$,

$$f^h(\mathbf{x}) = \sum_{i=1}^{n_{\text{Point}}} c_i \psi_i(\mathbf{x})$$

where $n_{\text{Point}}$ is the total number of nodes, the $c_i$'s are the degrees of freedom (the discrete solution) and the $\psi_i(\mathbf{x})$'s are the shape functions. The local approximation on each element is

$$f^h(\mathbf{x}) |_{\Omega_e} = \sum_{i=1}^{n_{\text{Node}}} c_i^{(e)} \psi_i^{(e)}(\mathbf{x})$$

where $n_{\text{Node}}$ is the number of nodes per element, the $c_i^{(e)}$'s are the local degrees of freedom, the $\psi_i^{(e)}(\mathbf{x})$'s are the localised shape functions and $\Omega_e$ is the element in question. Differentiating both sides of the above equation,

$$\left. \frac{\partial^j f^h}{\partial x_k \cdots \partial x_l} \right|_{\Omega_e} = \sum_{i=1}^{n_{\text{Node}}} c_i^{(e)} \frac{\partial^j \psi_i^{(e)}}{\partial x_k \cdots \partial x_l}.$$  (2)

The problem of calculating the derivatives of a finite element solution therefore translates directly into one of calculating the derivatives of the localised shape functions on each element. These localised shape functions are defined in terms of a basis as follows

$$\psi_i^{(e)}(\mathbf{x}(\xi)) \equiv \phi_i(\xi).$$

where the $\{\phi_i(\xi)\}$ is the basis defined on the master element domain, $\hat{\Omega}$. In this way the problem can be transferred into one in terms of the master element.

3.1 The Two–Dimensional Case

For a two dimensional problem

$$\left[ \frac{\partial f^h}{\partial x_1}, \frac{\partial f^h}{\partial x_2}, \frac{\partial^2 f^h}{\partial x_1^2}, \frac{\partial^2 f^h}{\partial x_2^2}, \frac{\partial^2 f^h}{\partial x_1 \partial x_2} \right] = \sum_{i=1}^{n_{\text{Node}}} c_i^{(e)} \left[ \frac{\partial \psi_i^{(e)}}{\partial x_1}, \frac{\partial \psi_i^{(e)}}{\partial x_2}, \frac{\partial^2 \psi_i^{(e)}}{\partial x_1^2}, \frac{\partial^2 \psi_i^{(e)}}{\partial x_2^2}, \frac{\partial^2 \psi_i^{(e)}}{\partial x_1 \partial x_2} \right]$$

(by equation (2)).  (3)

\(^1\)Notice that the symmetry of the stress tensor is preserved when introducing the approximation.
Applying the chain rule the first derivative of the basis with respect to the first variable is
\[
\frac{\partial \phi_i}{\partial \xi_1} = \frac{\partial \psi_i^{(e)}}{\partial x_1} \frac{\partial x_1}{\partial \xi_1} + \frac{\partial \psi_i^{(e)}}{\partial x_2} \frac{\partial x_2}{\partial \xi_1} = \left[ \frac{\partial x_1}{\partial \xi_1}, \frac{\partial x_2}{\partial \xi_1} \right] \begin{bmatrix} \frac{\partial \psi_i^{(e)}}{\partial x_1} \\ \frac{\partial \psi_i^{(e)}}{\partial x_2} \end{bmatrix}.
\]

The first derivative of the basis with respect to the second variable is
\[
\frac{\partial \phi_i}{\partial \xi_2} = \frac{\partial \psi_i^{(e)}}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial \psi_i^{(e)}}{\partial x_2} \frac{\partial x_2}{\partial \xi_2} = \left[ \frac{\partial x_1}{\partial \xi_2}, \frac{\partial x_2}{\partial \xi_2} \right] \begin{bmatrix} \frac{\partial \psi_i^{(e)}}{\partial x_1} \\ \frac{\partial \psi_i^{(e)}}{\partial x_2} \end{bmatrix}.
\]

The second derivative of the basis with respect to the first variable is
\[
\frac{\partial^2 \phi_i}{\partial \xi_1^2} = \frac{\partial}{\partial \xi_1} \left\{ \frac{\partial \psi_i^{(e)}}{\partial x_1} \frac{\partial x_1}{\partial \xi_1} + \frac{\partial \psi_i^{(e)}}{\partial x_2} \frac{\partial x_2}{\partial \xi_1} \right\} = \frac{\partial^2 \psi_i^{(e)}}{\partial x_1^2} \left( \frac{\partial x_1}{\partial \xi_1} \right)^2 + \frac{\partial^2 \psi_i^{(e)}}{\partial x_2 \partial x_1} \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_1} + \frac{\partial \psi_i^{(e)}}{\partial x_1} \frac{\partial^2 x_1}{\partial \xi_1^2} + \frac{\partial^2 \psi_i^{(e)}}{\partial x_2^2} \left( \frac{\partial x_2}{\partial \xi_1} \right)^2 + \frac{\partial^2 \psi_i^{(e)}}{\partial x_2 \partial x_1} \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2} + \frac{\partial \psi_i^{(e)}}{\partial x_1} \frac{\partial^2 x_2}{\partial \xi_1^2} + \frac{\partial^2 \psi_i^{(e)}}{\partial x_2 \partial x_1} \frac{\partial x_1}{\partial \xi_2} \frac{\partial x_2}{\partial \xi_2} + \frac{\partial \psi_i^{(e)}}{\partial x_1} \frac{\partial^2 x_2}{\partial \xi_2^2} + \frac{\partial^2 \psi_i^{(e)}}{\partial x_2^2} \left( \frac{\partial x_2}{\partial \xi_2} \right)^2 \right\}
\[
\left[ \begin{array}{c} \frac{\partial \psi_i^{(e)}}{\partial x_1} \\ \frac{\partial \psi_i^{(e)}}{\partial x_2} \end{array} \right].
\]

The second derivative of the basis with respect to the second variable is
\[
\frac{\partial^2 \phi_i}{\partial \xi_2^2} = \frac{\partial}{\partial \xi_2} \left\{ \frac{\partial \psi_i^{(e)}}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial \psi_i^{(e)}}{\partial x_2} \frac{\partial x_2}{\partial \xi_2} \right\}.
The cross derivative of the basis is

\[
\frac{\partial^2 \phi_i}{\partial \xi_1 \partial \xi_2} = \frac{\partial}{\partial \xi_1} \left\{ \frac{\partial \psi_i^{(e)}}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial \psi_i^{(e)}}{\partial x_2} \frac{\partial x_2}{\partial \xi_2} \right\} \\
= \frac{\partial^2 \psi_i^{(e)}}{\partial x_1^2} \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial^2 \psi_i^{(e)}}{\partial x_2^2} \frac{\partial x_2}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2} + \frac{\partial^2 \psi_i^{(e)}}{\partial x_1 \partial x_2} \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2} + \frac{\partial^2 \psi_i^{(e)}}{\partial x_1 \partial x_2} \frac{\partial x_2}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_2}
\]

\[
= \left[ \frac{\partial^2 x_1}{\partial \xi_1 \partial \xi_2}, \frac{\partial^2 x_2}{\partial \xi_1 \partial \xi_2}, \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_2}, \frac{\partial x_2}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2}, \left( \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2} + \frac{\partial x_1}{\partial \xi_2} \frac{\partial x_2}{\partial \xi_1} \right) \right].
\]
Collecting the above expressions together and re-expressing them in a vector–matrix form,

\[
\begin{bmatrix}
\frac{\partial \psi_i}{\partial \xi_1} \\
\frac{\partial \psi_i}{\partial \xi_2} \\
\ldots \\
\frac{\partial \psi_i}{\partial \xi_n}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} & 0 & 0 & 0 \\
\frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} & 0 & 0 & 0 \\
\ldots \\
\frac{\partial x_1}{\partial \xi_n} & \frac{\partial x_2}{\partial \xi_n}
\end{bmatrix}
\begin{bmatrix}
\psi_i^{(e)} \\
\psi_i^{(e)} \\
\psi_i^{(e)} \\
\psi_i^{(e)}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \phi_i}{\partial \xi_1} \\
\frac{\partial \phi_i}{\partial \xi_2} \\
\ldots \\
\frac{\partial \phi_i}{\partial \xi_n}
\end{bmatrix}
\begin{bmatrix}
\psi_i^{(e)} \\
\psi_i^{(e)} \\
\psi_i^{(e)} \\
\psi_i^{(e)}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_1}{\partial \xi_1} \\
\frac{\partial x_2}{\partial \xi_2} \\
\ldots \\
\frac{\partial x_n}{\partial \xi_n}
\end{bmatrix}.
\]

(4)

It follows from equation (4) that

\[
\begin{bmatrix}
\frac{\partial \psi_i^{(e)}}{\partial x_1} & \frac{\partial \psi_i^{(e)}}{\partial x_2} & \frac{\partial \psi_i^{(e)}}{\partial x_1^2} & \frac{\partial \psi_i^{(e)}}{\partial x_2^2} & \frac{\partial \psi_i^{(e)}}{\partial x_1 \partial x_2}
\end{bmatrix}
\begin{bmatrix}
\psi_i^{(e)} \\
\psi_i^{(e)} \\
\psi_i^{(e)} \\
\psi_i^{(e)} \\
\psi_i^{(e)}
\end{bmatrix}
= [Q(\xi)]^{-1} \mathbf{d}^{(e)}(\xi).
\]

This equation is the formula by which the much desired shape function derivatives are calculated. Substituting it into equation (3)

\[
\begin{bmatrix}
\frac{\partial f^{(e)}}{\partial x_1} & \frac{\partial f^{(e)}}{\partial x_2} & \frac{\partial f^{(e)}}{\partial x_1^2} & \frac{\partial f^{(e)}}{\partial x_2^2} & \frac{\partial f^{(e)}}{\partial x_1 \partial x_2}
\end{bmatrix}
\begin{bmatrix}
x_i(\xi)
\end{bmatrix}
= \sum_{i=1}^{nNode} c_i^{(e)} [Q(\xi)]^{-1} \mathbf{d}^{(e)}(\xi).
\]

(5)

This equation is the formula by which the first, second and cross derivatives of a finite element solution to a two-dimensional problem are calculated.

The Matrix Entries: The matrix entries may all be formulated by taking derivatives of the finite element mapping. Taking the opportunity to develop a systematic notation for the purposes of the algorithm simultaneously,

\[
x_i(\xi) = \sum_{k=1}^{nNode} \phi_k(\xi) (x_i | \text{node } k) = \sum_{k=1}^{nNode} \text{shape}(k, 1) * eCoord(k, i)
\]

(6)

where \((x_i | \text{node } k)\) is the \(i\)th coordinate of node \(k\), as is \(eCoord(k, i)\), the \(\phi_k(\xi)\)'s are the basis, as are the \(\text{shape}(k, 1)\)'s. The matrix entries are calculated according to

\[
\frac{\partial x_i}{\partial \xi_j} = \sum_{k=1}^{nNode} \frac{\partial \phi_k}{\partial \xi_j} (x_i | \text{node } k) = \sum_{k=1}^{nNode} \text{shape}(k, j + 1) * eCoord(k, i)
\]

\[
\frac{\partial^2 x_i}{\partial \xi_j^2} = \sum_{k=1}^{nNode} \frac{\partial^2 \phi_k}{\partial \xi_j^2} (x_i | \text{node } k) = \sum_{k=1}^{nNode} \text{shape}(k, j + 3) * eCoord(k, i)
\]

\[
\frac{\partial^2 x_i}{\partial \xi_i \partial \xi_2} = \sum_{k=1}^{nNode} \frac{\partial^2 \phi_k}{\partial \xi_i \partial \xi_2} (x_i | \text{node } k) = \sum_{k=1}^{nNode} \text{shape}(k, 6) * eCoord(k, i),
\]

(7)
where the definition of the \( \text{shape}(k, j) \)'s follows from the equations above.

**The Derivatives of the Basis:** Obtaining formulae for the various derivatives of the basis is an elementary exercise in differentiation. The resulting formulae in the particular instance of the 8–noded quadrilateral basis (appendix, page 21) are listed in the appendix on page 21. A combined structure–flow chart diagram of the algorithm which computes the derivatives of a finite element solution is given on page 9.

### 3.2 Some Test Examples

Finite element approximations for a number of simple, analytic surfaces were devised by evaluating the self same functions at the nodes of the test mesh depicted in Figure 2. A comparison of the various derivatives of the approximated surface with those of the analytic function itself confirmed the algorithm to be working.

**Test 1:** For the surface

\[
f(x_1, x_2) = 1,
\]

the first, second and crossed derivatives were obtained to specified precision (approximately 16 significant figures) at all thirteen nodes.

![Figure 2: Test Mesh](image)
Figure 1: Combined Structure–Flow Chart Diagram of an Algorithm which Computes the Derivatives of a Finite Element Solution Analytically.
Test 2: For the surface
\[ f(x_1, x_2) = x_1, \]
the first, second and crossed derivatives were obtained to specified precision (approximately 16 significant figures) at all thirteen nodes.

Test 3: For the surface
\[ f(x_1, x_2) = x_1^2 - 4x_1 + 3, \]
the first, second and crossed derivatives were obtained to specified precision (approximately 16 significant figures) at all thirteen nodes.

4 The Various Derivatives of \( Z(x_3, h) \)

The function which describes the attenuation with depth is
\[ Z(x_3, h) = \frac{\cosh(\kappa(h + x_3))}{\cosh(\kappa h)} \]
where the \( x_3 \) coordinate is measured from the mean water level, \( h \) is the depth below mean water level, \( \pi \) is the usual mathematical constant and \( \kappa \) is defined by the non-dimensional dispersion relation
\[ \frac{1}{\kappa} = \text{Tanh}(\kappa h). \]

Observing that \( h = h(x_1, x_2) \) and \( \kappa = \kappa(x_1, x_2) \), the first, second and cross derivatives are accordingly formulated in the appendix on page 23. At the sea bed where \( x_3 = -h \):

\[ Z \bigg|_{x_3=-h} = \frac{1}{\cosh(\kappa h)}, \]
\[ \frac{\partial Z}{\partial x_1} \bigg|_{x_3=-h} = \frac{-1}{\cosh(\kappa h)} \left( \frac{\partial h}{\partial x_1} + \frac{h}{\kappa} \frac{\partial \kappa}{\partial x_1} \right), \]
\[ \frac{\partial Z}{\partial x_2} \bigg|_{x_3=-h} = \frac{-1}{\cosh(\kappa h)} \left( \frac{\partial h}{\partial x_2} + \frac{h}{\kappa} \frac{\partial \kappa}{\partial x_2} \right), \]
\[ \frac{\partial Z}{\partial x_3} \bigg|_{x_3=-h} = 0, \]
\[ \frac{\partial^2 Z}{\partial x_1^2} \bigg|_{x_3=-h} = \frac{1}{\cosh(\kappa h)} \left[ (1 + \kappa^2) \left( \frac{\partial h}{\partial x_1} \right)^2 - \frac{\partial^2 h}{\partial x_1^2} + h \left( 2\kappa - 1 - \sinh(\kappa h) - \frac{1}{\kappa} \right) \right. \]
\[ \left. - \frac{1}{h\kappa} \frac{\partial h}{\partial x_1} \frac{\partial \kappa}{\partial x_1} + \frac{h}{\kappa} \left( \frac{2}{\kappa} + h\kappa \right) \left( \frac{\partial \kappa}{\partial x_1} \right)^2 - \frac{h}{\kappa} \frac{\partial^2 \kappa}{\partial x_1^2} \right], \]
\[ \frac{\partial^2 Z}{\partial x_2^2} \bigg|_{x_3=-h} = \frac{1}{\cosh(\kappa h)} \left[ (1 + \kappa^2) \left( \frac{\partial h}{\partial x_2} \right)^2 - \frac{\partial^2 h}{\partial x_2^2} + h \left( 2\kappa - 1 - \sinh(\kappa h) - \frac{1}{\kappa} \right) \right. \]
\[ \left. - \frac{1}{h\kappa} \frac{\partial h}{\partial x_2} \frac{\partial \kappa}{\partial x_2} + \frac{h}{\kappa} \left( \frac{2}{\kappa} + h\kappa \right) \left( \frac{\partial \kappa}{\partial x_2} \right)^2 - \frac{h}{\kappa} \frac{\partial^2 \kappa}{\partial x_2^2} \right]. \]
\[
\frac{\partial^2 Z}{\partial x_1 \partial x_2} \bigg|_{x_3=-h} = \frac{1}{\cosh(\kappa h)} \left[ \left( 1 + \kappa^2 \right) \frac{\partial h}{\partial x_1} \frac{\partial h}{\partial x_2} - \frac{\partial^2 h}{\partial x_1 \partial x_2} + \left( h \kappa - \frac{h}{\kappa} - \frac{1}{\kappa} \right) \right] \\
\frac{\partial^2 Z}{\partial x_1 \partial x_3} \bigg|_{x_3=-h} = \frac{\kappa}{\cosh(\kappa h)} \left( \kappa \frac{\partial h}{\partial x_1} + h \frac{\partial \kappa}{\partial x_1} \right), \\
\frac{\partial^2 Z}{\partial x_2 \partial x_3} \bigg|_{x_3=-h} = \frac{\kappa}{\cosh(\kappa h)} \left( \kappa \frac{\partial h}{\partial x_2} + h \frac{\partial \kappa}{\partial x_2} \right) \quad \text{and} \\
\frac{\partial^2 Z}{\partial x_3^2} \bigg|_{x_3=-h} = \frac{(\kappa)^2}{\cosh(\kappa h)}.
\]

5. The Nodal Values of \( \kappa(x_1, x_2) \) and its Various Derivatives

Calculating the wave number, \( \kappa \), for a given depth is standard procedure. The dispersion relation

\[
\frac{1}{\kappa} = \tanh(\kappa h)
\]

is conventionally solved using Newton’s method. The resulting iterative scheme is,

\[
\kappa^{i+1} = \kappa^i - \frac{\kappa^i \tanh(\kappa^i h) - 1}{\tanh(\kappa^i h) + h \kappa^i (1 - \tanh^2(\kappa^i h))}
\]

where the superscript \( i \) denotes the successive iteration from which a given solution was obtained. The initial guess usually taken is

\[
\kappa = \frac{2\pi}{\lambda_0},
\]

where \( \lambda_0 \) is deep water wave-length.

Once this has been accomplished for each of the \( n \) nodes belonging to a given element, there is no reason why these nodal values shouldn’t be regarded as a discrete solution in order to determine the derivatives. Substituting into equation (5)

\[
\left[ \frac{\partial \kappa^h}{\partial x_1}, \frac{\partial \kappa^h}{\partial x_2}, \frac{\partial^2 \kappa^h}{\partial x_1^2}, \frac{\partial^2 \kappa^h}{\partial x_1 \partial x_2}, \frac{\partial^2 \kappa^h}{\partial x_2^2} \right]_{\mathbf{x}(\xi)} = \sum_{j=1}^{n_{\text{Node}}} \kappa_{\text{node } j} \left[ Q(\xi) \right]^{-1} d^j(\xi).
\]

The derivatives of \( \kappa \) can, alternatively, be calculated by the implicit differentiation of the dispersion relation. Considering \( [Q(\xi)]^{-1} d^j(\xi) \) must be calculated at each node \( j \), the former method is the more efficient.
6 The Sea Bed at a Node

Because nodes do not necessarily coincide with individual points of bathymetry measurement, and for the purposes of taking derivatives, a “sea bed” needs to be interpolated locally. A straightforward fit of an $n$ degree polynomial to the $n$ data points nearest a node, the use of cubic splines and a local least squares fit were all considered as possible ways to interpolate bathymetry between individual points of bathymetry measurement.

The manner in which available data was collected proved to be a deciding factor in the final choice. While the use of cubic splines is fairly established in the modelling of known surfaces, the problem with unknown surfaces is that slope information at the “knots” is required. Such information is never available in the raw bathymetry data. A further factor to consider is that the actual data sampling intervals range anywhere from slightly, to highly, irregular. One advantage of the least squares method is that a large data set can be taken into account, even individual data points weighted according to their proximity.

The argument against fitting an $n$ degree polynomial exactly to the nearest $n$ points in the vicinity of a given node is that the use of a high degree polynomial will result in a totally fictitious model in cases where the actual surface is of “lower degree” than the polynomial used, alternatively, where the sampling intervals are poor. Fitting a low degree polynomial surface could result in the use of an unrepresentative data sample. The solution is therefore to fit a fairly simple, low degree polynomial surface to a larger data set. This can be accomplished using the least squares method. A method based on the least absolute value of the errors is preferable in theory, of course, but not in practice.

Bi–quadratic and bi–cubic surfaces were experimented with using the method of least squares. The former was decided to be the better choice. Irregular data was found to allow extreme cases of the “wiggle” effect in the bi–cubic case. A bi–cubic surface also requires a far greater, hence locally less relevant data set and its greater degree is therefore not necessarily an advantage. In a real–life data comparison between actual measured depths, the depths predicted using cubic splines and those predicted using a local, least squares, bi–quadratic fit, a limited inspection suggested the least squares bi–quadratic fit to be superior.

6.1 The Least Squares Fit of a Bi–Quadratic Function

A generalised bi–quadratic equation has the form

$$h(x, y) = c_1 + c_2 y + c_3 x + c_4 xy + c_5 y^2 + c_6 x^2$$

or when written as the dot product of two vectors,

$$h(x, y) = [1, y, x, xy, y^2, x^2] \cdot [c_1, c_2, c_3, c_4, c_5, c_6]. \quad (8)$$

A least squares fit makes, what is in one sense, an optimal choice of the constants, $c_1, c_2, \ldots, c_6$. “In one sense”, in that it minimises the summed squares of the errors at
the data points and not the summed absolute values of these errors. The sum of the squares of the errors, \( \epsilon \), is

\[
\epsilon = \sum_{i=1}^{n} \left( c_1 + c_2 y_i + c_3 x_i + c_4 x_i y_i + c_5 y_i^2 + c_6 x_i^2 - z_i \right)^2
\]

where the \( z_i \) are the \( n \) data points located at \((x_i, y_i)\), the points to which the bi-quadratic equation is to be fitted. In order to minimise \( \epsilon \) with respect to the unknown constants,

\[
\frac{\partial \epsilon}{\partial c_1} = 0 \Rightarrow \sum_{i=1}^{n} (c_1 + c_2 y_i + c_3 x_i + c_4 x_i y_i + c_5 y_i^2 + c_6 x_i^2) = \sum_{i=1}^{n} z_i
\]

\[
\frac{\partial \epsilon}{\partial c_2} = 0 \Rightarrow \sum_{i=1}^{n} y_i (c_1 + c_2 y_i + c_3 x_i + c_4 x_i y_i + c_5 y_i^2 + c_6 x_i^2) = \sum_{i=1}^{n} y_i z_i
\]

\[
\frac{\partial \epsilon}{\partial c_3} = 0 \Rightarrow \sum_{i=1}^{n} x_i (c_1 + c_2 y_i + c_3 x_i + c_4 x_i y_i + c_5 y_i^2 + c_6 x_i^2) = \sum_{i=1}^{n} x_i z_i
\]

\[
\frac{\partial \epsilon}{\partial c_4} = 0 \Rightarrow \sum_{i=1}^{n} x_i y_i (c_1 + c_2 y_i + c_3 x_i + c_4 x_i y_i + c_5 y_i^2 + c_6 x_i^2) = \sum_{i=1}^{n} x_i y_i z_i
\]

\[
\frac{\partial \epsilon}{\partial c_5} = 0 \Rightarrow \sum_{i=1}^{n} y_i^2 (c_1 + c_2 y_i + c_3 x_i + c_4 x_i y_i + c_5 y_i^2 + c_6 x_i^2) = \sum_{i=1}^{n} y_i^2 z_i
\]

\[
\frac{\partial \epsilon}{\partial c_6} = 0 \Rightarrow \sum_{i=1}^{n} x_i^2 (c_1 + c_2 y_i + c_3 x_i + c_4 x_i y_i + c_5 y_i^2 + c_6 x_i^2) = \sum_{i=1}^{n} x_i^2 z_i
\]

Re-expressing the above system of equations in vector–matrix form,

\[
\frac{\partial \epsilon}{\partial \mathbf{c}} = 0 \Rightarrow \begin{bmatrix} 1 & y_1 & x_1 & x_1 y_1 & y_1^2 & x_1^2 \\ y_1 & y_1^2 & x_1 y_1 & x_1 y_1^2 & y_1^3 & x_1^2 y_1 \\ x_1 & x_1 y_1 & x_1^2 & x_1^2 y_1 & x_1 y_1^2 & x_1^3 \\ x_1 y_1 & x_1 y_1^2 & x_1^2 y_1 & x_1^2 y_1^2 & x_1 y_1^3 & x_1^3 y_1 \\ y_1^2 & y_1^3 & x_1 y_1^2 & x_1 y_1^3 & y_1^4 & x_1^2 y_1^2 \\ x_1^2 & x_1^2 y_1 & x_1^2 y_1^2 & x_1^2 y_1^3 & x_1^2 y_1^4 & x_1^4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_n \\ x_1 & x_2 & \cdots & x_n \\ x_1 y_1 & x_2 y_2 & \cdots & x_n y_n \\ y_1^2 & y_2^2 & \cdots & y_n^2 \\ x_1^2 & x_2^2 & \cdots & x_n^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}
\]

Therefore

\[
P \mathbf{c} = \mathbf{Oz},
\]

where \( \mathbf{P} \) and \( \mathbf{O} \) take their respective definitions from the previous equation. Solving for \( \mathbf{c} \),

\[
\mathbf{c} = \mathbf{P}^{-1} \mathbf{Oz}.
\]
Substituting this result into equation (9),

\[ h(x, y) = [1, y, x, xy, y^2, x^2] \cdot P^{-1}Oz \]

where \( h(x, y) \) is the depth modelled locally by this least squares fitted, bi–quadratic equation.

### 6.2 The Various Derivatives of \( h(x, y) \)

The corresponding derivatives of the sea bed are:

\[
\begin{align*}
\frac{\partial h}{\partial x} &= [0, 0, 1, y, 0, 2x] \cdot P^{-1}Oz \\
\frac{\partial h}{\partial y} &= [0, 1, 0, x, 2y, 0] \cdot P^{-1}Oz \\
\frac{\partial^2 h}{\partial x^2} &= [0, 0, 0, 0, 0, 2] \cdot P^{-1}Oz \\
\frac{\partial^2 h}{\partial y^2} &= [0, 0, 0, 2, 0] \cdot P^{-1}Oz \\
\frac{\partial^2 h}{\partial x \partial y} &= [0, 0, 0, 1, 0, 0] \cdot P^{-1}Oz
\end{align*}
\]

### 6.3 The Sea Bed Normal

The components of the sea bed normal are:

\[
\begin{align*}
N_1(x, y) &= -\frac{\partial h}{\partial x} = -[0, 0, 1, y, 0, 2x] \cdot P^{-1}Oz \\
N_2(x, y) &= -\frac{\partial h}{\partial y} = -[0, 1, 0, x, 2y, 0] \cdot P^{-1}Oz \\
N_3(x, y) &= \frac{\partial h}{\partial h} = 1,
\end{align*}
\]

and the unit normal,

\[
n(x, y) = \frac{N(x, y)}{\|N(x, y)\|_2}.
\]

A combined structure–flow chart diagram of an algorithm to model the sea bed locally in the vicinity of a node by way of a least squares fitted bi–quadratic can be found on page 15.

### 6.4 Some Test Examples

Data with which to test the algorithm was generated by evaluating a few simple, analytic surfaces at the required number of points. A comparison of outputted bathymetries and
Figure 3: Combined Structure–Flow Chart Diagram of an Algorithm Used to Model the Sea Bed Locally in the Vicinity of a Node.
sea-bed normals with those of the corresponding analytic function, from which the data was generated, showed the algorithm to be working.

**Tests 1:** The trivial cases

\[ h(x, y) = c, \ c \text{ a constant} \]

were used to generate the input

\[
\begin{align*}
x & & 1 & & 1 & & 2 & & 1 & & 3 & & 2 \\
y & & 1 & & 2 & & 1 & & 3 & & 1 & & 2 \\
z & & c & & c & & c & & c & & c & & c & & c
\end{align*}
\]

The algorithm calculated both depth and normal correct to specified precision (approximately 16 significant figures).

**Test 2:** For a topography containing the arbitrarily selected bi-quadratic

\[ h(x, y) = x^2 + 2y^2 + 3xy + 4x + 5y + 6 \]

the input generated was

\[
\begin{align*}
x & & 1 & & 1 & & 2 & & 1 & & 3 & & 2 \\
y & & 1 & & 2 & & 1 & & 3 & & 1 & & 2 \\
z & & 21 & & 35 & & 31 & & 53 & & 43 & & 48
\end{align*}
\]

The algorithm calculated depth and normal correct to specified precision (approximately 16 significant figures).

**Test 3:** A real-life data comparison was made between actual measured depths, the depths predicted using a local, least squares, bi-quadratic fit and those predicted using cubic splines. A limited inspection suggested the least squares, bi-quadratic fit to be the superior choice.

The algorithm is therefore considered to adequately perform the tasks for which it was designed.

### 7 The Traction Acting on the Sea Bed

The surface force per unit area, exerted by the fluid and acting on the sea bed, is given by

\[ t = \sigma n \]
where \( \sigma \) is the stress tensor at the sea bed and \( n \) is the unit normal to the sea bed. In terms of the quantities discussed and formulated so far,

\[
t_1 = -p n_1 + 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial^2 f^h}{\partial x_1^2} \right\} \frac{1}{\cosh(kh)} n_1 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial f^h}{\partial x_1} \right\} \frac{-1}{\cosh(kh)} \left( \frac{\partial h}{\partial x_1} + \frac{h \, \partial k}{k \, \partial x_1} \right) n_1 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} f^h \right\} \frac{1}{\cosh(kh)} \left[ (1 + \kappa^2) \left( \frac{\partial h}{\partial x_1} \right)^2 - \frac{\partial^2 h}{\partial x_1^2} \right] \\
+ h \left( 2\kappa - 1 - \sinh(kh) - \frac{1}{\kappa} - \frac{1}{h\kappa} \right) \frac{\partial h}{\partial x_1} \frac{\partial k}{\partial x_1} n_1 \\
+ \frac{h \left( 2\kappa \kappa + h\kappa \right)}{k^2 + h^2} \left( \frac{\partial k}{\partial x_1} \right)^2 - \frac{h \, \partial^2 k}{k \, \partial x_1^2} \right] n_1 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial^2 f^h}{\partial x_2 \partial x_1} \right\} \frac{1}{\cosh(kh)} n_2 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial f^h}{\partial x_2} \right\} \frac{-1}{\cosh(kh)} \left( \frac{\partial h}{\partial x_2} + \frac{h \, \partial k}{k \, \partial x_2} \right) n_2 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial f^h}{\partial x_1} \right\} \frac{-1}{\cosh(kh)} \left( \frac{\partial h}{\partial x_2} + \frac{h \, \partial k}{k \, \partial x_2} \right) n_2 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} f^h \right\} \frac{1}{\cosh(kh)} \left[ (1 + \kappa^2) \left( \frac{\partial h}{\partial x_1} \right)^2 - \frac{\partial^2 h}{\partial x_1^2} \right] \\
- \frac{\partial^2 h}{\partial x_1 \partial x_2} + \left( h\kappa - \frac{h \, \partial k}{k \, \partial x_2} + \frac{\partial h}{\partial x_2} \frac{\partial k}{\partial x_2} \right) + \frac{\partial^2 h}{\partial x_1 \partial x_2} \right] n_2 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial^2 f^h}{\partial x_2 \partial x_1} \right\} \frac{1}{\cosh(kh)} n_3 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial f^h}{\partial x_2} \right\} \frac{-1}{\cosh(kh)} \left( \frac{\partial h}{\partial x_1} + \frac{h \, \partial k}{k \, \partial x_1} \right) n_3 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial f^h}{\partial x_1} \right\} \frac{-1}{\cosh(kh)} \left( \frac{\partial h}{\partial x_2} + \frac{h \, \partial k}{k \, \partial x_2} \right) n_3 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} f^h \right\} \frac{1}{\cosh(kh)} \left[ (1 + \kappa^2) \left( \frac{\partial h}{\partial x_1} \right)^2 - \frac{\partial^2 h}{\partial x_1^2} \right] \\
+ \left( h\kappa - \frac{h \, \partial k}{k \, \partial x_1} + \frac{\partial h}{\partial x_1} \frac{\partial k}{\partial x_1} \right) + \frac{\partial^2 h}{\partial x_1 \partial x_2} \right] n_3 \\
+ \frac{\partial^2 h}{\partial x_1 \partial x_2} - \frac{h \, \partial^2 k}{k \, \partial x_1 \partial x_2} \right] n_3 \\
\]

\[
t_2 = -p n_2
\]
\[
\begin{align*}
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial^2 f^h}{\partial x_2 \partial x_1} \right\} & \frac{1}{\cosh(\kappa h)} n_1 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial f^h}{\partial x_2} \right\} & -\frac{1}{\cosh(\kappa h)} \left( \frac{\partial h}{\partial x_1} + \frac{h \partial \kappa}{\kappa \partial x_1} \right) n_1 \\
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial f^h}{\partial x_1} \right\} & -\frac{1}{\cosh(\kappa h)} \left( \frac{\partial h}{\partial x_2} + \frac{h \partial \kappa}{\kappa \partial x_2} \right) n_1
\end{align*}
\]

\[
+ 2\mu \text{Re} \left\{ e^{-i\omega t} f^h \right\} \frac{1}{\cosh(\kappa h)} \left[ (1 + \kappa^2) \left( \frac{\partial h}{\partial x_1} \frac{\partial h}{\partial x_2} - \frac{\partial^2 h}{\partial x_1 \partial x_2} \right) + \left( h^2 \kappa - \frac{h}{\kappa} - \frac{1}{\kappa} \right) \left( \frac{\partial h}{\partial x_1} \frac{\partial \kappa}{\partial x_2} + \frac{h \partial \kappa}{\kappa \partial x_2 \partial x_1} \right) \right] n_1
\]

\[
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial^2 f^h}{\partial x_1^2} \right\} \frac{1}{\cosh(\kappa h)} n_2
\]

\[
+ 2\mu \text{Re} \left\{ e^{-i\omega t} \frac{\partial f^h}{\partial x_2} \right\} -\frac{1}{\cosh(\kappa h)} \left( \frac{\partial h}{\partial x_2} + \frac{h \partial \kappa}{\kappa \partial x_2} \right) n_2
\]

\[
+ 2\mu \text{Re} \left\{ e^{-i\omega t} f^h \right\} \frac{1}{\cosh(\kappa h)} \left[ (1 + \kappa^2) \left( \frac{\partial h}{\partial x_2} \right)^2 - \frac{\partial^2 h}{\partial x_2^2} \right] + h \left( 2\kappa - 1 - \sinh(\kappa h) - \frac{1}{\kappa} - \frac{1}{h\kappa} \right) \left( \frac{\partial h}{\partial x_2} \frac{\partial \kappa}{\partial x_2} + \frac{h \partial \kappa}{\kappa \partial x_2 \partial x_2} \right) + h \left( \frac{2}{\kappa} + \frac{1}{h\kappa} \right) \left( \frac{\partial \kappa}{\partial x_2} \right)^2 - \frac{h \partial^2 \kappa}{\kappa \partial x_2^2} \right] n_2
\]

\[
+ 2\mu \text{Re} \left\{ e^{-i\omega t} f^h \right\} \frac{\kappa}{\cosh(\kappa h)} \left( \frac{\partial h}{\partial x_2} + \frac{h \partial \kappa}{\kappa \partial x_2} \right) n_3
\]

\[
t_3 = -pm_3
\]

\[
+ 2\mu \text{Re} \left\{ e^{-i\omega t} f^h \right\} \frac{\kappa}{\cosh(\kappa h)} \left( \frac{\partial h}{\partial x_1} + \frac{h \partial \kappa}{\kappa \partial x_1} \right) n_1
\]

\[
+ 2\mu \text{Re} \left\{ e^{-i\omega t} f^h \right\} \frac{\kappa}{\cosh(\kappa h)} \left( \frac{\partial h}{\partial x_2} + \frac{h \partial \kappa}{\kappa \partial x_2} \right) n_2
\]

\[
+ 2\mu \text{Re} \left\{ e^{-i\omega t} f^h \right\} \frac{(\kappa)^2}{\cosh(\kappa h)} n_3.
\]

where \( p \) is the pressure, \( \mu \) is the viscosity, \( e \) and \( i \) denote the usual mathematical constants, \( \omega \) is a frequency, \( t \) is time, \( f^h \) is the finite element solution to the mild slope wave equation, \( x_3 \) is the vertical coordinate measured from mean water level, \( h \) is the depth below mean water level (with the exception of the superscript) and \( \kappa \) is the wave number. The derivatives \( \frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2} \) etc. denote the \( \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \) etc. derivatives formulated in Subsection 5.2 on page 14 (the variables \( x \) and \( y \) were used in place of \( x_1 \) and \( x_2 \) so as to avoid confusion with the first and second data points, \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) respectively).

A structure chart of the entire algorithm to compute tractions on the sea bed from a solution to the mild slope wave equation is given on page 13.
Forces at the Sea Bed . . .

Figure 4: Structure Chart of an Algorithm to Compute Tensions from a Solution to the Mild Slope Wave Equation.

1 see Figure 1 on page 9 for detail

2 see Figure 3 on page 15 for detail
8 Conclusions

The derivatives of a finite element solution can be successfully computed on each element using

\[
\begin{bmatrix}
\frac{\partial f^h}{\partial x_1}, & \frac{\partial f^h}{\partial x_2}, & \frac{\partial^2 f^h}{\partial x_1^2}, & \frac{\partial^2 f^h}{\partial x_1 \partial x_2}, & \frac{\partial^2 f^h}{\partial x_2^2}
\end{bmatrix}
\bigg|_{x(\xi)} = \sum_{i=1}^{n_{\text{Node}}} c_i^{(e)} [Q(\xi)]^{-1} d^i(\xi)
\] (10)

where \(Q(\xi)\) and \(d^i(\xi)\) are defined in equation (4) on page 7, \(c_i^{(e)}\) is the discrete solution on each element and \(n_{\text{Node}}\) is the number of nodes on each element.

A bi–quadratic least squares fit (used to model the sea bed locally in the vicinity of a node) can be calculated according to

\[h(x, y) = [1, y, x, xy, y^2, x^2] \cdot P^{-1}Oz\]

where \(P, O\) are the matrices defined on page 13 and \(z\) is the vector of known depths used (sample points) in the vicinity of the node in question. The various derivatives of this sea bed can be calculated using

\[
\begin{align*}
\frac{\partial h}{\partial x} &= [0, 0, 1, y, 0, 2x] \cdot P^{-1}Oz \\
\frac{\partial h}{\partial y} &= [0, 1, 0, x, 2y, 0] \cdot P^{-1}Oz \\
\frac{\partial^2 h}{\partial x^2} &= [0, 0, 0, 0, 2] \cdot P^{-1}Oz \\
\frac{\partial^2 h}{\partial y^2} &= [0, 0, 0, 2, 0] \cdot P^{-1}Oz \\
\frac{\partial^2 h}{\partial x \partial y} &= [0, 0, 0, 1, 0, 0] \cdot P^{-1}Oz.
\end{align*}
\]

The components of the normal are then

\[
\begin{align*}
N_1(x, y) &= -\frac{\partial h}{\partial x} = -[0, 0, 1, y, 0, 2x] \cdot P^{-1}Oz, \\
N_2(x, y) &= -\frac{\partial h}{\partial y} = -[0, 1, 0, x, 2y, 0] \cdot P^{-1}Oz, \\
N_3(x, y) &= \frac{\partial h}{\partial h} = 1
\end{align*}
\]

and the unit normal is

\[
n(x, y) = \frac{N(x, y)}{\|N(x, y)\|_2}.
\]

A bi–quadratic least squares fit would appear to be a superior method to model the sea bed locally in the vicinity of a node when compared to the more conventional approach which involves gridding and the use of cubic splines.

The formula to compute the traction on the sea bed is given on page 16 (in terms of the derivatives of a finite element solution to the mild slope wave equation and a least squares fitted bi–quadratic model of the sea bed in the vicinity of each node). Lift, drag and Stokes’ formula may all be calculated from elements incidental to it.
9 Appendix I

The 8–Noded Quadrilateral Basis

The basis used in conjunction with the 8–noded quadrilateral element is:

\[
\begin{align*}
\phi_1(\xi) &= \frac{1}{4}(\xi_1^2 - \xi_1)(\xi_2^2 - \xi_2) \\
\phi_2(\xi) &= \frac{1}{4}(\xi_1^2 + \xi_1)(\xi_2^2 - \xi_2) \\
\phi_3(\xi) &= \frac{1}{4}(\xi_1^2 + \xi_1)(\xi_2^2 + \xi_2) \\
\phi_4(\xi) &= \frac{1}{4}(\xi_1^2 - \xi_1)(\xi_2^2 + \xi_2) \\
\phi_5(\xi) &= -\frac{1}{2}(\xi_1^2 - 1)(\xi_2^2 - \xi_2) \\
\phi_6(\xi) &= -\frac{1}{2}(\xi_1^2 + \xi_1)(\xi_2^2 - 1) \\
\phi_7(\xi) &= -\frac{1}{2}(\xi_1^2 - 1)(\xi_2^2 - \xi_2) \\
\phi_8(\xi) &= -\frac{1}{2}(\xi_1^2 - \xi_1)(\xi_2^2 - 1)
\end{align*}
\]

The Derivatives of the 8–Noded Quadrilateral Basis

The first derivatives of the 8–noded quadrilateral basis with respect to the first variable are:

\[
\begin{align*}
shape(1, 2) &\equiv \frac{\partial \phi_1}{\partial \xi_1} = \frac{1}{4}(\xi_2 + 2\xi_1 - 2\xi_1\xi_2 - \xi_2^2) \\
shape(2, 2) &\equiv \frac{\partial \phi_2}{\partial \xi_1} = -\xi_1 + \xi_1\xi_2 \\
shape(3, 2) &\equiv \frac{\partial \phi_3}{\partial \xi_1} = \frac{1}{4}(-\xi_2 + 2\xi_1 - 2\xi_1\xi_2 + \xi_2^2) \\
shape(4, 2) &\equiv \frac{\partial \phi_4}{\partial \xi_1} = \frac{1}{2}(1 - \xi_2^2) \\
shape(5, 2) &\equiv \frac{\partial \phi_5}{\partial \xi_1} = \frac{1}{4}(\xi_2 + 2\xi_1 + 2\xi_1\xi_2 + \xi_2^2) \\
shape(6, 2) &\equiv \frac{\partial \phi_6}{\partial \xi_1} = -\xi_1 - \xi_1\xi_2 \\
shape(7, 2) &\equiv \frac{\partial \phi_7}{\partial \xi_1} = \frac{1}{4}(-\xi_2 + 2\xi_1 + 2\xi_1\xi_2 - \xi_2^2) \\
shape(8, 2) &\equiv \frac{\partial \phi_8}{\partial \xi_1} = \frac{1}{2}(\xi_2^2 - 1).
\end{align*}
\]
The first derivatives of the 8–noded quadrilateral basis with respect to the second variable are:

\[
\text{shape}(1, 3) \equiv \frac{\partial \phi_1}{\partial \xi_2} = \frac{1}{4}(\xi_1 + 2\xi_2 - 2\xi_1\xi_2 - \xi_1^2)
\]

\[
\text{shape}(2, 3) \equiv \frac{\partial \phi_2}{\partial \xi_2} = \frac{1}{2}(\xi_1^2 - 1)
\]

\[
\text{shape}(3, 3) \equiv \frac{\partial \phi_3}{\partial \xi_2} = \frac{1}{4}(-\xi_1 + 2\xi_2 + 2\xi_1\xi_2 - \xi_1^2)
\]

\[
\text{shape}(4, 3) \equiv \frac{\partial \phi_4}{\partial \xi_2} = -(\xi_2 - \xi_1\xi_2)
\]

\[
\text{shape}(5, 3) \equiv \frac{\partial \phi_5}{\partial \xi_2} = \frac{1}{4}(\xi_1 + 2\xi_2 + 2\xi_1\xi_2 + \xi_1^2)
\]

\[
\text{shape}(6, 3) \equiv \frac{\partial \phi_6}{\partial \xi_2} = \frac{1}{2}(1 - \xi_1^2)
\]

\[
\text{shape}(7, 3) \equiv \frac{\partial \phi_7}{\partial \xi_2} = \frac{1}{4}(-\xi_1 + 2\xi_2 - 2\xi_1\xi_2 + \xi_1^2)
\]

\[
\text{shape}(8, 3) \equiv \frac{\partial \phi_8}{\partial \xi_2} = (\xi_1\xi_2 - \xi_2).
\]

The second derivatives of the 8–noded quadrilateral basis with respect to the first variable are:

\[
\text{shape}(1, 4) \equiv \frac{\partial^2 \phi_1}{\partial \xi_1^2} = \frac{1}{2}(1 - \xi_2)
\]

\[
\text{shape}(2, 4) \equiv \frac{\partial^2 \phi_2}{\partial \xi_1^2} = \xi_2 - 1
\]

\[
\text{shape}(3, 4) \equiv \frac{\partial^2 \phi_3}{\partial \xi_1^2} = \frac{1}{2}(1 - \xi_2)
\]

\[
\text{shape}(4, 4) \equiv \frac{\partial^2 \phi_4}{\partial \xi_1^2} = 0
\]

\[
\text{shape}(5, 4) \equiv \frac{\partial^2 \phi_5}{\partial \xi_1^2} = \frac{1}{2}(1 + \xi_2)
\]

\[
\text{shape}(6, 4) \equiv \frac{\partial^2 \phi_6}{\partial \xi_1^2} = -(1 + \xi_2)
\]

\[
\text{shape}(7, 4) \equiv \frac{\partial^2 \phi_7}{\partial \xi_1^2} = \frac{1}{2}(1 + \xi_2)
\]

\[
\text{shape}(8, 4) \equiv \frac{\partial^2 \phi_8}{\partial \xi_1^2} = 0.
\]

The second derivatives of the 8–noded quadrilateral basis with respect to the second variable are:

\[
\text{shape}(1, 5) \equiv \frac{\partial^2 \phi_1}{\partial \xi_2^2} = \frac{1}{2}(1 - \xi_1)
\]

\[
\text{shape}(2, 5) \equiv \frac{\partial^2 \phi_2}{\partial \xi_2^2} = 0
\]
\[ \text{shape}(3, 5) \equiv \frac{\partial^2 \phi_3}{\partial \xi_2^2} = \frac{1}{2}(1 + \xi_1) \]
\[ \text{shape}(4, 5) \equiv \frac{\partial^2 \phi_4}{\partial \xi_2^2} = -(1 + \xi_1) \]
\[ \text{shape}(5, 5) \equiv \frac{\partial^2 \phi_5}{\partial \xi_2^2} = \frac{1}{2}(1 + \xi_1) \]
\[ \text{shape}(6, 5) \equiv \frac{\partial^2 \phi_6}{\partial \xi_2^2} = 0 \]
\[ \text{shape}(7, 5) \equiv \frac{\partial^2 \phi_7}{\partial \xi_2^2} = \frac{1}{2}(1 - \xi_1) \]
\[ \text{shape}(8, 5) \equiv \frac{\partial^2 \phi_8}{\partial \xi_2^2} = \xi_1 - 1. \]

The cross derivatives of the 8–noded quadrilateral basis are:

\[ \text{shape}(1, 6) \equiv \frac{\partial^2 \phi_1}{\partial \xi_1 \partial \xi_2} = \frac{1}{4}(1 - 2\xi_1 - 2\xi_2) \]
\[ \text{shape}(2, 6) \equiv \frac{\partial^2 \phi_2}{\partial \xi_1 \partial \xi_2} = \xi_1 \]
\[ \text{shape}(3, 6) \equiv \frac{\partial^2 \phi_3}{\partial \xi_1 \partial \xi_2} = \frac{1}{4}(2\xi_2 - 2\xi_1 - 1) \]
\[ \text{shape}(4, 6) \equiv \frac{\partial^2 \phi_4}{\partial \xi_1 \partial \xi_2} = -\xi_2 \]
\[ \text{shape}(5, 6) \equiv \frac{\partial^2 \phi_5}{\partial \xi_1 \partial \xi_2} = \frac{1}{4}(1 + 2\xi_1 + 2\xi_2) \]
\[ \text{shape}(6, 6) \equiv \frac{\partial^2 \phi_6}{\partial \xi_1 \partial \xi_2} = -\xi_1 \]
\[ \text{shape}(7, 6) \equiv \frac{\partial^2 \phi_7}{\partial \xi_1 \partial \xi_2} = \frac{1}{4}(2\xi_1 - 2\xi_2 - 1) \]
\[ \text{shape}(8, 6) \equiv \frac{\partial^2 \phi_8}{\partial \xi_1 \partial \xi_2} = \xi_2. \]

The Various Derivatives of \( Z(x_3, h) \)

Since \( Z(x_3, h) = \frac{\cosh(\kappa(h + x_3))}{\cosh(\kappa h)} \) and \( \frac{1}{\kappa} = \tanh(\kappa h) \),

\[
\frac{\partial Z}{\partial x_1} = \frac{\kappa}{\cosh(\kappa h)} \left( \sinh(\kappa(h + x_3)) - \frac{1}{\kappa} \cosh(\kappa(h + x_3)) \right) \frac{\partial h}{\partial x_1} \]
\[
+ \frac{h}{\cosh(\kappa h)} \left( \sinh(\kappa(h + x_3)) - \frac{1}{\kappa} \cosh(\kappa(h + x_3)) \right) \frac{\partial \kappa}{\partial x_1},
\]
\[
\frac{\partial Z}{\partial x_2} = \frac{\kappa}{\cosh(\kappa h)} \left( \sinh(\kappa(h + x_3)) - \frac{1}{\kappa} \cosh(\kappa(h + x_3)) \right) \frac{\partial h}{\partial x_2}
\]
\[
+ \frac{h}{\cosh(\kappa h)} \left( \sinh(\kappa(h + x_3)) - \frac{1}{\kappa} \cosh(\kappa(h + x_3)) \right) \frac{\partial \kappa}{\partial x_2}.
\]
\[ \frac{\partial Z}{\partial x_3} = \frac{\kappa \sinh(\kappa(h + x_3))}{\cosh(\kappa h)}, \]

\[ \frac{\partial^2 Z}{\partial x_1^2} = \frac{\kappa}{\cosh(\kappa h)} \left[ \left( \frac{1}{\kappa} + \kappa \right) \cosh(\kappa(h + x_3)) - 2\sinh(\kappa(h + x_3)) \right] \left( \frac{\partial h}{\partial x_1} \right)^2 + \frac{\kappa}{\cosh(\kappa h)} \left( \sinh(\kappa(h + x_3)) - \frac{1}{\kappa} \cosh(\kappa(h + x_3)) \right) \frac{\partial^2 h}{\partial x_1^2} + \frac{h}{\cosh(\kappa h)} \left[ \left( 2\kappa - 1 - \sinh(\kappa h) - \frac{1}{\kappa} \right) \cosh(\kappa(h + x_3)) + \left( 1 + \frac{2}{h} + \kappa \sinh(\kappa h) - \kappa \right) \sinh(\kappa(h + x_3)) \right] \frac{\partial h}{\partial x_1} \frac{\partial \kappa}{\partial x_1}, \]

\[ \frac{\partial^2 Z}{\partial x_2^2} = \frac{\kappa}{\cosh(\kappa h)} \left[ \left( \frac{1}{\kappa} + \kappa \right) \cosh(\kappa(h + x_3)) - 2\sinh(\kappa(h + x_3)) \right] \left( \frac{\partial h}{\partial x_2} \right)^2 + \frac{\kappa}{\cosh(\kappa h)} \left( \sinh(\kappa(h + x_3)) - \frac{1}{\kappa} \cosh(\kappa(h + x_3)) \right) \frac{\partial^2 h}{\partial x_2^2} + \frac{h}{\cosh(\kappa h)} \left[ \left( 2\kappa - 1 - \sinh(\kappa h) - \frac{1}{\kappa} \right) \cosh(\kappa(h + x_3)) + \left( 1 + \frac{2}{h} + \kappa \sinh(\kappa h) - \kappa \right) \sinh(\kappa(h + x_3)) \right] \frac{\partial h}{\partial x_2} \frac{\partial \kappa}{\partial x_2}, \]

\[ \frac{\partial^2 Z}{\partial x_3^2} = (\kappa)^2 \frac{\cosh(\kappa(h + x_3))}{\cosh(\kappa h)}, \]

\[ \frac{\partial^2 Z}{\partial x_1 \partial x_2} = \frac{\kappa}{\cosh(\kappa h)} \left[ \left( \frac{1}{\kappa} + \kappa \right) \cosh(\kappa(h + x_3)) - 2\sinh(\kappa(h + x_3)) \right] \frac{\partial h}{\partial x_1} \frac{\partial h}{\partial x_2} + \frac{\kappa}{\cosh(\kappa h)} \left( \sinh(\kappa(h + x_3)) - \frac{1}{\kappa} \cosh(\kappa(h + x_3)) \right) \frac{\partial^2 h}{\partial x_1 \partial x_2}. \]
\[ + \frac{h}{\cosh(\kappa h)} \left[ \left( \kappa - \frac{1}{\kappa} - \frac{1}{h\kappa} \right) \cosh(\kappa(h + x_3)) \right] \]
\[ + \left( \frac{1}{h} - \frac{1}{2} \right) \sinh(\kappa(h + x_3)) \left( \frac{\partial h}{\partial x_1} \frac{\partial \kappa}{\partial x_2} + \frac{\partial h}{\partial x_2} \frac{\partial \kappa}{\partial x_1} \right) \]
\[ + \frac{h}{\kappa \cosh(\kappa h)} \left[ \left( \frac{2}{\kappa} + h\kappa \right) \cosh(\kappa(h + x_3)) - (1 + h) \sinh(\kappa(h + x_3)) \right] \frac{\partial \kappa}{\partial x_1} \frac{\partial \kappa}{\partial x_2} \]
\[ + \frac{h}{\cosh(\kappa h)} \left( \sinh(\kappa(h + x_3)) - \frac{1}{\kappa} \cosh(\kappa(h + x_3)) \right) \frac{\partial^2 \kappa}{\partial x_1 \partial x_2} \]
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