Revisiting double Dirac delta potential

Zafar Ahmed\textsuperscript{1,4}, Sachin Kumar\textsuperscript{2}, Mayank Sharma\textsuperscript{3} and Vibhu Sharma\textsuperscript{3}

\textsuperscript{1} Nuclear Physics Division, Bhabha Atomic Research Centre, Mumbai 400085, India
\textsuperscript{2} Theoretical Physics Division, Bhabha Atomic Research Centre, Mumbai 400085, India
\textsuperscript{3} Amity Institute of Applied Sciences, Amity University, Noida, UP, 201313, India

E-mail: zahmed@barc.gov.in, Sachinv@barc.gov.in, svibhu876@gmail.com and svibhu876@gmail.com

Received 6 April 2016, revised 2 May 2016
Accepted for publication 19 May 2016
Published 10 June 2016

Abstract

We study a general double Dirac delta potential to show that this is the simplest yet still versatile solvable potential to introduce double wells, avoided crossings, resonances and perfect transmission \((T = 1)\). Perfect transmission energies turn out to be the critical property of symmetric and anti-symmetric cases wherein these discrete energies are found to correspond to the eigenvalues of a Dirac delta potential placed symmetrically between two rigid walls. For well(s) or barrier(s), perfect transmission (or zero reflectivity, \(R(E)\)) at energy \(E = 0\) is non-intuitive. However, this has been found earlier and called the ‘threshold anomaly’. Here we show that it is a critical phenomenon and we can have \(0 < R(0) < 1\) when the parameters of the double delta potential satisfy an interesting condition. We also invoke a zero-energy and zero curvature eigenstate \((\psi(x) = Ax + B)\) of the delta well between two symmetric rigid walls for \(R(0) = 0\). We resolve that the resonant energies and the perfect transmission energies are different and they arise differently.

Keywords: resonances, transmission coefficient, bound states, reflection coefficient, perfect transmission energies, zero reflection at zero energy

(Some figures may appear in colour only in the online journal)

1. Introduction

The general one-dimensional double Dirac delta potential (DDDP) is written as (see figures 1(a)–(c))

\(4\) Author to whom any correspondence should be addressed.
When $V_1 = V_2 = -\alpha$ and $b = a$ it becomes symmetric double delta potential [1–3], which is well known to have at most two discrete eigenvalues; one when $\frac{\hbar^2}{2m_1} > 1$ and two when $\frac{\hbar^2}{2m_2} < 1$. The symmetric DDDP has also been studied [4] as a scattering potential possessing the oscillatory transmission coefficient $T(E)$ as a function of energy. Using the potential (1), a subtle ‘threshold anomaly’ in the scattering from one-dimensional attractive potential wells has been revealed earlier [5]. According to this the reflection probability, becoming anomalous ($R(E = 0) = 0$) is directly related to whether the potential is at the threshold of possessing a bound state near $E = 0$. Here we show the critical nature of this effect. This attractive double delta potential has also been studied for an interesting effect that the Wigner time-delay [6] at small energies is very large [7] if the potential supports a bound state near $E = 0$ or if its strength is just enough to support another bound state.

Notwithstanding the versatility and simplicity of this potential (1), it has not been utilized fully in textbooks. In this paper, we show that this is the simplest potential to introduce double wells, the rare avoided crossing of two levels in one-dimension, perfect transmission and resonances. In addition, the utility of transmission ($\tau$) and reflection ($\rho$) amplitudes in extracting bound states, resonances and perfect transmission energies is not often discussed in textbooks [2, 3, 6, 9, 10]. Here, we first explain these connections in general and demonstrate the extraction of three discrete energy spectra of bound states, resonances and perfect transmission from $R = |\rho|^2$ and $T = |\tau|^2$ for the general double Dirac delta potential given as (1).

$$V(x) = V_1 \delta(x + b) + V_2 \delta(x - a).$$

\[\text{Figure 1. (a)–(d) Depiction of various cases of the double Dirac delta potential (1). In (e) and (f) are the hard-box potentials, where the Dirac delta barrier or well has been placed symmetrically between two rigid walls, respectively.}\]
2. Scattering coefficients and discrete energies

The one-dimensional, time-independent Schrödinger equation is written as

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}[E - V(x)]\psi(x) = 0.$$  \hspace{1cm} (2)

Let us define

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \quad E > 0; \quad p = \sqrt{-\frac{2mE}{\hbar^2}}, \quad E < 0.$$  \hspace{1cm} (3)

Consider a particle-wave incident on the potential from far left ($x < 0$) where $V(x) = 0$, it may be reflected back towards the left ($x < a$) and transmitted towards the far right ($x > a$), where $V(x)$ is zero again. Every textbook $[2, 3, 6, 9, 10]$ writes the general solution of $\psi(x)$ in this case as

$$\psi(x < 0) = Ae^{ikx} + Be^{-ikx}, \quad \psi(x > a) = Fe^{ikx}.$$  \hspace{1cm} (4)

For the region $0 < x < a$, the particular solution of the Schrödinger equation (a combination of two linearly independent solutions) for the given potential is juxtaposed between these two solutions. One then matches the wave function and its derivative at $x = a$ to obtain the reflection and transmission amplitudes usually as

$$\rho = \frac{B}{A}, \quad \tau = \frac{F}{A},$$  \hspace{1cm} (5)

where $A, B, F$ are functions of energy and mass of the particle and the potential parameters. By reversing the signs of the strengths of the potential barrier (e.g., $V_1, V_2$ in (1)) we obtain $\rho' = B'/A'$ and $\tau' = F'/A'$ for the well. Let us change $k$ to $ip$ in the equation $a la$ (4). So on the left, we have $\psi(x < -a) = A'e^{-p_1} + B'e^{p_1}$ and $\psi(x > a) = F'e^{p_1}$ on the right. In order to have bound states we demand the wave function to converge to zero as $x \to \pm \infty$. This requires $A' = 0$ at $k = \frac{\phi_0}{\hbar}$, $p_1 > 0$, which in turn are the poles of $\rho'$ and $\tau'$ at negative energies, $E_0 = \frac{\hbar \phi_0^2}{2m} < 0$. One may also find the negative energy poles of $T'(E)$ and $R'(E)$. We would like to caution that in [6] (on page 109 in figure 6.9) the negative energy poles mistakenly appear as spikes of height 1 in the graph of $T(E)$.

Students could be further instructed to find the poles of $\rho$ and $\tau$, which amounts to finding the zeros of $A$, this turns the solutions (4) into Gamow’s pioneering idea $[8, 9]$ of an outgoing-wave boundary condition at the exit of the potential

$$\psi(x < 0) = Be^{-ikx}, \quad \psi(x > a) = Fe^{ikx}.$$  \hspace{1cm} (6)

In case the potential possesses resonances, one gets the poles at $E = E_0 - i\Gamma_n/2$ or $k = k_n - i\Gamma_n$. The solution $\Psi(x, t) = \psi(x)e^{-iEt/\hbar}$ of the time-dependent Schrödinger equation for the resonant state can be written as

$$\Psi(x > a, t) = F\ e^{ik_n x} e^{k_n^2 \hbar} e^{-i\Gamma_n \hbar} e^{-\hbar/2},$$  \hspace{1cm} (7)

where $k_n, k_n^2 > 0$, so that $\Gamma_n = 4k_n^2 \rho_0 > 0$. One can see that the spatial part is an oscillating wave with growing amplitude (spatial catastrophe) for $x \to \infty$. Similarly one can get a spatial catastrophe on the left side $x \to -\infty$. Time-wise, $\Psi(x, t)$ is well known as Gamow’s decaying state. These states have explained then enigmatic phenomenon of alpha-decay from a nucleus $[8]$. Recalling that Hermitian Hamiltonians have real eigenvalues is misplaced here as we are not imposing the condition of bound state that
\[ \psi(x < 0) = e^{\beta x}, \psi(x > a) = e^{-\beta x}, \text{ where } \beta = \sqrt{-2mE/\hbar}. \]\[ \text{Also the spatial catastrophe } e^{\beta x} \text{ in (7) will be controlled [9] by the time-wise decaying part } e^{-\beta x/2}. \]

3. The versatile double Dirac delta potential

In the following we obtain the \( \rho \) and \( \tau \) from scattering states of the potential (1) \((b = 0 \text{ in figures 1(a)–(c)})\), we write the plane wave solution of (2) in different regions as

\[ \psi(x < 0) = Ae^{ikx} + Be^{-ikx}, \psi(0 < x < a) = Ce^{ikx} + De^{-ikx}, \psi(x > a) = Fe^{ikx}, \]

\[ v_j = \frac{2mV_j}{\hbar^2}, \quad \text{(8)} \]

as the potential (1) is zero except at two points \( x = 0, a \) where it is suddenly in finite and hence discontinuous. Normally, the solution of the Schrödinger equation needs to be both continuous and differentiable. However, if a potential has the Dirac delta discontinuity \((\text{say})\) at \( x = c \), such that \( V(x) = V(x) + P\delta(x - c) \), where \( V(x) \) is continuous at \( x = c \), the integration of (2) from \( x = c - \epsilon \) to \( x = c + \epsilon \) and then the limit as \( \epsilon \to 0 \) yields

\[ \frac{d\psi(x)}{dx} \bigg|_{x < c} - \frac{d\psi(x)}{dx} \bigg|_{x > c} = \frac{2mP}{\hbar^2} \psi(c). \quad \text{(9)} \]

This is the well known condition of momentum mismatch due to the Dirac delta function \([2, 3, 9, 10]\) at \( x = c \). So the condition of continuity and (9) at \( x = 0, a \) gives the following equations

\[ A + B = C + D, \quad ik\left[(C - D) - (A - B)\right] = v_1(A + B) \]

\[ Ce^{ika} + De^{-ika} = Fe^{ika}, \quad ik\left[Fe^{ika} - Ce^{ika} + De^{-ika}\right] = v_2Fe^{ika}. \quad \text{(10)} \]

Using these equations we obtain the reflection and transmission amplitudes for the potential (1) as

\[ \rho = \frac{B}{A} = \frac{2ik(v_1e^{-ika} + v_2e^{ika}) + 2iv_1v_2\sin ka}{(2ik - v_1)(2ik - v_2)e^{-ika} - v_1v_2e^{ika}}, \quad \text{(11)} \]

\[ \tau = \frac{F}{A} = \frac{-4k^2e^{-ika}}{2ik - v_1)(2ik - v_2)e^{-ika} - v_1v_2e^{ika}}. \quad \text{(12)} \]

We now propose to extract various discrete spectra from equations (11) and (12).

3.1. Bound states

Let us change in (11), (12) \( V_1, V_2, k \to -U_1, -U_2, ip \) which amounts to changing \( v_j \) to \( u_j \) (see 13 below). The poles of \( \rho \) and \( \tau \) are then given by

\[ (2p - u_1)(2p - u_2) = u_1u_2e^{-2\alpha}, \quad u_j = \frac{2mU_j}{\hbar^2}. \quad \text{(13)} \]

To find the bound state eigenvalues of the double delta wells (figure 1(c)) one has to solve this implicit equation numerically. Since this one-dimensional finite potential well satisfies the condition that \( \int_{-\infty}^{\infty} V(x)dx \) < \( \infty \) (finite) \([11]\), it will have at least one bound state eigenvalue. Next, on the left of equation (13), we have a quadratic (parabolic) function of \( p \) and on the right we have a decreasing exponential; consequently they can cut each other at most at two values of \( p \). So there can be at most two discrete eigenvalues in the double delta well (figure 1(c)). Let us first recover the well known results in the special cases.
When \( a = 0 \) this potential becomes a single delta well at \( x = 0 \) with the strength as \((U_1 + U_2)\) in this case from \((13)\), we get
\[
E = -\frac{m_1 (U_1 + U_2)^2}{\hbar^2}.
\]
Further if \( U_2 = 0 \), we get the well known single eigenvalue of the Dirac delta potential as
\[
E = -\frac{mV_0^2}{2\hbar^2} [2, 3, 9, 10].
\]
Next when \( U_1 = U_0, U_2 = -U_0 \), from \((13)\) we get the eigenvalue equation in this case as
\[
(4p^2 - u_0^2) = -u_0^2 e^{-2pa} < 0 \Rightarrow E > -\frac{mU_0^2}{2\hbar^2}.
\]
If we take \( u_2 \to \infty \), the potential \((1)\) becomes a delta well near a rigid wall (figure 1(d)), which is a well studied potential \([2, 3]\). For this case let us divide equation \((13)\) by \( u_2 \) and take the limit \( u_2 \to \infty \), we get the eigenvalue equation as
\[
e^{-2pa} = 1 - 2p/u_1,
\]
the single eigenvalue will occur only if \( u_1 a > 1 \). When \( u_2 \) is changed from 0 to \( \infty \) the single eigenvalue of \((1)\) (figure 1(d)) will vary from \( E = -\frac{mV_0^2}{2\hbar^2} \) to the root of equation \((16)\). So if the delta well is strong enough \((u_1 a > 1\) ) even the rigid wall perturbation at \( x = a \) near the delta well cannot remove the single bound level from the well.

Again if \( U_1 = U_0 = U_2 \) but \( a = 0 \), the whole expression in \((13)\) gets factored into two well known equations \([2, 3]\)
\[
e^{-pa} = 2p/u_0 - 1 > 0, \quad e^{-pa} = 1 - 2p/u_0 > 0
\]
The first of these equations \((17)\) always has a real root for any positive value of \( u_0 \) confirming an unconditional bound state in the potential. The second equation above will have one real root only when \( u_0 a > 2 \) \([2, 3]\), so the first excited state exists conditionally.

Notice that \( p = 0 \) is an unconditional root of \((13)\), so \( E = 0 \) can be mistaken to be an essential bound state of \((1)\) (figure 1(c)). Let us investigate equation \((13)\) for \( p \approx 0 \), ignoring \( p^2 \) and writing \( e^{-pa} \approx 1 - pa \), we get
\[
\frac{1}{u_1} + \frac{1}{u_2} = a,
\]
\[
\text{Figure 2. The variation of two levels of the double well (figure 1(c)) potential (1) as the distance between the well, } a, \text{ is varied when (a) } u_1 = 11 = u_2 \text{ (b) } u_1 = 11, u_2 = 12. \text{ In (a) the level } E_1 \text{ starts appearing for } a > 2/11 = 0.1818 \text{ and in (b) it appears for } a > 23/132 = 0.1742 \text{ as per equation (18). In (a) two levels merge to one level at } E = -30.25, \text{ whereas in (b) two levels saturate to } E = -36 \text{ and } E = -30.25 \text{ (the ground state eigenvalues two independent delta potentials with depths as 12 and 11, respectively). Here and in all figures below we have taken } 2m = 1 = \hbar^2.\]
meaning that when \( a > 1 / u_1 + 1 / u_2 \), the first excited state \( E_1 < 0 \) would start appearing near \( E = 0 \) in the potential (1) (figure 1(c)). When \( u_1 = u_0 = u_2 \) this condition becomes \( u_0 a > 2 \) see figure 2(a) for \( u_1 = 11, \ E_1 \) starts appearing when \( a > 2 / 11 \). In figure 2(b), \( u_1 = 11, \ u_2 = 12 \), notice that \( E_1 \) starts appearing when \( a > 23 / 132 \).

In figure 2, we show that the characteristic double well behaviour [6] of the potential (1) (figure 1(c)), in the symmetric case \( u_1 = u_0 = u_2, u_0 a > 2 \) defines the threshold for the appearance of the first excited state. As the distance \( a \) increases the two eigenvalues merge [1] to one (the single eigenvalue of the independent delta potential, \( E = -30.25 \)) in the asymmetric case \( U_1 = 11, \ U_2 = 12 \) the two levels do not merge, instead they become parallel as both levels saturate to distinct values \( (-36, -30.25) \). We find that for large values of \( a \)

\[
E_0 = -\frac{m}{2\hbar^2} \left( \max[U_1, U_2] \right)^2 \\
E_1 = -\frac{m}{2\hbar^2} \left( \min[U_1, U_2] \right)^2.
\]  

(19)

In equation (13), if we put a very large, we get a quadratic equation for \( p \) whose roots are \( u_1 / 2 \) and \( u_2 / 2 \), confirming equation (19). The asymmetric double well potential is often not discussed hence the question as to what the parallel levels in figure 2(b) correspond to, does not arise. We would like to remark that these parallel levels are actually the ground state eigenvalues of the independent wells of depth \( U_1 \) and \( U_2 \) (see equation (19)).

Further, if we take \( u_1 = 10 \) and vary \( u_2 \) for three cases \( a = 0.5, 0.9, 1 \) we see a gradual avoided crossing of two levels solid (dashed) curves denoted \( E_0 \left( E_1 \right) \) (see figure 3). In part (c), it is as though \( E_0 \) and \( E_1 \) have crossed; however, the dashed and solid nature of these curves

**Figure 3.** Demonstration of avoided crossings of two levels when \( u_1 = 10 \) and \( u_2 \) is varied. When (a) \( a = 0.5 \), (b) \( a = 0.9 \), (c) \( a = 1 \). The special (threshold) values of \( u_2 \) for which \( E_1 \) starts appearing are \( 5/2, 5/4, \) and \( 10/9 \) for (a), (b), (c), respectively.

**Figure 4.** Plots of \( T(E) \) to show that negative energy poles indicate bound states of the potential (a) \( v_1 = -5, \ v_2 = -5 \ (E_0 = -4.98, \ E_0 = -7.14) \), (b) \( v_1 = -5, \ v_2 = 5 \ (E_0 = -6.20) \) (c) \( v_1 = 5, \ v_2 = 5 \) (no bound states).
belies the crossing. In one dimension the avoided crossing of two level though allowed is not usually observed. Rare instances have been discussed in [12], we claim that the double well potential (1) (figure 1(c)) presents the simplest model of avoided crossing in one dimension.

As discussed above in section 2, the common negative energy poles of $T(E)$ and $R(E)$ yield the bound state of the potential. See figure 4, which depicts two, one and no pole in $T(E)$ when the DDDP (1) possesses 2, 1 and 0 bound states.

3.2. Discrete perfect transmission (zero reflection) energies

3.2.1. At zero energy for the attractive DDDP. The reflection amplitude $\rho$ for the attractive DDDP, figure 1(c), deserves special attention; by changing $v_1, v_2 \rightarrow -u_1, -u_2$ in (11), we write

$$\rho = \frac{B}{A} = \frac{-2ik(u_1e^{-ika} + u_2e^{ika}) + 2iu_1u_2\sin ka}{(2ik + u_1)(2ik + u_2)e^{-ika} - u_1u_2e^{ika}}. \quad (20)$$

At $E = 0$, $\rho$ is indeterminate ($0/0$). However, using L’Hospital’s rule, we get

$$\rho(0) = 0, \quad \text{when } u_1 = u_0 = u_2 \text{ and } u_0a = 2. \quad (21)$$

$$\rho(0) = \frac{u_2a(u_2a - 2)}{u_2^2a^2 - 2u_2a + 2} < 1, \quad \text{if } \frac{1}{u_1} + \frac{1}{u_2} = a \quad (u_1 \text{ is fixed}) \quad (22)$$

$$\rho(0) = \frac{a_{02}u_1^2 - u_1}{u_1 + u_2 - a_{02}u_1u_2} = -1, \quad \text{if } \frac{1}{u_1} + \frac{1}{u_2} \neq a. \quad (23)$$

It may be mentioned that such cases as in equations (21) and (22) cannot arise for $\rho$ of the DDDP barrier. These equations (21) and (22) are new and they lead to a surprising and non-intuitive undulatory (wavelike) result that $0 \leq R(0) < 1$, whereas (23) is usual and most common. The result that $0 \leq R(0) < 1$ has been observed earlier and has been called the threshold anomaly [4]. In light of the results (21) and (22) derived here we conclude that this is a critical phenomenon and in order to bring out this critical nature of $R(0)$ graphically, in figure 5 we show $R(E)$ for three cases when $u_0a = 1.99, 2, 2.01$. Notice the dramatic result $R(0) = 0$ in figure 5(b). In figures 6(a) and (b), we show that one can arrange to have $0 < R(0) < 1$. Zero or small reflection at zero energy implies that a wave packet with zero average kinetic energy, localized to one side of the potential, will spread in both directions. When the low energy components scatter against the potential, they may be transmitted, but this would appear simply as wave packet spreading.
3.2.2. At non-zero energies in DDDP. The zeros of $\rho$ in equation (11) are to be obtained as

$$2\kappa (v_1 e^{-i\kappa a} + v_2 e^{i\kappa a}) + 2iv_1v_2 \sin \kappa a = 0. \quad (24)$$

The perfect transmission energies of a square well/barrier are known [9, 13, 14] to be the eigenvalues of the corresponding of hard box potentials. So for the square potential of width $a$ and height/depth $V_0$, the perfect transmission occurs at energies $E = \pm V_0 + \frac{n^2\hbar^2}{2ma^2}$, which are the eigenvalues of the hard box potential of width $a$. We may see that the aforementioned discrete energies are also the eigenvalues of even parity states of the hard box potential of width $a$.

We find that perfect transmission for double delta potential occurs only when it is symmetric or anti-symmetric; four further interesting interesting cases arise here.

**Case (i).** When $v_1 = -v_2 = v_0$ (in figure 1(b)), we get $[4\kappa v_0 - 2iv_0^2] \sin \kappa a = 0$ implying $\kappa a = n\pi$ giving

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, 3 ... \quad (25)$$

calling the well known eigenvalues of infinitely deep well (hard-box) of width $a$.

**Case (ii).** When $v_1 = v_2 = v_0$ (in figure 1(a)) from equation (24), we get

$$\tan \kappa a = -\frac{2k_0}{v_0}, \quad E_n = \frac{\hbar^2k^2a}{2m}, \quad (26)$$

the roots of this equation are well known [10] as the eigenvalues of even parity states when the Dirac delta barrier is placed symmetrically between two rigid walls at $x = -a$ and $x = a$ (see figure 1(c)).

**Case (iii).** When $v_1 = v_2 = -u_0$, in this case from (24) one gets

$$\tan \kappa a = \frac{2\kappa}{u_0}, \quad E_n = \frac{\hbar^2k^2a}{2m}, \quad (27)$$

calling the eigenvalues [16] of the even parity states when the Dirac delta well is placed symmetrically between two rigid walls at $x = -a$ and $x = a$ (figure 1(f)). This hard-box potential becomes dramatically special when $u_0a = 2$ (21), it is then that $E = 0$ becomes the ground state eigenvalue only when the zero energy and zero-curvature solution [15, 16] of the Schrödinger equation is sought as $\psi(x) = Ax + B$. Here, we point out that this novel possibility of $E = 0$ as an eigenvalue of the hard-box potential (figure 1(f)) forces the surprising result that $R(0) = 0$, when critically $u_0a$ becomes 2. This completes the
connection of perfect transmission energies of symmetric and antisymmetric DDDP (1) with the hard-box potentials (figures 1(e) and (f)). Next, see figures 7 and 8 displaying the phenomenon of perfect transmission when the DDDP (1) is symmetric or antisymmetric. It may be remarked that in a previous study [4] of the perfect transmission of DDDP (1), the role of the definite parity of the potential (1) has been not been brought out. In figure 7, we have Dirac delta strengths as small \((\pm 5)\) and see energy oscillations in \(T(E)\), whereas in figure 8 for higher values of the strengths \((\pm 30)\), we have deep oscillations in \(T(E)\). These maxima in \(T(E)\) are often misunderstood as resonances. Like the cases of square well/barriers [9, 13, 14], for the double Dirac delta potential, we again find perfect transmission energies \(\epsilon_n\) where \(T(\epsilon_n) = 1\) are different from resonant energies \(E_n\) (see below).

Case (iv). Other cases that are essentially non-symmetric or asymmetric, the roots of (24) are complex to be denoted as \(E = \epsilon_n - i 4\gamma_n/2, \epsilon_n, \gamma_n > 0\). Remarkably in these cases \(T(\epsilon_n) = 1\) making the transmission imperfect, see table 1, for asymmetric cases.

3.3. Discrete complex energy resonances

As discussed above the complex energy resonances (Gamow’s decaying states) can be obtained from the poles of \(\rho\) and \(\tau\) as

\[
(2ik - v_1)(2ik - v_2) - v_1v_2e^{2ik\alpha}.
\]
Table 1. First four resonant energies \((E_n - i\Gamma_n/2)\) and perfect transmission energies \((\epsilon_n - i\gamma_n/2)\) and values of \(T(E_n)\) and \(T(\epsilon_n)\) for symmetric, antisymmetric and asymmetric double Dirac delta potential \((1)\) (figures \(1(a)–(c)\)). We have taken \(2m = 1 = h^2\).

| \(\nu_1, \nu_2, \alpha\) | I \((E_n - i\Gamma_n/2)\) | II \((\epsilon_n - i\gamma_n/2)\) | III \(T(E_n)\) | IV \(T(\epsilon_n)\) |
|----------------|----------------|----------------|----------------|----------------|
| 0.9997 | 15.66 – 9.98i | 25.66 – 5.38i | 109.90 – 43.37i | 187.27 – 63.17i |
| 0.9998 | 9.7495 | 0.9999 | 0.9999 | 994.96 |
| 0.9134 | 19.25 – 0.14i | 30.93 – 0.25i | 994.96 | 0.9945 |
| 19.2074 | 58.6851 | 994.96 | 994.96 | 996.89 |
| 9.8969 | 1 | 1 | 1 | 1 |
| 3.29, 1 | 7.82 – 4.74i | 34.50 – 17.75i | 81.66 – 34.42i | 149.01 – 53.30i |
| 0.8659 | 0.9999 | 0.9847 | 0.9987 | 992.7 |
| 9.82 – 0.81i | 39.23 – 0.25i | 88.77 – 0.31i | 157.86 – 0.42i |
| 0.9999 | 9999 | 9999 | 9999 | 9999 |
| 3.29, 1 | 7.91 – 4.69i | 34.63 – 17.57i | 81.81 – 34.12i | 149.16 – 52.89i |
| 0.9669 | 0.9670 | 0.9847 | 0.9927 | 9999 |
| 9.8969 | 39.4784 | 88.8264 | 157.91 |
| 3, 2, 1 | 3.97 – 1.79i | 21.41 – 113.23i | 58.50 – 26.14i | 115.81 – 43.91i |
| 0.8655 | 0.9381 | 0.9775 | 0.9900 | 9999 |
| 4.70 – 0.41i | 24.99 – 0.14i | 64.56 – 0.25i | 123.81 – 0.36i |
| 0.9999 | 0.9999 | 0.9999 | 0.9999 | 9999 |
| 3, 2, 1 | 4.01 – 1.77i | 21.52 – 11.11i | 58.64 – 25.90i | 115.96 – 43.56i |
| 0.8696 | 0.9387 | 0.9775 | 0.9900 | 9999 |
| 0.9999 | 0.9999 | 0.9999 | 0.9999 | 9999 |
| 4.729 | 25.0365 | 64.6169 | 123.867 |
| 1 | 1 | 1 | 1 | 1 |
| 30, 30, 1 | 8.68 – 0.10i | 34.88 – 0.80i | 78.93 – 2.54i | 141.28 – 5.56i |
| 0.9997 | 0.9992 | 0.9987 | 0.9983 | 9999 |
| 8.6880 | 34.9042 | 79.0282 | 141.5120 |
| 1 | 1 | 1 | 1 | 1 |
| 30, 29, 1 | 8.66 – 0.10i | 34.81 – 0.82i | 78.80 – 2.61i | 141.08 – 5.70i |
| 0.9986 | 0.9982 | 0.9977 | 0.9975 | 9999 |
| 8.76 – 0.003i | 34.83 – 0.002i | 78.90 – 0.07i | 141.32 – 0.15i |
| 0.9988 | 0.9990 | 0.9991 | 0.9993 |

Roots of this equation \((28)\) of the type \(k = K_n - iK_n'(E = \epsilon_n - i\gamma_n/2), \) \(\epsilon_n, \Gamma_n > 0\) are called resonances (see Table 1) which exist in (1) whether it is a double barrier (figure (1a)), a well and a barrier (figure (1b)) or a double well (figure (1c)) potential. Table 1 presents the first four resonances of (1) with complex discrete eigenvalues, \(\epsilon_n - i\gamma_n/2\) for all symmetric, antisymmetric and asymmetric cases. Notice that resonances (unlike perfect transmission) occur whether the potential is symmetric or not and \(T(E_n) \approx 1\). One may check that \(E_n \approx \epsilon_n\). Earlier, it has been argued \([17, 18]\) that when \(|\Gamma_n| \ll \epsilon_n, \epsilon_n\) can be well approximated with \(\epsilon_n\) i.e., \(\epsilon_n \approx \epsilon_n\). We would like to mention that this situation arises when the strengths of the wells or barriers are high. In this regard, the last two sections of Table 1 can be seen to support
this approximation. However, in principle, table 1 for the DDDP (1), once again [9, 13, 14], shows that resonance energies are different from the perfect transmission energies.

4. Conclusion

We would like to conclude that our discussion of the extraction of discrete eigenvalues from the scattering amplitudes/coefficients is instructive, and that the double Dirac delta potential (DDDP) presents a delightful example. It has been emphasized that the discrete spectrum not only consists of bound state eigenvalues, it also consists of resonant energies and perfect transmission energies. We have presented the DDDP as the simplest solvable model of double wells. The observed avoided crossing (AC) of two levels as the depth of one of the wells is varied slowly can be seen to be the simplest instance of this rare phenomenon of AC in one dimension. The correspondence of perfect transmission energies with the eigenvalues of hard-box potentials (figures 1(e) and (f)) is the second but more interesting instance after the square well/barrier. It is now desirable to examine the generality of such a connection of perfect transmission energies of symmetric potentials with the eigenvalues of their counterpart hard-box potentials. We also resolve that the occurrence of the surprising and non-intuitive result that $0 < R(0) < 1$ is a critical effect. The present recourse to the double Dirac delta potential provides a second example after the square well/barrier to see that resonant energies and the perfect transmission energies are different and they have different origins.

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