General-Relativistic Viscous Fluid Dynamics

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We present the first example of a relativistic viscous fluid theory that satisfies all of the following properties: (a) the system coupled to Einstein’s equations is causal and strongly hyperbolic; (b) equilibrium states are stable; (c) all leading dissipative contributions are present, i.e., shear viscosity, bulk viscosity, and thermal conductivity; (d) non-zero baryon number is included; (e) entropy production is non-negative in the regime of validity of the theory; (f) all of the above holds in the nonlinear regime without any simplifying symmetry assumptions; these properties are accomplished using a generalization of Eckart’s theory containing only the hydrodynamic variables, so that no new extended degrees of freedom are needed as in Mueller-Israel-Stewart theories. Property (b), in particular, follows from a more general result that we also establish, namely, sufficient conditions that when added to stability in the fluid’s rest frame imply stability in any reference frame obtained via a Lorentz transformation. All our results are mathematically rigorously established. The framework presented here provides the starting point for systematic investigations of general-relativistic viscous phenomena in neutron star mergers.

I. INTRODUCTION

Relativistic fluid dynamics has been successfully used as an effective description of long wavelength, long time phenomena in a multitude of different physical systems, ranging from cosmology [1] to astrophysics [2] and also high-energy nuclear physics [3]. In the latter, relativistic viscous fluid dynamics has played an essential role in the description of the dynamical evolution of the quark-gluon plasma formed in ultrarelativistic heavy-ion collisions [4] and also in the quantitative extraction of its transport properties (see, for instance, [5]). More recently, with the observation of binary neutron star mergers [6–8], the modeling of the different dynamical stages experienced by the hot and dense matter formed in these collisions requires extending of our current understanding of viscous fluids towards the strong gravity regime where general relativistic effects are important (see, e.g., [9–13]).

The ubiquitousness of fluid dynamics stems from the existence of general conservation laws (such as energy, momentum, and baryon number) and their consequences to systems where there is a large separation of scales, such that the macroscopic behavior of conserved quantities can be understood without precise knowledge of all the details that govern the system’s underlying microscopic properties [14]. Ideal fluid dynamics is the extreme situation where dissipative effects are neglected and the theory’s basic properties in this limit are reasonably well understood, both in a fixed background as well as when coupling to Einstein’s equations is taken into account [2, 15, 16].

A central postulate in special and general relativity is the concept of local causality, i.e., that the speed within which information can propagate in any system cannot be larger than the speed of light [17]. This implies that the solution of the equations of motion at a given space-time point $x$ are completely determined by the spacetime region that is in the past of and causally connected to $x$ [17–19]. It is well-known that causality holds for ideal fluids coupled to Einstein’s equations [20]. In fact, this property must hold in relativity regardless of whether dissipation is present or not [17]. While causality is typically not an issue for most matter models under reasonable assumptions [18], historically, ensuring causality has been a stumbling block in the construction of relativistic theories of fluids with viscosity [2], which was initiated by Eckart 80 years ago [21].

The work of Eckart [21], and Landau and Lifshitz [14], introduced dissipation in relativistic fluid dynamics using two very natural general principles:

(i) a gradient expansion around equilibrium states;
(ii) the 2nd law of thermodynamics.

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The former goes back to the beginning of the 20th century \[22\] and the main idea behind it is that, sufficiently close to equilibrium, one can systematically describe the dynamics of viscous fluids using only the variables present in ideal hydrodynamics (e.g., temperature, flow velocity, and chemical potential) and their gradients. In the covariant case, viscous corrections are formally organized as a series in powers of gradients \[1\] around equilibrium, whose first-order truncation then becomes a potential relativistic generalization of the Navier-Stokes (NS) equations \[14, 21\]. Constraints on the coefficients that appear in such truncated series are found using (ii), i.e., by imposing that a Lorentz 4-vector entropy current constructed using a covariant version of the first law of thermodynamics has non-negative divergence \[14, 24\].

However, it is important to note that additional assumptions were made in \[21\] and \[14\] that do not follow from (i) and (ii). First, the energy and baryon densities measured by a co-moving observer with the fluid were assumed to be identical to the corresponding quantities in equilibrium. While convenient, such assumption is not mandatory and it is not a consequence of (i) and (ii). Also, due to the equivalence between mass and energy in relativity, one expects that different definitions for the fluid velocity are possible. Indeed, Eckart defined the fluid velocity as the “velocity of matter” \[21\], constructed in terms of the conserved (baryon) current, while for Landau and Lifshitz the fluid velocity was defined by the condition that a co-moving observer should see no heat flux \[14\]. While these definitions coincide for an ideal fluid, in the viscous case once the nonlinearity of the equations of motion are taken into account those choices lead to different results. Clearly, such choices for the flow are not unique and they are not consequences of (i) and (ii).

The choices in \[14\] and \[21\] mentioned in the last paragraph, although natural at first sight, do not lead to fluid theories that can be consistently embedded and solved in general relativity. In fact, those theories violate causality \[22\] predict that equilibrium solutions in flat spacetime are unstable \[26\]. Causality violation is readily understood because the equations of motion in both Eckart’s and Landau-Lifshitz’ formulations are not hyperbolic. The lack of hyperbolicity makes such dynamical evolution equations incompatible with the postulates of relativity \[17\]. The instability found in such approaches near equilibrium \[26\] is related to their acausal behavior - the linearized equations of motion predict the existence of an unstable mode even at arbitrarily large wavelengths.

Mueller \[27\], Israel, and Stewart \[28\] proposed a potential solution to the problems found in the theories of Eckart and Landau and Lifshitz. The Mueller-Israel-Stewart (MIS) approach is based on the idea that the thermodynamic fluxes (shear-stress tensor, bulk scalar, and heat flow) that describe the viscous corrections should be treated as new dynamical variables extending the space of variables of conventional theories \[29, 30\]. Therefore, this formulation does not formally employ (i), even though its regime of validity concerns only near equilibrium states \[31\]. The new dynamical variables satisfy additional equations of motion, which are then solved together with the conservation laws, to describe how the fluxes approach their relativistic NS form. Those additional equations of motion were originally obtained by Israel \[24\] using an Ansatz for the out-of-equilibrium entropy current and imposing the validity of the second law of thermodynamics [thus, making use of (ii)]. Modern approaches use different methods that do not directly rely on entropy production, such as resummations of the gradient expansion \[32\], kinetic theory derivations using the method of moments \[33\], or anisotropic hydrodynamic formulations \[34–36\]. Other approaches for relativistic fluids include divergence-type theories \[37, 38\] and recent formulations \[39, 40\] inspired by Carter’s formalism.

While causality (and stability) have been known to hold (for appropriate values of the transport coefficients) in MIS in the linearized regime around equilibrium \[11, 12\], only very recently \[43–45\] relevant progress was made to understand causality in the nonlinear regime of such theories. Ref. \[44\] established for the first time a set of conditions needed for causality to hold in the nonlinear regime of MIS-like theories with shear and bulk viscosity at zero chemical potential. Earlier significant work in this regard also includes \[45, 46\].

So far, MIS formulations have been the main tool used in numerical simulations of the fluid dynamic behavior of the quark-gluon plasma formed in heavy-ion collisions (see, for instance, the review \[3\]). However, despite their current popularity, it is important to keep in mind some of the limitations and potential issues that appear in MIS-based approaches. First, these theories lack the degree of universality expected to hold in hydrodynamics as the equations of motion themselves change depending on the derivation. For instance, the equations of motion in \[32\] have different terms than in \[33\], which is explained by the different power-counting scheme employed in those works. Furthermore, such equations are only expected to describe the transient regime of dilute gases as their derivation is most naturally understood within kinetic theory \[28, 33\]. Therefore, their use in other types of systems, such as in strongly coupled relativistic fluids, is a priori not justified. In fact, it is known that MIS-like equations do not

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1. In the covariant formulation, one employs a combination of derivatives such that there are no time derivatives of the hydrodynamic fields in the rest frame of the fluid \[23\].

2. Anisotropic hydrodynamics is, in principle, more general than most approaches as it investigates the problem of small deviations around an anisotropic non-equilibrium state.
generally describe the complex transient regime of holographic strongly coupled gauge theories\cite{17,49} (see \cite{50} for the case of higher-derivative corrections).

Additionally, despite the new developments in \cite{43,44}, many uncertainties still remain when it comes to the nonlinear regime of MIS approaches\cite{3}. With the exception of the case where only bulk viscosity is included\cite{43}, a robust set of conditions ensuring the causality of the MIS equations is still unknown\cite{4}. Furthermore, it has been recently proven\cite{51} that causality violation may happen in finite time even in the purely bulk viscous case. Finally, it has also been known for a while\cite{48,52} that the MIS theory encounters challenges when describing shock-waves, an important limitation given the preponderance of shock-waves in fluid dynamics.

Another potential limitation of MIS theories concerns the initial-value problem (also known as the Cauchy problem), which must be locally well-posed, i.e., given initial data there must exist a unique solution launched by the data\cite{5}. This is a requirement of consistency for any physical theory. Aside from its theoretical value, local well-posedness is also important for the implementation of numerical codes\cite{53}. In all fairness, numerical simulations of the MIS equations have been studied for a long time\cite{54,50}, before any local well-posedness result was available, and in any case some statements concerning local well-posedness have by now been established in \cite{43,44}. There are, however, important caveats which we now discuss.

Matching numerical solutions with the proven-to-exist mathematical solutions typically requires local well-posedness for initial-data with square integrable derivatives up to order \(N\) (\(N\) is a fixed integer depending on the structure of the equations), i.e., the so-called Sobolev spaces\cite{6}. Local well-posedness in Sobolev spaces is also connected to more stable, thus reliable, numerical schemes\cite{53}. In contrast, local well-posedness of the MIS equations\cite{44} has been established for a more restrictive class of initial-data, namely, that of quasi-analytic functions\cite{7}, with exception of the situation where only bulk viscosity is present, in which case Sobolev local well-posedness has been shown\cite{43}. In other words, in the general case, the MIS equations have been shown to be only weakly hyperbolic, whereas connecting local well-posedness with numerical simulations requires strong hyperbolicity\cite{58}. The former leads to local well-posedness for quasi-analytic data, whereas the latter for data in Sobolev spaces (see section IV for a more detailed discussion). We remark that every strongly hyperbolic system is in particular weakly hyperbolic, but the converse is in general not true. In other words, being weakly hyperbolic is a necessary but not sufficient condition for strong hyperbolicity. Thus, one might still be able to show that the general MIS equations are strongly hyperbolic, but at the moment this is not known.

It could be argued that the above discussion on causality and strong hyperbolicity is too academic in that both numerical simulations and applications of the MIS theory have been relatively well-established for some time\cite{3}. In fact, many of the key insights derived from the MIS are based on numerical simulations and have been obtained before any causality or local well-posedness result had been known. However, these numerical simulations concern only the theory in flat spacetime. When considering coupling to Einstein’s equations, as is needed in particular for the study of neutron star mergers, one is at great risk of generating unreliable numerical data if the system is not strongly hyperbolic. In fact, the difficulties of solving Einstein’s equations numerically and the importance of having strongly hyperbolic formulations to do so are well-known\cite{2}. Coupling them to a matter sector that might violate causality only adds to the potential for spurious data. Not to mention that a theory that is not causal cannot by any stretch be correct in the relativistic regime regardless of how well numerical simulations seem to perform. Therefore, a deeper understanding of these issues is in order, especially when considering viscous simulations of neutron star mergers\cite{11,12}.

In sum, despite its undeniable success in advancing the understanding of the physics of the quark-gluon plasma, MIS theories still face many challenges when it comes to more general physical settings. Thus, we believe that it is extremely important to also consider alternative theories of relativistic viscous fluids. This is especially the case given that the study of viscous effects in neutron star mergers\cite{12,59,60} is poised to become a major topic in the coming years and, as mentioned, it is far from clear that the MIS approach is the correct one for this setting.

In this work we further develop the effective theory formalism proposed by Bemfica, Disconzi, Noronha, and Kovtun

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\begin{itemize}
\item[3] In this regard, the multitude of ways such equations can be formulated contributes to this uncertainty.
\item[4] Ref. \cite{13} establishes one set of conditions that is necessary for causality and another, larger set of conditions, that is sufficient for causality. But these conditions are not exhaustive and their wide physical applicability is not known.
\item[5] Local well-posedness more often concerns not only existence and uniqueness but also continuous dependence of solutions on the data. Here, we slightly abuse terminology and use the term to mean existence and uniqueness only, since these are the properties of more immediate physical relevance.
\item[6] We remind the reader that the Sobolev space \(H^N\) is the space of functions \(f\) with finite norm \(\|f\|_{H^N}^2 = \sum_{k=0}^{N} \int |\partial^k f|^2 dx\), where \(\partial^k\) represent all derivatives of order \(k\) and the derivatives are understood in a distributional sense \cite{55}.
\item[7] These are functions satisfying \(\partial^k f(x) \sim (k!)^s |x|^s\), \(s > 1\); \(s = 1\) corresponds to analytic functions.
\end{itemize}
(BDNK) in Refs. [61–63] to obtain a new first-order theory of general-relativistic\(^8\) viscous fluid dynamics at nonzero baryon density, based on (i) and (ii), which possesses the following properties:

1. The equations of motion are proven to be causal and strongly hyperbolic. In particular, one obtains local existence and uniqueness of solutions for the initial-value problem with Sobolev regular data.

2. All sources of viscous effects (shear, bulk, and thermal conductivity) are taken into account, as well as conservation of baryon density.

3. The second law of thermodynamics holds in the regime of validity of the theory.

4. General equilibrium states are stable in flat spacetime.

Properties 1 through 3 hold in the full nonlinear regime even when the fluid is dynamically coupled to Einstein’s equations. To the best of our knowledge, there is no other theory of relativistic fluid dynamics that simultaneously fulfills all the properties above. This constitutes a solution to the main problems concerning relativistic viscous fluid dynamics initiated by Eckart decades ago. As such, the new theory presented here provides the necessary framework to investigate dissipative effects in relativity such as in neutron star mergers and also in low energy heavy-ion collisions.

We also remark that the BDNK theories introduced in Refs. [61–63] have been shown to be locally well-posed in Sobolev spaces in Refs. [64, 65] without coupling to Einstein’s equations. More precisely, it was shown that a suitable rewriting of the equations as a first-order system, similar to what is done here, leads to strongly hyperbolic equations. While these theories differ from the one treated in this paper in that they do not include baryon density, it is not difficult to see that a minor modification of the proof here presented applies to these theories. This leads to strong hyperbolicity of a similar first-order system of equations and then to local well-posedness in Sobolev spaces also when coupling to Einstein’s equations is considered.

This paper is organized as follows. In the next section we formulate the new theory of relativistic fluid dynamics based on the BDNK approach. In Section III, we determine necessary and sufficient conditions that must be fulfilled by the parameters of the theory for causality to hold. In Section IV, we prove that the full nonlinear system of equations in general relativity is strongly hyperbolic, the solutions are unique, and the initial-value problem is well-posed in general relativity. A new theorem concerning the linear stability properties of relativistic fluids in flat spacetime is proven in Section V. We employ this theorem in Section VI to obtain conditions that ensure that the new theory presented here is stable. Our conclusions and outlook can be found in Section VII.

Definitions: The spacetime metric \(g_{\mu\nu}\) has a mostly plus signature \((- + + +\)). Greek indices run from 0 to 3, Latin indices from 1 to 3. The space-time covariant derivative is denoted as \(\nabla_{\mu}\). We use natural units: \(c = \hbar = k_B = 1\).

II. GENERAL-RELATIVISTIC VISCOUS FLUID DYNAMICS AT FIRST-ORDER

We consider a general-relativistic fluid described by an energy-momentum tensor \(T^{\mu\nu}\) and a timelike conserved current \(J^{\mu}\) associated with a global \(U(1)\) charge that we take to represent baryon number. In our approach, the equations of relativistic fluid dynamics are given by the conservation laws

\[
\nabla_{\mu}J^{\mu} = 0 \quad \text{and} \quad \nabla_{\mu}T^{\mu\nu} = 0,
\]

which are dynamically coupled to Einstein’s field equations

\[
R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi GT_{\mu\nu}.
\]

For the sake of completeness, we begin by recalling the case of a fluid in local equilibrium [2]. In this limit, one uses the following expressions in the conservation laws

\[
T^{\mu\nu} = \varepsilon u^{\mu}u^{\nu} + P \Delta^{\mu\nu} \quad \text{and} \quad J^{\mu} = nu^{\mu},
\]

where \(\varepsilon\) is the equilibrium energy density, \(n\) is the equilibrium baryon density, \(P = P(\varepsilon, n)\) is the thermodynamical pressure defined by the equation of state, and \(u^{\mu}\) is a normalized timelike vector (i.e., \(u_{\mu}u^{\mu} = -1\)) called the flow

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\(^8\) This means that we treat the full set of equations of motion, i.e. Einstein’s equations dynamically coupled to the viscous fluid equations.
velocity, and $\Delta_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$ is a projector onto the space orthogonal to $u^\mu$. The thermodynamical quantities in equilibrium are connected via the first law of thermodynamics $\varepsilon + P = T s + \mu n$, where $T$ is the temperature, $s$ is the equilibrium entropy density, and $\mu$ is the chemical potential associated with the conserved baryon charge $\Omega$. In local equilibrium, both $u_{\mu}T^{\mu\nu}$ and $J^\mu$ are proportional to $u^\mu$ and, thus, the flow velocity may be defined using either quantity [2].

The system of equations [1] and [2] for an ideal fluid [defined by [3]] is causal in the full nonlinear regime. Furthermore, given suitably defined initial data for the dynamical variables, solutions for the nonlinear problem exist and are unique. The latter properties establish that the equations of motion of ideal relativistic fluid dynamics are locally well-posed in general relativity [12, 16]. These properties also hold in the absence of a conserved current i.e., at zero chemical potential where $P = P(\varepsilon)$, and $u^\mu$ is defined by $u_{\mu}T^{\mu\nu}$ [14].

Let us now consider the effects of dissipation. Without any loss of generality, one may decompose the current and the energy-momentum tensor in terms of an arbitrary future-directed unit timelike vector $u^\mu$ as follows [66]:

$$J^\mu = \mathcal{N}u^\mu + \mathcal{J}^\mu$$

(4)

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + u^\mu Q^\nu + u^\nu Q^\mu + \mathcal{T}^{\mu\nu}$$

(5)

where $\mathcal{N} = -u_{\mu}J^\mu$, $\mathcal{E} = u_{\mu}u_{\nu}T^{\mu\nu}$, and $\mathcal{P} = \Delta_{\mu\nu}T^{\mu\nu}/3$ are Lorentz scalars while the vectors $\mathcal{J}^\nu = \Delta^{\nu\mu}J^\mu$, $Q^\nu = -u_{\mu}T^{\mu\nu}\Delta_{\nu}^{\nu}$, and the traceless symmetric tensor $\mathcal{T}^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}T^{\alpha\beta}$, with $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^{\mu\nu}_{\alpha}\Delta^{\nu\beta}_{\alpha} + \frac{1}{3}\Delta^{\mu\nu}_{\alpha\beta}\Delta^{\alpha\beta}\right)$, are all transverse to $u_{\nu}$. Observe that this decomposition is purely algebraic and simply expresses the fact that a vector and a symmetric two-tensor can be decomposed relatively to a future-directed unit timelike vector. The physical content of the theory is prescribed by relating the several components in this decomposition to physical observables, which will then evolve according to [1] and [5].

The general decomposition in Eqs. [1] and [5] expresses $\{J^{\mu}, T^{\mu\nu}\}$ in terms of 17 variables $\{\mathcal{E}, \mathcal{N}, \mathcal{P}, u^\mu, \mathcal{J}^\mu, Q^\nu, \mathcal{T}^{\mu\nu}\}$ and the conservation laws in Eq. [1] give 5 equations of motion for these variables. Therefore, additional assumptions must be made to properly define the evolution of the fluid. As mentioned before, the standard approach in Refs. [14, 21] assumes that $\mathcal{E} = \varepsilon$ and $\mathcal{N} = n$. The same assumption is usually made in MIS [28], though different prescriptions can be easily defined in the context of kinetic theory [33, 67, 68]. A further constraint is usually imposed on the transverse vectors, i.e., either $\mathcal{J}^\mu = 0$ or $Q^\mu = 0$ throughout the evolution. For instance, the former gives $J^\mu = u^\mu$ and $T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + u^\mu Q^\nu + u^\nu Q^\mu + \mathcal{T}^{\mu\nu}$, where $\Pi$ is the bulk viscous pressure (in equilibrium, $\Pi = 0$, $Q^\nu = 0$, and $\mathcal{T}^{\mu\nu} = 0$). In this case, in an extended variable approach such MIS [28], $\Pi$, $Q^\nu$, and $\mathcal{T}^{\mu\nu}$ obey additional equations of motion that must be specified and solved together with the conservation laws, whereas in the NS approach these quantities are expressed in terms of $u^\mu$, $\varepsilon$, and its derivatives.

In this paper we investigate the problem of viscous fluids in general relativity using the BDNK formulation of relativistic fluid dynamics [61, 62]. The formalism was first developed in the case of a conformal fluid at zero chemical potential in Ref. [61] and the extension to non-conformal fluids was performed in [62] and [63]. The BDNK approach applies the basic tenets behind the construction of effective theories [63, 69, 71] to formulate hydrodynamics as a classical effective theory that describes the near equilibrium, long time/long wavelength behavior of many-body systems in terms of the same variables $\{T, \mu, u^\mu\}$ already present in equilibrium. For completeness, we remind the reader that an effective theory is constructed to capture the most general dynamics among low-energy degrees of freedom that is consistent with the assumed symmetries. When this procedure is done using an action principle, the action must include all possible fields consistent with the underlying symmetries up to a given operator dimension and the coefficients of this expansion can then be computed from the underlying microscopic theory. These coefficients are ultimately constrained by general physical principles such as unitarity, CPT invariance, and vacuum stability. Analogously, in an effective field formulation of relativistic viscous hydrodynamics, the equations of motion must take into account all the possible terms in the constitutive relations up to a given order in derivatives that describe deviations from equilibrium. The coefficients that appear in this expansion can then be computed from the underlying microscopic theory (using, for instance, linear response theory [64]), being ultimately constrained by general physical

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9 We note that $u^\mu\nabla_\mu\varepsilon = 0$ and $u^\mu\nabla_\mu n = 0$ in global equilibrium. These are much stronger constraints on the dynamical variables than in the case of local equilibrium where, e.g. only the combination $u^\mu\nabla_\mu(\varepsilon + P)\nabla_\mu u^\mu$ vanishes.

10 Such statement can be rigorously proven by adapting the analysis done in [45].

11 General constraints on the variables in [4] and [5] may be imposed by considering, for instance, energy conditions [13]. In fact, we note that the dominant energy condition [13] imposes that $Y^\mu = -u_{\mu}T^{\mu\nu} = \varepsilon u^\nu + Q^\nu$ must be a future-directed non spacelike vector. This implies that $\mathcal{E} \geq 0$ (weak energy condition) and $\mathcal{E}^2 \geq \mathcal{Q}_\mu \mathcal{Q}^\mu$. Furthermore, by imposing that $J^\mu$ must be future-directed and timelike one finds that $\mathcal{N} \geq 0$ and $\mathcal{N}^2 \geq J_\mu J^\mu$. 


principles such as causality/hyperbolicity in the case of relativistic fluids\cite{17} and also by the fact that the equilibrium state must be stable, i.e. small disturbances from equilibrium in an interacting (unitary) many-body system should decrease with time\cite{72}.

In practice, the most general expressions for the constitutive relations that define the quantities in (\ref{4} and \ref{5}), truncated to first order in derivatives, are (following the notation in \ref{62})

\begin{align}
\mathcal{E} &= \varepsilon + \varepsilon_1 \frac{u^\alpha \nabla_\alpha T}{T} + \varepsilon_2 \nabla_\alpha u^\alpha + \varepsilon_3 u^\alpha \nabla_\alpha (\mu/T), \\
\mathcal{P} &= P + \pi_1 \frac{u^\alpha \nabla_\alpha T}{T} + \pi_2 \nabla_\alpha u^\alpha + \pi_3 u^\alpha \nabla_\alpha (\mu/T), \\
\mathcal{N} &= n + \nu_1 \frac{u^\alpha \nabla_\alpha T}{T} + \nu_2 \nabla_\alpha u^\alpha + \nu_3 u^\alpha \nabla_\alpha (\mu/T), \\
Q^\mu &= \theta_1 \frac{\Delta^\mu\nu \nabla_\nu T}{T} + \theta_2 u^\alpha \nabla_\alpha u^\mu + \theta_3 \Delta^\mu\nu \nabla_\nu (\mu/T), \\
J^\mu &= \gamma_1 \frac{\Delta^\nu\lambda \nabla_\lambda T}{T} + \gamma_2 u^\alpha \nabla_\alpha u^\mu + \gamma_3 \Delta^\nu\lambda \nabla_\lambda (\mu/T), \\
T^{\mu\nu} &= -2\eta \sigma^{\mu\nu},
\end{align}

where $\sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta} \nabla_\alpha u_\beta$ is the shear tensor. The transport parameters $\{\varepsilon_i, \pi_i, \theta_i, \nu_i, \gamma_i\}$ and the shear viscosity $\eta$ are functions of $T$ and $\mu$. Thermodynamic consistency of the equilibrium state (i.e., that $\varepsilon$, $P$, and $n$ have the standard interpretations of equilibrium quantities connected via well-known thermodynamic relations) imposes that $\gamma_1 = \gamma_2$ and $\theta_1 = \theta_2$\cite{62}. The final equations of motion for $\{T, \mu, u^\mu\}$, which are of second-order in derivatives, are found by substituting the expressions above in the conservation laws.

We note that while $T^{\mu\nu}$ and $J^\mu$ have unambiguous meaning (as they correspond to well defined expectation values of quantum operators), the way we choose to express those quantities in terms of the variables $T$, $\mu$, $u^\mu$ and their derivatives is intrinsically ambiguous in an out-of-equilibrium setting\cite{62}. This choice of hydrodynamic variables out of equilibrium defines what is called a hydrodynamic frame\cite{12}. As stressed in\cite{62}, it is of course impossible to not choose a hydrodynamic frame since the latter actually defines the meaning of the variables $\{T, \mu, u^\mu\}$ out of equilibrium. The most commonly used classes of hydrodynamic frames are those proposed by Eckart\cite{21} and Landau-Lifshitz\cite{14}.

In fact, in the regime of validity of the first-order theory, one may shift $\{T, \mu, u^\mu\}$ by adding terms that are of first-order in derivatives, shifting also the transport parameters $\{\varepsilon_i, \pi_i, \theta_i, \nu_i, \gamma_i\}$, without formally changing the physical content of $T^{\mu\nu}$ and $J^\mu$\cite{62}. However, there are combinations of the transport parameters that remain invariant under these field redefinitions. In fact, the shear viscosity $\eta$ and the combination of coefficients that give the bulk viscosity $\zeta$ and charge conductivity $\sigma$ are invariant under first-order field redefinitions, as explained in\cite{62}. Additional constraints among the transport parameters appear when the underlying theory displays conformal invariance, as discussed in\cite{61} at $\mu = 0$, and at finite chemical potential in\cite{62,73} (see also\cite{74}).

Different hydrodynamic frames may be considered as long as such choices match in equilibrium. As a matter of fact, hydrodynamic frames different than Landau-Lifshitz’ and Eckart’s were studied before BDNK in\cite{72,78}. However, clearly not all the choices are appropriate, as discussed in\cite{61,62}. For instance, as mentioned before, it is clear that causality must hold also for viscous fluids in general relativity. Therefore, causality must be a property of the full nonlinear system of equations (\ref{1} and \ref{2}), regardless of the choices employed to describe $T^{\mu\nu}$ and $J^\mu$ in and out of equilibrium. One also requires that the Cauchy problem be locally well-posed, as discussed in the Introduction. In addition, equilibrium states must be stable. Hence, solutions of the equations of motion (\ref{1} and \ref{2}) must also be at least linearly stable with respect to small disturbances around equilibrium in flat spacetime. In summary, physical choices for the meaning of $\{T, \mu, u^\mu\}$ out of equilibrium, and the corresponding coefficients that define the constitutive relations in a given hydrodynamic frame, must lead to equations of motion that are causal and locally well-posed in the full nonlinear regime, and also linearly stable about equilibrium.

Ref.\cite{73} investigated a class of hydrodynamic frames where $\varepsilon_3 = \pi_3 = \theta_3 = 0$. This corresponds to the case where there are non-equilibrium corrections to both the conserved current and the heat flux. This choice is useful when considering relativistic fluids where the net baryon density is not very large, as in high-energy heavy-ion collisions. Conditions for causality were derived and limiting cases were studied that strongly indicated that this choice of hydrodynamic frame is stable against small disturbances around equilibrium. Further studies are needed to better understand the nonlinear features of its solutions (well-posedness) and also the stability properties of this class of hydrodynamic frames at nonzero baryon density in a wider class of equilibrium states.

\footnote{We note that this meaning of the word frame has nothing to do with observers or Lorentz frames. Unfortunately, these terminologies are too widespread to be modified here.}
In this paper we consider another class of hydrodynamic frames that we believe can be more naturally implemented in simulations of the baryon rich matter formed in neutron star mergers or in low energy heavy-ion collisions. Our choice for the hydrodynamic frame is closer to Eckart’s as we define the flow velocity using the baryon current, i.e., \( J^\mu = n u^\mu \) holds throughout the evolution (\( \gamma_1 = \nu_1 = 0 \)). Clearly, this limits the domain of applicability of the theory to problems where there are many more baryons than anti-baryons so the net baryon charge is large.

In this case, it is more convenient to use \( \varepsilon \) and \( n \) as dynamical variables instead of \( T \) and \( \mu/T \) because the most general expressions for the Lorentz scalar contributions to the constitutive relations involve only linear combinations of \( w^\mu \nabla_\mu \varepsilon \) and \( \nabla_\mu u^\mu \), given that current conservation implies that the replacement \( u^\lambda \nabla_\lambda n = -n \nabla_\lambda u^\lambda \) is valid. For simplicity, we choose\(^\text{13}\) to parametrize the out of equilibrium corrections to the scalars as follows

\[
\begin{align*}
\mathcal{E} &= \varepsilon + \tau_\varepsilon \left[ u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P) \nabla_\lambda u^\lambda \right] \\
\mathcal{P} &= P - \zeta \nabla_\lambda u^\lambda + \tau_P \left[ u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P) \nabla_\lambda u^\lambda \right],
\end{align*}
\]

(7a)

(7b)

where \( \tau_\varepsilon \) and \( \tau_P \) have dimensions of a relaxation time and \( \zeta \) is the bulk viscosity transport coefficient. When evaluated on the solutions of the equations of motion, one can see that these quantities assume their standard form as in Eckart’s theory up to second order in derivatives because \( \mathcal{E} \sim \varepsilon + O(\partial^2) \) and \( \mathcal{P} = P - \zeta \nabla_\mu u^\mu + O(\partial^2) \) on shell\(^\text{14}\).

In fact, we remind the reader that in Eckart’s theory \([21]\) the energy-momentum tensor is given by \( T_{\mu\nu} = \varepsilon u_\mu u_\nu + (P - \zeta \nabla_\lambda u^\lambda) \Delta_{\mu\nu} - 2\eta \sigma_{\mu\nu} + u_\mu Q_\nu + u_\nu Q_\mu \), with heat flux \( Q_\mu = -\kappa T \left( u^\lambda \nabla_\lambda u_\mu + \Delta_{\mu\lambda} \nabla_\lambda T/T \right) \) where \( \kappa = (\varepsilon + P)^2 \sigma/(n^2 T) \) is the thermal conductivity coefficient. However, as remarked in \([62]\), in the domain of validity of the first-order theory one may rewrite the Eckart expression for the heat flux as \( Q_\mu = \sigma T \left( (\varepsilon + P) / n \right) \Delta_{\mu\lambda} \nabla_\lambda (\mu/T) \) plus second-order terms. This is done by noticing that \( (\varepsilon + P) u^\lambda \nabla_\lambda u^\mu + \Delta_{\mu\lambda} \nabla_\lambda P = 0 + O(\partial^2) \) on shell, which implies that one may write\(^\text{15}\)

\[
\frac{u^\lambda \nabla_\lambda u^\alpha + \Delta_{\alpha\lambda} \nabla_\lambda T}{T} = -\frac{n T}{\varepsilon + P} \Delta_{\alpha\lambda} \nabla_\lambda (\mu/T) + O(\partial^2).
\]

(8)

Therefore, one can always choose the coefficients such that the heat flux \( Q^\mu \) has the same physical content of Eckart’s theory plus terms that are of second order on shell. We use this to write the heat flux as

\[
Q_\mu = \sigma T \left( (\varepsilon + P) / n \right) \Delta_{\mu\lambda} \nabla_\lambda (\mu/T) + \tau_Q \left[ (\varepsilon + P) u^\lambda \nabla_\lambda u_\mu + \Delta_{\mu\lambda} \nabla_\lambda P \right],
\]

(9)

where \( \tau_Q \) has dimensions of a relaxation time.

We display below the equations that define the conserved current and the energy-momentum tensor that will be further investigated in this work:

\[
\begin{align*}
J^\mu &= n u^\mu \\
T^{\mu\nu} &= (\varepsilon + A) u^\mu u^\nu + (P + \Pi) \Delta^{\mu\nu} - 2\eta \sigma^{\mu\nu} + u^\mu Q^\nu + u^\nu Q^\mu \\
A &= \tau_\varepsilon \left[ u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P) \nabla_\lambda u^\lambda \right] \\
\Pi &= -\zeta \nabla_\lambda u^\lambda + \tau_P \left[ u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P) \nabla_\lambda u^\lambda \right] \\
Q^\nu &= \tau_Q (\varepsilon + P) u^\lambda \nabla_\lambda u^\nu + \beta_\varepsilon \Delta^{\nu\lambda} \nabla_\lambda \varepsilon + \beta_n \Delta^{\nu\lambda} \nabla_\lambda n
\end{align*}
\]

(10a)

(10b)

(10c)

(10d)

(10e)

where

\[
\begin{align*}
\beta_\varepsilon &= \tau_Q \left( \frac{\partial P}{\partial \varepsilon} \right)_{n} + \sigma T (\varepsilon + P) n \left( \frac{\partial (\mu/T)}{\partial \varepsilon} \right)_{n} \\
\beta_n &= \tau_Q \left( \frac{\partial P}{\partial n} \right)_{\varepsilon} + \sigma T (\varepsilon + P) \left( \frac{\partial (\mu/T)}{\partial n} \right)_{\varepsilon},
\end{align*}
\]

(11a)

(11b)

and \( \tau_\varepsilon, \tau_P, \) and \( \tau_Q \) quantify the magnitude of second order corrections to the out of equilibrium contributions to the energy-momentum tensor given by the energy density correction \( A \), the bulk viscous pressure \( \Pi \), and the heat flux

\(^{13}\) We note that, in practice, 8 out of the 14 parameters in \([\text{14}]\) (note that \( \theta_1 = \theta_2 \) and \( \gamma_1 = \gamma_2 \)) can be set using first-order field redefinitions \([\text{62}]\). One is then left with \( \eta, \zeta, \sigma \), and three other parameters.

\(^{14}\) We follow traditional terminology where a given quantity is said to be on shell when it is evaluated using the solutions to the equations of motion.

\(^{15}\) Using the standard thermodynamic relation \( \frac{d P}{d T} = \frac{d T}{d P} \partial \left( \frac{d P}{d T} \right) \).
\[ Q^\mu. \] The equations of motion for the fluid variables are obtained from the conservation laws and they can be written explicitly as
\[
\begin{align*}
  u^\lambda \nabla_\lambda n + n \nabla_\lambda u^\lambda &= 0, \quad (12a) \\
  u^\lambda \nabla_\lambda (\varepsilon + P) + \Delta^\lambda_a \nabla_\lambda P &= - (A + \Pi) u^\lambda \nabla_\lambda A - (A + \Pi) u^\lambda - \nabla_\mu Q^\mu + 2\eta \sigma_{\mu \nu} \sigma^{\mu \nu}, \quad (12b) \\
  \nabla_\mu u^\mu &= A + \Pi - \Delta^\lambda_a \nabla_\lambda A, \quad (12c)
\end{align*}
\]
where \( \omega_{\mu \nu} = \frac{1}{2} (\Delta^\lambda_a \nabla_\lambda u^\nu - \Delta^\nu_a \nabla_\lambda u^\mu) \) is the kinematic vorticity tensor. The equations above show that, on shell, \( A \sim 0 + \mathcal{O}(\partial^2) \), \( \Pi \sim - \nabla_\mu u^\mu + \mathcal{O}(\partial^2) \), and \( Q_\mu = \sigma T \frac{\varepsilon + P}{n} \Delta^\lambda_a \nabla_\lambda (\mu/T) + \mathcal{O}(\partial^2) \). Eqs. (11), (11), and (12) define a causal and stable generalization of Eckart’s theory that is fully compatible with general relativity, as we shall prove in the next sections. We remark that when one neglects the effects of a conserved current altogether, the theory reduces to the case studied in Refs. \[62, 63\]. For additional discussion about the case without a chemical potential, including far from equilibrium behavior and also the presence of analytical solutions, see Refs. \[79, 81\].

### A. Entropy Production

It is instructive to investigate how the second law of thermodynamics is obeyed in this general first-order approach. This was discussed in detail by Kovtun in \[62\] and, more recently, by other authors in Ref. \[82\].

The standard covariant definition of the entropy current based on the first law of thermodynamics \( TS^\mu = Pu^\mu - u_\nu T^{\mu \nu} - \mu J^\mu \) \[28\], together with (10), can be used to show that the entropy density measured by a co-moving observer is given by
\[
\frac{n}{\varepsilon + P} Q^\mu = s + \frac{A}{T} \frac{\nabla_\mu T}{T}. \quad (13)
\]

Given that in our system one finds that \( A = 0 + \mathcal{O}(\partial^2) \), one can see that the entropy density is maximized in equilibrium in the regime of validity of the first-order theory.

Furthermore, using Eqs. (11) and (12) one finds that the divergence of the entropy current is given by
\[
\nabla_\mu S^\mu = 2\eta \sigma_{\mu \nu} \sigma^{\mu \nu} - \frac{\Pi}{T} \nabla_\mu u^\mu + \frac{n}{\varepsilon + P} Q^\nu \Delta^\mu_a \nabla_\lambda (\mu/T) - \frac{Q^\mu}{T} \left[ u^\lambda \nabla_\lambda u_\nu + \frac{\Delta^\lambda_a \nabla_\lambda P}{\varepsilon + P} \right] - \frac{A}{T} \frac{u^\lambda \nabla_\lambda T}{T} \frac{T}{T}. \quad (14)
\]

It is crucial to note \[62\] that in a first-order approach \( \nabla_\mu \nabla_\lambda \mu \) can only be correctly determined up to second order in derivatives.\(^\text{16}\) This means that not all the terms in (14) actually contribute to this expression at second order. For instance, when evaluating (14) on shell one must keep in mind that the last two terms in (14) are already at least of third order and must, thus, be dropped. A similar argument can be used to show that the term \( II \nabla_\mu u^\mu = - \zeta (\Delta_\mu u^\nu)^2 + \mathcal{O}(\partial^3) \). Therefore, one can see that
\[
\nabla_\mu S^\mu = 2\eta \sigma_{\mu \nu} \sigma^{\mu \nu} + \frac{\zeta (\Delta_\mu u^\nu)^2}{T} + \sigma T \left[ \Delta^\mu_a \nabla_\lambda (\mu/T) \right] \left[ \Delta_\mu^\alpha \nabla_\lambda (\mu/T) \right] + \mathcal{O}(\partial^3), \quad (15)
\]
which is non-negative when \( \eta, \zeta, \sigma \geq 0 \). Hence, there are no violations of the second law of thermodynamics in the domain of validity of the first-order theory - higher order derivative terms \( \mathcal{O}(\partial^3) \) in the entropy production can only be understood by considering terms of higher order in derivatives in the constitutive relations in \( T^{\mu \nu} \) and \( J^\mu \), which is beyond the scope of the first-order approach.

### III. Causality

In order to determine the conditions under which causality holds in this theory, we need to understand the system’s characteristics. Our system is a mixed first-second order system of PDEs. While the principal part and characteristics

\(^\text{16}\) We remark that in this argument terms such as \( \nabla_\mu \nabla_\nu \phi \) and \( (\nabla_\mu \phi)(\nabla_\nu \phi) \), for any field \( \phi \), count as second order terms.
of systems of this form can be investigated using Leray’s theory [18, 83, 84], here it is simpler to transform our equations into a system where all equations are of second-order. We thus apply $u^\mu \nabla_\mu$ to (12a). In this case, the conservation laws coupled to Einstein’s equations (2) written in harmonic gauge, $g^{\mu\nu} \Gamma^\alpha_{\mu\nu} = 0$, read

\begin{align}
&u^\beta u^\alpha \partial_\alpha n + n \delta^\alpha_\beta u^\beta \partial_\beta u + \tilde{B}_1(n, u, g) \partial^2 g = B_1(\partial n, \partial u, \partial g), \\
&(\tau_e u^\alpha u^\beta + \beta_\epsilon \Delta^\alpha_\beta) \partial_\alpha \xi + \beta_n \Delta^\alpha_\beta \partial_\beta n + \rho(\tau_\xi + \tau_Q) u^\alpha \delta^\beta_\nu \partial_\beta u + \tilde{B}_2(\xi, n, u, g) \partial^2 g = B_2(\partial \xi, \partial n, \partial u, \partial g), \\
&(\beta_\epsilon + \tau_P) u^\alpha \Delta^\alpha_\beta \partial_\beta \xi + \beta_n u^\alpha \Delta^\alpha_\beta \partial_\beta n + C_{\nu}^{\alpha\beta} \partial^\alpha \xi + \tilde{B}_3(\xi, n, u, g) \partial^2 g = B_3^\nu(\partial \xi, \partial n, \partial u, \partial g), \\
&g^{\alpha\beta} \partial^2_\alpha \xi = B_4^{\mu\nu}(\partial \xi, \partial n, \partial u, \partial g),
\end{align}

where $\partial_\alpha = \partial / \partial x^\alpha$ (using standard partial derivatives), $\rho = (\varepsilon + P)$, and $A_{(\alpha B_\beta)} = (A_\alpha A_\beta + A_\beta A_\alpha)/2$. The remaining notation is as follows. We use $\partial^\ell \xi$ to indicate that a term depends on at most $\ell$ derivatives of $\xi$. A term of the form $B(\partial^{i_1} \phi_1, \ldots, \partial^{i_k} \phi_k) \partial^\ell \phi_i$, $i \in \{1, \ldots, k\}$, indicates an expression that is linear in $\partial^\ell \phi_i$, with coefficients depending on at most $\ell_1$ derivatives of $\phi_1, \ldots, \ell_k$ derivatives of $\phi_k$. For example, the term $17 \ (u^\mu \partial_\mu \xi + \partial_\mu u^\mu) g^{\alpha\beta} \partial_\alpha \xi \partial_\beta \xi$ would not be written as $B(\partial \xi, \partial n, \partial u, \partial g) \partial^2 g$. The terms $\tilde{B}$ above are top-order in derivatives of $g$ and thus belong to the principal part, although, as we will see, their explicit form is not needed for our argument, whereas the $B$ terms are lower order and do not contribute to the principal part. We have also defined

$$C_{\nu}^{\alpha\beta} = (\tau_P \rho - \zeta - \frac{\eta}{3}) \Delta^\alpha(\delta^\beta_\nu) + (\rho \tau_Q u^\alpha u^\beta - \eta \Delta^\alpha_\beta) \delta^\mu_\nu,$$

where $\rho = \varepsilon + P$. We notice that by taking $u^\mu \nabla_\mu$ of (12a) we are not introducing new characteristics in the system. This can be viewed from the characteristic determinant computed below which contains an overall factor of $u^\mu \xi_\mu$ to a power greater than one. Theorem I below establishes necessary and sufficient conditions for causality to hold in our system of equations. We show that the assumptions of Theorem I are not empty in section VI A. Throughout this paper, we use the following definition for the speed of sound $c_s$:

$$c_s^2 = \left( \frac{\partial P}{\partial \varepsilon} \right)_s = \left( \frac{\partial P}{\partial n} \right)_n + \frac{n}{\varepsilon + P} \left( \frac{\partial P}{\partial n} \right)_\varepsilon,$$

where $\bar{s}$ is the equilibrium entropy per particle. Also, we define

$$\kappa_s = \frac{nT}{(\varepsilon + P)^2} \left( \frac{\partial (\mu/T)}{\partial \varepsilon} \right)_s = \frac{nT}{(\varepsilon + P)^2} \left[ \frac{\partial (\mu/T)}{\partial n} \right]_n + T(\varepsilon + P) \left[ \frac{\partial (\mu/T)}{\partial n} \right]_\varepsilon.$$

**Theorem I.** Let $(\varepsilon, n, u^\mu, g_{\alpha\beta})$ be a solution to (2) and (12), with $u^\mu u_\mu = -1$, defined in a globally hyperbolic spacetime $(M, g_{\alpha\beta})$. Assume that:

(A1) $\rho = \varepsilon + P, \tau_\epsilon, \tau_Q, \tau_P > 0$ and $\eta, \zeta, \sigma \geq 0$.

Then, causality holds for $(\varepsilon, n, u^\mu, g_{\alpha\beta})$ if, and only if, the following conditions are satisfied:

\begin{align}
\rho \tau_Q &> \eta, \\
\tau_\epsilon \left( \rho c_s^2 \tau_Q + \zeta + \frac{4\eta}{3} + \sigma \kappa_s \right) + \rho \tau_P \tau_Q &> 4 \rho \tau_Q \left[ \tau_P \left( \rho c_s^2 \tau_Q + \sigma \kappa_s \right) - \beta_\epsilon \left( \zeta + \frac{4\eta}{3} \right) \right] > 0, \\
2 \rho \tau_\epsilon \tau_Q &> \tau_\epsilon \left( \rho c_s^2 \tau_Q + \zeta + \frac{4\eta}{3} + \sigma \kappa_s \right) + \rho \tau_P \tau_Q \geq 0, \\
\rho \tau_\epsilon \tau_Q + \sigma \kappa_s \tau_P &> \tau_\epsilon \left( \rho c_s^2 \tau_Q + \zeta + \frac{4\eta}{3} + \sigma \kappa_s \right) + \rho \tau_P \tau_Q (1 - c_s^2) + \beta_\epsilon \left( \zeta + \frac{4\eta}{3} \right).
\end{align}

The same result holds true for equations (12) if the metric is not dynamical.

**Proof.** The proof can be reduced to a computation of the characteristics of (10). We only consider the 10 independent components of the metric and, thus, this system of equations can be written in terms of a $16 \times 1$ column

---

17 A term of this form is not present in our system, we write it here only for illustration.
Mathematically, to see that
\[ \text{det}\left[ \begin{array}{cccc}
\tau, u^\alpha u^\beta \\
(\beta_c + \tau Q)u^\alpha (\Delta^\beta)^{\mu}
\end{array} \right] = 0 \]
where, to obtain (24a) we defined
\[ A \equiv \rho \tau_c \tau_Q, \]
\[ B \equiv -\tau_c \left( \rho c_2^2 \tau_Q + \zeta + \frac{4\eta}{3} + \sigma \kappa \right) - \rho \tau_p \tau_Q, \]
\[ C \equiv \tau_p \left( \rho c_2^2 \tau_Q + \sigma \kappa \right) - \beta_c \left( \zeta + \frac{4\eta}{3} \right), \]
and the fact that \( \beta_c + n \beta_n / \rho = \tau_Q c_3^2 + \sigma \kappa / \rho \). In the calculation performed above we used the well-known identity \( \text{det} (\delta^\mu_\nu + A^\mu B_\nu) = 1 + A^\mu B_\mu \) that follows from Silvester’s determinant theorem. In Eq. (24a) it becomes evident that assumption (A1) guarantees that \( v^\mu \neq 0 \), eliminating one of the possible acausal roots. From (24a) to (24b) we defined \( n_1 = 3, n_\pm = 1, c_1 = \frac{m}{\rho \tau_c}, \) and \( c_{\pm} = -\frac{B^+ \sqrt{B^+ - 4AC}}{2A} \). Note that since \( \xi_\alpha \xi_\alpha = -b^2 + (v \cdot v) \), besides the causal root \( b = u^\alpha \xi_\alpha = 0 \), the remaining roots in (24c) can be cast as \( b^2 = c_\alpha (v \cdot v) \). Then, (C1) demands that \( c_\alpha \in \mathbb{R} \) and (C2) that \( 0 \leq c_\alpha < 1 \) for causality.\(^{18} \) Therefore, the 6 roots related to \( c_1 \) are causal when (24a) is observed. As for the roots \( c_{\pm} \), they are real if \( B^2 - 4AC \geq 0 \), i.e., if the first inequality in (20b) holds. On the other hand, (C2) demands that \( c_{\pm} \geq 0 \) which is guaranteed if \( -B \geq 0 \) [second inequality in condition (20c)] together with \( C \geq 0 \) [second inequality of (20c)], and \( c_{\pm} < 1 \), which demands that \( 2A + B > 0 \) [first inequality in condition (20c)] and \( A + B + C > 0 \) [condition (20d)].
We observe that, although we employed the harmonic gauge to calculate the system’s characteristics, the causality established in Theorem I does not depend on any gauge choices. This follows from well-known properties of Einstein’s equations [19] and the geometric invariance of the characteristics [53]. See the end of Section IV.C for further comments in this direction.

The analysis above and the conditions we obtained for causality are valid in the full nonlinear regime of the theory. However, we remark in passing that the principal part concerning only the fluid equations would have exactly the same structure if one were to linearize the fluid dynamic equations about equilibrium with nonzero flow in Minkowski spacetime. This is a generic feature of the BDNK approach (at least, when truncated at first order), i.e., the analysis of the system’s characteristics, and thus of its causality properties, is the same in the nonlinear regime and in the linearization about a generic equilibrium state. This is not, however, a general feature of hydrodynamic models as it does not hold in MIS theories. In fact, as discussed at length in [43, 44], in MIS the thermodynamic fluxes explicitly enter in the calculation of the characteristics, but they are not present in the linear analysis.

IV. STRONG HYPERBOLICITY AND LOCAL WELL-POSEDNESS

In this section we investigate the initial-value problem for equations (2) and (12). The goal is to show that the system is causal and locally well-posed under very general conditions. First, we briefly discuss the initial data required to solve the system of equations. Then, we re-write our system as a first-order system. We show that this first-order system is diagonalizable in the sense of Proposition I. This means, in particular, that the system is strong hyperbolic according to the usual definition of the term, as in, e.g., [2, 20]. The importance of having strongly hyperbolic equations is due to its implications for the initial-value problem. As already mentioned, one is generally interested in evolution equations that are locally well-posed in Sobolev spaces. For equations with constant coefficients, local well-posedness in Sobolev spaces is equivalent to strong hyperbolicity [50]. For non-constant coefficients and nonlinear systems, such an equivalence does not hold [57, 58]. However, there remains a close connection between strong hyperbolicity and local well-posedness in Sobolev spaces. For most reasonable systems, once diagonalizability is available, one can use standard techniques to derive energy estimates which, in turn, can be used to prove local well-posedness. This is precisely the case for our system of equations. Even though our equations consist of a system of second order PDEs, we can use the diagonalized system of first-order equations to derive energy estimates. Once these estimates are available, we use a standard approximation argument as in [16, 90] to obtain local well-posedness (see Theorem II).

A. Initial data

Equations (12) are second order in $\varepsilon$, $n$, and $u^\mu$. Thus, initial data along a non-characteristic hypersurface consist of $\varepsilon$, $n$, $u^\mu$ and their first-order time derivatives. Clearly, the initial $u^\mu$ has to satisfy $u^\mu u^\mu = -1$. Also, it is important to note that Eq. (12a) is first-order and, thus, the initial-data cannot be arbitrary but must satisfy a compatibility condition ensuring that (12a) holds at $t = 0$. Therefore, one can use (12a) to write the time derivative of $n$ in terms of the time derivative of $u^\mu$ (this feature would also appear in Navier-Stokes theory in the Eckart hydrodynamic frame).

A natural choice to determine the initial conditions for the matter sector is to set an initial state that is within the regime of validity of the first-order theory and closely reproduces Eckart’s theory. First, one can directly extract $n$ and $u^\mu$ from $J^\mu$ at the initial spacelike hypersurface. Then, one sets the non-equilibrium correction to the energy density $\mathcal{A}$ in (10) to zero in the initial state, so then the initial value for $\varepsilon$ equals $T^{\mu\nu}u_\mu u_\nu$ and the first-order time derivative of $\varepsilon$ is defined in terms of the first-order time derivative of the flow velocity (plus spatial derivatives that are known in the initial state). Clearly, $\mathcal{A}$ will be different than zero later during the actual evolution, and its value can be used to check if the simulations remain within the regime of validity of the first-order approach (i.e., $|\mathcal{A}|/\varepsilon$ must remain smaller than unit). Finally, the time derivative of the flow velocity can be set by imposing that the second-order on shell term $(\varepsilon + P)u^\lambda \nabla_\lambda u^\nu + \Delta^\lambda \nabla_\lambda P$ vanishes. Hence, one can obtain the time derivative of the flow velocity and all the other required initial data in the regime of validity of the first-order approach, emulating Eckart’s theory as much as possible.

We recall that the initial-data for the gravitational sector has to further satisfy the well-known Einstein constraint equations. We briefly make some comments on this in section VII.
B. Diagonalization and Eigenvectors

In this section we write equations (2) and (12) as a first-order system, as discussed above. For this, we begin defining the variables $V = u^a \partial_a \varepsilon$, $\nu^\nu = \Delta^\nu_{\alpha} \partial_\alpha \varepsilon$, $W = u^a \partial_a n$, $\nu^\mu = \Delta^\mu_{\alpha} \partial_\alpha n$, $S^\mu = u^a \nabla_a u^\mu$, $S_\alpha^\nu = \Delta^\alpha_{\alpha} \nabla_\alpha u^\nu$, $F_{\mu \nu} = u^a \partial_a g_{\mu \nu}$, and $F_{\rho}^\mu = \Delta^{\rho \alpha} \partial_\alpha g_{\mu \nu}$. Then, the equations of motion can be cast as

$$
\tau_\alpha \partial_\alpha V + \tau_\sigma \partial_\sigma S^\nu + \tau_\nu \partial_\nu S^\sigma + \beta_\alpha \partial_\alpha W^\nu + \beta_\nu \partial_\nu W^\sigma = r_1, 
$$

$$
\tau_\rho \partial_\rho \nu^\mu - \Delta^\mu_{\alpha} \partial_\alpha V = r_2, 
$$

$$
u^\alpha \partial_\alpha W^\mu + n \Delta^\mu_{\alpha} \partial_\alpha S^\nu = r_3, 
$$

$$
u^\alpha \partial_\alpha S^\nu - \Delta^\alpha_{\alpha} \partial_\alpha \nu^\nu - \Lambda^\nu_{\alpha \beta} \partial_\alpha F_A - \gamma^\nu_{\delta \alpha} \partial_\alpha F^\delta_A = r_4, 
$$

$$u^\alpha \partial_\alpha W^\mu + \Delta^\alpha_{\beta} F^\beta_A = r_5, 
$$

$$u^\alpha \partial_\alpha \varepsilon = r_6, 
$$

$$u^\alpha \partial_\alpha n = r_7, 
$$

$$u^\nu \partial_\nu u^\mu = r_8, 
$$

$$u^\alpha \partial_\alpha g = r_9, 
$$

$$u^\sigma \partial_\sigma g_{\mu \nu} = r_{10}, 
$$

$$u^\sigma \partial_\sigma g_{\mu \nu} = r_{11}, 
$$

where $A = \sigma \beta$ for $\sigma \geq \beta$, i.e., $A$ takes the 10 independent values 00, 01, 02, 03, 11, 12, 13, 22, 23, 33 with repeated index $A$ summing from 00 to 33,

$$\Pi^\mu_{\nu \lambda \alpha} = -\eta (\Delta^\mu_{\nu \lambda \alpha} + \Delta^\nu_{\mu \lambda \alpha}) + \left( \rho \tau P - \zeta + \frac{2\mu}{3} \right) \Delta^\nu_{\mu \lambda \alpha},$$

$$\Lambda^\nu_{\mu \lambda \alpha} = \frac{1}{2} \left[ g^{\sigma \nu} (\Delta^\beta_{\alpha \lambda}) u^\sigma - u^\nu (\Delta^\beta_{\alpha \lambda}) g^{\nu \sigma} - u^\nu (\Delta^\beta_{\nu \lambda}) \Delta^\alpha_{\nu \sigma} \right] (2 - \delta_A),$$

$$\gamma^\nu_{\delta \alpha} = \frac{1}{2} u^{(\nu \delta \beta)} \Delta^\alpha_{\nu \delta} (2 - \delta_A).$$

By $\delta_A$ we mean the Kronecker delta in the sense that when $A = \sigma \beta$ then $\delta_A = \delta_{\sigma \delta}$, and zero otherwise. Also, the terms $r$ may be functions of the 95 variables. The equations in (20) were obtained as follows: Eqs. (26a) and (26f) come from the conservation law $\nabla_\nu T^{\mu \nu} = 0$ when projected into the directions parallel and perpendicular to $u^\nu$, respectively. Eqs. (26b), (26c), (26d), and (26g), correspond, respectively, to the identities $\nabla_\alpha \nabla_\beta \varepsilon - \nabla_\beta \nabla_\alpha \varepsilon = 0$, $\nabla_\alpha \nabla_\beta n - \nabla_\beta \nabla_\alpha n = 0$, $\nabla_\alpha \nabla_\beta w^\nu - \nabla_\beta \nabla_\alpha w^\nu = \nabla_\beta \nabla_\alpha \varepsilon$, terms of order zero in derivatives, and $\partial_\alpha \partial_\beta g_{\mu \nu} - \partial_\beta \partial_\alpha g_{\mu \nu} = 0$, all contracted with $u^\alpha \Delta^\beta_{\nu \lambda}$. Eq. (26) is the Einstein equation in the harmonic gauge, i.e., $g^{\alpha \nu} \partial_\alpha \partial_\beta g_{\mu \nu} = 0$ terms of lower order in derivatives, while (26h) - (26k) are the definitions of $V$, $W$ (also using the identity $u^\nu \nabla_\alpha n + n \nabla_\alpha u^\nu = W + n S_3^\nu$ to eliminate $W$ thoroughly), $S^\nu$, and $F_A$, respectively. We may now define the 95 × 1 column vectors $U$ and $R$ as

$$U = \begin{bmatrix} u_m \\ u_g \\ u_d \end{bmatrix}$$

and $R = (r_1, \cdots, r_{11})^T$, where $u_m = (V, \nu^\nu, \nu^\mu, S^\nu, S_1^\nu, S_2^\nu, S_3^\nu)^T \in \mathbb{R}^{29}$, $u_g = (F_A, F_A^1, F_A^2, F_A^3, F_A^4)^T \in \mathbb{R}^{50}$, and $u_d = (\varepsilon, n, \nu^\mu, g)^T \in \mathbb{R}^{16}$, to write the quasi-linear first order system (20) in matrix mode as

$$\mathbb{A}^A \partial_A U = R,\tag{29}$$

where $\mathbb{A}^A = A^A \oplus u^a I_{16}$ (being the direct sum). The matrix $A^A$ is split into the form

$$A^A = \begin{bmatrix} A_m & -L_A \\ 0_{50 \times 29} & A_g \end{bmatrix}$$

(30)
where

\[ A_m^\alpha = \begin{bmatrix}
\tau_x u^\alpha & \rho T Q u^\alpha \delta_v^\mu & \beta_c^\alpha u^\alpha \delta_v^\mu & \beta_n^\alpha u^\alpha \delta_v^\mu & \rho r_x u^\alpha \delta_v^\mu & \rho r_\zeta u^\alpha \delta_v^\mu & \rho r_\tau u^\alpha \delta_v^\mu & \rho r_\tau u^\alpha \delta_v^\mu \\
\tau P \Delta^\mu v & \rho T Q u^\alpha \delta_v^\mu & \beta_c^\alpha u^\alpha \delta_v^\mu & \beta_n^\alpha u^\alpha \delta_v^\mu & \rho r_x u^\alpha \delta_v^\mu & \rho r_\zeta u^\alpha \delta_v^\mu & \rho r_\tau u^\alpha \delta_v^\mu & \rho r_\tau u^\alpha \delta_v^\mu \\
- \Delta^\mu v & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4x1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4x4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{31}\]

while

\[ A_9^\alpha = \begin{bmatrix}
u^\alpha I_{10} & - \Delta^\mu I_{10} & - \Delta^\mu I_{10} & - \Delta^\mu I_{10} & - \Delta^\mu I_{10} \\
- \Delta^\mu I_{10} & u^\alpha I_{10} & 0 & 0 & 0 & 0 \\
- \Delta^\mu I_{10} & 0 & u^\alpha I_{10} & 0 & 0 & 0 \\
- \Delta^\mu I_{10} & 0 & 0 & u^\alpha I_{10} & 0 & 0 \\
- \Delta^\mu I_{10} & 0 & 0 & 0 & u^\alpha I_{10} & 0 \\
- \Delta^\mu I_{10} & 0 & 0 & 0 & 0 & u^\alpha I_{10}
\end{bmatrix}, \tag{32}\]

and

\[ L^\alpha = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{33}\]

We are now ready to establish that, when written as a first-order system as above, the equations of motion are strongly hyperbolic. In section VI A we show that the assumptions of Proposition I are not empty.

**Proposition I.** Consider the system (26). Assume that (A1) with \( \eta > 0 \) holds and that (20) in Theorem I holds in strict form, i.e., with > instead of \( \geq \). Let \( \xi \) be a timelike co-vector. Then:

(i) \( \det(\xi^\alpha \xi_\alpha) \neq 0 \); 

(ii) For any spacelike vector \( \zeta \), the eigenvalue problem \( (\xi_\alpha + \Lambda \xi_\alpha) \xi^\alpha V = 0 \) has only real eigenvalues \( \Lambda \) and a complete set of eigenvectors \( V \).

**Proof.** To prove (i) we may compute the determinant \( \det(\xi^\alpha \xi_\alpha) = \det(\xi^\alpha A_m^\alpha) \det(A_9^\alpha) (u^\alpha \xi_\alpha)^{10} \). Note that \( u^\alpha \xi_\alpha \neq 0 \) if \( \xi \) is timelike. We must then look into the matter and gravity sector in what follows. We again define \( b = u^\alpha \xi_\alpha \) and \( v^\mu = \Delta^\mu \xi_\alpha \), \( v \cdot v = \Delta^\mu \xi_\mu \xi_\nu \), and introduce

\[ \Xi^\mu = v_\lambda \Pi^\mu \lambda v^\lambda \xi_\alpha = - \eta (v \cdot v) \delta^\mu_\nu - \eta u^\mu \xi_\nu + \left( \rho r_\tau - \zeta + \frac{2\eta}{3} \right) v^\mu v_\nu \]  \tag{34}
to obtain

\[
\det(\xi_A A_m^a) = \det \begin{bmatrix}
\tau_c b_2 + \beta_a (v \cdot v) & b_2 (\rho \tau Q \xi \nu + \rho \tau \xi \nu) - n \xi_\alpha (v \cdot v) v_\nu \\
(\tau_p + \beta_x) v^\mu & \rho \tau Q b^2 \delta_{\nu}^\mu + \Xi_{\nu}^\mu - n \xi_\alpha v^\mu v_\nu \\
\end{bmatrix}
\]

\[
= b^{19} \det \left[ \tau_c b_2 + \beta_a (v \cdot v) \right] \det \left\{ \left[ \rho \tau Q b^2 - n (v \cdot v) \right] \delta_{\nu}^\mu - n \xi_\alpha v^\mu \right\}
\]

\[
+ \left( \rho \tau_p - \xi + \frac{2 \xi}{3} - n \xi_\alpha \right) v^\mu v_\nu \left( \tau_c b_2 + \beta_a (v \cdot v) \right) \left[ b^2 (\rho \tau Q \xi \nu + \rho \tau \xi \nu) - n \xi_\alpha (v \cdot v) v_\nu \right] v^\mu
\]

\[
= b^{19} \left[ \rho \tau Q b^2 - n (v \cdot v) \right]^3 \left\{ \tau_c b_2 + \beta_a (v \cdot v) \right\} \left[ \rho \tau Q b^2 - \frac{4 \xi}{3} (v \cdot v) + (\rho \tau_p - \xi - n \xi_\alpha) (v \cdot v) \right]
\]

\[
- (\tau_c + \beta_a) \left[ b^2 (\rho \tau Q + \rho \tau) - n \xi_\alpha (v \cdot v) \right] (v \cdot v)
\]

\[
= \rho^{4} \tau Q \tau_c b^{19} \prod_{a=1, \pm} \left[ b^2 - c_a (v \cdot v) \right]^{n_a},
\]

(35)

where, as we have obtained in (24), (25), and in the text below it, \( n_t = 3, n_\pm = 1, c_1 = \frac{-1}{n}, \) and \( c_\pm = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \).

It is worth mentioning that the assumptions of Proposition I guarantee that \( 0 < c_1, c_\pm < 1 \). We have used recursively in (35) the formula

\[
\det \begin{bmatrix}
A_{n \times n} & B_{n \times m} \\
C_{m \times n} & D_{m \times m} \\
\end{bmatrix} = \det(D) \det(A - BD^{-1}C)
\]

(36)

for any invertible square matrix \( D \). Under assumptions (A1), \( \eta > 0 \), and conditions (24) in the strict form, then one obtains that \( \det(\xi_A A_m^a) = 0 \) only if \( 0 \leq c_\alpha < 1 \) (with the equality holding only in the case \( a = 0 \)), i.e., the equation \( b_2^2 - c_a (v_\alpha \cdot v_\alpha) = 0 \) gives \( \xi_\alpha, \alpha \) such that \( \xi_\alpha \xi_\alpha = -b_2^2 + v_\alpha \cdot v_\alpha = (1 - c_a) v_\alpha \cdot v_\alpha > 0 \). Thus, if \( \xi \) is timelike, then (i) is guaranteed for the matter sector as well. As for the gravity sector one obtains that

\[
\det(\xi_A A_g^a) = \det \begin{bmatrix}
b I_{10} & -v_0 I_{10} & -v_1 I_{10} & -v_2 I_{10} & -v_3 I_{10} \\
-v_0^T I_{10} & b I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} \\
-v_1^T I_{10} & 0_{10 \times 10} & b I_{10} & 0_{10 \times 10} & 0_{10 \times 10} \\
-v_2^T I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & b I_{10} & 0_{10 \times 10} \\
-v_3^T I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} & b I_{10} \\
\end{bmatrix}
\]

\[
= \frac{1}{b^{10}} \det \begin{bmatrix}
(b^2 - v_\nu v_\nu) I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} \\
-v_0^T I_{10} & b I_{10} & 0_{10 \times 10} & 0_{10 \times 10} \\
-v_1^T I_{10} & 0_{10 \times 10} & b I_{10} & 0_{10 \times 10} \\
-v_2^T I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & b I_{10} \\
-v_3^T I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} \\
\end{bmatrix}
\]

\[
= (\alpha^a \xi_\alpha)^{30} (\xi_\alpha \xi_\alpha)^{10}.
\]

(37)

Again, note that if \( \xi \) is timelike, then \( \det(\xi_A A_g^a) \neq 0 \). This completes the proof of (i).

As for (ii), let us define \( \phi_\alpha = \xi_\alpha + A_\xi_\alpha \) and make the changes \( \xi \rightarrow \phi \) in the determinant calculations above. Then, the eigenvalues \( \Lambda \) are obtained from the roots of \( \det(\phi_\alpha z^{\alpha}) = \det(\phi_\alpha A_m^a) \det(\phi_\alpha A_g^a) / (u^\alpha \phi_\alpha)^{10} = 0 \). Note that the general form of the equations implies that the roots \( \phi_\alpha = -u_\alpha v^\beta \phi_\beta + \Delta_\alpha^\beta \phi_\beta \) obey

\[
(u^\alpha \phi_\alpha)^2 - \beta \Delta_\alpha^\beta \phi_\alpha \phi_\beta = 0,
\]

(38)
In the operations above we used the fact that
\[ \phi < 0, \]
where, from causality, in any of the above cases we have that \( \xi \), inequality and that
\[ \phi > 0, \]
where, since \( \phi \) eienvectors of \( \beta \Lambda = \alpha, \alpha \Lambda \)
m, this root has multiplicity 19. The eigenvector that obey
\[ (u^\alpha \xi_\alpha)(u^\beta \xi_\beta) - 2(u^\alpha \xi_\alpha)(u^\beta \xi_\beta)\Lambda^{\mu \nu} \mu \psi \nu \]
when
\[ m = \frac{\beta(\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)}{(u^\alpha \xi_\alpha)^2 - \beta \Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta}, \]
where, since \( \xi_\alpha \xi^\alpha < 0 \), then \( (u^\alpha \xi_\alpha)^2 - \beta \Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta > 0 \) because \( 0 \leq \beta \leq 1 \) and
\[ Z = \beta \left\{ (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)(u^\mu \xi_\mu)^2 + (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)(u^\mu \xi_\mu)^2 - 2(u^\alpha \xi_\alpha)(u^\beta \xi_\beta)\Delta^{\mu \nu} \mu \psi \nu \right\} - \beta \left\{ (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)(\Delta^{\mu \nu} \psi \psi \nu) - (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)^2 \right\} \]
\[ > \beta \left\{ (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)(u^\mu \xi_\mu)^2 + (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)(u^\mu \xi_\mu)^2 - 2(u^\alpha \xi_\alpha)(u^\beta \xi_\beta)\Delta^{\mu \nu} \mu \psi \nu \right\} - (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)(\Delta^{\mu \nu} \psi \psi \nu) + (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)^2 \]
\[ = \beta \left\{ (\xi^\alpha \xi_\alpha)(\xi^\beta \bar{\xi}_\beta) + [(u^\alpha \xi_\alpha)(u^\beta \bar{\xi}_\beta) - (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)^2 \right\} > 0. \]

In the operations above we used the fact that \( 0 \leq \beta \leq 1 \), \((\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)^2 \leq (\Delta^{\alpha \beta} \xi_\alpha \bar{\xi}_\beta)(\Delta^{\mu \nu} \psi \psi \nu)\)
from the Cauchy-Schwarz inequality and that \( \xi \) is timelike and \( \bar{\xi} \) spacelike. Thus, causality guarantees reality of the eigenvalues.

Now we turn to the problem of completeness of the set of eigenvectors. We begin by counting the linearly independent eigenvectors of \( \phi^{(m)}_{a, \alpha} A_m^{\alpha} \), where \( \phi^{(m)}_{a, \alpha} = \xi_\alpha + \Lambda^{(m)}_{a, \alpha} \) and \( \Lambda^{(m)}_{a} \) are the eigenvalues of the matter sector and are obtained by means of \( \{ 39 \} \) in the cases \( \beta = c_0 = 0 \) when \( a = 0 \) and \( \beta = c_a \) when \( a = 1, \pm \). Let us define an arbitrary vector
\[ v^{(m)} = \begin{bmatrix} F \\ G^\nu \\ H^\mu \\ I^\mu \\ J_0^\nu \\ J_1^\nu \\ J_2^\nu \\ J_3^\nu \end{bmatrix}, \]

Then, for each of the eigenvalues \( \Lambda^{(m)}_a \), \( a = 0, 1, \pm \), we must verify how many of the 29 variables in the vector \( \{ 41 \} \) are free parameters under the equation \( \phi^{(m)}_{a, \alpha} A_m^{\alpha} v_{(m)}^{(a)} = 0 \). In fact, this is the dimension of the null space of the matrix \( \phi^{(m)}_{a, \alpha} A_m^{\alpha} \) and corresponds to the number of linearly independent (LI) eigenvectors of \( \Lambda^{(m)}_a \). The eigenvectors are the following:

- \( \Lambda^{(m)}_0 \): this root has multiplicity 19. The eigenvector that obey \( \phi^{(m)}_{0, \alpha} A_m^{\alpha} v_{0}^{(m)} = 0 \) is
\[ v_{0}^{(m)} = \begin{bmatrix} 0 \\ 0_{4 \times 1} \\ H^\mu \\ I^\mu \\ J_0^\nu \\ J_1^\nu \\ J_2^\nu \\ J_3^\nu \end{bmatrix}, \]

where only 19 out of the 24 components \( H^\mu, I^\mu, J_\nu^\nu \) are free variables because of the \( 1 + 1 + 3 \) constraints \( \beta \phi^{(m)}_{0, \nu} H^\nu + \beta \phi^{(m)}_{0, \nu} I^\nu = 0, J_\lambda^\lambda = 0, \) and \( \Delta^{\mu \lambda} \phi^{(m)}_{0, \nu} J_\nu^\nu + \Delta^{\lambda \nu} \phi^{(m)}_{0, \nu} J_\nu^\nu = 0 \) (note that the last 4 equations are not all independent since the contraction with \( u_\mu \) is identically zero, resulting in 3 independent constraints). Thus, the multiplicity of \( \Lambda_0 \) equals the number of LI eigenvectors, i.e., 19.

- \( \Lambda^{(m)}_1 \pm \): in this case each of the two eigenvalues have multiplicity 3 since \( n_1 = 3 \) in \( \{ 35 \} \) (note that since we assumed here that \( \eta > 0 \), than \( c_1 \neq 0 \) and, thus, \( c_1 \neq c_0 \) and the eigenvalues are different from the case \( c_0 = 0 \).

We may perform some elementary row operations over the linear system \( \phi^{(m)}_{a, \alpha} A_m^{\alpha} u_1^{(m)} = 0 \) to obtain, by imposing
\[ b^2 - c_1(v \cdot v) = 0 \] (remember that \( b = u^\alpha \phi_\alpha \) and \( v^\alpha = \Delta^{\alpha \beta} \phi_\beta \) after the change \( \xi \rightarrow \phi \)),

\[
\begin{bmatrix}
\tau_\varepsilon b^2 + \beta_\varepsilon (v \cdot v) & b \rho \tau_Q \phi_\nu + b \rho \tau v_\nu - \frac{n \beta_\varepsilon (v \cdot v)}{b} v_\nu \\
0_{4 \times 1} & K_{\nu} v^\mu \\
-\nu^\mu & 0_{4 \times 4} \\
0_{4 \times 1} & \frac{n \nu \nu}{b} \\
0_{4 \times 1} & -v_0 \delta^\mu_\nu \\
0_{4 \times 1} & -v_1 \delta^\mu_\nu \\
0_{4 \times 1} & -v_2 \delta^\mu_\nu \\
0_{4 \times 1} & -v_3 \delta^\mu_\nu
\end{bmatrix}
\begin{bmatrix}
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)}
\end{bmatrix} =
\begin{bmatrix}
0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4}
\end{bmatrix}
\begin{bmatrix}
0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} \\
0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} \\
0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} \\
0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} \\
0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} \\
0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} \\
0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} \\
0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4} & 0_{4 \times 4}
\end{bmatrix}
\begin{bmatrix}
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)} \\
v_1^{(m)}
\end{bmatrix} = 0,
\]

where

\[
K_\nu = \left[ -\eta \xi_\nu + \left( \rho \tau_P - \zeta + \frac{2\eta}{3} - n \beta_\varepsilon \right) v_\nu \right] \left[ \tau_\varepsilon b^2 + \beta_\varepsilon (v \cdot v) \right] - \left( \tau_P + \beta_\varepsilon \right) [b^2 \rho \tau_Q \xi_\nu + b^2 \rho \tau v_\nu - n \beta_\varepsilon (v \cdot v) v_\nu].
\]

This enables us to find the eigenvectors

\[
\pm v_1^{(m)} = \begin{bmatrix}
F_\mu \\
G_\nu \\
H_{\nu} \\
I_{\mu} \\
J_{\nu} \\
K_{\nu} \\
L_{\mu} \\
M_{\nu}
\end{bmatrix},
\]

where, from the \( 29 + 29 = 58 \) components of the above eigenvectors (29 for \( \Lambda_1^{(m)+} \) and 29 \( \Lambda_1^{(m)-} \) cases), they are subjected to the following 26 + 26 constraints: \( 1 + 1 = 2 \) constraints

\[
[\tau_\varepsilon b^2 + \beta_\varepsilon (v \cdot v)] F_\mu + b_\nu \rho \tau_Q \pm \phi(1, \nu) G_\nu + b_\nu \rho \tau v_\nu G_\nu - \frac{\beta_\varepsilon (v \cdot v)}{b_\nu} v_\nu = 0,
\]

\( 1 + 1 = 2 \) constraints \( K_\nu v_\mu = 0, 4 + 4 = 8 \) constraints \( b_\nu H_\mu = v_\nu F_\mu, 4 + 4 = 8 \) constraints \( n v_\nu^4 v_\nu^4 G_\nu + b_\nu I_{\mu} = 0 \), and the 16 + 32 constraints \( b_\nu J_{\mu} = v_\nu G_\mu \), where \( \pm \phi^{(m)}_{1, \nu} = \pm \Lambda_1^{(m)} \phi_\nu + \phi_\alpha \) and \( b_\nu \) and \( v_\nu \) are defined in terms of \( \phi^{(m)}_{1, \nu} \). Hence, there is a total of \( 3 + 3 = 6 \) free parameters. Once again, the degeneracy equals the number of LI eigenvectors.

\( \Lambda_1^{(m)} \times : \) since there is no degeneracy in these four last eigenvalues and they are distinct from the others because \( c_\pm \neq 0 \) in the strict form of the inequalities in \( (20) \) and different among them, then one has 4 LI eigenvectors.

Thus, the system has \( 19 + 6 + 4 = 29 \) LI eigenvectors. Therefore, there is a complete set in \( \mathbb{R}^{29} \), namely, \( \{ v_b^{(m)} \}_{b=1}^{29} \) such that \( \phi^{(m)}_a A_m^a v_b^{(m)} = 0 \). Hence, we can use the 29 linearly independent set \( S^{(m)} = \{ V_b^{(m)} \}_{b=1}^{29} \) to verify that

\[
V_b^{(m)} = \begin{bmatrix}
v_b^{(m)} \\
v_b^{(m)} \\
v_b^{(m)} \\
v_b^{(m)} \\0_{66 \times 1}
\end{bmatrix}
\]

obeys \( (\zeta_\alpha + \Lambda_1^{(m)} \xi_\alpha) \mathcal{Q}^a V_b^{(m)} = 0 \).

Now, before we discuss the gravity sector \( \{ F_A, F_A^3 \} \), let us look at the sector containing the original fields \( \varepsilon, n, u^\nu \), and \( g_{\mu \nu} \). In this case, let us define

\[
V^{(d)} = \begin{bmatrix}
0_{79 \times 1} \\
v^{(d)}
\end{bmatrix},
\]

where \( v^{(d)} \) is a \( 16 \times 1 \) column vector. Then, \( (\zeta_\alpha + \Lambda_1^{(d)} \xi_\alpha) \mathcal{Q}^a V^{(d)} = 0 \) reduces to the eigenvalue problem \( u^\alpha \phi_\alpha I_{16} v^{(d)} = 0 \) whose eigenvalues are \( u^\alpha \phi_\alpha = 0 \), i.e., \( \Lambda^{(d)} = \zeta_\alpha u^\alpha / \xi_\alpha u^\alpha \). Thus, the eigenvectors may be any basis of \( \mathbb{R}^{16} \). Let \( \{ v_a^{(d)} \}_{a=1}^{16} \) be a basis of \( \mathbb{R}^{16} \). Then, the set \( S^{(d)} = \{ V_a^{(d)} \}_{a=1}^{16} \) is a linearly independent set of 16 eigenvectors of \( \phi_\alpha \mathcal{Q}^a \).
To finalize the eigenvector counting we have to analyze the sector containing $F_A$ and $F_A^3$. In this case, let us define

$$V^{(g)} = \begin{bmatrix} w \\ v^{(g)} \\ 0_{16 \times 1} \end{bmatrix},$$

(48)

where $w$ is some $29 \times 1$ columns vector while $v^{(g)}$ is a $50 \times 1$ columns vector. The eigenvalues of this sector are in (47) and are given by $\Lambda_0^{(g)} = u^2 \zeta_a/\nu^2 \xi_\beta$, coming from $u^2 \phi_0^{(g)} = 0$ (here $\phi_0^{(g)} = \zeta_a + \Lambda_0^{(g)} \xi_\alpha$) with multiplicity 30 and corresponding to $\beta = 0$, and the two roots $\pm \Lambda_1^{(g)}$ with multiplicity 10 each coming from $\pm \phi_1^{(g)} = -[u^2 \pm \phi_1^{(g)} \beta]_2 + \Delta \alpha^\beta \phi_1^{(g)} \phi_1^{(g)}$ = 0, which corresponds to $\beta = 1$, i.e., gravitational waves moving at the speed of light. Then, the eigenvalue problem $\phi_{a,\alpha}^{(g)} \Phi \Lambda v_a^{(g)} = 0$ reduces to the two equations

$$\phi_{a,\alpha}^{(g)} A_\alpha^a \Lambda w = L^a v^{(g)}_a,$$

$$\phi_{a,\alpha}^{(g)} A_\alpha^a \Lambda v_a^{(g)} = 0.$$

(49a)

(49b)

For the eigenvalues $\pm \Lambda_1^{(g)}$, one obtains that $\det[(\pm \phi_{1,\alpha}^{(g)} A_\alpha^a)_{m,n}] = 0$ because the root $\beta = 1$ has been eliminated from the matter sector (remember that $c_\alpha < 1$). Thus, there exists a solution of (49a) for each $v_a^{(g)}$ in (49b). One needs to count the number of linearly independent $v_a^{(g)}$ for $\Lambda_1^{(g)}$, i.e., the number of vectors in the basis of the kernel of $\Phi_{a,\alpha}^{(g)} A_\alpha^a$. In this case, after some elementary row operations [look at the second equality in (47) after setting $b^2 = v \cdot v$] one obtains that

$$\pm \phi_{1,\alpha}^{(g)} A_\alpha^a \sim \begin{bmatrix} 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} \\ -\Delta \alpha \pm \phi_{1,\alpha}^{(g)} I_{10} & (u^2 \pm \phi_{1,\alpha}^{(g)} I_{10} & 0_{10 \times 10} & 0_{10 \times 10} \\ -\Delta \alpha \pm \phi_{1,\alpha}^{(g)} I_{10} & 0_{10 \times 10} & (u^2 \pm \phi_{1,\alpha}^{(g)} I_{10} & 0_{10 \times 10} \\ -\Delta \alpha \pm \phi_{1,\alpha}^{(g)} I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & (u^2 \pm \phi_{1,\alpha}^{(g)} I_{10} \end{bmatrix},$$

(50)

which has 40 pivots and 10 independent variables (corresponding to the variables associated to the first 10 columns). Thus, there are 10 linearly independent vectors for each eigenvalue $\pm \Lambda_1^{(g)}$, i.e., there is a set $\{v_{1,b}^{(g)}\}_{b=1}^{10}$ of 20 linearly independent vectors with corresponding $w_1^{(g)} = [\pm \phi_{1,\alpha}^{(g)} A_\alpha^a]^{-1}L^a v_{1,b}^{(g)}$ coming from (49a) such that $S^{(g)}_1 = \{v_{1,b}^{(g)} \pm v_{1,b}^{(g)}\}_{b=1}^{10}$, where

$$\pm v_{1,b}^{(g)} = \begin{bmatrix} u_{1,b}^\pm \\ \pm v_{1,b}^{(g)} \\ 0_{16 \times 1} \end{bmatrix},$$

(51)

is a linearly independent set of 20 eigenvectors of $\phi_{1,\alpha}^{(g)} A_\alpha^a$.

As for the eigenvalue $\Lambda_0^{(g)}$, note that in this case $\det[\phi_{0,\alpha}^{(g)} A_\alpha^a] = 0$ because $\beta = c_0 = 0$ is also a root of this equation. Thus, for every solution $v_a^{(g)}$ in (49a), (49b) can be either undetermined or have infinite solutions. However, for any two different solutions, say, $w^1$ and $w^2$ for one $v_a^{(g)}$, the difference between $V_a^{(g)\dagger} - V_a^{(g)\dagger}$ corresponds to a vector in the space spanned by $S^{(g)}_m$, that lies in the Kernel of $\phi_{0,\alpha}^{(g)} A_\alpha^a$. Therefore, since we are counting the number of linearly independent eigenvectors, we must choose one particular solution $w_a$, if it exists, for each $v_a^{(g)}$. We begin by solving Eq. (49b). Let $\{L^1_a = u^1, L^2_a, L^3_a\}$ be a set of linearly independent vectors that are orthogonal to $\phi_{0,\alpha}^{(g)} = \zeta_a + \Lambda_0^{(g)} \xi_a$, to wit, $L^1_a \phi_{0,\alpha}^{(g)} = 0$ and $\{e_a\}_{a=1}^{10}$ be any basis of $\mathbb{R}^{10}$. Then, one may verify that the 30 linearly independent vectors

$$v_{0,ac}^{(g)} = \begin{bmatrix} 0_{10 \times 1} \\ L^1_a e_a \\ L^2_a e_a \\ L^3_a e_a \\ e_a \end{bmatrix}$$

(52)
We recall that we are interested in the diagonalization because it allows us to invoke known techniques to prove local well-posedness. If the system would fail the system would not be diagonalizable. Nor does it imply that local well-posedness, established in the next section, is already present in the system, the multiplicity of the characteristics would change. This does not mean that the multiplicity of the eigenvalues might change. This is because with equality one can have

This leads to the solution

and then, by inserting (52) and (54) into Eq. (49a), one finds that

This leads to the solution

and, thus, the set

is a linearly independent set of 30 eigenvectors of \( \hat{\phi}_0 \mathcal{G}^{(g)} \). Thus, \( \hat{\mathcal{S}} = \mathcal{S}^{(m)} \cup \mathcal{S}^{(d)} \cup \mathcal{S}_1^{(g)} \cup \mathcal{S}_0^{(g)} \) contains a complete set of eigenvectors \( V \) of \( \hat{\phi}_0 \mathcal{G}^{\alpha} V = 0 \) in \( \mathbb{R}^{95} \). This completes the proof.

We remark that the assumption that the inequalities hold in strict form is technical. If equality is allowed, then the multiplicity of the eigenvalues might change. This is because with equality one can have \( c_a = 0 \) for \( a = 1 \) or \( \pm \) and thus the characteristics defined by \( b^2 - c_a(v \cdot v) = 0 \) can degenerate into the characteristics \( b = 0 \). Since the latter is already present in the system, the multiplicity of the characteristics would change. This does not mean that the system would not be diagonalizable. Nor does it imply that local well-posedness, established in the next section, would fail. However, a different proof would be needed to show diagonalization in the case \( c_a = 0 \) in the cases.

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19 We recall that we are interested in the diagonalization because it allows us to invoke known techniques to prove local well-posedness. If the system is not diagonalizable, it remains possible that different techniques would lead to local well-posedness.
\[ a = 1 \text{ or } \pm. \] We believe that treating this very special case here would be a distraction from the main points of the paper. We also recall that already in the case of an ideal fluid, a different approach to local well-posedness has to be employed when the characteristics degenerate [91].

## C. Local well-posedness

In this section we establish the local existence and uniqueness of solutions to the nonlinear equations of motion in [2] and [12]. We begin by noticing that [12] used the normalization \( u^\mu u_\mu = -1 \) to project the divergence of \( T_{\mu \nu} \) and \( J^\mu \) onto the directions parallel and orthogonal to \( u^\mu \). In order to show that the condition \( u^\mu u_\mu = -1 \) is propagated by the flow, it is more convenient to work directly with [1] and [2]. In order to complete the system, we differentiate \( u^\mu u_\mu = -1 \) twice in the \( u^\mu \) direction,

\[
u \beta \nabla_\beta [u^\alpha \nabla_\alpha (u^\alpha u_\alpha)] = 0. \tag{57}\]

We also differentiate \( \nabla_\mu J^\mu = 0 \) once, as in section III

\[
u \nabla_\mu (\nabla_\nu J^\nu) = 0. \tag{58}\]

Observe that (57) and (58) imply that \( u^\mu u_\mu = -1 \) and \( \nabla_\mu J^\mu = 0 \) hold at later times if these hold at the initial time.

The main result of this section can be found below.

**Theorem II.** Let \( (\Sigma, \hat{g}_{\alpha \beta}, \hat{\kappa}_{\alpha \beta}, \hat{\varepsilon}, \hat{n}, \hat{\nu}, \hat{\mu}, \hat{\nu}') \) be an initial-data set for the system comprised of Einstein’s equations [1] and \( \nabla_\mu J^\mu = 0 \), where \( T_{\alpha \beta} \) and \( J^\mu \) are given in [10]. Assume that \( \hat{u}^\mu \hat{u}_\mu = -1, \hat{n} \geq C > 0 \), where \( C \) is a constant, and that \( \nabla_\mu J^\mu = 0 \) holds for the initial data. Assume (A1) with \( \eta > 0 \) and suppose that (20) of Theorem I hold in strict form and that the transport coefficients are analytic functions of their arguments. Finally, assume that \( \hat{g}_{\alpha \beta}, \hat{\kappa}_{\alpha \beta}, \hat{\varepsilon}, \hat{n}, \hat{\nu} \in H^N(\Sigma) \) and that \( \hat{\kappa}_{\alpha \beta}, \hat{\varepsilon}, \hat{n}, \hat{\nu} \in H^{N-1}(\Sigma) \), \( N \geq 5 \), \( H^N \) is the Sobolev space. Then, there exists a globally hyperbolic development of the initial data. This globally hyperbolic development is unique if taken to be the maximum globally hyperbolic development of the initial data.

**Proof.** As usual in studies of the initial-value problem for Einstein’s equations [12], we embed \( \Sigma \) into \( \mathbb{R} \times \Sigma \) and work in harmonic coordinates in the neighborhood of a point. Observe that we already know the system to be causal under our assumptions thus localization arguments are allowed.

The equations to be studied read

\[
u = u^\alpha u^\beta \partial^\alpha \partial^\beta n + nu^\alpha \delta_\alpha^\beta \partial^\beta u^\nu + \hat{B}_1(n, u, g) \partial^2 g = B_1(\partial n, \partial u, \partial g) \tag{59a}\]

\[
u u_\mu u^\alpha \partial_\alpha \partial_\beta u^\nu + \hat{B}_2(n, \varepsilon, u, g) \partial^2 g = B_2(\partial n, \partial \varepsilon, \partial u, \partial g), \tag{59b}\]

\[
u = \beta_\beta (u^{\alpha} \Delta^{\alpha \beta} + \Delta^{\mu (\alpha \beta)} u^\mu) \partial_\beta \partial_\nu + \rho(\varepsilon + \nu Q) w^\nu \Delta^{(\alpha \beta)} w^\mu + \tau Q \rho u^\alpha u^\beta \delta_\mu^\nu, \tag{59c}\]

\[
u = g^{\alpha \beta} \partial_\alpha \partial_\beta g_{\mu \nu} = B_4(\partial n, \partial \varepsilon, \partial u, \partial g), \tag{59d}\]

where

\[
u = \left( \tau P - \xi - \frac{\eta}{3} \right) \Delta^{(\alpha \beta)} - \eta \Delta^{\alpha \beta} \delta_\mu^\nu + \rho(\varepsilon + \nu Q) w^\nu \Delta^{(\alpha \beta)} w^\mu + \tau Q \rho u^\alpha u^\beta \delta_\mu^\nu, \tag{60a}\]

\[
u = u^\nu (\beta_\beta \Delta^{\alpha \beta} + \tau u^\alpha u^\beta) + (\beta_\beta + \nu P) \Delta^{(\alpha \beta)}, \tag{60b}\]

and the notation for the \( \hat{B}'s \) and \( B's \) follow the same construction as in Section III.

We can write (59a) in matrix form as

\[
u = A(\partial) \Psi + B(\partial \Psi) = 0, \tag{61}\]

where \( \Psi = (\varepsilon, u, g_{\mu \nu})^T \) is a 16 × 1 column vector (we count only the 10 independent \( g_{\mu \nu} \)), \( B(\partial \Psi) \) is also a 16 × 1 column vector containing the \( B' \)s, i.e., the lower order terms in derivatives of each equation, and

\[
u = A(\partial) = \begin{bmatrix} A(\partial) & b(\partial) \\ 0_{10 \times 6} & g^{\alpha \beta} \partial_\alpha \partial_\beta I_{10} \end{bmatrix}, \tag{62}\]
The $6 \times 10$ matrix $b(\partial)$ contains the terms $\tilde{b}\partial^2 g$ while

$$A(\partial) = \begin{bmatrix}
0 & \frac{u^\alpha u_\beta}{\partial_\alpha \partial_\beta} & \frac{n\delta_\alpha^\mu u_\beta}{\partial_\mu} & u_\nu u^\alpha u_\beta & \\
0 & 0 & \frac{n\delta_\alpha^\mu u_\beta}{\partial_\mu} & u_\nu u^\alpha u_\beta & \\
\Theta^{\mu \nu \rho}_{\alpha \beta \gamma} & \beta_n (u^\mu \Delta_{\alpha \beta} + \Delta_{\mu (\alpha \beta)}) & 0 & 0 & \\
\Theta^{\mu \nu \rho}_{\alpha \beta \gamma} & \beta_n (u^\mu \Delta_{\alpha \beta} + \Delta_{\mu (\alpha \beta)}) & 0 & 0 & \\
0 & \frac{n\delta_\alpha^\mu u_\beta}{\partial_\mu} & u_\nu u^\alpha u_\beta & \\
0 & 0 & \frac{n\delta_\alpha^\mu u_\beta}{\partial_\mu} & u_\nu u^\alpha u_\beta & \\
\end{bmatrix} \partial_\alpha \partial_\beta. \quad (63)$$

Let us compute the characteristic determinant of the system and its roots, i.e., $\det[A(\xi)] = \det[A(\xi)](\xi^\alpha \xi_\alpha)^{10} = 0$, where the substitution $\partial \to \xi$ takes place. The pure gravity sector has the roots $\xi^\alpha \xi_\alpha = 0$. As for the matter sector, by again defining $b = u^\alpha \xi_\alpha$, $v_\mu = \Delta_{\mu (\alpha \beta)} \xi_\alpha \xi_\beta$, and

$$\tilde{C}_\mu^\nu = \tilde{C}_\mu^{\nu \alpha \beta} \xi_\alpha \xi_\beta = [\tau_Q \rho b^2 - \eta(v \cdot v)] \delta_\mu^\nu + (\tau_Q \rho - \xi - \frac{\eta}{3}) v_\mu \xi_\nu + \rho(\tau_\varepsilon + \tau_Q) bv_\mu v_\nu, \quad (64a)$$

$$\tilde{D}_\mu^\nu = (\tau_Q \rho - \xi - \frac{\eta}{3} - n\beta_n) v_\mu \xi_\nu + [\tau_Q \rho b^2 - \eta(v \cdot v)] \delta_\mu^\nu, \quad (64b)$$

$$\tilde{E}_\mu^\nu = \tilde{E}_\mu^{\alpha \beta} \xi_\alpha \xi_\beta = [\beta_\varepsilon (v \cdot v) + \tau_\varepsilon b^2] u_\mu + (\beta_\varepsilon + \tau_\varepsilon) bv_\mu, \quad (64c)$$

where $\tilde{D}_\mu^\nu$ is the same as the one defined in (23), we obtain that (by carrying out some elementary row operations)

$$\det[A(\xi)] = \det \begin{bmatrix}
0 & b^2 & n\delta_\nu^\nu \xi_\nu \\
0 & 0 & b^2 \mu_\nu \\
\tilde{C}_\mu^\nu & n\delta_\nu^\nu \xi_\nu & \tilde{C}_\mu^\nu \\
\end{bmatrix} = \frac{b^3}{\tau_Q \rho b^2 - \eta(v \cdot v)} \cdot \det \begin{bmatrix}
0 & b \\
0 & 0 \\
\tilde{C}_\mu^\nu & n\delta_\nu^\nu \xi_\nu \\
\end{bmatrix} = \frac{b^3}{\tau_Q \rho b^2 - \eta(v \cdot v)} \cdot \det \begin{bmatrix}
\tau_\varepsilon b^2 + \beta_\varepsilon (v \cdot v) & \beta_n (v \cdot v) & \rho(\tau_\varepsilon + \tau_Q) bv_\nu \\
\beta_\varepsilon (v \cdot v) & \beta_n (v \cdot v) & \beta_\varepsilon bv_\nu \\
(\beta_\varepsilon + \tau + P) bv_\nu & \beta_\varepsilon bv_\nu & \beta_\varepsilon bv_\nu \\
\end{bmatrix} \cdot \tilde{D}_\mu^\nu. \quad (65)$$

The last determinant is the same as the one obtained in (24) and the result turns out to be

$$\det[A(\xi)] = -b^4 [\tau_\varepsilon \rho b^2 - \eta(v \cdot v)^2] \left[ A b^4 + B b^2 (v \cdot v) + C (v \cdot v)^2 \right] = -\rho^4 \tau_\varepsilon \rho b^2 (u^\alpha \xi_\alpha)^4 \prod_{\alpha = 1, \pm} \left( (u^\alpha \xi_\alpha)^2 - c_\alpha \Delta_{\alpha \beta} \xi_\alpha \xi_\beta \right)^{\tilde{n}_\pm}, \quad (66)$$

where $A$, $B$, $C$, and $c_\alpha$ are the same as the ones defined in (25) and below it in the text, while $\tilde{n}_\pm = n_\pm = 1$. Note that the characteristics are still the same as in section 11 as expected, although the multiplicity of the roots changed (and there was no reason for the multiplicities to be the same). We conclude that the characteristic determinant of the system is a product of strictly hyperbolic polynomials. We verify at once that the system is a Leray-Ohyama system 12, 22 for which the results of 23 (see also 23) apply. Thus, if the initial data is quasi-analytic20 we obtain quasi-analytic solutions.

Denote the initial-data set in the theorem by $\mathcal{D}$ and let $\mathcal{D}_\ell$ be a sequence of quasi-analytic initial-data converging to $\mathcal{D}$ in $H^N$ (see footnote 6 for the definition of $H^N$). Let $\Psi_\ell$ solutions corresponding to $\mathcal{D}_\ell$ (which exist from the foregoing). In order to finish the proof of the theorem, it suffices to show that $\Psi_\ell$ has a limit in $H^N$. The limit will then be a solution with the desired properties because we can pass to the limit in the equations since $s \geq 5$.

According to the arguments given in section 16.2 of 14 or in 64, 65, the diagonalized obtained in section 11 implies that $U$ defined in (28) admits a uniform bound in $H^{N-1}$, and uniform difference bounds in $H^{N-2}$ also holds. We apply these bounds to the vector $U_\ell$ corresponding to $\Psi_\ell$. We see at once that the uniform $H^{N-1}$ bounds for $U_\ell$ imply uniform $H^N$ bounds for $\Psi_\ell$, and the difference bounds imply that $\Psi_\ell$ is a Cauchy sequence in $H^{N-3}$, thus converging in this space. But low-norm convergence combined with high-norm boundedness implies that the limit is in fact in $H^N$ 67. \boxed{\square} 

20 See footnote 7.
We observe that a similar local well-posedness result holds for the fluid equations in a fixed background.

We recall that a standard tensorial argument [19] guarantees that the solution established in Theorem II is intrinsically defined, i.e., given the data, which is defined independently of coordinates or gauge choices, there exists a spacetime where Einstein’s equations are satisfied, and this spacetime is defined without any reference to coordinates or gauge choices – even if in the process of proving that this spacetime exists one has to work in a specific gauge and coordinate system. Therefore, even though we used the harmonic gauge in the proof, the existence of the solution is guaranteed for other choices as well. This logic is similar to showing that a map from a finite-dimensional vector space into itself is invertible: one can choose a basis, write the matrix of the linear transformation with respect to that basis, and compute its determinant. The map is invertible if and only if the determinant is non-zero, and this conclusion (the invertibility or not of the linear map) is independent of any basis choice – even if to show that the map is invertible we picked a basis and computed the determinant with respect to that basis.

We note, however, the following subtlety which is very relevant for numerical simulations. The fact that a unique solution is guaranteed to exist for given initial data, and that this solution is well-defined regardless of gauge choices, does not imply that such a solution can always be reconstructed from an arbitrary gauge. In other words, suppose we write the equations in a different gauge. If we can numerically integrate them, we will obtain the solution found in Theorem II written on that gauge (modulo numerical accuracy). However, it is possible that the gauge we chose is not adequate to solve the equations numerically, so that our numerical simulation will not produce a solution. This does not mean, of course, that solutions do not exist; it simply means that the guaranteed-to-exist solution given by Theorem II cannot be accessed from that specific gauge. To use again our analogy with determinants: suppose we computed the determinant on a basis $b_1$ and found it to be non-zero, but now we are interested in computing the determinant numerically using another basis $b_2$. Depending on the basis $b_2$ and the numerical algorithm we use, this might not be possible, which, of course, does not mean that the determinant is zero or ill-defined.

Thus, the practical matter of solving the equations numerically is not settled by an abstract existence and uniqueness result as Theorem II. Such theorems are naturally important as they provide the foundations on which numerical investigations can be built, i.e., it makes sense to look for solutions numerically because solutions do exist. But these theorems do not, in general, point to how to recover solutions numerically. That is why there is a great deal of work dedicated to writing Einstein’s equations in different forms and special gauges, even if basic existence results for Einstein’s equations coupled to most matter models are known, as reviewed in [2, 95].

V. A NEW THEOREM ABOUT LINEAR STABILITY

Any ordinary fluid\footnote{We only consider systems such that the equilibrium state is unique and has a finite correlation length. Therefore, in principle, our discussion does not apply to systems where the correlation length in equilibrium can become arbitrarily large, such as at a critical point.} must be stable against small deviations from the thermodynamic equilibrium state [14]. We recall that in equilibrium \( \beta_\mu = u_\mu/T \) must be a Killing vector, i.e. \( \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0 \), and also \( \nabla_\alpha (\mu/T) = 0 \). In Minkowski spacetime, equilibrium corresponds to a class of states\footnote{In this paper we neglect the constant thermal vorticity term, see [19] for a nice discussion of its physical content and consequences.} with constant \( T \) and \( \mu \) and background flow velocity \( u^\alpha = \gamma (1, \mathbf{v}) \) defined by a constant sub-luminal 3-velocity \( \mathbf{v} \) (where \( \gamma = 1/\sqrt{1 - \mathbf{v}^2} \)). In the local rest frame (LRF) \( \mathbf{v} = 0 \) and the background flow is simply \( u^\alpha = (1, 0, 0, 0) \). In a stable theory, small disturbances from the general equilibrium state \( T \to T + \delta T(t, \mathbf{x}), \mu \to \mu + \delta \mu(t, \mathbf{x}) \), and \( u^\alpha \to u^\alpha + \delta u^\alpha(t, \mathbf{x}) \) (with \( u_\mu \delta u^\mu = 0 \)) lead to small variations in the energy-momentum tensor and current, \( \delta T^\mu_{\nu}(t, \mathbf{x}) \) and \( \delta J^\mu(t, \mathbf{x}) \), which decay with time.

The standard theories from Eckart and Landau-Lifshitz are unstable, as shown by Hiscock and Lindblom many years ago [26]. This instability appears because such theories possess exponentially growing, hence \textit{unstable}, non-hydrodynamic modes\footnote{The frequency of a hydrodynamic mode, such as a sound wave, vanishes in a spatially uniform state. On the other hand, a non-hydrodynamic mode correspond to a collective excitation that possesses nonzero frequency even at zero wavenumber.}, which spoil linear stability around equilibrium even at vanishing wave number. For Landau-Lifshitz theory at zero chemical potential, this instability is only observed when considering a general equilibrium state with nonzero \( \mathbf{v} \) [26, 15, 98], while in the case of Eckart the instability already appears even when \( \mathbf{v} = 0 \). The lack of causality in these approaches implies that it is not sufficient to investigate only the static \( \mathbf{v} = 0 \) case in order to determine the stability properties of a general equilibrium state where \( \mathbf{v} \neq 0 \), even though such states are in principle connected via a simple Lorentz transformation.

The necessity to investigate the stability properties of general equilibrium states where \( \mathbf{v} \neq 0 \) makes linear stability analyses of viscous hydrodynamic theories very complicated. Already in the local rest frame, finding whether the linear modes of the system are stable requires determining the sign of the imaginary part of the roots of a high order
polynomial, which becomes a daunting task when \( v \neq 0 \) (see [73] and [99] for recent examples of how complicated a \( v \neq 0 \) analysis can become in BDNK and MIS theory, respectively).

We prove below a new theorem that gives sufficient conditions for causal fluid dynamic equations to be linearly stable against disturbances of a general equilibrium state with arbitrary background velocity. In this case, proving stability for the local rest frame implies stability in any other frame\(^{24}\) connected to the local rest frame via a Lorentz transformation. This general feature is expected to hold in any interacting relativistic system, i.e., no issues should appear if one simply observes a given system in another inertial frame. We then use this theorem in Section [Y1] to find conditions under which the hydrodynamic theory presented here is stable. We remark that our results can be used to establish stability at nonzero \( v \neq 0 \) in other theories as well, e.g. MIS, as long as the conditions discussed below are fulfilled.

### A. Transforming a second order system of linear differential equations into a first order one

We begin by showing how one may convert a system of linear second order PDE’s into a first order one, as this is needed for the theory discussed in this paper. Let the system of linearized second order PDE’s be given by

\[
\sum_b A(\partial)_b^a \delta \psi^b(X) = 0, \tag{67}
\]

where \( a \) and \( b \) runs from 1 to \( n \), \( A(\partial)_b^a \) are differential linear operators of order up to 2, and \( \delta \psi^1(X), \cdots, \delta \psi^n(X) \) are the perturbed fields (for instance, \( \delta x, \delta n, \) and etc). We suppose that (67) arises from the conservation laws

\[
-u_\alpha \partial_\beta T^\alpha_\beta = 0, \quad \Delta^\alpha_\beta \partial_\beta T^\alpha_\mu = 0, \quad \partial_\alpha \delta \rho = -u_\alpha u^\gamma \partial_\gamma \delta J_\beta + \Delta^\alpha_\beta \partial_\alpha \partial_\beta J_\mu = 0,
\]

where the first two come from \( \partial_\alpha \partial_\beta T^\alpha_\beta = 0 \), while the last equation appears only when \( J_\alpha^\mu \) is included. In this manner, the derivatives in the EOM’s shall always appear as combinations of \( u^\alpha \partial_\alpha \) and \( \Delta^\alpha_\beta \partial_\beta \). Thus, if the system in (67) has one or more second order equations, it can be rewritten as a first order system in the \( N = 5n \) new variables \( \delta \tilde{\psi}^a(X) = u^\alpha \partial_\alpha \psi^a(X) \) and \( \delta \tilde{\psi}^a(X) = \Delta^\alpha_\beta \partial_\alpha \psi^a(X) \). This definitions automatically lead (67) to \( n \) first order linear equations. It then remains to supplement with the \( 4n \) dynamical equations that misses. By means of the identity \( \partial_\alpha \partial_\beta \psi^a(\xi) - \partial_\beta \partial_\alpha \psi^a(\xi) = 0 \), one may find the extra \( 4n \) dynamical equations \( u^\alpha \partial_\alpha \tilde{\psi}^a(X) - \partial_\beta \partial_\alpha \psi^a(X) = 0 \), totaling the needed \( 5n \) first order dynamical equations, as required. In matrix form it becomes

\[
A^\alpha \partial_\alpha \delta \Psi(X) + B \delta \Psi(X) = 0, \tag{68}
\]

where \( A^\alpha \) and \( B \) are \( N \times N \) constant real matrices and \( \delta \Psi(X) \) is a \( N \times 1 \) column vector with entries \( \delta \tilde{\psi}^1(X), \delta \tilde{\psi}^1(X), \cdots, \delta \tilde{\psi}^n(X), \delta \tilde{\psi}^n(X) \). This ends the procedure. However, if one of the equations in (67) is already of first order but contains variables that have second order derivative in other equations, then one can eliminate this equation by using it as a constraint to eliminate one of the variables. For example, consider the case of the ideal current \( J^\mu = n u^\mu \). In this case, the conservation equation \( \partial_\mu J^\alpha = 0 \) becomes \( u^\alpha \partial_\alpha n(X) + n \partial_\alpha \delta n(X) = 0 \). If \( T^\mu_\nu \) has shear or bulk, for example, then the other equations must have second order derivatives of \( \delta \tilde{\nu}^\mu \). Thus, one must write \( \partial_\alpha \delta J^\alpha = 0 \) as \( \delta \tilde{\nu}^\mu + n \delta \tilde{\nu}^\mu = 0 \), where \( \delta \tilde{\nu}^\mu = \Delta^\alpha_\beta \partial_\alpha \tilde{\nu}^\mu \) and \( \delta \tilde{\nu}^\nu = u^\alpha \partial_\alpha n \). This is a zeroth order equation in the new variables and, therefore, is just a constraint. One may use this constraint in order to eliminate the variable \( \delta \tilde{\nu}^\nu \) in the other dynamical equations. Then, in this case one ends up with \( 5n - 1 \) dynamical equations in the \( 5n - 1 \) fields.

Finally, we remark that other approaches to viscous relativistic fluids, such as MIS, are already written in the format (68) in the linearized regime so the procedure to reduce the order of the equations of motion described above is not needed and one can skip directly to the part below.

### B. New linear stability theorem

To study linear stability, let us expand the perturbed fields in the Fourier modes \( K^\mu = (i \Gamma, k^i) \) by substituting \( \delta \Psi(X) \rightarrow \exp(iK_\mu X^\mu) \delta \Psi(K) = \exp(i \Gamma t + ik \cdot x') \delta \Psi(K) \) in (68). The result is

\[
iK_\mu A^\mu \delta \Psi(K) + B \delta \Psi(K) = 0. \tag{69}
\]

\(^{24}\) Note that the word frame here is used in the standard context in special relativity (inertial observer). It has nothing to do with the concept of a hydrodynamic frame discussed in previous sections, which concerned the definition of hydrodynamic variables out of equilibrium.
Since $K^\mu$ appears, as aforementioned, as combinations of $-u^\alpha K_\alpha = \gamma(i\Gamma - k_i v^i)$ and $\Delta^{\mu\nu} K_\mu K_\nu = (u^\mu K_\mu)^2 + \Gamma^2 + k^2$, where $k^2 = k_i k^i$, then the direction of $k^i$ is not relevant once one keeps $v^i$ arbitrary. Thus, we may write $K^\mu = -n^{\mu}n_\mu + \zeta \zeta K^\mu$, where $n_\mu$ is timelike and $\zeta$ is spacelike, with $n^{\mu}n_\mu = -1$, $n_\mu \zeta^\mu = 0$, and $\zeta^\mu \zeta_\mu = 1$, [for example, it is common to choose $K^\mu = (K^0, k, 0, 0)$ so that $n_\mu$ and $\zeta_\mu$ are $(-1, 0, 0, 0)$ and $(0, 1, 0, 0)$, respectively]. In this case we define $\Omega = n_\alpha K^\alpha$ and $\kappa = \zeta_\alpha K^\alpha$ such that $K^\mu = -\Omega n^\mu + \kappa \zeta^\mu$. Then, (69) can be written as

$$i\Omega(-n_\alpha A^\alpha)\delta \Psi(K) = -i\kappa n_\alpha A^\alpha \delta \Psi(K) - B \delta \Psi(K).$$

(70)

The general form of the co-vectors $n$ and $\zeta$ is $n_\alpha = \gamma_n(-1, c^i)$ for any $c^i$ such that $0 \leq c^i c_i < 1$ and where $\gamma_n = 1/\sqrt{1 - c^i c_i} \geq 1$, and $\zeta_\alpha = \gamma_\zeta(-d^i d_i) \geq 1$, where $d^i d_i = 1$ for an arbitrary unitary $d^i$ and where $\gamma_\zeta = 1/\sqrt{1 - (d^i c_i)^2} \geq 1$. From the Cauchy-Schwarz inequality $(d^i c_i)^2 \leq |c|^2$ (here $|c| = \sqrt{c^i c_i}$), then one obtains that

$$\gamma_n \geq \gamma_\zeta.$$  

(71)

Stability demands that the perturbed modes $\Gamma = \Gamma(k^i)$ are such that $\Gamma_R \leq 0$. Now, consider the eigenvalue problem

$$(\Lambda n_\alpha + \zeta_\alpha)A^\alpha \tau = 0,$$

(72)

where here $\Lambda$ is the eigenvalue associated with the right eigenvector $\tau$.

**Proposition II.** If (68) is causal, then the eigenvalues $\Lambda$ are real and lie in the range $[-1, 1]$ (as a consequence, the eigenvectors $\tau$ may be real). Furthermore, det($n_\alpha A^\alpha$) $\neq 0$.

**Proof.** Causality demands that the roots of $Q(\xi) = \text{det}(\zeta_\alpha A^\alpha)$ are such that (i) $\xi_0 = \xi_0(\xi_i) \in \mathbb{R}$ and that (ii) the curves $\xi_0$ lie outside or over the light-cone. In other words, $\xi^\alpha \zeta_\alpha \geq 1$. If one writes $\zeta_\alpha = \Lambda n_\alpha + \zeta_\alpha$, where $n$ and $\zeta$ are real, then condition (i) means that $\Lambda$ is real. On the other hand, since $n$ and $\zeta$ are orthonormal, then condition (ii) means that $\xi_\alpha \xi^\alpha = -\Lambda^2 + 1 \geq 0$, which demands that $\Lambda^2 \leq 1$, i.e., $\Lambda \in [-1, 1]$. Now, since $Q(\xi) = 0$ if and only if $\xi$ is spacelike or lightlike, this means that det($n_\alpha A^\alpha$) $\neq 0$.

**Theorem III.** Let (72) have a set of $N$ linearly independent real eigenvectors $\{r_1, \cdots, r_N\}$. If (68) is causal and stable in the local rest frame $\mathcal{O}$, then it is also stable in any other Lorentz frame $\mathcal{O}'$ connected to $\mathcal{O}$ by a Lorentz transformation.

**Proof.** From causality det($n_\alpha A^\alpha$) $\neq 0$ as far as $n$ is timelike. Thus, we can rewrite (70) as

$$i\Omega \delta \Psi(K) = -i\kappa(-n_\alpha A^\alpha)^{-1}\zeta_\alpha A^\beta \delta \Psi(K) - (-n_\alpha A^\alpha)^{-1}B \delta \Psi(K).$$

(73)

Since the eigenvalue problem (72) contains $N$ linearly independent vectors $r_a$, one may write (72) as

$$(-n_\alpha A^\alpha)^{-1}\zeta_\alpha A^\beta r_a = \Lambda_a r_a$$

(74)

and define the $N \times N$ invertible matrix $R = [r_1 \cdots r_N]$ whose columns are the eigenvectors $r_1, \cdots, r_N$ and the $N \times N$ matrix

$$L \equiv R^{-1} = \begin{bmatrix} 1_1 \\ \vdots \\ 1_N \end{bmatrix},$$

where the rows $1_a$ are the left eigenvectors of $(-n_\alpha A^\alpha)\zeta_\alpha A^\beta$ which, consequently, obey $1_a r_b = \delta_{ab}$ (because $RL = I_N$). Then, we can write

$$\delta \Psi(K) = RL \delta \Psi(K) = \sum_a c_a(K) r_a = R c,$$

(75)

where $c_a(K) = 1_a \delta \Psi(K)$ is a $c$-number and $c$ is the $N \times 1$ matrix

$$c = L \delta \Psi(K) = \begin{bmatrix} c_1(K) \\ \vdots \\ c_N(K) \end{bmatrix}. $$
Therefore, (73) becomes
\[ i\Omega Rc = -i\kappa RDc - (n_\alpha A^\alpha)^{-1} B Rc, \]
where \( D \) is the \( N \times N \) real diagonal matrix \( D = \text{diag}(\Lambda_1, \ldots, \Lambda_N) \) and, thus, \((n_\alpha A^\alpha)^{-1} \zeta_3 A^\beta R = RD\). By multiplying (76) by \( c^d R^{-1} \) from the left one obtains
\[ i\Omega |c|^2 = -i\kappa c^d Dc - c^d R^{-1} (n_\alpha A^\alpha)^{-1} B Rc. \]
Since \( D \) is real and diagonal (which gives \( c^d Dc \in \mathbb{R} \)), \( \Omega = \gamma_n (i\Gamma + c^d k_i) \), and \( \kappa = \zeta c (i\gamma d c\Gamma + \tilde{d} k_j) \), then
\[ \Gamma R^d (\gamma_n I_N + \zeta c \tilde{d} c J) \) \( c = -\Re[c^d R^{-1} (n_\alpha A^\alpha)^{-1} B Rc]. \]
On the other hand, note that \( \gamma_n I_N + \zeta c \tilde{d} c J \) is diagonal with elements
\[ (\gamma_n I_N + \zeta c \tilde{d} c J)_{aa} = \gamma_n + \zeta c \tilde{d} c J_a > 0 \]
because \( |\tilde{d} c J| \leq |c| < 1 \), \( \Lambda \in [-1, 1] \), and \( \gamma_n \geq \zeta c \) from (71). Hence, \( \gamma_n I_N + \zeta c \tilde{d} c J \) is a positive Hermitian matrix and \( c (\gamma_n I_N + \zeta c \tilde{d} c J) \) \( c > 0 \). The consequence is that \( \Gamma R \leq 0 \) if and only if
\[ \Re[c^d R^{-1} (n_\alpha A^\alpha)^{-1} B Rc] \geq 0. \]

Now, let \( \mathcal{O} \) be the LRF and \( \mathcal{O}' \) some other boosted frame. The connection between the two frames is given by the Lorentz transform \( t' = \gamma(t - v^i x_i), \) \( x_i' = \gamma(x_i - v^i t) \), and \( x_i = x_i^\perp \), where \( || \) and \( \perp \) stand for the components parallel and perpendicular to \( v^i \), respectively. This can be compactly written as \( X'' = \Lambda X' \). Thus, one obtains that \( K'' = \Lambda X' \), and \( \delta \Psi'(K') = M \delta \Psi(K) \) from the structure of (85) (where \( M \) is an \( N \times N \) invertible matrix), leading to \( \Lambda'' = \Lambda M A' M^{-1} \) and \( B' = MBM^{-1} \). In particular, \( \zeta A'' = M^{-1} \zeta A' M \) and \( n_\alpha A'' = M^{-1} (n_\alpha A') M \). From (72), these relations give \( \Gamma = M R \) with the same eigenvalue \( \Lambda \) in both frames. Then, since \( \delta \Psi(K) = Rc \) and \( \delta \Psi'(K') = R' c' = M R c \), one concludes that \( c = c' \), i.e., \( c_{\alpha}(K') = c_{\alpha}(K) \). Therefore, one arrives at the following identity:
\[ c^d R^{-1} (-n_\alpha A^\alpha)^{-1} B' R' c = c^d R^{-1} (-n_\alpha A^\alpha)^{-1} B Rc. \]
However, if the system is stable in the LRF, then (82) holds and, from (81), one automatically obtains that \( \Gamma R \leq 0 \), proving that the system is also stable in any other frame \( \mathcal{O}' \) obtained via a Lorentz transformation.

\[ \square \]

VI. CONDITIONS FOR LINEAR STABILITY

We now apply the theorem proved in the last section to determine conditions that ensure the stability of the hydrodynamic theory proposed in this paper. Let us first define
\[ D \equiv \rho c_\zeta^2 (\tau_e + \tau_Q) + \zeta + \frac{4\eta}{3} + \sigma \kappa \]
and
\[ E \equiv \sigma [p_s \kappa_s - c_{\kappa_{\zeta}}^2] = \sigma T \rho \left[ \left( \frac{\partial P}{\partial \tau_e} \right)_n \left( \frac{\partial \zeta}{\partial \tau_e} \right)_n - \left( \frac{\partial P}{\partial \eta} \right)_e \left( \frac{\partial \zeta}{\partial \eta} \right)_e \right], \]
where \( \kappa_s = (T \rho^2 / n) (\partial \zeta / \partial \eta)_s = \kappa_e + \kappa_n \), \( \kappa_e = (T \rho^2 / n) (\partial \zeta / \partial \eta)_n \), \( \kappa_n = (T \rho) (\partial \zeta / \partial \eta)_e \), and \( p_s = (\partial P / \partial \zeta)_n \). Standard thermodynamic identities imply that \( p_s \kappa_s - c_{\kappa_{\zeta}}^2 > 0 \), then \( E \geq 0 \) from (A1). By assuming the Cowling approximation \[ \text{(10)} \] with \( g_{\mu\nu} \approx g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \), we find that:

The system described by (12) is linearly stable if it is causal within the strict form of the inequalities in (20) together with the additional restriction \( \eta > 0 \) in (A1) and
\[ (\tau_e + \tau_Q) |B| \geq 2 \tau_s \tau_Q D \geq 2 c_{\tau} \zeta \tau_Q (\tau_e + \tau_Q), \]
\[ (\tau_e + \tau_Q) |B| D + \rho c_{\tau} \tau_Q (\tau_e + \tau_Q) E > \tau_s \tau_Q D^2 + \rho (\tau_e + \tau_Q)^3 C, \]
\[ c_{\zeta}^2 D - E \geq \rho c_{\tau}^2 (\tau_e + \tau_Q), \]
\[ (\tau_e + \tau_Q) [ |B| (c_{\zeta}^2 D - 2E) + 2c_{\tau} \rho \tau_Q E + CD ] > 2 c_{\tau} \rho (\tau_e + \tau_Q)^2 C + \tau_s \tau_Q D(c_{\zeta}^2 D - E), \]
\[ |B| D [ C(\tau_e + \tau_Q) + E \tau_s \tau_Q ] + 2 \rho \tau_Q (\tau_e + \tau_Q) CE > \rho C^2 (\tau_e + \tau_Q)^2 + \tau_s \tau_Q (CD^2 + \rho \tau_Q E^2) + B^2 E (\tau_e + \tau_Q), \]
where $B$ and $C$ are given in (20) with $|B| = -B > 0$ from (20c) in the strict form.

To prove the statement above, as before we may expand the perturbations $\delta \Psi = (\delta \varepsilon, \delta u^\mu, \delta n)$ in Fourier modes by means of the substitution $\delta \Psi(X) \to \exp[T(\Gamma + k^2)]\delta \Psi(K)$, where $K^\mu = (i\Gamma, k^i)$ is dimensionless due to the introduction of background temperature $T$ in the exponent. We begin by proving stability in the local rest frame, where the modes are the roots of the shear and sound polynomials

Shear channel: \[ \tau Q T^2 + \eta k^2 + \Gamma = 0, \] (85a)

Sound channel: \[ a_0 T^5 + a_1 \Gamma^4 + a_2 \Gamma^3 + a_3 \Gamma^2 + a_4 \Gamma + a_5 = 0, \] (85b)

where $k^2 = k^i k_i$ and

\[ a_0 = \tau_\epsilon \tau_Q, \] (86a)
\[ a_1 = \tau_\epsilon + \tau_Q, \] (86b)
\[ a_2 = 1 + k^2 |\bar{B}|, \] (86c)
\[ a_3 = k^2 \bar{D}, \] (86d)
\[ a_4 = c_s^2 k^2 + k^4 \bar{C}, \] (86e)
\[ a_5 = k^4 \bar{E}. \] (86f)

We defined the dimensionless quantities $\tau Q = T \tau Q, \tau_\epsilon = T \tau_\epsilon, \eta = T \eta / \rho, \bar{B} = (T^2 / \rho) B, \bar{C} = (T^2 / \rho) C, \bar{D} = (T / \rho) D$, and $\bar{E} = (T / \rho) E$. From the second inequality in (20c) in its strict form one obtains that $\bar{B} < 0$ (see the definition of $a_2$). The analysis of stability in the LRF goes as follows:

Shear stability conditions: The second order polynomial (85a) has two roots with $\Gamma_R \leq 0$ only if $\tau Q > 0$ and $\eta \geq 0$, which is in accordance with assumption (A1). One can see that $\tau Q$ clearly acts as a relaxation time $\tau_\epsilon$ for the shear channel, which ensures causality. In fact, the condition $\tau Q > 0$ is clear since the leading contribution to the non-hydrodynamic frequency in this channel goes as $1 / \tau Q$ at zero wavenumber.

Sound stability conditions: As for the sound channel in the rest frame, by means of the Routh-Hurwitz criterion [10], the necessary and sufficient conditions for $\Gamma_R < 0$ are (i) $a_0, a_1 > 0$, (ii) $a_1 a_2 - a_0 a_3 > 0$, (iii) $a_2 a_1 - a_0 a_4 - a_3 a_1 a_4 - a_0 a_5 > 0$, (iv) $(a_2 a_4 - a_0 a_5)[a_2 a_1 - a_0 a_4] - a_3 a_1 a_4 - a_0 a_5 a_2 - a_0 a_4 a_2 a_3 > 0$, and (v) $a_5 > 0$. Condition (i) is already satisfied from (A1). Condition (ii) corresponds to the first inequality in (85a), while (iii) is the second inequality in (85a) and (85b). Condition (iv) corresponds to (85c)–(85d). Given that $E \geq 0$, thus, when $E = 0$ and (i)–(iv) are observed, then $\Gamma_R \leq 0$, which is in accordance with stability. Also, if $k = 0$, then $\Gamma_R \leq 0$ (three zero roots and two negative roots) because $a_0, a_1, a_2 > 0$ from (A1). Hence, the system is linearly stable in the local rest frame.

We remark that our system displays three types of hydrodynamic modes and three non-hydrodynamic modes. In the small $k$ expansion that typically defines the linearized hydrodynamic regime, our shear channel gives a diffusive hydrodynamic mode with (real) frequency $\omega(k) = -i k^2 \eta / (\varepsilon + P) + \ldots$ while in the sound channel one finds proper sound waves with $\omega(k) = \pm c_s k - i k^2 T \Sigma / 2 + \ldots$ and also a heat diffusion mode with $\omega(k) = -i D k^2 + \ldots$, where $D \sim \sigma$, and $\Sigma = \Gamma_s (\eta, \zeta, \sigma)$ just as in Eckart theory (see Ref. [73] for their detailed expressions). Therefore, our theory has the same physical content of Eckart’s theory in the hydrodynamic regime. On the other hand, the shear channel has a non-hydrodynamic mode with frequency given by $\omega(k) = -i \tau_\epsilon / \bar{D}$ while the sound channel has two non-hydrodynamic modes with frequency $\omega(k) = -i / \tau_\epsilon + \ldots$ and $\omega(k) = -i / \tau Q + \ldots$ in the low $k$ limit. These non-hydrodynamic modes parameterize the UV behavior of the system in a way that ensures causality and stability, making sure that the theory is well defined (though, of course, not accurate) even outside the typical domain of validity of hydrodynamics.

The complete proof of linear stability demands an analysis of the linearized system around an equilibrium state at nonzero velocity. In this regard, we shall use the results presented in Sec. [13]. We first write the system in [12] as a first-order linear system of PDE’s. Then, since we already have proven causality and also linear stability in the LRF, it remains to be shown that the first order counterpart of [12] is diagonalizable in the sense of [74]. This is done below.

A first order system: following Sec. [1A] we may define $\delta V = u^o \partial_o \delta \varepsilon, \delta V^\mu = \Delta^{\mu o} \partial_o \delta \varepsilon, \delta W = u^o \partial_o \delta n, \delta W^\mu = \Delta^{\mu o} \partial_o \delta W, \delta S^\mu = u^o \partial_o \delta u^\mu, \delta S^\mu_\chi = \Delta^{\mu o}_\chi \partial_o \delta u^\mu$. Since the current is ideal, i.e., $J^\mu = nu^\mu$, then the linearized

25 The same role is played by the shear relaxation time coefficient $\tau_\epsilon$ present in MIS theory.
conservation equation $\partial_\mu J^\mu = \delta W + n\delta S'_\nu = 0$ enables us to eliminate $\delta W$ from the new system of equations. Hence, the first order equations become

\begin{align}
\tau_\varepsilon u^\alpha \partial_\alpha \delta V + \rho \tau_Q \partial_\mu \delta S^\mu + \beta_\varepsilon \partial_\alpha \delta V^\alpha + \beta_\varepsilon \partial_\alpha \delta W^\alpha + \rho \tau_\varepsilon u^\alpha \partial_\alpha \delta S'_\nu + \delta V + \rho \delta S'_\nu &= 0, \\
\tau_\varepsilon \partial_\mu \delta V^\alpha + \rho \tau_Q u^\mu \partial_\alpha \delta S^\mu + \beta_\varepsilon u^\mu \partial_\alpha \delta W^\mu + \beta_\varepsilon u^\mu \partial_\alpha \delta S'_\nu + \rho \tau_\varepsilon u^\mu \partial_\alpha \delta W^\mu + \rho \delta S^\mu &= 0, \\
u^\alpha \partial_\alpha \delta S^\nu - \Delta^\nu \partial_\alpha \delta S^\nu &= 0,
\end{align}

(87a)\quad(87b)\quad(87c)

where $p' = (\partial P/\partial n_\varepsilon)$ and

$$\Pi^\mu_{\nu\alpha} - \eta \left( \Delta^\mu \delta^\alpha_{\nu} + \Delta^\nu \delta^\alpha_{\mu} \right) + \left( \rho \tau_\varepsilon - \zeta + \frac{2\eta}{3} \right) \Delta^\mu \delta^\lambda_{\nu} = 0.$$ 

(88)

The supplemental equations $\{87c-87c\}$ come from the identities $\partial_\nu \partial_\mu \delta \varepsilon - \partial_\mu \partial_\nu \delta \varepsilon = 0, \partial_\nu \partial_\mu \delta \eta - \partial_\mu \partial_\nu \delta \eta = 0, \partial_\nu \partial_\mu \delta \nu - \partial_\mu \partial_\nu \delta \nu = 0$, respectively, when contracted with $u^\alpha \Delta^\beta_{\lambda}$. In particular, in Eq. $\{87d\}$ we have substituted $\delta W = -n\delta S'_\nu$ that comes from the conservation equation of $J^\mu$. Then, we may write $\{87\}$ in matrix form $A^\alpha \partial_\alpha \delta \Psi(X) + B\delta \Psi(X) = 0$, where $\delta \Psi(X)$ is the $29 \times 1$ column matrix with entries $\delta V, \delta S^\nu, \delta \nu^\nu, \delta \nu^\nu, \delta S'_\nu, \delta S^\nu, \delta S'_\nu, \delta S'_\nu$.

\begin{align}
A^\alpha &= \begin{bmatrix}
\tau_\varepsilon u^\alpha & \rho \tau_Q \partial_\mu \delta^\nu_{\mu} & \beta_\varepsilon \delta^\nu_{\mu} & \beta_\varepsilon \partial_\mu \delta^\nu_{\nu} & \beta_\varepsilon \partial_\mu \delta S^\nu & \rho \tau_\varepsilon \partial_\mu \delta^\nu_{\nu} \\
\tau_\varepsilon \partial_\mu \delta V^\alpha & \rho \tau_Q u^\mu \partial_\alpha \delta S^\mu & \beta_\varepsilon u^\mu \partial_\alpha \delta W^\mu & \beta_\varepsilon u^\mu \partial_\alpha \delta S'_\nu & \beta_\varepsilon u^\mu \partial_\alpha \delta W^\mu & \beta_\varepsilon u^\mu \partial_\alpha \delta S'_\nu \\
-\Delta^\mu \delta^\nu_{\mu} & 0_{1 \times 4} & u^\mu \delta^\nu_{\nu} & 0_{1 \times 4} & u^\nu \delta^\mu_{\nu} & 0_{1 \times 4} \\
0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
-\Delta^\mu \delta^\nu_{\mu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & -\Delta^\mu \delta^\nu_{\mu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & -\Delta^\mu \delta^\nu_{\mu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & -\Delta^\mu \delta^\nu_{\mu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4}
\end{bmatrix},
\end{align}

(89)

and

\begin{align}
B &= \begin{bmatrix}
1 & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & \rho \delta^0_{\nu} & \rho \delta^1_{\nu} & \rho \delta^2_{\nu} & \rho \delta^3_{\nu} \\
0_{1 \times 4} & \rho \delta^0_{\nu} & \rho \delta^1_{\nu} & \rho \delta^2_{\nu} & \rho \delta^3_{\nu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & \rho \delta^0_{\nu} & \rho \delta^1_{\nu} & \rho \delta^2_{\nu} & \rho \delta^3_{\nu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & \rho \delta^0_{\nu} & \rho \delta^1_{\nu} & \rho \delta^2_{\nu} & \rho \delta^3_{\nu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & \rho \delta^0_{\nu} & \rho \delta^1_{\nu} & \rho \delta^2_{\nu} & \rho \delta^3_{\nu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & \rho \delta^0_{\nu} & \rho \delta^1_{\nu} & \rho \delta^2_{\nu} & \rho \delta^3_{\nu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & \rho \delta^0_{\nu} & \rho \delta^1_{\nu} & \rho \delta^2_{\nu} & \rho \delta^3_{\nu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} \\
0_{1 \times 4} & \rho \delta^0_{\nu} & \rho \delta^1_{\nu} & \rho \delta^2_{\nu} & \rho \delta^3_{\nu} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4}
\end{bmatrix}.
\end{align}

(90)

We must now obtain the eigenvectors of $\{72\}$. However, note that $A^\alpha$ above is exactly the same as the matrix $A^\alpha_3$ in $\{31\}$ with the difference that now the coefficients of $A^\alpha$ are constants. We have already proven in Sec. $\{1\}$ that the matrix $A^\alpha_3$ in Eq. $\{72\}$ has real eigenvalues and a complete set of eigenvectors in $\mathbb{R}^{29}$. The same solution is true for $A^\alpha$ in $\{72\}$ if we change $\varepsilon_\alpha \rightarrow n_\alpha$ (and also $A^\alpha_3 \rightarrow A^\alpha$) in the results for the matter sector in Sec. $\{1\}$. Thus, the $29 \times 29$ matrix $(-n_\alpha A^\alpha_3)_{\xi_\beta A^3}$ is diagonalizable, completing the requirements from Theorem III. This shows that the theory is linearly stable in any other reference frame $\mathcal{O}'$ connected via a Lorentz transformation. Therefore, one then obtains stability in any equilibrium state.

A. Fulfilling the causality, local well-posedness, and linear stability conditions

We now give a simple example that illustrates that the set of linear stability conditions (and consequently, causality and well-posedness, since those are part of the linear stability conditions) is not empty. Let us analyze the case where $\tau_Q = \tau_\varepsilon$ and $\tau_\varepsilon = c_\varepsilon^2 \tau_\varepsilon$, assuming an equation of state $P = P(\varepsilon)$, with $c_\varepsilon^2 = p_\varepsilon = 1/2$. Also, assume that $\zeta + 4\eta/3 > 0$ (their specific values are not relevant as far as they are positive and $\eta > 0$ for the sake of the stability and well-posedness theorems). Then, one may easily verify that the causality conditions $\{20\}$ hold in their strict form, as required, and that the remaining conditions $\{31\}$ are also observed when $\rho \tau_\varepsilon = 8(\zeta + 4\eta/3)$, $\kappa_\varepsilon = \kappa_\eta/2 = 1/4$, and in the three different situations, namely, $\sigma/(\zeta + 4\eta/3) = 0, 1/4,$ and $1$. 
VII. CONCLUSIONS AND OUTLOOK

In this work, we presented the first example of a relativistic theory of viscous fluid dynamics that simultaneously satisfies the following properties: the system when coupled to Einstein’s equations is causal, strongly hyperbolic, and the solutions are well-posed (see the content of Theorem I and II); equilibrium states in flat spacetime are stable (consequence of Theorem III); all dissipative contributions (shear viscosity, bulk viscosity, and heat flow) are included; and finally the effects from nonzero baryon number are also taken into account. All of the above holds without any simplifying symmetry assumptions and are mathematically rigorously established.

This is accomplished in a natural way using a theory containing only the original hydrodynamic variables, which is different than other approaches where the space of variables is extended (such as in Mueller-Israel-Stewart theory). However, it is important to remark that the meaning of the hydrodynamic variables in our work is different than in standard approaches, such as [14] and [21]. In fact, in the context of the BDNK formalism put forward in [61–63], our formulation uses a definition for the hydrodynamic variables (i.e. our choice of hydrodynamic frame) that is not standard as there are nonzero out of equilibrium corrections to the energy density and there is heat flux even at zero baryon density. Despite these necessary differences (imposed by causality and stability), the theory still provides the simplest causal and strongly hyperbolic generalization of Eckart’s original theory [21], sharing the same physical properties in the hydrodynamic regime (for instance, both theories have the same spectrum of hydrodynamic modes). However, differently than Eckart’s approach, our formulation is fully compatible with the postulates of general relativity and its physical content in dynamical settings can be readily investigated using numerical relativity simulations. In fact, we hope that the framework presented here will provide the starting point for future systematic studies of viscous phenomena in the presence of strong gravitational fields, such as in neutron star mergers.

Motivated by the task of establishing stability in general equilibrium states in flat spacetime, in this work we also proved a new general result (see Theorem III) concerning the stability of relativistic fluids. In fact, we found conditions that causal relativistic fluids should satisfy such that stability around the static equilibrium state directly implies stability in any other equilibrium state at nonzero background velocity. Theorem III is very general and its regime of applicability goes beyond BDNK theories and it could also be relevant when investigating the stability properties of other sets of linear equations of motion as well.

Our theory can be used to understand how matter in general relativity starts to deviate from equilibrium. An immediate application is in the modeling of viscous effects in neutron star mergers. Our approach can be useful in simulations that aim at determining the fate of the hypermassive remnant formed after the merger of neutron stars, hopefully leading to a better quantitative understanding of their evolution and eventual gravitational collapse towards a black hole. Differently than any other approach in the literature, the new features displayed by our formulation and its strongly hyperbolic character make it a suitable candidate to be used in such simulations. This will be especially relevant also when considering how viscous effects may modify the gravitational wave signals emitted soon after the merger [12]. In this regard, we remark that previous simulations performed in Ref. [11] employed a formulation of relativistic viscous hydrodynamics where the key properties studied here (causality, hyperbolicity, and well-posedness) are not known to hold at the moment in the nonlinear regime.

Our work is applicable in the case of baryon rich matter, such as that formed in neutron star mergers or in low energy heavy-ion collisions. The latter include the experimental efforts in the beam energy scan program at RHIC [102], the STAR fixed-target program [102], the HADES experiment at GSI [103], the future FAIR facility at GSI [104], and also NICA [105]. High energy heavy-ion collisions, such as those studied at the LHC, involve a different regime than the one considered here where the net baryon number can be very small and, thus, that case is better understood using a different formulation such as the one proposed in [73], also in the context of the BDNK formalism.

In our approach, we only take into account first order derivative corrections to the dynamics. Therefore, the domain of validity of our theory is currently limited by the size of such deviations. Hence, further work is needed to extend our analysis, incorporating higher order derivative corrections, to get a better understanding of what happens as the system gets farther and farther from equilibrium. In this context, it would be interesting to investigate the large order behavior of the derivative expansion constructed a la BDNK. This is different than the standard gradient expansion since in BDNK the constitutive relations contain time derivatives even in the local rest frame of the fluid. The large order behavior of the relativistic gradient series has been recently the focus of several works [106,113], and it would be interesting to extend such analyses to include the type of theories investigated here.

There are a number of ways in which our work could be extended or improved. First, it would be useful to obtain a better qualitative understanding why some hydrodynamic frames (such as the Landau-Lifshitz frame or the Eckart frame) are not compatible with causality and stability in the BDNK approach, given that the situation is different in other formulations. In fact, the Landau frame seems to display no significant issues in the case of Mueller-Israel-Stewart even in the nonlinear regime at least at zero chemical potential, as demonstrated in [43]. Perhaps a more in depth investigation of how BDNK emerges in kinetic theory, going beyond the original work done in [61,63], can be useful in this regard. Also, it would be interesting to use the BDNK approach to investigate causality and stability
in more exotic cases, such as in relativistic superfluids. Furthermore, the inclusion of electromagnetic field effects in the dynamics of relativistic viscous fluids can also be of particular relevance, especially in the context of neutron star mergers [120] and high-energy heavy ion collisions [121]. This problem has been recently investigated using other formulations of viscous fluid dynamics, see for instance Refs. [122, 123]. Consistent modeling of relativistic viscous fluid dynamics coupled to electromagnetic fields can also be relevant to determine the importance of dissipative processes in the dynamics and radiative properties of slowly accreting black holes, as discussed in [122].

Further work needs to be done to understand the global in-time features of solutions of relativistic viscous fluid dynamics. For instance, one may investigate the presence of shocks, which is a topic widely investigated in the context of ideal fluids [18, 126–129] and was done in [51] for the MIT theory. The importance of hydrodynamic shocks has been recognized both in an astrophysical setting [122] as well as in study of jets in the quark-gluon plasma [130–142]. We also remark that one task that we have not done here was the construction of initial data for the full Einstein plus fluid system by solving the Einstein constraint equations. We believe that standard arguments to handle the constraints [18] will be applicable in our case. This will be investigated in detail in a future work.

We believe our work will also be relevant to give insight into the physics of turbulent fluids embedded in general relativity. The fact that the equations of motion of the viscous fluid must be hyperbolic in relativity stands in sharp contrast to the parabolic nature of the non-relativistic Navier-Stokes equations, usually employed in studies of turbulence. Recent work in Ref. [143] tackled the problem of turbulence in the relativistic regime using theories that are either known to be acausal (e.g. Eckart’s theory) or other approaches whose hyperbolic character in the nonlinear regime is not yet fully understood. Our formulation may be very useful in this regard, as it provides a simple hyperbolic generalization of Eckart’s theory that is fully compatible with general relativity.

In summary, in this paper we propose a new solution to the 80 year old problem initiated by Eckart concerning the motion of viscous fluids in relativity. Our approach is rooted in well-known physical principles and solid mathematics, displays a number of desired properties, and extends the state-of-the-art of the field in a number of ways. Potential applications of the formalism presented here spread across a numbers of areas, including astrophysics, nuclear physics, cosmology, and mathematical physics. This work establishes for the first time a common unifying framework, from heavy-ion collisions to neutron stars, that can be used to discover the novel properties displayed by ultradense baryonic matter as it evolves in spacetime.

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