Formalized procedure of transition to classical limit in application to the Dirac equation.

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Abstract

Classical model $S_{Dcl}$ of the Dirac particle $S_D$ is constructed. $S_D$ is the dynamic system described by the Dirac equation. For investigation of $S_D$ and construction of $S_{Dcl}$ one uses a new dynamic method: dynamic disquantization. This relativistic purely dynamic procedure does not use principles of quantum mechanics. The obtained classical analog $S_{Dcl}$ is described by a system of ordinary differential equations, containing the quantum constant as a parameter. Dynamic equations for $S_{Dcl}$ are determined by the Dirac equation uniquely. The dynamic system $S_{Dcl}$ has ten degrees of freedom and cannot be a pointlike particle, because it has an internal structure. Internal degrees of freedom appears to be described nonrelativistically. One discusses interplay between the conventional axiomatic methods and the dynamical methods of the quantum systems investigation. In particular, one discusses the reasons, why the internal degrees of freedom of the Dirac particle and their nonrelativistic character were not discovered during eighty years.

Key words: dynamical methods; classical Dirac particle; internal structure of Dirac particle; interplay between dynamical and axiomatical methods.

1 Introduction

In the framework of axiomatic presentation of quantum mechanics there exist no formal procedure of transition to classical approximation. The classical description is obtained from the quantum one, when we set, that the quantum constant $\hbar = 0$. Unfortunately, we cannot obtain the classical approximation for the Schrödinger particle $S_s$, setting $\hbar = 0$ in the action for the Schrödinger particle. This action has
the form

$$\mathcal{S}_S: \quad \mathcal{A}_S [\psi, \psi^*] = \int \left\{ \frac{i\hbar}{2} (\psi^* \partial_0 \psi - \partial_0 \psi^* \cdot \psi) - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi \right\} dt dx \quad (1.1)$$

where $\psi = \psi(t, x)$ is a complex one-component wave function, $\psi^* = \psi^*(t, x)$ is the quantity complex conjugate to $\psi$, and $m$ is the particle mass. The action (1.1) generates the dynamic equation

$$i\hbar \partial_0 \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi \quad (1.2)$$

The 4-current $j^k$ and the energy-momentum tensor $T^k_l$ are the canonical quantities associated with the action $\mathcal{A}_S [\psi, \psi^*]$. They are determined by the relations

$$j^k = \{ \rho, j \} = \frac{i}{\hbar} \left( \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi^*)} \psi^* - \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} \psi \right) = \left\{ \psi^* \psi, -\frac{i\hbar}{2m} (\psi^* \nabla \psi - \nabla \psi^* \cdot \psi) \right\} \quad (1.3)$$

$$T^k_l = \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi^*)} \partial_t \psi^* + \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} \partial_t \psi - \delta^k_l \mathcal{L}, \quad k, l = 0, 1, 2, 3 \quad (1.4)$$

where $\mathcal{L}$ is the Lagrangian density for the action (1.1)

$$\mathcal{L} = \frac{i\hbar}{2} (\psi^* \partial_0 \psi - \partial_0 \psi^* \cdot \psi) - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi \quad (1.5)$$

If we set $\hbar = 0$ in the action (1.1), the description degenerates. Such a degeneration is conditioned by the artificial use of the quantum constant in the action (1.1). The fact is that at the natural description of the Schrödinger particle $\mathcal{S}_S$ the action contains two independent constants: dynamical constant $b$ and quantum constant $\hbar$. The dynamical constant $b$ is simply an arbitrary constant of integration, which can take any nonvanishing value. At such a natural description the quantum constant $\hbar$ describes the quantum effects, and quantum description transits to the classical one, if one sets $\hbar = 0$.

The natural description is obtained from the action (1.1) by means of the transformation

$$\psi \to \Psi_b = |\psi| \exp \left( \frac{b}{\hbar} \log |\psi| \right), \quad \psi = |\Psi_b| \exp \left( -\frac{b}{\hbar} \log |\Psi_b| \right) \quad (1.6)$$

The natural description is carried out in terms of the wave function $\Psi_b$. The action has the form

$$\mathcal{S}_S: \quad \mathcal{A}_S [\Psi_b, \Psi_b^*] = \int \left\{ \frac{ib}{2} (\Psi_b^* \partial_0 \Psi_b - \partial_0 \Psi_b^* \cdot \Psi_b) - \frac{b^2}{2m} \nabla \Psi_b^* \nabla \Psi_b \right. \nonumber$$

$$\left. - \frac{\hbar^2}{2m} \left( \nabla |\Psi_b| \right)^2 \right\} dt dx \quad (1.7)$$
The dynamic equation takes the form
\[ ib \partial_0 \Psi_b = -\frac{b^2}{2m} \nabla^2 \Psi_b - \frac{\hbar^2}{8m} \left( \frac{\left( \nabla \rho \right)^2}{\rho^2} + 2 \nabla \frac{\nabla \rho}{\rho} \right) \Psi_b, \quad \rho \equiv \Psi_b^* \Psi_b \] (1.8)

Instead of (1.3), we obtain
\[ \rho = \Psi_b^* \Psi_b, \quad j = -\frac{ib}{2m} (\Psi_b^* \nabla \Psi_b - \nabla \Psi_b^* \cdot \Psi_b) \] (1.9)

We underline that the actions (1.4) and (1.7) describe the same dynamic system in different dependent variables. The action (1.7) contains two parameters \( b \) and \( \bar{h} \), whereas the action (1.1) contains only one parameter \( \bar{h} \).

Description in terms of \( \Psi_b \) is a natural description, because, the constant \( \bar{h} \) describes the quantum effects, and setting \( \bar{h} = 0 \) in the action (1.7), we obtain the action for the statistical ensemble of free classical particles (this fact has been proved in Appendix A)

\[ S_{\text{Sc}} : \quad A_{\text{Sc}} [\Psi_b, \Psi_b^*] = \int \left\{ \frac{ib}{2} (\Psi_b^* \partial_0 \Psi_b - \partial_0 \Psi_b^* \cdot \Psi_b) - \frac{b^2}{2m} \nabla \Psi_b^* \nabla \Psi_b + \frac{b^2}{2m} (\nabla |\Psi_b|)^2 \right\} dt dx \] (1.10)

If we identify parameters \( b \) and \( \bar{h} \) in (1.7) - (1.9), we obtain the artificial description (1.1) - (1.4), where setting \( \bar{h} = 0 \), we set simultaneously \( b = 0 \) and the description degenerates. At the artificial description the dynamic term and the quantum term of the dynamic equation are compensated, and the dynamic equation becomes to be linear. This is the main advantage of the artificial description.

The Dirac particle is the dynamic system \( S_D \), described by the Dirac equation. The action \( A_D \) for the dynamic system \( S_D \) has the form

\[ S_D : \quad A_D[\bar{\psi}, \psi] = c^2 \int (-mc\bar{\psi}\psi + i\frac{\hbar}{2} \bar{\psi} \gamma^i \partial_i \psi - i\frac{\hbar}{2} \partial_0 \bar{\psi} \gamma^0 \psi - \frac{e}{c} A_l \bar{\psi} \gamma^l \psi) d^4 x \] (1.11)

where \( m \) and \( e \) are respectively the mass and the charge of the Dirac particle, and \( c \) is the speed of the light. Here \( \psi \) is four-component complex wave function, \( \psi^* \) is the Hermitian conjugate wave function, and \( \bar{\psi} = \psi^* \gamma^0 \) is the conjugate one. The quantities \( \gamma^i, \ i = 0, 1, 2, 3 \) are \( 4 \times 4 \) complex constant matrices, satisfying the relation
\[ \gamma^i \gamma^k + \gamma^k \gamma^i = 2 g^{kl} I, \quad k, l = 0, 1, 2, 3. \] (1.12)

where \( I \) is the \( 4 \times 4 \) identity matrix, and \( g^{kl} = \text{diag}(e^{-2}, -1, -1, -1) \) is the metric tensor. The quantity \( A_k, k = 0, 1, 2, 3 \) is the electromagnetic potential. The action (1.11) generates dynamic equation for the dynamic system \( S_D \), known as the Dirac equation
\[ \gamma^l \left( -i \hbar \partial_l + \frac{e}{c} A_l \right) \psi + mc\psi = 0 \] (1.13)
and expressions for physical quantities: the 4-flux \( j^k \) of particles and the energy-momentum tensor \( T^k_l \)

\[
j^k = c^2 \bar{\psi} \gamma^k \psi, \quad T^k_l = \frac{ie^2}{2} \left( \bar{\psi} \gamma^k \partial_l \psi - \partial_l \bar{\psi} \cdot \gamma^k \psi \right)
\] (1.14)

Description of the Dirac particle is also artificial in the sense, that it is degenerate at \( \hbar = 0 \). Unfortunately, the transformation of the kind (1.6) is unknown in the case of the Dirac particle \( S_D \), and we are forced to look for another approach to the derivation of the classical approximation. We use dynamical methods of investigation. It means the investigation of the dynamic system \( S_D \) simply as a dynamic system without a use of quantum principles. Such an approach admits one to obtain classical approximation of \( S_D \) by means of some dynamical procedure, which does not contain a reference to the quantum constant \( \hbar \). This procedure (dynamical disquantization) is insensitive to the form of application of the quantum constant (natural or artificial). The dynamical disquantization is a special case of dynamical methods, applied to the investigation of quantum systems. We manifest application of dynamical methods in the example of the Schrödinger particle \( S_S \).

2 Dynamical methods of investigation

We use the mathematical technique, which is more developed, than conventional formalism of quantum mechanics. This technique supposes that all essential information on the quantum dynamical system is contained in the dynamic system itself. Such specific quantum concepts as the wave function and principles of quantum mechanics appear to be only the means of description. The wave function as the means of description may be applied to both quantum and classical dynamic systems. But the quantum principles may be applied only to quantum dynamic systems, because they contains some constraints, which are not satisfied for classical systems. The quantum system and classical system distinguish dynamically (in additional terms in the action), but not in the way of description. This fact becomes to be clear, when both systems are described in the same terms. For instance, the quantum system and the corresponding classical system may be described in terms of the wave function, or both systems may be described in terms of the particle position and momentum.

Progress in the development of the mathematical technique has a mathematical ground: integration of dynamic equations. This pure mathematical achievement has physical consequences. It appears that the quantum mechanics may be considered to be a statistical description of randomly moving particles. We underline that we investigate well known quantum systems, and all new results are corollaries of the more developed methods of investigation.

The dynamic system \( S_S \) is determined completely by dynamic equations (1.2) and expressions (1.3), (1.4) for the 4-current and the energy-momentum tensor. Only connection between the particle and the wave function is not described by
these relations. This connection is described by means of the relations

\[
\langle F(x, p) \rangle = B \int \text{Re} \{\psi^{*} F(x, \hat{p}) \psi \} \, dx, \quad \hat{p} = -\text{i} \hbar \nabla, \quad B = \left( \int \psi^{*} \psi \, dx \right)^{-1}
\]

which define the mean value \( \langle F(x, p) \rangle \) of any function \( F(x, p) \) of the particle coordinates \( x \) and momentum \( p \). Application of the rules (2.1) is restricted by some conditions. They demand that the dynamic equations be linear and the wave function be a vector in the Hilbert space of states. We shall refer to the relations (2.1) together with the restrictions imposed on its applications as the quantum principles, because von Neumann has shown [1], that the quantum mechanics can be deduced from relations of the type (2.1), provided they are valid for all observable quantities. Thus, the interpretation of the wave function is carried out on the basis of the quantum principles, which are something external with respect to the dynamic system \( S \).

In reality, the quantum principles are not necessary for interpretation of the wave function and properties of the dynamic system \( S \). It is sufficient to make a proper change of dynamic variables and to describe the dynamic system \( S \) in terms of the particle coordinates \( x \). Such a description does not contain the enigmatic wave function, whose meaning is unclear, and one does not need the quantum principles (2.1) for its interpretation. The Schrödinger particle \( S \) is a partial case of the generalized Schrödinger particle \( S_{gs} \), which is the dynamic system \( S_{gs} \), described by the action

\[
A_{gs}[\psi, \psi^{*}] = \int \left\{ \frac{\text{i} \hbar}{2} (\psi^{*} \partial_{0} \psi - \partial_{0} \psi^{*} \cdot \psi) - \frac{\hbar^{2}}{2m} \nabla \psi^{*} \nabla \psi + \frac{\hbar^{2}}{8m} \sum_{\alpha=1}^{3} (\nabla s_{\alpha})^{2} \rho \right\} \, d^{4}x
\]

\[
\rho \equiv \psi^{*} \psi, \quad s \equiv \frac{\psi^{*} \sigma \psi}{\rho}, \quad \sigma = \{\sigma_{\alpha}\}, \quad \alpha = 1, 2, 3
\]

Here \( \psi = \left( \begin{array}{c} \psi_{1} \\ \psi_{2} \end{array} \right) \), \( \psi^{*} = (\psi_{1}^{*}, \psi_{2}^{*}) \) is the two-component wave function, and \( \sigma_{\alpha} \) are the Pauli matrices. The 4-current is defined by the relation (1.3) with two-component wave function \( \psi \). In the case, when components \( \psi_{1} \) and \( \psi_{2} \) are linear dependent (for instance, \( \psi = \left( \begin{array}{c} \psi_{1} \\ 0 \end{array} \right) \)), the mean spin vector \( s \equiv \text{const} \), and the last term in the action (2.2) vanishes. In this case the dynamic system \( S_{gs} \) turns into the dynamic system (1.1).

One can show, that the dynamic system \( S_{gs} \) is another representation of the dynamic system \( E [S_{st}] \), i.e. the action for \( S_{gs} \) can be obtained from the action for the dynamic system \( E [S_{st}] \) by means of a proper change of variables [2].

The dynamic system \( E [S_{st}] \) is a statistical ensemble of stochastic particles \( S_{st} \).
It is described by the action

$$
E[S_{st}] : \quad A_{E[S_{st}]}[x, u_{st}] = \int \left\{ \frac{m}{2} \left( \frac{dx}{dt} \right)^2 + \frac{m}{2} u_{st}^2 - \frac{\hbar}{2} \nabla u_{st} \right\} dt d\xi \quad (2.4)
$$

where $u_{st} = u_{st}(t, x)$ is a vector function of arguments $t, x$ (not of $t, \xi$), and $x = x(t, \xi)$ is a 3-vector function of independent variables $t, \xi = \{\xi_1, \xi_2, \xi_3\}$. Dynamic equations for the dynamic system $E[S_{st}]$ are obtained as a result of variation of the action (2.4) with respect to dependent dynamic variables $x, u_{st}$. In the action (2.4) the variables $\xi$ label stochastic systems $S_{st}$, constituting the statistical ensemble. The operator $\nabla$ is defined in the space of coordinates $x$ by the relation

$$
\nabla = \{\partial_1, \partial_2, \partial_3\} \equiv \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\} \quad (2.5)
$$

The 3-vector $u_{st}$ describes the mean value of the stochastic component of the particle motion, which is considered to be a function of the variables $t, x$. The first term $\frac{m}{2} \left( \frac{dx}{dt} \right)^2$ describes the energy of the regular component of the stochastic particle motion. The second term $mu_{st}^2/2$ describes the energy of the random component of velocity. The components $\frac{dx}{dt}$ and $u_{st}$ of the total velocity are connected with different degrees of freedom, and their energies should be added in the expression for the Lagrange function density. The last term $-\hbar \nabla u_{st}/2$ describes interplay between the velocity $\frac{dx}{dt}$ of the regular component and the random one $u_{st}$.

The action (2.4) is a sum (integral) of actions for independent stochastic systems $S_{st}$, labelled by the parameters $\xi = \{\xi_1, \xi_2, \xi_3\}$. Any stochastic system $S_{st}$ is a stochastic particle, whose state is described by its coordinate $x(t)$. The action for the stochastic system $S_{st}$ is obtained from the action (2.4) for $E[S_{st}]$. It has the form

$$
S_{st} : \quad A_{S_{st}}[x, u_{st}] = \int \left\{ \frac{m}{2} \left( \frac{dx}{dt} \right)^2 + \frac{m}{2} u_{st}^2 - \frac{\hbar}{2} \nabla u_{st} \right\} dt \quad (2.6)
$$

where $x = x(t)$. In reality, the action (2.6) is not well defined mathematically, if $\hbar \neq 0$. It is only symbolic, because the operator (2.5) is defined in the vicinity of the point $x$, but not at the point $x$ itself. As a result the dynamic equations for the stochastic system $S_{st}$ do not exist, if $\hbar \neq 0$. This fact agrees with the stochasticity of $S_{st}$. By definition the system $S_{st}$ is stochastic, if there exist no dynamic equations for $S_{st}$. If we cut off interaction with the stochastic agent, setting $\hbar = 0$ in the action (2.6) (or remove two last terms), we obtain the well defined action for the free nonrelativistic deterministic particle $S_d$

$$
S_d : \quad A_{S_d}[x, u_{st}] = \int \left\{ \frac{m}{2} \left( \frac{dx}{dt} \right)^2 + \frac{m}{2} u_{st}^2 \right\} dt, \quad x = x(t) \quad (2.7)
$$

The Schrödinger particle $S_S$ (1.1) is a partial case of the dynamic system $E[S_{st}]$ (2.4), whereas the generalized Schrödinger particle $S_{gs}$ (2.2) coincide with the dynamic system $E[S_{st}]$ (2.4). The action (2.2) may be obtained from the action (2.4) mathematically by means of a proper change of variables. (see Appendix A).
Interpretation of the dynamic system \((2.4)\) is very simple, but dynamic equations for \(E[\mathcal{S}_{\text{st}}]\) are rather complicated. They have the form

\[
\frac{\delta A_{\varepsilon[\mathcal{S}_{\text{st}}]}}{\delta x} = -m\frac{d^2 x}{dt^2} + \nabla \left( \frac{m}{2} u_{\text{st}}^2 - \frac{\hbar}{2} \nabla u_{\text{st}} \right) = 0 \tag{2.8}
\]

\[
\frac{\delta A_{\varepsilon[\mathcal{S}_{\text{st}}]}}{\delta u_{\text{st}}} = m u_{\text{st}} \rho + \frac{\hbar}{2} \nabla \rho = 0, \tag{2.9}
\]

where \(\rho\) is the function of derivatives of \(x\) with respect to \(\xi = \{\xi_1, \xi_2, \xi_3\}\), determined by the relation

\[
\rho = \left[ \frac{\partial (x^1, x^2, x^3)}{\partial (\xi_1, \xi_2, \xi_3)} \right]^{-1} = \frac{\partial (x^1, x^2, x^3)}{\partial (\xi_1, \xi_2, \xi_3)} \tag{2.10}
\]

Resolving the relation \((2.9)\) with respect to \(u_{\text{st}}\) in the form

\[
uu_{\text{st}} = -\frac{\hbar}{2m} \nabla \ln \rho, \tag{2.11}
\]

and eliminating \(u_{\text{st}}\) from \((2.8)\), we obtain

\[
m\frac{d^2 x}{dt^2} = -\nabla U(\rho, \nabla \rho), \quad U(\rho, \nabla \rho) = \frac{\hbar^2}{8m} \left( \frac{(\nabla \rho)^2}{\rho^2} - 2 \frac{\nabla^2 \rho}{\rho} \right) \tag{2.12}
\]

Thus, dynamic equations, generated by the action \((2.4)\), describe the regular motion component of any particle \(\mathcal{S}_{\text{st}}\), as a motion in a very complicated potential field \(U\), depending on the distribution of all particles of the statistical ensemble \(E[\mathcal{S}_{\text{st}}]\). Of course, the trajectories \(x = x(t, \xi)\) do not describe the motion of individual stochastic particles. They describe only statistical average motion of stochastic particles. The situation reminds situation in the gas dynamics. The dynamic equations of the gas dynamics describe the motion of the ”gas particles”, which contain many molecules. Motion of the gas molecules is random and chaotic. It cannot be described by the gas dynamics equations, which describe only regular component of the molecule motion.

Note, that the term \(\frac{m}{2} u_{\text{st}}^2\) in \((2.6)\) looks as a kinetic energy, but according to \((2.11)\) it does not depend on the temporary derivative \(\dot{x}\), and in dynamic equations it acts as a potential energy.

The statistical ensemble \((2.4)\) may be considered to be some fluid. We may speak about the flow of the statistical ensemble \(E[\mathcal{S}_{\text{st}}]\), keeping in mind, that dynamic equation \((2.12)\) for the dynamic system \(E[\mathcal{S}_{\text{st}}]\) may be interpreted as hydrodynamic equation for some ”quantum” fluid.

On the contrary, the dynamic equations, generated by the action \((1.1)\), are linear and rather simple, whereas their interpretation is very complicated, because it uses the principles of quantum mechanics \((2.1)\). Thus, the description by means of the action \((2.4)\) admits a simple interpretation, but dynamic equations are very complicated for a solution.
If the action \((1.1)\) is a special case of the action \((2.4)\), it is reasonable to use the dynamic system \(\mathcal{E}[S_{st}]\) as starting point for the statement of the problem and for interpretation of the results obtained, whereas the dynamic system \(S_{S}\) will be used only for solution of dynamic equations, which have a simple form in terms of the wave function. Note, that according \((A.36)\) the action \((2.4)\) is transformed to the action \((1.7)\), containing two parameters: arbitrary dynamical constant \(b\) and quantum constant \(\bar{\hbar}\). It is a natural form of the action, generating nonlinear dynamic equation, if \(b \neq \bar{\hbar}\). Artificial form \((1.1)\) of the action is obtained after artificial identification of the dynamical constant with the quantum one.

Why was the statistical ensemble \(\mathcal{E}[S_{st}]\) as a starting point was not not used before? Why was the problem of the stochastic motion of microparticles stated in terms of enigmatic wave function? The answer is very simple. The connection between two different forms \((1.1)\) and \((2.4)\) of the action for the Schrödinger particle has not been known for a long time.

It is known, that the Schrödinger equation can be written in the hydrodynamical form \([3]\). D. Bohm \([4]\) used this circumstance for the hydrodynamic interpretation of quantum mechanics. But it was only interpretation of the quantum principles in the hydrodynamical terms. He failed to eliminate the quantum principles and the wave function from the foundation of the quantum mechanics, and the wave function remained to be an enigmatic object – the vector in the Hilbert space. One failed to connect the wave function with the hydrodynamic variables: the density \(\rho\) and the velocity \(v\). In more exact terms the connection between the wave function and hydrodynamic variables \(\rho, v\) was established, but it was a one-way connection. In the case of the irrotational flow the hydrodynamical variables can be expressed via the wave function \(\psi\), but one cannot do this in the case of the irrotational flow. Hence, one can transit from the description in terms of the wave function to the description in terms of \(\rho, v\), but one cannot transit from the hydrodynamic description in terms of \(\rho, v\) to a description in terms of \(\psi\), because, in general, the fluid flow is rotational, and the dynamic system \((2.4)\) cannot be described in terms of the one-component wave function.

Let us present the wave function in the form

\[ \psi = \sqrt{\rho}e^{i\varphi}, \]  

(2.13)

substitute it in the Schrödinger equation \((1.2)\) and separate the real and imaginary parts of the equation. We obtain two real equations

\[ \partial_0 \ln \rho = -\frac{\hbar}{m} \left( \nabla^2 \varphi + \nabla \ln \rho \nabla \varphi \right) \]  

(2.14)

\[ \partial_0 \varphi + \frac{\hbar}{2m} \left( \nabla \varphi \right)^2 = \frac{\hbar}{2m} \left( \frac{1}{2} \nabla^2 \ln \rho + \left( \frac{1}{2} \nabla \ln \rho \right)^2 \right) \]  

(2.15)

To obtain hydrodynamic equations, one needs to take gradient of the equation \((2.15)\) and introduce the velocity \(v = \{v^1, v^2, v^3\}\) by means of the relation

\[ v = \frac{\hbar}{m} \nabla \varphi \]  

(2.16)
We obtain

$$\partial_0 \rho + \partial_\alpha (\rho v^\alpha) = 0, \quad \partial_0 v^\alpha + v^\beta \partial_\beta v^\alpha = -\frac{1}{\rho} \partial_\beta P^{\alpha\beta}, \quad \alpha = 1, 2, 3 \quad (2.17)$$

where $P^{\alpha\beta}$ is the tension tensor

$$P^{\alpha\beta} = \frac{\hbar^2}{4m^2} \left( \frac{(\partial_\alpha \rho) \partial_\beta \rho}{\rho} - \partial_\alpha \partial_\beta \rho \right) \quad (2.18)$$

The hydrodynamic equations (2.17) are obtained as a result of differentiation of the equation (1.2), written in terms of the wave function. It means that to transit from the hydrodynamic equations (2.17) to the equation, written in terms of the wave function, one needs to integrate the hydrodynamic equations (2.17). Besides, in the case of the irrotational flow the wave function is presented in terms of $\rho$ and hydrodynamical potential $\varphi$. The same is valid in the general case, but the number of the hydrodynamical potentials is to be more than one, and it is necessary to introduce additional hydrodynamic (Clebsch) potentials.

The problem of integration of the hydrodynamical equations is rather complicated problem, which has been solved only in the end of eighties [5]. To solve this problem, it was necessary to develop a special Jacobian technique [2], which was used already by Clebsch [6, 7].

As soon as the hydrodynamic equations for the ideal fluid have been integrated, it becomes clear, that the wave function is simply a method of the ideal fluid description. The wave function $\psi$ ceases to be an enigmatic vector of the Hilbert space, whose meaning was obtained only via quantum principles. Now one can determine the chain of the dynamic variable transformations which turn the action (2.4) into the action (2.2) (for details see Appendix A). As a result the action (2.4) may be used as a starting point for the description of the quantum Schrödinger particle $S$. At such a description the quantum principles (2.1) are not needed, because they are only a tool for the interpretation of the wave function.

The statistical ensemble (2.4) as the starting point of the quantum description has a series of advantages over the action (1.1):

1. The statistical ensemble (2.4) is a very transparent construction founded on the simple physical idea, that the quantum particle is a stochastically moving particle.

2. It does not use quantum principles, which are nonrelativistic and cannot be extended properly to the relativistical case.

3. Statistical ensemble (2.4) is a more general construction, because the action (1.1) is a partial case of the action (2.4)

4. Description in terms of the dynamic system (2.4) is a statistical description. As any statistical description it contains two objects: the individual stochastic particle $S_{st}$ and the statistical average particle $\langle S_{st} \rangle$. Respectively there
are two kinds of measurements: individual measurement (S-measurement) produced over the individual particle $S_{\text{st}}$ and the massive measurement (M-measurement) produced over the statistical average particle $\langle S_{\text{st}} \rangle$. These measurements have different properties, and their identification is inadmissible.

The complexity of dynamic equations (2.12) is the only defect of the statistical description (2.4).

We underline that the transition from the action (2.4) as a starting point to the action (1.1) is motivated mathematically. No additional physical arguments have been used for the substantiation of the statistical ensemble (2.4) as a starting point of the quantum description.

The statistical description, founded on the action (2.4) leads to the statement that wave function describes a state of the statistical ensemble $E[\mathcal{S}_{\text{st}}]$, but not a state of a single quantum particle. Discussion of the question, what object is described by the wave function, has a long history. Some researchers [8] believe, that the wave function describes the state of a single quantum particle, whereas other ones [9, 10] believe that the wave function describes the state of the statistical ensemble. There is a long list of different opinions about this question, but we do not present them, because this problem is not a question of a belief. It can and must be solved on the basis of the mathematical formalism.

The problem is set as follows. What dynamic system is described by the action (1.1)? A single quantum particle, or a statistical ensemble of single particles? Let us go to the limit $\hbar \to 0$. Then the action (1.1) will describe the classical dynamic system $\mathcal{S}_{\text{cl}}$. If the dynamic system $\mathcal{S}_{\text{cl}}$ is a single classical particle, then the wave function describes the state of a single particle. If the dynamic system $\mathcal{S}_{\text{cl}}$ is a statistical ensemble of classical particles, then the wave function describes the state of a statistical ensemble of single particles.

Setting $\hbar = 0$ in (1.7), (1.8), we obtain

$$\mathcal{S}_{\text{cl}} : \quad \mathcal{A}_{\text{cl}} [\Psi_b, \Psi_b^*] = \int \left\{ \frac{ib}{2} \left( \Psi_b^* \partial_0 \Psi_b - \partial_0 \Psi_b^* \cdot \Psi_b \right) - \frac{b^2}{2m} \nabla \Psi_b^* \nabla \Psi_b - \frac{b^2}{2m} \left( \nabla \right| \Psi_b^* \left| \nabla \right| \Psi_b \right) \right\} dt dx$$

$$\quad + \frac{b^2}{2m} \left( \nabla \right| \Psi_b^* \left| \nabla \right| \Psi_b \right) dtdx \quad (2.19)$$

$$ib\partial_0 \Psi_b = -\frac{b^2}{2m} \nabla^2 \Psi_b + \frac{b^2}{8m} \left( \frac{(\nabla \rho)^2}{\rho^2} + 2 \nabla \frac{\nabla \rho}{\rho} \right) \Psi_b, \quad \rho \equiv \Psi_b^* \Psi_b \quad (2.20)$$

The action (2.19) describes the statistical ensemble of free classical particles and, hence, the wave function describes the statistical ensemble, but not a single particle. The action (2.19) may not describe a single classical particle, because the dynamic system (2.19) has infinite number of the freedom degrees. As far as the description (2.19) in terms of the wave function $\Psi_b$ is a limit $\hbar \to 0$ of the description in terms of the wave function $\psi$, the wave function $\psi$ in (1.1) may not describe a single quantum
Thus, the supposition that the wave function describes a state of a single particle is incompatible with the quantum mechanics formalism.

According to the Copenhagen interpretation of quantum mechanics the wave function $\psi$ describes the state of a single quantum particle, whereas the state of a classical particle is described by its position $x$ and its momentum $p$. It is supposed that the wave function is a specific quantum quantity, which has no classical analog. In accordance with this approach one may not go to the limit $\hbar \to 0$ in the action (1.1), because the action vanishes, and the description degenerates.

The transformation (1.6) changes only the scale of the wave function phase $\ln(\psi/|\psi|)$, and this change may be very slight. The wave function $\Psi_b$ is the valid wave function, which can be used, in particular, for calculation of average values by means of the relation (2.1). This calculation may be produced for any value of the constant $b$. At different values of the parameter $b$ the wave function $\Psi_b$ may describe the same state of $S$, because the state of the dynamic system does not determine the wave function uniquely, and the same state of $S$ may be described by different wave functions. From viewpoint of the statistical description (2.4) the wave function is not uniquely defined, because it is constructed of hydrodynamic potentials, i.e. it is a result of integration of uniquely defined velocity $v$. The parameter $b$ in the transformation (2.1) is a constant of integration.

We may set $b = \hbar$ in the relations (2.19), (2.20) and obtain a description of "classical particle " in the form containing the quantum constant $\hbar$.

\[
S_{S\text{cl}} : \quad \mathcal{A}_{S\text{cl}}[\psi, \psi^*] = \int \left\{ \frac{i\hbar}{2} (\psi^* \partial_0 \psi - \partial_0 \psi^* \cdot \psi) - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi + \frac{\hbar^2}{2m} (\nabla |\psi|)^2 \right\} dt dx \tag{2.21}
\]

\[
i\hbar \partial_0 \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{\hbar^2}{8m} \left( \frac{(\nabla \rho)^2}{\rho^2} + 2 \nabla \nabla \rho \right) \psi, \quad \rho \equiv \psi^* \psi \tag{2.22}
\]

The same result may be obtained from (2.19), (2.20) by means of the transformation inverse to the transformation (1.6). Formally the action (2.21) distinguishes from the action (1.1) in the last term, which describes a lack of quantum effects. The quantum constant in two first terms has no relation to quantum effects. The dependence on $\hbar$ is conditioned by a special choice of the arbitrary constant $b$.

The action (2.21) describes the dynamic system $S_{S\text{cl}} = \mathcal{E} [S_{d}]$ in the "quantum language", i.e. in terms of the wave function. The action

\[
S_{S\text{cl}} = \mathcal{E} [S_{d}] : \quad \mathcal{A}_{\mathcal{E}[S_{d}]}[\mathbf{x}] = \int \frac{m}{2} \left( \frac{d\mathbf{x}}{dt} \right)^2 dt d\xi \tag{2.23}
\]

where $\mathbf{x} = \mathbf{x}(t, \xi)$, describes the same dynamic system in the "classical language", i.e. in terms of classical variables $x, p$. In the same way the action (1.1) describes the dynamic system $S_S = \mathcal{E} [S_{st}]$ in quantum language, whereas the action (2.4)
describes the same dynamic system in the classical language. It is reasonable that the quantum system $S$ is described simpler in the quantum language, whereas the classical system $S_{cl} = \mathcal{E} [S_d]$ is described simpler in the classical language. However, it is not a reason for the statement that the quantum system is to be described in the quantum language (in terms of the wave function).

Two different description of the classical system $S_d$ can be used for interpretation of the rule (2.1) and for interpretation of the correspondence principle. The obtained results may be applied to the quantum system $S$, because the difference between the dynamic systems $S = \mathcal{E} [S_d]$ and $S_{cl} = \mathcal{E} [S_d]$, described respectively by actions (1.1) and (2.21), manifests itself only in the additional nonlinear term in the dynamic equation. The possibility of description $S = \mathcal{E} [S_d]$ and $S_{cl} = \mathcal{E} [S_d]$ in both languages (classical and quantum) shuts the door before the Copenhagen interpretation, where the wave function is supposed to describe a single particle. Thus, there is neither reason nor excuse for application of the Copenhagen interpretation.

Mathematical formalism of quantum mechanics deals with continuous dynamic systems: Schrödinger particle $S$, Dirac particle $S_D$, etc. From the viewpoint of the mathematical technique it is of no importance, whether $S$ is an individual particle, or it is a statistical ensemble of particles (statistically average particle). The Copenhagen interpretation meets the difficulties, when it tries to test predictions of mathematical formalism in single experiments. For instance, there exists the problem of the mechanism of the wave function reduction in the single experiment. Another problem concerns the two-slit experiment. How can an individual particle pass through two slits at once? Such problems of quantum interpretations are not wiredrawn. The physical journals publish discussions concerning problems of quantum interpretation. For instance, such a discussion was declared in 2002 by the journal Uspekhi Fizicheskich Nauk. These problems cannot be solved in the framework of the Copenhagen interpretation, which does not distinguish between the individual particle $S$ and the statistically average particle $\langle S \rangle$. Confusion of two different objects, having different properties generates difficulties and paradoxes. The wave function does not describe the state of individual particle $S$, and it is meaningless to ask, how the state of individual particle changes at a single measurement (S-measurement). At the massive experiment (M-measurement) we obtain a distribution $F ( R' )$ of the measured quantity $R$, but not a single value $R'$ of the measured quantity. At such a situation it is useless to ask, how the obtained result $R'$ influences on the the state of the statistical ensemble. Finally, we may define the third type of measurement (SM-measurement): the massive measurement of the quantity $R$ leading to a definite value $R'$ of the measured quantity $R$. The SM-measurement is the M-measurement leading to a distribution $F ( R' )$, accompanied by a selection of those particles, where result of S-measurement is $R'$. Uniting all particles with the measured value $R'$ in one statistical ensemble $\mathcal{E}_{R'}$, we can put the question about the wave function of $\mathcal{E}_{R'}$. Of course, the wave function $\psi_{R'}$ of $\mathcal{E}_{R'}$ does not coincide, in general, with the initial wave function $\psi$, and this change of the wave function is considered as a reduction of the wave function. The origin of the reduction is quite transparent. It is the selection, which is produced to obtain the
same value $R'$ of the measured quantity for all particles of the statistical ensemble. Thus, the problems of reduction are conditioned by the confusion of concepts of the individual particle $S$ and the statistical average particle $\langle S \rangle$, which takes place at the Copenhagen interpretation.

As concerns the particle, passing through two slits simultaneously, it is a reasonable property of the statistical average object. It is a pure statistical property, which has nothing to do with quantum properties. Individual particle can pass either through one slit, or through another, whereas the statistical average particle can pass through both slits simultaneously. (Compare, individual person is either a man, or a woman, whereas the statistical average person is a hermaphrodite (half-man half-woman), and there are no quantum mechanical properties here.)

## 3 Dynamic disquantization

The quantum language, i.e. the description, containing the quantum constant $\hbar$, may be used for a description of a classical dynamic system, because the quantum constant $\hbar$ may be used instead of the arbitrary dynamical constant $b$. Replacement of dynamical constant $b$ by the quantum constant is produced to make the dynamic equations to be linear. For instance, in the action (1.7) the quantum constant $\hbar$ is used naturally, i.e. in the sense that setting $\hbar = 0$, we suppress the quantum effects. In the action (1.1) for the same dynamic system the quantum constant $\hbar$ is used artificially in the sense that setting $\hbar = 0$, we do not suppress the quantum effects. Furthermore, setting $\hbar = 0$, we destroy any description. But the action (1.1) generates linear dynamic equation, and this circumstance is a reason for the artificial identification $b = \hbar$, when the *dynamical* constant $b$ is identified with the quantum constant $\hbar$.

Such an artificial identification may be produced in other quantum systems (for instance, in $S_D$ and $S_P$), and we cannot be sure, that setting $\hbar = 0$, we suppress the quantum effects. Besides, we cannot be sure that, using the transformation of the type (1.6), we can separate the quantum terms from dynamical and statistical ones.

We need a more effective formal dynamical procedure, which could suppress the stochastic terms. Let us compare dynamic equations (2.12) for $S_S = \mathcal{E} [S_{st}]$ written in the form

$$\frac{d\mathbf{p}}{dt} = -\nabla U(\rho, \nabla \rho), \quad \frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{m}, \quad U(\rho, \nabla \rho) = \frac{\hbar^2}{8m} \left(\frac{(\nabla \rho)^2}{\rho^2} - 2 \frac{\nabla^2 \rho}{\rho}\right)$$

(3.1)

with the dynamic equations for $S_{Scl} = \mathcal{E} [S_{cl}]$, which have the form

$$\frac{d\mathbf{p}}{dt} = 0, \quad \frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{m}$$

(3.2)

where $\mathbf{x} = \mathbf{x}(t, \xi), \mathbf{p} = \mathbf{p}(t, \xi)$. Dynamic equations (3.1), are the partial differential equations, because $\rho$ is defined by the relation (2.10), containing derivatives with respect to $\xi_\alpha, \alpha = 1, 2, 3$, whereas dynamic equations (3.2) are ordinary differential
equations. Equations (3.2) contain derivatives only in one direction in the space of independent variables \( \{t, \xi\} \), whereas equations (3.1) contain derivatives in different directions of the space of independent variables \( \{t, \xi\} \). This property is conserved at any change of independent dynamical variables, and, in particular, at the change \( \{t, \xi\} \to \{t, x\} \). If a system of partial differential equations contains derivative only in one direction of the space of independent variables, this system can be reduced to the system of ordinary differential equations by means of a proper change of variables.

Usually the classical particle (for instance, the classical Schrödinger particle) has two properties: (1) it does not contain the quantum constant \( \hbar \), (2) it has the finite number of the freedom degrees. We may imagine such a dynamic system, which contains the quantum constant \( \hbar \) and has definite number of the freedom degrees. Is such a dynamic system classical? We believe that such a dynamic system should be classified as classical, because it is deterministic dynamical system, but not a stochastic one. As concerns to dependence on the quantum constant \( \hbar \), it may describe, in general, not only stochastic effects. The quantum constant \( \hbar \) is an attribute of the space-time \( [11] \), and in principle it may appear as a parameter in deterministic classical systems, but not only in the stochastic ones. Besides, the dynamic system \( S_I \) with finite number of the freedom degrees is simpler for investigation, than the continuous dynamic system \( S_c \), having infinite number of the freedom degrees, because dynamic equations of \( S_I \) are the ordinary differential equations, whereas dynamic equations of \( S_c \) are partial differential equations. Transition to the classical approximation as a means of investigation of the quantum dynamic system, which is continuous, is justified only if the classical system is the dynamic system \( S_I \).

If we want to suppress the quantum effects, we must to reduce the system of partial differential equations to the system of ordinary differential equations. To make this, we should project derivatives in the space of independent variables onto some direction. Then the system will contain derivatives only in one direction, and hence it may be reduced to the system of ordinary differential equations. Onto what direction should derivatives in the system (3.1) be projected, to obtain the system (3.2)?

Such a direction is described by the 4-current \( j^k = \{\rho, j\} = \{j^k\} \), \( k = 0, 1, 2, 3 \) in the space-time. The projection should be made in the space of independent variables \( \{t, x\} \), i.e. in the space-time. It is convenient to choose dependent variables in such a way, that the 4-current \( j^k \) were one of dependent variables. We take the action (A.22) for the dynamic system \( S_S = E [S_{st}] \)

\[
A_{E[S_{st}]}[\varphi, \xi, j] = \int \left\{ \frac{m}{2} \frac{j^\alpha j^\alpha}{\rho} - b j^k (\partial_k \varphi + g^\alpha (\xi) \partial_k \xi_\alpha) - \frac{\hbar^2 (\nabla \rho)^2}{8 m \rho} \right\} d^4 x, \quad (3.3)
\]

where according to (A.15) and (A.19)

\[
j^k = \{\rho, j\} = \left\{ \rho, \frac{b \rho}{m} (\nabla \varphi + g^\alpha (\xi) \nabla \xi_\alpha) \right\} \quad (3.4)
\]
and $g^\alpha (\xi)$, $\alpha = 1, 2, 3$ are arbitrary functions of argument $\xi$.

The second term in the action (3.3) contains derivatives only in the direction of the 4-vector $j^k$. In the last term of (3.3) the derivatives are to be projected onto the vector $j^k$. We are to make the change

$$\partial_l \to \partial_{||l} = \frac{j^k j^l}{j^s j^s} \partial_k,$$  

$l = 0, 1, 2, 3$ \hspace{1cm} (3.5)

in the action (3.3). We obtain

$$\frac{(\nabla \rho)^2}{\rho} = \frac{(\partial_\alpha \rho)(\partial_\alpha \rho)}{\rho} \to \frac{j_\alpha j_\alpha (j^i \partial_i \rho)^2}{\rho (j^s j^s)^2} \hspace{1cm} (3.6)$$

$$j_\alpha j_\alpha = j^2 = \rho^2 v^2, \quad j^s j^s = c^2 \rho^2 - \rho^2 v^2$$

In the nonrelativistic approximation, when the velocity $|v| \ll c$, we obtain the following estimation

$$\frac{(\nabla \rho)^2}{\rho} \approx \frac{v^2 (j^i \partial_i \rho)^2}{c^4 \rho^3} \hspace{1cm} (3.7)$$

In the nonrelativistic approximation $c \to \infty$ the last term in the action (3.3) is to be neglected after the change (3.5). Thus, in the case of the Schrödinger particle $S_S$ the change (3.5) leads to a suppression of quantum effects.

We shall refer to the procedure (3.5) as the dynamical disquantization, because it transforms the Schrödinger particle $S_S = E [S_{st}]$ into the classical system $S_{Scl} = E [S_d]$. The dynamical disquantization is the relativistic dynamical procedure. It does not refer to the quantum constant and suppresses any stochasticity regardless of its origin. From here on we shall use the dynamical disquantization for the suppression of stochasticity in quantum systems.

Strictly, the dynamical disquantization is to be applied to the dynamic equations. But in many cases the application of the dynamical disquantization to the action leads to the same result, as its application to the dynamic equations.

## 4 Classical approximation of the Dirac particle

For transition to the classical approximation it is sufficient to make dynamical disquantization of the Dirac equation (1.13). Using expression (1.14) for 4-current $j^k$, we make this directly in (1.13). We obtain

$$-i\hbar \frac{\psi^* \gamma^0 \gamma_i \psi}{(\psi^* \gamma^0 \gamma^s \psi)(\psi^* \gamma^0 \gamma_s \psi)} \left(\psi^* \gamma^0 \gamma^k \psi\right) \left(\psi^* \gamma^i \gamma^k \psi\right) \gamma^l \partial_l \psi + \frac{e}{c} A_i \gamma^l \psi + mc \psi = 0 \hspace{1cm} (4.1)$$

or after transformation

$$-i\hbar \frac{d}{d\tau} \psi + \frac{e}{c} A_i \frac{j_k}{\sqrt{j^s j^s}} \gamma^k \gamma^l \psi + mc \frac{j_k}{\sqrt{j^s j^s}} \gamma^k \psi = 0, \quad \frac{d}{d\tau} \equiv \frac{j^k}{\sqrt{j^s j^s}} \partial_k \hspace{1cm} (4.2)$$
where \( j^k \) is determined by the relation (1.14) via the wave function \( \psi \).

Formally equations (4.2) form a system of ordinary differential equations for the dependent variables \( \psi \), considered as function of independent variable \( \tau \). Unfortunately, interpretation of equations (4.2) is difficult. It is more convenient to use the hydrodinamical variables, where the the components of the 4-current \( j^k \) are dependent variables. The transformation to the hydrodynamical variables may be easier carried out in the action (1.11).

Transforming the action (1.11), we use the mathematical technique [12, 13], where the wave function \( \psi \) is considered to be a function of hypercomplex numbers \( \gamma \) and coordinates \( x \). In this case the physical quantities are obtained by means of a convolution of expressions \( \psi^* O \psi \) with the zero divisor. This technique allows one to work without fixing the \( \gamma \)-matrices representation.

Using designations

\[
\gamma_5 = c\gamma^{0123} \equiv c\gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad (4.3)
\]

\[
\sigma = \{\sigma_1, \sigma_2, \sigma_3,\} = \{-i\gamma^2 \gamma^3, -i\gamma^3 \gamma^1, -i\gamma^1 \gamma^2\} \quad (4.4)
\]

we make the change of variables

\[
\psi = Ae^{i\varphi + \frac{i}{2} \gamma_5 \kappa} \exp \left( -\frac{i}{2} \gamma_5 \sigma \eta \right) \exp \left( \frac{i\pi}{2} \sigma n \right) \Pi \quad (4.5)
\]

\[
\psi^* = A\Pi \exp \left( -\frac{i\pi}{2} \sigma n \right) \exp \left( -\frac{i}{2} \gamma_5 \sigma \eta \right) e^{-i\varphi - \frac{i}{2} \gamma_5 \kappa} \quad (4.6)
\]

where (*) means the Hermitian conjugation, and the quantity

\[
\Pi = \frac{1}{4} (1 + c\gamma^0)(1 + z\sigma), \quad z = \{z^\alpha\} = \text{const}, \quad i\alpha = 1, 2, 3; \quad z^2 = 1 \quad (4.7)
\]

is the zero divisor (projector). The quantities \( A, \kappa, \varphi, \eta = \{\eta^\alpha\}, \ n = \{n^\alpha\}, \ \alpha = 1, 2, 3, \ n^2 = 1 \) are eight real parameters, determining the wave function \( \psi \). These parameters may be considered as new dependent variables, describing the state of dynamic system \( S_D \). The quantity \( \varphi \) is a scalar, and \( \kappa \) is a pseudoscalar. Six remaining variables \( A, \eta = \{\eta^\alpha\}, \ n = \{n^\alpha\}, \ \alpha = 1, 2, 3, \ n^2 = 1 \) can be expressed through the flux 4-vector \( j^l = \bar{\psi} \gamma^l \psi \) and spin 4-pseudovector

\[
S^l = i\bar{\psi} \gamma_5 \gamma^l \psi, \quad l = 0, 1, 2, 3 \quad (4.8)
\]

Because of two identities

\[
S^l S_l \equiv -j^l j_l, \quad j^l S_l \equiv 0. \quad (4.9)
\]

there are only six independent components among eight components of quantities \( j^l \) and \( S^l \).

Mathematical details of the dependent variables transformation can be found in [14], where the action is calculated for the case \( c = 1 \) and vanishing electromagnetic
field $A_l = 0$. As a result we have the following form of the action, written in the hydrodynamical form

$$S_D : \quad A_D[j, \varphi, \kappa, \xi] = \int \mathcal{L} d^4 x, \quad \mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}_{q1} + \mathcal{L}_{q2} \quad (4.10)$$

$$\mathcal{L}_{cl} = -mc \rho - \hbar j^i \partial_i \varphi - \frac{e}{c} A_l j^l - \frac{\hbar j^l}{2(1 + \xi z)} \varepsilon_{\alpha \beta \gamma} \xi^\alpha \partial_i \xi^\beta z^\gamma, \quad \rho \equiv \sqrt{j^i j_i} \quad (4.11)$$

$$\mathcal{L}_{q1} = 2mc \rho \sin^2 \left(\frac{\kappa}{2}\right) - \frac{\hbar}{2} S^l \partial_l \kappa, \quad (4.12)$$

$$\mathcal{L}_{q2} = \frac{\hbar (\rho + cj^0)}{2} \varepsilon_{\alpha \beta \gamma} \partial^\alpha \left(\frac{j^\beta}{\rho + cj^0}\right) \xi^\gamma - \frac{\hbar}{2(\rho + cj^0)} \varepsilon_{\alpha \beta \gamma} \left(\partial^0 j^\beta\right) j^\alpha \xi^\gamma \quad (4.13)$$

where $\varepsilon_{\alpha \beta \gamma}$ is the Levi-Chivita 3-pseudotensor. The Lagrangian density $\mathcal{L}$ is a function of 4-vector $j^l$, scalar $\varphi$, pseudoscalar $\kappa$, and the unit 3-pseudovector $\xi$, which is connected with the spin 4-pseudovector $S^l$ by means of the relations

$$\xi^\alpha = \rho^{-1} \left[S^\alpha - \frac{j^\alpha S^0}{(\rho + cj^0)}\right], \quad \alpha = 1, 2, 3; \quad \rho \equiv \sqrt{j^i j_i} \quad (4.14)$$

$$S^0 = j \xi, \quad S^\alpha = \rho \xi^\alpha + \frac{j \xi^\alpha}{\rho + cj^0}, \quad \alpha = 1, 2, 3 \quad (4.15)$$

Let us set for simplicity $A_l = 0$ and $c = 1$. Producing the dynamical disquantization (3.5) in (4.10) - (4.13), we obtain

$$A_{Dqu}[j, \varphi, \xi] = \int \left\{ -\kappa_0 m \rho - \hbar j^i \left(\partial_i \varphi + \frac{\varepsilon_{\alpha \beta \gamma} \xi^\alpha \partial_i \xi^\beta z^\gamma}{2(1 + \xi z)}\right) \right. \right.$$

$$\left. + \frac{\hbar j^k}{2(\rho + j_0) \rho} \varepsilon_{\alpha \beta \gamma} \left(\partial_k j^\beta\right) j^\alpha \xi^\gamma \right\} d^4 x \quad (4.16)$$

where $\kappa_0 = \pm 1$ is the solution of the dynamic equation $\delta A_{Dqu}/\delta \kappa = 0$, which does not contain derivatives, because the last term of (4.12) vanishes after dynamical disquantization (3.5) in virtue of the second identity (4.9).

We introduce the Lagrangian coordinates $\tau = \{\tau_0, \tau\} = \{\tau_i(x)\}$, $i = 0, 1, 2, 3$ as functions of coordinates $x$ in such a way that only coordinate $\tau_0$ changes along the direction $j^i$, i.e.

$$j^k \partial_k \tau_\mu = 0, \quad \mu = 1, 2, 3 \quad (4.17)$$

Considering the variables $\tau = \{\tau_0, \tau\}$ as independent variables in (4.16), we obtain after calculations (See mathematical details in [14])

$$A_{Dqu}[x, \xi] = \int \left\{ -\kappa_0 m \sqrt{j^i j_i} + \hbar \frac{(\xi \times \xi) z}{2(1 + \xi z)} + \hbar \frac{(\xi \times \dot{\xi}) \xi}{2 \sqrt{j^i j_i} \sqrt{j^i j_i}} \right\} d^4 \tau \quad (4.18)$$
where period means the total derivative $\dot{x}^s \equiv dx^s/d\tau_0$. The quantities $x = \{x^0, x_i\} = \{x^i\}$, $i = 0, 1, 2, 3$, and $\xi = \{\xi_\alpha\}, \alpha = 1, 2, 3$ are considered to be functions of the Lagrangian coordinates $\tau_0$, $\tau = \{\tau_1, \tau_2, \tau_3\}$. Here and in what follows the symbol $\times$ means the vector product of two 3-vectors. The quantity $z$ is the constant unit 3-vector (4.7). In fact, variables $x$ depend on $\tau$ as on parameters, because the action (4.18) does not contain derivatives with respect to $\tau_\alpha, \alpha = 1, 2, 3$. Lagrangian density of the action (4.18) does not contain independent variables $\tau$ explicitly. Hence, it may be written in the form

$$A_{\text{Dqu}}[x, \xi] = \int A_{\text{Dcl}}[x, \xi]d\tau, \quad d\tau = d\tau_1 d\tau_2 d\tau_3 \quad (4.19)$$

where

$$A_{\text{Dcl}}[x, \xi] = \int \left\{ -\kappa_0 mc\sqrt{\dot{x}^i \dot{x}_i} + \hbar \frac{(\dot{x} \times \dot{x})z}{2(1 + \xi z)} + \hbar \frac{(\dot{x} \times \ddot{x})\xi}{2\sqrt{\dot{x}^s \dot{x}_s (\sqrt{\dot{x}^0 \dot{x}_0} + \dot{x}^0)}} \right\} d\tau_0 \quad (4.20)$$

Here period means differentiation with respect to $\tau_0$. It is easy to see that the action (4.20) is invariant with respect to transformation $\tau_0 \rightarrow \tilde{\tau}_0 = F(\tau_0)$, where $F$ is an arbitrary monotone function.

The action (4.20) can be written in the relativistically covariant form

$$A_{\text{Dcl}}[x, \xi] = \int \left\{ -\kappa_0 m \sqrt{\dot{x}^i \dot{x}_i} - \frac{e}{c} A_i \dot{x}^i \right\} d\tau_0 + \hbar \frac{\varepsilon_{iklm} \xi^k f^l z^m}{2(1 - \xi^s z_s)} + \hbar \frac{Q \varepsilon_{iklm} \dot{x}^i \dot{x}^k f^l \dot{x}^m}{2} \quad (4.21)$$

$$Q = Q(\dot{x}, f) = \frac{1}{\sqrt{\dot{x}^s \dot{x}_s (\dot{x}^l f_l + \sqrt{\dot{x}^i \dot{x}_i})}}, \quad \dot{x}^l = \frac{dx^l}{d\tau_0} \quad (4.22)$$

where 4-vectors $f^k, z^k$ are defined respectively by relations

$$f^i = \{1, 0, 0, 0\} \quad (4.23)$$

$$z^k = \{0, z\} = \{0, z^1, z^2, z^3\}, \quad z^k z_k = -1, \quad \xi z = -\xi_l z^l \quad (4.24)$$

$$\xi^k = \{\xi_0, \xi\}, \quad \xi^i f_i = 0 \quad (4.25)$$

The quantities $f^k$ and $z^k$ are constant 4-vectors. They have the form (4.23) and (4.24) only in some coordinate system. In other coordinate systems their form is obtained as a result of corresponding transformation of relations (4.23) and (4.24). These 4-vectors describe some structures, existing in the space-time. The vector $z^k$ describes introduction of the projector (4.7). The 4-vector $z^k$ appears to be fictitious (see [14]).

However, the 4-vector $f^k$ is not fictitious. It describes existence of some special direction in the space-time. This direction describes separation of the space-time
into absolute time and absolute space. Having the vector $f^k$, one can assign absolute
time $t = x^k f_k$ and absolute spatial distance $r = \sqrt{(x^k f_k)^2 - x^k x_k}$ to any space-time
vector $x^k$. It means that the terms, containing the constant 4-vector $f^k$ are non-relativistic. This statement is in accordance with the theorem, proved by Anderson [15]. According to this theorem the symmetry group of dynamic equations, written
in the relativistically covariant form is determined by the symmetry group of absolute objects. The absolute objects are the quantities, which are the same for all solutions of the dynamic equations. The absolute objects are structures or force
fields external with respect to considered dynamic system. In the given case the
4-vector $f^k$ is such an absolute object. If we set (4.23) in (4.21), we return to the
action (4.18), which does not contain the absolute object $f^k$, but simultaneously the
action (4.18) appears to be written in the noncovariant form. In other words, the
relativistically covariant form of description is a method of separation of absolute
objects (or space-time structures) used at the description of the dynamic system.

Two first terms in the action (4.21) do not contain the 4-vector $f^k$. They describe
relativistically the motion of the classical Dirac particle as a whole. Two last terms
contain the 4-vector $f^k$ and the 4-pseudovector $\xi^k$. They describe internal degrees
of freedom of the classical Dirac particle. The description of the internal degrees of
freedom is nonrelativistic.

Dynamical methods of investigation and the formalized procedure of transition to
the classical approximation (3.5) admit one to discover two unexpected properties
of the Dirac particle: (1) the Dirac particle has an internal degrees of freedom,
(2) these degrees of freedom are described nonrelativistically. The last property
means that formally the Dirac equation is nonrelativistic equation, i.e. the set of all
solutions of the Dirac equation is not invariant with respect to the Lorentz group of
transformations.

In the case, when the electromagnetic field is absent $A_k = 0$, dynamic equations,
generated by the action (4.20) have the form (see details in [14])

\[
\begin{align*}
\frac{d}{d\tau_0} \left( -\kappa_0 m \frac{\dot{x}}{\sqrt{\dot{x}^s \dot{x}_s}} + \frac{\hbar Q}{2} (\xi \times \ddot{x}) - \frac{\hbar}{2} \frac{\partial Q}{\partial \dot{x}} (\dddot{x} \times \dot{x}) \xi + \frac{\hbar}{2} \frac{d}{d\tau_0} (Q(\xi \times \dddot{x})) \right) &= 0 \\
\frac{d}{d\tau_0} \left( \kappa_0 m \frac{\dot{x}^0}{\sqrt{\dot{x}^s \dot{x}_s}} - \frac{\hbar}{2} \frac{\partial Q}{\partial \dot{x}^0} (\dddot{x} \times \dot{x}) \xi \right) &= 0 \\
\dot{\xi} &= -(\dddot{x} \times \dot{x}) \times \xi Q,
\end{align*}
\]

(4.26) (4.27) (4.28)

where the quantity $Q$ is defined by the relation (4.22), (4.23), i.e.

\[
Q = Q(\dot{x}) = \left( \frac{1}{\sqrt{\dot{x}^s \dot{x}_s (\sqrt{\dot{x}^s \dot{x}_s} + \dot{x}^0)}} \right)^{-1}, \quad \dot{x}^s \dot{x}_s = \dot{x}^0 - \dot{x}^2
\]

(4.29)

World line of the classical Dirac particle, described by the dynamic equations (4.26),
\[ dx/dt = \left\{ \frac{\sqrt{\gamma^2 - 1}}{\gamma} \cos (\Omega t), \frac{\sqrt{\gamma^2 - 1}}{\gamma} \sin (\Omega t), 0 \right\}, \quad \Omega = \frac{2m}{\hbar \gamma^2} \]  

\[ x = \left\{ \frac{\hbar \gamma \sqrt{\gamma^2 - 1}}{2m} \sin \left( \frac{2m}{\hbar \gamma^2} t \right), \frac{\hbar \gamma \sqrt{\gamma^2 - 1}}{2m} \cos \left( \frac{2m}{\hbar \gamma^2} t \right), 0 \right\} \]  

where \( \gamma \geq 1 \) is an arbitrary constant (Lorentz factor), describing intensity of the circular motion of the classical Dirac particle.

Details of interpretation of solution (4.31) can be found in [14]. We shall not go into these details here, because we are interested mainly in interplay between the dynamical methods and conventional axiomatic methods of the quantum system investigation. In particular, we are interested in such questions. Is it possible to discover internal degrees of freedom of the Dirac particle by conventional methods? Can one discover the nonrelativistical character of the Dirac equation by conventional methods? If yes, then why have not these properties of the Dirac particle been discovered for eighty years? In other words, we want to compare effectiveness of dynamical methods and axiomatic ones. There was a reason for the statement of such questions.

## 5 On mistakes in application of the conventional investigation method to the Dirac particle

At first, the unexpected properties of the Dirac particle were presented in the paper [16]. Unfortunately, I failed to publish this paper in a physical journal, because of negative review of referees. Statements of the paper on internal degrees of freedom and especially on a nonrelativistical character of the Dirac equation met objections of the referees. These objections were of such a kind: (1) 'It is well known that the Dirac equation is relativistic', (2) 'Maybe, the author's calculations contain a mistake, because the Dirac equation agrees very well with the experiments', (3) 'I cannot imagine, that the Dirac equation is nonrelativistic, because it can be written in the relativistically covariant form'. Discussion and evaluation of the merits of the paper was absent in all reviews. The viewpoint of referees reflects the statistical average opinion of the scientific community, and it should be taken into account.

The paper was rejected despite my comments to the reviews of the referees. Then I decided that two new unexpected results were too many for one paper and divided the paper into two parts. In the paper [14] the internal degrees of freedom were discussed, whereas the paper [17] was devoted to discussion of the nonrelativistic character of the Dirac equation. Again the referees have not found mistakes or defects in my papers, but they recommended against publication of the papers.
If the referees do not want or cannot find mistakes in my papers, I have to search for mistakes in the conventional axiomatic method, which lead to the absence of internal degrees of freedom of the Dirac particle and to the relativistic character of the Dirac equation. The corresponding mistakes have been easily found. Discovery and discussion of these mistakes is presented below.

First about internal degrees of freedom of the Dirac particle. Conventionally the Pauli equation is considered to be a nonrelativistic approximation of the Dirac equation. The Pauli equation is a system of two first order complex differential equations for two complex dependent variables, whereas the Dirac equation is a system of four first order complex differential equations for four complex dependent variables. It means that in the classical approximation the classical Pauli particle $S_{Pcl}$ has less degrees of freedom, than the classical Dirac particle $S_{Dcl}$. The Pauli particle $S_{Pcl}$ is the nonrelativistic pointlike particle with spin. Is the classical Dirac particle $S_{Dcl}$ a relativistic particle with spin and additional degrees of freedom, or is the order of the Dirac system reduced indeed at the transition to the nonrelativistic approximation?

Why is the order of the system of dynamic equations reduced at the nonrelativistic approximation? It is reduced because coefficients before some time derivatives are of the order $c^{-2}$, which vanish at $c \to \infty$. In other words, the Dirac system of dynamic equations is a system of differential equations with small parameter before the highest derivative. One may not neglect these terms, because at high temporal frequency they may be of the same order as other terms, even if coefficients before the derivatives are small. Neglecting the terms with highest derivatives, we lose the high frequency solutions of dynamic equations. From formal viewpoint such a neglect of the terms with highest derivatives is a mathematical mistake, leading to a loss of the high frequency solutions.

The scrupulous analysis shows [18] that solution of the Dirac equation in the nonrelativistic approximation has the form

$$
\psi = \frac{\exp \left( -\frac{i}{2} \Omega t \right)}{\sqrt{2}} \begin{pmatrix} \psi_1 + \psi_2 \\ \psi_1 - \psi_2 \end{pmatrix}, \quad \Omega = \frac{2mc^2}{\hbar} \tag{5.1}
$$

where $\psi$ is the dependent variable of the action (1.11), $\psi_1$ and $\psi_2$ are two-component complex quantities, defined by the relations

$$
\psi_1 = u_+ (t, \mathbf{x}) + e^{i\Omega t} u_- (t, \mathbf{x}) \tag{5.2}
$$

$$
\psi_2 = -\frac{1}{2mc} \left( i\hbar \partial_\mu - \frac{e}{c} A_\mu \right) \sigma_\mu \left( u_+ (t, \mathbf{x}) + e^{i\Omega t} u_- (t, \mathbf{x}) \right) \tag{5.3}
$$

Here $\sigma_\mu, \mu = 1, 2, 3$ are the Pauli matrices, and $u_+ (t, \mathbf{x}), u_- (t, \mathbf{x})$ are two-component quantities, which are solutions of the Pauli equations with the different sign of the mass $m$.

$$
i\hbar \partial_0 u_+ = \hat{H}_P (m) u_+, \quad i\hbar \partial_0 u_- = \hat{H}_P (-m) u_- \tag{5.4}
$$
\[ H_P (m) = \frac{\pi_\mu \pi_\mu}{2m} + \frac{i e \hbar}{2m c} \varepsilon_{\mu \alpha \nu} \partial_\nu A_\mu \sigma_\alpha = \frac{\pi^2}{2m} + \frac{i e \hbar}{2mc} H \sigma \]  

(5.5)

\[ \pi = \{ \pi_1, \pi_2, \pi_3 \}, \quad \pi_\mu \equiv i \hbar \partial_\mu + \frac{e}{c} A_\mu, \quad \mu = 1, 2, 3 \]

Here \( H = \nabla \times A \) is the magnetic field.

The frequency \( \Omega = mc^2 / \hbar \) tends to \( \infty \), if \( c \to \infty \). In the nonrelativistic case, when \( \gamma \) is close to 1, the frequency \( \Omega \) coincides with the frequency \( \Omega \), which figures in relations (4.30), describing world line of the classical Dirac particle \( S_{Dcl} \). If we ignore high frequency part of solution, setting \( u_+ = 0 \) in relations (5.1) - (5.3), we obtain a solution of the Pauli equation. As far as the Dirac particle is a charged particle, the states, where \( u_+ \neq 0 \land u_- \neq 0 \), are unstable, because at such a state the 4-current \( j^k \) has components, oscillating with the frequency \( \Omega \). The Dirac particle emanates electromagnetic waves of the frequency \( \Omega \), until one of quantities \( u_+ \) or \( u_- \) becomes to be equal to zero [18]. As far as the time of transition to the stable state is very short, the Dirac particle exists practically either at the low frequency state \( \psi_{lf} \), when \( u_- = 0 \), or at the high frequency state \( \psi_{hf} \), when \( u_+ = 0 \). The state \( \psi_{lf} \) is associated with the particle, whereas the state \( \psi_{hf} \) is associated with the antiparticle. Formally the superposition of the states \( \psi_{lf} \) and \( \psi_{hf} \) is possible, but it is not realized for the charged Dirac particle because of its instability. This conclusion on the instability is valid also for the classical Dirac particle \( S_{Dcl} \), described by the relations (4.30), (4.31).

Thus, the Dirac particle \( S_D \), described by the action (1.11), has internal degrees of freedom, which has not been discovered because of a mathematical mistake in the transition to the nonrelativistic approximation, which carries out an interpretation of the Dirac particle. The internal degrees of freedom are not displayed in the nonrelativistic applications of the Dirac equation, and there was no necessity of looking for them. The transition to classical approximation must discover all degrees of freedom automatically, but only in the case, when the procedure of transition to the classical approximation is formalized. Conventionally the transition to the classical approximation was not formalized, and there are reason for this. It is accompanied by a series of additional suppositions, which always can be chosen in such a way to obtain the desirable result known in advance. First the procedure of transition to the classical approximation was formalized in [16], and the internal degrees of freedom of \( S_{Dcl} \) were discovered immediately.

The Dirac equation can be written in the relativistically covariant form. It is common practice to think, that it means that the Dirac equation a relativistic equation. In other words, it and has the Lorentz symmetry, i.e. the set of all its solutions is transformed to the same set of solutions at any Lorentz transformation. This opinion has been existing for many years, and we try to understand the reason of this viewpoint.

Formally the relativistic character of dynamic equations associates with the representation of these equations in the relativistically covariant form. However, this association is valid only at some additional conditions, which are fulfilled practically
always, and as a result these conditions are not mentioned usually in the conventional formulation of the relativistic invariance (i.e. compatibility of dynamic equations with the principles of relativity). Unfortunately, in the case of the Dirac equation these additional conditions are not fulfilled, and the Dirac equation appears to be formally nonrelativistic. In reality, only internal degrees of freedom are nonrelativistic. If these internal degrees of freedom are ignored, the Dirac particle appears to be relativistic.

The additional constraint in the formulation of the relativistical invariance changes the formulation. The correct formulation looks as follows. Symmetry of dynamic equations, written in the relativistically covariant form coincides with the symmetry of their absolute objects. The absolute objects are such quantities, which are the same for all solutions. Formally absolute objects are some structures or objects, which are external with respect to considered dynamic system. Usually such an absolute object is the metric tensor, which has the form \( g_{ik} = \text{diag}\{c^2, -1, -1, -1\} \).

The group of symmetry of \( g_{ik} \) is the Lorentz group, and the symmetry group of dynamic equations appears to be the Lorentz group. The Maxwell equations, the Klein-Gordon equation and many other dynamic equations for real dynamic systems contain only the metric tensor as an absolute object, and the formulation of relativistical invariance is simplified. It looks as follows. The Lorentz group is the symmetry group of dynamic equations, written in the relativistically covariant form. In such a form it used by most researchers.

The Dirac equation does not contain the metric tensor. Instead it contains the \( \gamma \)-matrices \( \gamma^i \), \( i = 0, 1, 2, 3 \). The \( \gamma \)-matrices form a matrix 4-vector, whose symmetry group is lower, than the Lorentz group. As a result the Dirac equation appears to have not a symmetry of the Lorentz group. In other words, the Dirac equation appears to be nonrelativistic equation.

What physical situation is behind this result? Why does the dynamic equation, written in the relativistically covariant form, become to be nonrelativistic, if it contains an absolute vector? To answer this question, we consider an example of a charged classical particle, moving in the given electromagnetic field \( F_{ik} \).

Dynamic equations for the relativistic particle may be written in the noncovariant form

\[
\frac{d}{dt} \frac{m\dot{x}^\mu}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}} = \frac{e}{c} F^{\mu 0} + \frac{e}{c} F^{\mu \nu} g_{\nu \beta} \dot{x}^\beta, \quad \mu = 1, 2, 3, \quad \dot{x} \equiv \frac{dx}{dt} \quad (5.6)
\]

and in the relativistically covariant form

\[
m \frac{d^2 x^k}{d\tau^2} = \frac{e}{c} F^{kl} g_{ls} \frac{dx^s}{d\tau}, \quad k = 0, 1, 2, 3 \quad (5.7)
\]

where \( \tau \) is the proper time, \( e, m \) are respectively the particle charge and the particle mass.

If the particle is nonrelativistic the dynamic equations are written in the noncovariant form

\[
m \frac{d^2 x^\mu}{dt^2} = \frac{e}{c} F^{\mu 0} + \frac{e}{c} F^{\mu \nu} g_{\nu \beta} \frac{dx^\beta}{dt}, \quad \mu = 1, 2, 3 \quad (5.8)
\]
Can the dynamic equations (5.8) for the nonrelativistic particle be written in the relativistically covariant form? The answer is yes, although most researchers believe that it is impossible. In the relativistically covariant form the dynamic equations (5.8) have the form

\[
m \frac{d}{d\tau} \left[ (l_k \dot{x}^k)^{-1} \dot{x}^i - \frac{1}{2} g^{ik} l_k (l_j \dot{x}^j)^{-1} \dot{x}^i g_{sl} \dot{x}^l \right] = \frac{e}{c} F^{il} g_{lk} \dot{x}^k, \quad i = 0, 1, 2, 3 \quad (5.9)
\]

where \( \dot{x}^k \equiv dx^k/d\tau \). The quantity \( l_k, k = 0, 1, 2, 3 \) is a constant timelike unit 4-vector

\[
g^{ik} l_i l_k = 1; \quad (5.10)
\]

Using the special choice of \( l_k = \{c, 0, 0, 0\} \) and substituting it in (5.2), it is easy to verify, that we obtain the dynamic equations (5.8) for \( i = 1, 2, 3 \). For \( i = 0 \) we obtain dynamic equation, which is a corollary of (5.8).

As far as dynamic equations for both relativistic and nonrelativistic particles can be written in the noncovariant form and in the relativistically covariant one, it is clear that the difference between the relativistic and nonrelativistic dynamic equations is not connected with form of dynamic equations. There is anything else, which distinguishes the relativistic conception from the nonrelativistic one.

It is well known that the difference lies in different space-time conceptions. In the Newtonian conception there is an absolute simultaneity and there are two invariant quantities: absolute time \( t \) and absolute space distance \( r \), whereas in the relativistic space-time conception there exists only one absolute quantity: the space-time interval \( s = \sqrt{c^2 t^2 - r^2} \). The Newtonian space-time \( S_N \) has seven-parametric continuous group of motion, whereas the Minkowski space-time \( S_M \) has ten-parametric continuous group of motion. Besides, the Newtonian space-time \( S_N \) may be considered to be the Minkowski space-time \( S_M \) with additional geometric structure \( \mathcal{L} \), given in it. In other words, \( S_N = S_M \wedge \mathcal{L} \). The additional structure \( \mathcal{L} \) is a specific timelike direction in \( S_M \), described by the constant timelike vector \( l_k \). Any hyperplane, orthogonal to \( l_k \), is a set of absolutely simultaneous events. Introduction of \( \mathcal{L} \) admits one to construct two invariants in \( S_M \wedge \mathcal{L} \)

\[
t = l_k x^k, \quad r = \sqrt{x^k x_k + (l_k x^k)^2} \quad (5.11)
\]

for any vector \( x^k \), whereas in \( S_M \) we have only one invariant \( s = \sqrt{x^k x_k} \). Construction of this invariant does not contain a reference to \( \mathcal{L} \).

The Newtonian space-time \( S_N \) considered as \( S_M \wedge \mathcal{L} \) admits only such motions of \( S_M \), which transform vector \( l_k \) into the same vector \( l_k \) and do not violate the structure \( \mathcal{L} \). The condition of the structure \( \mathcal{L} \) conservation at the space-time motion reduces the ten-parametric group of motion of \( S_M \) to seven-parametric group of motion of \( S_M \wedge \mathcal{L} \). In general, at the relativistically covariant description the absolute objects, introduced by Anderson [15], may be considered as the quantities, describing additional structures in \( S_M \). It means, that any system of dynamic equations may be written in the relativistically covariant form, provided the proper absolute
objects (additional structures) are introduced. Thus, to determine, whether the dynamic equations are compatible with the principles of relativity, we may write them in the relativistically covariant form and determine whether or not they contain absolute objects and what are properties of these absolute objects. If the dynamic equations contain the constant timelike vector $l_k$, we have nonrelativistic dynamic system, because $l_k$ describes the additional space-time structure, characteristic for the Newtonian space-time $S_N$ represented as $S_M \wedge L$.

Such an approach is convenient in the sense, that it does not contain a reference to the coordinate system, which is simply a method of description. Relativistic character of dynamic equation is connected directly with absence of additional space-time structures in $S_M$, but not with the relativistically covariant form of the dynamic equations, because any dynamic equations can be always written in the relativistically covariant form, provided the proper geometrical structure is introduced in $S_M$. The relativistically covariant representation of dynamic equations is necessary only for a discovery of additional geometrical structures in the Minkowski space-time $S_M$. Additional geometric structures are primary, whereas the relativistic covariance is secondary, because it admits one only to discover these structures if they takes place. The relativistic covariance in itself is indifferent with respect to the relativistic invariance of the dynamic equations. The additional structure is formally present in the equations (5.9) and described by the formal parameters $l_k$. When we use substitution $l_k = \{c, 0, 0, 0\}$ in (5.9), we obtain (5.8), where the structure $L$ is formally absent, because the formal parameters $l_k$ of $L$ are absent.

Thus, the relativistic invariance of the dynamic equations is connected with existence and properties of additional structures in $S_M$, whereas the relativistic covariance is only a method of discovery of these structures.

The relativistically covariant dynamic equation is relativistic, provided it does not contain a reference to some additional structure. However, such a formulation is unreliable, because the reservation of a reference to additional structure may be omitted by mistake. In this case the relativistic character of dynamic equations appears to be connected with the relativistic covariance of these equations, but not with the additional structure $L$ in $S_M$. It is this case that takes place in reality. As a result we have an associative mistake, when the relativistic invariance is associated with the relativistic covariance, although in reality the relativistic invariance is associated with an absence of additional geometrical structures in $S_M$. Relativistic covariant form of dynamic equations is only a condition, when existence of additional structure in $S_M$ can be discovered.

Thus, the internal degrees of freedom of the Dirac particle has not been discovered theoretically, because they cannot be obtained experimentally. All precise experiments with such a Dirac particle as electron are nonrelativistic (correction to the spectrum of hydrogen) and internal rigid degrees of freedom give only a negligible correction. As far as the internal degrees of freedom remained unknown, one cannot obtain experimentally nonrelativistic character of their description. Unfortunately, the quantum principles do not work in the relativistic region, and the conventional quantum theory cannot obtain those results, which has not been obtained experi-
mentally, because it needs additional hypotheses, having experimental basis. The dynamical methods are free of this defect. They work independently of experimental data and additional hypotheses.

Mathematical Appendix

A Transformation of the action for the statistical ensemble

Let us transform the action

\[ \mathcal{E}[\mathcal{S}_{st}] : \mathcal{A}[\mathcal{S}_{st}] [x, u_{st}] = \int \left\{ \frac{m \dot{x}^2}{2} - \frac{e}{c} A_0 - \frac{e}{c} \mathbf{A} \frac{dx}{dt} + \frac{m u_{st}^2}{2} - \frac{\hbar}{2} \nabla u_{st} \right\} dt d\xi \]  

(A.1)

for the statistical ensemble of stochastic particles, moving in the given electromagnetic field \( A = \{A_0, \mathbf{A}\} = \{A_0, A_1, A_2, A_3\} \). Here \( x = x(t, \xi), u_{st} = u_{st}(t, x) \) are dependent dynamic variables, and \( \nabla = \{\partial_1, \partial_2, \partial_3\} = \{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\} \). The variable \( x \) describes the regular component of the stochastic particle motion. The dynamic variable \( u_{st} \) is a function of \( t, x \) and depends on \( \xi \) via \( x \). The quantity \( u_{st} \) may be regarded as the mean velocity of the stochastic component, whereas \( x = x(t, \xi) \) describes the regular component of the particle motion. The last term in (A.1) describes influence of the stochasticity on the evolution of the regular component.

To eliminate variable \( u_{st} \), we should to solve dynamic equations \( \delta \mathcal{A}/\delta u_{st} = 0 \) with respect to \( u_{st} \). As far as \( u_{st} \) is a function of \( t, x \), we should go to independent variables \( t, x \) in the action (A.1). We obtain instead of (A.1)

\[ \mathcal{A}[\mathcal{S}_{st}] [\xi, u_{st}] = \int \left\{ \frac{m \dot{x}^2}{2} - \frac{e}{c} A_0 - \frac{e}{c} \mathbf{A} \frac{dx}{dt} + \frac{m u_{st}^2}{2} - \frac{\hbar}{2} \nabla u_{st} \right\} \rho(t, x) dt dx \]  

(A.2)

where \( \xi, u_{st} \) are dependent variables, whereas \( t, x \) are independent variables. Here \( \rho \) and \( \dot{x} = u \) are functions of \( \xi \), defined by the relations

\[ \rho = \frac{\partial (\xi_1, \xi_2, \xi_3)}{\partial (x^1, x^2, x^3)}, \quad \dot{x} \equiv u = \frac{\partial (x, \xi_1, \xi_2, \xi_3)}{\partial (t, \xi_1, \xi_2, \xi_3)} = \frac{1}{\rho} \frac{\partial (x, \xi_1, \xi_2, \xi_3)}{\partial (t, x^1, x^2, x^3)}. \]  

(A.3)

Variation of (A.2) with respect \( u_{st} \) gives

\[ \frac{\delta \mathcal{A}[\mathcal{S}_{st}]}{\delta u_{st}} = m u_{st} \rho + \frac{\hbar}{2} \nabla \rho = 0 \]  

(A.4)

Resolving the equation (A.4) with respect \( u_{st} \) in the form

\[ u_{st} = -\frac{\hbar}{2m} \nabla \ln \rho, \]  

(A.5)

we obtain instead of (A.2)

\[ \mathcal{A}[\mathcal{S}_{st}] [\xi] = \int \left\{ \frac{m}{2} \left( \frac{dx}{dt} \right)^2 - \frac{e}{c} A_0 - \frac{e}{c} \mathbf{A} \frac{dx}{dt} - \frac{\hbar^2}{8m} \frac{(\nabla \rho)^2}{\rho^2} \right\} \rho dt dx \]  

(A.6)
where $\rho$ and $\frac{d\xi}{dt}$ are functions of space-time derivatives of $\xi = \{\xi_1, \xi_2, \xi_3\}$, determined by the relations (A.3). The action (A.6) describes some ideal charged fluid with the internal energy per unit mass

$$U(\rho, \nabla \rho) = \frac{\hbar^2}{8m} \frac{(\nabla \rho)^2}{\rho^2} \quad (A.7)$$

Let us introduce new dependent variables $j = \{\rho, \rho u\} = \{j^k\}, \ k = 0, 1, 2, 3$ by means of relations (A.3). From formal viewpoint it is convenient to represent the hydrodynamic variables $j = \{\rho, \rho u\} = \{j^k\}, \ k = 0, 1, 2, 3$ in the form

$$j^k = \frac{\partial (x_k, \xi_1, \xi_2, \xi_3)}{\partial (x^0, x^1, x^2, x^3)} = \frac{\partial J}{\partial \xi_{0,k}}, \quad k = 0, 1, 2, 3 \quad (A.8)$$

where the Jacobian

$$J = \frac{\partial (\xi_0, \xi_1, \xi_2, \xi_3)}{\partial (x^0, x^1, x^2, x^3)} = \text{det} |\xi_{i,k}|, \quad \xi_{i,k} \equiv \partial_k \xi_i, \quad l, k = 0, 1, 2, 3 \quad (A.9)$$

is considered to be a function of variables $\xi_{i,k} \equiv \partial_k \xi_i, \ l, k = 0, 1, 2, 3$. The variable $\xi_0$ is the new dependent variable (temporal Lagrangian coordinate), which appears to be fictitious.

We introduce new dynamic variables by the Lagrange multipliers $p = \{p_k\}, \ k = 0, 1, 2, 3$, and obtain instead of (A.6)

$$A_{\varepsilon[S_{\varepsilon}]}[\xi, j, p] = \int \left\{ \frac{m}{2\rho} j^\alpha j^\alpha - \frac{e}{c} A_0 \rho - \frac{e}{c} A_{\alpha} j^\alpha - p_k \left( j^k - \frac{\partial J}{\partial \xi_{0,k}} \right) - \frac{\hbar^2}{8m} \frac{(\nabla \rho)^2}{\rho} \right\} d^4x \quad (A.10)$$

where $\xi = \{\xi_k\}, \ k = 0, 1, 2, 3$.

Variation of the action (A.10) with respect to $\xi_l$ leads to the dynamic equations

$$\frac{\delta A_{\varepsilon[S_{\varepsilon}]}[\xi, j, p]}{\delta \xi_l} = \partial_s \left( p_k \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \right) = 0, \quad l = 0, 1, 2, 3 \quad (A.11)$$

As far as the variable $\xi_0$ is fictitious, there are only three independent equations among four equations (A.11).

Using identities

$$\frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \equiv J^{-1} \left( \frac{\partial J}{\partial \xi_{0,k}} \frac{\partial J}{\partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,k}} \frac{\partial J}{\partial \xi_{l,k}} \right) \quad (A.12)$$

$$\frac{\partial J}{\partial \xi_{i,l}} \xi_{k,l} \equiv J \delta_{k,l}, \quad \partial_l \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{i,l}} \equiv 0 \quad (A.13)$$

and designations (A.8), we can eliminate the variables $\xi$ from the equations (A.11). We obtain

$$j^k \partial_l p_k - j^k \partial_k p_l = 0, \quad l = 0, 1, 2, 3 \quad (A.14)$$

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Variation of (A.10) with respect to $j^\beta$ and $j^0 = \rho$ gives respectively

$$p_\beta = m \frac{j^\beta}{\rho} - \frac{e}{c} A_\beta, \quad \beta = 1, 2, 3$$
\hspace{1cm} (A.15)

$$p_0 = -\frac{m}{2 \rho^2} j^\alpha j^\alpha - \frac{e}{m c} A_0 + \frac{\hbar^2}{8 m} \left( \frac{2 \nabla^2 \rho}{\rho} - \frac{(\nabla \rho)^2}{\rho^2} \right)$$
\hspace{1cm} (A.16)

Eliminating $p_\nu$ from the equations (A.14) by means of relations (A.15), (A.16), we obtain hydrodynamic equations for the ideal charged fluid in the conventional form

$$(\partial_\mu + v^\alpha \partial_\alpha) v^\mu = \frac{e}{mc} F_{\mu 0} + \frac{e}{mc} F_{\mu \alpha} v^\alpha - \frac{1}{m \rho} \partial_\mu p, \quad \mu = 1, 2, 3$$
\hspace{1cm} (A.17)

where the pressure $p$ and the electromagnetic field $F_{i k}$ are defined by the relations

$$p = \frac{\hbar^2}{8 m} \left( \frac{(\nabla \rho)^2}{\rho^2} - 2 \frac{\nabla^2 \rho}{\rho} \right), \quad F_{i k} = \partial_k A_i - \partial_i A_k, \quad i, k = 0, 1, 2, 3$$
\hspace{1cm} (A.18)

The wave function is constructed of potentials. The equations (A.17) does not contain potentials $\xi$ and $A_\alpha$, and they cannot be used for description of the fluid in terms of the wave function. To construct a description in terms of the wave function, we should not to eliminate potentials $\xi$ from the equations (A.11). Instead, we should integrate them. The dynamic equations (A.11) may be considered to be linear partial differential equations with respect to variables $p_\nu$. They can be solved in the form

$$p_k = b (\partial_k \varphi + g^\alpha (\xi) \partial_k \xi_\alpha), \quad k = 0, 1, 2, 3$$
\hspace{1cm} (A.19)

where $g^\alpha (\xi), \alpha = 1, 2, 3$ are arbitrary functions of the argument $\xi = \{\xi_1, \xi_2, \xi_3\}$, $b$ is an arbitrary real constant, and $\varphi$ is the variable $\xi_0$, which ceases to be fictitious.

One can test by the direct substitution that the relation (A.19) is the general solution of linear equations (A.11). Indeed, using (A.12) and the second identity (A.13), the equations (A.11) may be written in the form

$$\frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \partial_s p_k = J^{-1} \left( \frac{\partial J}{\partial \xi_{0,k} \partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s} \partial \xi_{l,k}} \right) \partial_s p_k = 0$$
\hspace{1cm} (A.20)

Substituting (A.19) in (A.20) and taking into account antisymmetry of the bracket in (A.20) with respect to indices $k$ and $s$, we obtain

$$J^{-1} \left( \frac{\partial J}{\partial \xi_{0,k} \partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s} \partial \xi_{l,k}} \right) \frac{\partial g^\alpha (\xi)}{\partial \xi_{\mu}} \xi_{\mu, s} \xi_{\alpha, k} = 0$$
\hspace{1cm} (A.21)

The relation (A.21) is the valid equality, as it follows from the first identity (A.13).

Let us substitute (A.19) in the action (A.10). Taking into account the first identity (A.13) and omitting the term

$$\frac{\partial J}{\partial \xi_{0,k}} \partial_k \varphi = \frac{\partial (\varphi, \xi_1, \xi_2, \xi_3)}{\partial (x^0, x^1, x^2, x^3)}$$
which does not contribute to the dynamic equation, we obtain

\[ E[S_{st}] : \quad A_{E[S_{st}]}[\varphi, \xi, j] = \int \left\{ \frac{m j^\alpha j^\alpha}{2} - \frac{e}{c} A_k j^k - j^k p_k - \frac{\hbar^2}{8m} \left( \nabla \rho \right)^2 \right\} d^4x, \]  

(A.22)

Here quantities \( p_k \) are determined by the relations (A.19).

The action in the form (A.22) is remarkable in the sense, that it contains information on initial values of the fluid velocities \( \mathbf{v} = j/\rho \). Dynamic equations, generated by the action (A.22), are partial differential equations, and one needs to give initial values for variables \( \varphi, \xi \).

But initial values for variables \( \varphi, \xi \) determine only labelling of the fluid particles, and they may be chosen universal. For instance, we may choose for all fluid flows

\[ \varphi (0, \mathbf{x}) = \varphi_{in} (\mathbf{x}) = 0, \quad \xi (0, \mathbf{x}) = \xi_{in} (\mathbf{x}) = \mathbf{x} \]  

(A.23)

Then the functions \( g(\xi) \) are determined by the initial values of the velocity \( \mathbf{v} (0, \mathbf{x}) = \mathbf{v}_{in} (\mathbf{x}) \) in the form \[ g(\xi) = \mathbf{v}_{in} (\xi) \]  

(A.24)

The initial value \( \rho (0, \mathbf{x}) = \rho_{in} (\mathbf{x}) \) of the density \( \rho \) may be also included in the action (A.22). It is necessary only to redefine the connection between the quantities \( j^k \) and \( \xi \), substituting the relations (A.8) by the relations \[ j^k = \rho_0 (\xi) \frac{\partial (x^k, \xi_1, \xi_2, \xi_3)}{\partial (x^0, x^1, x^2, x^3)}, \quad k = 0, 1, 2, 3 \]  

(A.25)

where \( \rho_0 (\xi) \) is an arbitrary function of \( \xi \). At the initial conditions (A.24) this arbitrary function is to be chosen in the form

\[ \rho_0 (\mathbf{x}) = \rho_{in} (\mathbf{x}) = \rho (0, \mathbf{x}) \]  

Now we eliminate the variables \( j = \{ j^1, j^2, j^3 \} \) from the action (A.22), using relation (A.15). We obtain

\[ A_{E[S_{st}]}[\rho, \varphi, \xi] = \int \left\{ -p_0 - \frac{e}{c} A_0 - \frac{(p_\beta + \xi A_\beta)}{2m} \left( \frac{p_\beta + \xi A_\beta}{2m} \right) - \frac{\hbar^2}{8m} \left( \nabla \rho \right)^2 \right\} \rho d^4x, \]  

(A.26)

where the quantities \( p_k, k = 0, 1, 2, 3 \) are determined by the relation (A.19).

Instead of dependent variables \( \rho, \varphi, \xi \) we introduce the \( n \)-component complex function \( \psi = \{ \psi_\alpha \}, \quad \alpha = 1, 2, \ldots, n \), which is defined by the relations \[ \psi_\alpha = \sqrt{\rho} e^{i\varphi} u_\alpha (\xi), \quad \psi^*_\alpha = \sqrt{\rho} e^{-i\varphi} u^*_\alpha (\xi), \quad \alpha = 1, 2, \ldots, n, \]  

(A.27)

\[ \psi^* \psi \equiv \sum_{\alpha=1}^{n} \psi^*_\alpha \psi_\alpha, \]  

(A.28)
where (*) means the complex conjugate. The quantities \( u_{\alpha}(\xi), \alpha = 1, 2, \ldots, n \) are functions of only variables \( \xi \), and satisfy the relations

\[
-\frac{i}{2} \sum_{\alpha=1}^{n} \left( u_{\alpha}^* \frac{\partial u_{\alpha}}{\partial \xi} - \frac{\partial u_{\alpha}^*}{\partial \xi} u_{\alpha} \right) = g^{\beta}(\xi), \quad \beta = 1, 2, 3, \quad \sum_{\alpha=1}^{n} u_{\alpha}^* u_{\alpha} = 1. \tag{A.29}
\]

The number \( n \) is such a natural number that the equations (A.29) admit a solution. In general, \( n \) depends on the form of the arbitrary integration functions \( g = \{g^{\beta}(\xi)\}, \beta = 1, 2, 3 \). The functions \( g \) determine vorticity of the fluid flow. If \( g = 0 \), equations (A.29) have the solution \( u_1 = 1, u_\alpha = 0, \alpha = 2, 3, \ldots n \). In this case the function \( \psi \) may have one component, and the fluid flow is irrotational.

In the general case it is easy to verify that

\[
\rho = \psi^* \psi, \quad \rho p_0 (\varphi, \xi) = -\frac{ib}{2} (\psi^* \partial_0 \psi - \partial_0 \psi^* \cdot \psi) \tag{A.30}
\]

\[
\rho p_\alpha (\varphi, \xi) = -\frac{ib}{2} (\psi^* \partial_\alpha \psi - \partial_\alpha \psi^* \cdot \psi), \quad \alpha = 1, 2, 3, \tag{A.31}
\]

The variational problem with the action (A.22) appears to be equivalent to the variational problem with the action functional

\[
A_{[S_{St}]}[\psi, \psi^*] = \int \left\{ \frac{ib}{2} (\psi^* \partial_0 \psi - \partial_0 \psi^* \cdot \psi) - \frac{e}{c} A_0 \rho \right. \]

\[
- \frac{\rho}{2m} \left( -\frac{ib}{2\rho} (\psi^* \nabla \psi - \nabla \psi^* \cdot \psi) + \frac{e}{c} A \right)^2 - \frac{\hbar^2}{8m} \frac{(\nabla \rho)^2}{\rho} \left. \right\} d^4x \tag{A.32}
\]

or

\[
A_{[S_{St}]}[\psi, \psi^*] = \int \left\{ \frac{ib}{2} (\psi^* \partial_0 \psi - \partial_0 \psi^* \cdot \psi) - \frac{e}{c} A_0 + \frac{ib^2}{2m\rho} (\psi^* \nabla \psi - \nabla \psi^* \cdot \psi)^2 \right. \]

\[
+ \frac{ib e}{2mc} A \left( \psi^* \nabla \psi - \nabla \psi^* \cdot \psi \right) - \frac{\hbar^2}{8m} \frac{\nabla \rho}{\rho} - \frac{\rho}{2m} \left( \frac{e}{c} A \right)^2 \left. \right\} d^4x \tag{A.33}
\]

For the two-component function \( \psi \) (\( n = 2 \)) the following identity takes place

\[
(\nabla \rho)^2 - (\psi^* \nabla \psi - \nabla \psi^* \cdot \psi)^2 \equiv 4 \rho \nabla \psi^* \nabla \psi - \rho^2 \sum_{\alpha=1}^{3} (\nabla s_\alpha)^2, \tag{A.34}
\]

\[
\rho \equiv \psi^* \psi, \quad s \equiv \frac{\psi^* \sigma \psi}{\rho}, \quad \sigma = \{\sigma_\alpha\}, \quad \alpha = 1, 2, 3, \tag{A.35}
\]

where \( \sigma_\alpha \) are the Pauli matrices. In virtue of the identity (A.34) the action (A.32) reduces to the form

\[
A_{[S_{St}]}[\psi, \psi^*]
\]
\[
= \int \left\{ \frac{ib}{2} \left( \psi^* \partial_0 \psi - \partial_0 \psi^* \cdot \psi \right) - \frac{e}{c} A_0 - \frac{1}{2m} \left( -ib \nabla \psi^* - \frac{e}{c} A \psi \right) \left( ib \nabla \psi - \frac{e}{c} A \psi \right) \right. \\
+ \frac{b^2 - \hbar^2}{8\rho m} (\nabla \rho)^2 + \frac{b^2}{8m} \sum_{\alpha=3}^{\alpha=3} (\nabla s_\alpha)^2 \rho \right\} \, d^4x, \tag{A.36}
\]

where \( s \) and \( \rho \) are defined by the relations (A.35). One should expect, that the two-component wave function describes the general case, because the number of real components of the two-component wave function coincides with the number of hydrodynamic variables \( \{\rho, j\} \). But this statement is not yet proved.

In the case of irrotational flow, when the two-component function \( \psi \) has linear dependent components, for instance \( \psi = \{\psi_1, 0\} \), the 3-vector \( s = \text{const} \), and the term containing 3-vector \( s \) vanishes. In the special case, when the electromagnetic potentials \( A_k = 0 \), the action (A.36) for \( E[S_{st}] \) coincides with the action (1.7) for \( S \).

Finally, if we choose the arbitrary constant \( b \) in the form \( b = \hbar \) and set \( A_k = 0 \), we obtain the action (1.1) for the Schrödinger particle.

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