Effects of Electromagnetic Field on the Structure of Massive Compact Objects

M. Z. Bhatti * and Z. Tariq †
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

This paper encompasses a set of stellar equations that administer the formation and evolution of self-gravitating, dissipative spherically symmetric fluid distributions having anisotropic stresses in the presence of electromagnetic field. The Riemann tensor is split orthogonally to procure five scalar functions named as structure scalars which are then utilized in the stellar equations. It is shown that some basic fluid properties such as energy density inhomogeneity, pressure anisotropy and heat flux are interlinked with the obtained scalars. Further, it is shown that all the solutions to Einstein equations can be written in terms of these five scalars keeping in view the static case.

Keywords: Structure scalars; Relativistic dissipative fluids; Electromagnetic field
PACS: 04.40.Nr; 04.20.G2; 41.20.

1 Introduction

The constituents of enormous stellar configurations are tied together by a combined force of gravitation which act on the system as a whole. Such a gravitational force is known as self-gravitational force and the celestial system is known as a self-gravitating system. Self-gravitational forces play pivotal role in the comprehension of star formation as they provide help to understand the time evolution of stellar systems. All the celestial bodies like

---

* mzaem.math@pu.edu.pk
† zohatariq24@yahoo.com
stars, galaxies and galaxy clusters would disintegrate and expand without this force. Self-gravitating fluids are classified with the help of certain physical variables like density, pressure etc. Such fluids have diversity of applications in relativistic astrophysics and cosmology.

Fundamental scalar functions that help in delineating the formation and evolution of self-gravitating fluid distributions are termed as structure scalars. They appear when the Riemann tensor is split orthogonally and are found to be interrelated with basic fluid attributes including energy density inhomogeneity, pressure anisotropy and dissipative flux etc. Such scalar functions have irrefutable physical significance and are proved to be one of the best ways, so far, to delineate the evolution of self-gravitating stellar configurations.

One of the main targets in modern astrophysics is to comprehend the consequences of electromagnetic field in the formation and evolution of the stellar configurations. Arbanil and Zanchin [1] analyzed the equilibrium configurations of static charged and uncharged spheres comprising relativistic polytropic fluid. They compared the obtained results with the spheres that have non-relativistic polytropic fluid. Weber [2] over-viewed the astrophysical phenomena related to strange quark matter. He discussed the possible observations associated with the states of matter inside compact stars. Negreiros et al. [3] showed that the electric field strength may increase as strange matter formulates a color super-conductor. Ivanov [4] studied Reissner-Nordström solution by considering the perfect fluid and found general formulas for de-Sitter solutions in the presence of electromagnetic field.

To gain insight into the time evolution of different celestial objects, structure scalars come in handy. A lot of work has been done by relativists in this regard. Herrera et al. [5] considered the extension of Lemaitre-Tolman-Bondi (LTB) space-times for the dissipative case and presented one of the symmetric properties of LTB. They also described LTB model by using certain scalar functions known as structure scalars. Herrera et al. [6] re-obtained the stellar evolution equations for dissipative, anisotropic spherically symmetric fluid and analyzed the condition for stability of shear-free condition. Herrera et al. [7] explored the physical meanings of scalar functions for dust cloud by including the cosmological constant and investigated the changes produced by certain factors in the inhomogeneity factor. Herrera et al. [8] worked out a set of equations for spherically symmetric self-gravitating dissipative fluids in terms of five fundamental scalar functions and presented that these are directly related to the dissipative flux, local pressure anisotropy and energy density etc. Herrera et al. [9] deployed a set of equations that govern dissipative spherically symmetric fluids. They emphasized on the relationship between the Weyl and shear tensors, the anisotropic factor and the density inhomogeneity.

Much efforts have been put forward by researchers in order to grasp the concept of complexity in the stellar systems. Calbet and López-Ruiz [10] derived the evolution equations for tetrahedral gas and presented that complexity does not exceed its minimum and maximum values. Lopez-Ruiz et al. [11] proposed a measure of complexity using probabilistic
approach and showed that it is applicable to a variety of physical situations. Herrera et al. [12] extended the notion of complexity of relativistic fluid distributions to the vacuum solution of the Bondi metric and found a link between vorticity and complexity. Herrera [13] propounded a novel notion for complexity of static spherically symmetric self-gravitating systems in the framework of general relativity. They also formulated the Einstein’s equations that fulfill the criteria of zero complexity. Crutchfield and Young [14] used statistical mechanics to delineate the complexity of non-linear mechanical systems.

The incorporation of heat dissipative flux is mandatory for the understanding of internal constitution and high temperature of stars. Researchers presented different solutions for different types of stellar systems including the effects of heat flux. Grammenos [15] presented a thermodynamical solution of Friedmann-like spherical stellar configuration for non-adiabatic collapse. Herrera et al. [16] analyzed the outcomes of thermal conduction within a relativistic fluid and concluded that its evolution depends directly on thermodynamical variables. Blandford et al. [17] showed that thermal effects in the outer crust of a neutron star give rise to magnetic field. de Oliveira et al. [18] propounded a collapsing radiating star model comprising isotropic shear-free fluid having heat flow in radial direction.

The exploration of the solutions of Einstein equations with different background metrics and matter fields have attracted the attention of many researchers [19]. Turimov et al. [20] proposed static axisymmetric solutions of Einstein field equations coupled with the static and axisymmetric phantom field. They found the validity of null energy condition for the case of phantom field while for the case of fundamental scalar field, it does not hold. Quevedo [21] inspected the Newtonian and relativistic multipole moments and concluded that the gravitational field of static and axisymmetric mass distributions can be delineated using this approach. Tolman [22] developed a scheme to obtain explicit solutions of the field equations.

The expansion scalar and shear stress tensors are used for better understanding of the behavior of fluids. A lot of research has been carried out using these tensors in combination with other fluid properties. Naidu et al. [23] analyzed the consequences of anisotropic pressure and heat flux on radiating spherically symmetric stars during its gravitational collapse. They concluded that the relaxation time for heat flux and shear stresses are significantly different. Ivanov [24] established a general form for the collapse of a charged spherical body having anisotropic stresses with shear and bulk viscosities. Herrera et al. [25] proposed expansion-free spherically symmetric fluid distributions. They presented the set of field equations as well as junction conditions for the case of anisotropic dissipative fluid configurations and concluded that cavity must be produced if the evolution is expansion-free. Herrera et al. [26] investigated the attributes of dissipative axially symmetric stellar structures with shear-free condition and concluded that the system progresses towards FLRW space-time in the absence of dissipation. The stability of relativistic interiors as well as the existence of self-gravitating structures has been discussed widely in literature [27].
The key objective of this work is to investigate the part played by the structure scalars in the formation and evolution of charged self-gravitating, dissipative spherically symmetric configurations having anisotropic stresses. The format of the paper is as follows. Section 2 states the Einstein equations in the presence of electromagnetic field along with the introduction of some physical variables which are used to delineate self-gravitating dissipative anisotropic fluid. Section 3 focuses on the orthogonal splitting of the Riemann tensor from which five structure scalars are procured. Section 4 lists down several alternatives to delineate the physical relevance of such scalar functions. In section 5, we abridges the whole discussion.

2 The General Formalism

This section enlists certain stellar equations using few physical variables that explain the behavior of self-gravitating anisotropic fluid under the influence of electromagnetic field [8].

2.1 Einstein-Maxwell Equations and Kinematical Quantities

We assume a spherically symmetric collapsing fluid having anisotropic stresses experiencing heat dissipation. The line element for such distribution is given by

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,$$

(1)

where $\nu$ and $\lambda$ are functions of temporal and radial coordinates. The electromagnetic energy-momentum tensor is as follows

$$S_{\alpha\beta} = \frac{1}{4\pi} \left( -F_{\alpha\delta}F_{\beta\delta} + \frac{1}{4}F_{\delta\omega}F_{\delta\omega}g_{\alpha\beta} \right).$$

The electromagnetic field tensor symbolized by $F_{\alpha\beta}$ is given as $F_{\alpha\beta} = \varphi_{\beta,\alpha} - \varphi_{\alpha,\beta}$ with the four-potential denoted as $\varphi^\alpha = \varphi(r)\delta_0^\alpha$ and the four-current density as $J^\alpha = \sigma(r)u^\alpha$. Here, $\sigma$ depicts the charge density whereas $\varphi$ is symbolized for scalar potential. The Einstein-Maxwell equations are given by

$$F^{\alpha\beta}_{;\beta} = \mu_0 J^\alpha, \quad F_{[\alpha\beta;\delta]} = 0,$$

where magnetic permeability is denoted by $\mu_0$. The non-zero components of the Maxwell field equations provide the following couple of equations

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial \varphi}{\partial r} \left[ -\frac{\lambda'}{2} - \frac{\nu'}{2} + \frac{2}{r} \right] = \mu_0 \sigma e^{\lambda^2 + \frac{\nu^2}{2}} \sqrt{1 - \omega^2}, \quad \dot{\phi}' - \phi' \left[ \frac{\dot{\nu}}{2} + \frac{\dot{\lambda}}{2} \right] = \mu_0 \sigma e^{\nu^2 + \frac{\phi^2}{2}} \sqrt{1 - \omega^2}.$$

(2)
Here, primed quantities depict that the derivative is taken with respect to the radial coordinate $r$. Integration of first equation in Eq. (2) yields

$$\frac{\partial \varphi}{\partial r} = \frac{se^{\frac{\nu + \lambda}{2}}}{r^2}, \quad \text{where} \quad s = \int_0^r \mu_0 \sigma r^2 e^{\lambda/2} dr. \quad (3)$$

The components of electromagnetic stress tensor that survive are

$$S_{00} = \frac{s^2 e^{\nu}}{8\pi r^4}, \quad S_{11} = -\frac{s^2 e^\lambda}{8\pi r^4}, \quad S_{22} = \frac{s^2}{8\pi r^2}, \quad S_{33} = \frac{s^2 \sin^2 \theta}{8\pi r^2}.$$

We procure the Einstein equations ($G_{\mu\nu} = \kappa (T_{\mu\nu} + S_{\mu\nu})$) to obtain

$$\kappa \left( T_{0}^0 + \frac{s^2}{8\pi r^4} \right) = \frac{1}{r^2} - \frac{e^{-\lambda}}{r^2} + \frac{\lambda}{r}, \quad (4)$$

$$\kappa \left( T_{1}^1 + \frac{s^2}{8\pi r^4} \right) = \frac{1}{r^2} - \frac{e^{-\lambda}}{r^2} - \frac{\nu'}{r}, \quad (5)$$

$$\kappa \left( T_{2}^2 - \frac{s^2}{8\pi r^4} \right) = e^{-\nu} \left( \frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda} \nu'}{4} \right) + e^{-\lambda} \left( \frac{\lambda'}{2r} - \frac{\nu''}{2} - \frac{\nu^2}{4} - \frac{\nu'}{2r} + \frac{\lambda \nu'}{4} \right), \quad (6)$$

$$\kappa T_{01} = \frac{\dot{\lambda}}{r}. \quad (7)$$

The pure locally Minkowski coordinates expressed as $(\tau, x, y, z)$ are given by \cite{8}

$$d\tau = e^{\nu/2} dt, \quad dy = r d\theta, \quad dx = e^{\lambda/2} dr, \quad dz = r \sin \theta d\phi.$$

The Minkowski components of energy-momentum tensor are written as

$$\bar{T}_{0}^0 = T_{0}^0, \quad \bar{T}_{1}^1 = T_{1}^1, \quad \bar{T}_{2}^2 = T_{2}^2, \quad \bar{T}_{3}^3 = T_{3}^3, \quad \bar{T}_{01} = e^{-(\nu + \lambda)/2} T_{01}.$$

From the perspective of a comoving observer, the covariant components of the energy-momentum tensor in Minkowski coordinates as presented earlier \cite{9} becomes

$$\begin{pmatrix}
\rho + \varepsilon & -q - \varepsilon & 0 & 0 \\
-q - \varepsilon & P_r + \varepsilon & 0 & 0 \\
0 & 0 & P_\perp & 0 \\
0 & 0 & 0 & P_\perp
\end{pmatrix},$$

so that the Lorentz transformation yields

$$T_{0}^0 = T_{0}^0 = \frac{\rho + P_r \omega^2}{1 - \omega^2} + \frac{2\omega q}{1 - \omega^2} + \frac{\varepsilon(1 + \omega)}{1 - \omega^2}, \quad (8)$$

5
The coordinate velocity represented by $\frac{d\omega}{dt}$ in the coordinate system $(t, r, \theta, \phi)$ is linked with \( \omega = e^{(\lambda - \nu)/2} (\frac{dr}{dt}) \). By making use of Eqs. (8)-(11) into Eqs. (11)-(17), we acquire

\[
\begin{align*}
T^1_1 &= T^1_1 = - \left( \frac{\rho \omega^2 + P_r}{1 - \omega^2} + \frac{2\omega q}{1 - \omega^2} + \frac{\varepsilon(1 + \omega)}{1 - \omega} \right), \\
T_{01} &= e^{-(\nu + \lambda)/2} T_{01} = - \left( \frac{\omega e^{(\nu + \lambda)/2}(\rho + P_r)}{1 - \omega^2} + \frac{q e^{(\nu + \lambda)/2}(1 + \omega^2)}{1 - \omega^2} + \frac{\varepsilon(1 + \omega) e^{(\nu + \lambda)/2}}{1 - \omega} \right), \\
T_2^2 &= T_2^2 = -P_\perp; \quad T_3^3 = T_3^3 = -P_\perp.
\end{align*}
\] (9)

In non-comoving coordinates, the four-velocity vector for our line element takes the form as

\[
\begin{align*}
\rho + P_r \omega^2 + 2\omega q + \frac{\varepsilon(1 + \omega)}{1 - \omega} + \frac{s^2}{8\pi r^4} &= \frac{1}{\kappa} \left( \frac{1 - e^{-\lambda}}{r} - \frac{\lambda e^{-\lambda}}{r^2} \right), \\
-\left( \rho \omega^2 + P_r + 2\omega q + \frac{\varepsilon(1 + \omega)}{1 - \omega} \right) + \frac{s^2}{8\pi r^4} &= \frac{1}{\kappa} \left( \frac{1 - e^{-\lambda}}{r} - \frac{\nu e^{-\lambda}}{r^2} \right), \\
- P_\perp - \frac{s^2}{8\pi r^4} &= \frac{1}{\kappa} \left[ e^{-\lambda} \frac{\dot{\lambda}}{r} + \frac{\dot{\lambda}}{4} + e^{-\lambda} \left( \frac{\lambda'}{2} - \frac{\nu''}{2} - \frac{\nu'}{2} + \frac{\lambda'}{4} \right) \right].
\end{align*}
\] (10)

Our next target is to carry out few kinematical quantities including four-acceleration, shear tensor and expansion scalar similar to those found in [8]. The non-zero components of four-acceleration as given in [29] in this non-comoving coordinate system are

\[
a_1 \omega = -a_0 e^{(\lambda - \nu)/2} = - \left( \frac{\omega^2 \omega'}{(1 - \omega^2)^2} + \frac{\omega \nu'}{2(1 - \omega^2)^2} + \frac{\omega^2 \lambda e^{(\lambda - \nu)/2}}{2(1 - \omega^2)} + \frac{\omega \dot{\omega} e^{(\lambda - \nu)/2}}{(1 - \omega^2)^2} \right). \quad (12)
\]

The shear tensor \( \sigma_{\alpha\beta} \) is defined as

\[
\sigma_{\alpha\beta} = u_{\alpha;\beta} + u_{\beta;\alpha} - u_{\alpha} a_{\beta} - u_{\beta} a_{\alpha} - \frac{2\Theta h_{\alpha\beta}}{3},
\]

where

\[
h_{\alpha\beta} = g_{\alpha\beta} - u_{\alpha} u_{\beta}, \quad \text{and} \quad \Theta = u_{\alpha}^{;\alpha}.
\]

The expansion scalar (\( \Theta \)) takes the following value [8]

\[
\Theta = \frac{\dot{\lambda} e^{-\nu/2}}{2\sqrt{(1 - \omega^2)}} + \frac{\omega \dot{\omega} e^{-\nu/2}}{(1 - \omega^2)^{3/2}} + \frac{\omega \nu' e^{-\lambda/2}}{2\sqrt{(1 - \omega^2)}} + \frac{\omega' e^{-\lambda/2}}{(1 - \omega^2)^{3/2}} + \frac{2\omega e^{-\lambda/2}}{r \sqrt{(1 - \omega^2)}}.
\]

6
The non-vanishing components of shear tensor are procured as

- \( \sigma_{00} = \omega^2 e^{\nu-\lambda} \sigma_{11}, \)
- \( \sigma_{01} = -\omega e^{(\nu-\lambda)/2} \sigma_{11}, \)
- \( \sigma_{11} = -\frac{2}{3(1-\omega^2)^{3/2}} \left[ \lambda e^{-\nu/2} + \frac{2\omega \dot{\omega} e^{\nu/2}}{(1-\omega^2)} + \omega \nu' e^{\lambda/2} + \frac{2\omega' e^{\lambda/2}}{(1-\omega^2)} - \frac{2\omega e^{\lambda/2}}{r} \right], \)
- \( \sigma_{22} = -\frac{2}{3} \left( 1 - \omega^2 \right)^{3/2} \sigma_{11}, \)
- \( \sigma_{33} = -\frac{2}{3} \left( 1 - \omega^2 \right)^{3/2} \sin^2 \theta \sigma_{11}. \)

The shear tensor can be written in an alternate form as

\[ \sigma_{\alpha\beta} = \sigma \left( s_\alpha s_\beta + \frac{1}{3} h_{\alpha\beta} \right), \]

where

\[ \sigma = -\frac{1}{\sqrt{(1-\omega^2)}} \left[ \lambda e^{-\nu/2} + \frac{2\omega \dot{\omega} e^{\nu/2}}{(1-\omega^2)} + \omega \nu' e^{\lambda/2} + \frac{2\omega' e^{\lambda/2}}{(1-\omega^2)} - \frac{2\omega e^{\lambda/2}}{r} \right]. \]

and

\[ s^\alpha = \left( \frac{\omega e^{-\nu/2}}{\sqrt{(1-\omega^2)}}, \frac{e^{-\lambda/2}}{\sqrt{(1-\omega^2)}}, 0, 0 \right), \tag{13} \]

which fulfills the following properties

\[ s^\alpha u_\alpha = 0, \quad s^\alpha s_\alpha = -1. \]

The energy-momentum tensor for non-comoving coordinate system is considered to be anisotropic and dissipative as

\[ T^\alpha_\beta = \tilde{\rho} u^\alpha u_\beta - \tilde{P} h^\alpha_\beta + \Pi^\alpha_\beta + \tilde{q} (s^\alpha u_\beta + s_\beta u^\alpha), \tag{14} \]

with

\[ \Pi^\alpha_\beta = \Pi \left( s^\alpha s_\beta + \frac{1}{3} h^\alpha_\beta \right), \quad \tilde{P} = \tilde{P}_r + \frac{2\tilde{P}_\perp}{3}, \quad \Pi = \tilde{P}_r - \tilde{P}_\perp, \quad \tilde{q}^\alpha = \tilde{q} s^\alpha, \]

\[ \tilde{q} = q + \varepsilon, \quad \tilde{\rho} = \rho + \varepsilon, \quad \tilde{P}_r = P_r + \varepsilon. \]

To procure the junction conditions, we assume charged-Vaidya spacetime as the exterior metric which is given by

\[ ds^2 = \left( \frac{1 - 2M(u)}{R} + \frac{Q^2}{R^2} \right) du^2 + 2dudR - R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]
here, the retarded time coordinate is symbolized by \( u \) while \( R \) represents the null coordinate. For the smooth matching of two metrics over the boundary surface \( r = r_{\Sigma} \), we call for the continuity of the first and second fundamental forms across the surface. This is done using the continuity of the line elements across the hypersurface, i.e., \((ds^2)_{\Sigma} = (ds^2)_{\Sigma} = (ds^2)_{\Sigma} \).

Next, the extrinsic curvature \( k_{\alpha\beta} \) defined as \( k_{\alpha\beta}^\pm = -\eta^\pm \left( \frac{\partial^2 \chi^{\pm \alpha}}{\partial \varepsilon_\alpha \partial \varepsilon_\beta} + \Gamma^{\pm \alpha}_{\mu \nu} \frac{\partial \chi^{\pm \mu}}{\partial \varepsilon_\alpha} \frac{\partial \chi^{\pm \nu}}{\partial \varepsilon_\beta} \right) \) must also be continuous for both the interior and exterior line elements, i.e., \( k_{\alpha\beta}^+ = k_{\alpha\beta}^- \) where \( \chi^{\sigma} \) and \( \varepsilon^{\sigma} \) represent the coordinates of \( \Sigma \) in the interior or exterior manifold and \( \eta_\sigma \) depicts the components of vector normal to \( \Sigma \). Following this procedure, certain conditions are obtained that are considered to be necessary and sufficient for smooth matching of exterior and interior regions of a particular astrophysical object. Consequently, we procure

\[
e^\nu \equiv \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right), \quad e^{-\lambda} \equiv \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right), \quad Q \equiv s, \quad P_r \equiv q, \quad (15)
\]

where \( \Sigma \) specifies that the manipulation has been made over the boundary surface. We noticed that in the absence of dissipative flux, the radial pressure vanishes.

2.2 The Riemann and the Weyl Tensor

The Riemann tensor utilizes Weyl tensor \( C^\rho_{\alpha\beta\mu} \), the Ricci tensor \( R_{\alpha\beta} \) and the scalar curvature \( R \) whose mathematical form is given by [8]

\[
R^\rho_{\alpha\beta\mu} = C^\rho_{\alpha\beta\mu} + \frac{1}{2} R^\rho_{\beta\mu} g_{\alpha\nu} + \frac{1}{2} R^{\mu\nu}_{\alpha\beta} \delta^\rho_{\nu} + \frac{1}{2} R^{\mu\nu}_{\alpha\beta} \delta^\rho_{\nu} - \frac{1}{2} R^{\rho}_{\mu\nu} g_{\alpha\beta} - \frac{1}{6} R \left( \delta^\rho_{\mu\nu} g_{\alpha\beta} - g_{\alpha\beta} \delta^\rho_{\mu\nu} \right). \quad (16)
\]

The Magnetic part for the Weyl tensor becomes zero because of spherical symmetry. Thus, the Weyl tensor can be completely expressed using its electric part \( E_{\alpha\beta} = C_{\alpha\beta\gamma\delta} u^\gamma u^\delta \) as

\[
C_{\mu\nu\kappa\lambda} = E^{\gamma\delta} u^\alpha u^\beta (g_{\mu\nu\alpha\beta} g_{\kappa\lambda\delta} - \eta_{\mu\nu\alpha\beta} \eta_{\kappa\lambda\delta}),
\]

with \( g_{\mu\nu\alpha\beta} \) being equal to \( g_{\mu\nu\alpha\beta} = g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha} \) and \( \eta_{\mu\nu\alpha\beta} \) symbolizes the Levi-Civita tensor. We noticed that \( E_{\alpha\beta} \) can also be expressed as

\[
E_{\alpha\beta} = E \left( s_{\alpha} s_{\beta} + \frac{1}{3} h_{\alpha\beta} \right), \quad (17)
\]

where the value for \( E \) already calculated in [8] is

\[
E = \frac{\dot{\lambda} e^{-\nu}}{4} + \frac{\dot{\lambda}^2 e^{-\nu}}{8} - \frac{\dot{\lambda} \dot{e} e^{-\nu}}{8} - \frac{\nu' e^{-\nu}}{4} - \frac{\nu'^2 e^{-\lambda}}{8} + \frac{\lambda' \nu e^{-\lambda}}{4r} - \frac{\nu' e^{-\lambda}}{4r} - \frac{\lambda e^{-\lambda}}{4r} - \frac{e^{-\lambda}}{2r^2} + \frac{1}{2r^2}. \quad (18)
\]

Also, the electric part of the Weyl tensor satisfies

\[
E^\alpha_{\alpha} = 0, \quad E_{\alpha\gamma} = E_{(\alpha\gamma)}, \quad E_{\alpha\gamma} u^\gamma = 0.
\]
2.3 The Mass Function and the Tolman Mass

This section incorporates two distinct and interesting definitions of the interior mass of the spherical body under the influence of electromagnetic field. The two masses are then interlinked using few relations which will be used afterwards in the physical description of the structure scalars which have influence of electromagnetic field over them [8].

2.3.1 The Mass Function

The mass function \( m \) for the line element given in Eq. (1) is defined by

\[
R^3_{232} = 1 - e^{-\lambda} = \frac{2m}{r} - \frac{\kappa s^2}{8\pi r^2}.
\]  

(19)

Using Eqs. (16), (17) and the field equations, we get

\[
\frac{3m}{r^3} = \frac{\kappa}{2} \left( \tilde{\rho} - \tilde{P}_r + \tilde{P}_\perp \right) + \frac{3\kappa s^2}{8\pi r^4} + E.
\]  

(20)

The above equation can be re-written in an alternate form as

\[
m = \frac{\kappa}{2} \int_0^r r^2 \left( T^0_0 + \frac{ss'}{4\pi r^3} \right) dr.
\]  

(21)

It is worth-noticing here that electromagnetism results in an increased value of the mass function. Utilizing the above equation along with Einstein equations, we procure another relation for mass function as [8]

\[
m = \frac{\kappa r^3}{6} \left( T^0_0 + T^1_1 - T^2_2 \right) + \frac{\kappa s^2}{16\pi r} + \frac{r^3 E}{3}.
\]  

(22)

Differentiating with respect to \( r \) and utilizing Eq. (21), it follows

\[
\left( \frac{r^3 E}{3} \right)' = -\frac{\kappa r^3}{6} \left( T^0_0 \right)' + \frac{\kappa s^2}{6} \left( T^2_2 - T^1_1 \right)' + \frac{\kappa s^2}{16\pi r^3}.
\]  

(23)

Integrating w.r.t. \( r \), we attain

\[
E = -\frac{\kappa}{2r^3} \int_0^r r^3 (T^0_0)' dr + \frac{\kappa}{2} \left( T^2_2 - T^1_1 \right) + \frac{3\kappa s^2}{16\pi r^3} \int_0^r \frac{s^2}{r^2} dr.
\]  

(24)

By substituting Eq. (24) in (22), we get

\[
m(r, t) = \frac{\kappa r^3}{6} (T^0_0) - \frac{\kappa}{6} \int_0^r r^3 (T^0_0)' dr + \frac{\kappa s^2}{16\pi r} + \int_0^r \frac{\kappa s^2}{16\pi r^2} dr.
\]  

(25)

We can write \( T^0_0 = \tilde{\rho} \) and \( T^1_1 = \Pi \). This can only be done if the following criteria is fulfilled [8].
1. If we consider the static regime, i.e. when $\omega$ along with all the time derivatives becomes zero.

2. If we consider the quasistatic regime, i.e., $\omega^2 \approx \dot{\omega} \approx \dot{\nu} \approx \dot{\lambda} \approx \ddot{\nu} \approx \ddot{\lambda} \approx 0$.

3. Just after the system leaves its state of equilibrium, i.e. $\omega \approx \dot{\nu} \approx \dot{\lambda} \approx 0$ but $\dot{\omega} \neq 0$.

Keeping in view these three cases, Eq. (24) become

$$E = -\frac{\kappa}{2r^3} \int_0^r r^3 (\tilde{\rho})' dr + \frac{\kappa \Pi}{2} + \frac{3\kappa}{16\pi r^3} \int_0^r \frac{s^2}{r^2} dr.$$  

We can see that this equation links the Weyl tensor with pressure anisotropy $\Pi$ and energy density inhomogeneity $\rho'$ in addition to a charge term appearing due to the existence of electromagnetic field. Now, Eq. (25) can take the following form

$$m(r, t) = -\frac{\kappa r^3}{6} \tilde{\rho} + \frac{\kappa}{6} \int_0^r r^3 (\tilde{\rho})' dr + \frac{\kappa s^2}{16\pi r} + \int_0^r \frac{\kappa s^2}{16\pi r^2} dr.$$  

This equation defines the mass function as a combination of homogeneous distribution of energy density, the change caused by inhomogeneity of energy density and charge terms appearing due to electromagnetic field. We can conclude from here that the electromagnetic field results in the increased mass of the spherical body under consideration.

### 2.3.2 Tolman Mass

To comprehend the energy composition of a spherical body, Tolman proposed a new definition known as Tolman mass. The mathematical formulation as given in [30] is

$$m_T = \frac{\kappa}{2} \int_0^r r^2 e^{(\nu+\lambda)/2} (T^0_0 - T^1_1 - 2T^2_2) dr + \frac{1}{2} \int_0^r r^2 e^{(\nu+\lambda)/2} \frac{\partial}{\partial t} \left[ \frac{\partial L}{\partial (g_{\alpha\beta} \sqrt{-g})} \right] g^{\alpha\beta} dr,$$

where $L$ signifies the gravitational Lagrangian density. The mass inside sphere having radius $r$ within the boundary $\Sigma$ is

$$m_T = \frac{\kappa}{2} \int_0^r r^2 e^{(\nu+\lambda)/2} (T^0_0 - T^1_1 - 2T^2_2) dr + \frac{1}{2} \int_0^r r^2 e^{(\nu+\lambda)/2} \frac{\partial}{\partial t} \left[ \frac{\partial L}{\partial (g_{\alpha\beta} \sqrt{-g})} \right] g^{\alpha\beta} dr.$$

After some manipulations, we acquire

$$m_T = e^{(\nu+\lambda)/2} \left[ m(r, t) - \frac{\kappa r^3 T^1_1}{2} + \frac{\kappa r^3}{2} \left( \frac{s^2}{8\pi r^4} \right) \right]. \quad (26)$$
Substituting the value of $T_1^1$ from Eq. (5) and $m$ from Eq. (19), we attain

$$m_T = e^{(\nu+\lambda)/2} \left[ \frac{\nu' e^{-\lambda r^2}}{2} + \frac{\kappa s^2}{8\pi r} \right].$$

(27)

It is prominent that electromagnetism results in the increased energy budget of the system. Under the consideration of static field, the gravitational acceleration for test particle turns out to be

$$a = \frac{m_T e^{-\nu/2}}{r^2} - \frac{\kappa s^2 e^{\lambda/2}}{8\pi r^3}.$$

Working out the derivative of Eq. (27) with respect to $r$ and utilizing field equations along with Eqs. (22) and (26), we acquire an alternative expression for $m_T$ as

$$rm_T' - 3m_T = r^3 e^{(\lambda-\nu)/2} \left[ \frac{\dot{\lambda}}{2} + \frac{\dot{\lambda} \dot{\nu}}{2} - \frac{\dot{\lambda} \dot{\nu}}{4} \right] + r^3 e^{(\lambda+\nu)/2} \left[ \frac{\kappa}{2} (T_1^1 - T_2^2) - E \right]$$

$$+ r \left( \frac{e^{(\lambda+\nu)/2} \kappa s^2}{8\pi r} \right)' - \frac{3\kappa}{8\pi} \int_0^r \frac{s^2 e^{(\lambda+\nu)/2}}{r^2} dr.$$

We integrate the above equation and get

$$m_T = (m_T)_{\Sigma} \left( \frac{r}{r_{\Sigma}} \right)^3 - r^3 \int_r^{r_{\Sigma}} \frac{e^{(\lambda-\nu)/2}}{2r} \left[ \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda} \dot{\nu}}{2} \right] dr - r^3 \int_r^{r_{\Sigma}} \frac{e^{(\lambda+\nu)/2}}{r} dr$$

$$\times \left[ \frac{\kappa}{2} (T_1^1 - T_2^2) - E \right] dr - r^3 \int_r^{r_{\Sigma}} \frac{1}{r^3} \left( \frac{\kappa e^{(\lambda+\nu)/2} s^2}{8\pi r} \right)' dr + r^3 \int_r^{r_{\Sigma}} \frac{3\kappa s^*}{8\pi r^4} dr,$$

(28)

with

$$s^* = \int_0^r \frac{s^2 e^{(\lambda+\nu)/2}}{r^2} dr.$$

Substituting the value of $E$ in the above equation, we attain

$$m_T = (m_T)_{\Sigma} \left( \frac{r}{r_{\Sigma}} \right)^3 - r^3 \int_r^{r_{\Sigma}} \frac{e^{(\lambda-\nu)/2}}{2r} \left[ \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda} \dot{\nu}}{2} \right] dr - r^3 \int_r^{r_{\Sigma}} \frac{e^{(\lambda+\nu)/2}}{r} dr$$

$$\times \left[ \frac{\kappa}{2} (T_1^1 - T_2^2) + \frac{\kappa}{2r^3} \int_0^r r^3 (T_0^0)' dr - \frac{3\kappa}{16\pi r^3} \int_0^r \frac{s^2}{r^2} dr \right] dr - r^3 \int_r^{r_{\Sigma}} \frac{1}{r^3}$$

$$\times \left( \frac{\kappa e^{(\lambda+\nu)/2} s^2}{8\pi r} \right)' dr + r^3 \int_r^{r_{\Sigma}} \frac{3\kappa s^*}{8\pi r^4} dr.$$
If we consider the three cases, which we have defined earlier, for the above equation, it will depict that the Tolman mass for spherical body having radius \( r \) within the boundary \( \Sigma \) can be described using homogeneous energy density \( \rho \), local pressure anisotropy \( \Pi \), changes induced from the non-equilibrium state of the system and terms arising due to the effect of electromagnetic field.

### 2.3.3 Structure and Evolution Equations

To gain insight into the formation and evolution of a self-gravitating fluid, a set of stellar equations can be used which in our case is given as \([8]\)

\[
\tilde{\rho}^* + (\tilde{\rho} + \tilde{P}_r)\Theta - \frac{2}{3} \left( \Theta + \frac{\sigma}{2} \right) \Pi + \tilde{q}^* + 2\tilde{q} \left( a + \frac{s^1}{r} \right) + \left( \frac{ss'}{4\pi r^4} - \frac{s^2}{\pi r^6} \right) = 0, \tag{29}
\]

\[
\tilde{P}_r^* + (\tilde{\rho} + \tilde{P}_r)a + \frac{2s^1\Pi}{r} - \frac{\tilde{q}}{3}(\sigma - 4\Theta) - \tilde{q}^* = 0, \tag{30}
\]

\[
\Theta^* + \frac{\Theta^2}{3} + \frac{\sigma^2}{6} - a^* - a^2 - \frac{2as^1}{r} + \frac{\kappa}{2}(\tilde{\rho} + 3\tilde{P}_r) - k\Pi + \frac{\kappa s^2}{8\pi r^4}, \tag{31}
\]

\[
\left( \frac{\sigma}{2} + \Theta \right)^* = -\frac{3\sigma s^1}{2r} + \frac{3k\tilde{q}}{2}, \tag{32}
\]

\[
a^* + a^2 + \frac{\sigma^*}{2} + \frac{\Theta\sigma}{3} - \frac{as^1}{r} - \frac{\sigma^2}{12} = -E - \frac{\kappa\Pi}{2} + \frac{\kappa s^2}{8\pi r^4}, \tag{33}
\]

\[
-\frac{3\kappa s^1\tilde{q}}{2r} = \left( \Theta + \frac{\sigma}{2} \right) \left[ \frac{\kappa \tilde{P}_r}{2} + \frac{3m}{r^3} + \frac{3\kappa s^2}{8\pi r^4} \right] + \left( E - \frac{\kappa\Pi}{2} + \kappa\tilde{\rho} + \frac{3\kappa s^2}{16\pi r^4} \right)^*, \tag{34}
\]

\[
\left( E + \frac{\kappa\tilde{\rho}}{2} - \frac{\kappa\Pi}{2} + \frac{\kappa s^2}{8\pi r^4} \right)^* = \frac{3s^1}{r} \left( \frac{\kappa\Pi}{2} - E - \frac{\kappa s^2}{8\pi r^4} \right) + \frac{\kappa\tilde{q}}{2} \left( \frac{\sigma}{2} + \Theta \right), \tag{35}
\]

\[
\frac{3m}{r^3} = \frac{\kappa\tilde{\rho}}{2} + \frac{\kappa}{2}(P_\perp - P_r) + E + \frac{3\kappa s^2}{16\pi r^4}, \tag{36}
\]

where, \( f^* = f_{\mu}s^\mu, f^* = f_{\mu}u^\mu \) and \( a^\mu = as^\mu \) and \( \frac{\sigma}{2} + \Theta = \frac{3\omega s^1}{r} \). All these stellar equations provide information relating to the formation and evolution of our proposed system. Equations (29) and (30) are the well-known conservation equations. Equation (31) represents the Raychaudhuri equation while (32) is derived using the Ricci identities. Utilizing Eq. (16) along with the field equations, we get (33) and by making use of the Weyl tensor in Bianchi identities, Eqs. (34) and (35) are produced. One of the forthcoming sections makes use of all these equations to describe the formation and evolution of self-gravitating systems in the form of structure scalars. As compared to the work done by Herrera et al. \([8]\), our results show that electromagnetism adds to the complexity of the stellar configuration.
3 The Orthogonal Splitting of the Riemann Tensor

Our aim is to split the Riemann tensor orthogonally to acquire certain scalar quantities. For this purpose, we use the following tensors already introduced in [8] as

\[ Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta} u^{\gamma} u^{\delta}, \quad Z_{\alpha\beta} = R_{\alpha\gamma\beta\delta} u^{\gamma} u^{\delta} = \frac{1}{2} \eta_{\alpha\gamma\epsilon\rho} R^{\epsilon\rho}_{\beta\delta} u^{\gamma} u^{\delta}, \]

\[ X_{\alpha\beta} = R^{*}_{\alpha\gamma\beta\delta} u^{\gamma} u^{\delta} = \frac{1}{2} \eta_{\epsilon\rho\alpha\gamma} R^{\epsilon\rho}_{\beta\delta} u^{\gamma} u^{\delta}, \]

where

\[ R^{*}_{\alpha\beta\gamma\delta} = \frac{1}{2} \eta_{\rho\gamma\alpha\beta} R^{\rho}_{\gamma\delta}. \]

Making use of the Einstein’s equations in Eq.(16) can be written as

\[ R_{\alpha\gamma} = C_{\alpha\gamma} + 2\kappa T^{[\alpha}_{\beta\gamma]} + 2\kappa E^{[\alpha}_{\beta\gamma]} + \kappa T \left[ \frac{1}{3} \delta_{\beta\gamma}^{\delta\eta} - \delta_{[\beta}^{\delta \gamma]} \right]. \] (37)

Substituting Eq.(14) in (37), the Riemann tensor is split as

\[ R_{\alpha\gamma} = R^{(I)}_{\alpha\gamma} + R^{(II)}_{\alpha\gamma} + R^{(III)}_{\alpha\gamma}, \]

where, we have

\[ R^{\alpha\gamma}_{(I)\beta\delta} = 2\kappa \left( \bar{\rho} + \frac{s^{2}}{8\pi r^{4}} \right) u^{[\alpha}_{\beta}\gamma\delta] - 2\kappa \left( \bar{P} + \frac{s^{2}}{24\pi r^{4}} \right) h^{[\alpha}_{\beta}\gamma\delta] + \kappa(\bar{\rho} - 3\bar{P}) \times \left[ \frac{1}{3} \delta_{\beta\gamma}^{\delta\eta} - \delta_{[\beta}^{\delta \gamma]} \right], \]

\[ R^{\alpha\gamma}_{(II)\beta\delta} = 2\kappa \left( \Pi - \frac{s^{2}}{4\pi r^{4}} \right) \left[ s^{[\alpha}_{\beta} s^{\gamma\delta]} + \frac{1}{3} h^{[\alpha}_{\beta}\gamma\delta] \right] + 2\kappa \left[ \bar{q} s^{[\alpha}_{\beta} u^{\gamma\delta]} + \bar{q} u^{[\alpha} s^{\gamma\delta]} - \epsilon_{\alpha\gamma\delta} \epsilon_{\beta\gamma\delta} E^{\mu\nu} \right], \]

\[ R^{\alpha\gamma}_{(III)\beta\delta} = 4u^{[\alpha}_{\beta} u^{\gamma\delta]} - \epsilon_{\alpha\gamma\delta} \epsilon_{\beta\gamma\delta} E^{\mu\nu}, \]

fulfilling the following relations

\[ \epsilon_{\alpha\gamma\beta} = u^{\nu} \eta_{\mu\nu\alpha\beta}, \quad \epsilon_{\alpha\gamma\beta} u^{\beta} = 0. \]

The vanishing occurs as a consequence of zero magnetic part of the Weyl tensor. Also, we have used [8]

\[ \epsilon^{\mu\nu\rho\gamma} \epsilon_{\nu\alpha\beta} = u^{\sigma} u^{\rho} \eta^{\mu\nu\gamma} \eta_{\sigma\nu\alpha\beta}, \quad \epsilon^{\mu\nu\rho\gamma} \epsilon_{\nu\alpha\beta} = \delta_{\alpha}^{\mu} h_{\beta}^{\rho} - \delta_{\alpha}^{\rho} h_{\beta}^{\mu} + u_{\alpha} \left( u^{\mu} \delta_{\beta}^{\rho} - \delta_{\beta}^{\mu} u^{\gamma} \right). \]
Contracting $\mu$ with $\alpha$, we get
\[ \epsilon^{\mu\nu\epsilon_{\nu\mu\beta}} = -2h_\beta^\gamma. \]
The explicit expressions for the tensors $X_{\alpha\beta}$, $Y_{\alpha\beta}$ and $Z_{\alpha\beta}$ are manipulated with field equations as
\[
X_{\alpha\beta} = \frac{\kappa}{3} \left[ \bar{\rho} + \frac{s^2 \kappa}{8\pi r^4} \right] h_{\alpha\beta} + \left[ \frac{\kappa \Pi}{2} - \frac{\kappa s^2}{8\pi r^4} \right] \times \left( s_\alpha s_\beta + \frac{1}{3} h_{\alpha\beta} \right) - E_{\alpha\beta}, \quad (38)
\]
\[
Y_{\alpha\beta} = \frac{\kappa}{6} \left[ \bar{\rho} + 3 \bar{\rho} + \frac{s^2}{4\pi r^4} \right] h_{\alpha\beta} + \left[ \frac{\kappa \Pi}{2} - \frac{\kappa s^2}{8\pi r^4} \right] \times \left( s_\alpha s_\beta + \frac{1}{3} h_{\alpha\beta} \right) + E_{\alpha\beta}, \quad (39)
\]
\[
Z_{\alpha\beta} = \frac{\kappa}{2} \bar{q} s^\epsilon \epsilon_{\alpha\epsilon\beta}. \quad (40)
\]
Bel superenergy denoted by $\bar{W}$ and super-Poynting vector denoted by $\bar{P}_\alpha$ as defined earlier in [8] are given as
\[
\bar{W} = \frac{1}{2} \left( X_{\alpha\beta} X^{\alpha\beta} + Y_{\alpha\beta} Y^{\alpha\beta} + Z_{\alpha\beta} Z^{\alpha\beta} \right), \quad \bar{P}_\alpha = \epsilon_{\alpha\beta\gamma} \left( Y_\beta^\gamma Z^{\beta\delta} - X_\beta^\gamma Z^{\beta\delta} \right).
\]
Substituting the values of the three tensors discussed above, we acquire
\[
\bar{W} = \frac{5\kappa^2 \bar{\rho}^2}{24} + \frac{\kappa^2 \bar{\rho} \bar{P}}{4} + \frac{3\kappa^2 \bar{P}}{8} + \frac{\kappa^2 \Pi^2}{6} + \frac{2E^2}{3} + \frac{\kappa^2 \bar{\rho} s^2}{16\pi r^4} - \frac{\kappa^2 s^2 \Pi}{12\pi r^4} + \frac{\kappa^2 s^2 \bar{q}^2}{64\pi^2 r^8} + \frac{\kappa^2 \bar{q} \bar{P}}{2},
\]
\[
\bar{P}_\alpha = \left( \frac{\kappa^2 \bar{\rho}}{2} + \frac{\kappa^2 \bar{q} \bar{P} r}{2} \right) s_\alpha.
\]
The last equation exhibits that if there is no heat flux, the super-Poynting vector becomes zero. Bel Robinson scalar ($W$) can be defined by the relation $W = E^{\alpha\beta}_{\alpha\beta}$. Since the magnetic part of the Weyl tensor vanishes in our case, we attain $W = \frac{4E^2}{3}$. Consequently, we can have
\[
\bar{W} - W = \frac{5\kappa^2 \bar{\rho}^2}{24} + \frac{\kappa^2 \bar{\rho} \bar{P}}{4} + \frac{3\kappa^2 \bar{P}}{8} + \frac{\kappa^2 \Pi^2}{6} + \frac{2E^2}{3} + \frac{\kappa^2 \bar{\rho} s^2}{16\pi r^4} - \frac{\kappa^2 s^2 \Pi}{12\pi r^4} + \frac{\kappa^2 s^2 \bar{q}^2}{64\pi^2 r^8} + \frac{\kappa^2 \bar{q} \bar{P}}{2}.
\]

### 3.1 Five Relevant Scalars

Now, we shall formulate five scalars previously calculated in [7] which will further be utilized in Eqs. (29) - (36). We noticed that $X_{\alpha\beta}$ and $Y_{\alpha\beta}$ can be split into two parts, i.e., trace and
tracefree scalar parts. Working out such parts for $X_{\alpha\beta}$, we attain

$$X_{\alpha\beta} = \frac{1}{3} TrX h_{\alpha\beta} + X_{<\alpha\beta>},$$

where $TrX = X^\alpha_\alpha$. Also,

$$X_{<\alpha\beta>} = h^{\mu\nu}_\alpha h^{\nu}_\beta \left(X_{\mu\nu} - \frac{1}{3} TrX h_{\mu\nu}\right),$$

with

$$TrX = X_T = k\rho + \frac{s^2}{8\pi r^4}.$$  \hfill(41)

Consequently, we can have

$$X_{\alpha\beta} = X_{TF} \left(s_\alpha s_\beta + \frac{h_{\alpha\beta}}{3}\right),$$

where

$$X_{TF} \equiv \left(\frac{\kappa\Pi}{2} - \frac{s^2\kappa}{8\pi r^4} - E\right).$$ \hfill(42)

Following the same steps, we acquire trace and tracefree parts for $Y_{\alpha\beta}$ as

$$TrY \equiv Y_T = \frac{\kappa}{2} \left(\rho + 3\bar{P} - 2\Pi + \frac{s^2}{4\pi r^4}\right).$$ \hfill(43)

We procure

$$Y_{\alpha\beta} = Y_{TF} \left(s_\alpha s_\beta + \frac{h_{\alpha\beta}}{3}\right),$$

here

$$Y_{TF} \equiv \left(\frac{\kappa\Pi}{2} - \frac{s^2\kappa}{8\pi r^4} + E\right).$$ \hfill(44)

Making use of the explicit expression for $Z_{\alpha\beta}$, we obtain another scalar function as

$$Z = \sqrt{Z_{\alpha\beta} Z^{\alpha\beta}} = \frac{\kappa\tilde{q}}{\sqrt{2}}.$$ \hfill(45)

We observe that in the presence of electromagnetic field, $X_{TF}$ and $Y_{TF}$ describe local pressure anisotropy along with an additional quantity representing the inclusion of charge as given below

$$\kappa\Pi - \frac{\kappa s^2}{4\pi r^4} = X_{TF} + Y_{TF}.$$ \hfill(46)
Now using these five scalar functions \((X_T, X_{TF}, Y_T, Y_{TF}, Z)\), we can rewrite Eqs. (29)-(36) as

\[
\frac{\kappa \bar{\rho}^*}{2} + \frac{1}{3} [X_T + X_{TF} + Y_T + Y_{TF}] \theta = \frac{1}{3} \left( \theta + \frac{\sigma}{2} \right) \left( X_{TF} + Y_{TF} + \frac{\kappa s^2}{4\pi r^4} \right) - \frac{\sqrt{2}Z^\dagger}{2} - \sqrt{2} \left[ Za + \frac{Zs^1}{r} \right],
\]

\[
\frac{\kappa \bar{P}_\alpha^\dagger}{2} + \frac{a}{3} [X_T + X_{TF} + Y_T + Y_{TF}] + \frac{s^1}{r} \left( X_{TF} + Y_{TF} + \frac{\kappa s^2}{8\pi r^4} \right) = \frac{\sqrt{2}Z}{3} (\sigma - 4\theta) - \frac{\sqrt{2}Z^\dagger}{2} + \frac{ss'e^{-\lambda}}{4\pi r^4},
\]

\[
\theta^* + \frac{\theta^2}{3} + \frac{\sigma^2}{6} - a^\dagger - a^2 - 2as^1 = -Y_T,
\]

\[
\left( \frac{\sigma}{2} + \theta \right)^\dagger = -3\frac{\sigma s^1}{2r} + \frac{3\sqrt{2}Z}{2},
\]

\[
a^\dagger + a^2 + \frac{\theta^*}{3} - \frac{\theta^2}{3} - \frac{\sigma^2}{r} - \frac{2as^1}{12} = -Y_{TF},
\]

\[
\frac{1}{3} \left[ (Y_T + Y_{TF}) - 2X_{TF} + X_T \right] \left( \frac{\sigma}{2} + \theta \right) + \left( \frac{X_T}{2} - X_{TF} \right)^* = \frac{-3s^1\sqrt{2}Z}{2r},
\]

\[
\left( \frac{\kappa \bar{\rho}}{2} - X_{TF} \right)^\dagger = 3\frac{s^1X_{TF}}{r} + \frac{\sqrt{2}Z}{2} \left( \frac{\sigma}{2} + \theta \right).
\]

\[
\frac{3m}{r^3} = \frac{X_T}{2} - X_{TF}.
\]

The Bel superenergy and super-Poynting vector take the following form

\[
\bar{W} = \frac{1}{6} (X_T^2 + Y_T^2) + \frac{1}{3} (X_{TF}^2 + Y_{TF}^2) + Z^2,
\]

\[
\bar{P}_\alpha = \frac{\sqrt{2}Z}{3} (X_T + X_{TF} + Y_T + Y_{TF}) s_\alpha.
\]

### 3.2 On the Physical Meaning of Structure Scalars

Following the work of Herrera et al. [8], this section is focused on the physical interpretation of five scalar functions procured in the previous section. Clearly, the trace part \(X_T\) of \(X_{\alpha\beta}\) deals with the homogeneous energy density of the system. The scalar \(Z\) administers the heat dissipative flux. The tracefree parts \(X_{TF}\) and \(Y_{TF}\) administers the local pressure anisotropy.
in the presence of charge. To comprehend the physical interpretation of scalar $Y_T$ and $Y_{TF}$, we make use of Eq.(28) and obtain the following result

$$m_T = (m_T)_{\Sigma} \left( \frac{r}{r_{\Sigma}} \right)^3 - r^3 \int_{r}^{r_{\Sigma}} \frac{e^{(\lambda - \nu)/2}}{2r} \lambda dr + r^3 \int_{r}^{r_{\Sigma}} \frac{e^{(\lambda + \nu)/2}}{r} dr \times \left[ Y_{TF} + \frac{s^2 \kappa}{r^4} \right] dr - r^3 \int_{r}^{r_{\Sigma}} \frac{1}{r^3} \left( \frac{\kappa e^{(\lambda + \nu)/2} s^2}{8\pi r} \right)' dr + r^3 \int_{r}^{r_{\Sigma}} \frac{3\kappa s^4}{8\pi r^4} dr.$$ 

Considering our proposed system to be in an equilibrium or quasi-equilibrium state, we acquire the following expression for the Tolman mass

$$m_T = \frac{\kappa}{2} \int_{0}^{r_{\Sigma}} r^2 e^{(\nu + \lambda)/2} (T_0^0 - T_1^1 - 2T_2^2) dr,$$

which can alternatively be written as

$$m_T = \int_{0}^{r} r^2 e^{(\nu + \lambda)/2} Y_T dr.$$

Thus, $Y_T$ can be used to describe the Tolman mass for a spherical object.

4 Static Spheres With Anisotropic Pressure

This section takes into account only the static spherically symmetric systems defined previously in [8]. Considering the line element defined in Eq.(1), we express three different substitutes to it, each one exhibiting different static spheres in terms of above mentioned structure scalars.

4.1 First Alternative

By making use of Eqs.(19) and (54), we obtain

$$e^{-\lambda} = 1 - \frac{2m}{r} + \frac{s^2 \kappa}{8\pi r^2}.$$ 

For a static sphere, using Eq.(49) and (51), we may define the gravitational acceleration as

$$a = \frac{r}{3s^1} (Y_{TF} + Y_T). \quad (55)$$
Alternatively, using Eqs. (12) and (13), we attain
\[ a = \frac{e^{-\lambda/2}v'}{2}, \quad s^1 = e^{-\lambda/2}. \] (56)

Substituting Eq. (56) into (55) and integrating w.r.t. radial coordinate, we get
\[ e^\nu = Ce^{\int \frac{2\nu}{r}(Y_{TF}+Y_T)[1-\frac{2r^2}{3}\left(\frac{X_T}{2} - X_{TF} + \frac{3\kappa s^2}{16\pi r^4}\right)]^{-1} (Y_{TF}+Y_T) dr}, \]
with \( C \) being the integration constant which one can easily determine from Eq. (15). Consequently, in the static case, the line element becomes
\[ ds^2 = Ce^{\int \frac{2\nu}{r}(Y_{TF}+Y_T)[1-\frac{2r^2}{3}\left(\frac{X_T}{2} - X_{TF} + \frac{3\kappa s^2}{16\pi r^4}\right)]^{-1} (Y_{TF}+Y_T) dr} dt^2 \\
- \left[ 1 - \frac{2r^2}{3} \left(\frac{X_T}{2} - X_{TF} + \frac{3\kappa s^2}{16\pi r^4}\right) \right]^{-1} dr^2 - r^2 \sin^2 \theta d\phi^2. \]

It is worth noting that all the static anisotropic fluid spheres can be fully expressed using the scalar functions \( \frac{X_T}{2} - X_{TF} \) and \( Y_{TF} + Y_T \).

### 4.2 Second Alternative

By making use of Eqs. (21), (41) and (54), we acquire
\[ m(r) = \frac{r^3}{3} \left[ \frac{m'}{r^2} - \frac{1}{r^2} \left( \frac{\kappa s^2}{16\pi r} \right)' - X_{TF} \right] + \frac{\kappa s^2}{16\pi r}. \]

Performing integration w.r.t. \( r \), we obtain
\[ m(r) = r^3 \left( \int \frac{X_{TF}}{r} dr + C_1 \right) + \frac{\kappa s^2}{16\pi r}. \]

Further, by making use of Eq. (19), we can write
\[ e^{-\lambda} = 1 - 2r^2 \left( \int \frac{X_{TF}}{r} dr + C_1 \right), \]
where the constant of integration \( C_1 \) can be easily determined from Eq. (15). Considering the static regime, the field equation (5) yields
\[ \frac{\kappa P_r}{2} = \frac{1}{2} \left( \frac{e^{-\lambda} - 1}{r^2} \right) + \frac{\nu' e^{-\lambda}}{2r} + \frac{\kappa s^2}{16\pi r^4}. \]
which on utilizing Eqs. (19), (43), (44), (55) and (56), may be re-written as
\[
\frac{\kappa P_r}{2} + \frac{m}{r^3} = \frac{\nu e^{-\lambda}}{2r} + \frac{\kappa s^2}{8\pi r^4} = Y_h,
\]
where
\[
Y_h = \frac{1}{3} (Y_T + Y_{TF}) + \frac{\kappa s^2}{8\pi r^4}.
\]
Upon integration, we found
\[
e^\nu = C_2 e^{\int 2r Y_h \left[1 - 2r^2 \int \frac{X_{TF}}{r} dr + C_1\right]^{-1} dr}.
\]
Here, again, $C_2$ symbolizes the integration constant which can be easily attainable from Eq. (15). The line element under these considerations takes the following form
\[
 ds^2 = C_2 e^{\int 2r Y_h \left[1 - 2r^2 \int \frac{X_{TF}}{r} dr + C_1\right]^{-1} dr} dt^2 - \left[1 - 2r^2 \int \frac{X_{TF}}{r} dr + C_1\right]^{-1} dr^2 - r^2 sin^2 \theta d\phi^2.
\]
From the last equation, we conclude that all the possible space-times exhibiting static anisotropic spheres can be expressed in terms of scalar functions $X_{TF}$ and $Y_h$.

### 4.3 Third Alternative

Here, we workout the line element which is expressible in terms of only the trace-free parts $X_{TF}$ and $Y_{TF}$ of $X_{\alpha\beta}$ and $Y_{\alpha\beta}$ respectively. Since we know that in the static case, we have
\[
 E = -\frac{e^{-\lambda}}{2} \left[ \frac{\nu''}{2} + \left( \frac{\nu'}{2} \right)^2 + \frac{\nu'}{2} \left( -\frac{\lambda'}{2} - \frac{1}{r} \right) + \frac{\lambda'}{2r} + 1 - \frac{e^\lambda}{r^2} \right]. \tag{57}
\]
Now, we introduce two new variables $y$ and $u$ as follows
\[
y = e^{-\lambda}, \quad \frac{\nu'}{2} = \frac{u'}{u}.
\]
Eq. (57) then takes the following form
\[
y' + 2y \left( \frac{u'' - \frac{u'}{r} + \frac{u}{r^2}}{u' - \frac{u}{r}} \right) = \frac{2u(1 - 2r^2 E)}{r^2 \left( u' - \frac{u}{r} \right)}.
\]
Performing integration w.r.t. $r$, we acquire
\[ y = e^{-\int k(r)dr} \left( \int e^{-\int k(r)dr} f(r)dr + c_1 \right), \] (58)
where, we have
\[ k(r) = 2 \frac{d}{dr} \left[ \ln \left( \frac{u'}{u} - \frac{u}{r} \right) \right], \quad f(r) = \frac{2u(1 - 2r^2E)}{r^2 \left( \frac{u'}{u} - \frac{u}{r} \right)}, \]
with $c_1$ being the constant of integration easily obtainable from junction conditions. In terms of original variables, we can have
\[ \frac{\nu'}{2} - 1 = \frac{e^{\lambda/2}}{r} \times \sqrt{(1 - 2Er^2) + c_1 r^2 e^{-\nu} + r^2 e^{-\nu} \int \frac{e^\nu}{r^2} (2r^2E)'dr}. \]

Now, introducing a new variable $z$ as follows
\[ e^\nu = e^{2 \int zdr} / r^2, \]
from which we attain the following result
\[ z(r) = \frac{\nu'}{2} + \frac{1}{r}. \] (59)

Making use of Eqs. (58) and (59), the newly introduced variable $z$ and the Weyl tensor $E$ are interlinked as
\[ z(r) = \frac{2}{r} + \frac{e^{\lambda/2}}{r} \sqrt{(1 - 2Er^2) + c_1 r^4 e^{-\int 2z(r)dr} + r^4 e^{-\int 2z(r)dr} \int \frac{e^\nu}{r^4} (2r^2E)'dr}. \] (60)

Next, utilizing field equations, we acquire
\[ \kappa \Pi - \frac{\kappa s^2}{4 \pi r^4} = e^{-\lambda} \left[ -\frac{\nu''}{2} - \left( \frac{\nu'}{2} \right)^2 + \frac{\nu'}{2r} + \frac{1}{r^2} \right] + \frac{\lambda e^{-\lambda}}{2} \left[ \frac{\nu'}{2} + \frac{1}{r} \right] - \frac{1}{r^2}. \]

Using the new variables $y$ and $z$, we attain
\[ y' + y \left[ \frac{2z'}{z} + 2z - \frac{6}{r} + \frac{4}{r^2 z} \right] = -\frac{2}{z} \left[ \kappa \Pi - \frac{\kappa s^2}{4 \pi r^4} + \frac{1}{r^2} \right]. \]
Performing integration w.r.t \( r \), the value of \( \lambda \) comes out to be

\[
e^{\lambda(r)} = \frac{z^2 e^{\int (2z + \frac{1}{z r^2}) \, dr}}{r^6 \left[ -2 \int \frac{z}{r^8} \left( \kappa \Pi r^2 - \frac{\kappa s^2}{4 \pi r^2} + 1 \right) e^{\int (2z + \frac{1}{z r^2}) \, dr} \, dr + C \right]},
\]

(61)

where \( C \) symbolizes integration constant. Using the scalars \( X_{TF} \) and \( Y_{TF} \), Eqs. (60) and (61) can also be written as

\[
z(r) = 2 + \frac{e^{\lambda/2}}{r} \sqrt{\left[ 1 - r^2 (Y_{TF} - X_{TF}) \right] + r^4 e^{-\int 2zd}\left( c_1 + \int [r^2 (Y_{TF} - X_{TF})] r^{-2} \int e^{2zd} \right)},
\]

(62)

and for \( \lambda \), we can have

\[
e^{\lambda(r)} = \frac{z^2 e^{\int (2z + \frac{1}{z r^2}) \, dr}}{r^6 \left[ -2 \int \frac{z}{r^8} \left( 1 + r^2 (Y_{TF} + X_{TF}) \right) e^{\int (2z + \frac{1}{z r^2}) \, dr} \, dr + C \right]},
\]

If we consider conformally flat fluids having anisotropic pressure, then \( Y_{TF} = X_{TF} \). Making use of this condition in Eq. (62), the value of \( z \) takes the following form

\[
z = \frac{2}{r} + \frac{e^{\lambda/2}}{r} \tanh \left( \int \frac{e^{\lambda/2}}{r} \, dr \right).
\]

5 Conclusion

Keeping in view the presence of electromagnetic field, a detailed study relating to self-gravitating dissipative spherically symmetric fluid is presented seeking the help of structure scalars that appear when we split the Riemann tensor orthogonally. We found five such scalars i.e., \( (X_T, X_{TF}, Y_T, Y_{TF}, Z) \) which reduces to two in number if we consider static and dissipation-less dust fluid with anisotropic stresses and only one for the case of static isotropic fluid distribution. We observe that \( Z \) and \( X_T \) delineate the dissipative flux and the energy density respectively. For dissipation-less fluid, the scalar \( X_{TF} \) administers the energy density inhomogeneity. The scalars \( Y_T \) and \( Y_{TF} \) appear in the definition of Tolman mass with \( Y_{TF} \) delineating the consequences of inhomogeneity of energy density and anisotropy of pressure on the Tolman mass of the proposed system and \( Y_T \) delineating the Tolman mass density. Considering the static case, we manipulated the Einstein equations as three ordinary differential equations with five unknowns. We have observed that the solutions of
the field equations can be completely characterized by these scalar functions in the presence of electromagnatic field like the charge free case. Some particular solutions have been illustrated to comprehend this argument and to signify the physical relevance of the obtained scalar functions.

We have found some solutions to Einstein-Maxwell equations in terms of structure scalars obtained from the splitting of the Riemann tensor representing static anisotropic spheres. It predicts about the formation and structure of stellar configurations including the fact that the electromagnetic field results in an increase in the mass of the static sphere (as predicted by set of equations in second alternative). From the astrophysical point of view, we have evaluated few results that are known to predict astrophysical activities taking place in the cosmos. One of the best examples is Tolman mass. Since, we know that expression for Tolman mass of a stellar system provides information about the total energy budget of that system, thus, the occurrence of the structure scalar $Y_{TF}$ in our calculated expression manifests that the presence of charge does affect the total energy of any stellar object. Moreover, the structural equations obtained above provide information about the formation, structure and gradual development of stellar objects. Structure scalars $(X_T, Y_T, X_{TF}, Y_{TF}, Z)$ appearing in these equations show that certain stellar configurations may exist that have electromagnetic fields around them. These results depict the effects of electromagnetic field on the structure of compact objects. So, we can say that the stellar equations we obtained using this formalism may represent the mathematics behind any static anisotropic astrophysical object out there in the cosmos.

Acknowledgments

The work of M. Z. Bhatti (PI) has been supported by National Research Project for Universities (NRPU), Higher Education Commission, Pakistan under research project No. 8769/Punjab/ NRPU/R&D/HEC/2017. Also, the authors would like to thank the Higher Education Commission, Islamabad, Pakistan for its financial support through the Indigenous Ph.D. Fellowship For 5000 Scholars, Phase-II, Batch-V.

References

[1] J. D. V. Arbañil and V. T. Zanchin, Phys. Rev. D 97, 104045 (2018).
[2] F. Weber, Prog. Part. Nucl. Phys. 54, 193 (2005).
[3] R. P. Negreiros, F. Weber, M. Malheiro and V. Usov, Phys. Rev. D 80, 083006 (2009).
[4] B. V. Ivanov, Phys. Rev. D 65, 104001 (2002).
[5] L. Herrera, A. Di Prisco, J. Ospino and J. Carot, Phys. Rev. D 82, 024021 (2010).
[6] L. Herrera, A. Di Prisco and J. Ospino, Gen. Relativ. Gravit. 42, 1585 (2010).
[7] L. Herrera, A. Di Prisco and J. Ibáñez, Phys. Rev. D 84, 107501 (2011).
[8] L. Herrera, J. Ospino, A. Di Prisco, E. Fuenmayor and O. Troconis, Phys. Rev. D 79, 064025 (2009).
[9] L. Herrera, A. Di Prisco, J. Martin, J. Ospino, N. O. Santos and O. Troconis, Phys. Rev. D 69, 084026 (2004).
[10] X. Calbet and R. López-Ruiz, Phys. Rev. E 63, 066116 (2001).
[11] R. Lopez-Ruiz, H.L. Mancini and X. Calbet, Phys. Lett. A 209, 321 (1995).
[12] L. Herrera, A. Di Prisco and J. Carot, Phys. Rev. D 99, 124028 (2019).
[13] L. Herrera, Phys. Rev. D 97, 044010 (2018).
[14] J. P. Crutchfield and K. Young, Phys. Rev. Lett. 63, 105 (1989).
[15] T. Grammenos, Astrophys. Space Sci. 31, 211 (1994).
[16] L. Herrera, J. L. Hernández-Pastora, J. Martin and J. Martinez, Class. Quantum Grav. 14, 2239 (1997).
[17] R. D. Blandford, J. H. Applegate and L. Hernquist, Mon. Not. R. Astron. Soc. 204, 1025 (1983).
[18] A. K. G. de Oliveira, N. O. Santos and C. A. Kolassis, Mon. Not. R. Astron. Soc. 216, 1001 (1985).
[19] Z. Yousaf, Eur. Phys. J. Plus 134, 245 (2019); Eur. Phys. J. Plus 132, 71 (2017); Mod. Phys. Lett. A 34, 1950333 (2019); Astrophys. Space Sci. 363, 226 (2018); M. Z. Bhatti, Z. Yousaf, M. Ilyas, J. Astrophys. Astr. 39, 69 (2018).
[20] B. Turimov, B. Ahmedov, M. Kološ and Z. Stuchlík, Phys. Rev. D 98, 084039 (2018).
[21] H. Quevedo, Phys. Rev. D 39, 2904 (1989).
[22] R. C. Tolman, Phys. Rev. 55, 364 (1939).
[23] N. F. Naidu, M. Govender and K. S. Govinder, Int. J. Mod. Phys. D 15, 1053 (2006).
[24] B. V. Ivanov, Eur. Phys. J. C 79, 520 (2019).
[25] L. Herrera, A. Di Prisco and J. Ospino, Gen. Relativ. Gravit. 42, 1585 (2010).
[26] L. Herrera, A. Di Prisco and J. Ospino, Phys. Rev. D 89, 127502 (2014).
[27] M. Z. Bhatti, Z. Yousaf and M. Yousaf, Phys. Dark Universe 28, 100501 (2020); M. Z. Bhatti and Z. Tariq, Phys. Dark Universe 28, 100482 (2020); Eur. Phys. J. Plus 134, 521 (2019); M. Z. Bhatti, Z. Yousaf and M. Nawaz, Int. J. Geomet. Meth. Mod. Phys. 17, 2050017 (2020); M.Z. Bhatti, Mod. Phys. Lett. A 33, 2050069 (2020); Eur. Phys. J. Plus 133, 431 (2018).
[28] L. Herrera, A. Di Prisco, J. Ospino and E. Fuenmayor, J. Math. Phys. 42, 2129 (2001).
[29] L. Herrera, J. Martin and J. Ospino, J. Math. Phys. 43, 4889 (2002).
[30] L. Herrera, A. Di Prisco, J. Hernández-Pastora and N. O. Santos, Phys. Lett. A 237, 113 (1998).