A note on non singular Einstein-Aether cosmologies

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\section*{Abstract}
An effective Lagrangian approach based on an extended Einstein-Aether (EA) model is presented and formulated in a generic non flat Friedman-Lemaître-Robertson-Walker (FLRW) space-time. For a flat FLRW space-time, a Friedmann equation similar to the one obtained in Quantum Loop Cosmology (QLC) is reproduced, with related non singular bounce solution. Finally, the Static Spherically Symmetric (SSS) solutions are investigated.

\section{Introduction}
The theory of General Relativity (GR), with a suitable cosmological positive constant and Dark Matter, describes remarkably well a large part of the history of the Universe, including the acceleration (or Dark Energy dominated) era. With the addition of cold dark matter, this is essentially the so called \textit{$\Lambda$CDM} model, or the standard cosmological model, which has been recently tested with high accuracy \cite{1,2}.

It is also well known that GR admits solutions corresponding to singular space-times, namely metrics whose scalar curvature invariants have singularities, or equivalently there exist geodesic incomplete metrics. It is believed that quantum corrections to GR might solve the singularity issue. In fact, in Quantum Loop Cosmology, an effective modified Friedmann equation has been obtained, whose solution in a flat Friedmann–Lemaître–Robertson–Walker (FLRW) space-times admits a \textit{bounce}, a solution without the GR Big Bang singularity \cite{3,4,5} (in this respect see also Ref. \cite{6}).

Our aim in this paper is to extend this effective approach to non flat FLRW space-times, namely
\begin{equation}
\begin{aligned}
ds^2 &= -N^2(t)dt^2 + a(t)^2 \left( \frac{d\rho^2}{1 - k \rho^2} + \rho^2 d\Omega_2^2 \right),
\end{aligned}
\end{equation}

where $a(t)$ is the scale factor, $N(t)$ is the lapse function, $k$ is the spatial curvature and $d\Omega_2$ is the two dimensional sphere metric.

When $k = 0$, and we are dealing with GR, with unconventional equation of state for the matter content, it is not difficult to propose models with bounce solutions at $t = 0$. In fact, consider the usual first Friedmann equation
\begin{equation}
\begin{aligned}
3H^2 &= \rho, \quad H(t) = \frac{\dot{a}}{a},
\end{aligned}
\end{equation}

and the matter conservation equation
\begin{equation}
\begin{aligned}
\dot{\rho} &= -3(\rho + p)H,
\end{aligned}
\end{equation}

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where we consider the units \(8\pi G_N = 1\), the dot denotes derivatives with respect to the time \(t\), \(H(t) = \dot{a}/a\) is the Hubble factor, \(\rho\) and \(P\) are respectively the energy density and pressure of the matter fluid. From the two equations above we can obtain the second Friedmann equation

\[
\dot{H} = -\frac{1}{2}(\rho + P).
\]  

(4)

As a result, assuming an equation of state

\[
P = \omega\rho,
\]  

(5)

and making use of Friedmann equations, we find

\[
H(t) = \frac{2}{3}\int \frac{1}{(1 + \omega)dt}.
\]  

(6)

In the \(\Lambda\)CDM model, the equation of state parameter \(\omega\) is constant and we obtain the GR singular Big Bang solution. However, assuming \(\omega\) time dependent we can obtain regular solutions. For example, considering (see for example \([7]\) and references therein),

\[1 + \omega(t) = c_1 - \frac{c_2}{t^\alpha},\]

(7)

we find

\[
H(t) = \frac{2}{3}\frac{(\alpha - 1)t^{\alpha-1}}{c_2 + (\alpha - 1)c_1t^\alpha},
\]  

(8)

where \(c_1, c_2\) and \(\alpha\) are fixed parameters. As a consequence, one has a bounce solution as soon as \(c_1, c_2 > 0\) and \(\alpha > 1\), such that \(H(0) = 0\) and \(\dot{H}(0) > 0\). However, one should note that the singularity here is present at \(t = 0\) in the pressure \(p\). In the case of flat \((k = 0)\) FLRW space-times, more realistic cosmological bounces models have been studied (see for example \([8, 9]\). In our approach, we will try to avoid the drawbacks related to the singularity of matter observables, working also in non flat FLRW space-times. We shall make use of a specific class of modified Lorentz-violating gravitational models called Einstein-Aether (EA) models \([10, 11]\). The original model contains four parameters which describe deviation from GR via the Aether vector field coupling with the metric. These parameters can be constrained using several experimental results, see for example references quoted in \([12]\). Furthermore, extended EA models have been proposed and investigated in several works, see for example Refs. \([13, 14, 15]\). In the original model, the problem of initial singularity has not been fully solved (see Ref. \([16, 17]\).

The content of this paper is the following. Firstly, in Sec. 2 we review the standard mimetic gravity theories in order to show similarities and differences with respect to the Einstein-Aether theory. Then we will present a specific extended EA model, and show that its associated generalized Friedmann equations, in general, may admit non-singular FLRW metrics as solutions. Moreover, some remarks on the existence of de Sitter (dS) solution and on the existence of Schwarzschild solution on Static Spherical Symmetric (SSS) space-time are also made.

Note that both the original EA model and the so-called mimetic gravitational model, and their extensions, which we will be reviewed in the next Section, contain vector and scalar fields with fixed four norm. In our approach, this constraint will be implemented using a Lagrangian multiplier approach.

If not otherwise stated, in this paper we will be making use of the convention \(c = 1\) and \(8\pi G_N = 1\).

## 2 Extended mimetic gravitational model

We start with a short review of mimetic gravitational models. We follow Refs. \([18, 19]\). The relevance of mimetic models stems from the fact that if one wants to work in \(D = 4\), and dealing with second order differential equations on FLRW space-time, one may use Horndeski model \([20]\), mimetic gravitational models \([21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]\), DOHST models \([39, 40]\), or Non Polynomial Gravity models \([41, 42]\). In this brief review, by means of the Lagrangian mini-superspace approach within the non flat FLRW space-times in Eq. \([1]\), we study an extended mimetic model introduced in Ref. \([43]\).
The action reads

$$I = \int_{\mathcal{M}} d^4 x \sqrt{-g} \left[ \frac{R}{2} + \lambda \left( X - \frac{1}{2} \right) + f[\phi(t)] \right] + I_m,$$

(9)

where $g$ is the determinant of the metric $g_{\mu\nu}$, $X \equiv -\frac{1}{2} g^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi$, $\lambda$ is a Lagrange multiplier field, $\phi$ is the mimetic scalar field, and $I_m$ is the usual matter-radiation action. The higher order differential term in $\phi$, depends on $\chi(\phi) = -\nabla^\mu \nabla_\mu \phi / 3$. With the metric $\left[ 1 \right]$, the action is a functional of $a(t)$, $N(t)$ and $\lambda(t)$. We assume $\phi = \phi(t)$, i.e., a dependency on $t$ only. We get the Lagrangian:

$$L = -6 \frac{\dot{a}^2 a}{N} + 6k a N + N a^3 f[\chi(\phi)] + N a^3 \lambda \left( \frac{\dot{\phi}^2}{2N^2} - \frac{1}{2} \right) + \frac{a^3 (\rho + p)}{N} - N a^3 (\rho - p),$$

(10)

where $\rho$ and $p$ are the energy density and pressure of matter and the dot denotes the time-derivative.

The variation with respect to the lapse function $N$, and considering $N = 1$ and $\phi = 1$ after the computation, gives the generalized first Friedmann equation

$$6 \left( H^2 + \frac{k}{a^2} \right) + f(H) - H \frac{df(H)}{dH} = 2\rho + 2\lambda - \frac{1}{3} \frac{d}{dt} \frac{df(H)}{dH},$$

(12)

where $f$ is now a function of $H$ because of Eq. $\left[ 11 \right]$ with $N = 1$. The variation with respect to the field $\phi$, leads after integration to $\left[ 13 \right]$.

$$\lambda = \frac{C}{a^3} + \frac{1}{6 \frac{d}{dt} \frac{df(H)}{dH}},$$

(13)

where $C$ is an integration constant mimicking the mimetic dark matter contribution. In the following, we put $C = 0$. Thus the first Friedmann equation $\left[ 14 \right]$ becomes

$$6 \left( H^2 + \frac{k}{a^2} \right) + f(H) - H \frac{df(H)}{dH} = 2\rho.$$

(14)

The variation with respect to the $a$ gives the generalized second Friedmann equation,

$$3 H^2 + 2 \dot{H} + \frac{f(H)}{2} - \frac{H}{2} \frac{df(H)}{dH} - \frac{1}{6} \frac{d}{dt} \frac{df(H)}{dH} = -p.$$

(15)

Deriving the Friedmann equation and making use of the above results, one gets the matter conservation equation

$$\dot{\rho} = -3H(\rho + p).$$

(16)

As a result, when $p = \omega \rho$ with $\omega$ constant, one obtains the well known solution

$$\rho(t) = \rho_0 a(t)^{-3(1+\omega)}.$$

(17)

We can make a choice for the arbitrary function $f$. For instance, consider the choice $\left[ 18 \right]$, $\left[ 19 \right]$.

$$f(H) = 6H^2 + \frac{12}{a^2} \left[ 1 - \sqrt{1 - \alpha^2 H^2} - \alpha H \arcsin (\alpha H) \right],$$

(18)

where $\alpha$ is a dimensional positive parameter. Since $f(H)$ goes to zero when $\alpha \rightarrow 0$, in this limit one recovers GR. Thus $f(H)$ may represent a “correction” to Einstein gravity.

From the first Friedmann equation $\left[ 14 \right]$ one has

$$\frac{6}{a^2} \left[ 1 - \sqrt{1 - H^2 \alpha^2} \right] = \rho - \frac{3k}{a^2},$$

(19)

which is equivalent to

$$3 H^2 = \left( \rho - \frac{3k}{a^2} \right) \left( 1 - \frac{(\rho - \frac{3k}{a^2})}{\rho_c} \right), \quad \text{where} \quad \rho_c = \frac{12}{a^2}.$$  

(20)
An alternative Lagrangian derivation within a mimetic approach, when \( k \) is not vanishing, is presented in Ref. [11] and references therein.

For \( k = 0 \), one has

\[
3H^2 = \rho \left( 1 - \frac{\rho}{\rho_c} \right).
\]

This is the QLC modified Friedmann equation in flat FLRW space-time. For an equation of state \( p = \omega \rho \), it admits a bounce solution. Here the critical density is given by \( \rho_c \). For other cosmological bounce solutions see Ref. [46, 47] and references therein. Furthermore, in the case \( \omega = -1 \), namely \( \rho = \rho_0 \), the above equation admits a flat \( k = 0 \) de Sitter solution.

Finally, we know that several mimetic models seem to be plagued by gradient and/or ghost instabilities, see for example Refs. [48, 49]. This is a reason why we are going to investigate an alternative with similar mimetic structure but different typology of main field, i.e. a vector and not a scalar like \( \phi \). These models are called AE extended models.

3 The Einstein–Aether extended model

In this section we propose an extended AE models defined on nonflat FLRW. We denote by \( u^\mu \) the time-like Aether 4-vector field. Both the original and the extended model depends on the invariant

\[
\mathcal{K} = c_1 (\nabla_\mu u^\nu)^2 + c_2 (\nabla_\mu u^\mu)^2 + c_3 \nabla_\mu u^\nu \nabla_\nu u^\nu + c_4 u^\alpha \nabla_\alpha u^\beta \nabla_\beta u_\mu.
\]

On a non flat FLRW, one has \( \mathcal{K} = 3\beta H^2 \) with \( \beta = c_1 + 3c_2 + c_3 \), where \( c_{1,2,3,4} \) are dimensional quantities. This justify the fact that, with regard to the background field equations of motion, we may consider the expansion \( \theta = -\nabla_\mu u^\mu = 3H \) as the main variable. In other words, \( \mathcal{K} = 3\theta^2/3 \). In the following we consider a generic space-time with \( \theta = -\nabla_\mu u^\mu \), and then study specific space-times applications.

The action for our AE extended model reads

\[
I = \int_M d^4x \sqrt{-g} \left[ \frac{R}{2} + \lambda (u_\mu u^\mu + 1) - f(\theta) \right] + I_m,
\]

where \( I_m \) is the matter action. In the original work of Ref.[10], \( f(\theta) = 3\mathcal{K} = \beta \theta^2 \). For completeness we present the equations of motion for the theory

\[
G_{\mu\nu} - \frac{1}{2} f(\theta) g_{\mu\nu} - \frac{1}{2\lambda} [2u_\mu u_\nu + g_{\mu\nu}(1 + u_\mu u^\mu)]
\]

\[
+ \frac{1}{2} g_{\mu\nu} \left[ \nabla_\sigma u^\sigma d^2 f(\theta) \right] + u^\sigma \nabla_\rho \nabla_\sigma u^\rho \frac{d^2 f(\theta)}{d\theta^2} = 0,
\]

\[
2\lambda u_\nu - g_{\rho\sigma} \nabla_\mu \nabla_\sigma u^\rho \frac{d^2 f(\theta)}{d\theta^2} = 0,
\]

\[
1 + u_\mu u^\mu = 0,
\]

which was obtained varying the action respectively with respect to the metric \( g^{\mu\nu} \), the vector field \( u^\mu \) and the mimetic scalar field \( \lambda \).

In the following sections we consider this theory on two different cases: the FLRW and static spherical symmetric space-times.

3.1 Friedmann–Lemaître–Robertson–Walker solutions

In this section we present the main results for a FLRW metric. In order to find the equations of motion, we can proceed in two ways. One way is evaluate the equations of motion \( \{24-26\} \). Here instead we use again the mini-superspace approach which was applied to the extended mimetic model in Sec. 2.

We consider \( u^\mu = (b, 0, 0, 0) \) the time-like Aether 4-vector, whose norm is given by \( u_\mu u^\mu = -N^2 b \). In the metric \( 1 \) with \( k \neq 0 \), we have

\[
\theta = -\nabla_\mu u^\mu = -\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} u^\mu) = 3H b + b \frac{N}{N} + \frac{d}{dt} b.
\]

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} u^\mu) = 3H b + b \frac{N}{N} + \frac{d}{dt} b,
\]
The related mini-superspace Lagrangian reads
\[
L = -\frac{\dot{a}^2 a}{N} + 6kaN - Na^3 f(\theta) + Na^3 \lambda \left(1 - N^2 b^2\right) + \frac{a^3 (\rho + p)}{N} - Na^3 (\rho - p). \tag{28}
\]

The variation with respect to \( \lambda \) leads to \( b = 1/N \). Therefore the Aether field norm is \( u_\mu u^\mu = -1 \), which is compatible with (29).

Making the variation with respect to \( N \), and then considering \( N = 1 \) and \( b = 1 \), we obtain the generalized Friedmann equation
\[
\frac{\dot{\theta}^2}{3} + \frac{3k}{a^2} - \frac{f(\theta)}{2} + \frac{\dot{\theta}}{2} \frac{d}{d\theta} f(\theta) = \rho + \lambda - \frac{d}{2dt} \frac{d f(\theta)}{d\theta}, \quad \text{where} \quad 3\frac{\dot{a}}{a} = 3H. \tag{29}
\]

The variation with respect to \( b \) leads to,
\[
2\lambda = \frac{d}{dt} \frac{df(\theta)}{d\theta}, \tag{30}
\]

such that equation (29) simply reads
\[
\frac{\dot{\theta}^2}{3} + \frac{3k}{a^2} - \frac{f(\theta)}{2} + \frac{\dot{\theta}}{2} \frac{d}{d\theta} f(\theta) = \rho, \quad \text{where} \quad 3\frac{\dot{a}}{a} = 3H. \tag{31}
\]

Finally, the variation with respect to \( a \) is the second Friedmann equation
\[
\frac{2\dot{\theta}}{3} - \frac{2k}{a^2} = -(\rho + p) - \frac{1}{2} \frac{d}{dt} \frac{d}{d\theta} f(\theta). \tag{32}
\]

From the equations above we can derive the matter conservation law
\[
\dot{\rho} = -3H (\rho + p). \tag{33}
\]

In fact, this equation follows by deriving the first Friedmann equation (29) with respect to the time \( t \) and making use of the second Friedmann equation (32). We note that the matter conservation law is identical both in GR and in the mimetic extended model considered in Sec. 2.

Consider now the following choice
\[
f(\theta) = \beta \theta^2. \tag{34}
\]

This is the same as considering \( f(\theta) = 3K \). With this choice the first Friedmann equation reads
\[
3H^2 \left(1 + \frac{\beta}{2}\right) + \frac{3k}{a^2} = \rho. \tag{35}
\]

We note that in order to obtain a positive matter energy density, we have to assume \( \beta + 2 > 0 \). This is the Friedmann equation for the original AE model, whose solutions have been investigated in Refs. [16, 17], where singular solutions have been found. In fact, as in GR, we can easily integrate the equation of motion when \( k = 0 \). In this case, making use of matter conservation law (33) with a barotropic matter fluid, i.e. \( p = \omega \rho \) with \( \omega \) constant, one has
\[
\dot{\rho} = -\sqrt{\frac{6}{\beta + 2} (1 + \omega) \rho^2}, \tag{36}
\]

the solution being
\[
\rho(t) = \frac{2(\beta + 2)}{3(1 + \omega) t^2}. \tag{37}
\]

Therefore we have the Big Bang singularity at \( t = 0 \), as in GR.
3.1.1 A regular bounce solution

In this section we present a regular bounce solution similar to the one made in the extended gravitational mimetic model in Eq. (13). The proposed function is

\[ f(\theta) = -\frac{2\theta^2}{3} - \frac{4}{3\alpha^2} \left[ 1 - \sqrt{1 - \alpha^2 \theta^2 - \alpha \theta \arcsin (\alpha \theta)} \right]. \] (38)

It should be noted that assuming the dimensional parameter \(\alpha\) very small, one has

\[ f(\theta) = \frac{1}{18} \alpha^2 \theta^4 + \ldots. \] (39)

Thus, with a small parameter \(\alpha\), \(f(\theta)\) represents a correction to the GR which starts with a second order contribution in the invariant \(K^2\).

Furthermore, with the \(f\) function choice \((38)\), the first Friedmann equation is

\[ 3H^2 = \left( \rho - \frac{3k}{\alpha^2} \right) - \frac{3\alpha^2}{4} \left( \rho - \frac{3k}{\alpha^2} \right)^2, \] (40)

where the critical density is \(\rho_c \equiv 4/3\alpha^2\). This is the same modified Friedmann equation obtained within the extended mimetic approach. Again, for \(k = 0\) one has

\[ 3H^2 = \rho \left( 1 - \frac{\rho}{\rho_c} \right). \] (41)

namely the QLC modified Friedmann equation in flat FLRW space-time.

For \(k = 0\) is also possible to integrate the matter conservation law. In fact, writing \(p = \omega \rho\), we obtain

\[ \dot{\rho} = -\sqrt{3(1 + \omega)} \rho \sqrt{\rho - \rho^2/\rho_c}, \] (42)

the solution being

\[ \rho(t) = \frac{\rho_c}{1 + \frac{\rho_c}{\rho} (1 + \omega)^2 t^2}. \] (43)

We note that the GR limit is recovered when \(\rho_c \to \infty\).

By means of Eq. (13) and matter conservation law, one gets the well known bounce solution for \(\omega \neq 1\),

\[ a(t) = \left( \frac{\rho_0}{\rho_c} + \frac{3}{4} \rho_0 (1 + \omega)^2 t^4 \right)^{-\frac{1}{3(1+\omega)}}, \] (44)

where \(\rho_0\) is an integration constant. Furthermore, in the case \(\omega = -1\), namely \(\rho = \rho_0\), the above equation \((14)\) admits a \(k = 0\) dS solution.

On the other side, when \(k \neq 0\), one has to make use of Eq. \((10)\), which may be rewritten as

\[ 3H^2 = \rho - \frac{3k}{\alpha^2} - \frac{(\rho - \frac{3k}{\alpha^2})^2}{\rho_c}. \] (45)

In general we can show that the Big-Bang singularity at \(t = 0\) is absent. Firstly, we give an example of exact solution. Consider the barotropic equation of state \(p = -\rho/3\), i.e. the equation of state parameter is \(\omega = -1/3\). As a result, \(\rho(t) = \rho_0 a(t)^{-2}\). It is convenient to introduce the quantity \(y(t) = a^2(t)\). In this case, equation \((10)\) becomes

\[ \frac{3}{4} y^2 = \left( \rho_0 - 3k \right) y - \left( \rho_0 - 3k \right)^2 \frac{1}{\rho_c}. \] (46)

The related solutions are

\[ y(t) \equiv a^2(t) = \frac{\rho_0 - 3k}{\rho_c} + \left[ C \pm \sqrt{\frac{\rho_0 - 3k}{3}} t \right]^2, \] (47)

where \(C\) is an arbitrary dimensionless constant of integration. Moreover we assume \(\rho_0 > 3k\) in order to obtain a real solution. These are regular bounce solutions, with \(a(0) \neq 0\). When \(C = 0\), the regular solutions become a unique symmetric bounce solution, namely

\[ a(t)^2 = \frac{\rho_0 - 3k}{\rho_c} + \frac{\rho_0 - 3k}{3} t^2. \] (48)
The related density is also regular. When \( \rho_c \) goes to infinity, one recovers the GR solution, admitting the Big Bang singularity.

For a generic \( \omega \), it is not easy to find an exact solution. Alternatively, one may start separating the variable in equation (44) with \( y = a^2 \), namely

\[
\int \frac{dy}{\sqrt{Y(y)}} = t ,
\]

(49)

where we used the matter conservation law and defined

\[
Y(y) = \frac{4 \rho_0}{3} \left( y^{1/2-3\omega/2} - \frac{\rho_0}{\rho_c} y^{-(1+3\omega)} - \frac{3k}{\rho_0} + \frac{6k}{\rho_c} y^{-\left(1/2+3/2\omega\right)} - \frac{9k^2}{\rho_c \rho_0} \right) .
\]

(50)

If \( k \) is not vanishing, the above integral is only exactly computable only for \( \omega = -1/3 \).

However, if we make an expansion around the critical point defined by \( Y(y_*) = 0 \), namely

\[
y_*^{1/2-3\omega/2} - \frac{\rho_0}{\rho_c} y_*^{-(1+3\omega)} - \frac{3k}{\rho_0} + \frac{6k}{\rho_c} y_*^{-\left(1/2+3/2\omega\right)} - \frac{9k^2}{\rho_c \rho_0} = 0 ,
\]

(51)

we can search for an approximate solution, valid for small \( t \), which is given by

\[
y(t) \simeq y_* + \frac{Y'}{4} t^2 ,
\]

(52)

where

\[
Y' = \frac{4 \rho_0}{3} \left( \frac{1-3\omega}{2} y_*^{1/2-3\omega/2} - \frac{\rho_0}{\rho_c} (1+3\omega) y_*^{-(1+3\omega)} - \frac{3k}{\rho_0} + \frac{3k(1+3\omega)}{\rho_c} y_*^{-\left(1/2+3/2\omega\right)} \right) .
\]

(53)

In the above equation, \( y_* \) is solution of the transcendental equation (51). It is easy to show that for \( \omega = -1/3 \), one gets the approximate solution related to the exact solution found before. If \( k = 0 \) one recovers \( y_* = (\rho_0/\rho_c)^{2/(3(1+\omega))] \), such that

\[
y(t) \simeq \left( \frac{\rho_0}{\rho_c} \right) \left( \frac{2}{2} \right)^{1/2} + \frac{1}{2} \rho_0 (1 + \omega) \left( \frac{\rho_0}{\rho_c} \right) \left( \frac{1+3\omega}{2} \right) t^2 ,
\]

(54)

which is consistent with the result in Eq. (44).

Thus, from Eq. (52) we can conclude that one has to deal with a regular symmetric bounce as soon as \( y_* > 0 \) and \( Y' > 0 \). In the limit \( \rho_c \to \infty \) and \( k = 0 \), \( \omega \neq -1 \), \( y_* = 0 \) and the Big-bang singularity obviously appears.

### 3.1.2 Cosmological perturbation theory

In this section we present the cosmological perturbation theory results in extended AE models within flat FLRW space-times, which has been also investigated in Refs. [30, 51]. We refer to the perturbed FRLW metric in Newtonian gauge

\[
d \tilde{s}^2 = -\left[ 1 + \Psi(t, \vec{r}) \right] dt^2 + a(t)^2 \left[ 1 + \Phi(t, \vec{r}) \right] \left( \frac{d\phi^2}{1-k\rho^2} + \rho^2 d\Omega_2^2 \right) ,
\]

(55)

where \( \Psi \) and \( \Phi \) are the non-homogeneous and non-isotropic Newtonian potentials.

Firstly, we notice that the perturbation of \( \theta \) and \( \mathcal{K} \) at first order are not the same. In fact the perturbation of \( \theta^2 \) at the first order is

\[
\delta \theta^2 = 18H^2 \delta u_0^2 + 6H \delta \dot{u}_0 + 6H \dot{\Psi} - 18H \dot{\Phi} ,
\]

(56)

where \( \delta \dot{u}_0 \) is the perturbation of the first component of the Aether 4-vector field. On the contrary, the first order perturbation of \( 3\mathcal{K} \) is given by

\[
3\delta \mathcal{K} = 6 \beta H^2 \delta \dot{u}_0 + 18 c_2 \dot{H} \delta u_0^2 + 6 c_2 H \dot{\Psi} - 6 \beta H \dot{\Phi} .
\]

(57)

Since at the background level we identified \( 3\mathcal{K} = \beta \theta^2 = (c_1 + 3c_2 + c_3) \theta^2 \), we can conclude that the theory formulated in terms of \( \mathcal{K} \) and \( \theta \) are equivalent at the first order perturbation level only if we consider \( c_1 + c_3 = 0 \), i.e. \( \beta = 3c_2 \).
We have generalized the propagation speed of gravitational waves to non flat FLRW spacetimes. The result is the same as in the flat case, namely
\[ c_T^2 = \frac{1}{1 + (c_1 + c_4) \frac{dr}{dt}}. \] (58)
As a result, if we require \( c_T^2 = 1 \), one has \( c_1 + c_4 = 0 \) such that \( \beta = 3c_2 \). Therefore, every model which uses \( \theta \) instead of \( K \) will always satisfy \( c_T^2 = 1 \) regardless of the choice of \( f(\theta) \).

Considering \( \beta = 3c_2 \), in accordance with [50] we obtain
\[ c_T^2 = \frac{c_2}{3(c_1 + c_4)} \left( 1 + \frac{\theta^2}{\theta^2} \right), \] (59)
which in general it is required to be positive to avoid gradient instabilities and smaller than the speed of light (\( c_T^2 < 1 \) in our units) to avoid superluminarity.

For example, in the case of the original AE model \( f(\theta) = \beta \theta^2 \), with \( c_1 + c_4 = 0 \) and \( \beta = 3c_2 \), we get \( c_T^2 = 1 \) and
\[ c_T^2 = \frac{2c_2}{3(c_1 + c_4)}. \] (60)
If we consider our choice
\[ f(\theta) = -\frac{2\theta^2}{3} - \frac{4}{3\alpha^2} \left[ 1 - \sqrt{1 - \alpha^2 \theta^2} - \alpha \theta \arcsin (\alpha \theta) \right], \] (61)
we unsurprisingly obtain again \( c_T^2 = 1 \) (see discussion around Eq. [58]) and
\[ c_T^2 = \frac{c_2}{(c_1 + c_4)} \left( 1 + \frac{1}{\sqrt{1 - \alpha^2 \theta^2} - 1} \right). \] (62)
Since \( \alpha \theta < < 1 \), one gets
\[ c_T^2 = \frac{2c_2}{(c_1 + c_4)} \left( 1 + \frac{\alpha^2 \theta^2}{6} + \ldots \right). \] (63)
Thus, one has a small correction to the original AE model.

However, it is known that in the original AE model, the weak field limit and non relativistic approximation lead an effective Newton constant given by [53],
\[ G_e = G_N \left( 1 - \frac{c_1 + c_4}{2} \right), \] (64)
where \( G_N \) is the usual Newton constant used in GR. As a consequence, one has to consider \( c_1 + c_4 \) very small and proportional to \( c_2 \).

### 3.2 Static Spherical Symmetric solutions

In this section, we briefly investigate the existence of Static Spherical Symmetric (SSS) solutions in a specific extended AE models.

We consider the SSS space-time in the following form,
\[ ds^2 = -A(r)^2 B(r) dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_2, \] (65)
where \( d\Omega_2 \) is the two dimensional sphere metric, and \( A \equiv A(r), B \equiv B(r) \) are functions of the radial coordinate \( r \) only. By assuming \( u^\mu \equiv u^\mu(r) \), the Lagrange multiplier constraint [29], among with the field equations [27], gives \( u^\mu = (0, \sqrt{-B}, 0, 0) \). Therefore, in general the vector field is imaginary on SSS space-times, similarly to the static case of mimetic gravity (see for example Ref. [34]). Thus we have
\[ \theta^2 = -\frac{1}{4r^2} \left( \frac{2rA/A + 2B + rB'}{B} \right)^2 < 0. \] (66)
As mentioned before, for the sake of simplicity, we shall restrict our AE static models to depend only on the scalar \( K = -\theta^2 \), \( \theta \) being an imaginary quantity. Furthermore, it is convenient to work with \( \theta \) instead of \( K \), but with \( f(\text{theta}) \) a real quantity.
From (24)-(25) we obtain the equations of motion

\[ 1 - B(r) - \frac{r dB}{dr} - r^2 f + r \sqrt{-B(r)} \frac{df}{d\theta} \left( 2 + \frac{r dA}{A dr} + \frac{r dB}{2B dr} \right) + \]

\[ + \frac{d^2 f}{d\theta^2} \left[ 2B - r \frac{dB}{dr} + \frac{r^2 B}{A^2} \left( \frac{dA}{dr} \right)^2 - \frac{r^2 dB}{2A dr} + \frac{r^2}{4B} \left( \frac{dB}{dr} \right)^2 - \frac{r^2 B}{A} \frac{d^2 A}{dr^2} - \frac{1}{2} \frac{r^2 dB}{dr^2} \right] = 0, \]

(67)

(68)

In the following we present two examples with different \( f(\theta) \) function choices.

### 3.2.1 Linear case

In this example we consider the Schwarzschild gauge with \( A(r) = 1 \), and \( f \) linear in \( \sqrt{K} \). Thus, our first example is \( f(\theta) = \gamma \theta \). We investigate this model in order to check our formalism. In fact, this model corresponds to the additive term \( \sqrt{K} \): since it’s a divergence of a 4-vector, it is a trivial correction to GR.

With this choice, the equations of motions reduce to the single equation

\[ 1 - B(r) - r \frac{dB}{dr} - r^2 f + r \sqrt{-B(r)} \frac{df}{d\theta} \left( 2 + \frac{r dA}{A dr} + \frac{r dB}{2B dr} \right) = 0. \]

(69)

Since

\[ \theta = -\frac{AB + r dB}{2r \sqrt{-B}}, \]

(70)

and \( f(\theta) = \gamma \theta \), the last two terms cancel and one has

\[ 1 - B(r) - r \frac{dB}{dr} = 0. \]

(71)

Therefore, the exact solution is given by the Schwarzschild solution

\[ B(r) = \left( 1 - \frac{C}{r} \right), \]

(72)

where \( C \) is a mass term.

### 3.2.2 Quartic case

In this section we consider a quadratic model in \( K \), mimicking our model in Eq. (38) with \( \alpha^2 \ll 1 \). The function \( f(\theta) \) given by (39). At the zeroth order in \( \alpha \) the model is GR, thus it admits the Schwarzschild solution. At the second order in \( \alpha \) one may look for a solution in the form,

\[ A^2 = 1 + \alpha^2 \bar{A}(r), \quad B(r) = 1 - \frac{C}{r} + \alpha^2 \bar{B}(r), \]

(73)

where we consider \( \bar{A} \) and \( \bar{B} \) as perturbations of the main functions. By solving the first equation of (67), we find that at the second order in \( \alpha \) the dependence on \( \bar{A}(r) \) disappears and the solution of \( \bar{B}(r) \) is given by

\[ \bar{B}(r) = \frac{C_0}{r} + \frac{1}{96r^2} \left[ 81C^2 - \frac{261C}{r^2} + \frac{249}{r} + \frac{3}{r - C} + \frac{4}{C} \log \left( \frac{r}{r - C} \right) \right], \]

(74)

where \( C_0 \) is a new integration constant we may choose to be vanishing. Finally, by using the second equation in (67) at the second order, one finds the expression for \( \bar{A}(r) \) through the
solution for $\tilde{A}(r)$,

$$
\tilde{A}(r) = \frac{1}{96C} \left\{ \frac{3C^2}{2(r - C)^4} + \frac{3(3 + C)}{(C - r)^2} + \frac{44 - 19C}{C(C - r)} + \frac{108C(5 + C)}{r^3} \\
+ \frac{54 - 72C}{r^2} + \frac{90 - 111C}{Cr} + \frac{C(16 + 3C)}{3(r - C)^3} \\
- \frac{2}{C^2(r - C)} \log \left( \frac{r}{r - C} \right) \left[ 23r(2C - 1) + C(27 - 44C) + 2(C - r) \log \left( \frac{r}{r - C} \right) \right] \right\},
$$

where we consider $C_0 = 0$. We note that these functions are badly divergent for $r \to C$, and their regime of validity is in the limit $\alpha^2C \ll r$, for which we recover the corrections to GR. Thus, these results cannot be used to find corrections to the GR horizon. Moreover, we note that for $r \to \infty$ we obtain $\tilde{A}, \tilde{B} \to 0$, i.e. the large $r$ asymptotically flat GR limit.

4 Conclusions

In this paper we have investigated an effective Lagrangian approach based on an extended Einstein-Aether model in a generic non flat FLRW space-time. Making a suitable choice for the Lagrangian, namely the Einstein-Hilbert one with the addition of non polynomial correction, a generalized Friedmann equation has been obtained. In the case of a flat FLRW space-time, we were able to reproduce the Quantum Loop Cosmology Friedmann equation. In the case of non flat FLRW space-times, the presence of non singular metrics has been found. Cosmological perturbations of the model has been discussed, showing its reliability. Finally, the Static Spherically Symmetric solutions have been investigated. We found that the Schwarzschild solution can be found with a suitable choice of Lagrangian. Moreover, with a different choice, a theory which consists in correction to Schwaerzshild can be obtained.

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