The first integrals and exact solutions of a two-component Belousov–Zhabotinskii reaction system

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Abstract. The popular Belousov–Zhabotinskii (BZ) system of equations for description of a two component reaction is considered. The Painlevé test is applied to determine integrability of this system. It is shown that the system of equations is nonintegrable in general case. The parameter values of the mathematical model are found for the case when the system of equations passes the Painlevé test. Simplest solutions of the system of equations are presented. Additional conditions are specified when the general solutions of the system can be found. These general solutions are found using the new generalized method for finding exact solutions and the first integrals. Two first integrals of the Belousov–Zhabotinskii reaction system are given at additional conditions on parameter values for the mathematical model.

1. Introduction
The reaction known as the Belousov–Zhabotinskii reaction was discovered by the russian biochemist Boris Belousov in 1951. He noted the first reaction with the Ce$^{3+}$/Ce$^{4+}$ couple as a catalyst and citric acid as a reductant. He observed that the color of the reaction solution oscillated between colorless and yellow and found that the frequency of oscillations increased with the rise of temperature. Belousov made two attempts to publish his discovery, but he failed. His work has been finally published only eight years later.

The study of this reaction continued when professor Simon Shnol invited his graduate student Anatoly Zhabotinskii to investigate the reaction mechanism. Zhabotinskii’s group conducted detailed studies of the reaction, including various options and the first mathematical model, which was able to show oscillatory behavior.

The first model of the reaction was obtained in 1967 by Zhabotinskii and Korzukhin. It was based on the selection of empirical relations correctly describing oscillations in the system. The mechanism proposed by Field and Noyes (Oregonator model) [1] is one of the simplest and at the same time is the most popular in the research works investigating behavior of the BZ reaction. This model shows simple oscillations similar to the experimentally observed, but it can not show more complex types of oscillations, such as complex-periodic and chaotic. The model of Showalter, Noyes and Bar-Eli (Modified Oregonator model) [2] was developed to simulate the complex-periodic and chaotic behavior of the BZ reaction.

A good and varied introduction to the reaction can be found in the book of articles edited by Field and Burger [3]. It also has several articles on chemical oscillators and wave phenomena.
Winfree [4] discusses some of the reaction’s properties, both temporal and spatial. A brief review of the detailed reaction and its properties is given by Tyson [5]. Some other questions corresponding to the study of the Belousov–Zhabotinskii reaction system were considered in works [6, 7, 8, 9, 10, 11].

In this paper we follow a simplified reaction mechanism (based on the Noyes–Field model) suggested by Murray [12, 13, 14]. The key chemical elements in the BZ reaction are

\[ X = \text{HBrO}_2, \quad Y = \text{Br}^-, \quad A = \text{BrO}_3^-, \quad P = \text{HBrO}, \] (1)

and the model reactions can be approximated by the sequence

\[ X + Y \rightarrow X + P, \quad X + Y \rightarrow 2P, \quad A + X \rightarrow 2X, \quad 2X \rightarrow P + A \] (2) with the known rate constants \( k_1, k_2, k_3, k_4. \)

So, applying the law of mass action to reactions (2) we get the second-order system of kinetics equations for the concentrations of the bromous acid and bromide ion, denoted by \( U \) and \( V \) [13], [14], [15]

\[
\frac{dU}{dt} = k_1 V - k_2 U V + k_3 U - k_4 U^2, \quad \frac{dV}{dt} = -k_1 V - k_2 U V, \] (3)

where \( t \) is the time and \( k_i \)'s are rate constants. We assume that reactants do not diffuse in the media. The compound \( \text{HBrO} \) does not appear in our analysis and the concentration \( A \) (\( \text{BrO}_3^- \)) is constant.

We use the variables and parameters further in the form

\[ U = \frac{k_3}{k_4} u, \quad V = \frac{k_3}{k_2} v, \quad t = \frac{t'}{k_3}, \quad \alpha = \frac{k_1}{k_2}, \quad \beta = \frac{k_2}{k_4}, \quad \delta = \frac{k_1}{k_3}. \] (5)

For simplicity we omit the primes and obtain the following system

\[
\frac{du}{dt} = \alpha v + u (1 - u - \beta v), \quad \frac{dv}{dt} = -\delta v - \beta u v. \] (6)

The rest of this work is organized as follows. In Section 2 we apply the Painlevé test to understand analytical properties of system (6). In Section 3 we found two partial cases of the general solution for the Belousov–Zhabotinskii system at \( \beta = 0 \) and \( \beta = 1 \). Using the standard simplest equation method we present one-parameter exact solutions of the Belousov–Zhabotinskii system in Section 4. The key moment of this Letter is given in Section 5, where we present a new generalized approach for finding exact solutions and first integrals of nonlinear ordinary differential equations. In this Section we find the general solutions of the Belousov–Zhabotinskii system for some parameter values of mathematical model and obtain the first integrals of this system.

2. An application of the Painlevé test to the Belousov–Zhabotinskii reaction system

Detailed investigation of the integrability and the Painlevé property for low-dimensional systems was suggested in paper [16]. Here we apply the Painlevé test for understanding the integrability of the Belousov–Zhabotinskii reaction system of equations because to the best of our knowledge
the Painlevé test was not used for consideration of this system. To apply the Painlevé test to (6) we transform the system to one equation. From the last equation of system (6) we get

\[ u = -\frac{\delta}{\beta} - \frac{v_t}{v}, \quad (\beta \neq 0). \] (8)

(The case \( \beta = 0 \) is considered in the next Section). Substituting (8) into the first equation of system (6) we have the second-order differential equation in the form

\[ vv_{tt} - \left(1 + \frac{1}{\beta}\right) v_t^2 - \left(1 + \frac{2\delta}{\beta}\right) v v_t + \beta v^2 v_t + \beta (\alpha + \delta) v^3 - \left(\delta + \frac{\delta^2}{\beta}\right) v^2 = 0. \] (9)

Let us apply the Painlevé test to study the integrability of equation (9) using three steps of the Kovalevsky algorithm [17, 18].

At the first step we look for the first member of the general solution expansion in the Laurent series substituting

\[ u = a_0 t^p. \] (10)

into equation (9). Note that (9) is autonomous and we keep in mind that we can change \( t \to t - t_0 \) in all results (where \( t_0 \) is an arbitrary constant).

Substituting (10) into the equation with the leading members

\[ vv_{tt} - \left(1 + \frac{1}{\beta}\right) v_t^2 + \beta v^2 v_t = 0 \] (11)

we get \( p = 1 \) and \( a_0 = \frac{\beta - 1}{\beta^2} \).

At the second step of the Painlevé test we substitute the expression in the form

\[ u = \frac{\beta - 1}{\beta^2 t} + u_j t^{1-j}, \] (12)

into the equation with the leading members (11) and look for linear expression with respect to \( u_j \). Here the coefficient \( u_j \) is one of coefficients for the general solution expansion in the Laurent series. As a result we obtain the Fuchs indices which are determined by formulas

\[ j_1 = -1, \quad j_2 = \frac{\beta - 1}{\beta}. \] (13)

Taking into account the values of the Fuchs indices we see that equation (9) does not pass the Painlevé test in the general case. However it can have the Painlevé property in case of

\[ \beta = \frac{1}{1-N}, \quad (N \neq 1). \] (14)

In this case the second Fuchs index will be integer.

Substituting the expansion in the form

\[ u(t) = \frac{\beta - 1}{\beta^2 t} + a_1 + a_2 t + a_3 t^2 + \ldots. \] (15)

into equation (9) and equating the expressions at different powers \( t \) to zero, we get

\[ a_1 = -\frac{\alpha \beta^2 + \beta^2 + \delta \beta^2 - 2\alpha \beta - \beta - \delta + \alpha}{2\beta^2}. \] (16)
At $\beta = -1$ the coefficient $a_1 = -2\alpha - 1$ but the coefficient $a_2$ is not determined. Equation (9) passes the Painlevé test only in two cases

$$\delta_1 = -2\alpha, \quad \delta_2 = -2\alpha - 1.$$  \hfill (17)

In case of $\beta = -\frac{1}{2}$ the coefficients $a_1$ and $a_2$ are determined. They take the form

$$a_1^{(1)} = -\frac{3}{2}\delta - \frac{9}{2}\alpha - \frac{3}{2}, a_2^{(1)} = -\frac{3}{4}\delta - \frac{9}{8}\delta^2 - \frac{21}{4}\alpha\delta - \frac{15}{4}\alpha - \frac{81}{8}\alpha^2 - \frac{1}{2}.$$  \hfill (18)

However the coefficient $a_3$ can not be found. The equation passes the Painlevé test when $\delta$ is equal to

$$\delta_3 = -3\alpha, \quad \delta_4 = -3\alpha - 1.$$  \hfill (19)

We can hope that this regularity is carried out for the next members of the expansion for the general solution.

One can see that it is difficult to determine the arbitrary constant $a_j$ at large $j$ because we need to calculate all previous coefficients. So we suggest checking the arbitrary constant at the third step of the Painlevé test to use the possible form of the first integral.

We know that the value of the Fuchs index obtained at the second step by Painlevé test corresponds to the order of poles for leading members of the first integral [19], [20]. Moreover it is well known that the Painlevé test is a necessary condition for integrability of nonlinear differential equations. And if we find the first integrals of equation (9) we will obtain the sufficient condition of integrability for this equation. We discuss this property in Section 5.

### 3. Solutions of the Belousov–Zhabotiskii system at $\beta = 0$ and $\beta = 1$

The second equation in the Belousov–Zhabotinskii system (6) at $\beta = 0$ can be easily solved. In this case we can write the equation for $u(t)$ in the form

$$\frac{du}{dt} = u - u^2 - \frac{C_1^2\delta^2}{4}e^{-\delta t},$$  \hfill (20)

where an arbitrary constant $C_1$ is taken in this form for convenience of calculations.

The general solution of equation (20) is expressed by formula

$$u = -\frac{\psi_1}{\psi},$$  \hfill (21)

where $\psi(t)$ is expressed via the Bessel function in the form

$$\psi(t) = e^{-\frac{1}{2}t}\left[C_2 J_{-\frac{1}{2}}\left(C_1 e^{\frac{\delta t}{2}}\right) + C_3 Y_{-\frac{1}{2}}\left(C_1 e^{\frac{\delta t}{2}}\right)\right].$$  \hfill (22)

In case of $\beta = 1$ from (9) we obtain the following equation

$$v v_{tt} - 2 v_t^2 - (1 + 2\delta) v v_t + v^2 v_t + (\alpha + \delta) v^3 - \left(\delta + \delta^2\right) v^2 = 0.$$  \hfill (23)

From 23 at $\delta = 0$ or $\delta = -1$ we obtain the first integral of equation 9 in the form

$$v_t + (1 + 2\delta) v + v^2 \ln(v) + (\alpha + \delta) v^2 t = C_4 v^2,$$  \hfill (24)

Additionally in case of $\alpha = -\delta$ we obtain the solution in the form of integral

$$\int \frac{dv}{C_4 v^2 - (1 + 2\delta) v - v^2 \ln v} = t + C_5,$$  \hfill (25)

where $C_5$ is an arbitrary constant.
4. One-parameter exact solutions of the Belousov–Zhabotinskii reaction system

At the present time there are a lot of approaches for finding exact solutions of nonlinear differential equations (see, for example, papers [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32]). However, their application to the Belousov–Zhabotinskii equation can give only one-parameter exact solutions. To find these exact solutions of equation (9) let us use the simplest equation method [22].

The general solution of equation (9) has the first order of pole and we can look for the exact solution of this equation in the form

\[ v(t) = a_1 Q(t), \]  

(26)

where \( Q(t) \) is a solution of the Riccati equation in the form

\[ Q_t - A Q + Q^2 = 0. \]  

(27)

Formula (26) was taken because the general solution of equation (27) also has the first order. Differentiating equation (27) in \( t \) we get that the solution of equation (27) also satisfies the following equation

\[ Q_{tt} - A^2 Q + 3 A Q^2 - 2 Q^3 = 0. \]  

(28)

Substituting (26) into equation (9) and taking into account the derivatives \( Q_t \) and \( Q_{tt} \) from equations (27) and (28) we obtain the polynomial in \( Q(t) \) in place of differential equation (9). Equating expressions in this polynomial at the same degrees of \( Q(t) \) to zero, we have the algebraic equations with respect to parameters \( a_1, A, \alpha, \beta \) and \( \delta \).

Solving these equations we obtain

\[ a_1 = \frac{\beta - 1}{\beta^2}. \]  

(29)

We also get two cases. One of them is

\[ \delta_1 = \frac{\alpha}{\beta} - \alpha - 1, \quad A_1 = 1 + \alpha - \frac{\alpha}{\beta} \]  

(30)

and the second one is the following

\[ \delta_2 = \frac{\alpha}{\beta} - \alpha, \quad A_2 = \alpha - \beta - \frac{\alpha}{\beta}. \]  

(31)

Substituting these values of parameters into equation (27) and into (26) we have exact solutions of equation (9).

In the first case at \( \delta = \delta_1 = \frac{\alpha}{\beta} - \alpha - 1 \) we have the solution of equation (9) in the form

\[ v_1(t) = \frac{(\beta - 1)(\alpha \beta + \beta - \alpha)}{\beta^3 + C_1 \beta^2 (\alpha \beta + \beta - \alpha) e^\frac{(\alpha - \beta - \alpha) t}{\beta}}. \]  

(32)

and in the case of \( \delta = \delta_2 = \frac{\alpha}{\beta} - \alpha \) the solution takes the form

\[ v_2(t) = \frac{(\beta - 1)(\alpha \beta - \beta^2 - \alpha)}{\beta^3 + C_1 \beta^2 (\alpha \beta - \beta^2 - \alpha) e^\frac{(\alpha + \beta^2 - \alpha) t}{\beta}}. \]  

(33)

Here \( C_1 \) is an arbitrary constant.

Solutions (32) and (33) are kinks in time.
Other solutions of equation (9) can be found in cases of $A_1 = 0$ ($\alpha = \frac{\beta}{1-\beta}$, $\delta_1 = 0$) and at $A_2 = 0$ ($\alpha = \frac{\beta^2}{\beta^2-1}$, $\delta_2 = -\beta$). These rational solutions of equations (27) and (9) take the form

$$Q(t) = \frac{1}{t + C_1}, \quad v(t) = \frac{\beta - 1}{\beta (t + C_1)}.$$  

Solutions (32), (33) and (34) have only one arbitrary constant of the second–order ordinary differential equation and are not the general solutions of equation (9).

5. First integrals and the general solutions of the Belousov–Zhabotiskii reaction system

Let us consider the generalization of the simplest equation that allows us to obtain the first integrals and the general solution of equation (9). First of all let us note that the leading members of equation (27) have the second order of pole. Moreover the Fuchs index for the general solution of equation (9) at $\beta = -1$ is equal to $N = 2$. It is known that the arbitrary coefficient in the expansion of the general solution in the Laurent series corresponds to the leading members of the first integral [19], [20]. Using this observation let us look for the first integral of equation (9) at $\beta = -1$ assuming [21]

$$v(t) = b_1 Q(t),$$  

where $Q(t)$ satisfies the equation in the form

$$Q_t = A Q - Q^2 + S(t).$$  

As this takes place $S(t)$ is an unknown function of $t$ that can be found.

Differentiating (36) in $t$ we have

$$Q_{tt} = A^2 Q - 3 A Q^2 + A S + 2 Q^3 - 2 Q S + S_t.$$  

Substituting (35) into equation (9) and taking into account (36) and (37), after equating expressions at the same degrees of $Q(t)$ to zero we have the algebraic system of equation. Solving this system of equations we get

$$b_1 = -2, \quad A = 2 \alpha + 1, \quad \delta_1 = -2 \alpha - 1, \quad \delta_2 = -2 \alpha.$$  

and

$$S_1 = C_1 e^{(2\alpha+1)t}, \quad S_2 = C_1 e^{2\alpha t}.$$  

The first integral at $\beta = -1$ and at $\delta = \delta_1 = -2 \alpha - 1$ is expressed by formula

$$v_t - (2 \alpha + 1) v - \frac{1}{2} v^2 = C_1 e^{(2\alpha+1)t}.$$  

and the first integral at $\beta = -1, \delta = \delta_2 = -2 \alpha$ takes the form

$$v_t - (2 \alpha + 1) v - \frac{1}{2} v^2 = C_1 e^{2\alpha t}.$$  

Equations (40) and (41) are reduced to the linear equations of the second order using the transformation

$$v(t) = \frac{\psi_2}{\psi}.$$  

In fact the general solutions of equation (9) can be reduced to the nonlinear second-order differential equation which was studied by Painlevé school more than century ago [33], [34], [35], [36], so this example has an illustrative purpose.

Let us note that equation (9) can be transformed to the Abel equation of the second order that can be solved at conditions (38).

Let us also demonstrate the application of our approach for the value of the Fuchs index $N = -3$. In this case $\beta = \frac{1}{4}$ and we search for the first integral assuming $v(t) = b_1 Q$, where $Q(t)$ satisfies the equation in the form

$$Q^{-5} Q_t - AQ^{-4} + Q^{-3} = S(t).$$

From (43) we get

$$Q_t = A Q - Q^2 + Q^5 S(t).$$

Differentiating (44) in $t$ we obtain the consequence of equation (44) in the form

$$Q_{tt} = 5 S^2 Q^9 - 7 S Q^6 + (6 A S + S_t) Q^5 + 2 Q^3 - 3 A Q^2 + A^2 Q.$$

Substituting (35) into equation (9) and using equations (44) and (45) we obtain a polynomial in $Q(t)$ again. Equating expressions at the same degrees $Q(t)$ to zero, after solving the algebraic system of equations we get

$$b_1 = -12, \quad \delta_1 = 3 \alpha - 1, \quad \delta_2 = 3 \alpha, \quad A_1 = 1 - 3 \alpha, \quad A_2 = -3 \alpha - \frac{1}{4},$$

and functions $S_1(t)$ and $S_2(t)$ in the form

$$S_1(t) = C_1 e^{(12 \alpha - 3) t}, \quad S_2(t) = C_1 e^{12 \alpha t}. $$

We have found that at $N = -3$ equation (9) passes the Painlevé test and, what is more, we get the sufficient condition for the Painlevé property at values of parameters (46). As a result we have the two first integrals for equation (9) at $\beta = \frac{1}{4}$.

The first integral at $\delta_1 = 3 \alpha - 1$ takes the form

$$v_t - (3 \alpha - 1) v - \frac{1}{12} v^2 = C_2 v^5 e^{(12 \alpha - 3) t},$$

and at $\delta_2 = 3 \alpha$ the first integral can be written as

$$v_t - \left(3 \alpha + \frac{1}{4}\right) v - \frac{1}{12} v^2 = C_2 v^5 e^{12 \alpha t},$$

where $C_2$ is an arbitrary constant. We can use this approach to check the third step of the Painlevé test for equation (9) in case of other values of the Fuchs index $j_2$.

Taking into consideration the observations mentioned above we can look for the first integral of equation (9) in the form

$$v^{\delta + 1} - v + A v^{\frac{1}{\beta}} + B v^{\frac{\beta - 1}{\beta}} = S_n(t),$$

where the parameters $A$ and $B$ are unknown. We can find them if we substitute derivatives $v_t$ and $v_{tt}$ into equation (9).

We obtain at $B = \frac{\beta^2}{\beta - 1}$ the following values of parameters $\delta$ and $A$

$$\delta_1 = \frac{\alpha}{\beta} - 1 - \alpha, \quad A_1 = 1 + \alpha - \frac{\alpha}{\beta}, \quad \delta_2 = \frac{\alpha}{\beta} - \alpha, \quad A_2 = \alpha - \beta - \frac{\alpha}{\beta}. $$
Using the values of parameters (51) and the form of first integrals (50) we get the two first integrals of equation (9) for two values $\delta$.

At $\delta_1 = \frac{\alpha}{\beta} - 1 - \alpha$ we have the first integral in form

$$v^{-\frac{\beta+1}{\beta}} v_t + \left(1 + \alpha - \frac{\alpha}{\beta}\right) v^{-\frac{1}{\beta}} + \frac{\beta^2}{\beta - 1} v^{\frac{\beta-1}{\beta}} = I_1 e^{\frac{(\alpha-\beta)(\beta-1)}{\beta^2} t}, \quad (\beta \neq 0; 1) \quad (52)$$

and at $\delta_2 = \frac{\alpha}{\beta} - \alpha$ we get in the form

$$v^{-\frac{\beta+1}{\beta}} v_t + \left(\alpha - \beta - \frac{\alpha}{\beta}\right) v^{-\frac{1}{\beta}} + \frac{\beta^2}{\beta - 1} v^{\frac{\beta-1}{\beta}} = I_2 e^{\frac{(\beta-1)\alpha}{\beta^2} t}, \quad (\beta \neq 0; 1) \quad (53)$$

where $I_1$ and $I_2$ are arbitrary constants.

These first integrals can also be written in the another form.

At $\delta = \delta_1 = \frac{\alpha}{\beta} - \alpha - 1$ the first integral takes the form

$$v_t + \left(1 + \alpha - \frac{\alpha}{\beta}\right) v + \frac{\beta^2}{\beta - 1} v^2 = I_1 v^{\frac{\beta+1}{\beta}} e^{\frac{(\beta-1)(\beta-\alpha)}{\beta^2} t}, \quad (\beta \neq 0; 1) \quad (54)$$

and at $\delta = \delta_2 = \frac{\alpha}{\beta} - \alpha$ we get the first integral in the form

$$v_t + \left(\alpha - \beta - \frac{\alpha}{\beta}\right) v + \frac{\beta^2}{\beta - 1} v^2 = I_2 v^{\frac{\beta+1}{\beta}} e^{\frac{(\beta-1)\alpha}{\beta^2} t}, \quad (\beta \neq 0; 1). \quad (55)$$

From the first integrals (54) and (55) we see that at $\delta = \delta_1$ in case of $\alpha < 1$ and $\alpha < \beta < 1$ we have a dissipative process for the first integral of the Belousov–Zhabotinski reaction system. In case of $\delta = \delta_2$ we observe the dissipation in the first integral at $\alpha > 0$ and $\beta < 1$ or at $\alpha < 0$ and $\beta > 1$.

In case of $\delta = \delta_1$ and $\alpha = \beta$ (or $\delta = \delta_2$ and $\alpha = 0$) the general solution of equation (9) is found from the first integrals (54) and (55) and is determined by the formula

$$t + C_6 = \int \frac{(\beta - 1) dv}{I_1 (1 - \beta) v^{\frac{\beta+1}{\beta}} + \beta^2 v^2 + \beta (1 - \beta) v} \quad (56)$$

where $C_6$ is an arbitrary constant.

**Table 1.** Exact solutions of the Belousov–Zhabotinski system at given parameters

| n | $\alpha$ | $\beta$ | $\delta$ | v(t) |
|---|---|---|---|---|
| 1 | $\alpha$ | 0 | $\delta$ | (18) |
| 2 | $\alpha$ | 1 | 0 | (23) |
| 3 | $\alpha$ | 1 | $\frac{\beta}{\alpha} - 1$ | (23) |
| 4 | $\alpha$ | $\beta$ | $\frac{\beta}{\alpha} - \alpha - 1$ | (30) |
| 5 | $\alpha$ | $\beta$ | $\frac{\beta}{\alpha} - \alpha$ | (31) |
| 6 | $\alpha$ | $\beta$ | 0 | (32) |
| 7 | $\frac{\alpha}{\beta}$ | $\beta$ | $-\beta$ | (32) |
| 8 | $\alpha$ | $\alpha$ | $-\alpha$ | (54) |
| 9 | 0 | $\beta$ | 0 | (54) |

Exact solutions of the two–component Belousov–Zhabotinski equation (9) are given at Table 1.
6. Conclusion
In this paper we have considered the integrability of the Belousov–Zhabotinskii reaction system using the Painlevé approach. We have found that this system could not be integrable in general case, and we tried to look for conditions on parameters of this system when it can be integrable. Using the Painlevé test we have obtained conditions on parameter $\beta$ of the mathematical model when the system can pass the Painlevé test. Taking into account the standard simplest equation method for finding exact solutions we have obtained two exact solutions of the considered system. Using the link between Fuchs indices and first integrals of nonlinear differential equations we suggested the modification in the third step of the Painlevé test for nonlinear differential equations. We have also found the first integrals of the Belousov–Zhabotinskii reaction system for some values of parameters $\delta$ taking into account the values of the Fuchs indices. It is worth to note that we used the Painlevé approach to find the first integrals of the Belousov–Zhabotinskii reaction system. But we obtained that there are the first integrals for values of parameters when equation (9) does not pass the Painlevé test.

Acknowledgments
This research was supported by Russian Science Foundation Grant No 18-11-00209 "Development of methods for investigation of nonlinear mathematical models”.

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