Unification of M- and F-Theory Calabi-Yau Fourfold Vacua

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Abstract

We consider splitting type phase transitions between Calabi-Yau fourfolds. These transitions generalize previously known types of conifold transitions between threefolds. Similar to conifold configurations the singular varieties mediating the transitions between fourfolds connect moduli spaces of different dimensions, describing ground states in M- and F-theory with different numbers of massless modes as well as different numbers of cycles to wrap various p-branes around. The web of Calabi-Yau fourfolds obtained in this way contains the class of all complete intersection manifolds embedded in products of ordinary projective spaces, but extends also to weighted configurations. It follows from this that for some of the fourfold transitions vacua with vanishing superpotential are connected to ground states with nonzero superpotential.

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1. Introduction

It has been a longstanding problem to formulate a dynamics on the collective moduli space of string theory which would allow to determine the physical ground state of the string. A first step in this direction would be to have a criterion which distinguishes between different vacua in an intrinsic manner. Such a criterion has recently been found by Witten \[1\] in the context of M– and F–theory \[2–7\]. Compactifications of these theories to three and four dimensions involve eight-dimensional manifolds, in particular Calabi-Yau fourfolds. This observation has sparked considerable interest in the previously little investigated class of Kähler fourfolds with vanishing first Chern class. Attention so far has focused mostly on orbifolds \[10–13\] and more general complete intersection spaces \[14–21\]. Witten’s observation shows that different ground states of F– and M–theory lead to non-vanishing (vanishing) superpotential because of the (non)existence of certain types of divisors in the internal Calabi–Yau space.

The natural question then arises whether the moduli space of Calabi-Yau fourfolds is connected so that M- and F-theory can conceivably ‘tunnel’ between these different types of ground states. A simple argument shows that this is to be expected, at least for certain types of fourfolds. Special among four-dimensional Calabi-Yau manifolds CY\(_4\) are fibered spaces for which the generic (quasi-)smooth fiber is a Calabi-Yau threefold CY\(_3\). For such fibrations we can use the known conifold transitions between threefolds \[22–24\], or more severe transitions described by operations on the toric data \[25,26\], to induce a transition in the fourfold by degenerating the fibers pointwise. The Hodge numbers of the fibers change in this process and therefore we expect to be able to link in this way the moduli spaces of cohomologically distinct fourfolds. The connectedness of the collective moduli space of Calabi-Yau threefolds therefore immediately implies the connectedness of at least some regions of the moduli space of fourfolds.

The class of CY\(_3\) fibered fourfolds is further distinguished because even though in F-theory on such spaces we are considering N=1 supersymmetric theories in D=4 we expect that for this type of manifolds many of the N=2 results carry over by a fiber-wise application via the adiabatic limit argument of \[27\] or the twist map construction of \[28,16\]. Using the duality results of \[2–7,29\] then makes it clear in particular that many of the physical aspects of the threefold transitions \[30–47\] will have F-theory and M-theory

\[\text{o Two notable exceptions are refs. }^{\text{8,9}}\]
counterparts, perhaps by utilizing heterotic string models based on the (0,2) Calabi-Yau threefolds considered in [48].

In the present paper we will take the first steps in this direction by showing that large classes of Calabi-Yau fourfolds are connected. In Section 2 we generalize to fourfolds the determinantal conifold transition between ordinary projective complete intersection Calabi-Yau threefolds [22] and its weighted extension [24]. Whereas in the case of threefolds the degenerations are rather mild, involving only conifold configurations with a finite number of nodes, the higher dimensional transitions we are going to describe proceed via singular varieties which involve degenerations for which the singular sets are two-dimensional, described in general by disconnected configurations of algebraic curves. Similar to the case of threefolds, however, the singular sets can be resolved in two different ways, either by deforming the degenerate variety along some complex modulus or by performing a small resolutions. Each of these ways to resolve the singularities leads to a (quasi-)smooth Calabi-Yau manifold with a different Hodge diamond. In Section 3 we describe a second splitting construction for fourfolds which is based on considerations of discriminantal varieties, discussed in the framework of ordinary projective complete intersection Calabi-Yau threefolds in [49].

The web of Calabi-Yau fourfolds obtained by these splitting constructions contains as a subset the class of all complete intersection manifolds embedded in products of ordinary projective spaces. This web is further extended by connecting it to the collective moduli space of weighted complete intersection spaces. All the constructions described in the present paper are independent of the fiber structure of the varieties involved and the spaces connected may or may not be fibered.

In the final Sections we apply splitting to define transitions between F&M-vacua on fourfolds and show that direct splits can connect ground states with zero superpotential to those with non-vanishing superpotential. It turns out that it is the small resolution of certain configurations of singular curves contained in the degenerate varieties which mediate the transitions that can generate divisors with the needed properties to give rise to a non-vanishing superpotential. This leads to the possibility that certain splitting type phase transitions between Calabi-Yau fourfolds lead to supersymmetry breaking.
2. Splitting Transitions between Fourfolds

Our focus in the following will be on complete intersection manifolds contained in configurations of the type

$$\mathbb{P}(k_1^1, \ldots, k_{n_1+1}^1) \times \mathbb{P}(k_1^2, \ldots, k_{n_2+1}^2) \times \cdots \times \mathbb{P}(k_1^F, \ldots, k_{n_F+1}^F)$$

such configurations describe the intersection of the zero locus of $N$ polynomials embedded in a product of weighted projective spaces, where $N = \left( \sum_{i=1}^{F} n_i - 4 \right)$ is the number of polynomials $p_a$ of F-degree $(d_1^a, \ldots, d_N^a)$. Even though our considerations can be applied to general intersection spaces our main interest is in manifolds for which the first Chern class

$$c_1(X) = \sum_{i=1}^{F} \left[ n_i + 1 - \sum_{a=1}^{N} d_i^a \right] h_i$$

vanishes. Here we denote by $h_i, i = 1, \ldots, F$ the pullback of the generators of $H^2(\mathbb{P}(k_1^i, \ldots, k_{n_i+1}^i))$. Useful for the following will be the remaining Chern classes of weighted complete intersection Calabi-Yau fourfolds

$$c_2(X) = \frac{1}{2} \left[ \sum_{a=1}^{N} \left( \sum_{i=1}^{F} d_i^a h_i \right)^2 - \sum_{i=1}^{F} \sum_{r=1}^{n_i+1} (k_r^i h_i)^2 \right]$$

$$c_3(X) = -\frac{1}{3} \left[ \sum_{a=1}^{N} \left( \sum_{i=1}^{F} d_i^a h_i \right)^3 - \sum_{i=1}^{F} \sum_{r=1}^{n_i+1} (k_r^i h_i)^3 \right]$$

$$c_4(X) = \frac{1}{4} \left[ \sum_{a=1}^{N} \left( \sum_{i=1}^{F} d_i^a h_i \right)^4 - \sum_{i=1}^{F} \sum_{r=1}^{n_i+1} (k_r^i h_i)^4 + 2c_2^2 \right].$$

It follows from the structure of $c_4$ that the Euler numbers of complete intersection fourfolds embedded in products of ordinary projective spaces is always positive.

2.1. Determinantal Splitting Transitions

There are several different types of transitions which one can construct between Calabi-Yau fourfolds. As the closest analog to the conifold transition one might consider the situation in which the fourfolds degenerate into varieties for which the singularities are...
again described by a finite number of singular points. Even in the case of threefolds it is not a simple matter in general to compute the resolution data and to check whether the resolved spaces are in fact again of Calabi-Yau type. A detailed investigation of the latter problem in the context of threefolds can be found in [50]. It is for this reason that the conifold transitions of splitting type are particularly simple - since they connect weighted complete intersection CYs per construction these global problems are resolved automatically. We therefore focus in the following on the four-dimensional generalization of the splitting construction.

\(\mathbb{P}_1\) \textit{Splits}

Consider the weighted complete intersection varieties of type (2.1). Introducing two vectors \(u, v\) such that \((u^i + v^i) = d_1^i\) and denoting the remaining \((F \times (N - 1))\)-matrix by \(M\), we write these spaces as \(Y[(u + v) M]\). The simplest kind of transition is the \(\mathbb{P}_1\)-split which is defined by

\[
X = Y[(u + v) M] \longleftrightarrow \mathbb{P}_1^Y \left[ \begin{array}{ccc} 1 & 1 & 0 \\ u & v & M \end{array} \right] = X_{\text{split}}. \quad (2.4)
\]

The split variety of the rhs is described by the polynomials of the original manifold and two additional polynomials, which we can write as

\[
p_1 = x_1 Q(y_i) + x_2 R(y_i)
\]

\[
p_2 = x_1 S(y_i) + x_2 T(y_i), \quad (2.5)
\]

where \(Q(y_i), R(y_i)\) are of multi-degree \(u\) and \(S(y_i), T(y_i)\) are of degree \(v\). In (2.5) we collectively denote the coordinates of the space \(Y\) by \(y_i\) whereas the \(x_i\) are the coordinates of the projective line \(\mathbb{P}_1\).

Insight into the precise relation of these two manifolds is obtained by comparing their Euler numbers, which can be obtained by Cherning. Since we are only interested in the local geometry of the transition we neglect for the moment possible orbifold singularities\(^\dagger\) and suppose that the ambient space is a product of ordinary projective spaces \(Y = \prod_{i=1}^F \mathbb{P}_{n_i}\). We denote the Kähler form of the split factor \(\mathbb{P}_1\) by \(H_0\) and the Kähler form of the \(i^{th}\) factor by \(H_i\). The Euler numbers are then obtained by integrating the top

\(^\dagger\) In general it is possible in weighted manifolds for hypersurface singularities to sit on top of orbifold singularities. In this case the following formulae have to be modified in analogy to the analysis of ref. [24] for threefolds.
form over the ambient space. Because the generators $H_i$ of $H^2(\mathbb{P}_{n_i})$ are normalized such that $\int_{\mathbb{P}_{n_i}} H_i^{n_i} = 1$ the coefficient of $H_0 \prod_i H_i^{n_i}$ is precisely the Euler characteristic. Using the formulae (2.3) one can then show that the difference between the Euler numbers of the two manifolds of the split (2.4) are related by

$$\chi(X_{\text{split}}) - \chi(X) = -3(u + v)u^2v^2 \prod a=2 \sum i=1 d^i_a H_i,$$

where we have abused notation by writing the first two components $N_a$ of the normal bundle $\mathcal{N} = \oplus_{a=0}^{N} N_a$ of the split manifolds as $c_1(N_0) = (H_0 + u)$ and $c_1(N_1) = (H_0 + v)$ with

$$u = \sum_{i=1}^{F} d^0_i H_i, \quad v = \sum_{i=1}^{F} d^1_i H_i.$$  

The result (2.6) shows that the split manifold describes a resolution of the determinantal variety in $Y[(u + v) \ M]$ defined by the original polynomials and the determinantal polynomial

$$Y[(u + v) \ M] \ni X^2 = \{p_{\text{det}} =QT - RS = 0, \quad p_a = 0, \quad a = 2, \ldots, N\}$$

which can be viewed as the projection $\pi : X_{\text{split}} \rightarrow X^2$ along the projective line $\mathbb{P}_1$.

To see this one notes that the hypersurface (2.8) is singular on the locus

$$\Sigma = Y[u \ u \ v \ v \ M]$$

which describes (generically) a curve because $Y[(u + v) \ M]$ is four-dimensional. The Euler number of this curve $\Sigma$ however can be determined via Cherning to be

$$\chi(\Sigma) \prod_{i=1}^{F} H_i^{n_i} = -(u + v)u^2v^2 \prod_{a=2}^{N} \sum_{i=1}^{F} d^i_a H_i,$$

hence we obtain the Euler number relation

$$\chi(X_{\text{split}}) = \chi(X) + 3 \chi(\Sigma).$$

We therefore see that we can smooth out the determinantal variety in two different ways: first by adding an appropriate deformation to

$$p_{\text{def}} = p_{\text{det}} + t \cdot p_{\text{trans}}$$
which deforms the non-transverse determinantal polynomial into a transverse polynomial $p_{\text{def}}$. In contrast to threefolds $[22, 23]$, where the singular locus of the splitting transition is formed by a number of nodes, i.e. to a conifold configuration, for fourfolds the singular locus is an algebraic curve, i.e. a real two-dimensional surface with, in general, several components. The important point however is that the singular set again admits a small resolution which, for fourfolds, involves the projective plane $\mathbb{P}_2$ instead of the projective line $\mathbb{P}_1$ of the threefold. Performing such a small resolution leads to the higher codimension split manifold. Thus we arrive at the same singular space by degenerating two distinct manifolds in different ways

$$X \rightarrow X^\sharp \leftarrow X_{\text{split}}.$$  

Put differently, we can start from a determinantal variety and smooth out the singularities in two distinct ways

$$X \leftrightarrow X^\sharp \rightarrow X_{\text{split}}.$$  

Important for the general picture is the following generalization of the determinantal $\mathbb{P}_1$ split.

$\mathbb{P}_n$ Splits
A similar discussion applies to the generalized $\mathbb{P}_n$-split

$$X = Y \left[ \sum_{a=1}^{n+1} u_a M \right] \leftrightarrow \mathbb{P}_n \left[ \begin{array}{cccc} 1 & u_1 & \cdots & u_{n+1} \\ u_1 & u_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{array} \right] = X_{\text{split}} \quad (2.12)$$

for which the Euler relation takes the form

$$\left( \chi(X_{\text{split}}) - \chi(X) \right) \prod_{i=1}^{F} H_i^{n_i}$$

$$= \frac{3}{10} \left[ 4 \sum_{a} u_a^3 \sum_{b < c} u_b u_c + 6 \sum_{a} u_a^2 \sum_{b < c < d} u_b u_c u_d ight. \left. + \sum_{a} u_a \left( 4 \sum_{b < c < d < e} u_b u_c u_d u_e - \sum_{\# \{b, c, d, e\} > 1} u_b u_c u_d u_e \right) \right] \prod_{\beta} \xi_{\beta} \quad (2.13)$$

where

$$u_a = \sum_i d^i_a H_i$$
for $a = 1, ..., (n+1)$ and $\xi_\beta$ are the corresponding columns of matrix $M$

$$\xi_\beta = \sum_i m_i^\beta H_i.$$  

Fourfold splits have a different local degeneration structure than threefold splits but they share certain features of these lower-dimensional counterparts. Most importantly the general $\mathbb{P}_n$ splits immediately allow to connect all complete intersection Calabi-Yau fourfolds embedded in products of ordinary projective spaces

$$
\begin{bmatrix}
    d_1^1 & d_2^1 & \cdots & d_N^1 \\
    d_1^2 & d_2^2 & \cdots & d_N^2 \\
    \vdots & \vdots & \ddots & \vdots \\
    d_1^F & d_2^F & \cdots & d_N^F
\end{bmatrix}
$$

(2.14)

to a particularly simple configuration, given by

$$
\begin{bmatrix}
    2 \\
    2 \\
    2 \\
    2 \\
    2
\end{bmatrix}_{1440}
$$

with Euler number $\chi = 1440$ which can be determined via Cherning. From Lefshetz’ hyperplane theorem we know that $h^{(1,1)} = 5$ and $h^{(2,1)} = 0$. The dimension of $H^{(3,1)}$ for this manifold can be determined by counting complex deformations with the result $h^{(3,1)} = 227$. From the Euler number we can then determine that final remaining Hodge number to obtain the complete Hodge diamond

$$
\begin{array}{cccc}
    & 1 & \\
0 & 0 & 0 & \\
0 & 5 & 0 & 0 \\
1 & 227 & 972 & 227 & 1.
\end{array}
$$

(2.16)

This result is consistent with the Hodge number constraint for Calabi-Yau fourfolds

$$44 + 4h^{(1,1)} + 4h^{(3,1)} - 2h^{(2,1)} - h^{(2,2)} = 0$$

(2.17)

pointed out in [15].

Starting from any of the configurations (2.14) one simply applies the $\mathbb{P}_1$ split to any projective factor with $n_i > 1$ until all corresponding $d_i^a = 1$ at which point the $\mathbb{P}_n$ is contracted.
As in the case of threefolds \([22]\) we also encounter the phenomenon of ineffective splitting in those situations when the determinantal variety is actually smooth. This can happen even though generically we have a higher dimensional singular set. An example which illustrates this is given by

\[
\begin{bmatrix}
\mathbb{P}_1 & 2 & 0 \\
\mathbb{P}_2 & 2 & 1 \\
\mathbb{P}_3 & 0 & 4 \\
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
\mathbb{P}_1 & 0 & 1 & 1 \\
\mathbb{P}_2 & 0 & 1 & 1 \\
\mathbb{P}_3 & 4 & 0 & 0 \\
\end{bmatrix}
\]

(2.18)

for which the relevant set is

\[
\begin{bmatrix}
\mathbb{P}_1 & 0 & 1 & 1 & 1 \\
\mathbb{P}_2 & 1 & 1 & 1 & 1 \\
\mathbb{P}_3 & 4 & 0 & 0 & 0 \\
\end{bmatrix} = \emptyset.
\]

(2.19)

A complicating feature of splitting transitions between fourfolds however is that in contradistinction to threefold splits there exists the possibility of nontrivial splits, or contractions, which connect manifold with the same Euler number. Whereas in the case of threefolds a split at constant Euler number is necessarily ineffective, providing different configurations of the same underlying manifold, it is clear from (2.11) that nontrivial fourfold splits can occur when the singular set consists of a configuration of tori. An example of such a transition at constant Euler number \(\chi = 396\) is given by

\[
\begin{bmatrix}
\mathbb{P}_2 & 3 & 0 \\
\mathbb{P}_4 & 1 & 4 \\
\end{bmatrix}^{396}
\leftrightarrow
\begin{bmatrix}
\mathbb{P}_1 & 0 & 1 & 1 \\
\mathbb{P}_2 & 3 & 0 & 0 \\
\mathbb{P}_4 & 1 & 1 & 3 \\
\end{bmatrix}^{396}
\]

(2.20)

Here the determinantal variety degenerates at the configuration

\[
\begin{bmatrix}
\mathbb{P}_2 & 3 & 0 & 0 & 0 & 0 \\
\mathbb{P}_4 & 1 & 1 & 1 & 3 & 3 \\
\end{bmatrix},
\]

(2.21)

describing nine tori.

2.2. Examples

The perhaps simplest example of a splitting transition is the split of the sextic

\[
\begin{bmatrix}
\mathbb{P}_5[6]^{2610} \\
\mathbb{P}_5^{2160} \\
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
\mathbb{P}_1 & 1 & 1 \\
\mathbb{P}_5 & 1 & 5 \\
\end{bmatrix}^{2160}
\]

(2.22)
where the smooth hypersurface can be defined by the Fermat polynomial

\[ p = \sum_i z_i^6 \]

and a transverse choice of the split configuration is provided by

\[
\begin{align*}
  p_1 &= x_1 y_1 + x_2 y_2 \\
  p_2 &= x_1 \left( y_2^6 + y_4^6 + y_6^6 \right) + x_2 \left( y_1^6 + y_3^6 + y_5^6 \right).
\end{align*}
\]

(2.23)

Again the subscripts indicate the Euler numbers, the latter of which can be obtained by resolving the singular set of the determinantal variety, given by the genus \( g = 76 \) curve \( \Sigma = \mathbb{P}^3[5,5] \). More precisely the split (2.22) connects the Hodge diamond of the sextic hypersurface

\[
\begin{array}{cccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 426 & 1752 & 426 & 1.
\end{array}
\]

(2.24)

with the Hodge diamond

\[
\begin{array}{cccc}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 0 \\
1 & 350 & 1452 & 350 & 1.
\end{array}
\]

(2.25)

of the codimension two complete intersection manifold of (2.22).

More interesting splits are obtained by following the strategy described in the introduction, i.e. by considering fourfolds which are CY3 fibered. Transitions of this type can be obtained as follows. Consider the weighted threefold splits of (24)

\[
\begin{align*}
\mathbb{P}(k_1,k_1,k_2,k_3,k_4)[d] & \leftrightarrow \mathbb{P}(1,1)_{(k_1,k_1,k_2,k_3,k_4)} \mathbb{P}(1,1,1,k_{k_1},k_{k_1},k_{k_2},k_{k_3},k_{k_4}) \\
& \quad \left[ \begin{array}{cc}
1 & 1 \\
1 & (d-k_1)
\end{array} \right]
\end{align*}
\]

(2.26)

with \( d = 2k_1+k_2+k_3+k_4 \). These threefolds can be used to construct CY3–fibered fourfolds via the twist map [28][16]. The generic fiber of such manifolds then is a quasismooth Calabi-Yau threefold. Let \( \ell = d/k_4 \in 2\mathbb{N} + 1 \). For the hypersurfaces of (2.26) this amounts to choosing the curve \( C_\ell = \mathbb{P}(2,1,1)[2\ell] \) and applying the twist map

\[
\begin{align*}
\mathbb{P}(2,1,1,2k_1,2k_2,2k_3,4,4,4)[d] & \rightarrow \mathbb{P}(2,1,1,2k_1,2k_2,2k_3,4,4,4)[2d]
\end{align*}
\]

(2.27)

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defined as
\[
((x_1, x_2, x_3), (y_1, y_2, y_3, y_4, y_5)) \mapsto \left( y_1, y_2, y_3, y_4, x_2 \sqrt{\frac{y_5}{x_1}}, x_3 \sqrt{\frac{y_5}{x_1}} \right).
\] (2.28)

For the codimension two threefold in (2.26) the twist map produces the complete intersection fourfolds
\[
\mathbb{P}_{(2,1,1)}[2\ell] \times \mathbb{P}_{(k_1, k_1, k_2, k_3, k_4)}^{(1,1)} \begin{bmatrix}
1 & 1 \\
k_1 & (d - k_1)
\end{bmatrix} \rightarrow \mathbb{P}_{(2k_1, 2k_1, 2k_2, 2k_3, k_4, k_4)}^{(1,1)} \begin{bmatrix}
1 & 1 \\
2k_1 & 2(d - k_1)
\end{bmatrix}.
\] (2.29)

From this we see that the twist map applied to threefolds which are connected via conifold transitions induces splitting transitions between fibered fourfolds
\[
\mathbb{P}_{(2k_1, 2k_1, 2k_2, 2k_3, k_4, k_4)}[2d] \leftrightarrow \mathbb{P}_{(2k_1, 2k_1, 2k_2, 2k_3, k_4, k_4)}^{(1,1)} \begin{bmatrix}
1 & 1 \\
2k_1 & 2(d - k_1)
\end{bmatrix}.
\] (2.30)

Of special interest in this context are fibrations for which the generic threefold fiber is itself a K3-fibration\(^\dagger\) whose generic fiber in turn is an elliptic fibration. Such fourfolds thus are particularly simple elliptic fibrations which are of use in F-theory. An example is given by the weighted split
\[
\mathbb{P}_{(8,8,4,2,1,1)}[24] \leftrightarrow \mathbb{P}_{(8,8,4,2,1,1)}^{(1,1)} \begin{bmatrix}
1 & 1 \\
8 & 16
\end{bmatrix},
\] (2.31)

where the lhs manifold is defined by the zero locus of the polynomial
\[
p = z_0^3 + z_1^3 + z_2^6 + z_3^{12} + z_4^{24} + z_5^{24} = 0
\] (2.32)

and the rhs by the equations
\[
p_1 = x_1 y_1 + x_2 y_2
\]
\[
p_2 = x_1 (y_2^2 + y_4^8 + y_6^{16}) + x_2 (y_1^2 + y_3^4 + y_5^{16}).
\] (2.33)

The determinantal variety
\[
p_{det} = y_1(y_2^2 + y_4^8 + y_6^{16}) - y_2(y_2^2 + y_4^8 + y_6^{16})
\] (2.34)

\(^\dagger\) Several lists identifying such examples among the class of hypersurfaces \[51\] have been described in \[12\] [24] [53] [54]. Reference \[54\] also contains a discussion of the much larger class of K3 fibrations described by hypersurfaces in toric varieties.
is singular on the locus $\Sigma = \mathbb{P}(4,2,1,1)[16,16]$, describing a smooth curve of genus $g = 385$.

The fibration type of the hypersurface of (2.31) has been discussed in [14], where also the Hodge diamond was determined to be $(h^{(1,1)} = 6, h^{(2,1)} = 1, h^{(3,1)} = 803, h^{(2,2)} = 3278)$. Both manifolds of this split have a nested fibration structure in which the Calabi-Yau fourfold CY$_4$ is a CY$_3$ fibration with threefolds which in turn are K3 fibrations with elliptic K3 fibers. This iterative fibration structure can be summarized in the diagram

$$
\begin{array}{ccc}
T^2 & \rightarrow & \text{K3} \\
\downarrow & & \downarrow \\
\mathbb{P}_1 & \rightarrow & \text{CY$_3$} \\
& & \downarrow \\
& & \text{CY$_4$} \\
\end{array}
$$

(2.35)

Using the twist map constructions described in [28,16] one finds the embedding structure for the hypersurface to be given by

$$
\mathbb{P}_2[3] \rightarrow \mathbb{P}(2,2,1,1)[6] \rightarrow \mathbb{P}(4,4,2,1,1)[12] \rightarrow \mathbb{P}(8,8,4,2,1,1)[24],
$$

(2.36)

whereas the codimension two space leads to the iterative structure

$$
\begin{array}{ccc}
\mathbb{P}_1 \\
\mathbb{P}_2
\end{array}
\begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix}
\rightarrow
\begin{array}{ccc}
\mathbb{P}(1,1) \\
\mathbb{P}(2,2,1,1)
\end{array}
\begin{bmatrix}
1 & 1 \\
2 & 4
\end{bmatrix}
\rightarrow
\begin{array}{ccc}
\mathbb{P}(1,1) \\
\mathbb{P}(4,4,2,1,1)
\end{array}
\begin{bmatrix}
1 & 1 \\
4 & 8
\end{bmatrix}
\rightarrow
\begin{array}{ccc}
\mathbb{P}(1,1) \\
\mathbb{P}(8,8,4,2,1,1)
\end{array}
\begin{bmatrix}
1 & 1 \\
8 & 16
\end{bmatrix}.
$$

(2.37)

This shows that the generic Calabi-Yau threefold fiber of the codimension two split is obtained from the hypersurface threefold fiber via the split

$$
\mathbb{P}(4,4,1,1,2)[12]^{(5,101)} \leftrightarrow \mathbb{P}(1,1)_{\mathbb{P}(4,4,1,1,2)} \begin{bmatrix}
1 & 1 \\
4 & 8
\end{bmatrix}^{(6,70)},
$$

(2.38)

where the hypersurface on the lhs is defined by the polynomial

$$
p = z_0^3 + z_1^3 + z_2^{12} + z_3^{12} + z_4^6
$$

and the codimension two variety on the rhs is defined by

$$
\begin{align*}
p_1 &= x_1 y_1 + x_2 y_2 \
p_2 &= x_1 (y_2^2 + y_4^8 + y_5^4) + x_2 (y_1^2 + y_3^8 + y_5^4). \end{align*}
$$

(2.39)

The determinantal threefold

$$
\mathbb{P}(4,4,1,1,2)[12] \ni X^\sharp = \{p_{\text{det}} = y_1^3 - y_2^3 + (y_1 y_3^8 - y_2 y_4^8) + (y_1 - y_2) y_5^4 = 0\}
$$

(2.40)

is singular at the $\mathbb{P}(1,1,2)[8,8] = 32$ points. Thus we see that the fiber degenerates at a number of points on the curve $\Sigma$ precisely when the determinantal fourfold variety degenerates at $\Sigma$. 

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3. Discriminantal Splitting

There are other simple types of transitions between fourfolds for which the global issues mentioned in the previous Section are under control as well.

Consider the following class of configurations

$$X_{\text{split}} = \mathbb{P}_1 \times Y \begin{bmatrix} 2 & 0 \\ u & M \end{bmatrix}$$

(3.1)

defined by the zero locus of the polynomials

$$p_1(x_i, y_k) = \sum_{ij} R_{ij}(y_k)x_i x_j$$

$$p_a(y_k) = 0, \quad a = 2, ..., N$$

(3.2)

where we again denote the coordinates of the ambient space $\mathbb{P}_1 \times Y$ by $(x_i, y_j)$. Adapting the threefold analysis of [49] to fourfolds shows that this space can be understood as the double cover of the space $Y[M]$ branched over a threefold $B \subset Y[M]$ except over the singular locus of this threefold.

More precisely, consider the discriminantal hypersurface in $Y[M]$ defined by the polynomial

$$p_{\text{dis}} = \sum R_{ij}(y) R_{kl}(y) \epsilon^{ik} \epsilon^{jl}$$

(3.3)

and let $\pi : X_{\text{split}} \to Y[M]$ be the projection of $X_{\text{split}}$ along the projective curve $\mathbb{P}_1$. For each of the points $y \in Y$ the inverse image $\pi^{-1}(y)$ then consists of

1. two points if $p_{\text{dis}} \neq 0$
2. one point if $p_{\text{dis}} = 0$ but at least one of the $R_{ij}$ is non-vanishing
3. a copy of $\mathbb{P}_1$ if all $R_{ij}$s are identically zero.

This shows that (3.1) is a double cover except for the vanishing locus of the discriminant

$$B = \{p_{\text{dis}} = 0\} \subset Y[M],$$

(3.4)

which describes a hypersurface of degree $2u$ in $Y[M]$. This discriminant locus is singular at the vanishing locus of all the $R_{ij}$s

$$\Sigma = Y[M \ u \ u \ u],$$

(3.5)

describing a curve (configuration) in $B$. Smoothing out this singularity by deforming the discriminant then provides an alternative way of resolving the singularity. Similar to the
situation encountered in the determinantal splitting and contraction transitions we can resolve the singular curve of the fourfold in two different ways. This then provides a second type of fourfold transition.

A particularly simple class is given by the discriminantal splits

\[ X = \mathbb{P}_{n+1}[2, u_2 \cdots u_{n-3}] \leftrightarrow \mathbb{P}_1 \left[ \begin{array}{cccc} 2 & 0 & \cdots & 0 \\ 1 & u_2 & \cdots & u_{n-3} \end{array} \right] = X_{\text{split}} \]  

(3.6)

for which

\[ \chi(X_{\text{split}}) - \chi(X) = 6 \prod_{a=2}^{n-3} u_a, \]  

(3.7)

leading to a discriminant locus which is singular at the curve \( \Sigma = \mathbb{P}_{n-3}[d_2 \cdots d_{n-3}] \). A concrete split of this type is described by

\[ \mathbb{P}_6[5, 2] \leftrightarrow \mathbb{P}_1 \left[ \begin{array}{c} 2 \\ 1 \end{array} \right] \mathbb{P}_5 \left[ \begin{array}{c} 0 \\ 5 \end{array} \right], \]  

(3.8)

with \( \Sigma = \mathbb{P}_2[5] \), a curve of genus \( g(\Sigma) = 6 \).

Even simpler are splits of discriminantal splits of the type

\[ \mathbb{P}_1 \mathbb{P}_{n+1} \left[ \begin{array}{c} 2 \\ n+1 \end{array} \right] \leftrightarrow \mathbb{P}_{(1, \ldots, n+1)}[2(n+1)] \]  

(3.9)

where the hypersurface on the rhs lives in a weighted \((n+1)\)-space.

More interesting however is that by adapting to fourfolds certain threefold isomorphisms constructed via fractional transformations discussed in [55,24] we can construct discriminantal transitions between weighted hypersurfaces, such as the split

\[ \mathbb{P}_{(1,1,2,2,2,4)}[12] \leftrightarrow \mathbb{P}_5[6]. \]  

(3.10)

In order to see this one first notes that we can rewrite the weighted hypersurface in (3.10) as

\[ \mathbb{P}_{(1,1,2,2,2,4)} \sim \mathbb{P}_{(1,1,1,1,1,2)} \left[ \begin{array}{c} 2 \\ 1 \end{array} \right]. \]  

(3.11)

This follows most easily by going to the Landau-Ginzburg phase in which the addition of trivial mass terms is irrelevant. Thus we can equivalently consider the Fermat potential in the configuration

\[ C_{(1,1,6,6,2,2,2,4)} \ni \left\{ \sum_{i=1}^{2}(x_i^{12} + y_i^2) + \sum_{j=3}^{5} x_j^6 + x_6^3 = 0 \right\} \]  

(3.12)
at central charge $c = 12$. Here we have denoted the coordinates in the weighted complex space by $(x_1, y_1, x_2, y_2, x_3, ..., x_6)$. Modding out two trivial $\mathbb{Z}_2$s and applying the corresponding fractional transformation, as explained in [55,24], we find a third representation of this theory provided by

$$\mathcal{C}_{(1,6,1,6,2,2,2,4)[12]} / \mathbb{Z}_2^2 \left[ \begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \end{array} \right] \sim \mathcal{C}_{(2,5,2,5,2,2,2,4)[12]}$$

(3.13)

with the fractional transform described by the potential

$$W = \sum_{i=1}^{2} (x_i^6 + x_i y_i^2) + \sum_{j=3}^{5} x_j^6 + x_6^3.$$  

(3.14)

The manifold phase of this Landau-Ginzburg theory can finally be seen to be described by the codimension two configuration of (3.11) by using the construction of [56].

Repeating now the analysis above we find that this codimension two complete intersection manifold is the double cover of $\mathbb{P}(1,1,1,1,2)[6]$ branched over the discriminant locus described by the threefold $B = \mathbb{P}(1,1,1,1,2)[6 \cdot 2]$, which is singular at the smooth genus four curve $\Sigma = \mathbb{P}(1,1,2)[6]$. Deforming the discriminant locus then leads to a variety which is isomorphic to the smooth sextic fourfold.

### 4. Superpotentials

Different types of Calabi-Yau fourfolds lend themselves for the compactification of various higher dimensional theories. If the fourfold $X$ admits an elliptic fibration

$$\mathbb{T}^2 \rightarrow X \downarrow \mathcal{B}$$

with fiber $\mathbb{T}^2$ and a threefold base $\mathcal{B}$, then $M$-theory on $X$ leads to type IIB string theory on the base [34]. If the base $\mathcal{B}$ in turn is fibered over a surface $\mathcal{S}$ with the generic fiber being a sphere $\mathbb{P}_1$, i.e. we have the structure

$$\mathbb{T}^2 \rightarrow X \downarrow \mathcal{B} \downarrow \mathcal{S}_o,
then IIB(\mathcal{B}) leads to the heterotic string compactified on an elliptically fibered Calabi-Yau threefold over \( S \).

According to the results of ref. [1] a superpotential in M-theory compactification on Calabi-Yau fourfolds is generated by five-branes wrapping around complex divisors \( D \subset X \) such that

\[
\chi(D, \mathcal{O}_D) = 1. \tag{4.1}
\]

A sufficient criterion for this to hold clearly is that the divisors contain no nontrivial holomorphic forms, \( h^{(i,0)} = 0, i > 0 \).

It follows from the structure of the second Chern class that algebraic divisors \( D \subset X \), described by polynomials in these manifolds, cannot generate a superpotential because for the holomorphic Euler number

\[
\chi(X, \mathcal{L}) = \sum_i (-1)^i \dim H^i(X, \mathcal{L}) \tag{4.2}
\]

for any line bundle \( \mathcal{L} \) on a manifold \( X \) one computes via Hirzebruch-Riemann-Roch

\[
\chi(\mathcal{L}) = \int_X \text{ch}(\mathcal{L}) \wedge Td(X) \tag{4.3}
\]

on a Calabi-Yau fourfold

\[
\chi(D, \mathcal{O}_D) = -\frac{1}{24} \int c_1(D)^2 \left( c_1(D)^2 + c_2(X) \right). \tag{4.4}
\]

Thus for manifolds in which all divisors are of this type, such as hypersurfaces in \( \prod_i \mathbb{P}_{n_i} \), no superpotential can be generated. In [17] a manifold was described which does contain the requisite divisors. The manifold can be represented as a complete intersection of the form

\[
\begin{pmatrix}
\mathbb{P}_1 & 1 & 1 \\
\mathbb{P}_1 & 2 & 0 \\
\mathbb{P}_2 & 3 & 0 \\
\mathbb{P}_2 & 0 & 3
\end{pmatrix} \tag{4.5}
\]

described by two polynomials whose degrees are described by the columns of this configuration. The manifolds of this deformation class are double elliptic fibrations which are also K3 fibered with fibers which in turn are elliptic. The relevant divisors identified in [17] can be described as blow-ups of \( \mathbb{P}_1 \times \mathbb{P}_2 \) along the curve described by the base locus of the K3 fibration. In the next Section we will describe splitting transitions, and their inverses, contractions, between the configuration (4.5) and two hypersurfaces in which precisely these divisors originate from small resolutions of the singular set of the determinantal variety connecting the smooth manifolds.
5. Generating a superpotential via splitting

Consider the manifold

\[
X = \begin{bmatrix}
\mathbb{P}_1 \\
\mathbb{P}_2 \\
\mathbb{P}_2
\end{bmatrix}
= \begin{bmatrix}
2 \\
3 \\
3
\end{bmatrix}. \tag{5.1}
\]

The Euler number of this space can be determined via Cherning From Lefshetz’ hyperplane theorem we know that \(h^{(1,1)} = 3\) and \(h^{(2,1)} = 0\). Furthermore we can determine \(h^{(3,1)} = 280\) by counting complex deformations. Plugging all this into the Euler number leads to the complete Hodge half-diamond

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 280 & 1176 & 280 & 1.
\end{array}
\tag{5.2}
\]

It follows from Lefshetz and the Todd formula that manifolds of this type, i.e. hypersurfaces embedded in products of ordinary projective spaces, do not lead to non-vanishing superpotential. However the manifold above can be split into one that does contain divisors which generate a superpotential. More precisely (5.1) is part of a sequence of splits which connects the manifold (4.5), which has been shown in [17] to lead to a superpotential with modular properties, with the sextic fourfold

\[
\begin{array}{cccc}
\mathbb{P}_5[6] & \leftrightarrow & \begin{bmatrix}
\mathbb{P}_1 \\
\mathbb{P}_5
\end{bmatrix} & \begin{bmatrix}
1 \\
5
\end{bmatrix} & \leftrightarrow & \begin{bmatrix}
\mathbb{P}_2 \\
\mathbb{P}_1 \\
\mathbb{P}_5
\end{bmatrix} & \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix} & \leftrightarrow & \begin{bmatrix}
\mathbb{P}_2 \\
\mathbb{P}_1 \\
\mathbb{P}_5
\end{bmatrix} & \begin{bmatrix}
2 \\
3 \\
3
\end{bmatrix} & \leftrightarrow & \begin{bmatrix}
\mathbb{P}_1 \\
\mathbb{P}_2 \\
\mathbb{P}_2
\end{bmatrix} & \begin{bmatrix}
1 & 1 \\
2 & 0 \\
3 & 0
\end{bmatrix} &= X_{\text{split}}. \tag{5.3}
\end{array}
\]

Both of these spaces are elliptic fibrations and the split manifold is also a K3-fibration with generic elliptic K3 fibers.

The determinantal hypersurface

\[
\begin{bmatrix}
\mathbb{P}_1 \\
\mathbb{P}_2 \\
\mathbb{P}_2
\end{bmatrix}
\begin{bmatrix}
2 \\
3 \\
3
\end{bmatrix} \ni X^2 = \{p_{\text{det}} = QT - RS = 0\} \tag{5.4}
\]

is singular at the locus

\[
\begin{bmatrix}
\mathbb{P}_1 \\
\mathbb{P}_2 \\
\mathbb{P}_2
\end{bmatrix}
\begin{bmatrix}
2 & 2 & 0 & 0 \\
3 & 3 & 0 & 0 \\
0 & 0 & 3 & 3
\end{bmatrix} = 9 \times \Sigma, \quad \tag{5.5}
\]

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where $\Sigma = \mathbb{P}_1 \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ and $\mathbb{P}_2[3 \ 3] = 9$ pts. The curve $\Sigma$ has Euler number $\chi(\Sigma) = -54$ and hence is of genus $g(\Sigma) = 28$. Thus the singular set has 9 different components and the splitting formula (2.11) becomes

$$\chi(X_{\text{split}}) = \chi(X) + 3 \cdot 9 \chi(\Sigma) = 288.$$ (5.6)

We see from this that it is precisely the small resolution of the curve $\Sigma$ which introduces the divisors in $X_{\text{split}}$ which are responsible for the superpotential.

The algebraic divisors

$$X \supset D = \begin{bmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{bmatrix} \begin{bmatrix} 2 & a_1 \\ 3 & a_2 \\ 3 & a_3 \end{bmatrix}$$

of the manifold $X$ are transformed by the splitting transition into the divisors

$$X_{\text{split}} \supset D_{\text{split}} = \begin{bmatrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & a_1 \\ 3 & 0 & a_2 \\ 0 & 3 & a_3 \end{bmatrix}.$$ (5.7)

On the singular determinantal variety this divisor degenerates into a number of points whose resolution is described by (5.8).

A similar discussion applies to the $\mathbb{P}_1$-split

$$X' = \begin{bmatrix} \mathbb{P}_2 \\ \mathbb{P}_3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}_{2016} \leftrightarrow \begin{bmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 4 \end{bmatrix}_{288} = X'_{\text{split}},$$ (5.9)

which connects the lhs elliptic fibration with Hodge numbers ($h^{(1,1)}(X') = 2, h^{(2,1)}(X') = 0, h^{(3,1)}(X') = 326, h^{(2,2)}(X') = 1356$) to the codimension two elliptic fibration of the rhs with $\chi(X'_{\text{split}}) = 288$. The determinantal variety $X'^{\sharp}$ is singular at the locus

$$\begin{bmatrix} \mathbb{P}_2 \\ \mathbb{P}_3 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} = 9 \times \Sigma,$$ (5.10)

where $\Sigma = \mathbb{P}_3[4 \ 4]$ is a genus $g(\Sigma) = 33$ curve, and therefore (2.11) leads to $\chi(X'_{\text{split}}) = 288$.

In this example the small resolution of the split transition replaces pointwise the curve $\Sigma \subset \mathbb{P}_3$ by the projective plane, thereby introducing the necessary divisor for a non-vanishing superpotential. On the polynomial divisors

$$X' \supset D = \begin{bmatrix} \mathbb{P}_2 \\ \mathbb{P}_3 \end{bmatrix} \begin{bmatrix} 3 & a_1 \\ 4 & a_2 \end{bmatrix},$$ (5.11)
which are split into
\[ X'_{\text{split}} \supset D_{\text{split}} = \begin{bmatrix} \mathbb{P}_1 & 1 & 1 & 0 \\ \mathbb{P}_2 & 3 & 0 & a_1 \\ \mathbb{P}_3 & 0 & 4 & a_2 \end{bmatrix}, \]

the small resolution of the curve \( \Sigma \) again translates into the resolution of a number of points.

The manifold \( X'_{\text{split}} \) can in fact be split and contracted at constant Euler number into the split manifold \([4.5]\) via

\[
\begin{aligned}
\mathbb{P}_1 & \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 4 \end{bmatrix} & \leftrightarrow & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} & \leftrightarrow & \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\
\mathbb{P}_2 & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix} & \leftrightarrow & \begin{bmatrix} 0 & 3 \\ 0 & 2 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.
\end{aligned}
\]

This sequence involves nontrivial determinantal varieties which degenerate at configurations of tori, as discussed in Section 2.

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