SOME INVARIANT SUBALGEBRAS ARE GRADED ISOLATED SINGULARITIES

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Abstract. In this note, we prove that the invariant subalgebra of the skew polynomial algebra \(k[x_0, x_1, \ldots, x_{n-1}]/(\{x_i x_j + x_j x_i \mid i \neq j\})\) under the action \(x_i \mapsto x_{i+1}(i \in \mathbb{Z}/n \mathbb{Z})\) is a graded isolated singularity, and thus a conjecture of Chan-Young-Zhang is true.

1. Introduction

Noncommutative graded isolated singularities are defined by Ueyama [9, Definition 2.2]. A noetherian connected graded algebra \(B\) is called a graded isolated singularity if the associated noncommutative projective scheme \(\text{Proj}(B)\) (in the sense of [1]) has finite global dimension. See [4, 3, 7, 6] for some examples of graded isolated singularities.

Let \(A\) be a noetherian Artin-Schelter regular algebra and \(G\) be a finite subgroup of \(\text{Aut}_{gr}(A)\). To prove a version of the noncommutative Auslander theorem, an invariant called the pertinency of the \(G\)-action on \(A\) is introduced in [3] and [2]. We recall it here.

The pertinency of the \(G\)-action on \(A\) [2, Definition 0.1] is defined to be \(p(A, G) := \text{GKdim}(A) - \text{GKdim}(A\#G/(e_0))\), where \((e_0)\) is the ideal of the skew group algebra \(A\#G\) generated by \(e_0 := 1\#|G| \sum_{g \in G} g\).

Then, by [8, Theorem 3.10], \(A^G\) is a graded isolated singularity if and only if \(p(A, G) = \text{GKdim}(A)\). Unlike in the commutative cases, it is difficult to determine when the invariant subalgebra is a graded isolated singularity.

Let \(k\) be an algebraically closed field of characteristic zero. Let \(A = k[x_0, \ldots, x_{n-1}](n \geq 2)\) be the \((-1)\)-skew polynomial algebra, which is generated by \(\{x_0, \ldots, x_{n-1}\}\) and subject to the relations \(x_i x_j = (-1)x_j x_i(\forall i \neq j)\).

Let \(G := C_n\) be the cyclic group of order \(n\) acting on \(A\) by permuting the generators of the algebra cyclically; namely, \(C_n\) is generated by \(\sigma = (012\cdots n-1)\) of order \(n\) that acts on the generators by \(\sigma x_i = x_{i+1}, \forall i \in \mathbb{Z}_n := \mathbb{Z}/n \mathbb{Z}\).

In [5, Theorem 0.4], Chan, Young and Zhang prove the following result on graded isolated singularities.

**Theorem 1.1.** If either 3 or 5 divides \(n\), then \(p(A, G) < \text{GKdim}A = n\). Consequently, the invariant subalgebra \(A^G\) is not a graded isolated singularity.

Based on this theorem and [5, Theorem 0.2], Chan, Young and Zhang give the following conjecture [5, Conjecture 0.5].

**Conjecture 1.2.** The invariant subalgebra \(A^G\) is a graded isolated singularity if and only if \(n\) is not divisible by 3 or 5.

To prove Conjecture 1.2 is true, it suffices to prove the following theorem, which is the main result in this note.

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Theorem 1.3. If $n$ is not divisible by 3 or 5, then $p(A, G) = \text{GKdim } A = n$. As a consequence, $A^G$ is a graded isolated singularity.

2. Preliminaries

Before giving a proof of Theorem 1.3, let us recall some notations and results in [5]. Let $\omega$ be a primitive $n$th root of unity. For any $\gamma = 0, 1, \ldots, n - 1 \in \mathbb{Z}_n$, let

$$b_{\gamma} := \frac{1}{n} \sum_{i=0}^{n-1} \omega^{i\gamma} x_i \in A \subseteq A^\#C_n.$$ 

Then $b_{\gamma}$ is an $\omega^{-\gamma}$-eigenvector of $\sigma$. Let

$$e_{\gamma} := \frac{1}{n} \sum_{i=0}^{n-1} (\omega^i \sigma)^i \in k C_n \subseteq A^\#C_n,$$

which are idempotent elements.

Suppose $\deg(x_i) = 1$ and $\deg(e_i) = 0$ for all $i \in \mathbb{Z}_n$. As usual, $[-, -]$ denotes the graded commutator of the graded ring $A^\#C_n$, that is, $[u, v] = uv - (-1)^{\deg(u) \deg(v)} vu$ for any homogeneous elements $u, v \in A^\#C_n$.

Lemma 2.1. [5, Lemma 1.1] The graded algebras $A$ and $A^\#C_n$ can be presented as

$$A \cong \frac{k \langle b_0, \ldots, b_{n-1} \rangle}{(b_0, b_k) - (b_i, b_{k-i})} \quad \text{and} \quad A^\#C_n \cong \frac{k \langle b_0, \ldots, b_{n-1}, e_0, \ldots, e_{n-1} \rangle}{(e_0 b_0 - b_0 e_0, e_0 e_0 - \delta_{ij} e_i, b_0, b_k - (b_i, b_{k-i})]}$$

respectively, where $\delta_{ij}$ is the Kronecker delta and indices are taken modulo $n$.

For each $j \in \mathbb{Z}_n$, let

$$c_j := [b_k, b_{j-k}] = b_kb_{j-k} + b_{j-k}b_k = \frac{2}{n^2} \sum_{i=0}^{n-1} \omega^{ij} x_i^2.$$ 

Then $c_j$ is an $\omega^{-j}$-eigenvector of $\sigma$.

For any vector $i = (i_0, \ldots, i_{n-1}) \in \mathbb{N}^n$, we use the following notations:

$$b^i = b_{i_0} \cdots b_{i_{n-1}} \quad \text{and} \quad c^i = c_{i_0} \cdots c_{i_{n-1}}.$$ 

Let $R_n$ be the subspace of $A$ spanned by the elements $b^i c^j$ such that $\sum_{s=0}^{n-1} (i_s + j_s)s = \gamma \mod n$; that is, $R_n$ consists of $\omega^{-\gamma}$-eigenvectors of $\sigma$. This gives an $R_n$-module decomposition

$$A = R_0 \oplus R_1 \oplus \cdots \oplus R_{n-1}.$$ 

Definition 2.2. (1) Let $\Phi_n := \{ k \mid c_k^{N_k} \in (e_0) \text{ for some } N_k \geq 0 \}$, where $(e_0)$ is the two-sided ideal of $A^\#C_n$ containing $e_0$.

(2) Let $\phi_2(n) := \{ k \mid 0 \leq k \leq n - 1, \gcd(k, n) = 2^w \text{ for some } w \geq 0 \}$.

(3) Let $\Psi_j^{[n]} := \{ i \mid c_i^N \in R_j A \text{ for some } N \geq 0 \}$.

(4) [5, Definition 5.2 and Lemma 5.3(1)] We say $n$ is admissible if, for any $i$ and $j, i \in \Psi_j^{[n]}$, or equivalently, $\text{GKdim}(A^\#C_n/(e_0)) = 0$.

(5) Let $\overline{A} := A/(c_k \mid k \in \Phi_n)$, and $\overline{\Psi}_j^{[n]} := \{ i \mid c_i^N \in \overline{R}_j \overline{A} \text{ for some } N \geq 0 \}$ where $\overline{R}_j = \frac{R_j}{(c_k \mid k \in \overline{\Phi}_n)} \subseteq \overline{A}$.

Let $\mathbb{Z}_n^\times$ be the set of invertible elements in $\mathbb{Z}_n$.

Lemma 2.3. (1) [5, Definition 6.1] $\Phi_n$ is a special subset of $\mathbb{Z}_n^\times$; that is, $k \in \Phi_n$ if and only if $\lambda k \in \Phi_n$ for all $\lambda \in \mathbb{Z}_n^\times$.

(2) [5, Proposition 2.3] $\phi_2(n) \subseteq \Phi_n$. 


The following proposition follows from the proof of [5, Proposition 6.6].

**Proposition 2.4.** Let \( n \geq 2 \) such that \( 3, 5 \mid n \). If \( 1, \ldots, n-1 \in \overline{\Phi}_1^n \), then \( 0 \in \overline{\Phi}_1^n \).

**Proposition 2.5.** [5, Proposition 6.8] Let \( n \geq 2 \). Suppose that

1. every proper factor of \( n \) is admissible, and
2. for each \( 0 \leq i \leq n-1, i \in \overline{\Phi}_1^n \).

Then \( n \) is admissible.

### 3. Proof of the Theorem 1.3

**Proof of Theorem 1.3.** We prove it by induction on \( n \). Assume that every proper factor of \( n \) is admissible. By Proposition 2.5, it suffices to prove that

\[
\text{for each } 0 \leq i \leq n-1, i \in \overline{\Phi}_1^n .
\]

If this is not true, that is, there is \( 0 \leq m \leq n-1 \) such that \( m \notin \overline{\Phi}_1^n \). Then we may assume that

1. \( m \neq 0 \), by Proposition 2.4;
2. \( m \mid n \), by Lemma 2.3 (2) as \( \mathbb{Z}_n^X \subseteq \phi_2(n) \subseteq \Phi_n \);
3. \( m > 5 \), by Lemma 2.3 (2) and assumption \( 3, 5 \mid n \).

Write \( n = mq \) with \( q > 1 \).

Since \( c_m \) is an eigenvector of \( \sigma \), then \( C_n \) acts on the localization \( A[c_m^{-1}] \), and \( A[c_m^{-1}] \# C_n \langle (e_0) \rangle \cong \langle A \# C_n \rangle \langle (e_0) \rangle \). Let

\[
\tilde{A} = \frac{\langle b_0, \ldots, b_{m-1} \rangle}{\langle [b_0, b_k] - [b_1, b_{k-1}] \mid l, k \in \mathbb{Z}_m \rangle}
\]

be a subalgebra of \( A \), and \( \tilde{R}_\gamma \) be the subspace of \( \tilde{A} \) spanned by the elements \( b^i c^j \) such that

\[
\sum_{s=0}^{m-1} (i_s + j_s)s = \gamma \mod m .
\]

For any \( b^i c^j = b_0^i \cdots b_{m-1}^{i_0} \cdots c_{m-1}^{j_1} \in \tilde{R}_1 \) with

\[
\sum_{s=0}^{m-1} (i_s + j_s)s = mk + 1 \text{ for some } k \geq 0,
\]

then there exists \( l > 0 \) such that \( (l-1)q \leq k < lq \). Hence \( b^i c^{j-k} \in \tilde{R}_1 A \), and \( b^i c^j \in \tilde{R}_1 A[c_m^{-1}] \).

It follows that

\[
\tilde{R}_1 \tilde{A} \subseteq \tilde{R}_1 A[c_m^{-1}] .
\]

Write \( \omega = \omega^a \). Note that \( \tilde{A} \cong k_1[\tilde{x}_0, \ldots, \tilde{x}_{m-1}] \) via \( b_\gamma \mapsto \frac{1}{m} \sum_{i=0}^{m-1} \tilde{x}^i \gamma \tilde{x}_i \). Then the cyclic group \( C_m \) of order \( m \) acts on \( \tilde{A} \) by permuting the generators of the algebra cyclically; namely, \( C_m \) is generated by \( \tilde{\sigma} = (012 \cdots m-1) \) of order \( m \) that acts on the generators by

\[
\tilde{\sigma} \tilde{x}_i = \tilde{x}_{i+1}, \forall i \in \mathbb{Z}_m .
\]

Then \( \tilde{R}_\gamma \) consists of \( \omega^{-\gamma} \)-eigenvectors of \( \tilde{\sigma} \). By assumption, \( m \) is admissible, so for any \( 0 \leq i \leq m - 1 \), there exists \( N_i \) such that

\[
c_i^{N_i} \in \tilde{R}_1 \tilde{A} \subseteq \tilde{R}_1 A[c_m^{-1}] .
\]

Let \( \Gamma \) be the right ideal \( \tilde{R}_1 A[c_m^{-1}] + \sum_{\exists N_k, c_k^{N_k} \in \tilde{R}_1 A[c_m^{-1}]} c_k A[c_m^{-1}] \) of \( A[c_m^{-1}] \). Next we prove that

\[
\Gamma = A[c_m^{-1}] .
\]

The following proof is quite similar to the proof of [5, Propostition 6.6].

Claim 1. Let \( 0 \leq j < \frac{m-1}{2} \). If \( c_m^s b_j \in \Gamma \) for some \( s > 0 \), then \( c_m^{s+1} b_{j+1} \in \Gamma \).
Proof of Claim 1. First of all, $b_{j+1}b_{m-j} \in \tilde{R}_1\tilde{A} \subseteq R_1A[c_m^{-1}]$ since $(j+1) + (m-j) = 1 \mod m$. Due to $c_m^s b_j \in \Gamma$, then
\[
\Gamma \ni [b_{j+1}b_{m-j}, c_m^s b_j] = c_m^s b_{j+1}b_{m-j}b_j - c_m^s b_jb_{j+1}b_{m-j} = c_m^s b_{j+1}b_{m-j}b_j + c_m^s b_{j+1}b_{m-j} - c_m^s c_{j+1}b_{m-j} = c_m^{s+1}b_{j+1} - c_m^s c_{j+1}b_{m-j}.
\]
Since there exists $N \in \mathbb{Z}$ such that $c_{N+1}^s b_{j+1} \in R_1\tilde{A}$ for $2j + 1 < m$, then $c_m^{s+1}b_{j+1} \in \Gamma$. \qed

Claim 2. Suppose that $m = 2k + 1$. If $c_m^s b_k \in \Gamma$, then $c_m^{s+2}b_{k+2} \in \Gamma$.

Proof of Claim 2. Note that $b_{k+1}b_{k+2}b_{m-1} \in \tilde{R}_1\tilde{A} \subseteq \Gamma$ as $(k+1) + (k+2) + (m-1) = 1 \mod m$.
\[
\Gamma \ni [c_m^s b_k, b_{k+1}b_{k+2}b_{m-1}] = c_m^s b_kb_{k+1}b_{k+2}b_{m-1} + c_m^s b_{k+1}b_{k+2}b_{m-1}b_k = c_m^s b_kb_{k+1}b_{k+2}b_{m-1} + c_m^s b_{k+1}b_{k+2}b_{m-1} - c_m^s b_{k+1}b_{k+2}b_{m-1} = c_m^s b_{k+1}b_{k+2}b_{m-1} + c_m^s \in \Gamma.
\]
Since $c_m+1c_m^{-1} \in R_1A$, then $c_m+1 \in R_1A[c_m^{-1}]$. Hence $c_m^{s+1}b_{k+2}b_{m-1} + c_m^s c_{3k}b_{k+1}b_{k+2} \in \Gamma$.
\[
\Gamma \ni [c_m^{s+1}b_{k+2}b_{m-1} + c_m^s c_{3k}b_{k+1}b_{k+2}, b_1] = c_m^{s+1}b_{k+2}b_{m-1}b_1 - c_m^{s+1}b_kb_{k+2}b_{m-1} + c_m^s c_{3k}b_{k+1}b_1b_{k+2} + c_m^s c_{3k}b_{k+1}b_1b_{k+2} - c_m^s c_{3k}b_{k+1}b_1b_{k+2} = c_m^{s+1}b_{k+2}b_{m-1} + c_m^s c_{3k}b_{k+1}b_1b_{k+2} - c_m^s c_{3k}b_{k+1}b_1b_{k+2}.
\]
By assumption $m > 5$, so $k > 2$. Since $k + 2 < k + 3 < 2k + 1 = m$, $c_{k+2}, c_{k+3} \in \Gamma$ by assumption. It follows that $c_m^{s+2}b_{k+2} \in \Gamma$. \qed

Claim 3. $c_m^{-1} \in \Gamma$.

Proof of Claim 3. Assume that $m$ is even. Starting with $b_1$, and applying Claim 1 ($\frac{m}{2} - 1$) times, we get $c_m^{-1}b_{\frac{m}{2}} \in \Gamma$. Hence $c_m^{-1} = [c_m^{-1}b_{\frac{m}{2}}, c_m^{-1}b_{\frac{m}{2}}] \in \Gamma$.

If $m = 2k + 1$ is odd, then by applying Claim 1 ($k - 2$) and ($k - 1$) times we get $c_m^{-2}b_{k-1}$ and $c_m^{-1}b_k \in \Gamma$ respectively. By applying Claim 2 we get $c_m^{k+1}b_{k+2} \in \Gamma$.

Therefore, $c_{2k} = [c_m^{k-2}b_{k-1}, c_m^{k+1}b_{k+2}] \in \Gamma$. \qed

By Claim 3, $\Gamma = A[c_m^{-1}]$. Recall that $\Gamma = R_1A[c_m^{-1}] + \sum_{\exists N, c_N^s \in R_1A[c_m^{-1}]} c_NA[c_m^{-1}]$. It is not difficult to see that $A[c_m^{-1}] = R_1A[c_m^{-1}]$. So there exists $N \geq 0$ such that $c_N^N \in R_1A$, which is a contradiction (as $m \notin \mathbb{N}$). This implies $\mathbb{N} = \{0, 1, \cdots, n-1\}$, that is, $n$ is admissible. Hence $\dim G(A\#C_n/(e_0)) = 0$, and $p(A, G) = n$. \qed

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