Electronic relaxation rates in metallic ferromagnets

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I. INTRODUCTION

Electronic relaxation rates contain important information about the excitations in a metallic system. The single-particle relaxation rate, $1/\tau$, determines the lifetime of quasi-particles as well as the thermal conductivity $\kappa = v_F c_V \tau / 3$: the transport relaxation rate, $1/\tau_T$, the electrical conductivity via the Drude formula $\sigma = n_e e^2 \tau_T / m_e$. Here $m_e$ and $n_e$ are the conduction electron effective mass and number density, respectively, $v_F$ is the Fermi velocity, and $c_V$ is the specific heat. There are various contributions to these relaxation rates, including those from the scattering of electrons by propagating, or particle-like, excitations. For instance, the coupling of longitudinal phonons to conduction electrons leads to the well-known Bloch $T^5$-behavior of the electrical resistivity; the corresponding effect in the single-particle relaxation rate is a $T^3$-law. In magnetically ordered phases, the coupling of the conduction electrons to any magnetic Goldstone modes contributes to the relaxation rates. In isotropic Heisenberg ferromagnets, the Goldstone modes are the ferromagnons with a frequency-momentum relation $\omega \sim k^2$. They have been found to contribute a $T^2$ term to the transport relaxation rate. In helimagnets, which have a helically modulated magnetic ground state, the corresponding Goldstone mode (the helimagnon) has been shown to lead to a term in the electrical resistivity that is proportional to $T^{5/2}$ in the low-temperature limit. In antiferromagnets, the corresponding contribution is known to be proportional to $T^3$. These results all hold for three-dimensional systems, which is the only physical dimension in which long-range magnetic order exists. For later reference we note, however, that the various power laws quoted above are dimensionality dependent. For instance, in a generic dimension $d > 2$ the contribution from ferromagnons to the resistivity is proportional to $T^{(d+1)/2}$.

In addition to the scattering by propagating excitations, there are contributions to the transport coefficients due to excitations with a continuous spectrum. The best known example is the one due to the Coulomb interaction between the electrons. In simple metals it leads to a $T^2$ contribution to both the single-particle rate and the transport rate, i.e., a lower power than the phonon contribution. However, since the relevant energy scale is the Fermi energy $\epsilon_F$ or Fermi temperature $T_F$ (we use units such that $\hbar = k_B = 1$), which is much larger than the Debye temperature, this dominates the phonon contribution only at very low temperatures. In metals that display ferromagnetism the latter statement is not necessarily true, due to Fermi surfaces that consist of multiple sheets, and the issue of both the temperature dependence and the prefactor of the Coulomb contribution to the electrical resistivity is complicated. These are old questions that recently have been revisited in the context of quantum criticality and exotic metals.

Another example is the scattering of electrons in ferromagnets by both longitudinal magnetization fluctuations and the so-called Stoner excitations in the transverse channel. The latter are dissipative, non-hydrodynamic transverse excitations in addition to the propagating spin waves. In a random-phase approximation, the contribution to the resistivity from these dissipative excitations, both longitudinal and transverse, was shown in Ref. to result in a $T^2$ behavior with a prefactor that is inversely proportional to the magnetization. This is qualitatively the same behavior these authors found for the scattering by magnons, and it agrees roughly with the trend observed in Fe, Co, and Ni. As we will see, this conclusion, as far as the magnons is concerned, is true only in a temperature window, but not at asymptotically low temperature. It should be stressed, however,
that this similarity is somewhat accidental and approximation dependent even in the regime where it holds. For instance, the power law of the magnon contribution is dimensionality dependent, as mentioned above, while the contribution from the dissipative excitations is not. Also, the prefactor of the former is essentially determined by the dispersion relation of the magnons, which is governed by very general principles, whereas the latter is dependent on many non-universal details. Nevertheless, the fact that various contributions of very different nature to the relaxation rates show a $T^2$ temperature dependence makes the interpretation of the experimentally observed $T^2$ behavior of the electrical resistivity in many ferromagnetic materials difficult. At the same time, the electrical resistivity is a basic physical property that is very useful, for instance, for tracking and identifying magnetic phase transitions, and establishing its behavior in the ferromagnetic phase as a benchmark is important.

In this paper we focus on the magnon contribution to the relaxation rates in ferromagnets and show that for this process the established result is qualitatively incorrect at asymptotically low temperatures; instead of a $T^2$ temperature dependence, the magnon contributions to both the electrical resistivity and the thermal resistivity display an exponential behavior. A problem with the established ferromagnetic result was first noted in Ref. [22] which showed that the results for the helimagnetic and ferromagnetic cases are not mutually consistent: If one considers the ferromagnetic limit of the helimagnetic ground state, by letting the wavelength of the helix go to infinity, one finds that the leading contribution to the relaxation rate, which would yield a power law, vanishes. What is left behind is an exponential behavior of the form

$$1/\tau_T \propto (T^2/\lambda) \exp(-T_0/T), \quad (1.1)$$

where the temperature scale $T_0$ depends on the conduction band splitting or “Stoner gap” $\lambda$ or, equivalently, the magnetization and on the Fermi energy $\epsilon_F$. This result is surprising, given that the relaxation rates due to magnetic Goldstone modes in both helimagnets and antiferromagnets show a power-law behavior. The purpose of this paper is to discuss this problem, and to elaborate on the brief remarks that were given in Appendix D of Ref. [22]. We will show that the asymptotic low-temperature behavior of both the transport relaxation rates due to magnons is indeed exponential of the form shown in Eq. (1.1), with $T_0 \approx Dk_F^2(\lambda/\epsilon_F)^2$ with $D$ the spin-wave stiffness, which itself depends on $\lambda$, and $k_F$ and $\epsilon_F$ the Fermi wave number and Fermi energy, respectively. This result holds in an asymptotic regime defined by $T \ll T_0$. However, in a sizable pre-asymptotic temperature window given by $T_0 \ll T \ll Dk_F^2$ one recovers the $T^2$ behavior found previously. The reason for the exponential asymptotic result is the fact that, in a ferromagnet, the Goldstone modes are purely transverse, and therefore couple only quasi-particles in different Stoner bands. The effective electron-electron-interaction due to ferromagnon exchange therefore describes purely inter-Stoner-band scattering, which leads to an activated process. In contrast, in helimagnets and antiferromagnets there is an intra-Stoner-band coupling which leads to a power law. This vanishes as the characteristic wave number of the magnetic order goes to zero in the ferromagnetic limit.

These results are valid for all metallic ferromagnets, whether or not the magnetism is caused by the conduction electrons themselves or by localized electrons in a different band. We will refer to such systems as “itinerant ferromagnets” and “localized-moment ferromagnets”, respectively. In the main body of the paper we will consider a very general model that does not depend on which of these two cases is realized, and that uses only very general properties of ferromagnets that follow from symmetry arguments. A more specific Stoner-type model for the case of itinerant ferromagnets is considered in an appendix.

This paper is organized as follows. In Sec. II we derive an effective action that describes an effective electron-electron interaction due to the exchange of ferromagnons. The effective action is valid for calculating relaxation rates to first order in the magnon propagator, and it holds for both itinerant and localized-moment ferromagnets. In Sec. III we use this model to calculate the single-particle relaxation time, and in Sec. IV we calculate the transport relaxation time, and hence the electrical conductivity, by evaluating the pertinent Kubo formula in an approximation that is equivalent to the Boltzmann equation. In Sec. V we discuss our results. In Appendix A we recall the Stoner-Moriya mean-field treatment of itinerant ferromagnets. In Appendix B we recall the cases of electron-electron and electron-phonon scattering in non-magnetic metals, and cast them in a language that illustrates why our general method works even in the case of itinerant ferromagnets.

II. EFFECTIVE ACTION

In this section we derive and motivate an effective action that is suitable for calculating the effects of long-range ferromagnetic order, and the associated Goldstone modes, on the electronic relaxation rates in a metallic ferromagnet.

A. Coupling of magnetic fluctuations to conduction electrons

Let $S_0[\bar{\psi}, \psi]$ be an action for conduction electrons in terms of fermionic spinor fields $\bar{\psi} = (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow)$ and $\psi = (\psi_\uparrow, \psi_\downarrow)$ that depend on a spin projection index $\sigma = (\uparrow, \downarrow) \equiv (+, -)$. The electronic spin density is given by

$$n_s(x) = \sum_{\sigma, \sigma'} \bar{\psi}_\sigma(x) \sigma_{\sigma\sigma'} \psi_{\sigma'}(x). \quad (2.1)$$
Here $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ denotes the Pauli matrices, and $x = (x, \tau)$ comprises the real-space position $x$ and the imaginary-time variable $\tau$. Now assume that the conduction electrons are subject to a magnetization $M(x)$ of unspecified origin. The magnetization will act as an effective magnetic field that couples to the conduction electrons via a Zeeman term. The action then reads

$$S[\tilde{\psi}, \psi] = S_0[\tilde{\psi}, \psi] + \Gamma_1 \int dx \, M(x) \cdot n_s(x) \, ,$$

with $\Gamma_1$ a coupling constant that dimensionally is an energy times a volume, or an inverse density of states. In a ferromagnetic state, the magnetization has a nonzero average value that we assume to be in the 3-direction, $\langle M_i(x) \rangle = \delta_{i3} m$. In a mean-field approximation that replaces $M$ by its average value the action then takes the form

$$S_{\lambda}[\tilde{\psi}, \psi] = S_0[\tilde{\psi}, \psi] + \lambda \int dx \, n_{s,3}(x) \, ,$$

where $\lambda = \Gamma_1 m$ is directly proportional to the average magnetization. Here we have chosen the sign of the action such that the partition function is given by

$$Z_{\lambda} = \int D[\tilde{\psi}, \psi] \, e^{S_{\lambda}[\tilde{\psi}, \psi]} \, .$$

$\lambda$ splits the conduction band into two sub-bands, one for each spin projection. We will refer to $\lambda$ as the Stoner gap but we emphasize that the physical situation we are considering is much more general than the one considered in the Stoner model. In particular, we do not necessarily assume that the conduction electrons themselves are the source of the magnetization.

Now consider fluctuations $\delta M$ of the magnetization. The action, Eq. (2.2), then reads

$$S[\tilde{\psi}, \psi] = S_{\lambda}[\tilde{\psi}, \psi] + \Gamma_1 \int dx \, \delta M(x) \cdot n_s(x) \, .$$

In addition we need an action that governs $\delta M$. If the latter is to describe the fluctuations of the physical magnetization, then this must be

$$S_{\text{fluc}}[\delta M] = -\frac{1}{2} \int dx \, dy \, \delta M_{s,i}(x) \chi_{ij}^{-1}(x, y) \delta M_{s,j}(y) \, ,$$

where $\chi_{ij}(x, y)$ is the physical magnetic susceptibility. In a ferromagnetic phase, the transverse $(i,j = 1, 2$ with our choice for the magnetization direction) components of $\chi_{ij}$ contain the ferromagnons, which are the Goldstone modes associated with the ferromagnetic order. The transverse part of $\chi_{ij}$ is thus singular in the limit of small frequencies and wave numbers. Adding Eqs. (2.4a) and (2.4b), and integrating out $\delta M$, we obtain a purely electronic effective action

$$S_{\text{eff}}[\tilde{\psi}, \psi] = S_{\lambda}[\tilde{\psi}, \psi] + S_{\text{ex}}[\tilde{\psi}, \psi] \, ,$$

with

$$S_{\text{ex}}[\tilde{\psi}, \psi] = \frac{\Gamma_1^2}{2} \int dx dy \, \delta n_{s,i}(x) \chi_{ij}(x, y) \delta n_{s,j}(y) \, .$$

If we use only the singular, transverse, part of $\chi_{ij}$, then $S_{\text{ex}}$ describes an effective electron-electron interaction mediated by an exchange of magnons.

**B. Effective action**

In order to make the effective action given by Eqs. (2.5) suitable for explicit calculations, we now specify $S_0$ and $\chi_{ij}$. The former in principle describes interacting electrons in a conduction band. However, the electron-electron interaction is not of any qualitative importance for our purposes, and we therefore take $S_0$ to describe noninteracting electrons with an energy-momentum relation $\epsilon_k$. We denote the chemical potential by $\mu$, and define $\xi_k = \epsilon_k - \mu$. $S_{\lambda}$ then reads

$$S_{\lambda}[\tilde{\psi}, \psi] = \sum_k \sum_\sigma \left[ i \omega_n - \epsilon_{\sigma}(k) \right] \tilde{\psi}_\sigma(k) \psi_\sigma(k) \, ,$$

with

$$\omega_{\pm}(k) = \xi_k \mp \lambda \, .$$

Here we see explicitly that the magnetization splits the conduction band into two Stoner bands whose Fermi surfaces (FS) are defined by

$$\omega_{\sigma}(p) \big{|}_{p \in \text{FS}, \sigma} = 0 \, ,$$

and we denote the density of states at the $\sigma$-Fermi surface and the corresponding Fermi wave number by $N_{F\sigma}^+$ and $k_{\text{F},\sigma}^+$, respectively. In the case of a parabolic band we have

$$k_{\text{F},\pm}^+ = k_F \sqrt{1 \pm \lambda/\epsilon_F} \, ,$$

$$N_{F,\pm}^+ = k_F^+ m_e/2\pi^2 \, .$$

The Green functions for the two Stoner bands are

$$G_{\lambda,\sigma}(p) = 1/(i\omega_n - \omega_{\sigma}(p)) \, ,$$

with $\omega_n = 2\pi T(n + 1/2)$ ($n$ integer) a fermionic Matsubara frequency.

The structure of the transverse magnetic susceptibility at small frequencies and wave numbers in an isotropic ferromagnet is entirely determined by symmetry arguments. The Goldstone modes of the spontaneously broken rotational symmetry in spin space are ferromagnons with a resonance frequency

$$\omega_0(k) = D(\lambda) k^2 \, ,$$

The spin-stiffness coefficient $D$ vanishes as $\lambda \rightarrow 0$. It has the dimensions of a diffusion coefficient, and is given by a magnetic energy scale divided by a microscopic wave
number scale squared, with the latter on the order of the Fermi wave number. In the Stoner-Moriya mean-field theory\textsuperscript{23} of itinerant ferromagnets the former is given by $\lambda$, and for nearly free electrons one obtains

$$D(\lambda) = \lambda/6k_F^2 \quad \text{(Stoner)} . \quad (2.11a)$$

In a Heisenberg spin model with exchange energy $J$ and lattice constant $a$ the corresponding result is\textsuperscript{25,29}

$$D = J a^2 \quad \text{(Heisenberg)} . \quad (2.11b)$$

If one takes into account mode-mode coupling effects that are not included in the mean-field theory one finds that $D(\lambda)$ is a nonanalytic function of $\lambda$\textsuperscript{30,31} The transverse magnetic susceptibility can be expressed in terms of simple poles that describe circularly polarized ferromagnons, viz.,

$$\chi(\pm, i\Omega) = \frac{K(\lambda)}{(2N_F1)^2} \frac{1}{\omega_0(\pm, \Omega)} . \quad (2.12)$$

The coefficient $K(\lambda)$ is dimensionally an inverse volume. It vanishes as $\lambda \to 0$; in the Stoner-Moriya mean-field theory it is given by (see Appendix A)

$$K(\lambda) = 4N_F\lambda \quad \text{(Stoner)} . \quad (2.13a)$$

In a Heisenberg spin model, one has\textsuperscript{32}

$$K = 2m \quad \text{(Heisenberg)} . \quad (2.13b)$$

The transverse susceptibility tensor takes the form

$$\chi_T(k) = \frac{1}{2} \left( \begin{array}{cc} \chi^+(k) + \chi^-(k) & i[\chi^+(k) - \chi^-(k)] \\ -i[\chi^+(k) - \chi^-(k)] & \chi^+(k) + \chi^-(k) \end{array} \right) \quad (2.14a)$$

where $k \equiv (k, i\Omega)$. Explicitly one has for small $k$ and $\Omega$

$$\chi_T(k) = \frac{K(\lambda)}{(2N_F1)^2} \frac{1}{\omega_0(k, \Omega)^2 - (i\Omega)^2} \left( \begin{array}{c} D(\lambda)k^2 - i(i\Omega) \\ i(i\Omega) \end{array} \right) \quad (2.14b)$$

In Appendix A we show how this structure emerges in an explicit model calculation.

The magnon exchange interaction, Eq. (2.5b), can now be written

$$S_{ex}[\psi, \bar{\psi}] = \frac{1}{2} \sum_{\sigma, \sigma'} \int_k \delta n_{\sigma\sigma'}(k) V_{\sigma\sigma'}(k) \delta n_{\sigma'\sigma}(-k) \quad \text{(2.15a)}$$

Here $\int_k \equiv (1/V) \sum_k T \int_{-\Omega}^{\Omega}$, and the effective potential is given by

$$V_{\sigma\sigma'}(k) = V_{\sigma\sigma'}(k) + V_{\sigma\sigma'}(-k) \quad \text{(2.15b)}$$

with

$$V_{\sigma\sigma'}(k) = (1 - \delta_{\sigma\sigma'}) \Gamma_1^2 \chi^\sigma(k) \quad \text{(2.15c)}$$

This effective interaction is shown diagrammatically in Fig. [1]. Notice that the exchange of magnons couples only electrons with opposite spin projections, i.e., it leads to inter-Stoner-band scattering only. This is in contrast to the case of helimagnets, where there is an intra-Stoner-band contribution whose prefactor is proportional to the square of the helical pitch wave number\textsuperscript{23}

We add a few remarks concerning the validity of this effective action. We have assumed that the conduction electrons are subject to a magnetization and magnetic fluctuations of unspecified origin whose dynamics are governed by the physical magnetic susceptibility. Integrating out these fluctuations leads to an effective action that is purely electronic. Since the feedback of the conduction electrons on the magnetic susceptibility has already been built into the effective action, the latter must not be used in ways that constitute, directly or indirectly, a renormalization of the susceptibility; doing so would constitute double counting. However, it is safe to use the effective action for perturbative calculations of any observable to first order in the effective potential given by $\Gamma_1^2 \chi$, and we will use it to calculate the quasiparticle and transport lifetimes to that order. We also note that the validity of this procedure is more obvious in cases where the magnetization is due to localized electrons in a band different from the conduction band than in the case of itinerant magnets. However, the coupling of the spin density to the magnetization fluctuations produced by the other electrons is still the same if all electrons are in the same band, and with the above caveats the effective action is still valid in that case. To illustrate this point we consider the ordinary Fermi-liquid contribution to the electronic relaxation rate, as well as the one due to phonons, in Appendix B, where we demonstrate that a reasoning for density fluctuations that is analogous to the one given above for magnetization fluctuations leads to the standard results for the relaxation rate in these cases.

### C. Energy scales

Before we use the effective action to calculate the single-particle and transport relaxation rates, let us discuss the relevant energy scales and their relation to experimentally observable quantities. Here we do so for the simple case of one conduction band; in Sec. [V] we will disc-
cuss the more complicated, and more realistic, situation that arises from the presence of several bands.

The most obvious fundamental magnetic energy scale is the Stoner gap $\lambda$, or the closely related exchange splitting $\delta E_{\text{ex}} = 2\lambda$. It can be measured by photoemission, and also obtained by band structure calculations. The smallest wave number that can be transferred by means of magnon exchange is $k_0 = \delta E_{\text{ex}}/v_F$. For a parabolic band this corresponds to $k_0 = k_F^+ - k_F^-$, but the above expression is more general. The smallest energy that can be transferred by magnon exchange is thus

$$T_0 = D k_0^2 \approx \frac{1}{4} D k_F^2 (\delta E_{\text{ex}}/\epsilon_F)^2 .$$  \hspace{1cm} (2.16)

The largest momentum transfer is given by $k_1 \approx 2k_F$, and we thus have another energy scale,

$$T_1 = 4Dk_F^2 .$$  \hspace{1cm} (2.17)

$T_1$ one expects to be close to the exchange splitting; within Stoner theory one has $T_1 = 2\lambda/3 = \delta E_{\text{ex}}/3$. Finally, the microscopic energy scale is given by the Fermi energy $\epsilon_F$, and we have a hierarchy of energy scales, viz., $T_0 \ll T_1 \ll \epsilon_F$. In particular, the ratio $T_0/T_1$ is given in terms of the Stoner gap in units of the microscopic energy,

$$T_0/T_1 \approx \frac{1}{4} (\lambda/\epsilon_F)^2 .$$  \hspace{1cm} (2.18a)

Alternatively, we can use $n_e/2N_F$ as the microscopic energy scale and express the ratio $T_0/T_1$ in terms of the magnetization $m$,

$$T_0/T_1 \approx \frac{1}{9} (m/n_e)^2 .$$  \hspace{1cm} (2.18b)

Within Stoner theory this relation holds for $\lambda/\epsilon_F \ll 1$, see Eq. (2.17), but as an order-of-magnitude estimate it is expected to hold much more generally.

We finally mention that crystal-field effects break spin-rotational invariance, which gives the magnons a small gap and leads to yet another energy scale that affects the relaxation rates at very low temperatures. The magnitude of this effect is highly material dependent, and we neglect it for simplicity.

III. SINGLE-PARTICLE RELAXATION RATE

We now calculate the single-particle inelastic relaxation rate due to the exchange of magnons. To linear order in the effective potential, Eq. (2.15a) yields two contributions to the electronic self energy $\Sigma$, which are shown in Fig. 2. Analytically, we have

$$\Sigma_\sigma(p) = \int k \sum_{\sigma'} \nu_{\sigma\sigma'}(k) G_{\lambda,\sigma'}(p + k)$$

$$= 2\Gamma_1^2 \int_k \chi_\sigma(k) G_{\lambda,-\sigma}(p + k) .$$  \hspace{1cm} (3.1)

FIG. 2: Self-energy contributions $\Sigma_\sigma(p)$ for the $\sigma$-spin Green function.

Here we have defined the self energy such that the full Green function $G$ is given by a Dyson equation

$$g^{-1}_\sigma(p) = G^{-1}_{\lambda,\sigma}(p) - \Sigma_\sigma(p) .$$  \hspace{1cm} (3.2)

Now we consider the single-particle relaxation rate $\Gamma$ for a spin-$\sigma$ quasiparticle, averaged over the Fermi surface:

$$\Gamma_\sigma(\epsilon) = \frac{1}{N_F^\sigma V} \sum_p \delta(\omega_{\sigma}(p)) \Sigma''_{\sigma}(p, \epsilon) ,$$  \hspace{1cm} (3.3)

where $\Sigma''_{\sigma}(p, \epsilon) = \text{Im} \Sigma(p, i\omega \rightarrow \epsilon + i0)$ is the spectrum of the self energy. Using a spectral representation for the effective potential and performing the Matsubara frequency sum in Eq. (3.1) we find

$$\Gamma_\sigma(\epsilon) = N_F^{-\sigma} \int du \left[ n_B(u) + n_F(u + \epsilon) \right] \sum_{\sigma'} \bar{\nu}'_{\sigma\sigma'}'(u)$$

$$= 2\Gamma_1^2 N_F^{-\sigma} \sum_{\sigma'} \left[ n_B(u) + n_F(u + \epsilon) \right] \bar{\chi}_{\sigma'}'(u) ,$$  \hspace{1cm} (3.4a)

where $n_B(u) = 1/(e^{u/T} - 1)$ and $n_F(u) = 1/(e^{u/T} + 1)$ are the Bose and Fermi distribution functions, respectively. Here we have defined

$$\bar{\nu}'_{\sigma\sigma'}(u) = \frac{1}{N_F^\sigma N_F'^{\sigma'} V^2} \sum_{k,p} \delta(\omega_{\sigma}(k)) \delta(\omega_{\sigma'}(p))$$

$$\times \nu''_{\sigma\sigma'}(k - p, u)$$  \hspace{1cm} (3.4b)

and analogously

$$\bar{\chi}_{\sigma'}'(u) = \frac{1}{N_F^\sigma N_F'^{\sigma'} V^2} \sum_{k,p} \delta(\omega_{\sigma}(k)) \delta(\omega_{\sigma'}(p)) \chi''_{\sigma'}(k - p, u) ,$$  \hspace{1cm} (3.4c)

with

$$\chi''_{\sigma}(k, u) = \mp K(\lambda) \pi \delta(\omega_0(k) + u)$$  \hspace{1cm} (3.4d)

Here we have used

$$\text{Im} \Sigma(p, i\omega \rightarrow \epsilon + i0) = \frac{1}{N_F^\sigma V} \sum_{\sigma'} \bar{\nu}'_{\sigma\sigma'}'(u)$$

$$= 2\Gamma_1^2 N_F^{-\sigma} \sum_{\sigma'} \left[ n_B(u) + n_F(u + \epsilon) \right] \bar{\chi}_{\sigma'}'(u) ,$$

where $\nu''_{\sigma\sigma'}(u) = \text{Im} \nu''_{\sigma\sigma'}(u)$ and $\chi''_{\sigma'}(u) = \text{Im} \chi''_{\sigma'}(u)$ are the spectral functions of the self energy.

Finally, we have

$$\bar{\nu}'_{\sigma\sigma'}(u) = \frac{1}{N_F^\sigma N_F'^{\sigma'} V^2} \sum_{k,p} \delta(\omega_{\sigma}(k)) \delta(\omega_{\sigma'}(p))$$

$$\times \nu''_{\sigma\sigma'}(k - p, u)$$  \hspace{1cm} (3.4b)

and analogously

$$\bar{\chi}_{\sigma'}'(u) = \frac{1}{N_F^\sigma N_F'^{\sigma'} V^2} \sum_{k,p} \delta(\omega_{\sigma}(k)) \delta(\omega_{\sigma'}(p)) \chi''_{\sigma'}(k - p, u) ,$$  \hspace{1cm} (3.4c)

with

$$\chi''_{\sigma}(k, u) = \mp K(\lambda) \pi \delta(\omega_0(k) + u)$$  \hspace{1cm} (3.4d)
the spectra of the susceptibilities $\chi_\pm$ defined in Eq. (2.12). We note the symmetry relation

$$N_F^+ \Gamma_+(\epsilon) = N_F^- \Gamma_-(\epsilon) ,$$

(3.4e)

which follows from the symmetry properties of $\tilde{\chi}_\sigma'(u)$.

Notice that the wave vectors $k$ and $p$ in Eq. (3.4a) are pinned to different Fermi surfaces as a result of the pure inter-Stoner-band scattering mentioned after Eq. (2.15b). The spectrum $\tilde{\chi}_\sigma'(u)$ will therefore be nonzero only for frequencies

$$T_0 \leq |u| \leq T_1 ,$$

(3.5)

with $T_0$ and $T_1$ given by Eqs. (2.16) and (2.17).

On the energy shell, $\epsilon = 0$, we obtain for the relaxation rate $1/\tau$ on the $\sigma$-Fermi surface

$$1/2\tau_\sigma = \Gamma_\sigma(\epsilon = 0) = \frac{\pi K}{2N_F^0 T_1} \int_{T_0/T}^{T_1/T} \frac{dx}{\sinh x}$$

$$= \frac{\pi K}{N_F^0 T_1} \times \left\{ \begin{array}{ll}
T e^{-T_0/T} & \text{if } T \ll T_0 \\
\frac{1}{2} T \ln(T/T_0) & \text{if } T_0 \ll T \leq T_1 \\
\frac{1}{2} \ln(T_1/T_0) T & \text{if } T \gg T_1 .
\end{array} \right.$$

(3.6)

For the thermal resistivity $\rho_{th} = 1/\kappa$ this implies

$$\rho_{th} = \frac{6}{v_F^0 e^2} \frac{\pi K}{T_1} \times \left\{ \begin{array}{ll}
T e^{-T_0/T} & \text{if } T \ll T_0 \\
\frac{1}{2} T \ln(T/T_0) & \text{if } T_0 \ll T \leq T_1 \\
\frac{1}{2} \ln(T_1/T_0) T & \text{if } T \gg T_1 .
\end{array} \right.$$

(3.7)

In Eq. (3.7) the prefactor is valid in the limit $\lambda \rightarrow 0$; more generally there are corrections of $O((\lambda/e_F)^2)$.

The second line in Eqs. (3.6) and (3.7) is valid to leading logarithmic accuracy only. We see that at asymptotically low temperatures the relaxation rate is exponentially small, and that in the pre-asymptotic temperature window $T_0 \ll T \ll T_1$ there is a logarithmic correction to the linear behavior. We will further discuss these results in Sec. V.

IV. TRANSPORT RELAXATION RATE

We now turn to the transport relaxation rate, which determines the electrical resistivity. The latter is the inverse of the electrical conductivity, which is given by the Kubo formulix:

$$\sigma_{ij}(i\Omega) = \frac{i}{i\Omega} [\pi_{ij}(i\Omega) - \pi_{ij}(i\Omega = 0)] ,$$

(4.1a)

where the tensor

$$\pi_{ij}(i\Omega) = -e^2 T \sum_{n_1,n_2} \frac{1}{V} \sum_k \langle v_i(k) v_j(p) \rangle$$

$$\times \langle \tilde{\psi}_{n_1,\sigma'}(k) \tilde{\psi}_{n_1+n,\sigma}(k) \tilde{\psi}_{n_2,\sigma'}(p) \tilde{\psi}_{n_2-n,\sigma'}(p) \rangle .$$

(4.1b)

is the current-current susceptibility or polarization function. Here $v(k) = \partial \epsilon_k/\partial k$, and the average is to be taken with the effective action, Eq. (2.5a). The four-fermion correlation function in Eq. (4.1b) is conveniently expressed in terms of the single-particle Green function

$$G_\sigma(p) = 1/(G_{\lambda,\sigma}(p) - \Sigma_\sigma(p))$$

(4.2)

and a vector vertex function $\Gamma_\sigma$ with components $\Gamma_\sigma$:

$$\pi_{ij}(i\Omega) = -ie^2 T \sum_{i\omega} \frac{1}{V} \sum_{p,\sigma} \langle \tilde{\psi}_{n,\sigma}(p) \rangle$$

$$\times \Gamma_\sigma(p,i\omega) \times \Gamma_\sigma(p,i\omega - i\Omega) .$$

(4.3)

Here we have assumed a quadratic dependence of $\epsilon_k$ on $k$ for simplicity. It is important to calculate the vertex function $\Gamma$ and the self energy $\Sigma$ in mutually consistent approximations. We use the familiar procedure that consists of a self-consistent Born approximation for the self energy, which to linear order in the potential $V$ is represented by Eq. (3.4), and a ladder approximation for the vertex function,

$$\Gamma_\sigma(p;i\omega,i\omega - i\Omega) = i \frac{p}{m_e} + \frac{T}{V} \sum_{k,i\Omega'} \sum_{\sigma'} V_{\sigma\sigma'}(k-p,i\Omega')$$

$$\times G_{\sigma'}(k,i\omega+i\Omega') G_{\sigma'}(k,i\omega - i\Omega + i\Omega')$$

$$\times \Gamma_{\sigma'}(k;i\omega + i\Omega',i\omega - i\Omega + i\Omega') .$$

(4.4)

We mention that umklapp processes, which are not explicitly considered here, are necessary in order to obtain a nonzero transport relaxation rate. In fact, in a Galilean invariant system the electrical resistivity vanishes due to momentum conservation and the contributions contained in our approximation are cancelled by terms not included in the ladder approximation. However, the above approximation is effectively valid in the presence of umklapp processes, as is the case for Coulomb scattering. If we define a scalar vertex function $\gamma$ by $\Gamma(p;i\omega,i\omega') = i(p/m_e)\gamma(p;i\omega,i\omega')$, then the Bethe-Salpeter equation for the latter becomes

$$\gamma_{\sigma}(p;i\omega,i\omega - i\Omega) = 1 + \frac{T}{V} \sum_{k,i\Omega'} \sum_{\sigma'} V_{\sigma\sigma'}(p-k,i\Omega')$$

$$\times \frac{p \cdot k}{p^2} G_{\sigma'}(k,i\omega+i\Omega') G_{\sigma'}(k,i\omega - i\Omega + i\Omega')$$

$$\times \gamma_{\sigma'}(k;i\omega - i\Omega',i\omega - i\Omega - i\Omega') .$$

(4.5)

The polarization and conductivity tensors are diagonal, $\sigma_{ij}(i\Omega) = \delta_{ij} \sigma(i\Omega)$, and the sum over Matsubara frequencies in Eq. (4.3) can be transformed into an integral along the real axis. In the limit of low temperature, the imaginary part of the self energy, which yields the relaxation rate, goes to zero as we have seen in the preceding subsection. The real part just renormalizes the Fermi energy. The relevant limit is thus the one of a vanishing self energy, and in this limit the leading contributions to the integral come from terms where the frequency arguments of the two Green functions lie on different sides of
the real axis. In the static limit, the Kubo formula for the conductivity \( \sigma = \lim_{\Omega \to 0} \text{Re} (i\Omega \to \Omega + i0) \), thus becomes

\[
\sigma = \frac{e^2}{3\pi n_e} \int_{-\infty}^{\infty} \frac{de}{4T} \frac{1}{\cosh^2(e/2T)} \frac{1}{V} \sum_p p^2 \\
\times \sum_\sigma |G_\sigma(p, \epsilon + i0)|^2 \gamma_\sigma(p; \epsilon + i0, \epsilon - i0). \tag{4.6}
\]

The pole of the Green function ensures that the dominant contribution from the momentum integral comes from the momenta that obey \( \omega_\sigma(p) = \epsilon \). Furthermore, since \( \epsilon \) scales as \( T \), for the leading \( T \)-dependence we can neglect all \( \epsilon \)-dependencies that do not occur in the form \( \epsilon/T \). Equation (4.6) then reduces to

\[
\sigma = \frac{e^2}{2m_e} \int_{-\infty}^{\infty} \frac{de}{4T} \frac{1}{\cosh^2(e/2T)} \sum_\sigma n_\sigma(\epsilon) \Lambda_\sigma(\epsilon). \tag{4.7a}
\]

Here \( n_\sigma \) is the density of the \( \sigma \)-spin electrons, \( \Gamma_\sigma \) is the single-particle rate defined by Eq. (3.3), and

\[
\Lambda_\sigma(\epsilon) = \frac{1}{N_F^\sigma} \sum_p \delta(\omega_\sigma(p)) \gamma_\sigma(p; \epsilon + i0, \epsilon - i0). \tag{4.7b}
\]

Using analogous arguments we find, from Eq. (4.5), that \( \Lambda_\sigma(\epsilon) \) obeys an integral equation

\[
\Lambda_\sigma(\epsilon) = 1 + \sum_\sigma N_F^\sigma \int_{-\infty}^{\infty} du \ W_{\sigma\sigma'}(u) \left[ n_B(u) + n_F(u + \epsilon) \right] \\
\times \frac{\Lambda_{\sigma'}(\epsilon + u)}{\Gamma_{\sigma'}(\epsilon + u)}. \tag{4.8a}
\]

where

\[
W_{\sigma\sigma'}(u) = \frac{1}{N_F^\sigma N_F^{\sigma'}} \frac{1}{V^2} \sum_{p,k} \sum_{\sigma \neq \sigma'} \delta(\omega_\sigma(k)) \delta(\omega_{\sigma'}(p)) \\
\times \gamma_{\sigma\sigma'}(k - p, u) k \cdot p / p^2 \tag{4.8b}
\]

with \( \gamma_{\sigma\sigma'}(k - p, u) \) the spectrum of the effective potential defined in Eq. (2.13).

Now we exploit the fact that \( k \) and \( p \) are pinned to the respective Fermi surfaces, and use the resulting identity

\[
k \cdot p = k_F^2 \left[ 1 - \omega_0(k - p)/2Dk_F^2 \right]
\]

to write

\[
W_{\sigma\sigma'}(u) = (k_F/k_F')^2 \left[ \tilde{\gamma}_{\sigma\sigma'}''(u) - \tilde{\gamma}_{\sigma\sigma'}''(u) \right] \tag{4.9}
\]

with \( \tilde{\gamma}_{\sigma\sigma'}'' \) from Eq. (3.4b), and

\[
\tilde{\gamma}_{\sigma\sigma'}''(u) = \frac{1}{N_F^\sigma N_F^{\sigma'}} \frac{1}{V^2} \sum_{k,p} \delta(\omega_\sigma(k)) \delta(\omega_{\sigma'}(p)) \frac{\omega_0(k - p)}{2Dk_F^2} \\
\times \gamma_{\sigma\sigma'}''(k - p, u). \tag{4.10}
\]

Note that the magnon frequency \( \omega_0 \) in Eq. (4.10) is equal to \( \pm u \) on account of the spectrum, and therefore \( \tilde{\gamma}_{\sigma\sigma'}''(u) \) has an extra factor of \( u \) compared to \( \tilde{\gamma}_{\sigma\sigma'}''(u) \).}

The integral equation for the vertex function \( \Lambda \) now reads

\[
\Lambda_\sigma(\epsilon) = 1 + \frac{(k_F/k_F')^2}{N_F^\sigma} \int_{-\infty}^{\infty} du \sum_{\sigma'} N_F^{\sigma'} \left[ \tilde{\gamma}_{\sigma\sigma'}''(u) - \tilde{\gamma}_{\sigma\sigma'}''(u) \right] \\
\times \left[ n_B(u) + n_F(u + \epsilon) \right] \frac{\Lambda_{\sigma'}(u + \epsilon)}{\Gamma_{\sigma'}(u + \epsilon)}, \tag{4.12}
\]

For the case of a spin-independent potential, Eq. (4.12) reduces to the integral equation familiar from the electron-phonon scattering problem; only the \( u \)-dependence of the kernel is different. This integral equation is usually solved in the seemingly uncontrolled approximation, which turns Eq. (4.12) into two coupled algebraic equations for \( \Lambda_\pm(\epsilon) \). Since the prefactor of the temperature dependence of the conductivity is approximation-dependent anyway, we can put \( \epsilon = 0 \) and use the temperature-dependent rates \( \Gamma_\sigma \equiv \Gamma_\sigma(\epsilon = 0) \) and vertices \( \Lambda_\sigma \equiv \Lambda_\sigma(\epsilon = 0) \) in the Kubo formula, Eq. (4.1a). \( \Lambda_\sigma \) then obeys

\[
\Lambda_+ = 1 + (k_F/k_F')^2 \left[ \Gamma_+ - \Gamma_+^{(2)} \right] \Lambda_- / \Gamma_+, \tag{4.13}
\]

\[
\Lambda_- = 1 + (k_F/k_F')^2 \left[ \Gamma_- - \Gamma_-^{(2)} \right] \Lambda_+ / \Gamma_-, \tag{4.13}
\]

\[\Gamma_\sigma^{(2)} = \frac{\pi K}{N_F^\sigma} \frac{T^2}{T_1} \int_{T_0/T}^{T_1/T} dx \frac{x}{\sinh x} \]

\[\chi_{\sigma\sigma'}(T) = \frac{\pi K}{N_F^\sigma} \frac{T^2}{T_1} \times \left\{ \begin{array}{ll} \frac{2\pi T}{T_1} \left( 1 + \frac{T}{T_0} \right) \exp(-T_0/T) & \text{if } T \ll T_0 \\ \frac{2\pi^2 T}{T_1} & \text{if } T_0 \ll T \ll T_1 \\ \frac{1}{T} & \text{if } T \gg T_1 \end{array} \right. \]
Comparing with Eq. ([3.6]) we see that for asymptotically small $T$, $\Gamma^{(2)}$ is proportional to $\Gamma$ with a small factor of proportionality $2T_0/T_1 \ll 1$, whereas for $T_0 \ll T_1$ it carries an additional factor of temperature.

We now solve the equations ([4.13]). Neglecting $\lambda/\epsilon_F \ll 1$ wherever it is not of qualitative importance, we find

$$\Lambda \pm = \frac{\Gamma\pm \pm \Gamma - \Gamma^{(2)}}{\pm \Gamma - \Gamma^{(2)} + \pm \Gamma - \Gamma^{(2)} - \pm \Gamma^{(2)}\Gamma^{(2)}}.$$  

Equation ([3.4e]) allows us to express $\Lambda\pm/\Gamma\pm$ entirely in terms of $\Gamma\sigma$ and $\Gamma^{(2)}\sigma$, Neglecting all prefactors that just give small corrections of $O(\lambda/\epsilon_F)$ to factors of $O(1)$ we finally obtain a transport relaxation time

$$\tau_{tr} = \frac{\Gamma - \Gamma^{(2)}/2}{\frac{4\hbar}{2T_1}(\Gamma)^2 + 2\Gamma(\Gamma^{(2)}) - (\Gamma^{(2)})^2},$$  

in terms of which the electrical conductivity is given by a Drude formula

$$\sigma = \frac{n_e e^2}{m_e} \tau_{tr},$$  

We see that in the preasymptotic temperature window $T_0 \ll T \ll T_1$ we recover the $T^2$ behavior that was obtained in Ref. [13] but for asymptotically low temperatures we obtain an exponentially small result that has the form of Eq. ([1.1]). We will discuss this result in the next section.

### V. DISCUSSION

To summarize our results, we have presented a very general theory of electron relaxation due to the exchange of magnons in metallic ferromagnets. The theory is valid for both itinerant ferromagnets, where the magnetization is due to the conduction electrons themselves, and for localized-moment ferromagnets, where the magnetization is due to localized spins in a different band. We have found that at asymptotically low temperatures, below a temperature scale $T_0$, both the single-particle relaxation rate and the transport relaxation rate are exponentially small. This behavior carries over to the magnon-exchange contributions to the thermal and electrical resistivities, which are determined by these respective rates. The exponential temperature dependence is a direct consequence of the split conduction band in a metallic ferromagnet. In a preasymptotic temperature regime $T_0 \ll T \ll T_1$, with $T_1$ close to the exchange splitting, we recover the $T^2$ behavior of the transport rate that was found in Ref. [13]. The single-particle rate is proportional to $T$ in this regime. For $T \gg T_1$ the two rates both show a linear temperature dependence.

We start our discussion of these results by recalling the physical reason for the exponential dependence at low temperatures. Figure 5 schematically shows the split conduction band (a), and the densities of states for the up (+) and down (-) spin electrons (b), for the case of a spherical Fermi surface. Since the magnons couple only electrons with opposite spin, the smallest transferable wave number is $k_0 = k_F^+ - k_F^- \approx \Delta E_{ex}/v_F$. Given the magnon dispersion relation, $\omega = Dk^2$, this translates into a smallest transferable energy $T_0 = Dk_0^2$, and since the magnon stiffness coefficient $D$ is itself roughly proportional to $\Delta E_{ex}$, we have $T_0 \propto (\Delta E_{ex})^3$. For temperatures $T \ll T_0$ the relaxation rates will thus show activated behavior with an activation energy $T_0$. The exponential behavior is multiplied by a power law that cannot be captured by elementary arguments. The largest momentum transfer is given by $k_1 = k_F^+ + k_F^- \approx 2k_F$, and the corresponding largest energy transfer is $T_1 = Dk_1^2 \approx \Delta E_{ex}$. $T_1$ is the fundamental magnetic energy scale, analogous
to the Debye temperature $\Theta_D$ in the case of electron-phonon coupling. $T_0$ has no analog in the electron-phonon problem. For $T \ll T_1$ the transport-relaxation rate is small compared to the single-particle rate by a factor of $T/T_1$. This is analogous to the electron-phonon case, where the corresponding factor is $(T/\Theta_D)^2$. The difference between our results and those of Ueda and Moriya, Ref. [15] can be traced to the fact that these authors neglected the exchange splitting in the final stages of their calculation. As a result, they obtained a $T^2$ behavior of the transport relaxation rate at low temperatures, which in fact is valid only for temperatures larger than $T_0$. Note that this discrepancy pertains to the magnon or spin-wave contribution to the electrical resistivity only. The contributions from dissipative excitations, which we have not discussed, have been found to be unaffected by the exchange splitting and proportional to $T^2$ even at asymptotically low temperatures.[15]

For the power-law behavior at $T \gg T_0$ the quadratic spectrum of the magnons is important, and also the coupling of the electrons to the magnetic fluctuations. Comparing with the case of helical magnets[10] we notice one important difference with respect to the latter. In either case the Goldstone mode is a phase fluctuation, but in the ferromagnon case the electron spin density couples directly to the phase, whereas in the helimagnon case the coupling is to the gradient of the phase. This is because in the helimagnon case the dominant low-$T$ contribution to the scattering rates comes from intra-Stoner-band scattering. Within a given band, the phase itself has no physical meaning, and the coupling therefore involves a gradient. In the ferromagnetic case, on the other hand, we deal with inter-Stoner-band scattering. The coupling therefore effectively is to the difference of two phases, which does have a physical meaning. We note in passing that this latter notion also manifests itself in a spin Josephson effect, see Ref. [17].

We now turn to estimates of the values of $T_1$ and $T_0$. To get an idea about the order of magnitude of these temperature scales, let us first consider the fictitious case of simple (i.e., single-conduction-band) metals with magnetic properties as in the classic “high-temperature” ferromagnets nickel, cobalt, and iron. The values of the exchange splitting in these materials, as determined by photoemission, are $\Delta E_{\text{ex}} \approx 0.25$ eV, 1.0 eV, and 2.0 eV, respectively.[18,19] Values for the spin-stiffness coefficient $D$ in meVÅ$^2$ obtained from neutron scattering are 364 for Ni, 500 for Co, and 281 for Fe.[20] With a generic value $k_F \approx 1Å^{-1}$ for the Fermi wave number, and $\epsilon_F \approx 10^5$ K for the Fermi energy, Eqs. (2.16) and (2.17) yield $T_1 \approx 10,000 - 20,000$ K for these materials, and $T_0 \approx 500$ mK for Ni, 10 K for Co, and 30 K for Fe. Estimates of the ratio $T_0/T_1$ using the relation (2.18) yields similar results. Notice that the prefactor $\pi K/N_F T_1$ in Eq. (4.18) is of order unity, so the prefactor of the $T^2$ behavior of the resistivity is larger than the Fermi-liquid $T^2$ contribution (see Eq. (B7)) by roughly a factor of $\epsilon_F/T_1 \approx 10$ in a single-band model.

Also of interest are weak ferromagnets, such as MnSi,[20] or NiAl, where $D \approx 23.5$ meVÅ$^2$ (MnSi) and $D \approx 70$ meVÅ$^2$ (NiAl), respectively. The magnetic moments, $0.4\mu_B$ per formula unit for MnSi[21] and $0.17\mu_B$ for NiAl[22] are about two thirds and one third, respectively, of that of Ni. Given the observed near-linear correlation between the magnetic moment and the exchange splitting[23] this suggests $\Delta E_{\text{ex}} \approx 0.17$ eV for MnSi and $\Delta E_{\text{ex}} \approx 0.07$ eV for NiAl. If we use again $k_F \approx 1Å^{-1}$ and $\epsilon_F \approx 10^5$ K, this yields $T_1 \approx 1,000$ K and $T_0 \approx 20$ mK for MnSi, and $T_1 \approx 2,800$ K and $T_0 \approx 10$ mK for NiAl.

In reality, all of these materials are transition metals, or compounds containing transition metals, with a complicated band structure and Fermi surfaces that consist of multiple sheets. One consequence of this is that the electron-electron scattering contribution to the electrical resistivity is likely much larger than a single-band model would imply, and it has been suggested that it makes the largest contribution to the observed $T^2$ behavior at low temperatures.[18] The reason is that different band edges have different distances from the common chemical potential, which in effect leads to different Fermi temperatures. Depending on whether or not the various scattering processes flip the electron spin, and whether or not they couple different sheets of the Fermi surface, the relaxation rates or the relaxation times may be additive, which leads a complicated structure of the overall resistivity. In addition, there are the contributions from the dissipative spin excitations, which also are proportional to $T^2$.[15] As a result, the low-temperature transport rate in Fe, Ni, and Co is about 100 times larger than one would expect from the Coulomb contribution in a single-band model with a single Fermi temperature of about $0.05 K T_1$, on the other hand, is largely unaffected by a complicated band structure: It is given by $D$ times the largest possible momentum transfer squared, see Sec. [11C] and in a good metal the latter is on the order of $2\pi/a$, with $a$ the lattice constant, which is close to the value of $2k_F$ for a single spherical Fermi surface.

FIG. 3: Fermi surfaces and associates Fermi wave numbers (a) and densities of states (b) for the up- and down-spin electrons. $\lambda$ is the Stoner gap, and $\Delta E_{\text{ex}}$ is the exchange splitting. $k_0$ and $k_1$ are the smallest and largest transferrable wave numbers, respectively. See the text for more explanation.
that yields the same electron density. The estimates of
the temperature scale $T_1$, which is the magnetic analog
of the Debye temperature for phonons, given above are
therefore model independent and depend only on the ex-
perimentally measured spin stiffness coefficient.

As a result, we expect the magnon contribution to the
electrical resistivity in Fe, Ni, and Co at temperatures
$T > T_0$ to be about an order of magnitude less than
the combined contribution from the Coulomb interaction
and the dissipative magnetic excitations. In MnSi and
Ni$_3$Al $T_1$ is much lower and the magnon scattering is ac-
cordingly stronger. However, the observed prefactors of
the $T^2$ term in the resistivity of MnSi and Ni$_3$Al are or-
ders of magnitude larger than even the ones in Fe, Ni,
and Co, and the same is true for the weak ferromagnet
ZrZn$_2$. The prefactor $\pi K/N_T T_1$ in Eq. (4.18) is ex-
pected to be of $O(1)$ not just in model calculations, but
also in real materials, since both $K$ and $T_1$ correlated
roughly with the magnetization. Given the above discus-
sion of the relatively narrow range of plausible values of
$T_1$, we conclude that the experimental value of the pre-
factor of the $T^2$ term in the electrical resistivity of weak
ferromagnets cannot be explained by electron-
magnon scattering. We emphasize again, however, that
these considerations do not take into account the scatter-
ing of electrons by dissipative magnetization fluctuations,
which lead to a $T^2$ contribution to the resistivity even at
low temperatures and whose prefactor is not as universal
as that of the magnon-exchange contribution. A corre-
sponding statement holds for the Coulomb contribution.

For $T > T_0$ the influence of the band structure is more
complicated. Consider the effective potential given by Eqs.
(3.4b - 3.4d). If the up-spin and down-spin electrons,
respectively, belong to different bands with different ef-
ective masses, then there will be a lower cutoff for the fre-
quency $u$ even in the limit of a vanishing Stoner gap,
$\lambda \to 0$. For magnon-exchange scattering between elec-
trons in Stoner-subbands of the same band, on the other
hand, the structure of the calculations in Secs. III and IV
is unchanged. We therefore expect different values of $T_0$
for the various scattering processes that involve electrons
on different sheets of the Fermi surface.

The following picture now emerges. With decreasing
temperature, contributions to the magnon-exchange part
of the electronic scattering rate will sequentially freeze
out as the temperature drops below a sequence of temper-
ature scales $T_0$. Rough estimates for the lowest of these
temperature scales have been given above; estimating the
higher ones requires a detailed analysis of the band struc-
ture. Below this lowest $T_0$ the magnon contribution to
both the transport rate and the single-particle rate will
be exponentially small, leaving the Coulomb contribution
and the one from dissipative magnetization fluctuations
as the most obvious candidates for a $T^2$ behavior. Exper-
imentally, this is expected to manifest itself in a distinct
temperature dependence of the prefactor of the $T^2$ term
in the electrical resistivity. It is desirable for the relevant
temperature scales to be small enough that phonon con-
tributions are negligible. In that respect, Fe, Ni, and Co
are not ideal. In MnSi and Ni$_3$Al, the helical nature of the magnetic
phase is expected to manifest itself on the temperature
scale given by $T_0$. This leaves Ni$_3$Al, or other true weak
ferromagnets as the most promising candidates for ob-
servering this consequence of the exchange splitting in a
metallic ferromagnet. We stress, however, that accord-
ing to the above discussion the magnon contribution to
the electrical resistivity in weak ferromagnets is likely
dwarfed by other contributions.

Another possible effect of a complicated band structure
is that there may be points or lines in reciprocal space
where the two Stoner band cross. This will weaken the
exponential suppression of the relaxation rates, but the
weakening will depend on the nature of the crossing.

We finally mention that the interplay of quenched dis-
order with the scattering processes discussed above con-
stitutes an interesting problem that is likely important
for a quantitative understanding of real materials. For
fairly strong disorder, $\lambda\tau_d \ll 1$ with $\tau_d$ the elastic scat-
tering time, the theory of Ref. 33 applies and it is easy
to see that there is no exponential suppression of the
magnon contribution to the relaxation rates at low tem-
perature. A complete discussion of disorder effects con-
stitutes a separate problem.

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Appendix A: Stoner model for itinerant
ferromagnets

In this appendix we show how to recover the Stoner-
Moriya result for itinerant ferromagnets within the
present formalism. Our starting point is a fermionic ac-
tion
\[ S[\tilde{\psi}, \psi] = S_0[\tilde{\psi}, \psi] + \frac{\Gamma_t}{2} \int dx \, n_s(x) \cdot n_e(x) , \]  
(A1)
with $\Gamma_t$ the spin-triplet interaction amplitude that is re-
sponsible for ferromagnetism. Our notation is the same as
in Sec. III.

1. Mean-field approximation

A simple mean-field approximation that describes fe-
romagnetic order, and its coupling to the electron spin
density, consists of replacing one of the spin density fields
in Eq. (A1) by its expectation value according to
\[ n_s^2 \approx 2 \langle n_s \rangle \cdot n_s - \langle n_s \rangle^2 . \]  
(A2)
If we take the magnetic order to be in the 3-direction,  
\langle n_{a,3} \rangle = \delta_{i3} \lambda / \Gamma_t, \]  
then this approximation amounts to replacing the action \( S \) with an action \( S_\lambda \) that describes electrons with no spin-triplet interaction subject to a magnetic field of strength \( \lambda \) in the 3-direction:

\[
S_\lambda[\bar{\psi}, \psi] = S_0[\bar{\psi}, \psi] + \lambda \int dx \: n_{a,3}(x). \tag{A3}
\]

The remaining question pertains to the action one should use to calculate \( \langle n_{a,3} \rangle \); this choice determines \( \lambda \). The usual self-consistent mean-field requirement stipulates that this average be determined by \( S_\lambda \) itself:

\[
\lambda = \Gamma_t \langle n_{a,3}(x) \rangle_\lambda = \frac{\Gamma_t}{Z_\lambda} \int D[\bar{\psi}, \psi] \: n_{a,3}(x) \: e^{S_\lambda[\bar{\psi}, \psi]} = \frac{\Gamma_t}{Z_\lambda} \log Z_\lambda. \tag{A4}
\]

For simplicity, we take \( S_0 \) to describe free electrons. That is, we neglect all electron-electron interactions that are not crucial for magnetism, and we assume a parabolic band; a generalization to band electrons is straightforward. The Green function corresponding to \( S_0 \) then is

\[
G_0(k, i\omega_n) = \frac{1}{i\omega_n - \xi_k}, \tag{A5}
\]

and the self-consistency condition, Eq. \( \text{[A4]} \), takes the form

\[
1 = -2\Gamma_t \int_p \frac{1}{G_0^{-2}(p) - \lambda^2}. \tag{A6}
\]

We recognize this as the equation of state of Stoner theory, with \( \lambda \) the Stoner gap. The condition for a nonzero solution for \( \lambda \) is \( 2N_F \Gamma_t > 1 \), and by performing the integral we find explicitly

\[
\lambda = 2N_F \Gamma_t \frac{\epsilon_p}{3} \left[ (1 + \lambda/\epsilon_F)^{3/2} - (1 - \lambda/\epsilon_F)^{3/2} \right]. \tag{A7a}
\]

If we recall that the magnetization is given as \( m = \lambda / \Gamma_t \) (see Eqs. \( \text{[2.2]} \) \( \text{[2.3a]} \)), we can write this result as

\[
m = 2N_F \lambda \left[ 1 + O((\lambda/\epsilon_F)^2) \right]. \tag{A7b}
\]

The action \( S_\lambda \) contains information about the long-range order, but does not contain any ferromagnetic fluctuations. It will serve as a building block for the effective action, and we will refer to it as the “reference ensemble”.

We now determine the spin susceptibility

\[
\chi_{\lambda,ij}(x, y) = \langle \delta n_{a,i}(x) \delta n_{a,j}(y) \rangle_{S_\lambda}, \tag{A8}
\]

associated with the reference ensemble. In terms of the reference-ensemble Green function

\[
G_\lambda(k) = \frac{G_0^{-1}(k)}{G_0^{-2}(k) - \lambda^2} \sigma_0 - \frac{\lambda}{G_0^{-2}(k) - \lambda^2} \sigma_3, \tag{A9}
\]

\( \chi \) can be written

\[
\chi_{\lambda,ij}(x, y) = -\text{tr} \: [\sigma_i G_\lambda(x, y) \sigma_j G_\lambda(y, x)] \tag{A10}
\]

where the trace is over the spin degrees of freedom. Evaluating the trace, and performing a Fourier transform, we find

\[
\chi_{\lambda,ij}(k) = \left( \begin{array}{ccc} f_1(k) & f_2(k) & 0 \\ -f_2(k) & f_1(k) & 0 \\ 0 & 0 & f_3(k) \end{array} \right), \tag{A11}
\]

where

\[
f_1(k) = -2 \int_p \frac{G_0^{-1}(p)G_0^{-1}(p - k) - \lambda^2}{G_0^{-2}(p) - \lambda^2 [G_0^{-2}(p - k) - \lambda^2]}, \tag{A12a}
\]

\[
f_2(k) = -2i\lambda \int_p \frac{G_0^{-1}(p) - G_0^{-1}(p - k)}{G_0^{-2}(p) - \lambda^2 [G_0^{-2}(p - k) - \lambda^2]}, \tag{A12b}
\]

\[
f_3(k) = -2 \int_p \frac{G_0^{-1}(p)G_0^{-1}(p - k) + \lambda^2}{G_0^{-2}(p) - \lambda^2 [G_0^{-2}(p - k) - \lambda^2]}, \tag{A12c}
\]

We note that

\[
f_1(k = 0) = 1 / \Gamma_t, \tag{A13a}
\]

\[
f_2(k = 0) = 0, \tag{A13b}
\]

with the first equality following from the equation of state, Eq. \( \text{[A6]} \).

## 2. Physical spin susceptibility, and Goldstone modes

The reference ensemble does not reflect the magnons that are the Goldstone modes of the spontaneously broken symmetry in the ferromagnetic phase. To describe the magnons we need a theory of fluctuations that is consistent with the treatment of the static magnetization. Quite generally, a Gaussian approximation for the order-parameter fluctuations is consistent with a mean-field treatment of the order parameter itself\[15\]. To determine the former we first note that the reference ensemble spin susceptibility \( \chi_\lambda \) corresponds to a Gaussian fluctuation action

\[
A_{\text{fluct}}[\delta n_a] = -\frac{1}{2} \int dxdy \: \delta n_{a,i}(x) \chi_{\lambda,ij}(x, y) \delta n_{a,j}(y) \tag{A14a}
\]

that generates \( \chi_\lambda \) via

\[
\chi_{\lambda,ij}(x, y) = \int D[\delta n_a] \: \delta n_{a,i}(x) \delta n_{a,j}(y) \: e^{-A_{\text{fluct}}[\delta n_a]} \tag{A14b}
\]

To this we need to add the fluctuation contribution from the original spin-triplet interaction in Eq. \( \text{[A1]} \). The Gaussian fluctuation action then reads

\[
A_{\text{fluct}}[\delta n_a] = -\frac{1}{2} \int dxdy \: \delta n_{a,i}(x) \chi_{ij}^{-1}(x, y) \delta n_{a,j}(y) \tag{A15a}
\]
with the physical spin susceptibility $\chi$ given by
\[
\chi^{-1}_{ij}(x, y) = \chi^{-1}_{ij}(x, y) - \delta_{ij} \Gamma_t . \tag{A15b}
\]

Inverting Eq. (A17a) we obtain the transverse physical
spin susceptibility in the form given in Eq. (2.14b), with
\[
N(k, i\Omega_n) = (f_1(k, i\Omega_n))^2 + (f_2(k, i\Omega_n))^2 . \tag{A16b}
\]
From Eqs. (A13) we see that $\chi^{-1}_T$ at zero frequency and wave number has two zero eigenvalues. These reflect the two Goldstone modes. Expanding to linear order in $i\Omega$ and to second order in $k$ we find explicitly
\[
\chi^{-1}_T(k, i\Omega_n) = \left( \frac{2N_F \Gamma_t}{2\epsilon_F} \right)^2 \left( \begin{array}{ccc}
f_1(k, i\Omega_n)/N(k, i\Omega_n) - \Gamma_t & -f_2(k, i\Omega_n)/N(k, i\Omega_n) \\
f_2(k, i\Omega_n)/N(k, i\Omega_n) & f_1(k, i\Omega_n)/N(k, i\Omega_n) - \Gamma_t \end{array} \right) . \tag{A16a}
\]
where
\[
f_2(k, \lambda) = \frac{-4\epsilon_F^3}{5\lambda^3} \left( 1 - \frac{3\lambda}{2\epsilon_F} \right) \left( 1 + \frac{\lambda}{\epsilon_F} \right)^{3/2} - \left( 1 + \frac{3\lambda}{2\epsilon_F} \right) \left( 1 - \frac{\lambda}{\epsilon_F} \right)^{3/2} 
\]
\[
f_1(k, \lambda) = \frac{\epsilon_F}{3\lambda} \left( 1 + \frac{\lambda}{\epsilon_F} \right)^{3/2} - \left( 1 - \frac{\lambda}{\epsilon_F} \right)^{3/2} . \tag{A17b}
\]
Physically, the Stoner gap is always small compared to the Fermi energy, and it therefore is useful to consider the limit of weak ferromagnets, $2N_F \Gamma_t \approx 1$ and $\lambda/\epsilon_F \ll 1$, where we have
\[
f_2(k, \lambda \to 0) = f_1(k, \lambda \to 0) = 1 + O(\lambda^2) . \tag{A18}
\]
Focusing on the transverse (T) channel ($i = 1, 2$), and performing a Fourier transform, we have
\[
\chi^{-1}_T(k, i\Omega_n) = \left( \frac{2N_F \Gamma_t}{2\epsilon_F} \right)^2 \left( \begin{array}{ccc}
f_1(k, i\Omega_n)/N(k, i\Omega_n) - \Gamma_t & -f_2(k, i\Omega_n)/N(k, i\Omega_n) \\
f_2(k, i\Omega_n)/N(k, i\Omega_n) & f_1(k, i\Omega_n)/N(k, i\Omega_n) - \Gamma_t \end{array} \right) . \tag{A16a}
\]
where
\[
N(k, i\Omega_n) = (f_1(k, i\Omega_n))^2 + (f_2(k, i\Omega_n))^2 . \tag{A16b}
\]
From Eqs. (A13) we see that $\chi^{-1}_T$ at zero frequency and wave number has two zero eigenvalues. These reflect the two Goldstone modes. Expanding to linear order in $i\Omega$ and to second order in $k$ we find explicitly
\[
\chi^{-1}_T(k, i\Omega_n) = \left( \frac{2N_F \Gamma_t}{2\epsilon_F} \right)^2 \left( \begin{array}{ccc}
f_1(k, i\Omega_n)/N(k, i\Omega_n) - \Gamma_t & -f_2(k, i\Omega_n)/N(k, i\Omega_n) \\
f_2(k, i\Omega_n)/N(k, i\Omega_n) & f_1(k, i\Omega_n)/N(k, i\Omega_n) - \Gamma_t \end{array} \right) . \tag{A16a}
\]
where
\[
f_2(k, \lambda) = \frac{-4\epsilon_F^3}{5\lambda^3} \left( 1 - \frac{3\lambda}{2\epsilon_F} \right) \left( 1 + \frac{\lambda}{\epsilon_F} \right)^{3/2} - \left( 1 + \frac{3\lambda}{2\epsilon_F} \right) \left( 1 - \frac{\lambda}{\epsilon_F} \right)^{3/2} 
\]
\[
f_1(k, \lambda) = \frac{\epsilon_F}{3\lambda} \left( 1 + \frac{\lambda}{\epsilon_F} \right)^{3/2} - \left( 1 - \frac{\lambda}{\epsilon_F} \right)^{3/2} . \tag{A17b}
\]

Appendix B: Single-particle scattering rate in a Fermi liquid due to Coulomb and electron-phonon interactions

As a further illustration of our arguments leading to an effective action for calculating relaxation rates, let us consider the well-known case of quasiparticle relaxation due to density fluctuations. To this end, we consider the very simple case of spinless, noninteracting electrons with action $S_0$, and add a statically screened Coulomb interaction
\[
S_{\text{int}} = \int_k \delta n(k) v_{sc}(k) \delta n(-k) . \tag{B1}
\]
Here $v_{sc}(k) = 4\pi e^2/(k^2 + \kappa^2)$, with $\kappa$ the screening wave number, and $n(k)$ is the Fourier transform of the electron number density $n(x) = \psi(x)\psi^\dagger(x)$. A finite average density is already built into $S_0$ via the chemical potential, so $S_0$ serves the purpose of the reference ensemble action $S_\lambda$ in Sec. II or Appendix A. Now we follow the logic of Sec. II A. A number density fluctuation $\delta N$ will couple to the field $\delta n(x)$ via the interaction $v_{sc}$ to produce an action
\[
S[\tilde{\psi}, \psi] = S_0[\tilde{\psi}, \psi] 
+ \int dx dy \delta N(x) \delta(\tau_x - \tau_y) v_{sc}(x - y) \delta n(y) . \tag{B2}
\]
and the density fluctuations are governed by a Gaussian action
\[
S_{\text{fluc}}[\delta N] = -\frac{1}{2} \int dx dy \delta N(x) \chi^{-1}(x-y) \delta n(y) . \tag{B3}
\]
with $\chi$ the physical density susceptibility. Integrating out the density fluctuations, we obtain an effective action
\[
S_{\text{eff}}[\tilde{\psi}, \psi] = S_0[\tilde{\psi}, \psi] + \frac{1}{2} \int_k \delta n(k) V(k) \delta n(-k) \tag{B4a}
\]
with an effective potential
\[ V(k) = (\nu_{sc}(k))^2 \chi(k). \]  
(B4b)

Now we calculate the single-particle relaxation rate as in Sec. III. We obtain
\[ \frac{1}{2 \tau} = \Gamma(\epsilon = 0) = 2N_F \int_{-\infty}^{\infty} du \bar{V}''(u) \frac{1}{\sinh(u/T)}, \]  
(B5a)

where
\[ \bar{V}''(u) = \frac{1}{(N_F V)^2} \sum_{k,p} \delta(\xi_k) \delta(\xi_p) \bar{V}''(k - p, u). \]  
(B5b)

From Eq. (B4b) we see that the spectrum of the potential V is given by the spectrum of the density susceptibility, which to lowest order in the screened Coulomb interaction is just the Lindhard function \( \chi_0 \). For \( |u| < (2k_F |k| - k^2)/2m_e \), the spectrum of the latter is
\[ \chi_0''(k, u) = \pi N_F u/v_F |k|, \]  
(B6)

with \( v_F \) the Fermi velocity. For the relaxation rate due to the electron-electron interaction we thus obtain the well-known Fermi-liquid result
\[ \frac{1}{2 \tau_{e-e}} = \frac{\pi}{4} \frac{T^2}{\epsilon_F}. \]  
(B7)

The above derivation is similar in spirit to the arguments given in Ref. 49. The point of this exercise is to demonstrate that our heuristic method of coupling density fluctuations to the appropriate fermion fields that we employed in Sec. II still works in this case where the density fluctuations are produced by the very electrons they couple to. To put the result for the effective interaction, Eq. (B4a), in context, consider a bare Coulomb interaction, \( \nu_c(k) = 4\pi e^2/k^2 \), and perform an RPA resumption to produce a dynamically screened Coulomb interaction
\[ V_{sc}(k) = \frac{\nu_c(k)}{1 + \nu_c(k) \chi_0(k)}. \]  
(B8)

To linear order in the frequency, the spectrum of the effective potential V coincides with the spectrum of \( V_{sc} \), and V therefore suffices to produce the leading low-temperature dependence of the relaxation rate. Our effective action thus captures the leading effects of the soft modes in the system (here, the soft particle-hole excitations that are reflected in the spectrum of the Lindhard function; in Secs. IV and V the magnons). Note that it does not suffice to produce static screening, which requires taking into account massive modes, which is why the above argument starts with a statically screened interaction. Also note that the effective interaction V is quadratic in the bare interaction \( \nu_{sc} \), in accordance with Fermi’s golden rule. Analogously, the effective interaction in Sec. III Eq. (2.15b), is quadratic in the coupling constant \( \Gamma_1 \). We also mention that the \( T^2 \) result, Eq. (B7), holds for any short-ranged interaction, with the prefactor proportional to the potential strength squared.

The above considerations assumed an electronic density fluctuation \( \delta n \) interacting with a density fluctuation \( \delta N \) created by all other electrons, in analogy with magnetization fluctuations in the case of an itinerant magnet. However, there is no reason why \( \delta N \) cannot be a density fluctuation extraneous to the electron system, in analogy to magnetization fluctuations due to electrons in a band other than the conduction band. For instance, if \( \delta N \) is an ionic density fluctuation, it will still couple to \( \delta n \) via a statically screened Coulomb interaction. Equations (B2) - (B5) remain formally valid, except that the susceptibility \( \chi \) now describes ionic density fluctuations, i.e., phonons. If we consider longitudinal phonons the susceptibility is the same as in a fluid and given by
\[ \chi''(k, u) = \pi \rho^2 \kappa \omega_L^2, \]  
(B9)

with \( \omega_L(k) = c |k| \) the longitudinal phonon frequency. Here \( \rho \) is the ionic number density, \( c \) is the longitudinal speed of sound, and \( \kappa = -(\partial V/\partial p)/V \), with \( V \) the system volume and \( p \) the pressure, is the compressibility. We thus have
\[ \bar{V}''(u) = \frac{\pi \rho^2 \kappa}{16} \frac{u^2}{c^2 k_F^2 N_F} \]  
(B10)

which leads to the familiar \( T^3 \) result for the single-particle scattering due to the electron-phonon interaction in metals,
\[ 1/\tau_{e-ph} = \frac{7\pi}{3} \frac{\zeta(3)}{n_e m_e c^2} \rho^2 \kappa T^3. \]  
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We note in passing that the same mode-mode coupling effects qualitatively change the nature of the quantum ferromagnetic transition, see Ref. [33].

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