On \((2k+1, 2k+3)\)-core partitions with distinct parts

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Abstract. In this paper, we are mainly concerned with the enumeration of \((2k+1, 2k+3)\)-core partitions with distinct parts. We derive the number and the largest size of such partitions, confirming two conjectures posed by Straub.

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1 Introduction

A partition \(\lambda\) of a positive integer \(n\) is defined to be a sequence of nonnegative integers \((\lambda_1, \lambda_2, \ldots, \lambda_m)\) such that \(\lambda_1 + \lambda_2 + \cdots + \lambda_m = n\) and \(\lambda_1 \geq \lambda_2 \cdots \geq \lambda_m\). The empty partition is denoted by \(\emptyset\). We write \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \vdash n\) and we say that \(n\) is the size of \(\lambda\), denoted by \(|\lambda|\). The Young diagram of \(\lambda\) is defined to be an up- and left-justified array of \(n\) boxes with \(\lambda_i\) boxes in the \(i\)-th row. The hook of each box \(B\) in \(\lambda\) consists of the box \(B\) itself and boxes directly to the right and directly below \(B\). The hook length of \(B\), denoted by \(h(B)\), is the number of boxes in the hook of \(B\).

For a partition \(\lambda\), the \(\beta\)-set of \(\lambda\), denoted by \(\beta(\lambda)\), is defined to be the set of hook lengths of the boxes in the first column of \(\lambda\). For example, Figure 1 illustrates the Young diagram and the hook lengths of a partition \(\lambda = (5, 3, 3, 1)\). The \(\beta\)-set of \(\lambda\) is \(\beta(\lambda) = \{8, 5, 4, 1\}\). Notice that a partition \(\lambda\) is uniquely determined by its \(\beta\)-set. Given a decreasing sequence of positive integers \((h_1, h_2, \ldots, h_m)\), it is easily seen that the unique partition \(\lambda\) with \(\beta(\lambda) = \{h_1, h_2, \ldots, h_m\}\) is

\[
\lambda = (h_1 - (m - 1), h_2 - (m - 2), \ldots, h_{m-1} - 1, h_m).
\]
Hence, we have

$$|\lambda| = \sum_{i=1}^{m} h_m - \binom{m}{2}.$$  \hspace{1cm} (1.1)

\begin{align*}
8 & \quad 6 & \quad 5 & \quad 2 & \quad 1 \\
5 & \quad 3 & \quad 2 \\
4 & \quad 2 & \quad 1 \\
1
\end{align*}

Figure 1: The Young diagram of $\lambda = (5, 3, 3, 1)$.

For a positive integer $t$, a partition is said to be a $t$-core partition, or simply a $t$-core, if it contains no box whose hook length is a multiple of $t$. Let $s$ be a positive integer not equal to $t$, we say that $\lambda$ is an $(s, t)$-core partition if it is simultaneously an $s$-core and a $t$-core. Anderson [5] showed that the number of $(s, t)$-core partitions is the rational Catalan number $\frac{1}{s+t} \binom{s+t}{s}$ when $s$ and $t$ are coprime to each other. The proof of Anderson’s theorem is through characterizing the $\beta$-sets of $(s, t)$-core partitions as order ideals of the poset $P_{s,t}$, where

$$P_{(s,t)} = \mathbb{N}^+ \setminus \{ n \in \mathbb{N}^+ \mid n = k_1 s + k_2 t \text{ for some } k_1, k_2 \in \mathbb{N} \}$$

whose partial order is fixed by requiring $x \in P_{s,t}$ to cover $y \in P_{s,t}$ if $x - y$ is either $s$ or $t$.

Simultaneous core partitions have been extensively studied. Results on the number, the largest size and the average size of such partitions could be found in [1, 3, 4, 6, 8, 10, 12, 14, 15]. Recently, Straub [11] and Xiong [13] independently showed that the number of $(s, s+1)$-core partitions into distinct parts is given by the Fibonacci number $F_{s+1}$, which verify a conjecture posed by Amdeberhan [2]. In [13], Xiong also obtained results on the the largest size and the average size of such partitions, which completely settle Amdeberhan’s conjecture concerning the enumeration of $(s, s+1)$-core partitions into distinct parts.

In this paper, we are mainly concerned with the number and the largest size of $(2k + 1, 2k + 3)$-core partition into distinct parts, which verify the following two conjectures posed by Straub [11].

\textbf{Conjecture 1.1} If $s$ is odd, then the number of $(s, s+2)$-core partitions into distinct parts equals $2^{s-1}$. 

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Conjecture 1.2 If \( s \) is odd, then the largest size of \((s, s + 2)\)-core partitions into distinct parts is given by \( \frac{(s^2-1)(s+3)(5s+17)}{384} \).

2 Proof of Conjecture 1.1

In this section, we aim to confirm Conjecture 1.1. We begin with some definitions and notations.

Throughout the article, we will follow the poset terminology given by Stanley [9]. Let \( P \) be a poset. For two elements \( x \) and \( y \) in \( P \), we say that \( y \) covers \( x \) if \( x < y \) and there exists no element \( z \in P \) such that \( x < z < y \). Let \( P \) be a graded poset. An element \( x \) in \( P \) is said to be of rank \( s \) if it covers an element of rank \( s - 1 \). Note that the elements of rank 0 in \( P \) are just the minimal elements. The Hasse diagram of a finite poset \( P \) is a graph whose vertices are the elements of \( P \), whose edges are the cover relations, and such that if \( y \) covers \( x \) then there is an edge connecting \( x \) and \( y \) and \( y \) is placed above \( x \). For example, in the Hasse diagram of \( P_{t,t+1} \), each element \( x \) of rank \( s \) covers exactly two elements \( x - t \) and \( x - t - 1 \) of rank \( s - 1 \) for all \( 1 \leq s \leq t - 1 \), and the elements of rank 0 are \( 1, 2, \ldots, t - 1 \). See Figure 2 for an illustration of the Hasse diagram of the poset \( P_{7,8} \). An order ideal of \( P \) is a subset \( I \) such that if any \( y \in I \) and \( x < y \) in \( P \), then \( x \in I \). Denote by \( J(P) \) the set of all order ideals of \( P \).

![Figure 2: The Hasse diagram of the poset \( P_{7,8} \).](image)

In the following lemma, we give a characterization of the poset \( P_{2k+1,2k+3} \).

Lemma 2.1 Let \( k \) be a positive integer. Then we have

\[
P_{2k+1,2k+3} = Q_k \cup Q_k',
\]

where

\[
Q_k = \{2i - 1 \mid 1 \leq i \leq k\}
\]
and
\[ Q'_k = \{ 2i + (s-1)(2k+3) \mid 1 \leq s \leq 2k, 1 \leq i \leq 2k+1-s \}. \]

**Proof.** First we aim to show that \( Q_k \cup Q'_k \subseteq P_{2k+1,2k+3} \). Assume to the contrary that there exists an element \( y \in Q_k \cup Q'_k \) such that \( y \) is not contained in \( P_{2k+1,2k+3} \). Choose \( y \) to be the smallest such element. Clearly, we have \( y = 2i + (s-1)(2k+3) \) for some \( 1 \leq s \leq 2k \) and \( 1 \leq i \leq 2k+1-s \). Since \( y \notin P_{2k+1,2k+3} \), we have that \( 2i + (s-1)(2k+3) = m(2k+1) + n(2k+3) \) for some nonnegative integers \( m \) and \( n \). We assert that \( n = 0 \). Otherwise, if \( n > 0 \), then \( s > 1 \). It follows that \( y - (2k+3) = 2i + (s-2)(2k+3) \in Q_k \) and \( y - (2k+3) = m(2k+1) + (n-1)(2k+3) \notin P_{2k+1,2k+3} \). This contradicts the fact that \( y \) is the smallest element that we choose. Hence we obtain that \( y = 2i + (s-1)(2k+3) = m(2k+1) \). That is \( 2i + 2s - 2 = (m - s + 1)(2k+1) \). Clearly, we have \( m > s + 2 \). Recall that \( i \leq 2k+1-s \). This yields that \( 2i + 2s - 2 \leq 4k \). Using the fact that \( m > s + 2 \), we have \( (m-s+1)(2k+1) \geq 4k+2 \). This implies that \( 2i + 2s - 2 < (m-s+1)(2k+1) \), which yields a contradiction with the relation \( 2i + 2s - 2 = (m - s + 1)(2k+1) \). Hence, we have \( Q_k \cup Q'_k \subseteq P_{2k+1,2k+3} \).

To complete the proof of the lemma, it remains to show that \( P_{2k+1,2k+3} \subseteq Q_k \cup Q'_k \). Let \( y \) be an element in \( P_{2k+1,2k+3} \). We proceed to show that \( y \in Q_k \cup Q'_k \). It is apparent that \( y = x + p(2k+3) \) for some \( 1 \leq x \leq 2k+2, x \neq 2k+1 \) and \( p \geq 0 \). We consider two cases.

**Case 1:** \( x \) is odd. That is \( y = 2i - 1 + p(2k+3) \) for some \( 1 \leq i \leq k \) and \( p \geq 0 \). If \( p = 0 \), then we have \( y \in Q_k \). Otherwise, we have \( y = 2i - 1 + p(2k+3) = 2(k+i+1) + (p-1)(2k+3) \). By setting \( m = k+i+1 \), we have \( y = 2m + (p-1)(2k+3) \). In order to show that \( y \in Q'_k \), it suffices to verify that \( p \leq 2k \) and \( m \leq 2k+1-p \). Assume to the contrary that \( p \geq 2k+2-m \). Since \( P_{2k+1,2k+3} \) is an order ideal and \( y \in P_{2k+1,2k+3} \), we have \( 2m + (2k+1-m)(2k+3) = (2k+3-m)(2k+1) \in P_{2k+1,2k+3} \). This contradicts the definition of \( P_{2k+1,2k+3} \). Thus, we have \( m \leq 2k+1-p \). Since \( m = k+i+1 \geq 1 \), we have \( m \geq 1 \). This yields that \( p \leq 2k \).

**Case 2:** \( x \) is even. That is \( y = 2i + p(2k+3) \) for some \( 1 \leq i \leq k+1 \) and \( p \geq 0 \). In order to show that \( y \in Q'_k \), it remains to verify that \( p \leq 2k-1 \) and \( i \leq 2k-p \). Assume to the contrary that \( p \geq 2k-i+1 \). By the definition of the poset \( P_{2k+1,2k+3} \) and \( y \in P_{2k+1,2k+3} \), we have that \( 2i + (2k-i+1)(2k+3) = (2k+3-i)(2k+1) \in P_{2k+1,2k+3} \). This contradicts the definition of the poset \( P_{2k+1,2k+3} \). Hence we have concluded that \( i \leq 2k-p \). Using the fact that \( i \geq 1 \), we have \( p \leq 2k-1 \).

Combining the above two cases, we obtain that \( P_{2k+1,2k+3} \subseteq Q_k \cup Q'_k \). This completes the proof. \( \blacksquare \)
Denote by $A_s = \{2i + (s - 1)(2k + 3) \mid 1 \leq i \leq 2k + 1 - s\}$ for $1 \leq s \leq 2k$. According to Lemma 2.1, we have $P_{2k+1,2k+3} = Q_k \cup A_1 \cup A_2 \cup \ldots \cup A_{2k}$. It is easily seen that each element $x$ of $A_{s+1}$ covers exactly two elements $x - (2k + 3)$ and $x - (2k + 1)$ in $A_s$ for all $1 \leq s \leq 2 - 1$. Moreover, each element $2k + 2i + 2$ covers exactly two elements $2i - 1$ and $2i + 1$ for all $1 \leq i \leq k - 1$, and the element $2k + 2$ covers the element $1$. Hence, the Hasse diagram of the poset $P_{2k+1,2k+3}$ can be easily constructed by obeying the above rules. For example, the Hasse diagram of the poset $P_{13,15}$ is shown in Figure 3.

Figure 3: The Hasse diagram of the poset $P_{13,15}$.

Denote by $M_{2k+1,2k+3}$ the poset obtained from $P_{2k+1,2k+3}$ by removing all the elements $y$ such that $y \geq 2k + 2$ and preserving the cover relation among the remaining elements. From the Hasse diagram of $P_{2k+1,2k+3}$, we immediately obtain the following characterization of the poset $M_{2k+1,2k+3}$.

**Lemma 2.2** Let $k$ be a positive integer. Define

$$T_k = \{2i - 1 + (s - 1)(2k + 3) \mid 1 \leq s \leq k, 1 \leq i \leq k + 1 - s\}$$

and

$$T'_k = \{2i + (s - 1)(2k + 3) \mid 1 \leq s \leq k, 1 \leq i \leq k + 1 - s\}.$$  

Then we have $M_{2k+1,2k+3} = T_k \cup T'_k$. 

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Relying on Lemma 2.2, we can get the Hasse diagrams of the poset $M_{2k+1,2k+3}$. Figure 4 illustrates the Hasse diagram of the poset $M_{13,15}$.

![Hasse diagram](image)

**Figure 4:** The Hasse diagram of the poset $M_{13,15}$.

In the following theorem, Anderson [5] established a correspondence between core partitions and order ideals of a certain poset by mapping a partition to its $\beta$-set.

**Theorem 2.3** Let $s$, $t$ be two coprime positive integers, and let $\lambda$ be a partition of $n$. Then $\lambda$ is an $s$-core (or $(s,t)$-core) partition if and only if $\beta(\lambda)$ is an order ideal of $P_s$ (or $P_{s,t}$).

The following theorem provides a characterization of the $\beta$-set of partitions into distinct parts [13].

**Theorem 2.4** The partition $\lambda$ is a partition into distinct parts if and only if there doesn’t exist $x, y \in \beta(\lambda)$ with $x - y = 1$.

An order ideal $I$ of $P$ is said to be nice if there doesn’t exist $x, y \in I$ with $x - y = 1$. Denote by $L(P)$ the set of nice order ideals of the poset $P$. Combining Theorems 2.3 and 2.4 we obtain the following result.

**Theorem 2.5** Let $s$, $t$ be two coprime positive integers, and let $\lambda$ be a partition of $n$. Then $\lambda$ is an $s$-core (or $(s,t)$-core) partition into distinct parts if and only if $\beta(\lambda)$ is a nice order ideal of $P_s$ (or $P_{s,t}$).

In order to get the enumeration of $(2k+1,2k+3)$-core partitions into distinct parts, we need the notions of Dyck path and free Dyck path. Recall that a Dyck path of order $n$ is a lattice path in $Z \times Z$ from $(0, 0)$ to $(2n, 0)$ using up steps $U = (1, 1)$ and down steps $D = (1, -1)$, and never lying below the $x$-axis [7].
Denote by $D_n$ the set of all Dyck paths of order $n$. It is well known that Dyck paths of order $n$ are counted by the $n$-th Catalan number $c_n = \frac{1}{n+1} \binom{2n}{n}$. A free Dyck path is just a Dyck path but without the restriction that it cannot go below the x-axis. Let $\mathcal{FD}_n$ denote the set of free Dyck paths from $(0,0)$ to $(2n,0)$. By simple arguments we have that $|\mathcal{FD}_n| = \binom{2n}{n}$.

We first describe a bijection $\phi$ between the set of order ideals of $P_{t,t+1}$ and the set of Dyck paths from $(0,0)$ to $(2t,0)$. To this end, we shall partition $J(P_{t,t+1})$ according to the smallest missing element of rank 0 in an order ideal. For $1 \leq i \leq t-1$, let $J_i(P_{t,t+1})$ denote the set of order ideals of $P_{t,t+1}$ such that $i$ is the smallest missing elements of rank 0. Let $J_t(P_{t,t+1})$ denote the set of order ideals containing all the minimal elements of $P_{t,t+1}$. Now we are in the position to construct a Dyck path from $I \in J_t(P_{t,t+1})$ recursively as follows.

- For $2 \leq i \leq t-1$ and an order ideal $I \in J_i(P_{t,t+1})$, we can decompose $I$ into three parts: one is $\{1,2,\ldots,i-1\}$, one is isomorphic to an order ideal $I_1$ of $J(P_{t-1,i})$, and one is isomorphic to an order ideal $I_2$ of $J(P_{t-i,t-i+1})$. Define $\phi(I) = U\phi(I_1)D\phi(I_2)$.

- For any order ideal $I \in J_1(P_{t,t+1})$, $I$ is isomorphic to an order ideal $I_1$ of $J(P_{t-1,1})$. Define $\phi(I) = UD\phi(I_1)$.

- For any order ideal $I \in J_t(P_{t,t+1})$, one can decompose $I$ into two parts: one is $1,2,\ldots,t-1$ and the other is isomorphic to an order ideal $I_1$ of $J(P_{t-1,t})$. Define $\phi(I) = U\phi(I_1)D$.

It is easy to check that the resulting path is a Dyck from $(0,0)$ to $(2t,0)$. Conversely, given a Dyck path $P$, by the first return decomposition, $P$ can be uniquely decomposed as $P = UP_1DP_2$, where $P_1$ and $P_2$ are (possibly empty) Dyck paths. Hence, the above procedure is reversible and the map $\phi$ is a bijection.

For a path $P = p_1p_2\ldots p_n$ where $p_i \in \{U,D\}$ for all $1 \leq i \leq n$, the reverse of the path, denoted by $\overline{P}$, is defined by $p_np_{n-1}\ldots p_1$. For example, the reverse of the path $P = UDDUD$ is given by $DUDDU$. It is easily seen that for a Dyck path $P$, its reverse $\overline{P}$ is a free Dyck path which lies weakly below the x-axis. Denote by $\hat{L}(M_{2k+1,2k+3})$ the set of order ideals in $L(M_{2k+1,2k+3})$ which do not contain the element 1.

**Lemma 2.6** Let $k$ be a nonnegative integer. There is a bijection between the set $\hat{L}(M_{2k+1,2k+3})$ and the set $\mathcal{FD}_k$. 

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Proof. First we describe a recursive map \( \psi \) from the set \( \hat{L}(M_{2k+1,2k+3}) \) to the set \( \mathcal{FD}_k \). Let \( I \in \hat{L}(M_{2k+1,2k+3}) \). For \( k = 0 \), let \( \psi(I) \) be the empty path. For \( k = 1 \), let \( \psi(I) = UD \) if \( I \) contains the element 2, and let \( \psi(I) = DU \), otherwise. Assume that for all \( 2 \leq j \leq k - 1 \) and an order ideal \( I' \in \hat{L}(M_{2j+1,2j+3}) \) which contains the element 2\( j \), we have \( \psi(I') \in \mathcal{FD}_j \) which ends with a down step. Similarly, for all \( 2 \leq j \leq k - 1 \) and an order ideal \( I' \in \hat{L}(M_{2j+1,2j+3}) \) which does not contain the element 2\( j \), we have \( \psi(I') \in \mathcal{FD}_j \) which ends with an up step. Now we construct a lattice path from \((0,0)\) to \((2k,0)\) as follows.

- If \( I \) contains the element 2\( k \), then let 2\( \ell \) be the largest missing even number satisfying that \( 0 \leq \ell \leq k - 1 \). In this case, one can decompose \( I \) into three parts: one is \( \{2\ell+2,2\ell+4,\ldots,2k\} \), one is isomorphic to an order ideal \( I' \) of \( J(P_{k-\ell,k-\ell+1}) \), and one is isomorphic to an order ideal \( I'' \) of \( \hat{L}(M_{2\ell+1,2\ell+3}) \). It is easily seen that \( 2\ell \notin I'' \). Set \( \psi(I) = \psi(I'')\phi(I') \).

- If \( I \) does not contain the element 2\( k \) but contains the element 2\( j \) for some \( 1 \leq j \leq k - 1 \), then let \( \ell \) be the largest integer such that \( I \) contains the element 2\( \ell \). In this case, one can decompose \( I \) into two parts: one is isomorphic to an order ideal \( I' \) of \( J(P_{k-\ell,k-\ell+1}) \), and the other is isomorphic to an order ideal \( I'' \) of \( \hat{L}(M_{2\ell+1,2\ell+3}) \). Clearly, \( I'' \) contains the element 2\( \ell \). Set \( \psi(I) = \psi(I'')\phi(I') \).

- If \( I \) does not contain the element 2\( j \) for all \( 1 \leq j \leq k \), then \( I \) is isomorphic to an order ideal \( I' \) of \( J(P_{k,k+1}) \). Set \( \psi(I) = \phi(I') \).

By induction hypothesise, one can easily check that the resulting path \( \psi(I) \) is a free Dyck path from \((0,0)\) to \((2k,0)\). Moreover, the path \( \psi(I) \) ends with a down step if \( I \) contains the element 2\( k \), and \( \psi(I) \) ends with an up step, otherwise. Thus, the map \( \psi \) is well defined, that is, we have \( \psi(I) \in \mathcal{FD}_k \) for any \( I \in M(P_{2k+1,2k+3}) \).

Conversely, for any free Dyck path \( P \) ending with a down step, it can be uniquely decomposed as

\[
P = P'P''
\]

where \( P' \) is a (possibly empty) free Dyck path ending with an up step and \( P'' \) is a Dyck path. Analogously, for any free Dyck path \( P \) ending with an up step, it can be uniquely decomposed as

\[
P = P'P''
\]

where \( P' \) is a (possibly empty) free Dyck path ending with a down step and \( P'' \) is the reverse of a Dyck path. Hence for any free Dyck path \( P \in \mathcal{FD}_k \), we can
recover an order ideal in \( \hat{L}(M_{2k+1,2k+3}) \) by reversing the above procedure. This implies that the map \( \psi \) is a bijection, which completes the proof.

By Lemma 2.6, we immediately get the following result on the cardinality of the set \( \hat{L}(M_{2k+1,2k+3}) \).

**Lemma 2.7** Let \( k \) be a nonnegative integer. The cardinality of the set \( \hat{L}(M_{2k+1,2k+3}) \) is given by \( \binom{2k}{k} \).

Let \( k \geq 1 \). Denote by \( \tilde{L}(M_{2k+1,2k+3}) \) the set of order ideals of \( L(M_{2k+1,2k+3}) \) which contain neither 1 nor \( 2k \). From the construction of the map \( \psi \), it is easy to check that the map \( \psi \) sends an order ideal \( I \in \tilde{L}(M_{2k+1,2k+3}) \) to a free Dyck path from \((0,0)\) to \((2k,0)\) ending with an up step. The latter is counted by \( \binom{2k-1}{k} \). Hence, we have the following result.

**Lemma 2.8** Let \( k \) be a positive integer. The number of nice order ideals of \( \hat{L}(M_{2k+1,2k+3}) \) is counted by \( \binom{2k-1}{k} \).

In the following theorem, we shall obtain the enumeration of the cardinality of the set \( L(M_{2k+1,2k+3}) \).

**Theorem 2.9** Let \( k \) be a nonnegative integer. The cardinality of the set \( L(M_{2k+1,2k+3}) \) is given by

\[
\sum_{i=0}^{k} \frac{1}{i+1} \binom{2i}{i} \binom{2k-2i}{k-i}.
\]

**Proof.** It is easily seen that the assertion holds for \( k = 0 \). For \( k > 0 \) and \( 0 \leq i \leq k \), denote by \( L_i(M_{2k+1,2k+3}) \) the set of order ideals \( I \) of \( L(M_{2k+1,2k+3}) \) such that \( 2i+1 \) is the smallest odd number missing from \( I \). For an order ideal \( I \in L_i(M_{2k+1,2k+3}) \) where \( 1 \leq i \leq k \), we can decompose it into three parts: one is \( \{1, 3, \ldots, 2i-1\} \), one is isomorphic to an order ideal of \( J(P_{i,i+1}) \), and one is isomorphic to an order ideal of \( \hat{L}(M_{2k-2i+1,2k-2i+3}) \). Thus, by Lemma 2.7, we deduce that

\[
|L_i(M_{2k+1,2k+3})| = c_i \binom{2k-2i}{k-i} = \frac{1}{i+1} \binom{2i}{i} \binom{2k-2i}{k-i}.
\]

For an order ideal \( I \in L_0(M_{2k+1,2k+3}) \), we can decompose it into two parts: one is \( \{2, 4, \ldots, 2k\} \) and the other is isomorphic to an order ideal of \( J(P_{k,k+1}) \). Thus, it follows that

\[
|L_0(M_{2k+1,2k+3})| = c_k = \frac{1}{k+1} \binom{2k}{k}.
\]
Hence, we have

\[ |L(M_{2k+1,2k+3})| = \sum_{i=1}^{k} |L_i(M_{2k+1,2k+3})| + |L_0(M_{2k+1,2k+3})| = \sum_{i=0}^{k} \frac{1}{i+1} \binom{2i}{i} \binom{2k-2i}{k-i} \]

as desired, which completes the proof.

Recall that \( J_i(P_{t,t+1}) \) is the set of order ideals in \( J(P_{t,t+1}) \) such that the \( i \) is the smallest missing element of rank 0. For \( t \geq 2 \) and \( 2 \leq i \leq t \), denote by \( J_i^*(P_{t,t+1}) \) the set of order ideals in \( J_i(P_{t,t+1}) \) in which exactly one element less than \( i \) is marked by a star. Let \( J^*(P_{t,t+1}) = \bigcup_{i=2}^{t} J_i^*(P_{t,t+1}) \).

**Lemma 2.10** For \( t \geq 1 \), we have \( |J^*(P_{t+1,t+2})| = \sum_{i=1}^{t} ic_i c_{t-i} \).

**Proof.** From the construction of the map \( \phi \), it is easily seen that the map \( \phi \) sends an order ideal in \( J^*(P_{t+1,t+2}) \) to a Dyck path from \((0,0)\) to \((2t+2,0)\) in which exactly one up step which is to the left of the first return point and to the right of the first up step is marked by a star. Hence, we have

\[ |J^*(P_{t+1,t+2})| = \sum_{i=1}^{t} ic_i c_{t-i}, \]

as desired. This completes the proof.

For \( k \geq 1 \), denote by \( S(P_{2k+1,2k+3}) \) the set of nice order ideals in \( J(P_{2k+1,2k+3}) \) which contain the element \( 2k+2 \). For \( 1 \leq m \leq k \) and \( l \leq k - m \), denote by \( S_{m,l}(P_{2k+1,2k+3}) \) the set of all order ideals \( I \in S(P_{2k+1,2k+3}) \) such that \( 2m+1 \) is the smallest missing odd element from \( I \) and \( 2m+2l \) is the largest missing even element of rank 0.

Before we deal with the enumeration of the order ideals in \( S(P_{2k+1,2k+3}) \), we need the following lemma.

**Lemma 2.11** For \( k \geq 1 \), we have \( \sum_{i=0}^{k-1} c_i \binom{2k-2i-1}{k-i} = \binom{2k}{k} - c_k \).

**Proof.** Notice that the number of free Dyck paths from \((0,0)\) to \((2k,0)\) whose steps are weakly above the \( x \)-axis is given by \( c_k \). Hence the number of free Dyck paths from \((0,0)\) to \((2k,0)\) which have at least one step below \( x \)-axis is given by \( \binom{2k}{k} - c_k \). On the other hand, let \( P = p_1p_2 \ldots p_{2k} \in DF_k \) where each \( p_i \in \{U,D\} \). If \( P \) has at least one step below the \( x \)-axis, then suppose that \( p_{2i+1} \) is the leftmost down step which is weakly below the \( x \)-axis. Then the section \( p_1p_2 \ldots p_{2i} \) is a Dyck path from \((0,0)\) to \((2i,0)\), and the remaining section of \( P \) is free Dyck path from
(2i, 0) to (2k, 0) which stars with a down step. Hence, the number of free Dyck paths from (0, 0) to (2k, 0) which have at least one down step weakly below the x-axis is given by \( \sum_{i=0}^{k-1} c_i \binom{2k-2i-1}{k-i} \). Hence, we have

\[
\sum_{i=0}^{k-1} c_i \binom{2k-2i-1}{k-i} = \binom{2k}{k} - c_k
\]
as desired. This completes the proof.

Now we are in the position to get the enumeration of nice order ideals of \( J(P_{2k+1,2k+3}) \) which contain the element 2k + 2. Let \( X_{k,\ell} \) denote the set of ordered pairs \((I', I'')\) where \( I' \in \widetilde{L}(M_{2\ell+1,2\ell+3}) \) and \( I'' \in J^*(P_{k-\ell+1,k-\ell+2}) \). Let \( X_k = \bigcup_{\ell=0}^{k-1} X_{k,\ell} \).

**Theorem 2.12** Let \( k \) be a positive integer. The cardinality of the set \( S(P_{2k+1,2k+3}) \) is given by \( \sum_{j=1}^{k} \frac{(2j)}{(j)} \binom{2k-2j}{k-j} \).

**Proof.** Obviously, the assertion holds for \( k = 1 \) since there is exactly one order ideal in \( S(P_{3,5}) \). For \( k \geq 2 \), we first describe a map \( \gamma \) from the set \( S(P_{2k+1,2k+3}) \) to the set \( X_k \). Let \( I \) be an order ideal in \( S_{m,\ell}(P_{2k+1,2k+3}) \) for some \( 1 \leq m \leq k \) and \( 0 \leq \ell \leq k - m \). Then we can decompose \( I \) into three parts: one is \( \{1, 3, \ldots, 2m - 1\} \), one is isomorphic to an order ideal \( I_1 \in \widetilde{L}(M_{2\ell+1,2\ell+3}) \), and one is isomorphic to an order ideal \( I_2 \in J(P_{k-\ell+1,k-\ell+2}) \). For example, Figure 5 illustrates the decomposition of an order ideal \( I \) of \( S_{1,3}(P_{13,15}) \). It can be decomposed into three parts: one is \( \{1\} \), one is order-isomorphic to an order ideal in \( \widetilde{L}(P_{7,9}) \) whose elements lie in the triangles \( A \) and \( B \), and one is order-isomorphic to an order ideal in \( J(P_{4,5}) \) whose elements lie in the triangle \( C \).

By assigning a star to the element \( k - m - \ell + 1 \), we can obtain a marked order ideal \( I_2' \in J^*(P_{k-\ell+1,k-\ell+2}) \) from \( I_2 \). Set \( \gamma(I) = (I_1, I_2') \). It is easily seen that the map \( \gamma \) is well defined. Conversely, given an ordered pair \((I', I'')\) \( X_{k,\ell} \) such that the element \( k - \ell - m + 1 \) is marked by a star, we can recover the order ideal \( I \) by reversing the above procedure. Hence, the map \( \gamma \) is a bijection between the set \( S(P_{2k+1,2k+3}) \) and the set \( X_k \). This yields that \( |S(P_{2k+1,2k+3})| = |X_k| \).

According to the definition of \( X_k \), we have

\[
|X_k| = \sum_{\ell=0}^{k-1} |X_{k,\ell}|
\]
for all \( k \geq 2 \). Notice that the cardinality of the set \( \tilde{L}(M_{1,3}) \) is given by 1. By Lemmas 2.8 and 2.10 we deduce that, for \( k \geq 2 \),

\[
|X_k| = \sum_{i=1}^{k} ic_i c_{k-i} + \sum_{\ell=1}^{k-1} \binom{2\ell-1}{\ell} \left( \sum_{i=1}^{k-\ell} ic_i c_{k-\ell-i} \right)
\]

\[
= \sum_{i=1}^{k} ic_i c_{k-i} + \sum_{i=1}^{k-1} ic_i \left( \sum_{\ell=1}^{k-i-1} \binom{2\ell-1}{\ell} c_{k-i-\ell} \right)
\]

\[
= \sum_{i=1}^{k} ic_i c_{k-i} + \sum_{i=1}^{k-1} ic_i \left( \sum_{m=0}^{k-i-1} c_m \binom{2k-2i-2m-1}{k-i-m} \right)
\]

\[
= \sum_{i=1}^{k} ic_i c_{k-i} + \sum_{i=1}^{k-1} ic_i \left( \binom{2k-2i}{k-i} - c_{k-i} \right) \quad \text{(by Lemma 2.11)}
\]

\[
= \sum_{i=1}^{k} \frac{i}{i+1} \binom{2i}{k-i}
\]

as desired. This completes the proof.

We are now ready to complete the proof of Conjecture 1.1.

**Proof of Conjecture 1.1.** Combining Theorems 2.9 and 2.12 we deduce that the number of \((2k+1, 2k+3)\)-core partitions into distinct parts is given by

\[
|L(M_{2k+1,2k+3})| + |S(P_{2k+1,2k+3})| = \sum_{i=0}^{k} \binom{2i}{i} \binom{2k-2i}{k-i}.
\]

Notice that

\[
\frac{1}{1-4x} = \left( \frac{1}{\sqrt{1-4x}} \right)^2 = \left( \sum_{n=0}^{\infty} \binom{2n}{n} x^n \right)^2.
\]
By comparing the coefficient of $x^k$ in the above formula, we have $\sum_{i=0}^{k} \binom{2i}{i} \binom{2k-2i}{k-i} = 2^{2k}$. This yields that

$$|L(M_{2k+1,2k+3})| + |S(P_{2k+1,2k+3})| = 2^{2k}. \quad (2.1)$$

Substituting $k = \frac{s-1}{2}$ in (2.1), we are led to a proof of Conjecture 1.1. □

### 3 Proof of Conjecture 1.2

In this section, we shall construct a partition $\kappa_k$ for any integer $k \geq 0$. It turns out that $\kappa_k$ is the unique $(2k+1, 2k+3)$-core partitions into distinct parts which has the largest size. This leads to a proof of Conjecture 1.2.

The following four lemmas give a characterization of the largest $(2k+1, 2k+3)$-core partitions with distinct parts whose $\beta$-set is contained in $L(M_{2k+1,2k+3})$. Denote by $B_s$ (resp. $B'_s$) the set of elements of rank $s$ in $T_k$ (resp. $T'_k$). By Lemma 2.2, we have $B_s = \{2i - 1 + (s - 1)(2k + 3) \mid 1 \leq i \leq k + 1 - s\}$ and $B'_s = \{2i + (s - 1)(2k + 3) \mid 1 \leq i \leq k + 1 - s\}$ for all $1 \leq s \leq k$.

**Lemma 3.1** Let $\lambda$ be a $(2k+1, 2k+3)$-core partitions into distinct parts which has the largest size. If $\beta(\lambda) \in L(M_{2k+1,2k+3})$ and $\beta(\lambda)$ contains an element $x$ in $B_s$ (resp. $B'_s$), then $\beta(\lambda)$ contains all the element $y \in B_s$ (resp. $y \in B'_s$) when $y > x$.

**Proof.** It suffices to show that $\beta(\lambda)$ also contains the element $x+2$ if $\beta(\lambda)$ contains an element $x$ in $B_s$ (resp. $B'_s$) and $x+2 \in B_s$ (resp. $x+2 \in B'_s$). If not, we choose $s$ to be the smallest integer for such $B_s$ (resp. $B'_s$) and let $x$ be the smallest such number once $s$ is determined. If $s \geq 1$, then replace $y$ by $y+2$ if $y \geq p$ for some $p \leq x$ and $p \in B_s$ (resp. $p \in B'_s$). Otherwise, replace $y$ by $y+1$ if $y \geq p$ for some $p \leq x$ and $p \in B_0$ (resp. $p \in B'_0$). By this process, we obtain a new nice order ideal $\beta'$ of $M_{2k+1,2k+3}$. It is easily seen that $\beta'$ has the same number of parts as $\beta(\lambda)$ and a larger sum of the elements. By relation (1.1), the size of the $(2k+1, 2k+3)$-core partition into distinct parts corresponding to $\beta'$ is larger than $\lambda$, which contradicts the assumption that $\lambda$ is a $(2k+1, 2k+3)$-core partitions into distinct parts which has the largest size. This completes the proof. □

Denote by $U_t$ the unique order ideal which contains all the elements of $P_{t,t+1}$. For $1 \leq i \leq 2k$, let $\beta_{i,0}$ denote the unique nice order ideal in $M_{2k+1,2k+3}$ which is isomorphic to $U_{k-\left\lceil \frac{i+1}{2} \right\rceil +1}$ and contains all the elements $i+2, i+4, \ldots, i+2(k-\left\lceil \frac{i+1}{2} \right\rceil)$. For $1 \leq i \leq 2k$ and $1 \leq j \leq k-\left\lceil \frac{i+1}{2} \right\rceil +1$, let $\beta_{i,j}$ be the union of $\beta_{i,0}$ and
the chain consisting of $i, i + 2k + 3, \ldots, i + (j - 1)(2k + 3)$. See Figure 6 for an illustration of the order ideal $\beta_{4,3}$. For $1 \leq j \leq k$, denote by $\gamma_{k,j}$ be the union of $\beta_{1,k}$ and the chain consisting of $2k+2, 2k+2+(2k+3), \ldots, 2k+2+(j-1)(2k+3)$.

It is easy to check that both $\beta_{i,j}$ and $\gamma_{k,j}$ are nice order ideals of $P_{2k+1,2k+3}$. Let $\lambda_{i,j}$ (resp. $\mu_{k,j}$) be the unique partition such that $\beta(\lambda_{i,j}) = \beta_{i,j}$ (resp. $\beta(\mu_{k,j}) = \gamma_{k,j}$). By Theorem 2.5, both $\lambda_{i,j}$ and $\mu_{k,j}$ are $(2k+1, 2k+3)$-core partitions with distinct parts.

**Lemma 3.2** Fix $k \geq 1$. Let $\lambda$ be a $(2k+1, 2k+3)$-core partition into distinct parts such that $\beta(\lambda) \in L(M_{2k+1,2k+3})$. If $\lambda$ if of maximum size, then there exist some integers $1 \leq i \leq 2k$ and $1 \leq j \leq k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1$ such that $\beta(\lambda_{i,j}) = \beta_{i,j}$.

**Proof.** Let $i$ be the minimal integer of rank 0 in $\beta(\lambda)$ and $j$ be the maximal integer such that $i + (j - 1)(2k + 3)$ is contained in $\beta(\lambda)$. We claim that $i + 2 + m(2k + 3)$ is contained in $\beta(\lambda)$ for all $0 \leq m \leq k - \left\lfloor \frac{i+1}{2} \right\rfloor - 1$. Assume to the contrary that the claim is not valid, that is, there exists some integer $s$ such that $\beta(\lambda)$ does not contain the element $i + 2 + s(2k + 3)$. We choose $s$ to be the smallest such integer. By Lemma 3.1, we have $s > j - 1$. By replacing the element $i + (j - 1)(2k + 3)$ by the element $i + 2 + s(2k + 3)$, we get a new nice order ideal $\beta'$ of $J(P_{2k+1,2k+3})$. It is easy to check that $\beta'$ has the same cardinality as $\beta(\lambda)$ and has a larger sum of the elements. By relation (1.1), the size of the partition corresponding to $\beta'$ is larger than that of $\lambda$. This yields a contradiction with the fact that $\lambda$ has the largest size, which completes the proof of the claim. Combining the claim and Lemma 3.1, we have $\beta(\lambda) = \beta_{i,j}$. This completes the proof.

**Lemma 3.3** For $1 \leq i \leq 2k$, we have $|\lambda_{i,j}| \leq |\lambda_{i,k - \left\lceil \frac{i+1}{2} \right\rceil + 1}$ for $1 \leq j \leq k - \left\lceil \frac{i+1}{2} \right\rceil + 1$, with the equality holding if and only if $j = k - \left\lceil \frac{i+1}{2} \right\rceil + 1$. Moreover, we have $|\lambda_{i,k - \left\lceil \frac{i+1}{2} \right\rceil + 1}| > |\lambda_{i+2,k - \left\lceil \frac{i+1}{2} \right\rceil}|$ for all $k \geq 3$ and $1 \leq i \leq 2k - 2$. 

![Figure 6: The order ideal $\beta_{4,3}$.](image-url)
Proof. By relation (1.1), the size of \( \lambda_{i,j} \) is given by

\[
|\lambda_{i,j}| = \sum_{h \in \beta_{i,j}} h - \left( \frac{\beta_{i,j}}{2} \right)
= \sum_{h \in \beta_{i,0}} h + \sum_{p=0}^{j-1} (i + p(2k + 3)) - \left( \frac{\beta_{i,0}}{2} + j \right)
= \sum_{h \in \beta_{i,0}} h - \left( \frac{\beta_{i,0}}{2} \right) + \sum_{p=0}^{j-1} (i + p(2k + 3)) - \left( \frac{\beta_{i,0}}{2} + j \right)
= |\lambda_{i,0}| + ij + \left( \frac{j}{2} \right)(2k + 2) - j|\beta_{i,0}|
\]  

(3.1)

By the definition of \( \beta_{i,0} \), we obtain that

\[
|\beta_{i,0}| = \left( k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1 \right).
\]

(3.2)

Hence, we have

\[
|\lambda_{i,j}| = |\lambda_{i,0}| + ij + \left( \frac{j}{2} \right)(2k + 2) - j \left( k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1 \right).
\]

(3.3)

In particular, we have

\[
|\lambda_{i,k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1} - |\lambda_{i,0}| = \begin{cases} 
(k + m + 1)\left( \frac{k-m+1}{2} \right) + 2m(k - m + 1) & \text{if } i=2m, \\
(k + m + 1)\left( \frac{k-m+1}{2} \right) + (2m - 1)(k - m + 1) & \text{if } i=2m-1,
\end{cases}
\]

and

\[
|\lambda_{i,1} - |\lambda_{i,0}| = \begin{cases} 
-\left( \frac{k-m+1}{2} \right) + 2m & \text{if } i=2m, \\
-\left( \frac{k-m+1}{2} \right) + 2m - 1 & \text{if } i=2m-1.
\end{cases}
\]

(3.4)

This implies that if \( i < 2k - 1 \), then

\[
|\lambda_{i,k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1}| > |\lambda_{i,1}|.
\]

(3.6)

Moreover, when \( i = 2k \) or \( 2k - 1 \), we have

\[
|\lambda_{i,k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1}| = |\lambda_{i,1}|.
\]

(3.7)

For fixed \( k \) and \( i \), we see that \( \lambda_{i,j} \) is a quadratic function of \( j \) with a positive leading coefficient. Hence the maximum value of \( \lambda_{i,j} \) is obtained at \( j = 1 \) or \( j = k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1 \) when \( j \) ranges over \( [1, k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1] \). In view of (3.6) and (3.7), we have \( |\lambda_{i,j}| \leq |\lambda_{i,k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1}| \) for \( 1 \leq j \leq k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1 \), with the equality holding if and only if \( j = k - \left\lfloor \frac{i+1}{2} \right\rfloor + 1 \). This completes the proof of the first part of the lemma.
From the definition of $\beta_{i+1,0}$, we have $\beta_{i,k-\lfloor \frac{i+1}{2} \rfloor +1} = \beta_{i-2,0}$ for all $3 \leq i \leq 2k$. In view of (3.4), we have $|\lambda_{i,k-\lfloor \frac{i+1}{2} \rfloor +1}| > |\lambda_{i,0}| = |\lambda_{i+2,k-\lfloor \frac{i+1}{2} \rfloor}|$ for all $k \geq 3$ and $1 \leq i \leq 2k-2$. This completes the proof.

**Lemma 3.4** Fix $k \geq 1$. Let $\lambda$ be a $(2k+1, 2k+3)$-core partition into distinct parts such that $\beta(\lambda) \in L(M_{2k+1,2k+3})$. If $\lambda$ is of maximum size, then $\beta(\lambda) = \beta_{2,k}$.

**Proof.** From lemmas 3.2 and 3.3, we have $\beta(\lambda) = \beta_{2,k}$ or $\beta_{1,k}$. According to the definition of $\beta_{1,k}$ and $\beta_{2,k}$, it is easy to verify that

$$|\beta_{1,k}| = |\beta_{2,k}|$$

and

$$\sum_{h \in \beta_{2,k}} h = \sum_{h \in \beta_{1,k}} h + \frac{k(k+1)}{2}.$$ 

By relation (1.1), we have

$$|\lambda_{2,k}| - |\lambda_{1,k}| = \frac{k(k+1)}{2} > 0.$$ (3.8)

Hence we have $\beta(\lambda) = \beta_{2,k}$ as desired, which completes the proof.

Now we proceed to deal with the largest $(2k+1, 2k+3)$-core partitions with distinct parts and its $\beta$-set is contained in $S(P_{2k+1,2k+3})$. Given positive integers $a \leq b$, we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$.

Recall that $Q_k = \{2i-1 \mid 1 \leq i \leq k\}$ and $A_s = \{2i + (s-1)(2k+3) \mid 1 \leq i \leq 2k+1-s\}$ for $1 \leq s \leq 2k$.

**Lemma 3.5** Let $\lambda$ be a $(2k+1, 2k+3)$-core partitions into distinct parts which has the largest size. If $\beta(\lambda) \in S(P_{2k+1,2k+3})$, then $\beta(\lambda)$ contains all the elements of $Q_k$.

**Proof.** If not, suppose that $2m+1$ is the smallest missing odd element of rank 0 from $\beta(\lambda)$ for some $1 \leq m \leq k-1$. Let $\ell$ be the largest integer such that the element $2m+2\ell$ is missing from $\beta(\lambda)$ once $m$ is determined. By the definition of $S_{m,\ell}(P_{2k+1,2k+3})$, we have $\beta(\lambda) \in S_{m,\ell}(P_{2k+1,2k+3})$. As in the proof of Theorem 2.12, $\beta(\lambda)$ can be decomposed into three parts $I_1, I_2, I_3$, where

- $I_1 = \{1, 3, \ldots, 2m-1\}$;
- $I_2$ is isomorphic to an order ideal of $\tilde{L}(M_{2\ell+1,2\ell+3})$;
• $I_3$ is isomorphic to an order ideal of $J(P_{k-\ell+1,k-\ell+2})$.

To be more precise, $I_2$ consists of the elements $y \in I$ such that $y \succeq p$ for some $p \in [2m+1, 2m+2\ell]$, whereas $I_1 \cup I_3$ consists of the elements $y \in I$ such that $y \succeq p$ for some $p \in [1, 2m] \cup [2m+2\ell+1, 2k+2]$.

We proceed to construct a new order ideal $\beta'$ of $P_{2k+1,2k+3}$ by the following procedure.

• Firstly, replace $I_1$ by $\{1, 3, \ldots, 2m+1\}$;

• Secondly, replace each element $y$ of $I_2$ by $y+1$;

• Finally, replace each element $y$ of $I_3$ by $y+2$.

Suppose that the cardinality of $I_3$ is given by $x$. It is easily seen that

$$|\beta'| = |\beta(\lambda)| + 1$$

and

$$\sum_{h \in \beta'} h = \sum_{h \in \beta(\lambda)} h + 2x + 2m + 1 + |\beta(\lambda)| - (x + m) = \sum_{h \in \beta(\lambda)} h + x + m + 1 + |\beta(\lambda)|.$$

By relation (1.1), the size of the $(2k+1, 2k+3)$-core partition into distinct parts corresponding to $\beta'$ is given by

$$\sum_{h \in \beta'} h - (|\beta'|/2) = \sum_{h \in \beta(\lambda)} h + |\beta(\lambda)| + x + m + 1 - (|\beta(\lambda)| + 1) = \sum_{h \in \beta(\lambda)} h - (|\beta(\lambda)|/2) + |\beta(\lambda)| + 1 + x + m - (|\beta(\lambda)|/2) + (|\beta(\lambda)|/2) = |\lambda| + x + m + 1.$$

This yields that the partition corresponding to $\beta'$ has a larger size than $\lambda$, which contradicts the assumption that $\lambda$ is a $(2k+1, 2k+3)$-core partitions into distinct parts which has the largest size. This completes the proof.

**Lemma 3.6** Fix $s \geq 1$. Let $\lambda$ be a $(2k+1, 2k+3)$-core partitions into distinct parts which has the largest size. If $\beta(\lambda) \in S(P_{2k+1,2k+3})$ and $\beta(\lambda)$ contains an element $x \in A_s$, then $\beta(\lambda)$ contains all the elements $y \in A_s$ when $y > x$.

**Proof.** It suffices to show that $\beta(\lambda)$ also contains the element $x+2$ if $\beta(\lambda)$ contains an element $x \in A_s$. If not, we choose $s$ to be the smallest such integer and let $x$ be the smallest such number once $s$ is determined. We can obtain a new nice order ideal $\beta'$ of $P_{2k+1,2k+3}$ by replacing each element $y$ by $y+2$ if $y \geq p$ such
that \( p \leq x \) and \( p \) is of rank \( s \). It is easily seen that \( \beta' \) has the same number of parts as \( \beta(\lambda) \) and a larger sum of the elements. By relation (1.1), the size of the \((2k+1, 2k+3)\)-core partition into distinct parts corresponding to \( \beta' \) is larger than \( \lambda \), which contradicts the assumption that \( \lambda \) is a \((2k+1, 2k+3)\)-core partitions into distinct parts which has the largest size. This completes the proof.  

\[ \text{Lemma 3.7} \]

Fix \( k \geq 1 \). Let \( \lambda \) be a \((2k+1, 2k+3)\)-core partition into distinct parts such that \( \beta(\lambda) \in S(P_{2k+1,2k+3}) \). If \( \lambda \) is of maximum size, then there exist some integer \( 1 \leq i \leq k \) such that \( \beta(\lambda) = \gamma_{k,i} \).

Proof. Let \( i \) be the maximal integer such that \( \beta(\lambda) \) contains the element \( 2k + 2 + (i - 1)(2k + 3) \). Let \( j \) be the maximal integer such that \( \beta(\lambda) \) contains the element \( 1 + (j - 1)(2k + 3) \) once \( i \) is determined. By Lemma 3.6, we have \( i < j \). We claim that \( \beta(\lambda) \) contains the element \( 1 + m(2k + 3) \) for all \( 0 \leq m \leq k \). If not, suppose that \( \ell \) is the smallest integer such that \( \beta(\lambda) \) does not contain the element \( 1 + \ell(2k + 3) \). It is easy to check that \( \ell > j - 1 \). By replacing the element \( 2k + 2 + (i - 1)(2k + 3) \) by the element \( 1 + \ell(2k + 3) \), we get a new nice order ideal \( \beta' \) of \( P_{2k+1,2k+3} \). It is easy to check that \( \beta' \) has the same cardinality as \( \beta(\lambda) \) and has a larger sum of the elements. By relation (1.1), the size of the partition corresponding to \( \beta' \) is larger than that of \( \lambda \). This yields a contradiction with the fact that \( \lambda \) has the largest size, which completes the proof of the claim. Combining the claim and Lemmas 3.5 and 3.6 we have \( \beta(\lambda) = \gamma_{k,i} \). This completes the proof.

\[ \text{Lemma 3.8} \]

Let \( k \geq 2 \). We have \( |\mu_{k,i}| \leq |\mu_{k,k}| \) for \( 1 \leq i \leq k \), with the equality holding if and only if \( i = k \).

Proof. By relation (1.1), the size of \( \mu_{k,i} \) is given by

\[
|\mu_{k,i}| = \sum_{h \in \gamma_{k,i}} h - \binom{|\gamma_{k,i}|}{2} = \sum_{h \in \beta_{1,k}} h + \sum_{p=0}^{i-1} (2k + 2 + p(2k + 3)) - \binom{|\beta_{1,k}| + i}{2} = \sum_{h \in \beta_{1,k}} h - \left( \binom{|\beta_{1,k}|}{2} + \sum_{p=0}^{i-1} (2k + 2 + p(2k + 3)) - \binom{|\beta_{1,k}| + i}{2} \right) + \binom{|\beta_{1,k}|}{2} + \binom{|\beta_{1,k}|}{2} - |\lambda_{1,k}| + (2k + 2) \binom{i+1}{2} - i|\beta_{1,k}|.
\]

By the definition of \( \beta_{1,k} \), we obtain that

\[
|\beta_{1,k}| = \binom{k+1}{2}.
\]
Hence, we have

$$|\mu_{k,i}| = |\lambda_{1,k}| + (2k + 2) \left( \frac{i + 1}{2} \right) - i \left( \frac{k + 1}{2} \right). \quad (3.11)$$

In particular, we have

$$|\mu_{k,k}| - |\lambda_{1,k}| = (k + 2) \left( \frac{k + 1}{2} \right) = \frac{k(k + 1)(k + 2)}{2} \quad (3.12)$$

and

$$|\mu_{k,1}| - |\lambda_{1,k}| = -\frac{k^2 - 3k - 4}{2}. \quad (3.13)$$

In view of (3.12) and (3.13), we deduce that

$$|\mu_{k,k}| - |\mu_{k,1}| = \frac{k(k+1)(k+2)}{2} + \frac{k^2 - 3k - 4}{2}$$

$$= \frac{k^3 + 4k^2 - k - 4}{2}$$

$$= \frac{(k-1)(k+1)(k+4)}{2}. \quad (3.14)$$

For $k \geq 2$, it is easy to verify that $\frac{(k-1)(k+1)(k+4)}{2} > 0$. Hence, for $k \geq 2$, we have

$$|\mu_{k,k}| > |\mu_{k,1}|. \quad (3.15)$$

For fixed $k$, we see that $\mu_{k,i}$ is a quadratic function of $i$ with a positive leading coefficient. Hence the maximum value of $\mu_{k,i}$ is obtained at $i = 1$ or $i = k$ when $i$ ranges over $[1, k]$. In view of (3.15), we have $|\mu_{k,i}| \leq |\mu_{k,k}|$ for $1 \leq j \leq k$, with the equality holding if and only if $i = k$. This completes the proof.

In view of (3.8) and (3.12), we have $|\mu_{k,k}| > |\lambda_{2,k}|$ for all $k \geq 2$. One can easily verify that $|\mu_{1,k}| > |\lambda_{2,k}|$ for $k = 1$. Hence, from Lemmas 3.3, 3.7, and 3.8, we deduce the following result.

**Theorem 3.9** Fix $k \geq 1$. Let $\lambda$ be a $(2k + 1, 2k + 3)$-core partition into distinct parts which has the largest size. Then we have $\lambda = \mu_{k,k}$.

Now we are ready to derive the largest size of $(2k + 1, 2k + 3)$-core partitions into distinct parts.

**Theorem 3.10** Let $k \geq 0$. The largest size of $(2k + 1, 2k + 3)$-core partitions into distinct parts is given by $\frac{k(k+1)(k+2)(5k+11)}{24}$. 

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Proof. Clearly, the assertion holds for $k = 0$. For $k \geq 1$, let $\lambda$ be a $(2k+1, 2k+3)$-core partition into distinct parts which has the largest size. By Theorem 3.9 we have $\lambda = \mu_{k,k}$. In view of (3.12), we have
\[
|\lambda| = |\mu_{k,k}| = |\lambda_{1,k}| + \frac{k(k+1)(k+2)}{2}.
\] (3.16)

By relation (1.1) and the definition of $\beta_{1,k}$, we have
\[
|\lambda_{1,k}| = \sum_{j=0}^{k-1}(k-j)(2j+1) + \sum_{j=1}^{k}j(k-j)(2k+3) - \left(\binom{k+1}{2}\right)
\] (3.17)

By simple computation, from (3.16) and (3.17), we can deduce that
\[
|\lambda| = |\mu_{k,k}| = \frac{k(k+1)(k+2)(5k+11)}{24},
\]
as desired.

Let $s$ be an odd positive integer. Substituting $k = \frac{s-1}{2}$ in Theorem 3.10 we are led to a proof of Conjecture 1.2.

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