Grothendieck-Lefschetz Theory, Set-Theoretic Complete Intersections and Rational Normal Scrolls

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Abstract

Using the Grothendieck-Lefschetz theory (see [12]) we prove a criterion to deduce that certain subvarieties of $\mathbb{P}^n$ of dimension $\geq 2$ are not set-theoretic complete intersections (see Theorem 1 of the Introduction). As applications we give a number of relevant examples. In the last part of the paper we prove that the arithmetic rank of a rational normal $d$-dimensional scroll $S_{n_1, \ldots, n_d}$ in $\mathbb{P}^N$ is $N - 2$, by producing an explicit set of $N - 2$ homogeneous equations which define these scrolls set-theoretically (see Theorem 2 of the Introduction).

Introduction

Let us start by recalling the following definition.

Definition. Let $Y$ be a closed irreducible subvariety of the projective space $\mathbb{P}^n$, and denote by $I_+(Y)$ the homogeneous prime ideal generated by all the homogeneous polynomials in $k[T_0, T_1, \ldots, T_n]$ (in $n + 1$ variables) that vanish at each point of $Y$. If $f_1, \ldots, f_r \in k[T_0, T_1, \ldots, T_n]$ are homogeneous polynomials, denote also by $V_+(f_1, \ldots, f_r)$ the locus of points of $\mathbb{P}^n$ where $f_1, \ldots, f_r$ vanish. The arithmetic rank of $Y$ in $\mathbb{P}^n$, denoted by $\text{ara}(Y)$, is the minimal number of homogeneous equations needed to define $Y$ set-theoretically in $\mathbb{P}^n$, i.e. $\text{ara}(Y)$ is the minimal natural number $r$ for which there exist $r$ homogeneous polynomials $f_1, \ldots, f_r \in k[T_0, T_1, \ldots, T_n]$ such that $V_+(f_1, \ldots, f_r) = Y$. By Nullstellensatz, $\text{ara}(Y)$ is the minimal number of homogeneous equations needed to define $Y$ set-theoretically in $\mathbb{P}^n$, i.e. $\text{ara}(Y)$ is the minimal natural number $r$ for which there exist $r$ homogeneous polynomials $f_1, \ldots, f_r \in k[T_0, T_1, \ldots, T_n]$ such that $I_+(Y) = \sqrt{(f_1, \ldots, f_r)}$. Clearly, $\text{ara}(Y) \geq \text{codim}_{\mathbb{P}^n}(Y)$. If $\text{ara}(Y) = \text{codim}_{\mathbb{P}^n}(Y)$, we say that $Y$ is a set-theoretic complete intersection in $\mathbb{P}^n$.

This paper has two main parts. In the first part we show how the Grothendieck-Lefschetz theory (see [12]) can be used to provide necessary conditions for a given subvariety $Y$ of dimension $d \geq 2$ of the projective space $\mathbb{P}^n$ (over an algebraically closed field of arbitrary characteristic) to be a set-theoretic complete intersection in $\mathbb{P}^n$. We shall illustrate this through a number of relevant examples. In the second part of the paper we show that the arithmetic rank of any rational normal scroll $S$ of dimension $\geq 2$ in $\mathbb{P}^N$ is $N - 2$, by exhibiting an explicit minimal set of $N - 2$ defining equations for $S$.

The paper is organized as follows. In Section 1 we recall some basic results from Grothendieck-Lefschetz theory that are going to be used in Section 2. We also recall two Lefschetz theorems (for singular cohomology and for étale cohomology) that will be used in Section 4.
Theorem 1 Let $Y$ be a closed irreducible subvariety of $\mathbb{P}^n$ of dimension $\geq 2$ over an algebraically closed field $k$ of characteristic $p \geq 0$.

i) If $Y$ is a set-theoretic complete intersection in $\mathbb{P}^n$ then $Y$ is algebraically simply connected, i.e. there are no non-trivial connected étale covers of $Y$.

ii) Assume that $p = 0$ and $Y$ normal. If $H^1(\mathcal{O}_Y) \neq 0$, then $Y$ is not a set-theoretic complete intersection in $\mathbb{P}^n$.

iii) Assume that $p > 0$ and $Y$ is normal. If $H^1(\mathcal{O}_Y) \neq 0$ and the Picard scheme $\text{Pic}^0_Y$ of $Y$ is reduced, then $Y$ is not a set-theoretic complete intersection in $\mathbb{P}^n$. (If for example $H^2(\mathcal{O}_Y) = 0$, then $\text{Pic}^0_Y$ is always reduced, see [11], Exposé 236, Proposition 2.10, ii)).

iv) Assume that $Y$ is a set-theoretic complete intersection in $\mathbb{P}^n$. Then the restriction map $\alpha : \text{Pic}(\mathbb{P}^n) \to \text{Pic}(Y)$ is injective and $\text{Coker}(\alpha)$ is torsion-free if $p = 0$, and has no $s$-torsion for every integer $s > 0$ which is prime to $p$, if $p > 0$.

v) Assume that there exists a line bundle $L$ on $Y$ and an integer $s \geq 2$ such that $\mathcal{O}_Y(1) \cong L^\otimes s$. If $p > 0$ assume moreover that $s$ is prime to $p$. Then $Y$ is not a set-theoretic complete intersection in $\mathbb{P}^n$.

vi) Assume that $Y$ is a set-theoretic complete intersection of dimension $\geq 3$. If $p = 0$ then the restriction map $\text{Pic}(\mathbb{P}^n) \to \text{Pic}(Y)$ is an isomorphism. If $p > 0$ and $Y$ is nonsingular, then $\text{Pic}(Y)/\mathbb{Z}[\mathcal{O}_Y(1)]$ is a finite $p$-group (and in particular, $\text{rank \ Pic}(Y) = 1$).

In some special cases, parts of Theorem 1 are known. To our best knowledge the approach to prove Theorem 1, which is based on Grothendieck-Lefschetz theory, is new. For instance, if $Y$ is a nonsingular closed subvariety of the complex projective space $\mathbb{P}^n_{\mathbb{C}}$, part ii) is an old result of Hartshorne [15], while part iii) is new. Part vi) is known in characteristic zero (see [23]). However, we give another proof based on a result of Grothendieck [12] (see also Theorem 1.4 below). In Section 3 we prove Theorem 1 using Grothendieck-Lefschetz theory [12], although parts ii) and iii) also require some basic results from the theory of Picard schemes, see Grothendieck [11].

In Section 3, we apply Theorem 1 to provide several examples (in any characteristic) of subvarieties $Y \subseteq \mathbb{P}^n$ of dimension $\geq 2$ that cannot be set-theoretic complete intersections in $\mathbb{P}^n$.

In Section 4 we determine the arithmetic rank of the rational normal scrolls. Specifically, given the integers $\{d, n_1, \ldots, n_d\}$ such that $d \geq 2$ and $n_i \geq 1$, $i = 1, \ldots, d$, let us consider the $d$-dimensional rational normal scroll

$$S_{n_1, \ldots, n_d} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_d))$$

embedded in $\mathbb{P}^N$ via the very ample complete linear system $|\mathcal{O}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_d))(1)|$, where $N := \sum_{i=1}^d n_i + d - 1$ and $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_d))$ is the projective bundle associated to the vector bundle $\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_d)$ over $\mathbb{P}^1$. 

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We prove the following result (see Theorem 4.1 and Corollary 4.2 below):

**Theorem 2** Under the above notation and assumptions, the arithmetic rank of \( S_{n_1,\ldots,n_d} \) in \( \mathbb{P}^N \) is \( N - 2 = \sum_{i=1}^d n_i + d - 3 \). In particular, \( S_{n_1,\ldots,n_d} \) is a set-theoretic complete intersection in \( \mathbb{P}^N \) if and only if \( \dim(S_{n_1,\ldots,n_d}) = 2 \).

The fact that the 2-dimensional rational normal scrolls \( S_{n_1,n_2} \) are set-theoretic complete intersections in \( \mathbb{P}^{n_1+n_2+1} \) was already known, see Valla [31] and Robbiano-Valla [26] in some special cases, and subsequently, Verdi [34] in general. Our approach provides in particular a new proof of the result of Verdi [34] for the two-dimensional rational normal scrolls. Moreover, in general our homogeneous equations are of lower degree than Verdi’s equations.

As far as the proof of Theorem 2 is concerned, we notice that the inequality \( \text{ara}(S_{n_1,\ldots,n_d}) \geq N - 2 \) is of topological nature. In fact, in characteristic zero this inequality is a consequence of a generalization (due to Lazarsfeld [16]) of a topological result of Sommese [30] (see Corollary 1.6 below), while in positive characteristics it follows from an analogous result in the étale cohomology, essentially due to Lyubeznik [17] (see Theorem 1.8 below).

So the problem is reduced to proving the reverse inequality \( \text{ara}(S_{n_1,\ldots,n_d}) \leq N - 2 \). And this is done by exhibiting \( N - 2 \) explicit homogeneous equations defining \( S_{n_1,\ldots,n_d} \) set-theoretically in \( \mathbb{P}^N \).

Throughout this paper we shall fix an algebraically closed field \( k \) of characteristic \( p \geq 0 \). All algebraic varieties that will occur will be defined over \( k \). The terminology and the notation used are standard, unless otherwise explicitly stated.

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1 Background material

In this section we recall some well-known theorems that will be used in the sequel. We start with some basic facts from Grothendieck-Lefschetz theory (see [12]).

**Definition 1.1 (Grothendieck [12])** Let \( Y \) be a closed subvariety of a projective variety \( X \). We say that the pair \( (X,Y) \) satisfies the Grothendieck-Lefschetz condition \( \text{Lef}(X,Y) \) if for every open subset \( U \) of \( X \) containing \( Y \) and for every vector bundle \( E \) on \( U \) the natural map \( H^0(U,E) \rightarrow H^0(X/Y, \pi^*E) \) is an isomorphism, where \( X/Y \) is the formal completion of \( X \) along \( Y \), \( \pi: X/Y \rightarrow U \) the canonical morphism, and \( \pi^*E \) is the pullback of \( E \). We also say that \( (X,Y) \) satisfies the **effective Grothendieck-Lefschetz condition** \( \text{Lef}(X,Y) \) if the Grothendieck-Lefschetz condition \( \text{Lef}(X,Y) \) holds and, moreover, for every formal vector bundle \( \mathcal{E} \) on \( X/Y \) there exists an open subset \( U \) of \( X \) and a vector bundle \( E \) on \( U \) such that \( \mathcal{E} \cong \pi^*E \).

**Theorem 1.2 (Grothendieck [12], or also [14], Theorem 1.5, page 172)** Let \( X \) be closed irreducible subvariety of \( \mathbb{P}^n \). Let \( Y \) be a complete intersection subscheme of \( X \) with \( r \) hyperplanes of \( \mathbb{P}^n \), and assume that \( \dim(Y) = \dim(X) - r \geq 2 \). If in addition \( Y \) is contained in the nonsingular locus of \( X \), then the effective Grothendieck-Lefschetz condition \( \text{Lef}(X,Y) \) holds.
Theorem 1.3 (Grothendieck [12], Expos´e X, Th´eor`eme 3.10) Let $Y$ be a closed subvariety of $\mathbb{P}^n$ such that the effective Grothendieck-Lefschetz condition $\text{Leff}(\mathbb{P}^n, Y)$ holds. Then $Y$ is algebraically simply connected, i.e. there are no non-trivial connected ´etale covers of $Y$.

Theorem 1.4 (Grothendieck [12], Expos´e XII, Corollary 3.7) Let $Y$ be a (non necessarily reduced or irreducible) subscheme of $\mathbb{P}^n$ of dimension $\geq 3$ which is a scheme-theoretic complete intersection in $\mathbb{P}^n$. Then the natural restriction map $\text{Pic}(\mathbb{P}^n) \to \text{Pic}(Y)$ is an isomorphism.

In the last section we shall also make use of two further Lefschetz type results. The first one regards the singular cohomology (see [16], (1.8)) and generalizes earlier results due to Sommese [30] and Newstead [22] and [23]. Instead the second one uses the ´etale cohomology.

Theorem 1.5 Let $X$ be a nonsingular projective variety over $\mathbb{C}$ of dimension $n \geq 2$, and let $E$ be an ample vector bundle of rank $e$ on $X$. Let $s \in \Gamma(X, E)$ be a global section of $S$ and let $Y = Z(s)$ be the zero locus of $s$. Then the natural restriction map of singular cohomology groups

$$H^i(X, \mathcal{O}) \to H^i(Y, \mathcal{O})$$

is an isomorphism for every $i < n - e$, and injective if $i = n - e$.

Using Theorem 1.5, the exponential sequences for $X$ and for $Y$ and Serre’s GAGA one immediately gets:

Corollary 1.6 Under the hypotheses of Theorem 1.5, assume that $n - e \geq 3$. Then the natural restriction map $\text{Pic}(X) \to \text{Pic}(Y)$ is an isomorphism.

Remark 1.7 Under the extra-hypothesis that $\text{dim}(Y) = n - e$, Theorem 1.5 was proved by Sommese in [30]. Actually, Lazarsfeld observed in [16] that essentially the same proof of Sommese also yields the general case when $\text{dim}(Y) \geq n - e$ (and in the last section we are going to use this result exactly under this more general assumption). On the other hand, Newstead proved various Lefschetz type results (for homotopy groups, and singular homology and cohomology groups) in the case where $E$ is a direct sum of line bundles of the form $\mathcal{O}_X(m)$, with $m > 0$, see [22] and [23].

The next theorem (which follows easily from some results of Lyubeznik [17]) takes care of the case when the characteristic of $k$ is arbitrary.

Theorem 1.8 Assume that $p > 0$ and let $Y$ be a nonsingular closed subvariety of $\mathbb{P}^N$ which is set theoretically given by $s$ equations, with $s \leq N - 3$. Then the restriction map $\alpha: \text{Pic}(\mathbb{P}^N) \to \text{Pic}(Y)$ is injective and $\text{Coker}(\alpha)$ is a finite torsion $p$-group.

Proof. It is a basic fact that the ´etale cohomological dimension of an affine variety $U$ is $\leq \text{dim}(U)$ (see e.g. Milne [18], Theorem 15.1; this result is in fact an ´etale analogue of a classical topological result of Andreotti and Frankel, see [1], cf. also Milnor [19], page ?). If instead $U$ is covered by $s$ open affine subsets, then this result plus repeated application of Mayer-Vietoris (see [18], Theorem 10.8) yield the fact that the ´etale cohomological
dimension of $U$ is $\leq \dim(U) + s - 1$. Applying this to $U := \mathbb{P}^N \setminus Y$ (which by hypothesis is covered by $s$ affines, namely the complements of the surfaces of the $s$ equations defining $Y$ set-theoretically) we get that the étale cohomological dimension of $\mathbb{P}^N \setminus Y$ is $\leq N + s - 1 \leq 2N - 4$. At this point we can apply Lemma 11.1 in [17] to get the conclusion. □

Remark 1.9 A Lefschetz type result similar to Theorem 1.8 but for fundamental group has been proved by Cutkovsky in [9].

2 Necessary conditions for set-theoretic complete intersections

We start with the following result:

Proposition 2.1 (cf. [2], p. 115) Let $Y$ be a closed subvariety of the projective irreducible variety $X$ over $k$, and assume that $p = 0$. Then for every formal line bundle $\mathcal{L} \in \text{Pic}(X/Y)$ such that $\mathcal{L}|_Y \cong M^\otimes s$, with $s \geq 2$ an integer and $M \in \text{Pic}(Y)$, there exists a formal line bundle $\mathcal{M} \in \text{Pic}(X/Y)$ such that $\mathcal{L} \cong M^\otimes s$ and $\mathcal{M}|_Y \cong M$. The same statement holds if $p > 0$, provided that $s$ is prime to $p$.

Proof. Since this result is going to be used later on in an essential way, for the convenience of the reader we include the proof. For every $n \geq 0$ consider the infinitesimal neighbourhood $Y(n) = (Y, \mathcal{O}_X/J^{n+1})$ of order $n$ of $Y$ in $X$. We have the inclusions of subschemes

$$Y(0) \subset Y(1) \subset Y(2) \subset \cdots \subset X.$$ 

Then giving a formal line bundle $\mathcal{L}$ on $X/Y$ amounts to giving a sequence $\{L_n\}_{n \geq 0}$, with $L_n \in \text{Pic}(Y(n))$ such that $L_{n+1}|_Y \cong L_n$ for every $n \geq 0$. The hypothesis says that $L_0 \cong M^\otimes s$ for some $M$ in $\text{Pic}(Y(0)) = \text{Pic}(Y)$. We shall construct by induction a formal line bundle $\mathcal{M} = \{M_n\}_{n \geq 0}$ in $\text{Pic}(X/Y)$ with the desired properties. Starting with $M_0 = M$, the induction step is the following:

Claim. Assume that for a fixed integer $n \geq 0$ there exists $M_n \in \text{Pic}(Y(n))$ such that $L_n \cong M_n^\otimes s$. Then there exists $M_{n+1} \in \text{Pic}(Y(n+1))$ such that $L_{n+1} \cong M_{n+1}^\otimes s$ and $M_{n+1}|_Y \cong M_n$.

Indeed, consider the exact sequence of cohomology

$$H^1(Y, J^{n+1}/J^{n+2}) \to \text{Pic}(Y(n+1)) \to \text{Pic}(Y(n)) \to H^2(Y, J^{n+1}/J^{n+2})$$

associated to the truncated exponential exact sequence

$$0 \to J^{n+1}/J^{n+2} \to \mathcal{O}_Y^*|_{Y(n+1)} \to \mathcal{O}_Y^*|_{Y(n)} \to 0,$$

where $J$ is the sheaf of ideals of $Y$ in $X$. To prove the claim, observe that in this cohomology sequence the extreme terms are vector spaces over $k$; in particular $H^2(Y, J^{n+1}/J^{n+2})$ has no torsion because $\text{char}(k) = 0$. Then the class of $M_n$ in

$$\text{Pic}(Y(n))/\text{Im}(\text{Pic}(Y(n+1)) \to \text{Pic}(Y(n))) \subseteq H^2(Y, J^{n+1}/J^{n+2})$$
is a torsion element of order dividing $s$. Since $H^2(Y,\mathcal{I}^{n+1}/\mathcal{I}^{n+2})$ has no torsion we infer that $M_n \in \text{Im}(\text{Pic}(Y(n+1)) \to \text{Pic}(Y(n)))$, i.e. there exists $N \in \text{Pic}(Y(n+1))$ such that $N|Y(n) \cong M_n$. Now

$$(L_{n+1} \otimes N^{\otimes(-s)})|Y(n) \cong L_n \otimes M_n^{\otimes(-s)} \cong \mathcal{O}_{Y(n)}.$$ 

Therefore $L_{n+1} \otimes N^{\otimes(-s)}$ is a line bundle on $Y(n+1)$ coming from the $k$-vector space $H^1(Y,\mathcal{I}^{n+1}/\mathcal{I}^{n+2})$. Since $\text{char}(k) = 0$ every element of such a $k$-vector space is divisible by $s$, whence

$$L_{n+1} \otimes N^{\otimes(-s)} \cong P^{\otimes s}, \quad \text{with } P \in \text{Pic}(Y(n+1)) \text{ such that } P|Y(n) \cong \mathcal{O}_{Y(n)}.$$ 

If we take $M_{n+1} = N \otimes P$ we get $L_{n+1} \cong M_{n+1}^{\otimes s}$ and $M_{n+1}|Y(n) \cong M_n$, which proves the claim.

The last assertion of the proposition comes from the above argument plus the observation that the $k$-vector space $H^2(Y,\mathcal{I}^{n+1}/\mathcal{I}^{n+2})$ over a field $k$ of characteristic $p > 0$ has no $e$-torsion and every element of the $k$-vector space $H^1(Y,\mathcal{I}^{n+1}/\mathcal{I}^{n+2})$ is (uniquely) divisible by $e$, for every $e > 0$ prime to $p$.

Here are two corollaries of Proposition 2.1.

**Corollary 2.2** Under the notation and hypotheses of Proposition 2.1, the abelian group $\text{Coker}(\text{Pic}(X/Y) \to \text{Pic}(Y))$ is torsion-free if $p = 0$, and has no $e$-torsion for every positive integer $e$ which is prime to $p$, if $p > 0$.

If $A$ is an abelian (multiplicative) group with neutral element $e$, we shall denote by $\text{Tors}(A)$ the torsion subgroup of $A$. Let $p \geq 2$ be a prime integer. Then we also set

$$\text{Tors}^p(A) := \{a \in A \mid \exists s > 0 \text{ such that } s \text{ is prime to } p \text{ and } a^s = e\}.$$ 

Clearly $\text{Tors}^p(A)$ is a subgroup of $A$. Then we also have:

**Corollary 2.3** Under the notation and hypotheses of Proposition 2.1 (with $p = \text{char}(k)$), assume furthermore that $X$ is nonsingular and $\text{Leff}(X,Y)$ holds. Then:

i) The abelian group $\text{Coker}(\text{Pic}(X) \to \text{Pic}(Y))$ is torsion-free if $p = 0$, and has no $s$-torsion for every positive integer $s$ which is prime to $p$ if $p > 0$. If in addition the restriction map $\alpha: \text{Pic}(X) \to \text{Pic}(Y)$ is injective (this is always the case if $X = \mathbb{P}^n$ and $\dim(Y) > 0$), then $\alpha$ induces an isomorphism $\text{Tors}(\text{Pic}(X)) \cong \text{Tors}(\text{Pic}(Y))$ if $p = 0$, and an isomorphism $\text{Tors}^p(\text{Pic}(X)) \cong \text{Tors}^p(\text{Pic}(Y))$ if $p > 0$.

ii) Assume in addition that $Y$ meets every hypersurface of $X$. Let $L$ be a line bundle on $X$ such that $L|Y \cong M^{\otimes s}$ for some $M \in \text{Pic}(Y)$ and $s \geq 2$ prime to $p$, if $p > 0$. Then there exists a line bundle $M' \in \text{Pic}(X)$ such that $M'|Y \cong M$ and $L \cong M'^{\otimes s}$.

**Proof.** i) The canonical restriction map $\text{Pic}(X) \to \text{Pic}(Y)$ factors as $\text{Pic}(X) \to \text{Pic}(X/Y) \to \text{Pic}(Y)$. By Corollary 2.2 it is enough to show that the map $\text{Pic}(X) \to \text{Pic}(X/Y)$ is surjective. To check this, let $\mathcal{L} \in \text{Pic}(X/Y)$ be an arbitrary formal line bundle. By $\text{Leff}(X,Y)$, there exists an open subset $U$ of $X$ containing $Y$ and a line bundle $L'$ on $U$ such that $L' \cong \mathcal{L}$. Since $X$ is nonsingular, $L'$ extends to a line bundle $L \in \text{Pic}(X)$. Then clearly $L = L' \cong \mathcal{L}$, which yields i).
To prove ii) observe that the hypotheses that $X$ is nonsingular and $Y$ meets every hypersurface of $X$ implies that the restriction map $\text{Pic}(X) \to \text{Pic}(U)$ is an isomorphism for every open subset $U$ containing $Y$. Then ii) follows immediately from Proposition 2.4. Indeed, by Proposition 2.4 and the fact that $\text{Leff}(X, Y)$ holds we infer that there exists a formal line bundle $M \in \text{Pic}(X_0)$ such that $M|_Y \cong M$ and the formal completion $\hat{\mathcal{L}}$ is isomorphic to $M^\otimes s$. By $\text{Leff}(X, Y)$ again we find an open neighbourhood $U$ of $Y$ in $X$ and a line bundle $M'' \in \text{Pic}(U)$ such that $L|U \cong M''^\otimes s$ and the formal completion $\hat{M}$ is isomorphic to $\mathcal{M}$ (in particular, $M''|Y \cong M$ and $L|U \cong M''^\otimes s$). Finally, since the restriction map $\text{Pic}(X) \to \text{Pic}(U)$ is an isomorphism, we can (uniquely) extend $M''$ to a line bundle $M' \in \text{Pic}(X)$ with the desired properties. \hfill \Box

**Remark 2.4** The hypothesis that $X$ is nonsingular is essential in Corollary 2.3. Indeed, fix $r \geq 2$ and $s \geq 2$, and take $X \subset \mathbb{P}^{(r+s)}$ the projective cone over the polarized variety $(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(s))$, e.g. the projective cone over the Veronese embedding $\mathbb{P}^r \hookrightarrow \mathbb{P}^{(r+s)-1}$. Take $Y$ the intersection of $X$ with the hyperplane at infinity. Then $Y \cong \mathbb{P}^r$, and since $Y$ is a hyperplane section of $X$ of dimension $r \geq 2$, by Grothendieck’s result (Theorem 1.2 above) the effective Grothendieck-Lefschetz condition $\text{Leff}(X, Y)$ holds. In this case $\text{Coker}(\text{Pic}(X) \to \text{Pic}(Y)) \cong \mathbb{Z}/s\mathbb{Z}$, and in particular, $\text{Coker}(\text{Pic}(X) \to \text{Pic}(Y))$ has torsion if $p = 0$. However, if $U := X \setminus \{p\}$, with $p$ the vertex of the cone $X$, then $U$ is nonsingular and the restriction map $\text{Pic}(U) \to \text{Pic}(Y)$ is an isomorphism.

**Remarks 2.5**

i) The earliest reference we are aware of, regarding the torsion-freeness of the cokernel of some natural restriction maps between singular cohomology groups, is [1]. Specifically, let $X$ be an $n$-dimensional nonsingular subvariety of the complex projective space $\mathbb{P}^N(\mathbb{C})$, and let $Y$ the proper intersection of $X$ with an hyperplane $H$ of $\mathbb{P}^N(\mathbb{C})$. Then part of the famous topological theorem on hyperplane sections asserts that if $n \geq 2$ then the canonical restriction map $H^{n-1}(X, \mathbb{Z}) \to H^{n-1}(Y, \mathbb{Z})$ is injective and its cokernel is torsion-free.

ii) Let $Y$ be a nonsingular (scheme-theoretic) complete intersection surface of $\mathbb{P}^N$ over a field $k$ of characteristic zero. Then Robbiano proved in [25] a criterion for a curve $C$ lying on $Y$ to be the scheme-theoretic intersection of $Y$ with a hyperplane $H$ of $\mathbb{P}^n$. In order to do that he used in an essential way the fact that the cokernel of the canonical map $\text{Pic}(\mathbb{P}^N) \to \text{Pic}(Y)$ is torsion-free.

**Lemma 2.6** Let $Y$ be a closed irreducible subvariety of $\mathbb{P}^n$ of dimension $d \geq 2$. If $Y$ is a set-theoretic complete intersection in $\mathbb{P}^n$ then the effective Grothendieck-Lefschetz condition $\text{Leff}(\mathbb{P}^n, Y)$ holds.

Proof. Let $f_1, \ldots, f_r \in k[T_0, T_1, \ldots, T_n]$ be homogeneous polynomials defining $Y$ in $\mathbb{P}^n$ as a set-theoretic complete intersection in $\mathbb{P}^n$, where $r = n - d$. Then we have $\sqrt{(f_1, \ldots, f_r)} = \mathcal{I}_Y$. If $Y'$ is the subscheme of $\mathbb{P}^n$ defined by the ideal $(f_1, \ldots, f_r)$ then $Y'$ is a scheme-theoretic complete intersection of $\mathbb{P}^n$ of dimension $\geq 2$, and hence by Theorem 1.2 $\text{Leff}(\mathbb{P}^n, Y')$ holds. Since $Y'_\text{red} = Y$, we infer that $\mathbb{P}^n/Y = \mathbb{P}^n/Y'$, and in particular, $\text{Leff}(\mathbb{P}^n, Y)$ also holds. \hfill \Box

Now we are ready to prove the main result of this section.

**Theorem 2.7** Let $Y$ be a closed irreducible subvariety of $\mathbb{P}^n$ of dimension $\geq 2$ over an algebraically closed field $k$ of characteristic $p \geq 0$. 

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i) If $Y$ is a set-theoretic complete intersection in $\mathbb{P}^n$ then $Y$ is algebraically simply connected, i.e. there are no non-trivial connected étale covers of $Y$.

ii) Assume that $p = 0$ and $Y$ normal. If $H^1(\mathcal{O}_Y) \neq 0$, then $Y$ is not a set-theoretic complete intersection in $\mathbb{P}^n$.

iii) Assume that $p > 0$ and $Y$ is normal. If $H^1(\mathcal{O}_Y) \neq 0$ and the Picard scheme $\text{Pic}^0_y$ of $Y$ is reduced, then $Y$ is not a set-theoretic complete intersection in $\mathbb{P}^n$. (If for example $H^2(\mathcal{O}_Y) = 0$, then $\text{Pic}^0_y$ is always reduced, see [11], Exposé 236, Proposition 2.10, ii)).

iv) Assume that $Y$ is a set-theoretic complete intersection in $\mathbb{P}^n$. Then the restriction map $\alpha: \text{Pic}(\mathbb{P}^n) \to \text{Pic}(Y)$ is injective and $\text{Coker}(\alpha)$ is torsion-free if $p = 0$, and has no $s$-torsion for every integer $s > 0$ which is prime to $p$, if $p > 0$.

v) Assume that there exists a line bundle $L$ on $Y$ and an integer $s \geq 2$ such that $\mathcal{O}_Y(1) \cong L^{\otimes s}$. If $p > 0$ assume moreover that $s$ is prime to $p$. Then $Y$ is not a set-theoretic complete intersection in $\mathbb{P}^n$.

vi) Assume that $Y$ is a set-theoretic complete intersection of dimension $\geq 3$. If $p = 0$ then the restriction map $\text{Pic}(\mathbb{P}^n) \to \text{Pic}(Y)$ is an isomorphism. If $p > 0$ and $Y$ is nonsingular, then $\text{Pic}(Y)/\mathbb{Z}[\mathcal{O}_Y(1)]$ is a finite $p$-group (and in particular, rank $\text{Pic}(Y) = 1$).

Proof. Part i) is a consequence of Lemma [2.6] and of [12], Exposé X, Théorème 3.10.

ii) Assume that $Y$ is a set-theoretic complete intersection in $\mathbb{P}^n$. Then by Lemma [2.6] $\text{Leff}(\mathbb{P}^n, Y)$ holds. Clearly, $Y$ meets every hypersurface of $\mathbb{P}^n$. Then by Corollary [2.3] i), we get $\text{Tors}(\text{Pic}(\mathbb{P}^n)) \cong \text{Tors}(\text{Pic}(Y))$. Since $\text{Pic}(\mathbb{P}^n) = \mathbb{Z}$, we get $\text{Tors}(\text{Pic}(Y)) = 0$. But, under our hypotheses, this is absurd because $\text{Pic}(Y)$ contains the subgroup $\text{Pic}^0(Y)$ of isomorphism classes of line bundles on $Y$ which are algebraically trivial. Then $\text{Pic}^0(Y)$ is the underlying set of the Picard scheme $\text{Pic}^0_Y$. The fact that $Y$ is normal implies that the Picard scheme $\text{Pic}^0_Y$ is proper over $k$, and in particular, $(\text{Pic}^0_Y)_{\text{red}}$ is an abelian variety (see [11], Exposé 236, Théorème 2.1, ii)). In fact, since the characteristic of $k$ is zero, by a theorem of Chevalley, the abelian scheme $\text{Pic}^0_Y$ is reduced (see [20], Lecture 25, Theorem 1) and the tangent space $T_{\text{Pic}^0_Y, 0}$ is isomorphic with $H^1(\mathcal{O}_Y)$ (see [20], Lecture 24), which by hypothesis is of dimension $q := h^0(\mathcal{O}_Y) > 0$. Then the abelian group $\text{Pic}^0(Y)$ contains a lot of torsion (for example, if $e \geq 2$ is an integer, the $e$-torsion subgroup of $\text{Pic}^0(Y)$ is isomorphic with $(\mathbb{Z}/e\mathbb{Z})^{2q} \neq 0$ (see [21], Chap. II, §7), which yields the desired contradiction because $\text{Tors}(\text{Pic}(\mathbb{P}^n)) = 0$.

Notice that ii) is also an easy consequence of i). In fact, take a non-trivial torsion element in $\text{Pic}^0(Y)$ of order $m \geq 2$ which is prime to $p$ if $p > 0$, i.e. a non-trivial line bundle $L \in \text{Pic}(Y)$ such that $L^{\otimes m} \cong \mathcal{O}_Y$ for some $m \geq 2$ (with $m$ is the least natural number with this property). Then $L$ produces the non-trivial connected cyclic étale cover $\tilde{Y} = \text{Spec}(\oplus_{i=0}^{m-1} L^{\otimes i})$, and, in particular, $Y$ is not algebraically simply connected.

iii) The proof in this case is almost identical with the proof of ii). The only difference is that in characteristic $p > 0$ the Picard scheme may not be reduced. But this possibility is ruled out by our hypothesis. Moreover, the $e$-torsion subgroup of a $q$-dimensional abelian
variety is still isomorphic with \((\mathbb{Z}/e\mathbb{Z})^2\), provided that \(p\) does not divide \(e\) (see [21], Chap. II, §7).

iv) Clearly, \(Y\) meets every hypersurface of \(\mathbb{P}^n\), and in particular the map \(\alpha\) is injective (Corollary [23] i)). Moreover by the proof of i), since \(Y\) is a set-theoretic complete intersection in \(\mathbb{P}^n\) of dimension \(\geq 2\), \(\text{Leff}(\mathbb{P}^n, Y)\) holds. Then the conclusion follows from Corollary [23] i).

v) We have \(\text{Coker}(\text{Pic}(\mathbb{P}^n) \to \text{Pic}(Y)) = \text{Pic}(Y)/\mathbb{Z}[\mathcal{O}_Y(1)]\). Then the conclusion follows from iv).

vi) The result follows (in arbitrary characteristic) from Theorem [1.8] while in characteristic zero – from an old result of Sommese (see [30], Proposition (1.16)). However, if \(p = 0\) we shall give another proof using Theorem 1.4 of Grothendieck.

By Lefschetz’s principle we may assume that \(k = \mathbb{C}\). Let \(f_1, \ldots, f_r \in \mathbb{C}[T_0, T_1, \ldots, T_n]\) be homogeneous equations defining \(Y\) set-theoretically in \(\mathbb{P}^n\), and set \(Y' := \text{Proj}(\mathbb{C}[T_0, T_1, \ldots, T_n])\). Since \(\sqrt{(f_1, \ldots, f_r)} = \mathcal{I}_+(Y)\), we have \(Y'_{\text{red}} = Y\), and in particular the underlying topological spaces of \(Y'\) and \(Y\) are the same. If \(F\) is an algebraic (e.g. a coherent) sheaf on an algebraic scheme \(Z\) over \(\mathbb{C}\), we shall denote by \(F^\text{an}\) the analytic sheaf associated to \(F\) in the sense of Serre’s GAGA (see [27]). Then we have the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z}_{Y'} & \to & \mathcal{O}^\text{an}_{Y'} & \to & \mathcal{O}^\text{an}_{Y'}^* & \to & 0 \\
& & \text{id} \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathbb{Z}_Y & \to & \mathcal{O}^\text{an}_Y & \to & \mathcal{O}^\text{an}_Y^* & \to & 0
\end{array}
\]

where the rows are the exponential exact sequences of \(Y'\) and \(Y\) (\(\mathbb{Z}_{Y'}\) and \(\mathbb{Z}_Y\) are the constant sheaves on \(Y'\) and on \(Y\) respectively with stalks \(\mathbb{Z}\) ) and the vertical arrows are the canonical restriction maps induced by the inclusion \(Y \subseteq Y'\). The above diagram yields the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
H^1(\mathcal{O}^\text{an}_{Y'}) & \to & H^1(\mathcal{O}^\text{an}_{Y'}^*) & \to & H^2(Y', \mathbb{Z}) & \to & H^2(\mathcal{O}^\text{an}_{Y'}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(\mathcal{O}^\text{an}_Y) & \to & H^1(\mathcal{O}^\text{an}_Y^*) & \to & H^2(Y, \mathbb{Z}) & \to & H^2(\mathcal{O}^\text{an}_Y)
\end{array}
\]

Since \(Y'\) is a scheme-theoretic complete intersection in \(\mathbb{P}^n\) of dimension \(\geq 3\) we get \(H^i(\mathcal{O}_{Y'}) = 0\) for \(i = 1, 2\). Then by Serre’s GAGA [27] we also get \(H^i(\mathcal{O}^\text{an}_{Y'}) = 0\) for \(i = 1, 2\). Moreover, Serre’s GAGA, also implies

\(H^1(\mathcal{O}^\text{an}_{Y'}^*) \cong H^1(\mathcal{O}_{Y'}^*) = \text{Pic}(Y')\).

Doing the same thing for the cohomology groups on the bottom row of the last diagram and observing that since \(Y\) is a set-theoretic complete intersection of dimension \(\geq 2\), by ii) we have \(H^1(\mathcal{O}_Y) = 0\), by putting things together we get the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \text{Pic}(Y') & \to & H^2(Y', \mathbb{Z}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Pic}(Y) & \to & H^2(Y, \mathbb{Z})
\end{array}
\]
From this diagram it follows that the canonical restriction map $\text{Pic}(Y') \to \text{Pic}(Y)$ is an isomorphism. Finally, since $Y'$ is a scheme-theoretic complete intersection of dimension $\geq 3$, by Theorem 1.4 of Grothendieck, the restriction map $\text{Pic}(\mathbb{P}^n) \to \text{Pic}(Y')$ is an isomorphism. Therefore the restriction map $\text{Pic}(\mathbb{P}^n) \to \text{Pic}(Y)$ is an isomorphism, i.e. $\text{Pic}(Y) = \mathbb{Z}[\mathcal{O}_Y(1)]$. 

**Remarks 2.8** i) If $k = \mathbb{C}$ and $Y$ is nonsingular, part ii) of Theorem 2.7 is an old result of Hartshorne (see [15], Corollary 8.6). Our proof of this more general result contained in ii) and in iii) is completely different from Hartshorne’s proof (loc. cit.).

ii) Newstead proved in [22] and in [23] topological Lefschetz theorems (for singular cohomology with coefficients in $\mathbb{Z}$) for submanifolds $Y$ of the complex projective space $\mathbb{P}^n$, which are defined by “not too many equations” in $\mathbb{P}^n$. As applications he gave several examples of submanifolds $Y$ of the complex projective space $\mathbb{P}^n$ which are not set-theoretic complete intersections.

iii) In Theorem 2.7 vi), the finite $p$-group $\text{Pic}(Y)/\mathbb{Z}[\mathcal{O}_Y(1)]$ may effectively be non-trivial (see Proposition 3.1 in the next section).

### 3 Examples of projective varieties that are not set-theoretic complete intersections

**The Veronese embedding.** The image of the $s$-fold Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^s$ (the rational normal curve of degree $s \geq 2$ in $\mathbb{P}^s$) is known to be a set-theoretic complete intersection in $\mathbb{P}^s$ (this is completely elementary, see [33], cf. also [26]), but not a scheme-theoretic complete intersection if $s \geq 3$ (because it is not subcanonical).

On the other hand, for every integers $r, s \geq 2$ consider the $s$-fold Veronese embedding $i: \mathbb{P}^r \hookrightarrow \mathbb{P}^{n(r,s)}$ over an algebraically closed field of characteristic $p \geq 0$, with $n(r,s) = \binom{r+s}{s} - 1$. Then $Y := i(\mathbb{P}^r)$ is not a set-theoretic complete intersection in $\mathbb{P}^{n(r,s)}$, provided that $p$ does not divide $s$, if $p > 0$. This follows immediately from Theorem 2.7 iv), because $\mathcal{O}_{\mathbb{P}^{n(r,s)}}(1)|_Y = \mathcal{O}_{\mathbb{P}^r}(s)$ and $s \geq 2$ and $s$ is prime to $p$, if $p > 0$. (These facts have already been noticed in [2], page 116.)

The situation when $p > 0$ and $p|s$ is rather interesting (in the sense that in some cases $Y$ may be a set-theoretic complete intersection). Precisely, one has the following result:

**Proposition 3.1 (Gattazzo [10])** Assume that $p > 0$ and $r \geq 2$, and let $s = p^m$ be a positive power of $p$. Then the image $Y$ of the Veronese embedding $i: \mathbb{P}^r \hookrightarrow \mathbb{P}^{n(r,s)}$ is a set-theoretic (but not a scheme-theoretic) complete intersection.

**Remarks 3.2** i) Assume that $p > 0$ and $r \geq 2$. Then by Proposition 3.1 the image $Y$ of the $p^m$-fold Veronese embedding $\mathbb{P}^r \hookrightarrow \mathbb{P}^{n(r,p^m)}$ is a set-theoretic complete intersection in $\mathbb{P}^{n(r,p^m)}$, with $n(r,p^m) = \binom{r+p^m}{p^m}$. On the other hand, in the hypotheses of Proposition 3.1, $\text{Coker}(\alpha) = \mathbb{Z}/p^m\mathbb{Z}$. This shows in particular that in Theorem 2.7 iv), $\text{Coker}(\alpha)$ may be a non-trivial finite $p$-group if $p > 0$ (compare with Corollary 2.8 i)), and also that Theorem 2.7 vi) is false in general in positive characteristic.

ii) From Proposition 3.1 and the above arguments we infer that $\text{Leff}(\mathbb{P}^{n(r,p^m)}, Y)$ does hold, where $Y$ is the image of the $p^m$-fold Veronese embedding $\mathbb{P}^r \hookrightarrow \mathbb{P}^{n(r,p^m)}$ over
an algebraically closed field $k$ of characteristic $p > 0$. This is in contrast with the case $p = 0$ when $\text{Eff}(\mathbb{P}^{m+n}, Y)$ never holds.

iii) Gattazzo proved in \cite{10} an even more general result than Proposition 3.1. Namely, he showed that also some projections of the $p^m$-fold Veronese embedding are set-theoretic complete intersections if $p > 0$. For instance, the projection $Y \subset \mathbb{P}^4$ of the Veronese surface in $\mathbb{P}^5$ from a general point of $\mathbb{P}^5$ is also a set-theoretic complete intersection in $\mathbb{P}^4$ if the characteristic of $k$ is 2.

**The Segre embedding.** Let $i: \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^{mn+m+n}$ be the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^n$, with $m, n \geq 1$ and $m + n \geq 3$. Assume that the ground field is $\mathbb{C}$. Then $Y := i(\mathbb{P}^m \times \mathbb{P}^n)$ is not a set-theoretic complete intersection in $\mathbb{P}^{mn+m+n}$. Indeed this follows from Theorem 2.7 vi), because $\text{Pic}(\mathbb{P}^{mn+m+n}) \cong \mathbb{Z}$ and $\text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \cong \mathbb{Z} \times \mathbb{Z}$ (see also \cite{16}). However, much more is known in this case. Namely, Bruns and Schwänzl proved in \cite{6} (see also \cite{7} and \cite{5}) that the arithmetic rank of the variety defined by the $(p \times q)$-matrix is $pq - t^2 + 1$. If we take $t = 2, p = m + 1$ and $q = n + 1$, we find that the arithmetic rank of $Y := i(\mathbb{P}^m \times \mathbb{P}^n)$ is $pq - t^2 + 1 = mn + m + n - 2$. Notice also that in the case when $m$ is arbitrary and $n = 1$ this result is also a consequence of Theorem 2 of the Introduction (cf. also Corollary 4.8 below).

**Examples of surfaces that are not set-theoretic complete intersections.**

i) **Surfaces with geometric genus zero.** Let $Y$ be any ruled nonrational surface (not necessarily minimal) over an algebraically closed field of arbitrary characteristic, i.e. $Y$ is birationally equivalent to $B \times \mathbb{P}^1$, with $B$ a nonsingular projective curve $B$ of genus $g > 0$. Consider an arbitrary projective embedding $Y \hookrightarrow \mathbb{P}^n$. We have $h^1(\mathcal{O}_Y) = g > 0$ and $H^2(\mathcal{O}_Y) = 0$. Therefore, by Theorem 2.7 i) and ii), $Y$ is not a set-theoretic complete intersection in $\mathbb{P}^n$. In particular, we obtain the following fact (proved in \cite{29} using ad hoc arguments: De Rham cohomology if $p = 0$ and the étale cohomology if $p > 0$): if $E \subset \mathbb{P}^2$ is an elliptic curve, then $Y := E \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^5$ (via the Segre embedding) is not a set-theoretic complete intersection in $\mathbb{P}^5$.

In fact, the same as above holds if, instead of taking a ruled nonrational surface, we take any nonsingular projective surface $X$ with geometric genus $p_g = h^2(\mathcal{O}_X) = 0$ and irregularity $q = h^1(\mathcal{O}_X) > 0$. For example a hyperelliptic surface $Y$; this is a surface with invariants $p_g = 0, q = 1, b_1 = b_2 = 2, \chi(\mathcal{O}_Y) = 0$ and Kodaira dimension $\kappa(Y) = 0$ (see e.g. \cite{3}). Such a surface $Y$ has the property that the Picard scheme is always reduced and the Albanese map $f: Y \rightarrow \text{Alb}(Y) = B$ has the following properties: $B$ is an elliptic curve, every fiber of $f$ is an elliptic curve, and there is a second elliptic fibration $Y \rightarrow \mathbb{P}^1$ (loc. cit.).

ii) **Enriques surfaces.** Let $Y$ be an Enriques surface embedded in $\mathbb{P}^n$ over $k$ and assume that $p \neq 2$. Then $Y$ is not a set-theoretic complete intersection in $\mathbb{P}^n$. Indeed, in this case $\text{Pic}(Y)$ contains a non-trivial element of order 2, namely the canonical class $\mathcal{O}_Y(K)$ (and in particular, is not algebraically simply connected because $\mathcal{O}_X(K)$ produces the cyclic non-trivial étale cover of $Y$ of degree 2). Then the conclusion follows from Theorem 2.7 i). Alternatively, $[\mathcal{O}_Y(K)] \not\in \text{Im}(\text{Pic}(\mathbb{P}^n) \rightarrow \text{Pic}(Y))$ (because $\text{Pic}(\mathbb{P}^n) = \mathbb{Z}$), whence $[\mathcal{O}_Y(K)]$ defines a non-trivial element of order 2 in $\text{Coker}(\text{Pic}(\mathbb{P}^n) \rightarrow \text{Pic}(Y))$. Since the characteristic of $k$ is $\neq 2$, the conclusion also follows from Corollary 2.3 i).
iii) **Ruled nonrational surfaces with rational singularities.** Let $X$ be a nonsingular ruled nonrational surface over an algebraically closed field $k$ of characteristic zero, and assume that $p \geq 0$ is arbitrary. Let $\pi: X \to B$ be the canonical ruled fibration, with $B$ a nonsingular projective curve of genus $g = h^1(\mathcal{O}_X) > 0$, and assume that there exists at least one degenerate fiber (i.e. reducible) fiber $\pi^{-1}(b)$. Fix $m \geq 1$ points $b_1, \ldots, b_m \in B$ such that the fiber $\pi^{-1}(b_i)$ is degenerate for every $i = 1, \ldots, m$. As is well known (see e.g. [4], Lemma 7), if for every $i = 1, \ldots, m$ we are given a closed connected curve $\mathcal{C} \neq Z_i \subseteq \pi^{-1}(b_i)$ is a closed connected curve of $\pi^{-1}(b_i)$, then there exists a birational morphism $f: X \to Y$, with $Y$ a normal projective surface such that:

- $f(Z_i)$ is a point of $y_i \in Y$, $i = 1, \ldots, m$, and the restriction

$$f' := f|Z_1 \cup \ldots \cup Z_m: X \setminus (Z_1 \cup \ldots \cup Z_m) \to Y \setminus \{y_1, \ldots, y_m\}$$

is a biregular isomorphism.

- The singularities $y_i \in Y$ are rational, $i = 1, \ldots, m$, i.e. $R^1f_*(\mathcal{O}_X) = 0$, and in particular, $h^1(\mathcal{O}_Y) = h^1(\mathcal{O}_X) > 0$ and $H^2(\mathcal{O}_Y) = H^2(\mathcal{O}_X) = 0$. Moreover, the point $y_i$ is effectively singular on $Y$ if $Z_i$ is not an exceptional curve of the first kind on $X$.

- The morphism $\pi: X \to B$ factors uniquely as $\pi = \pi' \circ f$, con $\pi': Y \to B$.

Now, let $Y \hookrightarrow \mathbb{P}^n$ be any projective embedding. Then by Theorem 2.7 ii) and iii) we deduce that $Y$ is not a set-theoretic intersection in $\mathbb{P}^n$.

**iv) Nonruled normal surfaces.** Let $B$ be a smooth projective curve of genus $g \geq 1$ over $k$. Fix $2g$ distinct points $x, y_1, \ldots, y_{2g-1} \in B$. Let $f: X \to B \times B$ be the blowing up morphism of $B \times B$ of centers the $2g - 1$ points $(x, y_1), \ldots, (x, y_{2g-1}) \in B \times B$, and let $C$ be the strict transform of the curve $\{x\} \times B$ via $f$. Clearly, $u := f|C$ yields an isomorphism $C \cong B$.

We shall show that there exists a birational morphism $g: X \to Y$, with $Y$ a normal projective surface, which blows down the curve $C$ to a point of $Y$. In order to do that, we firstly observe that $C^2 = 1 - 2g < 0$ (by the construction of the curve $C$).

On the other hand, in the commutative diagram

$$
\begin{array}{ccc}
H^1(\mathcal{O}_{B \times B}) & \longrightarrow & H^1(\mathcal{O}_{\{x\} \times B}) \\
\downarrow f^* & & \downarrow u^* \\
H^1(\mathcal{O}_X) & \longrightarrow & H^1(\mathcal{O}_C)
\end{array}
$$

the vertical maps are isomorphisms (since $f$ is the blowing up morphism of $B \times B$ of center finitely many nonsingular points), and the top horizontal arrow is surjective because the inclusion $\{x\} \times B \hookrightarrow B \times B$ is a section of the second projection of $B \times B$. It follows that the bottom horizontal map is also surjective. Let $\mathcal{O}_X(-C)$ be the ideal sheaf of $C$ in $\mathcal{O}_X$.

We claim that the restriction maps $H^1(\mathcal{O}_{(i+1)C}) \to H^1(\mathcal{O}_{iC})$ are isomorphisms for every $i \geq 1$, where $iC$ is the $i$-th infinitesimal neighbourhood of $C$ in $X$. This follows from the cohomology exact sequence

$$H^1(\mathcal{O}_X(-iC)/\mathcal{O}_X(-(i+1)C)) \to H^1(\mathcal{O}_{(i+1)C}) \to H^1(\mathcal{O}_{iC}) \to H^2(\mathcal{O}_X(-iC)/\mathcal{O}_X(-(i+1)C)),$$

if we show that the first and the last vector space are zero. The last vector space is clearly zero because $C$ is a curve. The first vector space is zero because $C^2 = 1 - 2g$ implies that $\deg(\mathcal{O}_X(-iC)/\mathcal{O}_X(-(i+1)C)) = i(2g - 1) \geq 2g - 1$ if $i \geq 1$. Recalling that the
bottom horizontal arrow in the above diagram is surjective, by induction we infer that the restriction maps

\[ H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_C) \]

are surjective for every \( i \geq 1 \).

Moreover, we claim that the Picard scheme \( \text{Pic}_X^0 \) is always reduced. In characteristic zero this holds by a very general theorem of Cartier (see [11], or also [20], Lecture 25). If instead \( p > 0 \), we have \( h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{B \times B}) = 2g \) (by Künneth), and since \( H^1(\mathcal{O}_X) \) is canonically identified with the tangent space to \( \text{Pic}_X^0 \) at the origin, we get \( \dim(\text{Pic}_X^0) \leq 2g \). Moreover, this inequality is strict if and only if \( \text{Pic}_X^0 \) is nonreduced. On the other hand, as is well known, the dual abelian variety of the abelian variety \( (\text{Pic}_X^0)_{\text{red}} \) is the Albanese variety \( \text{Alb}(X) = \text{Alb}(B \times B) \), which is isomorphic to \( \text{Alb}(B) \times \text{Alb}(B) \). Hence \( \dim(\text{Pic}_X^0)_{\text{red}} = 2g \) because \( \text{Alb}(B) \) is the Jacobian of \( B \) and its dimension is \( g \). Putting things together it follows that \( \dim(\text{Pic}_X^0) = 2g \) and \( \text{Pic}_X^0 \) is reduced.

Now, the surjectivity of \( \mathbb{I} \), the inequality \( C^2 < 0 \) and the fact that the Picard scheme \( \text{Pic}_X^0 \) is reduced allow us to apply Theorem 14.23 of [3] to deduce that (in arbitrary characteristic) there exists a birational morphism \( g : X \to Y \), with \( Y \) a normal projective surface such that:

- The image \( g(C) \) is a point of \( y \in Y \), and
- The restriction \( X \setminus C \to Y \setminus \{ y \} \) of \( g \) is a biregular isomorphism.

Notice that the projectivity of \( Y \) in the conclusion of Theorem 14.23 of [3] is the main point. The surface \( Y \) is going to be our example. We only need to show that \( H^1(\mathcal{O}_Y) \neq 0 \).

To see this, consider the canonical exact sequence in low degrees

\[ 0 \to H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X) \to H^0(Y, R^1f_*\mathcal{O}_X) = R^1g_*\mathcal{O}_X)_y \to 0 \]  (2)

associated to the spectral sequence

\[ E_2^{p,q} = H^p(Y, R^qg_*\mathcal{O}_X) \Rightarrow H^{p+q}(X, \mathcal{O}_X). \]

By Grothendieck-Zariski’s theorem on formal functions together with the fact (proved above) that \( H^1(\mathcal{O}_{iC}) \cong H^1(\mathcal{O}_C) \) for every \( i \geq 1 \), we have

\[ R^1g_*\mathcal{O}_X)_y = \lim_{i \in \mathbb{N}} H^1(\mathcal{O}_{iC}) \cong H^1(\mathcal{O}_C), \]

whence the exact sequence (2) becomes

\[ 0 \to H^1(\mathcal{O}_Y) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_C) \to 0. \]

Since \( h^1(\mathcal{O}_C) = h^1(\mathcal{O}_B) = g \) and \( h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{B \times B}) = 2g \), we get \( h^1(\mathcal{O}_Y) = g > 0 \).

Finally, let \( Y \hookrightarrow \mathbb{P}^n \) be an arbitrary projective embedding of \( Y \). Then by Theorem [27] ii) and iii) we deduce that \( Y \) is not a set-theoretic intersection in \( \mathbb{P}^n \). Notice that in this example the surface \( Y \) is birationally equivalent to an abelian surface if \( g = 1 \), and to a surface of general type if \( g \geq 2 \).

v) Nonnormal surfaces. Let \( C \) be an irreducible curve over \( k \), which is obtained from its normalization \( \tilde{C} \) by identifying \( n+1 \) distinct points \( P_0, \ldots, P_n \), with \( n \geq 1 \) (in the terminology of Serre [23], chap. IV, \( C \) is defined by the module \( \sum_{i=0}^n P_i \); in the classical terminology, the singularity of \( C \) is an ordinary \((n+1)\)-fold point with \((n+1)\) distinct
tangents). For instance, if \( n = 1 \) then \( C \) has just one singularity, which is an ordinary double point with distinct tangents. Then by Oort [24], Proposition (2.3), there is an exact sequence
\[
0 \to \mathcal{G}_m^{\oplus n} \to \text{Pic}_C^{0} \to \text{Pic}_C^{0} = 0
\]
of algebraic groups, where \( \mathcal{G}_m = k \setminus \{0\} \) is the multiplicative group of \( k \). Since \( \text{Tors}(\mathcal{G}_m) \neq 0 \), it follows that \( \text{Tors}(\text{Pic}(C)) \neq 0 \).

Now, let \( E \) be a vector bundle of rank \( r \geq 2 \) on \( C \) and consider the projective bundle \( Y := \mathbb{P}(E) \) associated to \( E \). Since \( \text{Pic}(Y) \cong \text{Pic}(C) \oplus \mathbb{Z} \), it follows that \( \text{Tors}(\text{Pic}(Y)) = \text{Tors}(\text{Pic}(C)) \). Let \( Y \hookrightarrow \mathbb{P}^N \) be any projective embedding of \( Y \). Then by Corollary (2.3 i), \( Y \) cannot be a set-theoretic complete intersection in \( \mathbb{P}^N \). This example has some interest because if we assume that the curve \( \tilde{C} \) is rational, then \( Y \) is a singular (nonnormal) rational projective variety of dimension \( r \geq 2 \).

4 The arithmetic rank of rational normal scrolls

Let \( E \) be an ample vector bundle of rank \( d \geq 2 \) over the projective line \( \mathbb{P}^1 \). By a well known theorem of Grothendieck, \( E \) can be written as a direct sum of line bundles
\[
E = \mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_d),
\]
and since \( E \) is ample, \( n_i > 0 \) for every \( i = 1, \ldots, d \). Let
\[
\mathbb{P}(E) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_d))
\]
be the projective bundle associated to \( E \). Since \( n_i > 0 \) for every \( i = 1, \ldots, d \), the tautological line bundle \( \mathcal{O}_{\mathbb{P}(E)}(1) \) is ample, and in fact, very ample. Consider the closed embedding \( i: \mathbb{P}(E) \hookrightarrow \mathbb{P}^N \) associated to the very ample complete linear system \( |\mathcal{O}_{\mathbb{P}(E)}(1)| \), with \( N := \sum_{i=1}^d n_i + d - 1 \). Then \( S_{n_1,\ldots,n_d} := i(\mathbb{P}(E)) = i(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_d)) \) is a nonsingular \( d \)-dimensional subvariety of \( \mathbb{P}^N \), which is known to be arithmetically Cohen-Macaulay in \( \mathbb{P}^N \); moreover, \( \text{Pic}(S_{n_1,\ldots,n_d}) \cong \text{Pic}(\mathbb{P}(E)) \cong \mathbb{Z} \oplus \mathbb{Z} \) (generated by the classes of \( \mathcal{O}_{\mathbb{P}(E)}(1) \) and \( \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \)), where \( \pi: \mathbb{P}(E) \to \mathbb{P}^1 \) is the canonical projection of \( \mathbb{P}(E) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_d)) \). The subvariety \( S_{n_1,\ldots,n_d} \) of \( \mathbb{P}^N \) is called the \( d \)-dimensional rational normal scroll.

The aim of this section is to prove the following result (see Theorem 2 of the Introduction):

**Theorem 4.1** Under the above notation and assumptions, the arithmetic rank of \( S_{n_1,\ldots,n_d} \) in \( \mathbb{P}^N \) is \( N - 2 = \sum_{i=1}^d n_i + d - 3 \).

**Corollary 4.2** Under the notation of Theorem 4.1, \( S_{n_1,\ldots,n_d} \) is a set-theoretic complete intersection in \( \mathbb{P}^N \) if and only if \( S_{n_1,\ldots,n_d} \) is a surface (i.e. \( d = 2 \)). In particular, the two dimensional rational normal scroll \( S_{n_1,n_2} \) is set-theoretic complete intersection in \( \mathbb{P}^{n_1+n_2+1} \), but not a scheme-theoretic complete intersection, unless \( n_1 = n_2 = 1 \).

**Proof.** The first part is a direct consequence of Theorem 4.1. For the last part we notice that the canonical class of \( S_{n_1,n_2} \) is given by
\[
\omega_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1)\oplus\mathcal{O}_{\mathbb{P}^1}(n_2))} = \pi^*\mathcal{O}_{\mathbb{P}^1}(n_1 + n_2 - 2) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1)\oplus\mathcal{O}_{\mathbb{P}^1}(n_2))}(-2),
\]
whence \( S_{n_1,n_2} \) is subcanonical in \( \mathbb{P}^{n_1+n_2+1} \) if and only if \( n_1 = n_2 = 1 \). \( \square \)
Remark 4.3 The fact that the rational normal scrolls \( S_{n_1,n_2} \) are set-theoretic complete intersections in \( \mathbb{P}^{n_1+n_2+1} \) was already known, see Valla [31] and Robbiano-Valla [20] in some special cases, and subsequently, Verdi [34] in general. In particular, our approach also reproduces (in a completely different way) the result of Verdi [34] for the two-dimensional rational normal scrolls. Moreover our method produces \( n_1+n_2-1 \) homogeneous equations defining \( S_{n_1,n_2} \) as set-theoretic complete intersection in \( \mathbb{P}^{n_1+n_2+1} \) which are in general of lower degrees with respect to the equations obtained in Verdi [34]. For example, if \( n_1 = n_2 = 2 \), we prove that \( S_{2,2} \) is the set-theoretic complete intersection of three hyperquadrics in \( \mathbb{P}^5 \), while Verdi needs two hyperquadrics and one hyperquartic.

The proof of Theorem 4.1 requires some preparation.

We first recall that the rational normal curve \( C_n \) of degree \( n \) in \( \mathbb{P}^n \), \( n \geq 1 \), is defined as the image of the Veronese map \( \nu_n : \mathbb{P}^1 \to \mathbb{P}^n \) sending \([\alpha, \beta]\) to \([\alpha^n, \alpha^{n-1}\beta, \ldots, \alpha\beta^{n-1}, \beta^n]\). It is well known that \( C_n \) may be realized as the locus of points which give rank one to the image of the Veronese map

\[
\begin{pmatrix}
X_0 & X_1 & \cdots & X_{n-1} \\
X_1 & X_2 & \cdots & X_n
\end{pmatrix}
\]

Further, in [26] Valla and Robbiano, by using Gr"obner bases theory, showed that \( C_n \) is the set-theoretic complete intersection of the \( n-1 \) hypersurfaces defined by the following polynomials

\[
E_i = E_i(X_0, \ldots, X_n) = \sum_{\alpha=0}^{i} (-1)^{\alpha} \binom{i}{\alpha} X_{i+1}^{-\alpha} X_{\alpha} X_i^\alpha, \quad i = 1, \ldots, n-1.
\]

Notice that it was Verdi who proved, see [33], for the first time that \( C_n \) is a set-theoretic complete intersection in \( \mathbb{P}^n \). However, her equations and methods are different from those used by Robbiano and Valla, who found slightly simpler equations.

For every integers \( d \geq 2 \) and \( n_1, n_2, \ldots, n_d > 0 \) as above, the \( d \)-dimensional rational normal scroll \( S_{n_1,\ldots,n_d} \) can also be described as the rank one determinantal variety associated to the matrix

\[
A = \begin{pmatrix}
X_{1,0} & X_{1,1} & \cdots & X_{1,n_1-1} & \vdots & \vdots & X_{d,0} & X_{d,1} & \cdots & X_{d,n_d-1} \\
X_{1,1} & X_{1,2} & \cdots & X_{1,n_1} & \vdots & \vdots & X_{d,1} & X_{d,2} & \cdots & X_{d,n_d}
\end{pmatrix}
\]

i.e. a matrix consisting of \( d \) blocks of sizes \( 2 \times n_1, \ldots, 2 \times n_d \) respectively, with each block a generic catalecticant matrix (see [13], pp. 105–109). These blocks correspond to the canonical decomposition

\[
H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) = \bigoplus_{i=1}^{d} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n_i)).
\]

Notice that a basis of the \( k \)-vector space \( H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n_i)) \) is \( k[T_{i,0}, T_{i,1}]_{n_i} \) is \( T_{i,0}^{n_1-1}T_{i,1} \), \( \ldots, T_{i,0}T_{i,1}^{n_1-1}, T_{i,1}^{n_i} \). Here \( \mathbb{P}^1 = \text{Proj}(k[T_{i,0}, T_{i,1}]) \), with \( T_{i,0} \) and \( T_{i,1} \) two independent variables over \( k \), \( i = 1, \ldots, d \), and \( k[T_{i,0}, T_{i,1}]_{n_i} \) is the \( k \)-vector space of all homogeneous polynomials in \( T_{i,0} \) and \( T_{i,1} \) of degree \( n_i \).

The homogeneous ideal \( \varphi := \mathcal{I}(S_{n_1,\ldots,n_d}) \) of \( S_{n_1,\ldots,n_d} \) in \( \mathbb{P}^N \) (generated by all homogeneous polynomials vanishing on \( S_{n_1,\ldots,n_d} \)) is thus the ideal generated by the \( 2 \times 2 \) minors of the matrix \( A \) in the polynomial ring \( k[X_{1,0}, \ldots, X_{1,n_1}, \ldots, X_{d,0}, \ldots, X_{d,n_d}] \).
We want to exhibit $N - 2 = \sum_{i=1}^{d} n_i + d - 3$ homogeneous equations defining $S_{n_1, \ldots, n_d}$ in $\mathbb{P}^N$ set-theoretically. In order to do it, the first step is to introduce a class of polynomials, which we call bridges and which will be crucial in order to detect the equations defining the rational normal scrolls. The bridges are defined in the following way.

Let $a$ and $b$ be positive integers and let $m$ be the least common multiple of $a$ and $b$. We can write $m = ap = bq$ and for every $\alpha = 0, \ldots, m$ we can divide $\alpha$ by $p$ and by $q$, thus getting

$$\alpha = cp + r = eq + f,$$

where $0 \leq r \leq p - 1$ and $0 \leq f \leq q - 1$.

In the polynomial ring $k[X_0, \ldots, X_a, Y_0, \ldots, Y_b]$ we consider the polynomial

$$B_{a,b}(X_0, \ldots, X_a, Y_0, \ldots, Y_b) := \sum_{\alpha=0}^{m} (-1)^{\alpha} \binom{m}{\alpha} X_{a-c}^{p-r} X_{a-c-1}^{r} Y_{e-f}^{q-f} Y_{e+1}^{f}.$$  (4)

We notice that if $\alpha = m$ then $c = a$, $r = 0$, $e = b$ and $f = 0$; in this case we let $X_{-1} = 1$ and $Y_{b+1} = 1$. The polynomial

$$B_{a,b}(X,Y) := B_{a,b}(X_0, \ldots, X_a, Y_0, \ldots, Y_b)$$

is called the bridge between $k[X_0, \ldots, X_a]$ and $k[Y_0, \ldots, Y_b]$ and it is homogeneous of degree $m/a + m/b = p + q$. When it is clear from the context, we shall simply write $B_{a,b}$ instead of $B_{a,b}(X,Y)$. The bridge $B_{a,b}$ has the following two relevant properties.

• **Property 1.** For every $u, s, t, v \in k$ we have

$$B_{a,b}(us^a, us^{a-1}t, \ldots, us^{a-1}t, us^b, vs^{b-1}t, \ldots, vs^{b-1}t, vs^b, vs^{b-1}t, \ldots, vs^{b-1}t, v^b) = 0.$$ 

Indeed, what we have to do is to replace in $B_{a,b}$ every $X_j$ by $us^{a-j}t^j$ and every $Y_h$ by $vs^{b-h}w^h$. We get:

$$\sum_{\alpha=0}^{m} (-1)^{\alpha} \binom{m}{\alpha} (us^{c(p-r)} + r(c+1)) + (q-f)(b-e) + f(b-e-1)(q-f) + f(e+1) =$$

$$\sum_{\alpha=0}^{m} (-1)^{\alpha} \binom{m}{\alpha} u^{p} s^{c(p-r)} + r(c+1) + (q-f)(b-e) + f(b-e-1)(q-f) + f(e+1) =$$

$$u^{p} s^{c(p-r)} + r(c+1) + (q-f)(b-e) + f(b-e-1)(q-f) + f(e+1) =$$

• **Property 2.** For every $s, t, z, w \in k$ we have

$$B_{a,b}(s^a, s^{a-1}t, \ldots, s^{a-1}t, z^b, z^{b-1}w, \ldots, z^{b-1}w, tw) = (tz - sw)^m.$$ 

This time we have to replace in $B_{a,b}$ every $X_j$ by $s^{a-j}t^j$ and every $Y_h$ by $z^{b-h}w^h$. We get:

$$\sum_{\alpha=0}^{m} (-1)^{\alpha} \binom{m}{\alpha} (s^{c(p-r)} + r(c+1)) + (q-f)(b-e) + f(b-e-1)(q-f) + f(e+1) =$$

$$\sum_{\alpha=0}^{m} (-1)^{\alpha} \binom{m}{\alpha} s^{c(p-r)} + r(c+1) + (q-f)(b-e) + f(b-e-1)(q-f) + f(e+1) =$$

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We notice that
\[{\text{us}^a : \text{us}^{a-1}t : \cdots : \text{ust}^{a-1} : \text{ut}^a : \text{vs}^b : \text{vs}^{b-1}t : \cdots : \text{vst}^{b-1} : \text{vt}^b}\}

are the parametric equations of the rational normal scroll \(S_{a,b}\) defined by the vanishing of the \((2 \times 2)\)-minors of the matrix

\[M_{a,b} = \begin{pmatrix} X_0 & X_1 & \cdots & X_{a-1} & Y_0 & Y_1 & \cdots & Y_{b-1} \\ X_1 & X_2 & \cdots & X_a & Y_1 & Y_2 & \cdots & Y_b \end{pmatrix}.
\]

Hence Property 1 implies that \(B_{a,b}\) is in the ideal generated by the \((2 \times 2)\)-minors of \(M_{a,b}\). Namely, \(S_{a,b} = \overline{W}\) where \(W\) is the set of points with coordinates (see e.g. [5])

\[W = \text{us}^a : \text{us}^{a-1}t : \cdots : \text{ust}^{a-1} : \text{ut}^a : \text{vs}^b : \text{vs}^{b-1}t : \cdots : \text{vst}^{b-1} : \text{vt}^b\].

By Property 1 we get that \(W\), and hence \(S_{a,b} = \overline{W}\) is contained in the zero-locus of \(B_{a,b}\). This implies that \(B_{a,b}\) is contained in the defining ideal of \(S_{a,b}\), which is the ideal generated by the \(2 \times 2\) minors of \(M_{a,b}\).

Notice that, given \(a\) and \(b\), while computing \(B_{a,b}\) we can avoid all the nasty euclidean divisions which appear in the definition itself. Better, one can do as follows. Let us consider the following list of monomials of degree \(p\) in the \(X_i\)'s:

\[\{X_p^a, X_{a-1}^a, \ldots, X_a^a, X_{a-1}^{p-1}, X_a^{p-1}, X_{a-2}^{p-1}, \ldots, X_{a-1}^{a-2}, \ldots, X_1^{a-1}, X_0^{p-1}, X_0^p\}.\]

In the same way one can write down the following list of monomials of degree \(q\) in the \(Y_i\)'s:

\[\{Y_q^q, Y_{q-1}^q, \ldots, Y_0, Y_{q-1}^q, \ldots, Y_1, Y_2^q, \ldots, Y_{b-1}, Y_{b-1}^{q-1}, Y_q^q\}.\]

The first list has \(ap + 1 = m + 1\) terms and the second, \(bq + 1 = m + 1\) terms. The bridge \(B_{a,b}\) is the sum of the products of the corresponding monomials in the two lists with appropriate binomial coefficients.

**Examples 4.4**

1. If \(a = 2, b = 4\), then \(m = 4, p = 2, q = 1\). The two lists are the following \(\{X_2^2, X_2X_1, X_1^2, X_1X_0, X_0^2\}\) and \(\{Y_0, Y_1, Y_2, Y_3, Y_4\}\).

Hence \(B_{2,4}(X, Y) = X_2^2Y_0 - (\binom{1}{1})X_2X_1Y_1 + (\binom{2}{2})X_1^2Y_2 - (\binom{3}{3})X_1X_0Y_3 + X_0^2Y_4\).

2. If \(a = 2, b = 3\), then \(m = 6, p = q = 2\). The two lists are the following \(\{X_2^2, X_2X_1, X_2X_0, X_1^2, X_1X_0, X_0^2\}\) and \(\{Y_0^2, Y_0Y_1, Y_1^2, Y_2, Y_2^2, Y_3, Y_3^2\}\). Hence \(B_{2,3}(X, Y) = X_2^2Y_0^2 - (\binom{1}{0})X_2X_1Y_0Y_1 + (\binom{2}{2})X_2X_1^2Y_1^2 - (\binom{3}{3})X_1X_0Y_3 + (\binom{4}{4})X_0^2Y_3^2 - (\binom{5}{5})X_1X_0^2Y_5 + X_0^3Y_5^2\).

3. If \(a = b\), then \(m = a, p = q = 1\). The two lists are \(\{X_a, X_{a-1}, \ldots, X_2, X_1, X_0\}\) and \(\{Y_0, Y_1, Y_2, \ldots, Y_{a-2}, Y_{a-1}, Y_a\}\). Hence \(B_{a,a}(X, Y) = \sum_{j=0}^{a}(-1)^j(\binom{a}{j})X_{a-j}Y_j\).

4. If \(a = 3, b = 4\), then \(m = 12, p = 4, q = 3\). Then we get

\[B_{3,4}(X, Y) = X_3^4Y_0^3 - 12X_3^2Y_0^2Y_1 + (\binom{12}{2})X_3^2X_2Y_0Y_1^2 - (\binom{12}{3})X_3X_2^2Y_0Y_1^3 + (\binom{12}{4})X_3Y_0^2Y_2 - (\binom{12}{3})X_3X_2Y_0Y_2^2 + (\binom{12}{6})X_3^2X_2^2Y_0Y_2^3 - (\binom{12}{4})X_3^2X_2^2Y_0Y_3 + (\binom{12}{4})X_2^2Y_0^2Y_3 + (\binom{12}{9})X_3X_2Y_0^3 + (\binom{12}{10})X_3^2X_2Y_0^3 - 12X_3^2Y_3^2Y_4 + 12X_3X_2Y_3^2Y_4 + X_3^3Y_4^3.\]
We are now ready to prove the main result of this section.

Proof of Theorem 4.1 We divide the proof in two steps.

Step 1. The following inequality holds:

\[
\text{ara}(S_{n_1,\ldots,n_d}) \leq N - 2 = \sum_{i=1}^{d} n_i + d - 3.
\]

To prove Step 1 it is enough to find \( \sum_{i=1}^{d} n_i + d - 3 \) homogeneous polynomials defining \( S_{n_1,\ldots,n_d} \) in \( \mathbb{P}^N \) set-theoretically.

Let us consider the polynomials \( \{F_{1,1},\ldots,F_{d,n_1}\} \) in \( k[X_{i,0},\ldots,X_{i,n_1}] \) whose corresponding equations define set-theoretically the rational normal curve \( C_{n_1} \) in \( \mathbb{P}^{n_1} \), see [1]. Similarly we consider the polynomials \( \{F_{2,1},\ldots,F_{d,n_2}\} \) and so on up to \( \{F_{d,1},\ldots,F_{d,n_d}\} \). This is a collection of \( \sum_{i=1}^{d} n_i - d \) polynomials in \( k[X_{i,0},\ldots,X_{i,n_1},\ldots,X_{d,0},\ldots,X_{d,n_d}] \) belonging to the homogeneous ideal \( \mathcal{I} := \mathcal{I}_+(S_{n_1,\ldots,n_d}) \).

We are going to find some \( 2d - 3 \) more equations. This will be achieved by considering the bridges \( B_{n_i,n_j} \) between \( k[X_{i,0},\ldots,X_{i,n_i}] \) and \( k[X_{j,0},\ldots,X_{j,n_j}] \) for every \( 1 \leq i < j \leq d \). If \( m_{i,j} = n_i p_{i,j} = n_j q_{i,j} \) is the least common multiple of \( n_i \) and \( n_j \), then \( B_{n_i,n_j} \) is homogeneous of degree \( p_{i,j} + q_{i,j} \).

By Property 1 of the bridges we have that for every \( 1 \leq i < j \leq d \) the polynomial \( B_{n_i,n_j} \) belongs to the ideal of the polynomial ring \( k[X_{i,0},X_{i,1},\ldots,X_{i,n_i},X_{j,0},X_{j,1},\ldots,X_{j,n_j}] \) generated by the \((2 \times 2)\)-minors of the matrix

\[
\begin{pmatrix}
X_{i,0} & X_{i,1} & \cdots & X_{i,n_i-1} & X_{j,0} & X_{j,1} & \cdots & X_{j,n_j-1} \\
X_{i,1} & X_{i,2} & \cdots & X_{i,n_i} & X_{j,1} & X_{j,2} & \cdots & X_{j,n_j}
\end{pmatrix}.
\]

In particular, \( B_{n_i,n_j} \in \mathcal{I} = \mathcal{I}_+(S_{n_1,\ldots,n_d}) \).

We associate a weight to the bridges by letting weight \( (B_{n_i,n_j}) := i + j \). Hence we have that \( B_{n_1,n_2} \) has weight 3, \( B_{n_1,n_3} \) has weight 4, \( B_{n_1,n_4} \) and \( B_{n_2,n_3} \) have weight 5, \( B_{n_1,n_5} \) and \( B_{n_2,n_4} \) have weight 6, \( B_{n_1,n_6}, B_{n_2,n_5} \) and \( B_{n_3,n_4} \) have weight 7 and so on. Notice that the possible weight for a bridge is an integer \( w \) such that \( 3 \leq w \leq 2d - 1 \). Now for every \( k = 3,\ldots,2d - 1 \), let \( r_k \) be the least common multiple of the numbers \( p_{i,j} + q_{i,j} \) when \( i + j = k \), i.e.

\[
r_k := \text{lcm}\{p_{i,j} + q_{i,j} \mid i + j = k\}.
\]

Further for every \( i \) and \( j \) such that \( i + j = k \) we let

\[
c_{i,j} := \frac{r_k}{p_{i,j} + q_{i,j}}.
\]

Finally for every \( k = 3,\ldots,2d - 1 \), we let

\[
G_k := \sum_{i+j=k} B_{n_i,n_j}^{c_{i,j}}.
\]

It is clear that \( G_k \) is an homogeneous polynomial of degree \( r_k \) for every \( k = 3,\ldots,2d - 1 \). The polynomials \( G_k \) are in \( \mathcal{I} \) because we have already seen that the bridges are in \( \mathcal{I} \).
For example we have $r_3 = p_{1,2} + q_{1,2}$ so that $c_{1,2} = 1$ and $G_3 = B_{n_1,n_2}$. Also $r_4 = p_{1,3} + q_{1,3}$ so that $c_{1,3} = 1$ and $G_4 = B_{n_1,n_3}$. Instead we have $r_5 = \text{lcm}(p_{1,4}+q_{1,4}, p_{2,3}+q_{2,3})$, so that
\[
    c_{1,4} = \frac{r_5}{p_{1,4}+q_{1,4}}, \quad c_{2,3} = \frac{r_5}{p_{2,3}+q_{2,3}}, \quad \text{and} \quad G_5 = B_{n_1,n_4} + B_{n_2,n_3}.
\]

Set
\[
    J = (F_{1,1}, \ldots, F_{n_1-1,1}, \ldots, F_{d,1}, \ldots, F_{d,n_d-1}, G_3, \ldots, G_{2d-1}).
\]

We are going to prove that the equations corresponding to these $\sum n_i - d + 2d - 3 = \sum n_i + d - 3$ homogeneous polynomial define set-theoretically the scroll $S_{n_1, \ldots, n_d}$.

In other words, it’s enough to prove the following
\[
    \wp = \sqrt{J}.
\] (5)

Clearly, $J \subseteq \wp$, so that $\sqrt{J} \subseteq \wp$. On the other hand, by Nullstellensatz, the reverse inclusion is equivalent with $\mathcal{V}_+(J) \subseteq \mathcal{V}_+(\wp)$. To prove this latter inclusion, let $P$ be an arbitrary point of $\mathcal{V}_+(J)$. We have to show that $P \in \mathcal{V}_+(\wp)$. Since $P \in \mathcal{V}_+(F_{1,1}, \ldots, F_{n_1-1,1}, \ldots, F_{d,1}, \ldots, F_{d,n_d-1})$, the coordinates of $P$ are of the following form
\[
    [t_1^{n_1}, t_1^{n_1-1}u_1, \ldots, t_1u_1^{n_1-1}, u_1^{n_1}; t_2^{n_2}, t_2^{n_2-1}u_2, \ldots, t_2u_2^{n_2-1}, u_2^{n_2}],
\]
or, in a compact way,
\[
    \{X_{i,j} = t_i^{n_i-j}u_i^j\}_{i=1,\ldots,n_i, j=0,\ldots,n_i}.
\]

Let us consider the matrix
\[
    D := \begin{pmatrix} t_1 & t_2 & \ldots & t_d \\ u_1 & u_2 & \ldots & u_d \end{pmatrix}
\]
and for every $1 \leq i < j \leq d$, let $\alpha_{i,j}$ be the $2 \times 2$ minor involving its i-th and j-th column.

We have $0 = G_3(P) = B_{n_1,n_2}(P)$, hence, by Property 2 of the bridges,
\[
    0 = B_{n_1,n_2}(t_1^{n_1}, t_1^{n_1-1}u_1, \ldots, t_1u_1^{n_1-1}, u_1^{n_1}; t_2^{n_2}, t_2^{n_2-1}u_2, \ldots, t_2u_2^{n_2-1}, u_2^{n_2}) = (u_1t_2-t_1u_2)^{m_1,2}.
\]

This implies $\alpha_{1,2} = 0$.

In the same way we have $0 = G_4(P) = B_{n_1,n_3}(P)$, hence
\[
    0 = B_{n_1,n_3}(t_1^{n_1}, t_1^{n_1-1}u_1, \ldots, t_1u_1^{n_1-1}, u_1^{n_1}; t_3^{n_3}, t_3^{n_3-1}u_3, \ldots, t_3u_3^{n_3-1}, u_3^{n_3}) = (u_1t_3-t_1u_3)^{m_1,3}.
\]

This implies $\alpha_{1,3} = 0$.

Further $0 = G_5(P) = (B_{n_1,n_4} + B_{n_2,n_3})(P)$, hence
\[
    0 = (B_{n_1,n_4} + B_{n_2,n_3})(P) = (B_{n_1,n_4}(P))^{c_{1,4}} + (B_{n_2,n_3}(P))^{c_{2,3}} =
\]
\[
= (B_{n_1,n_4}(t_1^{n_1}, t_1^{n_1-1}u_1, \ldots, t_1u_1^{n_1-1}, u_1^{n_1}; t_4^{n_4}, t_4^{n_4-1}u_4, \ldots, t_4u_4^{n_4-1}, u_4^{n_4}))^{c_{1,4}} +
\]
\[
+ (B_{n_2,n_3}(t_2^{n_2}, t_2^{n_2-1}u_2, \ldots, t_2u_2^{n_2-1}, u_2^{n_2}; t_3^{n_3}, t_3^{n_3-1}u_3, \ldots, t_3u_3^{n_3-1}, u_3^{n_3}))^{c_{2,3}} =
\]
\[
= (u_1t_4-t_1u_4)^{m_{1,4}c_{1,4}} + (u_2t_3-t_2u_3)^{m_{2,3}c_{2,3}}.
\]

This implies
\[
\alpha_{1,4}^{m_{1,4}c_{1,4}} + \alpha_{2,3}^{m_{2,3}c_{2,3}} = 0.
\]
In the same way, for every \( k = 3, \ldots, 2d - 1 \), we get
\[
0 = \sum_{i+j=k} \alpha_{i,j}^{m_{i,j}c_{i,j}} = \sum_{i+j=k} \epsilon_{i,j},
\]
where, for simplicity, we put \( \epsilon_{i,j} := m_{i,j}c_{i,j} \).

We claim that this implies \( \alpha_{i,j} = 0 \) for every \( 1 \leq i < j \leq d \).

To prove this claim we order the \( \alpha_{i,j} \)'s as follows:
\[
\alpha_{i,j} < \alpha_{h,k} \iff \begin{cases} 
  i + j < h + k, \\
  i + j = h + k \text{ and } i < h.
\end{cases}
\]
First observe that \( \alpha_{1,2} = \alpha_{1,3} = 0 \), so that we can argue by induction. Let us assume that \( \alpha_{1,3} < \alpha_{a,b} \) and that \( \alpha_{h,k} = 0 \) for every \( \alpha_{h,k} < \alpha_{a,b} \). One has
\[
\epsilon_{a,b}^{e_{a,b}+1} = \alpha_{a,b} \left( \sum_{i+j=a+b} \alpha_{i,j}^{e_{i,j}} \right) - \alpha_{a,b} \left( \sum_{i+j=a+b} \alpha_{i,j}^{e_{i,j}} \right).
\]
We only need to prove that if \( (i, j) \neq (a, b) \) and \( i + j = a + b \), then \( \alpha_{i,j} \alpha_{a,b} = 0 \). If \( i < a \), then \( \alpha_{i,j} < \alpha_{a,b} \) so that, by the inductive assumption \( \alpha_{i,j} = 0 \) and we are done. If, instead, \( i > a \), then \( j < b \) so that
\[
a < i < j < b.
\]

By Plücker’s relations, we have
\[
\alpha_{i,j} \alpha_{a,b} - \alpha_{a,j} \alpha_{i,b} + \alpha_{a,i} \alpha_{j,b} = 0.
\]

Since \( \alpha_{a,i} < \alpha_{a,b} \) and \( \alpha_{a,j} < \alpha_{a,b} \), we have \( \alpha_{a,i} = \alpha_{a,j} = 0 \) and the claim is proved.

As a consequence we get that the matrix \( D \) has rank one. But this clearly implies that the matrix
\[
\begin{pmatrix}
  t_1^n & t_1^{n-1}u_1 & \ldots & t_1u_1^{n-1} & \ldots & t_d^n & t_d^{n-1}u_d & \ldots & t_du_d^{n-1} \\
  t_1^{n-1}u_1 & t_1^{n-2}u_1^2 & \ldots & u_1^{n-1} & \ldots & t_d^{n-1}u_d & t_d^{n-2}u_d^2 & \ldots & u_d^{n-1}
\end{pmatrix}
\]

has also rank one. This means that the point \( P \) is in \( V_+(\varphi) \), which proves Step 1.

**Step 2.** The following inequality holds:
\[
\text{ara}(S_{n_1, \ldots, n_d}) \geq N - 2 = \sum_{i=1}^d n_i + d - 3. \tag{6}
\]

The proof of this step is topological. We first notice the fact that the cokernel of the canonical restriction map \( \alpha: \text{Pic}(\mathbb{P}^N) \to \text{Pic}(S_{n_1, \ldots, n_d}) \) is isomorphic to \( \mathbb{Z} \) (and this holds in arbitrary characteristic).

Now, assume by way of contradiction that \( \text{ara}(S_{n_1, \ldots, n_d}) \leq N - 3 \). If the characteristic \( p \) of the ground field \( k \) is 0, then by Lefschetz’s principle we can assume \( k = \mathbb{C} \). Since
ara\((S_{n_1\ldots,n_d})\) \leq N - 3, then by Corollary 1.6 the map \(\alpha\) must be an isomorphism. This is a contradiction because \(\text{Pic}(S_{n_1\ldots,n_d}) \cong \mathbb{Z} \times \mathbb{Z}\).

If instead \(p > 0\), the inequality \(ara(Y) \leq N - 3\), together with Theorem 1.8, imply that \(\text{Coker}(\alpha)\) is a finite \(p\)-group, which is again a contradiction. This proves Step 2 in arbitrary characteristic.

Notice that in Step 2 there is nothing to prove if \(\dim(S_{n_1\ldots,n_d}) = 2\) because in this case \(\text{codim}_{\mathbb{P}^N}(S_{n_1\ldots,n_d}) = N - 2\). If instead \(\dim(S_{n_1\ldots,n_d}) = 3\) then \(\text{codim}_{\mathbb{P}^N}(S_{n_1\ldots,n_d}) = N - 3\), and since \(\text{Pic}(S_{n_1\ldots,n_d}) \cong \mathbb{Z} \times \mathbb{Z}\), Step 2 is also a consequence of Theorem 2.7 vi).

Then Step 1 and Step 2 conclude the proof of Theorem 4.1.

In characteristic zero Corollary 1.6 and the proof of Step 2 yield actually the following result:

**Corollary 4.5** Under the hypotheses of Theorem 4.1, assume that the characteristic of \(k\) is zero and \(d \geq 3\). Then there exists no ample vector bundle \(F\) of rank \(\leq N - 3\) on \(\mathbb{P}^n\) and a global section of \(F\) vanishing precisely on \(S_{n_1\ldots,n_d}\). Moreover, this upper bound for the rank of \(F\) is optimal.

**Remark 4.6** Corollary 4.5 generalizes the following result noticed by Lazarsfeld in [16]: the image \(S_{1,1,1}\) of the Segre embedding \(\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5\) cannot be the zero locus of a global section of an ample vector bundle of rank two on \(\mathbb{P}^5\).

**Examples 4.7** 1. Let us consider the 2-dimensional scroll \(S_{n_1,n_2}\) in \(\mathbb{P}^{n_1+n_2+1}\); its defining ideal \(\mathcal{I} = \mathcal{I}_+ (S_{n_1,n_2})\) is generated by the \(2 \times 2\) minors of the matrix

\[
\begin{pmatrix}
X_0 & X_1 & \cdots & X_{n_1-1} & Y_0 & Y_1 & \cdots & Y_{n_2-1} \\
X_1 & X_2 & \cdots & X_{n_1} & Y_1 & Y_2 & \cdots & Y_{n_2}
\end{pmatrix}
\]

The proof of Theorem 4.1 shows in particular that \(S_{n_1,n_2}\) is set-theoretic complete intersection in \(\mathbb{P}^{n_1+n_2+1}\) via the following \(n_1 + n_2 - 1\) equations:

\[
F_{1,i}(X_0,\ldots,X_{n_1}) = \sum_{\alpha=0}^{i} (-1)^{i} \binom{i}{\alpha} X_{i+1}^{\alpha} X_{i+1}^{1-\alpha} X_\alpha, \quad i = 1,\ldots, n_1 - 1,
\]

\[
F_{2,j}(Y_0,\ldots,Y_{n_2}) = \sum_{\beta=0}^{j} (-1)^{j} \binom{j}{\beta} Y_{j+1}^{\beta} Y_{j+1}^{1-\beta} Y_\beta, \quad j = 1,\ldots, n_2 - 1,
\]

and the bridge \(B_{n_1,n_2}(X_0,\ldots,X_{n_1};Y_0,\ldots,Y_{n_2})\) (see the formula (4)). These equations are simpler and of lower degree than the equations found by Verdi in [31].

2. In order to give the idea of the size of the polynomials involved in our computation, we now explicitly write down the equations defining set-theoretically the scroll \(S_{2,2,3,4}\) in \(\mathbb{P}^{14}\).

The defining ideal of this scroll is the ideal \(\mathcal{I}\) generated by the \(2 \times 2\) minors of the matrix

\[
\begin{pmatrix}
X_0 & X_1 & Y_0 & Y_1 & Z_0 & Z_1 & Z_2 & T_0 & T_1 & T_2 & T_3 \\
X_1 & X_2 & Y_1 & Y_2 & Z_1 & Z_2 & Z_3 & T_1 & T_2 & T_3 & T_4
\end{pmatrix}
\]

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This scroll has dimension 4 and codimension 9. The main result of this section proves that the arithmetic rank is $\alpha(S_{2,3,4}) = 12$. Namely $\wp$ is the radical of the ideal generated by the following polynomials.

$$
\begin{align*}
X_0X_2 - X_1^2, & \quad Y_0Y_2 - Y_1^2, \quad Z_0Z_2 - Z_1^2, \quad Z_0Z_3 - 2Z_1Z_2Z_3 + Z_2^3 \\
T_0T_2 - T_1^2, & \quad T_0T_3^2 - 2T_1T_2T_3 + T_2^3, \quad T_0T_4^3 - 3T_1T_3T_4^2 + 3T_2T_3T_4 - T_3^4 \\
B_{2,2}(X,Y), & \quad B_{2,3}(X,Z), \quad B_{2,4}(X,T)^5 + B_{2,3}(Y,Z)^3, \quad B_{2,4}(Y,T), \quad B_{3,4}(Z,T).
\end{align*}
$$

**Corollary 4.8** Let $i: \mathbb{P}^{d-1} \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{d(r+1)-1}$ be the Segre-Veronese embedding given by the complete linear system $|\mathcal{O}_{\mathbb{P}^{d-1} \times \mathbb{P}^1}(1,r)|$. Then the subvariety $i(\mathbb{P}^{d-1} \times \mathbb{P}^1)$ is the set-theoretic intersection of $d(r+1) - 3$ homogeneous equations in $\mathbb{P}^{d(r+1)-1}$.

**Proof.** This is just Theorem 4.1 applied to $S_{r,r,...,r} = i(\mathbb{P}^{d-1} \times \mathbb{P}^1)$. □

**Remark 4.9** Using ad-hoc methods and assuming that the characteristic of $k$ is $\neq 2$, Varbaro proved Corollary 4.8 independently in the special case $r = 2$ (see [32], Theorem 3.11).

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