Kondo Length in Bosonic Lattices

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Motivated by the fact that the low-energy properties of the Kondo model can be effectively simulated in spin chains, we study the realization of the effect with bond impurities in ultracold bosonic lattices at half-filling. After presenting a discussion of the effective theory and of the mapping of the bosonic chain onto a lattice spin Hamiltonian, we provide estimates for the Kondo length as a function of the parameters of the bosonic model. We point out that the Kondo length can be extracted from the integrated real space correlation functions, which are experimentally accessible quantities in experiments with cold atoms.

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I. INTRODUCTION

The Kondo effect in metals arises from the interaction between magnetic impurities and conduction electrons, resulting in a net, low-temperature increase of the resistance. The Kondo effect has been initially studied in metals, like Cu, containing magnetic impurities, like Co atoms. A first arena for realizing Kondo effect in a controlled way in a solid state system has been instead provided by quantum dots in contacts with metallic leads, in which the electrons trapped within the dot can give rise to a net nonzero total spin interacting with the spin of conduction electrons from the leads, thus mimicking the behavior of a magnetic impurity in a metallic host. An alternative realization of Kondo physics is recovered within the universal, low energy-long distance physics of a magnetic impurity coupled to a gapless antiferromagnetic chain. A noticeable advantage of working with the spin chain realization of the Kondo effect is that a series of tools developed for spin systems, including entanglement witnesses and negativity, can be used to study the Kondo physics in these systems.

The large interest in the Kondo effect on both the experimental and the theoretical side is motivated by the fact that it provides one of the simplest examples of a many-body correlated system. In fact, due to the large amount of analytical and numerical tools developed to attack it, the Kondo effect has become a paradigmatic example of a strongly interacting system and a testing ground for a number of different many-body techniques. In addition to that, another important, long-lasting reason of interest in Kondo systems lies in that the multichannel "overscreened" version of the effect provides a remarkable realization of non-Fermi liquid behavior. Finally, the nontrivial properties of Kondo lattices provide a major arena in which to study many-body nonperturbative effects, related to heavy-fermion materials. A recent example of both theoretical and experimental activity on multichannel Kondo systems is provided by the topological Kondo model, based on the merging of several one-dimensional quantum wires with suitably induced and possibly controllable Majorana modes tunnel-coupled at their edges, and by recent proposals of realizing topological Kondo Hamiltonians in Y-junctions of XX and Ising chains and of Tonks-Girardeau gases.

The onset of the Kondo effect is set by the Kondo temperature \( T_K \), which emerges from the perturbative renormalization group (RG) approach as a scale at which the system crosses over towards the strongly correlated nonperturbative regime. The systematic implementation of RG techniques has clearly evidenced the scaling behavior characterizing the Kondo regime, which results in the collapse onto each other of the curves describing physical quantities in terms of the temperature \( T \), once \( T \) is rescaled by \( T_K \). Within scaling regime, another scale invariant quantity with the dimension of a length emerges, the Kondo screening length \( \xi_K \), given by \( \xi_K = \hbar v_F / k_B T_K \), where \( v_F \) is the Fermi velocity of conduction electrons and \( k_B \) is the Boltzmann constant. Physically, \( \xi_K \) defines the length scale over which the impurity magnetic moment is fully screened by the spin of conduction electrons, that is, the "size of the Kondo cloud". Despite the remarkable efforts paired in the last years to estimate \( \xi_K \) in various systems by using combinations of perturbative, as well as nonperturbative numerical methods, the Kondo length still appears quite an elusive quantity to directly detect, both in solid-state electronic systems as well as in spin chains. This makes it desirable to investigate alternative systems in which to get an easier experimental access to \( \xi_K \).

A promising route in this direction may be provided by the versatility in the control and manipulation of ultracold...
atoms\textsuperscript{27–29}. Indeed, in the last years several proposals of schemes in which features of the Kondo effect can be studied in these systems have been discussed. Refs.\textsuperscript{29,30} suggest to realize the spin-boson model using two hyperfine levels of a bosonic gas\textsuperscript{28}, or trapped ions arranged in Coulomb crystals\textsuperscript{30} (notice that in general the Kondo problem may be though of as a spin-1/2, system interacting with a fermionic bath\textsuperscript{31}). Ref.\textsuperscript{32} proposes to use ultracold atoms in multi-band optical lattices controlled through spatially periodic Raman pulses to investigate a class of strongly correlated physical systems related to the Kondo problem. Other schemes involve the use of ultracold fermions near a Feshbach resonance\textsuperscript{33}, or in superlattices\textsuperscript{34}. More recently, the implementation of a Fermi sea of spinless fermions\textsuperscript{35} or of two different hyperfine states of one atom species\textsuperscript{36} interacting with an impurity atom of different species confined by an isotropic potential has been proposed\textsuperscript{37}. The simulation of the $SU(6)$ Coqblin-Schrieffer model for an ultracold fermionic gas of Yb atoms with metastable states has been discussed, while alkaline-earth fermions with two orbitals were also at the heart of the recent proposal of simulating Kondo physics through a suitable application of laser excitations\textsuperscript{38}. Despite such an intense theoretical activity, including the investigation of optical Feshbach resonances to engineer Kondo-type spin-dependent interactions in Li-Rb mixtures\textsuperscript{39}, and the remarkable progress in the manipulation of ultracold atomic systems, such as alkaline-earth gases, up to now an experimental detection of features of Kondo physics and in particular of the Kondo length in ultracold atomic systems is still lacking.

In view of the observation that optical lattices provide an highly controllable setup in which it is possible to vary the parameters of the Hamiltonian and to accordingly add impurities with controllable parameters\textsuperscript{39,40}, in this paper we propose to study the Kondo length in ultracold atoms loaded on an optical lattice. Our scheme is based on the well-known mapping between the lattice Bose-Hubbard (BH) Hamiltonian and the XXZ spin-1/2 Hamiltonian\textsuperscript{41}, as well as on the Jordan-Wigner (JW) representation for the spin 1/2 operators, which allows for a further mapping onto a Luttinger liquid model\textsuperscript{42–44}. Kondo effect in Heisenberg spin-1/2 antiferromagnetic spin chains has been extensively studied\textsuperscript{45–47}, though mostly for side-coupled impurities (i.e., at the edge of the chain). For instance, in Ref.\textsuperscript{46}, the Kondo impurity is coupled to a single site of a gapless XXZ spin, while in Ref.\textsuperscript{48} a magnetic impurity is coupled at the end of a $J_1 - J_2$ spin-1/2 chain. At variance, in trapped ultracold atomic systems, it is usually difficult to create an impurity at the edge of the system. Accordingly, in this paper we propose to study the Kondo length at an extended (at least two links) impurity realized in the bulk of a cold atom system on a 1d optical lattice. In particular, we assume the lattice to be at half-odd filling, so to avoid the onset of a gapped phase that takes place at integer filling in the limit of a strong repulsive interaction between the particles. Since the real space correlation functions are quantities that one can measure in a real cold atom experiment, we address the issue of how to extract the Kondo length from the zeros of the integrated real space density-density correlators. Finally, we provide estimates for $\xi_K$ and show that, for typical values of the system parameters, it takes values within the reach of experimental detectability ($\sim$ tens of lattice sites).

The paper is organized as follows:

- In section \textbf{II} we provide the effective description of a system of ultracold atoms on a 1d optical lattice. We review its Luttinger liquid description and how to accordingly model impurities (and in particular bond impurities) in the lattice;
- In section \textbf{III} we derive the scaling equations for the Kondo running couplings and use them to estimate the corresponding Kondo length;
- In section \textbf{IV} we discuss how to numerically extract the Kondo length from the integrated real space density-density correlations and compare the results with the ones obtained in section \textbf{III};
- In section \textbf{V} we summarize and discuss our results.

Mathematical details of the derivation and reviews of known results in the literature are provided in the various appendices.

\textbf{II. EFFECTIVE MODEL HAMILTONIAN}

Based on the spin-1/2 XXZ spin-chain Hamiltonian description of (homogeneous, as well as inomogenous) interacting bosonic ultracold atoms at half-filling in a deep optical lattice, in this section we propose to model impurities in the spin chain by locally modifying the strength of the link parameters of the optical lattice, eventually resorting to a model describing two XXZ “half-spin chains”, interacting with each other via a local impurity. When the impurity is realized as a spin-1/2 local spin, such a system corresponds to a possible realization of the (two channel) Kondo effect in spin chains\textsuperscript{49–51}. Therefore, our mapping leads to the conclusion that spin chain Kondo effect may possibly realized and detected within bosonic cold atoms loaded onto a one-dimensional optical lattice.
A. Spin-1/2 description of the homogeneous Bose-Hubbard model on an open chain

We start by reviewing the mapping between the BH chain for large on-site interaction energy $U$, and the spin-1/2 XXZ spin chain. Specifically, we consider a system of interacting ultracold bosons on a deep one-dimensional lattice. This is described by the extended BH Hamiltonian,

$$H_{\text{BH}} = -\sum_{j=-\ell}^{\ell-1} t_{j,j+1}(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j) + \frac{U}{2} \sum_{j=-\ell}^{\ell} n_j(n_j - 1) + V \sum_{j=-\ell}^{\ell-1} n_j n_{j+1} - \mu \sum_{j=-\ell}^{\ell} n_j .$$

(1)

In Eq. (1), $b_j, b_j^\dagger$ are respectively the annihilation and the creation operator of a single boson at site $j$ (with $j = -\ell, \cdots, \ell$) and, accordingly, they satisfy the commutator algebra $[b_j, b_{j'}^\dagger] = \delta_{j,j'}$, all the other commutators being equal to 0. As usual, we set $n_j = b_j^\dagger b_j$. Moreover, $t_{j,j+1}$ is the hopping amplitude for bosons between nearest neighboring sites $j$ and $j + 1$, $U$ is the interaction energy between particles on the same site, $V$ is the interaction energy between particles on nearest-neighboring sites. Typically, for alkali atoms one has $V \ll U$ while, for dipolar gases on a lattice, $V$ may be of the same order as $U/2$. Throughout all the paper we take $U > 0$ and $V \geq 0$. In this section we assume $t_{j,j+1}$ to be uniform across the chain and equal to $t$. At variance, starting from the next section II B we realize an impurity in the chain by means of a pertinent modulation of the $t_{j,j+1}$'s in real space. In performing the calculations, we will be assuming open boundary conditions on the $2\ell+1$-site chain and we will set the average number of particles per site by fixing the filling

$$f = \frac{N_T}{N} ,$$

(2)

where $N_T$ is the total number of particles on the lattice ($N = 2\ell + 1$ is the number of sites).

In the large-$U$ limit, one may set up a mapping between the BH Hamiltonian in Eq. (1) and a pertinent spin-model Hamiltonian $H_S$, either describing an integer $\frac{3}{2}$, or an half-odd spin chain, depending on the value of $f$. For integer filling at large $U$, that is, for $f = n$ (with $n = 1, 2, \ldots$), corresponding to $\mu = \mu_0(n) = n(U + 2V) - U/2$ and $U \gg t$, a low-energy effective description of $H_{\text{BH}}$ is recovered by restricting the dynamics to the subspace $\mathcal{F}_1$ of the Fock space spanned by the states $\otimes_{j=1}^{N} |k_j\rangle$, with $|k_j\rangle$ being the eigenstate of $n_j$ belonging to the eigenvalue $k = n - 1, n, n + 1$. To actually construct the effective Hamiltonian restricted to $\mathcal{F}_1$, $H_{\text{spin-1}}$, one first defines $P_1$ to be the projector onto $\mathcal{F}_1$. Therefore, acting with $P_1$ on $H_{\text{BH}}$, one recovers $H_{\text{spin-1}}$ for a spin-1 chain, with the spin-1 operators at site $j$ defined as $S_j^+ = P_1(n_i - n)P_1, S_j^- = \frac{1}{\sqrt{n}} P_1 b_j^\dagger P_1$. As a result, one finds

$$H_{\text{spin-1}} = -J_1 \sum_{i=-\ell}^{\ell-1} (S_i^+ S_{i+1}^- + S_{i+1}^+ S_i^-) + V \sum_{i=-\ell}^{\ell-1} S_i^z S_{i+1}^z + \frac{U}{2} \sum_{i=-\ell}^{\ell} (S_i^z)^2 ,$$

(3)

with $J_1 \sim tn$. (Note that, in order for the mapping to be reliable, one has to have a large enough $n$, so that $n \pm 1 \approx n$). Relating on $H_{\text{spin-1}}$ in Eq. (3) and on the analysis of the phase diagram of spin-1 chains derived within the standard bosonization approach, the occurrence of Mott and Haldane gapped insulating phases for ultracold atoms on a lattice has been predicted and discussed.

At variance, at $U/t \gg 1$ and half-odd filling $f = n + 1/2$ (with $n = 1, 2, \ldots$), corresponding to setting the chemical potential so that $\mu = (U + 2V)(n + \frac{1}{2}) - \frac{U}{2}$, the effective low-energy Hamiltonian for the system is constructed by projecting $H_{\text{BH}}$ onto the subspace of the Hilbert space $\mathcal{F}_2$, spanned by the states $\otimes_{j=1}^{N} |n + \frac{1}{2} + \sigma\rangle_j$, with $\sigma = \pm \frac{1}{2}$. In this case, the large value of $U/t$ does not lead to a Mott insulating phase, as it happens for a generic value of $f$. Indeed, the degeneracy between the states $|n\rangle$ and $|n + 1\rangle$ at each site allows for restoring superfluidity, similarly to what happens in the phase model describing one-dimensional arrays of Josephson junctions at the charge-degenerate point. To formally introduce the mapping, we define $P_{1/2}$ to be the projector onto $\mathcal{F}_2$. Projecting $H_{\text{BH}}$ with $P_{1/2}$ and supplementing the projection with a Schrieffer Wolff (SW) summation, which takes into account virtual processes involving high-energy states, one eventually gets the effective spin-1/2 XXZ Hamiltonian given by

$$H_{\text{spin-1/2}} = -J \sum_{j=-\ell}^{\ell-1} (S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-) + J_\Delta \sum_{j=-\ell}^{\ell-1} S_j^z S_{j+1}^z ,$$

(4)
with the spin-1/2 operators $S_j^\alpha$ defined as
\begin{align}
S_j^+ &= \frac{1}{\sqrt{n+1}} P_{zj} b_j P_{zj}, \\
S_j^- &= \frac{1}{\sqrt{n+1}} P_{zj} b_j P_{zj}, \\
S_j^z &= P_{zj} [b_j^\dagger b_j - f] P_{zj}.
\end{align}

The parameters $J$ and $\Delta$ are given by
\begin{align}
J &= \tilde{J} \left[ 1 - \frac{2\tilde{J}}{U/\rho} \right], \\
\Delta &= \frac{\tilde{\Delta}}{1 - \frac{2\tilde{J}}{U/\rho}}
\end{align}

and
\begin{align}
\tilde{J} &= 2t \left( n + \frac{1}{2} \right), \\
\tilde{\Delta} &= \frac{V}{\tilde{J}} - \frac{\ell^2(2n^2 + 6n + 4)}{JU} - \frac{4\ell^2(n + 1)^2}{JU}, \\
\rho &= \frac{U(n + 1)}{2J} - \sqrt{\left[ \frac{U(n + 1)}{2J} \right]^2 + n + 2}.
\end{align}

Notice that the spin-1/2 Hamiltonian in Eq. (4) has to be supplemented with the condition that $\sum_j S_j^z = 0$, implying that physically acceptable states are only the eigenstates of $\sum_j n_j$ belonging to the eigenvalue $N_T$: this corresponds to singling out of the Hilbert space only the zero magnetization sector. As discussed in detail in Ref. [55], $H_{\text{spin-1/2}}$ provides an excellent effective description of the low-energy dynamics of the BH model at half-odd filling. Although the mapping is done in the large-$U$ limit, in Ref. [55] it is shown that it is in remarkable agreement with DMRG results also for $U/J$ as low as $\sim 3 - 5$ and for low values of $N_T$ such as $N_T N \sim 30$.

Accordingly, $H_{\text{spin-1/2}}$ in Eq. (4) has to be modified by adding a term $\sum_{j=\ell}^\ell \epsilon_j n_j$ to the right-hand side of Eq. (4). As soon as the potential energy scale is smaller than $U$, we expect that the mapping to be still valid (we recall that with a trapping parabolic potential typically $\epsilon_j = \Omega j^2$ with $\Omega \equiv m\omega^2\lambda^2/8$, $m$ being the atom mass, $\omega$ the confining frequency and $\lambda/2$ the lattice spacing\cite{57}). Yet, we stress that recent progresses in the realizations of potentials with hard walls\cite{38,40}, make the optical lattice realization of chains with open boundary conditions to lie within the reach of present technology.

Another point to be addressed is what happens slightly away from half-filling, that is, for $f = n + 1/2 + \varepsilon$, with $\varepsilon \ll 1$. In this case, one again recovers the effective Hamiltonian in Eq. (4), but now with the constraint on physically acceptable states given by $\langle 1/N \rangle \langle \sum_j S_j^z \rangle = \varepsilon$. Since keeping within a finite magnetization sector is equivalent to have a nonzero applied magnetic field\cite{56}, one has then to add to the right-hand side of Eq. (4) a term of the form $\mathcal{H} \sum_j S_j^z$: again, we expect that the mapping is valid as soon as that the magnetic energy is smaller than the interaction energy scale $U$, and, of course, that the system spectrum keeps gapless\cite{33,56}.

We now discuss how the mapping is modified in the presence of (bond) impurities in the parent BH Hamiltonian.

**B. Bond impurities: the effective weak link Hamiltonian and the “even-odd” effect**

Compared to possible practical realization of spin-chain models, optical lattices provide an highly controllable setup in which it is possible to vary the parameters of the Hamiltonian as well as to add impurities with tunable parameters\cite{39,40}. To spell out how a link defect may be created in an optical lattice, one may either think of pertinently modulating the lattice, so that the energy barriers among its wells vary inhomogeneously across the chain, or of inserting one, or more, extra laser beams, centered on the minima of the lattice potential. In this latter case, one makes the atoms feel a total potential given by $V_{\text{ext}} = V_{\text{opt}} + V_{\text{laser}}$, where the optical potential is given by $V_{\text{opt}} = V_0 \sin^2(kx)$,
with $k = 2\pi / \lambda$ and $\lambda = \lambda_0 / \sin(\theta/2)$, $\lambda_0$ being the wavelength of the lasers and $\theta$ the angle between the laser beams forming the main lattice (notice that the lattice spacing is $d = \lambda / 2$). For counterpropagating laser beams having the same direction, $\theta = \pi$ and $d = \lambda_0 / 2$, while $d$ can be enhanced by making the beams intersect at an angle $\theta \neq \pi$. $V_{\text{laser}}$ is the additional potential due to extra (blue-detuned) lasers: with one additional laser, centered at or close to an energy maximum of $V_{\text{opt}}$, say at $x \equiv x_0;1$ among the minima $x_0$ and $x_1$, the potential takes the form $V_{\text{laser}} \approx V_1 e^{-(x-x_0;1)^2/\sigma^2}$. When the width $\sigma$ is much smaller than the lattice spacing, the hopping rate between the sites $j = 0$ and $j = 1$ is reduced and no on-site energy term appear, as shown in panels a) and b) of Fig.1. Notice that we use a notation such that the $j$-th minimum corresponds to the minimum $x_j = jd$ in the continuum space.

When $x_{0;1}$ is equidistant from the lattice minima $x_0$ and $x_1$, corresponding to $x_{0;1} = \lambda / 4 = d/2$ and $\sigma < d$, then only the hopping $t_{0;1}$ is practically altered (see Fig.1h). When $x_{0;1}$ is displaced from $d/2$ one has an asymmetry and also a nearest neighboring link (e.g., $t_{-1;0}$ in Fig.1b) may be altered (an additional on-site energy $\epsilon_0$ is also present). With $d \sim 2 - 3 \mu m$, one should have $\sigma \lesssim 2 \mu m$, in order to basically alter only one link. Notice that barrier of few $\mu m$ can be rather straightforwardly implemented\textsuperscript{55,56} and recently a barrier of $\sim 2 \mu m$ has been realized in a Fermi gas\textsuperscript{66}.

As discussed in the following, this is the prototypical realization of a weak-link impurity in an otherwise homogenous spin chain\textsuperscript{67,68}.

In general, reducing the hopping rate between links close to each other may either lead to an effective weak link impurity, or to a spin-1/2 effective magnetic impurity, depending on whether the number of lattice sites between the reduced-hopping-amplitude links is even, or odd (see appendix A for a detailed discussion of this point). To “double” the construction displayed in panels a) and b) of Fig.1 to the one we sketch in panels c) and d) of Fig.1, we consider a potential of the form $V_{\text{laser}} = V_1 e^{-(x-x_{0;1})^2/\sigma^2} + V_2 e^{-(x-x_{-1;0})^2/\sigma^2}$ with $x_{-1;0}$ lying between sites $j = -1$ and $j = 0$: assuming again $\sigma \lesssim d$, when $V_1 = V_2$ and $x_{0;1} = -x_{-1;0} = d/2$ then only two links are altered, and in an equal way (the hoppings $t_{-1;0}$ and $t_{0;1}$ in Fig.1c), otherwise one has two different hoppings (again $t_{-1;0}$ and $t_{0;1}$ in Fig.1d). When $\sigma$ is comparable with $d$, apart from the variation of the hopping rates, on-site energy terms enter the Hamiltonian in Eq.1, giving rise to local magnetic fields in the spin Hamiltonian in Eq.1. Though this latter kind of “site defects” might readily accounted for within spin-1/2 XXZ framework, for simplicity we will not consider them in the following, and will only retain link defects, due to nonhomogeneities in the boson hopping amplitudes between nearest neighboring sites and in the interaction energy $V$. Correspondingly, the hopping amplitude $t_j, j+1$ in Eq.1 takes a dependence on the site $j$ also far form the region in which the potential $V_{\text{laser}}$ is centered.

In the followin, we consider inhomogeneous distributions of link parameters symmetric about the center of the chain (that is, about $j = 0$). Moreover, for the sake of simplicity, we discuss a situation in which two (symmetrically placed) inhomogeneities enclose a central region, whose link parameters may, or may not, be equal to the ones of the rest of the chain. We believe that, though experimentally challenging, this setup would correspond to the a situation in which the experimental detection of the Kondo length is neater. In fact, we note that all the experimental required ingredients are already available, as our setup requires two lasers with $\sigma \lesssim d$ (ideally, $\sigma \ll d$) and centered with similar precision.

As we stated above and discuss in detail in appendix A an “extended central region” as such can either be mapped onto an effective weak link, between two otherwise homogeneous “half-chains”, or onto an effective isolated spin-1/2 impurity, weakly connected to the two half-chains. In particular, in this latter case, the Kondo effect may arise, yielding remarkable nonperturbative effects and, eventually, “sewing together” the two half chains, even for a repulsive bulk interaction\textsuperscript{45,46}. Denoting by $G$ the region singled out by weakening one or more links, in order to build an effective description of $G$, we assume that the mapping onto a spin-1/2 XXZ-chain works equally well with the central region, and employ a SW summation, in order to trade the actual dynamics of $G$ for an effective boundary Hamiltonian, that describes the effective degrees of freedom of the central region interacting with the half chains. One is then led to the Hamiltonian in Eq.1, with link-dependent hopping rates $t_{j,j+1}$.

To fix the ideas, we consider $M$ links altered. Specifically, we mostly focus on $M = 2$ with only two hopping rates changed, corresponding to two blue-detuned lasers, and briefly comment on the more general case. To resort to the Kondo-like Hamiltonian for a spin-1/2 impurity embedded within a spin-1/2 XXZ-chain, we define the hopping rate to be equal to $t$ throughout the whole chain but between $j = -1$ and $j = 0$, where we assume it to be equal to $t_L$, and between $j = 0$ and $j = 1$, where we set it equal to $t_R$, (corresponding to panels c) and d) of Fig.1. On going through the SW transformation, one therefore gets the effective spin-1/2 Hamiltonian $H_s = H_{\text{bulk}} + H_K$, with $H_{\text{bulk}} = H_L + H_R$ and

$$H_L = -J \sum_{j=-\ell}^{\ell-2} (S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-) + J\Delta \sum_{j=-\ell}^{\ell-2} S_j^z S_{j+1}^z,$$

$$H_R = -J \sum_{j=1}^{\ell-1} (S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-) + J\Delta \sum_{j=1}^{\ell-1} S_j^z S_{j+1}^z. \quad (8)$$
The "Kondo-like" term is instead given by

$$H_K = -J'_L (S_{-1}^+ S_0^- + S_{-1}^- S_0^+) - J'_R (S_0^+ S_1^- + S_0^- S_1^+) + J'_L S_{-1}^z S_0^- + J'_R S_0^z S_1^-,$$

where $J'_\alpha = t_\alpha f$ and $J'_{z\alpha} \approx V - 3J'^2_\alpha/4U$ (with $\alpha = L, R$).

Our choice for $H_K$ corresponds to the simplest case in which $G$ contains an even number of links – or, which is the same, an odd number of sites, as schematically depicted in Fig. 2b). We see that the isolated site works as an isolated spin-1/2 impurity $S_G$, interacting with the two half chains via the boundary interaction Hamiltonian $H^{(1)}_B \equiv H_K$. The other possibility, which we show in Fig. 2a), corresponds to the case in which an odd number of links is altered and $G$ contains an even number of sites. In particular, in Fig. 2a) we have only one altered hopping coefficient. As expected and as we outline in the following, this latter case is basically equivalent to a simple weak link between the $R$- and the $L$- half chain, which is expected to realize the spin-chain version of Kane-Fisher physics of impurities in an interacting one-dimensional electronic system. In appendix A we review the effective low-energy description for a region $G$ containing an in principle arbitrary number of sites. In particular, we conclude that either the number of sites within $G$ is odd, and therefore the resulting boundary Hamiltonian takes the form of $H_K$ in Eq. (9), or it is even, eventually leading to a weak link Hamiltonian. Even though this latter case is certainly an interesting subject of investigation, we are mostly interested in the realization of effective magnetic impurities. Therefore, henceforth we will be using $H_s$ as the main reference Hamiltonian, to discuss the emergence of Kondo physics in our system.

III. RENORMALIZATION GROUP FLOW OF THE IMPURITY HAMILTONIAN PARAMETERS

In this section, we employ the RG approach to recover the low-energy long-wavelength physics of an impurity in an otherwise homogeneous chain. In Fig. 2 we provide a sketch of the two standard ways of realizing the impurity as an island containing either an even or odd number of spins. The former case is equivalent to a weak link in an otherwise homogeneous chain, originally discussed in Refs. [69,70] for electronic systems, and reviewed in detail in Ref. [71] in the specific context of spin chains. In this case, which we briefly review below, when $\Delta > 0$ in Eqs. (6)), the impurity corresponds to an irrelevant perturbation, which implies an RG flow of the system towards the fixed point corresponding to two disconnected chains, while for $\Delta < 0$ the weak link Hamiltonian becomes a relevant perturbation.
Though this implies the emergence of an "healing length " for the weak link as an RG invariant length scale, with a corresponding flow towards a fixed point corresponding to the two chains joined into an effectively homogenous single chain, there is no screening of a dynamical spinful impurity by the surrounding spin degrees of freedom and, accordingly, no screening cloud is detected in this case\textsuperscript{42}.

At variance, a dynamical effective impurity screening takes place in the case of an effective spin-1/2 impurity\textsuperscript{72}. In this latter case, at any $\Delta$ such that $-1 < \Delta < 1$, the perturbative RG approach shows that the disconnected-chain weakly coupled fixed point is ultimately unstable. In fact, the RG trajectories flow towards a strongly coupled fixed point, corresponding to healing the chain but, at variance with what happens at a weak link for $0 < \Delta \leq 1$, this time with the chain healing taking place through an effective Kondo-screening of the magnetic impurity\textsuperscript{45}. We now separately discuss the two cases.

A. Central region with an even number of sites

As we show in appendix B\textsuperscript{8} a central region containing an even number of sites is effectively described by the weak link Hamiltonian $V_{B}^{2}/J$ given by Eq. (A9). Such an Hamiltonian is fully parametrized in terms of the "longitudinal" coupling $\lambda_{z}$ and of the "transverse" coupling $\lambda_{\perp}$. To describe how the relative weight of the impurity interaction depends on the reference cutoff scale of the system, one has to recover the corresponding RG scaling equations for the running coupling strengths associated to $\lambda_{z}$ and to $\lambda_{\perp}$. This is readily done by resorting to the Abelian bosonization approach to spin chains\textsuperscript{45}, by means of which one substitutes $V_{B}^{2}/J$ with the bosonized Euclidean action $S_{E}^{B}$ in Eq. (111). In bosonizing the two side chains, we assume for the sake of simplicity that they are described by the same parameters and have the same length $\ell$. Therefore, in the bosonization framework they are described by the two bosonic fields $\Phi_{L,R}(x)$, whose dynamics is determined by the free action $S_{0}[\Phi_{L},\Phi_{R}] = S_{E}[\Phi_{L}] + S_{E}[\Phi_{R}]$, with $S_{E}[\Phi]$ given in Eq. (B2), together with the corresponding dual fields $\Theta_{L,R}(x)$. For weak bare impurity couplings, one sets the boundary conditions on the bosonic fields by referring to the disconnected fixed point, corresponding to $\lambda_{z} = \lambda_{\perp} = 1$. This corresponds to assuming Neumann boundary conditions for $\Phi_{L}(x,\tau)$ and for $\Phi_{R}(x,\tau)$ at both boundaries\textsuperscript{68,73–75}, that is

$$\frac{\partial \Phi_{\alpha}(0,\tau)}{\partial x} = \frac{\partial \Phi_{\alpha}(\ell,\tau)}{\partial x} = 0 \quad (\alpha = L, R) \quad ,$$

(10)

which implies the following mode expansion for $\Phi_{L,R}(x,\tau)$:

$$\Phi_{L,R}(x,\tau) = \sqrt{\frac{\pi}{\ell g}} \left\{ q_{\alpha,\perp} \left\{ -\frac{2\pi}{\ell} P_{L,R}(iu\tau) + i \sum_{n \neq 0}^{\ell} \frac{\alpha_{L,R}(n)}{n} \cos \left[ \frac{\pi n x}{\ell} \right] e^{-\frac{n\pi u}{\ell}} \right\} \right\} ,$$

(11)

with $g$ and $u$ being respectively the Luttinger parameter and the plasmon velocity of the Luttinger liquid model effectively describing the long-wavelength physics of the chain\textsuperscript{42,76,77} (see appendix B for the detailed definition of the parameters). The mode operators satisfy the commutation relations

$$[q_{\alpha}, P_{\alpha'}] = i \delta_{\alpha,\alpha'} \quad ,$$

$$[\alpha_{\alpha}(n), \alpha_{\alpha'}(n')] = n \delta_{n+n',0} \delta_{\alpha,\alpha'} \quad ,$$

(12)

all the other commutators being $= 0$. Because of the cross relations relating the derivatives of $\Phi_{L}$ and $\Phi_{R}$ to those of their dual fields $\Theta_{R}, \Theta_{L}$, one obtains that $\Theta_{L}(x,\tau)$ and $\Theta_{R}(x,\tau)$ must obey Dirichlet boundary conditions at both boundary, that is, they must be pinned at values independent of $\tau$, which yields the mode expansion

$$\Theta_{L,R}(x,\tau) = \sqrt{2g} \left\{ \theta_{L,R} \left\{ -\frac{2\pi x}{\ell} P_{L,R} + \sum_{n \neq 0}^{\ell} \frac{\alpha_{L,R}(n)}{n} \sin \left[ \frac{\pi n x}{\ell} \right] e^{-\frac{n\pi u}{\ell}} \right\} \right\} \quad ,$$

(13)
with $\theta_{L,R}$ constants.

The bosonized formula for the weak link Euclidean action $S_B^0$ in Eq. (11) readily allows for recovering the scaling dimensions of the various terms from standard Luttinger liquid techniques, once one has assumed the mode expansions in Eqs. (12,13,14,15). Specifically, one finds that the term $\propto \lambda_\perp$ has scaling dimension $h_\perp = \frac{1}{g}$, while the one $\propto \lambda_z$ has scaling dimension $h_\parallel = 2$. As we use the chain length $\ell$ as scaling parameter of the system, to keep in touch with the standard RG approach, we define the dimensionless running coupling strengths $L_\perp(\ell) = \left(\frac{\ell}{\ell_0}\right)^{1-\frac{1}{g}} \lambda_\perp$ and $L_\parallel(\ell) = \left(\frac{\ell}{\ell_0}\right)^{-\frac{1}{g}}$, with $\ell_0$ being a reference length scale (see below for the discussion on the estimate of $\ell_0$).

To leading order in the coupling strengths, we obtain the perturbative RG equations for the running parameters given by

$$\frac{dL_\perp(\ell)}{d\ln \left(\frac{\ell}{\ell_0}\right)} = \left[1 - \frac{1}{g}\right] L_\perp(\ell)$$

$$\frac{dL_\parallel(\ell)}{d\ln \left(\frac{\ell}{\ell_0}\right)} = -L_\parallel(\ell) .$$

Eqs. (14) encode the main result concerning the dynamics of a weak link in an otherwise uniform XXZ chain. Leaving aside the trivial case $g = 1$, corresponding to effectively noninteracting JW fermions, which do not induce any universal (i.e., independent of the bare values of the system parameters) flow towards a conformal fixed point, we see that the behavior of the running strengths on increasing $\ell$ is drastically different, according to whether $g < 1 (\Delta > 0)$, or $g > 1 (\Delta < 0)$. In the former case, both $h_\perp$ and $h_\parallel$ are $> 1$, which implies that $V_B^{2J}$ is an irrelevant perturbation to the disconnected fixed point. The impurity interaction strengths flow to zero in the low-energy, long-wavelength limit, that is, under RG trajectories, the system flows back towards the fixed point corresponding to two disconnected chains. At variance, when $g > 1$, $L_\perp(\ell)$ grows along the RG trajectories and the system flows towards a “strongly coupled” fixed point, which corresponds to the healed chain, in which the weak link has been healed within an effectively uniform chain obtained by merging the two side chains with each other. The healing takes place at a scale $\ell \sim \ell_{\text{Heal}}$, with

$$\ell_{\text{Heal}} \sim \ell_0 \left(\frac{1}{L(\ell_0)}\right)^{-\frac{\pi}{\Delta}} .$$

As we see from Eq. (15), defining $\ell_{\text{Heal}}$ requires introducing a nonuniversal, reference length scale $\ell_0$. $\ell_0$ is (the plasmon velocity times) the reciprocal of the high-energy cutoff $D_0$ of our system. To estimate $D_0$, we may simply require that we cutoff all the processes at energies at which the approximations we employed in appendix B to get the effective boundary Hamiltonians break down. This means that $D_0$ must be of the order of the energy difference $\delta E$ between the groundstate(s) and the first excited state of the central region Hamiltonian. From the discussion of appendix B we see that $\delta E \sim J$, which, since we normalized all the running couplings to $J$, implies $\ell_0 \sim d, d$ being the lattice step of the microscopic lattice Hamiltonian describing our spin system. It is important, before concluding this section, to stress once more the fact that, though an RG invariant length scale $\ell_{\text{Heal}}$ emerges already at a weak link between two chains with $\Delta < 0$, there is no screening cloud associated to this specific problem. Indeed, in the case of a weak link impurity, the healing of the chain is merely a consequence of repeated scattering off the Friedel oscillations due to backscattering at the weak link, which conspire to fully heal the impurity at a scale $\ell_{\text{Heal}}$. At variance, when there is an active spin-1/2 impurity, the density oscillations are no longer simply determined by the scattering by Friedel oscillations, but there is also the emergence of the Kondo screening cloud induced in the system. In the following section we discuss in detail this latter situation.

**B. Central region with an odd number of sites**

A region containing an odd number of sites typically has a twofold degenerate groundstate and, therefore, is mapped onto an effective spin-1/2 impurity $S_G$. The corresponding impurity Hamiltonian in Eq. (12) takes the form of the Kondo spin-chain interaction Hamiltonian for a central impurity in an otherwise uniform spin chains. Again, to discuss the RG flow of the impurity coupling strength, we resort to Eq. (13), corresponding to the bosonized spin Kondo Hamiltonian $H_K$ given by

$$H_K = \sum_{\alpha=L,R} \left\{ -J'_\alpha S^+_0 e^{-i2\Phi_{\alpha}(0)} + S^-_0 e^{i2\Phi_{\alpha}(0)} \right\} + J'_\alpha S^+_0 \frac{1}{\sqrt{2\pi}} \frac{\partial \Theta_{\alpha}(0)}{\partial x} .$$


To spell out the full RG flow of the impurity coupling strength, we derive the RG equations up to second order in the coupling strengths themselves. This can be done by means of standard techniques for Kondo effect in spin chains and, in particular, by considering the fusion rules between the various operators entering $H_K$ in Eq. (10). In doing so, in principle additional, weak link-like, operators describing direct tunneling between the two chains can be generated, such as, for instance, a term $\propto e^{-\Phi_L(0)-\Phi_R(0)}$, with scaling dimension $h_\Lambda = \frac{1}{g}$. However, one may safely neglect a term as such, since, for $g < 1$, it corresponds to an additional irrelevant boundary operator that has no effects on the RG flow of the running couplings appearing in $H_K$. For $g \geq 1$ it becomes marginal/relevant, but still subleading, compared to the terms $\propto J_\alpha$, as we discuss in the following and, therefore, it can again be neglected for the purpose of working our the RG flow of the boundary couplings. This observation effectively enables us to neglect operators mixing the $L$ and the $R$ couplings with each other and, accordingly, to factorize the RG equations for the running couplings with respect to the index $\alpha$.

More in detail, we define the dimensionless variables $G_\alpha(\ell)$ and $G_{z,\alpha}(\ell)$ as

$$G_\alpha(\ell) = \left( \frac{\ell}{\ell_0} \right)^{1-\frac{1}{2g}} \frac{J'}{J} \quad \text{and} \quad G_{z,\alpha}(\ell) = \frac{J'_{z,\alpha}}{J},$$  

again with $\alpha = L, R$.

The RG equations for the running couplings are given by

$$\frac{dG_\alpha(\ell)}{d\ln(\frac{\ell}{\ell_0})} = h_\alpha G_\alpha(\ell) + G_\alpha(\ell)G_{z,\alpha}(\ell)$$

$$\frac{dG_{z,\alpha}(\ell)}{d\ln(\frac{\ell}{\ell_0})} = G_{z,\alpha}(\ell)^2,$$

with $h_\alpha = 1 - 1/(2g)$. For the reasons discussed above, the RG equations in Eq. (13) for the $L$- and the $R$-coupling strengths are decoupled from each other. In fact, they are formally identical to the corresponding equations obtained for a single link impurity placed at the end of the chain (“Kondo side impurity”). At variance with this latter case, as argued by Affleck and Eggert in our specific case of a “Kondo central impurity” the scenario for what concerns the possible Kondo-like fixed points is much richer, according to whether $G_L(\ell_0) \neq G_R(\ell_0)$ (“asymmetric case”), or $G_L(\ell_0) = G_R(\ell_0)$ (“symmetric case”), as we discuss below.

To integrate Eqs. (13), we define the reduced variables $X_\alpha(\ell) \equiv G_\alpha(\ell)$ and $X_{z,\alpha}(\ell) = G_{z,\alpha}(\ell) + 1 - \frac{1}{2g}$ for $\alpha = L, R$ (since the equations for the two values of $\alpha$ are formally equal to each other, from now on we will understand the index $\alpha$). As a result, one gets

$$\frac{dX(\ell)}{d\ln(\frac{\ell}{\ell_0})} = X(\ell)X_z(\ell); \quad \frac{dX_z(\ell)}{d\ln(\frac{\ell}{\ell_0})} = X^2(\ell).$$

Eqs. (13) coincide with the RG equations obtained for the Kosterlitz-Thouless phase transition. To solve them, we note that the quantity

$$\kappa = X^2_z(\ell) - X^2(\ell),$$

is invariant along the RG trajectories. In terms of the microscopic parameters of the BH Hamiltonian one gets

$$\kappa = \kappa(\ell_0) = (V/J - 3J^2/(4UJ) + 1 - 1/(2g))^2 - (J'/J)^2.$$  

To avoid the onset of Mott-insulating phases, we have to assume that the interaction is such that $g > 1/2$. This implies $h_\alpha > 0$ and $X_z(\ell_0) > 0$: thus, we assume $X(\ell_0), X_z(\ell_0) > 0$. This means that the RG trajectories always lie within the first quarter of the $(X, X_z)$-parameter plane and, in particular, that the running couplings always grow along the trajectories.

Using the constant of motion in Eq. (20), Eqs. (13) can be easily integrated. As a result, one may estimate the RG invariant length scale $\ell_\alpha$ defined by the condition that, at the scale $\ell \sim \ell_\alpha$, the perturbative calculation breaks down (which leads us to eventually identify $\ell_\alpha$ with $\xi_K$). As this is signaled by the onset of a divergence in the running parameter $X(\ell)$, one may find the explicit formulas for $\ell_\alpha$, depending on the sign of $\kappa$, as detailed below:

- $\kappa = 0$. In this case, as the symmetry at $\ell = \ell_0$ between $K$ and $X_z$ is preserved along the RG trajectories, it is enough to provide the explicit solution for $X_z(\ell)(= X(\ell))$, which is given by

$$X_z(\ell) = \frac{X_z(\ell_0)}{1 - X_z(\ell_0)\ln(\frac{\ell}{\ell_0})}.$$
From Eq. (21), one obtains
\[ \ell_\ast \sim \ell_0 \exp \left[ \frac{1}{X_z(\ell_0)} \right], \tag{22} \]
which is the familiar result one recovers for the "standard" Kondo effect in metals82.

- \( \kappa < 0 \). In this case, the explicit solution of Eqs. (19) is given by
\[ X_z(\ell) = \sqrt{-\kappa} \tan \left\{ \text{atan} \left[ \frac{X_z(\ell_0)}{\sqrt{-\kappa}} \right] + \sqrt{-\kappa} \ln \left( \frac{\ell}{\ell_0} \right) \right\} \]
\[ X(\ell) = \sqrt{-\kappa + X_z^2(\ell)}, \tag{23} \]
which yields
\[ \ell_\ast \sim \ell_0 \exp \left[ \frac{\pi - 2 \text{atan}(X_z(\ell_0))}{2\sqrt{|\kappa|}} \right]. \tag{24} \]

- \( \kappa > 0 \). In this case one obtains
\[ X_z(\ell) = -\sqrt{\kappa} \left\{ \frac{[X_z(\ell_0) - \sqrt{\kappa}] \left( \frac{\ell}{\ell_0} \right)^2 \sqrt{\kappa}}{[X_z(\ell_0) - \sqrt{\kappa}] \left( \frac{\ell}{\ell_0} \right)^2 \sqrt{\kappa} - [X_z(\ell_0) + \sqrt{\kappa}]} \right\} \]
\[ X(\ell) = \sqrt{-\kappa + X_z^2(\ell)} \tag{25} \]

As a result, we obtain
\[ \ell_\ast \sim \ell_0 \left\{ \frac{X_z(\ell_0) + \sqrt{\kappa}}{X_z(\ell_0) - \sqrt{\kappa}} \right\}^{\frac{1}{2\sqrt{\kappa}}}. \tag{26} \]

To provide some realistic estimates of \( \ell_\ast \), in Fig. 3 we plot \( \ell_\ast/\ell_0 \) as a function of the repulsive interaction potential \( V \), keeping fixed all the other system parameters (see the caption for the numerical values of the various parameters). The two plots we show correspond to different values of \( J' \). Also, we note a remarkable decrease of \( \ell_\ast \) with \( V/J \), and, in particular, a finite \( \ell_\ast \) even at extremely small values of \( V \), which correspond to negative values of \( J'_\ast \) and, thus, to an apparently ferromagnetic Kondo coupling between the impurity and the chain. In fact, in order for the Kondo coupling to be antiferromagnetic, and, thus, to correspond to a relevant boundary perturbation, one has to either have both \( J' \) and \( J'_\ast \) positive, or the former one positive, the latter negative. In our case, the RG equations in Eqs. (19), show how the \( \beta \)-function for the running coupling \( X(= G) \) is proportional to \( X'_z G \), rather than to \( G_z G \). Thus, what matters here is the fact that \( X_z - G_z = 1 - \frac{1}{2\xi} \) > 0, which makes \( X_z(\ell_0) \) positive even though \( G_z(\ell_0) \) is negative. As a result, even when both \( J' \) and \( J'_\ast \) are negative as it may happen, for instance, if one starts from a BH model with \( V \sim 0 \), one may still recover a Kondo-like RG flow and find a finite \( \ell_\ast \), as evidenced by the plots in Fig. 3.

Being an invariant quantity along the RG trajectories, here \( \ell_\ast \) plays the same role as \( \xi_K \) in the ordinary Kondo effect, that is, once the RG trajectories for the running strengths are constructed by using the system size \( \ell \) as driving variable, all the curves are expected to collapse onto each other, provided at each curve \( \ell \) is reascaled by the corresponding \( \ell_\ast \).

In fact, in the specific type of system we are focusing onto, that is, an ensemble of cold atoms loaded on a pertinently engineered optical lattice, it may be difficult to vary \( \ell \) by, in addition, keeping the filling constant (not to affect the parameters of the effective Luttinger liquid model Hamiltonian describing the system). Yet, one may resort to a fully complementary approach in which, as we highlight in the following, the length \( \ell \), as well as the filling \( f \), are kept fixed and, taking advantage of the scaling properties of the Kondo RG flow, one probes the scaling properties by varying \( \ell_\ast \). Indeed, from our Eqs. (22, 24, 26), one sees that in all cases of interest, the relation between \( \ell_\ast \) and the microscopic parameters characterizing the impurity Hamiltonian is known. As a result, one can in principle arbitrarily tune \( \ell_\ast \) at fixed \( \ell \) by varying the tunable system parameter. As we show in the following, this provides an alternative way for probing scaling behavior, more suitable to an optical lattice hosting a cold atom condensate. In order to express the integrated RG flow equations for the running parameters as a function of \( \ell \) and \( \ell_\ast \), it is sufficient to integrate the differential equations in Eqs. (19) from \( \ell_\ast \) up to \( \ell \). As a result, one obtains
From Eqs. (27, 28, 29), one therefore concludes that, once expressed in terms of $\ell/\ell_*$, the integrated RG flow for the running coupling strengths only depends on the parameter $\kappa$. Curves corresponding to the same values of $\kappa$ just collapse onto each other, independently of the values of all the other parameters.

To conclude the discussion of the impurity dynamics in the RG framework, it is worth stressing that, strictly speaking, we have so far focused onto the RG flow describing each $G_\alpha, G_{z,\alpha}$, with $\alpha = L, R$, by completely ignoring the effects of the other set of couplings. In fact, since in our system the impurity is coupled to two different chains, it makes a substantial difference whether the bare coupling are symmetric, or not, namely, whether $J'_L = J'_R$ and $J'_{z,L} = J'_{z,R}$, or not. In the following, we concisely review the effect of the asymmetry versus symmetry in the bare coupling, as well as the stability of the putative Kondo fixed point one infers from the perturbative RG approach.

We pause here for an important comment. As discussed in, in the spin chain realization of the Kondo model, one exactly retrives the equation of the conventional Kondo effect at $g = 1/2$ only after adding a frustrating second-neighbor interaction, thus resorting to the so-called $J_1 - J_2$ model Hamiltonian. In principle, the same would happen for the XXX-spin chain with nearest-neighbor interaction only, except that, strictly speaking, the correspondence is exactly realized only in the limit of an infinitely long chains. In the case of finite chains, the presence of a marginally Umklapp irrelevant operator may induce finite-size violations from Kondo scaling which, as stated above, disappear in the thermodynamic limit. Yet, as this point is mostly of interest because it may affect the precision of numerical calculations, we do not address it here and refer to Ref.[8] for a detailed discussion of this specific topic.

C. Symmetric versus non-symmetric impurity coupling and strongly coupled Kondo fixed point

In the case of an effective spin-1/2 impurity, the RG flow, as well as the nature of the corresponding stable Kondo fixed point, deeply depends on whether the bare couplings between the impurity and the chains are symmetric, or not.
Importantly enough, while the nature of the Kondo fixed point may be quite different in the two cases (two-channel versus one-channel spin-Kondo fixed point), one can still expect to be able to detect the onset of the Kondo regime and to probe the corresponding Kondo length by looking at the density-density correlations in real space, though the correlations themselves behave differently in the two cases. We discuss at length about this latter point in the next section. Here, we rather discuss about the nature of the Kondo fixed point in the two different situations, starting with the case of symmetric couplings between the impurity and the chains.

When $J'_L = J' R$ and $J_{zL} = J_{zR}$, since, to leading order in the running couplings, there is no mixing between the $L$- and the $R$- coupling strengths, the RG flow for both sets of parameters is realized according to the solutions we show in section [I]. The $L-R$ symmetry is not expected to be broken all the way down to the strongly coupled fixed point which, consequently, we identify with the two-channel spin-chain Kondo fixed point, in which the impurity is healed and the two chains have effectively joined into a single uniform chain. Due to the $L-R$ symmetry, one can readily show that all the allowed boundary operators at the strongly coupled fixed point are irrelevant, leading to the conclusion that the two-channel spin-Kondo fixed point is stable, in this case.

Concerning the effects of the asymmetry, on comparing the scale dimensions of the various impurity boundary operators, one expects them to be particularly relevant if the asymmetry is realized in the transverse Kondo coupling strengths, that is, if one has $J'_L \gg J'_R$. We assume that this is the case which, moving to the dimensionless couplings, implies $G_L(\ell_0) \gg G_R(\ell_0)$. Due to monotonicity of the integrated RG curves, we expect that this inequality keeps preserved along the integrated flow, that is, $G_L(\ell) \gg G_R(\ell)$ at any scale $\ell \geq \ell_0$. In analogy with the standard procedure used with multichannel Kondo effect with non-equivalent channels, one defines $\ell_*$ as the scale at which the larger running coupling $G_L(\ell)$ diverges, which is the signal of the onset of the nonperturbative regime. Due to the coupling asymmetry, we then expect $G_R(\ell_*) \ll 1$, that is, at the scale $\ell \sim \ell_*$, the system may be regarded as a semi-infinite chain at the left-hand side, undergoing Kondo effect with an isolated magnetic impurity, weakly interacting with a second semi-infinite chain, at the right-hand side. To infer the effects of the residual coupling, one may assume that, at $\ell \sim \ell_*$, the impurity is "re-absorbed" in the left-hand chain, so that the novel scenario will consist of the left-hand chain, with one additional site, connected with a link of strength $\sim G_R(\ell_*)$ to the endpoint of the right-hand chain. Within bosonization approach, the weak link Hamiltonian is given by:

$$V_{B}^{Asym} \sim -G_R(\ell_*) e^{-\frac{1}{g}(|\Phi_L(0)|-|\Phi_R(0)|)} + \text{h.c.}$$

$V_{B}^{Asym}$ has scaling dimension $\frac{1}{g}$. Depending on whether $g > 1$, or $g < 1$, it can therefore be either relevant, or irrelevant (or marginal if $g = 1$). When relevant, it drives the system towards a fixed point in which the weak link is healed. When irrelevant, the fixed point corresponds to the two disconnected chains. In either case, the residual flow takes place after the onset of Kondo screening. We therefore conclude that Kondo screening takes place in the left-hand chain only and, accordingly, one expect to be able to probe $\ell_*$ by just looking at the real space density-density correlations in that chain only. From the above discussion we therefore conclude that Kondo effect is actually realized at a chain with an effective spin-1/2 impurity whether the impurity couplings to the chains are symmetric, or not, though the fixed point the system is driven to along the RG trajectories can be different in the two cases.

### IV. DENSITY-DENSITY CORRELATIONS AND MEASUREMENT OF THE KONDO LENGTH

In analogy to the screening length $\xi_K$ in the standard Kondo effect, in the spin chain realization of the effect, the screening length $\ell_*$ is identified with the typical size of a cluster of spins fully screening the moment of the isolated magnetic impurity, either lying at one side of the impurity itself (in the one-channel version of the effect-side impurity at the end of a single spin chain), or surrounding the impurity on both sides (two channel version of the effect-impurity embedded within an otherwise uniform chain).

So far, $\ell_*$ showed itself as quite an elusive quantity to experimentally detect, both in electronic Kondo effect, as well as in spin Kondo effect. In this section, we propose to prove $\ell_*$ in the effective spin-1/2 XXZ chain describing the BH model, by measuring the integrated real-space density-density correlation functions (local spin-spin susceptibility). The idea of inferring informations on the Kondo length by looking at the scaling properties of the real-space local spin susceptibility was first put forward in Ref. [87]. In the specific context of lattice model Hamiltonians, the integrated real-space correlations have been proposed as a tool to extract $\xi_K$ in a quantum dot, regarded as a local Anderson model, interacting with itinerant lattice spinful fermions. Specifically, letting $S_G$ denote the spin of the isolated spin-1/2 impurity and $S_j$ the spin operator in the site $j$, assuming that the impurity is located at one of the endpoints of the chain and that the whole model, including the term describing the interaction between $S_G$ and the spins of the chain, is spin-rotational invariant, one may introduce the integrated real-space correlation function $\Sigma(x)$, defined
\[ \Sigma(x) = 1 + \sum_{y=1}^{x} \left[ \frac{\langle S_G \cdot S_y \rangle}{\langle S_G \cdot S_G \rangle} \right] . \]  

The basic idea is that the first zero of \( \Sigma(x) \) one encounters in moving from the location of the impurity, identifies the portion of the whole chains containing the spins that fully screen \( S_G \). Once one has found the solution of the equation \( \Sigma(x = x_\ast) = 0 \), one therefore naturally identifies \( x_\ast \) with \( \ell_\ast \). It is important to stress that this idea equally applies whether one is considering the spin impurity at just one side of the chain (one-channel spin chain Kondo), or embedded within the chain (two-channel spin chain Kondo). Thus, while in the following we mostly consider the two-channel case, we readily infer that our discussion applies also to the one-channel case.

To adapt the approach of Ref. \[83\] to our specific case, first of all, since our impurity is located at the center of the chain, we have to modify the definition of \( \Sigma(x) \) so to sum over \( j \) running from \( -x \) to \( x \). In addition, in our case both the bulk spin-spin interaction, as well as the effective Kondo interaction with the impurity, are not isotropic in the spin space. This requires modifying the definition of \( \Sigma(x) \), in analogy to what is done in Ref. \[83\] in the case in which an applied magnetic field breaks the spin rotational invariance. Thus, to probe \( \ell_\ast \), we use the integrated \( z \)-component of the spin correlation function, \( \Sigma_z(x) \), defined as

\[ \Sigma_z(x) = 1 + \sum_{y=-x}^{x} \left[ \frac{\langle S_G^z S_y^z \rangle - \langle S_G^z \rangle \langle S_y^z \rangle}{\langle S_G^z S_G^z \rangle - \langle S_G^z \rangle^2} \right] . \]  

In general, estimating \( \ell_\ast \) from \( \Sigma_z(x) \) would require exactly computing the spin-spin correlation functions by means of a numerical technique, such as it is done in Ref. \[83\] – nevertheless one in general expect that the estimate of \( \ell_\ast \) obtained using perturbative RG differ by a factor order of 1 from the one obtained by nonperturbative, numerical means. For the purpose of showing the consistency between the estimate of \( \ell_\ast \) from the spin-spin correlation functions and the results from the perturbative analysis of section [III.B], one therefore expects it to be sufficient to resort to a perturbative (in \( J'_z, J' \)) calculation of \( \Sigma_z(x) \), eventually improved by substituting the bare coupling strengths with the running ones, computed at an appropriate scale. To leading order in the impurity couplings, we obtain

\[ \langle S_G^z S_y^z \rangle = -J'_z \int_0^\beta d\tau G_{z,z}(y,1;\tau|\ell) , \quad (y > 0) \]  

\[ \langle S_G^z S_y^z \rangle = -J'_z \int_0^\beta d\tau G_{z,z}(y,1;\tau|\ell) , \quad (y < 0) \]  

with the finite-\( \tau \) correlation function \( G_{z,z}(x,x';\tau|\ell) \) defined in Eq. \([32]\). To incorporate scale effects in the result of Eq. \([33]\), we therefore replace the bare impurity coupling strengths with the running ones we derived in section [III.B] computed at an appropriate length scale, which we identify with the size \( x \) of the spin cluster effectively contributing to impurity screening. Therefore, referring to the dimensionless running coupling \( X_z(\lambda) \) defined in Eqs. \([13]\), we obtain

\[ \Sigma_z(x) = 1 - \frac{8J'_z(x)}{\pi u} \sum_{y=1}^{x} \int_0^\infty dw G_{z,z} \left( y,1;\frac{\pi uw}{\ell} \right) \]  

\[ = 1 - 8\varphi(\Delta) \left[ X_z(x) + \frac{1}{2g} - 1 \right] \ell \sum_{y=1}^{x} \int_0^\infty dw G_{z,z} \left( y,1;\frac{\pi uw}{\ell} \right) , \]  

with \( \varphi(\Delta) \) given by

\[ \varphi(\Delta) = \frac{\arccos \left( \frac{\Delta}{2} \right)}{\pi^2 \sqrt{1 - \left( \frac{\Delta}{2} \right)^2}} . \]  

Remarkably, \( \varphi(\Delta) \to 1 \) as \( \Delta \to 0 \). In Fig. \[1\] we show \( \Sigma_z(x) \) vs. \( x \) (only the positive part of the graph) for two paradigmatic situations: in Fig. \[1a\] we consider the absence of nearest-neighbor "bare" density-density interaction \( (V = 0) \), while in Fig. \[1b\] we consider a rather large value of \( V \) \( (V/J \sim 3) \). From the analysis of Ref. \[53\] one sees that, even at \( V = 0 \), a nonzero attractive density-density interaction between nearest-neighboring sites of the chain is actually induced by higher order (in \( t/U \)) virtual processes, which implies that, for \( V = 0 \), \( g \) keeps slightly higher than \( 1 \). At variance, for finite \( V \), \( g \) can be either larger, or smaller than \( 1 \), as it is the case in the plot in Fig. \[1b\]. In both cases we see the effect of "Friedel-like" oscillations in the density-density correlation, which eventually conspire
to set $\Sigma_z(x)$ to 0 at a scale $x \sim \ell_*$ (see the caption of the figures for more details on the numerical value of the various parameters).

In general, Eq.\(\text{(}22\text{)}\) as to be regarded within the context of the general scaling theory for $\Sigma_z(x)$. In our specific case, at variance with what happens in the "standard" Kondo problem of itinerant electrons in a metal magnetically interacting with an isolated impurity\(\text{,}\) the boundary action in Eq.\(\text{(}12\text{)}\) contains terms that are relevant as the length scale grows. In general, in this case a closed-form scaling formula for physical quantities cannot be inferred from the perturbative results, due to the proliferation of additional terms generated at higher orders in perturbation theory\(,\) Nevertheless, here one can still recover a pertinently adapted scaling equation, as only dimensionless contributions to $S^\text{B}_L$ effectively contribute $\Sigma_z(x)$ to any order in perturbation theory. The point is that, as we are considering a boundary operator in a bosonized theory in which the fields $\Phi_{L,R}(x,\tau)$ obey Neumann boundary conditions at the boundary, the fields $\Theta_{L,R}(0,\tau)$ appearing in the bosonized formula for $S^\text{B}_L, S^\text{B} _R$ in Eqs.\(\text{(}13\text{)}\) are pinned at a constant for any $\tau$. As a result, the corresponding contribution to the boundary interaction reduces to the one in Eq.\(\text{(}12\text{)}\), which is purely dimensionless and, therefore, marginal. As for what concerns the contribution $\propto J_{L,R}$, it is traded for a marginal one once one uses as running couplings the rescaled variables $X_L$ and $X_R$, rather than $J'_{L,R}$. Now, from Eqs.\(\text{(}13\text{)}\) we see that the bosonization formula for $S^\text{B}_j$ contains a term that has dimension $d_1 = 1$ and a term with dimension $d_\text{2} = (2g)^{-1}$. Taking into account the dynamics of the degrees of freedom of the chains comprised over a segment of length $x$, we therefore may make the scaling ansatz for $\Sigma_z$ in the form

$$\Sigma_z[x, \ell, X_z, X] = \tilde{\omega}_0 \left[ \frac{x}{\ell}, X_z, X \right] + \ell^{1-g} \tilde{\omega}_1 \left[ \frac{x}{\ell}, X_z, X \right],$$

(36)

with $\omega_0, \omega_1$ scaling functions. Now, we note that, due to the existence of the RG invariant $\kappa$, which relates to each other the running parameters $X_z$ and $X$ along the RG trajectories (Eq.\(\text{(}20\text{)}\) in the perturbative regime), we may trade $\tilde{\omega}_{0,1} \left[ \frac{x}{\ell}, X_z, X \right]$ for two functions $\omega_{0,1}$ of only $\frac{x}{\ell}$ and $X$. As a final result, Eq.\(\text{(}36\text{)}\) becomes

$$\Sigma_z[x, \ell, X_z, X] = \omega_0 \left[ \frac{x}{\ell}, X_z(x) \right] + \ell^{1-g} \omega_1 \left[ \frac{x}{\ell}, X_z(x) \right].$$

(37)

Eq.\(\text{(}37\text{)}\) provides the leading perturbative approximation at weak boundary coupling, as it can be easily checked from the explicit formula in Eq.\(\text{(}22\text{)}\). Eq.\(\text{(}37\text{)}\) illustrates how the function we explicitly use in our calculation can be regarded as just an approximation to the exact scaling function for $\Sigma_z(x)$. A more refined analytical treatment might in principle be done by considering higher-order contributions in perturbation theory in $S^\text{B}_G$. Alternatively, one might resort to a fully numerical approach, similar to the one used in Ref.\(\text{.}\) Yet, due to the absence of an intermediate-coupling phase transition in the Kondo effect\(,\) in our opinion resorting to a more sophisticated approach would improve the quantitative relation between the microscopic "bare" system parameters and the ones in the effective low-energy long-wavelength model Hamiltonian, without affecting the main qualitative conclusion about the Kondo healing length and its effects.
For this reason, here we prefer to rely on the perturbative RG approach extended to the correlation functions which, as we show before, already provide reliable and consistent results on the effects of the emergence of $\ell_*$ on the physical quantities.

V. CONCLUSIONS

In this paper we have studied the measurement of the Kondo screening length in systems of ultracold atoms in deep optical lattices. Our motivation relies primarily on the fact that the detection of the Kondo screening length from experimentally measurable quantities in general appears to be quite a challenging task. For this reason, we proposed to perform the measurement in cold atom setups, whose parameters can be, in principle, tuned in a controllable way to desired values.

Specifically, after reviewing the mapping between the BH model at half-filling with inhomogenous hopping amplitudes onto a spin chain Hamiltonian with Kondo-like magnetic impurities, we have proposed to extract the Kondo length from a suitable quantity obtained by integrating the real space density-density correlation functions. The corresponding estimates we recover for the Kondo length are eventually found to assume values definitely within the reach of present experiments (~ tens of lattice sites for typical values of the system parameters.) Concerning this point, a comment is in order for quantum-optics oriented readers: in a typical measurement of the Kondo effect at a magnetic impurity in a conducting metallic host, one has access to the Kondo temperature $T_K$, by just looking at the scale at which the resistance (or the conductance, in experiments in quantum dots) bends upwards, on lowering $T$. The very existence of the screening length $\xi_K$ is just inferred from the emergence of $T_K$ and from the applicability of one-parameter scaling to the Kondo regime, which yields $\xi_K = \hbar v_F/k_B T_K$. However the latter relation stems for the validity of the RG approach and, ultimately, directly probing $\xi_K$ in solid-state samples would correspond to verifying the scaling in the Kondo limit, which is what makes it hard to actually perform the measurement. At variance, as we comment for solid-state oriented readers, in the ultracold gases systems we investigate here, one can certainly study dynamics (e.g., tilting the system) but a stationary flow of atoms cannot be (so far) established, so that the measure of $T_K$ may be an hard task to achieve. Rather surprisingly, as our results highlight, it is the Kondo length which can be more easily directly detected in ultracold gases and our corresponding estimates (order of tens of lattice sites) appear to be rather encouraging in this direction.

Several interesting issues deserve in our opinion further work: as first, it would be desirable to compare the perturbative results we obtain in this paper with numerical, nonperturbative findings in the Bose-Hubbard chain, to determine the corresponding correction to the value of $\ell_*$. Even more importantly, we mostly assumed that it is possible to alter the hopping parameters in a finite region without affecting the others. This led us to infer, for instance, the existence of the ensuing even-odd effect – however, having two lasers with $\sigma \ll d$ is a condition that may be straightforwardly implementable. In this case, one has to deal with generic space-dependent hopping amplitudes $t_{j,j+1}$. It would therefore be of interest to address, very likely within a fully numerical approach, the fate of the even-odd effect in the presence of a small modulation in space of the outer hopping terms. In particular, a per se theoretically interesting issue would be the competition between an extended nonlocal central region and the occurrence of magnetic and/or nonmagnetic impurities in the chain. Another point to be addressed is that an on-site nonuniform potential may in principle be present (event though its effect may be reduced by hard wall confining potentials) and an interesting task is to determine the interplay between the Kondo length and the length scale of such an additional potential.

In conclusion, we believe that our results show that the possible realization of the setup proposed in this paper could pave the way to the study of magnetic impurities and, in perspective, to the experimental implementation of ultracold realizations of Kondo lattices and detection of the Kondo length, providing, at the same time, a chance for studying several interesting many-body problems in a controllable way.

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Appendix A: Effective weak link- and Kondo-Hamiltonians for a spin-1/2 XXZ spin chain

In this appendix we review the description of a region $G$, singled out by weakening two links in a XXZ spin chain, in terms of an effective low-energy Hamiltonian $H_G$. In particular, we show how, depending on whether the number of sites contained within $G$ is odd, or even, either $H_G$ coincides with the Kondo Hamiltonian $H_K$ in Eq.(9), or it describes a weak link between two "half-chains".67,68.
In general, Kondo effect in spin-1/2 chains has been studied for an isolated magnetic impurity (the “Kondo spin”), which may either lie at the end of the chain (boundary impurity), or at its middle (embedded impurity)\cite{35,36}. In the former case, the impurity can be realized by “weakening” one link of the chain, in the latter case, instead, it can be realized by weakening two links in the body of the chain. Following the discussion in section [11] of the main text, here we mostly focus on the latter case. In general, in a spin chain, impurities may be realized as extended objects, as well, that is, by regions containing two, or more, sites. Whether the Kondo physics is realized, or not, does actually depends on whether the level spectrum of the isolated impurity takes, or not, a degenerate ground state.

A doubly degenerate ground state is certainly realized in an extended region with an odd number of sites, without explicit breaking of “spin inversion” symmetry (that is, in the absence of local “magnetic fields”). For instance, let us consider a central region realized by three sites \((j = -1, 0, 1)\), lying between the weak links. Let the central region Hamiltonian be given by

\[
H_{3J}^{\text{Middle}} = -J \left( S_{-1}^+ S_0^- + S_0^+ S_1^- + \text{h.c.} \right) + J^z \left( S_{-1}^z S_0^- + S_0^z S_1^+ \right),
\]

and let the central region be connected to the left-hand chain (which, as in the main text, we denote by the label \(L\)), and to the right-hand chain (denoted by the label \(R\)) with the coupling Hamiltonian

\[
H_{\text{Coupling}} = - \left( J'_L S_{1,L}^+ S_{-1}^- + J'_R S_{1,R}^+ S_{1}^- + \text{h.c.} \right) + \left( J'_{z,L} S_{1,L}^z S_{1}^- + J'_{z,R} S_{1,R}^z S_{1}^- \right).
\]

A simple algebraic calculation shows that the ground state of \(H_{3J}^{\text{Middle}}\) is doubly degenerate and consists of the spin-1/2 doublet given by

\[
\frac{1}{2} \phi_2 = \frac{1}{\sqrt{2}} \left( \sin \left( \frac{\theta}{2} \right) | \uparrow \downarrow \uparrow \rangle + | \downarrow \uparrow \uparrow \rangle \right) + \sqrt{2} \cos \left( \frac{\theta}{2} \right) | \uparrow \downarrow \uparrow \rangle, \quad (A3)
\]

and

\[
-\frac{1}{2} \phi_2 = \frac{1}{\sqrt{2}} \left( \sin \left( \frac{\theta}{2} \right) | \downarrow \uparrow \downarrow \rangle + | \uparrow \downarrow \downarrow \rangle \right) + \sqrt{2} \cos \left( \frac{\theta}{2} \right) | \downarrow \uparrow \downarrow \rangle, \quad (A4)
\]

with

\[
\cos(\theta) = \frac{J_z}{\sqrt{2 J^2 + J_z}}, \quad \cos(\theta) = \frac{\sqrt{2} J_z}{\sqrt{2 J^2 + J_z}}, \quad (A5)
\]

whose energy is given by \(E^\phi_2 = -J_z - \sqrt{J_z^2 + 2J^2}\). Defining an effective spin-1/2 operator for the central region, \(S_G\), as

\[
S_G^+ = \frac{1}{2} \phi_2 \langle -\frac{1}{2} |, \quad S_G^- = \frac{1}{2} \sum_{b=\pm 1} b | b \frac{1}{2} \rangle \phi_2 \langle b \frac{1}{2} |, \quad (A6)
\]

allows to rewrite \(H_{3J}^{\text{Middle}} + H_{\text{Coupling}}\) as

\[
V_B^{3J} = - \left[ |J'_L \sin(\theta) S_{1,L}^+ + J'_R S_{1,R}^+ \rangle S_G^- + |J'_L S_{1,L}^- + J'_R S_{1,R}^- \rangle S_G^+ \right] + \cos(\theta) |J'_{z,L} S_{1,L}^z + J'_{z,R} S_{1,R}^z \rangle S_G^- \quad (A7)
\]

Thus, we see that we got back to the spin-1/2 spin-chain Kondo Hamiltonian, with a renormalization of the boundary couplings, according to

\[
J'_{L(R)} \rightarrow J'_{L(R)} \sin(\theta) = \frac{\sqrt{2} J'_{L(R)} J_z}{\sqrt{2 J^2 + J_z}}, \quad J'_{z,L(R)} \rightarrow J'_{z,L(R)} \cos(\theta) = \frac{J'_{z,L(R)} J_z}{\sqrt{2 J^2 + J_z}}. \quad (A8)
\]

A local magnetic field \(h\) may break the ground state degeneracy, thus leading, in principle, to the breakdown of Kondo effect. However, in analogy to what happens in a Kondo dot in the presence of an external magnetic field\cite{7,8}, Kondo physics should survive, at least as long as \(h \ll E_K\), with \(E_K(\sim k_B T_K)\) being the typical energy scale associated to the onset of Kondo physics.

At variance, when the central region is made by an even number of sites, the groundstate is not degenerate anymore. As a consequence, the central region should be regarded as a weak link between two chains. For instance, we may consider the case in which the central region is made by two sites. Using for the various parameters the same symbols we used above, performing a SW resummation, we obtain the effective boundary Hamiltonian

\[
V_B^{3J} = -\lambda_1 \left( S_{L,1}^+ S_{R,1}^- + S_{R,1}^+ S_{L,1}^- \right) - \lambda_2 S_{L,1}^z S_{R,1}^- \quad (A9)
\]
with

$$\lambda_\perp \sim \frac{(J')^2}{J + 2J_z}, \quad \lambda_z \sim \frac{(J_z)^2}{2J} \quad .$$

(A10)

By means of an RG analysis similar to the one we performed in Section III we conclude that, out of the two contributions to $V_{B}^{1}$ in Eq. (A9), the one $\propto \lambda_\perp$ has scaling dimension $h_\perp = 1/g$, while the one $\propto \lambda_z$ has scaling dimension $h_z = 2$. As we highlight in the main text, the term $\propto \lambda_\perp$ can become relevant, provided $g > 17,68$.

Appendix B: bosonization approach to impurities in the XXZ spin chain

In this section we review the bosonization approach to the XXZ spin chain as it was originally developed in Refs. [15,46]. As a starting point, we consider a single, homogeneous spin-1/2 XXZ spin chain, with $\ell$ sites, obeying open boundary conditions at its endpoints, described by the model Hamiltonian $H_{XXZ}$, given by

$$H_{XXZ} = -J \sum_{j=1}^{\ell-1} \left( S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^- \right) + J_z \sum_{j=1}^{\ell-1} S_j^z S_{j+1}^z \quad .$$

(B1)

The low-energy, long-wavelength dynamics of such a chain is described in terms of a spinless, real bosonic field $\Phi(x, \tau)$ and of its dual field $\Theta(x, \tau)$. The imaginary time action for $\Phi$ is given by

$$S_{E}[\Phi] = \frac{g}{4\pi} \int_0^\beta d\tau \int_0^\ell dx \left[ \frac{1}{u} \left( \frac{\partial \Phi}{\partial \tau} \right)^2 + u \left( \frac{\partial \Phi}{\partial x} \right)^2 \right] \quad ,$$

(B2)

where the constants $g, u$ are given by

$$g = \frac{\pi}{2(\pi - \arccos(\frac{J_z}{J}))}, \quad u = v_f \left[ \frac{\pi}{2} \sqrt{1 - \left( \frac{J_z}{J} \right)^2} \right] \quad ,$$

(B3)

with $v_f = 2dJ$, $d$ being the lattice step, and $\Delta = J_z/J$. The fields $\Phi$ and $\Theta$ are related to each other by the relations $\frac{\partial \Phi(x, \tau)}{\partial x} = \frac{1}{2} \frac{\partial \Theta(x, \tau)}{\partial \tau}$, and $\frac{\partial \Theta(x, \tau)}{\partial x} = \frac{1}{2} \frac{\partial \Phi(x, \tau)}{\partial \tau}$. A careful bosonization procedure shows that, in addition to the free Hamiltonian in Eq. (B2), an additional Sine-Gordon, Umklapp interaction arises, given by

$$H_{LG}^{SG} = -G_U \int_0^\ell dx \cos[2\sqrt{2}\Theta(x)] \quad .$$

(B4)

Since the scaling dimension of $H_{LG}^{SG}$ is $h_U = 4g$, it will be always irrelevant within the window of values of $g$ we are considering here, that is, $1/2 < g$. In fact, $H_{LG}^{SG}$ becomes marginally irrelevant at the “Heisenberg point”, $g = 1/2$, which deserves special attention, though we do not consider it here. Within the continuous bosonic field framework, the open boundary conditions of the chain are accounted for by imposing Neumann-like boundary conditions on the field $\Phi(x, \tau)$ at both boundaries, that is

$$\frac{\partial \Phi(0, \tau)}{\partial x} = \frac{\partial \Phi(\ell, \tau)}{\partial x} = 0 \quad .$$

(B5)

Eq. (B5) implies the following mode expansions for $\Phi(x, \tau)$ and $\Theta(x, \tau)$

$$\Phi(x, \tau) = \sqrt{\frac{g}{2}} \left\{ q - \frac{i \pi u\tau}{\ell} \right\} P + \sum_{n \neq 0} \frac{\alpha(n)}{n} \cos \left[ \frac{\pi n x}{\ell} \right] e^{-\frac{\pi n u\tau}{\ell}}$$

$$\Theta(x, \tau) = \sqrt{2g} \left\{ \theta + \frac{\pi x}{\ell} P + \sum_{n \neq 0} \frac{\alpha(n)}{n} \sin \left[ \frac{\pi n x}{\ell} \right] e^{-\frac{\pi n u\tau}{\ell}} \right\}$$

(B6)

with the normal modes satisfying the algebra

$$[g, p] = i \quad , \quad [\alpha(n), \alpha(n')] = n \delta_{n+n',0} \quad .$$

(B7)
The bosonization procedure allows for expressing the spin operators in terms of the \( \Phi \)- and \( \Theta \)-fields. The result is

\[
S_j^+ \rightarrow \left\{ e(-1)^j e^{\frac{i}{\sqrt{2}} \Phi(x_j, \tau)} + be^{\frac{i}{\sqrt{2}} \Phi(x_j, \tau) + i\sqrt{2} \Theta(x_j, \tau)} \right\}
\]

\[
S_j^z \rightarrow \left[ \frac{1}{\sqrt{2\pi}} \frac{\partial \Theta(x_j, \tau)}{\partial x} + a(-1)^j \sin[\sqrt{2}\Theta(x_j)] \right]. \quad (B8)
\]

The numerical parameters \( a, b, c \) in Eq.\( (B8) \) depend only on the anisotropy parameter \( \Delta = J_z/J^0 \). While their actual values is not essential to the RG analysis in section \( III \) it becomes important when computing the real-space correlation functions of the chain within bosonization approach, in which case one may refer to the extensive literature on the subject, as we do in Section \( IV \).

To employ the bosonization approach to study an impurity created between the \( L \) and the \( R \) chain, we start by doubling the construction outlined above, so to separately bosonize the two chains with open boundary conditions (which is appropriate in the limit of a weak interaction strength for either \( H_K \) in Eq.\( (9) \), or \( V_B^z \) in Eq.\( (A9) \)). Therefore, on introducing two pairs of conjugate bosonic fields \( \Phi_L, \Theta_L \) and \( \Phi_R, \Theta_R \) to describe the two chains, the corresponding Euclidean action is given by

\[
S_E[\Phi_L, \Phi_R] = \frac{g}{4\pi} \int_0^\beta d\tau \int_0^\ell dx \sum_{X=L,R} \left[ \frac{1}{u} \left( \frac{\partial \Phi_X}{\partial \tau} \right)^2 + u \left( \frac{\partial \Phi_X}{\partial x} \right)^2 \right], \quad (B9)
\]

supplemented with the boundary conditions

\[
\frac{\partial \Phi_L(x,0)}{\partial x} = \frac{\partial \Phi_L(\ell,\tau)}{\partial x} = 0, \quad \frac{\partial \Phi_R(x,0)}{\partial x} = \frac{\partial \Phi_R(\ell,\tau)}{\partial x} = 0. \quad (B10)
\]

Taking into account the bosonization recipe for the spin-1/2 operators, Eqs.\( (B8) \), one obtains that, in the case in which \( G \) contains an even number of sites (and is, therefore, described by the prototypical impurity Hamiltonian \( V_B^z \)), the effective weak link impurity between the two chains is described by the Euclidean action

\[
S_G^B = -\lambda_\perp \int_0^\beta d\tau \left\{ e^{\frac{i}{\sqrt{2}} \Phi_L(\tau)} + e^{-\frac{i}{\sqrt{2}} \Phi_R(\tau)} \right\} + \frac{\lambda_z}{2\pi^2} \int_0^\beta d\tau \frac{\partial \Theta_L(\tau)}{\partial x} \frac{\partial \Theta_R(\tau)}{\partial x}, \quad (B11)
\]

with \( \Phi_{L,R}(\tau) \equiv \Phi_{L,R}(0, \tau) \), and \( \Theta_{L,R}(\tau) \equiv \Theta_{L,R}(0, \tau) \). Similarly, in the case in which \( G \) contains an odd number of sites, in bosonic coordinates, the prototypical Kondo Hamiltonian \( H_K \) yields to the Euclidean action given by

\[
S_G^B = -\int_0^\beta d\tau \left\{ [J_L e^{\frac{i}{\sqrt{2}} \Phi_L(\tau)} + J_R e^{-\frac{i}{\sqrt{2}} \Phi_R(\tau)}] S_G^F + \text{h.c.} \right\} + \frac{1}{\sqrt{2\pi}} \int_0^\beta d\tau \left\{ [J'_{L,R} \frac{\partial \Theta_L(\tau)}{\partial x} + J'_{R,L} \frac{\partial \Theta_R(\tau)}{\partial x}] S_G^F \right\}. \quad (B12)
\]

Eqs.\( (B11),(B12) \) provide the starting point to perform the impurity RG analysis of Section \( III \).

Appendix C: Bosonization results for the correlation functions between spin operators at finite imaginary time

In this appendix we provide the generalization of the equal-time spin-spin correlation functions on an open chain, derived in Ref.\( [90] \), to the case in which the spin operators are computed at different imaginary times \( \tau, \tau' \). As discussed in the main text, such a generalization is a necessary step in order to compute the contributions to the spin correlations due to the impurity interaction in \( S_G^B \). The starting point is provided by the finite-\( \tau \) bosonic operators over a homogeneous, finite-size chain of length \( \ell \), which we provide in Eqs.\( (B8) \) of the main text. Inserting those formulas for \( \Phi(x, \tau) \) and \( \Theta(x, \tau) \) in the bosonic formulas in Eqs.\( (B8) \) and computing the imaginary-time ordered
correlation functions $G_{+-}(x, x'; \tau|\ell) = \langle T_\tau S^+_{\ell}(x) S^-_{\ell}(x') \rangle$ and $G_{zz}(x, x'; \tau|\ell) = \langle T_\tau S^z_{\ell}(x) S^z_{\ell}(x') \rangle$, one obtains:

$$G^{+-}(x, x'; \tau|\ell) =$$

$$c^2(-1)^{x-x'}\left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x}{\ell} \right) \right\} \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x'}{\ell} \right) \right\} \frac{2\ell}{\pi} \sinh \left( \frac{\pi}{2\ell} u \tau + i(x-x') \right) \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi}{2\ell} u \tau + i(x+x') \right) \right\}$$

$$+ b^2 \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x}{\ell} \right) \right\} \frac{2\ell}{\pi} \sin \left( \frac{\pi x'}{\ell} \right) \frac{2\ell}{\pi} \sinh \left( \frac{\pi}{2\ell} u \tau + i(x-x') \right) \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi}{2\ell} u \tau + i(x+x') \right) \right\}$$

$$+ b c \text{sgn}(x-x') \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x}{\ell} \right) \right\} \frac{2\ell}{\pi} \sin \left( \frac{\pi x'}{\ell} \right) \frac{2\ell}{\pi} \sinh \left( \frac{\pi}{2\ell} u \tau + i(x-x') \right) \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi}{2\ell} u \tau + i(x+x') \right) \right\}$$

$$\times \left[ \left( -1 \right)^{x} \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x}{\ell} \right) \right\} ^{-g} - \left( -1 \right)^{x'} \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x'}{\ell} \right) \right\} ^{-g} \right] ,$$

as well as

$$G^{zz}(x, x'; \tau|\ell) = - \frac{g}{4 \ell^2} \left\{ \left\{ \frac{1 - \cosh \left( \frac{\pi u \tau}{\ell} \right) \cos \left( \frac{\pi (x-x')}{\ell} \right)}{1 + \cos^2 \left( \frac{\pi (x-x')}{\ell} \right) - 2 \cos \left( \frac{\pi (x-x')}{\ell} \right) \cosh \left( \frac{\pi u \tau}{\ell} \right) + \sinh^2 \left( \frac{\pi u \tau}{\ell} \right) \right\} \right\}$$

$$+ \left\{ \frac{1 - \cosh \left( \frac{\pi u \tau}{\ell} \right) \cos \left( \frac{\pi (x+x')}{\ell} \right)}{1 + \cos^2 \left( \frac{\pi (x+x')}{\ell} \right) - 2 \cos \left( \frac{\pi (x+x')}{\ell} \right) \cosh \left( \frac{\pi u \tau}{\ell} \right) + \sinh^2 \left( \frac{\pi u \tau}{\ell} \right) \right\} \right\}$$

$$+ \frac{a^2}{2} \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x}{\ell} \right) \right\} ^{-g} \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x'}{\ell} \right) \right\} ^{-g} \times$$

$$\left\{ \sinh \left( \frac{\pi}{2\ell} [u \tau + i(x-x')] \right) \right\} ^{-2g} - \left\{ \sinh \left( \frac{\pi}{2\ell} [u \tau + i(x+x')] \right) \right\} ^{-2g} \times$$

$$- \frac{a^2 g}{2\ell} \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x'}{\ell} \right) \right\} ^{-g} \times$$

$$\left\{ \coth \left( \frac{\pi}{2\ell} [u \tau + i(x+x')] \right) - \coth \left( \frac{\pi}{2\ell} [u \tau - i(x+x')] \right) \right\}$$

$$- \frac{a^2 g}{2\ell} \left\{ \frac{2\ell}{\pi} \sin \left( \frac{\pi x'}{\ell} \right) \right\} ^{-g} \times$$

$$\left\{ \coth \left( \frac{\pi}{2\ell} [u \tau + i(x-x')] \right) - \coth \left( \frac{\pi}{2\ell} [u \tau - i(x-x')] \right) \right\} .$$

As stated above, Eqs. (C1, C2) provide the finite-$\tau$ generalization of Eqs. (8a,8b) of Ref. [90], to which they reduce in the $\tau \rightarrow 0$ limit.

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