Lower Bounds on the Quantum Capacity and Highest Error Exponent of General Memoryless Channels

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Abstract

Tradeoffs between the information rate and fidelity of quantum error-correcting codes are discussed. Quantum channels to be considered are those subject to independent errors and modeled as tensor products of copies of a general completely positive linear map, where the dimension of the underlying Hilbert space is a prime number. On such a quantum channel, the highest fidelity of a quantum error-correcting code of length \( n \) and rate \( R \) is proven to be lower bounded by \( 1 - \exp[-nE(R) + o(n)] \) for some function \( E(R) \). The \( E(R) \) is positive below some threshold \( R_0 \), a direct consequence of which is that \( R_0 \) is a lower bound on the quantum capacity. This is an extension of the author’s previous result [M. Hamada, Phys. Rev. A, vol. 65, 052305, 2002; LANL e-Print, quant-ph/0109114, 2001]. While it states the result for the depolarizing channel and a slight generalization of it (Pauli channels), the result of this work applies to general discrete memoryless channels, including channel models derived from a physical law of time evolution.

Keywords

Completely positive linear maps, error exponent, fidelity, symplectic geometry, the method of types, quantum capacity, quantum error-correcting codes.

I. Introduction

Quantum error-correcting codes (also called quantum codes or codes in this work) have attracted much attention as schemes that protect quantum states from decoherence during quantum computation. Shor invented the first code and stated that the ultimate goal would be to define the quantum analog of Shannon’s channel capacity, and find encoding schemes which approach this capacity [1]. On quantum memoryless channels, several bounds on the quantum capacity are known [2], [3], [4], [5], [6]. Good surveys on this problem are given in the introductory section of [3] and in [4]. There is a conjecture that the known upper bound based on the notion called coherent information is tight [5, Section VI]. On the other hand, the existing lower bounds seem to have left much room for improvement. For example, there is a lower bound on the capacity of the so-called depolarizing channel which can be proved by a random coding argument that evaluates the average performance over the whole ensemble of standard quantum error-correcting codes [8], [9], or by an argument using an entanglement purification protocol [10]. Shor and Smolin [11], [12] argued that this bound is not tight showing the existence of quantum codes, which are, in a sense, analogous to classical concatenated codes [13], of performance beyond it for a limited class of very noisy channels. The present author recently strengthened the result on standard quantum error-correcting codes [8], [9] in another direction, namely, established exponential convergence of fidelity of codes used on slight generalizations of the depolarizing channel [14]. In other words, using these simple channels, he illustrated that certain results and ideas around the error exponent problem in classical information theory, which has been a central issue [13], [14], [15], [16], [17], [18], [19], can be extended to quantum channels. The classical error exponent problem is, roughly speaking, to determine the function \( E_{cl}(R,W) \) such that the decoding error probability \( P_{n}^{*} \) of the best code of length \( n \) and rate \( R \) behaves like \( P_{n}^{*} \approx \exp[-nE_{cl}(R,W)] \)
on a channel $W$. The $E_{cl}(R,W)$, which is called the reliability function or the highest achievable error exponent of a channel $W$, is positive below the capacity of $W$, and decreasing in $R$. See, e.g., [14], [15] for precise definitions of the reliability function, [16] for a recent development, and [15], [20] for history. There is no reason to employ codes of rates near the capacity exclusively because the less $R$ is, the greater $E_{cl}(R,W)$ is, and hence the less $P^*_n \approx \exp[-nE_{cl}(R,W)]$ is exponentially.

The goal of this work is to show such exponential convergence of the fidelity of quantum error-correcting codes on a much wider class of channels. The channels to be considered here are those subject to independent errors and modeled as tensor products of copies of a general completely positive (CP) linear map [21], [22]. Our channel class includes those derived from a physical law of time evolution, or from master (Lindblad) equations [9], [23], [24], [25], though it is stipulated that the Hilbert spaces underlying channel have dimensions of prime numbers. One example of such channels is the amplitude-damping channel, which has often been discussed in the context of quantum error correction [1], [24], [26]. Despite the fact that this channel has often been treated as a model of quantum noise suffered during quantum computation, it has been not known whether standard quantum error-correcting codes work reliably at a positive rate for all large enough code lengths on this channel.

This work was inspired by Matsumoto and Uyematsu [27], who tried to prove a lower bound on the quantum capacity of a general memoryless channel using standard quantum error-correcting codes. However, their proof turned out to be wrong unfortunately [R. Matsumoto and T. Uyematsu, 24th Symposium on Information Theory and Its Applications, Kobe, Hyogo, Japan, Dec. 7, 2001]. In fact, they used the inequality similar to that in Lemma 5 below, which allegedly held for the standard fidelity measure (minimum fidelity, denoted by $F(C)$ in this paper), in [27], but this fails as shown in Example 3 below. Moreover, their bound [27] is smaller than Preskill’s lower bound [9, Section 7.16.2] for the so-called Pauli channels in general. It may be said that their contribution lies in the use of the estimate due to Calderbank et al., which will be given in Lemma 6 below in a slightly different form, in the present context. This is what this work has inherited from [27]. Thus, the question of whether quantum error-correcting codes work reliably on general channels or not is yet to be answered, which this paper is concerned with from an information-theoretic viewpoint. Specifically, exponential convergence of the fidelity of codes on general memoryless channels is established. The proof to be presented below exploits existing information-theoretic techniques, such as the method of types [15], [28], [29], [30], as well as a previously unused property of standard quantum-error-correcting codes.

We remark that in the setting where classical messages are sent over quantum channels, the error exponent problem has been discussed by Burnashev and Holevo [31] and Holevo [32] while this paper is concerned with the problem of preserving or transmitting quantum states in the presence of quantum noise. Note also that the error exponents of quantum error-detecting codes, which do not correct errors but only detect errors, have been discussed by Ashikhmin et al. [33].

The rest of the paper is organized as follows. Section II presents the main result. In Section III, a performance measure for codes, which is called the minimum average fidelity, is introduced and it is argued that evaluating this measure gives a good estimate for the standard fidelity. Section IV reviews the standard quantum codes, and Section V gives bounds on the minimum average fidelity of codes. Finally, the main result is proved in Section VI, which is followed by a concluding section. Appendices are given to prove a proposition, two lemmas, and an inequality between the proposed bound and the previously known one.

**II. Main Result**

As usual, all possible quantum operations and state changes, including quantum channels, are described in terms of completely positive (CP) linear maps [21], [22], [2], [5]. In this work, only trace-preserving completely positive (TPCP) linear maps are treated. Given a Hilbert space $\mathcal{H}$ of finite dimension, let $L(\mathcal{H})$ denote the set of linear operators on $\mathcal{H}$. In general, every CP linear map $\mathcal{M} : L(\mathcal{H}) \to L(\mathcal{H})$ has an operator-sum representation $\mathcal{M}(\rho) = \sum_{i \in I} M_i \rho M_i^\dagger$ with some set of operators...
\{M_i \in \mathcal{L}(H)\}_{i \in \mathcal{I}}\), which is not unique \cite{22,2}. When \(\mathcal{M}\) is specified by a set of operators \(\{M_i\}_{i \in \mathcal{I}}\) in this way, we write \(\mathcal{M} \sim \{M_i\}_{i \in \mathcal{I}}\). Note that we can always have \(|\mathcal{I}|\) equal to \((\dim H)^2\), including null operators in \(\{M_i\}_{i \in \mathcal{I}}\) if necessary \cite{22}.

Hereafter, \(H\) denotes an arbitrarily fixed Hilbert space of dimension \(d\), which is a prime number. A quantum channel is a sequence of TPCP linear maps \(\{A_n : \mathcal{L}(H^{\otimes n}) \to \mathcal{L}(H^{\otimes n})\}\); the map \(A_n\) with a fixed \(n\) is also called a channel. We want a large subspace \(\mathcal{C} \subseteq H^{\otimes n}\) every state vector in which remains almost unchanged after the effect of a channel followed by the action of some suitable recovery process. The recovery process is again described as a TPCP linear map \(\mathcal{R} : \mathcal{L}(H^{\otimes n}) \to \mathcal{L}(H^{\otimes n})\). A pair \((\mathcal{C}, \mathcal{R})\) consisting of such a subspace \(\mathcal{C}\) and a TPCP linear map \(\mathcal{R}\) is called a code and its performance is evaluated in terms of the minimum fidelity \cite{23,1,3}.

\[
F(\mathcal{C}, \mathcal{R}_n) = \min_{|\psi\rangle \in \mathcal{C}} \langle \psi | \mathcal{R}_n (|\psi\rangle \langle |\psi|) |\psi\rangle,
\]

where \(\mathcal{R}_n\) denotes the composition of \(A_n\) and \(\mathcal{R}\). Throughout, bras \(\langle |\cdot|\) and kets \(|\cdot\rangle\) are assumed normalized. Sometimes, a subspace \(\mathcal{C}\) alone is called a code assuming implicitly some recovery operator. Let \(F_{n,k}^*(\mathcal{A}_n)\) denote the supremum of \(F(\mathcal{C}, \mathcal{R}_n)\) such that there exists a code \((\mathcal{C}, \mathcal{R})\) with \(\log_d \dim \mathcal{C} \geq k\), where \(n\) is a positive integer and \(k\) is a nonnegative real number. This paper gives an exponential lower bound on \(F_{n,k}^*(\mathcal{A}_n)\), where for simplicity we state the result in the case where the channel is memoryless, i.e., when \(A_n = A^{\otimes n}\) for some \(A : \mathcal{L}(H) \to \mathcal{L}(H)\); the channel \(\{A_n = A^{\otimes n}\}\) is referred to as the memoryless channel \(A\).

The codes to be proven to have the desired performance are symplectic (stabilizer or additive) codes \cite{34,33,36,37,35}. In designing these codes, the following basis of \(\mathcal{L}(H)\), which has some nice algebraic properties, is used. Fix an orthonormal basis (ONB) \(\{|0\rangle, \ldots, |d-1\rangle\}\) of \(H\). The 'error basis' is \(\mathcal{N} = \{N_{ij} = X^i Z^j\}_{i,j \in \mathcal{X}}\) where \(\mathcal{X} = \{0, \ldots, d-1\}^2\) and the unitary operators \(X, Z \in \mathcal{L}(H)\) are defined by

\[
X|j\rangle = |(j-1) \mod d\rangle, \quad Z|j\rangle = \omega^j |j\rangle
\]

with \(\omega\) being a primitive \(d\)-th root of unity \cite[Section IV-15]{10}. When \(d = 2\), the basis elements become \(I, X, XZ, Z\), which are the same as the identity and three Pauli operators up to a phase factor. As usual, the classical Kullback-Leibler information (informational divergence or relative entropy) is denoted by \(D\) and entropy by \(H\) \cite{15,28,30}. Specifically, for probability distributions \(P\) and \(Q\) on a finite set \(\mathcal{X}\), we define \(D(P||Q)\) by \(D(P||Q) = \sum_{x \in \mathcal{X}} P(x) \log_d [P(x)/Q(x)]\) and \(H(Q) = -\sum_{x \in \mathcal{X}} Q(x) \log_d Q(x)\). By convention, we assume \(\log(a/0) = \infty\) for \(a > 0\) and \(0 \log 0 = 0 \log(0/0) = 0\).

To state our result, we associate a probability distribution with a channel.

**Definition 1:** For a memoryless channel \(A : \mathcal{L}(H) \to \mathcal{L}(H)\), we define a probability distribution \(P_A = P_{A,N}\) on \(\mathcal{X}\) as follows. For an operator-sum representation \(A \sim \{A_u\}_{u \in \mathcal{X}}\), expand \(A_u\) in terms of the error basis \(N\) as \(A_u = \sum_{v \in \mathcal{X}} a_{uv} N_v\), \(u \in \mathcal{X}\). Then,

\[
P_A(v) = P_{A,N}(v) = \sum_{u \in \mathcal{X}} |a_{uv}|^2, \quad v \in \mathcal{X}.
\]

**Remarks:** With \(A\) and \(N\) fixed, the \(P_A\) does not depend on the choice of \(\{A_u\}_{u \in \mathcal{X}}\) while it depends on \(N\) as well as \(A\). That \(\sum_{v \in \mathcal{X}} P(v) = 1\) readily follows from the trace-preserving condition \(\sum_{u \in \mathcal{X}} A_u^\dagger A_u = I\) and the property of the basis \(N\) that \(N_u^\dagger N_v = I\) if and only if \(u = v\) \cite{37}.

This paper’s main result is the following one.

**Theorem 1:** Let integers \(n, k\) and a real number \(R\) satisfy \(0 \leq k \leq \lceil Rn \rceil\) and \(0 \leq R \leq 1\) (a typical choice is \(k = \lceil Rn \rceil\) for an arbitrarily fixed rate \(R\)). Then, for any memoryless channel \(A : \mathcal{L}(H) \to \mathcal{L}(H),\)
and for any choice of the basis \{\ket{0}, \ldots, \ket{d-1}\} and \(\omega\) which determine \(N\), we have

\[
F^*_n,(\mathcal{A}^\otimes n) \geq 1 - 2d^2(n+1)^{2(d^2-1)}d^{-nE(R,P,A,N)}
\]

where

\[
E(R, P) = \min_Q[D(Q||P) + |1 - H(Q) - R|^+],
\]

\(|x|^+ = \max\{x, 0\}\), the minimization with respect to \(Q\) is over all probability distributions on \(\mathcal{X} = \{0, \ldots, d-1\}^2\).

An immediate consequence of the theorem is that the quantum capacity \([2, 3, 4, 5]\) of \(\mathcal{A}\) is lower bounded by

\[
\max_N[1 - H(P_{A,N})],
\]

(3)

where the maximum is over all choices of the basis \{\ket{0}, \ldots, \ket{d-1}\} of \(H\) and the primitive \(d\)-th root of unity \(\omega\). To be precise, the capacity of \(\{A_n\}\) is defined as the supremum of achievable rates on \(\{A_n\}\), where a rate \(R\) is said to be achievable if there exists a sequence of codes \(\{(C_n, R_n)\}\) such that \(\liminf_n \log_d \dim C_n/n \geq R\) and \(\lim_n F(C_n, R_n, A_n) = 1\). To see the bound, observe that \(E(R, P)\) is positive for \(R < 1 - H(P)\) due to the basic inequality \(D(Q||P) \geq 0\) whenever equality occurs if and only if \(Q = P\) \([13]\). The bound \(1 - H(P_A)\) appeared earlier in Preskill \([9, Section 7.16.2]\) in the case where \(d = 2\) and \((a_{uv})\) is diagonal. The restriction of \((a_{uv})\) being diagonal also exists in this author’s previous result \([12]\). Namely, it treated channels of the form \(\mathcal{A} \sim \{\sqrt{P(u)}N_u\} \forall x\) with some probability distribution \(P\) on \(\mathcal{X}\), which are sometimes called Pauli channels especially for \(d = 2\).

Another direct consequence of the theorem is

\[
\liminf_{n \to \infty} -\frac{1}{n} \log_d \left[1 - F^*_n,(\mathcal{A}^\otimes n)\right] \geq \max_N E(R, P_{A,N}),
\]

(4)

where the range of the maximization is the same as that for \([3]\) above. This bound resembles the random coding exponent \(E_r(R, W)\) of a classical channel \(W\). As mentioned in \([12]\), the function \(E(R, P)\) is, in fact, the ‘slided’ random coding exponent \(E_r(R+1, W)\) of some simple classical channel \(W\), i.e., the additive channel defined by \(W(y|x) = P(y-x), x, y \in \mathcal{X} = \mathbb{Z}/d\mathbb{Z}\), which becomes the quaternary (completely) symmetric channel \([11]\) in the case where \(d = 2\) and \(A\) is the depolarizing channel. In \([12]\), one can find another form of \(E\), which is the translation of an older form of classical random coding exponent \(E_r\) known in the literature (see, e.g., \([13]\), pp. 168, 192-193, and \([13, 14]\)) and suitable for computing \(E(R, P_{A,N})\) numerically (Fig. 1; also Fig. 1 of \([12]\)).

It should be remarked that, for the obvious reason, the bounds in \([3]\) and \([4]\) actually can be replaced by

\[
\max_{U,N}[1 - H(P_{U,A,N})] \quad \text{and} \quad \max_{U,N} E(R, P_{U,A,N}),
\]

where \(UA\) denotes the composition of \(A\) and \(U\), the map \(U\) ranges over all TPCP ones on \(L(H)\), and the range of \(N\) is the same as above. The role of \(U\) is preprocessing before the recovery operation \(R\), so that restricting the range of \(U\) to the set of easily implementable ones, say, to that of all unitary maps of the form \(U(\rho) = U\rho U^\dagger\) with some unitary operator on \(H\), may be reasonable.

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1In the literature, \(\liminf_n \log_d \dim C_n/n \geq R\) is sometimes replaced by \(\limsup_n \log_d \dim C_n/n \geq R\) (e.g., \([5]\)). Note also that in the definition of the quantum capacity (for transmission of subspaces) by Barnum et al. \([3]\), a slightly more general setting is assumed, i.e., two Hilbert spaces \(H_s\) and \(H_s\) are used instead of \(H\), but our bound is also valid in their setting because we can put \(H_s = H_s = H\). Apart from this difference, there is a seemingly different definition of the quantum capacity using entanglement fidelity, but actually they are the same \([3]\).
The function $E(R, P) = E(R, p)$ in the case where $d = 2$ and $P((0, 0)) = 1 - p$, $P(u) = p/3$ for $u \neq (0, 0)$, $u \in X = \{0, 1\}^2$, which applies to the depolarizing channel.

In the case of the depolarizing channel, the relationship between this paper’s bound (or that of [12]) and the previously known bounds are best understood with Fig. 1, which depicts $E(R, P_A, N) = E(R, p)$ in the case where $d = 2$ and $P((0, 0)) = 1 - p$, $P(u) = p/3$ for $u \neq (0, 0)$, $u \in X = \{0, 1\}^2$, which applies to the depolarizing channel $A \sim \{\sqrt{1 - p}I, \sqrt{p/3}X, \sqrt{p/3}XZ, \sqrt{p/3}Z\}$. For this channel, the known bound $1 - H_1(p)$ [3, Fig. 8], [8], [9], where

$$H_1(p) = -p \log_2(p) - (1 - p) \log_2(1 - p) + p \log_2 3,$$

appears in Fig. 1 as the curve on which the surface $E(R, p)$ meets the horizontal $pR$-plane. The Shor-Smolin code [10], [4] has improved this lower bound slightly for a limited range of $p$ around the point $(p^*, 0, 0)$, where the lower bound $1 - H_1(p)$ vanishes [$1 - H_1(p^*) = 0$, $p^* \approx 0.1893$].

Maximization of the bound $E(R, P_A, N)$ or $1 - H(P_A, N)$ with respect to the basis $N$ and the TPCP map $U$ seems troublesome and is largely left untouched except for the following simple case.

**Proposition 1:** Let a channel $A \sim \{A_x\}_{x \in X}$ be given by $A_x = \sqrt{Q(x)}\tilde{N}_x$, $x \in X$, where $\tilde{N}_{(i,j)} = \tilde{X}^i\tilde{Z}^j$, $\tilde{X}$ and $\tilde{Z}$ are defined by

$$\tilde{X}|b_j\rangle = |b_{(j-1) \mod d}\rangle, \quad \tilde{Z}|b_j\rangle = \tilde{\omega}^j|b_j\rangle,$$

similarly to (2), with $\{|b_j\rangle\}$ and $\tilde{\omega}$ being an ONB of $\mathcal{H}$ and a primitive $d$-th root of unity, respectively, and $Q$ is a probability distribution on $X$. Then, the maximum of $1 - H(P_{U, A, N})$ with respect to $N$ and $U$, i.e., with respect to $\{|0\rangle, \ldots, |d - 1\rangle\}$, $\omega$ and $U$, where $U : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ ranges over all unitary maps, is achieved by $|j\rangle = |b_j\rangle$, $j = 0, \ldots, d - 1$, $\omega = \tilde{\omega}$, and $U = I$, where $I$ denotes the identity map on $\mathcal{L}(\mathcal{H})$. \hfill \Box

A proof is given in Appendix A.
Next, we consider general channels. In a setting where elaborated coding schemes that rely on purification protocols are allowed, the lower bound $1 - H_1(p')$, as well as the Shor-Smolin improvement on this, for a general channel $A$ with $d = 2$ was known before this work [3, 12, 10, the last paragraph], where

$$p' = 1 - \max_\eta \langle \eta | I \otimes A | (| \Phi^+ \rangle \langle \Phi^+ |) | \eta \rangle,$$  \hspace{1cm} (5)$$

$| \Phi^+ \rangle = 2^{-1/2}(|00\rangle + |11\rangle)$, and the maximum is over all completely entangled states $\eta$. We compare our bound with the bound $1 - H_1(p')$, which is ‘almost’ the best among those previously known in the sense that the known improvement outperforms this only if $1 - 0.8115 = 0.1885 \leq p' \leq 1 - 0.8094 = 0.1906$ and the difference between $1 - H_1(p')$ and the improved one is at most $10^{-2}$ [4, Fig. 8]. As is proved in Appendix B, for every basis $N$ defined with (2) for some $\{|0\rangle, |1\rangle\}$, where $d = 2$, there exists some unitary map $U$ satisfying

$$1 - H(P_{UA,N}) \geq 1 - H(p').$$  \hspace{1cm} (6)$$

Roughly speaking, the gain of this paper’s bound comes from the fact the bound has the form $1 - H(P_{UA,N}) = 1 - H((1 - p', p_1, p_2, p_3)) = 1 - h(p') - p' H((p_1/p', p_2/p', p_3/p'))$, and for a fixed $p' = p_1 + p_2 + p_3 > 0$, its minimum is $1 - H_1(p')$ (reached when $p_1 = p_2 = p_3$); Bennett et al.’s scheme [3] loses information on $M = [I \otimes A](| \Phi^+ \rangle \langle \Phi^+ |)$ by ‘twirling’ (a random bilateral rotation), which increases entropy of $M$ as high as to $H_1(p')$.

The next example illustrates the advantage of this work.

Example 1. Let us consider the amplitude-damping channel whose Kraus operators are

$$A_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{bmatrix} \quad \text{and} \quad A_{(1,0)} = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

in matrix form with respect to the basis $\{|0\rangle, |1\rangle\}$, where $d = 2$ and $0 \leq \gamma \leq 1$. This channel has often been discussed as a reasonable model in the context of quantum error correction [4, Section 3.4.2], [24, Chapter 8], [26] while to this author’s knowledge, it was not known if any positive rates were achievable by standard quantum error-correcting (stabilizer) codes on this channel. The $A_{(0,0)}$ and $A_{(1,1)}$ can be expanded, respectively, as

$$A_{(0,0)} = \frac{1 + \sqrt{1 - \gamma}}{2} I + \frac{1 - \sqrt{1 - \gamma}}{2} Z$$

and

$$A_{(1,0)} = \frac{\sqrt{\gamma}}{2} (X - XZ).$$

Regarding $A_{(0,1)} = A_{(1,1)}$ as the null operator, we have

$$P_A((0,0)) = (2 - \gamma + 2\sqrt{1 - \gamma})/4, \quad P_A((1,0)) = \gamma/4,$$

$$P_A((0,1)) = (2 - \gamma - 2\sqrt{1 - \gamma})/4, \quad P_A((1,1)) = \gamma/4.$$  

Hence, our lower bound to the quantum capacity of this channel is

$$1 - H(P_A) = 1 - h\left(\frac{\gamma}{2}\right) - \left(1 - \frac{\gamma}{2}\right) h\left(\frac{1}{2} + \frac{\sqrt{1 - \gamma}}{2 - \gamma}\right) - \frac{\gamma}{2}.$$  \hspace{1cm} (7)$$

This bound actually achieves the maximum of $1 - H(P_{A,N'})$ with respect to $N'$ as can be checked by a direct calculation and the concavity of entropy.

This bound, together with the previously known one $1 - H_1(p')$ with (5), is plotted in Fig. 2, where $p'$ is calculated in Appendix B, Example 4.

□
This paper's bound $f = 1 - H(P_A)$ in (7), drawn as solid line, and the previously known one $g = 1 - H_1(p')$, dotted line, with $p' = 1 - (2 - \gamma + 2\sqrt{1 - \gamma})/4$ for the amplitude-damping channel in Example 1. Shor and Smolin [10], [4] succeeded in improving $1 - H_1(p')$ by an amount less than $10^{-2}$ for some values of $p'$ with $1 - H_1(p') < 10^{-2}$.

III. Minimum Average Fidelity

The minimum fidelity given in (1) is the simplest criterion for design of quantum error correction schemes. A known substitute for the minimum fidelity is the entanglement fidelity [2]. It turns out that yet another criterion is useful to establish Theorem 1: We seek codes of large minimum average fidelity. The minimum average fidelity $F_a(C) = F_a(C, RA_n)$ of a code $(C, R)$ used on a channel $A_n : L(H^\otimes n) \to L(H^\otimes n)$ is defined by

$$F_a(C) = \min_{B} \frac{1}{K} \sum_{\psi \in B} F(\psi, RA_n)$$

where $F(\psi, RA_n) = |\langle \psi | RA_\alpha (|\psi\rangle \langle \psi|) \psi \rangle|$, $K$ is the dimension of $C$, and the minimization with respect to $B$ is taken over all ONBs of $C$. Note that the minimum exists since the minimization can be written as that of a continuous function defined on a compact set. According to Schumacher [2], any average fidelity, and hence the minimum average fidelity are not less than the entanglement fidelity.

Employing the minimum average fidelity may need an account. In the previous work [12], Theorem 1 was proved for memoryless channels of the form $A \sim \{\sqrt{P(u)} N_u\}_{u \in X}$. In this case, $F(C)$ is trivially lower bounded by the sum of probabilities of errors that are correctable by $C$. The major difficulty in analysis on general channels lies in the fact that this bound is no longer true in general. However, as we will see in the sequel, a similar bound holds for a properly chosen symplectic quantum code if we replace $F$ by the minimum average fidelity $F_a$. Furthermore, an estimate for $F_a(C)$ automatically gives one for $F(C)$ by the following lemma.

Lemma 1: Let the minimum average fidelity $F_a(C) = F_a(C, RA_n)$ of a code $(C, R)$ used on a channel $A_n : L(H^\otimes n) \to L(H^\otimes n)$ satisfy

$$1 - F_a(C) \leq G$$

for some constant $G$, and assume $C$ has dimension $K \geq 2$. Then, there exists a $\lfloor K/2 \rfloor$-dimensional subspace $D$ of $C$ whose minimum fidelity $F(D) = F(D, RA_n)$ fulfills

$$1 - F(D) \leq 2G.$$
Proof. Let a normalized vector \( \psi_1 \) minimize \( F(\psi) = \langle \psi | \mathcal{R} \mathcal{A}_n (|\psi\rangle \langle \psi|) | \psi \rangle \) among those in \( C (= C_0) \), and let \( C_1 \) be the orthogonal complement of \( \text{span}\{\psi_1\} \) in \( C \), which means \( C = C_1 \oplus \text{span}\{\psi_1\} \). Next, let \( \psi_2 \) minimize \( F(\psi) \) among those in \( C_1 \), and let \( C_2 \) be the orthogonal complement of \( \text{span}\{\psi_1, \psi_2\} \) in \( C \), which means \( C = C_2 \oplus \text{span}\{\psi_1, \psi_2\} \). Continue in the same way until we obtain \( \psi_{[K/2]} \) and \( C_{[K/2]} \). Put \( D = C_{[K/2]} \). We annex an arbitrarily chosen ONB \( \{\psi_{[K/2]+1}, \ldots, \psi_K\} \) of \( D \) to \( \{\psi_1, \ldots, \psi_{[K/2]}\} \) to form an ONB of \( C \). Now put \( e(\psi) = 1 - F(\psi) \). Then, by construction,

\[
1 - F(D) \leq e(\psi_{[K/2]}) \leq \frac{e(\psi_1) + \cdots + e(\psi_{[K/2]})}{[K/2]} \leq 2 \frac{e(\psi_1) + \cdots + e(\psi_K)}{K} \leq 2G,
\]

as promised. \( \Box \)

This lemma and its proof are analogous to those known in the classical information theory [13], p. 140. A similar idea was used by Barnum et al. [5], where they adopted entanglement fidelity in place of minimum average fidelity. This lemma means that a properly chosen subcode \( D \) of \( C \) works without any loss of asymptotic performance.

IV. Codes based on Symplectic Geometry

To prove the theorem, we use symplectic quantum codes, so that we shall recall basic facts on them in this section. We can regard the index of \( N_{(i,j)} = X^i Z^j \), \((i, j) \in \mathcal{X} \), as a pair of elements from the field \( F = \mathbb{F}_d = \mathbb{Z}/d\mathbb{Z} \), the finite field consisting of \( d \) elements. From these, we obtain a basis \( \mathbb{N}_n = \{N_x \mid x \in (\mathbb{F}^2)^n\} \) of \( L(\mathbb{H}^{\otimes n}) \), where \( N_x = N_{x_1} \otimes \cdots \otimes N_{x_n} \) for \( x = (x_1, \ldots, x_n) \in (\mathbb{F}^2)^n \). We write \( N_J \) for \( \{N_x \in \mathbb{N}_n \mid x \in J\} \) where \( J \subseteq (\mathbb{F}^2)^n \). The index of a basis element

\[(u_1, v_1), \ldots, (u_n, v_n) \in (\mathbb{F}^2)^n\]

can be regarded as the plain \( 2n \)-dimensional vector

\[x = (u_1, v_1, \ldots, u_n, v_n) \in \mathbb{F}^{2n} \]

We can equip the vector space \( \mathbb{F}^{2n} \) over \( \mathbb{F} \) with a symplectic bilinear form (symplectic pairing, or inner product), which is defined by

\[ (x, y)_{\text{sp}} = \sum_{i=1}^{n} u_i v'_i - v_i u'_i \] \hspace{1cm} (9)

for the above \( x \) and \( y = (u'_1, v'_1, \ldots, u'_n, v'_n) \in \mathbb{F}^{2n} \) [13], [14]. Given a subspace \( L \subseteq \mathbb{F}^{2n} \), let

\[ L^\perp = \{x \in \mathbb{F}^{2n} \mid \forall y \in L, (x, y)_{\text{sp}} = 0\}. \]

**Lemma 2:** [34], [36] Let a subspace \( L \subseteq \mathbb{F}^{2n} \) satisfy

\[ L \subseteq L^\perp \quad \text{and} \quad \dim L = n - m. \] \hspace{1cm} (10)

Choose a set \( J \subseteq \mathbb{F}^{2n} \) such that

\[ \{y - x \mid x \in J, y \in J\} \subseteq (L^\perp \setminus L)^c, \] \hspace{1cm} (11)
where the superscript $C$ denotes complement. Then, there exist $d^{n-m}$ subspaces of the form
\[
\{ \psi \in H^\otimes n \mid \forall M \in N_L, \ M\psi = \tau(M)\psi \}\tag{12}
\]
each of which has dimension $d^m$, where $\tau(M)$ are scalars, and hence eigenvalues of $M \in N_L$. The direct sum of these subspaces is the whole space $H^\otimes n$ and each subspace together with a suitable recovery operator serves as an $N_L$-correcting quantum code.

Remarks. A precise definition of $N_L$-correcting codes can be found in Section III of [26] and the above lemma has been verified with Theorem III.2 therein. Most constructions of quantum error-correcting codes relies on this lemma, which is valid even if $d$ is a prime other than two [37], [38], [39]; related topics have been discussed in [45], [46], [47]. In this paper, we call the quantum codes in Lemma 3 symplectic quantum codes or symplectic codes while Rains [39] indicates $L$ by the latter term. Symplectic codes are often called additive codes [35], [36] or stabilizer codes [34], [8], and the set $N_L$ in the lemma is called a stabilizer in the literature. \hfill \Box

The next lemma, which immediately follows from Lemma 2, will be used in the proof of Theorem 1 below.

Lemma 3: [34], [30] As in Lemma 2, assume a subspace $L \subseteq F^{2n}$ satisfies (10). In addition, let $J_0 \subseteq F^{2n}$ be a set satisfying
\[
\forall x, y \in J_0, \ [y - x \in L^\perp \Rightarrow x = y]. \tag{13}
\]
Then, the condition (11) is fulfilled, so that the $d^{n-m}$ codes of the form (12) are $d^m$-dimensional $N_{J_0}$-correcting codes. \hfill \Box

Assumption. When we speak of an $N_j$-correcting symplectic code $C$, the recovery operator $R$ for the code is always the one presented by Knill and Laflamme [26], proof of Theorem III.2.

Note that the $R$ is determined from $C$ and $J$ in general. In the present case where $C$ is a symplectic quantum code in Lemma 2 (or Lemma 4 below), the recovery operator $R$ can be written explicitly, viz., $R = \{\Pi_{0} \} \cup \{N_{r}\Pi_{r}\}_{r \in J_0}$, where $\Pi_{0}$ is the projection onto $N_{J_0}C = \{N_{r}\psi \mid \psi \in C\}$, and $\Pi_{0}$ is the projection onto the orthogonal complement of $\bigoplus_{r \in J_0} N_{r}C$ in $H^\otimes n$. The premise (13) of Lemma 3 can be restated as that $J_0$ is a set of representatives of cosets of $L^\perp$ in $F^{2n}$. When the code is used on a channel $A_n \sim \{\sqrt{P_n(x)}N_x\}$, a natural choice for $J_0$ would be a set consisting of representatives each of which maximizes the probability $P_n(x)$ in the coset $\{36\}$ since it is analogous to maximum likelihood decoding, which is an optimum strategy for classical coding (see Slepian [18] or any textbook of information theory). In the proof below, we choose another set of representatives, the classical counterpart of which (minimum entropy decoding) asymptotically yields the same performance as maximum likelihood decoding [15], [29].

V. Bound on Minimum Average Fidelity

A. Plan of Proof

Our strategy for proving Theorem 1 is to employ the random coding technique known in classical information theory [13], [14], [49], [15]. A typical random coding argument goes as follows. Suppose $F'(C)$ is a measure of performance, which is the minimum average fidelity in our case, of a code $C$ and we want to prove the existence of a code $C$ with $F'(C) \geq G$. We take some ensemble $E$ of codes, and evaluate the ensemble average $|E|^{-1} \sum_{C \in E} F'(C)$. If the average is lower bounded by $G$, then we can conclude at least one code $C$ in $E$ has performance not smaller than $G$. In what follows, we will use this proof method twice, that is, first, with $L$ fixed and $E$ being the set, say $E(L)$, of $d^{n-m}$ subspaces in Lemma 2 or 3 and second, with $E$ consisting of all $L$ satisfying (10).
B. Preskill’s Lower Bound on Fidelity

Preskill showed an interesting lower bounds on the minimum fidelity of a code used on quantum channels, which will be presented in a slightly different form here.

Lemma 4: For a channel $\mathcal{A}_n : \mathbb{L}(\mathbb{H}^\otimes n) \to \mathbb{L}(\mathbb{H}^\otimes n)$, an $N_J$-correcting code ($\mathcal{C} \subseteq \mathbb{H}^\otimes n$, $\mathcal{R}$) and any state $|\psi\rangle \in \mathcal{C}$, the fidelity $F(\psi) = \langle \psi | \mathcal{R} \mathcal{A}_n (|\psi\rangle \langle \psi|) |\psi\rangle$ is bounded by

$$F(\psi) \geq 1 - \sum_{x \in \mathcal{X}^n} \langle \psi | B_x^\dagger B_x |\psi\rangle$$

where $B_x = \sum_{y \in \mathcal{Y}} a_{xy} N_y$, $x \in \mathcal{X}^n$.

This is Preskill’s lower bound [9], Section 7.4.1, Eq. (7.58), and the above form can be obtained by rewriting the channel, which was described in terms of unitary evolution of a state of an enlarged system and a partial trace operation, into an operator-sum representation. In Appendix C, an alternative proof which uses only operator-sum representations is presented.

C. Minimum Average Fidelity Bound for Symplectic Codes

To evaluate the minimum average fidelity of codes, we first associate a sequence of probability distributions $\{P_{A_n}\}$ with the channel $\{\mathcal{A}_n\}$ on which codes are to be evaluated.

Definition 2: For each $n$, let $\mathcal{A}_n \sim \{A_x^{(n)}\}_{x \in \mathcal{X}^n}$, expand $A_x^{(n)} = \sum_{y \in \mathcal{Y}} a_{xy} N_y$, $x \in \mathcal{X}^n$, and define a probability distribution $P_{A_n}$ on $\mathcal{X}^n$ by

$$P_{A_n}(y) = \sum_x |a_{xy}|^2, \quad y \in \mathcal{X}^n.$$

That $\sum_{x \in \mathcal{X}^n} P_{A_n}(x) = 1$ readily follows, again, from the trace-preserving condition $\sum_{x \in \mathcal{X}^n} A_x^{(n)\dagger} A_x^{(n)} = I$ and the property of the basis $N_n$ that $N_x^\dagger N_y = I$ if and only if $x = y$ [37].

Example 2. Let $\{\mathcal{A}_n\}$ be a memoryless channel $\mathcal{A}_n = A_0^{\otimes n}$, $n = 1, 2, \ldots$. It is easy to see that

$$P_{A_n}(y_1, \ldots, y_n) = \prod_{i=1}^n P_A(y_i)$$

where $P_A = P$ has already appeared in Definition [1] □

The next is a result of the first application of random coding technique in this paper.

Lemma 5: As in Lemma [2], let a subspace $L \subseteq \mathbb{F}^{2^n}$ satisfy [10] and [11] with some $J \subseteq \mathbb{F}^{2^n}$, and let $\mathcal{A}_n : \mathbb{L}(\mathbb{H}^\otimes n) \to \mathbb{L}(\mathbb{H}^\otimes n)$ be a channel (TPCP linear map). With $L$, $J$ and $\mathcal{A}_n$ fixed, let $\mathcal{C}(L)$ achieve the maximum of $F_a(\mathcal{C}) = F_a(\mathcal{C}, \mathcal{R} \mathcal{A}_n)$ in $\mathcal{E}(L)$ (see Section [3]-A), i.e., the maximum among the $d^{n-m}$ symplectic codes associated with $L$ as in Lemma [2] or [3]. Then,

$$1 - F_a(\mathcal{C}(L)) \leq \sum_{x \notin J} P_{A_n}(x).$$

Proof. Taking the averages over an ONB $\mathcal{B}$ of a code $\mathcal{C}$ of both sides of the inequality in Lemma [1], we have

$$1 - \frac{1}{d^m} \sum_{\psi \in \mathcal{B}} F(\psi) \leq \frac{1}{d^m} \sum_{\psi \in \mathcal{B}} \sum_x \langle \psi | B_x^\dagger B_x |\psi\rangle,$$

This holds for all ONBs $\mathcal{B}$ of $\mathcal{C}$ including the worst one $\mathcal{B}_*(\mathcal{C})$, which is a minimizer for $F_a(\mathcal{C})$, so that

$$1 - F_a(\mathcal{C}) \leq \frac{1}{d^m} \sum_{\psi \in \mathcal{B}_*(\mathcal{C})} \sum_x \langle \psi | B_x^\dagger B_x |\psi\rangle.$$
With $L$ fixed, we have $d^{n-m}$ choices for $\mathcal{C}$. Taking the averages of both sides of the above inequality over these choices, we obtain

$$\frac{1}{d^{n-m}} \sum_{\mathcal{C}} [1 - F_a(\mathcal{C})] \leq \frac{1}{d^{n-m}} \sum_{\mathcal{C}} \frac{1}{d^m} \sum_{\psi \in B_x(\mathcal{C})} \sum_{x} \langle \psi | B_x^† B_x | \psi \rangle$$

$$= \frac{1}{d^n} \sum_{x} \sum_{\mathcal{C}} \sum_{\psi \in B_x(\mathcal{C})} \langle \psi | B_x^† B_x | \psi \rangle$$

$$= \frac{1}{d^n} \sum_{x} \mathrm{Tr} B_x^† B_x$$

$$= \frac{1}{d^n} \sum_{x} \sum_{y,z \in J^c} a_{xy}^* N_y a_{xz} N_z$$

$$= \sum_{x} \sum_{y \in J^c} |a_{xy}|^2$$

$$= \sum_{y \in J^c} P_{A_n}(y),$$

where we have used the fact that the $d^{n-m}$ subspaces $\mathcal{C}$ sum to $H^\otimes n$ orthogonally for the second equality, and the property of error basis $N_n$ that $\mathrm{Tr} N_y^† N_z = d^n \delta_{yz}$ for the fourth equality \[^{\text{[37]}}\] \(\square\). Hence, at least, one code $(\mathcal{C}, \mathcal{R})$ has the promised minimum average fidelity.

Example 3. To illustrate the difference between the minimum average fidelity $F_a$ and minimum fidelity $F$ as well as the significance of Lemma \[^{3}\] \(\square\), let us consider again the amplitude-damping channel discussed in Example 1 and evaluate some small codes on this channel. Let $n = 2$ and $m = 1$. In this example, we denote a vector $(u_1, v_1, u_2, v_2) \in \mathbb{F}^4$ simply by $u_1v_1u_2v_2$. Let $L = \{0000, 0101\}$. Then, $N_L = \{I \otimes I, Z \otimes Z\}$, and we have two symplectic codes $\mathcal{C}_0 = \text{span}\{00, 11\}$ and $\mathcal{C}_1 = \text{span}\{01, 10\}$, where $|00\rangle = |0 \rangle \otimes |0 \rangle$ and so on. It is easy to check that the cosets of $L^\perp$ in $\mathbb{F}^4$ are

$$L^\perp = \{0000, 0101, 1010, 1111, 0001, 0100, 1011, 1110\}$$

and

$$h_1 + L^\perp = \{1000, 1101, 0010, 0111, 1001, 1100, 0111, 1110, 1111\},$$

where $h_1 = 1000$. Let $\Pi_0$ and $\Pi_1$ denote the projections onto $\mathcal{C}_0$ and $\mathcal{C}_1$, respectively. Putting $J_0 = \{0000, h_1\}$ and $J = h_1 + L = \{0000, 0101, 1000, 1101\}$, we see that both $(\mathcal{C}_0, \mathcal{R}_0)$ and $(\mathcal{C}_1, \mathcal{R}_1)$, where $\mathcal{R}_0 \sim \{\Pi_0, N_{h_1}^† \Pi_1\}$ and $\mathcal{R}_1 \sim \{\Pi_1, N_{h_1}^† \Pi_0\}$, are $N_{J_0}$-correcting as well as $N_J$-correcting from Lemmas \[^{3}\] and \[^{4}\] or directly from Lemma \[^{2}\] (recall also the general form of $\mathcal{R}$ for a symplectic code was given in the the last paragraph of Section \[^{IV}\]). If we prepare an input state $|\psi\rangle = x|00\rangle + y|11\rangle \in \mathcal{C}_0$, then, the fidelity $F(\psi) = \langle \psi | \mathcal{R}_0 A^\otimes 2 (|\psi\rangle\langle \psi |) | \psi \rangle$ can be calculated as $1 - \gamma yy^*$. This implies the minimum fidelity is $F(\mathcal{C}_0) = 1 - \gamma$ while the minimum average fidelity is $F_a(\mathcal{C}_0) = 1 - \gamma/2$. In a similar way, evaluating $F(\mathcal{C}_1)$ results in $F(\mathcal{C}_1) = 1 - \gamma$ and $F_a(\mathcal{C}_1) = 1 - \gamma/2$. One the other hand, the bound in Lemma \[^{3}\] states $F_a(\mathcal{C}(L)) \geq 1 - \sum_{z \in J} P_{A^\otimes 2}(z) = \sum_{z \in J} P_A(z) = 1 - 3\gamma/4$, where $P_A^2$ is the product measure obtained from $P_A$ as in \[^{IV}\]. This is an example for which the inequality in Lemma \[^{3}\] is true but that with $F_a$ replaced by $F$ fails. \(\square\)
VI. Proof of Theorem I

We put $P = P_{AN}$. Since the bound in the theorem is trivial when $k \geq n-1$, we assume $m = k+1 < n$. What we want is a code $(\mathcal{D}, \mathcal{R})$ with dimension $d^k$ whose minimum fidelity is lower bounded by $1 - 2d^2(n + 1)^2(d^2 - 1)d^{-nE(R,P)}$. To show the existence of such a code, it is enough to prove

$$1 - F_n(\mathcal{C}(L)) \leq d^2(n + 1)^2(d^2 - 1)d^{-nE(R,P)} \quad (15)$$

for some $L$ with $\dim L = n - m = n - (k + 1)$ and some choice of $J_0$ in Lemma 3, where $\mathcal{C}(L)$ achieves the maximum of $F_n(\mathcal{C}) = F_n(\mathcal{C}, \mathcal{R}, \mathcal{A})$ among the $d^{n-m}$ symplectic codes associated with $L$ as in Lemma 3 since we have Lemma 4. Recall that the probability distribution $P_{An}$ for the memoryless channel $\mathcal{A}$ has a product form as in (14), which is denoted by $P_n$ in this proof.

We employ the method of types [15], [28], [29], [30], on which a few basic facts to be used are collected here. For $x = (x_1, \ldots, x_n) \in \mathcal{X}^n$, define a probability distribution $P_x$ on $\mathcal{X}$ by

$$P_x(u) = \frac{|\{i \mid 1 \leq i \leq n, x_i = u\}|}{n}, \quad u \in \mathcal{X},$$

which is called the type (empirical distribution) of $x$. With $\mathcal{X}$ fixed, the set of all possible types of sequences from $\mathcal{X}^n$ is denoted by $\mathcal{Q}_n(\mathcal{X})$ or simply by $\mathcal{Q}_n$. For a type $Q \in \mathcal{Q}_n$, $\mathcal{T}_Q^n$ is defined as $\{x \in \mathcal{X}^n \mid P_x = Q\}$. In what follows, we use

$$|\mathcal{Q}_n| \leq (n + 1)^{|\mathcal{X}| - 1}, \quad (16)$$

where $|\mathcal{X}| = d^2$ in the present case, and

$$\forall Q \in \mathcal{Q}_n, \ |\mathcal{T}_Q^n| \leq d^{nH(Q)}. \quad (17)$$

Note that if $x \in \mathcal{X}^n$ has type $Q$, then $P^n(x) = \prod_{a \in \mathcal{X}} P(a)^{nQ(a)} = \exp_d \{-n[H(Q) + D(Q||P)]\}$.

We apply Lemma 3 choosing $J_0$ as follows. Since $\dim L = n - m$, we have $\dim L^\perp = n + m$ [13], [50]. From each of the $d^{n-m}$ cosets of $L^\perp$ in $\mathbb{F}_2^n$, select a vector that minimizes $H(P_x)$, i.e., a vector $x$ satisfying $H(P_x) \leq H(P_y)$ for any $y$ in the coset. Let $J_0(L)$ denote the set of the $d^{n-m}$ selected vectors. This selection uses the idea of the minimum entropy decoder known in the classical information theory literature [29]. Let

$$A = \{L \subseteq \mathbb{F}_2^n \mid L \text{ linear, } L \subseteq L^\perp, \dim L = n - m\}$$

and for each $L \in A$, let $\mathcal{C}(L)$ be the best $N_{J_0(L)}$-correcting code in $\mathcal{E}(L)$. Putting

$$\mathcal{F} = \frac{1}{|A|} \sum_{L \in A} F_n(\mathcal{C}(L)), $$

we will show that $1 - \mathcal{F}$ is bounded from above by $d^2(n + 1)^2(d^2 - 1)d^{-nE(R,P)}$, which will ensure (13) for some $L$ and hence, establish the theorem by the argument at the beginning of this proof. This is our second application of the random coding method.

The $\{0, 1\}$-valued indicator function $1[T]$ equals 1 if the statement $T$ is true and equals 0 otherwise. From Lemma 3, we have

$$1 - \mathcal{F} \leq \frac{1}{|A|} \sum_{L \in A} \sum_{x \notin J_0(L)} P^n(x)$$

$$= \frac{1}{|A|} \sum_{L \in A} \sum_{x \in \mathbb{F}_2^n} P^n(x) 1[x \notin J_0(L)]$$

$$= \sum_{x \in \mathbb{F}_2^n} P^n(x) \frac{|B(x)|}{|A|}, \quad (18)$$
where we have put
\[ B(x) = \{ L \in A \mid x \notin J_0(L) \}, \quad x \in \mathbb{F}^{2n}. \]

The fraction \(|B(x)|/|A|\) is trivially bounded as
\[ \frac{|B(x)|}{|A|} \leq 1, \quad x \in \mathbb{F}^{2n}. \tag{19} \]

We use the next lemma, a proof of which is given in Appendix D.

**Lemma 6:** Let
\[ A(x) = \{ L \in A \mid x \in L^\perp \setminus \{0\} \}. \]

Then, \(|A(0)| = 0\) and
\[ \frac{|A(x)|}{|A|} = \frac{d^{n+m} - 1}{d^{2n} - 1} \leq \frac{1}{d^{n-m}}, \quad x \in \mathbb{F}^{2n}, \ x \neq 0. \tag{20} \]

**Remarks.** Note that \( A \) is not empty since any \((n - m)\)-dimensional subspace of
\[ \{(x_1, 0, x_3, 0, \ldots, x_{2n-1}, 0) \in \mathbb{F}^{2n} \mid x_1, x_3, \ldots, x_{2n-1} \in \mathbb{F}\} \]
is contained in \( A \). This lemma is essentially due to Calderbank et al. \cite{35} who have used it with \( A(x) \) replaced by \( \{ L \in A' \mid x \in L^\perp \setminus L \} \) for some \( A' \subseteq A \) to prove the Gilbert-Varshamov-type bound for quantum codes. Matsumoto and Uyematsu \cite{27} proved Lemma 6 with \( A(x) \) replaced by \( \{ L \in A \mid x \in L^\perp \setminus L \} \) using the Witt lemma explicitly \cite{43}, \cite{44}. The present definition of \( A(x) \) makes the argument easier. \qedsymbol

Since \( B(x) \subseteq \{ L \in A \mid \exists y \in \mathbb{F}^{2n}, H(P_y) \leq H(P_x), y - x \in L^\perp \setminus \{0\} \} \) from the design of \( J_0(L) \) specified above (cf. \cite{49}),
\[ |B(x)| \leq \sum_{y \in \mathbb{F}^{2n} : H(P_y) \leq H(P_x), y \neq x} |A(y - x)| \leq \sum_{y \in \mathbb{F}^{2n} : H(P_y) \leq H(P_x), y \neq x} |A|d^{-n+m}, \tag{21} \]
where we have used (20) for the latter inequality. Combining (18), (19) and (21), we can proceed as follows with the aid of the basic inequalities in (16) and (17) as well as the inequality \( \min\{a + b, 1\} \leq \]
This implies at least one $L$ satisfies (14), and the proof is complete owing to Lemma 1.

\[ \Box \]

**VII. Concluding Remarks**

This paper provided evidence, from an information theoretic viewpoint, that standard quantum error correction schemes work reliably in the presence of quantum noise, the effects of which are modeled as general completely positive linear maps. What is technically new is evaluating the minimum average fidelity over all eigenspaces of a stabilizer $N_L$, which yields a good estimate for the minimum fidelity of codes. The thus obtained fact (Lemma 3) allowed us to derive the main result in a manner familiar in information theory. Likewise, based on Lemma 3 and with another classical technique, a high-rate of codes. The thus obtained fact (Lemma 5) allowed us to derive the main result in a manner familiar in information theory. Likewise, based on Lemma 3 and with another classical technique, a high-rate of codes.

Although this paper’s lower bound on the capacity is the best among those known except for a few cases, it is important to recognize that this paper’s lower bound is not tight in general. In this sense, Shor and Smolin [14], [4] have gone further. Specifically, Shor and Smolin exploited the ‘degeneracy’ of error-correcting codes to present a lower bound on the capacity of the depolarizing channel $A \sim \{ \sqrt{1-p} I, \sqrt{p/3} X, \sqrt{p/3} X Z, \sqrt{p/3} Z \}$ such that their bound is positive while the bound $1 - H(P_A) = 1 - h(p) - p \log_2 3$ becomes negative for restricted values of $p$, where $h$ is the binary entropy function. The degeneracy concept is somewhat misleading because a single quantum code can be regarded as both degenerate and nondegenerate as is clearly understood from the next lemma, which is a refinement of Lemma 3.

**Lemma 7:** As in Lemma 3, assume a subspace $L \subseteq F^{2n}$ and $J_0$ satisfy (14) and (13), respectively. Put

\[ J = \{ z + w \mid z \in J_0, w \in L \}. \]

Then, the condition (14) is fulfilled, so that the $d^{n-m}$ codes of the form (12) are $d^m$-dimensional $N_j$-correcting codes.

If an $N_j$-correcting code is given and $\{ M | \psi \mid M \in N_j \}$ is not linearly independent for a state $| \psi \rangle$ in the code space, then the code is called degenerate [34]. The codes in Lemma 7 are nondegenerate $N_j$-correcting codes while they are degenerate $N_j$-correcting codes. In this paper, we have evaluated...
nondegenerate $N_{d_0}$-correcting codes with $|J_0| = d^{n-m}$, but actually $|J| = d^{2(n-m)}$ in this case. Hence, the codes can correct more errors than those evaluated in this paper. Suggestions for developing Shor and Smolin’s result can be found in the final section of [3].

Shor and Smolin’s result does not deny the possibility of the tightness of this paper’s bound for all channels. Extending this work’s result to the case of channels with memory of a Markovian nature is possible if second-order (or higher-order) types are used instead of the usual types [22]. It may be also interesting to ask whether the present approach will help us obtain bounds or improve the known ones for Gaussian quantum channels already discussed in the literature [53], [54], [55].

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Appendices

A. Proof of Proposition [4]

In this proof, we assume $d = 2$ for notational simplicity. The proof readily extends to the case where $d > 2$. First, we show that the maximum of $1 - H(P_{UAN})$ with the restriction $U = I$ is achieved by the indicated $N$. For $\mathcal{M} : L(H^2) \rightarrow L(H^2)$ and $4 \times 4$ matrices $M$ over $\mathbb{C}$, we write $\mathcal{M} \sim M$ if $M$ is the matrix of $\mathcal{M}$ with respect to the basis $\{|b_0b_0\rangle, |b_0b_1\rangle, |b_1b_0\rangle, |b_1b_1\rangle\}$, where $|b_0b_0\rangle = |b_0\rangle \otimes |b_0\rangle$ and so on. We use the next lemma due to Choi [22], [56].

Lemma 8: [22] A linear map $\mathcal{A} : L(H) \rightarrow L(H)$ is completely positive if and only if $[I \otimes \mathcal{A}](|\Phi^+\rangle\langle\Phi^+|)$ is positive, where $I$ is the identity map on $L(H)$, and

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|b_0b_0\rangle + |b_1b_1\rangle).$$

Moreover, if we represent $[I \otimes \mathcal{A}](|\Phi^+\rangle\langle\Phi^+|)$ as

$$[I \otimes \mathcal{A}](|\Phi^+\rangle\langle\Phi^+|) \sim \frac{1}{2} \sum_{x \in \mathcal{X}} a^*_x a_x$$

and rearrange the elements of $a_x = (\hat{\alpha}_{x,00}, \hat{\alpha}_{x,01}, \hat{\alpha}_{x,10}, \hat{\alpha}_{x,11}) \in \mathbb{C}^4$ into the matrix form

$$\hat{A}_x = \begin{bmatrix} \hat{\alpha}_{x,00} & \hat{\alpha}_{x,01} \\ \hat{\alpha}_{x,10} & \hat{\alpha}_{x,11} \end{bmatrix}, \quad x \in \mathcal{X},$$

we obtain an operator-sum representation of $\mathcal{A}$: $\mathcal{A} \sim \{A_x\}$, where $A_x : L(H) \rightarrow L(H)$ is the Hermitian adjoint operator of $\sum_{(i,j) \in \mathcal{X}} \hat{a}_{x,ij}|b_i\rangle\langle b_j|$, i.e., the adjoint of the operator whose matrix is $\hat{A}_x$, $x \in \mathcal{X}$. ◇

Remark. The correspondence $\xi : \mathbb{C}^4 \rightarrow L(H)$ that has sent $a_x$ to $A_x$ is explicitly written as

$$\xi(m_{00}, m_{01}, m_{10}, m_{11}) = \sum_{(i,j) \in \mathcal{X}} m^*_{ij} |b_j\rangle\langle b_i|.$$

If we define an inner product $\langle \cdot, \cdot \rangle$ on $L(H)$ by $\langle N, M \rangle = 2^{-1} \text{Tr} N^\dagger M$ (half the Hilbert-Schmidt inner product), then $\{N_x\}_{x \in \mathcal{X}}$ is an orthonormal basis with respect to this inner product, and hence $P = P_\mathcal{A}$ in Theorem [4] is rewritten as

$$P(y) = \sum_{x \in \mathcal{X}} |\langle N_y, A_x \rangle|^2.$$
In fact, one sees that $P(y)$ has a physical meaning as follows. If we define an inner product between $n = (n_{00}, n_{01}, n_{10}, n_{11})$ and $m = (m_{00}, m_{01}, m_{10}, m_{11})$ by $\langle n, m \rangle = 2^{-1} \sum_{z \in \mathcal{X}} n_z m_z^*$, then $\langle \xi(n), \xi(m) \rangle = \langle n, m \rangle$, so that we have

$$P(y) = \sum_{x \in \mathcal{X}} |\langle n_y, a_x \rangle|^2,$$

where $\xi(n_y) = N_y$. Now, imagine we perform the orthogonal measurement $\{2^{-1}n_y^* n_y \}_{y \in \mathcal{X}}$ on the system in the state (22). Then, we obtain the result $y$ with probability

$$\frac{1}{2} n_y^* \frac{1}{2} \sum_{x \in \mathcal{X}} a_x^* a_x n_y^\dagger$$

$$= \frac{1}{4} \sum_{x \in \mathcal{X}} n_y a_x^* a_x n_y^\dagger$$

$$= \sum_{x \in \mathcal{X}} |\langle n_y, a_x \rangle|^2$$

$$= P(y).$$

Then, from the property of von Neumann entropy [27], $H(P)$ is not smaller than the von Neumann entropy of the state (22) and equals it when $n_x$ is proportional to $a_x$ for each $x \in \mathcal{X}$, which is fulfilled by setting $|0\rangle = |b_0\rangle$ and $|1\rangle = |b_1\rangle$ (and $\omega = \tilde{\omega}$ for $d > 2$). To complete the proof, we have only to notice that any unitary map preserves the entropy of the state that it acts on, which implies $H(P)$ does not decrease by preprocessing of applying $I \otimes U$ to $[I \otimes A](|\Phi^+\rangle\langle\Phi^+|)$. □

**B. Comparison of Bounds**

In this appendix, we prove (6), which states that our bound $1 - H(P_{UA})$ is not smaller than the previously known one $1 - H_1(p')$, and then, calculate $1 - H_1(p')$ for the amplitude-damping channel as an example. Putting $|b_0\rangle = |0\rangle$ and $|b_1\rangle = |1\rangle$ (and hence viewing state vectors in terms of the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$), we shall use the argument in the previous appendix, which applies to general CP maps $A$ except the last paragraph.

First, we prove (6). As argued by Bennett et al. [3, p. 3830], every maximally entangled state can be represented, up to an overall phase factor, as the transpose of $(u+iv, w+iz, -w+iz, u-iv)$ with $u, v, w, z$ real, i.e., as $(x, y, -y^*, x^*)^t$, where $xx^* + yy^* = 1/2$. Suppose $|\eta\rangle = x|00\rangle + y|01\rangle - y^*|10\rangle + x^*|11\rangle$ achieves the maximum in (6). Then, putting $u = \sqrt{2}(x^*, y^*, -y, x)$, this maximum can be written as

$$\frac{1}{2} u^\dagger u \frac{1}{2} \sum_{s \in \mathcal{X}} a_s^* a_s u^\dagger$$

$$= \sum_{s \in \mathcal{X}} |\langle u, a_s \rangle|^2$$

$$= \sum_{s \in \mathcal{X}} |\langle U, a_s \rangle|^2$$

$$= \sum_{s \in \mathcal{X}} |2^{-1} \text{Tr} U A_s^\dagger|^2$$

$$= \sum_{s \in \mathcal{X}} |\langle I, U A_s^\dagger \rangle|^2$$

$$= P_{UA}((0, 0))$$
where $U = \xi(u)$, and $U(\rho) = U^\dagger \rho U$ (note that $U$ is unitary). Hence, $1 - H(P_{\mathcal{U}} A) \leq 1 - H_1(1 - P_{\mathcal{U}} A((0, 0)))$, and the inequality is strict unless $P_{\mathcal{U}} A((1, 0)) = P_{\mathcal{U}} A((0, 1)) = P_{\mathcal{U}} A((1, 1))$ by the property of the Shannon entropy $H$.

**Example 4.** We have calculated $1 - H(P_A)$ for the amplitude-damping channel in Example 1. For comparison, we compute $1 - H_1(p')$ with (3) for this channel. For the operator-sum representation in Example 1, we have $a_{(0,0)} = (1,0,0,\sqrt{1-\gamma})$ and $a_{(1,0)} = (0,0,\sqrt{\gamma},0)$. Hence, the maximized quantity in (3) can be calculated as

$$\frac{1}{2^2} - \frac{1}{2} \sum_{s=(0,0),(1,0)} a_s^* a_s u^\dagger = \gamma/4 + (1 - \gamma + \sqrt{1-\gamma})u^2 + (1 - \gamma - \sqrt{1-\gamma})v^2,$$

where $u = \text{Re } x$ and $v = \text{Im } x$. From the normalization constraint $0 \leq u^2 + v^2 \leq 1/2$, it follows that the maximum is $(2 - \gamma + 2\sqrt{1-\gamma})/4$ and hence, $p' = 1 - (2 - \gamma + 2\sqrt{1-\gamma})/4$. \hfill \Box

**C. Proof of Lemma 4**

We employ the recovery operator $\mathcal{R} \sim \{\mathcal{O}\} \cup \{R_r\}$ constructed in the proof of Theorem III.2 of [26] as well as the notation therein, where in the present case their $\{A_n\}$ are to be read $\{N_x\}$. Since the conditions (19) and (20) in Theorem III.2 of [26] can be restated without referring to the code basis $\{|0_L\}, \ldots, |(K-1)1\rangle\}$ (see, e.g., [37], [58]), we can assume $|\psi\rangle = |0_L\rangle$ without loss of generality. Suppressing the superscript of $A^{(n)}_x$ and using the relations $R_r = V_r \sum_i |\nu_i^r\rangle \langle \nu_i^r|$ and $V_r |\nu_i^r\rangle = |i_L\rangle$ [26], we have

$$F(\psi) = \sum_r \sum_x \langle 0_L | R_r A_x | 0_L \rangle \langle 0_L | A^\dagger_x R^\dagger_r | 0_L \rangle$$

$$= \sum_r \sum_x \langle \nu_i^r | A_x | 0_L \rangle \langle 0_L | A^\dagger_x | \nu_i^r \rangle$$

$$= \sum_x \langle 0_L | A^\dagger_x \Pi_0 A_x | 0_L \rangle,$$

where we have put $\Pi_i = \sum_r |\nu_i^r\rangle \langle \nu_i^r|, 0 \leq i \leq K - 1$. Also we put $\Pi_K = \mathcal{O} = I - \sum_{0 \leq i \leq K-1} \Pi_i$. Thus,

$$1 - F(\psi) = \sum_{1 \leq i \leq K} \sum_x \langle 0_L | A^\dagger_x \Pi_i A_x | 0_L \rangle$$

$$= \sum_{1 \leq i \leq K} \sum_x \sum_{y,z} a^*_{xy} a_{xz} \langle 0_L | N^\dagger_y \Pi_i N_z | 0_L \rangle$$

$$= \sum_{1 \leq i \leq K} \sum_x \sum_{y,z \in J^c} a^*_{xy} a_{xz} \langle 0_L | N^\dagger_y \Pi_i N_z | 0_L \rangle$$

$$= \sum_{1 \leq i \leq K} \sum_x \langle 0_L | B^\dagger_x \Pi_i B_x | 0_L \rangle$$

$$\leq \sum_x \langle 0_L | B^\dagger_x B_x | 0_L \rangle,$$

where $B_x = \sum_{y \in J^c} a_{xy} N_y$. \hfill \Box

**D. Proof of Lemma 5**

That $|A(0)| = 0$ is trivial. The lemma follows if we show that $|A(x)| = |A(y)|$ for any two distinct nonzero vectors $x$ and $y$. This is because if it is so, putting $M = |A(x)|$, $x \neq 0$, and counting the pair $(x, L)$ such that $x \in L^\perp$, $L \in A$ and $x \neq 0$ in two ways, we will have $(d^{2n} - 1)M = |A|(d^{m+m} - 1)$. To
prove $|A(x)| = |A(y)|$, we use the Witt lemma, which states that for a space $V$ with a nondegenerate (nonsingular) symplectic form and subspaces $U$ and $W$ of $V$, if an isometry (an invertible linear map that preserves the inner-product) $\alpha$ from $U$ to $W$ exists, then $\alpha$ can be extended to an isometry from $F^{2n}$ onto itself [14, p. 81], [13, Theorem 3.9]. First, note that any linear map from the space $\text{span}\{x\}$ to $\text{span}\{y\}$ preserve the symplectic inner product [9], which always equals 0 on these spaces. Among such maps, we choose the isometry $\alpha$ with $y = \alpha(x)$. Then, by the Witt lemma, $\alpha$ can be extended to $F^{2n}$. Since $L \in A(x)$ implies $\alpha(L) \in A(y)$, we have $|A(x)| \geq |A(y)|$; since $L \in A(y)$ implies $\alpha^{-1}(L) \in A(x)$, we have $|A(x)| \leq |A(y)|$. Hence, $|A(x)| = |A(y)|$, establishing the lemma. 

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