The partial $C^0$-estimate along a general continuity path and applications

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Abstract We establish a new partial $C^0$-estimate along a continuity path mixed with conic singularities along a simple normal crossing divisor and a positive twisted $(1,1)$-form on Fano manifolds. As an application, this estimate enables us to show the reductivity of the automorphism group of the limit space, which leads to a new proof of the Yau-Tian-Donaldson conjecture.

Keywords partial $C^0$-estimate, Kähler-Einstein metric, conic Kähler metric

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1 Introduction

Finding canonical metrics on Kähler manifolds is a central problem in Kähler geometry. In the 1970s in the celebrated work [41], Yau solved Calabi’s conjecture and established the existence of Ricci flat Kähler metrics on Kähler manifolds with $c_1(M) = 0$. Aubin [2] and Yau also established the existence of Kähler-Einstein metrics with negative Ricci curvature on Kähler manifolds with $c_1(M) < 0$. The main idea is to establish a priori estimates for the solutions to the family of complex Monge-Ampère equations along a continuity path. The remaining problem is the Fano case, i.e., $c_1(M) > 0$. Unlike the two cases above, Matsushima [24] and Futaki [16] showed that there are obstructions to the existence of Kähler-Einstein metrics on Fano manifolds. Thus there are Fano manifolds which do not admit Kähler-Einstein metrics.

To solve the Kähler-Einstein problem on Fano manifolds, Tian made crucial progress in [30] which first introduced the partial $C^0$-estimate. Let us recall the basic settings of this problem: Let $(M, \omega_0)$ be a Fano manifold with a Kähler metric $\omega_0 \in [2\pi c_1(M)]$, which satisfies that $\text{Ric}(\omega_0) = \omega_0 + \sqrt{-1} \partial \bar{\partial} h$, where $h$ is a smooth $\omega_0$-PSH (pluri-subharmonic) function on $M$. Suppose that $\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$ is the Kähler-Einstein metric on $M$. Then $\varphi$ satisfies the following complex Monge-Ampère equation:

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{h-\varphi} \omega_0^n,$$

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where $\varphi$ satisfies that $\int_M e^{b_1-t\varphi} \omega_0^n = V$. To solve this equation, a standard way as [2,41] is to establish the solution to the following continuous family of Monge-Ampère equations:

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n = e^{b_1-t\varphi_t} \omega_0^n$$  \hspace{1cm} (1.1)$$

for $t \in [0,1]$ with $\int_M e^{b_1-t\varphi} \omega_0^n = V$. Tian realized that it was impossible to derive the \textit{a priori} $C^0$-estimate of (1.1) directly as [41]. Instead, he noted that different embeddings of the Fano manifold into a projective space $\mathbb{CP}^{N_1}$ by the holomorphic sections $S_0, S_1, \ldots, S_{N_1}$ of the line bundle $K_M^{\ell}$ induce a family of Kähler metrics $\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi$ parameterized by the automorphism group of the projective space, where $\psi = \log \rho - \sup \log \rho$ with

$$\rho := \rho_{\omega, \ell} = \sum_{i=0}^{N_1} \|S_i\|^2_{\omega}$$  \hspace{1cm} (1.2)$$

and $\| \cdot \|_{\omega}$ is the Hermitian metric on $K_M^{\ell}$ induced by $\omega$ with the normalization condition $\int_M \|S_i\|^2_{\omega} \omega^n = 1$. Finally, he established the partial $C^0$-estimate and showed that the reductivity of the automorphism group of the manifold implies the existence of the Kähler-Einstein metric in the case of Fano surfaces. Later, in [31] Tian extended his idea and proposed $K$-stability and conjectured that this condition implies the existence of Kähler-Einstein metrics on Fano manifolds. In [31,33], Tian also pointed out that the partial $C^0$-estimate is the crucial step to solve the conjecture. Tian [35] in 2012 and Chen et al. [8–10] proved the partial $C^0$-estimate along the continuity path of conic Kähler-Einstein metrics and finally solved this folklore conjecture. Recently, Li et al. [21,22] solved the Yau-Tian-Donaldson conjecture in the case of singular Fano varieties.

As the most crucial part in the study of the Kähler-Einstein problem, the partial $C^0$-estimate has become an interesting topic in Kähler geometry. In [33], Tian introduced a partial $C^0$-conjecture in a more general form that there exists a uniform partial $C^0$-estimate for a family of Fano manifolds with uniform positive lower Ricci curvature bounds and fixed volumes. Besides the partial $C^0$-estimates in [9,10,35] along the continuity path of conic Kähler-Einstein metrics, Donaldson and Sun [15] and Tian [34] also considered the partial $C^0$-estimates on Kähler-Einstein manifolds. Besides those works, there are some works such as [11,19,28,40] which consider the partial $C^0$-estimates under different settings.

Another important ingredient in the study of the Kähler-Einstein problem is the conic Kähler metrics. As a natural generalization of Kähler-Einstein metrics, the conic Kähler-Einstein metrics were studied in [3,4,17,18,20,23,25,38,39] and played the important role in the solution to the Yau-Tian-Donaldson conjecture. In fact, the conic Kähler-Einstein metrics with deforming cone angles give rise to the continuity path which establishes the existence of the smooth Kähler-Einstein metric as soon as the cone angle attains $2\pi$.

In this paper, we consider a general continuity path $\{\omega_t\}_{t \in [0,T]}$ with $T \in (0,1]$ on the Fano manifold $M$ as follows:

$$\text{Ric}(\omega_t) = t\omega_t + (1-t) \left( \sum_{r=1}^{m} 2\pi b_r [D_r] + b_0 \omega_0 \right),$$  \hspace{1cm} (1.3)$$

where

- $\omega_0$ is a smooth positive $(1,1)$ form in $c_1(M)$;
- $D_1, \ldots, D_m$ are semi-ample irreducible divisors with simple normal crossings;
- $b_0, b_1 = \frac{p_0}{q}, \ldots, b_m = \frac{p_m}{q}$ are positive rational numbers less than 1, where $p_1, \ldots, p_n, q$ are the least integers in the definition of $b_1, \ldots, b_m$;
- $\sum_{r=1}^{m} b_r [D_r] \in (1-b_0)c_1(M)$;
- $\{\omega_t\} \in c_1(M)$ is a family of Kähler metrics with conic singularities along $D_r$.

Actually, this path (1.3) is a generalization to the path of conic Kähler-Einstein metrics in [8–10,35] and Aubin’s path in [28]. Our main result is the partial $C^0$-estimate along this general continuity path (1.3).

**Theorem 1.1.** For $l = l_i \to \infty$, there exist constants $c_i > 0$ such that for any $x \in M$, $t \in [0,T)$,

$$\rho_{\omega, \ell}(x) > c_i.$$  \hspace{1cm} (1.4)$$
By the partial $C^0$-estimate as above, similar to the previous works, as $t \to T$ there exists a family of automorphisms $\sigma_t \in G = SL(N_1 + 1)$ corresponding to $(M, \bigcup D_r, \omega_t)$, which give rise to elements in the stabilizer $G_\infty$ of the Gromov-Hausdorff limit $(M_\infty, D_\infty, \omega_\infty)$). Then we can show the following corollary.

**Corollary 1.2.** The Lie algebra $\eta_\infty$ of $G_\infty$ is reductive.

Finally, by the partial $C^0$-estimate and the corollary above, we have the following version of the Yau-Tian-Donaldson conjecture.

**Corollary 1.3.** Given a Fano manifold $M$ admitting no holomorphic vector fields, if $M$ is K-stable, there exists a Kähler-Einstein metric on $M$.

We want to point out some merits of our continuity path (1.3). On one hand, we could simplify the openness argument in [14] due to the existence of $\alpha_0$. On the other hand, in the study of the limit space structure as $t$ tends to $T$, in [12] they need to consider the currents and do complicated procedures to those currents; however, the conic singularities in our path (1.3) could carry the main part of the singular structure, as [35]. Thus the analysis of the limit structure could be simplified.

Let us briefly describe the main ideas of this paper. We mainly follow the steps in [35] combined with some ideas from [28]. First, we need to establish the geometric limit structure as $t \to T$. For this target, we need to approximate the conic metric in (1.3) by smooth metrics with uniform Ricci lower bounds, which was done by Shen [26] based on the techniques in [35]. Next, we make use of Cheeger-Colding-Tian theory combined with Carron's technique [5] as [28] to establish the limit structure, especially the structure of the tangent cone. Then we can modify the smooth convergence part in [35] and then establish the partial $C^0$-estimate. Finally, we follow [35] to complete the two corollaries.

## 2 The structure of the Gromov-Hausdorff limit space

On the Kähler manifold $M$ associated with a simple normal crossing divisor $D = \sum_{r=1}^m D_r$, if a positive $(1,1)$-current $\omega$ is a smooth metric out of $D$ and without loss of generality, in the local holomorphic coordinate chart around a point $p$ lying in the intersection of $D_1, \ldots, D_k$ where locally $D_r = \{z_r = 0\}$, $\omega$ is asymptotically to the model metric

$$\omega_{\text{cone}} := \sqrt{-1} \left( \sum_{r=1}^k \frac{dz_r \wedge \bar{dz}_r}{|z_r|^{2(1-\beta_r)}} + \sum_{r=k+1}^n dz_r \wedge \bar{dz}_r \right),$$

then we call that $\omega$ is a conic Kähler metric with cone angles $2\pi \beta_r$ along $D_r$ for $r = 1, \ldots, k$. To apply classical Cheeger-Colding-Tian theory [6,7] to the conic metrics, as [35], we need to approximate the conic metric defined in (1.3) by smooth Kähler metrics with uniform lower Ricci curvature bounds. By (1.3), it follows that $\text{Ric}(\omega_t) \geq \omega_t$, and then we have the following approximation theorem by Shen [26].

**Theorem 2.1** (See [26]). Given a Kähler manifold $(M, \omega_0)$ with $D = \sum_{r=1}^m D_r$ which has simple normal crossings with each component $D_r$ irreducible and semi-ample, a conic Kähler metric $\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$ with cone angles $2\pi \beta_r$ (0 < $\beta_r < 1, 1 \leq r \leq m$) along $D_r$, and $\varphi \in \text{PSH}(M, \omega_0)$ which is smooth on $M \setminus D$, if the conic Kähler metric $\omega$ satisfies that $\text{Ric}(\omega) \geq \mu \omega$, then for any $\delta > 0$, there exists a smooth Kähler metric $\omega_\delta$ in the same Kähler class to $\omega$ satisfying that $\text{Ric}(\omega_\delta) \geq \mu_\delta \omega$ which converges to $\omega$ in the Gromov-Hausdorff topology on $M$ and in the smooth topology outside $D$ as $\delta$ tends to 0.

By Theorem 2.1, for any sequence $t_i \to T$ and $\omega_i := \omega_{t_i}$, there exists a sequence of smooth Kähler metrics $\tilde{\omega}_i$ satisfying that

(A1) $[\tilde{\omega}_i] = [\omega_i] \in 2\pi c_1(M)$;

(A2) $\text{Ric}(\tilde{\omega}_i) \geq t_i \omega_i$;

(A3) the Gromov-Hausdorff distance $d_{GH}(\omega_i, \tilde{\omega}_i) \leq 1/i$.

Then by the Gromov compactness theorem, without loss of generality $(M, \tilde{\omega}_i)$ converge to a metric space $(M_\infty, d_\infty)$ in the Gromov-Hausdorff topology. In addition, it follows from (A3) that $(M, \omega_i)$ also converge to $(M_\infty, d_\infty)$ in the Gromov-Hausdorff topology. Similar to [35, Theorem 3.1] we have the following geometric description of $(M_\infty, d_\infty)$. 

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Theorem 2.2. There is a closed subset $S \subset M_\infty$ of Hausdorff codimension at least 2 such that $M_\infty \setminus S$ is a smooth Kähler manifold and $d_\infty$ is induced by a twisted Kähler-Einstein metric $\omega_\infty$ outside $S$ which satisfies that $\text{Ric}(\omega_\infty) = T\omega_\infty + (1 - T)\omega_0$. If $T < 1$, $\omega_\infty$ converge to $\omega_\infty$ in $C^\infty$ topology. If $T = 1$, the singular set $S$ has codimension at least 4 and $\omega_\infty$ extends to a smooth Kähler-Einstein metric on $M_\infty \setminus S$.

Proof. The proof is almost the same as [35]. We note that Tian and Wang [39] gave a general proof of the compactness theorem for conic Kähler-Einstein metrics. Set $R$ as the regular part of $M_\infty$, where the tangent cone is $\mathbb{R}^{2n}$. When $T = 1$, we can show that

$$\int_M |\text{Ric}(\tilde{\omega}_i) - \omega_i| \omega_i^n \leq \int_M (\text{Ric}(\omega_i) - t_i\omega_i)\omega_i^{n-1} + (1 - t_i) \int_M \tilde{\omega}_i^n = 2(1 - t_i) \int_M \tilde{\omega}_i^n \to 0,$$

which implies that $(M, \omega_i)$ composite an almost Kähler-Einstein sequence defined by Tian and Wang [37]. By [37], the singular set $S$ in $M_\infty$ has codimension at least 4 and $d_\infty$ is induced by the smooth Kähler-Einstein metric $\omega_0$ on $M_\infty \setminus S$.

When $T < 1$ away from divisors, similar to [35], we first show that $R$ is open. For any $x \in R$, there is a sequence of the points $x_i \subset (M, \omega_i)$ which converge to $x$. Then by the definition of $R$, the volumes of the balls $B_j(x_i, \omega_i)$ in $(M, \omega_i)$ are uniformly close to the Euclidean volumes and thus those balls should be away from divisors along which the metrics are asymptotically conic. In this case, by applying Proposition 2.3 below which is modified from [28], it follows that the Ricci curvature of the balls $B_j(x_i, \omega_i)$ is uniformly bounded. Thus it follows from Anderson’s $\epsilon$-regularity theorem [1] that there is a uniform curvature bound on $B_j(x_i, \omega_i)$ in $(M, \omega_i)$ which implies that $\omega_i$ smoothly converge to a twisted Kähler-Einstein metric $\omega_\infty$ on $M_\infty \setminus S$, and moreover it satisfies that $\text{Ric}(\omega_\infty) = T\omega_\infty + (1 - T)\omega_0$ and induces $d_\infty$. Furthermore, the stratifications of $S$ with codimension at least 2 are all the same. In fact, the proof shows that the limit of the divisors $D_\epsilon$ lies in $S$.

Let us recall some basic facts from [6]. The stratification of $S$ can be described as $S_0 \subset S_1 \subset \cdots \subset S_{2n-1}$, where $S_k$ denotes the subset of $S$ where no tangent cone can split off a factor $\mathbb{R}^{k+1}$. A tangent cone $C_x$ at $x \in M_\infty$ can be defined as the limit of $(M_\infty, r^{-1}_{j+1}d_\infty, x)$ for a subsequence $r_j \to 0$. By Theorem 2.2 when $T < 1$ the singular set $S = S_{2n-2}$ and when $T = 1$ the singular set $S = S_{2n-4}$.

To give a detailed characterization of the tangent cone, we need the following proposition modified from [28, Proposition 8], which also has been used in the proof of Theorem 2.2.

Proposition 2.3. By fixing $T_0, T_1 \in (0, 1)$, for any $t < T \in (T_0, T_1)$, there exists $R_0, \delta > 0$ depending on $T_0$ and $T_1$ such that if $I(B_{r_0}(p, \omega_\infty)) > 1 - \delta$, for $r_0 \leq R_0$ then

$$|\text{Ric}(\omega_t)| \leq 5$$
on $B_{r_0}(p, \omega_\infty)$, where $I(\Omega) := \inf_{B_r(x) \subset \Omega} \frac{\text{Vol}(B_r(x))}{\text{Vol}(B_r(0, \text{Vol}^{1/n}))}$.

Proof. First, note that as $t < T_1 < 1$, the cone angles of the corresponding conic metric $\omega_t$ along divisors will be strictly less than $2\pi(1 - \delta)$ for some $\delta > 0$. Thus we can rule out the possibility that the center of the ball lies on divisors and restrict the whole ball to the regular part. In the regular part where the path equation (1.3) is the same as the continuity path equation in [28], we can adapt the argument of [28, Proposition 8] to our case and the proof is concluded.

Now we can show the following structure theorem of the tangent cone, similar to [35] which is modified from [7].

Theorem 2.4. Let $C_x$ be a tangent cone of $M_\infty$ at $x \in S$. Then we have the following:

(C1) Each $C_x$ is regular outside a closed subcone $S_x$ of complex codimension at least 1. Such $S_x$ is the singular part of $C_x$.

(C2) $C_x = C^k \times S'_x$, in particular $S_{2k+1} = S_{2k}$ and we define $o$ to be the vertex of $C_x$.

(C3) There is a natural Kähler-Ricci flat metric $g_x$ whose Kähler form $\omega_x$ is $\sqrt{-1}\partial\bar{\partial}r_x^2$ on $C_x \setminus S_x$, where $r_x$ denotes the distance function from $o$. Also $g_x$ is a flat conic metric.

(C4) For any $x \in S_{2n-2}$ with $C_x = C^{n-1} \times C'_x$, $C'_x$ is a two-dimensional flat cone of angle $2\pi$. Moreover, there exist $\beta_0, \beta_1 \in (0, 1)$ such that $\beta_0 \leq \beta \leq \beta_1$. 

[Note: Further content not provided due to the nature of the request.]
Proof. As in [35], the proof of (C1)–(C3) can be modified from [7] directly. For (C4), we still need a modified slice argument in [7,35]. Since $(M,\omega_i)$ converge to $(M_\infty,\omega_\infty)$ there are $r_i \to 0$ and $x_i \in M$ such that $(M,r_i^{-2}\omega_i,\kappa_i)$ converge to the cone $\mathcal{C}$. Thus there are $\epsilon_i \to 0$ and the map

$$(\Phi_i, u_i) : B_{5/2}(x_i, r_i^{-2}\omega_i) \to \mathbb{C}^{n-1} \times \mathbb{R}_+$$

satisfying that

$$\max\{\text{lip}(\Phi_i), \text{lip}(u_i)\} \leq c(n), \quad \int_{|z| < 1, z \in \mathbb{C}^{n-1}} |V(z) - 2\pi \bar{\beta}| dz \wedge d\bar{z} \leq \epsilon_i,$$

where $V(z)$ is the volume of $\Sigma_z = \Phi_i^{-1}(z) \cap u_i^{-1}([0,1])$ with respect to $r_i^{-2}\omega_i$. As denoted by [35] we can assume that $\Phi_i$ is smooth along the divisor $D$.

Now write $\epsilon = \epsilon_i$ and $(\Phi, u) = (\Phi_i, u_i)$. By the estimate above, we can find a subset $B_i$ of $\{|z| < 1\} \subset \mathbb{C}^{n-1}$ with large measure such that for any $z \in B_i$, $\Sigma_z$ is transversal to $D = \sum D_r$ with its boundary converging to $\{z\} \times S^1_\beta$ as $i \to \infty$, where $S^1_\beta$ denotes the unit circle in $\mathbb{C}$ and $|V(z) - 2\pi \bar{\beta}| \leq C\tau$ for some uniform constant $C$.

Furthermore, writing $\Phi := (h_1 + \sqrt{-1}h_2, \ldots, h_{2n-3} + \sqrt{-1}h_{2n-2})$, by [7] we have the following estimates:

(i) $\int_{\Sigma_z} |(\nabla h_k, \nabla h_i)| - \delta_{ki} = o(1)$;
(ii) $\int_{\Sigma_z} |\text{hess} h_k| = o(1)$;
(iii) $\int_{\Sigma_z} |(\nabla h_k, \nabla^2)| = o(1)$;
(iv) $\int_{\Sigma_z} |\nabla (\nabla h_k, \nabla v^2)| = o(1)$.

Now $K_M^{-1}$ is restricted to a line bundle on $\Sigma_z$ with an induced Hermitian metric $h_z$ by $r_i^{-2}\omega_i$ whose curvature $\Omega$ is equal to

$$\text{Ric}(r_i^{-2}\omega_i) = t_i\omega_i + (1-t_i) \left( \sum_{r=1}^m 2\pi b_r i_z^r [D_r] + b_0 \alpha_0 \right),$$

where $t : \Sigma_z \to M$ denotes the embedding. Now define $\Sigma_{z,\delta} := \Sigma_z \setminus \bigcup B_\delta(p_s)$, where $p_s$ denote the points where the divisors $D_r$ transversally intersect with $\Sigma_z$ and $\delta$ is small enough. Suppose that each $D_r$ has $m_r$ intersections (counting multiplicities) with $\Sigma_z$ and define $\beta_{r,i} = 1 - (1-t_i) b_r$ to be the corresponding cone angles of $\omega_i$ along $D_r$. Setting $K$ as the Gaussian curvature on $\Sigma_z$, by the estimates of $\Phi$ above we can see that the second fundamental form $\Pi_{\Sigma_z}$ in $(M, r_i^{-2}\omega_i)$ is small, and it follows that

$$\int_{\Sigma_{z,\delta}} \text{Ric}(r_i^{-2}\omega_i) + o(1) = (1-t_i)b_0 \int_{\Sigma_{z,\delta}} r_i^2 \alpha_0 + t_i r_i^2 \int_{\Sigma_z} r_i^{-2}\omega_i + o(1).$$

It follows from the Gauss-Bonnet formula that

$$2\pi \left( \chi \left( \Sigma_z - \sum_{r} m_r \right) \right) = 2\pi \chi(\Sigma_{z,\delta}) = \int_{\Sigma_{z,\delta}} K + \int_{\partial \Sigma_{z,\delta}} H - 2\pi \sum_{r} m_r \beta_{r,i},$$

where $H$ denotes the geodesic curvature on the boundary. Let $i \to \infty$ and $\delta \to 0$. It follows that

$$1 - \bar{\beta} \geq \sum_{r} m_r (1 - \beta_r) + o(1),$$

where $\beta_r := 1 - (1-T)b_r$. Thus if there is at least one $m_r > 0$ we have the upper bound $\bar{\beta} \leq \bar{\beta}_1$ for some $\bar{\beta}_1 < 1$.

On the other hand, if all $m_r = 0$, i.e., $\Sigma_z$ has no intersection with all the divisors, we can adapt the argument in [28, Proposition 11]. Suppose that there are sequences of the points

$$\{y_i\} \subset S_{2n-2} = S \subset M_\infty$$

such that they have the tangent cones $C_{y_i}$ with cone angles $2\pi \gamma_i$ converging to $2\pi$. Note that as $S$ is closed we may assume that $y_i \to y \in S$. Then there are sequences of the points $x_i \in (M, \omega_i)$ and $r_i \to 0$ such that

$$\text{Vol}B_{r_i}^{-i}(x_i, 1) = \gamma_i \text{Vol}_{2\pi} B(0, 1) + o(1),$$

and the limit of a subsequence of those scaled balls around $x_i$ is the tangent cone $C_y$ where $y \in S_{2n-2}$. Thus as $\gamma_i \to 1$, by Proposition 2.3, $r_i^{-2}\omega_i$ has bounded Ricci curvature on the half ball $B_{1/2}(x_i, r_i^{-2}\omega_i)$.

However, by Cheeger-Colding-Tian theory [7] it follows that the singular set $S$ of the limit of this sequence $(B_{1/2}(x_i, r_i^{-2}\omega_i), r_i^{-2}\omega_i)$ has at least codimension 4, which is a contradiction to $y \in S_{2n-2}$. Thus $\gamma_i$ cannot be arbitrarily close to 1 and we establish a uniform upper bound $K \leq \beta_1$ for all possibilities. The lower bound $\beta_0$ can be derived from the standard Bishop-Gromov volume comparison theorem.

\[Q.E.D.\]

### 3 Smooth convergence

As in [35], we need to show that $\omega_i$ in the last section smoothly converge to $\omega_\infty$ outside a closed subset of codimension at least 2, which is crucial for the partial $C^0$-estimate. When $T < 1$ actually we have shown that the limit of divisors lies in the singular set $S$ as $\omega_i$ smoothly converge to $\omega_\infty$ outside $S$. However, when $T = 1$ the singular set $S$ has codimension at least 4 but each $\omega_i$ is smooth outside the divisors which have codimension 2, and thus the conclusion is not clear. In this section, we follow the strategy in [35] to show the conclusion of smooth convergence.

For our continuity path (1.3), we need to establish a new construction of Hermitian metrics on $K_{M}^{-1}$. According to the assumptions of (1.3), first as $\alpha_0 \in c_1(M)$, by Yau’s solution to Calabi’s conjecture [41], there exists a smooth Kähler metric $\omega_{\alpha_0} \in c_1(M)$ such that $\text{Ric}(\omega_{\alpha_0}) = \alpha_0$. Now set $\tilde{H}_{\alpha_0} := \omega^n_{\alpha_0}$ as a Hermitian metric on $K_M^{-1}$, and then it follows that

$$R(\tilde{H}_{\alpha_0}) = \text{Ric}(\omega_{\alpha_0}) = \alpha_0.$$

Next, recall that $\omega_i = \omega_{\alpha_i}$ in (1.3) is a conic metric with cone angles $2\pi \beta_{r,i}$ along $D_r$, where $\beta_{r,i} = 1 - (1 - t_i)b_r$. For simplicity, we skip the index $i$, and then the Hermitian metric $\tilde{H}_\omega := \omega^n$ defined on $K_M^{-1}$ has the order $\prod_r |S_r|^{-(1 - \beta_r)}$ along $D = \sum_r D_r$, where $S_r$ is the defining section of $D_r$. By the conditions that $b_r = \frac{\beta_r}{q}$ and $\sum_r b_r[D_r] \in (1 - b_0)c_1(M)$, we construct the following holomorphic section:

$$S := S_{p_1}^1 \otimes S_{p_2}^2 \otimes \cdots \otimes S_{p_r}^r \in H^0(M, q(1 - b_0)K_M^{-1}).$$

Then we can construct a Hermitian metric

$$H_\omega := \tilde{H}_{\alpha_0}(S, S)^{-c_1} \tilde{H}_\omega(S, S)^{c_2} \tilde{H}_\omega$$

(3.1)

on $K_M^{-1}$ with the constants $c_1$ and $c_2$ such that $H_\omega$ is nonsingular and its curvature $R(H_\omega) = \omega$. For those targets, as $S \in H^0(M, q(1 - b_0)K_M^{-1})$, we can show that

$$\tilde{H}_{\alpha_0}(S, S) \sim \prod_r |S_r|^{2p_r}, \quad \tilde{H}_\omega(S, S) \sim \prod_r |S_r|^{2p_r} \prod_r |S_r|^{2(1 - \beta_r)q(1 - b_0)}.$$  

Thus to make $H_\omega$ nonsingular for each $r$, it suffices to satisfy

$$-c_1p_r + c_2(p_r - q(1 - b_0)(1 - \beta_r)) - (1 - \beta_r) = 0.$$

On the other hand, for the second requirement, we need

$$R(H_\omega) = -c_1q(1 - b_0)R(\tilde{H}_{\alpha_0}) + (1 + c_2q(1 - b_0))R(\tilde{H}_\omega) - 2\pi(c_2 - c_1)\sum_r p_r[D_r]$$

$$= (1 + c_2q(1 - b_0))\text{Ric}(\omega) - 2\pi(c_2 - c_1)\sum_r p_r[D_r] - c_1q(1 - b_0)\alpha_0 = \omega.$$
To satisfy the equations above, it suffices to have
\[
\begin{align*}
q(c_2 - c_1) &= (1 - t)(1 + c_2q(1-b_0)), \\
c_1q(1-b_0) &= b_0(1-t)(1 + c_2q(1-b_0)).
\end{align*}
\] (3.2)
Thus we have
\[
c_1 = b_0c_2 = \frac{b_0(1-t)}{tq(1-b_0)}, \quad c_2 = \frac{1-t}{tq(1-b_0)},
\]
which gives us the required \( H_\omega \), which is called the associated Hermitian metric of \( \omega \). Moreover, for any \( \sigma \in H^0(M, K_M^{-1}) \), it follows that \( H_\omega(\sigma, \sigma) \) is bounded.

The remaining steps are almost the same as [35], so we only sketch the main steps in the following. First, by Theorem 2.1, similar to the argument in the proof of Lemma 3.3 in [35], we have uniform Sobolev inequalities with respect to all \( \omega_i \). By [35, Lemma 4.1], we also have the equations for \( \sigma \in H^0(M, K_M^{-1}) \),
\[
\begin{align*}
\Delta_i\|\sigma\|^2 &= \|\nabla\sigma\|^2 - n\|\sigma\|^2, \\
\Delta_i\|\nabla\sigma\|^2 &= \|\nabla^2\sigma\|^2 - (n+2)t\|\nabla\sigma\|^2,
\end{align*}
\]
where \( \|\cdot\| \) denotes the Hermitian metric on \( K_M^i \) induced by \( H_i = H_\omega \), \( \nabla \) denotes the covariant derivative of \( H_\omega \), and \( \Delta_i \) denotes the Laplacian of \( \omega_i \).

Lemma 3.1 (See [35, Corollary 4.2]). If \( \sigma_i \) is a sequence in \( H^0(M, K_M^{-1}) \) satisfying that \( \int_M \|\sigma\|^2 \omega_i^n = 1 \), then
\[
\sup_M (\|\sigma_i\|_i + t^{-1/2}\|\nabla\sigma_i\|_i) \leq Cn^{1/2}.
\] (3.3)
It follows from Lemma 3.1 that \( \|\sigma_i\| \) are uniformly continuous. Thus by taking a subsequence if necessary we may assume that \( \|\sigma_i\| \) converge to a Lipschitz function \( F_\infty \) as \( i \to \infty \) and \( \int_M F_\infty^n \omega_\infty^n = 1 \). We can see that \( F_\infty \) is not identical to \( 0 \). We only need to show that \( \omega_i \) converge to \( \omega_\infty \) away from \( F_\infty^{-1}(0) \cup S \) and \( F_\infty \) is equal to the square norm of a holomorphic section on \( M_\infty \).

If \( F_\infty(x) \neq 0 \) for some \( x \in M_\infty \), we can argue that for \( x_i \to x \) where \( x_i \in (M, \omega_i) \), the small balls \( B_r(x_i, \omega_i) \) are away from divisors. In this case, the volumes of those small balls are close to those of Euclidean balls. Then similar to the proof of Theorem 2.2, by [1] we can show the uniform curvature estimate and consequently the smooth convergence follows away from \( F_\infty^{-1}(0) \cup S \). Furthermore, we can show that \( M_\infty \setminus F_\infty^{-1}(0) \) is dense. In fact, if there exists an open set \( U \subset F_\infty^{-1}(0) \), then as \( \|\sigma_i\| \) is bounded from above, it follows that
\[
\lim_{i \to \infty} \int_M \log \left( \frac{1}{i} + \|\sigma_i\|_i^2 \right) \omega_i^n = -\infty.
\]
However, by direct computations, we have
\[
\sqrt{-1} \partial \bar{\partial} \log \left( \frac{1}{i} + \|\sigma_i\|_i^2 \right) \geq -i\omega_i.
\]
Thus by the standard Moser iteration and the uniform Sobolev inequality in [35, Lemma 3.3], it follows that
\[
\sup_M \log \left( \frac{1}{i} + \|\sigma_i\|_i^2 \right) \leq C \left( 1 + \int_M \log \left( \frac{1}{i} + \|\sigma_i\|_i^2 \right) \omega_i^n \right),
\]
which tends to \( -\infty \) as \( i \to \infty \). However, this contradicts the fact that \( \int_M \|\sigma\|^2 \omega_i^n = 1 \), and thus \( M_\infty \setminus F_\infty^{-1}(0) \) is dense.

As \( \omega_i \to \omega_\infty \) smoothly away from \( F_\infty^{-1}(0) \cup S \), \( \sigma_i \) converge to a holomorphic section \( \sigma_\infty \) on \( M_\infty \setminus F_\infty^{-1}(0) \cup S \) with \( F_\infty = \|\sigma_\infty\|_\infty \). As \( \sigma_\infty \) is uniformly bounded and holomorphic on the dense set, it could be extended to a holomorphic section on \( M_\infty \setminus S \). As \( \|\sigma_i\|_i = 0 \) on \( D \), the limit of \( D \) must lie in \( D_\infty = \{ F_\infty = 0 \} \). On the other hand, if \( D_\infty \) does not coincide with the limit of \( D \), there exist \( x \in D_\infty \) and \( r > 0 \) such that \( B_{2r}(x, d_\infty) \cap D_\infty \) is disjoint from the limit of \( D \). Then for sufficiently large \( i \), \( B_r(x, \omega_i) \)
is disjoint from $D$ and thus lies in the smooth part of $(M, \omega)$. By Proposition 2.3, the Ricci curvature on $B_{r/2}(x, \omega_i)$ is uniformly bounded and it follows from Cheeger-Colding-Tian theory \cite{7} that $S \cap B_r(x, d_\infty)$ is of complex codimension at least 2 and near a generic point $y \in B_r(x, d_\infty)$, $\sigma_\infty$ is holomorphic and defines $D_\infty$. By the smooth convergence of $(M, \omega_i)$ to $(M_\infty, d_\infty)$ near $y$, it follows that $\sigma_i(y) = 0$ which contradicts the assumption that $y$ is not in the limit of $D$. Thus $D_\infty$ must coincide with the limit of $D$.

If $T = 1$ as $S$ is of complex codimension at least 2 then $D_\infty = \{\sigma_\infty = 0\}$, which is a divisor of $K_M^{-l}$.

In summary, we have the following theorem as \cite[Theorem 4.3]{35}.

**Theorem 3.2.** $(M, \omega_i)$ converge to $(M_\infty, \omega_\infty)$ in the $C^\infty$-topology outside a closed subset $S \cup D_\infty$ where $S$ is of codimension at least 4, and $D$ converges to $D_\infty$ in the Gromov-Hausdorff topology. If $T < 1$, $S = S \cup D_\infty$. If $T = 1$, $S = S$ and $D_\infty$ is a divisor of $K_M^{-l}$.

### 4 The partial $C^0$-estimate

In this section, we follow \cite{35} to prove Theorem 1.1, i.e., establish the partial $C^0$-estimate. In fact by the last two chapters, it suffices to show the partial $C^0$-estimate (1.2) with respect to the sequence $\omega_i = \omega_{t_i}$ for $t_i \to T$.

By Theorem 3.2, we have shown that $\omega_i$ smoothly converge to $\omega_\infty$ outside $S$ when $T < 1$ or outside $S \cup D_\infty$ when $T = 1$. Meanwhile, any section $\sigma_i \in H^0(M, K_M^{1})$ which has the same support with $D$ satisfies $\int_M H_i(\sigma_i, \omega_i \alpha_i) = 1$ where $H_i = H_{\omega_i}$ was defined in the last chapter. Then $\sigma_i$ converge to $\sigma_\infty \in H^0(M_\infty, K_M^{-l})$ as $\omega_i$ converge smoothly to $\omega_\infty$. Moreover, $\sigma_\infty$ also has the same support with the divisor $D_\infty$ which is the limit of $D$.

In particular, choosing $\sigma$ as

$$S = S_1^0 \otimes S_2^0 \otimes \cdots \otimes S_p^0 \in H^0(M, q(1-b_0)K_M^{1})$$

normalized by $H_i$, we have the limit $\sigma_\infty \in H^0(M_\infty, q(1-b_0)K_M^{-l})$. Similar to the last section, we can define a Hermitian metric $H_\infty$ on $K_M^{-l}$ over $M_\infty \setminus S$ by

$$H_\infty = \hat{H}_{\omega_\infty}(\sigma_\infty, \sigma_\infty)^{-c_1} \hat{H}_{\omega_\infty}(\sigma_\infty, \sigma_\infty)^{c_2} \hat{H}_{\omega_\infty},$$

where

$$c_1 = b_0c_2 = \frac{b_0(1-T)}{Tq(1-b_0)}, \quad c_2 = \frac{1-T}{Tq(1-b_0)},$$

and $\hat{H}_{\omega_\infty}$ and $\hat{H}_{\omega_\infty}$ are defined by $\omega_\alpha^n$ and $\omega_\infty^n$ on $K_M^{-l}$. Note that $\omega_\alpha^n$ can be defined on $M_\infty \setminus S$ by the smooth convergence.

Similar to \cite[Lemmas 5.2 and 5.3]{35}, we can easily derive the following two lemmas.

**Lemma 4.1.** The Hermitian metrics $H_i$ converge to $H_\infty$ on $M_\infty \setminus S$ in the $C^\infty$-topology. Moreover, we have

$$H_\infty(\sigma_\infty, \sigma_\infty) < \infty$$

and

$$\int_{M_\infty} H_\infty(\sigma_\infty, \sigma_\infty) \omega_\alpha^n = 1.$$

**Lemma 4.2.** For any $l > 0$, if $\{\tau_i\}$ is any sequence of $H^0(M, lK_M^{-1})$ satisfying

$$\int_M H_i(\tau_i, \tau_i) \omega_i^n = 1,$$

then a subsequence $\tau_i$ converges to a section $\tau_\infty \in H^0(M_\infty, lK_M^{-1})$.

Now we can begin the proof of Theorem 1.1. The main steps are quite similar to those in \cite{35} so here we only sketch the main steps of the proof. First, by (1.2) and the fact that $||\sigma_i||$ and their covariant derivatives are uniformly bounded, applying the finite covering argument, we only need to show that for any $x \in M_\infty$, there are an $l = l_x$ and a sequence $x_i \in M$ such that $x_i \to x$ and

$$\inf_i \rho_{x_i, l}(x_i) > 0.$$  \hspace{1cm} (4.1)
Next, we need the following lemma on Hörmander’s $L^2$-estimate for the $\bar{\partial}$-operator on $(M, \omega_i)$, which follows from the $L^2$-estimate with respect to the approximating metrics constructed in Theorem 2.1.

**Lemma 4.3.** For any $l > 0$, given a $(0, 1)$ form $\zeta$ with the values in $K^{-l}_{M_i}$ with $\bar{\partial}\zeta = 0$, there is a smooth section $\vartheta$ of $K^{-l}_{M_i}$ such that $\bar{\partial}\vartheta = \zeta$ and moreover,

$$
\int_M \|\vartheta\|^2_{\omega_i} \leq \frac{1}{l + \epsilon_l} \int_M \|\zeta\|^2_{\omega_i},
$$

where $\|\cdot\|_1$ denotes the norm induced by $\omega_i$ as before.

Next, recall that in Theorem 2.4 we established the existence and basic properties of the tangent cones $C_x$ for $x \in (M_{\infty}, \omega_{\infty})$. Then by taking a subsequence if necessary, for $r_j \to 0$, we have a tangent cone $C_x$ at $x$ which is the Gromov-Hausdorff limit of $(M_{\infty}, r_j^{-2}\omega_{\infty}, x)$ satisfying

- (T1) each $C_x$ is regular outside a closed subcone $S_x$ of complex codimension at least 1, where $S_x$ is the singular set of $C_x$;
- (T2) there is a natural Kähler Ricci-flat metric $g_x$ on $C_x \setminus S_x$ which is also a conic metric, with the Kähler form $\omega_x = \sqrt{-1}\partial\bar{\partial}^2_x$ on the regular part of $C_x$ where $\rho_x$ denotes the distance function from the vertex $x$ of $C_x$.

Define $L_x = C_x \times \mathbb{C}$ to be the trivial line bundle over $C_x$ equipped with the Hermitian metric $e^{-\rho_x^2} |\cdot|^2$. Thus the curvature of this metric is just $\omega_x$. For any $\epsilon > 0$, put

$$
V(x; \epsilon) := \{y \in C_x \mid y \in B_{r^{-1}_{\epsilon}}(0, g_x) \setminus B_{r_{\epsilon}}(0, g_x), \ d(y, S_x) > \epsilon\}.
$$

Choose the sequence $r_j \to 0$ such that $r_j^{-2}$ are integers and $(M_{\infty}, r_j^{-2}\omega_{\infty}, x)$ converge to $(C_x, g_x, 0)$. By [7] for any $\epsilon, \delta > 0$, there exists a $j_0$ such that for any $j \geq j_0$, there is a diffeomorphism $\phi : V(x; \epsilon/4) \to M_{\infty} \setminus S$ satisfying

1. $d(x, \phi(V(x; \epsilon))) < 10\epsilon$ and $\phi(V(x; \epsilon)) \subset B_{(1+\epsilon^4)}(x)$, where $r = r_j$;
2. define $g_{\infty}$ to be the Kähler metric with the Kähler form $\omega_{\infty}$, and then

$$
\|r^{-2}\phi^* g_{\infty} - g_x\|_{C^0(V(x; \epsilon/2))} \leq \delta
$$

with respect to $g_x$.

From the settings above, we have the following lemma.

**Lemma 4.4** (See [35, Lemma 5.7]). Given any $\epsilon > 0$ and small $\delta > 0$, there are a sufficiently large $l = r^{-2}$ and a diffeomorphism $\phi : V(x; \epsilon/4) \to M_{\infty} \setminus S$ with the properties (1) and (2) above and an isomorphism $\psi$ from $C_x \times \mathbb{C}$ onto $K^{-l}_{M_{\infty}}$ over $V(x; \epsilon)$ commuting with $\phi$ satisfying

$$
\|\psi(1)\|^2 = e^{-\rho_x^2} \quad \text{and} \quad \|\nabla \psi\|_{C^1(V(x; \epsilon))} \leq \delta,
$$

where $\|\cdot\|$ denotes the induced norm on $K^{-l}_{M_{\infty}}$ by $\omega_{\infty}$ and $\nabla$ denotes the covariant derivative with respect to $\|\cdot\|$ and $e^{-\rho_x^2} |\cdot|^2$.

**Proof.** We sketch the main steps of the proof and refer to [35] for the details. By the approximation construction on the tangent cone above, we can cover $V(x; \epsilon')$ by finitely many geodesic balls $B_{s_\alpha}(y_\alpha)$ ($1 \leq \alpha \leq N$) satisfying $\bar{B}_{2s_\alpha}(y_\alpha)$ are strongly convex and contained in $C_x \setminus S_x$, $B_{s_\alpha/2}(y_\alpha)$ are mutually disjoint, and $s_\alpha \geq v_x d(y_\alpha, S_x)$ for some constant $v_x$.

For $l = r^{-2}$ and $\phi$ above, we can first construct the bundle morphism $\tilde{\psi}_\alpha$ over the balls $B_{2s_\alpha}(y_\alpha)$. Let $\gamma_y \subset B_{2s_\alpha}(y_\alpha)$ be the unique minimizing geodesic connecting $y_\alpha$ and $y \in B_{2s_\alpha}(y_\alpha)$. At $y_\alpha$ define $\tilde{\psi}_\alpha(1) \in L|_{\phi(y_\alpha)}$ such that

$$
\|\tilde{\psi}_\alpha(1)\|^2 = e^{-\rho_x^2(y_\alpha)},
$$

where $L = K^{-l}_{M_{\infty}}$. Then for $y \in U_\alpha := B_{2s_\alpha}(y_\alpha)$, set $a(y)$ and $\tau(\phi(y))$ as the parallel transportation of 1 and $\tilde{\psi}_\alpha(1)$ along $\gamma_y$ and $\phi(\gamma_y)$ with respect to the norms $e^{-\rho_x^2} |\cdot|^2$ and $\|\cdot\|$. Then we can define $\tilde{\psi}_\alpha(a(y)) = \tau(\phi(y))$. 
By this construction obviously the first condition in (4.3) holds for $\tilde{\psi}_{\alpha}$ over $U_{\alpha}$. For the derivative estimate in the second one, noting that $a: U_{\alpha} \mapsto U_{\alpha} \times \mathbb{C}$ and $\tau: U_{\alpha} \mapsto \phi^{*}L|_{U_{\alpha}}$ satisfy that $\tilde{\psi}_{\alpha}(a) = \tau$, we have

$$\nabla \tau = \nabla \tilde{\psi}_{\alpha}(a) + \tilde{\psi}_{\alpha}(\nabla a),$$

where the covariant derivative is taken with respect to the metrics defined above. By the definition of $\tilde{\psi}_{\alpha}$, it follows that $\nabla \tilde{\psi}_{\alpha}(y_{\alpha}) = 0$. At $y$, by taking the covariant derivative along $\gamma_{y}$, it follows that

$$\nabla_{T}\nabla_{X} \tau = \nabla_{T}(\nabla_{X} \tilde{\psi}_{\alpha}(a)) + \tilde{\psi}_{\alpha}(\nabla_{T} \nabla_{X} a),$$

where $T$ is the unit tangent of $\gamma_{y}$ and $X$ is a vector field along $\gamma_{y}$ with $[T, X] = 0$, and note that $\nabla_{T} \tilde{\psi}_{\alpha} = 0$ by the definition. Similarly, by switching $X$ and $T$, it follows that

$$\nabla_{X} \nabla_{T} \tau = \tilde{\psi}_{\alpha}(\nabla_{X} \nabla_{T} a).$$

By taking the difference of these two formulas, it follows from the curvature formula that

$$l_{\phi^{*}\omega_{\infty}}(T, X) \tilde{\psi}_{\alpha}(a) = \nabla_{T}(\nabla_{X} \tilde{\psi}_{\alpha}(a)) + \omega_{x}(T, X)a.$$

Note that as $l \to \infty$, $l_{\phi^{*}\omega_{\infty}}$ converges to $\omega_{x}$. It follows that $\nabla_{T}(\nabla_{X} \tilde{\psi}_{\alpha}(a))$ is quite small when $l$ is sufficiently large. As $\nabla_{X} \tilde{\psi}_{\alpha} = 0$ at $y_{\alpha}$, $\|\nabla \tilde{\psi}_{\alpha}\|_{C^{0}(U_{\alpha})}$ can be made sufficiently small. Higher derivative estimates could be concluded by induction.

Next, we need to modify each $\tilde{\psi}_{\alpha}$. By the construction above, for any $\alpha$ and $\gamma$, we have the transition function

$$\theta_{\alpha\gamma} = \tilde{\psi}_{\alpha}^{-1} \circ \tilde{\psi}_{\alpha}: U_{\alpha} \cap U_{\gamma} \mapsto S^{1}$$

by the first conclusion for $\tilde{\psi}_{\alpha}$ in (4.4). Those transition functions compose a closed cycle $\{\theta_{\alpha\gamma}\}$. From the derivative estimate of $\tilde{\psi}_{\alpha}$, each $\theta_{\alpha\gamma}$ is close to a constant. Thus we can multiply each $\tilde{\psi}_{\alpha}$ by a suitable unit function such that each $\theta_{\alpha\gamma}$ is a unit constant and all the derivative estimates for $\tilde{\psi}_{\alpha}$ are still quite small. Then the modified cycles $\theta_{\alpha\gamma}$ give rise to a flat bundle $F$ which essentially induces an isomorphism $\xi: F \mapsto K_{M_{\infty}}^{-1}$ over a neighborhood of $V(x; \epsilon')$ satisfying all the estimates in (4.4). The isomorphism $\xi$ induces an isomorphism $\xi^{k}: F^{k} \mapsto K_{M_{\infty}}^{-kl}$ for each $k$. Furthermore, choose $E_{\epsilon} \subset \partial B_{1}(o, g_{x}) \setminus S_{x}$ with

$$d_{\partial B_{1}(o, g_{x})}(E_{\epsilon}, S_{x} \cap \partial B_{1}(o, g_{x})) \geq \epsilon^2$$

such that the topology of $E_{\epsilon}$ depends only on $\epsilon$ and $S_{x}$. Then choose $\epsilon'$ and set

$$U(x; \epsilon', \epsilon) = \left\{ y \in C_{x} \left| \sqrt{\epsilon'} < d(x, y) < \epsilon^{-1}, \frac{y}{d(x, y)} \in E_{\epsilon} \right. \right\}$$

such that $U(x; \epsilon', \epsilon) \subset V(x, \epsilon')$.

The flat bundle $F|_{U(x; \epsilon', \epsilon)}$ is given by a representation

$$\rho: \pi_{1}(U(x; \epsilon', \epsilon), o) \mapsto H_{1}(E_{\epsilon}, Z) \mapsto S^{1}.$$

Note that $H_{1}(E_{\epsilon}, Z)$ is the sum of an abelian group of finite rank $m$ and a finite group of order $\nu$ which depend only on $\epsilon$ and $S_{x}$. Thus we can choose $k$ such that $F^{k}$ is essentially trivial on the scale of $\delta$, i.e., the corresponding transition functions are in a $\delta'$-neighborhood of the identity in $S^{1}$, where $\delta'$ depends on and is much smaller than $\delta$.

Replace the original $l$ by $kl$ and set $\epsilon'$ such that

$$k^{-1/2}V(x; \epsilon) := \{ y \in C_{x} \left| \epsilon < \sqrt{k}d(x, y) < \epsilon^{-1}, \sqrt{k}d(y, S_{x}) > \epsilon \right. \subset U(x; \epsilon', \epsilon) \}.$$

Correspondingly, redefine $\phi$ as the original $\phi$ composed with the scaling $y \mapsto k^{-1}y$ on $C_{x}$. Since $(M_{\infty}, k\tau_{j}^{-2}\omega_{\infty}, x)$ also converge to the cone $(C_{x}, y_{x}, x)$ all the properties and constructions before also follow. Moreover, the new flat bundle $F$ which is the $k$-th power of the original one will have transition functions $\delta'$-close to the identity for $\delta' \ll \delta$. Thus we can modify $\tilde{\psi}_{\alpha}$ slightly and finally we complete all the constructions satisfying the lemma.
Lemma 4.5 (See [35, Lemma 5.8]). For any $\bar{\epsilon} > 0$, there is a smooth function $\gamma_{\bar{\epsilon}}$ on $\mathcal{C}_x$ satisfying
\begin{enumerate}
\item $\gamma_{\bar{\epsilon}}(y) = 1$ if $d(y, \mathcal{S}_x) \geq \bar{\epsilon}$ where $d$ is the distance of $(\mathcal{C}_x, g_x)$;
\item $0 \leq \gamma_{\bar{\epsilon}} < 1$ and $\gamma_{\bar{\epsilon}}(y) = 0$ in a neighborhood of $\mathcal{S}_x$;
\item $|\nabla \gamma_{\bar{\epsilon}}| \leq C$ for some constant $C = C(\bar{\epsilon})$ and
\end{enumerate}
\[ \int_{B_{\delta}(x)} |\nabla \gamma_{\bar{\epsilon}}|^2 \omega_x^n \leq \bar{\epsilon}. \]

Proof. See [35] for the details of the proof. Note that $\mathcal{S}_x$ is the union of $\mathcal{S}_y^0$ and $\mathcal{S}_x$ where $\mathcal{S}_y^0$ is the part where all the points $y$ have a tangent cone of the form $\mathbb{C}^{n-1} \times \mathcal{C}_x'$ with the standard conic metric, and $\mathcal{S}_x$ is a subcone of complex codimension at least 2. In the simplest case $\mathcal{S}_x = \mathbb{C}^{n-1}$, we can set a cutoff function $\eta$ satisfying that $0 \leq \eta \leq 1, |\eta'| \leq 1$, and
\[ \eta(t) = 0 \quad \text{for } t > \log(-\log \delta) \quad \text{and} \quad \eta(t) = 1 \quad \text{for } t < \log(-\log \delta). \]

Then we can define the cutoff function $\gamma_{\bar{\epsilon}}(y) = 1$, if $\rho(y) \geq \bar{\epsilon}$, otherwise,
\[ \gamma_{\bar{\epsilon}}(y) = \eta \left( \log \left( - \log \left( \frac{\rho(y)}{\bar{\epsilon}} \right) \right) \right), \]
and we have
\[ \nabla \gamma_{\bar{\epsilon}}(y) = \frac{\eta' \nabla \rho(y)}{\rho(y) \log \left( \frac{\rho(y)}{\bar{\epsilon}} \right)}. \]

By the standard computations, it follows that
\[ \int_{B_{\delta}(x)} |\nabla \gamma_{\bar{\epsilon}}|^2 \omega_x^n \leq \frac{a_{n-1}}{\bar{\epsilon}^{2n-2}} \int_{S_3} \frac{dr}{r^2(-\log r)^2} \leq \frac{a_{n-1}}{\bar{\epsilon}^{2n-2}(-\log \delta)}, \]
where $a_{n-1}$ is the volume of the unit ball in $\mathbb{R}^{2n-2}$.

By choosing suitable constant $\delta$, we can make the integration less than $\bar{\epsilon}$, and moreover it follows that $|\nabla \gamma_{\bar{\epsilon}}| \leq C(\bar{\epsilon})$.

In general, recall that $(\mathcal{C}_x, g_x, o)$ is the limit of $(M_i, r_i^{-2} \omega_i, x_i)$. Thus there are $\delta_i$ tending to 0 and the diffeomorphisms
\[ \tilde{\phi}_i : V(x; \delta_i) \to M \setminus T_{\delta_i}(D), \quad T_{\delta_i}(D) = \{ z \mid d_i(z, D) \leq \delta_i \}, \]
where $d_i$ denotes the distance with respect to $r_i^{-2} \omega_i$ satisfying
\[ |r_i^{-2} \tilde{\phi}_i^* \omega_i - \omega_x|_{C^0(V(x, \delta_i))} \leq \delta_i. \]

Suppose that $l_i = r_i^{-2}$ are integers.

Now for $y \in \mathcal{S}_y^0$, by the assumption there are integers $k_j = s_j^{-2}$ such that $(\mathcal{C}_y, k_j g_x, y)$ converge to $(\mathcal{C}_y, g_\beta, o)$, where $\mathcal{C}_y = \mathbb{C}^{n-1} \times \mathcal{C}_x'$ with the flat conic metric $g_\beta$ as above. Thus there are the diffeomorphisms
\[ \partial_j : V(y; j^{-1}) \subset C_y \mapsto C_x \setminus \mathcal{S}_x \]
satisfying
\[ \| s_j^{-2} \partial_j^k \omega_z - \omega_0 \|_{C^0(V(y; j^{-1}))} \leq \frac{1}{j}. \]

Now it follows from the constructions above that for any \( \delta > 0 \), there exist \( i_\delta \) and \( j_\delta \) such that for any \( i \geq i_\delta, j \geq j_\delta \), we have \( \phi_i \cdot \sigma_j : V(y; j^{-1}) \to M \setminus T_k(D) \) satisfying
\[ \| k_j \sigma_j^* \phi_i^* \omega_i - \omega_{\beta} \|_{C^0(V(y; j^{-1}))} \leq \delta. \] (4.5)

Consider \( C_y \times C \) as a bundle over \( C_y \) with the norm \( e^{-|z'|^2-|z_n|^2} \), and take \( f_0 = \alpha_0, f_1 = \alpha_1 z_1, \ldots, f_n = \alpha_n z_n \), where \( \alpha_0, \ldots, \alpha_n > 0 \) are chosen such that
\[ \int_{C^{n-1} \times C_y} |f_k|^2 e^{-|z'|^2-|z_n|^2} \omega_0^0 = 1. \] (4.6)

Apply Lemma 4.4 to each \( f_k \). Then for sufficiently large \( i \) and \( j \), there exist the isomorphisms \( \psi_{i,j} : C_y \times C \to K^{-k_i} \) over \( V(y; j^{-1}) \) satisfying
\[ \| \psi_{i,j}(f_k) \|^2 = |f_k|^2 e^{-|z'|^2-|z_n|^2} \] and \[ \| \nabla \psi_{i,j} \|_{C^0(V(y; j^{-1}))} \leq \delta. \] (4.7)

We will see that the partial \( C^{0} \)-estimate follows from Lemma 4.5 later. Until now we have proved this lemma in the case where \( \mathcal{K} \) is a simple cone as above. Thus by the argument of the partial \( C^{0} \)-estimate in the following, we can show that there exist the holomorphic sections \( S_{i,j} \) of \( K^{-k_i} \) over \( M \) such that
\[ \sup_{V(y; j^{-1}) \cap B_{10}(o,g_j)} |\psi_{i,j}^* S_{i,j} - f_k| \leq \frac{\epsilon}{2}, \] (4.8)
where \( \epsilon \) tends to 0 as \( \delta \) tends to 0. Moreover by the Moser iteration in the proof of Lemma 3.1 in the same region, we have
\[ \| \nabla S_{i,j} \|_i \leq C \] (4.9)
with respect to the Hermitian norm associated with \( \omega_i \) in Section 3. By (4.8) combined with the fact \( f_0 = \alpha_0 > 0 \), it follows that there exists a \( c > 0 \) depending only on \( \alpha_0 \) such that
\[ \| S_{i,j}^0 \|_i \geq c \quad \text{on} \quad \hat{\phi}_i(\sigma_j(B_{10}(o, g_j))). \] (4.10)

Thus we can define a holomorphic map \( F_{i,j} : \hat{\phi}_i(\sigma_j(B_{10}(o, g_j))) \to C^n \) by
\[ F_{i,j} = \left( S_{i,j}^1(x) \right) \left( S_{i,j}^2(x) \right) = \cdots \left( S_{i,j}^n(x) \right). \] (4.11)

Then \( F_{i,j} \cdot \hat{\phi}_i \cdot \sigma_j \) smoothly converge to \( (f_1/f_0, \ldots, f_n/f_0) \) outside the singular set \( \{ z_n = 0 \} \) as \( j, i \to \infty \). Thus for sufficiently large \( i \) and \( j \), we have
\[ |F_{i,j} \cdot \hat{\phi}_i \cdot \sigma_j(z) - (f_1/f_0, \ldots, f_n/f_0)(z)| \leq \epsilon \] (4.12)
for any \( z \in U_j := \{(z', z_n) \in B_{10}(o, g_j) \mid |z_n|^2 > j^{-1} \} \subset V(y; j^{-1}). \) As \( i \) and \( j \) are sufficiently large we may assume \( B_{k^{\delta}r_i}(x_i, \omega_i) \subset \hat{\phi}_i(\sigma_j(B_{10}(o, g_j))) \) and moreover \( F_{i,j} \) is a holomorphic map from \( B_{k^{\delta}r_i}(x_i, \omega_i) \) onto its image containing \( B_{k^{\delta}r_i}(o, g_j) \) for sufficiently small \( \epsilon \). By (4.9), it follows that
\[ \sup_{B_{k^{\delta}r_i}(x_i, \omega_i)} |dF_{i,j}| \omega_i \leq C(s_j r_i)^{-2}, \] (4.13)
which implies that
\[ F_{i,j}^* \omega_0 \leq C(s_j r_i)^{-2} \omega_i, \] (4.14)
where \( \omega_0 \) is the Euclidean metric on \( C^n \).
Next, we need to show that for sufficiently large $j$, $F_{i,j}(D \cap B_{T_s, r}(x_i, \omega_j))$ converge to a local divisor $D^j \subset \mathbb{C}^n$ as $i \to \infty$ where $D = \sum_r D_r$. First, we need to bound the volume of $F_{i,j}(D \cap B_{T_s, r}(x_i, \omega_j))$. Recalling that $(C_s, s_j^{-2} g_s, y)$ converge to the standard cone $\mathbb{C}^{n-1} \times \mathbb{C}_0'$ with the metric $g_3$, for sufficiently large $i$ and $j$, $F_{i,j}$ maps $D \cap B_{B_{T_s, r}(x_i, \omega_j)}$ into a tubular neighborhood

$$T_{8, \epsilon} = \{(z', z_n) \mid |z'| < 8, |z_n| < \epsilon\}.$$ 

By the slicing argument in [7] (or Theorem 2.4) for each $z'$ with $|z'| < 7.5$, the complex line segment $\{(z', z_n) \mid |z_n| < 6\}$ intersects with $F_{i,j}(D \cap B_{T_s, r}(x_i, \omega_j))$ at $m_r(z')$ points (counted with multiplicity), where $m_r(z')$ satisfies that $1 - \beta_s \geq |z_n| (1 - \beta_s)$ as in Theorem 2.4. Moreover, for any such $z'$ there is $m > 0$ such that $\sum_r m_r(z') \leq m$.

Let $\tilde{\eta}: \mathbb{R} \to \mathbb{R}$ be a cutoff function satisfying $\tilde{\eta}(t) = 1$ for $t \leq 56$, $\tilde{\eta}(t) = 0$ for $t > 60$, $|\tilde{\eta}'(t)| \leq 1$ and $|\tilde{\eta}''(t)| \leq 2$. Then we have

$$\int_{F_{i,j}(D \cap B_{T_s, r}(x_i, \omega_j))} \tilde{\eta}|z'|^2(\sqrt{-1}dz' \wedge dz' + \sqrt{-1}d\bar{z}'|z_n|^2)^{n-1}$$

$$= \int_{F_{i,j}(D \cap B_{T_s, r}(x_i, \omega_j))} \tilde{\eta}|z'|^2((\sqrt{-1}dz' \wedge dz' + \sqrt{-1}d\bar{z}'|z_n|^2)^{n-1})$$

It follows that

$$\int_{F_{i,j}(D \cap B_{T_s, r}(x_i, \omega_j))} \omega_0^{n-1} \leq 200n.$$  \hfill (4.15)

By the argument in Section 3, we can show that the limit of $D$ coincides with $S_x$ modulo a subset of Hausdorff codimension at least 4 under the Gromov-Hausdorff convergence of $(M, r_i^{-2}\omega_i, x_i)$ to $(C_s, \omega_s, o)$ and thus $F_{\infty,j}(S_x \cap B_{T_s}(y, g_s))$ coincides with $D^j$.

By the monotonicity of the subvariety $F_{i,j}(D)$ combined with (4.14), as $i$ and $j$ are sufficiently large, it follows that

$$1 \leq \int_{F_{i,j}(D \cap B_{4, (o, \omega)})} \omega_0^{n-1} \leq \int_{D \cap B_{B_{T_s, r}(x_i, \omega_j)}} F_{i,j}^* \omega_0^{n-1} \leq C \int_{D \cap B_{B_{T_s, r}(x_i, \omega_j)}} (s_j r_i)^{2-2n}\omega_1^{n-1}.$$ 

Then as $i \to \infty$, it follows that

$$\frac{s_j^{2n-2}}{C} \leq \int_{S_x \cap B_{T_s, j}(y, \omega_s)} F_{\infty,j}^* \omega_0^{n-1} \leq H_{2n-2}(S_x \cap B_{6r_j}(y, \omega_s)).$$  \hfill (4.16)

where $H_{2n-2}$ denotes the $(2n - 2)$-dimensional Hausdorff measure with respect to $g_s$.

In summary, we have the following lemma.

**Lemma 4.6.** For any $\epsilon > 0$ small there is a $j_c$ such that for any $j \geq j_c$, the Lipschitz map $F_{\infty,j}$ maps $B_{T_s, j}(y, g_s)$ into $B_{T_s, o}(g_j)$ satisfying that

1. its image contains $B_{T_s, o}(g_j)$;
2. $F_{\infty,j}(S_x \cap B_{T_s, j}(y, g_s))$ is a local divisor $D^j \subset T_{8, \epsilon}$;
3. for any $\delta > 0$, there is a $\nu = \nu(\delta)$ such that $F_{\infty,j}^{-1}(T_{8, \nu}) \subset T_3(S_x \cap B_{T_s, j}(y, g_s))$.

**Proof.** We only need to show (3). If (3) is not true, then $F_{\infty,j}^{-1}(D^j \cap B_{B_6,5}(o, g_j))$ has at least two distinct components: One lies in $S_x$ and the other does not. Thus for sufficiently large $i$, the preimage $F_{i,j}^{-1}(F_{i,j}(D^j) \cap B_{B_6,5}(o, g_j))$ has at least two components. On the other hand, for sufficiently large $i$ and $j$ restricted on $B_{10s, r}(x_i, \omega_i) \setminus T_3(D)$, $F_{i,j}$ is biholomorphic onto its image. By (4.12) and (4.13), $F_{i,j}$ is very close to a coordinate map, i.e., almost separating points on $B_{B_6,5}(o, g_j)$. By (4.10), $B_{10s, r}(x_i, \omega_i)$ lies in some $\mathbb{C}^{N_i,j}$, and thus $F_{i,j}$ is one-to-one on $B_{T_s, r}(x_i, \omega_i)$ which leads to a contradiction. \hfill □
Next, for sufficiently large $i$ and $j$, there are uniformly bounded functions $\varphi_{i,j}$ on $B_{8\sqrt{r}_i}(x_i, \omega_i)$ satisfying
\[
(s_j r_i)^{-2} \omega_i = \sqrt{-1} \partial \bar{\partial} \varphi_{i,j}. \tag{4.17}
\]
The reason is that the construction $S^0_{i,j} \in K^{-k_1}_M$ is perturbed from the constant $a_0$, and thus $\|S^0_{i,j}\|_i$ is close to a uniform constant for sufficiently large $i$ and $j$ and note that
\[
-\sqrt{-1} \partial \bar{\partial} \log \|S^0_{i,j}\|^2 = k_j \omega_i = (s_j r_i)^{-2} \omega_i.
\]
Moreover by this observation, the volume of $D \cap B_{T_{s_j} r_i}(x_i, \omega_i)$ with respect to $(s_j r_i)^{-2} \omega_i$ is uniformly bounded. It follows from (4.15), (4.17) and the basic fact in pluripotential theory that
\[
\int_{D \cap B_{T_{s_j} r_i}(x_i, \omega_i)} \omega_i^{n-1} = \int_{D \cap B_{T_{s_j} r_i}(x_i, \omega_i)} (s_j r_i)^{2n-2}(\sqrt{-1} \partial \bar{\partial} \varphi_{i,j})^{n-1} \leq C(s_j r_i)^{2n-2}.
\]
As $i \to \infty$, we have
\[
\mathcal{H}_{2n-2}(S_x \cap B_{T_{s_j} (y_j, \omega_j)}) \leq C s_j^{2n-2}. \tag{4.18}
\]
Then by a covering argument, it follows that for any $R > 0$, there is a constant $C_R$ such that
\[
\mathcal{H}_{2n-2}(S_x \cap B_R (a, \omega_x)) \leq C_R. \tag{4.19}
\]

We also need the following key lemma.

**Lemma 4.7.** All the notation follows from above and assume that
(1) $\xi : \mathbb{R} \mapsto [0, 1]$ is a smooth function with $\xi(t) = 1$ for all $t \geq 8\epsilon$, and
(2) $f$ is a holomorphic function on $F_{\infty,j}(B_{2r_s}(y_j, \omega_j))$ such that $|f(z', z_n)| \leq |z_n|$ whenever $|z_n| \geq 8\epsilon$.

Then there is a uniform constant $C$ such that
\[
s_j^{2-2n} \int_{B_{2r_s}(y_j, \omega_j)} \left| \nabla (h \cdot F_{\infty,j}) \right|^2_{\omega_j} \omega_j^n \leq C \int_{F_{\infty,j}(B_{2r_s}(y_j, \omega_j))} \sqrt{-1} \partial h \wedge \bar{\partial} h \wedge (dz' \wedge d\bar{z'})^{n-1},
\]
where $h(z', z_n) = \xi(|f|^2(z', z_n))$.

**Proof.** It suffices to prove this inequality for all $F_{i,j}$ and then let $i \to \infty$. Let $\tilde{\eta} : \mathbb{R} \mapsto \mathbb{R}$ be a cutoff function such that $\tilde{\eta}(t) = 1$ for $t \leq 40$, $\tilde{\eta}(t) = 0$ for $t > 46$, $|\tilde{\eta}'| \leq 1$ and $|\tilde{\eta}''| \leq 2$. Then we have $\sqrt{-1} \partial \bar{\partial} \tilde{\eta}(|z'|^2) \leq 200n dz' \wedge d\bar{z'}$.

By the assumptions (1) and (2), we can see that $\tilde{\eta}(|z'|^2)|dh|^2$ vanishes near the boundary of $F_{i,j}(B_{2r_s}(x_i, \omega_i))$. It is easy to check that $\partial h \wedge \bar{\partial} h = 0$. It follows from those facts, (4.17) and the integration by parts that
\[
\int_{B_{2r_s}(x_i, \omega_i)} \tilde{\eta}(|z'|^2) |\nabla (h \cdot F_{i,j})|^2_{\omega_i} \omega_i^n
\]
\[
\leq C \int_{F_{i,j}(B_{2r_s}(x_i, \omega_i))} \sqrt{-1} \partial h \wedge \bar{\partial} h \wedge (dz' \wedge d\bar{z'})^{n-1}.
\]
Then the lemma follows by letting $i \to \infty$.  

To complete the whole construction of the cutoff function in Lemma 4.5, let $\tilde{\epsilon}$ be given. Fix a small $\epsilon_0 > 0$. Since $S_x$ has complex codimension at least 2, we can find a finite cover of $S_x \cap B_{2\epsilon^{-1}}(x, g_x)$ by the balls $B_{r_a}(y_a, g_x)$ ($a = 1, \ldots, l$) satisfying
(i) $y_a \in S_x$ and $2r_a \leq \epsilon_0$;
(ii) $B_{r_a/2}(y_a, g_x)$ are mutually disjoint;
(iii) $\sum_a r_a^{2n-3} \leq 1$;
(iv) the number of overlapping balls $B_{2r_a}(y_a, g_x)$ is uniformly bounded.
Define $\tilde{\eta}$ to be a cutoff function $\mathbb{R} \mapsto \mathbb{R}$ satisfying $0 \leq \tilde{\eta} \leq 1, |\tilde{\eta}'| \leq 2$, and moreover $\tilde{\eta}(t) = 1$ for $t > 1.6$ and $\tilde{\eta}(t) = 0$ for $t \leq 1.1$. Set $\chi = \prod_a \chi_a$ where $\chi_a(y) = \tilde{\eta}(\frac{d(y, y_a)}{r_a})$ if $y \in B_{2r_a}(y_a, g_x)$ and $\chi_a(y) = 1$ otherwise. Then $\chi$ vanishes on the closure of $B = \bigcup_a B_{r_a}(y_a, g_x)$ which contains $S_x \cap B_{r-1}(x, g_x)$. Furthermore, $\chi$ satisfies
\[
\int_{C_a} |\nabla \chi|^{2} \omega_x^n + \int_{B_{2r_a}(y_a, g_x)} |\nabla \chi_a|^{2} \omega_x^n \leq C \sum_{a} \int_{B_{2r_a}(y_a, g_x)} |\nabla \chi_a|^{2} \omega_x^n \leq C \sum_{a} \int_{B_{2r_a}(y_a, g_x)} \chi_a^{2n-2} \leq C \epsilon_0 \sum_{a} \chi_a^{2n-3} \leq C \epsilon_0. \tag{4.20}
\]

There is a finite cover of $S_x \cap B_{r-1}(x, g_x) \setminus B$ by the balls $B_{\delta a}(y_a, g_x)$ for which Lemma 4.6 holds ($b = 1, \ldots, N$). We can also assume that the number of overlapping balls $B_{\delta a}(y_a, g_x)$ is bounded by a uniform constant $K$. Choose the smooth functions $\zeta_b$ associated with the cover $\{B_{\delta a}(y_a, g_x)\}$ satisfying
(1) $0 \leq \zeta_b \leq 1, |\nabla \zeta_b| \leq C\delta^{-3}$;
(2) $\text{supp}(\zeta_b)$ is contained in $\{B_{\delta a}(y_a, g_x)\}$;
(3) $\sum_{a} \zeta_b \equiv 1$ near $S_x \cap B_{r-1}(x, g_x) \setminus B$.

Then $\{\zeta_b, 1 - \sum_a \zeta_a\}$ form a partition of the unit for the cover $\{B_{\delta a}(y_a, g_x)\}$ and $B_{r-1}(x, g_x)$. As the proof in the simplest case in the beginning, we denote by $\eta$ a cutoff function $\mathbb{R} \mapsto \mathbb{R}$ satisfying $0 \leq \eta \leq 1, |\eta'(t)| \leq 1$ and

$$\eta(t) = 0 \quad \text{for } t > \log(-\log \delta) \quad \text{and} \quad \eta(t) = 1 \quad \text{for } t < \log(-\log \delta).$$

For each $b$, let $F_b$ be the map from $B_{7\delta}(y_b, g_x)$ into $B_{7+\epsilon}(o, g_{3\delta})$ and $D_b \subset B_{7+\epsilon}(o, g_{3\delta})$ be the divisor given by Lemma 4.6. Let $\nu$ be that in Lemma 4.6(3) for $\delta = \epsilon$. It is easy to see that $\nu$ can be chosen independent of $b$. Let $10 \epsilon < \nu$ and $f_b$ be a local defining function of $D_b$ satisfying Lemma 4.7(2). We define a function $\gamma_{\epsilon, b}$ on $B_{7\delta}(y_b, g_x)$ as follows: If $|f_b(F_b(y))| \geq \epsilon/3$, put $\gamma_{\epsilon, b}(y) = 1$ and if $|f_b(F_b(y))| \leq \epsilon$, put

$$\gamma_{\epsilon, b}(y) = \eta \left( \log \left( -\log \left( \frac{|f_b(F_b(y))|}{\epsilon} \right) \right) \right). \tag{4.21}$$

Choosing $\delta$ sufficiently small, we can deduce from the beginning of the proof that

$$\int_{B_{\delta a}(y_a, g_x)} |\nabla \gamma_{\epsilon, b}|^{2} \omega_x^n \leq \epsilon_0 s_b^{2n-2}. \tag{4.22}$$

Moreover if $\epsilon \delta < \nu$, then by Lemma 4.6(3), $\gamma_{\epsilon, b}(y) = 1$ if $d(y, S_x) \geq \epsilon$.

Now combine all the constructions above and define

$$\gamma(y) = \chi(y) \left( 1 - \sum_b \zeta_b(y) + \sum_b \zeta_b(y) \gamma_{\epsilon, b}(y) \right). \tag{4.23}$$

Then $\gamma(y) = 1$ whenever $d(y, S_x) \geq \epsilon$ and vanishes in a neighborhood of $S_x$. It follows from (4.23) and (4.20) that

$$\int_{B_{r-1}(x, g_x)} |\nabla \gamma|^{2} \omega_x^n \leq C \left( \epsilon_0 + K \sum_b \int_{B_{\delta a}(y_a, g_x)} |\nabla \zeta_b(1 - \gamma_{\epsilon, b})|^{2} \omega_x^n \right).$$

Assuming that $\epsilon \leq \epsilon_0$, by (4.22) and (4.16) we have

$$\int_{B_{\delta a}(y_a, g_x)} \int_{B_{r-1}(x, g_x)} |\nabla \zeta_b(1 - \gamma_{\epsilon, b})|^{2} \omega_x^n \leq 4 \epsilon_0 s_b^{2n-2} \leq 4 \epsilon_0 \mathcal{H}_{2n-2}(S_x \cap B_{r-1}(x, g_x)).$$

Combining the estimates above and (4.19), we have

$$\int_{B_{r-1}(x, g_x)} |\nabla \gamma|^{2} \omega_x^n \leq C(1 + 4K^2 \mathcal{H}_{2n-2}(S_x \cap B_{r-1}(x, g_x))) \epsilon_0 \leq C(1 + 4K^2 C_{r-1}) \epsilon_0.$$

The proof of Lemma 4.5 is completed. □
Now we continue the proof of the partial $C^0$-estimate (4.1). Define a cutoff function $\eta$ satisfying

$$
\eta(t) = 1 \quad \text{for } t \leq 1, \quad \eta(t) = 0 \quad \text{for } t \geq 2 \quad \text{and} \quad |\eta'(t)| \leq 1.
$$

Let $\delta_0 > 0$ be determined later. Choose $\bar{\epsilon}$ such that $\gamma_{\bar{\epsilon}} = 1$ on $V(x; \delta)$. Then we choose $\epsilon$ such that $\delta_0 > 4 \epsilon$ and $V(x; \epsilon)$ contains the support of $\gamma_{\bar{\epsilon}}$. Now for any $y \in V(x; \epsilon)$, define

$$
\hat{\tau}(\phi(y)) = \eta(2\epsilon (\rho_x(y) + \rho_x(y)^{-1})) \gamma_{\bar{\epsilon}}(y) \tau(\phi(y)). \tag{4.24}
$$

It follows from the constructions above that

$$
\hat{\tau} = \tau \quad \text{on} \quad \phi(V(x; \delta_0)). \tag{4.25}
$$

Moreover, it follows from (4.4) and the fact $(M_\infty, r_j^{-2} \omega_\infty, x)$ converge to $(C_z, \omega_x, o)$ that

$$
\int_{M_\infty} \|\hat{\partial} \hat{\tau}\|_{2, \omega_\infty}^2 \lesssim \nu r^{2n-2}, \tag{4.26}
$$

where $r = r_j$ and $\nu = \nu(\delta, \epsilon)$ which could be sufficiently small as long as $\delta, \epsilon$ and $\bar{\epsilon}$ are small. Moreover, we also have

$$
\int_{M_\infty} \|\hat{\tau}\|_{2, \omega_\infty}^2 \lesssim 2^n \int_{V(x; \epsilon)} e^{-\rho_x^2 \omega_x^2} < C_0 r^{2n}. \tag{4.27}
$$

Now we can see that $\hat{\tau}$ is supported outside the singular set $S$ of $M_\infty$. Meanwhile we need to modify $\hat{\tau}$ to be supported outside $D_\infty$. If $T < 1$, we know that $D_\infty \subset S$, and then we just set $\hat{\tau} = \tau$. If $T = 1$, we could put $\rho = \|\sigma_\infty\|$ which was constructed in the last section. Similarly, let $\bar{\eta}$ be a cutoff function satisfying $0 \leq \bar{\eta} \leq 1$, $|\bar{\eta}'| \leq 1$ and

$$
\bar{\eta}(t) = 0 \quad \text{for } t > \log(\log(\log \bar{\epsilon})) \quad \text{and} \quad \bar{\eta}(t) = 1 \quad \text{for } t < \log(\log(\log \bar{\epsilon})).
$$

Define

$$
\bar{\tau}(z) = \bar{\eta}(\log(\log(\log \rho(z))) \hat{\tau}(z).
$$

Then $\bar{\tau}$ is supported away from $S \cup D_\infty$ and coincides with $\hat{\tau}$ wherever $\rho \geq \bar{\epsilon}$. When $\bar{\epsilon}$ is small enough, it follows from (4.26) that

$$
\int_{M_\infty} \|\hat{\partial} \bar{\tau}\|^2_{2, \omega_\infty} \lesssim 2^n \int_{V(x; \epsilon)} e^{-\rho_x^2 \omega_x^2} < C_0 r^{2n}. \tag{4.28}
$$

Now set $U(x; \epsilon)$ to be $\phi(V(x; \epsilon))$ if $T < 1$ and

$$
\phi(V(x; \epsilon)) \setminus \{z \mid d_\infty(z, D_\infty) \leq \epsilon \}
$$

if $T = 1$. If $\bar{\epsilon}$ is sufficiently small $\bar{\tau} = \tau$ on $U(x; \delta_0)$ and the support of $\bar{\tau}$ is contained in $U(x; \epsilon)$ if $\epsilon$ is small enough.

Recall that $(M \setminus D, \omega_i)$ converges to $(M_\infty \setminus (S \cup D_\infty, \omega_\infty)$ and the Hermitian metric $H_i$ defined on $K_{M_i}$ converges to $H_\infty$ on $M_\infty \setminus (S \cup D_\infty)$ in the $C^\infty$-topology. Thus for $\delta_1 \to 0$, we could define the diffeomorphisms

$$
\tilde{\phi}_i : M_\infty \setminus T_1(S \cup D_\infty) \mapsto M \setminus T_1(D)
$$

associated with the smooth isomorphisms

$$
F_i : K_{M_\infty} \mapsto K_M
$$

over $M_\infty \setminus T_1(S \cup D_\infty)$, where $T_1(S \cup D_\infty)$ and $T_1(D)$ are $\delta_1$-tubular neighborhoods of $S \cup D_\infty$ and $D$ with respect to $\omega_i$ and $\omega_\infty$. Moreover, we have

(C1) $\tilde{\phi}_i(M_\infty \setminus T_1(S \cup D_\infty)) \subset M \setminus T_1(D)$;
(C2) $\pi_\epsilon \circ F_i = \tilde{\phi}_i \circ \pi_\infty$, where $\pi_\epsilon$ and $\pi_\infty$ are corresponding projections;
(C3) $\|\tilde{\phi}_i^* \omega_i - \omega_\infty\|_{C^2(M_\infty \setminus T_1(S \cup D_\infty))} \lesssim \delta_1$.
We set large enough \( i \) such that \( U(x;\epsilon) \subset M_{\infty} \setminus T_i(S \cup D_{\infty}) \). Set \( \tilde{\tau}_i = F_i(\tau) \). Then it follows that
\[
\tilde{\tau}_i = F_i(\tau) \text{ on } \phi_i(U(x;\delta_0)) \text{ and moreover it follows from (4.28) that}
\[
\int_M \| \partial \tilde{\tau}_i \|^2 \omega_i^n \leq 3\nu r^{2n-2}. \tag{4.29}
\]
By the \( L^2 \)-estimate in Lemma 4.3, there exists a section \( v_i \) of \( K_M^{-1} \) such that \( \partial v_i = \partial \tilde{\tau}_i \) and
\[
\int_M \| v_i \|^2 \omega_i^n \leq \frac{1}{7} \int_M \| \partial \tilde{\tau}_i \|^2 \omega_i^n \leq 3\nu r^{2n}.
\]
Take \( \sigma_i = \tilde{\tau}_i - v_i \), which is a holomorphic section of \( K_M^{-1} \). Then it follows from above and (4.27) that
\[
\int_M \| \sigma_i \|^2 \omega_i^n \leq 2C_0 r^{2n}, \tag{4.30}
\]
where \( C \) is independent of \( i \). As \( \tilde{\tau}_i = F_i(\tau) \) on \( \tilde{\phi}_i(U(x;\delta_0)) \), and \( F_i \) is almost holomorphic by the condition \((C_2)\), it follows from (4.4) that \( \| \tilde{\phi}_i \| \leq c\delta \). Then the standard elliptic estimate implies that
\[
\sup_{\tilde{\phi}_i(U(x;\epsilon_0) \cap \phi(B_1(o, g_x)))} \| v_i \|^2 \leq C(\delta_0 r)^{-2n} \int_M \| v_i \|^2 \omega_i^n \leq C\delta_0^{-2n} \nu. \tag{4.31}
\]
As \( C \) is uniform, let \( \delta \) and \( \epsilon \) be small enough such that \( \nu = \nu(\delta, \epsilon) \) satisfies that \( 20C\nu \leq \delta_0^{2n} \). Then it follows from (4.4) and (4.31) that
\[
\| \sigma_i \| \geq \| F_i(\tau) \|_i - \| v_i \|_i > e^{-\frac{i}{2}} - \frac{1}{4} > \frac{1}{3} \text{ on } \tilde{\phi}_i(U(x;2\delta_0) \cap \phi(B_1(o, g_x))). \tag{4.32}
\]
On the other hand by the derivative estimate of the holomorphic sections in Lemma 3.1 and (4.30), we have
\[
\sup_M \| \nabla \sigma_i \|_i \leq C t^{n-1} \left( \int_M \| \sigma_i \|^2 \omega_i^n \right)^{\frac{1}{2}} \leq C'' r^{-1}. \tag{4.33}
\]
By the choice of \( \phi \), if \( \epsilon \) is much smaller than \( \delta_0 \) then for some \( u \in \partial B_1(o, g_x) \), we have
\[
d_{\infty}(x, \phi(\delta_0 u)) \leq d_{\infty}(x, \phi(\epsilon u)) + d_{\infty}(\phi(\epsilon u), \phi(\delta_0 u)) \leq 2\delta_0 r,
\]
where \( \phi(\epsilon u) \in \partial U(x;\epsilon) \). Then for sufficiently large \( i \), it follows that
\[
d_i(x_i, \tilde{\phi}_i(\phi(\delta_0 u))) \leq 5\delta_0 r.
\]
Then it follows from (4.32) and (4.33) that
\[
\| \sigma_i \|_i(x_i) \geq \frac{1}{3} - 5C'' \delta_0.
\]
Thus we can choose \( \delta_0 \leq \frac{1}{10C''} \), and then \( \| \sigma_i \|_i(x_i) \geq \frac{1}{10} \). Combining the fact (4.30), we can see that (4.1) holds, and consequently, Theorem 1.1 is completed.

Also by the partial \( C^0 \)-estimate, we have the following structure theorem as [35, Theorem 5.9].

**Theorem 4.8.** The Gromov-Hausdorff limit \( M_{\infty} \) is a normal variety embedded in some \( \mathbb{C}P^N \) whose singular set is a subvariety \( \mathcal{S} \) of complex codimension at least 2. If \( T < 1 \), \( \mathcal{S} \) is a subvariety consisting of a divisor \( D_{\infty} \) and a subvariety \( \mathcal{S} \) of complex codimension at least 2. If \( T = 1 \), \( \mathcal{S} = \mathcal{F} \). Moreover, \( D_{\infty} \) is the limit of \( D \) under the Gromov-Hausdorff convergence.
5 Reductivity of the automorphism group of the limit space

In this section, we follow [35, Lemma 6.9] to prove Corollary 1.2. As \( t_i \to T \), by taking a subsequence we may assume that \((M, D, \omega_t)\) converge to \((M_\infty, D_\infty, \omega_\infty)\). By the partial \(C^0\)-estimate in Theorem 1.1, similar to [35], we have

1. \( M \) is embedded in \( \mathbb{C}P^N \) through an orthonormal basis of \( H^0(M, K_M^{-1}) \) given by \( H_i = H_{\omega_i} \);
2. \( M_\infty \subset \mathbb{C}P^N \) is a normal subvariety with a divisor \( D_\infty \);
3. there are \( \sigma_i \in G = SL(N + 1, \mathbb{C}) \) such that \( (\sigma_i(M), \sigma_i(D)) \) converge to \((M_\infty, D_\infty)\).

It follows that the stabilizer \( G_\infty \) of \((\sigma_i(M), \sigma_i(D)) \) in \( G \) contains a nontrivial holomorphic subgroup. We want to show that the Lie algebra \( \eta_\infty \) of \( G_\infty \) is reductive. For this target, we also need two steps. First, we show that any holomorphic field in \( \eta_\infty \) is a complexification of a Killing field on \( M_\infty \). Then we can show that any Killing field can be extended to be the imaginary part of a holomorphic field on \( \mathbb{C}P^N \).

As in [35], let \( X \) be a holomorphic vector field on \( \mathbb{C}P^N \) which is tangent to \( M_\infty \). We need to show that there is a bounded function \( \theta_\infty \) such that \( i_X \omega_\infty = \sqrt{-1} \delta \theta_\infty \) on \( M_\infty \setminus \mathcal{S} \cup D_\infty \). Let \( \phi_s \) be a one-parameter subgroup of automorphisms generated by \( Y = ReX \) or \( ImX \). Then we have

\[
\phi_s^* \omega_\infty = \frac{1}{t} \omega_{FS} + \sqrt{-1} \delta \phi_s \psi_s.
\]

Let \( \phi_s \) act on (1.3) and note that \( \omega_i \) converge to \( \omega_\infty \) smoothly on \( M_\infty \setminus \mathcal{S} \cup D_\infty \). It follows that

\[
\text{Ric}(\phi_s^* \omega_\infty) = T \phi_s^* \omega_\infty + (1 - T) b_0 \alpha_0
\]

on \( M_\infty \setminus \mathcal{S} \cup D_\infty \). By comparing the equations in the cases of \( s \) and \( 0 \), and suitably by choosing \( \psi_s \), it follows that

\[
\phi_s^* \omega_\infty^n = (\omega_\infty + \sqrt{-1} \delta \phi_s \xi_s)^n = e^{-T \xi_s} \omega_\infty^n \quad \text{on} \quad M_\infty \setminus \mathcal{S} \cup D_\infty,
\]

where \( \xi_s = \psi_s - \psi_0 \). As in [35], first we can show that \( \psi_s \) or \( \xi_s \) is bounded and continuous by the partial \( C^0 \)-estimate. To see the continuity, we note that

\[
\omega_\infty = \frac{1}{t} \omega_{FS} + \sqrt{-1} \delta \psi_0,
\]

\[
\phi_s^* \omega_\infty = \frac{1}{t} \phi_s^* \omega_{FS} + \sqrt{-1} \delta \phi_s \psi_0 \circ \phi_s = \frac{1}{t} \omega_{FS} + \sqrt{-1} \delta \phi_s \psi_s.
\]

Thus we have

\[
\xi_s = \psi_s - \psi_0 = \phi_s \circ \psi_s - \psi_0 + \zeta_s,
\]

where \( \zeta_s \) is smooth on \( \mathbb{C}P^N \) and satisfies

\[
\phi_s^* \omega_{FS} = \omega_{FS} + \sqrt{-1} \delta \phi_s \zeta_s, \quad \zeta_0 = 0.
\]

By the partial \( C^0 \)-estimate a subsequence of \( \{\log \rho_{s,l}\} \) converges to \( t \psi_0 + c \) for some constant \( c \) as \( (M, \omega_t) \) converge to \((M_\infty, \omega_\infty)\). This follows from the definition of \( \rho_{s,l} \) in (1.2) and the fact \( \sqrt{-1} \delta \rho_{s,l} = \omega_{FS} - \omega_\infty \).

By the gradient estimate in Lemma 3.1, \( \rho_{s,l} \) are uniformly continuous for each fixed \( l \). Moreover, \( \rho_{s,l} \) are uniformly bounded by a positive constant. It follows that \( \psi_0 \) is continuous and thus \( \xi_s \) is continuous. We also see that \( |\xi_s| \leq \frac{1}{2} \) for sufficiently small \( s \).

It follows from (5.1) that

\[
- \sqrt{-1} \delta \xi_s \wedge \left( \sum_{i=0}^{n-1} \omega_{\infty}^i \wedge \phi_s^* \omega_\infty^{n-1-i} \right) = (1 - e^{-T \xi_s}) \omega_\infty^n.
\]

Recall that as \((M, \omega_t)\) converge to \((M_\infty, \omega_\infty)\), \( D \subset (M, \omega_t) \) converges to a divisor \( D_\infty \subset M_\infty \) modulo a singular set \( \mathcal{S} \) with complex codimension at least 2. By Section 3, we have a holomorphic section \( \tau_\infty \) whose zero set contains \( \mathcal{S} \cup D_\infty \). In particular, \( M_\infty \setminus \tau_\infty^{-1}(0) \) is contained in the regular part of \((M_\infty, \omega_\infty)\). Thus we can choose a cutoff function \( \tilde{\eta} : \mathbb{R} \to \mathbb{R} \) satisfying \( \tilde{\eta}(t) = 1 \) for \( t \geq 2 \), \( \tilde{\eta}(t) = 0 \) for \( t < 1 \), \( |\tilde{\eta}'(t)| \leq 1 \) and \( |\tilde{\eta}''(t)| \leq 4 \). For any \( \epsilon > 0 \), we can define \( \gamma_\epsilon(x) = \tilde{\eta}(\epsilon \log(- \log \|\tau_\infty\|^2_0(x))) \).
Multiplying (5.2) by $\gamma_\delta^2 \xi_s$ and integrating by parts, we have
\[
\int_{M_\infty} \gamma_\delta^2 \xi_s (1 - e^{-T \xi_s}) \omega_\infty = \int_{M_\infty} \sqrt{-1} \partial (\gamma_\delta^2 \xi_s) \wedge \partial \xi_s \wedge \left( \sum_{i=0}^{n-1} \omega_n^i \wedge \phi_s^i \omega_\infty^{n-i-1} \right) \\
\leq \frac{3}{4} \int_{M_\infty} \sqrt{-1} \gamma_\delta^2 \xi_s \wedge \partial \xi_s \wedge \omega_\infty^{n-1} \\
- \int_{M_\infty} \sqrt{-1} \xi_s \partial \gamma_\delta \wedge \partial \xi_s \wedge \left( \sum_{i=0}^{n-1} \omega_n^i \wedge \phi_s^i \omega_\infty^{n-i-1} \right).
\]

For the last term, recall that $\omega_\infty = \frac{1}{2} \omega_{FS} + \sqrt{-1} \partial \bar{\partial} \psi_0$ and $\phi_s^i \omega_\infty = \frac{1}{2} \omega_{FS} + \sqrt{-1} \partial \bar{\partial} \psi_s$, where $\psi_0$ and $\psi_s$ are uniformly bounded functions by the partial $C^0$-estimate. It follows from similar computations in Lemma 4.5 combined with standard pluri-potential theory (see [35, Lemma 6.10] for the details) that the last term tends to 0 as $\epsilon \to 0$. Thus we have the following inequality if $s$ is so small such that $|\xi_s| \leq \frac{1}{2}$:
\[
\frac{1}{n} \int_{M_\infty} |\nabla \xi_s|^2 \omega_\infty^{n-1} \leq \frac{1}{3} \int_{M_\infty} |\xi_s|^2 \omega_\infty.
\]
By using $\bar{\eta}$ above, we set $\bar{\gamma}_\delta(x) = 1 - \bar{\eta}(\delta^{-1} ||\tau_\infty||_0(x))$. Then $\gamma_\delta(x) = 1$ when $||\tau_\infty||_0(x) \leq \delta$ and has its support in the subset $E_\delta := \{ x \in M_\infty : ||\tau_\infty||_0(x) \leq 2\delta \}$.

Recall that $\xi_s = \psi_0 \circ \phi_s - \psi_0 + \zeta_s$ and note that $\zeta_s$ is defined on $CP^n$ by $\phi_s^i \omega_{FS} = \omega_{FS} + l \sqrt{-1} \partial \bar{\partial} \zeta_s$, and thus $|\zeta_s| \leq C_\delta s$. On the other hand, as $\psi_0$ is defined by $\omega_\infty = \frac{\omega_{FS}}{2} + l \sqrt{-1} \partial \bar{\partial} \psi_0$, $|\psi_0| \leq C'_\delta$ outside $E_{\delta/4}$ for some $C'_\delta$. By noting that $\phi_0$ is the identity map, for $t$ small it follows that
\[
|\zeta_s| \leq |\psi_0 \circ \phi_s - \psi_0| + |\zeta_s| \leq \left( C'_\delta \sup_{M_\infty \setminus E_{\delta/4}} + C_\delta \right) s \quad \text{on } M_\infty \setminus E_{\delta/2}.
\]
Then we have
\[
\int_{M_\infty \setminus E_{\delta/2}} |\xi_s|^2 \omega_\infty^{n-1} \leq C''_\delta s^2.
\]
By combining (5.3) and (5.4), it follows that
\[
\int_{M_\infty} |\nabla (\gamma_\delta \xi_s)|^2 \omega_\infty \leq \frac{2}{3} \int_{M_\infty} |\nabla \xi_s|^2 \omega_\infty^{n-1} + 10 \int_{M_\infty} |\nabla \gamma_\delta|^2 |\xi_s|^2 \omega_\infty \\
\leq 3n \left( \int_{E_{\delta/2}} |\gamma_\delta \xi_s|^2 \omega_\infty^3 + \int_{M_\infty \setminus E_{\delta/2}} |\xi_s|^2 \omega_\infty^3 \right) + s^2 C''_\delta \\
\leq 3n \int_{M_\infty} |\gamma_\delta \xi_s|^2 \omega_\infty^3 + s^2 C_\delta.
\]
Next, we need the following estimate of the first eigenvalue of $E_\delta$ that $\lambda_1(E_\delta) \geq 4n$ for $\delta$ sufficiently small, where
\[
\lambda_1(E) := \left\{ \frac{\int_E |\nabla v|^2 \omega_\infty^{n-1}}{\int_E |v|^2 \omega_\infty^{n-1}} \right\} \quad \text{if } v \in C^1(E \setminus S) \cap L^\infty(E), v \mid_{\partial E} = 0
\]
f for any open $E \subset M_\infty$ with nonempty boundary $\partial E \subset M_\infty \setminus S$. This estimate follows from the smooth approximation in Theorem 2.1, Croke and Li’s standard estimates (see the claim in [35, p.47]). Thus it follows from (5.4), (5.5) and the first eigenvalue estimate above that
\[
\int_{M_\infty} |\xi_s|^2 \omega_\infty^3 \leq 2 \left( \int_{M_\infty} |\gamma_\delta \xi_s|^2 \omega_\infty^3 + \int_{M_\infty} |(1 - \gamma_\delta) \xi_s|^2 \omega_\infty^3 \right) \\
\leq \frac{2C_\delta}{n} s^2 + 2 \int_{M_\infty \setminus E_{\delta/2}} |\xi_s|^2 \omega_\infty^3 \leq C_\delta s^2.
\]
Combining this with (5.3) and dividing it by $s^2$, we have
\[
\int_{M_\infty} (|s^{-1} \nabla \xi_s|^2 + |s^{-1} \xi_s|^2) \omega_\infty \leq 2C_\delta.
\]
Now assume $Y = \text{Re}X$ as the generator of $\phi_s$. Recalling that $\xi_s = \psi_0 \circ \phi_s - \psi_0 + \zeta_s$ and $\psi_0$ is smooth on $M_\infty \setminus \mathcal{S} \cup D_\infty$, we can see that $s^{-1}\xi_s$ converges pointwisely to $u$ on $M_\infty \setminus \mathcal{S} \cup D_\infty$ as $t \to 0$. Letting $t \to 0$, by (5.6) we have
\[
\int_{M_\infty} |u|^2 \omega_{\infty}^n \leq C. \tag{5.7}
\]
By differentiating (5.1), it follows that
\[
\int_{M_\infty} u \omega_{\infty}^n = 0.
\]
For any $q \geq 2$, multiplying (5.2) by $\gamma_s |\xi_s|^q\omega_{\infty}^n$, integrating by parts and letting $\epsilon \to 0$, we have
\[
\int_{M_\infty} |\nabla|s^{-1}\xi_s|^{q/2}|^2 \omega_{\infty}^n \leq \frac{12nTq^2}{q-1} \int_{M_\infty} |s^{-1}\xi_s|^2 \omega_{\infty}^n. \tag{5.8}
\]
By (5.7) and the Sobolev inequality, for any $q \geq 2$ there is a uniform constant $C_q$ satisfying
\[
\int_{M_\infty} |s^{-1}\xi_s|^q \omega_{\infty}^n \leq C_q.
\]
Furthermore, given any $\epsilon > 0$, it follows from above that
\[
\int_{E_\delta} |s^{-1}\xi_s|^q \omega_{\infty}^n \leq \text{Vol}(E_\delta)^{1/n} \left( \int_{M_\infty} |s^{-1}\xi_s|^q \omega_{\infty}^n \right)^{\frac{n-1}{n}} \leq \text{Vol}(E_\delta)^{1/n} (C_{\text{in}})^{\frac{n-1}{n}} \leq \frac{\epsilon}{3}
\]
for sufficiently small $\delta$. Let $s \to 0$. It follows that
\[
\int_{E_\delta} |u|^q \omega_{\infty}^n \leq \frac{\epsilon}{3}.
\]
On the other hand, as $s^{-1}\xi_s$ converges to $u$ outside $D_\infty \cup \mathcal{S}$ for $s$ sufficiently small, we have
\[
\int_{M_\infty \setminus E_\delta} |s^{-1}\xi_s - u|^q \omega_{\infty}^n \leq \frac{\epsilon}{3}.
\]
It follows from the estimates above that $s^{-1}\xi_s$ converges to $u$ in any $L^q$-norm. Then let $s \to 0$, and we have a similar inequality to (5.8) for $u$. By (5.7) and the standard Moser iteration we can show that $u$ is bounded.

Note that each $\psi_s$ is smooth outside $\mathcal{S} \cup D_\infty$ and satisfies
\[
\frac{1}{t} \omega_{FS} + \sqrt{-1}\partial \bar{\partial} \psi_s = \phi_s^* \omega_{\infty} = \frac{1}{t} \phi_s^* \omega_{FS} + \sqrt{-1}\partial \bar{\partial} \phi_s^* \psi_0.
\]
It follows that $\psi_s = \phi_s^* \psi_0 + \zeta_s$, where $\phi_s^* \omega_{FS} = \omega_{FS} + \sqrt{-1}\partial \bar{\partial} \zeta_s$. Note that $\zeta_s$ is a smooth function on $\mathbb{C}P^N$ as well as in $s$. Thus we have
\[
u = Y(\psi_0) + \theta_u, \quad \text{where } \theta_u = \frac{\partial \zeta_s}{\partial s} \bigg|_{s=0}.
\]
Similarly by taking $Y$ to be the imaginary part of $X$, we can get a bounded function $v = Y(\psi_0) + \theta_v$. Now set $\theta_\infty = u + \sqrt{-1}\partial \bar{\partial} u$ and $\theta = \theta_u + \theta_v$. Then $\theta_\infty = X(\psi_0) + \theta$ is bounded on $M_\infty$ and $i \partial \bar{\partial} \omega_{\infty} = \sqrt{-1}\partial \bar{\partial} \theta_\infty$ since $iX \omega_{\infty} = \sqrt{-1}\partial \bar{\partial} \theta$. Moreover, it holds that
\[
\int_{M_\infty} |\nabla \theta_\infty|^2 \omega_{\infty}^n < \infty \quad \text{and} \quad \int_{M_\infty} \theta_\infty \omega_{\infty}^n = 0.
\]
Next, we need to show that $\theta_\infty$ satisfies an eigenfunction equation in a weak sense. Similar to [35], by using a test function $\zeta$ and taking the derivative of $s$ to (5.1), it follows that
\[
\int_{M_\infty} X(\zeta) \omega_{\infty}^n = T \int_{M_\infty} \zeta \theta_\infty \omega_{\infty}^n.
\]
which implies that in the weak sense,

\[- \Delta_\infty \theta_\infty = T \theta_\infty \quad \text{on} \quad M_\infty.\]  

(5.9)

By Theorem 2.1, there are smooth Kähler metrics \( \tilde{\omega} \) with \( \text{Ric}(\tilde{\omega}) \geq t_1 \tilde{\omega} \) and converging to \( (M_\infty, \omega_\infty) \) in the Cheeger-Gromov topology, where \( t_1 \to T \) as \( i \to \infty \). Similar to [35], we can show that any bounded function \( \theta \) satisfying (5.9) is the limit of eigenfunctions \( \theta_i \) on \( M \) which satisfies \(- \Delta_i \theta_i = \lambda_i \theta_i \) with \( \lambda_i \to T \) as \( t_i \to T \), where \( \Delta_i \) denotes the Laplacian of \( \tilde{\omega}_i \). It follows from Bochner’s formula that \( \lambda_i \geq t_i \), and moreover,

\[
\int_M |\nabla \bar{\partial} \theta_i|^2 \tilde{\omega}_i^n = \int_M ((\Delta_i \theta_i)^2 - \text{Ric}(\partial \theta_i, \partial \theta_i)) \tilde{\omega}_i^n \leq (\lambda_i - t_i) \int_M |\partial \theta_i|^2 \tilde{\omega}_i^n = \lambda_i (\lambda_i - t_i).
\]

It follows that \( \nabla \bar{\partial} \theta = 0 \) and \( \bar{\partial} \theta \) induces a holomorphic vector field \( Z \) outside \( S \) of \( M_\infty \). Actually, (5.9) has real solutions which implies that the imaginary part \( Y \) of \( Z \) is a Killing field. Thus the Lie algebra \( \eta_\infty \) is the complexification of a Lie algebra of Killing fields. Finally, by the same argument as in [35], \( Z \) could be extended to a holomorphic vector field on the ambient \( \mathbb{C}P^N \), which implies that \( \eta_\infty \) is reductive. The proof of Corollary 1.2 is completed.

6 The Kähler-Einstein problem

To show the existence of the Kähler-Einstein metric on the Fano manifold, we can proceed along a continuity path which was first considered by Yau [41] and later developed by a lot of geometers. In the proof of Corollary 1.3 we choose (1.3) as our continuity path

\[
\text{Ric}(\omega_t) = t \omega_t + (1 - t) \left( \sum_{r=1}^m 2\pi b_r [D_r] + b_0 \alpha_0 \right). \tag{6.1}
\]

Choose a smooth Kähler metric \( \omega \in c_1(M) \) and suppose \( \omega_t = \omega + \sqrt{-1} \partial \bar{\partial} f_t \). We can proceed to transform (6.1) to a family of complex Monge-Ampère equations. As \( \omega \in c_1(M) \), there exists a smooth function \( f_1 \) such that

\[
\text{Ric}(\omega) = \omega + \sqrt{-1} \partial \bar{\partial} f_1 = t \omega + (1 - t) \omega + \sqrt{-1} \partial \bar{\partial} f_1.
\]

Next, as \( 2\pi \sum_r b_r [D_r] + b_0 \alpha_0 \in c_1(M) \), there exists a smooth function \( f_2 \) such that

\[
\omega = \sum_r b_r \Theta_r + b_0 \alpha_0 + \sqrt{-1} \partial \bar{\partial} f_2,
\]

where \( \Theta_r \) represents the curvature of some Hermitian metric \( \| \cdot \|_r \) defined on the holomorphic line bundle associated with \( D_r \). Combining all the equations above, we have

\[
(\omega + \sqrt{-1} \partial \bar{\partial} f_1)^n = \frac{e^{f_1 + (1-t) f_2 - t \varphi_t} \omega^n}{\prod_r \| S_r \|^{2b_r (1-t)}}, \tag{6.2}
\]

where \( \varphi_t \) satisfies

\[
\int_M \frac{e^{f_1 + (1-t) f_2 - t \varphi_t} \omega^n}{\prod_r \| S_r \|^{2b_r (1-t)}} = 1.
\]

Suppose that the solvable set \( I \subset [0,1] \) is the set of \( t \) such that (6.1) or (6.2) is solvable. Actually we could choose the suitable divisors \( D_r \), smooth (1,1) form \( \alpha_0 \) and coefficients \( b_r < 1 \) such that (6.1) is solvable at \( t = 0 \) (see, for example, [17]). Thus \( I \neq \emptyset \). Next, we want to show the following lemma.

**Lemma 6.1.** \( I \) is an open set in \( [0,1] \).
Proof. As the linear theory of conic metrics [14, 18] is not so clear in the case of simple normal crossing divisors, we may need some more work to go through this problem. In fact, we need to show that if (6.2) is solvable at \( t = t_0 \in [0, 1) \), then (6.2) is solvable for a small neighborhood around \( t_0 \). The main observation is that the Ricci curvature of \( \omega_t \) is strictly greater than \( t_0 \). In the case where \( \omega_t \) is a smooth continuity path, it is enough to show the openness by [30–32]. In our settings similar to [14], we need to show a Hölder estimate for the linearization \( L_{t_0} = \Delta_{t_0} + t_0 \) of (6.2) at \( t = t_0 \), where \( \Delta_{t_0} \) denotes the Laplacian of \( \omega_{t_0} \). The idea here is to apply the approximation in Theorem 2.1 and establish a uniform Hölder estimate for this family of linearizations.

By the proof of Theorem 2.1 in [26] (also see [35]), given a solution \( \varphi_{t_0} \) to (6.2) at \( t = t_0 \), for any \( \delta > 0 \), we can solve a family of approximated complex Monge-Ampère equations

\[
(\omega + \sqrt{-1} \partial \bar{\partial} \varphi_{t_0, \delta})^n = e^{f_1 + (1-t) f_2 - t_0 \varphi_{t_0, \delta}} \omega^n,
\]

(6.3)

where \( \varphi_{t_0, \delta} \) satisfies

\[
\int_M e^{f_1 + (1-t) f_2 - t_0 \varphi_{t_0, \delta}} \omega^n = 1.
\]

Then \( \omega_{t_0, \delta} \) converges to \( \omega_t \) in the \( C^\infty(M \setminus D) \) and Gromov-Hausdorff topology. Moreover, by the Hölder estimate of complex Monge-Ampère equations with conic singularities in [27, 36], it follows that \( \| \varphi_{t_0} \|_{C^{2,\alpha}(M)} \leq c_0 \) for some \( \alpha \in (0, 1) \) with respect to \( \omega_{t_0} \). Checking the details of the proof, we can see that those estimates only depend on the local geometry of the model conic metric without anything concerning the linear theory, and thus by the Gromov-Hausdorff convergence of \( \omega_{t_0, \delta} \) we can show that \( \| \varphi_{t_0, \delta} \|_{C^{2,\alpha}(M)} \leq c_0 \) for some \( \alpha \in (0, 1) \) with respect to \( \omega_{t_0, \delta} \).

Note that the linearization of (6.3) is \( L_{t_0, \delta} := \Delta_{t_0, \delta} + t_0 \), where \( \Delta_{t_0, \delta} \) denotes the Laplacian of the approximated metric \( \omega_{t_0, \delta} = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_{t_0, \delta} \). It follows from Theorem 2.1 that \( \text{Ric}(\omega_{t_0, \delta}) \geq t_0 \varphi_{t_0, \delta} \). Thus it follows from Bochner’s formula that the first eigenvalue \( \lambda_1(-\Delta_{t_0, \delta}) > t_0 \) and then \( L_{t_0, \delta} : C^\alpha(M) \to C^{2,\alpha}(M) \) is invertible for some \( \alpha \) less than the Hölder exponent in the \( C^{2,\alpha} \)-estimate of \( \varphi_{t_0} \). Above we want to claim that for this \( \alpha \) and \( \delta_0 \in (0, 1) \), there exists a uniform constant \( C_\alpha > 0 \) such that for any \( \psi \in C^{2,\alpha}(M) \) which satisfies that \( \int_M \psi \omega^n = 0 \) and \( \delta \in (0, \delta_0) \), it follows that

\[
\| \psi \|_{C^{2,\alpha}(M)} \leq C_\alpha \| L_{t_0, \delta} \psi \|_{C^\alpha(M)},
\]

(6.4)

where the norm is with respect to \( \omega_{t_0, \delta} \). If this claim is false, we can always find a sequence \( \delta_k \to 0 \) such that there exists a sequence of the functions \( \psi_k \) satisfying that

- \( \int_M \psi_k \omega^n_{t_0} = 0 \);
- \( \| \psi_k \|_{C^{2,\alpha}(M)} = 1 \) with respect to \( \omega_{t_0, \delta_k} \);
- \( \| L_{t_0, \delta_k} \psi_k \|_{C^\alpha(M)} \leq 1/k \) with respect to \( \omega_{t_0, \delta_k} \).

As \( \| \psi_k \|_{C^{2,\alpha}(M)} = 1 \) with respect to \( \omega_{t_0, \delta_k} \), combined with the fact that \( \omega_{t_0, \delta_k} \) converge to \( \omega_{t_0} \) in the Gromov-Hausdorff topology with the uniform Hölder norm, we can show that for any smaller \( \alpha' \in (0, \alpha) \), possibly by passing to a subsequence, \( \psi_k \) converge to \( \psi \) in the norm of \( C^{2,\alpha'}(M) \) such that \( \| \psi \|_{C^{2,\alpha}(M)} = 1 \) with respect to \( \omega_{t_0} \). Moreover, it follows from this convergence and \( \| L_{t_0, \delta_k} \psi \|_{C^\alpha(M)} \leq 1/k \) that \( \| \psi \|_{C^{2,\alpha}(M)} = 0 \) and thus \( (\Delta_{t_0} + t_0) \psi = 0 \).

Now setting \( M_\epsilon \) as the subset in \( M \) outside a tubular neighborhood of \( D = \sum_{r} D_r \) with scale \( \epsilon \), ignoring \( t_0 \) in the Laplacian and covariant derivatives below, by standard computations, we have

\[
0 = - \int_{M_\epsilon} (\Delta + t_0) \psi \cdot \Delta \psi + \int_{\partial M_\epsilon} (\Delta + t_0) \psi \ast \nabla \psi
= \int_{M_\epsilon} \nabla ((\Delta + t_0) \psi) \cdot \nabla \psi
= \int_{M_\epsilon} (\nabla \nabla \nabla \psi \cdot \nabla \psi - \text{Ric}(\nabla \psi, \nabla \psi) + t_0 |\nabla \psi|^2)
= \int_{\partial M_\epsilon} \nabla \nabla \psi \ast \nabla \psi - \int_{M_\epsilon} |\nabla \nabla \psi|^2 - \int_{M_\epsilon} (\text{Ric}(\nabla \psi, \nabla \psi) - t_0 |\nabla \psi|^2).
\]
With respect to $\omega_0$, since $\sqrt{-1}\partial\bar{\partial}\psi$ is $C^\alpha$, along $M_\epsilon$ the third order derivative $\nabla\nabla\nabla\psi$ has the order at most $e^{\alpha-1}$, so does $\nabla\nabla\psi$. On the other hand, the measure of $\partial M_\epsilon$ has the order of $\epsilon$ and $\nabla\psi$ is bounded, and thus the first integration over $\partial M_\epsilon$ on the right-hand side tends to 0 as $\epsilon$ tends to 0. As $\text{Ric}(\omega_0)$ is strictly greater than $t_0$, it follows that $\psi$ must be 0, which contradicts $\|\psi\|_{C^{2,\alpha}(M)} = 1$. Thus the claim (6.4) follows.

By this claim, as $\omega_{0,\epsilon}$ converges to $\omega_0$, in the Gromov-Hausdorff topology with the uniform $C^{2,\alpha}$ norm, (6.4) is also true for $\mathcal{L}_{t_0} = \Delta_{t_0} + t_0$. Thus the openness of $I$ follows from the inverse function theorem and this lemma is true. \hfill $\Box$

**Remark 6.2.** In fact, if there is no $\alpha_0$ in (6.1), we could show the openness result in the case where there are no holomorphic vector fields tangential to divisors, which generalizes Donaldson’s openness theorem [14] to the case of simple normal crossing divisors without use of linear theory.

It remains to show the closeness of the solvable set $I$. Suppose that we have a sequence $t_i \in I$ such that $t_i \to T$ but $T \notin I$. Then by the argument in [18, 35], $\|\varphi_{t_i}\|_{C^\alpha}$ must diverge to $\infty$. By Corollary 1.2 and geometrical invariant theory, there exists a $C^\ast$-subgroup $G_0 \subset G$ which degenerates $(M, D)$ to $(M_\infty, D_\infty)$. As $\|\varphi_{t_i}\|_{C^\alpha}$ diverges to $\infty$, by [31] the central fiber $(M_\infty, D_\infty)$ of this degeneration is not biholomorphic to $(M, D)$. We will derive a contradiction to $K$-stability.

Now let us recall the $K$-stability defined by Tian [31, 35]. First, assume that $\omega \in c_1(M)$ is a smooth Kähler metric on a Fano manifold $M$. For any holomorphic vector field $X$ on $M$, the Futaki invariant is defined by

$$f_M(X) := -n \int_M \theta_X(\text{Ric}(\omega) - \omega) \wedge \omega^{n-1},$$

(6.5)

where $i_X\omega = \sqrt{-1}\partial\bar{\partial}_X$ and this is a holomorphic invariant by Futaki [16]. Moreover, in [13] the Futaki invariant was extended to normal Fano varieties as follows: Assume $M \to \mathbb{C}P^N$ through a basis of $H^0(M, K_M^l)$ for sufficiently large $l$. Then for any algebraic subgroup $G_0 = \{\sigma(t)\}_{t \in \mathbb{C}}$ of $G = SL(N + 1, \mathbb{C})$ there is a unique limiting cycle $M_0 = \lim_{t \to 0} \sigma(t)(M) \subset \mathbb{C}P^N$. Let $X$ be the holomorphic vector field whose real part generates the action by $\sigma(\epsilon^{-s})$. If $M_0$ is normal, then (6.5) could be generalized to $M_0$ by [13]. Then we can define the $K$-stability as follows.

**Definition 6.3.** We say that $M$ is $K$-stable with respect to $K_M^{-l}$ if $\text{Re}(f_{M_0}(X)) \geq 0$ for any $\mathbb{C}^\ast$ subgroup $G_0 \subset SL(N + 1, \mathbb{C})$ with a normal $M_0$, and the equality holds if and only if $M_0$ is biholomorphic to $M$. We say that $M$ is $K$-stable if it is $K$-stable for all sufficiently large $l$.

We also need the Mabuchi energy and the twisted Mabuchi energy in the proof. Given a Fano manifold $(M, \omega)$ as before, suppose that $\{\varphi_s\}$ is a path in $\text{PSH}(M, \omega)$ connecting 0 and $\varphi$. Then the Mabuchi energy can be defined as follows:

$$M_\omega(\varphi) := -n \int_0^1 \int_M \varphi_s(\text{Ric}(\omega_{\varphi_s}) - \omega_{\varphi_s}) \wedge \omega_{\varphi_s}^{n-1} ds,$$

(6.6)

where $\omega_{\varphi_s} := \omega + \sqrt{-1}\partial\bar{\partial}\varphi_s$, and moreover this Mabuchi energy is independent of the choices of paths. Give the $C^\ast$-action $\{\sigma_s := \sigma(\epsilon^{-s})\}$ above and suppose

$$\frac{1}{s} \sigma_s^\ast \omega_{FS} = \omega + \sqrt{-1}\partial\bar{\partial}\psi_s.$$

It follows from [13] that

$$\lim_{s \to +\infty} \frac{d}{ds} M_\omega(\psi_s) = \text{Re}(f_{M_0}(X)),$$

(6.7)

where $X$ is the generator of $\{\sigma_s\}$.

Similarly we could define the twisted Mabuchi energy as [20, 26]:

$$M_{\omega, t}(\varphi) := -n \int_0^1 \int_M \varphi_s \left(\text{Ric}(\omega_{\varphi_s}) - t_0 \omega_{\varphi_s} - (1 - t) \left(2\pi \sum_r b_r [D_r] + b_0 \omega_0\right)\right) \wedge \omega_{\varphi_s}^{n-1} ds$$
Thus it follows from the fact to the argument in [13] we can also define the corresponding generalized Futaki invariant on the limit. We note that the twisted Mabuchi energy is also independent of the paths connecting 0 and $\varphi$. Similar to the argument in [13] we can also define the corresponding twisted Mabuchi energy $E_{\omega, t}$ and moreover similar to (6.7), if the $C^*$-action $\sigma_t$ preserves $(M_\infty, D_\infty)$, it follows that

$$
\lim_{s \to +\infty} \frac{d}{ds} M_{\omega, t}(\psi_s) = \operatorname{Re}(f_{M_\infty, D_\infty, t}(X)).
$$

To complete the proof, we need the following lemma which is modified from [13].

**Lemma 6.4.** $\operatorname{Re}(f_{M_\infty, D_\infty, t}(X)) \geq 0$ for all small enough $t > 0$ and $\operatorname{Re}(f_{M_\infty, D_\infty, t}(X)) = 0$.

**Proof.** Let us briefly describe the main ingredients in the proof and refer to [13, 20, 23] for more details. To show the first conclusion, we first need to show that the corresponding twisted Mabuchi energy $E_{\omega, t}$ is bounded from below. For this target, we could argue as in [20, 23] by use of the log $\alpha$-invariant. The log $\alpha$-invariant was introduced by Berman [3] which originally came from Tian’s $\alpha$-invariant [29]. In our case, we can define the log $\alpha$-invariant as follows:

$$
\alpha([\omega], D, t) := \sup \left\{ \alpha > 0 \left| \int_M \frac{\omega^{\sup \varphi} \omega_n \omega_n}{\prod_r \|S_r\|^{2b_r(1-t)}} < +\infty \text{ for any } \varphi \in \text{ PSH}(M, \omega) \right. \right\}.
$$

The existence of a positive $\alpha([\omega], D, t)$ follows from [3] and in our case follows from [23, Lemma 2.6]. Similar to [20, 23] as (6.2) is solvable at $t$, by setting $\varphi = \varphi_t$, for $\alpha \in (0, \alpha([\omega], D, t))$ there exists $C_{\alpha} > 0$ such that

$$
\log C_{\alpha} \geq \int_M e^{\sup \varphi} \omega_n^2 \rho_n \prod_r \|S_r\|^{2b_r(1-t)}
$$

$$
\geq \int_M \omega_n^2 \rho_n^2 + \int_M H_1 \omega_n^2 - \int_M \log \frac{\omega_n^2}{\omega_n^2 \omega_n^2}
$$

$$
\geq \alpha \int_M (\omega_n - \rho_n) + \int_M H_1 \omega_n^2 - M_{\omega, t}(\varphi) - t(I_\omega(\varphi) - J_\omega(\varphi)).
$$

Thus it follows from the fact $I_\omega(\varphi) \leq (n + 1)J_\omega(\varphi)$ that

$$
M_{\omega, t}(\varphi) \geq \left( \alpha - \frac{n}{n + 1} \right) I_\omega(\varphi) - C_{\alpha}.
$$
where $C_{\alpha,t}^\prime$ is a constant depending only on $\alpha$ and $t$. It follows that the twisted Mabuchi energy is bounded from below for small $t > 0$. Then it follows from (6.9) that $\text{Re}(f_{M_{\infty},D_{\infty,t}}(X)) \geq 0$; otherwise $M_{\alpha,t}(\psi_s)$ may decrease to $-\infty$ which is impossible. For the second conclusion, noting that as $X$ is generated by the degeneration of the metrics as $t_i \to T$, we could use a sequence $\phi_{t_i}$ which solves (6.2) at $t_i$ as the path when $i$ is large enough. Thus by the definition of the twisted Mabuchi energy and (6.9), it follows that $\text{Re}(f_{M_{\infty},D_{\infty},T}(X)) = 0$. 

To conclude the proof of Corollary 1.3, as in [35], by (6.6) and (6.8), we have the following relation:

$$(T - \tau)M_{\omega}(\psi_s) = (1 - \tau)M_{\omega,T}(\psi_s) - (1 - T)M_{\omega,T}(\psi_s),$$

where $\tau > 0$ is chosen small enough as in Lemma 6.4. Take the derivative with respect to $s$ and let $s \to +\infty$. It follows from (6.7) and (6.9) that

$$(T - \tau)\text{Re}(f_{M_{\infty}}(X)) = (1 - \tau)\text{Re}(f_{M_{\infty},D_{\infty},T}(X)) - (1 - T)\text{Re}(f_{M_{\infty},D_{\infty},T}(X)).$$

By Lemma 6.4, $\text{Re}(f_{M_{\infty},D_{\infty},T}(X)) = 0$ and on the other hand $\text{Re}(f_{M_{\infty},D_{\infty},0}(X)) \geq 0$ for small enough $\tau > 0$. Thus it follows that $\text{Re}(f_{M_{\infty}}(X)) \leq 0$ as $T > 0$ and recall that $M_{\infty}$ is not biholomorphic to $M$ as $\|\varphi_{t_i}\|_{C^0}$ diverges to infinity, which contradicts the $K$-stability that $\text{Re}(f_{M_{\infty},D_{\infty},0}(X)) \geq 0$ and the equality holds only if $M_{\infty}$ is biholomorphic to $M$. Thus we complete the proof of Corollary 1.3.

**Remark 6.5.** If we replace the $K$-stability by the $CM$-stability in Corollary 1.3 as in [35] we can also establish the existence of Kähler-Einstein metrics by quite similar arguments as in [35].

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