SLE AND SPIDERNETS

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Abstract. We regard SLE from a quantum probability point of view and approximate the underlying quantum process by the growth of a random graph, which arises from the comb product of a certain spidernet and its complement. We obtain a stronger result for the deterministic Loewner equation and continuous non-negative increasing driving functions.

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1. Introduction

The Loewner equation
\[
\frac{\partial g_t(z)}{\partial t} = \frac{1}{g_t(z) - U(t)}, \quad g_0(z) = z \in \mathbb{C}^+ := \{ z \in \mathbb{C} | \text{Im}(z) > 0 \},
\]
where \( U : [0, \infty) \to \mathbb{R} \) is measurable and bounded on compact intervals, is usually interpreted as describing a family \( (g_t)_{t \geq 0} \) of conformal mappings \( g_t : \mathbb{C}^+ \setminus K_t \to \mathbb{C}^+ \), where \( (K_t)_{t \geq 0} \) is a family of growing, bounded subsets \( K_t \subset \mathbb{C}^+ \), also called hulls.

The Schramm-Loewner evolution SLE(\( \kappa \)), \( \kappa \geq 0 \), is defined via \([1.1]\) with \( U(t) = \sqrt{\kappa/2}B_t \), where \( B_t \) is a standard Brownian motion. In this case, we describe the evolution of growing random hulls.

Besides this analytic-geometric view, we might regard equation \([1.1]\) also as an evolution equation for a family \( (\mu_t)_{t \geq 0} \) of probability measures on \( \mathbb{R} \) defined via
\[
\frac{1}{g_t^{-1}(z)} = \int_{\mathbb{R}} \frac{1}{z - u} \mu_t(du).
\]
This interpretation is justified by quantum probability theory: Such families \( (\mu_t)_{t \geq 0} \) arise as the distributions of certain quantum processes \( (X_t)_{t \geq 0} \) with monotonically independent increments. Here, a quantum process is simply a family of self-adjoint linear operators on a fixed Hilbert space. For the notions “distribution of \( X_t \)” and “monotone independence”, we refer to Section 3.

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For $U(t) \equiv 0$, the mappings $g_t$ are given as $g_t(t) = \sqrt{z^2 + 2t}$ and $K_t$ is the straight line segment between 0 and $\sqrt{2t}$. The corresponding measure $\mu_t$ is an arcsine distribution with mean 0 and variance $t$. In this case, the associated process $(X_t)$ is called a monotone Brownian motion.

| Conformal mappings | Growing sets | Distributions $\mu_t$ | Quantum process $(X_t)$ |
|--------------------|--------------|-----------------------|-------------------------|
| $g_t(z) = \sqrt{z^2 + 2t}$ | $K_t = [0, \sqrt{2t}]$ | $\frac{dx}{\pi \sqrt{2t-x^2}}, x \in (-\sqrt{2t}, \sqrt{2t})$ | monotone Brownian motion |

(1) Muraki constructed a monotone Brownian motion on a certain Fock space in [Mur97] (before he introduced the notion of monotone independence around the year 2000).

(2) In [AGO04, Theorem 5.1], the authors construct a sequence of undirected graphs $G_1, G_2, \ldots$, whose adjacency matrices $A_1, A_2, \ldots$ can be interpreted as a discrete approximation of a monotone Brownian motion. The graph $G_{n-1}$ is a subgraph of $G_n$, and $A_n$ can be regarded as self-adjoint operator on the Hilbert space $l^2(V_n)$, where $V_n$ denotes the vertex set of $G_n$. Thus, the growing graph $(G_n)_{n \in \mathbb{N}}$ can be thought of as a “monotone quantum random walk”, and the moments of $A_n$ (scaled in a suitable way) converge to the moments of a monotone Brownian motion.

It is natural to ask whether the constructions (1) and (2) can be extended to arbitrary driving functions $U$. In [Jek17, Theorem 6.8], the author generalizes (1) by constructing quantum processes with monotonically independent increments associated to (1.1) for any choice of $U$, which can then be used to construct a (classically random) quantum process $(X^\kappa_t)$ associated to SLE($\kappa$). In this paper we are concerned with (2).

**Outline of this work:**

In Section 2 we recall some facts about Loewner’s differential equation and we explain its relation to monotone probability theory in Section 3.

In Section 4 we find discrete approximations as in (2) via comb products of certain spidernets for the case of continuous non-negative increasing driving functions (Sections 4.1, 4.2). Asking for a construction (2) for SLE($\kappa$) does not make sense in the first place: consider the $k$--th moment $m(k)$ of $X^\kappa_t$, with $t > 0$, $\kappa > 0$, and $k \geq 2$ and odd. It is a random number whose distribution is symmetric with respect to 0, and the probability that $m(k) < 0$ is positive. However, as an adjacency matrix contains only 0 and 1 entries, it has only non-negative moments. Hence, we cannot approximate $X^\kappa_t$ by adjacency matrices with respect to convergence of moments. However, in Section 4.3 we find such an approximation with respect to weak convergence of the distributions.

We also look at some basic properties of the distribution of $X^\kappa_t$ in Section 5.
2. THE LOEWNER EQUATION AND SLE

For SLE and the Loewner equation, we refer the interested reader to the book [Law05].

The slit Loewner equation is given by
\[
\frac{\partial g_t(z)}{\partial t} = \frac{1}{g_t(z) - U(t)} \quad \text{for a.e. } t \geq 0, \quad g_0(z) = z \in \mathbb{C}^+ = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \},
\]
with a measurable function \( U : [0, \infty) \to \mathbb{R} \) which is bounded on compact intervals.

The solution yields a family \( (g_t)_{t \geq 0} \) of conformal mappings \( g_t : \mathbb{C}^+ \setminus K_t \to \mathbb{C}^+ \) with a strictly growing family \( (K_t)_{t \geq 0} \) of bounded sets, i.e. \( K_s \subseteq K_t \) whenever \( 0 \leq s < t \). The initial condition implies \( K_0 = \emptyset \).

Let \( f_t = g_t^{-1} \). The family \( (f_t)_{t \geq 0} \) is also called a decreasing Loewner chain. From (2.1) it follows that the family satisfies the following partial differential equation:
\[
\frac{\partial}{\partial t} f_t(z) = \frac{\partial}{\partial z} f_t(z) \cdot \frac{1}{f_t - U(t)} \quad \text{for a.e. } t \geq 0, \quad f_0(z) = z \in \mathbb{C}^+.
\]

Each \( f_t \) has hydrodynamic normalization. More precisely,
\[
f_t(z) = z - \frac{t}{z} + O(|z|^{-1})
\]
as \( |z| \to \infty \) in the sense of a non-tangential limit.

**Example 2.1.** For \( U(t) = u \in \mathbb{R} \), we obtain \( g_t(z) = \sqrt{(z - u)^2 + 2t} + u \) and \( f_t = \sqrt{(z - u)^2 - 2t} + u \), where the square roots are chosen such that the functions map into the upper half-plane \( \mathbb{C}^+ \). We have \( K_t = [u, u + \sqrt{2t}] \), i.e. we describe the growth of a straight line starting at \( u \). \( \star \)

**Remark 2.2.** Assume that \( K_t \) is a slit, i.e. \( K_t = \gamma([0, t]) \) for a simple curve \( \gamma \) as in the previous example. Then \( U \) is continuous and \( g_t \) can be extended continuously to the tip \( \gamma(t) \) of the slit \( K_t \) and we have \( U(t) = g_t(\gamma(t)) \).

Not every continuous \( U \) generates slits. However, if \( U \) is sufficiently smooth, then \( K_t \) is a slit, see [LMR10, Lim05, MR05]. \( \star \)

The celebrated Schramm-Loewner evolution can be defined as follows:

Let \( \kappa \geq 0 \). Then SLE(\( \kappa \)) is defined as the random family \( (K_t)_{t \geq 0} \) obtained by (2.1) with \( U(t) = \sqrt{\kappa/2} B_t \), where \( B_t \) is a standard Brownian motion. Fix some \( T > 0 \). Then the random hull \( K_T \) is a slit almost surely if and only if \( \kappa \in [0, 4] \).

The corresponding random growth process \( (K_t)_{t \geq 0} \) has shown to be the scaling limit of random curves from different statistical models, depending on the value of \( \kappa \):
- SLE(2): loop erased random walk,
- SLE(3): critical Ising model,
- SLE(4): harmonic explorer, contour lines of the discrete Gaussian free field,
- SLE(6): critical percolation,
- SLE(8): uniform spanning tree.

* SLE(\( \kappa \)) is usually defined via the equation \( \frac{\partial g_t(z)}{\partial t} = \frac{\kappa}{g_t(z)} \cdot \frac{\partial g_t(z)}{\partial z} \). For this reason, we use the constant \( \sqrt{\kappa/2} \) in our definition.
While the geometric interpretation of Loewner’s equation focuses on the growing sets \((K_t)_{t \geq 0}\) (or the mappings \((f_t)_{t \geq 0}\)), we now switch to a probabilistic point of view, which regards a family \((\mu_t)_{t \geq 0}\) of probability measures on \(\mathbb{R}\) instead.

Let \(\mu\) be a probability measure on \(\mathbb{R}\). The \(F\)-transform \(F_\mu\) of \(\mu\) is defined as the multiplicative inverse of the Cauchy transform of \(\mu\), i.e. as the mapping

\[
F : \mathbb{C}^+ \to \mathbb{C}^+, \quad F_\mu(z) := \left( \int_{\mathbb{R}} \frac{1}{z - u} \mu(du) \right)^{-1}.
\]

The measure \(\mu\) can be recovered from \(F\) via the Stieltjes-Perron inversion formula. We have the following simple characterization.

**Lemma 2.3.**

(a) A holomorphic function \(F : \mathbb{C}^+ \to \mathbb{C}\) is the \(F\)-transform of a probability measure \(\mu\) on \(\mathbb{R}\) if and only if \(F(\mathbb{C}^+) \subseteq \mathbb{C}^+\) and \(F'(\infty) = 1\) (as a nontangential derivative).

(b) Let \((f_t)\) be the solution to (2.2). Then, for every \(t \geq 0\), \(f_t = F_{\mu_t}\) for a probability measure \(\mu_t\) on \(\mathbb{R}\).

(c) Let \(\mu, \mu_n\), with \(n \in \mathbb{N}\), be probability measures on \(\mathbb{R}\). Then \(\mu_n \to \mu\) with respect to weak convergence if and only if \(F_{\mu_n} \to F_\mu\) locally uniformly on \(\mathbb{C}^+\).

**Proof.** Statement (a) follows from the Nevanlinna representation formula and [Maa92, Prop. 2.1], (b) follows from (a) and the hydrodynamic normalization (2.3), and (c) follows from [Maa92, Theorem 2.5].

**Remark 2.4.** Consider the more general Loewner equation

\[
\frac{\partial}{\partial t} f_t(z) = \frac{\partial}{\partial z} f_t(z) \cdot M(z, t) \quad \text{for a.e. } t \geq 0, \quad f_0(z) = z \in \mathbb{C}^+,
\]

where, for a.e. \(t \geq 0\), \(M(\cdot, t)\) has the form

\[
M(z, t) = a_t + \int_{\mathbb{R}} \frac{1 + xz}{x - z} \tau_t(dx),
\]

with \(a_t \in \mathbb{R}\) and \(\tau_t\) is a finite, non-negative Borel measure on \(\mathbb{R}\). Furthermore, \((z, t) \mapsto M(z, t)\) needs to satisfy certain regularity conditions. Again, the solution \((f_t)\) is a family of univalent mappings \(f_t : \mathbb{C}^+ \to \mathbb{C}^+\) with \(f_t(\mathbb{C}^+) \subseteq f_s(\mathbb{C}^+)\) for all \(0 \leq s \leq t\) and each \(f_t\) is the \(F\)-transform of a probability measure on \(\mathbb{R}\).

The following embedding result is proved in [FHS]: If \(F_\mu\) is univalent, then there exists \(T \geq 0\) and a function \(M(z, t)\) of the above form such that the solution \((f_t)\) of (2.4) satisfies \(f_T = F_\mu\).

**Example 2.5.** The arcsine distribution \(\mu_{A,t}\) with mean 0 and variance \(t\) is given by the density

\[
\frac{dx}{\pi \sqrt{2t - x^2}}, \quad x \in (-\sqrt{2t}, \sqrt{2t}).
\]

We have \(F_{\mu_{A,t}}(z) = \sqrt{z^2 - 2t}\), which are the mappings from Example 2.1 for \(u = 0\).

The following simple scaling relation will be useful later on.

**Lemma 2.6.** Let \(c, d > 0\) and let \(f_t = F_{\mu_t}\) be generated by the driving function \(U(t)\). Consider the scaled measures \(\nu_t(B) = \mu_{d,t}(c \cdot B)\). Let \(h_t = F_{\nu_t}\). Then \(h_t\) solves

\[
\frac{\partial}{\partial t} h_t(z) = \frac{\partial}{\partial z} h_t(z) \cdot \frac{d/c^2}{h_t(z) - U(d \cdot t)/c},
\]

**Proof.** We have

\[
h_t(z) = \left( \int_{\mathbb{R}} \frac{1}{z - u} \mu_{d,t}(c \cdot du) \right)^{-1} = \left( \int_{\mathbb{R}} \frac{1}{cz - u} \mu_{d,t}(du) \right)^{-1} = \left( \int_{\mathbb{R}} \frac{c}{cz - u} \mu_{d,t}(du) \right)^{-1} = f_{d,t}(cz)/c.
\]

Then (2.2) leads to

\[
\frac{\partial}{\partial t} h_t(z) = \frac{d}{c} \frac{\partial}{\partial t} f_{d,t}(cz) = \frac{d}{c} \frac{\partial}{\partial z} f_{d,t}(cz) \cdot \frac{1}{f_{d,t}(cz) - U(d \cdot t)} = \frac{\partial}{\partial z} h_t(z) \cdot \frac{d/c^2}{h_t(z) - U(d \cdot t)/c}.
\]
The reason why it makes sense to consider Loewner’s differential equation in this way is given by quantum probability theory, more precisely, by monotone increment processes.

### 3. Monotone Increment Processes

Let $H$ be a Hilbert space and denote by $B(H)$ the space of all bounded linear operators on $H$. In quantum probability theory, elements of $B(H)$ are regarded as non-commutative random variables in the following way.

Fix a unit vector $\xi \in H$. Then we can define a so called state $\Phi$ as the $\mathbb{C}$-linear mapping

$$
\Phi : B(H) \to \mathbb{C}, \quad \Phi(X) = \langle \xi, X\xi \rangle.
$$

Motivated by quantum mechanics, we can think of $\Phi(a)$ as the expectation of the quantum random variable $a \in B(H)$.

**Definition 3.1.** We call $(H, \xi)$ a quantum probability space.

Assume that $a \in B(H)$ is self-adjoint. Then there exists a unique probability measure $\mu$ on $\mathbb{R}$ such that the moments of $\mu$ are given by $\Phi(a^n)$, i.e. $\int_{\mathbb{R}} x^n \mu(dx) = \Phi(a^n)$ for all $n \in \mathbb{N}$. We call $\mu$ the distribution of $a$.

The notion of independence is of vital importance for classical probability theory. In a certain sense, there are only five suitable notions of independence in the non-commutative setting: tensor, Boolean, free, monotone and anti-monotone independence; see [Mur03].

In all five cases, independence of two elements $a, b \in B(H)$ is expressed algebraically by computation rules for mixed moments. We consider monotone independence, introduced by N. Muraki, which is a non-commutative independence, i.e. independence of two random variables is defined for the pair $(a, b)$ and not for the set $(a, b)$.

**Definition 3.2.** Let $X_1, \ldots, X_N \in B(H)$ be self-adjoint random variables in the quantum probability space $(H, \xi)$. The tuple $(X_1, X_2, \ldots, X_N)$ is called monotonically independent if

$$
\Phi(X_{i_1}^{p_1} \ldots X_{i_k}^{p_k} \ldots X_{i_m}^{p_m}) = \Phi(X_{i_1}^{p_1}) \cdot \Phi(X_{i_2}^{p_2} \ldots X_{i_{k-1}}^{p_{k-1}} X_{i_{k+1}}^{p_{k+1}} \ldots X_{i_m}^{p_m})
$$

for all $m \in \mathbb{N}$, $p_1, \ldots, p_m \in \mathbb{N}_0$, whenever $i_{k-1} < i_k > i_{k+1}$ (one of the inequalities is eliminated when $k = 1$ or $k = m$).

Assume that $(X, Y)$ is a pair of monotonically independent self-adjoint random variables. If $\alpha$ and $\beta$ are the distributions of $X$ and $Y$ respectively, then it can be shown that the distribution $\gamma$ of $Z = X + Y$ can be computed by

$$
F_{\gamma} = F_{\alpha} \circ F_{\beta},
$$

see, e.g., [Fra09] Theorem 3.10. This relation defines the additive monotone convolution $\alpha \triangleright \beta := \gamma$.

**Remark 3.3** (Literature). For quantum probability theory (including its important relations to random matrices), we refer the reader to introductions such as [Att, DNV92, Mey93, MS17].

The five notions lead to central limit theorems, the investigation of quantum stochastic processes with independent increments, and to quantum stochastic differential equations. The latter topics are treated in detail in the books [ABKL05] [BFGKT06].

Finally, we also refer to [Oba17], where the author shows how quantum probability theory can be applied to the spectral analysis of graphs. The different notions of independence appear in connection with certain products for graphs.

We now explain the relation of monotone independence to the Loewner equation. Let $(f_t)_{t \geq 0}$ be the solution to (2.2) and let $0 \leq s \leq t$. Then $f_t = f_s \circ f_{s,t}$ for some univalent function $f_{s,t}: \mathbb{C}^+ \to \mathbb{C}^+$, as the image domains $f_t(\mathbb{C}^+)$ are decreasing.

As $f_0$ is the identity, we have $f_t = f_0 \circ f_{0,t}$. We can apply Lemma 2.3 to see that we can write $f_{s,t} = F_{\mu_{s,t}}$ for a probability measure $\mu_{s,t}$ on $\mathbb{R}$. Hence, we have

$$
\mu_{0,t} = \mu_{0,s} \triangleright \mu_{s,t},
$$

which suggests that there might be an underlying family $(X_t)_{t \geq 0}$ of self-adjoint operators such that $X_0 = 0$, $X_s$ and $X_t - X_s$ are independent for $s \leq t$, and $\mu_{s,t}$ is the distribution of $X_t - X_s$. Equation (3.1) would then follow from

$$
X_t = X_s + (X_t - X_s).
$$
This leads us to the following definition.

**Definition 3.4.** Let \((H, \xi)\) be a quantum probability space and \((X_t)_{t \geq 0}\) a family of self-adjoint operators on \(H\) with \(X_0 = 0\). We call \((X_t)\) a **self-adjoint operator-valued additive monotone increment process (SAIP)** if the following conditions are satisfied:

(a) For every \(s \geq 0\), the mapping \(t \mapsto \mu_{s,t}\) is continuous w.r.t. weak convergence, where \(\mu_{s,t}\) denotes the distribution of \(X_t - X_s\).

(b) The tuples

\[
(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})
\]

are monotonically independent for all \(n \in \mathbb{N}\) and all \(t_1, \ldots, t_n \in \mathbb{R}\) s.t. \(0 \leq t_1 \leq t_2 \leq \cdots \leq t_n\).

We also write \(\mu_t\) instead of \(\mu_{0,t}\) for the distribution of \(X_t\).

**Example 3.5** (Monotone Brownian motion). Recall the arcsine distribution \(\mu_{A,t}\) with mean 0 and variance \(t\) from Example 2.5. The normalized distribution \(\mu_{1,t}\) is the monotone analogue of the normal distribution from classical probability, as it is the limit distribution in the central limit theorem of monotone probability theory, see [Mur00, Theorem 2].

A SAIP \((X_t)\) with distributions \(\mu_t = \mu_{A,t}\) is thus called a **monotone Brownian motion**. We have

\[
F_{\mu_{A,t}}(z) = \sqrt{z^2 - 2t}.
\]

These mappings simply describe the growth of a straight line starting at 0, see Example 2.1. In [Mur97], Muraki constructed a monotone Brownian motion on a certain Fock space.

The following result follows from [Jek17, Theorem 6.8].

**Theorem 3.6.** Let \((f_t)_{t \geq 0}\) be the solution to (2.2). Write \(f_t = f_s \circ f_{s,t}\) and define \(\mu_{s,t}\) by \(f_{s,t} = F_{\mu_{s,t}}\). Then there exists a SAIP \((X_t)_{t \geq 0}\) on a quantum probability space \((H, \xi)\) such that the distribution of \(X_t - X_s\) is given by \(\mu_{s,t}\).

The quantum probability space \((H, \xi)\) in this construction depends on the distributions \((\mu_{s,t})\). However, by using isomorphisms (and possibly embeddings into a larger Hilbert space), we can assume that \((H, \xi)\) is in fact independent of the distributions.

Now SLE can be treated as follows:

Let \((f_t)_{t \geq 0}\) be the solution to (2.2) for \(U(t) = \sqrt{\kappa/2} B_t\) and denote by \((\sigma_t)_{t \geq 0}\) the corresponding random measures, i.e. \(f_t = F_{\sigma_t}\).

Theorem 3.6 yields a mapping

\[
P : C([0, \infty), \mathbb{R}) \to \{\text{all SAIP on } (H, \xi)\},
\]

which we can use to push forward the probability measure on \(C([0, \infty), \mathbb{R})\), given by a Brownian motion scaled by \(\sqrt{\kappa/2}\), to obtain the random SAIP \((X^{\kappa}_t)_{t \geq 0}\) that realizes the measures \((\sigma_t)_{t \geq 0}\).
4. Approximation via spidernets

We now follow the work [AGO04] and modify its main result (Theorem 5.1), which can be interpreted as a discrete approximation of a monotone Brownian motion, a “monotone quantum random walk”, via adjacency matrices of certain graphs.

Let $V$ be a vertex set, finite or countable infinite, with a distinguished vertex $o \in V$.
Let $A : V \times V \to \mathbb{N}_0$ be a symmetric matrix.

We can interpret $A$ as the adjacency matrix of an undirected graph with vertex set $V$, possibly having loops, where $A_{xy} \neq 0$ if and only if $x \sim y$, i.e. $x$ and $y$ are connected by an edge. If $A_{xy} > 1$, then $x$ and $y$ are connected by several edges. Alternatively, we can think of weighted edges.

**Definition 4.1.** We define a graph as such a triple $(V, A, o)$.

In Sections 4.1 and 4.2, we will only deal with unweighted graphs without loops, i.e. $A_{xx} = 0$ for all $x \in V$ and $A_{xy} \in \{0, 1\}$ for all $x, y \in V$.

In Section 4.3, however, we will use the more general setting as $A$ will have non-negative integers on the diagonal, i.e. the graph will have weighted loops.

If $\sup \{\deg(v) | v \in V\} < \infty$, then $A$ can be regarded as a bounded self-adjoint operator on the Hilbert space $l^2(V)$, see [MW89, Theorem 3.1]. The distinguished vertex $o \in V$ enables us to regard $A$ as a quantum random variable on the quantum probability space $(l^2(V), \delta_o)$, where $\delta_o \in l^2(V)$ with $\langle \delta_o \rangle(o) = 1$, $\langle \delta_o \rangle(x) = 0$ for $x \neq o$.

**Example 4.2.** Let $V = \mathbb{Z}$ with $A_{jk} = 1$ if and only if $|j - k| = 1$ and 0 otherwise. Choose $o = 0$.

Then the distribution of $A$ within the probability space $(l^2(\mathbb{Z}), \delta_0)$ is given by the arcsine distribution with mean 0 and variance 2, see [AGO04, Section 6.1].

Let $G_1 = (V_1, A_1, o_1), G_2 = (V_2, A_2, o_2)$ be two graphs. Then the comb product $G_1 \triangleright G_2 = (V_3, A_3, o_3)$ (with respect to $o_2$) is defined as the graph with vertices $V_3 = V_1 \times V_2$, distinguished vertex $o_3 = (o_1, o_2)$, and

\[
A_{(xx')(yy')}(3) = A_{xx'}^1 \delta_{yy'2} \delta_{y'2} + \delta_{xx'} A_{yy'}^2.
\]

Here we use the symbol $\delta_{xy} = 1$ if $x = y$, $\delta_{xy} = 0$ if $x \neq y$. It can be verified that $(x, y) \sim (x', y')$ if and only if
- $x \sim x', x \neq x'$ and $y = y' = o_2$, or
- $x = x', y = y' = o_2$, and $x \sim x$ or $o_2 \sim o_2$, or
- $x = x'$ and $y \sim y'$, $(y, y') \neq (o_2, o_2)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{comb_product}
\caption{The comb product of two graphs.}
\end{figure}

If $\sup \{\deg(v) | v \in V_j\} < \infty$ for $j = 1, 2$, then the adjacency matrix $A^3$ of $G_1 \triangleright G_2$ acts on $l^2(V_1 \times V_2) \simeq l^2(V_1) \otimes l^2(V_2)$.

The following lemma is a slightly more general version of [AGO04, Theorem 3.1]. Its proof follows from definition (4.1) and by induction.
Lemma 4.3. Let \( G_1 = (V_1, A_1, o_1), \ldots, G_n = (V_n, A_n, o_n) \) be graphs. Denote by \( I^k \) the identity on \( l^2(V_k) \) and by \( P^k \) the projection from \( l^2(V_k) \) onto the subspace spanned by \( \delta_{o_k} \), i.e., \( (P^k(\psi))(y) = \delta_{y_0_k}(\psi(o_k)) \). Denote by \( B \) the adjacency matrix of the graph \( G_1 \triangleright G_2 \triangleright \ldots \triangleright G_n \). Then

\[
B = \sum_{j=1}^{n} I^1 \otimes \ldots \otimes I^{j-1} \otimes A^j \otimes P^{j+1} \otimes \ldots \otimes P^n.
\]

Assume that \( \sup\{\deg(v) \mid v \in V_j\} < \infty \) for all \( j = 1, \ldots, n \). Then the adjacency matrix \( B \) can be regarded as a quantum random variable in \( (l^2(V_1 \times \ldots \times V_n), \delta_{o_1} \otimes \ldots \otimes \delta_{o_n}) \). By [AG04, Proposition 4.1], the random variables \( (I^1 \otimes \ldots \otimes I^{j-1} \otimes A^j \otimes P^{j+1} \otimes \ldots \otimes P^n)_{j \in \{1, \ldots, n\}} \) are monotonically independent. Thus the distribution of \( B \) is given by the monotone convolution of the distributions of the summands in (4.2). Furthermore, it is easy to see that the moments of \( I^1 \otimes \ldots \otimes I^{j-1} \otimes A^j \otimes P^{j+1} \otimes \ldots \otimes P^n \) with respect to \( (l^2(V_1 \times \ldots \times V_n), \delta_{o_1} \otimes \ldots \otimes \delta_{o_n}) \) agree with the moments of \( A^j \) within \( (l^2(V_j), \delta_{o_j}) \). Thus we obtain:

Lemma 4.4. Assume that \( \sup\{\deg(v) \mid v \in V_j\} < \infty \) for all \( j = 1, \ldots, n \). Then the random variables \( (I^1 \otimes \ldots \otimes I^{j-1} \otimes A^j \otimes P^{j+1} \otimes \ldots \otimes P^n)_{j \in \{1, \ldots, n\}} \) are monotonically independent in the quantum probability space \( (l^2(V_1 \times \ldots \times V_n), \delta_{o_1} \otimes \ldots \otimes \delta_{o_n}) \). Let \( \mu_j \) be the distribution of \( A_j \) within \( (l^2(V_j), \delta_{o_j}) \). Then \( B \) has the distribution

\[
\mu_1 \triangleright \mu_2 \triangleright \ldots \triangleright \mu_n.
\]

We now construct special graphs whose distributions will be related to the Loewner equation. We denote by \( d(x, y) \) the length of the shortest walk within a graph connecting \( x \) and \( y \). For \( \varepsilon \in \{-1, 0, +1\} \), we define

\[
\omega_\varepsilon = \{y \in V \mid y \sim x, d(o, y) = d(o, x) + \varepsilon\}.
\]

Let \( a \in N, b \in N \setminus \{1\} \) and \( c \in N \) with \( c \leq b - 1 \). A **spidernet with data \( (a, b, c) \)**, see [NO07, Def. 4.25]), is a loop-free graph \( (V, A, o) \) with root \( o \in V \) such that

\[
\omega_{+1}(o) = a, \omega_{-1}(o) = \omega_0(o) = 0, \quad \text{and} \quad \omega_{+1}(x) = c, \omega_{-1}(x) = 1, \omega_0(x) = b - 1 - c
\]

for all \( x \in V \setminus \{o\} \) (and \( A_{xy} \in \{0, 1\} \) for all \( x, y \in V \)).

![Figure 3. Two spidernets with data (4,4,2).](image)

Lemma 4.5 (See Thm. 4.29 in [NO07]). The spectrum of the adjacency matrix of a spidernet w.r.t. the quantum probability space \( (l^2(V), \delta_o) \) is the free Meixner law \( m_{a,c,b-1-c} \).

The free Meixner law is described in [NO07, Section 4.5]. We will only need the following property.

Lemma 4.6. Let \( n \in N, u \in N_0 \). Then the distribution \( m_{2n,u} \) has F-transform \( \sqrt{(z-u)^2 - 4n + u} \). It has 0 mean and variance \( 2n \).
Proof. This can be easily verified by using the explicit formula [1006 Equation (B.1)]. □

We now combine the following two observations:

(A) On the one hand, by Lemma 4.5, \( m_{2n,n,u} \) is the distribution of a spidernet with data \( (2n, n + 1 + u, n) \); provided such a spidernet exists.

From looking at the \( 2n \) vertices with \( d(o,x) = 1 \), we get the necessary condition \( b - 1 - c = u \leq 2n - 1 \). Conversely, one can verify that for each \( n \in \mathbb{N} \) and every \( u \in \{0, ..., 2n - 1\} \) there exists a spidernet with data \( (2n, n + 1 + u, n) \). We denote by \( S_{n,u} \) a fixed spidernet with such data.

(B) On the other hand, we obtain \( F_{m_{2n,n,u}}(z) = \sqrt{(z-u)^2 - 4n + u} \) as the solution of the Loewner equation with \( U(t) = u \) at \( t = 2n \), see Example 2.1. Obviously, we can also write \( m_{2n,n,u} = \delta_{-u} \triangleright \mu_{A,2n} \triangleright \delta_u \).

Hence, approximating a driving function by piecewise constant driving functions is related to approximating the corresponding measures by distributions of spidernets.

![Figure 4. Left: The free Meixner law \( m_{4,2,0} \) is simply the arcsine distribution. Right: The density of \( m_{4,2,1} \) in \([1 - 2\sqrt{2}, 1 + 2\sqrt{2}]\) and its atom at \(-2\).](image)

4.1. Certain differentiable driving functions.

In this section we assume that \( U : [0, \infty) \to \mathbb{R} \) is continuously differentiable with \( U'(t) \geq 0 \) and \( U(0) = 0 \).

Let \( (f_t)_{t \geq 0} \) be the solution to the corresponding Loewner equation and denote by \( (\mu_t)_{t \geq 0} \) the probability measures with \( F_{\mu_t} = f_t \). Furthermore, let \( (X_t)_{t \geq 0} \) be a corresponding SAIP process.

Fix some \( T > 0 \). We would like to approximate \( (X_t)_{t \in [0,T]} \) by a discrete quantum process, where each random variable is the adjacency matrix of a graph. By means of the lemmas above, we can now proceed as follows.

As \( U' \) is continuous, we find \( B > 0 \) such that \( U'(t) \in [0,B] \) for all \( t \in [0,T] \).

We let \( n \in \mathbb{N} \) be large enough such that

\[
2n^2 - 1 \geq B \sqrt{2T} \sqrt{n} + 1. \tag{4.3}
\]

For \( k = 1, \ldots, n \), we define

\[
u_{n,k} = \left\lfloor \sqrt{2T} \sqrt{n} \cdot \frac{U(k/n \cdot T)}{t/n} \right\rfloor - \left\lfloor \sqrt{2T} \sqrt{n} \cdot \frac{U((k-1)/n \cdot T)}{t/n} \right\rfloor \in \{0, \ldots, \left\lfloor B \sqrt{2T} \sqrt{n} + 1 \right\rfloor \}
\]

Here, \( \lfloor x \rfloor \) denotes the largest \( m \in \mathbb{N}_0 \) with \( m \leq x \). Note that (4.3) implies that the spidernet \( S_n^{2\nu_{n,u},u_{n,k}} \) exists for all \( k = 1, \ldots, n \). We denote by \( V_{n,k} \) the vertex set and by \( o_{n,k} \) the root of \( S_n^{2\nu_{n,u},u_{n,k}} \).
Theorem 4.7. For $k = 1, \ldots, n$, let $C_{n,k}$ be the graph

$$C_{n,k} := S_{n^2,u_{n,1}} \triangleright S_{n^2,u_{n,2}} \triangleright \ldots \triangleright S_{n^2,u_{n,k}}.$$ 

Then $(C_{n,k})_{k=1}^n$ is an approximation of the quantum process $(X_t)_{t \in [0,T]}$ in the following sense:

(a) Let $A_{n,k}$ be the adjacency matrix of $C_{n,k}$. Denote by $\mu_{n,k}$ the distribution of $A_{n,k}$ with respect to the quantum probability space $(P(V_{n,1} \times \ldots \times V_{n,k}), \delta_{o_{n,1}} \otimes \ldots \otimes \delta_{o_{n,k}})$. Then

$$\lim_{n \to \infty} \mu_{n,[tn/T]}\left(\sqrt{2n^3/T} \cdot \cdot \right) = \mu_t(\cdot)$$

with respect to weak convergence for all $t \in [0,T]$. The limit also holds true with respect to the convergence of all moments.

(b) Consider the quantum probability space $(P(V_{n,1} \times \ldots \times V_{n,n}), \delta_{o_{n,1}} \otimes \ldots \otimes \delta_{o_{n,n}})$. Extend $A_{n,k}$ to $l^2(V_{n,1} \times \ldots \times V_{n,n})$ by $A_{n,k} := A_{n,k} \otimes P_{n,n+1} \otimes \ldots \otimes P_{n,n}$, where $P_{n,l}$ denotes the projection in $l^2(V_{n,l})$ onto $\delta_{o_{n,l}}$. Then the increments $(A_{n,1}, A_{n,2} - A_{n,1}, \ldots, A_{n,n} - A_{n,n-1})$ are monotonically independent.

Proof. Statement (b) follows directly from Lemmas 4.3 and 4.4.

Let $U_n : [0,2n^3] \to \mathbb{R}$ be the function which is constant on $u_{n,1}$ on $(0,2n^2]$, constant $u_{n,1} + u_{n,2}$ on $(2n^2, 4n^3]$, etc., and $U_n(0) = 0$.

Let $f_{n,t}$ be the solution to (2.2) with this driving function and define the measures $\alpha_{n,t}$ by $F_{\alpha_{n,t}} = f_{n,t}$. By Example 2.1 and Lemma 4.6 we have

$$\alpha_{n,2n^2} = m_{2n^2,u_{n,1}}.$$ 

Starting the Loewner equation (2.2) for $h_t$ at $t = 2n^2$ with initial value $h_{2n^2}(z) = z$ and driving function $U_n(t)$ yields the mappings $(h_t)$ that satisfy $f_{n,t+2n^2} = f_{n,2n^2} \circ h_t$. Obviously, $h_{2n^2} = F_{m_{2n^2,u_{n,1}}}$ and thus $\alpha_{n,4n^2} = m_{2n^2,u_{n,1}}$. By induction we obtain

$$\alpha_{n,2kn^2} = \triangleright_{j=1}^k m_{2n^2,u_{n,j}}.$$ 

On the other hand, Lemmas 4.3, 4.4, 4.5 imply

(4.4) $$\mu_{n,k} = \triangleright_{j=1}^k m_{2n^2,u_{n,j}}$$

for all $k = 1, \ldots, n$.

The function $V_n : [0,T] \to \mathbb{R}, V_n(t) := \sqrt{\frac{T}{2n^3}} \cdot U_n(t/T \cdot 2n^3)$ is constant on the intervals $(\frac{(k-1)T}{n}, \frac{kT}{n}]$, $k = 1, \ldots, n$. We have

(4.5) $$U(k/n \cdot T) - V_n(k/n \cdot T) = U(k/n \cdot T) - \sqrt{\frac{T}{2n^3}} \cdot U_n(k \cdot 2n^2) =$$

$$U(k/n \cdot T) - \sqrt{\frac{T}{2n^3}} \cdot \left[ \sqrt{2T} \cdot \sqrt{n} \cdot \frac{U(k/n \cdot T)}{\frac{T}{n}} \right] \leq \sqrt{\frac{T}{2n^3}}$$

and, obviously, $U(k/n \cdot T) - V_n(k/n \cdot T) \geq 0$.

Now let $t \in (\frac{(k-1)T}{n}, \frac{kT}{n}]$. As $U$ is monotonously increasing with $U'(t) \leq B$, we have

(4.6) $$|U(t) - V_n(t)| = |U(t) - V_n(kT/n)| \leq$$

$$|U(t) - U(kT/n)| + |U(kT/n) - V_n(kT/n)| \leq \frac{BT}{n} + \sqrt{\frac{T}{2n^3}}.$$ 

Hence, together with $U(0) = V_n(0) = 0$, we obtain

(4.7) $$\sup_{t \in [0,T]} |U(t) - V_n(t)| \to 0 \quad \text{as} \quad n \to \infty.$$ 

Let $(h_{n,t})_{t \in [0,T]}$ be the Loewner chain that corresponds to $V_n$. Define the measures $\nu_{n,t}$ by $h_{n,t} = F_{\nu_{n,t}}$. Note that $V_n$ has the form $V_n = U_n(d \cdot t) \cdot c$ with $d = c^2$. Hence, by Lemma 2.6 we have

$$\nu_{n,t}(M) = \alpha_{n,t/T \cdot 2n^3}(\sqrt{2n^3/T} \cdot M)$$
for all $t \geq 0$ and all Borel subsets $M \subset \mathbb{R}$. If $t$ has the form $t = kT/n, k = 1, \ldots, n$, then (4.4) gives

$$
(4.8) \quad \nu_{n,t}(M) = \mu_{n,k}(\sqrt{2n^3/T} \cdot M) = (\sup_{j=1}^{k} m_{2n^2, n^2, u_{n,j}})(\sqrt{2n^3/T} \cdot M)
$$

= $(\sup_{j=1}^{kn/T} m_{2n^2, n^2, u_{n,j}})(\sqrt{2n^3/T} \cdot M)$.

For every $t \in [0, T]$ we have $h_{n,t} \to f_t$ locally uniformly because of (4.7) and Law05 Proposition 4.47. By Lemma 2.3 (c) we have $\nu_{n,t} \to \mu_t$ with respect to weak convergence, or

$$
\mu_{n,[tn/T]}(\sqrt{2n^3/T} \cdot M) = (\sup_{j=1}^{[tn/T]} m_{2n^2, n^2, u_{n,j}})(\sqrt{2n^3/T} \cdot M) \to \mu_t. \quad (\forall t \in [0, T])
$$

It remains to show that this limit also holds with respect to convergence of all moments. The family $(V_n)_n$ is uniformly bounded on $[0, T]$, i.e. we find $L > 0$ such that $|V_n(t)| < L$ for all $n$ and $t \in [0, T]$. Now consider equation (2.1) on $\mathbb{R}$ with initial value $L$, i.e.

$$
\dot{x}(t) = \frac{1}{x(t) - V_n(t)}, \quad x(0) = L.
$$

Clearly, $\dot{x}(t) > 0$ and thus $x(t) \geq L$, which implies that the solution exists up to $t = T$. We also find a constant $M_1 > 0$ such that $x(t) < M_1$ independent of $n$. Hence $x(t) = h_{n,t}^{-1}(L) < M_1$ for all $t \in [0, T]$ and $n \in \mathbb{N}$. In other words, $1/h_{n,t}$ maps $[M_1, \infty)$ into $\mathbb{R}$, in fact, into $(0, 1/L)$, and, by the Stieltjes-Perron inversion formula, we conclude that $\text{supp}(\nu_{n,t}) \subset (-\infty, M_1]$ for all $t \in [0, T]$ and $n \in \mathbb{N}$. By considering the initial value $x(0) = -L$, we find $M_2 < 0$ such that $M_2 < x(T)$ and conclude that

$$
\text{supp}(\nu_{n,t}) \subset [M_2, M_1]
$$

for all $t \in [0, T]$ and $n \in \mathbb{N}$. Thus, weak convergence of $\nu_{n,t}$ is equivalent to convergence of all its moments. \hfill \Box

4.2. Non-negative increasing continuous functions.

Let $U : [0, T] \to \mathbb{R}$ be continuous and monotonically increasing with $U(0) = 0$.

Let $(\mu_t)_{t \in [0,T]}$ be the associated probability measures and $(X_t)_{t \in [0,T]}$ an associated SAIP.

Let $A_n : [0, T] \to \mathbb{R}$ be a sequence of continuously differentiable functions with $A_n(0) = 0, A_n'(t) \geq 0$ for all $t \in [0, T]$ such that

$$
\sup_{t \in [0,T]} |A_n(t) - U(t)| \to 0
$$

as $n \to \infty$.

Denote by $f_{n,t}$ the functions generated by $A_n$ and by $\beta_{n,t}$ the associated probability measures.

Then $f_{n,t} \to f_t$ locally uniformly for each $t \in [0, T]$ due to Law05 Proposition 4.47. Lemma 2.3 (c) implies $\beta_{n,t} \to \mu_t$ w.r.t. weak convergence for all $t \in [0, T]$.

Now we can approximate $\beta_{n,t}$ by distributions of adjacency matrices.

Let $B_n > 0$ be a bound for $A_n'$ on $[0, T]$. We may assume that

$$
(4.9) \quad B_n \leq \max\{\sqrt{n}, \frac{\sqrt{n}}{\sqrt{2T}}\}.
$$

For $k = 1, \ldots, n$, we define

$$
\begin{align*}
\nu_{n,k} = [\sqrt{2T}\sqrt{n} \cdot \frac{A_n(k/n \cdot T)}{T/n}] - [\sqrt{2T}\sqrt{n} \cdot \frac{A_n((k-1)/n \cdot T)}{T/n}] \in \{0, \ldots, \lfloor B_n\sqrt{2T}\sqrt{n} \rfloor + 1\}.
\end{align*}
$$

Due to (4.9) we have $u_{n,k} \leq n + 1$, which is $\leq 2n^2 - 1$ for $n \geq 2$.

Thus, for $n \geq 2$, we can define the graph $C_{n,k}$ just as we did in Theorem 4.7. Denote by $A_{n,k}$ its adjacency matrix and by $\mu_{n,k}$ its distribution. The discrete process $(A_{n,k})_{k=1,n}$ has again monotonically independent increments in the sense of Theorem 4.7 (b). Furthermore, we have:
Theorem 4.8.

\[ \lim_{n \to \infty} \mu_n,\lfloor tn/T \rfloor (\sqrt{2n^3/T} \cdot \cdot) = \mu_t(\cdot). \]

Proof. Define \( U_n \) and \( V_n \) just as in the proof of Theorem 4.7. Then we have

\[ \sup_{t \in [0,T]} |A_n(t) - V_n(t)| \leq \frac{B_n T}{n} + \sqrt{\frac{T}{2n^3}} \leq \frac{T}{\sqrt{n}} + \sqrt{\frac{T}{2n^3}} \to 0 \]

as \( n \to \infty \). We take a distance \( d_{w}(\cdot, \cdot) \) for probability measures on \( \mathbb{R} \) which is compatible with weak convergence (e.g. the Lévy-Prokhorov distance).

By following the proof of Theorem 4.7, we obtain

\[ d_{w}(\mu_n, \lfloor tn/T \rfloor (\sqrt{2n^3/T} \cdot \cdot), \beta_n, t) \to 0 \text{ as } n \to \infty. \]

Furthermore, we chose \( A_n \) such that \( d_{w}(\mu_t, \beta_n, t) \to 0 \text{ as } n \to \infty. \) Hence,

\[ d_{w}(\mu_n, \lfloor tn/T \rfloor (\sqrt{2n^3/T} \cdot \cdot), \mu_t) \to 0 \text{ as } n \to \infty. \]

\[ \square \]

4.3. SLE. Recall that \( (\sigma_t^\kappa)_{t \geq 0} \) denotes the random measure process associated to \( \text{SLE}(\kappa) \). Its corresponding quantum process is denoted by \( (X_t^\kappa)_{t \geq 0} \). We consider the special case

\[ (4.10) \]

\[ \kappa = \frac{1}{2}. \]

In Remark (4.10), we explain why this is the simplest choice for our purpose, and which other values for \( \kappa \) can be handled in the same way.

Consider a spidernet \( S \) with data (4.4.2). For \( N \in \mathbb{N} \), we consider the induced subgraph of \( S \) obtained by taking all vertices with \( d(x, o) < N \). This graph contains \( 1 + 2^2 + \ldots + 2^N = 2^{N+1} - 3 \) vertices. Next, we add some edges to this graph. Consider all \( v \in V \) with \( d(x, o) = N - 1 \). We add loops to all such \( v \) and additional edges within this set such that the new graph, denoted by \( S_{N}^{+1} \), becomes a 4-regular graph, i.e. each vertex has degree 4. Let \( V_N \) be the vertex set and \( A^N \) the adjacency matrix of \( S_{N}^{+1} \).

Figure 5. \( S_{3}^{+1} \).

Let \( B^N \) be the matrix defined via \( B_{xy}^N = 1 - A_{xy}^N \). The corresponding graph, denoted by \( S_{N}^{-1} \), is the graph complement of \( S_{N}^{+1} \) (except for the loops, which are usually excluded in graph complements); see the figure below for \( S_{2}^{+1} \) and \( S_{2}^{-1} \). For \( N \geq 3 \), also \( S_{N}^{-1} \) is a connected graph (but drawing \( S_{3}^{-1} \) is already unpleasant). We write \( \delta_o \) for the unit vector in \( l^2(V_N) = \mathbb{C}^{|V_N|} \) and suppress the dependence on \( N \).
Let $N \in \mathbb{N}$. Let $u_1, u_2, \ldots$ be a sequence of independent random variables, each with distribution $\mathbb{P}[u_k = +1] = \mathbb{P}[u_k = -1] = \frac{1}{2}$. Let $C_{N,n}$ be the random graph

$$C_{N,n} := S_{N}^{u_1} \triangleright S_{N}^{u_2} \triangleright \ldots \triangleright S_{N}^{u_n}.$$ 

Then $(C_{N,n})_{n \in \mathbb{N}}$ is a weak approximation of the quantum SLE process $(X_1^{1/2})$ in the following sense:

(a) Let $A_{N,n}$ be the adjacency matrix of $C_{N,n}$. Denote by $\mu_{N,n}$ the distribution of $A_{N,n}$ with respect to the quantum probability space $(\mathbb{P}(\mathbb{V}_N^o), \delta_o^{\otimes n})$ and let $T \geq 0$. Then

$$\lim_{n \to \infty} \lim_{N \to \infty} \mu_{N,[n^{1/4}]^2}(\sqrt{n} \cdot) = \sigma_{T}^{1/2}(\cdot)$$

in distribution with respect to the topology induced by weak convergence.

(b) Fix $N, n \in \mathbb{N}$. Consider the quantum probability space $(\mathbb{P}(\mathbb{V}_N^o), \delta_o^{\otimes n})$. Extend $A_{N,k}$ to $\mathbb{P}(\mathbb{V}_N^o)$ by $A_{N,k} := A_{N,k} \otimes P^N \otimes \ldots \otimes P^N$, where $P^N$ denotes the projection in $\mathbb{P}(\mathbb{V}_N^o)$ onto $\delta_o$. Then the increments $(A_1, A_2 - A_1, \ldots, A_n - A_{n-1})$ are monotonically independent.

**Proof.** Statement (b) follows directly from Lemmas 3.3 and 4.4. Let $\mu_{N,1}^+$ be the distribution of $A^N$. As $N \to \infty$ we have $\mu_{N}^+ \to m_{4,2,1}$ with respect to weak convergence due to Lemma 4.5 and [MW80, Lemma 4.11].

So, let $\lambda_1 \leq \ldots \leq \lambda_N = 4$ be the eigenvalues of $A^N$ with normalized eigenvectors $v_1, \ldots, v_N$, where we can choose $v_N = (1/\sqrt{N}, \ldots, 1/\sqrt{N})$, as $S_N^1$ is 4-regular. For $x \in \mathbb{R}$, denote by $\Delta_x$ the Dirac measure at $x$. Then we have the convergence

$$\mu_{N}^+ = \sum_{j=1}^{N} \Delta_{\lambda_j} \langle \delta_o, v_j \rangle^2 \to m_{4,2,1}.$$

Next we consider $S_N^{-1}$. As $S_N^1$ is 4-regular, the eigenvalues of $B^N$ are given by $-\lambda_{N-1}, \ldots, -\lambda_1, N - 4$ with eigenvectors $v_{N-1}, \ldots, v_1, v_N$. The distribution of $B^N$, denoted by $\mu_{N}^{-1}$, is thus given by

$$\mu_{N}^{-1} = \sum_{j=1}^{N-1} \Delta_{-\lambda_j} \langle \delta_o, v_j \rangle^2 + \Delta_{N-4} \langle \delta_o, v_N \rangle^2 = \sum_{j=1}^{N-1} \Delta_{-\lambda_j} \langle \delta_o, v_j \rangle^2 + \frac{1}{N} \Delta_{N-4}.$$ 

Hence, as $N \to \infty$,

$$\mu_{N}^{-1} \to m_{4,2,-1}, \tag{4.11}$$

where $m_{4,2,-1}$ is simply $m_{4,2,1}$ reflected, i.e. $m_{4,2,-1}(M) = m_{4,2,1}(-M)$ for every Borel subset $M \subseteq \mathbb{R}$.

Lemmas 4.3, 4.4, 4.5 yield

$$\mu_{N,n} \to m_{4,2,u_1} \triangleright \ldots \triangleright m_{4,2,u_n} \tag{4.12}$$

as $N \to \infty$ with probability 1.

Let $U : [0, \infty) \to \mathbb{R}$ be the function $U(t) = \sum_{j=1}^{[t/4]} u_j$. Then $U$ is constant $u_1$ in $[0,4)$, $u_1 + u_2$ in $[4,8)$, etc. Let $f_t$ be the solution to (2.2) with this driving function and define the measures $\alpha_t$ by
Starting the Loewner equation (2.2) for the metric space \( h_t \) at \( t = 4 \) with initial value \( h_4(z) = z \) and driving function \( U(t) \) yields the mappings \( h_t \) that satisfy \( f_{4+t} = f_4 \circ h_t \). Obviously, \( h_4 = F_{m_{4,2,u_2}} \) and thus \( \alpha_8 = m_{4,2,u_1} \). By induction we obtain
\[
\alpha_{4m} = \bigcup_{j=1}^{m} m_{4,2,u_j}
\]
for all \( m \in \mathbb{N} \). The function \( V(t) := U(4t) \) is just a normal random walk (with increments \( \pm 1 \)) and we know that \( V(nt)/\sqrt{n} \) converges in distribution to a Brownian motion \( B_t \) as random elements of the metric space \( C([0,T], \mathbb{R}) \) with the topology induced by the maximum norm. We need the driving function \( \sqrt{\kappa/2} B_t \), or \( B_{t \kappa/2} \), which has the same distribution. Hence we define \( U_n(t) = U(2\kappa \cdot nt)/\sqrt{n} \) and we have \( U_n(t) \to B_{t \kappa/2} \) as \( n \to \infty \).

Let \((h_{n,t})_{t \in [0,T]} \) and \((h_{t})_{t \in [0,T]} \) be the random Loewner chains that correspond to \( U_n \) and \( B_{t \kappa/2} \) respectively. Define the measures \( \nu_{n,t} \) by \( h_{n,t} F_{\nu_{n,t}} \). Note that \( U_n \) has the form \( U_n = U(d \cdot t)/c \) with \( d = c^2 \). Here we use assumption (4.10). Hence, by Lemma 2.6 we have
\[
\nu_{n,t}(B) = \alpha_{2nt}(\sqrt{n}B) = \alpha_{nt}(\sqrt{n}B)
\]
for all \( t \geq 0 \). If \( t \) has the form \( t = 2m/(\kappa n) = 4m/n, m \in \mathbb{N} \), then (4.13) gives
\[
\nu_{n,t}(B) = \alpha_{4m}(\sqrt{n}B) = (\bigcup_{j=1}^{m} m_{4,2,u_j})(\sqrt{n}B) = (\bigcup_{j=1}^{m} m_{4,2,u_j})(\sqrt{n}B).
\]
Finally, \( h_{n,T} \to h_T \) in distribution with respect to locally uniform convergence; see [RS17, Theorem 2.4]. By Lemma 2.3 (c), we have \( \nu_{n,T} \to \sigma^f_\kappa \) in distribution with respect to weak convergence, or
\[
(\bigcup_{j=1}^{m} m_{4,2,u_j})(\sqrt{n}B) \to \sigma^f_\kappa.
\]
The statement follows together with (4.12). \( \square \)

Remark 4.10. In Theorem 4.9 we chose a spidernet with data \((4,4,2)\). Recall that we denoted by \( S_{M,u}, M \in \mathbb{N}, u \in \{1,\ldots,2M-1\} \), a spidernet with data \((2M,M+1+u,M)\). Can we choose a general \( S_{M,u} \) instead of the special case \( M = 2, u = 1? \)

In the proof, the function \( U \) will then change by \( \pm u \) within the time interval \( 2M \). Thus we need to define \( V(t) = U(2Mt)/u \) and \( U_n = U(Mkn)/u(\sqrt{n}) \). The application of Lemma 2.6 now requires \( d = c^2 \), i.e. \( Mkn = u^2 n \) and we obtain \( \kappa = \frac{u^2}{M} \). Furthermore, we used that the spidernet is a regular graph. We have \( \text{deg}(a) = 2M \) and \( \text{deg}(x) = M + 1 + u \) for \( x \neq a \), i.e. we have a regular graph if \( u = M - 1 \), which requires \( M \geq 2 \). One can verify that the proof works in these cases too, i.e. for
\[
\kappa = \frac{(M-1)^2}{M}, u = M - 1, M \geq 2.
\]

Remark 4.11. As noted in the introduction, we cannot have convergence of the (scaled) moments of \( \mu_{N,n} \) to the moments of \( \sigma^f_\kappa \). Indeed, the first moment of \( B^N \) within \((l^2(V_N), \delta_0)\) is 1 and thus (4.11) is not true with respect to convergence of moments. \( \star \)
5. On the SLE measure

Fix some $T > 0$ and consider the random measure $\sigma^T_\kappa$ that corresponds to the random SLE($\kappa$) conformal mapping. Then we can define a two-step experiment by generating a Brownian motion $(B_t)_{t \in [0,T]}$ and then a random number $S^T_\kappa$ according to $\sigma^T_\kappa$. Denote the unconditional distribution of $S^T_\kappa$ by $\mu^T_\kappa$.

Figure 5 shows a numerical simulation of the distribution $\mu^T_\kappa$ for some values of $\kappa$ and $T = 1$.

Lemma (2.6) implies that $\sigma^T_{c^2} \cdot T (c \cdot B)$ has the same distribution as $\sigma^T_\kappa$ for every $T > 0$ and $c > 0$. This implies that $\mu^T_{c^2}(c \cdot B) = \mu^T_\kappa(B)$ and, with $c = 1/\sqrt{T}$, we get $\mu^T_1(B/\sqrt{T}) = \mu^T(B)$ for all Borel subsets $B \subseteq \mathbb{R}$.

**Figure 7.** Histograms with 10,000 points approximating the distribution $\mu^T_\kappa$ with $T = 1$ and $\kappa = 0, 1, 2, \text{ and } 8$.

**Theorem 5.1.** All moments of $\mu^T_\kappa$ exist.

**Proof.** Let $(f_t)_{t \geq 0}$ be the corresponding random Loewner chain. For each $T > 0$, the random mapping $f_T$ maps a compact interval $[L_T, R_T]$ onto $\partial (\mathbb{C}^+ \setminus f_T(\mathbb{C}^+)) \cap \mathbb{C}^+$ (in the sense of cluster sets). Furthermore, it extends continuously to $\mathbb{R} \setminus [L_T, R_T]$, which is mapped bijectively onto a subset of $\mathbb{R} \setminus \{0\}$, as $0$ is the starting point of the growing hulls (which are slits when $\kappa \in [0, 4]$). Thus we have $\text{Im}(1/f_T(x)) = 0$ for all $x \in \mathbb{R} \setminus [L_T, R_T]$ and, by the Stieltjes-Perron inversion formula, we conclude that $[L_T, R_T]$ contains the support of $\sigma^T_\kappa$.

Let $x(t)$ be the solution of the following initial value problem

$$
\frac{dx(t)}{dt} = \frac{1}{x(t) - \sqrt{\kappa/2}B_t}, \quad x(0) = x_0 > 0.
$$

Reparametrizing $x(t)$ by

$$
y(t) := (x(t) - \sqrt{\kappa/2}B_t)/(\sqrt{\kappa/2})
$$

gives

$$
dy(t) = \frac{2/\kappa}{y(t)} dt + dW_t, \quad y(0) = x_0\sqrt{2/\kappa},
$$

15
where \( W_t = -B_t \) is again a Brownian motion. Thus, \( y \) is a Bessel process.

For \( \kappa \in [0, 4] \), the solution exists for all \( t \geq 0 \) (see [Law05, Prop. 1.21]) and we have \( R_T = \lim_{x_0 \downarrow 0} x(T; x_0) \). In particular, we have \( R_T \leq x(T; 1) \).

For \( \kappa > 4 \), the solution \( x(t; x_0) \) exists up to the random time \( S(x_0) \), i.e. we have \( x(t; x_0) - \sqrt{\kappa/2} B_t \to 0 \), or \( y(t; x_0) \to 0 \) as \( t \to S(x_0) \).

If \( x_0 \) is big enough such that \( S(x_0) \geq T \), we have \( R_T \leq x(T; x_0) \). We put again \( x_0 = 1 \) and for the case \( S := S(1) < T \) we now define \( x(t) \) also for \( t \in [S, T] \) such that the inequality \( R_T \leq x(T; 1) \) still holds.

For \( t < S \), we have

\[
y(t) = \int_0^t \frac{2/\kappa}{y(s)} ds + W_t + y(0) \leq \int_0^t \frac{2/\kappa}{y(s)} ds + M_T + \sqrt{2/\kappa},
\]

where \( M_T := \max_{t \in [0, T]} B_t \geq 0 \). Define \( z(t) \) as the solution to the initial value problem

\[
\frac{d}{dt} z(t) = \frac{2/\kappa}{y(t)}, \quad z(0) = M_T + \sqrt{2/\kappa},
\]

i.e. \( z(t) = \sqrt{(M_T + \sqrt{2/\kappa})^2 + 4t/\kappa} \). Then we have \( y(t) \leq z(t) \) for all \( t \leq \min\{S, T\} \).

We have \( \lim_{t \downarrow S} y(t) = 0 \). Now we define \( y(t) \) for \( t \geq S \) as the solution to

\[
y(t) = \int_S^t \frac{2/\kappa}{y(s)} ds + W_t + \sqrt{2/\kappa} \leq \int_S^t \frac{2/\kappa}{y(s)} ds + \sqrt{(M_T + \sqrt{2/\kappa})^2 + 4S/\kappa},
\]

i.e. \( y(t) \) still satisfies the same differential equation but we jump from \( \lim_{t \downarrow S} y(t) = 0 \) to \( y(S) = \sqrt{2/\kappa} \). The solution is defined until some \( S' \). Note that, again, \( y(t) \leq \sqrt{(M_T + \sqrt{2/\kappa})^2 + 4T/\kappa} \) for \( S \leq t \leq S' \).

We continue this construction and thus we define \( y(t) \), and, by relation \( \ref{5.2} \), also \( x(t) \) for all \( t \in [0, T] \). This construction implies \( R_T \leq x(T) \), as \( x(t) \) satisfies equation \( \ref{5.1} \) piecewise on intervals and \( x(t) \geq \sqrt{\kappa/2} B_t \) for all \( t \in [0, T] \). (In other words, \( x(T) = x(T; x_0) \) for some large \( x_0 > 0 \).) We have

\[
y(T) \leq \sqrt{(M_T + \sqrt{2/\kappa})^2 + 4T/\kappa} \leq M_T + \sqrt{2/\kappa} + \sqrt{4T/\kappa}.
\]

Hence

\[
R_T \leq x(T) \leq \sqrt{\kappa/2} M_T + 1 + \sqrt{2T} + \sqrt{\kappa/2} B_T \leq 2 \sqrt{\kappa/2} M_T + 1 + \sqrt{2T}.
\]

We have

\[
\int_{\mathbb{R}} x^m \sigma_T^2(dx) = \int_{L_T}^{R_T} x^m \sigma_T^2(dx) \leq \max\{\|L_T\|^m, \|R_T\|^m\}
\]

for every \( m \in \mathbb{N} \). Denote by \( \beta \) the probability measure of a Brownian motion on the space \( C([0, \infty), \mathbb{R}) \).

In order to show the existence of

\[
\mathbb{E}[(S_T^\beta)^m] = \int_{C([0, \infty), \mathbb{R})} \int_{\mathbb{R}} x^m \sigma_T^2(dx; \omega) d\beta(\omega),
\]

it is sufficient to show that \( \max\{\|L_T\|^m, \|R_T\|^m\} \) is integrable w.r.t. \( \beta \). Due to symmetry, \( R_T \) and \( -L_T \) have the same distribution, and as \( L_T \leq R_T \), \( \ref{5.3} \) shows that \( \|L_T\|^m \) and \( \|R_T\|^m \) are both integrable. Note that the maximum \( M_T \) has the same distribution as \( \sqrt{T}|Z| \), where \( Z \) has standard normal distribution.

\[ \square \]

\textbf{Remark 5.2.} Estimate \( \ref{5.3} \) also shows that its moments uniquely determine the distribution \( \mu_T^\beta \), see [Bil95, Theorem 30.1]. \[ \star \]
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