A NOTE ON ALBERTI’S LUZIN-TYPE THEOREM FOR GRADIENTS

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ABSTRACT. We give a “soft” proof of Alberti’s Luzin-type theorem in [1] (G. Alberti, A Luzin-type theorem for gradients, J. Funct. Anal. 100 (1991)), using elementary geometric measure theory and topology. Applications to the $C^2$-rectifiability problem are also discussed.

1. INTRODUCTION

This paper is devoted to a new proof of the following theorem by G. Alberti [1]:

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^N$ be an open set with finite $N$-dimensional Lebesgue measure; $N \geq 2$. Let $v : \Omega \to \mathbb{R}^N$ be a Borel vectorfield. Then, for any $\epsilon > 0$, there exist an open set $A \subset \Omega$ and a function $\phi \in C_0^1(\Omega)$ such that $\mathcal{H}^N(A) \leq \epsilon \mathcal{H}^N(\Omega)$ and $v = \nabla \phi$ on $\Omega \sim A$.

Alberti’s theorem says that any Borel vectorfield is “nearly” — in the sense of Luzin [16] — the gradient of a scalar potential. It can also be interpreted as follows: any differential 1-form is “nearly” exact. The latter statement readily generalises to differential forms of arbitrary degree on Riemannian manifolds; see [17]. Proposition 2.3 for the Euclidean setting.

Various improvements and generalisations of Alberti’s theorem have been studied; cf. Moonens–Pfeffer [17] for an a.e.-version of Theorem 1.1 (namely, $\epsilon = 0$ therein) and extension to charges/flat cochains, Francos [10] for extension to higher-order derivatives, and David [4] to metric measure spaces, as well as the references cited therein.

Alberti’s original proof of Theorem 1.1 in [1] is constructive: one divides $\Omega$ into dyadic cubes, approximates $v$ by affine functions on the dyadic cubes at each level, smooths in “transition layers” via convolution, and iteratively corrects resulting errors at the next level.

Here we present an alternative proof using geometric measure theory and topology. We take a new perspective by looking at graphs of functions, rather than the functions per se, and finding approximations to the graphs by topological arguments. This approach is motivated by, among others, the seminal works of Giaquinta–Modica–Soucek [13, 14] on harmonic maps.

2. A ROUGH APPROXIMATION

Following Alberti’s original approach ([1], Lemma 7), we first establish

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^N$ be an open set with finite $N$-dimensional Lebesgue measure ($N \geq 2$), let $v : \Omega \to \mathbb{R}^N$ be a Borel vectorfield, and let $\eta, \epsilon, \vartheta$ be arbitrary positive numbers. There exist a compact set $K \subset \Omega$ and a function $\phi \in C_0^1(\Omega)$ such that $\mathcal{H}^N(\Omega \sim K) \leq \epsilon \mathcal{H}^N(\Omega)$, $\|\phi\|_{C^0(\Omega)} \leq \vartheta$, and $|v - \nabla \phi| \leq \eta$ on $K$.

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Here, as in Moonens–Pfeffer [17], we require \( \phi \) to satisfy the uniform smallness condition \( \| \phi \|_{C^0(\Omega)} \leq \theta \) in addition to Alberti’s original result. Roughly speaking, \( \phi \) is a “micro-oscillation”.

**Proof.** The arguments are divided into five steps.

**Step 1. Reduction to continuous, bounded vectorfields.**

Let \( \kappa \) be a small positive number to be specified later. Since \( v \) is Borel, there exists a finite number \( \Lambda = \Lambda(\kappa) \) such that \( B = B(\kappa) := \{ x \in \Omega : |v(x)| > \Lambda \} \) satisfies \( \mathcal{H}^N(B) < \kappa \). On the other hand, applying the classical Luzin theorem [16], we may find a Borel set \( B' \subset \Omega \) and a continuous vectorfield \( v_0 : \Omega \to \mathbb{R}^N \) such that \( \mathcal{H}^N(B') < \kappa \) and \( v_0 = v \) on \( \Omega \sim B' \). Let us set

\[
v_1 := \begin{cases} v_0 & \text{on } \Omega \sim B, \\ \Lambda|v_0|^{-1}v_0 & \text{on } B. \end{cases}
\]

Clearly, \( v_1 \) coincides with \( v \) on \( \Omega \sim (B \cup B') \), \( |v_1| \leq \Lambda \) on \( \Omega \), and \( v_1 \) is continuous on \( \Omega \).

Suppose now that Lemma 2.1 is already proved for the continuous, bounded vectorfield \( v_1 \); that is, we have a compact set \( K_1 \subset \Omega \) and a function \( \phi \in C^1_c(\Omega) \) satisfying \( \mathcal{H}^N(\Omega \sim K_1) \leq \epsilon \mathcal{H}^N(\Omega)/2 \) and \( |v_1 - \nabla \phi| \leq \eta \) on \( K_1 \). Then, for the compact set \( K := K_1 \sim [\text{int}(B \cup B')] \), we have \( \mathcal{H}^N(\Omega \sim K) \leq \epsilon \mathcal{H}^N(\Omega)/2 + 2\kappa \) and \( |v_1 - \nabla \phi| \leq \eta \) on \( K \). Thus one may conclude Lemma 2.1 for the Borel vectorfield \( v \) by taking \( \kappa := \epsilon \mathcal{H}^N(\Omega)/4 \).

**Step 2. Reduction to smooth, compactly supported vectorfields.**

For each sufficiently small \( a > 0 \), we denote as usual

\[
\Xi_a(\bullet) := a^{-N}\Xi_1(\bullet/a)
\]

where \( \Xi_1 \) is the standard mollifier on \( \mathbb{R}^N \), and

\[
\Omega_a := \{ x \in \Omega : \text{dist}(x, \mathbb{R}^N \sim \Omega) > a \}.
\]

For any continuous bounded \( v : \Omega \to \mathbb{R}^N \) we take

\[
v_2 := (v\chi_{\Omega_a}) \ast \Xi_{a/10} \in C^\infty_c(\Omega; \mathbb{R}^N)
\]

with \( \sigma > 0 \) to be specified. In fact, \( \text{spt}(v_2) \subseteq \Omega_{4a/5} \). Here, \( \chi_E \) is the characteristic function of set \( E \), and \( \ast \) is the convolution operator.

Suppose that Lemma 2.1 is already established for smooth, compactly supported vectorfields. Then, for a compact set \( K_2 \subset \Omega_{4a/5} \) and a function \( \phi_2 \in C^1_c(\Omega_{4a/5}) \), there hold \( |v_2 - \nabla \phi_2| \leq \eta \) on \( K_2 \) and \( \mathcal{H}^N(\Omega_{4a/5} \sim K_2) \leq \epsilon \mathcal{H}^N(\Omega_{4a/5})/2 \). Take \( K \equiv K_2 \) and \( \phi \equiv \text{extension-by-zero of } \phi_2 \). Clearly \( |v - \nabla \phi| \leq \eta \) on \( K \), and for the case \( \mathcal{H}^N(\Omega_{4a/5} \sim K) > 0 \) we have

\[
\mathcal{H}^N(\Omega \sim K) \leq \frac{\mathcal{H}^N(\Omega \sim K)}{\mathcal{H}^N(\Omega_{4a/5} \sim K)} \leq \frac{\mathcal{H}^N(\Omega_{4a/5} \sim K)}{\mathcal{H}^N(\Omega_{4a/5})} \leq \frac{\mathcal{H}^N(\Omega \sim K)}{2 \mathcal{H}^N(\Omega_{4a/5})}.
\]

Since \( \mathcal{H}^N(\Omega_a)/\mathcal{H}^N(\Omega) \nearrow 1 \) as \( a \searrow 0 \) for the open set \( \Omega \subset \mathbb{R}^N \), with \( \sigma \) sufficiently small, the last two factors in the final line in (2.1) can be chosen arbitrarily close to 1. Thus \( \mathcal{H}^N(\Omega \sim K) \leq \epsilon \mathcal{H}^N(\Omega) \). The above inequality holds trivially when \( \mathcal{H}^N(\Omega_{4a/5} \sim K) = 0 \), by shrinking \( \sigma > 0 \) if necessary. This proves Lemma 2.1 for the continuous bounded vectorfield \( v \).
Step 3. Reduction to vectorfields with rectifiable graphs.

In view of the preceding step, from now on we may assume that \( v \in C^\infty_c(\Omega; \mathbb{R}^N) \) for some fixed \( \sigma > 0 \). For the graph
\[
\Gamma := \text{graph}_\Omega(v) := \{(x, v(x)) : x \in \Omega\} \subset \Omega \times \mathbb{R}^N,
\]
we have
\[
\mathcal{H}^N(\Gamma) := \int_{\Omega} \sqrt{1 + |\nabla v|^2} \, d\mathcal{H}^N \leq M,
\]
where \( M \) is a finite number depending only on \( \|v\|_{C^1(\Omega; \mathbb{R}^N)} \) and the \( N \)-dimensional Lebesgue measure of \( \Omega \). Applying Federer’s structure theorem ([9, 18]), we get the decomposition
\[
\Gamma = Y \sqcup Z,
\]
where \( Y \) is \((\mathcal{H}^N, N)\)-rectifiable and \( Z \) is purely \( N \)-unrectifiable.

Denote by \( \text{pr} \) the projection of \( \Omega \times \mathbb{R}^N \) onto the first coordinate; we claim that
\[
\mathcal{H}^N(\text{pr}(Z)) = 0. \tag{2.2}
\]
Indeed, if it were false, there would be an open ball \( O \subset \text{pr}(Z) \subset \Omega \) with \( \mathcal{H}^N(O) > 0 \). By the boundedness of \( \Omega \) and that \( v \in C^\infty_c(\Omega; \mathbb{R}^N) \), the supremum of \( |\nabla v| \) over \( \Omega \) is finite; thus the following transversality result holds:
\[
\inf \left\{ \text{dist}(T_\xi \Gamma, \mathcal{V}) : \xi \in \Gamma, \mathcal{V} \in \text{Gr}(N, 2N) \text{ is orthogonal to the image of } \text{pr} \right\} \geq \epsilon_0 > 0. \tag{2.3}
\]
Throughout, \( \text{Gr}(N, 2N) \) is the Grassmannian manifold of \( N \)-planes in \( \mathbb{R}^{2N} \) with the natural topology: the distance between two \( N \)-planes is the operator norm of the difference of the corresponding projections. Denote by \( \mu \) the Haar measure on \( \text{Gr}(N, 2N) \) with a fixed normalisation.

By (2.3), there is a neighbourhood \( \mathcal{N} \subset \text{Gr}(N, 2N) \) such that
- \( \mathcal{N} \) contains the horizontal section \( \mathbb{R}^N \times \{0\} \);
- Each \( \Pi \in \mathcal{N} \) contains the origin of \( \mathbb{R}^{2N} \);
- \( \mu(\mathcal{N}) > 0 \); and
- \( \Gamma_O := \text{graph}_\Omega(v) \) is a graph over each \( N \)-plane in \( \mathcal{N} \).

Writing \( \text{pr}_\Pi \) for the projection of \( \Gamma_O \) onto \( \Pi \in \mathcal{N} \), we deduce that \( \{\text{pr}_\Pi : \Pi \in \mathcal{N}\} \) constitutes a family continuous bijections onto their images; furthermore, this family is continuous in \( \Pi \). Therefore, we obtain a continuous map:
\[
\mathcal{N} \ni \Pi \longmapsto \mathcal{H}^N(\text{pr}_\Pi(\Gamma_O)) \in \mathbb{R}_+.
\]
Noticing that \( \text{pr}_{\mathbb{R}^N \times \{0\}} \equiv \text{pr} \) and \( \mathcal{H}^N(O) > 0 \), we have found a neighbourhood of \( \mathbb{R}^N \times \{0\} \) of positive \( \mu \)-measure about \( \mathbb{R}^N \times \{0\} \), such that projections of \( \Gamma \) in these directions all have positive \( \mathcal{H}^N \)-measure. This contradicts the unrectifiability of \( Z \); hence the claim (2.2) follows.

From now on, one assumes that the graph of \( v \) is \((\mathcal{H}^N, N)\)-rectifiable.

Step 4. Reduction to vectorfields mapping between PL-manifolds.

Let \( v \in C^\infty_c(\Omega; \mathbb{R}^N) \) be such that \( \Gamma := \text{graph}_\Omega(v) \) is \((\mathcal{H}^N, N)\)-rectifiable and that \( \text{spt}(v) \in \Omega_\sigma \) for some \( \sigma > 0 \). In light of the statement of Lemma 2.1, one has the freedom of modifying \( v \) on arbitrarily \( \mathcal{H}^N \)-small sets. Thus, using the graphical structure of \( \Gamma \) and the definition of rectifiability, we can assume in the sequel that \( \Gamma \) is a \( C^1 \)-submanifold embedded in \( \mathbb{R}^{2N} \).
Now, the classical Cairn–Whitehead theorem ([3, 19]) implies that $\Gamma$ has an essentially unique triangulation $K_{\Gamma}$. As $\Gamma = \text{graph}_{\Omega}(v)$ for $v$ smooth, it induces a triangulation $K_{\sigma}$ of $\Omega_{\sigma} \supset \text{spt}(v)$ via the projection $\text{pr} : \Omega \times \mathbb{R}^N \to \Omega$. Hence, here and hereafter, we may view $v$ as a map between PL-manifolds, namely

$$v : |K_{\sigma}| \subset \mathbb{R}^N \to |K_{\Gamma}| \subset \mathbb{R}^{2N}.$$ 
Throughout, we use $|K|$ to denote the geometrical realisation of triangulation $K$.

**Step 5: Completion of the proof by simplicial approximation.**

Let $\eta > 0$ be arbitrary. By the simplicial approximation theorem ([15], Theorem 2C.1, p.177), one can find by taking successive barycentric subdivisions of $K_{\Omega_{\sigma}}$ (not relabelled) a simplicial map $v_3 : |K_{\Omega_{\sigma}}| \to |K_{\Gamma}|$ such that

$$\|v_3 - v\|_{C^0(\mathbb{R}^N; \mathbb{R}^N)} \leq \frac{\eta}{2}. \quad (2.4)$$

Here, as usual, we identify $v_3$ with its extension-by-zero defined on $\mathbb{R}^N$.

On each simplex $\tau$ of the corresponding barycentric subdivision, $v_3$ is equal to the linear combination of its values at the vertices. Since $v_3$ is a simplicial map, it equals to $\nabla \psi$ for some $\psi : |K_{\Omega_{\sigma}}| \to \mathbb{R}$ on each $N$-simplex $\tau$. Since $v_3$ is a $C^0$-map, $\psi$ is $C^1$ in the interior of any such $\tau$. Moreover, since $v$ is uniformly continuous on $\Omega_{\sigma}$, by refining the barycentric subdivisions we can make the oscillation of $v_3$ on any such $\tau$ arbitrarily small. By subtracting a constant, one may further assume that

$$\max \left\{ \|\psi\|_{C^0(\tau)} : \tau \text{ is an } N\text{-simplex of } K_{\Omega_{\sigma}} \right\} \leq \frac{\vartheta}{2}.$$ 

Finally, let us modify $\psi$ to obtain the desired map $\phi$, which is $C^1$ on the whole domain $\Omega$. Note that $\psi$ is $C^1$ except on the closure of the $(N-1)$-skeleton of $K_{\Omega_{\sigma}}$, which is a null set with respect to the $N$-dimensional Lebesgue measure. Take an open neighbourhood $O_2$ of the $(N-1)$-skeleton of arbitrarily small measure, e.g., $\mathcal{H}^N(O_2) \leq \epsilon \mathcal{H}^N(\Omega)$. A standard smoothing argument then yields $\phi \in C_2^1(\Omega)$ such that $\phi \equiv \psi$ on $\Omega_{\sigma} \sim O_2$ and that $\|\phi\|_{C^0(\Omega)} \leq \vartheta$. Thanks to (2.4), we can now complete the proof by suitably choosing $\rho$ and setting $K := \Omega \sim O_2$. \hfill \Box

The above proof is “soft”: only geometric measure theoretic and topological arguments are involved, but not hard analysis. It relies essentially on the graphical structure of $\Gamma$. With a little additional effort we can also recover the quantitative $L_p$-estimates in [1], Lemma 7.

3. **Rectifying the Error**

In this section, we explain how to deduce Theorem 1.1 (i.e., Theorem 1 in Alberti [1]) from Lemma 2.1. Our treatment is adapted from [1]; nevertheless, we shall establish an approximation theorem in the more general setting of functional analysis. By doing so, we emphasise the central rôle played by the extension property (3.3), which imposes severe difficulties for generalising Theorem 1.1 to function spaces with higher regularity; see Remark 3.2 below.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces consisting of Borel functions from an open subset $\Omega \subset \mathbb{R}^N$ (that is, $\mathcal{X} :=$ the completion of Borel functions on $\Omega$ with respect to $\| \cdot \|_{\mathcal{X}}$, and similarly for $\mathcal{Y}$), let $\mathcal{Y} \leq \mathcal{Y}'$ be a Banach subspace, and let $T : \mathcal{Y} \to \mathcal{X}$ be a bounded linear operator.
Suppose that there is a uniform constant \( C_0 > 0 \) such that
\[
C_0^{-1} \|\phi\|_{\mathcal{Y}} \leq \|\phi\|_{\mathcal{X}} + \|T\phi\|_{\mathcal{X}} \leq C_0 \|\phi\|_{\mathcal{Y}}.
\] (3.1)

For a subset \( \Omega' \subseteq \Omega \), we denote by \( \mathcal{X}(\Omega') \) the \( \|\cdot\|_{\mathcal{X}} \)-completion of Borel functions over \( \Omega' \), and similarly for \( \mathcal{Y}(\Omega'), \mathcal{Z}(\Omega') \). Also, suppose that

the norm topology of \( \mathcal{X}' \) is stronger than the pointwise topology on \( \Omega \),

and that \( \|\cdot\|_{\mathcal{X}} \) satisfies the extension property:

For each compact subset \( K \subseteq \Omega \), each \( f \in \mathcal{X}(K) \), and each \( s > 0 \), there exists \( \overline{f} \in \mathcal{X}'(\Omega) \) such that \( \overline{f}|K = f \) and \( \|\overline{f}\|_{\mathcal{X}} \leq (1 + s)\|f\|_{\mathcal{X}(K)} \). \( \text{(3.3)} \)

Then, we have:

**Theorem 3.1.** Let \( \mathcal{X}', \mathcal{Y}, \mathcal{Z}, \Omega, \) and \( T \) be as above (in particular, the hypotheses \( \text{(3.1)} \|\), \( \text{(3.2)} \), \( \text{(3.3)} \) hold); let \( v \in \mathcal{X}' \). Assume that for any triplet of positive numbers \( \{\eta, \epsilon, \vartheta\} \), there are a compact set \( K \subseteq \Omega \) and an element \( \phi \in \mathcal{Y} \) such that \( H^N(\Omega \sim K) \leq \epsilon H^N(\Omega), \|\phi\|_{\mathcal{X}} \leq \theta \), and \( \|v - T\phi\|_{\mathcal{X}(K)} \leq \eta \). Then, for arbitrary positive numbers \( \{\delta, \kappa\} \), we can find an open set \( A \subseteq \Omega \) and an element \( \tilde{\phi} \in \mathcal{Y} \) such that \( H^N(A) < \delta H^N(\Omega), \|\tilde{\phi}\|_{\mathcal{X}} \leq \kappa \), and that \( v = T\tilde{\phi} \) on \( \Omega \sim A \).

Theorem 3.1 states that, starting from a “rough” Lusin-type approximation result with three positive parameters \( \{\eta, \epsilon, \vartheta\} \), we can obtain a refined Lusin-type result which remains valid for \( \eta = 0 \); meanwhile, restrictions on the weaker norm \( \|\cdot\|_{\mathcal{X}} \) can be retained.

Assuming this, Alberti’s Theorem \( \text{(1.1)} \) can be deduced as an immediate corollary:

**Proof of Theorem \( \text{(1.1)} \)** Take \( \mathcal{X} = C^0_0(\Omega; \mathbb{R}^N), \mathcal{Y} = C^0_0(\Omega), \mathcal{Z} = C^0_0(\Omega), \) and \( T = \nabla : \mathcal{Y} \rightarrow \mathcal{X} \) as in Theorem 3.1. In this case, the extension property \( \text{(3.3)} \) follows from Tietze’s theorem (which even allows \( s = 0 \) in \( \text{(3.3)} \)), and assumptions of Theorem 3.1 on the rough approximation with parameters \( \{\eta, \epsilon, \vartheta\} \) are verified by Theorem 2.1.

**Proof of Theorem \( \text{(3.3)} \)** We construct an iteration scheme:

\[
\left\{(v_n, K_n, \varphi_n)\right\}_{n=0,1,2,\ldots} \subset \mathcal{X} \times \text{compact subsets of } \Omega \times \mathcal{Y}.
\]

Take \( v_0 := v, K_0 := \overline{\Omega} \), and \( \varphi_0 := 0 \).

Assume that \( (v_n, K_n, \varphi_n) \) has been constructed; we shall define \( (v_{n+1}, K_{n+1}, \varphi_{n+1}) \). First, by assumption, we can take a compact set \( K_{n+1} \) such that \( H^N(\Omega \sim K_{n+1}) \leq 2^{-n-1}\delta H^N(\Omega), \) and take \( \varphi_{n+1} \in \mathcal{Y} \) such that \( \|v_n - T\varphi_{n+1}\|_{\mathcal{X}(K_{n+1})} \leq 16^{-n}\eta \) as well as \( \|\varphi_{n+1}\|_{\mathcal{X}} \leq 2^{-n-1}\kappa \). Then, set \( v_{n+1} := v_n - T\varphi_{n+1} \in \mathcal{X}'(K_{n+1}) \). In view of the extension property \( \text{(3.3)} \), we may extend \( v_{n+1} \) to an element of \( \mathcal{X}' \equiv \mathcal{X}'(\Omega) \) (without relabelling) such that

\[
\|v_{n+1}\|_{\mathcal{X}} \leq 8^{-n}\eta.
\] (3.4)

Now, define \( \tilde{\phi} \) and \( A \) as follows:

\[
\begin{cases}
\tilde{\phi} := \sum_{j=1}^{\infty} \varphi_j; \\
A := \Omega \sim \bigcap_{j=1}^{\infty} K_j.
\end{cases}
\]

It is clear that \( A \) is open and \( H^N(A) \leq \sum_{j=1}^{\infty} H^N(\Omega \sim K_j) \leq \sum_{j=1}^{\infty} 2^{-j}\delta H^N(\Omega) = \delta H^N(\Omega) \).

Also, \( \tilde{\phi} \) is a well-defined element of \( \mathcal{Z} \) with \( \|\tilde{\phi}\|_{\mathcal{X}} \leq \kappa \). On the other hand, by \( \text{(3.4)} \) we know
that \( \{v_k\} \) is a Cauchy sequence in \( \mathcal{X} \), hence it converges to a limit, which must be 0. Since \( T\varphi_{n+1} = v_n - v_{n+1} \), it implies that \( \{T\varphi_n\} \) converges in \( \mathcal{X} \). Moreover, by (4.1) one obtains

\[
\sum_{j=1}^{N} \|\varphi_j\|_X \leq C_0 \sum_{j=1}^{N} \left( \|T\varphi_j\|_X + \|\varphi_j\|_X \right)
\]

\[
\leq C_0 \left( \|v\|_X + \sum_{j=1}^{N} (8^{-j-1} + 8^{-j})\eta + \sum_{j=1}^{N} 2^{-j}\kappa \right)
\]

\[
\leq C_0 \left( \|v\|_X + \frac{\eta}{2} + \kappa \right).
\]

Sending \( N \to \infty \), we deduce that \( \tilde{\phi} \in \mathcal{X} \); in fact, \( \|\tilde{\phi}\|_X \leq C_0(\|v\|_X + \eta/2 + \kappa) \).

Finally, let us prove that \( v = T\tilde{\phi} \) on \( \Omega \sim A \). Indeed,

\[
v - T\tilde{\phi} = v_0 - T\varphi_1 - T\varphi_2 - T\varphi_3 - \ldots
\]

where \( v_0 - T\varphi_1 = v_1 \) on \( K_1 \), \( v_0 - T\varphi_1 - T\varphi_2 = v_2 \) on \( K_1 \cap K_2 \), and so on. In view of the extension property (3.3), we have

\[
\|v_j\|_X \left( \bigcap_{i=1}^{N} \kappa_i \right) \leq (1 + s)4^{-j}\eta.
\]

Sending \( j \to \infty \) yields that \( \|v - T\tilde{\phi}\|_X = 0 \), where \( v - T\tilde{\phi} \) denotes the extension of \( v - T\tilde{\phi} \) from \( \Omega \sim A = \bigcap_{j=1}^{\infty} K_j \) to \( \Omega \), whose existence is ensured by (3.3). But \( v - T\tilde{\phi} \) is defined pointwise on \( \Omega \sim A \), so it must be zero thereon, thanks to (3.2). The proof is now complete. \( \square \)

**Remark 3.2.** The extension property (3.3) is crucial for our proof of Theorem 3.1. It is unclear whether an analogous result of Alberti’s theorem holds for \( \mathcal{X} = C^{k,\alpha}(\Omega; \mathbb{R}^N) \), \( \mathcal{Y} = C^{k+1,\alpha}(\Omega) \), \( \mathcal{X} = C^{k,\alpha}(\Omega) \), and \( T = \nabla \) for any \( k \geq 1 \), \( \alpha \in [0,1] \). There, (3.3) amounts to the hypotheses for Whitney extensions of Hölder/Lipschitz functions, which are not automatically verified as in the case of Tietze extension for continuous functions. This is reminiscent of S. Delladio’s works (see, e.g., [3]) on the higher-order rectifiability criteria on certain generalised fibre bundles.

4. Rectifiable \( N \)-currents that are non-\( C^2 \)-rectifiable: arbitrary \( N \)

An important problem in geometric measure theory concerns the \( C^2 \)-rectifiability of Legendrian currents, which are natural generalisations of graphs of Gauss maps on hypersurfaces with weaker regularity. Pioneered by Anzellotti–Serapioni [2], studies on the \( C^2 \)-rectifiability problem have been carried out by Delladio [3, 6] and Fu [11, 12], among many other researchers.

It was first observed in [12] that the \( C^2 \)-rectifiability problem is closely related to Alberti’s Theorem 3.1. In particular, Fu showed ([12], Proposition 1) that Alberti’s construction yields a rectifiable current on \( \mathbb{R}^3 \) which is not \( C^2 \)-rectifiable. For an \( N \)-dimensional Riemannian manifold \( \mathcal{M} \), we say that \( E \subset \mathcal{M} \) is a \( C^0 \)-rectifiable set if and only if there are Borel measurable sets \( \{E_0, E_1, E_2, \ldots\} \subset \mathcal{M} \) satisfying \( E = \bigcup_{j=0}^{\infty} E_0 \), \( \mathcal{H}^N(E_0) = 0 \), and for each \( j = 1, 2, \ldots \) there exists \( f_j \in C^0(\mathbb{R}^N; \mathcal{M}) \) such that \( E_j = \text{the image of } f_j \). Thus, rectifiable \( \equiv C^1 \)-rectifiable.

Combining ideas from [11, 12] and elements of contact geometry (cf. e.g. [7]), we can obtain a “natural” rectifiable current of arbitrary degree that is non-\( C^2 \)-rectifiable.

**Theorem 4.1.** Let \( (\mathcal{M}, g) \) be an arbitrary \( N \)-dimensional closed Riemannian manifold of with positive volume; \( N \geq 2 \). There is a rectifiable \( N \)-current \( \mathcal{I} \in \mathcal{R}_N(\mathcal{J}^1 \mathcal{M}) \) which is non-\( C^2 \)-rectifiable. Here \( \mathcal{J}^1 \mathcal{M} := T^* \mathcal{M} \times \mathbb{R} \) is the 1-jet space of \( \mathcal{M} \).
For this purpose, we need a generalisation of Alberti’s Theorem \[\|\] any differential form is “nearly” exact. Moonens–Pfeffer [17, Proposition 2.1] proved this on Euclidean spaces; their arguments readily generalise to manifolds by a partition of unity argument. We write out the details for completeness. Here and throughout, \(\mathcal{D}^k\) denotes the space of \(C^\infty\)-differential \(k\)-forms, and \(\text{Vol}_h\) is the volume measure induced by Riemannian metric \(h\).

**Lemma 4.2.** Let \((\Sigma, h)\) be a closed, smooth Riemannian manifold of dimension \(m \geq 2\) with \(\text{Vol}_h(\Sigma) > 0\), let \(\omega \in \mathcal{D}^k(\Sigma)\) for \(k \geq 1\), and let \(\epsilon > 0\) be arbitrary. There exist an open set \(A \subset \Sigma\) and a \(C^1\)-differential \((k-1)\)-form \(\gamma\) on \(\Sigma\), such that \(\text{Vol}_h(A) \leq \epsilon \text{Vol}_h(\Sigma)\) and \(\omega = d\gamma\) on \(\Sigma \sim A\).

**Proof of Lemma 4.2.** Let \(\{U_i\}_{i=1}^I\) be a finite smooth atlas for \(\Sigma\) with coordinate mappings \(\psi_i : U_i \to \mathbb{R}^m\), and let \(\{\chi_i\}_{i=1}^I\) be a smooth partition of unity subordinate to \(\{U_i\}\). Then we get

\[
\pi^{(i)} := (\psi_i)_\#(\omega \chi_i) \in \mathcal{D}^k(\mathbb{R}^m) \quad \text{for each } i \in \{1, 2, \ldots, I\},
\]

where \((\psi_i)_\#\) denotes the pushforward under \(\psi_i\). Write \(\{x^1, \ldots, x^m\}\) for the canonical coordinates on \(\mathbb{R}^m\) (fixed for all \(i\)), and \(\Lambda(m, k)\) for the set of multi-indices \(\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k\) such that \(1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_k \leq m\). Then, for each \(i \in \{1, 2, \ldots, I\}\), we can find smooth coefficient functions \(b^{(i)}_\lambda\) compactly supported on \(\psi_i(U_i) \subset \mathbb{R}^m\) such that

\[
\pi^{(i)}(x) = \sum_{\lambda \in \Lambda(m, k)} b^{(i)}_\lambda(x) dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_k} \quad \text{for } x \in \psi_i(U_i).
\]

Now, invoking Alberti’s Theorem \[\|\] and the canonical isomorphism between vectorfields and 1-forms, for any \(\epsilon' > 0\) and \(i \in \{1, 2, \ldots, I\}\) we can find an open set \(\Omega^{(i)} \subset \psi_i(U_i)\) (thanks to the finiteness of the indexing set \(\Lambda(m, k)\)) and a \(C^1(\psi_i(U_i))\)-map \(\phi^{(i, \lambda)}\), such that \(\mathcal{H}^m(\Omega^{(i)}) \leq \epsilon' \mathcal{H}^m(\psi_i(U_i))\) and \(b^{(i)}_\lambda dx^{\lambda_1} = d\phi^{(i, \lambda)}\) outside \(\Omega^{(i)}\). Therefore, we get

\[
\pi^{(i)} = d \left( \sum_{\lambda \in \Lambda(m, k)} \phi^{(i, \lambda)} dx^{\lambda_2} \wedge \cdots \wedge dx^{\lambda_k} \right) \quad \text{on } \psi_i(U_i) \sim \Omega^{(i)}.
\]

Take \(A := \bigcup_{i=1}^I \psi_i^{-1}(\Omega^{(i)})\): it is an open set in \(\Sigma\); also, by suitably choosing \(\epsilon'\) depending on \(\epsilon, h\), and \(\{\psi_i\}\), we can ensure that \(\text{Vol}_h(A) \leq \epsilon \text{Vol}_h(\Sigma)\). Furthermore, on \(\Sigma \sim A\) there holds

\[
\omega = \sum_{i=1}^I \omega \chi_i = \sum_{i=1}^I (\psi_i)_\# d \left( \sum_{\lambda \in \Lambda(m, k)} \phi^{(i, \lambda)} dx^{\lambda_2} \wedge \cdots \wedge dx^{\lambda_k} \right)
= d \left( \sum_{i=1}^I \psi_i^\# \left( \sum_{\lambda \in \Lambda(m, k)} \phi^{(i, \lambda)} dx^{\lambda_2} \wedge \cdots \wedge dx^{\lambda_k} \right) \right).
\]

The argument inside \(\{\cdots\}\) in the last line is clearly a \(C^1\)-differential \(k-1\)-form on \(\Sigma\). The proof is completed by calling it \(\gamma\). \(\square\)

**Proof of Theorem 4.1.** It is proved by Fu [11, Lemma 1.1; also see 12, Proposition 2] that if a rectifiable \(N\)-current \(\mathcal{J}\) is carried by a \(C^2\)-rectifiable set, and if \(\beta\) is a smooth differentiable form of degree \(\leq (N-1)\) such that \(\mathcal{J} \llcorner \beta = 0\), then \(\mathcal{J} \llcorner d\beta = 0\). To prove Theorem \[\|\], we shall construct \(\mathcal{J} \in \mathcal{R}_N(\mathcal{J}^1\mathcal{M})\) and \(\beta \in \mathcal{D}^1(\mathcal{J}^1\mathcal{M})\) such that \(\mathcal{J} \llcorner \beta = 0\) while \(\mathcal{J} \llcorner d\beta \neq 0\).
To begin with, let $\alpha \in \mathcal{D}^1(T^*\mathcal{M})$ be the tautological 1-form (a.k.a. the Liouville/canonical 1-form) on the cotangent bundle $T^*\mathcal{M}$. We endow $T^*\mathcal{M}$ with the Sasaki metric $\mathcal{g}$ associated to the Riemannian metric $g$ on $\mathcal{M}$. By assumption, $\text{Vol}_\mathcal{g}(T^*\mathcal{M}) = V_0 > 0$.

Then, by Lemma 1.2 for any $\varpi \in [0, 1]$ there exist an open set $U \subseteq T^*\mathcal{M}$ and a $C^1$-function $\phi : T^*\mathcal{M} \to \mathbb{R}$ such that $\text{Vol}_\mathcal{g}(U) \geq \varpi V_0 > 0$ and $d\phi = \alpha$ on $U$.

Set $\mathcal{K} := [[\text{graph}_U, \phi]]$, the current obtained by integrating on the graph of $\phi$ over $U$ with respect to the Sasaki metric. Since $U$ is an open set and $\phi$ is $C^1$, $\mathcal{K}$ is indeed a rectifiable $N$-current carried by the 1-jet space $\mathcal{J}^1\mathcal{M}$. Moreover, we take the natural contact form $\beta := dz - \alpha \in \mathcal{D}^1(\mathcal{J}^1\mathcal{M})$, where $dz$ is the obvious volume 1-form on the $\mathbb{R}$ factor of $\mathcal{J}^1\mathcal{M}$.

Clearly, $\mathcal{K} \mathcal{L} \beta$ equals $d\phi - \alpha$ restricted to $U$, which is zero by construction. On the other hand, the $n$-fold wedge product $(d\beta)^\wedge N \equiv \pm (d\alpha)^\wedge N$. But $d\alpha$ is the canonical symplectic form on $T^*\mathcal{M}$, so $(d\alpha)^\wedge N$ coincides the volume form on $(T^*\mathcal{M}, \mathcal{g})$. Thus, we have $\mathcal{K} \mathcal{L} (d\beta)^\wedge N = \pm \text{Vol}_\mathcal{g}(U) \neq 0$, which implies that $\mathcal{K} \mathcal{L} d\beta \neq 0$. The proof is now complete. \hfill $\Box$

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