Verification of the Violation of WWŻB Inequality Using Werner States

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Abstract. The generation and manipulation of entangled particles presents itself as one of the most important results in quantum mechanics. With the work from Bell, it was possible to prove the nonlocal nature of quantum mechanics, which is nowadays widely accepted. Apart from being possible to prove entanglement from Bell’s inequality, it is difficult to compute the system as it increases the number of particles. Such a system containing \( N \) qubits can be described by Werner-Wolf-Zukowski-Brukner (WWŻB) inequality. In this work, we show how to obtain the maximum of violation of WWŻB inequality using a Werner state, simplifying the problem considerably. We get two different results; one for a system containing an odd number of particles and other for a system containing an even number.

1. Introduction

In quantum key distribution (QKD) systems the security is based on the impossibility to copy a photon with an arbitrary quantum state [1]. However, due to losses that occur in the quantum channel (optical fiber), the distance to which key distribution can be achieved is limited, since it is not possible to use optical amplification to extend it [2]. Therefore, QKD can be performed up to a few hundreds of kilometers, even if we are able to significantly improve single-photon detectors [3, 4]. By using entangled-photon pairs, in theory it is possible to double the distance to which key distribution can be achieved, being extremely difficult to surpass 1000 km [5, 2]. Through the use of quantum repeaters it is expected to be possible to surpass this distance, but the problem is that they are are still in a preliminary stage of research [6]. As an alternative to quantum repeaters stands entanglement swapping [7], which promises to be helpful in extending the distance for quantum communications beyond 1000 km [8, 9, 10].

The entanglement of particles spatially separated is one of the most noticeable results of quantum mechanics. As a consequence, it generates nonclassical correlations between the particles of the entangled system. These are the correlations that Einstein, Podolsky and Rosen claimed to give rise to a paradox that shows the incompleteness of quantum mechanics, as a
local theory [11]. The nonlocal nature of quantum mechanics was only widely accepted after the demonstration of the experiment suggested by Bell [12, 13, 14]. Bell presented his experiment with an inequality which obeys to local theories but is incompatible with quantum mechanics, in a seminal result inside quantum information science [12]. But even if the original theory of Bell considered only an entangled state with two photons, if one wants to use an $N$-qubit quantum system where each part is allowed to choose independently between two dichotomic observables (i.e. each one has only two possible results), Bell’s theorem can be generalized to the so-called Werner-Wolf-Zukowski-Brukner (WWZB) inequality [15, 16].

In this work, we verify the maximum for violation of WWZB inequality using a Werner state [17]. This type of state can be interesting for quantum communications since it traduces a mixed state which includes noise and that approximates more accurately to what we obtain in the laboratory. We show in detail how to find the maximum for violation of WWZB inequality from the correlation function and evaluate it for odd and even systems of particles. Finally, we present our conclusions for this work.

2. Theoretical Description

2.1. WWZB Inequality and Werner States

In the conditions of the standard Bell experiment [13, 14], i.e. if we consider a scheme in which each local observer can choose between two dichotomic observables, the restrictions given by local theories can be expressed through the WWZB inequality as,

$$\left| \sum_{k_1,\ldots,k_N=-1,1} S(k_1,\ldots,k_N) \times \sum_{s_1,\ldots,s_N=1,2} k_{s_1}^{k_1-1} \ldots k_{s_N}^{k_N-1} E(s_1,\ldots,s_N) \right| \leq 2^N. \tag{1}$$

In Eq. (1), $S(k_1,\ldots,k_N) = \pm 1$ and represents an arbitrary function of the sum of the indices $k_1,\ldots,k_N$. The term $s_j$ traduces the two possible measurement configurations for the observer $j$ and $E(s_1,\ldots,s_N)$ is the correlation function [16]. Equation (1) compiles a set of $2^{2N}$ Bell inequalities for the correlation function and summarizes all possible local realistic constraints on this function for an $N$-qubit system.

Let us now define a Werner state, which will be used in the verification of the violation of WWZB inequality. This type of mixed state can be represented by the density matrix and can take the form:

$$\rho_W = v |\Psi_{\text{GHZ}}\rangle \langle \Psi_{\text{GHZ}}| + (1 - v) \rho_{\text{noise}}, \tag{2}$$

where $|\Psi_{\text{GHZ}}\rangle$ is a Greenberger-Horne-Zeilinger state that allows maximum entanglement and $v$ can be interpreted as the interferometric visibility observed in a multi-particle experiment [17]. In an experiment of interferometry this scaling parameter is directly related to the visibility of the interference pattern. Visitibilities lower than 1 generally traduce the existence of some noise contribution to the state. The term $\rho_{\text{noise}} = I/2^N$ represents the completely uncorrelated noise contribution, i.e. describes a totally mixed state where $I$ is an unitary matrix of size $2^N \times 2^N$. The aforementioned critical $v$ gives the threshold beyond which no local hidden variables model can resemble the quantum predictions [18].

2.2. Maximum of the Correlation Function

The correlation function for two particles can vary between -1 and 1 and reflects the classical correlation level that they present. However, it does not tells us anything about quantum correlations. Nevertheless, it is from the correlation function that we will be able to calculate the maximum of violation of WWZB inequality, and it is this quantity that will traduce the desired quantum correlations.
In the most general formulation, the correlation function can be seen as a function that depends on the spin state and the measurement directions. The correlation function can be written as,

$$E = \text{Tr} [\rho_W \Omega],$$

(3)

where $\rho_W$ from Eq. (2) can be written also as,

$$\rho_W = \begin{pmatrix}
a & 0 & \cdots & 0 & b \\
0 & c & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & c & 0 \\
b & 0 & \cdots & 0 & a
\end{pmatrix},$$

(4)

with $a = v^2 + \frac{1 - v^2}{2N}$, $b = \frac{v}{2}$ and $c = \frac{1 - v^2}{2N}$ [19]. The parameter $\Omega$ is the Kronecker product of Pauli vectors. For the plane $\hat{x} - \hat{z}$, the Pauli vector of the $i$th observer is $\sin(\theta_i)\sigma_x + \cos(\theta_i)\sigma_z$, where $\sigma_x$ and $\sigma_z$ are Pauli matrices. Therefore, $\Omega$ takes the form of a product of cosines and sines and can be expressed as,

$$\Omega = \begin{pmatrix}
\xi_1 \prod_{i=1}^{N} \cos \theta_i & \cdots & \cdots & \cdots & \prod_{i=1}^{N} \sin \theta_i \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \cdots & \xi_j \prod_{i=1}^{N} \cos \theta_i & \cdots & \vdots \\
\prod_{i=1}^{N} \sin \theta_i & \cdots & \cdots & \cdots & \xi_2 \prod_{i=1}^{N} \cos \theta_i
\end{pmatrix},$$

(5)

where $\xi_j$ is used to explicit the signal of the $j$th diagonal element and $\theta_i$ is the angle that defines the measurement direction of the observer $i$. In Eq. (5), $\xi$ is given by,

$$\xi = \bigotimes_{i=1}^{N} \begin{pmatrix} + \\ - \end{pmatrix}. $$

(6)

Note that the matrix in Eq. (5) shows only the elements necessary for the results to be presented in the next section, i.e., only the corner and diagonal elements are used. Also, the diagonal elements are always the same, $\prod_{i=1}^{N} \cos \theta_i$, where only the signal represented by $\xi$ changes.

Next, we can apply Eq. (6) to two different cases, i.e., when the number of particles is odd and when it is even. When the number of particles is odd we will have one result, which is the same for all odd cases; when the number of particles is even we will get another result, which is also the same for all even cases. This is due to the fact that Eq. (6) has always as many ‘+’ as ‘−’, but in the odd case it starts with ‘+’ and ends with ‘−’ and in the even case it starts and ends with ‘+’. As an example, we can use $N = 3$ to represent the odd case, and $N = 4$ to represent the even case.

Starting from the odd case, when $N = 3$, we get from Eq. (6),

$$\xi = (+ - + + + -),$$

(7)

which means that the diagonal of Eq. (5) is null. Therefore, solving Eq. (3), we obtain,

$$E = 2b \prod_{i=1}^{N} \sin \theta_i. $$

(8)
We can now calculate the maximum of $E$ from Eq. (8), which we call $L$, as,

$$L = \sqrt{(2b)^2} = v. \hspace{1cm} (9)$$

Moving now to the even case, when $N = 4$, Eq. (6) is written as,

$$\xi = (+ - - + + + - - + + + - - +). \hspace{1cm} (10)$$

Applying this result to Eq. (3), we now get,

$$E = 2a \prod_{i=1}^{N} \cos \theta_i + 2b \prod_{i=1}^{N} \sin \theta_i - 2c \prod_{i=1}^{N} \cos \theta_i, \hspace{1cm} (11)$$

whose solution for the maximum is found as,

$$L = \sqrt{(2(a-c))^2 + (2b)^2} = \sqrt{2}v. \hspace{1cm} (12)$$

From Eqs. (9) and (12) we can describe the maximum for violation of WWŽB inequality as,

$$L = \begin{cases} 
v, & \text{if } N \text{ is odd} \\
\sqrt{2}v, & \text{if } N \text{ is even} \end{cases} \hspace{1cm} (13)$$

When $L > 1$ it is observed the violation of WWŽB inequality in Eq. (1).

3. Numerical Results

In this section we present the results for the verification of the violation of WWŽB inequality, both when using an odd and an even number of particles.

In Fig. 1 are shown the results given by Eq. (13), where it is possible to distinguish between an even and an odd system of particles. First it can be seen that the odd system can be described

![Figure 1](image-url)
by local theories, since it does not violates the WWŽZB inequality in Eq. (1) for any value of $v$. From a system with an even number of particles, one can see that it violates WWŽZB inequality from $v = 1/\sqrt{2}$.

The result shown in Fig. 1, although relatively simple, tells us that $\rho_{\text{noise}}$ does not have any influence in the maximum of violation. This comes from the fact that the result of Eq. (13) is equal to the result for the GHZ state multiplied with the weight of the parameter $v$ [19].

In Fig. 2 is displayed the result from Eq. (1) for two particular cases, i.e., when $N = 3$ and when $N = 4$. The value of $v$ was set to 0.8, which is above the threshold of violation when $N$ is even. When $N = 3$, from Eq. (1) are obtained $2^{2^3} = 256$ inequalities and when $N = 4$ we obtain $2^{2^4} = 65536$, as shown in Fig. 2. From Eq. (13), in the even case, it is possible to determine that when $v = 0.8$, $\mathcal{L} = 0.8\sqrt{2} \approx 1.13$, which is the maximum that we observe in Fig. 2(b). We verified also that from the 65536 inequalities which can be obtained from Eq. (1) when $N = 4$, 1794 violate the classical boundary. Additionally, we determined that from the 1794 inequalities, 120 allow the maximum of violation. These results are important due to the following reason: if one wants to know if a given state cannot be written by a local theory, it is

![Figure 2. Values of WWŽZB inequalities when $N = 3$ (up) and $N = 4$ (down).](image-url)
necessary to find at least one inequality of Eq. (1) that violates the classical boundary. However, from Eq. (1) we observe that the number of inequalities increases exponentially with $N$, and as larger it is, more difficult is to find a solution for the system. This way, from Eq. (13), which is a single equation applicable to all cases, it is possible to obtain the same information in a much more easier way.

4. Conclusions
We have shown that entangled quantum states have a major role in QKD systems, mainly in the possibility that they allow to increase the distance of transmission by several orders.

Since Werner states are entangled states which include a term of noise, they traduce more accurately what is obtained in experiments. Thus, the understanding of their behavior is fundamental to improve the systems of quantum technologies. Also, we verified that these states can present a simple behavior, being possible to discriminate two cases, one for $N$ even and other for $N$ odd. In what concerns the parameter $v$, we have shown that it is the one that determines the threshold for violation of WWZB inequality when $N$ is even.

The generation of multipartite entangled quantum states is a key scientific endeavour only by itself, but also an enabling technology for quantum communication systems. Up to now it was possible to demonstrate the generation of eight-photon entanglement, which is a breakthrough achievement for science [20, 21].

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5. References
[1] Gisin N, Ribordy G, Tittel W and Zbinden H 2002 Rev. Mod. Phys. 74 145–195
[2] Takesue H 2011 NTT Tech. Review 9(9) 50–54
[3] Collins D, Gisin N and de Riedmatten H 2005 J. Mod. Opt. 52 735–753
[4] Wang S, Chen W, Guo J F, Yin Z Q, Li H W, Zhou Z, Guo G C and Han Z F 2012 Opt. Lett. 37 1008
[5] Takesue H and Miquel B 2009 Opt. Express 17 10748
[6] Sanguoard N, Simon C, de Riedmatten H and Gisin N 2011 Rev. Mod. Phys. 83 33–34
[7] Żukowski M, Zeilinger A, Horne M A and Ekert A K 1993 Phys. Rev. Lett. 71 4287–4290
[8] Abruzzo S, Bratzik S, Bernardes N K, Kampermann H, van Loock P and Bruß D 2013 Phys. Rev. A 87 052315
[9] Bratzik S, Abruzzo S, Kampermann H and Bruß D 2013 Phys. Rev. A 87 062335
[10] Khalique A and Sanders B C 2014 Phys. Rev. A 90(3) 032304
[11] Einstein A, Podolsky B and Rosen N 1935 Phys. Rev. 47(10) 777–780
[12] Bell J S 1964 Physics 1 195–200
[13] Freedman S J and Clauser J F 1972 Phys. Rev. Lett. 28 938–941
[14] Aspect A, Grangier P and Roger G 1982 Phys. Rev. Lett. 49 91–94
[15] Werner R F and Wolf M M 2001 Phys. Rev. A 64(3) 032112
[16] Żukowski M and Brukner Č 2002 Phys. Rev. Lett. 88 210401
[17] Werner R F 1989 Phys. Rev. A 40(8) 4277–4281
[18] Weinfurter H and Żukowski M 2001 Phys. Rev. A 64(1) 010102
[19] Martins I P, Almeida A J, Silva N A, André P S and Pinto A N 2014 Eur. Phys. J. D 68 228
[20] Huang Y F, Liu B H, Peng L, Li Y H, Li L, Li C F and Guo G C 2011 Nat. Commun. 2 546
[21] Yao X C, Wang T X, Xu P, Lu H, Pan G S, Bao X H, Peng C Z, Lu C Y, Chen Y A and Pan J W 2012 Nature Photon. 6 225–228