Magnetic structure of isospin-asymmetric QCD matter in neutron stars

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We study QCD under the influence of background magnetic fields and isospin chemical potentials using lattice simulations. This setup exhibits a sign problem which is circumvented using a Taylor-expansion in the magnetic field. The ground state of the system in the pion condensation phase is found to exhibit a pronounced diamagnetic response. We elaborate on how this diamagnetism may contribute to the pressure balance in the inner core of strongly magnetized neutron stars. In addition we show that the onset of pion condensation shifts to larger chemical potentials due to the enhancement of the charged pion mass for growing magnetic fields. Finally, we sketch the magnetic structure of the QCD phase diagram in the temperature-isospin chemical potential plane.

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1. INTRODUCTION

The elementary degrees of freedom of the high-energy phase of Quantum Chromodynamics (QCD) are confined quarks and gluons forming the quark-gluon plasma (QGP). One physical environment where the QGP may exist is the inner core of dense neutron stars – compact stellar objects created during the gravitational collapse of massive supernovae. A certain class of neutron stars (magnetars) exhibits intense magnetic fields of strengths up to $10^{10–11}$ T at the star surface. These extreme magnetic fields are presumably generated by a convective dynamo mechanism during the first few seconds of the collapse [1], and are believed to be responsible for the strong electromagnetic activity of the star in the form of gamma-ray and X-ray bursts [2]. Magnetic fields can induce a strong deformation of the star, leading to the radiation of gravitational waves [3]. During mergers of binary neutron star systems, magnetic fields can even be amplified drastically [4] and have a significant impact on the emitted gravitational signal [5]. Thus, an understanding of the behavior of neutron star matter in magnetic fields is desired in many respects.

While the magnetic fields at the surface of the star are typically determined using measurements of rotation periods and time derivatives thereof [6], the strength of the field in the deep interior of the star is highly uncertain. On general grounds the field $B$ is expected to become enhanced towards the star center, see, e.g., Ref. [7]. The maximal possible strength is estimated to be $B \approx 10^{14}$ T based on the equality of the gravitational and magnetic energies [6]. These extreme values for $B$ are in the regime where a competition between the electromagnetic and the strong forces takes place and the magnetic properties of the QGP as a medium become important.

An additional characteristic aspect of neutron stars is the isospin asymmetry that develops in their interior through protons converting into neutrons and neutrinos via electron capture. The core is thus described by a high baryonic density accompanied by a considerable isospin density. In the grand canonical approach to statistical physics, these densities are controlled by the corresponding chemical potentials. On the level of the (up and down) quark content, they are written as

$$\mu_B = 3(\mu_u + \mu_d)/2, \quad \mu_I = (\mu_u - \mu_d)/2.$$  \hfill (1)

To understand the structure of the dense and strongly magnetized inner core, an investigation of the combined effect of $\mu_I$ ($f = u, d$ labels the quark flavors) and $B$ on the ground state of QCD matter is necessary. In the strongly interacting regime this is only possible using non-perturbative approaches like lattice simulations.

To leading order in $B$, the magnetic response of the system is characterized by the magnetic susceptibility,

$$\chi = \frac{1}{V} \frac{\partial^2 F}{\partial (eB)^2} \bigg|_{B=0},$$  \hfill (2)

defined in terms of the free energy $F$ of the system ($V$ denotes the three-dimensional volume). Here we considered the magnetic field in units of the elementary charge $e > 0$. Note that the first derivative of $F$ at $B = 0$ vanishes due to parity symmetry. The sign of $\chi$ distinguishes between paramagnetism ($\chi > 0$) and diamagnetism ($\chi < 0$).

In this letter we calculate the magnetic susceptibility at nonzero isospin chemical potentials and discuss the structure of the $\mu_I - B - T$ phase diagram using numerical lattice QCD simulations. Our results indicate a strong diamagnetic response for large isospin chemical potentials. We also present an argument suggesting that $\chi$ is less affected by the baryonic chemical potential and, thus, $\mu_I$ is the most relevant control parameter for the magnetic response of QCD at nonzero densities. In addition, we discuss the impact of the diamagnetic nature of the isospin-asymmetric state on magnetars. Assuming typical magnetic field configurations, this diamagnetism delivers a considerable anisotropic contribution to the gravitational pressure in the core, which may have implications on, e.g., the convective processes in the interior of the star.
2. LATTICE SETUP AND OBSERVABLES

We consider QCD with two quark flavors, described by a two-component quark field $\psi = (\psi_u, \psi_d)$ and the fermion matrix

$$M = \begin{pmatrix} \not{D}(\mu, q_u) + m & \lambda \gamma_5 \\ -\lambda \gamma_5 & \not{D}(-\mu, q_d) + m \end{pmatrix} \tag{3}$$

where $q_f$ is the electric charge of the quark of flavor $f$, and we assumed a degenerate mass $m$. Here, $\not{D} = \gamma_\mu D_\mu$ is the Dirac operator with the SU(3) and U(1) covariant derivatives $D_\mu = \partial_\mu + igA_\mu + ig_f a_\mu$. The Abelian vector potential is chosen such that it generates a magnetic field in the $z$ direction, $a_0 = 0, \nabla \times \mathbf{a} = \mathbf{B} \parallel \mathbf{\hat{z}}$.

The term proportional to $\lambda$ is inserted in Eq. (3) in order to allow the observation of pion condensation, signalled by the expectation value $\pi \equiv \bar{\psi} u \gamma_5 \psi_d - \bar{\psi} d \gamma_5 \psi_u$. In a finite volume this spontaneous breaking cannot occur without such a small explicit breaking. The coefficient $\lambda$ will be extrapolated to zero at the end of the analysis. Incidentally, for $\mu_I \gtrsim m$ the smallest eigenvalue of the Hermitian operator $M^\dagger M$ (which is used in the simulation algorithm) equals $\lambda^2$, such that the system becomes ill-conditioned as $\lambda \to 0$. Nevertheless, for the values necessary to perform this extrapolation (see Sec. 3) this numerical problem still turned out to be feasible.

This theory is simulated on a symmetric $8^4$ lattice with spacing $a$ such that the spatial volume equals $V = L^3 = (8a)^3$. To allow for a cross-check of the algorithm and of the simulation code, we use the same lattice discretization as Ref. [8]: the plaquette gauge action and rooted staggered quarks. The inverse gauge coupling is $\beta \equiv 6/g^2 = 5.2$ and the quark mass in lattice units equals $ma = 0.025$. The lattice spacing is determined using the Wilson flow [9] via the $w_0$ scale proposed in Ref. [10] to be $a = 0.299(2)$ fm. The linear size of the system is $L \approx 2.4$ fm. Thus, $L^{-1} \approx 80$ MeV is well below the finite temperature deconfinement transition and the system may be approximated as being at zero temperature. Through fitting the pseudoscalar propagator we get for the pion mass in lattice units $m_\pi a = 0.402(5)$, giving $m_\pi \approx 260$ MeV.

Using the $\gamma_5$-hermiticity of the Dirac operator (for staggered quarks the role of $\gamma_5$ is played by $\eta_5 = (-1)^{n_u+n_d+n_s+n_s}$),

$$\gamma_5 \not{D}(\mu, q) \gamma_5 = \not{D}(-\mu, q), \tag{4}$$

one can prove that the fermionic action $S_f = -\log \det M$ is real and positive if the electric charges of the two flavors coincide. However, having $q_d = -2q_u = -e/3$ is essential to capture the fact that the particles excited by the isospin chemical potential are the charged pions. Therefore, in the presence of the magnetic field, $S_f$ becomes complex and cannot be simulated using conventional lattice Monte-Carlo methods.

We tackle this complex action problem by simulating at $\mu_I \neq 0$ and $B = 0$ – where $S_f$ is real and positive – and performing a (leading-order) Taylor-expansion in the magnetic field. This expansion involves the derivatives of the free energy with respect to $B$ – starting with $\chi$ of Eq. (2). One technical complication is that on a finite lattice with periodic boundary conditions, the magnetic flux $\Phi = eB \cdot L^2$ is quantized [11], making the derivative with respect to $B$ ill-defined. An advantageous strategy is to consider a modified magnetic field, for example one that is positive in one half and negative in the other half of the lattice [12]. $B = B \text{sign}(L/2 - x) \cdot \mathbf{\hat{z}}$. Then the total flux is zero and the derivative with respect to $B$, being a continuous variable, can be taken. This setup was tested to give reliable results at $\mu_I = 0$ where a direct simulation at $B \neq 0$ is also possible [13].

To study the thermodynamics of the system, besides the already defined magnetic susceptibility, Eq. (2), we also consider the chiral condensate, the pion condensate and the isospin density

$$\bar{\psi} \psi = -\frac{1}{V} \frac{\partial F}{\partial m}, \quad \pi = -\frac{1}{V} \frac{\partial F}{\partial \lambda}, \quad n_I = -\frac{1}{V} \frac{\partial F}{\partial \mu_I}. \tag{5}$$

Note that the magnetic susceptibility contains an additive divergence at zero temperature and zero density, such that its renormalization reads

$$\chi' = \chi|_{\mu_I \approx T = 0}, \tag{6}$$
as inherited from the renormalization of the electric charge [14]. This $B$-dependent divergence cancels from the observables of Eq. (5). For more details on this renormalization see, e.g., Ref. [13].

3. RESULTS

We begin by considering the observables of Eq. (5) at vanishing magnetic field. Fig. 1 shows the results at various values of the explicit symmetry breaking parameter $\lambda$ and a linear extrapolation to $\lambda = 0$ for each $\mu_I$. The difference to a quadratic extrapolation is used to define the systematic uncertainty, and is included in the errors shown in the plot.

We observe that the $\lambda = 0$ limit of $\pi$ vanishes in the low-$\mu_I$ region, whereas it increases drastically for high chemical potentials, signalling the condensation of pions. The onset of this condensation is predicted by chiral perturbation theory ($\chi$PT) to occur at $\mu_I = m_\pi/2$ [16], in good agreement with the lattice data. All other observables are also insensitive to $\mu_I$ (in the $\lambda \to 0$ limit) up to this onset value. This is in general referred to as the Silver Blaze phenomenon. The chiral condensate drops sharply above $\mu_I = m_\pi/2$, while the isospin density starts to grow at the condensation threshold. We note that our results are within statistical errors consistent with those of Ref. [8], up to the fact that in Ref. [8] a different normalization convention was used. Our normalization is chosen such that the results at $\mu_I = 0$ correspond to
two degenerate flavors. We note moreover that lattice discretization effects start to dominate for $a\mu_I \gtrsim 1$ (not visible in the plot): here all lattice sites become occupied and the isospin density saturates at $a^3 n_I^{\text{sat}} = 3/2$.

To perform the $\lambda$-extrapolation in a more effective manner, we consider $\chi$PT to describe the behavior of the observables for small $\lambda$ and for small $\mu_I$ [15]. This dependence involves two parameters: the pion mass and the chiral condensate at $\mu_I = \lambda = 0$, denoted by $G$,

$$\bar{\psi}\psi = G \cos \alpha, \quad \pi = G \sin \alpha, \quad n_I = \frac{4mG\mu_I}{m_\pi^2} \sin^2 \alpha, \quad \text{Eq. (7)}$$

where $\alpha$ is the vacuum angle, determined at the extremum of $\mathcal{F}$ by

$$\sin(\alpha - \phi) = \frac{4\mu_I^2}{m_\pi^2} \sin \alpha \cos \alpha, \quad \phi = \arctan \lambda/m. \quad \text{Eq. (8)}$$

The structure of the chiral Lagrangian remains the same for two-color QCD and for QCD with adjoint quarks. Thus, Eqs. (7) and (8) are also valid in these settings [17].

We carry out a simultaneous fit of all three observables, considering data points for all values of $\lambda$, up to $a\mu_I = 0.2$. Considering the pion mass as a free parameter of the fit, we obtain $am_\pi = 0.4053(1)$, consistent with our previous determination using the pseudoscalar propagator. The latter is indicated by the gray line in the figure. In fact, $\chi$PT predicts pion condensation and chiral symmetry restoration to proceed simultaneously such that $\pi^2 + \bar{\psi}\psi^2$ remains constant. The lattice data do not support this prediction for $\mu_I \gtrsim m_\pi/2$ ($\pi$ is underestimated by $\chi$PT) as was also realized in Ref. [8]. However, below $m_\pi/2$ – where the $\lambda$-dependence is most pronounced and, thus, the extrapolation cumbersome – the $\chi$PT prediction is in excellent agreement with the lattice data. This comparison also reveals that the linear $\lambda \to 0$ extrapolations of the lattice data using the available three $\lambda$ values – with the exception of the points just at the condensation threshold – are reliable. Note moreover that a true phase transition only appears in the thermodynamic limit, and in a finite volume the observables slightly deviate from the behavior dictated by $\chi$PT around the onset chemical potential.

We proceed by performing a similar, linear $\lambda \to 0$ extrapolation for the renormalized magnetic susceptibility. Subtracting the $\mu_I = 0$ contribution at each $\lambda$ turned out to be advantageous here as it makes the $\lambda \to 0$ extrapolation flatter. The results again show a Silver Blaze-type behavior up to $m_\pi/2$ and a rapid drop towards negative values beyond the onset of pion condensation. This implies that the QCD medium at zero temperature is diamagnetic for $\mu_I > m_\pi/2$. The diamagnetic response may be understood qualitatively from the fact that just above $m_\pi/2$ the system can be approximated as a dilute gas of pions. Pions are spinless and couple to the magnetic field only via their angular momentum. This coupling gives rise to a Landau-type diamagnetism, i.e., a negative magnetic susceptibility. Note that in the condensation phase, $\chi'$ is an order of magnitude larger than its value around the deconfinement temperature ($T \approx 150$ MeV) at $\mu_I = 0$ [13].
susceptibility for free quarks, giving [18]
\[
\chi^r \xrightarrow{\mu_1 \to \infty} \frac{1}{4\pi^2} \sum_f (q_f/e)^2 \cdot \log(\mu_f^2/\Lambda^2) > 0,
\]
where the prefactor is related to the QED \(\beta\)-function and \(\Lambda\) is a dimensionful scale (in the on-shell renormalization scheme of the free theory, \(\Lambda = m\)) [13]. Thus, the susceptibility must eventually turn positive as \(\mu_f\) increases. To explore the region where \(\chi^r\) crosses zero, further simulations on finer lattices are necessary.

4. INTERPRETATION

Let us discuss the strong diamagnetic response above \(\mu_{\pi} = m_{\pi}/2\) from a different point of view and consider how the charged pion mass responds to the magnetic field. Taking the pion as a point-like particle, the leading-order dependence reads
\[
m_{\pi}(B) = \sqrt{m_{\pi}^2(0) + eB},
\]
as a consequence of the lowest-Landau-level structure for a scalar particle. At \(T = 0\), the renormalized free energy can be calculated as the integral of the isospin density,
\[
-\frac{\mathcal{F}(B, \mu_1)}{V} = \int_0^{\mu_f} d\mu'_f n_1(B, \mu'_f).
\]
For \(B = 0\), taking into account the dependence \(n_1(\mu_f)\) from Fig. 1 (for \(\lambda \to 0\)), this implies that \(-\mathcal{F}\) is zero up to \(m_{\pi}/2\) and becomes positive above the threshold. Let us now switch on a weak magnetic field and consider the free energy up to \(\mathcal{O}((eB)^2)\). To this order, \(\mathcal{F}(B, \mu_1)\) still vanishes for chemical potentials up to the corresponding pion mass. However, due to Eq. (10), the Silver Blaze region expands as \(B\) grows. This can only be maintained if the surface \(-\mathcal{F}\) has a large negative curvature in the \(B\)-direction, i.e., through a large negative susceptibility (for an illustration see Fig. 3). Thus, the pion condensation phase must exhibit strong diamagnetism – in line with our results presented in Sec. 3.

Note that the key component in the above argumentation was the scalar nature of the pion, which allowed for a phase with direct Bose-Einstein condensation and, at the same time, implied an increase in the mass as \(B\) grows, Eq. (10). For the baryonic chemical potential \(\mu_B\), the excited particles are protons and neutrons. In this case the baryon density grows much slower \(n_B \propto (\mu_B^2 - m_B^2)^{3/2}\), and condensation can only occur via Cooper-pairing if the ground state is in the superfluid phase. Moreover, in the spin-half channel, the mass (to leading order) is independent of the magnetic field (both for neutrons and for protons), implying that the Silver Blaze region is also insensitive to \(B\). Thus, \(\chi^r\) is expected to be suppressed for \(\mu_B\) just above \(m_B\). Note that QCD corrections can induce a mild dependence of the proton mass on \(B\), and through that a nonzero value for \(\chi^r\), but due to the absence of direct condensation and due to the larger mass, these effects are not expected to produce a pronounced behavior like the one seen in Fig. 2.

To back up the picture described above, we also determined the lowest-order expansion coefficients of the observables in Eq. (5). In Fig. 4 we show the second derivative of the pion condensate with respect to \(eB\) and the reconstructed observable \(\pi(B)\) for a few values of the magnetic field. Again three values of \(\lambda\) are plotted, together with an extrapolation of the data to \(\lambda = 0\). The derivative is found to exhibit a pronounced dip around \(\mu_f = m_{\pi}/2\) and, as a consequence, the rise in \(\pi\) is shifted to higher isospin chemical potentials as \(B\) is increased. The lattice data for the scalar condensate and for the isospin density show similar trends.

![FIG. 3: Illustration of the negative of the free energy as a function of \(B\) and \(\mu_f\) for small magnetic fields. The Silver Blaze region \(\mathcal{F} = 0\) expands as \(B\) grows, implying a large negative magnetic susceptibility in the pion condensation phase.](image-url)

![FIG. 4: Second derivative of the pion condensate (upper panel, the symbols are similar as in Fig. 2) and the \(\lambda \to 0\) limit of the pion condensate at nonzero magnetic fields using the leading Taylor-expansion (lower panel). The gray vertical lines indicate the pion mass for each magnetic field (increasing from left to right).](image-url)
Therefore, the results are in qualitative agreement with the discussion above, namely that the magnetic field shifts the onset of pion condensation to higher isospin chemical potentials. On the quantitative level, the results suggest that this shift is less pronounced than the expectation based on Eq. (10). (Note that the Taylor-expansion in $B$ breaks down at the phase transition, where $F$ is non-analytic. Still, the reconstruction of $\pi(B)$ is expected to converge outside of the close vicinity of the onset isospin chemical potential.)

5. CONCLUSIONS

We have discussed the QCD phase diagram in the $\mu_I - B$ plane for the first time using lattice simulations. This setup has a complex action problem, which was circumvented through a Taylor-expansion in $B$ at nonzero isospin chemical potentials. We measured thermodynamic observables for a wide range of $\mu_I$ values, in the Silver Blaze region, through the onset of pion condensation at $\mu_I = m_\pi/2$, up to lattice saturation. The results indicate that the condensation threshold is shifted to higher values of $\mu_I$ as $B$ grows, in qualitative agreement with the dependence $m_\pi(B)$ of the pion mass on the magnetic field. This tendency explains the observed strong diamagnetic behavior of the system in the pion condensation phase. Moreover, we also presented an argument suggesting that the magnetic response of the QCD ground state is most sensitive to isospin chemical potentials, and the baryon chemical potential is not expected to play a dominant role in this respect.

Our results may have implications for the pressure balance in the interior of strongly magnetized neutron stars. The minimization of the free energy in an inhomogeneous magnetic field induces the force density,

$$ f_d = -\frac{1}{V} \nabla F = \chi B |\nabla |eB|, $$

Note that this force is only sensitive to the magnetic properties of the medium, and the energy $B^2/2$ of the magnetic field does not contribute to $f_d$ (assuming that $B$ is constant in time). Note moreover that since the free energy is a Lorentz-scalar, $f_d$ only depends on the magnitude of $B$. We adopt the poloidal magnetic field profile $B(r, \theta)$ of Ref. [7] for a rotating magnetar with radius $R = 10$ km and central field strength $1.5 \cdot 10^{14}$ T. The rotation axis is given by $\theta = 0$. Inserting $\chi^2 \approx -0.1$ for the susceptibility we obtain $f_d(r)$, see the curves in Fig. 5 for three fixed values of the polar angle $\theta$.

The so obtained force density is to be compared to the gradient of the isotropic pressure profile $p(r)$ in the star. For a first approximation, we take the simplified case of a star with constant density (see Eq. (3.157) of Ref. [19]), and consider typical values for the central pressure $p_c \approx 10^{14}$ Pa. The resulting gradient is also included in Fig. 5, indicating that in this case the diamagnetic effect amounts to up to 10% of the gravitational pressure gradient in the inner core $r \lesssim 3$ km. The curves for $f_d(r, \theta)$ also reveal that the diamagnetic force is anisotropic and tends to push material from the center towards the equator. Thus, we expect that the diamagnetism of isospin-asymmetric QCD matter plays a relevant role for the description of convective processes in the inner core. We mention that a similar mechanism in the case of heavy-ion collisions was discussed in Ref. [20].

![Fig. 5](image.png)

**FIG. 5:** The force generated by QCD diamagnetism (along various directions specified by the polar angle $\theta$) and 10% of the pressure gradient as functions of the radial coordinate.

![Fig. 6](image.png)

**FIG. 6:** Conjecture of the magnetic structure of the phase diagram in the $T - \mu_I$ plane. Regions with positive (negative) susceptibilities are represented by red (blue). The solid line represents a true phase transition. There may be additional phase transition lines in the interior of the diagram.

We conclude by sketching the magnetic nature of the QCD phase diagram in the $T - \mu_I$ plane. At $\mu_I = 0$, lattice simulations have shown that the susceptibility is positive for $T \gtrsim 120$ MeV, whereas it becomes slightly negative for lower temperatures [13]. This diamagnetic region is predicted by $\chi$PT and by the Hadron Resonance Gas model [13, 21], and also stems from the presence of charged pions. However, while at $T = 0$ and $\mu_I > m_\pi/2$, pions are created in abundance, at $T > 0$ they are induced merely by thermal fluctuations. Accordingly, the diamagnetic response at $T > 0$ is much weaker than...
the one in the pion condensation phase. Well above the
crossover transition temperature at $\mu_I = 0$, the dominant
degrees of freedom are quarks, giving rise to a strong paramagnetism, with $\chi^r \propto \log(T)$, similarly as in Eq. (9),
just with $\mu_I$ replaced by $T$ [13].

Based on this picture, our conjecture of the magnetic
phase diagram of QCD is shown in Fig. 6. The phase
transition at $\mu_I = m_\pi/2$ is expected to turn into an
analytic crossover as the temperature grows. Whether
there are additional phase transition lines is presently un-
clear. For example, the isospin density has been shown
to change the nature of the deconfinement transition in
8-flavor QCD [22]. To determine the detailed structure of
the interior of the phase diagram for the interesting case
of 2 or 2+1 flavors, further simulations are necessary.

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[1] R. C. Duncan and C. Thompson Astrophys. J. 392 (1992) L9.
[2] C. Thompson and R. C. Duncan Astrophys.J. 473 (1996) 322.
[3] C. Cutler Phys.Rev. D66 (2002) 084025, [gr-qc/0206051].
[4] D. Price and S. Rosswog Science 312 (2006) 719–22, [astro-ph/0603845].
[5] M. Anderson, E. W. Hirschmann, L. Lehner, S. L. Liebling, P. M. Motl, et al. Phys.Rev.Lett. 100 (2008) 191101, [arXiv:0801.4387].
[6] A. K. Harding and D. Lai Rept.Prog.Phys. 69 (2006) 2631, [astro-ph/0606794].
[7] M. Bocquet, S. Bonazzola, E. Gourgoulhon, and J. Novak Astron.Astrophys. 301 (1995) 757, [gr-qc/9503044].
[8] J. Kogut and D. Sinclair Phys.Rev. D66 (2002) 034505, [hep-lat/0202028].
[9] M. Lüscher JHEP 1008 (2010) 071, [arXiv:1006.4518].
[10] S. Borsányi, S. Dürre, Z. Fodor, C. Hoelbling, S. D. Katz, et al. JHEP 1209 (2012) 010, [arXiv:1203.4469].
[11] G. ’t Hooft Nucl. Phys. B153 (1979) 141.
[12] L. Levková and C. DeTar Phys.Rev.Lett. 112 (2014) 012002, [arXiv:1309.1142].
[13] G. Bali, F. Bruckmann, G. Endrődi, S. Katz, and A. Schäfer arXiv:1406.0269. 
[14] J. S. Schwinger Phys.Rev. 82 (1951) 664–679.
[15] K. Splittorff, D. Toublan, and J. Verbaarschot Nucl.Phys. B639 (2002) 524–548, [hep-ph/0204076].
[16] D. Son and M. A. Stephanov Phys.Rev.Lett. 86 (2001) 592–595, [hep-ph/0005225].
[17] J. Kogut, M. A. Stephanov, D. Toublan, J. Verbaarschot, and A. Zhitnitsky Nucl.Phys. B582 (2000) 477–513, [hep-ph/0001171].
[18] P. Elmfors, D. Persson, and B.-S. Skagerstam Astropart.Phys. 2 (1994) 299–326, [hep-ph/9312226].
[19] N. Glendenning, Compact Stars: Nuclear Physics, Particle Physics, and General Relativity. Second edition. Springer New York, 2000.
[20] G. Bali, F. Bruckmann, G. Endrődi, and A. Schäfer Phys.Rev.Lett. 112 (2014) 042301, [arXiv:1311.2559].
[21] C. Bonati, M. D’Elia, M. Mariti, F. Negro, and F. Sanfilippo Phys.Rev. D89 (2014) 054506, [arXiv:1310.8656].
[22] P. de Forcrand, M. A. Stephanov, and U. Wenger PoS LAT2007 (2007) 237, [arXiv:0711.0023].