Compatibility between shape equation and boundary conditions of lipid membranes with free edges

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Only some special open surfaces satisfying the shape equation of lipid membranes can be compatible with the boundary conditions. As a result of this compatibility, the first integral of the shape equation should vanish for axisymmetric lipid membranes, from which two theorems of non-existence are verified: (i) There is no axisymmetric open membrane being a part of torus satisfying the shape equation; (ii) There is no axisymmetric open membrane being a part of a biconcave discoidal surface satisfying the shape equation. Additionally, the shape equation is reduced to a second-order differential equation while the boundary conditions are reduced to two equations due to this compatibility. Numerical solutions to the reduced shape equation and boundary conditions agree well with the experimental data [A. Saitoh et al., Proc. Natl. Acad. Sci. USA 95, 1026 (1998)].

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I. INTRODUCTION

The elasticity and configuration of lipid vesicles have attracted much theoretical attention of physicists [1–4] since Helfrich proposed the spontaneous curvature model of lipid bilayers in his seminal work [5]. The shape equation to describe equilibrium configurations of lipid vesicles was derived in 1987 [6, 7] based on Helfrich’s model. There are two typical analytical solutions to the shape equations. One is a torus with a ratio (v2) of its two generation radii [8, 9]. Another is a vesicle with biconcave discoidal shape [10]. In fact, the latter solution does not correspond to a vesicle free of external force because of a logarithmic singularity in the solution [11–13].

The opening-up process of lipid vesicles by talin was observed by Saitoh et al. [14], which pushes us to investigate the shape equation and boundary conditions of lipid membranes with free exposed edges. This topic was discussed theoretically and numerically by several researchers [15–21]. Based on Helfrich’s model, the shape equation and boundary conditions were derived by Capovilla et al. [15, 16], Tu et al. [17, 18], and Yin et al. [19] in different forms. Wang and Du obtained various shapes of open membranes through numerical simulations by phase field method [20]. Using the area difference elasticity model, Umeda et al. derived the shape equation and boundary conditions and then compared their numerical results with the experiment [21]. They found that the line tension of the free edge of the open lipid membrane increases with decreasing the concentration of talin [21]. The above theoretical and numerical results can be generalized to investigate adhesions between lipid vesicles [22, 23], configurations of lipid vesicles with different lipid domains [18, 20, 24, 25], and vesicle formation [26]. However, the above theoretical researches [15–19] do not contain sufficient discussions on analytical solutions to the shape equation with the boundary conditions. Additionally, the authors merely compared their numerical results with the experimental ones qualitatively in their numerical work [20, 21]. It is still lack of the quantitative comparison between the numerical and experimental results. Two natural questions are led to: Can we find analytical solutions? At least, it is instructive to investigate the possibility of finding the analytical solutions. Can we use the numerical results to fit the experimental data quantitatively? We hope we can do that by taking the number of parameters as small as possible.

Generally speaking, the shape equation derived from Helfrich’s model is a fourth-order nonlinear differential equation, while the boundary conditions include three nonlinear equations describing the shapes of the free edges of lipid membranes. In principle, one can obtain the general solution with unknown constants to a linear differential equation, and then determine the unknown constants by using the linear boundary conditions. Thus there is no mathematical difficulty to find the solution satisfying both the linear differential equation and linear boundary conditions. However, the problem becomes more complicated if both the differential equation and boundary conditions are nonlinear. There is no general solution to a nonlinear differential equation in mathematics. Consequently, one can only conjecture some special solutions in a few cases. If we further consider the boundary conditions, only a few ones among the above known solutions can fit them. Therefore, it is quite helpful to investigate the feature of the special solutions that can satisfy both the nonlinear differential equation and the boundary conditions. Since it is very difficult to obtain solutions to the shape equation with the boundary conditions, we may first conjecture a surface satisfying the shape equation, and then find a curve in the surface satisfying the boundary conditions as an edge of the surface. However, one might not find any curve satisfying the boundary conditions for a given surface satisfying the
shape equation. Only some special ones among the surfaces satisfying the shape equation can admit the boundary conditions. The profound reason is that the points in the boundary curve should satisfy not only the boundary conditions, but also the shape equation because they also locate in the surface. In other words, there exist some additional constraints between the shape equation and the boundary conditions. These constraints which have not been touched in Refs. [15–21] are called the compatibility condition in this paper.

It is not a straightforward task to find the compatibility condition in general case. The axisymmetric lipid membranes with edges will give us some clues. The shape equation is reduced to a third-order differential equation in axisymmetric case [27,28]. Zheng and Liu proved that it was integrable [29], and could be further transformed into a second-order differential equation with an integral constant. In this paper, we will show that the compatibility condition is that this integral constant vanishes for axisymmetric membranes. Due to this compatibility, the shape equation is reduced to a second-order differential equation while the boundary conditions are reduced to two equations. The rest of this paper is organized as follows: In Sec. III, we present the general shape equation and boundary conditions of lipid membranes with free edges. In Sec. III, we discuss the compatibility between the shape equation and boundary conditions in axisymmetric case, and then verify two theorems of non-existence. In Sec. IV, we find some axisymmetric numerical solutions and compare them with experimental data quantitatively. A brief summary is given in the last section.

II. SHAPE EQUATION AND BOUNDARY CONDITIONS

A lipid membrane with a free edge is represented as an open surface with a boundary curve $C$. As shown in Fig. 1 we can construct an orthogonal right-handed frame $\{e_1, e_2, e_3\}$ at each point of the surface such that $e_3$ is the normal vector of the surface. For each point in the boundary curve, we take $e_2$ to be perpendicular to the tangent direction $e_1$ of the boundary curve and point at the side that the surface is located.

The free energy of the membrane may be expressed as

$$F = \int \left[ \left( \frac{k_c}{2} \right) (2H + c_0)^2 + \tilde{k}K \right] dA + \lambda A + \gamma L,$$  

(1)

where the first and second terms are the bending energy $\tilde{\cal R}$ and the surface energy of the membrane, respectively, while the third term is the line energy of the free exposed edge. $H$ and $K$ are the mean curvature and gaussian curvature of the surface, respectively. $dA$ is the area element of the surface. $A$ and $L$ are the total area of the surface and the total length of the boundary curve, respectively. $k_c$ and $\tilde{k}$ are the bending moduli. $c_0$ is the spontaneous curvature. $\lambda$ and $\gamma$ are the surface tension and line tension, respectively.

By calculating the variation of free energy (1), we can obtain [17]

$$\left( 2H + c_0 \right) \left( 2H^2 - c_0 H - 2K \right) - 2\tilde{\lambda} H + \nabla^2 (2H) = 0, \quad (2)$$

and

$$\left[ (2H + c_0) + \tilde{k}\kappa_n \right] C = 0, \quad (3)$$

$$\left[ -2 \partial H / \partial e_2 + \tilde{\gamma} \kappa_n + \tilde{k} d\tau_g / ds \right] C = 0, \quad (4)$$

$$\left[ (1/2)(2H + c_0)^2 + \tilde{k} K + \tilde{\lambda} + \tilde{\gamma} \kappa_g \right] C = 0. \quad (5)$$

where $\tilde{\lambda} \equiv \lambda / k_c, \quad \tilde{k} \equiv \tilde{k} / k_c, \quad \tilde{\gamma} \equiv \gamma / k_c$ are the reduced surface tension, reduced bending modulus, and reduced line tension, respectively. $\kappa_n, \kappa_g, \tau_g,$ and $ds$ are the normal curvature, geodesic curvature, geodesic torsion, and arc length element of the boundary curve, respectively. Equation (2) determines the equilibrium shape of the membrane, thus we call it shape equation. For a given surface satisfying the shape equation, Eqs. (3)–(5) determine the shape of the boundary curve and its position in the surface, thus we call them boundary conditions. Equation (2) expresses the normal force balance of the membrane. Equation (3) is the moment balance equation around $e_1$ at each point in curve $C$. Equations (4) and (5) are the force balance equations along $e_3$ and $e_2$ at each point in curve $C$, respectively. Thus, in general, the above four equations are independent of each other.

III. COMPATIBILITY BETWEEN THE SHAPE EQUATION AND BOUNDARY CONDITIONS

We have mentioned that only some special ones among the surfaces satisfying the shape equation (2) can admit the boundary conditions (3)–(5). What is the common
feature of these special surfaces? we will find this feature for axisymmetric surfaces.

When a planar curve AC shown in Fig. 2 revolves around z axis, an axisymmetric surface is generated. Let \( \psi \) represent the angle between the tangent line and the horizontal plane. Each point in the surface can be expressed as vector form \( \mathbf{r} = \{ \rho \cos \phi, \rho \sin \phi, z(\rho) \} \) where \( \rho \) and \( \phi \) are radius and azimuth angle that the point corresponds to. Introduce a notation \( \sigma \) such that \( \sigma = 1 \) if \( \mathbf{e}_1 \) is parallel to \( \partial \mathbf{r}/\partial \phi \), and \( \sigma = -1 \) if \( \mathbf{e}_1 \) is antiparallel to \( \partial \mathbf{r}/\partial \phi \) in the boundary curve generated by point C. The above equations (2) - (5) are transformed into

\[
(h - c_0) \left( \frac{h^2}{2} + \frac{c_0h}{2} - 2K \right) - \lambda h + \frac{\cos \psi}{\rho} (\rho \cos \psi h')' = 0, \quad (6)
\]

\[
\left[ h - c_0 + \tilde{k} \sin \frac{\psi}{\rho} \right]_C = 0, \quad (7)
\]

\[
\left[ -\sigma \cos \psi h' + \tilde{\gamma} \sin \frac{\psi}{\rho} \right]_C = 0, \quad (8)
\]

\[
\left[ \frac{k^2}{2} \left( \frac{\sin \psi}{\rho} \right)^2 + \tilde{k} K + \tilde{\lambda} - \sigma \eta \cos \psi \rho \right]_C = 0, \quad (9)
\]

where \( h \equiv \sin \psi/\rho + (\sin \psi)'/\rho \) and \( K \equiv \sin \psi(\sin \psi)'/\rho \). The ‘prime’ represents the derivative with respect to \( \rho \).

The shape equation (6) is a third-order differential equation. Following Zheng and Liu’s work [28], we can transform it into a second order differential equation

\[
\cos \psi h' + (h - c_0) \sin \psi \psi' - \tilde{\lambda} \tan \psi \\
+ \frac{\eta_0}{\rho \cos \psi} - \frac{\tan \psi}{2} (h - c_0)^2 = 0 \quad (10)
\]

with an integral constant \( \eta_0 \) (so called the first integral). The configuration of an axisymmetric open lipid membrane should satisfy the shape equation (10) or (10) and boundary conditions (7) - (9). In particular, the points in the boundary curve should satisfy not only the boundary conditions, but also the shape equation (10) because they also locate in the surface. That is, Eqs. (7) - (9) and (10) should be compatible with each other in the edge. Substituting Eqs. (7) - (9) into (10), we derive the compatibility condition to be

\[
\eta_0 = 0. \quad (11)
\]

No we will discuss two examples and verify two theorems of non-existence by considering the above compatibility condition.

First, let us consider a part of a torus shown in Fig. 3a generated by an arc expressed by \( \sin \psi = \alpha \rho + \beta \) with two non-vanishing constants \( \alpha \) and \( \beta \). Substituting it into the shape equation (10), we obtain \( c_0 = 0 \), \( \beta = \sqrt{2} \), \( \lambda = 0 \), and \( \eta_0 = -\alpha \). That is, the torus can be a solution to the shape equation. However, \( \eta_0 = -\alpha \neq 0 \) contradicts to the compatibility condition (11). Thus we arrive at:

**Theorem 1.** There is no axisymmetric open membrane being a part of torus generated by a circle expressed by \( \sin \psi = \alpha \rho + \sqrt{2} \).

Secondly, we consider a biconcave discoidal surface [10] generated by a planar curve expressed by \( \sin \psi = \alpha \rho \ln(\rho/\beta) \) with two non-vanishing constants \( \alpha \) and \( \beta \). To avoid the logarithmic singularity at two poles, we may dig two holes around the poles in the surface as shown in Fig. 3b. Substituting \( \sin \psi = \alpha \rho \ln(\rho/\beta) \) into the shape equation (10), we obtain \( \lambda = 0 \), \( \alpha = c_0 \), and \( \eta_0 = -2c_0 \). That is, the biconcave discoidal surface can be a solution to the shape equation. However, \( \eta_0 = -2c_0 \neq 0 \) contradicts to the compatibility condition (11). Thus we arrive at:

**Theorem 2.** There is no axisymmetric open membrane being a part of a biconcave discoidal surface generated by a planar curve expressed by \( \sin \psi = c_0 \rho \ln(\rho/\beta) \).

In the above discussion, the theorems of non-existence are deduced as natural corollaries of the compatibility condition. It does not mean that the proofs are unique. The other proofs are presented in Appendix A. In Ref. [30], the present author has proved that there is no open lipid membrane being a part of a constant mean curvature surface. These theorems reveal that it is almost hopeless to find analytical solutions to the shape...
equations with the boundary conditions. Thus we need to seek for numerical solutions.

IV. AXISYMMETRIC NUMERICAL SOLUTIONS

The compatibility condition leads to a more important result that the shape equation can be simplified as

$$\cos \psi \dot{h} + (h - c_0) \sin \psi \dot{\psi} - \ddot{\lambda} \tan \psi - \frac{\tan \psi}{2} (h - c_0)^2 = (12)$$

while the boundary conditions can be taken only two equations (7) and (9) because Eq. (8) is not independent of Eqs. (7), (14), and (12). However, it is still very difficult to obtain analytical solutions to Eq. (12) with boundary conditions (7) and (9). We will find axisymmetric numerical solutions and compare them with experimental data [14] in this section.

Because $\psi$ might be a multi-valued function of the independent variable $\rho$, the above equations are unsuitable for numerical solutions. Here we take the arc-length of curve AC in Fig. 2 as an independent variable. Then we have $\dot{\rho} = \cos \psi$ and $\dot{z} = \sin \psi$, where the ‘dot’ represents the derivative with respect to the arc-length. The shape equation can be transformed into

$$\ddot{\psi} = -\frac{\sin \psi}{2} \psi' - \frac{1}{\rho} \cos \psi \psi' + \frac{\sin 2\psi}{2 \rho} + \ddot{\lambda} \tan \psi + \frac{\tan \psi}{2} \left( \frac{\sin \psi}{\rho} \right)$$

while the boundary conditions become

$$\left[ \psi - c_0 + \left( 1 + \tilde{k} \right) \frac{\sin \psi}{\rho} \right]_C = 0,$$ (13)

and

$$\left[ \tilde{c} \frac{\sin \psi}{\rho} - \left( 1 + \frac{\tilde{k}}{2} \right) \ddot{\lambda} \left( \frac{\sin \psi}{\rho} \right)^2 + \tilde{\gamma} \cos \psi \right]_C = 0.$$ (14)

In fact, these equations can also be derived from the Lagrangian method as shown in Appendix B. In addition, we impose the initial conditions $z(0) = \rho(0) = \mu m$, and $\psi(0) = 0$. We can use the shooting method to find numerical solutions to Eq. (13) with boundary conditions [Eqs. (13) and (14)] and these initial conditions, and then fit the parameters ($\tilde{k}$, $c_0$, $\tilde{\lambda}$, $\tilde{\gamma}$) with experimental data. The basic idea is as below. For the given values of $\tilde{k}$, $c_0$, $\tilde{\lambda}$, $\tilde{\gamma}$ and $\psi(0)$, we can solve Eq. (13) with boundary conditions (14) and (15). Then we compare the graph of the solution to the outline of the open membrane in the experiment. Tune the values of the parameters until the graph of the solution and the outline of the membrane almost superpose each other. Thus we obtain a group of proper values of the parameters ($\tilde{k}$, $c_0$, $\tilde{\lambda}$, $\tilde{\gamma}$).

In the experiment [14], the hole of the lipid membrane is enlarged with increasing the concentration of talin, and vice versa. Talin molecules adhere to the edge of the membrane. Thus it is reasonable to assume that the line tension of the edge depends on the concentration of talin, while the bending moduli and spontaneous curvature of the membrane do not. That is, we should have the common values of $\tilde{k}$ and $c_0$ for a membrane at different concentrations of talin. This gives a constraint in our fitting. As shown in Fig. 4 our numerical results (solid, dash, and dot lines) obtained from Eqs. (13)-(15) agree well with the experimental data (squares, circles, and triangles extracted from Fig. 3.1 to K in Ref. [14]) of the outlines of an axisymmetric lipid membrane at different concentration of talin.
V. CONCLUSION

In the above discussion, we investigate the compatibility between shape equation and boundary conditions of lipid membranes with free edges. The main results obtained in this paper are as follows.

(i) The compatibility condition for axisymmetric lipid membranes with free edges is that the first integral of the shape equation \( \psi = \psi(a) \) should be vanishing, i.e., Eq. (A1).

(ii) Two theorems (Theorem 1 and Theorem 2 in Sec. III) of non-existence are verified as natural corollaries of the compatibility condition, which give two examples to reveal that one indeed might not find any curve satisfying the boundary conditions in a given surface satisfying the shape equation. These theorems also correct two flaws on analytical solutions in Ref. [17].

(iii) The shape equation of axisymmetric lipid membranes is reduced to Eq. (12). Then only two equations in boundary conditions are independent. This conclusion is the same as the case in Ref. [15] with vanishing \( k \) and \( \gamma \).

(iv) As shown in Fig. 4, the numerical solutions to the reduced shape equation \( (\ref{eq:12}) \) with boundary conditions \( (\ref{eq:7}) \) and \( (\ref{eq:9}) \) agree well with the experimental data \( (\ref{eq:14}) \). Finally, we would like to point out two difficulties that we have not fully overcome yet: (i) The compatibility condition between shape equation and boundary conditions for asymmetric (not axisymmetric) lipid membranes with edges is unclear. We do not even know whether it exists, much less what it is. (ii) We use the shooting method to find numerical solutions. But this method is not so efficient to the numerical solutions due to the complicated boundary conditions. A much more efficient method is expected. The above challenges should be addressed in the future work.

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Appendix A: Other proofs to theorems non-existence

The proofs can be divided into two classes in terms of different starting points. One is based on the stress analysis, another is based on the scaling argument.

1. Stress analysis

Capovilla et al. proposed the stress tensor in a lipid membrane and then derived the shape equation and boundary conditions from the stress tensor \( (\ref{eq:15}) \). Recently, they found that \( (\ref{eq:12}) \) the line integral

\[
\oint_{\Gamma} \mathbf{f}_a \cdot \mathbf{\hat{z}} = c, \tag{A1}
\]

where \( \Gamma \) is any circle perpendicular to the symmetric axis in an axisymmetric membrane. \( \mathbf{f}_a \) and \( \mathbf{\hat{z}} \) represent the normal of \( \Gamma \) tangent to the membrane surface and the stress in the membrane, respectively. \( \mathbf{\hat{z}} \) is the unit vector along the symmetric axis. \( c \) is a constant dependent on the topology and the curvature singularity of the membrane.

First, the constant \( c \) is non-vanishing for an axisymmetric torus free of curvature singularity, which implies that the stress in each circle \( \Gamma \) perpendicular to the symmetric axis in the torus surface cannot be zero. However, the stress in the free edges should be vanishing. Thus, we cannot find any \( \Gamma \) as a free edge of an axisymmetric open membrane being a part of the torus, i.e., theorem 1 is arrived at.

Secondly, there exist singularity points at two poles of the biconcave discoidal surface generated by a planar curve expressed by \( \sin \psi = \alpha \rho \ln(\alpha/\beta) \). The singularity results in a non-vanishing \( c \) \( (\ref{eq:12}) \) \( (\ref{eq:13}) \), and then non-zero stress in each circle \( \Gamma \) in the biconcave discoidal surface. Thus, we cannot find any \( \Gamma \) as a free edge of an axisymmetric open membrane being a part of the biconcave discoidal surface, i.e., theorem 2 is arrived at.

2. Scaling argument

The free energy \( (\ref{eq:11}) \) can be written in another form

\[
F = \int [(k_c/2)(2H)^2 + \tilde{k}K]dA \\
+ 2k_c c_0 \int H dA + (\lambda + k_c c_0^2/2)A + \gamma L. \tag{A2}
\]

Let us consider the scaling transformation \( r \rightarrow \Lambda r \), where the vector \( r \) represents the position of each point in the membrane and \( \Lambda \) is a scaling parameter \( (\ref{eq:16}) \). Under this transformation, we have \( A \rightarrow A^2, L \rightarrow \Lambda L, H \rightarrow \Lambda^{-1}H, \) and \( K \rightarrow \Lambda^{-2}K \). Thus, Eq. (A2) is transformed into

\[
F(\Lambda) = \int [(k_c/2)(2H)^2 + \tilde{k}K]dA \\
+ 2k_c c_0 \Lambda \int H dA + (\lambda + k_c c_0^2/2)\Lambda^2 A + \gamma \Lambda L \tag{A3}
\]

The equilibrium configuration should satisfy \( \partial F/\partial \Lambda = 0 \) when \( \Lambda = 1 \) \( (\ref{eq:10}) \). Thus we obtain

\[
2c_0 \int H dA + (2\lambda + c_0^2)A + \gamma L = 0. \tag{A4}
\]
This equation is an additional constraint for open membranes.

As shown in Sec. [11], if there exists an open membrane being a part of torus, then the shape equation (2) requires \( \hat{\lambda} = 0 \) and \( c_0 = 0 \), which contradicts the constraint (A4) because \( \xi L > 0 \). Thus we arrive at theorem 1.

Because Willmore surfaces satisfy the special form of Eq. (2) with vanishing \( \hat{\lambda} \) and \( c_0 \) [31], as a byproduct of the constraint (A4), we obtain a much stronger theorem of non-existence: There is no open membrane being a part of a Willmore surface.

Next, we turn to the biconcave discodal surface. If there exists an open membrane being a part of a biconcave discodal surface generated by a planar curve expressed by \( \sin\psi = c_0 \ln(\rho/\beta) \), the shape equation (2) requires \( \hat{\lambda} = 0 \). Substituting \( 2H = -c_0[1 + 2\ln(\rho/\beta)] \) into Eq. (B2), we will not obtain a contradiction. Thus theorem 2 cannot be deduced from the scaling argument.

### Appendix B: Derivation of the reduced shape equation and boundary conditions by using the Lagrange method

For the revolving surface generated by the planar curve shown in Fig. 2, Eq. (1) can be transformed into

\[
F/2\pi k_c = \int_0^{s_2} [\rho f^2/2 + \dot{k} \sin\psi \dot{\psi} + \dot{\lambda} \rho + \dot{\xi} \rho] ds, \quad (B1)
\]

with \( f = \sin\psi/\rho + \dot{\psi} - c_0 \). We should minimize \( F/2\pi k_c \) with the constraints \( \dot{\rho} = \cos\psi \) and \( \dot{z} = \sin\psi \), thus we construct an action \( S = \int_0^{s_2} \mathcal{L} ds \) with a Lagrangian [3]

\[
\mathcal{L} = \rho f^2/2 + \dot{k} \sin\psi \dot{\psi} + \dot{\lambda} \rho + \dot{\xi} \rho + \zeta (\dot{\rho} - \cos\psi) + \eta (\dot{z} - \sin\psi), \quad (B2)
\]

where \( \zeta \) and \( \eta \) are two Lagrange multipliers. In terms of the variational theory, we can derive

\[
\delta S = \int_0^{s_2} \delta \mathcal{L} ds - \mathcal{H}\delta s_2
= \int_0^{s_2} \left( \frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \psi} \right) \delta \dot{\psi} ds + \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi \bigg|_0^{s_2}
\]

+ \int_0^{s_2} \left( \frac{\partial \mathcal{L}}{\partial \rho} \right) \delta \rho ds + \frac{\partial \mathcal{L}}{\partial \rho} \dot{\rho} \bigg|_0^{s_2}
\]

+ \int_0^{s_2} \left( - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \frac{\partial \mathcal{L}}{\partial \psi} \right) \delta \dot{\psi} ds + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta \dot{\psi} \bigg|_0^{s_2}
\]

+ \int_0^{s_2} (\dot{\rho} - \cos\psi) \delta \dot{\psi} ds + \int_0^{s_2} (\dot{z} - \sin\psi) \delta \eta ds
- \mathcal{H}|_C \delta s_2 = 0, \quad (B3)

where the Hamiltonian \( \mathcal{H} = \dot{\psi} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \dot{\rho} \frac{\partial \mathcal{L}}{\partial \dot{\rho}} + \dot{z} \frac{\partial \mathcal{L}}{\partial \dot{z}} - \mathcal{L} \). Imposing \( \psi(0) = 0, \rho(0) = z(0) = 0 \mu m \), and substituting Eq. (B2) into Eq. (B3), we can obtain

\[
\begin{aligned}
\zeta \sin\psi - \dot{\xi} \rho = 0, \\
\eta = \text{constant}, \\
\dot{\rho} = \cos\psi, \\
\dot{z} = \sin\psi,
\end{aligned}
\]

with boundary conditions

\[
\begin{aligned}
\left[ f + \dot{x} \sin\psi \right]_C &= 0, \\
\zeta|_C + \dot{\xi} = 0, \\
\eta|_C &= 0, \\
\mathcal{H}|_C &= 0.
\end{aligned}
\]

Eq. (B8) is equivalent to boundary condition (14).

Because \( \mathcal{L} \) does not explicitly contain \( s \), \( \mathcal{H} \) is a constant. Combining Eqs. (B5), (B10), (B11) and the definition of \( \mathcal{H} \), we derive

\[
\zeta \cos\psi = \rho [f(f - 2\psi)/2 + \dot{\lambda}]. \quad (B12)
\]

From Eqs. (B14) and (B12) we can obtain the shape equation (13). Equation (B11) can be transformed into the boundary condition (15) with Eq. (B5). From Eqs. (B14) and (B15) we can also obtain the other boundary condition, which is not independent of the shape equation (13) and boundary conditions (14) and (15).

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