KIRILLOV THEORY FOR $C^*(G, \Omega)$

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Abstract.
Let $G$ be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$; let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let $\Omega$ be a locally compact second countable Hausdorff space with a continuous $G$ action, and let $C^*(G, \Omega)$ be the corresponding transformation group $C^*$ algebra. We construct a continuous surjective map $\phi$ from a quotient space, $\mathfrak{g}^* \times \Omega / \sim$, which is a homeomorphism from $\mathfrak{g}^* \times \Omega / \sim$ to $\text{Prim}(C^*(G, \Omega))$.

We also describe a character theory for $C^*(G, \Omega)$ which generalizes Kirillov character theory for $G$.

CHAPTER I
INTRODUCTION

Our primary references for this paper are Corwin and Greenleaf [3]; fundamental ideas were provided by Siegfried Echterhoff’s paper [7], and the original inspiration was Dana P. Williams’s paper [26].

Let $G$ be a locally compact group acting on a second countable locally compact Hausdorff space $\Omega$.

A basic question of $C^*$ theory is an explicit description of $\text{Prim}(C^*(G, \Omega))$. For $G$ a connected, simply connected nilpotent Lie group and $\Omega$ a point, a description of $\text{Prim}(C^*(G, \Omega))$ as a set was given in the classic paper of Kirillov [21] where he showed that the natural map from $\mathfrak{g}^*/G \mapsto \hat{G}$ was continuous with $\mathfrak{g}^*/G$ in the quotient topology. Kirillov conjectured that this was a homeomorphism; this was first proved by Brown [1]. When $\Omega$ is not
a single point, but $G$ is abelian, a complete description of $\text{Prim}(C^*(G, \Omega))$, including a natural description of the topology, was provided by Dana P. Williams [26]. In [7], for general $G$, Siegfried Echterhoff gave a complete description but required strong hypotheses for the action.

We define a map $\phi : \mathfrak{g}^* \times \Omega / \sim \mapsto \text{Prim}(C^*(G, \Omega))$, and, with a mild restriction upon the action, show that $\phi$ is a continuous open surjective map when $\mathfrak{g}^* \times \Omega$ is given the product topology. We further provide a homeomorphism $\psi$ from a quotient space $\sim$ of $\mathfrak{g}^* \times \Omega$ to $\text{Prim}(C^*(G, \Omega))$.

When $C^*(G, \Omega)$ is Type I, we provide a character theory corresponding to the Kirillov character theory of $G$.

**SECTIONS**

This paper is divided into three sections: Section one consists of preliminary results as well as setting notation. In section two we prove the main results of this paper,

1. An explicit parameterization of $\text{Prim}(C^*(G, \Omega))$, and,
2. a computation of the topology of $\text{Prim}(C^*(G, \Omega))$.

In section three, we generalize the Kirillov character formula for $C^*(G, \Omega)$ under stronger hypotheses of the action.

**CREDITS**

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Section 1, Preliminaries

Section 1.1

$C^*$ algebras and representation spaces

Definition 1.

Denote by $\Omega/G$ the orbit space with the quotient topology.

For $\phi \in C_0(\Omega)$, define $^s \phi$ by $^s \phi(x) = \phi(s^{-1} \cdot x)$.

Denote by $\mathcal{K}(G)$ the space of closed subgroups of $G$, given the relativized compact-open topology of Fell [10].

Let $\mathcal{Q}(G) = \{\langle H, T \rangle \mid H \in \mathcal{K}(G), T \in \text{Rep}(G)\}$.

Let $x \in \Omega$; by $G_x$ we denote the stabilizer of $x$ in $G$. These are assumed connected.

For $x \in \Omega$, $f \in g^*$, by $p_x$ we will denote a polarizing subalgebra of $g_x$ for $f$. Let $P_x = \exp(p_x)$.

Denote by $p$ an isotropic subalgebra, not necessarily polarizing. Let $P = \exp(p)$.

Also:

Define $\chi_{f,P_x}$ to be the character of $P_x = \exp(p_x)$ given by $\chi_{f,P_x}(\exp(X)) = e^{i f(X)}$.

Let $\tau_{f,x} = \text{ind}_{P_x}^{G_x}(\chi_{f,P_x})$ be an irreducible representation of $G_x$.

Let $\tau'_{f,x} = \text{ind}_{P}^{G_x} be(\chi_{f,P})$, a representation of $G_x$, not in general irreducible.

We assume representations non-degenerate.
Definition 2. A covariant representation $L = (V_L, M_L)$ of $(G, \Omega)$ on a Hilbert space $H_L$ consists of a uniformly bounded strongly continuous unitary representation $V$ of $G$ on $H_L$, and a norm-decreasing non-degenerate $*$-preserving representation of $C_0(\Omega)$, $M$ on $H_L$ such that $V(s)M(\phi)V(s^{-1}) = M(\phi)$.

Let $A$ be a fixed $C^*$ algebra. We use the hull-kernel topology on $\hat{A}$; see [24], Theorem 5.4.6.

On the set $\mathcal{I}(A)$ of closed ideals of $A$ we use the topology developed by Dana P. Williams on pg. 338 [26]. This is the topology that has as its sub-base the sets $\{O_J\}_{J \in \mathcal{I}(A)}$, where $O_J = \{I \in \mathcal{I}(A) \mid I \not\supset J\}$. On Prim$(A)$ this topology restricts to the usual hull-kernel (Jacobson) topology. One can see that this topology is almost Fell’s “inner hull kernel” topology; see [10].

Definition 3. Let $\pi$ and $\rho$ be representations of a locally compact group $G$.

The representation $\rho$ is weakly contained in $\pi$ if every positive definite matrix coefficient $\langle \rho(\xi), \xi \rangle$ can be approximated uniformly on compacta of $G$ by finite sums of positive definite matrix coefficients $\langle \pi(x)\xi', \xi' \rangle$. For $\rho$ weakly contained in $\pi$, we use the notation $\rho \prec \pi$. The spectrum of $\pi$ is the set of all representations weakly contained in $\pi$.

Let $H \subseteq G$ be a closed subgroup, and let $f_0$ be a non-negative, real-valued function in $C_c(G)$ that does not vanish at the identity element. For the remainder of this paper, let $\mu_H$ be the left Haar measure on $H$ defined by

$$\int_H f_0(t) d\mu_H(t) = 1.$$
Such a choice is referred to as a continuous (“smooth”) choice of Haar measures, and has the property that $H \mapsto \int_H f \, d\mu_H$ is continuous on $\mathcal{K}(G)$ for each $f \in C_c(G)$; see the theorem of [15], page 908.

**Lemma 1.** Assume $\{f_n\}_{n=1}^{\infty} \subseteq C_c(G)$ converges to $f \in C_c(G)$ in the inductive limit topology and $H_n \to H$ in $\mathcal{K}(G)$. Then

$$\int_{H_n} f_n \, d\mu_{H_n} \to \int_H f \, d\mu_H.$$

**Proof.**

Follows by an $\epsilon$--$\delta$ argument. \qed

**Definition 4.** Now define $\mathcal{E}(X)$ to be the union of all the spaces $C(A)$, where $A \in \mathcal{K}(S)$, thus the elements of $\mathcal{E}(X)$ are complex-valued functions $f$ such that for the domain of $f$, $D(f)$, we have $D(f) \in \mathcal{K}(S)$. Identifying a function with its graph we have $\mathcal{E}(X) \subseteq \mathcal{K}(S)$. We will always consider the set $\mathcal{E}(X)$ as being equipped with the (relativized) semicompact-open topology.

**Definition 5.** Let $X$ be an arbitrary locally compact space. Let $\mathcal{K}(X)$ denote the set of all closed subsets of $X$, and equip $\mathcal{K}(X)$ with the topology whose open sets have as basic open neighborhoods

$$U(C, \mathcal{F}) = \{F \in \mathcal{K}(X) \mid F \cap C = \emptyset, \; F \cap O \neq \emptyset \text{ for all } O \in \mathcal{F}\}$$

where $C$ is a compact subset of $X$ and $\mathcal{F}$ is a finite family of open sets of $X$. This topology is call the compact-open topology of $\mathcal{K}(X)$. We note $\mathcal{K}(X)$ equipped with the compact-open topology is a compact Hausdorff space, see [10], Theorem 1. \qed
Lemma 2. Let \( \{\langle K_i, T_i \rangle \} \) be a net of elements of \( Q(G) \) and \( \langle K, T \rangle \) an element of \( Q(G) \). Then \( \langle K_i, T_i \rangle \to \langle K, T \rangle \) if and only if, for each finite sequence \( \phi_1, \cdots, \phi_n \) of functions of positive type on \( K \) associated with \( \langle K, T \rangle \), and each subnet of \( \{\langle K_i, T_i \rangle \} \), there exists

1. a subnet \( \{\langle K'^{ij}, T'^{ij} \rangle \} \) of that subnet, and
2. for each \( j \) and each \( r = 1, \cdots, n \) a finite sum \( \phi^i_r \) of functions of positive type associated with \( \langle K'^{ij}, T'^{ij} \rangle \) such that \( \phi^i_r \to \phi_r \) in \( E_s(G) \) for each \( r \).

Proof.

See [13] Theorem 3.1’, page 439. □

Section 1.2

Nilpotent Lie algebras and groups, and their representation spaces

The purpose of this section is to give some basic information that is needed for this paper about nilpotent Lie algebras, groups, and the representation space.

Let \( G \) denote a Lie group and \( \mathfrak{g} \) its Lie algebra.

Definition 6. The adjoint representation, \( \text{ad} \) of \( \mathfrak{g} \) on \( \mathfrak{g} \) is defined as \( \text{ad}_x : \mathfrak{g} \to \text{GL}(\mathfrak{g}) \) by \( \text{ad}_x(y) = [x, y] \), for all \( y \in \mathfrak{g} \) (here \( [\cdot, \cdot] \) denotes Lie bracket on \( \mathfrak{g} \)).

Definition 7. The Lie algebra \( \mathfrak{g} \) is said to be nilpotent if \( \text{ad}_x \) is a nilpotent endomorphism of \( \mathfrak{g} \), for all \( x \in \mathfrak{g} \).

The Lie group \( G \) is nilpotent if \( \mathfrak{g} \) is nilpotent.
Now we briefly describe the representation theory of nilpotent Lie groups that we use. We follow the notation of Corwin and Greenleaf [3] and use it as our main reference.

Let $G$ be a connected, simply-connected nilpotent Lie group. Denote the dual space of $\mathfrak{g}$ by $\mathfrak{g}^*$. $G$ acts on $\mathfrak{g}^*$ by the coadjoint map, $\text{Ad}^*$: for $x \in G$, $Y \in \mathfrak{g}$, and $f \in \mathfrak{g}^*$, define $(\text{Ad}^*(x)f)(Y) = f(\text{Ad}(x^{-1})Y)$, $Y \in \mathfrak{g}$, $f \in \mathfrak{g}^*$, $x \in G$.

If $f \in \mathfrak{g}^*$. Define the coadjoint orbit of $f$ in $\mathfrak{g}^*$ to be $\text{Ad}^*(G)f$.

Let $f \in \mathfrak{g}^*$. A subspace $p \subseteq \mathfrak{g}$ is called isotropic if $f|_p = 0$. If $p$ is a maximally isotropic subspace of $\mathfrak{g}$ which is also a subalgebra, then $p$ is called a polarization, or a maximal subordinate subalgebra for $f$.

For any $X, Y \in \mathfrak{g}$, we use the Campbell-Baker-Hausdorff formula to form $\exp(X) \ast \exp(Y)$; see Corwin and Greenleaf [3], pg. 11.

We have the following result, which seems well known:

**Lemma 3.** Let $p$ be a polarizing subalgebra of $\mathfrak{g}$ for $f \in \mathfrak{g}^*$. We may define a one-dimensional representation $\chi_{f,p}$ of $P = \exp(p)$ by $\chi_{f,p}(\exp(X)) = e^{i \cdot f(X)}$.

We may induce the representation $\chi_{f,p}$ from $P$ to a representation $\pi_{f,p}$ of $G$; for details on induced representations, see [22] and [23].

Give $\mathfrak{g}^*/G$ the quotient topology and $\hat{G}$ the hull-kernel topology. By [1] and [21] we have the following important theorem:
Theorem 1.

(1) Let $f \in \mathfrak{g}^*$, and $\mathfrak{p}$ be a polarizing subalgebra for $f$. Let $\pi_{f,P} = \text{ind}^G_P(\chi_{f,P})$.

The representation $\pi_{f,P}$ is irreducible, and up to equivalence, every irreducible representation of $G$ is obtained this way.

(1) We have $\pi_{f,P} \cong \pi_{f',P'} \iff \exists g \in G$ such that $\text{Ad}^*(g)f = f'$.

(3) The induced map (the Kirillov map) $\mathfrak{g}^*/\text{Ad}^*(G) \mapsto \hat{G}$ is a homeomorphism.

Let $G$ be a nilpotent Lie group, $S$ a closed connected subgroup, having Lie algebras $\mathfrak{g}$ and $\mathfrak{s}$, respectively. Define $\mathfrak{s}^\perp$ to be the set of linear functionals in $\mathfrak{g}^*$ that are zero on $\mathfrak{s}$, i.e., $\mathfrak{s}^\perp = \{g \in \mathfrak{g}^* \mid g|_\mathfrak{s} = 0\}$.

Lemma 4. Let $G$ be a simply connected nilpotent Lie group, $S$ a closed connected subgroup; assume $f \in \mathfrak{g}^*$ satisfies $f([\mathfrak{s}, \mathfrak{s}]) = 0$ so $\chi_f(\exp(Y)) = e^{i\cdot f(Y)}$ is a one dimensional representation of $S$. The representation $W = \text{ind}^S_G(\chi_f)$ weakly contains $\{\pi_{f'} \in \hat{G} \mid f' \in f + \mathfrak{s}^\perp\}$, in fact, $\text{Sp}(W) = \text{Fell-closure}(\{\pi_{f'} \in \hat{G} \mid f' \in f + \mathfrak{s}^\perp\})$.

Proof.

A proof may be found in in [4], Theorem N.2.5.

Definition 8. Let $G$ a connected, simply connected nilpotent Lie group. By Kirillov, [21], the irreducible representations of $G$ are in one-to-one correspondence with $\text{Ad}^*$ orbits in $\mathfrak{g}^*$.

If $W \in \hat{G}$, we denote its correspondence orbit by $\Omega_W$.

The following result was proven by Joy [19].
Lemma 5 (Joy’s Lemma). Let $G$ be a real, connected, simply connected nilpotent Lie group. Let $\langle H_n, S_n \rangle \to \langle H, S \rangle$ in $Q(G)$. If $f \in \mathfrak{g}^*$ such that $f|_h \in \Omega_S$, then for every sub-sequence of $\{\langle H_n, S_n \rangle\}_{n=1}^{\infty}$, there is a sub-sequence $\{\langle H_{n_i}, S_{n_i} \rangle\}_{i=1}^{\infty}$ such that for each $i$, there exists $f_i \in \mathfrak{g}^*$ such that $f_i|_{h_{n_i}} \in \Omega_{S_{n_i}}$ and $f_i \to f$ in $\mathfrak{g}^*$.

Proof.

See [19], page 138. □

Section 1.3

Induced representations, and ideals of $C^*(G, \Omega)$

Let $G$ be a group, acting on a locally compact Hausdorff space $\Omega$.

Let $x \in \Omega$ and $G_x = \{g \in G \mid gx = x\}$ be the stability subgroup of $x$. Let $\rho_x : C_0(\Omega) \mapsto \mathbb{C}$ be the representation of $C_0(\Omega)$ given by evaluation at $x$; for $\phi \in C_0(\Omega)$, $\rho_x(\phi) = \phi(x)$.

Let $\mathfrak{g}_x$ be the Lie algebra of $G_x$.

Let $\tau$ be a representation of $G_x$ on the Hilbert space $H_{\tau}$. The pair $(\tau, \rho_x)$ forms a covariant pair for $C^*(G_x, \Omega)$ with the representation of $C^*(G_x, \Omega)$ defined as follows: for $v \in H_{\tau}$, $\phi \in C_0(\Omega)$, $x \in \Omega$, $g \in G_x$, define $\rho_x(\phi)(v) = \phi(x) \cdot v$.

We may induce the representation $(\tau, \rho_x)$ from $C^*(G_x, \Omega)$ to $C^*(G, \Omega)$. Define the induced representation of $(\tau, \rho_x)$ to be $L = (V, \rho_x)$, where $V = \text{ind}_{G_x}^G(\tau)$, and $\rho_x$ is a representation of $C_0(\Omega)$ on $H_V$, acting by $\rho_x(\phi)(g)(r) = \psi(r \cdot x)f(r)$. We use the notation $L = (V, M) = \text{ind}_{\langle S, \Omega \rangle}^{G, \Omega}(\tau, \rho_x)$. 
We now present a different, equivalent way, to induce representations of $C^*(G, \Omega)$.

Assume $H$ is a closed subgroup of $G$.

Assume that we have the $C^*$ algebra $C^*(G, \Omega)$, and that $\pi$ is a representation of $C^*(H, \Omega)$ acting on the Hilbert space $H_\pi$. Let $\xi$ and $\eta$ be arbitrary vectors in $H_\pi$. Define the induced representation, $L = \text{ind}_{(H, \Omega)}^{(G, \Omega)}(\pi)$ as follows:

Let $B = C_c(H, \Omega)$, and define a $C_c(H, \Omega)$-valued inner product on $C_c(G, \Omega)$ as follows:

$$\langle f, g \rangle_B(t, y) = \int_{s \in G} f(st, s \cdot y)g(st, s \cdot y)d\mu_G(s)$$

Define an inner product on $C_c(G, \Omega) \otimes H_\pi$ by

$$\langle f \otimes \xi, g \otimes \eta \rangle_L = \langle \pi(\langle g, f \rangle_B)\xi, \eta \rangle_\tau$$

Let $H_L$ be the completion of $C_c(G, \Omega) \otimes H_\pi$.

The induced representation $L$ of $h \in C^*(G, \Omega)$ acts on the class of $f \otimes \xi$ by $(h * f) \otimes \xi$; see [18], page 204, or [26], page 340.

**Proposition 1.** Let $G$ be a group acting on a locally compact Hausdorff space $\Omega$. Fix a point $x \in \Omega$. Let $\tau$ a representation of $G_x$, and $\rho_x$ be a point evaluation of $\Omega$. If $\tau$ is irreducible, then the induced representation $L = \text{ind}_{(G_x, \Omega)}^{(G, \Omega)}(\tau, \rho_x)$, of $C^*(G, \Omega)$, is an irreducible representation of $C^*(G, \Omega)$.

**Proof.**

See [26], Proposition 4.2 on pg. 344. $\Box$
Definition 9. Let \( L = (V, M) \) be a representation of \((H, \Omega), s \in G\). Let \( K = sHs^{-1} \), and let \( L^s = (V, M)^s = (V^s, M^s) \) be the covariant representation of \((K, \Omega)\) given by \( V^s(r) = V(s^{-1}rs) \), and \( M^s(\phi) = M(s\phi) \).

The following has second-countability as a hypothesis, true in our case.

Theorem 2. If \( S = \text{ind}_{(H, \Omega)}^{(G, \Omega)}(V, M) \) and \( T = \text{ind}_{(K, \Omega)}^{(G, \Omega)}((W, N)) \), then \( S \cong T \) if and only if for some \( s \in G \), we have \( T = \text{ind}_{(sKs^{-1}, \Omega)}^{(G, \Omega)}((V, M)^s) \).

Proof. See [15], Theorem 2.1. \( \square \)

We shortly prove an important proposition that enables us to separate certain kernels of \( C^*(G, \Omega) \). Several definitions are needed.

The following equivalence relation \( \sim_1 \) was motivated by Dana P. Williams [26], and reduces to his when \( G \) is abelian.

Definition 10. Define an equivalence relation \( \sim_1 \) on \( g^* \times \Omega \) as follows: \((f, x) \sim_1 (f', x')\) when:

1. There exists \( s \in G \) such that \( x' = s \cdot x \).
2. For some \( h \in g^\perp \), we have \( \text{Ad}^*(s)f = f' + h \).

We write \( \mathcal{O}_{(f, x)} \) for the equivalence class of the functional-point pair \((f, x)\).

Now we define a second quotient space \( \sim \) on \( g^* \times \Omega \) by \( \mathcal{O}_{(f, x)} \sim \mathcal{O}_{(h, y)} \iff \overline{\mathcal{O}_{(f, x)}} = \overline{\mathcal{O}_{(h, y)}} \). This is denoted \( g^* \times \Omega / \sim \).

To verify the above is an equivalence relation is a simple exercise.

Let \((V, M)\) be a covariant pair for \((G, \Omega)\). If \( M \) is given by \( \rho_x \), a point evaluation of a point \( x \in \Omega \), then for \( \phi \in C_0(\Omega) \) we have \( \rho_x(\phi)(r) = \phi(r \cdot x) \).
Proposition 2 which follows Lemma 6 does not depend upon $G$ being a nilpotent Lie group. It has been proven by Takesaki ([25], Theorem 7.2) when our orbit space $\Omega/G$ has a $T_0$ topology. It was proven in generality by Phil Green [18], pg. 210 Proposition 11, Part (ii). We present a different proof for our case.

As $\Omega$ is assumed separable, it is metrizable by [20], page 146 and Theorem 16, page 125. Hence $\Omega$ is $T_4$, and the Tietze Extension Theorem may be applied to $\Omega$; this is needed to prove the next lemma.

**Lemma 6.** Let $x \in \Omega$, $G_x = \text{the stabilizer of } x \text{ in } G$. Let $C \subseteq G/G_x$ be compact in $G/G_x$. If $f \in C_c(G/G_x)$ is supported on $C$, we may find a sequence of continuous functions $\{f_n\}_{n=1}^{\infty} \subseteq C_0(\Omega)$ with

$$f_n(y) \rightarrow \begin{cases} f(s) \text{ when } y = s \cdot x, & y \in C \cdot x \\ 0 & y \notin C \cdot x. \end{cases}$$

**Proof.**

We comment on the conclusion of this lemma. The function $h$ on $\Omega$ defined by

$$h(y) = \begin{cases} f(s) \text{ when } y = s \cdot x, & y \in C \cdot x \\ 0 & y \notin C \cdot x \end{cases}$$

is not, in general, a continuous function on $\Omega$. However, we may find a sequence of continuous functions limiting on $h$, so $h$ will be Borel.

The set $C \cdot x$ is closed and compact in $\Omega$ by sequential arguments. Now simply define $f'$ on $C \cdot x$ by $f'(s \cdot x) = f(s)$. Assume that $\{y_n\}_{n=1}^{\infty} \subseteq C \cdot x$, and
\[ y_n \rightarrow y \in C \cdot x. \] We need show that \( f'(y_n) \rightarrow f'(y) \). Assume that we have a collection \( \{ \{ g_n \}_{n=1}^{\infty}, g \} \subseteq C \) and \( y_n = g_n \cdot x \) and \( y = g \cdot x \), we may easily find such as \( C \) is compact in \( G \) and \( C \cdot x \) is compact in \( \Omega \). As \( C \) is compact, we may assume that \( g_n \rightarrow g' \), and as \( g' \cdot x = g \cdot x \), and \( \Omega \) is Hausdorff, we have \( g' = g \). As \( f \) is continuous on \( C \), we have that \( f' \) is continuous on \( C \cdot x \) is clear by brute force. We now employ the Tietze Extension Theorem to extend to all \( \Omega \).

Now we note that \( C \cdot x \) may not contain an open set. We find a sequence \( \{ C_n \}_{n=1}^{\infty} \) of nested compact neighborhoods with \( \cap_{n=1}^{\infty} C_n = C \cdot x \). For each \( n \), we use Urysohn’s Lemma to find a continuous function \( g_n \) which is 1 on \( C \cdot x \) and 0 on the closure of \( \widehat{C_n} \cdot x \). We set \( f_n = g_n f' \) and we are done. □

**Proposition 2.** Let \( x \) be any fixed point in \( \Omega \), \( \tau_1 \) and \( \tau_2 \) be two representations of \( G_x \) such that \((\tau_1, \rho_x)\) and \((\tau_2, \rho_x)\) do not have the same kernel as representations of \( C^*(G_x, \Omega) \). Defining \( L_1 = \text{ind}_{(G_x, \Omega)}^{(G, \Omega)}(\tau_1, \rho_x) \) and \( L_2 = \text{ind}_{(G_x, \Omega)}^{(G, \Omega)}(\tau_2, \rho_x) \), we have \( \ker(L_1) \neq \ker(L_2) \).

**Proof.**

We first define a new \( C^* \) algebra, that being \( C^*(G,G/G_x) \). We observe that our orbit space \((G/G_x)/G\) for \( C^*(G,G/G_x) \) is a \( T_0 \) space; it consists of one point. Hence \( C^*(G,G/G_x) \) is Type I, and we have a canonical homeomorphism between \( C^*(\widehat{G}/G/G_x) \) and \( \text{Prim}(C^*(G,G/G_x)) \).

We identify the point \( x \in \Omega \) with \( x' = e \in G/G_x \) in the new orbit space. We further observe that \( G_{x'} = G_x \) is clear. We use the same representations
\( \tau_1 \) and \( \tau_2 \) of \( G_{x'} \). Now define

\[ L_1' = \text{ind}^{(G_2, \Omega)}_{(G_{x'}, \Omega)}(\tau_1, \rho_{x'}) \]

and

\[ L_2' = \text{ind}^{(G_2, \Omega)}_{(G_{x'}, \Omega)}(\tau_2, \rho_{x'}) , \]

two irreducible representations of \( C^* (G, G/G_x) \), and that these have different kernels is clear by the above comments and the fact that our orbit space is now \( T_0 \).

Throughout we assume that \( \xi_i \) and \( \eta_i \) \( (i = 1, 2) \) are vectors in the space of \( \tau_i \) \( (i = 1, 2) \). The representation \( (\tau_i, \rho_{x'}) \) of \( C^* (G_{x'}, G/G_{x'}) \) we will refer to as \( \pi_i \).

Now, as \( \text{ker} (L_1') \neq \text{ker} (L_2') \), we may find an \( h' \in C^* (G, G/G_{x'}) \) with \( h' \in \text{ker} (L_1') \) and \( h' \notin \text{ker} (L_2') \). So we may assume that for all elementary tensors \( f' \otimes \xi_1 \), that \( L_1'(h')(f' \otimes \xi_1) = h' \ast f' \otimes \xi_1 = 0 \) (or more properly, the class of zero), and that for some elementary tensor \( f' \otimes \xi_2 \) that \( L_2'(h')(f' \otimes \xi_2) \neq 0 \).

We may further assume that \( f' \) is in the dense subset of continuous functions with compact support having the form

\[ f'(s, y) = \sum_{i=1}^l f'_{1,i}(s)f'_{2,i}(y) . \]

We may assume that each \( f'_{1,i} \) in the last displayed formula is a continuous function of compact support on \( G \), and \( f'_{2,i} \) is a continuous function of compact support on \( G/G_{x} \) for each \( i \), and \( f' \) and its component functions are collectively supported on \( C_1 \times C_2 \), where \( C_1 \) is a compact set in \( G \) and \( C_2 \) is a compact set in \( G/G_x \).
We will refer to all inner products of the $C^*$ algebra $C^*(G, G/G_x)$ as $\langle \cdot \cdot \rangle'_i$ to avoid confusion later on.

Re-iterating, we may assume by a scaling argument that

$$\langle h' \ast f' \otimes \xi_1, f' \otimes \xi_1 \rangle'_1 = 0,$$

and

$$\langle h' \ast f' \otimes \xi_2, f' \otimes \xi_2 \rangle'_2 = 1,$$

Now assume that $\{h'_n\}_{n=1}^\infty \subseteq C_c(G, G/G_x)$ and $h'_n \to h'$ in the topology of $C^*(G, G/G_x)$, and each $h'_n$ has the form

$$h'_n(s, y) = \sum_{i=1}^{N_n} \phi'_{n,i}(s) \psi'_{n,i}(y),$$

and each is supported on a set $C^n_1 \times C^n_2$, each $C^n_1$ a compact set in $G$ and each $C^n_2$ a compact set in $G/G_x$.

We here note that $h'_n \ast f'$ (each $n$) is a continuous function of compact support; this is proven on pages 32-33 of [8].

We have

$$\langle h'_n \ast f' \otimes \xi_1, f' \otimes \xi_1 \rangle'_1 \to 0$$

$$\langle h'_n \ast f' \otimes \xi_2, f' \otimes \xi_2 \rangle'_2 \to 1,$$

and more explicitly, when we “untwist” the formulas we have

$$\langle h'_n \ast f' \otimes \xi_i, f' \otimes \xi_i \rangle'_i = \langle \pi_i((f', (h'_n \ast f'))_{B_i}) \xi_i, \xi_i \rangle'_i =$$

$$\left\langle \pi_i \left( 2Gx(t) \int_{s \in G} \bar{f} (s, s \cdot y) (h'_n \ast f')(st, s \cdot y) d\mu_G(s) \right)(t, y) \right\rangle \xi_i, \xi_i \right>_i =$$

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\[
\int_{r \in G_x} \left( \gamma_{G_x}(r) \int_{s \in G} \overline{f^j}(s, s \cdot x') \right) (h'_n \ast f')(sr, s \cdot x') d\mu_G(s) \left\langle \tau_i(r) \xi_i, \xi_i \right\rangle d\mu_{G_x}(r) \xrightarrow{n \to \infty} \left\{
\begin{array}{cl}
0 & i = 1 \\
1 & i = 2.
\end{array}
\right.
\]

Now we return to \( C^*(G, \Omega) \). All inner products in this \( C^* \) algebra we refer to by \( \left\langle \cdot, \cdot \right\rangle_i \) (no \( 's \) so as to distinguish from those in \( C^*(G, G/G_x) \) ).

Again, we have the hypothesis that \( \tau_1 \not\cong \tau_2 \) as representations of \( G_x \). Let

\[
L_1 = \text{ind}_{(G_x, \Omega)}^{(G, \Omega)}(\tau_1, \rho_x),
\]

and

\[
L_2 = \text{ind}_{(G_x, \Omega)}^{(G, \Omega)}(\tau_2, \rho_x).
\]

We define a collection of functions \( \{h_{n,j}\}_{n,j=1}^\infty, \{f_j\}_{j=1}^\infty \) in \( C_c(G, \Omega) \) in a special way. Define

\[
f_j(s, y) = \sum_{i=1}^l f_{1,i}(s) f_{2,i,j}(y),
\]
each \( f_{1,i} = f'_{1,i} \); we do not change these functions. Now we choose \( f_{2,i,j} \) to satisfy \( f_{2,i,j}(s \cdot x) = f'_{2,i}(s \cdot x') \), so \( f_{2,i,j} \) “behaves the same as \( f'_{2,i} \)” on the specific set \( C_2 \cdot x \), extend to all \( \Omega \), and require

\[
f_{2,i,j}(y) \xrightarrow{j \to \infty} \left\{ \begin{array}{ll}
    f'_{2,i}(s \cdot x') & \text{when } y = s \cdot x, \ s \in C_2 \\
    0 & \text{otherwise,}
\end{array} \right.
\]

see Lemma 6 for specifics. We note that we have
\[ \lim_{j \to \infty} f_j(s, y) = \begin{cases} 
 f'(s, s \cdot x') & \text{when } y = s \cdot x, \ s \in C_2 \\
 0 & \text{otherwise}, \end{cases} \]

Similarly use Lemma 6 to define a sequence \( \{h_{n,j}\}_{n,j=1}^{\infty} \) with (for each fixed \( n \))
\[ h_{n,j}(s, y) \xrightarrow{j \to \infty} \begin{cases} 
 h'_n(s, s \cdot x') & \text{when } y = s \cdot x, \ s \in C_2 \\
 0 & \text{otherwise}. \end{cases} \]

For each \( n \) and all \( j \) (fixed \( n \)) we may assume that the functions \( f_j \) and \( h_{n,j} \) have support contained in the set \( S^n_1 \times S^n_2, S^n_1 \) compact in \( G \), and \( S^n_2 \) compact in \( \Omega \); see again the proof of Lemma 6. We also (for each fixed \( n \)) choose them uniformly bounded for all \( j \).

Now we re-write our old calculations with these new functions and follow the same steps now in the \( C^* \) algebra \( C^*(G, \Omega) \):

\[ \langle h_{n,j} * f_j \otimes \xi_i, f_j \otimes \xi_i \rangle_i = \]

(1)
\[ \int_{r \in G_x} \left( \int_{s \in G} f_j(s, s \cdot x)(h_{n,j} * f_j)(sr, s \cdot x)d\mu_G(s) \right) \langle \tau_i(r)\xi_i, \xi_i \rangle_i d\mu_{G_x}(r). \]

Now we observe the integral in \( s \) is over a compact set as \( f_j \) is of compact support, and by the way that we choose the functions \( h_{n,j} * f_j \) for all \( j \) these have support contained in another compact set we denote \( K_1 \times K_2 \). So we must have \( sr \in K_1 \), and as \( s \in S^n_1 \) is forced already, we have \( r \in ((S^n_2)^{-1} \cdot K_1) \cap G_x \) is forced, and this is compact in \( G_x \).

Now we observe we may apply the Bounded Convergence Theorem to the above integral, these functions are converging in \( j \) and for each \( n \) have been
chosen in a bounded fashion with compact support, hence are easily bounded above by an integrable function. So for each fixed \( n \), as we limit \( j \to \infty \), formula (1) above converges to

\[
\begin{align*}
\langle h_n \ast f' \otimes \xi_1, f' \otimes \xi_1 \rangle'_{1} & \quad (i = 1) \\
\langle h_n \ast f' \otimes \xi_2, f' \otimes \xi_2 \rangle'_{2} & \quad (i = 2),
\end{align*}
\]

where we remind the reader that these inner products are in our \( C^* \) algebra \( C^*(G, G/G_x) \). Furthermore, we may assume (fixed \( n \)) that for all \( j \geq n \)

\[
\left| \langle h_{n,j} \ast f_j \otimes \xi_1, f_j \otimes \xi_1 \rangle_{1} - \langle h_n' \ast f'_j \otimes \xi_1, f'_j \otimes \xi_1 \rangle'_{1} \right| < \frac{1}{n},
\]

\[
\left| \langle h_{n,j} \ast f_j \otimes \xi_2, f_j \otimes \xi_2 \rangle_{2} - \langle h_n' \ast f'_j \otimes \xi_2, f'_j \otimes \xi_2 \rangle'_{2} \right| < \frac{1}{n},
\]

where we have mixed inner products of \( C^*(G) \) and \( C^*(G, G/G_x) \) in the above.

Now we may choose the diagonal sequence \( \{h_{j,j}\}_{j=1}^\infty \) and the sequence \( \{f_j\}_{j=1}^\infty \) in the above integrals, and for \( i = 1 \) the sequence converges to 0, for \( i = 2 \) it converges to 1. As \( \xi_1 \) was arbitrary, we now have \( L_1(h_{j,j}) \to 0 \) and \( L_2(h_{j,j}) \not\to 0 \), showing that \( L_1 \) and \( L_2 \) cannot have the same kernel, the desired result. \( \square \)

**Corollary 1.** Assume the pairs \( (f_1, x) \) and \( (f_2, x) \) give rise to irreducible representations \( (\tau_1, \rho_x) \) and \( (\tau_2, \rho_x) \) of \( C^*(G_x, \Omega) \). These induce to equivalent irreducible representations of \( C^*(G, \Omega) \) if and only if \( \tau_1 \) and \( \tau_2 \) are the equivalent irreducible representations of \( G_x \). This says that \( (f_1, x) \) and \( (f_2, x) \) are in the same equivalence class mod \( \sim_1 \), and \( f_1|_{g_x} \) and \( f_2|_{g_x} \) are in the same \( G_x \) orbit in \( g_x^* \); see Definition 10.
For the remainder of this section, we use Siegfried Echterhoff [7] as our main reference.

**Definition 11.** Let $G$ be our connected, simply-connected nilpotent Lie group, and let $\mathcal{K}(G)$ be the space of closed subgroups of $G$. Let $\mathcal{N}$ be a locally compact space. Assume $H: \mathcal{N} \rightarrow \mathcal{K}(G)$ and $H(i) = H_i$ is a continuous map from $\mathcal{N}$ to $\mathcal{K}(G)$. Define:

$$\mathcal{N}^H = \{(i, x) \in \mathcal{N} \times G \mid x \in H_i\}$$

In this paper, we use $\mathcal{N} = \mathbb{N} \cup \infty$, the one-point compactification of the natural numbers.

The following results are from [7].

**Definition 12.**

Now let $G$ again be a nilpotent Lie group and $(G, \Omega)$ denote a covariant system. We make the space $C_c(\mathcal{N}^H, C_0(\Omega))$ into a normed *-algebra. Define multiplication, involution, and norms by

$$f \ast g(i, t, x) = \int_{s \in H_i} f(i, s, x)g(i, s^{-1}t, s^{-1} \cdot x) \mathrm{d}\mu_{H_i}(s)$$

$$f^*(i, t, x) = \overline{f(i, t^{-1}, t^{-1} \cdot x)}$$

$$\|f\|_1 = \sup_{i \in \mathcal{N}} \int_{s \in H_i} \sup_{x \in \Omega} |f(i, s, x)| \mathrm{d}\mu_{H_i}(s)$$

for $f, g \in C_c(\mathcal{N}^H, C_0(\Omega))$.
**Definition 13.** Denote by $L^1(N^H, C_0(\Omega))$ the completion of $C_c(N^H, C_0(\Omega))$ with respect to the above norm. Any covariant representation $\tau$ of the system $(H_i, C_0(K_i, C_0(\Omega)))$ defines a $^*$ - representation of $L^1(N^H, C_0(\Omega))$ by

$$ (i, \tau)(F) = \tau(F_i) $$

**Definition 14.** The representation space

$$ \mathcal{R}(C^*(N^H, C_0(\Omega))) = \{(i, \rho) \mid i \in \mathcal{N}, \rho \in \text{Rep}(H_i, C_0(\Omega))\} $$

with the relative topology of $\text{Rep}(C^*(N^K, C_0(\Omega)))$ is the subgroup representation space of $C^*(N^K, C_0(\Omega))$.

Let $C^*(N^H, C_0(\Omega))$ be the enveloping $C^*$ algebra of $L^1(N^H, C_0(\Omega))$. We call $C^*(N^H, C_0(\Omega))$ the subgroup algebra of $(N^H, C_0(\Omega))$.

A standard technique in nilpotent harmonic analysis is induction from codimension one subgroups. The next proposition extends this idea from the above-defined algebras to our transformation group $C^*$ algebras.

**Proposition 3.** Let $G$ be a nilpotent Lie group acting upon the locally compact Hausdorff space $\Omega$.

1. Assume we have two continuous maps, $k(n)$ and $h(n)$, from $\mathcal{N} = \mathbb{N} \cup \infty$ to $K(G)$. Also assume that for all $n$, $h(n) = H_n$ is codimension 1 in $k(n) = K_n$.

2. Assume we have a collection of representations $\{(i, \pi_i)\}$ in the space $\mathcal{R}(C^*(N^K, C_0(\Omega)))$, and, for each $i \in \mathbb{N}$, $\pi_i$ is induced from $(H_i, \Omega)$. 

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Assume for each \( i \in \mathbb{N} \) that \( \pi_i \) is irreducible on \( C^*(K_i, \Omega) \), and that each \( \pi_i \) corresponds to a functional-point pair \((f_i, x_i)\).

(3) Assume in the space \( \mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega))) \) that \((i, \pi_i) \to (\infty, \pi_\infty)\).

Then,

(1) By passing to a sub-sequence, we may choose a collection \( \{\sigma_i\}_{i=1}^\infty \), with each \( \sigma_i \) an irreducible representation of \( C^*(H_i, \Omega) \), and \( \sigma_\infty \), an irreducible representation of \( (H_\infty, \Omega) \), these representations satisfying:

\[
ker(\pi_i) = ker(\text{ind}_{(H_i, \Omega)}^{(K_i, \Omega)}(\sigma_i)),
\]

\[
\pi_\infty \prec \text{Ind}_{(H_\infty, \Omega)}^{(K_\infty, \Omega)}(\sigma_\infty), \text{ and } (i, \sigma_i) \to (\infty, \sigma_\infty) \text{ in } \mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega))).
\]

(2) For each \( i \), the functional-point pairs (Definition 10) corresponding to \( \sigma_i \) may be chosen in the same equivalence class mod \( \sim_1 \) as \((f_i, x_i)\).

Proof.

By assumption, for each \( i \in \mathbb{N} \), \( \pi_i \) is induced from \((H_i, \Omega)\), so assume that \( \pi_i = \text{ind}_{(H_i, \Omega)}^{(K_i, \Omega)}(\sigma_i) \). By Corollary 1, we may assume that the functional-point pairs (Definition 10) corresponding to \( \sigma_i \) are the same as those corresponding to \( \pi_i \).

The restriction map \( \mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega))) \to \mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega))) \) is continuous (Proposition 7 of [7]), so we have:
\[ (i, \pi|_{(H_i, \Omega)}) \rightarrow_{i \to \infty} (\infty, \pi|_{(H_\infty, \Omega)}). \]

Note by [3], Lemma 1.1.8 that \( H(i) \) is normal in \( K(i) \). By [26], Proposition 7 on pg. 70, for each \( i \), the restricted representation is equivalent to the following direct integral:

\[ (i, \pi_i|_{(H_i, \Omega)}) = (i, \int_{s \in K_i/H_i} \sigma_i^s). \]

Prior to finishing the proof, we present several facts particular to connected, simply connected nilpotent Lie groups. We observe our group representations are not generally irreducible, but are induced from irreducible representations of subgroups, hence are direct integrals of irreducibles, as are their restrictions to subgroups.

**Fact 1:** Let \( \pi = \pi_f \) be an infinite-dimensional irreducible representation of \( G \). Let \( G_0 \) be a codimension one subgroup of \( G \), and let \( f_0 = f|_{g_0} \) the restriction of \( f \) to \( g_0 \). Let \( \pi_0 = \pi_{f_0} \) be the irreducible representation of \( G_0 \) associated by Kirillov theory to \( f_0 \) on \( g_0 \).

By [3], Theorem 2.5.3, the representation \( \pi|_{G_0} \) is either a representation that induces directly to \( \pi \), or is the following direct integral:

\[ \pi|_{G_0} = \int_{s \in G/H} (\pi_0)^s, \]

and \( \pi = \text{ind}_{G_0}^G (\pi_0) \).

Let \( \sigma = \text{ind}_{G_0}^G (\pi_0) \). We have one of the following:

a) \( \sigma \cong \pi \)
b) $\sigma \cong \infty \cdot \pi$

**Fact 2:** Recall Lemma 2.4 on pg. 338 of [26]:

**Lemma.** Let $A$ be a $C^*$ algebra. Let $\{I_\alpha\}_{\alpha \in \Lambda}$ be a net of ideals of $A$, converging to $I$. Assume that each $I_\alpha$ is the intersection of a set of primitive ideals $F_\alpha \in \mathcal{K}(\text{Prim}(A))$, and $F$ corresponds to $I$. Then, given any $F' \supseteq F$, there is a subnet of primitive ideals of $A$, $\{I_\beta\}_{\beta \in \Lambda'}$, such that there are $F'_\beta \in F_\beta$, with $\{F'_\beta\}_{\beta \in \Lambda'}$ converging to $F'$ in $\text{Prim}(A)$.

Now we return to the proof of our proposition.

Note that $\pi_\infty$ may be one-dimensional, hence not induced from $H_\infty$.

By the first of the above two facts, choose an irreducible representation $\sigma'_\infty$ of $C^*(H_\infty, \Omega)$ with $\sigma'_\infty \prec \pi_\infty|_{(H_\infty, \Omega)}$ and $\pi_\infty \prec \text{ind}^{(K_n, \Omega)}_{(H_n, \Omega)}(\sigma'_\infty)$.

We may assume that the functional-point pair for $\sigma'_\infty$ is in the same $\sim_1$ class as that of $\pi_\infty$.

By an application of the second fact above, we may pass to a sub-sequence, and choose $\{(n, \sigma'_n)\}_{n=1}^\infty$ with $(n, \sigma'_n) \to (n, \sigma'_\infty)$.

By an application of Theorem 2 and Corollary 1, for all $n \in \mathbb{N}$, we have $\sigma'_n \cong \sigma^s_n$ for some $s_n \in H_n$. So the functional-point pairs associated to $\sigma_n$ and $\sigma'_n$ are in the same equivalence class mod $\sim_1$. □

**Lemma 7.** Assume that $G$ is a nilpotent Lie group acting on a locally compact Hausdorff space $\Omega$.

Assume we have a continuous map $h(i) = H_i$ from $\mathcal{N} = \mathbb{N} \cup \infty$ to $K(G)$, and $H_i \to H_\infty$.

Assume we have a collection of one-dimensional representations $\{\{\pi_i \mid i \in \mathcal{N}\}\}$, $\pi_i \in \text{Rep}(C^*(H_i, \Omega))$, each $\pi_i = (\chi_{f_i}, \rho_{x_i})$. 24
Assume in the space $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$ that $(i, \pi_i) \to (\infty, \pi_\infty)$.

Then:

(a) $x_n \to x$, and

(b) by passing to a sub-sequence, we may choose another collection $\{f'_i\}_{i=1}^\infty$

with $\chi_{f'_i} = \chi_f$.

Furthermore, for each $i \in \mathbb{N}$, $(f'_i, x_i)$ may be chosen in the same $\sim_1$ class of $g^* \times \Omega$ as $(f_i, x_i)$.

Proof.

For any $\phi \in C_c(G)$ and $\psi \in C_c(\Omega)$, we may consider $\phi \cdot \psi \in C^*(\mathcal{N}^H, C_0(\Omega))$

by setting $(\phi \cdot \psi)(i) = \phi|_{H_i} \cdot \psi$. By hypothesis,

$$
\pi_i(\phi|_{H_i} \cdot \psi) = \psi(x_i) \int_{s \in H_i} \phi(s) \chi_{f'_i}(s) d\mu_{H_i}(s)
\to \psi(x) \int_{s \in H} \phi(s) \chi_f(s) d\mu_H(s) = \pi_\infty(\phi|_{H_\infty} \cdot \psi).
$$

Let $e$ denote the identity element of the group. By the continuity of $\text{Res}_e^{H_\infty}$, the restriction map from $\mathcal{R}(C^*(\mathcal{N}, C_0(\Omega)))$ to $\Omega$, (Proposition 7 of [7]),

we have for $\phi \in C_0(\Omega)$, $\text{Res}_e^{H_\infty}(\pi_i)(\phi) = \phi(x_i)$, and $\phi(x_i) \to \phi(x)$ for all $\phi \in C_0(\Omega)$; consequently $x_i \to x$.

Choose $\psi$ identical one in a neighborhood of $x \in \Omega$. In Fell’s subgroup-pair topology [12] we have:

$$
\langle \chi_{f_i}, H_i \rangle \to \langle \chi_f, H_\infty \rangle.
$$
Pass to a sub-sequence, and use Lemma 5 to find a sequence \( \{f'_i\}_{i=1}^\infty \subseteq g^* \) with \( \chi(f'_i|_{H_i}) = \chi(f_i|_{H_i}) \) and \( f'_i \to f_\infty \) in \( g^* \) and by Lemma 5, each \( f'_i \) is in the same \( H_i \)-orbit as \( f_i \) and \((f'_i, x_i)\) is in the same \( \sim_1 \) equivalence class of \( g^* \times \Omega \). □

Now for our “big” lemma, important in the next section.

**Lemma 8.** Assume that we have a collection \( \{\{F_i, \}_{i=1}^\infty, F\} \) of primitive ideals of \( C^*(G, \Omega) \), with

\[
F_i = \ker(L_i) = \ker(\text{ind}_{(G_{x_i}, \Omega)}^{G, \Omega}(\tau_{f_i, x_i}, \rho_{x_i})) \quad \longrightarrow F = \ker(L) = \ker(\text{ind}_{(G_x, \Omega)}^{G, \Omega}(\tau_{f, x}, \rho_{x})).
\]

By passing to a sub-sequence we may choose:

\[
\{y_i\}_{i=1}^\infty \subseteq \Omega \text{ and } \{f'_i\}_{i=1}^\infty \subseteq g^*, \text{ with } y_i \to y \in \Omega, f'_i \to f',
\]

with

\[
\ker(L_i) = \ker(\text{ind}_{(G_{y_i}, \Omega)}^{G, \Omega}(\tau_{f'_i, y_i}, \rho_{y_i}))
\]

and

\[
\ker(L) = \ker(\text{ind}_{(G_x, \Omega)}^{G, \Omega}(\tau_{f', x}, \rho_{x})).
\]

Furthermore, the functional-point pairs \((f'_i, y_i)\) corresponding to \( \tau'_i \) may be chosen in the same \( \sim \) equivalence class as \((f_i, x_i)\), and \((f', y)\) in the same \( \sim \) equivalence class as \((f, x)\).

**Proof.**
By Lemma 4.5 of [26] we may assume that \( x_i \to x \) and that the \( \sim \) equivalence classes don’t change.

If an infinite sub-sequence of \( \{L_i\}_{i=1}^{\infty} \) consists of one-dimensional representations of \( C^*(G, \Omega) \), we may use the constant sequence \( G_{x_i} = G \) for all \( i \) and Lemma 7.

If no infinite sub-sequence of \( \{L_i\}_{i=1}^{\infty} \) consists of one-dimensional representations, we may assume that each is induced from a codimension one subgroup \( H_i \). We pass to a sub-sequence and assume that for the sequence \( \{p_i\}_{i=1}^{\infty} \) of polarizing subalgebras, we have \( \dim(p_i) \) constant.

By compactness of \( K(G) \), we may pass to a sub-sequence and assume that \( G_{x_i} \to S \subseteq G_x \) and \( p_i \to p \), where \( p \) may not be polarizing for the action of \( f \) on \( g_x \). Assume that a representation at least weakly containing \( L \) is induced from \( (H, \Omega) \).

We may use Proposition 3 to successively reduce our problem by one dimension, until we do have a sequence of one-dimensional representations and Lemma 7 may be applied to these. \( \square \)
THE TOPOLOGY ON PRIM($C^*(G, \Omega)$)

Lemma 9. Let $x \in \Omega$, $f \in \mathfrak{g}^*$ be given.

Let $\mathfrak{p}_x$ be polarizing for the restriction of $f$ to $\mathfrak{g}_x$, and let $\mathfrak{p}$ be isotropic (not necessarily polarizing) for the restriction of $f$ to $\mathfrak{g}_x$. Then $L' = (V', \rho_x) = \text{ind}_{(P, \Omega)}^{(G, \Omega)}(\chi_{f,P}, \rho_x)$ weakly contains $L = (V, M) = \text{ind}_{(P_x, \Omega)}^{(G, \Omega)}(\chi_{f,P_x}, \rho_x)$.

Proof.

Let $\chi_{f,P}$ be the character of $P$ determined by $f$, and let $\chi_{f,P_x}$ be the corresponding character of $P_x$. By Lemma 4, on the level of stabilizer subgroups, we have $\text{ind}_{(P_x, \Omega)}^{(G_x, \Omega)}(\chi_{f,P_x}, \rho_x) \prec \text{ind}_{(P, \Omega)}^{(G, \Omega)}(\chi_{f,P}, \rho_x)$. As induction preserves weak containment (Proposition 9, [18]), the conclusion is clear from this and “induction in stages”; see Proposition 8, pg. 207 of [18]. □

Definition 14. Define $\phi : \mathfrak{g}^* \times \Omega \mapsto \text{Prim}(C^*(G, \Omega))$ by

$$\phi(f, x) = \ker(\text{ind}_{(G_x, \Omega)}^{(G, \Omega)}(\tau_{f,x}, \rho_x)).$$

Lemma 10. $\phi$ is continuous in the product topology of $\mathfrak{g}^* \times \Omega$.

Proof.
Assume in the product topology of $\mathfrak{g}^* \times \Omega$ that $(f_n, x_n) \to (f, x)$. Denote the sequence of stability subgroups as $\{G_{x_n}\}_{n=1}^\infty$; we assume that $G_{x_n} \to S \subseteq G_x$.

Let $\{p_n\}_{n=1}^\infty$ denote the sequence of polarizing subalgebras of $\mathfrak{g}_{x_n}$ for the sequence of functionals $\{f_n\}_{n=1}^\infty$. As $f_n \to f$, we may assume that $p_n \to p$, where $p$ may be of lower dimension than a polarizing subalgebra $p_x$ of $\mathfrak{g}_x$ for $f|_{\mathfrak{g}_x}$.

Denote by $g$ the restricted functional $f|_p$. Denote by $\sigma$ the representation of $C^*(P, \Omega)$ given by the obvious character $\chi_g$ of $P$ and a point evaluation $\rho_x$ of $\Omega$.

For each $n$ denote by $\pi_n$ the representation of $C^*(P_{x_n}, \Omega)$ given by the pair $(\chi_{f_n}, \rho_{x_n})$, and by $\pi$ the representation of $C^*(P_x, \Omega)$ given by the pair $(\chi_f, \rho_x)$.

Define $L_n = \text{ind}_{(G_{x_n}, \Omega)}^{(G, \Omega)}(\pi_n)$, an irreducible representation of $C^*(G, \Omega)$, and $L = \text{ind}_{(P_x, \Omega)}^{(G, \Omega)}(\rho_x)$, also an irreducible representation of $C^*(G, \Omega)$.

Let $L' = \text{ind}_{(P, \Omega)}^{(G, \Omega)}(\sigma)$ be induced from $C^*(P, \Omega)$; this may not be irreducible.

Amending Echterhoff [7], Proposition 6 on pg. 69 slightly for our purposes, we have:

Let $(G, \Omega)$ be a covariant system, $N$ be the locally compact space $\mathbb{N} \cup \infty$, and $P(n) = p_{x_n}$ be a continuous map with $P(n) = p_n \to p = P(\infty)$ in $\mathcal{K}(G)$. Then the map

$$\text{Ind}^G_P : \mathcal{R}(C^*(N^P, \Omega)) \mapsto C^*(G, \Omega); \quad (n, \pi_n) \mapsto (n, \text{ind}_{(P_n, \Omega)}^{(G, \Omega)}(\pi_n))$$
is continuous.

Using this, we have \( L_n \rightarrow L' \), but as \( L \prec L' \) (Lemma 9), we are done. \( \square \)

**Definition 15.** We say that \( C^*(G, \Omega) \) is **EH regular** if:

1. \( C^*(G, \Omega) \) is quasi-regular,
2. for every \( P \in \text{Prim}(C^*(G, \Omega)) \), there is an \( x \in \Omega \) and an irreducible representation \( \tau \) of \( G_x \) such that \( P = \ker(\text{ind}^{(G, \Omega)}_{(G_x, \Omega)}(\tau, \rho_x)) \).

This has been established in our case by Jon Rosenberg and Elliot Goelman [17].

Remember our equivalence relation \( \mathfrak{g}^* \times \Omega / \sim \), as well as the equivalence class \( \mathcal{O}_{(f, x)} \) of \((f, x)\) in \( \mathfrak{g}^* \times \Omega \); see Definition 10.

**Definition 16.** Using \( \sim \) from Definition 10, define \( \psi : \mathfrak{g}^* \times \Omega / \sim \rightarrow \text{Prim}(C^*(G, \Omega)) \) by \( \psi(f, x) = \ker(\text{ind}^{(G, \Omega)}_{(G_x, \Omega)}(\tau_{f, x}, \rho_x)) \). We show that \( \psi \) factors through \( \sim \) in the next lemma.

**Lemma 11.** The map \( \psi \) is one-to-one onto on \( \mathfrak{g}^* \times \Omega / \sim \).

**Proof.**

Onto is clear by EH regularity; we show that \( \psi \) is 1-1.

When \( \mathcal{O}_{(f, x)} = \mathcal{O}_{(h, y)} \), that the primitive ideals defined by \((f, x)\) and \((h, y)\) are the same follows from the continuity of the map \( \phi \) (Lemma 10) and the fact that primitive ideal spaces are \( T_0 \).

Now assume that \( \psi(f, x) = \psi(h, y) \).

As \( G \cdot x = G \cdot y \), by Lemma 4.5 [26], we may find a sequence \( \{g_n\}_{n=1}^\infty \subseteq G \) such that \( g_n \cdot x \rightarrow y \).

Define \( f_n = \text{Ad}^*(g_n)f \), and \( x_n = g_xn \).
We have: $\psi(f_n, x_n) = \psi(f, x)$, and the sequence $\psi(f_n, x_n)$ always remains in the $\sim_1$ equivalence class of $(f, x)$, and $\psi(f_n, x_n) = \psi(h, y)$.

If a sub-sequence of the sequence $\psi((f_n, x_n))$ consists of kernels of one-dimensional representations of $C^*(G, \Omega)$, we may use Lemma 7 to get an equivalent sequence $(f_n, x_n)$ converging to $(h, y)$ with each $(f_n, x_n)$ still in the equivalence class of $(f, x)$.

Thus we may assume there is no sub-sequence of one-dimensional representations.

Note that $\{\psi(f_n, x_n)\}_{n=1}^{\infty}$ is a constant sequence of ideals equal to $\psi(h, y)$.

In Lemma 8 we showed that if we had a collection of primitive ideals, $
\{\{F_n, F\}_{n=1}^{\infty}\}$ of $C^*(G, \Omega)$, with

$$F_n = \ker(\text{ind}_{(G_x, \Omega)}^{(G, \Omega)}(\tau_{f_n, x_n}, \rho_{x_n})) \to F = \ker(\text{ind}_{(G, \Omega)}^{(G, \Omega)}(\tau_{f, x}, \rho_x)),$$

that, passing to a sub-sequence, we could choose:

$$\{y_n\}_{n=1}^{\infty} \subseteq \Omega \text{ and } \{f'_n\}_{n=1}^{\infty} \subseteq g^*, \text{ with } y_n \to y \in \Omega, f'_n \to f',$$

with $F_n = \ker(\text{ind}_{(G_{y_n}, \Omega)}^{(G, \Omega)}(\tau_{f'_n, y_n}, \rho_{y_n}))$ and $F = \ker(\text{ind}_{(G_y, \Omega)}^{(G, \Omega)}(\tau_{f'_y, \rho_y})).$ We may also choose the functional-point pairs corresponding to $\tau_{f'_n}$ and $\tau_{f'}$ in the same $\sim_1$ equivalence class as those corresponding to $\pi_n$.

Thus by choosing appropriate functional-point pairs in the equivalence class of $(f, x)$ we may realize $(h, y)$ as a subsequential limit, and the equivalence class closures are equal. \square

Now to characterize the primitive ideal space of $C^*(G, \Omega)$.
**Theorem 3.** The map $\psi$ is a homeomorphism from $g^* \times \Omega/ \sim$ to $\text{Prim}(C^*(G, \Omega))$.

**Proof.**

We follow the philosophy of Dana P. Williams’ Theorem 5.3 of [26].

We have the diagram:

$g^* \times \Omega \xrightarrow{q} g^* \times \Omega/ \sim \xrightarrow{\psi} \text{Prim}(C^*(G, \Omega))$

As $\phi$ (Lemma 10) and the natural map $q$ of $g^* \times \Omega$ to $g^* \times \Omega/ \sim$ are continuous, $\psi$ is continuous. By the last lemma $\psi$ is 1-1 onto.

Let $F$ be closed in $g^* \times \Omega$ and saturated with respect to $\sim$. We show that $\psi(F)$ is closed in $\text{Prim}(C^*(G, \Omega))$.

Assume $\{A_n\}_{n=1}^\infty \subseteq \psi(A)$ and $A_n \to A$. By EH regularity, assume that $A_n = \ker(L_n) = \ker(\text{ind}_{(G_{x_n}, \Omega)}^{(G, \Omega)}(\tau_{f_n,x_n}, \rho_{x_n}))$.

Pass to a sub-sequence, apply Lemma 8 to choose $\{f'_n\}_{n=1}^\infty \subseteq g^*, \{y_n\}_{n=1}^\infty \subseteq \Omega$

with $f'_n \to f$, $y_n \to x$, and

$$\ker(L_n) = \ker(\text{ind}_{(G_{y_n}, \Omega)}^{(G, \Omega)}(\tau_{f'_n,y_n}, \rho_{y_n})).$$

As $(f'_n, y_n) \to (f, x)$ in $g^* \times \Omega$, and $F$ is saturated and closed, we are done. □
CHAPTER 3

TRACES OF IRREDUCIBLE REPRESENTATIONS OF $C^*(G, \Omega)$

We give a character theory for $C^*(G, \Omega)$ with $G$ nilpotent Lie, analogous to Kirillov’s character theory. Our primary reference is Corwin and Greenleaf [3].

Section 3.1

Schwartz functions on $G$

Definition 17.

In $\mathbb{R}^n$, define multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$. Assume $\alpha_i \in \mathbb{N}_0^n$ and $\beta_i \in \mathbb{N}_0^n$, $\forall i$.

Let $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$, and $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$, a “polynomial coefficient differential operator” by $x^\beta D^\alpha$.

On $\mathbb{R}^n$, the Schwartz functions $S(\mathbb{R}^n)$ are those $C^\infty$ functions $f$ such that

$$\|x^\beta D^\alpha f\|_\infty < \infty,$$

for all multi-indices $\alpha, \beta$. The natural topology of $S(\mathbb{R}^n)$ is determined by these seminorms. Denote the polynomial coefficient differential operators on $\mathbb{R}^n$ by $\mathcal{P}(\mathbb{R}^n)$, and let $L \in \mathcal{P}$ be arbitrary.
On $G$, define $\mathcal{S}(G)$, the Schwartz functions on $G$, to be the $C^\infty$ functions on $G$ such that $\|L(f)\|_\infty$ is bounded for all $L \in \mathcal{P}(\mathbb{R}^n)$.

**Section 3.2**

**Traces on $C^\ast(G)$**

For $\pi$ a unitary infinite-dimensional irreducible representation of a nilpotent Lie group $G$, the operators $\pi(x)$ have no trace. However, for any $\pi \in \hat{G}$, there is a tempered distribution $\theta_\pi$ on $G$ that plays the role of the classical trace character $\theta_\pi(g) = \text{Tr}(\pi(g))$ for finite and compact groups. For $\pi \in \mathcal{S}(G)$, the Schwartz functions on $G$, the operator

$$\pi(\phi)(\xi) = \int_{s \in G} \phi(s)\pi(s)\xi \, d\mu_G(s) \quad (\xi \in H_\pi)$$

turns to be trace class; To wit:

**Theorem 4.** Let $\pi = \pi_l$ be an irreducible representation of a nilpotent Lie group, let $m$ be a polarization for $l$, and model $\pi$ in $L^2(\mathbb{R}^k)$ using any weak Malcev basis through $m$. If $\phi \in \mathcal{S}(G)$, then $\pi(\phi)$ is trace class and

$$\pi_\phi f(s) = \int_{\mathbb{R}^k} K_\phi(s,t)f(t) \, dt, \text{ for all } f \in L^2(\mathbb{R}^k)$$

where $K_\phi \in \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^k)$. Furthermore,

$$\theta_\pi(\phi) = \text{Tr}(\pi(\phi)) = \int_{\mathbb{R}^k} K_\phi(s,s) \, ds \text{ (absolutely convergent)}$$

and the functional $\theta_\pi$ is a tempered distribution on $\mathcal{S}(G)$.

**Proof.** See [3], Theorem 4.2.1, page 133. \qed
Definition 18. We may obtain explicit formulas for the kernel integral $K_\phi$, once an $f \in g^*$, a polarization $p$, and a weak Malcev basis ([3], Theorem 1.1.13, pg. 10) through $p$ are specified. Let $n = \dim(g)$. Let $P = \exp(p)$; assume $\dim(g/p) = k$. If $\{X_1, \ldots, X_n\}$ is the weak Malcev basis, let $p = n - k = \dim(p)$. Define polynomial maps $\gamma : \mathbb{R}^n \mapsto G$, $\alpha : \mathbb{R}^p \mapsto P$, $\beta : \mathbb{R}^k \mapsto G/P$ by

$$\gamma(s,t) = \exp(s_1 X_1) \cdots \exp(s_p X_p) \cdot \exp(t_1 X_{p+1}) \cdots \exp(t_k X_n)$$

$$\alpha(s) = \gamma(s,0), \quad \beta(t) = \gamma(0,t).$$

Let $d\mu_G$, $d\mu_P$, $d\mu_{G/P}$ be the invariant measures on $G$, $P$ and $G/P$ determined by Lebesgue measures $ds\,dt, ds, dt$, as in Theorems 1.2.10, 1.2.12 and 1.2.13 of [3].

We describe $\text{Tr}(\pi(\phi))$ in terms of integrals over coadjoint orbits in $g^*$. Given a Euclidean measure $dX$ on $g$, normalize measures on $g$ and $g^*$ so that Fourier inversion holds, and define the Euclidean Fourier transforms $\hat{h}$ (resp. $\mathcal{F}h$) of functions $h$ on $G$ (resp. $g$), as in [3], pg. 137.

Each coadjoint orbit $\mathcal{O}_f = \text{Ad}^*(G)f$ has an invariant measure $\mu$ that is unique up to scalar multiple, as $\mathcal{O}_f \cong G/R_f$, where

$$R_f = \text{Stab}_G(f) = \{x \in G \mid \text{Ad}^*(x)f = f\}$$

Section 3.3

Traces of irreducible representations of $C^*(G, \Omega)$
Here we characterize some operators on $C^*(G, \Omega)$ which are trace class. Assume $G$ is a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$.

Recall that we use left actions.

**Proposition 4.** Let $\phi \in \mathcal{S}(G)$, $\psi \in C_0(\Omega)$. Let $L = (V, M)$ be an irreducible representation of $C^*(G, \Omega)$ corresponding to $(f, x) \in \mathfrak{g}^* \times \Omega$. Let $k = \dim(\mathfrak{g}/\mathfrak{p}_x)$, where $\mathfrak{p}_x$ is polarizing for the action of $f$ in $\mathfrak{g}_x$. Assume the Hilbert space of $L$ is $L^2(\mathbb{R}^k)$. For $\psi \in C_0(\Omega)$ with $\psi(\cdot, x)|_{G/G_x} \in \mathcal{S}(G/G_x)$ and $\phi \in \mathcal{S}(G)$, then $L(\phi \cdot \psi)$ has kernel $K$,

$$K(r, s) = \psi(\exp(r) \cdot x) \int_{t \in \mathfrak{p}_x} \phi(\beta(r)t\beta(s)^{-1})e^{i \cdot f(\log(t))}d\mu_g(t).$$

**Proof.**

For $\phi \in \mathcal{S}(G)$, $\psi \in C_0(\Omega)$ such that $\psi(\cdot, x)|_{G/G_x} \in \mathcal{S}(G/G_x)$, $h \in L^2(\mathbb{R}^k)$ we have

$$(L(\phi \cdot \psi)h)(\exp(r)) = \psi(\exp(r) \cdot x) \int_{s \in \mathfrak{g}} \phi(\exp(s))h(\exp(s)^{-1}\exp(r))d\mu_g(s) =$$

(letting $s \to s^{-1}$, and our measure doesn’t change)

$$\psi(\exp(r) \cdot x) \int_{s \in \mathfrak{g}} \phi(\exp(s^{-1}))h(\exp(s)\exp(r))d\mu_g(s) =$$

(letting $s \to sr^{-1}$)

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\[
\psi(\exp(r) \cdot x) \int_{s \in \mathfrak{g}} \phi(\exp(r)(\exp(s^{-1}))h(\exp(s))d\mu_{\mathfrak{g}} = \\
\text{(splitting } \mathfrak{g} \text{ into } \mathfrak{g}/\mathfrak{p} \text{ and } \mathfrak{p}:)
\]

\[
\psi(\exp(r) \cdot x) \int_{s \in \mathfrak{g}/\mathfrak{p}} \int_{t \in \mathfrak{p}} \phi(\exp(r)(\exp(t^{-1})\exp(s^{-1}))h(\exp(s)\exp(t))d\mu_{\mathfrak{g}} = \\
\psi(\exp(r) \cdot x) \int_{s \in \mathfrak{g}/\mathfrak{p}} \int_{t \in \mathfrak{p}} \phi(\exp(r)(\exp(t^{-1})\exp(s^{-1}))e^{i \cdot f(t)}h(\exp(s))d\mu_{\mathfrak{g}}
\]

We now let

\[
K(r, s) = \psi(\exp(r) \cdot x) \int_{t \in \mathfrak{p}} \phi(\beta(r)\exp(t)\beta(s)^{-1})e^{i \cdot f(t)} d\mu_{\mathfrak{p}}(t),
\]

and the result is clear. \(\square\)

**Definition 19**

Let \(H\) be a Hilbert space; for some \(k \in \mathbb{N}\), we assume that \(H \cong L^2(\mathbb{R}^k)\).

For a function \(\psi \in C_0(\mathbb{R}^k)\), we denote the multiplication operator by \(\psi\) as \(M_\psi\).

**Lemma 12.** Let \(\rho\) be an irreducible representation of a subgroup \(H\) of \(G\), corresponding to the restriction of a functional \(f \in \mathfrak{g}^*\) to \(\mathfrak{h}\). Defining \(\pi = \text{ind}_{\mathfrak{h}}^{\mathfrak{g}}(\rho)\), we have that the operator \(T_{\psi, \phi} = M_\psi \cdot \pi(\phi)\) is trace class with a Schwartz kernel when \(\psi \in S(\mathbb{G}/H)\) and \(\phi \in S(G)\). Also, for fixed \(\psi\), \(T_{\psi, \phi}\) is tempered in \(\phi\).

**Proof.**

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We do this by induction on the codimension of \( H \) in \( G \). If \( \text{codim}(H) = 0 \), this is true by [3], Theorem 4.2.1.

Assume the lemma true for \( \text{codim}(H) = n \).

By [3], Theorem 1.1.3 we find a subgroup \( G_1 \subseteq G \) with \( \text{codim}(G_1) = 1 \) and \( H \subseteq G_1 \). The subgroup \( H \) is codimension \( n \) in \( G_1 \), and \( G_1 \) is normal in \( G \) by [3], Lemma 1.1.8.

Assume the Lie algebra of \( G_1 \) is \( g_1 \). Let \( g = g_1 \oplus (\mathbb{R} - \text{span}\{X\}) \); we have a smooth cross section for \( G/G_1 \) by \( \exp(\mathbb{R} \cdot X) \cong \mathbb{R} \).

Define \( \tau = \text{ind}^{G}_H(\rho) \), acting on the Hilbert space \( H_\tau \).

Assume \( \pi = \text{ind}^{G}_{G_1}(\tau) \) acts on \( H_\pi = L^2(\mathbb{R}) \otimes H_\tau \).

Let \( a \in G/G_1 \), \( f \in C^\infty(H_\pi) \). By \( f(a) \) we denote the element of \( H_\tau \) corresponding to \( f(a) \). For \( z \in G_1 \), by \( f(a)(z) \) we refer to the value of the \( H_\tau \)-valued function \( f(a) \) in \( H_\tau \) at the point \( z \) in \( G_1 \).

Let \( \phi_1 \in S(G/G_1) \), \( \phi_2 \in S(G_1) \), \( \psi_1 \in S(G/G_1) \), and \( \psi_2 \in S(G_1/H) \). Note that sums of elements of the form \( \phi_1 \cdot \phi_2 \) are dense in \( S(G) \), and that sums of the form \( \psi_1 \cdot \psi_2 \) are dense in \( S(G/H) \).

Let \( r \in G/G_1 \). Let \( f \in C^\infty(H_\pi) \). We have

\[
(\rho_{\psi_1 \cdot \psi_2} \pi(\phi_1 \cdot \phi_2)f)(r) = \rho_{\psi_1 \cdot \psi_2} \int_{s \in G/G_1} \int_{t \in G_1} \phi_1(s)\phi_2(t)\pi(s)\pi(t)f(r)ds\,dt = \rho_{\psi_1 \cdot \psi_2} \int_{s \in G/G_1} \int_{t \in G_1} \phi_1(s)\phi_2(t)\pi(s)f(t^{-1}r)ds\,dt = \]

(Note that \( r^{-1}tr \in G_1 \) by normality)

\[
\rho_{\psi_1 \cdot \psi_2} \int_{s \in G/G_1} \int_{t \in G_1} \phi_1(s)\phi_2(t)\pi(s)\tau(r^{-1}tr)f(r)ds\,dt = \]

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\[
\rho_{\psi_1 \psi_2} \int_{s \in G/G_1} \int_{t \in G_1} \phi_1(s)\phi_2(t)\tau(r^{-1}sts^{-1}r)f(s^{-1}r)dt\,ds = \\
\text{(Let } t \to s^{-1}rtr^{-1}s \text{ in the inner integral and set } s^{-1}r = a) \\
\]

(1) \[
\rho_{\psi_1} \int_{a \in G/G_1} \phi_1(ra^{-1}) \left[ \rho_{\psi_2} \int_{t \in G_1} \phi_2(ata^{-1})\tau(t)dt \right] f(a)\,da.
\]

Note that for fixed \( a \), by induction hypothesis the operator in the brackets of (1) above is a trace class operator on \( H_r \), and for fixed \( \psi_2 \), is tempered in \( \phi \). Employ [3], Proposition 1.2.8 and for some selection of polynomials \( \{P_i\}_{i=1}^n \) and at a fixed point \( y \in G_1 \) the inner integral of formula (1) equals

\[
\left( \rho_{\psi_2} \int_{t \in G_1} \phi_2(P_1(a,t), \ldots, P_n(a,t))\tau(t)f(a)dt \right)(y) = \\
\int_{z \in G_1} k_{\phi_2}^a(z, y)f(a)(z)dz.
\]

By the inductive hypothesis, we may find the integral kernel \( k_{\phi_2}^a(z, y) \) and it may be chosen Schwartz in \( y \) and \( z \).

Recall that for fixed \( a \), \( \text{Tr}(k_{\phi_2}^a) \) is tempered in \( \phi_2 \) by inductive hypothesis. By [27], Corollary 1, pg. 43 we know that there exists a seminorm \( p \) on \( G_1 \) such and a constant \( C \) such that that \( |\text{Tr}(k_{\phi_2}^a)| \leq C \cdot p(\phi_2^a) \), where \( \phi_2^a \) is defined as obvious.

Assume that we have multi-indices \( \alpha, \beta \in \mathbb{N}_0^n \), with \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \beta = (\beta_1, \ldots, \beta_n) \); define \( L = w_1^{\beta_1} \cdots w_n^{\beta_n} \cdot \frac{\partial^{(|\alpha|)}}{\partial w_1^{\alpha_1} \cdots \partial w_n^{\alpha_n}} \)
And we have for the seminorm $p$:

$$|p(\phi^a)| = \sup_{w \in \mathbb{R}^n} \{ |L(\phi_2 (P_1(a, w), \ldots, P_n(a, w)))| \} \leq |Q(a)|,$$

by Proposition 1.2.9 of [3] and the $n$ dimensional chain rule, where $Q$ is some polynomial.

So $\text{Tr}(k^a_{\phi_2}) = \int_{y \in G_1} k^a_{\phi_2}(y, y)dy$ grows no faster than polynomial in $a$, and $k^a_{\phi_2}$ grows no faster that polynomial in $a$.

For $a$ in a bounded set, $\phi_2(ata^{-1})$ is bounded by a $L^1$ function. Let $p_x$ be a polarizing subalgebra of $g_x$ with respect to the restriction of $f$ to $g_x$. By Proposition 4 we have:

$$k^a_{\phi_2}(r, s) = \psi_2(\exp(r) \cdot x) \int_{t \in P_x} \phi_2(\beta(r) \cdot ata^{-1} \cdot \beta(s)^{-1}) e^{i \cdot f(\log(t))} d\mu(t),$$

and we may differentiate in $a$ under the integral sign; infinite differentiability of $k^a_{\phi_2}$ in $a$ follows.

Choose a smooth splitting of $G$ into $G/G_1$ and $G_1$ by projections $p : G \mapsto G/G_1$ and $q : G \mapsto G_1$.

For our functions $\phi_1, \phi_2, \psi_1$ and $\psi_2$, define the integral kernel $K$ of $\rho_{\psi_1 \cdot \psi_2} \cdot \pi(\phi_1 \cdot \phi_2)$ by

$$K(x, y) = \rho_{\psi_1}(p(y)) \cdot \phi_1(p(y)p(x^{-1})) \cdot k^p(y)(q(x), q(y)).$$

Integrated against $f \in H_\pi$, this gives us $\rho_{\psi_1 \cdot \psi_2} \cdot \pi(\phi_1 \cdot \phi_2)(f)$:

$$(\rho_{\psi_1 \cdot \psi_2} \cdot \pi(\phi_1 \cdot \phi_2)f)(y) = (Kf)(y) = \int_{x \in G} K(x, y)f(x)dx =$$
\[
\rho \psi_1(p(y)) \int_{x \in G} \phi_1(p(y)p(x^{-1})) k_{\phi_2}^p(q(x), q(y)) f(x) dx = \\
\rho \psi_1(p(y)) \int_{a \in G/G_1} \int_{z \in G_1} \phi_1(p(y)a^{-1}) k_{a}^\phi(z, q(y)) f(a) (z) dz \, da.
\]

Note \(k_{\phi_2}^a\) is Schwartz on \(G_1\), and its integral and the integrals of its derivatives grow no faster than polynomial in \(a\).

Note \(G/G_1 \cong \mathbb{R}\); treat composition on \(G/G_1\) as addition on \(\mathbb{R}\). The functions \(\psi_1\) and \(\phi_1\) are both Schwartz on \(G/G_1\), and part of the integral kernel is \(\rho \psi_1(p(y)) \cdot \phi_1(p(y)p(x^{-1}))\). The other part \(k_{\phi_2}^a\) is Schwartz already.

By Peetre’s Inequality ([14], pg. 10), when \(\psi\) and \(\phi\) are both Schwartz on \(\mathbb{R}\), \(\psi(y)\phi(y + a)\) is Schwartz on \(\mathbb{R} \times \mathbb{R}\).

Similar properties follow for derivatives, and \(\psi(y)\phi(y + a)\) is Schwartz on \(\mathbb{R}^2\).

Our entire integral kernel is Schwartz, and the operator \(\rho \psi_1 \cdot \psi_2 \cdot \pi(\phi_1 \cdot \phi_2)\) is trace class by [3] Theorem A.3.9.

The kernel \(K\) is clearly tempered in \(\phi_1 \in \mathcal{S}(G/G_1)\); the final result follows by induction and density arguments. \(\square\)

Henceforth assume that \(\Omega/G\) is a \(T_0\) space. By Theorem 2.1 [8], \((G, \Omega)\) is Polish, and for any \(x \in \Omega\), we have \(G \cdot x \cong G/G_x\).

**Definition 20.** For any orbit \(G \cdot x\), define

\[\mathcal{A}_{G \cdot x} = \mathbb{R} - \text{span}\{ \phi \cdot \psi \mid \phi \in \mathcal{S}(G), \psi \in C_0(\Omega) \text{ and } \psi(\cdot \, x)|_{G/G_x} \in \mathcal{S}(G/G_x) \}\].
Theorem 5. Let $L = (V, M)$ be the irreducible representation of $C^*(G, \Omega)$ associated to the pair $(f, x) \in \mathfrak{g}^* \times \Omega$. The representation $L$ is trace class on $A_{G, \Omega}$, and for fixed $\psi$, is tempered in $\phi$.

Proof.

Note $L(\psi \cdot \phi) = M_\psi(\cdot, x) \cdot V(\phi)$, and apply Lemma 12 just proven. □

Definition 21. Remember Definition 18, where we defined maps $\alpha, \beta, \gamma$ of $\mathfrak{p}, \mathfrak{g}/\mathfrak{p}$, and $\mathfrak{g}$ (resp.) to $G$. Assume $f \in \mathfrak{g}^*$, and for $x \in \Omega$, $\mathfrak{g}_x$ is the Lie algebra of $G_x$. Assume $\dim(\mathfrak{g}_x) = l$. Define a new map $\delta : \mathbb{R}^l \mapsto G_x$ by $\delta(s) = \gamma(s_1, \ldots, s_l, 0)$.

Theorem 6. Let $L$ be a representation of $C^*(G, \Omega)$ corresponding to the functional-point pair $(f, x) \in \mathfrak{g}^* \times \Omega$.

Let $\mathfrak{p}_x$ be a polarizing subalgebra of $\mathfrak{g}_x$ with respect to $f|_{\mathfrak{g}_x}$; assume $k = \dim(\mathfrak{g}/\mathfrak{p}_x)$.

Assume $L$ acts on the Hilbert space $L^2(\mathbb{R}^k)$.

Let $O_L$ be the $\sim_1$ equivalence class of $(f, x)$ in $\mathfrak{g}^* \times \Omega$, specifically,

$$O_{(f, x)} = \{(l, y) \in \mathfrak{g}^* \times \Omega \mid \text{for some } s \in G, \text{we have}$$

$$l = Ad^*(s)f + h, \ y = s \cdot x, \ h \in \mathfrak{g}_x^\perp \}.$$  

Let $p, q$ be the natural projections from $O_L$ to $\mathfrak{g}^*$ and $\Omega$, respectively. Let $\phi \in \mathcal{S}(G), \ \psi(\cdot, x) \in \mathcal{S}(G/G_x)$. We have

$$\text{Tr}(L(\phi \cdot \psi)) = \int_{z \in O_L} \psi(q(z)) \hat{\phi}(p(z)) dz$$

for a particular choice of $G$-invariant measure $dz$ on $O_L$. 42
Proof.

We closely mimic the proof of Theorem 4.2.4 on pp. 138-41 of [3]. Assume that $p_x$ has dimension $p$, so $n = \dim(g) = p + k$.

We give $g$ a standard basis realization on $L^2(\mathbb{R}^k)$. By Proposition 4 above, and [3], Theorem A.3.9, we have

$$\text{Tr}(L) = \int_{s \in g/p_x} K(s, s) \text{d}s =$$

$$\int_{s \in g/p_x} \psi(\beta(s) \cdot x) \int_{t \in p_x} \phi(\beta(s)\exp(t)\beta(s)^{-1})e^{i \cdot f(t)} \text{d}\mu_{p_x}(t) \text{d}\mu_{g/p_x}(s).$$

(1)

Let $j$ be a subspace complementary to $p_x$ in $g$. It is easy to take $j = \mathbb{R}$-span of the last $k$ basis vectors of our Malcev basis through $p_x$. We have an additive splitting $H + X \in p_x \oplus j$ for each element in $g$. Let $dX, dH$ be arbitrarily assigned Euclidean measures on $j, p_x$; then we have that $dH \, dX$ is a Euclidean measure on $g$, which we use to define the above integrals.

For $\phi \in S(G), \ u \in \mathbb{R}^k \cong j \cong g_x/p_x$, we define

$$\phi_u(H, X) = \phi(\beta(u)\exp(H + X)\beta(u)^{-1}).$$

For each fixed $u$, this is a Schwartz function on $\mathbb{R}^p \times \mathbb{R}^k \cong p_x \times j$. Viewing $j^* \cong p_x^\perp$, define the Fourier transform

$$\mathcal{F}_\phi(f^\perp) = \int_{X \in j} \phi(X)e^{i \cdot f^\perp(X)} \text{d}f^\perp \quad \text{for } \phi \in S(Z), \ f^\perp \in p_x^\perp.$$
When the measures are suitably normalized, this is an $L^2$ isometry with inverse
\[
(F^{-1} \phi)(X) = \int_{f \perp \in p_x^\perp} \phi(f) e^{-i \cdot f(X)} df, \\
\]
taking the dual Euclidean measure $dl$ on $p_x^\perp$. We have
\[
\phi_u(H, 0) = (F^{-1} F \phi_u)(H, 0) = \int_{f \perp \in \mathbb{R}^\perp} e^{-i \cdot f(0)} F \phi_u(H, f) df = \\
\]
(2) \[
\int_{f \perp \in p_x^\perp} \int_{X \in z} \phi_u(H, X) e^{i \cdot f(X)} dX df < \infty
\]
We want to insert equation 1 into equation 2 and interchange some integrals, this will be done introducing and removing an ad hoc function which enables us to use Fubini’s Theorem. Let $\{w_j\}_{j=1}^\infty$ be a collection of functions on $p_x^\perp$ with $0 \leq w_j(f) \leq 1$, and $w_j \uparrow 1$ uniformly on compacta in $p_x^\perp$ and for all $j$,
\[
\int_{f \perp \in p_x^\perp} w_j(f) df < \infty
\]
As $F \phi_u(H, f)$ is Schwartz in both variables (remember that the Fourier transform of a Schwartz function is also Schwartz; see Proposition 4 of [27]), by dominated convergence we have
\[
\text{Tr}(L) = \int_{u \in g/p_x} \psi(\beta(u) \cdot x) \int_{H \in p_x} e^{i \cdot f(H)} \phi_u(H, 0) dH du = \\
\]
\[
\int_{u \in g/p_x} \psi(\beta(u) \cdot x) \int_{H \in p_x} \left[ \lim_{j \to \infty} \int_{f \perp \in p_x^\perp} \mathcal{F} \phi_u(H, f) w_j(f) df \right] e^{i \cdot f(H)} dH du = \\
\]
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\[
\int_{u \in \mathbb{G}/\mathbb{P}_x} \psi(\beta(u) \cdot x) \left[ \lim_{j \to \infty} \int_{H \in \mathbb{P}_x} \int_{f^\perp \in \mathbb{P}_x^\perp} \int_{X \in \mathfrak{z}} e^{i(f(H) + f^\perp(X))} \phi_u(H, X)w_j(f^\perp) \right. \\
\left. \quad \, dX \, df^\perp \, dH \right] du.
\]

Fubini’s Theorem may be applied to the innermost triple integral, when we re-arrange, we get

\[
\text{Tr}(L) = \\
\int_{u \in \mathbb{G}/\mathbb{P}_x} \psi(\beta(u) \cdot x) \left[ \lim_{j \to \infty} \int_{f^\perp \in \mathbb{P}_x^\perp} \int_{\{H, X\} \in \mathbb{P}_x \oplus \mathfrak{z}} e^{i(f(H) + f^\perp(X))} \phi_u(H, X) \right. \\
\left. \quad \, dX \, dH \, df^\perp \right] du = \\
\int_{u \in \mathbb{G}/\mathbb{P}_x} \psi(\beta(u) \cdot x) \left[ \int_{f^\perp \in \mathbb{P}_x^\perp} \int_{\{H, X\} \in \mathbb{P}_x \oplus \mathfrak{z}} e^{i(f(H) + f^\perp(X))} \phi_u(H, X) dX \, dH \, df^\perp \right] du,
\]

by the Dominated Convergence Theorem, as the integral over \( \mathbb{P}_x \oplus \mathfrak{z} \) is Schwartz in \( f^\perp \). The integral over \( \mathbb{P}_x^\perp \) amounts to integration over \( \mathfrak{z}^* \); by translation invariance in this integral, we may replace \( f^\perp(X) \) with \((f + f^\perp)(X)\). As \( f(H) = (f + f^\perp)(H) \) for all \( f^\perp \in \mathbb{P}_x^\perp \), all \( H \in \mathbb{P}_x \), we may write

\[
\text{Tr}(L) = \\
\int_{u \in \mathbb{G}/\mathbb{P}_x} \psi(\beta(u) \cdot x) \int_{f^\perp \in \mathbb{P}_x^\perp} \int_{\{H, X\} \in \mathbb{P}_x \oplus \mathfrak{z}} e^{i(f+f^\perp)(H)}e^{i(f+f^\perp)(X)} \\
\phi(\beta(u)\exp(H + X)\beta(u)^{-1}) \, dX \, dH \, df^\perp \, du = \\
\int_{u \in \mathbb{G}/\mathbb{P}_x} \psi(\beta(u) \cdot x) \int_{f^\perp \in \mathbb{P}_x^\perp} \int_{Y \in \mathbb{G}} e^{i(f+f^\perp)(Y)} \phi(\beta(u)\exp(Y)\beta(u)^{-1}) \, dY \, df^\perp \, du =
\]

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(letting $Y \to \text{Ad}(\beta(u^{-1}))(Y)$ in the last integral above)

$$
\int_{u \in \mathfrak{g}/\mathfrak{p}_x} \psi(\beta(u) \cdot x) \int_{f^+ \in \mathfrak{p}_x^+} \int_{Y \in \mathfrak{g}} e^{i(f + f^+)(\text{Ad}(\beta(u^{-1}))(Y))} \phi(\exp(Y))dY df^+ du = \\
\int_{u \in \mathfrak{g}/\mathfrak{p}_x} \psi(\beta(u) \cdot x) \int_{f^+ \in \mathfrak{p}_x^+} \int_{Y \in \mathfrak{g}} e^{i(\text{Ad}^*(\beta(u))(Y)f + f^+)} \phi(\exp(Y))dY df^+ du = \\
(3) \int_{u \in \mathfrak{g}/\mathfrak{p}_x} \psi(\beta(u) \cdot x) \int_{f^+ \in \mathfrak{p}_x^+} \tilde{\phi}(\text{Ad}^*(\beta(u)^{-1})(f + f^+)) df^+ du.
$$

We now split $\mathfrak{g}/\mathfrak{p}_x$ into $\mathfrak{g}/\mathfrak{g}_x$ and $\mathfrak{g}_x/\mathfrak{p}_x$, and as $\beta(t) \cdot x = x$ for all $t \in \mathfrak{g}_x/\mathfrak{p}_x$, and we get that formula (3) above equals

$$
\int_{u \in \mathfrak{g}/\mathfrak{g}_x} \psi(\beta(u) \cdot x) \int_{v \in \mathfrak{g}_x/\mathfrak{p}_x} \int_{f^+ \in \mathfrak{p}_x^+} \tilde{\phi}(\text{Ad}^*(\beta(u))\text{Ad}^*(\beta(v))(f + f^+)) df^+ dv du = \\
(4) \int_{u \in \mathfrak{g}/\mathfrak{g}_x} \psi(\beta(u) \cdot x) \int_{v \in \mathfrak{g}_x/\mathfrak{p}_x} \int_{f_1^+ \in \mathfrak{g}_x^+} \int_{f_2^+ \in \mathfrak{p}_x^+ / \mathfrak{g}_x^+} \tilde{\phi}(\text{Ad}^*(\beta(u))\text{Ad}^*(\beta(v))(f + f_1^+ + f_2^+)) df_1^+ df_2^+ dv du.
$$

We note that by using the integral in $f_2^+$ that we may now assume that $f$ is nonzero only on $\mathfrak{g}_x$.

We now work with the integral in $f_1^+$. If $R_f$ is the stabilizer of the functional $f$ restricted to $\mathfrak{g}_x$, there exists invariant measures $d\hat{p}$ and $d\hat{x}$ on $\mathfrak{p}_x/R_f$ and $\mathfrak{g}_x/\mathfrak{p}_x$ such that $d\hat{p}d\hat{x}$ is invariant measure on $\mathfrak{g}_x/R_f$. We know from
Proposition 3.1.18 of [3] that \( \text{Ad}^*(P_x)(f) = f + p_x^+/g_x^+ = (f + p_x^+)|_{g_x}, \) and that the natural diffeomorphism \( \Delta : P_x/R_f \mapsto \text{Ad}^*(P_x)(f) = (f + p_x^+)|_{g_x} \) is equivariant and measure preserving on \( (f + p_x^+) \\) (see Proposition 3.1.18 of [3]). We now note that

\[
q \circ \text{Ad}^*(p)(f_1^+) = q(\text{Ad}^*(p)(f_1^+) + (\text{Ad}^*(p)f - f)) = \text{Ad}^*(p)(f + f_1^+).
\]

As the linear part \( \text{Ad}^*(p)|_{p_x^+/g_x^+} \) of \( A(p) \) is unipotent, the operator \( A(p) \) preserves \( df_1^+ \), and by formula (1) above, \( \text{Ad}^*(p) \) preserves \( \nu \) on the affine space \( f + p_x^+/g_x^+ \). As \( \text{Ad}^*(p) \) is also measure preserving on \( P_x/R_f \), we have that under \( \Delta \), \( \nu \) is identified with an invariant measure on \( P_x/R_f \), which must be a scalar multiple of \( d\hat{\rho} \): \( (\Delta^{-1})^*\nu = c \cdot d\hat{\rho} \). Hence if \( \phi \in L^1(f + p_x^+/g_x^+) \) we must have

\[
\int_{f_1^+ \in p_x^+/g_x^+} \phi(f + f_1^+) df_1^+ = \int_{f' \in f + p_x^+/g_x^+} \phi(f') d\nu(f') = c \cdot \int_{\hat{\rho} \in P_x/R_f} \phi(\text{Ad}^*(\hat{\rho})f) d\hat{\rho}.
\]

Consequently, formula (4) above equals

\[
\int_{u \in g/g_x} \int_{v \in g_x/p_x} \int_{f_2^+ \in g_x^+} \int_{m \in p_x/r_f} \phi(\text{Ad}^*(\beta(u))\text{Ad}^*(\delta(v))(\text{Ad}^*(\alpha(m))f + f_2^+)) \cdot \frac{d\hat{\rho}}{dm} \cdot df_2^+ \cdot dv \cdot du =
\]

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(by Haar measure we move the $\text{Ad}^*(\alpha(m))$ to include $f_2^\perp$, we note that $\text{Ad}^*(\alpha(m))f_2^\perp$ is still zero on $g_x$)

$$
c \cdot \int_{u \in g / g_x} \int \int \hat{\phi}(\text{Ad}^*(\beta(u))\text{Ad}^*(\delta(v))(\text{Ad}^*(\alpha(m))(f + f_2^\perp)))
\hspace{2cm} dm \, df_2^\perp \, dv \, du =
$$

(now combining $v$ and $m$ into a single variable $y$)

$$
c \cdot \int_{u \in g / g_x} \int_{f_2^\perp \in g_2^\perp} \int \hat{\phi}(\text{Ad}^*(\beta(u))\text{Ad}^*(\delta(y))(f + f_2^\perp))dy \, df_2 \, du.
$$

This is our orbital integral. □
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